Intersections of finite families of finite index subfactors.

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Abstract

We prove that finiteness of the index of the intersection of a finite set of finite index subalgebras in a von Neumann algebra (with small centre) is equivalent to the finite dimensionality of the algebra generated by the conditional expectations onto the subalgebras.

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1 Introduction

Subfactor theory provides an entry point into a world of mathematics and physics containing large parts of conformal field theory, 2-dimensional statistical mechanical models, quantum groups, finitely presented groups and low dimensional topology. We cite two facts attesting to the effectiveness of the subfactor point of view. The first is the discovery of a polynomial invariant of links in [9] and the second is the use of subfactors by Wassermann in [22] to give a rigorous and entirely unitary definition of fusion of loop group representations.

But the very breadth of areas in this world means that one would not expect subfactors to yield the most effective technique for any of the particular topics mentioned above. As things stand for instance it would be foolish to expect subfactor ideas to have a big impact on finite group theory. It is more reasonable to expect subfactors to provide the machinery for theories close to the abstract structure of a subfactor itself. The most obvious such structure is an intermediate subfactor and the first author and Bisch in [2] led to the discovery of an entirely new algebra, in the spirit of the type A Hecke algebra, which they called the ”Fuss-Catalan” algebra. This success is a compelling argument for further work on the lattice of intermediate subfactors of a factor, which had been suggested in papers by Watatani ([23]) and Watatani and Sano ([20]). The main contribution of [20] was to introduce the angle operator between two subfactors as an invariant of a pair of subfactors and calculate several examples. The following question is hinted at but never mentioned explicitly in [20]:

Q: If the spectrum of the angle operator between two finite index subfactors is finite, is the intersection of finite index?

(Note that it follows from the elementary theory of the index that the angle condition is necessary for the intersection to have finite index.) The simplest non-trivial example of a subfactor is given by a pair $\Gamma_0 \subseteq \Gamma$ of discrete groups each having infinite non-identity conjugacy classes. The group von Neumann algebras give the factor and subfactor $M$ and $N$ and $[M : N] = [\Gamma : \Gamma_0]$. Of course the intersection of two finite index subgroups has finite index so this case is rather special. Indeed what is special is that the orthogonal projections (=conditional expectations) onto these subgroup subfactors all commute so that angles are all either 0 or $\pi/2$ so the angle operator has only two elements in its spectrum. Another simple construction of subfactors is as fixed point algebras. If $G$ and $H$ are two finite groups of automorphisms of a $\Pi_1$ factor $M$ then the fixed point algebras $M^G$ and $M^H$ are of finite index and the conditional expectations will not commute so that $Q$ becomes quite relevant. The simplest example is when $G$ and $H$ are are $\mathbb{Z}/2\mathbb{Z}$ so that the group generated by them is dihedral. It is easy to check that the answer to $Q$ is affirmative in this case. This example, and question $Q$, make perfect sense in Galois theory: if $E$ and $F$ are two subfields of a field $K$ with $[K : E]$ and $[K : F]$ finite there are trace maps $Tr_E$ and $Tr_F$ which play the role of the conditional expectations and one may ask if finite rank of the $\mathbb{Z}$-algebra generated by $Tr_E$ and $Tr_F$ guarantees that $[K : E \cap F] < \infty$. George Bergmann ([1]) answered this question in the affirmative in zero characteristic, thus reinforcing $Q$. He further extended it to a finite collection of subfields rather than
just two. Similarly we can extend $Q$ to $Q'$:

$Q'$ If the conditional expectations onto a finite family of finite index subfactors generate a finite dimensional algebra, is the intersection of the subfactors of finite index?

Of course there is no reason for the intersection of subfactors to be a factor so we must use some alternative definition of finite index to properly formulate $Q'$ in detail. Several equivalent definitions are available, the most general being the probabilistic one of Pimsner and Popa in [17]. In the case of finite von Neumann algebras one may use the property that the large algebra is a finitely generated left (or right) module over the small one. For properly infinite algebras we will use the endomorphism theory pioneered by Longo (see [12]). In this paper we will answer $Q'$ in the affirmative. Our proof in the properly infinite case uses the properties of type III factors and has applications to conformal field theories where type III factors appear naturally.

The paper is organized as follows: After introducing some basics of index theory and setting up notations in §2, in §3 we prove $Q'$ in Th. 3.1, and we give two applications in Cor. 3.2 and Th. 3.3. In §4 we first prove Th. 4.4 using two lemmas. An averaging technique in [6] plays a key role in the proof of Lemma 4.8. Th. 4.4 implies Cor. 4.9 which proves the extension of $Q'$. In §4.3, we describe the setting of conformal nets where the assumptions of Cor. 4.9 are naturally satisfied (cf. Lemma 4.12, Cor. 4.13 and Lemma 4.14), and apply Cor. 4.9 to a large class of conformal nets in Cor. 4.16.

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2 Preliminaries

Index theory of subfactors was initiated in [7] in the setting of finite factors. In this paper we will use the following more general definition from [17] and [16]:

**Definition 2.1.** Let $N \subset M$ be an inclusion of von Neumann algebras with a conditional expectation $E : M \rightarrow N$. The (probabilistic) index of $E$, denoted simply by $\text{Ind}E$, is defined by

$$\text{Ind}E = (\sup\{c \geq 0|E(m) \geq cm, \forall m \in M_+\})^{-1}.$$  

The inequality in the above definition will be referred to as Pimsner-Popa inequality. The inclusion $N \subset M$ has finite index if there exists a conditional expectation $E : M \rightarrow N$ such that $\text{Ind}E < \infty$.

Let $N \subset M$ be an inclusion of von Neumann algebras and $E : M \rightarrow N$ a normal faithful conditional expectation. Let $\varphi$ be a normal faithful state on $M$ such that $\varphi = \varphi \cdot E$. Let $\mathcal{H}$ be the Hilbert space of the GNS representation associated to $\varphi$, and let $\Omega \in \mathcal{H}$ be the vector such that $\varphi(m) = \langle m\Omega, \Omega \rangle$, $\forall m \in M$. Let $e$ be the Jones projection from $\mathcal{H}$ to $\overline{N\Omega}$.  

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Definition 2.2. A family of elements \( \{m_j\} \subset M \) satisfying the conditions:

1. \( E(m^*_jm_j) = \delta_{ij} f_j \) where \( f_j \) is a projection in \( N, \forall i, j \);
2. \( \sum_j m_j(e\mathcal{H}) = \mathcal{H} \)

is called an orthonormal basis of \( M \) over \( N \) via \( E \).

