Energy transfer in coupled nonlinear phononic waveguides: transition from wandering breather to nonlinear self-trapping

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Abstract. We consider, both analytically and numerically, the dynamics of stationary and slowly-moving breathers (localized short-wavelength excitations) in two weakly coupled nonlinear oscillator chains (nonlinear phononic waveguides). We show that there are two qualitatively different dynamical regimes of the coupled breathers: the oscillatory exchange of the low-amplitude breather between the phononic waveguides (wandering breather), and one-waveguide-localization (nonlinear self-trapping) of the high-amplitude breather. We also show that phase-coherent dynamics of the coupled breathers in two weakly linked nonlinear phononic waveguides has a profound analogy, and is described by a similar pair of equations, to the tunnelling quantum dynamics of two weakly linked Bose-Einstein condensates in a symmetric double-well potential (single bosonic Josephson junction). The exchange of phonon energy and excitations between the coupled phononic waveguides takes on the role which the exchange of atoms via quantum tunnelling plays in the case of the coupled condensates. On the basis of this analogy, we predict a new tunnelling mode of the coupled Bose-Einstein condensates in a single bosonic Josephson junction in which their relative phase oscillates around $\pi/2$. The dynamics of relative phase of two weakly linked Bose-Einstein condensates can be studied by means of interference, while the dynamics of the exchange of lattice excitations in coupled nonlinear phononic waveguides can be observed by means of light scattering.

1. Introduction

Nonlinear excitations (solitons, kink-solitons, intrinsically localized modes and breathers) can be created most easily in low-dimensional (1D and quasi-1D) systems [1-10]. We consider two linearly coupled nonlinear oscillator chains (with unit lattice period). We model the chains with the Fermi-Pasta-Ulam ($\beta$-FPU) Hamiltonian, which is one of the most simple and universal models of nonlinear lattices and which can be applied to a diverse range of physical problems [11]:

$$\mathcal{H} = \sum_n \sum_{i=1}^2 \frac{1}{2} p_{n}^{(i)2} + \frac{1}{2} f^{(i)} (u_{n+1}^{(i)} - u_{n}^{(i)})^2$$

$$+ \frac{1}{4} \beta (u_{n+1}^{(i)} - u_{n}^{(i)})^4 + \frac{1}{2} C (u_{n}^{(1)} - u_{n}^{(2)})^2. \tag{1}$$
Here $u^{(i)}_n$ is displacement of the $n$-th particle from its equilibrium position in the $i$-th chain, $p^{(i)}_n = \dot{u}^{(i)}_n$ is particle momentum, $l^{(i)}$, $\beta$ and $C$ are, respectively, intra-chain linear, nonlinear and inter-chain linear force constants. We assume that the coupling is weak, $C \ll l^{(i)}$, and do not include the nonlinear inter-chain interaction. The $\beta$-FPU Hamiltonian (1) describes, in particular, purely transverse particle motion [6,7]. Torsion dynamics of DNA double helix can also be approximated by Hamiltonian (1) [12].

2. Discussion

We are interested in the dynamics of high-frequency and therefore short-wavelength excitations of the coupled chains, with the wave numbers $k \sim \pi$ close to the Brillouin-zone boundary, when the displacements of the nearest-neighbor particles are mainly anti-phase. For this case we introduce scalar continuous envelope-functions for the particle displacements in the chains, $f^{(i)}_n = u^{(i)}_n (-1)^n$, $f^{(i)}_0 \equiv f(x)_i$, where $i = 1, 2$. The envelope functions $f(x)_i$ are assumed to be slowly varying on the interatomic scale, $\partial f_i / \partial x \ll f_i$, which allows one to write nonlinear partial differential equations for $f_i$, cf., e.g., Refs. [2,6,13]. Making in Eqs. (1) an expansion of the differences $u^{(i)}_{n+1} - u^{(i)}_n$ up to the second order, we get the following equations for $f_i$, $i = 1, 2$: 

$$
\ddot{f}_i + \omega_{mi}^2 f_i + \frac{\partial^2 f_i}{\partial x^2} + 6\beta f_i^3 - Cf_{3-i} = 0,
$$

where $\omega_{mi} = \sqrt{4l^{(i)} + C}$ is characteristic frequency slightly above the maximal phonon frequency in the $i$-th isolated chain.

