On domino insertion and Kazhdan–Lusztig cells in type $B_n$

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Summary. Based on empirical evidence obtained using the CHEVIE computer algebra system, we present a series of conjectures concerning the combinatorial description of the Kazhdan–Lusztig cells for type $B_n$ with unequal parameters. These conjectures form a far-reaching extension of the results of Bonnafé and Iancu obtained earlier in the so-called “asymptotic case”. We give some partial results in support of our conjectures.

1 Introduction and the main conjectures

Let $W$ be a Coxeter group, $\Gamma$ be a totally ordered abelian group and $L: W \to \Gamma_{\geq 0}$ be a weight function, in the sense of Lusztig [30, §3.1]. This gives rise to various pre-order relations on $W$, usually denoted by $\leq_L$, $\leq_R$ and $\leq_{LR}$. Let $\sim_L$, $\sim_R$ and $\sim_{LR}$ be the corresponding equivalence relations. The equivalence classes are called the left, right and two-sided cells of $W$, respectively. They were first defined by Kazhdan and Lusztig [26] in the case where $L$ is the length function on $W$ (the “equal parameter case”), and by Lusztig [29] in general. They play a fundamental role, for example, in the representation theory of finite or $p$-adic groups of Lie type; see the survey in [30, Chap. 0].

Our aim is to understand the dependence of the Kazhdan–Lusztig cells on the weight function $L$. We shall be interested in the case where $W$ is a finite Coxeter group. Then unequal parameters can only arise in type $I_2(m)$ (dihedral), $F_4$ or $B_n$. Now types $I_2(m)$ and $F_4$ can be dealt with by computational methods; see [14]. Thus, as far as finite Coxeter groups are concerned, the real issue is to study type $B_n$ with unequal parameters. And in any case, this is the most important case with respect to applications to finite classical groups (unitary, symplectic, and orthogonal). Quite recently, new connections be-
It is a conjecture that between Kazhdan–Lusztig cells in type $B_n$ and the theory of rational Cherednik algebras appeared in the work of Gordon and Martino [22].

The purpose of this paper is to present a series of conjectures which would completely and explicitly determine the Kazhdan–Lusztig cells in type $B_n$ for any positive weight function $L$. We will also establish some relative results in support of these conjectures. So let now $W = W_n$ be a Coxeter group of type $B_n$, with generating set $S_n = \{ t, s_1, \ldots, s_{n-1} \}$ and Dynkin diagram as given below; the “weights” $a, b \in F_{>0}$ attached to the generators of $W_n$ uniquely determine a weight function $L = L_{a,b}$ on $W_n$.

If $b$ is “large” with respect to $a$, more precisely, if $b > (n - 1)a$, then we are in the “asymptotic case” studied in [5] (see also [3, Prop. 5.1 and Cor. 5.2] for the determination of the exact bound). In general, we expect that the combinatorics governing the cells in type $B_n$ are provided by the “domino insertion of a signed permutation into a 2-core”; see [27], [28], [33] (see also §3). Having fixed $r \geq 0$, let $\delta_r$ be the partition with parts $(r, r - 1, \ldots, 0)$ (a 2-core). Let $P_r(n)$ be the set of partitions $\lambda \vdash (1^2 r (r + 1) + 2n)$ such that $\lambda$ has 2-core $\delta_r$. Then the domino insertion with respect to $\delta_r$ gives a bijection from $W_n$ onto the set of all pairs of standard domino tableaux of the same shape $\lambda \in P_r(n)$. We write this bijection as $w \rightarrow (P^r(w), Q^r(w))$; see [27, §2] for a detailed description.

The following conjectures have been verified for $n \leq 6$ by explicit computation using CHEVIE [18] and the program Coxeter developed by du Cloux [7].

**Conjecture A.** Assume that $\Gamma = \mathbb{Z}$, $a = 2$ and $b = 2r + 1$ where $r \geq 0$. Then the following hold.

(a) $w, w' \in W_n$ lie in the same Kazhdan–Lusztig left cell if and only if $Q^r(w) = Q^r(w')$.

(b) $w, w' \in W_n$ lie in the same Kazhdan–Lusztig right cell if and only if $P^r(w) = P^r(w')$.

(c) $w, w' \in W_n$ lie in the same Kazhdan–Lusztig two-sided cell if and only if all of $P^r(w)$, $Q^r(w)$, $P^r(w')$, $Q^r(w')$ have the same shape.

**Remark 1.1.** The 2-core $\delta_r$, the set of partitions $P_r(n)$, and the parameters $a = 2$, $b = 2r + 1$ (where $\Gamma = \mathbb{Z}$) naturally arise in the representation theory of the finite unitary groups $GU_N(q)$, where $N = \frac{1}{2}r(r + 1) + 2n$. The Hecke algebra of type $B_n$ with parameters $q^{2r+1}, q^2, \ldots, q^2$ appears as the endomorphism algebra of a certain induced cuspidal representation. The irreducible representations of this endomorphism algebra parametrize the unipotent representations of $GU_N(q)$ indexed by partitions in $P_r(n)$; see [6, §13.9]. In this
case, Conjecture A(c) is somewhat more precise than [30, Conj. 25.3 (b)] (see §3.3 for more details).

Conjecture A. Let \( r \geq 0 \) and assume that \( a, b \) are any elements of \( \Gamma_{>0} \) such that \( ra < b < (r+1)a \). Then the statements in Conjecture A still hold. That is, the Kazhdan-Lusztig (left, right, two-sided) cells for this choice of parameters coincide with those obtained for the special values \( a = 2 \) and \( b = 2r+1 \) (where \( \Gamma = \mathbb{Z} \)).

Remark 1.2. Assume we are in the setting of Conjecture A or A\(^+\). If \( w \in W_n \), let \( \lambda(w) \in \mathcal{P}_r(n) \) denote the shape of \( P_r(w) \) (or \( Q_r(w) \)). Let \( \preceq \) denote the dominance order on partitions. The following property of the pre-order \( \preceq_{LR} \) has been checked for \( n \leq 4 \) by using CHEVIE [18]:

\[
\lambda(w) \preceq \lambda(w') \quad \text{if and only if} \quad w \preceq_{LR} w'
\]

Remark 1.3. Assume that the statement concerning the left cells in Conjecture A (or A\(^+\)) is true. Since \( P_r(w^{-1}) = Q_r(w) \) (see for instance [27, Lemma 7]), this would imply that the statement concerning the right cells is also true. However, it is not clear that the partition into two-sided cells easily follows from the knowledge of the partitions into left and right cells. Indeed, it is conjectured (but not proved in general) that the relation \( \sim_{LR} \) is generated by \( \sim_L \) and \( \sim_R \). This would follow from Lusztig’s Conjectures (P4), (P9), (P10) and (P11).

Remark 1.4. If \( b > (n-1)a \) ("asymptotic case"), then domino insertion is equivalent to the generalized Robinson–Schensted correspondence in [5, §3] (see Theorem 3.13). Thus, Conjectures A and A\(^+\) holds in this case [5, Th. 7.7], [3, Cor. 3.6 and Rem. 3.7]. Also, the refinement (c\(^+\)) proposed in Remark 1.2 holds in this case if \( w \) and \( w' \) have the same \( t \)-length [3, Th. 3.5 and Rem. 3.7] (the \( t \)-length of an element \( w \in W_n \) is the number of occurrences of \( t \) in a reduced decomposition of \( w \)).

Remark 1.5. Assume that Conjectures A and A\(^+\) hold. Then we also conjecture that the Kazhdan–Lusztig basis of the Iwahori–Hecke algebra \( H_n \) associated to \( W_n \) and the weight function \( L_{a,b} \) is a cellular basis in the sense of Graham–Lehrer [23]. See Subsection 2.2 for a more precise statement and applications to the representation theory of non-semisimple specialisations of \( H_n \).

We define the equivalence relation \( \simeq_r \) on elements of \( W_n \) as follows: we write \( w \simeq_r w' \) if and only if \( Q_r(w) = Q_r(w') \). An equivalence class for the relation \( \simeq_r \) is called a left \( r \)-cell. In other words, left \( r \)-cells are the fibers of the map \( Q_r \). Similarly, we define right \( r \)-cells as the fibers of the map \( P_r \) and two-sided \( r \)-cell as the fibers of the map \( \lambda : W_n \to \mathcal{P}_r(n) \).

Conjectures A and A\(^+\) deal with the Kazhdan–Lusztig cells for parameters such that \( ra < b < (r+1)a \). The next conjecture is concerned with the Kazhdan–Lusztig cells whenever \( b \in \mathbb{N}^* a \).
**Conjecture B.** Assume that $b = ra$ for some $r \geq 1$. Then the Kazhdan–Lusztig left (resp. right, resp. two-sided) cells of $W_n$ are the smallest subsets of $W_n$ which are at the same time unions of left (resp. right, resp. two-sided) $(r-1)$-cells and left (resp. right, resp. two-sided) $r$-cells.

