GROMOV COMPACTNESS IN TROPICAL GEOMETRY
AND IN NON-ARCHIMEDEAN ANALYTIC GEOMETRY

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ABSTRACT. Gromov’s compactness theorem for pseudo-holomorphic curves is a foundational result in symplectic geometry. In this article, we prove the analogs of Gromov’s compactness theorem in tropical geometry, in non-Archimedean analytic geometry, and for the tropicalization of the moduli stack of analytic stable maps. As intermediate steps, we introduce a notion of Kähler structures using metrizations of virtual line bundles to ensure the boundedness of the moduli spaces in question. We construct the moduli stack of non-Archimedean analytic stable maps using formal stacks, Artin’s representability criterion and the geometry of stable curves. Our main tools are Berkovich spaces, formal models, balancing conditions, vanishing cycles and quantifier eliminations. The results are expected to provide the foundations for subsequent works on enumerative non-Archimedean analytic geometry.

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1. Introduction

Let us recall Gromov’s compactness theorem in symplectic geometry.

**Theorem 1.1** (Symplectic Gromov compactness (cf. [Gro85], [Pan94], [Kon95])). Let $X$ be a symplectic manifold equipped with a tame almost complex structure and $K \subset X$ a compact subset. The moduli space of stable maps into $K$ with uniform bounds on genus and area is compact and Hausdorff.

The theorem above has an algebraic version proposed by Maxim Kontsevich.

**Theorem 1.2** (Algebraic Gromov compactness ([Kon95]§1.3.1)). Let $X$ be a projective scheme of finite type over a field. The moduli stack of stable maps into $X$ with uniform bounds on genus and degree is a proper algebraic stack.

These two theorems are foundational results in enumerative geometry, such as in the theory of Gromov-Witten invariants, Floer homology, etc. Roughly speaking, when one wants to count certain geometrical objects, the first question to ask is whether there are a finite number of such objects. This question can often be translated into the compactness of the moduli space parameterizing those objects. The compactness is also an essential condition for the construction of virtual fundamental cycles.

Counting curves in algebraic varieties is an old subject. It gives non-trivial invariants, displays rich geometrical phenomena, and is intimately related with fundamental concepts in string theory, such as BPS states. Let us look at the example of counting curves in a complex algebraic surface. It is difficult to visualize how curves of real dimension two sit in the ambient space of real dimension four. The idea of tropical geometry, as developed by Grigory Mikhalkin [Mik06] along with many other mathematicians, enables us to cut down half of the real dimensions. So the previous scenario becomes piecewise linear tropical curves of real dimension one sitting in the ambient tropical space of real dimension two. Problems of enumerative algebraic geometry can thus be translated into enumerative tropical geometry, and then be solved by combinatorial arguments.

Although one often studies tropical varieties formed out of algebraic varieties, the tropicalization procedure is in fact analytic in nature. It applies to a family of complex algebraic varieties rather than a single one. Given a family of complex algebraic varieties $\mathcal{X}$ over the formal power series ring $\mathbb{C}[t]$, its
generic fiber $\mathcal{X}_\eta$ can be endowed with the structure of a non-Archimedean analytic space\(^1\) over the field of Laurent power series $\mathbb{C}((t))$. Vladimir Berkovich\([\text{Ber99}]\) proved that if $\mathcal{X}$ is semi-stable, the space $\mathcal{X}_\eta$ admits a strong deformation retraction onto a skeleton $S_X \subset \mathcal{X}_\eta$ which is homeomorphic to the incidence complex of the special fiber $\mathcal{X}_s$. We call $S_X$ the Clemens polytope and regard it as the tropicalization of the space $\mathcal{X}_\eta$. Under the deformation retraction, analytic curves in $\mathcal{X}_\eta$ project to tropical curves in $S_X$. Therefore, non-Archimedean analytic geometry provides the bridge between the complex analytic world and the tropical world. From a slightly different point of view, counting non-Archimedean analytic curves can also be interpreted as counting families of complex algebraic curves.

We study in this article the analogs of Gromov’s compactness theorem in the contexts mentioned above. A direct motivation of our work stems from the speculations in\([\text{KS01}]\) §3.3 which relate the SYZ fibration for Calabi-Yau manifolds with the deformation retraction in the previous paragraph.

Let us describe briefly our main results while precise definitions are given in the body of the article.

**Theorem 1.3** (Tropical Gromov compactness (Theorem 5.12)). Let $S$ be a Clemens polytope equipped with a simple Kähler structure. The moduli space of tropical curves in $S$ with a uniform bound on degree is a compact finite polyhedral complex.

Let $k$ denote a complete discrete valuation field, $k^o$ the ring of integers, and $\tilde{k}$ the residue field.

**Theorem 1.4** (Non-Archimedean analytic Gromov compactness (Theorem 6.46)). Let $\mathcal{X}$ be a strictly semi-stable formal scheme over $k^o$ equipped with a Kähler structure. Let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}_\eta, A)$ denote the moduli stack of $n$-pointed genus $g$ $k$-analytic stable maps into the generic fiber $\mathcal{X}_\eta$ with degree bounded by a real number $A$. Then $\overline{\mathcal{M}}_{g,n}(\mathcal{X}_\eta, A)$ is a compact strictly $k$-analytic stack. It is a proper strictly $k$-analytic stack if $\mathcal{X}$ is proper.

Now we assume that the residue field $\tilde{k}$ is of characteristic zero\(^2\).

Let $\tau_M : \overline{\mathcal{M}}_{g,n}(\mathcal{X}_\eta, A) \to M_n(S_X, A)$ denote the tropicalization map that sends $n$-pointed genus $g$ $k$-analytic stable maps to $n$-pointed tropical curves in $S_X$. Let $\overline{\mathcal{M}}_{t,g,n}(\mathcal{X}_\eta, A)$ denote the image of $\tau_M$.

**Theorem 1.5** (Continuity Theorem (Theorem 7.1, Corollary 7.6)). The tropicalization map $\tau_M$ is continuous.

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\(^1\) In precise terms, we use the theory of $k$-analytic spaces developed by Vladimir Berkovich\([\text{Ber90}, \text{Ber93}]\) restricted to the case of strictly $k$-analytic spaces.

\(^2\) The only place where we use the zero characteristic assumption is in the proof of Theorem 7.1. A purely analytic proof of the theorem without using formal models would help remove this restriction.
Theorem 1.6 (Analytic-tropical Gromov compactness (Theorem 7.13, Corollary 7.14)). The image \( \overline{M}_{g,n}(X_{\eta}, A) \) of the tropicalization map \( \tau_M \) is a compact finite polyhedral complex in \( M_n(S_X, A) \).

Let us outline our approaches.

In Section 2, we describe the basic settings of global tropical geometry. The central object is a strictly semi-stable formal scheme \( X \) (Definition 2.2, 2.3) over the ring of integers \( k^0 \). We construct a continuous, proper, surjective map \( \tau \) from the generic fiber \( X_{\eta} \) to the incidence complex \( S_X \) of the special fiber \( X_s \). We call \( S_X \) the Clemens polytope and regard it as the tropicalization of \( X_{\eta} \) with respect to the formal model \( X \). We prove the functoriality of tropicalization in Proposition 2.10.

The map \( \tau \) sends \( k \)-analytic curves in \( X_{\eta} \) to tropical curves in \( S_X \) with distinguished geometrical properties, called balancing conditions. It is a generalization of the classical balancing conditions ([NS06], [BPR11]). Balancing conditions in the global setting were studied in [Yu13a] using \( k \)-analytic cohomological arguments. In Section 4, we give another proof using the theory of semi-stable reduction of \( k \)-analytic curves reviewed in Section 4.1. The new approach via models leads to a stronger statement (Theorem 4.10). Its proof witnesses the interplay between four geometrical objects of different nature: the formal scheme \( X \), the scheme \( X_s \), the \( k \)-analytic space \( X_{\eta} \) and the simplicial complex \( S_X \). The main observation is that the tropical weights can be read out directly from certain intersection numbers concerning the special fibers via the functor of vanishing cycles (Proposition 4.12).

In order to make sure that various moduli spaces in question are of finite type, we need to impose certain positivity conditions, i.e. analogs of Kähler metrics of complex geometry. Basic notions of virtual line bundles, metrizations and curvatures are defined in Section 3.1. They are specialized to curves in Section 3.2, where we define the degrees of virtual line bundles. In Section 3.3, we study the functoriality of curvature for metrized virtual line bundles.

In Section 5, we define Kähler structures on the formal model \( X \) and explain how they give rise to a combinatorial notion of simple Kähler structures on the Clemens polytope \( S_X \). We prove the tropical Gromov compactness theorem (Theorem 5.12) in Section 5.1 using purely combinatorial arguments based on the previous work [Yu13b]. Combined with Section 5.2, it serves to control the complexity of the tropical curves obtained from \( k \)-analytic curves in \( X_{\eta} \) with bounded degrees.

We prove the non-Archimedean analytic Gromov compactness theorem (Theorem 6.46) in Section 6. In Section 6.1, we introduce the notion of formal stacks and \( k \)-analytic stacks. We prove that the generic fiber of a proper formal stack is a proper \( k \)-analytic stack in Theorem 6.14. In Section 6.2, we use Artin’s representability criterion to construct the algebraic moduli stack of stable maps in full generality without the projectivity assumption (Theorem 6.20). In Section 6.3, we prove that the moduli stack of \( k \)-analytic
stable maps into $X_\eta$ is a paracompact strictly $k$-analytic stack by exhibiting it as the generic fiber of the formal stack of formal stable maps into the formal model $\mathfrak X$ (Theorem 6.40). The proof uses formal GAGA to reduce to the algebraic situation, followed by a careful study on the geometry of stable curves inspired by the work of de Jong on alterations ([dJ96], [dJ97]). Then the non-Archimedean analytic Gromov compactness theorem is reduced to the properness of the algebraic stack of stable maps into the special fiber $X_s$, which is settled in the last part of Section 6.3.

In Section 7, we study the tropicalization of the moduli stack of $k$-analytic stable maps (compare [ACP12]). In Section 7.1, we prove the continuity of the map from the moduli space of $k$-analytic stable maps to the moduli space of tropical curves using the balancing conditions in Section 4.2 and the formal models of families of stable maps developed in Section 6.3. The continuity combined with quantifier eliminations from the model theory of rigid subanalytic sets ([Lip93]), we prove in Section 7.2 the analytic-tropical Gromov compactness theorem (Corollary 7.14).

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2. Basic settings of global tropical geometry

In this section, we describe the basic settings of global tropical geometry (cf. [KT02], [BFJ11]).

Let $k$ denote a complete valuation field, $k^\circ$ the ring of integers, $k^{\text{co}}$ the maximal ideal of $k^\circ$, and $\bar{k}$ the residue field.

For an adic ring $A$, the symbol $\text{Spf}(A)$ denotes the formal spectrum of $A$. For an affinoid $k$-algebra $A$, the symbol $\text{Sp}(A)$ denotes the Berkovich spectrum of $A$.

For $n \geq 0$, $0 \leq d \leq n$, $m = (m_0, \ldots, m_d) \in \mathbb{Z}_{>0}^{d+1}$ and $a \in k^{\text{co}} \setminus 0$, put

$$\mathcal{G}(n,d,m,a) = \text{Spf} \left( k^\circ \{ T_0, \ldots, T_d, S^{\pm}_{d+1}, \ldots, S^{\pm}_n \} /(T_0^{m_0} \cdots T_d^{m_d} - a) \right).$$

When $m = (1, \ldots, 1)$, we denote $\mathcal{G}(n,d,m,a)$ simply by $\mathcal{G}(n,d,a)$.

**Definition 2.1.** A formal scheme $\mathfrak X$ is said to be (locally) finitely presented over $k^\circ$ if it is a (locally) finite union of open affine subschemes of the form

$$\text{Spf} \left( k^\circ \{ T_0, \ldots, T_n \} /(f_1, \ldots, f_m) \right).$$

**Definition 2.2.** Let $\mathfrak X$ be a formal scheme finitely presented over $k^\circ$. $\mathfrak X$ is said to be generalized strictly semi-stable if every point $x$ of $\mathfrak X$ has an open affine neighborhood $\mathcal{U}$ such that the structural morphism $\mathcal{U} \to \text{Spf} k^\circ$ factorizes through an étale morphism $\phi : \mathcal{U} \to \mathcal{G}(n,d,m,a)$ for some $0 \leq d \leq n$. 

$n, m = (m_0, \ldots, m_d) \in \mathbb{Z}_{>0}^{d+1}$ and $a \in k^\circ \setminus 0$, where $m_i$ does not equal to the characteristic of the residue field $k$ for any $0 \leq i \leq d$. The generic fiber $X_\eta$ is a $k$-analytic space in the sense of Berkovich, and the special fiber $X_s$ is a scheme of finite type over the residue field $k$. We denote the reduction map by $\pi: X_\eta \to X_s$ (see [Ber94]). We call the formal scheme $X$ a formal model of its generic fiber $X_\eta$.

Let $X$ be a generalized strictly semi-stable formal scheme over $k^\circ$. Let $\{ D_i \mid i \in I_X \}$ denote the finite set of irreducible components of the special fiber $X^{\text{red}}_s$ with its reduced structure. Denote by $\mult_i$ the multiplicity of $D_i$ in the special fiber $X_s$. For any non-empty subset $I \subset I_X$, we put $D_I = \bigcap_{i \in I} D_i$.

**Definition 2.3.** A generalized strictly semi-stable formal scheme $X$ is said to be strictly semi-stable if its special fiber $X_s$ is reduced and if the strata $D_I$ are all irreducible.

**Definition 2.4.** The Clemens polytope $S_X$ is the finite simplicial sub-complex of the simplex $\Delta^{I_X}$ such that $\Delta^I$ is a face of $S_X$ if and only if $D_I \neq \emptyset$.

One constructs a continuous, proper, surjective map $\tau: X_\eta \to S_X$ as follows.

**Definition 2.5.** A real-valued function on the Clemens polytope $S_X$ is said to be simple if it is affine on every simplicial face of $S_X$. We denote the vector space of simple functions on $S_X$ by $\text{Simp}_X$.

**Definition 2.6.** A vertical divisor on $X$ is a Cartier divisor on $X$ supported on the special fiber $X_s$.

Let $\text{Div}_0(X)^*_\mathbb{R}$ denote the vector space of vertical $\mathbb{R}$-divisors on $X$. It has a basis $(e_i)_{i \in I_X}$ indexed by the set of irreducible components of $X^{\text{red}}_s$. Let $(e^*_i)_{i \in I_X}$ denote the dual basis of the dual vector space $\text{Div}_0(X)^*_\mathbb{R}$. The dual basis gives rise to an identification

$$\text{Div}_0(X)^*_\mathbb{R} \simeq \mathbb{R}^{I_X}. \tag{2.2}$$

Let $D$ be a vertical effective Cartier divisor on $X$. For any point $x \in X_\eta$, assume that the divisor $D$ is cut out by some function $u$ near the point of reduction $\pi(x)$. We let $\varphi^0_D(x) = \text{val}(u(x))$, where $\text{val}(\cdot)$ denotes the valuation map. In this way, we obtain a continuous function $\varphi^0_D$ on $X_\eta$. The map $D \mapsto \varphi^0_D$ extends by linearity to a map from $\text{Div}_0(X)^*_\mathbb{R}$ to the space of continuous functions $C^0(X_\eta)$ on $X_\eta$.

**Proposition 2.7.** Let $\tau: X_\eta \to \text{Div}_0(X)^*_\mathbb{R}$ be the evaluation map defined by $\langle \tau(x), D \rangle = \varphi^0_D(x)$, for any $x \in X_\eta$, $D \in \text{Div}_0(X)^*_\mathbb{R}$. We have

(i) The image of $\tau$ is homeomorphic to the Clemens polytope $S_X$ by a piecewise linear map;

(ii) For any vertical $\mathbb{R}$-divisor $D = \sum a_i D_i \in \text{Div}_0(X)^*_\mathbb{R}$, there exists a unique simple function $\varphi_D$ on $S_X$ such that $\varphi^0_D = \varphi_D \circ \tau$.
(iii) The map $D \to \varphi_D$ gives an isomorphism of vector spaces $\text{Div}_0(\mathcal{X})_\mathbb{R} \simeq \text{Simp}_{\mathcal{X}}$.

Proof. Let $\mathcal{U}$ be an open affine subscheme of $\mathcal{X}$ equipped with an étale morphism $\phi: \mathcal{U} \to \mathcal{S}(n, d, m, a)$ as in Definition 2.2. A vertical effective Cartier divisor $D$ in this chart is given by some function $f = T_0^{a_0} \cdots T_d^{a_d}$. So we have

$$\varphi_D^0(x) = a_0 \text{val}(T_0(x)) + \cdots + a_d \text{val}(T_d(x))$$
on the affinoid domain $\mathcal{U}_\eta \subset \mathcal{X}_\eta$. Therefore, the map $\tau$ restricted to $\mathcal{U}_\eta$ is given by

$$\tau: x \mapsto (\text{val}(T_0(x)), \ldots, \text{val}(T_d(x))).$$

Now the statements (i) and (ii) are easily verified on this chart. The statement (iii) follows from (ii). \qed

Remark 2.8. The homeomorphism between the Clemens polytope $S_{\mathcal{X}}$ and the image of the map $\tau$ in $\text{Div}_0(\mathcal{X})^*_\mathbb{R}$ gives a canonical embedding

$$S_{\mathcal{X}} \subset \text{Div}_0(\mathcal{X})^*_\mathbb{R}.$$ 

From now on, we always regard $S_{\mathcal{X}}$ as being embedded in $\text{Div}_0(\mathcal{X})^*_\mathbb{R}$. Therefore, we obtain a one-to-one correspondence between simple functions on $S_{\mathcal{X}}$ and linear functions on $\text{Div}_0(\mathcal{X})^*_\mathbb{R}$. For a simple function $\varphi$ on $S_{\mathcal{X}}$, we denote the corresponding linear function again by $\varphi$. For the simple function $\varphi_D$ in Proposition 2.7(ii), we have

$$\varphi_D(e_i^*) = a_i.$$ 

From the inclusion (2.4) and the identification (2.2), the embedding $S_{\mathcal{X}} \subset \text{Div}_0(\mathcal{X})^*_\mathbb{R} \simeq \mathbb{R}^k$ equips $S_{\mathcal{X}}$ with an integral linear structure induced by the ambient Euclidean space (see [Yu13a] Section 2). For any point $v \in S_{\mathcal{X}}$, we denote the tangent space of $S_{\mathcal{X}}$ at $v$ by $T_vS_{\mathcal{X}}$ and the integral lattice inside it by $T_vS_{\mathcal{X}}(\mathbb{Z})$.

In the case where the formal scheme $\mathcal{X}$ is strictly semi-stable, as a special case of [Ber99], we have an inclusion map $\theta: S_{\mathcal{X}} \hookrightarrow \mathcal{X}_\eta$ and a strong deformation retraction from $\mathcal{X}_\eta$ to $S_{\mathcal{X}}$. So the Clemens polytope $S_{\mathcal{X}}$ can be viewed as the skeleton of $\mathcal{X}_\eta$ associated to the formal model $\mathcal{X}$. In the generalized case, neither the inclusion map nor the deformation retraction still exists.

Example 2.9. Let $\mathcal{X} = \text{Spf} \ k^\circ \{X\}/(X^2 - a^2)$ for some $a \in k^\circ \setminus \{0\}$. The generic fiber $\mathcal{X}_\eta = \text{Spf} \ k\{X\}/(X^2 - a^2)$ consists of two points, while the Clemens polytope $S_{\mathcal{X}}$ consists of only one point.

We regard the Clemens polytope $S_{\mathcal{X}}$ as the tropicalization of the $k$-analytic space $\mathcal{X}_\eta$ with respect to the formal model $\mathcal{X}$. The following proposition shows the functoriality of tropicalization.

Proposition 2.10. Let $\mathcal{X}, \mathcal{Y}$ be two generalized strictly semi-stable formal schemes over $k^\circ$ and $S_{\mathcal{X}}, S_{\mathcal{Y}}$ the corresponding Clemens polytopes. Let
\( \tau_X: X_\eta \to S_X \) and \( \tau_\mathfrak{Y}: \mathfrak{Y}_\eta \to S_\mathfrak{Y} \) be the maps defined in Proposition 2.7. Let \( f: X \to \mathfrak{Y} \) be a morphism of formal schemes. There exists a continuous map \( S_f: S_X \to S_\mathfrak{Y} \) such that the diagram

\[
\begin{array}{ccc}
X_\eta & \xrightarrow{f} & \mathfrak{Y}_\eta \\
\downarrow \tau_X & & \downarrow \tau_\mathfrak{Y} \\
S_X & \xrightarrow{S_f} & S_\mathfrak{Y}
\end{array}
\]

commutes. Moreover, the map \( S_f \) is affine on every simplicial face of \( S_X \).

**Proof.** The pullback map \( f^*: \text{Div}_0(\mathfrak{Y})_\mathbb{R} \to \text{Div}_0(X)_\mathbb{R} \) induces a linear map \( S_f: \text{Div}_0(\mathfrak{Y})_\mathbb{R} \to \text{Div}_0(X)_\mathbb{R} \) by duality. By the canonical embeddings \( S_X \subset \text{Div}_0(X)_\mathbb{R}^{*} \) and \( S_\mathfrak{Y} \subset \text{Div}_0(\mathfrak{Y})_\mathbb{R}^{*} \) (cf. the inclusion (2.4)), it suffices to show the commutativity of the following diagram

\[
\begin{array}{ccc}
X_\eta & \xrightarrow{f_\eta} & \mathfrak{Y}_\eta \\
\downarrow \tau_X & & \downarrow \tau_\mathfrak{Y} \\
\text{Div}_0(X)_\mathbb{R}^{*} & \xrightarrow{S_f} & \text{Div}_0(\mathfrak{Y})_\mathbb{R}^{*}
\end{array}
\]

In other words, it suffices to show that

\[
(2.6) \quad \varphi_{f^*(D)}^0(x) = \varphi_D^0(f_\eta(x))
\]

for any point \( x \in X_\eta \) and any vertical effective divisor \( D \in \text{Div}_0(\mathfrak{Y}) \). Let \( \pi_\mathfrak{Y}: \mathfrak{Y}_\eta \to \mathfrak{Y} \), denote the reduction map. Assume that the divisor \( D \) is cut out by some function \( u \) near the point of reduction \( \pi_\mathfrak{Y}(f_\eta(x)) \). Then the identity (2.6) is equivalent to the obvious identity

\[
\text{val} \left( (u \circ f)(x) \right) = \text{val} \left( u(f_\eta(x)) \right).
\]

\( \square \)

### 3. Metrizations of virtual line bundles

#### 3.1. Basic notions.

In order to ensure that various moduli spaces of stable maps are of finite type, we need to impose certain positivity conditions, i.e. analogs of Kähler metrics of complex geometry. A non-Archimedean analytic analogue of Kähler geometry was proposed in [KT02]. Our definitions are inspired by the suggestions in loc. cit. We fix a generalized strictly semistable formal scheme \( X \) over the ring of integers \( \mathbb{k}^\circ \) throughout this section.

We denote by \( PL_X \) the sheaf of functions on the Clemens polytope \( S_X \) whose germs at every point are germs of simple functions on \( S_X \).

**Definition 3.1.** For a smooth variety \( V \) defined over the residue field \( \hat{k} \), let \( \text{Div}(V) \) denote the abelian group of divisors in \( V \) and \( \text{Div}^0(V) \) the subgroup consisting of divisors whose intersection number with any proper curve in \( V \) is zero. We denote \( N^1(V) = \text{Div}(V)/\text{Div}^0(V) \). An element \( D \in N^1(V) \) is said to be **nef** if its intersection number with any proper curve in \( V \) is
non-negative. It is said to be \emph{ample} if it has a representative which is an ample divisor.

