Exit laws from large balls of (an)isotropic random walks in random environment

Erich Baur∗
Erwin Bolthausen†

Abstract

We study exit laws from large balls in $\mathbb{Z}^d$, $d \geq 3$, of random walks in an i.i.d. random environment which is a small perturbation of the environment corresponding the simple random walk. Under a centering condition on the measure governing the environment, we prove that the exit laws are close to those of a symmetric random walk, whose underlying nearest neighbor transition kernel we identify as a small perturbation of the simple random walk kernel. We obtain three types of estimates: Difference estimates between the two exit measures and difference estimates between smoothed versions, both in total variation distance, as well as local results comparing exit probabilities on boundary segments. The results are sufficient to deduce transience of the random walks in random environment.

Our work includes the results on isotropic random walks in random environment which appeared in Bolthausen and Zeitouni [7]. Since several proofs in [7] turned out to be incomplete, a somewhat different approach was given in the first author’s thesis [1]. Here, we revisit this approach and extend it to certain anisotropic random walks in random environment. This has to our knowledge not yet been considered before, and it should be seen as a further step towards a fully perturbative theory of random walks in random environment.

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∗Universität Zürich. Email: erich.baur@math.uzh.ch.
†Universität Zürich. Email: eb@math.uzh.ch.
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0 Introduction and main results

Our model of random walks in random environment

Consider the integer lattice $\mathbb{Z}^d$ with unit vectors $e_i$, whose $i$th component equals 1. We let $P$ be the set of probability distributions on $\{\pm e_i : i = 1, \ldots, d\}$. Given a probability
measure $\mu$ on $\mathcal{P}$, we equip $\Omega = \mathcal{P}^{2d}$ with its natural product $\sigma$-field $\mathcal{F}$ and the product measure $\mathbb{P} = \mu^{\otimes 2d}$. Each element $\omega \in \Omega$ yields transition probabilities of a nearest neighbor Markov chain on $\mathbb{Z}^d$, the random walk in random environment (RWRE for short), via

$$p_\omega(x, x + e) = \omega_x(e), \quad e \in \{\pm e_i : i = 1, \ldots, d\}.$$  

We write $P_{x,\omega}$ for the “quenched” law of the canonical Markov chain $(X_n)_{n \geq 0}$ with these transition probabilities, starting at $x \in \mathbb{Z}^d$.

We study asymptotic properties of the RWRE in dimension $d \geq 3$ when the underlying environments are small perturbations of the fixed environment $\omega_x(\pm e_i) = 1/(2d)$ corresponding to simple random walk.

- Let $0 < \varepsilon < 1/(2d)$. We say that $A_0(\varepsilon)$ holds if $\mu(P_\varepsilon) = 1$, where

$$P_\varepsilon = \{q \in \mathcal{P} : |q(\pm e_i) - 1/(2d)| \leq \varepsilon \text{ for all } i = 1, \ldots, d\}.$$  

The perturbative behavior concerns the behavior of the RWRE when $A_0(\varepsilon)$ holds for small $\varepsilon$. However, even for arbitrarily small $\varepsilon$, such walks can behave in very different manners. This motivates a further “centering” restriction on $\mu$.

- We say that $A_1$ holds if $\mu$ is invariant under all $d$ reflections $O_i : \mathbb{R}^d \to \mathbb{R}^d$ mapping the unit vector $e_i$ to its inverse, i.e. $O_ie_i = -e_i$ and $O_i e_j = e_j$ for $j \neq i$. In other words, the laws of $(\omega_0(O_i e_i))_{|e_i|=1}$ and $(\omega_0(e_i))_{|e_i|=1}$ coincide, for each $i = 1, \ldots, d$.

Condition $A_1$ is weaker than the isotropy condition introduced by Bricmont and Kupiainen [23], which requires that $\mu$ is invariant under all orthogonal transformations $O : \mathbb{R}^d \to \mathbb{R}^d$ fixing the lattice $\mathbb{Z}^d$. This stronger condition was also assumed in Bolthausen and Zeitouni [7], in the first author’s thesis [1] and in a similar form in Sznitman and Zeitouni [23], who consider isotropic diffusions.

Our main results

Write $V_L = \{y \in \mathbb{Z}^d : |y| \leq L\}$ for the discrete ball of radius $L$. Given $\omega \in \Omega$, denote by $\Pi_L = \Pi_L(\omega)$ the exit distribution from $V_L$ of the random walk with law $P_{x,\omega}$, i.e.

$$\Pi_L(x, z) = P_{x,\omega}(\tau_L = z),$$

where $\tau_L = \inf\{n \geq 0 : X_n \notin V_L\}$. For probability measures $\nu_1$ and $\nu_2$, we let $||\nu_1 - \nu_2||_1$ be the total variation distance between $\nu_1$ and $\nu_2$. Denote by $E$ the expectation with respect to $\mathbb{P}$, and let $p_\nu(\pm e_i) = p_\nu(x, x \pm e_i) = 1/(2d)$ be the transition kernel of simple random walk.

**Proposition 0.1.** Assume $A_1$. There is $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, under $A_0(\varepsilon)$ the limit

$$2p_\infty(\pm e_i) = \lim_{L \to \infty} \sum_{y \in \mathbb{Z}^d} E[\Pi_L(0, y)] \frac{|y_i|^2}{|y|^2}$$

exists for $i = 1, \ldots, d$. Moreover, $||p_\infty - p_0||_1 \to 0$ as $\varepsilon \downarrow 0$.  

From now on, \( p_{\infty} \) is always given by the limit above. The proposition suggests that for large radii \( L \), the RWRE exit measure should be close to that of a symmetric random walk with transition kernel \( p_{\infty} \). Write \( \pi_{L}^{(p)}(x, \cdot) \) for the exit distribution from \( V_{L} \) of a random walk with homogeneous nearest neighbor kernel \( p \), started at \( x \in \mathbb{Z}^{d} \). Recall that we assume \( d \geq 3 \).

**Theorem 0.1.** Assume A1. For \( \delta > 0 \) small enough, there exists \( \varepsilon_{0} = \varepsilon_{0}(\delta) > 0 \) such that if A0(\( \varepsilon \)) is satisfied for some \( \varepsilon \leq \varepsilon_{0} \), then

\[
\mathbb{P} \left( \sup_{x \in V_{L/5}} \| (\Pi_{L} - \pi_{L}^{(p_{\infty})})(x, \cdot) \|_{1} > \delta \right) \leq \exp \left( - (\log L)^{2} \right).
\]

The difference in total variation of the exit laws of the RWRE and the random walk with kernel \( p_{\infty} \) does not tend to zero, due to localized perturbations near the boundary. However, with an additional smoothing, convergence occurs. Let \( \rho \) be a random variable with a smooth density compactly supported in \((1, 2)\).

**Theorem 0.2.** There exists \( \varepsilon_{0} > 0 \) such that if A0(\( \varepsilon \)) is satisfied for some \( \varepsilon \leq \varepsilon_{0} \), then for any \( \eta > 0 \), we can find \( L_{\eta} \) and a smoothing radius \( m_{\eta} \) such that for \( m \geq m_{\eta}, L \geq L_{\eta} \),

\[
\mathbb{P} \left( \sup_{x \in V_{L/5}} \| (\Pi_{L} - \pi_{L}^{(p_{\infty})})\pi_{p-m}^{(p_{\infty})}(x, \cdot) \|_{1} > \eta \right) \leq \exp \left( - (\log L)^{2} \right).
\]

**Remark 0.1.** (i) As an easy consequence of the last theorem, if one increases the smoothing scale with \( L \), i.e. if \( m = m_{L} \uparrow \infty \) (arbitrary slowly) as \( L \to \infty \), then

\[
\sup_{x \in V_{L/5}} \| (\Pi_{L} - \pi_{L}^{(p_{\infty})})\pi_{p-m}^{(p_{\infty})}(x, \cdot) \|_{1} \to 0 \quad \mathbb{P}\text{-almost surely.}
\]

(ii) The particular form of the smoothing kernel \( \pi_{p-m}^{(p_{\infty})} \) is not important. However, this form is useful for our inductive proof.

Our methods enable us to compare the exit measures in a more local way. Denote by \( \partial V_{L} = \{ y \in \mathbb{Z}^{d} : d(y, V_{L}) = 1 \} \) the outer boundary of \( V_{L} \). For positive \( t \) and \( z \in \partial V_{L} \) let \( W_{t}(z) = V_{t}(z) \cap \partial V_{L} \), where \( V_{t}(z) = z + V_{t} \). Then \( |W_{t}(z)| \) is of order \( t^{d-1} \). Dropping the center point \( z \in \partial V_{L} \) from the notation, we obtain

**Theorem 0.3.** Assume A1. There exist \( \varepsilon_{0} > 0 \) and \( L_{0} > 0 \) such that if A0(\( \varepsilon \)) is satisfied for some \( \varepsilon \leq \varepsilon_{0} \), then for \( L \geq L_{0} \), there exists an event \( A_{L} \in \mathcal{F} \) with \( \mathbb{P}(A_{L}^{c}) \leq \exp\left( - (1/2)(\log L)^{2} \right) \) such that on \( A_{L} \), the following holds true. If \( 0 < \eta < 1 \) and \( x \in V_{\eta L} \), then

(i) For \( t \geq L/(\log L)^{15} \) and every set \( W_{t} \) as above, there exists \( C = C(\eta) \) with

\[
\Pi_{L}(x, W_{t}) \leq C \pi_{L}^{(p_{\infty})}(x, W_{t}).
\]
There exists a symmetric nearest neighbor kernel \( p_L \) such that for \( t \geq L/(\log L)^6 \),

\[
\Pi_L(x, W_t) = \pi_L^{(p_L)}(x, W_t) \left( 1 + O \left( (\log L)^{-5/2} \right) \right).
\]

Here, the constant in the \( O \)-notation depends only on \( d \) and \( \eta \).

We give one possible choice for the kernel \( p_L \) in (6). Our results can also be used to deduce transience of the RWRE.

**Corollary 0.1.** Assume A1. There exist \( \varepsilon_0 \) such that if A0(\( \varepsilon \)) is satisfied for some \( \varepsilon \leq \varepsilon_0 \), then on \( \mathbb{P} \)-almost all \( \omega \in \Omega \) the RWRE \( (X_n)_{n \geq 0} \) is transient.

This paper is inspired by the work of Bolthausen and Zeitouni [7]. There, Theorems 0.1 and 0.2 appeared in a similar form for the case of isotropic RWRE in dimension \( d \geq 3 \). However, as it is explained in the first author’s thesis [1], even for such random walks it was of great interest to develop a somewhat new approach. This was achieved in [1], where also a result on sojourn times in balls is proved.

Here, the main difficulty stems from the fact that the kernel \( p_\infty \) is not explicitly computable and depends in a complicated way on \( \mu \). We will not first prove the existence of \( p_\infty \) and then deduce our results about exit measures - in fact, it will be a side effect of our multiscale analysis of exit laws that \( p_\infty \) exists and is the right object of comparison. The idea of its construction starts with the observation that if the statements of Theorems 0.1 and 0.2 are true for some kernel \( p_\infty \), then the averaged exit distribution on a global scale will be the same as the exit distribution of the random walk with kernel \( p_\infty \), when \( L \to \infty \). Therefore, it is natural to choose for any scale \( L \) a symmetric transition kernel \( p_L \) which has the property that the covariance matrix of the averaged exit distribution from \( V_L \), scaled down by \( L^2 \) is the covariance matrix of \( p_L \) (in fact, we will choose \( p_L \) in a slightly different way, see (6) for the precise definition). The difficult task is then to show that \( p_\infty = \lim_{L \to \infty} p_L \) exists. In the isotropic case, this problem is absent since one can choose \( p_L = p_0 \) for every \( L \).

Let us comment on some further literature which is relevant for our study. For a detailed survey on RWRE, the reader is invited to consult the lecture notes of Sznitman [20], [22] and Zeitouni [25], [26], and also the overview article of Bogachev [8].

Assuming A0(\( \varepsilon \)) for small \( \varepsilon \) and the stronger isotropy condition that was mentioned in the beginning, Bricmont and Kupiainen [9] prove a (quenched) invariance principle, showing that in dimensions \( d \geq 3 \), the RWRE is asymptotically Gaussian, on \( \mathbb{P} \)-almost all environments. A continuous counterpart, isotropic diffusions in a random environment which are small perturbations of Brownian motion, has been investigated by Sznitman and Zeitouni in [23]. They prove transience and a full quenched invariance principle in dimensions \( d \geq 3 \).

Our centering condition A1 excludes so-called ballistic behavior, i.e. the regime where the limit velocity \( v = \lim_{n \to \infty} X_n/n \) is an almost sure constant vector different from zero. Ballistic behavior has been studied extensively, for example in Kalikow [12], Sznitman [18], [19], [21], Bolthausen and Sznitman [5], or more recently Berger [2] and Berger, Drewitz, Ramírez [3].
In the perturbative regime when \( d \geq 3 \), Sznitman \cite{Sznitman} shows that some strength of the mean local drift \( \mu = \mathbb{E}[\sum_{e=1}^{\infty} \omega_0(e)] \) is enough to deduce ballisticity. However, as examples in Bolthausen, Sznitman and Zeitouni \cite{BSZ} for dimensions \( d \geq 7 \) demonstrate, ballisticity can also occur with \( \mu = 0 \), and one can even construct examples exhibiting ballistic behavior with \( \mu = -cm \) and \( c > 0 \). Note that our condition A1 implies \( \mu = 0 \).

Concerning non-ballistic behavior, much is known for the class of balanced RWRE when \( \mathbb{P}(\omega_0(e_i) = \omega_0(-e_i)) = 1 \) for all \( i = 1, \ldots, d \). Employing the method of environment viewed from the particle, Lawler proves in \cite{Lawler} that for \( \mathbb{P} \)-almost all \( \omega \), \( X_{[n]} / \sqrt{n} \) converges in \( \mathbb{P}_0,\omega \)-distribution to a non-degenerate Brownian motion with diagonal covariance matrix, even in the non-perturbative regime. Moreover, the RWRE is recurrent in dimension \( d = 2 \) and transient when \( d \geq 3 \), see \cite{Lawler}. Recently, within the i.i.d. setting, diffusive behavior has been shown in the mere elliptic case by Guo and Zeitouni \cite{GuoZeitouni} and in the non-elliptic case by Berger and Deuschel \cite{BD}.

For a better reading, a rough overview over this paper is given in Section 1.4.

\section{Basic notation and main techniques}

\subsection{Basic notation}

Our purpose here is to cover the most relevant notation which will be used throughout this text. Further notation will be introduced later on when needed.

\textbf{Sets and distances}

We let \( \mathbb{N} = \{0,1,2,3,\ldots\} \) and \( \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} \). For a set \( A \), its complement is denoted by \( A^c \). If \( A \subset \mathbb{R}^d \) is measurable and non-discrete, we write \( |A| \) for its \( d \)-dimensional Lebesgue measure. Sometimes, \( |A| \) denotes the surface measure instead, but this will be clear from the context. If \( A \subset \mathbb{Z}^d \), then \( |A| \) denotes its cardinality.

For \( x \in \mathbb{R}^d \), \( |x| \) is the Euclidean norm. If \( A, B \subset \mathbb{R}^d \), we set \( d(A,B) = \inf \{|x-y| : x \in A, y \in B\} \) and \( \text{diam}(A) = \sup \{|x-y| : x, y \in A\} \). Given \( L > 0 \), let \( V_L = \{x \in \mathbb{Z}^d : |x| \leq L\} \), and for \( x \in \mathbb{Z}^d \), \( V_L(x) = V_L + x \). For Euclidean balls in \( \mathbb{R}^d \) we write \( C_L = \{x \in \mathbb{R}^d : |x| < L\} \) and for \( x \in \mathbb{R}^d \), \( C_L(x) = x + C_L \).

If \( V \subset \mathbb{Z}^d \), then \( \partial V = \{x \in V^c \cap \mathbb{Z}^d : d(\{x\},V) = 1\} \) is the outer boundary, while in the case of a non-discrete set \( V \subset \mathbb{R}^d \), \( \partial V \) stands for the usual topological boundary of \( V \) and \( \overline{V} \) for its closure. For \( x \in \mathbb{R}^d \), \( \partial V(x) = L - |x| \). Finally, for \( 0 \leq a < b \leq L \), the “shell” is defined by \( \text{Sh}_L(a,b) = \{x \in V_L : a \leq d_L(x) < b\} \), \( \text{Sh}_L(b) = \text{Sh}_L(0,b) \).

\textbf{Functions}

If \( a, b \) are two real numbers, we set \( a \wedge b = \min\{a, b\} \), \( a \vee b = \max\{a, b\} \). The largest integer not greater than \( a \) is denoted by \( \lfloor a \rfloor \). As usual, set \( 1/0 = \infty \). For us, \( \log \) is
the logarithm to the base $e$. For $x, z \in \mathbb{R}^d$, the Delta function $\delta_x(z)$ is defined to be equal one for $z = x$ and zero otherwise. If $V \subset \mathbb{Z}^d$ is a set, then $\delta_V$ is the probability distribution on the subsets of $\mathbb{Z}^d$ satisfying $\delta_V(V') = 1$ if $V' = V$ and zero otherwise.

Given two functions $F, G : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, we write $FG$ for the (matrix) product $FG(x, y) = \sum_{u \in \mathbb{Z}^d} F(x, u) G(u, y)$, provided the right hand side is absolutely summable. $F^k$ is the $k$th power defined in this way, and $F^0(x, y) = \delta_x(y)$. $F$ can also operate on functions $f : \mathbb{Z}^d \to \mathbb{R}$ from the left via $Ff(x) = \sum_{y \in \mathbb{Z}^d} F(x, y)f(y)$.

We use the symbol $1_W$ for the indicator function of the set $W$. By an abuse of notation, we denote again by $1_{P^c}$ the constant function $\mathbb{1}_{P}$. A typical function we have in mind is the constant function $\mathbb{1}_{P}$.

Let $L > 0$. Putting $U_L = \{ x \in \mathbb{R}^d : L/2 < |x| < 2L \}$, we define

$$\mathcal{M}_L = \left\{ \psi : U_L \to (L/10, 5L), \psi \in C^4(U_L), \| D^i \psi \|_{U_L} \leq 10 \text{ for } i = 1, \ldots, 4 \right\}.$$

We will mostly interpret functions $\psi \in \mathcal{M}_L$ as maps from $U_L \cap \mathbb{Z}^d \subset \mathbb{R}^d$. A typical function we have in mind is the constant function $\psi \equiv L$.

### Transition probabilities and exit distributions

Given (not necessarily nearest neighbor) transition probabilities $p = (p(x, y))_{x, y \in \mathbb{Z}^d}$, we write $P_{x,p}$ for the law of the canonical Markov chain $(X_n)_{n \geq 0}$ on $((\mathbb{Z}^d)^\mathbb{N}, \mathcal{G})$, $\mathcal{G}$ the $\sigma$-algebra generated by cylinder functions, with transition probabilities $p$ and starting point $X_0 = x$. $P_{x,p}$ -a.s. The expectation with respect to $P_{x,p}$ is denoted by $E_{x,p}$. The simple random walk kernel $p_c(x, x \pm e_i) = 1/(2d)$ will play a prominent role. Clearly, every $p \in \mathcal{P}$ gives rise to a homogeneous nearest neighbor kernel, which, by a small abuse of notation, we denote again by $p$.

If $V \subset \mathbb{Z}^d$, we denote by $\tau_V = \inf \{ n \geq 0 : X_n \notin V \}$ the first exit time from $V$, with $\inf \emptyset = \infty$, whereas $T_V = \tau_{V^c}$ is the first hitting time of $V$. Given $x, z \in \mathbb{Z}^d$ and $p, V$ as above, we define

$$\text{ex}_V(x, z; p) = P_{x,p}(\tau_V = z).$$

Notice that for $x \in V^c$, $\text{ex}_V(x, z; p) = \delta_x(z)$.

Recall the definitions of the sets $\mathcal{P}$ and $\mathcal{P}_c$ from the introduction. For $0 < \kappa < 1/(2d)$, let

$$\mathcal{P}_\kappa = \{ p \in \mathcal{P}_c : p(e_i) = p(-e_i), i = 1, \ldots, d \},$$
i.e. \( \mathcal{P}_\kappa^s \) is the subset of \( \mathcal{P}_\kappa \) which contains all symmetric probability distributions on \( \{ \pm e_i : i = 1, \ldots, d \} \). At various places, the parameter \( \kappa \) bounds the range of the symmetric transition kernels we work with.

For \( p \in \mathcal{P} \), we write
\[
\pi_V^{(p)}(x, z) = \exp V(x, z; p),
\]
and given \( \omega \in \Omega \), we set
\[
\Pi_V(x, z) = \exp V(x, z; p_\omega).
\]

Here, \( \Pi_V \) should be understood as a random exit distribution, but we suppress \( \omega \) in the notation.

### Coarse grained random walks

Fix once for all a probability density \( \varphi \in C^\infty(\mathbb{R}_+, \mathbb{R}_+) \) with compact support in \( (1, 2) \). Given a nonempty subset \( W \subset \mathbb{Z}^d \), \( x \in W \) and \( m_x > 0 \), the image measure of the rescaled density \( (1/m_x)\varphi(t/m_x)dt \) under the mapping \( t \mapsto V_t(x) \cap W \) defines a probability distribution on (finite) sets containing \( x \). If \( \psi = (m_x)_{x \in W} \) is a field of positive numbers, we obtain in this way a collection of probability distributions indexed by \( x \in W \), a coarse graining scheme on \( W \).

Now if \( p = (p(x, y))_{x, y \in \mathbb{Z}_d} \) is a collection of transition probabilities on \( W \), we obtain coarse grained transitions corresponding to \( (\psi, p) \) by setting
\[
p_{CG}^{(p)}(x, \cdot) = \frac{1}{m_x} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{m_x} \right) \exp V_t(x) \cap W(x, \cdot; p)dt, \quad x \in W. \tag{1}
\]

If \( p \in \mathcal{P} \), we write \( \hat{\pi}^{(p)}_\psi \) instead of \( p_{CG}^{(p)} \). Note that for every choice of \( W \) and \( \psi \), \( \hat{\pi}^{(p)}_\psi \) defines a probability kernel.

### Coarse graining schemes in the ball

For the motion in the ball \( V_L \), we use a particular field \( \psi \). Once for all, define
\[
s_L = \frac{L}{(\log L)^3} \quad \text{and} \quad r_L = \frac{L}{(\log L)^{15}}.
\]

Our schemes in the ball are indexed by a parameter \( r \), which can either be a constant \( \geq 100 \), but much smaller than \( r_L \), or, in most of the cases, \( r = r_L \). We fix a smooth function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying
\[
h(x) = \begin{cases} 
  x & \text{for } x \leq \frac{1}{2} \\
  1 & \text{for } x \geq 2
\end{cases},
\]
such that \( h \) is strictly monotone and concave on \( (1/2, 2) \), with first derivative bounded uniformly by 1. Define \( h_{L,r} : \overline{C_L} \to \mathbb{R}_+ \) by
\[
h_{L,r}(x) = \frac{1}{20} \max \left\{ s_L h \left( \frac{d_L(x)}{s_L} \right), r \right\}. \tag{2}
\]
Figure 1: The coarse graining scheme in $V_L$. In the bulk $\{ x \in V_L : d_L(x) \geq 2s_L \}$, the exit distributions are taken from balls of radii between $(1/20)s_L$ and $(1/10)s_L$. When entering $\text{Sh}_L(2s_L)$, the coarse graining radii start to shrink, up to the boundary layer $\text{Sh}_L(r)$, where the exit distributions are taken from intersected balls $V_t(x) \cap V_L$, $t \in [(1/20)r, (1/10)r]$.

Since we mostly work with $r = r_L$, we use the abbreviation $h_L = h_{L,r_L}$. We write $\hat{\Pi}_{L,r}$ for the coarse grained RWRE transition kernel associated to $(\psi = (h_{L,r}(x))_{x \in V_L}, \omega)$,

$$\hat{\Pi}_{L,r}(x, \cdot) = \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{h_{L,r}(x)} \right) \Pi_{V_t(x) \cap V_L}(x, \cdot) dt,$$

and $\hat{\pi}^{(p)}_{L,r}$ for that coming from symmetric random walk with transition kernel $p \in \mathcal{P}$, where in the definition $\Pi$ is replaced by $\pi^{(p)}$. Note that $\hat{\Pi}_{L,r}$ is a random transition kernel, depending on the environment $\omega$. For convenience, we set $\hat{\Pi}_{L,r}(x, \cdot) = \hat{\pi}^{(p)}_{L,r}(x, \cdot) = \delta_x(\cdot)$ for $x \in \mathbb{Z}^d \setminus V_L$. By the strong Markov property, the exit measures from the ball $V_L$ remain unchanged under these transition kernels, i.e.

$$\text{ex}_{V_L}(x, \cdot ; \hat{\Pi}_{L,r}) = \Pi_L(x, \cdot) \quad \text{and} \quad \text{ex}_{V_L}(x, \cdot ; \hat{\pi}^{(p)}_{L,r}) = \pi_L^{(p)}(x, \cdot).$$

**Remark 1.1.** (i) Later on, we will also work with slightly modified transition kernels $\tilde{\Pi}$ and $\tilde{\pi}^{(p)}$, which depend on the environment. We elaborate on this in Section 4.3. (ii) Due to the lack of the last smoothing step outside $V_L$, we need to zoom in near the boundary in order to handle non-smoothed exit distributions in Section 5.3. The parameter $r$ allows us to adjust the step size in the boundary region. (iii) For every choice of $r$,

$$h_{L,r}(x) = \begin{cases} 
\frac{d_L(x)}{20} & \text{for } x \in V_L \text{ with } r_L \leq d_L(x) \leq s_L/2 \\
\frac{s_L}{20} & \text{for } x \in V_L \text{ with } d_L(x) \geq 2s_L
\end{cases}.$$

**Abbreviations**

If it is clear from the context which transition kernel $p$ we are working with, we often drop the sub- or superscript $p$ from notation. Then, for example, we write $\pi_V$ for $\pi_{V}^{(p)}$, $P_x$ instead of $P_{x,p}$ or $E_x$ for $E_{x,p}$. Given transition probabilities $p_\omega$ coming from an environment $\omega$, we use the notation $P_{x,\omega}$, $E_{x,\omega}$. 

In order to avoid double indices, we usually write $\pi_L$ instead of $\pi_V$, $\Pi_L$ for $\Pi_V$ and $\tau_L$ for $\tau_V$ if $V = V_L$ is the ball around zero of radius $L$.

Many of our quantities, e.g. the transition kernels $\hat{\Pi}_{L,r}$ and $\hat{\pi}_{L,r}$, are indexed by both $L$ and $r$. While we always keep the indices in the statements, we normally drop both of them in the proofs.

Finally, we will often use the abbreviations $d(y,B)$ for $d(\{y\},B)$, $T_x$ for $T_{\{x\}}$ and $\mathbb{P}(A;B)$ for $\mathbb{P}(A \cap B)$.

Some words about constants, $O$-notation and large $L$ behavior

All our constants are positive. They only depend on the dimension $d \geq 3$ unless stated otherwise. In particular, constants do not depend on $L$, on $\delta$, on $\omega$ or on any point $x \in \mathbb{Z}^d$, and they are also independent of the parameter $r$.

At some places, the reader might have the impression that constants depend on the transition kernel $p$. However, we only work with $p \in \mathcal{P}_s$, and $\kappa$ can be chosen (arbitrarily) small. Such kernels $p$ are therefore small perturbations of the simple random walk kernel $p_o$, and since all dependencies emerge in a continuous way (in $p$), we may always assume that constants are uniform in $p$.

We use $C$ and $c$ for generic positive constants whose values can change in different expressions, even in the same line. In the proofs, we often use other constants like $K,C_1,c_1$; their values are fixed throughout the proofs. Lower-case constants usually indicate small (positive) values.

Given two functions $f,g$ defined on some subset of $\mathbb{R}$, we write $f(t) = O(g(t))$ if there exists a positive $C > 0$ and a real number $t_0$ such that $|f(t)| \leq C|g(t)|$ for $t \geq t_0$.

If a statement holds for “$L$ large (enough)”, this means that there exists $L_0 > 0$ depending only on the dimension such that the statement is true for all $L \geq L_0$. This applies analogously to the expressions “$\delta$ (or $\varepsilon$, or $\kappa$) small (enough)”.

The reader should always keep in mind that we are interested in asymptotics when $L \to \infty$ and the perturbation parameter $\varepsilon$ is arbitrarily small, but fixed. Even though some of our statements are valid only for large $L$ and $\varepsilon$ (or $\delta$, or $\kappa$) sufficiently small, we do not mention this every time.

1.2 Perturbation expansion for Green’s functions

Our approach of comparing the RWRE exit distribution with that of an appropriate symmetric random walk is based on a perturbation argument. Namely, the resolvent equation allows us to express Green’s functions of the RWRE in terms of Green’s functions of homogeneous random walks. More generally, let $p = (p(x,y))_{x,y \in \mathbb{Z}^d}$ be a family of finite range transition probabilities on $\mathbb{Z}^d$, and let $V \subset \mathbb{Z}^d$ be a finite set. The corresponding Green’s kernel or Green’s function for $V$ is defined by

$$g_V(p)(x,y) = \sum_{k=0}^{\infty} (1_V p)^k (x,y).$$
The connection with the exit measure is given by the fact that for \( z \notin V \), we have

\[
g_V(p)(\cdot, z) = \text{ex}_V(\cdot, z; p).
\]

Now write \( g \) for \( g_V(p) \) and let \( P \) be another transition kernel with corresponding Green’s function \( G \) for \( V \). With \( \Delta = 1_V (P - p) \), we have by the resolvent equation

\[
G - g = g\Delta G = G\Delta g.
\]

In order to get rid of \( G \) on the right hand side, we iterate (3) and obtain

\[
G - g = \sum_{k=1}^{\infty} (g\Delta)^k g,
\]

provided the infinite series converges, which is always the case in our setting. Writing (4) as

\[
G = g \sum_{k=0}^{\infty} (\Delta g)^k,
\]

replacing the rightmost \( g \) by \( g(x, \cdot) = \delta_x(\cdot) + 1_V pg(x, \cdot) \) and reordering terms, we get

\[
G = g \sum_{m=0}^{\infty} (Rg)^m \sum_{k=0}^{\infty} \Delta^k,
\]

where \( R = \sum_{k=1}^{\infty} \Delta^k p \).

Two Green’s functions for the ball \( V_L \) will play a particular role: The (coarse grained) RWRE Green’s function corresponding to \( \hat{\Pi}_{L,r} \), which we denote by \( \hat{G}_{L,r} \), and the Green’s function corresponding to \( \hat{\pi}_{L,r} \), denoted by \( \hat{g}_{L,r} \). A “goodified” version of \( \hat{G}_{L,r} \) will be introduced in Section 2.

1.3 Main technical statement

Basically, we will read off our main results from Proposition 1.1. The latter involves a technical condition, which we will propagate from one level to the next. This condition depends on the deviation \( \delta \) (cf. Theorem 0.1) and on a parameter \( L_0 \geq 3 \) which will finally be chosen sufficiently large.

Recall the coarse graining schemes on \( V_L \). Even though we use the “final” kernel \( p_\infty \) in the formulation of our main theorems, we will work in the proofs with intermediate kernels \( p_L \) depending on the radius of the ball. More precisely, we assign to each \( L > 0 \) the symmetric transition kernel (\( i = 1, \ldots, d \))

\[
p_L(\pm e_i) = \begin{cases} 
1/(2d) & \text{for } 0 < L \leq L_0 \\
\frac{1}{2} \sum_{y \in \mathbb{Z}^d} \mathbb{E} \left[ \hat{\Pi}_{L,r}(0, y) \right] \frac{y^2}{|y|^2} & \text{for } L > L_0.
\end{cases}
\]

Since \( h_{L,r}(0) = s_L/20 \), the definition of \( p_L \) does not depend on the parameter \( r \). In words, for radii \( 0 < L \leq L_0 \), \( p_L \) agrees with the simple random walk kernel \( p_0 \), while
for $L > L_0$ the kernel $p_L$ is defined as an average of variances of normalized mean exit distributions from balls of radii $t \in [(1/20)s_L, (1/10)s_L]$.

We still need some notation. For $x \in \mathbb{Z}^d$, $\psi : \partial V_i(x) \rightarrow (0, \infty)$ and $p, q \in \mathcal{P}$, define

$$D_{t,p,\psi,q}(x) = \left\| \left( \Pi_{V_i(x)} - \pi_{V_i(x)}^{(p)} \right) \pi_{\psi}(x, \cdot) \right\|_1,$$

$$D_{t,p}(x) = \left\| \left( \Pi_{V_i(x)} - \pi_{V_i(x)}^{(p)} \right) (x, \cdot) \right\|_1.$$ 

If $\psi \equiv m$ is constant, we write $D_{t,p,m,q}$ instead of $D_{t,p,\psi,q}$. We usually drop $x$ from the notation if $x = 0$. Further, let

$$D^*_t = \sup_{x \in V_{i/2}} \left\| \left( \Pi_{V_i} - \pi_{V_i}^{(p)} \right) \pi_{\psi}(x, \cdot) \right\|_1,$$

$$D^*_{t,p} = \sup_{x \in V_{i/2}} \left\| \left( \Pi_{V_i} - \pi_{V_i}^{(p)} \right) (x, \cdot) \right\|_1.$$ 

With $\delta > 0$, we set for $i = 1, 2, 3$

$$b_i(L, p, \psi, q, \delta) = \mathbb{P} \left( \left\{ (\log L)^{-9+9(i-1)/4} < D^*_{L,p,\psi,q} \leq (\log L)^{-9+9i/4} \right\} \cap \left\{ D^*_{L,p} \leq \delta \right\} \right),$$

and

$$b_4(L, p, \psi, q, \delta) = \mathbb{P} \left( \left\{ D^*_{L,p,\psi,q} > (\log L)^{-3+3/4} \right\} \cup \left\{ D^*_{L,p} > \delta \right\} \right).$$

Put $\iota = (\log L_0)^{-7}$, and let us now formulate

**Condition C1**

Let $\delta > 0$ and $L_1 \geq L_0 \geq 3$. We say that **C1$(\delta, L_0, L_1)$** holds if

- For all $3 \leq L \leq 2L_0$, all $\psi \in \mathcal{M}_L$ and all $q \in \mathcal{P}_r$,

$$\mathbb{P} \left( \left\{ D^*_{L,p,\psi,q} \geq \log (\log L)^{-9} \right\} \cup \left\{ D^*_{L,p} > \delta \right\} \right) \leq \exp (-(\log (2L_0))^2).$$

- For all $L_0 < L \leq L_1$, $L' \in [L, 2L]$, $\psi \in \mathcal{M}_{L'}$, and $q \in \mathcal{P}_r$,

$$b_i(L', p_L, \psi, q, \delta) \leq \frac{1}{4} \exp \left( -((3 + i)/4)(\log L')^2 \right)$$

for $i = 1, 2, 3, 4$.

