Infinite-Time Singularity Type of the Kähler–Ricci Flow

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Abstract For the Kähler–Ricci flow on a compact Kähler manifold with semi-ample canonical line bundle, we prove that the singularity type at infinity does not depend on the choice of the initial metric. We also provide new simple proofs for some existing classification results on infinite-time singularity type of the Kähler–Ricci flow.

Keywords Kähler–Ricci flow · Semi-ample canonical line bundle · Singularity type

Mathematics Subject Classification Primary 53C44

1 Introduction

We study the singularity type of long time solution of the Kähler–Ricci flow on compact Kähler manifolds. Thanks to the maximal existence time theorem of the Kähler–Ricci flow [3,17,20], the existence of long time solution is equivalent to nefness of the canonical line bundle. In this paper, motivated by the Abundance Conjecture (which predicts that if the canonical line bundle of an algebraic manifold is nef, then it is semi-ample), we will further restrict our discussions on \( n \)-dimensional Kähler manifold, denoted by \( X \), with semi-ample canonical line bundle \( K_X \), and so the Kodaira dimension \( \text{kod}(X) \in \{0, 1, \ldots, n\} \).

Given an arbitrary Kähler metric \( \omega_0 \) on \( X \), let \( \omega = \omega(t) \) \( t \in [0, \infty) \) be the solution to the Kähler–Ricci flow

\[
\partial_t \omega = -\text{Ric}(\omega) - \omega
\]
running from $\omega_0$. Recall from [9] that the infinite-time singularities of the Kähler–Ricci flow are divided into two types IIb and III. Precisely, we say a long time solution of the Kähler–Ricci flow (1.1) is of type IIb if
\[
\sup_{X \times [0, \infty)} |Rm(\omega(t))|_{\omega(t)} = \infty
\]
and of type III if
\[
\sup_{X \times [0, \infty)} |Rm(\omega(t))|_{\omega(t)} < \infty.
\]

There are nice results in the study of infinite-time singularity type of the Kähler–Ricci flow, see e.g., [4, 5, 8, 19]. In particular, for the Kähler–Ricci flow on Kähler manifold with semi-ample canonical line bundle, Tosatti-Zhang [19] classified the infinite-time singularity type for many cases. More precisely, if we let
\[
f : X \rightarrow X_{can} \subset \mathbb{C}P^N
\]
be the semi-ample fibration with connected fibers induced by pluricanonical system of $X$, where $X_{can}$, a $kod(X)$-dimensional irreducible normal projective variety, is the canonical model of $X$, and $V \subset X_{can}$ be the singular set of $X_{can}$ together with the critical values of $f$, and $X_y = f^{-1}(y)$ be a smooth fiber for $y \in X_{can} \setminus V$, then [19, Theorems 1.5, 1.6] reads

**Theorem 1.1** [19, Theorems 1.5, 1.6] Let $X$ be a compact Kähler manifold with semi-ample $K_X$ and consider a solution of the Kähler–Ricci flow (1.1).

(1) Suppose $kod(X) = 0$.
- If $X$ is a finite quotient of a torus, then the solution is of type III.
- If $K_X$ is not a finite quotient of a torus, then the solution is of type IIb.

(2) Suppose $kod(X) = n$.
- If $K_X$ is ample, then the solution is of type III.
- If $K_X$ is not ample, then the solution is of type IIb.

(3) Suppose $0 < kod(X) < n$.
- If $X_y$ is not a finite quotient of a torus, then the solution is of type IIb.
- If $X_y$ is a finite quotient of a torus and $V = \emptyset$, then the solution is of type III.

(4) Suppose $n = 2$ and $kod(X) = 1$, then the solution is of type III if and only if the only singular fibers on $f$ are of type $m I_0$, $m > 1$.

Note that if $n = dim(X) = 1$, then the conclusions in items (1) and (2) of Theorem 1.1 were already shown by Hamilton [8]; the first part of item (2) of Theorem 1.1 was contained in Cao [3] and Tsuji [20]; and in the second part of Theorem 1.1(3), if in particular $X_y$ is a torus and also $V = \emptyset$, then that conclusion was first proved by Fong-Zhang [4, Sect. 5] by assuming $X$ is projective and the initial Kähler class is rational, and the projectivity and rationality assumptions were removed by Hein–Tosatti in [10] (also see [5, 16] for certain special case, i.e., $X$ is a product). Moreover, when $n = 2$, since Abundance Conjecture holds for Kähler surfaces, we can just assume $K_X$ is nef.
and a complete classification of singularity type is given by combining items (1), (2), and (4) of Theorem 1.1.

