The symmetry, period and Calabi-Yau dimension of finite dimensional mesh algebras

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Abstract

Within the class of finite dimensional mesh algebras, we identify those which are symmetric and those whose stable module category is Calabi-Yau. We also give, in combinatorial terms, explicit formulas for the \(\Omega\)-period of any such algebra, where \(\Omega\) is the syzygy functor, and for the Calabi-Yau Frobenius and the stable Calabi-Yau dimensions, when they are defined.

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1 Introduction

A \(\text{Hom}\) finite triangulated \(K\)-category \(\mathcal{T}\), with suspension functor \(\sum: \mathcal{T} \rightarrow \mathcal{T}\), is called Calabi-Yau (see [27]), when there is a natural number \(n\) such that \(\sum^n\) is a Serre functor (i.e. \(D\text{Hom}_\mathcal{T}(X, -)\)) and \(\text{Hom}_\mathcal{T}(-, \sum^n X)\) are naturally isomorphic as cohomological functors \(\mathcal{T}^{\text{op}} \rightarrow K-\text{mod}\). In such a case, the smallest natural number \(m\) such that \(\sum^m\) is a Serre functor is called the Calabi-Yau dimension (\(\text{CY-dimension}\) for short) of \(\mathcal{T}\). Calabi-Yau triangulated categories appear in many fields of Mathematics and Theoretical Physics. In Representation Theory of algebras, the concept plays an important role in the study of cluster algebras and cluster categories (see [26]).

When \(\Lambda\) is a self-injective finite dimensional (associative unital) algebra, its stable module category \(\Lambda-\text{mod}\) is a triangulated category and the Calabi-Yau condition on this category naturally appears and has been deeply studied (see, e.g., [15], [7], [18], [13], [24], [25]...). The concept is related with that of Frobenius Calabi-Yau algebra, as defined by Ersland and Schedler ([18]). The algebra \(\Lambda\) is called Calabi-Yau Frobenius when \(\Omega^r_{\Lambda^e}(\Lambda)\) is isomorphic to \(D(\Lambda) = \text{Hom}_K(\Lambda, K)\) as \(\Lambda\)-bimodule, for some integer \(r \geq 0\). If the algebra \(\Lambda\) is Calabi-Yau Frobenius, then \(\Lambda-\text{mod}\) is Calabi-Yau and the Calabi-Yau dimension of this category is less or equal than the smallest \(r\) such that \(\Omega^r_{\Lambda^e}(\Lambda)\) is isomorphic to \(D(\Lambda)\), a number which is called here the Calabi-Yau Frobenius dimension of \(\Lambda\). In general, it is not known whether these two numbers are equal.

A basic finite dimensional algebra \(\Lambda\) is self-injective precisely when there is an isomorphism of \(\Lambda\)-bimodules between \(D(\Lambda)\) and the twisted bimodule \(\underline{1}_{\Lambda^e}\), for some automorphism \(\eta\) of \(\Lambda\). This automorphism is uniquely determined up to inner automorphism and is called the Nakayama automorphism of \(\Lambda\) (see section 2 for more details). Then the problem of deciding when \(\Lambda\) is Calabi-Yau Frobenius is a more general problem, namely, to determine under which conditions \(\Omega^r_{\Lambda^e}(\Lambda)\) is isomorphic to a twisted bimodule \(\underline{1}_{\Lambda^e}\), for some automorphism \(\varphi\) of \(\Lambda\), which is then determined up to inner automorphism. By a result of Green-Snashall-Solberg ([22]), this condition
on a finite dimensional algebra forces it to be self-injective. When \( \varphi \) is the identity (or an inner automorphism), the algebra \( \Lambda \) is called \textit{periodic} and the problem of determining the self-injective algebras which are periodic is widely open. However, there is a lot of work in the literature were several classes of periodic algebras have been identified (see, e.g., [9], [10], [12]). Even when an algebra \( \Lambda \) is known to be periodic, it is usually hard to calculate explicitly its \textit{period}, that is, the smallest of the integers \( r > 0 \) such that \( \Omega^r_\Lambda(\Lambda) \) is isomorphic to \( \Lambda \) as a bimodule.

Another interesting problem in the context of finite dimensional self-injective algebras is that of determining when such an algebra is weakly symmetric or symmetric. An algebra is \textit{symmetric} when \( D(\Lambda) \) is isomorphic to \( \Lambda \) as a \( \Lambda \)-bimodule. This is equivalent to saying that the \textit{Nakayama functor} \( DHom_\Lambda(-, \Lambda) \cong D(\Lambda) \otimes_\Lambda - : \Lambda - \text{Mod} \to \Lambda - \text{Mod} \) is naturally isomorphic to the identity functor. The algebra is weakly symmetric when this functor just preserves the iso-classes of simple modules.

In this paper we tackle the problems mentioned above for a special class of finite dimensional self-injective algebras, which has deserved a lot of attention in recent times. Following [10], if \( \Delta \) is one of the Dynkin quivers \( \mathbb{A}_r, \mathbb{D}_r \) of \( \mathbb{E}_n \) \((n = \text{6, 7, 8})\), an \textit{m-fold mesh algebra of type} \( \Delta \) is the mesh algebra of the stable translation quiver \( \mathbb{Z}\Delta/G \), where \( G \) is a weakly admissible group of automorphisms of \( \mathbb{Z}\Delta \) (see subsections 3.1 and 3.2 for definitions and details). By a result of Dugas ([13] Theorem 3.1), the \textit{m-fold mesh algebras} are precisely the mesh algebras of translation quivers which are finite dimensional. This class of algebras properly contains the stable Auslander algebras of all standard representation-finite self-injective algebras (see [13]) and, also, the Auslander algebras of several hypersurface singularities (see [16] Section 8). In fact, due to the classification by Amiot [1] of the standard algebraic triangulated categories of finite type, we know that the \textit{m-fold mesh algebras} are precisely the Auslander algebras of these triangulated categories. Then any such triangulated category can be identified with the category \( \Lambda - \text{proj of finitely generated projective} \Lambda \)-modules, where \( \Lambda \) is an \textit{m-fold mesh algebra}, when taking as suspension functor in \( \Lambda - \text{proj} \) the functor \( \Omega^3_\Lambda(\Lambda) \otimes - \). Similarly the Serre functor on this latter category is identified with the Nakayama functor \( D(\Lambda) \otimes_\Lambda - \). It is well-known (see [10] or [13]) that \( \Omega^3_\Lambda(\Lambda) \cong \mu_{A_1} \), for an automorphism \( \mu \) which has finite order as an outer automorphism, so that in particular, all the algebras in the class are periodic, a result proved in [9] Section 6 even before the class of algebras was introduced. Therefore the explicit identification of the automorphisms \( \eta \) and \( \mu \) for all \textit{m-fold mesh algebra}, which we obtain in this paper (see Theorem 4.2 and Corollary 5.3) translates to an explicit description of the Serre functor and a quasi-inverse of the suspension functor for any algebraic triangulated category of finite type. From previous papers it seems that only the action of these functors on objects was known.

A particular case of standard algebraic triangulated category of finite type is the stable category \( \Lambda - \text{mod} \) of a representation-finite selfinjective algebra \( \Lambda \). Within the class of \textit{m-fold mesh algebras} \( \Lambda \) which are the Auslander algebra of such a stable category, those which are Calabi-Yau-Frobenius and those for which \( \Lambda - \text{mod} \) is Calabi-Yau have been completely determined. The task was initiated in in [7] and [15], but the main part of the work was done in [13] and [25]. In the first of these two papers, Dugas identified such an algebra by the type \((\Delta, f, t)\) of the original representation-finite selfinjective algebra \( \Lambda \), as defined by Asashiba ([1]) inspired by the work of Riedtmann (25). He completed the task when \( t = 1 \) or 3, and also in many cases with \( t = 2 \). In fact the author even described a relationship between the Calabi-Yau condition of \( \Lambda - \text{mod} \) and of \( \Lambda - \text{mod} \) (see [13] Proposition 2.1). The remaining cases for \( t = 2 \) have been recently settled by Ivanov-Volkow (25). On the question of periodicity, the explicit calculation of the period of an \textit{m-fold mesh algebra} has been also done in a few cases. From the papers [29], [17] and [8] we know that the period is 6 for all preprojective algebras of generalized Dynkin type. In [12] and [13] the period is calculated when \( \Lambda \) is the stable Auslander algebra of a standard representation-finite self-injective algebra of type \((\Delta, f, t)\), when \( t = 1 \) or when this type is equal to \((\mathbb{D}_4, f, 3)\), \((\mathbb{D}_n, f, 2)\), with \( n > 4 \) and \( f > 1 \) odd, or \((\mathbb{E}_6, f, 2)\). In fact, the author uses these results to calculate the period of the original finite type selfinjective algebra \( \Lambda \).

We now explain the main results of our paper. Let \( B = B(\Delta) \) be the mesh algebra of the translation quiver \( \mathbb{Z}\Delta \), where \( \Delta \) is one of the Dynkin quivers mentioned above, and let \( G \) be a weakly admissible automorphism of \( \mathbb{Z}\Delta \) which is viewed also as an automorphism of \( B \). The algebra \( B \) is graded Quasi-Frobenius (see definition 5), which roughly means that \( B \) and its category of graded modules behave as a self-injective finite dimensional algebra and its category of modules. The crucial result of our paper is Theorem 4.2 which explicitly defines a graded Nakayama automorphism
of \( B \) which commutes with the elements of \( G \), for any choice of \((\Delta, G)\). Since each \( m \)-fold mesh algebra \( \Lambda \) is isomorphic to the orbit category \( B/G \), the consequence is that one derives an explicit definition of a graded Nakayama automorphism, for each \( m \)-fold mesh algebra. We then use the key Lemma 4.3 which determines when a \( G \)-invariant graded automorphism of \( B \) induces an inner automorphism of \( \Lambda = B/G \). Using the extended type \((\Delta, m, t)\) (see definition 5) to identify an \( m \)-fold mesh algebra, we get, expressed in terms of this type, the main results of the paper, all referred to \( m \)-fold mesh algebras:

1. An identification of all weakly symmetric and symmetric algebras in the class (Theorem 4.6);
2. An explicit formula for the period of any algebra in the class (Proposition 5.6 when \( \Delta = \Lambda_2 \), and Theorem 5.10 for all the other cases).
3. An identification of the precise relation between the stable Calabi-Yau dimension and the Calabi-Yau Frobenius dimension of an \( m \)-fold algebra, showing that both dimensions may differ when \( \Delta = \Lambda_2 \), but are always equal when \( \Delta \neq \Lambda_r \), for \( r = 1, 2 \) (Propositions 5.10 and 5.14).
4. A criterion for an \( m \)-fold mesh algebra to be stably Calabi-Yau, together with the identification in such case of the stable Calabi-Yau dimension (Proposition 5.12 for the case \( \Delta = \Lambda_2 \), Corollary 5.18 for characteristic 2, and Theorem 5.19 for all other cases).

We now describe the organization of the paper. In section 2 we recall from \cite{3} the concept of pseudo-Frobenius graded algebra with enough idempotents and the corresponding results which we need in the rest of the paper.

In section 3, we recall the definition of the mesh algebra of a stable translation quiver and give a list of essentially known properties (Proposition 3.1) for the case of the mesh algebra \( B = B(\Delta) \) of \( \mathbb{Z}\Delta \), when \( \Delta \) is a Dynkin quiver. We then recall the definition of an \( m \)-fold mesh algebra and their known properties, and introduce the notion of extended type of such an algebra, which plays a crucial role in the rest of the paper. With the idea of simplifying some calculations, we end the section by performing a change of relations which, roughly speaking, transforms sums of paths of length 2 into differences.

In section 4 we give the explicit definition of the \( G \)-invariant graded Nakayama automorphism of \( B \) and give the crucial Lemma 4.3 mentioned above. We then give the list of all weakly symmetric and symmetric \( m \)-fold mesh algebras.

In the final section 5 we first calculate explicitly the initial part of a '\( G \)-invariant' minimal projective resolution of \( B \) as a graded \( B \)-bimodule, We prove in particular that \( \Omega^3_{B^r}(B) \) is always isomorphic to \( \mu B_1 \), for a graded automorphism \( \mu \) of \( B \) which is in the centralizer of \( G \) and which is explicitly calculated. Then the induced automorphism \( \bar{\mu} \) of \( \Lambda = B/G \) has the property that \( \Omega^3_{\Lambda}(\Lambda) \cong \bar{\mu}_1 \Lambda_1 \) and this is fundamental in the rest of the paper. We then calculate the period of all \( m \)-fold mesh algebras and find the precise relation between the Calabi-Yau Frobenius condition of \( \Lambda \), in the sense of \cite{18}, and the condition that \( \Lambda \mod \) be a Calabi-Yau category. We end the paper by giving necessary and sufficient conditions for a mesh algebra to be stably Calabi-Yau and, when this is the case, we calculate explicitly the Calabi-Yau dimension of \( \Lambda \mod \).

2 Pseudo-Frobenius graded algebras with enough idempotents

This section is devoted to compiling from \cite{3} information concerning the class of pseudo-Frobenius graded algebras with enough idempotents, which will very useful in the subsequent sections. The reader is referred to that paper for proofs and further details.

Let \( A \) be an associative algebra over a field \( K \). Such an algebra is said to be an \emph{algebra with enough idempotents} when there is a family \((e_i)_{i \in I}\) of nonzero orthogonal idempotents, called \emph{distinguished family}, such that \( \oplus_{i \in I} e_i A = A = \oplus_{i \in I} A e_i \). If in addition, \( H \) is a fixed abelian group with additive notation, an \emph{\( H \)-graded algebra with enough idempotents} will be an algebra with enough idempotents \( A \), together with an \( H \)-grading \( A = \oplus_{h \in H} A_h \), such that one can choose a distinguished family of orthogonal idempotents which are homogeneous of degree 0. Frequently,
we will interpret such an algebra as a (small) graded $K$-category with $I$ as set of objects, where $e_i A e_j$ is the set of morphisms from $i$ to $j$ and where the composition of morphisms is just the anti-multiplication: $b \circ a = ab$. Then the concept of functor between such algebras makes sense and will be used sometimes.

Throughout this section by $A$ we will mean an $H$-graded algebra with enough idempotents on which a distinguished family of orthogonal idempotents is fixed. All considered left (resp. right) $A$-modules are supposed to be unital, i.e., $A M = M$ (resp. $M A = M$) for any left (resp. right) $A$-module $M$. We will also denote by $A - Gr$ (resp. $Gr - A$) the category of ($H$-)graded unital left (resp. right) modules, where the morphisms are the graded homomorphisms of degree 0.

The enveloping algebra of $A$ is the algebra $A^e = A \otimes A^{op}$, where if $a, b \in A$ we will denote by $a \otimes b^o$ the corresponding element of $A^e$. This is also an $H$-graded algebra with enough idempotents where $(A \otimes A^{op})_h = \oplus_{s+t = h} A_s \otimes A^{op}_t$. The distinguished family of orthogonal idempotents with which we will work is the family $(e_i \otimes e_j^o)_{(i,j) \in I \times I}$. A left graded $A^e$-module $M$ will be identified with a graded $A$-bimodule by putting $a x b = (a \otimes b^o)x$, for all $x \in M$ and $a, b \in A$. Similarly, a right graded $A^e$-module is identified with a graded $A$-bimodule by putting $axb = x(b \otimes a^o)$, for all $x \in M$ and $a, b \in A$. In this way, we identify the three categories $A^e - Gr$, $Gr - A^e$ and $A - Gr - A$, where the last one is the category of graded unitary $A$-bimodules, which we will simply name ‘graded bimodules’.

Recall that if $M = \oplus_{h \in H} M_h$ is a graded $A$-module and $k \in H$ is any element, then we get a graded module $M[k]$ having the same underlying ungraded $A$-module as $M$, but where $M[k]_h = M_{k+h}$ for each $h \in H$. If $M$ and $N$ are graded left $A$-modules, then $\text{HOM}_A(M,N) := \oplus_{h \in H} \text{Hom}_{A-Gr}(M,N[h])$ has a structure of graded $K$-vector space, where the homogeneous component of degree $h$ is $\text{HOM}_A(M,N)_h := \text{Hom}_{A-Gr}(M,N[h])$, i.e., $\text{HOM}_A(M,N)_h$ consists of the graded homomorphisms $M \rightarrow N$ of degree $h$.

The graded algebras we are interested in have some extra properties. For reader’s convenience we recall the following definitions.

**Definition 1.** Let $A = \oplus_{h \in H} A_h$ be a graded algebra with enough idempotents. It will be called locally finite dimensional when $e_i A_h e_j$ is finite dimensional, for all $(i, j, h) \in I \times I \times H$. Such a graded algebra $A$ will be called graded locally bounded when the following two conditions hold:

1. For each $(i, h) \in I \times H$, the set $I^{(i,h)} = \{ j \in I : e_i A_h e_j \neq 0 \}$ is finite.
2. For each $(i, h) \in I \times H$, the set $I_{(i,h)} = \{ j \in I : e_j A_h e_i \neq 0 \}$ is finite.

Observe that these definitions do not depend on the distinguished family $(e_i)$ considered.

**Definition 2.** A locally finite dimensional graded algebra with enough idempotents $A = \oplus_{h \in H} A_h$ will be called weakly basic when it has a distinguished family $(e_i)_{i \in I}$ of orthogonal homogeneous idempotents of degree 0 such that:

1. $e_i A_0 e_i$ is a local algebra, for each $i \in I$.
2. $e_i A e_j$ is contained in the graded Jacobson radical $J^{gr}(A)$, for all $i, j \in I$, $i \neq j$.

It will be called basic when, in addition, $e_i A_h e_i \subseteq J^{gr}(A)$, for all $i \in I$ and $h \in H \setminus \{0\}$.

We will use also the term ‘(weakly) basic’ to denote any distinguished family $(e_i)_{i \in I}$ of orthogonal idempotents satisfying the above conditions.

**Definition 3.** Let $V = \oplus_{h \in H} V_h$ and $W = \oplus_{h \in H} W_h$ be graded $K$-vector spaces, where the homogeneous components are finite dimensional, and let $d \in H$ be any element. A bilinear form
\((-,-): V \times W \to K\) is said to be of degree \(d\) if \((V_h, W_k) \neq 0\) implies that \(h+k = d\). Such a form will be called nondegenerate when the induced maps \(W \to D(V) \ (w \mapsto (-,w))\) and \(V \to D(W) \ (v \mapsto (v,-))\) are bijective.

Note that, in the above situation, if \((-,-) : V \times W \to K\) is a nondegenerate bilinear form of degree \(d\), then the bijective map \(W \to D(V)\) given above gives an isomorphism of graded \(K\)-vector spaces \(W[d] \xrightarrow{\sim} D(V)\) (resp. \(V[d] \xrightarrow{\sim} D(W)\)).

The following concept is fundamental for us.

**Definition 4.** Let \(A = \bigoplus_{h \in H} A_h\) be a weakly basic graded algebra with enough idempotents. A bilinear form \((-,-) : A \times A \to K\) is said to be a graded Nakayama form when the following assertions hold:

1. \((ab,c) = (a,bc), \) for all \(a, b, c \in A\)
2. For each \(i \in I\) there is a unique \(\nu(i) \in I\) such that \((e_i A, A e_{\nu(i)}) \neq 0\) and the assignment \(i \mapsto \nu(i)\) defines a bijection \(\nu : I \to I\).
3. There is a map \(h : I \to H\) such that the induced map \((-,-) : e_i A e_j \times e_j A e_{\nu(i)} \to K\) is a nondegenerate graded bilinear form of degree \(h_i = h(i)\), for all \(i, j \in I\).

The bijection \(\nu\) is called the Nakayama permutation and \(h\) will be called the degree map. When \(h\) is a constant map and \(h(i) = h\), we will say that \((-,-) : A \times A \to K\) is a graded Nakayama form of degree \(h\).

 Recall that a Quillen exact category \(E\) (e.g. an abelian category) is said to be a Frobenius category when it has enough projectives and enough injectives and the projective and the injective objects are the same in \(E\).

**Definition 5.** A weakly basic locally finite dimensional graded algebra \(A\) with enough idempotents is called graded pseudo-Frobenius (resp. graded Quasi-Frobenius) if it admits a graded Nakayama form \((-,-) : A \times A \to K\) (resp. both \(A - Gr\) and \(Gr - A\) are Frobenius categories).

A graded Quasi-Frobenius algebra \(A\) is always graded pseudo-Frobenius and the converse is true whenever \(A\) is graded locally Noetherian i.e., whenever \(A e_i\) and \(e_i A\) satisfies the ACC on graded submodules, for each \(i \in I\). Note then that, for a finite dimensional algebra \(A\), viewed as a graded algebra concentrated in zero degree, the notions of self-injective, graded pseudo-Frobenius and graded Quasi-Frobenius coincide.

When \(A\) is a graded locally bounded pseudo-Frobenius algebra, its graded Nakayama form \((-,-)\) induces an automorphism of (ungraded) algebras \(\eta \in Aut(A)\) such that \(D(A) \cong 1_A \eta\). This automorphism is unique, up to inner automorphism, and called Nakayama automorphism. It is given by the rule \((a,-) = (-, \eta(a))\), for every \(a \in A\), and turns out to be an automorphism of graded algebras whenever the associated map \(h : I \to H\) takes constant value \(h\). Indeed, in this latter case we get an isomorphism of graded algebras \(D(A) \cong 1_A \eta[h]\).

The following result gives a handy criterion, in the locally Noetherian case, for \(A\) to be graded Quasi-Frobenius.

**Corollary 2.1.** Let \(A = \bigoplus_{h \in H} A_h\) be a weakly basic locally Noetherian graded algebra with enough idempotents. The following assertions are equivalent:

1. The following two conditions hold:

   (a) For each \(i \in I\), \(A e_i\) and \(e_i A\) have a simple essential socle in \(A - Gr\) and \(Gr - A\), respectively

   (b) There are bijective maps \(\nu, \nu' : I \to I\) such that \(\text{Soc}_{gr}(e_i A) \cong \frac{e_i A}{e_i A e_{\nu'(i)}} [h_i]\) and \(\text{Soc}_{gr}(A e_i) \cong \frac{A e_i}{(J_{gr}(A)e_{\nu'(i)})[h'_i]}\), for certain \(h_i, h'_i \in H\)

2. \(A\) is graded Quasi-Frobenius
The following result shows that if $A$ is a split graded pseudo-Frobenius algebra, then all possible graded Nakayama forms for $A$ come in similar way. Recall from \([3]\) if $A$ is such an algebra, then $\text{Soc}_{gr}(A) = \text{Soc}_{gr}(A_0)$. Recall also that if $V = \oplus_{h \in H} V_h$ is a graded vector space, then its support, denoted $\text{Supp}(V)$, is the set of $h \in H$ such that $V_h \neq 0$.

**Proposition 2.2.** Let $A$ be a split pseudo-Frobenius graded algebra and $(e_i)_{i \in I}$ a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:

1. All graded Nakayama forms for $A$ have the same Nakayama permutation. It assigns to each $i \in I$ the unique $\nu(i) \in I$ such that $e_i \text{Soc}_{gr}(A)e_{\nu(i)} \neq 0$.

2. If $h_i \in \text{Supp}(e_i \text{Soc}_{gr}(A))$, then $\dim(e_i \text{Soc}_{gr}(A)h_i) = 1$

3. For a bilinear form $(-, -) : A \times A \to K$, the following statements are equivalent:

   (a) $(-, -)$ is a graded Nakayama form for $A$

   (b) There exists an element $h = (h_i) \in \prod_{i \in I} \text{Supp}(e_i \text{Soc}_{gr}(A))$ and a basis $B_i$ of $e_i A h_i e_{\nu(i)}$ for each $i \in I$, such that:

   i. $B_i$ contains a (unique) element $w_i$ of $e_i \text{Soc}_{gr}(A)h_i$

   ii. If $a, b \in \bigcup_{i,j} e_i A e_j$ are homogeneous elements, then $(e_i A h_k e_j) = 0$ unless $j = \nu(i)$ and $h + k = h_i$

   iii. If $(a, b) \in e_i A h_k$ for all $i \in I$, we call $(-, -)$ the graded Nakayama form of $A$ of degree $h$ associated to $B$.

**Definition 6.** Let $A = \oplus_{h \in H} A_h$ be a split pseudo-Frobenius graded algebra, with $(e_i)_{i \in I}$ as weakly basic distinguished family of idempotents and $\nu : I \to I$ as Nakayama permutation. Given a pair $(B, h)$ consisting of an element $h = (h_i)_{i \in I} \in \prod_{i \in I} \text{Supp}(e_i \text{Soc}_{gr}(A))$ and a family $B = (B_i)_{i \in I}$, where $B_i$ is a basis of $e_i A_h e_{\nu(i)}$ containing an element of $e_i \text{Soc}_{gr}(A)$, for each $i \in I$, we call graded Nakayama form associated to $(B, h)$ to the bilinear form $(-, -) : A \times A \to K$ determined by the conditions b.ii and b.iii of last proposition. When $h$ is constant, i.e., there is $h \in H$ such that $h_i = h$ for all $i \in I$, we will call $(-, -)$ the graded Nakayama form of $A$ of degree $h$ associated to $B$.

