Area-Preserving Diffeomorphisms, $w_{\infty}$ Algebras and $w_{\infty}$ Gravity †

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ABSTRACT

The $w_{\infty}$ algebra is a particular generalization of the Virasoro algebra with generators of higher spin 2, 3, ..., $\infty$. It can be viewed as the algebra of a class of functions, relative to a Poisson bracket, on a suitably chosen surface. Thus, $w_{\infty}$ is a special case of area-preserving diffeomorphisms of an arbitrary surface. We review various aspects of area-preserving diffeomorphisms, $w_{\infty}$ algebras and $w_{\infty}$ gravity. The topics covered include a) the structure of the algebra of area-preserving diffeomorphisms with central extensions and their relation to $w_{\infty}$ algebras, b) various generalizations of $w_{\infty}$ algebras, c) the structure of $w_{\infty}$ gravity and its geometrical aspects, d) nonlinear realizations of $w_{\infty}$ symmetry and e) various quantum realizations of $w_{\infty}$ symmetry.

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1. Introduction

Area-preserving diffeomorphisms of a surface $\Sigma$, denoted by $\text{SDiff}(\Sigma)$, arise in diverse areas of theoretical and mathematical physics. For example, since they are the canonical transformations which preserve the Poisson bracket, they naturally arise in theory of dynamical systems [1]. In the context of high energy physics, the area-preserving diffeomorphisms of a 2-sphere, $\text{SDiff}(S^2)$, has been encountered in the theory of relativistic membranes [2]. It arises as a subgroup of the diffeomorphisms of the 3-dimensional world-volume in a light-cone gauge. It was observed that (in the case of spherical and toroidal membranes at least) the structure constants of the area-preserving diffeomorphisms are those of $SU(N)$ in the $N \to \infty$ limit (in the case of a torus there are the additional global diffeomorphisms) [2]. This fact was used in regularizing the quantum theory of the supermembrane, by considering the $SU(N)$ theory and then taking the $N \to \infty$ limit [4].

Area-preserving diffeomorphisms of a 2-torus, $\text{SDiff}(T^2)$, has been studied as the analog of the Virasoro symmetry [5]. It turns out that the generators of the Virasoro algebra (with or without central charge) can be constructed as a linear combination of infinitely many $\text{SDiff}(T^2)$ generators [6]. Later it was realised that the Virasoro algebra can be embedded in the area-preserving diffeomorphisms of the 2-plane, $\text{SDiff}(R^2)$, in a manifest manner [7]. In this case, all the other generators of the algebra form a representation of the Virasoro algebra and thus can be viewed as generators with conformal spins $3, 4, \ldots, \infty$. This observation, opened the way for borrowing many of the techniques used in the study of Virasoro algebra and the 2D conformal field theories.

There is a great deal of arbitrariness in choosing basis functions in order to exhibit the structure constants of the $\text{SDiff}$ algebras explicitly. The problem becomes more acute when one deals with noncompact surfaces such as $R^2$ or a cylinder, in which case one usually works with basis functions which diverge at infinity. Nonetheless, there is a particular choice of basis elements appropriate for a surface of cylindrical topology [8], which, though divergent at infinity, it yields a specific set of structure constants. Although, this particular algebra is referred to as $w_\infty$ or $w_{1+\infty}$ algebra (in the latter case a spin 1 generator is included), $\text{SDiff}$ algebras for other kinds of surfaces, or for the same surface but with a different choice of basis elements are also referred to loosely as $w_\infty$ algebras. Often it is not so clear which one of these algebras are isomorphic into each other, especially in the case of noncompact surfaces, and when infinitely many redefinitions of the generators is required. At any rate, the terminology of $w_\infty$ will be used to refer to a wide class of higher spin algebras in two dimensions which contain the Virasoro algebra, and are connected to certain area-preserving diffeomorphisms in a way to be specified carefully case by case, if needed.

† The generalization of this group for higher dimensional extended objects, which are known as the $p$-branes, is the volume-preserving diffeomorphisms of a $p$ dimensional manifold [3].
In $w_\infty$ algebras the commutator of a spin $s$ generator with a spin $s'$ generator yields a spin $(s + s' - 2)$ generator, and a central extension can arise only in the commutator of two spin 2 generators. Denoting the Betti number of the surface $\Sigma$ by $b_1$, the algebra $\text{SDiff}(\Sigma)$ admits $b_1$ independent central extensions [9]. An extension of $w_\infty$ in which the commutator a spin $s$ generator with a spin $s'$ generator yields all spins between spin 2 and spin $(s + s' - 2)$ arise was constructed explicitly in Refs. [8,10]. This is referred to as $W_\infty$ algebra and it does admit central extensions in all spin sectors.

A new field theoretic realization of $w_\infty$ was found by constructing the so called $w_\infty$ gravity action [11], which is the analog of the Polyakov’s bosonic string action, incorporating now the coupling of all higher spin world-sheet fields to matter fields. Interestingly enough, when quantized, the $w_\infty$ symmetry of this model deforms into $W_\infty$ symmetry [12]. In fact, it has been shown that this is a rather general phenomenon which arises even in simpler quantum mechanical systems in one dimension [13]. A geometric understanding of $w_\infty$ gravity is still lacking, but interesting attempts have been made in this direction [14]. The $w_\infty$ transformation rules for a scalar field have also been interpreted in the context of a nonlinear realization of $w_\infty$ [11,15], which we shall discuss later.

There are a number of other areas in which $w_\infty$ symmetry arises. For example, a 2D sigma model based on an area-preserving diffeomorphisms of a 2-surface turns out to have field equations that are closely related to the self-dual gravity equations in 4D [16]. This is rather interesting, because it suggests a 2D conformal field theory approach to an interesting 4D quantum gravity problem. In fact, self-dual gravity equations have been shown to arise in string theories with local $N = 2$ supersymmetry on the world-sheet, and a target spacetime of $(2, 2)$ signature [17]. Furthermore, it has been shown that the loop group of $w_\infty$ arises as a symmetry group in the $N = 2$ superstring theory [17]. More recently, the loop algebra of area-preserving diffeomorphisms of a 2-plane has been discovered as the algebra of an infinite set of spin 1 primary fields constructed out of a matter field and the Liouville field [18]. A possible connection between the $c = 1$ bosonic string theory where this symmetry arises and the $N = 2$ superstring theory has been suggested [17]. Finally, let us mention that the area-preserving diffeomorphisms have also been encountered as symmetry groups in the study of higher spin field theories in $2 + 1$ dimensions [18].

In this brief review, we shall first describe the algebraic structure of $\text{SDiff}(\Sigma)$ and $w_\infty$. After we summarize various generalizations of these algebras, we shall turn to their classical field theoretic realizations. In particular, we shall describe the (super) $w_\infty$ gravity, nonlinear realizations and various quantum realizations of $w_{1+\infty}$ symmetry.

2. The structure of area-preserving diffeomorphisms and $w_\infty$ algebras

In two dimensions the area-form is the same as a nondegenerate closed 2-form, i.e. a symplectic 2-form. Hence, the area-preserving diffeomorphisms are the same as the symplectic diffeomorphisms, namely those diffeomorphisms which leave the symplectic form on
the surface invariant. Unless stated otherwise we shall consider compact orientable surfaces. Denoting the symplectic form by $\Omega = \Omega_{ab} \, d\sigma^a \, d\sigma^b$ where $\sigma^a$, $a = 1, 2$ are the coordinates of the surface, it must satisfy the condition $L_\xi \Omega = 0$, where $L_\xi$ denotes the Lie derivative along a vector $\xi^a$. The most general solution to this condition takes the form

$$\xi^a = \Omega^{ab} (\partial_b \Lambda + \omega_b) \quad (2.1)$$

where $\Lambda(\sigma)$ is an arbitrary function and $\omega_a$ is a harmonic 1-form, i.e. it is curl-free but it can not be written as a derivative of a scalar globally. As is well known, on a genus $g$ surface there are $2g$ independent such harmonic 1-forms. Let us expand $\omega_a = c^r \omega_a^{(r)}$, $r = 1, ..., 2g$ where $c_r$ are arbitrary constants and $\omega_a^{(r)}$, are harmonic 1-forms which we normalize as

$$\int_\Sigma \omega^{(r)} \wedge \omega^{(s)} = \delta_{rs}.$$ We can represent the generators of the area-preserving diffeomorphisms as follows

$$L_\Lambda = \Omega^{ab} \partial_b \Lambda \partial_a, \quad P_r = \Omega^{ab} \omega_b^{(r)} \partial_a, \quad r = 1, ..., 2g \quad (2.2)$$

These generators obey the algebra

$$[L_\Lambda_1, L_\Lambda_2] = L_{\Lambda_{12}}, \quad (2.3a)$$

$$[P_r, L_\Lambda] = L_{\Lambda r}, \quad (2.3b)$$

$$[P_r, P_s] = L_{\Lambda rs}, \quad (2.3c)$$

where the parameters appearing on the right-hand sides are given by

$$\Lambda_{12} = \Omega^{ab} \partial_b \Lambda_1 \partial_a \Lambda_2 \quad (2.4a)$$

$$\Lambda_r = \Omega^{ab} \omega_b^{(r)} \partial_a \Lambda \quad (2.4b)$$

$$\Lambda_{rs} = \Omega^{ab} \omega_b^{(r)} \omega_a^{(s)} \quad (2.4c)$$

From (2.4a) it is clear that $\text{SDiff}(\Sigma)$ algebra is the algebra of functions on $\Sigma$ relative to a Poisson bracket defined by $\{A, B\} := \Omega^{ab} \partial_b A \partial_a B$.

