Evaluating the Pixel Array Method as Applied to Partial Differential Equations

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Abstract

The Pixel Array (PA) Method, originally introduced by Spivak et. al., is a fast method for solving nonlinear or linear systems. One of its distinguishing features is that it presents all solutions within a bounding box, represented by a plot whose axes are the values of “exposed variables”. Here we develop a set-theoretic variant of the PA method, named the Pixel Array Solution-Set (PASS) method, that gives PA access to “hidden variables” whose values are not displayed on plot axes. We evaluate the effectiveness of PASS at numerically finding steady states for several partial differential equations. We discretize several one-dimensional solved reaction-diffusion equations, such as the Fisher equation and the Benjamin-Bona-Mahany equation, using finite differences. Then, we run PASS on each equation, and determine whether it successfully finds all boundary conditions for which a numerical steady state might exist. Then we verify whether the steady states found by PASS are correct. Finally, we discuss the benefits and weaknesses of PASS.

1 Introduction

Many physical systems are modeled using partial differential equations (PDEs), which describe the relationship between changing variables in the system. Examples of PDEs include the reaction diffusion equation, which is used to model chemical reactions, population dynamics, and plasma physics among other physical phenomena; and the Benjamin–Bona–Mahony equation, which models long surface gravity waves (BBM, 2016).

Due to the ubiquity of PDEs, entire fields of research are dedicated to solving them, both analytically and numerically. In particular, numerically solving PDEs has widespread applications in computational fluid dynamics, aerodynamics, and other engineering subjects.

Equations that have not been solved analytically are often analyzed using numerical methods. For instance, unsolved forms of the reaction diffusion equation:

\[ u_t = D(x)u_{xx} + R(u) + f(x, t) \]  \hspace{1cm} (1)

Are examined using numerical methods in (Baeumer et al., 2008; Liu et al., 1996; Reitz, 1981). These methods usually require initial and boundary conditions to be satisfied, and provide one numerical solution.
This is where the Pixel Array (PA) method (Spivak et al., 2016) becomes relevant. Unlike other numerical methods, it provides all solutions to a system of equations within a bounding box and is often faster than quasi-Newton methods. Because partial differential equations can be converted to a system of equations using discrete approximation methods, the PA method can be applied to partial differential equations. However, the PA method is quite new, and thus the goal of this paper is to establish its accuracy at numerically solving PDEs.

To test and also use the PA method, we needed to adapt it so as to produce a matrix of full solution sets, rather than just boolean values indicating where solutions exist. We refer to this adapted method as the Pixel Array Solution-Set (PASS) method. The “bounding box” in this case consists of a pair of ranges for boundary values, one on each side of the one-dimensional PDE system; of course a higher-dimensional tensor of solution sets would be appropriate for a higher-dimensional system. Using the PASS method, we were able to accurately find steady states for a variety of well-established PDEs.

The plan of the paper is as follows. In Section 2 we provide a brief review of the finite differences method, as well as describe our modification of the PA method. In Section 3 we use finite differences to discretize the heat equation, the local Fisher equation, the Benjamin-Bona-Mahony equation, and the Sine-Gordon equation. Then we use PASS to calculate numerical solutions. As all the equations were previously solved, we can use them to demonstrate the effectiveness of PASS.

2 Methods

The methods we use to numerically solve for steady states are described below. We first briefly illustrate our discretization method, finite differences. Then we describe the difference between the Pixel Array method of (Spivak et al., 2016) and the Pixel Array Solution-Set (PASS) method developed here. Finally, we demonstrate how using PASS with discretized partial differential equations lets us generate steady states.

