Accelerated methods for weakly-quasi-convex optimization problems

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Abstract
We provide a quick overview of the class of $\alpha$-weakly-quasi-convex problems and its relationships with other problem classes. We show that the previously known Sequential Subspace Optimization method retains its optimal convergence rate when applied to minimization problems with smooth $\alpha$-weakly-quasi-convex objectives. We also show that Nemirovski’s conjugate gradients method of strongly convex minimization achieves its optimal convergence rate under weaker conditions of $\alpha$-weak-quasi-convexity and quadratic growth. Previously known results only capture the special case of 1-weak-quasi-convexity or give convergence rates with worse dependence on the parameter $\alpha$.

Keywords Non-convex minimization · First-order methods · Accelerated methods

1 Introduction

One of the most well-studied optimization problem classes is the class of minimization of convex functions with Lipschitz continuous derivatives. In this case, one of the most important implications of the convexity is its unimodality: the local minima of the object function are also global minima and comprise a convex subset.

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However, there exist conditions weaker than convexity which still imply the
unimodality of the objective function and the convergence of particular first-order
minimization methods to a global minimum. One example of such condition is
star-convexity, which is sufficient to derive the convergence rate of the cubic
regularization of the Newton method (Nesterov and Polyak 2006). Besides, the
recent works (Kleinberg et al. 2018; Zhou et al. 2019) allows us to think that loss
function for neural methods around minima for some modern neural networks is
star convex. Another example is weak-quasi-convexity, which is a further general-
ization of star-convexity.

The example of weakly-quasi-convex but not star-convex function was shown
in Hardt et al. (2016). The authors considered recurrent neural network without
non-linear activation for approximation of sequence generated by unknown model
from the same class. The quadratic loss function was chosen. It was approximated
by so-called idealized risk, and this new loss function is weakly-quasi-convex
under natural assumption about parameters of unknown model. However, in
the mentioned work the problem is solved using the stochastic gradient descent
method, which is not optimal even in the convex case.

Another important example was provided in the recent work (Wang and Wibi-
sono 2023). In this work, authors consider the generalized linear model – com-
bination of linear layer with non-linear activation. The paper contains proof that
square loss minimization problem is quasar-convex for some typical non-linear
activations like Leaky-ReLU, logistic functions and quadratic functions.

Some first-order methods can be applied to star-convex or $\alpha$-weakly-quasi-con-
 vex problems without any modifications. The most basic examples of such meth-
ods are the gradient descent method and the stochastic gradient descent method
(Hardt et al. 2016; Gower et al. 2020). However, at the time of writing, no accel-
 erated first-order methods have been shown to achieve optimal convergence rates
in the general weakly-quasi-convex setting. The purpose of this work is to con-
struct such methods.

In this paper we obtain Theorem 2 devoted to Sequential Subspace Optimiza-
tion (SESOP) method. It indicates that this method has optimal convergence rate
for weakly-quasi-convex functions. The main disadvantage of this method is that at
each iteration we have to solve a possibly difficult auxiliary problem in $\mathbb{R}^3$. Nev-
evertheless, this method works for any smooth weakly-quasi-convex function and does
not require any additional information about function like Lipschitz constant for
gradient.

Another considered method is Nemirovski’s Conjugate Gradient method. It is
well-known that it has optimal convergence rate for strongly convex minimization if
one use restarts technique. Theorem 3 presents a generalized result for the class of $\alpha$
-weakly-quasi-convex functions satisfying the quadratic growth condition and with
Lipschitz continuous gradients.
Since the first version of this paper became available online, some new results have been established. In Nesterov et al. (2020) we present an accelerated first-order method with provable convergence in the weakly-quasi-convex case. However, the convergence rate established for that method has worse dependence on parameter of weakly-quasi-convexity than the methods studied in this work. In Hinder et al. (2020) a worst-case lower bound on the complexity (in terms of the required number of gradient evaluations) of a first order method is established. This bound coincides with the complexity bound established for the SESOP method in this work. The authors also present a novel first-order method with a complexity optimal up to a logarithmic factor.

The last section of this paper is devoted to numerical experiments. Note, that SESOP method does not require any function parameters and it can be considered as method that is adaptive to Lipschitz constant of gradient. Nevertheless, the adaptive to parameter of strong convexity methods are not known now. In the last section, we provide some experiments evidencing that SESOP can work faster than known optimal methods for strong convex functions. We compare its with well-known Nesterov accelerated gradient descent and considered Conjugate Gradients method with restarts. It is important, that these methods require knowledge about parameter of strong convexity. Let us highlight that we demonstrate that SESOP converges better in some special cases not only for iterations but for time too.