We will give a combinatorially more precise version of Conjecture B in §4.

**Remark 1.6.** (a) If $r \geq n$ then, since the left $r$-cells and the left $(r-1)$-cells coincide, then the Conjecture B holds (“asymptotic case”, see Remark 1.4).

(b) There is one case which is not covered by Conjectures A, A$^+$ or B: it is when $b > ra$ for every $r \in \mathbb{N}$. But this case is exactly the case which is dealt with in [5, Th. 7.7] (and [3, Cor. 3.6] for the determination of two-sided cells) and it leads to the same partition into left and two-sided cells as the case where $(a, b) = (2, 2n-1)$ for instance (see Remark 1.4).

(c) The fundamental difference between the cases where $b \in \{a, 2a, \ldots, (n-1)a\}$ and $b \notin \{a, 2a, \ldots, (n-1)a\}$ is already appearant in [30, Chap. 22], where the “constructible representations” are considered. Conjecturally, these are precisely the representations given by the various left cells of $W$. By [30, Chap. 22], the constructible representations are all irreducible if and only if $b \notin \{a, 2a, \ldots, (n-1)a\}$.

(d) Again, in Conjecture B, the statement concerning left cells is equivalent to the statement concerning right cells. However, the statement concerning two-sided cells would then follow if one could prove that the relation $\sim_{LR}$ is generated by the relations $\sim_L$ and $\sim_R$.

(e) Conjectures A$^+$ and B are consistent with analogous results for type $F_4$ (see [14] as far as Conjecture A$^+$ is concerned; Geck also checked that an analogue of Conjecture B holds in type $F_4$).

In Section 2, we will discuss representation-theoretic issues related to Conjecture A. In Sections 3 and 4, we will present a number of partial results in support of our conjectures.

## 2 Leading matrix coefficients and cellular bases

Let $W$ be a finite Coxeter group with generating set $S$. Let $\Gamma$ be a totally ordered abelian group. Let $L: W \to \Gamma$ be a weight function in the sense of Lusztig [30, §3.1]. Thus, we have $L(ww') = L(w) + L(w')$ for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$ where $l: W \to \mathbb{N}$ is the usual length function with respect to $S$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$). Let $A = \mathbb{Z}[\Gamma]$ be the group ring of $\Gamma$. It will be denoted exponentially: in other words, $A = \bigoplus_{\gamma \in \Gamma} \mathbb{Z}v^\gamma$ and $v^\gamma v^{\gamma'} = v^{\gamma+\gamma'}$. If $\gamma_0 \in \Gamma$, let $A_{>\gamma_0} = \bigoplus_{\gamma > \gamma_0} \mathbb{Z}v^\gamma$. We define similarly $A_{\geq \gamma_0}$, $A_{<\gamma_0}$ and $A_{\leq \gamma_0}$.

Let $H = H_A(W, S, L)$ be the corresponding Iwahori–Hecke algebra. Then $H$ is free over $A$ with basis $(T_w)_{w \in W}$; the multiplication is given by the rule...
$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ T_{sw} + (u^{L(s)} - v^{-L(s)}) T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$

where $w \in W$ and $s \in S$. For basic properties of $W$ and $H$, we refer to [21].

2.1 Leading matrix coefficients

We now recall the basic facts concerning the leading matrix coefficients introduced in [12]. First, since $\Gamma$ is an ordered group, the ring $A$ is integral. Similarly, the group algebra $R[\Gamma]$ is integral; we denote by $K = R(\Gamma)$ its field of fractions.

Extending scalars from $A$ to the field $K$, we obtain a finite dimensional $K$-algebra $H_K = K \otimes_A H$, with basis $(T_w)_{w \in W}$. It is well-known that $H_K$ is split semisimple and abstractly isomorphic to the group algebra of $W$ over $K$; see, for example, [19, Remark 3.1]. Let Irr$(H_K)$ be the set of irreducible characters of $H_K$. We write this set in the form

$Irr(H_K) = \{ \chi_\lambda \mid \lambda \in \Lambda \}$,

where $\Lambda$ is some finite indexing set. If $\lambda \in \Lambda$, we denote by $d_\lambda$ the degree of $\chi_\lambda$. We have a symmetrizing trace $\tau: H_K \to K$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$; see [21, §8.1]. The fact that $H_K$ is split semisimple yields that

$$\tau = \sum_{\lambda \in \Lambda} \frac{1}{c_\lambda} \chi_\lambda \quad \text{where } 0 \neq c_\lambda \in K.$$ 

The elements $c_\lambda$ are called the Schur elements. There is a unique $a(\lambda) \in \Gamma_{\geq 0}$ and a positive real number $r_\lambda$ such that

$$c_\lambda \in r_\lambda v^{-2a(\lambda)} + A_{\geq -2a(\lambda)};$$

see [12, Def. 3.3]. The number $a(\lambda)$ is called the $a$-invariant of $\chi_\lambda$. Using the orthogonal representations defined in [12, §4], we obtain the leading matrix coefficients $c_i^{j, \lambda} \in \mathbb{R}$ for $\lambda \in \Lambda$ and $1 \leq i, j \leq d_\lambda$. See [12, §4] for further general results concerning these coefficients.

Following [19, Def. 3.3], we say that

- $H$ is integral if $c_i^{j, \lambda} \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $1 \leq i, j \leq d_\lambda$;
- $H$ is normalized if $r_\lambda = 1$ for all $\lambda \in \Lambda$.

The relevance of these notions is given by the following result.

Theorem 2.1 (See [12, §4] and [19, Lemma 3.8]). Assume that $H$ is integral and normalized.

(a) We have $c_i^{j, \lambda} \in \{0, \pm 1\}$ for all $w \in W$, $\lambda \in \Lambda$ and $1 \leq i, j \leq d_\lambda$. 
(b) For any $\lambda \in \Lambda$ and $1 \leq i, j \leq d_\lambda$, there exists a unique $w \in W$ such that $c_{w,\lambda}^{ij} \neq 0$; we denote that element by $w = w_\lambda(i, j)$. The correspondence $(\lambda, i, j) \mapsto w_\lambda(i, j)$ defines a bijective map
\[
\{(\lambda, i, j) \mid \lambda \in \Lambda, 1 \leq i, j \leq d_\lambda\} \rightarrow W.
\]

(c) For a fixed $\lambda \in \Lambda$ and $1 \leq k \leq d_\lambda$,
(i) $\Sigma_{\lambda,k} := \{w_\lambda(i, k) \mid 1 \leq i \leq d_\lambda\}$ is contained in a left cell;
(ii) $\mathcal{R}_{\lambda,k} := \{w_\lambda(k, j) \mid 1 \leq j \leq d_\lambda\}$ is contained in a right cell.

Remark 2.2. Assume that Lusztig’s conjectures (P1)–(P15) in [30, §14.2] hold for $\mathcal{H}$. Assume also that $\mathcal{H}$ is normalized and integral. Combining [15, Corollary 4.8] and [19, Lemma 3.10], we conclude that the sets $\Sigma_{\lambda,k}$ and $\mathcal{R}_{\lambda,k}$ are precisely the left cells and the right cells of $W$, respectively.

Now let $W = W_n$ be the Coxeter group of type $B_n$ as in Section 1; let $\mathcal{H}_n$ be the associated Iwahori–Hecke algebra with respect to the weight function $L = L_{a,b}$ where $a, b \geq 0$.

**Proposition 2.3.** Assume that $a > 0$ and $b \notin \{a, 2a, \ldots, (n - 1)a\}$. Then $\mathcal{H}_n$ is integral and normalized.

**Proof.** The fact that $\mathcal{H}_n$ is normalized follows from the explicit description of $a(\lambda)$ in [30, Prop. 22.14]. To show that $\mathcal{H}_n$ is integral we follow once more the discussion in [19, Example 3.6] where we showed that $\mathcal{H}_n$ is integral if $b > (n - 1)a$. So we may, and we will, assume from now on that $b < (n - 1)a$. Since $b \notin \{a, 2a, \ldots, (n - 1)a\}$, there exists a unique $r \geq 0$ such that $ra < b < (r + 1)a$. Given $\lambda \in \Lambda$, let $S^\lambda$ be the Specht module constructed by Dipper–James–Murphy [8]. There is a non-degenerate $\mathcal{H}_n$-invariant bilinear form $\langle \ , \ \rangle_\lambda$ on $S^\lambda$. Let $\{f_i \mid t \in T_\lambda\}$ be the orthogonal basis constructed in [8, Theorem 8.11], where $T_\lambda$ is the set of all standard bitableaux of shape $\lambda$.