We now assume that $G$ is a group acting on $A$ as a group of graded automorphisms (of degree 0) which permutes the $e_i$. That is, if $\text{Aut}^{gr}(A)$ denotes the group of graded automorphisms of degree 0 which permute the $e_i$, then we have a group homomorphism $\varphi : G \to \text{Aut}^{gr}(A)$. We will write $a^g = \varphi(g)(a)$, for each $a \in A$ and $g \in G$.

The following definition will be needed for our purposes.

**Definition 7.** Let $A = \oplus_{h \in H} A_h$ be a graded pseudo-Frobenius algebra and $G$ be a group acting on $A$ as graded automorphisms. A graded Nakayama form $(-, -) : A \times A \to K$ will be called $G$-invariant when $(a^g, b^g) = (a, b)$ for all $a, b \in A$ and all $g \in G$.

We say that a group $G$ as above acts freely on objects when $g(i) \neq i$, for all $i \in I$ and $g \in G \setminus \{1\}$. In such case one can consider the orbit category $A/G$. The objects of this category are the $G$-orbits $[i]$ of indices $i \in I$ and the morphisms from $[i]$ to $[j]$ are formal sums $\sum_{g \in G} [a_g]$, where $[a_g]$ is the $G$-orbit of an element $a_g \in e_i A e_{g(j)}$. This definition does not depend on $i, j$, but just on the orbits $[i], [j]$. The anticomposition of morphisms extends by $K$-linearity the following rule. If $a, b \in \bigcup_{i,j} e_i A e_j$ and $[a], [b]$ denote the $G$-orbits of $a$ and $b$, then $[a] \cdot [b] = 0$, in case $[t(a)] \neq [i(b)]$, where $t(a)$ and $i(b)$ denote the terminus vertex of $a$ and the initial vertex of $b$, and $[a] \cdot [b] = [a b]$, in case $[t(a)] = [i(b)]$, where $g$ is the unique element of $G$ such that $g(i(b)) = t(a)$.

We now recall a result from \([3]\) concerning the preservation of the pseudo-Frobenius condition via the canonical projection $\pi : A \to A/G$. with takes $a \sim [a]$.

**Proposition 2.3.** Let $A = \oplus_{h \in H} A_h$ be a (split basic) locally bounded graded pseudo-Frobenius algebra, with $(e_i)_{i \in I}$ as weakly basic distinguished family of orthogonal homogeneous idempotents, and let $G$ be a group which acts on $A$ as graded automorphisms which permute the $e_i$ and which acts freely on objects. If there exists a $G$-invariant graded Nakayama form $(-, -) : A \times A \to K$, then $A/G$ is a (split basic) locally bounded graded pseudo-Frobenius algebra whose graded Nakayama form is induced from $(-, -)$. 

6
Under the assumptions of last proposition, it is known that the functor $\pi$ is a covering functor, that is, it is surjective on objects (i.e., vertices) and induces bijective maps $\oplus_{i\in I}e_iAhe_i \to \oplus_{i\in I}e_iAhe_i$ and $\oplus_{i\in I}e_iAhe_i \to \oplus_{i\in I}e_iAhe_i$ for each $(i,j,h) \in I \times J \times H$. Furthermore, the pushdown functor $\pi_\Lambda : A \to \Lambda - Gr$ which takes $Ae_i \mapsto \Lambda e_i$ is exact (see, e.g., [11] and [9] for further details).

The following result ensures that, in the split case, $G$-invariant graded Nakayama forms always exist, a fact which allows to apply last proposition.

**Corollary 2.4.** Let $A = \oplus_{h \in H}A_h$ be a split graded pseudo-Frobenius algebra with Nakayama permutation $\nu$ and let $G$ be a group of graded automorphisms of $A$ which permute the $e_i$ and acts freely on objects. The equality $\nu(g(i)) = g(\nu(i))$ holds, for all $g \in G$ and $i \in I$. Moreover, there exist an element $h = (h_i)_{i \in I} \in \prod_{i \in I} \text{Supp}(e_i\text{Soc}_{gr}(A))$ and a basis $B_i$ of $e_iA_h e_{\nu(i)}$, for each $i \in I$, satisfying the following properties:

1. $h_i = h_{g(i)}$, for all $i \in I$

2. $g(B_i) = B_{g(i)}$ and $B_i$ contains an element of $e_i\text{Soc}_{gr}(A)$, for all $i \in I$

In such case, letting $B = \bigcup_{i \in I} B_i$, the graded Nakayama form associated to the pair $(B,h)$ (see definition [2]) is $G$-invariant.

**Remark 2.5.** The basis of the previous corollary is constructed as follows. We fix a subset $I_0 \subseteq I$ representing the $G$-orbits of objects and, for each $i \in I_0$, we fix an $h_i \in \text{Supp}(e_i\text{Soc}_{gr}(A))$ and a basis $B_i$ of $e_iA_h e_{\nu(i)}$ containing an element $w_i \in e_i\text{Soc}_{gr}(A)$. Then, for each $j \in I$, we define $B_j = g(B_i)$ where $i \in I_0$ and $g \in G$ are the unique elements such that $j = g(i)$.

When $A$ is split locally bounded pseudo-Frobenius, we have that $\eta \circ g = g \circ \eta$, for all $g \in G$, and hence, the Nakayama automorphism $\bar{\eta}$ of $\Lambda = B/G$ is induced from $\eta$, i.e., $\bar{\eta}([a]) = [\eta(a)]$, for each $a \in \bigcup_{i,j} e_iAe_j$.

### 3 The mesh algebra of a Dynkin quiver

#### 3.1 Stable translation quivers

Recall that a quiver or oriented graph is a quadruple $Q = (Q_0, Q_1, i, t)$, where $Q_0$ and $Q_1$ are sets, whose elements are called vertices and arrows respectively, and $i, t : Q_1 \to Q_0$ are maps. If $a \in Q_1$ then $i(a)$ and $t(a)$ are called the origin (or initial vertex) and the terminus of $a$.

Given a quiver $Q$, a path in $Q$ is a concatenation of arrows $p = a_1a_2...a_r$ such that $t(a_k) = i(a_{k+1})$, for all $k = 1, \ldots, r$. In such case, we put $i(p) = i(a_1)$ and $t(p) = t(a_r)$ and call them the origin and terminus of $p$. The number $r$ is the length of $p$ and we view the vertices of $Q$ as paths of length 0. The path algebra of $Q$, denoted by $KQ$, is the $K$-vector space with basis the set of paths, where the multiplication extends by $K$-linearity the multiplication of paths. This multiplication is defined as $pq = 0$, when $t(p) \neq i(q)$, and $pq$ is the obvious concatenation path, when $t(p) = i(q)$. The algebra $KQ$ is an algebra with enough idempotents, where $Q_0$ is a distinguished family of orthogonal idempotents. If $a \in Q_0$ is a vertex, we will write it as $e_i$, when we view it as an element of $KQ$.

A stable translation quiver is a pair $(\Gamma, \tau)$, where $\Gamma$ is a locally finite quiver (i.e. given any vertex, there is a finite number of arrows having it as origin or terminus) and $\tau : \Gamma_0 \to \Gamma_0$ is a bijective map such that for any $x, y \in \Gamma_0$, the number of arrows from $x$ to $y$ is equal to the number of arrows from $\tau(y)$ to $x$. The map $\tau$ will be called the Auslander-Reiten translation. Throughout the rest of the work, whenever we have a stable translation quiver, we will also fix a bijection $\sigma : \Gamma_1(x,y) \to \Gamma_1(\tau(y), x)$ called a polarization of $(\Gamma, \tau)$. Note that, from the definition of $\sigma$, one gets that $\tau$ can be extended to a graph automorphism of $\Gamma$ by setting $\tau(a) = \sigma(\tau(a))a$, for all $a \in \Gamma_1$. If $K\Gamma$ denotes the path algebra of $\Gamma$, then the mesh algebra of $\Gamma$ is $K(\Gamma) = K\Gamma/\Gamma_0$, where $\Gamma_0$ is the ideal of $K\Gamma$ generated by the so-called mesh relations $r_x$, where $r_x = \sum_{a \in \Gamma_1, t(a) = \tau(a)} \sigma(\tau(a))a$, for each $x \in \Gamma_0$. Therefore $K(\Gamma)$ is canonically a positively $(\mathbb{Z})$-graded algebra with enough idempotents, where the grading is induced by the path length, and $\tau$ becomes a graded automorphism of $K(\Gamma)$.

The typical example of stable translation quiver is the following. Given a locally finite quiver $\Delta$, the stable translation quiver $\mathbb{Z}\Delta$ will have as set of vertices $\mathbb{Z}\Delta_0 := \mathbb{Z} \times \Delta_0$. Moreover, for each
arrow \( \alpha : x \rightarrow y \) in \( \Delta_1 \), we have arrows \((n, \alpha) : (n, x) \rightarrow (n, y)\) and \((n, \alpha)’ : (n, y) \rightarrow (n+1, x)\) in \(\mathbb{Z}\Delta_1\). Finally, we define \(\tau(n, x) = (n-1, x)\), for each \((n, x) \in \mathbb{Z}\Delta_0\), and \(\sigma(n, \alpha) = (n-1, \alpha)’\) and \(\sigma[(n, \alpha)’] = (n, \alpha)\).

In general, different quivers \(\Delta\) and \(\Delta’\) with the same underlying graph give non-isomorphic translation quivers \(\mathbb{Z}\Delta\) and \(\mathbb{Z}\Delta’\). However, when \(\Delta\) is a tree, e.g. when \(\Delta\) is any of the Dynkin quivers \(A_n, D_{n+1}, E_6, E_7, E_8\), the isoclass of the translation quiver \(\mathbb{Z}\Delta\) does not depend on the orientation of the arrows.

A group of automorphism \(G\) of a stable translation quiver \((\Gamma, \tau)\) is a group of automorphism of \(\Gamma\) which commute with \(\tau\) and \(\sigma\). Such a group is called weakly admissible when \(x^{\pm} \cap (gx)^{\pm} = \emptyset\), for each \(x \in \Gamma_0\) and \(g \in G\{1\}\), where \(x^{\pm} := \{y \in \Gamma_0 : \Gamma_1(x, y) \neq \emptyset\}\). In such a case, when \(G\) acts freely on objects, the orbit quiver \(\Gamma/G\) inherits a structure of stable translation quiver, with the AR translation \(\tau\) mapping \([x] \rightarrow [\tau(x)]\), for each \(x \in \Gamma_0 \cup \Gamma_1\). Moreover, the group \(G\) can be interpreted as a group of graded automorphisms of the mesh algebra \(K\Gamma\) and \(K\Gamma/G\) is canonically isomorphic to the mesh algebra of \(\Gamma/G\).

### 3.2 The mesh algebra of a Dynkin quiver. Basic properties

Throughout this section \(\Delta\) will be one of the Dynkin quivers \(A_n, D_{n+1} (n \geq 3)\) or \(E_n (n = 6, 7, 8)\), and \(\mathbb{Z}\Delta\) will be the associated translation quiver. Its path algebra will be denoted by \(K\mathbb{Z}\Delta\) and we will put \(B = K(\mathbb{Z}\Delta)\) for the mesh algebra.

When \(\Delta = A_{2n-1}, E_6\) or \(D_{n+1}\), with \(n > 3\), the underlying unoriented graph admits a canonical automorphism of order 2. Similarly, \(D_4\) admits an automorphism of order 3. In each case, the automorphism \(\rho\) extends to an automorphism of \(\mathbb{Z}\Delta\) with the same order. In the case of \(A_{2n}\), the canonical automorphism of order 2 of the underlying graph extends to an automorphism of \(\mathbb{Z}\Delta\), but this automorphism has infinite order. It is still denoted by \(\rho\) and it plays, in some sense, a similar role to the other cases. This automorphism of \(\mathbb{Z}A_{2n}\) is obtained by applying the symmetry with respect to the central horizontal line and moving half a unit to the right. Note that we have \(\rho^2 = \tau^{-1}\).

Although the orientation in \(\Delta\) does not change the isomorphism type of \(\mathbb{Z}\Delta\), in order to numbering the vertices of \(\mathbb{Z}\Delta\) we need to fix an orientation in \(\Delta\). Below we fix such an orientation, and then give the corresponding definition of the automorphism \(\rho\) of \(\mathbb{Z}\Delta\) mentioned above.

1. If \(\Delta = A_{2n}\):

\[
\begin{array}{c}
1 \\
\downarrow \rho
\end{array}
\begin{array}{c}
2 \\
\ldots
\end{array}
\begin{array}{c}
n \\
\uparrow \rho
\end{array}

\]

then \(\rho(k, i) = (k + i - n, 2n + 1 - i)\).

2. If \(\Delta = A_{2n-1}\):

\[
\begin{array}{c}
1 \\
\downarrow \rho
\end{array}
\begin{array}{c}
2 \\
\ldots
\end{array}
\begin{array}{c}
n - 1 \\
\uparrow \rho
\end{array}

\]

then \(\rho(k, i) = (k + i - n, 2n - i)\).

3. \(\Delta = D_{n+1}\):

\[
\begin{array}{c}
0 \\
\downarrow \rho
\end{array}
\begin{array}{c}
2 \\
\ldots
\end{array}
\begin{array}{c}
n \\
\uparrow \rho
\end{array}
\begin{array}{c}
1 \\
\downarrow \rho
\end{array}

\]

with \(n > 3\), then \(\rho(k, 0) = (k, 1)\), \(\rho(k, 1) = (k, 0)\) and \(\rho\) fixes all vertices \((k, i)\), with \(i \neq 0, 1\).

4. If \(\Delta = D_4\):
then $\rho$ fixes the vertices $(k, 2)$ and, for $k$ fixed, it applies the 3-cycle $(013)$ to the second component of each vertex $(k, i)$.

5. If $\Delta = \mathbb{E}_6$:

\[
\begin{array}{cccccc}
0 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
5 & \leftarrow & 4 & \leftarrow & 3 & \leftarrow \\
\end{array}
\]

then $\rho(k, i) = (k + i - 3, 6 - i)$ for all $i \neq 0$ and $\rho(k, 0) = (k, 0)$.

6. If $\Delta = \mathbb{E}_7$:

\[
\begin{array}{cccccc}
0 & \rightarrow & 6 & \rightarrow & 5 & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
7 & \leftarrow & 6 & \leftarrow & 5 & \leftarrow \\
\end{array}
\]

7. If $\Delta = \mathbb{E}_8$:

\[
\begin{array}{cccccc}
0 & \rightarrow & 7 & \rightarrow & 6 & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
8 & \leftarrow & 7 & \leftarrow & 6 & \leftarrow \\
\end{array}
\]

The following facts are well-known (cf. [10][Section 1.1] and [19][Section 6.5]).

**Proposition 3.1.** Let $\Delta$ be a Dynkin quiver, $\tilde{\Delta}$ be its associated graph, $c_\Delta$ be its Coxeter number and $B = K(\mathbb{Z}\Delta)$ be the mesh algebra of the stable translation quiver $\mathbb{Z}\Delta$. The following assertions hold:

1. Each path of length $> c_\Delta - 2$ in $\mathbb{Z}\Delta$ is zero in $B$.

2. For each $(k, i) \in \mathbb{Z}\Delta_0$, there is a unique vertex $\nu(k, i) \in (\mathbb{Z}\Delta)_0$ for which there is a path $(k, i) \rightarrow \ldots \rightarrow \nu(k, i)$ in $\mathbb{Z}\Delta$ of length $c_\Delta - 2$ which is nonzero in $B$. This path is unique, up to sign in $B$.

3. If $(k, i) \rightarrow \ldots \rightarrow (m, j)$ is a nonzero path then there is a path $q : (m, j) \rightarrow \ldots \rightarrow \nu(k, i)$ such that $pq$ is a nonzero path (of length $c_\Delta - 2$).

4. The assignment $(k, i) \mapsto \nu(k, i)$ gives a bijection $\nu : (\mathbb{Z}\Delta)_0 \rightarrow (\mathbb{Z}\Delta)_0$, called the Nakayama permutation.

5. The vertex $\nu(k, i)$ is given as follows:

   (a) If $\Delta = \mathbb{A}_r$, with $r = 2n$ or $2n - 1$, (hence $c_\Delta = r + 1$), then $\nu(k, i) = \rho r^{1-n}(k, i) = (k + i - 1, r + 1 - i)$.

   (b) If $\Delta = \mathbb{D}_{n+1}$ (hence $c_\Delta = 2n$), then

   i. $\nu(k, i) = \tau^{1-n}(k, i) = (k + n - 1, i)$, in case $n + 1$ is even.
where \( \rho \) is a m-fold mesh algebra following concept, which will be used later on in the paper. \( \rho = G \) by applying Proposition 2.3 since the cyclic group \( G \) is the mesh algebra of \( \Lambda \) isomorphic to \( \Lambda = \mathbb{Z} \) and are also periodic (see [9]).

By conditions 2 and 3 of Proposition 3.1, we have that \( \dim(Soc(\rho)) \leq l \), where \( l = \rho \tau^{-1} \) and \( l = \rho \tau^{-2} \).

Moreover, by Proposition 2.2, we know that \( B \) has essential simple (graded and ungraded) socle, which is isomorphic to \( \mathbb{S}(\nu^{-1}(k,i)[l]) \) as graded left B-module. Then all conditions of Corollary 3.1 are satisfied, with \( \nu' = \nu^{-1} \).

Moreover, by Proposition 2.2 we know that \( B \) admits a graded Nakayama form with constant degree function \( h \) such that \( h(k,i) = l \), for all \( (k,i) \in \Gamma_0 \).

3.3 m-fold mesh algebras

When \( \Gamma = \mathbb{Z} \Delta \), with \( \Delta \) a Dynkin quiver, it is known that each weakly admissible group \( G \) of automorphisms is infinite cyclic (see [28], [1]) and below is the list of the resulting stable translation quivers \( \mathbb{Z} \Delta / G \) that appear, where a generator of \( G \) is given in each case (see [13]). We will denote by \((-)\) the ‘subgroup generated by’ and, in each case, the following automorphism \( \rho \) is always that of the list preceding Proposition 3.1:

\[
\begin{align*}
\Delta^{(m)} = \mathbb{Z} \Delta / (\tau^m), & \quad \text{for } \Delta = A_n, D_n, E_n. \\
\mathbb{P}^{(m)} & = \mathbb{Z} A_2n-1 / (\rho \tau^m). \\
\mathbb{C}^{(m)} & = \mathbb{Z} D_{n+1} / (\rho \tau^m). \\
\mathbb{F}_4^{(m)} & = \mathbb{Z} E_6 / (\rho \tau^m). \\
\mathbb{G}_2^{(m)} & = \mathbb{Z} D_4 / (\rho \tau^m). \\
\mathbb{I}^{(m)} & = \mathbb{Z} A_{2n} / (\rho \tau^m).
\end{align*}
\]

As shown by Dugas (see [13], Section 3), they are the only stable translation quivers with finite-dimensional mesh algebras. These mesh algebras are isomorphic to \( \Lambda = B / G \) in each case, where \( B \) is the mesh algebra of \( \mathbb{Z} \Delta \). Abusing of notation, we will simply write \( \Lambda = \mathbb{Z} \Delta / (\varphi) \). These algebras are called m-fold mesh algebras and are known to be self-injective, a fact that can be easily seen by applying Proposition 2.3 since the cyclic group \( G \) acts freely on objects, i.e., on \( \mathbb{Z} \Delta / 0 \). They are also periodic (see [11]).

Note that, except for \( \mathbb{I}^{(m)} \), each generator of the group \( G \) in the above list is of the form \( \rho \tau^m \), where \( \rho \) is an automorphism of order 1 (i.e. \( \rho = id_{2A_n} \)), 2 or 3. This leads us to introduce the following concept, which will be used later on in the paper.

Definition 8. Let \( \Lambda = \mathbb{Z} \Delta / (\rho \tau^m) \) be an m-fold mesh algebra of a Dynkin quiver, possibly with \( \rho = id_{2A_n} \). The extended type of \( \Lambda \) will be the triple \( (\Delta, m, t) \), where \( t \) is the order of \( \rho \), in case \( \Lambda \neq \mathbb{I}^{(m)} \), and \( t = 2 \) when \( \Lambda = \mathbb{I}^{(m)} \).

We warn the reader that the commonly used type for stable Auslander algebras of representation-finite self-injective algebras (see [4], [13], [25]) does not coincide with the here defined extended type.
3.4 A change of presentation

For calculation purposes, it is convenient to modify the mesh relations. We want that if \((k, i) \in (\mathbb{Z}\Delta)_0\) is a vertex which is the end of exactly two arrows, then the corresponding mesh relation changes from a sum to a difference. When \(\Delta = \mathbb{D}_{n+1}\) and we consider the three paths \((k, 2) \rightarrow (k, i) \rightarrow (k + 1, 2) \ (i = 0, 1, 3)\), we want that the path going through \((k, 3)\) is the sum of the other two. Finally, when \(\Delta = \mathbb{E}_{n} \ (n = 6, 7, 8)\) and we consider the three paths \((k, 3) \rightarrow (k, i) \rightarrow (k + 1, 3) \ (i = 0, 4)\) and \((k, 3) \rightarrow (k + 1, 2) \rightarrow (k + 1, 3)\), we want that the one going through \((k, 0)\) is the sum of the other two. This can be done by selecting an appropriate subset \(X \subset (\mathbb{Z}\Delta)_1\) and applying the automorphism of \(K\mathbb{Z}\Delta\) which fixes the vertices and all the arrows not in \(X\) and changes the sign of the arrows in \(X\). But we want the same phenomena to pass from \(B\) to \(\Lambda = B/G\), for any weakly admissible group of automorphisms \(G\) of \(\mathbb{Z}\Delta\). This forces us to impose the condition that \(X\) be \(G\)-invariant, i.e., that \(g(X) = X\) for each \(g \in G\).

**Proposition 3.3.** Let \(\Delta\) be a Dynkin quiver, \(K\mathbb{Z}\Delta\) be the path algebra of \(\mathbb{Z}\Delta\), let \(I\) be the ideal of \(K\mathbb{Z}\Delta\) generated by the mesh relations and let \(G\) be the group of automorphisms of \(\mathbb{Z}\Delta\) generated by \(\rho\) and \(\tau\), whenever \(\rho\) exists, and just by \(\tau\) otherwise. Let \(X \subset (\mathbb{Z}\Delta)_1\) be the set of arrows constructed as follows:

1. If \(\Delta \neq \mathbb{A}_{2n-1}, \mathbb{D}_4\) and \(X'\) is the set of arrows given in the following list, then \(X\) is the union of the \(G\)-orbits of elements of \(X'\):
   
   (a) When \(\Delta = \mathbb{A}_{2n}\), \(X' = \{(0, i) \rightarrow (0, i + 1) : 1 \leq i \leq n - 1 \text{ and } i \not\equiv n \ (\text{mod } 2)\}\).
   
   (b) When \(\Delta = \mathbb{D}_{n+1}\), with \(n > 3\), \(X' = \{(0, i) \rightarrow (0, i + 1) : 2 \leq i \leq n - 2 \text{ and } i \equiv 0 \ (\text{mod } 2)\}\).
   
   (c) When \(\Delta = \mathbb{E}_6\), \(X' = \{(0, 2) \rightarrow (0, 3)\}\).
   
   (d) When \(\Delta = \mathbb{E}_n \ (n = 7, 8)\), \(X' = \{(0, 2) \rightarrow (0, 3), (0, 4) \rightarrow (1, 3), (0, 6) \rightarrow (1, 5)\}\).

2. If \(\Delta = \mathbb{D}_4\) and \(G = \langle \tau^m \rangle\), then \(X\) is the union of the \(\langle \tau \rangle\)-orbits of the arrows \((0, 2) \rightarrow (0, 3)\).

3. If \(\Delta = \mathbb{A}_{2n-1}, \mathbb{D}_4\), the given set \(X\) is \(G\)-invariant, for all choices of the weakly admissible group of automorphisms \(G\). When \(\Delta = \mathbb{A}_{2n-1} \text{ or } \mathbb{D}_4\), \(X\) is \(G\)-invariant for the respective group \(G\).

Moreover, let \(s : X \rightarrow \mathbb{Z}_2\) be the signature map, where \(s(a) = 1 \text{ exactly when } a \in X\), and let \(\varphi : K\mathbb{Z}\Delta \rightarrow K\mathbb{Z}\Delta\) be the unique graded algebra automorphism which fixes the vertices and maps \(a \mapsto (-1)^{s(a)}a\), for each \(a \in (\mathbb{Z}\Delta)_1\). Then \(\varphi(I)\) is the ideal of \(K\mathbb{Z}\Delta\) generated by the relations mentioned in the paragraph preceding this proposition.

