Lower bounds for non-trivial travelling wave solutions of equations of KdV type

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Abstract

We prove that if a solution of an equation of KdV type is bounded above by a travelling wave with an amplitude that decays faster than a given linear exponential then it must be zero. We assume no restrictions neither on the size nor in the direction of the speed of the travelling wave.

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1. Introduction and main results

In this paper, we will continue our study initiated in [2] on solutions of dispersive systems that decay like a linear exponential. In particular, we extend the results proved in that paper for travelling wave solutions of nonlinear Schrödinger (NLS) equations to the case of equations of Korteweg–de Vries (KdV) type.

More concretely we shall consider solutions of equations of the form

\[(\partial_t + \partial_x^3)u = a(u)\partial_x u, \quad x \in \mathbb{R} \quad t \in \mathbb{R}, \tag{1.1}\]

with \(a\) regular and such that

\[|a(s)| \leq M_1 (|s| + |s|^j), \quad j = 1, 2, 3, \ldots. \tag{1.2}\]

Our main result is the following one.

**Theorem 1.** Assume \(u \in C^1(\mathbb{R} : H^1(\mathbb{R}))\) is a real solution of (1.1)–(1.2) with \(a(u)\) also real. Then there exist \(\lambda_0 \geq 1 \text{ and } c_0 > 0 \text{ such that if for some } \lambda > 0, b \in \mathbb{R}\)

\[\sup_{t \geq 0} \|u(\cdot, t)\|_{L^2}^2 \leq M_2, \tag{1.3}\]
\[
\sup_{t \geq 0} \int e^{i|x-\beta t|} |u(x, t)|^2 \, dx \leq M_3, \quad (1.4)
\]
\[
\sup_{t \geq 0} \int e^{2i|x-\beta t|} |u(x, t)|^2 < +\infty
\]
and
\[
\lambda \geq \max \left\{ \lambda_0, c_0 M_1^2 \left( 1 + \left[ M_3 + M_2^{1/2} M_1^{1/2} \right]^{1/2} \right) \right\},
\]
then
\[
u \equiv 0.
\]

Relevant examples of nonlinear potentials are the pure power ones
\[
a(u) = -ju^j, \quad j = 1, 2, 3, \ldots, \quad (1.6)
\]
and the completely integrable ones
\[
a(u) = \alpha_1 u + \alpha_2 u^2, \quad (1.7)
\]
that are usually known as Gardner equations, where the cases \((\alpha_1, \alpha_2) = (1, 0)\) and \((\alpha_1, \alpha_2) = (0, 1)\) are the KdV and modified KdV equation, respectively.

**Remark 1.** For (1.6) there is a scaling symmetry
\[
u_j(x, t) = \lambda^{2j} \nu \left( \lambda x, \lambda^2 t \right), \quad \lambda > 0,
\]
and an explicit family of travelling waves
\[
\lambda^{2j} \varphi_j \left( \lambda(x - \lambda^2 t) \right), \quad \text{with } \varphi_j = \left( \frac{j + 2}{2} \right) \frac{1}{\text{sech} \left( \frac{j}{2} x \right)}.
\]
Hence in this case we have travelling wave solutions that propagate to the right with velocity \(\lambda^2\).

Also given any \(j\) and using the scaling invariance we can always choose \(\lambda\) small enough such that there exists a non-trivial solution that verifies (1.3), (1.4) and (1.5).

Regarding negative velocities we recall the existence of breather solutions in the case of Gardner equations for some specific choices of the parameter \(\alpha_1\) and \(\alpha_2\). We consider for simplicity the modified KdV equation; and specifically we take \(a(u) = -6u^2\). Then, with this normalization the soliton solutions are
\[
u(x, t) = \lambda \text{ sech} \left( \lambda(x - \lambda^2 t) \right), \quad (1.8)
\]
and the breather solutions are [11]
\[
u(x, t) = 2\lambda \text{ sech} \left( \lambda(x + \gamma t) \right) \left\{ \frac{\cos [\Phi(x, t)] - \frac{\lambda}{\mu} \sin \Phi(x, t) \tanh \left( \lambda(x + \gamma t) \right)}{1 + \left( \frac{\lambda}{\mu} \right)^2 \sin^2 \Phi(x, t) \sech^2 \left( \lambda(x + \gamma t) \right)} \right\}, \quad (1.9)
\]
with
\[
\gamma = 3\mu^2 - \lambda^2, \quad \delta = \mu^2 - 3\lambda^2 \quad \text{and } \quad \Phi = \mu(x + \delta t) - \tan^{-1} \left( \frac{\lambda}{\mu} \right).
\]

