Quasifinite representations of the Lie superalgebra of quantum pseudo differential operators

Carina Boyallian* and Vanesa Meinardi

February 1, 2008

Abstract

In this paper we extend general results obtained in [1] for quasifinite highest weight representations of $\mathbb{Z}$-graded Lie algebras to $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebras, and we apply these to classify the irreducible quasifinite highest weight modules of the Lie superalgebra of quantum pseudo-differential operators.

1 Introduction

The study of Lie superalgebras and its representations plays an important role in Conformal Field Theory and Supersymmetries in physics.

The main difficulty to develop a suitable representation theory for certain $\mathbb{Z}$-graded Lie superalgebras lies in the fact that the graded subspaces

*The first author was supported in part by Conicet, ANPCyT, Agencia Cba Ciencia, and Secyt-UNC (Argentina).
of some highest weight modules over these Lie superalgebras are infinite dimensional, in spite of having a natural principal gradation and a triangular decomposition. However, most of the physical theories usually require that these subspaces have finite dimension, to which we refer as *quasifiniteness*.

In [2], Kac and Radul developed a powerful machinery and begun the systematic study of quasifinite representations of the Lie algebra of differential operators on the circle and the Lie algebra of quantum pseudo differential operators.

Following this work, there have been later developments and many extensions as in ([3],[4],[1],[5], etc). Moreover, the results in [2], were extended to the study for the quasifinite representations of the Lie superalgebra of differential operators on the supercircle in [6] and its subalgebras in [7].

In [1], they developed a general theory that characterize the quasifinite highest weight representation of any $\mathbb{Z}$-graded Lie algebra, under some mild conditions.

In the first part of this article we extend this results for Lie superalgebras. Then using this, we classify the quasifinite highest weight representation of the Lie superalgebra of quantum pseudo differential operators. Observe that the extension of these results are useful to simplify some computations made in [7] and [8].

In order to give a realization of these representations in terms of tensor products of quasifinite representations of the Lie superalgebra of infinite matrices with a finite number of non-zero diagonals with coefficients in the truncated polynomials, we need to characterize them, using the extension of the results in [1].
2 Quasifinite representations of graded Lie superalgebras

Recall that a superalgebra is a $\mathbb{Z}_2$-graded algebra. A Lie superalgebra is a superalgebra $g = g_0 \oplus g_1$, $(\mathfrak{U}, \mathfrak{T} \in \mathbb{Z}_2)$, with multiplication given by a super bracket $[,]$ satisfying:

$$[a, b] = -(-1)^{\bar{a}\bar{b}}[b, a]$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]],$$

for all $a \in g_{\bar{a}}$, $b \in g_{\bar{b}}$ with $\bar{a}$ and $\bar{b} \in \mathbb{Z}_2$.

Let $g$ be a $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebra over $\mathbb{C}$, namely

$$g = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} g_j$$

and $[g_i, g_j] \subseteq g_{i+j}$ with $i, j \in \frac{1}{2}\mathbb{Z}$.

The $\frac{1}{2}\mathbb{Z}$-gradation of a Lie superalgebra is consistent with the $\mathbb{Z}_2$-gradation if

$$g_0 = \bigoplus_{i \in \mathbb{Z}} g_i$$

and $g_1 = \bigoplus_{i \in \mathbb{Z} + \frac{1}{2}} g_i$.

For a $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebra $g$, set

$$g_+ = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}_{>0}} g_i, \quad g_- = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}_{>0}} g_{-i}.$$ 

In this section $g$ will denote a consistent $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebra.

**Definition 2.1.** A subalgebra $p$ of $g$ is called parabolic if it contains $g_0 \oplus g_+$ as a proper subalgebra, that is,

$$p = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} p_j, \quad p_j = g_j \quad \text{for} \quad j \geq 0 \quad \text{and} \quad p_{-j} \neq 0 \quad \text{for some} \quad j > 0.$$ 

We assume the following properties on $g$:

$(SP_1)$ $g_0$ is commutative,

$(SP_2)$ If $a \in g_{-k}$ $(k > 0)$, and $[a, g_{\frac{1}{2}}] = 0$, then $a = 0$. 

3
Remark 2.2. As an immediate consequence of the definition of parabolic subalgebra and condition $(SP_2)$, if $p$ is any parabolic subalgebra of $g$ with $p_{-k} \neq 0$ ($k > 0$), then $p_{-k+\frac{1}{2}} \neq 0$.

Given $a \in g_{-\frac{1}{2}}, a \neq 0$, we define $p^a = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} p^a_j$, where

\[
p^a_j = g_j \quad \text{for all} \quad j \geq 0,
\]
\[
p^a_{-\frac{1}{2}} = \sum \{ \cdots [[a, g_0], g_0], \cdots \}
\]
and $p^a_{-k-\frac{1}{2}} = [p^a_{-\frac{1}{2}}, p^a_{-k}]$. \hspace{1cm} (2.1)

We have the following Lemma, whose proof we shall omit since it is identical to the proof of Lemma 2.2 in [1] with the obvious modifications.

Lemma 2.3. Let $a \in g_{-\frac{1}{2}}, a \neq 0$. Then:

(a) $p^a$ is the minimal parabolic superalgebra containing $a$.

(b) $g^0 := [p^a, p^a] \cap g_0 = [a, g_{\frac{1}{2}}]$.

Remark 2.4. The examples of parabolic subalgebras considered in [7] and [8] motivate the following definition.

Definition 2.5. (a) A parabolic subalgebra $p$ is called non-degenerate if $p_{-j}$ has finite codimension in $g_{-j}$ for all $j > 0$.

(b) A non-zero element $a \in g_{-\frac{1}{2}}$ is called non-degenerate if $p^a$ is non-degenerate.

In order to study quasifinite representations of graded Lie superalgebras we recall some definitions and notions.

A $g$-module $V$ is called $\frac{1}{2} \mathbb{Z}$-graded if

\[
V = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} V_j \quad \text{and} \quad g_i V_j \subseteq V_{i+j} \quad (i, j \in \frac{1}{2} \mathbb{Z}),
\]
and $V$ is called quasifinite if $\dim V_j < \infty$ for all $j$. 

4
Given $\lambda \in g_0^*$, a highest weight module is a $\frac{1}{2}\mathbb{Z}$-graded $g$-module $V(g, \lambda) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} V_j$ defined by the following properties:

- $V_0 = \mathbb{C} v_{\lambda}$ where $v_{\lambda}$ is a non-zero vector,
- $hv_{\lambda} = \lambda(h)v_{\lambda}$ for $h \in g_0$,
- $g_+ v_{\lambda} = 0$,
- $\mathcal{U}(g_-) v_{\lambda} = V(g, \lambda)$. (2.2)

Here and further $\mathcal{U}(s)$ stands for the universal enveloping superalgebra of the Lie superalgebra $s$.

A non-zero vector $v \in V(g, \lambda)$ is called singular if $g_+ v = 0$.

The Verma module is constructed as follows

$$M(g, \lambda) = \mathcal{U}(g) \bigotimes_{\mathcal{U}(g_0 \oplus g_+)} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda := \mathbb{C} c_{\lambda}$, is the 1-dimensional $g_0 \oplus g_+$-module given by $hc_{\lambda} = \lambda(h)c_{\lambda}$ if $h \in g_0$ and $g_+ c_{\lambda} = 0$, and the action of $g$ on $M(g, \lambda)$ is induced by the left multiplication in $\mathcal{U}(g)$.

Any highest module $V(g, \lambda)$ is a quotient of $M(g, \lambda)$. The "smallest" among the $V(g, \lambda)$ is the unique irreducible module $L(g, \lambda)$ (which is the quotient of $M(g, \lambda)$ by its maximal graded submodule).

