BCOV invariant and blow-up

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Abstract

Bershadsky, Cecotti, Ooguri and Vafa constructed a real-valued invariant for Calabi–Yau manifolds, which is now called the BCOV invariant. In this paper, we extend the BCOV invariant to such pairs \((X,D)\), where \(X\) is a compact Kähler manifold and \(D\) is a pluricanonical divisor on \(X\) with simple normal crossing support. We also study the behavior of the extended BCOV invariant under blow-ups. The results in this paper lead to a joint work with Fu proving that birational Calabi–Yau manifolds have the same BCOV invariant.

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Introduction

In this paper, we consider a real-valued invariant for Calabi–Yau manifolds equipped with Ricci flat metrics, which is now called the BCOV torsion. The BCOV torsion was introduced by...
Bershadsky, Cecotti, Ooguri and Vafa [BCOV93, BCOV94] as the stringy genus-one partition function of $N = 2$ superconformal field theory. Their work extended the mirror symmetry conjecture of Candelas, de la Ossa, Green and Parkes [COGP91]. Fang and Lu [FL05] used BCOV torsion to study the moduli space of Calabi–Yau manifolds.

The BCOV torsion is an invariant on the B-side. Its mirror on the A-side is conjecturally the genus-one Gromov–Witten invariant. Though genus $\geq 2$ Gromov–Witten invariants have been intensively studied recently, there is no rigorously defined genus $\geq 2$ invariant on the B-side.

The BCOV invariant is a real-valued invariant for Calabi–Yau manifolds, which could be viewed as a normalization of the BCOV torsion. Fang, Lu and Yoshikawa [FLY08] constructed the BCOV invariant for Calabi–Yau threefolds and established the asymptotics of the BCOV invariant (of Calabi–Yau threefolds) for one-parameter normal crossings degenerations. They also confirmed the (B-side) genus-one mirror symmetry conjecture of Bershadsky, Cecotti, Ooguri and Vafa [BCOV93, BCOV94] for quintic threefolds.

Eriksson, Freixas i Montplet and Mourougane [EFM21] constructed the BCOV invariant for Calabi–Yau manifolds of arbitrary dimension and established the asymptotics of the BCOV invariant for one-parameter normal crossings degenerations. In another paper [EFM22], they confirmed the (B-side) genus-one mirror symmetry conjecture of Bershadsky, Cecotti, Ooguri and Vafa [BCOV93, BCOV94] for Calabi–Yau hypersurfaces of arbitrary dimension, which is compatible with the results of Zinger [Zin08, Zin09] on the A-side.

For a Calabi–Yau manifold $X$, we denote by $\tau(X)$ the logarithm of the BCOV invariant of $X$ defined in [EFM21].

Yoshikawa [Yos06, Conjecture 2.1] conjectured that for a pair of birational projective Calabi–Yau threefolds $(X, X')$, we have $\tau(X') = \tau(X)$. Eriksson, Freixas i Montplet and Mourougane [EFM21, Conjecture B] conjectured the following higher-dimensional analogue.

**Conjecture 0.1.** For a pair of birational projective Calabi–Yau manifolds $(X, X')$, we have

$$\tau(X') = \tau(X). \quad (0.1)$$

Let $X$ and $X'$ be projective Calabi–Yau threefolds defined over a field $L$. Let $T$ be a finite set of embeddings $L \hookrightarrow \C$. For $\sigma \in T$, we denote by $X_\sigma$ (respectively, $X'_\sigma$) the base change of $X$ (respectively, $X'$) to $\C$ via the embedding $\sigma$. We denote by $D^b(X_\sigma)$ (respectively, $D^b(X'_\sigma)$) the bounded derived category of coherent sheaves on $X_\sigma$ (respectively, $X'_\sigma$). Maillot and Rössler [MR12, Theorem 1.1] showed that if one of the following conditions holds:

(a) there exists $\sigma \in T$ such that $X_\sigma$ and $X'_\sigma$ are birational;
(b) there exists $\sigma \in T$ such that $D^b(X_\sigma)$ and $D^b(X'_\sigma)$ are equivalent;

then there exist a positive integer $n$ and a non-zero element $\alpha \in L$ such that

$$\tau(X'_\sigma) - \tau(X_\sigma) = \frac{1}{n} \log |\sigma(\alpha)| \quad \text{for all } \sigma \in T. \quad (0.2)$$

Although a result of Bridgeland [Bri02, Theorem 1.1] showed that condition (a) implies condition (b), Maillot and Rössler gave separate proofs for conditions (a) and (b).

Let $X$ be a Calabi–Yau threefold. Let $Z \hookrightarrow X$ be a $(-1, -1)$-curve. Let $X'$ be the Atiyah flop of $X$ along $Z$, which is also a Calabi–Yau threefold. We assume that both $X$ and $X'$ are compact and Kähler. The current author [Zha22, Corollary 0.5] showed that

$$\tau(X') = \tau(X). \quad (0.3)$$
In other words, Conjecture 0.1 holds for three-dimensional Atiyah flops. The proof of (0.3) consists of two key ingredients:

(i) we extend the BCOV invariant from Calabi–Yau manifolds to certain ‘Calabi–Yau pairs’, more precisely, we consider manifolds equipped with smooth reduced canonical divisors;
(ii) we study the behavior of the extended BCOV invariant under blow-ups.

To fully confirm Conjecture 0.1 following this strategy, it is necessary to further extend the BCOV invariant as well as the blow-up formula. This is exactly the purpose of this paper. We consider pairs consisting of a compact Kähler manifold and a canonical divisor with rational coefficients on the manifold with simple normal crossing support and without component of multiplicity \( \leq -1 \). We construct the BCOV invariant of such pairs and establish a blow-up formula for our BCOV invariant.

In the joint work with Fu [FZ20], we use the results in this paper together with a factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [AKMW02, Theorem 0.3.1] to confirm Conjecture 0.1 in full generality.

Let us now give more detail about the matter of this paper.

**BCOV torsion.** We use the notation in (0.23) and (0.24). Let \( X \) be an \( n \)-dimensional compact Kähler manifold. Let \( H^{p,\bullet}_{\text{dR}}(X) \) be the de Rham cohomology of \( X \). Let \( H^k_{\text{dR}}(X) = \bigoplus_{p+q=k} H^{p,q}(X) \) be the Hodge decomposition. Set

\[
\lambda_p(X) = \det H^{p,\bullet}(X) = \bigotimes_{q=0}^n (\det H^{p,q}(X))^{(-1)^q} \quad \text{for } p = 0, \ldots, n,
\]

\[
\lambda_{\text{tot}}(X) = \bigotimes_{k=1}^{2n} (\det H^k_{\text{dR}}(X))^{(-1)^k} = \bigotimes_{p=1}^n \left( \lambda_p(X) \otimes \overline{\lambda_p(X)} \right)^{(-1)^p}.
\]

(0.4)

Let \( H^{k}_{\text{Sing}}(X, \mathbb{C}) \) be the singular cohomology of \( X \) with coefficients in \( \mathbb{C} \). We identify \( H^k_{\text{dR}}(X) \) with \( H^k_{\text{Sing}}(X, \mathbb{C}) \) (see (1.121)). For \( k = 0, \ldots, 2n \), let

\[
\sigma_{k,1}, \ldots, \sigma_{k,b_k} \in \text{Im}(H^k_{\text{Sing}}(X, \mathbb{Z}) \to H^k_{\text{Sing}}(X, \mathbb{R})) \subseteq H^k_{\text{dR}}(X)
\]

be a basis of the lattice. Set

\[
\sigma_X = \bigotimes_{k=1}^{2n} (\sigma_{k,1} \wedge \cdots \wedge \sigma_{k,b_k})^{(-1)^k} \in \lambda_{\text{tot}}(X),
\]

(0.6)

which is well-defined up to \( \pm 1 \).

Let \( \omega \) be a Kähler form on \( X \). Let \( \| \|_{\lambda_p(X),\omega} \) be the Quillen metric (see §1.4) on \( \lambda_p(X) \) associated with \( \omega \). Let \( \| \|_{\lambda_{\text{tot}}(X),\omega} \) be the metric on \( \lambda_{\text{tot}}(X) \) induced by \( \| \|_{\lambda_p(X),\omega} \) via (0.4). Set

\[
\tau_{\text{BCOV}}(X, \omega) = \log \| \sigma_X \|_{\lambda_{\text{tot}}(X),\omega},
\]

(0.7)

which we call the unnormalized BCOV invariant of \((X, \omega)\).

**BCOV invariant.** For a compact complex manifold \( X \) and a divisor \( D \) on \( X \), we denote

\[
D = \sum_{j=1}^l m_j D_j,
\]

(0.8)

where \( m_j \in \mathbb{Z} \setminus \{0\} \), \( D_1, \ldots, D_l \subset X \) are mutually distinct and irreducible. We call \( D \) a divisor with simple normal crossing support if \( D_1, \ldots, D_l \) are smooth and transversally intersect. Let \( d \)
be a non-zero integer. We assume that \( D \) is of simple normal crossing support and \( m_j \neq -d \) for \( j = 1, \ldots, l \). For \( J \subseteq \{1, \ldots, l\} \), we denote
\[
\begin{align*}
  w^J_d &= \prod_{j \in J} \frac{-m_j}{m_j + d}, \\
  D_J &= X \cap \bigcap_{j \in J} D_j, \\
  w^\emptyset_d &= 1, \\
  D^\emptyset &= X.
\end{align*}
\]
(0.9)

See [FZ20, §4] for an interpretation of this construction.

Now let \( X \) be a compact Kähler manifold. Let \( K^d_X \) be the canonical line bundle over \( X \). Let \( \gamma \in \mathcal{M}(X, K^d_X) \) be an invertible element.

**Definition 0.2.** We call \((X, \gamma)\) a \(d\)-Calabi–Yau pair if:

(i) \( \text{div}(\gamma) = \sum_{j=1}^l m_j D_j \) is of simple normal crossing support;

(ii) \( m_j \neq -d \) for \( j = 1, \ldots, l \).

Here are some examples of \(d\)-Calabi–Yau pairs.

(a) If \( X \) is a compact Kähler Calabi–Yau manifold and \( 0 \neq \gamma \in H^0(X, K^d_X) \), then \((X, \gamma)\) is a \(d\)-Calabi–Yau pair.

(b) If \((X, \gamma)\) is a \(d\)-Calabi–Yau pair with \( d > 0 \) and \( Y \subseteq X \) transversally intersects with \( \text{div}(\gamma) \) in the sense of Definition 1.1, then \((\text{Bl}_Y X, f^* \gamma)\) is a \(d\)-Calabi–Yau pair, where \( f : \text{Bl}_Y X \to X \) is the blow-up along \( Y \).

Now we assume that \((X, \gamma)\) is a \(d\)-Calabi–Yau pair. Let \( w^J_d \) and \( D_J \) be as in (0.9). Let \( \omega \) be a Kähler form on \( X \). Recall that \( \tau_{\text{BCOV}}(\cdot, \cdot) \) was constructed in (0.7). The BCOV invariant of \((X, \gamma)\) is defined as
\[
\tau_d(X, \gamma) = \sum_{J \subseteq \{1, \ldots, l\}} w^J_d \tau_{\text{BCOV}}(D_J, \omega|_{D_J}) + \text{correction terms},
\]
(0.10)
where the correction terms are Bott–Chern-type integrations (see Definition 3.2 and (3.10)). We construct \( \tau_d(X, \gamma) \) and show that it is independent of \( \omega \).

We can further extend our construction to canonical divisors with rational coefficients. We consider a pair \((X, D)\), where \( X \) is an \( n \)-dimensional compact Kähler manifold, \( D \) is a canonical divisor with rational coefficients on \( X \) such that:

(i) \( D \) is of simple normal crossing support;

(ii) each component of \( D \) is of multiplicity \( > -1 \).

**Definition 0.3.** Let \( d \) be a positive integer such that \( dD \) is a divisor with integer coefficients. Let \( \gamma \) be a meromorphic section of \( K^d_X \) such that \( \text{div}(\gamma) = dD \). We define
\[
\tau(X, D) = \tau_d(X, \gamma) + \frac{\chi_d(X, dD)}{12} \log \left( (2\pi)^{-n} \int_{X \setminus |D|} \|\gamma\|^{1/d} \right),
\]
(0.11)
where \( \chi_d(\cdot, \cdot) \) is defined in Definition 1.3, \( |D| \) is defined in (0.25), \( \|\gamma\|^{1/d} \) is the unique positive volume form on \( X \setminus |D| \) whose \( d \)th tensor power equals \( i^{n^2d} \gamma \). By Propositions 3.3, 3.4, the BCOV invariant \( \tau(X, D) \) is well-defined, i.e. independent of \( d \) and \( \gamma \).

Our BCOV invariant differs from the one defined in [EFM21] by a topological invariant. More precisely, if \( X \) is a Calabi–Yau manifold, the logarithm of the BCOV invariant of \( X \)
defined in [EFM21] is equal to
\[ \tau(X, \emptyset) + \frac{\log(2\pi)}{2} \sum_{k=0}^{2n} (-1)^k k(n-k)b_k(X), \tag{0.12} \]
where \( b_k(X) \) is the \( k \)th Betti number of \( X \). The sum of Betti numbers in (0.12) comes from our choice of the \( L^2 \)-metric (see (1.70)) and the identification between singular cohomology and de Rham cohomology (see (1.121)).

**Curvature formula.** Let \( \pi : \mathcal{X} \to S \) be a holomorphic submersion. We assume that \( \pi \) is locally Kähler in the sense of [BGS88b, Definition 1.25], i.e. for any \( s \in S \), there exists an open subset \( s \in U \subseteq S \) such that \( \pi^{-1}(U) \) is Kähler. For \( s \in S \), we denote \( X_s = \pi^{-1}(s) \).

Let \( \left( \gamma_s \in \mathcal{M}(X_s, K^d_{X_s}) \right)_{s \in S} \) be a holomorphic family. We assume that \( \left( X_s, \gamma_s \right) \) is a \( d \)-Calabi–Yau pair for any \( s \in S \). We assume that there exist \( l \in \mathbb{N}, m_1, \ldots, m_l \in \mathbb{Z} \setminus \{0, -d\} \) and \( \left( D_{j,s} \subseteq X_s \right)_{j \in \{1, \ldots, l\}, s \in S} \) such that
\[ \text{div}(\gamma_s) = \sum_{j=1}^{l} m_j D_{j,s} \quad \text{for} \ s \in S. \tag{0.14} \]

For \( J \subseteq \{1, \ldots, l\} \) and \( s \in S \), let \( D_{J,s} \subseteq X_s \) be as in (0.9) with \( X \) replaced by \( X_s \) and \( D_j \) replaced by \( D_{j,s} \). We assume that \( \left( D_{J,s} \right)_{s \in S} \) is a smooth holomorphic family for each \( J \).

Let \( \tau_d(X, \gamma) \) be the function \( s \mapsto \tau_d(X_s, \gamma_s) \) on \( S \). Let \( w^J_d \) be as in (0.9). Let \( H^\bullet(D_j) \) be the variation of Hodge structure associated with \( \left( D_{J,s} \right)_{s \in S} \). Let \( \omega_{H^\bullet(D_j)} \in \Omega^{1,1}(S) \) be its Hodge form (see [Zha22, §1.2]).

**Theorem 0.4.** The following identity holds:
\[ \frac{\partial \partial}{2\pi i} \tau_d(X, \gamma) = \sum_{J \subseteq \{1, \ldots, l\}} w^J_d \omega_{H^\bullet(D_j)}. \tag{0.15} \]

**Blow-up formula.** Let \( (X, \gamma) \) be a \( d \)-Calabi–Yau pair in the sense of Definition 0.2 with \( d > 0 \).

Let \( Y \subseteq X \) be a connected complex submanifold such that \( Y, D_1, \ldots, D_l \) transversally intersect (in the sense of Definition 1.1). We assume that \( m_j > 0 \) for \( j \) satisfying \( Y \subseteq D_j \). Let \( r \) be the codimension of \( Y \subseteq X \). Let \( q \) be the number of \( D_j \) containing \( Y \). Then we have \( q \leq r \). Without loss of generality, we assume that
\[ Y \subseteq D_j \quad \text{for} \ j = 1, \ldots, q; \ Y \not\subseteq D_j \quad \text{for} \ j = q + 1, \ldots, l. \tag{0.16} \]

Let \( f : X' \to X \) be the blow-up along \( Y \). Let \( D'_j \subseteq X' \) be the strict transformation of \( D_j \subseteq X \). Set \( E = f^{-1}(Y) \). Let \( f^*\gamma \in \mathcal{M}(X', K_{X'}) \) be the pull-back of \( \gamma \). We denote \( D' = \text{div}(f^*\gamma) \). We denote
\[ m_0 = m_1 + \cdots + m_q + rd - d. \tag{0.17} \]
We have (cf. [MM07, Proposition 2.1.11])
\[ D' = m_0 E + \sum_{j=1}^{l} m_j D'_j. \tag{0.18} \]
Hence, \( (X', f^*\gamma) \) is a \( d \)-Calabi–Yau pair.
Set

\[ D_Y = \sum_{j=q+1}^{l} m_j(D_j \cap Y), \quad D_E = \sum_{j=1}^{l} m_j(D'_j \cap E). \] (0.19)

Then \( D_Y \) (respectively, \( D_E \)) is a divisor on \( Y \) (respectively, \( E \)) with simple normal crossing support.

We identify \( \mathbb{C}P^r \) with \( \mathbb{C} \cup \mathbb{C}P^{r-1} \). Let \((z_1, \ldots, z_r) \in \mathbb{C}^r \) be the coordinates. Let \( \gamma_{r,m_1,\ldots,m_q} \in \mathcal{M}(\mathbb{C}P^r, K_{\mathbb{C}P^r}^d) \) be such that

\[ \gamma_{r,m_1,\ldots,m_q}|_{\mathbb{C}^r} = (dz_1 \wedge \cdots \wedge dz_r)^d \prod_{j=1}^{q} z_j^{m_j}. \] (0.20)

Let \( H_k \subseteq \mathbb{C}P^r \) be the closure of \( \{z_k = 0\} \subseteq \mathbb{C}^r \). Let \( H_\infty = \mathbb{C}P^{r-1} \subseteq \mathbb{C}P^r \). We have

\[ \text{div}(\gamma_{r,m_1,\ldots,m_q}) = -(m_1 + \cdots + m_q + rd + d)H_\infty + \sum_{j=1}^{q} m_j H_j. \] (0.21)

Thus, \((\mathbb{C}P^r, \gamma_{r,m_1,\ldots,m_q})\) is a \( d \)-Calabi–Yau pair.

**Theorem 0.5.** The following identities hold:

\[ \chi_d(X', f^*\gamma) - \chi_d(X, \gamma) = 0, \]
\[ \tau_d(X', f^*\gamma) - \tau_d(X, \gamma) = \chi_d(E, D_E)\tau_d(\mathbb{C}P^1, \gamma_{1,m_0}) - \chi_d(Y, D_Y)\tau_d(\mathbb{C}P^r, \gamma_{r,m_1,\ldots,m_q}), \] (0.22)

where \( \chi_d(\cdot, \cdot) \) is given by Definition 1.3.

The proof of Theorem 0.5 is based on:

(i) the deformation to the normal cone introduced by Baum, Fulton and MacPherson [BFM75, §1.5];
(ii) the immersion formula for Quillen metrics due to Bismut and Lebeau [BL91];
(iii) the submersion formula for Quillen metrics due to Berthomieu and Bismut [BB94];
(iv) the blow-up formula for Quillen metrics due to Bismut [Bis97];
(v) the relation between the holomorphic torsion and the de Rham torsion established by Bismut [Bis04].

We remark that the Quillen metric can be extended to orbifolds, and the immersion formula and the submersion formula still hold (see [Ma05, Ma21]).

**Notation.** For a complex vector space \( V \), we denote

\[ \det V = \Lambda^\dim V, \] (0.23)

which is a complex line. For a complex line \( \lambda \), we denote by \( \lambda^{-1} \) the dual of \( \lambda \). For a graded complex vector space \( V^\bullet = \bigoplus_{k=0}^{m} V^k \), we denote

\[ \det V^\bullet = \bigotimes_{k=0}^{m} (\det V^k)(-1)^k. \] (0.24)

For a complex manifold \( X \) and a divisor \( D = m_1 D_1 + \cdots + m_l D_l \) on \( X \), where \( m_1, \ldots, m_l \in \mathbb{Z}\setminus\{0\} \), \( D_1, \ldots, D_l \) are mutually distinct and irreducible, we denote

\[ |D| = D_1 \cup \cdots \cup D_l \subseteq X, \] (0.25)

which we call the support of \( D \).
For a complex manifold $X$, we denote by $\Omega^{p,q}(X)$ the vector space of $(p,q)$-forms on $X$. We denote by $\mathcal{O}_X$ the analytic coherent sheaf of holomorphic functions on $X$. We denote by $\Omega^p_X$ the analytic coherent sheaf of holomorphic $p$-forms on $X$. For a complex vector bundle $E$ over $X$, we denote by $\Omega^{p,q}(X,E)$ the vector space of $(p,q)$-forms on $X$ with values in $E$. We denote by $\mathcal{O}_X(E)$ the analytic coherent sheaf of holomorphic sections of $E$. For an analytic coherent sheaf $F$ on $X$, we denote by $H^q(X,F)$ the $q$th cohomology of $F$. We denote $H^q(X,E) = H^q(X,\mathcal{O}_X(E))$. We denote $H^p(X) = H^p(X,\Omega^p_X)$. We denote by $\mathcal{O}_X^\ast$ the analytic coherent sheaf of holomorphic functions on $X$.

1. Preliminaries

1.1 Divisor with simple normal crossing support

For $I \subseteq \{1, \ldots, n\}$, we denote
\[
\mathbb{C}^n_I = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i = 0 \text{ for } i \in I\} \subseteq \mathbb{C}^n. \tag{1.1}
\]

Let $X$ be an $n$-dimensional complex manifold.

**Definition 1.1.** For closed complex submanifolds $Y_1, \ldots, Y_l \subseteq X$, we say that $Y_1, \ldots, Y_l$ transversally intersect if for any $x \in X$, there exists a holomorphic local chart $\mathbb{C}^n \supseteq U \xrightarrow{\varphi} X$ such that:

(i) $0 \in U$ and $\varphi(0) = x$;

(ii) for each $k$, either $\varphi^{-1}(Y_k) = \emptyset$ or $\varphi^{-1}(Y_k) = U \cap \mathbb{C}^n_{I_k}$ for certain $I_k \subseteq \{1, \ldots, n\}$.

Let $D$ be a divisor on $X$. We denote
\[
D = \sum_{j=1}^l m_j D_j, \tag{1.2}
\]
where $m_j \in \mathbb{Z}\setminus\{0\}$, $D_1, \ldots, D_l \subseteq X$ are mutually distinct and irreducible.

**Definition 1.2.** We call $D$ a divisor with simple normal crossing support if $D_1, \ldots, D_l$ are smooth and transversally intersect.

For $J \subseteq \{1, \ldots, l\}$, let $w^J_d$ and $D_J$ be as in (0.9), let $\chi(D_J)$ be the topological Euler characteristic of $D_J$.

**Definition 1.3.** If $D$ is a divisor with simple normal crossing support, we define
\[
\chi_d(X,D) = \sum_{J \subseteq \{1, \ldots, l\}} w^J_d \chi(D_J). \tag{1.3}
\]

Moreover, if there is a meromorphic section $\gamma$ of a holomorphic line bundle over $X$ such that $\text{div}(\gamma) = D$, we define
\[
\chi_d(X,\gamma) = \chi_d(X,D). \tag{1.4}
\]

Now we assume that $D$ is a divisor with simple normal crossing support. Let $L$ be a holomorphic line bundle over $X$ together with $\gamma \in \mathcal{M}(X,L)$ such that $\text{div}(\gamma) = D$. Let $\gamma^{-1} \in \mathcal{M}(X,L^{-1})$ be the inverse of $\gamma$. 

