QUANTUM MONTGOMERY IDENTITY AND SOME QUANTUM INTEGRAL INEQUALITIES

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Abstract. We discover a new version of the celebrated Montgomery identity via quantum integral operators and establish certain quantum integral inequalities of Ostrowski type by using this identity. Relevant connections of the results obtained in this work with those deduced in earlier published papers are also considered.

1. Introduction

The following inequality is named the Ostrowski type inequality [25].

**Theorem 1.** [9]. Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ and $f' \in L[a,b]$. If $|f'(x)| < M$ where $x \in [a,b]$, then the following inequality holds:

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| 
\leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{2} \right]
$$

for all $x \in [a,b]$.

To prove the Ostrowski type inequality above, the following famous Montgomery identity is very useful, see [21]:

$$
f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \int_a^x t-a \frac{f'(t) \, dt}{b-a} + \int_x^b \frac{t-b}{b-a} f'(t) \, dt,
$$

where $f(x)$ is a continuous function on $[a,b]$ with a continuous first derivative in $(a,b)$.

By changing variable, the Montgomery identity (1.2) can be expressed in the following way:

$$
f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt = (b-a) \int_0^1 K(t) f'(tb + (1-t)a) \, dt,
$$

where

$$
K(t) = \begin{cases} 
  t, & t \in \left[0, \frac{x-a}{b-a}\right], \\
  t-1, & t \in \left(\frac{x-a}{b-a}, 1\right].
\end{cases}
$$

A number of different identities of the Montgomery type were investigated and many Ostrowski type inequalities were obtained by using these identities. For example, through the framework of Montgomery’s identity, Cerone and Dragonir [3] developed a systematic study which produced some novel inequalities. By introducing some parameters, Budak and Sarikaya [7] as well as Özdemir et al. [26] established the generalized Montgomery-type identities for differentiable mappings and certain generalized Ostrowski-type inequalities, respectively. Then in [1], Aljinović...
presented another simpler generalization of the Montgomery identity for fractional integrals by utilizing the weighted Montgomery identity. Further the generalized Montgomery identity involving the Ostrowski type inequalities in question with applications to local fractional integrals can be found in [27]. For more related results considering the different Montgomery identities, The interested reader is referred, for example, to [2, 13, 14, 19, 28] and the references therein.

However, to the best of our knowledge, the quantum Montgomery type identity has not obtained so far. This paper aims to investigate, by setting up a quantum Montgomery identity and by the help of this identity, some new quantum integral inequalities such as Ostrowski type, midpoint type, etc. We shall deal with mappings whose derivatives in absolute value are quantum differentiable convex mappings.

Throughout this paper, let $0 < q < 1$ be a constant. It is known that quantum calculus constructs in a quantum geometric set. That is, if $qx \in A$ for all $x \in A$, then the set $A$ is called quantum geometric.

Suppose that $f(t)$ is an arbitrary function defined on the interval $[0, b]$. Clearly, for $b > 0$, the interval $[0, b]$ is a quantum geometric set. The quantum derivative of $f(t)$ is defined with the following expression:

$$ D_q f(t) := \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \neq 0, $$

$$ D_q f(0) := \lim_{t \to 0} D_q f(t). $$

Note that

$$ \lim_{q \to 1^{-}} D_q f(t) = \frac{df(t)}{dt}, $$

if $f(t)$ is differentiable.

The definite quantum integral of $f(t)$ is defined as:

$$ \int_{0}^{b} f(t) \, d_q t = (1 - q) \sum_{n=0}^{\infty} q^n f(q^n b) $$

and

$$ \int_{c}^{b} f(t) \, d_q t = \int_{0}^{b} f(t) \, d_q t - \int_{c}^{b} f(t) \, d_q t, $$

where $0 < c < b$, see [3, 12].

Note that if the series in right-hand side of (1.6) is convergence, then $\int_{0}^{b} f(t) \, d_q t$ is exist, i.e., $f(t)$ is quantum integrable on $[0, b]$. Also, provided that if $\int_{0}^{b} f(t) \, dt$ converges, then one has

$$ \lim_{q \to 1^{-}} \int_{0}^{b} f(t) \, d_q t = \int_{0}^{b} f(t) \, dt. \; \; (3, \text{ page 6}) $$

These definitions are not sufficient in establishing integral inequalities for a function defined on an arbitrary closed interval $[a, b] \subset \mathbb{R}$. Due to this fact, Tariboon and Ntouyas in [30, 31] improved these definitions as follows:

**Definition 1.** [31, 30]. For a continuous function $f : [a, b] \to \mathbb{R}$, the $q$-derivative of $f$ at $t \in [a, b]$ is characterized by the expression:

$$ aD_q f(t) = \frac{f(t) - f(qt + (1 - q)a)}{(1 - q)(t - a)}, \quad t \neq a, $$

$$ aD_q f(a) = \lim_{t \to a} aD_q f(t). $$
The function \( f \) is said to be \( q \)-differentiable on \([a, b]\), if \( _aD_q f(t) \) exists for all \( t \in [a, b] \).