**Definition 3.2.** Let $\Delta^I, I \subset I_\mathbb{X}$ be a face of the Clemens polytope $S_\mathbb{X}$. For any simple function $\varphi$ on $S_\mathbb{X}$, we define the \emph{derivative} of $\varphi$ with respect to $\Delta^I$
\[
\partial_I \varphi = \sum_{i \in I_\mathbb{X}} \varphi(e_i^*) \cdot [D_i]_{D_I} \in N^1(D_{\mathcal{I}})_\mathbb{R},
\]
where $[D_i]$ denotes the divisor class of $D_i$, $\varphi(e_i^*)$ is defined by the embedding in Remark 2.8, and $N^1(D_{\mathcal{I}})_\mathbb{R} = N^1(D_\mathcal{I}) \otimes \mathbb{R}$. The function $\varphi$ is said to be \emph{linear} (resp. \emph{convex}, \emph{strictly convex}) along the open simplicial face $3(\Delta^I)$ corresponding to $I$ if $\partial_I \varphi$ is trivial (resp. nef, ample) in $N^1(D_{\mathcal{I}})_\mathbb{R}$. Let $\text{LinLoc}(\varphi)$ (resp. $\text{ConvLoc}(\varphi)$, $\text{SConvLoc}(\varphi)$) be the union of the open simplicial faces in $S_\mathbb{X}$ along which $\varphi$ is linear (resp. convex, strictly convex).

For any subset $U \subset S_\mathbb{X}$, the restriction $\varphi|_U$ is said to be \emph{linear} (resp. \emph{convex}, \emph{strictly convex}) if $\text{LinLoc}(\varphi)$ (resp. $\text{ConvLoc}(\varphi)$, $\text{SConvLoc}(\varphi)$) contains $U$.

We denote by $\text{Lin}_\mathbb{X}$ (resp. $\text{Conv}_\mathbb{X}$, $\text{SConv}_\mathbb{X}$) the subsheaf of $\text{PL}_\mathbb{X}$ whose germs are germs of linear (resp. convex, strictly convex) functions. The sheaf $\text{Lin}_\mathbb{X}$ acts on the sheaf $\text{PL}_\mathbb{X}$ (resp. $\text{Conv}_\mathbb{X}$, $\text{SConv}_\mathbb{X}$) via additions
\[
\psi \mapsto (\varphi \mapsto \varphi + \psi),
\]
where $\psi$ is a local section of $\text{Lin}_\mathbb{X}$ and $\varphi$ is a local section of $\text{PL}_\mathbb{X}$ (resp. $\text{Conv}_\mathbb{X}$, $\text{SConv}_\mathbb{X}$).

**Definition 3.3.** A \emph{virtual line bundle} $L$ on the $k$-analytic space $\mathbb{X}_\eta$ with respect to the formal model $\mathbb{X}$ is a torsor over the sheaf $\text{Lin}_\mathbb{X}$. A simple (resp. convex, strictly convex) metrization $\hat{L}$ of a virtual line bundle $L$ is a global section of the sheaf $\text{Lin}_\mathbb{X} \otimes L$ (resp. $\text{Conv}_\mathbb{X} \otimes L$, $\text{SConv}_\mathbb{X} \otimes L$), where the tensor product is taken over the sheaf $\text{Lin}_\mathbb{X}$.

In concrete terms, a simple metrization $\hat{L}$ gives for each $i \in I_\mathbb{X}$, a germ of a simple function $\varphi_i$ at the vertex $i$ of $S_\mathbb{X}$ up to addition by linear functions. So we obtain a collection of numerical classes $\partial_i \varphi_i \in N^1(D_i)_\mathbb{R}$ for every $i \in I_\mathbb{X}$.

**Definition 3.4.** We call such a collection of numerical classes $\partial_i \varphi_i \in N^1(D_i)_\mathbb{R}$ for every $i \in I_\mathbb{X}$ the \emph{curvature} of the metrized virtual line bundle $\hat{L}$, which we denote by $c(\hat{L})$.

**Lemma 3.5.** The curvature $c(\hat{L})$ has the following compatibility property. For any simplicial face $\Delta^I, I \subset I_\mathbb{X}$ and any two vertices $i, j \in I$, we have
\[
(\partial_i \varphi_i)|_{D_I} = (\partial_j \varphi_j)|_{D_I}.
\]

\footnote{The open simplicial face of a simplicial face is defined to be the relative interior of the simplicial face.}
Proof. Since $L$ is a torsor over the sheaf $\text{Lin}_X$, the simple functions $\varphi_i$ and $\varphi_j$ differ by a simple function which is linear along the open simplicial face $(\Delta^1)^p$. Therefore, we obtain the claimed identity. \hfill \Box

3.2. Degree of virtual line bundles on a curve. We assume in this section that our generalized strictly semi-stable formal scheme $X$ is proper, irreducible and one-dimensional. For simplicity, we assume moreover that the stratum $D_I$ is irreducible for any subset $I \subset I_X$, which can always be achieved by making admissible blowups. In this case, the Clemens polytope $S_X$ is a connected one dimensional simplicial complex (i.e. a graph).

We fix an order on the set of vertices $I_X$. For two vertices $i$ and $j$, we write $i \prec j$ if $i$ and $j$ are connected by an edge and if $i$ is inferior to $j$ with respect to the fixed order. For every vertex $i$ of $S_X$, let $U_i$ be the union of the vertex $i$ and all the open edges whose closures contain $i$. It is an open neighborhood of the vertex $i$. If $e_{ij}$ is an edge with two endpoints $i$ and $j$, then $U_i \cap U_j$ is the interior of the edge $e_{ij}$.

Lemma 3.6. We have
\begin{align*}
H^0\left(\text{Lin}_{X|U_i}\right) &\simeq \mathbb{R}^{d(i)}, \quad H^q\left(\text{Lin}_{X|U_i}\right) = 0 \quad (3.1) \\
H^0\left(\text{Lin}_{X|U_j \cap U_k}\right) &\simeq \mathbb{R} \oplus \mathbb{R}, \quad H^q\left(\text{Lin}_{X|U_j \cap U_k}\right) = 0 \quad (3.2)
\end{align*}
for any $q \geq 1$, any vertex $i$, and any pair of vertices $j, k$ such that $j \prec k$. Here $d(i)$ denotes the number of edges connected to the vertex $i$.

Proof. For a pair of vertices $j \prec k$, the stratum $D_{j,k} = D_j \cap D_k$ is a point by our assumptions. Since a point does not have any interesting divisors on it, the sheaf $\text{Lin}_{X|U_j \cap U_k}$ is isomorphic to the sheaf $\mathcal{P}_{X|U_j \cap U_k}$. The latter sheaf is a constant sheaf of 2-dimensional real vector spaces on $U_j \cap U_k$. So we have the isomorphisms (3.2).

The sheaf $\text{Lin}_{X|U_i}$ is more complicated. For every vertex $j$ which is connected to $i$ by an edge, we denote by $\iota_j$ the inclusion map $U_i \cap U_j \hookrightarrow U_i$. We have the adjunction morphism
$$\text{Lin}_{X|U_i} \rightarrow \iota_j^* \iota_j^*(\text{Lin}_{X|U_i}) \simeq \iota_j^*(\mathbb{R}^2_{U_i \cap U_j}).$$
Taking direct sums over all vertices $j$ that are connected to $i$, we obtain a morphism
$$\text{Lin}_{X|U_i} \rightarrow \bigoplus_{j \prec i} \iota_j^*(\mathbb{R}^2_{U_i \cap U_j}),$$
whose cokernel is a skyscraper sheaf of rank $d(i)$ concentrated at vertex $i$. Let us denote this skyscraper sheaf by $\mathbb{R}^{d(i)}_i$. We obtain a short exact sequence of constructible sheaves on $U_i$
$$0 \rightarrow \text{Lin}_{X|U_i} \rightarrow \bigoplus_{j \prec i} \iota_j^*(\mathbb{R}^2_{U_i \cap U_j}) \rightarrow \mathbb{R}^{d(i)}_i \rightarrow 0.$$The cohomology groups are then easily calculated via the associated long exact sequence, and we obtain the isomorphisms (3.1). \hfill \Box
Proposition 3.7. We have an isomorphism
\[ \text{deg}: H^1(\text{Lin}_X) \xrightarrow{\sim} \mathbb{R}. \]

Proof. By Lemma 3.6, the cohomology of the sheaf \( \text{Lin}_X \) can be calculated by the following Čech complex with respect to the covering \( \{U_i, i \in I_X\} \)
\[ 0 \to \prod_{i \in I_X} H^0(\text{Lin}_{X|U_i}) \xrightarrow{d^0} \prod_{j < k} H^0(\text{Lin}_{X|U_j \cap U_k}) \to 0. \]

Let \( n_E \) denote the number of edges of \( S_X \). By Lemma 3.6, we have isomorphisms
\[ \prod_{i \in I_X} H^0(\text{Lin}_{X|U_i}) \simeq \mathbb{R}^{2n_E}, \]
\[ \prod_{j < k} H^0(\text{Lin}_{X|U_j \cap U_k}) \simeq \mathbb{R}^{2n_E}. \]
The differential \( d^0 \) is defined as follows. An element of \( \prod_{i \in I_X} H^0(\text{Lin}_{X|U_i}) \) is a collection of linear functions \( \{ \varphi_i \mid i \in I_X \} \) on the open subsets \( U_i \). For any pair of vertices \( j, k \) such that \( j < k \), let \( d^0(\{ \varphi_i \mid i \in I_X \})_{jk} \) denote the component of \( d^0(\{ \varphi_i \mid i \in I_X \}) \) in \( H^0(\text{Lin}_{X|U_j \cap U_k}) \). We put
\[ d^0(\{ \varphi_i \mid i \in I_X \})_{jk}(j) = \varphi_j(j) - \varphi_k(j), \]
\[ d^0(\{ \varphi_i \mid i \in I_X \})_{jk}(k) = \varphi_j(k) - \varphi_k(k). \]
The irreducibility of the formal model \( X \) implies that the graph \( S_X \) is connected. Since \( \text{Ker} d^0 \) consists of constant functions on \( S_X \), we have \( \dim_{\mathbb{R}} \text{Ker} d^0 = 1 \). So \( \dim_{\mathbb{R}} \text{Im} d^0 = 2n_E - 1 \), and
\[ \dim_{\mathbb{R}} H^1(\text{Lin}_X) = 1. \]

Let \( \text{deg} \) be the map
\[ \text{deg}: \prod_{j < k} H^0(\text{Lin}_{X|U_j \cap U_k}) \to \mathbb{R} \]
defined as follows. An element of the left hand side is a collection of linear functions \( \{ \varphi_{jk} \mid j < k \} \) on all open edges. For any pair of vertices \( j, k \) such that \( j < k \), let
\[ \delta \varphi_{jk} = \text{mult}_j \cdot \varphi_{jk}(e^*_j) - \text{mult}_k \cdot \varphi_{jk}(e^*_k), \]
and let
\[ \text{deg}(\{ \varphi_{jk} \mid j < k \}) = \sum_{j < k} \delta \varphi_{jk}. \]

Lemma 3.8. We have \( \text{deg} \circ d^0 = 0 \).

Proof. Let \( \{ \varphi_i \mid i \in I_X \} \) be an element of \( \prod_{i \in I_X} H^0(\text{Lin}_{X|U_i}) \). The linearity of the function \( \varphi_i \) at the vertex \( i \) implies that
\[ \partial_i \varphi_i = \sum_{j \in I_X} \varphi_i(e^*_j) \cdot [D_j]|_{D_i} = 0 \in N^1(D_i)_{\mathbb{R}}. \]
If the vertices $i$ and $j$ are not connected by an edge, $[D_j]_{|D_i} = 0$. So we have
\[ \partial_i \varphi_i = \varphi_i(e_i^*) \cdot [D_i]_{|D_i} + \sum_{j \prec i} \varphi_i(e_j^*) \cdot [D_j]_{|D_i} = 0, \]
where $j \dashrightarrow i$ means that the vertex $j$ and the vertex $i$ are connected by an edge. Since
\[ 0 = \sum_{j \in I_X} \text{mult}_j \cdot [D_j]_{|D_i} = \text{mult}_i \cdot [D_i]_{|D_i} + \sum_{j \prec i} \text{mult}_j \cdot [D_j]_{|D_i}, \]
and
\[ [D_j]_{|D_i} = 1 \in N^1(D_i)_{\mathbb{R}} \quad \text{for} \quad j \dashrightarrow i, \]
we obtain that
\[ (3.5) \quad \varphi_i(e_i^*) \cdot \sum_{j \prec i} \text{mult}_j = \text{mult}_i \cdot \sum_{j \prec i} \varphi_i(e_j^*). \]
Therefore, we have
\[
\begin{align*}
(deg \circ d^i)(\{\varphi_i \mid i \in I_X\}) &= deg(\{\varphi_j - \varphi_k \mid j < k\}) \\
&= \sum_{j < k} \left( \text{mult}_j \cdot (\varphi_j(e_j^*) - \varphi_k(e_k^*)) - \text{mult}_k \cdot (\varphi_j(e_j^*) - \varphi_k(e_k^*)) \right) \\
&= \sum_{i \in I_X} \left( - \varphi_i(e_i^*) \cdot \sum_{j \prec i} \text{mult}_j + \text{mult}_i \cdot \sum_{j \prec i} \varphi_i(e_j^*) \right) = 0
\end{align*}
\]
\[ \square \]

To conclude, Lemma 3.8 and the equality (3.3) imply that the map $deg$ induces an isomorphism
\[ deg : H^1(Lin_X) \xrightarrow{\sim} \mathbb{R}. \]
\[ \square \]

**Definition 3.9.** The **degree** of a virtual line bundle $L$ on the $k$-analytic curve $X_\eta$ with respect to the formal model $X$ is the corresponding element in $H^1(Lin_X)$, which we denote by $deg(L)$.

**Remark 3.10.** By Proposition 3.7, $deg(L)$ is a real number.

**Remark 3.11.** Recall that we have the map $\tau : X_\eta \to S_X$ defined in Proposition 2.7, so we can pullback sheaves on $S_X$ to sheaves on $X_\eta$ by $\tau$. In the case where the formal model $X$ is strictly semi-stable, the map $\tau$ is a strong deformation retraction. So we have $H^1(Lin_X) \simeq H^1(\tau^* Lin_X)$. This implies the following. If we have two proper irreducible one-dimensional strictly semi-stable formal models $X$ and $X'$ over $k'$ such that $X_\eta \simeq X'_\eta =: X$. Denote by $\tau : X \to S_X$, $\tau' : X \to S'_X$ the retraction maps. Let $L$ and $L'$ be virtual line bundles on $X$ with respect to the formal models $X$ and $X'$ respectively, such that $\tau^* L \simeq \tau'^* L'$. Then we have $deg(L) = deg(L')$. In short, the degree of a virtual line bundle on a $k$-analytic curve does not depend on the choice of strictly semi-stable models.
As in the case of complex geometry, the degree of a virtual line bundle can also be calculated via the curvature of its metrizations. Let $\hat{\mathcal{L}}$ be a simple metrization of the virtual line bundle $\mathcal{L}$. By Definition 3.4, the curvature $c(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$ is a collection of numerical classes $\partial_i \varphi_i \in N^1(D_i)_{\mathbb{R}}$ for every $i \in I_X$.

**Definition 3.12.** The degree $\deg(c(\hat{\mathcal{L}}))$ of the curvature $c(\hat{\mathcal{L}})$ is the sum
\[
\sum_{i \in I_X} \text{mult}_i \cdot \deg \partial_i \varphi_i.
\]

**Proposition 3.13.** For a virtual line bundle $\mathcal{L}$ and a simple metrization $\hat{\mathcal{L}}$ of $\mathcal{L}$, we have
\[
\deg(c(\hat{\mathcal{L}})) = \deg(L).
\]

**Proof.** Assume that the virtual line bundle $\mathcal{L}$ is given by a collection of linear functions $\{ \varphi_{jk} \mid j \prec k \}$ on all open edges, which we regard as transition functions. By definition,
\[
\deg(L) = \deg(\{ \varphi_{jk} \mid j \prec k \}) = \sum_{j \prec k} \delta \varphi_{jk} = \sum_{j \prec k} (\text{mult}_j \cdot \varphi_{jk}(e^*_k) - \text{mult}_k \cdot \varphi_{jk}(e^*_j)).
\]

On the other hand, assume that the simple metrization $\hat{\mathcal{L}}$ is given by a collection of simple functions $\{ \varphi_i \mid i \in I_X \}$ such that $\varphi_j - \varphi_k = \varphi_{jk}$ for any pair of vertices $j \prec k$. As in the deduction of the equality (3.5), we have
\[
\text{mult}_i \cdot \deg \partial_i \varphi_i = -\varphi_i(e^*_i) \sum_{j \prec i} \text{mult}_j + \text{mult}_i \cdot \sum_{j \succ i} \varphi_i(e^*_j).
\]

Therefore,
\[
\deg(c(\hat{\mathcal{L}})) = \sum_{i \in I_X} \text{mult}_i \cdot \deg \partial_i \varphi_i
\]
\[
= \sum_{i \in I_X} \left( -\varphi_i(e^*_i) \sum_{j \prec i} \text{mult}_j + \text{mult}_i \cdot \sum_{j \succ i} \varphi_i(e^*_j) \right)
\]
\[
= \sum_{j \prec k} \left( \text{mult}_j \cdot (\varphi_j(e^*_k) - \varphi_k(e^*_j)) - \text{mult}_k \cdot (\varphi_j(e^*_j) - \varphi_k(e^*_j)) \right)
\]
\[
= \sum_{j \prec k} \left( \text{mult}_j \cdot \varphi_{jk}(e^*_k) - \text{mult}_k \cdot \varphi_{jk}(e^*_j) \right)
\]
\[
= \deg L.
\]
3.3. Functoriality of the definition of curvature. Let \( f: \mathcal{X} \to \mathcal{Y} \) be a morphism of generalized strictly semi-stable formal schemes over \( k^0 \). Let \( \{ D_i \mid i \in \mathcal{I}_X \} \) and \( \{ D_j \mid j \in \mathcal{I}_Y \} \) denote respectively the set of irreducible components of \( X^\text{red} \) and \( \mathcal{Y}^\text{red} \) with their reduced structures. Let \( c(\tilde{L}) \) be the curvature of a metrized virtual line bundle \( \tilde{L} \) on \( \mathcal{Y}_q \) with respect to the formal model \( \mathcal{Y} \). By definition, \( c(\tilde{L}) \) is a collection of numerical classes \( c_j' \in N^1(D_j)_{\mathbb{R}} \) for every \( j \in \mathcal{I}_Y \). The morphism \( f: \mathcal{X} \to \mathcal{Y} \) induces a morphism of \( \tilde{k} \)-schemes \( f_s: \mathcal{X}_s \to \mathcal{Y}_s \) and a morphism \( f^s_\ast: X^\text{red}_s \to \mathcal{Y}^\text{red}_s \) between their reduced structures. We define the pullback \( f^s_\ast c(\tilde{L}) \) of the curvature \( c(\tilde{L}) \) as follows. For each \( i \in \mathcal{I}_X \), suppose that the image \( f^s_\ast(D_i) \) is contained in some \( D_j \) for \( j \in \mathcal{I}_Y \). Let \( c_i = (f^s_\ast)^\ast c_j' \in N^1(D_i)_{\mathbb{R}} \). By Lemma 3.5, the numerical class \( c_i \) does not depend on the choice of \( j \in \mathcal{I}_Y \). Then the pullback \( f^s_\ast c(\tilde{L}) \) is defined to be the collection of numerical classes \( c_i \in N^1(D_i)_{\mathbb{R}} \) for all \( i \in \mathcal{I}_X \).

By Proposition 2.10, the morphism \( f \) induces a map \( S_f: S_X \to S_Y \) which is affine on every simplicial face of \( S_X \).

Lemma 3.14. The pullback of a simple (resp. linear) function on \( S_Y \) by \( S_f \) is a simple (resp. linear) function on \( S_X \).

Proof. Let \( \varphi \) be a simple function on \( S_Y \) and let \( S_f^\ast \varphi \) be the pullback of \( \varphi \) by \( S_f \). The fact that \( S_f \) is affine on every simplicial face of \( S_X \) implies that \( S_f^\ast \varphi \) is a simple function on \( S_X \). For any \( I \subset \mathcal{I}_X \), let \( D_I = \bigcap_{i \in \mathcal{I}_X} D_i \) denote the corresponding stratum of \( X^\text{red} \) with its reduced structure. Suppose that the image \( f^s_\ast(D_I) \) is contained in the stratum \( D_J \) of \( \mathcal{Y}^\text{red} \) for some \( J \subset \mathcal{I}_Y \). Let \( (f^s_\ast)^\ast(D_I) \) denote the pullback of \( \partial_J \varphi \) by \( (f^s_\ast)^\ast(D_I): D_I \to D_J \). It suffices to prove the following lemma. \( \square \)

Lemma 3.15. We have \( (f^s_\ast)^\ast(D_I) \partial_J \varphi = \partial_I (S_f^\ast \varphi) \).

Proof. Proposition 2.7 gives us the one-to-one correspondence between vertical \( \mathbb{R} \)-divisors on a formal model \( \mathcal{X} \) and simple functions on the Clemens polytope \( S_X \). By linearity and by the definition of the derivatives \( \partial_I, \partial_J \), it suffices to show that for any vertical effective divisor \( D \in \text{Div}_0(\mathcal{Y}) \), we have \( (f^s_\ast)^\ast(D_I) = (f^\ast[D])_{\mid D_I} \). The identify above follows from the commutative diagram

\[
\begin{array}{ccc}
D_I & \xrightarrow{i^\text{red}} & D_J \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}.
\end{array}
\]

\( \square \)

Lemma 3.14 ensures that the notions of pullback of virtual lines bundles and pullback of their metrizations are well-defined. Moreover, Lemma 3.15
Proposition 3.16. Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of generalized strictly semi-stable formal schemes over $k^0$, $L$ a virtual line bundle on $\mathcal{Y}_\eta$ with respect to the formal model $\mathcal{Y}$, and $\hat{L}$ a metrization of $L$. The pullback of the curvature of $\hat{L}$ equals the curvature of the pullback of $L$, i.e.

$$f^*_c(\hat{L}) = c(f^*(L)).$$

4. Balancing conditions via semi-stable reduction

4.1. Semi-stable reduction and skeletons of $k$-analytic curves. We review the notion of formal $k$-analytic spaces and its relations with formal schemes following [Duc]. We assume in this section that the non-Archimedean field $k$ is either discretely valued or algebraically closed.

Given a $k$-analytic space $X$, we want to construct a formal model for $X$, i.e. a formal scheme $\mathcal{X}$ locally finitely presented over $k^0$ whose generic fiber $\mathcal{X}_\eta$ is isomorphic to $X$. When $X = \text{Sp}_B(A)$ for a $k$-affinoid algebra $A$, a good candidate for the formal model $\mathcal{X}$ is the formal spectrum $\text{Spf}(\mathcal{O}(\mathcal{X}))$, where $\mathcal{O}(\mathcal{X}) = \{ f \in \mathcal{A} \mid |f|_{\sup} \leq 1 \}$. We also denote $\mathcal{O}(\mathcal{X})^0 = \{ f \in \mathcal{A} \mid |f|_{\sup} < 1 \}$. However, such constructions can not be glued together for non-affinoid $k$-analytic spaces in general. One has to specify a suitable covering $\mathcal{U}$ of $X$ by affinoid spaces in the beginning. In order to ensure that one can glue together the formal models corresponding to different elements in the covering $\mathcal{U}$ along open subschemes rather than arbitrary subschemes, we resort to the following property.

**Definition 4.1 ([Duc](6.3.1)).** Let $X$ be a $k$-affinoid space. An analytic domain $Y$ in $X$ is called special if $Y$ is a finite union of affinoid domains in $X$ of the form

$$\mathcal{D}(f) := \{ \xi \in X \mid |f(\xi)| = |f|_{\sup} \}$$

for some analytic function $f$ on $X$.

**Definition 4.2 ([Duc](6.3.2)).** Let $X$ be a $k$-analytic space. A formal strictly $k$-affinoid atlas on $X$ is a $G$-covering $\mathcal{U}$ of $X$ by strictly $k$-affinoid domains such that for any two elements $V, W$ in $\mathcal{U}$, the intersection $V \cap W$ is special in both $V$ and $W$. A formal strictly $k$-analytic space is a pair $(X, \mathcal{U})$ consisting of a $k$-analytic space $X$ and a formal strictly $k$-affinoid atlas $\mathcal{U}$ on $X$.

