Bounding the difference of two singular moduli

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Abstract

For a fixed singular modulus $\alpha$, we give an effective lower bound of norm of $x - \alpha$ for another singular modulus $x$ with large discriminant. We then generalize this result for $\Phi_m(x, \alpha)$, where $\Phi_m(X, Y) \in \mathbb{Z}[X, Y]$ is the $m$-th modular polynomial.

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1 Introduction

Let $\mathbb{H}$ be the upper half plane, a point $\tau \in \mathbb{H}$ is called a CM-point if $\text{End}(E_\tau)$ is an order in an imaginary quadratic field, where $E_\tau$ is the elliptic curve over $\mathbb{C}$ corresponding to $\tau$. It is well-known that $\tau \in \mathbb{H}$ is CM if and only if $\tau$ is algebraic number of degree 2. We call $j(\tau)$ a singular modulus if $\tau \in \mathbb{H}$ is CM. From the classical CM-theory, we know that every singular modulus is an algebraic integer. We call $j(\tau)$ a singular unit if it is a singular modulus and an algebraic unit.

In [6], Habegger proved that there is at most finitely many singular units. However his proof is ineffective. After this, in [2], Bilu, Habegger and Kühne proved that there are no singular units. Indeed, their method can be generalized to give an effective bound of norm of difference between two singular moduli, that is exactly what we do in this paper.

Before stating our result, let us fix some notations. Given a singular modulus $x = j(\tau)$, the discriminant of $x$ is defined to be the discriminant of the order $\text{End}(E_\tau)$ in an imaginary quadratic number field. We know that, in this case, $\text{End}(E_\tau)$ is isomorphic to $\mathcal{O}_\Delta := \mathbb{Z}[(\Delta + \sqrt{\Delta})/2]$. By the CM-theory, the singular moduli of a given discriminant form a Galois orbit over $\mathbb{Q}$ of cardinality...
equal to the class number \( C(\Delta) \) of \( \mathcal{O}_\Delta \). For a number field \( K \), and \( \alpha \in K \), we denote \( N_{K/Q}(\alpha) \) the absolute norm of \( \alpha \), and denote \( h(\alpha) \) the absolute height of \( \alpha \).

In this paper, we are going to prove the following result:

**Theorem 1.1** Let \( \alpha, x \) be two singular moduli of discriminants \( \Delta_\alpha, \Delta \) respectively, and \( K = Q(\alpha, x) \).

1. If \( \Delta_\alpha \neq -3, -4 \) and \( |\Delta| \geq \max\{ e^{3.12} |C(\Delta_\alpha)| \Delta_\alpha, e^{h(\alpha)^3}, 10^{15} \cdot |C(\Delta_\alpha)|^{1/20} \} \), then
   \[
   \log |N_{K/Q}(x - \alpha)| > \frac{|\Delta|^{1/2}}{2}.
   \]

2. If \( \Delta_\alpha = -4 \), i.e. \( \alpha = 1728 \), and \( |\Delta| \geq 10^{15} \), then
   \[
   \log |N_{K/Q}(x - 1728)| > \frac{2|\Delta|^{1/2}}{5}.
   \]

3. If \( \Delta_\alpha = -3 \), i.e. \( \alpha = 0 \), and \( |\Delta| \geq 10^{15} \), then
   \[
   \log |N_{K/Q}(x)| > \frac{|\Delta|^{1/2}}{20}.
   \]

In this theorem, the bound is effective.

The idea of proving Theorem 1.1 is from [2]. Set \( \zeta_1 = e^{2\pi i/3} \) and \( \zeta_6 = e^{\pi i/3} \), let \( \mathcal{F} \) be the standard fundamental domain in the Poincaré plane, that is, the open hyperbolic triangle with vertices \( \zeta_3, \zeta_6, \) and \( i\infty \), together with the geodesics \( [i, \zeta_6] \) and \( [\zeta_6, i\infty] \). Given \( \varepsilon \in (0, 1/4) \), and a point \( \tau \in \mathcal{F} \), denote by \( C_\varepsilon(\tau, \Delta) \) the number of singular moduli of discriminant \( \Delta \) which can be written \( j(z) \) where \( z \in \mathbb{H} \) satisfies \( |z - \tau| < \varepsilon \). Firstly, we give an effective upper bound of \( C_\varepsilon(\tau, \Delta) \), see Corollary 4.2. Then by using this bound and the lower bound for the difference of two singular moduli from [1], we manage to give an upper bound for the height of difference, see Corollary 4.2. The lower bound for height of difference comes from [2], see Section 5. With these two bounds, by estimating each term in the both sides, we deduce Theorem 1.1.

Let us remark, since Bilu, Habegger and Kühne [2] have given most of results we need for the case where \( \tau = \zeta_6 \), i.e. \( \Delta_\alpha = -3 \) in Theorem 1.1 (3), we will use their result directly and focus mainly on the case where \( \tau \neq \zeta_6 \).

There are other works about the norm of difference between two singular moduli. In fact, Gross and Zagier [5] stated explicit formula for absolute norm of difference between two singular moduli. With their works, Li [9] also managed to give a bound of norm of difference between two singular moduli, his bound is a strictly positive number, which allows him to prove a generalized version of the main result of Bilu, Habegger and Kühne [2]. Even more, he gave a bound for \( \log |N_{K/Q}(\Phi_m(x, \alpha))| \), where \( \Phi_m(X, Y) \in \mathbb{Z}[X, Y] \) is the \( m \)-th modular polynomial. However, it is not clear how his bound behaves as \( \Delta \to -\infty \).

We can generalize our main result to give a bound for \( N_{K/Q}(\Phi_m(x, \alpha)) \) when the discriminant \( \Delta \) of \( x \) is sufficiently large. Recall the definition of \( \Phi_m(X, Y) \). For \( z_1, z_2 \in \mathbb{H} \),

\[
\Phi_m(j(z_1), j(z_2)) = \prod_{\gamma \in \text{SL}_2(\mathbb{Z}) \setminus D_m} (j(z_1) - j(\gamma z_2)),
\]

where

\[
D_m := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = m \right\}.
\]

We have the following corollary from Theorem 1.1.

**Corollary 1.2** Keep the notations in Theorem 1.1. Let \( m \geq 1 \) be an integer. If \( \Delta \) is sufficiently large (in terms of \( \alpha \) and \( m \)), then

\[
\log |N_{K/Q}(\Phi_m(x, \alpha))| > \frac{|\Delta|^{1/2}}{20}.
\]
Our result requires to fix a singular modulus \( \alpha \), one generalization of this work is to give an explicit lower bound for \( \log |N_{K/Q}(x - \alpha)| \) when both \( \Delta_\alpha \) and \( \Delta \) vary.