Clearly, if \( a = 0 \) in (1.9), then \( _aD_q f(t) = D_q f(t) \), where \( D_q f(t) \) is familiar quantum derivatives given in (1.5).

**Definition 2.** [31, 30]. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then the quantum definite integral on \([a, b]\) is delineated as

\[
\int_a^b f(t) \, _aD_q t = (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n) a)
\]

and

\[
\int_a^b f(t) \, _aD_q t = \int_a^b f(t) \, _aD_q t - \int_a^c f(t) \, _aD_q t,
\]

where \( a < c < b \).

Clearly, if \( a = 0 \) in (1.10), then

\[
\int_0^b f(t) \, _aD_q t = \int_0^b f(t) \, d_q t,
\]

where \( f_0^b f(t) \, d_q t \) is familiar definite quantum integrals on \([0, b]\) given in (1.6).

Definition 1 and Definition 2 have actually developed previous definitions and have been widely used for quantum integral inequalities. There is a lot of remarkable papers about quantum integral inequalities based on these definitions, including Kunt et al. [17] in the study of the quantum Hermite–Hadamard inequalities for mappings of two variables considering convexity and quasi-convexity on the coordinates, Noor et al. [22] in quantum Ostrowski-type inequalities for quantum differentiable convex mappings, quantum estimates for Hermite–Hadamard inequalities via convexity and quasi-convexity, quantum analogues of Iyengar type inequalities for some classes of preinvex mappings, as well as Tunc et al. [32] in the Simpson-type inequalities for convex mappings via quantum integral operators. For more results related to the quantum integral operators, the interested reader is directed, for example, to [5, 16, 18, 29, 34] and the references cited therein.

In [6], Alp et al. proved the following inequality named quantum Hermite–Hadamard type inequality. Also in [33], Zhang et al. proved the same inequality with the fewer assumptions and shorter method.

**Theorem 2.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function with \( 0 < q < 1 \). Then we have

\[
\int_a^b f\left(\frac{qa + b}{1 + q}\right) \leq \frac{1}{b - a} \int_a^b f(t) \, _aD_q t \leq \frac{qf(a) + f(b)}{1 + q}.
\]

2. **Main Results**

Firstly, we shall discuss the assumptions of being continuous of the function \( f(t) \) in the Definition 1 and the Definition 2. Also, in this conditions, we want to discuss if the similar cases with (1.6), (1.8) and (1.9) could be exist.

Considering the Definition 1 it is unnecessary that the function \( f(t) \) is being continuous on \([a, b]\). Indeed, for all \( t \in [a, b] \), \( qt + (1 - q) a \in [a, b] \) and \( f(t) - f(qt + (1 - q) a) \in \mathbb{R} \). It means that \( \frac{(t - qt + (1 - q) a)}{(1 - q)(t - a)} \in \mathbb{R} \) exists for all \( t \neq a \), i.e., the Definition 1 should be as follows:

**Definition 3.** (Quantum derivative on \([a, b]\)) An arbitrary function \( f(t) \) defined on \([a, b]\) is called quantum differentiable on \((a, b)\) with the following expression:

\[
_aD_q f(t) \, \in \mathbb{R}, \ t \neq a
\]
and quantum differentiable on \( t = a \), if the following limit exists:
\[
_a D_q f(a) = \lim_{t \to a} \frac{df(t)}{dt}
\]

**Lemma 1.** (Similar case with (1.5)) Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function. Then we have

\[
(2.2) \quad \lim_{q \to 1^-} \int_a^b f(t) \frac{df}{dt} \, dt = \int_a^b f(t) \, dt.
\]

**Proof.** If \( f(t) \) is differentiable on \([a, b]\), then we have that

\[
\lim_{q \to 1^-} \int_a^b f(t) \frac{df}{dt} \, dt = \lim_{q \to 1^-} \frac{f(t) - f(qt + (1 - q)a)}{(1 - q)(t - a)}
\]

Thus, for \( q \to 1^- \), we have

\[
\int_a^b f(t) \frac{df}{dt} \, dt = \lim_{q \to 1^-} \int_a^b f(t) \frac{df}{dt} \, dt = \int_a^b f(t) \, dt
\]

\[
= \lim_{q \to 1^-} \int_a^b f(t) \frac{df}{dt} \, dt = \int_a^b f(t) \, dt.
\]

\[\square\]

Considering Definition 2, it is unnecessary that the function \( f(t) \) is being continuous on \([a, b]\). Undoubtedly, it is not difficult to construct an example for a discontinuous function that is quantum integrable on \([a, b]\).