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We denote by \((T^*X \oplus T^*X)^{\otimes k}\) the kth tensor power of \(T^*X \oplus T^*X\). We denote

\[ E_k^\pm = (T^*X \oplus T^*X)^{\otimes k} \otimes L^\pm. \]

In particular, we have \(E_0^\pm = L^\pm\). Let \(\nabla_{E_k^\pm}\) be a connection on \(E_k^\pm\).

Let \(L_j\) be the normal line bundle of \(D_j \hookrightarrow X\).

**Definition 1.4.** We define \(\text{Res}_{D_j}(\gamma) \in \mathcal{M}(D_j, L \otimes L_j^{-m_j})\) as follows:

\[
\text{Res}_{D_j}(\gamma) = \begin{cases} 
\frac{1}{m_j!} \left( \nabla_{E_j}^{m_j-1} \cdots \nabla_{E_0}^\gamma \right) |_{D_j} & \text{if } m_j > 0, \\
\frac{1}{|m_j|!} \left( \left( \nabla_{E_j}^{m_j} \cdots \nabla_{E_0}^{\gamma-1} \right) |_{D_j} \right)^{-1} & \text{if } m_j < 0.
\end{cases}
\]

Here \(\text{Res}_{D_j}(\gamma)\) is independent of \((\nabla_{E_k^\pm})_{k \in \mathbb{N}}\).

For \(j \in \{1, \ldots, l\}\), we have

\[
\text{div}(\text{Res}_{D_j}(\gamma)) = \sum_{k \in \{1, \ldots, l\} \setminus \{j\}} m_k(D_j \cap D_k).
\]

For distinct \(j, k \in \{1, \ldots, l\}\), we have

\[
\text{Res}_{D_j \cap D_k}(\text{Res}_{D_j}(\gamma)) = \text{Res}_{D_j \cap D_k}(\text{Res}_{D_k}(\gamma)) \\
\in \mathcal{M}(D_j \cap D_k, L \otimes L_j^{-m_j} \otimes L_k^{-m_k}).
\]

**1.2 Some characteristic classes**

For an \((m \times m)\)-matrix \(A\), we define

\[
\text{ch}(A) = \text{Tr}[e^A], \quad \text{Td}(A) = \det \left( \frac{A}{\text{Id} - e^{-A}} \right), \quad c(A) = \det(\text{Id} + A).
\]

We have

\[
c(tA) = 1 + \sum_{k=1}^m t^k c_k(A),
\]

where \(c_k(A)\) is the kth elementary symmetric polynomial of the eigenvalues of \(A\).

Let \(V\) be an \(m\)-dimensional complex vector space. Let \(R \in \text{End}(V)\). Let \(V^*\) be the dual of \(V\). Let \(R^* \in \text{End}(V^{\ast})\) be the dual of \(R\). For \(r = 1, \ldots, m\), we construct \(R_r \in \text{End}(\Lambda^r V^*)\) by induction,

\[
R_1 = -R^*, \quad R_r = R_1 \wedge \text{Id}_{\Lambda^{r-1} V^*} + \text{Id}_{V^*} \wedge R_{r-1}.
\]

We use the convention \(\Lambda^0 V^* = \mathbb{C}\) and \(R_0 = 0\).

Let \(\lambda_1, \ldots, \lambda_m\) be the eigenvalues of \(R\). For \(p \in \mathbb{N}\) and \(F\) a polynomial of \(\lambda_1, \ldots, \lambda_m\), we denote by \(\{F\}_p\) the component of \(F\) of degree \(p\).
Proposition 1.5. The following identities hold:

\[ Td(R) \left( \sum_{r=0}^{m} (-1)^r \text{ch}(R_r) \right) = c_m(R), \]

\[ \left\{ Td(R) \left( \sum_{r=1}^{m} (-1)^r r \text{ch}(R_r) \right) \right\}^{[\leq m]} = -c_{m-1}(R) + \frac{m}{2} c_m(R), \]  

\[ \left\{ Td(R) \left( \sum_{r=2}^{m} (-1)^r r(r-1) \text{ch}(R_r) \right) \right\}^{[m]} = \frac{1}{6} (c_1 c_{m-1})(R) + \frac{m(3m - 5)}{12} c_m(R). \]  

Proof. Note that the eigenvalues of \( R_r \) are given by \(((-1)^r \lambda_{j_1} \cdots \lambda_{j_r})_{1 \leq j_1 < \cdots < j_r \leq m}\), we have

\[ Td(R) = \prod_{j=1}^{m} \frac{\lambda_j}{1 - e^{-\lambda_j}}, \quad \sum_{r=0}^{m} (-1)^r t^r \text{ch}(R_r) = \prod_{j=1}^{m} (1 - te^{-\lambda_j}). \]  

Taking \( t = 1 \) in (1.14), we obtain the first identity in (1.13).

Taking the derivative of the second identity in (1.14) at \( t = 1 \), we obtain

\[ \sum_{r=0}^{m} (-1)^r r \text{ch}(R_r) = - \left( \sum_{j=1}^{m} \frac{e^{-\lambda_j}}{1 - e^{-\lambda_j}} \right) \prod_{j=1}^{m} (1 - e^{-\lambda_j}). \]  

From the first identity in (1.14), (1.15) and the identity

\[ \frac{e^{-\lambda_j}}{1 - e^{-\lambda_j}} = \lambda_j^{-1} - \frac{1}{2} + \frac{1}{12} \lambda_j + \cdots, \]  

we obtain the second identity in (1.13).

Taking the second derivative of the second identity in (1.14) at \( t = 1 \), we obtain

\[ \sum_{r=0}^{m} (-1)^r r(r-1) \text{ch}(R_r) = \left( \left( \sum_{j=1}^{m} \frac{e^{-\lambda_j}}{1 - e^{-\lambda_j}} \right)^2 - \sum_{j=1}^{m} \left( \frac{e^{-\lambda_j}}{1 - e^{-\lambda_j}} \right)^2 \right) \prod_{j=1}^{m} (1 - e^{-\lambda_j}). \]  

From the first identity in (1.14), (1.16) and (1.17), we obtain the third identity in (1.13). This completes the proof. \( \square \)

For an \((m \times m)\)-matrix \( A \), we define

\[ Td'(A) = \frac{\partial}{\partial t} \left. Td(A + t \text{Id}) \right|_{t=0}. \]  

Proposition 1.6. We have

\[ \left\{ Td'(R) \left( \sum_{r=0}^{m} (-1)^r \text{ch}(R_r) \right) \right\}^{[m]} = \frac{m}{2} c_m(R), \]

\[ \left\{ Td'(R) \left( \sum_{r=0}^{m} (-1)^r r \text{ch}(R_r) \right) \right\}^{[m]} = \frac{1}{12} (c_1 c_{m-1})(R) + \frac{m^2}{4} c_m(R). \]  

Proof. Let \( c_k' \) be as in (1.18) with \( Td \) replaced by \( c_k \). We have

\[ c_1'(R) = m, \quad c_2'(R) = (m - 1)c_1(R). \]  

On the other hand, we have

\[ \left\{ Td(R) \right\}^{[\leq 2]} = 1 + \frac{1}{2} c_1(R) + \frac{1}{12} (c_1^2(R) + c_2(R)). \]
By (1.20) and (1.21), we have

\[
\left\{ \frac{\text{Td}'(R)}{\text{Td}(R)} \right\} \left[ \leq 1 \right] = \frac{m}{2} - \frac{1}{12} c_1(R).
\]

(1.22)

From (1.13) and (1.22), we obtain (1.19). This completes the proof. \( \square \)

### 1.3 Chern form and Bott–Chern form

Let \( S \) be a compact Kähler manifold. We denote

\[
Q^S = \bigoplus_{p=0}^{\dim S} \Omega^{p,p}(S),
\]

(1.23)

\[
Q^{S,0} = \bigoplus_{p=1}^{\dim S} (\partial \Omega^{p-1,p}(S) + \bar{\partial} \Omega^{p,p-1}(S)) \subseteq Q^S.
\]

Let \( E \) be a holomorphic vector bundle over \( S \). Let \( g^E \) be a Hermitian metric on \( E \). Let \( R^E \in \Omega^{1,1}(S, \text{End}(E)) \) be the curvature of the Chern connection on \((E, g^E)\). Recall that \( c(\cdot) \) was defined in (1.10). The total Chern form of \((E, g^E)\) is defined by

\[
c(E, g^E) = c\left( -\frac{R^E}{2\pi i} \right) \in Q^S.
\]

(1.24)

The total Chern class of \( E \) is defined by

\[
c(E) = \left[ c(E, g^E) \right] \in H^{\text{even}}_{\text{dR}}(S),
\]

(1.25)

which is independent of \( g^E \).

Let \( E' \subseteq E \) be a holomorphic subbundle. Let \( E'' = E/E' \). We have a short exact sequence of holomorphic vector bundles over \( S \),

\[
0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0,
\]

(1.26)

where \( \alpha \) (respectively, \( \beta \)) is the canonical embedding (respectively, projection). We have

\[
c(E) = c(E')c(E'').
\]

(1.27)

Let \( g^{E'} \) be a Hermitian metric on \( E' \). Let \( g^{E''} \) be a Hermitian metric on \( E'' \). The Bott–Chern form [BGS88a, §1f)]

\[
\tilde{c}(g^{E'}, g^E, g^{E''}) \in Q^S/Q^{S,0}
\]

(1.28)

is such that

\[
\frac{\partial \bar{\partial}}{2\pi i} \tilde{c}(g^{E'}, g^E, g^{E''}) = c(E, g^E) - c(E' \oplus E'', g^{E'} \oplus g^{E''})
\]

\[
= c(E, g^E) - c(E', g^{E'})c(E'', g^{E''}).
\]

(1.29)

Let \( \alpha^*g^E \) be the Hermitian metric on \( E' \) induced by \( g^E \) via the embedding \( \alpha : E' \to E \). Let \( \beta_*g^E \) be the quotient Hermitian metric on \( E'' \) induced by \( g^E \) via the surjection \( \beta : E \to E'' \). We denote

\[
\tilde{c}(E', E, g^E) = \tilde{c}(\alpha^*g^E, g^E, \beta_*g^E).
\]

(1.30)

Let \( \beta^*g^{E''} \) be the Hermitian pseudometric on \( E \) induced by \( g^{E''} \) via the surjection \( \beta : E \to E'' \). For \( \varepsilon > 0 \), set

\[
g^E_\varepsilon = g^E + \frac{1}{\varepsilon} \beta^*g^{E''}.
\]

(1.31)
We equip $Q^S \subseteq \Omega^\bullet(S)$ with the compact-open topology. We equip $Q^S/Q^{S,0}$ with the quotient topology.

**Proposition 1.7.** As $\varepsilon \to 0$,

$$c(E, g_{\varepsilon}^E) \to c(E', \alpha^* g_{\varepsilon}^E) c(E'', g_{\varepsilon}^{E''}) \quad \text{and} \quad \bar{c}(E', E, g_{\varepsilon}^E) \to 0. \quad (1.32)$$

**Proof.** We follow the proof of [BGS88a, Theorem 1.29].

Let $\text{pr} : S \times \mathbb{C} \to S$ be the canonical projection. Let

$$\tilde{\alpha} : \text{pr}^* E' \to \text{pr}^* E \quad (1.33)$$

be the pull-back of $\alpha : E' \to E$. Let $(s, z) \in S \times \mathbb{C}$ be coordinates. Let $\sigma \in H^0(S \times \mathbb{C}, \mathbb{C})$ be the holomorphic function $\sigma(s, z) = z$. Let

$$\tilde{\sigma} : \text{pr}^* E' \to \text{pr}^* E' \quad (1.34)$$

be the multiplication by $\sigma$. Set

$$\mathcal{E}' = \text{pr}^* E', \quad \mathcal{E} = \text{Coker}(\tilde{\alpha} \oplus \tilde{\sigma} : \text{pr}^* E' \to \text{pr}^* E \oplus \text{pr}^* E'). \quad (1.35)$$

We get a short exact sequence of holomorphic vector bundles over $S \times \mathbb{C}$,

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0, \quad (1.36)$$

where $\mathcal{E}' \to \mathcal{E}$ is induced by the embedding $0 \oplus \text{Id}_{\text{pr}^* E'} : \text{pr}^* E' \hookrightarrow \text{pr}^* E \oplus \text{pr}^* E'$, and $\mathcal{E} \to \mathcal{E}'' := \text{Coker}(\mathcal{E}' \to \mathcal{E})$ is the canonical projection. For $z \in \mathbb{C}$, let

$$0 \to \mathcal{E}'_z \to \mathcal{E}_z \to \mathcal{E}''_z \to 0 \quad (1.37)$$

be the restriction of (1.36) to $S \times \{z\}$. For $z \neq 0$, let

$$\phi_z : E \to \mathcal{E}_z = \text{Coker}(\alpha \oplus z \text{Id}_{\mathcal{E}'_z} : \mathcal{E}' \to E \oplus E') \quad (1.38)$$

be the isomorphism induced by the embedding $\text{Id}_E \oplus 0 : E \hookrightarrow E \oplus E'$. We obtain a commutative diagram

$$\begin{array}{c}
0 \longrightarrow E' \longrightarrow E' \\
\downarrow \quad \downarrow \\
0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}_z \\
\downarrow \quad \downarrow \\
0 \longrightarrow \mathcal{E}'_z \longrightarrow \mathcal{E}_z \longrightarrow \mathcal{E}''_z \longrightarrow 0
\end{array} \quad (1.39)$$

where the vertical maps are induced by $\phi_z$. Let

$$\phi_0 : E' \oplus E'' \to \mathcal{E}_0 = \text{Coker}(\alpha \oplus 0 : E' \to E \oplus E') = E'' \oplus E' \quad (1.40)$$

be the obvious isomorphism. We obtain a commutative diagram

$$\begin{array}{c}
0 \longrightarrow E' \longrightarrow E' \oplus E'' \longrightarrow E'' \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathcal{E}'_0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}''_0 \longrightarrow 0
\end{array} \quad (1.41)$$

where the vertical maps are induced by $\phi_0$. 

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We can construct a Hermitian metric \(g^E\) on \(E\) such that
\[
\phi_z^* g^E = |z|^2 g^E + \beta^* g^{E''} \quad \text{for } z \neq 0, \quad \phi_0^* g^E = \alpha^* g^E \oplus g^{E''}. \tag{1.42}
\]
To show that \(g^E\) is a smooth metric, we consider the metric \(g^{pr*E \oplus pr^*E'}\) on \(pr^*E \oplus pr^*E'\) defined by
\[
g^{pr*E \oplus pr^*E'}|_{S \times \{z\}} = (1 + |z|^2)(g^E \oplus \alpha^* g^E). \tag{1.43}
\]
We can directly verify that \(g^E\) is the quotient metric induced by \(g^{pr*E \oplus pr^*E'}\) via the canonical projection \(pr^*E \oplus pr^*E' \to \mathcal{E}\).

By (1.39) and (1.42), for \(\varepsilon = |z|^2 > 0\), we have
\[
c(\mathcal{E}_z, g^{E^\varepsilon}) = c(E, g^E), \quad c(\mathcal{E}_z, g^{E^\varepsilon}) = \tilde{c}(E', E, g^E). \tag{1.44}
\]
By [BGS88a, Theorem 1.29 iii)], (1.41) and (1.42), we have
\[
c(\mathcal{E}_0, g^{E^0}) = c(E', \alpha^* g^E)c(E'', g^{E''}), \quad \tilde{c}(\mathcal{E}_0, \mathcal{E}_z, g^{E^\varepsilon}) = 0. \tag{1.45}
\]
On the other hand, by [BGS88a, Theorem 1.29 ii)], we have
\[
\lim_{z \to 0} c(\mathcal{E}_z, g^{E^\varepsilon}) = c(\mathcal{E}_0, g^{E^0}), \quad \lim_{z \to 0} \tilde{c}(\mathcal{E}_z, \mathcal{E}_z, g^{E^\varepsilon}) = \tilde{c}(\mathcal{E}_0, \mathcal{E}_0, g^{E^0}). \tag{1.46}
\]
From (1.44)–(1.46), we obtain (1.32). This completes the proof. \(\square\)

Remark 1.8. We can also prove Proposition 1.7 by applying the arguments in [BB94, (4.67)–(4.70) and (4.75)–(4.81)], which show that the connection of \(\mathcal{E}\) converges to a triangular 2 \(\times\) 2 matrix with diagonal elements given by the connections of \(E'\) and \(E''\) as \(\varepsilon \to 0\). Though [BB94, (4.67)–(4.70) and (4.75)–(4.81)] work with tangent bundles, the argument equally holds in our case (because the connections under consideration are Chern connections).

Let \(F \subseteq E\) be a holomorphic subbundle. Set \(F' = \alpha^{-1}(F) \subseteq E', \ F'' = \beta(F) \subseteq E''\).

**Proposition 1.9.** If \(F' = E'\), as \(\varepsilon \to 0\),
\[
\tilde{c}(F, E, g^E) \to c(E', \alpha^* g^E)\tilde{c}(F', E', g^{E''}). \tag{1.47}
\]
If \(F'' = E''\), as \(\varepsilon \to 0\),
\[
\tilde{c}(F, E, g^E) \to c(E'', \alpha^* g^E)\tilde{c}(F', E', g^{E''}). \tag{1.48}
\]
**Proof.** We use the notation from the proof of Proposition 1.7. Set
\[
\mathcal{F} = \text{Coker}(\hat{\alpha} \oplus \hat{\sigma}|_{pr^*F'} : pr^*F' \to pr^*F \oplus pr^*F') \subseteq \mathcal{E}. \tag{1.49}
\]
For \(z \in \mathbb{C}\), let \(\mathcal{F}_z\) be the restriction of \(\mathcal{F}\) to \(S \times \{z\}\).

For \(z \neq 0\), we have \(\phi_z(F) = \mathcal{F}_z \subseteq \mathcal{E}_z\). By (1.42), for \(\varepsilon = |z|^2 > 0\), we have
\[
\tilde{c}(\mathcal{F}_z, \mathcal{E}_z, g^{E^\varepsilon}) = \tilde{c}(F, E, g^E). \tag{1.50}
\]
We have \(\phi_0(F) = F' \oplus F'' \subseteq E' \oplus E'' = \mathcal{E}_0\). By (1.42), we have
\[
\tilde{c}(\mathcal{F}_0, \mathcal{E}_0, g^{E^0}) = \tilde{c}(F' \oplus F'', E' \oplus E'', \alpha^* g^E \oplus g^{E''}). \tag{1.51}
\]
By [BGS88a, Theorem 1.29], we have
\[
\tilde{c}(F' \oplus F'', E' \oplus E'', \alpha^* g^E \oplus g^{E''}) = c(E', \alpha^* g^E)\tilde{c}(F', E', g^{E''}) \quad \text{if } F' = E', \tag{1.52}
\]
\[
\tilde{c}(F' \oplus F'', E' \oplus E'', \alpha^* g^E \oplus g^{E''}) = c(E'', g^{E''})\tilde{c}(F', E', \alpha^* g^E) \quad \text{if } F'' = E''. \tag{1.53}
\]
On the other hand, by [BGS88a, Theorem 1.29 ii)], we have
\[
\lim_{z \to 0} \tilde{c}(\mathcal{F}_z, \mathcal{E}_z, g^{E^\varepsilon}) = \tilde{c}(\mathcal{F}_0, \mathcal{E}_0, g^{E^0}). \tag{1.54}
\]
From (1.50)–(1.53), we obtain (1.47) and (1.48). This completes the proof. \(\square\)
Recall that $\text{Td}()$ was defined in (1.10). The Bott–Chern form [BGS88a, §1f]
\[
\text{Td}(g^{E'}, g^E, g^{E''}) \in Q^S/Q^{S,0} \quad (1.54)
\]
is such that
\[
\frac{\partial \partial}{2\pi i} \text{Td}(g^{E'}, g^E, g^{E''}) = \text{Td}(E, g^E) - \text{Td}(E', g^{E'})\text{Td}(E'', g^{E''}). \quad (1.55)
\]

**Proposition 1.10.** Propositions 1.7 and 1.9 hold with $c(\cdot)$ replaced by $\text{Td}(\cdot)$.

Recall that $\text{ch}(\cdot)$ was defined in (1.10). The Bott–Chern form [BGS88a, §1f]
\[
\text{ch}(g^{E'}, g^E, g^{E''}) \in Q^S/Q^{S,0} \quad (1.56)
\]
is such that
\[
\frac{\partial \partial}{2\pi i} \text{ch}(g^{E'}, g^E, g^{E''}) = \text{ch}(E', g^{E'}) - \text{ch}(E, g^E) + \text{ch}(E'', g^{E''}). \quad (1.57)
\]

For another Hermitian metric $\hat{g}^E$ on $E$, let
\[
\text{ch}(\hat{g}^E, g^E) \in Q^S/Q^{S,0} \quad (1.58)
\]
be the Bott–Chern form [BGS88a, §1f] such that
\[
\frac{\partial \partial}{2\pi i} \text{ch}(\hat{g}^E, g^E) = \text{ch}(E, \hat{g}^E) - \text{ch}(E, g^E). \quad (1.59)
\]
The following proposition is a direct consequence of the construction of the Bott–Chern form [BGS88a, §1f]).

**Proposition 1.11.** For another Hermitian metric $\hat{g}^E$ (respectively, $\hat{g}^{E'}$, $\hat{g}^{E''}$) on $E$ (respectively, $E'$, $E''$), we have
\[
\text{ch}(\hat{g}^{E'}, \hat{g}^E, \hat{g}^{E''}) = \text{ch}(\hat{g}^{E'}, g^E, g^{E''}) + \text{ch}(\hat{g}^{E''}, g^{E'}) - \text{ch}(g^E, g^{E''}) + \text{ch}(\hat{g}^E, g^{E'}). \quad (1.60)
\]
For $a, b > 0$, we have
\[
\text{ch}(ag^E, bg^E) = \text{ch}(E, g^E)(\log b - \log a). \quad (1.61)
\]

For $(g^E_t)_{t \in \mathbb{R}}$ a smooth family of Hermitian metrics on $E$, the map $t \mapsto \text{ch}(g^E_t, g^E_0)$ is continuous. In particular, we have
\[
\text{ch}(g^E_t, g^E_0) \to 0 \quad \text{as} \quad t \to 0. \quad (1.62)
\]
Let $E^*$ be the dual of $E$. Following [BB94, §1a]), for $p = 0, \ldots, \dim E$ and $s = 0, \ldots, p - 1$, set
\[
I^p_s = \{ u \in \Lambda^p E^* : u(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in E', v_{s+2}, \ldots, v_p \in E \}. \quad (1.63)
\]
For convenience, we denote $I^p_0 = \Lambda^p E^*$ and $I^p_{-1} = 0$. We obtain a filtration
\[
\Lambda^p E^* = I^p_0 \hookrightarrow I^p_{-1} \hookrightarrow \cdots \hookrightarrow I^p_{-1} = 0. \quad (1.64)
\]

For $r = 0, \ldots, \dim E''$ and $s = 0, \ldots, \dim E'$, we denote $E_{r,s} = \Lambda^s E'^* \otimes \Lambda^r E^{''*}$. We have a short exact sequence of holomorphic vector bundles over $S$,
\[
0 \to I^{r+s}_{s-1} \to I^{r+s}_s \to E_{r,s} \to 0. \quad (1.65)
\]
Recall that $g^E_\varepsilon$ was defined in (1.31). Let $g^{\Lambda^p E^*}_\varepsilon$ be the Hermitian metric on $\Lambda^p E^*$ induced by $g^E_\varepsilon$. Let $g^{I^{r+s}_s}_\varepsilon$ be the restriction of $g^{\Lambda^p E^*}_\varepsilon$ to $I^{r+s}_s$. Let $g^{E_{r,s}}_\varepsilon$ be the quotient metric on $E_{r,s}$ induced by $g^{I^{r+s}_s}_\varepsilon$ via the surjection $I^{r+s}_s \to E_{r,s}$.

