THE NODAL CUBIC AND QUANTUM GROUPS AT
ROOTS OF UNITY

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Abstract. In a recent article, the coordinate ring of the nodal cubic was given the structure of a quantum homogeneous space. Here the corresponding coalgebra Galois extension is expressed in terms of quantum groups at roots of unity, and is shown to be cleft. Furthermore, the minimal quotient extensions are determined.

1. Introduction

In [10], the coordinate ring $B$ of the nodal cubic is embedded as a right coideal subalgebra into a Hopf algebra which is free over $B$. In this paper, we determine a minimal Hopf algebra $A$ with this property, and show that the Galois extension $B \subseteq A$ is cleft. The main result is:

**Theorem 1.** Let $k$ be a field containing a primitive third root $r$ of unity, and $A$ be the Hopf algebra over $k$ generated by $a, b, x$ and $y$ satisfying

$ba = ab, \quad ya = ay, \quad bx = xb, \quad yx = xy, \quad by = -yb,$

$x + axa^2 + a^2xa - a + 1 = 0, \quad x^2 + axa^2 + xaxa^2 + x + axa^2 = 0,$

$y^2 = x^2 + x^3, \quad b^2 = a^3 = 1, \quad F^3 = 0,$

where $F := xa + (r + 1)ax + \frac{r+2}{3} (a - a^2)$, and

$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b,$

$\Delta(x) = 1 \otimes x + x \otimes a, \quad \Delta(y) = 1 \otimes y + y \otimes b.$

Let $p, q \in k$ satisfy $p^2 = q^2 + q^3$, and $B, C \subseteq A$ be the subalgebras generated by $x + qa, y + pb$ and by $a, b, F$, respectively. Then we have:

1. The subalgebra $B$ is a right coideal and isomorphic to the coordinate ring of the nodal cubic, and $C$ is a Hopf subalgebra.
2. Multiplication in $A$ defines an isomorphism $C \otimes B \cong A$.
3. If $\text{char}(k) \neq 2$ and $0 \neq I \subseteq A$ is a Hopf ideal, then $B \cap I \neq 0$.

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Notice that $B \subseteq A$ is a coalgebra Galois extension. The coalgebra in question is $A/AB^+$, where $B^+ = B \cap \ker \varepsilon$. Since $AB^+ \neq B^+ A$, this is not a (Hopf) algebra quotient of $A$. As such, the Hopf subalgebra $C$ in Theorem 1 is only isomorphic to $A/AB^+$ as coalgebras. That $B \subseteq A$ is a cleft extension follows immediately from Theorem 1 (2), cf. [5, Proposition 2.3]. Part (2) of Theorem 1 also implies that $A$ is free and hence faithfully flat as a right $B$-module. Thus $B$ is a quantum homogeneous space in the sense of [16].

We remark that the Hopf algebra $u_r(\mathfrak{sl}_2) \otimes u_{-1}(\mathfrak{sl}_2)$ is obtained as a quotient of $A$ by adding the relation $y^2 = x^2 + x^3 = 0$. Here, $u_r(\mathfrak{sl}_2)$ is the Frobenius-Lusztig kernel associated to $U_r(\mathfrak{sl}_2)$ (see e.g. [8, Definition VI.5.6] or [9, Section 3.3.1]) and by a slight abuse of notation, $u_{-1}(\mathfrak{sl}_2)$ denotes Sweedler’s 4-dimensional Hopf algebra. The other simple root vector in $u_r(\mathfrak{sl}_2)$ is the class of

$$E := xa - rax + \frac{1 - r}{3}(a - a^2).$$

The classification of Hopf algebras in terms of their Gelfand-Kirillov dimension has been the focus of recent interest, see e.g. [4, 7, 12, 18]. Prompted by a question from Ken Brown, we show in Proposition 6 that $A$ is Noetherian of Gelfand-Kirillov dimension one, but is not semiprime.

In [3], the construction of quantum homogeneous space structures is extended to more general decomposable plane curves. Hence an interesting question for future research is whether the results in this note also extend to these curves.

