MINIMAL PRÜFER-DRESS RINGS AND PRODUCTS OF IDEMPOTENT MATRICES

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Abstract. We investigate a special class of Prüfer domains, firstly introduced by Dress in 1965. The minimal Dress ring $D_K$, of a field $K$, is the smallest subring of $K$ that contains every element of the form $1/(1 + x^2)$, with $x \in K$. We show that, for some choices of $K$, $D_K$ may be a valuation domain, or, more generally, a Bézout domain admitting a weak algorithm. Then we focus on the minimal Dress ring $D$ of $\mathbb{R}(X)$: we describe its elements, we prove that it is a Dedekind domain and we characterize its non-principal ideals. Moreover, we study the products of $2 \times 2$ idempotent matrices over $D$, a subject of particular interest for Prüfer non-Bézout domains.

Introduction

It is well-known that the class $\mathcal{P}$ of Prüfer domains is as important as large, and that several natural questions related to $\mathcal{P}$ are still open. (A most significant one is whether every Bézout domain is an elementary divisor ring; see [8] Ch. III.6.)

A nice subclass of $\mathcal{P}$ was discovered by Dress in the 1965 paper [4]. Let $K$ be a field; a subring $D$ of $K$ is said to be a Prüfer-Dress ring (or simply a Dress ring) if $D$ contains every element of the form $1/(1 + x^2)$ for $x \in K$.

Of course, many important Prüfer domains (the integers $\mathbb{Z}$, to name one) are not Dress rings. Nonetheless, the result in [4] that any Dress ring is actually Prüfer (see the first section) is quite useful. Indeed, it clearly furnishes a method for constructing relevant examples of Prüfer domains (e.g., see [10], [16]).

For any assigned field $K$, we may consider the minimal Dress ring $D_K$ of $K$, namely $D_K = \mathbb{Z}_X[1/(1 + x^2) : x \in K]$ where $\mathbb{Z}_X$ is the prime subring of $K$. In this paper we mainly deal with minimal Dress rings $D_K$. In the first section we give some general results. We show that $D_K$ may coincide with $K$ (e.g., for $K = \mathbb{R}$), and that $D_K$ may be a valuation domain or a pull-back of a valuation domain. For $K = \mathbb{R}(A)$, $A$ a set of indeterminates, we also examine the relations between $D_K$ and the Prüfer domains investigated by Schütting [15] (1979). Recall that Schütting was the first to prove the

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existence of Prüfer domains \( R \) containing finitely generated ideals that are neither principal nor two-generated (see also [12] for a neat discussion on this subject). As a by-product of the results in [15], we get that also some minimal Dress rings admit \( n \)-generated ideals that are not 2-generated.

In the second section we focus on the minimal Dress ring \( D \) of \( K = \mathbb{R}(X) \). We describe the elements of \( D \), prove that it is a Dedekind domain, and characterize its non-principal ideals in terms of their generators. We also show, in Remark 2.6, that \( D \) is a simple example of a Dedekind domain of square stable range one, that does not have 1 in the stable range (cf. [11] for these notions).

Finally, in the third section we examine the products of idempotent \( 2 \times 2 \) matrices over \( D \). This kind of questions have raised considerable interest, both in the commutative and non-commutative setting. The literature dedicated to these products includes [7], [13], [14], [1], [5], [3]. We say that \( R \) satisfies property (ID\(_2\)) if every \( 2 \times 2 \) singular matrix over \( R \) is a product of idempotent matrices. A natural conjecture, proposed in [14] and investigated in [3], states that, if \( R \) satisfies (ID\(_2\)), then \( R \) is a Bézout domain (the converse is not true, not even for PID’s; see [14]). Note that in [3] it is shown that an integral domain \( R \) satisfying (ID\(_2\)) is necessarily Prüfer.

In this paper we prove, in Theorem 3.3, that if \( p, q \) are elements of \( D \) satisfying some conditions on their degrees and roots (recall that \( p, q \) are rational functions), then the matrix \[
\begin{pmatrix}
p & q \\
0 & 0
\end{pmatrix}
\] is a product of idempotent matrices.

The question whether \( D \) does not satisfy (ID\(_2\)), in accordance with the conjecture, remains open. We just observe that to verify that a Prüfer domain does not satisfy (ID\(_2\)) is always a challenging problem; for instance, see [2] and [3] to get an idea of the difficulty. As a matter of fact, in those papers, it is shown that matrices as above are not products of idempotent matrices, for suitable choices of \( p, q \).

1. Dress rings

Let \( K \) be a (commutative) field; a subring \( D \) of \( K \) is said to be a Prüfer-Dress ring (or simply a Dress ring) if \( D \) contains every element of the form
\[
\frac{1}{1 + x^2}, \quad x \in K.
\]

Of course, the existence of a Dress ring in \( K \) implies that the polynomial \( X^2 + 1 \in K[X] \) is irreducible; in particular, if \( K \) has characteristic \( p \neq 0 \), then \( p \equiv 3 \) modulo 4.

A Dress ring \( D \) is always a Prüfer domain, since every ideal of \( D \) generated by two elements is invertible, a sufficient condition to be Prüfer (see [9], Theorem 22.1). Indeed, for any ideal \((a, b)\) in \( D \) we have \( (a, b)^2 = (a^2 + b^2)D \). This easily derives from the formula, observed by Dress [4],
\[
\frac{2x}{1 + x^2} - \frac{y^2 - z^2}{y^2 + z^2} \quad \text{if } y = x + 1, z = x - 1.
\]

In fact, for \( x = b/a \) or \( x = a/b \), using (1) we get \( a^2/(a^2 + b^2) \in D \), \( b^2/(a^2 + b^2) \in D \), respectively. From (2) we get \( ab/(a^2 + b^2) \in D \) (recall that \( 1/2 \in \mathbb{R} \)).
It follows that \((a, b)^2 = (a^2, ab, b^2) \subseteq (a^2 + b^2)D\), and the reverse inclusion is trivial. From the above formula it also follows that \(K\) is the field of fractions of \(D\).

