Practical Prescribed-Time Event-Triggered Bipartite Consensus of Linear Multi-Agent Systems

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ABSTRACT In this paper, the practical prescribed-time bipartite consensus problem of linear multi-agent systems with a connected structurally balanced signed graph is investigated by using event-triggered mechanism. Novel event-triggered control law and triggering condition are proposed for each agent, in which the sampled states of agent and its neighbors are needed. By using the algebraic graph theory and Lyapunov stability theory, it is verified that the practical bipartite consensus can be achieved at fully prespecified time with an adjustable neighborhood range. Moreover, a theoretical discussion is provided to illustrate that Zeno behavior can be avoided during the whole time span. The feasibility of the presented control strategy is validated by a simulation result.

INDEX TERMS Practical prescribed-time consensus, bipartite consensus, event-triggered control, multi-agent systems.

I. INTRODUCTION

Over the past decade, the consensus problem of multi-agent systems has become one of the hottest topics due to its broad potential applications [1], [2]. In most of the previous works, continuous communication among neighboring agents is usually required in the process of achieving consensus. However, in the practical applications, agents are usually powered by limited energy sources and bandwidth. Sufficient communication resources and ideal communication environments cannot be guaranteed. Moreover, high communication frequency is not conducive to the service life of the agent. As a consequence, resource-efficient control strategies are urgently needed. Event-triggered control was then applied to the consensus problem of multi-agent systems [3]. Under an event-triggered control mechanism, the communication between neighbors is usually executed when a pre-designed triggering condition is satisfied. The consumption of communication resources would be reduced by intermittent communication with varying periods.

Numerous works have been devoted to the research for event-triggered control of multi-agent systems in recent years. For instance, event-triggered control was investigated for agents with integrator-type dynamics [3], [4], general linear dynamics [5], [6], and nonlinear dynamics [7]–[9]. It should be mentioned that the aforementioned works mainly focused on the asymptotic consensus control. However, as pointed out in [10], finite-time consensus control is preferable when higher precision and faster convergence rate are demanded. In this context, it is meaningful to investigate finite-time consensus under event-triggered control mechanisms. Different event-triggered control laws and triggering conditions were developed to achieve finite-time consensus for multiple integrator-like systems [11], [12], general linear systems [13] and nonlinear systems [14]. It is worthy mentioning that the state-dependent triggering condition of each agent presented in [11], [12] contains the continuous information of its neighbors, which is undesirable for saving resources. Therefore, the authors in [13] proposed a triggering condition without continuous inter-neighboring communication. However, under the control strategies proposed in [11]–[14], the settling time is depended on the initial conditions and would be going to adequately large with large initial states in a serious situation.

To overcome the aforementioned problem of finite-time consensus, the fixed-time control was thus introduced into the
consensus problem of integrator-type systems [15] and nonlinear systems [16], [17]. Furthermore, the event-triggered fixed-time consensus problem of multi-agent systems with uncertain nonlinear dynamics were considered in [18], [19]. Note that the upper bound of the settling time in fixed-time consensus control [15]–[19] is independent of initial conditions but is subject to certain restrictions. For making the settling time explicitly preassigned, the authors in [20] presented a prescribed-time control method for agents with high-order chain integrator dynamics to achieve leader-following consensus. Moreover, prescribed-time consensus and containment control of first-order multi-agent systems were investigated in [21]. The authors in [22] addressed the prescribed-time consensus problem for general linear multi-agent systems, because many physical systems are modeled by linear dynamics in real practice.

Note that most of the above literatures were focused on the multi-agent systems with cooperative communication links. However, competition is another inherent relationship among agents. In fact, both competition and cooperation usually exist in many complex practical systems [23]. A real group, which may be in social networks [24] or biological systems [25], often has divergences in the process of moving towards the goal. In engineering scenarios, there also exist similar phenomenons like bidirectional flying of unmanned air vehicles. As a result, great efforts have been made to the bipartite consensus of multi-agent systems with cooperation interactions [26]–[29], where all agents converge to a value with the same modulus but opposite signs. The authors in [27] studied the bipartite consensus problem of first-order multi-agent systems with intermittent interaction. The bipartite consensus problem of linear multi-agent systems with input saturation was examined in [28], where the low gain feedback technique was used. Moreover, the authors in [29] investigated the bipartite output containment control of linear heterogeneous multi-agent systems with three control protocols based on a distributed feed-forward approach.

Besides of the results of asymptotic bipartite consensus, finite-time bipartite consensus of first-order multi-agent systems with directional link failure was studied in [30]. The adaptive finite-time bipartite consensus protocol was designed for second-order multi-agent systems with external disturbances in [31], where the leaderless case and leader-follower case were discussed respectively. Moreover, fixed-time bipartite consensus of first-order multi-agent systems was successfully investigated in [32].

Driven by the demands for resource saving, event-triggered bipartite consensus control was applied to multi-agent systems with integrator-type dynamics [33], [34]. The bipartite containment consensus problem of linear multi-agent systems was studied in [35], where an observer-based event-triggered control mechanism was proposed. Most recently, the authors in [36] constructed event-triggered control ensuring prescribed-time bipartite consensus of first-order multi-agent systems, where the settling time is completely preassigned and inter-neighboring communication among agents is discontinuous. However, the results of agents with single-integrator dynamics cannot be extended directly to general linear multiple agents. To the best of our knowledge, there have been few efforts dedicated to the prescribed-time bipartite consensus for general linear multi-agent systems via event-triggered control, which is the main motivation of this paper. The contribution and novelty of our paper is twofold.

(i) A new practical event-triggered prescribed-time bipartite consensus control strategy for linear multi-agent systems is proposed, in which the settling time is fully prespecified and irrelevant to initial conditions and any other parameters. Besides, the communication resources would be saved due to the event-triggered control strategy and the state-dependent triggering condition based on sampled neighbor agents’ information.

(ii) The communication topology studied in this paper is more general and significant, since cooperative and competitive interactions coexist in it.