Lemma 2.3. Let \( M \subset B(\mathcal{H}) \) be a von Neumann algebra represented standardly on a Hilbert space \( \mathcal{H} \) with a cyclic separating vector \( \Omega \) and denote by \( \varphi \) the vector state on \( M \) with \( \varphi(m) = \langle m\Omega, \Omega \rangle \). Let \( N \subset M \) be a von Neumann algebra and assume that \( E : M \rightarrow N \) is a faithful conditional expectation with \( \varphi(E(m)) = \varphi(m), \forall m \in M \). Let \( e \) be the Jones projection from \( \mathcal{H} \) to \( N\Omega \), and \( J \) the canonical conjugation of \( M \) with respect to \( \Omega \). Denote by \( \langle M, e \rangle \) the von Neumann algebra generated by \( M, e \). Then:

1. \( MeM \) is weakly dense in \( \langle M, e \rangle \);
2. \( \langle M, e \rangle = JN^*J \);
3. \( N \subset M \) has finite index if and only if \( M \subset \langle M, e \rangle \) has finite index;
4. \( N \subset M \) has finite index if and only if \( N \otimes B \subset M \otimes B \) has finite index where \( B \) is a factor.

Proof (1), (2) is contained in [7]. (3) follows by 1.2 of Page 9 of [16]. As for (4), note that if \( E : M \otimes B \rightarrow N \otimes B \) is a conditional expectation which verifies Pimsner-Popa inequality for some constant \( c > 0 \), then \( E(M \otimes 1) = (N \otimes 1) \), hence by restriction \( N \subset M \) has finite index. On the other hand if \( N \subset M \) has finite index and assume that \( E : M \rightarrow N \) is a conditional expectation which verifies Pimsner-Popa inequality for some constant \( c' > 0 \), by Th. 1.1.6 of [16] there is an orthonormal basis \( \{m_i\} \) of \( M \) over \( N \) via \( E \), such that \( \sum_j m_j m^*_j \) is bounded. By definition \( \{m_i \otimes 1\} \) is also an orthonormal basis of \( M \otimes B \) over \( N \otimes B \) via \( E \otimes id \), and hence by Th. 1.1.6 of [16] again \( N \otimes B \subset M \otimes B \) has finite index.

3 Finite type von Neumann algebra case

Let \( M \) be a direct sum of finitely many finite factors with faithful trace \( tr \). Let \( \mathcal{P} \) be a finite set of (unital) finite index subalgebras of \( M \) and for each \( P \in \mathcal{P} \) let \( e_P \) be the projection from \( L^2(M, tr) \) onto \( L^2(P, tr) \). Let \( \mathcal{F} = \{e_P : P \in \mathcal{P}\} \).

Theorem 3.1. If \( N = \bigcap_{P \in \mathcal{P}} P \) then

\[ [M : N] < \infty \iff \dim\{\mathcal{F}'\} < \infty. \]

Proof (only if) If \( [M : N] < \infty \) then \( \dim\{N' \cap \langle M, e_N \rangle\} < \infty \) by reduction to the factor case. But \( \mathcal{F}' \subseteq N' \cap \langle M, e_N \rangle \).

If the key thing to establish is that \( Me_N M \) is contained in a finitely generated left \( M \)-module contained in \( \langle M, e_N \rangle \).

Since \( \mathcal{F}' \) is finite dimensional and \( e_N = \inf_{f \in \mathcal{F}} f \), \( e_N \) is a polynomial in the \( \{f \in \mathcal{F}\} \). Thus \( Me_N M \subseteq \sum_{w \in \mathcal{W}} MwM \) where \( \mathcal{W} \) is a finite set of words on \( \{f \in \mathcal{F}\} \). Now let \( w \in \mathcal{W} \) be \( f_1 f_2 \cdots f_k \). Then for each \( i \), \( Mf_i M \) (which is contained in \( \langle M, e_N \rangle \))
Since the central support of \( p_2 \) as in the previous paragraph \( \ell \) and \( L \) in \( M \) coupling constant, With hypotheses as above, \( F \).

Choose positive numbers \( a \) \( [M, a] \) bound on \( Jp \) Neumann algebra

Since \( F \).

Proof

\[ M_{e_N} M \subseteq \sum_{i=1}^{n} Mr_i \]

for \( r_i \in \langle M, e_N \rangle \).

Now consider \( \langle M, e_N \rangle \). It has a faithful normal trace \( Tr \) so we may consider its action on \( L^2(\langle M, e_N \rangle, Tr) \). By contradiction we suppose \( \langle M, e_N \rangle \) is not a finite von Neumann algebra.

First suppose \( M_* \) is separable, then so is \( L^2(\langle M, e_N \rangle, Tr) \) so by proposition 3.14 of [21] there is a cyclic vector for \( M_{e_N} M \) in \( L^2(\langle M, e_N \rangle, Tr) \), call it \( \Omega \). Then \( \{ r_i \Omega \} \) is a finite set which is cyclic for \( M \). By the coupling constant, \( M' \) is finite (on \( L^2(\langle M, e_N \rangle, Tr) \)) and \( \langle M, e_N \rangle \subseteq JM'J \).

The nonseparable case requires a little more care. We know that \( \langle M, e_N \rangle \) is a semifinite von Neumann algebra whose center, being the same as the center of \( N \), has only countably many mutually orthogonal projections. Thus we may, by proposition 1.40 of [21], find an infinite, \( \sigma \)-finite projection \( p \in \langle M, e_N \rangle \) of central support 1 with \( p = \sum_n p_n \) where \( p_n \) are mutually orthogonal projections with \( Tr(p_n) < \infty \). Choose positive numbers \( a_n \) with \( \sum_n a_n^2 Tr(p_n) < \infty \). Then \( \Omega = \sum_n a_n p_n \) is a vector in \( L^2(\langle M, e_N \rangle, Tr) \) and it is a separating vector for \( p\langle M, e_N \rangle P \) on \( pL^2(\langle M, e_N \rangle, Tr) \).

Since the central support of \( p \) is 1, the action of \( \langle M, e_N \rangle \) on \( L^2(\langle M, e_N \rangle, Tr) \) is faithful so as in the previous paragraph \( \{ r_i \Omega \} \) is a finite set which is cyclic for \( M \). Hence by the coupling constant, \( M' \) is finite (on \( L^2(\langle M, e_N \rangle, Tr) \) and it contains the infinite von Neumann algebra \( Jp\langle M, e_N \rangle J \) — a contradiction. (Note that \( J \) is the canonical involution on \( L^2(\langle M, e_N \rangle, Tr) \).

If we know that \( N \) and \( M \) are \( \Pi_1 \) factors with \( [M : N] < \infty \), the proof of 3.1 gives a bound on \( [M : N] \) in terms of \( \dim A \) (where we set \( A = F'' \)) and the individual \( [M : P] \) for \( P \in \mathcal{P} \). To give an explicit (but rather crude) bound, let \( L = \max_{P \in \mathcal{P}} [M : P] \) and \( \ell \) be the length of the longest word \( w \) in some basis of \( A \) consisting of words on \( F \).

