ON BLOWUP SOLUTIONS TO THE FOCUSING INTERCRITICAL NONLINEAR FOURTH-ORDER SCHRÖDINGER EQUATION

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ABSTRACT. In this paper we study dynamical properties of blowup solutions to the focusing intercritical (mass-supercritical and energy-subcritical) nonlinear fourth-order Schrödinger equation. We firstly establish the profile decomposition of bounded sequences in $H^b \cap H^2$. We also prove a compactness lemma and a variational characterization of ground states related to the equation. As a result, we obtain the $H^b$-concentration of blowup solutions with bounded $H^b$-norm and the limiting profile of blowup solutions with critical $H^b$-norm.

1. Introduction

Consider the Cauchy problem for the focusing intercritical nonlinear fourth-order Schrödinger equation

$$
\begin{cases}
  i\partial_t u - \Delta^2 u = -|u|^\alpha u, & \text{on } [0, \infty) \times \mathbb{R}^d , \\
  u(0) = u_0 ,
\end{cases}
$$

(1.1)

where $u$ is a complex valued function defined on $[0, \infty) \times \mathbb{R}^d$ and $2_* < \alpha < 2^*$ with

$$2_* := \frac{8}{d}, \quad 2^* := \begin{cases} 
  \infty & \text{if } d = 1, 2, 3, 4 , \\
  \frac{8}{d-4} & \text{if } d \geq 5. 
\end{cases}
$$

(1.2)

The fourth-order Schrödinger equation was introduced by Karpman [18] and Karpman-Shagalov [19] taking into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations are of the form

$$i\partial_t - \Delta^2 u + \epsilon \Delta u = \mu |u|^\alpha u, \quad u(0) = u_0 ,
$$

(1.3)

where $\epsilon \in \{0, \pm 1\}$, $\mu \in \{\pm\}$ and $\alpha > 0$. The equation (1.1) is a special case of (1.3) with $\epsilon = 0$ and $\mu = -1$. The study of nonlinear fourth-order Schrödinger equations (1.3) has attracted a lot of interest in the past several years (see [26], [27], [13], [14], [16], [22], [23], [24], [8], [9] and references therein).

The equation (1.1) enjoys the scaling invariance

$$u_{\lambda}(t, x) := \lambda^{\frac{4}{\alpha}} u(\lambda^4 t, \lambda x), \quad \lambda > 0.
$$

It means that if $u$ solves (1.1), then $u_{\lambda}$ solves the same equation with initial data $u_{\lambda}(0, x) = \lambda^{\frac{4}{\alpha}} u_0(\lambda x)$. A direct computation shows

$$\|u_{\lambda}(0)\|_{H^b} = \lambda^{\gamma + \frac{4}{\alpha} - \frac{d}{2}} \|u_0\|_{H^b} .$$

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From this, we define the critical Sobolev exponent

$$\gamma_c := \frac{d}{2} - \frac{4}{\alpha}. \quad (1.4)$$

We also define the critical Lebesgue exponent

$$\alpha_c := \frac{2d}{d - 2\gamma_c} = \frac{d\alpha}{4}. \quad (1.5)$$

By Sobolev embedding, we have $H^{\gamma_c} \hookrightarrow L^{\alpha_c}$. The local well-posedness for (1.1) in Sobolev spaces was studied in [7, 8] (see also [26] for $H^2$ initial data). It is known that (1.1) is locally well-posed in $H^\gamma$ for $\gamma \geq \max\{\gamma_c, 0\}$ satisfying for $\alpha > 0$ not an even integer,

$$\lceil \gamma \rceil \leq \alpha + 1, \quad (1.6)$$

where $\lceil \gamma \rceil$ is the smallest integer greater than or equal to $\gamma$. This condition ensures the nonlinearity to have enough regularity. Moreover, the solution enjoys the conservation of mass

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u_0),$$

and $H^2$ solution has conserved energy

$$E(u(t)) = \frac{1}{2} \int |\Delta u(t,x)|^2 dx - \frac{1}{\alpha + 2} \int |u(t,x)|^{\alpha + 2} dx = E(u_0).$$

In the subcritical regime, i.e. $\gamma > \gamma_c$, the existence time depends only on the $H^\gamma$-norm of initial data. There is also a blowup alternative: if $T$ is the maximal time of existence, then either $T = \infty$ or

$$T < \infty, \quad \lim_{t \to T} \|u(t)\|_{H^\gamma} = \infty.$$

It is well-known (see e.g. [26]) that if $\gamma_c < 0$ or $0 < \alpha < \frac{8}{d}$, then (1.1) is globally well-posed in $H^2$. Thus the blowup in $H^2$ may occur only for $\alpha \geq \frac{8}{d}$. Recently, Boulanger-Lenzmann established in [5] blowup criteria for (1.3) with radial data in $H^2$ in the mass-critical ($\gamma_c = 0$), mass and energy intercritical ($0 < \gamma_c < 2$) and energy-critical ($\gamma_c = 2$) cases. This naturally leads to the study of dynamical properties of blowup solutions such as blowup rate, concentration and limiting profile, etc.

In the mass-critical case $\gamma_c = 0$ or $\alpha = \frac{8}{d}$, the study of $H^2$ blowup solutions to (1.1) is closely related to the notion of ground states which are solutions of the elliptic equation

$$\Delta^2 Q + Q - |Q|^2 Q = 0.$$

Fibich-Ilan-Papanicolaou in [10] showed some numerical observations which implies that if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution exists globally; and if $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, then the solution may blow up in finite time. Later, Baruch-Fibich-Mandelbaum in [2] proved some dynamical properties such as blowup rate, $L^2$-concentration for radial blowup solutions. In [30], Zhu-Yang-Zhang established the profile decomposition and a compactness result to study dynamical properties such as $L^2$-concentration, limiting profile with minimal mass of blowup solutions in general case (i.e. without radially symmetric assumption). For dynamical properties of blowup solutions with low regularity initial data, we refer the reader to [32] and [9].

In the mass and energy intercritical case $0 < \gamma_c < 2$, there are few works concerning dynamical properties of blowup solutions to (1.1). To our knowledge, the only paper addressed this problem belongs to [31] where the authors studied $L^{\gamma_c}$-concentration of radial blowup solutions. We also refer to [3] for numerical study of blowup solutions to the equation.

The main purpose of this paper is to show dynamical properties of blowup solutions to (1.1)
with initial data in $\dot{H}^{\gamma_c} \cap \dot{H}^2$. The main difficulty in this consideration is the lack of conservation of mass. To study dynamics of blowup solutions in $\dot{H}^{\gamma_c} \cap \dot{H}^2$, we firstly need the local well-posedness. For data in $\dot{H}^2$, the local well-posedness is well-known (see e.g. [26]). However, for data in $\dot{H}^{\gamma_c} \cap \dot{H}^2$ the local theory is not a trivial consequence of the one for $\dot{H}^2$ data due to the lack of mass conservation. We thus need to show a new local theory for our purpose, and it will be done in Section 2. It is worth noticing that thanks to Strichartz estimates with a “gain” of derivatives, we can remove the regularity requirement (1.6). However, we can only show the local well-posedness in dimensions $d \geq 5$, the one for $d \leq 4$ is still open. After the local theory is established, we need to show the existence of blowup solutions. In [5], the authors showed blowup criteria for radial $H^2$ solutions to (1.3). In their proof, the conservation of mass plays a crucial role. In our setting, the lack of mass conservation makes the problem more difficult. We are only able to prove a blowup criteria for negative energy radial solutions with an additional condition

$$\sup_{t \in [0,T]} \|u(t)\|_{\dot{H}^{\gamma_c}} < \infty.$$ (1.7)

This condition is also needed in our results for dynamical properties of blowup solutions. We refer to Section 4 for more details. To study blowup dynamics for data in $\dot{H}^{\gamma_c} \cap \dot{H}^2$, we establish the profile decomposition for bounded sequences in $\dot{H}^{\gamma_c} \cap \dot{H}^2$. This is done by following the argument of [15] (see also [12]). With the help of this profile decomposition, we study the sharp constant to the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{\alpha+2,2}} \leq A_{GN} \|f\|_{H^{\gamma_c}} \|f\|_{\dot{H}^2}^2.$$ (1.8)

It follows (see Proposition 3.2) that the sharp constant $A_{GN}$ is attained at a function $U \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ of the form

$$U(x) = aQ(\lambda x + x_0),$$

for some $a \in \mathbb{C}^*$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$, where $Q$ is a solution to the elliptic equation

$$\Delta^2 Q + (-\Delta)^\gamma c Q - |Q|^{\alpha} Q = 0.$$

Moreover,

$$A_{GN} = \frac{\alpha + 2}{2} \|Q\|_{H^{\gamma_c}}^{-\alpha}.$$ 

The profile decomposition also gives a compactness lemma, that is for any bounded sequence $(v_n)_{n \geq 1}$ in $\dot{H}^{\gamma_c} \cap \dot{H}^2$ satisfying

$$\lim_{n \to \infty} \sup_{n \to \infty} \|v_n\|_{H^2} \leq M, \quad \lim_{n \to \infty} \|v_n\|_{L^{\alpha+2}} \geq m,$$

there exists a sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } \dot{H}^{\gamma_c} \cap \dot{H}^2,$$

for some $V \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ satisfying

$$\|V\|_{H^{\gamma_c}}^\alpha \geq \frac{2}{\alpha + 2} \frac{m^{\alpha+2}}{M^2} \|Q\|_{H^{\gamma_c}}^\alpha.$$ 

As a consequence, we show that the $\dot{H}^{\gamma_c}$-norm of blowup solutions satisfying (1.7) must concentrate by an amount which is bounded from below by $\|Q\|_{\dot{H}^{\gamma_c}}$ at the blowup time. Finally, we show the limiting profile of blowup solutions with critical norm

$$\sup_{t \in [0,T]} \|u(t)\|_{\dot{H}^{\gamma_c}} = \|Q\|_{\dot{H}^{\gamma_c}}.$$

The plan of this paper is as follows. In Section 2, we give some preliminaries including Strichartz estimates, the local well-posedness for data in $\dot{H}^{\gamma_c} \cap \dot{H}^2$ and the profile decomposition of bounded
sequences in $\dot{H}^\infty \cap \dot{H}^2$. In Section 3, we use the profile decomposition to study the sharp Gagliardo-Nirenberg inequality (1.8). The global existence and blowup criteria will be given in Section 4. Section 5 is devoted to the blowup concentration, and finally the limiting profile of blowup solutions with critical norm will be given in Section 6.

2. Preliminaries

2.1. Homogeneous Sobolev spaces. We firstly recall the definition and properties of homogeneous Sobolev spaces (see e.g. [11, Appendix], [29, Chapter 5] or [4, Chapter 6]). Given $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, the generalized homogeneous Sobolev space is defined by

$$
\dot{W}^{\gamma, q} := \{ u \in \mathcal{S}' \mid \| u \|_{\dot{W}^{\gamma, q}} := \| |\nabla|^\gamma u \|_{L^q} < \infty \},
$$

where $\mathcal{S}$ is the subspace of the Schwartz space $\mathcal{S}_{\gamma}$ consisting of functions $\phi$ satisfying $D^\beta \hat{\phi}(0) = 0$ for all $\beta \in \mathbb{N}^d$ with $\gamma$ the Fourier transform on $\mathcal{S}$, and $\mathcal{S}'$ is its topology dual space. One can see $\mathcal{S}'$ as $S'/\mathcal{P}$ where $\mathcal{P}$ is the set of all polynomials on $\mathbb{R}^d$. Under these settings, $\dot{W}^{\gamma, q}$ are Banach spaces. Moreover, the space $\mathcal{S}_{\gamma}$ is dense in $\dot{W}^{\gamma, q}$. In this paper, we shall use $\dot{H}^\gamma := \dot{W}^{\gamma, 2}$. We note that the spaces $\dot{H}^{\gamma_1}$ and $\dot{H}^{\gamma_2}$ cannot be compared for the inclusion. Nevertheless, for $\gamma_1 < \gamma < \gamma_2$, the space $\dot{H}^\gamma$ is an interpolation space between $\dot{H}^{\gamma_1}$ and $\dot{H}^{\gamma_2}$.

2.2. Strichartz estimates. In this subsection, we recall Strichartz estimates for the fourth-order Schrödinger equation. Let $I \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the mixed norm

$$
\| u \|_{L^p(I, L^q)} := \left( \int_I \left( \int_{\mathbb{R}^d} |u(t, x)|^q dx \right)^{\frac{p}{q}} \right)^{\frac{1}{p}},
$$

with a usual modification when either $p$ or $q$ are infinity. We also denote for $(p, q) \in [1, \infty]^2$,

$$
\gamma_{p, q} = \frac{d}{2} - \frac{d}{q} - \frac{4}{p}.
$$

**Definition 2.1.** A pair $(p, q)$ is called **Schrödinger admissible**, for short $(p, q) \in S$, if

$$(p, q) \in [2, \infty]^2, \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

A pair $(p, q)$ is call **biharmonic admissible**, for short $(p, q) \in B$, if

$$(p, q) \in S, \quad \gamma_{p, q} = 0.$$

We have the following Strichartz estimates for the fourth-order Schrödinger equation.