Let us now consider the quantum version of the above algebra whose generators we shall denote by $\hat{L}_\Lambda$ and $\hat{P}_r$. The central extension of such an algebra was determined in Ref. [9]. The result takes the form

$$[\hat{L}_\Lambda_1, \hat{L}_\Lambda_2] = \hat{L}_{\Lambda_{12}} + \int_\Sigma d^2 \sigma \alpha \Omega^{ab} \omega_b (\Lambda_1 \partial_a \Lambda_2) \quad (2.5a)$$

$$[\hat{P}_r, \hat{L}_\Lambda] = \hat{L}_{\Lambda r} - 2 \int_\Sigma d^2 \sigma \alpha \Omega^{ab} \omega_b \omega_a^{(r)} \Lambda \quad (2.5b)$$

$$[\hat{P}_r, \hat{P}_s] = \hat{L}_{\Lambda rs} + \int_\Sigma d^2 \beta \Omega^{ab} \omega_b^{(r)} \omega_a^{(s)} \quad (2.5c)$$

where $\alpha$ and $\beta$ are two arbitrary densities.
In practice, it is very useful to expand the generators of this algebra in a suitable basis and to find the structure constants explicitly. The choice of basis depends on the topology and geometry of the surface \( \Sigma \) and on the type of functions we wish to expand on it. These issues arise in the very rich and fascinating subject of harmonic analysis. An extensive discussion of this subject is beyond the scope of this paper. We shall be content by giving few examples here. Let us consider first a flat two torus \( T^2 \) defined by a square lattice with side \( 2\pi \). We shall take the symplectic structure to be the Levi-Civita symbol \( \Omega^{ab} = \epsilon^{ab} \) with \( \epsilon^{12} = -\epsilon^{21} = 1 \). It is natural to choose as basis functions \( e^m \cdot \sigma \) where \( n = (n_1, n_2) \) and \( n_1, n_2 \) are integers. Note that this is a complete and orthonormal basis, and furnish a unitary representation of the torus group \( U(1) \times U(1) \). In terms of these basis functions we may expand the parameters \( \Lambda(\sigma) \) as

\[
\Lambda(\sigma) = \sum_n \Lambda_n e^{i n \cdot \sigma}, \quad (2.6)
\]

The torus has genus 1, so there are two independent harmonic 1-forms. We can parametrize the most general harmonic 1-form as \( \omega_a = (a_1, a_2) \), where the components \( a_1 \) and \( a_2 \) are arbitrary constants. Defining the Fourier components of the \( \text{SDiff} (T^2) \) generators as follows

\[
\hat{L}_n = \sum_n \Lambda_n \hat{L}_n, \quad (2.7)
\]

we find that (2.5) reduces in this special case to [9]

\[
[\hat{L}_n, \hat{L}_m] = n \times m \hat{L}_{n+m} + a \times n \delta_{n+m,0} \quad (2.8a)
\]

\[
[\hat{P}_r, \hat{L}_n] = n_r \hat{L}_n + b_r \delta_{n,0} \quad (2.8b)
\]

\[
[\hat{P}_r, \hat{P}_s] = c\epsilon_{rs}, \quad (2.8c)
\]

where \( n \times m \equiv \epsilon^{ab} n_a m_b \) and \( b_r, c \) are constants given by \( \beta_r = \epsilon_{rs} c_s \Lambda_0 \int d^2 \sigma \alpha \) and \( c = \int d^2 \sigma \beta \), and \( r, s = 1, 2 \) since \( b_1 = 2 \) for a torus. Note that \( L_0 \) does not occur in the commutation relations (2.8). However, it can be introduced on the right hand side of (2.8c) as \( \epsilon_{rs} L_0 \). It would be a central charge commuting with all the generators. Moreover, the central extension on the right hand side of (2.8c) could be absorbed in a redefinition of \( L_0 \) [9].

An interesting fact about the algebra (2.8a) is that, the Virasoro algebra with a central extension can be obtained from it as a infinite linear combination of the form \( L_N = \sum c_N^n L_n \).

The exact form of the constant coefficients \( c_N^n \) can be found in Ref. [6].

A super extension of \( \text{SDiff}(T^2) \) does exist [9,4]. For completeness, we reproduce it here:

\[
[L_n, L_m] = n \times m L_{n+m},
\]

\[
[L_n, G_m] = n \times m G_{n+m},
\]

\[
\{G_n, G_m\} = L_{n+m}, \quad (2.9)
\]
where $G_n$ are, of course, the superpartners of the bosonic generators $L_n$. Note that, not merely is there no central extension possible in the $\{G, G\}$ anticommutator, but also the central term that could be present in the $[L, L]$ commutator in the bosonic case must now be absent by Jacobi identities.

Another simple example of a compact, orientable surface is a 2-sphere $S^2$. As basis functions it is natural to choose the spherical harmonics $Y^\ell_m(\theta, \phi)$. They furnish a representation of the rotation group SO(3), and they form a complete and orthonormal set. Let us furthermore take the symplectic structure to be $\Omega^{ab} = \sin^{-1}\theta \epsilon^{ab}$, and expand the parameters and the generators of SDiff($S^2$) as $\Lambda = \sum_\ell, m \Lambda^\ell_m Y^\ell_m$ and $L_\Lambda = \sum_\ell, m \Lambda^\ell_m L^\ell_m$. Since there exist no harmonic 1-forms on $S^2$, there will be no non-trivial central extension and the generators $L^\ell_m$ obey the following classical SDiff($S^2$) algebra

$$[L^\ell_m, L^j_n] = c^\ell_j_k(m, n)L^k_{m+n} \tag{2.10}$$

where $c^\ell_j_k(m, n)$ are the structure constants, which are essentially the Clebsch-Gordan coefficients of SO(3) and can be written as

$$c^\ell_j_k(m, n) = \int d\theta d\phi (\epsilon^{ab} \partial_b Y^\ell_m \partial_a Y^j_n) Y^r_{-m-n} \tag{2.11}$$

If we take the surface $\Sigma$ to be a 2-sphere with north and south poles removed, $S^2 \setminus \pm 1$, which is topologically equivalent to a cylinder, then there will be a nontrivial central extension. Its form is somewhat complicated, and it can be found in Ref. [19].

We next consider some examples of noncompact surfaces. In the case of 2-hyperboloid, $H^2$, we can expand the generators of SDiff($H^2$) in a basis closely related to the spherical harmonics. In such a basis the structure constants of SDiff($H^2$) are essentially the Clebsh-Gordan coefficients of SO($2,1$). Since SO($2,1$) is contained as a subalgebra of SDiff($H^2$), if we identify it with the Lorentz group in $2+1$ dimensions, then the possibility of interpreting SDiff($H^2$) as a higher spin algebra in $2+1$ dimensions arises. If we take two copies of SDiff($H^2$), we can then identify the subgroup SO($2,1$) $\oplus$ SO($2,1$) with the anti de Sitter group in $2+1$ dimensions, and thus obtain its infinite dimensional extension as a higher spin algebra [20]. This algebra is very similar to the infinite dimensional extension of the AdS group SO(3,2) in four dimensions obtained in [21].

