Minkowski Spacetime and QED from Ontology of Time

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We investigate the consequences of a rigorous ontology of time. Based on the existence in time as an absolute and exclusive primary fact, we construct spacetime as an adequate but not fundamental interpretation of the properties of dynamics. From 3 axioms we derive a general Hamiltonian and prove that a geometrical interpretation of symplectic dynamics is possible given the corresponding Lie algebra is isomorphic to the representation of a Clifford algebra. On this basis we derive the dimensionality of spacetime, the form of Lorentz transformations and fundamental laws of physics as the Lorentz force, Maxwell electrodynamics and finally the Dirac equation.

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I. INTRODUCTION AND FOUNDATION

Pablo Echenique-Rubba recently emphasized the relevance of ontological commitments in physics. Physics has a special and intimate connection to the concept of time: it is founded on certain assumptions that have to be agreed unless the possibility of science and specifically physics is denied. One of these assumptions is the repeatability (of experiments) at any time, i.e. the homogeneity of time and the (re-) productivity of experimental setups. The general measurability of quantities is another assumption, which implies the existence of constant rulers, i.e. objects that are constant in time. It cannot be denied that these ontological assumptions are (part of) the fundament of physics as a scientific discipline.

This essay is an attempt to minimize the number of ontological commitments, an attempt to sketch a consistent derivation of fundamental laws of physics from an ontology of time. The introduction of a new or modified ontology requires to question the roots of what we usually consider to be certain, i.e. some excellence in unlearning. With the words of John Maynard Keynes: “The difficulty lies, not in the new ideas, but in escaping from the old ones, which ramify [...] into every corner of our minds.”

To doubt established theories and to question their basic concepts ought to be routine exercise in science: if we believe in progress in science than we have to admit that there is always the possibility of a simpler and clearer theory or interpretation. However, the more developed a scientific discipline becomes, the harder it will be to re-think it’s fundamental assumptions. The structure of spacetime is such a fundament of physics, the (lack of) understanding of quantization is another.

Before Einstein, the physical concepts of space and time were merely a precise formulation of what we might call “experienced” or “apparent” spacetime. But Einstein’s special relativity opened Pandora’s box and the question if “true” spacetime has 4, 10 or even 26 dimensions - some of which are “compactified” - is intensely debated since (24). Not only higher dimensionalities are debated - it also seems possible to describe the 3-dimensional world as a 2-dimensional hologram\footnote{Electronic mail: christian-baumgarten@gmx.net} - yet another idea that questions the ontological status of spacetime and reality as a whole. A (rough and random) survey of publications on the question of spacetime dimensionality results that most\footnote{Electronic mail: christian-baumgarten@gmx.net} (though not all\cite{18,19}) authors discuss the question of the dimensionality of space, while time apparently is considered to have a different status.

There is no a priori reason why it should be impossible to consider worlds (or at least computer simulations of worlds) with nearly arbitrary space dimension. But it remains unclear, how to “simulate” a world without a time dimension. Without time, the universe looses its character and a mere sculpture remains. The inability to imagine something is certainly no proof that it does not exist - but even if time would not exist (in the way that we experience it), the fact remains that the implicit ontology of physics is based on the existence of time. It was an idea of Sir Hamilton to develop algebra as the science of pure time\cite{20} and it was his deep belief that “that the intuition of time is more deep-seated in the human mind than the intuition of space.”\cite{21}. The situation is slightly different with space: If we declare that it does not exist or that it has more (or less) than 3 dimensions, we only have to explain the fact, that apparently the world as we experience it, has 3+1 dimensions. But as far as we can see there is no (onto-) logical problem.

We have (ad hoc) no strong reason at hand, why time should be one-dimensional. But we tend to assume that even if we would live in a world with multiple time dimensions, we might not be able to notice. If time was more than one-dimensional, it remains unclear how the multiple dimensions could be experienced and hence the experienced time might still be one-dim.: Though space has 3 dimensions, a path through space has a one-dimensional parameterization.

In special relativity the concepts of space and time are merged to what is called “Minkowski spacetime”. There are (at least) two interpretations of Minkowski
spacetime: It can be understood as the statification of time\textsuperscript{5} or as a dynamization of space\textsuperscript{26}. We consider spacetime to be the subordinated concept compared to time. There are indeed physical reasons for our suspicion, problems or paradoxes that could not be solved in a completely satisfactory way. To name a few: the problem of finite self-energy of point charges, the problem of non-radiating orbital electrons, Bell’s theorem about locality and causality and finally the form of quantum mechanics: the wave function in “real” space can as well be replaced by its Fourier transform - the wave function in momentum space - and there is no fundamental reason to prefer either. The role of space as a Fourier transform of the distribution in “momentum space” suggests that space might emerge from dynamics\textsuperscript{24}. If we take such considerations serious, then we have to admit that spacetime could be appearance or interpretation - a construction of mind that helps to understand and express the relations between perceptions in an organized manner. If spacetime would not be fundamental, then all physical paradoxes that are based on the a priori existence of spacetime are likely paradox, because of this erroneous ontological basis. This essay is an attempt to further explore this idea. Insofar it is a gedankenexperiment: How might physics look like if we assume that spacetime only appears to exist at the same ontological level as time? If this hypothesis is not absurd, then even the apparent dimensionality of spacetime might depend on the considered type of interaction and the physical framework that we look at. In this case, spacetime might have no fixed dimensionality at all.

Hence we formulate a working hypothesis: spacetime does not exist a priori - it appears. It is not fundamental but a construction. In the words of Mark Van Raamsdonk: “Everything around us the whole three-dimensional physical world is an illusion born from information encoded elsewhere, on a two-dimensional chip”\textsuperscript{25} or in the words of Albert Einstein: “Spacetime does not claim existence on its own, but only as a structural quality of the field”\textsuperscript{26}.

In this essay we derive fundamental laws of physics from a rigorous reasoning based on time. We do not present a new concept of time nor do we offer a (new) non-cyclic definition of what time is. Instead we investigate what existence in time physically means. And we will describe how the structure of spacetime emerges from a geometrical interpretation of fundamental algebraic properties of (classical) dynamics\textsuperscript{1}.

We believe that it is not sufficient to describe the “laws of physics” - including structure and dimensionality of spacetime. These laws are not by chance, they have to be by reason. It is not enough to “discover” the laws of nature - one has to derive and explain their form. We intent to keep the number of concepts minimal. That is, we are committed to Ockham’s razor.

Therefore the concept of a (charged or uncharged) “point mass” for instance can not be regarded as fundamental. Though the concept of a point mass seems to be sufficiently abstract\textsuperscript{1} in many aspects, it implies various concepts like inertia, space, velocity, acceleration and indirectly even the concept of force. It is our vision that the concept of a point mass should be a result of the theory and not its starting point. Instead of inventing presumptions about the nature of space, we claim that also space (or Minkowski spacetime) has to be constructed and explained. Again: the concept of spacetime should be the result of a fundamental theory rather than part of the primary assumptions.

Besides the idea of time we consider the concept of the degree of freedom (DOF) as abstract and general enough to be regarded as fundamental. Physics is a science that enables to predict the values of quantities. A degree of freedom is essentially a quantity that exists in time and its existence becomes manifest by variation: a DOF is nothing but a variable, i.e. a quantity changing with time. Hence the fundamental entities in our considerations are variables and the most we can say about them is to postulate their existence and to postulate the literal meaning of this naming: Their existence implies nothing but variation. One might be attempted to ask what varies. A typical answer in physics would be “field strength” or “amplitude”. It is obvious that this type of answer offers a new name but has few explanatory power. However the inability to specify what varies is not a problem, it is a logical implication of fundamentality.

Even stronger: It is not only impossible to tell what varies, it is an intrinsic property of truly fundamental variables, that they can not be directly measured. Here is an ultimate limit of (experimental) physics: A measurement requires a reference, i.e. something constant of the same physical dimension. We measure length in “meter” and a measurement of length is a comparison of an unknown and “variable” quantity with a constant quantity (“the meter”). If we accept that existence is essentially being in time, any constant that could serve as reference must be generated from variation. The constants that are necessary for reference can therefore only emerge from dynamics: they are constants of motion (COMs). Thus, if a variable is considered to be fundamental, it is impossible to construct a reference to measure it, since this would require constants of motion from “more” fundamental variables. But then the more fundamental variable could not be measured etcpp. It follows that (in a world ruled by time) fundamental variables are not directly measurable. There must be a level at which it is not possible to assign physical meaning to dynamic variables\textsuperscript{2}. However this does not imply the impossibility of a classical description of the dynamics of these variables.

\textsuperscript{1} The idea that geometry might be understood as an aspect of dynamics is not new. See for instance Ref. (\textsuperscript{26}).

\textsuperscript{2} It might be possible to assign either a logical or - more likely -
Macroscopic (i.e. measurable) variables of cause exist - otherwise the concept of measurement would be meaningless. But they are based on references that are obtained from COMs: the prototype meter is an object composed of atoms and molecules which are composed of “elementary particles”. The motion of these components is stable and the length of the reference object “meter” is in consequence a constant of motion of the particles it is composed of. But at the level of these particles - or at the level of their constituents - at some level we find pure variation. Acuminated we claim that if it is possible to measure a variable, it can not be fundamental.

All fundamental variables are equal or equivalent. As long as we have no forceful argument to introduce different types of variables, Ockham’s razor requires them to be all ontologically identical. We might compare this to the bits or bytes in a computer RAM: They are all equal. They only differ by their address, i.e. by their position in address space. Hence all fundamental variables are either all real or all imaginary. The latter might be preferable to (and we will) explain what structure and structure function of the variables, the structure of the dynamics. If objects do not change their structure (i.e. their “identity”) in some interaction, then the involved physical processes apparently must be structure preserving. If a system is defined and understood by its structure, then its continuous existence requires - besides a continuous variation of its constituents (the variables) - that the structure of the variations must be preserved. At some point we have to (and we will) explain what structure and structure preservation formally means.

The paper is organized as follows: In Sec. II we derive the Hamiltonian equations of motion (EQOM) and the basic properties of the symplectic unit matrix from the concept of time. In Sec. III constants of motion are introduced with the help of Lax Pairs. We rediscover important algebraic objects that guide us to a possible reason for the construction principles of spacetime: The congruence of the algebraic structure of (skew-) Hamiltonian matrices with the basic elements (i.e. generators) of Clifford algebras. We describe the basic measurable entities in a world based on time: (Second) moments of the fundamental variables. Since second moments can be represented by expectation values of matrix operators, the relations between these matrices are the relations between the observables (i.e. the laws of physics).

In Sec. IV we derive the conditions for the emergence of a geometric space from symplectic dynamics. Observables are expectation values of matrix operators and hence spacetime geometry should be represented by a system of basic matrices with specific properties. We will show that the conditions are (only) fulfilled by matrix systems that represent Clifford algebras $Cl_{p,q}$ with specific choices of $p$ and $q$.

In Sec. V we describe symplectic transformations as structure preserving transformations. These transformations are the basis for the geometry of spacetime. And we explore the meaning of structure defining transformations.

In Sec. VI we give a short overview over the basic properties of Clifford algebras in general and especially of $Cl_{N−1,1}(\mathbb{R})$. We derive conditions that limit the possible dimensionality of spacetime. And we argue why apparent spacetime is isomorphic to $Cl_{3,1}(\mathbb{R})$.

In Sec. VII we describe Lorentz transformations - boosts and rotations - as structure preserving (symplectic) transformations. We present an interpretation of the Dirac matrix system called the “electromechanical equivalence” (EMEQ). Guided by the EMEQ we derive the Lorentz force and Maxwell’s equations. We give arguments why momentum and energy should be related to spatial- and time-derivatives and finally we sketch the path towards the Dirac equation.

Sec. VIII finalizes the discussion and a summary in given in Sec. IX.

II. FUNDAMENTALS

A. Variables

Existence in time implies that the variables are changing with time. In contrast to variables observables are functions of variables that can be measured and we need to explain how references for the observables emerge that are constant in time. The axioms of our ontologically based physics are:

1. Existence happens in time\(^3\). The “time” that we have in mind does not have to be identical to the time that an observer would measure using a clock, but we assume that there is a (monoton) functional relationship.

\(^3\) We consider here time as experienced, i.e. a single time dimension. Max Tegmark has argued that only one time dimension is possible\(^2\). Further below we will find another (algebraic) argument for this statement (see Eq. [10]).
2. Time is manifest by variation. Hence any fundamental entity varies. Physical entities are quantities. Hence all primary physical entities are “variables”.

3. Measurements require (constant) references, i.e. a physical world requires constants.

From these axioms we deduced that

1. There are no other constants than constants of motion.

2. Fundamental variables can not be directly measured.

The value of a single variable varies. Hence a single variable can not generate constants of motion. Therefore we start with an arbitrary number \( k > 1 \) of dynamical variables. The axioms suggest the existence of one (or a set of) function(s) \( \mathcal{H}(\psi_1, \psi_2, \ldots, \psi_k) = \text{const} \), where \( \psi \) is a list of variables. Hence we may write (with the presumption \( \mathcal{H} = \text{const} \) we imply here \( \frac{\partial \mathcal{H}}{\partial \psi} = 0 \): \[ \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial \psi_1} \dot{\psi}_1 + \frac{\partial \mathcal{H}}{\partial \psi_2} \dot{\psi}_2 + \cdots + \frac{\partial \mathcal{H}}{\partial \psi_k} \dot{\psi}_k = 0 \, , \] or in vector notation: \[ \frac{d\mathcal{H}}{dt} = (\nabla_\psi \mathcal{H})^T \cdot \dot{\psi} = 0 \, . \] (1)

The (simplest) general solution is given by \[ \dot{\psi} = S (\nabla_\psi \mathcal{H}) \, , \] (3) where \( S \) is a \( k \times k \) skew-symmetric real matrix, i.e. \( S^T = -S \). Then there exists a non-singular matrix \( Q \) such that \[ Q^T S Q = \text{diag}(\eta_0, \eta_0, \eta_0, \ldots, 0, 0, 0) \, . \] (4)

This change of variables is used to find (or recognise or define) the “natural” variables. We call such transformations structure defining (see below and App. (C)). According to the axioms the matrix has full rank \( 2n \leq k \), i.e. all variables vary and hence we remove all constants from the state vector such that \[ Q^T S Q = \text{diag}(\eta_0, \eta_0, \eta_0, \ldots, \eta_0) = \gamma_0 \, , \] (6)

and rewrite Eq. (2) accordingly

\[ \frac{d\mathcal{H}}{dt} = (Q^{-1} \nabla \mathcal{H})^T Q^T S Q(Q^{-1} \nabla \mathcal{H}) = 0 \]

\[ = (Q^{-1} \nabla \mathcal{H})^T \gamma_0 (Q^{-1} \nabla \mathcal{H}) = 0 \, . \] (7)

The matrix \( \gamma_0 \) has the even dimension \( 2n \times 2n \): in any time-like physical world the dynamical variables come in pairs: This is a consequence of the axioms and not a postulate. It is therefore sensible to refer to a pair of variables when we speak of a degree of freedom. We call \( \psi_{2j} = q_j \) the \( j \)-th canonical coordinate and \( \psi_{2j+1} = p_j \) the \( j \)-th canonical momentum, i.e. \( \psi = (q_1, p_1, \ldots, q_n, p_n)^T \).

But this nomenclature is purely formal, if the variables in \( \psi \) are fundamental. The matrix \( \gamma_0 \) is called the symplectic unit matrix and the equations of motion (EQOM) have Hamilton’s form:

\[ \dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \]

or, in vector notation:

\[ \dot{\psi} = \gamma_0 \nabla_\psi \mathcal{H} \, . \] (9)

Whenever we have constants (of motion), we can derive Hamilton’s EQOM in some way. Up to this point, the idea of a space of a specific dimension is not necessary - nor is it necessary to refer to space at all. In fact, what we introduced here is the phase space of the abstract and intrinsically unmeasurable basic variable list \( \psi \). Fundamental variables - as represented by the state vector \( \psi \) - are abstract and not directly measurable because on the level of these variables there are no constants and hence no references. Just formally we call them “coordinates” and “momenta”. However, the Hamiltonian formalism does not require that variables are unmeasurable and therefore the Hamiltonian formalism can be applied in any system with dynamical constants. The physical properties of macroscopic systems are defined as averages of a huge number of fundamental variables. Most of them are only approximately constant - usually effects like friction or thermal radiation are neglected in classical mechanics. On the fundamental level intended here, there are no such effects and the equations are supposed to hold strictly. If the variables are fundamental, they are by definition not macroscopic and the amount of substance can not “diffuse away” into a more fundamental level.