**Example 1.** Let \( 0 < q < 1 \) be a constant, and we define

\[ A := \{q^n(2 + (1 - q^n)(-1)) : n = 0, 1, 2, \ldots \} \subset [-1, 2], \]

\[ f : [-1, 2] \to \mathbb{R} \text{ and } f(t) := \begin{cases} 
1, & t \in A, \\
0, & t \in [-1, 2] \setminus A. 
\end{cases} \]

Clearly, the function \( f(t) \) is not continuous on \([-1, 2]\).

On the other hand

\[ \int_{-1}^2 f(t) \, dt = (1 - q)(2 - (-1)) \sum_{n=0}^{\infty} q^n f(2^n + (1 - q^n)(-1)) = 3(1 - q) \sum_{n=0}^{\infty} q^n = 3 \frac{1}{1 - q} = 3, \]

i.e., the function \( f(t) \) is quantum integrable on \([-1, 2]\).

Similarly, the Definition 2 should be described in the following way.

**Definition 4.** (Quantum definite integral on \([a, b]\)) Let \( f : [a, b] \to \mathbb{R} \) be an arbitrary function. Then the quantum definite integral on \([a, b]\) is delineated as

\[
(2.3) \quad \int_a^b f(t) \, dt = (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(2^n b + (1 - q^n)(-1))
\]

If the series in right hand-side of (2.3) is convergent, then \( \int_a^b f(t) \, dt \) is exist, i.e., \( f(t) \) is quantum integrable on \([a, b]\).
Lemma 2. (Similar case with (1.8)) Let \( f : [a, b] \to \mathbb{R} \) be an arbitrary function. Then, provided that if \( f_a^b \) converges, then we have

\[
\lim_{q \to 1^-} \int_a^b f(t) \, a_dq t = \int_a^b f(t) \, dt.
\]

Proof. If \( f_a^b \) converges, then \( \int_0^1 f(tb + (1-t)a) \, dt \) also converges. Using (1.8), we have that

\[
\lim_{q \to 1^-} \int_a^b f(t) \, a_dq t = \lim_{q \to 1^-} \left[ (1-q) (b-a) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n) a) \right]
\]

\[
= (b-a) \lim_{q \to 1^-} \int_0^1 f(tb + (1-t)a) \, a_dq t
\]

\[
= (b-a) \int_0^1 f(tb + (1-t)a) \, dt
\]

\[
= \int_a^b f(t) \, dt.
\]

\[\square\]

We next present a important quantum Montgomery identity, which is similar with the identity in (1.8).

Lemma 3. (Quantum Montgomery identity) Let \( f : [a, b] \to \mathbb{R} \) be an arbitrary function with \( \quad D_q f \) is quantum integrable on \([a, b]\), then the following quantum identity holds:

\[
f(x) - \frac{1}{b-a} \int_a^b f(t) \, a_dq t = (b-a) \int_0^1 K_q(t) \, aD_q f(tb + (1-t) a) \, a_dq t,
\]

where

\[
K_q(t) = \begin{cases} \frac{qt}{b-a}, & t \in \left[0, \frac{b-a}{b-a}\right], \\ \frac{qt}{b-a} - 1, & t \in \left[\frac{b-a}{b-a}, 1\right]. \end{cases}
\]