The paper is organised as follows: the next section contains the proof of (1) and (2) in the main theorem, and explains the relation between this paper and previous results from [10]. Section 3 contains an analogue of Theorem 1 which does not require the base field $k$ to contain a primitive third root of unity. However, the link to small quantum groups is lost. The final section discusses the Hopf algebra $A$ from the perspective of the classification of pointed Hopf algebras, i.e. we describe the braided vector space of skew primitive elements. This is then used to prove Theorem 1 (3). We also describe the Nichols algebra of $A$.

2. THE CLEFT EXTENSION $B \subseteq A$

Throughout this section, let $k$ be a field containing a primitive third root of unity $r$, and let $p, q \in k$ be such that $p^2 = q^2 + q^3$. 

2.1. **On the origin of relations.** We begin by pointing out that the defining relations in [10] can be obtained from Manin’s approach to quantum groups [13]. To that end, one determines the universal Hopf algebra coacting by a given formula.

**Proposition 2.** Assume that $U$ is a Hopf algebra, $\rho : B \rightarrow B \otimes U$ is a right coaction on $B = k[s,t]/\langle s^2 + s^3 - t^2 \rangle$ and that we have

$$\rho(s) = 1 \otimes x + s \otimes a, \quad \rho(t) = 1 \otimes y + t \otimes b$$

for elements $a, b, x, y \in U$. Then $B$ is a $U$-comodule algebra if and only if the following relations hold in $U$:

$$ba = ab, \quad ya = ay, \quad bx = xb, \quad yx = xy, \quad by = -yb,$$

$$a^2 x + axa + xa^2 + a^2 - a^3 = x^2 a + xax + ax^2 + xa + ax = 0,$$

$$y^2 = x^2 + x^3, \quad b^2 = a^3.$$

**Proof.** The coaction $\rho$ turns $B$ into a $U$-comodule algebra (i.e. $\rho$ is an algebra map) if and only if the elements $\rho(s)$ and $\rho(t)$ of $B \otimes U$ satisfy the defining relations of $B$,

$$\rho(s)\rho(t) = \rho(t)\rho(s), \quad \rho(s)^2 + \rho(s)^3 = \rho(t)^2.$$

Inserting the explicit formulas of $\rho(s)$ and $\rho(t)$ shows that the stated commutation relations in $U$ are sufficient for $\rho$ to be an algebra map. That they are also necessary follows from the freeness of $B \otimes U$ as a $B$-module.

From now on, we denote by $U$ the universal Hopf algebra with the above properties. Note that the coassociativity of $\rho$ implies that $a, b$ are necessarily group-like (and hence invertible) and $x, y$ are $(1, a)$-respectively $(1, b)$-primitive. Thus $U$ is generated as an algebra by $a, a^{-1}, b, b^{-1}, x, y$ satisfying the above relations. Furthermore, being generated by group-likes and skew primitives, $U$ is a pointed Hopf algebra.

The commutation relations in Proposition 2 together with the relation $p^2 = q^2 + q^3$ imply that

$$(y + pb)^2 = (x + qa)^2 + (x + qa)^3.$$

Hence, by a slight abuse of notation, we hereby identify $s$ and $t$ with $x + qa \in U$ and $y + pb \in U$, respectively, and view $B$ as a subalgebra of $U$. We address the choice of the parameters $(p, q)$ and the connection to [10] in Section 2.3.
2.2. Reduction of the Hopf algebra. Our aim is to find minimal quotient Hopf algebras of $U$ which contain $B$ as a quantum homogeneous space. The Hopf algebra $A$ in Theorem 1 is obtained from $U$ by adding the relations $b^2 = a^3 = 1$ and $F^3 = 0$, where

$$F := xa + (r + 1)ax + \frac{r + 2}{3}(a - a^2).$$

That the former relation can be added is straightforward: as $x$ commutes with $b$ and $y$ commutes with $a$, the element $b^2 = a^3$ is a central group-like, and the PBW basis of $U$ presented in [10] shows that $B$ embeds into the resulting quotient.

The following lemma establishes the commutation relations between $F$ and the generators $x + qa, y + pb$ of $B$. This is used to show that the class of $F^3$ spans a Hopf ideal in $\tilde{A} := U/\langle a^3 - 1 \rangle$ and that this ideal intersects $B$ trivially. Hence, taking the quotient by this ideal yields a Hopf algebra which contains $B$. This is precisely the Hopf algebra $A$ from Theorem 1.