Similarly to Proposition 1.7, we have the following proposition.
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Proposition 1.12. As $\varepsilon \to 0$,

$$\tilde{c}_0\left(g_{\varepsilon}^{I+}, g_{\varepsilon}^{I+s}, g_{\varepsilon}^{E_{\varepsilon}}\right) \to 0. \quad (1.66)$$

Proof. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be as in (1.36). Let $\mathcal{I}^p_p \subseteq \Lambda^p \mathcal{E}^*$ be as in (1.63) with $E$ replaced by $\mathcal{E}$ and $E'$ replaced by $\mathcal{E}'$. We denote $\mathcal{E}_{\varepsilon} = \Lambda^q \mathcal{E}'' \otimes \Lambda^s \mathcal{E}''$. We have a short exact sequence of holomorphic vector bundles over $S \times \mathbb{C}$,

$$0 \to \mathcal{I}^{p+q}_s \to \mathcal{I}^{q+s}_s \to \mathcal{E}_{\varepsilon} \to 0. \quad (1.67)$$

Proceeding in the same way as in the proof of Proposition 1.7 with (1.36) replaced by (1.67), we obtain (1.66). This completes the proof. \qed

1.4 Quillen metric

Let $X$ be an $n$-dimensional compact Kähler manifold. Let $E$ be a holomorphic vector bundle over $X$. Let $\bar{\partial}^E$ be the Dolbeault operator on

$$\Omega^{0,\bullet}(X, E) = \mathcal{C}^\infty(X, \Lambda^\bullet(T^*X) \otimes E). \quad (1.68)$$

For $q = 0, \ldots, n$, we have $H^q(X, E) = H^q(\Omega^{0,\bullet}(X, E), \bar{\partial}^E)$. Set

$$\lambda(E) = \det H^\bullet(X, E) := \prod_{q=0}^n \left(\det H^q(X, E)\right)^{(−1)^q}. \quad (1.69)$$

Let $g^{TX}$ be a Kähler metric on $TX$. Let $g^E$ be a Hermitian metric on $E$. Let $\langle \cdot, \cdot \rangle_{\Lambda^\bullet(T^*X) \otimes E}$ be the Hermitian product on $\Lambda^\bullet(T^*X) \otimes E$ induced by $g^{TX}$ and $g^E$. Let $dv_X$ be the Riemannian volume form on $X$ induced by $g^{TX}$. For $s_1, s_2 \in \Omega^{0,\bullet}(X, E)$, set

$$\langle s_1, s_2 \rangle = (2\pi)^{-n} \int_X \langle s_1, s_2 \rangle_{\Lambda^\bullet(T^*X) \otimes E} dv_X, \quad (1.70)$$

which we call the $L^2$-product.

Let $\bar{\partial}^{E,*}$ be the formal adjoint of $\bar{\partial}^E$ with respect to the Hermitian product (1.70). The Kodaira Laplacian on $\Omega^{0,\bullet}(X, E)$ is defined by

$$\square^E = \bar{\partial}^E \bar{\partial}^{E,*} + \bar{\partial}^{E,*} \bar{\partial}^E. \quad (1.71)$$

Let $\square^E_q$ be the restriction of $\square^E$ to $\Omega^{0,q}(X, E)$.

By the Hodge theorem, we have

$$\ker(\square^E_q) = \{ s \in \Omega^{0,q}(X, E) : \bar{\partial}^E s = 0, \bar{\partial}^{E,*} s = 0 \}. \quad (1.72)$$

Still by the Hodge theorem, the following map is bijective:

$$\ker(\square^E_q) \to H^q(X, E)$$

$$s \mapsto [s]. \quad (1.73)$$

Let $| \cdot |_{\lambda(E)}$ be the $L^2$-metric on $\lambda(E)$ induced by the metric (1.70) via (1.69) and (1.73).

Let $\text{Sp}(\square^E_q)$ be the spectrum of $\square^E_q$, which is a multiset.\(^1\) For $z \in \mathbb{C}$ with $\text{Re}(z) > n$, set

$$\theta(z) = \sum_{q=1}^n (-1)^{q+1} q \sum_{\lambda \in \text{Sp}(\square^E_q), \lambda \neq 0} \lambda^{-z}. \quad (1.74)$$

\(^1\) A multiset allows for multiple instances for each of its elements.
By [See67], the function $\theta(z)$ extends to a meromorphic function of $z \in \mathbb{C}$, which is holomorphic at $z = 0$.

The following definition is due to Quillen [Qui85] and Bismut, Gillet and Soulé [BGS88b, §1d]).

**Definition 1.13.** The Quillen metric on $\lambda(E)$ is defined by
\[
\|\cdot\|_{\lambda(E)} = \exp \left( \frac{1}{2} \theta'(0) \right) \|\cdot\|_{\lambda(E)}.
\]

**Remark 1.14.** Denote $\chi(X, E) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, E)$. For $a > 0$, if we replace $g^E$ by $ag^E$, then $\|\cdot\|_{\lambda(E)}$ is replaced by $a^{(X,E)/2} \|\cdot\|_{\lambda(E)}$.

### 1.5 Analytic torsion form

Let $\pi : X \to Y$ be a holomorphic submersion between Kähler manifolds with compact fiber $Z$.

Let $E$ be a holomorphic vector bundle over $X$. Let $R^* \pi_* E$ be the derived direct image of $E$, which is a graded analytic coherent sheaf on $Y$. We assume that $R^* \pi_* E$ is a graded holomorphic vector bundle. Let $H^\bullet(Z, E)$ be the fiberwise cohomology. More precisely, its fiber at $y \in Y$ is given by $H^\bullet(Z_y, E|_{Z_y})$. We have a canonical identification $R^* \pi_* E = H^\bullet(Z, E)$. We have the Grothendieck–Riemann–Roch formula,
\[
\chi(H^\bullet(Z, E)) := \sum_j (-1)^j \chi(H^j(Z, E)) = \int_Z \text{Td}(TZ) \chi(E) \in H^\text{even}_{\text{dR}}(Y).
\]

Let $\omega \in \Omega^{1,1}(X)$ be a Kähler form. Let $g^{TZ}$ be the Hermitian metric on $TZ$ associated with $\omega$. Let $g^E$ be a Hermitian metric on $E$. Let $g^{H^\bullet(Z, E)}$ be the $L^2$-metric on $H^\bullet(Z, E)$ associated with $g^{TZ}$ and $g^E$ via (1.73).

We use the notation in (1.23). Let $\chi(H^\bullet(Z, E), g^{H^\bullet(Z, E)}) \in Q^Y$ be the Chern character form of $(H^\bullet(Z, E), g^{H^\bullet(Z, E)})$. We introduce $\text{Td}(TZ, g^{TZ}) \in Q^X$ and $\chi(E, g^E) \in Q^X$ in the same way.

Bismut and Köhler [BK92, Definition 3.8] defined the analytic torsion forms. The analytic torsion form associated with $(\pi : X \to Y, \omega, E, g^E)$ is a differential form on $Y$, which we denote by $T(\omega, g^E)$. Moreover, we have
\[
T(\omega, g^E) \in Q^Y.
\]

We sometimes view $T(\omega, g^E)$ as an element in $Q^Y/Q^{Y, 0}$. By [BK92, Theorem 3.9], we have
\[
\frac{\partial \bar{\partial}}{2\pi i} T(\omega, g^E) = \chi(H^\bullet(Z, E), g^{H^\bullet(Z, E)}) - \int_Z \text{Td}(TZ, g^{TZ}) \chi(E, g^E).
\]

The identity (1.78) is a refinement of the Grothendieck–Riemann–Roch formula (1.76).

For $y \in Y$, let $\theta_y(z)$ be as in (1.74) with $(X, g^{TX}, E, g^E)$ replaced by $(Z_y, g^{TZ_y}, E|_{Z_y}, g^E|_{Z_y})$. Let $\theta'(0)$ be the function $y \mapsto \theta'_y(0)$ on $Y$. By the construction of the analytic torsion forms, we have
\[
\{T(\omega, g^E)\}^{(0,0)} = \theta'(0) \in \mathcal{C}^\infty(Y),
\]
where $\{\cdot\}^{(0,0)}$ means the component of degree $(0, 0)$.

Let $F$ be a holomorphic vector bundle over $Y$. Let $\pi^* F$ be its pull-back via $\pi$, which is a holomorphic vector bundle over $X$. Let $g^F$ be a Hermitian metric on $F$. Let $g^{E \otimes \pi^* F}$ be the Hermitian metric on $E \otimes \pi^* F$ induced by $g^E$ and $g^F$. Let
\[
T(\omega, g^{E \otimes \pi^* F}) \in Q^Y
\]
be the analytic torsion form associated with $(\pi : X \to Y, \omega, E \otimes \pi^* F, g^{E \otimes \pi^* F})$. 

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The following proposition is a direct consequence of the construction of the analytic torsion forms.

**Proposition 1.15.** The following identity holds in $Q^Y/Q^{Y,0}$:

$$T(\omega, g^E \otimes \pi^* F) = \text{ch}(F, g^F) T(\omega, g^E).$$  (1.81)

For $p = 0, \ldots, \dim Z$, let $g^{\Lambda^p(T^* Z)}$ be the metric on $\Lambda^p(T^* Z)$ induced by $g^{T Z}$. Let

$$T(\omega, g^{\Lambda^p(T^* Z)}) \in Q^Y$$  (1.82)

be the analytic torsion form associated with $(\pi : X \to Y, \omega, \Lambda^p(T^* Z), g^{\Lambda^p(T^* Z)})$.

**Theorem 1.16** (Bismut [Bis04, Theorem 4.15]). The following identity holds in $Q^Y/Q^{Y,0}$,

$$\sum_{p=0}^{\dim Z} (-1)^p T(\omega, g^{\Lambda^p(T^* Z)}) = 0.$$  (1.83)

### 1.6 Properties of the Quillen metric

In this subsection, we state several results describing the behavior of the Quillen metric under submersion, resolution, immersion and blow-up.

**Submersion.** Let $\pi : X \to Y, Z, E$ and $H^\bullet(Z, E)$ be as in §1.5. We assume that $X$ and $Y$ are compact. We further assume that the Leray spectral sequence for $E$ and $\pi$ degenerates at $E_2$, i.e.

$$H^q(X, E) \simeq \bigoplus_{j+k=q} H^j(Y, H^k(Z, E)) \quad \text{for } q = 0, \ldots, \dim X.$$  (1.84)

We denote

$$\det H^\bullet(Y, H^\bullet(Z, E)) = \bigotimes_{k=0}^{\dim Z} \left( \det H^\bullet(Y, H^k(Z, E)) \right)^{(-1)^k}$$

$$= \bigotimes_{j=0}^{\dim Y} \bigotimes_{k=0}^{\dim Z} \left( \det H^j(Y, H^k(Z, E)) \right)^{(-1)^{j+k}}.$$  (1.85)

Let

$$\sigma \in \det H^\bullet(X, E) \otimes \left( \det H^\bullet(Y, H^\bullet(Z, E)) \right)^{-1}$$  (1.86)

be the canonical section induced by (1.84).

Let $\omega_X \in \Omega^{1,1}(X)$ and $\omega_Y \in \Omega^{1,1}(Y)$ be Kähler forms. For $\varepsilon > 0$, set

$$\omega_\varepsilon = \omega_X + \frac{1}{\varepsilon} \pi^* \omega_Y.$$  (1.87)

Let $g^E$ be a Hermitian metric on $E$.

Let $g^{TX}_\varepsilon$ be the metric on $TX$ associated with $\omega_\varepsilon$. Let $$
\| \cdot \|_{\det H^\bullet(X, E), \varepsilon} (1.88)
$$

be the Quillen metric on $\det H^\bullet(X, E)$ associated with $g^{TX}_\varepsilon$ and $g^E$. Let $g^{TY}$ be the metric on $TY$ associated with $\omega_Y$. Let $g^{TZ}$ be the metric on $TZ$ associated with $\omega_X|_Z$. Let $g^{H^\bullet(Z, E)}$ be the
$L^2$-metric on $H^\bullet(Z, E)$ associated with $g^{TZ}$ and $g^E$. For $k = 0, \ldots, \dim Z$, let

$$\|\cdot\|_{\det H^\bullet(Y, H^k(Z, E))}$$

be the Quillen metric on $\det H^\bullet(Y, H^k(Z, E))$ associated with $g^{TY}$ and $g^{H^k(Z, E)}$. Let

$$\|\cdot\|_{\det H^\bullet(Y, H^\bullet(Z, E))}$$

be the metric on $\det H^\bullet(Y, H^\bullet(Z, E))$ induced by the Quillen metrics (1.89) via (1.85). Let $\|\sigma\|_\varepsilon$ be the norm of $\sigma$ with respect to the metrics (1.88) and (1.90).

We use the notation in (1.23). Let $\text{Td}(TY, g^{TY}) \in Q^Y$ be the Todd form of $(TY, g^{TY})$. Let $T(\omega, g^E) \in Q^Y$ (1.91) be the analytic torsion form (see §1.5) associated with $(\pi : X \to Y, \omega_X, E, g^E)$.

Recall that $\text{Td}'(\cdot)$ was defined by (1.18).

**Theorem 1.17** (Berthomieu and Bismut [BB94, Theorem 3.2]). As $\varepsilon \to 0$,

$$\log\|\sigma\|_\varepsilon^2 + \int_Y \text{Td}'(TY) \int_Z \text{Td}(TZ) \text{ch}(E) \log \varepsilon \to \int_Y \text{Td}(TY, g^{TY}) T(\omega, g^E).$$

**Resolution.** Let $X$ be a compact Kähler manifold. Let

$$0 \to E^0 \to E^1 \to E^2 \to 0$$

be a short exact sequence of holomorphic vector bundles over $X$. Let $\sigma \in \bigotimes_{k=0}^2 (\det H^\bullet(X, E^k))(-1)^{k+1}$ be the canonical section induced by the long exact sequence induced by (1.93).

Let $g^{TX}$ be a Kähler metric on $TX$. For $k = 0, 1, 2$, let $g^{E^k}$ be a Hermitian metric on $E^k$. Let

$$\|\cdot\|_{\det H^\bullet(X, E^k)}$$

be the Quillen metric on $\det H^\bullet(X, E^k)$ associated with $g^{TX}$ and $g^{E^k}$. Let $\|\sigma\|$ be the norm of $\sigma$ with respect to the metrics (1.95).

We use the notation in (1.23). Let $\text{Td}(TX, g^{TX}) \in Q^X$ be the Todd form of $(TX, g^{TX})$. Let $\text{ch}(E^k, g^{E^k}) \in Q^X$ be the Chern character form of $(E^k, g^{E^k})$. Let

$$\tilde{\text{ch}}(g^{E^\bullet}) \in Q^X/Q^{X,0}$$

be the Bott–Chern form [BGS88a, §1f] such that

$$\frac{\partial \bar{\partial}}{2\pi i} \tilde{\text{ch}}(g^{E^\bullet}) = \sum_{k=0}^2 (-1)^k \text{ch}(E^k, g^{E^k}).$$

**Theorem 1.18** (Bismut, Gillet and Soulé [BGS88b, Theorem 1.23]). The following identity holds:

$$\log\|\sigma\|^2 = \int_X \text{Td}(TX, g^{TX}) \tilde{\text{ch}}(g^{E^\bullet}).$$

**Immersion.** Let $X$ be a compact Kähler manifold. Let $Y \subseteq X$ be a complex submanifold of codimension one. Let $i : Y \hookrightarrow X$ be the canonical embedding. Let $F$ be a holomorphic vector bundle over $Y$. Let $v : E_1 \to E_0$ be a map between holomorphic vector bundles over $X$ which,
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together with a restriction map \( r : E_0|_Y \rightarrow F \), provides a resolution of \( i_* \mathcal{O}_Y(F) \). More precisely, we have an exact sequence of analytic coherent sheaves on \( X \),

\[
0 \rightarrow \mathcal{O}_X(E_1) \overset{\nu}{\rightarrow} \mathcal{O}_X(E_0) \overset{r}{\rightarrow} i_* \mathcal{O}_Y(F) \rightarrow 0.
\]

Let

\[
\sigma \in \left( \det H^\bullet(X, E_1) \right)^{-1} \otimes \det H^\bullet(X, E_0) \otimes \left( \det H^\bullet(Y, F) \right)^{-1}
\]

be the canonical section induced by the long exact sequence induced by \( (1.99) \).

Let \( \omega \in \Omega_{1,1}(X) \) be a Kähler form. For \( k = 0, 1 \), let \( g^{E_k} \) be a Hermitian metric on \( E_k \). Let \( g^F \) be a Hermitian metric on \( F \). Assume that there is an open neighborhood \( Y \subseteq U \subseteq X \) such that \( v|_{X \setminus U} \) is isometric, i.e.

\[
g^{E_1|_{X \setminus U}} = v^* g^{E_0|_{X \setminus U}}.
\]

Let \( g^{TX} \) be the metric on \( TX \) associated with \( \omega \). For \( k = 0, 1 \), let

\[
\|\cdot\|_{\det H^\bullet(X, E_k)}
\]

be the Quillen metric on \( \det H^\bullet(X, E_k) \) associated with \( g^{TX} \) and \( g^{E_k} \). Let \( g^{TY} \) be the metric on \( TY \) associated with \( \omega|_Y \). Let

\[
\|\cdot\|_{\det H^\bullet(Y, F)}
\]

be the Quillen metric on \( \det H^\bullet(Y, F) \) associated with \( g^{TY} \) and \( g^F \). Let \( \|\sigma\| \) be the norm of \( \sigma \) with respect to the metrics \( (1.102) \) and \( (1.103) \).

The following theorem is a direct consequence of the immersion formula due to Bismut and Lebeau [BL91, Theorem 0.1] and the anomaly formula due to Bismut, Gillet and Soulé [BGS88b, Theorem 1.23].

**Theorem 1.19.** We have

\[
\log\|\sigma\|^2 = \alpha(U, \omega|_U, v|_U, g^{E^\bullet|_U}, r, g^F),
\]

where \( \alpha(U, \omega|_U, v|_U, r|_U, g^{E^\bullet}, g^F) \) is a real number determined by

\[
U, \quad \omega|_U, \quad v|_U : E_1|_U \rightarrow E_0|_U, \quad g^{E^\bullet|_U}, \quad r : E_0|_Y \rightarrow F, \quad g^F.
\]

More precisely, given

\[
\tilde{Y} \subseteq \tilde{U} \subseteq \tilde{X}, \quad \tilde{\omega}, \quad \tilde{v} : \tilde{E}_1 \rightarrow \tilde{E}_0, \quad \tilde{r} : \tilde{E}_0|_{\tilde{Y}} \rightarrow \tilde{F}, \quad g^{E^\bullet}, \quad g^F
\]

satisfying the same properties that

\[
Y \subseteq U \subseteq X, \quad \omega, \quad v : E_1 \rightarrow E_0, \quad r : E_0|_Y \rightarrow F, \quad g^{E^\bullet}, \quad g^F
\]

satisfy, if there is a biholomorphic map \( U \rightarrow \tilde{U} \) inducing an isomorphism between the restrictions of the data above to \( U \) and \( \tilde{U} \), then

\[
\log\|\sigma\|^2 = \log\|\tilde{\sigma}\|^2,
\]

where

\[
\tilde{\sigma} \in \left( \det H^\bullet(\tilde{X}, \tilde{E}_1) \right)^{-1} \otimes \det H^\bullet(\tilde{X}, \tilde{E}_0) \otimes \left( \det H^\bullet(\tilde{Y}, \tilde{F}) \right)^{-1}
\]

is the canonical section, and \( \|\tilde{\sigma}\| \) is its norm with respect to the Quillen metrics.

**Remark 1.20.** The real number \( \alpha(U, \omega|_U, v|_U, r|_U, g^{E^\bullet}, g^F) \) depends continuously on the input data.

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Blow-up. Let $X$ be a compact Kähler manifold. Let $Y \subseteq X$ be a complex submanifold of codimension $r \geq 2$. Let $f : X' \to X$ be the blow-up along $Y$. Let $E$ be a holomorphic vector bundle over $X$. Let $f^*E$ be the pull-back of $E$ via $f$, which is a holomorphic vector bundle over $X'$. Applying spectral sequence, we obtain a canonical identification

$$H^\bullet(X', f^*E) = H^\bullet(X, E).$$

(1.110)

Let

$$\sigma \in \left( \det H^\bullet(X, E) \right)^{-1} \otimes \det H^\bullet(X', f^*E)$$

(1.111)

be the canonical section induced by (1.110).

Let $\omega \in \Omega^{1,1}(X)$ and $\omega' \in \Omega^{1,1}(X')$ be Kähler forms. Assume that there are open neighborhoods $Y \subseteq U \subseteq X$ and $f^{-1}(Y) \subseteq U' \subseteq X'$ such that

$$f^{-1}(U) = U', \quad f^*(\omega|_{X \setminus U}) = \omega'|_{X \setminus U'}.$$ 

(1.112)

For the existence of such $\omega$ and $\omega'$, see the proof of [Voi02, Proposition 3.24]. Let $g^E$ be a Hermitian metric on $E$.

Let $g^{TX}$ be the metric on $TX$ associated with $\omega$. Let

$$\|\cdot\|_{\det H^\bullet(X, E)}$$

(1.113)

be the Quillen metric on $\det H^\bullet(X, E)$ associated with $g^{TX}$ and $g^E$. Let $g^{TX'}$ be the metric on $TX'$ associated with $\omega'$. Let

$$\|\cdot\|_{\det H^\bullet(X', f^*E)}$$

(1.114)

be the Quillen metric on $\det H^\bullet(X', f^*E)$ associated with $g^{TX'}$ and $f^*g^E$. Let $\|\sigma\|$ be the norm of $\sigma$ with respect to the metrics (1.113) and (1.114).

The following theorem is a direct consequence of the blow-up formula due to Bismut [Bis97, Theorem 8.10].

**Theorem 1.21.** We have

$$\log \|\sigma\|^2 = \alpha(U, \omega|_U, U', \omega'|_{U'}, E|_U, g^E|_U),$$

(1.115)

where $\alpha(U, \omega|_U, U', \omega'|_{U'}, E|_U, g^E|_U)$ is a real number determined by

$$U, \quad \omega|_U, \quad U', \quad \omega'|_{U'}, \quad E|_U, \quad g^E|_U.$$ 

(1.116)

**Remark 1.22.** The real number $\alpha(U, \omega|_U, U', \omega'|_{U'}, E|_U, g^E|_U)$ depends continuously on the input data.

### 1.7 Topological torsion and BCOV torsion

Let $X$ be an $n$-dimensional compact Kähler manifold. For $p = 0, \ldots, n$, set

$$\lambda_p(X) = \det H^{p, \bullet}(X) := \bigotimes_{q=0}^n (\det H^{p, q}(X))^{(-1)^q}.$$ 

(1.117)

Set

$$\eta(X) = \det H^\bullet_{\text{dR}}(X) := \bigotimes_{k=0}^{2n} (\det H^k_{\text{dR}}(X))^{(-1)^k}$$

$$= \bigotimes_{p=0}^n (\lambda_p(X))^{(-1)^p}.$$ 

(1.118)
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Set

\[
\lambda(X) = \bigotimes_{0 \leq p, q \leq n} \left( \det H^{p,q}(X) \right)^{(-1)^{p+q}p} = \bigotimes_{p=1}^{n} (\lambda_p(X))^{(-1)^p},
\]

(1.119)

\[
\lambda_{\text{tot}}(X) = \bigotimes_{k=1}^{2n} \left( \det H^k_{\text{dR}}(X) \right)^{(-1)^k} = \lambda(X) \otimes \lambda(\bar{X}).
\]

(1.120)

The identities in (1.119) appeared in [Kat14]. They were applied to the theory of BCOV invariant by Eriksson, Freixas i Montplet and Mourougane [EFM21].