$H^2$ is topologically equivalent to a disk and hence its first Betti number is zero. Consequently, SDiff($H^2$) does not admit a nontrivial central extension. However, if we remove the origin, the resulting surface $H^2 \setminus \{0\}$ is topologically equivalent to a cylinder and the associated area-preserving diffeomorphism algebra will have a nontrivial central extension. In using the general formula (2.5) to compute the central extension, however, care must be exercised in choosing an integration measure such that the relevant integrals are well defined. In what follows we shall mainly concentrate on the classical area-preserving diffeomorphisms.
The surfaces of interest are those whose area preserving diffeomorphisms can be represented in a basis such that the algebra of area-preserving diffeomorphisms are manifestly an infinite dimensional extension of the Virasoro algebra, i.e. such that the generators of SDiff(Σ) decompose in a simple manner under the Virasoro group. To this end let us first consider the area-preserving diffeomorphisms of the upper half plane \( R^2_+ \), with coordinates \( x, y \). A choice of basis which would be suitable for expansions of square integrable functions on the plane, would be the unitary representation of the Euclidean group in two dimensions, \( E_2 \). These are essentially the Bessel functions. However, in such a basis the embedding of the Virasoro algebra would not be manifest. To make the connection with the Virasoro algebra manifest, a more suitable choice of basis is \( y^{\ell+1}x^{\ell+m+1} \) where \( \ell, m \) are integers [2,22]. To avoid the singularities at the origin, we may remove the origin, and thus consider \( R^2 \setminus \{0\} \), which is topologically equivalent to a cylinder. Note that, the basis functions diverge as \( x, y \to \infty \), and they are not orthonormal either. However, as we will be interested in functions which admit a Taylor expansion we shall not be concerned about these properties of the basis functions. Let us proceed by choosing the symplectic structure to be \( \Omega^{ab} = \delta^{ab} \), and expanding the generators and the parameters as \( \Lambda = -\sum_{\ell,m} \Lambda^\ell_m y^{\ell+1}x^{\ell+m+1} \) and \( L_\Lambda = \sum_{\ell,m} \Lambda^\ell_m v^\ell_m \). In this basis, the generators \( v^\ell_m \) take the form

\[
v^\ell_m = y^{\ell+1}x^{\ell+m+1} \left[ -\frac{\partial}{\partial x} + (\ell + m + 1)\frac{\partial}{\partial y} \right], \quad m \geq -\ell - 1 \quad (2.12)
\]

and they obey the following SDiff(\( R^2 \)) algebra

\[
[ v^\ell_m, v^j_n ] = [(j + 1)(\ell + m + 1) - (\ell + 1)(j + n + 1)]v^{\ell+j}_{m+n} \quad (2.13)
\]

Clearly \( v^0_m \) generate the Virasoro algebra without central extension:

\[
[ v^0_m, v^0_n ] = (m - n)v^0_{m+n} \quad (2.14)
\]

Furthermore, the generators \( v^\ell_m \) form a representation of the Virasoro algebra since

\[
[ v^0_m, v^\ell_n ] = [(\ell + 1)n - m]v^\ell_{m+n} \quad (2.15)
\]

This is to be compared with the commutation rule between the Virasoro generators \( L_n \) and the Fourier modes of a spin \( s \) conformal field \( w^s_m \) given by

\[
[ L_n, w^s_m ] = [(s - 1)n - m]w^s_{m+n} \quad (2.16)
\]

Therefore the generators \( v^\ell_m \) can be viewed as the Fourier modes (labelled by \( m \)) of a conformal field of conformal spin \((\ell + 2)\).

Another choice of basis functions for SDiff(\( R^2 \)) considered in Ref. [23] is \( x^{s+m}y^{s-m} \). Using the expansions \( \Lambda = -\sum \Lambda^s_s x^{s+m}y^{s-m} \) and \( L_\Lambda = \sum \Lambda^s_m v^s_m \), one finds that SDiff(\( R^2 \)) algebra in this basis takes the form

\[
[ v^s_m, v^t_n ] = [(t - n)(s + m) - (s - m)(t + n)]v^{s+t-1}_{m+n} \quad (2.17)
\]
In this basis \( \frac{1}{2} v^1_n \) obeys the Virasoro algebra, and by commuting it with \( v^s_m \) we find that \( v^s_m \) can be viewed as the Fourier modes (labelled by \( m \)) of a conformal field of spin \((s + 1)\). Note that in order to avoid negative powers of \( x \) and \( y \) in the basis functions, we must keep the generators \( v^s_m \) with \(-s \geq m \leq s\). Note also that in order to have integer powers of \( x \) and \( y \), \( s, n \) must be both integers or both half integers.

There are other choices of basis functions which give rise to algebras with a somewhat different interpretation of the generators as Fourier modes of conformal fields. For example, using the polar coordinates \( r, \theta \), consider the basis functions \( r^{\ell+2} e^{im\theta} \). Furthermore, choosing the symplectic structure to be \( \Omega^{ab} = r^{-1} e^{ab} \), and using the expansions \( \Lambda = -i \sum \Lambda_m^\ell r^{\ell+2} e^{im\theta} \) and \( L_A = \sum_{\ell,m} \Lambda_m^\ell v^\ell_m \), where \( \ell, m \) are integers, we find that \( \text{SDiff}(R^2 \setminus \{0\}) \) in this basis takes the form

\[
[v^\ell_m, v^j_n] = [(j + 2)m - (\ell + 2)n]v^{\ell+j}_{m+n},
\]

where \( \ell \geq -1 \) and \(-\infty < m < \infty \). In this basis, \( \frac{1}{2} v^0_m \) obey the Virasoro algebra. Using the notation \( L_m = \frac{1}{2} v^0_m \), from (2.18) we obtain

\[
[L_m, v^\ell_n] = \left[ \frac{1}{2}(\ell + 2) - n \right] v^\ell_n
\]

Compared with (2.16), this implies that \( v^\ell_n \) can be viewed as the Fourier modes (labeled by \( n \)) of a conformal field of spin \(((\ell + 4)/2\). We shall call this algebra the twisted \( w_\infty \) algebra.

Finally, let us describe a somewhat more convenient choice of basis to describe the algebra of area-preserving diffeomorphisms of a surface with the topology of a cylinder, \( R \times S^1 \). Let the coordinates of the cylinder be \( 0 \leq x \leq 2\pi \) and \( \infty < y < \infty \). A suitable choice of basis functions is \( y^{\ell+1} e^{imx} \) [8]. Choosing the symplectic structure to be \( \Omega^{ab} = e^{ab} \) and using the expansions \( \Lambda = -i \sum \Lambda_m^\ell y^{\ell+1} e^{imx} \) and \( L_A = \sum_{\ell,m} \Lambda_m^\ell v^\ell_m \), we find that \( \text{SDiff}(R \times S^1) \) algebra takes the form

\[
[v^i_m, v^j_n] = [(j + 1)m - (i + 1)n]v^{i+j}_{m+n},
\]

where \( \ell \geq -1 \) and \(-\infty < m < \infty \). Evidently \( v^0_m \) obeys the Virasoro algebra and from the commutator

\[
[v^0_n, v^\ell_m] = [(\ell + 1)n - m]v^\ell_{m+n}
\]

we see that \( v^\ell_m \) are the Fourier modes (labelled by \( m \)) of a conformal field of spin \((\ell + 2)\). In particular \( v^{-1}_m \) has spin 1. This algebra is usually referred to as \( w_{1+\infty} \) algebra. If we exclude the spin 1 generator, we still have a closed algebra, known as the \( w_\infty \) algebra.

Experience with Virasoro algebra suggests that in order to have a nontrivial unitary representation of \( w_{1+\infty} \) it should admit a central extension. Starting directly from (2.15) and searching for central extensions by means of checking the Jacobi identities, one finds
that it is allowed only in the spin 2 sector. Denoting the central extension that arises in the commutator of $v_m^i$ with $v_n^j$ by $c^{ij}(m, n)$, it takes the familiar form

$$c^{ij}(m, n) = \frac{c}{12}(m^3 - m)\delta^{i,0}\delta^{j,0}\delta_{m+n,0},$$

(2.22)

where $c$ is an arbitrary constant. An extension of $w_{1+\infty}$ which does contain a central extension in all spin sectors exists, and it is referred to as $W_{1+\infty}$. It turns out that $w_{1+\infty}$ algebra can be viewed as a contraction of the latter. Alternatively, $w_{1+\infty}$ can be interpreted as the classical limit of the quantum algebra $W_{1+\infty}$.

$w_{1+\infty}$ algebra admits two natural subalgebras. One of them, which we shall denote by $w^+_1$ has the generators $v_m^\ell$ with the restriction $\ell \geq -\ell - 1$, $\infty \leq m \leq \infty$, and the other one, $w^-_1$ generated by $v_m^\ell$ with the restriction $\ell \leq -\ell - 1$, $\infty \leq m \leq \infty$. Another useful subalgebra is the Cartan subalgebra. There are a number of ways of choosing it. For example, $v_0^n$ are infinitely many mutually commuting generators, thus forming the Cartan subalgebra $\dagger$. Another set of mutually commuting generators are $v_{-\ell+1}$ and $v_{\ell+1}$. These algebras will play a role when we discuss the nonlinear realizations of $w_{1+\infty}$.

