THE SITE $R^+_G$ FOR A PROFINITE GROUP $G$

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ABSTRACT. Let $G$ be a non-finite profinite group and let $G - \text{Sets}_{df}$ be the canonical site of finite discrete $G$-sets. Then the category $R^+_G$, defined by Devinatz and Hopkins, is the category obtained by considering $G - \text{Sets}_{df}$ together with the profinite $G$-space $G$ itself, with morphisms being continuous $G$-equivariant maps. We show that $R^+_G$ is a site when equipped with the pretopology of epimorphic covers. Also, we explain why the associated topology on $R^+_G$ is not subcanonical, and hence, not canonical. We note that, since $R^+_G$ is a site, there is automatically a model category structure on the category of presheaves of spectra on the site. Finally, we point out that such presheaves of spectra are a nice way of organizing the data that is obtained by taking the homotopy fixed points of a continuous $G$-spectrum with respect to the open subgroups of $G$.

1. Introduction

Let $G$ be a profinite group that is not a finite group. Let $R^+_G$ be the category with objects all finite discrete left $G$-sets together with the left $G$-space $G$. The morphisms of $R^+_G$ are the continuous $G$-equivariant maps. Since $G$ is not finite, the object $G$ in $R^+_G$ is very different in character from all the other objects of $R^+_G$. In this paper, we show that $R^+_G$ is a site when equipped with the pretopology of epimorphic covers.

As far as the author knows, the category $R^+_G$ is first defined and used in the paper [Devinatz and Hopkins, 2004], by Ethan Devinatz and Mike Hopkins. Let $G_n$ be the profinite group $S_n \rtimes \text{Gal}(F_{p^n}/F_p)$, where $S_n$ is the $n$th Morava stabilizer group. In [Devinatz and Hopkins, 2004, Theorem 1], Devinatz and Hopkins construct a contravariant functor $\mathbf{F}$ that is a presheaf - that is, a presheaf -

$$\mathbf{F} : (R^+_G)^{\text{op}} \to (\mathcal{E}_\infty)_{K(n)},$$

to the category $(\mathcal{E}_\infty)_{K(n)}$ of $K(n)$-local commutative $S$-algebras (see [Elmendorf et. al., 1997]), where $K(n)$ is the $n$th Morava $K$-theory (see [Rudyak, 1998, Chapter 9] for an exposition of $K(n)$). The functor $\mathbf{F}$ has the properties that, if $U$ is an open subgroup of $G_n$, then $\mathbf{F}(G_n/U) = E_n^{dhU}$, and $\mathbf{F}(G_n) = E_n$, where $E_n$ is the $n$th Lubin-Tate spectrum (for salient facts about $E_n$ and its importance in homotopy theory, see [Devinatz and Hopkins, 1995, Introduction]), and $E_n^{dhU}$ is a spectrum that behaves like the $U$-homotopy fixed point spectrum of $E_n$ with respect to the continuous $U$-action. Since $\text{Hom}_{R^+_G}(G_n, G_n) \cong G_n$, functoriality implies that $G_n$ acts on $E_n$ by maps of commutative $S$-algebras. In Section 5, we will give several related examples of presheaves of spectra that illustrate the utility of the category $R^+_G$.

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The pretopology of epimorphic covers on a small category $C$ is the pretopology $K$ given by all covering families $\{f_i: C_i \to C | i \in I\}$ such that $\phi: \coprod_{i \in I} C_i \to C$ is onto, where $C_i, C \in C, f_i \in \text{Mor}_C(C_i, C)$, and $I$ is some indexing set. (Of course, one must prove that these covering families actually give a pretopology on $C$.) We note that we do not require that $\phi$ be a morphism in $C$; for our purposes, $C = R^+_G$ and we only require that $\phi$ be an epimorphism in the category of all $G$-sets (so that $\phi$ does not have to be continuous). This assumption is important for our work, since, for example, $G \coprod G$ is not in $R^+_G$.

The pretopology $K$ is a familiar one. For example, for a profinite group $G$, $K$ is the standard basis used for the site $G - \text{Sets}_{df}$ of finite discrete $G$-sets ([Jardine, 1997, pg. 206]). However, there is an important difference between $R^+_G$ and $G - \text{Sets}_{df}$: the latter category is closed under pullbacks, but it is easy to see that $R^+_G$ does not have all pullbacks (this point will be discussed later). But in a category with pullbacks, the canonical topology, the finest topology in which every representable presheaf is a sheaf, is given by all covering families of universal effective epimorphisms (see Expose IV, 4.3 of [Demazure, 1970]). This implies that $G - \text{Sets}_{df}$ is a site with the canonical topology when equipped with pretopology $K$. However, due to the lack of sufficient pullbacks, we cannot conclude that $K$ gives $R^+_G$ the canonical topology. In fact, we will show that $K$ does not generate the canonical topology, since $K$ does not yield a subcanonical topology.

Note that $R^+_G$ is built out of the two subcategories $G - \text{Sets}_{df}$ and the groupoid $G$. Since each of these categories is a site via $K$ (for $G$, this is verified in Lemma 2 below), it is natural to think that $R^+_G$ is also a site via $K$. Our main result (Theorem 3.1), verifies that this is indeed the case.

As discussed earlier, $F$ is a presheaf of spectra on the site $R^+_G$. More generally, there is the category $\text{PreSpt}(R^+_G)$ of presheaves of spectra on $R^+_G$. Furthermore, since $R^+_G$ is a site, the work of Jardine (e.g., [Jardine, 1987], [Jardine, 1997]) implies that $\text{PreSpt}(R^+_G)$ is a model category. We recall the definition of this model category in Section 5.