Another natural generalization is to give a non-trivial lower bound for \( v_p(N_{K/Q}(x - \alpha)) \) in terms of their discriminants, where \( v_p \) is the \( p \)-adic discrete valuation. The motivation for this is to find all singular \( S \)-units (that is, singular moduli that are \( S \)-units). Recall that, given a finite set \( S \) of prime numbers, an \( S \)-unit is an algebraic number whose denominator and numerator are composed of prime ideals dividing primes from \( S \). Recently, Campagna \(^8\) showed that, if \( S \) is the set of rational primes congruent to 1 modulo 3, then there are no singular \( S \)-units. Herrero, Menares and Rivera-Letelier \(^8\) proved that, given a singular modulus \( \alpha \), there are only finitely many singular moduli \( x \) such that \( x - \alpha \) is an \( S \)-unit. In particular, if \( \alpha = 0 \), there are only finitely many singular \( S \)-units. However, their proof is ineffective. We expect a non-trivial lower bound for \( v_p(N_{K/Q}(x - \alpha)) \) can provide an effective method to calculate all singular moduli \( x \) such that \( x - \alpha \) is an \( S \)-unit for a given singular modulus \( \alpha \) and a finite set \( S \) of primes.

2 General setting

For a number field \( K, x \in K \), we denote by \( N_{K/Q}(x) \) the absolute norm of \( x \). Let \( \Delta \) be a negative integer satisfying \( \Delta \equiv 0, 1 \mod 4 \) and

\[ \mathcal{O}_\Delta = \mathbb{Z}[\Delta + \sqrt{\Delta}/2], \]

the imaginary quadratic order of discriminant \( \Delta \). We suppose that \( D \) is the discriminant of \( \mathbb{Q}(\sqrt{\Delta}) \), and \( f = [\mathcal{O}_D : \mathcal{O}_\Delta] \) is the conductor of \( \mathcal{O}_\Delta \), so we have \( \Delta = f^2 D \). We also denote the class number of the order \( \mathcal{O}_\Delta \) by \( \mathcal{C}(\Delta) \), since \( h \) is used for height of an algebraic number. For further uses, we define the modified conductor \( \tilde{f} \) of \( \mathcal{O}_\Delta \) by

\[ \tilde{f} = \begin{cases} f, & D \equiv 1 \mod 4, \\ 2f, & D \equiv 0 \mod 4. \end{cases} \]

On the other hand, let \( \mathcal{F} \) be the standard fundamental domain in the Poincaré plane, that is, the open hyperbolic triangle with vertices \( \zeta_3, \zeta_6 \), and \( \infty \), together with the geodesics \([i, \zeta_6]\) and \([\zeta_6, \infty]\); here \( \zeta_3 = e^{2\pi i/3} \) and \( \zeta_6 = e^{\pi i/3} \). Then the Klein \( j \)-invariant \( j : \mathbb{H} \to \mathbb{C} \) induces a bijection \( j : \mathcal{F} \to \mathbb{C} \).

For each \( \text{CM-point} \ \tau \) in the standard fundamental domain \( \mathcal{F} \), i.e. quadratic imaginary number in \( \mathcal{F} \), the discriminant \( \Delta_\tau \) of \( \tau \) is defined to be the discriminant of the primitive polynomial of \( \tau \) over \( \mathbb{Z} \), it is also the discriminant of the order \( \text{End}(\mathbb{C}/\Lambda_\tau) \), i.e. \( \text{End}(\mathbb{C}/\Lambda_\tau) = \mathcal{O}_{\Delta_\tau} \), where \( \Lambda_\tau \) is the lattice generated by 1 and \( \tau \). Since the \( j \)-invariant \( j : \mathcal{F} \to \mathbb{C} \) is a bijection, we call \( \Delta_\tau \), the discriminant of \( \alpha = j(\tau) \), also denoted by \( \Delta_\alpha \).

By classical CM-theory, we know that \( \mathbb{Q}(\sqrt{\Delta_\tau}, j(\tau)) \) is the ring class field of \( \mathbb{Q}(\sqrt{\Delta_\tau}) \) for the order \( \mathcal{O}_{\Delta_\tau} \), hence \( \mathbb{Q}(\sqrt{\Delta_\tau}, j(\tau))/\mathbb{Q}(\sqrt{\Delta_\tau}) \) is Galois and \( \mathcal{C}(\Delta_\tau) = [\mathbb{Q}(\sqrt{\Delta_\tau}, j(\tau)) : \mathbb{Q}(\sqrt{\Delta_\tau})] = [\mathbb{Q}(j(\tau)) : \mathbb{Q}] \).

For \( n \in \mathbb{N}^+ \), we denote

\[ \omega(n) = \sum_{d|n} 1, \quad \sigma_0(n) = \sum_{d|n} 1, \quad \sigma_1(n) = \sum_{d|n} d. \]

3 An Estimate for \( C_\varepsilon(\tau, \Delta) \)

For each \( \tau \in \mathcal{F} \) and \( \varepsilon \in (0, 1/4) \), we define

\[ S_\varepsilon(\tau, \Delta) = \{ z \in \mathbb{H} \mid z \text{ is an imaginary quadratic number of discriminant } \Delta \text{ and } |z - \tau| < \varepsilon \}, \]

\[ C_\varepsilon(\tau, \Delta) = \# S_\varepsilon(\tau, \Delta), \]

here \# means the cardinality of a set.
Let $S_\Delta$ be the set of primitive positive definite forms of discriminant $\Delta$, that is, a quadratic form $ax^2 + bxy + cy^2 \in S_\Delta$ if $a, b, c \in \mathbb{Z}$ and 

$$a > 0, \quad \gcd(a, b, c) = 1, \quad \Delta = b^2 - 4ac < 0$$

For $ax^2 + bxy + cy^2 \in S_\Delta$, we set 

$$\tau(a, b, c) = \frac{b + \sqrt{\Delta}}{2a},$$

then the map $ax^2 + bxy + cy^2 \mapsto \tau(a, b, c)$ defines a bijection from $S_\Delta$ to the set of imaginary number of discriminant $\Delta$.

We will prove the following theorem and corollary:

**Theorem 3.1** Let $\tau \in F$ and $\varepsilon \in (0, 1/4)$, then 

$$C_\varepsilon(\tau, \Delta) \leq F \times \left( \frac{48 + 16\sqrt{3}\sigma_1(\bar{f})}{3}\frac{|\Delta|^{1/2}\varepsilon}{f} + \frac{12 + 4\sqrt{3}}{3}|\Delta|^{1/2}\varepsilon + \frac{8|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}}\sigma_0(\bar{f})\varepsilon + 2 \right),$$

where 

$$F = F(\Delta) = \max\{2^{\omega(a)} : a \leq |\Delta|^{1/2} \}.$$  

(1)

**Corollary 3.2** In the set-up of Theorem 3.1, assume that $|\Delta| \geq 10^{14}$. Then 

$$C_\varepsilon(\tau, \Delta) \leq F \times \left( 46.488|\Delta|^{1/2}\varepsilon^2 \log \log |\Delta|^{1/2} + 7.752|\Delta|^{1/2}\varepsilon + 2 \right)$$

3.1 Some lemmas

We say that $d \in \mathbb{Z}$ is a quadratic divisor of $n \in \mathbb{Z}$ if $d^2 | n$. We denote by $\gcd_2(m, n)$ the greatest common quadratic divisor of integers $m$ and $n$.