**Proof.** The \(G\)-invariance of \(X\) is clear. In order to prove that \(\varphi(I)\) is as indicated, note that the mesh relation \(\sum_{t(a) = (k, i)} \sigma(a)a\) is mapped onto \(\sum_{t(a) = (k, i)}(-1)^{\sigma(a)}\sigma(a)a\), with the signature \(s(p)\) of a path defined as the sum of the signature of its arrows. The result will follow from the verification of the following facts, which are routine:

i) If \((k, i)\) is the terminus of exactly two arrows \(a\) and \(b\), then the set \(X \cap \{a, b, \sigma(a), \sigma(b)\}\) has only one element.

ii) When \(\Delta = \mathbb{D}_{n+1}\), with \(n > 3\), and \(a : (k - 1, 3) \rightarrow (k, 2)\), \(b : (k - 1, 0) \rightarrow (k, 2)\) and \(c : (k - 1, 1) \rightarrow (k, 2)\) are the three arrows ending at \((k, 2)\), then \(X \cap \{a, b, c, \sigma(a), \sigma(b), \sigma(c)\} = \{\sigma(a)\}\).
When \( \Delta = \mathbb{E}_n \) (\( n = 6, 7, 8 \)) and \( a : (k, 2) \to (k, 3), b : (k - 1, 0) \to (k, 3) \) and \( c : (k - 1, 4) \to (k, 3) \) are the three arrows ending at \( (k, 2) \), then \( s(\sigma(b)b) \neq 1 = s(\sigma(a)a) = s(\sigma(c)c) \).

\( \square \)

**Corollary 3.4.** With the terminology of the previous proposition, the mesh algebra is isomorphic as a graded algebra to \( B' := KZ\Delta/\varphi(I) \) and, in each case, the ideal \( \varphi(I) \) is \( G \)-invariant. In particular, \( G \) may be viewed as group of graded automorphisms of \( B' \) and \( \varphi \) induces an isomorphism \( B/G \overset{\cong}{\to} B'/G \).

**Proof.** Since \( \varphi \) is a graded automorphism of \( KZ\Delta \) it induces an isomorphism \( B = KZ\Delta/I \overset{\cong}{\to} KZ\Delta/\varphi(I) = B' \). If we view \( G \) as a group of graded automorphisms of \( KZ\Delta \), then the fact that \( X \) is \( G \)-invariant implies that \( \varphi \circ g = g \circ \varphi \), for each \( g \in G \). From this remark the rest of the corollary is clear.

\( \square \)

**Remark 3.5.** When \( \Delta = \mathbb{D}_4 \) and \( G = (\rho r^m) \), one cannot find a \( G \)-invariant set of arrows \( X \) as in the above proposition guaranteeing that, for each \( k \in \mathbb{Z} \), the path \( (k - 1, 2) \to (k - 1, 3) \to (k, 2) \) is the sum of the other two paths from \( (k - 1, 2) \) to \( (k, 2) \). This is the reason for the following convention.

**Convention 3.6.** From now on in this paper, the term ‘mesh algebra’ will denote the algebra \( KZ\Delta/\varphi(I) \) given by Corollary \([2.4]\) or just \( KZ\mathbb{D}_4/I \) in case \( (\Delta, G) = (\mathbb{D}_4, (\rho r^m)) \). This ‘new’ mesh algebra will be still denoted by \( B \).

## 4 The Nakayama automorphism. Symmetric \( m \)-fold mesh algebras

### 4.1 The Nakayama automorphism of the mesh algebra of a Dynkin quiver

The quiver \( \mathbb{Z} \Delta \) does not have double arrows and, hence, if \( a : x \to y \) is an arrow, then there exists exactly one arrow \( \nu(x) \to \nu(y) \), where \( \nu \) is the Nakayama permutation. This allows us to extend \( \nu \) to an automorphism of the stable translation quiver \( \mathbb{Z} \Delta \) and, hence, also to an automorphism of the path algebra \( K\mathbb{Z} \Delta \). Moreover, due to the (new) mesh relations (see Proposition \([4.3]\) and the paragraph preceding it), we easily see that if \( I' \) is the ideal of \( K\mathbb{Z} \Delta \) generated by those mesh relations, then \( \nu(I') = I' \). Note also from Proposition \([3.3]\) that, as an automorphism of the quiver \( \mathbb{Z} \Delta \), we have that \( \nu = \tau^k \) or \( \nu = \rho r^k \), for a suitable natural number \( k \). It follows that if \( G \) is any weakly admissible automorphism of \( \mathbb{Z} \Delta \), then \( \nu \circ g = g \circ \nu \) for all \( g \in G \). All these comments prove:

**Lemma 4.1.** Let \( \Delta \) be a Dynkin quiver, \( B = B(\Delta) \) be its associated mesh algebra and \( G \) be a weakly admissible group of automorphisms of \( \mathbb{Z} \Delta \). The Nakayama permutation \( \nu \) extends to a graded automorphism \( \nu : B \longrightarrow B \) such that \( \nu \circ g = g \circ \nu \), for all \( g \in G \).

The following result is fundamental for us.

**Theorem 4.2.** Let \( \Delta \) be a Dynkin quiver with the labeling of vertices and the orientation of the arrows of subsection \([3.2]\) and let \( G = (\varphi) \) be a weakly admissible group of automorphisms of \( \mathbb{Z} \Delta \). If \( \eta \) is the graded automorphism of \( B \) which acts as the Nakayama permutation on the vertices and acts on the arrows as indicated in the following list, then \( \eta \) is a Nakayama automorphism of \( B \) such that \( \eta \circ g = g \circ \eta \), for all \( g \in G \).

1. When \( \Delta = \mathbb{A}_n \) and \( \varphi \) is arbitrary, \( \eta(\alpha) = \nu(\alpha) \) for all \( \alpha \in (\mathbb{Z} \Delta)_1 \)
2. When \( \Delta = \mathbb{D}_{n+1} \):
   
   - (a) If \( n + 1 \geq 4 \) and \( \varphi = \tau^m \) then:
     
     i. \( \eta(\alpha) = -\nu(\alpha) \), whenever \( \alpha : (k, i) \longrightarrow (k, i+1) \) is an upward arrow with \( i \in \{2, \ldots, n-1\} \).
ii. \( \eta(\alpha) = \nu(\alpha) \), wherever \( \alpha : (k, i) \rightarrow (k + 1, i - 1) \) is downward arrow with \( i \in \{3, ..., n\} \).

iii. \( \eta(\varepsilon_i) = (-1)^i \nu(\varepsilon_i) \), for the arrow \( \varepsilon_i : (k, 2) \rightarrow (k, i) \) \( i = 0, 1 \),

iv. \( \eta(\varepsilon'_i) = (-1)^{i+1} \nu(\varepsilon'_i) \), for the arrow \( \varepsilon'_i : (k, i) \rightarrow (k + 1, 2) \) \( i = 0, 1 \).

(b) If \( n + 1 > 4 \) and \( \varphi = \rho \tau^m \) then:

i. \( \eta(\alpha) = -\nu(\alpha) \), whenever \( \alpha \) is an upward arrow as above or \( \alpha : (k, i) \rightarrow (k+1, i-1) \) is downward arrow as above such that \( k \equiv -1 \) (mod \( m \)).

ii. \( \eta(\alpha) = \nu(\alpha) \), whenever \( \alpha : (k, i) \rightarrow (k + 1, i - 1) \) is downward arrow such that \( k \not\equiv -1 \) (mod \( m \)).

iii. For the remaining arrows, if \( q \) and \( r \) are the quotient and remainder of dividing \( k \) by \( m \), then

\[
\eta(\varepsilon_i) = (-1)^q \nu(\varepsilon_i) \quad (i = 0, 1).
\]

\[
\eta(\varepsilon'_i) = (-1)^{q+1} \nu(\varepsilon'_i), \text{ when } r \neq m - 1, \text{ and } \eta(\varepsilon'_i) = (-1)^q \nu(\varepsilon'_i) \text{ otherwise}
\]

(c) If \( n + 1 = 4 \) and \( \varphi = \rho \tau^m \) (see Convention \([5,0]\)), then:

i. \( \eta(\varepsilon_i) = \nu(\varepsilon_i) \), whenever \( \varepsilon_i : (k, 2) \rightarrow (k, i) \) \( i = 0, 1, 3 \)

ii. \( \eta(\varepsilon'_i) = -\nu(\varepsilon'_i) \), whenever \( \varepsilon'_i : (k, i) \rightarrow (k + 1, 2) \) \( i = 0, 1, 3 \).

3. When \( \Delta = \mathbb{E}_6 \):

(a) If \( \varphi = \tau^m \) then:

i. \( \eta(\alpha) = \nu(\alpha) \)

ii. \( \eta(\alpha') = -\nu(\alpha') \).

iii. \( \eta(\beta) = (-1)^q \nu(\beta) \).

iv. \( \eta(\beta') = (-1)^{q+1} \nu(\beta') \).

v. \( \eta(\gamma) = (-1)^q \nu(\gamma) \)

vi. \( \eta(\gamma') = \nu(\gamma') \), when either \( q \) is odd and \( r \neq m - 1 \) or \( q \) is even and \( r = m - 1 \), and

\( \eta(\gamma') = -\nu(\gamma') \) otherwise.

vii. \( \eta(\delta) = -\nu(\delta) \)

viii. \( \eta(\delta') = \nu(\delta') \).

ix. \( \eta(\varepsilon) = -\nu(\varepsilon) \)

x. \( \eta(\varepsilon') = -\nu(\varepsilon') \), when \( r = m - 1 \), and \( \eta(\varepsilon') = \nu(\varepsilon') \) otherwise.

4. When \( \Delta = \mathbb{E}_7 \), \( \varphi = \tau^m \), and then:

i. \( \eta(\alpha) \) is given as in 3.(a) for any arrow a contained in the copy of \( \mathbb{E}_6 \).

ii. \( \eta(\zeta) = \nu(\zeta) \) and \( \eta(\zeta') = -\nu(\zeta') \), where \( \zeta : (k, 5) \rightarrow (k, 6) \) and \( \zeta' : (k, 6) \rightarrow (k + 1, 5) \).

5. When \( \Delta = \mathbb{E}_8 \), \( \varphi = \tau^m \), and then:

i. \( \eta(\alpha) \) is given as in 4 for any arrow a contained in the copy of \( \mathbb{E}_7 \).

ii. \( \eta(\theta) = \nu(\theta) \) and \( \eta(\theta') = -\nu(\theta') \), where \( \theta : (k, 6) \rightarrow (k, 7) \) and \( \theta' : (k, 7) \rightarrow (k + 1, 6) \).
Proof. Let $\nu$ be the Nakayama permutation of $\mathbb{Z} \Delta$ (see Proposition 3.1). By the proof of Corollary 2.2, we know that $\text{Soc}_{gr}(e_{(k,i)}B) = \text{Soc}(e_{(k,i)}B)$ is one-dimensional and concentrated in degree $l = c_\Delta - 2$, for each $(k,i) \in \mathbb{Z} \Delta_0$. By applying Proposition 2.2 after taking a nonzero element $w_{(k,i)} \in e_{(k,i)} \text{Soc}_{gr}(B)$, for each $(k,i) \in (\mathbb{Z} \Delta)_0$, we can take the graded Nakayama form $(-, -) : B \times B \to K$ of degree $l$ associated to $B = (B_{(k,i)})_{(k,i) \in \mathbb{Z} \Delta_0}$ (see definition 0), where $B_{(k,i)} = \{ w_{(k,i)} \}$ is a basis of $e_{(k,i)} B e_{(k,i)}$, for each $(k,i) \in \mathbb{Z} \Delta_0$. It is clear that the so obtained graded Nakayama form will be $G$-invariant whenever $B \subset \bigcup_{(k,i) \in \mathbb{Z} \Delta_0} B_{(k,i)}$ is $G$-invariant. The existence of a $G$-invariant basis is guaranteed by Corollary 2.4. Moreover, in such case, recall that the associated Nakayama automorphism $\eta$ satisfies that $\eta g = g \eta$, for all $g \in G$ (see Remark 2.3). The canonical way of constructing such a $G$-invariant basis $B$ is also given in that remark. Namely, we select a set $I'$ of representatives of the $G$-orbits of vertices and a element $0 \neq w_{(k,i)} \in e_{(k,i)} \text{Soc}_{gr}(B)$, for each $(k,i) \in I'$. Then $B = \{ g(w_{(k,i)}) : g \in G, (k,i) \in I' \}$ is a $G$-invariant basis as desired. However, if we choose $B$ to be $\tau$-invariant, then it is $G$-invariant for $G = (\tau^m)$. So, in order to construct $B$, we will only need to consider the cases $\varphi = \tau$ and $\varphi = \rho \tau^m$.

Once the $G$-invariant basis $B$ of $\text{Soc}_{gr}(B) = \text{Soc}(B)$ has been described, the strategy to identify the action of the associated Nakayama automorphism $\eta$ on the arrows is very simple. Given an arrow $a$, we take a path $q : t(a) \to \ldots \to \nu(i(a))$ of length $l - 1$ such that $aq$ is a nonzero path. Then we have $aq = (-1)^{u(a)} w_{t(a)}$, so that, by definition of the graded Nakayama form associated to $B$, we have an equality $(\alpha, q) = (-1)^{u(a)}$. Since the quiver $\mathbb{Z} \Delta$ does not have double arrows we know that $\eta(\alpha) = \lambda(\alpha) \nu(\alpha)$, for some $\lambda(\alpha) \in K^\ast$. In particular we know that $\nu \nu(\alpha)$ is a nonzero path (of length $l$) because $(q, \eta(\alpha)) = (\alpha, q) \neq 0$. If we have an equality $\nu \nu(\alpha) = (-1)^{v(\alpha)} w_{t(\alpha)}$ in $B$, then it follows that $(-1)^{v(\alpha)} = (\alpha, q) = (q, \eta(\alpha)) = \lambda(\alpha)(q, \nu(\alpha)) = \lambda(\alpha)(-1)^{v(\alpha)}$. Then we get $\lambda(\alpha) = (-1)^{v(\alpha)}$ and the task is reduced to calculate the exponents $u(\alpha)$ and $v(\alpha)$ in each case. Taking into account that we have $\eta \circ g = g \circ \eta$, for each $g \in G$, it is enough to calculate $u(\alpha)$ and $v(\alpha)$ just for the arrows starting at a vertex of $I'$.

To construct $B$ when $\Delta = A_n$ has no problem, for all paths of length $l = c_\Delta - 2$ from $(k,i)$ to $\nu(k,i)$ are equal in $B$. So in this case the choice of $w_{(k,i)}$ will be the element of $B$ represented by a path from $(k,i)$ to $\nu(k,i)$ and $B = \{ w_{(k,i)} : (k,i) \in (\mathbb{Z} \Delta)_0 \}$ is $G$-invariant for any choice of $\varphi$.

On what concerns the explicit calculations for the cases when $\Delta$ is either $\bigoplus_{n+1}$ or $E_r$ ($r = 6, 7, 8$), they can be found in the appendix given at the end of this paper.

\end{proof}

4.2 Some important auxiliary results

Recall that a walk in a quiver $Q$ between the vertices $i$ and $j$ is a finite sequence $i = i_0 \leftrightarrow i_1 \leftrightarrow \ldots \leftrightarrow i_{r-1} \leftrightarrow i_r = j$, where each edge $i_k \leftrightarrow i_{k+1}$ is either an arrow $i_k \to i_{k+1}$ or an arrow $i_k \to i_{k-1}$. We write such a walk as $\alpha_1^{e_1} \ldots \alpha_r^{e_r}$, where the $\alpha_i$ are arrows and $e_i$ is 1 or $-1$, depending on whether the corresponding edge is an arrow pointing to the right or to the left.

We will need the following concept from [23].

Definition 9. Let $Q$ be a (not necessarily finite) quiver. An acyclic character on $Q$ (over the field $K$) is a map $\chi : Q_1 \to K^\ast$ such that if $p = \alpha_1^{e_1} \ldots \alpha_r^{e_r}$ and $q = \beta_1^{f_1} \ldots \beta_s^{f_s}$ are two walks of length $> 0$ between any given vertices $i$ and $j$, then $\prod_{1 \leq j \leq r} \chi(\alpha_i)^{e_i} = \prod_{1 \leq j \leq s} \chi(\beta_j)^{f_j}$.

Notice that if $A$ is a graded algebra with enough idempotents, $G \subseteq \text{Aut}^{gr}(A)$ is a group acting freely on objects such that $A/G$ is finite dimensional and $f : A \to A$ is a graded automorphism commuting with the elements in $G$, then the assignment $[a] \mapsto [f(a)]$, with $a \in \bigcup_{i,j \in I} e_i A e_j$, determines a graded automorphism $\bar{f}$ of $A/G$.

The following general result will be very useful.

Lemma 4.3. Let $A = \bigoplus_{n \geq 0} A_n$ be a basic positively $\mathbb{Z}$-graded pseudo-Frobenius algebra with enough idempotents such that $e_i A e_i \cong K$, for each $i \in I$, let $G$ be a group of graded automorphisms of $A$ acting freely on objects such that $A = A/G$ is finite dimensional and let $f : A \to A$ be a graded automorphism that fixes all idempotents $e_i$ and satisfies that $f \circ g = g \circ f$, for all $g \in G$.

Then the following assertions are equivalent:

1. $\bar{f}$ is an inner automorphism of $\Lambda$. 


2. There is a map \( \lambda : I \rightarrow K^* \) such that \( f(a) = \lambda(i(a))^{-1} \lambda(t(a))a \), for all \( a \in \bigcup_{i,j \in I} e_i A e_j \), and \( \lambda \circ g_I \lambda = \lambda \), for all \( g \in G \).

Proof. 1) \( \implies \) 2) Let \( \lambda : I \rightarrow K^* \) be any map and \( \chi : A \rightarrow A \) be the (graded) automorphism which is the identity on objects and maps \( a \sim \lambda(i(a))^{-1} \lambda(t(a))a \), for each \( a \in \bigcup_{i,j \in I} e_i A e_j \).

If now \( f \) is as in the statement and \( \tilde{f} \) is inner, the goal is to find a map \( \lambda \) as in the previous paragraph such that \( \chi_a = f \) and \( \lambda \circ g_I \lambda = \lambda \), for all \( g \in G \).

We know from Proposition 2.2 that \( \Lambda \) is a split basic graded algebra. So it is given by a finite quiver with relations whose set of vertices is (in bijection with) the set \( I/G = \{ [i] : i \in I \} \) of \( G \)-orbits of elements of \( I \). From Proposition 10 and Theorem 12 we get a map \( \lambda : I/G \rightarrow K^* \) such that the assignment \( [a] \sim \lambda([i(a)])^{-1} \lambda([t(a)])[a] \), where \( a \in \bigcup_{i,j \in I} e_i A e_j \), is a (graded) inner automorphism of \( \Lambda \) such that \( \lambda^{-1} \circ \tilde{f} \) is the inner automorphism \( \iota = \iota_{\Lambda - x} \) of \( \Lambda \) defined by an element of the form \( 1 - x \), where \( x \in J(\Lambda) \). In our situation, the equality \( J(\Lambda) = \oplus_{n>0} \Lambda_n \) holds, so that \( x \) is a sum of homogeneous elements of degree \( >0 \). But \( \iota = \lambda^{-1} \circ \tilde{f} \) is also a graded automorphism, so that we have that \( \iota(\Lambda_n) = (1-x)\Lambda_n(1-x)^{-1} = \Lambda_n \). If \( y \in \Lambda_n \) then the \( n \)-th homogeneous component of \((1-x)y(1-x)^{-1} = y \). It follows that \( \iota \) is the identity on \( \Lambda_n \), for each \( n \geq 0 \). Therefore we have \( \iota = id_\Lambda \), so that \( \tilde{f} = u \).

Let now \( \pi : A \rightarrow \Lambda = A/G \) be the \( G \)-covering functor and let \( \lambda \) be the composition map \( I \xrightarrow{\pi} I/G \xrightarrow{\lambda} K^* \). By definition, we have that \( \lambda \circ g = \lambda \), for all \( g \in G \). As a consequence, the associated automorphism \( \chi_\lambda : A \xrightarrow{\lambda} A \) defined above has the property that \( \chi_\lambda([a]) = \pi(u[a]) = \tilde{f}([a]) = [f(a)] \), for each \( a \in \bigcup_{i,j \in I} e_i A e_j \). Since \( f \) is the identity on objects we immediately get that \( f = \chi_\lambda \) as desired.

2) \( \implies \) 1) The map \( \lambda \) of the hypothesis satisfies that \( \chi_a = f \). It then follows that \( \chi_\lambda = \tilde{f} \), where \( \chi_\lambda : \Lambda \rightarrow \Lambda \) maps \( [a] \sim \lambda([i(a)])^{-1} \lambda([t(a)])[a] \), for each \( a \in \bigcup_{i,j \in I} e_i A e_j \). Note that \( \chi_\lambda \) is well defined because \( \lambda \circ g_I \lambda = \lambda \), for all \( g \in G \). It turns out that \( \chi_\lambda \) is the inner automorphism of \( \Lambda \) defined by the element \( \sum_{[i] \in I/G} \lambda([i])^{-1} e_{[i]} \).

In the rest of the paper, for any \( m \)-fold mesh algebra \( \Lambda \), we shall denote by \( \text{Im}^g(\Lambda) \) the subgroup of \( \text{Aut}^g(\Lambda) \) consisting of those graded automorphisms which are inner.

### Proposition 4.4
Let \( \Delta \) be a Dynkin quiver, let \( G \) be a weakly admissible group of automorphisms of \( Z\Delta \), let \( \Lambda = B(\Delta) \) be the associated graded Nakayama automorphism of \( B = B(\Delta) \). The images of \( \tau, \tilde{\nu} \) and \( \tilde{\eta} \) by the canonical group homomorphism \( \text{Aut}^g(\Lambda) \rightarrow \text{Aut}^g(\Lambda)/\text{Im}^g(\Lambda) \) are all in the center of \( \text{Aut}^g(\Lambda)/\text{Im}^g(\Lambda) \).

Proof. Due to the fact that the quiver of \( \Lambda \) (i.e. \( Q = Z\Delta/G \)) does not have double arrows, each graded automorphism \( \varphi \) of \( \Lambda \) induces an automorphism \( \bar{\varphi} \) of \( Q \). This automorphism extends to an automorphism on the path algebra \( KQ \) which respects the mesh relations. Therefore we can look at \( \bar{\varphi} \) as a graded automorphism of \( \Lambda \) as well. Since \( \bar{\varphi} \) and \( \tau \) commute when viewed as automorphisms of the quiver \( Q \), we get that they also commute as graded automorphisms of \( \Lambda \). On the other hand, if \( (-,-) : \Lambda \times \Lambda \rightarrow K \) is a graded Nakayama form whose associated Nakayama automorphism is \( \bar{\eta} \), then the map \( [-,-] : \Lambda \times \Lambda \rightarrow K \) given by \( [a,b] = (\varphi(a), \varphi(b)) \) is a graded Nakayama form whose associated Nakayama automorphism is \( \varphi^{-1} \circ \bar{\eta} \circ \varphi \). But all Nakayama automorphisms of \( \Lambda \) are equal, up to composition by an inner automorphism. We conclude that \( \varphi^{-1} \circ \bar{\eta} \circ \varphi \circ \bar{\varphi}^{-1} \in \text{Im}^g(\Lambda) \), so that the statement about \( \bar{\eta} \) is proved. Moreover, the Nakayama permutation of \( Q_0 \) associated to \( \varphi^{-1} \circ \bar{\eta} \circ \varphi \) "is\" \( \varphi^{-1} \circ \bar{\nu} \circ \varphi \), which, by Proposition 22, implies that \( \bar{\nu} \circ \varphi = \bar{\varphi} \circ \bar{\nu} \), an equality that may be seen as an equality of (graded) automorphisms of \( \Lambda \).

From the previous paragraph we also deduce that each \( \varphi \in \text{Aut}^g(\Lambda) \) decomposes as \( \varphi = \bar{\varphi} \circ \bar{\chi} \), where \( \bar{\chi} \) is an automorphism of \( \Lambda \) which is the identity on vertices of \( Q \). This means that we also have a map \( \xi : Q_1 = \frac{Z\Delta_1}{G} \rightarrow K^* \) such that \( \bar{\chi}([a]) = \xi([a])[a] \), for each \( a \in Z\Delta_1 \). Denoting by \( \xi \) the composition \( Z\Delta_1 \xrightarrow{\pi} \frac{Z\Delta_1}{G} \xrightarrow{\bar{\xi}} K^* \), we have that \( \xi \circ g_{Z\Delta_1} = \xi \), for all \( g \in G \), and \( \bar{\chi}([a]) = \xi([a])[a] \), for all \( a \in Z\Delta_1 \). We then get a \( G \)-invariant graded automorphism \( \chi = \xi^2 \) of \( \Lambda \) which fixes the vertices, maps \( a \sim \xi(a)a \), for all \( a \in Z\Delta_1 \), and has the property that \( \bar{\chi}([a]) = \xi([a]) \), for all \( a \in Z\Delta_1 \).