2 Preliminaries

Throughout this paper we will be dealing with the global minimization problem

\[ f(x) \to \min_{x \in \mathbb{R}^n}. \]

\[ f : \mathbb{R}^n \to \mathbb{R} \] is assumed to be differentiable and \( L \)-smooth:

\[ \| \nabla f(y) - \nabla f(x) \| \leq L \| y - x \| \quad \forall x, y \in \mathbb{R}^n. \]

We will also assume that the solution set \( \mathcal{X}^* \) is not empty and denote \( f^* = \min_{x \in \mathbb{R}^n} f(x) \). \( \| \cdot \| \) denotes the Euclidean norm, \( \langle \cdot, \cdot \rangle \) denotes the scalar product defined as \( \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \).

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1 The first version appears in 2017 as arXive paper https://arxiv.org/pdf/1710.00797.pdf. We did not publish it since we try to eliminate auxiliary subspace optimization. In 2018 motivated by our arXive paper it was done in Hinder et al. (2020). We lost the interest for publishing. But in the last few years we fulfilled different experiments with SESOP type algorithms and have observed that these algorithms, which use subspace optimization (not line search!), converge for smooth strongly convex problems much faster than standard accelerated methods, which required as input smoothness and strong convexity constants. SESOP does not require any input information. To the best of our knowledge this observation shed a light on the classical problem – to develop accelerated method that is adaptive in strong convexity parameter. The solution could be – to use two or three dimensional subspace optimization at each iteration. Based on this new observation we understand that subspace optimization could be an advantage in practice and decide to publish our arXive paper to draw attention to all these issues...
3 Conditions

In our work we will be studying a generalization of convexity called $\alpha$-weak-quasi-convexity, as defined in Hardt et al. (2016).

**Definition 1** A differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be $\alpha$-weakly-quasi-convex ($\alpha$-WQC) with respect to $x^* \in X^*$ with parameter $\alpha \in (0, 1]$ if for all $x \in \mathbb{R}^n$

$$\alpha(f(x) - f^*) \leq \langle \nabla f(x), x - x^* \rangle.$$ 

Note, that $\alpha$-weak-quasi-convexity guarantees that any local minimizer of $f$ is also a global minimizer. This condition controls the distance between the graph of the function and the tangent plane to the function’s graph constructed at any point. Any convex function attaining its minimum is also 1-WQC, but the converse is generally not true, even for functions of one variable. The function $f(x) = |x|(1 - e^{-|x|})$ is an example of a non-convex 1-WQC function.

To weaken strong convexity we will be using the quadratic growth (QG) condition (Anitescu 2000; Bonnans and Ioffe 1995).

**Definition 2** A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to satisfy the quadratic growth condition if for some $\mu > 0$ and for all $x \in \mathbb{R}^n$

$$f(x) - f^* \geq \frac{\mu}{2} \| x - P(x) \|^2,$$

where $P(x)$ is the projection of $x$ onto $X^*$.

Note that the same condition appears in Nesterov and Polyak (2006) as a property of the solution set. The solution set $X^*$ is called globally non-degenerate if it satisfies the inequality in the definition of QG.

Though this condition shares some similarities with strong convexity, a non-strongly convex function may still satisfy it. Function $f(x) = (\|x\|^2 - 1)^2$ serves as an example. Nesterov and Polyak (2006).

3.1 Relationship with other conditions

Naturally, $\alpha$-weak-quasi-convexity and the QG condition are not the only ways to weaken convexity and strong convexity. What follows is a short list of other similar conditions and their relationships with the ones used in this paper.

Let us define the Polyak–Łojasiewicz condition – another condition used to replace strong convexity in convergence arguments. Note, one of the recent nice application of this condition [11].

**Definition 3** A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to satisfy the Polyak–Łojasiewicz condition if for some $\mu > 0$ and for all $x \in \mathbb{R}^n$
\[ \frac{1}{2} \| \nabla f(x) \|^2 \geq \mu(f(x) - f^*). \]

As shown in Karimi et al. (2016), QG is weaker than the the Polyak–Łojasiewicz condition.

Following (Nesterov and Polyak 2006), we define star-convexity.

**Definition 4** A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called star-convex if for any \( x^* \in X^* \) and any \( x \in \mathbb{R}^n \) we have

\[ f(\lambda x^* + (1 - \lambda)x) \leq \lambda f(x^*) + (1 - \lambda)f(x) \quad \forall x \in \mathbb{R}^n, \forall \lambda \in [0, 1]. \]

This definition is ideologically similar to that of \( \alpha \)-WQC in the way it restricts convexity to the direction towards the solution set \( X^* \). In fact, for differentiable functions and with \( \alpha = 1 \) these two definitions are equivalent.

**Lemma 1** Let function \( f : \mathbb{R}^n \to \mathbb{R} \) be differentiable. Then it is 1-WQC if and only if it is star-convexity.