Using the recursion formula in [9, Prop. 3.8], it is straightforward to show that, for each basis element $f_i$, there exist integers $s_{ti}, a_{ti}, b_{ij}, c_{tk}, d_{it} \in \mathbb{Z}$ such that $a_{ti} \geq 0$, $b_{ij} \geq 0$, and
\[
\langle f_i, f_i \rangle_\lambda = v^{2s_{ti}} \prod_{j}(1 + v^{2a_{ti} + \cdots + v^{2d_{tj}a}}) \prod_{k}(1 + v^{2(b+c_{tk}a)}) \prod_{l}(1 + v^{2(b+d_{it}a)}).
\]

In [19, Example 3.6], we noticed that we also have $b + c_{tk}a > 0$ and $b + d_{it}a > 0$ if $b > (n - 1)a$, and this allowed us to deduce that $\mathcal{H}_n$ is integral in that case. Now, if we only assume that $ra < b < (r + 1)a$, then $b + c_{tk}a$ and $b + d_{it}a$ will no longer be strictly positive, but at least we know that they cannot be zero. Thus, there exist $h_{ti}, h'_{ti}, m_{tk}, m'_{ti} \in \mathbb{Z}$ such that
\[
\prod_{k}(1 + v^{2(b+c_{tk}a)}) = v^{2h_{ti}} \prod_{k}(1 + v^{2m_{tk}}) \quad \text{where } m_{tk} > 0,
\]
\[
\prod_{l}(1 + v^{2(b+d_{it}a)}) = v^{2h'_{ti}} \prod_{l}(1 + v^{2m'_{it}}) \quad \text{where } m'_{ti} > 0.
\]
Hence, setting
\[ \tilde{f}_t := v^{-s_a - h_t + h'_t} \cdot \left( \prod_j (1 + v^{2a + \cdots + v^{2h_{ij}}} \cdot (1 + v^{2a})) \cdot (1 + v^{2b_t a}) \right) \cdot f_t, \]
we obtain \( \langle \tilde{f}_t, \tilde{f}_t \rangle_{\lambda} \in 1 + v \mathbb{Z}[v] \) for all \( t \). We can then proceed exactly as in [19, Example 3.6] to conclude that \( H_n \) is integral. \[ \square \]

The above result, in combination with Theorem 2.1, provides a first approximation to the left and right cells of \( W_n \). By Remark 2.2, the sets \( L_{\lambda,k} \) and \( R_{\lambda,k} \) should be precisely the left and right cells, respectively. In this context, Conjecture A would give an explicit combinatorial description of the correspondence \( (\lambda, i, j) \mapsto w_\lambda(i, j) \).

2.2 Cellular bases

Let us assume that we are in the setting of Conjecture A. As announced in Remark 1.5, we believe that then the Kazhdan–Lusztig basis of \( H_n \) will be cellular in the sense of Graham–Lehrer [23]. To state this more precisely, we have to introduce some further notation. Let \( (C_w)_{w \in W} \) be the Kazhdan–Lusztig basis of \( H_n \); the element \( C_w \) is uniquely determined by the conditions that \( C_w = C_w \) and \( C_w \equiv T_w \mod H_{n,>0} \), where \( H_{n,>0} = \sum_{w \in W_n} A_{>0} T_w \) and the bar denotes the ring involution defined in [30, Lemma 4.2]. Furthermore, let \( * : H_n \rightarrow H_n \) be the unique anti-automorphism such that \( T_w^* = T_{w^{-1}} \) for all \( w \in W_n \). We also have \( C_w^* = C_{w^{-1}} \) for any \( w \in W_n \).

Now assume that \( a > 0 \) and \( b \notin \{a, 2a, \ldots, (n-1)a\} \). If \( b < (n-1)a \), let \( r \geq 0 \) be such that \( ra < b < (r+1)a \). If \( b > (n-1)a \), let \( r \) be any natural number greater than or equal to \( n-1 \).

We set \( A_r := P_r(n) \) and consider the partial order on \( A_r \) given by the dominance order \( \trianglelefteq \) on partitions. For \( \lambda \in A_r \), let \( M_r(\lambda) \) denote the set of standard domino tableaux of shape \( \lambda \). If \( (S, T) \in M_r(\lambda) \times M_r(\lambda) \), let \( C_r(S, T) := C_w \) where \( (S, T) = (P^r(w), Q^r(w)) \).

**Conjecture C.** With the above notation, \( (A_r, M_r, C_r, *) \) is a cell datum in the sense of Graham–Lehrer [23, Def. 1.1].

The existence of a cellular structure has strong representation-theoretic applications. For the remainder of this section, assume that Conjecture C is true. Let \( \theta : A \rightarrow k \) be a ring homomorphism into a field \( k \). Extending scalars from \( A \) to \( k \), we obtain a \( k \)-algebra \( H_{n,k} := k \otimes_A H_n \) which will no longer be semisimple in general. The theory of cellular algebras [23] provides, for every \( \lambda \in A_r \), a cell module \( S^\lambda \) of \( H_{n,k} \), endowed with an \( H_{n,k} \)-equivariant bilinear form \( \phi^\lambda \). We set
\( D^\lambda := S^\lambda / \text{rad} \phi^\lambda \) for every \( \lambda \in \Lambda_r \).

Let \( \Lambda_r^\circ := \{ D^\lambda \mid \lambda \in \Lambda_r \text{ such that } \phi^\lambda \neq 0 \} \). Then we have
\[
\text{Irr}(H_{n,k}) = \{ D^\lambda \mid \lambda \in \Lambda_r^\circ \}; \quad \text{see Graham–Lehrer [23, Thm 3.4].}
\]

Thus, we obtain a natural parametrization of the irreducible representations of \( H_{n,k} \) by the set \( \Lambda_r^\circ \subseteq \Lambda_r \).

**Remark 2.4.** Assume that \( b > (n - 1)a > 0 \). Then Conjecture C holds by [16, Cor. 6.4]. In this case, the set \( \Lambda_r^\circ \) is determined explicitly by Dipper–James–Murphy [8] and Ariki [1]. Finally, Iancu–Pallikaros [25] show that the cell modules \( S^\lambda \) are canonically isomorphic to the Specht modules defined by Dipper–James–Murphy [8].

**Remark 2.5.** Now consider arbitrary values of \( a, b \) such that \( a > 0 \) and \( b \not\in \{a, 2a, \ldots, (n - 1)a\} \). Then, assuming that the conjectured relation \((c^+)\) in Remark 1.2 holds, a description of the set \( \Lambda_r \) follows from the results of Geck–Jacon [20] on canonical basic sets. Indeed, one readily shows that the set \( \Lambda_r^\circ \) coincides with the canonical basic set determined by [20]. Thus, by the results of [20], we have explicit combinatorial descriptions of \( \Lambda_r^\circ \) in all cases. Note that these descriptions heavily depend on \( a, b \) and \( \theta : A \to k \).

It is shown in [17] that, if \( a = 2 \) and \( b = 1 \) or 3, then the sets \( \Lambda_r^\circ \) parametrize the modular principal series representations of the finite unitary groups.

### 3 Domino insertion

The aim of this section is to describe the domino insertion algorithm and to provide some theoretical evidences for Conjecture A. For this purpose we will see \( W_n \) as the group of permutations \( w \) of \( \{-1, -2, \ldots, -n\} \cup \{1, 2, \ldots, n\} \) such that \( w(-i) = -w(i) \) for any \( i \). The identification is as follows: \( t \) corresponds to the transposition \((1, -1)\) and \( s_i \) to \((i, i + 1)(-i, -i - 1)\). If \( r \leq n \), we identify \( W_r \) with the subgroup of \( W_n \) generated by \( S_r = \{t, s_1, s_2, \ldots, s_{r-1}\} \). The symmetric group of degree \( n \) will be denoted by \( S_n \): when necessary, we shall identify it in the natural way with the subgroup of \( W_n \) generated by \( \{s_1, s_2, \ldots, s_{n-1}\} \). Let \( t_1 = t \) and, if \( 1 \leq i \leq n - 1 \), let \( t_{i+1} = s_i t_i s_i \). As a signed permutation, \( t_i \) is just the transposition \((i, -i)\).

**Remark 3.1.** Since we shall be interested in various descent sets of elements of \( W_n \), we state here for our future needs the following two easy facts. Let \( w \in W_n \). Then the following hold.