In [Davis, 2006], the author showed that, given a continuous $G$-spectrum $Z$, then, for any open subgroup $U$ of $G$, there is a homotopy fixed point spectrum $Z^{hU}$, defined with respect to the continuous action of $U$ on $Z$. In Examples 5.7 and 5.8, we see that there is a presheaf that organizes in a functorial way the following data: $Z, Z^{hU}$ for all $U$ open in $G$, and the maps between these spectra that are induced by continuous $G$-equivariant maps between the $G$-spaces $G$ and $G/U$. Thus, $\text{PreSpt}(R^+_G)$ is a natural category within which to work with continuous $G$-spectra. It is our hope that the model category structure on $\text{PreSpt}(R^+_G)$ can be useful for the theory of homotopy fixed points for profinite groups, though we have not yet found any such applications.

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2. Preliminaries

Before we prove our main results, we first collect some easy facts which will be helpful later. As stated in the Introduction, $G$ always refers to an infinite profinite group. (If the profinite group $G$ is finite, then $R_G^+ = G - \text{Sets}_{df}$ and there is nothing to prove.)

2.1. Lemma. Let $f : C \to G$ be any morphism in $R_G^+$ with $C \neq \emptyset$. Then $C = G$.

Proof. Choose any $c \in C$ and let $f(c) = \gamma$. Choose any $\delta \in G$. Then

$$\delta = (\delta \gamma^{-1}) \gamma = (\delta \gamma^{-1}) \cdot f(c) = f((\delta \gamma^{-1}) \cdot c),$$

by the $G$-equivariance of $f$. Thus, $f$ is onto and $|\text{im}(f)| = \infty$, so that $C$ cannot be a finite set.

2.2. Lemma. For a topological group $G$, let $G$ be the groupoid with the single object $G$ and morphisms the $G$-equivariant maps $G \to G$ given by right multiplication by some element of $G$. Then $G$ is a site with the pretopology $\mathcal{K}$ of epimorphic covers.

Proof. Any diagram $G \xrightarrow{f} G \xleftarrow{g} G$, where $f$ and $g$ are given by multiplication by $\gamma$ and $\delta$, respectively, can be completed to a commutative square

$$
\begin{array}{ccc}
G & \xrightarrow{f'} & G \\
\downarrow{g'} & & \downarrow{g} \\
G & \xleftarrow{f} & G,
\end{array}
$$

where $f'$ and $g'$ are given by multiplication by $\delta^{-1}$ and $\gamma^{-1}$, respectively. This property suffices to show that $G$ is a site with the atomic topology, in which every sieve is a covering sieve if and only if it is nonempty. It is easy to see that the only nonempty sieve of $G$ is $\text{Mor}_G(G, G)$ itself. Thus, the only covering sieve of $G$ is the maximal sieve. Since every morphism of $G$ is a homeomorphism, in the pretopology $\mathcal{K}$, the collection of covers is exactly the collection of all nonempty subsets of $\text{Mor}_G(G, G)$. Then it is easy to check that $\mathcal{K}$ is the maximal basis that generates the atomic topology.

Observe that if $f : G \to G$ is a morphism in $R_G^+$, then by $G$-equivariance, $f$ is the map given by multiplication by $f(1)$ on the right. As mentioned earlier, we have

2.3. Lemma. The category $G - \text{Sets}_{df}$, a full subcategory of $R_G^+$, is closed under pullbacks.

Proof. The pullback of a diagram in $G - \text{Sets}_{df}$ is formed simply by regarding the diagram as being in the category $T_G$ of discrete $G$-sets. The category $T_G$ is closed under pullbacks, as explained in [Mac Lane and Moerdijk, 1994, pg. 31].
We recall the following useful result and its proof.

2.4. **Lemma.** Let $X$ be any finite set in $R_G^+$, We write $X = \bigsqcup_{i=1}^{n} \overline{x_i}$, the disjoint union of all the distinct orbits $\overline{x_i}$, with each $x_i$ a representative. Then $X$ is homeomorphic to $\bigsqcup_{i=1}^{n} G/U_i$, where $U_i = G_{x_i}$ is the stabilizer in $G$ of $x_i$.

**Proof.** Let $f : G/U_i \to \overline{x_i}$ be given by $f(\gamma U_i) = \gamma \cdot x_i$. Since $X$ is a discrete $G$-set, the stabilizer $U_i$ is an open subgroup of $G$ with finite index, so that $G/U_i$ is a finite set. Then $f$ is open and continuous since it is a map between discrete spaces. Also, it is clear that $f$ is onto. Now suppose $\gamma U_i = \delta U_i$. Then $\gamma^{-1}\delta \in U_i$, so that $(\gamma^{-1}\delta) \cdot x_i = (\gamma^{-1}) \cdot (\delta \cdot x_i) = x_i$. Thus, $\gamma \cdot x_i = \delta \cdot x_i$ and $f$ is well-defined. Assume that $\gamma \cdot x_i = \delta \cdot x_i$. Then $\gamma^{-1}\delta \in G_{x_i}$ so that $f$ is a monomorphism.

2.5. **Lemma.** Let $X$ be any finite discrete $G$-set in $R_G^+$ and let $\psi : G \to X$ be any $G$-equivariant function. Then $\psi$ is a morphism in $R_G^+$.

**Proof.** As in Lemma 2.4, we identify $X$ with $\bigsqcup_{i=1}^{n} G/U_i$. Since $\psi$ is $G$-equivariant and $\psi(\gamma) = \gamma \cdot \psi(1)$, $\psi$ is determined by $\psi(1)$. Let $\psi(1) = \delta U_j$ for some $\delta \in G$ and some $j$. Then for any $\gamma$ in $G$,

$$\gamma U_j = (\gamma \delta^{-1}\delta) U_j = (\gamma \delta^{-1}) \cdot \psi(1) = \psi(\gamma \delta^{-1}),$$

so that $\text{im} \psi = G/U_j$. Since $X$ is discrete, $\psi$ is continuous, if, for any $x \in X$, $\psi^{-1}(x)$ is open in $G$. It suffices, by the identification, to let $x = \gamma U_j$, for any $\gamma \in G$. Then

$$\psi^{-1}(\gamma U_j) = \{ \zeta \in G | \psi(\zeta) = \gamma U_j \} = \{ \zeta \in G | \zeta \cdot (\delta U_j) = \gamma U_j \} = \{ \zeta \in G | \delta^{-1}\zeta^{-1} \gamma \in U_j \} = \gamma U_j \delta^{-1}.$$

Since $U_j$ is open and multiplication on the left or the right is always a homeomorphism in a topological group, we see that $\psi^{-1}(x)$ is an open set in $G$.