We will use the following lemmas.

**Lemma 3.3** ([2], Lemma 2.4) Let $a$ be a positive integer and $\Delta$ a non-zero integer. Then the set of $b \in \mathbb{Z}$ satisfying $b^2 \equiv \Delta \mod{a}$ consists of at most $2^{\omega(a)} + 1$ residue classes modulo $a/\gcd_2(a, \Delta)$.

**Lemma 3.4** Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, and $m$ a positive integer. Then every residue class modulo $m$ has at most $(\beta - \alpha)/m + 1$ elements in the interval $[\alpha, \beta]$.

**Lemma 3.5** Let $\tau \in F$, and $\varepsilon \in (0, 1/4)$, and let $ax^2 + bxy + cy^2 \in S_\Delta$ be such that $|\tau(a, b, c) - \tau| < \varepsilon$. Then 

$$\frac{|\Delta|^{1/2}}{2(\text{Im} \tau + \varepsilon)} < a < \frac{|\Delta|^{1/2}}{2(\text{Im} \tau - \varepsilon)},$$

(2)

$$2a(\text{Re} \tau - \varepsilon) < b < 2a(\text{Re} \tau + \varepsilon).$$

(3)

**Proof**

Set $z = \tau(a, b, c)$, then from $|z - \tau| < \varepsilon$, we have 

$$|\text{Im} z - \text{Im} \tau| < \varepsilon, \quad |\text{Re} z - \text{Re} \tau| < \varepsilon,$$

that is, 

$$\left| \frac{|\Delta|^{1/2}}{2a} - \text{Im} \tau \right| < \varepsilon, \quad \left| \frac{b}{2a} - \text{Re} \tau \right| < \varepsilon,$$

so we have (2) and (3).
3.2 Proof of Theorem 3.1

Set
\[ I = \left( \frac{|\Delta|^{1/2}}{2(\text{Im } \tau + \varepsilon)} : \frac{|\Delta|^{1/2}}{2(\text{Im } \tau - \varepsilon)} \right), \]
\[ \tau(a, b, c) = \frac{b + \sqrt{\Delta}}{2a}. \]

By Lemma ??, if \( \tau(a, b, c) \in S_c(\tau, \Delta) \), then \( a \in I \) and \( b \in (2a(\text{Re } \tau - \varepsilon), 2a(\text{Re } \tau + \varepsilon)) \).

For a fixed \( a \), by Lemma 3.3 and Lemma 3.4 and \( \omega(a/\gcd(a, \Delta)) \leq \omega(a) \), there are at most \( (4\varepsilon \gcd(a, \Delta) + 1) \cdot 2^{\omega(a) + 1} \) possible \( b \)'s. Since \( \varepsilon < 1/4, \text{Im } \tau \geq \sqrt{3}/2 \), then \( \frac{|\Delta|^{1/2}}{\sqrt{3}/2 - 1/2} \leq |\Delta|^{1/2} \).

Hence
\[ C_\varepsilon(\tau, \Delta) \leq 8\varepsilon \sum_{a \in I \cap \mathbb{Z}} \gcd_2(a, \Delta) \cdot 2^{\omega(a)} + 2 \sum_{a \in I \cap \mathbb{Z}} 2^{\omega(a)} \]
\[ \leq 8\varepsilon F \sum_{a \in I \cap \mathbb{Z}} \gcd_2(a, \Delta) + 2F \#(I \cap \mathbb{Z}). \]

Note that
\[ \sum_{a \in I \cap \mathbb{Z}} \gcd_2(a, \Delta) \leq \sum_{d^2|\Delta} d \cdot \#(I \cap d^2\mathbb{Z}), \]
and the length of \( I \) is
\[ \frac{|\Delta|^{1/2}}{2(\text{Im } \tau + \varepsilon)} - \frac{|\Delta|^{1/2}}{2(\text{Im } \tau - \varepsilon)} = \frac{\varepsilon}{(\text{Im } \tau + \varepsilon)(\text{Im } \tau - \varepsilon)} \]
\[ \leq \frac{|\Delta|^{1/2}}{\sqrt{3}/2 - 1/2} \leq \frac{2\sqrt{3}}{3}|\Delta|^{1/2}. \]

When \( d > \frac{|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}} \), we have \( \frac{|\Delta|^{1/2}}{2(\text{Im } \tau - \varepsilon)} < d^2 \). Combine this with Lemma 3.3 we have
\[ \#(I \cap d^2\mathbb{Z}) \leq \begin{cases} \frac{6 + 2\sqrt{3}|\Delta|^{1/2}}{3} \varepsilon + 1 & \text{if } d \leq \frac{|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}}, \\ 0 & \text{if } d > \frac{|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}}. \end{cases} \]

Since \( \Delta/\bar{f}^2 \) is square-free, so for a positive integer \( d \), \( d^2 \mid \Delta \) if and only if \( d \mid \bar{f} \), hence
\[ \sum_{d^2|\Delta} d \cdot \#(I \cap d^2\mathbb{Z}) \leq \sum_{d \mid \bar{f}} d \left( \frac{6 + 2\sqrt{3}|\Delta|^{1/2}}{3} \varepsilon + 1 \right) \]
\[ \leq \frac{6 + 2\sqrt{3}}{3} |\Delta|^{1/2} \varepsilon \sum_{d \mid \bar{f}} 1/d + \sum_{d \mid \bar{f}} d \]
\[ \leq \frac{6 + 2\sqrt{3}}{3} \sigma_1(\bar{f}) |\Delta|^{1/2} \varepsilon + \frac{|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}} \sigma_0(\bar{f}). \]

Again, by Lemma 3.4 we have
\[ \#(I \cap \mathbb{Z}) \leq \frac{6 + 2\sqrt{3}}{3} |\Delta|^{1/2} \varepsilon + 1. \]
Hence,

\[ C_\varepsilon(\tau, \Delta) \leq 8\varepsilon F \times \left( \frac{6 + 2\sqrt{3}}{3} \frac{\sigma_1(\tilde{f})}{f} |\Delta|^{1/2 \varepsilon} + \frac{|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}} \sigma_0(\tilde{f}) \right) + 2F \times \left( \frac{6 + 2\sqrt{3}}{3} |\Delta|^{1/2 \varepsilon} + 1 \right) \]

\[ \leq F \times \left( \frac{48 + 16\sqrt{3}}{3} \frac{\sigma_1(\tilde{f})}{f} |\Delta|^{1/2 \varepsilon^2} + \frac{12 + 4\sqrt{3}}{3} |\Delta|^{1/2 \varepsilon} + \frac{8|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}} \sigma_0(\tilde{f}) \varepsilon + 2 \right). \]

