Noncommuting Coordinates in the Landau problem

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Abstract
Basic ideas about noncommuting coordinates are summarized, and then coordinate noncommutativity, as it arises in the Landau problem, is investigated. I review a quantum solution to the Landau problem, and evaluate the coordinate commutator in a truncated state space of Landau levels. Restriction to the lowest Landau level reproduces the well known commutator of planar coordinates. Inclusion of a finite number of Landau levels yields a matrix generalization.

1 Introduction
Because of its relevance to string theory, the idea of noncommuting spatial coordinates has gained much attention recently, but the idea actually predates string theory. Coordinate noncommutativity, defined by the equation

\[ [x^i, x^j] = i\theta^{ij} \]  

where \( \theta^{ij} \) is a constant, anti-symmetric two-index object, implies a coordinate uncertainty relation, resulting in a discretization of space itself (similar to the familiar quantum phase space) and the elimination of spatial singularities. The idea was suggested by Heisenberg in the 1930s as a way to eliminate divergences in quantum field theory arising from the assumption of point interactions between fields and matter, and the first paper on the subject appeared some years later [1]. Since then, much attention has been given to the study of quantum field theory on noncommutative spaces. Because quantum mechanics is just the one-particle nonrelativistic sector of quantum field theory, it is also relevant to study the quantum mechanics of particles in...
noncommutative spaces, and to understand the transition between the commutative and noncommutative regimes.

There exists a well known phenomenological realization of noncommuting coordinates in the realm of quantum mechanics: a charged particle in an external magnetic field so strong that projection to the lowest Landau level is justified. A charged particle in an external magnetic field is effectively confined to a two-dimensional space perpendicular to the field, which becomes noncommuting when motion is projected onto the lowest Landau level. It is interesting to note that this example is similar to the instances of noncommuting coordinates which arise in string theory in that both are characterized by the presence of a strong background magnetic-like field. Because the lowest Landau level is the only physically realized example of noncommuting coordinates, it should be useful to our understanding of noncommutative spaces in physics to know precisely how noncommutativity arises in this system. To this end, a quantum solution to the Landau problem is reviewed, and then the coordinate commutator is calculated after a projection to a truncated space of Landau levels.

2 The Landau Problem

We consider a charged \( e \) and massive \( m \) particle in a constant magnetic field \( B \), which we chose to point along the \( z \) axis without loss of generality. This is called the Landau problem because the eigenstates and eigenvalues were investigated for the first time by Landau. The Lagrangian describing the particle’s motion is

\[
L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z). \tag{2}
\]

From this, we determine the Hamiltonian for the particle in the usual way, with the result

\[
H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 \tag{3}
\]

where the canonical momentum \( \vec{p} \) of the particle is no longer the usual \( m\vec{v} \) but is equal to \( m\vec{v} + \frac{e}{c}\vec{A} \). The Hamiltonian is thus simply \( H = \frac{1}{2}m\vec{v}^2 \), which is what we expect since the magnetic field does no work and thus cannot contribute to the energy.

To solve the eigenstates and eigenenergies of this problem, we must prescribe a vector potential \( \vec{A} \), satisfying \( \nabla \times \vec{A} = B\hat{z} \). We chose \( \vec{A} = (0, XB, 0) \). Substituting this gauge choice into the Hamiltonian, we have

\[
H = \frac{1}{2m}(p_x^2 + p_y^2 + (p_y - \frac{eXB}{c})^2). \tag{4}
\]
Because $H$ commutes with both $p_z$ and with $p_y$, we can write the eigenvalue equations $p_z |\Psi\rangle = \hbar k_z |\Psi\rangle$ and $p_y |\Psi\rangle = \hbar k_y |\Psi\rangle$, where $|\Psi\rangle$ denotes the eigenstates of the full Hamiltonian, and $\hbar k_z$ and $\hbar k_y$ are the eigenvalues of their respective momentum operators. Using these eigenvalues, the Hamiltonian can be written as

$$H = \frac{1}{2m}(\hbar k_z)^2 + \frac{1}{2m} \left( p_x^2 + \left( \frac{eB}{c} \right)^2 \left( x - \frac{e\hbar k_y}{eB} \right) \right).\tag{5}$$