As an immediate consequence of Theorem 1.1, one has

**Corollary 1.2** [19] In the cases covered by Theorem 1.1, the singularity type does not depend on the choice of the initial metric.

The classification given by Theorem 1.1 (and hence Corollary 1.2) is almost complete (for Kähler manifold with semi-ample canonical line bundle) but leaves open the case when

\[(\star) \ 0 < \kod(X) < n (n \geq 3) \text{ and the general fiber is a finite quotient of a torus and} \ V \neq \emptyset.\]

In general, it is expected (see [19, Section 1]) that the singularity type in case (\(\star\)) is also independent of the initial metric. More generally, a conjecture raised by Tosatti in [18, Conjecture 6.7] predicts

**Conjecture 1.3** [18, Conjecture 6.7] For the Kähler–Ricci flow on any compact Kähler manifold with nef canonical line bundle, the singularity type at infinity does not depend on the choice of the initial metric.

In this note, we shall partially conform the conjectures mentioned in the above last two paragraphs. Our main result can be stated as follows.

**Theorem 1.4** Let \(X\) be a compact Kähler manifold with semi-ample \(K_X\). Then the singularity type of the Kähler–Ricci flow (1.1) on \(X\) does not depend on the choice of the initial metric.

Note that the Abundance Conjecture holds for any 3-dimensional compact Kähler manifold (see [2]). Therefore, Theorem 1.4 implies the following

**Corollary 1.5** Let \(X\) be a 3-dimensional compact Kähler manifold with nef \(K_X\). Then the singularity type of the Kähler–Ricci flow (1.1) on \(X\) does not depend on the choice of the initial metric.

As we have seen from Corollary 1.2, in those cases covered by Theorem 1.1, our Theorem 1.4 is not new. In fact, Theorem 1.4 is only new in case (\(\star\)), for which a classification of singularity type is still open. However, we would like to point out that, while the original proof of Theorem 1.1 (and hence Corollary 1.2) in [19] is based on analysis on the blowup limits of the Kähler–Ricci flow and a nice observation which relates singularity type to an algebraic condition (i.e., existence of some special rational curves), our proof of Theorem 1.4 takes a totally different (and more elementary) approach, which does not involve the classification results, e.g., Theorem 1.1, on singularity type and hence also provides an alternative proof for Corollary 1.2. In fact, our arguments in this note will only use the maximum principle, and so are purely analytic.

Theorem 1.4 can help to classify the singularity type. For example, by applying Theorem 1.4, we can give a simple proof for item (1) of Theorem 1.1, see Example 4.1 in Section 4. Moreover, by the argument for Theorem 1.4, we will provide a simple proof for item (2) of Theorem 1.1. To be more precise, let us state the harder part as follows.
Theorem 1.6 (Second part of Theorem 1.1 (2)) On a compact Kähler manifold X with semi-ample K_X and kod(X) = n, if there exists a type III solution ω = ω(t) to the Kähler–Ricci flow (1.1), then K_X is ample.

Theorem 1.6 was first proved by Tosatti-Zhang [19], whose argument involves the existence of some special rational curves guaranteed by Moishezon [12] and Kawamata [11]. Later, an analytic proof for Theorem 1.6 was given in Guo [7, Theorem 1.2] by using Cheeger–Colding’s theory on Ricci limit space. In this note, we shall provide a simple analytic proof for Theorem 1.6, which will only involve the maximum principle argument, see Example 4.4 in Sect. 4 for more details.

In the remaining part of this note, we will recall some necessary properties of the Kähler–Ricci flow in Sect. 2 and then give a proof of Theorem 3.1 in Sect. 3. In Sect. 4, we will present some related examples, which in particular contain new simple proofs for items (1) and (2) of Theorem 1.1.

2 Properties of the Kähler–Ricci Flow

We collect some necessary properties of the Kähler–Ricci flow.