3. **The proof of the main theorem**

With these lemmas in hand, we are ready for

3.1. **Theorem.** For any profinite group $G$, the category $R_G^+$ equipped with the pretopology $K$ of epimorphic covers is a small site.

Before proving the theorem, we first make some remarks about pullbacks in $R_G^+$ and how this affects our proof. In a category $\mathcal{C}$ with sufficient pullbacks, to prove that a pretopology is given by a function $K$, which assigns to each object $C$ a collection $K(C)$ of families of morphisms with codomain $C$, one must prove the stability axiom, which says the following: if $\{ f_i : C_i \to C | i \in I \} \in K(C)$, then for any morphism $g : D \to C$, the family of pullbacks

$$\{ \pi_L : D \times_C C_i \to D | i \in I \} \in K(D).$$

Let us examine what this axiom would require of $R_G^+$.
3.2. Example. The map $G \to \ast$ forms a covering family and so the stability axiom requires that $G \times \{\ast\} G = G \times G$ be in $R_G^+$. 

3.3. Example. Let $C$ be any finite discrete $G$-set with more than one element and with trivial $G$-action, $g : G \to C$ any morphism, and consider the cover

$$\{f_i : C_i \to C \mid i \in I\} \in K(C),$$

where $C_j = C$ and $f_j : C \to C$ is the morphism mapping $C$ to $g(1)$, for some $j \in I$. Because the action is trivial, $f_j$ is $G$-equivariant. There certainly exist covers of $C$ of this form, since one could let $f_k = \text{id}_C$, for some $k \neq j$ in $I$, and then let the other $f_i$ be any morphisms with codomain $C$. Then the stability axiom requires that $G \times_C C$ exists in $R_G^+$, but this is impossible, since

$$G \times_C C = \{(\gamma, c) \mid g(\gamma) = f_j(c)\} = \{(\gamma, c) \mid \gamma \cdot g(1) = g(1)\} = G \times (1) \times C = G \times C.$$

Thus, the stability axiom for a pretopology must be altered so that one still obtains a topology. We list the correct axioms for our situation below. They are taken from [Mac Lane and Moerdijk, 1994, Exercise 3, pg. 156].

1. If $f : C' \to C$ is an isomorphism, then $\{f : C' \to C\} \in K(C)$.

2. (stability axiom) If $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, then for any morphism $g : D \to C$, there exists a cover $\{h_j : D_j \to D \mid j \in J\} \in K(D)$ such that for each $j$, $g \circ h_j$ factors through some $f_i$.

3. (transitivity axiom) If $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, and if for each $i \in I$ there is a family $\{g_{ij} : D_{ij} \to C_i \mid j \in I_i\} \in K(C_i)$, then the family of composites $\{f_i \circ g_{ij} : D_{ij} \to C \mid i \in I, j \in I_i\}$ is in $K(C)$.

**Proof of Theorem 3.1.** It is clear that the pretopology of epimorphic covers satisfies axiom (1) above. Also, it is easy to see that axiom (3) holds. Indeed, using the above notation, choose any $c \in C$. Then there is some $c_i \in C_i$ for some $i$, such that $f_i(c_i) = c$. Similarly, there must be some $d_{ij} \in D_{ij}$ for some $j$, such that $g_{ij}(d_{ij}) = c_i$. Hence, $(f_i \circ g_{ij})(d_{ij}) = f_i(c_i) = c$, so that $\coprod_{i,j} D_{ij} \to C$ is onto. This verifies (3). We verify (2) by considering five cases.

**Case (1):** Suppose that $D$ and each of the $C_i$ are finite sets in $R_G^+$. By Lemma 2.1, $C$ must be a finite set. Consider the cover

$$\{\pi_L(i) : D \times_C C_i \to D \mid i \in I\},$$

where $\pi_L(i)$ is the obvious map and $g \circ \pi_L(i)$ factors through $f_i$ via the canonical map $\pi_R(i)$. Now choose any $d \in D$ and let $g(d) = c \in C$. Then there exists some $i$ such that
This shows that \( g_\lambda = C \) for each \( d, c_i \in D \times C \). Thus, \( (d, c_i) \in D \times C \), so that \( \prod_i D \times C \to D \) maps \( (d, c_i) \) to \( d \) and is therefore an epimorphism. This shows that \( \{\pi_L(i)\} \) in \( K(D) \).

**Case (2):** Suppose that \( D = G \) and each \( C_i \) is a finite set in \( R_G^c \). By Lemma 2.1, \( C \) is a finite set and we identify it with \( \prod_{i=1}^n G/U_i \), where \( U_i = G_{x_i} \), the stabilizer of \( x_i \) in \( G \). The map \( g \) is determined by \( g(1) = \delta U_k \) for some \( \delta \in G \) and some stabilizer \( U_k \). Since \( \prod_i C_i \to C \) is onto and \( \text{im}(g) = G/U_k \), there exists some \( c_i \in C_i \) such that \( f_i(c_i) = U_k \).

Since \( C_i \) is a finite set, we can identify \( c_i \) with some \( \mu G_z \), where \( \mu \in G \) and \( G_z \) is the stabilizer of some element \( z \in C_i \).