Lemma 3. In $\tilde{A} = U/\langle a^3 - 1 \rangle$, we have:

1. The (class of) $F$ satisfies
   $$aF = x^2Fa, \quad bF = Fb, \quad (y + pb)F = F(y + pb),$$
   $$(x + qa)F = rF(x + qa) + \frac{3q + 1}{3}(r + 2)aF + \frac{r - 1}{3}F + \frac{1}{3}(a - 1),$$
   and
   $$\Delta(F) = a \otimes F + F \otimes a^2.$$

2. The element $F^3 \in \tilde{A}$ is central and primitive.

3. The set
   $$\{a^ib^jF^l \mid i \in \{0, 1, 2\}, j \in \{0, 1\}, l \in \mathbb{N}\}$$
   is a right $B$-module basis of $\tilde{A}$.

Proof. The commutation relations and the coproduct of $F$ (and consequently those of $F^3$) are computed in a straightforward manner, yielding (1) and (2).

The basis in (3) of $\tilde{A}$ as a right $B$-module is derived from its $k$-linear monomial basis

$$\{a^ib^jF^l(x + qa)^m(y + pb)^n \mid i, n \in \{0, 1, 2\}, j \in \{0, 1\}, l, m \in \mathbb{N}\}.$$ 

That this is a basis follows from Bergman’s diamond lemma [2]. To apply the latter, we first derive a new presentation of $\tilde{A}$ in terms of generators $a, b, F, x + qa$ and $y + pb$ and consider the lexicographic monomial ordering with $a < b < F < x + qa < y + pb$. The complete
set of relations satisfied by these generators consists of: the relations in (1), the relations in Proposition [2] and the relations $a^3 = 1$ and $F = xa + (r + 1)ax + \frac{r+2}{3}(a - a^2)$, all rewritten in terms of the new generators. Then, one has to verify that there are no ambiguities in the relations. This follows by a straightforward computation, since written in the form of explicit reduction rules, those relations are:

$$ba = ab, \quad b^2 = a^3 = 1,$$

$$Fa = raF, \quad Fb = bF,$$

$$(x + qa)a = -(r + 1)a(x + qa) + F + \frac{r+2}{3}((1 + 3q)a^2 - a),$$

$$(x + qa)b = b(x + qa),$$

$$(x + qa)F = rF(x + qa) + \frac{1+3q}{3}(r + 2)aF + \frac{r-1}{3}F + \frac{1}{3}(a - 1),$$

$$(y + pb)a = a(y + pb), \quad (y + pb)b = -b(y + pb) + 2pb^2,$$

$$(y + pb)F = F(y + pb), \quad (y + pb)(x + qa) = (x + qa)(y + pb),$$

$$(y + pb)^2 = (x + qa)^2 + (x + qa)^3. \quad \Box$$

Remark 4. An alternative argument is to observe that the relations presented above also characterize $\hat{A}$ as a quotient of an iterated Ore extension. Starting from the group algebra of $\mathbb{Z}_3 \times \mathbb{Z}_2$ (generated by $a$ and $b$), one adds $F$ followed by $x + qa$ and $y + pb$, subject to skew-commutation relations given in $\mathcal{R}$. It is well-known that an Ore extension $R[t; \sigma, \delta]$ is the ring generated over $R$ by one additional generator $t$, subject to skew-commutation relations $ta = \sigma(a)t + \delta(a)$, for all $a \in R$, and that the monomials $t^i$ form a basis as an $R$-module (which is a particular case of Bergman’s diamond lemma), see e.g. [15] 1.2 for more details. Then $\hat{A}$ is simply the quotient of such an iterated Ore extension by the relation $(y + pb)^2 = (x + qa)^2 + (x + qa)^3$.

We also note, that compared to the proposition in [10] p. 657, besides the choice of another lexicographic order, the role of the independent variable $xa$ in the monomial basis is played by $F$.

We are now ready to prove the first parts of Theorem 1.