By the first paragraph of this proof, we know that \( \tau \) and \( \bar{\nu} \) commute with the (graded) automorphisms of \( \Lambda \) given by quiver automorphisms of \( Q \). In order to end the proof, we just need to
Both expressions are equal to 1 since \( \xi \circ g_{\Delta a} = \xi \) for all \( g \in G \). Moreover, by Proposition \([23]\) we know that \( \nu = \rho^k \circ \tau^l \), where \( k \in \{0,1\} \) and \( l > 0 \). Our task is hence reduced to check that \( \psi \circ \chi \circ \psi^{-1} \circ \chi^{-1} \) is inner, when \( \psi \in \{\rho,\tau\} \) and \( \chi = \chi^2 \), for some map \( \xi \) as above. Note that \( \psi \circ \chi \circ \psi^{-1} \circ \chi^{-1} \) is an automorphism of \( B \) which fixes the vertices, commutes with all elements of \( G \) and maps \( a \to \chi(a)^{-1} \chi(\psi^{-1}(a)) \), for each \( a \in \Delta a \). Then Lemma \([23]\) can be used. Note that the compatibility of \( \chi \) with the mesh relations implies that, for each \((k,i) \in \Delta a_0\), the product \( \xi(\sigma(a))\xi(a) \) is constant on the set of arrows ending at \((k,i)\). It follows easily that the map \( \xi : \Delta a_1 \to K^* \), which takes \( a \to \xi(a)^{-1} \xi(\psi^{-1}(a)) \), is an acyclic character of \( \Delta \). By \([23]\) Proposition \(10\) and its proof, there is a map \( \lambda = \Delta a_0 \to K^* \) such that \( \lambda^{-1}_0 \lambda(a) = \xi(a)^{-1} \xi(\psi^{-1}(a)) \), for all \( a \in \Delta a_1 \).

By Lemma \([23]\) our task is reduced to prove that such a map \( \lambda \) has the property that \( \lambda \circ g_{\Delta a_0} = \lambda \), for all \( g \in G \). We claim that if \( G = \langle \varphi \rangle \) and we have that \( \lambda \circ \varphi \)(r, j) = \( \lambda(r, j) \) for one vertex \((r, j)\), then \( \lambda \circ g_{\Delta a_0} = \lambda \), for all \( g \in G \). Indeed, suppose that this is the case. By definition of \( \lambda \), if \( \lambda \circ \varphi \) and \( \lambda \) act the same on the origin or the terminus of a given arrow \( a \), then they act the same both on \( i(a) \) and \( t(a) \). Then, by the connectedness of \( \Delta a \), we get that \( \lambda \circ g_{\Delta a_0} = \lambda \) as desired. We then pass to prove, for all possiblilities of the extended type \((\Delta, m, t)\), that there is a \((r, j) \in \Delta a_0 \) such that \( \lambda \circ \varphi \)(r, j) = \( \lambda(r, j) \), where \( \varphi \) is either \( \tau^m \), when \( t = 1 \), or \( \rho \circ \tau^m \), when \( t = 2 \). Indeed, except when \((\Delta, m, t) = (k_2n, m, 2)\), we always have a \( j \in \Delta a_0 \) such that \( \rho(a, j) = (k, j) \), for all \( k \in \mathbb{Z} \) (here we consider also the case when \( t = 1 \), and hence \( \rho = \Delta a_2 \)). In particular, we have \( \varphi(0, j) = \tau^m(0, j) = (m, j) \). We fix such a vertex and consider the canonical path \( \varphi(0, j) = (m, j) \to (m + 1, j) \to \cdots \to (0, j) \), and by definition of \( \lambda \), we then have an equality

\[
\lambda(0, j) = \varphi(0, j)^{-1} \xi(\sigma(\varphi^{-1}(0))) \xi(\sigma(\varphi^{-1}(1))) \cdots \xi(\sigma(\varphi^{-1}(j))) = \lambda(0, j).
\]

We need to prove that the equality \( \prod_{0 \leq k \leq m-1} \xi(\sigma(\ddots(\sigma(\varphi^{-1}(0))))\cdots(\ddots(\sigma(\varphi^{-1}(j))))) = 1 \) holds. Note that, acting on arrows, both \( \rho \) and \( \tau \) commute with \( \sigma \). For \( \psi = \tau \), we then get that

\[
\prod_{0 \leq k \leq m-1} \xi(\sigma(\ddots(\sigma(\varphi^{-1}(0))))\cdots(\ddots(\sigma(\varphi^{-1}(j))))) = \prod_{0 \leq k \leq m-1} \xi(\sigma(\ddots(\sigma(\varphi^{-1}(0))))\cdots(\ddots(\sigma(\varphi^{-1}(j))))) = \prod_{0 \leq k \leq m-1} \xi(\sigma(\varphi^{-1}(j))) = 1.
\]

When \( G = \langle \tau^m \rangle \), this expression is equal to 1 since \( \xi \circ g_{\Delta a_1} = \xi \), for all \( g \in G \). On the other hand, if \( G = \langle \rho \tau^m \rangle \) (and hence \( \Delta \neq k_2n \)), then \( \rho(\sigma^{-2}(a)) \) and \( \sigma^2(a) \) are arrows ending at \((r, j)\), for each integer \( r \geq 0 \), due to the choice of \( j \). It follows that \( \xi(\sigma^{-2}(\rho(a))) \xi(\sigma^2(\rho(a))) = \xi(\sigma^{-2}(\rho(a))) \xi(\sigma^2(a)) \), for each \( r \geq 0 \) (see the third paragraph of this proof). We then get \( \prod_{0 \leq k \leq m-1} \xi(\sigma(a)) \prod_{0 \leq k \leq m-1} \xi(\tau(a)) = 1 \) also in this case.

We finally consider the case when \((\Delta, t) = (k_2n, 2)\). Similarly to the other cases, we consider the canonical path

\[
\varphi(0, n) = \rho \tau^m(0, n) = (m, n + 1) \to (m + 1, n) \to \cdots \to (n, 0).
\]

With the argument used above, we need to check that

\[
\prod_{0 \leq k \leq m-2} \xi(\sigma^k(a)) = 1.
\]

Note that we have \( \rho(\sigma(a)) = \sigma^{-1}(a) \), for each \( k \in \mathbb{Z} \). As a consequence, we get:

\[
\prod_{0 \leq k \leq m-2} \xi(\sigma^k(a)) = \xi(\sigma(a))^{-1} \cdot \xi(\sigma^{-1}(a)) \cdot \xi(\sigma^{-2}(a)) \cdots \cdot \xi(\sigma^{-m}(a)) = 1.
\]

Both expressions are equal to 1 since \( \xi \circ g_{\Delta a_1} = \xi \), for all \( g \in G = \langle \rho \tau^m \rangle \).
Proposition 4.5. Let $\Lambda$ be the $m$-fold mesh algebra of extended type $(\Delta, m, t)$ and let $H(\Delta, m, t)$ be the set of integers $s$ such that $\eta^s\nu^{-s}$ is an inner automorphism of $\Lambda$. Then $H(\Delta, m, t)$ is a subgroup of $\mathbb{Z}$ and the following assertions hold:

1. If $\text{char}(K) = 2$ or $\Delta = \Lambda_r$, then $H(\Delta, m, t) = \mathbb{Z}$.

2. If $\text{char}(K) \neq 2$ and $\Delta \neq \Lambda_r$, then $H(\Delta, m, t) = \mathbb{Z}$, when $m + t$ is odd, and $H(\Delta, m, t) = 2\mathbb{Z}$ otherwise.

Proof. Theorem 4.2 gives a map $\xi : \mathbb{Z}\Delta_1 \rightarrow K^*$, where $\xi(a) = (-1)^{u(a)}$, for some $u(a) \in \mathbb{Z}_2$, such that $\xi \circ g_{\mathbb{Z}\Delta_1} = \xi$ and if $\chi = \chi^\xi$, with the terminology of the proof of Proposition 4.3 then $\eta := \nu \circ \chi$ is a $G$-invariant graded Nakayama automorphism of $B$. By the last mentioned proposition, we conclude that $H(\Delta, m, t) = \{ s \in \mathbb{Z} : \bar{\chi}^s \in \text{Inn}(\Lambda) \}$. It is clearly a subgroup of $\mathbb{Z}$, which contains $2$ since $\chi \circ \chi = \text{id}_B$. Moreover, when $\text{char}(K) = 2$ or $\Delta = \Lambda_r$, we have $\chi = \text{id}_B$ so that $H(\Delta, m, t) = \mathbb{Z}$ in this case. Our task reduces to determine, for the remaining cases, when $\bar{\chi} \in \text{Inn}(\Lambda)$.

Using Lemma 4.3 and arguing as in the proof of Proposition 4.2 we have a map $\lambda : \mathbb{Z}\Delta_0 \rightarrow K^*$ such that $\lambda|_{\mathbb{Z}\Delta_0}, \lambda(a) = \xi(a) = (-1)^{u(a)}$, for all $a \in \mathbb{Z}\Delta_1$, and we need to determine when $\lambda \circ \varphi_{\mathbb{Z}\Delta_0} = \lambda$, where $\varphi$ is the canonical generator of $G$. We emulate the proof of the previous proposition and fix a $j \in \Delta_0$ such that $\rho(0, j) = (0, j)$. By the proof of the mentioned proposition, it is enough to show that $\lambda(\varphi(0, j)) = \lambda(0, j)$. We have a path

$$\varphi(0, j) = (-m, j) \xrightarrow{\sigma^{m-1}(a)} \ast \xrightarrow{\sigma^{m-2}(a)} (-m + 1, j) \rightarrow \ldots \rightarrow (-1, j) \xrightarrow{\sigma(a)} \ast \rightarrow (0, j),$$

which, by the definition of $\lambda$, implies that

$$\lambda(0, j) = \prod_{0 \leq k \leq 2m-1} (-1)^{u(a^k(a))}\lambda_{\varphi(0, j)} = (-1)^{\sum_{0 \leq k \leq 2m-1} u(a^k(a))}\lambda_{\varphi(0, j)}.$$

When $t = 1$, so that $\varphi = \tau^m$, Theorem 4.2 gives that, for all choices of $\Delta$, we have $u(b) \neq u(\sigma(b))$, for all $b \in \mathbb{Z}\Delta_1$. It follows that $\sum_{0 \leq k \leq 2m-1} u(a^k(a)) = m$. Similarly, when $t = 3$ (whence $\Delta = \mathbb{D}_4$), we have $j = 2$ and, looking at Theorem 4.2 one gets that, for any choice of the arrow $a$ ending at $(0, 2)$, the equality $\sum_{0 \leq k \leq 2m-1} u(a^k(a)) = m$ also holds. In both cases, we conclude that the equality $\lambda_{\varphi(0, j)} = \lambda_{(0, j)}$ holds exactly when $m$ is even.

Finally, suppose that $t = 2$, so that $\varphi = \rho^m$. We take as arrow $a$ ending at $(0, j)$ the following:

1. When $\Delta = \mathbb{D}_{n+1}$, with $n + 1 > 4$, we have $j = 2$ and take $a : (-1, 3) \rightarrow (0, 2)$;

2. When $\Delta = \mathbb{E}_6$, we have $j = 3$ and take $a = \epsilon' : (-1, 0) \rightarrow (0, 3)$.

Using Theorem 4.2 we see that in the family of exponents $\{ u(\sigma^r(a)) : r = 0, 1, \ldots, 2m - 1 \}$ there are exactly $m + 1$ which are equal to $1$. It follows that $\lambda_{\varphi(0, j)} = \lambda_{(0, j)}$ holds exactly when $m$ is odd.

\[
\square
\]

4.3 Symmetric and weakly symmetric $m$-fold mesh algebras

The only result of this subsection identifies all the weakly symmetric and symmetric $m$-fold mesh algebras.

Theorem 4.6. Let $\Lambda$ be an $m$-fold mesh algebra of extended type $(\Delta, m, t)$. If $\Lambda$ is weakly symmetric then $t = 1$ or $t = 2$ and, when $\text{char}(K) = 2$ or $\Delta = \Lambda_r$, such an algebra is also symmetric. Moreover, the following assertions hold:

1. When $t = 1$, $\Lambda$ is weakly symmetric if, and only if, $\Delta$ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$ and $m$ is a divisor of $\frac{\Delta - 1}{2} - 1$. When $\text{char}(K) \neq 2$, such an algebra is symmetric if, and only if, $m$ is even.
2. When \( t = 2 \) and \( \Delta \neq A_{2n} \), \( \Lambda \) is weakly symmetric if, and only if, \( m \) divides \( \frac{n}{2} - 1 \) and, moreover, the quotient of the division is odd, in case \( \Delta = A_{2n-1} \), and even, in case \( \Delta = D_{2r} \).

When \( \text{char}(K) \neq 2 \), such an algebra is symmetric if, and only if, \( \Delta = A_{2n-1} \) or \( m \) is odd.

3. When \( (\Delta, m, t) = (A_{2n}, m, 2) \), i.e. \( \Lambda = L_{m}^{(n)} \), the algebra is (weakly) symmetric if, and only if, \( 2m - 1 \) divides \( 2n - 1 \).

**Proof.** The algebra \( \Lambda \) is weakly symmetric if, and only if, the automorphism \( \tilde{\nu} : \Lambda \rightarrow \Lambda \) induced by \( \nu \) is the identity on vertices. We identify the vertices of the quiver of \( \Lambda \) as \( G \)-orbits of vertices of \( \mathbb{Z}\Delta_{0} \), where \( G \) is the weakly admissible group of automorphism considered in each case. If we take care to choose a vertex \( (k, i) \) which is not fixed by \( \rho \), then the equality \( \tilde{\nu}[(k, i)] = [(k, i)] \) holds exactly when there is a \( \nu(k, i) = g(k, i) \). But if \( \tilde{G} \) denotes the group of automorphisms generated by \( \rho \) and \( \tau \), then \( \tilde{G} \) acts freely on the vertices not fixed by \( \rho \). Since \( G \subseteq \tilde{G} \) and \( \nu \in \tilde{G} \) (see Proposition \([3] \)) the equality \( \nu(k, i) = g(k, i) \) implies that \( \nu = g \). Therefore the algebra \( \Lambda \) is weakly symmetric if, and only if, \( \nu \) belongs to \( G \).

On the other hand, \( \Lambda \) is symmetric if, and only if, \( \tilde{\nu} : \Lambda \rightarrow \Lambda \) is an inner automorphism. By Lemma \([1,3] \) this is equivalent to saying that \( \Lambda \) is weakly symmetric and \( \bar{\eta} \circ \tilde{\nu}^{-1} \) is an inner automorphism of \( \Lambda \). That is, \( \Lambda \) is symmetric if, and only if, \( \Lambda \) is weakly symmetric and \( H(\Delta, m, t) = \mathbb{Z} \). As a consequence, once the weakly symmetric \( m \)-fold mesh algebras have been identified, the part of the theorem referring to symmetric algebras follows directly from Proposition \([1,5] \).

If \( t = 3 \), then \( \Delta = D_{4} \), \( G = (\rho r m) \), with \( r \) acting on vertices as the 3-cycle (013), and \( \nu = \tau^{-2} \). It is impossible to have \( \tau^{-2} \in G \) and therefore \( \Lambda \) is never weakly symmetric in this case.

If \( t = 1 \), then \( G = (\tau m) \). If we assume that \( \Delta \neq D_{2r}, E_{7}, E_{6} \) and then \( \nu = \rho r^{-1} n \), for some integer \( n \). Again it is impossible that \( \nu \in G \) and, hence, \( \Lambda \) cannot be weakly symmetric. On the contrary, suppose that \( \Delta \) is one of the Dynkin quivers \( D_{2r}, E_{7}, \) or \( E_{6} \). Then \( \nu = \tau^{-1} n \), with \( n = \frac{\rho r}{2} \), and \( \nu \) belongs to \( G \) if, and only if, there is an integer \( r \) such that \( \tau^{-1} n = (\tau m)^{r} \), which is equivalent to saying that \( n - 1 = -mr \) since \( \tau \) has infinite order. Then \( \Lambda \) is weakly symmetric in this case if, and only if, \( m \) divides \( n - 1 \).

Suppose now that \( t = 2 \) and \( \Delta \neq A_{2n} \). Then \( G = (\rho r m) \) and, except when \( \Delta = D_{2r} \), we have that \( \nu = \rho r^{-1} n \), where \( n = \frac{\rho r}{2} \). Assume that \( \Delta \neq D_{2r} \). Then \( \nu \) is in \( G \) if, and only if, there is an integer \( r \) such that \( \rho r^{-1} n = (\rho r m)^{r} \). Note that then \( r \) is necessarily odd. If follows that \( \Delta \) is weakly symmetric if, and only if, \( m \) divides \( n - 1 \) and the quotient \( \frac{n - 1}{m} \) is an odd number. But the condition on \( \frac{n - 1}{m} \) to be odd is superfluous when \( \Delta = D_{2r+1} \) or \( E_{6} \) because \( n \) is even in both cases.

Consider now the case in which \( (\Delta, t) = (D_{2r}, 2) \). Then \( \nu = \tau^{-1} n \), where \( n = \frac{\rho r}{2} = 2r - 1 \). Then \( \nu \) is in \( G \) if, and only if, there is an integer \( s \) such that \( \tau^{-1} n = (\rho r m)^{s} \). This forces \( s \) to be even. We then get that \( \Lambda \) is weakly symmetric if, and only if, \( m \) divides \( n - 1 \) and the quotient \( \frac{n - 1}{m} \) is even.

Finally, let us consider the case when the extended type is \( (A_{2n}, m, 2) \). In this case \( \rho^{2} = \tau^{-1} \) and \( \nu = \rho r^{-1} n \). Then \( \nu \) is in \( G \) if, and only if, there is an integer \( r \) such that \( \rho r^{-1} n = (\rho r m)^{r} \). This forces \( r = 2s + 1 \) to be odd, and then \( \rho r^{-1} m(2s + 1) = (\rho r m)^{2s+1} = \rho r^{-1} n \). Then \( \Lambda \) is weakly symmetric if, and only if, there is an integer \( s \) such that \( (2m - 1)s = 1 - m - n \). That is, if and only if \( 2m - 1 \) divides \( m + n - 1 \), which is equivalent to saying that \( 2m - 1 \) divides \( 2(m + n - 1) - (2m - 1) = 2n - 1 \). \( \square \)

5 The period and the stable Calabi-Yau dimension of an \( m \)-fold mesh algebra

5.1 The minimal projective resolution of the regular bimodule

**Lemma 5.1.** Let \( \Delta \) be a Dynkin quiver and \( B = B(\mathbb{Z}\Delta) \) be its associated mesh algebra. For any weakly admissible group of automorphisms \( G \) of \( \mathbb{Z}\Delta \), there is a basis \( B \) of \( G \) consisting of paths which is \( G \)-invariant (i.e. \( g(B) = B \) for all \( g \in G \)).

**Proof.** The way of constructing the basis \( B \) is analogous to the way in which a \( G \)-invariant basis of \( \text{soc}(B) \) was constructed (see the initial paragraphs of the proof of Theorem \([4, 2] \)). The task is reduced to find, for each vertex \( (k, i) \) in a fixed subset \( I' \subset \mathbb{Z}\Delta_{0} \) of representatives of the \( G \)-orbits, a basis of \( e_{(k, i)} B \) consisting of paths. Since the existence of this basis is clear the result follows. \( \square \)
Suppose that \((-,-): B \times B \rightarrow K\) is a \(G\)-invariant graded Nakayama form for \(B\). Given a basis \(B\) as in last lemma, its (right) dual basis with respect to \((-,-)\) will be the basis \(B^* = \bigcup_{(k,i) \in (\mathbb{Z}_\Delta)_0} B^* e_{v(k,i)}\), where \(B^* e_{v(k,i)}\) is the (right) dual basis of \(e_{v(k,i)}B\) with respect to the induced graded bilinear form \((-,-): e_{v(k,i)}B \times B e_{v(k,i)} \rightarrow K\). By the graded condition of this bilinear form, \(B^*\) consists of homogeneous elements. By the \(G\)-invariance of \((-,-)\) and \(B\), we immediately get that \(B^*\) is \(G\)-invariant. On what concerns the minimal projective resolution of \(B\) as a bimodule, we will need to fix a basis \(B\) as given by last lemma and use it and its dual basis to give the desired resolution.

By a classical argument for unital algebras, also valid here (see, e.g., [8] or [13]), we know that if \(B'\) is the original mesh algebra, i.e., \(K\mathbb{Z}\Delta/\mathbb{I}\) where \(I\) is the ideal generated by \(r_{(k,i)} = \sum_{t(a) = (k,i)} \sigma(a)a\) with \((k,i) \in \mathbb{Z}\Delta_0\), then the initial part of the minimal projective resolution of \(B'\) as a graded \(B'\)-bimodule has the following shape

\[
Q^{-2} \xrightarrow{R'} Q^{-1} \xrightarrow{\delta} Q^0 \xrightarrow{u} B' \rightarrow 0
\]

where \(Q^0 = \bigoplus_{(k,i) \in (\mathbb{Z}_\Delta)_0} B^* e_{v(k,i)} \otimes e_{v(k,i)} B')[0]\), \(Q^{-1} = \bigoplus_{a \in (\mathbb{Z}\Delta)_1} B' e_i(a) \otimes e_t(a) B')[-1]\), \(Q^{-2} = \bigoplus_{(k,i) \in (\mathbb{Z}_\Delta)_0} B' e_{\tau(k,i)} \otimes e_{v(k,i)} B')[-2]\), and

1. \(u\) is the multiplication map,
2. \(\delta\) is the only homomorphism of \(B'\)-bimodules such that, for all \(a \in (\mathbb{Z}_\Delta)_1\),

\[
\delta(e_i(a) \otimes e_t(a)) = a \otimes e_t(a) - e_i(a) \otimes a;
\]
3. \(R'\) is the only homomorphism of \(B'\)-bimodules such that, for all \((k,i) \in (\mathbb{Z}\Delta)_0\),

\[
R'(e_{\tau(k,i)} \otimes e_{v(k,i)}) = \sum_{t(a) = (k,i)} [\sigma(a) \otimes e_{v(k,i)} + e_{\tau(k,i)} \otimes a]
\]

We will slightly modify this resolution bearing in mind that, for simplification purposes, we are working with the mesh algebra \(B\) given by the new relations as in Section 3.4. We point out that this change only accounts for the difference in the description \(R'\). Indeed, considering the canonical algebra isomorphism \(\varphi = \varphi^{-1}: K\mathbb{Z}\Delta \xrightarrow{\cong} K\mathbb{Z}\Delta\), given in Proposition 3.3 and the induced isomorphism of graded algebras \(B \xrightarrow{\cong} B'\), it is routine to check that, up to isomorphism, the initial part of the minimal graded projective resolution of \(B\) as a \(B\)-bimodule is given by

\[
Q^{-2} \xrightarrow{R} Q^{-1} \xrightarrow{\delta} Q^0 \xrightarrow{u} B \rightarrow 0,
\]

where \(R\) is the only homomorphism of \(B\)-bimodules such that, for all \((k,i) \in (\mathbb{Z}_\Delta)_0\),

\[
R(e_{\tau(k,i)} \otimes e_{v(k,i)}) = \sum_{t(a) = (k,i)} (-1)^{s(a)} [\sigma(a) \otimes e_{v(k,i)} + e_{\tau(k,i)} \otimes a]
\]

where \(s: \mathbb{Z}_\Delta_1 \rightarrow \mathbb{Z}_2\) is the associated signature map given in Proposition 3.3 which we assume to be the empty set when \((\Delta,G) = (\mathbb{D}_4, (\rho^m)^n))\), and the signature of a path is the sum of the signatures of its arrows.

Next we identify the elements generating \(\text{Ker}(R)\).

**Proposition 5.2.** Let \(\Delta\) be a Dynkin quiver and let \(B\) be the associated mesh algebra. Denote by \(\tau'\) the graded automorphism of \(B\) which acts as \(\tau\) on vertices and maps \(a \mapsto (-1)^{s(a)+s(\tau(a))}a\), for each \(a \in (\mathbb{Z}_\Delta)_1\). If for each \((k,i) \in (\mathbb{Z}_\Delta)_0\) we consider the homogeneous elements of \(Q^{-2}\) given by

\[
\xi_{(k,i)} = \sum_{x \in e_{v(k,i)}B} (-1)^{\text{deg}(x)} \xi'(x) \otimes x^*,
\]

then \(\bigoplus_{(k,i) \in \mathbb{Z}_\Delta_0} B \xi_{(k,i)} = \text{Ker}(R) = \bigoplus_{(k,i) \in \mathbb{Z}_\Delta_0} \xi_{(k,i)} B\).
Proof. Let us denote by \( h \) the induced isomorphism of graded algebras \( B \cong B' \) and by \( f \) its inverse. We put \( B' = h(B) \), where \( B \) is the \( G \)-invariant basis of \( B \) given by the previous lemma. The mentioned classical arguments also show that the elements \( \xi_{(k,i)} = \sum_{x \in \epsilon(k,i)} B'(-1)^{\deg(x)} \tau(x) \otimes x' \), with \( (k,i) \in (\mathbb{Z} \Delta)_{0} \), are in \( \text{Ker}(f) \) (see [11]).

From the equalities \( f(\tau(x)) = \tau'(f(x)) \) and \( f(x') = f(x)^* \), for all \( x \in B' \), and the fact that \( f(B') = B \) we immediately get that \( \xi(k,i) = f(\xi_{(k,i)}) = \sum_{y \in \epsilon(k,i), B} (-1)^{\deg(y)} \tau(y) \otimes y' \). Therefore the \( \xi(k,i) \) are elements of \( L := \text{Ker}(f) \).