3.3 Proof of Corollary 3.2

The following lemma estimate \( \sigma_0(\tilde{f}) \) and \( \sigma_1(\tilde{f}) \) in terms of \(|\Delta|\):

**Lemma 3.6 ([2], Lemma 2.8)** For \(|\Delta| \geq 10^{14}\), we have

\[ \sigma_0(\tilde{f}) \leq |\Delta|^{0.192}, \]

\[ \sigma_1(\tilde{f})/\tilde{f} \leq 1.842 \log \log |\Delta|^{1/2}. \]

With this lemma, we have

\[ \frac{48 + 16\sqrt{3}}{3} \sigma_1(\tilde{f})/f \leq \frac{48 + 16\sqrt{3}}{3} \cdot 1.842 \log \log |\Delta|^{1/2} \leq 46.488 \log \log |\Delta|^{1/2}, \]

\[ \frac{8|\Delta|^{1/4}}{(\sqrt{3} - 1)^{1/2}} \sigma_0(\tilde{f}) \leq \frac{8}{(\sqrt{3} - 1)^{1/2}} |\Delta|^{0.442} \leq \frac{8}{10^{0.812} |\Delta|^{1/2}} \leq 1.442 |\Delta|^{1/2}. \]

\[ \frac{12 + 4\sqrt{3}}{3} + 1.442 \leq 7.752 \]

With these bounds and Theorem 3.1, we have Corollary 3.2.

4 An Upper Bound for the Height of the difference of Singular Moduli

Let \( \alpha = j(\tau), x = j(z) \) be two different singular moduli with \( \tau, z \in \mathcal{F} \), and \( \Delta_\alpha, \Delta = \Delta_x \) be their discriminants respectively. Let \( K = \mathbb{Q}(x - \alpha) \), \( d = [K : \mathbb{Q}] \), then we have \( K = \mathbb{Q}(\alpha, x) \), see [4, Theorem 4.1]. Hence we can assume that \( d = sC(\Delta_\alpha) \), where \( s = [K : \mathbb{Q}(\alpha)] \). We suppose that the set of embeddings of \( K \) to \( \mathbb{C} \) is \( \{ \sigma_1, \ldots, \sigma_d \} \). For each \( k \), set \( \alpha_k = \sigma_k(\alpha) = j(\tau_k) \) with \( \tau_k \in \mathcal{F} \), and set \( x_k = \sigma_k(x) = j(z_k) \) such that \( z_k \in \mathbb{H} \) is the nearest point to \( \tau_k \) among \( \text{SL}_2(\mathbb{Z})z_k \) with respect to the absolute valuation on \( \mathbb{C} \). Then \( \alpha_k \neq x_k \) for each \( k \), and we have

\[ h(x - \alpha) = h((x - \alpha)^{-1}) = \frac{1}{d} \sum_{k=1}^{d} \log^+ |x_k - \alpha_k|^{-1} + \frac{1}{d} \log |N_{K/\mathbb{Q}}(x - \alpha)|, \]  

(4)

where \( \log^+(\cdot) = \log \max\{1, \cdot\} \).

In this section, we are going to prove that following theorem and corollary:

**Theorem 4.1** Let \( \alpha = j(\tau), x = j(z) \) be two different singular moduli with \( \tau, z \in \mathcal{F} \), and \( \Delta_\alpha, \Delta = \Delta_x \) their discriminants respectively. Let \( K = \mathbb{Q}(x - \alpha) \), \( d = [K : \mathbb{Q}] \),

(1) if \( \tau \neq i, \zeta_6 \) and \( 0 < \varepsilon < \min\left\{ \frac{1}{|\Delta_\alpha|^{1/2}}, 10^{-8} \right\} \), then

\[ h(x - \alpha) \leq \sum_{1 \leq k \in C(\Delta_\alpha)} 4 \frac{C_k(\tau_k, \Delta)}{d} \log(\max\{|\Delta|, |\Delta_\alpha|\}) + \log(\varepsilon^{-1}) + 2 \log |\Delta_\alpha| - 7.783 \]

\[ + \frac{1}{d} \log |N_{K/\mathbb{Q}}(x - \alpha)|; \]
(2) if \( \tau = i \) and \( 0 < \varepsilon \leq 7 \cdot 10^{-3} \), then
\[
h(x - 1728) \leq 2 \frac{C(i, \Delta)}{C(\Delta)} \log |\Delta| + 2 \log e^{-1} - 9.9 + \frac{1}{C(\Delta)} \log |N_{K/Q}(x - 1728)|.
\]

We don’t discuss the case where \( \tau = \zeta_6 \), since the bound for this case in the following corollary can be get directly from [2].

**Corollary 4.2** In the setup of Theorem 4.1, assume that \( |\Delta| \geq 10^{14} \),

(1) if \( \tau \neq i, \zeta_6 \), then
\[
h(x - \alpha) \leq \frac{8A|\Delta_\alpha|}{C(\Delta)} + \log \frac{A|\Delta_\alpha|}{d} + 4 \log |\Delta_\alpha| + 0.33 + \frac{1}{d} \log |N_{K/Q}(x - \alpha)|;
\]

(2) if \( \tau = i \), then
\[
h(x - 1728) \leq \frac{4A}{C(\Delta)} + 2 \log \frac{A|\Delta|^{1/2}}{C(\Delta)} - 2.68 + \frac{1}{C(\Delta)} \log |N_{K/Q}(x - 1728)|;
\]

(3) if \( \tau = \zeta_6 \), then
\[
h(x) \leq \frac{12A}{C(\Delta)} + 3 \log \frac{A|\Delta|^{1/2}}{C(\Delta)} - 3.77 + \frac{1}{C(\Delta)} \log |N_{K/Q}(x)|,
\]

where \( A = F \log \max\{|\Delta|, |\Delta_\alpha|\} \) and \( F \) is defined in Theorem 3.1.

**4.1 Proof of Theorem 4.1**

The following lemmas and theorems are needed.