These breathers can be seen as wave packets with an amplitude and a frequency determined by the parameters \(\lambda\) and \(\mu\). The velocity of the amplitude is \(\gamma = 3\mu^2 - \lambda^2\) and it can take any real value.
Remark 2. Note that in the above examples the decay rate of the amplitude is just given by \( \lambda \) as in the statement of our theorem. Choosing as before \( \lambda \) small enough we also conclude the sharpness of our result for negative velocities.

Remark 3. Breather solutions also exist for complex KdV equations. For example if we choose \( a(u) = -|u|^2 \) with \( u(x, t) \in \mathbb{C} \) it is easy to check (see for example [7]) that

\[
 u(x, t) = \sqrt{3} e^{-i \mu (3 \lambda^2 - \mu^2 + i \mu x)} \psi_\lambda (x - (\lambda^2 - 3 \mu^2) t),
\]

with \( \psi_\lambda (x) = \lambda \psi_1 (\lambda x) \) and \( \psi_1 (x) = \sqrt{2} \text{sech} x \). The arguments needed for the proof of theorem 1 also work for this complex KdV (i.e. \( a(u) = -|u|^2 \)).

There are three fundamental ingredients in the proof of theorem 1. The first one is that the \( L^2 \) norm is a conserved quantity. This property is needed to conclude that \( u(x, 0) = 0 \) if for some sequence of times \( t_n \) we have that

\[
 \lim_{t_n \to \infty} \| u(\cdot, t_n) \|_{L^2} = 0.
\]

The second ingredient is Kato’s theory in [5] concerning the persistent property of solutions to the initial value problem with data that satisfies linear exponential decay. Kato’s result is used to prove that if (1.5) holds then a similar inequality is also true for \( \partial^k_x u \) with \( k = 1, 2, 3 \). We require this property since instead of working with the equation (1.1)–(1.2) we use the one satisfied by \( f = e^{i \theta} u \) with \( \theta(x, t) \) an appropriate Carleman weight that grows almost linearly at infinity.

The final ingredient is the convexity of \( H(t) = \| f \|_{L^2}^2 = \langle f, f \rangle \). As done in [2, 3] we estimate \( H(t) \) and \( \dot{H}(t) \) using integration by parts, therefore we need \( f \) and its spatial derivatives in \( L^2 \). At this point we closely follow the arguments in [2] but the algebra, as it can be expected, turns out to be more complicated in the KdV setting than in the NLS one. The details can be found in the proof of claim 1 in section 2. In this claim we establish the positivity of the commutator that appears in the computation of \( \dot{H}(t) \)—see also [9, 10].

There is still another difference with respect to [2]. For generalized KdV (1.1)–(1.2) the nonlinear potential can not be treated in a perturbation way and some structure is needed (see the proof of claim 2 in section 2). This structure still holds in the complex KdV equation that we mentioned above.

The use of Carleman weights to obtain positive commutators is a standard technique in elliptic theory for obtaining lower bounds of eigenfunctions of a Schrödinger operator—see [1, 4, 8]. In the time evolution setting some modifications of this technique are needed and an important step in our approach is the use of the two identities (2.3) and (2.4). These identities also appear in [2].

The rest of the paper, that is to say section 2, is devoted to the proof of theorem 1.

2. Proof of theorem 1

From the results by Kato in [5] we can assume that (1.5) is also satisfied possibly with another \( \lambda \) for \( \partial^k_x u(x, t) \) with \( k = 1, 2, 3 \).