For \( \mathbb{A} = \mathbb{Z}, \mathbb{R}, \mathbb{C} \), we denote by \( H^\bullet_{\text{Sing}}(X, \mathbb{A}) \) the singular cohomology of \( X \) with coefficients in \( \mathbb{A} \). For \( k = 0, \ldots, 2n \), let

\[
\sigma_{k,1}, \ldots, \sigma_{k,b_k} \in \text{Im}(H^k_{\text{Sing}}(X, \mathbb{Z}) \to H^k_{\text{Sing}}(X, \mathbb{R}))
\]

be a basis of the lattice. We fix a square root of \( i \). In what follows, the choice of square root is irrelevant. We identify \( H^k_{\text{dR}}(X) \) with \( H^k_{\text{Sing}}(X, \mathbb{C}) \) as follows:

\[
H^k_{\text{dR}}(X) \to H^k_{\text{Sing}}(X, \mathbb{C})
\]

\[
[\alpha] \mapsto \left[ a \mapsto (2\pi i)^{-k/2} \int_a \alpha \right],
\]

(1.121)

where \( \alpha \) is a closed \( k \)-form on \( X \) and \( a \) is a \( k \)-chain in \( X \). Then \( \sigma_{k,1}, \ldots, \sigma_{k,b_k} \) form a basis of \( H^k_{\text{dR}}(X) \). Set

\[
\epsilon_X = \bigotimes_{k=0}^{2n} \sigma_k^{-1} \in \eta(X), \quad \sigma_X = \bigotimes_{k=1}^{2n} \sigma_k^{-1} \in \lambda_{\text{tot}}(X),
\]

(1.122)

which are well-defined up to \( \pm 1 \).

Let \( \omega \) be a Kähler form on \( X \). Let \( \| \cdot \|_{\lambda_p(X), \omega} \) be the Quillen metric on \( \lambda_p(X) \) associated with \( \omega \). Let \( \| \cdot \|_{\eta(X)} \) be the metric on \( \eta(X) \) induced by \( \| \cdot \|_{\lambda_p(X), \omega} \) via (1.118). The same calculation as in [Zha22, Theorem 2.1] together with the first identity in Proposition 1.5 shows that \( \| \cdot \|_{\eta(X)} \) is independent of \( \omega \).

**Definition 1.23.** We define

\[
\tau_{\text{top}}(X) = \log \| \epsilon_X \|_{\eta(X)}.
\]

(1.123)

Indeed \( \| \cdot \|_{\eta(X)} \) is the classical Ray–Singer metric up to a normalization. Later, we use this fact to show that \( \tau_{\text{top}}(X) = 0 \).

Let \( \| \cdot \|_{\lambda(X), \omega} \) be the metric on \( \lambda(X) \) induced by \( \| \cdot \|_{\lambda_p(X), \omega} \) via the first identity in (1.119). Let \( \| \cdot \|_{\lambda_{\text{tot}}(X), \omega} \) be the metric on \( \lambda_{\text{tot}}(X) \) induced by \( \| \cdot \|_{\lambda(X), \omega} \) via the second identity in (1.119).

**Definition 1.24.** We define

\[
\tau_{\text{BCOV}}(X, \omega) = \log \| \sigma_X \|_{\lambda_{\text{tot}}(X), \omega}.
\]

(1.124)

For \( p = 0, \ldots, n \), let \( g^\omega_{\Lambda^p(T^*X)} \) be the metric on \( \Lambda^p(T^*X) \) induced by \( \omega \). Let \( g^\omega_{\Omega^{p,q}(X)} \) be the \( L^2 \)-metric on \( \Omega^{p,q}(X) \). More precisely, \( g^\omega_{\Omega^{p,q}(X)} \) is defined by (1.70) with \( (E, g^E) \) replaced by \( (\Lambda^p(T^*X), g^\omega_{\Lambda^p(T^*X)}) \). Let \( g^\omega_{H^{p,q}(X)} \) be the \( L^2 \)-metric on \( H^{p,q}(X) \). More precisely, \( g^\omega_{H^{p,q}(X)} \) is induced by \( g^\omega_{\Omega^{p,q}(X)} \) via the Hodge theorem. Let \( \| \cdot \|_{\eta(X), \omega} \) be the metric on \( \eta(X) \) induced by \( (g^\omega_{H^{p,q}(X)})_{0 \leq p, q \leq n} \) via (1.117) and (1.118).
We refer the reader to the proof of [Voi02, Proposition 3.18].

Proof. Let $\square_p$ be as in (1.71) with $(\Omega^{0,\bullet}(X,E),\bar{\partial}^E,\bar{\partial}^E)\otimes g^E$ replaced by $(\Omega^{p,\bullet}(X),\bar{\partial},g^\omega)$ with $\omega$ the Fubini–Study metric. More precisely, $\omega$ is the Kähler form on $\mathbb{P}^n$ such that for any $\mathcal{L}$, we have $\int_{\mathbb{P}^n} \omega^{n-k} = 1$, where $b_k$ is the $k$th Betti number of $\mathbb{P}^n$. From (1.127) and (1.128), we obtain $|\epsilon_X|_{\Omega(X),\omega} = 1$, which is equivalent to the second equality in (1.125). This completes the proof.

2. Several properties of the BCOV torsion

2.1 Kähler metric on projective bundle

For a complex vector space $V$, we denote by $\mathbb{P}(V)$ the set of complex lines in $V$. Then $\mathbb{P}(V)$ is complex manifold.

Let $Y$ be an $m$-dimensional compact Kähler manifold. Let $N$ be a holomorphic vector bundle over $Y$ of rank $n$. Let $\mathcal{L}$ be the trivial line bundle over $Y$. Set

$$X = \mathbb{P}(N \oplus \mathcal{L}).$$

(2.1)

Let $\pi : X \to Y$ be the canonical projection. For $y \in Y$, we denote $Z_y = \pi^{-1}(y)$, which is isomorphic to $\mathbb{C}P^n$. Let $\omega_{\mathbb{C}P^n}$ be the Kähler form on $\mathbb{C}P^n$ associated with the Fubini–Study metric. More precisely, $-i\omega_{\mathbb{C}P^n}$ is equal to the curvature of the tautological line bundle over $\mathbb{C}P^n$ equipped with the standard metric.

Lemma 2.1. There exists a Kähler form $\omega$ on $X$ such that for any $y \in Y$, there exists an isomorphism $\phi_y : \mathbb{C}P^n \to Z_y$ such that $\phi_y^*(\omega|_{Z_y}) = \omega_{\mathbb{C}P^n}$.

Here $(\phi_y)_{y \in Y}$ is merely a set of maps parameterized by $y \in Y$. It is not even required to depend continuously on $y$.

Proof. We refer the reader to the proof of [Voi02, Proposition 3.18].

Let $s \in \{1,\ldots,n\}$. We assume that there are holomorphic line bundles $L_1,\ldots,L_s$ over $Y$ together with a surjection between holomorphic vector bundles,

$$N \to L_1 \oplus \cdots \oplus L_s.$$  

(2.2)
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For \( k = 1, \ldots, s \), let \( N \to L_k \) be the composition of (2.2) and the canonical projection \( L_{1} \oplus \cdots \oplus L_{s} \to L_{k} \). Set

\[
N_{k} = \text{Ker}(N \to L_{k}) \subseteq N, \quad X_{k} = \mathbb{P}(N_{k} \oplus \mathcal{K}) \subseteq X, \quad X_{0} = \mathbb{P}(N) \subseteq X. \tag{2.3}
\]

Let \( [\xi_0 : \cdots : \xi_n] \) be homogenous coordinates on \( \mathbb{C}P^n \). For \( k = 0, \ldots, n \), we denote \( H_k = \{ \xi_k = 0 \} \subseteq \mathbb{C}P^n \).

**Lemma 2.2.** There exists a Kähler form \( \omega \) on \( X \) such that for any \( y \in Y \), there exists an isomorphism \( \phi_y : \mathbb{C}P^n \to Z_y \) such that \( \phi_y^* (\omega|_{Z_y}) = \omega_{\mathbb{C}P^n} \) and \( \phi_y^{-1} (X_y \cap Z_y) = H_k \) for \( k = 0, \ldots, s \).

**Proof.** Let \( N^* \) be the dual of \( N \). We have \( L_{1}^{-1} \oplus \cdots \oplus L_{s}^{-1} \hookrightarrow N^* \). Let \( g^{N*} \) be a Hermitian metric on \( N^* \) such that \( L_{1}^{-1}, \ldots, L_{s}^{-1} \subseteq N^* \) are mutually orthogonal. Let \( g^N \) be the dual metric on \( N \).

Now, proceeding in the same way as in the proof of [Voi02, Proposition 3.18], we obtain \( \omega \) satisfying the desired properties. This completes the proof. \( \square \)

### 2.2 Behavior under adiabatic limit

We use the notation in § 2.1. By Lemma 2.1, there exists a Kähler form \( \omega_X \) on \( X \) such that for any \( y \in Y \), there exists an isomorphism \( \phi_y : \mathbb{C}P^n \to Z_y \) such that

\[
\phi_y^* (\omega_X|_{Z_y}) = \omega_{\mathbb{C}P^n}. \tag{2.4}
\]

Let \( \omega_{Z_y} = \omega_X|_{Z_y} \). Note that \( (Z_y, \omega_{Z_y})_{y \in Y} \) are mutually isometric, we omit the index \( y \) as long as there is no confusion. Let \( \omega_Y \) be a Kähler form on \( Y \). For \( \varepsilon > 0 \), set

\[
\omega_{\varepsilon} = \omega_X + \frac{1}{\varepsilon} \pi^* \omega_Y. \tag{2.5}
\]

We denote

\[
(c_1c_{m-1})(Y) = \int_Y c_1(TY)c_{m-1}(TY). \tag{2.6}
\]

Let \( \chi(\cdot) \) be the topological Euler characteristic. Recall that \( \tau_{BCOV}(\cdot, \cdot) \) was defined in Definition 1.24.

**Theorem 2.3.** As \( \varepsilon \to 0 \),

\[
\tau_{BCOV}(X, \omega_{\varepsilon}) - \frac{1}{12} \chi(Z)(m\chi(Y) + (c_1c_{m-1})(Y)) \log \varepsilon
\]

\[
\to \chi(Z)\tau_{BCOV}(Y, \omega_Y) + \chi(Y)\tau_{BCOV}(Z, \omega_Z). \tag{2.7}
\]

**Proof.** The proof consists of several steps.

Recall that \( \eta(\cdot) \) was constructed in (1.118) and \( \lambda_{\text{tot}}(\cdot) \) was constructed in (1.119).

**Step 1.** We construct two canonical sections of

\[
\lambda_{\text{tot}}(X) \otimes (\lambda_{\text{tot}}(Y))^{-\chi(Z)} \otimes (\eta(Y))^{-m\chi(Z)}. \tag{2.8}
\]

For \( p = 0, \ldots, m + n \) and \( s = 0, \ldots, p - 1 \), set

\[
P^p_s = \{ u \in \Lambda^p(T^*X) : u(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in TZ, v_{s+2}, \ldots, v_p \in TX \}. \tag{2.9}
\]

For convenience, we denote \( P^p_0 = \Lambda^p(T^*X) \) and \( P^p_{-1} = 0 \). We obtain a filtration

\[
\Lambda^p(T^*X) = P^p_0 \hookrightarrow P^p_{m-1} \hookrightarrow \cdots \hookrightarrow P^p_{-1} = 0. \tag{2.10}
\]

For \( r = 0, \ldots, m \) and \( s = 0, \ldots, n \), we denote

\[
E_{r, s} = \Lambda^s(T^*Z) \otimes \pi^s\Lambda^r(T^*Y). \tag{2.11}
\]

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We have a short exact sequence of holomorphic vector bundles over $X$, 

$$0 \to I_{s-1}^{r+s} \to I_s^{r+s} \to E_{r,s} \to 0.$$  

(2.12)

Let 

$$\alpha_{r,s} \in (\det H^\bullet(X, I_{s-1}^{r+s}))^{-1} \otimes (\det H^\bullet(X, I_s^{r+s}) \otimes (\det H^\bullet(X, E_{r,s}))^{-1}. \quad (2.13)$$

be the canonical section induced by the long exact sequence induced by (2.12).

Let $H^\bullet\bullet(Z)$ be the fiberwise cohomology. As $Z \simeq \mathbb{CP}^n$, we have 

$$H^p,q(Z) = \mathbb{C} \quad \text{for } p = 0, \ldots, n, \quad H^p,q(Z) = 0 \quad \text{for } p \neq q. \quad (2.14)$$

Applying spectral sequence while using (2.11) and (2.14), we obtain 

$$H^s(X, E_{r,s}) \simeq H^{r,q-s}(Y, H^{s,s}(Z)) := H^{q-s}(Y, \Lambda^r(T^*Y) \otimes H^{s,s}(Z)). \quad (2.15)$$

Let 

$$\beta_{r,s} \in \det H^\bullet(X, E_{r,s}) \otimes (\det H^{r,s}(Y, H^{s,s}(Z)))^{-(1)^s} \quad (2.16)$$

be the canonical section induced by (2.15).

We have a generator of lattice, 

$$\delta_s \in H^2_{\text{Sing}}(\mathbb{CP}^n, \mathbb{Z}) \subseteq H^2_{\text{Sing}}(\mathbb{CP}^n, \mathbb{R}) \subseteq H^2_{\text{Sing}}(\mathbb{CP}^n, \mathbb{C}). \quad (2.17)$$

We identify $H^2_{\text{Sing}}(\mathbb{CP}^n, \mathbb{C})$ with $H^2_{\text{dR}}(\mathbb{CP}^n) = H^{s,s}(\mathbb{CP}^n)$ (see (1.121)). Since $H^{s,s}(Z) = H^{s,s}(\mathbb{CP}^n) = H^2_{\text{Sing}}(\mathbb{CP}^n, \mathbb{C})$ is a trivial line bundle over $Y$, we have an isomorphism (cf. [GH94, p. 607]) 

$$H^r\bullet(Y) \to H^r\bullet(Y, H^{s,s}(Z)) = H^r\bullet(Y) \otimes H^{s,s}(\mathbb{CP}^n)$$

$$u \mapsto u \otimes \delta_s. \quad (2.18)$$

Let 

$$\gamma_{r,s} \in (\det H^r\bullet(Y, H^{s,s}(Z)))^{-1} \otimes (\det H^r\bullet(Y))^{-(1)^s} \quad (2.19)$$

be the canonical section induced by (2.18). By (2.13), (2.16) and (2.19), we have 

$$\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s} \in (\det H^r\bullet(X, I_{s-1}^{r+s}))^{-1} \otimes (\det H^r\bullet(X, I_s^{r+s}) \otimes (\det H^r\bullet(Y))^{-(1)^s}. \quad (2.20)$$

Recall that $\lambda(\cdot)$ was defined in (1.119). By (1.119) and (2.10), we have 

$$\lambda(X) = \bigotimes_{p=1}^{m+n} (\det H^\bullet(X, \Lambda^p(T^*X)))^{-(1)^p}$$

$$= \bigotimes_{p=1}^{m+n} (\det H^\bullet(X, I_p^r))^{-(1)^p}$$

$$= \bigotimes_{p=0}^{m} \bigotimes_{s=0}^{n} ((\det H^\bullet(X, I_{s-1}^{r+s}))^{-1} \otimes \det H^\bullet(X, I_s^{r+s}))^{-(1)^r+s(r+s)}. \quad (2.21)$$

On the other hand, by (1.118), (1.119) and the identities 

$$n + 1 = \chi(Z), \quad \sum_{s=0}^{n} s = \frac{n(n + 1)}{2} = \frac{n}{2} \chi(Z), \quad (2.22)$$

we have 

$$\bigotimes_{r=0}^{m} \bigotimes_{s=0}^{n} (\det H^r\bullet(Y))^{-(1)^r(r+s)} = (\lambda(Y))^{\chi(Z)} \otimes (\eta(Y))^{n \chi(Z)/2}. \quad (2.23)$$

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By (2.20), (2.21) and (2.23), we have
\[
\prod_{r=0}^{m} \prod_{s=0}^{n} (\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s})^{(-1)^{r+s}(r+s)} \in \lambda(X) \otimes (\lambda(Y))^{-\chi(Z)} \otimes (\eta(Y))^{-n\chi(Z)/2}.
\] (2.24)

By (1.119) and (2.24), we have
\[
\prod_{r=0}^{m} \prod_{s=0}^{n} (\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s})^{(-1)^{r+s}(r+s)} \otimes \prod_{r=0}^{m} \prod_{s=0}^{n} (\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s})^{(-1)^{r+s}(r+s)}
\in \lambda_{\text{tot}}(X) \otimes (\lambda_{\text{tot}}(Y))^{-\chi(Z)} \otimes (\eta(Y))^{-n\chi(Z)},
\] (2.25)
where $\bar{\cdot}$ is the conjugation.

Let $\sigma_X \in \lambda_{\text{tot}}(X)$, $\sigma_Y \in \lambda_{\text{tot}}(Y)$ and $\epsilon_Y \in \eta(Y)$ be as in (1.122). Obviously, we have
\[
\sigma_X \otimes \sigma_Y^{-\chi(Z)} \otimes \epsilon_Y^{-n\chi(Z)} \in \lambda_{\text{tot}}(X) \otimes (\lambda_{\text{tot}}(Y))^{-\chi(Z)} \otimes (\eta(Y))^{-n\chi(Z)}.
\] (2.26)

**Step 2.** We show that
\[
\prod_{r=0}^{m} \prod_{s=0}^{n} (\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s})^{(-1)^{r+s}(r+s)} \otimes \prod_{r=0}^{m} \prod_{s=0}^{n} (\alpha_{r,s} \otimes \beta_{r,s} \otimes \gamma_{r,s})^{(-1)^{r+s}(r+s)} = \pm \sigma_X \otimes \sigma_Y^{-\chi(Z)} \otimes \epsilon_Y^{-n\chi(Z)}.
\] (2.27)

Let $Z(-1)$ be the inverse of the Tate twist, which is a Hodge structure of pure weight two. For $j \in \mathbb{N}$, we denote by $Z(-j)$ its $j$th tensor power. We have canonical identifications of Hodge structures,
\[
H^{2j}_{\text{Sing}}(\mathbb{C}P^n, Z) = Z(-j) \quad \text{for } j = 0, \ldots, n,
\]
\[
H^k_{\text{Sing}}(X, Z) = \bigoplus_{j=0}^{n} H^{k-2j}_{\text{Sing}}(Y, Z) \otimes H^{2j}_{\text{Sing}}(\mathbb{C}P^n, Z)
\] (2.28)
\[
= \bigoplus_{j=0}^{n} H^{k-2j}_{\text{Sing}}(Y, Z) \otimes Z(-j).
\]

Complexifying (2.28) and applying Hodge decomposition, we obtain
\[
H^{j,j}(\mathbb{C}P^n) = \mathbb{C} \quad \text{for } j = 0, \ldots, n,
\]
\[
H^{p,q}(X) = \bigoplus_{j=0}^{n} H^{p-j,q-j}(Y) \otimes H^{j,j}(\mathbb{C}P^n) = \bigoplus_{j=0}^{n} H^{p-j,q-j}(Y).
\] (2.29)

We use the identifications in (2.28) and (2.29) until the end of Step 2.

**Claim.** For complex vector spaces $A$ and $B$, the canonical identification $\det A \otimes \det B \otimes (\det(A \oplus B))^{-1} = \mathbb{C}$ is such that the canonical section of $\det A \otimes \det B \otimes (\det(A \oplus B))^{-1}$ is identified with $1 \in \mathbb{C}$.

Recall that $I^{r+s}_s$ was defined in (2.9) and $E_{r,s}$ was defined in (2.11). We have
\[
H^q(X, I^{r+s}_s) = \bigoplus_{j=0}^{s} H^{r+s-j,q-j}(Y), \quad H^q(X, E_{r,s}) = H^{r,q-s}(Y).
\] (2.30)

By (2.30), we have
\[
H^q(X, I^{r+s}_s) = H^q(X, I^{r+s}_{s-1}) \oplus H^q(X, E_{r,s}).
\] (2.31)
Applying the claim in the last paragraph to (2.31), we obtain
\[(\det H^\bullet(X, I_{s-1}^+))^{-1} \otimes \det H^\bullet(X, I_s^+) \otimes (\det H^\bullet(X, E_{r,s}))^{-1} = \mathbb{C}, \quad \alpha_{r,s} = 1. \tag{2.32}\]

A similar argument shows that
\[
\det H^\bullet(X, E_{r,s}) \otimes (\det H^r(Y, H^{s,s}(Z)))^{-(−1)^s} = \mathbb{C}, \quad \beta_{r,s} = 1,
\]
\[
(\det H^r(Y, H^{s,s}(Z)))^{(−1)^s} \otimes (\det H^\bullet(Y))^{−(−1)^s} = \mathbb{C}, \quad \gamma_{r,s} = 1.
\tag{2.33}\]

Using (1.119), (1.121) and (2.28), we can show that
\[
\lambda_{tot}(X) \otimes (\lambda_{tot}(Y))^{-\chi(Z)} \otimes (\eta(Y))^{-n\chi(Z)} = \mathbb{C},
\]
\[
\sigma_X \otimes \sigma_Y^{-\chi(Z)} \otimes \epsilon_Y^{-n\chi(Z)} = \pm 1.
\tag{2.34}\]

From (2.32)–(2.34), we obtain (2.27).

**Step 3.** We introduce several Quillen metrics.

- Let \(g_{TX}^r\) be the metric on \(TX\) induced by \(\omega_e\).
- Let \(g_{\Lambda^p(T^*X)}^r\) be the metric on \(\Lambda^p(T^*X)\) induced by \(g_{TX}^r\).
- Let \(g_{I_s^+}^p\) be the metric on \(I_s^+\) induced by \(g_{\Lambda^p(T^*X)}^r\) via (2.10).
- Let \(g^{TY}\) be the metric on \(TY\) induced by \(\omega_Y\).
- Let \(g^{\Lambda^r(T^*Y)}\) be the metric on \(\Lambda^r(T^*Y)\) induced by \(g^{TY}\).
- Let \(g^{TZ}\) be the metric on \(TZ\) induced by \(\omega_Z = \omega_e|_Z\).
- Let \(g^{\Lambda^s(T^*Z)}\) be the metric on \(\Lambda^s(T^*Z)\) induced by \(g^{TZ}\).
- Let \(g^{E_{r,s}}\) be the metric on \(E_{r,s}\) induced by \(g^{\Lambda^r(T^*Y)}\) and \(g^{\Lambda^s(T^*Z)}\) via (2.11).

Let
\[
\|\cdot\|_{\det H^\bullet(X, I_s^+), \varepsilon}
\tag{2.35}\]
be the Quillen metric on \(\det H^\bullet(X, I_s^+)\) associated with \(g_{TX}^r\) and \(g_{I_s^+}^p\). Let
\[
\|\cdot\|_{\det H^\bullet(X, E_{r,s}), \varepsilon}
\tag{2.36}\]
be the Quillen metric on \(\det H^\bullet(X, E_{r,s})\) associated with \(g_{TX}^r\) and \(g^{E_{r,s}}\). Recall that \(\alpha_{r,s}\) was defined by (2.13). Let \(\|\alpha_{r,s}\|_{\varepsilon}\) be the norm of \(\alpha_{r,s}\) with respect to the metrics (2.35) and (2.36).

- Let \(g^{\Omega^{s,s}(Z)}\) be the \(L^2\)-metric on \(\Omega^{s,s}(Z)\) induced by \(g^{TZ}\) (see (1.70)).
- Let \(g^{H^{s,s}(Z)}\) be the metric on \(H^{s,s}(Z)\) induced by \(g^{\Omega^{s,s}(Z)}\) via the Hodge theorem.