For simplicity we denote $V(g, \lambda) = V(\lambda)$, $M(g, \lambda) = M(\lambda)$ and $L(g, \lambda) = L(\lambda)$.

Now, let $p = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} p_j$, be a parabolic subalgebra of $g$ and let $\lambda \in g_0^*$ be such that $\lambda|_{g_0 \cap [p,p]} = 0$. Then the $g_0 \oplus g_+$-module $\mathbb{C}_\lambda = \mathbb{C} c_{\lambda}$ extends to $p$ by letting $p_j \cdot c_{\lambda} = 0$ for $j < 0$, and we may construct the highest weight module

$$M(g, \lambda, p) = \mathcal{U}(g) \bigotimes_{\mathcal{U}(p)} \mathbb{C}_\lambda,$$

which is called the generalized Verma module.
We also require the following condition on $g$:

$(SP_3)$ If $p$ is a non-degenerate parabolic subalgebra of $g$, then there exists a non-degenerate element $a$ such that $p^a \subseteq p$.

**Remark 2.6.** The examples considered in [7] and [8] satisfy the properties $(SP_1)$, $(SP_2)$ y $(SP_3)$.

The main result of this section is the following Theorem, whose proof we shall omit since it is completely analogous to the one in [1].

**Theorem 2.7.** Let $g$ be a Lie superalgebra that satisfies $(SP_1)$, $(SP_2)$ and $(SP_3)$. The following conditions on $\lambda \in g_0^*$ are equivalent:

1. $M(\lambda)$ contains a singular vector $av_\lambda \in M(\lambda)_{-\frac{1}{2}}$ where $a$ is non-degenerate;
2. There exist a non-degenerate element $a \in g_{-\frac{1}{2}}$ such that $\lambda([g_{\frac{1}{2}},a]) = 0$.
3. $L(\lambda)$ is quasifinite.
4. There exist a non-degenerate element $a \in g_{-\frac{1}{2}}$ such that $L(\lambda)$ is the irreducible quotient of the generalized Verma module $M(g,\lambda,p^a)$.

**3 The Lie superalgebra of quantum pseudo differential operators**

Let $q \in \mathbb{C}^\times$ and $|q| \neq 1$. Now, $T_q$ denote the following operator on $\mathbb{C}[z,z^{-1}]$:

$$T_q(f(z)) = f(qz)$$

Let $\mathcal{G}_q^{as}$ denote the associative algebra of all operators on $\mathbb{C}[z,z^{-1}]$ of the form

$$E = \sum_{k \in \mathbb{Z}} e_k(z)T_q^k$$

where $e_k(z) \in \mathbb{C}[z,z^{-1}]$ and the sum is finite.
We write such an operator as a linear combination of operators of the form 
\( z^k f(T_q) \), where \( f \) is a Laurent polynomial in \( T_q \) and \( k \in \mathbb{Z} \). The product in \( \mathfrak{G}_q^{as} \) is given by
\[
(z^m f(T_q))(z^k g(T_q)) = z^{m+k} f(q^k T_q) g(T_q).
\]

Denote by \( M(1|1) \) the set of \( 2 \times 2 \) supermatrix
\[
\begin{pmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{pmatrix},
\]
where \( f_{ij} \in \mathbb{C} \), viewed as the associative superalgebra of linear transformations of the complex \( (1|1) \)-dimensional superspace \( \mathbb{C}^{1|1} \).

We denote \( M_{ij} \) the \( 2 \times 2 \) matrix with 1 in the \( ij \)-place and 0 everywhere else. Declaring \( M_{11}, M_{22} \) even and \( M_{12}, M_{21} \) odd elements, we endow \( M(1|1) \) with a \( \mathbb{Z}_2 \)-gradation.

We denote by \( S\mathfrak{G}_q^{as} \) the associative superalgebra of \( 2 \times 2 \) supermatrices with entries in \( \mathfrak{G}_q^{as} \) namely
\[
S\mathfrak{G}_q^{as} = \mathfrak{G}_q^{as} \otimes M(1|1),
\]
and the product is given by the usual matrix multiplication. Let \( S\mathfrak{G}_q \) denote the corresponding Lie superalgebra where the Lie superbracket is explicitly given by:
\[
[z^n f(T_q) M_{ij}, z^m g(T_q) M_{rs}] = (z^n f(T_q) M_{ij})(z^m g(T_q) M_{rs})
- (-1)^{|M_{ij}| |M_{rs}|} (z^m g(T_q) M_{rs})(z^n f(T_q) M_{ij})
= z^{n+m} \left( f(q^n T_q) g(T_q) \delta_{ji} M_{is}
- (-1)^{|M_{ij}| |M_{rs}|} g(q^n T_q) f(T_q) \delta_{si} M_{rj} \right), \quad (3.1)
\]
where \(|M|\) denotes the parity of \( M \). Now, introduce the linear map \( Str_0 : S\mathfrak{G}_q \to \mathbb{C} \) as
\[
\begin{pmatrix}
  f_{11}(T_q) & f_{12}(T_q) \\
  f_{21}(T_q) & f_{22}(T_q)
\end{pmatrix} = \left(f_{11}(T_q)\right)_0 - \left(f_{22}(T_q)\right)_0,
\]

where \((f(T_q))_0 = f_0\) if \(f(T_q) = \sum_k f_k T^k_q\), \((f_k \in \mathbb{C})\). We should notice that \(\text{Str}_0\) has the following property:

\[
\text{Str}_0(f(T_q) M_{ij} g(T_q) M_{kl}) = (-1)^{|M_{ij}||M_{kl}|} \text{Str}_0(g(T_q) M_{kl} f(T_q) M_{ij}).
\]

Thus, define a one-dimensional central extension \(\widetilde{S}\mathfrak{g}_q\) of \(S\mathfrak{g}_q\) with the following super bracket:

\[
[z^r F(T_q), z^s G(T_q)] = \left[(z^r F(T_q)), (z^s G(T_q))\right] + \psi_{\sigma, \text{Str}_0}(z^r F(T_q), z^s G(T_q)) C,
\]

where \(C\) is the central charge, and the super 2-cocycle \(\psi_{\sigma, \text{Str}_0}\) is given by

\[
\psi_{\sigma, \text{Str}_0}(z^r F(T_q) M_{ij}, z^s g(T_q) M_{kl}) = \text{Str}_0 \left((1 + \sigma + \cdots + \sigma^{r-1})(\sigma^{-r}(f(T_q) M_{ij}) g(T_q) M_{kl})\right)
\]

\[
= -(-1)^r \sum_{m=0}^{r-1} \left(f(q^{-r+m} T_q) g(q^m T_q)\right)_0 \delta_{kj} \delta_{il}.
\]

(3.2)

if \(r = -s > 0\) and 0 otherwise. Here \(\sigma\) is the automorphism of \(\mathfrak{g}^{as}_q\) given by

\[
\sigma(f(T_q) M_{ij}) = f(q T_q) M_{ij} \quad (\text{cf. with (1.3.1) in [2]}).
\]

The principal \(\frac{1}{2}\mathbb{Z}\)-gradation in \(\widetilde{S}\mathfrak{g}_q\) is given by

\[
\widetilde{S}\mathfrak{g}_q = \bigoplus_{n \in \mathbb{Z}} (\widetilde{S}\mathfrak{g}_q)_n,
\]

\[
(\widetilde{S}\mathfrak{g}_q)_n = \{z^n (f_{11}(T_q) M_{11} + f_{22}(T_q) M_{22}) + \delta_{n,0} C : f_{i i} \in \mathbb{C}[w, w^{-1}] \text{ with } i = 1, 2\},
\]