Proof. By the Definition 3 \( f(t) \) is quantum differentiable on \((a, b)\) and \( \quad D_q f \) is exist. Since \( \quad D_q f \) is quantum integrable on \([a, b]\), by the Definition 3 the quantum integral for the right-side of (2.5) is exist. Let us start calculating the integral for the right-side of (2.5). With the help of (2.1) and (2.3), we have that

\[
(b-a) \int_0^1 K_q(t) \, aD_q f(tb + (1-t) a) \, a_dq t
\]

\[
= (b-a) \left[ \int_0^{\infty} \frac{qt}{b-a} aD_q f(tb + (1-t) a) \, a_dq t + \int_0^1 (qt - 1) aD_q f(tb + (1-t) a) \, a_dq t \right]
\]

\[
= (b-a) \left[ \int_0^{\infty} \frac{qt}{b-a} aD_q f(tb + (1-t) a) \, a_dq t + \int_0^1 (qt - 1) aD_q f(tb + (1-t) a) \, a_dq t - \int_0^{\infty} (qt - 1) aD_q f(tb + (1-t) a) \, a_dq t \right]
\]

\[
= (b-a) \left[ \int_0^{\infty} (qt - 1) aD_q f(tb + (1-t) a) \, a_dq t + \int_0^{\infty} aD_q f(tb + (1-t) a) \, a_dq t \right]
\]
\[
(b - a) \left[ \int_0^1 q f (tb + (1 - t) a) \, dq_t - \int_0^1 q f (tb + (1 - t) a) \, dq_t \right] + \int_0^1 q f (tb + (1 - t) a) \, dq_t = \left(1 - q\right) \sum_{n=0}^\infty q^n f (q^n b + (1 - q^n) a) - \sum_{n=0}^\infty q^n f (q^n b + (1 - q^n) a) - \sum_{n=0}^\infty f (q^n b + (1 - q^n) a)
\]

\[
= \frac{1}{1-q} \left[ q \left( \sum_{n=0}^\infty q^n f (q^n b + (1 - q^n) a) - \sum_{n=0}^\infty q^n f (q^n b + (1 - q^n) a) - \sum_{n=0}^\infty f (q^n b + (1 - q^n) a) \right) + \sum_{n=0}^\infty f \left( q^n b + (1 - q^n) a \right) \right]
\]

\[
= q \left( \int_0^1 q f (tb + (1 - t) a) \, dq_t - \int_0^1 q f (tb + (1 - t) a) \, dq_t \right) + \int_0^1 q f (tb + (1 - t) a) \, dq_t
\]

which completes the proof.
Remark 1. If one takes limit $q \to 1^-$ on the Quantum Montgomery identity in (2.5), one has the Montgomery identity in (1.3).

The following calculations of quantum definite integrals are used in next result:

\[ \int_0^{\frac{x-a}{b-a}} q \, dt = q (1-q) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( \frac{x-a}{b-a} \right)^n \]

\[ = q (1-q) \left( \frac{x-a}{b-a} \right)^2 \frac{1}{1-q^2} \]

\[ = \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2 , \]

\[ \int_0^{\frac{x-a}{b-a}} q^2 \, dt = q (1-q) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( \frac{x-a}{b-a} \right)^n \]

\[ = q (1-q) \left( \frac{x-a}{b-a} \right)^3 \frac{1}{1-q^3} \]

\[ = \frac{q}{1+q+q^2} \left( \frac{x-a}{b-a} \right)^3 , \]

\[ \int_0^{\frac{1}{1-q}} (1-qt) \, dt = \int_0^{\frac{x-a}{b-a}} (1-qt) \, dt \]

\[ = \left[ (1-q) \sum_{n=0}^{\infty} q^n \left( 1-qq^n \right) \right] \]

\[ - \left( 1-q \right) \frac{x-a}{b-a} \sum_{n=0}^{\infty} q^n \left( 1-qq^n \frac{x-a}{b-a} \right) \]

\[ = \left[ (1-q) \left( \frac{1}{1-q} - \frac{x-a}{1-q} \left( \frac{1}{1-q} - \frac{x-a}{1-q} \right) \right) \right] \]

\[ = \frac{1}{1+q} - \frac{x-a}{b-a} \left( 1 - \frac{q}{1+q} \frac{x-a}{b-a} \right) \]

\[ = \frac{1}{1+q} \left( 1 - \frac{b-x}{b-a} \right) \left( \frac{1}{1+q} + \frac{q}{1+q} \frac{b-x}{b-a} \right) \]

\[ = \frac{1}{1+q} \left( 1 - \frac{b-x}{b-a} \right) \left( \frac{1}{1+q} + \frac{q}{1+q} \frac{b-x}{b-a} \right) \]

\[ = \left[ \frac{1}{1+q} \frac{x-a}{b-a} - \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2 \right] = \frac{q}{1+q} \left( \frac{b-x}{b-a} \right)^2 , \]
and