Let
\[
\|\cdot\|_{\det H^r(X, H^{s,s}(Z))}
\tag{2.37}\]
be the Quillen metric on \(\det H^r(X, H^{s,s}(Z)) = \det H^\bullet(Y, \Lambda^r(T^*Y) \otimes H^{s,s}(Z))\) associated with \(g^{TY}\) and \(g^{\Lambda^r(T^*Y)} \otimes g^{H^{s,s}(Z)}\). Recall that \(\beta_{r,s}\) was defined by (2.16). Let \(\|\beta_{r,s}\|_{\varepsilon}\) be the norm of \(\beta_{r,s}\) with respect to the metrics (2.36) and (2.37). Let
\[
\|\cdot\|_{\det H^r(X, Y)}
\tag{2.38}\]
be the Quillen metric on \(\det H^r(X, Y) = \det H^\bullet(Y, \Lambda^r(T^*Y))\) associated with \(g^{TY}\) and \(g^{\Lambda^r(T^*Y)}\). Recall that \(\gamma_{r,s}\) was defined by (2.19). Let \(\|\gamma_{r,s}\|\) be the norm of \(\gamma_{r,s}\) with respect to the metrics (2.37) and (2.38).
By (1.119) and (2.10), we have
\[
\sigma_X \in \lambda_{tot}(X) = \bigotimes_{p=1}^{m+n} (\det H^*(X,I_p)^{(−1)p})^{−1} \otimes \bigotimes_{p=1}^{m+n} (\det H^*(X,I_p)^{(−1)p})^{−1}.
\]

Let \(\|\sigma_X\|_\epsilon\) be the norm of \(\sigma_X\) with respect to the metrics (2.35) with \(s = p\). By (1.118) and (1.119), we have
\[
\epsilon_Y \in \eta(Y) = \bigotimes_{r=0}^{m} (\det H^r(Y)^{(−1)r}),
\]
\[
\sigma_Y \in \lambda_{tot}(Y) = \bigotimes_{r=1}^{m} (\det H^r(Y)^{(−1)r}) \otimes \bigotimes_{r=1}^{m} (\det H^r(Y)^{(−1)r}).
\]

Let \(\|\epsilon_Y\|\) be the norm of \(\epsilon_Y\) with respect to the metrics (2.38). Let \(\|\sigma_Y\|\) be the norm of \(\sigma_Y\) with respect to the metrics (2.38). By (2.27), we have
\[
\sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s}(r+s) \left( \log \|\alpha_{r,s}\|_\epsilon^2 + \log \|\beta_{r,s}\|_\epsilon^2 + \log \|\gamma_{r,s}\|_\epsilon^2 \right) = \log \|\sigma_X\|_\epsilon - \chi(Z) \log \|\sigma_Y\| - n\chi(Z) \log \|\epsilon_Y\|.
\]

On the other hand, by Definition 1.23 and Proposition 1.25, we have
\[
\log \|\epsilon_Y\| = 0.
\]

By Definition 1.24, (2.41) and (2.42), we have
\[
\tau_{BCOV}(X,\omega_\epsilon) = \chi(Z)\tau_{BCOV}(Y,\omega_Y) + \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s}(r+s) \left( \log \|\alpha_{r,s}\|_\epsilon^2 + \log \|\beta_{r,s}\|_\epsilon^2 + \log \|\gamma_{r,s}\|_\epsilon^2 \right).
\]

**Step 4.** We estimate \(\log \|\alpha_{r,s}\|_\epsilon^2\).

Recall that \(I_{s+r}^+\) was defined in (2.9), \(E_{r,s}\) was defined in (2.11), \(g^+_r\) and \(g^{-r}_r\) were defined at the beginning of Step 3. Let \(g_{E_{r,s}}^+\) be quotient metric on \(E_{r,s}\) induced by \(g^+_r\) via the surjection \(I_{s+r}^+ \to E_{r,s}\). Note that \(g^+_r\) is induced by \(\omega_\epsilon\). By (2.5), as \(\epsilon \to 0\),
\[
\epsilon^{-r} g_{E_{r,s}}^+ \to g_{E_{r,s}}^-.
\]

We use the notation from (1.23). Let
\[
\tilde{T}_{r,s,\epsilon} = \tilde{\text{ch}}(g^+_r,g_{E_{r,s}}^-) \in Q^{X}/Q^{X,0}
\]
be the Bott–Chern form (1.56) with \(0 \to E' \to E \to E'' \to 0\) replaced by (2.12) and \((g^E',g^E,g^{E''})\) replaced by \((g^+_1,g_{E_{r,s}}^+,g_{E_{r,s}}^-)\). Let
\[
\tilde{T}_{r,s,\epsilon} = \tilde{\text{ch}}(g^+_r,g_{E_{r,s}}^-) \in Q^{X}/Q^{X,0}
\]
be the Bott–Chern form (1.56) with \(0 \to E' \to E \to E'' \to 0\) replaced by (2.12) and \((g^E',g^E,g^{E''})\) replaced by \((g^+_1,g_{E_{r,s}}^+,g_{E_{r,s}}^-)\). By Proposition 1.11 and (2.44), as \(\epsilon \to 0\),
\[
T_{r,s,\epsilon} - \tilde{T}_{r,s,\epsilon} - \text{ch}(E_{r,s},g_{E_{r,s}}^-)r \log \epsilon = \text{ch}(g_{E_{r,s}}^+,g_{E_{r,s}}^-) - \text{ch}(E_{r,s},g_{E_{r,s}}^-)r \log \epsilon \to 0.
\]
On the other hand, by Proposition 1.12, as $\varepsilon \to 0$,
\[ T_{r,s,\varepsilon} \to 0. \tag{2.48} \]

By (2.47) and (2.48), as $\varepsilon \to 0$,
\[ T_{r,s,\varepsilon} - \text{ch}(E_{r,s}, g_{E_{r,s}}) r \log \varepsilon \to 0. \tag{2.49} \]

Applying Theorem 1.18 to the short exact sequence (2.12), we obtain
\[ \log \|\alpha_{r,s}\|^2_{\varepsilon} = \int_X Td(TX, g_{TX}) T_{r,s,\varepsilon}. \tag{2.50} \]

By Proposition 1.10, as $\varepsilon \to 0$,
\[ Td(TX, g_{TX}) \to \pi^* Td(TY,g_{TY}) Td(TZ, g_{TZ}). \tag{2.51} \]

On the other hand, by the Grothendieck–Riemann–Roch formula (1.76), (2.11) and (2.14), we have
\[ \int_X \pi^* Td(TY,g_{TY}) Td(TZ, g_{TZ}) \text{ch}(E_{r,s}, g_{E_{r,s}}) = \int_Y Td(TY) \text{ch}(\Lambda^r(T^sY)). \tag{2.52} \]

By (2.49)–(2.52), as $\varepsilon \to 0$,
\[ \log \|\alpha_{r,s}\|^2_{\varepsilon} - (-1)^s \int_Y Td(TY) \text{ch}(\Lambda^r(T^sY)) \log \varepsilon \to 0. \tag{2.53} \]

By Proposition 1.5, (2.22) and (2.53), as $\varepsilon \to 0$,
\[ \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} (r + s) \log \|\alpha_{r,s}\|^2_{\varepsilon} \]
\[ - \left( \frac{m(3m + 3n + 1)}{12} \chi(Y) + \frac{1}{6} (c_1 c_{m-1})(Y) \right) \chi(Z) \log \varepsilon \to 0. \tag{2.54} \]

Step 5. We estimate $\log \|\beta_{r,s}\|^2_{\varepsilon}$.

Let
\[ T_{r,s} \in Q^Y \tag{2.55} \]
be the Bismut–Kähler analytic torsion form (see §1.5) associated with $(\pi : X \to Y, \omega_X, E_{r,s}, g_{E_{r,s}})$. Applying Theorem 1.17 with $E = E_{r,s}$, as $\varepsilon \to 0$,
\[ \log \|\beta_{r,s}\|^2_{\varepsilon} + \int_Y Td'(TY) \int_Z Td(TZ) \text{ch}(E_{r,s}) \log \varepsilon \to \int_Y Td(TY, g_{TY}) T_{r,s}. \tag{2.56} \]

Similarly to (2.52), we have
\[ \int_Y Td'(TY) \int_Z Td(TZ) \text{ch}(E_{r,s}) = (-1)^s \int_Y Td'(TY) \text{ch}(\Lambda^r(T^sY)). \tag{2.57} \]

Applying Proposition 1.15 with $E = E_{0,s}$ and $F = \Lambda^r(T^sY)$, we obtain
\[ T_{r,s} = \text{ch}(\Lambda^r(T^sY), g_{\Lambda^r(T^sY)}) T_{0,s} \text{ modulo } Q^Y. \tag{2.58} \]
By (2.56)–(2.58), as \( \varepsilon \to 0 \),
\[
\log \| \beta_{r,s} \|_2^2 + (-1)^s \int_Y Td'(TY) \text{ch}(\Lambda^r(T^*Y)) \log \varepsilon
\]
\[
- \int_Y Td(TY, g^TY) \text{ch}(\Lambda^r(T^*Y), g^{\Lambda^r(T^*Y)}) T_{0,s}.
\]
(2.59)

On the other hand, by Theorem 1.16, we have
\[
\sum_{s=0}^{n} (-1)^s T_{0,s} = 0 \text{ modulo } Q_{Y}.
\]
(2.60)

By Propositions 1.5, 1.6, (2.22), (2.59) and (2.60), as \( \varepsilon \to 0 \),
\[
\sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} (r + s) \log \| \beta_{r,s} \|_\varepsilon^2 + \left( \frac{m(m+n)}{4} \chi(Y) + \frac{1}{12} (c_1 c_{m-1})(Y) \right) \chi(Z) \log \varepsilon
\]
\[
- \int_Y c_m(TY, g^TY) \sum_{s=0}^{n} (-1)^s s T_{0,s}
\]
\[
= \int_Y c_m(TY, g^TY) \sum_{s=0}^{n} (-1)^s s \{ T_{0,s} \}^{(0,0)}.
\]
(2.61)

where \( \{ \cdot \}^{(0,0)} \) means the component of degree \((0,0)\).

Step 6. We calculate \( \log \| \gamma_{r,s} \|_2^2 \).

Recall that \( H^{s,s}(Z) \) is a trivial line bundle over \( Y \). Recall that \( g^{H^{s,s}(Z)} \) was constructed in the paragraph above (2.37). By our assumption (2.4), \( g^{H^{s,s}(Z)} \) is a constant metric. Recall that \( \delta_s \in H^{s,s}(Z) \) was constructed in (2.17). Let \( |\delta_s| \) be the norm of \( \delta_s \) with respect to \( g^{H^{s,s}(Z)} \), which is a constant function on \( Y \). In the following, we do not distinguish between a constant function and its value. We denote \( \chi_r(Y) = \sum_{q=0}^{m} (-1)^q \dim H^{r,q}(Y) \). By Remark 1.14, we have
\[
\log \| \gamma_{r,s} \|_2^2 = (-1)^s \chi_r(Y) \log |\delta_s|^2.
\]
(2.62)

Let \( \epsilon_Z \in \eta(Z) \) be as in (1.122). We have
\[
\epsilon_Z = \pm \bigotimes_{s=0}^{n} \delta_s.
\]
(2.63)

Let \( |\epsilon_Z| \) be the norm of \( \epsilon_Z \) with respect to the metrics \( g^{H^{s,s}(Z)} \). By Proposition 1.25 and (2.63), we have
\[
\sum_{s=0}^{n} \log |\delta_s|^2 = \log |\epsilon_Z|^2 = 0.
\]
(2.64)

Let \( \sigma_Z \in \lambda_{\text{tot}}(Z) \) be as in (1.122). We have
\[
\sigma_Z = \pm \bigotimes_{s=1}^{n} \delta_s^{2s}.
\]
(2.65)

Let \( |\sigma_Z| \) be the norm of \( \sigma_Z \) with respect to the metrics \( g^{H^{s,s}(Z)} \). By (2.65), we have
\[
\sum_{s=0}^{n} s \log |\delta_s|^2 = \log |\sigma_Z|.
\]
(2.66)
By (2.62), (2.64), (2.66) and the identity \( \sum_{r=0}^{n} (-1)^r \chi_r(Y) = \chi(Y) \), we have

\[
\sum_{r=0}^{n} \sum_{s=0}^{n} (-1)^{r+s} (r+s) \log \|\gamma_{r,s}\|^2 = \chi(Y) \log |\sigma_Z|.
\]

(2.67)

**Step 7.** We conclude.

By (2.43), (2.54), (2.61) and (2.67), as \( \varepsilon \to 0 \),

\[
\tau_{BCOV}(X,\omega_\varepsilon) - \frac{1}{12} \chi(Z) (m\chi(Y) + (c_1 c_{m-1})(Y)) \log \varepsilon
\]

\[
\to \chi(Z) \tau_{BCOV}(Y,\omega_Y) + \chi(Y) \log |\sigma_Z|
\]

\[
+ \int_Y c_m(TY, g^{TY}) \sum_{s=0}^{n} (-1)^s s \{T_{0,s}\}^{(0,0)}.
\]

(2.68)

Let \( \theta_s(z) \) be as in (1.74) with \( (X,\omega) \) replaced by \( (Z,\omega_Z) \) and \( (E,g^E) \) replaced by \( (\Lambda^*(T^*Z), g^{\Lambda^*(T^*Z)}) \). By Definition 1.13, 1.24, we have

\[
\tau_{BCOV}(Z,\omega_Z) = \log |\sigma_Z| + \sum_{s=0}^{n} (-1)^s s \theta'_s(0).
\]

(2.69)

By (2.4), all the terms in (2.69) are constant functions on \( Y \). By (1.79), we have

\[
\{T_{0,s}\}^{(0,0)} = \theta'_s(0).
\]

(2.70)

From (2.68)–(2.70), we obtain (2.7). This completes the proof.

**Remark 2.4.** The key ingredient in the proof of Theorem 2.3 is [BB94, Theorem 3.2], which is a consequence of [BB94, Theorem 3.1]. Of course, we can replace [BB94, Theorem 3.2] by [BB94, Theorem 3.1] in our proof to obtain a formula for \( \tau_{BCOV}(X,\omega_X) \). However, because [BB94, Theorem 3.1] involves a Bott–Chern form, the formula obtained will be far from clean.

### 2.3 Behavior under blow-ups

The following lemma is direct consequence of Bott formula [Bot57] (see also [OSS11, p. 5]).

**Lemma 2.5.** Let \( L \) be the holomorphic line bundle of degree one over \( \mathbb{C}P^n \). For \( k = 1, \ldots, n \) and \( s = 1, \ldots, k \), we have

\[
H^* (\mathbb{C}P^n, \Lambda^k (T^* \mathbb{C}P^n) \otimes L^s) = 0.
\]

(2.71)

Let \( X \) be an \( n \)-dimensional compact Kähler manifold. Let \( Y \subseteq X \) be a closed complex submanifold. Let \( f : X' \to X \) be the blow-up along \( Y \). Let \( Y \subseteq U \subseteq X \) be an open neighborhood of \( Y \). Set \( U' = f^{-1}(U) \). Let \( \omega \) be a Kähler form on \( X \). Let \( \omega' \) be a Kähler form on \( X' \) such that

\[
\omega'|_{X \setminus U} = f^*(\omega|_{X \setminus U}).
\]

(2.72)

For the existence of such \( \omega' \), see the proof of [Voi02, Proposition 3.24].

**Theorem 2.6.** We have

\[
\tau_{BCOV}(X',\omega') - \tau_{BCOV}(X,\omega) = \alpha(U,U',\omega|_U,\omega'|_{U'}),
\]

(2.73)

where \( \alpha(U,U',\omega|_U,\omega'|_{U'}) \) is a real number determined by \( U, U', \omega|_U \) and \( \omega'|_{U'} \).

**Proof.** The proof consists of several steps.
Step 0. We introduce several pieces of notation. We denote \( D = f^{-1}(Y) \). Let \( i : D \hookrightarrow X' \) be the canonical embedding. Let \( \mathcal{I} \subseteq \mathcal{O}_{X'} \) be the ideal sheaf associated with \( D \). More precisely, for open subset \( U \subseteq X' \), we have

\[
\mathcal{I}(U) = \{ \theta \in \mathcal{O}_{X'}(U) : \theta|_{U \cap D} = 0 \}.
\] (2.74)

For \( p = 0, \ldots, n \), there exist holomorphic vector bundles over \( X' \) linked by holomorphic maps

\[
f^* \Lambda^p(T^*X) = F^p_0 \to F^p_1 \to \cdots \to F^p_n = \Lambda^p(T^*X')
\] (2.75)
such that for \( s = 0, \ldots, p-1 \),

- the induced map \( \partial_{X'}(F^p_{s+1}) \to \partial_{X'}(F^p_s) \) is injective;
- we have \( \mathcal{I} \otimes \partial_{X'}(F^p_s) \hookrightarrow \partial_{X'}(F^p_{s+1}) \hookrightarrow \partial_{X'}(F^p_s) \).

Set

\[
\mathcal{F}_s^p = \partial_{X'}(F^p_s)/\partial_{X'}(F^p_{s+1}).
\] (2.76)

Then we have a commutative diagram of analytic coherent sheaves on \( X' \),

\[
\begin{array}{cccccc}
0 & \longrightarrow & \partial_{X'}(F^p_{s+1}) & \longrightarrow & \partial_{X'}(F^p_s) & \longrightarrow & \mathcal{F}_s^p & \longrightarrow & 0 \\
& & & \downarrow & & \uparrow \iota_s \partial_{D}(F^p_s|_D) & \\
& & & & & 0 & & & 
\end{array}
\] (2.77)

where the first row is exact. Now we briefly explain the existence of these \( F^p_s \). We have

\[
\mathcal{I} \otimes \partial_{X'}(\Lambda^p(T^*X')) \hookrightarrow \partial_{X'}(f^* \Lambda^p(T^*X)) \hookrightarrow \partial_{X'}(\Lambda^p(T^*X')).
\] (2.78)

For \( s = 0, \ldots, p \), let \( \mathcal{F}_s^p \) be the sub-sheaf of \( \partial_{X'}(\Lambda^p(T^*X')) \) generated by \( \mathcal{I} \otimes \partial_{X'}(\Lambda^p(T^*X')) \) and \( \partial_{X'}(f^* \Lambda^p(T^*X)) \). Then the desired properties hold with \( \partial_{X'}(F^p_s) \) replaced by \( \mathcal{F}_s^p \). It remains to show that each \( \mathcal{F}_s^p \) is given by a holomorphic vector bundle. Let \( r \) be the codimension of \( Y \hookrightarrow X \). Let \( N_Y \) be the normal bundle of \( Y \hookrightarrow X \). Let \( \pi : D = \mathbb{P}(N_Y) \to Y \) be the canonical projection. Let \( (y_0, y_1, \ldots, y_{n-r}, z_1, \ldots, z_{r-1}) \in \mathbb{C}^n \) be local coordinates on a neighborhood of \( x \in D \) such that:

- \( (y_1, \ldots, y_{n-r}) \) are the coordinates on \( Y \);
- \( (z_1, \ldots, z_{r-1}) \) are the coordinates on the fiber of \( \pi : D \to Y \);
- \( D \subseteq X' \) is given by the equation \( y_0 = 0 \).

Then the image of \( \partial_{X'}(f^*T^*X) \hookrightarrow \partial_{X'}(T^*X') \) is generated by

\[
dy_0, dy_1, \ldots, dy_{n-r}, y_0 dz_1, \ldots, y_0 dz_{r-1}.
\] (2.79)

As a consequence, the image of \( \mathcal{F}_s^p \hookrightarrow \partial_{X'}(\Lambda^p(T^*X')) \) is generated by

\[
y_0^{\min\{s,|I|\}} \prod_{i \in I} dy_i \otimes \bigotimes_{j \in J} dz_j
\] (2.80)

with \( I \subseteq \{0, 1, \ldots, n-r\} \) and \( J \subseteq \{1, \ldots, r-1\} \) satisfying \(|I| + |J| = p\). Each term in (2.80) yields a holomorphic line bundle. Hence, \( \mathcal{F}_s^p \) is given by a holomorphic vector bundle, which we denote by \( F^p_s \).
Let $TD \to \pi^*TY$ be the derivative of $\pi$. Set
\[ T^V D = \text{Ker}(TD \to \pi^*TY) \subseteq TD \subseteq TX'|_D. \] (2.81)

Set
\[ I^p_s = \{ \alpha \in \Lambda^p(T^*X')|_D : \alpha(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in T^V D, \ldots, v_p \in TX'|_D \}. \] (2.82)

We obtain a filtration of holomorphic vector bundles over $D$,
\[ \Lambda^p(T^*X')|_D = I^p_p \supseteq I^p_{p-1} \supseteq \cdots \supseteq I^p_0. \] (2.83)

Let $N_D$ be the normal line bundle of $D \to X'$. From the calculation in local coordinates, we see that
\[ G^p_s = i_* \mathcal{O}_D(N^{-s}_D \otimes (I^p_p/I^p_s)) \quad \text{for } s = 0, \ldots, p-1. \] (2.84)

For convenience, we denote
\[ G^p_s = N^{-s}_D \otimes (I^p_p/I^p_s). \] (2.85)

Then we obtain a short exact sequence
\[ 0 \to G^p_{s+1} \to G^p_s \to i_* \mathcal{O}_D(G^p_s) \to 0. \] (2.86)

**Step 1.** We show that
\[ H^q(D, G^p_0) = \bigoplus_{k=1}^{r-1} H^{k,k}(\mathbb{C}P^{r-1}) \otimes H^{p-k,q-k}(Y), \] (2.87)
\[ H^q(D, G^p_s) = 0 \quad \text{for } s = 1, \ldots, p-1. \]

Set
\[ J^p_s = \{ \alpha \in \Lambda^p(T^*D) : \alpha(v_1, \ldots, v_p) = 0 \text{ for any } v_1, \ldots, v_{s+1} \in T^V D, \ldots, v_p \in TD \}. \] (2.88)

Let $\phi : \Lambda^p(T^*X')|_D \to \Lambda^p(T^*D)$ be the canonical projection. By (2.82) and (2.88), we have
\[ J^p_p = \phi(I^p_p) \subseteq \Lambda^p(T^*D). \] (2.89)

By (2.83) and (2.89), we have a filtration of holomorphic vector bundles over $D$,
\[ \Lambda^p(T^*D) = J^p_p \supseteq J^p_{p-1} \supseteq \cdots \supseteq J^p_0. \] (2.90)

We also have
\[ J^p_{k+1}/J^p_k = \pi^*(\Lambda^{p-k}(T^*Y)) \otimes \Lambda^k(T^V,sD), \] (2.91)
and a short exact sequence of holomorphic vector bundles over $D$,
\[ 0 \to N^{-1}_D \otimes J^p_{k-1} \to J^p_k \to J^p_k \to 0. \] (2.92)

Combining (2.91) and (2.92), we obtain a short exact sequence,
\[ 0 \to N^{-1}_D \otimes \pi^*(\Lambda^{p-k-1}(T^*Y)) \otimes \Lambda^k(T^V,sD) \to I^p_k/I^p_{k-1} \]
\[ \to \pi^*(\Lambda^{p-k}(T^*Y)) \otimes \Lambda^k(T^V,sD) \to 0. \] (2.93)

By (2.85) and (2.93), $G^p_0$ admits a filtration with factors
\[ (N^{-s-\epsilon}_D \otimes \pi^*(\Lambda^{p-k-\epsilon}(T^*Y)) \otimes \Lambda^k(T^V,sD))_{\epsilon=0,1,k=s+1,\ldots,p}. \] (2.94)

We remark that $\pi : D \to Y$ is a $\mathbb{C}P^{r-1}$-bundle and the restriction of $N^{-1}_D$ to the fiber of $\pi : D \to Y$ is a holomorphic line bundle of degree one. Applying spectral sequence while using
BCOV invariant and blow-up

Lemma 2.5, we see that the cohomology of the holomorphic vector bundles in (2.94) vanishes unless \( \epsilon = s = 0 \). Hence, we obtain the second identity in (2.87). This argument also shows that

\[
H^q(D, G^*_0) = H^q(D, J^p_p/I^p_0) = H^q(D, J^p_p/J^p_0).
\]

(2.95)

Using spectral sequence and (2.91), we obtain

\[
H^q(D, J^p_k/J^p_{k-1}) = H^{p,k}(\mathbb{CP}^{r-1}) \otimes H^{p-k,q-k}(Y).
\]

(2.96)

On the other hand, it is classical that

\[
H^q(D, J^p_p) = H^q(D, \Lambda^p(T^*D)) = \bigoplus_{k=0}^{r-1} H^{k,k}(\mathbb{CP}^{r-1}) \otimes H^{p-k,q-k}(Y).
\]

(2.97)

From (2.95)–(2.97), we obtain the first identity in (2.87).

