HIGHER DIMENSIONAL ALGEBRAIC FIBERINGS FOR PRO-$p$ GROUPS

DESSISLAVA H. KOCHLOUKOVA

Abstract. We prove some conditions for higher dimensional algebraic fibering of pro-$p$ group extensions and we establish corollaries about incoherence of pro-$p$ groups. In particular, if $G = K \rtimes \Gamma$ is a pro-$p$ group, $\Gamma$ a finitely generated free pro-$p$ group with $d(\Gamma) \geq 2$, $K$ a finitely presented pro-$p$ group with $N$ a normal pro-$p$ subgroup of $K$ such that $K/N \simeq \mathbb{Z}_p$ and $N$ not finitely generated as a pro-$p$ group, then $G$ is incoherent (in the category of pro-$p$ groups). Furthermore we show that if $K$ is a free pro-$p$ group with $d(K) = 2$ then either $\text{Aut}_p(K)$ is incoherent (in the category of pro-$p$ groups) or there is a finitely presented pro-$p$ group, without non-procyclic free pro-$p$ subgroups, that has a metabelian pro-$p$ quotient that is not finitely presented i.e. a pro-$p$ version of a result of Bieri-Strebel does not hold.

1. Introduction

For a pro-$p$ group $G$ we denote by $K[[G]]$ the completed group algebra of $G$ over the ring $K$, where $K$ is the field with $p$ elements $\mathbb{F}_p$ or the ring of the $p$-adic numbers $\mathbb{Z}_p$. By definition a pro-$p$ group $G$ is of type $FP_m$ if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a projective resolution where all projectives in dimension $\leq m$ are finitely generated $\mathbb{Z}_p[[G]]$-modules. Note that $G$ is of type $FP_1$ if and only if $G$ is finitely generated as a pro-$p$ group. And $G$ is of type $FP_2$ if and only if $G$ is finitely presented as a pro-$p$ group i.e. $G \simeq F/R$, where $F$ is a free pro-$p$ group with a finite free basis $X$ and $R$ is the smallest normal pro-$p$ subgroup of $F$ that contains some fixed finite set of relations of $G$. It is interesting to note that for abstract (discrete) groups the abstract versions of the properties $FP_2$ and finite presentability do not coincide [3].

In this paper we develop a pro-$p$ version of some of the results on algebraic fibering of abstract group extensions developed by the author and Vidussi in [14] and in the case of results on incoherence we prove results stronger than the ones proved in the abstract case. The results in [14] generalise the main results of Friedl and Vidussi in [9] and the main results of Kropholler and Walsh in [16]. The proofs of the results of [9], [14] and [16] use the Bieri-Strebel-Neumann-Renz $\Sigma$-invariants introduced in [5] and [6]. In [11] King suggested a $\Sigma$-invariant in the case of metabelian pro-$p$ groups [11]. We will use the King invariant in the proof of Proposition 3.4 but the rest of the results in this paper would have homological proofs independant from the King invariant.

Theorem 1.1. Let $1 \to K \to G \to \Gamma \to 1$ be a short exact sequence of pro-$p$ groups such that $G$ and $K$ are of type $FP_{n_0}$, $\Gamma^{ab}$ is infinite and there is a normal pro-$p$ subgroup $N$ of $K$ such that $G' \cap K \subseteq N$, $K/N \simeq \mathbb{Z}_p$ and $N$ is of type $FP_{n_1}$. Then there is a normal pro-$p$ subgroup $M$ of $G$ such that $G/M \simeq \mathbb{Z}_p$, $M \cap K = N$ and $M$ is of type $FP_{n_0}$. Furthermore if $K$, $G$ and $N$ are of type $FP_{\infty}$, then $M$ can be chosen of type $FP_{\infty}$.

We call a discrete pro-$p$ character of $G$ a non-trivial homomorphism of pro-$p$ groups $\alpha : G \to H$ such that $H \simeq \mathbb{Z}_p$. Then the theorem could be restated as : assume that $G$ and $K$ are of type $FP_{n_0}$, $\Gamma^{ab}$ is infinite and there is a discrete pro-$p$ character $\alpha$ of $G$ such that $\alpha|_K \neq 0$, $\text{Ker}(\alpha) \cap K = N$ is of type $FP_{n_0-1}$. Then there exists a discrete pro-$p$ character $\mu$ of $G$ such that $M = \text{Ker}(\mu)$ is of type $FP_{n_0}$ and $\mu|_K = \alpha|_K$, in particular $M \cap K = N$.

Key words and phrases. algebraic fibering, pro-$p$ groups, coherence, homological type $FP_m$. 

1
There is a lot in the literature on coherent abstract groups but very little is known for coherent pro-$p$ groups. Similar to the abstract case a pro-$p$ group $G$ is coherent (in the category of pro-$p$ groups) if every finitely generated pro-$p$ subgroup of $G$ is finitely presented as a pro-$p$ group i.e. is of type $FP_2$. We generalise this concept and define that a pro-$p$ group $G$ is $n$-coherent if any pro-$p$ subgroup of $G$ that is of type $FP_n$ is of type $FP_{n+1}$. Thus a pro-$p$ group is 1-coherent if and only if it is coherent (in the category of pro-$p$ groups).

**Corollary 1.2.** Let $K$, $Γ$ and $G = K × Γ$ be pro-$p$ groups, where $Γ$ is finitely generated free pro-$p$ but not pro-$p$ cyclic. Suppose that $K$ is of type $FP_{n+1}$ and there is a normal pro-$p$ subgroup $N$ of $K$ such that $G' ∩ K ⊆ N$, $K/N ≅ Z_p$ and $N$ is of type $FP_{n-1}$ but is not of type $FP_n$. Then there is a normal pro-$p$ subgroup $M$ of $G$ such that $G/M ≅ Z_p$, $M ∩ K = N$ and $M$ is of type $FP_n$ but is not of type $FP_{n+1}$. In particular $G$ is not $n_0$-coherent.

As in the case of Theorem 1.1 Corollary 1.2 can be restated in terms of discrete pro-$p$ characters.

For a free abstract group its rank is the minimal number of generators. Since for pro-$p$ groups $G$ finite rank is used for a notion different from the one adopted for abstract groups, we write $d(G)$ for the minimal number of generators of $G$. It is known that abstract (free finite rank)-by-$Z$ groups are coherent [8]. There is a conjecture suggested by Wise and independently by Kropholler and Walsh that an abstract (free of finite rank $\geq 2$)-by-(free of finite rank $\geq 2$) group is incoherent, see [16]. In [16] Kropholler and Walsh proved that (free of rank 2)-by-(free of finite rank $\geq 2$) abstract group is incoherent. The proof uses significantly that for a free abstract group $F_2$ of rank 2 we have that $Out(F_2) \cong GL_2(Z)$ and some explicit calculations with a finite generating set of a subgroup of finite index in $GL_2(Z)$ were used. Such an approach would not work for pro-$p$ groups since by Romankov’s result in [23] the automorphism group of a free pro-$p$ group $G$, where $2 \leq d(G) < \infty$, is not finitely generated as a topological group. Still a pro-$p$ version of the Kropholler-Walsh result holds and it is a particular case of Corollary 1.4 that follows from the following quite general theorem.

**Theorem 1.3.** Let $G = K × Γ$ be a pro-$p$ group with $K$ a finitely presented pro-$p$ group such that there is a normal pro-$p$ subgroup $N$ of $K$ such that $K/N ≅ Z_p$ and $N$ is not finitely generated, $Γ$ a finitely generated free pro-$p$ group with $d(Γ) \geq 2$. Then $G$ is incoherent (in the category of pro-$p$ groups).

The class of pro-$p$ groups $L$ was first considered by the author and Zalesskii in [15]. This class of groups contains all finitely generated free pro-$p$ groups. It shares many properties with the class of abstract limit groups and it is defined using extensions of centralizers. There are many open questions about the class of pro-$p$ groups $L$. For example, by Wilton’s result from [25] every finitely generated subgroup of an abstract limit group is a virtual retract, but the pro-$p$ version of this result is still an open problem. In order to prove Corollary 1.4 we show in Proposition 3.6 that the abelianization of any non-trivial pro-$p$ group from $L$ is always infinite. The same argument can be adapted for the class of abstract limit groups.