2.1 Finite Differences

To apply the PASS method to a PDE, we need to remove all derivatives from the equation. We choose to do this with finite differences, which uses secant lines to approximate derivatives. Using finite differences, the approximate second derivative with respect to $x$ is:

$$u_{xx} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1})$$

(2)

Where $h$ is the length of a discrete spatial interval in $x$, and $u_i$ corresponds to numerical solution value in the spatial interval $i$. As an example, we apply finite differences to the reaction-diffusion equation (1). Note that because we only care about steady states, any derivatives with respect to time $t$ are set to 0. We get:

$$0 = \frac{D}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + R(u_i) + \hat{f}(x_i)$$

(3)

Equations like (3) will be called discrete steady state conditions. We will be solving systems of that equation for all $i$ with the PASS method. If the discrete steady state condition is satisfied for all $u_i$, then the values of $u_i \forall i$ in order form a numerical steady state.
2.2 PASS: Modifying the Pixel Array method to pass along solution sets

Each equation in the discretized system corresponds to an array in both the Pixel Array (PA) and PASS method. In (Spivak et al., 2016), the PA method is executed with boolean k-dimensional matrices or tensors, each of which represents one equation or inequality. Each of the k dimensions in a given tensor specifies one variable of the corresponding equation, and each row or column in that dimension represents an interval of values the variable can assume. For each possible combination of k variable values, the corresponding entry in the matrix is “True” (1) if the equation is satisfied somewhere in that range of values and “False” (0) otherwise. To represent this, we present Figure 1a, which is a function graph, and figure (Fig. 1b) as its corresponding pixel array with vertical dimension y and horizontal dimension x. Observe that the plot and pixel array are turned to their sides, to match conventions described in (Spivak et al., 2016).

![Function Graph and Pixel Array](image)

(a) An example plot of a function, oriented so its axes match that of a matrix

(b) A low-resolution pixel array corresponding to the function plot.

In Figure (1), each of the cells in Figure (1a) corresponds to a pixel in Figure (1b). For instance, the function \( y = x^2 \) goes through the cell with \( x \in [1, 2], y \in [2, 3] \), and therefore its associated pixel, the third column in the second row, has the value 1.

It is useful to set up a naming convention which relates the tensor entries and the parts of the plot they correspond to. We thus refer to a tensor entry by the value exactly at the center of its range; e.g., in the above example our pixel would be \((1.5, 2.5)\). When we wish to refer to all of an entry’s possible values as opposed to just the center value, we refer to it as a subcube. We say that the bin-size—the range of each variable defining subcube—for this pixel is \((1, 1)\). For simplicity, we will assume that the bin sizes of all variables are equal, for every system we discuss.

Suppose each equation in the system has been plotted as a tensor, as above. To solve the simultaneous system of equations, these tensors are multiplied together by contracting various edges. The resulting tensor will then indicate all combinations of the remaining variables for which there exists a solution to the system.

What differentiates the PASS method from the PA method is that we are interested in finding numerical steady states, not just determining their existence for certain boundary conditions. Thus, the boolean definition of a pixel array is too limited. This is resolved by
introducing a new data type, the solutionSet. Each entry of our modified pixel array is of this data type. A solutionSet is a finite set of tuples of any finite length. In our work, the lengths of each tuple in a given solutionSet turn out to be the same, e.g. \{ (0, 1, 2), (1, 1, 1), (2, 1, 0) \}, because they will represent the set of solutions—in certain selected variables—to some system of equations.

To get some intuition for the solutionSet data type, consider the homogeneous heat equation:

\[ u_t - D(x)u_{xx} = 0 \]

With \( u_t = 0 \) and \( D(x) \equiv 1 \), the equation becomes

\[ u_{xx} = 0, \] (4)

otherwise known as the one-dimensional Laplace equation. After being processed by finite differences and multiplying both sides by \( h^2 \neq 0 \) we get:

\[ u_{i+1} - 2u_i + u_{i-1} = 0 \] (5)

The corresponding tensor has three dimensions, corresponding to the variables \( u_{i+1}, u_i, \) and \( u_{i-1} \) in Eq. (5). Each tensor entry corresponds to a subcube; let \((x, y, z)\) denote the center point of that subcube. The value in that entry is the solution-set for the equation Eq. (5). That is, the value will be the empty set \( \{ \} \) if the subcube centered at \((x, y, z)\) does not contain a solution \((u_{i+1}, u_i, u_{i-1})\) where value of each \( u_j \) is within the subcube, and it will be the one-element set \( \{(x, y, z)\}\) if there exists such a \((u_{i+1}, u_i, u_{i-1})\).