**Proof** \( \Rightarrow \)

Let us assume that \( f(x) \) is not star-convex:

\[ \exists \lambda \in (0, 1), x \in \mathbb{R}^n \text{ s.t. } f(\lambda x^* + (1 - \lambda)x) - \lambda f(x^*) - (1 - \lambda)f(x) > 0. \]

By maximizing the LHS of the above inequality over \( \lambda \in (0, 1) \) we get that for some \( \lambda^* \in (0, 1) \) and \( z = \lambda^* x^* + (1 - \lambda^*)x \)

\[ \langle \nabla f(z), x - x^* \rangle = f(x) - f(x^*) \]

and

\[ f(z) > \lambda^* f(x^*) + (1 - \lambda^*)f(x). \]

Now we note that \( x - x^* = \frac{z - x^*}{1 - \lambda^*} \) and \( f(z) - f(x^*) > (1 - \lambda^*)(f(x) - f(x^*)). \) This in turn implies that \( \langle \nabla f(z), z - x^* \rangle < f(z) - f(x^*) \), so \( f(x) \) is not 1-WQC.

\( \Leftarrow \) If \( f \) is star-convex, then

\[ f(x^*) - f(x) \geq \frac{f(x + \lambda(x^* - x)) - f(x)}{\lambda} \quad \forall \lambda \in (0, 1), x \in \mathbb{R}^n. \]

Taking the limit \( \lambda \to +0 \), we obtain \( f(x^*) - f(x) \geq \langle \nabla f(x), x^* - x \rangle \), so \( f(x) \) is 1-WQC. \( \square \)

Another condition was recently introduced in Csiba and Richtárik (2017) to generalize convexity. Called the weak PL inequality, in our notation it may be defined as follows.
Definition 5 A function $f(x)$ is said to satisfy the weak PL inequality with respect to $x^* \in X^*$ if for some $\mu > 0$ and for all $x \in \mathbb{R}^n$

$$\sqrt{\mu}(f(x) - f^*) \leq \|\nabla f(x)\|\|x - x^*\|.$$ 

It immediately follows from the Cauchy-Schwarz inequality that the weak PL inequality is weaker than $\alpha$-WQC.

## 4 Gradient descent method

One of the first questions arising whenever a new condition is proposed to replace convexity is whether it’s sufficient to guarantee the convergence of the gradient descent method. Fortunately, this is the case with $\alpha$-WQC.

### Algorithm 1 Gradient descent

Require: $f, x_0, T$

1. for $k = 0$ to $T - 1$ do
2. \hspace{1em} $x_{k+1} \leftarrow x_k - \frac{1}{L} \nabla f(x_k)$
3. end for
4. return $x_T$

It is well known that for a convex $L$-smooth objective $f$ the gradient descent method generates a sequence $\{x_k\}$ such that

$$f(x_k) - f^* = O\left(\frac{LR^2}{k}\right),$$

where $R = \|x_0 - x^*\|$. We will now provide proof of a similar result for $\alpha$-WQC objectives.

**Theorem 1** Let the objective function $f : \mathbb{R}^n \to \mathbb{R}$ be $L$-smooth and $\alpha$-WQC with respect to $x^* \in X^*$. Then the sequence $\{x_k\}$ generated by the gradient descent method satisfies

$$f(x_k) - f^* \leq \frac{LR^2}{\alpha(k + 1)},$$

where $R = \|x_0 - x^*\|$.

**Proof** Any $L$-smooth function $f$ satisfies the following inequality:

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2 \quad \forall x, y \in \mathbb{R}^n.$$ 

Setting $x = x_k, y = x_{k+1}$, we obtain
\[ f(x_{k+1}) - f(x_k) \leq -\frac{1}{2L} \| \nabla f(x_k) \|^2. \]  

(1)

This shows that the sequence \( \{ f(x_k) \}_{k=0}^{\infty} \) is non-increasing. On the other hand, we have

\[
\frac{1}{2} \| x_{k+1} - x^* \|^2 = \frac{1}{2} \| x_k - x^* \|^2 - \frac{1}{L} \langle \nabla f(x_k), x_k - x^* \rangle + \frac{1}{2L^2} \| \nabla f(x_k) \|^2.
\]

Taking into account this equality and the gradient descent guarantee (1), we get

\[
\langle \nabla f(x_k), x_k - x^* \rangle \leq \frac{L}{2} \| x_k - x^* \|^2 - \frac{L}{2} \| x_k - x^* \|^2 + f(x_k) - f(x_{k+1}).
\]

Further, denote \( \varepsilon_k = f(x_k) - f^* \). The above inequality combined with the definition of \( \alpha \)-WQC shows that

\[
\varepsilon_k \leq \frac{1}{\alpha} \langle \nabla f(x_k), x_k - x^* \rangle \leq \frac{1}{\alpha} \left[ \frac{L}{2} \| x_k - x^* \|^2 - \frac{L}{2} \| x_{k+1} - x^* \|^2 + \varepsilon_k - \varepsilon_{k+1} \right].
\]

Summing it up for \( k = 0, \ldots, T \) results in

\[
\sum_{k=0}^{T} \varepsilon_k \leq \frac{1}{\alpha} \left[ \frac{LR^2}{2} + \frac{L}{2} \| x_{T+1} - x^* \|^2 + \varepsilon_0 - \varepsilon_{T+1} \right] \leq \frac{1}{\alpha} \left[ \frac{LR^2}{2} + \varepsilon_0 \right].
\]