\begin{align}
\int_{\frac{x-a}{b-a}}^{1} (t - qt^2) \, dq \, dt \\
= \int_{0}^{1} (t - qt^2) \, dq \, dt - \int_{0}^{\frac{x-a}{b-a}} (t - qt^2) \, dq \, dt \\
= \left[ (1 - q) \sum_{n=0}^{\infty} q^n (q^n - qq^{2n}) \right] \\
= \left[ (1 - q) \left( \frac{1}{t^{q+1} - \frac{q}{1+t^{q+1}}} \right) \right] \\
= \left[ (1 - q) \left( \frac{1}{t^{q+1} - \frac{q}{1+t^{q+1}}} \right) \right] \\
= \left[ \frac{1}{(1+q)(1+q+q^2)} \left( \frac{x-a}{b-a} \right)^2 \right].
\end{align}

Let us introduce some new quantum integral inequalities by the help of quantum power mean inequality and Lemma 3.

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be an arbitrary function with \( a_D f \) is quantum integrable on \( [a, b] \). If \( |a_D f| \), \( r \geq 1 \) is a convex function, then the following quantum integral inequality holds:

\begin{align}
(2.10) \quad & \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dq \right| \\
\leq & \quad (b-a) \left[ K_1 \left( a, b, x, q \right) \left| a_D f(a)^r K_2 \left( a, b, x, q \right) \right| + K_3 \left( a, b, x, q \right) \left| a_D f(b)^r K_4 \left( a, b, x, q \right) \right| \right]^{\frac{1}{r}}
\end{align}

for all \( x \in [a, b] \), where

\begin{align}
K_1 \left( a, b, x, q \right) & = \int_{\frac{x-a}{b-a}}^{1} qt \, dq \, dt = \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2, \\
K_2 \left( a, b, x, q \right) & = \int_{0}^{\frac{x-a}{b-a}} qt^2 \, dq \, dt = \frac{q}{1+q+q^2} \left( \frac{x-a}{b-a} \right)^3, \\
K_3 \left( a, b, x, q \right) & = \int_{\frac{x-a}{b-a}}^{1} qt - qt^2 \, dq \, dt = K_1 \left( a, b, x, q \right) - K_2 \left( a, b, x, q \right), \\
K_4 \left( a, b, x, q \right) & = \int_{\frac{x-a}{b-a}}^{1} (1 - qt) \, dq \, dt = \frac{q}{1+q} \left( \frac{b-x}{b-a} \right)^2, \\
K_5 \left( a, b, x, q \right) & = \int_{\frac{x-a}{b-a}}^{1} (t - qt^2) \, dq \, dt = \left[ \frac{1}{(1+q)(1+q+q^2)} \left( \frac{x-a}{b-a} \right)^2 \right].
\end{align}
and

\[ K_6(a, b, x, q) = \int_{\frac{a}{b-a}}^{1} (1 - qt - t^2) \, \partial t \quad \text{is} \quad K_4(a, b, x, q) - K_5(a, b, x, q). \]

**Proof.** Using convexity of \( |a D_q f|^r \), we have that

\[ |a D_q f (tb + (1 - t) a)|^r \leq t |a D_q f (a)|^r + (1 - t) |a D_q f (b)|^r. \]

By using Lemma 3, quantum power mean inequality and (2.11), we have that

\[ (2.12) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \partial t \right| \]

\[ \leq (b - a) \int_0^1 |K_q(t)| \left| a D_q f (tb + (1 - t) a) \right| \partial t \]

\[ \leq (b - a) \left[ \int_{\frac{a}{b-a}}^{1} qt \left| a D_q f (tb + (1 - t) a) \right| \partial t \right. \]

\[ + \left. \int_{\frac{a}{b-a}}^{1} (1 - qt) \left| a D_q f (tb + (1 - t) a) \right| \partial t \right] \]

\[ \leq (b - a) \left[ \left( \int_{\frac{a}{b-a}}^{1} qt \partial t \right)^{1 - \frac{1}{r}} \right. \]

\[ \times \left( \int_{\frac{a}{b-a}}^{1} qt \left| a D_q f (tb + (1 - t) a) \right|^r \partial t \right)^{\frac{1}{r}} \]

\[ \leq (b - a) \left[ \left( \int_{\frac{a}{b-a}}^{1} qt \partial t \right)^{1 - \frac{1}{r}} \right. \]

\[ \times \left( \int_{\frac{a}{b-a}}^{1} qt \left[ t |a D_q f (a)|^r + (1 - t) |a D_q f (b)|^r \right] \partial t \right)^{\frac{1}{r}} \]