Tensor multiplication involves multiplication and addition. For boolean values, multiplication is as usual, e.g. \( 1 \times 0 = 0 \), and addition is almost as usual, the exception being that \( 1 + 1 = 1 \); this is the usual Boolean semiring, which is used in the PA method. For the PASS method, we need to define multiplication and addition of the solutionSet data type. This data type will store the solutions to the local discrete steady state condition.

The multiplication of solutionSets is defined to be the Cartesian product of the tuples, and addition is defined to be set union. So if two theoretical solutionSets, \( \{(1),(2),(3)\} \) and \( \{(3,4)\} \) were multiplied, the resultant tuple would be \( \{(1,3),(2,3),(3,3),(1,4),(2,4),(3,4)\} \). If they were added, we would get \( \{(1),(2),(3),(4)\} \). This defines a commutative semiring, whose additive unit is the empty set \( \{\} \) and whose multiplicative unit is \( \{()\} \). Thus for example, when a solutionSet is multiplied by \( \{\} \), the result is zero. When a solutionSet is added to \( \{\} \) or multiplied by \( \{()\} \), the solutionSet remains the same.

With this definition of multiplication, the generalized matrix multiplication algorithm remains the same as it was in (Spivak et al., 2016).

### 2.3 Producing Solutions

We will now demonstrate how the above definition of the solutionSet data type allows us to compute numerical steady states to a partial differential equation. Specifically, we will show how the numerical steady states are stored within solutionSets, and expanded as more multiplications are performed.

We show this by example. Once again, consider the heat equation (4) and its discrete version (5). As discussed in Section 2.2, all entries in the pixel array with dimensions \( u_{i+1}, u_i, \) and \( u_{i-1} \) contain the single solution \( (u_i) \) if \( u_i \) satisfies Eq. (5). This is true for all partitions \( i \).
With all the matrices defined, we will multiply them using the general matrix multiplication algorithm described in Section 2.2, but to do so we must determine which variables are exposed. We resolve this by introducing the **Wiring Diagram** (WD) for this system; see Fig. 2. As described in (Spivak et al., 2016), the WD lets us connect matrices by their variables and lets us decide which variables we want to be exposed when we have finished multiplying.

In Fig. 2, the black dots (and the lines connecting them to circles) are called *links*, while the circles are called *cells* (in (Spivak et al., 2016) they were called *packs*). Each cell represents one equation, its ports represent the variables of that equation, and the links represent the dependencies between variables. For example, because the equation for $u_i$ in Eq. (5) depends on $u_{i+1}$ and $u_{i-1}$, there are links connecting the cells representing $u_{i+1}$, $u_{i-1}$, and $u_i$.

The last thing to explain about Fig. 2 is the fact that we have both hidden boundaries, cells 0 and 6, and the exposed boundaries, cells 1 and 5. The boundary conditions for the heat equation reference the exposed boundaries. At first, we did not include hidden boundaries in our research. However, we found that in that case, we get many discontinuous or anomalous steady states due to the spatial discretization of equations with a left-right asymmetry. For example, this problem does not arise for the heat equation, because our equation is symmetric (it treats $u_{i+1}$ and $u_{i-1}$ the same way). By adding the hidden boundaries, the anomaly is partially handled.

Fig. 3 represents a multiplication of two 3-dimensional tensors $M_1$ and $M_2$, the result of which is a 4-dimensional tensor. It is given by contracting along the links connecting $M_1$ and $M_2$, which represent shared dimensions. Performing multiple such tensor multiplications, one obtains the tensor multiplication indicated by Fig. 2.

Now that we have explained how to multiply matrices and determine exposed variables and boundary conditions, we proceed to compute the value of a general solutionSet. By the definition of solutionSet multiplication defined in Section 2.2, if the entry $\{(u_1)\}$ is multiplied by the entry $\{(u_2)\}$, the resulting solutionSet is $\{(u_1, u_2)\}$. Recall from Section 2.2 that the entry $\{(u_1)\}$ would only occur if Eq. (5) were satisfied for that particular value. The same can be said for the entry $\{u_2\}$. Therefore, the product $\{(u_1, u_2)\}$ would only be produced if their values satisfied in Eq. (5) for both $i = 1$ and $i = 2$. This means that the final value within the

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5The dimension of a tensor is indicated in the diagram by its number of ports.
solutionSet is a tuple of numerical steady states, because each local numerical solution fulfills
the discrete steady state condition.