Let

\[ f : X \to X_{\text{can}} \subset \mathbb{CP}^N \]

be the semi-ample fibration with connected fibers induced by pluricanonical system, where \( X_{\text{can}} \), a \( \text{kod}(X) \)-dimensional irreducible normal projective variety, is the canonical model of \( X \). Let \( \chi \) be a multiple of Fubini–Study metric on \( \mathbb{CP}^N \) such that \( f^*\chi \) is a smooth semi-positive representative of \( -2\pi c_1(X) \) and \( \Omega \) a fixed smooth positive volume form on \( X \) with \( \sqrt{-1} \partial \bar{\partial} \log \Omega = f^*\chi \). Given an arbitrary Kähler metric \( \omega_0 \) on \( X \). We set \( \omega_t := e^{-t}\omega_0 + (1 - e^{-t})f^*\chi \) and reduce the Kähler–Ricci flow to the following parabolic complex Monge–Ampère equation of \( \varphi = \varphi(t) \),

\[ \partial_t \varphi = \log \frac{e^{(n-k)t}(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} - \varphi \]

with \( \varphi(0) = 0 \), where \( n = \text{dim}(X) \), \( k = \text{kod}(X) \) and \( \omega = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi \) is the solution to the Kähler–Ricci flow (1.1) with \( \omega(0) = \omega_0 \).

Lemma 2.1 [15, 23] There exists a uniform constant \( C \) such that on \( X \times [0, \infty) \),

\[ |\varphi| + |\partial_t \varphi| \leq C. \tag{2.1} \]

The (2.1) is proved in [23] when \( \text{kod}(X) = n \) and in [15, Sect. 2] when \( 0 \leq \text{kod}(X) \leq n \).

We will also need Shi's estimates [14] (also see [16, Theorem 2.14])

Lemma 2.2 Let \( \omega \) be a solution to the Kähler–Ricci (1.1) on \( X \) with type III singularity, then there exists a uniform constant \( C \) such that on \( X \times [0, \infty) \),

\[ |\varphi| + |\partial_t \varphi| \leq C. \]
Here \( g(t) \) is the Riemannian metric associated to \( \omega(t) \).

3 Proof of Theorem 1.4

We now give a proof of our main result Theorem 1.4. To this end, we will prove the following key observation, which implies Theorem 1.4 immediately and may be interesting in itself.

\[ \text{Theorem 3.1} \quad \text{Let} \ X \ \text{be a compact Kähler manifold with semi-ample} \ K_X. \ \text{Assume that there exists a Kähler metric} \ \tilde{\omega}_0 \ \text{such that the Kähler–Ricci flow} \ (1.1) \ \text{running from} \ \tilde{\omega}_0 \ \text{develops type III singularity. Then the Kähler–Ricci flow} \ (1.1) \ \text{running form any} \ Kähler \ \text{metric on} \ X \ \text{develops type III singularity.} \]

To prove Theorem 3.1, the basic idea is to compare two solutions of the Kähler–Ricci flow by using the Schwarz Lemma arguments and then to bound the curvature by using the maximum principle.

\[ \text{Proof of Theorem 3.1} \quad \text{Assume we are given a long time solution} \ \tilde{\omega} = \tilde{\omega}(t) \ \text{of the Kähler–Ricci flow} \ (1.1) \ \text{running from a Kähler metric} \ \tilde{\omega}_0 \ \text{on} \ X, \ \text{which is of type III singulary, i.e., for some uniform constant} \ C \geq 1, \]

\[ \sup_{X \times [0, \infty)} |Rm(\tilde{\omega}(t))|_{\tilde{\omega}(t)} \leq C. \quad (3.1) \]

Define \( \tilde{\omega}_t, \tilde{\phi} \) in the same manner as in last section (see Lemma 2.1).

Let \( \omega_0 \) be an arbitrary Kähler metric on \( X \) and \( \omega \) the solution of the Kähler–Ricci flow (1.1) running from \( \omega_0 \). We now show that \( \omega \) is of type III singulary.

In the following, we use \( g_{i\bar{j}}, R_{i\bar{j}k\bar{l}}, R_{i\bar{j}}, \nabla, \Gamma^i_{jk} \), etc. to denote the local components, curvature, connection, Christoffel symbols, etc. of \( \omega \) and \( \tilde{g}_{i\bar{j}}, \tilde{R}_{i\bar{j}k\bar{l}}, \tilde{R}_{i\bar{j}}, \tilde{\nabla}, \tilde{\Gamma}^i_{jk} \), etc. to denote those of \( \tilde{\omega} \).