From now on, \(K\) will always denote a field not containing square roots of \(-1\). For any ring \(R\), we denote by \(R^*\) its multiplicative subgroup of units.

We are interested in the minimal Dress ring \(D_K\) of \(K\), namely
\[
D_K = \mathbb{Z}_\chi[1/(1 + x^2) : x \in K],
\]
where \(\chi\) is the characteristic of \(K\) and \(\mathbb{Z}_\chi = \mathbb{Z}/\mathbb{Z}\) is the prime subring of \(K\). (Recall that \(\chi \neq 1, 2\) modulo 4.)

We recall now an equivalent definition of minimal Dress rings, in terms of intersections of special valuation domains. Given a valuation \(v\) on the field \(K\), we will denote by \(V_v\) the corresponding valuation domain and by \(\mathfrak{M}_v\) the maximal ideal of \(V_v\).

Dress [3] proved that \(D_K\) is the intersection
\[
D_K = \bigcap_{v \in B} V_v,
\]
where \(B\) is the family of the valuations over \(K\) such that \(\sqrt{-1} \notin V_v/\mathfrak{M}_v\). Note that \(B \neq \emptyset\), since, by the standing assumption \(\sqrt{-1} \notin K\), at least the trivial valuation lies in \(B\).

For the sake of completeness, we verify the equality [3]. If \(v \in B\), then for every \(x \in K\) we get \(v(1 + x^2) \leq 0\), hence \(1/(1 + x^2) \in V_v\). It follows that \(V_v \supseteq \mathbb{Z}_\chi[1/(1 + x^2) : x \in K] = D_K\), hence \(D_K \subseteq \bigcap_{v \in B} V_v\). Conversely, being a Prüfer domain, \(D_K\) is integrally closed, hence it is the intersection of its valuation overrings. Take any valuation overring \(V\) of \(D_K\), with maximal ideal \(\mathfrak{M}\). Assume, for a contradiction, that \(\sqrt{-1} \in V/\mathfrak{M}\). Then there exists \(y \in V\) such that \(1 + y^2 \in \mathfrak{M}\), so \(1/(1 + y^2) \notin V\). However \(1/(1 + y^2) \in D_K \subset V\), impossible. We have got the reverse inclusion \(D_K \supseteq \bigcap_{v \in B} V_v\).

Now we examine the basic examples of minimal Dress rings.

**Proposition 1.1. Let \(D = D_K\) be the minimal Dress ring in the field \(K\).**

(i) If \(K\) is an ordered field such that every positive element is a square, then \(D = K\).

(ii) If \(K = \mathbb{Q}\), then \(D = \mathbb{Z}_S\) where \(S\) is the multiplicatively closed subset of \(\mathbb{Z}\) generated by the primes \(p \equiv 1\) modulo 4. \(D\) is an Euclidean domain that is not a valuation domain.

**Proof.** (i) By definition, \(D = \mathbb{Z}[1/(1 + x^2) : x \in \mathbb{R}]\). Take any \(\alpha \in K\) with \(0 < \alpha < 1\). Then \(1/\alpha = 1 + x^2\) for a suitable \(x \in \mathbb{R}\), and therefore \(\alpha \in D\). Now take any \(\beta \in K\), and choose an integer \(m\) such that \(0 < \beta/m < 1\). Then \(\beta/m \in D\), and so \(\beta \in D\), as well. We conclude that \(D = K\).

(ii) Now \(D = \mathbb{Z}[1/(1 + x^2) : x \in \mathbb{Q}] = \mathbb{Z}[a^2/(a^2 + b^2) : a, b \in \mathbb{Z} \setminus \{0\}]\). Take any prime number \(p\) and assume that \(1/p \in D\). Then
\[
1/p = \prod_{i=1}^{m}(a_i^2 + b_i^2)
\]
for suitable integers \(m, a_i, b_i\). It follows that \(p\) divides \(\prod_{i}(a_i^2 + b_i^2)\). As well-known, for instance by the properties of the Gaussian integers, this is
possible if and only if $p$ is a sum of two squares, if and only if $p \equiv 1 \pmod{4}$. Conversely, assume that the prime number $p$ is a sum of two squares, say $p = a^2 + b^2$. Then we get

$$1/p = \frac{1}{a^2 + b^2} = \frac{\lambda a^2}{a^2 + b^2} + \frac{\mu b^2}{a^2 + b^2} \in D,$$

for suitable integers $\lambda, \mu$, that exist since $a^2, b^2$ are coprime. We easily conclude that $D = \mathbb{Z}_S$, as in the statement.

In particular, $D$ is Euclidean, since it is a localization of the Euclidean domain $\mathbb{Z}$, and is not a valuation domain, since, for instance, both $3/7$ and $7/3$ do not lie in $D$. □

The following two propositions apply, respectively, when $K$ is either $\mathbb{R}((X))$ or $\mathbb{Q}((X))$, the fields of Laurent series ($X$ an indeterminate).

**Proposition 1.2.** Let $K$ be an Henselian field with respect to the valuation $v$, $V$ the valuation domain of $v$, $\mathfrak{M}$ its maximal ideal. If $V/\mathfrak{M}$ is an ordered field such that every positive element is a square, then $V$ is the minimal Dress ring in $K$.