**Corollary 3.2.** With hypotheses as above,

\[ [M : N] \leq L^\ell \dim A. \]

**Proof** Since \( \langle M, e_N \rangle \subseteq \sum_w M_w M \) it suffices to bound \( \dim_M(\tilde{M}w\tilde{M}) \) for each \( w \). But since all indices are finite one may take the Connes tensor product of the \( L^2(MfM) \) instead of the algebraic one and \( \dim_M(\tilde{M}w\tilde{M}) \leq L^{\text{length}(w)} \) follows from the multiplicativity of the \( M \)-dimension under tensor product and the fact that \( MfM \) is just a basic construction for a subactor in \( \mathcal{P} \), so its \( M \)-dimension is bounded by \( L \).
Using the formalism of planar algebras (cf. [8]) the first author has found the slightly better bound (when $|P| = 2$),

$$[M : N] \leq \frac{L^\ell \dim A}{4}.$$ 

In practice the bound seems to be a lot sharper.

Given a subfactor it is always possible to perturb it (for instance conjugating it by a unitary) to obtain a pair of subfactors to which 3.1 can be applied. We have only carried this analysis out in one case, namely the locally trivial subfactors coming from automorphisms which can be thought of as perturbations of the tensor product subfactor. We obtain the following result which does not seem easy to prove by other means.

**Theorem 3.3.** Let $\mathcal{A}$ be a finite set of automorphisms of the finite factor $M$. Then $N^\mathcal{A} = \{x \in N | \alpha(x) = x \forall \alpha \in \mathcal{A}\}$ is of finite index in $N$ iff the spectrum of the operator

$$T = \sum_{\alpha, \beta \in \mathcal{A}} \alpha \beta^{-1}$$

is finite.

**Proof** Let $M$ be the finite direct sum of factors $\oplus_{\alpha \in \mathcal{A}} N$ where we take a copy of $N$ for each $\alpha$, and let $P$ and $Q$ be the two subfactors $P = \{\oplus_{\alpha} x | x \in N\}$ and $Q = \{\oplus_{\alpha} \alpha(x) | x \in N\}$. Then $P \cap Q = \{\oplus_{\alpha} x | x \in N^\mathcal{A}\}$ so that the index of $N^\mathcal{A}$ in $N$ is finite iff the index of $P \cap Q$ in $N$ is finite. Use the obvious trace on $M$ to form conditional expectations. Identify $P$ with $N$ in the obvious way. We have

$$E_P(\oplus_{\alpha} x_\alpha) = \frac{\sum_{\alpha} x_\alpha}{|\mathcal{A}|}$$

and

$$E_Q(\oplus_{\alpha} x_\alpha) = \oplus_{\beta \in \mathcal{A}} \beta(\frac{\sum_{\alpha} \alpha^{-1}(x_\alpha)}{|\mathcal{A}|}).$$

So we see that $E_P E_Q E_P$ is, up to a multiple, our operator $T$. Now the algebra generated by two idempotents $p$ and $q$ is finite dimensional iff the algebra generated by $pq p$ is finite dimensional, which in this context is the same as the finiteness of the spectrum of $pq p$. So by 3.1 we are done.

The result appears to be non-trivial especially if the automorphisms $\alpha$ are inner, in which case it states that the algebra generated by $Ad(u)$ for some finite set of unitaries $u$ is finite dimensional iff $\sum_{u,v} Ad(\overline{uv})$ has finite spectrum. Note that it is NOT true that the algebra generated by $u$ is finite dimensional iff $\sum_{u,v} uv^*$ has finite spectrum.
4 General factor case (with separable predual)

4.1 Preliminaries on Sectors

Let $M$ be a properly infinite factor. The sectors of $M$ are given by

$$\text{Sect}(M) = \text{End}(M)/\text{Inn}(M),$$

namely $\text{Sect}(M)$ is the quotient of the semigroup of the endomorphisms of $M$ modulo the equivalence relation: $\rho, \rho' \in \text{End}(M), \rho \sim \rho'$ iff there is a unitary $u \in M$ such that $\rho'(x) = u\rho(x)u^*$ for all $x \in M$.

$\text{Sect}(M)$ is a $*$-semiring (there is an addition, a product and an involution) equivalent to the Connes correspondences (bimodules) on $M$ up to unitary equivalence. If $\rho$ is an element of $\text{End}(M)$ we shall denote by $[\rho]$ its class in $\text{Sect}(M)$. The operations are:

**Addition** (direct sum): Let $\rho_1, \rho_2, \ldots, \rho_n \in \text{End}(M)$. Choose a non-degenerate $n$-dimensional Hilbert $H$ space of isometries in $M$ and a basis $v_1, \ldots, v_n$ for $H$. Here $H$ is non-degenerate means that the isometries $v_1, \ldots, v_n$ verifies $\sum_i v_i v_i^* = 1$.

Then

$$\rho(x) \equiv \sum_{i=1}^n v_i \rho_i(x) v_i^*, \quad x \in M,$$

is an endomorphism of $M$. The definition of the direct sum endomorphism $\rho$ does not depend on the choice of $H$ or on the basis, up to inner automorphism of $M$, namely $\rho$ is a well-defined sector of $M$.

**Composition** (monoidal product). The usual composition of maps

$$\rho_1 \cdot \rho_2(x) = \rho_1(\rho_2(x)), \quad x \in M,$$

defined on $\text{End}(M)$ passes to the quotient $\text{Sect}(M)$. Let $\rho \in \text{End}(M)$ and $\varepsilon$ be a normal faithful conditional expectation $\varepsilon : M \to \rho(M)$. We define a number $d_\varepsilon \geq 1$ (possibly $\infty$) by:

$$d_\varepsilon^{-2} := \text{Max}\{t \in [0, +\infty) | \varepsilon(m_+) \geq tm_+, \forall m_+ \in M_+\}$$

(Pimsner-Popa inequality in [17]).

We define

$$d(\rho) = \text{Min}_\varepsilon \{d_\varepsilon\},$$

where the minimum is taken over $\varepsilon$ with $d_\varepsilon < \infty$ (otherwise we put $d(\rho) = \infty$). $d(\rho)$ is called the dimension of $\rho$. We say that $\rho$ has finite index if $d(\rho) < \infty$. It is clear from the definition that the dimension of $\rho$ depends only the sector $[\rho]$. The following properties of the dimension can be found in [12].

**Lemma 4.1.** (1) If $[\rho] = [\rho_1] + [\rho_2]$, then $d(\rho) = d(\rho_1) + d(\rho_2)$;

(2) $d(\rho_1 \rho_2) = d(\rho_1)d(\rho_2)$. 


For $\lambda, \mu \in \text{End}(M)$, we will use $\text{Hom}(\lambda, \mu)$ denote the vector space of intertwiners from $\lambda$ to $\mu$, i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a \in M, a\lambda(x) = \mu(x)a$ for any $x \in M$. A sector $\lambda$ is said to be irreducible if the vector space $\text{Hom}(\lambda, \lambda)$ has dimension one.