Let us summarize this condition in words.

The first point controls the total variation distance of the RWRE exit measure to the exit measure of simple random walk on balls of radii $3 \leq L \leq 2L_0$. Note that the bound on the probability is given in terms of $L_0$, for all such $L$.

The second point concerns radii $L_0 < L \leq L_1$ and gives control over the deviation of the RWRE exit measure from that of a symmetric random walk with kernel $p_L$. It also includes a continuity property of RWRE exit measures when $L'$ varies (note that we use $p_L$ on the left hand side, not $p_{L'}$), which will be crucial to compare the distance between two kernels for different radii, see Lemma 2.2.

The main technical statement of this paper is
Section 4.

This important concept is first explained in Section 2 and then further developed in it is convenient to “goodify” the environment, that is to replace bad points by good ones.

Our coarse grained transition kernels are given by exit distributions from smaller balls inside $V_L$, and we obtain our results by transferring inductively information on smaller scales to the scale $L$. The notion of good and bad points, introduced in Section 2, allows us to classify the exit behavior on smaller scales. If inside $V_L$ all points are good, then the estimates on smaller balls can be transferred to a (globally smoothed) estimate on the larger ball $V_L$ (Lemma 5.4). But bad points can appear, and in fact we have to distinguish four different levels of badness (Section 2.3). When bad points are present, it is convenient to “goodify” the environment, that is to replace bad points by good ones. This important concept is first explained in Section 2 and then further developed in Section 4.

### 1.4 A short reading guide

**Proposition 1.1.** Assume A1. For $\delta > 0$ small enough, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ with the following property: If $\varepsilon \leq \varepsilon_0$ and A0($\varepsilon$) holds, then

(i) There exists $L_0 = L_0(\delta)$ such that for $L_1 \geq L_0$,

\[ C_1(\delta, L_0, L_1) \Rightarrow C_1(\delta, L_0, L_1(\log L_1)^2). \]

(ii) There exist $L_0 = L_0(\delta)$ and sequences $l_n, m_n \to \infty$ with the following property: If $L_1 \geq l_n$ and $L_1 \leq L \leq L_1(\log L_1)^2$, then for $m \geq m_n$, $q \in \mathcal{P}_i$,

\[ C_1(\delta, L_0, L_1) \Rightarrow \left( \mathbb{P}\left( D_{L,p_L,m,q}^* > 1/n \right) \leq \exp\left( - (\log L)^2 \right) \right). \]

**Remark 1.2.** (i) It is important to realize that for every choice of $\delta$ and $L_0$, we can obtain that $C_1(\delta, L_0, L_1)$ is fulfilled, simply by choosing the perturbation $\varepsilon$ small enough. This observation provides us with the base step of the induction in Proposition 1.1 (i): Once we know that $C_1(\delta, L_0, L_1)$ propagates for properly chosen $\delta$ and $L_0$, we can choose $\varepsilon$ so small such that $C_1(\delta, L_0, L)$ holds for all $L \geq L_0$.

(ii) The number $\varepsilon$ defined above $C_1$ bounds the range of symmetric transition kernels $q$ from which smoothing kernels $\hat{\pi}_\psi^{(q)}$ are built. In Lemma 2.2 we will see that under $C_1(\delta, L_0, L_1)$, for $L \leq L_1(\log L_1)^2$, the kernels $p_L$ are elements of $\mathcal{P}_i$.

(iii) If $C_1(\delta, L_0, L_1)$ is satisfied, then for any $3 \leq L \leq L_1$ and for all $L' \in [L, 2L]$, all $\psi \in \mathcal{M}_{L'}$ and all $q \in \mathcal{P}_i$,

\[ \mathbb{P}\left( \{ D_{L',p_L,\psi,q}^* > (\log L')^{-9} \} \cup \{ D_{L',\psi,L}^* > \delta \} \right) \leq \exp\left( - (\log L')^2 \right). \]

For the rest of this paper, if we write “assume $C_1(\delta, L_0, L_1)$”, this means that we assume $C_1(\delta, L_0, L_1)$ for some $\delta > 0$ and some $L_1 \geq L_0$, where $\delta$ and $L_0$ can be chosen arbitrarily small respectively large.
However, for the globally smoothed estimate, we only have to deal with the case
where all bad points are enclosed in a comparably small region - two or more such
regions are too unlikely (Lemma 2.3). Some special care is required for the worst class
of bad points in the interior of the ball. For environments containing such points, we
slightly modify the coarse graining scheme inside $V_L$, as described in Section 4.3.

In Lemma 5.5, we prove the smoothed estimates on environments with bad points
and show that the degree of badness decreases by one from one scale to the next.

For exit measures where no or only a local last smoothing step is added (Section 5.3,
Lemmata 5.6 and 5.7, respectively), bad points near the boundary of $V_L$ are much
more delicate to handle, since we have to take into account several possibly bad regions.
However, they do not occur too frequently (Lemma 2.4) and can be controlled by capacity
arguments.

All these estimates require precise bounds on coarse grained Green’s functions, which
are developed in Section 4. Basically, we show that on environments with no bad points,
the coarse grained RWRE Green’s function for the ball can be estimated from above by
the analogous quantity coming from simple random walk (or some symmetric perturba-
tion).

In Section 3, we present various bounds on hitting probabilities for both symmetric
random walk and Brownian motion, and difference estimates of smoothed exit measures.
One main difficulty comes from the fact that we have to work with a whole family
($p_L$) of nearest neighbor transition kernels. For example, we have to control the total
variation distance of exit measures corresponding to two different kernels. Here, the
crucial statement is Lemma 3.4, which is formulated in terms of Brownian motion and
then transferred to random walks via coupling arguments.

The statements from Sections 5.2 and 5.2 are finally used in Section 6 to prove the
main results. In the appendix we prove the main statements from Section 3, as well as
a local central limit theorem for the coarse grained symmetric random walk.

2 Transition kernels and notion of badness

Here, we look closer at the family of kernels defined in (6) and introduce the concept of
“good” and “bad” points. Further, we define “goodified” transition kernels and prove two
estimates ensuring that we do not have to consider environments with bad points that
are widely spread out in the ball or densely packed in the boundary region.

2.1 Some properties of the kernels $p_L$

The first general statement exemplifies how to extract information about a symmetric
kernel $p \in \mathcal{P}_\kappa$, $0 < \kappa < 1/(2d)$, out of the corresponding exit measure on $\partial V_L$.

**Lemma 2.1.** For $i = 1, \ldots, d$,

$$p(e_i) = \frac{1}{2} \sum_{y \in \partial V_L} \pi^{(p)}_L(0, y) \left( \frac{y_i}{L} \right)^2 + O(L^{-1}).$$
2.1 Some properties of the kernels $p_L$

**Proof:** Recall that under $P_{0,p}$, $(X_n)_{n \geq 0}$ denotes the canonical random walk on $\mathbb{Z}^d$ with transition kernel $p$ starting at the origin. Let $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$ be the filtration up to time $m$, and denote by $X_{n,i}$ the $i$th component of $X_n$. Due to the symmetry of $p$, the process $X_{n,i}^2 - 2p(e_i)n$, $n \geq 0$, is a martingale with respect to $\mathcal{G}_n$. By the optional stopping theorem,

$$E_{0,p}[X_{\tau_{L},i}^2] = 2p(e_i)E_{0,p}[\tau_L].$$

Since $X_{\tau_{L,1}}^2 + \ldots + X_{\tau_{L,d}}^2 = (L + O(1))^2$, it follows that $E_{0,p}[\tau_L] = (L + O(1))^2$, and the claim is proved. \hfill $\square$

Now let us turn to $p_L$.

**Lemma 2.2.** Assume C1($\delta, L_0, L_1$). There exists a constant $C > 0$ such that

(i) For $3 \leq L \leq L_1(\log L_1)^2$,

$$||p_{sL/20} - p_L||_1 \leq C(\log L)^{-9}.$$  

In particular, if $L_0$ is sufficiently large, we have $p_L \in \mathcal{P}_c^8$.

(ii) Let $3 \leq L \leq L_1$, and $L' \in [L/2, L]$. Then

$$||p_{L'} - p_L||_1 \leq C(\log L)^{-9}.$$

**Proof:** (i) By Lemma 2.1 writing $p$ for $p_{sL/20}$ and $\hat{\pi}$ for $\hat{\pi}_{L,r}$, for each $i$,

$$2p(e_i) = \sum_y \hat{\pi}^{(p)}(0,y) \frac{y_i^2}{|y|^2} + O(s_L^{-1}). \quad (7)$$

Thus, using the definition of $p_L$ in the first equality, the fact that the variance of the sum of two independent random variables equals to the sum of their variances in the second equality and condition C1 in the last step,

$$|2p(e_i) - 2p_L(e_i)| = \sum_y \left( \hat{\pi}^{(p)} - E[\hat{\Pi}] \right)(0,y) \frac{y_i^2}{|y|^2} + O(s_L^{-1})$$

$$= \sum_y \left( \hat{\pi}^{(p)} - E[\hat{\Pi}] \right) \hat{\pi}^{(p)}(0,y) \frac{y_i^2}{|y|^2} + O(s_L^{-1})$$

$$\leq C(\log L)^{-9}.$$  

Repeating the last steps, we see that $||p_L - p_o|| \leq \iota = (\log L_0)^{-7}$, and (i) follows.

(ii) We have by Lemma 2.1 for $\psi \in \mathcal{M}_L$,

$$||p_{L'} - p_L||_1 = \sum_{y \in \partial V_L} \left( \pi_{L'}^{(p_L)} - \pi_L^{(p_L)} \right)(0,y) \left( \frac{y_i}{L} \right)^2 + O(L^{-1})$$

$$\leq \frac{1}{L^2} \sum_{y \in \partial V_L} \left( \pi_{L'}^{(p_L)} - \pi_L^{(p_L)} \right) \hat{\pi}_\psi^{(p_L)}(0,y) y_i^2 + O(L^{-1})$$

$$\leq C \left( ||(\pi_{L'}^{(p_L)} - \pi_L^{(p_L)}) \hat{\pi}_\psi^{(p_L)}(0,\cdot)||_1 \right) + O(L^{-1}).$$
Since under C1(δ, L, 0, L1), using \( p_L \in \mathcal{P}_r \),
\[
\left\| \left( \hat{\pi}^{(p_L)} - \hat{\pi}^{(pl)} \right) \psi(x, \cdot) \right\|_1 \leq C \left( \mathbb{E} \left[ D_{L,pl,\psi,pl} \right] + \mathbb{E} \left[ D_{L,pl,\psi,pl} \right] \right) \leq C (\log L)^{-9},
\]
the second claim is proved. \( \square \)

### 2.2 Good and bad points

We shall partition the grid points inside \( V_L \) according to their influence on the exit behavior. Recall the assignment (8). We say that a point \( x \in V_L \) is **good** (with respect to \( L, \delta > 0 \) and \( r, 100 \leq r \leq r_L \)) if

- For all \( t \in [h_{L,r}(x), 2h_{L,r}(x)] \), with \( q = ph_{L,r}(x) \), \( \left\| \left( \Pi_{V_L(x)} - \hat{\pi}^{(q)}_{V_L(x)} \right) (x, \cdot) \right\|_1 \leq \delta. \)
- If \( d_L(x) > 2r \), then additionally
  \[
  \left\| \left( \hat{\Pi}_{L,r} - \hat{\pi}^{(q)}_{L,r} \right) \hat{\pi}^{(q)}_{L,r}(x, \cdot) \right\|_1 \leq (\log h_{L,r}(x))^{-9}.
  \]

A point \( x \in V_L \) which is not good is called **bad**. We denote by \( \mathcal{B}_{L,r} = \mathcal{B}_{L,r}(\omega) \) the set of all bad points inside \( V_L \) and write \( \mathcal{B}_L = \mathcal{B}_{L,r_L} \) for short. Furthermore, set \( \mathcal{B}_{L,r}^0 = \mathcal{B}_{L,r} \cap \text{Sh}_{L}(r) \) and \( \mathcal{B}^*_L = \mathcal{B}_{L,r} \cup \mathcal{B}_L = \mathcal{B}_{L,r}^0 \cup \mathcal{B}_L \). Of course, the set of bad points depends also on \( \delta \), but we do not indicate this.

**Remark 2.1.** (i) For the coarse graining scheme associated to \( r = r_L \), we have by definition \( \mathcal{B}_{L,r_L}^* = \mathcal{B}_L \). When performing the non-smoothed estimates in Section 5.3, we work with constant \( r \). In this case, \( \mathcal{B}_{L,r}^* \) can contain more points than \( \mathcal{B}_L \).

(ii) Assume \( L \) large. If \( x \in V_L \) with \( d_L(x) > 2r \), then the function \( h_{L,r}(x + \cdot) \), defined in [2], lies in \( \mathcal{M}_t \) for each \( t \in [h_{L,r}(x), 2h_{L,r}(x)] \). Thus, for all \( x \in V_L \), we can use C1(δ, L, 0, L1) to control the event \( \{ x \in \mathcal{B}_{L,r} \} \), provided \( 2h_{L,r}(x) \leq L_1 \).

It is difficult to obtain estimates for the RWRE in the presence of bad points. For all environments, we therefore introduce “goodified” transition kernels. Write \( p \) for \( p_{sL/20} \). Then the goodified transition kernels are defined as follows.

\[
\hat{\Pi}^{q}_{L,r}(x, \cdot) = \begin{cases} 
\hat{\Pi}_{L,r}(x, \cdot) & \text{for } x \in V_L \setminus \mathcal{B}^*_L, \\
\hat{\pi}^{(p)}_{L,r}(x, \cdot) & \text{for } x \in \mathcal{B}^*_L.
\end{cases}
\]

We write \( \hat{G}^{q}_{L,r} \) for the corresponding (random) Green’s function. Note that the transition kernel \( q \) used in the definition of a good point \( x \in V_L \) does depend on the location of \( x \) inside the ball, whereas the goodifying-procedure uses the same transition kernel \( p \) for all points (which agrees with \( q \) for \( x \in V_L \) with \( d_L(x) \geq 2s_L \), since in this region \( h_{L,r} \equiv (1/20)s_L \)). Goodified transition kernels and Green’s functions will play a major role from Section 4 onwards.
2.3 Bad regions in the case $r = r_L$

The next lemma shows that with high probability, all bad points with respect to $r = r_L$ are contained in a ball of radius $4h_L(x)$. Let

$$D_L = \{V_{4h_L(x)}(x) : x \in V_L\}.$$  

We will look at the events $\text{OneBad}_L = \{B_L \subset D \text{ for some } D \in D_L\}$ and $\text{ManyBad}_L = (\text{OneBad}_L)^c$. It is also useful to define the set of good environments, $\text{Good}_L = \{B_L = \emptyset\} \subset \text{OneBad}_L$.

**Lemma 2.3.** Assume $C_1(\delta, L_0, L_1)$. Then for $L$ with $L_1 \leq L \leq L_1(\log L_1)^2$, 

$$\mathbb{P}(\text{ManyBad}_L) \leq \exp\left(-\frac{19}{10}(\log L)^2\right).$$

**Proof:** Let $x \in V_L$ with $d_L(x) > 2r_L$. Set $q = p_{h_L}(x)$, and $\Delta = \hat{\pi}_L(x, \cdot) - \hat{\pi}_L^{(q)}(x, \cdot)$.

Using $\frac{1}{20}r_L \leq h_L(x) \leq s_L \leq L_1/2$, we have

$$\mathbb{P}(x \in B_L) = \mathbb{P}\left(\{||\Delta(x, \cdot)||_1 > (\log h_L(x))^{-9}\} \cup \{||\Delta(x, \cdot)||_1 > \delta\}\right)$$

$$\leq \mathbb{P}\left(\bigcup_{t \in [h_L(x), 2h_L(x)]} \{D_{t,q,h_L}(x) > (\log h_L(x))^{-9}\} \cup \{D_{t,q}(x) > \delta\}\right)$$

$$\leq Cs_L d \exp\left(- (\log(r_L/20))^2\right),$$

and a similar estimate holds in the case $d_L(x) \leq 2r_L$. On the event $\text{ManyBad}_L$, there exist $x, y \in B_L$ with $|x - y| > 2h_L(x) + 2h_L(y)$. But for such $x, y$, the events $\{x \in B_L\}$ and $\{y \in B_L\}$ are independent, whence for $L$ large

$$\mathbb{P}(\text{ManyBad}_L) \leq C L^2 d s_L^2 \left[\exp\left(- (\log(r_L/20))^2\right)\right]^2 \leq \exp\left(- (19/10)(\log L)^2\right).$$

The estimate is good enough for our inductive procedure, so we only have to deal with the case where all possibly bad points are enclosed in a ball $D \in D_L$. However, inside $D$ we need to look closer at the degree of badness.

We say that $\omega \in \text{OneBad}_L$ is **bad on level $i$, $i = 1, 2, 3$**, if the following holds:
• For all \( x \in V_L \), for all \( t \in [h_L(x), 2h_L(x)] \), with \( q = p_{h_L(x)} \),
\[
\left\| \left( \Pi_{V_i(x)} - \pi^{(q)}_{V_i(x)} \right)(x, \cdot) \right\|_1 \leq \delta.
\]

• For all \( x \in V_L \) with \( d_L(x) > 2r_L \), additionally
\[
\left\| \left( \hat{\Pi}_{L,r_L} - \hat{\pi}^{(q)}_{L,r_L} \right) \hat{\pi}^{(q)}_{L,r_L}(x, \cdot) \right\|_1 \leq (\log h_L(x))^{-9+9i/4}.
\]

• There exists \( x \in B_L(\omega) \) with \( d_L(x) > 2r_L \) such that
\[
\left\| \left( \hat{\Pi}_{L,r_L} - \hat{\pi}^{(q)}_{L,r_L} \right) \hat{\pi}^{(q)}_{L,r_L}(x, \cdot) \right\|_1 > (\log h_L(x))^{-9+9(i-1)/4}.
\]

If \( \omega \in \text{OneBad}_L \) is neither bad on level \( i = 1, 2, 3 \) nor good, we call \( \omega \) bad on level 4. In this case, \( B_L(\omega) \) contains “really bad” points. We write \( \text{OneBad}^{(i)}_L \subset \text{OneBad}_L \) for the subset of all those \( \omega \) which are bad on level \( i = 1, 2, 3, 4 \). Observe that
\[
\text{OneBad}_L = \text{Good}_L \cup \bigcup_{i=1}^{4} \text{OneBad}^{(i)}_L.
\]

On \( \text{Good}_L \), \( \hat{\Pi}^q_{L,r_L} = \hat{\Pi}_{L,r_L} \) and therefore \( \hat{G}^q_{L,r_L} = \hat{G}_{L,r_L} \).

### 2.4 Bad regions when \( r \) is a constant

When estimating non-smoothed exit measures, we cannot stop the refinement of the coarse graining in the boundary region \( \text{Sh}_L(r_L) \). Instead, we will choose \( r \) as a (large) constant. However, now it is no longer true that essentially all bad points are contained in one single region \( D \in D_L \). For example, if \( x \in V_L \) such that \( d_L(x) \) is of order \( \log L \), we only have a bound of the form
\[
\mathbb{P}(x \in B_{L,r}) \leq \exp\left(-c(\log \log L)^2\right),
\]
which is clearly not enough to get an estimate as in Lemma \ref{lem:shadowing}. We therefore choose a different strategy to handle bad points within \( \text{Sh}_L(r_L) \). We split the boundary region into layers of an appropriate size and use independence to show that with high probability, bad regions are rather sparse within those layers. Then the Green’s function estimates of Corollary \ref{cor:green-function} will ensure that on such environments, there is a high chance to never hit points in \( B_{L,r}^0 \) before leaving the ball.

To begin with the first part, fix \( r \) with \( r \geq r_0 \geq 100 \), where \( r_0 = r_0(d) \) is constant that will be chosen below. Let \( L \) be large enough such that \( r < r_L \), and set \( J_1 = J_1(L) = \left\lfloor \frac{\log(r_L/r)}{\log 2} \right\rfloor + 1 \). We define layers \( \Lambda_0 = \text{Sh}_L(2r) \) and \( \Lambda_j = \text{Sh}_L(r2^j, r2^{j+1}) \), \( 1 \leq j \leq J_1 \). Then,
\[
\text{Sh}_L(2r_L) \subset \bigcup_{0 \leq j \leq J_1} \Lambda_j \subset \text{Sh}_L(4r_L).
\]
Let \( j \in \mathbb{N} \). For \( k \in \mathbb{Z} \), consider the interval \( I_k^{(j)} = (kr2^j, (k+1)r2^j] \cap \mathbb{Z} \). We divide \( \Lambda_j \) into subsets by setting \( D_k^{(j)} = \Lambda_j \cap (I_{k_1} \times \ldots \times I_{k_d}), \) where \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \). Denote by \( Q_{j,r} \) the set of these subsets which are not empty. Setting \( N_{j,r} = |Q_{j,r}|, \) it follows that
\[
\frac{1}{C} \left( \frac{L}{r2^j} \right)^{d-1} \leq N_{j,r} \leq C \left( \frac{L}{r2^j} \right)^{d-1}.
\]
We say that a set \( D \in Q_{j,r} \) is bad if \( B_{L,r}^q \cap D \neq \emptyset \). As we want to make use of independence, we partition \( Q_{j,r} \) into disjoint sets \( Q_{j,r}^{(1)}, \ldots, Q_{j,r}^{(R)} \), such that for each \( 1 \leq m \leq R \) we have
- \( d(D, D') > 4 \max_{x \in \Lambda_j} h_{L,r}(x) \) for all \( D \neq D' \in Q_{j,r}^{(m)} \),
- \( N_{j,r}^{(m)} = |Q_{j,r}^{(m)}| \geq \frac{N_{j,r}}{2R} \).

Notice that \( R \in \mathbb{N} \) can be chosen to depend on the dimension only. Then the events \( \{D \text{ is bad}\} \), \( D \in Q_{j,r}^{(m)} \), are independent. Further, if \( L_1 \leq L \leq L_1 (\log L_1)^2 \), it follows that under \( C1(\delta, L_1) \),
\[
\Pr(D \text{ is bad}) \leq C (r2^j)^{2d} \exp \left( -\left( \log(r2^j/20) \right)^2 \right) \leq \exp \left( -(\log r + j)^{5/3} \right) = p_{j,r},
\]
for all \( r \geq r_0 \) and \( j \in \mathbb{N} \), if \( r_0 \) is big enough. Let \( Y_{j,r} \) and \( Y_{j,r}^{(m)} \) be the number of bad sets in \( Q_{j,r} \) and \( Q_{j,r}^{(m)} \), respectively. For \( r \geq 5 \), we have \( p_{j,r} \leq (\log r + j)^{-3/2} \leq 1/2 \). A standard large deviation estimate for Bernoulli random variables yields
\[
\Pr \left( Y_{j,r}^{(m)} \geq (\log r + j)^{-3/2} N_{j,r}^{(m)} \right) \leq \exp \left( -N_{j,r}^{(m)} I \left( (\log r + j)^{-3/2} \mid p_{j,r} \right) \right),
\]
with \( I(x \mid p) = x \log(x/p) + (1-x) \log((1-x)/(1-p)) \). By enlarging \( r_0 \) if necessary, we get \( I \left( (\log r + j)^{-3/2} \mid p_{j,r} \right) \geq 2R (\log r + j)^{1/7} \) for \( r \geq r_0 \), whence
\[
\Pr \left( Y_{j,r} \geq (\log r + j)^{-3/2} N_{j,r} \right) 
\leq R \max_{m=1, \ldots, R} \Pr \left( Y_{j,r}^{(m)} \geq (\log r + j)^{-3/2} N_{j,r}^{(m)} \right) 
\leq R \exp \left( -(\log r + j)^{1/7} N_{j,r} \right)
\leq R \exp \left( -\frac{1}{C} (\log r + j)^{1/7} \left( \frac{L}{r2^j} \right)^{d-1} \right) 
\leq \exp \left( -(\log r + j)^{1/7} (\log L)^{29} \right),
\]
for \( r_0 \leq r < r_L \), \( 0 \leq j \leq J_1(L) \) and \( L \) large enough. In particular,

\[
\sum_{0 \leq j \leq J_1(L)} \mathbb{P}\left( Y_{j,r} \geq (\log r + j)^{-3/2} N_{j,r} \right) \leq \exp\left(-\left(\log L\right)^{28}\right).
\]

Therefore, setting

\[\text{BdBad}_{L,r} = \bigcup_{0 \leq j \leq J_1(L)} \{ Y_{j,r} \geq (\log r + j)^{-3/2} N_{j,r} \},\]

we have proved the following

**Lemma 2.4.** There exists a constant \( r_0 > 0 \) such that if \( r \geq r_0 \), then \( \text{C1}(\delta, L_0, L_1) \) implies that for \( L_1 \leq L \leq L_1(\log L_1)^2 \),

\[\mathbb{P}(\text{BdBad}_{L,r}) \leq \exp\left(-\left(\log L\right)^{28}\right)\].

### 3 Some important estimates

In this section, we collect estimates on symmetric random walks with kernel \( p \in \mathcal{P}_\kappa \) and on \( d \)-dimensional Brownian motion with (diagonal) covariance matrix given by

\[\Lambda_p = (2dp(e_i)\delta_{i,j})_{i,j=1}^d.\]  

(9)

We can safely use the same letter as for the layers defined in the foregoing section, since it will always be clear from the context what is meant. The following statements hold for small \( \kappa \), meaning that there exists \( 0 < \kappa_0 < 1/(2d) \) such that for \( 0 < \kappa \leq \kappa_0 \), the statements hold true. All constants are then uniform in \( p \in \mathcal{P}_\kappa \).

#### 3.1 Hitting probabilities

The first two lemmata concern symmetric random walk. The proofs are provided in the appendix.

**Lemma 3.1.** Let \( p \in \mathcal{P}_\kappa \), and let \( 0 < \eta < 1 \).

(i) There exists \( C = C(\eta) > 0 \) such that for all \( x \in V_{\eta L}, z \in \partial V_L \),

\[C^{-1}L^{-d+1} \leq \pi_{L}^{(p)}(x,z) \leq CL^{-d+1}.
\]

(ii) There exists \( C = C(\eta) > 0 \) such that for all \( x, x' \in V_{\eta L}, z \in \partial V_L \),

\[\left| \pi_{L}^{(p)}(x,z) - \pi_{L}^{(p)}(x',z) \right| \leq C|x - x'|L^{-d}.
\]

A good control over hitting probabilities is given by
Lemma 3.2. Let $a \geq 1$ and $x, y \in \mathbb{Z}^d$ with $x \notin V_a(y)$. There exists a constant $C > 0$ such that for $p \in \mathcal{P}_k^s$,

(i) \[ P_{x,p}(T_{V_a(y)} < \infty) \leq C \left( \frac{a}{|x-y|} \right)^{d-2}. \]

(ii) There exists $C > 0$, independent of $a$, such that when $|x-y| > 7a$,

\[ P_{x,p}(T_{V_a(y)} < \tau_L) \leq C \frac{a^{d-2}\max\{a,d_L(y)\}\max\{1,d_L(x)\}}{|x-y|^d}. \]

(iii) There exists $C > 0$ such that for all $x \in V_L$, $z \in \partial V_L$,

\[ C^{-1} \frac{d_L(x)}{|x-z|^d} \leq \pi_L^{(p)}(x,z) \leq C \frac{\max\{1,d_L(x)\}}{|x-z|^d}. \]

We need to compare exit laws of random walks with different kernels $p \in \mathcal{P}_k^s$, and we need difference estimates on smoothed exit measures. In this direction, it is easier to work with Brownian motion and then transfer the results back to the discrete setting. Let us first introduce some additional notation. Let $p \in \mathcal{P}_k^s$. For a domain $U \subset \mathbb{R}^d$ with smooth boundary and $x \in U$, denote by $\pi_U^{(p)}(x,dz)$ the exit measure from $U$ of $d$-dimensional Brownian motion $W_t$ started at $x$, with diffusion (or covariance) matrix $\Lambda_p$ defined in \[ \mathbb{F} \], i.e. $\mathbb{E}[(W_1-x)^2] = \Lambda_p$. In the case $U = C_L$, we simply write $\pi_L^{(p)}(x,dz)$. By a small abuse of notation, we also write $\pi_U^{(p)}(x,z)$ (or $\pi_L^{(p)}(x,z)$ if $U = C_L$) for the (continuous version of the) density with respect to surface measure on $U$.

In particular, $\pi_U^{B(p_o)}$ is the exit measure from $U$ of standard $d$-dimensional Brownian motion with covariance matrix $I_d$. Its density $\pi_L^{B(p_o)}(x,z)$ is given by the Poisson kernel

\[ \pi_L^{B(p_o)}(x,z) = \frac{1}{d\alpha(d)L} \frac{L^2 - |x|^2}{|x-z|^d}, \tag{10} \]

where $\alpha(d)$ is the volume of the unit ball. For general $p \in \mathcal{P}_k^s$, there is no explicit expression for the kernel $\pi_L^{(p)}(x,z)$. However, we have

Lemma 3.3. There exists $C > 0$ such that for $p \in \mathcal{P}_k^s$ and $x \in C_L$, $z \in \partial C_L$,

(i) \[ C^{-1} \frac{d_L(x)}{|x-z|^d} \leq \pi_L^{(p)}(x,z) \leq C \frac{d_L(x)}{|x-z|^d}. \]

(ii) For $k \in \mathbb{N}$,

\[ C^{-1} \frac{d_L(x)}{|x-z|^{d+k}} \leq \nabla_x \pi_L^{(p)}(x,z) \leq C \frac{d_L(x)}{|x-z|^{d+k}}. \]
From this lemma, we can directly read off the corresponding statements of Lemma 3.1 for Brownian motion with covariance matrix $\Lambda_p$, $p \in P_s^\kappa$. Clearly, also Lemma 3.2 has a direct analog. In fact, part (iii) is reformulated for Brownian motion in the last lemma. For the results corresponding to (i) and (ii), one can follow the proof of Lemma 3.2 in the appendix, replacing the random walk estimates by those for Brownian motion. These analogous results will be used in the appendix.

The following important lemma controls the difference of two Brownian exit densities on $\partial C_L$, when the corresponding diffusion matrices are close together.

**Lemma 3.4.** There exists $C > 0$ such that for $p, q \in P_s^\kappa$, for all $x \in C_{(2/3)L}$ and $z \in \partial C_L$,

$$\left| (\pi_{L}^{B(p)} - \pi_{L}^{B(q)})(x, z) \right| \leq C \|q - p\|_1 L^{-(d-1)}.$$

The proof involves techniques from the theory of elliptic PDEs and is given in the appendix, as well as the proof of the foregoing lemma. It should be pretty clear that differences of exit probabilities of symmetric random walks can be bounded in the same way, i.e.

$$\left| (\pi_{L}^{(p)} - \pi_{L}^{(q)})(x, z) \right| \leq C \|q - p\|_1 L^{-(d-1)}.$$

However, it seems more difficult to prove this, and in any case, we will only need a weaker form, which can be readily deduced from the last lemma and a coupling argument given in the appendix.

**Lemma 3.5.** There exists $C > 0$ such that for $p, q \in P_s^\kappa$, for large $L$, $\psi \in \mathcal{M}_L$, any $x \in \mathbb{Z}^d$ in the domain of $\psi$ and any $z \in \mathbb{Z}^d$,

$$\left| (\hat{\pi}_{\psi}^{(p)} - \hat{\pi}_{\psi}^{(q)})(x, z) \right| \leq C \|q - p\|_1 L^{-(d-1)}.$$

In particular, for $x \in V_L$ with $d_L(x) > (1/10)r$,

$$\| (\hat{\pi}_{L, r}^{(p)} - \hat{\pi}_{L, r}^{(q)})(x, \cdot) \|_1 \leq C \max \{ h_{L, r}(x)^{-1/4}, \|q - p\|_1 \}.$$

**Proof:** By comparing $\hat{\pi}_{\psi}^{(p)}$ to the analogous Brownian quantity $\hat{\pi}_{B(p)}^{(p)}$ defined in [61], the first claim follows from Lemma 3.4, Lemma 7.2 (vii) from the appendix and the triangle inequality. The second statement is a special case of the first one, with the choice $\psi(x) = h_{L, r}(x)$. The restriction to $x$ with $d_L(x) > (1/10)r$ ensures that all exit distributions are taken from balls that lie completely inside $V_L$. □

Let us finish this part by proving the following useful estimate.

**Lemma 3.6.** Let $a > 0$, $l, m \geq 1$ and $x \in \mathbb{Z}^d$. Set $R_l = V_l \backslash V_{l-1}$, $\alpha = \max \{ \|x| - l\|, a \}$. Then for some constant $C = C(m) > 0$

$$\sum_{y \in R_l} \frac{1}{(a + |x - y|)^m} \leq C \left\{ \begin{array}{ll} l^{d-(m+1)} \max \{ \log(l/\alpha), 1 \} & \text{for } 1 \leq m < d - 1 \\ \alpha^{d-(m+1)} & \text{for } m = d - 1 \\ \alpha^{d-(m+1)} & \text{for } m \geq d \end{array} \right.$$
**Proof:** If \( \alpha > l \), then the left-hand side is bounded by
\[
C l^{d-1} \alpha^{-m} \leq C \max \{ \alpha^{d-(m+1)}, l^{d-(m+1)} \}.
\]
If \( \alpha \leq l \), we set \( A_k = \{ y \in R_l : |x - y| \in [(k - 1)\alpha, k\alpha) \} \). Then, for all \( k \geq 1 \),
\[
\max_{y \in A_k} \frac{1}{(a + |x - y|)^m} \leq 2^m k^{-m} \alpha^{-m}.
\]
Since for \( k\alpha \leq l/10 \) we have \( |A_k| \leq C \alpha^{k\alpha} \), the claim then follows from
\[
\sum_{y \in R_l} \frac{1}{(a + |x - y|)^m} \leq C \left( \sum_{1 \leq k \leq \lfloor l/(10\alpha) \rfloor} \frac{\alpha(k\alpha)^{d-2}}{(k\alpha)^m} \right) + C l^{d-1} l^{-m}
\]
\[
\leq C \alpha^{d-(m+1)} \sum_{1 \leq k \leq \lfloor l/(10\alpha) \rfloor} k^{d-(m+2)} + C l^{d-(m+1)}.
\]

### 3.2 Smoothed exit measures

In order to obtain difference estimates for smoothed exit distributions of a symmetric random walk, we will compare them to the corresponding quantities of Brownian motion.