**Proposition 4.3 ([Duc](6.3.9)).** Let $(X, \mathcal{U})$ be a formal strictly $k$-analytic space. Under the assumption that $X$ is reduced, one can construct a formal model $\mathfrak{X}^\mathcal{U}$ for $X$ according to the formal strictly $k$-affinoid atlas $\mathcal{U}$. We have $\mathfrak{X}^\mathcal{U}_\eta \simeq X$ and $\mathfrak{X}^\mathcal{U} \simeq (\mathfrak{X}^\mathcal{U}_s)^{\text{red}}$, where $\mathfrak{X}^\mathcal{U}$ denotes the gluing of the affine $\widehat{k}$-schemes of the form $\text{Spec}(\mathcal{O}(V)^0/\mathcal{O}(V)^{00})$ over all affinoid charts $V \in \mathcal{U}$.
**Definition 4.4.** Let $C$ be a compact quasi-smooth strictly $k$-analytic curve. A finite set of type II points $S^0 \subset C$ is called **semi-stable** if every connected component of $C \setminus S^0$ is either an open disc or an open annulus. Let $S$ denote the convex hull of $S^0$ in $C$. The set $S^0$ is called **strictly semi-stable** if moreover the graph $S$ has no self-loops and if there is at most one edge between every two distinct vertices.

**Theorem 4.5.** Let $C$ be a compact quasi-smooth strictly $k$-analytic curve. The generalized (strictly) semi-stable formal models of $C$ over $k^0$ are in bijection with the (strictly) semi-stable sets of type II points in $C$.

**Proof.** The theorem follows from the detailed preparations in [Duc]. Given a generalized semi-stable formal model $C$ of $C$, we put

$$S^0 = \pi_{\mathcal{E}}^{-1}\{\text{generic points of the irreducible components of } \mathcal{E}_s^{\text{red}}\},$$

where $\pi_{\mathcal{E}} : \mathcal{E}_\eta \to \mathcal{E}_s^{\text{red}}$ denotes the reduction map. For the other direction, given a set of type II points $S^0 \subset C$, one can construct a formal strictly $k$-affinoid atlas $U_{S^0}$ on $C$ following [Duc](6.3.15). Roughly, the elements in $U_{S^0}$ are of the form $C \setminus U$, where $U$ is a finite union of connected components of $C \setminus S$ subject to certain conditions. By Proposition 4.3, we obtain a formal model $\mathcal{E}$ of $C$ corresponding to the atlas $U_{S^0}$. One concludes using the result of Bosch and Lüttkebohmert [BL85] Proposition 2.2 and 2.3, which says that a closed point $x \in C^{\text{red}}$ is non-singular (resp. ordinary double) if and only if $\pi_{\mathcal{E}}^{-1}(x)$ is isomorphic to an open disc (resp. an open annulus). □

In order to obtain (strictly) semi-stable formal models of the curve $C$ rather than generalized (strictly) semi-stable formal models, one has to allow finite extensions of the ground field $k$.

**Corollary 4.6** ([Duc](6.4.3)). Let $C$ be a compact quasi-smooth strictly $k$-analytic curve and let $S^0$ be a (strictly) semi-stable set of type II points in $C$. There exists a finite separable extension $k'$ of the field $k$, such that the generalized (strictly) semi-stable formal model of $C := C \times_k k'$ over the ring of integers $(k')^0$ corresponding to the set of type II points $S^0 := S^0 \times_k k'$ is (strictly) semi-stable.

### 4.2. Vanishing cycles and balancing conditions.

Let $k^s$ be a separable closure of $k$, $k^s$ its completion, and $\tilde{k}$ its residue field. Let $\mathfrak{X}$ be a strictly semi-stable formal scheme over $k^0$, and let $\mathfrak{X}_s, \mathfrak{X}_\eta, I_\mathfrak{X}, S_\mathfrak{X}, \pi : \mathfrak{X}_\eta \to \mathfrak{X}_s, \tau : \mathfrak{X}_\eta \to S_\mathfrak{X}$ be as in Section 2. Here we restrict ourselves to strictly semi-stable formal schemes rather than generalized ones, because the sheaf of vanishing cycles for the latter has non-trivial monodromy, which complicates greatly the story.

Let $C$ be a compact strictly $k$-analytic curve, and let $f : C \to \mathfrak{X}_\eta$ be a $k$-analytic morphism. Let $C'$ denote the image $(\tau \circ f)(C) \subset S_\mathfrak{X}$.

**Proposition 4.7.** The image $C' = (\tau \circ f)(C)$ is a finite polyhedral complex of dimension less than or equal to one (i.e. a graph) embedded in $S_\mathfrak{X}$. 
Proof. Choose a finite covering of the formal model \(X\) by affine open subschemes of the form \(U\) as in Definition 2.2. It suffices to prove that the image of \(f^{-1}(U_{\eta})\) under the map \((\tau \circ f)\) is a finite polyhedral complex of dimension less than or equal to one in \(S_X\). Using the explicit description of the map \(\tau : X_{\eta} \rightarrow S_X\) in (2.3), this follows from [Duc12] Theorem 3.2(1) (see also Section 7.2 Theorem 7.11). \(\square\)

Remark 4.8. Proposition 4.7 is also a consequence of Proposition 2.10 via formal models (see Proposition 4.13, Theorem 6.40).

After a suitable subdivision of the graph \(C^t\) if necessary, we can associate every edge \(e\) of \(C^t\) with a tropical weight \(w_e \in \mathbb{Z}^{I_X}\) up to multiplication by \(-1\), which is parallel to the edge \(e\) sitting inside \(S_X \subset \mathbb{R}^{I_X}\) (see [Yu13a] §5). Let us recall briefly the definition of tropical weights. It is a direct generalization of the classical case. The inverse image \((\tau \circ f)^{-1}(e^0)\) of an open edge \(e^0\) of \(C^t\) is a disjoint union of open annuli in \(C\). For each open annulus \(A\) above, we will define a weight \(w_A \in \mathbb{Z}^{I_X}\). The tropical weight \(w_e\) would be the sum of \(w_A\) over all open annuli in \((\tau \circ f)^{-1}(e^0)\). Fix \(i \in I_X\) and let \(p_i : \mathbb{R}^{I_X} \rightarrow \mathbb{R}\) be the projection to the \(i\)th coordinate. By (2.3), the map \(p_i \circ \tau \circ f|_A\) is given by the valuation of a certain invertible function \(f_i\) on \(A\). Assume that the annulus \(A\) is given by \(c_1 < |z| < c_2\) and write \(f_i = \sum_{m \in \mathbb{Z}} f_{i,m} z^m\), where \(f_{i,m} \in k\). Since \(f_i\) is invertible on \(A\), there exists \(m_i \in \mathbb{Z}\) such that \(|f_{i,m_i}| r^{m_i} > |f_{i,m}| r^m\) for all \(m \neq m_i, c_1 < r < c_2\). We set the \(i\)th component \(w_{i,A}\) of the weight \(w_A\) to be \(m_i\).

Definition 4.9. The graph \(C^t\) equipped with the tropical weights is called the tropical curve obtained from the morphism \(f : C \rightarrow X_{\eta}\).

The local shape of the tropical curve \(C^t\) sitting inside \(S_X\) satisfies distinguished geometrical properties, which we referred to as generalized balancing conditions ([Yu13a] Theorem 1.1). Using the theory of semi-stable reduction for \(k\)-analytic curves in Section 4.1, we can give another proof of our previous result. The new proof via models is essential for the applications of balancing conditions in the study of the tropicalization of the moduli stack of \(k\)-analytic stable maps (see Sections 5.2, 7.1).

Let \(v\) be a vertex of the tropical curve \(C^t\). Assume that \(v\) sits in the interior of the face \(\Delta_{I_v}\) of \(S_X\) corresponding to a subset \(I_v \subset I_X\). We denote the sum of tropical weights

\[\sigma_v = \sum_{e \ni v} \tilde{w}_e \in T_v S_X(\mathbb{Z}),\]

where the sum is taken over all edges of \(C^t\) which contain \(v\) as an endpoint, and \(\tilde{w}_e\) is the representative of the tropical weight \(w_e\) that points away from \(v\).

Let \(D_{I_v}\) denote the stratum of the special fiber \(X_s\) corresponding to the face \(\Delta_{I_v}\) and let \(\overline{D}_{I_v} = D_{I_v} \times \tilde{k}^s\). Put \(J = \{ j \mid D_{I_v \cup \{j\}} \neq \emptyset \}\). We have a
natural inclusion
\[ T_v S_X \subset \text{Ker} \left( \mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z} \right). \]

Let \( \alpha \) be the map
\[ \alpha: \{ \text{proper curves in } \overline{D}_v \} \to \text{Ker} \left( \mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z} \right) \]
\[ K \mapsto \left( \deg \mathcal{O}(D_j)|_K, \ j \in J \right), \]
which sends each proper curve \( K \) in \( \overline{D}_v \) to the degrees of the pullback of the line bundles \( \mathcal{O}(D_j) \) to \( K \) for every \( j \in J \).

**Theorem 4.10.** Under the assumptions that
\begin{itemize}
  \item[(i)] The curve \( C \) is compact and quasi-smooth;
  \item[(ii)] The vertex \( v \) does not lie in the image of the boundary \((\tau \circ f)(\partial C)\),
\end{itemize}
the sum of weights \( \sigma_v \) viewed in \( \text{Ker} \left( \mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z} \right) \) by the inclusion (4.1) lies in the image of the map \( \alpha \) defined in (4.2).

Now we discuss the proof of Theorem 4.10. We fix a prime number \( l \) different from the characteristic of the residue field \( \tilde{k} \). Using the inclusion (4.1), the sum of weights \( \sigma_v \) induces a linear map by duality
\[ \sigma_v^*: \text{Coker} \left( \mathbb{Q}_l \xrightarrow{\Delta} \mathbb{Q}_l^J \right)(-1) \to \mathbb{Q}_l(-1), \]
where \( \Delta \) denotes the diagonal map and the symbol \((-1)\) denotes the Tate twist.

Let \( \mathbb{Q}_l, X \) denote the constant sheaf with values in \( \mathbb{Q}_l \) on a space \( X \) endowed with \( \acute{e} \)tale topology whenever it makes sense, and let \( R\Psi \) and \( R\Phi \) denote the derived functors of the sheaf of nearby cycles and vanishing cycles respectively. Let \( X_\sigma = X_\circ \times \tilde{k}^\circ \). We have the following exact triangle
\[ \mathbb{Q}_l, X_\sigma \to R\Psi \mathbb{Q}_l, X_\circ \to R\Phi \mathbb{Q}_l, X_\circ \xrightarrow{+1}. \]

Let \( j: \overline{D}_v \to X_\sigma \) denote the closed immersion. We apply \( j^* \) to (4.4) then take global sections, and let \( \alpha^* \) denote the boundary map
\[ \alpha^*: R^1\Gamma(j^* R\Phi \mathbb{Q}_l, X_\circ) \to H^2_{\acute{e}t}(j^* \mathbb{Q}_l, X_\sigma). \]

Recall that we have the following calculation of the sheaf of vanishing cycles for a strictly semi-stable formal scheme \( X \) over \( k^\circ \).

**Theorem 4.11** ([RZ82], [Yu13a] Corollary 3.2). We have an isomorphism
\[ R^1\Gamma (j^* R\Phi \mathbb{Q}_l, X_\circ) \simeq \text{Coker} \left( \mathbb{Q}_l \xrightarrow{\Delta} \mathbb{Q}_l^J \right)(-1), \]
where \( \Delta \) denotes the diagonal map. Moreover, the boundary map
\[ \alpha^*: \text{Coker} \left( \mathbb{Q}_l \xrightarrow{\Delta} \mathbb{Q}_l^J \right)(-1) \to H^2_{\acute{e}t}(j^* \mathbb{Q}_l, X_\circ) \simeq H^2_{\acute{e}t}(\overline{D}_v, \mathbb{Q}_l) \]
is induced by the cycle class map in \( \acute{e} \)tale cohomology.

Therefore, by duality, Theorem 4.10 is reduced to the following statement.
Proposition 4.12. If we allow a finite separable extension of the ground field \(k\), there exists a smooth projective curve \(C\) over the residue field \(\bar{k}\) and a morphism of \(\bar{k}\)-schemes \(g: C \to D_{\bar{k}}\), such that the map \(\sigma^*_v\) defined in (4.3) is equal to the composition of the following maps
\[
R^1\Gamma\left(j^*R\Phi_{\bar{Q}_l,X_v}\right) \xrightarrow{\alpha} H^2_{\text{ét}}(j^*Q_{l,X}^v) \xrightarrow{\Sigma} H^2_{\text{ét}}(C \times \bar{k}, \bar{Q}_l) \xrightarrow{\bar{g}_*} \bar{Q}_l(-1),
\]
where the sum \(\Sigma\) is taken over all connected components of \(C\).

Proof. In order to use the theory of semi-stable reduction for the \(k\)-analytic curve \(C\), we need the following technical preparation, which follows from [Yu13a] Proposition 5.1.

Proposition 4.13. There exists a finite set of type II points \(S^0\) in \(C\) which satisfies the following conditions:

(i) The set \(S^0\) is strictly semi-stable in the sense of Definition 4.4;
(ii) Let \((C, U_{S^0})\) be the corresponding formal strictly \(k\)-analytic curve (see the proof of Theorem 4.5), and let \((X_\eta, U_X)\) be the formal strictly \(k\)-analytic space induced by the formal model \(X\). Then the \(k\)-analytic morphism \(f: C \to X_\eta\) is moreover a morphism of formal \(k\)-analytic spaces.

Let \(C\) be the generalized strictly semi-stable formal model over \(k^0\) associated to the strictly semi-stable set \(S^0\) (see Theorem 4.5). Since making finite field extensions of the ground field \(k\) does not change the tropical curve \(C^t\) sitting inside \(S_X\), we can assume that the formal model \(C\) is strictly semi-stable by Corollary 4.6. By Proposition 4.13(ii) we obtain a morphism of formal schemes \(f: C \to X_\eta\), such that \(f_\eta \simeq f\).

Let \(S_C\) be the skeleton of \(C\) associated to the formal model \(C\). By construction, it is isomorphic to the convex hull of \(S^0\) in \(C\). We denote by \(\tau_C: C \to S_C\) the retraction map defined in Proposition 2.7. By Proposition 2.10, the induced map \(S_j: S_C \to S_X\) is affine on each edge of the graph \(S_C\), and moreover we have
\[
\tau \circ f = S_j \circ \tau_C.
\]

Let \(v_0\) be a vertex of \(S_C\) which maps to \(v\) under \(S_j\), and let \(C_{v_0}^0\) be the irreducible component of \(C_s\) corresponding to \(v_0\). The \(k\)-scheme \(C_{v_0}^0\) is proper by the assumption (ii) in Theorem 4.10.

Let \(J_C\) be the set of irreducible components of \(C_s\) that intersects \(C_{v_0}^0\) non-trivially, \(J_C^0 = J_C \setminus \{C_s^0\}\), and let \(\pi_C: C \to C_s\) denote the reduction map. Then
\[
A_{v_0} = \left\{ \pi_C^{-1}(C_s^0 \cap C_s') \mid C_s' \in J_C^0 \right\}
\]
is a set of open annuli embedded in our \(k\)-analytic curve \(C\). The closure of each annulus in \(A_{v_0}\) has one boundary point equal to \(v_0\). We choose the orientations for these annuli to be the ones that point away from \(v_0\). Let
\[
\sigma_{v_0} = \sum_{A \in A_{v_0}} \tilde{w}_A \in T_v S_X(Z),
\]
where $\tilde{w}_A$ denotes the tropical weight contributed by the annulus $A$. We have

$$\sigma = \sum_{v_0} \sigma_{v_0},$$

where the sum is taken over all vertices of the graph $S_\xi$ that maps to $v$ by $S_f$. Using the inclusion (4.1), by duality, $\sigma_{v_0}$ induces a map

$$\sigma^{*}_{v_0}: \text{Coker} \left( \mathbb{Q}_l \xrightarrow{\Delta} \mathbb{Q}_l \right)(-1) \rightarrow \mathbb{Q}_l(-1).$$

Let $\xi = \xi \times \tilde{k}^s$, $\xi^{\nu} = \xi^{\nu} \times \tilde{k}^s$, and denote by $j_C: \xi^{\nu} \rightarrow \xi$ the closed immersion. For any étale sheaf $F$ on $\xi$, we have the adjunction morphism

$$\sigma^{*}_{v_0}: \text{Coker} \left( \mathbb{Q}_l \xrightarrow{\Delta} \mathbb{Q}_l \right)(-1) \rightarrow \mathbb{Q}_l(-1).$$

Let $C_s = C_s \times \tilde{k}^s$, $C^{\nu}_s = C^{\nu}_s \times \tilde{k}^s$, and denote by $j: C^{\nu}_s \rightarrow C_s$ the closed immersion. For any étale sheaf $F$ on $C_s$, we have the adjunction morphism

$$j^{\ast}_C \rightarrow j^{\ast}_C F.$$ 

Lemma 4.14. The assumption that $v$ lies in the interior of the face $\Delta^I$ implies that the image of $C^{\nu}_s$ under the map $f_s$ is contained in $D^{I}_I$.

Proof. Let $\pi: X_\eta \rightarrow X_s$ be the reduction map. By the construction of the retraction map $\tau: X_\eta \rightarrow S_X$, it is easy to see that $\tau(x)$ lies in the interior of a face $\Delta^I$ of $S_X$ if and only if $\pi(x)$ lies in the open stratum

$$D^I_I := D_I \setminus \bigcup_{I \subseteq I'} D_{I'}. $$

The lemma follows from the above observation. □

By Lemma 4.14, the sheaf $f_s^{\ast}j^{\ast}_C F$ is supported on $\overline{D_I} \subset X_s$. Therefore, we obtain a morphism

$$f_s^{\ast} \rightarrow f_s^{\ast}j^{\ast}_C F.$$ 

By [Ber94] Corollary 4.5(ii), we obtain a morphism

$$R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, X_\eta \right) \rightarrow R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right).$$

Applying $j^{\ast}_C$, we obtain a morphism

$$j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, X_\eta \right) \rightarrow j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right).$$

Substituting $\xi^{\nu}$ by $\mathbb{Q}_l^{\nu}$ in (4.5), we obtain a morphism

$$j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right) \rightarrow j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right).$$

Combining with (4.6) and taking global sections, we obtain a map

$$f^{\ast}_\Phi: R^{\ast}_C \left( j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, X_\eta \right) \right) \rightarrow R^{\ast}_C \left( j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right) \right).$$

Similarly, we have maps

$$f^{\ast}_\Psi: R^{\ast}_C \left( j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, X_\eta \right) \right) \rightarrow R^{\ast}_C \left( j^{\ast}_C R^1 \Gamma \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right) \right),$$

and

$$f^{\ast}_s: H^2_\et \left( j^{\ast}_C \mathbb{Q}_l, X_\eta \right) \rightarrow H^2_\et \left( j^{\ast}_C \mathbb{Q}_l, C_\eta \right).$$
They form the following commutative diagram by functoriality.

\[(4.7)\]

\[\begin{array}{ccc}
R^1\Gamma(j^* R\Psi Q_l, x_n) & \xrightarrow{\beta^*} & R^1\Gamma(j^* R\Phi Q_l, x_n) \\
\uparrow f^*_s & & \uparrow f^*_s \\
R^1\Gamma(j^*_C R\Psi Q_l, x_n) & \xrightarrow{\beta^*_C} & R^1\Gamma(j^*_C R\Phi Q_l, x_n) \\
\end{array}\]

\[\xrightarrow{\alpha^*_C} \xrightarrow{H^2_{\alpha}} (j^*_C Q_l, x_n) \xrightarrow{\sim} Q_l(-1).\]

By Theorem 4.11, we have

\[R^1\Gamma(j^* R\Phi Q_l, x_n) \simeq \operatorname{Coker}(Q_l \xrightarrow{\Delta} Q_l^l)(-1).\]

Using the inclusion (4.1), for each annulus \(A \in A^x\), the tropical weight \(\tilde{w}_A \in T_x S_1(Z)\) induces a map

\[\tilde{w}_A^*: R^1\Gamma(j^* R\Phi Q_l, x_n) \to Q_l(-1).\]

The sum of the maps \(\tilde{w}_A^*\) over all annuli \(A \in A^x\) is the map

\[\sigma^*_{\omega_0}: R^1\Gamma(j^* R\Phi Q_l, x_n) \to Q_l(-1)\]

induced by the sum of weights \(\sigma_{\omega_0} \in T_x S_1(Z)\).

Let \(A\) be an open annulus in \(A^x\), \(p_c \in C\), the point \(\pi_1(A)\), and \(j_{p_c}: \{p_c\} \hookrightarrow C\) the inclusion map. By Theorem 4.11, we have isomorphisms

\[R^1\Gamma(j^*_p R\Phi Q_l, x_n) \xrightarrow{\sim} R^1\Gamma(j^*_p R\Phi Q_l, x_n) \simeq \operatorname{Coker}(Q_l \xrightarrow{\Delta} Q_l \oplus Q_l)(-1).\]

Let \(s\) be the projection map

\[R^1\Gamma(j^*_p R\Phi Q_l, x_n) \simeq \operatorname{Coker}(Q_l \xrightarrow{\Delta} Q_l \oplus Q_l)(-1) \to Q_l(-1)\]

\[(x_0, x_1) \mapsto x_1 - x_0,\]

where the component \(x_0\) corresponds to the irreducible component \(C_1^x\). Let \(r_C\) be the restriction map

\[R^1\Gamma(j^*_C R\Phi Q_l, x_n) \to R^1\Gamma(j^*_p R\Phi Q_l, x_n).\]

**Lemma 4.15.** The composition \(s \circ r_C \circ f^*_A\) of the following morphisms

\[R^1\Gamma(j^* R\Phi Q_l, x_n) \xrightarrow{f^*_A} R^1\Gamma(j^*_C R\Phi Q_l, x_n) \xrightarrow{r_C} R^1\Gamma(j^*_p R\Phi Q_l, x_n) \xrightarrow{s} Q_l(-1)\]

is equal to the map

\[\tilde{w}_A^*: R^1\Gamma(j^* R\Phi Q_l, x_n) \to Q_l(-1).\]

**Proof.** Let \(p = f_s(p_c) \in \mathcal{X}_s\), \(j_p: \{p\} \hookrightarrow \mathcal{X}\) the closed immersion, and

\[r: R^1\Gamma(j^* R\Phi Q_l, x_n) \to R^1\Gamma(j^*_p R\Phi Q_l, x_n)\]

the restriction map. We have a commutative diagram

\[(4.8)\]

\[\begin{array}{ccc}
R^1\Gamma(j^* R\Phi Q_l, x_n) & \xrightarrow{f^*_A} & R^1\Gamma(j^*_C R\Phi Q_l, x_n) \\
\downarrow r & & \downarrow r_C \\
R^1\Gamma(j^*_p R\Phi Q_l, x_n) & \xrightarrow{f^*_p} & R^1\Gamma(j^*_p R\Phi Q_l, x_n). \\
\end{array}\]
It follows from the cohomological interpretations of tropical weights in [Yu13a] §5 that the composition $s \circ f^*_\Phi \circ r$ is equal to the map $\tilde{w}^*_A$. We conclude our lemma by the commutativity of (4.8).

Lemma 4.16. The map $\sigma^{*}_{v_0} : R^1 \Gamma(j^* R \Phi Q_{l,x_n}) \to \mathbb{Q}_l(-1)$ is equal to the composition $\alpha^*_C \circ f^*_\Phi$ in the diagram (4.7).

Proof. By Theorem 4.11, we have the isomorphism

$$R^1 \Gamma(j^*_C R \Phi Q_{l,x_n}) \simeq \text{Coker} \left( \mathbb{Q}_l \xrightarrow{\Delta} Q^f_l \right) (-1),$$

and the map $\alpha^*_C$ in the diagram (4.7) is induced by the cycle class map. So $\alpha^*_C$ is the sum of the maps $s \circ r_C$ that we considered in Lemma 4.15 over all annuli $A \in A^{v_0}$. On the other hand the map

$$\sigma^{*}_{v_0} : R^1 \Gamma(j^* R \Phi Q_{l,x_n}) \to \mathbb{Q}_l(-1)$$

is the sum of the maps

$$\tilde{w}^*_A : R^1 \Gamma(j^* R \Phi Q_{l,x_n}) \to \mathbb{Q}_l(-1)$$

over all annuli $A \in A^{v_0}$. Therefore Lemma 4.16 follows from Lemma 4.15.

Combining Lemma 4.16 and the commutativity of the diagram (4.7), we obtain the following lemma.