**Lemma 4.2.** Let $N \subset M$ be a properly infinite subalgebra and assume that there exists a normal faithful conditional expectation $E : M \to N$. Suppose that $M$ is represented standardly on a Hilbert space $H$ with a cyclic separating vector $\Omega$, and the vector state $\omega(m) = \langle m\Omega, \Omega \rangle$ on $M$ verifies $\omega(E(m)) = \omega(m), \forall m \in M$. Let $e$ be the Jones projection from $H$ onto $N\Omega$ and $M_1 := \langle M, e \rangle$ be the von Neumann algebras generated by $M, e$. Then:

1. Let $v \in M_1$ be an isometry with $vv^* = e$. Then for every $m_1 \in M_1$, there is a unique element, denoted by $\gamma(m_1) \in N$, such that $vm_1 = \gamma(m_1)v$, and $\gamma \in \text{End}(M_1)$;

2. We will use the same notation $\gamma$ to denote its restriction to $M$. Then $N \subset M$ has finite index if and only if $\gamma(M) \subset M$ has finite index.

**Proof** The first part is Prop. 2.9 of [14], and the second part follows from (3) of Lemma 2.3.

**Remark 4.3.** The endomorphism $\gamma$ in (2) of Lemma 4.2 is called canonical endomorphism for $N \subset M$ in [14].

### 4.2 General factor case

**Theorem 4.4.** Let $M$ be a factor with separable predual, and let $N \subset M$, $E : M \to N, \mathcal{H}, \Omega$ and $e$ be as in Lemma 2.3. If $e = \sum_{1 \leq i \leq k} m_i R_i$ with $m_i \in M, R_i \in B(\mathcal{H})$ such that $R_i m = \rho_i(m) R_i, \forall m \in M$ and each $\rho_i \in \text{End}(M)$ has finite index, $i = 1, \ldots, k$. Then $N \subset M$ has finite index.

**Remark 4.5.** Note that each $\rho_i \in \text{End}(M)$ has finite index, $i = 1, \ldots, k$ is a necessary condition when $N \subset M$ are properly infinite von Neumann algebras, since by Lemma 4.2 $e = vv^* = \gamma(v^*) v, vm = \gamma(m)v, \forall m \in M$ for any pair $N \subset M$ as in Lemma 4.2, including the case when $N \subset M$ has infinite index. Also note that the conditions on Jones projection $e$ in Theorem 4.4 are similar to that of Theorem 3.1, but neither theorem implies the other.

The proof of this theorem is divided into following steps consisting of two lemmas.

First note that replacing $N, M, E, e, m_i, R_i, \rho_i$ by $N \otimes B, M \otimes B, E \otimes \text{id}, e \otimes 1, m_i \otimes 1, R_i \otimes 1, \rho_i \otimes \text{id}$ respectively if necessary where $B$ is a type III factor with separable dual, by (4) of Lemma 2.3 it is enough to prove the theorem for $N \otimes B \subset M \otimes B$. So we can assume that $M$ is a type III factor, and $N$ is a type III von Neumann algebra. Let $v, \gamma$ be as in Lemma 2.3 such that $e = vv^* = \gamma(v^*) v, vm = \gamma(m)v, \forall m \in M$. Note that since each $\rho_i \in \text{End}(M)$ has finite index, $i = 1, \ldots, k$, $\rho_i$ can be decomposed into sum of finitely many irreducible sectors. Hence we can assume that $e = \sum_{1 \leq i \leq n} \sum_{1 \leq a \leq l(i)} m_{ia} R_{ia}$ with $m_{ia} \in M, R_{ia} \in B(\mathcal{H})$ such that $R_{ia} m = \rho_i(m) R_{ia}, \forall m \in M$ and each $\rho_i \in \text{End}(M)$ is irreducible, has finite index,
1 \leq \alpha \leq l(i) < \infty is a label, and \([\rho_i] \neq [\rho_j], \text{ if } i \neq j, i, j = 1, \ldots, n. Moreover, replacing \{R_{i\alpha}\} by a maximal linearly independent subset over \mathbb{C} if necessary, we can assume that \{R_{i\alpha}\} are linearly independent over \mathbb{C}.

**Definition 4.6.** Let \(V \subset M\) be a vector space consisting of all finite linear combinations of elements \(s_i\) with the property that \(s_i^*v = \sum_{1 \leq \alpha \leq l(i)} c_{\alpha}R_{i\alpha}, c_{\alpha} \in \mathbb{C}, i = 1, \ldots, n.\) Denote by \(W\) the vector space spanned by \(\{R_{i\alpha}\}\).

**Lemma 4.7.** \(V\) is a finite dimensional Hilbert space with natural inner product \(\langle s, t \rangle = t^*s, \) and when \(V\) is not zero we can choose an orthonormal basis \(\{s_{i\beta}\}\) with the property \(s_{i\beta} \in \text{Hom}(\rho_i, \gamma), s_{i\beta}^*s_{l\beta'} = \delta_{i\beta'}\delta_{\beta'}\).

**Proof** Let \(W\) be the vector space spanned by \(\{R_{i\alpha}\}\). Then \(W\) is a vector space with dimension \(\prod_{1 \leq i \leq n} l(i)\). Let \(F : V \to W\) be a conjugate linear map defined by \(F(s) = s^*v\). We claim that \(F\) is one-to-one: if \(s^*v = 0\), then \(s^*vv^* = s^*e = 0\), and so \(s^*\Omega = 0\) which implies that \(s = s^* = 0\) since \(\Omega\) is separating for \(M\). Hence \(\text{dim}V \leq \text{dim}W\).

Assume that \(s_i \in V, s_i^*v = \sum_{1 \leq \alpha \leq l(i)} c_{\alpha}R_{i\alpha}, c_{\alpha} \in \mathbb{C}.\) Then

\[
s_i^*vm = s_i^*\gamma(m)v = \rho_i(m)s_i^*v
\]

by the intertwining property of \(R_{i\alpha}\), hence \(s_i^*\gamma(m)vv^* = \rho_i(m)s_i^*vv^*, \forall m \in M\) and using the separating property of \(\Omega\) for \(M\) again we have \(s_i^* \in \text{Hom}(\gamma, \rho_i).\) Since \(\rho_i\) are irreducible, \(\text{Hom}(\rho_i, \rho_j) = \delta_{ij}\mathbb{C}, i, j = 1, \ldots, n.\) It follows that \(\langle s, t \rangle = t^*s\) is an inner product, and the last part of the lemma follows. \(\Box\)

Let \(s = \sum_{i\beta} s_{i\beta}s_{i\beta}^*\) where \(s_{i\beta}\) is the orthonormal basis as in Lemma 4.7. When \(V\) is zero we set \(s = 0.\) By construction we have \(st = t, \forall t \in V,\) in fact \(s\) is the left support of \(V\) in \(M.\)

**Lemma 4.8.** \(s = 1.\)

**Proof** Let us compute

\[
\gamma(v^*)(1 - s)v = \gamma(v^*)v - \sum_{i\beta} \gamma(v^*)s_{i\beta}s_{i\beta}^*v = \sum_{i\alpha} m'_{i\alpha}R_{i\alpha} - \sum_{i\beta} \gamma(v^*)s_{i\beta}s_{i\beta}^*v
\]