**Lemma 4.3** In the setup of Theorem 4.1,

1) if \( \text{Im } \tau \geq 1.3 \), then there exists \( z' \in \mathbb{H} \) with \( x = j(z') \) such that
\[
|x - \alpha| \geq e^{2.67} \min\{0.4|z' - \tau|, 0.04\};
\]

2) if \( \text{Im } \tau \leq 1.3 \) and \( \tau \neq i, \zeta_6 \), then there exist \( z' \in \mathbb{H} \) with \( x = j(z') \) such that
\[
|x - \alpha| \geq \min\{5 \cdot 10^{-7}, 800|\Delta_\alpha|^{-4}, 2400|\Delta_\alpha|^{-2}|z' - \tau|\}.
\]

**Proof**

Combine Proposition 4.1 and Proposition 4.2 in [1].

**Theorem 4.4** ([1] Theorem 1.1) In the setup of Theorem 4.1, we have
\[
|x - \alpha| \geq 800 \max\{|\Delta|, |\Delta_\alpha|\}^{-4}.
\]

**Lemma 4.5** For \( i \neq z \in \mathcal{F} \) with discriminant \( \Delta \), we have
\[
|j(z) - 1728| \geq 20000 \min\{|z - i|, 0.01\}^2,
\]
\[
|j(z) - 1728| \geq 2000|\Delta|^{-2}.
\]
Proof

Combine Proposition 3.7 and Corollary 5.3 in [1].

We start to prove Theorem 1.1 (1). Let \( \tau_k, z_k, \alpha_k, x_k \) be as the beginning of this section. Then we have

\[
\sum_{k=1}^{d} \log^+ |x_k - \alpha_k|^{-1} = \sum_{1 \leq k \leq d, z_k \in S_i(\tau_k, \Delta)} \log^+ |x_k - \alpha_k|^{-1} + \sum_{1 \leq k \leq d, z_k \notin S_i(\tau_k, \Delta)} \log^+ |x_k - \alpha_k|^{-1}
\]

For the first sum, by Theorem 3.4, each term in the sum has

\[
\log^+ |x_k - \alpha_k|^{-1} \leq \max\{0, 4 \log(\max\{|\Delta|, |\Delta_\alpha|\}) - \log(800)\} \leq 4 \log(\max\{|\Delta|, |\Delta_\alpha|\}),
\]

so we have

\[
\sum_{1 \leq k \leq d, z_k \in S_i(\tau_k, \Delta)} \log^+ |x_k - \alpha_k|^{-1} \leq \sum_{1 \leq k \leq C(\Delta_\alpha)} 4C_{\varepsilon}(\tau_k, \Delta) \log(\max\{|\Delta|, |\Delta_\alpha|\}). \tag{5}
\]

For the second sum, we claim that if \( |z_k - \tau_k| \geq \varepsilon \), then

\[
|x_k - \alpha_k| \geq 2400|\Delta_\alpha|^{-2}\varepsilon.
\]

In fact, we can replace \( \tau \) by \( \tau_k \) and \( z \) by \( z_k \) in Lemma 4.3 by the choice of \( z_k \), i.e. \( z_k \) is the nearest point to \( x_k \) among \( \text{SL}_2(\mathbb{Z})z_k \in \mathbb{H} \) with respect to the absolute valuation, then

\[
|x_k - \alpha_k| \geq \min\{e^{2.6\varepsilon}, 0.4\varepsilon, 5 \cdot 10^{-7}, 800|\Delta_\alpha|^{-4}, 2400|\Delta_\alpha|^{-2}\varepsilon\}.
\]

Notice that \( |\Delta_\alpha| \geq 7 \) and \( \varepsilon < \min\{\frac{1}{|\Delta_\alpha|}, 10^{-8}\} \), then

\[
2400|\Delta_\alpha|^{-2}\varepsilon \leq 800|\Delta_\alpha|^{-4},
\]

\[
2400|\Delta_\alpha|^{-2}\varepsilon \leq \frac{2400}{49} \cdot 10^{-8} < 5 \cdot 10^{-7},
\]

\[
2400|\Delta_\alpha|^{-2}\varepsilon \leq \frac{2400}{49} \cdot 1410 \varepsilon \leq e^{2.6\varepsilon} \cdot 0.4\varepsilon,
\]

so we have our claim. Hence

\[
\log^+ |x_k - \alpha_k|^{-1} \leq \log \left( \frac{|\Delta_\alpha|^2}{2400} \varepsilon^{-1} \right) \leq \log(\varepsilon^{-1}) + 2 \log |\Delta_\alpha| - 7.783,
\]

\[
\sum_{1 \leq k \leq d, z_k \notin S_i(\tau_k, \Delta)} \log^+ |x_k - \alpha_k|^{-1} \leq d(\log(\varepsilon^{-1}) + 2 \log |\Delta_\alpha| - 7.783). \tag{6}
\]

Combine (5), (6) and the equality (1), we have the bound in (1).

For Theorem 1.1 (2), the proof is similar as above. Since \( j(\tau) = 1728 \), then \( d = C(\Delta) \) and

\[
\sum_{k=1}^{C(\Delta)} \log^+ |x_k - 1728|^{-1} = \sum_{1 \leq k \leq C(\Delta), z_k \in S_i(\tau, \Delta)} \log^+ |x_k - 1728|^{-1} + \sum_{1 \leq k \leq C(\Delta), z_k \notin S_i(\tau, \Delta)} \log^+ |x_k - 1728|^{-1}
\]

For the first sum, by Lemma 4.5

\[
\log^+ |x_k - 1728|^{-1} \leq \max\{0, 2 \log |\Delta| - \log 2000\} \leq 2 \log |\Delta|,
\]

\[
\sum_{1 \leq k \leq C(\Delta), z_k \in S_i(\tau, \Delta)} \log^+ |x_k - 1728|^{-1} \leq 2C_{\varepsilon}(i, \Delta) \log |\Delta|.
\]
For the second sum, since $\varepsilon \leq 7 \cdot 10^{-3}$, $\varepsilon^{-2} > 20000$ and $|z_k - i| \geq \varepsilon$, we have

$$|x_k - 1728|^{-1} \leq 20000^{-1} \min\{\varepsilon, 0.01\}^{-2} = 20000^{-1}\varepsilon^{-2},$$

$$\log^+ |x_k - 1728|^{-1} \leq \max\{0, 2 \log \varepsilon^{-1} - \log(20000)\} \leq 2 \log \varepsilon^{-1} - 9.9,$$

$$\sum_{1 \leq k \leq C(\delta)} \log^+ |x_k - 1728|^{-1} \leq C(\delta)(2 \log \varepsilon^{-1} - 9.9).$$

Hence, as above, we have

$$h(x - 1728) \leq 2 \frac{C(i, \Delta)}{C(\Delta)} \log |\Delta| + 2 \log \varepsilon^{-1} - 9.9 + \frac{1}{C(\Delta)} \log |N_K/Q(x - 1728)|.$$

4.2 Proof of Corollary 4.2

We will use the following lemmas from [2].