By \( L \)-smoothness of \( f \), we have \( \varepsilon_0 \leq \frac{LR^2}{2} \). Since the sequence \( \{ \varepsilon_k \} \) is non-increasing, we have

\[(T + 1)\varepsilon_T \leq \frac{LR^2}{\alpha},\]

which is exactly the statement of the theorem. \( \square \)

5 Subspace optimization

In 2005 Guy Narkiss and Zibulevsky (2005) presented a first-order method with optimal (up to a multiplicative constant independent of the problem) convergence rate for smooth convex problems. In this section, we will demonstrate that this method retains its convergence rate for \( \alpha \)-WQC \( L \)-smooth functions. The proof of this fact only slightly differs from the original proof in Narkiss and Zibulevsky (2005).

Let \( D_k \) be an \( n \times 3 \) matrix (\( n \) is the dimensionality of the objective’s domain), the columns of which are the following vectors:
where

\[ \omega_i = \begin{cases} 
1, & i = 0, \\
\frac{1}{2} + \sqrt{\frac{1}{4} + \omega_{i-1}^2}, & i > 0.
\end{cases} \]

These matrices will determine the subspaces over which we will minimize our objective. With \( D_k \) defined this way, the algorithm takes the following form:

**Algorithm 2 SESOP(\( f, x_0, T \))**

**Require:** The objective function \( f \), initial point \( x_0 \), number of iterations \( T \)

1. for \( k = 0 \) to \( T - 1 \) do
2. \( \tau_k \leftarrow \arg\min_{\tau \in \mathbb{R}^3} f(x_k + D_k \tau) \)
3. \( x_{k+1} \leftarrow x_k + D_k \tau_k \)
4. end for
5. return \( x_T \)

**Theorem 2** Let the objective function \( f \) be \( L \)-smooth and \( \alpha \)-WQC with respect to \( x^* \in \mathcal{X}^* \). Then the sequence \( \{x_k\}_k \) generated by the SESOP method satisfies

\[ f(x_k) - f^* \leq \frac{2LR^2}{\alpha^2 k^2}, \]

where \( R = \|x^* - x_0\| \).

**Proof** Since \( \nabla f(x_k) \) belongs to the set of directions generated by \( D_k \), we can use the following guarantee of gradient descent with fixed step length for \( L \)-smooth functions:

\[ f(x_{k+1}) = \min_{s \in \mathbb{R}^3} f(x_k + D_k s) \leq f\left(x_k - \frac{1}{L} \nabla f(x_k)\right) \leq f(x_k) - \frac{\|\nabla f(x_k)\|^2}{2L}. \quad (2) \]

The definition of \( \alpha \)-WQC may be rewritten as follows:

\[ f(x_k) - f^* \leq \frac{1}{\alpha} \langle \nabla f(x_k), x_k - x^* \rangle. \quad (3) \]
By the construction of $x_k$, we have that $x_k$ is a minimizer of $f$ on the subspace containing the directions $x_k - x_{k-1}$ and $x_{k-1} - x_0$. It means that $\nabla f(x_k) \perp x_k - x_0$, which in turn allows us to write the following inequality instead of (3):

$$f(x_k) - f^* \leq \frac{1}{\alpha}(\nabla f(x_k), x_0 - x^*),$$

Take a weighted sum over $k = 0, \ldots, T - 1$ for some $T \in \mathbb{N}$ with weights $\omega_k$ defined above.

$$\sum_{k=0}^{T-1} \omega_k (f(x_k) - f^*) \leq \frac{1}{\alpha} \left( \sum_{k=0}^{T-1} \omega_k \nabla f(x_k), x_0 - x^* \right) \leq \frac{1}{\alpha} \left\| \sum_{k=0}^{T-1} \omega_k \nabla f(x_k) \right\| R. \quad (4)$$

Since $x_k$ is also a minimizer on the subspace containing $x_{k-1} + \sum_{k=0}^{k-1} \omega_k \nabla f(x_k)$, we have that $\nabla f(x_k) \perp \sum_{k=0}^{k-1} \omega_k \nabla f(x_k)$. Using (2) and the Pythagorean theorem, we get

$$\left\| \sum_{k=0}^{T-1} \omega_k \nabla f(x_k) \right\|^2 = \sum_{k=0}^{T-1} \omega_k^2 \|\nabla f(x_k)\|^2 \leq 2L \sum_{k=0}^{T-1} \omega_k^2 (f(x_k) - f(x_{k+1})).$$

Note that our choice of $\omega_k$ is equivalent to choosing the greatest $\omega_k$ satisfying

$$\omega_k = \begin{cases} 1, & k = 0 \\ \omega_k^2 - \omega_{k-1}^2, & k > 0. \end{cases}$$