Proposition 5. The subalgebra $B \subseteq A$ is a right coideal, and we have:

1. The elements $a$, $b$ and $F$ generate a Hopf subalgebra $C \subseteq A$, such that $C \cong A/AB^+$ as coalgebras, where $B^+ := B \cap \ker \varepsilon$.
2. $A \cong C \otimes B$ as left $C$-comodules and right $B$-modules. In particular, $A$ is free, hence faithfully flat, as a $B$-module.
Proof. In view of (1) and (3) of Lemma 3, the elements $a$, $b$ and $F$ generate a Hopf subalgebra $C$ of $A$ with vector space basis
$$\{a^ib^lF^j \mid i, l \in \{0, 1, 2\}, j \in \{0, 1\}\}.$$ The elements $(x + qa - q)$ and $(y + pb - p)$ generate the left ideal $AB^+$. This is also a coideal, so there is a unique coalgebra structure on $A/AB^+$ for which the quotient map $A \to A/AB^+$ is a coalgebra map. Composing with the embedding $C \to A$ yields a coalgebra map $C \to A/AB^+$ which is easily seen to be bijective, using the basis from Lemma 3. This proves (1).

Multiplication in $A$ defines a right $B$-linear map $C \otimes B \to A$ which is bijective in view of Lemma 3(3). The identification $C \cong A/AB^+$ turns $A$ into a left $C$-comodule whose coaction $\lambda: A \to C \otimes A$ is the coproduct of $A$ followed by the quotient map on the first component. Since $B$ is a right coideal subalgebra, it is contained in
$$A^{\text{co}C} = \{b \in A \mid \lambda(b) = 1 \otimes b\},$$ which implies that the multiplication $C \otimes B \to A$ is left $C$-colinear. □

2.3. On the parameters $p$ and $q$. Note that in [10], the choice of $p$, $q$ is incorporated in the defining relations of the Hopf algebra. However, if $A(p,q)$ denotes the Hopf algebras defined there (with generators labelled by the subscript $(p, q)$), then
$$x(0,0) \mapsto x(p,q) - qa(p,q), \quad y(0,0) \mapsto y(p,q) - pb(p,q),$$
$$a(0,0) \mapsto a(p,q), \quad b(0,0) \mapsto b(p,q)$$
extends to Hopf algebra isomorphisms $A(0,0) \to A(p,q)$. Using these, we here embed the quantum homogeneous spaces studied in [10] all into $A(0,0)$. The latter is simply the universal Hopf algebra coacting in the given way on $B$ described in Proposition 2.

In this way, the elements $x + qa$ and $y + pb$ generate for any choice of $(p, q)$, with $p^2 = q^2 + q^3$, a right coideal subalgebra. As $A$-comodule algebras, these are all isomorphic to each other (and to the coordinate ring of the nodal cubic).

2.4. Relation to small quantum groups. Note that if we denote
$$E := xa - r ax + \frac{1 - r}{3}(a - a^2), \quad K := a^2,$$
then the following relations are satisfied
$$KE = r^2 EK, \quad [E, F] = \frac{K - K^2}{r - r^2}.$$
These are the defining relations of the quantum universal enveloping algebra $U_r(\mathfrak{sl}_2)$. Furthermore, observe that $x$ can be written in terms of $E$, $F$ and $K$ as follows

$$x = \frac{1 - r^2}{3} FK + \frac{1 - r}{3} EK + \frac{r - r^2}{3} (K^2 - K).$$

Thus, we obtain a presentation of $A$ as a quotient of $U_r(\mathfrak{sl}_2) \otimes U_{-1}(\mathfrak{sl}_2)$ (to be able to specify $U_q(\mathfrak{sl}_2)$ at $q = -1$, we refer to the non-restricted version using integral forms, see e.g. [6, §9.2]). The image of $U_r(\mathfrak{sl}_2) \otimes 1$ is the subalgebra generated by $a$ and $x$ and the image of $1 \otimes U_{-1}(\mathfrak{sl}_2)$ is the subalgebra generated by $b$ and $y$.

Adding the relation $E^3 = 0$ is equivalent to adding the relation $x^2 + x^3 = 0$ and yields the small quantum group $u_r(\mathfrak{sl}_2) \otimes u_{-1}(\mathfrak{sl}_2)$. Note, finally, that expressing $A$ in this way also reveals the Casimir element

$$\Omega := EF + \frac{r^2K + rK^2}{(r - r^2)^2} = (xa)^2 - a^2x - a^2x^2 + \frac{1}{3},$$

which is central.