Firstly, by a direct computation we have

\[ \Delta_\omega tr_\omega \tilde{\omega} = g^{bp} \tilde{g}^a_{p\bar{q}} \tilde{g}_{b\bar{q}} R_{ab} - g^{ij} \tilde{g}^p_{p\bar{q}} \tilde{R}_{ij\bar{p}\bar{q}} + g^{ij} \tilde{g}^{p\bar{q}} \tilde{g}^{ba} \nabla_i \tilde{g}_{p\bar{b}} \nabla_j \tilde{g}_{a\bar{q}}. \quad (3.2) \]

On the other hand,

\[ \partial_t tr_\omega \tilde{\omega} = \partial_t (g^{ij} \tilde{g}_{ij}) \]
\[ = -g^{jp} \tilde{g}^{qi} \tilde{g}_{ij} (-R_{p\bar{q}} - g_{p\bar{q}}) + g^{ij} (-\tilde{R}_{ij} - \tilde{g}_{ij}) \]
\[ = g^{jp} \tilde{g}^{qi} \tilde{g}_{ij} R_{p\bar{q}} - tr_\omega (Ric(\tilde{\omega})). \quad (3.3) \]

By combining (3.2) and (3.3), we get the evolution of \( tr_\omega \tilde{\omega} \):

\[ (\partial_t - \Delta_\omega) tr_\omega \tilde{\omega} = -tr_\omega (Ric(\tilde{\omega})) + g^{ij} \tilde{g}^{p\bar{q}} \tilde{R}_{ij\bar{p}\bar{q}} - g^{ij} \tilde{g}^{p\bar{q}} \tilde{g}^{ba} \nabla_i \tilde{g}_{p\bar{b}} \nabla_j \tilde{g}_{a\bar{q}}. \]

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By assumption, the curvature of $\tilde{\omega}$ is uniformly bounded (see (3.1)), so we can find some uniform constant $C \geq 1$ such that

$$(\partial_t - \Delta_\omega)tr_\omega \tilde{\omega} \leq Ctr_\omega \tilde{\omega} + C(tr_\omega \tilde{\omega})^2 - g^{ji}g^{p\bar{q}}g^{ba}\nabla_i g_{p\bar{b}}\nabla_j \tilde{g}_{a\bar{q}}.$$  (3.4)

Consequently, there exists two uniform positive constants $C_0$ and $C$ such that (see [3,21])

$$(\partial_t - \Delta_\omega)\log tr_\omega \tilde{\omega} \leq C_0tr_\omega \tilde{\omega} + C.$$  (3.5)

On the other hand, by Lemma 2.1 we have

$$(\partial_t - \Delta_\omega)\tilde{\phi} = \partial_t \tilde{\phi} - tr_\omega \tilde{\omega} + tr_\omega \tilde{\omega}_t \leq -tr_\omega \tilde{\omega} + tr_\omega \tilde{\omega}_t + C$$

and

$$(\partial_t - \Delta_\omega)\phi = \partial_t \phi - n + tr_\omega \omega_t \geq tr_\omega \omega_t - C.$$  (3.6)

Now, for the constant $C_0$ in (3.5), we fix a constant $A \geq C_0 + 1$ such that

$$(C_0 + 1)\tilde{\omega}_0 \leq A\omega_0$$

and so

$$(C_0 + 1)\tilde{\omega}_t \leq A\omega_t.$$  (3.7)

Then we get

$$(\partial_t - \Delta_\omega)((C_0 + 1)\tilde{\phi} - A\phi) \leq -(C_0 + 1)tr_\omega \tilde{\omega} + C$$

and hence,

$$(\partial_t - \Delta_\omega)(\log tr_\omega \tilde{\omega} + (C_0 + 1)\tilde{\phi} - A\phi) \leq -tr_\omega \tilde{\omega} + C$$

By applying the maximum principle to (3.6) and Lemma 2.1 we have, for some uniform constant $C \geq 1$,

$$tr_\omega \tilde{\omega} \leq C$$

on $X \times [0, \infty)$. Note that Lemma 2.1 implies that one can find a constant $C \geq 1$ such that

$$C^{-1}\tilde{\omega}^n \leq \omega^n \leq C\tilde{\omega}^n$$

on $X \times [0, \infty)$. So,

$$tr_\omega \omega \leq \frac{1}{(n-1)!}(tr_\omega \tilde{\omega})^{n-1}\frac{\omega^n}{\tilde{\omega}^n} \leq C.$$  (3.7)
Consequently, we have a uniform constant $C \geq 1$ such that
\begin{equation}
C^{-1} \tilde{\omega} \leq \omega \leq C \tilde{\omega}.
\end{equation}
on $X \times [0, \infty)$.