**Proof.** Up to isomorphism, we may assume $V/\mathfrak{M} \subseteq \mathbb{R}$. Then the field $K$ has characteristic zero and $\mathbb{Q} \subseteq V$. Let $D_K$ be the minimal Dress ring in $K$. We firstly prove that $V \subseteq D_K$. We start verifying that $\mathfrak{M} \subseteq D_K$. Take any $0 \neq x \in \mathfrak{M}$. We look for $z \in K$ such that $x = z/(1 + z^2)$. Consider the polynomial $\phi = Y^2 - Y + x^2 \in V[Y]$, that, modulo $\mathfrak{M}$, has the nonzero simple root $1 + \mathfrak{M}$. Since $V$ is Henselian, $\phi$ has a root $y \in V$. Setting $z = x/y$, we readily see that $z$ is the element we were looking for. Since $x$ was arbitrary, we conclude that $\mathfrak{M} \subseteq D$. Now take any unit $\eta \in V^*$. Since $\mathbb{Q} \subseteq V$ and the positive elements of $V/\mathfrak{M}$ are squares, we may pick an integer $m$ such that $\eta/m \equiv 1/(1 + z^2)$ modulo $\mathfrak{M}$, for some $z \in V$ (see Proposition 1.1(i)). We conclude that $\eta \in D$, since $\mathfrak{M} \subseteq D$.

Let us verify the reverse inclusion. Since every element of $K \setminus V$ is the inverse of an element of $\mathfrak{M}$, it suffices to show that $1/x \notin D_K$ for every $x \in \mathfrak{M}$. We start noting that, since $V$ is a valuation domain, every element $\alpha = 1/(1 + z^2)$, $z \in K$ may be written either as $\alpha = 1/(1 + f^2)$, $f \in V$, or $\alpha = g^2/(1 + g^2)$, $g \in V$. As a consequence, any $r \in D_K$ has the form $r = f/\prod_i (1 + g_i^2)$, for suitable $f, g_i \in V$. We now assume, for a contradiction, that $1/x = f/\prod_i (1 + g_i^2) \in D_K$, for suitable $f, g_i \in V$. It follows that $\prod_i (1 + g_i^2) \equiv 0$ modulo $\mathfrak{M}$, against the hypothesis $V/\mathfrak{M} \subseteq \mathbb{R}$. □

Following [14], we say that an integral domain $R$ admits a weak (Euclidean) algorithm, if for any $a, b \in R$ there exists a (finite) sequence of divisions that starts with $a, b$ and terminates with last remainder zero. Necessarily, such an $R$ is a Bézout domain.

**Proposition 1.3.** Let $K$ be an Henselian field with respect to the valuation $v$, $V$ the valuation domain of $v$, $\mathfrak{M}$ its maximal ideal. Let $D_K$ be the minimal Dress ring in $K$. If $V/\mathfrak{M} \cong \mathbb{Q}$, then $D_K = \mathbb{Z}_S + \mathfrak{M}$, where $S$ is the multiplicatively closed subset of $\mathbb{Z}$ generated by the primes $p \equiv 1 \pmod{4}$. $D_K$ is not a valuation domain, nor Noetherian, but admits a weak algorithm. In particular, $D_K$ is a Bézout domain.
Proof. Since $V/\mathfrak{M} \cong \mathbb{Q}$, the field $K$ has characteristic zero and $\mathbb{Q} \subset V$. Since $V$ is Henselian, in the same way as in Proposition 1.2 we may show that $\mathfrak{M} \subset D_K$. Moreover, like in Proposition 1.1(ii) we see that $\mathbb{Z}_S \subset D_K$. It follows that $\mathbb{Z}_S + \mathfrak{M} \subseteq D_K$.

Let us verify the reverse inclusion. Take any $r \in D_K$; we may assume that $r \notin \mathfrak{M}$. Since $V/\mathfrak{M} \cong \mathbb{Q}$, the element $r$ may be written as $r = a + \delta$, where $a \in \mathbb{Q}$ and $\delta \in \mathfrak{M}$. Hence, to prove that $D_K \subseteq \mathbb{Z}_S + \mathfrak{M}$, it suffices to verify that $a \in \mathbb{Z}_S$. Since $a \in D_K$, we get $a = f/\prod_i (1 + g_i^2)$, for some $f, g_i \in V$ (cf. the proof of Proposition 1.2). Again using $V/\mathfrak{M} \cong \mathbb{Q}$, we get $a \equiv b/\prod_i (1 + c_i^2)$ modulo $\mathfrak{M}$, for some $b, c_i \in \mathbb{Q}$, and therefore $a = b/\prod_i (1 + c_i^2)$, since $\mathbb{Q} \cap \mathfrak{M} = 0$. So $a \in \mathbb{Z}_S$, by Proposition 1.1(ii).

Moreover, $D_K$ is not a valuation domain, since $3/7, 7/3 \in \mathbb{Q} \setminus D_K$, and $D_K$ is not Noetherian, since the $D_K$-ideal $\mathfrak{M}$ is not finitely generated. To get more examples, we remark that any $D_K$-ideal generated by the set \( \{x/p^n : n > 0\} \), where $0 \neq x \in \mathfrak{M}$ and $p \equiv 3 \mod 4$, is not finitely generated. Finally, in view of Proposition 6.4 of [14], $D_K$ admits a weak algorithm, since $\mathbb{Z}_S$ is Euclidean; in particular $D_K$ is a Bézout domain. \( \square \)

We remark that all the above examples of minimal Dress rings satisfy property ID$_2$, recalled in the introduction (the last one since it admits a weak algorithm, cf. [14]).