Returning to (4) and denoting $\epsilon_k = f(x_k) - f^*$, we get

$$S = \sum_{k=0}^{T-1} \omega_k \epsilon_k \leq \left( \frac{2LR^2}{\alpha^2} \sum_{k=0}^{T-1} \omega_k^2 (\epsilon_k - \epsilon_{k+1}) \right)^{-1/2} =$$

$$= \sqrt{\frac{2LR^2}{\alpha^2} \sqrt{\epsilon_0 \omega_0^2 + \epsilon_1 (\omega_1^2 - \omega_0^2) + \ldots + \epsilon_{T-1} (\omega_{T-1}^2 - \omega_{T-2}^2) - \epsilon_T \omega_{T-1}^2} =$$

$$= \sqrt{\frac{2LR^2}{\alpha^2} \sqrt{\epsilon_0 \omega_0^2 + \sum_{k=1}^{T-1} \epsilon_k \omega_k - \epsilon_T \omega_{T-1}^2} =$$

$$= \sqrt{\frac{2LR^2}{\alpha^2} \sqrt{S - \epsilon_T \omega_{T-1}^2}.}$$

Rewriting that, we get

$$\omega_{T-1}^2 \epsilon_T \leq S - \frac{\alpha^2 S^2}{2LR^2}. \quad (5)$$
Note that $\omega_0 = 1$ and for $k \geq 1$ inequality $\omega_k = \frac{1}{2} + \sqrt{\frac{1}{4} + \omega_{k-1}^2} \geq \frac{1}{2} + \omega_{k-1}$ holds. Consequently, by induction, we obtain $\omega_k \geq \frac{k+1}{2}$. Maximizing the right-hand side of (5) over $S$ and using the fact that $\omega_k \geq \frac{k+1}{2}$, we obtain

$$\varepsilon_T \leq \frac{2LR^2}{\alpha^2 T^2}.$$ 

\Box

6 Nemirovski’s conjugate gradients method

Consider a quadratic minimization problem

$$\phi(x) = \frac{1}{2} \langle x, Ax \rangle + \langle b, x \rangle \rightarrow \min_{x \in \mathbb{R}^n},$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $\mu\|x\| \leq \|Ax\| \leq L\|x\|$. The last pair of inequalities means that $\phi(x)$ is $L$-smooth and strongly convex. Denote by $x^*$ the solution to this problem, by $x_0$ the initial point and by $x_k$ the points generated by the conjugate gradients method. In Nemirovsky and Yudin (1979) the optimal convergence rate of the conjugate gradients method for quadratic objectives was attributed to its following five properties.

1. $\phi(x_{k+1}) \leq \phi(x_k) - \frac{1}{2L} \nabla \phi(x_k)$;
2. $\nabla \phi(x_k) \perp \sum_{i=0}^{k-1} \nabla \phi(x_i)$;
3. $\nabla \phi(x_k) \perp x_k - x_0$;
4. $\langle \nabla \phi(x_k), x^* - x_k \rangle \leq \phi(x^*) - \phi(x_k)$;
5. $\phi(x_0) - \phi(x^*) \geq \frac{2}{\alpha^2} \|x_0 - x^*\|^2$.

These five properties are enough to derive the convergence rate of the conjugate gradients method. A method of strongly convex optimization was then constructed to possess the same five properties.
The gradient descent step in this algorithm leads to the first property for the sequence $\hat{x}_k$. The next two are guaranteed by the subspace minimization step. However, the final two properties are direct consequences of the strong convexity of the objective. Now note that these 2 properties are practically the very definitions of 1-WQC and quadratic growth. This suggests that this algorithm can be generalized to the $\alpha$-WQC setting.

**Theorem 3** Let $f$ be an $L$-smooth and $\alpha$-WQC with respect to $P(x_0)$ (the projection of $x_0$ onto $X^*$) function satisfying the quadratic growth condition with constant $\mu > 0$. Then $CG(f, x_0, T)$ returns $x_T$ such that

$$f(x_T) - f^* \leq \frac{3}{4} (f(x_0) - f^*),$$

where

$$T = \left\lceil \frac{4}{3\alpha} \sqrt{\frac{L}{\mu}} \right\rceil.$$  

**Proof** Denote $x^* = P(x_0)$. Assume $\epsilon_T > \frac{3}{4}\epsilon_0$, which also implies $\epsilon_k > \frac{3}{4}\epsilon_0$ for $k = 1, \ldots, T$. Note, that we have the following inequality for the points $\{x\}_j^T$ generated by SESEOP:

$$f(x_0) \geq f(x_1) \geq f(\hat{x}_1) \geq f(x_2) \geq \cdots \geq f(x_T).$$

Therefore, our assumption implies that $\epsilon_0 \neq 0$.