Then define the cover to be \( \{\lambda: G \to G\} \), where \( \lambda(\gamma) = \gamma \delta^{-1} \). Define \( \alpha_i: G \to C_i \) to be the \( G \)-equivariant map given by \( 1 \mapsto \mu G_z \). By Lemma 2.5, \( \alpha_i \) is continuous and is a morphism in \( R_G^c \). Since \( \lambda \) is a homeomorphism, the cover \( \{\lambda\} \) is in \( K(D) \). Now,

\[
(g \circ \lambda)(1) = g(\delta^{-1}) = \delta^{-1} \cdot g(1) = U_k = \mu \cdot f_i(G_z) = \mu \cdot f_i(\mu^{-1} \cdot \alpha_i(1)) = (f_i \circ \alpha_i)(1).
\]

This shows that \( g \circ \lambda \) factors through \( f_i \) via \( \alpha_i \).

**Case (3):** Suppose not all the \( C_i \) are finite sets and that \( D = G \). Also, assume that \( C = G \). This implies that \( C_i = G \) for all \( i \in I \). Choose any \( k \in I \), let \( \alpha_k = \text{id}_C \), and define \( \lambda: G \to G \) to be multiplication on the right by \( f_k(1)g(1)^{-1} \). Then the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\text{id}_G} & G \\
\downarrow{\lambda} & & \downarrow{f_k} \\
G & \xrightarrow{g} & G
\end{array}
\]

is commutative, since

\[
(g \circ \lambda)(1) = g(f_k(1)g(1)^{-1}) = f_k(1)g(1)^{-1} \cdot g(1) = f_k(1) = (f_k \circ \alpha_k)(1).
\]

Thus, \( g \circ \lambda \) factors through \( f_k \) via \( \alpha_k \), so that the stability axiom is verified by letting the covering family be \( \{\lambda\} \).

**Case (4):** Suppose that not all the \( C_i \) are finite sets, \( D = G \), and \( C \) is a finite set. With \( C \) as in Lemma 2.4, let \( g(1) = \delta U_k \subseteq C \), as in Case (2). Then there exists some \( l \) such that \( f_l(c_l) = U_k \), for some \( c_l \in C_l \). Now we consider two subcases.

**Case (4a):** Suppose that \( C_l \) is a finite set in \( R_G^c \). Just as in Case (2), we construct maps \( \lambda \) and \( \alpha_l \), so that \( g \circ \lambda \) factors through \( f_l \) via \( \alpha_l \) and \( \{\lambda\} \in K(D) \).

**Case (4b):** Suppose that \( C_l = G \). By \( G \)-equivariance, \( f_l(1) = c_l^{-1}U_k \). Then define \( \lambda: G \to G \) by \( 1 \mapsto \delta^{-1} \) and \( \alpha_l: G \to G \) by \( 1 \mapsto c_l \). Then \( g \circ \lambda \) factors through \( f_l \) via \( \alpha_l \), since

\[
(g \circ \lambda)(1) = g(\delta^{-1}) = \delta^{-1} \cdot g(1) = U_k = f_l(c_l) = (f_l \circ \alpha_l)(1).
\]

Thus, the cover \( \{\lambda\} \), as a homeomorphism, is in \( K(D) \). This completes Case (4).

Now we consider the final possibility, **Case (5):** suppose that not all of the \( C_i \) are finite sets and suppose that \( D \) is a finite set. This implies that \( C \) is a finite set. This case is more difficult than the others because the cover consists of more than one morphism and it combines the previous constructions. For each \( d \in D \), we make a choice of some
$c_l \in C_l$ for some $l$, such that $c_l$ is in the preimage of $g(d)$ under $\coprod C_i \to C$. Then write $D = D_{df} \coprod D_G$, where $D_{df}$ is the set of all $d$ such that the corresponding $C_l$ is in a finite set, and $D_G$ is the set of all $d$ such that the corresponding $C_l = G$. Now consider the cover $\{h_d : D_d \to D | d \in D = D_{df} \coprod D_G\}$, where

$$D_d = \begin{cases} D \times_C C_d & \text{if } d \in D_{df}, \\ G & \text{if } d \in D_G. \end{cases}$$

If $d \in D_{df}$, then $h_d = \pi_L$ and $\alpha_d : D \times_C C_d \to C_d$ is the canonical map $\pi_R$; it is clear that the required square commutes. Now suppose $d \in D_G$. Then there exists $c_l \in C_l = G$ for some $l$, such that $g(d) = f_l(c_l)$. We write $f_l(1) = \theta U_k \in C$ for some $\theta \in G$ and for some stabilizer $U_k$. Then we define $\alpha_d : G \to C_l = G$ by $1 \mapsto \theta^{-1}$. Also, we define $h_d : G \to D$ by $1 \mapsto (\theta^{-1}c_l^{-1}) \cdot d$. Lemma 2.5 shows that $h_d$ is a morphism in $R^+_G$. Then we have the required commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha_d} & G \\
\downarrow{h_d} & & \downarrow{f_l} \\
D & \xrightarrow{g} & C, \end{array}$$

since

$$(g \circ h_d)(1) = g((\theta^{-1}c_l^{-1}) \cdot d) = (\theta^{-1}c_l^{-1}) \cdot g(d) = (\theta^{-1}c_l^{-1}) \cdot f_l(c_l) = f_l(\theta^{-1}) = (f_l \circ \alpha_d)(1).$$

The only remaining detail is to show that $\{h_d\} \in K(D)$; that is, we must show that $\phi : \coprod_D D_d \to D$ is an epimorphism. Let $d$ be any element in $D$. Suppose $d \in D_{df}$. Then, using our choice above, there exists some $c_l \in C_l$, a finite set for some $l$, such that $f_l(c_l) = g(d)$. Then $(d, c_l) \in D \times_C C_l$ and $\phi(d, c_l) = \pi_L(d, c_l) = d$. Now suppose $d \in D_G$. With $c_l$ and $\theta$ as above, $c_\theta \in D_d = G$ and $\phi(c_\theta) = h_d(c_\theta) = (c_\theta) \cdot h_d(1) = d$. Therefore, $\phi$ is an epimorphism.