Let \( p, q \in P^s_\kappa \), and let \( \psi = (m_x)_{x \in \mathbb{R}^d} \) be field of positive real numbers. The smoothed exit distribution from \( V_L \) of the random walk (with respect to \( p, q, \psi \)) is defined as
\[
\phi_{L,p,\psi,q}^L(x,z) = \pi_{L,p}^L(x,y) \frac{1}{m_y} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{m_y} \right) \pi_{V_L}(y,z) dt.
\]
For Brownian motion, the smoothing step is defined analogous to (1), namely
\[
\hat{\pi}_\psi^B(x,dz) = \frac{1}{m_x} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{m_x} \right) \pi_{C_1}(x,dz) dt.
\]

The smoothed exit distribution from \( C_L \) is then given by
\[
\phi_{L,p,\psi,q}^B(x,dz) = \pi_{L,p}^B \hat{\pi}_\psi^B(x,dz)
\]
\[
= \int_{\partial C_L} \pi_{L,p}^B(x,dy) \frac{1}{m_y} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{m_y} \right) \pi_{C_1}(y,dz) dt.
\]

By \( \phi_{L,p,\psi,q}^B(x,z) \) we denote the density of \( \phi_{L,p,\psi,q}^B(x,dz) \) with respect to \( d \)-dimensional Lebesgue measure. For the proof of the next lemma, we refer to the appendix.

**Lemma 3.7.** There exists \( C > 0 \) such that for \( p, q \in P^s_\kappa \) and \( \psi \in \mathcal{M}_L \).

(i)
\[
\sup_{x \in V_L} \sup_{z \in \mathbb{Z}^d} \left| \left( \phi_{L,p,\psi,q}^L - \phi_{L,p,\psi,q}^B \right)(x,z) \right| \leq C L^{-(d+1/4)}.
\]
(ii) \[ \sup_{z \in \mathbb{R}^d} \| D^i \phi_{L,p,\psi,q}^B (\cdot, z) \|_{C_L} \leq C L^{-(d+i)}, \quad i = 0, 1, 2, 3. \]

(iii) \[ \sup_{x,x' \in \mathcal{V}_L \cup \partial \mathcal{V}_L} \sup_{z \in \mathbb{Z}^d} | \phi_{L,p,\psi,q}(x, z) - \phi_{L,p,\psi,q}(x', z) | \leq C \left( L^{-(d+1/4)} + |x - x'| L^{-(d+1)} \right). \]

The next proposition will be applied at the end of the proof of Lemma 5.4. At this point, the symmetry condition \( A1 \) comes into play. We give a general formulation in terms of a signed measure \( \nu \). Let us introduce the following notation. For \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d, i = 1, \ldots, d, \) put \( x^{(i)} = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_d). \)

**Proposition 3.1.** Let \( p,q \in P_s^\kappa \) and \( l > 0 \). Consider a measure \( \nu \) on \( \mathcal{V}_L \) with total mass zero satisfying \( \nu(x) = \nu(x^{(i)}) \) for all \( x \) and all \( i = 1, \ldots, d \). Then there is a constant \( C > 0 \) such that for \( y' \in \mathcal{V}_L \) with \( \mathcal{V}_l(y') \subset \mathcal{V}_L \) and all \( z \in \mathbb{Z}^d, \psi \in \mathcal{M}_L, \)

\[ \left| \sum_{y \in \mathcal{V}_l(y')} \nu(y - y') \phi_{L,p,\psi,q}(y, z) - \sum_{y \in \mathcal{V}_l(y')} \nu(y) \phi_{L,p,\psi,q}^B(y, z) \right| \leq C ||\nu||_1 \left( L^{-(d+1/4)} + \left( \frac{l}{L} \right)^2 L^{-d} \right). \]

**Proof:** We simply write \( \phi \) for \( \phi_{L,p,\psi,q} \) and \( \phi^B \) for \( \phi_{L,p,\psi,q}^B \). Since the proof is the same for all \( y' \in \mathcal{V}_L \) with \( \mathcal{V}_l(y') \subset \mathcal{V}_L \), we can assume \( y' = 0 \). By Lemma 3.7 (i),

\[ \left| \sum_{y} \nu(y) \phi(y, z) - \sum_{y} \nu(y) \phi^B(y, z) \right| \leq C ||\nu||_1 L^{-(d+1/4)}. \]

Taylor’s expansion gives

\[ \sum_{y} \nu(y) \phi^B(y, z) = \sum_{y} \nu(y) \left[ \phi^B(y, z) - \phi^B(0, z) \right] = \sum_{y} \nu(y) \nabla_y \phi^B(0, z) \cdot y + \frac{1}{2} \sum_{y} \nu(y) y \cdot H_x \phi^B(0, z) y + R(\nu, 0, z), \]

(11)

where \( \nabla_y \phi^B \) is the gradient, \( H_x \phi^B \) the Hessian of \( \phi^B \) with respect to the first variable, and \( R(\nu, 0, z) \) is the remainder term. Due to the symmetry condition on \( \nu \), the first summand on the right side of (11) vanishes, and for the second and third summand one can use Lemma 3.7 (ii). \( \square \)
Remark 3.1. In [7] and [1], it is assumed that $\mu$, the measure governing the environment, is isotropic. This leads to consider a measure $\nu$ that is invariant not only under $x \mapsto x^{(i)}$, but also under $x \mapsto x^{+(i,j)}$, where for $i < j$,

$$x^{+(i,j)} = (x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_d).$$

In this case, the choice $p = p_o$ results in the sharper bound

$$\left| \sum_{y \in \mathcal{V}(y')} \nu(y - y') \phi_{L, p, \psi, q}(y, z) \right| \leq C \|\nu\|_1 \left( L^{-(d+1/4)} + \left( \frac{1}{L} \right)^3 L^{-d} \right),$$

see Proposition 3.1 in [1]. It is then clear from the proof of Lemma 5.4 that one can work with $p_L = p_o$ for all radii $L$, i.e. the (isotropic) RWRE exit measure approaches that of simple random walk.

4 Green’s functions for the ball

One main task of our approach aims at developing good estimates on Green’s functions for the ball of both coarse grained (goodified) RWRE as well as coarse grained symmetric random walk in the perturbative regime. The main result is Lemma 4.2. For coarse grained symmetric random walk, the estimates on hitting probabilities of the last section together with Proposition 4.2 yield the right control.

On a certain class of environments, we need to modify the transition kernels in order to ensure that bad points are not visited too often by the coarse grained random walks. This modification will be described in Section 4.3.

We work with the same convention concerning the parameter $\kappa$ as in Section 3.

4.1 A local central limit theorem

Let $p \in \mathcal{P}_\kappa$ and $m \geq 1$. Denote by $\hat{\pi}_m = \hat{\pi}_m(p)$ the coarse grained transition probabilities on $\mathbb{Z}^d$ belonging to the field $\psi = (m_x)_{x \in \mathbb{Z}^d}$, where $m_x = m$ is chosen constant in $x$ (cf. [1]). We constantly drop $p$ from notation. Notice that $\hat{\pi}_m$ is centered, and the covariances satisfy

$$\sum_{y \in \mathbb{Z}^d} (y_i - x_i)(y_j - x_j) \hat{\pi}_m(x, y) = \lambda_{m, i} \delta_{i,j},$$

where for large $m$, $C^{-1} < \lambda_{m, i}/m^2 < C$ for some $C > 0$. Define the matrix

$$\Lambda_m = (\lambda_{m, i} \delta_{i,j})_{i,j=1}^d,$$

and let for $x \in \mathbb{Z}^d$

$$\mathcal{J}_m(x) = |\Lambda_m^{-1/2} x|. $$
Proposition 4.1 (Local central limit theorem). Let \( p \in \mathcal{P}_\kappa \), and let \( x, y \in \mathbb{Z}^d \). For \( m \geq 1 \) and all integers \( n \geq 1 \),
\[
(\hat{s}_m)^n(x, y) = \frac{1}{(2\pi n)^{d/2}} \det \Lambda_{m}^{1/2} \exp \left( -\frac{\mathcal{J}^2_m(x-y)}{2n} \right) + O \left( m^{-d} n^{-(d+2)/2} \right).
\]

For the corresponding Green’s function \( \hat{g}_{m,\mathbb{Z}^d}(x, y) = \sum_{n=0}^{\infty} (\hat{s}_m)^n(x, y) \) we obtain

Proposition 4.2. Let \( p \in \mathcal{P}_\kappa \). Let \( x, y \in \mathbb{Z}^d \), and assume \( m \geq m_0 > 0 \) large enough.

(i) For \( |x - y| < 3m \),
\[
\hat{g}_{m,\mathbb{Z}^d}(x, y) = \delta_x(y) + O(m^{-d}).
\]

(ii) For \( |x - y| \geq 3m \), there exists a constant \( c(d) > 0 \) such that
\[
\hat{g}_{m,\mathbb{Z}^d}(x, y) = \frac{c(d) \det \Lambda_{m}^{1/2}}{\mathcal{J}_m(x-y)^{d-2}} + O \left( \frac{1}{|x-y|^d} \left( \log \frac{|x-y|}{m} \right)^d \right).
\]

Note that the constants in the \( O \)-notation are independent of \( n, m \) and \( |x - y| \).

In our applications, \( m \) will be a function of \( L \). Although these results look rather standard, we cannot directly refer to the literature because we have to keep track of the \( m \)-dependency. We give a proof of both statements in the appendix. The last proposition will be used to estimate the Green’s function for the ball \( V_L \), \( \hat{g}_m(x, y) = \sum_{n=0}^{\infty} (1_{V_L} \hat{s}_m)^n(x, y) \). Clearly, \( \hat{g}_m \) is is bounded from above by \( \hat{g}_{m,\mathbb{Z}^d} \).

4.2 Estimates on coarse grained Green’s functions

As we will show, the perturbation expansion enables us to control the goodified Green’s function \( \hat{G}_{L,r}^q \) essentially in terms of \( \hat{g}_{L,r}^{(p)} \), where \( p \) is the kernel corresponding to the radius \( s_L/20 \), stemming from the assignment (6).

The first step in controlling the Green’s function is provided by the following lemma.

Lemma 4.1. Assume \( \mathbf{C1}(\delta, L_0, L_1) \), let \( L_1 \leq L \leq L_1 \log L_1^2 \) and put \( p = p_{s_L/20} \). Then for all \( x \in V_L \setminus \text{Sh}_L(2r) \), with \( H(x) = \max \{ L_0, h_{L,r}(x) \} \),
\[
\left\| (\hat{\Pi}^g_{L,r} - \hat{\pi}^{(p)}_{L,r}) \hat{\pi}^{(p)}_{L,r}(x, \cdot) \right\|_1 \leq C \min \left\{ \log(s_L/H(x)) \log(H(x))^{-9}, (\log H(x))^{-8} \right\},
\]
\[
\left\| (\hat{\Pi}^g_{L,r} - \hat{\pi}^{(q)}_{L,r}) \hat{\pi}^{(q)}_{L,r}(x, \cdot) \right\|_1 \leq 2\delta.
\]

Proof: For \( x \in B^c_{L,r} \), both left sides are zero. Now let \( x \in V_L \setminus (\text{Sh}_L(2r) \cup B^c_{L,r}) \), and set \( q = p_{h_{L,r}(x)} \). By the triangle inequality,
\[
\left\| (\hat{\Pi}^g - \hat{\pi}^{(p)}) \hat{\pi}^{(p)}(x, \cdot) \right\|_1 \leq \left\| (\hat{\Pi}^g - \hat{\pi}^{(q)}) \hat{\pi}^{(q)}(x, \cdot) \right\|_1 + \left\| (\hat{\pi}^{(p)} - \hat{\pi}^{(q)})(x, \cdot) \right\|_1
\]
\[
\leq \left\| (\hat{\Pi}^g - \hat{\pi}^{(q)}) \hat{\pi}^{(q)}(x, \cdot) \right\|_1 + 2 \sup_{y \in V_L \setminus \text{Sh}(r)} \left\| (\hat{\pi}^{(p)} - \hat{\pi}^{(q)})(y, \cdot) \right\|_1
\]
\[
\leq C \left( (\log H(x))^{-9} + \| p - q \|_1 \right),
\]
where in the last line we used that $x$ is good and Lemma 3.3. Now, with $K = \lfloor (\log 2)^{-1} \log(s_{L}/H(x)) \rfloor$, Lemma 2.2 shows

$$
\|p - q\|_{1} \leq C \sum_{i=1}^{K} (\log(2^{-1}s_{L}))^{-9} \leq C \min \left\{ K(\log H(x))^{-9}, (\log H(x))^{-8} \right\}.
$$

This proves the claim for the smoothed difference. For the non-smoothed difference,

$$
\left\| (\hat{\Pi} - \hat{\pi}(p))(x, \cdot) \right\|_{1} \leq \left\| (\hat{\Pi} - \hat{\pi}(q))(x, \cdot) \right\|_{1} + \left\| (\hat{\pi}(p) - \hat{\pi}(q))(x, \cdot) \right\|_{1}.
$$

Since $x$ is good, the first term is bounded by $\delta$, and, by what we have just seen, the second term is bounded by $\delta$ as well if we choose $L_0$ (and so $L$) large enough. \qed

**Remark 4.1.** As the reader should notice, the choice of the parameter $r$ depends on $\delta$. See also the preliminary remarks of Section 5.3.

Recall that in the goodifying procedure introduced in Section 2, “bad” exit distributions inside $V_L$ are replaced by such of a symmetric random walk with one-step distribution $p = p_{\epsilon_{L/20}}$. For this $p$ and good points $x$ within the boundary region $Sh_L(2r)$, we would like to use at least an estimate of the form

$$
\left\| (\hat{\Pi}_{L,r} - \hat{\pi}_{L,r}(p))(x, \cdot) \right\|_{1} \leq C\delta.
$$

However, exit measures at points $x$ inside $Sh_L((1/10)r)$ are taken from intersected balls $V_t(x) \cap V_L$. We therefore work in this (and only in this) section with slightly modified transition kernels $\hat{\Pi}_{L,r}, \tilde{\pi}_{L,r}, \tilde{\Pi}_{L,r}$ in the enlarged ball $V_{L+r}$, taking the exit measure in $Sh_L(2r)$ from uncut balls $V_t(x) \subset V_{L+r}$, $t \in [h_{L,r}(x), 2h_{L,r}(x)]$.

Now, to make things precise, for $q \in \mathcal{P}_L^x$, setting $h_{L,r}(x) = (1/20)r$ for $x \notin C_L$, we let $\tilde{\pi}_{L,r}^{(q)}$ be the coarse grained symmetric random walk kernel under $\tilde{\psi} = (h_{L,r}(x))_{x \in V_{L+r}}$,

$$
\tilde{\pi}_{L,r}^{(q)}(x, \cdot) = \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{h_{L,r}(x)} \right) \tilde{\pi}_{V_t(x) \cap V_{L+r}}^{(q)}(x, \cdot) dt.
$$

For the corresponding RWRE kernel, we forget about the environment on $V_{L+r} \backslash V_L$ and set

$$
\tilde{\Pi}_{L,r}^{(q)}(x, \cdot) = \begin{cases} 
\frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{h_{L,r}(x)} \right) \Pi_{V_t(x)}(x, \cdot) dt & \text{for } x \in V_L \\
\tilde{\pi}_{L,r}^{(q)}(x, \cdot) & \text{for } x \in V_{L+r} \backslash V_L
\end{cases}
$$

For $p = p_{\epsilon_{L/20}}$ and all good $x \in V_L$ we now have $\| (\hat{\Pi}_{L,r}^{(p)} - \tilde{\pi}_{L,r}^{(p)})(x, \cdot) \|_1 \leq \delta$ provided $\epsilon$ is small enough, while for $x \in V_{L+r} \backslash V_L$, the difference vanishes anyway. The goodified version of $\tilde{\Pi}_{L,r}^{(p)}$ is then obtained in an analogous way to (8),

$$
\tilde{\Pi}_{L,r}^{(p)}(x, \cdot) = \begin{cases} 
\tilde{\Pi}_{L,r}^{(p)}(x, \cdot) & \text{for } x \notin B_{L,r}^x \\
\tilde{\pi}_{L,r}^{(p)}(x, \cdot) & \text{for } x \in B_{L,r}^x
\end{cases}
$$
Clearly, for \( x \in V_L \setminus \text{Sh}_L(2r) \), the first statement of Lemma 4.1 holds with the left side there replaced by
\[
\left\| (\tilde{\Pi}_{L,r}^g - \tilde{\pi}_{L,r}^{(p)})\tilde{\pi}_{L,r}^{(p)}(x, \cdot) \right\|_1.
\]
But thanks to the modified transition kernels, we now have
\[
\left\| (\tilde{\Pi}_{L,r}^g - \tilde{\pi}_{L,r}^{(p)})(x, \cdot) \right\|_1 \leq 2\delta
\]
for all \( x \in V_L \). Indeed, one just has to notice that Lemma 4.1 can now also be applied to points \( x \in \text{Sh}_L(2r) \), with the same proof.

We write \( \tilde{G}_{L,r}, \tilde{g}_{L,r} \) and \( \tilde{G}_{L,r}^g \) for the Green’s functions on \( V_{L+r} \) corresponding to \( \tilde{\Pi}_{L,r}, \tilde{\pi}_{L,r} \) and \( \tilde{\Pi}_{L,r}^g \). Note
\[
\tilde{G}_{L,r} \leq \tilde{G}_{L,r}^g, \quad \tilde{g}_{L,r} \leq \tilde{g}_{L,r}^g, \quad \tilde{G}_{L,r}^g \leq \tilde{G}_{L,r}^g \quad \text{pointwise on } V_{L+r} \times (V_{L+r} \setminus \partial V_L).
\]
Since we do not have exact expressions for \( \tilde{g}_{L,r} \) or \( \tilde{G}_{L,r} \), we construct a (deterministic) kernel \( \Gamma_{L,r} \) that bounds the Green’s functions from above. For \( x \in V_{L+r} \), set
\[
\tilde{d}(x) = \max \left( \frac{d_{L+r}(x)}{2}, 3r \right), \quad a(x) = \min \left( \tilde{d}(x), s_L \right).
\]
Further, let
\[
\Gamma_{L,r}^{(1)}(x, y) = \frac{\tilde{d}(x)\tilde{d}(y)}{a(y)^2(a(y) + |x - y|)^d}, \quad \Gamma_{L,r}^{(2)}(x, y) = \frac{1}{a(y)^2(a(y) + |x - y|)^{d-2}}.
\]
The kernel \( \Gamma_{L,r} \) is defined as the pointwise minimum
\[
\Gamma_{L,r} = \min \left\{ \Gamma_{L,r}^{(1)}, \Gamma_{L,r}^{(2)} \right\}.
\]
We cannot derive pointwise estimates on the Green’s functions in terms of \( \Gamma_{L,r} \), but we can use this kernel to obtain upper bounds on neighborhoods \( U(x) = V_{a(x)}(x) \cap V_{L+r} \).

Call a function \( F : V_{L+r} \times V_{L+r} \rightarrow \mathbb{R}_+ \) a positive kernel. Given two positive kernels \( F \) and \( G \), we write \( F \preceq G \) if for all \( x, y \in V_{L+r} \),
\[
F(x, U(y)) \leq G(x, U(y)),
\]
where \( F(x, U) \) stands for \( \sum_{y \in U} F(x, y) \). Further, we write \( F \asymp 1 \), if there is a constant \( C > 0 \) such that for all \( x, y \in V_{L+r} \),
\[
\frac{1}{C} F(x, y) \leq F(\cdot, \cdot) \leq CF(x, y) \quad \text{on } U(x) \times U(y).
\]
We adopt this notation to positive functions of one argument: For \( f : V_{L+r} \rightarrow \mathbb{R}_+ \), \( f \asymp 1 \) means that for some \( C > 0 \), \( C^{-1}f(x) \leq f(\cdot) \leq Cf(x) \) on any \( U(x) \subset V_{L+r} \).

Finally, given \( 0 < \eta < 1 \), we say that a positive kernel \( A \) on \( V_{L+r} \) is \( \eta \)-smoothing, if for all \( x \in V_{L+r} \), \( A(x, U(x)) \leq \eta \), and \( A(x, y) = 0 \) whenever \( y \notin U(x) \).

Now we are in the position to formulate our main statement of this section. Recall our convention concerning constants: They only depend on the dimension unless stated otherwise.
Lemma 4.2.

(i) There exists a constant $C_1 > 0$ such that for all $q \in \mathcal{P}^e_\kappa$,
\[
\hat{g}^{(q)}_{L,r} \leq C_1 \Gamma_{L,r} \quad \text{and} \quad \tilde{g}^{(q)}_{L,r} \leq C_1 \Gamma_{L,r}.
\]

(ii) There exists a constant $C > 0$ such that for $\delta > 0$ small,
\[
\hat{G}^q_{L,r} \leq C \Gamma_{L,r} \quad \text{and} \quad \tilde{G}^q_{L,r} \leq C \Gamma_{L,r}.
\]

Remark 4.2. (i) Thanks to (12), it suffices to show the bounds for $\tilde{g}^{L,r}$ and $\tilde{G}^q_{L,r}$. For later use, we keep track of the constant in part (i) of the lemma.

(ii) We will later apply part (i) with $q = p_L$. From Lemma 2.2 we know that we can assume $p_L \in \mathcal{P}^e_\kappa$ for every choice of $\kappa > 0$, if $L_0$ is large.

We first prove part (i), which is a straightforward consequence of the estimates on hitting probabilities in Section 3 and the next lemma.

Lemma 4.3. There exists a constant $C > 0$ such that for all $q \in \mathcal{P}^e_\kappa$, for all $x \in V_{L+r}$ and $y \in V_L$ with $d_L(y) \geq 4s_L$,
\[
\tilde{g}^{(q)}_{L,r}(x,y) \leq C \left\{ \begin{array}{ll}
\frac{1}{s^L_{\delta} \max\{|x-y|, s_L\}^{d-2}}, & y \neq x \\
1 & y = x
\end{array} \right.
\]

Proof: The underlying one-step transition kernel is always given by $q \in \mathcal{P}^e_\kappa$, which we therefore omit from notation. For example, $\hat{g}_m = \hat{g}^{(q)}_m$, $P_x = P_{x,q}$, and so on. If $x = y$, then the claim follows from transience of simple random walk. Now assume $x \neq y$, and always $d_L(y) \geq 4s_L$. Consider first the case $|x-y| \leq s_L$. Let $\hat{g}_m$ be defined as in the beginning of Section 4.1. Recall our coarse graining scheme. With $m = s_L/20$ we have
\[
\hat{g}(x,y) \leq \hat{g}_m(x,y) + \sup_{v \in Sh_L(2s_L)} P_v \left( T_{V_{sL}}(y) < \tau_{V_{L+r}} \right) \sup_{w \neq y, |w-y| \leq s_L} \tilde{g}(w,y).
\]

Since
\[
\sup_{v \in Sh_L(2s_L)} P_v \left( T_{V_{sL}}(y) < \tau_{V_{L+r}} \right) < 1
\]
uniformly in $L$, it follows from Proposition 4.2 that
\[
\hat{g}(x,y) \leq C \sup_{w \neq y, |w-y| \leq s_L} \hat{g}_m(w,y) \leq C \sup_{w \neq y, |w-y| \leq s_L} \tilde{g}(w,y) \leq \frac{C s^L_{\delta}}{s^L_{\delta} |x-y|^{d-2}}.
\]

If $|x-y| > s_L$ we use Lemma 3.2 (i) and the first case to get
\[
\hat{g}(x,y) \leq P_x \left( T_{V_{sL}}(y) < \infty \right) \sup_{w \neq y, |w-y| \leq s_L} \tilde{g}(w,y) \leq \frac{C}{s^L_{\delta} |x-y|^{d-2}}.
\]
Proof of Lemma 4.2 (i): It suffices to prove the bound for $\tilde{g}$. First we show that there exists a constant $C > 0$ such that for all $y \in V_{L+r}$,

$$\sup_{x \in V_{L+r}} \tilde{g}(x, U(y)) \leq C. \quad (14)$$

At first let $d_{L+r}(y) \leq 6r$. Then $U(y) \subset Sh_{L+r}(10r)$. We claim that even

$$\sup_{x \in V_{L+r}} \tilde{g}(x, Sh_{L+r}(10r)) \leq C. \quad (15)$$

for some $C > 0$. Indeed, if $z \in Sh_{L+r}(10r)$, then $\tilde{\pi}(z, \cdot)$ is an (averaging) exit distribution from balls $V_l(z) \cap V_{L+r}$, where $l \geq r/20$. Using Lemma 3.1 (i), we find a constant $k_1 = k_1(d)$ such that after any $z \in Sh_{L+r}(10r)$, $V_{L+r}$ is left after $k_1$ steps with probability $> 0$, uniformly in $z$. This together with the strong Markov property implies (15). Next assume $6r < d_{L+r}(y) \leq 6s_L$. Then $U(y) \subset S(y) = Sh_{L+r}(\frac{1}{2}d_{L+r}(y), 2d_{L+r}(y))$. We claim that

$$\sup_{x \in V_{L+r}} \tilde{g}(x, S(y)) \leq C. \quad (16)$$

For $z \in S(y)$, $\tilde{\pi}(z, \cdot)$ is an averaging exit distribution from balls $V_l(z)$, where $l \geq d_{L+r}(y)/240$. By Lemma 3.1 (i), we find some small $0 < c < 1$ and a constant $k_2(c, d)$ such that after $k_2$ steps, the walk has probability $> 0$ to be in $Sh_{L+r}(\frac{1-c}{2}d_{L+r}(y))$, uniformly in $z$ and $y$. But starting in $Sh_{L+r}(\frac{1-c}{2}d_{L+r}(y))$, an iterative application of Lemma 3.1 (i) shows that with probability $> 0$, the ball $V_{L+r}$ is left before $S(y)$ is visited again. Therefore (16) and hence (14) hold in this case. At last, let $d_{L+r}(y) > 6s_L$. Then $d_L(w) \geq 4s_L$ for $w \in U(y)$. Estimating

$$\tilde{g}(x, w) \leq 1 + \sup_{v \neq w} \tilde{g}(v, w),$$

we get with part (i) that

$$\sup_{w \in U(y)} \tilde{g}(x, w) \leq 1 + \frac{C}{s_L}. $$

Summing over $w \in U(y)$, (14) follows. Finally, note that for any $x \in V_{L+r}$,

$$\tilde{g}(x, U(y)) \leq P_x \left(T_{U(y)} < \tau_{V_{L+r}} \right) \sup_{w \in U(y)} \tilde{g}(w, U(y)).$$

Now $\tilde{g} \leq CT$ follows from (14) and the hitting estimates of Lemma 3.2. \hfill \Box

Let us now explain our strategy for proving part (ii). By version (5) of the perturbation expansion, we can express $G^y_{L,r}$ in a series involving $\tilde{g}_{L,r}$ and differences of exit measures. The Green’s function $\tilde{g}_{L,r}$ is already controlled by means of $\Gamma_{L,r}$. Looking at (5), we thus have to understand what happens if $\Gamma_{L,r}$ is concatenated with certain smoothing kernels. This will be the content of Proposition 4.3.
4.2 Estimates on coarse grained Green’s functions

We start with collecting some important properties of \( \Gamma_{L,r} \), which will be used throughout this text. Define for \( j \in \mathbb{N} \)

\[
L_j = \{ y \in V_L : j \leq d_L(y) < j + 1 \}, \\
E_j = \{ y \in V_{L+r} : \tilde{d}(y) \leq 3jr \}.
\]

**Lemma 4.4** (Properties of \( \Gamma_{L,r} \)).

(i) Both \( \tilde{d} \) and \( a \) are Lipschitz with constant \( 1/2 \). Moreover, for \( x, y \in V_{L+r} \),

\[
a(y) + |x - y| \leq a(x) + \frac{3}{2} |x - y|.
\]

(ii) \( \Gamma_{L,r} \asymp 1 \).

(iii) For \( 0 \leq j \leq 2s_L, x \in V_{L+r} \),

\[
\sum_{y \in L_j} \left( \max \left\{ 1, \frac{\tilde{d}(x)}{a(y)} \right\} \frac{1}{(a(y) + |x - y|)^d} \right) \leq C \frac{1}{j \lor r}.
\]

(iv) For \( 1 \leq j \leq \frac{1}{3} s_L \),

\[
\sup_{x \in V_{L+r}} \Gamma_{L,r}(x, E_j) \leq C \log(j + 1),
\]

and for \( 0 \leq \alpha < 3 \),

\[
\sup_{x \in V_{L+r}} \Gamma_{L,r}(x, Sh_L(s_L, L/(\log L)^\alpha)) \leq C (\log \log L)(\log L)^{6-2\alpha}.
\]

(v) For \( x \in V_{L+r} \), in the case of constant \( r \),

\[
\Gamma_{L,r}(x, V_L) \leq C \max \left\{ \frac{\tilde{d}(x)}{L}(\log L)^6, \left( \frac{\tilde{d}(x)}{r} \right)^\wedge \log L \right\}.
\]

In the case \( r = r_L \),

\[
\Gamma_{L,r_L}(x, V_L) \leq C \max \left\{ \frac{\tilde{d}(x)}{L}(\log L)^6, \left( \frac{\tilde{d}(x)}{r_L} \right)^\wedge \log \log L \right\}.
\]

**Proof:** (i) The second statement is a direct consequence of the Lipschitz property, which in turn follows immediately from the definitions of \( \tilde{d} \) and \( a \).

(ii) As for \( y' \in U(y) \), \( \frac{1}{2} a(y) \leq a(y') \leq \frac{3}{2} a(y) \) and similarly with \( a \) replaced by \( \tilde{d} \), it suffices to show that for \( x' \in U(x), y' \in U(y) \),

\[
\frac{1}{C} (a(y) + |x - y|) \leq a(y') + |x' - y'| \leq C (a(y) + |x - y|). \tag{17}
\]
First consider the case $|x - y| \geq 4 \max \{a(x), a(y)\}$. Then

$$a(y) + |x - y| \leq 2a(y') + 2( |x - y| - a(x) - a(y)) \leq 2(a(y') + |x' - y'|).$$

If $|x - y| \leq 4a(y)$ then

$$a(y) + |x - y| \leq 5a(y) \leq 5a(y) + |x' - y'| \leq 10(a(y') + |x' - y'|),$$

while for $|x - y| \leq 4a(x)$, using part (i) in the first inequality,

$$a(y) + |x - y| \leq a(x) + \frac{3}{2}|x - y| \leq 7a(x) \leq 14(a(y') + |x' - y'|).$$

This proves the first inequality in (17). The second one follows from

$$a(y') + |x' - y'| \leq \frac{5}{2}a(y) + a(x) + |x - y| \leq \frac{7}{2}(a(y) + |x - y|).$$

(iii) If $j \leq 2s_L$ and $y \in L_j$, then $a(y)$ is of order $j \lor r$. By Lemma 3.6 we have

$$\sum_{y \in L_j} \frac{1}{(j \lor r + |x - y|)^d} \leq C \min \left\{ \frac{1}{j \lor r}, \frac{1}{d_{L+r}(x) - (j + r)} \right\}.$$

It remains to show that

$$\max \left\{ 1, \frac{\tilde{d}(x)}{j \lor r} \right\} \min \left\{ \frac{1}{j \lor r}, \frac{1}{d_{L+r}(x) - (j + r)} \right\} \leq C \frac{1}{j \lor r}.$$

(18) If $\tilde{d}(x) \leq (j \lor 3r)$, this is clear. If $\tilde{d}(x) > (j \lor 3r)$, (18) follows from $|d_{L+r}(x) - (j + r)| \geq \tilde{d}(x)/2$.

(iv) If $\tilde{d}(y) \leq 3jr$, then $d_L(y) \leq 6jr$. Estimating $\Gamma$ by $\Gamma^{(1)}$, we get

$$\Gamma(x, E_j) \leq C \sum_{i=0}^{6jr} \sum_{y \in L_i} \frac{\tilde{d}(x)}{a(y) (a(y) + |x - y|)^d}.$$

Now the first assertion of (iv) follows from (iii). The second is proved similarly, so we omit the details.

(v) Set $B = \{ y \in V_L : \tilde{d}(y) \leq s_L \lor 2\tilde{d}(x) \}$. For $y \in V_L \setminus B$, it holds that $a(y) = s_L$ and $|x - y| \geq \tilde{d}(y)$. Therefore,

$$\Gamma(x, V_L \setminus B) \leq \Gamma^{(1)}(x, V_L \setminus B) \leq \frac{\tilde{d}(x)}{s_L^2} \sum_{y \in V_L \setminus B} \frac{1}{(s_L + |y|)^{d-1}} \leq C \frac{\tilde{d}(x)}{L} (\log L)^6.$$

Furthermore,

$$\Gamma(x, B) \leq \sum_{i=0}^{2s_L} \sum_{y \in L_i} \frac{\tilde{d}(x)}{a(y) (a(y) + |x - y|)^d} + \frac{1}{s_L^2} \sum_{y \in V_L : s_L \leq a(y) \leq 2\tilde{d}(x)} \frac{1}{(s_L + |x - y|)^{d-2}}.$$
Lemma 3.6 bounds the second term by $C(\bar{d}(x)/L)(\log L)^6$. For the first term, we use twice part (iii) and once Lemma 3.6 to get

$$\sum_{i=0}^{2s_L} \sum_{y \in L_i} \bar{d}(x) \frac{1}{a(y) (a(y) + |x - y|)^d} \leq C \sum_{i=0}^{5r} \frac{1}{i \wedge r} + C \min \left\{ \bar{d}(x) \frac{2s_L}{i^2}, \sum_{i=5r}^{2s_L} \frac{1}{i} \right\}.$$ 

This proves (v). \qed

**Proposition 4.3** (Concatenating). Let $F, G$ be positive kernels with $F \preceq G$.

(i) If $A$ is $\eta$-smoothing and $G \asymp 1$, then for some constant $C = C(d, G) > 0$,

$$FA \preceq C\eta G.$$ 

(ii) If $\Phi$ is a positive function on $V_{L+r}$ with $\Phi \asymp 1$, then for some $C = C(d, \Phi) > 0$,

$$F\Phi \leq CG\Phi.$$ 

**Proof:** (i) As $a$ is Lipschitz with constant $1/2$, we can choose $K = K(d)$ points $y_k$ out of the set $M = \{y' \in V_{L+r} : U(y') \cap U(y) \neq \emptyset\}$ such that $M$ is covered by the union of the $U(y_k), k = 1, \ldots, K$. Since $A(y', U(y)) \neq 0$ implies $y' \in M$, we then have

$$FA(x, U(y)) = \sum_{y' \in M} F(x, y') \sum_{y'' \in U(y)} A(y', y'') \leq \eta \sum_{k=1}^{K} F(x, U(y_k))$$

$$\leq \eta \sum_{k=1}^{K} G(x, U(y_k)).$$

Using $G \asymp 1$, we get $G(x, U(y_k)) \leq C|U(y_k)|G(x, y)$. Clearly $|U(y_k)| \leq C|U(y)|$, so that

$$FA(x, U(y)) \leq CK\eta|U(y)|G(x, y).$$

A second application of $G \asymp 1$ yields the claim.