(a) If \( 1 \leq i \leq n - 1 \), then \( \ell(ws_i) > \ell(w) \) if and only if \( w(i) < w(i + 1) \).
(b) If \( 1 \leq i \leq n \), then \( \ell(w t_i) > \ell(w) \) if and only if \( w(i) > 0 \).
3.1 Partitions and Tableaux

We refer to [27, 33] for further details of the material in this section. We shall assume some familiarity with (standard) Young tableaux.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l) \geq 0)$ be a partition of $n = |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We will not distinguish between a partition $\lambda$ and its Young diagram (often denoted $D(\lambda)$). Our Young diagrams will be drawn in the English notation so that the boxes are upper-left justified. When $\lambda$ and $\mu$ are partitions satisfying $\mu \subset \lambda$ we will use $\lambda/\mu$ to denote the shape corresponding to the set-difference of the diagrams of $\lambda$ and $\mu$. We call $\lambda/\mu$ a domino if it consists of exactly two squares sharing an edge.

The 2-core (or just core) $\tilde{\lambda}$ of a shape $\lambda$ is obtained by removing dominoes from $\lambda$, keeping the shape a partition, until this is no longer possible. The partition $\tilde{\lambda}$ does not depend on how these dominoes are removed. Every 2-core has the shape of a staircase $\delta_r = (r, r - 1, \ldots, 0)$ for some integer $r \geq 0$.

We denote the set of partitions by $P$ and the set of partitions with 2-core $\delta_r$ by $P_r$. The set of all partitions $\lambda$ satisfying the conditions:

$\tilde{\lambda} = \delta_r$ and $|\lambda| = |\delta_r| + 2n$

will be denoted $P_r(n)$. Note that $P = \cup_{r,n} P_r(n)$ is a disjoint union.

A (standard) domino tableau $D$ of shape $\lambda \in P_r(n)$ consists of a tiling of the shape $\lambda/\tilde{\lambda}$ by dominoes and a filling of the dominoes with the integers $\{1, 2, \ldots, n\}$, each used exactly once, so that the numbers are increasing when read along either the rows or columns. The value of a domino is the number written inside it. We will denote by dom$_i$ the domino with the value $i$ inside. We will also write sh($D$) = $\lambda$ for the shape of $D$. An equivalent description of the domino tableau $D$ is as the sequence of partitions $\{\tilde{\lambda} = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^n = \lambda\}$, where sh(dom$_i$) = $\lambda^i/\lambda^{i-1}$. If the values of the dominoes in a tableau $D$ are not restricted to the set $\{1, 2, \ldots, n\}$ (but each value occurs at most once), we will call $D$ an injective domino tableau.

We now describe a number of operations on standard Young and domino tableaux needed in the sequel. One may obtain a standard Young tableau $T = T(D)$ from a domino tableau $D$ by replacing a domino with the value $i$ in $D$ by two boxes containing $\bar{i}$ and $i$ in $T$. The boxes are placed so that $T$ is standard with respect to the order $1 < 1 < 2 < 2 < \cdots$. If $D$ has shape $\lambda$ then $T(D)$ will have shape $\lambda/\tilde{\lambda}$. Suppose now that $Y$ is a standard Young tableau of shape $\lambda$ filled with letters smaller than any of the letters occurring in $D$. Define $T_Y(D)$ by “filling” in the empty squares in $T(D)$ with the tableau $Y$.

Let $T$ be a standard Young tableau and $i$ a letter occurring in $T$. The conversion process proceeds as follows (see [24, 33]). Replace the letter $i$ in $T$ with another letter $j$. The resulting tableau may not be standard, so we repeatedly swap $j$ with its neighbours until the tableau is standard. We say that the value $i$ has been converted to $j$. 

Domino insertion and Kazhdan–Lusztig cells 9
Now let $T$ be any standard Young tableau filled with barred $\bar{i}$ and non-barred letters $i$. Define $T^\text{neg}$ by successively converting barred letters $\bar{i}$ to negative letters $-i$, starting with the smallest letters. The main fact that we shall need is that the operation “neg” is invertible. We refer the reader to [33] for a full discussion of these operations.

3.2 The Barbasch-Vogan domino insertion algorithm

The Robinson-Schensted correspondence establishes a bijection

$$\pi \leftrightarrow (P(\pi), Q(\pi))$$

between permutations $\pi \in \mathfrak{S}_n$ and pairs of standard tableaux with the same shape and size $n$ (see [34]). Domino insertion generalizes this by replacing the symmetric group with the hyperoctahedral group. It depends on the choice of a core $\delta_r$, and establishes a bijection between $W_n$ and pairs $(P_r, Q_r)$ of standard domino tableaux of the same shape $\lambda \in P_r(n)$. There are in fact many such bijections but we will be concerned only with the algorithm introduced by Barbasch and Vogan [2] and later given a different description by Garfinkle [10]. We now describe this algorithm following the more modern expositions [27, 33].

Let $D$ be an injective domino tableau with shape $\lambda$ such that $i > 0$ is a value which does not occur in $D$. We describe the insertion $E = D \leftarrow i$ (or $E = D \leftarrow -i$) of a horizontal (vertical) domino with value $i$ into $D$. Let $D_{<i} \subset D$ denote the sub-domino tableau of $D$ containing all dominoes with values less than $i$. If $\lambda$ has a 2-core $\tilde{\lambda}$, then we will always assume that $\tilde{\lambda} \subset \text{sh}(D_{<i})$. Let $E_{\leq i}$ be the domino tableau obtained from $D_{<i}$ by adding an additional vertical domino in the first column or an additional horizontal domino in the first row labeled $i$.

For $j > i$ we define $E_{\leq j}$, supposing that $E_{\leq j-1}$ is known. If $D$ contains no domino labeled $j$ then $E_{\leq j} = E_{\leq j-1}$; otherwise let dom$_j$ denote the domino in $D$ labeled $j$. Let $\mu = \text{sh}(E_{\leq j-1})$. We now distinguish four cases:

1. If $\mu \cap \text{dom}_j = \emptyset$ do not touch, then we set $E_{\leq j} = E_{\leq j-1} \cup \text{dom}_j$.
2. If $\mu \cap \text{dom}_j = (k, l)$ is exactly one square in the $k$-th row and $l$-th column, then we add a domino containing $j$ to $E_{\leq j-1}$ to obtain the tableau $E_{\leq j}$ which has shape $\mu \cup \text{dom}_j \cup (k+1, l+1)$.
3. If $\mu \cap \text{dom}_j = \text{dom}_j$ is horizontal, then we bump the domino $\text{dom}_j$ to the next row, by setting $E_{\leq j}$ to be the union of $E_{\leq j-1}$ with an additional (horizontal) domino with value $j$ one row below that of $\text{dom}_j$.
4. If $\mu \cap \text{dom}_j = \text{dom}_j$ is vertical, then we bump the domino $\text{dom}_j$ to the next column, by setting $E_{\leq j}$ to be the union of $E_{\leq j-1}$ with an additional (vertical) domino with value $j$ one column to the right of $\text{dom}_j$.

Finally we let $E = \lim_{j \to \infty} E_{\leq j}$. 

Let \( w = w(1)w(2) \cdots w(n) \in W_n \) be a hyperoctahedral permutation written in one-line notation. Thus, for each \( i \), we have \( w(i) \in \{ \pm 1, \pm 2, \ldots, \pm n \} \); furthermore, \( |w(1)||w(2)| \cdots |w(n)| \in S_n \) is a usual permutation. Let \( \delta_r \) be a 2-core assumed to be fixed. Then the insertion tableau \( P^r(w) \) is defined as \( (\delta_r \leftarrow w(1)) \leftarrow w(2) \cdots \leftarrow w(n) \). The sequence of shapes obtained in the process defines another standard domino tableau called the recording tableau \( Q^r(w) \) of \( w \in W_n \). The insertion tableau \( P^r(w) \) can of course be defined for any sequence \( w = w(1)w(2) \cdots w(n) \) such that \( |w(i)| \neq |w(j)| \) for \( i \neq j \).

The following theorem is due to Barbasch-Vogan [2] and Garfinkle [10] when \( r = 0, 1 \) and extended by van Leeuwen [28] to larger cores.

**Theorem 3.2.** Fix \( r \geq 0 \). The domino insertion algorithm defines a bijection between \( w \in W_n \) and pairs \((P, Q)\) of standard domino tableaux of the same shape lying in \( P^r(n) \). This bijection satisfies the equality \( P^r(w) = Q^r(w^{-1}) \).

It is easy to see that the bijectivity in Theorem 3.2 together with Conjecture \( A^+ \) would imply that the relevant left cell representations are irreducible. This is consistent with the conjecture that left cell representations for \( W_n \) are irreducible for “generic parameters” and in particular if \( b \notin \{ a, 2a, \ldots, (n-1)a \} \) (see Proposition 2.3).