Let $\mathcal{A}$ be a set of indeterminates over $\mathbb{R}$, $K = \mathbb{R}(\mathcal{A})$ the corresponding field of rational functions. If $f, g \in \mathbb{R}[\mathcal{A}]$, we define the degree of $f/g$ in the natural way, namely: $\deg(f/g) = \deg(f) - \deg(g)$. We remark the following useful, readily verified property: if $f_1, g_1, f_2, g_2 \in \mathbb{R}[\mathcal{A}]$ are sums of squares, then $\deg(f_1/g_1 + f_2/g_2) = \sup\{\deg(f_1/g_1), \deg(f_2/g_2)\}$.

Recall that a field $F$ is formally real if $1 + \sum x_i^2 \neq 0$ for every choice of $x_i \in F$ (in particular $\sqrt{-1} \notin F$). A valuation $v$ on the field $K$ is said to be formally real if the residue field $V_v/\mathfrak{M}_v$ is formally real. The existence of formally real valuations on the field $K$ implies that $K$ is formally real; in particular, $\sqrt{-1} \notin K$. In the 1979 paper [15] Schülling considered the integral domain

$$R_K = \bigcap_{v \in C} V_v,$$

where $C$ is the set of the formally real valuations on $K$. From the equality [3] it immediately follows that $R_K$ contains the minimal Dress ring $D_K$. In fact, if $v$ is a formally real valuation, then $\sqrt{-1} \notin V_v/\mathfrak{M}_v$, hence $V_v \supset D_K$. So the integral domain $R_K$ is a Dress ring in our sense, hence it is a Prüfer domain; it satisfies several interesting properties (see Chapter II of [4] for a detailed description).

The main Schülling’s purpose was to exhibit examples of Prüfer domains that admit finitely generated ideals that are not 2-generated. Indeed, if $K = \mathbb{R}(X_1, \ldots, X_n)$, it was proved that the fractional ideal $(1, X_1, \ldots, X_n)$ of $R_K$ cannot be generated by less then $n + 1$ elements (see [15] and [12]). This result is deep and difficult to prove: techniques of algebraic geometry are required.
If $K = \mathbb{R}[X_1, \ldots, X_n]$ with $n \geq 2$ we conclude that also $D_K$ admits finitely generated ideals ideals that are not 2-generated. For instance, if $n = 2$ the fractional $D_K$-ideal $(1, X_1, X_2)$ is not 2-generated over $D_K$, otherwise the fractional $R_K$-ideal would be 2-generated over $R_K \supset D_K$, against Schütting results in [15].

**Proposition 1.4.** In the above notation, let $K = \mathbb{R}(X_1, \ldots, X_n)$.

(i) If $n = 1$, then $R_K$ coincides with $D_K$;

(ii) if $n \geq 2$, then $R_K$ properly contains $D_K$.

**Proof.** (i) Let $K = \mathbb{R}(X)$. By (3) $R_K = D_K$ if and only if for every valuation overring $V$ of $D_K$, the residue field $V/\mathfrak{m}$ is formally real. Assume, for a contradiction, that there exist $r_1, \ldots, r_m \in V \setminus \mathfrak{m}$ such that $1 + \sum_{i=1}^{m} r_i^2 \in \mathfrak{m}$. We may write $r_i = f_i/g$, where $f_i, g \in \mathbb{R}[X]$. Let $v$ be the valuation determined by $V$; recall that $\sqrt{-1} \notin V/\mathfrak{m}$. We firstly consider the case where $v(X) \geq 0$, hence $v(g) \geq 0$, and the above equality yields $\phi = g^2 + \sum_{i=1}^{m} f_i^2 \in \mathfrak{m}$. Factorizing $\phi$ in $\mathbb{R}[X]$, we may write $\phi = \alpha \prod_j (1 + s_j^2) \in \mathfrak{m}$, for suitable $\alpha > 0$ and linear polynomials $s_j$. We reached a contradiction, since $\sqrt{-1} \notin V/\mathfrak{m}$ implies that $1 + s_j^2 \notin \mathfrak{m}$, for all $j$.

It remains to examine the special case where $v(X) < 0$. Under the present circumstances $v = -\deg$, hence $f_i/g \in V \setminus \mathfrak{m}$ implies $\deg f_i = \deg g$ and $\deg \phi = 2 \deg g$. Therefore $v(\phi/g^2) = -\deg(\phi/g^2) = 0$, so $1 + \sum_{i=1}^{m} r_i^2 \notin \mathfrak{m}$, another contradiction.

(ii) Let $K = \mathbb{R}(X_1, \ldots, X_n)$ with $n \geq 2$. The Schütting ring $R_K$ properly contains $D_K$ if there exists a valuation overring $V$ of $D_K$ such that $V/\mathfrak{m}$ is not formally real. Let us consider the irreducible polynomial $f = 1 + X_1^2 + \cdots + X_n^2 \in \mathbb{R}[X_1, \ldots, X_n]$. Let $w$ be the rank-one discrete valuation determined by $f$, i.e., the valuation that extends the assignments $w(f) = 1$ and $w(g) = 0$ if $f$ does not divide $g \in \mathbb{R}[X_1, \ldots, X_n]$. By the definition, $w$ is not a formally real valuation, hence the valuation domain $V$ determined by $w$ does not contain $R_K$. To get our conclusion, it suffices to prove that $V$ is a valuation overring of $D_K$. Assume, for a contradiction, that $\sqrt{-1} \in V/\mathfrak{m}$, say $a^2/b^2 + 1 \in \mathfrak{m}$, with $a, b \in \mathbb{R}[X_1, \ldots, X_n]$, $b$ coprime with $f$. Then there exist $c, d \in K[X_1, \ldots, X_n]$, $d$ coprime with $f$, such that $d(a^2 + b^2) = fcb^2$. By coprimality we get $a^2 + b^2 = fe$, for some $e \in \mathbb{R}[X_1, \ldots, X_n]$. Factorizing the last equality in $\mathbb{C}[X_1, \ldots, X_n]$, we derive that $f$ divides $a + ib$ (and $a - ib$, as well) in $\mathbb{C}[X_1, \ldots, X_n]$, since $f$ is irreducible in $\mathbb{C}[X_1, \ldots, X_n]$.