The rest of this paper is organized as follows. In Section II, some preliminaries and problem statements are briefly presented. The primary theoretical results are addressed in Section III. Section IV provides the simulation example, while Section V concludes the paper.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. NOTATIONS

Denote \( \mathbb{R}^n \) as the \( n \)-dimension Euclidean space and \( \mathbb{R}^{n \times m} \) the \( n \times m \) real matrix space. \( I_n \) is the \( n \times 1 \) column vector with all entries equal to one and \( I_p \) is the \( n \times n \) identity matrix. \(|\cdot|\) and \( \|\cdot\| \) stand for the modulus of a scalar and the Euclidean norm of a vector, respectively. Let \( \text{sign}(\cdot) \) denote the sign function. The Kronecker product of matrices \( A \) and \( B \) is denoted by \( A \otimes B \). By \( \lambda_{\text{min}}(\cdot) \) and \( \lambda_{\text{max}}(\cdot) \), we denote the smallest and largest eigenvalue of the matrix.

B. GRAPH THEORY

The communication topology of the multi-agent systems with coopetition relationship is normally modeled by a signed graph \( \mathcal{G}(A) = (V, E, A) \). \( V = \{1, 2, \cdots, N\} \) is the set of nodes and \( E \subseteq V \times V \) is the set of edges, in which an edge is represented by a pair of distinct nodes. If \( (i, j) \in E \), it indicates that \( i \) can receive the information from \( j \) and \( j \) is called as a neighbor of \( i \). Moreover, the neighbor set of node \( i \) is defined as \( N_i = \{j \in V | (j, i) \in E\} \). The adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \) is defined by \( a_{ij} = 0 \) and \( a_{ij} \) is equal to the weight associated with edge \( (i, j) \). Clearly, \( a_{ij} \) is positive or negative signifies that the interaction between node \( i \) and \( j \) is collaborative or antagonistic. When there is no communication between \( i \) and \( j \), \( a_{ij} = 0 \).

For an undirected graph \( \mathcal{G}(A) \), \( (i, j) \in E \) if and only if \( (j, i) \in E \), and meanwhile \( a_{ij} = a_{ji} \) holds. A path from node \( i \) to node \( j \) is a sequence of edges \( \{(i, k), (k, l), \cdots, (r, j)\} \) where all nodes \( i, k, l, \cdots, r, j \) are distinct. If there exists a path between every pair of nodes in \( \mathcal{G}(A) \), then \( \mathcal{G}(A) \) is called connected [37]. Furthermore, The Laplacian matrix
where $T$ is the settling time and $c$ is a positive constant that can be adjusted to the appropriate range.

In this work, an event-triggered control strategy will be designed for system (1) such that the states of all the agents achieve practical prescribed-time bipartite consensus. For each agent, the construction of the control law requires only the sampled information of itself and its neighbors, which makes the update of the control law occurs when the triggering condition is satisfied. Meanwhile, a state-dependent triggering condition is designed based on sampled states such that the verification of it is independent of continuous communication.

In order to solve the problem, the following formulas are needed in this paper.

\begin{align*}
\text{Lemma 3 (Young’s Inequality [38]):} & \quad \text{For any vectors } x, y \in \mathbb{R}^n \text{ and } \epsilon > 0, \text{ it holds that } x^T y \leq \frac{\epsilon x^T x}{2} + \frac{\epsilon y^T y}{2}. \\
\text{Lemma 4:} & \quad \text{For three constants } \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R} \text{ that satisfy } \\
& \quad \epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0, \text{ and } \epsilon_1 \neq \epsilon_2, \epsilon_3 = \min(\epsilon_1, \epsilon_2), \text{ we have} \\
& \quad 0 \leq \frac{1}{\epsilon_2 - \epsilon_1} |(e^{-\epsilon_1 t} - e^{-\epsilon_2 t})| \leq \frac{1}{|\epsilon_2 - \epsilon_1|} e^{-\epsilon_3 t}, \quad t \in [0, +\infty).
\end{align*}

\[ \text{Proof: For } t > 0, \text{ if } \epsilon_1 < \epsilon_2, \text{ it is clear that } 0 \leq (e^{-\epsilon_1 t} - e^{-\epsilon_2 t})/(\epsilon_2 - \epsilon_1) \leq e^{-\epsilon_3 t}/|\epsilon_2 - \epsilon_1| \text{. Similarly, the conclusion holds if } \epsilon_1 > \epsilon_2. \]

### III. MAIN RESULTS

In this section, we address the control strategy and discuss the consensus performance.

#### A. EVENT-TRIGGERED CONTROL MECHANISM

For achieving control objectives, a time-varying scaling function $\mu(t)$ similar to the one defined in [21] is proposed as

\[ \mu(t) = \begin{cases} 
\frac{T^h}{(T - t)^h}, & t \in [0, T), \\
1, & t \in [T, \infty),
\end{cases} \]

where $h > 0$, $T$ is the desired settling time and $\tilde{T}$ is a larger value than $T$. For $m > 0$, it is easy to observe that $\mu^m(t) - \mu^{-m}(t)$ is monotonically increasing (decreasing) on $[0, T)$ with $\mu^m(0) - \mu^{-m}(0) = 1$ and $\lim_{t \to \tilde{T}^+} \mu^m(t) = +\infty$ $(\lim_{t \to \tilde{T}^-} \mu^m(t) = 0)$.

The event-triggered control law for each agent is designed as

\[ u_i(t) = -\left(\beta + \gamma \mu(t) K \sum_{j \in N_i} a_{ij} \left( e^{\tilde{\Delta} \left( -\gamma \theta_i \right)} x_j(t_j^l) \right) - \text{sign}(a_{ij}) e^{\tilde{\Delta} \left( -\gamma \theta_i \right)} x_i(t_i^l) \right), \quad t \in [t_{i, l}, t_{i, l+1}^l) \]

(3)

in which the parameters are specified as follows.