Set

\[
\lambda(G^*_0) = \bigotimes_{p=1}^n (\det H^\bullet(D, G^*_p))^{-1} \otimes \lambda(G^*_0) = \lambda(G^*_0) \otimes \lambda(G^*_0).
\]

(2.98)

Recall that \( \lambda_{\text{tot}}(X) \) was defined in (1.119).

**Step 2.** We construct two canonical sections of

\[
(\lambda_{\text{tot}}(X))^{-1} \otimes \lambda_{\text{tot}}(X') \otimes (\lambda_{\text{tot}}(G^*_0))^{-1}
\]

(2.99)

and show that they coincide up to \( \pm 1 \).

Let

\[
\mu_{p,s} \in (\det H^\bullet(X', F^p_{s+1}))^{-1} \otimes \det H^\bullet(X', F^p_s) \otimes (\det H^\bullet(D, G^*_p))^{-1}
\]

(2.100)

be the canonical section induced by the long exact sequence induced by (2.86). Indeed, by (2.87), we have

\[
\mu_{p,s} \in (\det H^\bullet(X', F^p_{s+1}))^{-1} \otimes \det H^\bullet(X', F^p_s) \quad \text{for } s \neq 0.
\]

(2.101)

Set

\[
\mu_p = \bigotimes_{s=0}^{p-1} \mu_{p,s} \in (\det H^\bullet(X', F^p_p))^{-1} \otimes \det H^\bullet(X', F^p_p) \otimes (\det H^\bullet(D, G^*_p))^{-1}
\]

\[
= (\det H^\bullet(X', f^\bullet \Lambda^p(T^*X)))^{-1} \otimes \det H^{p,\bullet}(X') \otimes (\det H^\bullet(D, G^*_p))^{-1}.
\]

(2.102)

We remark that \( f_\ast \mathcal{O}_{X'} = \mathcal{O}_X \) and \( R^p f_\ast \mathcal{O}_{X'} = 0 \). Using spectral sequence, we obtain a canonical identification

\[
H^{p,\bullet}(X) = H^\bullet(X', f^\bullet \Lambda^p(T^*X)).
\]

(2.103)

Let

\[
\nu_p \in (\det H^{p,\bullet}(X))^{-1} \otimes \det H^\bullet(X', f^\bullet \Lambda^p(T^*X))
\]

(2.104)

be the canonical section induced by (2.103).

By (2.102) and (2.104), we have

\[
\mu_p \otimes \nu_p \in (\det H^{p,\bullet}(X))^{-1} \otimes \det H^{p,\bullet}(X') \otimes (\det H^\bullet(D, G^*_0))^{-1}.
\]

(2.105)

By (1.119), (2.98) and (2.105), we have

\[
\bigotimes_{p=1}^n (\mu_p \otimes \nu_p)^{-1} \in (\lambda(X))^{-1} \otimes \lambda(X') \otimes (\lambda(G^*_0))^{-1},
\]

(2.106)
and
\[ \bigotimes_{p=1}^{n} (\mu_p \otimes \nu_p)^{(-1)p} \bigotimes_{p=1}^{n} (\mu_p \otimes \nu_p)^{(-1)p} \in (\lambda^p_{\text{tot}}(X))^1 \otimes \lambda^p_{\text{tot}}(X') \otimes (\lambda^p_{\text{tot}}(G_0^*))^1. \] 

(2.107)

We have the Hodge decomposition
\[ H^j_{\text{dR}}(Y) = \bigoplus_{p+q=j} H^{p,q}(Y). \] 

(2.108)

Let \( b_k \) be the \( k \)-th Betti number of \( Y \). By (2.87), (2.98) and (2.108), we have
\[ \lambda^p_{\text{tot}}(G_0^*) = \bigotimes_{k=1}^{r-1} 2k+2n-2r \bigotimes_{j=2k}^{b_j} \left( (\det H^2_{\text{dR}}(\mathbb{CP}^{r-1}))^{b_j-2k} \otimes \det H^{2k}_{\text{dR}}(Y) \right)^{(-1)^j}. \] 

(2.109)

Let
\[ \delta_j \in H^j_{\text{Sing}}(\mathbb{CP}^{r-1}, \mathbb{Z}) \subseteq H^j_{\text{Sing}}(\mathbb{CP}^{r-1}, \mathbb{C}) = H^j_{\text{dR}}(\mathbb{CP}^{r-1}) \] 

be a generator of \( H^j_{\text{Sing}}(\mathbb{CP}^{r-1}, \mathbb{Z}) \). Let
\[ \tau_{j,1}, \ldots, \tau_{j,b_j} \in \text{Im}(H^2_{\text{Sing}}(Y, \mathbb{Z}) \rightarrow H^2_{\text{Sing}}(Y, \mathbb{R})) \subseteq H^2_{\text{dR}}(Y) \] 

be a basis of the lattice. We denote \( \tilde{\tau}_j = \tau_{j,1} \wedge \cdots \wedge \tau_{j,b_j} \in \det H^2_{\text{dR}}(Y) \). Set
\[ \sigma_{G_0^*} = \bigotimes_{k=1}^{r-1} 2k+2n-2r \bigotimes_{j=2k}^{b_j} \left( \delta_{2k}^{b_j-2k} \otimes \tau_{j-2k} \right)^{(-1)^j} \in \lambda^p_{\text{tot}}(G_0^*). \] 

(2.112)

Let \( \sigma_X \in \lambda^p_{\text{tot}}(X) \) and \( \sigma_{X'} \in \lambda^p_{\text{tot}}(X') \) be as in (1.122). Obviously, we have
\[ \sigma_{X}^{-1} \otimes \sigma_{X'} \otimes \sigma_{G_0^*}^{-1} \in (\lambda^p_{\text{tot}}(X))^{-1} \otimes \lambda^p_{\text{tot}}(X') \otimes (\lambda^p_{\text{tot}}(G_0^*))^{-1}. \] 

(2.113)

We have a canonical identification (cf. [Voi02, Théorème 7.31])
\[ H^j_{\text{Sing}}(X', \mathbb{Z}) = H^j_{\text{Sing}}(X, \mathbb{Z}) \oplus \bigoplus_{k=1}^{r-1} H^2_{\text{Sing}}(\mathbb{CP}^{r-1}, \mathbb{Z}) \otimes H^{2k}_{\text{Sing}}(Y, \mathbb{Z}), \] 

which induces an isomorphism of Hodge structures. Similarly to Step 2 in the proof of Theorem 2.3, using (2.114), we can show that
\[ \bigotimes_{p=1}^{n} (\mu_p \otimes \nu_p)^{(-1)p} \bigotimes_{p=1}^{n} (\mu_p \otimes \nu_p)^{(-1)p} = \pm \sigma_{X}^{-1} \otimes \sigma_{X'} \otimes \sigma_{G_0^*}^{-1}. \] 

(2.115)

**Step 3.** We introduce Quillen metrics.

Let \( g^{TX} \) be the metric on \( TX \) induced by \( \omega \). Let \( g^{A^p(T^*X)} \) be the metric on \( A^p(T^*X) \) induced by \( g^{TX} \). Let
\[ \|\cdot\|_{\det H^p_{\bullet}(X)} \] 

be the Quillen metric on \( \det H^p_{\bullet}(X) = \det H^*(X, A^p(T^*X)) \) associated with \( g^{TX} \) and \( g^{A^p(T^*X)} \).

Let \( g^{TX'} \) be the metric on \( TX' \) induced by \( \omega' \). Let \( g^{A^p(T^*X')} \) be the metric on \( A^p(T^*X') \) induced by \( g^{TX'} \). Let
\[ \|\cdot\|_{\det H^p_{\bullet}(X')} \] 

be the Quillen metric on \( \det H^p_{\bullet}(X') = \det H^*(X', A^p(T^*X')) \) associated with \( g^{TX'} \) and \( g^{A^p(T^*X')} \).
Let
\[
\|\cdot\|_{\det H^\bullet(X', f^* \Lambda^p(T^*X))} \tag{2.118}
\]
be the Quillen metric on \(\det H^\bullet(X', f^* \Lambda^p(T^*X))\) associated with \(g^{TX'}\) and \(f^* g^{\Lambda^p(T^*X)}\).

Let \(g^{TD}\) and \(g^{ND}\) be the metrics on \(TD\) and \(ND\) induced by \(g^{TX'}\). Let \(g^{F_0^{p}}\) be the metric on \(F^{p}_{0}\) induced by \(g^{\Lambda^p(T^*X)}\) via (2.83). Let \(g^{G^{p}_{s}}\) be the metric on \(G^{p}_{s}\) induced by \(g^{ND}\) and \(g^{F^{p}_{s}}\) via (2.85). Let
\[
\|\cdot\|_{\det H^\bullet(D, G^{p}_{s})} \tag{2.119}
\]
be the Quillen metric on \(H^\bullet(D, G^{p}_{s})\) associated with \(g^{TD}\) and \(g^{G^{p}_{s}}\). By the second identity in (2.87), we have a canonical identification \(\det H^\bullet(D, G^{p}_{s}) = \mathbb{C}\) for \(s \neq 0\). However, the metric (2.119) with \(s \neq 0\) is not necessarily the standard metric on \(\mathbb{C}\).

We remark that
\[
\Lambda^p(T^*X')|_{X'\setminus U'} = F^{p}_{s}|_{X'\setminus U'} = f^* \Lambda^p(T^*X)|_{X'\setminus U'} \quad \text{for} \ s = 0, \ldots, p. \tag{2.120}
\]
We equip \(F^{p}_{s}\) with Hermitian metric \(g^{F^{p}_{s}}\) such that
\[
g^{F^{p}_{0}} = g^{\Lambda^p(T^*X')}, \quad g^{F^{p}_{s}} = f^* g^{\Lambda^p(T^*X)}, \tag{2.121}
\]
\[
g^{F^{p}_{s+1}}|_{X'\setminus U'} = g^{F^{p}_{s}}|_{X'\setminus U'} \quad \text{for} \ s = 0, \ldots, p - 1.
\]
Our assumption (2.72) implies \(g^{\Lambda^p(T^*X')}|_{X'\setminus U'} = f^* (g^{\Lambda^p(T^*X)}|_{X'\setminus U'})\), which guarantees the existence of \(g^{F^{p}_{s}}\) satisfying (2.121). Let
\[
\|\cdot\|_{\det H^\bullet(X', F^{p}_{s})} \tag{2.122}
\]
be the Quillen metric on \(\det H^\bullet(X', F^{p}_{s})\) associated with \(g^{TX'}\) and \(g^{F^{p}_{s}}\). We remark that \(H^\bullet(X', F^{p}_{0}) = H^p\bullet(X')\) and
\[
\|\cdot\|_{\det H^\bullet(X', F^{p}_{0})} = \|\cdot\|_{\det H^p\bullet(X')} \tag{2.123}
\]
Recall that \(\mu_{p,s}\) was defined in (2.100). Let \(\|\mu_{p,s}\|\) be the norm of \(\mu_{p,s}\) with respect to the metrics (2.119) and (2.122).

Recall that \(\nu_{p}\) was defined in (2.104). Let \(\|\nu_{p}\|\) be the norm of \(\nu_{p}\) with respect to the Quillen metrics (2.116) and (2.118).

Recall that \(\sigma_{G^{*}_{s}}\) was defined in (2.112). By (2.98) and the second identity in (2.87), we can and do view \(\sigma_{G^{*}_{s}}\) as the section of
\[
\lambda_{\text{tot}}(G^{*}_{s}) := \bigotimes_{p=1}^{n} \bigotimes_{s=0}^{p-1} (\det H^\bullet(D, G^{p}_{s}))^{(-1)^{pp}} \bigotimes_{p=1}^{n} \bigotimes_{s=0}^{p-1} (\det H^\bullet(D, G^{p}_{s}))^{(-1)^{pp}}. \tag{2.124}
\]
Let \(\|\sigma_{G^{*}_{s}}\|_{\lambda_{\text{tot}}(G^{*}_{s})}\) be the norm of \(\sigma_{G^{*}_{s}} \in \lambda_{\text{tot}}(G^{*}_{s})\) with respect to the metrics (2.119).

Let \(\|\sigma_{X'}\|_{\lambda_{\text{tot}}(X')}\) be the norm of \(\sigma_{X'}\) with respect to the metrics (2.116). Let \(\|\sigma_{X'}\|_{\lambda_{\text{tot}}(X')}\) be the norm of \(\sigma_{X'}\) with respect to the metrics (2.117). By (2.102) and (2.115), we have
\[
\log\|\sigma_{X'}\|_{\lambda_{\text{tot}}(X')} - \log\|\sigma_{X}\|_{\lambda_{\text{tot}}(X)} = \log\|\sigma_{G^{*}_{s}}\|_{\lambda_{\text{tot}}(G^{*}_{s})} = \
\sum_{p=1}^{n} (-1)^{p} p \left( \log\|\nu_{p}\|^{2} + \sum_{s=0}^{p-1} \log\|\mu_{p,s}\|^{2} \right). \tag{2.125}
\]
By Definition 1.24 and (2.125), we have
\[ \tau_{\text{BCOV}}(X', \omega') - \tau_{\text{BCOV}}(X, \omega) = \log \|\sigma_{G_0}^*\|_{\lambda_{\text{tot}}(G_0^*)} + \sum_{p=1}^{n} (-1)^p p \left( \log \|\nu_p\|^2 + \sum_{s=0}^{p-1} \log \|\mu_{p,s}\|^2 \right). \] (2.126)

**Step 4.** We conclude.

For ease of notation, we denote
\[ \alpha_{p,s} = \log \|\mu_{p,s}\|^2. \] (2.127)

Applying Theorem 1.19 to the short exact sequence (2.86) while using the second line in (2.121), we see that \( \alpha_{p,s} \) is determined by \( (U', \omega'|_{U'}, g^{F_p}|_{U'}, g^{F_p}|_{U'}) \). We denote
\[ \alpha_p = \sum_{s=0}^{p-1} \alpha_{p,s}. \] (2.128)

We remark that for \( s = 1, \ldots, p-1 \), the contributions of the metric \( \|\cdot\|_{\det H^*_{(X', F_p')}} \) (see (2.122)) to \( \alpha_{p,s-1} \) and \( \alpha_{p,s} \) cancel with each other. Thus, \( \alpha_p \) is independent of \( (g^{F_p})_{s=1, \ldots, p-1} \). Hence, \( \alpha_p \) is determined by \( (U', \omega'|_{U'}, g^{F_p}|_{U'}, g^{F_p}|_{U'}) \). Now, applying the first line in (2.121), we see that \( \alpha_p \) is determined by \( (U, U', \omega|_{U'}, \omega'|_{U'}) \).

For ease of notation, we denote
\[ \beta_p = \log \|\nu_p\|^2. \] (2.129)

Applying Theorem 1.21 with \( E = \Lambda^p(T^*X) \) while using (2.72), we see that \( \beta_p \) is determined by \( (U, U', \omega|_{U'}, \omega'|_{U'}) \).

By (2.126)–(2.129), we have
\[ \tau_{\text{BCOV}}(X', \omega') - \tau_{\text{BCOV}}(X, \omega) = \log \|\sigma_{G_0}^*\|_{\lambda_{\text{tot}}(G_0^*)} + \sum_{p=1}^{n} (-1)^p p (\alpha_p + \beta_p). \] (2.130)

Here:
- the section \( \sigma_{G_0}^* \in \lambda_{\text{tot}}(G_0^*) \) is determined by \( D \subseteq U' \) and its normal bundle;
- the Quillen metric \( \|\cdot\|_{\lambda_{\text{tot}}(G_0^*)} \) is determined by \( \omega'|_{U'} \);
- the real number \( \alpha_p \) is determined by \( (U, U', \omega|_{U'}, \omega'|_{U'}) \);
- the real number \( \beta_p \) is determined by \( (U, U', \omega|_{U'}, \omega'|_{U'}) \).

In conclusion, the right-hand side of (2.130) is determined by \( (U, U', \omega|_{U'}, \omega'|_{U'}) \). This completes the proof.

Let \( \pi : \mathcal{Y} \to \mathbb{C} \) be a holomorphic submersion between complex manifolds. Let \( \mathcal{Y} \subseteq \mathcal{Y} \) be a closed complex submanifold. We assume that \( \pi|_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C} \) is a holomorphic submersion with compact fiber. For \( z \in \mathbb{C} \), we denote \( U_z = \pi^{-1}(z) \) and \( Y_z = U_z \cap \mathcal{Y} \). Assume that for any \( z \in \mathbb{C} \), \( U_z \) can be extended to a compact Kähler manifold. More precisely, there exist a compact Kähler manifold \( X_z \) and a holomorphic embedding \( i_z : U_z \to X_z \) whose image is open. Here \( \{X_z : z \in \mathbb{C}\} \) is just a set of complex manifolds parameterized by \( \mathbb{C} \). The topology of \( X_z \) may vary as \( z \) varies. We identify \( U_z \) with \( i_z(U_z) \subseteq X_z \). Let \( f_z : X'_z \to X_z \) be the blow-up along \( Y_z \).

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Set $U'_z = f^{-1}_z(U_z) \subseteq X'_z$. Let
\[
(\omega_z \in \Omega^{1,1}(X_z))_{z \in \mathbb{C}}, \quad (\omega'_z \in \Omega^{1,1}(X'_z))_{z \in \mathbb{C}}
\]
be Kähler forms. We assume that $(\omega_z|_{U_z})_{z \in \mathbb{C}}$ and $(\omega'_z|_{U'_z})_{z \in \mathbb{C}}$ are smooth families. We further assume that
\[
\omega'_z|_{X'_z \setminus U'_z} = f'_z(\omega_z|_{X \setminus U_z}) \quad \text{for } z \in \mathbb{C}.
\]

**Theorem 2.7.** The function $z \mapsto \tau_{\text{BCOV}}(X'_z, \omega'_z) - \tau_{\text{BCOV}}(X_z, \omega_z)$ is continuous.

**Proof.** We proceed in the same way as in the proof of Theorem 2.6. Each object constructed becomes a function of $z \in \mathbb{C}$. In particular, the identity (2.130) becomes
\[
\tau_{\text{BCOV}}(X'_z, \omega'_z) - \tau_{\text{BCOV}}(X_z, \omega_z) = \log\|\sigma_{G_0^*}\|_{\lambda(\mathfrak{g}^*_0)} + \sum_{p=1}^n (-1)^p p(\alpha_{p,z} + \beta_{p,z}).
\]

From Remarks 1.20 and 1.22 and the last paragraph in the proof of Theorem 2.6, we see that each term on the right-hand side of (2.133) is a continuous function of $z$. This completes the proof.

\[\square\]

3. **BCOV invariant**

3.1 **Several meromorphic sections**

Let $X$ be a compact complex manifold. Let $K_X$ be the canonical line bundle of $X$. Let $d$ be a non-zero integer. Let $K^d_X$ be the $d$th tensor power of $K_X$. We assume that there is an invertible element $\gamma \in \mathcal{M}(X, K^d_X)$. We denote
\[
\text{div}(\gamma) = D = \sum_{j=1}^l m_j D_j,
\]
where $m_j \in \mathbb{Z} \setminus \{0\}$, $D_1, \ldots, D_l \subseteq X$ are mutually distinct and irreducible. We assume that $D$ is of simple normal crossing support (see Definition 1.2).

For $J \subseteq \{1, \ldots, l\}$, let $D_J \subseteq X$ be as in (0.9). For $j \in J \subseteq \{1, \ldots, l\}$, let $L_{J,j}$ be the normal line bundle of $D_J \hookrightarrow D_{J \setminus \{j\}}$. Set
\[
K_J = K^d_X|_{D_J} \otimes \bigotimes_{j \in J} L_{J,j}^{-m_j} = K^d_{D_J} \otimes \bigotimes_{j \in J} L_{J,j}^{-m_j-d},
\]
which is a holomorphic line bundle over $D_J$. In particular, we have $K_0 = K^d_X$.

Recall that $\text{Res}(\cdot)$ was defined in Definition 1.4. By (1.9), there exist
\[
(\gamma_J \in \mathcal{M}(D_J, K_J))_{J \subseteq \{1, \ldots, l\}}
\]
such that
\[
\gamma_0 = \gamma, \quad \gamma_J = \text{Res}_{D_J}(\gamma_{J \setminus \{j\}}) \quad \text{for } j \in J \subseteq \{1, \ldots, l\}.
\]
By (1.8), we have
\[
\text{div}(\gamma_J) = \sum_{j \notin J} m_j D_{J \cup \{j\}}.
\]

3.2 **Construction of BCOV invariant**

We use the notation from § 3.1. We further assume that $X$ is Kähler and $m_j \neq -d$ for $j = 1, \ldots, l$. Then $(X, \gamma)$ is a $d$-Calabi–Yau pair (see Definition 0.2).
Let $\omega$ be a Kähler form on $X$. Let $| \cdot |_{K_{D_j}}$ be the metric on $K_{D_j}$ induced by $\omega$. Let $| \cdot |_{L_{J,j}}$ be the metric on $L_{J,j}$ induced by $\omega$. Let $| \cdot |_{K_j}$ be the metric on $K_j$ induced by $| \cdot |_{K_{D_j}}$ and $| \cdot |_{L_{J,j}}$ via (3.2).

We use the notation from (1.23). For $J \subseteq \{1, \ldots, l\}$, let $|J|$ be the number of elements in $J$, let $g_{\omega}^{TD_j}$ be the metric on $TD_j$ induced by $\omega$, let $c_k(TD_j, g_{\omega}^{TD_j}) \in Q^{D_j}$ be $k$th Chern form of $(TD_j, g_{\omega}^{TD_j})$. Let $n = \dim X$. Set

$$a_J(\gamma, \omega) = \frac{1}{12} \int_{D_j} c_{n-|J|}(TD_j, g_{\omega}^{TD_j}) \log |\gamma|_{K_j, \omega}^{2/d}.$$  \hspace{1cm} (3.6)

We consider the short exact sequence of holomorphic vector bundles over $D_j$,

$$0 \to TD_j \to TD_{j\setminus J}\mid_{D_j} \to L_{J,j} \to 0.$$  \hspace{1cm} (3.7)

Let

$$\tilde{c} \left( TD_j, TD_{j\setminus J}\mid_{D_j}, g_{\omega}^{TD_{j\setminus J}\mid_{D_j}} \right) \in Q^{D_j}/Q^{D_j,0}.$$  \hspace{1cm} (3.8)

be the Bott–Chern form (1.30) with $0 \to E' \to E \to E''$ replaced by (3.7) and $g^E$ replaced by $g_{\omega}^{TD_{j\setminus J}\mid_{D_j}}$. Set

$$b_{J,j}(\omega) = \frac{1}{12} \int_{D_j} \tilde{c} \left( TD_j, TD_{j\setminus J}\mid_{D_j}, g_{\omega}^{TD_{j\setminus J}\mid_{D_j}} \right).$$  \hspace{1cm} (3.9)

Let $w_d'$ be as in (0.9). Recall that $\tau_{BCOV}(-, -)$ was defined in Definition 1.24. For ease of notation, we denote $\tau_{BCOV}(D_j, \omega) = \tau_{BCOV}(D_j, \omega|_{D_j})$. We define

$$\tau_d(X, \gamma, \omega) = \sum_{J \subseteq \{1, \ldots, l\}} w_d' \left( \tau_{BCOV}(D_j, \omega) - a_J(\gamma, \omega) - \sum_{j \in J} \frac{m_j + d}{d} b_{J,j}(\omega) \right).$$  \hspace{1cm} (3.10)

**Theorem 3.1.** The real number $\tau_d(X, \gamma, \omega)$ is independent of $\omega$.

**Proof.** Let $(\omega_s)_{s \in \mathbb{CP}^1}$ be a smooth family of Kähler forms on $X$ parameterized by $\mathbb{CP}^1$. It is sufficient to show that $\tau_d(X, \gamma, \omega_s)$ is independent of $s$.