B. The Hamiltonian

We introduced the “Hamiltonian” as (an arbitrary) constant of motion. Typically there are several constants of motion. Hence the function \( \mathcal{H} \) is not yet well-defined. In the following we assume that \( \mathcal{H} \) is positive (semi-) definite with respect to the variables \( \psi \). This restriction is neither arbitrary nor weak: If the constructed constant of motion \( \mathcal{H} \) is a reference of measurable things, then there must be a Hamiltonian function \( \mathcal{H} \) that reflects the amount of something, for instance the amount.
of substance. Existence must be represented by a positive (semi-) definite Hamiltonian.

We write the “Hamiltonian” $\mathcal{H}$ as a Taylor series of the $2n$ variables:

$$\mathcal{H}(\psi) = \mathcal{H}_0 + \varepsilon^T \psi + \frac{1}{2!} \psi^T A \psi + \frac{1}{3!} B_{ijk} \psi_i \psi_j \psi_k + \ldots,$$

where $\varepsilon$ is a $2n$-dimensional vector and $A$ is a symmetric $2n \times 2n$ matrix and $B$ is a tensor that is symmetric in all indices. We assume that $\mathcal{H}$ has a local minimum somewhere. If we truncate to second order, then an offset $\tilde{\psi} = \psi - \psi_0$ of $\psi_0 = -A^{-1} \varepsilon (\psi = \psi - \psi_0)$ enables to get rid of the linear term. If higher orders are kept, the calculation of the shift $\psi_0$ is more involved. But in any (regular) case, if a local minimum exist, then we can choose the minimum to coincide with $\psi = 0$. The Hamiltonian EQOM (9) can then be written as a product of a matrix $F(\psi)$ and the vector $\psi$:

$$\dot{\psi} = \gamma_0 \nabla_{\psi} \mathcal{H} = F(\psi) \psi.$$  

(11)

III. CONSTANTS OF MOTION

Observables and constants of motion can be expressed by (the matrix of) second moments $\sigma$ as follows:

$$\sigma = \psi \psi^T,$$

(12)

so that for the matrix $S = \sigma \gamma_0$ holds:

$$\dot{S} = \dot{\sigma} \gamma_0 = \dot{\psi} \psi^T \gamma_0 + \psi \dot{\psi}^T \gamma_0$$

$$= F(\psi) \psi \psi^T \gamma_0 + \psi \psi^T F^T(\psi) \gamma_0$$

$$= F(\psi) S - S F(\psi) \gamma_0.$$  

(13)

We are especially interested in those processes, where

$$\gamma_0 F^T(\psi) \gamma_0 = F(\psi),$$

(14)

so that

$$\dot{S} = F(\psi) S - S F(\psi)$$

(15)

holds. That is, if $F$ and $S$ commute, then the matrix $S$ is constant, while $\psi$ varies. If also the matrix $F$ is constant (i.e. bilinear Hamiltonian in $\psi$), then the symmetry of $A$ results in a matrix $F = \gamma_0 A$, for which Eq. (13) holds.

It follows from Eq. (15) that $S$ and $F$ form a so-called “Lax Pair”, for which

$$Tr(S^k) = \text{const},$$

(16)

holds for any natural number $k$. Hence the bilinear form (12) is the basis for a set of constants as given by (16). The average of the bilinear form equals a matrix of second moments, if summed over an ensemble of $N$ systems:

$$\sigma = \frac{1}{N} \sum_{k=1}^{N} \psi_k \psi_k^T,$$

(17)

or - which is a complementary description - if we use a “density” $\rho(\psi)$ to describe the distribution of states in the phase space of $\psi$. In this case the matrix of second moments is given by

$$\sigma = \int \rho(\psi) \psi \psi^T d^n\psi \over \int \rho(\psi) d^n\psi.$$  

(18)

Eq. (15) holds also for a single “phase space trajectory” $\psi(t)$. However, the eigenvalues of the matrix of second moments $\sigma$ depend on whether they are computed for a single vector $\psi$ (which results in a matrix $\sigma$ with vanishing determinant and vanishing eigenvalues), for two linear independent vectors $(\psi_1, \psi_2)$ ($\sigma$ has vanishing determinant, but two non-zero eigenvalues) or $\geq 2n$ linear independent vectors $\psi_k$, as in Eq. (17). In the latter case, $\sigma$ is non-singular.

Since $S$ is by definition the product of a symmetric matrix $\sigma$ and the skew-symmetric matrix $\gamma_0$, it follows that

$$Tr(S) = 0.$$  

(19)

so that the simplest meaningful constants are given as

$$Tr(S^2) = \text{const}.$$  

(20)

In the following we restrict ourselves to linear systems (or systems that are linear in good approximation), where the Hamiltonian is second order in the fundamental variables $\psi$:

$$\mathcal{H}(\psi) \approx \frac{1}{2} \psi^T A \psi.$$  

(21)

Eq. (21) describes an $n$-dimensional harmonic oscillator. The truncation of the Hamiltonian to second order is not arbitrary - it guarantees stability. It is well-known, that non-linearities yield (in many or even most cases) instable and chaotic behavior. Therefore if we want to establish a stable system of references it is nearby to use strictly linear systems. This restriction is per se not problematic, since the equations that we are going to “derive” are linear.

The constants that are used as references are the quadratic forms (10). If a reference is based on a quadratic form, the variable compared with the reference (the measured variable) also must be a quadratic form. For this reason we consider that the “dynamical

6 In accelerator physics the eigenvalues of $S = \sigma \gamma_0$ are the “emittance” (times the unit imaginary). It is evident, that the position of a single particle at time $t$ does not define a phase space area.

7 Furthermore we refer to the Kustaanheimo-Stiefel (KS) transformation and appropriate generalizations that allow to map (for instance) the Kepler problem to the harmonic oscillator.

This shows that the harmonic oscillator is a surprisingly versatile model for stable systems.
variables” $q_i$ and $p_i$ cannot be measured directly, but only (functions of) quadratic forms based on these variables. We call these (functions of) quadratic forms **observables**.

If we plug this result into Eq. (2), we obtain:

$$\dot{\psi} = \gamma_0 A \psi = F \psi ,$$

(22)

Since $A$ is symmetric and $\gamma_0$ is skew-symmetric, the matrix $F = \gamma_0 A$ has zero trace. Furthermore one finds that Eq. (14) is fulfilled. Such matrices are called “Hamiltonian” or sometimes “infinitesimally symplectic”. We believe that these names are misleading, the former mainly because the matrix $F$ does not appear in the Hamiltonian and the latter since $F$ is neither symplectic nor infinitesimal. Therefore we use the term **symplex** (plural **symplexes**). A symplex is a matrix (operator) $S$ that holds

$$S^T = \gamma_0 S \gamma_0 .$$

(23)

A **cosymplex** (or “antisymplex” or “skew-Hamiltonian” matrix) is a matrix $C$ that holds

$$C^T = - \gamma_0 C \gamma_0 .$$

(24)

A symplex $S$ can always be written as a product of $\gamma_0$ and a skew-symmetric matrix. The sums of (co-) symplexes are (co-) symplexes, i.e. the superposition principle holds. Hence (co-) symplexes form a linear vector space and any (co-) symplex can be written as a linear combination of “basic” (co-) symplexes. The algebra of (co-) symplexes is the Lie algebra $sp(2n)$.

If we denote symplexes by $S$ and cosymplexes by $C$ (optionally with subscript), then it is easy to prove that the commutator of two symplexes is a cosymplex:

$$[S_1 S_2 + S_2 S_1]^T = S_2^T S_1^T + S_1^T S_2^T = \gamma_0 S_2 \gamma_0 \gamma_0 S_1 \gamma_0 + \gamma_0 S_1 \gamma_0 \gamma_0 S_2 \gamma_0 = - \gamma_0 (S_2 S_1 + S_1 S_2) \gamma_0 .$$

(25)

The following rules are obtained:

| symplex | cosymplex |
|---------|-----------|
| $S_1 S_2 - S_2 S_1$ | $S_1 S_2 + S_2 S_1$ |
| $C_1 C_2 - C_2 C_1$ | $C_1 C_2 + C_2 C_1$ |
| $C S + S C$ | $C S - S C$ |
| $S^{2n+1}$ | $C^n$ |

(26)

The relations Eq. (26) define the structure of a specific algebra, called the Lie algebra $sp(2n)$ of the symplectic group, denoted by $Sp(2n)$. Furthermore they explicitly raise specific rules for symplectic spaces, as we will explain further below in more detail. The matrix of observables $S$ of Eq. (10) is a symplex like the matrix of generators $F$. According to Eq. (26) odd powers of $S$ are symplexes with zero trace. Hence the non-zero constants of motion given by Eq. (10) are of even order.

Since the matrix $A$ is symmetric (Eq. (24)), the maximal number $\nu$ of free parameters of a symplex for $n$ degrees of freedom (DOF) is

$$\nu_s = \frac{2n(2n+1)}{2} .$$

(27)

The basis of the matrix $F$ must include $\nu_s$ basic symplexes. A real matrix basis is completed by $\nu_c$ cosymplexes:

$$\nu_c = \frac{2n(2n-1)}{2} .$$

(28)

### A. Measurable Quantities: Observables

Usually the second moments are assumed to be average values of $N$ identical systems as in Eq. (17), but the equations hold equivalently for a single system. However it should be noted that for a vector of $2n$ fundamental variables, $2n$ linear independent “samples” are required, if the matrix $S = \sigma \gamma_0$ is supposed to have a non-vanishing determinant. Later we will come back to this point.

We can define the $\sigma$-matrix as well using a (normalized) probability density $\rho$:

$$\sigma = \int \rho(\psi) \psi \psi^T d^{2n} \psi \equiv \langle \psi \psi^T \rangle ,$$

(29)

with the normalization according to:

$$1 = \int \rho(\psi) d^{2n} \psi ,$$

(30)

i.e. the density $\rho(\psi)$ is defined as a function of the fundamental variables or phase space variables.

We introduced the (“S-matrix”) $S = \sigma \gamma_0$ and derived Eq. (15). Let the **adjunct** vector $\bar{\psi}$ be defined by

$$\bar{\psi} \equiv \psi^T \gamma_0 ,$$

(31)

so that $S = \langle \psi \bar{\psi} \rangle$ and the expectation value of an operator $O$:

$$\langle O \rangle \equiv \langle \psi \bar{\psi} \rangle .$$

(32)

Now consider that $O$ is a cosymplex, i.e. can be written as $\gamma_0 B$ with some skew-symmetric matrix $B$, then the expectation value of $O = \gamma_0 B$ vanishes:

$$\langle O \rangle = \langle \psi^T \gamma_0 B \psi \rangle = - \langle \psi^T B \psi \rangle = 0 .$$

(33)
Hence only symplexes represent measurable properties of (closed) systems. Cosymplexes \( C \) only yield a non-vanishing result when they appear between different (i.e. linearly independent) states:

\[
\check{\phi} C \psi \neq 0 \Rightarrow \phi \neq \lambda \psi ,
\]

with some arbitrary (non-zero) factor \( \lambda \). Vice versa - if two (normalized) states \( \phi \) and \( \psi \) may not “generate” non-vanishing expectation values of the cosymplex type, then it follows that:

\[
\phi = \pm \psi .
\]

Since reality emerges from the totality of all fundamental variables, we expect that the coefficients of the matrix \( F \) are functions of (internal or external) measurable quantities, i.e. of some symplex \( S \) generated from fundamental variables. And since cosymplexes have a vanishing expectation value, they can not contribute. We will use this important result in Sec. V in the derivation of Maxwell’s equations. Furthermore, if a “particle” preserves its structure in a physical process, then the involved “forces” must be representable by symplexes. Otherwise the evolution is time would not be symplectic or structure preserving. And vice versa: If the particle structure (particle type) is transformed in a process, then there must be a contribution from cosymplexes as for instance axial vector currents as in the V-A theory of weak interaction.

This means that if we focus on a closed system, then (the expectation values of) all cosymplexes vanish.

The properties of “open” and “closed” have only a relative meaning which depends on the context. If we consider two separate “particles” described by state vectors \( \psi \) and \( \phi \) and let them get in contact, then we would for instance combine the state vectors into a larger state vector \( \Psi = (\psi, \phi)^T \). In App. \( A \) we give the explicit form of general \( 2n \times 2n \)-symplexes. It is shown that non-diagonal sub-blocks may per se have arbitrary terms. Hence the question, if a (sub-) matrix is a symplex or a cosymplex depends on the context. However all quantities - including the matrix \( F \) - must by some means result from “fundamental variables”, i.e. from state vectors. Also a static electromagnetic field has its origin in charges and currents of “particles” that are described by state vectors. If reality is described by the totality of the fundamental variables, then also the symplex \( F \) must somehow be generated from \( \psi \). Since the matrix of observables \( S = \sigma \gamma_0 \) is a symplex as well, the functional relationship might be simple: consider (for instance) a Hamiltonian of the form

\[
\mathcal{H} \propto \psi^T \sigma^{-1} \psi ,
\]

which implies \( F \propto \gamma_0 \sigma^{-1} \). A Boltzmann distribution \( \rho(\psi) \propto \exp\left(-\frac{\mathcal{H}}{kT}\right) \) then is identical to a multivariate Gaussian density distribution. In general the functional relationship between \( F \) and \( S \) might be more involved, but obviously there is a kind of duality between the matrix of observables \( S = \sigma \gamma_0 \) and the matrix of generators \( F = \gamma_0 A \) that we tried to express by Eq. (36). More details on this duality can be found in.

The time derivative of an expectation value is (assuming \( O \) does not explicitly depend on time):

\[
\langle \dot{O} \rangle = \langle \dot{\psi} O \psi \rangle + \langle \psi \dot{O} \psi \rangle = \langle \psi^T \gamma_0 O \psi \rangle + \langle \psi \gamma_0 F \psi \rangle = \langle \psi^T \gamma_0 F \gamma_0 O \psi \rangle + \langle \psi \gamma_0 F \psi \rangle = \langle \psi (O F - F O) \psi \rangle
\]

This general law connects the commutator of an operator \( O \) and \( F \) with its time derivative. This is the linear form of the Poisson bracket and it is completely classical.

IV. FROM LIE ALGEBRA TO SPACETIME GEOMETRY

The total set of (co-) symplexes is a Lie algebra (called \( sp(2n) \)). In the following we derive conditions for the emergence of spacetime structure from \( sp(2n) \). First of all we have to identify objects to represent (unit) vectors in space and time directions, we need a definition of length (a norm) and of orthogonality. \( \gamma_0 \) suggests itself as the natural choice of the “unit vector” in the “time direction”, since time is the only dimension we introduced so far - and \( \gamma_0 \) is a symplex.

From the (anti-) commutation properties described by Eq. (26) we conclude that any fundamental and simple system of (co-) symplexes requires that the basic (co-) symplexes either commute or anticommute: Only in this case the product is a pure symplex or cosymplex, i.e. observable or not. If we consider pure (co-) symplexes \( A \) and \( B \), then

\[
(AB)^T = B^T A^T = \pm \gamma_0 BA \gamma_0 .
\]

It follows that only if \( AB = \pm BA \), the product of pure (co-) symplexes is again a pure (co-) symplex.

Usually vector norms in Euclidean spaces represent the flat length of a vector, i.e. a property that is invariant under rotations and translations. Hence it is nearby to use an invariant quantity for the definition of a norm. We therefore use Eq. (16) with \( k = 2 \) to define the norm, i.e.:

\[
\| A \| = \left( \frac{1}{2n} \text{Tr}(A^2) \right)^{\frac{1}{2}} .
\]

where the division by \( 2n \) is introduced to let the unit matrix have unit norm. The matrix \( \gamma_0 \) hence has a norm of \( -1 \), i.e. the norm (39) is not positive definite.