(3.3)

and

\[
(\widetilde{S}\mathfrak{g}_q)_{n + 1/2} = \{z^n f_{12}(T_q) M_{12} + z^{n+1} f_{21}(T_q) M_{21} : f_{ij} \in \mathbb{C}[w, w^{-1}],
\]

\[
i, j \in \{1, 2\} \text{ with } i \neq j\}.
\]

(3.4)
4 Quasifinite Representations of $\widehat{SG}_q$

Let $V(\lambda)$ be a highest weight module over $\widehat{SG}_q$ with highest weight $\lambda$. The highest weight vector $v_\lambda \in V(\lambda)$ is characterized via the principal gradation as $(\widehat{SG}_q)_\alpha v_\lambda = 0$ for $\alpha \geq 1/2$ and $(\widehat{SG}_q)_0 v_\lambda \in \mathbb{C} v_\lambda$. Explicitly, these conditions are written as:

$$
\begin{align*}
z^n f(T_q) M_j v_\lambda &= 0 \quad \text{with } n \geq 1, \; f(w) \in \mathbb{C}[w, w^{-1}], \; i = j \; \text{or} \; i = 1, \; j = 2; \\
z^{n+1} f_{21}(T_q) M_{21} v_\lambda &= 0 \quad \text{with } n \geq 0, \; f_{21}(w) \in \mathbb{C}[w, w^{-1}]; \\
f_{12}(T_q) M_{12} v_\lambda &= 0 \quad \text{with } f_{12}(w) \in \mathbb{C}[w, w^{-1}]; \\
(T_q^s M_{ii}) v_\lambda &= \lambda(T_q^s M_{ii}) v_\lambda \quad \text{with } s \in \mathbb{Z}, \; i = 1, 2.
\end{align*}
$$

Consider $p = \bigoplus_{\alpha \in \mathbb{Z}/2} p_\alpha$ a parabolic subalgebra of $\widehat{SG}_q$. Thus $p_\alpha = (\widehat{SG}_q)_\alpha$ for all $\alpha \geq 0$ and $p_\alpha \neq 0$ for some $\alpha < 0$. Observe that for each $j \in \mathbb{N}$ we have

$$
p_{-j} = \{ z^{-j} (f_{11}(T_q) M_{11} + f_{22}(T_q) M_{22}) + \delta_{j,0} C : f_{ii}(w) \in I_{-j}^i \text{ with } i = 1, 2 \}$$

$$
p_{-j+1/2} = \{ z^{-j} f_{12}(T_q) M_{12} + z^{-j+1} f_{21}(T_q) M_{21} : f_{rs}(w) \in I_{-j}^{rs} \text{ with } r, s \in \{1, 2\}, \; r \neq s \},
$$

where $I_{-j}^r$ with $r, s \in \{1, 2\}$ are subspaces of $\mathbb{C}[w, w^{-1}]$. Since $[\widehat{SG}_q]_0, p_{-\alpha} \subseteq p_{-\alpha}$ with $\alpha \in \mathbb{Z}/2$, it is easy to check that $I_{-k}^{rs}$ satisfies

$$
A_k^{rs} I_{-k}^{rs} \subseteq I_{-k}^{rs} \quad \text{with} \quad r, s \in \{1, 2\},
$$

where $A_k^{rs} = \{ f(q^{-k} w) - f(w) : f(w) \in \mathbb{C}[w, w^{-1}] \}$ if $r, s \in \{1, 2\}$ and $r = s$ or $r = 1$ and $s = 2$, and $A_k^{21} = \{ f(q^{-k+1} w) - f(w) : f(w) \in \mathbb{C}[w, w^{-1}] \}$.

**Lemma 4.1.** (a) $I_{-k}^{rs}$ is an ideal for all $k \in \mathbb{N}$ and $r, s \in \{1, 2\}$.

(b) If $I_{-k}^{rs} \neq 0$ then it has finite codimension in $\mathbb{C}[w, w^{-1}]$.

**Proof.** Since $|q| \neq 1$, observe that $A_k^{rs} = \mathbb{C}[w, w^{-1}]$ for all $k \geq \frac{1}{2}$, $r, s \in \{1, 2\}$. Then $I_{-k}^{rs}$ is an ideal. Let $b_{-k}^{rs}$ be the monic polynomials that generate the corresponding ideals $I_{-k}^{rs}$, therefore $\dim(\mathbb{C}[w, w^{-1}]/(b_{-k}^{rs} = I_{-k}^{rs})) < \infty$. \qed
Proposition 4.2. (a) any non-zero element of \((\hat{\mathfrak{S}}_{q})_{-1/2}\) is non-degenerate. 
(b) Any parabolic subalgebra of \(\hat{\mathfrak{S}}_{q}\) is non-degenerate.
(c) Let \(d = z^{-1}b_{12}(T_{q})M_{12} + b_{21}(T_{q})M_{21} \in (\hat{\mathfrak{S}}_{q})_{-1/2}\). Then:

\[
\left(\hat{\mathfrak{S}}_{q}\right)^d_0 = [(\hat{\mathfrak{S}}_{q})_{1/2}, d] = \{ f(T_{q})b_{21}(T_{q})I + g(q^{-1}T_{q})b_{12}(T_{q})M_{22} + b_{12}(qT_{q})g(T_{q})M_{11} - (g(q^{-1}T_{q})b_{12}(T_{q}))_0 C : f(w), g(w) \in \mathbb{C}[w, w^{-1}] \}.
\] (4.1)

Proof. Let \(0 \neq d \in (\hat{\mathfrak{S}}_{q})_{-1/2}\). Then, by Lemma [4.1](b), part (a) follows. Let \(p\) be any parabolic subalgebra of \(\hat{\mathfrak{S}}_{q}\), using Remark [2.2] we get \(p_{-1/2} \neq 0\). Then, using (a) and \(p^d \subseteq p\) for any non-zero \(d \in p_{-1/2}\), we obtain (b). Let \(d = z^{-1}b_{12}(T_{q})M_{12} + b_{21}(T_{q})M_{21} \in (\hat{\mathfrak{S}}_{q})_{-1/2}\), and \(a = f(T_{q})M_{12} + z g(T_{q})M_{21} \in (\hat{\mathfrak{S}}_{q})_{1/2}\), with \(b_{ij}(w)\), \(f(w)\) and \(g(w) \in \mathbb{C}[w, w^{-1}]\), where \(i, j \in \{1, 2\}\) with \(i \neq j\). Then

\[
[a, d] = [f(T_{q})M_{12}, b_{21}(T_{q})M_{21}] + [zg(T_{q})M_{21}, z^{-1}b_{12}(T_{q})M_{12}]
\]

\[
= f(T_{q})b_{21}(T_{q})M_{11} + b_{21}(T_{q})f(T_{q})M_{22}
\]

\[
+ g(q^{-1}T_{q})b_{12}(T_{q})M_{22} + b_{12}(qT_{q})g(T_{q})M_{11} - (g(q^{-1}T_{q})b_{12}(T_{q}))_0 C
\]

\[
= f(T_{q})b_{21}(T_{q})I + g(q^{-1}T_{q})b_{12}(T_{q})M_{22} + b_{12}(qT_{q})g(T_{q})M_{11}
\]

\[
- (g(q^{-1}T_{q})b_{12}(T_{q}))_0 C;
\] (4.2)

finally, part c) follows by Lemma [2.3](b).