As in [2] we shall work with \( f = e^{i \theta} u \) with \( \theta = \theta(x, t) \). More concretely \( \theta(x, t) = \varphi(r), \quad r = |x - bt| \) for some regular even \( \varphi \) that will be fixed later on and that it grows at most linearly at infinity. As in [3] we have

\[
 e^{i \theta} \left( \partial_t + \partial_x^3 \right) e^{-i \theta} f = (S_\lambda + A_\lambda) f;
\]

\[
 S_\lambda = -3 \lambda \partial_x \left( \partial_x \theta \partial_x \right) + \left( -\lambda^3 (\partial_x \theta)^3 - \lambda \partial_x \theta - \lambda \partial_x \theta \right);
\]

\[
 A_\lambda = \lambda^2 \partial_x \theta - \lambda \partial_x \theta.
\]
\[ A_\lambda = \partial_t + \partial_x^3 + 3\lambda^2 (\partial_x^2 \partial_x + \partial_x^3) = \partial_x + \tilde{A}_\lambda; \]
\[ A^*_\lambda = -A_\lambda \quad \tilde{A}^*_\lambda = -\tilde{A}_\lambda \quad S^*_\lambda = S_\lambda. \tag{2.1} \]

As in [2] we shall use two identities that hold for solutions of
\[ \partial_t f = - (S_\lambda + \tilde{A}_\lambda) f + F, \tag{2.2} \]
with
\[ \tilde{A}^*_\lambda = -\tilde{A}_\lambda S^*_\lambda = S_\lambda. \]

First observe that from (2.2)
\[ \frac{d}{dt} \langle f, f \rangle = -2 \langle S_\lambda f, f \rangle + 2 \langle F, f \rangle, \tag{2.3} \]
and, recalling that in our case \( A_\lambda = \partial_t + \tilde{A}_\lambda \), we have
\[ \frac{d}{dt} \langle S_\lambda f, f \rangle = -\langle (S_\lambda A_\lambda - A_\lambda S_\lambda) f, f \rangle - 2\langle S_\lambda f, S_\lambda f \rangle + 2 \langle F, S_\lambda f \rangle. \tag{2.4} \]

Finally, we shall use the following Carleman weight \( \phi_0 \):
\[ \phi_0 \in C^6(\mathbb{R}), \quad \text{even and positive,} \]
\[ \phi_0(r) = r \quad \text{if } 0 \leq r \leq 3/2 \quad \text{and} \quad \phi_0(r) = 2 - \frac{\log 2}{4 \log r} \quad \text{if } r \geq 2, \tag{2.6} \]
\[ 0 < \phi_0'(r) \leq 1 \quad \text{and it is a decreasing function for } r > 3/2, \tag{2.7} \]
there exists \( c_0 > 0 \) such that
\[ \left| \frac{d^k \phi_0}{dr^k}(r) \right| \leq c_0 \phi_0''(r) \quad k = 3, 4, 5, 6. \tag{2.8} \]

The proof of the theorem will follow from the next three claims. Recall that \( f = e^{\phi(r)} u \) with \( r = |x - bt| \). If \( b \geq \frac{3}{2} \lambda^2 \) we shall choose \( \varphi = \frac{1}{4} \varphi_0 \). Otherwise we shall take \( \varphi = \varphi_0 \). The reason behind these choices will become clear in the proof of claim 1 where different cases will be considered.

**Claim 1.** There exist \( \lambda_0 > 1, A_0 > 0 \) such that for all \( \lambda \geq \lambda_0 \)
\[ \langle (S_\lambda A_\lambda - A_\lambda S_\lambda) f, f \rangle \geq A_0 \int \phi_0''(\lambda^3 f^2 + \lambda^2 f_x^2). \tag{2.10} \]
The proof of this claim is long and it is postponed.

**Claim 2.** There exists a constant $C > 0$ which depends on (1.5) and on $M_1$, $M_2$ and $M_3$ given in (1.2), (1.3) and (1.4) such that

$$\left| \langle a(u) \left( \partial_x f - \lambda \varphi'(x - bt) f \right), f \rangle \right| \leq \lambda C. \quad (2.11)$$

**Remark 4.** In the above inequality (2.11) the structure of the nonlinear term plays a role. This makes a difference with respect to the NLS equation, see [2].