- $\beta > 0$ is a scalar control gain satisfying

\[ \beta > \frac{1}{2\lambda_2} \]

(4)

with $\lambda_2$ defined in Lemma 2;
where $Q$ is the unique positive-definite solution of the algebraic Riccati equation [39]
\[
A^T Q + QA - QBB^T Q + I = 0; \quad (6)
\]

**•** For agent $i(j)$, $t_{i_0}^j(t_{i_0}^j)$ is the latest triggering instant before $t$ and $x_j(t_{i_0}^j(x_j(t_{i_0}^j)))$ denotes the sampled states at time $t_{i_0}^j(t_{i_0}^j)$.

Before giving the triggering condition, we introduce the state measurement error of agent $i (i \in \mathcal{V})$ as
\[
e_i(t) = \tilde{x}_i(t) - x_i(t), \quad t \in [t_{i_0}^j, t_{i_0}^j+1), \quad (7)
\]

with $\tilde{x}_i(t) \triangleq e^{(t-t_{i_0}^j)}x_i(t_{i_0}^j)$.

The triggering instant $t_{i_0}^j$ is determined by
\[
t_{i_0}^j+1 \triangleq \inf \left\{ t > t_{i_0}^j : f(t, e_i(t), \tilde{x}_i(t), \tilde{x}_j(t)) \geq 0 \right\},
\]

where $f(t, e_i(t), \tilde{x}_i(t), \tilde{x}_j(t)) \geq 0$ is called the triggering condition. Furthermore, $f(t, e_i(t), \tilde{x}_i(t), \tilde{x}_j(t))$ is given as
\[
f(t, e_i(t), \tilde{x}_i(t), \tilde{x}_j(t)) = \alpha_i\|K\|^2\|e_i(t)\|^2 - \frac{\kappa \beta}{3\beta + \gamma \mu(t)} \sum_{j \in \mathcal{N}_i} |a_{ij}|(\tilde{x}_i(t) - \text{sign}(a_{ij})\tilde{x}_j(t)) - \eta \varsigma(t), \quad (8)
\]

where $\alpha_i \triangleq \sum_{j \in \mathcal{N}_i} |a_{ij}|$, $\Gamma \triangleq \begin{pmatrix} QBB^T Q, \kappa \end{pmatrix}$. $\kappa$ and $\eta$ are positive constants to be designed. Meanwhile, $\varsigma(t)$ is defined as
\[
\varsigma(t) = \begin{cases} 
\mu^{-1-\frac{1}{2}}(t), & t \in [0, T), \\
\bar{e}^{-\beta t}, & t \in [T, \infty),
\end{cases} \quad (9)
\]

with $\theta > 0$. Without loss of generality, we assume $t_0^j = 0$, $i \in \mathcal{V}$.

In view of (7), the controller (3) changes into
\[
u_i(t) = -\beta K \sum_{j \in \mathcal{N}_i} |a_{ij}|(x_i(t) + e_i(t)) - \gamma \mu(t)K \sum_{j \in \mathcal{N}_i} |a_{ij}|(\tilde{x}_i(t) - \text{sign}(a_{ij})\tilde{x}_j(t)). \quad (10)
\]

Combining (1) and (10), we obtain the following closed-loop dynamics
\[
\dot{x}(t) = (I_N \otimes A - \beta L \otimes BK)x(t) - \beta(L \otimes BK)e(t) - \gamma \mu(t)(L \otimes BK)\tilde{x}(t), \quad (11)
\]

where $x(t)$, $\tilde{x}(t)$ and $e(t)$ are stack vectors defined as $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$, $\tilde{x}(t) = [\tilde{x}_1^T(t), \tilde{x}_2^T(t), \ldots, \tilde{x}_N^T(t)]^T$ and $e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T$, respectively.

**Remark 1:** Considering the definition of $\mu(t)$, the choice of $\tilde{T}$ is aimed to prevent the function from becoming infinite at the desired settling time $T$. Moreover, the increasing speed of the function can be controlled by adjusting the value of $h$.

**Remark 2:** It can be seen from (3) that the control gain $\beta + \gamma \mu(t)$ is time-varying on $[0, T)$, in which the scaling function $\mu(t)$ would play an important role in achieving practical consensus at the prespecified time $T$. Since $\mu(t)$ is monotonically increasing on $[0, T)$, the control input can grow rapidly to meet the requirement of the convergence rate. Moreover, $\mu(t)$ remains $1$ for $t \geq T$, which makes the control law collapse into the conventional event-triggered asymptotic consensus protocol and guarantees the desired consensus performance over $[T, \infty)$. Especially, $\mu(t)$ does not become infinity as time approaches $T$ because of $\tilde{T} > T$.

**Remark 3:** Since the final convergence value of linear multi-agent systems is a dynamic expression related to $e^{N}$ rather than a fixed constant, the state estimation model $e^{(t-t_{i_0}^j)}x_i(t_{i_0}^j)$ was applied to estimate the actual states more accurately.

**Remark 4:** It is worth noting that both the state-dependent term and the time-dependent term are included in the triggering condition (8)-(9), in order to reduce the communication frequency and avoid Zeno behavior. Since the sampled states are included in (8), the implementation of the triggering condition do not use continuous inter-neighboring communication. A time-varying function $\varsigma(t)$ in (9) is designed based on $\mu(t)$ and an exponential function, which is necessary for the consensus performance and the exclusion of Zeno behavior.

**B. CONSENSUS ANALYSIS**

Some intermediate variables are firstly introduced to facilitate the analysis.

Along the path of Lemma 1, we denote $z_i(t) = d_i x_i(t)$ and $\tilde{z}_i(t) = d_i \tilde{x}_i(t)$ for $i \in \mathcal{V}$. Recalling the definition of $e_i(t)$, we have $e_i(t) = d_i e_i(t)$. Moreover, the state vectors are given as $z_i(t) = [z_i^1(t), z_i^2(t), \ldots, z_i^N(t)]^T$, $\tilde{z}_i(t) = [\tilde{z}_i^1(t), \tilde{z}_i^2(t), \ldots, \tilde{z}_i^N(t)]^T$ and $e_i(t) = [e_i^1(t), e_i^2(t), \ldots, e_i^N(t)]^T$. Obviously, $z_i(t) = (D \otimes I_n)x_i(t)$ and $e_i(t) = (D \otimes I_n)e_i(t)$ are valid. Meanwhile, similar to (11), we have the dynamics of $z(t)$ presented as
\[
\dot{z}(t) = (I_N \otimes A - \beta L \otimes BK)z(t) - \beta(L \otimes BK)e(t) - \gamma \mu(t)(L \otimes BK)\tilde{z}(t), \quad (12)
\]

with $L$ proposed in Lemma 1.