A functional \(\lambda \in (\hat{\mathfrak{S}}_{q})^*_0\) is described by its labels

\[
\Delta_{l,i} = -\lambda(T_{q}^{l}M_{ii}),
\]

where \(i = 1, 2\), and \(l \in \mathbb{Z}\), and the central charge \(c = \lambda(C)\). We shall consider the generating series

\[
\Delta_{\lambda,i}(x) = \sum_{l \in \mathbb{Z}} x^{-l} \Delta_{l,i} \quad i = 1, 2.
\]
Recall that a quasipolynomial is a linear combination of functions of the form \( p(x)e^{\alpha x} \) where \( p(x) \in \mathbb{C}[x] \) and \( \alpha \in \mathbb{C} \). A formal power series is a quasipolynomial if and only if it satisfies a non-trivial linear differential equation with constant coefficients. We also have the following well known result.

**Theorem 4.3.** Given a quasipolynomial \( q(x) \) and a polynomial \( B(x) = \prod_i (x - A_i) \), let \( b(x) = \prod_i (x - a_i) \) where \( a_i = e^{A_i} \). Then \( b(x)(\sum_n q(n)x^{-n}) = 0 \) if and only if \( B(d/dx)q(x) = 0 \).

Now we state the main result of this article.

**Theorem 4.4.** An irreducible highest weight module \( L(\widehat{S\mathfrak{g}}, \lambda) \) is quasifinite if and only if one of the following equivalent conditions hold:

(i) There exist two monic non-zero polynomials \( b_{12}(x), b_{21}(x) \) such that

\[
\begin{align*}
    b_{12}(x)(\Delta_{\lambda,1}(q^{-1}x) + \Delta_{\lambda,2}(x) - c) &= 0, \\
    b_{21}(x)(\Delta_{\lambda,1}(x) + \Delta_{\lambda,2}(x)) &= 0.
\end{align*}
\]  

(ii) There exist quasipolynomials \( P_{12}(x) \) and \( P_{21}(x) \) such that \( P_{21}(0) = P_{12}(0) + c \) and \( (n \in \mathbb{Z}, n \neq 0) \):

\[
\begin{align*}
    P_{21}(n) &= \Delta_{n,1} + \Delta_{n,2}, \\
    P_{12}(n) &= \Delta_{n,1}q^n + \Delta_{n,2}.
\end{align*}
\]  

**Proof.** From Theorem 2.7 (2), we have that \( L(\widehat{S\mathfrak{g}}, \lambda) \) is quasifinite if and only if exist \( d \in (S\mathfrak{g})_{-1/2} \) non-degenerate such that \( \lambda([\widehat{S\mathfrak{g}}_{1/2}, d]) = 0. \) But by la Proposition 4.2(c) this is equivalent to

\[
\begin{align*}
0 &= \lambda(f(T_q)b_{21}(T_q)I) \quad \text{and} \\
0 &= \lambda(g(q^{-1}T_q)b_{12}(T_q)E_{22} + b_{12}(qT_q)g(T_q)E_{11}) - (g(q^{-1}T_q)b_{12}(T_q))_0 c, \\
\end{align*}
\]  

(4.5)
for all \( f(w) \) and \( g(w) \) in \( \mathbb{C}[w, w^{-1}] \). In particular for \( f(w) = w^s \) and \( g(w) = (qw)^r \), with \( r, s \in \mathbb{Z} \), we have

\[
0 = \lambda(T_q^s b_{21}(T_q) I) ,
0 = \lambda(T_q^r b_{12}(T_q) E_{22} + b_{12}(qT_q)(qT_q)^r E_{11}) - ((T_q)^r b_{12}(T_q))_0 c. \tag{4.6}
\]

Writing \( b_{12}(w) = \sum_i \beta_{12}^j w^j \) and \( b_{21}(w) = \sum_i \gamma_{21}^i w^i \) with \( \beta_{12}^j \) and \( \gamma_{21}^i \in \mathbb{C} \),

\[
0 = - \sum_i \gamma_{21}^i \lambda(T_q^{s+i} I) = \sum_i \gamma_{21}^i (\Delta_{s+i,1} + \Delta_{s+i,2}). \tag{4.7}
\]

and

\[
0 = - \sum_j \beta_{12}^j \lambda(T_q^{r+j} E_{22} + (qT_q)^{r+j} E_{11}) - \delta_{r,-j} \beta_{12}^j c = \sum_j \beta_{12}^j (q^{r+j} \Delta_{r+j,1} + \Delta_{r+j,2}) - \delta_{r,-j} \beta_{12}^j c \tag{4.8}
\]

Multiplying (4.7) by \( x^{-s} \) and adding over \( s \in \mathbb{Z} \),

\[
0 = \sum_{i,s} \gamma_{21}^i (\Delta_{s+i,1} x^{-s-i} + \Delta_{s+i,2} x^{-s-i}) x^i = \sum_i \gamma_{21}^i \sum_s (\Delta_{s+i,1} x^{-s-i} + \Delta_{s+i,2} x^{-s-i}) x^i = b_{21}(x)(\Delta_{\lambda,1}(x) + \Delta_{\lambda,2}(x)). \tag{4.9}
\]

Similarly, multiplying (4.8) by \( x^{-r} \) and adding over \( r \in \mathbb{Z} \),

\[
0 = \sum_{j,r} [\beta_{12}^j (q^{r+j} \Delta_{r+j,1} x^{-r-j} + \Delta_{r+j,2} x^{-r-j}) x^j - \delta_{r,-j} \beta_{12}^j x^{-r} c] = \sum_j \beta_{12}^j [\sum_r (q^{r+j} \Delta_{r+j,1} x^{-r-j} + \Delta_{r+j,2} x^{-r-j}) - c] x^j = b_{12}(x)(\Delta_{\lambda,1}(q^{-1} x) + \Delta_{\lambda,2}(x) - c). \tag{4.10}
\]

Thus we proved the first part. The equivalence between (i) and (ii) follows from Theorem 4.3. \( \square \)
4.1 Interplay between $S\mathfrak{g}_q$ and $\mathfrak{gl}_{\infty|\infty}[m]$

Given a non-negative integer $m$, consider the algebra of truncated polynomials $R = R_m = \mathbb{C}[t]/(t^{m+1})$, and let $M_{\infty}[m]$ be the associative algebra consisting of matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with $a_{ij} \in R_m$ such that $a_{ij} = 0$ for $|i - j| > > 0$. We denote by $\mathfrak{gl}_{\infty}[m]$ the Lie algebra obtained from $M_{\infty}[m]$ by taking the usual commutator.

Define the associative superalgebra $M_{\infty|\infty}[m] = M_{\infty}[m] \otimes M(1|1)$ with the induced $\mathbb{Z}_2$–graded structure from $M(1|1)$.

Denote $\mathfrak{gl}_{\infty|\infty}[m]$ the Lie superalgebra obtained from $M_{\infty|\infty}[m]$ by taking the usual super commutator. One may have two different ways of looking at $\mathfrak{gl}_{\infty|\infty}[m]$. First we may regard $\mathfrak{gl}_{\infty|\infty}[m] = \bigoplus_{i,j=1}^{\infty} \mathfrak{gl}_{\infty}[m] M_{ij}$, that is,

$$ \mathfrak{gl}_{\infty|\infty}[m] = \begin{bmatrix} \mathfrak{gl}_{\infty}[m] & \mathfrak{gl}_{\infty}[m] \\ \mathfrak{gl}_{\infty}[m] & \mathfrak{gl}_{\infty}[m] \end{bmatrix}. $$

One also may identify

$$ \mathfrak{gl}_{\infty|\infty}[m] = \{(a_{ij})_{i,j \in \mathbb{Z}/2} : a_{ij} \in R_m \text{ and } a_{ij} = 0 \text{ for } |i - j| > > 0\}. \quad (4.11) $$

Under this identification, the $\mathbb{Z}_2$–graded structure is given by

$$ |E_{ij}| = \begin{cases} 0, & \text{if } i - j \in \mathbb{Z} \\ \bar{1}, & \text{if } i - j \in \mathbb{Z} + 1/2, \end{cases} $$

where $E_{ij}$ denotes, as always, the infinite matrix with one in the $ij$ entry and 0 elsewhere.

