LETTER

Solitons solutions of nonlinear Schrödinger equation in the left-handed metamaterials by three different techniques

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Keywords: exact traveling-wave solution, soliton solutions, nonlinear Schrodinger equation, Dark, Bright solitons

Abstract

This paper, derives the exact traveling-wave solution and soliton solutions of nonlinear Schrodinger equation (NLSE) with higher-order nonlinear terms of Left-handed metamaterials (LHMs), the authors apply three different methods, namely: csch function method, the simplest equation method and the simplest equation method. The results obtained are Dark, Bright solitons and other solutions, which are well known in optics metamaterials and LHMs.

1. Introduction

It is well known that the partial differential equation (PDEs) of the non-linear Schrodinger equation with high-order nonlinear terms are near the complex physics phenomena which are concerned many fields from physics to biology etc [1–17]. Recently, some effective methods for getting solitons solutions in LHMs and optics has attracted many researchers attention because of soliton theory which is a very important and fascinating area of research in nonlinear left-handed metamaterials and optics. Houwe Alphonse et al [4] studied optical solitons in left-handed metamaterials. M Mirzazadeh et al [5] reported solitons to generalized resonant dispersive nonlinear Schrödinger’s equation with power law nonlinearity. I. V. Shadrivov et al [6] studied Spatial solitons in left-handed metamaterials. Ekici Mehmet et al [7] investigated optical solitons in birefringent fibers with Kerr nonlinearity. Biswas et al [8] obtained bright and dark solitons for MMs. Anjan Biwas et al [9] demonstrated the existence of singular solitons in optical metamaterials by ansatz method and simplest equation approach. Alphonse HOUWE et al obtained solitons of the perturbed nonlinear Schrödinger equation in the nonlinear left-handed transmission lines [18]. In this perspective, many methods for obtaining exact solutions of NLSE was investigated, such as Tan-sech method [10, 11], Exponential rational function method [12], the sine-cosine method [13, 14], the modified simple equation method [15], and so on.

In [16], N Taghizadeh et al used the first integral method to find exact soliton solution of the nonlinear Schrodinger equation and Ma and Chen [19] is used Direct search method to obtain exact solutions of the same nonlinear Schrodinger equation. This cubic nonlinear Schrodinger equation [16, 19], which is similar to that obtained in a left-handed transmission lines loaded with a varactor is in the following form:

\[ iu_t + au_{xx} + c|u|^2u = 0, \]  

(1)

Where \( u = u(x, t) \) is the complex-valued function of two real variables \( x, t \). \( a \) is the group velocity dispersion and the term \( c \) is the nonlinearity coefficient. The index \( m > 0 \), is the full nonlinearity parameter. For \( a = p = 1 \), \( c = q = \mu \), and \( m = 1 \) correspond to the non-linear Schrodinger equation and have been discussed [19].
Solitary waves of nonlinear Schrödinger’s equation in the left-handed metamaterials can pave the way for relevant studies, e.g., modulational instability. In the present paper, to befall many exact solutions and solitons solutions of this model of equation (1), the authors have used three integration schemes. They are csch method, the exp(−φ(ξ))-Expansion method and the simplest equation method that will uncover solitons solutions to the model. The beginning hypothesis is the traveling-wave transformation. The elaboration are all recorded in the upcoming section.

2. Traveling wave assumption

The solution of equation (1) is supposed to be

\[ u(x, t) = e^{θ(x, t)}U(ξ), \]

where \( U(ξ) \) is the amplitude component of the wave and \( ξ = x - vt \), while \( v \) is its speed. Here \( θ(x, t) = -kx + ωt + θ₀ \) represents the phase component of the soliton. The parameters \( ω, k \) and \( θ₀ \) are respectively the inverse pulse width, the frequency and the phase constant.