Lemma 4.17. The map $\sigma^{*}_{v_0} : R^1 \Gamma(j^* R \Phi Q_{l,x_n}) \to \mathbb{Q}_l(-1)$ is equal to the composition of the following morphisms

$$R^1 \Gamma(j^* R \Phi Q_{l,x_n}) \xrightarrow{\alpha^*_C} H^2_{et}(j^* Q_{l,x_n}) \xrightarrow{\iota^*_C} H^2_{et}(C^{v_0}_\mathcal{Y}, \mathbb{Q}_l) \xrightarrow{\sim} \mathbb{Q}_l(-1).$$

We conclude our proof of Proposition 4.12 by taking $C$ to be the disjoint union of $C^{v_0}_\mathcal{Y}$ over all vertices $v_0$ of the graph $S_\mathcal{Y}$ which map to the point $v \in S_X$ by the map $S_f$.

5. GROMOV COMPACTNESS IN TROPICAL GEOMETRY

5.1. Tropical Kähler manifolds and combinatorics. In this section, we introduce among others the notion of simple Kähler structures on Clemens polytopes. Its relation with Kähler structures on non-Archimedean analytic spaces will be discussed in Section 5.2. Then we discuss the proof of the tropical Gromov compactness theorem (Theorem 5.12), which is totally combinatorial in nature. The prototype of these considerations appeared in [Yu13b], where similar problems were studied in the setting of “local” tropical geometry, and the account is totally elementary. We will borrow some combinatorial tricks invented there to shorten the exposition of the proof of our current problem.

Definition 5.1. Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{Z}^n \setminus \{0\}$ by multiplication by $-1$. We denote the quotient by $W$. For any $w \in W$, we define its norm $|w| = \sqrt{\sum (w^i)^2}$ for any representative $(w^1, \ldots, w^n) \in \mathbb{Z}^n \setminus \{0\}$. 

**Definition 5.2.** An ordinary tropical curve $G$ in an open subset $U \subset \mathbb{R}^n$ is a finite one-dimensional polyhedral complex $G$ embedded in $U$ such that

(i) $G$ is closed in $U$ as a topological subspace;

(ii) Each edge $e$ of $G$ is equipped with a weight $w_e \in W$ parallel to the direction of $e$ inside $\mathbb{R}^n$.

(iii) We require that the ordinary balancing condition holds, i.e. for any vertex $v$ of $G$, we have $\sum_{e \ni \tilde{v}} w_e = 0$, where the sum is taken over all edges containing $v$ as an endpoint, and $\tilde{w_e}$ is the representative of $w_e$ that points away from $v$.

An ordinary tropical curve $G$ is said to be simple if each vertex of $G$ is at least 3-valent.

Let $K$ be the $n$-simplex obtained as the convex hull of the $n + 1$ points $(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ inside $\mathbb{R}^n$, and let $K^o$ be the interior of $K$. Let $G$ be an ordinary tropical curve in $K^o$, and let $\overline{G}$ denote the closure of $G$ in $\mathbb{R}^n$.

**Definition 5.3.** The boundary multiplicity of an ordinary tropical curve $G$ in $K^o$ is the sum

$$\sum_{e \text{ edge of } \overline{G}, e \cap \partial K \neq \emptyset} |w_e|.$$

**Theorem 5.4.** Let $A$ be a positive real number. There exists an integer $N$ such that for any ordinary simple tropical curve $G$ in $K^o$ whose boundary multiplicity is bounded by $A$, the number of vertices of $G$, the number of edges of $G$, and the norms of the weights of all the edges of $G$ are all bounded by $N$.

**Proof.** Regarding the boundary $\partial K$ as a simplicial complex of dimension $n - 1$, we denote by $(\partial K)^{n-2}$ its skeleton of dimension $n - 2$. In [Yu13b], an ordinary tropical curve $G$ in $K^o$ was called saturated if $\overline{G} \cap (\partial K)^{n-2} = \emptyset$ and if $\overline{G}$ intersects $\partial K \setminus (\partial K)^{n-2}$ perpendicularly. In this case, Theorem 5.4 is a consequence of Proposition 4.1 and Proposition 2.4 in [Yu13b]. For the general case, we can imitate the saturation trick introduced in Section 6 loc. cit. \[\square\]

Now we turn to the setting of global tropical geometry as introduced in Section 2. Let $\mathfrak{X}$ be a strictly semi-stable formal scheme over $k^\circ$, $I_X$ the set of irreducible components of the special fiber $\mathfrak{X}_s$, and $S_X \subset \mathbb{R}^{I_X}$ the Clemens polytope.

**Definition 5.5.** For any subset $I \subset I_X$, put $J_I = \{ j \in I_X \mid D_{I \cup \{j\}} \neq \emptyset \}$. Let $\mathbb{Z}/2\mathbb{Z}$ act on $\text{Ker} (\mathbb{Z}^{|J_I|} \xrightarrow{\Sigma} \mathbb{Z})$ by multiplication by $-1$. We denote the quotient by $W_I$. For any $w \in W_I$, we define its norm $|w| = \sqrt{\sum (w^j)^2}$ for any representative $(w^j) \in \text{Ker} (\mathbb{Z}^{|J_I|} \xrightarrow{\Sigma} \mathbb{Z})$. 


Definition 5.6. A tropical curve $G$ in the Clemens polytope $S_X$ is a connected finite one-dimensional\footnote{We allow $G$ to be a point in $S_X$ as a degenerate situation.} polyhedral complex embedded in $S_X$ such that

(i) $G$ is closed in $S_X$ as a topological subspace;

(ii) For each edge $e$ of $G$, assume that the interior of $e$ is contained in the interior of a face $\Delta^{I_e}$ for some $I_e \subset I_X$. Then the edge $e$ is equipped with a weight $w_e \in W_{I_e}$ parallel to the direction of $e$ inside $S_X \subset \mathbb{R}^{I_X}$.

Definition 5.7. Let $G$ be a tropical curve in the Clemens polytope $S_X$ and $v$ a vertex of $G$. Assume that $v$ sits in the interior of a face $\Delta^{I_v}$ for some $I_v \subset I_X$. We define the sum of weights around the vertex $v$ to be

$$\sigma_v = \sum_{e \ni v} \tilde{w}_e,$$

where the sum is taken over all edges containing $v$ as an endpoint, and $\tilde{w}_e$ is the representative of $w_e$ that points away from $v$. The sum makes sense via the embeddings

$$\text{Ker}\left(\mathbb{Z}^{I_v} \xrightarrow{\Sigma} \mathbb{Z}\right) \hookrightarrow \text{Ker}\left(\mathbb{Z}^{J_{I_v}} \xrightarrow{\Sigma} \mathbb{Z}\right), \quad \text{for all } e \ni v.$$

Definition 5.8. Let $G$ be a tropical curve in the Clemens polytope $S_X$. A vertex $v$ of $G$ is said to be of type I if the sum of weights $\sigma_v \in T_v S_X(\mathbb{Z})$ is zero. Otherwise it is said to be of type II. The tropical curve $G$ is said to be simple if all 2-valent vertices of $G$ are of type II.

Remark 5.9. From any (ordinary) tropical curve $G$, one can obtain a unique simple (ordinary) tropical curve $G^s$ by removing the redundant vertices.

Definition 5.10. A simple Kähler structure $\omega$ on the Clemens polytope $S_X$ is a collection of positive numbers $\omega_{I,j}$ for every face $\Delta^I, I \subset I_X$ and every vertex $j \in J_I$, such that $\omega_{I,j} > \omega_{I',j}$ whenever $I \supset I'$ and $j \in J_I$. We also allow $\omega_{I,j}$ to take the value $+\infty$.

Definition 5.11. Let $G$ be a tropical curve in the Clemens polytope $S_X$ equipped with a simple Kähler structure $\omega$. Let $v$ be a vertex of $G$ sitting in the interior of a face $\Delta^{I_v}$ for some $I_v \subset I_X$. The local degree of $G$ at $v$ with respect to $\omega$ is by definition the real number

$$|\sigma_v|_\omega := \max_{j \in J_{I_v}} \omega_{I_v,j} \cdot |\sigma^j_v|,$$

where $(\sigma^j_v)_{j \in J_{I_v}} \in \text{Ker}\left(\mathbb{Z}^{J_{I_v}} \xrightarrow{\Sigma} \mathbb{Z}\right)$ is any representative of the sum of weights $\sigma_v \in W_{I_v}$. The tropical degree $\deg_\omega(G)$ of the tropical curve $G$ with respect to the simple Kähler structure $\omega$ is the sum of the local degrees over all vertices of $G$.

Theorem 5.12. Let $S_X$ be a Clemens polytope equipped with a simple Kähler structure $\omega$. Fix a positive real number $A$. Denote by $M(S_X, A)$ the set of...
tropical curves in $S_X$ whose tropical degrees with respect to $\omega$ are bounded by $A$. Then $M(S_X, A)$ has the structure of a compact finite polyhedral complex.

Proof. We begin with several lemmas on the bounds of various quantities.

Lemma 5.13. There exists an integer $N$, such that for any tropical curve $G \in M(S_X, A)$, the number of type II vertices of $G$ and the norms of the sum of tropical weights around all type II vertices are bounded by $N$.

Proof. It follows directly from Definition 5.11. □

Lemma 5.14. There exists an integer $N$, such that for any tropical curve $G \in M(S_X, A)$, any edge $e$ of $G$, the norm of the weight $|w_e|$ is bounded by $N$.

Proof. We fix $i \in I_X$ and think of it as a direction in the Euclidean space $\mathbb{R}^{I_X}$. We would like to show that for any edge $e_0$ of our tropical curve $G$, the $i^{th}$ component $|w_{e_0}^i|$ is bounded. We borrow the ideas from [Yu13b] Section 3. A path $P$ starting from the edge $e_0$ with direction $i$ is by definition a chain of consecutive edges $e_0, e_1, \ldots, e_l$ of the tropical curve $G$ such that the projection to the $i^{th}$ coordinate $\mathbb{R}^{I_X} \to \mathbb{R}$ restricted to $P$ is injective. Let $m = |w_{e_0}^i|$. We can construct a collection of $m$ paths $P_1, \ldots, P_m$ starting from the edge $e_0$ as follows. We assign to each edge $e$ of our tropical curve $G$ an integer $c_i(e)$ called the capacity (in the $i^{th}$ direction). Initially we set $c_i(e) = |w_e^i|$ for all edges $e$ of $G$. To construct the path $P_1$, we start with the edge $e_0$, and we decrease the capacity $c_i(e_0)$ by 1. Suppose we have constructed a chain of edges $e_0, e_1, \ldots, e_j$. Let $v'$ be the endpoint of the edge $e_j$ with larger $i^{th}$ coordinate. We choose $e_{j+1}$ to be an edge of $G$ such that

(i) $v'$ is the endpoint of $e_{j+1}$ with smaller $i^{th}$ coordinate;
(ii) $|w_{e_{j+1}}^i| \neq 0$;
(iii) The capacity $c_i(e_{j+1})$ is positive.

If such an edge does not exist, we stop. Otherwise we choose $e_{j+1}$ as our next edge in the path $P_1$, decrease the capacity $c_i(e_{j+1})$ by 1 and iterate the procedure until we stop. As a result we obtain the path $P_1$. The other paths $P_2, \ldots, P_m$ are constructed exactly in the same way. Now the main observation is that each path $P_j$ must end on a vertex of type II. Moreover, for any vertex $v$ of type II, the number of paths ending on $v$ cannot be greater than $|\sigma^i_v|$. Therefore, the number of paths $m$ can be bounded thanks to Lemma 5.13. □

Lemma 5.15. There exists an integer $N$, such that for any tropical curve $G \in M(S_X, A)$, the number of vertices of $G$ is bounded by $N$.

Proof. Let $F$ be a simplex of the Clemens polytope $S_X$, $F^\circ$ its interior, and $\partial F$ its boundary. It suffices to show that the number of vertices of $G$ in $F^\circ$ can be bounded. Assume that $\dim F > 0$, otherwise there is nothing to prove. It follows from the definition that the vertices in $G_{F^\circ} \cap \partial F$ are
all of type II. If $G_{|F^o}$ were an ordinary tropical curve in $F^o$ (see Definition 5.2), then it follows from Theorem 5.4 and Lemma 5.13 that the number of vertices of $G$ in $F^o$ can be bounded. The observation is that in the general case, $G_{|F^o}$ is not far from an ordinary tropical curve in $F^o$. The vertices of $G_{|F^o}$ which do not satisfy the ordinary balancing condition (see Definition 5.2(iii)) are all of type II. Therefore, their number can be bounded thanks to Lemma 5.13. Now we can modify the graph $G_{|F^o}$ to be an ordinary tropical curve in $F^o$ by adding some rays to these unbalanced vertices. The number and the multiplicities of the rays needed can be bounded by Lemma 5.13. So we can reduce the general case to the case where $G_{|F^o}$ is an ordinary tropical curve in $F^o$.

Now we can conclude the following.

Lemma 5.16. There exists an integer $N$, such that for any tropical curve $G \in M(S_\mathcal{X}, A)$, the number of vertices, the number of edges, and the norms of the weights of all the edges of $G$ are all bounded by $N$.

Proof. Lemma 5.14 gives the bound on the norm of weights. Lemma 5.15 gives the bound on the number of vertices. The bound on the number of edges follows from the bound on the number of vertices.

Lemma 5.16 above implies that there are only a finite number of combinatorial types of the tropical curves in the set $M(S_\mathcal{X}, A)$. The moduli space of tropical curves in $M(S_\mathcal{X}, A)$ of a fixed combinatorial type has the structure of an open subset with polyhedral boundary in a Euclidean space. Indeed, if the tropical curves of our fixed combinatorial type had only vertices but no edges, the vertices could be moved freely in $S_\mathcal{X}$ without constraints. Then every edge adds some constraints which can be described by a finite number of affine equations and inequalities. All the constraints cut out some polyhedral region in a Euclidean space. We take such regions to be the open faces of the polyhedral structure of $M(S_\mathcal{X}, A)$. Incidence relations between different faces correspond to degenerations of combinatorial types of the tropical curves. We observe that the tropical curves sharing the same combinatorial type have the same tropical degree with respect to the simple Kahler structure $\omega$. Moreover, when degenerations of combinatorial types occur, the tropical degree either decreases or remains the same. This implies that the polyhedral structure on $M(S_\mathcal{X}, A)$ is finite and compact.

Remark 5.17. The topology on the set $M(S_\mathcal{X}, A)$ is induced by the polyhedral structure. There are more direct ways to introduce topology on the set $M(S_\mathcal{X}, A)$ without referring to polyhedral structures. For example, one can introduce the notion of parameterized tropical curves. A parameterized tropical curve into $S_\mathcal{X}$ is a piecewise affine map from a metrized graph to $S_\mathcal{X}$ plus tropical weights on every edge compatible with the integral linear structure on $S_\mathcal{X}$. Let $M(S_\mathcal{X}, A)_p$ denote the set of parameterized tropical curves into
$S_X$ whose tropical degrees with respect to the simple Kähler structure $\omega$ are bounded by $A$, and let $M(S_X, A)_p \to M(S_X, A)$ be the map forgetting the parameterizations. There is an obvious way to equipped the set $M(S_X, A)_p$ with a topology. Then we can define the topology on the set $M(S_X, A)$ as the quotient topology. Another possible way is to introduce the notion of tropical differential forms on the Clemens polytope $S_X$ and define the topology on the set $M(S_X, A)$ as the weak topology with respect to integration of tropical 1-forms over tropical curves. One can check that as long as we bound the number of vertices of the tropical curves, the various topologies are all equivalent.

Remark 5.18. We will also consider tropical curves with marked points. An $n$-pointed tropical curve $(G, \{s_i\})$ in the Clemens polytope $S_X$ is by definition a tropical curve $G$ in $S_X$ together with $n$ non-necessarily distinct points $s_1, \ldots, s_n$ on $G$. Denote by $M_n(S_X, A)$ the set of $n$-pointed tropical curves $S_X$ whose tropical degrees are bounded by $A$. The same reasonings of Theorem 5.12 show that the set $M_n(S_X, A)$ also carries the structure of a compact finite polyhedral complex.

5.2. Non-Archimedean analytic Kähler manifolds. It is difficult to imagine a genuine analog of Kähler metrics in non-Archimedean analytic geometry whose curvature is everywhere positive. Nevertheless, the following definition is a good approximation and would suffice for our purposes.

Definition 5.19. Let $\mathfrak{X}$ be a generalized strictly semi-stable formal scheme over $k^\circ$. A Kähler structure $\hat{L}$ on $\mathfrak{X}$ is a virtual line bundle $L$ on $\mathfrak{X}_\eta$ with respect to the formal model $\mathfrak{X}$ equipped with a strictly convex metrization $\hat{L}$. Given a strictly semi-stable formal scheme $\mathfrak{X}$ over $k^\circ$ equipped with a Kähler structure $\hat{L}$, we can associate to it a simple Kähler structure $\omega$ on the Clemens polytope $S_X$ as follows.

Let $\Delta^I$ be a face of $S_X$ for some $I \subset I_X$ and let $i \in I_X$ be a vertex. The curvature of the metrized virtual line bundle $\hat{L}$ at the vertex $i$ is an ample class $\partial_i \varphi_i$ in $N^1(D_i)_\mathbb{R}$ by Definition 3.4. By restriction to the stratum $D_I$, we get an ample class $\partial_I \varphi_i$ in $N^1(D_I)_\mathbb{R}$. Let $j$ be an element in $J_I$, and let

$$\omega_{I,j} = \min \left\{ C \cdot (\partial_I \varphi_i) \mid C \in \overline{NE}(D_I), |C \cdot \mathcal{O}(D_j)|_{D_I} = 1 \right\},$$

where $\overline{NE}(D_I)$ denotes the closure of the cone of effective proper curves in the stratum $D_I$.

For any finite separable field extension $k \subset k'$, we obtain a finite separable extension of the residue fields $\hat{k} \subset \hat{k}'$. Let $C'$ be an element in $\overline{NE}(D_I \times \hat{k}')$. We can average it by the action of $\text{Gal}(k'/\hat{k})$ and obtain an element $C$ in $\overline{NE}(D_I)$ such that $C \cdot D = C' \cdot (D \times \hat{k}')$ for any divisor $D$ on $D_I$. This implies
that
\[ \omega_{I,j} = \min \{ C \cdot (\partial_I \varphi_i) \mid C \in \overline{NE}(D_I), |C \cdot \mathcal{O}(D_j)| = 1 \} = \min \{ C \cdot (\partial_I \varphi_i) \mid C \in \overline{NE}(D_I), |C \cdot \mathcal{O}(D_j)| = 1 \}, \]
where \( \overline{D}_I = D_I \times \hat{k}^s \).

The ampleness of the class \( \partial_I \varphi_i \) in \( N^1(D_I)_{\mathbb{R}} \) implies that \( \omega_{I,j} \) is a positive real number. Lemma 3.5 shows that \( \omega_{I,j} \) does not depend on the choice of \( i \in I \). The fact that \( \omega_{I,j} > \omega_{I',j} \) for any \( I \supset I' \) follows from the inclusion \( \overline{NE}(D_I) \subset \overline{NE}(D_{I'}) \). So the collection
\[ \omega = \{ \omega_{I,j} \mid I \subset I_X, j \in J_I \} \]
is a simple Kähler structure on the Clemens polytope \( S_X \) (Definition 5.10).

**Definition 5.20.** The simple Kähler structure \( \omega \) constructed above is called the simple Kähler structure on the Clemens polytope \( S_X \) induced by the Kähler structure \( \tilde{L} \) on \( X \).

Now let \( f: C \to X_\eta \) be a morphism from a proper smooth irreducible \( k \)-analytic curve to \( X_\eta \). The degree of the morphism \( f \) is defined as follows. By Proposition 4.13, we can pick a strictly semi-stable formal model \( \mathcal{C} \) of \( C \) and a morphism \( \tilde{f}: \mathcal{C} \to X \) such that \( \tilde{f}_\eta \simeq f \), if we allow a finite extension of the ground field \( k \). Then \( f^*L \) is a virtual line bundle on \( C \) with respect to the formal model \( \mathcal{C} \). We define \( \deg f = \deg f^*L \) (see Definition 3.9). It does not depend on the choice of formal models by Remark 3.11.

Let \( C^t \) denote the tropical curve in the Clemens polytope \( S_X \) obtained from the morphism \( f: C \to X_\eta \) (see Definition 4.9). As a topological space \( C^t = (\tau \circ f)(C) \). We explained in the beginning of Section 4.2 that \( C^t \) is a tropical curve in the Clemens polytope \( S_X \) in the sense of Definition 5.6. We would like to relate the degree of the morphism \( f \) with the tropical degree (Definition 5.11) of the tropical curve \( C^t \).

Let \( S_\mathcal{C} \) be the Clemens polytope of the formal model \( \mathcal{C} \) and let \( S_l: S_\mathcal{C} \to S_X \) be the continuous piecewise affine map induced by the morphism \( f: \mathcal{C} \to X \) (see Proposition 2.10). For any vertex \( v_0 \) of \( S_\mathcal{C} \), let \( \mathcal{C}^v_0 \) denote the irreducible component of \( \mathcal{C}_s \) corresponding to \( v_0 \). Let \( v \) be a vertex of \( C^t \) sitting in the interior of a face \( I_v \subset I_X \), \( \sigma_v \) the sum of weights around \( v \), and let \( V(S_\mathcal{C})_v \) be the set of vertices of the graph \( S_\mathcal{C} \) which map to the point \( v \in S_X \) by the map \( S_l \). Define the map \( \sigma^*_v \) as (4.3). Lemma 4.17 shows that the map \( \sigma^*_v: R^1\Gamma(j^*R\Phi_{q,l,x_\eta}) \to \mathbb{Q}(-1) \) is equal to the sum of the composition of the following maps
\[ R^1\Gamma(j^*R\Phi_{q,l,x_\eta}) \overset{\alpha^*}{\to} H^2_\alpha(j^*\hat{\mathcal{O}}_{q,l,x_\eta}) \overset{I^*}{\to} H^2_\alpha(\mathcal{C}^v_0, \mathbb{Q}) \overset{\sim}{\to} \mathbb{Q}(-1) \]
over all vertices \( v_0 \) in \( V(S_\mathcal{C})_v \). In other words, we have shown the following lemma.
Lemma 5.21. The sum of weights $\sigma_v \in \text{Ker} \left( \mathbb{Z}^{J_{\ell_\nu}} \xrightarrow{\sum} \mathbb{Z} \right)$ is equal to the sum of $\sigma_{v_0} \in \text{Ker} \left( \mathbb{Z}^{J_{\ell_\nu}} \xrightarrow{\sum} \mathbb{Z} \right)$ over all $v_0 \in V(\mathcal{E}_\nu)$, where the $j^{th}$ component $\sigma_{v_0}^j$ of $\sigma_{v_0}$ is the degree of the line bundle $(\tilde{f}_s|_{\mathcal{E}_0})^* \mathcal{O}(D_j)$ on the curve $\mathcal{E}_v^{\nu}$.

Let us call the degree of the line bundle $(\tilde{f}_s|_{\mathcal{E}_0})^* (\partial_{I_{\ell,v}} \varphi_j)$ on the curve $\mathcal{E}_v^{\nu}$ the local degree of $\tilde{f} : \mathcal{E} \to \mathcal{X}$ at the irreducible component $\mathcal{C}_v^{\nu}$. By the construction of the simple Kähler structure $\omega$ on the Clemens polytope $S_\mathcal{X}$ induced by the Kähler structure $\tilde{L}$ on $\mathcal{X}$ (Definition 5.20), the local degree of $\tilde{f} : \mathcal{E} \to \mathcal{X}$ at the irreducible component $\mathcal{C}_v^{\nu}$ is at least $\omega_{I_{\ell,v}} \cdot (\sigma_v^j)$ for any $j \in J_{\ell,v}$. Taking maximum over $j \in J_{\ell,v}$, we obtain that the local degree of $\tilde{f} : \mathcal{E} \to \mathcal{X}$ at the irreducible component $\mathcal{C}_v^{\nu}$ is at least $|\sigma_v^j|$ (Definition 5.11). Moreover, since

$$\sum_{v_0 \in V(\mathcal{E}_\nu)} |\sigma_{v_0}|_\omega \geq \sum_{v_0 \in V(\mathcal{E}_\nu)} \sigma_{v_0} \geq |\sigma_v|_\omega,$$

we have proved the following proposition.

Proposition 5.22. The sum of the local degrees of $\tilde{f} : \mathcal{E} \to \mathcal{X}$ at the irreducible components $\mathcal{C}_v^{\nu}$ over all vertices $v_0 \in V(\mathcal{E}_\nu)$ is greater than or equal to the local degree of the tropical curve $C^t$ at the vertex $v \in S_\mathcal{X}$.