**Corollary 1.4.** Let $G = K × Γ$ be a pro-$p$ group with $K$ a non-abelian pro-$p$ group from the class $L$, $Γ$ a finitely generated free pro-$p$ group with $d(Γ) \geq 2$. Then $G$ is incoherent (in the category of pro-$p$ groups). In particular if $K$ is a finitely generated free pro-$p$ group with $d(K) \geq 2$ then $G$ is incoherent (in the category of pro-$p$ groups).

For a finite rank free pro-$p$ group $F$ the structure of $Aut(F)$ was studied first by Lubotsky in [18]. $Aut(F)$ is a topological group with a pro-$p$ subgroup of finite index. In [10] Gordon proved that
the automorphism group of an abstract free group of rank 2 is incoherent. Unfortunately we could not prove a pro-$p$ version of this result but still it would hold if the group of outer pro-$p$ automorphisms of a free pro-$p$ group of rank 2 contains a free non-procyclic pro-$p$ subgroup. For a free abstract group $F_2$ of rank 2 we have that $Out(F_2) \cong GL_2(\mathbb{Z})$ and since $SL_2(\mathbb{Z})$ is isomorphic to the free amalgamated product of $C_2$ and $C_6$ over a copy of $C_2$ it follows easily that $GL_2(\mathbb{Z})$ contains a free non-cyclic abstract group (or use the Tits alternative), hence $Out(F_2)$ contains a free non-cyclic abstract group. Nevertheless the group $GL_2^1(\mathbb{Z}_p) = \ker(GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{F}_p))$ does not contain a free pro-$p$ non-procyclic pro-$p$ subgroup, since it is $p$-adic analytic and so there is an upper limit on the number of generators of finitely generated pro-$p$ subgroups [7]. For related results on non-existance of free pro-$p$ subgroups in matrix groups see [1], [2], [26].

Let $G$ be a finitely generated pro-$p$ group. Define $Aut_0(G) = \ker(Aut(G) \to Aut(G/G^*))$, where $G^*$ is the Frattini subgroup of $G$. Then $Aut_0(G)$ is a pro-$p$ subgroup of $Aut(G)$ of finite index.

**Corollary 1.5.** Suppose that $K$ is a free pro-$p$ group with $d(K) = 2$. If $Out(K)$ contains a pro-$p$ free non-procyclic subgroup then $Aut_0(K)$ is incoherent (in the category of pro-$p$ groups).

By the Bieri-Strebel results in [4] for a finitely presented abstract group $H$ that does not contain free non-cyclic abstract subgroups, every metabelian quotient of $H$ is finitely presented. It is an open question whether a pro-$p$ version of the Bieri-Strebel result holds i.e. whether if $G$ is a finitely presented pro-$p$ group without free non-procyclic pro-$p$ subgroups then every metabelian pro-$p$ quotient of $G$ is finitely presented as a pro-$p$ group. Note that by the King classification of the finitely presented metabelian pro-$p$ groups in [12] every pro-$p$ quotient of a finitely presented metabelian pro-$p$ group is finitely presented pro-$p$. Using Corollary 1.5 and some ideas introduced by Romankov in [22], [23] we prove the following result.

**Corollary 1.6.** Suppose that $K$ is a free pro-$p$ group with $d(K) = 2$. Then either $Aut_0(K)$ is incoherent (in the category of pro-$p$ groups) or the pro-$p$ version of the Bieri-Strebel result does not hold.

**Acknowledgments** The author was partially supported by Bolsa de produtividade em pesquisa CNPq 305457/2021-7 and Projeto temático FAPESP 18/23690-6.

2. Preliminaries

2.1. **Homological finiteness properties for pro-$p$ groups.** Let $G$ be a pro-$p$ group. By definition

$$Z_p[[G]] = \lim_{\to \mathbb{Z}/p\mathbb{Z}}[G/U],$$

where the inverse limit is over all $i \geqslant 1$ and $U$ open subgroups of $G$. And

$$F_p[[G]] = Z_p[[G]]/pZ_p[[G]] = \lim_{\to \mathbb{F}_p[[G/U]]}$$

where the inverse limit is over all open subgroups $U$ of $G$.

By definition $G$ is of type $FP_m$ if the trivial $Z_p[[G]]$-module $Z_p$ has a projective resolution where all projectives in dimension $\leqslant m$ are finitely generated $Z_p[[G]]$-modules. By [11] for a pro-$p$ group the following conditions are equivalent:

1) $G$ is of type $FP_m$;
2) $H_i(G, Z_p)$ is a finitely generated (abelian) pro-$p$ group for $i \leqslant m$;
3) $H_i(G, F_p)$ is finite for $i \leqslant m$;
4) for $K$ either $\mathbb{F}_p$ or $\mathbb{Z}_p$ and $N$ a normal pro-$p$ subgroup of $G$ such that $K[[G]]$ is left and right Noetherian the homology groups $H_i(N,K)$ are finitely generated as $K[[G/N]]$-modules for $i \leq m$, where the $G/N$ action is induced by the conjugation action of $G$ on $N$.

The equivalence of the above conditions is a corollary of the fact that $\mathbb{Z}_p[[G]]$ and $\mathbb{F}_p[[G]]$ are local rings. Here $H_i(N,\mathbb{Z}_p)$ and $H_i(N,\mathbb{F}_p)$ are the standard homology groups of pro-$p$ groups with coefficients in the trivial pro-$p$ $\mathbb{Z}_p[[G]]$-modules $\mathbb{Z}_p$ and $\mathbb{F}_p$, for more on homology groups see [21].

2.2. The King invariant. Let $Q$ be a finitely generated abelian pro-$p$ group and let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Denote by $\mathbb{F}[[t]]^\times$ the multiplicative group of invertible elements in $\mathbb{F}[[t]]$. Consider

$$T(Q) = \{ \chi : Q \to \mathbb{F}[[t]]^\times \mid \chi \text{ is a continuous homomorphism} \},$$

where $\mathbb{F}[[t]]^\times$ is a topological group with topology induced by the topology of the ring $\mathbb{F}[[t]]$, given by the sequence of ideals $(t) \supseteq (t^2) \supseteq \ldots \supseteq (t^i) \supseteq \ldots$. Note that since $\chi$ is continuous we have that $\chi(Q) \subset 1 + t\mathbb{F}[[t]]$.

For $\chi \in T(Q)$ there is a unique continuous ring homomorphism

$$\overline{\chi} : \mathbb{Z}_p[[Q]] \to \mathbb{F}[[t]]$$

that extends $\chi$.

Let $A$ be a finitely generated pro-$p$ $\mathbb{Z}_p[[Q]]$-module. In [12] King defined the following invariant

$$\Delta(A) = \{ \chi \in T(Q) \mid \text{ann}\mathbb{Z}_p[[Q]](A) \subseteq \text{Ker}(\overline{\chi}) \}.$$

In [12] King used the notation $\Xi(A)$, that we here substitute by $\Delta(A)$.

Let $P$ be a pro-$p$ subgroup of $Q$. Define $T(Q,P) = \{ \chi \in T(Q) \mid \chi(P) = 1 \}$.

Theorem 2.1. [12, Thm B], [12, Lemma 2.5] Let $Q$ be a finitely generated abelian pro-$p$ group. Let $A$ be a finitely generated pro-$p$ $\mathbb{Z}_p[[Q]]$-module.

a) Then $A$ is finitely generated as an abelian pro-$p$ group if and only if $\Delta(A) = \{1\}$.

b) If $P$ is a pro-$p$ subgroup of $Q$ then $T(Q,P) \cap \Delta(A) = \Delta(A/[A,P])$. In particular $A$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[P]]$-module if and only if $T(Q,P) \cap \Delta(A) = \{1\}$.

We state the classification of the finitely presented metabelian pro-$p$ groups given by King in [12].

Theorem 2.2. [12] Let $1 \to A \to G \to Q \to 1$ be a short exact sequence of pro-$p$ groups, where $G$ is a finitely generated pro-$p$ group and $A$ and $Q$ are abelian pro-$p$ groups. Then $G$ is a finitely presented pro-$p$ group if and only if $\Delta(A) \cap \Delta(A)^{-1} = \{1\}$.