Two elements of an algebra are said to be orthogonal, if their inner product vanishes. Since we have no inner product defined (yet), we use an alternative geometric definition which we consider to be equivalent in flat spacetime - the Pythagorean theorem. Two vectors \( \vec{v} \) and \( \vec{w} \) are orthogonal, if

\[
\| \vec{v} + \vec{w} \|^2 = \| \vec{v} \|^2 + \| \vec{w} \|^2 ,
\]
which implies (in usual vector notation) that the inner product vanishes:

\[(\vec{v} + \vec{w})^2 = \vec{v}^2 + \vec{w}^2 + 2\vec{v} \cdot \vec{w} \, . \tag{41}\]

For two elements \(A\) and \(B\) of the Lie algebra \(sp(2n)\) the analog expression is given by

\[
\|A + B\|^2 = \frac{1}{2n} \text{Tr}((A + B)^2) = \frac{1}{2n} \text{Tr}(A^2 + B^2 + AB + BA) = \|A\|^2 + \|B\|^2 + \frac{1}{2n} \text{Tr}(AB + BA) \, . \tag{42}\]

As we argued above, all elements of the algebra should either commute or anticommute. Those elements \(A\) and \(B\) that anticommute are then orthogonal, so that:

\[
\text{Tr}(AB + BA) = 0 \, . \tag{43}\]

We therefore identify the anticommutator with the inner product:

\[
A \cdot B \equiv \frac{AB + BA}{2} \, . \tag{45}\]

Given the anticommutator of \(\gamma_k\) and \(\gamma_0\) vanishes, then we find (using \(\gamma_0^2 = -1\)):

\[
\gamma_k \gamma_0 + \gamma_0 \gamma_k = 0 \\
\gamma_k \gamma_0^2 + \gamma_0 \gamma_k \gamma_0 = 0 \\
\gamma_k = \gamma_0 \gamma_k \gamma_0 \, . \tag{46}\]

so that \(\gamma_k\) is a symplex, if (and only if) it is symmetric and a cosymplex, if it is skew-symmetric. Using the Frobenius norm\footnote{Since the anticommutator does not necessarily result a scalar, one may distinguish between inner product and scalar product. The scalar product can then be defined by the trace of the inner product: \[(A \cdot B)_S \equiv \frac{1}{2n} \text{Tr}(AB + BA) \, , \tag{44}\] where the index “S” indicates the scalar part.} we know, that symmetric unit “vectors” \(A\) square to \(+1\), since

\[
\text{Tr}(A^2) = \text{Tr}(AA^T) = \sum_{i,j} a_{ij}^2 \geq 0 \, . \tag{47}\]

We therefore need to find two types of measurable unit vectors: \(\gamma_0\) that represents time and squares to \(-1\) and a number of symmetric \(\gamma_k\) (anticommuting with \(\gamma_0\) and with each other) that square to \(+1\). We interpret the latter as unit vectors in space-like directions. Skew-symmetric elements that are orthogonal to \(\gamma_0\) are cosymplexes and have vanishing expectation values.

This can be interpreted as follows: \textit{there is only a single measurable time direction.} To summarize: if a (flat) spacetime emerges from dynamics, all symplexes that represent orthogonal directions have to anticommute. Therefore the obvious choice of the \textit{inner product} or \textit{scalar product} of two elements of a dynamically generated geometric algebra is the anticommutator. The unit vector in the time-direction squares to \(-1\) and all vector-type symplexes \(\gamma_x\) that are orthogonal to \(\gamma_0\) are symmetric and square to \(+1\).

Given that we can find a (sub-) set of \(N\) orthogonal symplexes \(\gamma_k, k \in [0, 1, \ldots, N-1]\) in \(sp(2n)\), then the product of two of them holds:

\[
\gamma_i \gamma_j = \frac{\gamma_0 + \gamma_i \gamma_j - \gamma_0 \gamma_j \gamma_0 + \gamma_i \gamma_0 \gamma_j \gamma_0}{2} = \frac{2 \gamma_i + \gamma_0 \gamma_j - \gamma_0 \gamma_i}{2} \, . \tag{48}\]

for \(i \neq j\). It is easy to prove that the product \(\gamma_i \gamma_j\) with \(i \neq j\) is orthogonal to both \(-\gamma_i\) and \(-\gamma_j\). Therefore and due to its anti-symmetry it is nearly to define the \textit{exterior product} by the commutator of \(A\) and \(B\):

\[
A \wedge B = \frac{AB - BA}{2} \, . \tag{49}\]

An algebraic system of \(N\) pairwise anti-commuting elements (that square to \(\pm 1\)) represents a Clifford algebra \(Cl_{p,q}\), \(N = p + q\). \(p\) is the number of unit vectors that square to \(+1\) and \(q\) the number of unit vectors that square to \(-1\). We found that in our case \(q = 1\). \textit{If (as we presumed) spacetime emerges from the dynamics of fundamental variables, then the apparent geometry of spacetime is described by the (representation of) a Clifford algebra \(Cl_{N-1,1}\). The unit vectors are called \textit{generators} of the Clifford algebra.}

There are \(\binom{N}{k}\) possible combinations (without repetition) of \(k\) elements from a set of \(N\) generators. Then there are \(\binom{N}{2}\) products of 2 basic matrices (named \textit{bivectors}), \(\binom{N}{3}\) products of 3 basic matrices (named \textit{trivectors}) and so on. The product of all \(N\) basic matrices is called \textit{pseudoscalar}. The \(N\) anti-commuting generators are the (unit) \textit{vectors}. The total number of all \(k\)-vectors then is\footnote{The case \(k = 0\) can be identified with the unit matrix.}

\[
\sum_{k=0}^{N} \binom{N}{k} = 2^N \, . \tag{50}\]

As we have no reason to allow more elements in the algebra than \(2^N\), the vector space \(sp(2n)\) and \(Cl_{N-1,1}\) should have the same dimension:

\[
2^N = (2n)^2 \, , \tag{51}\]

which - if fulfilled - is nothing but the requirement of \textit{completeness}.

If such a basis of anticommuting matrices exists, we know that

\[
\gamma_k^2 = \pm 1 \quad \text{for all } k \in [0, \ldots, 4n^2 - 1] \\
\gamma_k = \pm \gamma_k^T \quad \text{for all } k \in [0, \ldots, 4n^2 - 1] \\
\gamma_k \gamma_j = \pm \gamma_j \gamma_k \quad \text{for all } j \neq k, \in [0, \ldots, 4n^2 - 2] \, . \tag{52}\]
From Eqs. [52] above we readily conclude:
\[
\gamma_0 \gamma_k \gamma_0 = \pm \gamma_k^T, \\
\gamma_k \gamma_0 \gamma_k^T = \pm \gamma_0,
\]
so that all \(\gamma_k\) are either sympleics or cosympleics and they are either symplectic (\(\gamma_0 \gamma_k \gamma_0^T = \gamma_k\)) or “cosymplectic” (\(\gamma_k \gamma_0 \gamma_k^T = -\gamma_0\)). Since \(\text{Det}(AB) = \text{Det}(A)\text{Det}(B)\) and since \(\gamma_k^T = \pm 1\) it follows that all basic matrices have a determinant of \(\pm 1\).

Given such a matrix system exists for some \(n\) and \(N\), then it has remarkable symmetry properties. Note that these properties could be derived without reference to the specific form (or size) of the matrices. If (as we demand) Eq. [51] holds, then the system is a complete basis of the vector space of real \(2^n \times 2^n\)-matrices and therefore these matrices \(M\) can be written as a linear combination of the \(\gamma_k\) according to:
\[
M = \sum_{k=0}^{(2n)^2-1} m_k \gamma_k,
\]
so that with the signature \(s_j\) defined by \(s_j \equiv \text{Tr}(\gamma^2)/2n\) we have
\[
\frac{1}{4n} \text{Tr}(M \gamma_j + \gamma_j M) = \frac{1}{4n} \text{Tr}\left(\sum_{k=0}^{2n-1} m_k (\gamma_j \gamma_k + \gamma_k \gamma_j)\right) = s_j m_j.
\]

Given that \(\gamma_k \in [0, \ldots, N - 1]\) are the generating elements of the algebra, then the vector \(A = \sum_{k=0}^{N-1} a_k \gamma_k\) squares to a scalar:
\[
A^2 = \sum_{k=0}^{N-1} s_k a_k^2 = \sum_{k=1}^{N-1} a_k^2 - a_0^2.
\]

We have proven that a dynamical system generates a flat Minkowski spacetime with clean observables if and only if it is a representation of a Clifford algebra in which all generating elements are sympleics. The idea to relate Clifford algebras and symplectic spaces is not new (see for instance Ref. [41,42] and references therein), however (to the knowledge of the author) it has not been shown yet that the concept of a geometric algebra can be derived from the idea of a dynamically emergent spacetime.

Usually Clifford algebras are introduced by a mere definition (and not by a derivation) of a set of anticommuting abstract objects and in a second stage the question is raised, what known mathematical objects (matrices, quaternions or octonions) could possibly be used to represent the unit elements. We argue more physically: if it is possible to construct a geometry from a given classical dynamical system (of coupled harmonic oscillators) then it is isomorphic to the representation of a Clifford algebra. The structure of spacetime follows from dynamics instead of dynamics from spacetime.

V. STRUCTURE PRESERVATION

A. Transfer Matrices and Symplectic Transformations

The solution of the Hamiltonian EqOM (Eq. [22]) for the Taylored Hamiltonian (with constant matrix \(A\)) are given by the matrix exponential:
\[
\psi(t) = \exp \left( F(t - t_0) \right) \psi(t_0) = M(t, t_0) \psi(t_0).
\]
The matrix \(M(t, t_0) = \exp \left( F(t - t_0) \right)\) is usually called transfer matrix and it is well-known to be symplectic if \(F\) is a symplex (“Hamiltonian”), that is, \(M\) fulfills the following equation:
\[
M \gamma_0 M^T = \gamma_0, \\
M^{-1} = -\gamma_0 M^T \gamma_0, \\
M^{-1} \gamma_0 M = -\gamma_0.
\]

Since the matrix exponential of \(F\) is quite complicated in the general form of Eq. [51] it is convenient to analyze instead the transformations that are generated by the basic sympleics \(\gamma_j\). This can easily be done, since all \(\gamma_j\) square to \(\pm 1\):
\[
\begin{align*}
R_j &= \exp (\gamma_j \varepsilon) = \exp \left( \gamma_j^2 \varepsilon \right) \\
R_j &= \sum_{k=0}^{\infty} \frac{(\gamma_j^2)^k \varepsilon^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(\gamma_j^2)^{2k+1} \varepsilon^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(\gamma_j^2)^k \varepsilon^{2k}}{(2k)!} + \gamma_j \sum_{k=0}^{\infty} \frac{(\gamma_j^2)^k \varepsilon^{2k+1}}{(2k+1)!} \\
&= \begin{cases} 
1 \cos (\varepsilon) + \gamma_j \sin (\varepsilon) & \text{for } \gamma_j^2 = -1 \\
1 \cosh (\varepsilon) + \gamma_j \sinh (\varepsilon) & \text{for } \gamma_j^2 = 1 \\
1 \cos (\varepsilon) - \gamma_j \sin (\varepsilon) & \text{for } \gamma_j^2 = -1 \\
1 \cosh (\varepsilon) - \gamma_j \sinh (\varepsilon) & \text{for } \gamma_j^2 = 1 
\end{cases}
\end{align*}
\]
The solutions are the sum of the unit matrix and \(\gamma_j\) with (hyperbolic) (co-) sine coefficients. From Eqs. [59] we find that skew-symmetric sympleics (those that square to \(-1\)) are generators of orthogonal transformations, since
\[
\begin{align*}
R_j &= 1 \cos (\varepsilon) + \gamma_j \sin (\varepsilon) \\
R_j^T &= 1 \cos (\varepsilon) - \gamma_j \sin (\varepsilon) \\
\Rightarrow &= R_j R_j^T = 1.
\end{align*}
\]

Hence skew-symmetric sympleics are generators of (symplectic) rotations. The symplectic transformations generated by symmetric sympleics are called boosts. As we will show in more detail in Sec. [314] it is legitimate to call the structure preserving property of symplectic transformations the principle of (special) relativity since Lorentz transformations of “spinors” \(\psi\) in Minkowski spacetime are a subset of the possible symplectic transformations for two DOFs.

More generally the structure of the matrix exponential of a symplex \(F\) is:
\[
M(t) = \exp(F t) = C + S,
\]
where the (co-) symplex $S$ (C) is given by:
\[
S = \sinh(Ft) \\
C = \cosh(Ft),
\]
(62)
since (the linear combination of) all odd powers of a symplex are symplexes and all even powers are cosymplexes. The inverse transfer matrix $M^{-1}(t)$ is then given by:
\[
M^{-1}(t) = M(-t) = C - S.
\]
(63)

For the matrix of second moments $\sigma$ and $S = \sigma\gamma_0$

\[
\sigma(t) = \psi(t)\psi^T(t) = M(t, t_0)\psi(t_0)\psi(t_0)^T M^T(t, t_0)
\]
(64)
so that the evolution of $S$ in time is a symplectic similarity transformation. Hence all eigenvalues of $S$ are constants.

From Eq. (67) it is obvious that the column vectors of the transfer matrix are the solutions for initial conditions $\psi(0)$ that are equal to unit vectors. The first column $\psi_1(t)$ of $M(t, t_0)$ is given by:
\[
\psi_1(t) = M(t, t_0)\begin{pmatrix}1, 0, 0, 0\end{pmatrix}^T.
\]
(65)
Furthermore we note that if a symplex $F$ is symplectically similar to $\gamma_0$, then it follows from Eq. (68):
\[
F = M\gamma_0 M^{-1} = MM^T\gamma_0,
\]
(66)
so that $F$ equals the matrix $S = \sigma\gamma_0$ if the second moments are the average over the orthogonal unit vectors as exemplified by Eq. (63):
\[
\sigma = MM^T = \sum \psi_k(t)\psi_k^T(t).
\]
(67)
This is another example for the mentioned duality between observables and generators (see Eq. (60) and Ref. (38)).

B. Cosymplexes as Generators of Transformations

The definition of what we consider to be a symplex or a co-symplex depends on the choice of the symplectic unit matrix $\gamma_0$. Eq. (69) holds also for (skew-) symmetric cosymplexes as it holds for symplexes. Consider a skew-symmetric cosymplex $\gamma_j$ (which we know to anticommute with $\gamma_0$), then the matrix exponential is an orthogonal similarity transformation, so that the transformed matrix $\tilde{\gamma}_0$ is skew-symmetric and squares to the negative unit matrix:
\[
\tilde{\gamma}_0^T = (R_j\gamma_0 R_j^T)^T = R_j\gamma_0^T R_j^T
\]
(68)
\[
= -R_j \gamma_0 R_j^T = -\tilde{\gamma}_0
\]
\[
\tilde{\gamma}_0^2 = (R_j\gamma_0 R_j^T)^2 = R_j \gamma_0^2 R_j^T
\]
\[
= -R_j \gamma_0 R_j^T = -1
\]
(69)
The transformed matrix differs from $\gamma_0$, but has essentially the same properties.

If we consider on the other hand a symmetric cosymplex $\gamma_j$ (which we know to commute with $\gamma_0$), then the matrix exponential is also symmetric and commutes with $\gamma_0$ so that:
\[
(R_j\gamma_0 R_j^T)^T = R_j\gamma_0^T R_j^T = -R_j^2 \gamma_0
\]
\[
(R_j\gamma_0 R_j^T)^2 = R_j \gamma_0^2 R_j^T = -R_j^2 1
\]
(70)
The transformed matrix is still antisymmetric but it does not square to the (negative) unit matrix.

One may conclude that skew-symmetric cosymplexes are generators of time-like rotations: They change the “orientation” of $\gamma_0$, the system is rotated towards a new “time direction” $\tilde{\gamma}_0 = R\gamma_0 R^T$, but remains structurally similar (though not the same).

The physical meaning of such non-symplectic transformations can be related to a variety of phenomena. In Ref. (37) for instance non-symplectic transformation have been shown to be related to the appearance of vector potentials and to the transformation into curvilinear coordinate systems. In App. C we give an example which is directly related to the structure of a problem. Furthermore we explicitly give a matrix that transforms between the six possible choices for $\gamma_0$ of the Dirac algebra in App. C.

VI. THE DIMENSIONALITY OF SPACETIME

A. Clifford Algebras

Sets of (anti-) commuting matrices as we derived them above are known as generators of Clifford algebras. The inner products of the basic elements are the elements of the metric tensor \[1\]. The vector space constructed in this way is Euclidean only for diagonal metric tensors with signatures $s_1 = s_2 = \cdots = s_N = 1$, i.e. Euclidean unit vectors square to +1. Therefore we call generators that square to +1 spatial or space-like and generators that square to −1 time-like \[12\]. A Clifford algebra with $p$ space-like and $q$ time-like generator matrices is indicated by $Cl_{p,q}$.

From the dimensional requirement
\[
2^N = (2n)^2 = 4n^2.
\]
we conclude that such matrix systems can only be complete for even $N$ ($N = 2M$) in order to obtain integer

\[1\] Even though it is unclear at this point in what sense they are a tensor
\[12\] This is only meaningful for real representations, since the case of complex representations allows to switch the sign of matrices by the multiplication with the unit imaginary. We restrict ourselves to purely real representations where the number of variables is well-defined.
Any real matrix system that fulfills the above requirements is based on a number of degrees of freedom which is a power of 2. The simplest system should hence be the one for which \( M = 1 \) \((N = 2, n = 1)\) holds, i.e. it should consist of \(2 \times 2\)-matrices.