Corollary 5.23. The degree of the morphism $f : C \to \mathcal{X}_\eta$ with respect to the Kähler structure $\tilde{L}$ on $\mathcal{X}$ is greater than or equal to the tropical degree of the corresponding tropical curve $C^t$ with respect to the simple Kähler structure $\omega$ on the Clemens polytope $S_\mathcal{X}$ induced by $\tilde{L}$.

Proof. The corollary follows from Proposition 5.22 and Proposition 3.13. □

Remark 5.24. Corollary 5.23 combined with Lemma 5.16 controls the complexity of the tropical curves obtained from $k$-analytic curves into $\mathcal{X}_\eta$ with bounded degrees. In other words, given a positive real number $A$, there exists an integer $N$, such that for any non-Archimedean field extension $k'$ of $k$, any proper smooth irreducible $k'$-analytic curve $C$, any $k'$-analytic morphism $f : C \to \mathcal{X}_\eta \times k'$ with degree bounded by $A$, the number of vertices, the number of edges and the norms of the weights of all the edges of the associated tropical curve $C^t$ are all bounded by $N$. We remark that this finiteness property can also be obtained later as a consequence of Corollary 6.45 and Lemma 5.21.

6. Gromov compactness in non-Archimedean analytic geometry

6.1. Non-Archimedean analytic stacks. Moduli spaces often carry the structure of a stack. The notion of Deligne-Mumford stacks was introduced by Deligne and Mumford [DM69] in their study of the moduli space of stable curves. It was later generalized by Artin [Art74] to the current notion of algebraic stacks (also referred to as Artin stacks). Since the general notion of stacks makes sense for any Grothendieck site, one can define the analog of
algebraic stacks in the context of non-Archimedean analytic geometry. We hope that our considerations would serve as a motivation for further studies on $k$-analytic stacks. We begin by reviewing the relations between formal schemes, rigid analytic spaces à la Tate and $k$-analytic spaces à la Berkovich.

**Theorem 6.1** ([Ray74], [Ber93]1.6.1). We have equivalences of categories between

(i) the category of formal schemes finitely (resp. locally finitely) presented and flat over $k^\circ$, localized by admissible formal blowups,

(ii) the category of quasi-separated rigid analytic spaces with a finite (resp. locally finite) admissible affinoid covering,

(iii) the category of compact (resp. paracompact) strictly $k$-analytic spaces.

We introduce some standard terminologies for convenience.

**Definition 6.2.** Let $\mathcal{C}$ be a category, and let $X, Y$ be two functors from $\mathcal{C}^{\text{op}}$ to the category of sets. A morphism $f: X \to Y$ is called representable if for any object $V$ in $\mathcal{C}$, any morphism $V \to Y$, the fiber product $X \times_Y V$ is representable, where $V$ is regarded as a functor from $\mathcal{C}^{\text{op}}$ to the category of sets by the Yoneda embedding. Let $P$ be a property of morphisms in the category $\mathcal{C}$. We say that a representable morphism $f: X \to Y$ satisfies the property $P$, if for any object $V$ in $\mathcal{C}$, any morphism $V \to Y$, the morphism $X \times_Y V \to V$ satisfies the property $P$.

Let us introduce the analogs of algebraic spaces in the contexts of formal schemes and $k$-analytic spaces.

Denote by $\mathcal{C}_{\text{formal}}$ the category of formal schemes locally finitely presented over $k^\circ$.

**Definition 6.3.** A formal algebraic space $\mathcal{X}$ (locally) finitely presented over $k^\circ$ is a functor

$$\mathcal{X}: \mathcal{C}_{\text{formal}}^{\text{op}} \to \text{(sets)}$$

with the following properties

(i) The functor $\mathcal{X}$ is a sheaf for the étale topology;

(ii) The diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{k^\circ} \mathcal{X}$ is representable;

(iii) There is a formal scheme $\mathcal{U}$ (locally) finitely presented over $k^\circ$ and an étale surjective morphism $\mathcal{U} \to \mathcal{X}$.

Denote by $\mathcal{C}_{\text{rig}}$ the category of paracompact strictly $k$-analytic spaces.

**Definition 6.4.** A compact (resp. paracompact) strictly $k$-analytic étale space $X$ is a functor

$$X: \mathcal{C}_{\text{rig}}^{\text{op}} \to \text{(sets)}$$

with the following properties

(i) The functor $X$ is a sheaf for the Tate-étale topology.$^5$

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$^5$We refer to [CT09], [CT10] for the topologies on the category $\mathcal{C}_{\text{rig}}$ and the theories of descent.
(ii) The diagonal morphism $X \to X \times_k X$ is representable;
(iii) There is a compact (resp. paracompact) strictly $k$-analytic space $U$
and a quasi-étale surjective morphism $U \to X$.

Remark 6.5. It is an important question whether a $k$-analytic étale space is
representable by a $k$-analytic space (cf. [CT09], [FK13] Equivalence theorem).

Let us introduce the analogs of algebraic stacks in the contexts of formal
schemes and $k$-analytic spaces.

**Definition 6.6.** A formal stack $\mathcal{X}$ (locally) finitely presented over $k^\circ$ is a
stack in groupoids over the category $C_{\text{formal}}$ equipped with the étale topol-
yogy, such that the diagonal $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is representable by formal algebraic
spaces (locally) finitely presented over $k^\circ$, and that there exists a formal
scheme $\mathcal{U}$ (locally) finitely presented over $k^\circ$ and a smooth surjective mor-
phism $\mathcal{U} \to \mathcal{X}$. It is said to be of Deligne-Mumford type if one can require
the covering $\mathcal{U} \to \mathcal{X}$ to be étale.

**Definition 6.7.** A compact (resp. paracompact) strictly $k$-analytic stack $X$
is a stack in groupoids over the category $C_{\text{rig}}$ equipped with the Tate-étale
topology, such that the diagonal $X \to X \times_k X$ is representable by compact
(resp. paracompact) strictly $k$-analytic étale spaces, and that there exists a
compact (resp. paracompact) strictly $k$-analytic space $U$ and a quasi-smooth
surjective morphism $U \to X$. It is said to be of Deligne-Mumford type if
one can require the covering $U \to X$ to be quasi-étale.

Remark 6.8. By definition, a paracompact strictly $k$-analytic stack is comp-
act if and only if it admits a finite affinoid quasi-smooth covering.

Remark 6.9. An equivalent and technically useful way of defining a stack is
to present it as a quotient of a scheme by an equivalence groupoid. This is
called a presentation of a stack (see [Sta13] Tags 04T3, 04TJ). The notion
of presentations of algebraic stacks has immediate analogs for formal stacks
and $k$-analytic stacks, so we do not include the details.

Let $(\cdot)^\text{an}$ denote the analytification functor from the category of schemes
locally of finite type over $\text{Spec} k$ to the category of paracompact strictly $k$-
analytic spaces, let $(\cdot)_s$ denote the special fiber functor from the category
of formal schemes locally finitely presented over $\text{Spf} k^\circ$ to the category of
schemes locally of finite type over $\text{Spec} k$, and let $(\cdot)_\eta$ denote the generic
fiber functor from the category of formal schemes locally finitely presented
over $\text{Spf} k^\circ$ to the category of paracompact strictly $k$-analytic spaces. Using
presentations of stacks, the analytification functor $(\cdot)^\text{an}$ can be extended to
a functor from the category of algebraic stacks locally of finite type over
$\text{Spec} k$ to the category of paracompact strictly $k$-analytic stacks, and the
same holds for the special fiber functor $(\cdot)_s$ and the generic fiber functor
$(\cdot)_\eta$. 
Theorem 6.14. Let \( \mathfrak{X} \) be a formal stack locally finitely presented over \( k^0 \) and let \( \mathfrak{X}_\eta \) be the corresponding paracompact strictly \( k \)-analytic stack. Given a strictly \( k \)-analytic space \( T \) and a morphism \( t: T \to \mathfrak{X}_\eta \), contrary to the non-stacky situation in Theorem 6.1, it is not always possible to find a formal model \( \mathfrak{T} \) for \( T \) such that the morphism \( t: T \to \mathfrak{X}_\eta \) extends to a morphism \( t: \mathfrak{T} \to \mathfrak{X} \). It is nonetheless possible after some quasi-smooth covering of \( T \) (or quasi-étale covering in the case of Deligne-Mumford stacks).

Indeed, let \( \mathfrak{X}^0 \to \mathfrak{X} \) be a smooth covering of \( \mathfrak{X} \). Then \( \mathfrak{X}^0_\eta \to \mathfrak{X}_\eta \) is a quasi-smooth covering of \( \mathfrak{X}_\eta \). Let \( T^0 = T \times_{\mathfrak{X}_\eta} \mathfrak{X}^0_\eta \). Then \( T^0 \to T \) is quasi-smooth and the classical theory of formal models can be applied to the morphism \( t^0: T^0 \to \mathfrak{X}^0_\eta \) in order to get a formal model \( \mathfrak{T}^0 \to \mathfrak{X}^0_\eta \). However, not every morphism \( \mathfrak{T}^0 \to \mathfrak{X}^0 \) gives rise to a morphism \( T \to \mathfrak{X}_\eta \). We also have to take into account the descent data. Let \( \mathfrak{X}^1 = \mathfrak{X}^0 \times_\mathfrak{X} \mathfrak{X}^0 \) and \( T^1 = T^0 \times_T T^0 \). The descent datum is encoded in the morphism \( t^1: T^1 \to \mathfrak{X}^1_\eta \).

The theory of formal models implies that one can find a formal model \( \mathfrak{T}^1 \) of \( T^1 \) such that the morphism \( t^1: T^1 \to \mathfrak{X}^1_\eta \) extends to a morphism \( \mathfrak{T}^1 \to \mathfrak{X}^1 \).

To conclude, a formal model for the morphism \( t: T \to \mathfrak{X}_\eta \) consists of two morphisms of formal schemes \( t^0: \mathfrak{T}^0 \to \mathfrak{X}^0 \) and \( t^1: \mathfrak{T}^1 \to \mathfrak{X}^1 \).

Now we study the properness of morphisms of stacks.

Definition 6.11. A morphism \( f: \mathfrak{X} \to \mathfrak{Y} \) of formal stacks locally finitely presented over \( k^0 \) is said to be proper if the induced morphism \( f_s: \mathfrak{X}_s \to \mathfrak{Y}_s \) of algebraic stacks is proper.

Definition 6.12. A separated morphism \( f: X \to Y \) of paracompact strictly \( k \)-analytic stacks is said to be proper if there exists an affinoid quasi-smooth covering \( \{ Y_i \}_{i \in I} \) of \( Y \) and finite affinoid quasi-smooth coverings \( \{ U_{i,j} \}_{j \in J_i} \) and \( \{ V_{i,j} \}_{j \in J_i} \) of the \( k \)-analytic stacks \( X \times_Y Y_i \) such that \( V_{i,j} \subset \text{Int}(U_{i,j}/Y_i) \), where \( \text{Int}(U_{i,j}/Y_i) \) denotes the relative interior.

Remark 6.13. In algebraic geometry, the definition of properness is the combination of being of finite type, separated and universally closed. Definition 6.12 seems quite different from the algebraic situation, because \( k \)-analytic spaces can have boundaries. So the properness of \( k \)-analytic stacks is essentially the combination of being of finite type, separated and without boundary, while the topological properness (i.e. universal closedness) is automatically ensured by being of finite type.

Now assume that the non-Archimedean field \( k \) is of discrete valuation.

Theorem 6.14. Let \( f: \mathfrak{X} \to \mathfrak{Y} \) be a proper morphism of formal stacks locally finitely presented over \( k^0 \). Then the induced morphism \( f_\eta: \mathfrak{X}_\eta \to \mathfrak{Y}_\eta \) is a proper morphism of paracompact strictly \( k \)-analytic stacks.

Proof. The question being local on the base, we can assume that \( \mathfrak{Y} \) is affine without loss of generality. In this case, \( \mathfrak{X} \) is a formal stack finitely presented over \( k^0 \), and \( \mathfrak{X}_s \) is a scheme of finite type over \( \tilde{k} \). By [Ols05], there exists
a quasi-projective scheme \( X^0 \) over \( \mathcal{Y}_s \) and a proper surjective morphism \( X^0 \to \mathfrak{X}_s \). Since by assumption, the induced morphism \( f_s : \mathfrak{X}_s \to \mathcal{Y}_s \) is a proper morphism of algebraic stacks of finite type over \( \bar{k} \), the scheme \( X^0 \) is projective over \( \mathcal{Y}_s \). Since the morphism \( f_s \) is in particular separated, the scheme \( X^0 \) is also projective over \( \mathfrak{X}_s \) (cf. [Gro60] 5.5.12(v)).

We assume that \( X^0 \) is a closed subscheme of a projective space \( P^N_{X_s} \) over \( \mathfrak{X}_s \). Let \( \mathfrak{X}^0 \) denote the formal completion of \( P^N_{X} \) along \( X^0 \). By [Ber96], \( \mathfrak{X}^0 \) is a special formal scheme over \( k^0 \) and we have isomorphisms

\[
\mathfrak{X}^0_s \xrightarrow{\sim} X^0, \\
\mathfrak{X}^0_\eta \xrightarrow{\sim} \pi^{-1}(X^0),
\]

where \( \pi_p : P^N_{X_s} \to P^N_{\mathfrak{X}_s} \) denotes the reduction map. The anti-continuity of the reduction map \( \pi_p \) implies that the generic fiber \( \mathfrak{X}^0_\eta \) is a \( k \)-analytic open subspace of \( P^N_{X_s} \). Denote by \( p \) the composition of the open immersion \( \mathfrak{X}^0_\eta \to P^N_{X_s} \) and the projection \( P^N_{X_s} \to \mathfrak{X}_\eta \). The morphism \( p \) is smooth, thus open.

Since the formal stack \( \mathfrak{X} \) is finitely presented over \( k^0 \), its generic fiber \( \mathfrak{X}_\eta \) is a compact strictly \( k \)-analytic stack. Let \( q : A \to \mathfrak{X}_\eta \) be a finite affinoid quasi-smooth covering of \( \mathfrak{X}_\eta \), where \( A \) is a compact strictly \( k \)-analytic space.

For every point \( j \in A \), let \( x \) be a point in \( \mathfrak{X}^0_\eta \) such that \( q(j) = p(x) \). Since \( \mathfrak{X}^0_\eta \simeq X^0 \) is proper over \( \mathfrak{Y}_s \), by [Tem00] Theorem 4.1, there exists two affinoid neighborhoods \( U_j, V_j \) of the point \( x \) in \( \mathfrak{X}^0_\eta \) such that \( V_j \subset \operatorname{Int}(U_j/\mathfrak{Y}_\eta) \). Let \( V'_j \) denote the image of the morphism \( A \times_{\mathfrak{X}_s} V_j \to A \). The openness of the morphism \( p : \mathfrak{X}^0_\eta \to \mathfrak{X}_\eta \) implies that \( V'_j \) is a neighborhood of the point \( j \) in \( A \). By the compactness of \( A \), there exists a finite set of points \( J \subset A \) such that \( \{V'_j\}_{j \in J} \) is a finite covering of \( A \). Then \( \{U_j\}_{j \in J} \) and \( \{V_j\}_{j \in J} \) are two finite affinoid quasi-smooth coverings of the \( k \)-analytic stack \( \mathfrak{X}_\eta \) such that \( V_j \subset \operatorname{Int}(U_j/\mathfrak{Y}_\eta) \). Moreover, the separatedness of the morphism \( f_\eta : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta \) follows immediately from the separatedness of \( f : \mathfrak{X} \to \mathfrak{Y} \). So we have proved the properness of the morphism \( f_\eta \).

6.2. Algebraicity of the moduli stacks of stable maps. The notion of stable maps was introduced by Kontsevich in [Kon95] in order to compactify the moduli space of maps from smooth projective curves to projective varieties and provide an algebraic-geometric foundation for Gromov-Witten theory. In this section, we aim to generalize the theory to the non-projective case. This generality will be important in the study of \( k \)-analytic stable maps in Section 6.3.

Fix integers \( g \) and \( n \), which denote the genus and the number of marked points respectively. Fix a locally Noetherian scheme \( S \) and denote by \( \mathcal{S}ch/S \) the category of schemes over \( S \). Let \( X \) be a scheme locally of finite presentation over \( S \).

**Definition 6.15** (cf. [Kon95]§1.1, [AO01]§2). Let \( T \) be an \( S \)-scheme. An \( n \)-pointed genus \( g \) (algebraic) stable map \((C \to T, \{s_i\}, f)\) into \( X \) over \( T \)
consists of a morphism $C \to T$, a morphism $f : C \to X$ and $n$ morphisms $s_i : T \to C$ such that

(i) The morphism $C \to T$ is a proper flat family of curves;
(ii) The geometric fibers of $C \to T$ are reduced with at worst double points as singularities, and are of arithmetic genus $g$;
(iii) The $n$ morphisms $s_i : T \to C$ are disjoint sections of $C \to T$ which land in the smooth locus of $C \to T$;
(iv) (Stability condition) For any geometric fiber $C_t$ of $C \to T$, every irreducible component of $C_t$ of genus 0 (resp. 1) which maps to a point under $f$ must have at least 3 (resp. 1) special points on its normalization, where special points mean either marked points or the points coming from the double points.

**Definition 6.16.** If we remove the condition (iv) of Definition 6.15, we call the triple $(C \to T, \{s_i\}, f)$ an $n$-pointed genus $g$ pre-stable map into $X$ over $T$.

**Definition 6.17.** A morphism of stable maps $(C \to T, \{s_i\}, f) \to (C' \to T', \{s'_i\}, f')$ is a commutative diagram

\[
\begin{array}{ccc}
C & \longrightarrow & C' \\
\downarrow & & \downarrow \\
T & \longrightarrow & T'
\end{array}
\]

inducing an isomorphism $C \sim \to C' \times_T T'$, compatible with the sections $s_i, s'_i$ and the morphisms $f, f'$.

**Definition 6.18.** Let $T$ be an $S$-scheme. An $n$-pointed genus $g$ (algebraic) stable (resp. pre-stable) curve $(C \to T, \{s_i\})$ over $T$ is an $n$-pointed genus $g$ stable (resp. pre-stable) map $(C \to T, \{s_i\}, f)$ into $S$ over $T$ with $f$ being the structural morphism $C \to S$.

In the classical case where the base $S$ is a field and $X$ is projective over $S$, one fixes an element $\beta$ in the group of 1-dimensional cycles modulo homological equivalence. Denote by $\overline{M}_{g,n}(X, \beta)$ the category, fibered in groupoids over the category $\text{Sch}/S$, of $n$-pointed genus $g$ stable maps into $X$ with image homologous to $\beta$. The main property of this category is described by the following theorem.

**Theorem 6.19 ([Kon95]).** When $S$ is a field and $X$ is a projective scheme over $S$, the category $\overline{M}_{g,n}(X, \beta)$ is a proper algebraic stack.

More details can be found in [BM96], [FP97] and [AO01]. For the applications to $k$-analytic stable maps in non-Archimedean analytic geometry, we need to work over a more general base scheme $S$ and weaken the projectivity assumption. The statement of Theorem 6.19 consists of two parts: algebraicity and properness. We discuss algebraicity in this section and properness at the end of Section 6.3.
In the above-mentioned literatures, the algebraicity is shown by projective methods which rely on the assumption that $X$ is projective over $S$. More precisely, the projectivity of $X$ induces projective embeddings of the pre-stable curves at the source via very ample line bundles. The moduli stack of stable maps is then realized as a quotient stack of a subscheme of a certain Hilbert scheme. In order to prove the algebraicity without the projectivity assumption, we use Artin’s deformation theoretic criterion of representability ([Art74]).

**Theorem 6.20.** Let $X$ be a scheme locally of finite presentation over a locally Noetherian base scheme $S$, and let $\mathcal{M}_{g,n}(X)$ denote the category, fibered in groupoids over the category $\text{Sch}/S$, of $n$-pointed genus $g$ stable maps into $X$. Then $\mathcal{M}_{g,n}(X)$ is an algebraic stack locally of finite presentation over $S$.

**Proof.** The idea behind the theorem is that a stable map can be specified by a finite number of variables and equations thanks to the stability condition. More precisely, we have the following remark.

**Remark 6.21.** Let $(C \to T, \{s_i\}, f)$ be an $n$-pointed genus $g$ stable map into $X$ over $T$. Let $X_T = X \times_S T$, $f_T: C \to X_T$, and

$$\mathcal{L} = \bigoplus_{m \geq 0} f_T^* \left( \omega_{C/T} \left( \sum S_i \right) \right) \otimes^m,$$

where $\omega_{C/T}$ denotes the relative canonical sheaf and $S_i$ denotes the image of the section $s_i$. Then $\mathcal{L}$ is a quasi-coherent sheaf of graded algebras over $X_T$ and every direct summand of $\mathcal{L}$ is coherent. The stability condition implies that the invertible sheaf $\omega_{C/T} \left( \sum S_i \right)$ is relatively ample over $X_T$. Therefore, via the relative Proj construction, we recover the total space of the family of curves $C \simeq \text{Proj}_{X_T}(\mathcal{L})$, as well as the morphisms $C \to T$ and $f: C \to X$. Moreover, the sections $\{s_i\}$ can be determined by their graphs, which can be encoded as quotients of the structural sheaf of $C$. To conclude, the data of an $n$-pointed genus $g$ stable map into $X$ over $T$ can be encoded entirely in terms of coherent sheaves. Conversely, assume that we are given a quasi-coherent sheaf $\mathcal{L}$ of graded algebras over $X_T$ which is a direct sum of coherent sheaves, and that we are also given $n$ sections of the morphism $\text{Proj}_{X_T}(\mathcal{L}) \to T$. In order to check if they come from an $n$-pointed genus $g$ stable map into $X$ over $T$ by the constructions above, it suffices to verify the following conditions: First, $\text{Proj}_{X_T}(\mathcal{L})$ is flat over $T$; Second, every geometric fiber over $T$ is an $n$-pointed genus $g$ stable map into $X$ over an algebraically closed field. These two conditions are preserved with respect to various descent situations.

We study the properties of the category $\mathcal{M}_{g,n}(X)$ in Lemmas 6.22 - 6.28.

**Lemma 6.22** (sheaf property). The functor $\mathcal{M}_{g,n}(X)$ is a sheaf on the category $\text{Sch}/S$ for the $fpgc$-topology.
Proof. Let \( \pi : T' \to T \) be an fpqc-covering and let \( (C' \to T', \{ s'_i \}, f') \) be a stable map into \( X \) over \( T' \) equipped with descent data relative to \( \pi \). Denote \( X_T = X \times_S T, X_{T'} = X \times_S T' \). By base change, we obtain an fpqc covering \( \pi_X : X_{T'} \to X_T \) and a morphism \( C' \to X_{T'} \) equipped with descent data relative to \( \pi_X \). Denote by \( S'_i \) the image of \( s'_i \). As in Remark 6.21, the stability condition implies that the invertible sheaf \( \omega_{C'/T'}(\sum S'_i) \) is relatively ample over \( X_{T'} \). Therefore, the morphism \( C' \to X_{T'} \) descends to a morphism \( C \to X_T \) by [Gro63] Chapitre VIII Proposition 7.8. So the morphism \( f' : C' \to X \) descends to a morphism \( f : C \to X \) as well. The sections \( s'_i : T' \to C' \) descend to sections \( s_i : T \to C \) by Theorem 5.2 loc. cit. The properties for \( (C' \to T', \{ s'_i \}, f') \) to be a stable map also descend effectively to the triple \( (C \to T, \{ s_i \}, f) \) by Remark 6.21. So we have proved the sheaf property of \( \overline{M}_{g,n}(X) \) for the fpqc-topology. \( \square \)

Remark 6.23. One may try to descend the family \( C' \to T' \) of pre-stable curves with respect to the fpqc-covering \( T' \to T \) in the beginning. However, this is not possible in the category of schemes instead of algebraic spaces. The stability condition ensures the sheaf property in Lemma 6.22. Without the stability condition, one would have to allow the total space \( C \) to be an algebraic space in the definition of stable maps (see Definition 6.15).