In order to deal with the amplitude and phase of the coupled nonlinear excitations, it is useful to introduce complex wave fields $\Psi(x, t)_i$ for each chain, cf. Ref. [13]:

$$f(x, t)_i = \frac{1}{2} [\Psi(x, t)_i + \Psi(x, t)_i^*].$$

Then in resonance approximation we get from Eqs. (2) and (3) the following coupled equations for $\Psi(x, t)_i$, $i = 1, 2$:

$$
\frac{1}{2} \left( \frac{\partial^2 \Psi_i}{\partial t^2} + \frac{\partial^2 \Psi_i}{\partial x^2} + \omega_{mi}^2 \Psi_i \right) + 6\beta |\Psi_i|^2 \Psi_i = \frac{C}{2} \Psi_{3-i}.
$$

(and complex-conjugated equations for $\Psi_i^*$). This approximation, in which we neglect the higher harmonics, is valid for the dispersive system under consideration due to weakness of nonresonant interaction between the mode with fundamental frequency and its third harmonic, cf. [6,13].

To describe a slowly-moving breather, wandering between two weakly coupled nonlinear chains with positive (repulsive) anharmonic force constant $\beta$, we assume the following form for the complex fields $\Psi_1$ and $\Psi_2$:

$$
\Psi_1 = \Psi_{max} \frac{\exp[i(kx - \omega t)]}{\cosh[\lambda_1(x - Vt)]} \cos \Theta \exp(-i \frac{\Delta}{2}),
$$

$$
\Psi_2 = \Psi_{max} \frac{\exp[i(kx - \omega t)]}{\cosh[\lambda_2(x - Vt)]} \sin \Theta \exp(i \frac{\Delta}{2}),
$$

where $\omega, V \ll 1$ and $k \ll 1$ are, respectively, frequency, slow velocity and small wavenumber related with the moving breather, $\lambda_1, \lambda_2$ describe inverse localization lengths; $\Delta = \Delta(t - kx/\omega)$ stands for the relative phase of the coupled chains, while the parameter $\Theta = \Theta(t - kx/\omega)$ describes the population imbalance of the two chains $Z = (n_1 - n_2)/(n_1 + n_2) = \cos 2\Theta$, where $n_i = |\Psi_i|^2$ is local density of excitations in the $i$-th chain, and $n_1 + n_2 = \Psi_{max}^2 = \text{const.}$
Using Eqs. (4), (5) and (6), after some algebra we obtain dispersion equations for the introduced parameters,
\begin{equation}
\omega^2 = \frac{1}{2}(\omega_{m1}^2 + \omega_{m2}^2) + 3\beta\Psi_{max}^2 - k^2 - C \frac{\cos \Delta}{\sin(2\Theta)},
\end{equation}
\begin{equation}
\lambda_1^2 = 6\beta\Psi_{max}^2 \cos^2 \Theta, \lambda_2^2 = 6\beta\Psi_{max}^2 \sin^2 \Theta, \quad V = \frac{\partial \omega}{\partial k},
\end{equation}
and evolution equations for \(\Theta\) and \(\Delta\):
\begin{equation}
\dot{\Theta} = \frac{C}{2\omega} \sin \Delta,
\end{equation}
\begin{equation}
\dot{\Delta} = \frac{1}{2\omega}(\omega_{m1}^2 - \omega_{m2}^2) + \frac{3\beta\Psi_{max}^2}{\omega} \cos(2\Theta) + \frac{C}{\omega} \cos \Delta \cot(2\Theta).
\end{equation}