As we mentioned in Sec. III only those elements \( \gamma_k \) that are symplectic have a non-vanishing expectation value \( \psi^\dagger \gamma_k \psi \) and are therefore observables. Hence we restrict ourselves to Clifford algebras with generators that can be represented by symplectic - including \( \gamma_0 \) as the direction of time. Since any generator that anticommutes with \( \gamma_0 \) is symmetric and therefore squares to \(+1\), we have \( q = 1 \), i.e. the Clifford algebras are of the type \( C_{1,1}^p(q) \).

Clifford algebras with an even number \( N \) of generators have a pseudoscalar \( \gamma_n = \prod_{k=0}^{N-1} \gamma_k \) that anticommutes with all basic elements and therefore commutes with all bivectors. The pseudoscalar \( \gamma_n \) is a symmetry operator: All Clifford algebras of dimension \( N \) have the same number of \( k \)-vectors and \( N - k \)-vectors, and multiplication with the pseudoscalar flips between them. If we want to establish a correspondence between a Clifford algebra and spacetime geometry, there are certain aspects that have to be considered. If we have a basis of generators \( \gamma_0, \gamma_1, \ldots, \gamma_{N-1} \) of a Clifford algebra \( C^{-1}_{N-1,1} \) with metric

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \text{Diag}(-1,1,\ldots,1),
\]

then we know that \( \gamma_0 \gamma_k \) with \( k = 1, 2, \ldots N - 1 \) are the generators of boosts ("boosters") and \( \gamma_j \gamma_k \) with \( j \neq k \) and \( j, k \neq 0 \) are the generators of spatial rotations ("rotators"). We come back to this in the next subsection.

B. The Number of Space Dimensions

Spacetime is a “construction” based on (Hamiltonian) dynamics and on the geometric interpretation of the matrix operators that represent the observables. The possible numbers of spacetime dimensions \( N = p + q \) is given by the representation theory of real matrices\(^{13}\):

\[
p - q = 0 \text{ or } 2 \text{ mod } 8.
\]

Since \( \gamma_0 \) is the only possible skew-symmetric (time-like) symplex-generator of the Clifford algebra, time is one-dimensional \( q = 1 \) and therefore there are \( p = N - 1 \) space dimensions so that:

\[
N - 2 = 0 \text{ or } 2 \text{ mod } 8.
\]

Hence emergent spacetimes exist only with \( 1 + 1, 3 + 1, 9 + 1, 11 + 1, 17 + 1, 19 + 1, 25 + 1, 27 + 1, \ldots \) dimensions, corresponding to symplectic of size \( 2 \times 2, 4 \times 4, 32 \times 32, 64 \times 64, \ldots \).

Symplectic of size \( 2 \times 2 \) may only have real or pure imaginary eigenvalues and are therefore not able to represent the general case. Therefore the \( 2 \times 2 \) case can not be generalized and must be rejected as too simple. Certainly there are multiple arguments, why spacetime should have \( 3 + 1 \) dimensions. In the framework we present here, we can mention the following reasons:

1. Simplicity. Ockham’s razor demands to use the lowest possible number of assumptions and the simplest possible structure.

2. The eigenvalues of \( 2n \times 2n \) matrices over the reals are (for \( n > 1 \)) pairs of complex conjugate eigenvalues. Since \( N \) is even, all representations for \( N > 1 \) have matrix dimensions that are multiples of 4 and can therefore always be block-diagonalized to \( 4 \times 4 \)-blocks. Therefore higher dimensional cases based on \( 4M \times 4^M \)-matrices can be split into \( M \) objects with representation \( 4 \times 4 \), i.e. into \( M \) particles in \( C_{1,1}^{3,1} \).

3. The Pauli and the Dirac algebra are the only cases where the state vector \( \psi \) has the same dimension as the spacetime constructed from it, i.e. the only case where \( 2n = N \) so that

\[
2^N = N^2.
\]

The number of variables of the spinor should be equal of higher than the dimension of the object constructed from it - certainly not less.

4. In 3 dimensions the number of rotation axes equals the number of spatial directions since one can form 3 rotator pairs \( \gamma_j \gamma_k \) (both space-like vectors) without repetition from 3 spatial directions. Both - rotators and boosters - mutually anti-commute with each other and hence generate an orthogonal space of the same dimension. In more than 3 space dimensions this changes: The number of boosts stays the same, but the number of (symplectic, i.e. binary) rotators grows quadratically with the number of spatial directions \( N - 1 \) since one can select \( \binom{N-1}{2} = (N-1)(N-2)/2 \) rotators. For \( N - 1 = 4 \) space dimensions this gives 6 rotators and not all of them are “orthogonal” to each other. In more than 3 space dimensions some rotators commute, for instance in a 4-dimensional space the rotators \( \gamma_1 \gamma_2 \) and \( \gamma_3 \gamma_4 \) commute\(^{15}\). This means that (dynamically generated) spaces with more than 3 dimensions are topologically not “homogeneous”. More precisely: \( D = 3 \) is the only considered space dimension in which all axis have the same “relation” to all others.

\(^{13}\) A Clifford algebra also can be defined for odd \( N \), but these algebras are not isomorphic to a representation in form of \( 2n \times 2n \) real matrices.
5. In Sec. [VII C] we will show how electrodynamics can be derived with this approach. In higher-dimensional spacetimes the arguments that allow for this derivation might not be valid anymore (see also App. [A1]).

More arguments for the 3-dimensionality of space can be found in the literature, one recent for instance in Ref. [46], where the possibility of the existence of a hydrogen atom is discussed for different spacetime dimensionalities. We prefer arguments with objects or structures that we have already derived or defined. The arguments given above do not require an a priori existence of spacetime. They do not even require that spacetime has “in reality” a specific dimensionality. In fact, the arguments are (at least to some degree) epistemological nature, as we are concerned with the apparent dimensionality of spacetime - since in our gedankenexperiment spacetime is not “real” but emergent. We believe that with respect to this ontological view the arguments given above are sufficiently strong.

In Sec. [III] we raised the question of measurability. We emphasized that all unit vectors have to be symplcices in order to be measurable. Since all basic elements (unit vectors) anticommute, also the bi-vectors - i.e. the products of two unit vectors - are symplcices. Geometrically bi-vectors are “oriented faces”. However products of 3 orthogonal symplcices - which would correspond to a volume - are cosymplcices and therefore have vanishing expectation values (see also App. [A1]). It follows that in dynamically emerging spaces “objects” can have positions (or directions), but no measurable volume. I.e. fundamental (measurable) objects could be characterized as point-like particles.

C. The Real Dirac Matrices

In what has been sketched above, we convinced ourselves that the case of real 4 × 4-matrices that represent a geometric (Clifford) algebra is unique. It is not only the “smallest” but as far as we can see the most fundamental algebra that allows for construction of spacetime. And it is the only algebra that allows for the construction of spacetime as we experience it. With respect to the basic variables ψ it represents systems with 2 DOFs. Also in this respect it can be regarded as “fundamental” as it describes the smallest possible oscillatory system with (internal) coupling or “interaction” [14].

An appropriate choice of the basic matrices of the Clifford algebra $Cl_{1,1}(R)$ are the well known Dirac matrices. Ettore Majorana was the first who discovered a system of Dirac matrices that contains exclusively imaginary (or exclusively real) values[15]. We call these matrices the “real Dirac matrices” (RDMs) and since we are committed to the reals, we will use the RDMs instead of the conventional system [16].

Often the term “Dirac matrices” is used in more restrictive way and designates only four matrices, i.e. the generators of the Clifford algebra [17]. We define the RDMs as a complete system of matrices and include all elements of the Clifford algebra generated by $\gamma_{\mu}$:

$$
\gamma_{14} = \gamma_0 \gamma_1 \gamma_2 \gamma_3; \quad \gamma_{15} = 1
$$

$$
\gamma_4 = \gamma_0 \gamma_1; \quad \gamma_7 = \gamma_{14} \gamma_0 \gamma_1 = \gamma_2 \gamma_3
$$

$$
\gamma_5 = \gamma_0 \gamma_2; \quad \gamma_8 = \gamma_{14} \gamma_0 \gamma_2 = \gamma_3 \gamma_1
$$

$$
\gamma_6 = \gamma_0 \gamma_3; \quad \gamma_9 = \gamma_{14} \gamma_0 \gamma_3 = \gamma_1 \gamma_2
$$

$$
\gamma_{10} = \gamma_{14} \gamma_0 = \gamma_1 \gamma_2 \gamma_3
$$

$$
\gamma_{11} = \gamma_{14} \gamma_1 = \gamma_0 \gamma_2 \gamma_3
$$

$$
\gamma_{12} = \gamma_{14} \gamma_2 = \gamma_0 \gamma_3 \gamma_1
$$

$$
\gamma_{13} = \gamma_{14} \gamma_3 = \gamma_0 \gamma_1 \gamma_2
$$

There are $2n! = 24$ possible permutations of the variables in the state vector, but since a swap of the two DOFs does not change the form of $\gamma_0$, there are 12 possible basic systems of RDMs[18]. Since $\gamma_0$ is antisymmetric, there are only six possible choices for $\gamma_0$. Each of them allows to chose between two different sets of the “spatial” matrices[19]. The explicit form of the RDMs is given in Ref. [44,35]. The four basic RDMs are anti-commuting

$$
\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2 g_{\mu \nu} 1,
$$

and the “metric tensor” $g_{\mu \nu}$ has the form[18]

$$
g_{\mu \nu} = \text{Diag}(-1,1,1,1).$$

Any real 4 × 4 matrix $A$ can be written as a linear combination of RDMs according to

$$
A = \sum_{k=0}^{15} a_k \gamma_k ,
$$

where a quarter of the trace $\text{Tr}(A)$ equals the scalar component $a_{15}$ - since all other RDMs have zero trace. The RDM-coefficients $a_k$ are given by the scalar product

$$
a_k = A \cdot \gamma_k = \frac{\text{Tr}(\gamma_k^2)}{4} \text{Tr}(A \gamma_k + \gamma_k A).
$$

Any real 4 × 4 Hamiltonian matrix (symplex) $F$ can be written as a linear combination in this basis

$$
F = \sum_{k=0}^{9} f_k \gamma_k.
$$

14 The role of the Dirac matrices as generators of the symplectic group $Sp(4,R)$ has been described before, see for instance Ref. [45].
Without further explanation it might look like a jolly contingency that the (real) Dirac matrices have these geometric properties. And it is indeed fascinating how elegant the geometric algebra (which is based on the Dirac matrices) fits to the needs of spacetime geometry. In Refs. the authors write: “Historically, there have been many attempts to produce the appropriate mathematical formalism for modeling the nature of physical space, such as Euclid’s geometry, Descartes’ system of Cartesian coordinates, the Argand plane, Hamilton’s quaternions, Gibbs’ vector system using the dot and cross products. We illustrate however, that Clifford’s geometric algebra (GA) provides the most elegant description of physical space.”

We have shown that there is deeper reason for this elegance. Geometric algebra (GA) - if introduced by abstract mathematical anticommuting objects leave a mark of “artificial construction”. In mathematics it is possible to define objects with almost arbitrary properties: As long as a system is self-consistent, everything is possible. But without a physical reasoning the introduction of merely defined objects remains unsatisfactory. The missing link, the physical reason for the congruence of GA with euclidean geometry in 3 + 1 dimensions is that spacetime emerges from dynamics in the form of the Dirac algebra.

This is most obvious if one considers the “representation” of the Clifford algebra Cl_{3,1} in form of the RDMs: Any physical interaction of two DOF (i.e. two canonical pairs) in time carries the structure of geometric algebra since it is a Hamiltonian process based on 4 × 4-matrices. The 4 × 4-matrices obtain their structure from the symmetry of A and the symplectic unit matrix γ₀. We will carve this out in more details in Sec. VII.

VII. LORENTZ TRANSFORMATIONS AS SYMPLECTIC TRANSFORMATIONS

Structure preservation refers to Eq. (1). That is: the transformation of the skew-symmetric matrix S to γ₀ is the exact contrary of structure preserving - it is structure defining. Only transformations that preserve the form of γ₀ are called structure preserving. And since γ₀ emerges from Hamilton’s equations, we can say that structure preserving transformations are canonical. Linear structure preserving transformations are called symplectic.

Using Eq. 55 it is easy to show that symplectic transformations are structure preserving, i.e. a transformation of the dynamical variables ψ that does not change the form of the equations, neither it changes the form of γ₀. For the measurements given by S = σγ₀ we expect a similarity transformation. The linear transformation R is given as

\[ \tilde{\psi} = R \psi. \]  

so that the measurements are given by

\[
\begin{align*}
S &= \langle \psi \psi^T \rangle \gamma_0 \\
\tilde{S} &= \langle \tilde{\psi} \tilde{\psi}^T \rangle \gamma_0 \\
&= (R \langle \psi \psi^T \rangle R^T) \gamma_0 \\
&= -R (\langle \psi \psi^T \rangle \gamma_0 \gamma_0 R^T \gamma_0) \\
&= -RS \gamma_0 R^T \gamma_0 \\
&= RS R^{-1} \gamma_0 \\
\end{align*}
\]

so that the conditions are fulfilled if R is symplectic (fulfills Eq. 55). Then it is easy to show that symplectic transformations are structure preserving, i.e. a symplex F transformed by a symplectic similarity transformation R remains a symplex:

\[
\begin{align*}
\tilde{F} &= RF R^{-1} \\
F' &= (R^{-1})^T F^T R^T \\
&= (-\gamma_0 R \gamma_0) (\gamma_0 F \gamma_0) (-\gamma_0 R^{-1} \gamma_0) \\
&= \gamma_0 R F R^{-1} \gamma_0 \\
&= \gamma_0 F' \gamma_0.
\end{align*}
\]

If we compare this result with Eq. (61), then we recognize that a transformation is symplectic if it is possible to find a (constant) Hamiltonian from which the transformation can be derived, i.e. a transformation is symplectic, if it can be the result of a possible evolution in time.

As we show in the following, it is legitimate to call the structure preserving property of symplectic transformations the principle of (special) relativity since Lorentz transformations (LTs) of spinors in Minkowski spacetime are a subset of the possible symplectic transformations for two DOFs. It is well-known that the matrix exponential of a “Hamiltonian matrix” (i.e. a symplex) is a symplectic transformation. It remains to be shown which symplectes generate LTs. Consider we transform a matrix \(X = E \gamma_0 + P_x \gamma_1 + P_y \gamma_2 + P_z \gamma_3\) using \(R_4 = \exp (\gamma_4 \varepsilon / 2)\), then we obtain after decomposition into the RDM-coefficients:

\[
\begin{align*}
X' &= \tilde{R}_4 X R_4^{-1} \\
X' &= E' \gamma_0 + P_x' \gamma_1 + P_y' \gamma_2 + P_z' \gamma_3 \\
E' &= E \cosh (\varepsilon) + P_x \sinh (\varepsilon) \\
P_x' &= P_x \cosh (\varepsilon) + E \sinh (\varepsilon) \\
P_y' &= P_y \\
P_z' &= P_z
\end{align*}
\]

Using the usual parametrization in Minkowski space where \(\varepsilon\) is the “rapidity” \((\beta = \tanh (\varepsilon), \gamma = \cosh (\varepsilon), \beta \gamma = \sinh (\varepsilon))\), we find

\[
\begin{align*}
E' &= \gamma E + \beta \gamma P_x \\
P_x' &= \gamma P_x + \beta \gamma E
\end{align*}
\]

which is evidently identical to a Lorentz boost along x. Same transformation, but \(X = E_x \gamma_4 + E_y \gamma_5 + E_z \gamma_6 + \ldots\).
\[ B_x \gamma_7 + B_y \gamma_8 + B_z \gamma_9: \]

\[
\begin{align*}
X' &= RXR^{-1} \\
E'_x &= E_x \\
B'_x &= B_x \\
E'_y &= E_y \cosh(\varepsilon) + B_z \sinh(\varepsilon) \\
E'_z &= E_z \cosh(\varepsilon) - B_y \sinh(\varepsilon) \\
B'_y &= B_y \cosh(\varepsilon) - E_z \sinh(\varepsilon) \\
B'_z &= B_z \cosh(\varepsilon) + E_y \sinh(\varepsilon)
\end{align*}
\] (87)

which is again a Lorentz boost along \(x\), but now it shows the transformation behavior of electromagnetic fields. Obviously the algebraic structure of the Dirac matrices implies a transformation behavior according to the Lorentz transformations. Indeed the transformations as described above are isomorphic to the Lorentz transformation of the Dirac equation\(^{2,3}\), suggesting that the vector of variables \(\psi\) is essentially the Dirac spinor.