Example Let $A = \mathbb{F}_p[[s]]$, $Q = \mathbb{Z}_p$, $G = A \rtimes Q$, where $\mathbb{Z}_p$ has a generator $b$ and $b$ acts via conjugation on $A$ by multiplication with $1+s$. Since $\text{ann}\mathbb{Z}_p[[Q]](A) = p\mathbb{Z}_p[[Q]] \subseteq \text{Ker}(\overline{\chi})$ for any $\chi \in T(Q)$, we conclude that $\Delta(A) = T(Q) = \Delta(A)^{-1}$. Hence by Theorem 2.2 $G$ is not finitely presented.
3. Proofs

We start by citing a result on abstract groups. We recall first that an abstract group $G$ is of type $FP_m$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution where all projectives in dimension $\leq m$ are finitely generated. An abstract group $G$ is of (homotopical) type $F_n$ if there is a classifying space $K(G,1)$ with finite $n$-skeleton. If $n \geq 2$ then $G$ is of type $F_n$ if and only if it is of type $FP_n$ and is finitely presented (as an abstract group). The homotopical part of Proposition 3.1 was proved by Kuckuck in [17] and the homological part of Proposition 3.1 was proved by the author and Lima in [13]. The former has a geometric proof and the latter an algebraic one.

**Proposition 3.1.** [17], [13] Let $n \geq 1$ be a natural number, $A \hookrightarrow B \twoheadrightarrow C$ a short exact sequence of groups with $A$ of type $F_n$ (resp. of type $FP_n$) and $C$ of type $F_{n+1}$ (resp. of type $FP_{n+1}$). Assume there is another short exact sequence of groups $A \hookrightarrow B_0 \twoheadrightarrow C_0$ with $B_0$ of type $F_{n+1}$ (resp. of type $FP_{n+1}$) and that there is a group homomorphism $\theta : B_0 \to B$ such that $\theta|_A = id_A$, i.e. there is a commutative diagram of homomorphisms of groups

$$
\begin{array}{ccc}
A & \xrightarrow{id_A} & B_0 \\
\downarrow \theta & & \downarrow \nu \\
A & \xrightarrow{id_A} & B \\
\end{array}
$$

Then $B$ is of type $F_{n+1}$ (resp. of type $FP_{n+1}$).

We prove a pro-$p$ version of the above proposition. Recall that the property $FP_m$ for pro-$p$ groups was discussed in Section 2.1

**Lemma 3.2.** Let $n \geq 1$ be a natural number, $A \hookrightarrow B \twoheadrightarrow C$ a short exact sequence of pro-$p$ groups with $A$ of type $FP_n$ and $C$ of type $FP_{n+1}$. Assume there is another short exact sequence of pro-$p$ groups $A \hookrightarrow B_0 \twoheadrightarrow C_0$ with $B_0$ of type $FP_{n+1}$ and that there is a homomorphism of pro-$p$ groups $\theta : B_0 \to B$ such that $\theta|_A = id_A$, i.e. there is a commutative diagram of homomorphisms of groups

$$
\begin{array}{ccc}
A & \xrightarrow{id_A} & B_0 \\
\downarrow \theta & & \downarrow \nu \\
A & \xrightarrow{id_A} & B \\
\end{array}
$$

Then $B$ is of type $FP_{n+1}$.

**Proof.** Consider the LHS-spectral sequence

$$
E^2_{i,j} = H_i(C_0,H_j(A,\mathbb{F}_p))
$$

that converges to $H_{i+j}(B_0,\mathbb{F}_p)$. Similarly there is the LHS spectral sequence

$$
\tilde{E}^2_{i,j} = H_i(C,H_j(A,\mathbb{F}_p))
$$

that converges to $H_{i+j}(B,\mathbb{F}_p)$.

Since $A$ is of type $FP_n$ we have that $H_j(A,\mathbb{F}_p)$ is finite for all $j \leq n$. Then there is a pro-$p$ subgroup $C_1$ of finite index in $C$ such that $C_1$ acts trivially on $H_j(A,\mathbb{F}_p)$ for every $j \leq n$. Since $C$ is of type $FP_{n+1}$ we have that $C_1$ is of type $FP_{n+1}$. Then

$$
H_i(C_1,H_j(A,\mathbb{F}_p)) \cong \bigoplus H_i(C_1,\mathbb{F}_p) \text{ is finite for } j \leq n, i \leq n + 1,
$$

5
where we have $\dim_{\mathbb{F}_p} H_j(A, \mathbb{F}_p)$ direct summands. Since $C_1$ has finite index in $C$ we deduce that
\[ \hat{E}^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \] is finite for $j \leq n, i \leq n + 1$.

hence by the convergence of the second spectral sequence we obtain that $H_k(B, \mathbb{F}_p)$ is finite for $k \leq n$.

Note that we have shown that if $i + j = n + 1, i \neq 0$ then $\hat{E}^2_{i,j}$ is finite, hence $\hat{E}^\infty_{i,j}$ is finite. By the convergence of the spectral sequence there is a filtration of $H_{n+1}(B, \mathbb{F}_p)$
\[ 0 = F_{-1}(H_{n+1}(B, \mathbb{F}_p)) \subseteq \cdots \subseteq F_i(H_{n+1}(B, \mathbb{F}_p)) \subseteq F_{i+1}(H_{n+1}(B, \mathbb{F}_p)) \]
\[ \subseteq \cdots \subseteq F_{n+1}(H_{n+1}(B, \mathbb{F}_p)) = H_{n+1}(B, \mathbb{F}_p) \]
where $F_i(H_{n+1}(B, \mathbb{F}_p))/F_{i-1}(H_{n+1}(B, \mathbb{F}_p)) \simeq \hat{E}^\infty_{i,n+1-i}$.

Thus $H_{n+1}(B, \mathbb{F}_p)$ is finite if and only if $\hat{E}^\infty_{0,n+1}$ is finite.

Note that since any differential that comes out from $\hat{E}^r_{0,n+1}$ is zero we have that $\hat{E}^\infty_{0,n+1}$ is a quotient of $\hat{E}^2_{0,n+1} = H_0(C, H_{n+1}(A, \mathbb{F}_p))$, thus there is a map $\mu : H_0(C, H_{n+1}(A, \mathbb{F}_p)) \to H_{n+1}(B, \mathbb{F}_p)$
with image that equals $\hat{E}^\infty_{0,n+1}$. Thus $B$ is of type $FP_{n+1}$ if and only if $\text{Im}(\mu)$ is finite.

Similarly there is a map $\mu_0 : H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) \to H_{n+1}(B_0, \mathbb{F}_p)$
with image that equals $\hat{E}^\infty_{0,n+1}$ and such that $B_0$ is of type $FP_{n+1}$ if and only if $\text{Im}(\mu_0)$ is finite. Since $B_0$ is of type $FP_{n+1}$ we conclude that $\text{Im}(\mu_0)$ is finite.

The naturality of the LHS spectral sequence implies that we have the commutative diagram
\[
\begin{array}{ccc}
H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) & \xrightarrow{\rho} & H_0(C, H_{n+1}(A, \mathbb{F}_p)) \\
\downarrow{\iota_0} & & \downarrow{\mu} \\
H_{n+1}(B_0, \mathbb{F}_p) & \xrightarrow{\rho_0} & H_{n+1}(B, \mathbb{F}_p)
\end{array}
\]
where the maps $\rho$ and $\rho_0$ are induced by $\nu$. Recall that the action of $B_0$ on $A$ via conjugation induces an action of $B_0$ on $H_{n+1}(A, \mathbb{F}_p)$ where $A$ acts trivially and this induces the action of $C_0$ on $H_{n+1}(A, \mathbb{F}_p)$ that is used to define $H_0(C_0, H_{n+1}(A, \mathbb{F}_p))$. Similarly the action of $B$ on $A$ via conjugation induces an action of $B$ on $H_{n+1}(A, \mathbb{F}_p)$ where $A$ acts trivially and this induces the action of $C$ on $H_{n+1}(A, \mathbb{F}_p)$ that is used to define $H_0(C, H_{n+1}(A, \mathbb{F}_p))$. Recall that the map $\rho : H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) \to H_0(C, H_{n+1}(A, \mathbb{F}_p))$ from the commutative diagram is induced by $\nu$. If $\nu$ is surjective then $\rho$ is an isomorphism; if $\nu$ is injective then $\rho$ is surjective. Since every homomorphism $\nu$ is composition of one epimorphism followed by one monomorphism we conclude that $\rho$ is always surjective. Then $\text{Im}(\mu) = \text{Im}(\mu \circ \rho) = \text{Im}(\rho_0 \circ \mu_0)$ is a quotient of $\text{Im}(\mu_0)$

Since $\text{Im}(\mu_0)$ is finite we conclude that $\text{Im}(\mu)$ is finite. Hence $B$ is of type $FP_{n+1}$ as required.