**Proposition 2.2** (Strichartz estimates [6, 7]). Let $\gamma \in \mathbb{R}$ and $u$ be a weak solution to the inhomogeneous fourth-order Schrödinger equation, namely

$$
u(t) = e^{it\Delta^2} u_0 + \int_0^t e^{i(t-s)\Delta^2} F(s) ds,
$$

for some data $u_0$ and $F$. Then for all $(p, q)$ and $(a, b)$ Schrödinger admissible with $q < \infty$ and $b < \infty$,

$$
\| |\nabla|^\gamma u \|_{L^p(I, L^q)} \lesssim \| |\nabla|^\gamma + \gamma_{p, q} \gamma_{p, q} \| u_0 \|_{L^2} + \| |\nabla|^\gamma_{p, q} \gamma_{p, q} \cdot F \|_{L^p(I, L^q)},
$$

(2.3)

where $(a, a')$ and $(b, b')$ are conjugate pairs.

Note that the estimates (2.3) are exactly the ones given in [25] or [26] where the authors considered $(p, q)$ and $(a, b)$ are either sharp Schrödinger admissible, i.e.

$$(p, q) \in [2, \infty]^2, \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$
or biharmonic admissible. We refer to [6] or [7] for the proof of Proposition 2.2. Note that instead of using directly a dedicate dispersive estimate of [1] for the fundamental solution of the homogeneous fourth-order Schrödinger equation, one uses the scaling technique which is similar to the one of wave equation (see e.g. [20]).

We also have the following consequence of Strichartz estimates (2.3).

**Corollary 2.3.** Let $\gamma \in \mathbb{R}$ and $u$ be a weak solution to the inhomogeneous fourth-order Schrödinger equation (2.2) for some data $u_0$ and $F$. Then for all $(p, q)$ and $(a, b)$ biharmonic admissible satisfying $q < \infty$ and $b < \infty$,

$$
\|u\|_{L^p(\mathbb{R}^q, L^q)} \lesssim \|u_0\|_{L^2} + \|F\|_{L^{q'}(\mathbb{R}, L^{q'})},
$$

and

$$
|||\nabla|\gamma|u||_{L^p(\mathbb{R}^q, L^q)} \lesssim |||\nabla|\gamma|u_0||_{L^2} + |||\nabla|\gamma|^{-1}F||_{L^2(\mathbb{R}, L^{q'})}.
$$

Note that the estimates (2.5) is important to reduce the regularity requirement of the nonlinearity (see Subsection 2.4).

In the sequel, for a space time slab $I \times \mathbb{R}^d$ we define the Strichartz space $\dot{B}^0(I \times \mathbb{R}^d)$ as a closure of $\mathcal{S}_0$ under the norm

$$
\|u\|_{\dot{B}^0(I \times \mathbb{R}^d)} := \sup_{(p, q) \in \dot{B}} \|u\|_{L^p(I, L^q)}.
$$

For $\gamma \in \mathbb{R}$, the space $\dot{B}^\gamma(I \times \mathbb{R}^d)$ is defined as a closure of $\mathcal{S}_0$ under the norm

$$
\|u\|_{\dot{B}^\gamma(I \times \mathbb{R}^d)} := |||\nabla|\gamma|u||_{\dot{B}^0(I \times \mathbb{R}^d)}.
$$

We also use $\dot{N}^0(I \times \mathbb{R}^d)$ to denote the dual space of $\dot{B}^0(I \times \mathbb{R}^d)$ and

$$
\dot{N}^\gamma(I \times \mathbb{R}^d) := \{u : |\nabla|\gamma|u| \in \dot{N}^0(I \times \mathbb{R}^d)\}.
$$

To simplify the notation, we will use $\dot{B}^\gamma(I)$, $\dot{N}^\gamma(I)$ instead of $\dot{B}^\gamma(I \times \mathbb{R}^d)$ and $\dot{N}^\gamma(I \times \mathbb{R}^d)$. By Corollary 2.3, we have

$$
\|u\|_{\dot{B}^0(\mathbb{R})} \lesssim \|u_0\|_{L^2} + \|F\|_{\dot{N}^0(\mathbb{R})},
$$

and

$$
\|u\|_{\dot{B}^\gamma(\mathbb{R})} \lesssim \|u_0\|_{H^\gamma} + |||\nabla|\gamma|^{-1}F||_{L^2(\mathbb{R}, L^{q'})}.
$$

### 2.3. Nonlinear estimates

We next recall nonlinear estimates to study the local well-posedness for (1.1).

**Lemma 2.4** (Nonlinear estimates [17]). Let $F \in C^k(\mathbb{C}, \mathbb{C})$ with $k \in \mathbb{N}\setminus\{0\}$. Assume that there is $\alpha > 0$ such that $k \leq \alpha + 1$ and

$$
|D^jF(z)| \lesssim |z|^{|a+1-j|}, \quad z \in \mathbb{C}, j = 1, \ldots, k.
$$

Then for $\gamma \in [0, k]$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\alpha}{q}$, there exists $C = C(d, \alpha, \gamma, r, p, q) > 0$ such that for all $u \in \mathcal{S}$,

$$
|||\nabla|\gamma|F(u)||_{L^r} \leq C\|u\|_{L^p}^\alpha \|\nabla|\gamma|u||_{L^q}.
$$

Moreover, if $F$ is a homogeneous polynomial in $u$ and $\pi$, then (2.8) holds true for any $\gamma \geq 0$.

The proof of Lemma 2.4 is based on the fractional Leibniz rule (or Kato-Ponce inequality) and the fractional chain rule. We refer the reader to [17, Appendix] for the proof.
2.4. Local well-posedness. In this subsection, we recall the local well-posedness for (1.1) with initial data in \(H^2\) and in \(\dot{H}^{\gamma_c} \cap \dot{H}^2\) respectively. The case in \(H^2\) is well-known (see e.g. [26]), while the one in \(\dot{H}^{\gamma_c} \cap \dot{H}^2\) needs a careful consideration.

**Proposition 2.5** (Local well-posedness in \(H^2\) [26]). Let \(d \geq 1\), \(u_0 \in H^2\) and \(0 < \alpha < 2^*\). Then there exist \(T > 0\) and a unique solution \(u\) to (1.1) satisfying
\[
u \in C([0, T), H^2) \cap L^p_{\text{loc}}([0, T), W^{2, q}),
\]
for any biharmonic admissible pairs \((p, q)\) satisfying \(q < \infty\). The time of existence satisfies either \(T = \infty\) or \(T < \infty\) and \(\lim_{t \to T} \|u\|_{\dot{H}^2} = \infty\). Moreover, the solution enjoys the conservation of mass and energy.

**Proposition 2.6** (Local well-posedness in \(\dot{H}^{\gamma_c} \cap \dot{H}^2\)). Let \(d \geq 5\), \(0 < \alpha < 2^*\) and \(u_0 \in \dot{H}^{\gamma_c} \cap \dot{H}^2\). Then there exist \(T > 0\) and a unique solution \(u\) to (1.1) satisfying
\[
u \in C([0, T), \dot{H}^{\gamma_c} \cap \dot{H}^2) \cap L^p_{\text{loc}}([0, T), \dot{W}^{\gamma_c, q} \cap \dot{W}^{2, q}),
\]
for any biharmonic admissible pairs \((p, q)\) satisfying \(q < \infty\). The existence time satisfies either \(T = \infty\) or \(T < \infty\) and \(\lim_{t \to T} \|u(t)\|_{\dot{H}^{\gamma_c}} + \|u(t)\|_{\dot{H}^2} = \infty\). Moreover, the solution enjoys the conservation of energy.

**Remark 2.7.**
- When \(\gamma_c = 0\), Proposition 2.6 is a consequence of Proposition 2.5 since \(H^0 = L^2\) and \(L^2 \cap H^2 = H^2\).
- In [8], a similar result holds with an additional regularity assumption \(\alpha \geq 1\) if \(\alpha\) is not an even integer. Thanks to Strichartz estimate with a “gain” of derivatives (2.7), we can remove this regularity requirement.

**Proof of Proposition 2.6.** We firstly choose
\[
n = \frac{2d}{d + 2 - (d - 4)\alpha}, \quad n^* = \frac{2d}{d + 4 - (d - 4)\alpha}, \quad m^* = \frac{8}{(d - 4)\alpha - 4}.
\]
It is easy to check that
\[
\frac{d + 2}{2d} = \frac{(d - 4)\alpha}{2d} + \frac{1}{n}, \quad \frac{1}{m} = \frac{1}{n^*} \cdot \frac{1}{d^*}, \quad d = \frac{4}{m^*} + \frac{d}{n^*}.
\] (2.9)
In particular, \((m^*, n^*)\) is a biharmonic admissible and
\[
\theta := \frac{1}{2} \cdot \frac{1}{m^*} = 1 - \frac{(d - 4)\alpha}{8} > 0.
\] (2.10)
Consider
\[
X := \left\{ u \in \dot{B}^{\gamma_c}(I) \cap \dot{B}^{2}(I) : \|u\|_{\dot{B}^{\gamma_c}(I)} + \|u\|_{\dot{B}^{2}(I)} \leq M \right\},
\]
equipped with the distance
\[
d(u, v) := \|u - v\|_{\dot{B}^{0}(I)},
\]
where \(I = [0, \tau]\) and \(M, \tau > 0\) to be chosen later. By Duhamel’s formula, it suffices to prove that the functional
\[
\Phi(u)(t) := e^{i\Delta t^2}u_0 + i \int_0^t e^{i(t-s)\Delta} |u(s)|^\alpha u(s)ds
\]
is a contraction on \((X, d)\). By Strichartz estimate (2.7),
\[
\|\Phi(u)\|_{\dot{B}^{2}(I)} \lesssim \|u_0\|_{\dot{H}^2} + \|\nabla (|u|^\alpha u)\|_{L^2(I, L^{\frac{2d}{d + 2}})}.
\]
By Lemma 2.4,
\[
\|\nabla (|u|^\alpha u)\|_{L^2(I, L^{\frac{2d}{d + 2}})} \lesssim \|u\|_{L^{\infty}(I, L^{\frac{2d}{d + 2}})} \|\nabla u\|_{L^2(I, L^\infty)}.
\]
We now estimate $\|\Phi(u)\|_{B^{\gamma_c}(I)}$. To do so, we separate two cases $\gamma_c \geq 1$ and $0 < \gamma_c < 1$. In the case $\gamma_c \geq 1$, we estimate as above to get

$$\|\Phi(u)\|_{B^{\gamma_c}(I)} \lesssim \|u_0\|_{H^{\gamma_c}} + \|I^\theta\|_{B^{\alpha}(I)} \|u\|_{B^{\gamma_c}(I)}.$$ 

In the case $0 < \gamma_c < 1$, we choose

$$p = \frac{8(\alpha + 2)}{\alpha(d - 4)}, \quad q = \frac{d(\alpha + 2)}{d + 2\alpha}, \quad (2.11)$$

and choose $(m, n)$ so that

$$\frac{1}{p'} = \frac{1}{m} + \frac{\alpha}{p}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{n}.$$ 

It is easy to check that $(p, q)$ is biharmonic admissible and $n = \frac{dq}{d - 2q}$. The later fact gives the Sobolev embedding $W^{2,q} \hookrightarrow L^n$. By Strichartz estimate (2.6),

$$\|\Phi(u)\|_{B^{\gamma_c}(I)} \lesssim \|u_0\|_{H^{\gamma_c}} + \|\nabla\nabla^{\gamma_c} (|u|^\alpha u)\|_{L^{p'}(I, L^{n'})}.$$ 

By Lemma 2.4,

$$\|\nabla\nabla^{\gamma_c} (|u|^\alpha u)\|_{L^{p'}(I, L^{n'})} \lesssim \|u\|_{L^p(I, L^n)} \|\nabla\nabla^{\gamma_c} u\|_{L^m(I, L^n)}$$

$$\lesssim |I|^{\frac{d - 1}{d - 2}} \|\Delta u\|_{L^p(I, L^n)} \|\nabla\nabla^{\gamma_c} u\|_{L^m(I, L^n)}$$

$$\lesssim |I|^\theta \|u\|_{B^{\alpha}(I)} \|u\|_{B^{\gamma_c}(I)}.$$ 

In both cases, we have

$$\|\Phi(u)\|_{B^{\gamma_c}(I)} \lesssim \|u_0\|_{H^{\gamma_c}} + |I|^\theta \|u\|_{B^{\alpha}(I)} \|u\|_{B^{\gamma_c}(I)}.$$ 

Therefore,

$$\|\Phi(u)\|_{B^{\gamma_c}(I) \cap B^{\alpha}(I)} \lesssim \|u_0\|_{H^{\gamma_c} \cap H^2} + |I|^\theta \|u\|_{B^{\alpha}(I)} \|u\|_{B^{\gamma_c}(I) \cap B^{\alpha}(I)}.$$ 