If \( S_{(m,j)} = B_{e(m,j)}/J(B)e(m,j) \) is the simple graded left module concentrated in degree zero associated to the vertex \( (m,j) \), then the induced sequence

\[
Q^{-2} \otimes_{B} S_{(m,j)} \to Q^{-1} \otimes_{B} S_{(m,j)} \to Q^{0} \otimes_{B} S_{(m,j)} \to S_{(m,j)} \to 0
\]

is the initial part of the minimal projective resolution of \( S_{(m,j)} \). It is easy to see that the pushdown functor \( \pi_{a} : B - Gr \to \Delta - Gr \) preserves and reflects simple objects. When applied to the last resolution, we then get the minimal projective resolution of the simple \( \Lambda \)-module \( S_{(m,j)} \), where \( \Lambda \) is viewed as the orbit category \( B/G \) and where \( (m,j) \) denotes the \( G \)-orbit of \( (m,j) \). But we know that \( \Omega^{1}_{B}(S_{(m,j)}) \) is a simple \( \Lambda \)-module (see, e.g., [13]). It follows that \( \Omega^{1}_{B}(S_{(m,j)}) \) is a graded simple left \( B \)-module. Moreover we have an isomorphism \( Q^{-2} \otimes_{B} S_{(m,j)} \cong B_{e(m,j)}[-2] \in B - Gr \). By definition of the Nakayama permutation, we have that \( \text{Soc}_{\text{gr}}(B_{e(m,j)}) \cong S_{\nu^{-1}((m,j))}[-c_{\Delta} + 2] \). Then we have an isomorphism \( \Omega^{1}_{B}(S_{(m,j)}) \cong S_{\nu^{-1}((m,j))}[-c_{\Delta}] \), for all \( (m,j) \in \mathbb{Z} \Delta_{0} \). Considering the decomposition \( B/J(B) = \oplus_{(m,j) \in \mathbb{Z} \Delta_{0}} S_{(m,j)} \), we then get that \( L/LJ(B) \cong B/J(B) \) is isomorphic to \( B/J(B)[-c_{\Delta}] \) as a graded left \( B \)-module. Due to the fact that \( J(B) = J^{\text{gr}}(B) \) is nilpotent, we know that every left or right graded \( B \)-module has a projective cover. By taking projective covers in \( B - Gr \) and bearing in mind that \( L \) is projective on the left and on the right, we then get that \( L \cong B[-c_{\Delta}] \). With a symmetric argument, one also gets that \( BL \cong B[-c_{\Delta}] \). In particular, \( BL = \Omega^{1}_{B}(B) \) decomposes as a direct sum of indecomposable projective graded \( B \)-modules, all of them with multiplicity \( 1 \).

Note now that we have equalities \( e_{\nu^{-1}((k,i))} B_{e((k,i))} = \xi_{\nu^{-1}((k,i))} \otimes B_{e((k,i))} \), for all \( (k,i) \in \mathbb{Z} \Delta_{0} \). This gives surjective homomorphisms \( Be_{\nu^{-1}((k,i))}[-c_{\Delta}] \) and \( e_{(k,i)} B[-c_{\Delta}] \) respectively. It follows that \( \rho \) and \( \lambda \) are injective and, hence, they are isomorphisms. We then get that \( N := \oplus_{(k,i) \in \mathbb{Z} \Delta_{0}} B_{e((k,i))} \) is a graded submodule of \( BL \) isomorphic to \( BL \) and, hence, it is injective in \( B - Gr \) since this category is Frobenius. We then get that \( N \) is a direct summand of \( BL \) which is isomorphic to \( BL \). Since \( \text{End}_{B^{-Gr}}(B_{(k,i)}) \cong K \) for each vertex \( (k,i) \), Azumaya’s theorem applies (see [2] [Theorem 12.6]) and we can conclude that \( L = \oplus_{(k,i) \in \mathbb{Z} \Delta_{0}} B_{e((k,i))} \) for otherwise the decomposition of \( BL \) as a direct sum of indecomposable modules would contain summands with multiplicity \( > 1 \). By a symmetric argument, we get that \( L = \oplus_{(k,i) \in \mathbb{Z} \Delta_{0}} \xi_{(k,i)} B \).

\[ \square \]

**Proposition 5.3.** Let \( \Delta \) be a Dynkin quiver, let \( G \) be a weakly admissible group of automorphisms of \( B \) and fix a \( G \)-invariant graded Nakayama form and its associated Nakayama automorphism \( \eta \) (see Proposition 4.2). Assume that \( X \) is the \( G \)-invariant set of arrows given in Proposition 3.3 which we assume to be the empty set when \( (\Delta, G) = (\mathbb{D}_{4}, \langle \rho^{m} \rangle) \) and with respect to which we calculate the signature of arrows. Finally, let \( \kappa \) and \( \vartheta \) be the graded automorphisms of \( B \) which fix the vertices and act on arrows as:

1. \( \kappa(a) = -a \)
2. \( \vartheta(a) = (-1)^{s(\tau^{-1}(a)) + s(a)} a \),

for all \( a \in (\mathbb{Z} \Delta)_{1} \). If we put \( \xi = \sum_{(k,i) \in \mathbb{Z} \Delta_{0}} \xi_{(k,i)} \) and \( \mu' = \kappa \circ \eta \circ \tau^{-1} \circ \vartheta \), then we have an equality

\[ b \xi = \mu' b \]

for each element \( b \in \bigcup_{(k,i), (m,j) \in \mathbb{Z} \Delta_{0}} e_{(k,i)} B_{e(m,j)} \). Moreover, \( \mu' \circ g = g \circ \mu' \) for all \( g \in G \), and there exists an isomorphism of graded \( B \)-bimodules \( \Omega_{B \tau}(B) \cong \mu' B_{1}[-c_{\Delta}] \).
Proof. First note that, for any of the choices of the set $X$, the sum $s(\sigma^{-1}(a)) + s(\sigma(a)) + s(\tau^{-1}(a)) + s(a)$ in $\mathbb{Z}_2$ is constant when $a$ varies on the set of arrows ending at a given vertex $(k, i) \in (\mathbb{Z}\Delta)_0$. This implies that $\vartheta$ either preserves the relation $\sum_{a \in \Delta} (-1)^{s(\sigma(a))} s(\sigma(a)) a$ or multiplies it by $-1$. Then $\vartheta$ is a well-defined automorphism of $B$. Moreover, the $G$-invariant condition of the set of arrows $X$ implies that the sum $s(\tau^{-1}(a)) + s(a)$ in $\mathbb{Z}_2$ is $G$-invariant. This shows that $\vartheta \circ g = g \circ \vartheta$, for all $g \in G$. This implies that $\mu' \circ g = g \circ \mu'$ since we have $\kappa \circ g = g \circ \kappa$, for all $g \in G$.

All throughout the rest of the proof, a $G$-invariant basis $B$ of $B$ consisting of paths in $\mathbb{Z}\Delta$ is fixed, with respect to which the $\xi_{k(i)}$ are calculated. We shall prove that $a \xi_{\tau^{-1}(i(a))} = \xi_{\tau^{-1}(i(a))} \mu'(a)$, for all $a \in \mathbb{Z}\Delta_1$. Once this is proved, one easily shows by induction on $\deg(b)$ that if $b \in \bigcup_{(k, i), (m, j) \in \mathbb{Z}\Delta_0} e_{(k, i)} e_{(m, j)}$ is a homogeneous element with respect to the length grading, then the equality $b \xi_{\tau^{-1}(i(b))} = \xi_{\tau^{-1}(i(b))} \mu'(b)$ holds. It follows from this that the assignment $b \mapsto b \xi_{\tau^{-1}(i(b))}$ extends to an isomorphism of $B$-bimodules $1_B \xrightarrow{\mu, -1} \Omega^1_B(B)$, which actually induces an isomorphism of graded $B$-bimodules $\mu^* \Omega^1_B(B) \cong \Omega^1_B(B)$, when we view $\Omega^1_B(B)$ as a graded sub-bimodule of $Q^2 = (\otimes_{(k, i) \in \mathbb{Z}\Delta_0} \mathbb{B}_{e_{(k, i)}} \otimes e_{(k, i)} B)$. We have an equality:

$$a \xi_{\tau^{-1}(i(a))} = \sum x \in e_{\tau^{-1}(i(a))} B (-1)^{\deg(x)} a \tau'(-1) \otimes x^*.$$  

But we have $\tau'(-1) = (-1)^{s(\tau^{-1}(a))} + s(a) a$, so that

$$a \tau'(x) = (-1)^{s(\tau^{-1}(a))} + s(a) a \tau'(-1) a \tau'(x) = (-1)^{s(\tau^{-1}(a))} + s(a) \tau'(-1) a x.$$

Note that we have $\tau^{-1}(a) x = \sum y \in e_{\tau^{-1}(i(a))} B (-1)^{\deg(x)}(-1)^{s(\tau^{-1}(a))} + s(a) \tau'(-1) a x, y^* \otimes y^* \otimes x^*$

On the other hand, a direct calculation shows that $\mu'(a) = (-1)^{s(\tau^{-1}(a))} + s(a) (\eta \circ \tau^{-1})(a)$, for each $a \in (\mathbb{Z}\Delta)_1$. Then we have another equality

$$\xi_{\tau^{-1}(i(a))} \mu'(a) = \sum y \in e_{\tau^{-1}(i(a))} B (-1)^{\deg(y)}(-1)^{s(\tau^{-1}(a))} + s(a) \tau'(y) \otimes y^* \otimes (\eta \circ \tau^{-1})(a).$$

But we have an equality

$$y^* \otimes \tau^{-1}(a) = \sum x \in e_{\tau^{-1}(i(a))} B (x, y^* \otimes \tau^{-1}(a)) x^* = \sum x \in e_{\tau^{-1}(i(a))} B (xy^*, \eta(\tau^{-1}(a))) x^* = \sum x \in e_{\tau^{-1}(i(a))} B (\tau^{-1}(a), xy^*) x^* = \sum x \in e_{\tau^{-1}(i(a))} B (\tau^{-1}(a), y) x^* = \sum x \in e_{\tau^{-1}(i(a))} B (\tau^{-1}(a), y^*) x^*,$$

using that $(\cdot, \cdot)$ is a graded Nakayama form and that $\eta$ is its associated Nakayama automorphism. We then get

$$\xi_{\tau^{-1}(i(a))} \mu'(a) = \sum y \in e_{\tau^{-1}(i(a))} B \sum x \in e_{\tau^{-1}(i(a))} B (-1)^{\deg(y)}(-1)^{s(\tau^{-1}(a))} + s(a) \tau'(y) \otimes y^* \otimes x^*$$

Bearing in mind that $\deg(y) = \deg(\tau^{-1}(a) x) = \deg(x) + 1$ whenever $(\tau^{-1}(a) x, y^*) \neq 0$ we readily see that the second members of the equalities (!) and (!!) are equal. We then get $a \xi_{\tau^{-1}(i(a))} = \xi_{\tau^{-1}(i(a))} \mu'(a)$, as desired.

△

Remark 5.4. Note that, except when $(\Delta, G) = (\mathbb{A}_{2n-1}, \langle \rho^m \rangle)$, the automorphism $\vartheta$ of last proposition is the identity since $X = \tau(X)$.

Also, notice that whenever $(\Delta, G) \neq (\mathbb{A}_{2n-1}, \langle \rho^m \rangle)$, it is always possible to choose a map $\lambda : (\mathbb{Z}\Delta)_0 \rightarrow K^*$, taking values in $\{-1, 1\}$, such that $\lambda_{(a)} = -\lambda_{(a)},$ for all $a \in (\mathbb{Z}\Delta)_1$ and $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$. Indeed, if $\Delta \neq \mathbb{D}_{n+1}$, we define $\lambda(k, i) = (-1)^i$ for each $(k, i) \in (\mathbb{Z}\Delta)_0$, and if $\Delta = \mathbb{D}_{n+1}$, we put $\lambda(k, i) = (-1)^i$, when $i \neq 0$, and $\lambda(k, 0) = -1$. Then, the map $\psi : B \rightarrow B$, taking $b \mapsto \lambda_{(b)} b$, for any homogeneous element $b \in \bigcup_{(k, i), (m, j) \in (\mathbb{Z}\Delta)_0} e_{(k, i)} B e_{(m, j)}$, clearly defines an isomorphism of $B$-bimodules $\psi : \mathbb{B}_1 \rightarrow 1 \mathbb{B}_1$. This means that, in this case, the automorphism $\lambda \tau^{-1} \vartheta$ is another possibility for the twist in $\Omega^1_B(B)$. 

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Crucial for our goals is that what has been done in the last two propositions is 'G-invariant', which gives the following consequence.

**Corollary 5.5.** Let Δ be a Dynkin quiver, B the corresponding mesh algebra, G a weakly admissible group of automorphisms of \( \mathbb{Z}\Delta \) and let \( \Lambda = B/G \) be the associated m-fold mesh algebra. We denote by \( \mu \) the graded automorphism of \( B \) given by \( \nu \tau^{-1} \) when \( (\Delta, G) = (A_2 n, (\nu r^m)) \) and \( \tau^{-1} \vartheta \) when \( (\Delta, G) = (A_{2k-1}, (\nu r^m)) \) and \( \tau^{-1} \) otherwise. Then \( \mu \) induces a graded automorphism \( \bar{\mu} : \Lambda \to \Lambda \) of \( \Lambda \) and there is an isomorphism of graded \( \Lambda \)-bimodules \( \Omega^\Lambda_{\nu} (\Lambda) \cong \bar{\mu} \Lambda_1 [-c_\Delta] \), where \( c_\Delta \) is the Coxeter number.

**Proof.** We fix a \( G \)-invariant basis of \( B \) as in Lemma 5.4 and a \( G \)-invariant graded Nakayama form \((-,-) : B \times B \to K. \) If we interpret \( \Lambda = B/G \) as the orbit category and \([x] \) denotes the \( G \)-orbit of \( x \), for each \( x \in \bigcup_{(k,i),(m,j)} e_{(k,i)} B e_{(m,j)} \), note that the \( G \)-orbits of elements of \( B \) form a basis \( \bar{B} \) of \( \Lambda \) consisting of homogeneous elements in \( \bigcup_{(k,i),(m,j)} \mathbb{Z}\Delta_0/\mathbb{G} e_{(k,i)} \Lambda e_{(m,j)} \). Moreover, if \( \bar{B}^* \) is the right dual basis of \( \bar{B} \) with respect \((-,-) \), then \( \bar{B}^* = \{[x^*] : [x] \in \bar{B} \} \) is the right dual basis of \( \bar{B} \) with respect to the graded Nakayama form \((-,-) \to \Lambda \times \Lambda \to K \) induced from \((-,-) \) (see Proposition 2.3).

By taking into account the change of presentation of \( \Lambda \) and [13] [Section 4], we see that the initial part of the minimal projective resolution of \( \Lambda \) as a graded \( \Lambda \)-bimodule is of the form

\[
P^{-2} \xrightarrow{\bar{R}} P^{-1} \xrightarrow{p_0} \Lambda \to 0,
\]

where \( P^{-2} = \bigoplus_{(k,i) \in \mathbb{Z}\Delta_0/\mathbb{G} e_{(k,i)} \Lambda e_{(k,i)}} \) and we have equalities \( \bigoplus_{(k,i) \in \mathbb{Z}\Delta_0/\mathbb{G} \bar{e}_{(k,i)} \Lambda} \bar{e}_{(k,i)} = \text{Ker} (\bar{R}) = \bigoplus_{(k,i) \in \mathbb{Z}\Delta_0/\mathbb{G} \bar{e}_{(k,i)} \Lambda} \bar{e}_{(k,i)} \Lambda \), where \( \bar{e}_{(k,i)} = \sum_k e_{(k,i)} \delta \circ (\tau^r (x)) \otimes x^* \), for each \( (k,i) \in \mathbb{Z}\Delta_0/\mathbb{G} \).

On the other hand, by Proposition 5.3, we get that the automorphism \( \mu \) defined as above satisfies \( \varrho \circ \mu = \varrho \circ \mu \), for all \( \varrho \in G \), and hence, induces a graded automorphism \( \bar{\mu} : \Lambda \to \Lambda \) which maps \( \mu \) to \( \bar{\mu} (\mu) \). In case \( \mu = k \circ \eta \circ \tau^{-1} \circ \vartheta \), we get the equality \( \bar{\mu} \xi_{(k^r, (\nu k^r (b)))} = \xi_{(k^r, (\nu k^r (b)))} \mu (b) \), for each homogeneous element \( \bar{e} \in \bigcup_{(k,i),(m,j)} \mathbb{Z}\Delta_0/\mathbb{G} e_{(k,i)} \Lambda e_{(m,j)} \), from the corresponding equality in the proof of the previous proposition, just by replacing the homogeneous elements of \( \bar{B} \) by their orbits. We leave to the reader the routine verification. It then follows that the assignment \( \bar{e} \) gives an isomorphism of graded \( \Lambda \)-bimodules \( \Omega^\Lambda_{\nu} (\Lambda) \cong \Lambda_1 [\bar{\mu} \Lambda_1 [-c_\Delta] \)有这样的属性.

The different descriptions for the automorphism \( \mu \) of \( \Lambda \) given in the statement are valid using Remark 5.4.

\( \square \)

### 5.2 The period of an m-fold mesh algebra

This section is devoted to compute the \( \Omega \)-period of an \( m \)-fold mesh algebra \( \Lambda \). That is, the smallest of the positive integers \( r \) such that \( \Omega^\Lambda_{\nu} (\Lambda) \) is isomorphic to \( \Lambda \) as a \( \Lambda \)-bimodule. We need to separate the case of Loewy length 2, for which the period has already been computed (see, e.g., [14]), from the rest. We point out that the only connected self-injective algebras of Loewy length 2 are the \( m \)-fold mesh algebras \( \mathbb{A}_{2}^{(m)} \) and \( \mathbb{L}_{1}^{(m)} \) and that each of these is precisely the path algebra of a cyclic quiver modulo paths of length 2.

**Proposition 5.6.** Let \( \Lambda \) be a connected self-injective algebra of Loewy length 2. The following assertions hold:

1. If \( \text{char}(K) = 2 \) or \( \Lambda = \mathbb{A}_{2}^{(m)} \), i.e. \( |Q_0| \) is even, then the period of \( \Lambda \) is \( |Q_0| \).

2. If \( \text{char}(K) \neq 2 \) and \( \Lambda = \mathbb{L}_{1}^{(m)} \), i.e. \( |Q_0| \) is odd, then the period of \( \Lambda \) is \( 2|Q_0| \).

For the remaining cases we will need the following:

**Lemma 5.7.** Let \( \Lambda = B/G \) be an \( m \)-fold mesh algebra, with \( \Delta \neq \mathbb{A}_1, \mathbb{A}_2 \), and let \( r \geq 0 \) be an integer. The equality \( \text{dim}(\Omega^\Lambda_{\nu} (\Lambda)) = \text{dim}(\Lambda) \) holds if, and only if, \( r \in 2\mathbb{Z} \).

**Proof.** The 'if' part is well-known. For the 'only if' part, note that we have the following formulas for the dimensions of the syzygies:
1. $\dim(\Omega^i(\Lambda)) = \dim(\oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda) - \dim(\Lambda) = \sum_{i \in Q_0} \dim(\Lambda e_i)(\dim(e_i \Lambda) - 1)$, whenever $r \equiv 1 \pmod{3}$

2. $\dim(\Omega^i(\Lambda)) = \dim(\oplus_{i \in Q_0} \Lambda e_{i(i)} \otimes e_i \Lambda) - \dim(\mu_1 \Lambda) = \sum_{i \in Q_0} \dim(\Lambda e_{i(i)})(\dim(e_i \Lambda) - 1) = \sum_{i \in Q_0} \dim(\Lambda e_i)(\dim(e_i \Lambda) - 1)$, whenever $r \equiv 2 \pmod{3}$

For $r \equiv 1, 2 \pmod{3}$ the equality $\dim(\Omega^i(\Lambda)) = \dim(\Lambda)$ can occur if, and only if, $\dim(e_i \Lambda) = 2$, for each $i \in Q_0$. But this can only happen when the Loewy length is 2, which is discarded.

By the previous lemma, we know that $\dim(\Omega^i(\Lambda)) \neq \dim(\Lambda)$ whenever $r \not\equiv 3 \pmod{3}$. Due to the existence of an automorphism $\bar{\mu}$ of $\Lambda$ satisfying that $\Omega^1(\Lambda) \cong \Lambda$ as $\Lambda$-bimodules (see Proposition 5.9), in order to calculate the $\Omega$-period of $\Lambda$, we just need to control the positive integers $r$ such that $\bar{\mu}^r$ is inner. For the sake of simplicity, we shall divide the problem into two steps. We begin by identifying the smallest $u \in \mathbb{N}$ such that $(\bar{\nu} \circ \bar{\tau}^{-1})^u = \text{Id}_\Lambda$, that is, the smallest $u$ such that $\bar{\mu}^u$ acts as the identity on vertices. This is the content of the next result.

**Lemma 5.8.** Let $\Lambda = K(\mathbb{Z} \Delta)/\langle \varphi \rangle$ be an $m$-fold mesh algebra of extended type $(\Delta, m, t)$ and let us put $u := \min\{r \in \mathbb{Z}^+ \mid (\bar{\nu} \circ \bar{\tau}^{-1})^r = \text{Id}_\Lambda\}$. The following assertions hold:

1. If $t = 1$ then:
   
   (a) $u = \frac{2m}{\gcd(m, e_{\Delta})}$, whenever $\Delta$ is $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$;
   
   (b) $u = \frac{m}{\gcd(m, e_{\Delta})}$, whenever $\Delta$ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$.

2. If $t = 2$ then:
   
   (a) $u = \frac{2m}{\gcd(2m, m+c_{\Delta})}$, whenever $\Delta$ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$;
   
   (b) $u = \frac{2m}{\gcd(2m, m+c_{\Delta})}$, whenever $\Delta$ is $\mathbb{D}_{2r}$;
   
   (c) $u = \frac{2m}{\gcd(2m, m+1-\tau_{\Delta})}$, whenever $\Delta = \mathbb{A}_{2n}$.

3. If $t = 3$ (hence $\Lambda = K(\mathbb{Z} \Delta)/\langle \rho r^m \rangle$), then $u = m$.

**Proof.** The argument that we used for $\nu$ in the first paragraph of the proof of Theorem 4.6 is also valid for $\bar{\nu} \circ \bar{\tau}^{-1}$. Then $\bar{\nu} \circ \bar{\tau}^{-1} = \text{Id}_\Lambda$ if, and only if, $(\nu \circ \tau^{-1})^r \in G$.

When $\Delta$ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{n+1}$, with $n + 1$ odd, or $\mathbb{E}_6$, the Nakayama permutation is $\nu = \rho r^{1-n}$, where $n = \frac{c_{\Delta}}{2}$. Then $(\nu \circ \tau^{-1})^r = \rho r^{1-n}$. If $t = 1$ this automorphism is in $G$ if, and only if, $r = 2r'$ is even and $\tau^{-nr} = \tau^{-2nr}$ is equal to $(\rho r^m)^v = \rho^r m^v$, for some $v \in \mathbb{Z}$. This happens exactly when $2nr' \in m\mathbb{Z}$ and the smallest $r'$ satisfying this is $u' = \frac{m}{\gcd(2m, n)}$. We then get that $u = 2u' = \frac{2m}{\gcd(2m, m+c_{\Delta})} = \frac{2m}{\gcd(m, e_{\Delta})}$. Suppose that $t = 2$. Then $(\nu \circ \tau^{-1})^r = \rho r^{\tau-nr}$ is in $G = \langle \rho r^m \rangle$ if, and only if, there is $v \in \mathbb{Z}$ such that $v \equiv r \pmod{2}$ and $-nr = mv$. This is equivalent to saying that there is $k \in \mathbb{Z}$ such that $-nr = m(r+2k)$ or, equivalently, that $(m+n)r \in 2m\mathbb{Z}$. The smallest $r$ satisfying this property is $u = \frac{m}{\gcd(2m, m+n)} = \frac{2m}{\gcd(2m, m+c_{\Delta})}$. This proves 1.a, except for $\Delta = \mathbb{A}_{2n}$, and 2.a.

Suppose next that $\Delta$ is $\mathbb{D}_{n+1}$, with $n + 1$ even, $\mathbb{E}_7$ or $\mathbb{E}_8$. Then $\nu = \tau^{-1-n}$, where $n = \frac{c_{\Delta}}{2}$, so that $(\nu \circ \tau^{-1})^r = \tau^{-nr}$. When $t = 1$, this automorphism is in $G = \langle \tau^m \rangle$ if, and only if, $nr \in m\mathbb{Z}$. The smallest $r$ satisfying this property is $u = \frac{m}{\gcd(m, n)} = \frac{m}{\gcd(m, e_{\Delta})}$. On the other hand, if $t = 2$ then $\tau^{-nr}$ is in $G = \langle \rho r^m \rangle$ if, and only if, there is $v = 2u' \in 2\mathbb{Z}$ such that $-nr = mv = 2nr'$. The smallest $r$ satisfying this property is $u = \frac{2m}{\gcd(2m, m+n)} = \frac{2m}{\gcd(2m, m+c_{\Delta})}$. This proves 1.b and 2.b.