4. The site $R^+_G$ does not have the canonical topology

Now that we have established that $R^+_G$ is a site with pretopology $\mathcal{K}$, we begin working to show that, contrary to what typically happens with this pretopology, it does not give the canonical topology. We start with a definition.

4.1. DEFINITION. If $T$ is some collection of morphisms with codomain $C$, where $C$ is an object in the category $\mathcal{C}$, then $(T)$ denotes the sieve generated by $T$. Thus,

$$(T) = \{f \circ g | f \in T, \ \text{dom}(f) = \text{cod}(g)\}.$$  

4.2. LEMMA. Let $K$ be a pretopology on a category $\mathcal{C}$. Let $J$ be the Grothendieck topology generated by $K$. Then for any $C \in \mathcal{C}$, $J(C)$ consists exactly of all $(R) \cup (T)$ such that $R \in K(C)$ and $T$ is some collection of morphisms with codomain $C$. 
PROOF. Let $S$ be a covering sieve of $C$. Then there exists some $R \in K(C)$ such that $R \subset S$. We will prove that $S = (R) \cup (S)$, verifying the forward inclusion. To prove equality it suffices to show that $(R) \cup (S) \subset S$. If $f \in (R)$, $f = g \circ h$ for some $g \in R$ and some $h$ with $\text{dom}(g) = \text{cod}(h)$. Since $g \in S$, $f \in S$. Similarly, if $f \in (S)$, then $f \in S$. Now consider any family of morphisms $(R) \cup (T)$ as described in the statement of the lemma. Since $R \subset (R) \cup (T)$, $(R) \cup (T) \in J(C)$ if it is a sieve. Since $(R)$ and $(T)$ are sieves, it is clear that $(R) \cup (T)$ is also a sieve.

This result is useful for understanding the topology of a site, when the site is defined in terms of a pretopology. For example, $G-\text{Sets}_{df}$ is a site by the pretopology $K$ and its category of sheaves of sets is equivalent to the category of sheaves on the site $S(G)$ consisting of quotients of $G$ by open subgroups (the morphisms are the $G$-equivariant maps), where $S(G)$ is given the atomic topology (see [Mac Lane and Moerdijk, 1994, Chapter 3, Section 9]). Thus, one might ask if $G-\text{Sets}_{df}$ also has the atomic topology. However, the lemma allows us to see that $K$ generates a topology that is coarser than the atomic topology. To see this, let $X = G/U$ and $Y = G/U \coprod G/U$, where $U$ is a proper open subgroup of $G$. (Since $G$ is an infinite profinite group, the canonical way of writing $G$ as an inverse limit guarantees the existence of such a $U$.) We define $f: X \to Y$ by $f(U) = U$, where $U$ lives in the factor on the left; $f$ is the left inclusion. Now consider the sieve $S = \{\{f\}\}$. Clearly, $S$ does not contain an epimorphic cover, since $\text{im}(\coprod_{g \in S}(\text{dom}(g)) \to Y) = G/U$. The lemma indicates that every sieve of $G-\text{Sets}_{df}$ must contain an epimorphic cover, so that $S$ is not a sieve for $Y$ in the topology generated by $K$.

Now we consider the site $R^+_G$ with the pretopology $K$ of epimorphic covers. We use $\text{Hom}_G(X,Y)$ to denote continuous $G$-equivariant maps between continuous $G$-sets $X$ and $Y$. Recall that a presheaf of sets $P$ on a site $(C,J)$ is a sheaf, if for each object $C \in C$ and each covering sieve $S \in J(C)$, the diagram

$$P(C) \xrightarrow{e} \prod_{f \in S} P(\text{dom}(f)) \xrightarrow{p} \prod_{a} P(\text{dom}(g))$$

is an equalizer of sets, where the second product is over all $f,g$, with $f \in S$, $\text{dom}(f) = \text{cod}(g)$. Here, $e$ is the map $e(x) = \{P(f)(x)\}_{f}$, $p$ is given by

$$\{x^f\}_{f} \mapsto \{x^f g\}_{f,g},$$

and $a$ is given by

$$\{x^f\}_{f} \mapsto \{P(g)(x^f)\}_{f,g} = \{x^f \circ g\}_{f,g}.$$

Recall that a representable presheaf of $R^+_G$ is any presheaf which, up to isomorphism, has the form of $\text{Hom}_G(-,C)$ for some $C \in R^+_G$. Also, the Yoneda embedding

$$R^+_G \to \text{Sets}^{(R^+_G)^{op}}, \quad C \mapsto \text{Hom}_G(-,C)$$

is a full and faithful functor, so that one can identify $C$ with an object of $\text{Sets}^{(R^+_G)^{op}}$. We now consider which objects of $R^+_G$ yield sheaves of sets on $R^+_G$.

Noting that the empty set is a discrete $G$-set, we have

$$\text{Hom}_G(X,Y,\emptyset) = \{\emptyset\},$$

where $\emptyset$ is the unique $G$-set having no elements. But this means $\emptyset$ is a sheaf of sets on $R^+_G$.
4.3. Lemma. The presheaf \( \text{Hom}_G(-, \emptyset) \) is a sheaf of sets on the site \( R^+_G \).

Proof. Let \( \bullet : \emptyset \to X \) denote the vacuous map, for any \( X \in R^+_G \). Since \( \bullet : \emptyset \to \emptyset \) is vacuously an epimorphism, \( \{\bullet\} \) is the unique covering sieve for \( \emptyset \). Let \( C = \emptyset \). Then the desired equalizer diagram has the form

\[
\text{Hom}_G(\emptyset, \emptyset) = \{\bullet\} \xrightarrow{e} \{\bullet\} \xrightarrow{p} \emptyset.
\]

It is clear that this is an equalizer diagram.