Thus, it is easy to see that the bipartite consensus of the system (11) has the same meaning as the conventional consensus of the system (12).

Denote $\xi_i(t) = z_i(t) - (1/N)\sum_{j=1}^N z_j(t), i \in \mathcal{V}$, such that
\[
\xi_i(t) - \dot{\xi}_i(t) = z_i(t) - \dot{z}_i(t).
\]

Subsequently, with the stack vector $\xi(t) = [\xi_1^T (t), \xi_2^T (t), \ldots, \xi_N^T (t)]^T$, we can write $\xi(t)$ in a compact form as
\[
\dot{\xi}(t) = (M \otimes I_n)\xi(t), \quad (13)
\]

with $M = I_N - (1/N)11^T$. It is not hard to see that $M \tilde{L} = \tilde{L} \tilde{M}$ and $\dot{\xi}(t)$ satisfies
\[
\dot{\xi}(t) = (I_N \otimes A - \beta L \otimes BK)\xi(t) - \beta(L \otimes BK)e_i(t) - \gamma \mu(t)(L \otimes BK)\tilde{z}(t). \quad (14)
\]
With above analyses in the context, we are now in a position to present our main results.

**Theorem 1:** Consider the multi-agent system (1) with communication graph $G(A)$ under Assumptions 1 and 2. The control law is proposed as (3), in which the control gain scalars $\beta$ and $\gamma$ satisfy $1/(2\kappa_2) \leq \beta \leq \gamma$. Moreover, the triggering condition in (8)-(9) is presented with $0 < \kappa \leq 1$, $0 < \eta \leq V(0)/(4N\gamma \lambda_{\text{max}}(Q))$, and $\theta > 0$. Consequently, the practical prescribed-time bipartite consensus of the system (1) is achieved.

**Proof:** Consider the following Lyapunov function candidate $V(t) = \frac{1}{2} \xi^T(t)(I_N \otimes Q)\xi(t)$, which is evidently positive-definite. The time derivative of $V(t)$ along the trajectory of the dynamics (14) is

$$
\dot{V}(t) = \xi^T(t)(I_N \otimes QA - \beta \tilde{L} \otimes QBK)\xi(t)
- \beta z^2(t)(\tilde{L} \otimes QBK)e_c(t)
- y\mu(t)\xi^T(t)(\tilde{L} \otimes QBK)\xi(t).
$$

By noting (13), we have

$$
\xi^T(t)(\tilde{L} \otimes QBK)e_c(t)
= z^T(t)(\tilde{M} \otimes QBK)e_c(t)
= (\tilde{z}(t) - e(t))^T(\tilde{L} \otimes QBK)e_c(t)
= z^T(t)(\tilde{L} \otimes QBK)e_c(t) - e^T(t)(\tilde{L} \otimes QBK)e_c(t).
$$

and

$$
\xi^T(t)(\tilde{L} \otimes QBK)\tilde{z}(t)
= z^T(t)(\tilde{L} \otimes QBK)\tilde{z}(t) - e^T(t)(\tilde{L} \otimes QBK)\tilde{z}(t).
$$

Then, substituting (16) and (17) into (15), one gets

$$
\dot{V}(t) = \xi^T(t)(I_N \otimes QA - \beta \tilde{L} \otimes QBK)\xi(t)
+ \beta z^2(t)(\tilde{L} \otimes QBK)e_c(t)
- y\mu(t)z^T(t)(\tilde{L} \otimes QBK)\tilde{z}(t)
+ y\mu(t) - \beta e^T(t)(\tilde{L} \otimes QBK)\tilde{z}(t).
$$

Since $K = B^TQ$, $\bar{a}_{ij} = \tilde{a}_{ij}$ and $\tilde{a}_{ij} = |a_{ij}|$, it is not difficult to verify that

$$
e^T(t)(\tilde{L} \otimes QBK)e_c(t)
= \sum_{i=1}^{N} e_{ij}(t)QBK^TQ \sum_{j=1}^{N} \bar{a}_{ij}(e_{ij}(t) - e_{ij}(t))
= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(e_{ij}(t) - e_{ij}(t))\Gamma(e_{ij}(t) - e_{ij}(t))
\leq \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}e^2_{ij}(t) + \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}e^2_{ij}(t)\Gamma e_{ij}(t)
= 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}e^2_{ij}(t)\Gamma e_{ij}(t)
= 2 \sum_{i=1}^{N} \alpha_{i}K^2\|e_{ij}(t)\|^2
$$

and

$$
\xi^T(t)(\tilde{L} \otimes QBK)\tilde{z}(t)
= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(\tilde{z}_i(t) - \tilde{z}_j(t))^T \Gamma(\tilde{z}_i(t) - \tilde{z}_j(t)).
$$

where $\Gamma = QBBK^TQ$ and $\alpha_{i} = \sum_{j \in N_i} |a_{ij}|$ are defined in (8). Moreover, Young’s Inequality (Lemma 3) is used in (19) to get the inequality.