Recall that a pro-$p$ HNN extension is called proper if the canonical map from the base group to the pro-$p$ HNN extension is injective.
Lemma 3.3. Let $G = \langle A, t \mid K = K \rangle$ be a proper pro-$p$ HNN extension. Suppose that $A, K$ are pro-$p$ groups of type $FP_m$ and $M$ is a normal pro-$p$ subgroup of $G$ such that $G/M \cong \mathbb{Z}_p$, $K \not\subseteq M$ and $M \cap A$ is of type $FP_m$. Then the following holds:

(a) $M$ is of type $FP_m$ if and only if $M \cap K$ is of type $FP_{m-1}$;

(b) if $M$ is of type $FP_{m+1}$ then $M \cap K$ is of type $FP_m$.

Proof. The proper pro-$p$ HNN extension gives rise to the exact sequence of $\mathbb{F}_p[[G]]$-modules

$$0 \to \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \to \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \to \mathbb{F}_p \to 0$$

Note that since $K \not\subseteq M$ we have that $M \backslash G/K = G/MK$ is a proper pro-$p$ quotient of $G/M \cong \mathbb{Z}_p$, hence is finite. Similarly $M \backslash G/A = G/MA$ is finite. Note that there is an isomorphism of (left) $\mathbb{F}_p[[M]]$-modules

$$\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \cong \left( \bigoplus_{t \in M \backslash G/K} \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \right) \cong \bigoplus_{t \in M \backslash G/K} \mathbb{F}_p[[M \cap tKt^{-1}]] \mathbb{F}_p$$

Similarly there is an isomorphism of (left) $\mathbb{F}_p[[M]]$-modules

$$\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \cong \left( \bigoplus_{t \in M \backslash G/A} \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \right) \cong \bigoplus_{t \in M \backslash G/A} \mathbb{F}_p[[M \cap tAt^{-1}]] \mathbb{F}_p$$

The short exact sequence (1) gives rise to a long exact sequence in pro-$p$ homology

$$\ldots \to H_{m+1}(M, \mathbb{F}_p) \to H_m(M, \mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \to H_m(M, \mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \to H_m(M, \mathbb{F}_p)$$

$$\to H_{m-1}(M, \mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \to \ldots \to H_1(M, \mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \to H_1(M, \mathbb{F}_p)$$

Note that

$$H_i(M, \mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \cong H_i(M, \bigoplus_{t \in M \backslash G/K} \mathbb{F}_p[[M \cap tKt^{-1}]] \mathbb{F}_p) \cong \bigoplus_{t \in M \backslash G/K} H_i(M \cap tKt^{-1}, \mathbb{F}_p)$$

Similarly

$$H_i(M, \mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \cong \bigoplus_{t \in M \backslash G/A} H_i(M \cap tAt^{-1}, \mathbb{F}_p)$$

Then the long exact sequence could be rewritten as

$$\ldots \to H_{m+1}(M, \mathbb{F}_p) \to \bigoplus_{t \in M \backslash G/K} H_m(M \cap K, \mathbb{F}_p) \to \bigoplus_{t \in M \backslash G/A} H_m(M \cap A, \mathbb{F}_p) \to H_m(M, \mathbb{F}_p)$$

$$\to \bigoplus_{t \in M \backslash G/K} H_{m-1}(M \cap K, \mathbb{F}_p) \to \ldots \to \bigoplus_{t \in M \backslash G/A} H_1(M \cap A, \mathbb{F}_p) \to H_1(M, \mathbb{F}_p)$$

Since $M \cap A$ is of type $FP_m$ we have that $H_i(M \cap A, \mathbb{F}_p)$ is finite for $i \leq m$. Combining with $M \backslash G/A$ is finite, we conclude that $\bigoplus_{t \in M \backslash G/A} H_i(M \cap A, \mathbb{F}_p)$ is finite for $i \leq m$.

(a) Note that $M$ is of type $FP_m$ if and only if $H_i(M, \mathbb{F}_p)$ is finite for $i \leq m$. By the above long exact sequence together with the fact that $M \backslash G/K$ is finite, $H_i(M, \mathbb{F}_p)$ is finite for $i \leq m$ if and only if $\bigoplus_{t \in M \backslash G/K} H_i(M \cap K, \mathbb{F}_p)$ is finite for $i \leq m - 1$ i.e. $M \cap K$ is of type $FP_{m-1}$.

(b) If $M$ is of type $FP_{m+1}$ then $H_{m+1}(M, \mathbb{F}_p)$ is finite and since $H_m(M \cap A, \mathbb{Z}_p)$ is finite by the long exact sequence $H_m(M \cap K, \mathbb{F}_p)$ is finite. We already know by (a) that $M \cap K$ is of type $FP_{m-1}$, hence $M \cap K$ is of type $FP_m$. □

For a pro-$p$ group $G$ with a subset $S$ denote by $\langle S \rangle$ the pro-$p$ subgroup of $G$ generated by $S$. 7
Proposition 3.4. Let \( Q \cong \mathbb{Z}_p^2 = \langle x, y \rangle \) and \( A \) be a finitely generated pro-p \( \mathbb{Z}_p[[Q]] \)-module. Suppose that for \( H = \langle x \rangle \) we have that \( A \) is finitely generated as a pro-p \( \mathbb{Z}_p[[H]] \)-module. Let \( H_j = \langle xy^{-p^j} \rangle \). Then there is \( j_0 > 0 \) such that for every \( j \geq j_0 \) we have that \( A \) is finitely generated as \( \mathbb{Z}_p[[H_j]] \)-module.

Proof. By Theorem 2.1 if \( P \) is a pro-p subgroup of \( Q \) then \( A \) is finitely generated as \( \mathbb{Z}_p[[P]] \)-module if and only if \( T(Q, P) \cap \Delta(A) = \{1\} \). Let

\[
J = \text{ann}_{\mathbb{Z}_p[[Q]]}(A).
\]

Since \( A \) is finitely generated as a pro-p \( \mathbb{Z}_p[[H]] \)-module for every \( \chi \in T(Q, H) \setminus \{1\} \) we have that \( J \not\subseteq \text{Ker}(\chi) \).

Let \( \mu_j \in T(Q, H_j) \setminus \{1\} \). We aim to show that for sufficiently big \( j \) we have that \( \mu_j \not\in \Delta(A) \). Then by Theorem 2.1 \( A \) is finitely generated as \( \mathbb{Z}_p[[H_j]] \)-module.