In the derivation of Eqs. (9) and (10), it was assumed explicitly that the ratio \(\frac{\cosh|\lambda_1(x-Vt)|}{\cosh|\lambda_2(x-Vt)|}\) is equal to one. The latter is valid for small-amplitude breathers with long localization lengths, \(\lambda_1, \lambda_2 \ll 1\). In this case the above assumption, which is exact for the central region of the breathers, \(x-Vt \approx 0\), will be (approximately) valid for a large number of particles, which form weakly localized wandering breather in weakly coupled nonlinear chains. It is also assumed in the considered approximation that the shifts of wandering breather frequency \(\omega\), Eq. (7), caused by weak inter-chain coupling \(C\) and by nonlinearity \(\beta\Psi_{max}^2\), as well as the characteristic frequency difference \(|\omega_{m1} - \omega_{m2}|\), are all relatively small. It is worth to mention that equations similar to Eqs. (9) and (10) were first derived (for \(\omega_{m1} = \omega_{m2}\)) in Ref. [14] for the description of energy exchange between two weakly coupled classical anharmonic oscillators.

Equations (9) and (10) can be written in an equivalent form for population imbalance \(Z\) and relative phase \(\Delta\), when \(Z = \cos 2\Theta\) and \(\sqrt{1-Z^2} = \sin 2\Theta\):
\begin{equation}
\dot{Z} = -\frac{C}{\omega} \sqrt{1-Z^2} \sin \Delta,
\end{equation}
\begin{equation}
\dot{\Delta} = \frac{1}{2\omega}(\omega_{m1}^2 - \omega_{m2}^2) + \frac{3\beta\Psi_{max}^2}{\omega} \sin(2\Theta) + \frac{C}{\omega} \sqrt{1-Z^2} \cos \Delta,
\end{equation}
where \(Z, \Delta\) are canonically conjugate, \(\dot{Z}=-\frac{\partial H}{\partial \Delta}, \dot{\Delta}=-\frac{\partial H}{\partial Z}\), with the effective Hamiltonian \(H=3\beta\Psi_{max}^2 Z^2 - C \sqrt{1-Z^2} \cos \Delta + \frac{C}{2\omega} (\omega_{m1}^2 - \omega_{m2}^2)\). The very same equations for \(Z\) and \(\Delta\), which are equivalent to Eqs. (9) and (10) for \(\Theta\) and \(\Delta\), were obtained in [15] in the mean-field theory of tunnelling dynamics of two weakly coupled Bose-Einstein condensates, which were later used in the analysis of the experimental realization of a single bosonic Josephson junction [16]. Therefore generic evolution equations (11) and (12) (written in corresponding units) for the excitation-exchange dynamics of wandering breather in two weakly coupled nonlinear phononic waveguides or tunnelling dynamics of two weakly coupled Bose-Einstein condensates in a double-well potential do not explicitly depend on the "source" equations: \(\beta\)-FPU Eqs. (1) for the wandering breather or Gross-Pitaevskii equations for the weakly linked Bose-Einstein condensates [15].

For two identical chains, with \(\omega_{m1}=\omega_{m2}=\omega_m\), the ansatz
\begin{equation}
\cos \Delta = A(t)/\sin(2\Theta),
\end{equation}
where \(A=0\) for \(\sin(2\Theta)=0\), gives us from Eqs. (9) and (10) that
\begin{equation}
\cos \Delta = -\frac{3\beta\Psi_{max}^2}{2C} \sin(2\Theta) = -\frac{3\beta\Psi_{max}^2}{2C} \sqrt{1-Z^2}.
\end{equation}

Figure 1. Phase portrait for population imbalance $Z$ versus relative phase $\Delta$ of wandering breather in two weakly coupled nonlinear chains or two weakly coupled Bose-Einstein condensates in a symmetric double-well potential, which is given by Eq. (14). Lines 1 - 6 correspond, respectively, to $\kappa = 2, 1.25, 1, 0.8, 0.5$ and 0.1. Line 3 describes the separatrix mode.