Similarly, by (2.6),

$$\|\Phi(u) - \Phi(v)\|_{B^{\gamma_c}(I)} \lesssim \|u|^\alpha u - |v|^\alpha v\|_{L^{p'}(I, L^{n'})},$$

where $(p, q)$ is as in (2.11). We estimate

$$\|u|^\alpha u - |v|^\alpha v\|_{L^{p'}(I, L^{n'})} \lesssim \left(\|u\|_{L^{p}(I, L^n)}^\alpha + \|v\|_{L^{p}(I, L^n)}^\alpha\right) \|u - v\|_{L^{m}(I, L^n)}$$

$$\lesssim \|I|^\theta \left(\|\Delta u\|_{L^{p}(I, L^n)}^\alpha + \|\Delta v\|_{L^{p}(I, L^n)}^\alpha\right) \|u - v\|_{L^{m}(I, L^n)}$$

$$\lesssim \|I|^\theta \left(\|u\|_{B^{\alpha}(I)}^\alpha + \|v\|_{B^{\alpha}(I)}^\alpha\right) \|u - v\|_{B^{\gamma_c}(I)}.$$ 

This shows that for all $u, v \in X$, there exists $C > 0$ independent of $\tau$ and $u_0 \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ such that

$$\|\Phi(u)\|_{\dot{H}^{\gamma_c}(I)} + \|\Phi(u)\|_{\dot{B}^{\alpha}(I)} \leq C\|u_0\|_{H^{\gamma_c} \cap H^2} + C\tau^\theta M^{\alpha + 1},$$

$$d(\Phi(u), \Phi(v)) \leq C\tau^\theta M^\alpha d(u, v). \quad (2.12)$$
If we set $M = 2C\|u_0\|_{\dot{H}^{\gamma} \cap \dot{H}^2}$ and choose $\tau > 0$ so that
\[ C\tau^\theta M^\alpha \leq \frac{1}{2}, \]
then $\Phi$ is a strict contraction on $(X, d)$. This proves the existence of solution
\[ u \in \dot{B}^{\gamma}(I) \cap \dot{B}^2(I). \]
The time of existence depends only on the $\dot{H}^{\gamma} \cap \dot{H}^2$-norm of initial data. We thus have the blow-up alternative. The conservation of energy follows from the standard approximation. The proof is complete. □

**Corollary 2.8** (Blowup rate). Let $d \geq 5$, $0 < \alpha < 2^*$ and $u_0 \in \dot{H}^\gamma \cap \dot{H}^2$. Assume that the corresponding solution $u$ to (1.1) given in Proposition 2.6 blows up at finite time $0 < T < \infty$. Then there exists $C > 0$ such that
\[ \|u(t)\|_{\dot{H}^{\gamma} \cap \dot{H}^2} > \frac{C}{(T-t)^{\frac{\theta}{\alpha}}}, \tag{2.13} \]
for all $0 < t < T$.

**Proof.** Let $0 < t < T$. If we consider (1.1) with initial data $u(t)$, then it follows from (2.12) and the fixed point argument that if for some $M > 0$,
\[ C\|u(t)\|_{\dot{H}^{\gamma} \cap \dot{H}^2} + C(\tau - t)^\theta M^{\alpha+1} \leq M, \]
then $\tau < T$. Thus,
\[ C\|u(t)\|_{\dot{H}^{\gamma} \cap \dot{H}^2} + C(\tau - t)^\theta M^{\alpha+1} > M, \]
for all $M > 0$. Choosing $M = 2C\|u(t)\|_{\dot{H}^{\gamma} \cap \dot{H}^2}$, we see that
\[ (T-t)^\theta \|u(t)\|_{\dot{H}^{\gamma} \cap \dot{H}^2} > C. \]
This implies
\[ \|u(t)\|_{\dot{H}^{\gamma} \cap \dot{H}^2} > \frac{C}{(T-t)^{\frac{\theta}{\alpha}}}, \]
which is exactly (2.13) since $\frac{\theta}{\alpha} = \frac{8-(d-4)\alpha}{8\alpha} = \frac{2-\gamma}{4}$. The proof is complete. □

**2.5. Profile decomposition.** The main purpose of this subsection is to prove the profile decomposition related to the focusing intercritical NL4S by following the argument of [15] (see also [12]).

**Theorem 2.9** (Profile decomposition). Let $d \geq 1$ and $2_* < \alpha < 2^*$. Let $(v_n)_{n \geq 1}$ be a bounded sequence in $\dot{H}^\gamma \cap \dot{H}^2$. Then there exist a subsequence of $(v_n)_{n \geq 1}$ (still denoted $(v_n)_{n \geq 1}$), a family $(x_n^j)_{j \geq 1}$ of sequences in $\mathbb{R}^d$ and a sequence $(V_j)_{j \geq 1}$ of $\dot{H}^\gamma \cap \dot{H}^2$ functions such that
- for every $k \neq j$,
\[ |x_n^k - x_n^j| \to \infty, \quad \text{as } n \to \infty, \tag{2.14} \]
- for every $l \geq 1$ and every $x \in \mathbb{R}^d$,
\[ v_n(x) = \sum_{j=1}^l V_j(x - x_n^j) + v_n^l(x), \]
with
\[ \limsup_{n \to \infty} \|v_n^l\|_{L^q} \to 0, \quad \text{as } l \to \infty, \tag{2.15} \]
for every $q \in (\alpha_c, 2 + 2^*)$, where $\alpha_c$ is given in (1.5). Moreover,

$$\|v_n\|^2_{H^{\gamma_c}} = \sum_{j=1}^{l} \|V^j\|^2_{H^{\gamma_c}} + \|v_n^d\|^2_{H^{\gamma_c}} + o_n(1), \quad (2.16)$$

$$\|v_n\|^2_{H^2} = \sum_{j=1}^{l} \|V^j\|^2_{H^2} + \|v_n^d\|^2_{H^2} + o_n(1), \quad (2.17)$$

as $n \to \infty$.

**Remark 2.10.** In the case $\gamma_c = 0$ or $\alpha = 2$, Theorem 2.9 is exactly Proposition 2.3 in [30] due to the fact $\dot{H}^0 = L^2$ and $L^2 \cap \dot{H}^2 = H^2$.

**Proof of Theorem 2.9.** Since $\dot{H}^{\gamma_c} \cap \dot{H}^2$ is a Hilbert space, we denote $\Omega(v_n)$ the set of functions obtained as weak limits of sequences of the translated $v_n(\cdot + x_n)$ with $(x_n)_{n \geq 1}$ a sequence in $\mathbb{R}^d$. Denote

$$\eta(v_n) := \sup \{\|v\|_{H^{\gamma_c}} + \|v\|_{H^2} : v \in \Omega(v_n)\}.$$

Clearly,

$$\eta(v_n) \leq \limsup_{n \to \infty} \|v_n\|_{H^{\gamma_c}} + \|v_n\|_{H^2}.$$

We shall prove that there exist a sequence $(V^j)_{j \geq 1}$ of $\Omega(v_n)$ and a family $(x^j_n)_{j \geq 1}$ of sequences in $\mathbb{R}^d$ such that for every $k \neq j$,

$$|x^k_n - x^j_n| \to \infty, \quad \text{as } n \to \infty,$$

and up to a subsequence, the sequence $(V^j)_{j \geq 1}$ can be written as for every $l \geq 1$ and every $x \in \mathbb{R}^d$,

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x^j_n) + v^l_n(x),$$

with $\eta(v^l_n) \to 0$ as $l \to \infty$. Moreover, the identities (2.16) and (2.17) hold as $n \to \infty$.

Indeed, if $\eta(v_n) = 0$, then we can take $V^j = 0$ for all $j \geq 1$. Otherwise we choose $V^1 \in \Omega(v_n)$ such that

$$\|V^1\|_{H^{\gamma_c}} + \|V^1\|_{H^2} \geq \frac{1}{2}\eta(v_n) > 0.$$

By the definition of $\Omega(v_n)$, there exists a sequence $(x^1_n)_{n \geq 1} \subset \mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + x^1_n) \rightharpoonup V^1 \text{ weakly } \text{in } H^{\gamma_c} \cap \dot{H}^2.$$

Set $v^1_n(x) := v_n(x) - V^1(x - x^1_n)$. We see that $v^1_n(\cdot + x^1_n) \rightharpoonup 0$ weakly in $\dot{H}^{\gamma_c} \cap \dot{H}^2$ and thus

$$\|v^1_n\|^2_{H^{\gamma_c}} = \|V^1\|^2_{H^{\gamma_c}} + \|v^1_n\|^2_{H^{\gamma_c}} + o_n(1),$$

$$\|v^1_n\|^2_{H^2} = \|V^1\|^2_{H^2} + \|v^1_n\|^2_{H^2} + o_n(1),$$

as $n \to \infty$. We now replace $(v_n)_{n \geq 1}$ by $(v^1_n)_{n \geq 1}$ and repeat the same process. If $\eta(v^1_n) = 0$, then we choose $V^j = 0$ for all $j \geq 2$. Otherwise there exist $V^2 \in \Omega(v^1_n)$ and a sequence $(x^2_n)_{n \geq 1} \subset \mathbb{R}^d$ such that

$$\|V^2\|_{H^{\gamma_c}} + \|V^2\|_{H^2} \geq \frac{1}{2}\eta(v^1_n) > 0,$$

and

$$v^1_n(\cdot + x^2_n) \rightharpoonup V^2 \text{ weakly } \text{in } \dot{H}^{\gamma_c} \cap \dot{H}^2.$$

Set $v^2_n(x) := v^1_n(x) - V^2(x - x^2_n)$. We thus have $v^2_n(\cdot + x^2_n) \rightharpoonup 0$ weakly in $\dot{H}^{\gamma_c} \cap \dot{H}^2$ and

$$\|v^2_n\|^2_{H^{\gamma_c}} = \|V^2\|^2_{H^{\gamma_c}} + \|v^2_n\|^2_{H^{\gamma_c}} + o_n(1),$$