In this section we focus on the minimal Dress rings $D$ of the field of rational functions $\mathbb{R}(X)$. Our aim is to give a complete description of the elements of this ring. We will also prove that $D$ is a Dedekind domain, i.e. a Noetherian Prüfer domain, and we characterize the non-principal ideals of $D$.

Let $\Gamma = \{\alpha \prod_i \gamma_i\}$, where the $\gamma_i$ are monic degree-two polynomials irreducible over $\mathbb{R}[X]$ and $0 \neq \alpha$ is a real number. Of course, $\Gamma$ coincides with the set of the polynomials in $\mathbb{R}[X]$ that have no roots in $\mathbb{R}$. In what
follows we also use the notation $\Gamma^+ = \{ f \in \mathbb{R}[X] : f(r) > 0, \forall r \in \mathbb{R} \}$, and, correspondingly, $\Gamma^- = \{- f : f \in \Gamma^+ \}$.

It is easy to characterize the elements of $D$.

**Proposition 2.1.** Let $D$ be the minimal Dress ring of $\mathbb{R}[X]$. Then

$$D = \{ f/\gamma : f \in \mathbb{R}[X], \gamma \in \Gamma, \deg f \leq \deg \gamma \}.$$  

**Proof.** Since $\mathbb{R} \subset \mathbb{R}(X)$, we get $\mathbb{R} \subset D$, and therefore $D = \mathbb{R}[a^2/(a^2 + b^2)]$, where $a, b \in \mathbb{R}[X]$. Say $f/\gamma \in D$, for coprime polynomials $f, \gamma \in \mathbb{R}[X]$. It is then clear that $\gamma \in \Gamma$, since in $\mathbb{R}[X]$ a sum of two squares lies in $\Gamma$. Since the generators $a^2/(a^2 + b^2)$ have degree $\leq 0$, it readily follows that $\deg(f/\gamma) \leq 0$. Hence $D \subseteq \{ f/\gamma : \gamma \in \Gamma, \deg f \leq \deg \gamma \}$. To verify the reverse inclusion, it suffices to show that, for any assigned $\gamma \in \Gamma$ of degree $2\gamma$, say, the rational function $X^m/\gamma$ lies in $D$, for $0 \leq m \leq 2\gamma$. We can write $\gamma = \alpha \prod_{i=1}^{\gamma}(1 + (r_i X - s_i)^2)$, for suitable real numbers $\alpha \neq 0, r_i \neq 0, s_i$. Then

$$(1 + (r_i X - s_i)^2), (r_i X - s_i)/(1 + (r_i X - s_i)^2)$$ and $(r_i X - s_i)^2/(1 + (r_i X - s_i)^2)$ lie in $D$, hence easy computations show that $X/(1 + (r_i X - s_i)^2) \in D$, $X^2/(1 + (r_i X - s_i)^2) \in D$. We readily conclude that $X^m/\gamma \in D$. \hfill $\Box$

From the preceding characterization, we immediately see that $u \in D^*$ if and only if $u = \gamma_1/\gamma_2$, where $\gamma_1, \gamma_2 \in \Gamma$ and $\deg \gamma_1 = \deg \gamma_2$. Moreover, any $r \in D$ is associated to an element of the form $f/\gamma$, where $f \in \mathbb{R}[X]$ is a product of linear factors.

**Proposition 2.2.** Let $D$ be the minimal Dress ring of $\mathbb{R}(X)$. Let $J = (r_i : i \in \Lambda)$ be an ideal of $D$. Then $J^2 = (r_i^2 : i \in \Lambda)$.

**Proof.** It is clear that $J^2 \supseteq (r_i^2 : i \in \Lambda)$. Conversely, take a typical generator $r_j r_k$ of $J^2$. We know that $r_j r_k \in (r_j, r_k)^2 = (r_j^2 + r_k^2)D \subseteq (r_i^2 : i \in \Lambda)$. We conclude that $(r_i^2 : i \in \Lambda) \supseteq J^2$. \hfill $\Box$

**Theorem 2.3.** The minimal Dress ring $D$ of $\mathbb{R}(X)$ is a Dedekind domain.  

**Proof.** Since $D$ is a Prufer domain, in order to verify that it is actually a Dedekind domain, it suffices to show that $D$ is Noetherian.

Let $J \neq 0$ be an ideal of $D$. We will prove that $J^2$ is a principal ideal, so $J$ is invertible, and therefore finitely generated.

Say $J = (r_i : f_i/\gamma_i : i < \Lambda)$, where $f_i \in \mathbb{R}[X], \gamma_i \in \Gamma, \deg(r_i) \leq 0$. Possibly replacing $J$ with an isomorphic ideal, we may safely assume that the $f_i$ are coprime, i.e., there exists a finite subset $A$ of $\Lambda$ such that $1 = \gcd(f_j : j \in A)$. By Proposition 2.2 we know that $J^2 = (r_i^2 : i \in \Lambda)$.