Lemma 4.6 ([2] Lemma 3.5) Assume that $|\Delta| \geq 10^{14}$. Then $F \geq |\Delta|^{0.34/\log \log(|\Delta|^{1/2})}$ and $F \geq 18.54 \log \log(|\Delta|^{1/2}).$

Lemma 4.7 ([2] Lemma 3.6) For $\Delta \neq -3, -4$, we have

$$C(\Delta) \leq \pi^{-1}|\Delta|^{1/2}(2 + \log |\Delta|).$$

To prove Corollary 4.2 (1), by Corollary 3.2, we have

$$\sum_{1 \leq k \leq C(\Delta)} \frac{4C(\tau_k, \Delta)}{d} \log \max\{|\Delta|, |\Delta_{\alpha}|\} \leq \frac{4AC(\Delta_{\alpha})(46.488|\Delta|^{1/2}\varepsilon^2 \log \log |\Delta|^{1/2} + 7.752|\Delta|^{1/2}\varepsilon + 2)}{d}. \quad (7)$$

We can take $\varepsilon = 0.0003 \cdot \frac{d}{AC(\Delta_{\alpha})|\Delta|^{1/2}|\Delta_{\alpha}|^2},$ then $\varepsilon \leq \min\left\{\frac{1}{3|\Delta_{\alpha}|^2}, 10^{-8}\right\}.$ Indeed, $F \geq 256$ if $|\Delta| \geq 10^{14},$ and by Lemma 4.6 and Lemma 4.7, we have

$$0.0003 \cdot \frac{d}{AC(\Delta_{\alpha})|\Delta|^{1/2}|\Delta_{\alpha}|^2} \leq \frac{6 + 3 \log(10^{14})}{10000 \pi \log(10^{14})} \cdot \frac{1}{256} \leq \frac{1}{3};$$

$$0.0003 \cdot \frac{d}{AC(\Delta_{\alpha})|\Delta|^{1/2}|\Delta_{\alpha}|^2} \leq \frac{6 + 3 \log(10^{14})}{490000 \pi \log(10^{14})} \cdot \frac{1}{256} \leq 10^{-8}.$$

We estimate each term in the left of (7) with our $\varepsilon:$

$$\frac{46.488AC(\Delta_{\alpha})|\Delta|^{1/2}\varepsilon^2 \log \log |\Delta|^{1/2}}{d} \leq 36 \cdot 10^{-8} \cdot 46.488 \cdot \frac{d \log \log |\Delta|^{1/2}}{AC(\Delta_{\alpha})|\Delta|^{1/2}|\Delta_{\alpha}|^4} \leq 36 \cdot 10^{-8} \cdot 46.488 \log \log |\Delta|^{1/2} \cdot C(\Delta) \cdot \frac{|\Delta_{\alpha}|^4}{F} \cdot \frac{|\Delta|^{1/2} \log |\Delta|}{18.54 \cdot \pi \log(10^{14})} \leq \frac{0.0003 \cdot 31.008|\Delta_{\alpha}|^{-2}}{\Delta_{\alpha}} \leq 0.0005;$$

$$\frac{7.752AC(\Delta_{\alpha})|\Delta|^{1/2}\varepsilon}{d} \leq 0.0003 \cdot 31.008|\Delta_{\alpha}|^{-2} \leq 0.0005.$$
With above, we have
\[
\begin{align*}
  h(x - \alpha) &\leq \frac{8AC(\Delta)}{d} + \log(\frac{AC(\Delta)}{d}\frac{|\Delta|^{1/2}|\Delta|^2}{d}) + 2\log|\Delta| + 0.001 + \log\left(\frac{10000}{3}\right) - 7.783 \\
  &\quad + \frac{1}{d}\log|\mathcal{N}_{K/Q}(x - \alpha)| \\
  &\leq \frac{8AC(\Delta)}{d} + \log(\frac{AC(\Delta)}{d}\frac{|\Delta|^{1/2}}{d}) + 4\log|\Delta| + 0.33 + \frac{1}{d}\log|\mathcal{N}_{K/Q}(x - \alpha)|.
\end{align*}
\]

For Corollary 4.2 (2), the proof is similar. We set \( \varepsilon = 0.3\frac{C(\Delta)}{4|\Delta|^{1/2}} \), then \( \varepsilon \leq 7 \cdot 10^{-3} \). Indeed, since \( |\Delta| \geq 10^{14} \), so \( F \geq 256 \), hence
\[
0.3\frac{C(\Delta)}{|\Delta|^{1/2}} \leq 0.3\frac{C(\Delta)}{|\Delta|^{1/2} \log|\Delta|} \cdot \frac{1}{F} \leq 0.3\frac{2 + \log(10^{14})}{\pi \log(10^{14})} \cdot \frac{1}{256} \leq 5 \cdot 10^{-4}.
\]

By Corollary 3.2 Theorem 4.1(2), Lemma 4.6 and Lemma 4.7 we have
\[
\begin{align*}
  h(x - 1728) &\leq 2\frac{C_2(i, \Delta)}{C(\Delta)} \log|\Delta| + 2\log\varepsilon^{-1} - 9.9 + \frac{1}{C(\Delta)}\log|\mathcal{N}_{K/Q}(x - 1728)| \\
  &\leq 2\frac{A(46.888\Delta^{1/2} \varepsilon^2 \log|\Delta|^{1/2} + 7.752|\Delta|^{1/2} \varepsilon + 2)}{C(\Delta)} + 2\log\varepsilon^{-1} - 9.9 \\
  &\quad + \frac{1}{C(\Delta)}\log|\mathcal{N}_{K/Q}(x - 1728)| \\
  &\leq 2 \cdot 46.888 \cdot 0.3^2 \frac{\log\log|\Delta|^{1/2}}{C(\Delta)} + 2 \cdot 0.3 \cdot 7.752 + 4A + 2 \log\frac{A|\Delta|^{1/2}}{C(\Delta)} + 2 \cdot 46.888 \cdot 0.3^2 \frac{2 + \log(10^{14})}{18.54 \cdot \pi \log(10^{14})} - 2.84 \\
  &\quad + \frac{1}{C(\Delta)}\log|\mathcal{N}_{K/Q}(x - 1728)| \\
  &\leq 4A + 2 \log\frac{A|\Delta|^{1/2}}{C(\Delta)} - 2.68 + \frac{1}{C(\Delta)}\log|\mathcal{N}_{K/Q}(x - 1728)|.
\end{align*}
\]

For Corollary 4.2 (3), see [2, Corollary 3.2], without assuming that \( x \) is a singular unit, we add the term \( \frac{1}{C(\Delta)}\log|\mathcal{N}_{K/Q}(x)| \).

