On the Ultraviolet to Infrared Evolution of Chiral Gauge Theories

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We discuss the ultraviolet (UV) to infrared (IR) evolution of asymptotically free chiral gauge theories. Various types of IR behavior are considered, including confinement without spontaneous chiral symmetry breaking, and formation of bilinear fermion condensates that preserve or dynamically break the chiral gauge invariance. We compare different schemes for the study of this evolution and the resultant IR structure of the theories, including a conjectured inequality based on the counting of degrees of freedom. We note some new patterns of UV to IR evolution.

I. INTRODUCTION

Chiral gauge theories, in which the left- and right-handed chiral components of the (massless) fermions couple differently to the gauge field, play a key role in the standard model and in some efforts to extend the standard model. In the latter case, chiral gauge theories without elementary scalar fields are often explored. They can be constructed to be free of gauge anomalies, and with the number of fermions limited, they are asymptotically free, typically becoming strongly coupled in the infrared. They have been envisioned to play a role in the generation of flavor hierarchies, dynamical supersymmetry breaking, and quark and lepton substructure.

In this paper we revisit asymptotically free chiral gauge theories, investigating their evolution from the ultraviolet to the infrared. Since the strength of the gauge interaction grows as the renormalization-group (RG) scale $\mu$ decreases, there are, a priori, several possible types of behavior that can set in. Some early studies of strongly coupled chiral gauge theories include \cite{1-6}. We focus on theories that obey the ’t Hooft anomaly matching conditions \cite{1}.

These allow, for example, confinement without breaking of the global or gauged chiral symmetries. This would produce a set of massless, gauge-singlet, spin 1/2 composite fermions \cite{2}, although there is no dynamical basis that we know of for the formation of these bound states. Another type of infrared behavior, involving the formation of bilinear fermion condensates, is also possible. Because of the chiral nature of the gauge-fermion coupling, these condensates typically break the gauge symmetry \cite{3-7}. Yet another possibility is that the IR evolution may be controlled by a weak IR fixed point, with neither confinement nor spontaneous chiral symmetry breaking. The theory is then in the conformal window, occuring when the number of massless fermions is sufficiently large.

If the IR coupling is strong enough to trigger one of the first two types of behavior, there remains little understanding of which might occur. This contrasts with vectorial gauge theories, where a number of general properties have been established by a combination of continuum and lattice methods. For chiral gauge theories, however, the presence of fermion doubling on the lattice has made it challenging to construct a lattice implementation that could be used to study their strong-coupling behavior. It seems worthwhile therefore to re-examine the UV to IR evolution of these theories making use of various continuum field-theoretic methods.

In addition to the ’t Hooft anomaly matching conditions \cite{1}, there is the old most-attractive-channel (MAC) criterion for formation of bilinear fermion condensates \cite{8} based on single gauge-boson exchange. There also exist schemes for constraining RG flow based on the counting of degrees of freedom and the intuitive notion that flow to the IR should result in a thinning of this count. Of the various implementations of this idea, the use of the finite-temperature free energy to define the degree-of-freedom count \cite{8,9} leads to a conjectured inequality that can usefully constrain the IR behavior. For a class of chiral gauge theories, we examine the consequences of this inequality and compare to the MAC criterion.

In Section II, we review the general framework and methods employed. In Section III, we describe the features of a simple chiral gauge theory which we analyze with these methods. In Section IV, we describe a general class of theories in which a set of massless fermions with vector-like gauge couplings is added to the theory of Section III. In Section V, we summarize the possible phases of this general class. We present some conclusions in Section VI.

II. GENERAL FRAMEWORK AND METHODS

A. Beta Function

We focus on theories with gauge group $G = \text{SU}(N)$, and running gauge coupling $\alpha(\mu) = g(\mu)^2/(4\pi)$. The $\beta$ function, $\beta_\alpha = d\alpha/dt$, where $dt = d\ln \mu$, has the loop
where we restrict to $b_1 > 0$ to insure asymptotic freedom. The coefficients $b_\ell$ for $\ell \geq 3$ are scheme-dependent; in the MS scheme they have been calculated up to four-loop order $[10, 11]$. If $b_2 < 0$, an infrared zero can appear at two loops given by $\alpha_{IR,2\ell} = -4\pi b_1/b_2 [12]$. This a reliable result if $b_1/b_2 < 1$, a possibility depending on the number of massless fermions $[13]$. The phenomena of main interest here, bilinear fermion condensate formation and/or the appearance of massless composite fermions, are anticipated only at strong coupling, where the loop expansion is not reliable.

### B. MAC Criterion

If UV to IR evolution produces fermion condensates, an old scheme for discriminating among possible breaking patterns is the most-attractive-channel (MAC) criterion. For two chiral fermions in representations $R_1$ and $R_2$ of the gauge group, a rough measure of the likelihood of a condensation channel of the form $R_1 \times R_2 \rightarrow R_{\text{cond.}}$ is taken to be

$$\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{\text{cond.}}) ,$$

where $C_2(R)$ is the quadratic Casimir invariant for the representation $R [14]$.