Let us now take $\Delta = \mathbb{A}_{2n}$. Then $\nu = \rho r^{-1-n}$, so that $(\nu \circ \tau^{-1})^r = \rho r^{\tau-nr}$. If $t = 1$, this automorphism is in $G = \langle \tau^m \rangle$ if, and only if, $r = 2r'$ is even and there exists $v \in \mathbb{Z}$ such that $\rho^{2r'} \tau^{-2nr} = \tau^{-2(n+1)r'}$ is equal to $\tau^{mv}$. This is equivalent to saying that $(2n+1)r' \in m\mathbb{Z}$. The smallest $r'$ satisfying this property is $u' = \frac{2m}{\gcd(m, 2n+1)}$. We then get $u = \frac{2m}{\gcd(m, 2n+1)} = \frac{2m}{\gcd(m, e_{\Delta})}$, which completes 1.a. When $t = 2$, the automorphism $\rho r^{-nr}$ is in $G = \langle \rho r^m \rangle$ if, and only if, there exists $v \in \mathbb{Z}$ such that $v \equiv r \pmod{2}$ and $\rho r^{\tau-nr} = \rho^r \tau^{mv}$. This is in turn equivalent to the existence of an integer $k$ such that $\rho^r \tau^{nr} = \rho^{r+2k} \tau^{m(r+2k)} = \rho^{r-k} \tau^{nr+2mk}$. That is, if and only
if \(-nr = (2m-1)k + mr\). This happens exactly when \((m+n)r \in (2m-1)\mathbb{Z}\). The smallest \(r\) satisfying this property is \(u = \frac{2m-1}{\gcd(m+n,2m-1)}\). But we have that \(\gcd(m+n,2m-1) = \gcd(2m-1,2n+1)\), so that 2.c holds.

Finally, if \(t = 3\), and hence \(\Delta = D_4\), then \(\nu = \tau^{-2}\), so that \((\nu \tau^{-1})^r = \tau^{-3r}\) is in \(G = (\rho \tau^m)\) if, and only if, there is \(v = 3v' \in 3\mathbb{Z}\) such that \(-3r = 3mv'\). This happens exactly when \(r \in 3\mathbb{Z}\), which implies that \(u = m\) in this case.

**Lemma 5.9.** Let \(\Lambda\) be an \(m\)-fold algebra of extended type \((\mathbb{A}_2, m, 2)\) and let \(T\) be the subgroup of \(Z\) consisting of the integers \(s\) such that \(\tilde{\mu}^s\) and \((\tilde{\nu} \circ \tilde{\tau}^{-1})^s\) are equal, up to composition by an inner automorphism of \(\Lambda\). Then \(T = 2\mathbb{Z}\), when \(\text{char}(K) \neq 2\), and \(T = \mathbb{Z}\), when \(\text{char}(K) = 2\).

**Proof.** We can take \(\eta = \nu\) in this case (see Theorem 4.2). By Corollary 5.5 we can assume that there is a \(G\)-invariant involutive automorphism \(h\) of \(B\) which fixes the vertices and acts on arrows as \(h(a) = (-1)^{u(a)} a\), for each \(a \in Z\Delta_1\), where \(u(a) \in \mathbb{Z}_2\), such that either \(\mu = h \circ \nu \circ \tau^{-1}\) or \(\mu = \nu \circ \tau^{-1} \circ h\). Indeed the first situation, with \(h = \kappa\), appears when \(\Delta = \mathbb{A}_2n\), and the second situation appears, with \(h = \vartheta\), when \(\Delta = \mathbb{A}_{2n-1}\). Then, using Proposition 4.4 we get that \(T\) consists of the \(s \in Z\) such that \(h^s \in \text{Inn}(\Lambda)\), where \(h\) is the automorphism of \(\Lambda = B/G\) induced by \(h\). The involutive condition of \(h\) implies that \(2 \in T\) and, when \(\text{char}(K) = 2\), also \(1 \in T\). The proof hence reduces to check that \(h \notin \text{Inn}(\Lambda)\), when \(\text{char}(K) \neq 2\). Proceeding as in the proof of Proposition 4.4 we have a map \(\lambda : Z\Delta_0 \rightarrow K^*\) such that \(\lambda_i^{-1} a = h(a)\) (equivalently, \(\lambda_i^{-1} \lambda_j(a) = (-1)^{u(a)}\)), for all \(a \in Z\Delta_1\). Due to Lemma 4.3 we just need to prove \(\lambda \circ \rho \tau^m_{\Delta\Delta_0} \neq \lambda\), where \(G = \langle \rho \tau^m \rangle\). We study the possible situations:

a) When \(\Delta = \mathbb{A}_{2n}\): Here we have \(h = \kappa\), so that \(\lambda_i(a) = -\lambda_j(a)\), for all \(a \in Z\Delta_1\). It follows that \(\lambda_k(i,j) = -\lambda_{k(i,j)+1}\), for all \((k, i) \in Z\Delta_0\) with \(i < 2n\). But then \(\lambda_{\rho \tau^m(k,n)} = \lambda_{\rho(k-m,n)} = \lambda_{(k-m,n+1)} = \lambda_{(k,n+1)} = -\lambda_{(k,n)}\).

b) When \(\Delta = \mathbb{A}_{2n-1}\) and \(m\) is odd: (see Proposition 8.3): Here \(h = \vartheta\), so that \(\lambda_i(a) = (-1)^{s(\tau^{-1}(a)+s(a))}\lambda_i(a)\), for all \(a \in Z\Delta_1\). But \((-1)^{s(\tau^{-1}(a)+s(a))} = \text{equal to } -1\), when \(a\) is an upward arrow in the 'south' hemisphere or a downward arrow in the 'north' hemisphere, and it is equal to 1 otherwise. We then get that \(\lambda_{k,n} = \lambda_{(k,n+1)}\) and \(\lambda_{k,n} = \lambda_{(k+n,n-j)}\), for all \(j = 0, 1, ..., n-1\), so that \(\lambda_{\rho(k,i)} = \lambda_{(k,i)}\), for all \((k, i) \in Z\Delta_0\). It also follows that \(\lambda_{k,i} = -\lambda_{(k,i)}\), for all \((k, i) \in Z\Delta_0\). We then get that \(\lambda_{\rho \tau^m(k,i)} = \lambda_{(k+i)} = (-1)^{m} \lambda(k,i) = -\lambda\).

c) When \(\Delta = \mathbb{A}_{2n-1}\) and \(m\) is even: We again have \(\lambda_i(a) = (-1)^{s(\tau^{-1}(a)+s(a))}\lambda_i(a)\), for all \(a \in Z\Delta_1\). We now consider the arrow \(a : (0,n) \rightarrow (0, n + 1)\). Using Proposition 3.3 we see that we have the following formulas for the integers \(k \in \{0, 1, ..., m-1\}:

1. \(s(\sigma^{-2k}(a)) + s(\tau^{-1}(\sigma^{-2k}(a))) = 1\) exactly when \(k \neq m-1\);
2. \(s(\sigma^{-2k-1}(a)) + s(\tau^{-1}(\sigma^{-2k-1}(a))) = 1\) exactly when \(k = m - 2, m - 1\).

Considering the path

\[(0,n) \xrightarrow{\sigma^{-1}} (1,n) \rightarrow ... \rightarrow (m-1,n) \xrightarrow{\sigma^{-2(m-1)}} (m-1,n + 1) \xrightarrow{\sigma^{-2(m-1)}-1} (m,n) = \rho \tau^m(0,n)\]

and arguing as in the proof of Proposition 4.3 we conclude that \(\lambda_{(0,n)} = (-1)^{m+1} \lambda_{\rho \tau^m(0,n)} = -\lambda_{(0,n)}\). The proof is finished since \(\rho \tau^m = (\rho \tau^m)^{-1} \in G\).

We are now ready to describe explicitly the period of any \(m\)-fold mesh algebra.

**Theorem 5.10.** Let \(\Lambda\) be an \(m\)-fold mesh algebra of extended type \((\Delta, m, t)\), where \(\Delta \neq \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3\), let \(\pi = \pi(\Lambda)\) denote the period of \(\Lambda\) and, for each positive integer \(k\), denote by \(O_2(k)\) the biggest of the natural numbers \(r\) such that \(2^r\) divides \(k\). If \(\text{char}(K) = 2\) then \(\pi = 3u\), where \(u\) is the positive integer of Lemma 5.8. When \(\text{char}(K) \neq 2\), the period of \(\Lambda\) is given as follows:
1. If \( t = 1 \) then:

(a) When \( \Delta = \mathbb{A}_r, D_{2r-1} \) or \( \mathbb{E}_6 \), the period is \( \pi = \frac{6m}{\gcd(m,c_{\Delta})} \).

(b) When \( \Delta = D_{2r}, E_7 \) or \( E_8 \), the period is \( \pi = \frac{3m}{\gcd(m,c_{\Delta})} \), when \( m \) is even, and \( \pi = \frac{6m}{\gcd(m,c_{\Delta})} \) when \( m \) is odd.

2. If \( t = 2 \) then:

(a) When \( \Delta = \mathbb{A}_{2n-1}, D_{2r-1} \) or \( \mathbb{E}_6 \), the period is \( \pi = \frac{6m}{\gcd(2m,m+\frac{1}{2})} \), when \( O_2(m) \neq O_2(\frac{c_{\Delta}}{2}) \), and \( \pi = \frac{12m}{\gcd(2m,m+\frac{1}{2})} \) otherwise.

(b) When \( \Delta = D_{2r} \), the period is \( \pi = \frac{6m}{\gcd(2m,m+\frac{1}{2})} = \frac{6m}{\gcd(2m,2r-1)} \).

(c) When \( \Delta = \mathbb{A}_{2n} \), i.e. \( \Lambda = \mathbb{I}^{(m)} \), the period is \( \pi = \frac{6(2m-1)}{\gcd(2m-1,2n+1)} \).

3. If \( t = 3 \) then \( \pi = 3m \), when \( m \) is even, and \( 6m \), when \( m \) is odd.

Proof. Let \( u > 0 \) be the integer of Lemma 5.8. Then \( u\mathbb{Z} \) consists of the integers \( r \) such that \( \bar{\nu}^r = \bar{\eta}^r \), or equivalently \( (\bar{\nu}\circ \bar{\eta}^{-1})^r = id_{\Lambda} \), as automorphisms of \( \Lambda \). If \( \pi \) is the period of \( \Lambda \), then, by Lemma 5.7, we know that \( \pi = 3v \), where \( v \) is the smallest of the positive integers \( s \) such that \( \bar{\mu}^s \in \text{Inn}(\Lambda) \). These integers \( s \) obviously form a subgroup \( S = S(c_{\Delta},m) \) of \( \mathbb{Z} \), and then \( v\mathbb{Z} = S \). This subgroup is the intersection of \( u\mathbb{Z} \) with the subgroup \( T \) consisting of the integers \( r \) such that \( \bar{\mu}^r \) and \( (\bar{\nu}\circ \bar{\eta}^{-1})^r \) are equal, up to composition by an inner automorphism of \( \Lambda \). When \( (\Delta,m,t) = (\mathbb{A}_r,m,2) \), by Lemma 5.9, we get that \( v\mathbb{Z} = u\mathbb{Z} \cap 2\mathbb{Z} \), when \( \text{char}(K) \neq 2 \), and \( v\mathbb{Z} = u\mathbb{Z} \cap \mathbb{Z} = u\mathbb{Z} \), when \( \text{char}(K) = 2 \). This automatically gives 2.c and the part of characteristic 2 in this case. We claim that it also gives the formula in 2.a for \( \Delta = \mathbb{A}_{2n-1} \). Indeed, by Lemma 5.8, we have \( u = \frac{2m}{\gcd(2m,m+n)} \) in this case. But the biggest power of 2 which divides \( 2m \) divides \( 2n \) is a divisor of \( \gcd(2m,m+n) \) if, and only if, \( O_2(m) = O_2(n) \). Then the equality 2.a for \( \mathbb{A}_{2n-1} \) follows automatically.

When \( (\Delta,m,t) \neq (\mathbb{A}_r,m,2) \), by Proposition 5.3 and the subsequent remark, we can take \( \bar{\mu} = \bar{\eta} \circ \bar{\eta}^{-1} \). Then Proposition 1.4 implies that \( S \) consists of the integers \( s \) such that \( \bar{\eta}^s \) and \( \bar{\eta}^s \) are equal, up to composition by an inner automorphism of \( \Lambda \). We then get that \( S = u\mathbb{Z} \cap H(\Delta,m,t) \) (see Proposition 1.8). Therefore, Proposition 1.6 tells us that \( v = u \), when either \( H(\Delta,m,t) = \mathbb{Z} \) or \( u \) is even, and \( v = 2u \) otherwise. This completes the assertion for characteristic 2. We next check that, together with Proposition 1.6, it also gives all the remaining formulas of the theorem.

For the quivers \( \Delta \) in 1.a we always have that \( H(\Delta,m,t) = \mathbb{Z} \) when \( \Delta = \mathbb{A}_r \), and also in the other two cases when \( m \) is even. But if \( m \) is odd, then automatically \( u = \frac{2m}{\gcd(2m,m+n)} \) is even.

For the quivers in 1.b, we always have that \( n = \frac{c_{\Delta}}{2} \) is odd. Therefore \( u \) is even exactly when \( m \) is even. The reader should note that the case when \( t = 1 \) is also covered in Section 5 of [12].

For the quivers in 2.a which are not \( \mathbb{A}_{2n-1} \), we have that \( H(\Delta,m,t) = \mathbb{Z} \) exactly when \( m \) is odd. But \( \frac{c_{\Delta}}{2} \) is even, so that \( O_2(m) \neq O_2(\frac{c_{\Delta}}{2}) \) in that case. As we did above in the case \( (\Delta,m,t) = (\mathbb{A}_{2n-1},m,2) \), in case \( m \) even, we have that \( u = \frac{2m}{\gcd(2m,m+n)} \) is odd if, and only if, \( O_2(m) = O_2(\frac{c_{\Delta}}{2}) \). Then the formula in 2.a is true also for the cases different from \( \mathbb{A}_{2n-1} \).

For 2.b, we have that \( \frac{c_{\Delta}}{2} \) is odd, which implies that \( u \) is always even, and then the formula in 2.b is true.

Finally, when \( t = 3 \), we have that \( H(\Delta,m,t) = \mathbb{Z} \), exactly when \( m \) is even, and then the formula in 3) is automatic.

\[ \square \]

### 5.3 Inner and stably inner automorphisms

From now on, in this paper, we will denote by \( \Lambda \rightarrow \text{mod} \) the category of finite dimensional \( \Lambda \)-modules and, given an automorphism \( \phi \) of an algebra \( \Lambda \), the functor \( \phi : \Lambda \rightarrow \Lambda \rightarrow \text{mod} \rightleftharpoons \text{mod} \) will be denoted by \( \phi(-) \). Recall from [25] that an automorphism \( \sigma \) of \( \Lambda \) is stably inner if the functor \( \sigma(-) : \Lambda \rightarrow \text{mod} \rightarrow \text{mod} \) is naturally isomorphic to the identity functor. In particular, each inner automorphism is stably inner.
Lemma 5.11. Let $\Lambda = KQ/I$ be a finite dimensional selfinjective algebra, where $I$ is an admissible ideal of $KQ$ which is homogeneous with respect to the grading by path length, and consider the induced grading on $\Lambda$. Suppose that the Loewy length of $\Lambda$ is greater or equal than 4. A graded automorphism of $\Lambda$ is inner if, and only if, it is stably inner.

Proof. Let $\varphi$ be a stably inner graded automorphism of $\Lambda$. Let $l$ be the Loewy length of $\Lambda$. If $J = J(\Lambda) = J^r(\Lambda)$ is the Jacobson radical and $\text{Soc}^n(\Lambda) = \text{Soc}_{gr}^n(\Lambda)$ is the $n$-socle of $\Lambda$ (i.e. $\text{Soc}^0(\Lambda) = 0$ and $\text{Soc}^{n+1}(\Lambda)/\text{Soc}^n(\Lambda)$ is the socle of $\Lambda/\text{Soc}^n(\Lambda)$, for all $n \geq 0$), then we have $J^n = \text{Soc}^{l-n}(\Lambda) = \oplus_{k \geq n} A_k$, for all $n \geq 0$.

We then have $\text{Soc}^5(\Lambda) \subseteq J^2$ since $l \geq 4$. By Corollary 2.11 of [23], we have a map $\lambda : Q_0 \rightarrow K^*$ such that $\varphi(a) - \lambda^{-1}_{1(a)} \lambda_{1(a)} a \in J(\Lambda)^2$, for all $a \in Q_1$. If we define $\chi : \Lambda \rightarrow \Lambda$ as in the proof of Lemma 4.3, we get that $\chi$ is an inner automorphism of $\Lambda$ such that $(\varphi \circ \chi^{-1})(a) - a \in J(\Lambda)^2$, for all $a \in Q_1$. But $\varphi \circ \chi^{-1}$ is a graded automorphism since so are $\varphi$ and $\chi$. It then follows that $(\varphi \circ \chi)(a) = a$, for all $a \in Q_1$, which implies that $\varphi \circ \chi = \text{id}_\Lambda$, and so $\varphi = \chi$ is inner.

Recall that $\Lambda$ is a Nakayama algebra if each left or right indecomposable projective $\Lambda$-module is uniserial. We will need the following properties of self-injective algebras of Loewy length 2.

Proposition 5.12. Let $\Lambda = KQ/KQ_{>2}$ be selfinjective algebra such that $J(\Lambda)^2 = 0$ and suppose that $\Lambda$ does not have any semisimple summand as an algebra. The following assertions hold:

1. $\Lambda$ is a Nakayama algebra and $Q$ is a disjoint union of oriented cycles, with relations all the paths of length 2.
2. $\Lambda$ is a finite direct product of $m$-fold mesh algebras of Dynkin graph $\Delta = A_2$.
3. A graded automorphism $\varphi$ of $\Lambda$ is stably inner if, and only if, it fixes the vertices.
4. $\varphi$ is inner if, and only if, it fixes the vertices and there exists an acyclic character $\chi : Q_1 \rightarrow K^*$ such that $\varphi(a) = \chi(a)a$, for each arrow $a \in Q_1$.
5. If the quiver $Q$ is connected with $n$ vertices (whence an oriented cycle with $Q_0 = \mathbb{Z}_n$), then $\text{End}_{\Lambda^*}(\Lambda)$ is isomorphic to the $\Lambda$-bimodule $\bar{\mu} \Lambda_1$, where $\bar{\mu}$ is the automorphism acting on vertices as the $n$-cycle $(12...n)$ and on arrows as $\bar{\mu}(a_i) = -a_{i+1}$, where $a_i : i \rightarrow i + 1$ for each $i \in \mathbb{Z}_n$.

Proof. Assertion 1 is folklore. But $A_2^{(m)} = \mathbb{Z}A_2/\langle \sigma^m \rangle$ is the connected Nakayama algebra of Loewy length 2 with $2m$ vertices while $L_1^{(m)} = \mathbb{Z}A_2/\langle \rho \sigma^m \rangle$ is the one with $2m - 1$ vertices. Then assertion 2 is clear.

The only indecomposable objects in the stable category $\Lambda - \text{mod}$ are the simple modules, all of which have endomorphism algebra isomorphic to $K$. It follows that each additive self-equivalence $F : \Lambda - \text{mod} \xrightarrow{\cong} \Lambda - \text{mod}$ such that $F(S) \cong S$, for each simple module $S$, is naturally isomorphic to the identity. Since each automorphism $\varphi$ of $\Lambda$ induces the self-equivalence $F = \varphi(-)$, assertion 3 is clear.

Assertion 4 follows directly from [23][Theorem 12], taking into account that the only inner graded automorphism induced by an element $1 - x$, with $x \in J$, is the identity (see the proof of Lemma 4.3).

Suppose now that $Q$ is connected and has $n$ vertices, so that $\Lambda$ is either an $m$-fold mesh algebra of type $A_2^{(m)}$, and then $n = 2m$, or $L_1^{(m)}$, and then $n = 2m - 1$. By the explicit definition of the minimal projective resolution of $\Lambda$ as a bimodule (see [13]), we get that $\Omega_{\Lambda^*}(\Lambda)$ is generated as a $\Lambda$-bimodule by the elements $x_i = a_i \otimes e_{i+1} - e_i \otimes a_i$ ($i \in \mathbb{Z}_n$). But we have $\oplus_{i \in \mathbb{Z}_n} \Lambda x_i = \Omega_{\Lambda^*}(\Lambda) = \oplus_{i \in \mathbb{Z}_n} x_i \Lambda$. Moreover, if $\bar{\mu}$ is the automorphism mentioned in assertion 5 and $x = \sum_{i \in \mathbb{Z}_n} x_i$, then we have $yx = \bar{\mu}(y)$, whenever $y$ is either a vertex or an arrow. It then follows that the assignment $y \sim yx$ gives an isomorphism of $\Lambda$-bimodules $1_{\Lambda^{(m-1)}} \xrightarrow{\cong} \Omega_{\Lambda^*}(\Lambda)$.

□
5.4 The stable Calabi-Yau dimension of an \( m \)-fold mesh algebra

In case \( \Lambda \) is a self-injective algebra, Auslander formula (see [6], Chapter IV, Section 4) says that one has a natural isomorphism \( \text{DHom}_\Lambda(X, -) \cong \text{Ext}_\Lambda^1(-, \tau X) \), where \( \tau : \Lambda - \text{mod} \to \Lambda - \text{mod} \) is the Auslander-Reiten (AR) translation. Moreover, \( \tau = \Omega^2 \mathcal{N} \), where \( \mathcal{N} = \text{DHom}_\Lambda(-, \Lambda) \cong D(\Lambda) \otimes_\Lambda - : \Lambda - \text{mod} \to \Lambda - \text{mod} \) is the Nakayama functor (see [6]). As a consequence, as shown in [15], the stable category \( \Lambda - \text{mod} \) is naturally isomorphic to the functor \( - \to \Omega \) shown in [15], the stable category in general, it is not known if the equality holds. We discuss now this problem for \( D \)-modules.

Proposition 5.14. Let \( \Lambda \) be a connected self-injective algebra of Loewy length 2. Then \( \Lambda \) is always a stably Calabi-Yau algebra and the following equalities hold:

1. If \( \text{char}(K) = 2 \) or \( \Lambda = \mathbb{A}_2^{(m)} \), i.e. \( |Q_0| \) is even, then \( CY - \text{dim}(\Lambda) = CYF - \text{dim}(\Lambda) = 0 \).
2. If \( \text{char}(K) \neq 2 \) and \( \Lambda = \mathbb{L}_1^{(m)} \), i.e., \( |Q_0| \) odd, then \( CY - \text{dim}(\Lambda) = 0 \) and \( CYF - \text{dim}(\Lambda) = 2m - 1 = |Q_0| \).

Proof. By Proposition 5.12, we know that \( \Omega_{\Lambda_1}^{k+1} : \Lambda - \text{mod} \to \Lambda - \text{mod} \) is naturally isomorphic to \( \eta^{-1} : \Lambda - \text{mod} \to \Lambda - \text{mod} \), and only if \( \Omega_{\Lambda_1}^{k+1}(\Lambda) \) and \( \varphi_\eta \Lambda_1 \) are isomorphic \( \Lambda \)-bimodules, for some stably inner automorphism \( \varphi \) of \( \Lambda \).

We are now able to calculate the stable and Frobenius Calabi-Yau dimension of self-injective algebras of Loewy length 2.

Proposition 5.13. Let \( \Lambda \) be a connected self-injective algebra of Loewy length 2. Then \( \Lambda \) is always a stably Calabi-Yau algebra and the following equalities hold:

1. If \( \text{char}(K) = 2 \) or \( \Lambda = \mathbb{A}_2^{(m)} \), i.e. \( |Q_0| \) is even, then \( CY - \text{dim}(\Lambda) = CYF - \text{dim}(\Lambda) = 0 \).
2. If \( \text{char}(K) \neq 2 \) and \( \Lambda = \mathbb{L}_1^{(m)} \), i.e., \( |Q_0| \) odd, then \( CY - \text{dim}(\Lambda) = 0 \) and \( CYF - \text{dim}(\Lambda) = 2m - 1 = |Q_0| \).

Proposition 5.14. Let \( \Lambda \) be an \( m \)-fold mesh algebra of Dynkin type \( \Delta \) different from \( \mathbb{A}_r \), for \( r = 1, 2, 3 \). Then \( \Lambda \) is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. In such case the equality \( CY - \text{dim}(\Lambda) = CYF - \text{dim}(\Lambda) \) holds.
Proof. By Corollary 3.2 we know that the Loewy length of $\Lambda$ is $c_\Delta - 1$, where $c_\Delta$ is the Coxeter number. The Dynkin graphs $\Delta = A_r$ properties, with $r = 1, 2, 3$, are the only ones for which $c_\Delta - 1 \leq 3$. So $\Lambda$ has Loewy length $4$ in our case. Note that if $\Omega_{\Lambda}^{k+1}(\Lambda)$ is isomorphic to a twisted bimodule $\varphi \Lambda_1$, then we have $\dim(\Omega_{\Lambda}^{k+1}(\Lambda)) = \dim(\Lambda)$. By Lemma 5.7 we know that then $k + 1 \in 3\mathbb{Z}$.