Let $s = \sum_{j \in A} r_j^2$. Since $\deg(r_i) \leq 0$ for every $i \in \Lambda$, we may choose $s$ of maximal degree. Let us verify that $J^2 = sD$. It suffices to show that $r_m = (f_m/\gamma_m)^2 \in sD$ for every generator $r_m$ of $J^2$. Say $s = g/\gamma^2$, where $\gamma = \prod_{j \in A} \gamma_j$ and $g = \sum_{j \in A} f_j^2\gamma_j^2/\gamma^2$. Note that $g$ is a sum of squares and has no roots in $\mathbb{R}$, since the $f_j$ have no common root in $\mathbb{R}$. We get

$$\frac{(f_m/\gamma_m)^2}{s} = \frac{f_m^2 \gamma_m^2}{g \gamma_m^2} \in D.$$  

In fact $g \gamma_m^2 \in \Gamma$, since $g$ has no roots in $\mathbb{R}$. Moreover $\deg(f_m^2/\gamma_m^2) \leq \deg s$, otherwise the choice $s_1 = s + (f_m/\gamma_m)^2$ would contradict the assumption
that \( s \) has maximal degree. (Recall that \( \deg(s + r_m^2) = \sup(\deg(s), \deg(r_m^2)) \), since \( s \) is a sum of squares.) \( \square \)

Since \( D \) is a Dedekind domain, all its ideals are generated by two elements. The next proposition gives a characterization of the ideals over \( D \) that are exactly 2-generated.

**Proposition 2.4.** Let \( f/\gamma \) and \( g/\gamma \) be elements of \( D \), with \( f, g \in \mathbb{R}[X] \), \( \gamma \in \Gamma \). Let \( M = \gcd(f, g) \), say \( f = Mf' \), \( g = Mg' \). Then the ideal \( (f/\gamma, g/\gamma) \) is not principal if and only if \( s = \sup(\deg f', \deg g') \) is odd.

**Proof.** It suffices to prove that the fractional ideal \((f', g')\) is exactly 2-generated as a \( D \)-module if and only if \( s = \sup(\deg f', \deg g') \) is odd. In any case, \( f'^2 + g'^2 \in \Gamma \) and \( \deg(f'^2 + g'^2) = 2s \), since \( f', g' \) are coprime.

We firstly consider the case when \( \deg(f'^2 + g'^2) = 2s \) with \( s \) odd. Assume by contradiction that \((f', g') = hD \) for some \( h \in \mathbb{R}(X) \). It follows that \( (f', g')^2 = (f'^2 + g'^2)D = h^2D \), hence \( f'^2 + g'^2 = uh^2 \), where \( u \in D^* \). It follows that the rational function \( h \) has neither zeros nor poles in \( \mathbb{R} \), hence \( h = \gamma/\delta \), with \( \gamma, \delta \in \Gamma \). Therefore \( h \) has even degree, say \( \deg h = 2n \). Then \( u = (f'^2 + g'^2)/h^2 \) shows that \( \deg u = 2s - 4n \neq 0 \). But this is impossible, since \( u \) is a unit of \( D \), hence \( \deg u = 0 \).

Conversely, let us assume that \( s \) is even. If \( s = 0 \), both \( f' \) and \( g' \) lie in \( \mathbb{R} \) and \((f', g') = D \). If \( s > 0 \), take \( h \in \Gamma \) of degree \( s \), possible since \( s \) is even. Then \( f'/h \) and \( g'/h \) are both elements of \( D \), so \((f', g') \subseteq hD \).

Moreover, let
\[
\begin{align*}
u &= \frac{f'^2 + g'^2}{h^2}.
\end{align*}
\]
Then the choice of the degrees shows that \( \deg u = 0 \), so \( u \in D^* \). From the preceding equality we get
\[
\begin{align*}h &= \frac{(f'^2 + g'^2)}{uh} = u^{-1}f'h + u^{-1}g'h,\end{align*}
\]
where \( u^{-1}f'/h, u^{-1}g'/h \in D \), whence \( h \in (f', g') \). We have got the reverse inclusion \( hD \subseteq (f', g') \). \( \square \)

**Remark 2.5.** One may consider the minimal Dress rings \( D_K \) of \( K = \mathbb{R}(A) \), where \( A \) is any set of indeterminates. However, the property of being Noetherian is specific of the Dress ring of \( \mathbb{R}(X) \). In fact, by Schülling’s result in \[15\], generalized by Olberding and Roitman in \[12\], if \( A \) contains more than one indeterminate, then \( D_K \) contains finitely generated ideals that are not 2-generated. Therefore \( D_K \) cannot be Noetherian, otherwise, being a Prüfer domain, it should be Dedekind, hence all its ideals should be generated by two elements.

**Remark 2.6.** It is worth noting that the minimal Dress ring \( D \) of \( \mathbb{R}(X) \) is a simple example of a Dedekind domain of square stable range one, but without 1 in the stable range. We refer to \[11\] for a thorough investigation on these notions. The ring \( D \) appears to be easier than their examples, based on Theorem 5 of Swan’s paper \[10\].