The gradient descent guarantee

$$f(x_{k+1}) \leq f(\hat{x}_k) - \frac{\|\nabla f(\hat{x}_k)\|^2}{2L}$$

leads us to
\[ \| \nabla f(\hat{x}_k) \|^2 \leq 2L(f(\hat{x}_k) - f(x_{k+1})) \leq 2L(f(x_k) - f(x_{k+1})). \] (6)

Telescoping (6) for \( k = 0, \ldots, T - 1 \), we obtain
\[ \sum_{k=0}^{T-1} \| \nabla f(\hat{x}_k) \|^2 \leq 2L(\epsilon_0 - \epsilon_{T+1}) \leq \frac{L}{2} \epsilon_0. \] (7)

By the definition of \( \hat{x}_k \), \( \nabla f(\hat{x}_k) \perp \hat{x}_k - x_0 \). This allows us to use \( \alpha \)-WQC in the following way:
\[ \langle \nabla f(\hat{x}_k), x^* - x_0 \rangle = \langle \nabla f(\hat{x}_k), x^* - \hat{x}_k \rangle \leq \alpha (f^* - f(\hat{x}_k)) \leq -\frac{3\alpha}{4} \epsilon_0. \] (8)

Now telescoping (8) for \( k = 0, \ldots, T - 1 \) and using the Cauchy-Schwarz inequality, one gets
\[ -\| q_T \| \| x^* - x_0 \| \leq \langle q_T, x^* - x_0 \rangle < -\frac{3T \alpha}{4} \epsilon_0. \]

This inequality will allow us to obtain an upper bound on \( T \), which contradicts the theorem’s statement. All that remains is to get upper bounds on \( \| q_T \| \) and \( x^* - x_0 \).

Again, by definition of \( \hat{x}_k \), \( \nabla f(\hat{x}_k) \perp q_k \). By the Pythagorean theorem and (8),
\[ \| q_T \| = \left( \sum_{k=0}^{T-1} \| \nabla f(\hat{x}_k) \|^2 \right)^{\frac{1}{2}} \leq \sqrt{\frac{L}{2} \epsilon_0}. \]

Quadratic growth, on the other hand, implies the following upper bound
\[ \| x_0 - x^* \| \leq \sqrt{\frac{2}{\mu} \epsilon_0}. \]

Finally,
\[ -\sqrt{\frac{2}{\mu} \epsilon_0} \sqrt{\frac{L}{2} \epsilon_0} < -\frac{3T \alpha}{4} \epsilon_0, \]

or
\[ T < \frac{4}{3\alpha} \sqrt{\frac{L}{\mu}}. \]

This contradicts our choice of \( T = \left\lfloor \frac{4}{3\alpha} \sqrt{\frac{L}{\mu}} \right\rfloor \). \( \square \)

This result shows that if \( f \) were \( \alpha \)-WQC with respect to \( P(x) \ \forall x \in \mathbb{R}^n \), we would be able to apply a restarting technique to this method. To be more precise, under such
circumstances it is possible to achieve an accuracy of $\epsilon$ by performing $\log \frac{1}{\epsilon} \frac{L}{\mu}$ cycles of
\[
\frac{4}{3a} \sqrt{\frac{L}{\mu}}
\] iterations and using the output of each cycle as input for the next one. This means that by using Nemirovski’s conjugate gradients method we may get a point $y$ such that $f(y) - f^* \leq \epsilon$ in $O\left(\frac{1}{a} \sqrt{\frac{L}{\mu}} \log \frac{1}{\epsilon}\right)$ iterations.

Note, that many problems can be reduced to the system on nonlinear equations $g(x) = 0$. This system can be reduced to typically non-convex optimization problem
\[
\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{2} \|g(x)\|^2.
\]
If $\lambda_{\min} \left( \frac{\partial g}{\partial x} \left( \frac{\partial g}{\partial x} \right)^T \right) \geq \mu$, i.e. (see Definition 3)
\[
f(x) - f^* = f(x) \leq \frac{1}{2\mu} \|\nabla f(x)\|^2,
\]
it’s well known (Nesterov and Polyak 2006; Karimi et al. 2016; Gasnikov 2017) that under additional smoothness assumptions standard non-accelerated iterative methods (Gradient Descent, Cubic Regularized Newton method etc.) converge as if $f$ to be $\mu$-strongly convex function. For accelerated methods such results are not known. So we was motivated to find such additional sufficient conditions that guarantee convergence for properly chosen accelerated methods. In this section we observe that such a condition could be $\alpha$-weakly-quasi-convexity of $f$ (see Definition 1).

7 Numerical experiments

In this section, we present the results of numerical experiments for the SESOP and the CG methods. In this paper we’ve proved that these methods converge for weakly-quasi convex functions. But their practical efficiency could be demonstrated also for (strongly) convex problems. In different numerical experiments, we observe that these methods converges significantly faster than accelerated algorithms without line-search. So one of the reason is related with spectral properties of Hessian $\nabla^2 f(x^*)$: different accelerated line-search methods (CG-type methods) have this property (Gasnikov 2017). But we’ve observed that the SESOP algorithm (due to subspace minimization) does not require restarts (like CG-type methods) for strongly convex problems. To the best of our knowledge, the SESOP was not applied earlier for strongly convex problems. We’ve observed that the SESOP is the first such method that (1) is fully adaptive (doesn’t require any input information); (2) allows CG-type acceleration related with $\nabla f(x^*)$ properties; (3) works for both convex problems and strongly convex problems (without restarts).