**Proof.** Recall that

$$u = e^{-\lambda \varphi(x - bt)} f,$$

so that from (1.3) $u$ is in $L^2$. Also observe that

$$e^{\lambda \varphi} \partial_x u = \partial_x f - \lambda \varphi' f.$$

Hence by integration by parts

$$\int a(u) \left( \partial_x f - \lambda \varphi' f \right) f \, dx = \int a(u) e^{\lambda \varphi} \partial_x u e^{\lambda \varphi} u \, dx = - \int e^{2\lambda \varphi} 2\lambda \varphi' u^2 a(u) \, dx,$$

with

$$\alpha(s) = \frac{1}{s^2} \int_0^s a(s') s' \, ds'.$$

The claim easily follows from the boundedness of $\varphi_0'$ and (1.2)–(1.5).

**Claim 3.** There exists a universal constant $C_1$ such that

$$\| a(u) (f_x - \lambda \varphi' f) \|_{L^2} \leq C_1 \left( M_2^{1/2} M_3^{1/2} + M_3 \right)^{1/2} + \int \varphi_0'' \left( \lambda^3 f^2 + \lambda f_x^2 \right) \, dx. \quad (2.12)$$

**Proof.** We have

$$e^{\frac{1}{2}|y-bt|} u^2 \leq \int_{-\infty}^{\infty} \left| \frac{d}{dy} \left( e^{\frac{1}{2}|y-bt|} u^2 (y) \right) \right| \, dy \leq 2 \| u_x \|_{L^2} \left\| e^{\frac{1}{2}|y-bt|} u \right\|_{L^2} + \frac{1}{2} \left\| e^{\frac{1}{2}|y-bt|} u \right\|_{L^2}^2 \leq 2 \left( M_2 M_3 \right)^{1/2} + \frac{1}{2} M_3. \quad (2.13)$$

Also there is a universal constant $C_0$ such that

$$e^{-\frac{1}{2}r} \leq C_0 \varphi_0'(r). \quad (2.14)$$

Then we obtain from (2.13)

$$|a(u)|^2 \leq C_0 \varphi_0'(2 \left( M_2 M_3 \right)^{1/2} + \frac{1}{2} M_3). \quad (2.15)$$

As we already said, either $\varphi = \frac{1}{4} \varphi_0$, or $\varphi = \varphi_0$. Hence

$$|f_x - \lambda \varphi' f|^2 \leq 2|f_x|^2 + 2\lambda^2 \sup \varphi_0' \| f \|_2^2 \leq C_0 (|f_x|^2 + \lambda^2 |f|^2), \quad (2.16)$$

for another universal constant $C_0$. Also recall that $\lambda \geq 1$. The claim follows from (2.15) and (2.16).

We complete the proof of the theorem before we prove claim 1. We follow the arguments in [2], and more concretely those in propositions 1 and 2 of that paper.
Take $\lambda \geq \lambda_0$ and such that
\[ C_1 \frac{M_1^2}{\lambda} \left[ \left( \frac{M_2^1}{2} \right)^{1/2} + M_3 \right]^{1/2} + 1 \leq A_0 \frac{A_0}{\lambda}. \]

Recall (2.5) so that (2.2) is satisfied with
\[ F = a(u)(f_x - \lambda \phi'(x - bt)f). \]

Then, from (2.10) and (2.12) we obtain
\[ \langle [S_2; A_2] f, f \rangle + 2 \langle S_2 f, S_2 f \rangle - 2 \langle F, S_2 f \rangle \geq \langle S_2 f, S_2 f \rangle + A_0^{\frac{1}{2}} \int \phi''^0 (\lambda^3 f^2 + \lambda^2 f_x^2) \] (2.17)

Our first step is to use the above inequality to obtain a uniform estimate for $\langle S_2 f, S_2 f \rangle$.

Take for any $n \in \mathbb{N}$, $\eta_n(t)$ defined on $[n - \frac{1}{2}, n + \frac{1}{2}]$ as
\[ \eta_n(t) = \frac{1}{2} - |t - n|. \]

Note that $\eta_n(n \pm \frac{1}{2}) = 0$, so that the right-hand side of (2.4) is lower bounded using (2.17) by
\[ \int \eta_n \| S_2 f \|_{L^2}^2 \, dt. \]

On the other hand $|\eta'_n| = 1$ and $\eta''_n = -\delta(t - n) \leq 0$. Then, using (1.5) and (2.11), the right-hand side of (2.3) is bounded above by $\lambda C$ with $C$ depending on $M_1, M_2, M_3$, and the bound in (1.5), but not on $n$.

