Exact solutions to the Jaynes-Cummings model without the rotating-wave approximation

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Abstract – By using tunable extended bosonic coherent states, the JC model without the rotating-wave approximation can be mapped to a polynomial equation with a single variable. The solutions to this polynomial equation recover exactly all eigenvalues and eigenfunctions of the model for all coupling strengths and detunings, which can be readily applied to recent circuit quantum electrodynamic systems operating in the ultra-strong coupling regime.

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Introduction. – The Jaynes-Cummings (JC) model [1] describes the interaction of a two-level atom (qubit) with a single bosonic mode. It is a fundamental one in quantum optics. Based on the assumption of near-resonance and relatively weak atom-cavity coupling, the rotating-wave approximation (RWA) is usually employed, and analytically exact solutions can be trivially obtained. Recently, the JC model has also been closely related to condensed-matter physics. It has been recently applied to some solid-state systems, such as the Josephson charge qubit coupling to an electromagnetic resonator [2], the superconducting quantum interference device coupled with a nanomechanical resonator [3-5], and, most recently, the LC resonator inductively coupled to a superconducting qubit [6-10]. In traditional quantum optics where the coupling between the two-level “natural” atom and the single bosonic mode is quite weak, RWA is the most useful approximation. However, in circuit quantum electrodynamic (QED), the artificial atoms may interact very strongly with on-chip resonant circuits [8-10], the RWA cannot describe well the strong-coupling regime [8]. Therefore, the JC model without the RWA is the focus of current interests [11-22].

However, due to the inclusion of the counter-rotating terms, the Bosonic number is not conserved, the Bosonic Fock space has infinite dimensions, so any solution without the RWA is highly nontrivial. In the recent years, several non-RWA approaches have been proposed in the Dicke model [13], the quantum Zeno effect [23], and the spin-boson model [24]. Especially, by using extended bosonic coherent states, the present authors have solved the Dicke model without the RWA exactly in the numerical sense [13]. The most simple $N = 1$ Dicke model is just the JC model. These numerically exact solutions to the JC model are also described in detail in refs. [19,20]. To the best of our knowledge, analytical exact solutions are still lacking in the literature to date.

In this paper, we propose an analytical method to solve exactly the JC model without the RWA by means of tunable extended bosonic coherent states. The correlations among bosons are added step by step until further corrections do not change the results. By solving the Schrödinger equation, the expanded coefficients can be related through recurrence equations. A polynomial equation with only a single variable will be then derived. The solutions to this polynomial equation can give exactly all eigenfunctions and eigenvalues of the JC model without the RWA for arbitrary parameters. The paper is organized as follows: In the second section we describe in detail the model and the scheme to exactly solve the JC model. Then the results are presented and discussed in the third section. The conclusion is given in the last section.

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Model and method. — Without the RWA, the Hamiltonian of a qubit interacting with a single bosonic mode reads \( (\hbar = 1) \)
\[
H_0 = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + g (a^\dagger + a) \sigma_x,
\]
where \( a^\dagger \) and \( a \) are the bosonic annihilation and creation operators of the cavity, \( \Delta \) and \( \omega \) are the frequencies of the atom and cavity, \( g \) is the atom-cavity coupling constant, and \( \sigma_k (k = x, y, z) \) is the Pauli matrix of the two-level atoms. For convenience, we can write a transformed Hamiltonian with a rotation around the \( y \)-axis by an angle \( \frac{\pi}{2} \):
\[
H = -\frac{\Delta}{2} \sigma_x + a^\dagger a + g(a^\dagger + a) \sigma_z,
\]
in unit of \( \omega = 1 \). Associated with this Hamiltonian is a conserved parity \( \Pi \), such that \( \{ H, \Pi \} = 0 \), which is given by
\[
\Pi = \sigma_x \exp(\im \hat{N}),
\]
where \( \hat{N} = a^\dagger a \) is the bosonic number operator. \( \Pi \) has eigenvalue \( +1 \) or \( -1 \), depending on whether the excitation number is even or odd, so the system has the corresponding even or odd parity.