$$\|v^2_n\|^2_{H^2} = \|V^2\|^2_{H^2} + \|v^2_n\|^2_{H^2} + o_n(1),$$

where $\Omega(v^2_n)$ due
as $n \to \infty$. We claim that
$$|x_n^1 - x_n^2| \to \infty, \quad \text{as } n \to \infty.$$ 
In fact, if it is not true, then up to a subsequence, $x_n^1 - x_n^2 \to x_0$ as $n \to \infty$ for some $x_0 \in \mathbb{R}^d$. Since
$$v_n^1(x + (-1)^n x_n^1) = v_n^1(x + (x_n^2 - x_n^1) + x_n^1),$$
and $v_n^1(- x_n^1)$ converges weakly to 0, we see that $V^2 = 0$. This implies that $\eta(v_n^1) = 0$ and it is a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family $(x_n^1)_{j \geq 1}$ of $\tilde{H}^{\infty} \cap \tilde{H}^2$ functions satisfying the claim above. Furthermore, the convergence of the series $\sum_{j \geq 1} \|V_j\|_{H^{\infty}}^2 + \|V_j\|_{H^2}^2$ implies that
$$\|V_j\|_{H^{\infty}}^2 + \|V_j\|_{H^2}^2 \to 0, \quad \text{as } j \to \infty.$$ 
By construction, we have
$$\eta(v_n^j) \leq 2 (\|V^{j+1}\|_{H^{\infty}} + \|V^{j+1}\|_{H^2}),$$
which proves that $\eta(v_n^j) \to 0$ as $j \to \infty$. To complete the proof of Theorem 2.9, it remains to show (2.15). To do so, we introduce for $R > 1$ a function $\hat{\chi}_R \in \mathcal{S}$ satisfying $\hat{\chi}_R : \mathbb{R}^d \to [0, 1]$ and
$$\hat{\chi}_R(\xi) = \begin{cases} 1 & \text{if } 1/R \leq |\xi| \leq R, \\ 0 & \text{if } |\xi| \leq 1/2R \lor |\xi| \geq 2R. \end{cases}$$
We write
$$v_n^j = \chi_R \ast v_n^j + (\delta - \chi_R) \ast v_n^j,$$
where $\ast$ is the convolution operator. Let $q \in (\alpha_c, 2 + 2\ast)$ be fixed. By Sobolev embedding and the Plancherel formula, we have
$$\| (\delta - \chi_R) \ast v_n^j \|_{L^q} \lessapprox \| (\delta - \chi_R) \ast v_n^j \|_{H^\beta} \lessapprox \left( \int |\xi|^{2\beta} |(1 - \hat{\chi}_R(\xi)) \hat{v}_n^j(\xi)|^2 \, d\xi \right)^{1/2} \lessapprox \left( \int_{|\xi| \leq 1/R} |\xi|^{2\beta} |\hat{v}_n^j(\xi)|^2 \, d\xi \right)^{1/2} + \left( \int_{|\xi| \geq R} |\xi|^{2\beta} |\hat{v}_n^j(\xi)|^2 \, d\xi \right)^{1/2} \lessapprox R^{\gamma \beta} |\hat{v}_n^j|_{H^{\infty}} + R^{\beta - 2} |\hat{v}_n^j|_{H^2},$$
where $\beta = \frac{d}{2} - \frac{d}{q} \in (\gamma_c, 2)$. On the other hand, the H"older interpolation inequality implies
$$\| \chi_R \ast v_n^j \|_{L^q} \lessapprox \| \chi_R \ast v_n^j \|_{L^\infty}^{2\alpha_c} \| \chi_R \ast v_n^j \|_{L^\infty}^{1 - 2\alpha_c} \lessapprox \| v_n^j \|_{H^{\infty}}^{2\alpha_c} \| \chi_R \ast v_n^j \|_{L^\infty}^{1 - 2\alpha_c}.$$ 
Observe that
$$\limsup_{n \to \infty} \| \chi_R \ast v_n^j \|_{L^\infty} = \sup_{x_n} \limsup_{n \to \infty} |\chi_R \ast v_n^j(x_n)|.$$ 
Thus, by the definition of $\Omega(v_n^j)$, we infer that
$$\limsup_{n \to \infty} \| \chi_R \ast v_n^j \|_{L^\infty} \leq \sup \left\{ \int \chi_R(-x) v(x) \, dx : v \in \Omega(v_n^j) \right\}.$$ 
By the Plancherel formula, we have
$$\left| \int \chi_R(-x) v(x) \, dx \right| = \left| \int \hat{\chi}_R(\xi) \hat{v}(\xi) \, d\xi \right| \lessapprox \|\| \chi_R \|_{H^{\infty}} \|v\|_{\tilde{H}^{\infty}} \lessapprox R^{\frac{d}{2} \cdot \frac{\gamma_c}{2}} \eta(v_n^j).$$
We thus obtain for every $l \geq 1$, 
\[
\limsup_{n \to \infty} \|v_n^l\|_{L^s} \lesssim \limsup_{n \to \infty} \|\delta - \chi_R\| v_n^l + \limsup_{n \to \infty} \|\chi_R\| v_n^l \lesssim R^{-\epsilon} \|v_n^l\|_{\dot{H}^\gamma} + R^{3-2} \|v_n^l\|_{\dot{H}^2} + \|v_n^l\|_{\dot{H}^{\gamma}} \left[ R^2 \eta(v_n^l)^{1 - \frac{2}{\alpha}} \right].
\]
Choosing $R = \left[ \eta(v_n^l)^{-1} \right]^{\frac{1}{\beta - \epsilon}}$ for some $\epsilon > 0$ small enough, we see that
\[
\limsup_{n \to \infty} \|v_n^l\|_{L^s} \lesssim \eta(v_n^l)^{(\beta - \gamma)\epsilon} \|v_n^l\|_{\dot{H}^\gamma} + \eta(v_n^l)^{(2 - \beta)\epsilon} \|v_n^l\|_{\dot{H}^2} + \eta(v_n^l)^{(1 - \frac{2}{\alpha})\epsilon} \|v_n^l\|_{\dot{H}^{\gamma}}.
\]
Letting $l \to \infty$ and using the fact that $\eta(v_n^l) \to 0$ as $l \to \infty$ and the uniform boundedness in $\dot{H}^\gamma \cap \dot{H}^2$ of $(v_n^l)_{l \geq 1}$, we obtain
\[
\limsup_{n \to \infty} \|v_n^l\|_{L^s} \to 0, \quad \text{as} \quad l \to \infty.
\]
The proof is complete. \qed

3. Variational analysis

Let $d \geq 1$ and $2_\ast < \alpha < 2$. We consider the variational problems
\[
A_{GN} := \max \{ H(f) : f \in \dot{H}^\gamma \cap \dot{H}^2 \}, \quad H(f) := \|f\|_{\dot{H}^{\gamma}}^{\alpha + 2} \div \|f\|_{\dot{H}^{\gamma}}^2 \|f\|_{\dot{H}^2}^2,
\]
\[
B_{GN} := \max \{ K(f) : f \in L^\alpha \cap \dot{H}^2 \}, \quad K(f) := \|f\|_{L^\alpha}^{\alpha + 2} \div \|f\|_{L^\alpha}^2 \|f\|_{\dot{H}^2}^2.
\]
Here $A_{GN}$ and $B_{GN}$ are respectively sharp constants in the Gagliardo-Nirenberg inequalities
\[
\|f\|_{L^{\alpha + 2}}^{\alpha + 2} \leq A_{GN} \|f\|_{\dot{H}^{\gamma}}^{\alpha + 2} \|f\|_{\dot{H}^2}^2,
\]
\[
\|f\|_{L^{\alpha + 2}}^{\alpha + 2} \leq B_{GN} \|f\|_{L^\alpha}^{\alpha + 2} \|f\|_{\dot{H}^2}^2.
\]
Let us start with the following observation.

**Lemma 3.1.** If $g$ and $h$ are maximizers of $H(f)$ and $K(f)$ respectively, then $g$ and $h$ satisfy
\[
A_{GN} \|g\|_{\dot{H}^{\gamma}}^\alpha \Delta g + \frac{\alpha}{2} A_{GN} \|g\|_{\dot{H}^2}^\alpha \Delta^2 g - \frac{\alpha + 2}{2} \|g\|^\alpha g = 0, \tag{3.1}
\]
\[
B_{GN} \|h\|_{L^\alpha}^\alpha \Delta^2 h + \frac{\alpha}{2} B_{GN} \|h\|_{L^\alpha}^{\alpha - \alpha_0} \|h\|_{\dot{H}^2}^\alpha h - \frac{\alpha + 2}{2} \|h\|^\alpha h = 0, \tag{3.2}
\]
respectively.

**Proof.** If $g$ is a maximizer of $H$ in $\dot{H}^\gamma \cap \dot{H}^2$, then $g$ must satisfy the Euler-Lagrange equation
\[
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} H(g + \epsilon \phi) = 0,
\]
for all $\phi \in S_0$. A direct computation shows
\[
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \|g + \epsilon \phi\|_{L^{\alpha + 2}}^{\alpha + 2} = (\alpha + 2) \int \text{Re} \,(|g|^\alpha g \overline{\phi}) dx,
\]
\[
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \|g + \epsilon \phi\|_{\dot{H}^{\gamma}}^\alpha = \alpha \|g\|_{\dot{H}^{\gamma}}^{\alpha - 2} \int \text{Re} \,(\Delta^\gamma g \overline{\phi}) dx,
\]
and
\[
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \|g + \epsilon \phi\|_{\dot{H}^2}^2 = 2 \int \text{Re} \, (\Delta^2 g \overline{\phi}) dx.
\]
We thus get
\[
(\alpha + 2) \|g\|_{\dot{H}^{\gamma}}^{\alpha} \|g\|_{\dot{H}^2}^2 |g|^\alpha g - \alpha \|g\|_{L^{\alpha + 2}}^{\alpha + 2} \|g\|_{\dot{H}^{\gamma}}^{\alpha - 2} \|g\|_{\dot{H}^2}^2 \Delta^\gamma g - 2 \|g\|_{L^{\alpha + 2}}^{\alpha + 2} \|g\|_{\dot{H}^{\gamma}} \Delta^2 g = 0.
\]
Dividing by $2 \|g\|_{H^\gamma}^\gamma \|g\|_{H^2}^2$, we obtain (3.1). The proof of (3.2) is similar using the fact that

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \|h + \epsilon \phi\|_{L^\infty}^\alpha = \alpha \|h\|_{L^\infty}^{\alpha-\alpha\epsilon} \int \text{Re} (|h|^{\alpha\epsilon-2}h\phi)dx.$$ 

The proof is complete. \hfill \Box

We next use the profile decomposition given in Theorem 2.9 to obtain the following variational structure of the sharp constants $A_{GN}$ and $B_{GN}$.

**Proposition 3.2 (Variational structure of sharp constants).** Let $d \geq 1$ and $2_* < \alpha < 2^*$. 

- The sharp constant $A_{GN}$ is attained at a function $U \in H^\gamma \cap H^2$ of the form
  $$U(x) = aQ(\lambda x + x_0),$$
  for some $a \in \mathbb{C}^*, \lambda > 0$ and $x_0 \in \mathbb{R}^d$, where $Q$ is a solution to the elliptic equation
  $$\Delta^2 Q + (-\Delta)^\gamma Q - |Q|^\alpha Q = 0. \quad (3.3)$$
  Moreover,
  $$A_{GN} = \frac{\alpha + 2}{2} \|Q\|_{H^\gamma}^{-\alpha}.

- The sharp constant $B_{GN}$ is attained at a function $V \in L^\alpha \cap H^2$ of the form
  $$V(x) = bR(\mu x + y_0),$$
  for some $b \in \mathbb{C}^*, \mu > 0$ and $y_0 \in \mathbb{R}^d$, where $R$ is a solution to the elliptic equation
  $$\Delta^2 R + |R|^{\alpha - 2} R - |R|^\alpha R = 0. \quad (3.4)$$
  Moreover,
  $$B_{GN} = \frac{\alpha + 2}{2} \|R\|_{L^\infty}^{-\alpha}.$$

**Proof.** We only give the proof for $A_{GN}$, the one for $B_{GN}$ is treated similarly using the Sobolev embedding $H^\gamma \hookrightarrow L^\alpha$. We firstly observe that $H$ is invariant under the scaling

$$f_{\mu,\lambda}(x) := \mu f(\lambda x), \quad \mu, \lambda > 0.$$

Indeed, a simple computation shows

$$\|f_{\mu,\lambda}\|_{L^{\alpha+2}}^{\alpha+2} = \mu^{\alpha+2} \lambda^{-d} \|f\|_{L^{\alpha+2}}^{\alpha+2}, \quad \|f_{\mu,\lambda}\|_{H^\gamma}^\alpha = \mu^\alpha \lambda^{-4} \|f\|_{H^\gamma}^\alpha.$$

We thus get $H(f_{\mu,\lambda}) = H(f)$ for any $\mu, \lambda > 0$. Moreover, if we set $g(x) = \mu f(\lambda x)$ with

$$\mu = \left( \frac{\|f\|_{H^\gamma}^{\frac{\alpha+2}{2}}}{\|f\|_{H^2}^{\frac{\alpha}{2}}} \right)^{\frac{1}{\alpha \lambda}}, \quad \lambda = \left( \frac{\|f\|_{H^\gamma}^{\frac{\alpha}{2}}}{\|f\|_{H^2}^{\frac{\alpha}{2}}} \right)^{\frac{1}{\alpha \lambda}},$$

then $\|g\|_{H^\gamma} = \|g\|_{H^2} = 1$ and $H(g) = H(f)$. Now let $(v_n)_{n \geq 1}$ be the maximizing sequence such that $H(v_n) \to A_{GN}$ as $n \to \infty$. After scaling, we may assume that $\|v_n\|_{H^\gamma} = \|v_n\|_{H^2} = 1$ and $H(v_n) = \|v_n\|_{L^{\alpha+2}} \to A_{GN}$ as $n \to \infty$. Since $(v_n)_{n \geq 1}$ is bounded in $H^\gamma \cap H^2$, it follows from the profile decomposition given in Theorem 2.9 that there exist a sequence $(V^j)_{j \geq 1}$ of $H^\gamma \cap H^2$ functions and a family $(x_n^j)_{j \geq 1}$ of sequences in $\mathbb{R}^d$ such that up to a subsequence,

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x),$$
and \((2.15)\) and the identities \((2.16), (2.17)\) hold. In particular, we have for any \(l \geq 1\),

\[
\sum_{j=1}^{l} \|V^j\|^2_{H^\infty} \leq 1, \quad \sum_{j=1}^{l} \|V^j\|^2_{H^2} \leq 1,
\]

and

\[
\limsup_{n \to \infty} \|v_n\|_{L^{\alpha+2}}^{\alpha+2} \to 0, \quad \text{as} \quad l \to \infty.
\]

We have

\[
A_{GN} = \lim_{n \to \infty} \|v_n\|_{L^{\alpha+2}}^{\alpha+2} = \limsup_{n \to \infty} \left( \sum_{j=1}^{l} V^j(\cdot - x_n^j) \right)^{\alpha+2}
\]

\[
\leq \limsup_{n \to \infty} \left( \sum_{j=1}^{l} \|V^j(\cdot - x_n^j)\|_{L^{\alpha+2}} + \|v_n\|_{L^{\alpha+2}} \right)^{\alpha+2}
\]

\[
\leq \limsup_{n \to \infty} \left( \sum_{j=1}^{\infty} \|V^j(\cdot - x_n^j)\|_{L^{\alpha+2}} \right)^{\alpha+2}.
\]