## 5 Lower Bounds for the Height of a Singular Modulus

We have these propositions from [2]:

**Proposition 5.1 ([2] Proposition 4.1)** Let \( x \) be a singular modulus of discriminant \( \Delta \). Assume that \( |\Delta| \geq 16 \). Then
\[
h(x) \geq \frac{\pi |\Delta|^{1/2} - 0.01}{C(\Delta)}.
\]

**Proposition 5.2** Let \( x \) be a singular modulus of discriminant \( \Delta \). Then
\[
h(x) \geq \frac{3}{\sqrt{5}} \log|\Delta| - 9.79;
\]
\[
h(x) \geq \frac{1}{4\sqrt{5}} \log|\Delta| - 5.93.
\]
Proof

The first one see [2, Proposition 4.3], the second one see [7, Lemma 14 (ii)]

We can use the inequality $h(x - \alpha) \geq h(x) - h(\alpha) - \log 2$ and the results above to give the lower bounds of $h(x - \alpha)$ for an fixed $\alpha$.

6 Proof of Theorem 1.1 (1)

As the set-up in Section 4, Proposition 5.1 and 5.2 allow us to give lower bounds of the height of $x - \alpha$:

$$h(x - \alpha) \geq h(x) - h(\alpha) - \log 2 \geq \pi|\Delta|^{1/2} - 0.01 - h(\alpha) - \log 2,$$

$$h(x - \alpha) \geq h(x) - h(\alpha) - \log 2 \geq \frac{3}{\sqrt{5}} \log |\Delta| - h(\alpha) - 9.79 - \log 2.$$  

For Theorem 1.1 (1), recall the upper bound of $x - \alpha$ in Corollary 4.2 (1) when $|\Delta| \geq 10^{14}$:

$$h(x - \alpha) \leq \frac{8AC(\Delta_x)}{d} + \log\left(\frac{A|\Delta_y|^{1/2}}{d}\right) + 4 \log |\Delta_y| + 0.33 + \frac{1}{d} \log |\mathcal{N}_{K/\mathbb{Q}}(x - \alpha)|.$$  (10)

Throughout the proof of Theorem 1.1 (1), denote the discriminant of a singular modulus $x = j(z)$ by $\Delta$, and we assume that $X = |\Delta| \geq \max\{e3.12(C(\Delta)|\Delta_n|^{1/\log(\alpha)})^3, 10^{15}, C(\Delta)^9\}$. Hence $|\Delta| \geq |\Delta_n|$, since $h(\alpha) \geq 0$.

6.1 The main inequality

Recall that $A = F \log \max\{|\Delta|, |\Delta_n|\} = F \log X$. Minding 0.01 in (8) we deduce from (10) the inequality

$$\frac{8AC(\Delta_n)}{dY} + \frac{\log A + C}{Y} + \frac{\log(X^{1/2}/d)}{Y} + \frac{\log |\mathcal{N}_{K/\mathbb{Q}}(x - \alpha)|}{dY} \geq 1.$$  (11)

Note that $C > 3.11 > 0$, $\log A \geq 0$ because $C \geq 4 \log 7 + \frac{1}{4\sqrt{5}} \log X - 9.78$ in the second term of the left-hand side in (11). Similarly, in the 1st term and 4th term we may replace $Y$ by $\pi X^{1/2}/C(\Delta)$, and in the 3rd term we may replace $X^{1/2}/C(\Delta)$ by $\pi^{-1}Y$. Notice that $d \geq C(\Delta)$, we obtain

$$\frac{8AC(\Delta_n)}{\pi X^{1/2}} + \frac{\log A + C}{\sqrt{5} \log X - 9.78} + \frac{\log(\pi^{-1}Y)}{Y} + \frac{\log |\mathcal{N}_{K/\mathbb{Q}}(x - \alpha)|}{\pi X^{1/2}} \geq 1.$$  (12)

To obtain a lower bound of $\log |\mathcal{N}_{K/\mathbb{Q}}(x - \alpha)|$, we will bound from above each of the three terms in its left-hand side.

From the results in [2, Section 5.2 and Section 5.3], we know that, when $X \geq 10^{15}$,

$$\log A \leq \frac{\log 2}{2} \log \log X - c_1 - \log 2 + \log \log X,$$  (13)

where $c_1 < 1.1713142$. 

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6.2 Bound the first term in (12)

From above, easy to know that when \(X \geq 10^{15}\), we have

\[
\frac{\log(AX^{-1/2})}{\log X} \leq u_0(X),
\]

where

\[
u_0(X) = \frac{\log 2}{2} \frac{1}{\log X - c_1 - \log 2} + \frac{\log \log X}{\log X} - \frac{1}{2}
\]

which is decreasing for \(X \geq 10^{15}\). Hence

\[
\frac{\log(AX^{-1/2})}{\log X} \leq u_0(X) \leq u_0(10^{15}) \leq -0.1908,
\]

so

\[
8\pi C(\Delta_\tau) \pi X^{1/2} \leq 8\pi C(\Delta_\tau) \pi X^{1/2} < 0.0035,
\]

since \(X \geq C(\Delta_\tau)^6 \cdot 10^{15}\).

6.3 Bound the second term in (12)

Obviously, by (13)

\[
\frac{\log A + C - \log X - 9.78}{\sqrt{\log X}} \leq u_1(X) u_2(X),
\]

where

\[
u_1(X) = \frac{\log 2}{2} \frac{1}{\log X - c_1 - \log 2} + \frac{\log \log X + C}{\log X},
\]

\[
u_2(X) = \left(\frac{3}{\sqrt{5}} \frac{9.78}{\log X}\right)^{-1},
\]

which are decreasing for \(X \geq 10^{10}\).

Since \(X \geq e^{3.12(0.1\pi|\Delta_\alpha|^{4} e^{2h(\alpha)})^{3}} = e^{3C}\), we have

\[
\frac{\log \log X + C}{\log X} \leq 0.6.
\]

Indeed, \(g(x) = \log x - 0.6x + C\), which is decreasing for \(x > 5/3\). Let \(x_0 = 3C > 9.33 \geq 5/3\), since \(C > 3.11\). Hence

\[
g(x) \leq g(x_0) = \log 3 + \log C - 0.8C \leq \log 3 + \log(3.11) - 0.8 \cdot 3.11 < 0.
\]

With this we have

\[
u_1(X)u_2(X) \leq \left(\frac{\log 2}{2} \frac{1}{\log \log(10^{15}) - 1.1713142 - \log 2 + 0.6} \cdot u_2(10^{15}) < 0.7621.
\]

6.4 Bound the third term in (12)

For this term, we directly use the bound from [2, subsection 5.5]

\[
\frac{\log(\pi^{-1}Y)}{Y} < 0.0672.
\]