It is known that non-accelerated methods (such as the Steepest descent) converge for strong convex methods with speed $f(x_k) - f^* \leq C q^k$ for $q = \frac{k-1}{k+1}$ depending on conditional number $\kappa = \frac{L}{\mu}$ and don’t require any input information (such as Lipschitz constant $L$ or constant of strong convexity $\mu$). At the same time, accelerated methods (such as Nesterov method d’Aspremont et al. 2021) converge with speed $f(x_k) - f^* \leq C q_{acc}^k$ for $q_{acc} = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}$, but they require at least $\mu$ as an input.
(or instead of $\mu$ another additional information, like $f^*$Gasnikov 2017; Barré et al. 2020). So, our purpose is to compare the SESOP method with the accelerated methods that require the parameter $\mu$ as an input.

Note, that for quadratic problems standard the Conjugate gradient (CG) method (d’Aspremont et al. 2021) doesn’t require $\mu$ as an input. And so for quadratic problem we don’t have advantages of using the SESOP instead of the CG. That is why for our experiments we’ve chosen the following non-quadratic function popular in many applications (Anderson et al. 1992; Peyré et al. 2019) (soft-max):

$$f(x) = \log\left(\sum_{i=1}^{m} \exp(\langle a_i, x \rangle)\right) + \frac{\mu}{2} \|x\|^2. \quad (9)$$

It is a $\mu$-strongly convex function with $L = (\lambda_{\text{max}}(AA^T) + \mu)$-Lipschitz gradient constant. Moreover, we have conditional number $\kappa = \frac{L}{\mu} = \frac{\lambda_{\text{max}}(AA^T)}{\mu} + 1$ that approaches to 1 when $\mu \to \infty$. We will vary parameter $\mu$ in our experiments.

In all our experiments the dimension $n = 300$, parameter $m = 200$. Components of the matrix $A$ are generated one time independently of a standard normal distribution. The subproblems for the SESOP and the CG on each iteration were being solved through the CVXPY package (Diamond and Boyd 2016) with accuracy $10^{-7}$.

Firstly, we want to compare the practice speed of the SESOP method with estimates $Cq_k$ and $Cq_k^{\text{acc}}$. On the Fig. 1 we show this comparison. The green line is non-accelerated theoretical speed $(f(x_0) - f^*)q_k$, the orange is accelerated $(f(x_0) - f^*)q_k^{\text{acc}}$. We can see that the SESOP method is significantly better than the theoretical speed for accelerated methods. Let’s compare these speeds for different parameters $\mu$ and different conditional numbers correspondingly.

We will stop the SESOP method after approaching function value accuracy $10^{-8}$ and will compare parameter $q_{\text{practice}} = \left(\frac{f_k - f^*}{f_0 - f^*}\right)^{\frac{1}{2}}$, where $f_k = f(x_k)$, with $q$ and $q_{\text{acc}}$ for different conditional number. The results are presented in Fig. 2. We can see that $q_{\text{practice}}$ is significantly less than $q_{\text{acc}}$ for all conditional numbers. It means that the SESOP method converges for this function significantly faster than the theoretical speed for accelerated methods for all conditional numbers. Moreover, the significant gain we can see for the small conditional numbers that are less than $10^2$.

Finally, we want to compare the SESOP method with proposed in this article the CG method (with and without restarts) and Nesterov method in the following form Bubeck et al. (2015):

$$x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k),$$

$$y_{k+1} = x_{k+1} + \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} (x_{k+1} - x_k).$$
The results of this comparison for different conditional number is presented in Figs. 3 and 4. We can see that the SESOP method is better than other methods in both cases. Moreover, in Fig. 3 we can see that the SESOP method converges significantly better than other accelerated methods. So, it approaches accuracy $10^{-7}$ after 150 iterations when the nearest method (CG with restarts) approaches only $10^{-4}$. Also, we can see that the SESOP method and CG converges significantly faster for high conditional number (see Fig. 4).
Besides, the CG method without restarts works significantly worse than CG with restarts in both cases.

Further, we compare time of the considered methods. Again, we take the problem of minimization (9) and compare two methods. The first is Nesterov method with proven linear convergence rate for strong convex function. The second is researched in this article SESOP method. We demonstrate that it can achieve convergence rate in sense of iteration in practice as accelerated method. Nevertheless, it requires solving a low-dimensional problem on each iteration. Further, we demonstrate that SESOP method can demonstrate the same or better time than accelerated methods for some problems with specific structure.