By the elementary inequality

\[
\left| \sum_{j=1}^{l} a_j \right|^{\alpha+2} \leq \sum_{j=1}^{l} |a_j|^{\alpha+2} \leq C \sum_{j \neq k} |a_j| |a_k|^{\alpha+1},
\]

we have

\[
\int \left( \sum_{j=1}^{l} V^j(x - x_n^j) \right)^{\alpha+2} dx \leq \sum_{j=1}^{l} \int |V^j(x - x_n^j)|^{\alpha+2} dx + C \sum_{j \neq k} \int |V^j(x - x_n^j)||V^k(x - x_n^j)|^{\alpha+1} dx
\]

\[
\leq \sum_{j=1}^{l} \int |V^j(x - x_n^j)|^{\alpha+2} dx + C \sum_{j \neq k} \int |V^j(x + x_n^k - x_n^j)||V^k(x)|^{\alpha+1} dx.
\]

Using the pairwise orthogonality \((2.14)\), the Hölder inequality implies that \(V^j(\cdot + x_n^k - x_n^j) \to 0\)

in \(H^\infty \cap H^2\) as \(n \to \infty\) for any \(j \neq k\). This leads to the mixed terms in the sum \((3.6)\) vanish as \(n \to \infty\). This shows that

\[
A_{GN} \leq \sum_{j=1}^{\infty} \|V^j\|^2_{L^{\alpha+2}}.
\]

By the definition of \(A_{GN}\), we have

\[
\frac{\|V^j\|^2_{L^{\alpha+2}}}{A_{GN}} \leq \|V^j\|^2_{H^\infty} \|V^j\|^2_{H^2}.
\]

This implies that

\[
1 \leq \frac{\sum_{j=1}^{\infty} \|V^j\|^2_{L^{\alpha+2}}}{A_{GN}} \leq \sup_{j \geq 1} \|V^j\|^2_{H^\infty} \sum_{j=1}^{\infty} \|V^j\|^2_{H^2}.
\]

Since \(\sum_{j \geq 1} \|V^j\|^2_{H^\infty}\) is convergent, there exists \(j_0 \geq 1\) such that

\[
\|V^{j_0}\|^2_{H^\infty} = \sup_{j \geq 1} \|V^j\|^2_{H^\infty}.
\]
By (3.5), we see that
\[ 1 \leq \|V^{j_0}\|_{H^\gamma_c}^2 \sum_{j=1}^{\infty} \|V^j\|_{H^2}^2 \leq \|V^{j_0}\|_{H^\gamma_c}^2. \]
It follows from (3.5) that \( \|V^{j_0}\|_{H^\gamma_c} = 1 \). This shows that there is only one term \( V^{j_0} \) is non-zero, hence
\[ \|V^{j_0}\|_{H^\gamma_c} = \|V^{j_0}\|_{H^2} = 1, \quad \|V^{j_0}\|_{L^{\alpha+2}} = A_{GN}. \]
It means that \( V^{j_0} \) is the maximizer of \( H \) and Lemma 3.1 shows that
\[ A_{GN} \Delta^2 V^{j_0} + \frac{\alpha}{2} A_{GN} (-\Delta)^\gamma V^{j_0} - \frac{\alpha + 2}{2} |V^{j_0}|^\alpha V^{j_0} = 0. \]
Now if we set \( V^{j_0}(x) = aQ(\lambda x + x_0) \) for some \( a \in \mathbb{C}, \lambda > 0 \) and \( x_0 \in \mathbb{R}^d \) to be chosen shortly, then \( Q \) solves (3.3) provided that
\[ |a| = \left( \frac{2\lambda^4 A_{GN}}{\alpha + 2} \right)^{\frac{1}{2}}, \quad \lambda = \left( \frac{\alpha}{2} \right)^{\frac{1}{\alpha+2}}. \]  
(3.8)
This shows the existence of solutions to the elliptic equation (3.3). We now compute the sharp constant \( A_{GN} \) in terms of \( Q \). We have
\[ 1 = \|V^{j_0}\|_{H^\gamma_c}^\alpha = |a|^\alpha \lambda^{-d} \|Q\|_{H^\gamma_c}^\alpha = \frac{2A_{GN}}{\alpha + 2} \|Q\|_{H^\gamma_c}^\alpha. \]
This implies \( A_{GN} = \frac{\alpha + 2}{2} \|Q\|_{H^\gamma_c}^{-\alpha}. \) The proof is complete. \( \Box \)

**Remark 3.3.** Using (3.8) and the fact
\[ 1 = \|V^{j_0}\|_{H^\gamma_c}^\alpha = |a|^\alpha \lambda^{-d} \|Q\|_{H^\gamma_c}^\alpha, \]
\[ 1 = \|V^{j_0}\|_{H^2}^2 = |a|^2 \lambda^{4-d} \|Q\|_{H^2}^2, \]
\[ A_{GN} = \|V^{j_0}\|_{L^{\alpha+2}}^\alpha = |a|^\alpha \lambda^{-d} \|Q\|_{L^{\alpha+2}}^\alpha, \]
a direct computation shows the following Pohozaev identities
\[ \|Q\|_{H^\gamma_c}^2 = \frac{\alpha}{2} \|Q\|_{H^\gamma_c}^2 = \frac{\alpha}{\alpha + 2} \|Q\|_{L^{\alpha+2}}^{\alpha+2}. \]  
(3.9)
Another way to see above identities is to multiply (3.3) with \( \overline{Q} \) and \( x \cdot \nabla \overline{Q} \) and integrate over \( \mathbb{R}^d \) and perform integration by parts. Indeed, multiplying (3.3) with \( \overline{Q} \) and integrating by parts, we get
\[ \|Q\|_{H^2}^2 + \|Q\|_{H^\gamma_c}^2 - \|Q\|_{L^{\alpha+2}}^{\alpha+2} = 0. \]  
(3.10)
Multiplying (3.3) with \( x \cdot \nabla \overline{Q} \), integrating by parts and taking the real part, we have
\[ \left( 2 - \frac{d}{2} \right) \|Q\|_{H^2}^2 + \left( \gamma_c - \frac{d}{2} \right) \|Q\|_{H^\gamma_c}^2 + \frac{d}{\alpha + 2} \|Q\|_{L^{\alpha+2}}^{\alpha+2} = 0. \]  
(3.11)
From (3.10) and (3.11), we obtain (3.9). To see (3.11), we claim that for \( \gamma \geq 0, \)
\[ \Re \int (-\Delta)^\gamma Q x \cdot \nabla \overline{Q} dx = \left( \gamma - \frac{d}{2} \right) \|Q\|_{H^\gamma_c}^2. \]  
(3.12)
In fact, by Fourier transform,
\[
\text{Re} \int (-\Delta)^j Q x \cdot \nabla Q dx = \text{Re} \int \mathcal{F}(-\Delta)^j Q \mathcal{F}^{-1}[x \cdot \nabla Q] d\xi \\
= \text{Re} \int \mathcal{F}(-\Delta)^j Q \mathcal{F}[x \cdot \nabla Q] d\xi \\
= \text{Re} \int |\xi|^{2j} \mathcal{F}(Q) (\left( -d \mathcal{F}(Q) - \xi \cdot \nabla \mathcal{F}(Q) \right) d\xi \\
= -d\|Q\|^2_{H^j} - \text{Re} \int |\xi|^{2j} \mathcal{F}(Q) \xi \cdot \nabla \mathcal{F}(Q) d\xi. \tag{3.13}
\]

Here we use the fact that \( \mathcal{F}(x_j\partial_x u) = i\partial_\xi \mathcal{F}(\partial_x u) = i\partial_\xi (i\xi_j \mathcal{F}(u)) = -\mathcal{F}(u) - \xi_j \partial_\xi \mathcal{F}(u) \). By integration by parts,
\[
\text{Re} \int |\xi|^{2j} \mathcal{F}(Q) \xi \cdot \nabla \mathcal{F}(Q) d\xi = (-2\gamma - d)\|Q\|^2_{H^\gamma} - \text{Re} \int |\xi|^{2j} \mathcal{F}(Q) \mathcal{F}(Q) d\xi,
\]
or
\[
\text{Re} \int |\xi|^{2j} \mathcal{F}(Q) \xi \cdot \nabla \mathcal{F}(Q) d\xi = \left( -\gamma - \frac{d}{2} \right)\|Q\|^2_{H^\gamma}.
\]

This together with (3.13) shows (3.12), and (3.11) follows.

The Pohozaev identities (3.9) imply in particular that
\[
H(Q) = \|Q\|^{\alpha + 2}_{L^{\alpha + 2}} + \|[Q]_{H_\gamma}^\alpha \|Q\|^2_{H^2} = \frac{\alpha + 2}{2} \|Q\|^{-\alpha}_{H_\gamma} = A_{GN}, \quad E(Q) = 0.
\]

Similarly, we have
\[
\|R\|^2_{L^{\alpha}} = \frac{\alpha}{2} \|R\|^2_{H^2} = \frac{\alpha}{\alpha + 2} \|R\|^{\alpha + 2}_{L^{\alpha + 2}}.
\]

In particular,
\[
K(R) = \|R\|^{\alpha + 2}_{L^{\alpha + 2}} + \|[R]_{L_\gamma}^\alpha \|R\|^2_{H^2} = \frac{\alpha + 2}{2} \|R\|^{-\alpha}_{L_\gamma} = B_{GN}, \quad E(R) = 0.
\]

**Definition 3.4 (Ground state).**

- We call **Sobolev ground states** the maximizers of \( H \) which are solutions to (3.3). We denote the set of Sobolev ground states by \( G \).
- We call **Lebesgue ground states** the maximizers of \( K \) which are solutions to (3.4). We denote the set of Lebesgue ground states by \( H \).

Note that by Lemma 3.1, if \( g, h \) are Sobolev and Lebesgue ground states respectively, then
\[
A_{GN} = \frac{\alpha + 2}{2} \|g\|^{-\alpha}_{H_\gamma}, \quad B_{GN} = \frac{\alpha + 2}{2} \|h\|^{-\alpha}_{L_\gamma}.
\]

This implies that Sobolev ground states have the same \( H_\gamma \)-norm, and all Lebesgue ground states have the same \( L_\gamma \)-norm. Denote
\[
S_{gs} := \|g\|_{H_\gamma}, \quad \forall g \in G, \tag{3.14}
\]
\[
L_{gs} := \|h\|_{L_\gamma}, \quad \forall h \in H. \tag{3.15}
\]

In particular, we have the following sharp Gagliardo-Nirenberg inequalities
\[
\|f\|^{\alpha + 2}_{L^{\alpha + 2}} \leq A_{GN} \|f\|^{\alpha}_{H_\gamma} \|f\|^2_{H^2}, \tag{3.16}
\]
\[
\|f\|^{\alpha + 2}_{L^{\alpha + 2}} \leq B_{GN} \|f\|^{\alpha}_{L_\gamma} \|f\|^2_{H^2}, \tag{3.17}
\]

with
\[
A_{GN} = \frac{\alpha + 2}{2} S_{gs}^{-\alpha}, \quad B_{GN} = \frac{\alpha + 2}{2} L_{gs}^{-\alpha}.
\]

We next give another application of the profile decomposition given in Theorem 2.9.
Theorem 3.5 (Compactness lemma). Let \( d \geq 1 \) and \( 2_s < \alpha < 2^* \). Let \((v_n)_{n \geq 1}\) be a bounded sequence in \( H^{\gamma_c} \cap \dot{H}^2 \) such that
\[
\limsup_{n \to \infty} \|v_n\|_{\dot{H}^2} \leq M, \quad \limsup_{n \to \infty} \|v_n\|_{L^{\alpha+2}} \geq m.
\]

- Then there exists a sequence \((x_n)_{n \geq 1}\) in \( \mathbb{R}^d \) such that up to a subsequence,
  \[ v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^{\gamma_c} \cap \dot{H}^2, \]
  for some \( V \in H^{\gamma_c} \cap \dot{H}^2 \) satisfying
  \[
  \|V\|_{H^{\gamma_c}}^\alpha \geq \frac{2}{\alpha + 2} \frac{m^{\alpha+2}}{M^2 S_{\alpha}^\Omega_x}. \tag{3.18}
  \]
- Then there exists a sequence \((y_n)_{n \geq 1}\) in \( \mathbb{R}^d \) such that up to a subsequence,
  \[ v_n(\cdot + y_n) \rightharpoonup W \text{ weakly in } \alpha^c \cap \dot{H}^2, \]
  for some \( W \in L^{\alpha^c} \cap \dot{H}^2 \) satisfying
  \[
  \|W\|_{L^{\alpha^c}}^\alpha \geq \frac{2}{\alpha + 2} \frac{m^{\alpha+2}}{M^2 L_{\alpha}^x}. \tag{3.19}
  \]

Remark 3.6. The lower bounds (3.18) and (3.19) are optimal. In fact, if we take \( v_n = Q \in \mathcal{G} \) in the first case and \( v_n = R \in \mathcal{H} \) in the second case, then we get the equalities.