Since this is a measure of attractiveness arising from single-gauge-boson exchange, its utility is uncertain, but the MAC criterion is that where bilinear fermion condensates can occur and there are several possible channels, the one that takes place has the largest value of $\Delta C_2$. (In a vectorial gauge theory such as QCD, this implies that the only condensation is $R \times \bar{R} \rightarrow 1$, preserving the gauge symmetry.) A reasonable guess then is that as $\mu$ decreases and $\alpha(\mu)$ increases, condensation will first occur, in the MAC, when $\alpha(\mu)$ becomes of order $\alpha_{cr}$ given by

$$\frac{\alpha_{cr} \Delta C_2(R)}{2} \sim O(1) .$$

### C. Conjectured Inequality Concerning UV versus IR Degrees of Freedom

A general notion about RG flow to the IR, as first applied to second-order phase transitions and critical phenomena, is the thinning of degrees of freedom. For two-dimensional conformal field theories (CFTs), Zamolodchikov proved that the central charge, $c$, which may be considered to count the degrees of freedom in the theory, decreases as a consequence of this flow; that is, the new CFT to which the theory flows in the IR has a smaller value of $c$ than the UV CFT $[15]$.

In four space-time dimensions, an approach using finite temperature $T$ as the RG scale and the thermodynamic free energy $F(T)$ to count the degrees of freedom, can provide significant constraints on the UV to IR flow of a quantum field theory. The quantity $f(T)$, given by

$$F(T) \equiv f(T) \frac{\pi^2}{90} T^4 ,$$

is defined to count a single, massless bosonic degree of freedom as 1 when the theory is free. In this case,

$$f = 2N_V + \frac{7}{4}N_F + N_S ,$$

where $N_V$ is the number of massless gauge fields, $N_F$ is the number of massless chiral components of fermion fields, and $N_S$ is the number of scalar fields. Thus, for an asymptotically free, vectorial $SU(N)$ gauge theory with $N_F$ Dirac fermions transforming according to the fundamental representation, $f_{UV} \equiv f(\infty) = 2(N^2 - 1) + (7/2)N_FN_S$.

A conjectured inequality $[8]$ is that for an asymptotically free theory,

$$f_{IR} \equiv f(0) \leq f_{UV} \equiv f(\infty) .$$

No asymptotically free counterexample has yet been identified, although there are theories for which $f(T)$ is not monotonic $[8]$.$[9], [16]$. A further possibility is that for an asymptotically free theory,

$$f_{IR} \equiv f(0) \leq f_{UV} \equiv f(\infty) .$$

Another approach, following Ref. $[15]$, has investigated the possibility that for some quantity measuring the number of degrees of freedom in four space-time dimensions, one could prove either (i) that it is a non-increasing function of the RG flow from the UV to the IR or (ii), a weaker condition, that its value at a UV
fixed point is greater than its value at an IR fixed point. Some progress has been achieved by focusing on the coefficient $a$, entering the trace of the energy-momentum tensor, $\langle \theta^\mu_\nu \rangle = c W_{\sigma \lambda \mu \nu} W^{\lambda \lambda \mu \nu} - a E_\mu$, where $W_{\sigma \lambda \mu \nu}$ is the Weyl tensor and $E_\mu$ is the four-dimensional Euler density, whose integral is the Euler characteristic of the given spacetime manifold [17, 18]. For a free theory,

$$a = 62N_V + \frac{11}{2}N_F + N_S,$$

(2.7)

where $N_V$, $N_F$, and $N_S$ denote the number of massless vector, chiral fermion, and scalar fields. For the example of an asymptotically free vectorial SU($N$) gauge theory with $N_f$ massless Dirac fermions in the fundamental representation, $a_{UV} = 62(N^2 - 1) + 11N_fN$.

Both $a$ and $f$ are normalized to weight a single scalar field with coefficient 1. A peculiar feature of $a$ is the relative largeness of the coefficient, 62, emerging from a detailed computation, multiplying $N_V$ in Eq. (2.7). By comparison, the coefficient 2 multiplying $N_V$ in Eq. (2.5) which is applicable when the effective theory is weakly coupled, simply counts the two transverse gauge boson degrees of freedom. In the case of $a$, the large gauge boson term, which is present in the UV for an asymptotically free theory, means that the inequality constraint $a_{IR} \leq a_{UV}$ is not difficult to satisfy.

Yet another approach to describing the IR properties of quantum field theories involves compactification of the $\mathbb{R}^4$ space to $\mathbb{R}^3 \times S^1$, with periodic boundary conditions for all fields, and associated analysis of topological properties [19]. A degree-of-freedom counting inequality could perhaps be developed in this framework. In the present paper, we will focus on the thermal inequality $f_{IR} \leq f_{UV}$. Although it is still conjectural, it makes a clear connection between the Euclidean momentum scale and temperature in its weighting of coefficients of $N_V$, $N_F$, and $N_S$, and, moreover, it can be quite constraining.

D. Lattice Studies of Vectorial Theories

Lattice studies have been carried out to investigate the IR behavior of many vectorial, non-supersymmetric SU($N$) gauge theories. All results are so far compatible with the inequality [20]. Investigations of vectorial SU($2$) gauge theories, currently underway [21], are particularly interesting with respect to the inequality. Here, the reality of the fermion representations leads to a larger global symmetry, and typically therefore to a larger count of IR degrees of freedom when this symmetry is broken.