(ii) We can find a constant $K = K(d)$ and a covering of $V_{L+r}$ by neighborhoods $U(y_k)$, $y_k \in V_{L+r}$, such that every $y \in V_{L+r}$ is contained in at most $K$ many of the sets $U(y_k)$. Using $\Phi \asymp 1$, it follows that for $x \in V_{L+r}$,

$$F\Phi(x) = \sum_{y \in V_{L+r}} F(x, y)\Phi(y) \leq C \sum_{k=1}^{\infty} F(x, U(y_k))\Phi(y_k) \leq C \sum_{k=1}^{\infty} G(x, U(y_k))\Phi(y_k)$$

$$\leq C \sum_{k=1}^{\infty} \sum_{y \in U(y_k)} G(x, y)\Phi(y) \leq CK \sum_{y \in V_{L+r}} G(x, y)\Phi(y).$$

In terms of our specific kernel $\Gamma_{L,r}$, we obtain
Proposition 4.4. Let $A$ be $\eta$-smoothing, and let $F$ be a positive kernel satisfying $F \preceq \Gamma_{L,r}$.

(i) There exists a constant $C_2 > 0$ not depending on $F$ such that

$$FA \preceq C_2 \eta \Gamma_{L,r}.$$ 

(ii) If additionally $A(x,y) = 0$ for $x \notin V_L$ and $A(x,U(x)) \leq (\log a(x))^{-15/2}$ for $x \in V_L \setminus \mathcal{E}_1$, then there exists a constant $C_3 > 0$ not depending on $F$ such that for all $x, z \in V_{L+r}$,

$$FAG_{L,r}(x,z) \preceq C_3 \eta^{1/2} \Gamma_{L,r}(x,z).$$

Proof: (i) This is Proposition 4.3 (i) with $G = \Gamma$.

(ii) We set $B = V_L \setminus \mathcal{E}_1$ and split into

$$FA = F_{1\mathcal{E}_1}A\Gamma + F_{1B}A\Gamma.$$ (19)

Let $x, z \in V_{L+r}$ be fixed, and consider first $F_{1\mathcal{E}_1}A\Gamma(x,z)$. Using $\Gamma \preceq 1$, $A\Gamma(y,z) \leq C\eta \Gamma(y,z)$. As $\Gamma(\cdot,z) \preceq 1$ and $F_{1\mathcal{E}_1} \preceq \Gamma_1\mathcal{E}_2$, we get by Proposition 4.3 ii)

$$F_{1\mathcal{E}_1}A\Gamma(x,z) \preceq C\eta \Gamma_{1\mathcal{E}_2}\Gamma(x,z).$$

Setting $\mathcal{E}_2^1 = \{y \in \mathcal{E}_2 : |y - z| \geq |x - z|/2\}$, $\mathcal{E}_2^2 = \mathcal{E}_2 \setminus \mathcal{E}_2^1$, we split further into

$$\Gamma_{1\mathcal{E}_2}\Gamma = \Gamma_{1\mathcal{E}_2^1}\Gamma + \Gamma_{1\mathcal{E}_2^2}\Gamma.$$

If $y \in \mathcal{E}_2^1$, then $\Gamma(y,z) \leq C\Gamma(x,z)$. By Lemma 4.4 (iv), $\Gamma(x,\mathcal{E}_2) \leq C$. Together we obtain

$$\Gamma_{1\mathcal{E}_2^1}\Gamma(x,z) \preceq C\Gamma(x,z).$$

If $y \in \mathcal{E}_2^2$, then $\Gamma(x,y) \leq C\frac{a(z)^2}{a(x)^2}\Gamma(x,z)$ and $\Gamma^{(1)}(y,z) \leq C\frac{r^3}{a(z)^3}\Gamma^{(1)}(z,y)$, whence

$$\Gamma_{1\mathcal{E}_2^2}\Gamma(x,z) \preceq C\Gamma(x,z)\Gamma^{(1)}(z,\mathcal{E}_2) \leq C\Gamma(x,z).$$

We therefore have shown that

$$F_{1\mathcal{E}_1}A\Gamma(x,z) \preceq C\eta \Gamma(x,z).$$

To handle the second summand of (19), set $\sigma(y) = \min \left\{ \eta, (\log a(y))^{-15/2} \right\}$, $y \in V_{L+r}$.

Clearly, $1_B A\Gamma(y,z) \preceq C\sigma(y)\Gamma(y,z)$ and $F_{1B} \preceq \Gamma_1V_L$. Furthermore, $\sigma(\cdot)\Gamma(\cdot,z) \preceq 1$, so that by Proposition 4.3 ii)

$$F_{1B}A\Gamma(x,z) \preceq C\Gamma_1V_L\sigma\Gamma(x,z).$$

Consider $D^1 = \{y \in V_L : |y - z| \geq |x - z|/2\}$, $D^2 = V_L \setminus D^1$ and split into

$$\Gamma_1V_L\sigma\Gamma = \Gamma_1D^1\sigma\Gamma + \Gamma_1D^2\sigma\Gamma.$$
If \( y \in D^1 \), then \( \Gamma(y, z) \leq C \max \left\{ 1, \frac{\hat{d}(y)}{d(x)} \right\} \Gamma(x, z) \), implying \( \Gamma 1_{D^1} \sigma \Gamma(x, z) \leq C \eta^{1/2} \Gamma(x, z) \) if we prove

\[
\sum_{y \in V_L} \max \left\{ 1, \frac{\hat{d}(y)}{d(x)} \right\} \Gamma(x, y) \sigma(y) \leq C \eta^{1/2}.
\] (20)

To this end, we treat the summation over \( S^1 = \{ y \in V_L : d_L(y) \leq 2s_L \} \) and \( S^2 = V_L \setminus S_1 \) separately. If \( y \in S^2 \), then \( a(y) = s_L \). Estimating \( \Gamma \) by \( \Gamma^{(1)} \) and \( \hat{d}(y), \hat{d}(x) \) simply by \( L \), we get

\[
\sum_{y \in S^2} \max \left\{ 1, \frac{\hat{d}(y)}{d(x)} \right\} \Gamma(x, y) \sigma(y) \leq \frac{C}{(\log L)^{3/2}} \sum_{y \in V_L} \frac{1}{(s_L + |y|)^d} \leq \frac{C \log \log L}{(\log L)^{3/2}}.
\] (21)

If \( y \in S^1 \), we estimate \( \Gamma \) again by \( \Gamma^{(1)} \) and split the summation into the layers \( L_j, j = 0, \ldots, 2s_L \). On \( L_j \), \( \sigma(y) \leq C \min \left\{ \eta, (\log(j + 1))^{-15/2} \right\} \). Thus, by Lemma 4.4 (iii),

\[
\sum_{y \in S^1} \max \left\{ 1, \frac{\hat{d}(y)}{d(x)} \right\} \Gamma(x, y) \sigma(y)
\leq C \sum_{j=0}^{2s_L} \sum_{y \in L_j} \max \left\{ 1, \frac{\hat{d}(x)}{a(y)} \right\} \frac{\min \left\{ \eta, (\log(j + 1))^{-15/2} \right\}}{(a(y) + |x - y|)^d}
\leq C \sum_{j=0}^{2s_L} \min \left\{ \eta, (\log(j + 1))^{-15/2} \right\} \leq C \eta^{1/2}.
\]

Together with (21), we have proved (20). It remains to bound the term \( \Gamma 1_{D^2} \sigma \Gamma(x, z) \). But if \( y \in D^2 \), then

\[
a(y) + |x - y| \geq a(y) + \frac{1}{2} |x - z| \geq a(z) - \frac{1}{2} |y - z| + \frac{1}{2} |x - z| \geq \frac{1}{4} (a(z) + |x - z|),
\]

whence \( \Gamma(x, y) \leq C \frac{a(z)^2}{a(y)^2} \max \left\{ 1, \frac{\hat{d}(y)}{d(z)} \right\} \Gamma(x, z) \). Using Lemma 4.4 (i), we have

\[
\frac{a(z)^2}{a(y)^2} \Gamma(y, z) \leq C \Gamma(z, y),
\]

so that \( \Gamma 1_{D^2} \sigma \Gamma(x, z) \leq C \eta^{1/2} \Gamma(x, z) \) follows again from (20).

\[\square\]

Now we have collected all ingredients to finally prove part (ii) of our main Lemma 4.2.

**Proof of Lemma 4.2 (ii):** As already remarked, we only have to prove the statement involving \( G^a \). The perturbation expansion (5) yields

\[
\tilde{G}^a = \tilde{g} \sum_{m=0}^{\infty} (R \tilde{g})^m \sum_{k=0}^{\infty} \Delta^k,
\]
where $\Delta = 1_{V_L} (\tilde{\Pi}^g - \tilde{\pi})$, $R = \sum_{k=1}^{\infty} \Delta^k \tilde{\pi}$. With the constants $C_1$ of Lemma 4.2 (i) and $C_2, C_3$ of Proposition 4.4 we choose

$$\delta \leq \frac{1}{32} \left( \frac{1}{C_2 \vee C_1^2 C_3^2} \right).$$

From Lemma 4.2 (i) and Proposition 4.4 (i) with $A = |\Delta|$, $\eta = 2\delta$ we then deduce that $\tilde{g}|\Delta| \leq (C_1/2)\Gamma$, and, by iterating,

$$\sum_{k=1}^{\infty} \tilde{g}|\Delta|^{k-1} \leq 2C_1\Gamma.$$

Furthermore, by part (ii) of Proposition 4.4 with $A = |\Delta|\tilde{\pi}$ and Lemma 4.2 (i),

$$\sum_{k=1}^{\infty} \tilde{g}|\Delta|^{k-1} |\Delta| \tilde{\pi} | \tilde{g} \leq (C_1/2)\Gamma.$$

Iterating this procedure shows that for $m \in \mathbb{N}$,

$$\tilde{g}(|R|\tilde{g})^m \leq C_1 2^{-m}\Gamma.$$

Finally, by a further application of Proposition 4.4 (i),

$$\tilde{g} \sum_{m=0}^{\infty} (|R|\tilde{g})^m \sum_{k=0}^{\infty} |\Delta|^k \leq 4C_1\Gamma.$$

This proves the lemma.

4.3 Modified transitions on environments bad on level 4

We shall now describe an environment-depending second version of the coarse graining scheme, which leads to modified transition kernels $\tilde{\Pi}_{L,r}$, $\tilde{\Pi}_{g L,r}$, $\tilde{\pi}_{L,r}$ on “really bad” environments. Again, we write $p = p_{x_L/20}$ for short.

Assume $\omega \in \text{OneBad}_L$ is bad on level 4, with $\mathcal{B}_L(\omega) \subset V_{L/2}$. Then there exists $D = V_{4h_L(z)}(z) \subset \mathcal{D}_L$ with $\mathcal{B}_L(\omega) \subset D$, $z \in V_{L/2}$. On $D$, $c r_L \leq h_{L,r}(\cdot) \leq C r_L$. By Lemma 4.2 and the definition of $\Gamma_{L,r}$, it follows easily that we can find a constant $K_1 \geq 2$, depending only on $d$, such that whenever $|x - y| \geq K_1 h_{L,r}(y)$ for some $y \in \mathcal{B}_L$, we have

$$\hat{G}^g_{L,r}(x, \mathcal{B}_L) \leq C \Gamma_{L,r}(x, D) \leq \frac{1}{10}. \quad (22)$$

On such $\omega$, we let $t(x) = K_1 h_{L,r}(x)$ and define on $V_L$,

$$\tilde{\Pi}_{L,r}(x, \cdot) = \begin{cases} \exp_{\nu_L(z)}(x, \cdot) \hat{\Pi}_{L,r} & \text{for } x \in \mathcal{B}_L \\ \hat{\Pi}_{L,r}(x, \cdot) & \text{otherwise} \end{cases}.$$
4.3 Modified transitions on environments bad on level 4

By replacing $\hat{\Pi}$ by $\tilde{\pi}$ on the right side, we define $\tilde{\pi}_{L,r}(x, \cdot)$ in an analogous way, for all $q \in P^{\kappa}$. More precisely,

$$\tilde{\pi}^{(q)}_{L,r}(x, \cdot) = \begin{cases} \text{ex}_{V_L(x)} \left( x, \cdot ; \tilde{\pi}^{(q)}_{L,r} \right) & \text{for } x \in B_L \\
\tilde{\pi}^{(q)}_{L,r}(x, \cdot) & \text{otherwise} \end{cases}.$$

Note that $\tilde{\pi}^{(q)}_{L,r}$ depends on the environment. We work again with a goodified version of $\tilde{\Pi}_{L,r}$,

$$\tilde{\Pi}^q_{L,r}(x, \cdot) = \tilde{\Pi}^q_{L,r}(x, \cdot) \begin{cases} \text{ex}_{V_L(x)} \left( x, \cdot ; \tilde{\Pi}^q_{L,r} \right) & \text{for } x \in B_L \\
\tilde{\Pi}^q_{L,r}(x, \cdot) & \text{otherwise} \end{cases}.$$

For all other environments falling not into the above class, we change nothing and put $\tilde{\Pi}_{L,r} = \tilde{\Pi}_{L,r}$, $\tilde{\Pi}^q_{L,r} = \tilde{\Pi}^q_{L,r}$, $\tilde{\pi}_{L,r} = \tilde{\pi}_{L,r}$. This defines $\tilde{\Pi}_{L,r}$, $\tilde{\Pi}^q_{L,r}$ and $\tilde{\pi}_{L,r}$ on all environments. We write $\tilde{G}_{L,r}$, $\tilde{G}^q_{L,r}$, $\tilde{g}_{L,r}$ for the Green’s functions corresponding to $\tilde{\Pi}_{L,r}$, $\tilde{\Pi}^q_{L,r}$ and $\tilde{\pi}_{L,r}$.

**Some properties of the new transition kernels**

The following observations can be read off the definition and will be tacitly used below.

- On environments which are good or bad on level at most 3, the new kernels agree with the old ones, and so do their Green’s functions, i.e. $\tilde{G}_{L,r} = \tilde{G}_{L,r}$ and $\tilde{G}^q_{L,r} = \tilde{G}^q_{L,r}$. On $\text{Good}_L$ with the choice $r = r_L$, we have equality of all four Green’s functions.

- If $\omega$ is not bad on level 4 with $B_L \subset V_{L/2}$, then

$$1_{V_L} (\tilde{\Pi}_{L,r} - \tilde{\Pi}^q_{L,r}) = 1_{V_L} (\tilde{\Pi}_{L,r} - \tilde{\Pi}^q_{L,r}) = 1_{B_L} (\tilde{\Pi} - \tilde{\pi}^{(p)}).$$

This will be used in Section 5.
In contrast to $\hat{\pi}_{L,r}$, the kernel $\tilde{\pi}_{L,r}$ depends on the environment, too. However, $\hat{\Pi}_{L,r}$, $\hat{\Pi}_{L,r}^g$ and $\tilde{\pi}_{L,r}$ do not change the exit measure from $V_L$, i.e. for example,

$$\text{ex}_{V_L} \left( x, \cdot ; \hat{\Pi}_{L,r}^g \right) = \text{ex}_{V_L} \left( x, \cdot ; \tilde{\pi}_{L,r} \right).$$

The old transition kernels are finer in the sense that the (new) Green’s functions $\tilde{G}$, $\hat{G}^g$, $\tilde{g}$ are pointwise bounded from above by $\tilde{G}$, $\hat{G}^g$ and $\tilde{g}$, respectively. In particular, we obtain with the same constants as in Lemma 4.2.

Lemma 4.5.

(i) For all $q \in \mathcal{P}^s$, $g^{(q)}_{L,r} \preceq C_1 \Gamma_{L,r}$.

(ii) For $\delta > 0$ small, $\tilde{G}^g_{L,r} \preceq C \Gamma_{L,r}$.

For the new goodified Green’s function, we have

Corollary 4.1. There exists a constant $C > 0$ such that

(i) On OneBad$_L$, if $B_L \cap Sh_L(r_L) = \emptyset$ or for general $B_L$ in the case $r = r_L$,

$$\sup_{x \in V_L} \tilde{G}^g_{L,r}(x, B_L) \leq C.$$

On OneBad$_L$, if $B_L \not\subset V_{L/2}$, then, with $t = d(B_L, \partial V_L)$,

$$\sup_{x \in V_L/5} \tilde{G}^g_{L,r}(x, B_L) \leq C \left( \frac{s_L \wedge (t \vee r_L)}{L} \right)^{d-2}.$$

(ii) On (BdBad$_{L,r}$)$^c$, $\sup_{x \in V_{L/3}} \tilde{G}^g_{L,r}(x, B_L^3) \leq C (\log r)^{-1/2}$.

(iii) For $\omega \in \text{OneBad}_L$ bad on level at most 3 with $B_L \cap Sh_L(r_L) = \emptyset$, or for $\omega$ bad on level 4 with $B_L \subset V_{L/2}$, putting $\Delta = 1_{V_L} (\hat{\Pi}_{L,r} - \hat{\Pi}_{L,r}^g)$,

$$\sup_{x \in V_L} \sum_{k=0}^{\infty} \left\| \tilde{G}^g_{L,r} 1_{B_L} (x, \cdot ) \right\|_1 \leq C.$$

Proof: (i) The set $B_L$ is contained in a neighborhood $D \in \mathcal{D}_L$. As $\tilde{G}^g \preceq C \Gamma$, we have

$$\tilde{G}^g (x, B_L) \preceq C \Gamma^{(2)}(x, D).$$

From this, the first statement of (i) follows. Now let $x$ be inside $V_{L/5}$, and $B_L \not\subset V_{L/4}$. If the midpoint $z$ of $D$ can be chosen to lie inside $V_L \setminus Sh(r_L)$, $a(\cdot)/h_L(z)$ and $h_L(z)/a(\cdot)$
are bounded on \(D\). Then, the second statement of (i) is again a consequence of (23). If \(z \in \text{Sh}(r_L)\), we have
\[
\tilde{G}^g(x, B_L) \leq C \Gamma^{(1)}(x, D) \leq C \sum_{j=0}^{2r_L} \sum_{y \in L_j \cap D} \frac{L}{a(y)L^d} \leq C \left(\log L \right)^{d-1}.
\]
(ii) Recall the notation of Section 2.4. In order to bound \(\sup_{x \in V_{2L/3}} \tilde{G}^g(x, B_{\partial L, r})\), we look at the different bad sets \(D_{j,r} \in Q_{j,r}\) of layer \(\Lambda_j\), \(0 \leq j \leq J_1\). Estimating \(\tilde{G}^g\) by \(\Gamma^{(1)}\), we have
\[
\tilde{G}^g(x, D_{j,r}) \leq C(r^2)^{d-1} L^{-d+1}.
\]
On \((\text{BdBad}_{L,r})^c\), the number of bad sets in layer \(\Lambda_j\) is bounded by
\[
C(\log r + j)^{-3/2}(L/(r2^j))^{d-1}.
\]
Therefore,
\[
\tilde{G}^g(x, B_{L,r}^j \cap \Lambda_j) \leq C(\log r + j)^{-3/2}.
\]
Summing over \(0 \leq j \leq J_1\), this shows
\[
\tilde{G}^g(x, B_{L,r}^j) \leq C(\log r)^{-1/2}.
\]
(iii) Assume \(\omega \in \text{Good}_L\) or \(\omega\) is bad on level \(i = 1, 2, 3\). Then \(1_{B_L} \Delta = 1_{B_L}(\hat{\Pi} - \hat{\pi})\). Further, if \(B_L \cap \text{Sh}(r_L) = \emptyset\), we have \(\|\tilde{G}^g 1_{B_L} \Delta(x, \cdot)\|_1 \leq C \delta\). By choosing \(\delta\) small enough, the claim follows. If \(\omega\) is bad on level 4 and \(B_L \subset V_{L/2}\), we do not gain a factor \(\delta\) from \(\|1_{B_L} \Delta(y, \cdot)\|_1\). However, thanks to our modified transition kernels, using (22), \(\|1_{B_L} \Delta \tilde{G}^g 1_{B_L}(y, \cdot)\|_1 \leq 1/5\) (recall that \(\tilde{G}^g \leq \hat{G}^g\) pointwise), so that (ii) follows in this case, too.

\[\square\]

**Remark 4.3.** All \(\delta_0 > 0\) and \(L_0\) appearing in the next sections are understood to be chosen in such a way that if we take \(\delta \in (0, \delta_0]\) and \(L \geq L_0\), then the conclusions of Lemmata 4.2, 4.5 and Corollary 4.1 are valid.

### 5 Exit distributions from the ball

In this part, we prove the main estimates on exit measures that are required to propagate Condition C1 \((\delta, L_0, L_1)\). First, in Section 5.1 we collect some preliminary results involving the kernel \(p_L\). Then, in Section 5.2 we estimate the total variation norm of the globally smoothed difference \(D^*_{L,p,L} \psi, q\), while in Section 5.3 we prove the estimates for the non-smoothed quantity \(D^*_{L,p,L}\).
We work with both the original kernels $\hat{\Pi}, \hat{\Pi}^g, \hat{\pi}$ as well as with the modified kernels $\hat{\Pi}, \hat{\Pi}^g, \hat{\pi}$ from Section 4.3. For the goodified exit measure from $V_L$ we write

$$\Pi^g_L = \text{ex}_{V_L} (x, \cdot; \hat{\Pi}^g_{L,r}) = \text{ex}_{V_L} (x, \cdot; \hat{\Pi}^g_{L,r}).$$

Throughout this part, for reasons of readability, we put $p = p_{sL/20}$.

## 5.1 Preliminaries

We start with a generalization of Lemma 4.1 which forms one of the key steps in transferring Condition C1 from one level to the next.

**Lemma 5.1.** Assume $C1(\delta, L_0, L_1)$, and let $L_1 \leq L \leq L_1(\log L_1)^2$ Then for all $x \in V_L \setminus \text{Sh}_L(2r)$, with $H(x) = \max\{L_0, h_{L,r}(x)\}$,

$$\left\| (\hat{\Pi}^g_{L,r} - \hat{\pi}^{(p)}(x, \cdot)) \right\|_1 \leq C \min\{\log(s_L/H(x)) (\log H(x))^{-9}, (\log H(x))^{-8}\},$$

$$\left\| (\hat{\Pi}^g_{L,r} - \hat{\pi}^{(p)}(x, \cdot)) \right\|_1 \leq 3\delta.$$

**Proof:** Let $\Delta = 1_{V_L} (\hat{\Pi}^g - \hat{\pi}^{(p)})$. With $B = V_L \setminus \text{Sh}_L(2r)$,

$$1_B\Delta = 1_B (\hat{\Pi}^g - \hat{\pi}^{(p)}) + 1_B (\hat{\pi}^{(p)} - \hat{\pi}^{(pL)}).$$

Using Lemma 4.1, the first term is bounded in total variation by $2\delta$. For the second term, Lemma 2.2 in combination with Lemma 3.5 yield the bound $C(\log H(x))^{-9}$. Similarly,

$$1_B\Delta \hat{\pi}^{(pL)} = 1_B \left[ (\hat{\Pi}^g - \hat{\pi}^{(p)}) \hat{\pi}^{(p)} + \hat{\pi}^{(p)} (\hat{\pi}^{(p)} - \hat{\pi}^{(pL)}) + (\hat{\pi}^{(p)} - \hat{\pi}^{(pL)}) \hat{\pi}^{pL} + \hat{\Pi}^g (\hat{\pi}^{(pL)} - \hat{\pi}^{(p)}) \right].$$

Here, the last three terms on the right are bounded in total variation by $C(\log H(x))^{-9}$, and for the first one can use Lemma 4.1. \qed

The next lemma is useful for the globally smoothed exit distributions.

**Lemma 5.2.** Assume $C1(\delta, L_0, L_1)$, and let $L_1 \leq L \leq L_1(\log L_1)^2$. Put $\Delta = 1_{V_L} (\hat{\Pi}^g_{L,r} - \hat{\pi}^{(pL)}_{L,r})$. Then, for some $C > 0$, for all $\psi \in M_L$, $q \in P^n$, with $\phi_{L,pL,\psi,q} = \hat{\pi}^{(pL)}(x,y,\cdot) \hat{\pi}^{(pL)}(y,z,\cdot)$ as in Section 3,

$$\sup_{x \in V_L} \sup_{z \in \mathbb{Z}^d} |\Delta \phi_{L,pL,\psi,q}(x,z)| \leq C(\log L)^{-12} L^{-d}.$$

**Proof:** Write $\hat{\pi}$ for $\hat{\pi}^{(pL)}$, $\phi$ for $\phi_{L,pL,\psi,q}$. Using $\Delta \phi = \Delta \hat{\pi} \phi$ and the fact that $\Delta \hat{\pi}(x,\cdot)$ sums up to zero,

$$|\Delta \phi(x,z)| = \left| \sum_{y \in V_L \cup V_L} \Delta \hat{\pi}(x,y) \left( \phi(y,z) - \phi(x,z) \right) \right| \leq \left\| \Delta \hat{\pi}(x,\cdot) \right\|_1 \sup_{y,|\Delta \hat{\pi}(x,y)| > 0} \left| \phi(y,z) - \phi(x,z) \right|. $$
For $x \in V_L \setminus \mathrm{Sh}_L(2r_L)$, we have by Lemma 5.1 $\|\Delta \hat{\pi}(x, \cdot)\|_1 \leq C \log(s_L/h_L(x))(\log L)^{-9}$. Further, notice that $|\Delta \hat{\pi}(x, y)| > 0$ implies $|y - x| \leq Ch_L(x)$. Bounding $|\phi(y, z) - \phi(x, z)|$ by Lemma 3.7 (iii), the statement follows for those $x$. If $x \in \mathrm{Sh}_L(2r_L)$, we simply bound $\|\Delta \hat{\pi}(x, \cdot)\|_1$ by 2. Now we can restrict the supremum to those $y \in V_L$ with $|x - y| \leq 3r_L$, so the claim follows again from Lemma 3.7 (iii). \hfill \Box

We defined the kernel $p_L$ in terms of averaged variances of $E[\hat{\Pi}_{L,r}]$. Combined with Lemma 2.1, this shows that the covariances of $\hat{\pi}_{L,r}$ agree with those of $E[\hat{\Pi}_{L,r}]$ up to an error of order $O(s_L^{-1})$. The same holds true with $\hat{\Pi}_{L,r}$ replaced by $\hat{\Pi}_{L,r}^p$.

**Lemma 5.3.** Assume $C1(\delta, L_0, L_1)$. There exists a constant $C = C(d)$ such that for $L_1 \leq L \leq L_1(\log L)^2$, we have

$$\sum_y \left( E[\hat{\Pi}_{L,r}^p] - \hat{\pi}_{L,r}(p_L) \right)(0, y) \frac{y_i^2}{|y|^2} \leq C(\log L)^3 L^{-1}$$

for all $i = 1, \ldots, d$.

**Proof:** Note that under $C1(\delta, L_0, L_1)$,

$$E \left[\|\hat{\Pi} - \hat{\Pi}^p\|(0, \cdot)\right] \leq 2P(0 \in B_L) \leq C \exp \left(- (1/2)(\log L)^2\right).$$

Therefore,

$$2p_L(e_i) = \sum_y E[\hat{\Pi}^p](0, y) \frac{y_i^2}{|y|^2} + O \left(\exp(- (1/2)(\log L)^2)\right).$$

On the other hand, as in (7) with $p$ replaced by $p_L$

$$2p_L(e_i) = \sum_y \hat{\pi}(p_L)(0, y) \frac{y_i^2}{|y|^2} + O(s_L^{-1}),$$

and the statement follows. \hfill \Box

### 5.2 Globally smoothed exits

Our objective here is to establish the estimates for the smoothed difference $D_{L,p_L,\psi,q}^\ast$. In this section, we only work with coarse graining schemes corresponding to the choice $r = r_L$. Lemma 5.4 compares the “goodified” smoothed exit distribution from the ball of radius $L$ with that of symmetric random walk with transition kernel $p_L$. In particular, it provides an estimate for $D_{L,p_L,\psi,q}^\ast$ on $\mathrm{Good}_L$.

**Lemma 5.4.** Assume $A1$. For every (small) constant $c_0$, there exist $\delta_0 > 0$ and $L_0 \in \mathbb{N}$ such that if $\delta \in (0, \delta_0]$ and $L_1 \geq L_0$, then $C1(\delta, L_0, L_1)$ implies that for $L_1 \leq L \leq L_1(\log L)^2$, for all $\psi \in \mathcal{M}_L$ and all $q \in \mathcal{P}_\psi$,

$$P \left( \sup_{x \in V_L} \|\Pi_L^p - \pi_{L,r}(p_L)\hat{\pi}_\psi^q(x, \cdot)\|_1 \geq c_0(\log L)^{-9} \right) \leq \exp \left(- (\log L)^{7/3}\right).$$
Proof: Clearly, the claim follows if we show that
\[
\sup_x \sup_{z \in \mathbb{Z}^d} \mathbb{P} \left( \left| (\Pi^g - \pi^{(p_L)}) \hat{\pi}^{(q)}_\psi (x, z) \right| \geq c_0 (\log L)^{-9} L^{-d} \right) \leq \exp \left( -(\log L)^{5/2} \right). 
\] (24)

We use the abbreviations \( \phi = \pi^{(p_L)} \hat{\pi}^{(q)}_\psi \), \( \Delta = 1_{V_L} (\Pi^g - \pi^{(p_L)}) \), \( \hat{g} = \hat{g}^{(p_L)} \) and \( \hat{\pi} = \hat{\pi}^{(p_L)} \).