Proof of Theorem 3.5. As in the proof of Proposition 3.2, we only consider the first case, the second case is similar using the Sobolev embedding \( H^{\gamma_c} \hookrightarrow L^{\alpha^c} \). According to Theorem 2.9, there exist a sequence \((V^j)_{j \geq 1}\) of \( H^{\gamma_c} \cap \dot{H}^2 \) functions and a family \((x^j_n)_{n \geq 1}\) of sequences in \( \mathbb{R}^d \) such that up to a subsequence, the sequence \((v_n)_{n \geq 1}\) can be written as
\[
v_n(x) = \sum_{j=1}^l V^j(x - x^j_n) + v^j_n(x),
\]
and (2.15), (2.16), (2.17) hold. This implies that
\[
m^{\alpha+2} \leq \limsup_{n \to \infty} \|v_n\|_{L^{\alpha+2}}^{\alpha+2} = \limsup_{n \to \infty} \left\| \sum_{j=1}^l V^j(\cdot - x^j_n) + v^j_n \right\|_{L^{\alpha+2}}^{\alpha+2}
\]
\[
\leq \limsup_{n \to \infty} \left( \left\| \sum_{j=1}^l V^j(\cdot - x^j_n) \right\|_{L^{\alpha+2}} + \|v^j_n\|_{L^{\alpha+2}} \right)^{\alpha+2}
\]
\[
\leq \limsup_{n \to \infty} \left( \sum_{j=1}^l \|V^j(\cdot - x^j_n)\|_{L^{\alpha+2}}^\alpha \right)^{\alpha+2}. \tag{3.20}
\]
By the elementary inequality (3.7) and the pairwise orthogonality (2.14), the mixed terms in the sum (3.20) vanish as \( n \to \infty \). We thus get
\[
m^{\alpha+2} \leq \sum_{j=1}^\infty \|V^j\|_{L^{\alpha+2}}^{\alpha+2}.
\]
We next use the sharp Gagliardo-Nirenberg inequality (3.16) to estimate
\[
\sum_{j=1}^\infty \|V^j\|_{L^{\alpha+2}}^{\alpha+2} \leq \frac{\alpha + 2}{\alpha} \frac{1}{S_{\alpha}^\Omega} \sup_{j \geq 1} \|V^j\|_{H^{\gamma_c}}^\alpha \sum_{j=1}^\infty \|V^j\|_{H^2}^2. \tag{3.21}
\]
By \((2.17)\), we infer that
\[
\sum_{j=1}^{\infty} \|V_j^j\|^2_{\dot{H}^2} \leq \limsup_{n \to \infty} \|v_n\|^2_{\dot{H}^2} \leq M^2.
\]
Therefore,
\[
\sup_{j \geq 1} \|V_j^j\|^2_{\dot{H}^\infty} \geq \frac{2}{\alpha + 2} \frac{m^{\alpha+2}}{M^2} S^{\alpha}_{22}.
\]
Since the series \(\sum_{j \geq 1} \|V_j^j\|^2_{\dot{H}^\infty}\) is convergent, the supremum above is attained. In particular, there exists \(j_0\) such that
\[
\|V_j^j\|^\alpha_{\dot{H}^\infty} \geq \frac{2}{\alpha + 2} \frac{m^{\alpha+2}}{M^2} S^{\alpha}_{22}.
\]
By a change of variables, we write
\[
v_n(x + x_n^j) = V_j^j(x) + \sum_{i \leq j \leq n, i \neq j_0} V_i^j(x + x_n^j - x_i^j) + \tilde{v}_n^j(x),
\]
where \(\tilde{v}_n^j(x) := v_n(x + x_n^j)\). The pairwise orthogonality of the family \((x_n^j)_{j \geq 1}\) implies
\[
V_j^j(x + x_n^j - x_j^j) \to 0 \text{ weakly in } \dot{H}^\infty \cap \dot{H}^2,
\]
as \(n \to \infty\) for every \(j \neq j_0\). We thus get
\[
v_n(x + x_n^j) \to V_j^j + \tilde{v}^j, \quad \text{as } n \to \infty, \tag{3.22}
\]
where \(\tilde{v}^j\) is the weak limit of \((\tilde{v}_n^j)_{n \geq 1}\). On the other hand,
\[
\|\tilde{v}^j\|_{L^{\alpha+2}} \leq \limsup_{n \to \infty} \|\tilde{v}_n^j\|_{L^{\alpha+2}} = \limsup_{n \to \infty} \|v_n^l\|_{L^{\alpha+2}} \to 0, \quad \text{as } l \to \infty.
\]

By the uniqueness of the weak limit \((3.22)\), we get \(\tilde{v}^l = 0\) for every \(l \geq j_0\). Therefore, we obtain
\[
v_n(x + x_n^j) \to V_j^j.
\]
The sequence \((x_n^j)_{n \geq 1}\) and the function \(V_j^j\) now fulfill the conditions of Theorem 3.5. The proof is complete. \(\square\)

4. Global existence and blowup

We firstly use the sharp Gagliardo-Nirenberg inequality \((3.16)\) to show the following global existence.

**Proposition 4.1** (Global existence in \(\dot{H}^\infty \cap \dot{H}^2\)). Let \(\hat{d} \geq 5\) and \(2_* < \alpha < 2^*\). Let \(u_0 \in \dot{H}^\infty \cap \dot{H}^2\) and the corresponding solution \(u\) to \((1.1)\) defined on the maximal time \([0,T)\). Assume that
\[
\sup_{t \in [0,T)} \|u(t)\|_{\dot{H}^\infty} < S_{22}^{\alpha}.
\]
Then \(T = \infty\), i.e. the solution exists globally in time.

**Proof.** By the sharp Gagliardo-Nirenberg inequality \((3.16)\), we bound
\[
E(u(t)) = \frac{1}{2} \|u(t)\|^2_{\dot{H}^2} - \frac{1}{\alpha + 2} \|u(t)\|^{\alpha + 2}_{L^{\alpha+2}} \\
\geq \frac{1}{2} \left( 1 - \left( \frac{\|u(t)\|_{\dot{H}^\infty}}{S_{22}^{\alpha}} \right)^\alpha \right) \|u(t)\|^2_{\dot{H}^2}.
\]
Thanks to the conservation of energy and the assumption \((4.23)\), we obtain \(\sup_{t \in [0,T)} \|u(t)\|_{\dot{H}^2} < \infty\). By the blowup alternative given in Proposition 2.6 and \((4.23)\), the solution exists globally in time. The proof is complete. \(\square\)
We also have the following global well-posedness result.

**Proposition 4.2.** Let \( d \geq 5 \) and \( 2^* < \alpha < 2^* \). Let \( u_0 \in H^\infty \cap \dot{H}^2 \) and the corresponding solution \( u \) to (1.1) defined on the maximal time \([0, T]\). Assume that

\[
S_{gs} \leq \sup_{t \in [0, T]} \|u(t)\|_{H^\infty} < \infty, \quad \sup_{t \in [0, T]} \|u(t)\|_{L^\infty} < L_{gs}. \tag{4.24}
\]

Then \( T = \infty \), i.e. the solution exists globally in time.

The proof is similar to the one of Proposition 4.1 by using the shap Gagliardo-Nirenberg inequality (3.17).

We next recall blowup criteria for \( H^2 \) solutions to the equation (1.1) due to [5].

**Proposition 4.3 (Blowup in \( H^2 \) [5]).** Let \( d \geq 2 \), \( 2^* < \alpha < 2^* \), \( \alpha \leq 8 \) and \( u_0 \in H^2 \) be radial. Assume that

\[
E(u_0)M(u_0)\sigma < E(Q)M(Q)^\sigma, \quad \|u_0\|_{H^2} \|u_0\|_\dot{L}^2 > \|Q\|_{H^2} \|Q\|_\dot{L}^2,
\]

where

\[
\sigma := \frac{2 - \gamma_c}{\gamma_c} = \frac{8 - (d - 4)\alpha}{d\alpha - 8}. \tag{4.25}
\]

Then the corresponding solution \( u \) to (1.1) blows up in finite time.

**Remark 4.4.**
- The restriction \( \alpha \leq 8 \) comes from the radial Sobolev embedding (or Strauss’s inequality). An analogous restriction on \( \alpha \) appears in the blowup of \( H^1 \) solutions for the nonlinear Schrödinger equation.
- Note that if \( E(u_0) < 0 \), then the assumption \( E(u_0)M(u_0)^\sigma < E(Q)M(Q)^\sigma \) holds trivially.

If we assume \( u_0 \in H^\infty \cap \dot{H}^2 \), then the above blowup criteria does not hold due to the lack of mass conservation. Nevertheless, we have the following blowup criteria for initial data in \( H^\infty \cap \dot{H}^2 \).

**Proposition 4.5 (Blowup in \( \dot{H}^\infty \cap \dot{H}^2 \)).** Let \( d \geq 5 \), \( 2^* < \alpha < 2^* \), \( \alpha < 4 \) and \( u_0 \in \dot{H}^\infty \cap \dot{H}^2 \) be radial satisfying \( E(u_0) < 0 \). Assume that the corresponding solution \( u \) to (1.1) defined on a maximal interval \([0, T]\) satisfies

\[
\sup_{t \in [0, T]} \|u(t)\|_{\dot{H}^\infty} < \infty. \tag{4.26}
\]

Then the solution \( u \) to (1.1) blows up in finite time.

**Proof.** Let \( \theta : [0, \infty) \to [0, \infty) \) be a smooth function such that

\[
\theta(r) = \begin{cases} 
  r^2 & \text{if } r \leq 1, \\
  0 & \text{if } r \geq 2,
\end{cases}
\]

and \( \theta''(r) \leq 2 \) for \( r \geq 0 \).

For \( R > 0 \) given, we define the radial function \( \varphi_R : \mathbb{R}^d \to \mathbb{R} \) by

\[
\varphi_R(x) = \varphi_R(r) := R^2 \theta(r/R), \quad |x| = r. \tag{4.27}
\]

By definition, we have

\[
2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi''_R(r)}{r} \geq 0, \quad 2d - \Delta \varphi_R(x) \geq 0, \quad \text{for all } r \geq 0 \text{ and all } x \in \mathbb{R}^d,
\]

and

\[
\|\nabla^j \varphi_R\|_{L^\infty} \lesssim R^{2 - j}, \quad \text{for } j = 0, \cdots, 6,
\]

and also,

\[
\text{supp}(\nabla^j \varphi_R) \subset \begin{cases} 
  \{|x| \leq 2R\} & \text{for } j = 1, 2, \\
  \{R \leq |x| \leq 2R\} & \text{for } j = 3, \cdots, 6.
\end{cases}
\]
Let $u \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ be a solution to (1.1). We define the localized virial action associated to (1.1) by

$$M_{\varphi_R}(t) := 2 \int \nabla \varphi_R(x) \cdot \text{Im} (\overline{\varphi(t,x)} \nabla u(t,x)) dx. \quad (4.28)$$

We firstly show that $M_{\varphi_R}(t)$ is well-defined. To do so, we need the following estimate

$$\|u\|_{L^2(|x| \leq R)} \lesssim R^{\gamma_c} \|u\|_{L^{\alpha_c}(|x| \leq R)} \lesssim R^{\gamma_c} \|u\|_{H^{\alpha_c}(|x| \leq R)}, \quad (4.29)$$

which follows easily by Hölder’s inequality and the Sobolev embedding. Here $\gamma_c$ and $\alpha_c$ are given in (1.4) and (1.5) respectively. Since $\nabla \varphi_R$ is supported in $|x| \leq R$, the Hölder inequality together with (4.29) imply

$$|M_{\varphi_R}(t)| \lesssim \|\nabla \varphi_R\|_{L^\infty} \|u(t)\|_{L^2(|x| \leq R)} \|\nabla u(t)\|_{L^2(|x| \leq R)}$$

$$\lesssim \|\nabla \varphi_R\|_{L^\infty} \|u(t)\|_{L^2(|x| \leq R)}^{3/2} \|\Delta u(t)\|_{L^2(|x| \leq R)}^{1/2}$$

$$\lesssim R^{3\gamma_c/2} \|\nabla \varphi_R\|_{L^\infty} \|\Delta u(t)\|_{H^{\alpha_c}(|x| \leq R)}^{3/2} \|u(t)\|_{H^{\alpha_c}(|x| \leq R)}^{1/2}.$$  

Note that in the case $\theta(r) = r^2$ and $\varphi_R(x) = |x|^2$, we have formally the virial law (see e.g. [5]):

$$M_{||x||^2}(t) = \frac{d}{dt} \left( 4 \int x \cdot \text{Im} (\overline{\varphi(t,x)} \nabla u(t,x)) dx \right) = 16 \|\Delta u(t)\|_{L^2}^2 - \frac{4d\alpha}{\alpha + 2} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \quad (4.30)$$

$$= 4d\alpha E(u(t)) - 2(d\alpha - 8) \|\Delta u(t)\|_{L^2}^2.$$

We have the following variation rate of the virial action (see e.g. [5, Lemma 3.1] or [24, Proposition 3.1]):

$$M'_{\varphi_R}(u(t)) = \int \Delta^3 \varphi_R |u|^2 dx - 4 \sum_{j,k} \int \partial_{jk}^2 \Delta \varphi_R \text{Re} (\partial_j \overline{\varphi(t)} \partial_k u(t)) dx + 8 \sum_{j,k,l} \int \partial_{jk}^2 \varphi_R \text{Re} (\partial_j \overline{\varphi(t)} \partial_k \partial_l u(t)) dx$$

$$- 2 \int \Delta^2 \varphi_R |\nabla u|^2 dx - \frac{2\alpha}{\alpha + 2} \int \Delta \varphi_R |u|^{\alpha+2} dx. \quad (4.31)$$