We view the terms involved in (3.10) as smooth functions on $\mathbb{CP}^1$, i.e.

$$\tau_d(X, \gamma, \omega) : s \mapsto \tau_d(X, \gamma, \omega_s),$$
$$\tau_{BCOV}(D_j, \omega) : s \mapsto \tau_{BCOV}(D_j, \omega_s),$$

etc.

We view $TD_j$ and $L_{J,j}$ as holomorphic vector bundles over $D_j \times \mathbb{CP}^1$. Let $g_{\omega}^{TD_j}$ and $g_{\omega}^{L_{J,j}}$ be metrics on $TD_j$ and $L_{J,j}$ induced by $(\omega_s)_{s \in \mathbb{CP}^1}$. More precisely, the restrictions $g_{\omega}^{TD_j}|_{D_j \times \{s\}}$ and $g_{\omega}^{L_{J,j}}|_{D_j \times \{s\}}$ are induced by $\omega_s$. By [Zha22, Theorem 1.6], we have

$$\frac{d}{d\theta} \tau_{BCOV}(D_j, \omega) = \frac{1}{12} \int_{D_j} c_{n-|J|}(TD_j, g_{\omega}^{TD_j})c_1(TD_j, g_{\omega}^{TD_j}).$$  \hspace{1cm} (3.11)
Similarly to [Zha22, (2.9)], by the Poincaré–Lelong formula, (3.2), (3.5) and (3.6), we have

\[
\overline{\partial} \partial \frac{1}{2\pi i} a_J(\gamma, \omega) = \frac{1}{12} \int_{D_J} c_{n-|J|}(TD_J, g^*_\omega)(-c_1(K_J, |\cdot|_{K_J, \omega}) + \delta_{\text{div}(\gamma_J)})
\]

\[
= \frac{1}{12} \int_{D_J} c_{n-|J|}(TD_J, g^*_\omega) c_1(TD_J, g^*_\omega)
\]

\[
+ \sum_{j \in J} \frac{m_j + d}{12d} \int_{D_J} c_{n-|J|}(TD_J, g^*_\omega) c_1(L_{J,j}, |\cdot|_{L_{J,j}, \omega})
\]

\[
+ \sum_{j \notin J} \frac{m_j}{12d} \int_{D_{J \cup \{j\}}} c_{n-|J|}(TD_J, g^*_\omega).
\]

(3.13)

Similarly to [Zha22, (2.10)], by (1.29), (1.30) and (3.9), we have

\[
\overline{\partial} \partial \frac{1}{2\pi i} b_{J,j}(\omega) = \frac{1}{12} \int_{D_{J \cup \{j\}}} c_{n-|J|+1}(TD_{J \cup \{j\}}, g^*_\omega) \]

\[
- \frac{1}{12} \int_{D_J} c_{n-|J|}(TD_J, g^*_\omega) c_1(L_{J,j}, g^*_\omega).
\]

(3.14)

By (3.12)–(3.14), we have

\[
\overline{\partial} \partial \left( \tau_{\text{BCOV}}(D_J, \omega) - a_J(\gamma, \omega) - \sum_{k \in J} \frac{m_j + d}{d} b_{J,j}(\omega) \right)
\]

\[
= - \sum_{j \in J} \frac{m_j + d}{12d} \int_{D_J} c_{n-|J|+1}(TD_{J \cup \{j\}}, g^*_\omega) - \sum_{j \notin J} \frac{m_j}{12d} \int_{D_{J \cup \{j\}}} c_{n-|J|}(TD_J, g^*_\omega).
\]

(3.15)

From (0.9), (3.10) and (3.15), we obtain \( \overline{\partial} \partial \tau_d(X, \gamma, \omega) = 0 \). Hence, \( s \mapsto \tau_d(X, \gamma, \omega_s) \) is constant on \( \mathbb{C}P^1 \). This completes the proof.

\[\square\]

**DEFINITION 3.2.** The BCOV invariant of \((X, \gamma)\) is defined by

\[
\tau_d(X, \gamma) = \tau_d(X, \gamma, \omega).
\]

(3.16)

By Theorem 3.1, \( \tau_d(X, \gamma) \) is well-defined.

**PROPOSITION 3.3.** For a non-zero integer \( r \), let \( \gamma^r \in \mathcal{M}(X, K_X^r) \) be the \( r \)th tensor power of \( \gamma \). Then \((X, \gamma^r)\) is a \( rd \)-Calabi–Yau pair and

\[
\tau_{rd}(X, \gamma^r) = \tau_d(X, \gamma).
\]

(3.17)

*Proof.* Once we replace \( \gamma \) by \( \gamma^r \), each \( \gamma_J \) is replaced by \( \gamma^r_J \). We can directly verify that

\[
\tau_{rd}(X, \gamma^r, \omega) = \tau_d(X, \gamma, \omega).
\]

(3.18)

From Definition 3.2 and (3.18), we obtain (3.17). This completes the proof.

\[\square\]

Recall that \( \chi_d(\cdot, \cdot) \) was defined in Definition 1.3.

**PROPOSITION 3.4.** For \( z \in \mathbb{C}^* \), we have

\[
\tau_d(X, z\gamma) = \tau_d(X, \gamma) - \frac{\chi_d(X, D)}{12} \log |z|^{2/d}.
\]

(3.19)
Proof. Once we replace $\gamma$ by $z\gamma$, each $\gamma_J$ is replaced by $z\gamma_J$. By (3.6), we have

\[ a_J(z\gamma, \omega) - a_J(\gamma, \omega) = \frac{\chi(D_J)}{12} \log |z|^{2/d}. \tag{3.20} \]

By Definition 1.3, (3.10) and (3.20), we have

\[ \tau_d(X, z\gamma, \omega) - \tau_d(X, \gamma, \omega) = -\frac{\chi_d(X, D)}{12} \log |z|^{2/d}. \tag{3.21} \]

From Definition 3.2 and (3.21), we obtain (3.19). This completes the proof. \qed

Proof of Theorem 0.4. As $\pi : X \to S$ is locally Kähler, for any $s_0 \in S$, there exist an open subset $s_0 \in U \subseteq S$ and a Kähler form $\omega$ on $\pi^{-1}(U)$. For $s \in U$, we denote $\omega_s = \omega|_{X_s}$. Similarly to the proof of Theorem 3.1, we view the terms involved in (3.10) as smooth functions on $U$.

Though the fibration $\pi^{-1}(U) \to U$ is not necessarily trivial, the identities (3.13) and (3.14) still hold. On the other hand, by [Zha22, Theorem 1.6], we have

\[ \frac{\partial}{\partial \tau} \mathcal{BCOV}_J(D_J, \omega) = \omega_H^{\bullet}(D_J) + \frac{1}{12} \int_{D_J} c_n - |J| (TD_J, g^{TD_J}_\omega) c_1(TD_J, g^{TD_J}_\omega). \tag{3.22} \]

By (0.9), (3.10), (3.13), (3.14) and (3.22), we have

\[ \frac{\partial}{\partial \tau} \tau_d(X, \gamma, \omega) \bigg|_U = \sum_{J \subseteq \{1, \ldots, l\}} w_d^{J} \omega_H^{\bullet}(D_J). \tag{3.23} \]

From Definition 3.2 and (3.23), we obtain (0.15). This completes the proof. \qed

3.3 BCOV invariant of projective bundle

Let $Y$ be a compact Kähler manifold. Let $N$ be a holomorphic vector bundle of rank $r \geq 2$ over $Y$. Let $\mathcal{L}$ be the trivial line bundle over $Y$. Set

\[ X = \mathbb{P}(N \oplus \mathcal{L}). \tag{3.24} \]

Let $\pi : X \to Y$ be the canonical projection.

Let $q \in \{0, \ldots, r\}$. Let $(L_k)_{k=1, \ldots, q}$ be holomorphic line bundles over $Y$. We assume that there is a surjection between holomorphic vector bundles

\[ N \to L_1 \oplus \cdots \oplus L_q. \tag{3.25} \]

Let $N^*$ be the dual of $N$. Taking the dual of (3.25), we obtain

\[ L_1^{-1} \oplus \cdots \oplus L_q^{-1} \hookrightarrow N^*. \tag{3.26} \]

Let $d, m_1, \ldots, m_q$ be positive integers. Let

\[ \gamma_Y \in \mathcal{M}(Y, (K_Y \otimes \det N^*)^d \otimes L_1^{-m_1} \otimes \cdots \otimes L_q^{-m_q}) \tag{3.27} \]

be an invertible element. We assume that

- $\text{div}(\gamma_Y)$ is of simple normal crossing support;
- $\text{div}(\gamma_Y)$ does not possess component of multiplicity $-d$.

Denote $m = m_1 + \cdots + m_q$. Let $S^m N^*$ be the $m$th symmetric tensor power of $N^*$. By (3.26) and (3.27), we have

\[ \gamma_Y \in \mathcal{M}(Y, (K_Y \otimes \det N^*)^d \otimes S^m N^*). \tag{3.28} \]

Let $\mathcal{N}$ be the total space of $N$. We have

\[ X = \mathcal{N} \cup \mathbb{P}(N), \quad K_X|_{\mathcal{N}} = \pi^*(K_Y \otimes \det N^*). \tag{3.29} \]
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We may view a section of $S^m N^*$ as a function on $\mathcal{N}$. By (3.28) and (3.29), $\gamma_Y$ may be viewed as an element of $\mathcal{M}(\mathcal{N}, K^d_X)$. Let

$$\gamma_X \in \mathcal{M}(X, K^d_X)$$

be such that $\gamma_X|_\mathcal{N} = \gamma_Y$.

For $j = 1, \ldots, q$, let $N \to L_j$ be the composition of the map (3.25) and the canonical projection $L_1 \oplus \cdots \oplus L_q \to L_j$. Set

$$N_j = \text{Ker}(N \to L_j), \quad X_j = \mathbb{P}(N_j \oplus \mathcal{H}) \subseteq X, \quad X_\infty = \mathbb{P}(N) \subseteq X.$$  

(3.31)

We denote

$$\text{div}(\gamma_Y) = \sum_{j=q+1}^l m_j Y_j,$$  

(3.32)

where $Y_j \subseteq Y$ are mutually distinct and irreducible. For $j = q + 1, \ldots, l$, set

$$X_j = \pi^{-1}(Y_j) \subseteq X.$$  

(3.33)

Denote

$$m_\infty = -m_1 - \cdots - m_q - rd - d.$$  

(3.34)

Note that:

- $X$ is locally the product of an open subset of $Y$ and $\mathbb{C}P^r$;
- $\gamma_X$ is locally the product of a $d$-canonical section on an open subset of $Y$ and $\gamma_{r,m_1,\ldots,m_q}$ defined in (0.20);

we have

$$\text{div}(\gamma_X) = \pi^* \text{div}(\gamma_Y) + m_\infty X_\infty + \sum_{j=1}^q m_j X_j = m_\infty X_\infty + \sum_{j=1}^l m_j X_j,$$  

(3.35)

which is of simple normal crossing support. Hence, $(X, \gamma_X)$ is a $d$-Calabi–Yau pair.

For $y \in Y$, we denote $Z_y = \pi^{-1}(y)$. Let $K_{Y,y}$ be the fiber of $K_Y$ at $y \in Y$. We have

$$K_X|_{Z_y} = K_{Z_y} \oplus \pi^* K_{Y,y}.$$  

(3.36)

For $y \in Y \setminus \bigcup_{j=q+1}^l Y_j$, there exist $\gamma_{Z_y} \in \mathcal{M}(Z_y, K^d_{Z_y})$ and $\eta_y \in K^d_{Y,y}$ such that

$$\gamma_X|_{Z_y} = \gamma_{Z_y} \oplus \pi^* \eta_y.$$  

(3.37)

Then $(Z_y, \gamma_{Z_y})$ is a $d$-Calabi–Yau pair, which is independent of $y$ up to isomorphism. We may omit the index $y$ as long as there is no confusion. We remark that $(Z, \gamma_Z)$ is isomorphic to $(\mathbb{C}P^r, \gamma_{r,m_1,\ldots,m_q})$ constructed in the paragraph containing (0.20).

Recall that $\chi_d(\cdot, \cdot)$ was defined in Definition 1.3.

**Lemma 3.5.** The following identity holds:

$$\chi_d(Z, \gamma_Z) = 0.$$  

(3.38)
Proof. Set
\[
f(t) = t^{r-q} \prod_{j \in \{1, \ldots, q, \infty\}} \left(t - \frac{m_j}{m_j + d}\right).
\] (3.39)

For \( J \subseteq \{1, \ldots, q, \infty\} \), let \( w_d^J \) be as in (0.9). By (1.3), (1.4) and the fact that \( \chi(\mathbb{C}P^k) = k + 1 \), we have
\[
\chi_d(Z, \gamma_Z) = \sum_{J \subseteq \{1, \ldots, q, \infty\}} w_d^J (r + 1 - |J|) = f'(1).
\] (3.40)

On the other hand, we have
\[
f'(1) = r - q + \sum_{j \in \{1, \ldots, q, \infty\}} \left(1 - \frac{m_j}{m_j + d}\right)^{-1}
= \frac{m_1 + \cdots + m_q + m_\infty}{d} + r + 1.
\] (3.41)

From (3.34), (3.40) and (3.41), we obtain (3.38). This completes the proof. \( \square \)

Theorem 3.6. The following identity holds:
\[
\tau_d(X, \gamma_X) = \chi_d(Y, \gamma_Y) \tau_d(Z, \gamma_Z).
\] (3.42)

Proof. The proof consists of several steps.

Step 0. We introduce several pieces of notation.

We denote \( A = \{q + 1, \ldots, l\} \) and \( B = \{1, \ldots, q, \infty\} \). For \( I \subseteq A \) and \( J \subseteq B \), set
\[
Y_I = Y \cap \bigcap_{j \in I} Y_j, \quad X_{I,J} = X \cap \bigcap_{j \in I \cup J} X_j,
\]
\[
X_I = X_{I,\emptyset}, \quad X_J = X_{\emptyset, J}.
\] (3.43)

For \( y \in Y \) and \( J \subseteq B \), set
\[
Z_{J,y} = Z_y \cap X_J.
\] (3.44)

Note that \( Z_{J,y} \) is independent of \( y \) up to isomorphism, we may omit the index \( y \) as long as there is no confusion. We remark that \( \pi|_{X_{I,J}} : X_{I,J} \to Y_I \) is a fibration with fiber \( Z_J \).

Let \( \omega_X \) be a Kähler form on \( X \) such that Lemma 2.2 holds. Let \( \omega_Y \) be a Kähler form on \( Y \). For \( \varepsilon > 0 \), set
\[
\omega_\varepsilon = \omega_X + \frac{1}{\varepsilon} \pi^* \omega_Y.
\] (3.45)

For \( I \subseteq A \), \( J \subseteq B \) and \( j \in (A \cup B)\setminus(I \cup J) \), let \( a_{I,J}(\gamma_X, \omega_\varepsilon) \) and \( b_{I,J,j}(\omega_\varepsilon) \) be as in (3.6) and (3.9) with \( (X, \gamma, \omega) \) replaced by \( (X, \gamma_X, \omega_\varepsilon) \) and \( J \) replaced by \( I \cup J \). Let \( w_d^I \) be as in (0.9) with \( J \) replaced by \( I \). By Definition 3.2, (0.9) and (3.10), we have
\[
\tau_d(X, \gamma_X) = \sum_{I \subseteq A} \sum_{J \subseteq B} w_d^I w_d^J \tau_{BCOV}(X_{I,J}, \omega_\varepsilon)
- \sum_{I \subseteq A} \sum_{J \subseteq B} w_d^I w_d^J a_{I,J}(\gamma_X, \omega_\varepsilon)
- \sum_{I \subseteq A} \sum_{J \subseteq B} \sum_{j \in I \cup J} w_d^I w_d^J m_j + \frac{d}{d} b_{I,J,j}(\omega_\varepsilon).
\] (3.46)

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Step 1. We estimate $\tau_{\text{BCOV}}(X_{I,J}, \omega_\varepsilon)$.

For $y \in Y$, we denote $\omega_{Z_y} = \omega_X|_{Z_y}$. As $\omega_X$ satisfies Lemma 2.2, for any $J \subseteq B$, $(Z,\omega_{Z_y})_{y \in Y}$ are mutually isometric. We may omit the index $y$ as long as there is no confusion. For ease of notation, we denote

$$\tau_{\text{BCOV}}(Y, \omega_Y) = \tau_{\text{BCOV}}(Y, \omega_Y|_{Y_1}), \quad \tau_{\text{BCOV}}(Z, \omega_Z) = \tau_{\text{BCOV}}(Z, \omega_Z|_{Z_j}). \tag{3.47}$$

For $I \subseteq A$ and $J \subseteq B$, by Theorem 2.3, as $\varepsilon \to 0$,

$$\tau_{\text{BCOV}}(X_{I,J}, \omega_\varepsilon) = \frac{\chi(Z_j)}{12} \left( \dim(Y_1)\chi(Y_1) + c_1c_{\dim(Y_1)-1}(Y_1) \right) \log \varepsilon$$

$$- \chi(Z_j)\tau_{\text{BCOV}}(Y, \omega_Y) + \chi(Y_1)\tau_{\text{BCOV}}(Z, \omega_Z). \tag{3.48}$$

On the other hand, by Lemma 3.5, (1.3) and (1.4), we have

$$\sum_{I \subseteq A} w_d^I \chi(Y_1) = \chi_d(Y, \gamma_Y), \quad \sum_{J \subseteq B} w_d^J \chi(Z_j) = 0. \tag{3.49}$$

By (3.48) and (3.49), as $\varepsilon \to 0$,

$$\sum_{I \subseteq A} \sum_{J \subseteq B} w_d^I w_d^J \tau_{\text{BCOV}}(X_{I,J}, \omega_\varepsilon) \to \chi_d(Y, \gamma_Y) \sum_{J \subseteq B} w_d^J \tau_{\text{BCOV}}(Z, \omega_Z). \tag{3.50}$$

Step 2. We estimate $a_{I,J}(\gamma_X, \omega_\varepsilon)$.

For $I \subseteq A$ and $J \subseteq B$, let $K_{I,J}$ be as in (3.2) with $(X, \gamma)$ replaced by $(X, \gamma_X)$ and $J$ replaced by $I \cup J$. Then $K_{I,J}$ is a holomorphic line bundle over $X_{I,J}$. Let

$$\gamma_{I,J} \in \mathcal{M}(X_{I,J}, K_{I,J}) \tag{3.51}$$

be as in (3.4) with $(X, \gamma)$ replaced by $(X, \gamma_X)$ and $J$ replaced by $I \cup J$.

Let $U \subseteq Y$ be a small open subset. Set $\mathcal{U} = \pi^{-1}(U)$. Recall that $\gamma_Z \in \mathcal{M}(Z, K^d_Z)$ was constructed in the paragraph containing (3.36). We fix an identification $\mathcal{U} = U \times Z$ such that there exists $\eta \in \mathcal{M}(U, K^d_Y)$ satisfying

$$\gamma_X|_{\mathcal{U}} = \text{pr}_1^* \eta \otimes \text{pr}_2^* \gamma_Z, \tag{3.52}$$

where $\text{pr}_1 : U \times Z \to U$ and $\text{pr}_2 : U \times Z \to Z$ are canonical projections.

For $I \subseteq A$, let $K_I$ be as in (3.2) with $(X, \gamma)$ replaced by $(U, \eta)$. Then $K_I$ is a holomorphic line bundle over $U \cap X_I$. Let

$$\eta_I \in \mathcal{M}(U \cap X_I, K_I) \tag{3.53}$$

be as in (3.4) with $(X, \gamma)$ replaced by $(U, \eta)$. For $J \subseteq B$, let $K_J$ be as in (3.2) with $(X, \gamma)$ replaced by $(Z, \gamma_Z)$. Then $K_J$ is a holomorphic line bundle over $Z_J$. Let

$$\gamma_J \in \mathcal{M}(Z_J, K_J) \tag{3.54}$$

be as in (3.4) with $(X, \gamma)$ replaced by $(Z, \gamma_Z)$. By the constructions of $K_{I,J}$ and $\gamma_{I,J}$ in the paragraph containing (3.51), we have

$$K_{I,J}|_{U \cap X_{I,J}} = \text{pr}_1^* K_I \otimes \text{pr}_2^* K_J, \quad \gamma_{I,J}|_{U \cap X_{I,J}} = \text{pr}_1^* \eta_I \otimes \text{pr}_2^* \gamma_J. \tag{3.55}$$

For $I \subseteq A$ and $J \subseteq B$, let $g_{X_{I,J}} (\text{respectively}, g^U_{I,J}, g^{Z_J})$ be the metric on $TX_{I,J}$ (respectively, $TY_I, TZ_J$) induced by $\omega_\varepsilon$ (respectively, $\omega_Y, \omega_Z$), let $| \cdot|_{K_{I,J}, \varepsilon}$ (respectively, $| \cdot|_{K_I}, | \cdot|_{K_J}$) be the norm on $K_{I,J}$ (respectively, $K_I, K_J$) induced by $\omega_\varepsilon$ (respectively, $\omega_Y, \omega_Z$) in the same
way as in the paragraph above (3.6). We denote
\[ a_{I,J}(\mathcal{U}, \gamma_X, \omega_\varepsilon) = \frac{1}{12} \int_{U \cap X_{I,J}} c(TX, g_{TX}^\varepsilon) \log |\gamma_{I,J}|_{K_{I,J,\varepsilon}}^{2/d}. \] (3.56)

Recall that \( \omega_\varepsilon \) was defined in (3.45). As \( g_{TX}^{\varepsilon X_{I,J}} \) is induced by \( \omega_\varepsilon \), by Proposition 1.7, as \( \varepsilon \to 0 \).
\[ c(TX_{I,J}, g_{TX}^{\varepsilon X_{I,J}}) \to c(TZ_{I,J}, g_{TZ_{I,J}}^{\varepsilon}) \pi^* c(TY_{I,J}, g_{TY_{I,J}}^{\varepsilon}). \] (3.57)

Recall that \( \eta_I, \gamma_J \) and \( \gamma_{I,J} \) are linked by (3.55). As \( |\cdot|_{K_{I,J,\varepsilon}} \) is induced by \( \omega_\varepsilon \), as \( \varepsilon \to 0 \),
\[ \log |\gamma_{I,J}|_{K_{I,J,\varepsilon}}^2 \to \left( \text{dim}(Y) + \sum_{j \in I} m_j \right) \log \varepsilon \to \log |\gamma_J|_{K_J}^2 + \log |\eta|_{K_I}^2. \] (3.58)

Let \( a_J(\gamma_Z, \omega_Z) \) be as in (3.6) with \( (X, \gamma, \omega) \) replaced by \( (Z, \gamma_Z, \omega_Z) \). More precisely,
\[ a_J(\gamma_Z, \omega_Z) = \frac{1}{12} \int_{Z_J} c(TZ_J, g_{TZ_J}^{\varepsilon}) \log |\gamma_Z|_{K_J}^{2/d}. \] (3.59)

By (3.56)–(3.59), as \( \varepsilon \to 0 \),
\[ a_{I,J}(\mathcal{U}, \gamma_X, \omega_\varepsilon) = \frac{\chi(Z_J)}{12} \left( \text{dim}(Y) + \frac{1}{d} \sum_{j \in I} m_j \right) \log \varepsilon \int_{U \cap Y_I} c(TY_I, g_{TY_I}^{\varepsilon}) \]
\[ - \frac{\chi(Z_J)}{12} \int_{U \cap Y_I} c(TY_I, g_{TY_I}^{\varepsilon}) \log |\gamma_I|_{K_I}^{2/d} + a_J(\gamma_Z, \omega_Z) \int_{U \cap Y_I} c(TY_I, g_{TY_I}^{\varepsilon}). \] (3.60)