Lemma 6.24 (limit preserving). The functor \( \overline{M}_{g,n}(X) \) on the category \( \mathcal{S}ch/S \) is limit preserving, i.e. the canonical map
\[
\lim_{\rightarrow} \overline{M}_{g,n}(X)(T_i) \to \overline{M}_{g,n}(X)\left( \lim_{\rightarrow} T_i \right)
\]
is an equivalence of groupoids where \( \{ T_i \} \) is a directed inverse system of affine schemes.

Proof. The lemma says that any \( n \)-pointed genus \( g \) stable map into \( X \) over \( \lim_{\rightarrow} T_i \) is uniquely determined by an \( n \)-pointed genus \( g \) stable map into \( X \) over \( T_i \) for some \( i \) large enough. This follows from the properties of schemes of finite presentation (cf. [Gro66] Theorem 8.8.2). \( \square \)

Lemma 6.25 (Rim-Schlessinger condition for small affine pushouts (cf. [Sta13] Tag 07WP)). Let \( T, T_1, T_2, T_{12} \) be spectra of local Artinian rings of finite type over \( S \). Assume that \( T \to T_1 \) is a closed immersion and that
\[
\begin{array}{ccc}
T & \to & T_1 \\
\downarrow & & \downarrow \\
T_2 & \to & T_{12} = T_1 \coprod_T T_2
\end{array}
\]
is a diagram of pushout in the category \( \mathcal{S}ch/S \). Then the canonical map
\[
\overline{M}_{g,n}(X)(T_{12}) \xrightarrow{\sim} \overline{M}_{g,n}(X)(T_1) \times_{\overline{M}_{g,n}(X)(T_1)} \overline{M}_{g,n}(X)(T_2)
\]
is an equivalence of groupoids.
Proof. Using Remark 6.21, the pushout property for stable maps follows from the pushout property for quasi-coherent sheaves (cf. [Sta13] Tag 08LQ, Tag 08IW).

□

**Lemma 6.26** (effectiveness of formal objects). Let \( \hat{A} \) be a complete local algebra over \( S \), with maximal ideal \( m \), and with residue field of finite type over \( S \). Then the canonical map

\[
\mathcal{M}_{g,n}(X)(\hat{A}) \longrightarrow \lim_l \mathcal{M}_{g,n}(X)(\hat{A}/m^l)
\]

is an equivalence of groupoids.

Proof. Given a formal object \( \left\{ (C^{(l)} \to \text{Spec } \hat{A}/m^l, \{s_i^{(l)}\}, f^{(l)}) \right\} \) on the right hand side, using Remark 6.21, by Grothendieck’s existence theorem for formal sheaves, one can construct a triple \( (\hat{C} \to \text{Spec } \hat{A}, \{\hat{s}_i\}, \hat{f}) \) that induces the formal object by restrictions. It remains to see that the triple \( (\hat{C} \to \text{Spec } \hat{A}, \{\hat{s}_i\}, \hat{f}) \) is a stable map into \( X \) over \( \text{Spec } \hat{A} \). This follows from the fact that the stability condition (Definition 6.15(iv)) for a pre-stable map \( (C \to T, \{s_i\}, f) \) into \( X \) over \( T \) is an open condition on the base \( T \). So we have shown that the canonical map in the statement of the Lemma is essentially surjective. The full faithfulness is also implied by Grothendieck’s existence theorem along the same reasonings.

□

**Lemma 6.27** (local quasi-separation). Let \( (C \to T, \{s_i\}, f) \) be an element of \( \mathcal{M}_{g,n}(X)(T) \) and let \( \phi \) be an automorphism of the element. If \( \phi \) induces the identity in \( \mathcal{M}_{g,n}(X)(t) \) for a dense set of points \( t \in T \) of finite type, then \( \phi \) is the identity.

Proof. If \( \phi \) induces the identity in \( \mathcal{M}_{g,n}(X)(t) \) for a dense set of points \( t \in T \) of finite type, then \( \phi \) is the identity on a dense set of points of finite type on \( C \). Therefore, \( \phi \) is the identity because \( C \) is separated over \( T \).

□

The following lemma studies the deformation theory and the obstruction theory of the category \( \mathcal{M}_{g,n}(X) \).

**Lemma 6.28** (deformation and obstruction). Let \( A \) be an \( S \)-algebra, \( M \) an \( A \)-module, and \( x = (C \to \text{Spec } A, \{s_i\}, f) \) an element of \( \mathcal{M}_{g,n}(X)(\text{Spec } A) \). Denote by \( A[M] \) the \( A \)-algebra whose underlying \( A \)-module is \( A \oplus M \) and whose multiplication is given by \((a,m) \cdot (a',m') = (aa', am + a'm) \). We consider the deformation situation \( A[M] \to A, x \). Denote by \( L \) the relative cotangent complex \( L_{C/X \times_S \text{Spec } A} \) and by \( N \) the normal sheaf of the union of the images of the sections \( \{s_i\} \) in \( C \). We have

(i) The module of infinitesimal automorphisms\(^6\) \( \text{Aut}_x(M) \) sits in the following exact sequence

\[
0 \longrightarrow \text{Aut}_x(M) \longrightarrow \text{Hom}(L, \mathcal{O}_C \otimes_A M) \longrightarrow H^0(C, N \otimes M).
\]

\(^6\)See [Sta13] Tag 07Y9 for the definition.
(ii) The module of infinitesimal deformations $D_x(M)$ sits in the following exact sequence
\[
\text{Hom}(L, \mathcal{O}_C \otimes_A M) \rightarrow H^0(C, N \otimes M) \rightarrow D_x(M) \rightarrow \text{Ext}^1(L, \mathcal{O}_C \otimes_A M) \rightarrow 0.
\]

(iii) The module of obstructions $\mathcal{O}_x(M)$ can be given as $\text{Ext}^2(L, \mathcal{O}_C \otimes_A M)$.

Proof. If we ignore the sections $\{s_i\}$ and denote $\mathfrak{m} = (C/\text{Spec} \, A, f)$, then the module of infinitesimal automorphisms $\text{Aut}_x(M)$, the module of infinitesimal deformations $D_x(M)$ and the module of obstructions $\mathcal{O}_x(M)$ are given by $\text{Hom}(L, \mathcal{O}_C \otimes A)$, $\text{Ext}^1(L, \mathcal{O}_C \otimes A)$ and $\text{Ext}^2(L, \mathcal{O}_C \otimes A)$ respectively.

Now we take the sections $\{s_i\}$ into consideration. The module of infinitesimal automorphisms $\text{Aut}_x(M)$ consists of those elements of $\text{Aut}_x(M)$ that fix the sections $\{s_i\}$, so the statement (i) in the lemma is true. The module of infinitesimal deformations $D_x(M)$ is the extension of the module $D_x(M)$ by the module of infinitesimal deformations of the sections $H^0(C, N \otimes M)$. So the statement (ii) is true. Finally, since the sections $\{s_i\}$ land in the smooth locus of $C \rightarrow T$, infinitesimal deformations of the sections are unobstructed. So we have $\mathcal{O}_x(M) = \mathcal{O}_x(M)$, i.e. (iii) is true. \qed

Now we can verify Artin’s representability criterion for algebraic stacks as stated in [Art74] Theorem 5.3\textsuperscript{7}. Lemma 6.22 and Lemma 6.24 show that $\mathcal{M}_{g,n}(X)$ is a limit preserving stack on the category $\text{Sch}/S$. Lemma 6.25 verifies the first part of condition (1) of Theorem 5.3 loc. cit. (see [Art74] (2.3) Condition (S1')). Lemma 6.26 verifies condition (2). Using the properties of cotangent complexes ([Ill72]) and by the five lemma, Lemma 6.28 verifies condition (3) and the remaining parts of condition (1). Lemma 6.27 verifies condition (4). To conclude, we have proved that the category $\mathcal{M}_{g,n}(X)$ is an algebraic stack locally of finite presentation over $S$. \qed

Remark 6.29. Theorem 6.20 remains true if we replace $X$ by an algebraic space locally of finite presentation over $S$. The proof carries over using analogous theories on algebraic spaces. One can even allow $X$ to be a Deligne-Mumford stack over $S$ due to interests in Gromov-Witten invariants for orbifolds. However, in this more general setting, the notion of stable maps should be replaced by the notion of twisted stable maps (see [AV02] for the case where the Deligne-Mumford stack $X$ admits a projective coarse moduli scheme).

Remark 6.30. If we assume moreover that $X$ is separated over $S$, then using valuative criterion of separatedness, the algebraic stack $\mathcal{M}_{g,n}(X)$ is also separated over $S$ (cf. [Kon95]§1.3.1).

Remark 6.31. In characteristic zero, the stability condition (Definition 6.15(iv)) ensures that a stable map has no infinitesimal automorphisms. So $\mathcal{M}_{g,n}(X)$

\textsuperscript{7}The article [Art74] has the assumption that the base scheme $S$ is of finite type over an excellent Dedekind domain. It is generalized to the locally Noetherian case in [Sta13] Tag 07SZ. The generality of Artin’s original version is already enough for our purposes.
is a Deligne-Mumford stack in characteristic zero. In general, one can only ensure that the diagonal morphism of $\mathcal{M}_{g,n}(X)$ is quasi-finite.

### 6.3. Non-Archimedean analytic moduli stacks of stable maps

In this section, we study the representability of the moduli stack of formal stable maps and the moduli stack of non-Archimedean analytic stable maps.

Definition 6.15 and Definition 6.18 can be carried verbatim to the case of formal schemes and $k$-analytic spaces as follows.

**Definition 6.32.** Let $X$, $T$ be formal schemes over $k^\circ$ (resp. $k$-analytic spaces). An $n$-pointed genus $g$ formal (resp. $k$-analytic) stable map $(C \to T, \{s_i\}, f)$ into $X$ over $T$ consists of a morphism $C \to T$, a morphism $f : C \to X$ and $n$ morphisms $s_i : T \to C$ such that Conditions (i)-(iv) of Definition 6.15 are satisfied.

**Definition 6.33.** Let $T$ be a formal scheme over $k^\circ$ (resp. a $k$-analytic space). An $n$-pointed genus $g$ formal (resp. $k$-analytic) stable curve $(C \to T, \{s_i\})$ over $T$ is an $n$-pointed genus $g$ formal (resp. $k$-analytic) stable map $(C \to T, \{s_i\}, f)$ into $\text{Sp} k^\circ$ (resp. $\text{Sp}_B k$) with $f$ being the structural morphism.

These definitions are compatible with respect to the analytification functor $(\cdot)^\text{an}$, the special fiber functor $(\cdot)_s$ and the generic fiber functor $(\cdot)_\eta$ in the following sense.

**Lemma 6.34.** Let $X$ be an algebraic variety over $\text{Spec} k$, $A$ a strictly $k$-affinoid algebra and $(C \to \text{Spec} A, \{s_i\}, f)$ an $n$-pointed genus $g$ algebraic stable map into $X$ over $\text{Spec} A$. Then the triple $(C^\text{an} \to \text{Sp}_B A, \{s_i^\text{an}\}, f^\text{an})$ obtained by applying the analytification functor $(\cdot)^\text{an}$ is an $n$-pointed genus $g$ $k$-analytic stable map into $X^\text{an}$ over $\text{Sp}_B A$.

**Proof.** Conditions (i)-(iii) of Definition 6.15 present no problem. Condition (iv) follows from the fact that a geometric point of the $k$-analytic spectrum $\text{Sp}_B A$ is in particular a geometric point of $\text{Spec} A$.

**Lemma 6.35.** Let $\mathcal{X}$ be a formal scheme locally finitely presented over $k^\circ$, $A$ a finitely presented topological $k^\circ$-algebra and $(C \to \text{Sp} f A, \{s_i\}, f)$ an $n$-pointed genus $g$ formal stable map into $\mathcal{X}$ over $\text{Sp} f A$. Then the triple $(C_s \to \text{Spec}(A \otimes_{k^\circ} \tilde{k}), \{(s_i)_s\}, f_s)$ obtained by applying the special fiber functor $(\cdot)_s$ is an $n$-pointed genus $g$ algebraic stable map into $\mathcal{X}_s$ over $\text{Spec}(A \otimes_{k^\circ} \tilde{k})$.

**Proof.** Conditions (i)-(iii) of Definition 6.15 present no problem. Condition (iv) follows from the fact that a geometric point of the formal spectrum $\text{Sp} f A$ is in particular a geometric point of $\text{Spec}(A \otimes_{k^\circ} \tilde{k})$.

**Lemma 6.36.** Let $\mathcal{X}$ be a formal scheme locally finitely presented over $k^\circ$, $A$ a finitely presented topological $k^\circ$-algebra and $(C \to \text{Sp} f A, \{s_i\}, f)$ an $n$-pointed genus $g$ formal stable map into $\mathcal{X}$ over $\text{Sp} f A$. Then the triple $(C_\eta \to \text{Sp}_B(A \otimes_{k^\circ} k, \{(s_i)_\eta\}, f_\eta)$ obtained by applying the generic fiber functor $(\cdot)_\eta$ is an $n$-pointed genus $g$ $k$-analytic stable map into $\mathcal{X}_\eta$ over $\text{Sp}_B(A \otimes_{k^\circ} k)$.
We regard the triple \( X \) over a base scheme \( C \) map into stable curves if we take into account the additional marked points. Therefore, \( C_{\gamma} \) get smoothened or remains a double point. Let \( C \) irreducible components of \( C \) theory of pointed stable curves, \( C \) non-Archimedean field \( T \) is a morphism \( T' \to T \) where \( T' = \text{SpB}(k') \) for some algebraically closed non-Archimedean field \( k' \). Denote by \((k')^\circ\) the ring of integers of \( k' \) and let \( T' = \text{SpB}(k'^\circ) \). Let \( (C' \to T', \{s_i'\}, \hat{f}') \) be the pullback of the formal stable map \( (C \to T, \{s_i\}, f) \) along the morphism \( T' \to T \). It suffices to show that the triple \((C_n' \to T_n', \{(s'_i)_n\}, \hat{f}'_n)\) obtained by applying the generic fiber functor \((\cdot)_n\) verifies the stability condition.

Denote by \((C'_s \to T'_s, \{(s'_i)_s\}, \hat{f}'_s)\) the \( n \)-pointed genus \( g \) algebraic stable map into \( X_s \) over Spec \( k' \) obtained by applying the special fiber functor \((\cdot)_s\). We regard the triple \((C'_n \to T'_n, \{(s'_i)_n\}, \hat{f}'_n)\) as an infinitesimal deformation of \((C'_s \to T'_s, \{(s'_i)_s\}, \hat{f}'_s)\). As \( C'_s \) deforms into \( C'_n \), a double point may either get smoothened or remains a double point. Let \( C^0 \) be the union of the irreducible components of \( C'_n \) which map to a point under \( f'_n \) and let \( C^0_n \) be the union of the irreducible components of \( C'_n \) which map to a point under \( f'_n \). Let \( C^1 = \pi_{C'}(C^0) \) where \( \pi_{C'} : C'_n \to C'_s \) denotes the reduction map. Since an infinitesimal deformation of a non-constant map remains non-constant, \( C^1 \) is the union of some irreducible components of \( C^0 \). Let \( D \) denote the set of double points of the pre-stable curve \( C'_s \). Let \( D^0 \) denote the subset of points of \( D \cap C^0 \) which are non-singular points on \( C^0 \), and let \( D^1 \) denote the subset of points of \( D \cap C^1 \) which are non-singular points on \( C^1 \). Since \((C'_s \to T'_s, \{(s'_i)_s\}, \hat{f}'_s)\) is an \( n \)-pointed genus \( g \) algebraic stable map into \( X_s \), \( C^0 \) is a disjoint union of pointed stable curves if we add the points in \( D^0 \) as marked points. Therefore, \( C^1 \) is a disjoint union of pointed stable curves if we add the points in \( D^1 \) as marked points. As \( C'_s \) deforms into \( C'_n \), by construction, the part \( C^1 \subset C'_s \) does not merge with other irreducible components, and it gives rise to the part \( C^0_n \subset C'_n \). By the deformation theory of pointed stable curves, \( C^0_n \) is a disjoint union of \( k' \)-analytic pointed stable curves if we take into account the additional marked points. Therefore, we have proved that the triple \((C'_n \to T'_n, \{(s'_i)_n\}, \hat{f}'_n)\) is a \( k \)-analytic stable map into \( X_n \) over \( T'_n = \text{SpB}(k') \).

Let us introduce some notations. Fix integers \( g \) and \( n \). For a scheme \( X \) over a base scheme \( S \), we denote by \( \text{M}_{g,n}(X/S) \) the moduli stack of \( n \)-pointed genus \( g \) algebraic stable maps into \( X \) over \( S \). When there is no confusion about the base \( S \), we write \( \text{M}_{g,n}(X) \) as before. We have shown in Theorem 6.20 that \( \text{M}_{g,n}(X/S) \) is an algebraic stack locally of finite presentation over \( S \) if \( X \) is locally of finite presentation over \( S \). For a formal scheme \( X \) over \( k^\circ \), we denote by \( \text{M}_{g,n}(X) \) the moduli stack of \( n \)-pointed genus \( g \) formal stable maps into \( X \) over Spec \( k^\circ \). For a \( k \)-analytic space \( X \), we denote by \( \text{M}_{g,n}(X) \) the moduli stack of \( n \)-pointed genus \( g \) \( k \)-analytic stable maps into \( X \). We will study in the following the representability of the latter two stacks.

We begin with a lemma on the functorial property of moduli stacks.
Lemma 6.37. Let $X$ be a scheme over a base scheme $S$. For any scheme $S'$ over $S$, we have an isomorphism

$$\overline{M}_{g,n}((X \times_S S')/S') \xrightarrow{\sim} \overline{M}_{g,n}(X/S) \times_S S'.$$

Proof. One has to exhibit a functorial equivalence of groupoids

$$\overline{M}_{g,n}((X \times_S S')/S')(T) \xrightarrow{\sim} (\overline{M}_{g,n}(X/S) \times_S S')(T)$$

for any $S'$-scheme $T$. Indeed, an object of the left hand side is an $n$-pointed genus $g$ stable map $(C \to T, \{s_i\}, f)$ into $X \times_S S'$ over $T$. It corresponds to a morphism $T \to \overline{M}_{g,n}(X/S)$ by the projection $X \times_S S' \to X$ because $T$ can also be viewed as an $S$-scheme. So we obtain a morphism $T \to \overline{M}_{g,n}(X/S) \times_S S'$ by the universal property of fiber products. For the other direction, an object of the right hand side is an $S'$-morphism $T \to \overline{M}_{g,n}(X/S)$ by composing with the projection. This corresponds to an $n$-pointed genus $g$ stable map $(C \to T, \{s_i\}, f)$ into $X \times_S S'$ over $T$, i.e. an object of the left hand side. It is clear that the correspondences we constructed above give an equivalence of groupoids.

We assume from now on that the non-Archimedean field $k$ is of discrete valuation. Let $\pi$ be a uniformizing parameter. Put $S_m = \text{Spec}(k^o/\pi^m).$ Since a formal scheme locally finitely presented over $k^o$ is the inductive limit of schemes locally finitely presented over $S_m$, the representability of the moduli stack $\overline{M}_{g,n}(\mathfrak{X})$ can be established as an inductive limit of algebraic stacks.

Proposition 6.38. Let $\mathfrak{X}$ be a formal scheme locally finitely presented over $k^o$ and put $X_m = \mathfrak{X} \times_{k^o} S_m$. We have an isomorphism of stacks over the category $\mathcal{C}_{\text{formal}}$

$$\lim_m \overline{M}_{g,n}(X_m/S_m) \xrightarrow{\sim} \overline{M}_{g,n}(\mathfrak{X}).$$

Therefore, the stack $\overline{M}_{g,n}(\mathfrak{X})$ is a formal stack locally finitely presented over $k^o$.

Proof. One has to exhibit a functorial equivalence of groupoids

$$\left(\lim_m \overline{M}_{g,n}(X_m/S_m)\right)(\mathfrak{T}) \xrightarrow{\sim} \overline{M}_{g,n}(\mathfrak{X})(\mathfrak{T})$$

for any formal scheme $\mathfrak{T} \in \mathcal{C}_{\text{formal}}$. Indeed, an object of the left hand side is a morphism $\mathfrak{T} \to \lim_m \overline{M}_{g,n}(X_m/S_m)$. It corresponds to a compatible sequence of the morphisms $t_m: T_m \to \overline{M}_{g,n}(X_m/S_m)$ in the sense that $t_m = t_{m+1} \times_{S_{m+1}} S_m$, where $T_m = \mathfrak{T} \times_{k^o} S_m$. By functoriality, the sequence of morphisms $t_m$ corresponds to a compatible sequence of $n$-pointed genus $g$ algebraic stable maps into $X_m$ over $T_m$, which is the same as an $n$-pointed genus $g$ formal stable map into $\mathfrak{X}$ over $\mathfrak{T}$. So it corresponds to a morphism...
It is clear that the correspondence we constructed above gives an equivalence of groupoids. □

When a $k$-analytic space $X^{\text{an}}$ is the analytification of a proper algebraic variety $X$ over $k$, the representability of the moduli stack $\mathcal{M}_{g,n}(X^{\text{an}})$ can be established as the analytification of the algebraic moduli stack thanks to the GAGA principle.

**Theorem 6.39.** Let $X$ be a proper algebraic variety over $k$. We have an isomorphism of stacks over the category $\mathcal{C}_{\text{rig}}$

$$(\mathcal{M}_{g,n}(X))^{\text{an}} \cong \mathcal{M}_{g,n}(X^{\text{an}}),$$

where $(\cdot)^{\text{an}}$ denotes the analytification functor. Therefore, the stack $\mathcal{M}_{g,n}(X^{\text{an}})$ is a paracompact strictly $k$-analytic stack.

**Proof.** One has to exhibit a functorial equivalence of groupoids

$$(\mathcal{M}_{g,n}(X))^{\text{an}}(T) \cong \mathcal{M}_{g,n}(X^{\text{an}})(T)$$

for any strictly $k$-analytic space $T \in \mathcal{C}_{\text{rig}}$. By the sheaf property of the stacks, it suffices to show for $T = \text{Sp}_B A$ for any strictly $k$-affinoid algebra $A$. An object of the left hand side is a morphism $\text{Sp}_B A \to (\mathcal{M}_{g,n}(X))^{\text{an}}$. So we obtain a morphism $\text{Spec} A \to \mathcal{M}_{g,n}(X)$, which gives an $n$-pointed genus $g$ algebraic stable map into $X$ over $\text{Spec} A$. By Lemma 6.34, the analytification of this algebraic stable map gives a $k$-analytic stable map into $X^{\text{an}}$ over $\text{Sp}_B A$, i.e. an object of the right hand side.

For the other direction, let $(C \to T, \{s_i\}, f)$ be an $n$-pointed genus $g$ $k$-analytic stable map into $X^{\text{an}}$ over $\text{Sp}_B A$. Denote by $S_i$ the image of the sections $s_i$. Put $X_{T}^{\text{an}} = X^{\text{an}} \times_k T$, $f_T : C \to X_{T}^{\text{an}}$. The stability condition implies that the line bundle $\omega_C/T(\sum S_i)$ is relatively ample over $X_{T}^{\text{an}}$. Let $\mathcal{L}$ denote the $k$-analytic quasi-coherent sheaf of graded algebras

$$\mathcal{L} = \bigoplus_{m \geq 0} f_{T*} \left( \omega_C/T(\sum S_i) \right)^{\otimes m}$$

over $X_{T}^{\text{an}}$. It is a direct sum of $k$-analytic coherent sheaves. By the relative $k$-analytic GAGA principle (cf. [Con06] Example 3.2.6, [Köp74] §5-§6), $\mathcal{L}$ is isomorphic to the analytification of an algebraic quasi-coherent sheaf of graded algebras $\mathcal{L}_{\text{alg}}$ over $X_T$. Using the relative Proj construction, the sheaf $\mathcal{L}_{\text{alg}}$ gives us a family of algebraic curves $C_{\text{alg}} = \text{Proj}_{X_T}(\mathcal{L}_{\text{alg}})$ over $\text{Spec} A$ and a morphism from the $C_{\text{alg}}$ to $X$. Algebraization of the sections $\{s_i\}$ follows from the same relative $k$-analytic GAGA principle. So we have obtained an $n$-pointed genus $g$ algebraic stable map into $X$ over $\text{Spec} A$, thus a morphism $\text{Spec} A \to \mathcal{M}_{g,n}(X)$. By the universal property of analytification, we obtain a morphism from $T$ to $(\mathcal{M}_{g,n}(X))^{\text{an}}$. It is clear that the correspondences we constructed above give an equivalence of groupoids. □

In general, a paracompact strictly $k$-analytic space $X$ is far from being the analytification of an algebraic $k$-variety, but it does always admit a formal
model over $k^0$ (cf. Theorem 6.1). Therefore, in order to establish the representability of the moduli stack $\overline{M}_{g,n}(X)$ in this useful generality, we need to resort to the moduli stack of formal stable maps, whose representability has been shown in Proposition 6.38.