Since $\varphi_R(x) = |x|^2$ for $|x| \leq R$, we use (4.30) to have

$$M'_{\varphi_R}(t) = 16 \|\Delta u(t)\|_{L^2}^2 - \frac{4d\alpha}{\alpha + 2} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} - 16 \|\Delta u(t)\|_{L^2(|x| > R)}^2 + \frac{4d\alpha}{\alpha + 2} \|u(t)\|_{L^{\alpha+2}(|x| > R)}^{\alpha+2}$$

$$+ \int_{|x| > R} \Delta^3 \varphi_R |u(t)|^2 dx - 4 \sum_{j,k} \int_{|x| > R} \partial_{jk}^2 \Delta \varphi_R \text{Re} (\partial_j \overline{\varphi(t)} \partial_k u(t)) dx$$

$$+ 8 \sum_{j,k,l} \int_{|x| > R} \partial_{jk}^2 \varphi_R \text{Re} (\partial_j \overline{\varphi(t)} \partial_k \partial_l u(t)) dx$$

$$- 2 \int_{|x| > R} \Delta^2 \varphi_R |\nabla u(t)|^2 dx - \frac{2\alpha}{\alpha + 2} \int_{|x| > R} \Delta \varphi_R |u(t)|^{\alpha+2} dx$$

$$= 4d\alpha E(u(t)) - 2(d\alpha - 8) \|\Delta u(t)\|_{L^2}^2 + \int_{|x| > R} \Delta^3 \varphi_R |u(t)|^2 dx$$

$$- 4 \sum_{j,k} \int_{|x| > R} \partial_{jk}^2 \Delta \varphi_R \text{Re} (\partial_j \overline{\varphi(t)} \partial_k u(t)) dx - 2 \int_{|x| > R} \Delta^2 \varphi_R |\nabla u(t)|^2 dx.$$
\begin{align*}
+ 8 \sum_{j,k,l} \int_{|x| > R} \partial^2_j \varphi_R \Re (\partial^2_j \overline{\varphi(t)} \partial^2_l u(t)) dx - 16 \| \Delta u(t) \|^2_{L^2(|x| > R)} & \\
+ \frac{2 \alpha}{\alpha + 2} \int_{|x| > R} (2d - \Delta \varphi_R) |u(t)|^{\alpha + 2} dx.
\end{align*}

By the choice of \( \varphi_R \), the assumption (4.26) and (4.29), we bound
\begin{align*}
\left| \int_{|x| > R} \Delta^3 \varphi_R |u(t)|^2 dx \right| & \lesssim R^{-4} \| u(t) \|^2_{L^2(|x| \leq R)} \lesssim R^{-2(2 - \gamma)\epsilon} \| u(t) \|^2_{L^2} \lesssim R^{-2(2 - \gamma)}, \\
\left| \int_{|x| > R} \partial^2_j \Delta \varphi_R \partial_j \overline{\varphi(t)} \partial_k u(t) dx \right| & \lesssim R^{-2} \| \nabla u \|^2_{L^2(|x| \leq R)} \lesssim R^{-2(2 - \gamma)\epsilon} \| u(t) \|_{H^{\gamma} \infty} \| \Delta u(t) \|_{L^2} \lesssim R^{-2(2 - \gamma)} \| \Delta u(t) \|_{L^2}, \\
\left| \int_{|x| > R} \Delta^2 \varphi_R |\nabla u(t)|^2 dx \right| & \lesssim R^{-2(2 - \gamma)\epsilon} \| u(t) \|_{H^{\gamma} \infty} \| \Delta u(t) \|_{L^2} \lesssim R^{-2(2 - \gamma)} \| \Delta u(t) \|_{L^2}.
\end{align*}

Using the fact
\[ \partial^2_{jk} = \left( \delta_{jk} - \frac{x_j x_k}{r^2} \right) \frac{\partial}{\partial r} + \frac{x_j x_k}{r^2} \partial^2_r, \]
a calculation combined with integration by parts yields
\begin{align*}
\sum_{j,k,l} \int \partial^2_j \varphi_R \partial^2_j \overline{\varphi(t)} \partial^2_l u(t) dx &= \int \varphi''_R \partial^2_l u(t)^2 + \frac{d - 1}{r^2} \varphi'_R |\partial_r u(t)|^2 dx \\
&= 2 \int |\Delta u(t)|^2 - (2 - \varphi''_R) |\partial^2_l u(t)|^2 - \left( 2 - \frac{\varphi'_R}{r} \right) \frac{d - 1}{r^2} |\partial_r u(t)|^2 dx \\
& \leq 2 \| \Delta u(t) \|^2_{L^2}.
\end{align*}

Here we use the identity
\[ \| \Delta u(t) \|^2_{L^2} = \int |\partial^2_l u(t)|^2 + \frac{d - 1}{r^2} |\partial_r u(t)|^2 dx. \]

Thus,
\[ 8 \sum_{j,k,l} \int_{|x| > R} \partial^2_j \varphi_R \Re (\partial^2_j \overline{\varphi(t)} \partial^2_l u(t)) dx - 16 \| \Delta u(t) \|^2_{L^2(|x| > R)} \leq 0. \]

We obtain
\[ M'_R(\varphi_R(t) \leq 4d \alpha E(u(t)) - 2(d \alpha - 8) \| \Delta u(t) \|^2_{L^2} + O \left( R^{-2(2 - \gamma)\epsilon} + R^{-2(2 - \gamma)} \| \Delta u(t) \|_{L^2} \right) \\
+ \frac{2 \alpha}{\alpha + 2} \int_{|x| > R} (2d - \Delta \varphi_R) |u(t)|^{\alpha + 2} dx. \]

We now estimate the last term of the above inequality. To do so, we use the argument of [21].
Consider for \( A > 0 \) the annulus \( \mathcal{C} = \{ A < |x| \leq 2A \} \), we claim that for any \( \epsilon > 0 \),
\[ \| u(t) \|_{L^{\alpha + 2}(\mathcal{C})}^\alpha \leq \epsilon \| \Delta u(t) \|_{L^2(\mathcal{C})} + C(\epsilon) A^{-2(2 - \gamma)\epsilon}. \]
To see this, we use the radial Sobolev embedding (see e.g. [28]) and (4.29) to estimate
\[ \|u(t)\|_{L^{α+2}} \lesssim \left( \sup_C \|u(t,x)\| \right)^\alpha \|u(t)\|_{L^2(C)}^2 \]
\[ \lesssim A^{-\frac{(d-1)\alpha}{4} + \frac{d}{8}} \|\nabla u(t)\|_{L^2(C)}^{\frac{d}{8}} \|u(t)\|_{L^2(C)}^{\frac{3d}{4} + \frac{d}{8}} \]
\[ \lesssim A^{-\frac{(d-1)\alpha}{4} + \frac{d}{8}} \|\Delta u(t)\|_{L^2(C)}^{\frac{d}{8}} \|u(t)\|_{L^2(C)}^{\frac{3d}{4} + \frac{d}{8}} \]
\[ \lesssim A^{-\vartheta} \|\Delta u(t)\|_{L^2(C)}, \]
where
\[ \vartheta = \frac{(d - 1)\alpha}{2} - \left( \frac{3\alpha}{4} + 2 \right) \gamma_c = 2(2 - \gamma_c)\frac{4 - \alpha}{4} > 0. \]

By the Young inequality, we have for any \( \epsilon > 0 \),
\[ \|u(t)\|_{L^{α+2}} \lesssim \epsilon \|\Delta u(t)\|_{L^2(C)} + \epsilon^{-\frac{d\alpha}{4}} A^{-\frac{d\alpha}{8}} = \epsilon \|\Delta u(t)\|_{L^2(C)} + C(\epsilon)A^{-2(2 - \gamma_c)}. \]

This shows the claim above. Note that the condition \( \alpha < 4 \) is crucial to show (4.32). We now write
\[ \int_{|x| > R} \|u(t)\|_{L^{α+2}} dx = \sum_{j=0}^\infty \int_{2^j R < |x| \leq 2^{j+1} R} \|u(t)\|_{L^{α+2}} dx, \]
and apply (4.32) with \( A = 2^j R \) to get
\[ \int_{|x| > R} \|u(t)\|_{L^{α+2}} dx \leq \epsilon \sum_{j=0}^\infty \|\Delta u(t)\|_{L^2(2^j R < |x| \leq 2^{j+1} R)} + C(\epsilon) \sum_{j=0}^\infty (2^j R)^{-2(2 - \gamma_c)} \]
\[ \leq \epsilon \|\Delta u(t)\|_{L^2(|x| > R)} + C(\epsilon)R^{-2(2 - \gamma_c)}. \]
Since \( 2d - \varphi_R \|_{L^\infty} \lesssim 1 \), we obtain for any \( \epsilon > 0 \),
\[ \int_{|x| > R} (2d - \varphi_R) \|u(t)\|_{L^{α+2}} dx \lesssim \epsilon \|\Delta u(t)\|_{L^2(|x| > R)} + C(\epsilon)R^{-2(2 - \gamma_c)}. \]

Therefore,
\[ \mathcal{M}_\varphi(t) \leq 4d\alpha E(u(t)) - 2(2d - 8)\|\Delta u(t)\|_{L^2}^2 + O\left( R^{-2(2 - \gamma_c)} + R^{-2(2 - \gamma_c)} \|\Delta u(t)\|_{L^2} \right) \]
\[ + \epsilon \|\Delta u(t)\|_{L^2} + C(\epsilon)R^{-2(2 - \gamma_c)} \].

By taking \( \epsilon > 0 \) small enough and \( R > 0 \) large enough depending on \( \epsilon \), the conservation of energy implies
\[ \mathcal{M}_\varphi(t) \leq 2d\alpha E(u_0) - \delta \|\Delta u(t)\|_{L^2}^2, \]
for all \( t \in [0, T) \), where \( \delta := d\alpha - 8 > 0 \). With (4.34) at hand, the finite time blowup follows by a standard argument (see e.g. [5]). \( \square \)

5. Blowup concentration

**Theorem 5.1** (Blowup concentration). Let \( d \geq 5 \) and \( 2_{*} < \alpha < 2^*. \) Let \( u \in \dot{H}^{\gamma_c} \cap \dot{H}^2 \) be such that the corresponding solution \( u \) to (1.1) blows up at finite time \( 0 < T < \infty \). Assume that the solution satisfies
\[ \sup_{t \in [0, T)} \|u(t)\|_{\dot{H}^{\gamma_c}} < \infty. \]
Let $a(t) > 0$ be such that
\[ a(t)\|u(t)\|_{\dot{H}^{\gamma_c}} \to \infty, \] (5.2)
as $t \uparrow T$. Then there exist $x(t), y(t) \in \mathbb{R}^d$ such that
\[ \lim_{t \uparrow T} \inf_{T \leq t \leq T + \delta} \int_{|x - y(t)| \leq a(t)} |(\Delta)^{\frac{\gamma}{2}} u(t, x)|^2 \, dx \geq S^2_{gs}, \] (5.3)
and
\[ \lim_{t \uparrow T} \inf_{T \leq t \leq T + \delta} \int_{|x - y(t)| \leq a(t)} |u(t, x)|^{\alpha} \, dx \geq L^2_{gs}. \] (5.4)

**Remark 5.2.**

- The restriction $d \geq 5$ comes from the local well-posedness and blowup results. The result still holds true for dimensions $d \leq 4$ provided that one can show local well-posedness and blowup in such dimensions.
- By the blowup rate given in Corollary 2.8 and the assumption (5.1), we have
\[ \|u(t)\|_{\dot{H}^2} > \frac{C}{(T - t)^{\frac{2 - \alpha}{4}}} \]
for $t \uparrow T$. Rewriting
\[ \frac{1}{a(t)\|u(t)\|_{\dot{H}^{\gamma_c}}} = \frac{\sqrt{T - t}}{a(t)\|u(t)\|_{\dot{H}^{\gamma_c}}} \]
we see that any function $a(t) > 0$ satisfying $\frac{\sqrt{T - t}}{a(t)} \to 0$ as $t \uparrow T$ fulfills the conditions of Theorem 5.1.