One can naturally expand the wave function in the photonic Fock states like in ref. [25], which is also a starting point of the standard numerically diagonalization (ED) scheme. We here propose a different ansatz for the wave function in terms of tunable coherent states [26]
\[
|\Psi_\pm \rangle = \left( \sum_{n=0}^{M} c_n (a^\dagger)^n \exp(\alpha a^\dagger) \right) |0\rangle \pm \left( \sum_{n=0}^{M} c_n (-a^\dagger)^n \exp(-\alpha a^\dagger) \right) |0\rangle,
\]
where \( c_n \) is the expansion coefficient, \( \alpha \) is the eigenvalue of the coherent state and will be determined later, and \( \Psi_+ (\Psi_-) \) is the eigenfunction of even (odd) parity with eigenvalue \(+1(-1)\). The Schrödinger equation then gives
\[
[a^\dagger a + g (a^\dagger + a)] \sum_{n=0}^{M} c_n (a^\dagger)^n \exp(\alpha a^\dagger) |0\rangle = \frac{\Delta}{2} \sum_{n=0}^{M} c_n (-a^\dagger)^n \exp(-\alpha a^\dagger) |0\rangle = E^{\pm} \sum_{n=0}^{M} c_n (a^\dagger)^n \exp(\alpha a^\dagger) |0\rangle.
\]
By using \([a, a^\dagger] = 1\), equating the coefficients of the terms of \((a^\dagger)^m \exp(\alpha a^\dagger)|0\rangle\) on both sides give the following identities for any \( m \):
\[
(m + \alpha g) c_m + (\alpha + g) c_{m-1} + g (m + 1) c_{m+1} = \frac{\Delta}{2} \sum_{j=0}^{M} \frac{(2\alpha)^j}{j!} c_{m-j} = E^{\pm} c_m.
\]
By careful inspection of eq. (4), one can find that \( c_0 \) is flexible in the Schrödinger equation where the normalization for the eigenfunction is not necessary, so we select \( c_0 = 1.0 \). The linear term in \( a^\dagger \) in the Fock space can be also determined by the value of \( \alpha \) in the pure coherent state \( \exp(\alpha a^\dagger) |0\rangle \). It is useless that both \( c_1 \) and \( \alpha \) contribute to the linear term, so we can set \( c_1 = 0 \). Once the first two terms are fixed, the coefficients of the other terms higher than \( a^\dagger \) should be determined by solving the Schrödinger equation. The constant term yields
\[
E^{\pm} = \alpha g \mp \frac{\Delta}{2}.
\]
Inserting eq. (7) into eq. (6), we have the following recurrence equation:
\[
c_{m+1} = \frac{1}{(m+1)g} \left[ (m \pm \Delta) c_m + (\alpha + g) c_{m-1} \right] \\
\mp(-1)^m \frac{\Delta}{2} \sum_{j=0}^{M} \frac{(2\alpha)^j}{j!} c_{m-j}.
\]
The crucial ingredient in the ansatz equation (4) is that the expansion could be truncated at finite \( M \). This requires that the terms higher than \((a^\dagger)^M \exp(\alpha a^\dagger)|0\rangle\) can be sufficiently small and can be neglected completely. So we may set \( c_{M+1} = 0 \), which gives
\[
\left( M \pm \frac{\Delta}{2} \right) c_m + (\alpha + g) c_{m-1} = 0.
\]
Note that all coefficients \( c_\ell \) can be expressed by one variable \( \alpha \) through eq. (8), so this is a one-variable polynomial equation of degree \( M \). This is the central equation in this paper, whose roots could give the exact solutions to the JC model without the RWA.

To obtain the true exact results, in principle, the truncated number \( M \) should be taken to infinity. Fortunately, this is not necessary. It is found that finite terms in state (4) are sufficient to give exact results in the whole coupling range. Typically, the convergence is assumed to be achieved if the results are determined within very small relative errors when the truncated number \( M \) increases further. We like to stress here that increasing the value of \( M \) would almost not bring additional effort to solve eq. (9) in an ordinary PC. The precision for the results is only limited to the machine accuracy for all cases tested.