The identification between the two presentations of $\mathfrak{gl}_{\infty|\infty}[m]$ is given by $(i, j \in \mathbb{Z})$

$$ E_{ij} M_{11} = E_{i,j}, $$
$$ E_{ij} M_{22} = E_{i-1/2,j-1/2}, $$
$$ E_{ij} M_{12} = E_{i,j-1/2}, $$
$$ E_{ij} M_{21} = E_{i-1/2,j}. \quad (4.12) $$
Under this identification, the Lie superalgebra \( \mathfrak{gl}_{\infty}[m] \) is equipped with a natural \( \frac{1}{2}\mathbb{Z} \)-gradation

\[
\mathfrak{gl}_{\infty}[m] = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} (\mathfrak{gl}_{\infty}[m])_r
\]

where \((\mathfrak{gl}_{\infty}[m])_r\) is the completion of the linear span of \( E_{ij} \) with \( j - i = r \). This is the \textit{principal gradation} of \( \mathfrak{gl}_{\infty}[m] \).

Choose a branch of \( \log q \), and let \( \tau = \log q 2\pi i \). Then any \( s \in \mathbb{C} \) is uniquely written as \( s = q^a, a \in \mathbb{C}/\tau^{-1}\mathbb{Z} \).

Take \( s = q^a \in \mathbb{C} \) and let \( R^\infty = R^\infty \bigoplus R^\infty \theta = t^a R[t, t^{-1}] \bigoplus \theta t^a R[t, t^{-1}] \) with \( \theta \) an odd indeterminate. Consider the following basis in \( R^\infty \),

\[
\{ v_i = t^{-i+a}, v_{i-\frac{1}{2}} = t^{-i+a}\theta, i \in \mathbb{Z} \}.
\]

The Lie superalgebra \( \mathfrak{gl}_{\infty}[m] \) acts on \( R^\infty \) by letting \( E_{ij} v_k = \delta_{jk} v_i \) with \( i, j, k \in \frac{1}{2}\mathbb{Z} \). The Lie superalgebra \( S\mathfrak{g}_q \) acts on \( R^\infty \) as quantum pseudo-differential operators. In this way we obtain a family of embeddings \( \varphi_s^m \) of \( S\mathfrak{g}_q \) into \( \mathfrak{gl}_{\infty}[m] \) given by

\[
\begin{align*}
\varphi_s^m(t^k f_{11}(T_q) M_{11}) &= \sum_{j \in \mathbb{Z}} f_{11}(sq^{-j+t}) E_{j-k,j}, \\
\varphi_s^m(t^k f_{21}(T_q) M_{21}) &= \sum_{j \in \mathbb{Z}} f_{21}(sq^{-j+t}) E_{j-k-\frac{1}{2},j}, \\
\varphi_s^m(t^k f_{12}(T_q) M_{12}) &= \sum_{j \in \mathbb{Z}} f_{12}(sq^{-j+t}) E_{j-k,j-\frac{1}{2}}, \\
\varphi_s^m(t^k f_{22}(T_q) M_{22}) &= \sum_{j \in \mathbb{Z}} f_{22}(sq^{-j+t}) E_{j-k-\frac{1}{2},j-\frac{1}{2}}. 
\end{align*}
\]

(4.13)

Note that the principal gradation on \( \mathfrak{gl}_{\infty}[m] \) is compatible with that on \( S\mathfrak{g}_q \) under the map \( \varphi_s^m \) and observe that the embedding \( \varphi_s^m \) restricted to the \( \mathfrak{g}_q M_{11} \) coincides with (6.2.1) in [2].

Denote by \( \mathcal{O} \) the algebra of all holomorphic functions on \( \mathbb{C}^\times \) with topology of uniform convergence on compact sets. We define a completion \( S\mathfrak{g}_q^\mathcal{O} \).
of the associative superalgebra of quantum pseudo-differential operators by considering quantum pseudo-differential operators of infinite order of the form $z^k f(T_q)M_{ij}$, where $f \in \mathcal{O}$. The embedding $\varphi_{s}^{[m]}$ extends naturally to $S\mathfrak{G}_q^{\mathcal{O}}$.

Define

$$I_{s}^{[m]} = \{ f \in \mathcal{O} : f^{(i)}(sq^n) = 0 \text{ for all } n \in \mathbb{Z}, i = 0, \cdots, m \}$$

and

$$J_{s}^{[m]} = \bigoplus_{i,j=1}^{2} \bigoplus_{k \in \mathbb{Z}} z^k I_{s}^{[m]}M_{ij}.$$ 

Therefore, it follows by the Taylor formula for $\varphi_{s}^{[m]}$ that

$$\ker \varphi_{s}^{[m]} = J_{s}^{[m]}.$$ 

Now, fix $\bar{s} = (s_1, \cdots, s_n) \in \mathbb{C}^n$ such that if we write each $s_i = q^{a_i}$, we have

$$a_i - a_j \notin \mathbb{Z} + \tau^{-1}\mathbb{Z} \text{ for } i \neq j,$$ 

(4.14)

and fix $\bar{m} = (m_1, \cdots, m_n) \in \mathbb{Z}_+^n$.

Let $M_{\infty|\infty}[\bar{m}] = \bigoplus_{i=1}^{n} M_{\infty|\infty}[m_i]$. Consider the homomorphism

$$\varphi_{s}^{[\bar{m}]} = \bigoplus_{i=1}^{n} \varphi_{s_i}^{[m_i]} : S\mathfrak{G}_q^{\mathcal{O}_{as}} \longrightarrow M_{\infty|\infty}[\bar{m}].$$

It is well known that for every discrete sequence of points in $\mathbb{C}$ and a nonnegative integer $m$ there exists $f(w) \in \mathcal{O}$ having prescribed values of its first $m$ derivatives. Thus, due to this fact and condition (4.14) the following Proposition follows.

**Proposition 4.5.** We have the exact sequence of $\frac{1}{2}\mathbb{Z}$-graded associative superalgebras, provided that $|q| \neq 1$:

$$0 \longrightarrow J_{s}^{[\bar{m}]} \longrightarrow S\mathfrak{G}_q^{\mathcal{O}_{as}} \xrightarrow{\varphi_{s}^{[\bar{m}]} \bigoplus_{i=1}^{n} \varphi_{s_i}^{[m_i]}} M_{\infty|\infty}[\bar{m}] \longrightarrow 0$$

where $J_{s}^{[\bar{m}]} = \bigcap_{i=1}^{n} J_{s_i}^{[m_i]}$. 

15
Consider the following super 2-cocycle on $\mathfrak{gl}_{\infty}[m]$ with values in $R_m$:

$$C(A, B) = \text{Str}([J, A], B), \quad A, B \in \mathfrak{gl}_{\infty}[m],$$

where $J = \sum_{r \leq 0} E_{r,r}$, and for a matrix $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl}_{\infty}[m]$, $\text{Str}(A) := \sum_{r \in \frac{1}{2}\mathbb{Z}} (-1)^{2r} a_{rr}$.

Note that $C(A, B)$ is well defined for all $A$ and $B$ in $\mathfrak{gl}_{\infty}[m]$. Denote by

$$\widehat{\mathfrak{g}l}_{\infty}[m] = \left\{ (a_{ij})_{i,j \in \mathbb{Z}/2} : a_{ij} \in \mathbb{R}^\mathbb{m} y a_{ij} = 0 \text{ para } |i - j| >> 0 \right\} \bigoplus R_m,$$

the corresponding central extension. The $\frac{1}{2}\mathbb{Z}$-gradation of this Lie superalgebra extends from $\mathfrak{gl}_{\infty}[m]$ by letting $\text{gr}(R_m) = 0$.