According to the above analysis and using Young’s Inequality, we obtain that

$$
(\gamma\mu(t) - \beta)e^2(t)(\tilde{L} \otimes QBK)\xi(t)
= \frac{\gamma\mu(t) - \beta}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(e_{ij}(t) - e_{ij}(t))^T \Gamma(\tilde{z}_i(t) - \tilde{z}_j(t))
\leq \frac{\gamma\mu(t) - \beta}{2} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(e_{ij}(t) - e_{ij}(t))^T \Gamma e_{ij}(t)
- e_{ij}(t) + \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(\tilde{z}_i(t) - \tilde{z}_j(t))^T \Gamma(\tilde{z}_i(t) - \tilde{z}_j(t)) \right)
\leq \frac{\gamma\mu(t) - \beta}{2} \left( \sum_{i=1}^{N} \alpha_{i}K^2\|e_{ij}(t)\|^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(\tilde{z}_i(t) - \tilde{z}_j(t))^T \Gamma(\tilde{z}_i(t) - \tilde{z}_j(t)) \right)
$$

since $0 < \beta \leq \gamma$ and $\mu(t) \geq 1$.

Substituting (19), (20) and (21) into (18) yields

$$
\dot{V}(t) \leq \xi^T(t)(I_N \otimes QA - \beta \tilde{L} \otimes \Gamma)\xi(t)
- \frac{\beta}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(\tilde{z}_i(t) - \tilde{z}_j(t))^T \Gamma(\tilde{z}_i(t) - \tilde{z}_j(t))
+ \frac{3\beta + \gamma\mu(t)}{2} \sum_{i=1}^{N} \alpha_{i}K^2\|e_{ij}(t)\|^2.
$$

In light of the definition of $e_{ij}(t)$ and $\tilde{z}_i(t)$, it can be seen from the triggering condition (8) that

$$
\alpha_{i}K^2\|e_{ij}(t)\|^2
= \alpha_{i}K^2\|e_{ij}(t)\|^2
\leq \frac{\kappa\beta}{3\beta + \gamma\mu(t)} \sum_{j \in N_i} |a_{ij}|(\tilde{z}_i(t) - \text{sign}(a_{ij})\tilde{x}_j(t))^T
\times \Gamma(\tilde{z}_i(t) - \text{sign}(a_{ij})\tilde{x}_j(t)) + \eta\xi(t)
= \frac{\kappa\beta}{3\beta + \gamma\mu(t)} \sum_{j=1}^{N} \bar{a}_{ij}(\tilde{z}_i(t) - \tilde{z}_j(t))^T \Gamma(\tilde{z}_i(t) - \tilde{z}_j(t))
+ \eta\xi(t).
$$

Thus, (22) changes into

$$
\dot{V}(t) \leq \xi^T(t)(I_N \otimes QA - \beta \tilde{L} \otimes \Gamma)\xi(t)
- \frac{(1 - \kappa)\beta}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij}(\tilde{z}_i(t) - \tilde{z}_j(t))^T \Gamma e_{ij}(t)
$$

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\[
\begin{align*}
&\times (\dot{z}_i(t) - \dot{z}_j(t)) + \frac{3\beta + \nu \mu(t)}{2} \sum_{i=1}^{N} \eta \zeta_i(t) \\
&= \xi^T(t)(L_N \otimes QA - \beta \bar{L} \otimes \Gamma)\xi(t) \\
&\quad - \frac{(1 - \kappa)\beta}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{a}_{ij} \|K(\dot{z}_i(t) - \dot{z}_j(t))\|^2 \\
&\quad + \frac{3\beta + \nu \mu(t)}{2} N \eta \zeta(t) \\
&\leq \xi^T(t)(L_N \otimes QA - \beta \bar{L} \otimes \Gamma)\xi(t) \\
&\quad + \frac{3\beta + \nu \mu(t)}{2} N \eta \zeta(t),
\end{align*}
\]

in which we have used the condition \(0 < \kappa \leq 1\) to get the last inequality.

Since Assumption 1 holds, it is easy to see from Lemma (2) that \(\xi^T(t)(L_N \otimes \Gamma)\xi(t) \geq \lambda_2 \xi^T(t)(L_N \otimes \Gamma)\xi(t)\). Considering \(\beta \geq 1/(2\lambda_2)\) and substituting (6) into (23), we obtain

\[
\hat{V}(t) \leq \frac{1}{2} \xi^T(t)(L_N \otimes (A^T Q + QA - QBB^T Q))\xi(t) \\
\quad + \frac{3\beta + \nu \mu(t)}{2} N \eta \zeta(t) \\
\leq \frac{1}{2} \omega V(t) + \frac{3\beta + \nu \mu(t)}{2} N \eta \zeta(t),
\]

with \(\omega \triangleq 1/\lambda_{\text{max}}(Q)\).

Subsequently, we will investigate the practical prescribed-time bipartite consensus in the following two steps.

**Step 1.** Considering the time bounded on \(t \in [0, T]\), recalling the definition of \(\mu(t)\) and \(\zeta(t)\), (24) can be rewritten as

\[
\hat{V}(t) \leq \frac{1}{2} \omega V(t) + \frac{3\beta + \nu \mu(t)}{2} N \eta \mu^{-1/(1-\kappa)}(t) \\
\leq \frac{1}{2} \omega V(t) + \frac{3\beta + \nu \mu(t)}{2} N \eta \mu^{-1/(1-\kappa)}(t) \\
= \frac{1}{2} \omega V(t) + 2N \eta \gamma \frac{\bar{h}(\frac{1}{1-\kappa})}{T - t} \\
= \frac{1}{2} \omega V(t) - 2N \eta \gamma \frac{1}{T} t + 2N \eta \gamma.
\]

According to the comparison theorem [40], we obtain that

\[
V(t) \leq e^{-\frac{1}{2} \omega t} V(0) - \frac{2N \eta \gamma}{T} \int_0^t e^{-\frac{1}{2} \omega(t-\tau)} d\tau \\
+ 2N \eta \gamma \int_0^t e^{-\frac{1}{2} \omega(t-\tau)} d\tau
\]

with \(t \in [0, T]\).