\[ \leq (b - a) \left[ \left( \int_{\frac{a}{b-a}}^{1} qt \partial t \right)^{1 - \frac{1}{r}} \right. \]

\[ \times \left( \int_{\frac{a}{b-a}}^{1} qt \left[ t |a D_q f (a)|^r + (1 - t) |a D_q f (b)|^r \right] \partial t \right)^{\frac{1}{r}} \]

\[ \leq (b - a) \left[ \left( \int_{\frac{a}{b-a}}^{1} qt \partial t \right)^{1 - \frac{1}{r}} \right. \]

\[ \times \left( \int_{\frac{a}{b-a}}^{1} qt \left[ t |a D_q f (a)|^r + (1 - t) |a D_q f (b)|^r \right] \partial t \right)^{\frac{1}{r}} \]

Using (2.4)–(2.9) in (2.12), we obtain the desired result in (2.10). This ends the proof. \(\square\)

**Corollary 1.** In Theorem 3, the following inequalities are held by the following assumptions:
(1) $r = 1;$

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \\
\leq (b-a) \left[ \|aD_qf(a)\| K_2(a, b, x, q) + \|aD_qf(b)\| K_3(a, b, x, q) \right],
\]

(2) $r = 1$ and $|aD_qf(x)| < M$ for all $x \in [a, b]$ (a quantum Ostrowski type inequality, see [22, Theorem 3.1]);

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \\
\leq M (b-a) \left[ K_2(a, b, x, q) + K_3(a, b, x, q) + K_5(a, b, x, q) \right] + M (b-a) \left[ K_1(a, b, x, q) + K_4(a, b, x, q) \right] \\
\leq M (b-a) \left[ \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2 + \frac{q}{1+q} \left( \frac{b-x}{b-a} \right)^2 \right] \\
\leq \frac{qM}{b-a} \left[ (x-a)^2 + (b-x)^2 \right],
\]

(3) $r = 1, |aD_qf(x)| < M$ for all $x \in [a, b]$ and $q \to 1^-$ (Ostrowski inequality [11]);

(4) $r = 1$ and $x = \frac{qa+b}{1+q}$ (a new quantum midpoint type inequality);

\[
|f \left( \frac{qa+b}{1+q} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt| \\
\leq (b-a) \left[ \|aD_qf(a)\| K_2 \left( a, b, \frac{qa+b}{1+q}, q \right) + \|aD_qf(b)\| K_3 \left( a, b, \frac{qa+b}{1+q}, q \right) \right] \\
\leq (b-a) \left[ \frac{q}{(1+q)^2(1+q+q^2)} + \frac{q^2+q^3}{(1+q)^2(1+q+q^2)} \right] \\
\leq (b-a) \left[ \frac{q^3}{(1+q)^2(1+q+q^2)} + \frac{2q}{(1+q)^2(1+q+q^2)} \right] \\
\leq (b-a) \left[ \frac{q}{(1+q)^2(1+q+q^2)} + \frac{2q}{(1+q)^2(1+q+q^2)} + \frac{2q^3+q^4+q^5}{(1+q)^2(1+q+q^2)} \right],
\]

(5) $r = 1, x = \frac{2a+b}{1+q}$ and $q \to 1^-$ (a midpoint type inequality, see [15, Theorem 2.2]);

\[
|f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \frac{(b-a)(\|f'(a)\| + \|f'(b)\|)}{8},
\]
(6) $r = 1$ and $x = \frac{a+b}{2}$ (a new quantum midpoint type inequality);

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dq(t) \right| \\
\leq (b-a) \left[ \left| a D_q f(a) \right| K_2(a, b, \frac{a+b}{2}, q) + \left| a D_q f(b) \right| K_3(a, b, \frac{a+b}{2}, q) \right] \\
\leq (b-a) \left[ \left| a D_q f(a) \right| K_5(a, b, \frac{a+b}{2}, q) + \left| a D_q f(b) \right| K_6(a, b, \frac{a+b}{2}, q) \right] \\
\leq (b-a) \left[ \left| a D_q f(a) \right| \frac{g}{8(1+q)(1+q+q^2)} + \left| a D_q f(b) \right| \frac{q+q^2+2q^3}{8(1+q)(1+q+q^2)} \right] \\
\leq (b-a) \left[ \left| a D_q f(a) \right| \frac{6}{4q+4q^2+4q^3-6} + \left| a D_q f(b) \right| \frac{6}{8(1+q)(1+q+q^2)} \right],
\]