2.5. Ring-theoretic properties of $A$. Several recent articles classify prime Hopf algebras of Gelfand-Kirillov dimension one under additional assumptions, see e.g. [4, 12, 18]. Answering a question of Ken Brown, we remark that the Hopf algebra $A$ does not fall into this class:

**Proposition 6.** The algebra $A$ is Noetherian of Gelfand-Kirillov dimension one, but neither regular nor semiprime.

**Proof.** The algebra $A$ is a finitely generated module over $B$, which is Noetherian and of Gelfand-Kirillov dimension one. Hence so is $A$ ([11, Proposition 5.5]).

As $B$ is the coordinate ring of a singular curve, evaluation in the singularity defines a $B$-module of infinite projective dimension. This is the restriction of the trivial $A$-module (the ground field on which $A$ acts via $\varepsilon$), which therefore has infinite projective dimension (using that $A$ is a free $B$-module).

The Casimir element $\Omega$ has minimal polynomial

$$t^3 - \frac{1}{3} t + \frac{2}{27} = \left( t - \frac{1}{3} \right)^2 \left( t + \frac{2}{3} \right).$$

Therefore the ideal $I$ generated by $(\Omega - \frac{1}{3}) (\Omega + \frac{2}{3})$ is nilpotent and $A$ is not semiprime. □
3. A variant for arbitrary $k$

In the case where the field $k$ does not contain a primitive third root of unity, the element $F \in \tilde{A}$ is no longer well-defined and the relation to the theory of small quantum groups is lost. However, in this section we modify the definition of $A$ and obtain a variant of Theorem 1 that holds for arbitrary fields.

3.1. The element $c$. The following analogue of Lemma 3 is straightforward and provides a substitute for the element $F^3$:

**Lemma 7.** In $\tilde{A}$, we define
\[
c := (xa)^3 + 2a(xa)^2x + a(xa)^2 - 3a^2(xa)x^2 - 2a^2(xa)x - (xa)x + ax^2 + ax - 2x^2 - 2x.
\]

Then we have:

1. The element $c$ is central and primitive.
2. The set
\[
\{a^ib^j(xa)^l c^m \mid i, l \in \{0, 1, 2\}, j \in \{0, 1\}, m \in \mathbb{N}\}
\]

is a right $B$-module basis of $\tilde{A}$.

Just as in the previous section, the quotient Hopf algebra $\tilde{A}/\tilde{A}c$ contains $B$ as right coideal subalgebra and thus provides a variant of $A$ defined for general ground fields. For the remainder of this section, $A$ denotes this quotient.

We note that the element $x^2 + x^3$ (which is the same as $y^2$) is another central and primitive element in $\tilde{A}$. Note that in the case where there is a primitive third root of unity $r$, we have the following relation
\[
c = F^3 + (3r - 6)(x^2 + x^3).
\]

3.2. The cleaving map. Unlike in the previous section, we do not have a Hopf subalgebra $C \subseteq A$ isomorphic to $A/AB^+$. We instead explicitly define a cleaving map, i.e., a map $A/AB^+ \to A$ which is left $A/AB^+$-colinear and convolution invertible. The existence of such a map is equivalent to $A$ being isomorphic as left $A/AB^+$-comodules and right $B$-modules to $A/AB^+ \otimes B$ (see [3, Proposition 2.3]).

**Proposition 8.** The subalgebra $B \subseteq A$ generated by $x + qa$ and $y + pb$ is a right coideal of $A$ and there exists a cleaving map $\gamma: A/AB^+ \to A$.

**Proof.** The first claim is immediate. Consider the canonical projection $\pi: A \to A/AB^+$. In view of Lemma 7, the set
\[
\{[a^ib^j(xa)^l] \mid i, l \in \{0, 1, 2\}, j \in \{0, 1\}\}
\]
is a \( k \)-linear basis of \( A/AB^+ \). We define a splitting \( \gamma : A/AB^+ \to A \) of \( \pi \) by
\[
\gamma([a^i b^j]) = a^i b^j, \quad \gamma([a^i b^j x a]) = a^i b^j x a, \\
\gamma([a^i b^j (x a)^2]) = a^i b^j ((x a)^2 - axa(x + qa - q)).
\]
It is straightforward to check that \( \gamma \) is left \( A/AB^+ \)-colinear on the above mentioned basis.