3. Generalizations of the $w_\infty$ Algebras

There are a number of extensions of the area-preserving diffeomorphisms. Among them are the $N = 1$ [24] and $N = 2$ [22] supersymmetric extensions of $w_\infty$. The $N = 2$ super $w_\infty$ algebra takes the form [22,25]

$$[v_m^i, v_n^j] = [(j + 1)m - (i + 1)n]v^{i+j} + \frac{c}{8}(m^3 - m)\delta^{i,0}\delta^{j,0}\delta_{m+n,0}$$

$$[v_m^i, J_{n-1}^j] = [jm - (i + 1)n]J_{m+n-1}^{i+j-1}$$

$$\{G_r^\alpha, G_s^\beta\} = 2v_{r+s}^{\alpha+\beta} - 2[(\beta + \frac{1}{2})r - (\alpha + \frac{1}{2})s]J_{r+s}^{\alpha+\beta-1} + \frac{c}{2}(r^2 - \frac{1}{4})\delta^{\alpha,0}\delta^{\beta,0}\delta_{r+s,0}$$

$$[v_m^i, G_r^\alpha] = [(\alpha + \frac{1}{2})m - (i + 1)r]G_{m+r}^{\alpha+i}$$

$$[v_m^i, G_r^\alpha] = [(\alpha + \frac{1}{2})m - (i + 1)r]G_{m+r}^{\alpha+i}$$

$$[J_{m-1}^i, G_r^\alpha] = G_{m+r}^{i+\alpha}$$

$$[J_{m-1}^i, G_r^\alpha] = -G_{m+r}^{i+\alpha}$$

$$[J_{m-1}^i, J_{n-1}^j] = \frac{c}{2}m\delta^{i,0}\delta^{j,0}\delta_{m+n,0},$$

(3.1)

where the notation is self explanatory. In fact, this is the algebra of symplectic diffeomorphisms on a $(2, 2)$ superplane, i.e. a plane of two bosonic and two fermionic dimensions [24].

$\dagger$ It is interesting to note that, since in particular $[v_0^0, v_0^n] = 0$, and $v_0^0$ is the usual Hamiltonian $H = L_0 = v_0^0$, we can interpret $v_n^0$, $n \geq 1$ as infinitely many conserved quantities that commute with the Hamiltonian.
The $N = 1$ super $w_{\infty}$ algebra can be obtained from the above algebra by truncation, or directly as an algebra of the symplectic diffeomorphisms of a $(2, 1)$ superplane [24, 22]. For $i = j = \alpha = \beta = 0$, the algebra (3.1) reduces to the well known $N = 2$ superconformal algebra.

Yet another extension is called the topological $w_{\infty}$ algebra denoted by $w_{\infty}^{\text{top}}$ [26]. It is obtained from the $N = 2$ super $w_{\infty}$ algebra by a twisting procedure introduced by Witten [27]. The idea is to identify one of the fermionic generators as the nilpotent BRST charge $Q$, and to define bosonic generators which can be written in the form $\hat{\psi}_m = \{Q, \text{something} \}$. This is the higher-spin generalization of the property that holds for the energy-momentum tensor of a topological field theory. A suitable candidate for the BRST charge is $Q = -\bar{G}_0^{0, \frac{1}{2}}$. We then define the generators of $w_{\infty}^{\text{top}}$ to be $G_i^{m+\frac{1}{2}}$ and define $\hat{\psi}_m$ as $\hat{\psi}_m = -\{Q, G^{i+\frac{1}{2}}_m \}$. It can be easily shown that these generators obey the algebra

$$\begin{align*}
[\hat{\psi}_m, \hat{\psi}_n] &= [(j + 1)m - (i + 1)n] \hat{\psi}_m^{i+j-\ell} \\
[\hat{\psi}_m, G^{j}_{n+\frac{1}{2}}] &= [(j + 1)m - (i + 1)n] G_i^{j-\ell}_{m+n+\frac{1}{2}} \\
\{G_i^{m+\frac{1}{2}}, G_j^{j}_{n+\frac{1}{2}} &\} = 0
\end{align*}$$

Note that the structure constants for $[\hat{\psi}_m, G^{j}_{n+\frac{1}{2}}]$ are the same as those for $[\hat{\psi}_m, \hat{\psi}_n]$, and that the algebra is centerless. A field theoretic realization of $W_{\infty}^{\text{top}}$ is given in Ref. [26].

Another interesting extension of $w_{\infty}$ involves a Kac-Moody sector. It can be obtained directly by a suitable contraction of the $W_{1+\infty}$ algebra with $SUN$ symmetry found in Ref. [28]. The generators of the algebra are $v_i^m$ and $J_i^a$ where $i$ is an integer such that $i \geq -1$ and $a = 1, ..., N^2 - 1$ labels the adjoint representation of $SU(N)$ and they have the commutation relations

$$\begin{align*}
[v_m^i, v_n^j] &= [(j + 1)m - (i + 1)n]v^{i+j}_{m+n} + \frac{c}{12}(m^3 - m)\delta^{i,0}\delta^{j,0}\delta_{m+n,0} \\
[v_m^i, J_n^{j,a}] &= [(j + 1)m - (i + 1)n]J_{m+n}^{i+j,a} \\
[J_m^i, J_n^{j,a}] &= \frac{1}{2} f^{abc} J_{m+n}^{i+j+1,c} + \frac{1}{16} km\delta^{i+1,0}\delta^{j,0}\delta_{m+n,0}
\end{align*}$$

where $f^{abc}$ are the structure constants of $SU(N)$, and the central extension, $c$, of the Virasoro algebra, and the level, $k$, of the Kac-Moody algebra are related to each other by the Jacobi identity: $c = Nk$. The generators of the above algebra (without central extension) can be represented as follows

$$\begin{align*}
v_m^\ell &= -iy^\ell e^{imx} \left[ (\ell + 1)\frac{\partial}{\partial x} - imy\frac{\partial}{\partial y} \right] \\
J_m^{\ell,a} &= -it^a y^{\ell+1} e^{imx}
\end{align*}$$
where \( t^a \) are the generators of \( SU(N) \). It would be interesting to find a field theoretic realization of this algebra.

There also exists an infinite dimensional generalization of \( w_\infty \) algebra which is related to the symplectic diffeomorphisms in four dimensions. The generators are labelled as \( v_{m}^{\ell,\vec{k}} \) where \( \vec{k} = (k_1, k_2) \) and they obey the algebra [29]

\[
[V_{m}^{\ell,\vec{k}}, V_{n}^{j,\vec{\ell}}] = [(j+1)m - (\ell+1)n]V_{m+n}^{\ell+j,\vec{k}+\vec{\ell}} + \vec{k} \times \vec{\ell} \ V_{m+n}^{\ell+j+1,\vec{k}+\vec{\ell}},
\]

(3.5)

Another infinite dimensional extension of the \( w_\infty \) algebra is the loop algebra of \( w_\infty \). For example, the loop algebra of \( \text{SDiff}(R^2) \) in the basis (2.17) is

\[
[v_{m}^{s}(\sigma), v_{n}^{t}(\sigma')] = \left[(t-n)(s+m) - (s-m)(t+n)\right]v_{m+n}^{s+t-1}(\sigma)\frac{\partial}{\partial \sigma}\delta(\sigma - \sigma')
\]

(3.6)

This concludes the brief survey of some of the salient features of the area-preserving diffeomorphisms, \( w_{1+\infty} \) algebras and their generalizations. We now turn to their field theoretic realizations.