**Theorem 6.40.** Assume that the non-Archimedean field $k$ is of discrete valuation. Let $X$ be a formal scheme locally of finite presentation over $k^0$. We have an isomorphism of stacks over the category $C_{rig}$

$$\left(\overline{M}_{g,n}(X)\right)_\eta \cong \overline{M}_{g,n}(X_\eta),$$

where $(\cdot)_\eta$ denotes the generic fiber functor. Therefore, the stack $\overline{M}_{g,n}(X_\eta)$ is a paracompact strictly $k$-analytic stack.

**Proof.** One has to exhibit a functorial equivalence of groupoids

$$\left(\overline{M}_{g,n}(X)\right)_\eta(T) \xrightarrow{\sim} \overline{M}_{g,n}(X_\eta)(T)$$

for any $T \in C_{rig}$. By the sheaf property of the stacks, it suffices to show for $T = \text{Sp}_B A$ for any strictly $k$-affinoid algebra $A$.

An object of the left hand side is a morphism $t: T \to \left(\overline{M}_{g,n}(X)\right)_\eta$. Let $\overline{M}_{g,n}(X)^0 \to \overline{M}_{g,n}(X)$ be a smooth covering of the formal stack $\overline{M}_{g,n}(X)$, and let $\overline{M}_{g,n}(X)^1 = \overline{M}_{g,n}(X)^0 \times_{\overline{M}_{g,n}(X)} \overline{M}_{g,n}(X)^0$. Using Remark 6.10, we obtain two morphisms of formal schemes $t^0: \Sigma^0 \to \overline{M}_{g,n}(X)^0$, $t^1: \Sigma^1 \to \overline{M}_{g,n}(X)^1$, where $T^0 := \Sigma^0_\eta$ is a quasi-smooth covering of $T$ and $T^1 := \Sigma^1_\eta$ is isomorphic to $T^0 \times_T T^0$. By Proposition 6.38, the morphism $t^0$ gives an $n$-pointed genus $g$ formal stable map $\left(\mathcal{C} \to \Sigma^0, \{s_i\}, f\right)$ into $X$ over $\Sigma^0$ and the morphism $t^1$ gives a family of isomorphisms of formal stable maps parameterized by $\Sigma^1$. They do not constitute descent data because $\Sigma^1$ is not an equivalence groupoid over $\Sigma^0$. Nevertheless, they become effective descent data once passing to the generic fibers. More precisely, applying Lemma 6.36, we obtain a $k$-analytic stable map $(\mathcal{C}_\eta \to \Sigma^0_\eta, \{(s_i)_\eta\}, f_\eta)$ into $X_\eta$ over $\Sigma^0_\eta$ and a family of isomorphisms of $k$-analytic stable maps into $X_\eta$ parameterized by $T^1 = \Sigma^1_\eta$. They descend to an $n$-pointed genus $g$ $k$-analytic stable map into $X_\eta$ over $T$, which is an object of the right hand side.

For the other direction, an object of the right hand side is an $n$-pointed genus $g$ $k$-analytic stable map $(C \to T, \{s_i\}, f)$ into $X_\eta$ over $T$. By the theory of formal models (cf. Theorem 6.1), we can obtain a formal model $\mathcal{C}$ for $C$, a formal model $\Sigma$ for $T$ such that the morphisms of $k$-analytic spaces $C \to T$, $f: C \to X_\eta$ and $\{s_i: T \to C\}$ extend to morphisms of formal schemes $\mathcal{C} \to \Sigma$, $f: \mathcal{C} \to \mathcal{X}$ and $\{s_i: \Sigma \to \mathcal{C}\}$. Now our aim is to modify the triple $(\mathcal{C} \to \Sigma, \{s_i\}, f)$ in order to obtain a formal stable map into $\mathcal{X}$. We borrow some ideas from the proof of de Jong’s alteration theorem ([dJ96], [dJ97]). For clarity, we decompose our reasonings into 5 steps.

**Step 1 (algebraize).** For any morphism of formal schemes $\Sigma^0 \to \Sigma$, we denote by $(\mathcal{C}^0 \to \Sigma^0, \{s_i\}^0, f^0)$ the pullback of the triple $(\mathcal{C} \to \Sigma, \{s_i\}, f)$
to $\mathcal{S}$. Since $\mathcal{E} \to \mathcal{S}$ is a family of formal curves, there exists an affine Zariski covering $\mathcal{T}^0 \to \mathcal{S}$ such that the morphism $\mathcal{E}^0 \to \mathcal{T}^0$ is projective. Assume $\mathcal{T}^0 = \text{Spf} \mathcal{A}^0$ for some topological algebra $\mathcal{A}^0$ finitely presented over $k^0$. Put $\mathcal{T}^0_{\text{alg}} = \text{Spec} \mathcal{A}^0$. By formal GAGA, the projectivity implies that the morphism $\mathcal{E}^0 \to \mathcal{T}^0$ is the completion of a family of algebraic curves $\mathcal{C}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}$ along the special fibers over $\text{Spec} \bar{k}$. Again by formal GAGA, we can algebraize the sections $\{s_i^0\}$ as well. So we obtain a family of pointed algebraic curves $\left(\mathcal{C}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}; \{s_i^0\}\right)$.

Step 2 (rigidify). Replacing $\mathcal{T}^0_{\text{alg}}$ by an étale covering if necessary, we can add more disjoint sections $\{s_i^0\}$ such that every irreducible component of every geometric fiber of the morphism $\mathcal{C}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}$ meets at least three sections. Denote by $n'$ the total number of sections.

Step 3 (kill monodromies). If we take the generic fiber of the family of pointed algebraic curves $\left(\mathcal{C}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}; \{s_i^0\} \cup \{s_i^0\}\right)$, i.e. take fiber product with Spec $k$ over Spec $k^0$, we obtain an $n'$-pointed genus $g$ algebraic stable curve over $\mathcal{T}^0_{\text{alg},k} := \mathcal{T}^0_{\text{alg}} \times k = \text{Spec} (\mathcal{A}^0 \otimes_{k^0} k)$. This gives us a morphism $t_C$ from $\mathcal{T}^0_{\text{alg},k}$ to the Deligne-Mumford stack $\mathcal{M}_{g,n'}$ of $n'$-pointed genus $g$ algebraic stable curves over $k$. Let $M^0 \to \mathcal{M}_{g,n'}$ be a representable étale covering of the stack, and let $\mathcal{T}^0_{\text{alg},k} = \mathcal{T}^0_{\text{alg},k} \times_{\mathcal{M}_{g,n'}} M^0$. Denote by $t^0_C$ the morphism $\mathcal{T}^0_{\text{alg},k} \to M^0$ obtained from base change. We obtain an $n'$-pointed genus $g$ algebraic stable curve over $\mathcal{T}^0_{\text{alg},k}$ as the pullback of the universal family over $\mathcal{M}_{g,n'}$ to $M^0$ then to $\mathcal{T}^0_{\text{alg},k}$.

Step 4 (extend families of stable curves). By the properness of the Deligne-Mumford stack $\mathcal{M}_{g,n'}$, one can assume that the scheme $M^0$ is of finite type over $k$. Let $\overline{M}$ be the normalization of $\mathcal{M}_{g,n'}$ in the function field of $M^0$. Then $\overline{M}$ is a proper scheme by the properness of $\mathcal{M}_{g,n'}$. Put $\mathcal{T}^0$ to be the closure of the image of the morphism $\mathcal{T}^0_{\text{alg},k} \to \mathcal{T}^0_{\text{alg}} \times \overline{M}$. We obtain a proper morphism $\mathcal{T}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}$ and a morphism $\mathcal{T}^0_{\text{alg}} \to \overline{M}$. The latter morphism corresponds to an $n'$-pointed genus $g$ algebraic stable curve $\mathcal{C}^0_{\text{alg}}$ over $\mathcal{T}^0_{\text{alg}}$ as the pullback of the universal family over $\mathcal{M}_{g,n'}$ to $\overline{M}$ then to $\mathcal{T}^0_{\text{alg}}$.

Step 5 (extend morphisms of stable curves). Put $\mathcal{E}^0 \times_k \mathcal{T}^0_{\text{alg}} \times k \to \mathcal{E}^0_{\text{alg}} \times \mathcal{T}^0_{\text{alg}} \times k$. We would like to extend it to a morphism from $\mathcal{C}^0_{\text{alg}}$ to $\mathcal{C}^0_{\text{alg}}$ over $\mathcal{T}^0_{\text{alg}}$. We use the Hom scheme $\text{Mor} := \text{Mor}_{\mathcal{T}^0_{\text{alg}}} \left(\mathcal{C}^0_{\text{alg}}, \mathcal{C}^0_{\text{alg}}\right)$ parameterizing morphisms from $\mathcal{C}^0_{\text{alg}}$ to $\mathcal{C}^0_{\text{alg}}$ over $\mathcal{T}^0_{\text{alg}}$. By construction, we have a morphism $\mathcal{C}^0_{\text{alg}} \times k \to \mathcal{E}^0_{\text{alg}} \times \mathcal{T}^0_{\text{alg}} \times k$ corresponds to a morphism $t_M: \mathcal{T}^0_{\text{alg}} \times k \to \text{Mor}$. Let $\mathcal{T}^0_{\text{alg}}$ be the closure of the image of $t_M$ in the scheme $\text{Mor}$. We would like to show that the morphism $\mathcal{T}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}$ is proper. Indeed, the properness can be verified using the valuative criterion. Let $T'$ be a trait and let $T' \to \mathcal{T}^0_{\text{alg}}$ be a morphism that sends the generic point $\xi$ of $T'$ to the generic fiber $\mathcal{T}^0_{\text{alg}} \times k$. Then we would like to extend it to a morphism $\mathcal{T}^0_{\text{alg}} \to \mathcal{T}^0_{\text{alg}}$ such that the morphism $\mathcal{E}^0 \to \mathcal{T}^0_{\text{alg}}$ is projective.
of $\hat{\mathcal{C}}_0^{\prime}$. Let $\hat{\mathcal{C}}' \to T'$ and $\mathcal{C}' \to T'$ denote the pullback of $\hat{\mathcal{C}}_0^{\prime}$ and $\mathcal{C}_0^{\prime}$ respectively along the morphism $T' \to \hat{\mathcal{C}}_0^{\prime}$. We have a morphism $\hat{\mathcal{C}}'_{\xi} \to \mathcal{C}'_{\xi}$ over the generic point $\xi$ of $T'$. Moreover, every irreducible component of every geometric fiber of the morphism $\mathcal{C}' \to T'$ has at least three marked points by Step 2. In the case where the generic fiber $\hat{\mathcal{C}}'_{0}$ is smooth, the total space $\hat{\mathcal{C}}'$ is regular. Let $G$ denote the closure of the graph of the morphism $\hat{\mathcal{C}}_{\xi} \to \mathcal{C}'_{\xi}$ inside $\hat{\mathcal{C}}' \times_{T'} \mathcal{C}'$. We can apply de Jong’s three point lemma to the normalizations of $G$ and $\mathcal{C}'$ (cf. [dJ96] 4.18, [AO00] 4.9, Exercise 5.3, [AO01]). So the morphism $\hat{\mathcal{C}}'_{\xi} \to \mathcal{C}'_{\xi}$ can be extended to a morphism $\hat{\mathcal{C}}' \to \mathcal{C}'$ thanks to the three point condition. For the general case, by the deformation theory of pointed stable curves, any double point of $\hat{\mathcal{C}}'_{\xi}$ is a deformation of a double point on the special fiber $\hat{\mathcal{C}}_{s}'$. We can make a base change of $T'$ so that all the double points split. Denote by $\hat{\mathcal{C}}'$ the normalization of $\hat{\mathcal{C}}'$. Then the generic fiber $(\hat{\mathcal{C}}')_{\xi}$ is smooth. Applying the argument above to $\hat{\mathcal{C}}'$, the morphism $(\hat{\mathcal{C}}')_{\xi} \to \mathcal{C}'_{\xi}$ over $T'_{\xi}$ extends to a morphism $\hat{\mathcal{C}}' \to \mathcal{C}'$ over $T'$. It descends to a morphism $\hat{\mathcal{C}}' \to \mathcal{C}'$ over $T'$ by continuity. Using the valuative criterion of properness, we have shown that the morphism $\hat{\mathcal{C}}_0^{\prime} \to \mathcal{C}_0^{\prime}$ is proper. Therefore, by replacing $\hat{\mathcal{C}}_0^{\prime}$ with $\hat{\mathcal{C}}_0^{\prime}$, and by replacing $\hat{\mathcal{C}}_0^{\prime}$ and $\mathcal{C}_0^{\prime}$ with their pullbacks, the morphism $\hat{\mathcal{C}}_0^{\prime} \times k \to \mathcal{C}_0^{\prime} \times k$ can be successfully extended to a morphism $\hat{\mathcal{C}}_0^{\prime} \to \mathcal{C}_0^{\prime}$.

Step 6 (analytify). Let $\hat{\mathcal{C}}_0^{\prime}, \hat{\mathcal{C}}_0^{\prime}, \mathcal{C}_0^{\prime}$ be the completions of the schemes $\hat{\mathcal{C}}_0^{\prime}, \mathcal{C}_0^{\prime}, \hat{\mathcal{C}}_0^{\prime}$ respectively along their special fibers over Spec $\hat{k}$. Then $\{\hat{\mathcal{C}}_0^{\prime} \to \hat{\mathcal{C}}_0^{\prime}, \{s_j^{\prime}\} \cup \{s_j^{\prime}\}\}$ is a formal stable curve, where $\{s_j^{\prime}\}$ are the additional sections. Let $\hat{\mathcal{C}}^{\prime}$ be the composition of the morphisms

$$\hat{\mathcal{C}}_0^{\prime} \to \mathcal{C}_0^{\prime} \to \mathcal{C} \to \hat{x}.$$ 

If we take into account the additional sections $\{s_j^{\prime}\}$, we have constructed a formal stable map $\hat{\mathcal{C}}_0^{\prime} \to \hat{\mathcal{C}}_0^{\prime}, \{s_j^{\prime}\} \cup \{s_j^{\prime}\}, \hat{\mathcal{C}}^{\prime}$ into $\hat{x}$ over $\hat{\mathcal{C}}_0^{\prime}$. Now remove these auxiliary sections which we added in Step 2 and then contract the non-stable components, we obtain an $n$-pointed genus $g$ formal stable map into $\hat{x}$ over $\hat{\mathcal{C}}_0^{\prime}$ which we denote by $\hat{\mathcal{C}}^{\prime} \to \hat{x}, \{s_j^{\prime}\}$. If we apply the generic fiber functor $\hat{\cdot}$ to it, we obtain the pullback of the $k$-analytic stable map $C \to T, \{s_i\}$ along the quasi-étale covering $\hat{T}_0^{\prime} \to T$. The formal stable map $\hat{\mathcal{C}}^{\prime} \to \hat{\mathcal{C}}_0^{\prime}, \{s_j^{\prime}\}, \hat{\mathcal{C}}^{\prime}$ gives a morphism $\hat{\mathcal{C}}^{\prime} \to \mathcal{M}_{g,n}(\hat{x})$ by Proposition 6.38. Since the generic fiber of the formal stable map $\hat{\mathcal{C}}^{\prime} \to \hat{\mathcal{C}}_0^{\prime}, \{s_j^{\prime}\}, \hat{\mathcal{C}}^{\prime}$ into $\hat{x}$ over $\hat{\mathcal{C}}_0^{\prime}$ descends to a $k$-analytic stable map into $\mathcal{M}_{g,n}(\hat{x})$. 

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over $T$, the generic fiber of the morphism $\eta^0 : \overline{X}^0 \to \overline{\mathcal{M}}_{g,n}(\mathfrak{X})$ descends to a morphism $f : T \to (\overline{\mathcal{M}}_{g,n}(\mathfrak{X}))_\eta$, which is an object of the left hand side.

Finally it is not difficult to check that the correspondences we constructed above give an equivalence of groupoids. 

Now we specialize to the case where $\mathfrak{X}$ is a generalized strictly semi-stable formal scheme over $k^0$ equipped with a Kähler structure $\hat{L}$ (Definition 5.19). Let $C$ be a proper smooth irreducible $k$-analytic curve and $f : C \to \mathfrak{X}_\eta$ a $k$-analytic morphism. The degree of the morphism $f$ with respect to the Kähler structure $\hat{L}$ was defined in Section 5.2. If the $k$-analytic curve $C$ is not smooth but with ordinary double points, we define the degree of the morphism $f$ to be the degree after taking the normalization. Similarly, if $f : C \to \mathfrak{X}_s$ is a morphism from an algebraic pre-stable curve $C$ over the residue field $\tilde{k}$ to $\mathfrak{X}_s$, we define $\deg f = \deg f^*(\hat{L})$ (see Definition 3.12). If $\tilde{f} : \mathfrak{C} \to \mathfrak{X}$ is a morphism from a formal pre-stable curve $\mathfrak{C}$ over $k^0$ to $\mathfrak{X}$, we define $\deg \tilde{f} = \deg \tilde{f}_s$.

Fix a positive real number $A$. Denote by $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}_\eta,A)$ the moduli stack of $n$-pointed genus $g$ $k$-analytic stable maps into $\mathfrak{X}_\eta$ with degree bounded by $A$. Similarly, we define the moduli stacks $\overline{\mathcal{M}}_{g,n}(\mathfrak{X},A)$ and $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}_s,A)$. Proposition 6.38 and Theorem 6.40 imply the following isomorphisms

\[
\overline{\mathcal{M}}_{g,n}(\mathfrak{X},A)_s \simeq \overline{\mathcal{M}}_{g,n}(\mathfrak{X}_s,A),
\overline{\mathcal{M}}_{g,n}(\mathfrak{X},A)_\eta \simeq \overline{\mathcal{M}}_{g,n}(\mathfrak{X}_\eta,A).
\]

**Proposition 6.41.** The moduli stack $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}_s,A)$ is an algebraic stack of finite type over the residue field $\tilde{k}$. It is a proper algebraic stack if the formal model $\mathfrak{X}$ is proper.

**Proof.** For every $i \in I_{\mathfrak{X}}$, the Kähler structure $\hat{L}$ implies that the irreducible component $D_i$ is quasi-projective. It follows from the boundedness of Hilbert schemes that there exists an integer $N_i^0$ such that for any stable map $(C \to \text{Spec} \tilde{k}, \{s_i\}, f)$ into $D_i$ of genus bounded by $g$ and degree bounded by $A$, the number of irreducible components of $C$ must be bounded by $N_i^0$.

Now we introduce some definitions for the purpose of the proof.

**Definition 6.42.** A *decomposition datum* $\mathcal{D}$ is a collection of the following non-negative integers:

- $N_i$ for all $i \in I_{\mathfrak{X}}$, such that $N_i \leq N_i^0$,
- $g_{ij}$, $n_{ij}$ for all $i \in I_{\mathfrak{X}}$, $j \in \{1, \ldots, N_i\}$,
- $n_{ij}^{i'j'}$ for all $i, i' \in I_{\mathfrak{X}}$, $j \in \{1, \ldots, N_i\}$, $j' \in \{1, \ldots, N_{i'}\}$, such that $n_{ij}^{i'j'} = 0$ whenever $i = i'$, and $n_{ij}^{i'j'} = n_{ij'}^{i'j}$ for all $i, j, i', j'$.

Given a decomposition datum $\mathcal{D}$, we associate to it a graph $G_\mathcal{D}$ as follows. The vertices of $G_\mathcal{D}$ are labeled by pairs of integers $(i, j)$ for all $i \in I_{\mathfrak{X}}$, $j \in \{1, \ldots, N_i\}$. The number of edges between the vertex $(i, j)$ and the vertex $(i', j')$ is given by $n_{ij}^{i'j'}$. Let $g(G_\mathcal{D})$ denote the genus of the graph $G_\mathcal{D}$, i.e. the rank of the first homology group.
Definition 6.43. A decomposition datum $D$ is said to be of type $(g, n)$ if

(i) the associated graph $G_D$ is connected,
(ii) $g(G_D) + \sum_{i,j} g_{ij} = g$, and
(iii) $\sum_{i,j} n_{ij} = n$.

It follows from the definitions that

Lemma 6.44. There is only a finite number of decomposition data of type $(g, n)$.

Given a decomposition datum $D$ of type $(g, n)$, we consider the following construction. For every $i \in I_{X}, j \in \{1, \ldots, N_i\}$, put $n'_{ij} = n_{ij} + \sum_{i',j'} n'_{ij}'$.

Let $\overline{\mathcal{M}}_{g_{ij}, n'_{ij}}(D_i, A)$ denote the moduli stack of $n'_{ij}$-pointed genus $g_{ij}$ stable maps into the irreducible component $D_i$ of degree bounded by $A$. The Kähler structure $\widehat{L}$ implies that the irreducible component $D_i$ is quasi-projective. If the formal model $\mathfrak{X}$ is proper, then $D_i$ is projective. In this case, $\overline{\mathcal{M}}_{g_{ij}, n'_{ij}}(D_i, A)$ is a proper algebraic stack over the residue field $\overline{k}$ by Theorem 6.19. If we don’t assume the properness of $\mathfrak{X}$, then $\overline{\mathcal{M}}_{g_{ij}, n'_{ij}}(D_i, A)$ is an open sub-stack of a proper algebraic stack. So it is an algebraic stack of finite type over $\overline{k}$.

Let us label the $n'_{ij}$ marked points as follows:

(1)\(ij, (2)ij, \ldots, (n_{ij})ij\),
and (1)$^i_j$'j', (2)$^i_j$'j', \ldots, (n$^i_j$'j') for all $i \in I_{X}, j \in \{i, \ldots, N_i\}$.

Let $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, A)_{\text{dec}}$ denote the moduli stack of collections of $n'_{ij}$-pointed genus $g_{ij}$ stable maps into $D_i$ for every $i \in I_{X}, j \in \{1, \ldots, N_i\}$ which satisfy the following conditions:

(i) The marked point with label $(l)^{i'}_j$ maps into the stratum $D_{\{i,i'\}} = D_i \cap D_{i'}$;
(ii) The marked point with label $(l)^{i'}_j$ and the marked point with label $(l)^{i}_j$ map to the same point in the stratum $D_{\{i,i'\}}$;
(iii) If we denote by $A_{ij}$ the degree of the $n'_{ij}$-pointed genus $g_{ij}$ stable map into $D_i$, then $\sum A_{ij} \leq A$.

The morphism

$$\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, A)_{\text{dec}} \to \prod_{i,j} \overline{\mathcal{M}}_{g_{ij}, n'_{ij}}(D_i, A)$$

is a closed embedding of finite presentation. Therefore, it is an algebraic stack of finite type over $\overline{k}$. It is proper of $\mathfrak{X}$ is proper.

Furthermore, by construction, we have a finite surjective morphism from $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, A)_{\text{dec}}$ to the moduli stack $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, A)$. Theorem 6.20 shows that $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, A)$ is an algebraic stack locally of finite presentation over $\overline{k}$. Therefore, the moduli stack $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, A)$ is an algebraic stack of finite type over the residue field $\overline{k}$. It is a proper algebraic stack if $\mathfrak{X}$ is proper. □
Corollary 6.45. The moduli stack $\overline{M}_{g,n}(X, A)$ is a formal stack finitely presented over $k^\circ$. It is proper if $X$ is proper.

Theorem 6.46. Assume that the non-Archimedean field $k$ is of discrete valuation and that the formal model $X$ is equipped with a Kähler structure $\hat{L}$. Then the moduli stack $\overline{M}_{g,n}(X_\eta, A)$ is a compact strictly $k$-analytic stack. If we assume moreover that the formal model $X$ is proper, then $\overline{M}_{g,n}(X_\eta, A)$ is a proper strictly $k$-analytic stack.