This derivation did not require any reference to the speed of light, to time dilation or with alike. Instead we derived the LT's exclusively from the requirement that we are able to make measurements in a world with time in which spacetime is a construction. Maybe for historical reasons, textbooks usually argue with the “constancy of the speed of light” in the attempt to explain relativity. In view of the author, this is logically misleading, since the “constancy” of the speed of light is neither the cause nor the reason for, but a consequence of (special) relativity. Logically we have to derive electrodynamics and hence the speed of light from relativity and not vice versa - unless we can show how and why the Lorentz transformations can be derived from electrodynamics and electrodynamics independently from first principles. However - as often declared - it is not possible to derive electrodynamics from relativity alone\(^{51}\). But if we understand spacetime as emergent from symplectic dynamics, then there are additional restrictions and Maxwell’s equations can indeed be derived. It is an experimental fact that electrodynamical interactions are structure preserving. No fermion can be transformed into another by pure electromagnetic fields. Certainly there is pair production and particle-antiparticle annihilation, but it is conventional wisdom to identify the antiparticle with the particle going backwards in time. This only strengthens our argument, that electromagnetic fields are structure preserving. They have to be derived and understood in the context of dynamically emergent spacetime and structure preservation.

### A. The Electromechanical Equivalence (EMEQ)

The transformation properties \(^{55}\) and \(^{57}\) suggest the introduction of the electromechanical equivalence (EMEQ), as presented in Refs. \(^{34,35}\). In those papers the EMEQ was used as a merely formal tool that allows to obtain an meaningful order of symplectic transformations. Here we argue substantially and we derive Maxwell’s equations from it.

In Sec. \(^{33}\) we have then seen that the fundamental solution of the EQM is given by a symplectic transfer matrix. In order to produce constants of motion, the transfer matrix should represent a strongly stable system which means that all of its eigenvalues lie on the unit circle in the complex plane, i.e. the matrix \(F\) should have exclusively purely imaginary eigenvalues. We already started to clarify the structure of \(F\) by introducing the RDMs and by saying that the RDMs are a representation of the Clifford algebra \(Cl_{3,1}\). Therefore we have four basic elements and six bi-vectors which are symplexes. The other 6 members of the group of the RDMs are cosymplexes. The analysis of the transformation properties indicates how we should interpret the coefficients of the algebra. Now we explicitely introduce a nomenclature as follows. Starting with Eq. \(^{31}\) we suggest to “associate” the \(f_k\) as follows:

\[
\begin{align*}
f_0 &= \mathcal{E} & \text{Energy} \\
f_1, f_2, f_3^T &= \bar{P} & \text{Momentum} \\
f_4, f_5, f_6^T &= \vec{E} & \text{electric field} \\
f_7, f_8, f_9^T &= \vec{B} & \text{magnetic field}
\end{align*}
\] (88)

With this nomenclature the eigenvalues \(\pm i \omega_1, \pm i \omega_2\) of \(F\) are given as:

\[
\begin{align*}
\omega_1 &= \sqrt{K_1 + 2\sqrt{K_2}} \\
\omega_2 &= \sqrt{K_1 - 2\sqrt{K_2}} \\
K_1 &= -\text{Tr}(F^2)/4 = \mathcal{E}^2 + \vec{B}^2 - \vec{E}^2 - \bar{P}^2 \\
K_2 &= \text{Tr}(F^4)/16 - K_1^2/4 \\
&= (\mathcal{E} \cdot \vec{B} - \bar{P} \times \vec{E})^2 - (\vec{E} \cdot \vec{B})^2 - (\bar{P} \cdot \vec{B})^2 \\
\text{Det}(F) &= \omega_1^2 \omega_2^2 = K_1^2 - 4K_2
\end{align*}
\] (89)

Bi-vectors form the so-called even subalgebra of the Dirac algebra\(^{51}\). We have shown that any Clifford algebra \(Cl_{p,q}\) which is able to represent spacetime, must have an even number of generators \(N\). This holds also for the Dirac algebra. Even dimensional algebras have even subalgebras, which means that even elements generate exclusively even elements, whereas odd elements can be used to generate the full algebra. This might be interpreted in the following way: Fermions can generate the electromagnetic fields (alone) to generate fermions. Consequently there should be (differential-) equations that allow to derive the bi-vector fields \(\vec{E}\) and \(\vec{B}\) from vectors, but not vice versa. This means, that the description of a free particle without external fields should not refer to bi-vectors. Hence

\(^{19}\) Elements are said to be even, if they do not change sign when all basic elements \(\gamma_0, \ldots, \gamma_7\) change sign. They are odd otherwise. I.e. bi-vectors are even, vectors and tri-vectors are odd.
it is natural to distinguish the vector and bi-vector components and for vanishing bi-vectors ($\vec{E} = \vec{B} = 0$) one finds:

$$
\begin{align*}
K_1 &= \vec{E}^2 - \vec{B}^2 \\
K_2 &= 0 \\
\omega_1 &= \omega_2 = \sqrt{K_1} = \sqrt{\vec{E}^2 - \vec{B}^2}.
\end{align*}
$$

(90)

At the end of Sec. [III] we argued that (in a world based on time) a reference has to be understood as a quadratic form. Here we have an example. The mass is a constant of motion and has the physical meaning and unit of frequency. It follows that the “time” parameter as we introduced it in Eqs. (8), has to be identified with the eigenfrequency. The eigenfrequency that defines the scale of the eigentime is the mass and hence a massive particle is described by an inertial frame of reference combined with an oscillator of constant frequency (i.e. a clock). This is the deeper meaning of Einstein’s clock attached to all inertial frames. Einstein certainly had not a specific clock in mind, but rather the Einstein’s clock attached to all inertial frames. Einstein didn’t exist. Instead of an abstract clock in an abstract a-priori spacetime are meaningless in a deep sense: They don’t exist. Instead of an abstract clock in an abstract inertial frame we discover that the proper description of a particle is essentially an oscillator in its inertial frame.

For vanishing vector components ($\vec{E} = 0 = \vec{B}$) we have

$$
\begin{align*}
K_1 &= \vec{B}^2 - \vec{E}^2 \\
K_2 &= - (\vec{E} \cdot \vec{B}),
\end{align*}
$$

(91)

which are the well-known relativistic invariants of the electromagnetic field. It follows that the frequencies are only real-valued for $\vec{E} \cdot \vec{B} = 0$. An analysis of the transformations generated by the bi-vectors yields that the elements $\gamma_4, \gamma_5$ and $\gamma_6$ are responsible for Lorentz boosts, while $\gamma_7, \gamma_8$ and $\gamma_9$ are generators of (spatial) rotations. This corresponds to our physical intuition since (charged) particles are accelerated by electric fields but their “trajectories” are bended (i.e. rotated) in magnetic fields. It is therefore no surprise that - using the EMEQ - the derivation of the Lorentz force is straightforward (see Sec. [VII B]).

Furthermore we note that we have vector-elements associated with $\vec{E}$ and $\vec{B}$, but no elements associated directly with the spacetime coordinates. For a representation of a single elementary particle this makes sense insofar as the dynamics of a single particle may not refer to absolute positions (unless via the bi-vector fields). Spacetime coordinates do not refer to particle properties, but to the relativ position (relation) of particles. Hence as long as we refer to a single object based on fundamental variables, we observe the momentum and energy of a particle as it behaves in an external electromagnetic field.

As we already mentioned in Sec. [III], the symplex $S = \sigma \gamma_0$ is singular, if we use only one single real spinor $\psi$ to define

$$
\sigma = (\psi \psi^T \gamma_0),
$$

(92)

where the angles imply some (unspecified) sort of average. Since the eigenvalues of the symplex $S$ yield the “frequency” or “mass” $m = \sqrt{\vec{E}^2 - \vec{B}^2}$ of the structure which we identify with a particle, we need at least $N \geq 4$ linear independent spinors to obtain a massive fermion. In the classical Dirac theory, the electron spinor is a linear combination of two complex spinors, i.e. also requires 4 real spinors. From this point of view inertial mass might be interpreted statistically - as a property that can only be derived using averaging - either over samples or in time. The idea that mass might be a statistical phenomenon is neither new nor extraordinarily exotic, see for instance Ref. (54,55) and references therein. But here we do not refer to entropy but rather to dynamical or algebraic properties of real spinors.

B. The Lorentz Force

From what we said above we will derive the Lorentz force equations. The 4-momentum of a particle (fermion) is defined by vectors. The vector components of $S$ are associated with the 4-momentum:

$$
\vec{P} = \vec{E} \gamma_0 + P_x \gamma_1 + P_y \gamma_2 + P_z \gamma_3 = \vec{E} \gamma_0 + \vec{P} \gamma_7,
$$

(93)

where $\vec{E}$ is the energy and $\vec{P}$ the momentum. Accordingly fields are associated with the bi-vectors of the “force” matrix $\vec{F}$. Then we use Eq. (13) and obtain the Lorentz force equations

$$
\frac{d\vec{P}}{d\tau} = \vec{P} = \frac{q}{2m} (\vec{F} \vec{P} - \vec{P} \vec{F})
$$

(94)

where $\tau$ is the proper time and $\frac{q}{2m}$ is a relative scaling factor. In the lab frame time $dt = \gamma d\tau$ Eq. (94) yields (setting $c = 1$):

$$
\begin{align*}
\frac{d\vec{E}}{dt} &= \frac{q}{m} \vec{P} \vec{E} \\
\frac{d\vec{B}}{dt} &= \frac{q}{m} (\chi \vec{F} \vec{E} + \vec{P} \times \vec{B}) \\
\chi &= \frac{q}{m} (m \gamma \vec{E} + m \gamma \vec{v} \times \vec{B}) \\
\frac{d\vec{v}}{dt} &= \frac{q}{m} \vec{v} \vec{E} \\
\frac{d\vec{v}}{dt} &= \frac{q}{m} \left( \vec{E} + \vec{v} \times \vec{B} \right).
\end{align*}
$$

(95)

C. Electrodynamics from Symplectic Spacetime

The space- and time- coordinates are now written as a 4-vector

$$
\vec{X} = \gamma_0 t + \vec{v} \cdot \vec{x},
$$

(96)
and we will try to express the fields $\vec{E}$ and $\vec{B}$ - which were originally understood as functions of the eigentime $\tau$, are now parameterized by $\vec{x}$ and $t$:

$$\begin{align*}
\vec{E}(\tau) & \rightarrow \vec{E}(\vec{x}, t) \\
\vec{B}(\tau) & \rightarrow \vec{B}(\vec{x}, t).
\end{align*}$$

(97)

We derived the form of the Lorentz transformations without any reference to the so-called constancy of the speed of light. We used exclusively classical dynamics and the ontology of time, i.e. from the assumption that space does not exist a priori, but is a construction. The construction suggests that the bi-vector “fields” are dependent on quantities with the transformation properties of vectors, i.e. they are generated by “particles”. Now we are going to extend the dynamics of (single) variables to the dynamics of “fields”, depending on vector type parameters. We denote the coefficients of these vectors $(t, \vec{x})$ and keep the question open, how these vectors are related to $(\vec{E}, \vec{F})$. Hence we follow Einstein and introduce spacetime as a property of fields.26

There have been several attempts to “derive” Maxwell’s equations in the past.62,63 None of these succeeded in finding its way into standard textbooks, in which usually only the Lorentz covariance of Maxwell’s equations is treated. A derivation of Maxwell’s equations from the condition of Lorentz covariance alone is considered to be impossible.63 But our ansatz is superior insofar as we constructed an algebraic framework that allows for additional symplectic constraints. Symplectic constraints are well-known in different context and led (for instance) to the non-squeezing theorem, i.e. to the constraints are well-known in different context and led for additional symplectic constraints. Symplectic constraints are well-known in different context and led (for instance) to the non-squeezing theorem, i.e. to the constraints are well-known in different context and led for additional symplectic constraints.

It is important to keep track of the transformation properties with respect to symplectic transformations and hence the bi-vectors fields can only be expressed (for instance in form of a Taylor series) by algebraic expressions that respect the appropriate transformation properties. According to Eq. (98) the field matrix is given by

$$\begin{align*}
\mathbf{F} &= E_x \gamma_4 + E_y \gamma_5 + E_z \gamma_6 + B_x \gamma_7 + B_y \gamma_8 + B_z \gamma_9 .
\end{align*}$$

(98)

Written as a Taylor series, the first terms must have the following form in order to be linear in $\mathbf{X}$ and to yield a bi-vector:

$$\mathbf{F} = \mathbf{F}_0 + (\mathcal{D}\mathbf{F}) \mathbf{X} - \mathbf{X} (\mathcal{D}\mathbf{F}) + \cdots ,$$

(99)

where $\mathcal{D}\mathbf{F}$ is an appropriate derivative taken at $\mathbf{X} = 0$ and must be a vector. Alternative forms of the linear term either include an “axial” vector (cosymplex) $\mathbf{V}$ or a pseudoscalar, both having a vanishing expectation value:

$$\begin{align*}
\mathbf{V} &= V_i \gamma_{10} + V_x \gamma_{11} + V_y \gamma_{12} + V_z \gamma_{13} \\
\mathbf{F}_1 &= (\mathbf{V} \mathbf{X} + \mathbf{XV})/2
\end{align*}$$

(100)

Since we can exclude the appearance of cosymplexes in structure preserving interactions, the first order term has the form of Eq. (99).

We will now analyze first and second order expressions that can be constructed with respect to the transformation properties. We skip constant terms for the moment since we want to analyze the relations of the partial derivatives.

Linear terms can exclusively be expressed by the commutator of two vector quantities, i.e. $\mathbf{X}$ and another (constant) 4-vector $\mathbf{J} = \rho_0 \gamma_0 + \vec{j}_0 \cdot \vec{\gamma}$:

$$\begin{align*}
\mathbf{F}_1 &= (\gamma_0 \vec{E} + \gamma_{14} \gamma_0 \vec{B}) \cdot \vec{\gamma} = \frac{4\pi}{3} \frac{1}{2} (\mathbf{XJ} - \mathbf{JX}) ,
\end{align*}$$

(101)

so that we obtain

$$\begin{align*}
\vec{E} &= \frac{4\pi}{3} (\vec{j}_0 t - \rho_0 \vec{x}) \\
\vec{B} &= \frac{4\pi}{3} (\vec{x} \times \vec{j}_0)
\end{align*}$$

(102)

and find that these linear terms fulfill the Maxwell equations:

$$\begin{align*}
\vec{\nabla} \cdot \vec{E} &= 4\pi \rho_0 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0 \\
\vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= 4\pi \vec{j}_0
\end{align*}$$

(103)

We continue with a second order term:

$$\mathbf{F}_2 = \frac{1}{4} (\mathbf{XJXJ} - \mathbf{JXJX}) ,$$

(104)

and it also fulfills Maxwell’s equations:

$$\begin{align*}
\vec{\nabla} \cdot \vec{E} &= 4\pi \rho_1 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0 \\
\vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= 4\pi \vec{j}_1 ,
\end{align*}$$

(105)

where

$$\begin{align*}
\rho_1 &= -\vec{j}_0^2 t - 3 t \rho_0^2 + 4 (\vec{j}_0 \cdot \vec{x}) \rho_0 \\
\vec{j}_1 &= 4 (\vec{j}_0 \cdot \vec{x} - \rho_0 t) \vec{j}_0 + (\rho_0^2 - \vec{j}_0^2) \vec{x} .
\end{align*}$$

(106)

Both orders fulfill the continuity equation:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0 .$$

(107)

So far this is not a derivation of Maxwell’s equations, but it shows that the transformation properties of bi-vectors and the emergence or construction of spacetime imply Maxwell’s equations. There is a method to express this ideas mathematically by the use of differential operators.