III. A SIMPLE THEORY

We first review the UV to IR evolution of a familiar SU($N$) chiral gauge theory [5]. The chiral gauge symmetry forbids fermion mass terms in the Lagrangian. Without loss of generality, we write all fermions as left-handed.

A. Structure

The fermion content consists of a symmetric, rank-2 tensor, $\psi^{ab}_L = \psi^{(ab)}_L$, together with $N + 4$ copies (flavors) of the conjugate fundamental representation, $\chi_{a,i,L}$, $i = 1,...,N + 4$. Here and below, $a, b, ..$ are SU($N$) gauge indices with $N \geq 2$, and $i, j, ...$ are copy indices. That is, the fermion content is:

$$\psi^{(ab)}_L, \chi_{a,i,L}, i = 1,...,N + 4.$$  

(3.1)

With a normalization convention in which the contribution of a left-handed fermion in the fundamental representation of SU($N$) to the triangle anomaly is 1, the contribution of left-handed fermions in the rank-2 symmetric and antisymmetric representation is ($N + 4$). Thus, the fermion content yields theories that are free of chiral anomalies in gauged currents.

The one- and two-loop $\beta$ function coefficients are

$$b_1 = 3N - 2, \quad b_2 = \frac{1}{2}(13N^2 - 30N + 1 + 12N^{-1}).$$  

(3.2)

Because of asymptotic freedom, the $f_{UV}$ function is given by

$$f_{UV} = 2(N^2 - 1) + \frac{7}{4} \left[ \frac{N(N + 1)}{2} + (N + 4)N \right].$$  

(3.3)

The classical global flavor symmetry group is U($1$) $\otimes$ U($N + 4$)$_F$. Expressing U($N + 4$)$_F$ = SU($N + 4$)$_F$ $\otimes$ U($1$)$_F$, we recall that the U($1$) and U($1$)$_F$ are both rendered anomalous by instantons, but one can construct a linear combination, denoted $\hat{U}(1)$, that is conserved in the presence of instantons. Hence, the actual (non-anomalous) global flavor group of this theory is

$$G_f = SU(N + 4) \hat{F} \otimes \hat{\mathbb{U}}(1).$$  

(3.4)

B. Evolution

1. Confinement without Chiral Symmetry Breaking

A possibility is that the gauge interaction confines and produces gauge-singlet, massless composite fermions. The ’t Hooft conditions, that the anomalies associated with the global chiral flavor symmetries must be the same for the fundamental fields and for the composites, are satisfied if the latter are the composite fermions

$$B_{ij} = \tilde{F}_{a,i}S^{ab}\tilde{F}_{b,j} - (i \leftrightarrow j),$$  

(3.5)

transforming according to the conjugate-antisymmetric, rank-2 tensor representation of the global SU($N + 4$) symmetry group.
With only massless composite fermions in the spectrum, the low-energy effective theory (EFT) consists of interaction operators only of dimension-6 and higher, allowing $f_{IR}$ to be computed in the free theory:

$$f_{IR,SYM} = \frac{7}{4} \left( \frac{(N+4)(N+3)}{2} \right).$$

(3.6)

Hence, $f_{UV} - f_{IR,SYM}$ is non-negative for $N \geq 1.607$, that is, for all physical values of $N$.

2. Chiral Symmetry Breaking

An alternate possibility is that as $\mu$ decreases to some scale $\Lambda_N$, the gauge interaction grows sufficiently strong to produce bilinear fermion condensates. The MAC is $N \Rightarrow N\Lambda N$ scale $\Lambda N$ breaking the SU($N$) following interaction operators only of dimension-6 and higher, allowing $f_{IR}$ to be computed in the free theory:

$$\Delta C_2(S \times \bar{F} \rightarrow F) = C_2(S) = \frac{(N+2)(N-1)}{N}. \quad (3.7)$$

The condensate then has the form $\langle \psi^{ab}_L T C \chi_{b,i,L} \rangle$. Taking $a = 1$ and $i = 1$, it is

$$\sum_{b=1}^{N} \psi^{1b}_L T C \chi_{b,1,L}, \quad (3.8)$$

breaking the SU($N$) gauge symmetry to SU($N-1$) and the global SU($N+4$)$_F$ symmetry to SU($N+3$)$_F$. The fermions in this condensate, $\psi^{ab}_L$ and $\chi_{b,1,L}$, gain dynamical masses of order $\Lambda N$, and the $2N-1$ gauge bosons in the coset SU($N$)/SU($N-1$) gain masses of order $g\Lambda N$, where $g$ is the SU($N$) gauge coupling at the scale $\Lambda_N$.

The residual SU($N-1$) gauge theory is then capable of self breaking to SU($N-2$), and so on sequentially. Suppose, for simplicity, that $N = 3$. After the first breaking, there remains an SU($2$) gauge theory with three chiral fermions in the symmetric tensor $\psi^{ab}_L$, along with 18 chiral fermions in $\chi_{b,1,L}$, along with $2N-1$ gauge bosons in the coset SU($N$)/SU($N-1$) gain masses of order $g\Lambda_N$, where $g$ is the SU($N$) gauge coupling at the scale $\Lambda_N$.