By the perturbation expansion,
\[
(\Pi^g - \pi^{(p_L)}) \hat{\pi}^{(q)}_\psi = \hat{G}^g \Delta \phi. 
\] (25)

Set \( S = Sh_L(2L/(\log L)^2) \) and write
\[
\hat{G}^g \Delta \phi = \hat{G}^g 1_S \Delta \phi + \hat{G}^g 1_{S^c} \Delta \phi. 
\] (26)

Using \( \hat{G}^g \lesssim CT \), Lemma 4.4 (iv) (with \( r = r_L \)) and Lemma 5.2 yield the estimate
\[
|\hat{G}^g 1_S \Delta \phi (x, z)| \leq \sup_{x \in V_L} \hat{G}^g(x, S) \sup_{y \in V_L} |\Delta \phi(y, z)| \leq (\log L)^{-19/2} L^{-d}
\]
for \( L \) large. It remains to bound \( |\hat{G}^g 21_{S^c} \Delta \phi (x, z)| \). With \( B = V_L \setminus Sh_L(2r_L) \),
\[
\hat{G}^g = \hat{g} 1_B \Delta \hat{G}^g + \hat{g} 1_{B^c} \Delta \hat{G}^g + \hat{g}. 
\]

By replacing successively \( \hat{G}^g \) in the first summand on the right-hand side,
\[
\hat{G}^g 1_{S^c} \Delta \phi = \left( \sum_{k=0}^{\infty} (\hat{g} 1_B \Delta)^k \hat{g} + \sum_{k=0}^{\infty} (\hat{g} 1_B \Delta)^k \hat{g} 1_{B^c} \Delta \hat{G}^g \right) 1_{S^c} \Delta \phi 
\]
\[
= F 1_{S^c} \Delta \phi + F 1_{B^c} \Delta \hat{G}^g 1_{S^c} \Delta \phi, 
\]
where \( F = \sum_{k=0}^{\infty} (\hat{g} 1_B \Delta)^k \hat{g} \). With \( R = \sum_{k=1}^{\infty} (1_B \Delta)^k \hat{\pi} \), expansion (5) shows
\[
F = \hat{g} \sum_{m=0}^{\infty} (R \hat{g})^m \sum_{k=0}^{\infty} (1_B \Delta)^k. 
\]

From the proof of Lemma 4.2 (ii) we learn that \( |F| \leq CT \). By Lemma 4.4 (iv), (v) and again Lemma 5.2, we see that for large \( L \), uniformly in \( x \in V_L \) and \( z \in \mathbb{Z}^d \),
\[
|F 1_{B^c} \Delta \hat{G}^g 1_{S^c} \Delta \phi (x, z)| \leq CT (x, Sh_L(2r_L)) \sup_{v \in Sh_L(3r_L)} \Gamma(v, S^c \cap V_L) \sup_{w \in V_L} |\Delta \phi(w, z)| 
\]
\[
\leq (\log L)^{-11} L^{-d}. 
\]

Thus, the second summand of (27) is harmless. However, with the first summand one has to be more careful. With \( \xi = \hat{g} \sum_{k=0}^{\infty} (1_B \Delta)^k 1_{S^c} \Delta \phi \), we have
\[
F 1_{S^c} \Delta \phi = \xi + \hat{g} \sum_{m=0}^{\infty} (R \hat{g})^m R \xi = \xi + F 1_B \Delta \hat{\pi} \xi. 
\]
Clearly, \(|F_1 B \Delta \hat{\pi}(x, y)| \leq C (\log L)^{-3}\), so it remains to estimate \(\xi(y, z)\), uniformly in \(y\) and \(z\). Set \(N = N(L) = \lfloor \log \log L \rfloor\). For small \(\delta\), the summands of \(\xi\) with \(k \geq N\) are readily bounded by

\[
\sup_{y \in V_L} \sup_{z \in \mathbb{Z}^d} \sum_{k = N}^{\infty} |\hat{g}(B \Delta)^k 1_{S^c} \Delta \phi(y, z)| \leq C (\log L)^6 \sup_{k = N}^{\infty} \delta^k (\log L)^{-12} L^{-d} \leq (\log L)^{-10} L^{-d}.
\]

Now we look at the summands with \(k < N\). Since the coarse grained walk cannot bridge a gap of length \(L/(\log L)^2\) in less than \(N\) steps, we can drop the kernel \(1_B\). Defining \(S' = \text{Sh}_L (3L/(\log L)^2)\), we thus have

\[
\hat{g}(B \Delta)^k 1_{S^c} \Delta \phi = \hat{g} 1_{S'} \Delta^k 1_{S^c} \Delta \phi + \hat{g} 1_{S^c} \Delta^k 1_{S^c} \Delta \phi.
\]

The first summand is bounded in the same way as \(\hat{G} 1_{S^c} \Delta \phi\) from (26). Further, we can drop the kernel \(1_{S'}\) in the second summand. Therefore, (24) follows if we show

\[
\sup_{x \in V_L} \sup_{z \in \mathbb{Z}^d} \mathbb{P} \left( \sum_{k = 1}^{N} \hat{g} 1_{S^c} \Delta^k \phi(x, z) \left| \frac{1}{2} c_0 (\log L)^{-5} L^{-d} \right. \right) \leq \exp \left( -\log L)^{5/2} \right).
\]

For \(j \in \mathbb{Z}\), consider the interval \(I_j = (j N s_L, (j + 1) N s_L) \subset \mathbb{Z}\). We divide \(S^c \cap V_L\) into subsets \(W_j = (S^c \cap V_L) \cap (I_{j1} \times \ldots \times I_{jd})\), where \(j = (j_1, \ldots, j_d) \in \mathbb{Z}^d\). Let \(J\) be the set of those \(j\) for which \(W_j \neq \emptyset\). Then we can find a constant \(K\) depending only on the dimension and a disjoint partition of \(J\) into sets \(J_1, \ldots, J_K\), such that for any \(1 \leq l \leq K\),

\[
j, j' \in J_l, j \neq j' \implies d(W_j, W_{j'}) > N s_L. \tag{27}
\]

For \(x \in V_L, z \in \mathbb{Z}^d\), we set

\[
\xi_j = \xi_j(x, z) = \sum_{y \in W_j} \sum_{k = 1}^{N} \hat{g}(x, y) \Delta^k \phi(y, z),
\]

and further \(t = t(d, L) = (1/2) c_0 (\log L)^{-5} L^{-d}\). Assume that we can prove

\[
\left| \sum_{j \in J} \mathbb{E}[\xi_j] \right| \leq \frac{t}{2}. \tag{28}
\]

Then

\[
\mathbb{P} \left( \left| \sum_{j \in J} \xi_j \right| \geq t \right) \leq \mathbb{P} \left( \left| \sum_{j \in J} \xi_j - \mathbb{E}[\xi_j] \right| \geq \frac{t}{2} \right) \leq K \max_{1 \leq l \leq K} \mathbb{P} \left( \left| \sum_{j \in J_l} \xi_j - \mathbb{E}[\xi_j] \right| \geq \frac{t}{2K} \right).
\]

Due to (27), the random variables \(\xi_j - \mathbb{E}[\xi_j], j \in J_l\), are independent and centered. Hoeffding’s inequality yields, with \(||\xi_j||_\infty = \sup_{\omega \in \Omega} |\xi_j(\omega)|\),

\[
\mathbb{P} \left( \left| \sum_{j \in J_l} \xi_j - \mathbb{E}[\xi_j] \right| \geq \frac{t}{2K} \right) \leq 2 \exp \left( -c \frac{L^{-2d} (\log L)^{-(18+2/5)}}{\sum_{j \in J_l} ||\xi_j||_\infty^2} \right) \tag{29}
\]
for some constant $c > 0$. In order to control the sup-norm of the $\xi_j$, we use the estimates

$$g(x, W_j) \leq CN^d s_L^d \left( \frac{C N^d s_L^d}{s_L^2 + d(x, W_j)} \right)^{d-2} = CN^d \left( \frac{1 + d(x, W_j)}{s_L} \right)^{2-d},$$

and, by Lemmata 5.1 and 5.2, for $y \in W_j$, $|\Delta^k \phi(y, z)| \leq C \delta^{k-1} k (\log L)^{-12} L^{-d}$. Altogether we arrive at

$$||\xi_j||_{\infty} \leq C \left( 1 + \frac{d(x, W_j)}{s_L} \right)^{2-d} N^d (\log L)^{-12} L^{-d},$$

uniformly in $z$. If we put the last display into (29), we get, using $d \geq 3$ in the last line,

$$\mathbb{P}\left( \left| \sum_{j \in J} \xi_j - \mathbb{E}[\xi_j] \right| \geq \frac{t}{2K} \right) \leq 2 \exp\left( -c \frac{(\log L)^{6-2/5}}{N^4} \sum_{r=1}^{C/(\log L)^3/N} r^{-d+3} \right) \leq 2 \exp\left( -c \frac{(\log L)^{3-2/5}}{N^3} \right).$$

It follows that for $L$ large enough, uniformly in $x$ and $z$,

$$\mathbb{P}\left( \left| \sum_{j \in J} \xi_j \right| \geq \frac{t}{2} c_0 (\log L)^{-9} L^{-d} \right) \leq \exp\left( - (\log L)^{5/2} \right).$$

It remains to prove (28). We have

$$\left| \sum_{j \in J} \mathbb{E}[\xi_j] \right| \leq \sum_{y \in S^c} g(x, y) \sum_{y' \in V_L} \mathbb{E} \left[ \sum_{k=1}^{N} \Delta^k \hat{\pi}(y, y') \right] \phi(y', z).$$

The sum over the Green’s function is estimated by $g(x, S^c) \leq C(\log L)^6$. We treat the summands with $k \geq 2$ and $k = 1$ differently. For each summand with $k \geq 2$, we use Proposition 3.1 applied to $\nu(\cdot) = \mathbb{E}[\Delta^k \hat{\pi}(y, y + \cdot)]$. Recalling Lemma 5.1, we obtain by choosing $\delta$ small enough, uniformly in $y \in S^c$,

$$\left| \sum_{y' \in V_L} \mathbb{E}[\Delta^k \hat{\pi}(y, y')] \phi(y', z) \right| \leq C \delta^{k-1} (\log L)^{-9} (ks_L)^2 L^{-(d+2)} \leq (c_0/8)(1/2)^{k-1}(\log L)^{-15} L^{-d}.$$

For the summand $\nu(\cdot) = \mathbb{E}[\Delta^1 \hat{\pi}(y, y + \cdot)]$ corresponding to $k = 1$, the proof of Proposition 3.1 shows

$$\left| \sum_{y' \in V_L} \mathbb{E}[\Delta^1 \hat{\pi}(y, y')] \phi(y', z) \right| \leq \left| \frac{1}{2} \sum_{i=1}^{d} \partial^2 \hat{\phi}(y, z) \sum_{y'} \nu(y')(y_i)^2 \right| + C(\log L)^{-18} L^{-d}.$$

By translational invariance and Lemma 5.3 in the last step,

$$\left| \sum_{y'} \nu(y')(y_i)^2 \right| = \left| \sum_{y} \left( \mathbb{E}[\hat{\Phi}^y] - \hat{\pi} \right) \hat{\pi}(0, y)y_i^2 \right| = \left| \sum_{y} \left( \mathbb{E}[\hat{\Phi}^y] - \hat{\pi} \right) (0, y)y_i^2 \right| \leq C L/(\log L)^3.$$

Together with Lemma 3.7 (ii), this shows (28) and completes the proof. \qed
Remark 5.1. The reader should notice that for \( y \in S^c \), the signed measure \( \nu \) fulfills the requirement of Proposition 3.1. Indeed, after \( N = \lceil \log \log L \rceil \) steps away from \( y \), the coarse grained walks are still in the interior part of \( V_L \), where the coarse graining radius did not start to shrink. Therefore, the symmetry condition of Lemma 5.4 carries over to the signed measure \( \mathbb{E}[\sum_{k=1}^{N}(1_{V_L}(\hat{\Pi} - \hat{\pi}))^k \hat{\pi}(y, y + \cdot)] \). Replacing \( \hat{\Pi} \) by \( \hat{\Pi}^g \) does not destroy the symmetries of this measure, so that Proposition 3.1 can be applied to \( \nu \).

Next, we estimate \( D^*_{L,pl,\psi,q} \) on environments with bad points. Again, we make the choice \( r = r_L \).

Lemma 5.5. In the setting of Lemma 5.4 with a possibly smaller value of \( \delta_0 \) and a larger \( L_0 \), we have for \( i = 1, 2, 3, 4 \),

\[
\mathbb{P} \left( \sup_{x \in V_L/5} \left| \left| (\Pi - \pi^{(pl)}_L) \hat{\pi}^g (x, \cdot) \right| \right|_1 > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_L \right) \leq \exp \left( -\left( \log L \right)^{7/3} \right).
\]

Proof: For the ease of readability, we write \( \hat{\pi} \) for \( \hat{\pi}_\psi \) and let \( \phi = \pi^{(pl)} \hat{\pi} \). By the triangle inequality,

\[
|| (\Pi - \pi^{(pl)}) \hat{\pi}^g (x, \cdot) ||_1 \leq || (\Pi - \Pi^g) \hat{\pi}^g (x, \cdot) ||_1 + || (\Pi^g - \pi^{(pl)}) \hat{\pi} (x, \cdot) ||_1.
\]

The second summand on the right can be bounded by Lemma 5.4. For the first term we have, with \( \Delta = 1_{V_L}(\hat{\Pi} - \hat{\Pi}^g) \),

\[
(\Pi - \Pi^g) \hat{\pi} = \hat{G}^g 1_{B_L} \Delta \hat{\pi}.
\]

Note that since we are on OneBad, the set \( B_L \) is contained in a small region. First assume that \( B_L \subset \text{Sh}_L(L/(\log L)^{10}) \). Then \( \sup_{x \in V_L/5} \hat{G}^g (x, B_L) \leq C(\log L)^{-10} \) by Corollary 1.1 which bounds the first summand of (30). Next assume \( \omega \) bad on level 4 and \( B_L \not\subset V_{L/2} \). Then \( \sup_{x \in V_L/5} \hat{G}^g (x, B_L) \leq C(\log L)^{-3} \) by the same corollary, which is enough for this case.

It remains to consider the cases \( \omega \) bad on level at most 3 with \( B_L \not\subset \text{Sh}_L(L/(\log L)^{10}) \), or \( \omega \) bad on level 4 with \( B_L \subset V_{L/2} \). We expand

\[
(\Pi - \Pi^g) \hat{\pi} = \left( \hat{G}^g 1_{B_L} \Delta \Pi \right) \hat{\pi} = \sum_{k=1}^{\infty} \left( \hat{G}^g 1_{B_L} \Delta \right)^k \Pi^g \hat{\pi} \leq \sum_{k=1}^{\infty} \left( \hat{G}^g 1_{B_L} \Delta \right)^k \phi + \sum_{k=1}^{\infty} \left( \hat{G}^g 1_{B_L} \Delta \right)^k (\Pi^g - \pi^{(pl)}) \hat{\pi} \leq F_1 + F_2.
\]

By Corollary 4.1

\[
|| F_1 (x, \cdot) ||_1 \leq \sum_{k=0}^{\infty} ||(\hat{G}^g 1_{B_L} \Delta)^k (x, \cdot) ||_1 \sup_{v \in V_L} \sup_{w \in B_L} || \Delta \phi (w, \cdot) ||_1 \leq C \sup_{w \in B_L} || \Delta \phi (w, \cdot) ||_1.
\]
Proceeding as in Lemma 5.2 for $w \in B_L$, 
\[
||\Delta \phi(w, \cdot)||_1 \leq ||\Delta \hat{\pi}(p_L)(w, \cdot)||_1 \sup_{w': |\Delta \hat{\pi}(p_L)(w, w')| > 0} ||\phi(w', \cdot) - \phi(w, \cdot)||_1.
\]  
For the second factor on the right, Lemma 3.7 (iii) yields the bound $C(\log L)^{-3}$. If $\omega$ is bad on level 4, we simply bound the first factor by 2. If $\omega$ is bad on level at most 3, we have on $B_L$ the equality $\Delta = \Pi - \hat{\pi}(p)$. With $p' = p_{hL}(w)$, the triangle inequality gives, for every $i = 1, 2, 3$, 
\[
||\Delta \hat{\pi}(p_L)(w, \cdot)||_1 \leq ||(\Pi - \hat{\pi}(p'))\hat{\pi}(p')(w, \cdot)||_1 + ||(\hat{\pi}(p) - \hat{\pi}(p'))(w, \cdot)||_1 + \sup_{y \in V_L \setminus Sh_L(r_L)} ||(\hat{\pi}(p_L) - \hat{\pi}(p'))(y, \cdot)||_1.
\]
By definition, the first summand is bounded by $C(\log L)^{-9+\eta/4}$. For the second and third summand, we use Lemma 3.5 and the fact that by Lemma 2.2, $||p' - p_L||_1 \leq C(\log \log L)(\log L)^{-9}$, and similarly for $||p - p'||_1$. For all $i = 1, 2, 3, 4$, we arrive at $||F_1(x, \cdot)||_1 \leq C(\log L)^{-12+\eta/4}$. For $F_2$, we obtain once more with Corollary 4.1 
\[
||F_2(x, \cdot)||_1 \leq C \sup_{y \in V_L} ||(\Pi^\eta - \hat{\pi}(p_L))\hat{\pi}\psi(y, \cdot)||_1.
\]
This term is again estimated by Lemma 5.4 (with $c_0$ small enough), which completes the proof. \hfill \Box

### 5.3 Non-smoothed and locally smoothed exits

Here, we aim at bounding the total variation distance of the exit measures without additional smoothing (Lemma 5.6), as well as in the case where a kernel of constant smoothing radius $s$ is added (Lemma 5.7).

Throughout this part, we work with transition kernels coming from coarse graining schemes with constant parameter $r$. We always assume $L$ large enough such that $r < r_L$. The right choice of $r$ depends on the deviations $\delta$ and $\eta$ we are shooting for and will become clear from the proofs. In either case, we choose $r \geq r_0$, where $r_0$ is the constant from Section 2.4. The value of $r$ will then also influence the choice of the perturbation $\varepsilon_0$ in Lemma 5.6 and the smoothing radius $\ell$ in Lemma 5.7 respectively.

We recall the partition of bad points into the sets $B_L, B_{L,r}, B^0_{L,r}, B^*_{L,r}$ and the classification of environments into $\text{Good}_L, \text{OneBad}_L$ and $\text{BdBad}_L, \text{BdBad}_{L,r}$ from Section 2.

The bounds for $\text{ManyBad}_L$ (Lemma 2.3) and for $\text{BdBad}_{L,r}$ (Lemma 2.4) ensure that we may restrict ourselves to environments $\omega \in \text{OneBad}_L \cap (\text{BdBad}_{L,r})^c$. For such environments, we introduce two disjoint random sets $Q_{L,r}^1(\omega), Q_{L,r}^2(\omega) \subset V_L$ as follows:

- If $B_L(\omega) \subset V_{L/2}$, set $Q_{L,r}^1(\omega) = B_L(\omega)$ and $Q_{L,r}^2(\omega) = B^0_{L,r}(\omega)$.
- If $B_L(\omega) \not\subset V_{L/2}$, set $Q_{L,r}^1(\omega) = \emptyset$ and $Q_{L,r}^2(\omega) = B^*_{L,r}(\omega)$.

Of course, on $\text{Good}_L$, we have $Q_{L,r}^1(\omega) = \emptyset$ and $Q_{L,r}^2(\omega) = B^0_{L,r}(\omega)$.

Recall that we write $p$ for $p_{s,L/20}$. 

\[\]
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**Lemma 5.6.** There exists \( \delta_0 > 0 \) such that if \( \delta \in (0, \delta_0] \), there exist \( \varepsilon_0 = \varepsilon_0(\delta) > 0 \) and \( L_0 = L_0(\delta) > 0 \) with the following property: If \( \varepsilon \leq \varepsilon_0 \) and \( L_1 \geq L_0 \), then \( A_0(\varepsilon) \), \( C_1(\delta, L_0, L_1) \) imply that for \( L_1 \leq L \leq L_1(\log L_1)^{2} \),

\[
P \left( \sup_{x \in V_{L/5}} ||\Pi_{L} - \pi_{L}^{(p_{L})}(x, \cdot)||_{1} > \delta \right) \leq \exp \left( -\frac{9}{5}(\log L)^{2} \right).
\]

**Proof:** We choose \( \delta_0 > 0 \) according to Remark 4.3 and take \( \delta \in (0, \delta_0] \). The right choice of \( \varepsilon_0 \) and \( L_0 \) will be clear from the course of the proof. From Lemmata 2.3 and 2.4 we learn that if we take \( L_1 \geq L_0 \) for \( L_0 \) large and \( L_1 \leq L \leq L_1(\log L_1)^{2} \), then under \( C_1(\delta, L_0, L_1) \)

\[
P(\text{ManyBad}_{L} \cup \text{BdBad}_{L,r}) \leq \exp \left( -\frac{9}{5}(\log L)^{2} \right).
\]

Therefore, the claim follows if we show that on \( \text{OneBad}_{L} \cap (\text{BdBad}_{L,r})^{c} \), we have for all sufficiently small \( \varepsilon \) and all large \( L, x \in V_{L/5} \),

\[
|||\Pi - \pi^{(p_{L})}(x, \cdot)|||_{1} \leq \delta.
\]

Let \( \omega \in \text{OneBad}_{L} \cap (\text{BdBad}_{L,r})^{c} \). We use the partition of \( B_{L,r}^{\ast} \) into the sets \( Q^1, Q^2 \) described above. With \( \Delta = 1_{V_{L}}(\tilde{\Pi} - \tilde{\Pi}) \), we have inside \( V_{L} \),

\[
\Pi = \tilde{G}^g 1_{Q^1} \Delta \Pi + \tilde{G}^g 1_{Q^2} \Delta \Pi + \Pi^g.
\]

By replacing successively \( \Pi \) in the first summand on the right-hand side, we arrive at

\[
\Pi = \sum_{k=0}^{\infty} \left( \tilde{G}^g 1_{Q^1} \Delta \right)^{k} \Pi^g + \sum_{k=0}^{\infty} \left( \tilde{G}^g 1_{Q^1} \Delta \right)^{k} \tilde{G}^g 1_{Q^2} \Delta \Pi.
\]

Since with \( \Delta' = 1_{V_{L}}(\tilde{\Pi} - \tilde{\Pi}^{(p_{L})}) \), \( \Pi^g = \pi^{(p_{L})} + \tilde{G}^g \Delta' \pi^{(p_{L})} \), we obtain

\[
\Pi - \pi^{(p_{L})} = \sum_{k=1}^{\infty} \left( \tilde{G}^g 1_{Q^1} \Delta \right)^{k} (\pi^{(p_{L})}) + \sum_{k=0}^{\infty} \left( \tilde{G}^g 1_{Q^1} \Delta \right)^{k} \tilde{G}^g 1_{Q^2} \Delta \Pi + \sum_{k=0}^{\infty} \left( \tilde{G}^g 1_{Q^1} \Delta \right)^{k} \tilde{G}^{g} \Delta' \pi^{(p_{L})}
\]

\[
= F_1 + F_2 + F_3.
\]

We will now prove that each of the three parts \( F_1, F_2, F_3 \) is bounded by \( \delta/3 \). If \( Q^1 \neq \emptyset \), then \( Q^1 = B_{L} \subset V_{L/2} \) and \( Q^2 = B_{L,r}^{\prime} \). Using Corollary 4.1 in the second and Lemma 3.1 (ii) in the third inequality,

\[
||F_1(x, \cdot)||_{1} \leq \sum_{k=0}^{\infty} ||(\tilde{G}^g 1_{B_{L}} \Delta)^{k}(x, \cdot)||_{1} \sup_{y \in V_{L}} \tilde{G}^{g}(y, B_{L}) \sup_{z \in B_{L}} ||\Delta' \pi^{(p_{L})}(z, \cdot)||_{1}
\]

\[
\leq C \sup_{z \in V_{L/2}} ||\Delta' \pi^{(p_{L})}(z, \cdot)||_{1} \leq C(\log L)^{-3} \leq C(\log L_0)^{-3} \leq \delta/3
\]
for \( L_0 = L_0(\delta) \) large enough, \( L \geq L_0 \). Regarding \( F_2 \), we have in the case \( Q^1 \neq \emptyset \) by Corollary 4.1 (ii)
\[
\|F_2(x, \cdot)\|_1 \leq C \sup_{y \in \mathcal{B}_{L,r}} \hat{G}^g(y, B_{L,r}^g) \leq C (\log r)^{-1/2}.
\]

On the other hand, if \( Q^1 = \emptyset \), then \( B_L \) is outside \( V_{L/3} \), so that by Corollary 4.1 (i), (ii)
\[
\|F_2(x, \cdot)\|_1 \leq 2\hat{G}^g(x, B_{L,r}^g \cup B_L) \leq C \left((\log L)^{-3} + (\log r)^{-1/2}\right).
\]

Altogether, for all \( L \geq L_0 \), by choosing \( r = r(\delta) \) and \( L_0 = L_0(\delta, r) \) large enough,
\[
\|F_2(x, \cdot)\|_1 \leq C \left((\log L_0)^{-3} + (\log r)^{-1/2}\right) \leq \delta/3.
\]

It remains to handle \( F_3 \). Once again with Corollary 4.1 (iii) for some \( C_3 > 0 \),
\[
\|F_3(x, \cdot)\|_1 \leq C_3 \sup_{y \in \mathcal{B}_{L/3}} \left\| \hat{G}^g \Delta'_{\pi(pL)}(y, \cdot) \right\|_1.
\]

We write
\[
\hat{G}^g \Delta'_{\pi(pL)} = \hat{G}^g 1_{V_L \setminus Q^1} \Delta'_{\pi(pL)} + \hat{G}^g 1_{Q^1} \Delta'_{\pi(pL)}.
\]

As in (33), we deduce that
\[
\|\hat{G}^g 1_{Q^1} \Delta'_{\pi(pL)}\|_1 \leq C_3^{-1} \delta/21.
\]

for \( L_0 \) large enough, \( L \geq L_0 \). Concerning the first term of (35), we note that on \( V_L \setminus Q^1 \), by definition,
\[
\Delta'_{\pi(pL)} = (\hat{\Pi}^g - \tilde{\pi}(pL)) \pi(pL).
\]

For \( z \in V_L \setminus (\text{Sh}_L(2r_L) \cup Q^1) \), we obtain with Lemma 5.1
\[
\|\Delta'_{\pi(pL)}(z, \cdot)\|_1 \leq C (\log \log L)(\log L)^{-9}.
\]

Since \( \hat{G}^g(y, V_L) \leq C (\log L)^6 \), it follows that
\[
\sup_{y \in \mathcal{B}_{L/3}} \left\| \hat{G}^g 1_{V_L \setminus (Q^1 \cup \text{Sh}_L(2r_L))} \Delta'_{\pi(pL)}(y, \cdot) \right\|_1 \leq C (\log L)^{-2} \leq C_3^{-1} \delta/21
\]

for \( L \) large. Recall the definition of the layers \( \Lambda_j \) from Section 2.4. For \( z \in \Lambda_j, 1 \leq j \leq J_1 \), we have again by Lemma 5.1
\[
\|\Delta'_{\pi(pL)}(z, \cdot)\|_1 \leq C (\log r + j)^{-8}.
\]

Using Lemma 4.4 (iii), it follows that \( \hat{G}^g(y, \Lambda_j) \leq C \) for some constant \( C \), independent of \( r \) and \( j \). Thus,
\[
\sup_{y \in \mathcal{B}_{L/3}} \left\| \hat{G}^g 1_{\bigcup_{j=1}^{J_1} \Lambda_j} \Delta'_{\pi(pL)}(y, \cdot) \right\|_1 \leq C (\log r)^{-7} \leq C_3^{-1} \delta/21
\]
5.3 Non-smoothed and locally smoothed exits

if \( r \) is chosen large enough. For the first layer \( \Lambda_0 \), there is a constant \( C_0 \) with

\[
\sup_{y \in V_{2L/3}} \left\| \tilde{G}^g 1_{\Lambda_0} \Delta'(\pi^{(p_L)}(y, \cdot)) \right\|_1 \leq C_0 \sup_{z \in \Lambda_0} \left\| \Delta'(z, \cdot) \right\|_1.
\]

Now, with \( p_0 \equiv 1/(2d) \) denoting the transition kernel of simple random walk,

\[
\left\| \Delta'(z, \cdot) \right\|_1 \leq \left\| (\tilde{\Pi} - \tilde{\pi}^{(p_0)})(z, \cdot) \right\|_1 + \left\| (\tilde{\pi}^{(p_0)} - \hat{\pi}^{(p_L)})(z, \cdot) \right\|_1 + \left\| (\tilde{\Pi} - \hat{\Pi})(z, \cdot) \right\|_1. \tag{39}
\]

Concerning the second summand of \( 39 \), recall that by Lemma \( 2.2 \) \( \| p_0 - p_L \|_1 \leq (\log L_0)^{-7} \). Therefore, choosing \( L_0 \) large enough \(( r \) is now fixed), we can guarantee that

\[
\left\| (\tilde{\pi}^{(p_0)} - \hat{\pi}^{(p_L)})(z, \cdot) \right\|_1 \leq C_0^{-1} C_3^{-1} \delta / 21,
\]

uniformly in \( z \in \Lambda_0 \). The third summand of \( 39 \) vanishes if \( z \in \Lambda_0 \setminus B^*_L \). For \( z \in B^*_L \),

\[
\left\| (\tilde{\Pi} - \hat{\Pi})(z, \cdot) \right\|_1 = \left\| (\tilde{\pi}^{(p)} - \hat{\pi})(z, \cdot) \right\|_1 \leq \left\| (\tilde{\Pi} - \hat{\pi}^{(p_0)})(z, \cdot) \right\|_1 + \left\| (\tilde{\pi}^{(p_0)} - \hat{\pi}^{(p)})(z, \cdot) \right\|_1.
\]

The last term on the right is bounded in the same way as the second term of \( 39 \). For the first summand on the right of the inequality \(( and also for the first summand of \( 39 \)), we may simply choose \( \varepsilon_0 = \varepsilon_0(\delta, r) \) small enough such that for \( \varepsilon \leq \varepsilon_0 \),

\[
\sup_{z \in \Lambda_0} \left\| (\tilde{\Pi} - \hat{\pi}^{(p_0)})(z, \cdot) \right\|_1 \leq C_0^{-1} C_3^{-1} \delta / 21.
\]

Altogether we have shown that \( \| F_{\tilde{g}}(x, \cdot) \|_1 \leq \delta / 3 \), and the claim is proved. \( \square \)

**Remark 5.2.** As the proof shows, we do not have to assume \( C1(\delta, L_0, L_1) \) for the desired deviation \( \delta \). We could instead assume \( C1(\delta', L_0, L_1) \) for some arbitrary \( 0 < \delta' \leq \delta_0 \). However, \( L_0 \) depends on the chosen \( \delta \). This observation will be useful in the next lemma.

**Lemma 5.7.** There exists \( \delta_0 > 0 \) with the following property: For each \( \eta > 0 \), there exist a smoothing radius \( \delta_0 = \delta_0(\eta) \) and \( L_0 = L_0(\eta) \) such that if \( L_1 \geq L_0 \), \( l \geq l_0 \) and \( C1(\delta, L_0, L_1) \) holds for some \( \delta \in (0, \delta_0] \), then for \( L_1 \leq L \leq L_1(\log L_1)^2 \) and \( \psi \equiv l \), for all \( q \in \mathcal{P}_L^\phi \)

\[
\mathbb{P} \left( \sup_{x \in V_{L/5}} \left\| (\Pi_L - \pi_L^{(p_L)}(x, \cdot)) \right\|_1 > \eta \right) \leq \exp \left( -\frac{9}{5} \log L \right).
\]

**Proof:** The proof is based on a modification of the computations in the foregoing lemma. Let \( \delta_0 \) be as in Lemma \( 5.6 \). We fix an arbitrary \( 0 < \delta \leq \delta_0 \) and assume \( C1(\delta, L_0, L_1) \) for some \( L_1 \geq L_0 \), where \( L_0 = L_0(\eta) \) will be chosen later. In the following, “good” and “bad” is always to be understood with respect to \( \delta \). Again, for \( L_1 \leq L \leq L_1(\log L_1)^2 \),

\[
\mathbb{P} (\text{ManyBad}_L \cup \text{BdBad}_{L,r}) \leq \exp \left( -\frac{9}{5} \log L \right).
\]
For \( \omega \in \text{OneBad}_L \cap (\text{BdBad}_{L,r})^c \), we use the splitting [32] of \( \Pi - \pi^{(q)}_L \) into the parts \( F_1, F_2, F_3 \). For the summands \( F_1 \) and \( F_2 \), we do not need the additional smoothing by \( \hat{\pi}^{(q)}_\psi \), since by [33]

\[
\|F_1(x, \cdot)\|_1 \leq C(\log L)^{-3} \leq \eta/3,
\]

and by [34]

\[
\|F_2(x, \cdot)\|_1 \leq C \left( (\log L)^{-3} + (\log r)^{-1/2} \right) \leq \eta/3,
\]

if \( L \geq L_0 \) and \( r \), \( L_0 \) are chosen large enough, depending on \( d \) and \( \eta \). We turn to \( F_3 \). By [36], [37] and [38] we have, with (recall that \( \Delta' = 1_{V_L}(\Pi^g - \hat{\pi}^{(q)}_\psi) \)) as in the proof of Lemma 5.6, writing again \( \hat{\pi}_\psi \) for \( \hat{\pi}^{(q)}_\psi \),

\[
\|F_3\hat{\pi}_\psi(x, \cdot)\|_1 \leq C \left( \sup_{y \in V_L \cup \partial V_L} \left| \| \hat{G}^{g} 1_{V_L \setminus \Lambda_0} \Delta' \pi^{(p_L)}_L(y, \cdot) \|_1 \right| + \sup_{z \in \Lambda_0} \| \Delta' \pi^{(p_L)}_L \hat{\pi}_\psi(z, \cdot) \|_1 \right)
\leq C \left( \log L)^{-3} + (\log r)^{-8} + \sup_{z \in \Lambda_0} \| \Delta' \pi^{(p_L)}_L \hat{\pi}_\psi(z, \cdot) \|_1 \right)
\leq \eta/6 + C_1 \sup_{z \in \Lambda_0} \| \Delta' \pi^{(p_L)}_L \hat{\pi}_\psi(z, \cdot) \|_1,
\tag{40}
\]

if \( L \geq L_0 \) and \( r \), \( L_0 \) are sufficiently large. Regarding the second summand of (40), set \( m = 3r \) and define for \( K \in \mathbb{N} \)

\[
\vartheta_K(z) = \min \{ n \in \mathbb{N} : |X_n^z - z| > Km \} \in [0, \infty],
\]

where \( X_n^z \) denotes symmetric random walk with law \( P_{z,p_L} \). By the invariance principle for symmetric random walk, we can clearly choose \( K \) so large such that

\[
\max_{z \in V_L : d_L(z) \leq m} P_{z,p_L} (\vartheta_K(z) \leq \tau_L) \leq \frac{\eta}{24C_1}
\]

uniformly in \( L \geq L_0 \) and \( q \in P^0_L \), where \( C_1 \) is the constant from (40). If \( z \in \Lambda_0 \), \( z' \in V_L \cup \partial V_L \) with \( \Delta'(z, z') \neq 0 \), we have \( d_L(z') \leq m \) and \( |z - z'| \leq m \). Thus, using Lemma 7.2 (iii) of the appendix with \( \psi \equiv l \),

\[
C_1 \sup_{z \in \Lambda_0} \| \Delta' \pi^{(p_L)}_L \hat{\pi}_\psi(z, \cdot) \|_1
\leq C_1 \sup_{z \in \Lambda_0} \| \Delta'(z, z') \left( \sum_{w \in \partial V_L : |z' - w| > Km} \pi^{(p_L)}_L(z', w) + \sum_{w \in \partial V_L : |z' - w| \leq Km} \pi^{(p_L)}_L(z', w) \right) \left( \hat{\pi}_\psi(w, \cdot) - \hat{\pi}_\psi(z, \cdot) \right) \|_1
\leq \eta/6 + C(K + 1) m \log l \leq \eta/3,
\]

if we choose \( l = l(\eta, r) \) large enough. This proves the lemma. \( \Box \)
6 Proofs of the main results

Proof of Proposition 1.1: We take \( \delta \) small enough and choose \( L_0 = L_0(\delta) \) large enough according to Remark 4.3 and the statements of Section 5. In the course of this proof, we might enlarge \( L_0 \) further. (ii) is a consequence of Lemma 5.7, so we have to prove (i). Let \( L_1 \geq L_0 \), and assume that condition \( \text{C1}(\delta, L_0, L_1) \) holds. Then the first point of \( \text{C1}(\delta, L_0, L_1(\log L_1)^2) \) is trivially fulfilled. Now let \( L_1 < L \leq L_1(\log L_1)^2 \) and consider first \( L' = L \) and \( i = 1, 2, 3 \). Take \( \psi \in \mathcal{M}_L, q \in \mathcal{P}_L^* \). For simplicity, write \( D_L^* \) for \( D_{L,p_L,\psi,q} \). Then by Lemma 2.3,

\[
b_i(L, p_L, \psi, q, \delta) \leq \mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4} \right) \leq \mathbb{P} \left( \text{ManyBad}_{L} \right) + \mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_{L} \right) \leq \exp \left( -\frac{19}{10}(\log L)^2 \right) + \mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_{L} \right).
\]

For the last summand, we have by Lemmata 5.4, 5.5, under \( \text{C1}(\delta, L_0, L_1) \),

\[
\mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_{L} \right) \leq \mathbb{P} \left( D_L^* > (\log L)^{-9}; \text{Good}_{L} \right) + \sum_{j=1}^{4} \mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_{L}^{(j)} \right) \leq \exp \left( -(\log L)^{7/3} \right) + \sum_{j=1}^{i} \mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_{L}^{(j)} \right) + \sum_{j=i+1}^{4} \mathbb{P} \left( \text{OneBad}_{L}^{(j)} \right) \leq 4 \exp \left( -(\log L)^{7/3} \right) + C L^d s_L^d \exp \left( -((3 + i + 1)/4) (\log (r_L/20))^2 \right).
\]

Therefore, for \( L \) large,

\[
\mathbb{P} \left( D_L^* > (\log L)^{-9+9(i-1)/4}; \text{OneBad}_{L} \right) \leq \frac{1}{8} \exp \left( -((3 + i)/4) (\log L)^2 \right),
\]

and

\[
b_i(L, p_L, \psi, q, \delta) \leq \frac{1}{4} \exp \left( -((3 + i)/4) (\log L)^2 \right).
\]

For the case \( i = 4 \), notice that

\[
b_4(L, p_L, \psi, q, \delta) \leq \mathbb{P} \left( D_L^* > (\log L)^{-9/4} \right) + \mathbb{P} \left( D_{L,p_L}^* > \delta \right).
\]

The first summand can be estimated as the corresponding terms in the case \( i = 1, 2, 3 \), while for the last term we use Lemma 5.6.