Now let \( C \) be a nonempty finite set in \( G - \text{Sets}_{df} \). Let \( S \) be any covering sieve of \( C \). There must exist a morphism in \( S \) with domain equal to a nonempty object in \( R^+_G \). Therefore, since \( \emptyset \times Z = \emptyset \) for any space \( Z \), we have

\[
\text{Hom}_G(C, \emptyset) = \emptyset \xrightarrow{e} \emptyset \xrightarrow{p} \emptyset.
\]

Since the equalizer must exist and the vacuous map \( \bullet : \emptyset \to \emptyset \) is the unique map with codomain \( \emptyset \), this must be an equalizer diagram.

Finally, letting \( C = G \), we get

\[
\text{Hom}_G(G, \emptyset) = \emptyset \xrightarrow{e} \prod_{f \in \text{Hom}_G(G,G)} \emptyset = \emptyset \xrightarrow{p} \emptyset.
\]

Again, this is an equalizer diagram.

To prove the next theorem, we need the following lemma.

4.4. Lemma. If \( G \) is a compact topological group, \( U \) an open subgroup of \( G \), and \( X \neq \emptyset \) a finite discrete \( G \)-set, then

\[
\text{Hom}_G(G/U, X) \cong \{ x \in X \mid U < G_x \},
\]

where \( G_x \) is the stabilizer of \( x \) in \( G \).

Proof. Let \( f : G/U \to X \). It is clear that \( f \) is \( G \)-equivariant if and only if it is completely determined by \( f(U) \) in the obvious way. Since \( U \) is an open subgroup, it has finite index in \( G \), so that \( G/U \) is a discrete space. Thus, any \( G \)-equivariant map \( G/U \to X \) is continuous. The key is that \( f \) is well-defined if and only if \( U < G_{f(U)} \). To see this, first assume that \( f \) is well-defined; let \( \gamma \in U \). Then \( \gamma U = U \), so that \( \gamma \cdot f(U) = f(\gamma U) = f(U) \). Hence, \( \gamma \in G_{f(U)} \) and \( U < G_{f(U)} \). Now suppose that \( U < G_{f(U)} \) and take any \( \gamma U = \delta U \). This implies that \( \gamma^{-1}\delta \in U \) and hence, in \( G_{f(U)} \). Thus, \( (\gamma^{-1}\delta) \cdot f(U) = f(U) \), so that \( \gamma \cdot f(U) = \delta \cdot f(U) \). Equivariance gives \( f(\gamma U) = f(\delta U) \) and \( f \) is well-defined. Thus,

\[
\text{Hom}_G(G/U, X) \cong \{ f(U) \in X \mid U < G_{f(U)} \}.
\]
Henceforth, let $\mathcal{J}$ denote the topology of $R^+_G$ generated by $\mathcal{K}$.

4.5. THEOREM. Let $X$ be any object in $R^+_G$ that is not a finite discrete trivial $G$-set, where $G$ is an infinite profinite group. Then the presheaf $\text{Hom}_G(-, X)$ is not a sheaf of sets on the site $R^+_G$.

PROOF. Suppose $\text{Hom}_G(-, X)$ is a sheaf of sets on the site $R^+_G$. The equalizer condition says that for every object $C \in R^+_G$ and for every covering sieve $S \in \mathcal{J}(C)$,

$$\text{Hom}_G(C, X) \cong \left\{ \{h^f\}_{f} \mid h^f \circ g = h^f \circ g, f, g, f \in S, \text{dom}(f) = \text{cod}(g) \right\},$$

where for $f \in S$, $h^f \in \text{Hom}_G(\text{dom}(f), X)$. We will construct an example of some $C$ and $S$ such that this sheaf condition fails to be true with $X$ as above.

Let $C \in G-\text{Sets}_{dg}$; we identify $C$ with $\coprod_{i=1}^n G/U_i$, where each $U_i$ is an open subgroup of $G$. For each $i$, define $f_i : G \to C$ by $1 \mapsto U_i$. Thus, $\text{im}(f_i) = G/U_i$ and $\{f_i\}$ is an epimorphic cover of $C$. The preceding lemma tells us that $S = \{(f_i)\}$ is a covering sieve of $C$. For this $S$, we will examine the sheaf condition. Let $S = S' \cup S''$, where $S' = \{f_i\}$ and $S''$ is the complement of $S'$ in $S$. Thus, every $k \in S''$ has the form $k = f_i \circ g$ for some $g$ with $\text{dom}(f_i) = \text{cod}(g)$. Then

$$\left\{ \{h^f\}_{f} \mid h^f \circ g = h^f \circ g, f, g, f \in S, \text{dom}(f) = \text{cod}(g) \right\}$$

$$= \left\{ \{h^f_i\}_{f_i} \times \{h^k\}_{k \in S''} \mid h^f \circ g = h^f \circ g, f, g, f \in S, \text{dom}(f) = \text{cod}(g) \right\}$$

$$= \left\{ \{h^f_i\}_{f_i} \times \{h^f \circ g\}_{f \circ g \in S''} \mid h^f_i \circ g = h^f \circ g, f, g, f \in S, \text{dom}(f) = \text{cod}(g) \right\}$$

$$= \{ \{h^f_i\}_{f_i} \times \{h^f \circ g\}_{f \circ g \in S''} \mid h^f_i \in \text{Hom}_G(G, X), f_i \in S', g, \text{dom}(f_i) = \text{cod}(g) \}.$$

We verify the last equality. Suppose $h^f_i$ is any morphism in $\text{Hom}_G(G, X)$. Now take any $f$ and $g$ with $f \in S$ and $\text{dom}(f) = \text{cod}(g)$. If $f = f_i \in S'$, then $h^f \circ g = h^f_i \circ g = h^f_i \circ g$, by construction. Now suppose $f \in S''$. Then $f = f_i \circ k$ for some $k : G \to G$. Thus,

$$h^f \circ g = h^f_i \circ (k \circ g) = (h^f_i \circ k) \circ g = h^f_i \circ k \circ g = h^f \circ g.$$