4. \( w_\infty \) Gravity and Supergravity

Before we describe field theoretic realizations of \( w_\infty \) symmetry, it is useful to review a simple field theoretic realization of the Virasoro symmetry, and to formulate it in a language that lends itself readily to a \( w_\infty \) generalization. To this end consider the Lagrangian

\[
\mathcal{L} = \frac{1}{4} \sqrt{-h} \ h^{ij} \partial_i \phi \partial_j \phi,
\]

(4.1)

where \( h^{ij} \) is the inverse of the worldsheet metric \( h_{ij} \), \( (i, j = 0, 1) \), \( h = \text{deth}_{ij} \) and \( \phi \) is a real scalar. This Lagrangian clearly possesses the 2D diffeomorphism and Weyl symmetries. It is convenient to parametrize the metric as follows [25,14]

\[
h_{ij} = \Omega \left( \begin{array}{cc}
 2h_{++} & 1 + h_{++} h_{--} - 2h_{--} \\
 1 + h_{++} h_{--} - 2h_{--} & 2h_{--}
\end{array} \right),
\]

(4.2)

where \( \Omega \) is an arbitrary function which drops out in the action, and the light-cone coordinates are defined by \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \). In terms of these variables the Lagrangian (4.1) becomes

\[
\mathcal{L} = \frac{1}{2}(1 - h_{++} h_{--})^{-1} \left( \partial_+ \phi - h_{++} \partial_- \phi \right) \left( \partial_- \phi - h_{--} \partial_+ \phi \right).
\]

(4.3)

It turns out to be very useful to rewrite this Lagrangian in the following first order form

\[
\mathcal{L} = -\frac{1}{2} \partial_+ \phi \partial_- \phi - J_+ J_- + J_+ \partial_- \phi + J_- \partial_+ \phi - \frac{1}{2} h_{--} J_+^2 - \frac{1}{2} h_{++} J_-^2,
\]

(4.4)
where \( J_{\pm} \) are auxiliary fields which obey the field equations
\[
\begin{align*}
J_+ &= \partial_+ \phi - h_{++} J_- , \\
J_- &= \partial_- \phi - h_{--} J_+ , 
\end{align*}
\]  
(4.5)

These equations define a set of nested covariant derivatives [30]. Solving for \( J_{\pm} \) and substituting into (4.4) indeed yields (4.3). Thus the two Lagrangians are classically equivalent, though in principal they may be quantum inequivalent. The action of the Lagrangian (4.4) is invariant under 2D diffeomorphism transformations, with a general parameter \( k_+ (x^+, x^-) \), given by [11]
\[
\begin{align*}
\delta \phi &= k_+ J_- \\
\delta h_{++} &= \partial_+ k_+ - h_{++} \partial_- k_+ + k_+ \partial_- h_{++} \\
\delta h_{--} &= 0 \\
\delta J_- &= \partial_- (k_+ J_-) \\
\delta J_+ &= 0 ,
\end{align*}
\]  
(4.6)

and 2D diffeomorphisms with parameters \( k_- (x^+, x^-) \), which can be obtained from the above transformations by changing \( + \leftrightarrow - \) everywhere.

The \( W \) symmetric generalization of (4.4) is now remarkably simple. With the further generalization to the case in which the fields \( \phi \) and \( J_{\pm} \) take their values in the Lie algebra of \( SU(N) \) the answer can be written as follows [11]
\[
\begin{align*}
\mathcal{L} = & \operatorname{tr} \left( -\frac{1}{2} \partial_+ \phi \partial_- \phi - J_+ J_- + J_+ \partial_- \phi + J_- \partial_+ \phi \right) \\
& - \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_+ A_{\ell+2} J_-^{\ell+2} - \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_- A_{\ell+2} J_+^{\ell+2} .
\end{align*}
\]  
(4.7)

Note that \( A_{0+} = h_{++} \) and \( A_{0-} = h_{--} \). The equations of motion for the auxiliary fields now reads
\[
\begin{align*}
J_+ &= \partial_+ \phi - \sum_{\ell \geq 0} A_+ A_{\ell+1} J_-^{\ell+1} , \\
J_- &= \partial_- \phi - \sum_{\ell \geq 0} A_- A_{\ell+1} J_+^{\ell+1} .
\end{align*}
\]  
(4.8)

The Lagrangian (4.7) possesses the \( w_\infty \) symmetry with parameters \( k_+ (x^+, x^-) \) that gener-
\[12\]
alize (4.6) as follows [11]

\[ \delta \phi = \sum_{\ell \geq -1} k_{+\ell} J_{-}^{\ell+1} \]

\[ \delta A_{+\ell} = \partial_{+} k_{+\ell} - \sum_{j=0}^{\ell} [(j + 1) A_{+j} \partial_{-} k_{+(\ell-j)} - (\ell - j + 1) k_{+(\ell-j)} \partial_{-} A_{+j}] \]

\[ \delta A_{-\ell} = 0 \]

\[ \delta J_{-} = \sum_{\ell \geq -1} \partial_{-} [k_{+\ell} (J_{-})^{\ell+1}] \]

\[ \delta J_{+} = 0, \]

and \( W \) transformations with parameters \( k_{+\ell}(x^+, x^-) \) which can be obtained from above by the replacement \(+ \leftrightarrow -\) everywhere. It is important to note that we must set \( k_{-1} = -\frac{1}{N} \sum_{\ell \geq 1} k_{\pm\ell} \text{tr} J_{+}^{\ell+1}, \) (4.10)

to ensure the tracelessness \( \delta \phi \) and \( \delta J_{\pm} \) in the transformation rules above. The Lagrangian (4.8) has also Stueckelberg type shift symmetries which arise due to the fact that for \( SU(N) \), only \((N - 1)\) Casimirs of the form \( \text{tr}(J_{\pm})^{\ell+2} \) are really independent, while the rest can be factorize into products of these Casimirs. For a further discussion of these symmetries, see Ref. [11].

There exists an interesting chiral truncation of the \( w \) gravity theory discussed above. It is achieved by setting \( A_{+\ell} = 0 \). In that case from (4.10) we have \( J_{+} = \partial_{+} \phi \) and \( J_{-} = \partial_{-} \phi - \sum_{\ell \geq 0} A_{-\ell} \text{tr}(\partial_{+} \phi)^{\ell+1} \). In this case, it is more convenient to work in second order formalism. Thus, substituting for \( J_{\pm} \) into the Lagrangian (4.7), we obtain [11]

\[ \mathcal{L} = \frac{1}{2} \text{tr} \partial_{+} \phi \partial_{-} \phi - \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_{\ell} \text{tr}(\partial_{+} \phi)^{\ell+2}, \] (4.11)

where we have used the notation \( A_{-\ell} = A_{\ell} \). This Lagrangian has the following symmetry [11]

\[ \delta \phi = \sum_{\ell \geq -1} k_{\ell} (\partial_{+} \phi)^{\ell+1} \]

\[ \delta A_{\ell} = \partial_{-} k_{\ell} - \sum_{j=0}^{\ell+1} [(j + 1) A_{j} \partial_{+} k_{\ell-j} - (\ell - j + 1) k_{\ell-j} \partial_{+} A_{j}] \] (4.13)

The Lagrangian (4.11) has also the appropriate Stueckelberg symmetry. Using this symmetry one can obtain [11] the chiral \( W_{3} \) gravity of Ref. [31]. Note that the interaction term in this
Lagrangian has the form of a gauge field × conserved current \( \frac{1}{\ell+2} \text{tr}(\partial_+ \phi)^{\ell+2} \). It is important to note that the OPE of these currents do not form a closed algebra, while they do close with respect to Poisson bracket. Hence, the \( w_\infty \) symmetry described above is a classical symmetry, as expected.

The \( w_\infty \) gravity with \( N = 2 \) super \( w_\infty \) symmetry has also been constructed [32]. For readers’ convenience we summarize the result of Ref. [32] for chiral \( N = 2 \) super \( w_\infty \) here.

Consider two real scalar superfields \( \phi \) and \( \bar{\phi} \). Let the superspace coordinates be \( Z = (z, \theta) \), and define the covariant derivatives \( D = \partial_\theta - \theta \partial \) and \( \bar{D} = \partial_\bar{\theta} - \bar{\theta} \partial \). The currents which classically generate the \( N = 2 \) super \( w_\infty \) algebra are [32]

\[
\delta \phi = \sum_{\ell=-1,0,...}^\infty k_\ell D\phi(\partial_\phi)^{\ell+1} + \sum_{\ell=-\frac{1}{2},\frac{1}{2},...}^\infty \left[ k_\ell(\partial_\phi)^{\ell+\frac{3}{2}} - \frac{1}{2} Dk_\ell D\phi(\partial_\phi)^{\ell+\frac{3}{2}} \right]
\]

(4.16)

The transformation rules for the matter fields are [32]

\[
\delta \bar{\phi} = \sum_{\ell=-1,0,...}^\infty \left\{ - k_\ell(\partial_\phi)^{\ell+1} D\bar{\phi} - (\ell + 1) D[k_\ell D\phi(\partial_\phi)^{\ell+1} D\bar{\phi}] \right\}
\]

\[
+ \sum_{\ell=-\frac{1}{2},\frac{1}{2},...}^\infty \left\{ \frac{1}{2} Dk_\ell(\partial_\phi)^{\ell+\frac{3}{2}} - (\ell + \frac{3}{2}) D[k_\ell(\partial_\phi)^{\ell+\frac{3}{2}} D\bar{\phi}] \right\}
\]

\[
+ \frac{1}{2}(\ell + \frac{1}{2}) D[k_\ell D\phi(\partial_\phi)^{\ell-\frac{1}{2}} D\bar{\phi}] \}
\]