We have computational evidences for Conjectures A, \( A^+ \) and B: they were checked for \( n \leq 6 \) by using CHEVIE [18] and Coxeter [7]. In the rest of this section, we shall give theoretical evidences for Conjecture A and \( A^+ \) (induction of cells, multiplication by the longest element, link to [30, Conj. 25.3], asymptotic case, quasi-split case, right descent sets, coplactic relations).

### 3.3 Conjecture A and Lusztig’s Conjecture 25.3

There is an alternative description (in the case where \( r = 0, 1 \), it is in fact the original description of Barbasch and Vogan) of domino insertion. As we will now explain, it is related to [30, Conj. 25.3]. Let us fix in this subsection a Coxeter group \((W, S)\) of type \( A_{2n+r(r+1)}/2-1 \). Let \( \sigma \) be the unique non-trivial automorphism of \( W \) such that \( \sigma(S) = S \). If \( J \) is a subset of \( S \), we denote by \( W_J \) the parabolic subgroup of \( W \) generated by \( J \) and let \( w_J \) denote the longest element of \( W_J \).

Let \( I \) be the unique connected (when we view it as a subdiagram of the Dynkin diagram of \((W, S)\)) subset of \( S \) of cardinality \( r(r + 1)/2 - 1 \) (or 0 if \( r = 0 \)) such that \( \sigma(I) = I \):

```
\( n \)  \( \cdots \)  \( \bullet \)  \( \cdots \)  \( \bullet \)  \( \cdots \)  \( n \)
```

\( r(r+1)/2-1 \)
Let \( \mathcal{W} \) denote the subgroup of \( W \) consisting of all elements \( w \) such that \( wW_Iw^{-1} = W_I \) and \( w \) has minimal length in \( wW_I \) (see [30, §25.1]). If \( \Omega \) is a \( \sigma \)-orbit in \( S \setminus I \), we set \( s_\Omega = w_{1\cup \Omega}w_I \). If \( 0 \leq i \leq n-1 \), let \( \Omega_i \) denote the orbit of \( \sigma \) in \( S \setminus I \) consisting of elements which are separated from \( I \) by \( i \) nodes in the Dynkin diagram. Then \( \{ \Omega_0, \Omega_1, \ldots, \Omega_{n-1} \} \) is the set of orbits of \( \sigma \) in \( S \setminus I \). Moreover, there is a unique morphism of groups \( i_r : W_n \to \mathcal{W}^r \) that sends \( t \) to \( s_{\Omega_0} \) and \( s_i \) to \( s_{\Omega_i} \) (for \( 1 \leq i \leq n-1 \)). It is an isomorphism of groups (see [30, §25.1]).

The morphism \( i_r \) can be described explicitly in the language of signed permutations. First identify \( W \) with the permutation group of the following \( 2n + r(r-1)/2 \) elements (ordered according to the ordering of \( S \)):

\[
\{ -n < -(n-1) < \cdots < -1 < 0_1 < 0_2 < \cdots < 0_{r(r-1)/2} < 1 < 2 < \cdots n \}
\]

so that the subgroup \( W_I \) (which is isomorphic to \( \mathfrak{S}_{r(r+1)/2} \)) acts on the elements \( \{ 0_1, 0_2, \ldots, 0_{r(r-1)/2} \} \). Let \( w = w(1)w(2)\cdots w(n) \in W_n \). Then the two-line notation of \( i_r(w) \) is given by

\[
\begin{pmatrix}
-n & \cdots & -1 & 0_1 & \cdots & 0_{r(r-1)/2} & 1 & 2 & \cdots & n \\
-w(n) & \cdots & -w(1) & 0_1 & \cdots & 0_{r(r-1)/2} & w(1) & w(2) & \cdots & w(n)
\end{pmatrix}.
\]

Now, let \( c_0 \) denote the two-sided cell of \( W_I \) which has “shape” \( \delta_r \). If \( w, w' \in \mathfrak{S}_n \), we write \( w \simeq_{\mathfrak{S}} w' \) if \( Q(w) = Q(w') \) (the equivalence relation \( \simeq_{\mathfrak{S}} \) defines the Robinson-Schensted left cells of \( \mathfrak{S}_n \), which coincide with the Kazhdan-Lusztig left cells [26, §5]).

**Theorem 3.3.** Fix \( x \in c_0 \). Let \( w \in W_n \), \( r \geq 0 \) and \( \pi = i_r(w)x \in \mathfrak{S}_{2n+r(r+1)/2} \). Then we have

\[
(T_{P(x)}(P^r(w)))^{\text{neg}} = P(\pi) \quad \text{and} \quad (T_{Q(x)}(Q^r(w)))^{\text{neg}} = Q(\pi).
\]

Since \( \text{neg} \) is invertible, in particular \( w \simeq_{\mathfrak{S}} w' \) if and only if \( i_r(w)x \simeq_{\mathfrak{S}} i_r(w')x \).

For the construction of \( T_{P(x)}(P^r(w)) \) in Theorem 3.3 we are using the ordering \( 0_1 < 0_2 < \cdots < 0_{r(r-1)/2} < 1 < 2 < \cdots < n \). In the case where \( r = 0 \) or 1, Theorem 3.3 is essentially [28, Theorem 4.2.3] with different notation. To generalize the result to all \( r \geq 0 \) we follow the approach of [33].

**Proof.** For the case \( r = 0 \) the theorem is exactly [33, Theorem 32]. We now explain, assuming familiarity with [33], how to extend the result to larger cores. It is shown in [33, Lemma 31] that a domino insertion \( D \leftarrow i \) can be imitated by *doubly mixed insertion*, denoted \( P_{m_i} \). The proof of [33, Lemma 31] is local, and remains valid when we replace \( D_{<i} \) by any Young tableau of the same shape, filled with “small” letters. More precisely, their proof shows that \( T_{P(x)}(P^r(w)) \) can be obtained by doubly mixed insertion of a “biword” \( w^{\text{dup}} \) (explicitly defined in [33]) into \( P(x) \). Thus one has
(2) \[ TP(x)\left(P_r(w)\right) = P_m\cdot (x \sqcup w_{\text{dup}}), \]

where \( x \sqcup b \) denotes the word obtained from concatenating \( a \) and \( b \). In the notation of [33], \( x \) here is a biword with no bars so that \( P_m\cdot (x) = P(x) \).

Now [33, Theorem 21 and Proposition 14] connect doubly mixed insertion with usual Schensted insertion via the equation

\[ P_m\cdot (u)_{\text{neg}} = P\left(u^{\text{inv neg inv neg}}\right). \]

The operation denoted “inv neg inv neg” in [33] applied to \( x \sqcup w_{\text{dup}} \) coincides with our inclusion \( \iota_r(w) \). Combining (2) and (3) one obtains

\[ (TP(x)\left(P_r(w)\right))_{\text{neg}} = P(\pi). \]

The statement about recording tableau is obtained analogously, or by using the equation \( Q(\pi) = P(\pi^{-1}) = P(x^{-1}\iota_r(w)^{-1}) = P(\iota_r(w^{-1})x^{-1}) \).

**Remark 3.4.** Note that the last statement of Theorem 3.3 does not depend on the choice of \( x \in c_0 \).

**Corollary 3.5.** If \( r \geq 0 \) and if \( (a, b) = (2, 2r + 1) \), then Conjecture A(\( c \)) agrees with [30, Conj. 25.3] for the case \( (W, S, I, \sigma) \) described above.

### 3.4 Longest element

Let \( w_0 \) denote the longest element of \( W_n \): it is equal to \( t_1t_2\ldots t_n \) (or to \(-1 -2 \ldots -n \) in the one line notation). It is a classical result that two elements \( x \) and \( y \) in \( W_n \) satisfy \( x \sim_\mathcal{L} y \) if and only if \( w_0x \sim_\mathcal{L} w_0y \). The next result shows that the relations \( \simeq_r \) share the same property.

**Proposition 3.6.** Let \( r \geq 0 \) and let \( x, y \in W_n \). Then \( x \simeq_r y \) if and only if \( w_0x \simeq_r w_0y \).