Take \( a, b \in D \) such that \((a, b) = D \). Then, from \( D = (a, b)^2 = (a^2 + b^2)D \), it follows that \( a^2 + b^2 \in D^* \), and therefore \( D \) has square stable range one.
Let us show that $D$ does not have 1 in the stable range. Take the following elements of $D$: $a = X/\gamma$, $b = (X^2 - 1)/\gamma$, where $\gamma = 1 + X^2$. Then $D = (a, b)$, since $a^2 + b^2$ is a unit of $D$. Let us show that $a + bz \notin D^*$, for every $z \in D$. Let us assume, for a contradiction, that $a + bz \in D^*$, where $z = f/\delta$, for suitable $f \in \mathbb{R}[X]$ and $\delta \in \Gamma^+$. Being a unit, $a + bz$ has no roots in $\mathbb{R}$. Equivalently, $f_1 = X\delta + (X^2 - 1)f$ has no roots. However, $f_1(1) = \delta(1) > 0$, and $f_1(-1) = -\delta(-1) < 0$, so $f_1$ does have roots. We reached a contradiction.

3. Matrices over $D$ that are products of idempotent matrices

Proposition 2.4 and Theorem 2.3 show that $D$, the minimal Dress ring of $\mathbb{R}(X)$, is a Dedekind domain which is not a principal ideal domain. Hence we expect that $D$ does not satisfy property $(ID_2)$, in support of the conjecture recalled in the introduction. This difficult question is left open. Actually, in this section we will focus on properties of the entries of $2 \times 2$ singular matrices over $D$, that guarantee factorization into products of idempotent.

We start with some considerations that hold over an arbitrary integral domain $R$.

It is easy to prove that a singular nonzero matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is idempotent if and only if $d = 1 - a$ (cf. [14]).

Now we remark some useful factorizations into idempotents.

\begin{align*}
(4) \quad \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ 0 & 1 \end{pmatrix} ; \quad \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 - p & 0 \end{pmatrix} \\
(5) \quad \begin{pmatrix} p & rp \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 - p & 0 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}
\end{align*}

A pair of elements $p, q$ of $R$ is said to be an idempotent pair if $(p, q)$ is the first row of an idempotent matrix. In this case we get the factorization

\begin{align*}
(6) \quad \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & 1 - p \end{pmatrix},
\end{align*}

where $rq = p(1 - p)$.

We note that, obviously, every matrix similar to an idempotent matrix is idempotent, hence a singular matrix $S$ is a product of idempotent matrices if and only if any matrix similar to $S$ is a product of idempotents.

**Lemma 3.1.** Let $R$ be a domain, $p, q \in R$. The matrix $A = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if and only if such is $B = \begin{pmatrix} q & p \\ 0 & 0 \end{pmatrix}$.

**Proof.** Assume that $A$ is a product of idempotents. We get

\begin{align*}
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = S.
\end{align*}

Hence $S$ is also a product of idempotents. However $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} S$. 

hence $B$ is a product of idempotents. The argument is reversible. \hfill $\square$

Now we get back to $D$, the minimal Dress ring of $\mathbb{R}(X)$. We will provide explicit factorizations into idempotent matrices of particular classes of singular matrices of the form \( \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \), where the entries $p, q \in D$ satisfy rather natural mutual relations in terms of their roots and degrees.

The next lemma is crucial.

**Lemma 3.2.** Let $x, y$ be two non-zero polynomials in $\mathbb{R}[X]$ with $\deg(x) = \deg(y)$.

(a) If $y(u) > 0$ (or $y(u) < 0$) for every $u$ root of $x$, then there exists $\beta \in \Gamma$ such that $\delta = x^2 + y\beta \in \Gamma^+$, $\deg x - 1 \leq \deg \beta \leq \deg x = \deg \delta/2$.

(b) If $x(z) > 0$ (or $x(z) < 0$) for every $z$ root of $y$, then there exists $\eta \in \Gamma$ such that $\delta = x\eta + y^2 \in \Gamma^+$ and $\deg x - 1 \leq \deg \eta \leq \deg(y) = \deg \delta/2$.

**Proof.** We will prove (a), since (b) is analogous, exchanging the roles of $x$ and $y$. We distinguish the cases $x \in \Gamma$ and $x \notin \Gamma$.

**FIRST CASE:** $x \in \Gamma$.

Since $x$ has no roots in $\mathbb{R}$, then $x^2$ is always positive. Now take $\beta' \in \Gamma$ of degree equal to either $\deg x$ or $\deg x - 1$, in accordance with the parity of the degree of $x$, and such that the leading coefficient of $x^2 - \beta'y$ is positive. Then $\lim_{X \to \pm\infty} (x^2 - \beta'y) = +\infty$, hence there exists $M > 0$ such that $x^2 - \beta'y > 0$ outside the interval $I = [-M, M]$. Let $k > 0$ be the minimum of $x^2$ and $h > 0$ the maximum of $|\beta'y|$ in the interval $I$, and pick an integer $m \geq 2$ such that $k/mh < 1$. Then, inside the interval $I$, we get $|\beta'y|/mh < 1$, and

\[
x^2 \geq k > k|\beta'y|/mh.
\]

Therefore, $x^2 - k\beta'y/mh > 0$ inside the interval $I$. Moreover, outside the interval $I$, we have $x^2 - k\beta'y/mh > 0$, as well. This is trivial for the points where $k\beta'y/mh < 0$, and, for the points where $k\beta'y/mh > 0$, we get $x^2 - k\beta'y/mh > x^2 - \beta'y > 0$, since $k/mh < 1$. So the polynomial $\beta = -\beta'k/mh$ satisfies our requirements.