Results and discussion. — To have a sense about this method, we perform the first-order approximation (FOD) by considering \( M = 2 \). The only one coefficient to be determined is \( c_2 \), which is easily obtained by eq. (8): \( c_2 = -\frac{1}{2g} [(1 \pm \Delta) \alpha + g] \). The nonlinear equation then is given by
\[
\mp \Delta g \alpha^2 - (1 \pm \Delta) \alpha - g = 0.
\]
The corresponding energies are
\[ E^+ = -\frac{(1 + \Delta)}{2\Delta} \left( 1 + \sqrt{1 - \frac{4\Delta g^2}{(1 + \Delta)^2}} \right) - \frac{\Delta}{2}, \]
\[ E^- = -\frac{(1 - \Delta)}{2\Delta} \left( 1 + \sqrt{1 + \frac{4\Delta g^2}{(1 - \Delta)^2}} \right) + \frac{\Delta}{2}. \]

Note that the lower eigenvalue for the even parity (+), i.e. eq. (13) with plus sign in the RHS, increases with \( g \). It is physically unreasonable, because the ground-state (GS) energy should decrease with the qubit-cavity coupling. The corresponding coefficient \( c_3 \) is considerably larger than \( c_0 = 1 \), indicating that the wave function is not converging. So this solution should be omitted. The GS energy in the FOD is then analytically given by eq. (13) with minus sign.

The GS energies in the FOD with different ratios of \( \Delta/\omega \) are plotted in fig. 1 with solid circles. A generalized RWA (GRWA) for the JC model was performed recently [12] and the derived expression for the energy levels is significantly more accurate than that with RWA. The GS energies by the GRWA with the same ratios of \( \Delta/\omega \) are also listed in fig. 1 as solid curves for comparison. To show the accuracy, we also collect the GS energies by the numerically ED in bosonic Fock states \( |\alpha\rangle^m \rangle |0\rangle \) indicated by open circles.

It is interesting that the GS energies by the present FOD are much closer to the numerical ED ones for zero, large positive and negative detuning than those by GRWA. The relative difference for the GS energy of the present FOD and the numerical ED is less than \( 10^{-3} \) for \( g < 0.5 \). One may be deeply impressed by this good agreement, because the present FOD is only a preliminary approximation. Note that the maximum coupling constant in the ultrastrong-coupling regime in the QED circuit reported recently [8,9] is around \( g = 0.12 \). When only the GS energy is concerned in these systems, the present very simple analytical expression eq. (13) with minus sign in the RHS should be very helpful.

Furthermore, if we increase the truncated number \( M \), we would obtain more accurate energy levels. On the basis of the Abel-Ruffini theorem that the general solution in radicals is impossible to polynomial equations of degree five or higher [27], the most accurate analytically expressions of the energy levels of the system might be obtained by setting \( M = 4 \), which is however not shown here due to the complexity. Naively speaking, one could have \( 7 \) eigenstates for \( M = 4 \). Actually it is not that case. For the energy expression, eq. (7), only real roots for \( \alpha \) in eq. (9) are reasonable, some complex roots should be omitted.

Most importantly, for arbitrary model parameters, the saturation calculation can be arrived at if \( M \) is large enough, which results in exact solutions. It is very crucial to obtain the real roots of eq. (9) for sufficient large \( M \) where the general analytical expression is not available. To achieve this goal, we plot a two-dimensional view of the solutions for the one-variable polynomial equation eq. (9) for \( g = 0.1, 0.5, \) and 1 are presented in fig. 2. The real roots are just the crossing points of the curve \( y = f(\alpha) \) and the straight line \( y = 0 \).

It is observed universally that if \( M \) is an even (odd) number for even (odd) parity, the lowest eigenvalue for even (odd) parity is physically unreasonable, because the expansion coefficients in the wave function (4) are not converging. This characteristic has shown up in the case of \( M = 2 \) discussed above. Except for this lowest eigenvalue, it will be confirmed later that all other obtained eigenvalues of the model are true eigenvalues of the model. So for given model parameters \( g \) and \( \Delta \), we choose \( M \) to be an odd (even) number for even (odd) parity, so that the physically unreasonable solution does not appear.

In fig. 2, we choose \( M = 59 \) for the even parity and \( M = 60 \) for the odd parity, which is sufficiently large to ensure the convergence. The value of \( f(\alpha) \) is very large in the small-\( \alpha \) regime, and decreases quickly as \( \alpha \) increases. The real roots are clearly shown in this two-dimensional plot and can be obtained easily by Maple in the practical calculation. The number of real roots is considerably less

Fig. 1: (Color online) Comparisons of the GS energies by the present FOD (solid circles), GRWA (solid curves), and ED (open circles) as a function of \( g \) at \( \Delta/\omega = 0.5 \) (top), \( \Delta/\omega = 1 \) (middle), and \( \Delta/\omega = 1.5 \) (bottom).
we increase $M$ to the JC model. Actually, for every model parameters, roots for $\alpha$ coupling strength the total energy of this system should decrease with the energy levels, eq. (7), this is physically reasonable, because decreases with the coupling strength $\alpha$ with 13 roots for the even parity and 14 roots for the odd parity.