Note that \(\gamma(v^*)s_{i\beta} \in M, s_{i\beta}^*v \in W\) by Definition 4.6. So

\[
\gamma(v^*)(1 - s)v = \sum_{i\alpha} m'_{i\alpha}R_{i\alpha}
\]

for some \(m'_{i\alpha} \in M.\) If the left hand side above is non-zero, then there is at least one \(m'_{i\alpha} \neq 0,\) and we will derive a contradiction by an averaging trick as on Page 46 of [6]. Since \(M\) is a type III factor, \(m'_{i\alpha} \neq 0,\) as on Page 46 of [6] we can find \(a, b \in M\) such that \(E_{\rho_i}(am'_{i\alpha}\rho_i(b)) = 1,\) where \(E_{\rho_i} : M \to \rho_i(M)\) is a normal faithful conditional expectation. Let \(R\) be an injective simple subfactor in \(M\) as on Page 46 of [6] (This is
where the separability of $M$, is used), and let $u$ be an unitary element of $R$. Multiply both sides of equation (1) by $\rho_i(u^*)a$ on the left and $bu$ on the right, and use the intertwining properties of $R_{\alpha\alpha}, s$ and $v$ we get:

$$\rho_i(u^*)a\gamma(v^*)\gamma(b)\gamma(u)(1-s)v = \rho_i(u^*)am_{\alpha\alpha}\rho_i(b)\rho_i(u)R_{\alpha\alpha} + \sum_{(j,\beta)\neq(i,\alpha)}\rho_i(u^*)am_{j\beta}\rho_j(b)\rho_j(u)R_{j\beta}.$$  

Averaging by an invariant mean over the unitary elements of $R$ (which is amenable since $R$ is injective) as on Page 46 of [6] we get

$$t(1-s)v = c_{\alpha\alpha}R_{\alpha\alpha} + \sum_{(j,\beta)\neq(i,\alpha)}c_{j\beta}R_{j\beta},$$

where $t$ is in $M$, and $c_{j\beta} \in M$ satisfies $c_{j\beta}\rho_j(x) = \rho_i(x)c_{j\beta}, \forall x \in R$, and $E_{\rho_i}(c_{\alpha\alpha}) = 1$ since $E_{\rho_i} : M \rightarrow \rho_i(M)$ is normal. Since $\rho_j, \gamma$ is in $\rho_i(M)$, $\gamma(1-s)v = 0$. Multiply this equation on the left by $\gamma(m_1)\gamma(v)$ and on the right by $m_2v^*$, and note that $(1-s)\in \rho_i(M)$, we get $\gamma(m_1m_2)(1-s)v = 0, \forall m_1, m_2 \in M$.

By (1) of Lemma 2.3, 1 in the weak closure of $MeM$, therefore $(1-s)e = 0$. Using the fact that $\Omega$ is separating for $M$ and $e\Omega = \Omega$, we conclude that $s = 1$.

**The end of the proof of Theorem 4.4:**

By Lemma 4.8, $s = \sum_{j\beta}s_{j\beta}s_{j\beta}^* = 1$, and each $s_{j\beta} \in \text{Hom}(\rho_j, \gamma)$ is an isometry. It follows that $[\gamma] = \bigoplus_{1 \leq j \leq n}l(j)'[\rho_j]$ where $0 \leq l(j)' < \infty, j = 1, ..., n$. By Lemma 4.2 $d(\gamma) = \sum_{1 \leq j \leq n}l(j)'d(\rho_j) < \infty$, and by Lemma 4.2 $N \subset M$ has finite index.

**Corollary 4.9.** Let $N_1, ... N_n$ be von Neumann subalgebras of $M$ where $M$ is a factor with separable predual, $N = N_1 \cap N_2 ... \cap N_n$, and each $N_i \subset M$ has finite index, $i = 1, ..., n$. Assume that:

1. There is a $\varphi$ which is a normal faithful state on $M$ invariant under normal faithful condition expectations $E_i : M \rightarrow N_i$ and $E : M \rightarrow N$. Let $e_1, e$ be the corresponding Jones projection in $B(L^2(M, \varphi))$;

2. $e = e_1 \wedge e_2 \wedge ... \wedge e_n$.

Then $N \subset M$ has finite index if and only if $e_1, ..., e_n$ generate a finite dimensional algebra.
Proof The only if part follows from Page 2 of [16] and the assumption that \( M \) is a factor. Let us prove the if part. By replacing \( M, N_i, e_i, \varphi \) by \( M \otimes B, N_i \otimes B, e_i \otimes 1, e \otimes 1, \varphi \otimes \varphi_1 \) respectively if necessary where \( B \) is a type III factor with separable predual and \( \varphi_1 \) is a normal faithful state on \( B \), by (4) of Lemma 2.3 it is enough to prove the if part by assuming that \( M \) is a type III factor, and \( N_i, i = 1, 2, ..., n, N \) are type III von Neumann algebras.

Assume that \( e_1, ..., e_n \) generate a finite dimensional algebra. By assumption (2) there exists a non-commutative polynomial \( f \) such that \( e = f(e_1, ..., e_n) \). Let \( v_i \) be the isometries in \( B(L^2(M, \varphi)) \) as in Lemma 4.2 such that \( v_i v_i^* = e_i, v_i m = \gamma_i(m) v_i, \forall m \in M \). Note that for each \( i = 1, ..., n \) \( \gamma_i \in \text{End}(M) \) has finite index by Lemma 2.3 since \( N_i \subset M \) has finite index by assumption. Using \( e_i = v_i v_i^* = \gamma_i(v_i^*) v_i, \gamma_i(v_i^*) \in M \), and the intertwining properties \( v_i, i = 1, ..., n \), it follows that

\[
e = \sum_{1 \leq j \leq k} m_j R_j
\]

where \( m_j \in M \) and \( R_j \in B(L^2(M, \varphi)) \), \( R_j(m) = \rho_j(m) R_j, \forall m \in M \) and \( \rho_j \in \text{End}(M) \) is a finite compositions of \( \gamma_i, i = 1, ..., n \), and hence of finite index by Lemma 4.1. By Theorem 4.4 the proof is complete.

Remark 4.10. Note that when \( M \) is a type II\(_1\) factor in Cor. 4.9, one can take \( \varphi \) to be the trace on \( M \), then assumption (1) holds trivially, and assumption (2) holds by [19]. So we obtain another proof of Th.3.1 when \( M \) is a type II\(_1\) factor with separable predual, and this in fact partially inspired simpler proofs in section 3 for finite type von Neumann algebra case. We will see a large class of examples in the case when \( M \) is type III where assumptions (1) and (2) hold in Cor. 4.16.

4.3 Applications to conformal nets

4.3.1 Preliminaries on conformal nets

By an interval of the circle we mean an open connected non-empty subset \( I \) of \( S^1 \) such that the interior of its complement \( I' \) is not empty. We denote by \( \mathcal{I} \) the family of all intervals of \( S^1 \).