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The SU($2$) gauge theory can then fully self-break, perhaps at a lower scale. We form a symmetric combination of two $\chi_{b,1,L}$, mixing SU($2$) and flavor labels as the chiral fermions that pair with the three $\psi^{ab}_L$, and, with no loss of generality, we choose the flavor indices $i = 2, 3$ for these. The condensate is thus

$$\langle \sum_{a,b=2}^{3} \psi^{ab}_L T C \chi_{\{b,i\},L} \rangle, \quad (3.9)$$

where $\chi_{\{b,i\},L}$ is the symmetrized two-index combination. The remaining massless chiral fields are the $\chi_{b,i,L}$ ($b = 1, i = 2, 3$) and the single remaining, antisymmetrized $\chi_{\{b,i\},L}$, forming an antisymmetric tensor under a global SU($3$). There are also the 12 chiral fermions $\chi_{b,i,L}$ ($b = 1, 2, 3, i = 4, ..., 7$), transforming as an anti-fundamental under the global group SU($3$) × SU($4$).

For general $N$, after all breakings, the SU($N$) gauge group is fully broken, and the unbroken global symmetry is $SU'_{N}$. We have

$$G_{I} = SU'(N) \otimes SU(4) \otimes U(1)'. \quad (3.10)$$

There are $N(N-1)/2$ massless elementary chiral fermions transforming as the antisymmetric tensor representation of $SU'(N)$ and $4N$ massless elementary chiral fermions transforming as the anti-fundamental representation of $SU'(N) \otimes SU(4)$, for a total of $N(N+1)/2$ massless chiral fermions.. There are also $8N + 1$ composite NGBs. This is the difference between the number of generators for the initial local-plus-global symmetry group, $SU(N) \otimes G_{I}$, and the final (global) symmetry group $G_{I}'$, minus the $N^2$ - 1 absorbed to give mass to the gauge bosons of the original SU($N$) gauge theory.

The EFT describing these massless degrees of freedom includes interaction operators only of dimension $d > 4$, allowing $f_{IR}$ to be computed in the free theory. One finds

$$f_{IR,S\times F} = 8N + 1 + \frac{7}{4} \left[ \frac{(N-1)}{N} + 4N \right]. \quad (3.11)$$

Hence, $f_{UV} - f_{IR,S\times F}$ is nonnegative for $N \geq 2.0558$. We also note that $f_{IR,SYM} < f_{IR,S\times F}$ for all integer $N$, possibly favoring massless composite fermion formation.

Before discussing a generalization of this theory, we note that one could also consider a theory with an SU($N$) gauge group with $N \geq 5$ containing a massless fermion content consisting of a rank-2 antisymmetric representation $\psi^{[ij]}$ and $N-4$ copies of the conjugate fundamental representation, denoted $\bar{F}$. This theory is free of anomalies in gauged currents and satisfies the 't Hooft matching conditions. As with the present theory, there are several possible types of IR behavior, including (i) confinement without spontaneous chiral symmetry breaking, yielding massless composite fermion and (ii) formation of fermion condensates breaking gauge and global symmetries. In the case (ii), for SU($5$), the MAC is $A \times A \rightarrow F$; for SU($6$) there are two equally attractive MACs, $A \times A \rightarrow A$ and $A \times F \rightarrow F$, while for SU($N$) with $N \geq 7$, the MAC is $A \times F \rightarrow F$.

IV. A GENERAL CLASS OF THEORIES

We now consider a one-parameter family of chiral gauge theories formed by the addition to the theory of Eq. 6.1 of a vectorlike sector consisting of $p$ copies of the pair of fermions $\{ F + \bar{F} \}$. [5].
A. Structure of the General Class

The fermionic content is

\[ \psi_L^{(ab)}, \quad \chi_{a+L}, \quad i = 1, ..., N + 4 + p \]

\[ \omega_{p+L}^j, \quad j = 1, ..., p. \]  

(4.1)

The classical global flavor symmetry group is \( U(1) \otimes U(N+4+p) \otimes U(p) \). It can equivalently be written as \( U(1) \otimes SU(N+4+p) \otimes U(p) \otimes SU(p) \otimes U(1) \). The \( U(1) \) factor groups are rendered anomalous by \( SU(N) \) instantons, but one can form two linear combinations that are invariant. Hence, the actual (non-anomalous) global flavor group is

\[ G_f^p = SU(N+4+p) \otimes SU(p) \otimes \bar{U}(1) \otimes \bar{U}(1)'. \]  

(4.2)

The one-loop coefficient in the \( \beta \) function is

\[ b_1^p = 3N - 2 - \frac{2p}{3}. \]  

(4.3)

In the asymptotically free range, \( p < \frac{9N}{2} - 3 \), one can identify the free UV degrees of freedom, with the result

\[ f_{\text{UV}}^p = f_{\text{UV}} + \frac{7}{4}(2pN) \]

\[ = 2(N^2 - 1) + \frac{7}{4}\left[ \frac{N(N+1)}{2} + (N+4+2p)N \right]. \]  

(4.4)

The two-loop \( \beta \)-function coefficient is

\[ b_2^p = \frac{13N^2}{2} - 15N + \frac{1}{2} + 6N^{-1} + p\left(-\frac{13N}{3} + N^{-1}\right). \]  

(4.5)

B. Evolution of the General Class

1. Non-Abelian Coulomb Phase

For a range of \( p \) values below \( 9/2 - 3 \), the two-loop \( \beta \) function has a non-trivial zero (an IR fixed point) at

\[ \alpha_{\text{IR},2}\pi = \frac{8\pi N(9N - 6 - 2p)}{-39N^2 + 90N^2 - 3N - 36 + p(26N^2 - 6)}, \]  

(4.6)

a reliable result providing \( 9/2 - 3/N - p/N \ll 1 \) so that the fixed point is weak. In this range,

\[ f_{\text{IR},\text{NAC}}^p = f_{\text{UV}}^p + \text{perturbative corrections}, \]  

(4.7)

with the leading correction small and negative. Thus \( f_{\text{UV}}^p > f_{\text{IR},\text{NAC}}^p \).