**SECOND CASE:** $x \notin \Gamma$.

Let $u_1 < u_2 < \cdots < u_n$ be the distinct roots of $x$. Now we pick $\beta' \in \Gamma$ of degree equal to either $\deg x$ or $\deg x - 1$, in accordance with the parity of the degree of $x$, and such that the leading coefficient of $x^2 - \beta'y$ is positive, and $\beta'y(u_i) < 0$ for $1 \leq i \leq n$. As in the First Case, we may take an interval $I = [-M, M]$ such that $x^2 - \beta'y > 0$ outside $I$; under the present circumstances, we also choose $M > 0$ such that $-M < u_1, u_n < M$. For $1 \leq i \leq n$, since $\beta'y(u_i) < 0$, we may choose disjoint open intervals $I_i$ containing $u_i$, such that $\beta'y < 0$ in $I_i$. We also require that $-M < \inf I_i$ and $\sup I_i < M$. For $1 \leq i < n$, let $J_i$ be the closed interval whose end points are $\sup I_i$ and $\inf I_{i+1}$; we also define $J_0 = [-M, \inf I_1]$, and $J_n = [\sup I_n, M]$.

Now, for $0 \leq i \leq n$, let $k_i$ be the minimum of $x^2$ and $h_i$ the maximum of $|\beta'y|$ in $J_i$. So, arguing as in the First Case, we get

\[
x^2 - k_i\beta'y/mh_i > 0
\]

inside the interval $J_i$, where $m > 0$ is an integer enough large to work for all $i$. We conclude that there exists a real number $s \in [0, 1]$ such that
However note that $x^2 - s\beta'y > 0$ in $\bigcup_i J_i$. We conclude that $x^2 - s\beta'y > 0$ in $I = [-M, M]$, and, arguing as in the First Case, it is positive also outside $I$. We conclude that the polynomial $\beta = -s\beta'$ satisfies our requirements. \[\square\]

**Theorem 3.3.** Let $p$ and $q$ be two elements of $D$. Then the matrix $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices if one of the following holds:

(i) $\deg p \geq \deg q$ and $q(u) > 0$ (or $q(u) < 0$) for every $u$ root of $p$

(ii) $\deg q \geq \deg p$ and $p(z) > 0$ (or $p(z) < 0$) for every $z$ root of $q$.

**Proof.** Recall that, by Lemma 3.1, $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotents if and only if such is $\begin{pmatrix} q & p \\ 0 & 0 \end{pmatrix}$. Hence we may safely assume that (i) holds. Moreover, note that

$$(1 - 1) \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} (1 1) = \begin{pmatrix} p & p + q \\ 0 & 0 \end{pmatrix}.$$

Hence, in case $\deg(p) > \deg(q)$, it suffices to prove that $\begin{pmatrix} p & p + q \\ 0 & 0 \end{pmatrix}$ is a product of idempotents. Note also that $p + q$ keeps the same sign on every root $u$ of $p$. Say $p = x/\gamma$, $q = y/\gamma$, with $x, y \in \mathbb{R}[X]$, $\gamma \in \Gamma$.

As a further reduction, we may assume that $\deg p = \deg q$. Say $p = x/\gamma$, $q = y/\gamma$, with $x, y \in \mathbb{R}[X]$, $\gamma \in \Gamma$.

Since $1 - x^2/\gamma = y\beta/\gamma$, $T = \begin{pmatrix} x^2/\delta & yx/\delta \\ x\beta/\delta & y\beta/\delta \end{pmatrix}$ is an idempotent matrix over $D$. Hence, by (1) and the factorization

$$(x/\gamma \ y/\gamma) = (\delta/\gamma \eta \ 0) (x/\tau \ y/\tau).$$

we conclude that $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices over $D$. \[\square\]

**Corollary 3.4.** In the above notation, let $p = x/\gamma$ and $q = y/\gamma$ be two elements of $D$. If both $\deg x$ and $\deg y$ are $\leq 1$, then $\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.
Proof. We may assume \( x \neq 0 \neq y \) are nonzero. If \( x \) and \( y \) have no common roots, then our statement follows from Theorem 3.3, since either (i) or (ii) trivially holds when the degrees are \( \leq 1 \). Otherwise, we get \( y = rx \), for some \( r \in \mathbb{R} \), hence \( \Box \) yields the desired conclusion.

It is worth showing that, in case of “small degrees of the numerators”, we get factorizations into idempotents even when \( p, q \) have common roots.

**Proposition 3.5.** In the above notation, let \( p = x/\gamma, q = y/\gamma \in D \) with \( \deg x = \deg y = 2 \). If \( M = \gcd(x, y) \notin \mathbb{R} \), then \( A = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \) is a product of idempotent matrices.

**Proof.** If \( \deg M = 2 \), the factorization follows from (5).

If \( \deg M = 1 \), we may assume, up to a linear change of coordinates, that \( M = X \). Take \( r \in \mathbb{R} \) such that \( x + ry = sX \), where \( 0 \neq s \in \mathbb{R} \). Take \( \delta \in \Gamma \) such that \( \delta - x \) has degree 1, and consider the polynomial \( z = s^{-1}(\delta - x)x/X \in \mathbb{R}[X] \). Then the matrix \( \begin{pmatrix} x/\delta & sX/\delta \\ z/\delta & 1 - x/\delta \end{pmatrix} \) is idempotent, hence \( B = \begin{pmatrix} x/\delta & sX/\delta \\ 0 & 0 \end{pmatrix} \) is a product of idempotents. However, analogously to (7), we may verify that \( A \) is similar to \( B = \begin{pmatrix} x/\delta & y/\delta \\ 0 & 0 \end{pmatrix} \). Finally, since \( \deg \delta = 2 \), the equality \( A = \begin{pmatrix} \delta/\gamma & 0 \\ 0 & 0 \end{pmatrix} \) shows that \( A \) is a product of idempotents (cf., the proof of Theorem 3.3). \( \Box \)

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