If there is a $k$ such that $\Omega_{\Lambda}^{k+1}(\Lambda) \cong \varphi \Lambda_1$, for some inner or stably inner automorphism $\varphi$, then $k = 3s - 1$, for some integer $s > 0$. But we know that $\Omega_{\Lambda}^k(\Lambda) \cong \bar{\mu} \Lambda_1$, where $\bar{\mu}$ is a graded automorphism of $\Lambda$. We then have that $\Omega_{\Lambda}^k(\Lambda) \cong \varphi \Lambda_1$, for some stably inner (resp. inner) automorphism $\varphi$ if, and only if, $\bar{\mu}^s \bar{\eta}^{-1}$ is a stably inner (resp. inner) automorphism of $\Lambda$. The proof is finished using Lemma 5.11 since $\bar{\mu}^s \bar{\eta}^{-1}$ is a graded automorphism. \hfill $\Box$

The proof of last proposition shows that if $\Lambda$ is not of type $A_r$ $(r = 1, 2)$, then the algebra $\Lambda$ will be stably Calabi-Yau (resp. Calabi-Yau Frobenius) if, and only if, there exists an integer $s > 0$ such that $\bar{\mu}^s \bar{\eta}^{-1}$ is stably inner (resp. inner). A necessary condition for this is that $\bar{\mu}^s \bar{\eta}^{-1}$ fixes the vertices. So, as a first step to characterize the stably Calabi-Yau (resp. Calabi-Yau Frobenius) condition of $\Lambda$, we shall identify the positive integers $s$ such that $\bar{\mu}^s$ and $\bar{\eta}$ have the same action on vertices.

**Definition 10.** Let $\Lambda$ be an $m$-fold mesh algebra of type $\Delta \neq A_1, A_2$, with quiver $Q$. We will define the following sets of positive integers:

1. $\mathbb{N}_{CY}(\Lambda)$ consists of the integers $s > 0$ such that $\bar{\mu}^s$ and $\bar{\eta}$ have the same action on vertices.

2. $\bar{\mathbb{N}}_{CY}(\Lambda)$ consists of the integers $s > 0$ such that $\bar{\mu}^s \bar{\eta}^{-1}$ is an inner automorphism. Equivalently, it is the set of integers $s > 0$ such that $\Omega_{\Lambda}^s(\Lambda)$ is isomorphic to $\bar{\eta} \Lambda_1$ as a $\Lambda$-bimodule.

**Remark 5.15.** Under the hypotheses of last definition, we clearly have $\bar{\mathbb{N}}_{CY}(\Lambda) \subseteq \mathbb{N}_{CY}(\Lambda)$. Moreover $\Lambda$ is Calabi-Yau Frobenius if, and only if, $\bar{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$. In this latter case we have $CYF - \dim(\Lambda) = 3r - 1$, where $r = \min(\bar{\mathbb{N}}_{CY}(\Lambda))$, and this number is equal to $CY - \dim(\Delta)$ when $\Delta \neq A_3$. Note also that if $\bar{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \neq \emptyset$, then $CY - \dim(\Lambda) = CYF - \dim(\Lambda)$ since the fact that $\bar{\mu}^s \bar{\eta}^{-1}$ be stably inner implies that $s \in \mathbb{N}_{CY}(\Lambda)$.

We first identify $\mathbb{N}_{CY}(\Lambda)$ for any $m$-fold mesh algebra of Loewy length $> 2$.

**Proposition 5.16.** Let $\Lambda$ be an $m$-fold mesh algebra of extended type $(\Delta, m, t)$, where $\Delta \neq A_1, A_2$. The following assertions hold:

1. When $t = 1$, the set $\mathbb{N}_{CY}(\Lambda)$ is nonempty if, and only if, the following condition is true in each case:
   
   (a) $\gcd(m, c_\Delta) = 1$, when $\Delta$ is $A_r$, $\mathbb{D}_{2r-1}$ or $E_6$. Then $\mathbb{N}_{CY}(\Lambda) = \{s \geq 0: 2s' + 1 \equiv 0 \pmod{m}, c_\Delta s' \equiv -1 \pmod{m}\}$
   
   (b) $\gcd(m, \frac{c_\Delta}{2}) = 1$, when $\Delta$ is $\mathbb{D}_{2r}$, $E_7$ or $E_8$. Then $\mathbb{N}_{CY}(\Lambda) = \{s' > 0: \frac{c_\Delta}{2}(s' - 1) \equiv -1 \pmod{m}\}$.

2. When $t = 2$, the set $\mathbb{N}_{CY}(\Lambda)$ is nonempty if, and only if, the following condition is true in each case:
   
   (a) $\gcd(2m, m + \frac{c_\Delta}{2}) = 1$, when $\Delta$ is $A_{2n-1}$, $\mathbb{D}_{2r-1}$ or $E_6$. Then $\mathbb{N}_{CY}(\Lambda) = \{s > 0: (m + \frac{c_\Delta}{2})(s - 1) \equiv -1 \pmod{2m}\}$, and this set consists of even numbers.
   
   (b) $\gcd(m, \frac{c_\Delta}{2}) = \gcd(m, 2r - 1) = 1$, when $\Delta = \mathbb{D}_{2r}$. Then $\mathbb{N}_{CY}(\Lambda) = \{s > 0: (2r - 1)(s - 1) \equiv -1 \pmod{2m}\}$ and this set consists of even numbers.
   
   (c) $\gcd(2m - 1, 2n + 1) = 1$, when $\Delta = A_{2n}$. Then $\mathbb{N}_{CY}(\Lambda) = \{s > 0: (m + n)(s - 1) \equiv -1 \pmod{2m-1}\}$.

3. If $t = 3$ (and hence $\Delta = \mathbb{D}_4$), then $\mathbb{N}_{CY}(\Lambda) = \emptyset.$
Proof. Note that $\mu$ acts on vertices as $\nu \tau^{-1}$, where $\nu$ is the Nakayama permutation and $\tau$ the Auslander-Reiten translation of $B$. Viewing the vertices of the quiver of $A$ as $G$-orbits of vertices in $Z\Delta$, we get that $s$ is in $N_{\text{CY}}(A)$ if, and only if, $(\nu \tau^{-1})^*([(k, i)]) = \nu([(k, i)])$, equivalently $\nu^{*1} \tau^{-s}([(k, i)]) = [(k, i)]$, for each $G$-orbit $[(k, i)]$. Now the argument in the first paragraph of the proof of Theorem 4.9 can be applied to the automorphism $\nu^{s-1} \tau^{-s}$. We then get that $s$ is in $N_{\text{CY}}(A)$ if, and only if, $\nu^{s-1} \tau^{-s} \in G$. We use this to identify the set $N_{\text{CY}}(A)$ for all possible extended types, and the result will be derived from that.

If $t = 3$ and so $\Delta = D_4$, then we know that $\nu = \tau^{-2}$. It follows that $s$ is in $N_{\text{CY}}(A)$ if, and only if, $\tau^{-2(s-1)} \tau^{-s} = (\rho \tau m)^q$, for some $q \in \mathbb{Z}$, where $\rho$ is the automorphism of order 3 of $D_4$. By the free action of the group $\langle \rho, \tau \rangle$ on vertices not fixed by $\rho$, necessarily $q \in 3\mathbb{Z}$ and $2 - 3s = mq$, which is absurd. Then assertion 3 follows.

Suppose first that $\Delta \neq A_2n$. If $\Delta$ is $A_{2n-1}$, $D_{2n-1}$ or $E_6$, then $\nu = \tau^{1-n}$, where $n = \frac{\rho \tau m}{2}$. Then $\nu^{s-1} \tau^{-s} = \rho^{s-1} \tau^{-1} \tau^{(1-n)(s-1)} \tau^{-s} = \rho^{s-1} \tau^{-[n(s-1)+1]}$. When $t = 1$, we have that $G = \langle \tau^m \rangle$ and, hence, the automorphism $\nu^{s-1} \tau^{-s}$ is in $G$ if, and only if, there is $q \in \mathbb{Z}$ such that $\rho^{s-1} \tau^{-[n(s-1)+1]} = (\tau^m)^q$. This happens if, and only if, $s - 1 = 2s'$ is even and there is $q \in \mathbb{Z}$ such that $-2ns' - 1 = -n(s-1) - 1 = s' \equiv 0 \pmod{2}$. Therefore $s$ exists if, and only if, $gcd(m, c_{\Delta}) = gcd(m, 2n) = 1$. In this case $N_{\text{CY}}(A) = \{s > 0 : (m + \frac{\rho \tau m}{2}) (s - 1) \equiv -1 (\mod 2m)\}$ which proves 2.a.

Suppose next that $\Delta = D_{2r}, E_7$ or $E_8$, so that $\nu = \tau^{1-n}$, where $n = \frac{\rho \tau m}{2}$. Then $\nu^{s-1} \tau^{-s} = \tau^{(1-n)(s-1)} \tau^{-s} = \tau^{-[n(s-1)+1]}$. When $t = 1$, this automorphism is in $G = \langle \tau^m \rangle$ if, and only if, there is $q \in \mathbb{Z}$ such that $-n(s-1) - 1 = mq$. Then $s$ exists if, and only if, $gcd(m, 2n) = 1$. In this case $N_{\text{CY}}(A) = \{s > 0 : \frac{\rho \tau m}{2} (s - 1) \equiv -1 (\mod 2m)\}$, which proves 1.b. When $t = 2$, whence $\Delta = D_{2r}$, the automorphism $\nu^{s-1} \tau^{-s}$ is in $G = \langle \rho \tau^m \rangle$ if, and only if, there is an even integer $q = 2q'$ such that $-n(s-1) - 1 = 2mq'$. Then $s$ exists if, and only if, $gcd(2m, n) = 1$. But $n = 2r - 1$ is odd in this case. Then $gcd(2m, n) = 1$ if, and only if, $gcd(m, 2r - 1) = gcd(m, n) = 1$. On the other hand, note that $s - 1$ is necessarily odd, which implies that $N_{\text{CY}}(A) \subset 2\mathbb{Z}$. This completes the proof of 2.b.

Suppose now that $\Delta = A_{2n}$, so that $\rho^2 = \tau^{-1}$. Here $\nu = \tau^{1-n}$ and $\nu^{s-1} \tau^{-s} = \rho^{s-1} \tau^{-[n(s-1)+1]}$. When $t = 1$, this automorphism is in $G = \langle \tau^m \rangle$ if, and only if, $s - 1 = 2s'$ is even and $\tau^{-s} \tau^{-[2s'+1]} = (\tau^m)^q$, for some integer $q$. That is, $s$ exists if, and only if, there are $s' \geq 0$ and $q \in \mathbb{Z}$ such that $mq + (2s' + 1)q = 0$. Therefore $s$ exists if, and only if, $gcd(m, 2n) = 1$. In this case $N_{\text{CY}}(A) = \{s > 0 : \frac{\rho \tau m}{2} (s - 1) \equiv -1 (\mod 2m)\}$, which proves 1.a. When $t = 2$, the automorphism $\nu^{s-1} \tau^{-s}$ is in $G = \langle \rho \tau^m \rangle$ if, and only if, there is $q \in \mathbb{Z}$ such that $q \equiv s - 1 (\mod 2)$ and $\rho^{s-1} \tau^{-[n(s-1)+1]} = \rho^q \tau^{mq}$. This is equivalent to the existence of an integer $k$ such that $\rho^{s-1} \tau^{-[n(s-1)+1]} = \rho^{s-1+2k} \tau^{mq}$. Canceling $\rho^{s-1}$, we see that the condition is equivalent to the existence of an integer $k$ such that $-n(s-1) - 1 = m(s-1) + (2m - 1)k$ or, equivalently, such that $(m+n)(s-1) + (2m - 1)k = 0$. Then $s$ exists if, and only if, $gcd(m+n, 2m-1) = 1$, which is turn equivalent to saying that $gcd(2m-1, 2n+1) = 1$ since $(2m-1) + (2n+1) = 2(m+n)$. This proves 2.c and the proof is complete.

We now want to identify $\hat{N}_{\text{CY}}(A)$. The following is our crucial tool.

**Lemma 5.17.** Let $\Delta$ be a Dynkin quiver different from $A_1, A_2, B$ be its associated mesh algebra, $A = B/G$ be an $m$-fold mesh algebra of extended type $(\Delta, m, t)$ and let $\eta$ be a $G$-invariant graded Nakayama automorphism of $B$. If $s$ is an integer in $N_{\text{CY}}(A)$, then the following assertions are equivalent:

1. $s$ is in $\hat{N}_{\text{CY}}(A)$ (see definition 10).
2. There is a map $\lambda : \mathbb{Z}\Delta_0 \rightarrow K^*$ such that:

   (a) $\mu^*(a) = \lambda_{\eta(a)} \lambda_\mu(a) (\nu^{s-1} \tau^{-s} (a))$, for all $a \in (Z\Delta)_1$, where $\mu$ is the graded automorphism of Proposition 5.3.
(b) \(\lambda \circ g_{\Delta_0} = \lambda\), for all \(g \in G\).

If \((\Delta, m, t) \neq (\mathbb{A}_{2n-1}, m, 2)\), then these conditions are also equivalent to:

3. There is a map \(\lambda : \mathbb{Z}_{\Delta_0} \rightarrow K^*\) satisfying condition 2.b such that \((-1)^s \eta^{-1}(a) = \lambda^{-1}_{i(a)} \eta(a) \nu^{s-1}(a)\), for all \(a \in (\mathbb{Z}_{\Delta})_1\).

If \((\Delta, t) \neq (\mathbb{A}_r, 2)\) then the conditions are also equivalent to:

4. \(s - 1 \in H(\Delta, m, t)\) (see Proposition 4.3).

Proof. The first paragraph of the proof of Proposition 5.16 says that \(s \in \mathbb{N}_{CY}(\Lambda)\) if, and only if, \(\nu^{s-1} \tau^{-s} \in G\). The goal is to give necessary and sufficient conditions on such an integer \(s\) so that \(\bar{\eta}\) and \(\bar{\tau}^{-1}\) commute, up to composition by an inner automorphism of \(\Lambda\). By Lemma 4.3, this last condition is equivalent to saying that there is a map \(\lambda : \mathbb{Z}_{\Delta_0} \rightarrow K^*\) satisfying 2.b such that \((-1)^s \eta^{-1}(a) = \lambda^{-1}_{i(a)} \eta(a) \nu^{s-1} \tau^{-s}(a)\), for each \(a \in (\mathbb{Z}_{\Delta})_1\).

Putting \(b = \tau^{-s}(a)\) and defining \(\tilde{\lambda} : (\mathbb{Z}_{\Delta})_0 \rightarrow K^*\) by the rule \(\tilde{\lambda}(i) = \lambda(i) \nu^{s-1}(i)\), we get that \((-1)^s \eta^{-1}(b) = \lambda^{-1}_{i(b)} \lambda_{t(b)} \nu^{s-1}(b)\), for all \(b \in (\mathbb{Z}_{\Delta})_1\). Then assertions 2 and 3 are equivalent.

Finally, when \((\Delta, t) \neq (\mathbb{A}_r, 2)\), Corollary 5.13 says that we can choose \(\mu = \eta \tau^{-1}\). Then the proof of the equivalence of assertions 2 and 3, taken for \(\kappa = id_B\), shows that assertion 2 holds if, and only if, there is a map \(\lambda : \mathbb{Z}_{\Delta_0} \rightarrow K^*\) satisfying 2.b and such that \(\eta^{-1}(b) = \lambda^{-1}_{i(b)} \lambda_{t(b)} \nu^{s-1}(b)\), for all \(b \in (\mathbb{Z}_{\Delta})_1\). This is equivalent to saying that \(s - 1 \in H(\Delta, m, t)\).

The following is now a consequence of Proposition 5.16 and the foregoing lemma.

Corollary 5.18. Let \(\Lambda\) be an \(m\)-fold mesh algebra over a field of characteristic 2, with \(\Delta \neq \mathbb{A}_1\).

The algebra is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. When in addition \(\Delta \neq \mathbb{A}_2\), this is in turn equivalent to saying that \(\mathbb{N}_{CY}(\Lambda) \neq \emptyset\). Moreover, the following assertions hold:

1. When the Loewy length of \(\Lambda\) is \(\leq 2\), i.e. \(\Delta = \mathbb{A}_2\), the algebra is always Calabi-Yau Frobenius and \(CY - \dim(\Lambda) = CYF - \dim(\Lambda) = 0\).

2. When \(\Delta \neq \mathbb{A}_2\), we have \(CY - \dim(\Lambda) = CYF - \dim(\Lambda) = 3m - 1\), where \(m = \min(\mathbb{N}_{CY}(\Lambda))\) (see Proposition 7.10).

Proof. The case of Loewy length 2 is covered by Proposition 5.13. So we assume \(\Delta \neq \mathbb{A}_2\) in the sequel. If \(\Lambda\) is stably Calabi-Yau, then \(\mathbb{N}_{CY}(\Lambda) \neq \emptyset\). But, when \(\text{char}(K) = 2\), the \(G\)-invariant graded Nakayama automorphism of Theorem 4.2 is \(\eta = \nu\). In addition, the automorphisms \(\vartheta\) and \(\kappa\) of Proposition 5.3 are the identity. Then, in order to prove the equality \(\mathbb{N}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)\), one only needs to prove that if \(s \in \mathbb{N}_{CY}(\Lambda)\) then condition 2 of last lemma holds. But this is clear, by taking as \(\lambda\) any constant map.

We are now ready to give, for \(\text{char}(K) \neq 2\), the precise criterion for an \(m\)-fold mesh algebra to be stably Calabi-Yau, and to calculate \(CY - \dim(\Lambda)\) in that case.

Theorem 5.19. Let us assume that \(\text{char}(K) \neq 2\) and let \(\Lambda\) be the \(m\)-fold mesh algebra of extended type \((\Delta, m, t)\), where \(\Delta \neq \mathbb{A}_1, \mathbb{A}_2\). We adopt the convention that if \(a, b, k\) are fixed integers, then \(av \equiv b \pmod{k}\) means that \(v\) is the smallest non-negative integer satisfying the congruence. The algebra is Calabi-Yau Frobenius if, and only if, it is stably Calabi-Yau. Moreover, we have \(CY - \dim(\Lambda) = CY - \dim(\Lambda)\) and the following assertions hold:

1. If \(t = 1\) then

(a) When \(\Delta = \mathbb{A}_r, \mathbb{D}_{2r-1}\) or \(\mathbb{E}_6\), the algebra is stably Calabi-Yau if, and only if, \(\gcd(m, c_\Delta) = 1\). Then \(CY - \dim(\Lambda) = 6u + 2\), where \(c_\Delta u \equiv -1 \pmod{m}\).
(b) When $\Delta$ is $D_{2r}$, $E_7$ or $E_8$, the algebra is stably Calabi-Yau if, and only if, $\gcd(m, \frac{u}{2}) = 1$. Then:
   
i. $CY - \text{dim}(\Lambda) = 3u + 2$, where $\frac{u}{2} \equiv -1 \pmod{m}$, whenever $m$ is even;
   
i. $CY - \text{dim}(\Lambda) = 6u + 2$, where $\epsilon_{\Delta} \equiv -1 \pmod{m}$, whenever $m$ is odd;

2. If $t = 2$ then

   (a) When $\Delta = A_{2n-1}$, $D_{2r-1}$ or $E_6$, the algebra is stably Calabi-Yau if, and only if, $\gcd(2m, m + \frac{2u}{t}) = 1$. Then $CY - \text{dim}(\Lambda) = 3u + 2$, where $(m + \frac{2u}{t})u \equiv -1 \pmod{2m}$.

   (b) When $\Delta = D_{2r}$, the algebra is stably Calabi-Yau if, and only if, $\gcd(m, 2r - 1) = 1$ and $m$ is odd. Then $CY - \text{dim}(\Lambda) = 3u + 2$, where $(2r - 1)u \equiv -1 \pmod{2m}$.

   (c) When $\Delta = A_{2n}$, the algebra is stably Calabi-Yau if, and only if, $\gcd(2m - 1, 2n + 1) = 1$. Then $CY - \text{dim}(\Lambda) = 6u - 1$, where $(m + n)(2u - 1) \equiv -1 \pmod{2m - 1}$

3. If $t = 3$ then the algebra is not stably Calabi-Yau.

Proof. By Proposition 5.14, we know that, when $\Delta \neq A_3$, the algebra $\Lambda$ is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius and the corresponding dimensions are equal. From our arguments below it will follow that, when $\Delta = A_3$, we always have $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda)$, and then $CY - \text{dim}(\Lambda) = CYF - \text{dim}(\Lambda)$ also in this case (see Remark 5.13).

Our arguments will give an explicit identification of $\tilde{N}_{CY}(\Lambda)$ in terms of $N_{CY}(\Lambda)$. Then $CY - \text{dim}(\Lambda)$ will be $3v - 1$, where $v = \min(N_{CY}(\Lambda))$.

From Propositions 5.10 and 5.14, we know that, when $t = 3$, the algebra is never stably Calabi-Yau. So we assume in the sequel that $t \neq 3$.

Suppose first that $(\Delta, m, t) \neq (A_r, m, 2)$. Then Lemma 5.17 tells us that $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda) \cap (H(\Delta, m, t) + 1)$, where $H(\Delta, m, t) + 1 = \{ s \in Z : s - 1 \in H := H(\Delta, m, t) \}$. By Proposition 5.16 we get in these cases that the equality $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda)$ holds whenever $m + t$ is odd. We now examine the different cases:

1.a) If $\Delta = A_r$ then $H = Z$. When $\Delta$ is $D_{2r-1}$ or $E_6$, the Coxeter number $c_\Delta$ is even. If $N_{CY}(\Lambda) \neq \emptyset$ then $\gcd(m, c_\Delta) = 1$, so that $m$ is odd and $H = 2Z$. But then $N_{CY}(\Lambda) = N_{CY}(\Lambda) \cap (2Z + 1)$, which is equal to $N_{CY}(\Lambda)$ due to Proposition 5.16. So $\Lambda$ is stably Calabi-Yau if, and only if, $\gcd(m, c_\Delta) = 1$. Then $CY - \text{dim}(\Lambda) = 3(2u + 1) - 1 = 6u + 2$, where $2u + 1 = \min(N_{CY}(\Lambda))$.

1.b) We need to consider the case when $m$ is odd. In this case $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda) \cap (2Z + 1)$ is properly contained in $N_{CY}(\Lambda)$. However, we claim that if $N_{CY}(\Lambda) \neq \emptyset$ then $\tilde{N}_{CY}(\Lambda) \neq \emptyset$, which will prove that $\Lambda$ is stably Calabi-Yau if, and only if, $\gcd(m, \frac{2u}{t}) = 1$ using Proposition 5.10. Indeed, we need to prove that if $\gcd(m, \frac{2u}{t}) = 1$, then there is an integer $u' \geq 0$ such that $2u' + 1 \in N_{CY}(\Lambda)$ or, equivalently, that $\frac{2u}{t} - 1 \equiv -1 \pmod{m}$. But this is clear for if $m$ is odd then also $\gcd(m, c_\Delta) = 1$. Now the formulas in 1.b.i) and 1.b.ii) come directly from putting $s = u + 1$ and $s = 2u + 1$ and using the fact that $\frac{2u}{t} - 1 \equiv -1 \pmod{m}$.

2.a) Suppose first that $\Delta$ is $D_{2r-1}$ or $E_6$. In this case $\frac{2u}{t}$ is even. Then $\gcd(2m, m + \frac{2u}{t}) = 1$ implies that $m$ is odd and, hence, that $H = Z$. So in this case $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda)$ and the formula for $CY - \text{dim}(\Lambda)$ comes from putting $s = 1 + u$, where $(m + \frac{2u}{t})u \equiv -1(\text{mod} 2m)$.

Suppose next that $(\Delta, m, t) = (A_{2n-1}, m, 2)$, i.e. $\Delta = \Delta_n^{(m)}$. Here $\eta = \nu$. Then condition 2 of Lemma 5.17 can be rephrased by saying that $\mu^* \circ (\nu \circ \hat{\sigma})^*$ is equal, up to composition by an inner automorphism of $\Lambda$. This proves that $N_{CY}(\Lambda) = N_{CY}(\Lambda) \cap 2Z$ due to Lemma 5.9. But Proposition 5.10 tells us that then $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda)$. The formula for $CY - \text{dim}(\Lambda)$ is calculated as in the other two quivers of 2.a.

2.b) If $N_{CY}(\Lambda) \neq \emptyset$ then $\gcd(m, 2r - 1) = 1$. If $m$ is odd then $H = Z$. If $m$ is even, then $H = 2Z$ which implies that $\tilde{N}_{CY}(\Lambda) = N_{CY}(\Lambda) \cap (2Z + 1)$. But this is the empty set due to Proposition 5.10. The formula for $CY - \text{dim}(\Lambda)$ in the case when $m$ is odd follows again from putting $s = 1 + u$ and $(2r - 1)u \equiv -1(\text{mod} 2m)$.