The fixed-point strength increases monotonically with decreasing \( p \), becoming inaccessible via perturbation theory when \( 9/2 - 3/N - p/N = O(1) \). It is expected to reach a strength necessary for confinement and/or symmetry breaking at some critical value \( p_{\text{cr}} = O(N) \), not precisely known.

In the following we assume that there exists a \( p_{\text{cr}} \) below which the theory leaves the non-Abelian Coulomb phase. There are then several possibilities for its IR behavior.

2. Confinement With No Chiral Symmetry Breaking

One possibility is confinement without spontaneous chiral symmetry breaking producing a set of massless composite fermions transforming according to (i) the conjugate of the antisymmetric rank-2 tensor representation of the \( SU(N+4+p) \) global symmetry group and the singlet representation of the \( SU(p) \) group; (ii) the fundamental representation of \( SU(N+4+p) \) and \( SU(p) \); and (iii) the singlet representation of \( SU(N+4+p) \) and the conjugate of the symmetric representation of \( SU(p) \).

The EFT consists of interaction operators only of dimension 6 and higher, so that the free theory may be used to determine \( f_{\text{IR},\text{SYM}}^p \). Thus

\[ f_{\text{IR},\text{SYM}}^p = \frac{7}{4}\left[ \frac{(N+4+p)(N+3+p)}{2} + p(N+4+p) \right] + \frac{p(p+1)}{2}. \]  

(4.8)

Hence, \( f_{\text{UV}}^p - f_{\text{IR},\text{SYM}}^p \) is nonnegative providing

\[ p \leq -2 + \left[ \frac{15N^2 + 7N + 6}{14} \right]^{1/2}. \]  

(4.9)

The inequality (4.9) thus predicts that for \( p \) in this range, the interaction can confine and produce massless composite fermions, whereas it is forbidden for larger \( p \)-values.

3. Chiral Symmetry Breaking in the \( S \times F \rightarrow F \) Channel

There are various possibilities for IR behavior that involve dynamical chiral symmetry breaking. These can be classified according to several criteria, including

(i) their attractiveness measure \( \Delta C_2 \),

(ii) whether they also break the \( SU(N) \) gauge symmetry,

(iii) the value of \( f_{\text{IR}} \), which depends on the details of the sequential formation of condensates and breaking of global (and possibly gauge) symmetries at corresponding scales.

The most attractive channel (MAC) for fermion condensation is the \( S \times \bar{F} \rightarrow F \) channel, with measure
\[ \Delta C_2(S \times \bar{F} \rightarrow F) \]. The attractiveness for this channel may be compared with that for the channel \( F \times \bar{F} \rightarrow 1 \).
The difference between the \( \Delta C_2 \) values for these channels is
\[
\Delta C_2(S \times \bar{F} \rightarrow F) - \Delta C_2(F \times \bar{F} \rightarrow 1) = \Delta C_2(S) - 2\Delta C_2(F) = \frac{N - 1}{N} > 0. \tag{4.10}
\]

Thus, if the gauge interaction causes the formation of fermion condensates for \( p < p_{cr} \), rather than confining without chiral symmetry breaking, the MAC criterion points to condensation in the \( S \times \bar{F} \rightarrow F \) channel. In this case, we denote the scale where this condensation occurs as \( \Lambda_N \). Here, the gauge symmetry is broken from SU(\( N \)) to SU(\( N - 1 \)).

If \( N \geq 3 \), the same MAC argument implies that in the effective theory for scales below \( \Lambda_N \), there will again be condensation in the \( S \times \bar{F} \rightarrow F \) channel at a lower scale \( \Lambda_{N-1} \), breaking SU(\( N - 1 \)) to SU(\( N - 2 \)), and so forth. At each scale, the fermions involved in the condensate will gain dynamical masses. Since the \( N + 4 \) \( \chi_{a,i,L} \), \( i = 1, ..., N + 4 \), suffice to break the SU(\( N \)) symmetry completely, the additional \( p \) vectorlike pairs of fermions \( F, \bar{F} \) remain in the IR. Hence, the number of massless chiral components of fermions in the IR EFT is calculated by simply adding these to the number \( N(\Lambda_{N-1})/2 + 4N \) in the \( p = 0 \) theory, for a total of \( N(N - 1)/2 + 4N + 2pN = N(N + 7 + 4p)/2 \). Correspondingly, the final global symmetry group is
\[
G_f^p = SU'(N) \otimes SU(4+p) \otimes SU(p) \otimes U(1)'_p \otimes U(1)''_p. \tag{4.11}
\]

The number of NGBs is \( 2N(4+p) + 1 \). This is the difference between the number of generators of the initial local+global symmetry group SU(\( N \)) \( \otimes G_f^p \), and the final (global) symmetry group \( G_f^p \), minus the \( N^2 - 1 \) absorbed to give masses to the gauge bosons of the original SU(\( N \)) gauge theory.