Therefore we have the following

**Lemma 4.6.** The $\mathbb{C}$-linear map $\hat{\varphi}_s[m] : \hat{\mathbb{S}}_q \to \hat{\mathfrak{g}l}_{\infty}[m]$ defined by

$$\hat{\varphi}_s[m](\varepsilon^r T^k M_{ij}) = \varphi_s[m](\varepsilon^r T^k M_{ij}) \quad \text{with} \quad r \neq 0,$$

$$\hat{\varphi}_s[m](C) = 1 \in R_m,$$

$$\hat{\varphi}_s[m](T^k M_{ii}) = \varphi_s[m](T^k M_{ii}) - (-1)^i \frac{q^{ak}}{1-q^k} \sum_{j=0}^{m} (k \log q) \frac{t^j}{j!} \quad \text{if} \quad k \neq 0,$$

$$\hat{\varphi}_s[m](T^k M_{ij}) = \varphi_s[m](T^k M_{ij}) \quad \text{for all} \quad k, \quad \text{and} \quad i \neq j,$$

$$\hat{\varphi}_s[m](M_{ii}) = \varphi_s[m](M_{ii})$$

is a homomorphism of Lie superalgebras.

We shall need the following Proposition, whose proof is completely similar to Proposition 4.3 in [2].

**Proposition 4.7.** Let $V$ be a quasifinite $\hat{\mathbb{S}}_q$-module. Then the action of $\hat{\mathbb{S}}_q$ on $V$ naturally extends to the action of $(\hat{\mathbb{S}}_q)^0$ on $V$ for any $k \neq 0$.

We return now to the $\frac{1}{2}\mathbb{Z}$-graded complex Lie superalgebra $\hat{\mathfrak{g}l}_{\infty}[m]$.

An element $\lambda \in (\hat{\mathfrak{g}l}_{\infty}[m])_0^*$ is characterized by its labels.
\[ \lambda_k^{(j)} = \lambda(t^j E_{kk}), \quad k \in \frac{1}{2}\mathbb{Z}, \quad j = 0 \cdots, m \] (4.15)

and central charges
\[ c_j = \lambda(t^j) \quad j = 0, \cdots, m. \] (4.16)

As usual, we have the irreducible highest weight \( \widehat{\mathfrak{gl}}_{\infty|\infty}[m] \)-module \( L(\widehat{\mathfrak{gl}}_{\infty|\infty}[m], \lambda) \) associated to \( \lambda \). We will prove the following

**Theorem 4.8.** The \( \widehat{\mathfrak{gl}}_{\infty|\infty}[m] \)-module \( L(\lambda) \) is quasifinite if and only if for each \( l = 0 \cdots, m \) all but finitely many \( k \in \frac{1}{2}\mathbb{Z} \),
\[ \lambda_k^{(l)} + \lambda_{k-rac{1}{2}}^{(l)} + \delta_{k, \frac{1}{2}} c_l = 0. \] (4.17)

**Remark 4.9.** The case \( m = 0 \) of this Theorem was proved in [8]. However using Theorem 2.7, even this proof can be simplified.

In order to apply Theorem 2.7 we need to show that the superalgebra \( \widehat{\mathfrak{gl}}_{\infty|\infty}[m] \) satisfies conditions \((SP_1)\), \((SP_2)\) and \((SP_3)\) introduced in Section 2.

The fact that \( (\widehat{\mathfrak{gl}}_{\infty|\infty}[m])_0 \) is commutative is straightforward, thus we have \((SP_1)\).

Let us check \((SP_2)\). Take \( a = \sum_{j \in \frac{1}{2}\mathbb{Z}} a_j E_{j+k,j} \in (\widehat{\mathfrak{gl}}_{\infty|\infty}[m])_{-k} \) with \( k \in \frac{1}{2}\mathbb{Z}, k > 0 \), such that \([a, b] = 0 \) for all \( b \in (\widehat{\mathfrak{gl}}_{\infty|\infty}[m])_{\frac{1}{2}} \). In particular for \( b = E_{s-rac{1}{2},s} \) for any \( s \in \frac{1}{2}\mathbb{Z} \). Thus we have that
\[ 0 = a_{s-rac{1}{2}} E_{s-rac{1}{2}+k,s} - (-1)^{2k} a_{s-k} E_{s-rac{1}{2},s-k}. \]

Since \( k > 0 \), \( E_{s+rac{1}{2}-k,s} \) and \( E_{s+rac{1}{2},s+k} \) are linearly independent, then \( a_{s-rac{1}{2}} = 0 \) and \( a_{s-k} = 0 \) for all \( s \in \frac{1}{2}\mathbb{Z} \). We conclude that \( a = 0 \) proving \((SP_2)\).

**Remark 4.10.** Take \( a = \sum_{j \in \frac{1}{2}\mathbb{Z}} a_j(t) E_{j+rac{1}{2},j} \in (\widehat{\mathfrak{gl}}_{\infty|\infty}[m])_{-\frac{1}{2}} \) and recall that by definition
\[ p_{-\frac{1}{2}} = \sum_j [\cdots [[a, (\widehat{\mathfrak{gl}}_{\infty|\infty}[m])_0], (\widehat{\mathfrak{gl}}_{\infty|\infty}[m])_0] \cdots]. \]
So letting \( b = t^s E_{ii} \in (\hat{\mathfrak{gl}}_{\infty|\infty}[m])_0 \), we have in particular that
\[
[a, b] = a_i(t)t^s E_{i,i+\frac{1}{2},i} - a_{i-\frac{1}{2}}(t)t^s E_{i,i-\frac{1}{2}} \in \mathfrak{p}^{a}_{-\frac{1}{2}}.
\]
Then, for arbitrary \( k \)
\[
[a, b], E_{kk} = \delta_{ik}(a_k(t)t^s E_{k,k+\frac{1}{2},k} + a_{k-\frac{1}{2}}(t)t^s E_{k,k-\frac{1}{2}})
\]
\[
- \delta_{i+\frac{1}{2},k} a_{k-\frac{1}{2}}(t)t^s E_{k,k-\frac{1}{2}} - \delta_{i-\frac{1}{2},k} a_k(t)t^s E_{k,k+\frac{1}{2},k} \in \mathfrak{p}^{a}_{-\frac{1}{2}}.
\]
Choosing \( k = i + \frac{1}{2} \), we show that \( a_i(t)t^s E_{i,i+\frac{1}{2}} \in \mathfrak{p}^{a}_{-\frac{1}{2}} \) for all \( i \in \frac{1}{2} \) and \( s = 0, \cdots, m \).

Let \( I_{a_i(t)}, i \in \frac{1}{2}\mathbb{Z} \), the ideal of \( R_m \) generated by the corresponding \( a_i(t) \).
Thus we have shown that
\[
\prod I_{a_i(t)} E_{i,i+\frac{1}{2}} \subseteq \mathfrak{p}^{a}_{-\frac{1}{2}}. \tag{4.18}
\]
Computing the bracket between \( a \) and an arbitrary element in \((\hat{\mathfrak{gl}}_{\infty|\infty}[m])_0\) it is easy to show that equality holds in (4.18). Now, since \( \mathfrak{p}^{a}_{-\frac{1}{2}} = [\mathfrak{p}^{a}_{-\frac{1}{2}}, \mathfrak{p}^{a}_{-\frac{1}{2}}] \), inductively, it is straightforward to show that
\[
\prod I_{a_i(t)} E_{i+k,i} \subseteq \mathfrak{p}^{a}_{-k}, \tag{4.19}
\]
for all \( k \in \frac{1}{2}\mathbb{N} \).