Fig. 3  Comparison of the SESOP method, CG method with optimal restart from Theorem 3 and without restarts and Nesterov method. The conditional number $\kappa \sim 10^4$

Fig. 4  Comparison of the SESOP method, CG method with optimal restart from Theorem 3 and without restarts and Nesterov method. The conditional number $\kappa \sim 400$

Besides, the CG method without restarts works significantly worse than CG with restarts in both cases.
We will use the Ellipsoid method for internal problem. Note, that this method can achieve accuracy $\varepsilon$ through $O\left(\log \frac{1}{\varepsilon}\right)$ operations. Besides, in the work (Kuruzov and Stonyakin 2021) it was shown that SESOP method is robust to inaccurate solution of internal problem on each iteration.

Ellipsoid method needs gradient at a point $\tau \in \mathbb{R}^3$ on each iteration. The gradient is 3-dimensional vector, and it can be calculated efficiently in some special cases. In particular, in the problem the problem (9) we have $\nabla_{\tau} f(x + \tau D) = (AD)^\top \text{softmax}(Ax + AD\tau) + \mu(D^\top x + D^\top D\tau)$, where $\text{softmax}(x) = \frac{\exp(x_i)}{\sum_{j=1}^n \exp(x_j)}$. If vectors $Ax, D^\top x$ and matrices $D^\top D, AD$ then the complexity of one gradient with respect to $\tau$ is $O(m)$. In this case the complexity of each iteration in SESOP method is $O\left(mn + m \log \frac{1}{\varepsilon}\right)$. In the case, when required accuracy is not high ($\log \frac{1}{\varepsilon} \ll n$) it is near to the complexity of one gradient calculation with respect to $x$ - $O(mn)$.

In the table 1, we can see comparison (9) for different $m$ of Nesterov’s method and SESOP with Ellipsoid method for internal problems. We run this method for several values $m$. The stopping criterion for all methods is $\|\nabla f(x)\| \leq 10^{-7}$. We can see that SESOP method has better performance when gradient with respect to $\tau$ can be computed significantly faster than gradient with respect to $x$ in initial space. Moreover, the considered method does not require additional information about parameters of strong convexity or smoothness, unlike Nesterov method.

### Table 1 Comparison of time (s) for Nesterov method and SESOP with Ellipsoid method

| m   | Nesterov | SESOP |
|-----|----------|-------|
| 10  | 3.1      | 0.7   |
| 100 | 3.8      | 3.1   |
| 1000| 18.7     | 19.4  |

8 Discussion

Even though the SESOP and the CG methods presented above are optimal in terms of the amount of iterations required to achieve the desired accuracy, each iteration involves solving a subproblem over $\mathbb{R}^2$ or $\mathbb{R}^3$. However, since all the conditions replacing convexity and strong convexity in our paper involved some global minimizer $x^*$, which may not belong to the domain of any of these subproblems, they may be considered to be general non-convex optimization problems. Not only are such problems much more difficult than convex ones, the above convergence analyses relied on these subproblems to be solved exactly. This means that these theoretically optimal procedures can not be efficiently implemented directly.

The first draft of this paper motivated two different collectives to improve the SESOP, that required to solve difficult subproblems. In the paper (Bu and Mesbahi 2020) authors at each iteration have to solve a subproblem over $\mathbb{R}^1$. In the paper (Hinder et al. 2020) authors at each iteration have an exact solution of a subproblem. Moreover,
in Hinder et al. (2020) it was shown that our estimate from Theorem 2 is tight (i.e. corresponds to the lower bound from Hinder et al. (2020)). So these papers (joint with our paper) developed optimal (up to a numerical constant) methods in terms of gradient oracle calls for the class of $\alpha$-WQC target functions $f$, see Definition 1.

Note, that the considered in this paper classes of non-convex functions that can be optimized almost as convex ones. The searching of such classes of functions is one of the most popular direction in modern non-convex optimization, see recent survey (Danilova et al. 2020) and references there in. Here we just mention additional several classes: all local minima are good (Ge et al. 2016), class of function that can be considered as noisy convex quadratic functions (Bazarova et al. 2022). Note also, that the class of unimodal functions (functions that has unique extremum that is global minima) in terms of convergence in function value is far from the class of convex function (Gasnikov 2017).

Finally, let us discuss application of such methods for distributed optimization. The distributed optimization problem $\min_x \sum_{i=1}^m f_i(x)$ where functions $f_i$ are written on $m$ different nodes, can be reformulated as $\min_{X: WX=0} \sum_{i=1}^m f_i(x_i)$. For this problem we can construct the dual problem $\min_Y [F^*(WY) := \sum_{i=1}^m f_i^*(\{WY\}_i)]$. We can apply SESOP method for this problem. In this case, each iteration of SESOP will have the following form $x_{k+1} = \arg\min_Y F^*(WY_k + WD_k x)$. Note, that after one communication over variable $Y_k$ and matrix of directions $D_k$ the method requires solving only a distributed low-dimensional problem. It can allow to significantly decrease communication complexity when initial problem has high dimension.

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Author contributions SG and AG wrote the main manuscript text and IK provided results of numerical experiments. All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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