Combining (3.8) and (3.1), we can modify the well-known arguments for the Kähler–Ricci flow to bound the curvature of $\omega$ (see e.g., [3, 13, 16]). For completeness, we present some details here. Firstly, define a tensor $\Psi = (\Psi_{ij})$ by $\Psi_{ij} := \Gamma_{ij} - \tilde{\Gamma}_{ij}$ and $S = |\Psi|_{\omega}^2$. Then
\begin{equation}
S = g^{ji}g^{lk}g^{qp}\tilde{\nabla}_i g_{j\bar{k}}\nabla_j g_{l\bar{p}}.
\end{equation}

By (3.8) we find that the last term in (3.4)
\begin{equation}
g^{ji}g^{lp}g^{ba}\nabla_i g_{j\bar{p}}\nabla_j g_{a\bar{q}}
= g^{ji}g^{lp}g^{ba}(\nabla_i - \tilde{\nabla}_i)g_{j\bar{p}}(\nabla_j - \tilde{\nabla}_j)g_{a\bar{q}}
\geq C^{-1}S.
\end{equation}

Then we easily see from (3.4) that, for some uniform constant $C \geq 1$,
\begin{equation}
(\partial_t - \Delta_\omega)tr_\omega \tilde{\omega} \leq C - C^{-1}S.
\end{equation}

We need to compute the evolution of $S$. Firstly, recall [16, (2.38)]:
\begin{equation}
\Delta_\omega S = 2Re\left( g^{ji}g^{lp}g^{ba}\Delta_\omega \Psi_{ij}^k \Psi_{lj}^l + |\nabla \Psi|_{\omega}^2 + |\tilde{\nabla} \Psi|_{\omega}^2 \\
+ R^{ji}g^{lp}g_{kl}(\Psi_{ij}^k \Psi_{lj}^l) + g^{ji}R^{lp}g_{kl}\Psi_{ij}^k \Psi_{lj}^l - g^{ji}g^{lp}R_{kl}\Psi_{ij}^k \Psi_{lj}^l \right).
\end{equation}

Secondly, we have
\begin{equation}
\partial_t \Psi_{ij}^k = \partial_t \Gamma_{ij}^k - \partial_t \tilde{\Gamma}_{ij}^k = -\nabla_i R_{lp}^k + \tilde{\nabla}_i \tilde{R}_{lp}^k.
\end{equation}

Note that in (3.12) both $\partial_t \Gamma_{ij}^k$ and $\partial_t \tilde{\Gamma}_{ij}^k$ are tensors, we can compute them in normal coordinates with respect to $\omega$ and $\tilde{\omega}$, respectively.

Recall [16, (2.43), (2.44)]:
\begin{equation}
\nabla_b \Psi_{ij}^k = \tilde{\nabla}_b \tilde{\Gamma}_{ij}^k = \tilde{\nabla}_b \tilde{\nabla}_i R_{lp}^k.
\end{equation}

and
\begin{equation}
\Delta_\omega \Psi_{ij}^k = \nabla_b \tilde{\nabla}_b \tilde{\Gamma}_{ij}^k - \nabla_i R_{lp}^k.
\end{equation}

Combining (3.12) and (3.14) gives
\begin{equation}
\partial_t \Psi_{ij}^k = \Delta_\omega \Psi_{ij}^k + \tilde{\nabla}_i \tilde{R}_{lp}^k - \nabla_b \tilde{R}_{ibp}^k.
\end{equation}
and hence
\[
\partial_t S = \partial_t |\Psi|^2_\omega
= 2 \text{Re} \left( g^{ji} \tilde{g}^p g_{k\bar{l}} \left( \Delta_\omega \Psi^k_{ip} + \tilde{\nabla}_i \tilde{R}^k_p - \nabla^k \tilde{R}_{ibp} \right) \bar{\Psi}^l_{jq} \right) + S
+ R^{ji} \tilde{g}^p g_{k\bar{l}} \Psi^k_{ip} \bar{\Psi}^l_{jq} + g^{ji} \tilde{R}^p g_{k\bar{l}} \Psi^k_{ip} \bar{\Psi}^l_{jq} - g^{ji} \tilde{g}^p R_{k\bar{l}} \Psi^k_{ip} \bar{\Psi}^l_{jq}. 
\]  
(3.16)