**Proof.** This follows from the easy fact that \( P^r(w_0x) \) (resp. \( Q^r(w_0x) \)) is the conjugate (that is, the transpose with respect to the diagonal) of \( P^r(x) \) (resp. \( Q^r(x) \)), and similarly for \( y \). \( \square \)

### 3.5 Induction of cells

Let \( m \leq n \). Let \( X^m_n \) denote the set of elements \( w \in W_n \) which have minimal length in \( wW_m \). It is a cross-section of \( W_n/W_m \). By Remark 3.1, an element \( x \in W_n \) belongs to \( X^m_n \) if and only if \( 0 < x(1) < x(2) < \cdots < x(m) \). A theorem of Geck [13] asserts that, if \( C \) is a Kazhdan-Lusztig left cell of \( W_m \) (associated with the restriction of \( L_{a,b} \) to \( W_m \)), then \( X^m_nC \) is a union of Kazhdan-Lusztig left cells of \( W_n \). The next result show that the same hold if we replace Kazhdan Lusztig left cell by left \( r \)-cell.
Proposition 3.7. Let \( r \geq 0 \). If \( C \) is a left \( r \)-cell of \( W_m \), then \( X^m_n C \) is a union of left \( r \)-cells of \( W_n \).

Proof. Let \( w, w' \in W_m \) and \( x, x' \in X^m_n \) be such that \( xw \succeq_r x'w' \) (in \( W_n \)). We must show that \( w \succeq_r w' \) (in \( W_m \)). For the purpose of this proof, we shall denote by \( (P^r_n(w), Q^r_n(w)) \) (resp. \( (P^r_m(w), Q^r_m(w)) \)) the pair of standard domino tableaux obtained by viewing \( w \) as an element of \( W_n \) (resp. of \( W_m \)). Then, since \( x \) is increasing on \( \{1, 2, \ldots, m\} \) and takes only positive values, the dominoes filled with \( \{1, 2, \ldots, m\} \) in the recording tableau \( Q^r_n(xw) \) are the same as the one in the recording tableau \( Q^r_m(w) \). In particular, \( Q^r_m(w) \) is obtained from \( Q^r_n(xw) \) by removing the dominoes filled by \( \{m+1, m+2, \ldots, n\} \).

Similarly, \( Q^r_m(w') \) is obtained from \( Q^r_n(x') \) by removing the dominoes filled by \( \{m+1, m+2, \ldots, n\} \). Since \( Q^r_n(xw) = Q^r_n(x'w') \) by hypothesis, we have that \( Q^r_m(w) = Q^r_m(w') \). In other words, \( w \succeq_r w' \) in \( W_m \). \( \square \)

Corollary 3.8. Let \( r \geq 0 \) and let \( x \) and \( y \) be two elements of \( W_m \). Then \( x \simeq_r y \) in \( W_m \) if and only if \( x \simeq_r y \) in \( W_n \).

The previous corollary shows that it is not necessary to make the ambient group precise when one studies the equivalence relation \( \simeq_r \).

Geck’s result [13] is valid for any Coxeter group and any parabolic subgroup. We shall investigate now the analogue of Proposition 3.7 for the parabolic subgroup \( S_n \) of \( W_n \). We denote by \( X(n) \) the set of elements \( w \in W_n \) which have minimal length in \( wS_n \). It is a cross-section of \( W_n/S_n \). By Remark 3.1, an element \( w \in W_n \) belongs to \( X(n) \) if and only if \( w(1) < w(2) < \cdots < w(n) \).

Proposition 3.9. Let \( r \geq 0 \) and let \( C \) be a Robinson-Schensted left cell of \( S_n \). Then \( X(n) C \) is a union of domino left cells for \( \simeq_r \).

Proof. Let \( w, w' \in S_n \) and \( x, x' \in X(n) \) be such that \( xw \simeq_r x'w' \) in \( W_n \). We must show that \( w \simeq \circ \, w' \). It is well known that for two words \( a_1a_2\cdots a_k \) and \( b_1b_2\cdots b_k \) one has \( Q(a_1a_2\cdots a_k) = Q(b_1b_2\cdots b_k) \Rightarrow Q(a_1a_2\cdots a_k) = Q(b_1b_2\cdots b_k) \) for any \( 1 \leq j \leq l \leq k \). Indeed this is Geck’s result [13] for \( S_n \).

By Theorem 3.3, we have \( Q(xw)c) = Q(x'(w')c) \) for any \( c \in S_0(X) \). Treating \( t_r(xw)c \) as a word using (1), we thus have

\[
Q(x(1)w(2) \cdots w(n)) = Q(x'(1)w'(2) \cdots w'(n)).
\]

But \( x(1) < x(2) < \cdots < x(n) \) so that this is equivalent to \( Q(w) = Q(w') \). \( \square \)

If \( x \in X_m \) and if \( w, w' \in W_m \) are such that \( w \sim L \, w' \) in \( W_m \) then a result of Lusztig [30, Prop. 9.13] asserts that \( wx^{-1} \sim L \, w'x^{-1} \). The next result shows that the same statement holds if we replace \( \sim_L \) by \( \sim_r \).

Proposition 3.10. Let \( r \geq 0 \), \( x \in X_m \) and \( w, w' \in W_m \) be such that \( w \sim_x w' \). Then \( wx^{-1} \sim_r w'x^{-1} \).
Proof. Let us use here the notation of the proof of Proposition 3.7. So we assume that $P_m(w \leftarrow a) = P_m(w \leftarrow a)$ and we must show that $P_n(xw \leftarrow a) = P_n(xw \leftarrow a)$. Let $D = ((\ldots(D_r \leftarrow xw) \leftarrow xw(2) \ldots) \leftarrow xw(m))$ and $D' = ((\ldots((D_r \leftarrow xw) \leftarrow xw(2)) \ldots) \leftarrow xw(m))$. Since $w^{-1}(i) = i$ if $i \geq m + 1$, we have $P_n(xw) = ((\ldots((D \leftarrow x(m + 1)) \leftarrow x(m + 2)) \ldots) \leftarrow x(n))$ and similarly for $P_n(xw)$. Therefore, we only need to show that $D = D'$. But, since $w^{-1}$ stabilizes $\{1, 2, \ldots, m\}$ and since $x$ is increasing on $\{1, 2, \ldots, m\}$ and takes only positive values, it follows from the domino insertion algorithm that $D$ is obtained from $P_n(xw)$ by applying $x$, or in other words replacing the domino dom$_{1}(i)$ by dom$_{1}(i)$. Similarly, $D'$ is obtained from $P_n(xw)$ by applying $x$. Since $P_n(xw) = P_n(xw)$ by hypothesis, we get that $D = D'$, as desired. □

As for Geck’s result, Lusztig’s result [30, Prop. 9.13] is valid for any parabolic subgroup of any Coxeter group. The next result is the analogue of Proposition 3.10 for the parabolic subgroup $\mathfrak{S}(n)$ of $W_n$.

**Proposition 3.11.** Let $r \geq 0$, $x \in X(n)$ and $w, w' \in \mathfrak{S}(n)$ be such that $w \succeq w'$. Then $wx^{-1} \succeq w'x^{-1}$.

Proof. We must show that $P_r(xw) = P_r(xw')$ knowing that $P(w^{-1}) = P(w^{-1})$. For $u \in W_n$, denote by $u_{\mathfrak{S}}$ the word $u(1) u(2) \cdots u(n)$ and $u_{\mathfrak{S}}$ the word $-u(n) - u(n-1) \cdots - u(1)$. The equation $P(w^{-1}) = P(w^{-1})$ gives

\begin{equation}
P((xw')_{\mathfrak{S}}) = P((xw')_{\mathfrak{S}}) \quad \text{and} \quad P((xw')_{-\mathfrak{S}}) = P((xw')_{-\mathfrak{S}}),
\end{equation}

where we are comparing pairs of standard Young tableaux filled with the set of letters \{$(1), (2), \ldots, (n)$\} (resp. \{$(1), (2), \ldots, (n)$\}).

By Theorem 3.3, it is enough to show that $P(r_c(xw) \leftarrow c) = P(r_c(xw) \leftarrow c)$ for some fixed $c \in \mathfrak{S}(n)$. Using (1), we may write $r_c(xw) \leftarrow c$ in one-line notation as the concatenation $(xw)_{\mathfrak{S}} \leftarrow c \leftarrow (xw)_{\mathfrak{S}}$. It is well known that if $a, a', b$ are words such that $P(a) = P(a')$ then one has $P(a \sqcup b) = P(a' \sqcup b)$ (this is Lusztig’s result [30, Prop. 9.13] for the symmetric group). Combining this with (4) we obtain $P((xw)_{-\mathfrak{S}} \sqcup c \sqcup (xw)_{\mathfrak{S}}) = P((xw)_{-\mathfrak{S}} \sqcup c \sqcup (xw)_{\mathfrak{S}})$, as desired. □

**Remark 3.12.** The Propositions 3.7, 3.9, 3.10 and 3.11 generalize [4, Prop. 4.8] (which corresponds to the asymptotic case).

### 3.6 Asymptotic case, quasi-split case

We now prove Conjecture A for $r = 0, 1$ and $r \geq n - 1$.