From now on, motion in the $z$ direction is suppressed since it is not quantized, and only planar motion in the $x$-$y$ plane is retained. The second term in the Hamiltonian is just a shifted harmonic oscillator with angular frequency $\omega = \frac{eB}{cm}$. The eigenstates $|\Psi\rangle$ are thus labeled by the number of oscillator quanta $n$ and by $k_y$. The associated eigenenergies are

$$E_n = \hbar \frac{eB}{mc} \left( n + \frac{1}{2} \right).\tag{6}$$

$k_y$ is free, so each $n$ indexes an energy eigenstate, called a Landau level, which is infinitely degeneracy in $k_y$. Note that adjacent Landau levels are separated by energy $\hbar \frac{eB}{mc}$. The eigenfunctions in coordinate space are

$$\langle x, y|n, k_y\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ik_y y} \phi_n(x - \frac{e\hbar k_y}{eB})\tag{7}$$

where $\phi_n$ are the normalized harmonic oscillator wavefunctions. From now on, we let $k \equiv \hbar k_y$ and label the eigenstates $|n, k\rangle$.

### 3 Projection to the lowest Landau level

Since the separation between the states $|n, k\rangle$ is $\mathcal{O}(B/m)$, if the magnetic field is strong, only the lowest Landau level $|0, k\rangle$ is relevant. The higher states are essentially decoupled to infinity. The large $B$ limit is equivalent to the small $m$ limit. So the Lagrangian can be modified to describe only the lowest Landau level by setting $m$ to zero in (2). With this modification, and the addition of a potential $V(x, y)$ to represent impurities in the plane, the Lagrangian becomes

$$L_{\text{LL}} = \frac{e}{c} B x \dot{y} - V(x, y).\tag{8}$$
This has the same form as \( L = p\dot{q} - H(p, q) \), and thus we recognize \( \frac{eB}{c}x \) and \( y \) as canonical conjugates, with the corresponding commutator:

\[
\left[ \frac{eB}{c}x, y \right] = -i\hbar
\]

\[
\Rightarrow [x, y] = -i\frac{\hbar c}{eB}
\]

So we see that by restricting the particle to the first Landau level, the space it moves in no longer obeys the standard Heisenberg algebra. This is called the “Peierls substitution” \[2\].

We can verify this result in a less heuristic fashion, by directly calculating the matrix elements of the coordinate commutator \[3\].

\[
\langle n, k | [x, y] | n', k' \rangle = \langle n, k | xy | n', k' \rangle - \langle n, k | yx | n', k' \rangle
\]

\[
= \langle n, k | xy | n', k' \rangle - \langle n', k' | yx | n, k \rangle^* \tag{12}
\]

\[
= f(nk, n'k') - f^*(n'k', nk) \tag{13}
\]

where

\[
f(nk, n'k') = \langle n, k | xy | n', k' \rangle \tag{14}\]

We evaluate \( f \) by inserting a complete set of states in the product \( xy \):

\[
f(nk, n'k') = \sum_{m,q} \langle n, k | x | m, q \rangle \langle m, q | y | n', k' \rangle. \tag{15}\]

If we restrict the particle’s world to include only the lowest Landau level, then we should only include the lowest Landau level (with its infinite degeneracy) in the intermediate state sum of this calculation. Additionally, since we’re pretending that the world is confined to the lowest Landau level, it only makes sense to calculate this matrix element for \( n = n' = 0 \). To evaluate that element, we must calculate

\[
f(0k, 0k') = \int dq \langle 0, k | x | 0, q \rangle \langle 0, q | y | 0, k' \rangle. \tag{16}\]

To evaluate each matrix element, we explicitly write the plane wave functions and the harmonic oscillator wavefunctions. These expressions are readily evaluated by
integration over the $x$-$y$ plane.

$$\langle 0, k | x | 0, q \rangle = \int dx dy \frac{1}{\sqrt{2\pi\hbar}} e^{-iky/\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{iqy/\hbar} e^{-m\omega (x-ck)^2} e^{-m\omega (x-cq)^2}$$

$$= \delta(k-q) \frac{cq}{eB} \tag{17}$$

$$\langle 0, q | y | 0, k' \rangle = \int dx dy \frac{1}{\sqrt{2\pi\hbar}} e^{-i\omega y/\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{i\omega y/\hbar} e^{-m\omega (x-ck')^2} e^{-m\omega (x-cq)^2}$$

$$= i\hbar \delta'(q-k') \tag{18}$$

where the prime denotes differentiation with respect to the argument.