The IR EFT again consists of interaction operators only of dimension \( d > 4 \), so that the free theory determines \( f_{IR} \):
\[
f_{IR,S\times F}^p = 2N(4+p) + 1 + \frac{7}{4} \left( \frac{N(N - 1)}{2} + 4N + 2pN \right), \tag{4.12}
\]
behaving linearly with \( p \) as does \( f_{UV}^p \), but with a larger slope. The linearity of the NGB count with respect to \( p \) is notable, requiring a large unbroken global symmetry.

For this UV to IR evolution, \( f_{UV}^p - f_{IR,S\times F}^p \) is always negative for \( N = 2 \), but for \( N \geq 3 \), it is nonnegative (in accord with the conjectured inequality) for
\[
p \leq \frac{15N^2 - 25N - 12}{8N}. \tag{4.13}
\]

For \( N = 3 \), for example, \( p \leq 2 \).

4. **Chiral Symmetry Breaking in the \( F \times \bar{F} \rightarrow 1 \) Channel Followed by Confinement with no Further Symmetry Breaking**

   Given the uncertainties in the strong-coupling physics involved, it is important to consider other possible condensation channels. A natural one is the \( F \times \bar{F} \rightarrow 1 \) channel, involving the \( p \) fermions in the vectorlike subsector of the theory. Although it is less attractive than the \( S \times \bar{F} \rightarrow F \) channel, it has the important feature that it does not break the SU(\( N \)) gauge symmetry. The condensate for this channel is
\[
\langle \chi_{a,i,L}C\omega_{a,L}^p \rangle, \quad i = N + 4 + 1, ..., N + 4 + p \tag{4.14}
\]

This condensation pattern was studied in [16]. We denote the scale where it occurs as \( \Lambda_V \). It breaks the \( G_f^p \) global symmetry to \( SU(N+4)F \otimes SU(p)_V \otimes U(1)'_1 \otimes U(1)'_2 \), \( \tag{4.15} \)
leading to \( p(2N + p + 8) \) NGBs. The fermions involved in the condensate (4.14) get dynamical masses of order \( \Lambda_V \), and the low-energy effective field theory is then the \( p = 0 \) theory.

The further infrared evolution of this theory has been reviewed in Section III. The possibility of confinement without further chiral symmetry breaking is favored by \( f_{IR} \) minimization over further chiral symmetry breaking (along with gauge symmetry breaking). Massless composites are then formed, the EFT includes interaction operators only of dimension \( d > 4 \), and the function \( f_{IR} \) is [16]
\[
f_{IR,F \times \bar{F}, SYM}^p = p(2N + p + 8) + \frac{7}{4} \left( \frac{(N + 4)(N + 3)}{2} \right) \tag{4.16}
\]

Hence, \( f_{IR}^p - f_{F \times \bar{F}, SYM}^p \) is nonnegative, and in accord with the conjectured inequality (2.6) providing
\[
p \leq \frac{1}{4} \left[ 3N - 4 + \sqrt{69N^2 - 68N + 56} \right] \tag{4.17}
\]
For \( N = 3 \), for example, \( p \leq 3.687 \), allowing the physical values \( p = 0, 1, 2, 3 \).

5. **Chiral Symmetry Breaking in the \( F \times \bar{F} \rightarrow 1 \) Channel Followed by \( S \times \bar{F} \rightarrow F \) Chiral Symmetry Breaking**

   Finally, if after the condensation of the \( p \) vector-like pairs in the channel \( F \times \bar{F} \rightarrow 1 \), the residual, \( p = 0 \) theory breaks the chiral and gauge symmetries via \( S \times \bar{F} \rightarrow F \), as described in Section IIIB2, then the gauge symmetry is broken and the remaining global symmetry is
\[
G_f^p = SU'(N) \otimes SU(4) \otimes SU(p)_V \otimes U(1)'_1 \otimes U(1)'_2. \quad \tag{4.18}
\]
The IR EFT again consists of interaction operators only of dimension $d > 6$. The resultant $f_{IR}$ counts the massless degrees of freedom, and is

$$f^p_{IR,F \times \bar{F},S \times \bar{F}} = (2pN + p^2 + 8p) + (8N + 1) + \frac{7}{4} \left[ \frac{1}{2} N(N - 1) + 4N \right]. \quad (4.19)$$

Hence, $f^p_{UV} - f^p_{IR,F \times \bar{F},S \times \bar{F}}$ is nonnegative for

$$p \leq \frac{1}{4} \left[ 3N - 16 + \sqrt{69N^2 - 196N + 208} \right]. \quad (4.20)$$

For $N = 3$, for example, $p \leq (-7 + \sqrt{241})/4 = 2.131$, allowing the physical values $p = 1, 2$.