Since $h^f_i \circ g$ is determined by $h^f_i$ and $f_i \circ g$, we see that the set

$$\left\{ \{h^f_i\}_{f_i} \times \{h^f \circ g\}_{f \circ g \in S''} \mid h^f_i \in \text{Hom}_G(G, X), f_i \in S', g, \text{dom}(f_i) = \text{cod}(g) \right\}$$

is isomorphic to the set

$$\left\{ \{h^f_i\}_{f_i} \mid h^f_i \in \text{Hom}_G(G, X), f_i \in S' \right\} = \text{Hom}_G(G, X)^n,$$

where $\text{Hom}_G(G, X)^n$ is the $n$-fold Cartesian product of $\text{Hom}_G(G, X)$. Now, there is an isomorphism $\text{Hom}_G(G, X)^n \cong X^n$. Therefore, for $\text{Hom}_G(-, X)$ to be a sheaf, it must be that $\text{Hom}_G(C, X) \cong X^n$ for every $C \in G-\text{Sets}_{dg}$. If $X = G$ and $C \neq \emptyset$ is in $G-\text{Sets}_{dg}$, then $\text{Hom}_G(C, G) = \emptyset$, whereas, since $|C| \geq 1$, $n \geq 1$ and $X^n = G^n$. Thus, $\text{Hom}_G(-, G)$ is not a sheaf.
Now we consider $X \neq G$ and assume that $\text{Hom}_G(C, X) \cong X^n$ for every $C \in G-\text{Sets}_{df}$. This implies that

$$X^n \cong \text{Hom}_G(C, X) \cong \text{Hom}_G(\prod_{i=1}^n G/U_i, X) \cong \prod_{i=1}^n \text{Hom}_G(G/U_i, X) \cong \prod_{i=1}^n \{x \in X | U_i < Gx\} \subset X^n.$$ 

Therefore, it must be that $\{x \in X | U_i < Gx\} = X$, for all $i = 1, \ldots, n$. Thus, $U_i < Gx$ for all $x \in X$ and each $i$. Now let us write $X \cong \biguplus_{i=1}^n G/G_{x_i}$, where each $x_i$ is a representative from a distinct orbit of $X$. Let $C$ be a trivial $G$-set so that every stabilizer of $c \in C$ in $G$ is equal to $G$. This implies that $G < G_{x_j}$ for all $j$. Thus, each $G_{x_j} = G$. This indicates that $X$ must be a trivial $G$-set. This contradiction shows that every $X$ violates the sheaf condition for some $C$ and $S$.

This result immediately yields

4.6. COROLLARY. For an infinite profinite group $G$, the site $R^+_G$ with the pretopology $\mathcal{K}$ of epimorphic covers is not subcanonical.

Proof. There exists a proper open subgroup $U$ of $G$ satisfying $[G : U] > 1$. Thus, the representable presheaves $\text{Hom}_G(-, G)$ and $\text{Hom}_G(-, \biguplus_{i=1}^n G/U)$, for any $n \geq 1$, are not sheaves.

Since a canonical topology is, by definition, subcanonical, we obtain

4.7. COROLLARY. For an infinite profinite group $G$, the site $R^+_G$, with the pretopology $\mathcal{K}$, is not canonical.

The next result is an elementary fact about profinite groups that helps us understand “how often” representable presheaves fail to be sheaves in $R^+_G$ and what such “failing” presheaves can look like, based on what we know from Theorem 4.5.

4.8. LEMMA. If $G$ is an infinite profinite group, then $G$ contains an infinite number of distinct proper open subgroups.

Proof. We have already seen that $G$ has at least one proper open subgroup. Suppose that $G$ has only a finite number of distinct proper open subgroups. Then $G$ has a finite number of distinct proper open normal subgroups $N_1, \ldots, N_k$. Since $G$ is profinite, $N = \bigcap_{i=1}^k N_i = \{1\}$. Because $N$ is an open subgroup with finite index, it has uncountable order. This contradiction gives the conclusion.

4.9. REMARK. Since any topology finer than $\mathcal{J}$ would contain the covering sieve $\{(f_i)\}$ that was the key to Theorem 4.5, no topology finer than $\mathcal{J}$ can be subcanonical.

5. Presheaves of spectra on the site $R^+_G$

Let $\textbf{Ab}$ be the category of abelian groups, and let $\text{Spt}$ denote the model category of Bousfield-Friedlander spectra of pointed simplicial sets. We refer to the objects of $\text{Spt}$
as simply “spectra.” Now that \( R_G^+ \) is a site, we can consider the category \( \text{PreSpt}(R_G^+) \) of presheaves of spectra on the site \( R_G^+ \). By applying the work of Jardine ([Jardine, 1987], [Jardine, 1997, Section 2.3]), \( \text{PreSpt}(R_G^+) \) is a model category. We recall the critical definitions that give the model category structure and then we state Jardine’s result, when it is applied to \( R_G^+ \).

5.1. Definition. Let \( (R_G^+)\text{op} \to \text{Spt} \) be a presheaf of spectra. Then, for each \( n \in \mathbb{Z} \),

\[
\pi_n(P) : (R_G^+)\text{op} \to \text{Ab}, \quad C \mapsto \pi_n(P(C)),
\]

is a presheaf of abelian groups. Then the associated sheaf \( \tilde{\pi}_n(P) \) of abelian groups is the sheafification of \( \pi_n(P) \).