Fig. 2: (Color online) Schematic view of the solutions for the one-variable polynomial equation (9) for $g = 0.1$ (top panel), $0.5$ (middle panel), and $1.0$ (bottom panel). The left panel is for even parity and the right one for odd parity. The truncation numbers are $M = 59$ for even parity and $M = 60$ for odd parity. The insets show an enlarged view.

than $M$. E.g. for $g = 0.1$, we obtain a total of 27 real roots with 13 roots for the even parity and 14 roots for the odd parity, which are corresponding to 27 energy levels. Note that the roots for $\alpha$ for the same energy levels generally decrease with the coupling strength $g$. According to the energy levels, eq. (7), this is physically reasonable, because the total energy of this system should decrease with the coupling strength $g$.

Once the roots are at hand, one natural question is whether the energy levels and the wave functions corresponding to these roots are really the exact ones to the JC model. Actually, for every model parameters, we increase $M$ by 2 in each step until the values of the roots for $\alpha$ are not modified. Our criterion is that, if the relative difference for $\alpha$ is less than $10^{-3}$, so is the relative difference for the energy by eq. (7), we think $M$ is large enough to give the exact solutions to the JC model.

The exact solutions are also confirmed by the convergence of the expanded coefficients in the wave function (4), which are displayed in fig. 3, where the absolute value of coefficients $c_n$ normalized to the maximum value of $|\{c\}|$ for $g = 0.1$, 0.5, and 1.0 for the GS state and the first 5 excited states are plotted. It is observed that the coefficients $c_n$ vanish after $n = 40$ for $g = 0.1$, 20 for $g = 0.5$, and 10 for $g = 1$. It follows that $M = 40$ for $g = 0.1$, $M = 20$ for $g = 0.5$, and $M = 10$ for $g = 1$ are large enough to get the exact solutions for these 6 eigenstates. Further increase of $M$ could not change the wave function at all and therefore the saturated calculation is truly reached. In practical calculations, if we choose the truncation numbers $M = 59$ for even parity and $M = 60$ for odd parity, we can get exact solutions for the first more than 20 eigenstates for $g \geq 0.1$, which might be practically very useful when applied to the circuit QED system [8,9].

It is also interesting to note from fig. 3 that the necessary truncation number decreases as the coupling strength increases, indicating that the present approach can be applied to the JC model in the arbitrary ultrastrong coupling regime. In sharp contrast with the present approach, in the numerically ED in the bosonic Fock states, the dimension of the truncated subspace increases considerably with the coupling constant $g$, due to more photons being involved. In the present approach, infinite photons have been already included in the bosonic coherent states in wave function (4).

Note that the exact solution to the JC Hamiltonian (2) without the first term $-\frac{\gamma}{2} \sigma_z$ can be derived readily. By introducing the new operators [19]

$$A = a + g, \quad B = a - g,$$

we can observe that the linear term for the bosonic operator is removed, and only the number operators $A^\dagger A$ and $B^\dagger B$ are left. Therefore we can readily obtain the eigenfunctions

$$|m\rangle = \left( |m\rangle_A \pm (-1)^m |m\rangle_B \right)$$

with

$$|m\rangle_A = \frac{(a^\dagger + g/\omega)^m |0\rangle_A}{\sqrt{m!}}, \quad |m\rangle_B = \frac{(a^\dagger - g/\omega)^m |0\rangle_B}{\sqrt{m!}},$$

and the eigenvalues $E_m^\pm = m - g^2$ for the $m$-th excited state. The eigenstates are twofold degenerate in this case.
In wave function (16), the truncation number is just fixed to be \( m \) for the \( m \)-th excited state. When the term \( -\frac{1}{2} \sigma_x \) sets in, additional terms are needed to add to the \( m \)-th wave function, eq. (16), for the \( m \)-th excited state, and the displacement in the shift operators in eq. (15) should also be modified, which forms the ansatz equation (4). The stronger the coupling strength is, the fewer additional terms are needed, which is just demonstrated in fig. 3. It follows that the present approach is especially suited to the JC model in the strong-coupling regime, which highlights the advantage of this method.