**Theorem 3.13.** Conjecture A is true for $r \geq n - 1$. 

Proof. Let \( r \geq n - 1 \), and let \( D \) be a domino tableau with shape \( \lambda \in \mathcal{P}_r(n) \). The dominoes \( \{ \text{dom}_i \mid i \in \{1, 2, \ldots, n\} \} \) can be decomposed into the two disjoint collections \( \mathcal{D}_+ = \{ \text{dom}_i \mid \text{dom}_i \text{ is horizontal} \} \) and \( \mathcal{D}_- = \{ \text{dom}_i \mid \text{dom}_i \text{ is vertical} \} \) such that all the dominoes in \( \mathcal{D}_+ \) lie strictly above and to the right of all the dominoes in \( \mathcal{D}_- \). We call a tableau satisfying this property segregated. If the collection of dominoes \( \mathcal{D}_+ \) is left justified, and each domino replaced by a single box, one obtains a usual Young tableau. Similarly, if the dominoes \( \mathcal{D}_- \) are justified upwards and changed into boxes, one obtains a usual Young tableau.

In other words, \( D \) can be thought of as a union of two usual tableau \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) so that the union of the values in \( \mathcal{D}_+ \) and in \( \mathcal{D}_- \) is the set \( \{1, 2, \ldots, n\} \). To be consistent with the remaining discussion we in fact define \( \mathcal{D}_- \) to be the conjugate (reflection in the main diagonal) of the Young tableau obtained from the dominoes \( \mathcal{D}_- \), as described above.

Domino insertion is compatible with this decomposition so that the following diagram commutes:

\[
\begin{array}{ccc}
w & \xrightarrow{\text{domino insertion}} & (P^r(w), Q^r(w)) \\
[w_+, w_-] & \xrightarrow{\text{RS insertion}} & [P^r_+(w), Q^r_+(w)), (P^r_-(w), Q^r_-(w))] \\
\end{array}
\]

Here \( w_+ \) denotes the subword of \( w \) consisting of positive letters and \( w_- \) denotes the subword consisting of negative letters, with the minus signs removed. In [5], it is shown for \( r \geq n - 1 \) and \( w, w' \in W_n \) that \( w \sim_L w' \) if and only if \( Q(w_+) = Q(w'_+ \) and \( Q(w_-) = Q(w'_- \) holds for \( \sim_R \) and \( \sim_{LR} \). Since \( Q(w_+) = Q^r_+(w) \) and \( Q(w_-) = Q^r_-(w) \), we have \( w \sim_L w' \) if and only if \( Q^r(w) = Q^r(w') \), establishing Conjecture A in this case for left cells. A similar argument works for right cells and two-sided cells, using also the classification of two-sided cells in [3]. \( \square \)

**Theorem 3.14.** Conjecture A is true if \( a = 2b \) or if \( 3a = 2b \).

Proof. In [29], Lusztig determined the left cells of \( W_n \) with parameters \( b = (2r + 1)a/2 \) for \( r \in \{0, 1\} \) as follows. When \( r \in \{0, 1\} \) we have \( I = \emptyset \) in the notation of Section 3.3. The equal parameter weight function \( L \) on \( \mathcal{S}_{2n+r} \) restricts to the weight function \( L_{b,a} \) on \( \tau_r(W_n) \), where \( b = (2r + 1)a/2 \). Lusztig [29, Theorem 11] shows that each left cell of \( W_n \) is the intersection of a left cell of \( \mathcal{S}_{2n+r} \) with \( \tau_r(W_n) \). Thus \( w \simeq_r w' \) in \( W_n \) if and only if \( \tau_r(w) \simeq \tau_r(w') \) in \( \mathcal{S}_{2n+r} \). When \( r \in \{0, 1\} \) there is no need for the element \( x \in c_0 \) in Theorem 3.3, so one obtains Conjecture A for \( r \in \{0, 1\} \). \( \square \)

### 3.7 Right descent sets

If \( r \geq 0 \), let \( S_n^{(r)} = \{ s_1, s_2, \ldots, s_{n-1} \} \cup \{ t_1, \ldots, t_r \} \) (if \( r \geq n \), then \( S_n^{(r)} = S_n^{(n)} \)).

If \( w \in W_n \), let
be the extended right descent set of \( w \). The following proposition is easy:

**Proposition 3.15.** Let \( x \) and \( y \) be two elements of \( W_n \). Then:

(a) If \( x \succeq_r y \), then \( \mathcal{R}_n^{(r+1)}(x) = \mathcal{R}_n^{(r+1)}(y) \).
(b) If \( r > ra \) and if \( x \sim_L y \), then \( \mathcal{R}_n^{(r+1)}(x) = \mathcal{R}_n^{(r+1)}(y) \).

**Proof.** If \( r \geq n - 1 \), then statements (a) and (b) are equivalent by Theorem 3.13. But, in this case, (b) has been proved in [4, Prop. 4.5]. So let us assume from now on that \( r < n - 1 \). We shall prove (a) and (b) together. Let us set

\[
\mathcal{R}_s(x) = \{ s \in \{s_1, \ldots, s_{n-1}\} \mid \ell(ws) < \ell(w) \},
\]

\[
\mathcal{R}_t^{(r)}(x) = \{ s \in \{t_1, \ldots, t_{r}\} \mid \ell(ws) < \ell(w) \}.
\]

Then \( \mathcal{R}_n^{(r)}(x) = \mathcal{R}_s(x) \cup \mathcal{R}_t^{(r)}(x) \).

Write \( x = uw' \) and \( y = v'y' \), with \( u, v \in X_{n+1}^r \) and \( x', y' \in W_{r+1} \). Since \( \ell(uw') = \ell(u) + \ell(x') \) (and similarly for \( uw's \) for any \( s \in W_r \)), we have that \( \mathcal{R}_t^{(r)}(x) = \mathcal{R}_r^{(r)}(x') \). Similarly, \( \mathcal{R}_t^{(r)}(y) = \mathcal{R}_t^{(r)}(y') \). But, if \( x \) and \( y \) satisfy (a) or (b), then \( \mathcal{R}_t^{(r)}(x') = \mathcal{R}_t^{(r)}(y') \): indeed, this follows from the fact that (a) and (b) have been proved in the asymptotic case and, in case (a), from Proposition 3.7 and, in case (b), from [13].

Now it remains to show that \( \mathcal{R}_s(x) = \mathcal{R}_s(y) \) if \( x \) and \( y \) satisfy (a) or (b). In case (b), this follows from [30, Lemma 8.6]. So assume now that \( x \succeq_r y \). Write \( x = u's \) and \( y = v't \), with \( u, v \in X(n) \) and \( s, t \in \mathcal{S}_n \). As in the previous case, we have \( \mathcal{R}_s(x) = \mathcal{R}_s(s) \) and \( \mathcal{R}_s(y) = \mathcal{R}_s(t) \). Moreover, by Proposition 3.9, we have \( s \succeq_r t \). It is well-known that it implies that \( \mathcal{R}_s(s) = \mathcal{R}_s(t) \). □

**Remark 3.16.** Proposition 3.15 (a) can also be deduced from [33, Lemma 33] or [27, Lemma 9].

### 3.8 Coplactic relations

If \( x \) and \( y \) are two elements of \( W_n \) such that \( \ell(x) \leq \ell(y) \), then we write \( x \succeq_r y \) if there exists \( s \in S_n^{(0)} \) and \( s' \in S_n^{(r)} \) such that \( y = sx \) and \( \ell(s'x) < \ell(x) < \ell(y) < \ell(s'y) \). If \( \ell(x) \geq \ell(y) \), then we write \( x \succeq_r y \) if \( y \succeq_r x \). Let \( \equiv_r \) denote the reflexive and transitive closure of \( \succeq_r \).

**Remark 3.17.** (a) If \( x \equiv_r y \), then \( \ell(x) = \ell(y) \).

(b) If \( r' \geq r \) and if \( x \equiv_r y \), then \( x \equiv_{r'} y \) (indeed, if \( x \succeq_r y \), then \( x \succeq_{r'} y \); this just follows from the fact that \( S_n^{(r')} \subseteq S_n^{(r''')} \)). Moreover, the relations \( \equiv_n \) and \( \equiv_r \) are equal if \( r \geq n \).