Figure 2. Energy of slowly-moving wandering breather close to the separatrix, versus time and site. The breather is initially excited in chain 1, (a), with immovable chain 2, (b), with velocity $V=0.1$ and amplitude $\Psi_{max}=0.26$, for $l=1$, $\beta=1$, $C=0.1$. Separatrix solution of Eq. (15) corresponds to $\Psi_{max}=0.2582$. Separatrix-like dynamics of inter-chain energy exchange is established for $t \geq 3000$.

This exact solution of Eqs. (9) and (10), (11) and (12) conserves the effective Hamiltonian:

$$H = \frac{3\beta\Psi_{max}^2}{2C}.$$ The important feature of this solution is that the relative phase $\Delta$ is self-locked to the value $\pi/2$ modulo $\pi$ by the total population imbalance $|Z| = 1$ of the two coupled chains. The phase portrait of Eq. (14) in the $Z$-$\Delta$ plane is given by

$$\cos^2 \Delta + \kappa^2 Z^2 = \kappa^2,$$

where $\kappa = 3\beta\Psi_{max}^2/2C$, see figure 1.

Finally, for two identical weakly coupled chains with $l^{(1)} = l^{(2)} = 1$ and $\omega_m \approx 2$ we get from Eqs. (9), (10) and (14) the nonlinear physical-pendulum equation (with the elliptic modulus $\kappa = 3\beta\Psi_{max}^2/2C$) for $\delta = 2\Delta - \pi$:

$$\ddot{\delta} + \frac{C^2}{4} \sin \delta = 0.$$ (15)

We solve this equation with the initial conditions $\delta(0) = 0$, $\dot{\delta}(0) = \frac{3}{2}\beta\Psi_{max}^2$ and $\Theta(0) = 0$ (or $Z(0)=1$), which correspond to $\Delta(0) = \frac{\pi}{2}$ and zero complex field in the second chain at $t=0$ ($\Psi_2(0) = 0$). The latter condition is realized in our simulations.

Regime of wandering breather and complete inter-chain energy exchange is realized for $\kappa \ll 1$, when

$$\Delta \approx \frac{\pi}{2} + \frac{3\beta\Psi_{max}^2}{2C} \sin \left[ \left( C - \frac{(3\beta\Psi_{max}^2)^2}{16C} \right) \frac{\tilde{t}}{2} \right], \quad Z \approx \cos \left[ \left( C - \frac{(3\beta\Psi_{max}^2)^2}{16C} \right) \frac{\tilde{t}}{2} \right].$$ (16)

where $\tilde{t} = t - \frac{k}{2}x$. Regime of nonlinear self-trapping of the breather is realized for $\kappa \gg 1$, when

$$\Delta \approx \frac{\pi}{2} + \frac{3}{4}\beta\Psi_{max}^2 \tilde{t}, \quad Z = 1 - \frac{2C^2}{(3\beta\Psi_{max}^2)^2} \sin^2 \left( \frac{3}{4}\beta\Psi_{max}^2 \tilde{t} \right).$$ (17)

It corresponds to relatively weak inter-chain energy exchange and is similar to the macroscopic quantum self-trapping of Bose-Einstein condensate, observed in a single bosonic Josephson junction [16].
3. Conclusion

Both regimes of nonlinear dynamics of coupled phase-coherent breathers, including the separatrix mode, were confirmed in our simulations with direct numerical solution of Eqs. (1) for the β-FPU model. Similar results were obtained for the stationary and slowly-moving coupled breathers. Numerical simulation of spatio-temporal dynamics of slowly-moving, with $V = 0.1$, wandering breather in two identical β-FPU chains close to the separatrix is shown in figure 2. Separatrix-like dynamics is well established for the later delay time $t \geq 3000$ (when time is measured in units in which $\omega_m = \sqrt{4 + C} \approx 2$).

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