It follows from (3.11) and (3.16) that
\[
(\partial_t - \Delta_\omega) S = S - |\nabla \Psi|^2_\omega - |\bar{\nabla} \Psi|^2_\omega + 2 \text{Re} \left( g^{ji} \tilde{g}^p g_{k\bar{l}} \left( \tilde{\nabla}_i \tilde{R}^k_p - \nabla^k \tilde{R}_{ibp} \right) \bar{\Psi}^l_{jq} \right).
\]  
(3.17)

Note that, by Lemma 2.2 and (3.8), \( \tilde{\nabla}_i \tilde{R}^k_p \) is uniformly bounded with respect to \( \omega \). Moreover,
\[
\nabla^k \tilde{R}_{ibp} = g^{ka} \nabla_a \tilde{R}_{ibp}
= g^{ka} (\nabla_a - \tilde{\nabla}_a) \tilde{R}_{ibp} + g^{ka} \tilde{\nabla}_a \tilde{R}_{ibp}
= \Psi * \text{Rm}(\tilde{\omega}) + g^{ka} \tilde{\nabla}_a \tilde{R}_{ibp}.
\]  
(3.18)

where \( \Psi * \text{Rm}(\tilde{\omega}) \) means some linear combination of products of \( \Psi \) and \( \text{Rm}(\tilde{\omega}) \) by contraction using \( \omega \). Then, combining Lemma 2.2, (3.8), Cauchy inequality and above facts, we conclude from (3.17) that, for some uniform constant \( C \geq 1 \),
\[
(\partial_t - \Delta_\omega) S \leq CS + C - |\nabla \Psi|^2_\omega - |\bar{\nabla} \Psi|^2_\omega. 
\]  
(3.19)

Using (3.10) and (3.19), we find a sufficiently large constant \( A \) such that, for some constant \( C \geq 1 \),
\[
(\partial_t - \Delta_\omega) (S + A \text{tr}_\omega \tilde{\omega}) \leq -S + C. 
\]  
(3.20)

By applying the maximum principle to (3.20), we find a constant \( C \geq 1 \) such that
\[
S \leq C.
\]  
(3.21)

Then, by noting from (3.13), (3.1), and (3.8) that
\[
|\tilde{\nabla} \Psi|^2_\omega = \left| \tilde{R}^k_{ibp} - R^k_{ibp} \right|^2_\omega \geq \frac{1}{2} |\text{Rm}(\omega)|^2_\omega - C,
\]  
(3.22)

we know that (3.19) implies
\[
(\partial_t - \Delta_\omega) S \leq C - \frac{1}{2} |\text{Rm}(\omega)|^2_\omega. 
\]  
(3.23)
Recall [16, (2.61)], for points where $|Rm(\omega)|_\omega > 0$,

$$(\partial_t - \Delta_\omega)|Rm(\omega)|_\omega \leq C|Rm(\omega)|_\omega^2 - \frac{1}{2}|Rm(\omega)|_\omega. \quad (3.24)$$

Therefore, for sufficiently large constant $A$, we have

$$(\partial_t - \Delta_\omega)(|Rm(\omega)|_\omega + AS) \leq -\frac{1}{2}|Rm(\omega)|_\omega + C$$

and hence, by maximum principle, we have a constant $C \geq 1$ such that

$$\sup_{X \times [0, \infty)} |Rm(\omega)|_\omega \leq C.$$

Theorem 3.1 is proved.

4 Examples

Given Theorem 1.4, we can check the singularity type if the Kähler–Ricci flow admits some special solutions. We show several examples as follows.

Example 4.1 (An alternative proof for item (1) of Theorem 1.1) Let $X$ be a Calabi–Yau manifold. Thanks to Yau [21], we can fix a Ricci-flat metric $\omega_{CY}$ on $X$. Then it is easy to see that $\omega(t) = e^{-t}\omega_{CY}$ is the solution to the Kähler–Ricci flow (1.1) with initial metric $\omega(0) = \omega_{CY}$. Then

$$|Rm(\omega(t))|_{\omega(t)} = e^t|Rm(\omega_{CY})|_{\omega_{CY}}$$

and hence, $\omega(t)$ develops type III singularity, i.e.,

$$\sup_{X \times [0, \infty)} |Rm(\omega(t))|_{\omega(t)} < \infty,$$

if and only if $\omega_{CY}$ is flat. Therefore, by Theorem 1.4, the Kähler–Ricci flow running from any Kähler metric on $X$ develops type III singularity if and only if there exists a flat metric on $X$, and so if and only if $X$ is a finite quotient of a torus. This is exactly item (1) of Theorem 1.1.