It remains to show that for all \( L \) with \( L_1 < L \leq L_1(\log L_1)^2 \) and all \( L' \in (L, 2L] \), for all \( \psi \in \mathcal{M}_{L'} \) and all \( q \in \mathcal{P}_L^* \), all \( i = 1, 2, 3, 4 \),

\[
b_i(L', p_L, \psi, q, \delta) \leq \frac{1}{4} \exp \left( -((3 + i)/4) (\log L')^2 \right).
\]
This however is easily deduced from the estimates above, by a slight change of the coarse graining scheme. Indeed, defining for \( L' \in (L, 2L] \) the coarse graining function \( \tilde{h}_{L',r} : \mathcal{C}_{L'} \to \mathbb{R}_+ \) as
\[
\tilde{h}_{L',r}(x) = \frac{1}{20} \max \left\{ s_L h \left( \frac{d_{L'}(x)}{s_{L'}} \right), r \right\},
\]
it follows that \( \tilde{h}_{L',r}(x) = h_{L,r}(0) = s_L/20 \) for \( x \in V_{L'} \) with \( d_{L'}(x) \geq 2s_{L'} \). With an analogous definition of good and bad points within \( V_{L'} \), using the coarse graining function \( \tilde{h}_{L',r} \) instead of \( h_{L,r} \) and the transition kernels corresponding to \( \tilde{h}_{L',r} \), clearly all the statements of Sections 3 and 4 remain true. Moreover, we can use the kernel \( p_L \) to obtain the results of Section 5 for the radius \( L' = L \), noticing that in the proofs at most the constants change. Arguing now exactly as above for the choice \( L' = L \), we conclude that also the second point of condition \( \text{C1}(\delta, L_0, L_1(\log L_1)^2) \) holds true, provided \( L_0 \) is large, \( L_1 \geq L_0 \).

**Proof of Proposition 0.1:** According to Proposition 1.1 (i), for \( \delta, \varepsilon > 0 \) small and \( L_0 = L_0(\delta) \) large, conditions \( \text{A1}, \text{A0}(\varepsilon) \) and \( \text{C1}(\delta, L_0, L_0) \) imply \( \text{C1}(\delta, L_0, L_0) \) for all \( L \geq L_0 \). By shrinking \( \delta \) if necessary, we may further guarantee that \( \text{C1}(\delta, L_0, L_0) \) holds true, as it was explained below the statement of the proposition. Therefore, we can assume that \( \text{C1}(\delta, L_0, L) \) is satisfied for all \( L \geq L_0 \). By Lemma 2.2, the limit \( \lim_{L \to \infty} p_L(e_i) \) exists for each \( i = 1, \ldots, d \). Now let \( l = s_L/20 \). Using the definition of \( p_L \) in the first, Condition \( \text{C1} \) in the second and fourth and Lemma 2.1 in the third equality,
\[
2p_L(e_i) = \sum_y \mathbb{E} \left[ \Pi_{L,r}(0, y) \right] \frac{|y|^2}{y^2} = \sum_y \tilde{\pi}_{L,r}(0, y) \frac{|y|^2}{y^2} + O((\log L)^{-9})
\]
\[
= \sum_y \pi_i(0, y) \frac{|y|^2}{y^2} + O((\log L)^{-9}) = \sum_y \mathbb{E} [\Pi_i(0, y)] \frac{|y|^2}{y^2} + O((\log L)^{-9}).
\]
From this, the first statement of the proposition follows. Moreover, \( p_\infty \in \mathcal{P}_1 \), and recalling that we may first choose \( L_0 \) as large as we wish and then choose \( \varepsilon \) sufficiently small, we see that \( ||p_\infty - p_0||_1 \to 0 \) as \( \varepsilon \downarrow 0 \).

**Proof of Theorem 0.1:** As in the proof of Proposition 0.1, if the parameters are appropriately chosen, \( \text{C1}(\delta/2, L_0, L) \) holds true for all \( L \geq L_0 \). This implies
\[
\mathbb{P} \left( D_{L,p_L}^* > \delta/2 \right) \leq \exp \left( -(\log L)^2 \right). \tag{41}
\]
With \( p_\infty \) defined in Proposition 0.1, we obtain by the triangle inequality
\[
D_{L,p_\infty}^* = \sup_{x \in V_L} ||(\Pi - \pi(p_\infty)) (x, \cdot)||_1 \leq D_{L,p_L}^* + \sup_{x \in V_L} ||(\pi(p_\infty) - \pi(p_L)) (x, \cdot)||_1. \tag{42}
\]
The claim therefore follows if we show that the second summand is bounded by \( \delta/2 \) if \( L \geq L_0 \) and \( L_0 \) is large enough. To prove this bound, we move inside \( V_L \) according to the
coarse grained transition kernels $\hat{\pi}^{(p_L)}_{L,r}$ and $\hat{\pi}^{(p_{\infty})}_{L,r}$, respectively, where similarly to Section 5.3, $r$ is a large but fixed number. First, we recall from the proof of Proposition 0.1 that $p_{\infty} = \lim_{L \to \infty} p_L$. We therefore deduce from Lemma 2.2
\[
||p_{\infty} - p_L||_1 \leq C \sum_{k=0}^{\infty} (\log(2^k L))^{-9} \leq C (\log L)^{-8}.
\] (43)
This implies by Lemma 3.5 that for $x \in V_L$ with $d_L(x) > (1/10)r$,
\[
||((\hat{\pi}^{(p_L)} - \hat{\pi}^{(p_{\infty})})_L(x, \cdot))||_1 \leq C \max\{h_{L,r}(x)^{-1/4}, (\log L)^{-8}\}.
\] (44)
Now by the perturbation expansion, with $\Delta = 1_{V_L}(\hat{\pi}^{(p_L)} - \hat{\pi}^{(p_{\infty})})$,
\[
\hat{\pi}^{(p_{\infty})} - \hat{\pi}^{(p_L)} = \hat{g}^{(p_L)}(\Delta)\hat{\pi}^{(p_{\infty})} = \hat{g}^{(p_L)}1_{V_L \setminus Sh_L(2rL)}\Delta\hat{\pi}^{(p_{\infty})} + \hat{g}^{(p_L)}1_{Sh_L(2rL)}\Delta\hat{\pi}^{(p_{\infty})}.
\]
Since $\hat{g}^{(p_L)}(x, V_L) \leq C\Gamma(x, V_L) \leq C (\log L)^6$, we obtain with (44)
\[
\sup_{x \in V_L} ||\hat{g}^{(p_L)}1_{V_L \setminus Sh_L(2rL)}\Delta\hat{\pi}^{(p_{\infty})}(x, \cdot)||_1 \leq C(\log L)^{-2} \leq \delta/4,
\]
if $L$ is large enough. Further, as below (38), for $x \in V_L/5$, using again (44),
\[
||\hat{g}^{(p_L)}1_{Sh_L(2rL)}\Delta\hat{\pi}^{(p_{\infty})}(x, \cdot)||_1
\leq \left||\hat{g}^{(p_L)}1_{\bigcup_{j=1}^{J_1} A_j}\Delta(x, \cdot)||_1 + ||\hat{g}^{(p_L)}1_{A_0}\Delta(x, \cdot)||_1\right|
\leq Cr^{-1/4} + C_0 \sup_{z \in A_0} ||\Delta(z, \cdot)||_1.
\]
We choose $r$ so large such that $Cr^{-1/4} \leq \delta/8$. Now that $r$ is fixed, the difference over the first layer $Sh_L(2r)$ is also bounded by $C_0^{-1}\delta/4$ if the difference between $p_L$ and $p_{\infty}$ is small enough, that is if $L$ is large enough. This proves that the second summand of (42) is bounded by $\delta/2$, for $L \geq L_0$ and $L_0$ large. Applying (41), we conclude that
\[
\mathbb{P}\left(D_{L,p_{\infty}}^r - \delta\right) \leq \mathbb{P}\left(D_{L,p_L}^r - \delta/2\right) \leq \exp\left(-\deg L^2\right).
\]
\[\square\]

Since Theorem 0.2 is proved in a similar way, using the second part of Proposition 1.1, we may safely omit the details and turn now to the proof of the local estimates. **Proof of Theorem 0.3**: We choose $\delta > 0$ small and $L_0(\delta)$ large enough according to Proposition 1.1 such that $A0(\varepsilon)$ and $A1$ imply $C1(\delta, L_0, L)$ for all $L \geq L_0$. Further, we set $r = r_L$. Recall the definition of $\text{Good}_L$ from Section 2. We put $A_L = \text{Good}_L$ and let $p = p_{aL/20}$. Note that similar to Lemma 2.3 if $L \geq L_0$,
\[
\mathbb{P}(A_L^r) \leq \exp\left(-\frac{1}{2}(\log L)^2\right).
\]
For the rest of the proof, take $\omega \in A_L$. On such environments, $G$ equals $G^g$ by our choice $r = r_L$. 
Figure 5: On the proof of Theorem 0.3 (ii). There, $t \geq L/(\log L)^6 > l = L/(\log L)^{17/2}$. If the walk exits $V_L$ through $\partial V_L \setminus W_t$, it cannot enter $U_l(W_t)$ in one step with $\hat{\pi}_l$.

Now let us prove part (i). Observe that $W_t$ can be covered by $K|W_t|^{(d-1)}$ many neighborhoods $V_{\eta L}(y)$, $y \in \text{Sh}(r_L)$, as defined in Section 4.2, where $K$ depends on the dimension only. In particular, $\Gamma(x, W_t) \leq C(t/L)^{d-1}$. Applying Lemma 4.2 (ii), we deduce that

$$\Pi_L(x, W_t) = \hat{G}^g (x, W_t) \leq C(t/L)^{d-1}.$$ 

From Lemma 3.1 (i) we know that if $x \in V_{\eta L}$, then for some constant $c = c(d, \eta)$,

$$\pi^{(p_\circ)}(x, z) \geq cL^{-(d-1)}.$$ 

Together with the preceding equation, this shows (i).

(ii) If not explicitly mentioned otherwise, the underlying one-step transition kernel is in the following given by $p_L$ defined in (6), which we omit from notation. Set $l = (\log L)^{13/2}r$ and consider the smoothing kernel $\hat{\pi}_\psi$ with $\psi \equiv l \in \mathcal{M}_t$. Let

$$U_l(W_t) = \{ y \in \mathbb{Z}^d : d(y, W_t) \leq 2l \}.$$ 

We claim that

$$\Pi(x, W_t) - \pi(x, W_{t+6l}) \leq (\Pi - \pi) \hat{\pi}_\psi(x, U_l(W_t)), \quad \text{(45)}$$

$$\pi(x, W_{t-6l}) - \Pi(x, W_t) \leq (\pi - \Pi) \hat{\pi}_\psi(x, U_l(W_{t-6l})). \quad \text{(46)}$$

Concerning the first inequality,

$$\Pi \hat{\pi}_\psi(x, U_l(W_t)) \geq \sum_{y \in W_t} \Pi(x, y) \hat{\pi}_\psi(y, U_l(W_t)) = \Pi(x, W_t),$$

since $\hat{\pi}_\psi(y, U_l(W_t)) = 1$ for $y \in W_t$. Also,

$$\pi \hat{\pi}_\psi(x, U_l(W_t)) = \sum_{y \in W_{t+6l}} \pi(x, y) \hat{\pi}_\psi(y, U_l(W_t)) \leq \pi(x, W_{t+6l}).$$
since \( \hat{\pi}_\psi(y, U_l(W_l)) = 0 \) for \( y \in \partial V_L \setminus W_{l+6t} \). This proves (45), while (46) is entirely similar. In the remainder of this proof, we often write \( |F(x, y)| \) for \( |F(x, y)| \). If we show

\[
|\langle \pi - \Pi \rangle \hat{\pi}_\psi(D(x, U_l))| \leq O \left( (\log L)^{-\frac{5}{2}} \right) \pi(x, W_l),
\]

then by (45),

\[
\Pi(x, W_l) \leq \pi(x, W_{l+6t}) + O(\log L)^{-\frac{5}{2}) \pi(x, W_l) \\
= \pi(x, W_l) + \pi(x, W_{l+6t} \setminus W_l) + O(\log L)^{-\frac{5}{2}) \pi(x, W_l) \\
= \pi(x, W_l) \left( 1 + O \left( \max \{l/t, (\log L)^{-\frac{5}{2}} \} \right) \right) \\
= \pi(x, W_l) \left( 1 + O((\log L)^{-\frac{5}{2}}) \right).
\]

On the other hand, by (46) and still assuming (47),

\[
\Pi(x, W_l) \geq \pi(x, W_{l-6t}) - O((\log L)^{-\frac{5}{2}) \pi(x, W_l) = \pi(x, W_l) \left( 1 - O((\log L)^{-\frac{5}{2}}) \right),
\]

so that indeed

\[
\Pi(x, W_l) = \pi(x, W_l) \left( 1 + O((\log L)^{-\frac{5}{2}}) \right),
\]

provided we prove (47). In that direction, set \( B = V_L \setminus Sh_L(5r) \) and write, with \( \Delta = 1_{V_L} (\hat{\Pi}^g - \hat{\pi}(p_{x,l}) \),

\[
(\pi - \Pi) \hat{\pi}_\psi = \hat{G}^g \Delta \hat{\pi}_\psi = \hat{G}^g 1_B \Delta \hat{\pi}_\psi + \hat{G}^g 1_{Sh_L(5r)} \Delta \hat{\pi}_\psi.
\]

Looking at the first summand we have

\[
|\hat{G}^g 1_B \Delta \hat{\pi}_\psi| \leq (\hat{G}^g 1_B (\Delta \hat{\pi} | \pi) \leq C(\log \log L)(\log L)^{-9}.
\]

By Lemma 5.1 we have

\[
||1_B \Delta \pi(x, \cdot)||_1 \leq C(\log \log L)(\log L)^{-9}.
\]

Further, a slight modification of the proof of Proposition 4.4 (ii) shows

\[
\hat{G}^g 1_B (|\Delta \hat{\pi}| \mid \Gamma(x, z) \leq C(\log L)^{-\frac{5}{2}) \Gamma(x, z).
\]

Together with \( \pi \leq Ct \) and \( \pi(x, z) \geq c(d, \eta) L^{-(d-1)} \) this yields the bound

\[
\hat{G}^g 1_B (\Delta \hat{\pi} | \pi) \leq C(\log L)^{-\frac{5}{2}) \Gamma(x, W_l) \leq C(\log L)^{-\frac{5}{2}) \pi(x, W_l).
\]

To obtain (47), it remains to handle the second summand of (48), i.e. we have to bound

\[
|\hat{G}^g 1_{Sh_L(5r)} \Delta \hat{\pi}_\psi| \leq (\hat{G}^g 1_{Sh_L(5r)} \Delta \hat{\pi}_\psi | \pi(x, U_l(W_l))).
\]

We abbreviate \( S = Sh_L(5r) \) and split into

\[
\hat{G}^g 1_S \Delta \hat{\pi}_\psi(x, U_l(W_l)) \\
= \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L} \Delta \pi(y, z)(\hat{\pi}_\psi(z, U_l(W_l)) - \hat{\pi}_\psi(y, U_l(W_l))) \\
= \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in W_{l+6t}} \Delta \pi(y, z)(\hat{\pi}_\psi(z, U_l(W_l)) - \hat{\pi}_\psi(y, U_l(W_l))) \\
- \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L \setminus W_{l+6t}} \Delta \pi(y, z)(\hat{\pi}_\psi(y, U_l(W_l))).
\]
First note that since $\hat{\pi}_\psi(y, z') = 0$ if $|y - z'| > 2l$, 
\[
\left| \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L \setminus W_{t+6l}} \Delta \pi(y, z) \hat{\pi}_\psi(y, U_i(W_i)) \right| 
\leq (\hat{G}^g 1_{U_2(W_i) \cap S}|\Delta \pi)| (x, \partial V_L \setminus W_{t+6l}).
\]
For $y \in U_2(W_i) \cap S$, we apply Lemma 3.2 (iii) together with Lemma 3.6 and obtain 
\[
|\Delta \pi|(y, \partial V_L \setminus W_{t+6l}) \leq \sup_{y' : d(y', U_2(W_i) \cap S) \leq r} \pi(y', \partial V_L \setminus W_{t+6l}) \leq C r^l \leq C (\log L)^{-13/2}.
\]
Since $\hat{G}^g \leq \Gamma$ and $\pi(x, z) \geq cL^{-d-1}$, $G^g(x, U_2(W_i) \cap S) \leq C \pi(x, W_i)$, and thus 
\[
(\hat{G}^g 1_{U_2(W_i) \cap S}|\Delta \pi)| (x, \partial V_L \setminus W_{t+6l}) \leq C (\log L)^{-13/2} \pi(x, W_i).
\]
It remains to bound 
\[
\left| \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in W_{t+6l}} \Delta \pi(y, z)(\hat{\pi}_\psi(z, U_i(W_i)) - \hat{\pi}_\psi(y, U_i(W_i))) \right|.
\]
Set $D^1(y) = \{z \in W_{t+6l} : |z - y| \leq l(\log l)^{-4} \}$. If $D^1(y) \neq \emptyset$, then $d(y, W_i) \leq 7l$. Using Lemma 7.2 for the difference of the smoothing steps and the standard estimate on $\hat{G}^g$, 
\[
\left| \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in D^1(y)} |\Delta \pi(y, z)||\hat{\pi}_\psi(z, U_i(W_i)) - \hat{\pi}_\psi(y, U_i(W_i))| \right| 
\leq \frac{C \int D^1 \frac{t^d}{L^{d-r} (\log l)^{-3}} \leq C (\log L)^{-5/2} \pi(x, W_i)}.
\]
The region $W_{t+6l} \setminus D^1(y)$ we split into $B_0(y) = \{z \in W_{t+6l} : |z - y| \in (l(\log l)^{-4}, t)\}$, and 
\[
B_i(y) = \{z \in W_{t+6l} : |z - y| \in (it, (i + 1)t)\}, \quad i = 1, 2, \ldots, [2L/t].
\]
Furthermore, let 
\[
S_i = \{y \in S : B_i(y) \neq \emptyset\}, \quad i = 0, 1, \ldots, [2L/t].
\]
Then 
\[
\left| \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in W_{t+6l} \setminus D^1(y)} |\Delta \pi(y, z)| \right| \leq C \sum_{i=0}^{[2L/t]} \hat{G}^g(x, S_i) \sup_{y \in S_i} |\Delta \pi|(y, B_i(y)).
\]
If $i \geq 1$ and $y \in S_i$, then by Lemma 3.2 (iii) 
\[
|\Delta \pi|(y, B_i(y)) \leq \sup_{y' : |y' - y| \leq r} \pi(y', B_i(y)) \leq C r^{ \frac{t^{d-1}}{(it)^d} \leq C r^{ \frac{r}{t^d}}}
\]
while in the case $i = 0$, using the same lemma and additionally Lemma 3.6,
\[
\sup_{y':|y'-y| \leq r} \pi(y', B_i(y)) \leq Cr \sum_{z \in \partial V_L} \frac{1}{((1/2)t (\log t)^{-4} + |y - z|)^d} \\
\leq C \frac{r (\log t)^4}{t} \leq C (\log L)^{-5/2}.
\]

For the Green’s function, we use the estimates
\[
\hat{G}^g(x, S_0) \leq C \frac{t^{d-1}}{L^{d-1}}, \quad \hat{G}^g(x, \cup_{i \geq \lceil (1/10)L/t \rceil} S_i) \leq C,
\]
while for $i = 1, 2, \ldots, \lfloor (1/10)L/t \rfloor$, it holds that $|S_i| \leq Cr(i t)^{d-2}t$, whence
\[
\hat{G}^g(x, S_i) \leq C I^{d-2} t^{d-1} \frac{L^{d-1}}{L^{d-1}}.
\]

Altogether, we obtain
\[
\sum_{i=0}^{\lfloor 2L/t \rfloor} \hat{G}^g(x, S_i) \sup_{y \in S_i} |\Delta \pi|(y, B_i(y)) \\
\leq C \left( (\log L)^{-5/2} \frac{t^{d-1}}{L^{d-1}} + \left( \frac{r t^{d-1}}{t L^{d-1}} \sum_{i=1}^{\lfloor (1/10)L/t \rfloor} \frac{1}{i^2} \right) + \frac{t^{d-1} r}{L^{d-1} L} \right) \\
\leq C (\log L)^{-5/2} \frac{t^{d-1}}{L^{d-1}}.
\]

This finishes the proof of part (ii).

Let us finally show how to obtain transience of the RWRE.

**Proof of Corollary 0.1** As in the proof of Theorem 0.3, we assume that $C_1(\delta, L_0, L)$ holds true for all $L \geq L_0$. Fix numbers $\rho \geq 3, \alpha \in (0, (4\rho)^{-1})$ to be specified below. With these parameters and $n \geq 1$, we put $p = p_{\alpha \rho^n}$ and set
\[
q_{n, \alpha, \rho} = \hat{\pi}^{(p)}_{\psi},
\]
where $\psi = (m_x)_{x \in \mathbb{Z}^d}$ is chosen constant in $x$, namely $m_x = \alpha \rho^n$. Define
\[
A_n = \bigcap_{|x| \leq \rho^{3/2}} \bigcap_{t \in [\alpha \rho^n, 2\alpha \rho^n]} \{D_{t, p, \psi, p}(x) \leq (\log t)^{-9} \}.
\]

Under $C_1$, we then have
\[
\mathbb{P}(A_n^c) \leq C \alpha^d \rho^{(d+1)n^{3/2}} \exp \left( - (\log (\alpha \rho^n))^2 \right).
\]

Therefore, for any choice of $\alpha, \rho$ it holds that
\[
\sum_{n=1}^{\infty} \mathbb{P}(A_n^c) < \infty,
\]
whence by Borel-Cantelli
\[ P \left( \liminf_{n \to \infty} A_n \right) = 1. \] (49)

We denote the coarse-grained RWRE transition kernel by
\[ Q_{n,\alpha,\rho}(x, \cdot) = \frac{1}{\alpha \rho^n} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{\alpha \rho^n} \right) \Pi_V(x, \cdot) dt. \]

If \( |x| \leq \rho^{n/2} \), we have on \( A_n \)
\[ \| (Q_{n,\alpha,\rho} - q_{n,\alpha,\rho}) q_{n,\alpha,\rho}(x, \cdot) \|_1 \leq (\log(\alpha \rho^n))^{-9} \leq C(\alpha, \rho)n^{-9}. \]

Now assume \( |x| \leq \rho^n + 1 \). For \( N \) fixed and \( \omega \in A_n \), it follows that for \( 1 \leq M \leq N \)
\[ \left\| \left( (Q_{n,\alpha,\rho})^M - (q_{n,\alpha,\rho})^M \right) q_{n,\alpha,\rho}(x, \cdot) \right\|_1 \leq C(\alpha, \rho)Mn^{-9}. \] (50)

For fixed \( \omega \), let \( (\xi_k)_{k \geq 0} \) be the Markov chain running with transition kernel \( Q_{n,\alpha,\rho} \). Clearly, \( (\xi_k)_{k \geq 0} \) can be obtained by observing the basic RWRE \( (X_k)_{k \geq 0} \) at randomized stopping times. Then
\[
P_{x,\omega}(\xi_{N-1} \in V_{\rho^{n+1}+2\alpha \rho^n}) \\
\leq (Q_{n,\alpha,\rho})^{N-1} q_{n,\alpha,\rho}(x, V_{\rho^{n+1}+4\alpha \rho^n}) \\
\leq \left\| \left( (Q_{n,\alpha,\rho})^{N-1} - (q_{n,\alpha,\rho})^{N-1} \right) q_{n,\alpha,\rho}(x, \cdot) \right\|_1 + (q_{n,\alpha,\rho})^N (x, V_{2\rho^n+1}).
\]

Using Proposition 4.1 we can find \( N = N(\alpha, \rho) \in \mathbb{N} \), depending not on \( n \), such that for any \( x \) with \( |x| \leq \rho^n + 1 \), it holds that \( (q_{n,\alpha,\rho})^N(x, V_{2\rho^n+1}) \leq 1/10 \). With (50), we conclude that for such \( x, n \geq n_0(\alpha, \rho, N) \) large enough and \( \omega \in A_n \),
\[ P_{x,\omega}(\xi_{N-1} \in V_{\rho^{n+1}+2\alpha \rho^n}) \leq C(\alpha, \rho)Nn^{-9} + 1/10 \leq 1/5. \] (51)

On the other hand, if \( x \) is outside \( V_{\rho^{n-1}+2\alpha \rho^n} \),
\[
P_{x,\omega}(\xi_M \in V_{\rho^{n-1}+2\alpha \rho^n} \text{ for some } 0 \leq M \leq N-1) \\
\leq \sum_{M=1}^{N-1} (Q_{n,\alpha,\rho})^M q_{n,\alpha,\rho}(x, V_{\rho^{n-1}+4\alpha \rho^n}) \\
\leq \sum_{M=1}^{N-1} \left\| \left( (Q_{n,\alpha,\rho})^M - (q_{n,\alpha,\rho})^M \right) q_{n,\alpha,\rho}(x, \cdot) \right\|_1 + \sum_{k=2}^{N} (q_{n,\alpha,\rho})^k (x, V_{2\rho^{n-1}}).
\]

If \( \rho^n - 1 \leq |x| \), then \( (q_{n,\alpha,\rho})^k(x, V_{2\rho^{n-1}}) = 0 \) as long as \( k \leq (1 - 3/\rho)/(2\alpha) \). By first choosing \( \rho \) large enough, then \( \alpha \) small enough and estimating the higher summands again with Proposition 4.1 we deduce that for such \( x \) and all large \( n \),
\[ \sum_{k=1}^{\infty} (q_{n,\alpha,\rho})^k(x, V_{2\rho^{n-1}}) \leq 1/10. \]
Together with (50), we have for large $n$, $\omega \in A_n$ and $\rho^n - 1 \leq |x| \leq \rho^n + 1$,

$$P_{x,\omega}(\xi_M \in V_{\rho^{n+1}+2\alpha\rho^n}) \text{ for some } 0 \leq M \leq N - 1 \leq C(\alpha, \rho)N^2n^{-\theta} + 1/10 \leq 1/5. \quad (52)$$

Let $B$ be the event that the walk $(\xi_k)_{k \geq 0}$ leaves $V_{\rho^{n+1}+2\alpha\rho^n}$ before reaching $V_{\rho^{n-1}+2\alpha\rho^n}$. From (51) and (52) we deduce that $P_{x,\omega}(B) \geq 3/5$, provided $n$ is large enough, $\omega \in A_n$ and $\rho^n - 1 \leq |x| \leq \rho^n + 1$. But on $B$, the underlying basic RWRE $(X_k)_{k \geq 0}$ clearly leaves $V_{\rho^{n+1}}$ before reaching $V_{\rho^{n-1}}$. Hence if $\omega \in \{\lim \inf A_n\}$, there exists $m_0 = m_0(\omega) \in \mathbb{N}$ such that

$$P_{x,\omega}(\tau_{V_{\rho^n+1}} < \tau_{V_{\rho^n-1}}) \geq 3/5$$

for all $n \geq m_0$, $x$ with $|x| \geq \rho^n - 1$ (of course, we may now drop the constraint $|x| \leq \rho^n + 1$). From this property, transience easily follows. Indeed, for $m, M, k \in \mathbb{N}$ satisfying $M > m \geq m_0$ and $0 \leq k \leq M + 1 - m$, set

$$h_M(k) = \sup_{x: |x| \geq \rho^m + k - 1} P_{x,\omega}(\tau_{V_{\rho^n+1}} < \tau_{V_{\rho^m}}).$$

Then $h_M$ solves the difference inequality

$$h_M(k) \leq \frac{2}{5} h_M(k - 1) + \frac{3}{5} h_M(k + 1)$$

with boundary conditions $h_M(0) = 1$, $h_M(M + 1 - m) = 0$. Further, by either applying a discrete maximum principle or by a direct computation, we see that $h_M \leq \overline{h}_M$, where $\overline{h}_M$ is the solution of the difference equality

$$\overline{h}_M(k) = \frac{2}{5} \overline{h}_M(k - 1) + \frac{3}{5} \overline{h}_M(k + 1) \quad (53)$$

with boundary conditions $\overline{h}_M(0) = 1$, $\overline{h}_M(M + 1 - m) = 0$. Solving (53), we get

$$\overline{h}_M(k) = \frac{1}{1 - (3/2)^{M+1-m}} + \frac{1}{1 - (2/3)^{M+1-m}} \left(\frac{2}{3}\right)^k.$$
Letting $M \to \infty$, we deduce that for $|x| \geq \rho^{n+k}$,

$$P_{x,\omega} (T_{V_{\rho, m}} < \infty) \leq \lim_{M \to \infty} \tilde{T}_M(k) = \left( \frac{2}{3} \right)^k. \quad (54)$$

Together with (49), this proves that for almost all $\omega \in \Omega$, the random walk is transient under $P_{x,\omega}$.

7 Appendix

7.1 Some difference estimates

In this section we collect some difference estimates of (non)-smoothed exit distributions which are mainly needed to prove Lemma 3.7 (i) and (iii). The first technical lemma connects the exit distributions of a symmetric random walk with one-step distribution $p \in \mathcal{P}_k^x$ to those of Brownian motion with covariance matrix $\Lambda_p$. As always, $\kappa > 0$ can be chosen arbitrarily small.

**Lemma 7.1.** Let $p \in \mathcal{P}_k^x$, and let $\beta, \eta > 0$ with $3\eta < \beta < 1$. For large $L$, there exists a constant $C > 0$ such that for $A \subset \mathbb{R}^d$, $A^p = \{y \in \mathbb{R}^d : d(y, A) \leq L^\beta\}$ and $x \in V_L$ with $d_L(x) > L^\beta$, the following holds.

(i) $\pi^{B(p)}_L(x, A) \leq \pi^{B(p)}_L(x, A^\beta) \left(1 + CL^{-(\beta - 3\eta)}\right) + L^{-(d+1)}$.

(ii) $\pi^{B(p)}_L(x, A) \leq \pi^{B(p)}_L(x, A^\beta) \left(1 + CL^{-(\beta - 3\eta)}\right) + L^{-(d+1)}$.