---

20 In first order the two partials of the induction law vanish separately, i.e. $\vec{\nabla} \times \vec{E} = 0 = \partial_t \vec{B}$. 
Since the aim is a “construction of spacetime”, the appropriate generalized derivative is of vector-type, i.e. we define the covariant derivative by
\[
\partial \equiv -\partial_t \gamma_0 + \partial_x \gamma_1 + \partial_y \gamma_2 + \partial_z \gamma_3 .
\] (108)
The non-abelian nature of matrix multiplication requires to distinguish differential operators acting to the right and to the left, i.e. we have \( \partial \) as defined in Eq. 108 and \( \bar{\partial} \) which is written to the right of the operand (thus indicating the order of the matrix multiplication) so that
\[
\begin{align*}
\bar{\partial} \mathbf{F} &\equiv -\partial_t \mathbf{F} \gamma_0 + \partial_x \mathbf{F} \gamma_1 + \partial_y \mathbf{F} \gamma_2 + \partial_z \mathbf{F} \gamma_3 \\
\bar{\partial} \mathbf{F} &\equiv -\gamma_0 \partial_t \mathbf{F} + \gamma_1 \partial_x \mathbf{F} + \gamma_2 \partial_y \mathbf{F} + \gamma_3 \partial_z \mathbf{F} .
\end{align*}
\] (109)

In order to keep the transformation properties of derivative expressions transparent, we distinguish between the commutative
\[
\partial \wedge \mathbf{A} \equiv \frac{1}{2} \left( \bar{\partial} \mathbf{A} - \mathbf{A} \bar{\partial} \right) 
\] (110)
and the anti-commutative
\[
\partial \cdot \mathbf{A} \equiv \frac{1}{2} \left( \bar{\partial} \mathbf{A} + \mathbf{A} \bar{\partial} \right) 
\] (111)
derivative. Then we find:
\[
\begin{align*}
\frac{1}{2} \left( \bar{\partial} \text{ vector} - \text{ vector} \bar{\partial} \right) &\Rightarrow \text{bi-vector} \\
\frac{1}{2} \left( \bar{\partial} \text{ vector} + \text{ vector} \bar{\partial} \right) &\Rightarrow \text{scalar} = 0 \\
\frac{1}{2} \left( \bar{\partial} \text{ bi-vector} - \text{ bi-vector} \bar{\partial} \right) &\Rightarrow \text{vector} \\
\frac{1}{2} \left( \bar{\partial} \text{ bi-vector} + \text{ bi-vector} \bar{\partial} \right) &\Rightarrow \text{axial vector} = 0
\end{align*}
\] (112)

Since (in case of the Dirac algebra) any symplex is the sum of a vector and a bivector, we find (for this special case):
\[
\frac{1}{2} \left( \bar{\partial} \text{ symplex} + \text{ symplex} \bar{\partial} \right) = \partial \cdot \text{ symplex} = 0
\] (113)

Now we return to Eq. 99 and find that \( \mathcal{D} \mathbf{F} \) must be a vector and therefore has (in “first order”) the form
\[
\mathcal{D} \mathbf{F} = \frac{1}{2} \left( \bar{\partial} \mathbf{F} - \mathbf{F} \bar{\partial} \right) = 4 \pi \mathbf{J} ,
\] (114)
which is nothing but a definition of the vector current
\[
\mathbf{J} = \rho \gamma_0 + j_x \gamma_1 + j_y \gamma_2 + j_z \gamma_3 .
\] (115)

Written explicitly in components, Eq. 114 is given by
\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 4 \pi \rho \\
\nabla \times \mathbf{B} - \partial_t \mathbf{E} &= 4 \pi \mathbf{j}
\end{align*}
\] (116)

We define a vector-potential \( \mathbf{A} = \phi \gamma_0 + \mathbf{A}_x \gamma_1 + \mathbf{A}_y \gamma_2 + \mathbf{A}_z \gamma_3 \) which has (to first order) the form
\[
\mathbf{A}_1 = \frac{1}{4} \left( \mathbf{X} \mathbf{F}_0 - \mathbf{F}_0 \mathbf{X} \right) ,
\] (117)
where \( \mathbf{F}_0 \) is the constant term of Eq. 99. It is easily verified that \( \mathbf{F}_0 = \frac{1}{2} \left( \bar{\partial} \mathbf{A}_0 - \mathbf{A}_0 \bar{\partial} \right) \) and hence the electromagnetic (bi-vector-) fields of the general symplex \( \mathbf{F} \) is given by
\[
\mathbf{F} = \frac{1}{2} \left( \bar{\partial} \mathbf{A} - \mathbf{A} \bar{\partial} \right) ,
\] (118)
or explicitly in components:
\[
\begin{align*}
\mathbf{E} &= -\nabla \phi - \partial_t \mathbf{A} \\
\mathbf{B} &= \nabla \times \mathbf{A} .
\end{align*}
\] (119)

It is well-known that the homogeneous Maxwell equations are a direct consequence of Eq. 118:
\[
\begin{align*}
\bar{\partial} \mathbf{F} + \mathbf{F} \bar{\partial} &= \frac{1}{8 \pi} \left( \bar{\partial}^2 \mathbf{F} - \partial_t \mathbf{F} \bar{\partial} + \bar{\partial} \mathbf{F} \partial_t - \mathbf{F} \bar{\partial} \right) = 0 \\
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{B} + \partial_t \mathbf{B} &= 0
\end{align*}
\] (120)
since the squared operators are scalars and commute with \( \mathbf{A} \). Accordingly the continuity equation is a direct consequence of Eq. 114
\[
\bar{\partial} \mathbf{J} + \mathbf{J} \bar{\partial} = \frac{1}{8 \pi} \left( \bar{\partial}^2 \mathbf{F} - \partial_t \mathbf{F} \bar{\partial} + \bar{\partial} \mathbf{F} \partial_t - \mathbf{F} \bar{\partial} \right) = \partial_t \rho + \nabla \mathbf{j} = 0 .
\] (121)

If one assumes the validity of the Lorentz gauge according to Eq. 112
\[
\frac{1}{2} \left( \bar{\partial} \mathbf{A} + \mathbf{A} \bar{\partial} \right) = \partial_t \phi + \nabla \mathbf{A} = 0 ,
\] (122)
then one obtains the wave equation of the vector potential \( \mathbf{A} \):
\[
4 \pi \mathbf{J} = -\bar{\partial}^2 \mathbf{A} = \left( \partial_t^2 - \nabla^2 \right) \mathbf{A} .
\] (123)

In a “current free region” the fields have to fulfill the homogeneous wave equation:
\[
\bar{\partial}^2 \mathbf{F} = \left( \nabla^2 - \partial_t^2 \right) \mathbf{F} = 0 .
\] (124)

It follows that plane electromagnetic fields “in vacuum” hold \( \omega^2 = k^2 \) so that the fields have no “eigenfrequency” and no dispersion.

\[\text{21} \] The commutative derivative of the trivial first order form \( \mathbf{A} \propto \mathbf{x} \) vanishes and is therefore useless in this context.
Note that Eqs. (120) and (121) are of the same form. The continuity equation (121) describes charge as something with the properties of a substance.

In summary: We identified Lorentz transformations as symplectic transformations. The generators of these transformations have been identified as electric and magnetic fields. It follows that electrodynamics is structure preserving and hence the description of electromagnetic fields excludes terms that are not structure preserving. With these “additional conditions” that we were able to derive Maxwell’s equations if we interpret the bi-vectors of the matrix $F$ as functions of vector-type spacetime coordinates of $Cl_{3,1}$. In this case, the bi-vector fields in $F$ fulfill a wave equation that describes the propagation of electromagnetic waves, i.e. of light in the constructed spacetime. Then we find that electromagnetic waves “in vacuum” fulfill the relation $\omega^2 = \vec{k}^2$ and hence we conclude the constancy of the speed of light. Strictly speaking we did not derive Maxwell’s equations from special relativity, but both theories are the result of a spacetime that emerges from symplectic dynamics.

In principle above equations are not restricted to 3 space dimensions, but hold also in the higher-dimensional cases. However, if spacetime had 10 or 12 dimensions, the connections between commutators and anti-commutators are way more complex and we would have to consider additional “channels”: In 3-dimensional space, the symplex $F$ is composed exclusively of components of the vector and bi-vector type. In higher-dimensional space (10- or 12-dimensional spacetime), the general symplex $F$ consists (besides vectors and bi-vectors) of n-vectors with $n \in \{5, 6, 9, 10, \ldots\}$ (see App. A 1).

D. The Density

From the combination of Eq. (97) and Eq. (123) it follows that also the (current-) density has also to be constructed as a function of space and time:

$$\rho \rightarrow \rho(\vec{x}, t)$$
$$\vec{j} \rightarrow \vec{j}(\vec{x}, t).$$

and it is clear that we have to normalize the density by

$$\int \rho(\vec{x}, t) \, d^3x = \text{const} = Q,$$  \hspace{1cm} (126)

where $Q$ is the “charge”. Especially the relation between the density $\rho(\vec{x}, t)$ and the “phase space density” $\tilde{\rho}(\psi)$ according to Eq. (29) requires some attention. It is clear that the naive assumption

$$\rho(\vec{x}, t) = \tilde{\rho}(\psi(\vec{x}, t))$$

(127)

can not be applied directly. According to Eq. (29) we have

$$1 = \int \tilde{\rho}(\psi) \, d^4\psi = \int \tilde{\rho}(\psi(\vec{x}, t)) \sqrt{g} \, d^3x,$$  \hspace{1cm} (128)

where $g$ is the appropriate Gramian determinant. We assume in the following that the normalization has been adjusted accordingly.

If spacetime was no construction, we could be sure that the wave function $\psi(\vec{x}, t)$ is well-defined and single valued for any coordinate of Minkowski spacetime. However, if spacetime emerges from dynamics, then it might be the other way around: The spacetime coordinates might be functions of the phase space position $t(\psi, \vec{x}(t))$. This includes the possibility that a) different phase space positions are mapped to the same spacetime-position (overlaps of wavefunctions) and b) that space and time coordinates are not unique, i.e. the particles appears “instantaneously” at different “locations” of spacetime. Finally this might indeed be the reason why $\psi$ has to be interpreted as a “probability density”.

Of cause these considerations are already part of quantum mechanics: The “distribution” in space can equivalently be described as the Fourier transform of a distribution in momentum space.

E. The Wave Equations

According to Eq. (123) the vector potential fulfills a wave equation - and also the (electromagnetic) fields can be described by waves. Solutions of wave equations are usually analyzed with the help of the Fourier transformation. This means that that the solutions can be written as superpositions of plane waves, i.e.

$$\phi(\vec{x}, t) = \int \tilde{\phi}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{x} - \omega t)} \, d^3k$$

$$\tilde{\phi}(\vec{k}, \omega) = \int \phi(\vec{x}, t) e^{-i(\vec{k} \cdot \vec{x} - \omega t)} \, d^3x,$$  \hspace{1cm} (129)

where we skipped normalization constants for simplicity. One important feature of the Fourier transform is the replacement of nabla operator $\vec{V}$ with the wave vector $i\vec{k}$ and of the time derivative with the frequency $\partial_t \rightarrow -i\omega$. The applicability of the Fourier transform seems to support our scepticism with respect to the ontological status of spacetime. In any case, it is interesting to write Maxwell’s equations as conditions for wave functions (we skip the tilde as it is usually clear from the context, if we refer to the fields or their Fourier transform):

$$i \vec{k} \cdot \vec{B} = 0$$
$$i \vec{k} \times \vec{E} - i\omega \vec{B} = 0$$
$$i \vec{k} \cdot \vec{E} = 4\pi \rho$$
$$i \vec{k} \times \vec{B} + i\omega \vec{E} = 4\pi \vec{j}.$$

The second of Eq. (130) implies that all (single) solutions also fulfill $\vec{E} \cdot \vec{B} = 0$. The homogeneous part of Eq. (130) are structurally an exact copy of the expressions that we obtain from Eq. (59) with the condition $K_2 = 0$:

$$\vec{E} \cdot \vec{B} - \vec{P} \times \vec{E} = 0$$
$$\vec{P} \cdot \vec{B} = 0$$
$$\vec{E} \cdot \vec{B} = 0$$  \hspace{1cm} (131)
In Ref.\textsuperscript{22} we made intense use of the condition $\vec{P} \cdot \vec{B} = 0$ and $\vec{E} \cdot \vec{B} = 0$ which have to be fulfilled in order to decouple the matrix $F$. The third term $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}$ was not required to vanish for decoupling. However, if the corresponding Maxwell equation would not yield zero, then we had to deal with a non-zero but divergence-free magnetic current density $4 \pi \vec{j}_m = \partial_t \vec{B} + \vec{\nabla} \times \vec{E}$ - which has not been found to date. But there is another more abstract reason, why we may restrict the solutions to the case of $K_2 = 0$; only then the symplex $F$ is a symplectical similarity transformation of the pure time direction as described by Eq.\textsuperscript{136} - \textsuperscript{24}. The third scalar product $\vec{P} \cdot \vec{E}$ (and the corresponding field equation $\vec{\nabla} \cdot \vec{E}$) is in the general case non-zero and we emphasize that this is another correspondence to the symplectic decoupling formalism (Ref.\textsuperscript{23}). In both cases this term is not required to vanish. It is also remarkable, that the so-called duality rotations of the electromagnetic field do not fit into this approach\textsuperscript{23}. Electric and magnetic fields can not be exchanged or mixed. Though both are bi-vector fields ("second rank tensors"), the special role of the time coordinate breaks the suspected symmetry. Furthermore the only generator that might be used for such a rotation is the pseudo-scalar, i.e. a cosymplex. Insofar symplectic electrodynamics as derived above apparently has a higher explanatory power than the conventional formalism.

If we summarize (and extend) the stability conditions $K_2 \geq 0$ and $K_1 \geq 2 \sqrt{K_2}$ (Ref.\textsuperscript{25}) by the condition $K_2 = 0$, then it is appropriate (or even mathematically inevitable) to use this striking structural similarity and to postulate the equivalence of $\vec{k}$ and $\vec{P}$ ($\omega$ and $\vec{E}$). That is - we claim that $\vec{P} \sim \vec{k}$ ($\vec{E} \sim \omega$) and hence that the momentum and energy equal via the Fourier transform the spatial and time derivative, respectively. There is still an optional proportionality factor. But for the same reason that we did not refer to the speed of light, we also skip the proportionality factor $h$, since it has no physical significance, but depends on the choice of units\textsuperscript{26}. The EQOM of $\psi(x,t)$ must result in a current density (Eq.\textsuperscript{139}) that fulfills the continuity equation. The simplest possible solution is known to be the Dirac equation. At first sight Eq.\textsuperscript{22} does not appear to be similar to the Dirac equation, but with the help of the Fourier transform, the picture changes. The eigenvalues of $\mathbf{F}$ for a "particle" in a field-free region are given by $\pm i \sqrt{\mathbf{E}^2 - \mathbf{P}^2} = \pm i m$. Together with Eq.\textsuperscript{132} we can now flip the "account" of Eq.\textsuperscript{22} by the replacement of operators and eigenvalues (and vice versa):

$$
(\vec{E} \gamma_0 + P_x \gamma_1 + P_y \gamma_2 + P_z \gamma_3) \psi = \dot{\psi} = \frac{d\psi}{dt} \quad (i \partial_t \gamma_0 - i \partial_x \gamma_1 - i \partial_y \gamma_2 - i \partial_z \gamma_3) \psi = \pm i m \psi \\
(\partial_t \gamma_0 - \partial_x \gamma_1 - \partial_y \gamma_2 - \partial_z \gamma_3) \psi = \pm m \psi.
$$

Apart from "missing" unit imaginary, which is "hidden" in the definition of the Dirac matrices $\gamma_\mu$, Eq.\textsuperscript{137} is the Dirac equation. The unitary transformation which transforms to the conventional Form of the Dirac matrices ($\tilde{\gamma}_\mu$) is explicitly given in App.\textsuperscript{24}. However - the unitary transformation does not change the signs of the metric tensor. If we desire to have $\vec{\gamma}_0 = 1$, we still need to multiply the transformed matrices by the unit imaginary, so that:

$$
(-i \partial_t \tilde{\gamma}_0 + i \partial_x \tilde{\gamma}_1 + i \partial_y \tilde{\gamma}_2 + i \partial_z \tilde{\gamma}_3) \psi = \pm m \psi.
$$

\textsuperscript{22} Only for $K_2 > 0$ there are two different frequencies while for $K_2 = 0$ the eigenfrequencies are identical as in case of $\gamma_0$.

\textsuperscript{23} Concerning the electro-magnetic duality rotation see for instance the short theoretical review article of J.A. Mignaco and the latest experimental result\textsuperscript{25}. It is easy to show that "in vacuum" we also have $\vec{E}^2 - \vec{B}^2 = 0$ so that also $K_1 = 0$ and there are no eigenfrequencies of electromagnetic waves. Furthermore they do not generate inertial frames of reference.