After changing the variables, and substituting equation (2) into equation (1), and separating the real and imaginary parts it is obtained:

\[ (ν + 2k)U' = 0, \]

and

\[ -ωU + aU'' + cU^{2m+1} = 0, \]

from equation (3) leads to the speed wave of the soliton:

\[ ν = -2k, \]

Now, multiplying equation (4) by \( U' \) and integrating once with zero constant gives

\[ U'' - \frac{ω}{a}U^2 + \frac{c}{a(m + 1)}U^{2m+2} = 0, \]

assume

\[ U^2 = V, \]

equation (6) can be written as follows

\[ V'' - \frac{4ω}{a}V^2 + \frac{4c}{a(m + 1)}V^{m+2} = 0, \]

3. Application

In this section, three different integrations tools will be applied to befall exact solution and soliton solutions

3.1. csch function method

The solutions of many nonlinear equation can be expressed in form

\[ V(ξ) = Acsch^m(μξ), \]

and admits the following derivative

\[ V'(ξ) = -Aτμcsch^m(μξ).coth(μξ), \]

where \( A, τ \) and \( μ \) are parameters to be determined, \( μ \) is the wave number.

Substituting equations (10) and (9) into the reduced equation equation (8), it is obtained

\[ A^2τ^2μ^2csch^{2τ}(μξ) + A^2τ^2μ^2csch^{2τ+2}(μξ) - \frac{4ω}{a}A^2τ^2μ^2csch^{2τ}(μξ) \]

\[ + \frac{4c}{a(m + 1)}A^{m+2}τ^{m+2}μ^{m+2}csch^{m+2}(μξ) = 0, \]

To Balance the terms of the csch functions to find \( τ \).

\[ 2τ + 2 = (m + 2)τ, \]

To obtain the system of algebraic equation with the unknowns \( A \) and \( μ \), all terms are collected in equation (11) with the same power in \( csch^k(μξ) \) and set to zero.
\[ A^2 \left( \frac{1}{m} \right)^2 \mu^2 + \frac{4c}{a(m+1)} A^{m+2} \left( \frac{1}{m} \right)^{m+2} \mu^{m+2} = 0, \]  

(12)

and

\[ \left( \frac{1}{m} \right)^2 \mu^2 - \frac{4\omega}{a} = 0, \]  

(13)

Solving the system of equations equation (12) and equation (13) result is:

\[ A = \frac{1}{2} \left[ \frac{-at(m+1)}{m^2} \right]^{\frac{1}{2}}, \]  

(14)

and

\[ \mu = \pm 2 \sqrt{\frac{\omega}{a}} m, \]  

(15)

then, if \( m > 2 \),

\[ V(\xi) = \frac{1}{2} \left[ \frac{-at(m+1)}{m^2} \right]^{\frac{1}{2}} \text{csch}^2(\pm 2 \sqrt{\frac{\omega}{a}} m \xi), \]  

(16)

and therefore

\[ u(x, t) = e^{i(\kappa x + \omega t + \phi_i)} \sqrt{\pm 1 + \frac{1}{2} \left[ \frac{-at(m+1)}{m^2} \right]^{\frac{1}{2}} \text{csch}^2(\pm 2 \sqrt{\frac{\omega}{a}} m(x - vt))}, \]  

(17)

3.2. The \( \exp(-\phi(\xi)) \)-Expansion method

The key step is to suppose that the solution of equation (8) can be expressed by a rational polynomial as the following:

\[ V(\xi) = \sum_{i=0}^{N} a_i \exp(-\phi(\xi))^i, \]  

(18)

the parameter \( N \), it obtained by balancing the highest-order linear term with the nonlinear term, where \( i = 0, 1, \ldots, N \) and \( \phi(\xi) \) satisfies the following ordinary differential

\[ \phi'(\xi) = \Omega \exp(\phi(\xi)) + \exp(-\phi(\xi)), \]  

(19)

we balance \( V^{m+2} \) with \( V^1 \) to obtain \( (N+1)^2 = (m+2)N \)

\[ N = \frac{m \pm \sqrt{(m^2 - 4)}}{2}, \]  

(20)

The constraint condition is \( m^2 - 4 > 0 \).