A net \( \mathcal{A} \) of von Neumann algebras on \( S^1 \) is a map

\[
I \in \mathcal{I} \rightarrow \mathcal{A}(I) \subset B(\mathcal{H})
\]

from \( \mathcal{I} \) to von Neumann algebras on a fixed separable Hilbert space \( \mathcal{H} \) that satisfies:

A. Isotony. If \( I_1 \subset I_2 \) belong to \( \mathcal{I} \), then

\[
\mathcal{A}(I_1) \subset \mathcal{A}(I_2).
\]

If \( E \subset S^1 \) is any region, we shall put \( \mathcal{A}(E) \equiv \bigvee_{E \supset I \in \mathcal{I}} \mathcal{A}(I) \) with \( \mathcal{A}(E) = \mathbb{C} \) if \( E \) has empty interior (the symbol \( \bigvee \) denotes the von Neumann algebra generated).

The net \( \mathcal{A} \) is called local if it satisfies:
B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$ then
\[ [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}, \]
where brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if in addition satisfies the following properties C,D,E,F:

C. *Möbius covariance.* There exists a non-trivial strongly continuous unitary representation $U$ of the Möbius group $\text{Möb}$ (isomorphic to $\text{PSU}(1,1)$) on $\mathcal{H}$ such that
\[ U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gL), \quad g \in \text{Möb}, \; I \in \mathcal{I}. \]

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of $U$ (conformal Hamiltonian), denoted by $L_0$ in the following, is positive.

E. *Existence of the vacuum.* There exists a unit $U$-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and $\Omega$ is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

By the Reeh-Schlieder theorem $\Omega$ is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning
\[ \Delta^I_t = U(\Lambda_I(2\pi t)), \quad J_I = U(r_I). \]
Here $\Lambda_I$ is a canonical one-parameter subgroup of $\text{Möb}$ and $U(r_I)$ is a antiunitary acting geometrically on $\mathcal{A}$ as a reflection $r_I$ on $S^1$.

This implies *Haag duality*:
\[ \mathcal{A}(I)' = \mathcal{A}(I') \quad I \in \mathcal{I}, \]
where $I'$ is the interior of $S^1 \setminus I$.

F. *Irreducibility.* $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff $\Omega$ is the unique $U$-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are III$_1$-factors with separable predual in Connes classification of type III factors.

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation $U$ of $\text{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\text{Möb}$ such that for all $I \in \mathcal{I}$ we have
\[ U(\varphi)\mathcal{A}(I)U(\varphi)^* = \mathcal{A}(\varphi I), \quad \varphi \in \text{Diff}(S^1), \]
\[ U(\varphi)xU(\varphi)^* = x, \quad x \in \mathcal{A}(I), \; \varphi \in \text{Diff}(I'). \]
where $\text{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of $S^1$ and $\text{Diff}(I)$ the subgroup of diffeomorphisms $g$ such that $\varphi(z) = z$ for all $z \in I'$.

Note that if $\varphi \in \text{Diff}(I)$, then $U(\varphi) \in \mathcal{A}(I)$ by $\mathbf{G}$ and Haag duality.

Next we recall some definitions from [11]. Recall that $\mathcal{I}$ denotes the set of intervals of $S^1$. Let $I_1, I_2 \in \mathcal{I}$. We say that $I_1, I_2$ are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of $I$ in $S^1$. When $I_1, I_2$ are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in [26]. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible Möbius covariant net. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in $S^1$ where $I_3, I_4$ are disjoint intervals. Let

$$\mathcal{A}(E) := A(I_1) \vee A(I_2), \quad \hat{\mathcal{A}}(E) := (A(I_3) \vee A(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is split if $A(I_1) \vee A(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $A(I_1) \otimes A(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is strongly additive if $A(I_1) \vee A(I_2) = A(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from $I$.

**Definition 4.11.** [11, 15] $\mathcal{A}$ is said to be completely rational if $\mathcal{A}$ is split, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of $E$ by Prop. 5 of [11]) is denoted by $\mu_\mathcal{A}$ and is called the $\mu$-index of $\mathcal{A}$. If the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is infinity for some $E \in \mathcal{I}_2$, we define the $\mu$-index of $\mathcal{A}$ to be infinity.

Note that, by recent results in [15], every irreducible, split, local conformal net with finite $\mu$-index is automatically strongly additive. Hence we have modified the definition in [11] by dropping the strong additivity requirement in the above definition. Also note that if $\mathcal{A}$ is completely rational, then $\mathcal{A}$ has only finitely many irreducible covariant representations by [11].

Let $\mathcal{A}$ be a conformal net. By a conformal subnet (cf. [13]) we shall mean a map

$$I \in \mathcal{I} \rightarrow \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$, which is isotonic

$$\mathcal{B}(I_1) \subset \mathcal{B}(I_2), I_1 \subset I_2,$$

and Möbius covariant with respect to the the representation $U$, namely

$$U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(g.I)$$

for all $g \in \text{Möb}$ and $I \in \mathcal{I}$. Note that by Lemma 13 of [13] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{A}(I) \rightarrow \mathcal{B}(I)$ such that $E$ preserves the vector state given by the vacuum of $\mathcal{A}$.

**Lemma 4.12.** Let $\mathcal{B} \subset \mathcal{A}$ be a conformal subnet and assume that $U(\text{Diff}(I)) \subset \mathcal{B}(I), \forall I \in \mathcal{I}$, and $\mathcal{B}$ is completely rational. Then:

1. $\mathcal{B} \subset \mathcal{A}$ is irreducible, i.e., $\mathcal{B}(I)' \cap \mathcal{A}(I) = \mathbb{C}, \forall I \in \mathcal{I}$;
2. $\mathcal{B}(I) \subset \mathcal{A}(I)$ has finite index $\forall I \in \mathcal{I}$, and $\mathcal{A}$ is completely rational.
Proof Let $p \in \mathcal{B}(I)' \cap \mathcal{A}(I)$. Since $\mathcal{B}(I') \subset \mathcal{A}(I)$, we have $p \in (\mathcal{B}(I) \vee \mathcal{B}(I'))'$. Since $\mathcal{B}$ is completely rational, by strong additivity $p \in (\vee_{I \in \mathcal{I}} \mathcal{B}(I))'$, and so $p \in (\vee_{I \in \mathcal{I}} U(\text{Diff}(I)))'$ by assumption. Notice that $\text{Diff}(I), I \in \mathcal{I}$ generates $\text{Diff}(S^1)$, and so $p$ commutes with $U(\text{Diff}(S^1))$, and so $p\Omega$ is an eigenvector of the conformal Hamiltonian with eigenvalue 0. By $\mathbf{F}$ it follows that $p\Omega = x\Omega, x \in \mathbb{C}$, therefore $p = x$ since $\Omega$ is separating for $\mathcal{A}(I)$, proving (1). (2) follows from (1), Prop. 2.3 of [10] and Th. 24 of [13]. ■

**Corollary 4.13.** Let $\mathcal{A}$ be a conformal net, and $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ be conformal subnets of $\mathcal{A}$. Let $\mathcal{B}$ be the conformal subnet of $\mathcal{A}$ such that $\mathcal{B}(I) = \mathcal{B}_1(I) \cap \mathcal{B}_2(I) ... \cap \mathcal{B}_n(I), \forall I \in \mathcal{I}$. Assume that:

1. $U(\text{Diff}(I)) \subset \mathcal{B}_i(I), \forall I \in \mathcal{I}, i = 1, ..., n$;
2. Each $\mathcal{B}_i$ is completely rational, $i = 1, ..., n$;
3. $e = e_1 \wedge e_2 \cdot \cdot \cdot \wedge e_n$, where $e$ (resp. $e_i, i = 1, ..., n$) are Jones projections onto $\mathcal{B}(I)\Omega$ (resp. $\mathcal{B}_i(I)\Omega$, $i = 1, ..., n$).