The gauge fields transform as follows [32]

\[
\delta A_\ell = \bar{D}k_\ell + \sum_{j=\frac{1}{2},1,...} \left[ (\frac{1}{2} - j) A_j \partial_{-j+\frac{5}{2}} k_\ell \right]
\]

(4.17a)

\[
+ \frac{1}{2}(-1)^{2j} D\bar{A}_j Dk_{\ell-j+\frac{5}{2}} + (\ell - j + 3) \partial A_j k_{\ell-j+\frac{5}{2}} \]
\]

for \( \ell = -\frac{1}{2}, \frac{1}{2}, ... \), and

\[
\delta A_\ell = \bar{D}k_\ell - 2 \sum_{j=1,2,...}^{\ell+\frac{3}{2}} A_j k_{\ell-j+\frac{5}{2}} + \sum_{j=\frac{1}{2},\frac{3}{2},...} \left[ (\frac{1}{2} - j) A_j \partial_{-j+\frac{5}{2}} k_\ell \right]
\]

(4.17b)

\[
+ \frac{1}{2}(-1)^{2j} D\bar{A}_j Dk_{\ell-j+\frac{5}{2}} + (\ell - j + 3) \partial A_j k_{\ell-j+\frac{5}{2}} \]
\]

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for \( \ell = -1, 0, \ldots \). The spin \( \frac{1}{2} \) transformations can be included as in Ref. [32]. The nonchiral version of this theory has also been constructed [32].

Going back to the bosonic version of nonchiral \( w_\infty \) gravity, an interesting geometric formulation of it has been given by Hull [14]. Consider the case of a single real scalar \( \phi \). According to Ref. [14], the symmetry algebra of nonchiral \( w_\infty \) gravity is a subalgebra of the more general set of transformations

\[
\delta \phi = \sum_{n=2}^{\infty} \lambda^{i_1i_2\cdots i_{n-1}}(x) \partial_{i_1} \phi \partial_{i_2} \phi \cdots \partial_{i_{n-1}} \phi \equiv \Lambda(x^i, y_i), \tag{4.18}
\]

where \( y_i = \partial_i \phi \) and \( \lambda^{i_1i_2\cdots i_{n-1}}(x)(n = 2, \ldots) \) are the infinitesimal parameters which are symmetric tensors on the worldsheet. It is easy to show that these transformations satisfy the Poisson bracket algebra on the phase space with coordinates \( x^i, y_i \). The action proposed in Ref. [14] is

\[
S = \int d^2 x \tilde{F}(x, y) \tag{4.19}
\]

where \( \tilde{F}(x, y) \) has the expansion

\[
\tilde{F}(x, y) = \sum_{n=2}^{\infty} \frac{1}{n} \tilde{h}^{i_1i_2\cdots i_n}(x) y_{i_1} y_{i_2} \cdots y_{i_n}, \tag{4.20}
\]

where \( \tilde{h}^{i_1i_2\cdots i_n}(x) \) are gauge fields which are tensor densities on the world-sheet. Invariance of this action under the transformations (4.18) requires the imposition of the following constraint on the parameter \( \Lambda(x, y) \) [14]

\[
\epsilon_{ik} \epsilon_{jl} \frac{\partial^2 \Lambda}{\partial y_i \partial y_j} \frac{\partial^2 \tilde{F}}{\partial y_k \partial y_l} = 0 \tag{4.21}
\]

The expansion of this equation gives an infinite number of algebraic constraints on the parameters. The first such constraint is \( \lambda^{ij} \tilde{h}_{ij} = 0 \). Other constraints relate the trace of a rank \( \geq 3 \) parameter to the products of lower rank parameters and gauge fields. The invariance of the action under the transformations (4.18), in addition to the constraint (4.21), also requires that the gauge fields transform as [14]

\[
\delta \tilde{h}^{i_1\cdots i_p} = \sum_{n=2}^{p} \left[ - \frac{(p - n + 1)(n - 1)}{(p - 1)} \partial_j \left( \tilde{h}^{j(i_1\cdots i_{n-1} \lambda^{i_n\cdots i_p})} - \lambda^{j(i_1\cdots i_{p-n} \tilde{h}^{i_p\cdots i_{n+1}})} \right) 
- (n - 1) \tilde{h}^{j(i_1\cdots i_{n-1}} \partial_j \lambda^{i_n\cdots i_p)} + (p - n + 1) \lambda^{i_1\cdots i_{p-n+1}} \partial_j \tilde{h}^{i_{p-n+2} \cdots i_p)} \right] \tag{4.22}
\]

It turns out that a gauge invariant condition can be imposed on the gauge fields which reduces at the linearized level to the constraint present in \( w_\infty \) gravity. This constraint is [14]

\[
\det \left( \frac{\partial^2 \tilde{F}(x, y)}{\partial y_i \partial y_j} \right) = -1 \tag{4.23}
\]
The expansion of this equation gives an infinite number of algebraic constraints on the gauge fields. The first two constraints are \( \det (\tilde{h}^{ij}) = -1 \) and \( \tilde{h}^{ijk} \tilde{h}_{ij} = 0 \). Other constraints relate the trace of rank \( \geq 4 \) gauge fields to the products of lower rank gauge fields. The procedure for solving the constraints (4.21) and (4.22) in terms of unconstrained gauge fields and gauge parameters, as well as the attendant generalized Weyl symmetries have been outlined in Ref. [14].

The equation (4.23) has a nice geometrical interpretation. With an appropriate identification of \( \tilde{F}(x, y) \) with a Kahler potential, (4.23) can be interpreted as the Monge-Ampere equation for a Kahler metric of a self-dual four-manifold [14]. Other relations between \( w_\infty \) and self-dual geometry have been discussed in [7,16]

5. Nonlinear Realizations of \( w_\infty \)

The field theoretic realization of \( w_{1+\infty} \) in terms of a single real scalar as given in (4.12) can be understood within the framework of nonlinear realizations [11]. Denoting the coordinates of a cylinder by \( (x^+, y) \), the scalar field \( \phi(x^+, x^-) \) can be considered as parametrizing the coset space \( w_{1+\infty}/w_\infty \). The generators of this coset are \( v_{m}^{-1} = -ime^{im\theta} \partial_y \), and therefore we can choose the coset representative to be \( e^{\partial_y \phi} \). If we denote the coset generators by \( K \), and the subalgebra generators by \( H \), then one can see from (2.19) that the structure of the algebra is \([H,H] \subset H, [H,K] \subset H + K, [K,K] = 0\). Thus the coset \( w_{1+\infty}/w_\infty \) is not a symmetric space, and the K generators do not even form a linear representation of \( H \) (except when the subalgebra element lies in the Virasoro subalgebra). Although we can apply the general theory of non-linear realizations to this situation, we should not be surprised to find that some of the features are non-standard. In particular, the action of \( H \) on the coset will in general be non-linear.

Acting on the coset representative with a general \( G \) transformation \( g \), one has

\[ ge^{\partial_y \phi} = e^{\partial_y \phi} h, \]  

where \( h \) is an element of the subalgebra \( H \). For an infinitesimal transformation \( g = 1 + \delta g \), one therefore finds

\[ \partial_+ \delta \phi = \left( e^{-\partial_y \phi} \delta g e^{\partial_y \phi} \right)_{C/H} \]  

Taking \( \delta g = k_\ell (x^+ y) \ell+1 \) we find that \( \partial_+ \delta \phi = \partial_+ [k_\ell(\partial_+ \phi)^{\ell+1}] \). Hence we have

\[ \delta \phi = k_\ell (x^+ (\partial_+ \phi)^{\ell+1} \]  

which is indeed the global \( w_{1+\infty} \) transformation discussed in the previous section. This result can also be derived in the Poisson bracket language. Corresponding to (5.2) we have

\[ \delta \phi = (e^{-Ad_\phi} \delta g) \mid_{y=0} \]

\[ = (\delta g + \{\phi, \delta g\} + \frac{1}{2!}\{\phi, \{\phi, \delta g\}\} + \cdots) \mid_{y=0} \]  

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Setting \( y = 0 \) amounts to restriction to the coset direction. To generalize this construction to the case of the non-chiral bosonic \( w_\infty \) model in the light cone gauge, we consider two copies of the cylinder with coordinates \((x^+, y)\) and \(x^-, \tilde{y}\), so that we have a total of four coordinates \((x^+, x^-, y, \tilde{y})\). The coordinates \(y\) and \(\tilde{y}\) play the rôle of “momenta” conjugate to \(x^+\) and \(x^-\), and so one can think of the four-dimensional space as being the cotangent bundle \(T^*\Sigma\) of the 2-torus \(\Sigma\). The area-preserving transformations on the two cylinders may be described by introducing the symplectic form

\[
\Omega = dx^+ \wedge dy - dx^- \wedge d\tilde{y},
\]

and using this to define a Poisson bracket \(\{f, g\} = \Omega^{ij} \partial_i f \partial_j g\) in the four-dimensional space. The symplectic diffeomorphisms which leave this form invariant are \(\delta x^i = \{x^i, \delta g\}\), and the global non-chiral \(w_\infty\) transformations correspond to restricting \(\delta g\) to have the form \(\delta g = k_-(x^+)y^{\ell+1} + k_+(x^-)\tilde{y}^{\ell+1}\). Their action on \(\phi\) is given by (5.4), where \(\text{Ad}_\phi \delta g\) is now taken to be \(\Omega^{ij} \partial_i \phi \partial_j \delta g\) [11].