As we projected the bivector elements (fields) of the general symplex $\mathbf{F}$ into spacetime, we need to do the same with the phase space variables $\psi$ and thus obtain the "wave function" or spinor $\psi(x,t)$. A normalization might be written as

$$
\int \psi(x,t)^T \psi(x,t) d^3 x = 1.
$$

The phase space density function $\rho$ (and optionally the Gramian, see Eq.\textsuperscript{128}) might be "included" into the spinor function, such that

$$
\psi = \sqrt{\rho} \psi.
$$

Since the electric current must be a vector, the simplest Ansatz for the current density of a "particle" described by $\psi$ is \textsuperscript{24}

$$
\rho = -\psi \gamma_0 \psi \quad j_x = \psi \gamma_1 \psi \quad j_y = \psi \gamma_2 \psi \quad j_z = \psi \gamma_3 \psi
$$

\textsuperscript{24} This definition looks quite similar to the definition of the momentum by the EMEQ: Mass and charge density are proportional in case of "point particles".
VIII. ANYTHING ELSE?

A. Once More: Cosymplices

Much of what has been derived above is a consequence of the distinction between symplexes and cosymplices (or “Hamiltonian” and “Skew-Hamiltonian”) matrices and their expectation values. One central argument was that the generators of the considered Clifford algebras have to be symplexic since only sympleics have non-vanishing expectation values. Only symplexes are generators of symplectic transformations and hence only symplexic represent forces with measurable effects in time. One might argue that some entity might exist though its expectation value and its obvious consequences remain zero. Maybe it has hidden or indirect effects? For instance the algebra of 2\(n \times 2\) (co-) symplexic matrices have expectation values. One central argument was that the distinction between symplexes and cosymplices (or

\[
\begin{align*}
\zeta_0 &= \eta_3 \otimes \eta_3 \otimes \eta_0 \\
\zeta_1 &= \eta_3 \otimes \eta_3 \otimes \eta_1 \\
\zeta_2 &= \eta_3 \otimes \eta_1 \otimes \eta_2 \\
\zeta_3 &= \eta_3 \otimes \eta_2 \otimes \eta_1 \\
\zeta_4 &= \eta_1 \otimes \eta_0 \otimes \eta_2 \\
\zeta_5 &= \eta_2 \otimes \eta_0 \otimes \eta_2.
\end{align*}
\]

\[\langle [\zeta_\mu, \zeta_\nu] \rangle = \begin{pmatrix}
0 & E_x & E_y & E_z & 0 & 0 \\
-E_x & 0 & B_z & -B_y & 0 & 0 \\
-E_y & -B_x & 0 & B_z & 0 & 0 \\
-E_z & B_x & B_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & -\lambda & 0
\end{pmatrix}
\]

Hence the two additional elements are (in average) “decoupled” from the first four (and all other symplexic). The commutator of the two extra cosymplices is a symplex and hence yields a non-zero field value \(\lambda\), which acts only between the two additional time-like dimensions. Considerations like these are the background for the claim that time might indeed be multidimensional - but it is unlikely that we would be able to notice.

B. Higher-Dimensional Spaces

If spacetime is an emergent phenomenon, then the question arises whether other (high-dimensional) spacetimes may not emerge “in parallel”. We have no strong argument against this idea: Indeed the strongest evidence for a 3+1-dimensional spacetime that we have at hand, is based on light and electrodynamics. Chemistry is based almost exclusively on electromagnetic interactions: our world is the world of electrons. What we see, is light emitted, reflected and absorbed by electrons. If we touch a solid object, then electrons are “touching” electrons. Though we can experimentally investigate the weak and strong forces, gravitation and so on - but whatever we “see” with our own eyes, is to any degree of approximation light emitted by electrons.

Assume that the 9 + 1-dimensional spacetime emerges in parallel by the properties of other (high-dimensional) fields - then it still remains questionable, if and how observers would interpret the 9 + 1-dimensional spacetime. Certainly we can not expect that 9+1-dimensional is just “3 times of the same”, since the algebraic features (for instance of rotations) are different: An essential feature of 3 dimensions is that the spatial rotations do not commute. In more than 3 dimensions we can form commuting rotators, say \(\gamma_1 \gamma_2\) and \(\gamma_3 \gamma_4\). Hence in 9 dimensions we sort out 3 sets of 3 non-commuting rotators. Consider we use

1) \(\gamma_2 \gamma_3, \gamma_3 \gamma_1, \gamma_1 \gamma_2\)
2) \(\gamma_5 \gamma_6, \gamma_5 \gamma_6, \gamma_4 \gamma_5\)
3) \(\gamma_8 \gamma_9, \gamma_9 \gamma_8\)

We then find that any two rotators from different groups commute\(^{25}\). On the level of observables we derive from Eq. 15 that if two observables commute they are decoupled. Presumably observers grown up in a 3+1 dimensional world would not interpret the 9 + 1 dimensional spacetime correctly. Instead of “seeing” a 9+1-dimensional spacetime “parallel” to our “electronic” 3+1-dimensional spacetime, we would likely see 3 objects located in 3 + 1 dimensions. An ontological commitment to the idea that spacetime dimensionality must be a unique property of the world (and not of the interaction), might results in “conceptional blindness”. Though this is beyond the scope of this paper - a 9+1-dimensional world might fit to the description of hadrons composed of 3 quarks.

\(^{25}\) Dynamically emerging spacetimes have their own properties, which are determined by the structure of the corresponding symplectic Clifford algebra. One can not simply add another dimension as in arbitrary dimensional Euclidean spaces.
IX. SUMMARY

Based on the ontology of time, we introduced pure variables, which are defined exclusively by the property of variation. Measurability requires reference to constants. Since pure constants may not exist in a world of time they have to be constructed from dynamics. Looking back we can identify pure variables with the spinors of the Dirac equation. If we include a density function, then the spinors may represent “probability amplitudes”. It is nearby to assume a connection of the presented concept of fundamental variables to the “Zitterbewegung” of the Dirac electron\cite{52,71}.

Main thesis of our gedankenexperiment is the emergence of spacetime: if it is a construction based on dynamics then we showed that the algebraic construction of geometry based on symplexes necessarily leads to matrix-representations of Clifford algebras. Most (if not all) arguments that we used in the derivation are related to specific (algebraic) symmetries. We believe that these symmetries are less obvious if the conventional form of the Dirac matrices is used. Furthermore the use of complex numbers in the conventional form of the Dirac theory wrongly suggests that we are in a non-classical domain. We have shown that some of the apparent differences between quantum and classical mechanics are of ontological nature and lose much of their significance in the light of the ontology of existence in time.

The idea that the elements of Clifford algebras are related to Minkowski spacetime is not new and has been described (for instance) by D. Hestenes in various publications on spacetime algebra\cite{52, 72}: “The Dirac matrices are no more and no less than matrix representations of an orthonormal frame of spacetime vectors and thereby they characterize spacetime geometry”. However (to our knowledge) Hestenes never argued why Minkowski spacetime should have these properties. According to our ansatz Clifford algebras are not by chance the optimal mathematical representation of spacetime, but by reason: spacetime is brought into existence through Hamiltonian dynamical structures that are isomorphic to the Clifford algebra $Cl_{3,1}$. The emergence of spacetime is the reason why the Dirac algebra is the appropriate mathematical tool to represent spacetime\cite{52}.

Furthermore we gave several arguments for the (apparent) dimensionality of spacetime and we have shown that Lorentz transformations are (in this framework) structure preserving (i.e. symplectic) transformations. 3 central arguments where given in preparation of the derivation of Lorentz force and electrodynamics in form of Maxwell’s equations: The first argument is based on the difference between even and odd elements of the algebra, the second on the fact that the expectation values of cosymplices must vanish in symplectic dynamics (in order be structure preserving) and the third one on the transformation properties. The wave equations follow from Maxwell’s equations and the comparison with the structure of decoupling then guided us to the identification of momentum-energy with spacetime derivatives and hence to the Dirac equation.

We do not claim to derive quantum mechanics from classical arguments. To be precise: Even a rigorous derivation of the (fundamental) relativistic equation of motion of quantum mechanics - the Dirac equation - does not automatically imply a “derivation” of quantum mechanics (QM). Though it is rarely explicitly mentioned, it is known that the equations of motion used in quantum mechanics are (taken as such) classical\cite{52}. This has been confirmed in what we carved out in this essay. However, on the basis of the presented ontology of time we found an explanation, why the quantum-mechanical wavefunction is sort of unphysical: Impossible to be directly measured and even without a well-defined dimensional unit. These are exactly the properties of fundamental variables that we derived from the ontology of time. If our ontology is correct, we will never really know what the wave function is. But if our ontology is correct, then we at least know, why.

ACKNOWLEDGMENTS

Mathematica® has been used for part of the symbolic calculations. Additional software has been written in “C” and compiled with the GNU® C++ compilers on different Linux distributions. XFig 3.2.4 has been used to generate the figures, different versions of IATEX and GNU®-emacs for editing and layout.

Appendix A: (Co-) Symplexes for higher-dimensional Spacetimes

1. Which $k$-Vectors are (Co-) Symplexes

Given that we have a set of $N$ pairwise anticommuting (co-) symplexes $S_i$ ($C_i$), which can be regarded as generators of real Clifford algebras, then the type are the $k$-products (i.e. bivectors, trivectors etc.) can be calculated as follows:

\[
(S_1 S_2 S_3 \dotsc S_k)^T = S_k^T S_{k-1}^T \dotsc S_1^T = (-1)^{k-1} \gamma_0 S_k S_{k-1} S_{k-2} \dotsc S_1 \gamma_0 \quad (A1)
\]

The number of permutations that is required to reverse the order of $k$ matrices is $k(k-1)/2$, so that

\[
(S_1 S_2 S_3 \dotsc S_k)^T = (-1)^s \gamma_0 (S_1 S_2 S_3 \dotsc S_k) \gamma_0 , \quad (A2)
\]

with $s$ given by:

\[
s = k - 1 + k(k - 1)/2 = \frac{k^2 + k - 2}{2} . \quad (A3)
\]

If we consider $k$ cosymplices $C_i$ instead, we obtain $k$ more sign reversals, so that

\[
(C_1 C_2 C_3 \dotsc C_k)^T = (-1)^s \gamma_0 (C_1 C_2 C_3 \dotsc C_k) \gamma_0 , \quad (A4)
\]
TABLE I. Signs for products of \( k \) anti-commuting \((\text{co-})\) symplices according to Eq. (A3) and (A5). The “plus” signs correspond to symplices: Products of 2, 5, 6, 9, 10, \ldots\) symplices are again symplices.

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|
| \( s \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| \((-1)^s\) + | + | - | + | - | + | - | + | - | + | - | + | - |
| c | 1 | 4 | 8 | 13 | 19 | 26 | 34 | 45 | 53 | 64 | 76 | 89 |
| \((-1)^s\) + | + | - | + | - | + | - | + | - | + | - | + | - |

with \( c \) given by:

\[
c = 2k - 1 + k(k - 1)/2 = \frac{k^2 + 3k - 2}{2}.
\]  

(A5)

If \( s \) and \( c \) are even (odd), respectively, then the products are \((\text{co-})\) symplices. Tab. I lists the resulting signs for \( k = 2 \ldots 10 \). Clifford algebras that are derived from sets of generator with more than 4 basic symplices will hence result in penta- and hexavectors as “observables”. Another conclusion from Tab. I is that in \( 9 + 1 \)-dimensional spacetime, the pseudoscalar is the product of all 10 generators and therefore a symplex. This allows (in principle) for a scalar field. Since spatial rotators are combinations of two spatial generators, in \( 9 + 1 \)-dimensional spacetime there are \( \binom{9}{2} = 36 \) spatial rotations. A derivation of the corresponding “Maxwell’s equations” for such higher-dimensional spaces (if possible or not), lies beyond the scope of this paper, but if all bivector boosts have a corresponding electric field and all bivector rotations a corresponding magnetic field component, then we should expect 9 electric and 36 magnetic field components in \( 9 + 1 \) dimensional spacetime.

2. The real Pauli matrices and the structure of \((\text{co-})\) symplices for \( n \) DOFs

For one DOF and an appropriate treatment of the \( 2 \times 2 \)-blocks of the general case we introduce the real Pauli matrices \((\text{RPMs})\) according to Eq. (14) is:

\[
F = \begin{pmatrix}
D_1 & A_{12} & A_{13} & \cdots & A_{1n} \\
-\tilde{A}_{12} & D_2 & A_{23} & \cdots & A_{2n} \\
-\tilde{A}_{13} & -\tilde{A}_{23} & D_3 & \cdots & A_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\tilde{A}_{1n} & -\tilde{A}_{2n} & -\tilde{A}_{3n} & \cdots & D_n
\end{pmatrix}
\]  

(A7)

where the \( 2 \times 2 \) matrices \( D_k \) on the diagonal must be \( 2 \times 2 \)-symplices and can therefore be written as

\[
D_k = d_k^0 \eta_0 + d_k^1 \eta_1 + d_k^2 \eta_2.
\]  

(A8)

If the blocks \( A_{ij} \) above the diagonal have the general form

\[
A = \sum_{k=0}^{3} a_k \eta_k,
\]  

(A9)

then Eq. (14) fixes the form of the corresponding blocks below the diagonal \( \tilde{A}_{ij} \) to

\[
-\tilde{A}_{ij} = \eta_0 A_{ij}^T \eta_0
\]

\[
= -\eta_0 (a_{ij}^0 \eta_0 + a_{ij}^1 \eta_1 + a_{ij}^2 \eta_2 + a_{ij}^3 \eta_3)^T \eta_0
\]

\[
= \eta_0 (-a_{ij}^0 \eta_0 + a_{ij}^1 \eta_1 + a_{ij}^2 \eta_2 + a_{ij}^3 \eta_3) \eta_0
\]

\[
= a_{ij}^0 \eta_0 + a_{ij}^1 \eta_1 + a_{ij}^2 \eta_2 - a_{ij}^3 \eta_3
\]  

(A10)

From this one can count that each subblock \( A_{ij} \) has two antisymmetric and two symmetric coefficients. We count \( n(n-1)/2 \) subblocks \( A_{ij} \). Together with \( n \) diagonal subblocks with each having one antisymmetric coefficients, we should have

\[
\nu_s^2 = n(n-1) + n = n^2
\]  

(A11)

antisymmetric symplices. The number of independent antisymmetric parameters is \( \nu_s^2 = \nu_s^a + \nu_s^c = n(2n-1) = 2n^2 - n \). Therefore we have

\[
\nu_s^c = n(2n-1) - n^2 = n^2 - n
\]  

(A12)

antisymmetric cosymplecs. Since we have as many cosymplecs as we have antisymmetric matrix elements \( \nu_s = \nu_s^a \), the number of symmetric cosymplecs \( \nu_s^c \) is:

\[
\nu_s^c = \nu_c - \nu_s^a = 2n^2 - n - (n^2 - n) = n^2
\]  

(A13)

The number \( \nu_s^c \) of symmetric symplices is:

\[
\nu_s^c = \nu_s^c - \nu_s^a = 2n(2n+1)/2 - n^2 = n^2 + n.
\]  

(A14)

\( \tilde{A}_{ij} \) is called the symplectic conjugate of \( A_{ij} \). If \( S \) and \( C \) are the symplex and cosymplex-part of a matrix \( A = C + S \), then the symplectic conjugate is

\[
\tilde{A} = -\gamma_0 A^T \gamma_0 = C - S.
\]  

(A15)

As well-known one can quickly derive that

\[
\tilde{A} B = \tilde{B} \tilde{A}.
\]  

(A16)
If \( \mathbf{A} \) is written “classically” as
\[
\mathbf{A} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]
then \( \eta_0 \mathbf{A}^T \eta_0 \) is given by:
\[
\eta_0 \mathbf{A}^T \eta_0 = \begin{pmatrix}
-a_{22} & a_{12} \\
a_{21} & -a_{11}
\end{pmatrix},
\]
so that
\[
\mathbf{A} (\eta_0 \mathbf{A}^T \eta_0) = (a_{12} a_{21} - a_{11} a_{22}) \mathbf{1} = -\text{Det}(\mathbf{A}) \mathbf{1}.
\]

A matrix \( \mathbf{A} \) is symplectic, if \( \mathbf{A} \mathbf{A}^T = \mathbf{1} \). But we can say in any case, that \( \mathbf{A} \mathbf{A}^T \) is a co-symplect:
\[
\mathbf{A} \mathbf{A}^T = (\mathbf{C} + \mathbf{S})(\mathbf{C} - \mathbf{S}) = \mathbf{C}^2 - \mathbf{S}^2 + \mathbf{SC} - \mathbf{CS}
\]

since squares of (co-) symplices as well as the commutator of symplex and cosymplex are cosyplices.