As a consequence, we obtain
\[ \int \eta_n \| S_2 f \|_{L^2}^2 \, dt \leq \lambda C. \]

Hence there exists a sequence of times $T_n \to \infty$ such
\[ \sup_n \| S_2 f(\cdot, T_n) \|_{L^2}^2 < +\infty. \] (2.18)

The second and last step is to obtain a global space-time estimate for $f$ and therefore for $u$. For proving it we use again identities (2.3) and (2.4) with $F = a(u)(f_x - \lambda \phi'(x - bt)f)$ and with $\eta$ defined on $[0, T_n]$ regular and positive, and such that $\eta(0) = 0$ and $\eta \equiv 1$ if $t \geq 1$.

With this choice of $\eta$ the right-hand side of (2.3) is bounded using (1.5) and (2.11). For the right-hand side of (2.4) we use (2.17) and (2.18). Hence we obtain that
\[ \frac{A_0}{2} \int_1^\infty \int \phi''^0 (\lambda^3 f^2 + \lambda^2 f_x^2) \, dx \, dt < +\infty, \]

and as a consequence
\[ \int_1^\infty \int e^{2\lambda|x-bt|} |u(x,t)|^2 \, dx \, dt < +\infty. \]

This implies that there exists a sequence of times $t_n$ such that
\[ \lim_{t_n \to \infty} \| u(\cdot, t_n) \| = 0. \]

But from the $L^2$ conservation law $u_{t_0} \equiv 0$. The result follows from the uniqueness property of the initial value problem in $H^1$ given in [6].

In order to complete the proof of theorem 1 it remains to prove claim 1.

**Proof of claim 1.** We shall consider five different cases depending on the sign of $b$ and the relation between $b$ and $\lambda^2$. In what follows $C_0$ will denote a universal constant that can change from line to line. Only a finite number of choices of these constants will be made.
Recall (2.1). Hence, using the notation $\partial_x f = f_x$,

$$
\langle [S; A] f, f \rangle = \langle (S A - A S) f, f \rangle
= 9\lambda \int \varphi''(r) f_{xx}^2 + 6\lambda b \int \varphi''(r) f_x^2 + 18\lambda^3 \int (\varphi'(r))^2 \varphi''(r) f_x^2
- 6\lambda \int \varphi'''(r) f_x^2 + 9\lambda^5 \int (\varphi'(r))^4 \varphi''(r) f_x^2
- 3\lambda^3 \int (\varphi''(r) f_x^2 - 18\lambda^3 \int \varphi'(r) \varphi''(r) f_x^2
- 3\lambda^3 \int \varphi''(r) \varphi''(r) f_x^2 + \lambda \int \varphi''(r) f_x^2 - 2\lambda b \int \varphi''(r) f_x^2
- 6\lambda^3 b \int \varphi'(r) \varphi''(r) f_x^2
= \oplus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus + \ominus.
$$

(2.19)

Note that

$$
\oplus + \ominus + \ominus = \lambda \int \left(3\lambda^2 (\varphi'(r))^2 - b\right)^2 \varphi''(r) f_x^2.
$$

(2.20)

Case 1. $\frac{b}{\lambda^2} \geq \frac{1}{2}$. In this case we shall take $\varphi = \frac{1}{4} \varphi_0$.

From (2.19) and (2.20) we have $(\varphi_0' \leq 2)$

$$
\oplus + \ominus + \ominus \geq \frac{1}{16} \lambda^2 b^2 \int \varphi'' f_x^2.
$$

(2.21)

On the other hand,

$$
\oplus + \ominus + \ominus + \ominus + \ominus \leq \int \varphi'' f_x^2 \{ C_0 \lambda^3 + \lambda C_0 + \lambda b C_0 \}.
$$

(2.22)

Hence taking $\lambda^1_0$ large enough we obtain from (2.21)–(2.22) and for $\lambda > \lambda^1_0$ a bound for

$$
\lambda b^2 \int \varphi'' f_x^2 = \frac{\lambda b^2}{4} \int \varphi_0'' f_x^2 \geq \frac{9}{16} \lambda^5 \int \varphi_0'' f_x^2.
$$

(2.23)