The commutator can then be obtained:

$$\langle 0, k | [x, y] | 0, k' \rangle = -i\frac{\hbar e}{eB} \langle 0, k | 0, k' \rangle \tag{21}$$

which is consistent with (10).

\section{4 Projection to higher Landau levels}

One could now pretend that the world is restricted to the lowest $N+1$ Landau levels ($n = 0, 1, ..., N$). In this case, we cannot obtain the commutator $[x, y]$ heuristically by modifying the Lagrangian and reading off a canonical pair, but our method of explicitly calculating the relevant matrix elements is still valid. To evaluate $f$ we include the lowest $N+1$ levels (each with its infinite degeneracy) in the intermediate state sum.

$$f(nk, n'k') = \sum_{m=0}^{N} \int dq \langle n, k | x | m, q \rangle \langle m, q | y | n', k' \rangle. \tag{22}$$

This is defined for $n$ and $n'$ less than or equal to $N$. Each matrix element is evaluated by explicitly writing the coordinate space plane wavefunctions, and representing the harmonic oscillator wavefunctions as $\phi_n(x)$. The expressions are simplified by exploiting the orthonormality of the harmonic oscillator wavefunctions and integrating
over the plane.

\[
\langle n, k | x | m, q \rangle = \int dx dy \frac{1}{\sqrt{2\pi\hbar}} e^{-iky/\hbar} \phi_n(x - \frac{ck}{eB} x) \\
\frac{1}{\sqrt{2\pi\hbar}} e^{iqy/\hbar} \phi_m(x - \frac{cq}{eB})
\]

(23)

= \delta(q - k) \delta_{nm} \frac{ck}{eB} + \delta(q - k) \langle n | x | m \rangle

(24)

\[
\langle m, q | y | n', k' \rangle = \int dx dy \frac{1}{\sqrt{2\pi\hbar}} e^{-iqy/\hbar} \phi_n(x - \frac{cq}{eB}) y \\
\frac{1}{\sqrt{2\pi\hbar}} e^{ik'y/\hbar} \phi_{n'}(x - \frac{ck'}{eB})
\]

(25)

= \hbar i \delta(q - k') \delta_{nn'} - \frac{c}{eB} \delta(q - k') \langle n' | p | m \rangle

(26)

By evaluating the sums and integrals of (22), one can find that the matrix elements of the coordinate commutator vanish unless \( n = n' = N \). For this case, we get the result

\[
\langle N, k | [x, y] | N, k' \rangle = -i\hbar \frac{c}{eB} (N + 1) \delta(k - k').
\]

(27)

So, for example, the matrix representation of the coordinate commutator for a particle confined to the two lowest Landau levels is

\[
[x, y] = \begin{pmatrix} 0 & 0 \\ 0 & -2i\hbar \frac{c}{eB} \end{pmatrix}
\]

(28)

If we expanded our world to include another Landau level, we would find that the matrix element \( \langle 2, k | [x, y] | 2, k' \rangle \) vanishes because of the additional \( m = 2 \) term in the sum in (22). Thus we have shown that the phenomena of noncommutative space is not specific to the lowest Landau level but can be obtained by projecting to an arbitrary finite number of Landau levels. [4]

## 5 Motion in the symmetric gauge

By choosing a different gauge, we can verify this result, as well as come upon it much more elegantly. Substituting the symmetric gauge \( \vec{A} = \frac{B}{2} (-y, x) \) into the Hamiltonian describing the planar motion, we find

\[
H = \frac{1}{2m} \left( (p_x + \frac{eB}{2c} y)^2 + (p_y - \frac{eB}{2c} x)^2 \right)
\]

(29)

= \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2m} \left( \frac{eB}{2mc} \right)^2 (x^2 + y^2) - \frac{eB}{2mc} L.
\]