Let

\[
\overline{\mu}_j : \mathbb{Z}_p[[Q]] \to \mathbb{F}[[t]]
\]

be the continuous ring homomorphism induced by \( \mu_j \). Since \( \overline{\mu}_j(H_j) = 1 \) we have

\[
\overline{\mu}_j(x) = \mu_j(y^{p^j}).
\]

Let \( \chi \in T(Q, H) \setminus \{1\} \) be such that

\[
\chi(y) = \mu_j(y)
\]

and

\[
\overline{\chi} : \mathbb{Z}_p[[Q]] \to \mathbb{F}[[t]]
\]

be the continuous ring homomorphism induced by \( \chi \). Recall that \( \chi \in T(Q, H) \) implies that \( \chi(x) = 1 \). Then there is \( \lambda \in J \) such that \( \overline{\chi}(\lambda) \neq 0 \). Note that \( \lambda \in \mathbb{Z}_p[[Q]] = \mathbb{Z}_p[[t_1, t_2]] \), where \( x = 1 + t_1, y = 1 + t_2 \) and since \( \chi(y) = \mu_j(y) \) we have

\[
0 \neq \overline{\chi}(\lambda) = \overline{\chi}(\lambda|_{t_1=0}) = \overline{\mu}_j(\lambda|_{t_1=0}).
\]

Note that

\[
\overline{\mu}_j(t_2) = \overline{\mu}_j(1 + t_2) - \overline{\mu}_j(1) \in 1 + t\mathbb{F}[[t]] - 1 = t\mathbb{F}[[t]]
\]

hence \( \overline{\mu}_j(t_2)^{p^j} \in t^{p^j}\mathbb{F}[[t]] \). This together with the condition \( \mu_j(x) = \mu_j(y^{p^j}) \) implies

\[
\overline{\mu}_j(\lambda) = \overline{\mu}_j(\lambda|_{t_1=t_2^{p^j}}) \in \overline{\mu}_j(\lambda|_{t_1=0}) + t^{p^j}\mathbb{F}[[t]].
\]

Suppose that

\[
0 \neq \overline{\mu}_j(\lambda|_{t_1=0}) \in f t^m + t^{m+1}\mathbb{F}[[t]]
\]

where \( f \in \mathbb{F} \setminus \{0\}, m \geq 0 \). Then choose \( j_0 > 0 \) such that \( p^{j_0} > m \) and this implies that for \( j \geq j_0 \) we have \( \overline{\mu}_j(\lambda) \neq 0 \). Hence \( \mu_j \not\in \Delta(A) \). \( \square \)

Proposition 3.5. Let \( G \) be a pro-p group with a normal pro-p subgroup \( G_0 \) such that \( G/G_0 \cong \mathbb{Z}_p^2 \). Let \( S \) be a normal pro-p subgroup of \( G \) such that \( G/S \cong \mathbb{Z}_p \), \( G_0 \subseteq S \) and \( S \) is of type \( \text{FP}_m \) for some \( m \geq 1 \). Then there is a normal pro-p subgroup \( S_0 \) of \( G \) such that \( G/S_0 \cong \mathbb{Z}_p, S \neq S_0, G_0 \subseteq S_0 \) and \( S_0 \) is of type \( \text{FP}_m \).
Proof. Note that since $S$ is a pro-$p$ group of type $FP_m$ and $G/S \simeq \mathbb{Z}_p$ is a pro-$p$ group of type $FP_\infty$, hence of type $FP_m$, we can conclude that $G$ is a pro-$p$ group of type $FP_m$. Set

$Q = G/G_0 = \langle x, y \rangle$, where $H = S/G_0 = \langle x \rangle$.

Since $Q = G/G_0$ is a finitely generated abelian pro-$p$ group and $G$ is of type $FP_m$ we conclude that $A_i = H_i(G_0, \mathbb{Z}_p)$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[Q]]$-module for $i \leq m$. Since $S$ is a pro-$p$ group of type $FP_m$ we conclude that $A_i$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H]]$-module. Then by Proposition 3.3 for sufficiently big $j$ we have that $A_i$ is finitely generated as a pro-$p$ $\mathbb{Z}_p[[H_j]]$-module, where $H_j = \langle xy^{-p^j} \rangle \leq Q$, for every $i \leq m$. Then we define $S_0$ as the preimage in $G$ of one such $H_j$. □

Proofs of Theorem 1.1

There is a commutative diagram where the lines are short exact sequences of pro-$p$ groups

\[
\begin{array}{cccc}
K & \subset & \Pi & \rightarrow F_n \\
id_K & \downarrow & \pi & \downarrow \\
K & \subset & G & \rightarrow \Gamma
\end{array}
\]

where $F_n$ is the free pro-$p$ group with a free basis $s_1, \ldots, s_n$. Define

$\Pi = \prod_{i=1}^{n} \prod_{k=1}^{2} \prod_{j=1}^{n_i} \Pi_{ij}$

where $\prod_{i} \prod_{k=1}^{2} \prod_{j=1}^{n_i} \Pi_{ij}$ is the amalgamated free product in the category of pro-$p$ groups, and each $\Pi_i = K \rtimes \langle s_i \rangle$, $\langle s_i \rangle \simeq \mathbb{Z}_p$. Note that since $K$ is normal in $\Pi$ and $\Pi/K \simeq \Pi_1/K \prod_{i=1}^{n} \Pi_{ij}/K$ is a free pro-$p$ product we conclude that $\prod_{i=1}^{n} \prod_{k=1}^{2} \prod_{j=1}^{n_i} \Pi_{ij}$ embeds in $\Pi$ for every $1 \leq i \leq n$.

Recall that $\Gamma^{ab}$ is infinite, hence the image in $\Gamma^{ab}$ of at least one $\pi(s_i)$ has infinite order. Without loss of generality we can assume that the image of $\pi(s_1)$ in $\Gamma^{ab}$ has infinite order. In particular $\Pi_1 \simeq \pi(\Pi_1)$ is an isomorphism. Note that $[K, s_1] \subseteq G' \cap K \subseteq N$, hence $\Pi_1 \subseteq N$. We have $N \subseteq K \subseteq \Pi_1$ where $K/N \simeq \mathbb{Z}_p$, $\Pi_1/K \simeq \mathbb{Z}_p$, this together with the inclusion $\Pi_1 \subseteq N$ implies that $\Pi_1/N \simeq \mathbb{Z}_p^2$.

By assumption $K$ is of type $FP_{m_0}$. By Proposition 3.5 there is $S_0$ a normal pro-$p$ subgroup of $\Pi_1$ such that $N \subseteq S_0$, $S_0$ is of type $FP_{m_0}$, $S_0 \neq K$ and $\Pi_1/S_0 \simeq \mathbb{Z}_p$.

Recall that $\Pi_1 \simeq \pi(\Pi_1)$. Let

$\mu : G \rightarrow \mathbb{Z}_p$

be a homomorphism of pro-$p$ groups such that $\text{Ker}(\mu) \cap \pi(\Pi_1) = S_0$ i.e. $\text{Ker}(\mu) \cap \pi(\Pi_1) = \pi(S_0)$. This is possible since $\Pi_1/N \simeq \mathbb{Z}_p^2$ is abelian and $G' \cap K \subseteq N \subseteq S_0$. Note that $K \nsubseteq S_0$, hence $\mu(K) \neq 0$.

Consider the epimorphism of pro-$p$ groups

$\chi = \mu \circ \pi : \Pi \rightarrow \mathbb{Z}_p$.

Note that $\chi(K) \neq 0$, $\text{Ker}(\chi) \cap \Pi_1 = S_0$ is of type $FP_{m_0}$ and $\text{Ker}(\chi) \cap K = S_0 \cap K = N$ is of type $FP_{m_0-1}$. Then we view $\Pi_1 \prod_{k=1}^{2} \Pi_{ij}$ as a proper HNN extension

$\langle \Pi_1, s_2 \mid K^{x_2} = K \rangle$. 

with a pro-

base group \( \Pi_1 \), associated pro-

subgroup \( K \) and stable letter \( s_2 \). Then by Lemma 3.3 a)

\[
Ker(\chi) \cap (\Pi_1 \coprod_k \Pi_2) \text{ is of type } FP_{n_0}.
\]

We view \( \Pi_1 \coprod_k \Pi_2 \coprod_k \Pi_3 \) as a proper HNN extension with a base pro-

subgroup \( \Pi_1 \coprod_k \Pi_2 \), associated pro-

subgroup \( K \) and stable letter \( s_3 \) then by Lemma 3.3 a)

\[
Ker(\chi) \cap (\Pi_1 \coprod_k \Pi_2 \coprod_k \Pi_3) \text{ is of type } FP_{n_0}.
\]

Then repeating this argument several times we deduce that \( Ker(\chi) \) is of type \( FP_{n_0} \).

By construction \( Ker(\mu) \) is a quotient of \( Ker(\chi) \). If \( n_0 = 1 \) then \( Ker(\chi) \) is finitely generated (as a pro-

subgroup), then any pro-

quotient of \( Ker(\chi) \) is finitely generated (as a pro-

group). In particular \( Ker(\mu) \) is finitely generated (as a pro-

group).