(7) $|a D_q f(x)| < M$ for all $x \in [a, b]$ (a quantum Ostrowski type inequality, see [22] Theorem 3.1);

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dq(t) \right| \\
\leq (b-a) M \left[ K_1^{1+\frac{1}{r}}(a, b, x, q) \left| K_2(a, b, x, q) + K_3(a, b, x, q) \right| ^{\frac{1}{r}} + K_4^{1+\frac{1}{r}}(a, b, x, q) \left| K_5(a, b, x, q) + K_6(a, b, x, q) \right| ^{\frac{1}{r}} \right] \\
\leq (b-a) M \left[ K_1(a, b, x, q) + K_4(a, b, x, q) \right] \\
\leq (b-a) M \left[ \frac{q}{1+q} \left( \frac{x-a}{b-a} \right)^2 + \frac{q}{1+q} \left( \frac{b-x}{b-a} \right)^2 \right] \\
\leq \frac{qM}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{1+q} \right],
\]

(8) $x = \frac{qa+b}{1+q}$ (a new quantum midpoint type inequality);

\[
\left| f \left( \frac{qa+b}{1+q} \right) - \frac{1}{b-a} \int_a^b f(t) \, dq(t) \right| \\
\leq (b-a) \left[ K_1^{1-\frac{1}{r}}(a, b, \frac{qa+b}{1+q}, q) \left[ \left| a D_q f(a) \right| ^r K_2(a, b, \frac{qa+b}{1+q}, q) \right] ^{\frac{1}{r}} + K_4^{1-\frac{1}{r}}(a, b, \frac{qa+b}{1+q}, q) \right] \\
\leq (b-a) \left[ \left[ \frac{q}{1+q} \right] ^{1-\frac{1}{r}} \left| a D_q f(a) \right| ^r \frac{q}{(1+q)(1+q+q^2)} + \left[ \frac{q^3}{(1+q)(1+q+q^2)} \right] ^{1-\frac{1}{r}} \left| a D_q f(b) \right| ^r \frac{2q}{(1+q)(1+q+q^2)} \right],
\]
Theorem 4.

Finally, we give the following calculated quantum definite integrals used as the next Theorem 4.

\[
(9) \quad x = \frac{aq + b}{1 + q} \quad \text{and} \quad q \to 1^- \quad \text{(a midpoint type inequality, see [6 Corollary 17])};
\]

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| 
\leq (b - a) \frac{1}{2^{q + \frac{1}{2}}} \left[ \left( |f'(a)|^\tau \frac{q}{1 + q} + |f'(b)|^\tau \frac{q}{1 + q} \right) \frac{1}{\tau} \right],
\]

\[
(10) \quad x = \frac{a + b}{1 + q} \quad \text{(a new quantum midpoint type inequality)};
\]

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| 
\leq (b - a) \left[ K_1 \left( a, b, \frac{a + b}{1 + q}, q \right) + K_2 \left( a, b, \frac{a + b}{1 + q}, q \right) \right] \leq (b - a) \left( \frac{q}{4(1 + q)} \right)^{1 - \frac{1}{\tau}} \left[ \left( |f'(a)|^\tau \frac{q}{1 + q} + |f'(b)|^\tau \frac{q}{1 + q} \right) \frac{1}{\tau} \right],
\]

\[
(11) \quad x = \frac{a + qb}{1 + q} \quad \text{(a new quantum midpoint type inequality)};
\]

\[
\left| f \left( \frac{a + qb}{1 + q} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| 
\leq (b - a) \left[ K_1 \left( a, b, \frac{a + qb}{1 + q}, q \right) + K_2 \left( a, b, \frac{a + qb}{1 + q}, q \right) \right] \leq (b - a) \left( \frac{q}{(1 + q)^{r}} \right)^{1 - \frac{1}{\tau}} \left[ \left( |f'(a)|^\tau \frac{q}{1 + q} + |f'(b)|^\tau \frac{q}{1 + q} \right) \frac{1}{\tau} \right],
\]

Finally, we give the following calculated quantum definite integrals used as the next Theorem 4.

\[
(2.13) \quad \int_0^{\frac{x - a}{b - a}} t \, dq \, dt = (1 - q) \frac{1}{b - a} \sum_{n=0}^\infty q^n \left( q^n \frac{x - a}{b - a} \right) 
= (1 - q) \left( \frac{x - a}{b - a} \right)^2 \frac{1}{1 - q^2} 
= \frac{1}{1 + q} \left( \frac{x - a}{b - a} \right)^2,
\]
By using Lemma 3, quantum Hölder inequality and (2.14), the following quantum integral inequality holds:

\[ \int_a^b (1 - t) \, d_q t = \int_0^1 t \, d_q t - \int_0^{\frac{x-a}{b-a}} t \, d_q t \]