In order to check \((SP_3)\) first we will describe the non-degenerate elements of \( \hat{\mathfrak{gl}}_{\infty|\infty}[m] \) in the following

**Lemma 4.11.** An element \( a = \sum_j a_j(t)E_{j,j+\frac{1}{2}} \in (\hat{\mathfrak{gl}}_{\infty|\infty}[m])_{-\frac{1}{2}} \) is non-degenerate if and only if \( a_j(t) \in \mathbb{C} - \{0\} \) for all but finitely many \( j \in \frac{1}{2}\mathbb{Z} \).

**Proof.** Suppose that \( a \) is non-degenerate, that is, \( \mathfrak{p}^{a}_{-\frac{1}{2},j} \) have finite codimension in \((\hat{\mathfrak{gl}}_{\infty|\infty}[m])_{-\frac{1}{2}} \) for all \( j \in \frac{1}{2}\mathbb{N} \). In particular \( \mathfrak{p}^{a}_{-\frac{1}{2}} \) have finite codimension in \((\hat{\mathfrak{gl}}_{\infty|\infty}[m])_{-\frac{1}{2}} \). Thus, since equality holds in (4.18), \( a_i(t) \in \mathbb{C} - \{0\} \) for all but finitely many \( i \in \frac{1}{2}\mathbb{Z} \).

The converse statement follows immediately from (4.18) and (4.19). \( \square \)
Let us check \((SP_3)\). Let \(p\) non-degenerate. In particular \(p_{-\frac{1}{2}}\) have finite codimension on \((\hat{\mathfrak{gl}}_{\infty|\infty}[m])_{-\frac{1}{2}}\). Thus \(p_{-\frac{1}{2}} = \prod_i I_i E_{i+\frac{1}{2},i}\) with \(I_i \subseteq R_m\) a subspace such that \(I_i = R_m\) for all but finitely many \(i\). Let \(K\) be such finite set. Thus, by Lemma 4.11 \(a = \sum_{j \in K^c} E_{j-\frac{1}{2},j} \in p_{-\frac{1}{2}}\) is non-degenerate and by definition \(p^a \subseteq p\).

Now we can prove Theorem 4.8.

**Proof.** Let \(\lambda \in \left(\hat{\mathfrak{gl}}_{\infty|\infty}[m]\right)_0^*\). By Theorem 2.7 and Lemma 4.11, \(L(\lambda)\) is quasifinite if and only if there exists \(a = \sum_{i} a_i(t) E_{i+\frac{1}{2},i} \in \left(\hat{\mathfrak{gl}}_{\infty|\infty}[m]\right)_{-\frac{1}{2}}\) with \(a_i \in \mathbb{C} \setminus \{0\}\) for all but finitely many \(i\) and \(\lambda([a,b]) = 0\) for all \(b \in \left(\hat{\mathfrak{gl}}_{\infty|\infty}[m]\right)_{\frac{1}{2}}\).

Suppose that \(L(\lambda)\) is quasifinite, thus an element \(a\) as above exists. Let \(I = \{i \in \frac{1}{2}\mathbb{Z} : a_i \not\in \mathbb{C} \setminus \{0\}\}\). Note that \(|I| < \infty\). Consider \(k \in I^c, |k| \gg 0\) such that if \(k \in I^c\), then \(k - \frac{1}{2} \in I^c\).

Take \(b = t^l E_{k-\frac{1}{2},k} \in \left(\hat{\mathfrak{gl}}_{\infty|\infty}[m]\right)_{\frac{1}{2}}\) with \(l = 0, \cdots, m\).

Since \([a, b] = a_{k-\frac{1}{2}} t^l \left(E_{kk} + E_{k-\frac{1}{2},k-\frac{1}{2}} + \delta_{k,\frac{1}{2}}\right)\), then \(\lambda([a, b]) = 0\) implies

\[
0 = \lambda \left(t^l E_{k,k} + t^l E_{k-\frac{1}{2},k-\frac{1}{2}} + t^l \delta_{k,\frac{1}{2}}\right)
= \lambda^{(l)}_{k} + \lambda^{(l)}_{k-\frac{1}{2}} + \delta_{k,\frac{1}{2}} c_l
\quad \text{for all } k \in I^c, \text{ with } |k| \gg 0.
\]

Hence, quasifiniteness of the \(\hat{\mathfrak{gl}}_{\infty|\infty}[m]\)-module \(L(\lambda)\) implies (4.17).

Conversely, assume that (4.17) holds. Denote by \(I\) the finite set where this condition is not satisfied. Let \(0 \ll N\) such that if \(i \in I^c\) and \(|i| > N\), then \(i \pm \frac{1}{2} \in I^c\). Set \(a = \sum_{|i| > N} E_{i,i-\frac{1}{2}} \in \hat{\mathfrak{gl}}_{\infty|\infty}[m]_{-\frac{1}{2}}\). By Lemma 4.11 \(a\) is non-degenerate.

Consider an arbitrary element \(b = \sum_j b_j(t) E_{j-\frac{1}{2},j} \in \hat{\mathfrak{gl}}_{\infty|\infty}[m]_{\frac{1}{2}}\). Write
each \( b_j(t) = \sum_{t=0}^{m} B_j^t t^i \). Then

\[
\lambda([a, b]) = \lambda \left( \sum_{|i| > N} b_j(t)(E_{i,i-\frac{1}{2}} + E_{i,i}) + \sum_{|i| > N} b_j(t)C(E_{i,i-\frac{1}{2}}, E_{i,i}) \right)
\]

\[
= \sum_{t=0}^{m} \sum_{|i| > N} B_j^t \left( \lambda(t^i E_{i,i-\frac{1}{2}} E_{i,i-\frac{1}{2}}) + \delta_i \right)
\]

\[
= 0
\]

finishing the proof. \( \square \)

Given \( \vec{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N \), we define \( \hat{\mathfrak{gl}}_{\infty}[\vec{m}] = \bigoplus_{i=1}^{N} \hat{\mathfrak{gl}}_{\infty}[m_i] \).

By Proposition 4.5 we have a surjective Lie superalgebra homomorphism

\[
\hat{\varphi}^{[\vec{m}]} : \hat{\mathfrak{S}}_{\tilde{q}} \rightarrow \hat{\mathfrak{gl}}_{\infty}[\vec{m}],
\]

Choose a quasifinite \( \lambda_i \in \left( \hat{\mathfrak{gl}}_{\infty}[m_i] \right)^* \) and let \( L(\hat{\mathfrak{gl}}_{\infty}[m_i], \lambda_i) \) be the corresponding irreducible \( \hat{\mathfrak{gl}}_{\infty}[m_i] \)-module. Then

\[
L(\hat{\mathfrak{gl}}_{\infty}[\vec{m}], \vec{\lambda}) = \bigotimes_{i=1}^{n} L(\hat{\mathfrak{gl}}_{\infty}[m_i], \lambda_i)
\]

where \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \) is an irreducible \( \hat{\mathfrak{gl}}_{\infty}[\vec{m}] \)-module. Using the homomorphism \( \hat{\varphi}^{[\vec{m}]} \), we regard \( L(\hat{\mathfrak{gl}}_{\infty}[\vec{m}], \vec{\lambda}) \) as a \( \hat{\mathfrak{S}}_{\tilde{q}} \)-module, which we denote by \( L_{\tilde{s}}^{[\vec{m}]}(\vec{\lambda}) \).