Then, it is not difficult to verify that \(\int_0^t e^{-\frac{1}{2} \omega(t-\tau)} d\tau = 2(1 - e^{-\frac{1}{2} \omega t})/\omega \) and \(\int_0^t e^{-\frac{1}{2} \omega(t-\tau)} d\tau = 2(t + 2e^{-\frac{1}{2} \omega(t-\tau)})/\omega - 2/\omega\). Consequently, it leads to

\[
V(t) \leq \left( V(0) - \frac{8N \eta \gamma}{\omega^2 T} - \frac{4N \eta \gamma}{\omega} \right) e^{-\frac{1}{2} \omega t} \\
\quad - \frac{4N \eta \gamma}{\omega T} t + \frac{8N \eta \gamma}{\omega^2 T} + \frac{4N \eta \gamma}{\omega}.
\]

Since \(0 < T < \bar{T}, 0 < \eta \leq \omega V(0)/(4N \gamma)\) and \(e^{-\frac{1}{2} \omega T} \leq e^0 = 1\), we have \(\lim_{t \to T} V(t) \leq V(0) - 4N \eta \gamma T/(\omega \bar{T}) \leq c_1\), where \(c_1\) is a positive constant. Thus, by the definition of \(\dot{\xi}(t)\) and \(V(t)\), we obtain that

\[
\lim_{t \to \bar{T}} \left\| \dot{z}(T) - \frac{1}{N} \sum_{j=1}^{N} z_j(T) \right\| \leq \| \dot{\xi}(T) \| \leq \frac{2c_1}{\lambda_{\text{min}}(Q)}.
\]

**Step 2.** Considering the time bounded on \(t \in [T, \infty)\), it can be seen from the definition of \(\mu(t)\) and \(\zeta(t)\) that

\[
\dot{V}(t) \leq -\frac{1}{2} \omega V(t) + \frac{3\beta + \nu \mu(t)}{2} N \eta \gamma e^{-\theta t}.
\]

Similar to the analysis in **Step 1**, we have

\[
V(t) \leq e^{-\frac{1}{2} \omega t} V(0) + \frac{(3\beta + \nu \mu(t)) N \eta \gamma}{2} \int_{0}^{T} e^{-\frac{1}{2} \omega(t-\tau)} e^{-\theta \tau} d\tau
\]

\[
\leq e^{-\frac{1}{2} \omega T} V(0) + \frac{(3\beta + \nu \mu(t)) N \eta \gamma}{2} (e^{-\theta T} - e^{-\frac{1}{2} \omega T})
\]

\[
\leq \left( V(0) + \frac{(3\beta + \nu \mu(t)) N \eta \gamma}{|\omega - 2\theta|} \right) e^{-\psi t}
\]

with \(\psi = \min(\theta, \omega/2)\), where Lemma 4 is applied to get the last inequality. Subsequently, it is obvious that \(\lim_{t \to \infty} V(t) = 0\), which means that

\[
\lim_{t \to \infty} \left\| \dot{z}(t) - \frac{1}{N} \sum_{j=1}^{N} z_j(t) \right\| = 0.
\]

Result from analyses of the above two steps, it can be concluded that the asymptotic consensus of (12) is achieved, which implies that the bipartite consensus of (11) is achieved. As stated in Definition 1 and 2, the conditions of the practical prescribed-time bipartite consensus of system (1) are satisfied. The proof is completed. \(\square\)

**Remark 5:** In light of the definition of \(\mu(t)\), it can be seen from (2) that the settling time \(T\) can be set arbitrarily. Moreover, from (25), the state disagreement will converge to a neighborhood of the origin at \(T\), where the range of the neighborhood relies on \(c_1\). It is worth noting that the value of \(c_1\) can be adjusted by the designed parameters \(\gamma\) and \(\eta\). In other words, by choosing appropriate \(\gamma\) and \(\eta\), we would obtain a desired convergence neighborhood at \(T\).

**Theorem 2:** Under the event-triggered control strategy (3) and (8)-(9), the multi-agent system (1) will not exhibit Zeno behavior.

**Proof:** For agent \(i\), the derivative of \(e_i(t)\) is

\[
\dot{e}_i(t) = A\tilde{x}_i(t) - \dot{x}_i(t) = Ae_i(t) + (\beta + \nu \mu(t)) B K \sum_{j \in N_i} |a_{ij}| (\tilde{x}_j(t) - \text{sign}(a_{ij}) \tilde{x}_j(t)).
\]

Considering the convergence of the states, it is easy to see that \(\|\tilde{x}_i(t) - \text{sign}(a_{ij}) \tilde{x}_j(t)\|\) has a upper bound \(\tilde{X}_i\) between two
adjacent triggering instant of $i$. Thus, it leads to
\[
\|\dot{e}_i(t)\| \leq \|A\|\|e_i(t)\| + c_2(\beta + \gamma \mu(t))
\] (26)

with $c_2 \triangleq \|BK\| \sum_{j \in N_i} |a_{ij}| \bar{x}_j$.

Then, it can be seen from the triggering condition (8) that
\[
\alpha_i \|K\|^2 \|e_i(t)\|^2 \\
\leq \frac{\kappa \beta}{\kappa \beta (3 \beta + \gamma \mu(t))} \sum_{j \in N_i} |a_{ij}| \|\bar{x}_j(t)\|^2 - \text{sign}(a_{ij}) \bar{x}_j(t)) + \eta \varsigma(t)
\]
and
\[
\|e_i(t)\| \leq \sqrt{\frac{\kappa \beta}{\alpha_i (3 \beta + \gamma \mu(t))} \sum_{j \in N_i} |a_{ij}| \|\bar{x}_j(t)\|^2 + \frac{\eta \varsigma(t)}{\alpha_i \|K\|^2}}
\]

with $c_3 \triangleq \sum_{j \in N_i} |a_{ij}| \bar{x}_j^2$.

Substituting (27) into (26), we get
\[
\|\dot{e}_i(t)\| \leq \|A\| \sqrt{\frac{\kappa \beta}{\alpha_i (3 \beta + \gamma \mu(t))} \sum_{j \in N_i} |a_{ij}| \|\bar{x}_j(t)\|^2 + \frac{\eta \varsigma(t)}{\alpha_i \|K\|^2}}
\]

\[+ c_2(\beta + \gamma \mu(t)).
\] (28)

Like the approach used in the consensus analysis, we need to analyze the time range $t \in [0, T]$ and $t \in [T, \infty)$ respectively.