Thus, by multiplying all the matrices together, we calculate solutionSets of numerical
steady states. Each element within a single solutionSet satisfies the discrete steady state
condition locally. Furthermore, with the PASS method, we can calculate steady states for
all possible pairs of left and right boundary values at once, in some sense parallelizing that
computation.

3 Testing

We analyze the effectiveness of PASS by using it to produce steady states for 4 different
partial differential equations. In Section 3.1 we analyze the heat equation. In Section 3.2 we
test on the Fisher equation. Then, in Section 3.3, we examine the Benjamin-Bona-Mohany
equation. Finally, in Section 3.4, we look at the Sine-Gordon equation.

3.1 Homogeneous Heat Equation

We first consider the homogeneous heat equation as described by Eq. (4), whose discrete
steady state condition is Eq. (5). It is well known that steady states to the homogeneous heat
equation are linear between the two boundary conditions, and this fact can also be verified by
directly solving the equation.

Let us first check if PASS successfully finds all boundary conditions for which there
are steady states. For the heat equation, any pair of boundary conditions should have an
associated steady state. For possible boundary values in the range $[0, 2]$, our associated pixel
array is:

![Figure 4: The pixel array associated with the heat equation. Yellow indicates that at least one
numerical steady state was found for the boundary conditions indicated by the axes.]

This is our desired result. Now let us check whether the numerical steady states found by
PASS are sufficiently similar to the true, perfectly linear, steady state. One can prove that, for
any number of cells and any range, as the number of bins tends to infinity, the numerical
solution given by the PASS method converges to the true solution. Indeed, this is a direct
consequence of Theorem 3.4.1 in (Spivak et al., 2016).

With a finite number of bins, we can also theoretically bound the 'distance' between the
true solution and the results of the PASS method. Suppose there are $n$ cells, and for each
cell $1 \leq i \leq n$, let $u_i$ be the true solution to the heat equation at $x_i$ (these should be a linear
function of $i$) and let $v_i$ denote the numerical solution. By the definition of bins and the $L^\infty$
norm, the coordinates at any vertex of a subcube are $\pm \frac{b}{2}$ away from the center coordinates.
For a numerical solution to the heat equation to exist within the subcube, the value of equation Eq. (5) must be within \( \pm \epsilon \) of 0 for some \( \epsilon > 0 \). We calculate \( \epsilon \) by finding the maximum magnitude of the gradient within the subcube, and then multiplying that by the “radius,” or largest distance to any boundary, of the subcube. This calculation gives \( \epsilon = \frac{3b\sqrt{2}}{2} \). Thus the maximum difference from 0 at the center that still permits some point in the subcube to have the value 0 is \( \frac{3b\sqrt{2}}{2} \). In general, plotting a pixel array by comparing an expression’s value at the center of a subcube to an \( \epsilon \) will be called the epsilon method.

We can also calculate a functional bound of how much our solution could differ from the true steady state. We define this as the square root of the sum of the squared differences \( (v_i - u_i)^2 \) for all \( i \). After some calculation, we get:

\[
L = \epsilon \left[ \frac{n}{2} \right]^{5/2}
\]

where \( L \) is the modified \( L_2 \) norm. \( L \) clearly goes to 0 as \( b \) goes to 0.

We run the PASS method\(^2\) for a few combinations of \((b, n)\) to verify the bound in (6). The program partitions the spatial dimension into \( n = 8 \) parts (which we will call partitions) and a temperature mesh of \( b = .05 \), giving 21 bins for each cell. Using the single boundary value pair \([1.0, 0.1]\), the PASS method generated a total of 10 steady state approximations. 3 of those numerical steady states generated by PASS are shown in Fig. 5. As one can see, they are all approximately linear—a perfectly linear steady state is the true steady state of the the equation.