A \( 2n \times 2n \)-cosymplex \( \mathbf{C} \) has according to Eq. (24) the form
\[
\mathbf{C} = \begin{pmatrix}
\mathbf{E}_1 & \mathbf{B}_{12} & \mathbf{B}_{13} & \cdots & \mathbf{B}_{1n} \\
\mathbf{B}_{12} & \mathbf{E}_2 & \mathbf{B}_{23} & \cdots & \mathbf{B}_{2n} \\
\mathbf{B}_{13} & \mathbf{B}_{23} & \mathbf{E}_3 & \cdots & \mathbf{B}_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{B}_{1n} & \mathbf{B}_{2n} & \mathbf{B}_{3n} & \cdots & \mathbf{E}_n
\end{pmatrix}
\]

where the \( 2 \times 2 \) matrices \( \mathbf{E}_k \) on the diagonal must be \( 2 \times 2 \)-cosymplexes and are hence proportional to the unit matrix
\[
\mathbf{E}_k = e_k \mathbf{1}.
\]

3. Which \( k \)-Vectors of \( \text{Cl}_{N-1,1} \) are (Anti-) Symmetric

Possible dimensionalities for emergent spacetimes are given by Eq. (73). Since all but one generator are symmetric, the analysis of how many \( k \)-vectors are (anti-) symmetric, is reasonably simple. The total number of \( k \)-vectors is given by \( \binom{N}{k} \), the number \( \mu_k \) of \( k \)-vectors generated only with spatial elements, hence is \( \binom{N-1}{k} \).

For \( k \)-vectors that are generated exclusively from spatial basis vectors, one finds:
\[
(S_1 S_2 S_3 \cdots S_k)^T = S_k^T S_{k-1}^T S_{k-2}^T \cdots S_1^T
\]
\[
= (-1)^a S_1 S_2 S_3 \cdots S_k,
\]
where first step is possible as all \( S \) are spatial basis vectors and hence symmetric. The second step reflects the number of permutations that are required to reverse the order of \( k \) anticommuting elements:
\[
a = k(k - 1)/2.
\]

If one of the symplices equals \( \gamma_0 \) (i.e. is anti-symmetric), then we have
\[
(S_1 S_2 S_3 \cdots S_k)^T = S_k^T S_{k-1}^T S_{k-2}^T \cdots S_1^T
\]
\[
= -S_k S_{k-1} S_{k-2} \cdots S_1
\]
\[
= (-1)^a S_1 S_2 S_3 \cdots S_k \]
of a “particle”, i.e. we assumed that:

\[ E = -\bar{\psi} \gamma_0 \psi \propto \psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 \]
\[ P_x = \bar{\psi} \gamma_1 \psi \propto -\psi_0^2 + \psi_1^2 + \psi_2^2 - \psi_3^2 \]
\[ P_y = \bar{\psi} \gamma_2 \psi \propto 2(\psi_0 \psi_2 - \psi_1 \psi_3) \]
\[ P_z = \bar{\psi} \gamma_3 \psi \propto 2(\psi_0 \psi_1 + \psi_2 \psi_3) \]  

(B1)

Eq. (B1) is (up to factor) practically identical to the regularization transformation of Kustaanheimo and Stiefel (KST). At the same time we introduced a scaling factor \( \gamma \) to transform between the eigentime \( \tau \) and the time of an observer \( t \) - which is again a similarity to the use of the KST in celestial mechanics, where a Sundman transformation

\[ \frac{dt}{d\tau} = f(q, p) \]  

(B2)

is used. Despite these remarkable similarities, there are also significant differences, since in contrast to the KST, we do not increase the number of variables and we do not transform coordinates to coordinates. The “spinor” \( \psi \) was introduced as two canonical pairs (i.e. 2 coordinates and the 2 canonical momenta) and is used in Eq. (B1) to parameterize a 4-momentum vector, while the KST uses 4 “fictious” coordinates to parameterize 3 cartesian coordinates.

Appendix C: Non-Symplectic Transformations

1. Simple LC-Circuit

The key concept of symplectic transformations was mentioned to be the structure preservation. In the following we exemplify the meaning of structure preservation by giving examples for non-symplectic transformations. Consider a simple LC-circuit as shown in Fig. 1:

![Fig. 1. Simple LC-circuit.](image)

If we restrict ourselves to the simplest case (i.e. two capacitors, two inductors), the structure of the only non-trivial way to couple two LC-circuits is shown if Fig. 2. The equations of motion can be derived directly from the drawing using the general relations \( \dot{U}_C = I_C/C \) for an ideal capacitor and \( \dot{I}_L = U/L \) for the ideal inductor and Kirchhoff’s rule:

\[ \dot{I}_1 = U/L_1 \]
\[ \dot{U} = -(I_1 + I_3)/C_1 \]
\[ I_3 = (U - U_1)/L_2 \]
\[ \dot{U}_1 = I_3/C_2 \]
\[ 0 = I_1 + I_2 + I_3 \]
\[ U = U_1 + U_2 \]

(C6)

Quite obviously these equations fail to describe the correct relations for capacitors and inductors, which are:

\[ C \dot{U} = I_C \]
\[ L \dot{I}_L = U = -L \dot{I}_C \]  

(C3)

We note that the mistake is a wrong scaling of the variables. We recall that the product of the coordinate and the corresponding conjugate momentum should have the dimension of an action, while voltage times current results in a quantity with the dimension of power. Thus we introduce a scaling factor for the current and write:

\[ H = \frac{C}{2} U^2 + \frac{L}{2 a^2} P^2, \]  

(C4)

so that the canonical momentum is now \( P = a I_C \) and the EQOM are:

\[ \dot{U} = \frac{\partial H}{\partial I_C} = \frac{a}{L} P = \frac{a}{L} I_C \]
\[ \dot{P} = -\frac{\partial H}{\partial U} = -C U \]
\[ I_C = \frac{a}{L} U \]  

(C5)

The comparison with Eq. (C3) then yields \( a = L C \). But note that the scaling changes the product \( p_i q_i \), and hence is not a symplectic (structure preserving), but a non-symplectic (structure defining) transformation.

2. Coupled LC-Circuits
The energy sum again yields a Hamiltonian of the diagonal form:

\[ \mathcal{H} = \frac{C_1}{2} U^2 + \frac{L_1}{2} I_1^2 + \frac{C_2}{2} U_1^2 + \frac{L_2}{2} I_2^2 + \frac{L_3}{2} I_3^2. \quad (C7) \]

We use the voltages and currents of the Hamiltonian to define the state vector \( \phi = (U, U_1, I_1, I_3) \) and we find for its derivative:

\[ \dot{\phi} = \mathbf{F} \phi \]

\[ = \begin{pmatrix} 0 & 0 & -1/C_1 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L_1 & 0 & 0 & 0 \\ 1/L_2 & -1/L_2 & 0 & 0 \end{pmatrix} \phi. \quad (C8) \]

The matrix \( \mathbf{F} \) represents the structural properties of the LC-circuit. The matrix is not a symplectic and cannot be transformed into a symplex by any symplectic transformation. \textit{This does not mean} that the dynamics of the system cannot be derived from a Hamiltonian - it simply means that the transformation matrix which is required to map the system to a Hamiltonian system, is not structure preserving but structure defining.

In case of two coupled LC-circuits, it is likewise not sufficient to use scaling factors to obtain the canonical momenta and even if it was, the eigen-frequencies of the circuit cannot be guessed anymore. The square of the matrix from Eq. \( \text{[C8]} \) is given by:

\[ \ddot{\phi} = \begin{pmatrix} -\frac{L_1 + L_2}{C_1 L_1 L_2} & \frac{1}{C_1 L_2} & 0 & 0 \\ \frac{1}{C_2 L_1} & -\frac{1}{C_2 L_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{C_1 L_1} & -\frac{1}{C_1 L_2} \\ 0 & 0 & -\frac{1}{C_1 L_2} & -\frac{1}{C_1 C_2 L_2} \end{pmatrix} \phi. \quad (C9) \]

Obviously the state vector \( \phi \) is not composed of canonical variables. The Lagrangian \( \mathbf{L} \) is given by

\[ \mathbf{L} = \frac{L_1}{2} I_1^2 + \frac{L_2}{2} I_2^2 - \frac{C_1}{2} U^2 - \frac{C_2}{2} U_1^2. \quad (C10) \]

Eq. \( \text{[C8]} \) can be used to replace the currents by the derivatives of the voltages:

\[ I_1 = -C_1 \dot{U} - C_2 \dot{U}_1 \\ I_3 = C_2 \dot{U}_1 \]

\[ \mathbf{L} = \frac{L_1}{2} (C_1 \dot{U} + C_2 \dot{U}_1)^2 + \frac{L_2}{2} C_2^2 \dot{U}_1^2 - \frac{C_1}{2} U^2 - \frac{C_2}{2} U_1^2, \]

so that for the coordinates \( q_1 = U \) and \( q_2 = U_1 \), the canonical momenta are:

\[ p_1 = \frac{\partial \mathbf{L}}{\partial \dot{U}} = L_1 C_1 (C_1 \dot{U} + C_2 \dot{U}_1) \]

\[ = -L_1 C_1 I_1 \]

\[ p_2 = \frac{\partial \mathbf{L}}{\partial \dot{U}_1} = L_1 C_2 (C_1 \dot{U} + C_2 \dot{U}_1) + L_2 C_2 \dot{U}_1 \]

\[ = -L_1 C_2 I_1 + L_2 C_2 I_3 \]

so that the transformation matrix from the state vector \( \phi \) to the canonical variables \( \psi \) is given by:

\[ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -L_1 C_1 & 0 \\ 0 & 0 & -L_1 C_2 & L_2 C_2 \end{pmatrix} \begin{pmatrix} U \\ U_1 \\ I_1 \\ I_3 \end{pmatrix} \quad (C13) \]

The transformed matrix \( \tilde{\mathbf{F}} = \mathbf{T} \mathbf{F} \mathbf{T}^{-1} \) then is:

\[ \tilde{\mathbf{F}} = \begin{pmatrix} 0 & 0 & \frac{L_1+L_2}{C_1^2 L_1 L_2} & \frac{1}{C_1 C_2 L_2} \\ 0 & 0 & -\frac{1}{C_1 C_2 L_2} & \frac{1}{C_2^2 L_2} \\ -C_1 & 0 & 0 & 0 \\ 0 & -C_2 & 0 & 0 \end{pmatrix} \quad (C14) \]

The Hamiltonian can then be expressed in the canonical coordinates as:

\[ \mathcal{H} = \frac{1}{2} \psi^T \mathbf{A} \psi \]

\[ \mathbf{A} = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & \frac{L_1+L_2}{C_1^2 L_1 L_2} & \frac{1}{C_1 C_2 L_2} \\ 0 & 0 & -\frac{1}{C_1 C_2 L_2} & \frac{1}{C_2^2 L_2} \end{pmatrix}, \quad (C15) \]

where \( \mathbf{A} = \gamma_0 \tilde{\mathbf{F}} \). As was shown in Ref.\textsuperscript{25}, the Hamiltonian formulation (for stable oscillating systems) is always “similar” to the case of completely decoupled oscillators, i.e. the non-symplectic transformation maps the structure of Fig. 2 to two separate systems as shown in Fig. 4. The transformation \( \mathbf{T} \) is not symplectic and may serve as an example for a structure defining transformation in contrast to symplectic structure preserving transformations.

### 3. Nonsymplectic Transformations within the Dirac Algebra

In the following we present an orthogonal (non-symplectic) transformation that enables to “rotate the time direction”, i.e. to transform from \( \gamma_0 \) (as we used it here) to other representations of \( \gamma_0 \). We define the (normalized) Hadamard-matrix \( \mathbf{H}_4 \) according to

\[ \mathbf{H}_4 = \frac{\sqrt{2-\gamma_1-\gamma_4+\gamma_3}}{(\gamma_1+\gamma_3)(\gamma_2-\gamma_1)} \]

\[ = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad (C16) \]

\[ \gamma_0 \mathbf{H}_4 \gamma_0 = \mathbf{H}_4 \]

\[ \mathbf{H}_4 \gamma_0 \mathbf{H}_4 = -\gamma_0 \]
$H_4$ is a symplex and it is antisymplectic. Furthermore we define a “shifter” matrix $X$ according to

$$X = \frac{-\gamma_0 + \gamma_2 + \gamma_6 + \gamma_7}{2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{(C17)}$$

Some properties of the $X$-matrix are

$$\begin{align*}
\gamma_0 X \gamma_0 &= X^T \\
X^T \gamma_0 X &= -\gamma_7 \\
X^T \gamma_8 X &= \gamma_{14} \\
X^T \gamma_9 X &= -\gamma_{10} \\
\end{align*} \quad \text{(C18)}$$

Note that $X$ is a symplex and hence could be used as a force matrix. In this case we have a 4-th order differential equation of the form:

$$\begin{align*}
\psi' &= X \psi \\
\psi'' &= \psi, \\
\end{align*} \quad \text{(C19)}$$

since one finds $X^4 = 1$. The product of these two matrices $R_6 = H_4 X$ is an orthogonal matrix that transforms cyclic through all possible basis systems:

$$R_6 = H_4 X = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \quad \text{(C20)}$$

The most relevant properties are

$$\begin{align*}
R_6^T R_6 &= R_6 R_6^T = 1 \\
R_6^T \gamma_{10} R_6 &= -\gamma_8 \\
R_6^T \gamma_7 R_6 &= -\gamma_{14} \\
R_6^T \gamma_9 R_6 &= \gamma_{10}, \\
\end{align*} \quad \text{(C21)}$$

so that $R_6$ can be iteratively used to switch through all possible systems:

$$\begin{align*}
R_6^T \gamma_0 R_6 &= \gamma_7 \\
(R_6^T)^2 \gamma_0 (R_6)^2 &= -\gamma_{14} \\
(R_6^T)^3 \gamma_0 (R_6)^3 &= -\gamma_9 \\
(R_6^T)^4 \gamma_0 (R_6)^4 &= -\gamma_{10} \\
(R_6^T)^5 \gamma_0 (R_6)^5 &= \gamma_8 \\
(R_6^T)^6 \gamma_0 (R_6)^6 &= \gamma_0. \\
\end{align*} \quad \text{(C22)}$$

Finally one finds that $(R_6)^6 = 1$. But $R_6$ is neither symplectic nor is it a symplex. Expressed by the $\gamma$-matrices, $R_6$ is given by

$$4 R_6 = 1 + \gamma_0 + \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 - \gamma_9 - \gamma_{10} - \gamma_{11} - \gamma_{12} - \gamma_{13} - \gamma_{14}. \quad \text{(C23)}$$

Another matrix with equivalent properties is given by

$$\tilde{R}_6 = H_4 X^T. \quad \text{(C24)}$$

These transformations exemplifies part of the claim (i.e. the use of a specific (though arbitrary) form for $\gamma_0$) in Eq. 3.

### 4. Transformation to the Conventional Dirac Algebra

According to the fundamental theorem of the Dirac matrices\footnote{1}, we expect that any system of Dirac matrices is a unitary transform of any other system. Only, if we flip the sign of the metric tensor, an additional multiplication with the unit imaginary is required.

If we define the following unitary matrix $U$:

$$U = \frac{1}{2} \begin{pmatrix} 1 & i & i & -1 \\ -i & 1 & -1 & i \\ i & -1 & 1 & -i \\ 1 & i & -1 & -i \end{pmatrix}. \quad \text{(C25)}$$

then it is quickly verified that

$$\tilde{\gamma}_\mu = i U \gamma_\mu U^\dagger, \quad \text{(C26)}$$

where $\mu \in \{0 \ldots 3\}$ and $\tilde{\gamma}_\mu$ are the conventional Dirac matrices footnote. The explicit form of the real Dirac matrices is given for instance in Refs. (44\textsuperscript{-}45). Using $\tilde{\gamma}_\mu$, the other matrices of the Clifford algebra are quickly constructed. However, since we multiplied by the unit imaginary, it is clear that the conventional Dirac algebra is a rep of $Cl_{1,3}$.

### Appendix D: Graphical Representation of Dirac Matrices

Fig. 3 illustrates the geometric interpretation of the RDMs.

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FIG. 3. The group structure of the RDMs (i.e. the Clifford algebra \(Cl_{3,1}\)) can be represented by a tetrahedron: The vertices represent the “basic” simplices \(\gamma_0, \ldots, \gamma_3\), the edges the bi-vectors \(\gamma_0 \cdots \gamma_9\), the surfaces the components of the axial vector \(\gamma_0 \cdots \gamma_3\) and the volume the pseudoscalar \(\gamma_4\). Another graphical representation has been given by Good-manson.\(^{22}\)

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