(1) - For \( m = 2 \), the integer \( N = 1 \), and the rational polynomial becomes

\[ V(\xi) = a_0 + a_1 \exp(-\phi(\xi)) , \]  

(21)

Substituting equation (21) along with equation (19) into equation (8) involve the results of the system algebraic equation as follows by aid of Maple

Result 1: \( a_0 = a_0, a_1 = a_1, \quad \Omega = \Omega, \quad \lambda = \lambda, \quad c = \frac{-a_1^2 \Omega^2 + 4\omega + 2a_1^2}{a_0^2} \)

The following exact analytical solution can be obtained from equation (19) [21]

(1) If \( \Omega \neq 0 \) and \( \lambda^2 - 4\Omega > 0 \), we gain

\[ \phi(\xi) = \ln \left\{ \frac{\sqrt{\lambda^2 - 4\Omega}}{2\Omega} \tanh \left( \frac{\sqrt{\lambda^2 - 4\Omega}}{2\Omega} (\xi + D) - \frac{\lambda}{2\Omega} \right) \right\}, \]  

(22)

here \( D \) is integral constant, therefore
$$V(\xi) = a_0 + a_1 \left[ -\frac{1}{2} \frac{\sqrt{\lambda^2 - 4\Omega}}{2\Omega} \tanh \left( \frac{\sqrt{\lambda^2 - 4\Omega}}{2\Omega} (\xi + D) - \frac{\lambda}{2\Omega} \right) \right],$$  \hspace{1cm} (23)

recall equation (7) and equation (2), is obtained the following soliton solution

$$u(x, t) = e^{i(kx - \omega t + \theta_0)} \sqrt{a_0 + a_1 \left[ -\frac{1}{2} \frac{1}{\tanh \left( \frac{\sqrt{\lambda^2 - 4\Omega}}{2\Omega} (x - vt + D) - \frac{\lambda}{2\Omega} \right) \right]^2},$$  \hspace{1cm} (24)

(2) If $\Omega = 0$ and $\lambda^2 - 4\Omega < 0$, we have

$$\phi(\xi) = \ln \left\{ -\frac{\sqrt{-\lambda^2 + 4\Omega}}{2\Omega} \tan \left( \frac{\sqrt{-\lambda^2 + 4\Omega}}{2} (\xi + D) - \frac{\lambda}{2\Omega} \right) \right\},$$  \hspace{1cm} (25)

recall equation (7) and equation (2), we gain the following trigonometric solution

$$u(x, t) = e^{i(kx - \omega t + \theta_0)} \sqrt{a_0 + a_1 \left[ -\frac{1}{2} \frac{1}{\tanh \left( \frac{\sqrt{-\lambda^2 + 4\Omega}}{2\Omega} (x - vt + D) - \frac{\lambda}{2\Omega} \right) \right]^2},$$  \hspace{1cm} (26)

(3) If $\Omega = 0$, $\lambda = 0$ and $\lambda^2 - 4\Omega = 0$, the following traveling-wave solution can be had

$$u(x, t) = e^{i(kx - \omega t + \theta_0)} \sqrt{a_0 + a_1 \frac{-\lambda^2 (x - vt + D)}{2\lambda (x - vt + D) + 4}},$$  \hspace{1cm} (27)

Result 2: $a_0 = 0$, $a_1 = a_0$, $\Omega = 0$, $\lambda = \lambda$, $c = c$ here, as $\Omega = 0$, two cases are presented

(1) If $\Omega = 0$ and $\lambda^2 - 4\Omega > 0$, we gain

$$u(x, t) = e^{i(kx - \omega t + \theta_0)} \sqrt{a_0 + a_1 \frac{\lambda}{\exp(\lambda (x - vt + D)) - 1}},$$  \hspace{1cm} (28)

(2) If $\lambda = 0$ and $\lambda^2 - 4\Omega = 0$, the result is

$$u(x, t) = e^{i(kx - \omega t + \theta_0)} \sqrt{a_0 + a_1 \frac{-1}{x - vt + D}},$$  \hspace{1cm} (29)

3.3. The simplest equation method

The demarche is to suppose that $V(\xi)$ satisfies the Bernoulli and Riccati equations method [22, 23]. The step is to introduce the solution $V(\xi)$ of equation (8) in the following finite series form

$$V(\xi) = \sum_{i=0}^{N} a_i \phi(\xi)^i,$$  \hspace{1cm} (30)

where $a_i$ are real constants with $a_N \neq 0$, and $N$ is a positive integer to be determined. $\phi(\xi)$ satisfies the following ordinary differential equation

$$\phi'(\xi) = \rho - A\phi(\xi) + B\phi^2(\xi),$$  \hspace{1cm} (31)

Where $\rho$, $A$ and $B$ are independent on $\xi$, and will be determined later.