The coradical of \( A/AB^+ \) is spanned by the group-like elements. The restriction of \( \gamma \) to the coradical is convolution invertible, with convolution inverse given by \( [a^i b^j] \mapsto a^{-i} b^{-j} \). Hence, by [17, Proposition 6.2.2.], the map \( \gamma \) itself is convolution invertible. \( \square \)

Remark 9. The cleftness of \( A \) also follows from the abstract result [14, Theorem 1.3 (4)] of Masuoka, which also applies to \( \tilde{A} \).

4. Minimality of \( A \)

To prove the minimality of \( A \) as stated in Theorem 11(3), we need to describe its space of skew primitive elements. In this section, we resume the notation of Section 2, by denoting \( A = \tilde{A}/\tilde{A}F^3 \).

4.1. Skew primitives in \( \tilde{A} \) and \( A \). We start by proving the following lemma:

Lemma 10. Let \( H \) be a Hopf algebra, \( h \in H \) be a central primitive element and assume that \( H \) is free as a \( k[h] \)-module. Then the only skew primitive elements in the ideal \( Hh \) are the scalar multiples of \( h \).

Proof. By the hypothesis on \( H \), we can derive a vector space basis of \( H \) of the form \( \{v_j h^n\}_{j \in J, n \in \mathbb{N}} \), and we can assume without loss of generality that \( v_0 = 1 \).

Let \( a \in H \) be such that
\[
\Delta(a h) = 1 \otimes a h + a h \otimes g
\]
for some group-like element \( g \in H \). Using the vector space basis, there are unique \( w_{j n} \in H \) such that \( \Delta(a) = \sum_{j, n} v_j h^n \otimes w_{j n} \). Since \( \varepsilon(h) = 0 \), we have
\[
a = \sum_j \varepsilon(v_j) w_{j 0} = \sum_{j, n} \varepsilon(w_{j n}) v_j h^n
\]
and we obtain
\[
\sum_{j, n} v_j h^n \otimes w_{j n} h + v_j h^{n+1} \otimes w_{j n} = \Delta(a) \Delta(h) =
\]
\[
= \Delta(a h) = 1 \otimes a h + \sum_{j, n} \varepsilon(w_{j n}) v_j h^{n+1} \otimes g.
\]
By considering the first tensor component and \(n = 0\), one observes
\[
\sum_j v_j \otimes w_{j0} h = 1 \otimes ah
\]
which implies that \(w_{00} = a\) and \(w_{j0} = 0\), for \(j > 0\). We used the hypothesis that \(H\) is a free \(k[h]\)-module in the fact that \(h\) is not a zero divisor. Similarly, for all \(j, n \in \mathbb{N}\), one gets
\[
w_{jn+1} h = \varepsilon(w_{jn}) g - w_{jn}.
\]
It follows in particular that
\[
w_{jn} = 0, \quad \forall j > 0, n \geq 0.
\]
Thus \(a\) and hence \(ah\) are in fact elements of \(k[h]\):
\[
ah = \sum_n \varepsilon(w_{0n}) h^{n+1}.
\]
As \(k[h] \subseteq H\) is isomorphic to the universal enveloping algebra of the 1-dimensional Lie algebra, the only (skew) primitive elements are scalar multiples of \(h\).

We now describe the Yetter-Drinfel’d module of skew primitives in \(\tilde{A}\). Let \(P_{(1,g)}(\tilde{A})\) denote the vector space of \((1, g)\)-skew primitive elements, for a fixed group-like \(g \in \tilde{A}\). Since this always contains \(k(g - 1)\) as a subspace, we are interested in classifying the quotient spaces
\[
V_g(\tilde{A}) := P_{(1,g)} / k(g - 1),
\]
which is the goal the next proposition.

**Proposition 11.** The vector spaces \(V_1(\tilde{A}), V_a(\tilde{A})\) and \(V_b(\tilde{A})\) have linear bases given by
\[
V_1(\tilde{A}) : \{[x^2 + x^3], [F^3]\}, \quad V_a(\tilde{A}) : \{[x], [axa^2]\}, \quad V_b(\tilde{A}) : \{[y]\}.
\]
All other \(V_g(\tilde{A})\) are trivial.