2.c) It remains to consider the case $(\Delta, m, t) = (A_{2n}, m, 2)$, i.e. $\Lambda = \Delta_n^{(m)}$. We use condition 3 of Lemma 5.17. If $\lambda : Z\Delta_0 \to K^*$ is any map such that $(-1)^s \eta^s - 1(a) = \lambda_{i(a)}^{-1} \lambda_{i(a)} \nu^s(a)$, then $\lambda_{i(a)}^{-1} \nu^s(a) = (-1)^s$ since $\eta^s(a) = \nu^s(a)$, for all $a \in (Z\Delta)_1$. It follows that $\lambda_{i(a)}^{-1}$ =
Indeed, using the description of this last proposition, we need to see that the diophantic equation of Theorem 4.2, for each choice of a Dynkin graph $\Delta \in \{D_{n+1},E_r\}$ $(n > 3, r = 6,7,8$ and of $\varphi \in \{\tau, pr^m\}$, we shall give a convenient subset $I' \subset \mathbb{Z}\Delta_0$ which is a set of representatives of the $\varphi$-orbits, and we will then give a nonzero element $w_{(k,i)} \in e_{(k,i)}\text{Soc}_{gr}(B)$, for each $(k, i) \in I'$. Finally, we will use these elements to find, for each $a \in \mathbb{Z}\Delta_1$, the exponents $u(a)$ and $v(a)$ needed for the explicit formula of $\eta(a)$, as indicated in the mentioned proof.

1. When $\Delta = D_{n+1}$ with $n > 3$:

To simplify the notation, we shall denote by $u$, $v$ and $w$, respectively, each of the paths of length 2

\[
(r, 2) \rightarrow (r, 0) \rightarrow (r + 1, 2),
\]

\[
(r, 2) \rightarrow (r, 1) \rightarrow (r + 1, 2),
\]

\[
(r, 2) \rightarrow (r, 3) \rightarrow (r + 1, 2),
\]

with no mention to $r$. Then a composition of those paths $(r, 2) \rightarrow (r + 1, 2) \rightarrow \ldots \rightarrow (r + i, 2)$ will be denoted as a (noncommutative) monomial in the $u, v, w$.

We will need also to name the paths that we will use. Concretely:

i) $\gamma_{(k,i)}$ is the downward path $(k, i) \rightarrow \ldots \rightarrow (k + i - 2, 2)$, with the convention that $\gamma_{(k,2)} = e_{(k,2)}$.

ii) $\delta_{(m,j)}$ is the upward path $(m, 2) \rightarrow \ldots \rightarrow (m, j)$, with the convention that $\delta_{(m,2)} = e_{(m,2)}$.

iii) $\varepsilon_{(k,j)}$ is the arrow $(k, 2) \rightarrow (k, j)$ and $\varepsilon'_{(k,j)}$ is the arrow $(k, j) \rightarrow (k + 1, 2)$, for $j = 0, 1$.

(a) If $\varphi = \tau$, we will take the canonical slice $I' := \{(0, i) : i \in \Delta_0\}$.

Our choice of the $w_{(0,i)}$ is then the following:

(a) $w_{(0,i)} = \gamma_{(0,i)}wuw\ldots\delta_{(n-1,i)}$ whenever $i = 2, \ldots, n$.

(b) $w_{(0,0)} = \varepsilon'_{(0,0)}vwuv\ldots\varepsilon_{(0,0)}$

(c) $w_{(0,1)} = \varepsilon'_{(0,1)}vwuv\ldots\varepsilon_{(0,1)}$

(note that, for $j = 0, 1$, the vertex $\nu(0, j)$ depends on whether $n + 1$ is even or odd).

Now, we will use the letter $a$ to denote an upward arrow $(k, i) \rightarrow (k, i + 1)$, with $i = 2, \ldots, n - 1$, and the letter $\beta$ to denote a downward arrow $(k, i) \rightarrow (k + 1, i - 1)$ with $i = 3, \ldots, n$. We will also consider the arrows $\varepsilon_{j} := \varepsilon_{(k,j)} : (k, 2) \rightarrow (k, j)$ and $\varepsilon'_{j} := \varepsilon'_{(k,j)} : (k, j) \rightarrow (k + 1, 2)$, for $j = 0, 1$. In all cases we consider that the origin of each arrow is a vertex of $I'$. We will now create a table, where, for each of these arrows $a$, the path $p_a$ will be a path of length $l - 1$ from $t(a)$ to $\nu(i_a)$ such that $ap_a \neq 0$ in $B$. Then $u(a)$, $v(a)$ will be elements of $\mathbb{Z}_2$ such that $ap_a = (-1)^{u(a)}w_{i(a)}$ and $p_a v(a) = (-1)^{v(a)}w_{i(a)}$. The routine verification of these equalities is left to the reader.
When $\Delta = D_2$, we denote by $\varphi'$ exchanging the roles of 0 and 1. It follows that the action of the group of automorphisms generated by $\rho$, $\eta$, $\nu$, and $\tau$ for $\varphi = \rho^m$, we will take $I' = \{(k, i) : i \in \Delta_0 \text{ and } 0 \leq k < m\}$ and we will put $w_{(i,j)} = \rho^{-k}(w_{(0,i)})$, for each $(k, i) \in I'$, where $w_{(0,i)}$ is defined as in the previous case.

The symmetric equality is true when exchanging the roles of 0 and 1. It follows that $\eta(z_0) = \nu(z_0)$ (resp. $\eta(z_1) = -\nu(z_0)$) when the origin of $z_0$ (resp. $z_1$) is in $I'(q)$, with $q$ even, and $\eta(z_0) = -\nu(z_0)$ (resp. $\eta(z_1) = -\nu(z_0)$) otherwise. That is, we have $\eta(z_1) = -\nu(z_0)$. A similar argument shows that if $k \neq -1 \mod m$ and $z'_{j} : (k, j) \rightarrow (k + 1, 2)$, then we have $\eta(z'_j) = -\nu(z'_j)$. Finally, if $z'_j : ((q + 1)m - 1, j) \rightarrow ((q + 1)m, 2)$ we get that $\eta(z'_j) = -\nu(z'_j)$, which shows that the equalities in 2.b.iii hold.

2. When $\Delta = D_4$:

(a) If $\varphi = \tau^n$, the formulas for $\Delta = D_{n+1}$ with $n > 3$ and $\varphi = \tau^n$ are still valid in this case.

(b) If $\varphi = \rho^m$

We slightly divert from the previous case (see Convention 3.6). We take $w_{(0,0)} = \varepsilon_0^2 \varepsilon_1^2 \varepsilon_0$ and $w_{(0,2)} = \varepsilon_0 \varepsilon_1^2 \varepsilon_1$. Due to the fact that all nonzero paths from $(0, 2)$ to $\nu(0, 2) = (2, 2)$ are equal, up to sign, in $B$ we know that the action of $(\rho)$ on those paths is trivial. The base $B$ will be the union of the orbits of $w_{(0,0)}$ and $w_{(0,2)}$ under the action of the group of automorphisms generated by $\rho$ and $\tau$.

Recall that, in this case, the mesh arrows are the original ones: $r_{(k,i)} = \sum_{\alpha(a)=(k,i)} \sigma(a) a$. Note that if $\varepsilon : (k, i) \rightarrow (k + 1, 2)$, for $i = 0, 1, 3$, then we have $w_{(k,i)} = \varepsilon_0 \varepsilon_1 \varepsilon_0 \varepsilon_1$ and $w_{(k,2)} = \varepsilon_0 \varepsilon_1 \varepsilon_0 \varepsilon_1$. The corresponding table is then given as

| $\varepsilon_0$ | $\varepsilon_1$ | $\varepsilon_0 \varepsilon_1$ | $\varepsilon_0 \varepsilon_1$ |
|----------------|----------------|----------------|----------------|
| $\alpha$ : $(0, i) \rightarrow (0, i + 1)$ | $\beta$ : $(0, i) \rightarrow (1, i - 1)$ | $\gamma_0$ : $(0, 0) \rightarrow (1, 2)$ | $\delta_1$ : $(0, 1) \rightarrow (1, 2)$ |
| $\gamma_0$ : $(0, i+1) \rightarrow (0, i + 1)$ | $\gamma_1$ : $(1, i-1) \rightarrow (1, i-1)$ | $\gamma_2$ : $(1, i-1) \rightarrow (1, i-1)$ | $\gamma_3$ : $(0, 0) \rightarrow (0, 0)$ |
| $\gamma_0^2$ : $(0, 0) \rightarrow (1, 2)$ | $\delta_1$ : $(0, 1) \rightarrow (1, 2)$ | $\delta_2$ : $(0, 2) \rightarrow (1, 0)$ | $\delta_3$ : $(0, 1) \rightarrow (0, 1)$ |
| $\gamma_0^3$ : $(0, 0) \rightarrow (1, 0)$ | $\delta_2$ : $(0, 2) \rightarrow (0, 1)$ |
| $\varepsilon_0 : (0, 0) \rightarrow (1, 2)$ | $\varepsilon_1 : (0, 2) \rightarrow (0, 1)$ |

3. When $\Delta = E_n$ with $n = 6, 7, 8$:

For the sake of simplicity, we will write any path as a composition of arrows in $\{\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta', \zeta, \zeta', \theta, \theta' \varepsilon, \varepsilon'\}$ whenever they exist and assuming that each arrow is considered in the appropriate slice so that the composition makes sense.

Also, we denote by $u$, $w$ and $v$, respectively, each of the paths of length 2.
\[(k, 3) \rightarrow (k, 0) \rightarrow (k + 1, 3)\]
\[(k, 3) \rightarrow (k + 1, 2) \rightarrow (k + 1, 3)\]
\[(k, 3) \rightarrow (k, 4) \rightarrow (k + 1, 3)\]

with no mention to \(k\). Then any path \((k, 3) \rightarrow \ldots \rightarrow (k + r, 3)\) is equal in \(B\) to a monomial in \(u, v, w\), with the obvious sense of 'monomial'. Also, \(\beta' \beta = w, \gamma' \gamma = v\) and \(\varepsilon \varepsilon' = u\). It is important to keep in mind that \(u = v + w\).

A. When \(\Delta = E_6\):

(a) If \(\varphi = \tau\), we will take the canonical slice \(I' := \{(0, i) : i \in \Delta_0\}\).

With the abuse of notation of omitting \(k\) when showing a vertex \((k, i)\) in the diagrams below, we then take:

i. \(w_{(0,0)}\) is the path

\[0 \rightarrow 3 \rightarrow 3 \rightarrow 0\]

ii. \(w_{(0,1)}\) is the path

\[\begin{array}{ccc}
0 & \rightarrow 3 & \rightarrow 3 & \rightarrow 0 \\
& u & & w \\
\end{array}\]

iii. \(w_{(0,2)}\) is the path

\[\begin{array}{ccc}
1 & \rightarrow 2 & \rightarrow 3 \\
& & v^2 w \\
\end{array}\]

iv. \(w_{(0,3)}\) is the path

\[\begin{array}{ccc}
2 & \rightarrow 3 & \rightarrow 3 \\
& & u \cup u \cup w \\
\end{array}\]

v. \(w_{(0,4)}\) is the path

\[\begin{array}{ccc}
3 & \rightarrow 3 & \rightarrow 0 \\
& & \downarrow \\
\end{array}\]

and

\[\begin{array}{ccc}
2 & \rightarrow 3 & \rightarrow 0 \\
& & \downarrow \\
\end{array}\]
vi. $w_{(0,5)}$ is the path

Using the mesh relations, one gets, among others, the equalities $u^2 = w^3 = v^3 = 0$, $vw = wvw$, $vw^2v = -vwv^2 - v^2wv$ and $vwvw = -wvwv$.

Then, the table is the following:

| a          | $p_a$            | $u(a)$ | $v(a)$ |
|------------|------------------|--------|--------|
| $\alpha : (0,1) \rightarrow (0,2)$ | $\beta v^2 w^\gamma \delta$ | 0      | 0      |
| $\beta : (0,2) \rightarrow (0,3)$ | $vwvw^\gamma$ | 0      | 0      |
| $\gamma : (0,3) \rightarrow (0,4)$ | $\gamma^\prime vwvw$ | 0      | 0      |
| $\delta : (0,4) \rightarrow (0,5)$ | $\delta^\prime vw^\gamma \varepsilon^\beta$ | 1      | 0      |
| $\varepsilon : (0,3) \rightarrow (0,0)$ | $\varepsilon^\prime vwvw$ | 1      | 0      |
| $\alpha^\prime : (0,2) \rightarrow (1,1)$ | $\alpha \beta v^2 w^\gamma$ | 1      | 0      |
| $\beta^\prime : (0,3) \rightarrow (1,2)$ | $\beta vwvw$ | 1      | 0      |
| $\gamma^\prime : (0,4) \rightarrow (1,3)$ | $vwvw^\beta^\prime$ | 0      | 1      |
| $\delta^\prime : (0,5) \rightarrow (1,4)$ | $\gamma^\prime w^2 v^\beta \varepsilon^\alpha^\prime$ | 0      | 0      |
| $\varepsilon^\prime : (0,0) \rightarrow (1,3)$ | $vwvw\varepsilon$ | 0      | 0      |

From this table the equalities in 3.a follow.

(b) If $\varphi = \rho \tau^m$, we will consider the slice $T = \{(0,i) : i = 0,3,4,5\} \cup \{(1,2),(2,1)\}$, which is $\rho$-invariant, and then take $I' = \{\tau^{-k}(r,i) : (r,i) \in T$ and $0 \leq k < m\}$.

The paths $w_{(0,i)} (i = 0,3,4,5)$ will be as in the case $\varphi = \tau$, and we will define below the paths $w_{(1,2)}$ and $w_{(2,1)}$. Then we will take $w_{\tau^{-k}(r,j)} = \tau^{-k}(w_{(r,j)})$, for all $(r,j) \in T$ and $0 \leq k < m$.

i. $w_{(1,2)}$ is the path

and

ii. $w_{(2,1)}$ is the path
Arguing as in the case of $D_{n+1}$, we see that the values $u(a)$ and $v(a)$ are the ones in the last table, when $i(a), t(a) \in I'$. We then need only to give those values for the arrows $a$ with origin in $I'$ and terminus not in $I'$. We have the table:

| $a$ | $p_a$ | $u(a)$ | $v(a)$ |
|-----|-------|--------|--------|
| $\alpha : (m + 1, 1) \to (m + 1, 2)$ | $\beta vw^3 = 0$ | $0$ | $0$ |
| $\beta : (m, 2) \to (m, 3)$ | $vww$ | $0$ | $1$ |
| $\gamma' : (m - 1, 4) \to (m, 3)$ | $wvw$ | $0$ | $0$ |
| $\delta' : (m - 1, 5) \to (m, 4)$ | $\gamma' w^2 v^2 w^2 \alpha'$ | $0$ | $0$ |
| $\varepsilon' : (m - 1, 0) \to (m, 3)$ | $vww$ | $0$ | $1$ |

We have used in the construction of this table the fact that $w_{(k,2)} = \beta vwvw\gamma$ and $w_{(k,4)} = \gamma' wvwv\beta'$, for all $k \in \mathbb{Z}$, while $w_{(2,3)} = vwwv$ and $w_{(2r+1,3)} = vwwv$.

Note that, with the labeling of vertices that we are using, we have that $\rho(k, i) = (k + i - 3, 6 - i)$ for each $i \neq 0$ and $\rho(k, 0) = (k, 0)$. For each $q \in \mathbb{Z}$, we put $I'(q) := (\rho r^{-m})^q(I')$. When passing from a piece $I'(q)$ to $I'(q + 1)$ by applying $\rho r^{-m}$, an arrow $\alpha$ is transformed in an arrow $\delta'$ and an arrow $\delta'$ in an arrow $\alpha$. From the last two tables we then get that $\eta(\alpha) = \nu(\alpha)$ and $\eta(\delta') = \nu(\delta')$, for all arrows of the type $\alpha$ or $\delta'$ in $\mathbb{Z}\Delta$.

The argument of the previous paragraph can be applied to the pair of arrows $(\gamma, \beta')$ instead of $(\alpha, \delta')$ and we get from the last two tables that $\eta(\gamma) = \nu(\gamma)$ (resp. $\eta(\beta') = -\nu(\beta')$) when $\gamma$ (resp. $\beta'$) has its origin in $I'(q)$, with $q$ even, and $\eta(\gamma) = -\nu(\gamma)$ (resp. $\eta(\beta') = \nu(\beta')$) otherwise. From this the formulas in 3.b concerning $\gamma$ and $\beta'$ are clear.

Next we apply the argument to the pair of arrows $(\delta, \alpha')$ and get that $\eta(\delta) = -\nu(\delta)$ (resp. $\eta(\alpha') = -\nu(\alpha')$), for all arrows of type $\delta$ or $\alpha'$ in $\mathbb{Z}\Delta$.

An arrow of type $\varepsilon$ (resp. $\varepsilon'$) is transformed on one of the same type when applying $\rho r^{-m}$. It then follows that $\eta(\varepsilon) = -\nu(\varepsilon)$, for any arrow of type $\varepsilon$. It also follows that $\eta(\varepsilon') = -\nu(\varepsilon')$, when the origin of $\varepsilon'$ is $(k, 0)$ with $k \equiv -1 \pmod{m}$, and $\eta(\varepsilon') = \nu(\varepsilon')$ otherwise.

We finally apply the argument to the pair of arrows $(\beta, \gamma')$. If we look at the two pieces $I'(0)$ and $I'(1)$, then from the last two tables we see that if $\beta : (k, 2) \to (k, 3)$, with $(k, 3) \in I'(0) \cup I'(1)$, then $\eta(\beta) = \nu(\beta)$, when $k \in \{1, 2, ..., m-1, 2m\}$, and $\eta(\beta) = -\nu(\beta)$, when $k \in \{m, m+1, ..., 2m-1\}$. We then get that $\eta(\beta) = (-1)^q \nu(\beta)$, where $q$ is the quotient of dividing $k$ by $m$. By doing the same with $\gamma' : (k, 4) \to (k + 1, 3)$, we see that $\eta(\gamma') = -\nu(\gamma')$, when $k \in \{0, 1, ..., m-2, 2m-1\}$, and $\eta(\gamma') = \nu(\gamma')$, when $k \in \{m-1, m, ..., 2m-2\}$. If now $k \in \mathbb{Z}$ is arbitrary, then we obtain that $\eta(\gamma') = \nu(\gamma')$ if, and only if, $k \notin \bigcup_{i \in \mathbb{Z}}(2tm - 2, (2t + 1)m - 1)$. Equivalently, when $q$ is odd and $r \neq m - 1$ or $q$ is even and $r = m - 1$.

B. When $\Delta = E_7$: We consider the canonical slice $I' = \{(0, i) : i \in \Delta_0\}$. Using the same notation as when $\Delta = E_6$, we have in this case, among others, the equalities $u^2 = w^3 = v^4 = 0$, $vww = wvw - v^3$, and $vwvw = -vwww$. Then, we consider
i. \(w_{(0,0)}\) is the path

\[
\begin{array}{ccc}
0 & \rightarrow & 3 \\
& & \text{vwvwv}
\end{array}
\]

ii. \(w_{(0,1)}\) is the path

\[
\begin{array}{ccc}
3 & \rightarrow & 3 \\
& & \text{vw} \text{vw} \text{vw} \text{v}
\end{array}
\]

iii. \(w_{(0,2)}\) is the path

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
& & \text{vwv}
\end{array}
\]

iv. \(w_{(0,3)}\) is the path

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
& & \text{vwv}
\end{array}
\]

v. \(w_{(0,4)}\) is the path

\[
\begin{array}{ccc}
4 & \rightarrow & 3 \\
& & \text{vwv}
\end{array}
\]

vi. \(w_{(0,5)}\) is the path

\[
\begin{array}{ccc}
5 & \rightarrow & 4 \\
& & \text{vwv}
\end{array}
\]

vii. \(w_{(0,6)}\) is the path

\[
\begin{array}{ccc}
6 & \rightarrow & 5 \\
& & \text{vwv}
\end{array}
\]

Hence, we get the following table:
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\(a\) & \(p_a\) & \(u(a)\) & \(v(a)\) \\
\hline
\(\alpha : (0, 1) \to (0, 2)\) & \(\beta^2 w^3 w v w v^2 \alpha'\) & 0 & 0 \\
\(\beta : (0, 2) \to (0, 3)\) & \(w v w v w v^2 \beta'\) & 0 & 0 \\
\(\gamma : (0, 3) \to (0, 4)\) & \(\gamma^2 w v w v w v\) & 0 & 0 \\
\(\delta : (0, 4) \to (0, 5)\) & \(\delta' \gamma^2 w v w v^2 \gamma\) & 0 & 1 \\
\(\zeta : (0, 5) \to (0, 6)\) & \(\zeta' \delta' \gamma^2 w v w v^3 \gamma \delta\) & 0 & 0 \\
\(\varepsilon : (0, 3) \to (0, 0)\) & \(\varepsilon' w v w v w v\) & 0 & 1 \\
\(\alpha' : (0, 2) \to (1, 1)\) & \(\alpha' \beta w v w v^2 \beta'\) & 0 & 1 \\
\(\beta' : (0, 3) \to (1, 2)\) & \(\beta w v w v w v\) & 0 & 0 \\
\(\gamma' : (0, 4) \to (1, 3)\) & \(w v w v w v w v\gamma\) & 0 & 1 \\
\(\delta' : (0, 5) \to (1, 4)\) & \(\delta' \gamma^2 w v w v^3 \gamma \delta\) & 0 & 0 \\
\(\zeta' : (0, 6) \to (1, 5)\) & \(\zeta' \delta' \gamma^2 w v w v^3 \gamma \delta \zeta\) & 0 & 1 \\
\(\varepsilon' : (0, 0) \to (1, 3)\) & \(\varepsilon' w v w v w v\) & 0 & 0 \\
\hline
\end{tabular}
\end{center}

From this table the equalities in 4 follow.

C. When \(\Delta = E_8\):

As in the previous case, we consider the canonical slice and follow the same notation. In addition, note that we have the equalities \(v^2 = w^3 = v^5 = 0, v w v = w v w - v^3, (v w)^3 = (w v)^3 + v w v^4 - v^4 w v, (v w)^6 = (w v)^6 + (w v)^3 v w v^4 - v^4 w v^2 w v^4\) and \((w v)^7 = -(w v)^7\). Then, considering the following paths

i. \(w_{(0,0)}\) is the path

\[
\begin{array}{ccc}
0 & \rightarrow & 3 \\
& (v w)^6 v & \\
& 3 & \rightarrow & 0
\end{array}
\]

ii. \(w_{(0,1)}\) is the path

iii. \(w_{(0,2)}\) is the path

iv. \(w_{(0,3)}\) is the path

v. \(w_{(0,4)}\) is the path

vi. \(w_{(0,5)}\) is the path
vii. $w_{(0,6)}$ is the path

viii. $w_{(0,7)}$ is the path

we obtain the table below:

| $a$            | $p_a$                          | $u(a)$ | $v(a)$ |
|----------------|--------------------------------|--------|--------|
| $\alpha : (0,1) \to (0,2)$ | $\beta v^2 (vw)^2 \beta' \alpha'$ | 0      | 0      |
| $\beta : (0,2) \to (0,3)$    | $(vw)^3 u \beta'$             | 0      | 0      |
| $\gamma : (0,3) \to (0,4)$   | $\gamma' w (vw)^3$            | 0      | 0      |
| $\delta : (0,4) \to (0,5)$   | $\delta' \gamma' wv^2 (vw)^3 w \gamma'$ | 0      | 1      |
| $\zeta : (0,5) \to (0,6)$    | $\zeta' \delta' \gamma' wv^3 (vw)^3 w \gamma \delta$ | 0      | 0      |
| $\theta : (0,6) \to (0,7)$   | $\theta' \zeta' \delta' \gamma' wv^4 w v^2 w \gamma \delta \zeta$ | 0      | 0      |
| $\varepsilon : (0,3) \to (0,0)$ | $\varepsilon' w (vw)^3$       | 0      | 1      |
| $\alpha' : (0,2) \to (1,1)$  | $\alpha \beta (vw)^2 v \gamma \beta'$ | 0      | 1      |
| $\beta' : (0,3) \to (1,2)$   | $\beta (vw)^2 v$              | 1      | 0      |
| $\gamma' : (0,4) \to (1,3)$  | $(vw)^3 v \gamma$             | 0      | 1      |
| $\delta' : (0,5) \to (1,4)$  | $\gamma' w^2 (vw)^3 \gamma \delta$ | 0      | 0      |
| $\zeta' : (0,6) \to (1,5)$   | $\delta' \gamma' w^2 (vw)^4 w \gamma \delta \zeta$ | 0      | 1      |
| $\theta' : (0,7) \to (1,6)$  | $\zeta' \delta' \gamma' wv^3 w v^2 w \gamma \delta \zeta \theta$ | 0      | 1      |
| $\varepsilon' : (0,0) \to (1,3)$ | $(vw)^3 \varepsilon$          | 0      | 0      |

From this table the equalities in 5 follow.
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