Firstly, for $t \in [0, T)$, we have $\mu(t) \leq \mu(T)$ since $t \leq T < T$, where $t_i$ is latest sampling instant of agent $i$ before $t$. Therefore, in view of the definition of $\varsigma(t)$, it can be concluded from (28) that
\[
\|e_i(t)\| \leq \sqrt{\frac{\kappa \beta}{\alpha_i (3 \beta + \gamma \mu(t))} \sum_{j \in N_i} |a_{ij}| \|\bar{x}_j(t)\|^2 + \frac{\eta \varsigma(t)}{\alpha_i \|K\|^2}}
\]

\[+ c_2(\beta + \gamma \mu(t)).
\]

Hence, a lower bound of the inter-event interval of agent $i$ is determined by $\tau_i = t - t_i$, which is given by
\[
\Psi_1 \tau_i \geq \sqrt{\frac{\kappa \beta}{\alpha_i (3 \beta + \gamma \mu(t))} \sum_{j \in N_i} |a_{ij}| \|\bar{x}_j(t)\|^2} + \frac{\eta \varsigma(t)}{\alpha_i \|K\|^2} + c_2(\beta + \gamma \mu(t)).
\]

Consequently, it can be concluded that the solution $\tau_i$ is strictly positive, which means that Zeno behavior is avoided for $t \in [0, T]$.

Next, we analyze the case of $t \in [T, \infty)$, in which $\mu(t) = 1$ and $\varsigma(t) = e^{-\eta t}$. As a consequence, we get the following formula from (28) that
\[
\|e_i(t)\| \leq \sqrt{\frac{\kappa \beta c_3}{\alpha_i (3 \beta + \gamma)} + \eta e^{-\eta t}}
\]

\[+ c_2(\beta + \gamma \mu(t)).
\]

Similarly, a lower bound $\tau_i$ is solved with
\[
\Psi_2 \tau_i \geq \sqrt{\frac{\kappa \beta c_3}{\alpha_i (3 \beta + \gamma)} + \eta e^{-\eta t}} \\
\geq \Psi_2(t - t_i).
\]

Obviously, the solution $\tau_i$ is strictly positive. Furthermore, we can conclude that Zeno behavior is not exhibited for $t \in [T, \infty)$. The proof is completed.

Remark 6: The design of the function $\varsigma(t)$ in (9) is not unique, which means that other functions satisfying the control requirements can be found. Since $\mu(t)$ is included in (24), which is monotonically increasing for $t \in [0, T)$, the function $\varsigma(t)$ should contain $\mu^{-\beta}(t)$ with $\bar{m} \geq 1$. For $t \in [T, \infty)$, the function $\varsigma(t)$ should be monotonically decreasing to zero or equal to zero to guarantee the asymptotic convergence. Considering the above conditions, we can give an example of $\varsigma(t)$ as
\[
\varsigma(t) = \begin{cases} 
\mu^{-\bar{m} - \frac{1}{\bar{\eta}}}(t), & t \in [0, T), \\
0, & t \in [T, \infty).
\end{cases}
\]

With this function, the proof of Theorem 1 and 2 can also be implemented through aforementioned steps. However, such $\varsigma(t)$ will lead to smaller inter-event intervals than that of (9), since $\mu^{-\bar{m} - \frac{1}{\bar{\eta}}}(t) \geq \mu^{-\bar{m} - \frac{1}{\eta}}(t)$ and $e^{-\eta t} \geq 0$. Hence, $\varsigma(t)$ is preferable to other functions.

Remark 7: Along with the development of event-triggered control, the comparison between event-triggered control and sampled-data control [41] has also been investigated. For example, in [42], the authors proposed an event-triggered control scheme for discrete-time linear systems with Gaussian white noise disturbances. They proved that the event-triggered scheme proposed outperforms (in a quadratic cost sense) sampled-data control for the same average transmission rate. For the multi-agent systems under event-triggered control, accurate inter-event time is difficult to be calculated due to the aperiodic property of the inter-event times as well as the coupling effects among agents. However, in the simulation part of the revision, we have added a comparison between the two control schemes in order to demonstrate the advantage of event-triggered control to some extent.
IV. SIMULATION

In this section, a simulation example is provided to verify the theoretical results. A group of six agents with general linear dynamics is considered, in which the constant matrices are given as $A = [-2, 1; -0.1, 0.5]$ and $B = [2; 1]$. The communication graph among six agents is described by Fig. 1, which satisfies Assumption 1.

According to Fig. 1, the smallest positive eigenvalue of the Laplacian matrix is calculated as $\lambda_2 = 1.3649$. Thus, we choose $\beta = 0.4$ and $\gamma = 0.5$. By solving the algebraic Riccati equation (6), we have $K = [0.4008, 1.2105]$ and $Q = [0.2083, -0.0158; -0.0158, 1.2420]$. Meanwhile, we get $\lambda_{max}(Q) = 0.8050$. The parameters in (2) are set as $h = 0.5$ and $\overline{T} = 3$. By noting Remark 5, we choose $T = 2.9$. Initial values of the agents are given as $x_1 = [4; 3], x_2 = [5; 2], x_3 = [6; 1], x_4 = [7; 0], x_5 = [8; -1], x_6 = [4; -2]$. Then, we choose $\eta = 0.6$. Other parameters in the triggering condition (8)-(9) are selected as $\kappa = 0.5$ and $\theta = 1.2$ satisfying the conditions in Theorem 1.

The trajectories of the states are presented in Fig. 2, which illustrates that the practical bipartite consensus is achieved in the prespecified time 2.9s. Fig. 3 shows the inter-event intervals of agents, such that Zeno behavior do not exhibit. In details, the total triggering times of agents are 35, 17, 21, 23, 27 and 20, respectively.