**Proof of Theorem 5.1.** Let $(t_n)_{n \geq 1}$ be a sequence such that $t_n \uparrow T$ and $g \in G$. Set
\[ \lambda_n := \left( \frac{\|g\|_{\dot{H}^2}}{\|u(t_n)\|_{H^{\gamma_c}}} \right)^{\frac{1}{2 - \gamma_c}} \quad \text{and} \quad v_n(x) := \lambda_n^\frac{2}{\gamma_c} u(t_n, \lambda_n x). \]
By the blowup alternative and the assumption (5.1), we see that $\lambda_n \to 0$ as $n \to \infty$. Moreover, we have
\[ \|v_n\|_{H^{\gamma_c}} = \|u(t_n)\|_{H^{\gamma_c}} < \infty, \]
uniformly in $n$ and
\[ \|v_n\|_{\dot{H}^{\gamma_c}} = \lambda_n^{2 - \gamma_c} \|u(t_n)\|_{\dot{H}^2} = \|g\|_{\dot{H}^2}, \]
and
\[ E(v_n) = \lambda_n^{2(2 - \gamma_c)} E(u(t_n)) = \lambda_n^{2(2 - \gamma_c)} E(u_0) \to 0, \quad \text{as} \ n \to \infty. \]
This implies in particular that
\[ \|v_n\|_{L^{\alpha+2}_t} \to \frac{\alpha + 2}{2} \|g\|_{\dot{H}^2}, \quad \text{as} \ n \to \infty. \]
The sequence $(v_n)_{n \geq 1}$ satisfies the conditions of Theorem 3.5 with
\[ m^{\alpha+2} = \frac{\alpha + 2}{2} \|g\|_{\dot{H}^2}, \quad M^2 = \|g\|_{\dot{H}^2}. \]
Therefore, there exists a sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,
\[ v_n(\cdot + x_n) = \lambda_n^\frac{2}{\gamma_c} u(t_n, \lambda_n \cdot + x_n) \to V \text{ weakly in } \dot{H}^{\gamma_c} \cap \dot{H}^2, \]
Proof. We only prove Item 1, Item 2 is treated similarly. Since $E(u) = 0$, we have
\[ \|u\|_{H^\gamma}^2 \geq \frac{2}{\alpha + 2} \|u\|_{L^{\alpha+2}}^{\alpha+2}. \]
Thus
\[ H(u) = \frac{\|u\|_{L^{\alpha+2}}^{\alpha+2}}{\|u\|_{H^\gamma}^2} \geq \frac{\alpha + 2}{2} \frac{\|u\|_{H^\gamma}^\alpha}{\|u\|_{H^\gamma}^2} = \frac{\alpha + 2}{2} S_{g^s}^{-\alpha} = A_{GN}. \]

6. Limiting profile with critical norms

Let us start with the following characterization of solution with critical norms.

**Lemma 6.1.** Let $d \geq 1$ and $2_\ast < \alpha < 2^\ast$.
- If $u \in H^\gamma \cap H^2$ is such that $\|u\|_{H^\gamma} = S_{g^s}$ and $E(u) = 0$, then $u$ is of the form
  \[ u(x) = e^{i\theta} \lambda \frac{x}{|x|} g(\lambda x + x_0), \]
  for some $g \in G$, $\theta \in \mathbb{R}$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$.
- If $u \in L^{\alpha} \cap H^2$ is such that $\|u\|_{L^{\alpha}} = L_{g^s}$ and $E(u) = 0$, then $u$ is of the form
  \[ u(x) = e^{i\theta} \mu \frac{x}{|x|} h(\mu x + y_0), \]
  for some $h \in H$, $\partial \in \mathbb{R}$, $\mu > 0$ and $y_0 \in \mathbb{R}^d$.

**Proof.** We only prove Item 1, Item 2 is treated similarly. Since $E(u) = 0$, we have
\[ \|u\|_{H^2}^2 = \frac{2}{\alpha + 2} \|u\|_{L^{\alpha+2}}^{\alpha+2}. \]
This yields in particular that $\|u\|_{\dot{H}^{\gamma_\text{c}}} = S_{gs} = \|g\|_{\dot{H}^{\gamma_\text{c}}}$, we have $|a| = \lambda^{\frac{2}{\alpha}}$. This shows the result. 

We now have the following limiting profile of blowup solutions with critical norms.

**Theorem 6.2 (Limiting profile with critical norms).** Let $d \geq 5$ and $2_* < \alpha < 2$. Let $u_0 \in \dot{H}^{\gamma_\text{c}} \cap \dot{H}^2$ be such that the corresponding solution $u$ to (1.1) blows up at finite time $0 < T < \infty$.

- **Assume that**
  \[ \sup_{t \in [0, T)} \|u(t)\|_{\dot{H}^{\gamma_\text{c}}} = S_{gs}. \]  \( (6.1) \)

  Then there exist $g \in \mathcal{G}, \theta(t) \in \mathbb{R}, \lambda(t) > 0$ and $x(t) \in \mathbb{R}^d$ such that
  \[ e^{i \theta(t) \lambda^\frac{4}{\alpha} (t)} u(t, \lambda(t) \cdot + x(t)) \rightarrow g \text{ strongly in } \dot{H}^{\gamma_\text{c}} \cap \dot{H}^2 \text{ as } t \uparrow T. \]

- **Assume that**
  \[ \sup_{t \in [0, T)} \|u(t)\|_{\dot{H}^{\gamma_\text{c}}} < \infty, \quad \sup_{t \in [0, T)} \|u(t)\|_{L^{\alpha_\text{c}}} = L_{gs}. \]  \( (6.2) \)

  Then there exist $h \in \mathcal{H}, \vartheta(t) \in \mathbb{R}, \mu(t) > 0$ and $y(t) \in \mathbb{R}^d$ such that
  \[ e^{i \theta(t) \mu^\frac{4}{\alpha} (t)} u(t, \mu(t) \cdot + y(t)) \rightarrow h \text{ strongly in } L^{\alpha_\text{c}} \cap \dot{H}^2 \text{ as } t \uparrow T. \]

**Proof.** We only give the proof for the first case, the second case is similar. We will show that for any $(t_n)_{n \geq 1}$ satisfying $t_n \uparrow T$, there exist a subsequence still denoted by $(t_n)_{n \geq 1}, g \in \mathcal{G}$, sequences of $\theta_n \in \mathbb{R}, \lambda_n > 0$ and $x_n \in \mathbb{R}^d$ such that

\[ e^{i \theta_n(t_n, \lambda_n \cdot + x_n)} u(t_n, \lambda_n \cdot + x_n) \rightarrow g \text{ strongly in } \dot{H}^{\gamma_\text{c}} \cap \dot{H}^2 \text{ as } n \rightarrow \infty. \]  \( (6.3) \)

Let $(t_n)_{n \geq 1}$ be a sequence such that $t_n \uparrow T$. Set

\[ \lambda_n := \left( \frac{\|Q\|_{\dot{H}^2}}{\|u(t_n)\|_{\dot{H}^2}} \right)^{\frac{1}{2-\gamma_\text{c}}}, \quad v_n(x) := \lambda_n^\frac{2}{\alpha} u(t_n, \lambda_n x), \]

where $Q$ is as in Proposition 3.2. By the blowup alternative and (6.1), we see that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we have

\[ \|v_n\|_{\dot{H}^{\gamma_\text{c}}} = \|u(t_n)\|_{\dot{H}^{\gamma_\text{c}}} \leq S_{gs} = \|Q\|_{\dot{H}^{\gamma_\text{c}}}, \]  \( (6.4) \)

and

\[ \|v_n\|_{\dot{H}^2} = \lambda_n^{2-\gamma_\text{c}} \|u(t_n)\|_{\dot{H}^2} = \|Q\|_{\dot{H}^2}, \]  \( (6.5) \)

and

\[ E(v_n) = \lambda_n^{2(2-\gamma_\text{c})} E(u(t_n)) = \lambda_n^{2(2-\gamma_\text{c})} E(u_0) \rightarrow 0, \text{ as } n \rightarrow \infty. \]

This yields in particular that

\[ \|v_n\|_{L^{\alpha_\text{c} + 2}} \rightarrow \frac{\alpha + 2}{2} \left\|Q\right\|_{\dot{H}^2}^2, \text{ as } n \rightarrow \infty. \]  \( (6.6) \)

The sequence $(v_n)_{n \geq 1}$ satisfies the conditions of Theorem 3.5 with

\[ m^{\alpha_\text{c} + 2} = \frac{\alpha + 2}{2} \left\|Q\right\|_{\dot{H}^2}^2, \quad M^2 = \left\|Q\right\|_{\dot{H}^2}^2. \]

Therefore, there exists a sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,

\[ v_n(\cdot + x_n) = \lambda_n^\frac{2}{\alpha} u(t_n, \lambda_n \cdot + x_n) \rightarrow V \text{ weakly in } \dot{H}^{\gamma_\text{c}} \cap \dot{H}^2, \]
as } n \to \infty \text{ with } \|V\|_{\dot{H}^{\gamma_c}} \geq S_{gs}. \text{ Since } v_n(\cdot + x_n) \to V \text{ weakly in } \dot{H}^{\gamma_c} \cap \dot{H}^2 \text{ as } n \to \infty, \text{ the semi-continuity of weak convergence and } (6.4) \text{ imply}
\|V\|_{\dot{H}^{\gamma_c}} \leq \liminf_{n \to \infty} \|v_n\|_{\dot{H}^{\gamma_c}} \leq S_{gs}.

This together with the fact \|V\|_{\dot{H}^{\gamma_c}} \geq S_{gs} \text{ show that}
\|V\|_{\dot{H}^{\gamma_c}} = S_{gs} = \lim_{n \to \infty} \|v_n\|_{\dot{H}^{\gamma_c}}.

Therefore
\vspace{1em}
\[ v_n(\cdot + x_n) \to V \text{ strongly in } \dot{H}^{\gamma_c} \text{ as } n \to \infty. \]

On the other hand, the Gagliardo-Nirenberg inequality (3.16) shows that \( v_n(\cdot + x_n) \to V \) strongly in \( L^{\alpha+2} \) as \( n \to \infty \). Indeed, by (6.5),
\[ \|v_n(\cdot + x_n) - V\|_{L^{\alpha+2}}^{\alpha+2} \leq \|v_n(\cdot + x_n) - V\|_{\dot{H}^{\gamma_c}}^\alpha \|v_n(\cdot + x_n) - V\|_{\dot{H}^2} \]
\[ \leq (\|Q\|_{\dot{H}^2} + \|V\|_{\dot{H}^2}^{\alpha/2})^2 \|v_n(\cdot + x_n) - V\|_{\dot{H}^{\gamma_c}} \]
\[ \to 0 \text{ as } n \to \infty. \]

Moreover, using (6.6) and (6.7), the sharp Gagliardo-Nirenberg inequality (3.16) yields
\[ \|Q\|_{\dot{H}^2}^2 = \frac{2}{\alpha + 2} \lim_{n \to \infty} \|v_n\|_{L^{\alpha+2}}^{\alpha+2} = \frac{2}{\alpha + 2} \|V\|_{L^{\alpha+2}}^{\alpha+2} \leq \left( \frac{\|V\|_{\dot{H}^{\gamma_c}}}{S_{gs}} \right)^\alpha \|V\|_{\dot{H}^2}^2 = \|V\|_{\dot{H}^2}^2, \]

or \( \|Q\|_{\dot{H}^2} \leq \|V\|_{\dot{H}^2} \). By the semi-continuity of weak convergence and (6.5),
\[ \|V\|_{\dot{H}^2} \leq \liminf_{n \to \infty} \|v_n\|_{\dot{H}^2} = \|Q\|_{\dot{H}^2}. \]

Therefore,
\[ \|V\|_{\dot{H}^2} = \|Q\|_{\dot{H}^2} = \lim_{n \to \infty} \|v_n\|_{\dot{H}^2}. \quad (6.8) \]

Combining (6.7), (6.8) and using the fact \( v_n(\cdot + x_n) \to V \) weakly in \( \dot{H}^{\gamma_c} \cap \dot{H}^2 \), we conclude that
\[ v_n(\cdot + x_n) \to V \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2 \text{ as } n \to \infty. \]

In particular, we have
\[ E(V) = \lim_{n \to \infty} E(v_n) = 0. \]

This shows that there exists \( V \in \dot{H}^{\gamma_c} \cap \dot{H}^2 \) such that
\[ \|V\|_{\dot{H}^{\gamma_c}} = S_{gs}, \quad E(V) = 0. \]

By Lemma 6.1, there exists \( g \in \mathcal{G} \) such that \( V(x) = e^{i\theta} \lambda_n^{\frac{\gamma}{4}} g(\lambda x + x_0) \) for some \( \theta \in \mathbb{R}, \lambda > 0 \) and \( x_0 \in \mathbb{R}^d \). Thus
\[ v_n(\cdot + x_n) = \lambda_n^{\frac{\gamma}{4}} u(t_n, \lambda_n \cdot + x_n) \to V = e^{i\theta} \lambda_n^{\frac{\gamma}{4}} g(\lambda \cdot + x_0) \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2 \text{ as } n \to \infty. \]

Redefining variables as
\[ \overline{\lambda}_n := \lambda_n \lambda^{-1}, \quad \overline{\tau}_n := \lambda_n \lambda^{-1} x_0 + x_n, \]
we get
\[ e^{-i\theta} \overline{\lambda}_n^{\frac{\gamma}{4}} u(t_n, \overline{\lambda}_n \cdot + \overline{\tau}_n) \to g \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2 \text{ as } n \to \infty. \]

This proves (6.3) and the proof is complete. \( \square \)

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