6.5 Summing up

We can combine the above estimates and bound

\[
\frac{\log |N_{K/Q}(x - \alpha)|}{\pi X^{1/2}} \leq 1 - (0.0035 + 0.7621 + 0.0672) = 0.1672,
\]

so

\[
\log |N_{K/Q}(x - \alpha)| > \frac{|\Delta|^{1/2}}{2}.
\]
7 Proof of Theorem 1.1 (2)

As in the last section, we assume that \( |\Delta| \geq 10^{15} \). By inequality (8), (9) and Corollary 4.2 (2), we have

\[
\frac{4A}{C(\Delta)} + 2 \log \left( \frac{AX^{1/2}}{C(\Delta)} \right) + C + \frac{1}{C(\Delta)} \log |\mathcal{N}_{K/Q}(x - 1728)| \geq Y
\]

where

\[
C = h(1728) + \log 2 - 2.68 + 0.01 = \log(3456) - 2.67 > 0,
\]

\[
Y = \max \{ \frac{\pi X^{1/2}}{C(\Delta)}, \frac{3}{\sqrt{5}} \log X - 9.78 \}.
\]

We rewrite this as

\[
\frac{4A}{C(\Delta)} + \frac{2 \log A + C}{Y} + \frac{2 \log(X^{1/2}/C(\Delta))}{Y} + \frac{\log |\mathcal{N}_{K/Q}(x - 1728)|}{C(\Delta)Y} \geq 1.
\]

Hence,

\[
\frac{4A}{\pi X^{1/2}} + \frac{2 \log A + C}{\sqrt{5} \log X - 9.78} + \frac{2 \log(\pi^{-1}Y)}{Y} + \frac{\log |\mathcal{N}_{K/Q}(x - 1728)|}{\pi X^{1/2}} \geq 1.
\]

Using the similar method to estimate each term when \( X \geq 10^{15} \), we have

\[
\frac{4A}{\pi X^{1/2}} < 0.0018,
\]

\[
\frac{2 \log A + C}{\sqrt{5} \log X - 9.78} < 0.7337,
\]

\[
\frac{\log(\pi^{-1}Y)}{Y} < 0.0672,
\]

\[
\frac{\log |\mathcal{N}_{K/Q}(x - 1728)|}{\pi X^{1/2}} \geq 1 - (0.0018 + 0.7337 + 2 \cdot 0.0672) = 0.1301.
\]

Hence,

\[
\log |\mathcal{N}_{K/Q}(x - 1728)| \geq 0.1301 \pi X^{1/2} \geq \frac{2|\Delta|^{1/2}}{5}.
\]

8 Proof of Theorem 1.1 (3)

As before, we assume that \( X = |\Delta| \geq 10^{15} \). By Proposition 5.1, Proposition 5.2 and Corollary 4.2 (3), we have

\[
\frac{12A}{C(\Delta)} + 3 \log \left( \frac{AX^{1/2}}{C(\Delta)} \right) - 3.76 + \frac{1}{C(\Delta)} \log |\mathcal{N}_{K/Q}(x)| \geq Y,
\]

where

\[
Y = \max \{ \frac{\pi X^{1/2}}{C(\Delta)}, \frac{3}{\sqrt{5}} \log X - 9.78 \}.
\]

We rewrite this as

\[
\frac{12A/C(\Delta)}{Y} + \frac{3 \log A - 3.76}{Y} + \frac{3 \log(X^{1/2}/C(\Delta))}{Y} + \frac{\log |\mathcal{N}_{K/Q}(x)|/C(\Delta)}{Y} \geq 1.
\]

Note that \( 3 \log A - 3.76 > 0 \) because \( A \geq \log X \geq \log(10^{15}) > 30 \). Hence, we obtain

\[
\frac{12A}{\pi X^{1/2}} + \frac{3 \log A - 3.76}{\sqrt{5} \log X - 9.78} + \frac{3 \log(\pi^{-1}Y)}{Y} + \frac{\log |\mathcal{N}_{K/Q}(x)|}{\pi X^{1/2}} \geq 1.
\]

From the results in [2, Page 23 to Page 25], we know that, when \( X \geq 10^{15} \),

\[
AX^{-1/2} < 0.0014,
\]

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\[
\begin{align*}
3 \log A - 3.76 &< 0.7734, \\
\frac{\log X}{\sqrt{e}} - 9.78 &< 0.0672.
\end{align*}
\]

We can combine the above estimates and bound \(\frac{\log |N_{K/Q}(x)|}{\pi X^{1/2}}\) by
\[
\frac{\log |N_{K/Q}(x)|}{\pi X^{1/2}} > 1 - (12\pi^{-1} \cdot 0.0014 + 0.7734 + 3 \cdot 0.0672) > 0.019,
\]
so
\[
\log |N_{K/Q}(x)| > \frac{|\Delta|^{1/2}}{20}.
\]

## 9 Proof of Corollary 1.2

We know that the degree of \(\Phi_m(X, Y)\) at \(Y\) is \(\deg_Y \Phi_m(X, Y) = \sigma_1(m)\). Assume that \(\alpha = j(\tau)\), and \(\{\gamma_1, \cdots, \gamma_{\sigma_1(m)}\}\) is a set of representatives of \(\text{SL}_2(\mathbb{Z}) \backslash D_m\). We set \(\alpha_i = j(\gamma_i(\tau))\), which are also singular moduli.

Let \(L = K(\alpha_1, \cdots, \alpha_{\sigma_1(m)})\) and \(K_i = \mathbb{Q}(x, \alpha_i)\). We have \([K_i(\alpha) : K] \leq \deg_Y \Phi(X, Y) = \sigma_1(m)\), and
\[
N_{K_i(\alpha)/\mathbb{Q}}(x - \alpha_i) = N_{K_i/\mathbb{Q}}(x - \alpha_i)^{|K_i(\alpha) : K_i|}.
\]
Then, when \(\Delta\) is large enough, by Theorem 1.1 we have
\[
\begin{align*}
\log |N_{K/Q}(\Phi_m(x, \alpha))| &= \log |N_{L/Q}(\prod_{i=1}^{\sigma_1(m)} (x - \alpha_i))^{1/[L:K]}| \\
&= \frac{1}{[L : K]} \sum_{i=1}^{\sigma_1(m)} \log |N_{L/Q}(x - \alpha_i)| \\
&= \frac{[L : K]}{[L : K]} \sum_{i=1}^{\sigma_1(m)} \log |N_{K_i/Q}(x - \alpha_i)| \\
&= \frac{[K_i(\alpha) : K_i]}{[K_i(\alpha) : K]} \sum_{i=1}^{\sigma_1(m)} \log |N_{K_i/Q}(x - \alpha_i)| \\
&\geq \frac{1}{\sigma_1(m)} \sum_{i=1}^{\sigma_1(m)} |\Delta|^{1/2} \\
&= \frac{|\Delta|^{1/2}}{20}.
\end{align*}
\]

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