Moreover, to illustrate the advantage of event-triggered control in the resource utilization, we make a comparison between it and sampled-data control. The state trajectories under sampled-data control are depicted in Fig. 4 with a period as 0.1s. Thus, the total sampling time is 40 during the whole process. However, the evolution of the trajectories under the event-triggered control (3) and (8)-(9) is similar to that under the sampling control, while the sampling times in the former is less than that in the latter. With these results, it can be concluded that the proposed event-triggered control strategy can reduce the sampling frequency.

V. CONCLUSION

In this paper, we have investigated the practical prescribed-time bipartite consensus problem of linear multi-agent systems associated with a connected structurally balanced signed graph. A novel event-triggered control strategy was designed for each agent, in which the sampled states of itself and its neighbors are needed. Moreover, a scaling function was applied in the control law, which acts a vital part in the prescribed-time consensus. The state-dependent triggering condition was designed based on the scaling function and the sampled states of agent and its neighbors, such that the verification of it is independent of continuous inter-neighboring communication. Under the control law and the triggering condition provided in this paper, the practical prescribed-time bipartite consensus is proved to be achieved. In addition, we have demonstrated that Zeno behavior is avoided in the whole time span. Our future work will focus on the linear multi-agent systems associated with directed graphs.

REFERENCES

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” Proc. IEEE, vol. 95, no. 1, pp. 215–233, Jan. 2007.
[2] J. Qu, Z. Ji, and Y. Shi, “The graphical conditions for controllability of multiagent systems under equitable partition,” IEEE Trans. Cybern., early access, Aug. 4, 2020, doi: 10.1109/TCYB.2020.3004851.
B. Ning, Q.-L. Han, and Q. Lu, “Fixed-time leader-following consensus,” J. Franklin Inst., vol. 352, no. 12, pp. 8566–8581, Dec. 2015.

J. Han, H. Zhang, X. Liang, and R. Wang, “Distributed impulsive control for heterogeneous multi-agent systems based on event-triggered scheme,” J. Franklin Inst., vol. 356, no. 10, Nov. 2019.

X. Chen, F. Hao, and B. Ma, “Periodic event-triggered cooperative control of multiple non-holonomic wheeled mobile robots,” IET Control Theory Appl., vol. 11, no. 6, pp. 890–899, Apr. 2017.

C. Peng, J. Zhang, and Q.-L. Han, “Consensus of multi-agent systems with nonlinear dynamics using an integrated Sampled-Data-Based event-triggered communication scheme,” IEEE Trans. Cybern., vol. 48, no. 4, pp. 1110–1123, Apr. 2018.

A. Zhang, D. Zhou, P. Yang, and M. Yang, “Event-triggered finite-time consensus with fully continuous communication for second-order multi-agent systems,” Int. J. Control, Autom. Syst., vol. 17, no. 4, pp. 836–846, Apr. 2019.

J. Wang, Y. Zhang, X. Li, and Y. Zhao, “Finite-time consensus for nonholonomic multi-agent systems with disturbances via event-triggered integral sliding mode controller,” J. Franklin Inst., vol. 357, no. 12, pp. 7779–7795, Aug. 2020.

C. Du, X. Liu, W. Ren, P. Lu, and H. Liu, “Finite-time consensus for linear multi-agent systems via event-triggered strategy without continuous communication,” IEEE Trans. Control Netw. Syst., vol. 7, no. 1, pp. 19–29, Mar. 2020.

Y. Dong and J.-G. Xian, “Finite-time event-triggered consensus for nonlinear multi-agent networks under directed network topology,” IET Control Theory Appl., vol. 11, no. 15, pp. 2458–2464, Oct. 2017.

B. Tian, H. Lu, Y. Zuo, and W. Yang, “Fixed-time leader-follower output feedback consensus for second-order multi-agent systems,” IEEE Trans. Cybern., vol. 49, no. 4, pp. 1455–1450, Apr. 2019.

H. Hong, W. Yu, G. Wén, and X. Yu, “Distributed robust fixed-time consensus for nonlinear and disturbed multi-agent systems,” IEEE Trans. Syst., Man, Cybern., Syst., vol. 47, no. 7, pp. 1464–1473, Jul. 2017.

B. Ning, Q.-L. Han, and Q. Lu, “Fixed-time leader-following consensus for multiple wheeled mobile robots,” IEEE Trans. Cybern., vol. 50, no. 10, pp. 4381–4392, Oct. 2020, doi: 10.1109/TCYB.2019.2955453.

J. Liu, Y. Yu, Q. Wang, and C. Sun, “Fixed-time event-triggered consensus control for multi-agent systems with nonlinear uncertainties,” Neurocomputing, vol. 260, pp. 497–504, Oct. 2017.

Z. Guo and G. Chen, “Event-triggered fixed-time cooperative tracking control for uncertain nonlinear second-order multi-agent systems under directed network topology,” J. Franklin Inst., vol. 357, no. 6, pp. 3345–3364, Apr. 2020.

Y. Wang and Y. Song, “Leader-following control of high-order multi-agent systems under directed graphs: Pre-specified fixed time approach,” Automatica, vol. 87, pp. 113–120, Jan. 2018.

Y. Wang, Y. Song, D. J. Hill, and M. Krstic, “Prescribed-time consensus and containment control of networked multi-agent systems,” IEEE Trans. Cybern., vol. 49, no. 4, pp. 1138–1147, Apr. 2019.

Y. Fan, Y. Liu, C. Wen, W. Ren, and Z. Chen, “Designing distributed specified-time consensus protocols for linear multiagent systems over directed graphs,” IEEE Trans. Autom. Control, vol. 64, no. 7, pp. 2945–2952, Jul. 2019.

S. Wasserman and K. Faust, Social Network Analysis: Methods and Applications. Cambridge, U.K.: Cambridge Univ. Press, 1994.

D. Easley and J. Kleinberg, “Networks, crowds, and markets: Reasoning about a highly connected world,” J. R. Stat. Soc. Ser. A, Stat. Soc., vol. 175, no. 4, pp. 1073–1073, Oct. 2012.