**Proof.** Since \(x^2 + x^3\) and \(F^3\) are central and primitive, the quotient of \(\tilde{A}\) by the ideal generated by these two elements is a Hopf quotient \(D\), which is finite-dimensional. The image of any skew primitive element of \(\tilde{A}\) is skew primitive in \(D\). The group-like and skew primitive elements in \(D\) are determined by straightforward computation. The canonical projection identifies the group-likes with those of \(\tilde{A}\) itself. The skew primitives in the quotient are spanned by the residue classes of \(x, axa^2\) and \(y\), together with \(g - 1\), for \(g\) group-like.

It remains to determine the skew primitives contained in the ideal \(\tilde{A}(x^2 + x^3) + \tilde{A}F^3\). These are obtained by a two-fold application of Lemma 10, first with \(h = x^2 + x^3\) and subsequently with \(h = F^3\). □
When \( k \) contains a primitive third root of unity, Proposition \([11]\) of course follows as well from the presentation of \( \tilde{A} \) as a quotient of \( U_r(\mathfrak{sl}_2) \otimes U_{-1}(\mathfrak{sl}_2) \).

The computation in Proposition \([11]\) carries over nicely to \( A = \tilde{A}/\tilde{A}F^3 \). Define \( V_g(A) \) analogously as above.

**Corollary 12.** The vector spaces \( V_1(A), V_a(A) \) and \( V_b(A) \) have linear bases given by

\[
V_1(A) : \{[x^2 + x^3]\}, \quad V_a(A) : \{[x], [axa^2]\}, \quad V_b(A) : \{[y]\}.
\]

All other \( V_g(A) \) are trivial.

**Proof.** Using the basis from Lemma \([7]\) we obtain a splitting of \( \tilde{A} \to A \) as a coalgebra map, which embeds \( A \) as the subcoalgebra spanned over \( k \) by \([a^ib^j(x + qa)^r(y + pb)^s] | i, l \in \{0, 1, 2\}, j, s \in \{0, 1\}, r \in \mathbb{N}\]. Hence, \( A \) and \( \tilde{A} \) have the same group-like elements and under this embedding, \( P_{1,g}(A) = P_{1,g}(\tilde{A}) \cap A \).

4.2. **Proof of Theorem \([1]\)(3).** Now we can complete the proof of our main theorem.

**Proposition 13.** Assume \( \text{char}(k) \neq 2, B \subseteq A \) is as in Theorem \([2]\) and \( \rho : A \to H \) is a surjective Hopf algebra map with \( \ker \rho \cap B = 0 \). Then \( \rho \) is an isomorphism.

**Proof.** It is sufficient to verify that \( \rho \) is injective on each space \( P_{g,h}(A) \) for fixed group-like elements \( g, h \in A \); according to \([17, \text{Proposition 4.3.3}]\) (recall that \( A \) is pointed), it follows that \( \rho \) is injective, and thus an isomorphism.

First, the map \( \rho \) induces a group homomorphism \( \mathbb{Z}_3 \times \mathbb{Z}_2 \to G \), where \( G \) is the group of group-likes in \( H \). If the kernel of this homomorphism is a non-trivial subgroup, then it contains either \( a \) or \( b \). If \( \rho(b) = 1 \), then \( by + yb = 0 \) implies \( \rho(y) = 0 \) as \( \text{char}(k) \neq 2 \). Similarly, if \( \rho(a) = 1 \), then the commutation relations between \( a \) and \( x \) yield \( \rho(x) = 0 \). In both cases, we conclude that

\[
(y + pb)^2 - p^2 = y^2 = x^2 + x^3 \in \ker \rho \cap B = 0,
\]

a contradiction. Hence, the induced group homomorphism is injective. Since group-like elements are linearly independent, this means that the restriction of \( \rho \) to the coradical of \( A \) is injective (the coradical of \( A \) is simply the group algebra of \( \mathbb{Z}_3 \times \mathbb{Z}_2 \)).