The largest modified \( L_2 \) norm among PASS’s solutions was 1.46. When \( b = .05 \) and \( n = 8 \) are substituted in equation Eq. (6), we get 3.39, which is greater than 1.46. Thus, all steady states’ norms are smaller than the bound determined by equation Eq. (6). This bound also holds for the steady states given by other pairs of boundaries. Furthermore, in figure Fig. 5, we see that the numerical steady states are approximately linear, giving visual confirmation that our approximations are close to the steady states. The runtime under these conditions was about 48 seconds in Python 3 for 441 boundary value pairs on a 2016 Macbook.

### 3.2 Fisher-KPP Equation

Now, we consider the classical Fisher-KPP equation (Berestycki et al., 2009). In one dimension, the equation is:

\[
u_t = \nu_{xx} + \mu \nu (1 - \nu)
\]

Where \( \mu \) represents the intrinsic growth rate of a species (Baeumer et al., 2008) when used in the field of population dynamics. Its corresponding discrete equation for finding steady states is:

\[
0 = \frac{1}{h^2}(\nu_{i+1} - 2\nu_i + \nu_{i-1}) + \mu \nu_i (1 - \nu_i)
\]

The only bounded steady states of this equation are \( \nu \equiv 1 \) and \( \nu \equiv 0 \) (Berestycki et al., 2009), for all values of \( \mu > 0 \) and spatial interval size \( h > 0 \).\(^2\)

\(^2\)The reader can find the code for this method at https://github.com/cynliu98/Pixel_Array_Python, called PASS.py, in the archives folder.
Our method of plotting the pixel arrays for this equation is slightly different than plotting
the array for the heat equation. It is essentially a stricter form of the epsilon method as defined
in the previous section. An outline of the method is:

1. Let the discrete reaction diffusion expression be \( f(u_{i+1}, u_i, u_{i-1}) \), and calculate its value.

2. Round that expression to the nearest bin, and round 0 to the nearest expanded bin, where
   possible bin values \( b_j \) are written as \( b_j = mj + c \) for \( j \in \mathbb{Z}, j \in [0, r] \) where \( r \) is the
   resolution, and expanded bin values are just values of \( b_j \) with unbounded, integer \( j \). If
   those rounded values are the same, then the corresponding entry in the Pixel Array is
   nonempty.

This method obviously converges as the size of bins \( b \to 0 \), because \( f(u_{i+1}, u_i, u_{i-1}) \) and 0
must round to the same expanded bin and smaller \( b \) restricts the range of values that could
round to the same bin. This is true for any fixed spatial interval size \( h \), including as \( h \to 0 \).

We run the program with 41 bins, \( b = .05 \) in the range \([0, 2]\) and 10 cells, rounding up.
We first test 3 values of \( h, h = 1, h = .25, \) and \( h = .1, \) all with \( \mu = 1 \). Because there are far
significantly fewer steady states (just \( u \equiv 1 \) and \( u \equiv 0 \)), we check for two things in evaluating
the PASS method. Firstly, we see if PASS gives \( u \equiv 1 \) and \( u \equiv 0 \) for all possible parameters.
Secondly, we check the number and value of false positives given by PASS. We present Tables
1(a) and 1(b) as results:

The two steady states reported for all values of \( n \) listed above were \( u_i \equiv 0 \) and \( u_i \equiv 1 \) as
desired.

| \( n \) | Number of Steady States |
|-------|-------------------------|
| 4     | 2                       |
| 6     | 2                       |
| 8     | 2                       |
| 16    | 2                       |
| 32    | 2                       |

| \( \mu \) | Number of Steady States |
|---------|-------------------------|
| .2      | 8                       |
| .5      | 4                       |
| 2       | 2                       |
| 5       | 2                       |

Table 1: Number of steady states in terms of \( n \) and \( \mu \).
For $n = 16$, we then tested $\mu = .2, .5, 2,$ and $\mu = 5$ ($\mu = 1$ was used in Table 1(a)).

The results are in Table 1(b). Once again, $u_i \equiv 0$ and $u_i \equiv 1$ were found for all parameters. All of the additional steady states found for $\mu = .2$ and $\mu = .5$ were also constant-valued. For $\mu = .2$ the addition constant steady states were $u_i \equiv .05, .10, .9, .95, 1.05,$ and 1.1. For $\mu = .5$ we had $u_i \equiv .05$ and 1.05. Observe that all of these values are clustered around 0 and 1.