V. SUMMARY OF THE GENERAL CLASS OF THEORIES

The class of theories considered here has several possibilities for UV to IR evolution. For $p/N$ sufficiently close to the upper limit $9/2 - 3/N$:

(i) to a non-Abelian Coulomb phase with an IR fixed point of the gauge coupling and particle content consisting of the elementary gauge fields and fermions;

Then, for $p \leq$ some critical value $p_{cr}(N)$, there are several strong-coupling possibilities:

(ii) to a phase with confinement, massless composite fermion formation and no gauge or chiral symmetry breaking;

(iii) to a phase with sequential condensation in the respective $S \times \bar{F} \to F$ (MAC) channels, breaking the SU($N$) gauge symmetry completely, leaving a set of massless elementary fermions and massless composite NGBs in the infrared EFT;

(iv) to a phase with condensation of the $p$ vectorlike fermions in the channel $F \times \bar{F} \to 1$ followed by confinement with massless composite fermion formation, no further chiral symmetry breaking, and no gauge-symmetry breaking, so that the IR EFT consists of the massless composite fermions together with massless NGBs;

(v) to a phase with condensation of the $p$ vectorlike fermions in the channel $F \times \bar{F} \to 1$, followed by condensation in the $S \times \bar{F} \to F$ channel, again breaking the SU($N$) gauge symmetry completely, so that the IR particle content consists of massless NGBs and massless elementary fermions

We note that each of these phases has either elementary or composite massless fermions, but not both.

A. $N = 3$

To discuss these possibilities further, we first consider the case $N = 3$. In Fig. 1, we plot $f_{IR}$ for each of the strong coupling phases, along with $f_{UV}$. For $p/N$ near $3.5$, the theory is in the non-Abelian Coulomb phase (i), with $f_{IR,NAC} \leq f_{UV}$, the difference being computable in perturbation theory. There is no confinement and no symmetry breaking. This $f_{IR}$ curve is not shown explicitly in Fig. 1.

As $p$ is decreased, other IR phases become possible. With $p$ still relatively large, the MAC phase (iii) (red) with a large unbroken global symmetry $\bar{S} \times \bar{F}$ and no composite fermions has the smallest $f_{IR}$, but here $f_{IR,S \times \bar{F}} > f_{UV}$. For smaller $p$, when $f_{IR,S \times \bar{F}} \leq f_{UV}$, this phase does not give the smallest $f_{IR}$.

As $p$ is decreased further, the first phase to allow $f_{IR} \leq f_{UV}$, for $p \leq 3.69$, is phase (iv) (dark blue) with condensation of the $p$ vector-like fermions followed by confinement with massless composite fermion formation. There is no gauge symmetry breaking. Throughout the range $p \leq 3.69$, $f_{IR}$ for this phase is the smallest among the strong coupling phases, but the breaking channel is not the MAC.

As $p$ is decreased still further, other phases are allowed by the inequality $f_{IR} \leq f_{UV}$. In particular, phase (iii) (red) involving sequential breaking in the MAC channel $S \times \bar{F} \to F$, breaking the gauge symmetry completely, satisfies the inequality for $p \leq 2$, although it does not give the smallest $f_{IR}$.

Phase (v) (black) is compatible with the inequality for $p \leq 2.131$, but it never minimizes $f_{IR}$. Phase (ii) (green), with confinement but no symmetry breaking, respects the inequality for $p \leq 1.4$, but leads to the smallest $f_{IR}$ only when $p = 0$.

A plot of $f_{IR}$ values for $N > 3$ is similar.

B. Large-$N$ Limit

It interesting to consider the limit $N \to \infty$ with $r = p/N$ and $\xi(\mu) = a^2(\mu)N$ held fixed. The upper bound on $r$ from asymptotic freedom is $r < 9/2$, and the two-loop IR fixed point is at

$$\xi_{IR,2\ell} = \frac{8\pi(9 - 2r)}{13(2r - 3)}. \quad (5.1)$$

The reduced quantity $\bar{f} \equiv f/N^2$ is finite in this limit. We have

$$\bar{f}_{UV} = \frac{37}{8} \left( 1 - \frac{7r}{2} \right). \quad (5.2)$$

The theory is in the non-Abelian Coulomb phase (i) for $r$ near $9/2$, with $f_{IR,NAC}$ perturbatively below $\bar{f}_{UV}$.

Phase (ii) with confinement, massless-composite-fermion formation and no symmetry breaking, gives

$$\bar{f}_{IR,sym} = \frac{7}{8} \left( 1 + 4r + 4r^2 \right), \quad (5.3)$$
satisfying the inequality for \( r \leq 15/14 \).

Phase (iii), with sequential condensation in the respective \( S \times \bar{F} \to F \) (MAC) channels, gives

\[
\bar{f}_{IR,S \times \bar{F}} = \frac{7}{8} + \frac{11r}{2}, \tag{5.4}
\]

satisfying the inequality for \( r \leq 15/8 \).

Finally, with condensation of the \( p \) vectorlike fermions in the channel \( F \times \bar{F} \to 1 \), followed by either confinement without further chiral symmetry breaking (phase (iv)) or further sequential condensation in the \( S \times \bar{F} \) channels (phase(v)), we have

\[
\bar{f}_{IR,F \times \bar{F},SYM} = \bar{f}_{IR,F \times \bar{F}} = r(2 + r) + \frac{7}{8}. \tag{5.5}
\]

Interestingly, \( f_{IR} \) is the same for these two phases even though the term \( 7/8 \) counts composite fermions in one case and residual elementary fermions in the other. The inequality is satisfied for \( r < (3 + \sqrt{39})/4 \).