**Theorem 4.12.** Consider the embedding \( \hat{\varphi}^{[\vec{s}]} : \hat{\mathfrak{S}}_{\tilde{q}} \rightarrow \hat{\mathfrak{gl}}_{\infty}[\vec{m}] \) where \( \vec{s} = (s_1, \ldots, s_N) \), with \( s_i = q^{a_i} \in \mathbb{C} \) such that \( a_i - a_j \not\in \mathbb{Z} + \tau^{-1}\mathbb{Z} \) if \( i \neq j \), and let \( V \) be a quasifinite \( \hat{\mathfrak{gl}}_{\infty}[\vec{m}] \)-module. Then any \( \hat{\mathfrak{S}}_{\tilde{q}} \)-submodule of \( V \) is a \( \hat{\mathfrak{gl}}_{\infty}[\vec{m}] \)-submodule as well. In particular, the \( \hat{\mathfrak{S}}_{\tilde{q}} \)-modules \( L_{\tilde{s}}^{[\vec{m}]}(\vec{\lambda}) \) are irreducible.
Proof. Let $U$ be a $(\frac{1}{2}\mathbb{Z}\text{-graded})$ $\widehat{\mathfrak{S}}_q$-submodule of $V$. $U$ is a quasifinite $\widehat{\mathfrak{S}}_q$-module as well, hence by Proposition 4.7, it can be extended to $(\widehat{\mathfrak{S}}_q)$ for any $k \neq 0$. But the map $\varphi_s^{[m]}$ is surjective for any $k \neq 0$. Thus $U$ is invariant with respect to all members of the principal gradations $\left(\mathfrak{gl}_\infty[[\vec{m}]]\right)_k$ with $k \neq 0$. Since $\widehat{\mathfrak{gl}_\infty[[\vec{m}]]}$ coincides with its derived algebra, this proves the theorem. \hfill \Box

By Proposition 4.7 and Theorem 4.12, the $\widehat{\mathfrak{S}}_q$-modules $L_s^{[m]}(\vec{\lambda})$ are irreducible quasifinite highest weight modules. Let us calculate the labels $\Delta_{m,s,\lambda,k,i}$, with $i = 1, 2$, of the highest weight and the central charge $c$ of the $\widehat{\mathfrak{S}}_q$-modules $L_s^{[m]}(\lambda)$. We have $(k \neq 0)$

$$\Delta_{m,s,\lambda,k,1} = \lambda(\varphi_s^{[m]}(T^k q M_{11})) = \sum_{l=0}^{m} \frac{(k \log q)^l}{l!} \left[ \sum_j (s^j q^{-j})^k \lambda_j^{(l)} + \frac{q^{ak}}{1 - q^k c_1} \right], \quad (4.20)$$

$$\Delta_{m,s,\lambda,k,2} = \lambda(\varphi_s^{[m]}(T^k q E M_{22})) = \sum_{l=0}^{m} \frac{(k \log q)^l}{l!} \left[ \sum_j (s^j q^{-j})^k \lambda_j^{(l)} - \frac{q^{ak}}{1 - q^k c_1} \right] \quad (4.21)$$

and for $k = 0$

$$\Delta_{m,s,\lambda,0,i} = \lambda(\varphi_s^{[m]}(E_{ii})) = \begin{cases} \sum_j \lambda(E_{jj}) = \sum_j \lambda_j^{(0)} & \text{if } i = 1 \\ \sum_j \lambda(E_{j-\frac{1}{2}j-\frac{1}{2}}) = \sum_j \lambda_j^{(0)} & \text{if } i = 2. \end{cases}$$

The following Theorem shows that any irreducible quasifinite highest weight module $L(\widehat{\mathfrak{S}}_q, \lambda)$ can be obtained in a unique way. The proof of this result follows by the same argument used in Theorem 4.8 in [2] using the formulas above.

**Theorem 4.13.** Let $L = L(\widehat{\mathfrak{S}}_q, \lambda)$ be an irreducible quasifinite highest weight module with central charge $c$ and

$$\Delta_{n,1} + \Delta_{n,2} = P_{21}(n) \quad \text{and} \quad \Delta_{n,2} = \frac{P_{12}(n) - P_{21}(n)}{q^n - 1}$$

21
for \( n \neq 0 \), where \( P_{12}(x) \) and \( P_{21}(x) \) are quasipolynomial such that \( P_{12}(0) - P_{21}(0) = c \). We write \( P_{ij}(x) = \sum_{a \in \mathbb{C}} P_{ij,a}(x \log q) q^{ax} \) where \( P_{ij,a}(x \log q) \) are polynomials. We decompose the set \( \{ a \in \mathbb{C} : P_{ij,a}(x \log q) \neq 0 \} \) in a disjoint union of congruence classes \( \text{mod } \mathbb{Z} + \tau^{-1} \mathbb{Z} \). Let \( S = \{ a, a - k_1, a - k_2, \ldots \} \) be such a congruence class, let \( m = \max_{a \in S} \deg P_{ij,a}(x \log q) \), and let

\[
h^{(l)}_{k_r - \frac{1}{2}} = \left( \frac{d}{dx} \right)^l P_{21,a-k_r}(0) \quad \text{and} \quad h^{(l)}_{k_r} = \left( \frac{d}{dx} \right)^l P_{12,a-k_r}(0).
\]

We associate to \( S \) the \( \hat{\mathfrak{gl}}_{\infty|\infty}[m] \)-module \( L[m](\lambda_S) \) with the central charges

\[
c_l = \sum_{k_r} (h^{(l)}_{k_r - \frac{1}{2}} - h^{(l)}_{k_r}), \quad (4.22)
\]

and labels

\[
\lambda^{(l)}_i = \sum_{k_r > i} (\tilde{h}^{(l)}_{k_r} - h^{(l)}_{k_r - \frac{1}{2}}) \quad \text{and} \quad \lambda^{(l)}_{i - \frac{1}{2}} = \sum_{k_r \geq i} \left( h^{(l)}_{k_r - \frac{1}{2}} - \tilde{h}^{(l)}_{k_r+1} \right) \quad (4.23)
\]

for \( i \in \mathbb{Z} \) and \( \tilde{h}^{(l)}_k = h^{(l)}_k + \delta_{k,0}c_l \). Then the \( \hat{\mathfrak{S}}_q \)-module \( L \) is isomorphic to the tensor product of all the modules \( L[m](\lambda_S) \).

**Bibliography**

[1] V. G. Kac and J. I. Liberati, *Unitary quasifinite representations of \( W_{\infty} \),* Letters Math. Phys., 53 (2000), 11-27.

[2] V. G. Kac and A. Radul, *Quasifinite highest weight modules over the Lie algebra of differential operators on the circle,* Comm. Math. Phys. 157 (1993), 429-457.

[3] C. Boyallian, V. Kac, J. Liberati and C. Yan, *Quasifinite highest weight modules over the Lie algebra of matrix differential operators on the circle,* Journal of Math. Phys. 39 (1998), 2910-2928.
[4] V. G. Kac, W. Wang and C. Yan, *Quasifinite representations of classical Lie subalgebras of $W_{1+\infty}$* Adv. Math. **139** (1998), 56-140.

[5] C. Boyallian, J. Liberati, *On modules over matrix pseudo-differential operators*, Letters in Math. Phys. **60** (2002), 73-85.

[6] H. Awata, M. Fukama, M. Matsuo y S. Odake, *Quasifinite highest weight modules over the super $W_{\infty+\infty}$*, Comm. Math. Phys. **70** (1995), 151-179.

[7] S. Cheng and W. Wang, *Lie subalgebras of the differential operators on the supercircle*, Publ. Res. Inst. Math. Sci. **39**, No.3, (2003)545-600.

[8] N. Lam, R. B. Zhang *Quasi-finite representations, free field realizations, and character formulae of Lie superalgebras of infinite rank*, math.QA/0311096.

[9] V. Kac, *Infinite-dimensional Lie algebras*, 3rd edition, Cambridge University Press, Cambridge, 1990.