Proof. The theorem follows from Corollary 6.45 and Theorem 6.14. \qed

7. Tropicalization of the non-Archimedean analytic moduli stack

Every $k$-analytic stable map into the $k$-analytic space $X_\eta$ gives rise to a tropical curve in the Clemens polytope $S_X$. So we obtain a map from the moduli space of $k$-analytic stable maps to the moduli space of tropical curves. We call this map the tropicalization map of the moduli space of $k$-analytic stable maps. In Section 7.1, we prove the continuity of this tropicalization map using the balancing conditions in Section 4.2 and the formal models of families of stable maps developed in Section 6.3. In Section 7.2, we prove that the image of this tropicalization map is a compact finite polyhedral complex using the continuity plus quantifier eliminations in the model theory of rigid subanalytic sets.

7.1. Continuity and compactness. Let $X$ be a strictly semi-stable formal scheme over $k^\circ$ equipped with a Kähler structure $\hat{L}$. Let $\omega$ be the simple Kähler structure on the Clemens polytope $S_X$ induced by the Kähler structure $\hat{L}$ (Definition 5.20).

Fix a positive real number $A$. Let $M_n(S_X, A)$ denote the moduli space of $n$-pointed tropical curves whose tropical degrees with respect to $\omega$ are bounded by $A$ (see Remark 5.18). Let $k'$ be a non-Archimedean field extension of $k$. Let $(C \to \Sp_B K, \{s_i\}, f)$ be an $n$-pointed genus $g$ stable map into $X_\eta$ over $\Sp_B k'$ of degree bounded by $A$, and let $C^t$ denote the corresponding tropical curve in the Clemens polytope $S_X$ (Definition 4.9). By Corollary 5.23, the tropical degree of $C^t$ with respect to the simple Kähler structure $\omega$ is bounded by $A$. Moreover, we have $n$ marked points $\{(\tau \circ f)(s_i)\}$ on $C^t$. So the tropical curve $C^t$ is a point in the moduli space $M_n(S_X, A)$.

Now let $T$ be a paracompact strictly $k$-analytic space and let $(C \to T, \{s_i\}, f)$ be an $n$-pointed genus $g$ $k$-analytic stable map into $X_\eta$ over $T$. By the arguments above, we obtain a set theoretic map $\tau_T$ from $T$ to $M_n(S_X, A)$.

Assume that the characteristic of the residue field $k$ is zero.

Theorem 7.1. The map $\tau_T \colon T \to M_n(S_X, A)$ is continuous with respect to the Berkovich topology on $T$ and the topology on $M_n(S_X, A)$ induced by its finite polyhedral structure.
Proof. In order to prove the theorem, we need another description of the topology on the polyhedral complex $M_n(S_X, A)$.

**Definition 7.2.** For any vertex $p$ of a simplicial complex $S$, we denote by $\text{Star}(p)$ the star around $p$ inside $S$, i.e. the union of open faces of $S$ whose closure contains the vertex $p$.

**Definition 7.3.** A *subdivision datum* $\mathcal{D} = (\mathcal{S}_\mathcal{X}, P, P_1, \{i_p\}, \{w_p\}, \{q_i\})$ is a collection of the following objects:

- A simplicial subdivision $\mathcal{S}_\mathcal{X}$ of the Clemens polytope $S_X$.
- A subset $P$ of the set of vertices of $\mathcal{S}_\mathcal{X}$.
- A subset $P_1$ of $P$.
- For each $p \in P_1$, a choice of another vertex $i_p$ of $\mathcal{S}_\mathcal{X}$ and an integer $w_p$.
- A choice of $n$ vertices $q_1, \ldots, q_n$ of $\mathcal{S}_\mathcal{X}$.

Given a subdivision datum $\mathcal{D}$, let $U(\mathcal{D})$ be the subset of $M_n(S_X, A)$ consisting of $n$-pointed tropical curves $(G, \{s_i\})$ which satisfy the following conditions:

(i) For $i = 1, \ldots, n$, the marked point $s_i$ lies in $\text{Star}(q_i)$.
(ii) The graph $G$ is contained in the union $\bigcup_{p \in P} \text{Star}(p)$.
(iii) For every $p \in P$, the intersection $G \cap \text{Star}(p)$ is non-empty.
(iv) For every $p \in P_1$, every vertex $v$ of $G$ inside $\text{Star}(p)$, we denote by $(\sigma_v)^i_p$ the component in the direction $i_p$ of the sum of weights $\sigma_v$ around the vertex $v$. We require that the sum $\sum_{v \in \text{Star}(p)} (\sigma_v)^i_p$ equals $w_p$ for every $p \in P_1$.

A subdivision datum $\mathcal{D} = (\mathcal{S}_\mathcal{X}, P, P_1, \{i_p\}, \{w_p\}, \{q_i\})$ is said to be good if the subset $U(\mathcal{D})$ is an open subset of $M_n(S_X, A)$.

**Lemma 7.4.** The open subsets $U(\mathcal{D})$ for all good subdivision data form a base for the topology on $M_n(S_X, A)$.

Proof. First we identify $n$-pointed tropical curves in $M_n(S_X, A)$ with the same underlying graphs and marked points, and we denote by $M_n(S_X, A)_u$ the quotient space. The moduli space $M_n(S_X, A)$ is a finite polyhedral complex, so is $M_n(S_X, A)_u$. Moreover, the quotient topology on $M_n(S_X, A)_u$ is equivalent to the restriction of the Gromov-Hausdorff topology for subsets in $S_X$. Therefore, the open subsets of the form $u\left(U(\mathcal{D} = (\mathcal{S}_\mathcal{X}, P, P_1 = \emptyset, \emptyset, \emptyset, \{q_i\}))\right)$ form a base for the topology on $M_n(S_X, A)_u$, where $u$ denotes the forgetful map from $M_n(S_X, A)$ to $M_n(S_X, A)_u$.

In order to take into account the multiplicity of an edge $e$ of a tropical curve $G$, we make a sufficiently fine simplicial subdivision $\mathcal{S}_\mathcal{X}$ of $S_X$ and choose a vertex $p$ of $\mathcal{S}_\mathcal{X}$ very close the edge $e$ such that the intersection $G \cap \text{Star}(p)$ is an open interval of $e$ and it does not ramify under small deformations of $G$. Let $P_1$ be the collection of such vertices $p$ for every edge of $G$, and we specify suitable directions $\{i_p\}$ and weights $\{w_p\}$ according
to the tropical weights of the edges. Then we can have control on the multiplicities of every edge of the tropical curve $G$. So we have proved the lemma.

In order to prove the continuity of the map $\tau_T: T \to M_n(S_X, A)$, it suffices to show that for any point $b \in T$ and any good subdivision datum $\mathcal{D} = (\mathcal{S}_X, P, P_1, \{i_p\}, \{w_p\}, \{q_i\})$ such that the associated open subset $U(\mathcal{D})$ contains the point $\tau_T(b) \in M_n(S_X, A)$, the inverse image $\tau_T^{-1}(U(\mathcal{D}))$ is a neighborhood of $b$ in $T$.

We fix $b$ and $\mathcal{D}$ as above. If we allow a finite extension of the ground field $k$ and a further simplicial subdivision of $\mathcal{S}_X$, we can find a strictly semistable formal model $\overline{\mathcal{X}}$ for the $k$-analytic space $\mathcal{X}$ such that the Clemens polytope $S_T$ associated to $\overline{\mathcal{X}}$ is isomorphic to $S_X$ (see [KKMSD73] §II, III). Here we use the assumption that the residue field $\overline{k}$ is of characteristic 0. By the proof of Theorem 6.40, replacing $T$ by a quasi-étale covering if necessary, one can find a formal model $t: \mathcal{S}_X \to \mathcal{M}_{g,n}(\overline{\mathcal{X}}, A)$ for the morphism $t: T \to \mathcal{M}_{g,n}(\mathcal{X}_Y, A)$, i.e. an $n$-pointed genus $g$ formal stable map $(\mathcal{C} \to \mathcal{S}_X, \{a_i\}, f)$ into $\overline{\mathcal{X}}$ over $\mathcal{T}$. Applying the special fiber functor $(\cdot)_s$, we obtain a morphism $(\cdot)_s: \mathcal{T}_s \to \mathcal{M}_{g,n}((\mathcal{T}_s), A)$, i.e. an $n$-pointed genus $g$ algebraic stable map $(\mathcal{C}_s \to \mathcal{T}_s, \{(a_i)_s\}, (f)_s)$ into $\mathcal{T}_s$ over $\mathcal{T}_s$.

For every vertex $p$ of $\mathcal{T}_s$, denote by $\mathcal{D}_p$ the corresponding irreducible component of the special fiber $\mathcal{X}_s$. For every vertex $p \in P_1$, we construct as follows a locus $\mathcal{T}_s^p \subset \mathcal{T}_s$ and a stable map $(\mathcal{C}_s^p \to \mathcal{T}_s^p, \{s_i^p\}, (f)_s|_{\mathcal{T}_s^p})$ into the irreducible component $\mathcal{D}_p$ over the locus $\mathcal{T}_s^p$.

Let $b_s$ be a geometric point of $\mathcal{T}_s$ over the reduction of the point $b \in T$. Denote by $\mathcal{C}_{s,b}$ the fiber of the morphism $\mathcal{X}_s \to \mathcal{T}_s$ over the point $b_s \in \mathcal{T}_s$. Denote by $B_p$ be the following set of double points of $\mathcal{C}_{s,b}$: a double point $\nu$ of $\mathcal{C}_{s,b}$ belongs to $B_p$ if and only if one side of $\nu$ of the curve $\mathcal{C}_{s,b}$ is mapped into $\mathcal{D}_p$ under $(f)_s$ while the other side is not. Let $I_p$ be the set of irreducible components of $\mathcal{C}_{s,b}$ which are mapped into $\mathcal{D}_p$ under $(f)_s$. Let $\mathcal{T}_s^p \cap \mathcal{T}_s^0 = \mathcal{T}_s^p$ be the locus where none of the double points in $B_p$ gets smoothened. Over a point $b'$ in $\mathcal{T}_s^p$, it makes sense to distinguish the part $\mathcal{C}_{s,b'}$ of the fiber $\mathcal{C}_{s,b}$ which is the deformation of the union of the irreducible components in $I_p$. Let $\mathcal{T}_s^{p,0} \subset \mathcal{T}_s^p$ be the locus containing the points $b'$ over which the part $\mathcal{C}_{s,b'}$ of the fiber $\mathcal{C}_{s,b}$ is mapped into $\mathcal{D}_p$ under $(f)_s$. Let $\mathcal{C}_s^p$ be the union of the parts $\mathcal{C}_{s,b'}$ over all $b' \in \mathcal{X}_s^p$. We add sections to the former double points to the family of curves $\mathcal{C}_s^p \to \mathcal{X}_s^p$ and we obtain a stable map $(\mathcal{C}_s^p \to \mathcal{T}_s^p, \{s_i^p\}, (f)_s|_{\mathcal{T}_s^p})$ into $\mathcal{D}_p$ over $\mathcal{T}_s^p$. By construction, the locus $\mathcal{T}_s^0$ is closed in $\mathcal{T}_s$. Let $\mathcal{X}_s^{p,D} \subset \mathcal{X}_s^p$ be the locus for the family of curves $\mathcal{C}_s^p \to \mathcal{T}_s^p$ over which the degree of the pullback of the line bundle $O(\mathcal{D}_p)$ equals $w_p$. The flatness of the family of curves $\mathcal{C}_s^p \to \mathcal{T}_s^p$ ensures that $\mathcal{X}_s^{p,D}$ is a connected component of $\mathcal{X}_s^p$. In particular, $\mathcal{T}_s^{p,D}$ is closed in $\mathcal{T}_s$.

Now, let $\mathcal{T}_s^{p,D} \subset \mathcal{T}_s$ be the locus of $\mathcal{T}_s$ parameterizing $n$-pointed genus $g$ algebraic stable maps $(\mathcal{C}', \{s_i'\}, f')$ into $\mathcal{T}_s$ satisfying the following conditions:
(i) For $i = 1, \ldots, n$, the marked point $s'_i$ is mapped into the irreducible component $D_{q_i}$ under $f'$.
(ii) The image $f'(C')$ is contained in the union $\bigcup_{p \in P} D_p$.
(iii) For every $p \in P$, the intersection $f'(C') \cap D_p$ is non-empty.

Lemma 7.5. The locus $T^D_{s,0}$ is closed in $T_s$.

Proof. It follows from the properness of the family of curves $C_s \to T_s$ that the locus of $T_s$ over which Conditions (i) and (iii) are satisfied is closed in $T_s$.

Let $C^0_s$ be the inverse image $f_s^{-1}(X_s \setminus \bigcup_{p \in P} D_p)$. $C^0_s$ is open by the continuity of $f_s$ for the Zariski topology. The image of $C^0_s$ under the morphism $C_s \to T_s$ is open in $T_s$ by flatness. The complement of this image is exactly the locus of $T_s$ over which Condition (ii) is satisfied. So we have proved our lemma. $\square$

Therefore, the intersection
$$T^D_s := T^D_{s,0} \cap \bigcap_{p \in P} T^p_{s,D}$$
is closed in $T_s$. Lemma 5.21 implies that
$$\pi^{-1}_T(T^D_s) \subset \tau^{-1}_T(U(D)),$$
where $\pi_T : T \simeq \mathcal{T}_s \to T_s$ denotes the reduction map. By the anti-continuity of the reduction map, the set $\pi^{-1}_T(T^D_s)$ is open in $T$ for the Berkovich topology. Since $b \in \pi^{-1}_T(T^D_s)$ by construction, we have proved that $\tau^{-1}_T(U(D))$ is a neighborhood of $b$ in $T$. Using Lemma 7.4, we conclude that the map $\tau : T \to M_n(S_X, A)$ is continuous. $\square$

Corollary 7.6. The tropicalization map $\tau_M : \overline{M}_{g,n}(X_\eta, A) \to M_n(S_X, A)$ is a continuous map.

Corollary 7.7. Let $\overline{M}_{g,n}(X_\eta, A)$ denote the image of the tropicalization map $\tau_M : \overline{M}_{g,n}(X_\eta, A) \to M_n(S_X, A)$. Then $\overline{M}_{g,n}(X_\eta, A)$ is a compact subset of $M_n(S_X, A)$.

Proof. It follows from Theorem 6.46 and Corollary 7.6. $\square$

7.2. Polyhedrality via quantifier eliminations. In Corollary 7.7, we defined $\overline{M}_{g,n}(X_\eta, A)$ to be the subset of $n$-pointed tropical curves in $S_X$ induced by $n$-pointed genus $g$ stable maps into $X_\eta$ with degree bounded by $A$. In this section, we will show the polyhedral nature of the set $\overline{M}_{g,n}(X_\eta, A)$ using quantifier eliminations from model theory as well as the continuity theorem of Section 7.1. The model theoretic approach is inspired by Antoine Ducros’s work [Duc12] (I am very grateful for his explanations.). The model theory of algebraically closed valued fields is used in loc. cit. after algebraization of $k$-analytic situations. However, since we will not deal with a single $k$-analytic variety but with $k$-analytic families of $k$-analytic varieties, it is not
easy to apply the standard algebraization techniques. So we resort to the model theory of rigid subanalytic sets developed by Lipshitz [Lip93].

Due to technical difficulties, Lipshitz didn’t work directly with Tate algebras but with a slightly later algebra $k\langle X \rangle_{[\rho]}$ of analytic functions which are in particular functions convergent over the unit polydisc that is closed in the directions $X$ and open in the directions $\rho$. The language $\mathcal{L}^D_{an}$ of the theory includes among others a function symbol for each analytic function in $k\langle X \rangle_{[\rho]}$. Let us introduce some definitions that are directly related to our applications following the expositions in [Mar13] §0.5.1.

Since tropicalization is not sensitive to the extension of ground fields, we assume that our non-Archimedean field $k$ is algebraically closed. Let $k^\circ$ denote the ring of integers and $\Gamma = |k^*|$ the value group.

**Definition 7.8.** The algebra of $D$-functions $f : (k^\circ)^n \to k$ is defined inductively as follows:

(i) If $f \in k\langle X_1, \ldots, X_n \rangle$, then $f : (k^\circ)^n \to k$ is a $D$-function, where $k\langle X_1, \ldots, X_n \rangle$ denotes the Tate algebra.

(ii) If $f, g : (k^\circ)^n \to k$ are $D$-functions, then $D_0(f, g)$ and $D_1(f, g)$ are $D$-functions $(k^\circ)^n \to k$ where

$$D_0(f, g) : (k^\circ)^n \to k$$

$$x \mapsto \begin{cases} 
\frac{f(x)}{g(x)} & \text{if } |f(x)| \leq |g(x)| \\
0 & \text{otherwise}
\end{cases}$$

$$D_1(f, g) : (k^\circ)^n \to k$$

$$x \mapsto \begin{cases} 
\frac{f(x)}{g(x)} & \text{if } |f(x)| < |g(x)| \\
0 & \text{otherwise}
\end{cases}$$

(iii) Let $f_1, \ldots, f_l, g_1, \ldots, g_m$ be $D$-functions $(k^\circ)^n \to k$. Assume that $|f_i(x)| \leq 1$ and $|g_j(x)| < 1$ for all $x \in (k^\circ)^n$, $1 \leq i \leq l$, $1 \leq j \leq m$. Let $F \in k\langle X \rangle_{[\rho]}$ where $X$ denotes a vector of $l$ variables and $\rho$ denotes a vector of $m$ variables. Then

$$F(f_1, \ldots, f_l, g_1, \ldots, g_m) : (k^\circ)^n \to k$$

$$x \mapsto F(f_1(x), \ldots, f_l(x), g_1(x), \ldots, g_m(x))$$

is a $D$-function.

**Definition 7.9.** A subset $S$ of $(k^\circ)^m \times \Gamma^n$ is called subanalytic if it is a Boolean combination of subsets of the form

$$\{(x, \gamma) \in (k^\circ)^m \times \Gamma^n \mid |f(x)|^{\gamma^u} c \leq |g(x)|\} \cup \{(x, \gamma) \in (k^\circ)^m \times \Gamma^n \mid |f(x)|^{\gamma^u} c > |g(x)|\} \quad (7.1)$$

where $f, g$ are $D$-functions, $u \in \mathbb{Z}^n$ and $c \in \Gamma$.

**Remark 7.10.** Subanalytic sets are exactly the sets definable by first order formulas without quantifiers in the language $\mathcal{L}^D_{an}$. 
The following theorem has many variants in the literature with different proofs (cf. [BG84] Theorem A, [Ber96] Corollary 6.2.2, [Duc12] Theorem 3.2(1)). It is now a consequence of quantifier eliminations in the language $L_{D_{an}}$ ([Lip93] Theorem 3.8.2) and the explicit description of definable sets in Definition 7.9.

**Theorem 7.11** ([Mar13] Theorem 4.3.1). Let $Y$ be a $k$-affinoid space and let $S \subset Y$ be a subanalytic set as in Definition 7.9. Let $h_1, \ldots, h_n$ be $n$ $D$-functions on $S$. Then $(\text{val} \, h_1, \ldots, \text{val} \, h_n)(S) \cap \mathbb{R}^n$ is a finite polyhedral complex in $\mathbb{R}^n$ of dimension less than or equal to the dimension of $Y$, where we put $\text{val} \, 0 = \infty$.

We generalize Theorem 7.11 to the relative case. Let $p: Y \to T$ be a morphism of $k$-affinoid spaces, and let $h$ be a morphism from $Y$ to the $k$-analytic torus $G^n_{m,k}$. Let $G$ be the definable set

$$G = \left\{ (t, v) \in T(k) \times \mathbb{R}^n \mid \exists y \in Y(k) \left( (p(y) = t) \wedge (|h|(y) = v) \right) \right\}.$$ 

Using quantifier eliminations in the language $L_{D_{an}}$, the set $F$ can be written as a Boolean combination of the subsets of the form (7.1). Let $q_1, \ldots, q_N$ be the finite collection of $D$-functions that are involved in the definition of the set $G$. The image

$$(\text{val} \, q_1, \ldots, \text{val} \, q_N)(T)$$

is the space of parameters describing the polyhedral complexes of the form $\text{val}(h(Y_t)) \subset \mathbb{R}^n$ for all $t \in T$. Since $\text{val}(h(Y_t))$ is contained in $\mathbb{R}^n$, it is enough to consider the intersection

$$(\text{val} \, q_1, \ldots, \text{val} \, q_N)(T) \cap \mathbb{R}^N$$

as the parameter space. It follows from Theorem 7.11 that this intersection is a finite polyhedral complex in $\mathbb{R}^N$ of dimension less than or equal to the dimension of $T$. Now it is not difficult to deduce the following proposition.

**Proposition 7.12.** Let $X$ be a generalized strictly semi-stable formal scheme over $k^\circ$ equipped with a Kähler structure. Fix a positive number $A$. Let $T$ be a compact strictly $k$-analytic space and let $(C \to T, \{s_i\}, f)$ be an $n$-pointed genus $g$ $k$-analytic stable map into $X$ over $T$. As in Section 7.1, let $\tau_T: T \to M_n(S_X, A)$ denote the set-theoretically defined tropicalization map, and let $u$ denote the forgetful map from $M_n(S_X, A)$ to $M_n(S_X, A)_u$ forgetting the multiplicities of the edges of the tropical curves. Then the image of $T$ under the composition $u \circ \tau_T$ is a compact finite polyhedral complex in $M_n(S_X, A)_u$ of dimension less than or equal to the dimension of $T$.

**Proof.** We can assume that $T$ is a $k$-affinoid space. Choose a finite covering of the formal model $X$ by affine open sub-schemes of the form $U$ as in Definition 2.2. Let $\{Y_i\}$ be a finite affine covering of the inverse image $f^{-1}(U_i)$. Using the explicit description of the map $\tau: X_\eta \to S_X$ in (2.3), we apply the model-theoretic arguments above to the morphisms $Y_i \to T$. Therefore, the
image of $T$ under the composition $u \circ \tau_T$ is a finite polyhedral complex in $M_n(S_X, A)_u$ of dimension less than or equal to the dimension of $T$. Finally, the compactness of the image follows from the continuity of the map $u \circ \tau_T$ (Theorem 7.1).

\[\Box\]

**Theorem 7.13.** We use the settings in Proposition 7.12 and assume that the formal model $\mathfrak{X}$ is strictly semi-stable. Then the image of $T$ under the map $\tau_T: T \to M_n(S_X, A)$ is a compact finite polyhedral complex in $M_n(S_X, A)$.

**Proof.** We can assume that $T$ is a $k$-affinoid space. Let $u: M_n(S_X, A) \to M_n(S_X, A)_u$ be the forgetful map as in Proposition 7.12. Let $O$ be an open face\(^8\) of the polyhedral complex $M_n(S_X, A)_u$. Let $O_1, \ldots, O_m$ denote the connected components of the inverse image $u^{-1}(O)$. They are open faces of the polyhedral complex $M_n(S_X, A)$. It suffices to show that the intersection $\tau_T(T) \cap O_i$ is finite polyhedral for all $i = 1, \ldots, m$. Let $T_O \subset T$ denote the inverse image $(u \circ \tau_T)^{-1}(O)$. By Theorem 7.1, the restriction $(\tau_T)|_{T_O}: T_O \to u^{-1}(O)$ is a continuous map. Therefore, for any $i = 1, \ldots, m$, the inverse $T_{O_i} := ((\tau_T)|_{T_O})^{-1}(O_i)$ is a union of connected components of $T_O$. The model-theoretic arguments above imply that the map $u \circ \tau_T: T \to M_n(S_X, A)_u$ is given by the norms of a finite collection of $D$-functions $\{q_j\}$. Therefore, the subspace $T_O \subset T$ is a subanalytic set. Since the subspace $T_{O_i}$ is a union of connected components of $T_O$, it is also subanalytic. Restricting the $D$-functions $\{q_j\}$ to $T_{O_i}$, we prove that the image $\tau_T(T_{O_i}) = \tau_T(T) \cap O_i$ is finite polyhedral. Finally, the compactness of the image $\tau_T(T)$ follows from Theorem 7.1.

\[\Box\]

**Corollary 7.14.** Let $\overline{\mathfrak{M}}_{g,n}(\mathfrak{X}_n, A)$ denote the image of the tropicalization map $\tau_M: \overline{\mathfrak{M}}_{g,n}(\mathfrak{X}_n, A) \to M_n(S_X, A)$. Then $\overline{\mathfrak{M}}_{g,n}(\mathfrak{X}_n, A)$ is a compact finite polyhedral complex in $M_n(S_X, A)$.

**Proof.** The corollary follows from Theorem 7.13 and Theorem 6.46.

\[\Box\]

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\[^8\] An open face is by definition the relative interior of a face of the polyhedral complex.
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