We also have

$$
\ominus \leq 6 C_0 \lambda \int \varphi'' f_x^2.
$$

Then using $\ominus$ and taking $\lambda^1_0$ large enough we obtain a bound of

$$
\lambda b \int \varphi'' f_x^2.
$$

(2.24)

Finally, $\ominus$ gives a bound for $(f_{xx})^2$ and $\ominus$ is positive but worse than $\ominus$. Putting everything together we obtain for $\lambda > \lambda^1_0$ that

$$
\langle [S; A] f, f \rangle \geq A^1_0 \int \varphi_0' \left( \lambda (f_{xx})^2 + \lambda b f_x^2 + \lambda^2 f_x^2 \right) dx
\geq A^1_0 \int \varphi_0' \left( \lambda^3 f^2 + \lambda^2 f_x^2 \right) dx,
$$

(2.25)

for some $A^1_0 > 0$.

Case 2. $0 \leq \frac{b}{\lambda^2} \leq \frac{1}{2}$, and the integrals in $\ominus, \ominus, \ldots, \ominus$ will be considered in the region $r \geq 1$. We take $\varphi = \varphi_0$. 

Then
\[ 1 + 2 + 3 \geq 3g \int_{r \geq 1} \psi_0^2 f^2, \tag{2.26} \]
while the absolute value of the integral in the region \( r \geq 1 \) of \( 1 + 2 + 3 + 4 \) is upper bounded by
\[ \int_{r \geq 1} \psi_0^2 f^2 \left( C_0 \lambda^3 + C_0 \lambda + C_0 \lambda^2 \right) \tag{2.27} \]
Hence taking \( \lambda_0 \geq \lambda_0^I \) for some \( \lambda_0^I \) big enough we obtain from (2.26)–(2.27) a bound for
\[ \lambda^5 \int_{r \geq 1} \psi_0^2 f^2, \tag{2.28} \]
Using this time \( 1 \) instead of \( 2 \) (that is also positive) and \( 1 \) and (2.28) we obtain for the region \( r \geq 1 \) a lower bound for the commutator of the type
\[ A^I_0 \int_{r \geq 1} \psi_0^2 \left( \lambda (f_{xx})^2 + \lambda^3 f^2_x + \lambda^5 f^2 \right) \, dx, \tag{2.29} \]
that is better than what we need.

**Case 3.** \( 0 \leq \frac{b}{M^2} \leq \frac{1}{3} \) and \( r \leq 1 \). We take \( \varphi = \varphi_0 \).
We need a cut off function \( \eta \) such that \( \eta(r) = r \) if \( r \leq 1 \), \( \eta(r) \equiv 0 \) if \( r \geq 3/2 \) and \( \frac{d^2 \eta}{dr^2} \leq C_0 \) if \( k = 0, 1, 2 \). Then for \( r = |x - bt| \)
\[ \int \eta(r) f^2(x) \, dx = \frac{1}{6\lambda^2} \int (3\lambda r^2 - b) (\eta f^2)_{xx} \, dx \]
\[ = \frac{1}{6\lambda^2} \int (3\lambda r^2 - b) \left\{ (2f f_{xx} + 2f_x^2) \eta + \eta'' f^2 + 2\eta f f' \right\}. \tag{2.30} \]
Note that \( b \geq 0 \), so we get, using Cauchy–Schwarz,
\[ \int \eta \left( f^2 + \frac{b}{3\lambda^2} f_x^2 \right) \leq \frac{1}{6\lambda^2} \left( \frac{1}{2} \int (3\lambda^2 r^2 - b)^2 f^2 \eta + 2 \int (f_{xx})^2 \eta + 6\lambda \int r^2 f_x^2 \eta \right) + C_0 \int_{1 \leq r \leq 3/2} f^2 + \lambda f^2 + \frac{1}{\lambda} f_x^2 \right). \tag{2.30} \]
From the definition of \( \varphi_0 \) we have that the terms \( 1, 2, 4, 5 \) and \( 6 \) are zero and that for \( r \leq 3/2 \) we have that \( 1 + 2 + 4 \) gives
\[ \lambda \int_{r \leq 3/2} (3\lambda^2 r^2 - b)^2 \psi_0^2 f^2. \]
Therefore we just have to take care of \( 6 \). Using (2.30) we obtain
\[ 3\lambda^3 \int_{r \leq 1} f^2 \leq 3\lambda^3 \int f^2 \eta \]
\[ \leq \frac{1}{4} \lambda \int_{r \leq 3/2} (3\lambda r^2 - b)^2 f^2 + \lambda \int_{r \leq 3/2} (f_{xx})^2 + 3\lambda^2 \int r^2 f_x^2 \eta \]
\[ + C_0 \left( \lambda^4 \int_{1 \leq r \leq 3/2} f^2 + \lambda^2 \int_{1 \leq r \leq 3/2} f_x^2 \right) \]
\[ = I_1 + I_2 + I_3 + I_4. \]
\( I_1 \) is bounded by \( 1 + 2 + 4 \) because \( \psi_0^2 = 1 \) if \( r \leq 3/2, \) \( I_2 \) by \( 3 \), \( I_3 \) by \( 5 \) and \( I_4 \) by the bound (2.29) obtained in case 2, and that as we said was better than what we needed in terms
of powers of \( \lambda \). As a consequence for \( \lambda \geq \lambda_0^{\text{III}} \) and for some \( \lambda_0^{\text{III}} \) large enough we obtain a lower bound for the commutator of the type
\[
\int_{r \leq 1} \varphi_0'' \left( \lambda^3 f^2 + \lambda f_{xx}^2 \right) \, dx.
\] (2.31)