Now for the general case i.e. \( n_0 \geq 2 \) we will apply Lemma 3.2. Write \( \overline{Ker(\chi)} \) for the image of \( Ker(\chi) \) in \( F_n \) and \( \overline{Ker(\mu)} \) for the image of \( Ker(\mu) \) in \( \Gamma \). By construction \( Ker(\chi) \cap K = N = Ker(\mu) \cap K \).

By assumption \( N \) is of type \( FP_{n_0-1} \) and we have already shown that \( Ker(\chi) \) is of type \( FP_{n_0} \). By construction \( \mu(K) \neq 0 \), hence \( K.Ker(\mu) \neq Ker(\mu) \) and since \( G/Ker(\mu) \cong \mathbb{Z}_p \) we deduce that \( K.Ker(\mu) \) has finite index in \( G \) and so \( \overline{Ker(\mu)} \) has finite index in \( \Gamma \). Since in the short exact sequence of pro-

groups

\[
1 \to K \to G \to \Gamma \to 1
\]

we have that \( G \) and \( K \) are pro-

subgroups of type \( FP_{n_0} \) (it suffices that \( K \) is of type \( FP_{n_0-1} \)) we deduce that \( \Gamma \) is of type \( FP_{n_0} \). Then \( \overline{Ker(\mu)} \) is a pro-

subgroup of type \( FP_{n_0} \). Then we can apply Lemma 3.2 for the commutative diagram

\[
\begin{array}{ccc}
N = Ker(\chi) \cap K & \xrightarrow{id} & Ker(\chi) \\
\downarrow{\pi_{|Ker(\chi)}} & & \downarrow{\pi_{|Ker(\chi)}} \\
N = Ker(\mu) \cap K & \xrightarrow{id} & Ker(\mu)
\end{array}
\]

to deduce that \( Ker(\mu) \) is a pro-

subgroup of type \( FP_{n_0} \). Finally we set \( M = Ker(\mu) \).

**Proof of Corollary 1.2**

We define \( M \) as in the proof of Theorem 1.1 for \( \Gamma = F_n \) and \( \pi \) the identity map, \( \mu = \chi \). Thus \( M = Ker(\chi) = Ker(\mu) \) is a normal subgroup of \( G \), \( G/M \cong \mathbb{Z}_p \) and \( M \) is of type \( FP_{n_0} \). We view

\[
G = \Pi = \Pi_1 \coprod_k \Pi_2 \coprod_k \ldots \coprod_k \Pi_n
\]

as a proper HNN extension with a base pro-

subgroup \( A = \Pi_1 \coprod_k \Pi_2 \coprod_k \ldots \coprod_k \Pi_{n-1} \), associated pro-

subgroup \( K \) and stable letter \( s_n \). By the proof of Theorem 1.1 \( A \cap M = A \cap Ker(\chi) \) is of type \( FP_{n_0} \). Suppose that \( M \) is of type \( FP_{n_0+1} \). By Lemma 3.3 b) \( N = M \cap K \) is of type \( FP_{n_0} \), a contradiction. Hence \( M \) is not of type \( FP_{n_0+1} \). This completes the proof of the corollary.

**Proof of Theorem 1.3**

We claim that there is a finitely generated non-procyclic pro-

subgroup \( \Gamma_0 \) of \( \Gamma \) such that \( \Gamma_0 \) acts trivially on the abelianization \( K^{ab} = K/K' \) via conjugation. Let \( T = tor(K/K') \) be the torsion part of
$K^{ab}$. Then $V = K^{ab} / T \simeq \mathbb{Z}_p^d$, where $d \geq 1$. Note that the conjugation action of $\Gamma$ on $V \simeq \mathbb{Z}_p^d$ induces a homomorphism

$$\rho : \Gamma \to GL_d(\mathbb{Z}_p).$$

Note that $\text{Im}(\rho)$ is a pro-$p$ subgroup of $GL_d(\mathbb{Z}_p)$, hence is $p$-adic analytic and there is an upper bound on the number of generators of any finitely generated pro-$p$ subgroup of $\text{Im}(\rho)$ [7]. Hence $\rho$ is not injective. Alternatively we can use the main result of [11] to deduce that $\rho$ is not injective. Thus $\text{Ker}(\rho)$ is a non-trivial normal pro-$p$ subgroup of $\Gamma$ and we can choose $\Gamma_0$ any non-procyclic finitely generated pro-$p$ subgroup of $\text{Ker}(\rho)$.

Set $G_0 = K \rtimes \Gamma_0$. Then by Corollary [1.2] there is a normal pro-$p$ subgroup $M$ of $G_0$ such that $G_0 / M \simeq \mathbb{Z}_p$ and $M$ is not of type $FP_2$, i.e. is not finitely presented as a pro-$p$ group. Thus $G_0$ is incoherent (in the category of pro-$p$ groups). This completes the proof.

We recall the definition of the class of pro-$p$ groups $\mathcal{L}$. It uses the extension of centraliser construction. We define inductively the class $\mathcal{G}_n$ of pro-$p$ groups by setting $\mathcal{G}_0$ as the class of all finitely generated free pro-$p$ groups and a group $G_n \in \mathcal{G}_n$ if there is a decomposition $G_n = G_n-1 \amalg A$, where $\mathcal{G}_n-1 \in \mathcal{G}_{n-1}$, $C$ is self-centralised procyclic subgroup of $G_{n-1}$ and $A$ is a finitely generated free abelian pro-$p$ group such that $C$ is a direct summand of $A$. The class $\mathcal{L}$ is defined as the class of all finitely generated pro-$p$ subgroups $G$ of $G_n$ where $\mathcal{G}_n \in \mathcal{G}_n$ for $n \geq 0$. The minimal $n$ such that $G \leq G_n \in \mathcal{G}_n$ is called the weight of $G$.

**Proposition 3.6.** Let $K \in \mathcal{L}$ be a non-trivial pro-$p$ group. Then $K^{ab} = K / K'$ is infinite.

**Proof.** Let $K \in \mathcal{L}$ have weight $n$. Suppose that $K^{ab}$ is finite. And $n$ is the smallest possible with $K^{ab}$ finite. By [24] Thm. B] $K$ is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups $\Delta$, where each edge group is trivial or $\mathbb{Z}_p$ and each vertex group is either a non-abelian limit pro-$p$ group of weight at most $n - 1$ or a finitely generated abelian pro-$p$ group.

Let $\hat{G}$ be the underlying graph of the finite graph of groups $\Delta$. If it is not a tree then $K$ decomposes as a pro-$p$ HNN extension, hence the stable letter generates an infinite procyclic subgroup of $K^{ab}$, a contradiction.

We can assume that $|V(\hat{G})|$ is the smallest possible. Then we have a decomposition as an amalgamated pro-$p$ free product $K = K_0 \amalg G_v$, where $K_0$ is the fundamental pro-$p$ group of the subgraph of pro-$p$ groups $\Delta_0$ of $\Delta$ such that its underlying graph $\Gamma_0$ is obtained from $\Gamma$ by removing the edge $e_0$ and its vertex $v_0$ and we have that $e_0$ is the unique edge in $\Gamma$ that has $v_0$ as a vertex. Note that by [20] every amalgamated free pro-$p$ product with procyclic amalgamation is proper. Since the class $\mathcal{L}$ is closed under finitely generated pro-$p$ subgroups $K_0 \in \mathcal{L}$ and by the minimality of $|V(\Gamma)|$ and $n$ we have that $K^{ab}$ and $G_v^{ab}$ are infinite. If we write $t(M)$ for the torsion free rank of an abelian finitely generated pro-$p$ group $M$ then $t(K^{ab}) \geq t(K_0^{ab}) + t(G_v^{ab}) - t(C_0) \geq 1 + 1 - t(C_0) \geq 1,$

so $K^{ab}$ cannot be finite. \hfill $\square$

**Proof of Corollary 1.4** By Proposition 3.6 $K^{ab}$ is infinite. Let $N$ be a normal pro-$p$ subgroup of $K$ such that $K / N \simeq \mathbb{Z}_p$. By part (4) from the main theorem of [15] we have that $N$ is not finitely generated as a pro-$p$ group. Then we can apply Theorem 1.3.