It is instructive to consider some other nonlinear realizations of \(w_{1+\infty}\) involving more than a single scalar field. We have considered elsewhere [15] the nonlinear realization based on the coset \(w_{1+\infty}^\dagger/\text{Vir}^+\), where \(w_{1+\infty}^\dagger\) is generated by \(v^m_\ell\) with \(m \geq -\ell - 1\), and \(\text{Vir}^+\) is generated by \(v^0_m\) with \(m \geq 0\), i.e. the positive modes of the Virasoro algebra. In this construction we view the coordinate \(x^+\) itself as a coset parameter associated with the Virasoro generator \(L_{-1} = v^0_{-1}\). We choose the coset representative

\[
k = e^{-x^+ v^0_0} \prod_{\ell \neq 0} e^{-\phi(\ell)},
\]

where we have used the notation \(\phi(\ell) = \sum_{m=-\ell-1}^{\infty} \phi_m \ell \phi^\ell_m\). Note that there are infinitely many initial Goldstone fields \(\phi_m^\ell\) corresponding to the coset generators \(v^m_\ell, \ell \neq 0\). Excluding the generator \(v_{0}^{-1}\) which does not correspond to a Goldstone field but to the coordinate \(x^+\), all the remaining generators of the coset form a linear representation of the subalgebra. We can construct the Cartan-Maurer form as follows

\[
\mathcal{P} = k^{-1} dk = E^0_{-1} v^0_{-1} + \sum_{\ell \neq 0} E^\ell_m v^\ell_m + \sum_{m \geq 0} \omega^0_m v^0_m,
\]

where \(E_m^\ell\) and \(\omega_m^0\) are all 1-forms, i.e. \(E_m^\ell = dx^+ E_m^\ell\). Next we look for a maximum set of covariant constraints with which we can eliminate the inessential Goldstone fields. Such a constraint is

\[
E_m^\ell = 0, \quad \text{for} \quad \ell \neq 0
\]

As a consequence of we find that

\[
\begin{align*}
E^0_{-1} &= -dx^+, \\
\omega^0_m &= 0 \\
\phi_m^\ell &= -\frac{1}{\ell + m + 1} \partial_+ \phi^\ell_{m-1}, \quad \text{for} \quad m \geq -\ell
\end{align*}
\]
The last relation shows that only the Goldstone fields $\phi^\ell_{-\ell-1}$ (the “edge fields”) survive the constraints, and all the other Goldstone fields can be expressed in terms of their derivatives.

The transformation rules for the surviving Goldstone fields can be derived as follows. The action of the group $w_{1+\infty}^\uparrow$ on a coset representative, which we shall generically denote $e^{-\phi(x)}$, is given by

$$ge^{-\phi(x)} = e^{-\phi(x')} h,$$

where $h$ is an element of the divisor subgroup $\text{Vir}^\uparrow$. For infinitesimal transformations, we have

$$e^{\phi(x)+\delta\phi(x)} (1 + \delta h) e^{-\phi(x)} = 1 + \delta h,$$

where

$$\delta\phi(x) \equiv \phi'(x') - \phi(x) = \phi'(x) + \delta x^+ \partial_+ \phi(x) - \phi(x)$$

$$\equiv \delta\phi(x) + \delta x^+ \partial_+ \phi(x).$$

Projecting (5.11) into the coset direction yields the formula

$$\left. \left( e^\delta e^{-\phi} \right) \right|_{G/H} = \left. \left( e^\delta g e^{-\phi} \right) \right|_{G/H}.$$

Upon the use of (5.9), the variation (5.12) simplifies to

$$\delta\phi^\ell = \delta\phi^\ell - [\phi^\ell, v_{-1}^0] \delta x^+.$$ Substituting this result in (5.13) we find that all the $\delta x^+$ terms cancel pairwise except the $-\delta x^+$ in the $v_{-1}^0$ direction. Studying (5.13) level by level in spin, the transformation rules for all the Goldstone fields can now be derived. To this end it is useful to consider the first two levels of dressing of $\delta g$ which we parametrize as follows

$$\delta g = \sum_{\ell,m} \alpha^\ell_m v^\ell_m,$$

where the $\alpha^\ell_m$ are constant parameters. Consider a spin-$\ell$ transformation with parameter $\alpha^\ell \equiv \sum_m \alpha^\ell_m v^\ell_m$. Defining

$$e^{x^+ v_{-1}^0} \alpha^\ell \equiv \sum_m \beta^\ell_m (x^+) v^\ell_m,$$

we find that

$$\beta^\ell_{m+1} (x^+) = \frac{-1}{\ell + m + 2} \partial_+ \beta^\ell_m (x^+).$$

Note that this is the same relation as that satisfied by the fields $\phi^\ell_m$. Proceeding on to the next level of dressing, we define

$$\sum_{\ell,m} e^{\phi_{-1}^\ell} \beta^\ell_m (x^+) v^\ell_m e^{-\phi_{-1}^\ell} = \sum_{\ell,m} \gamma^\ell_m (\phi_{-1}^\ell) v^\ell_m$$
The field dependent parameters $\gamma^\ell_m$ can be straightforwardly computed in terms of $\beta^\ell_{m+1}(x^+)$, and as a consequence one finds that
\begin{equation}
\gamma^\ell_{-\ell-1} = \frac{1}{(\ell+1)!} \frac{\delta^{\ell+1} \gamma_0^{-1}}{\delta y^{\ell+1}},
\end{equation}
where we have introduced the notation $y \equiv -\partial_+ \phi_0^{-1}$. In terms of these quantities, Eq. (5.13) yields the results
\begin{align}
\delta x^+ &= -\gamma_0^{-1}, \quad (5.20a) \\
\delta \phi_0^{-1} &= -\gamma_0^{-1}. \quad (5.20b) \\
\delta \phi_2^{-2} &= -\gamma_2^{-2} - 2\phi_2^{-2} \partial_+ \gamma_0^{-1} + \partial_+ \phi_2^{-2} \gamma_0^{-1} \quad (5.20c) \\
\delta \phi_3^{-3} &= -\gamma_3^{-3} + \gamma_1^{-1} \partial_+ \phi_3^{-3} - 3\partial_+ \gamma_1^{-1} \phi_3^{-3} + \gamma_2^{-1} \partial_+ \phi_2^{-2} - \partial_+ \gamma_2^{-1} \phi_2^{-2} \quad (5.20d) \\
&\vdots
\end{align}
Note that only fields and parameters corresponding to the left edge, i.e. $\phi^\ell_{-\ell-1}$ occur in these results. Moreover, Eq. (5.20b) is precisely the $w_{1+\infty}$ transformation rule of Eq. (4.12), after identifying $\phi$ and $k^\ell$ with $(-\phi_0^{-1})$ and $\beta^\ell_{-\ell-1}$, respectively.

It may be verified that the action $L_0 = \frac{1}{2} \partial_+ \phi_0^{-1} \partial_- \phi_0^{-1}$ transforms by a total derivative under the full set of $w^\ell_{1+\infty}$ transformations $\delta \phi_0^{-1} = -\sum_{\ell=-1}^\infty k^\ell \gamma^{\ell+1}$. Free second-order scalar actions involving the higher left-edge Goldstone fields $\phi^\ell_{-\ell-1}$ cannot be made because they do not have Lorentz weight zero, since the Lorentz weight of $\phi^\ell_{-\ell-1}$ is $-(\ell+1)$. One may, however, couple the higher Goldstone fields to currents built from $\phi_0^{-1}$. One way to this would be to construct a composite $w_{1+\infty}$ connection $A^\ell_{-\ell-1}$ built out of the “edge scalars”, and to use it in the $w_{\infty}$ gravity action of Ref. [11], thus obtaining the global $w_{1+\infty}$ symmetric Lagrangian
\begin{equation}
L = \frac{1}{2} \partial_+ \phi_0^{-1} \partial_- \phi_0^{-1} - \sum_{\ell=0}^\infty \frac{1}{\ell+2} A^\ell_{-\ell-1}(\phi)(-\partial_+ \phi_0^{-1})^{\ell+2}.
\end{equation}
The construction of the composite connection requires further work. The idea is essentially as outlined above, though, the construction may have to be based on a different coset space than the one considered here [15].