Finally, from
\[
\lambda^2 \int f_x^2 \eta = -\lambda^2 \int (f_x f \eta' + ff'' \eta) = \lambda^2 \int \left( \frac{1}{2} f^2 \eta'' - ff'' \eta \right),
\]
and (2.31) we prove that there exists \( A_0^{\text{III}} \) and a lower bound of the type
\[
A_0^{\text{III}} \int_{r \leq 1} \varphi_0'' \left( \lambda^3 f^2 + \lambda^2 f_x^2 + \lambda (f_{xx})^2 \right).
\] (2.32)

Case 4. \( b \leq 0 \) and \( r \geq 1 \). We take \( \varphi = \varphi_0 \).

If \( b \leq 0 \), (1) comes with the wrong sign while (2) appears with the good one. Therefore we need another identity. We have
\[
(1) + (2) + (3) = 9\lambda \int \varphi_0'' (f_{xx})^2 + 6\lambda b \left( \int \varphi_0'' f f_{xx} + \varphi_0''' f_x f \right) + \lambda b^2 \int \varphi_0 f^2
\]
\[
= 9\lambda \int \varphi_0'' \left( f_{xx} - \frac{b}{3} f \right)^2 + 3\lambda b \int \varphi_0' f^2.
\] (2.33)

Note that for \( r \geq 1 \) the absolute value of the last term of the above identity
\[
|3\lambda b \int \varphi_0' f^2|,
\]

is upper bounded by
\[
C_0 \int_{r \geq 1} \varphi_0'' f^2 \left( \lambda |b| + |\lambda^3 + \lambda \right).
\]

But from (1) and (2) we obtain a lower bound of the type
\[
(\lambda^5 + |b| \lambda^3) \int_{1 \leq r} \varphi_0'' f^2,
\]

that can be taken in the region \( 1 \leq r \leq \frac{3}{2} \) as
\[
(9\lambda^5 + 6|b| \lambda^3) \int_{1 \leq r \leq \frac{3}{2}} \varphi_0'' f^2.
\]

Also (2) gives the bound needed for (2). Altogether we obtain for \( \lambda \geq \lambda_0^{\text{IV}} \) with \( \lambda_0^{\text{IV}} \) large enough, that the commutator is lower bounded by
\[
A_0^{\text{IV}} \int_{r \geq 1} \varphi_0'' \left( (\lambda^5 + |b| \lambda^3) f^2 + \lambda^2 f_x^2 \right) + \int_{1 \leq r \leq \frac{3}{2}} \varphi_0'' \left( (8\lambda^5 + 5|b| \lambda^3) f^2 + 17\lambda^3 f_x^2 \right)
\] (2.34)

for some \( A_0^{\text{IV}} \) and that is better than what we need.

Case 5. \( b \leq 0 \) and \( r \leq 1 \). We take \( \varphi = \varphi_0 \).