**Proof of Corollary 1.5** Let $F$ be a finitely generated free non-procyclic pro-$p$ group that embeds as a closed subgroup of $\text{Out}(K)$. Note that $G = K \rtimes F$ is a pro-$p$ group embeds as a closed subgroup of $\text{Aut}(K)$ and by Theorem 1.3 $G$ is incoherent (in the category of pro-$p$ groups).
Proof of Corollary 1.6 We recall first some results from [18]. Let $G$ be a finitely generated pro-$p$ group and $\text{Aut}(G)$ denote all continuous automorphisms of $G$ (which coincide with the abstract automorphisms of $G$). Denote $\text{Inn}(G)$ the group of the internal automorphisms. The group $\text{Aut}(G)$ is a profinite group.

Lemma 3.7. [18] a) Let $G$ be a finitely generated pro-$p$ group and $G^*$ be the Frattini subgroup of $G$ i.e. the intersection of all maximal open subgroups of $G$. Then $Ker(\text{Aut}(G) \to \text{Aut}(G/G^*))$ is a pro-$p$ subgroup of $\text{Aut}(G)$ of finite index.

b) Let $F$ be a finitely generated free pro-$p$ group and $N$ be a characteristic pro-$p$ subgroup of $F$. Then the map $\text{Aut}(F) \to \text{Aut}(F/N)$, obtained by taking the induced automorphisms, is surjective.

We set $\text{Aut}_0(G) = Ker(\text{Aut}(G) \to \text{Aut}(G/G^*))$ and $\text{Out}_0(G) = \text{Aut}_0(G)/\text{Inn}(G)$.

Lemma 3.8. Suppose $K$ is a free pro-$p$ group, $d(K) = 2$ and $M$ is the maximal pro-$p$ metabelian quotient of $K$. Then $\text{Out}(M)$ contains a finitely generated pro-$p$ subgroup $H$ such that $H$ has a metabelian pro-$p$ quotient that is not finitely presented (as a pro-$p$ group).

Lemma 3.8 implies Corollary 1.6 If $\text{Out}(K)$ contains a pro-$p$ free non-procyclic subgroup we can apply Corollary 1.5. Then we can assume that $\text{Out}(K)$ does not contain a pro-$p$ free non-procyclic subgroup. We can further assume that the pro-$p$ version of the Bieri-Strebel result holds otherwise Corollary 1.6 holds i.e. if a finitely presented pro-$p$ group does not contain a free non-procyclic pro-$p$ subgroup then any metabelian pro-$p$ quotient of that group is a finitely presented pro-$p$ group.

Let $H$ be a pro-$p$ subgroup of $\text{Out}(M)$ as in Lemma 3.8. Since $\text{Aut}_0(M)$ has finite index in $\text{Aut}(M)$ without loss of generality we can assume that $H \subseteq \text{Out}_0(M)$. The epimorphism of pro-$p$ groups $\text{Aut}_0(K) \to \text{Aut}_0(M)$ induces an epimorphism of pro-$p$ groups $\text{Out}_0(K) \to \text{Out}_0(M)$. Then there is a finitely generated pro-$p$ subgroup $\tilde{H}$ of $\text{Out}_0(K)$ that maps surjectively to $H$, in particular $\tilde{H}$ has a metabelian pro-$p$ quotient that is not finitely presented (as a pro-$p$ group). Then by the previous considerations $\tilde{H}$ is not a finitely presented pro-$p$ group.

Note that $\text{Inn}(K) \cong K$. Consider the short exact sequence $1 \to K \to \text{Aut}_0(K) \to \text{Out}_0(K) \to 1$ and let $H_0$ be the preimage of $\tilde{H}$ in $\text{Aut}_0(K)$. Then there is a short exact sequence

$$1 \to K \to H_0 \to \tilde{H} \to 1$$

of pro-$p$ groups. Since $K$ is a finitely generated pro-$p$ group we have that $H_0$ is a finitely generated pro-$p$ group and $H_0$ is not finitely presented otherwise $\tilde{H}$ would be a finitely presented pro-$p$ group, a contradiction. Thus $\text{Aut}_0(K)$ is incoherent (in the category of pro-$p$ groups).

Proof of Lemma 3.8 Here we use significantly ideas introduced in [22]. We fix $x_1, x_2$ a generating set of $M$. Define

$$I\text{Aut}(M) = \{ \varphi \in \text{Aut}(M) \mid \varphi \text{ induces on } M/M' \text{ the identity map} \},$$

where $\text{Aut}(M)$ denotes continuous automorphisms of $M$. In fact every abstract automorphism of a finitely generated pro-$p$ group is a continous one. Then there is a short exact sequence of profinite groups

$$1 \to I\text{Aut}(M) \to \text{Aut}(M) \to \text{Aut}(M^{ab}) = GL_2(\mathbb{Z}_p) \to 1.$$
By [22] there is a Bachmut embedding $\beta$ of $IAut(M)$ in $GL_2(\mathbb{Z}_p[[M^a]])$, where $M^a$ is the abelianization of $M$ i.e. the maximal pro-$p$ abelian quotient of $M$. By definition
\[ \beta(p) = (\partial(x_i^p)/\partial x_i), \]
where we use the notations from [22], thus $Aut(M)$ in this proof acts on the right, $\partial(x_i^p)/\partial x_i = \partial/\partial x_j(x_i^p)$ and
\[ \partial/\partial x_j : M \to \mathbb{Z}_p[[M^a]] \]
are the Fox derivatives defined by
\[ \partial/\partial x_j(1) = 0, \partial/\partial x_j(g_1 g_2) = \partial/\partial x_j(g_1) + \bar{g}_1 \partial/\partial x_j(g_2), \partial/\partial x_j(x_i) = \delta_{i,j} \]
the Kroniker symbol, where $\bar{g}_1$ is the image of $g_1 \in M$ in $M^a$. Define $\det(p) = \det(\beta(p))$. By [22]
\[ \det(IAut(M)) = 1 + \Delta =: P \]
is a multiplicative abelian group, where $\Delta$ is the unique maximal ideal of $\mathbb{Z}_p[[M^a]]$, and the $GL_2(\mathbb{Z}_p)$-action via conjugation on the abelianization of $IAut(M)$ induces an action on $\det(IAut(M)) = P$. Then we have a short exact sequence of profinite groups
\[ 1 \to P \to Aut(M)/\text{Ker}(\det) \to GL_2(\mathbb{Z}_p) \to 1. \]
Consider the pro-$p$ group
\[ GL_1^1(\mathbb{Z}_p) = \text{Ker}(GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{F}_p)) \]
Let $Q$ be the maximal pro-$p$ quotient of $P$ that has exponent $p$. Then there is a pro-$p$ subgroup $T$ of $Aut(M)/\text{Ker}(\det)$ and a short exact sequence of pro-$p$ groups
\[ 1 \to P \to T \to GL_2^1(\mathbb{Z}_p) \to 1 \]
and a pro-$p$ quotient $T_0$ of $T$ together with a short exact sequence of pro-$p$ groups
\[ 1 \to Q \to T_0 \to GL_2^1(\mathbb{Z}_p) \to 1. \]
By [23]
\[ P^p \cap (1 + p\Delta) = 1 + p^2\Delta \]
and for $\delta \in \Delta$ using $[\delta]$ for the image of $1 + p\delta$ in $Q$ we have that
\[ [\delta_1][\delta_2] = [\delta_1 + \delta_2]. \]
Thus the multiplicative subgroup of $Q$ generated by $\{[\delta] \mid \delta \in \Delta\}$ could be identified with the additive group that is the image of $\Delta$ mod $p$ i.e. with the augmentation ideal $s_1 \mathbb{F}_p[[s_1, s_2]] + s_2 \mathbb{F}_p[[s_1, s_2]]$ of $\mathbb{F}_p[[s_1, s_2]]$, where $s_i$ is the image of $x_i - 1$ in $\mathbb{Z}_p[[M^a]]$.
Consider now $\varphi_2 \in Aut(M)$ given by
\[ \varphi_2 = \rho^p, \text{ where } \rho(x_1) = x_1 x_2, \rho(x_2) = x_2 \]
and $\varphi_1 \in IAut(M)$ such that
\[ \det(\varphi_1) = 1 + ps_1. \]
Note that $\varphi_1$ is not uniquely determined and that the image of $\varphi_2$ in $GL_2(\mathbb{Z}_p)$ is in $GL_1^1(\mathbb{Z}_p)$. Hence the profinite subgroup $\Gamma$ of $Aut(M)$ generated by $\varphi_1, \varphi_2$ is in fact a pro-$p$ group. Let
\[ \Gamma_0 = \langle \psi_1, \psi_2 \rangle \]
be the image of $\Gamma$ in $T_0$, where $\psi_i$ is the image of $\varphi_i$ in $T_0$. Thus $\Gamma_0$ is a pro-$p$ group.
By [22] Prop. 4.4 for every $\varphi \in IAut(M)$ for $\varphi' = \rho^{-1}\varphi \rho, h' = \det(\beta(\varphi'))$ and $h = \det(\beta(\varphi))$ we have that $h'$ is obtained from $h$ applying the substitution $s_1 \to s_1 + s_2 + s_1 s_2$. 