(30)
where $L = xp_y - yp_x$ is the angular momentum in the $x$-$y$ plane. In this gauge, the system looks like a two-dimensional harmonic oscillator with an additional interaction $-\frac{eB}{2mc} L$. Since we are dealing here only with harmonic oscillator wavefunctions and not plane waves, it is clear that this problem will most easily solved by introducing harmonic oscillator creation and annihilation operators. To this end, we define:

$$a = \frac{1}{2} \sqrt{\frac{eB}{2\hbar c}} (x - iy) + \frac{i}{2} \sqrt{\frac{2c}{eB\hbar}} (p_x - ip_y)$$

$$b = \frac{1}{2} \sqrt{\frac{eB}{2\hbar c}} (x + iy) + \frac{i}{2} \sqrt{\frac{2c}{eB\hbar}} (p_x + ip_y).$$

satisfying the following commutation relations

$$[a, a^\dagger] = [b, b^\dagger] = 1$$

with all other commutators vanishing. $(a, a^\dagger)$ and $(b, b^\dagger)$ are two pairs of independent harmonic oscillator operators. We can now write $L$ and $H$ in terms of these operators:

$$L = \hbar (a^\dagger a - b^\dagger b)$$

$$H = \frac{\hbar eB}{2mc} (a^\dagger a + b^\dagger b + 1) - \hbar \frac{eB}{2mc} (a^\dagger a - b^\dagger b)$$

$$= \frac{\hbar eB}{mc} (b^\dagger b + \frac{1}{2}).$$

Eigenstates of the Hamiltonian are labeled by the number $j$ of excitation quanta of the oscillator $a$, and the number $n$ of excitation quanta of the oscillator $b$:

$$a^\dagger a |n, j\rangle = j |n, j\rangle,$$

$$b^\dagger b |n, j\rangle = n |n, j\rangle.$$
Then the coordinate commutator is
\[
[x, y] = \frac{i \hbar c}{2eB}[\alpha + \alpha^\dagger, \alpha - \alpha^\dagger] \tag{41}
\]
\[
= -i \frac{\hbar c}{eB}[\alpha, \alpha^\dagger]. \tag{42}
\]
It we evaluate this exactly, we find that the commutator vanishes as expected.

Now using this new basis we can explicitly evaluate matrix elements for an incomplete state space. We will denote the eigenstates $|n, j\rangle$ explicitly as product states: $|n\rangle|j\rangle$. To determine the commutator, we need to evaluate the matrix elements:
\[
\langle n|\langle j|\alpha\alpha^\dagger|l\rangle|j'\rangle|n'\rangle - \langle n|\langle j|\alpha^\dagger\alpha|l\rangle|j'\rangle|n'\rangle \equiv (1) - (2). \tag{43}
\]
We evaluate each term separately by inserting intermediate states.
\[
(1) = \sum_{m=0, l=0}^{m=N, l=\infty} \langle n|\langle j|\alpha|l\rangle|m\rangle\langle m|\langle l|\alpha^\dagger|j'\rangle|n'\rangle \tag{44}
\]
\[
(2) = \sum_{m=0, l=0}^{m=N, l=\infty} \langle n|\langle j|\alpha^\dagger|l\rangle|m\rangle\langle m|\langle l|\alpha|j'\rangle|n'\rangle \tag{45}
\]
Summing over $l$, one readily finds
\[
(1) - (2) = \delta_{nn'}\delta_{jj'}(1 + \sum_{m=1}^{N+1} m\delta_{nm} - \sum_{m=0}^{N-1} (m + 1)\delta_{nm}) \tag{46}
\]
Since $m$ never gets larger than $N$, we can write this as
\[
= \delta_{nn'}\delta_{jj'}(1 + \sum_{m=1}^{N} m\delta_{nm} - \sum_{m=0}^{N-1} (m + 1)\delta_{nm}) \tag{47}
\]
\[
= \delta_{nn'}\delta_{jj'}(1 + N\delta_{nN} - \sum_{m=0}^{N-1} \delta_{nm}) \tag{48}
\]
Let us analyze this expression. It clearly vanishes for off diagonal elements. For $n = n' \neq N$, it also vanishes because the sum will yield unity since $n$ is necessarily less than $N$. For $n = n' = N$, we obtain our earlier result:
\[
\langle N|\langle j|[x, y]|j'\rangle|N\rangle = -i \frac{\hbar c}{eB}(N + 1)\langle j|j'\rangle. \tag{49}
\]
Suppose we had let $N \to \infty$, then (46) would have been

$$= \delta_{nn'}\delta_{jj'}(1 + \sum_{m=1}^{\infty} m\delta_{nm} - \sum_{m=0}^{\infty} (m + 1)\delta_{nm})$$

$$= \delta_{nn'}\delta_{jj'}(1 - \delta_{n0} - \sum_{m=1}^{\infty} \delta_{nm})$$

which clearly vanishes for any choice of $n$. Noncommutativity in the Landau problem is clearly associated with a truncated state space.

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References

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