Let \( f : P \to Q \) be a morphism of presheaves of spectra on \( R_G^+ \). Then \( f \) is a weak equivalence if the induced map \( \tilde{\pi}_n(P) \to \tilde{\pi}_n(Q) \) of sheaves is an isomorphism, for all \( n \in \mathbb{Z} \). The map \( f \) is a cofibration if \( f(C) \) is a cofibration of spectra, for all \( C \in R_G^+ \). Also, \( f \) is a global fibration if \( f \) has the right lifting property with respect to all morphisms which are weak equivalences and cofibrations.

5.2. Theorem. [Jardine, 1997, Theorem 2.34] The category \( \text{PreSpt}(R_G^+) \), together with the classes of weak equivalences, cofibrations, and global fibrations, is a model category.

Now we give some interesting examples of presheaves of spectra on the site \( R_G^+ \).

5.3. Example. In the Introduction, we saw that the Devinatz-Hopkins functor \( \mathbf{F} \) is an example of an object in \( \text{PreSpt}(R_G^{G_n}) \).

For the next example, if \( X \) is a spectrum, then, for each \( k \geq 0 \), we let \( X_k \) be the \( k \)th pointed simplicial set constituting \( X \), and, for each \( l \geq 0 \), \( X_{k,l} \) is the pointed set of \( l \)-simplices of \( X_k \).

5.4. Example. Let \( X \) be a discrete \( G \)-spectrum (see [Davis, 2006] for a definition of this term), so that each \( X_{k,l} \) is a pointed discrete \( G \)-set. If \( C \in R_G^+ \), then let \( \text{Hom}_G(C, X) \) be the spectrum, such that

\[
\text{Hom}_G(C, X)_k = \text{Hom}_G(C, X_k),
\]

where

\[
\text{Hom}_G(C, X)_{k,l} = \text{Hom}_G(C, X_{k,l}) = \text{Hom}_G(C, X_{k,l}).
\]

Above, the set \( X_{k,l} \) is given the discrete topology, since it is naturally a discrete \( G \)-set. Then \( \text{Hom}_G(\cdot, X) \) is an object in \( \text{PreSpt}(R_G^+) \). It is easy to see that if \( U \) is an open subgroup of \( G \), then \( \text{Hom}_G(G/U, X) \cong X^U \), the \( U \)-fixed point spectrum of \( X \). Also, \( \text{Hom}_G(G, X) \cong X \).

Now we recall part of [Behrens and Davis, 2005, Proposition 3.3.1], since this result (and its corollary) will be helpful in our next example. We note that this result is only a slight extension of [Jardine, 1997, Remark 6.26]: if \( U \) is normal in \( G \), then the lemma below is an immediate consequence of Jardine’s remark.
5.5. Lemma. Let $X$ be a discrete $G$-spectrum. Also, let $f : X \to X_{f,G}$ be a trivial cofibration, such that $X_{f,G}$ is fibrant, where all this takes place in the model category of discrete $G$-spectra (see [Davis, 2006]). If $U$ is an open subgroup of $G$, then $X_{f,G}$ is fibrant in the model category of discrete $U$-spectra.

5.6. Corollary. Let $X$ and $U$ be as in the preceding lemma. Then $X^{hU} = (X_{f,G})^U$.

Proof. Let $f$ be as in the above lemma. Since $f$ is $G$-equivariant, it is $U$-equivariant. Also, since $f$ is a trivial cofibration in the model category of discrete $G$-spectra, it is a trivial cofibration in the model category of spectra. The preceding two facts imply that $f$ is a trivial cofibration in the model category of discrete $U$-spectra. By the lemma, $X_{f,G}$ is fibrant in this model category. Thus, $X^{hU} = (X_{f,G})^U$.

5.7. Example. Let $X$ be a discrete $G$-spectrum. Then $\text{Hom}_G(-, X_{f,G})$ is a presheaf in $\text{PreSpt}(R_G^+)$. In particular, notice that

$$\text{Hom}_G(G/U, X_{f,G}) \cong (X_{f,G})^U = X^{hU}$$

and

$$\text{Hom}_G(G, X_{f,G}) \cong X_{f,G} \simeq X.$$

5.8. Example. For any unfamiliar concepts in this example, we refer the reader to [Davis, 2006]. Let $Z = \text{holim}_i Z_i$ be a continuous $G$-spectrum, so that $\{Z_i\}_{i \geq 0}$ is a tower of discrete $G$-spectra, such that each $Z_i$ is a fibrant spectrum. Then

$$P(-) = \lim_i \text{Hom}_G(-, (Z_i)_{f,G}) \in \text{PreSpt}(R_G^+),$$

where

$$P(G/U) \cong \lim_i ((Z_i)_{f,G})^U = \lim_i (Z_i)^{hU} = Z^{hU}$$

and

$$P(G) \cong \lim_i (Z_i)_{f,G} \simeq Z.$$

References

Mark J. Behrens and Daniel G. Davis, *The homotopy fixed point spectra of profinite Galois extensions*, 33 pp., current version of manuscript is available at http://dgdavis.web.wesleyan.edu, 2005.

Daniel G. Davis, *Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action*, J. Pure Appl. Algebra 206 (2006), no. 3, 322–354.

Michel Demazure, *Topologies et faisceaux*, Schémas en groupes. SGA 3, I: Propriétés générales des schémas en groupes, Springer-Verlag, Berlin, 1970, pp. xvi+564.

Ethan S. Devinatz and Michael J. Hopkins, *The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts*, Amer. J. Math. 117 (1995), no. 3, 669–710.
Ethan S. Devinatz and Michael J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47.

A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole.

J. F. Jardine, *Stable homotopy theory of simplicial presheaves*, Canad. J. Math. **39** (1987), no. 3, 733–747.

J. F. Jardine, *Generalized étale cohomology theories*, Birkhäuser Verlag, Basel, 1997.

Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Springer-Verlag, New York, 1994, A first introduction to topos theory, Corrected reprint of the 1992 edition.

Yuli B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, With a foreword by Haynes Miller.

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