To obtain different exact solution and other solutions dependent of the parameters $\rho$, $A$ and $B$ two cases will be present
If we surmise \( m = 2 \), equation (8) becomes:

\[
Vr^2 = \frac{4\omega}{a} V^2 + \frac{4\epsilon}{3a} V^4 ,
\]  

(32)

By balancing the linear term of highest order derivatives with the highest order nonlinear term in equation (32), leads to \((N - 1)^2 = 0\), and \(N = 1\)

Then equation (30) becomes:

\[ V(\xi) = a_0 + a_1 \phi(\xi), \]  

(33)

Now, substituting equation (33) into equation (32) along with equation (31), the algebraic equations obtained are

\[
B^2 + \frac{2}{3} \cot^4 \alpha = 0, \\
-2 AB + \frac{8}{3} \cot \alpha \cot^3 \alpha = 0, \\
-4 \frac{\omega \alpha^2}{a} + A^2 - 2 \rho B + 4 \epsilon a_0 \alpha^2 = 0, \\
-8 \frac{\omega \alpha^2}{a} + 2 \rho A + \frac{8}{3} \epsilon a_0 \alpha^2 = 0, \\
\rho^2 + \frac{2}{3} \cot^4 \alpha - 4 \frac{\omega \alpha^2}{a} = 0,
\]

Solving this system with the aid of Maple gives

\[
a_0 = a_0, \quad a_1 = a_1, \quad \rho = \pm \frac{1}{2} \frac{\sqrt{\alpha \cot \alpha + 6 \omega \alpha^2 \cot^2 \alpha}}{\sqrt{-6 \alpha \cot \alpha}}, \quad A = \pm \frac{1}{2} \frac{\sqrt{\alpha \cot \alpha + 6 \omega \alpha^2 \cot^2 \alpha}}{\sqrt{-6 \alpha \cot \alpha}},
\]

\[
B = \pm \frac{1}{2} \sqrt{-6 \alpha \cot \alpha^2}.
\]

3.3.1. The Bernoulli equation

When \( \rho = 0, \) and \( A \neq 0, \) \( B \neq 0, \) the simplest equation method becomes Bernoulli equation. Therefore the following new traveling-wave and wavefront solutions of equation (1) are obtained [23] Case 1: For \( A > 0 \) and \( B < 0, \) equation (31) have the following solution

\[
\phi(\xi) = \frac{A \exp [A(\xi + \xi_0)]}{\sqrt{1 - B \exp [A(\xi + \xi_0)]}},
\]  

(34)

then substituting equation (34) and equation (33) in equation (2), is obtained

\[
u(x, t) = e^{\rho \theta} \sqrt{a_0 - a_1} \frac{A \exp [A(x - vt + \xi_0)]}{\sqrt{1 - B \exp [A(x - vt + \xi_0)]}} \]

(35)

Case 1: For \( A < 0 \) and \( B > 0, \) equation (31) have the following solution

\[
\phi(\xi) = \frac{A \exp [A(\xi + \xi_0)]}{\sqrt{1 + B \exp [A(\xi + \xi_0)]}},
\]  

(36)

then substituting equation (34) and equation (33) in equation (2), the result is

\[
u(x, t) = e^{\rho \theta} \sqrt{a_0 - a_1} \frac{A \exp [A(x - vt + \xi_0)]}{\sqrt{1 + B \exp [A(x - vt + \xi_0)]}} \]

(37)

where \( \xi_0 \) is the integration constant

However, if \( \frac{1}{2} \sqrt{-6 \alpha \cot \alpha} = -1 \) i.e \( B = -1, \) Solution of equation (31) becomes For \( A > 0, \)

\[
\phi(\xi) = \frac{A}{2} \left[ 1 + \tanh \left( \frac{A}{2} (\xi + \xi_0) \right) \right],
\]  