Then $\mathcal{B}$ is completely rational if and only if $e_1, ..., e_n$ generate a finite dimensional algebra.

**Proof** Note that by definition and assumption (1) we have $U(\text{Diff}(I)) \subset \mathcal{B}(I), \forall I \in \mathcal{I}$. By Lemma 4.12 and assumptions (1) and (2) each $\mathcal{B}_i(I) \subset \mathcal{A}(I)$ has finite index, and $\mathcal{A}$ is completely rational. By Lemma 4.12 again we have that $\mathcal{B}$ is completely rational if and only if $\mathcal{B}(I) \subset \mathcal{A}(I)$ has finite index. We note that $\mathcal{A}(I)$ is a type $III_1$ factor with separable predual as stated in $\mathbf{F}$. The assumption (1) of Corollary 4.9 in this case follows by the remark after the definition of conformal subnets. The corollary now follows from Corollary 4.9. ■

A large class of conformal subnets verifying assumptions (1), (2) and (3) of Cor. 4.13 come from cosets and orbifolds (cf. [24] and [25]). Let us recall some definitions. Let $G$ be a simply connected compact Lie group. By Th. 3.2 of [3], the vacuum positive energy representation of the loop group $LG$ (cf. [18]) at level $k$ on a Hilbert space $\mathcal{H}$, denoted by $\pi^0$, gives rise to an irreducible conformal net denoted by $\mathcal{A}_G$ when $k$ is fixed. We will use $\Omega$ to denote the vacuum vector. Note that $\mathcal{A}_G(I) = \pi^0(L_I G)''$ where $L_I G$ are these elements of $LG$ which are equal to identity of $G$ on $I'$. By Th. 13.4.2 of [18] (also cf. [4]) there is a projective unitary representation $U$ of $\text{Diff}(S^1)$ on $\mathcal{H}$ such that $U(\varphi)\pi^0(f)U(\varphi)^* = \pi^0(f \cdot \varphi^{-1})$ for any $f \in LG$.

Let $H \subset G$ be a simply connected Lie subgroup. We define a conformal subnet $\mathcal{A}_{H,G,H}$ of $\mathcal{A}$ by $\mathcal{A}_{H,G,H}(I) := \pi^0(L_I H)'' \vee (\pi^0(L_I H)') \cap \pi^0(L_I G)'$ where $L_I G$ is defined in [18]. Recall from §3 of [24] that $H \subset G$ is cofinite if $\mathcal{A}_{H,G,H}(I) \subset \mathcal{A}_G(I)$ has finite index for some $I \in \mathcal{I}$. Note that Conjecture 2.13 of [24] implies that any such $H \subset G$ is cofinite. See Cor. 3.4 of [24] for a list of inclusions which have been proved to be cofinite.

When $\Gamma$ is a finite subgroup of $G$, we denote by $\mathcal{A}^\Gamma$ a conformal subnet of $\mathcal{A}_G$ such that $\mathcal{A}^\Gamma(I) = \{m \in \mathcal{A}_G(I) | \pi^0(h)m = m\pi^0(h), \forall h \in \Gamma\}$. This is an example of orbifold construction in [25].
Lemma 4.14. Let $H_1, \ldots, H_l$ be simply connected Lie subgroups of $G$, and let $\Gamma_{i+1}, \ldots, \Gamma_n$ be finite subgroups of $G$. Let $B_i = \mathcal{A}_{H_i G H_i}$, $i = 1, \ldots, l$ and $B_j = \mathcal{A}^{\Gamma_j}$, $j = l + 1, \ldots, n$ be conformal subnets of $\mathcal{A}_G$ as above, and let $\mathcal{B}$ be the conformal subnet of $\mathcal{A}$ such that $\mathcal{B}(I) = B_1(I) \cap B_2(I) \cap \cdots \cap B_n(I), \forall I \in \mathcal{I}$. Then:

1. $U(\text{Diff}(I)) \subseteq B_i(I), \forall I \in \mathcal{I}, i = 1, \ldots, n$;
2. $e = e_1 \wedge e_2 \cdots \wedge e_n$, where $e$ (resp. $e_i, i = 1, \ldots, n$) are Jones projections onto $\overline{B(I)\Omega}$ (resp. $\overline{B_i(I)\Omega}, i = 1, \ldots, n$).

Proof Ad (1): When $B_j = \mathcal{A}^{\Gamma_j}$, $j = l + 1, \ldots, n$ (1) holds trivially since $U(\text{Diff}(S^1))$ commutes with $\Gamma_j$. Let us assume that $B_i = \mathcal{A}_{H_i G H_i}$. Let $\varphi \in \text{Diff}(I)$. By Th. 13.4.2 of [18] (also cf. [4]) we have $U(\varphi)\pi^0(f)U(\varphi)^* = \pi^0(f \cdot \varphi^{-1})$ for any $f \in LG$, and by the remark after $G$, $U(\varphi) \in \mathcal{A}_G(I)$. Apply the same argument to $LH$ we conclude that there is a unitary element $\tilde{\varphi} \in \pi^0(L_I H)$ such that $\tilde{\varphi}\pi^0(f)\tilde{\varphi}^* = \pi^0(f \cdot \varphi^{-1}) = U(\varphi)\pi^{0}(f)U(\varphi)^*$ for any $f \in L_I H$. It follows that $\tilde{\varphi}^*U(\varphi) \in \pi^0(L_I H)^\prime \cap \pi^0(L_I G)^\prime$, hence $U(\varphi) \in \mathcal{A}_{H_i G H_i}(I) = B_i$.