Next, we reprove item (2) of Theorem 1.1 by using the arguments in this note. To this end, we first give a general lemma.

Lemma 4.2 Let $X$ be a compact Kähler manifold with semi-ample $K_X$. Assume there exists a Kähler metric $\hat{\omega}$ such that the Kähler–Ricci flow (1.1), $\omega = \omega(t)$, running from $\hat{\omega}$ develops type III singularity. Then there exists a uniform positive constant $C$ such that, on $X \times [0, \infty)$, we have

$$\omega \leq C\hat{\omega}.$$
Proof The proof is similar to arguments in Sect. 3. First, we have

\[ \Delta_\hat{\omega}tr_\hat{\omega}\omega = g^{bp}\tilde{g}^{qa}g_{p\bar{q}}\hat{R}_{\bar{a}b} - g^{ji}\tilde{g}^{\bar{a}p}R_{i\bar{j}p\bar{q}} + g^{ji}\tilde{g}^{\bar{a}p}g^{ba}\hat{\nabla}_i g_{\bar{p}b}\hat{\nabla}_{j}\tilde{g}^{\bar{a}q} \]

and

\[ \partial_t tr_\hat{\omega}\omega = -tr_\hat{\omega}(Ric(\omega)) - tr_\hat{\omega}\omega. \]

Combining the above two equations gives

\[ (\partial_t - \Delta_\hat{\omega})tr_\hat{\omega}\omega = -tr_\hat{\omega}(Ric(\omega)) - tr_\hat{\omega}\omega - g^{bp}\tilde{g}^{qa}g_{p\bar{q}}\hat{R}_{\bar{a}b} + g^{ji}\tilde{g}^{\bar{a}p}R_{i\bar{j}p\bar{q}} - g^{ji}\tilde{g}^{\bar{a}p}g^{ba}\hat{\nabla}_i g_{\bar{p}b}\hat{\nabla}_{j}\tilde{g}^{\bar{a}q}. \]

Since \( \hat{\omega} \) is a fixed metric and \( \omega \) is of type III singularity, one easily finds a uniform constant \( C \geq 1 \) such that

\[ (\partial_t - \Delta_\hat{\omega})tr_\hat{\omega}\omega \leq Ctr_\hat{\omega}\omega + C(\partial_t - \Delta_\hat{\omega})\log tr_\hat{\omega}\omega \leq Ctr_\hat{\omega}\omega + C. \]

On the other hand, if we define \( \hat{\omega}_t = e^{-t}\hat{\omega} + (1 - e^{-t})f^*\chi, \hat{\phi} = \hat{\phi}(t) \) as in Sect. 2, then we have, for some uniform constant \( C \geq 1 \),

\[ (\partial_t - \Delta_\hat{\omega})\hat{\phi} = \partial_t\hat{\phi} - tr_\hat{\omega}\omega + tr_\hat{\omega}\hat{\omega}_t \leq -tr_\hat{\omega}\omega + C, \]

where we have used Lemma 2.1 and the fact that, for some uniform constant \( C \geq 1, \hat{\omega}_t \leq C\hat{\omega} \) on \( X \times [0, \infty) \). Then, as in Sect. 3, we can choose a sufficiently large constant \( A \) such that

\[ (\partial_t - \Delta_\hat{\omega})(\log tr_\hat{\omega}\omega + A\hat{\phi}) \leq -tr_\hat{\omega}\omega + C. \]

Now, by the maximum principle we see that

\[ tr_\hat{\omega}\omega \leq C. \]

Lemma 4.2 is proved.

Remark 4.3 An immediate consequence of Lemma 4.2 is that, under the assumption of Lemma 4.2, the diameter along the flow is uniformly bounded from above. Then, since type III singularity implies uniform bound for Ricci curvature, we can conclude some results on the Gromov–Hausdorff convergence of the flow. For example, by using arguments in [6] (also see [22, Sect. 4]) we can conclude that every Gromov–Hausdorff limit \( (M, d_M) \) along the flow (i.e., there exists a time sequence \( t_i \to \infty \) with \( (X, \omega(t_i)) \to (M, d_M) \) in Gromov–Hausdorff topology) contains an open dense subset homeomorphic to \( X_{can} \setminus V \). Here \( X_{can} \) and \( V \) are the same as in Sect. 1.
We are ready to give the second example.