Assume now that \( \rho(v) = 0 \), where \( v \) is \((g, h)\)-primitive. Without loss of generality, we assume \( g = 1 \) (as we can replace \( v \) by \( g^{-1}v \)). We now proceed on a case-by-case basis for possible values of \( h \), using the description of the spaces \( V_h(A) \) given by Proposition \([11]\).
For \( h \neq 1, a, b \), we have \( V_4(A) = 0 \) and thus \( P_{(1,1)} = k(h-1) \).
Therefore, \( \rho(v) = 0 \) implies \( v = 0 \), since we have already shown that \( \rho \)
is injective on the coradical of \( A \). For \( h = 1 \), we have \( P_{(1,1)} = k(x^2 + x^3) \),
and hence \( \rho(v) = 0 \) would contradict \( \ker \rho \cap B = 0 \), as seen above. For \( h = b \),
we have \( P_{(1,b)} = k(b-1) \oplus ky \), hence \( \rho(v) = 0 \) would imply \( \rho(y) = \lambda(\rho(b) - 1) \) for some \( \lambda \in k \) (otherwise \( \rho(b-1) = 0 \), which we have ruled out).
Thus combining this with \( \rho(yb + by) = 0 \), yields a contradiction. Similarly, for \( h = a \), we have \( P_{(1,a)} = k(a-1) \oplus kx \oplus k(axa^2) \).
Assume without loss of generality that \( \rho(axa^2) = \lambda(\rho(a) - 1) + \mu \rho(x) \), for some \( \lambda, \mu \in k \) (by conjugating with \( a \) if necessary).
Plugging this formula into \( xa^2 + axa + a^2x + a^2 - 1 = 0 = ax^2 + xax + x^2a + ax + xa \) again yields a contradiction.

Completely analogously one proves the minimality of the variants of \( A \) in which the relation \( F^3 = 0 \) is replaced by \( c + \lambda(x^2 + x^3) = 0 \), for \( \lambda \in k \) (such as the one considered in Section 4.1), and that these cover all minimal quotient Hopf algebras of \( U \) containing \( B \) as right coideal subalgebra.

4.3. The Nichols algebra of \( \tilde{A} \). From the point of view of the classification of pointed Hopf algebras, it is a natural task to describe the Nichols algebra of a given Hopf algebra, for which one uses the Yetter-Drinfel’d module of skew primitives \([1]\). For the sake of completeness, we present the Nichols algebra of \( \tilde{A} \) in this final subsection as a corollary of Section 4.1.

Let \( B(V) \) denote the Nichols algebra of a braided vector space \( (V, \chi) \).
If \( V = V' \oplus V'' \) as braided vector space, then \( B(V) \cong B(V') \otimes B(V'') \).
In particular, this applies to the case of \( V = V_1(\tilde{A}) \oplus V_a(\tilde{A}) \oplus V_b(\tilde{A}) \)
with respect to the Yetter-Drinfel’d braiding \( \chi \). This is the flip \( \tau \) in \( V_1(\tilde{A}) \) and \( -\tau \) in \( V_b(\tilde{A}) \), yielding a symmetric algebra and an exterior algebra as Nichols algebras for these components, respectively.

The only non-trivial component to discuss is \( V_a(\tilde{A}) \), especially when \( k \) does not contain a primitive third root of unity as the braiding on \( V_a(\tilde{A}) \) is then not of diagonal type: if \( u := [x] \), \( v := [axa^2] \), then the Yetter-Drinfel’d braiding is given by

\[
\begin{align*}
  u \otimes u &\mapsto v \otimes u, \quad v \otimes u &\mapsto v \otimes v, \\
  u \otimes v &\mapsto -u \otimes u - v \otimes u, \quad v \otimes v &\mapsto -u \otimes v - v \otimes v.
\end{align*}
\]

The Nichols algebra \( B(V_a(\tilde{A})) \) can be computed directly by determining the kernels of the quantum symmetrisers - it has the defining relations

\[
\begin{align*}
  u^3 & = v^3 = 0, \\
  u^2v + uvu + vu^2 & = u^2 + uv + v^2 = 0.
\end{align*}
\]
and is 9-dimensional with a basis given by
\[ \{1, u, v, u^2, vu, v^2, v^2u, v^2uv, vuv^2\} \].

In total, the Nichols algebra of \( \tilde{A} \) is the tensor product of a polynomial ring in two variables with an exterior algebra with one generator and with the Nichols algebra \( B(V_\alpha(\tilde{A})) \) described in the previous paragraph.

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