Ad (2): Note by definition $\overline{B(I)\Omega} \subseteq \cap_{1 \leq i \leq n} \overline{B(I)\Omega}$, and so it is sufficient to show that for any eigenvector $\psi$ of $L_0$ with eigenvalue $m \geq 0$ in $\cap_{1 \leq i \leq n} \overline{B(I)\Omega}$, $\psi$ is also in $\overline{B(I)\Omega}$. By Reeh-Schlieder theorem it is sufficient to show that $\psi \in \overline{\cup_{I \in \mathcal{I}} B(I)\Omega}$. The proof is essentially contained on Page 22 of [24] as follows: Choose two smooth functions $f_1(z)$ and $f_2(z)$ on the unit circle, with support $f_1 \subseteq I_1 \subseteq \mathcal{I}$, support $f_2 \subseteq I_2 \subseteq \mathcal{I}$ and $f_1 + f_2 = 1$. Then $\psi = V(\psi, z^{-1}\Omega) = V(\psi, z^{-1}f_1)\Omega + V(\psi, z^{-1}f_2)\Omega$ where $V(\psi, \cdot)$ are the smeared vertex operators as defined on Page 11 of [24]. By the same proof as in Prop. 2.11 of [24], $V(\psi, z^{-1}f_1)$ is a closed operator affiliated with with $B_i(I_1), i = 1, \ldots, n$, and so $V(\psi, z^{-1}f_1)$ is a closed operator affiliated with $B(I_1)$, it follows that $V(\psi, z^{-1}f_1)\Omega \in \overline{B(I_1)\Omega}$. Similarly $V(\psi, z^{-1}f_2)\Omega \in \overline{B(I_2)\Omega}$, and we conclude that $\psi \in \overline{\cup_{I \in \mathcal{I}} B(I)\Omega}$.

Remark 4.15. Due to Lemma 4.14, we conjecture that assumption (3) of Cor. 4.13 is always satisfied.

We note that by [26] which is based on [22], $\mathcal{A}_G$ is completely rational if $G = SU(N_1) \times SU(N_2) \times \cdots \times SU(N_k)$, and it has been conjectured that $\mathcal{A}_G$ is completely rational for all $G$.

Corollary 4.16. Let $H_1, \ldots, H_l$ be simply connected Lie subgroups of $G$, and let $\Gamma_{i+1}, \ldots, \Gamma_n$ be finite subgroups of $G$. Let $B_i = \mathcal{A}_{H_i G H_i}$, $i = 1, \ldots, l$ and $B_j = \mathcal{A}^{\Gamma_j}$, $j = l + 1, \ldots, n$ be conformal subnets of $\mathcal{A}_G$ as described before Lemma 4.14, and let $\mathcal{B}$ be the conformal subnet of $\mathcal{A}$ such that $\mathcal{B}(I) = B_1(I) \cap B_2(I) \cap \cdots \cap B_n(I), \forall I \in \mathcal{I}$. Assume that each $H_i \subseteq G, i = 1, 2, \ldots, l$ is cofinite and $\mathcal{A}_G$ is completely rational. Then $\mathcal{B}(I)$ is completely rational if and only if $e_1, \ldots, e_n$ generate a finite dimensional algebra where $e_i, i = 1, \ldots, n$ are Jones projections from $\mathcal{H}$ onto $\overline{B_i(I)\Omega}, i = 1, \ldots, n$.

Proof Note that since we assume that each $H_i \subseteq G, i = 1, 2, \ldots, l$ is cofinite, $\mathcal{B}_i(I) \subseteq \mathcal{A}_G(I)$ has finite index for $i = 1, \ldots, n$. By Th. 24 of [13] each $\mathcal{B}_i, i = 1, \ldots, n$ is completely rational. Hence assumption (2) of Cor. 4.13 is satisfied. Note that
assumptions (1) and (3) of Cor. 4.13 are satisfied thanks to Lemma 4.14. Hence the corollary is proved by Cor.4.13.

We note that the nature of the algebra generated by $e_1,\ldots,e_n$ in Cor. 4.16 can in principle be determined by the representation theory information about pairs $LH_i \subset LG$ and $\Gamma_j \subset LG$. In the case $n = 2$, it is well-known that $e_1,e_2$ generate a finite dimensional algebra if and only if the “angle operator” $e_1 e_2 e_1$ has finite spectrum. It is an interesting question to see if one can obtain new examples of completely rational conformal nets by using Cor. 4.16.

References

[1] G. Bergmann. Private communication(2001).

[2] D.Bisch and V. Jones Algebras associated to intermediate subfactors. Invent. Math. 128 (1997) 89-158.

[3] J. Fröhlich and F. Gabbiani, Operator algebras and Conformal field theory, Comm. Math. Phys., 155, 569-640 (1993).

[4] R. Goodman and N. Wallach, Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle, J. Reine Angew. Math 347 (1984) 69-133.

[5] F. Goodman, P. de la Harpe and V. Jones Coxeter graphs and towers of algebras., MSRI Publications 14. Springer-Verlag (1989).

[6] M. Izumi, R. Longo & S. Popa, A Galois correspondence for compact groups of automorphisms of von Neumann Algebras with a generalization to Kac algebras, J. Funct. Analysis, 155, 25-63 (1998).

[7] V. F. R. Jones, Index for subfactors, Invent. Math. 72 (1983) 1–25.

[8] V. F. R. Jones, Planar Algebras, I, math.OA/0309199.

[9] V.F.R. Jones, A polynomial invariant for knots via von Neumann algebras. Bulletin of the Amer. Math. Soc. 12 (1985), 103-112.

[10] Y. Kawahigashi & R. Longo, Classification of local conformal nets. Case $c < 1$, math-ph/0201015, to appear in Ann. Math.

[11] Y. Kawahigashi, R. Longo & M. Müger, Multi-interval subfactors and modularity of representations in conformal field theory, Commun. Math. Phys. 219 (2001) 631–669.

[12] R. Longo, Minimal index and braided subfactors, J. Funct. Anal. 109 (1992), 98-112.

[13] R. Longo, Conformal subnets and intermediate subfactors, Commun. Math. Phys. 237 n. 1-2 (2003), 7–30.

[14] R. Longo & K.-H. Rehren, Nets of subfactors, Rev. Math. Phys. 7 (1995) 567–597.

[15] R. Longo & F. Xu, Topological sectors and a dichotomy in conformal field theory, math.OA/0309366, Commun. Math. Phys. (in press)

[16] S. Popa, Classification of subfactors and their endomorphisms, CBMS No. 86, 1995.

[17] M. Pimsner, & S. Popa, Entropy and index for subfactors, Ann. Scient. Ec. Norm. Sup. 19 (1986), 57–106.

[18] A. Pressley and G. Segal, “Loop Groups” Oxford University Press 1986.
[19] Christian F. Skau, *Finite subalgebras of a von Neumann Algebra*, J. Funct. Anal. 25 (1977), 211–235.

[20] T. Sano and Y. Watatani, *Angles between two subfactors*. J. Operator Theory 32 (1994), no. 2, 209–241.

[21] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, (1979).

[22] A. Wassermann, *Operator algebras and Conformal field theories III*, Invent. Math. 133 (1998), 467-538.

[23] Y. Watatani, *Lattice structure of intermediate subfactors*, Quantum and non-commutative analysis (Kyoto, 1992), (Kluwer), 331–333.

[24] F. Xu, *Algebraic coset conformal field theories*, Commun. Math. Phys. 211 (2000) 1-43.

[25] F. Xu, *Algebraic orbifold conformal field theories*, Proceedings of National Academy of Sci. USA, Vol. 97, no. 26, 14069-14073.

[26] F. Xu, *Jones-Wassermann subfactors for disconnected intervals*, Commun. Contemp. Math. 2 (2000) 307–347.