Example 4.4 (An alternative proof of item (2) of Theorem 1.1) For the first part, assume $X$ be a compact Kähler manifold with ample $K_X$. Thanks to Aubin [1] and Yau [21], there exists a unique Kähler–Einstein metric $\omega_{KE}$ satisfying

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}.$$ 

Then it is easy to see that $\omega(t) \equiv \omega_{KE}$ is the solution to the Kähler–Ricci flow (1.1) with initial metric $\omega_{KE}$. Obviously, this solution develops type III singularity. Then, by Theorem 1.4, the Kähler–Ricci flow running from any Kähler metric on $X$ develops type III singularity. This is exactly the first part of item (2) of Theorem 1.1, which was first proved by Cao [3] and Tsuji [20].

Now, let us look at the second part. We need to show that, on compact Kähler manifold $X$ with semi-ample $K_X$ and $\text{kod}(X) = n$, if there exists a solution $\omega = \omega(t)$ to the Kähler–Ricci flow (1.1) with type III singularity, then $K_X$ is ample. To see this, we first apply Lemma 4.2 to see that $\omega(t) \leq C\hat{\omega}$ for some fixed Kähler metric $\hat{\omega}$ on $X$ and constant $C \geq 1$. On the other hand, since $\text{kod}(X) = n$, Lemma 2.1 gives

$$\omega(t)^n \geq C^{-1}\hat{\omega}^n,$$

and so, as in (3.7) we have

$$\text{tr}_{\omega(t)\hat{\omega}} \leq C$$

for some uniform constant $C \geq 1$. In conclusion, we have proved that, for some uniform constant $C \geq 1$, one has

$$C^{-1}\hat{\omega} \leq \omega \leq C\hat{\omega}. \quad (4.2)$$

Having (4.2), one can use the well-known arguments in the Kähler–Ricci flow (see e.g., [3, 13, 16]) to obtain higher order estimates of $\omega(t)$ and then, by the Arzela–Ascoli theorem we can choose a time sequence $t_i \to \infty$ such that $\omega(t_i) \to \omega_\infty$ in $C^\infty(X, \hat{\omega})$-topology, where $\omega_\infty$ is a smooth Kähler metric on $X$ and $\omega_\infty \in -2\pi c_1(X)$. Therefore, having a smooth positive representative, $K_X$ is ample by Kodaira embedding theorem. The proof is completed.

The last example is a special case of item (3) of Theorem 1.1.

Example 4.5 Let $Y$ be a Calabi–Yau manifold and $Z$ a compact Kähler manifold with ample $K_Y$. Set $X = Y \times Z$. It was first proved in [5, 16] (also see [4, 10]) that the Kähler–Ricci flow running from any Kähler metric on $X$ develops type III singularity if $Y$ is a torus. Moreover, according to item (3) of Theorem 1.1, the Kähler–Ricci
flow on $X$ develops type III singularity if and only if $Y$ is a finite quotient of a torus. We can apply our Theorem 1.4 to reprove this result. Indeed, as before, we only need to check some special solution. Precisely, if we let $\omega_X = \omega_Y + \omega_Z$, where $\omega_Y$ is a Calabi–Yau metric on $Y$ and $\omega_Z$ is the Kähler–Einstein metric on $Z$, then the solution of the Kähler–Ricci flow (1.1) running from $\omega_X$ is given by

$$\omega(t) = e^{-t} \omega_Y + \omega_Z.$$ 

Then we have

$$|Rm(\omega(t))|_{\omega(t)}^2 = e^{2t} |Rm(\omega_Y)|_{\omega_Y}^2 + |Rm(\omega_Z)|_{\omega_Z}^2,$$

from which we see that $\omega(t)$ develops type III singularity if and only if $\omega_Y$ is flat. Therefore, by Theorem 1.4, the Kähler–Ricci flow (1.1) running from any Kähler metric on $X$ develops type III singularity if and only if there exists a flat metric on $Y$, and so if and only if $Y$ is a finite quotient of a torus.

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