### 3.3 Benjamin-Bona-Mahony

The Benjamin-Bona-Mahony equation (BBM, 2016) models long surface gravity waves—waves with large wavelength are caused by gravity. The equation is

$$u_t + u_x + uu_x - u_{xxt} = 0$$  \hspace{1cm} (9)

This equation was completely solved in 1979 by Olver (Olver, 1979). The discrete form for solving steady states is:

$$\frac{u_{i+1} - u_i}{h}(1 + u_i) = 0$$  \hspace{1cm} (10)

From the equation, we can see that all constant-valued solutions are steady states of the equation. For $n = 16, h = .05,$ and $u_i \in [0, 2]$ with a resolution of .05, the pixel array associated with finding all numerical steady states to this system is:

![Figure 6: The pixel array associated with the Benjamin-Bona-Mahony equation. Yellow indicates at least one steady state was found for the boundary conditions, purple otherwise.](image)

One can verify from the output of the program that the numerical steady state produced for each boundary condition is indeed constant-valued and equal to the value at the boundaries.

### 3.4 Sine-Gordon

Finally, we look at the Sine-Gordon equation, which is:

$$u_{tt} - u_{xx} + \sin u = 0$$  \hspace{1cm} (11)

Soliton solutions of this equation are $u = \pm 4 \tan^{-1} \left( \pm \frac{x-Ut}{\sqrt{1-U^2}} \right)$ (N.F. Smyth, 1999). In the limit case of $t \to \pm \infty$, for $U \neq 0, 1,$ $u$ approaches the constant steady states $x = \pm 4\pi/2 = \pm 2\pi$
or $x = 0$ (depending on the sign of $\frac{x-UL}{\sqrt{1-U^2}}$). Both of these steady state solutions are numerical steady states of the discretized form of equation 11:

$$\sin u_i - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$  \hspace{1cm} (12)

However, discretization gives us false positive steady states. In fact, it gives us infinitely many false positives: $u_i \equiv k\pi$ for all integers $k \forall i$. It is not obvious from observation whether there are even more numerical steady states. Thus, we run PASS on the discrete equation, and get, for $n = 8, b = .2, u_i \in [0, 7]$:

![Figure 7: The pixel array associated with Sine-Gordon equation.](image)

From the pixel array, we see that there are numerical solutions indicated only for homogenous boundary values at 0 and $\approx \pi, 2\pi$. The output of the PASS program confirms that the steady states at those boundaries are constant.

4 Discussion and Conclusion

The PASS method successfully finds all steady states to various partial differential equations within a bounding box. If exact steady states cannot be found, approximate steady states that are similar to theoretical ones are found. Thus, PASS is effective at approximating numerical steady states of partial differential equations.

The primary novel benefit of the PASS method is the parallelizing the computation of many steady states, and the visualization of many steady states at once in the resultant pixel array. By doing so, it can facilitate the discovery and confirmation of patterns.

The PASS method has the same limitations as the original PA method, such as producing false positives and visualization of complex systems, and some additional limitations associated with discretization. As noted in section Section 3.4, discretization methods like finite difference may produce equations with more numerical steady states than the original. This issue may be resolved with more advanced discretization techniques such as finite element.

Because PASS does a lot of array manipulation, it can be very slow, despite the original PA method being very efficient. This is because the product of solutionSets can result in an exponentially increasing number of steady states per pixel. Thus, even small systems analyzed with PASS might cause Python to crash if there are many steady states (such as with
the heat equation). This problem can be mitigated by using PASS in conjunction with the PA method. In other words, because the PASS method is able to extract solution sets, it can be executed after running the more efficient boolean pixel array method to locate the areas of interest on which to run PASS.

The PASS method may be extended to find steady states of complex PDEs and even two- or three-dimensional systems. This may done by adjusting the steady state condition, the dimensions of our pixel arrays, editing plotting code for those arrays. These are changed based on the discretization and wiring diagram of whatever new equation we are examining. However, because many changes are required, using PASS might less practical for people compared to the usual PA method.

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