To show the exact nature of the solutions directly, we compare the present saturation solutions with those by numerical ED in bosonic Fock states in tables 1, 2 and 3. We only list the energies of the GS and the first 8 excited states for three typical detunings \( \Delta/\omega = 1, 0.5, \) and 1.5 at different coupling strength where the RWA is usually invalid. It is very important to observe that the present results are the same as those by ED in all cases. We should point out here that the dimension of the truncated subspace in the numerical ED is up to \( 10^5 \) at strong coupling, large computer memory is needed to store the matrix elements, it takes several hours of CPU time to perform the diagonalization. In the present approach, the dimension of the truncated subspace is only around 60, and the results are obtained immediately with any PC.

### Table 1: The first 9 low-lying energy levels \( E_i, (i = 0, 1, \ldots, 8) \) in the case of resonance \( \Delta/\omega = 1 \).

| \( E_i \) | Present | ED | Present | ED | Present | ED |
|---|---|---|---|---|---|---|
| \( E_0 \) | -0.505012531 | -0.505012531 | -0.633294235 | -0.633294235 | -1.14794573 | -1.14794573 |
| \( E_1 \) | 0.395102298 | 0.395102298 | -0.120023834 | -0.120023834 | -1.01017830 | -1.01017830 |
| \( E_2 \) | 0.594847069 | 0.59484707 | 0.695393717 | 0.695393717 | -0.231722500 | -0.231722500 |
| \( E_3 \) | 1.35388915 | 1.35388915 | 0.82530520 | 0.82530520 | 0.133435454 | 0.133435454 |
| \( E_4 \) | 1.63600849 | 1.63600849 | 1.58705309 | 1.58705309 | 0.927043866 | 0.927043866 |
| \( E_5 \) | 2.32235857 | 2.32235857 | 1.93553948 | 1.93553948 | 1.10480946 | 1.10480946 |
| \( E_6 \) | 2.66745885 | 2.66745885 | 2.54858735 | 2.54858735 | 1.84278099 | 1.84278099 |
| \( E_7 \) | 3.29593219 | 3.29593219 | 2.94783100 | 2.94783100 | 2.14361945 | 2.14361945 |
| \( E_8 \) | 3.69385838 | 3.69385838 | 3.90961541 | 3.90961541 | 2.94392772 | 2.94392772 |

### Table 2: The first 9 low-lying energy levels \( E_i, (i = 0, 1, \ldots, 8) \) for large detuning \( \Delta/\omega = 0.5 \).

| \( E_i \) | Present | ED | Present | ED | Present | ED |
|---|---|---|---|---|---|---|
| \( E_0 \) | -0.256681491 | -0.256681491 | -0.425996230 | -0.425996230 | -1.05412447 | -1.05412447 |
| \( E_1 \) | 0.756227984 | 0.756227984 | -0.135825244 | -0.135825244 | -0.986090220 | -0.986090220 |
| \( E_2 \) | 1.21812615 | 1.21812615 | 0.741771730 | 0.741771730 | -0.110627502 | -0.110627502 |
| \( E_3 \) | 1.76842053 | 1.76842053 | 0.760071830 | 0.760071830 | 0.0868714030 | 0.0868714030 |
| \( E_4 \) | 2.20649908 | 2.20649908 | 1.64729780 | 1.64729780 | 0.967028390 | 0.967028390 |
| \( E_5 \) | 2.77998652 | 2.77998652 | 1.83251690 | 1.83251690 | 1.04043102 | 1.04043102 |
| \( E_6 \) | 3.19542589 | 3.19542589 | 2.64925491 | 2.64925491 | 1.92078003 | 1.92078003 |
| \( E_7 \) | 3.79099791 | 3.79099791 | 2.85156863 | 2.85156863 | 2.07700324 | 2.07700324 |
| \( E_8 \) | 4.18484822 | 4.18484822 | 3.65576197 | 3.65576197 | 2.96634670 | 2.96634670 |

### Table 3: The first 9 low-lying energy levels \( E_i, (i = 0, 1, \ldots, 8) \) for large detuning \( \Delta/\omega = 1.5 \).