**Proof:** (i) Set $L' = L + L^n$, $L'' = L + 2L^n$ and denote by $A^\beta$ the image of $\{y \in \partial C_L : d(y, A) \leq L^\beta/2\}$ on $\partial C_{L'}$ under the map $y \mapsto (L'/L)y$. We write $\pi^B_L$ for $\pi^{B(p)}_L$. Then, denoting by $P^B_L$ the law of a $d$-dimensional Brownian motion $W_t$ with covariance $\Lambda_p$, conditioned on $W_0 = x$, and by $\hat{\tau}_L = \inf \{t \geq 0 : W_t \notin C_L\}$ the first exit time from $C_L$,

$$\pi^B_{L'}(x, A^\beta) \leq \pi^B_L(x, A^\beta) + P^B_x \left( W_{\hat{\tau}_L} \in A^\beta, W_{\hat{\tau}_L} \notin A^\beta \right).$$

Let us first assume that we have already proved

$$P^B_x \left( W_{\hat{\tau}_L} \in A^\beta, W_{\hat{\tau}_L} \notin A^\beta \right) \leq CL^{\eta - \beta} \pi^B_{L'}(x, A^\beta), \quad (55)$$

so that

$$\pi^B_L(x, A^\beta) \geq \pi^B_{L'}(x, A^\beta) \left(1 - CL^{-(\beta - \eta)}\right). \quad (56)$$

As a consequence of [24], Theorem 2, for each $k \in \mathbb{N}$ there exists a positive constant $C_1 = C_1(k)$ such that for each integer $n \geq 1$, one can construct on the same probability space a Brownian motion $W_t$ with covariance matrix $d^{-1}\Lambda_p$ as well as a symmetric random walk $X_n$ with one-step probability $p$, both starting in $x$ and satisfying (with $\mathbb{Q}$ denoting the probability measure on that space)

$$\mathbb{Q} \left( \max_{0 \leq m \leq n} |X_m - W_m| > C_1 \log n \right) \leq C_1 n^{-k}. \quad (57)$$
Choose \( k > (2/5)(d + 1) \) and let \( C_1(k) \) be the corresponding constant. The following arguments hold for sufficiently large \( L \). By standard results on the oscillation of Brownian paths,

\[
\mathbb{Q} \left( \sup_{0 \leq t \leq L^{5/2}} |W_{[t]} - W_t| > (5/2)C_1 \log L \right) \leq (1/3)L^{-(d+1)}. \tag{58}
\]

With

\[
B_1 = \left\{ \sup_{0 \leq t \leq L^{5/2}} |X_{[t]} - W_t| \leq 5C_1 \log L \right\},
\]

we deduce from (57) and (58) that

\[
\mathbb{Q} \left( B_1^c \right) \leq (2/3)L^{-(d+1)}.
\]

Let \( B_2 = \{ \tilde{\tau}_{L'} \lor \tau_{L''} \leq L^{5/2} \} \). We claim that

\[
\mathbb{Q} \left( B_2^c \right) \leq (1/3)L^{-(d+1)}. \tag{59}
\]

By the central limit theorem, one finds a constant \( c > 0 \) with \( \mathbb{Q} \left( \tau_{L''} \leq (L'')^2 \right) \geq c \) for \( L \) large. By the Markov property, we obtain \( \mathbb{Q} \left( \tilde{\tau}_{L'} > L^{5/2} \right) \leq (1 - c) L^{1/3} \). The probability \( \mathbb{Q} \left( \tilde{\tau}_{L'} > L^{5/2} \right) \) decays in the same way, and (59) follows. Since \( \pi_B^p \) is unchanged if the Brownian motion is replaced by a Brownian motion with covariance \( d^{-1} \Lambda_p \), we have (P \(_x \) denotes the law of \( X_n \))

\[
\pi_{L'}^B (x, A^\beta) \geq \mathbb{Q} \left( X_{\tau_L} \in A, W_{\tilde{\tau}_{L'}} \in A^\beta \right) \geq P_x (X_{\tau_L} \in A) - \mathbb{Q} \left( X_{\tau_L} \in A, W_{\tilde{\tau}_{L'}} \notin A^\beta, B_1 \cap B_2 \right) - L^{-(d+1)}.
\]

Let \( U = \left\{ z \in \mathbb{Z}^d : d(z, (\partial C_L) \setminus A^\beta) \leq 5C_1 \log L \right\} \). Then

\[
\mathbb{Q} \left( X_{\tau_L} \in A, W_{\tilde{\tau}_{L'}} \notin A^\beta, B_1 \cap B_2 \right) \leq P_x (X_{\tau_L} \in A, T_U < \tau_{L''}).
\]

By the strong Markov property,

\[
P_x (X_{\tau_L} \in A, T_U < \tau_{L''}) \leq P_x (X_{\tau_L} \in A) \sup_{y \in A} P_y (T_U < \tau_{L''}).
\]

Further, there exists a constant \( c > 0 \) such that for \( y \in A \) and \( z \in U \), we have \( |y - z| \geq cL^\beta \) and \( d_{L''}(z) \leq d_{L''}(y) \leq 2L^n \). Therefore, an application of first Lemma \( 3.2 \) (ii) and then Lemma \( 3.6 \) yields

\[
P_y (T_U < \tau_{L''}) \leq CL^{2n} \sum_{z \in U} \frac{1}{|y - z|^d} \leq CL^{2n}(\log L)L^{-\beta} \leq CL^{-(\beta - 3\eta)},
\]

uniformly in \( y \in A \). Going back to (60), we arrive at

\[
\pi_{L'}^B (x, A^\beta) \geq \pi_L (x, A) \left( 1 - CL^{-(\beta - 3\eta)} \right) - L^{-(d+1)}.
\]
Together with \([56]\), this shows (i), but we still have to prove \([55]\). First, by Lemma \(3.3\) (i) (all integrals are surface integrals)

\[
P^B_x \left( W_{\tau_L} \in A^\beta, W_{\tau_L} \notin A^\beta \right) \leq \int_{\partial C_L \setminus A^\beta} \pi^B_L(x, dy) \pi^B_L(y, A^\beta) \leq C d_L(x) L^\eta \int_{\partial C_L \setminus A^\beta} \frac{1}{|x-y|^d} \int_{A^\beta} \frac{1}{|y-z|^d} dz dy.
\]

Fix \(z \in A^\beta \subset \partial C_L\) and put

\[
D_1 = \{ y \in \partial C_L \setminus A^\beta : |y-z| > |x-z|/2 \}, \quad D_2 = \partial C_L \setminus (A^\beta \cup D_1).
\]

Then, using \(d_L(x) \geq L^\beta\) and Lemma \(3.6\) in the last step,

\[
\int_{D_1} \frac{1}{|x-y|^d} \frac{1}{|y-z|^d} dy \leq \frac{C}{|x-z|^d} \int_{D_1} \frac{1}{|x-y|^d} dy \leq \frac{C L^\beta - \eta}{|x-z|^d}.
\]

For the integral over \(D_2\) we obtain the same bound, using that \(|x-y| \geq (1/2)|x-z|\) for \(y \in D_2\), the fact that \(|y-z| \geq c L^\beta\) if \(y \in \partial C_L \setminus A^\beta, z \in A^\beta\), and Lemma \(3.6\). Altogether,

\[
P^B_x \left( W_{\tau_L} \in A^\beta, W_{\tau_L} \notin A^\beta \right) \leq \frac{C d_L(x) L^{\eta-\beta}}{|x-z|^d}.
\]

Integrating over \(z \in A^\beta\), we obtain with the lower bound of Lemma \(3.3\) (i),

\[
P^B_x \left( W_{\tau_L} \in A^\beta, W_{\tau_L} \notin A^\beta \right) \leq C L^{\eta-\beta} \pi^B_{L'} \left( x, A^\beta \right),
\]

as claimed.

(ii) One can follow the same steps, interchanging the role of Brownian motion and the random walk. \(\square\)

Again, let \(p \in \mathcal{P}_r\). We write \(\hat{\pi}^B_{\psi}(p, x, z)\) for the density of \(\hat{\pi}^B_{\psi}(p, dz)\) with respect to \(d\)-dimensional Lebesgue measure, i.e. for \(\psi = (m_x)_{x \in \mathbb{R}^d},\)

\[
\hat{\pi}^B_{\psi}(p, x, z) = \frac{1}{m_x} \varphi \left( \frac{|z-x|}{m_x} \right) \pi^B_{C_{|z-x|}} \left( 0, z-x \right).
\]

(61)

For ease of notation, we write in the following \(\hat{\pi}^B_{\psi}\) for \(\hat{\pi}^B_{\psi}(p)\) and \(\hat{\psi}\) for \(\hat{\psi}_{\psi}(p)\).

**Lemma 7.2.** Let \(p \in \mathcal{P}_r\). There exists a constant \(C > 0\) such that for large \(L\), \(\psi = (m_y) \in \mathcal{M}_L, x, x' \in \{(2/3)L \leq |y| \leq (3/2)L\} \cap \mathbb{Z}^d\) and any \(z, z' \in \mathbb{Z}^d,\)

(i) \(\hat{\pi}_{\psi}(x, z) \leq C L^{-d}\).

(ii) \(\hat{\pi}^B_{\psi}(x, z) \leq C L^{-d}\).

(iii) \(|\hat{\pi}_{\psi}(x, z) - \hat{\pi}_{\psi}(x', z)| \leq C|x-x'| L^{-(d+1)} \log L\).
(iv) \(|\hat{\pi}_\psi(x, z) - \hat{\pi}_\psi(x', z')| \leq C|z - z'|L^{-(d+1)}\log L.\)

(v) \(|\hat{\pi}_B^B(x, z) - \hat{\pi}_B^B(x', z)| \leq C|x - x'|L^{-(d+1)}\log L.\)

(vi) \(|\hat{\pi}_B^B(x, z) - \hat{\pi}_B^B(x', z')| \leq C|z - z'|L^{-(d+1)}\log L.\)

(vii) \(|\hat{\pi}_\psi(x, z) - \hat{\pi}_B^B(x, z)| \leq L^{-(d+1/4)}.\)

**Corollary 7.1.** In the situation of the preceding lemma,

(i) \[
|\hat{\pi}_\psi(x, z) - \hat{\pi}_\psi(x', z')| 
\leq C \min \{|x - x'|L^{-(d+1)}\log L, |x - x'|L^{-(d+1)} + L^{-(d+1/4)}\}.
\]

(ii) \[
|\hat{\pi}_\psi(x, z) - \hat{\pi}_\psi(x, z')| 
\leq C \min \{|z - z'|L^{-(d+1)}\log L, |z - z'|L^{-(d+1)} + L^{-(d+1/4)}\}.
\]

**Remark 7.1.** The condition on \(x\) and \(x'\) in the lemma is only to ensure that both points lie in the domain of \(\psi\).

**Proof.** Combine (iii)-(vii). \(\Box\)

**Proof of Lemma 7.2.** (i) and (ii) follow from the definitions of \(\hat{\pi}_\psi\) and \(\hat{\pi}_B^B\) together with part (i) of Lemma 3.1 and Lemma 3.3, respectively.

(iii), (iv) We can restrict ourselves to the case \(|x - x'| = 1\) as otherwise we take a shortest path connecting \(x\) with \(x'\) inside \(\{(2/3)L \leq |y| \leq (3/2)L\}\) and apply the result for distance one \(O(|x - x'|)\) times. We have

\[
\hat{\pi}_\psi(x, z) - \hat{\pi}_\psi(x', z) = \left(1 - \frac{m_x}{m_{x'}}\right)\hat{\pi}_\psi(x, z) + \frac{1}{m_{x'}} \int_{\mathbb{R}_+} \left(\varphi\left(\frac{t}{m_x}\right) - \varphi\left(\frac{t}{m_{x'}}\right)\right) \pi_{V\psi}(x, z) dt
\]

\[
+ \frac{1}{m_{x'}} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{m_{x'}}\right) \left(\pi_{V\psi}(x, z) - \pi_{V\psi}(x', z)\right) dt
\]

\[
= I_1 + I_2 + I_3.
\]

Using the fact that \(\psi \in \mathcal{M}_L\) and part (i) for \(\hat{\pi}_\psi(x, z)\), it follows that \(|I_1| \leq CL^{-(d+1)}\).

Using additionally the smoothness of \(\varphi\) and, by Lemma 3.1 (i), \(|\pi_{V\psi}(x, z)| \leq CL^{-(d-1)}\), we also have \(|I_2| \leq CL^{-(d+1)}\). It remains to handle \(I_3\). By translation invariance of the random walk, \(\pi_{V\psi}(x, z) = \pi_{V\psi}(0, z - x)\). In particular, both (iii) and (iv) will follow if we prove that

\[
\left|\int_{\mathbb{R}_+} \varphi\left(\frac{t}{m_{x'}}\right) \left(\pi_{V\psi}(0, z - x) - \pi_{V\psi}(0, z - x')\right) dt\right| \leq CL^{-d} \log L \quad (62)
\]
for $x,x'$ with $|x - x'| = 1$. By definition of $\mathcal{M}_L$, $m_x \in (L/10, 5L)$. We may therefore assume that $L/10 < |y - z| < 10L$ for $y = x, x'$. Due to the smoothness of $\varphi$ and the fact that the integral is over an interval of length at most 2, (62) will follow if we show

$$\left| \int_{L/10}^{10L} (\pi_{V_t}(0, z - x) - \pi_{V_t}(0, z - x')) dt \right| \leq CL^{-d} \log L.$$

We set $J = \{ t > 0 : z - x \in \partial V_t \}$ and $J' = \{ t > 0 : z - x' \in \partial V_t \}$, where

$$t' = t'(t) = \left| \frac{(z - x)}{|z - x|} - (x' - x) \right|.$$

$J$ is an interval of length at most 1, and $J'$ has the same length up to order $O(L^{-1})$. Furthermore, $|J \Delta J'|$ is of order $O(L^{-1})$, and $\left| \frac{d}{dt} t' \right| = 1 + O(L^{-1})$. Using that both $\pi_{V_t}(0, z - x)$ and $\pi_{V_t}(0, z - x')$ are of order $O(L^{-(d-1)})$, it therefore suffices to prove

$$\left| \int_{J \cap J'} (\pi_{V_t}(x, z) - \pi_{V_t}(x', z)) dt \right| \leq CL^{-d} \log L. \tag{63}$$

Write $V$ for $V_t(x)$ and $V'$ for $V_t(x')$. By a first exit decomposition,

$$\pi_{V}(x, z) \leq \pi_{V'}(x, z) + \sum_{y \in V \setminus V'} P_{x,y} (T_y < \tau_V) \pi_V(y, z).$$

By Lemma 3.1 (ii), we can replace $\pi_{V'}(x, z)$ by $\pi_{V'}(x', z) + O(L^{-d})$. For $y \in V \setminus V'$ we have by Lemma 3.2 (ii) $\pi_{V}(y, z) = O(|y - z|^{-d})$ and $P_{x,y} (T_y < \tau_V) = O(L^{-(d-1)})$, uniformly in $t \in J \cap J'$. Further, using $|x - x'| = 1$, we have with $r = |z - x|

$$\bigcup_{t \in J \cap J'} (V \setminus V') \subset V_t(x) \setminus V_{r-2}(x') \subset x + Sh_r(3),$$

and for any $y \in Sh_r(3)$, it follows by a geometric consideration that

$$\int_{J \cap J'} 1_{\{ y \in V \setminus V' \}} dt \leq C \frac{|y - z|}{L}.$$

Altogether, applying Lemma 3.6 in the last step,

$$\int_{J \cap J'} \pi_{V}(x, z) dt \leq \int_{J \cap J'} \pi_{V'}(x', z) dt + O(L^{-d}) + CL^{-(d-1)} \sum_{y \in x + Sh_r(3)} \frac{1}{|y - z|^d} \frac{|y - z|}{L} \leq \int_{J \cap J'} \pi_{V'}(x', z) dt + CL^{-d} \log L.$$

The reverse inequality, proved in the same way, then implies (63).

(v) and (vi) are the analogous statements of (iii) and (iv) for Brownian motion with
covariance matrix $\Lambda_q$ and can be proved in the same way.

(vii) Fix $\alpha = 2/3$, $\beta = 1/3$, and let $0 < \eta < 1/40$. Set $A = C_{L,\alpha}(z)$ and $A^z = A \cap \mathbb{Z}^d$. By part (iv), we have

$$\hat{\pi}_\psi(x, z) \leq \frac{1}{|A^z|} \hat{\pi}_\psi(x, A^z) + CL^{-(d+1-\alpha)} \log L.$$  \hfill (64)

Further,

$$\hat{\pi}_\psi(x, A^z) = \frac{1}{m_x} \int_{L/10}^{10L} \varphi \left( \frac{t}{m_x} \right) \pi_{V(x)}(x, A^z) \, dt.$$ \hfill (65)

By Lemma 7.1 (i), it follows that for $t \in (L/10, 10L)$

$$\pi_{V(x)}(x, A^z) \leq \pi_{V(x)}^B(x, A^z) (1 + CL^{-(\beta - 3\eta)}) + CL^{-(d+1)},$$

where $A^\beta = C_{L,\alpha+L,\beta}(z)$ and the constant $C$ is uniform in $t$. If we plug the last line into (65) and use part (ii) and (vi), we arrive at

$$\hat{\pi}_\psi(x, A^z) \leq \hat{\pi}_\psi^B(x, A^\beta) (1 + CL^{-(\beta - 3\eta)}) + CL^{-(d+1)}.$$ \hfill (66)

Notice that in our notation, $|A|$ is the volume of $A$, while $|A^z|$ is the cardinality of $A^z$. From Gauss we have learned that $|A| = |A^z| + O(L^{(d-1)\alpha})$. Going back to (64), this implies

$$\hat{\pi}_\psi(x, z) \leq \hat{\pi}_\psi^B(x, z) + L^{-(d+1/4)},$$

as claimed. To prove the reverse inequality, we can follow the same steps, replacing the random walk estimates by those of Brownian motion and vice versa. \hfill \Box

### 7.2 Proof of Lemma 3.7

For simplicity, let us write $\phi$ for $\phi_{L,\alpha,\psi,q}$ and $\phi^B$ for $\phi_{L,\alpha,\psi,q}^B$.

**Proof of Lemma 3.7** (i) Set $\alpha = 2/3$, $\beta = 1/3$ and $\eta = d(x, \partial V_L)$. Choose $y_1 \in \partial V_L$ such that $|x - y_1| = \eta$. First assume $\eta \leq L^\beta$. The following estimates are valid for $L$ large. First,

$$\phi(x, z) = \sum_{y \in \partial V_L, \, |y - y_1| \leq L^\alpha} \pi_L^{(p)}(x, y)\hat{\pi}_\psi^{(q)}(y, z) + \sum_{y \in \partial V_L, \, |y - y_1| > L^\alpha} \pi_L^{(p)}(x, y)\hat{\pi}_\psi^{(q)}(y, z) = I_1 + I_2.$$

For $I_2$, notice that $|y - y_1| > L^\alpha$ implies $|y - x| > L^\alpha/2$. Using Lemmata 7.2 (i), 3.2 (iii) in the first and Lemma 3.6 in the second inequality, we have

$$I_2 \leq C\eta L^{-d} \sum_{y \in \partial V_L, \, |y - y_1| > L^\alpha} \frac{1}{|x - y|^d} \leq C\eta L^{-(d+\alpha)} \leq L^{-(d+1/4)}.$$ \hfill (66)


For $I_1$, we first use Lemma 7.2 part (iii) to deduce
\[ \hat{\pi}_\psi(y, z) \leq \hat{\pi}_\psi(y_1, z) + CL^{-(d+1)} \log L. \]

Therefore by part (vii),
\[ I_1 \leq \hat{\pi}_\psi(y_1, z) + L^{-(d+1/4)} \leq \hat{\pi}_\psi^B(y_1, z) + 2L^{-(d+1/4)}. \]

A similar argument as in (66), using Lemma 3.3 (i), yields
\[ \int_{y \in \partial C_L| |y-y_1| > L^\alpha} \pi^B_L(x, dy) \leq L^{-1/4}. \]

Using Lemma 7.2 (ii) in the first and (v) in the second inequality, we conclude that
\[ \hat{\pi}_\psi^B(y_1, z) \leq \hat{\pi}_\psi^B(y_1, z) \int_{y \in \partial C_L| |y-y_1| \leq L^\alpha} \pi^B_L(x, dy) + CL^{-(d+1/4)} \leq \phi^B(x, z) + CL^{-(d+1/4)}. \]

Now we look at the case $\eta > L^\beta$. We take a cube $U_1$ of radius $L^\alpha$, centered at $y_1$, and set $W_1 = \partial V_L \cap U_1$. Then we can find a partition of $\partial V_L \backslash W_1$ into disjoint sets $W_i = \partial V_L \cap U_i$, $i = 2, \ldots, k_L$, where $U_i$ is a cube such that for some $c_1, c_2 > 0$ depending only on $d$,
\[ c_1 L^{\alpha(d-1)} \leq |W_i| \leq c_2 L^{\alpha(d-1)}. \]

For $i \geq 2$, we fix an arbitrary $y_i \in W_i$. Let $W_i^\beta = \{ y \in \mathbb{R}^d : d(y, W_i) \leq L^\beta \}$. Applying first Lemma 7.2 (iii) and then Lemma 7.1 (i) gives
\[ \phi(x, z) \leq \sum_{i=1}^{k_L} \pi^B_L(x, W_i) \hat{\pi}_\psi^B(y_i, z) + L^{-(d+1/4)} \leq \sum_{i=1}^{k_L} \pi^B_L(x, W_i^\beta) \hat{\pi}_\psi^B(y_i, z) \left(1 + L^{-1/4}\right) + L^{-(d+1/4)}. \]

As the $W_i^\beta$ overlap, we refine them as follows: Set $\hat{W}_1 = W_1^\beta \cap \partial C_L$, and split $\partial C_L \backslash \hat{W}_1$ into a collection of disjoint measurable sets $\hat{W}_i \subset \partial C_L \cap W_i^\beta$, $i = 2, \ldots, k_L$, such that $\bigcup_{i=1}^{k_L} \hat{W}_i = \partial C_L$ and $|\hat{W}_i| \leq c_1 L^{\alpha(d-2)+\beta}$ for some $C_1 = C_1(d)$. By construction we can find constants $c_3, c_4 > 0$ such that $|\hat{W}_i| \geq c_3 L^{\alpha(d-1)}$ and, for $i = 2, \ldots, k_L$,\[ \inf_{y \in W_i} |x-y| \geq c_4 \sup_{y \in \hat{W}_i} |x-y|, \]
which implies by Lemma 3.3 (i) that
\[
\sup_{y \in W_i} \pi_{L}^{B(p)}(x, y) \leq c_4^{-1} \inf_{y \in W_i} \pi_{L}^{B(p)}(x, y).
\]
For \(i = 1, \ldots, k_L\) we then have
\[
\pi_{L}^{B(p)}(x, W_i^\beta) \leq \pi_{L}^{B(p)}(x, \tilde{W}_i) \left(1 + C_1c_3^{-1}L^{\beta-a}\right) \leq \pi_{L}^{B(p)}(x, \tilde{W}_i) \left(1 + L^{-1/4}\right).
\]
Plugging the last line into (67),
\[
\phi(x, z) \leq \sum_{i=1}^{k_L} \pi_{L}^{B(p)}(x, \tilde{W}_i)^{(q)}(y, z) \left(1 + L^{-1/4}\right) + L^{-(d+1/4)}.
\]
A reapplication of Lemma 7.2 (iii), (vii) and then (ii) yields
\[
\phi(x, z) \leq \sum_{i=1}^{k_L} \int_{\tilde{W}_i} \pi_{L}^{B(p)}(x, dy) \pi_{\psi}^{(q)}(y, z) + L^{-(d+1/4)}
\]
\[
\leq \sum_{i=1}^{k_L} \int_{\tilde{W}_i} \pi_{L}^{B(p)}(x, dy) \pi_{\psi}^{(q)}(y, z) \left(1 + L^{-1/4}\right) + L^{-(d+1/4)}
\]
\[
= \phi_{L}^{B}(x, z) + CL^{-(d+1/4)}.
\]
The reverse inequality in both the cases \(\eta \leq L^\beta\) and \(\eta > L^\beta\) is obtained similarly.

(ii) Let \(\psi = (m_y)_y \in M_L\) and \(z \in \mathbb{Z}^d\). For \(y \in \mathbb{R}^d\) with \(L/2 < |y| < 2L\) we set
\[
g(y, z) = \frac{1}{m_y} \varphi \left(\frac{|z-y|}{m_y}\right) \pi_{L}^{B(p)}(0, z-y).
\]
(68)

Then
\[
\phi_{L}^{B}(x, z) = \int_{\partial C_L} \pi_{L}^{B(p)}(x, dy) g(y, z).
\]
Choose a cutoff function \(\chi \in C^\infty(\mathbb{R}^d)\) with compact support in \(\{x \in \mathbb{R}^d : 1/2 < |x| < 2\}\) such that \(\chi \equiv 1\) on \(\{2/3 \leq |x| \leq 3/2\}\). Setting \(m_v = 1\) for \(v \notin \{L/2 < |x| < 2L\}\), we define
\[
\tilde{g}(y, z) = g(Ly, z) \chi(y), \quad y \in \mathbb{R}^d.
\]
By Brownian scaling,
\[
\tilde{g}(y, z) = \frac{1}{m_{Ly}} \varphi \left(\frac{|z-Ly|}{m_{Ly}}\right) \frac{1}{|z-Ly|^{d-1}} \pi_{C_L}^{B(p)} \left(0, \frac{z-y}{|z-y|}\right) \chi(y).
\]
Notice that \(\tilde{g}(\cdot, z) \in C^4(\mathbb{R}^d)\), with \(\tilde{g}(y, z) = 0\) if \(|z-Ly| \notin (L/5, 10L)\) or \(|y| \notin (1/2, 2)\). Let \(\mathcal{L} = 2dp(e_1)\partial^2/\partial x_1^2 + \ldots + 2dp(e_d)\partial^2/\partial x_d^2\). Then \(u(\bar{x}, z) = \phi_{L}^{B}(x, z), \quad x = L\bar{x}\), solves
\[
\left\{ \begin{array}{l}
\mathcal{L}u(\cdot, z) = 0, \quad \text{in } C_1 \\
u(\cdot, z) = \tilde{g}(\cdot, z), \quad \text{on } \partial C_1.
\end{array} \right.
\]
By Corollary 6.5.4 of Krylov [14], \( u(\cdot, z) \) is smooth on \( C_1 \). Write

\[
|u(\cdot, z)|_k = \sum_{i=0}^{k} \left| \frac{\partial^i u(\cdot, z)}{\partial \cdot^i} \right|_{C_1}.
\]

By Theorem 6.3.2 in the same book, there exists \( C > 0 \) independent of \( z \) such that

\[
|u(\cdot, z)|_3 \leq C|\tilde{g}(\cdot, z)|_4.
\]

A direct calculation shows that \( \sup_{z \in \mathbb{R}^d} |\tilde{g}(\cdot, z)|_4 \leq CL^{-d} \). Now the claim follows from

\[
||D^{i} \phi|_{B(\cdots, z)}||_{C_1} = L^{-i}||D^{i} u(\cdot, z)||_{C_1}.
\]

(iii) Let \( x, x' \in V_L \cup \partial V_L \). Choose \( \tilde{x} \in V_L \) next to \( x \) and \( \tilde{x}' \in V_L \) next to \( x' \). Then \( |\tilde{x} - x| = 1 \) if \( x \in \partial V_L \) and \( \tilde{x} = x \) otherwise. By the triangle inequality,

\[
|\phi(x, z) - \phi(x', z)| \leq |\phi(x, z) - \phi(\tilde{x}, z)| + |\phi(\tilde{x}, z) - \phi(\tilde{x}', z)| + |\phi(\tilde{x}', z) - \phi(x', z)|. \tag{69}
\]

By parts (i) and (ii) combined with the mean value theorem, we get for the middle term

\[
|\phi(\tilde{x}, z) - \phi(\tilde{x}', z)| \leq |\phi(\tilde{x}, z) - \phi^{B}(\tilde{x}, z)| + |\phi^{B}(\tilde{x}, z) - \phi^{B}(\tilde{x}', z)| + |\phi^{B}(\tilde{x}', z) - \phi(x', z)|
\]

\[
\leq C \left( L^{-(d+1/4)} + |x - x'| L^{-(d+1)} \right).
\]

If \( x \in \partial V_L \), then \( \phi(x, z) = \hat{\pi}_\psi(x, z) \), so that we can write the first term of (69) as

\[
|\phi(x, z) - \phi(\tilde{x}, z)| = \left| \sum_{y \in \partial V_L} \pi_L(\tilde{x}, y) \left( \hat{\pi}_\psi(y, z) - \hat{\pi}_\psi(x, z) \right) \right|.
\]

Set \( A = \{ y \in \partial V_L : |x - y| > L^{1/4} \} \). Then by Lemmata 3.2 (iii) and 3.6

\[
\pi_L(\tilde{x}, A) \leq C \sum_{y \in A} \frac{1}{|x - y|^d} \leq CL^{-1/4}.
\]

For all \( y \in \partial V_L \), we have by Lemma 7.2 (i) \( |\hat{\pi}_\psi(y, z) - \hat{\pi}_\psi(x, z)| \leq CL^{-d} \). If \( y \in \partial V_L \setminus A \), then part (iii) gives \( |\hat{\pi}_\psi(y, z) - \hat{\pi}_\psi(x, z)| \leq CL^{-(d+3/4)} \log L \). Altogether,

\[
|\phi(x, z) - \phi(\tilde{x}, z)| \leq CL^{-(d+1/4)}.
\]

The third term of (69) is treated in exactly the same way. \( \Box \)
7.3 Proofs of Lemmata 3.1 and 3.2

We let $V_L^{(p)}(y) = \{ x \in \mathbb{Z}^d : |\Lambda_p^{-1/2} (x-y)| \leq L \}$, and $V_L^{(p)} = V_L^{(p)}(0)$. Note that

$$V_{(1+2dk)-1/2L} \subset V_L \subset V_{(1-2dk)-1/2L}. \quad (70)$$

We will make use of

Lemma 7.3.

(i) Let $0 < l < L$ and $x \in V_L^{(p)}$ with $l < |\Lambda_p^{-1/2}x| < L$. Then

$$P_{x,p} \left( \tau_{V_L^{(p)}} < T_{V_L^{(p)}} \right) = \frac{l^{-d+2} - |\Lambda_p^{-1/2}x|^{-d+2} + O(l^{-d+1})}{l^{-d+2} - L^{-d+2}}.$$

(ii) There exists $C > 0$ such that for all $\theta \in \mathbb{R}^d$ with $|\theta| = 1$ and $l \geq 0$,

$$P_{0,p}(X_n \cdot \theta \geq -l \text{ for all } 0 \leq n \leq \tau_L) \leq C (l+1)L^{-1}.$$

Proof: (i) For the case of simple random walk, that is $p = p_o$, this is Proposition 1.5.10 of [16]. For the case of general $p \in \mathcal{P}_\eta$, one can use Proposition 6.3.1 of [17] for the Green’s function, and then the proof is exactly the same.

(ii) This is a version of the gambler’s ruin estimate, see for example [17], Exercise 7.5.

We turn to the proof of Lemma 3.1. In the following, we write $P_x$ for $P_{x,p}$.

Proof of Lemma 3.1: (i) $\pi_L(\cdot, z)$ is $p$-harmonic inside $V_L$, i.e. for $x \in V_L$,

$$\pi_L(x,z) = \sum_{e \in \mathbb{Z}^d : |e| = 1} p(e)\pi_L(x+e,z).$$

Applying a discrete Harnack inequality, as, for example, provided by Theorem 6.3.9 in the book of Lawler and Limic [17], we obtain $C^{-1} \pi_L(0, z) \leq \pi_L(\cdot, z) \leq C \pi_L(0, z)$ on $V_n L$, for some $C = C(d, \eta)$, and it remains to show that $\pi_L(0, z)$ has the right order of magnitude. Note that we cannot directly apply Lemma 6.3.7 in the same book, since we look at the exit distribution from $V_L$, not from $V_L^{(p)}$. However, by a last-exit decomposition as in Lemma 6.3.7, with $g_{V_L}(x,y) = \sum_{k=0}^{\infty} (1_{V_L}^k)(x,y)$ and $\tilde{\tau}_A = \inf\{ n \geq 1 : X_n \notin A \}$,

$$\pi_L(0,z) = \sum_{y \in V_{L/2}}^{} g_{V_L}(0,y) P_z \left( X_{\tilde{\tau}_{V_L \setminus V_{L/2}}} = y \right).$$

Using (70), we have for $y \in V_{L/2}$, putting $L_1 = (1+2dk)^{-1/2}L$ and $L_2 = (1-2dk)^{-1/2}L$,

$$g_{V_{L_1}}(0,y) \leq g_{V_{L/2}}(0,y) \leq g_{V_{L_2}}(0,y).$$

For $y \in V_{L/2} \cap \partial(V_L \setminus V_{L/2})$ both outer Green’s functions are of order $L^{-d+2}$, by Proposition 6.3.5 of [17]. Now by Lemma 7.3 (ii)

$$P_z \left( X_{\tilde{\tau}_{V_L \setminus V_{L/2}}} \in V_{L/2} \right) \leq P_0 \left( X_n \cdot \theta \geq 0 \text{ for all } 0 \leq n \leq \tau_{L/2} \right) \leq CL^{-1},$$

7.3 Proofs of Lemmata [3.1] and [3.2] 69
which proves that $\pi_L(0,z) \leq C L^{-d+1}$. For the lower bound, if $\kappa$ is small enough, we find an ellipsoid $V_L^{(p)}(y)$ with $L_3 \geq (9/10)L$, centered at some $y \in V_L$ and lying completely inside $V_L$ such that $z \in \partial V_L^{(p)}(y) \cap \partial V_L$. Also, $V_{L_3/5}(y) \subset V_{L/2}$ if $\kappa$ is small. Therefore,

$$P_{\pi_L}(X_{\tau_L} \in V_{L/2}) \geq P_{\pi_L}(X_{\tau_{L_3/5}}(y) \in V_{L_3/5}(y)) .$$

With positive probability, the random walk starting at $z$ enters $V_{L_3}^{(p)}(y)$ in the next step and then visits a point $w$ with $d_L(w) \geq 1$, staying inside $V_{L_3}^{(p)}(y)$. Thus

$$P_{\pi_L}(X_{\tau_{L_3/5}}(y) \in V_{L_3/5}(y)) \geq c P_w \left( T_{V_{L_3/5}(y)} < \tau_{V_{L_3}^{(p)}(y)} \right) \geq c L^{-1},$$

where the last step follows from bounding the expression obtained in Lemma 7.3(i). With the estimate on $g_{\kappa L}$, this proves that $\pi_L(0,z)$ is bounded from below by $C^{-1} L^{-d+1}$, and (i) follows.

(ii) By the triangle inequality,

$$|\pi_L(x,z) - \pi_L(x',z)| \leq C \max_{u,v \in V_{\kappa L}; |u-v| \leq 1} |\pi_L(u,z) - \pi_L(v,z)| .$$

For $u \in V_{\kappa L}$, the function $\pi_L(u,\cdot,z)$ is $p$-harmonic inside $V_{(1-\kappa)L}$. The claim now follows from 177 Theorem 6.3.8, (6.19), together with (i).

Before we start with the proof of Lemma 7.2, we prove a further auxiliary lemma, which already includes the upper bound of part (iii).

**Lemma 7.4.** Let $x \in V_L$, $y \in \partial V_L$, and set $t = |x-y|$. Then

(i) 
\[ P_x(X_{\tau_L} = y) \leq C d_L(x)^{-d+1} . \]

(ii) 
\[ P_x(X_{\tau_L} = y) \leq C \max \left\{ 1, d_L(x) \right\} \max_{x' \in \partial V_{L/3}(y) \cap V_L} P_{x'}(X_{\tau_L} = y) . \]

(iii) 
\[ P_x(X_{\tau_L} = y) \leq C \max \left\{ 1, d_L(x) \right\} \frac{1}{|x-y|^d} . \]

**Proof:** (i) We can assume that $s = d_L(x) \geq 6$. If $s' = \lfloor s/3 \rfloor$, then $\partial V_{s'}(x) \subset V_{L-s'}$. Using Lemma 7.3(ii), we compute for any $y' \in V_L$ with $|y-y'| = 1$, $\theta = -y'/|y'|$,

\[ P_{y'}(T_{\partial V_{s'}(x)} < \tau_L) \leq P_0(X_n \cdot \theta \geq -1 \text{ for all } 0 \leq n \leq \tau_{V_{s'}}) \leq C s^{-1} . \]
By Lemma 3.2 (i) it follows that uniformly in \( z \in \partial V_{x'}(x) \),
\[
P_z(T_x < \tau_L) \leq P_z(T_x < \infty) \leq C(s')^{-d+2} \leq C_{S^{-d+2}}.
\]
Thus, by the strong Markov property at \( T_{\partial V_{x'}(x)} \),
\[
P_{y'}(T_x < \tau_L) \leq C S^{-d+1}.
\]
Since by time reversibility of symmetric random walk
\[
P_x(X_{\tau_L} = y) = \sum_{y' \in V_L, \ |y'-y|=1} P_x(X_{\tau_L} = y, X_{\tau_L-1} = y') = \frac{1}{2d} \sum_{y' \in V_L, \ |y'-y|=1} P_{y'}(T_x < \tau_L),
\]
the claim is proved.