By (3.49) and (3.60), as \( \varepsilon \to 0 \),
\[ \sum_{I \subseteq A} \sum_{J \subseteq B} w_{I,J}^d w_{I,J}^d a_{I,J}(U, \gamma_X, \omega_\varepsilon) \to \sum_{J \subseteq B} w_{I,J}^d a_J(\gamma_Z, \omega_Z) \sum_{I \subseteq A} \int_{U \cap Y_I} c(TY_I, g_{TY_I}^{\varepsilon}). \] (3.61)

The left-hand side of (3.61) yields a measure on \( X \),
\[ \mu_\varepsilon : \mathcal{U} \to \sum_{I \subseteq A} \sum_{J \subseteq B} w_{I,J}^d a_{I,J}(U, \gamma_X, \omega_\varepsilon), \] (3.62)

The right-hand side of (3.61) yields a measure on \( Y \),
\[ \nu : U \mapsto \sum_{J \subseteq B} \int_{U \cap Y_J} w_{I,J}^d \int_{U \cap Y_I} c(TY_I, g_{TY_I}^{\varepsilon}). \] (3.63)

The convergence in (3.61) is equivalent to the following: as \( \varepsilon \to 0 \),
\[ \pi_* \mu_\varepsilon \to \nu. \] (3.64)

By (3.49) and (3.62)–(3.64), as \( \varepsilon \to 0 \),
\[ \sum_{I \subseteq A} \sum_{J \subseteq B} w_{I,J}^d a_{I,J}(\gamma_X, \omega_\varepsilon) = \mu_\varepsilon(X) \to \nu(Y) = \chi_d(Y, \gamma_Y) \sum_{J \subseteq B} w_{I,J}^d a_J(\gamma_Z, \omega_Z). \] (3.65)

**Step 3.** We estimate \( b_{I,J}(\omega_\varepsilon) \).

First we consider the case \( j \in I \). We denote \( I' = I \setminus \{j\} \). By (3.9), we have
\[ b_{I,J}(\omega_\varepsilon) = \frac{1}{12} \int_{X_{I,J}} \tilde{c}(TX_{I,J}, TX_{I,J}, g_{TX_{I,J}}^{\varepsilon}|_{X_{I,J}}). \] (3.66)

By Proposition 1.9, as \( \varepsilon \to 0 \),
\[ \tilde{c}(TX_{I,J}, TX_{I,J}, g_{TX_{I,J}}^{\varepsilon}|_{X_{I,J}}) \to c(TZ_{I,J}, g_{TZ_{I,J}}^{\varepsilon}) \pi^* \tilde{c}(TY_{I,J}, TY_{I,J}, g_{TY_{I,J}}^{\varepsilon}|_{Y_I}). \] (3.67)
By (3.66) and (3.67), as $\varepsilon \to 0$,
\[
b_{I,J,j}(\omega_\varepsilon) \to \frac{\chi(Z_j)}{12} \int_{Y_I} \tilde{c}(TY_I, TY_I|_{Y_I}, g^{TY_I}|_{Y_I}). \tag{3.68}
\]
By (3.49) and (3.68), as $\varepsilon \to 0$,
\[
\sum_{I \subseteq A} \sum_{J \subseteq B} \sum_{j \in I} w_d^I w_d^J m_j + \frac{d}{d} b_{I,J,j}(\omega_\varepsilon) \to 0. \tag{3.69}
\]
Now we consider the case $j \in J$. We denote $J' = J \setminus \{j\}$. By (3.9), we have
\[
b_{I,J,j}(\omega_\varepsilon) = \frac{1}{12} \int_{X_{I,J}} \tilde{c}(TX_{I,J}, TX_{I,J}|_{X_{I,J}}, g^{TX_{I,J}}|_{X_{I,J}}). \tag{3.70}
\]
By Proposition 1.9, as $\varepsilon \to 0$,
\[
\tilde{c}(TX_{I,J}, TX_{I,J}|_{X_{I,J}}, g^{TX_{I,J}}|_{X_{I,J}}) \to \tilde{c}(TZ_J, TZ_J|_{Z_J}, g^{TZ_J}|_{Z_J}) \pi^*(TY_I, g^{TY_I}). \tag{3.71}
\]
Let $b_{I,J}(\omega_Z)$ be as in (3.9) with $(X, \gamma, \omega)$ replaced by $(Z, \gamma_Z, \omega_Z)$. More precisely,
\[
b_{I,J}(\omega_Z) = \frac{1}{12} \int_{Z_J} \tilde{c}(TZ_J, TZ_J|_{Z_J}, g^{TZ_J}|_{Z_J}). \tag{3.72}
\]
By (3.70)–(3.72), as $\varepsilon \to 0$,
\[
b_{I,J,j}(\omega_\varepsilon) \to \chi(Y_J)b_{I,J}(\omega_Z). \tag{3.73}
\]
By (3.49) and (3.73), as $\varepsilon \to 0$,
\[
\sum_{I \subseteq A} \sum_{J \subseteq B} \sum_{j \in J} w_d^I w_d^J m_j + \frac{d}{d} b_{I,J,j}(\omega_\varepsilon) \to \chi_d(Y, \gamma_Y) \sum_{J \subseteq B} \sum_{j \in J} w_d^J m_j + \frac{d}{d} b_{I,J}(\omega_Z). \tag{3.74}
\]
Step 4. We conclude.

Taking $\varepsilon \to 0$ on the right-hand side of (3.46) and applying (3.50), (3.65), (3.69) and (3.74), we obtain
\[
\tau_d(X, \gamma_X) = \chi_d(Y, \gamma_Y) \sum_{J \subseteq B} w_d^J \left( \tau_{BCOV}(Z_J, \omega_Z) - a_J(\gamma_Z, \omega_Z) - \sum_{j \in J} \frac{m_j + d}{d} b_{I,J}(\omega_Z) \right). \tag{3.75}
\]
On the other hand, by Definition 3.2 and (3.10), we have
\[
\tau(Z, \gamma_Z) = \sum_{J \subseteq B} w_d^J \left( \tau_{BCOV}(Z_J, \omega_Z) - a_J(\gamma_Z, \omega_Z) - \sum_{j \in J} \frac{m_j + d}{d} b_{I,J}(\omega_Z) \right). \tag{3.76}
\]
From (3.75) and (3.76), we obtain (3.42). This completes the proof. \hfill $\square$

### 3.4 Proof of Theorem 0.5

Now we are ready to prove Theorem 0.5.

**Proof of Theorem 0.5.** The proof consists of several steps.

**Step 1.** Following [BFM75, §1.5], we introduce a deformation to the normal cone.

Let $\mathcal{X} \to X \times \mathbb{C}$ be the blow-up along $Y \times \{0\}$. Let $\Pi : \mathcal{X} \to \mathbb{C}$ be the composition of the canonical projections $\mathcal{X} \to X \times \mathbb{C}$ and $X \times \mathbb{C} \to \mathbb{C}$. For $z \in \mathbb{C}^*$, we denote
\[
X_z = \Pi^{-1}(z). \tag{3.77}
\]
Let $\mathcal{O}$ be the trivial line bundle over $Y$. Recall that $N_Y$ is the normal bundle of $Y \hookrightarrow X$. Recall that $X'$ is the blow-up of $X$ along $Y$. The variety $\Pi^{-1}(0)$ consists of two irreducible
components: $\Pi^{-1}(0) = \Sigma_1 \cup \Sigma_2$ with $\Sigma_1 \simeq \mathbb{P}(N_Y \oplus \mathcal{K})$ and $\Sigma_2 \simeq X'$. We denote
$$X_0 = \Sigma_1. \quad (3.78)$$
For $j = 1, \ldots, l$, let $D_j \subseteq \mathcal{X}$ be the closure of $D_j \times \mathbb{C}^* \subseteq \mathcal{X}$. For $z \in \mathbb{C}$, we denote
$$D_{j,z} = D_j \cap X_z. \quad (3.79)$$
Let $\mathcal{Y} \subseteq \mathcal{X}$ be the closure of $Y \times \mathbb{C}^* \subseteq \mathcal{X}$. For $z \in \mathbb{C}$, we denote
$$Y_z = \mathcal{Y} \cap X_z. \quad (3.80)$$
See Figure 1.

Let $g^{TX}$ be a Hermitian metric on $TX$. Let $d(\cdot, \cdot): X \times X \to \mathbb{R}$ be the geodesic distance associated with $g^{TX}$. For $x \in X$, we denote
$$d_Y(x) = \inf_{y \in Y} d(x, y). \quad (3.81)$$
For $z \in \mathbb{C}^*$, set
$$U_z = \{ x \in X : d_Y(x) < |z| \} \times \{ z \} \subseteq X_z. \quad (3.82)$$
We identify the fiber of $\mathcal{K}$ with $\mathbb{C}$. For $v \in N_Y$ and $s \in \mathbb{C}$ such that $(v, s) \neq (0, 0)$, we denote by $[v : s]$ the image of $(v, s)$ in $\mathbb{P}(N_Y \oplus \mathcal{K})$. Let $| \cdot |$ be the norm on $N_Y$ induced by $g^{TX}$. Set
$$U_0 = \{ [v : s] \in \mathbb{P}(N_Y \oplus \mathcal{K}) : |v| < |s| \} \subseteq X_0. \quad (3.83)$$
For $\varepsilon > 0$ small enough, we have smooth families
$$(U_z)_{|z| < \varepsilon}, \quad (Y_z)_{|z| < \varepsilon}, \quad (U_z \cap D_{j,z})_{|z| < \varepsilon} \quad \text{with } j = 1, \ldots, l. \quad (3.84)$$
We remark that $Y_z \subseteq U_z$ for $z \in \mathbb{C}$.

Let $\mathcal{F}: \mathcal{X}' \to \mathcal{X}$ be the blow-up along $\mathcal{Y}$. For $z \in \mathbb{C}$, we denote
$$X'_z = \mathcal{F}^{-1}(X_z). \quad (3.85)$$
Set
$$f_z = \mathcal{F}|_{X'_z}: X'_z \to X_z, \quad (3.86)$$
which is the blow-up along $Y_z$. For $z \in \mathbb{C}$, set
$$D'_{0,z} = f_z^{-1}(Y_z) \subseteq X'_z. \quad (3.87)$$
For $z \in \mathbb{C}$ and $j = 1, \ldots, l$, let $D'_{j,z} \subseteq X'_z$ be the strict transformation of $D_{j,z} \subseteq X_z$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Deformation to the normal cone.}
\end{figure}
For \( z \in \mathbb{C} \), set
\[
U'_z = f^{-1}_z(U_z).
\] (3.88)

For \( \varepsilon > 0 \) small enough, we have smooth families
\[
(U'_z|_{|z|<\varepsilon}, \ (U'_z \cap D'_j, z)|_{|z|<\varepsilon} \text{ with } j=0,\ldots,l.
\] (3.89)

We remark that \( D'_0 \subseteq U'_z \) for \( z \in \mathbb{C} \).

**Step 2.** We introduce a family of meromorphic pluricanonical sections.

Denote
\[
m = m_1 + \cdots + m_q,
\] (3.90)

which is the vanishing order of \( \gamma \) on \( Y \). Recall that \( r \) is the codimension of \( Y \hookrightarrow X \). Recall that \( \gamma \in \mathcal{M}(X, K^d_X) \). For \( z \neq 0 \), we identify \( X_z \) with \( X \) in the obvious way. For \( z \neq 0 \), set
\[
\gamma_z = z^{-m-rd}\gamma \in \mathcal{M}(X_z, K^d_X).
\] (3.91)

There is a unique \( \gamma_0 \in \mathcal{M}(X_0, K^d_{X_0}) \) such that for \( \varepsilon > 0 \) small enough,
\[
(\gamma_z|_{|z|<\varepsilon}
\] (3.92)
is a smooth family. Now we briefly explain the existence of \( \gamma_0 \). We take a holomorphic local chart
\[
\varphi : \mathbb{C}^n \ni V \to X
\] (3.93)
such that:

- \( 0 \in V \) and \( \varphi(0) \in Y \);
- \( \varphi^{-1}(Y) = \{(z_1,\ldots,z_n) \in V : z_1 = \cdots = z_r = 0\} \);
- \( \varphi^*\gamma = \theta(z_1,\ldots,z_n)z_1^{m_1}\cdots z_r^{m_r}(dz_1 \wedge \cdots \wedge dz_n)^d \), where \( \theta \) is a holomorphic function on \( V \) such that \( \theta(0,\ldots,0,z_{r+1},\ldots,z_n) \neq 0 \) for generic \( z_{r+1},\ldots,z_n \).

For \( z \neq 0 \), let \( \varphi_z : V \to X_z \) be the composition of \( \varphi : V \to X \) and the identification \( X = X_z \). We take a holomorphic local chart
\[
\phi : \mathbb{C}^n \times \{ z \in \mathbb{C} : |z| < \varepsilon \} \ni W \to \mathcal{X}
\] (3.94)
such that for \( 0 < |z| < \varepsilon \):

- \( \phi(z_1,\ldots,z_n,z) \in \varphi_z(V) \subseteq X_z \);
- \( \varphi_z^{-1}(\phi(z_1,\ldots,z_n,z)) = (zz_1,\ldots,zz_r,z_{r+1},\ldots,z_n) \).

Then a direct calculation yields
\[
z^{-m-rd}\phi^*\gamma = \theta(zz_1,\ldots,zz_r,z_{r+1},\ldots,z_n)z_1^{m_1}\cdots z_r^{m_r}(dz_1 \wedge \cdots \wedge dz_n)^d
\]
\[
\to \theta(0,\ldots,0,z_{r+1},\ldots,z_n)z_1^{m_1}\cdots z_r^{m_r}(dz_1 \wedge \cdots \wedge dz_n)^d
\] (3.95)
as \( z \to 0 \). Moreover, the calculation above shows that the hypothesis in § 3.3 holds with \((X, \gamma_X)\) replaced by \((X_0, \gamma_0)\). In particular, \((X_0, \gamma_0)\) is a d-Calabi–Yau pair.

**Step 3.** We introduce a family of Kähler forms.

Let \( \mathcal{U} \subseteq \mathcal{X} \) be such that \( \mathcal{U} \cap X_z = U_z \) for any \( z \in \mathbb{C} \). Then \( \mathcal{U} \) is an open subset of \( \mathcal{X} \). Set \( \mathcal{U}' = \mathcal{F}^{-1}(\mathcal{U}) \subseteq \mathcal{X}' \). We have \( \mathcal{U}' \cap X'_z = U'_z \) for any \( z \in \mathbb{C} \).
Let $\omega$ be a Kähler form on $\mathcal{X}$. Let $\omega'$ be a Kähler form on $\mathcal{X}'$ such that
\[ \omega'|_{\mathcal{X}\setminus\mathcal{Y}} = \mathcal{F}^*(\omega|_{\mathcal{X}\setminus\mathcal{Y}}). \] (3.96)
For $z \in \mathbb{C}$, set
\[ \omega_z = \omega|_{X_z}, \quad \omega'_z = \omega'|_{X'_z}. \] (3.97)
By (3.86), (3.96) and (3.97), we have
\[ \omega'_z|_{X'_z \setminus U'_z} = f^*_z(\omega_z|_{X_z \setminus U_z}) \quad \text{for } z \in \mathbb{C}. \] (3.98)
For $\varepsilon > 0$ small enough, we have smooth families
\[ (\omega_z|_{U_z})_{|z| < \varepsilon}, \quad (\omega'_z|_{U'_z})_{|z| < \varepsilon}. \] (3.99)

**Step 4.** We show that the function $z \mapsto \tau_d(X'_z, f^*_z \gamma_z) - \tau_d(X_z, \gamma_z)$ is continuous at $z = 0$.

Denote
\[ m_0 = m_1 + \cdots + m_q + (r - 1)d. \] (3.100)
For $z \in \mathbb{C}$, by (3.79), (3.86), (3.87) and (3.92), we have
\[ \text{div}(\gamma_z) = \sum_{j=1}^l m_j D_{j,z}, \quad \text{div}(f_j^* \gamma_z) = \sum_{j=0}^l m_j D'_j,z. \] (3.101)
Here $D_{j,0}$ and $D'_{j,0}$ may be empty for certain $j$. Let $(D_{j,z})_{J \subseteq \{1, \ldots, l\}}$ be as in (0.9) with $X$ replaced by $X_z$ and $D_j$ replaced by $D_{j,z}$. Let $(D'_{j,z})_{J \subseteq \{0, \ldots, l\}}$ be as in (0.9) with $X'$ replaced by $X'_z$ and $D_j$ replaced by $D'_{j,z}$. By Definition 3.2 and (3.10), we have
\[
\tau_d(X'_z, f^*_z \gamma_z) - \tau_d(X_z, \gamma_z)
= \sum_{0 \in J \subseteq \{0, \ldots, l\}} w_d^0 \left( \tau_{BCOV}(D'_0|z, \omega'_z) - a_0(f^*_z \gamma_z, \omega'_z) - \sum_{j \in J} \frac{m_j + d}{d} b_{j,0}(\omega'_z) \right)
\quad - \sum_{J \subseteq \{1, \ldots, l\}} w_d^J a_0(f^*_z \gamma_z, \omega'_z) - a_0(\gamma_z, \omega_z))
\quad - \sum_{J \subseteq \{1, \ldots, l\}} \sum_{j \in J} w_d^J \frac{m_j + d}{d} (b_{j,j}(\omega'_z) - b_{j,j}(\omega_z))
\quad + \sum_{J \subseteq \{1, \ldots, l\}} w_d^J \left( \tau_{BCOV}(D'_j|z, \omega'_z) - \tau_{BCOV}(D_{j,z}, \omega_z) \right). \] (3.102)

For $0 \in J \subseteq \{0, \ldots, l\}$, we have $D'_{j,z} \subseteq U'_j$. Thus,
\[ (D'_{j,z})_{z \in \mathbb{C}} \] (3.103)
is a smooth family. Hence, the first summation in (3.102) is continuous at $z = 0$.

For $J \subseteq \{1, \ldots, l\}$, we denote
\[ D_{j,z} = D_{j,z}^0 \cup D_{j,z}^\infty \] (3.104)
such that each irreducible component of $D_{j,z}^0$ (respectively, $D_{j,z}^\infty$) lies in (respectively, does not lie in) $Y_z$. As $D_{j,z}^0 \subseteq Y_z \subseteq U_z$, the family
\[ (D_{j,z}^0)_{z \in \mathbb{C}} \] (3.105)
BCOV invariant and blow-up

is smooth. On the other hand, we have

\[ D_{J,z}^{\text{ex}} = f_z(D_{J,z}') . \]  

(3.106)

Moreover, the map \( f_z|_{D_{J,z}'} : D_{J,z}' \to D_{J,z}^{\text{ex}} \) is the blow-up along \( D_{J,z}^{\text{ex}} \cap Y_z \).

Recall that

\[ K_J, \quad \gamma_J, \quad g_{\omega J}^{TD_J}, \quad | \cdot |_{K_J, \omega} \]  

(3.107)

were constructed in §§3.1 and 3.2 for a d-Calabi–Yau pair \((X, \gamma)\) together with a Kähler form \(\omega\) on \(X\). Let

\[ K_{J,z}, \quad \gamma_{J,z}, \quad g_{\omega z}^{TD_{J,z}}, \quad | \cdot |_{K_{J,z}, \omega_z} \]  

(3.108)

be as in (3.107) with \((X, \gamma)\) replaced by \((X_z, \gamma_z)\) and \(\omega\) replaced by \(\omega_z\). Let

\[ K_{J,z}', \quad \gamma_{J,z}', \quad g_{\omega z}'^{TD_{J,z}'}, \quad | \cdot |_{K_{J,z}', \omega_z'} \]  

(3.109)

be as in (3.107) with \((X, \gamma)\) replaced by \((X', f_z^* \gamma_z)\) and \(\omega\) replaced by \(\omega_z\). By (3.6), (3.98), (3.104) and (3.106), for \(J \subseteq \{1, \ldots, l\}\), we have

\[
\begin{align*}
    a_J(f_z^* \gamma_z, \omega_z') - a_J(\gamma_z, \omega_z) &= \frac{1}{12} \int_{D_{J,z}' \cap U_z^J} c_{n-|J|} \left( T D_{J,z}', g_{\omega_z}'^{TD_{J,z}'} \right) \log | \gamma_{J,z}' |_{K_{J,z}', \omega_z'}^{2/d} \\
    &\quad - \frac{1}{12} \int_{D_{J,z}^{\text{ex}} \cap U_z^J} c_{n-|J|} \left( T D_{J,z}, g_{\omega z}^{TD_{J,z}} \right) \log | \gamma_{J,z} |_{K_{J,z}, \omega_z}^{2/d} \\
    &\quad - \frac{1}{12} \int_{D_{J,z}^{\text{in}}} c_{n-|J|} \left( T D_{J,z}, g_{\omega z}^{TD_{J,z}} \right) \log | \gamma_{J,z} |_{K_{J,z}, \omega_z}^{2/d},
\end{align*}
\]

(3.110)

By (3.89), each integration in (3.110) depends continuously on \(z\). Thus, the second summation in (3.102) is continuous at \(z = 0\). The same argument shows that the third summation in (3.102) is continuous at \(z = 0\).

By (3.104), we have the obvious identity

\[
\tau_{\text{BCOV}}(D_{J,z}', \omega_z') - \tau_{\text{BCOV}}(D_{J,z}, \omega_z) = \tau_{\text{BCOV}}(D_{J,z}', \omega_z') - \tau_{\text{BCOV}}(D_{J,z}^{\text{ex}}, \omega_z) - \tau_{\text{BCOV}}(D_{J,z}^{\text{in}}, \omega_z). \]

(3.111)

As the families in (3.99) are smooth, by Theorem 2.7 and (3.98), the function \(z \mapsto \tau_{\text{BCOV}}(D_{J,z}', \omega_z') - \tau_{\text{BCOV}}(D_{J,z}^{\text{ex}}, \omega_z)\) is continuous at \(z = 0\). As the families in (3.99) and (3.105) are smooth, the function \(z \mapsto \tau_{\text{BCOV}}(D_{J,z}^{\text{in}}, \omega_z)\) is continuous at \(z = 0\). Hence, the fourth summation in (3.102) is continuous at \(z = 0\).

Step 5. We conclude.

By Step 4, we have

\[
\lim_{z \to 0} \left( \tau(X'_z, f_z^* \gamma_z) - \tau(X_z, \gamma_z) \right) = \tau(X'_0, f_0^* \gamma_0) - \tau(X_0, \gamma_0). \]

(3.112)

On the other hand, by Proposition 3.4 and (3.91), for \(z \neq 0\), we have

\[
\begin{align*}
    \tau_d(X_z, \gamma_z) &= \frac{\chi_d(X, \gamma)}{12} \log |z|^{-(m+rd)/d}, \\
    \tau_d(X'_z, f_z^* \gamma) &= \frac{\chi_d(X', f_z^* \gamma)}{12} \log |z|^{-(m+rd)/d}.
\end{align*}
\]

(3.113)
Note that \((m + rd)/d > 0\), by (3.112) and (3.113), we have
\[
\chi_d(X', f^*\gamma) - \chi_d(X, \gamma) = 0,
\]
\[
\tau_d(X', f^*\gamma) - \tau_d(X, \gamma) = \tau_d(X'_0, f^*_0\gamma_0) - \tau_d(X_0, \gamma_0).
\] (3.114)

Note that \(X_0\) is a \(\mathbb{C}P^r\)-bundle over \(Y_0 \simeq Y\), by Theorem 3.6, we have
\[
\tau_d(X_0, \gamma_0) = \chi_d(Y, D_Y)\tau_d(\mathbb{C}P^r, \gamma_r, m_1, \ldots, m_q).
\] (3.115)

Recall that \(E = f^{-1}(Y)\). Note that \(X'_0\) is a \(\mathbb{C}P^1\)-bundle over \(D'_0, 0 \simeq E\), by Theorem 3.6, we have
\[
\tau_d(X'_0, f^*_0\gamma_0) = \chi_d(E, D_E)\tau_d(\mathbb{C}P^1, \gamma_1, m_0).
\] (3.116)

From (3.114)–(3.116), we obtain (0.22). This completes the proof. □

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