(2.15)

\[ = \frac{1}{1 + q} \left( 1 - \left( \frac{x-a}{b-a} \right)^2 \right) \]

and

(2.16)

\[ \int_a^b (1 - t) \, d_q t = \int_0^1 (1 - t) \, d_q t - \int_0^{\frac{x-a}{b-a}} (1 - t) \, d_q t \]

\[ = \frac{q}{1 + q} - \frac{x-a}{b-a} + \frac{1}{1 + q} \left( \frac{x-a}{b-a} \right)^2. \]

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be an arbitrary function with \( aD_q f \) is quantum integrable on \([a, b]\). If \( |aD_q f|^r \), \( r > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) is a convex function, then the following quantum integral inequality holds:

(2.17)

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, d_q t \right| \\
\leq (b - a) \left[ \left( \int_0^{\frac{x-a}{b-a}} q t \, d_q t \right)^{\frac{1}{p}} \right. \\
\times \left( |aD_q f(a)|^r \left[ \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2 \right] \right)^{\frac{1}{q}} \\
\left. + |aD_q f(b)|^r \left[ \frac{q}{1+q} - \frac{x-a}{b-a} + \frac{1}{1+q} \left( \frac{x-a}{b-a} \right)^2 \right] \right]^\frac{1}{p}
\]

for all \( x \in [a, b] \).

**Proof.** By using Lemma 3 quantum Hölder inequality and (2.17), we have that

(2.18)

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, d_q t \right| \\
\leq (b - a) \int_0^1 |K_q(t)| \, |aD_q f(tb + (1-t)a)| \, d_q t
\]
The authors declare that they have no competing interests.

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Conclusion

Utilizing mappings whose first derivatives absolute values are quantum differentiable convex, we establish some quantum integral inequalities of Ostrowski type in terms of the discovered quantum Montgomery identity. Furthermore, we investigate the important relevant connections between the results obtained in this work with those introduced in earlier published papers. Many sub-results can be derived from our main results by considering the special variable value for \( x \in [a, b] \), some fixed value for \( r \), as well as \( q \to 1^- \). It is worthwhile to mention that certain quantum inequalities presented in this work generalize parts of the very recent results given by Alp et al. (2018) and Noor et al. (2016). With these contributions, we hope to motivate the interested researchers to explore this fascinating field of the quantum integral inequality based on the techniques and ideas developed in this article.

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Remark 2. In Theorem 4, many different inequalities could be derived similarly to Corollary 3.

3. Conclusion

Using (2.13)-(2.10) in (2.18), we obtain the desired result in (2.17). This ends the proof.

\[
\begin{align*}
\leq & (b-a) \left[ \int_0^{\frac{t}{q}} \partial t \left| D_q f(t + (1-t)a) \right| \partial_q t \\
+ \int_{\frac{t}{q}}^{\frac{1}{q}} (1-qt) \left| D_q f(t + (1-t)a) \right| \partial_q t \\
\right. \\
& \left. \left( \int_0^{\frac{t}{q}} (qt)^p \partial_q t \right)^{\frac{p}{p-1}} \right] \\
\leq & (b-a) \left[ \left( \int_0^{\frac{t}{q}} \left| D_q f(t + (1-t)a) \right| \partial_q t \right)^{\frac{p}{p-1}} \\
& \left. \left( \int_0^{\frac{1}{q}} (1-qt)^p \partial_q t \right)^{\frac{p}{p-1}} \right] \\
& \left. \left( \int_0^{\frac{1}{q}} \left| D_q f(t + (1-t)a) \right| \partial_q t \right)^{\frac{p}{p-1}} \right] \\
\leq & (b-a) \left[ \left( \int_0^{\frac{t}{q}} \left| D_q f(t + (1-t)a) \right| \partial_q t \right)^{\frac{p}{p-1}} \\
& \left. \left( \int_0^{\frac{1}{q}} (1-qt)^p \partial_q t \right)^{\frac{p}{p-1}} \right] \\
& \left. \left( \int_0^{\frac{1}{q}} \left| D_q f(t + (1-t)a) \right| \partial_q t \right)^{\frac{p}{p-1}} \right] \\
\end{align*}
\]
Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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