(ii) We may assume that \( t = |x-y| > 100d \) and \( d_L(x) < t/100 \). Choose a point \( x' \) outside \( V_L \) such that \( V_{t/10}(x') \cap V_L = \emptyset \) and \( |x-x'| \leq d_L(x) + t/10 + \sqrt{d} \). Then \( |x-x'| \leq t/5 \). Furthermore, since \( |x'-y| \geq 4t/5 \),
\[
(V_{t/4}(x') \cup \partial V_{t/4}(x')) \cap V_{t/3}(y) = \emptyset.
\]
We apply twice the strong Markov property and obtain
\[
P_x(X_{\tau_L} = y) \leq P_x(\tau_{V_{t/4}(x')} < \tau_L) \max_{z \in \partial V_{t/3}(y) \cap V_L} P_z(X_{\tau_L} = y).
\]
Arguing much as in (i), Lemma 7.3 (ii) shows
\[
P_x(\tau_{V_{t/4}(x')} < \tau_L) \leq C \max \{ \frac{1}{t}, d_L(x) \},
\]
which concludes the proof of part (ii).

(iii) By (ii) it suffices to prove that for some constant \( K \) and for all \( l \geq 1 \)
\[
\max_{z \in \partial V_{t/3}(y) \cap V_L} P_z(X_{\tau_L} = y) \leq Kl^{-d+1}.
\] (71)
Let \( c_1 \) and \( c_2 \) be the constants from (i) and (ii), respectively. Define \( \eta = 3^{-d} c_2^{-1} \) and
\[
K = \max \{ 3^{d(d-1)} c_2^{-d-1}, c_1 \eta^{-d+1} \}.
\]
For \( l \leq 3d c_2 \) there is nothing to prove since \( Kl^{-d+1} \geq 1 \). Thus let \( l > 3d c_2 \), and choose \( l_0 \) with \( l_0 < l \leq 2l_0 \). Assume that (71) is proved for all \( l' \leq l_0 \). We show that (71) also holds for \( l \). For \( z \) with \( d_L(z) \geq \eta l \), it follows from (i) that
\[
P_z(X_{\tau_L} = y) \leq c_1 \eta^{-d+1} l^{-d+1} \leq Kl^{-d+1}.
\]
If \( 1 \leq d_L(z) < \eta l \), then by (ii) and the fact that \( l/3 \leq l_0 \)
\[
P_z(X_{\tau_L} = y) \leq c_2 \max \{ \frac{1}{|z-y|}, d_L(z) \} \max_{z' \in \partial V_{t/9}(y) \cap V_L} P_z(X_{\tau_L} = y)
\]
\[
\leq c_2 3 \eta K (l/3)^{-d+1} \leq Kl^{-d+1}.
\]

7.3 Proofs of Lemmata 3.1 and 3.2
If \( d_L(z) < 1 \), then again by (i)
\[
P_z(X_{\tau_L} = y) \leq c_2 3l^{-1} K (l/3)^{-d+1} \leq Kl^{-d+1}.
\]
This proves the claim. \( \square \)

**Proof of Lemma 3.2:** (i) follows from Proposition 6.4.2 of [17].

(ii) We consider different cases. If \(|x - y| \leq d_L(y)/2\), then \( d_L(x) \geq d_L(y)/2 \) and thus by Lemma 3.2 (i)
\[
P_x \left( T_{V_a(y)} < \tau_L \right) \leq P_x \left( T_{V_a(y)} < \tau_L \right) \leq C \left( \frac{a}{|x - y|} \right)^{d-2} \leq C \frac{a^{d-2} d_L(y) d_L(x)}{|x - y|^d}.
\]

For the rest of the proof we assume that \(|x - y| > d_L(y)/2\). Set \( a' = d_L(y)/5 \). First we argue that in the case \( 1 \leq a \leq a' \), we only have to prove the bound for \( a' \). Indeed, if \( d_L(y)/6 \leq a < a' \), we get an upper bound by replacing \( a \) by \( a' \). For \( 1 \leq a < d_L(y)/6 \), the strong Markov property together with Lemma 3.2 (i) yields
\[
P_x \left( T_{V_a(y)} < \tau_L \right) \leq \max_{z \in \partial(\mathbb{Z}^d \setminus \mathcal{V}_a(y))} P_z \left( T_{V_a(y)} < \tau_L \right) \leq C \left( \frac{a}{a' - 1} \right)^{d-2} \frac{(a')^{d-2} d_L(y) \max\{1, d_L(x)\}}{|x - y|^d} \leq C \frac{a^{d-2} d_L(y) \max\{1, d_L(x)\}}{|x - y|^d}.
\]

Now we prove the claim for \( a = d_L(y)/5 \). We take a point \( y' \in \partial V_L \) closest to \( y \). If \(|x - z| \geq |x - y|/2\) for all \( z \in V_a(y') \), then by Lemma 7.4 (iii)
\[
\max_{z \in V_a(y')} P_x (X_{\tau_L} = z) \leq C 2^d \frac{\max\{1, d_L(x)\}}{|x - y|^d}.
\]

As a subset of \( \mathbb{Z}^d \), \( V_a(y') \cap \partial V_L \) contains on the order of \( d_L(y)^{d-1} \) points. Therefore, by Lemma 3.1 (i), we deduce that there exists some \( \delta > 0 \) such that
\[
\min_{x' \in V_a(y)} P_{x'} (X_{\tau_L} \in V_a(y')) \geq \delta.
\]

We conclude that
\[
\frac{a^{d-1} \max\{1, d_L(x)\}}{|x - y|^d} \geq c \min_{x' \in V_a(y)} P_{x'} (X_{\tau_L} \in V_a(y')) \geq c P_x (X_{\tau_L} \in V_a(y'), T_{V_a(y)} < \tau_L)
\]
\[
= c \sum_{x' \in V_a(y)} P_x (X_{T_{V_a(y)}} = x', T_{V_a(y)} < \tau_L) P_{x'} (X_{\tau_L} \in V_a(y'))
\]
\[
\geq c \delta \cdot P_x (T_{V_a(y)} < \tau_L).
\]

On the other hand, if \(|x - z| < |x - y|/2\) for some \( z \in V_a(y') \), then
\[
|x - y| \leq |x - z| + |z - y'| + |y' - y| \leq 2 d_L(y) + |x - y|/2
\]
and thus
\[ d_L(y)/2 < |x - y| \leq 4 d_L(y). \] (73)

If \( d_L(x) \geq 4 d_L(y)/5 \), we use Lemma 3.2 (i) again. For \( d_L(x) < 4 d_L(y)/5 \), we get by Lemma 7.3 (ii)
\[
P_x \left( T_{V_{L-4d_L(y)/5}} < \tau_L \right) \leq C \frac{\max\{1, d_L(x)\}}{d_L(y)}.
\]

Together with (73), this proves the claim in this case. Altogether, we have proved the bound for \( 1 \leq a \leq d_L(y)/5 \). It remains to handle the case \( \max\{1, d_L(y)/5\} \leq a \). If \( z \in V_{6a}(y) \), we have that
\[ |x - y| \leq |x - z| + 6a \]
and thus, using \( |x - y| > 7a \),
\[ |x - y| \leq 7|x - z|. \]

Therefore Lemma 7.4 (iii) yields
\[
\max_{z \in V_{6a}(y)} P_x (X_{\tau_L} = z) \leq C \frac{\max\{1, d_L(x)\}}{|x - z|^d} \leq 7^d C \frac{\max\{1, d_L(x)\}}{|x - y|^d}.
\]

Again by Lemma 3.1 (i), we find some \( \delta > 0 \) such that
\[
\min_{x' \in V_{a}(y)} P_{x'} (X_{\tau_L} \in V_{6a}(y)) \geq \delta.
\]

A similar argument as in (72), with \( V_a(y') \) replaced by \( V_{6a}(y) \), finishes the proof of (ii).

(iii) It only remains to prove the lower bound. Let \( t = |x - y| \). First assume \( t \geq L/2 \).

By replacing \( V_L \) and \( V_{2L/3} \) by appropriate ellipsoids as in the proof of the lower bound of Lemma 3.1 (i), we deduce with part (i) of Lemma 7.3 that
\[
P_x \left( T_{V_{2L/3}} < \tau_L \right) \geq c \frac{d_L(x)}{t}.
\]

The claim then follows from the strong Markov property and Lemma 3.1 (i). Now assume \( t < L/2 \).

Let \( x' \in V_L \) such that \( V_t(x') \subset V_L \) and \( y \in \partial V_t(x') \). If \( d_L(x) > t/2 \), there is by Lemma 3.1 (i) a strictly positive probability to exit the ball \( V_{t/2}(x) \) within \( V_{2t/3}(x') \). Since by the same lemma,
\[
\inf_{z \in V_{2t/3}(x')} P_z (\tau_L = y) \geq ct^{-(d-1)},
\] (74)
we obtain the claim in this case again by applying the strong Markov property. Finally, assume \( d_L(x) \leq t/2 \). Once more by Lemma 7.3 (i),
\[
P_x \left( T_{V_{L-2t/3}} < \tau_L \right) \geq c \frac{d_L(x)}{t},
\]
and
\[
\mathbb{P}_x (\tau_L = y) \geq \mathbb{P}_x \left( \tau_L = y, T_{L-2t/3} < \tau_L, T_{V_{2t/3}(x')} < \tau_L \right) \\
\geq \frac{c}{t} \mathbb{P}_x \left( \tau_L = y | T_{L-2t/3} < \tau_L, T_{V_{2t/3}(x')} < \tau_L \right) \\
\times \mathbb{P}_x \left( T_{V_{2t/3}(x')} < \tau_L | T_{L-2t/3} < \tau_L \right).
\]

By a simple geometric consideration and again Lemma 3.1 (i), the second probability on the right side is bounded from below by some \( \delta > 0 \), and the first probability has already been estimated in (74).

### 7.4 Proofs of Lemmata 3.3 and 3.4

**Proof of Lemma 3.3** By Brownian scaling, we may restrict ourselves to the case \( L = 1 \). Let
\[
E_p = \{ y \in \mathbb{R}^d : |\Lambda_{p}^{1/2}y| < 1 \},
\]
and choose \( x \in C_1 \), \( z \in \partial C_1 \). Set \( x' = \Lambda_{p}^{-1/2}y \), \( x'' = \Lambda_{p}^{-1/2}x \) and \( z' = \Lambda_{p}^{-1/2}z \), \( z'' = \Lambda_{q}^{-1/2}z \). If \( \kappa \) is small, both \( x' \) and \( x'' \) are contained in \( C_{3/4} \subset C_{4/5} \subset E_p \cap E_q \).

For the rest of the proof, we write \( \pi_{E_p}^B (x, \cdot) \) instead of \( \pi_{E_p}^{B(\gamma)} (x, \cdot) \), and similarly \( \pi_{E_q}^B \) for \( \pi_{E_q}^{B(\gamma)} \). If \( \gamma \) is a parametrization of the unit sphere \( \partial C_1 \), then \( \Lambda_{p}^{-1/2} \circ \gamma \) and \( \Lambda_{q}^{-1/2} \circ \gamma \) are parametrizations of \( E_p \) and \( E_q \), respectively. Since the coefficients of the covariance matrices satisfy \((\Lambda_{p})_{i,j} = (\Lambda_{q})_{i,j} + O(\eta)\), we obtain by Brownian scaling
\[
|\pi_{C_1}^{B(p)} (x, z) - \pi_{C_1}^{B(q)} (x, z)| \leq C |\pi_{E_p}^B (x', z') - \pi_{E_q}^B (x'', z'')| + O(\eta).
\]

Clearly
\[
|z' - z''| = |(\Lambda_{p}^{-1/2} - \Lambda_{q}^{-1/2}) z| \leq C\eta,
\]
and also \( |x' - x''| \leq C\eta \). By the derivative estimate of Lemma 3.3 (ii),
\[
|\pi_{E_p}^B (x', z'') - \pi_{E_q}^B (x'', z'')| \leq C\eta.
\]

With (76), the claim will therefore follow if we show that
\[
|\pi_{E_p}^B (x, z') - \pi_{E_q}^B (x, z'')| \leq C\eta.
\]

**Proof of Lemma 3.4** We can assume \( L = 1 \). Let \( \eta = ||q - p||_1 \). Define \( E_p \) as in (75), and similarly \( E_q \). Let \( x \in C_{2/3} \), \( z \in \partial C_1 \), and put \( x' = \Lambda_{p}^{-1/2}x \), \( x'' = \Lambda_{q}^{-1/2}x \) and \( z' = \Lambda_{p}^{-1/2}z \), \( z'' = \Lambda_{q}^{-1/2}z \). If \( \kappa \) is small, both \( x' \) and \( x'' \) are contained in \( C_{3/4} \subset C_{4/5} \subset E_p \cap E_q \).
uniformly in \( x \in C_{3/4}, \ z' \in \partial E_p, \ z'' \in \partial E_q \) with \( |z' - z''| \leq C\eta \). In this direction, recall that the Green’s function on \( \mathbb{R}^d \) of standard \( d \)-dimensional Brownian motion is given by

\[
\Phi(x, y) = \frac{c_d}{|x - y|^{d-2}},
\]

where \( c_d = \Gamma(d/2 - 1)/(2\pi^{d/2}) \) (cf. [17], p. 241). The Green’s function of standard Brownian motion killed outside \( E_p \) is given by (see e.g. Evans [10])

\[
\Phi(x, y) - \Phi_x^{(p)}(y), \quad x, y \in E_p, \ x \neq y,
\]

where the corrector function \( \Phi_x^{(p)} \) solves the Dirichlet problem \((x \text{ is fixed})\)

\[
\begin{cases}
\Delta \Phi_x^{(p)} = 0, & \text{in } E_p \\
\Phi_x^{(p)}(\cdot) = \Phi(x, \cdot), & \text{on } \partial E_p.
\end{cases}
\] (78)

Furthermore, the density \( \pi_{E_p}^B(x, z') \) with respect to surface measure on \( \partial E_p \) is the normal derivative of the Green’s function in the direction of the inward unit normal vector \( \nu_p = \nu_p(z') \) on \( \partial E_p \), i.e.

\[
\pi_{E_p}^B(x, z') = \partial_{z'} \left( \Phi(x, z') - \Phi_x^{(p)}(z') \right) = \nabla_{z'} \left( \Phi(x, z') - \Phi_x^{(p)}(z') \right) \cdot \nu_p(z'), \quad z' \in \partial E_p.
\]

With \( \nu_q \) denoting the inward unit normal on \( \partial E_q \), we therefore have to show that

\[
|\partial_{z'} \left( \Phi(x, z') - \Phi_x^{(p)}(z') \right) - \partial_{z''} \left( \Phi(x, z'') - \Phi_x^{(q)}(z'') \right)| \leq C\eta
\]

uniformly in \( x \in C_{3/4}, \ z' \in \partial E_p, \ z'' \in \partial E_q \) with \( |z' - z''| \leq C\eta \). First, note that

\[
|\partial_{z'} \Phi(x, z') - \partial_{z''} \Phi(x, z'')| \\
\leq |\nabla_{z'} \Phi(x, z') - \nabla_{z''} \Phi(x, z'')| + |\nabla_{z'} \Phi(x, z') \cdot (\nu_p(z') - \nu_q(z''))|.
\]

Using (77) and

\[
\partial_{z'} \Phi(x, z') = c_d(2 - d) \frac{(x - z')_i}{|x - z'|^d},
\]
we easily obtain
\[ |\nabla_{z'} \Phi(x, z') - \nabla_{z''} \Phi(x, z'')| \leq C \eta. \]
Moreover, with \( \Lambda = \Lambda_q^{-1/2} \Lambda_p^{1/2} \), we have
\[ \nu_q(z'') = \frac{\Lambda^{-1} \nu_p(z')}{|\Lambda^{-1} \nu_p(z')|.} \]
The coefficients of the diagonal matrix \( \Lambda \) are of order \( 1 + O(\eta) \), which shows
\[ |\nabla_{z'} \Phi(x, z') \cdot (\nu_p(z') - \nu_q(z''))| \leq C |\nu_p(z') - \nu_q(z'')| \leq C \eta. \quad (79) \]

It remains to prove that
\[ |\partial_{\nu_q} \Phi^{(p)}_x (z') - \partial_{\nu_q} \Phi^{(q)}_x (z'')| \leq C \eta. \quad (80) \]

First recall that \( \Phi^{(p)}_x \) solves the Dirichlet problem \([78]\). The boundary function \( \Phi(x, \cdot) \) is smooth in a tubular neighborhood of the boundary, with bounded derivatives up to arbitrary order. By multiplying with an appropriate smooth cutoff function equals 1 near the boundary, we obtain a smooth function in \( \mathbb{R}^d \). By Corollary 6.5.4 of Krylov \([14]\), we see that \( \Phi^{(p)}_x \) is a smooth function in \( E_p \), and similarly \( \Phi^{(q)}_x \) is smooth in \( E_q \).

Now, with \( \Lambda \) as before, we have equality of the sets \( \Lambda E_p = E_q \). Fix \( x \in C_{3/4} \subset E_p \cap E_q \) and let
\[ u(y) = \Phi^{(q)}_x (\Lambda y) - \Phi^{(p)}_x (y), \quad y \in E_p. \]
With \( f(\cdot) = \Delta_y \Phi^{(q)}_x (\Lambda \cdot) \) and \( g(\cdot) = \Phi(x, \Lambda \cdot) - \Phi(x, \cdot) \), \( u \) solves
\[
\begin{cases}
\Delta u = f, & \text{in } E_p \\
u = g, & \text{on } \partial E_p .
\end{cases}
\]
Recalling that the coefficients of \( \Lambda \) are of order \( 1 + O(\eta) \), we use harmonicity of \( \Phi^{(q)}_x \) and boundedness of the derivatives to obtain
\[ ||D^0 f||_{E_p} + ||D^1 f||_{E_p} \leq C \eta. \]
The function \( g \) is smooth in a tubular neighborhood \( U \) of \( \partial E_p \), and a similar (explicit) calculation as above gives
\[ \sum_{i=0}^{3} ||D^i g||_U \leq C \eta. \]
We extend \( g \) to the interior of \( E_p \) such that \( \sum_{i=0}^{3} ||D^i g||_{E_p} \leq C \eta \). Then, applying a Schauder estimate as given by Theorem 6.3.2 of Krylov \([14]\), we deduce that the derivatives of \( u \) up to order 2 are uniformly bounded by \( C \eta \). But, similar as above \([79]\),
\[
|\partial_{\nu_q} \Phi^{(p)}_x (z') - \partial_{\nu_q} \Phi^{(q)}_x (z'')| \leq |\nabla_{z'} \Phi^{(p)}_x (z') - \nabla_{z''} \Phi^{(q)}_x (z'')| + C \eta
\leq |\nabla_{z'} u(z')| + C \eta
\leq C \eta,
\]
where in the next to last step we used \( z'' = \Lambda z' \) and \( |\nabla_{z''} \Phi^{(q)}_x (z'') - \nabla_{z'} \Phi^{(q)}_x (z'')| \leq C \eta. \) \( \square \)
7.5 Proofs of Propositions 4.1 and 4.2

Since \( \hat{\pi}_m(x, y) = \hat{\pi}_m(0, y - x) \), it suffices to look at \( \hat{\pi}_m(x) = \hat{\pi}_m(0, x) \) and \( \hat{g}_{m, z_d}(x) = \hat{g}_{m, z_d}(0, x) \). Recall the definitions of \( \lambda_{m,i} \) and \( \Lambda_m \) from Section 4.1.

**Proof of Proposition 4.1.** For bounded \( m \), that is \( m \leq m_0 \) for some \( m_0 \), the result is a special case of Theorem 2.1.1 in [17]. Also, for \( n \leq n_0 \) and all \( m \), the statement follows from Lemma 7.2 (i). We therefore have to prove the proposition only for large \( n \) and \( m \). To this end, let

\[
B_m = [-\sqrt{\lambda_{m,1}} \pi, \sqrt{\lambda_{m,1}} \pi] \times \cdots \times [-\sqrt{\lambda_{m,d}} \pi, \sqrt{\lambda_{m,d}} \pi],
\]

and for \( \theta \in B_m \) set

\[
\phi_m(\theta) = \sum_{y \in \mathbb{Z}^d} e^{i\theta \Lambda_m^{-1/2} y} \hat{\pi}_m(y).
\]

The Fourier inversion formula gives

\[
\hat{\pi}_m^n(x) = \frac{1}{(2\pi)^d \det \Lambda_m^{1/2}} \int_{B_m} e^{-ix \cdot \Lambda_m^{-1/2} \theta} [\phi_m(\theta)]^n d\theta.
\]

We decompose the integral into

\[
(2\pi)^d \det \Lambda_m^{1/2} n^{d/2} \hat{\pi}_m^n(x) = I_0(n, m, x) + \ldots + I_3(n, m, x),
\]

where, with \( \beta = \sqrt{n} \theta \),

\[
I_0(n, m, x) = \int_{\mathbb{R}^d} e^{-ix \cdot \Lambda_m^{-1/2} \beta / \sqrt{n}} e^{-|\beta|^2 / 2} d\beta,
\]

\[
I_1(n, m, x) = \int_{|\beta| \leq n^{1/4}} e^{-ix \cdot \Lambda_m^{-1/2} / \sqrt{n}} \left( [\phi_m(\beta / \sqrt{n})]^n - e^{-|\beta|^2 / 2} \right) d\beta,
\]

\[
I_2(n, m, x) = -\int_{|\beta| > n^{1/4}} e^{-ix \cdot \Lambda_m^{-1/2} / \sqrt{n}} e^{-|\beta|^2 / 2} d\beta,
\]

\[
I_3(n, m, x) = n^{d/2} \int_{n^{-1/4} < |\theta|, \theta \in B_m} e^{-ix \cdot \Lambda_m^{-1/2} \theta} [\phi_m(\theta)]^n d\theta.
\]

By completing the square in the exponential, we get

\[
I_0(n, m, x) = (2\pi)^d / 2 \exp \left( -J_m^2(x) \right).
\]

For \( I_1 \) and \( |\beta| \leq n^{1/4} \), we expand \( \phi_m \) in a series around the origin,

\[
\phi_m(\beta / \sqrt{n}) = 1 - |\beta|^2 / 2n + |\beta|^4 O \left( n^{-2} \right),
\]

\[
\log \phi_m(\beta / \sqrt{n}) = -|\beta|^2 / 2n + |\beta|^4 O \left( n^{-2} \right).
\]
Therefore, 
\[ [\phi_m(\beta/\sqrt{n})]^n = e^{-|\beta|^2/2} \left( 1 + |\beta|^4 O \left( n^{-1} \right) \right), \]
so that
\[ |I_1(n, m, x)| \leq O \left( n^{-1} \right) \int_{|\beta| \leq n^{1/4}} e^{-|\beta|^2/2} |\beta|^4 d\beta = O \left( n^{-1} \right). \]

Similarly, \( I_2 \) is bounded by
\[ |I_2(n, m, x)| \leq C \int_{n^{1/4}}^{\infty} r^{d-1} e^{-r^2/2} dr = O \left( n^{-1} \right). \]

Concerning \( I_3 \), we follow closely [7], proof of Proposition B1, and split the integral further into
\[ n^{-d/2} I_3(n, m, x) = \int_{n^{-1/4} < \theta < n} + \int_{a < |\theta| \leq A} + \int_{A < |\theta| \leq m^\alpha} + \int_{m^\alpha < |\theta|, \theta \in B_m} \]
\[ = (I_{3,0} + I_{3,1} + I_{3,2} + I_{3,3}) (n, m, x), \]
where \( 0 < a < A \) and \( A \in (0, 1) \) are constants that will be chosen in a moment, independently of \( n \) and \( m \). By (81), we can find \( a > 0 \) such that for \( |\beta| < a \sqrt{n} \), \( \log \phi_m(\theta) \leq -|\theta|^2/3 \) (recall that \( \beta = \sqrt{n} \theta \)). Then
\[ |I_{3,0}(n, m, x)| \leq C \int_{n^{-1/4}}^{\infty} r^{d-1} e^{-nr^2/3} dr = O \left( n^{-(d+2)/2} \right). \]
As a consequence of Lemma 3.1 (i) and of our coarse graining, it follows that for any \( 0 < a < A \), one has for some \( 0 < \rho = \rho(a, A) < 1 \), uniformly in \( m \) (and \( p \in \mathcal{P}_n^s \)),
\[ \sup_{a \leq |\theta| \leq A} |\phi_m(\theta)| \leq \rho. \]

Using this fact,
\[ |I_{3,1}(n, m, x)| \leq CA^d \rho^n = O \left( n^{-(d+2)/2} \right). \]
To deal with the last two integrals is more delicate since we have to take into account the \( m \)-dependency. First,
\[ |I_{3,2}(n, m, x)| \leq \int_{A < |\theta| \leq m^\alpha} |\phi_m(\theta)|^n d\theta. \]

We bound the integrand pointwise. Since \( \hat{\pi}_m(\cdot) \) is invariant under the maps \( e_i \mapsto -e_i \), it suffices to look at \( \theta \) with all components positive. Assume \( \theta_1 = \max \{ \theta_1, \ldots, \theta_d \} \). Set \( M = [2\pi \sqrt{\lambda_{m,1}}/\theta_1] \) and \( K = [5m/M] \). Notice that \( \hat{\pi}_m(x) \geq 0 \) implies \( |x| < 2m \). By taking \( A \) large enough, we can assume that on the domain of integration, \( M \leq m \). First,
\[ \phi_m(\theta) = \sum_{(x_2, \ldots, x_d)} \exp \left( \frac{i}{\sqrt{\lambda_{m,1}}} \sum_{s=2}^d x_s \theta_s \right) \sum_{j=1}^K \sum_{x_1 = -2m + (j-1)M}^{-2m + j M - 1} \exp \left( \frac{ix_1 \theta_1}{\sqrt{\lambda_{m,1}}} \right) \hat{\pi}_m(x). \]
Inside the \( x_1 \)-summation, we write for each \( j \) separately
\[
\hat{\pi}_m(x) = \hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)}) + \hat{\pi}_m(x^{(j)}),
\]
where \( x^{(j)} = (-2m + (j - 1)M, x_2, \ldots, x_d) \). By Corollary 7.1,
\[
|\hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)})| \leq C \left| \frac{x_1 + 2m - (j - 1)M}{m} \right|^{1/2} m^{-d}.
\]

Thus,
\[
\left| -2m+jM-1 \sum_{x_1=-2m+(j-1)M} \exp \left( \frac{ix_1 \theta_1}{\sqrt{\lambda_{m,1}}} \right) (\hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)})) \right| \leq C \theta_1^{-3/2} m^{-d+1},
\]
and
\[
\left| \sum_{j=1}^{K} \sum_{x_1=-2m+(j-1)M} \exp \left( \frac{ix_1 \theta_1}{\sqrt{\lambda_{m,1}}} \right) (\hat{\pi}_m(x) - \hat{\pi}_m(x^{(j)})) \right| \leq C \theta_1^{-1/2} m^{-d+1}.
\]

On our domain of integration, \( 0 < (\theta_1/\sqrt{\lambda_{m,1}}) \leq C m^{-\alpha} < 2\pi \) for large \( m \). Therefore,
\[
\left| \sum_{j=1}^{K} \hat{\pi}_m(x^{(j)}) \sum_{x_1=-2m+(j-1)M} \exp \left( \frac{ix_1 \theta_1}{\sqrt{\lambda_{m,1}}} \right) \right| \leq C K m^{-d} \left| \frac{1 - \exp(i\theta_1 M/\sqrt{\lambda_{m,1}})}{1 - \exp(i\theta_1/\sqrt{\lambda_{m,1}})} \right| \leq C|\theta|m^{-d},
\]
and altogether for sufficiently large \( A, m \) and \( n \),
\[
\int_{A<|\theta|\leq m^\alpha} |\phi_m(\theta)|^n \, d\theta \leq C_1^n \int_{A<|\theta|\leq m^\alpha} \left( \frac{1}{\sqrt{|\theta|}} + \frac{|\theta|}{m} \right)^n \, d\theta = O \left( n^{-(d+2)/2} \right).
\]

For \( I_{3,3} \) we again assume all components of \( \theta \) positive and \( \theta_1 = \max\{\theta_1, \ldots, \theta_d\} \). Since
\[
\hat{\pi}_m(x) = \sum_{y=-2m}^{x_1} \left( \hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d) \right),
\]
we have
\[
|\phi_m(\theta)| \leq C m^{d-1} \sum_{x_1=-2m}^{2m} \exp \left( \frac{ix_1 \theta_1}{\sqrt{\lambda_{m,1}}} \right) \sum_{y=-2m}^{x_1} \left( \hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d) \right) \leq C m^{d-1} \sum_{y=-2m}^{2m} \left| \hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d) \right| \sum_{x_1=y}^{2m} \exp \left( \frac{ix_1 \theta_1}{\sqrt{\lambda_{m,1}}} \right).
\]
The sum over the exponentials is estimated by $Cm/|\theta|$, so that again with Corollary 7.1,
\[ |\phi_m(\theta)| \leq C_2 m^{1/2} |\theta|^{-1}. \]
Hence, for $\alpha$ close to 1 and large $n, m$,
\[
\int_{m^{\alpha} < |\theta|, \theta \in B_m} |\phi^n_m(\theta)| \, d\theta \leq C_2^n m^{n/2+\alpha(d-n)} = O\left(n^{-(d+2)/2}\right).
\]

\[\square\]

For Proposition 4.2, we need a large deviation estimate.

**Lemma 7.5 (Large deviation estimate).** Let $p \in \mathcal{P}_k^\circ$. There exist constants $c_1, c_2 > 0$ such that for $|x| \geq 3m$,
\[ \hat{\pi}_m^n(x) \leq c_1 m^{-d} \exp\left(-\frac{|x|^2}{c_2 nm^2}\right). \]

**Proof:** Write $\mathbb{P}$ for $\mathbb{P}_0, \hat{\pi}_m$ and $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$, and denote by $X_n^j$ the $j$th component of the random walk $X_n$ under $\mathbb{P}$. For $r > 0$,
\[
\sum_{y : |y| \geq r} \hat{\pi}_m^n(y) \leq \sum_{j=1}^d \mathbb{P}(|X_n^j| \geq d^{-1/2}r) \\
\leq 2d \max_{j=1,\ldots,d} \mathbb{P}(X_n^j \geq d^{-1/2}r).
\]
We claim that
\[ \mathbb{P}(X_n^j \geq d^{-1/2}r) \leq \exp\left(-\frac{r^2}{8d nm^2}\right). \]
By the martingale maximal inequality for all $t, \lambda > 0$,
\[ \mathbb{P}(X_n^j \geq \lambda) \leq e^{-t\lambda} \mathbb{E}\left[\exp(tX_n^j)\right] = e^{-t\lambda} \left(\mathbb{E}\left[\exp(tX_1^j)\right]\right)^n. \]
Since $X_1^j \in (-2m, 2m)$ and $x \to e^{tx}$ is convex, it follows that
\[ \exp(tX_1^j) \leq \frac{1}{2} \frac{(2m - X_1^j)}{2m} e^{-2tm} + \frac{1}{2} \frac{(2m + X_1^j)}{2m} e^{2tm}. \]
Therefore, using symmetry of $X_1^j$,
\[ \mathbb{E}\left[\exp(tX_n^j)\right] \leq \left(\frac{1}{2} e^{-2tm} + \frac{1}{2} e^{2tm}\right)^n = \cosh^n(2tm) \leq e^{2nt^2m^2}, \]
and
\[ \mathbb{P}(X_n^j \geq d^{-1/2}r) \leq e^{-td^{-1/2}r} e^{2nt^2m^2}. \]
Putting $t = r/(4\sqrt{d}nm^2)$ we get

$$P(X_n^j \geq d^{-1/2}r) \leq \exp\left(-\frac{r^2}{8dnm^2}\right).$$

From this it follows that

$$\hat{\pi}_m(x) = \sum_{y : |y| \geq |x| - 2m} \hat{\pi}_m^{-1}(y)\hat{\pi}_m(x - y) \leq \frac{c_1}{m^d} \exp\left(-\frac{|x| - 2m)^2}{8d(n-1)m^2}\right) \leq \frac{c_1}{m^d} \exp\left(-\frac{|x|^2}{c_2nm^2}\right).$$

**Proof of Proposition 4.2:**

(i) This follows from Proposition 4.1.

(ii) We set

$$N = N(x, m) = \frac{|x|^2}{m^2} \left(\log \frac{|x|^2}{m^2}\right)^{-2}.$$

We split $\hat{g}_{m, Z^d}(x)$ into

$$\hat{g}_{m, Z^d}(x) = \sum_{n=1}^{\infty} \hat{\pi}_m^n(x) = \sum_{n=1}^{[N]} \hat{\pi}_m^n(x) + \sum_{n=[N]+1}^{\infty} \hat{\pi}_m^n(x).$$

For the first sum on the right, we use the large deviation estimate from Lemma 7.5

$$\sum_{n=1}^{[N]} \hat{\pi}_m^n(x) \leq c_1 m^{-d} \sum_{n=1}^{[N]} \exp\left(-\frac{|x|^2}{c_2nm^2}\right) = O\left(|x|^{-d}\right).$$

In the second sum, we replace the transition probabilities by the expressions obtained in Proposition 4.1. The error terms are estimated by

$$\sum_{n=[N]+1}^{\infty} O\left(m^{-d}n^{-(d+2)/2}\right) = O\left(|x|^{-d} \left(\log \frac{|x|^2}{m^2}\right)^d\right).$$

Putting $t_n = 2n\mathcal{J}_m^{-2}(x)$, we obtain for the main part

$$\sum_{n=[N]+1}^{\infty} \frac{1}{(2\pi n)^{d/2} \det \Lambda_m^{1/2}} \exp\left(-\frac{\mathcal{J}_m^2(x)}{2n}\right) = \frac{\mathcal{J}_m^{-d+2}(x)}{2\pi^{d/2} \det \Lambda_m^{1/2}} \sum_{n=[N]+1}^{\infty} t_n^{-d/2} \exp(-1/t_n)(t_n - t_{n-1}) = \frac{\mathcal{J}_m^{-d+2}(x)}{2\pi^{d/2} \det \Lambda_m^{1/2}} \int_0^{\infty} t^{-d/2} \exp(-1/t) dt + O\left(|x|^{-d}\right).$$

This proves the statement for $|x| \geq 3m$ with

$$c(d) = \frac{1}{2\pi^{d/2}} \int_0^{\infty} t^{-d/2} \exp(-1/t) dt.$$
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