In Fig. 2 we plot \( \bar{f} \) for the various phases as functions of \( r \). While the two curves for phases (iv) and (v) have collapsed to one, the qualitative picture is otherwise similar to the finite-\( N \) case.

The common curve for phases (iv) and (v) satisfies the inequality for the largest range of \( p/N \) values, and provides the smallest \( f_{IR} \) throughout this range. However, as in the finite-\( N \) case, the condensation channel for these phases is not the MAC, this being phase (iii) with sequential condensation in the channel \( S \times \bar{F} \to F \).

VI. DISCUSSION AND CONCLUSIONS

We have examined a class of SU(\( N \)) chiral gauge theories including \( p \) additional, vector-like fermions. In addition to a weak-coupling, non-Abelian Coulomb phase (i), present for large enough \( p/N \), we have described a set of four possible strong-coupling phases. We have classified these phases with reference to the degree-of-freedom-counting inequality \( f_{IR} \leq f_{UV} \).

One phase, (ii), breaks no symmetries and includes massless composite fermions. Another phase, (iii), new to this paper, involves sequential breaking in the maximally-attractive (MAC) channel, breaks the gauge symmetry entirely, and leaves a large unbroken global symmetry. This phase, together with two others, (iv), and (v), involves chiral symmetry breaking through bilinear fermion condensation, leaving unabsorbed massless Nambu-Goldstone bosons. Phase (iii), in which the SU(\( N \)) gauge symmetry is broken sequentially, is the MAC among these three. Two phases, (ii) and (iv), involve the formation of massless composite fermions, and leave the gauge symmetry unbroken. Two others, (iii) and (v), break it completely, and leave residual massless, elementary fermions.

Phase (iv), in which the vector-like fermions condense among themselves and the chirally coupled fermions form massless composite fermions, leaving the SU(\( N \)) gauge symmetry unbroken, satisfies the inequality \( f_{IR} \leq f_{UV} \) for the largest range of \( p/N \) values, and minimizes \( f_{IR} \) throughout this range. But it is not the MAC for condensation. (In the limit \( N \to \infty \) with \( r \equiv p/N \) held fixed, phases (iv) and (v) give the same value for \( f_{IR} \) ) Phase (ii), with no symmetry breaking and massless-composite-fermion formation, satisfies the inequality for small \( p/N \) values, but minimizes \( f_{IR} \) only at \( p = 0 \). These results are all shown in Figs. 1 and 2.

This survey of four possible strong-coupling phases is based on the conjectured inequality \( f_{IR} \leq f_{UV} \), along with the (possibly unreliable) MAC criterion, deriving from single-gauge-boson exchange. Phase (iii), involving MAC condensation, satisfies the inequality only for small \( p/N \), and when it does, it does not minimize \( f_{IR} \). In the absence of strong-coupling computations or proven, restrictive constraints in addition to 'tHooft anomaly matching, it remains unknown which of the strong-coupling phases is realized for various values of \( p \). While there is no dynamical basis that we know of for the formation of massless composite fermions in chiral gauge theories, as in phases (ii) and (iv), the formation of massless Nambu-Goldstone bosons, as in phases (iii) and (v), can be realized in various dynamical schemes. It is also an approximate feature of real-world QCD. The quantity \( f_{IR} \) for phases (iii) and (v) is shown in the red and black curves of Figs. 1 and 2.

This discussion of possibilities can perhaps provide a helpful template for the further study of the class of chiral gauge theories considered here, as well as other theories.

This research was partially supported by the grants DE-FG02-92ER-40704 (T.A.) and NSF-PHY-09-69739 (R.S.). One of us (T.A.) would like to acknowledge the hospitality of the Aspen Center for Physics while this paper was being completed.

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FIG. 1: Plot of $\bar{f} \equiv f/N^2$ for various types of IR behavior, as functions of $p/N$, for the case $N = 3$. The curves (with color in the online version) are $\bar{f}^p_{UV}$ (light blue), (ii) $\bar{f}^p_{IR,SYM}$ (green), (iii) $\bar{f}^p_{IR,S\times\bar{F}}$ (red), (iv) $\bar{f}^p_{IR,F\times\bar{F},SYM}$ (dark blue), and (v) $\bar{f}^p_{IR,F\times\bar{F},S\times\bar{F}}$ (black). The curve for $f_{IR,NAC}$ in phase (i), not shown, is perturbatively close to the curve for $\bar{f}^p_{UV}$. At $p/N = 1.2$, the curves, from bottom to top, are (iv), (UV), (iii), (v), and (ii).
FIG. 2: Plot of $\bar{f} \equiv f/N^2$ for various types of IR behavior, as functions of $r \equiv p/N$, in the large-$N$ limit (with color in the online version). The curves are $f_{p,UV}^p$ (light blue), (ii) $f_{p,SYM}^p$ (green), (iii) $f_{IR,S\times P}^p$ (red), (iv and v) $f_{p, F \times S \times P}^p = f_{IR,F \times S \times P}^p$ (black). The curve for $f_{IR,NAC}$ in phase (i), not shown, is perturbatively close to the curve for $f_{p,UV}^p$. At $r = 1.2$, from bottom to top, the curves are (iv and v), (iii), (UV), and (ii).