| \( E_i \) | Present | ED | Present | ED | Present | ED |
|---|---|---|---|---|---|---|
| \( E_0 \) | -0.75409629 | -0.75409629 | -0.856475589 | -0.856475589 | -1.277551560 | -1.277551560 |
| \( E_1 \) | 0.232066923 | 0.232066923 | -0.203235835 | -0.203235835 | -1.06857920 | -1.06857920 |
| \( E_2 \) | 0.768900004 | 0.768900004 | 0.600664430 | 0.600664430 | -0.357181745 | -0.357181745 |
| \( E_3 \) | 1.20197785 | 1.20197785 | 0.959696469 | 0.959696469 | 0.130321638 | 0.130321638 |
| \( E_4 \) | 1.78994281 | 1.78994281 | 1.47931569 | 1.47931569 | 0.871878630 | 0.871878630 |
| \( E_5 \) | 2.18237074 | 2.18237074 | 2.04745904 | 2.04745904 | 1.19903842 | 1.19903842 |
| \( E_6 \) | 2.80958033 | 2.80958033 | 2.45896800 | 2.45896800 | 1.76995826 | 1.76995826 |
| \( E_7 \) | 3.16398011 | 3.16398011 | 3.00884499 | 3.00884499 | 2.18521205 | 2.18521205 |
| \( E_8 \) | 3.82785600 | 3.82785600 | 3.53491412 | 3.53491412 | 2.95127586 | 2.95127586 |
The only numerical sense is just to solve one polynomial equation with a single variable, which is actually very easy to carry out within many simple numerical methods. One can also just plot a two-dimensional diagram to figure out the solutions readily. To the best of our knowledge, this is the first analytically exact solution to the JC model without the RWA for arbitrary coupling strengths and detunings, which is very simple and can be easily employed.

We would like to stress that the present approach is essentially different from that proposed in refs. [19,20]. In these papers, the eigenvalue of the coherent state $\alpha$ is fixed to be $g$ and there is no any recurrence relation for expansion coefficients. Instead, two new shifted operators are introduced, and numerical ED in the Fock space of these new operators is performed. However, in the present paper, the eigenvalue of the coherent state $\alpha$ can be tunable, and not any numerical ED is employed.

Finally, we unfold the exact nature of the ansatz equation (4) in the mathematical sense. Suppose that we have $\alpha$, the solution of the polynomial equation (9), $a$ priori, we can perform the following unitary transformation for Hamiltonian (2):

$$H^S = e^S He^{-S},$$
$$S = \alpha(a^† - a)\sigma_z. \quad (18)$$

The wave function for this transformed Hamiltonian can be in principle expressed in terms of the Fock states of the $a$ operator

$$|\psi\pm\rangle = \left(\sum_{n=0}^{M} f_n (a^†)^n |0\rangle \pm \sum_{n=0}^{M} f_n (-a)^n |0\rangle \right). \quad (19)$$

where $\pm$ denotes the parity, $M$ is the truncated number, and $f_n$ is another set of the expansion coefficient. Then the wave function for the Hamiltonian (2) becomes

$$|\psi_{\pm}\rangle' = e^S |\psi_{\pm}\rangle = e^{-\alpha^2/2} \left(\sum_{n=0}^{M} f_n (a^† - \alpha)^n e^{\alpha a^†}|0\rangle \pm \sum_{n=0}^{M} f_n (-a - \alpha)^n e^{-\alpha a}|0\rangle \right). \quad (20)$$

This is exactly the same as eq. (4) by setting $c_m = e^{-\alpha^2/2} \sum_{n=m}^{M} f_n \frac{m!}{(n-m)!} (-\alpha)^n f_n$. Therefore the wave function (4) is complete and its exact nature can be ensured convincingly as long as the saturated calculation is reached.

Conclusion. — In this paper, the JC model without the RWA can be mapped to a polynomial equation with a single variable by tunable extended coherent states. The solutions to this polynomial equation recover exactly all eigenvalues and eigenfunctions of the model for all coupling strengths and detunings. This method is specially suited for the strong-coupling circuit quantum electrodynamic systems, where the conventional numerical ED is considerably time-consuming. The idea proposed here may also be useful and extended to solve other spin-boson systems and electron-phonon coupling systems.

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