6. Quantum Realizations of $w_{\infty}$

The realization of $w_{1+\infty}$ in terms of the currents
\begin{equation}
v^\ell(z) = \frac{1}{\ell+2} (\partial_+ \phi)^{\ell+2}
\end{equation}
is necessarily a classical one, since in commuting two of these currents which involves an
operator product expansion, there will be terms coming from multiple Wick contractions of
the basic building blocks \( \partial \phi \), giving rise to terms that violate the closure of the algebra.
The single contraction, on the other hand, corresponds to evaluating the Poisson bracket
of the two currents, evidently gives a closed algebra. If one modifies the currents \((6.1)\) as

\[ V_\ell(z) = \sum c_{mnp}^\ell (\sqrt{\hbar})^m (\partial^n \phi)^p \]

where \( c_{mnp}^\ell \) are a set of constant coefficients, then the algebra will of course close on these currents. (On dimensional grounds \((n + 1)p + m = 2\ell + 2\)).

Although this closure may in some sense be considered as trivial, in fact there is a way of
choosing the coefficients (which amounts to using a certain basis for the algebra) in such
a way that the currents \( V_\ell(z) \) have a definite transformation property under an \( SL(2, R) \)
and moreover they are quasi-primary fields with respect to a natural Virasoro subalgebra.
The resulting algebra is the \( W_{1+\infty} \) algebra. At the level of quantum field theories this
phenomenon amounts to the deformation of the classical \( w_{1+\infty} \) symmetry to a quantum
\( W_{1+\infty} \) symmetry \([12]\). The former can be obtained in the \( \hbar \to 0 \) limit of the latter one. This
phenomenon has been found in the study of quantum \( w_\infty \) gravity as well as the study of a
quantum mechanical system on a circle \([13]\).

Turning our attention to the issue of constructing a quantum realization of \( w_{1+\infty} \), one
possibility is to construct the \( w_{1+\infty} \) currents in terms of b-c ghost systems. As a first step,
one constructs the BRST charge \([33]\)

\[ Q_{gh} = \oint f^{ij}(\partial, \partial)(- \frac{1}{2} c_i c_j b_{i+j}), \quad (6.2) \]

where \( b_i(z) \) and \( c_i(z) \) are anticommuting ghosts satisfying

\[ b_i(z)c_j(w) \sim \frac{\delta_{ij}}{z-w}, \quad (6.3) \]

or equivalently, the anticommutator \( \{b_i, c_j\} = \delta_{ij} \) and \( f^{ij}(m, n) = (j + 1)m - (i + 1)n \) are
the structure constants of \( w_{1+\infty} \). The notation \( f^{ij}(\partial, \partial) \) indicates that the Fourier-mode
index \( m \) is replaced by a partial derivative that acts on \( c_i \) only, and the index \( n \) is replaced
by a partial derivative that acts on \( c_j \) only. From \( Q_{gh} \), we can derive the ghostly quantum
realization \( v^i_{gh} \) of \( w_{1+\infty} \)

\[ v^i_{gh} = \{Q_{gh}, b_i\}, \quad (6.4) \]

This formula yields \([33]\)

\[ v^i_{gh}(z) = \sum_{j \geq 0} (i + j + 2) \partial c_j b_{i+j} + (j + 1)c_j \partial b_{i+j}, \quad (6.5) \]

which indeed generate the \( w_\infty \) algebra:

\[ v^i_{gh}(z)v^j_{gh}(w) \sim \frac{i + j + 2}{(z-w)^2} v_i^{i+j}(w) + \frac{i + 1}{z-w} \partial v^i_{gh}(w) + \delta^{i0} \delta^{j0} \frac{c/2}{(z-w)^4}. \quad (6.6) \]
The operator terms on the right-hand side come from single contractions. The central term in the spin-2 sector (the only one that occurs in the $w_\infty$ algebra) has a central charge that is formally divergent. After zeta-function regularization, one finds $c = 2$ [34,33].

Using the structure constants of the topological algebra $w_\infty^{\text{top}}$ given in (3.2), by means of the method outlined above, one can also construct its ghostly quantum realization as follows [26]

$$\hat{\gamma}^i_{\text{gh}}(z) = (i + j + 2)\partial c_j b_{i+j} + (j + 1)c_j \partial b_{i+j} - (i + j + 2)\partial b_j \beta_{i+j} - (j + 1)\gamma_j \partial \beta_{i+j},$$

$$G^i_{\text{gh}}(w) = (i + j + 2)\partial c_j \beta_{i+j} + (j + 1)c_j \partial \beta_{i+j};$$

(6.7)

where $\beta, \gamma$ are the commuting ghost fields, and summation over $j$ is understood. A detailed description of topological $w_{1+\infty}$ is given in Ref. [26].

Finally, let us mention a quantum realization of the loop algebra of SDiff($R^2$) due to Witten [23], which has emerged in the study of string theory with two dimensional target spacetime. The theory is characterized by the stress tensor

$$T_{zz} = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial \phi)^2 + \sqrt{2}\phi - 2b\partial c + c\partial b$$

(6.8)

where $X$ is the matter field, $\phi$ is the Liouville field and $b, c$ are the ghost fields, all of which obey the usual OPE rules. A quantum realization of of the loop algebra of SDiff($R^2$) is given by [23]

$$\nu^n_{\text{h}}(z) = \left(e^{-i\sqrt{2}\phi}\right)^{s-n} e^{i\sqrt{2}X} e^{\sqrt{2}(1-s)\phi}$$

(6.9)

Acting on a polynomials in [23]

$$x = \left(cb + \frac{i}{\sqrt{2}}(\partial X - i\partial \phi)\right) e^{\sqrt{2}(X+i\phi)}$$

$$y = \left(cb - \frac{i}{\sqrt{2}}(\partial X + i\partial \phi)\right) e^{-i\sqrt{2}(X-i\phi)},$$

(6.10)

the generators defined in (6.9) obey the algebra (3.6). In particular $v_{1/2}^1$ acts like $\frac{\partial}{\partial y}$ and $v_{1/2}^{-1}$ acts like $\frac{\partial}{\partial x}$, and hence commute, on the plane. However, as operators they do not commute, yielding a central extension term [23].

7. Comments

The subject of $w_\infty$ algebras is a rapidly growing one. Here we have attempted to give some of the salient features of this subject, and necessarily have omitted a number of topics, some of which are briefly mentioned in the introduction. Many of these topics are relevant in one way or another to the question of how to construct new string theories with higher world-sheet, and possibly higher spacetime symmetries.
The $w_\infty$ algebra is a certain $N \to \infty$ limit $[35]$ of the $W_N$ algebra $[36]$. Some work has already been done on string theories based on $W_N$ symmetry $[37,38,39]$. A typical feature which has arisen is that although one expects a higher slope and therefore higher spin massless fields in the spectrum $[40]$, it turns out that due to the necessary presence of a background charge at least for one of the scalars in the theory, the slope is pushed back to a value in such a way that there are no massless higher spin fields after all $[38,39]$. At the end, one finds the spectrum of the usual string and new massive trajectories. In this author’s opinion this state of affairs is somewhat disappointing, because the symmetry enlargement one intuitively expects is buried in the complexities of the massive trajectories, if at all there. Of course, there may be new field theoretic realizations of $W$-algebras still to be discovered where the situation may be dramatically different.

A string theory based on $w_\infty$, or its quantum deformation $W_\infty$ has not been considered so far. It would be very useful to accumulate field theoretic realizations of these symmetries which might play a role in constructing a sensible $w_\infty$ string theory. Such a theory may as well look like a topological field theory, since there would be infinitely many physical state conditions to satisfy. We would like to speculate, however, that there may exist a $w_\infty$ string theory where only a tower of higher spin massless fields in target spacetime would arise in the spectrum, corresponding to the infinitely many higher spin world-sheet symmetries, and that the usual string theory with its massive states may arise as a result of some sort of “spontaneous” symmetry breaking. Clearly a lot remains to be done, and there could be some surprises ahead in the search for a string theory (or possibly a theory of higher extended objects, such as supermembranes) with higher symmetries than those which have been realized so far.

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