Recall that by construction $det(\beta(\varphi_1)) = 1 + ps_1$. Then the action of $\psi_2$ on $\psi_1 = [s_1]$ by conjugations is induced by applying the substitution $s_1 \rightarrow s_1 + s_2 + s_1s_2$ exactly $p$-times, thus gives the substitution $s_1 \rightarrow (1 + s_1)(1 + s_2)^p - 1$. Similarly the action of $\psi_2^k$ on $\psi_1 = [s_1]$ by conjugation is induced by applying the substitution $s_1 \rightarrow s_1 + s_2 + s_1s_2$ exactly $pk$-times, thus gives the substitution $s_1 \rightarrow (1 + s_1)(1 + s_2)^{pk} - 1$. As explained above we can move to additive notation and work in the augmentation ideal $s_1F_p[[[s_1,s_2]]] + s_2F_p[[[s_1,s_2]]]$ of $F_p[[[s_1,s_2]]]$. This implies that the normal pro-$p$ subgroup $A$ of $\Gamma_0$ generated by $\psi_1$ can be identified with an additive subgroup of $s_1F_p[[[s_1,s_2]]] + s_2F_p[[[s_1,s_2]]]$ that contains $(1 + s_1)(1 + s_2)^{pk} - 1$ for $k \geq 0$, in particular $A$ is infinite.

Note that $\Gamma_0 \simeq A \rtimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is generated by $\psi_2$. We view $A$ as a $F_p[[t]]$-module via the conjugation action of $\psi_2 = 1 + t$. Furthermore $A$ is a pro-$p$ cyclic $F_p[[t]]$-module, with a generator $\psi_1$. Since every proper $F_p[[t]]$-module quotient of $F_p[[t]]$ is a finite additive group, we deduce that $A \simeq F_p[[t]]$. Then by the example after Theorem 2.2 $\Gamma_0$ is not a finitely presented pro-$p$ group.

Note that the image $W$ of $M \simeq \text{Inn}(M)$ in $T_0$ is inside $Q$ and since $M$ is a finitely generated pro-$p$ group and $Q$ is an abelian pro-$p$ group of finite exponent $p$ then $W$ and consequently $\Gamma_0 \cap W$ are finite. Since $\Gamma_0 \cap W$ is finite $\Gamma_0/(\Gamma_0 \cap W)$ is not a finitely presented pro-$p$ group. Actually examining the structure of $\Gamma_0$ it is easy to see that any finite normal subgroup of $\Gamma_0$ is trivial, in particular $\Gamma_0 \cap W = 1$. Finally $\Gamma_0 \simeq \Gamma_0/(\Gamma_0 \cap W)$ is a metabelian pro-$p$ quotient of a 2-generated pro-$p$ group $H \leq \text{Out}(M)$. This completes the proof of the lemma.

References

[1] Y. Barnea, M. Larsen, A non-abelian free pro-$p$ group is not linear over a local field, J. Algebra 214 (1999), no. 1, 338 - 341
[2] D. E-C. Ben-Ezra, E. Zelmanov, On pro-$2$ identities of $2 \times 2$ linear groups, Trans. Amer. Math. Soc. 374 (2021), no. 6, 4093 - 4128
[3] M. Bestvina, N. Brady, Morse theory and finiteness properties of groups, Inventiones Mathematicae, 129 (1997), 445 - 470
[4] R. Bieri, R. Strebel, Valuations and finitely presented metabelian groups, Proc. London Math. Soc. (3) 41(1980) 439–464.
[5] R. Bieri, W. D. Neumann, R. Strebel, A geometric invariant of discrete groups, Inventiones mathematicae, 90 (1987), 451 - 477
[6] R. Bieri, B. Renz. Valuations on free resolutions and higher geometric invariants of groups. Commentarii Mathematici Helvetici, 3 (1988), v. 63., 464 - 497
[7] J. Dixon, M. P. F. Du Sautoy, A. Mann, and D. Segal, “Analytic Pro-$p$ Groups,” London Math. Soc. Lecture Note Ser., Vol. 157, Cambridge Univ. Press, Cambridge, UK, 1991
[8] M. Feign, M. Handel, Mapping tori of free group automorphisms are coherent, Ann. Math. 149 1061–1077 (1999)
[9] S. Friedl, S. Vidussi, BNS invariants and algebraic fibrations of group extensions, Journal of the Institute of Mathematics of Jussieu, to appear, arXiv:1912.10524
[10] C. McA. Gordon, Artin groups, 3-manifolds and coherence, Bol. Soc. Mat. Mexicana (3) 10 (2004), 193–198
[11] J. D. King, Homological finiteness conditions for pro-$p$ groups, Comm. Algebra 27 (1999), no. 10, 4969 - 4991
[12] J. D. King, A geometric invariant for metabelian pro-$p$ groups, J. London Math. Soc. (2) 60 (1999), no. 1, 83 - 94
[13] D. H. Kochloukova, F. Lima, Homological finiteness properties of fibre products, Q. J. Math. 69 (2018), no. 3, 835 - 854
[14] D. H. Kochloukova, S. Vidussi, Higher dimensional algebraic fiberings of group extensions, preprint, arXiv:2205.05246
[15] D. H. Kochloukova, P. A. Zalesskii, On pro-$p$ analogues of limit groups via extensions of centralizers, Math. Z. 267 (2011), no. 1 - 2, 109 - 128
[16] R. Kropholler, G. Walsh, Incoherence and fibering of many free-by–free groups, Ann. Inst. Fourier (Grenoble) (to appear), arXiv:1910.09601
[17] B. Kuckuck, Subdirect products of groups and the $n - (n + 1) - (n + 2)$ conjecture, Q. J. Math. 65 (2014), 1293 - 1318
[18] A. Lubotzky, Combinatorial group theory for pro-$p$-groups, J. Pure Appl. Algebra 25 (1982), 311 - 325
[19] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields, Grundlehren der Mathematischen Wissenschaften, Vol. 323, Springer-Verlag, Berlin 2000
[20] L. Ribes, On amalgamated products of profinite groups, Math. Z. 123 (1971), 357 - 364
[21] L. Ribes, P. Zalesskii, Protfinite Groups, 2nd ed., Springer, Berlin, 2010
[22] V. A. Roman'kov, Generators for the automorphism groups of free metabelian pro-$p$ groups, Sibirski Matematicheski Zhurnal, Vol. 33, No. 5, 145 - 158, 1992
[23] V. A. Romankov, Infinite generation of groups of automorphisms of free pro-p-groups, Sibirsk. Mat. Zh. 34 (1993), no. 4, 153–159
[24] I. Snopce, P. A. Zalesskii, Subgroup properties of pro-p extensions of centralizers, Selecta Math. (N.S.) 20 (2014), no. 2, 465 - 489
[25] H. Wilton, Hall’s theorem for limit groups, Geom. Funct. Anal. 18 (2008), no. 1, 271 - 303
[26] A. N. Zubkov, Nonrepresentability of a free nonabelian pro-p-group by second-order matrices, Sibirsk. Mat. Zh. 28 (1987), no. 5, 64 - 69

Department of Mathematics, State University of Campinas (UNICAMP), 13083-859, Campinas, SP, Brazil

Email address: desi@unicamp.br