(c) If \( r \geq n - 1 \), then \( x \equiv_r y \) if and only if \( x \equiv_{n-1} y \). Let us prove this statement. By (b) above, we only need to show that, if \( x \succeq_n y \), then \( x \succeq_{n-1} y \). For this, we may assume that \( \ell(y) > \ell(x) \). So there exists \( i \in \{1, 2, \ldots, n-1\} \)
and \( s' \in S_n^{(0)} \) such that \( y = s_x \) and \( \ell(s'x) < \ell(x) < \ell(s)x < \ell(s's_i x) \). If \( s' \in S_n^{(n-1)} \) then we are done. So we may assume that \( s' = t_n \). Therefore, the first inequality says that \( x^{-1}(n) < 0 \) and the last inequality says that \( x^{-1}s_i(n) > 0 \). This implies that \( i = n - 1 \). Consequently, we have \( x^{-1}(n) < 0 \) and \( x^{-1}(n-1) > 0 \). But, the middle inequality says that \( x^{-1}(n-1) < x^{-1}(n) \), so we obtain a contradiction with the fact that \( s' = t_n \).

(d) If \( b > ra \) and if \( r \geq n - 1 \), then it follows from Theorem 3.13, from [5, Prop. 3.8] and from (a), (b) and (c) above that the relations \( \sim_L, \simeq_r \) and \( \equiv_r \) coincide.

**Proposition 3.18.** Let \( x \) and \( y \) be two elements of \( W_n \) such that \( x \equiv_r y \). Then the following hold:

(i) \( x \simeq_r y \).

(ii) If \( b > ra \), then \( x \sim_L y \).

**Proof.** We may, and we will, assume that \( x \sim_r y \). By symmetry, we may also assume that \( \ell(y) > \ell(x) \). We shall prove (i) and (ii) together. There exists \( s \in S_n^{(0)} \) and \( s' \in S_n^{(r)} \) such that \( y = sx \) and \( \ell(s'x) < \ell(x) < \ell(y) < \ell(s'y) \).

Two cases may occur:

- If \( s' \in S_n^{(0)} \), then write \( x = x'u^{-1} \) and \( y = y'v^{-1} \) with \( x', y' \in S_n \) and \( u, v \in X(n) \). Then \( y' = sx', u = v \) and \( \ell(s'x') < \ell(x') < \ell(y') < \ell(s'y') \). It is well-known that it implies that \( Q(x') = Q(y') \) (Knuth relations), so \( x' \simeq_{\Theta} y' \) and \( x' \sim_L y' \). Therefore, since moreover \( u = v \), it follows from Proposition 3.11 (resp. [30, Prop. 9.13]) that \( x \simeq_r y \) (resp. \( x \sim_L y \)).

- If \( s' \notin S_n^{(0)} \) then we write \( s = s_i \) and \( s' = t_j \). Then the relations \( y = sx \) and \( \ell(s'x) < \ell(x) < \ell(y) < \ell(s'y) \) imply that \( x^{-1}(j) < 0 \) and \( x^{-1}s_i(j) > 0 \). In particular, \( s \) and \( s' \) belong to \( W_{r+1} \). Now, write \( x = x'u^{-1} \) and \( y = y'v^{-1} \) with \( x', y' \in W_{r+1} \) and \( u, v \in X_n^{(r+1)} \). Then \( y' = sx', u = v \) and \( \ell(s'x') < \ell(x') < \ell(y') < \ell(s'y') \). By Remark 3.17 (d), this implies that \( x' \simeq y' \) and, if \( b > ra \), that \( x' \sim_L y' \). Therefore, since moreover \( u = v \), it follows from Proposition 3.11 (resp. [30, Prop. 9.13]) that \( x \simeq_r y \) (resp. \( x \sim_L y \)).

Even if we have both \( \ell_j(x) = \ell_j(y) \) and \( x \simeq_r y \) we do not necessarily have \( x \equiv_r y \). For example, let \( r = 0 \), \( n = 6 \) and take \( x = 5 6 1 4 2 -3 \) and \( y = 5 6 -1 4 3 2 \).

### 4 Cycles and Conjecture B

**4.1 Open and closed cycles**

We now describe a more refined combinatorial structure of domino tableaux introduced by Garfinkle [10]. We will mostly follow the setup of [31].
Let $D$ be a domino tableau with shape $\lambda \in \mathcal{P}_r(n)$. We call a square $(i, j) \in D$ variable if $i + j$ and $r$ have the same parity, otherwise we call it fixed. If the domino $\text{dom}_i$ contains the square $(k, l)$ we write $D(k, l) = i$.

Now let $(k, l)$ be the fixed square of dom$_i$. Suppose that dom$_i$ occupies the squares $\{(k, l), (k + 1, l)\}$ or $\{(k, l - 1), (k, l)\}$. We define a new domino $\text{dom}'_i$ by letting it occupy the squares

1. $\{(k, l), (k - 1, l)\}$ if $i < D(k - 1, l + 1)$,
2. $\{(k, l), (k, l + 1)\}$ if $i > D(k - 1, l + 1)$.

Otherwise dom$_i$ occupies the squares $\{(k, l), (k, l + 1)\}$ or $\{(k - 1, l), (k, l)\}$. We define a new domino dom$'_i$ by letting it occupy the squares

1. $\{(k, l), (k, l - 1)\}$ if $i < D(k + 1, l - 1)$,
2. $\{(k, l), (k + 1, l)\}$ if $i > D(k + 1, l - 1)$.

Now define the cycle $c = c(D, i)$ of $D$ through $i$ to be the smallest union $c$ of dominoes satisfying that (i) dom$_i \in c$ and (ii) dom$_j \in c$ if dom$_j \cap$ dom$'_k \neq \emptyset$ or dom$'_j \cap$ dom$_k \neq \emptyset$ for some dom$_k \in c$. If $c$ is a cycle of $D$ we let $M(D, c)$ be the domino tableau obtained from $D$ by replacing each domino dom$_i \in c$ by dom$'_i$. We call this procedure moving through $c$.

**Theorem 4.1 ([10]).** Let $D$ be a domino tableau and $c$ a cycle of $D$. Then $M(D, c)$ is a standard domino tableau. Furthermore, if $C$ is a set of cycles of $D$ then the tableau $M(D, C)$ obtained by moving through each $c \in C$ is defined unambiguously.

We call a cycle $c$ closed if $M(D, c)$ has the same shape as $D$; otherwise we call $c$ open. Note that each (non-trivial) cycle $c$ is in one of two positions, so that moving through is an invertible operation.

### 4.2 Evidence for Conjecture B

The notion of open and closed cycles allows us to state a combinatorially more precise version of Conjecture B.

**Conjecture D.** Assume that $b = ra$ for some $r \geq 1$. Then the following hold for any $w, w' \in W_n$:

(a) $w \sim_L w'$ if and only if $Q^{-1}(w) = M(Q^{-1}(w'), C)$ for a set $C$ of open cycles.
(b) $w \sim_R w'$ if and only if $P^{-1}(w) = M(P^{-1}(w'), C)$ for a set $C$ of open cycles.
(c) $w \sim_{LR} w'$ if and only if some tableau with shape equal to $\text{sh}(P^{-1}(w)) = \text{sh}(Q^{-1}(w'))$ can be obtained from a tableau with shape $\text{sh}(P^{-1}(w')) = \text{sh}(Q^{-1}(w'))$ by moving through a set of open cycles.
Remark 4.2. Each cycle $c$ of a domino tableau $D$ is in one of two positions, and by Theorem 4.1 they can be moved independently. Thus Conjecture D would imply that every left cell for the parameters $b = ra$ would be a union of $2^d$ left cells for the parameters $b = \frac{(2r-1)a}{2}$. Here $d$ is equal to the number of open cycles, which do not change the shape of the core, in one of the $Q$-tableaux in Conjecture D. This is consistent with the fact that, if $b = ra$ with $r \geq 1$, then the number of irreducible components of a constructible representation is a power of 2 (see [30, Chap. 22]; see also with [29, (12.1)] for the equal parameter case).

We have the following theorem of Pietraho, obtained via a careful study of the combinatorics of cycles.

**Theorem 4.3 (Pietraho [32]).** Conjecture B and Conjecture D are equivalent.

Some special cases of Conjectures B and D are known. The case $b = a$ or $r = 1$ is known as the equal parameter case and is closely connected with the classification of primitive ideals of classical Lie algebras.

**Theorem 4.4 (Garfinkle [11]).** Conjecture D is true for $r = 1$.

The asymptotic case follows from [5, 3].

**Theorem 4.5.** Conjecture D holds for $r \geq n$.

*Proof.* Let $D$ be a domino tableau with shape $\lambda \in \mathcal{P}_q(n)$ such that $q \geq n - 1$. Then moving through any cycle of $D$ changes the shape of the core of $D$. Thus (for left cells) the condition $Q^q(w) = M(Q^q(w'), C)$ in Conjecture D is the same as the condition $Q^q(w) = Q^q(w')$. This agrees with the classification given in [5]. A similar argument works for right cells and two-sided cells, also using the classification in [3]. ☐

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