(38)

and for \( A < 0, \)

\[
\phi(\xi) = \frac{A}{2} \left[ 1 - \tanh \left( \frac{A}{2} (\xi + \xi_0) \right) \right],
\]  

(39)

then substituting equation (38) or equation (39) and equation (33) in equation (2), the following solutions of equation (1) is obtained For \( A > 0, \)

\[
u(x, t) = e^{t(\xi - vt + \theta_0)} \sqrt{a_0 - a_1} \frac{A}{2} \left[ 1 + \tanh \left( \frac{A}{2} (x - vt + \xi_0) \right) \right] \]

(40)
and for $A < 0$, it is gained

$$u(x, t) = e^{i(-kx + \omega t + \theta_0)} \sqrt{a_0 + a_1} \left[ \frac{A}{2} (x - vt) \right],$$

(41)

where $\xi_0$ is the integration constant.

### 3.3.2. The Riccati equation

When $\rho \neq 0$, and $A = 0, B = 1$, the simplest equation method becomes the Riccati equation. Therefore the following new traveling-wave and trigonometric solutions of equation (1) are obtained [22]

**Case 1:** For $\rho < 0$ equation (31) have the following solutions

$$\phi(\xi) = -\sqrt{-\rho} \tanh(\sqrt{-\rho}(\xi + \xi_0)), \quad (42)$$

and

$$\phi(\xi) = -\sqrt{-\rho} \coth(\sqrt{-\rho}(\xi + \xi_0)), \quad (43)$$

then substituting equation (42) or equation (43) and equation (33) in equation (2), the following solutions of equation (1) is obtained

$$u(x, t) = e^{i(-kx + \omega t + \theta_0)} \sqrt{a_0 - a_1} \sqrt{-\rho} \tanh(\sqrt{-\rho}(x - vt + \xi_0)), \quad (44)$$

and

$$u(x, t) = e^{i(-kx + \omega t + \theta_0)} \sqrt{a_0 - a_1} \sqrt{-\rho} \coth(\sqrt{-\rho}(x - vt + \xi_0)), \quad (45)$$

where $\xi_0$ is the integration constant.

**Case 2:** For $\rho > 0$ equation (31) have the following solutions

$$\phi(\xi) = \sqrt{\rho} \tan(\sqrt{\rho}(\xi + \xi_0)), \quad (46)$$

and

$$\phi(\xi) = -\sqrt{\rho} \cot(\sqrt{\rho}(\xi + \xi_0)), \quad (47)$$

then substituting equation (46) or equation (47) and equation (33) in equation (2), the following solutions of equation (1) is obtained

$$u(x, t) = e^{i(-kx + \omega t + \theta_0)} \sqrt{a_0 + a_1} \sqrt{\rho} \tan(\sqrt{\rho}(x - vt + \xi_0)), \quad (48)$$

and

$$u(x, t) = e^{i(-kx + \omega t + \theta_0)} \sqrt{a_0 - a_1} \sqrt{\rho} \cot(\sqrt{\rho}(x - vt + \xi_0)), \quad (49)$$

where $\xi_0$ is the integration constant.
4. Some graphical representations

In this part of the paper, the application of the results obtained above are illustrated. Figures 1–5 are the graphical representation of equation (41). By varying the parameters $k, a_1, a, c, v, \omega$, one arrives at graphic representations well known in LHMs and optical fiber from the different graphical representations above, the solitons solutions (dark, bright) and other solutions obtained by the simple equation method. The results obtained are comparable to those well known in [18, 24].

5. summary

In this study, the authors apply successfully three different methods namely: csch function method, the $\exp(-\phi(\xi))$-Expansion method and simplest equation method to construct soliton solutions and other
solutions to the nonlinear Schrödinger equation (1). The results obtained are dark, bright and singular 1-soliton solutions. Note that the first two integrations failed to find known solitons. In the future, this model can be studied from a different perspectives. Subsequently, the model will be consider perturbation terms and spatio-temporal dispersion. Certainly abundant 1-solitons solutions and other solutions will be obtained. These results will be later disposable.

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