As in case 3 we need the cut off \( \eta \equiv 1 \) if \( r \leq 1 \) and \( \eta \equiv 0 \) if \( r \geq 3/2 \). We have \( r = |x - bt| \)
\[
\int f_x^2 \eta = -\int r \partial_x \left( f_x^2 \right) \eta - \int r f_x^2 \eta' \]
\[
= -\int 2rf_x f_{xx} \eta - \int r f_x^2 \eta' \]
\[
= -2 \int \eta r \left( f_{xx} - \frac{b}{3} f \right) f_x - 2b \int \eta r f_x - \int r \eta f_x^2.
\]
Hence using Cauchy–Schwarz and some integration by parts
\[
\int f^2 \eta - b \int f^2 \eta \leq \frac{1}{\lambda} \int \eta \left( f_{xx} - b f \right)^2 + \lambda \int \eta r^2 f^2 + \int r |\eta'| \left( \frac{|b|}{3} f^2 + f_x^2 \right). \tag{2.35}
\]

Also for \( r \leq 1 \) the terms ③, ④, ⑤, ⑥, ⑦ are zero. Therefore only ① remains as a negative term. We have
\[
\int f^2 \eta = \frac{1}{2} \int r^2 (f^2)_{xx} \eta - 2 \int r \eta' f f_x - \frac{1}{2} \int r^2 \eta'' f^2
\]
\[
= \int r^2 (f_x^2 + f f_{xx}) \eta - \frac{1}{2} \int r \eta' f f_x - \frac{1}{2} \int r^2 \eta'' f^2
\]
\[
= \int r^2 f_x^2 \eta + \int r^2 f \left( f_{xx} - b \frac{2}{3} f \right) \eta + b \int r^2 f^2 \eta
\]
\[
- 2 \int r \eta' f f_x - \frac{1}{2} \int r^2 \eta'' f^2.
\]

Then, using Cauchy–Schwarz
\[
\int f^2 \eta - b \int f^2 \eta \leq \int r^2 \eta f_x^2 + \frac{\lambda^2}{2} \int r^4 \eta f^2 + \frac{1}{2} \int \eta \left( f_{xx} - b \frac{2}{3} f \right)^2
\]
\[
+ C_0 \int_{|r| \leq 3/2} (1 + \lambda) f^2 + \frac{1}{\lambda} f_x^2.
\]

Then, for \( r \leq 1 \) the absolute value of ① is upper bounded by
\[
3\lambda^3 \left( \int_{r \leq 1} f^2 \eta + \lambda \int r^2 f^2 \right) \leq 3\lambda^3 \int r^2 \eta f_x^2 + 3 \int r^2 \eta f^2 + \lambda \int \eta \left( f_{xx} - b \frac{2}{3} f \right)^2
\]
\[
+ C_0 \int_{|r| \leq 3/2} \lambda^4 f^2 + \lambda^2 f_x^2
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]

Hence, we bound \( I_1 \) with ①. Note that even in the region \( 1 \leq |r| \leq 3/2 \) there is no problem using (2.34). For \( I_2 \) we use ① and (2.34) because \( \frac{4}{7} r^4 \leq 8 \) if \( r \leq 3/2 \). For \( I_3 \) we use (2.33) (\( \psi_0^V = 0 \) if \( r \leq 3/2 \)). Finally for \( I_4 \) we use (2.34).

As a conclusion we obtain a lower bound for the commutator of the type
\[
\lambda^3 \int_{r \leq 1} f^2 + \lambda \int r^2 f^2 + \lambda \int \eta \left( f_{xx} - b \frac{2}{3} f \right)^2
\]
\[
+ \lambda \int \eta r^2 f_x^2.
\]

Hence using (2.35) we obtain a bound of \( \lambda^2 \int f_x^2 \eta \). Therefore there exists \( A_0^V \) such that the commutator is bounded below by
\[
A_0^V \int_{\frac{1}{2} \leq r \leq 1} \psi_0^V \left( \lambda^3 f^2 + \lambda^2 f_x^2 \right). \tag{2.36}
\]

Taking \( A_0 = \max\{ A_0^I, A_0^II, A_0^III, A_0^V \} \) we conclude the proof of claim 1 using (2.25),
(2.29), (2.31), (2.34) and (2.36). Note that a bound of \( \lambda \int \psi_0^V (f_{xx})^2 \) is also obtained.

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