Dynamic hyperuniform states

Gustavo Castillo¹, Nicolás Mujica², Néstor Sepúlveda², Juan Carlos Sobarzo², and Rodrigo Soto²

¹Instituto de Ciencias de la Ingeniería, Universidad de O’Higgins, Rancagua, Chile and
²Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile

Hyperuniform states are the most efficient way to fill up space for disordered systems. In these states the particle distribution is disordered at the short scale but becomes increasingly uniform when looked at large scales. Hyperuniformity appears in several systems, but all reported cases correspond static or quasistatic states. Here, we show that a vibrated granular layer, close to the liquid-to-solid transition, displays dynamic hyperuniformity.

Prior to the transition, patches of the solid phase form, with length scales and mean lifetimes that diverge critically at the transition point. When reducing the wavenumber, density fluctuations encounter increasingly more patches that block their propagation, resulting in a density structure factor that tends to zero for small wavenumbers, which is a signature of hyperuniformity. A simple model demonstrates that this coupling of a density field to a highly fluctuating scalar friction field gives rise to dynamic hyperuniform states.

How can space be filled more homogeneously with an ensemble of hard spheres? Recently, hyperuniform systems have been identified as the most efficient way to fill up space for disordered configurations. These exotic particles distributions are disordered at short distances, as liquids, and more and more uniform when looked at large scales, just as regular, ordered lattices do. Hyperuniform states present the suppression of large wavelength fluctuations and have been observed in jammed granular [1, 2] and colloidal packings [3], in block-copolymer assemblies [4], in quasicrystals [5] and even in the patterns of photoreceptive cells in chicken eyes [6]. Recently, these states have even been observed in non-equilibrium transitions [7, 8], were the final configurations are basically static. However, no dynamic state has been identified so far to present hyperuniformity. Here, we show that a vibrated granular layer, close to the liquid-to-solid transition, displays dynamic hyperuniformity. A simple model demonstrates that it results from coupling density to a highly fluctuating scalar friction field.

It is possible to characterize the decay of particle correlations with distance by measuring the average number of particles $\langle n \rangle$ and its variance $\sigma_n^2 \equiv \langle n^2 \rangle - \langle n \rangle^2$ in boxes of different sizes. In condensed matter, under normal conditions, correlations decay rapidly and above a certain length, $\sigma_n^2 \propto \langle n \rangle$. This is however, not always the case, for example this occurs naturally in regular lattices, where fluctuations take place only at the boundaries. But it also appears in disordered systems, where it has been reported that for large systems $\sigma_n^2 \propto \langle n \rangle^{\beta/2}$.

When $\beta > 2$, the system is said to present giant density fluctuations and it has been reported to take place in several dynamic non-equilibrium systems as in vibrated nematic granular layers [9] or in active matter swarms [10]. The opposite case, when $\beta < 2$, corresponds to hyperuniformity. As the system grows, number fluctuations increase slower than the usual linear behavior, $\sigma_n^2 \propto \langle n \rangle$, and density fluctuations become suppressed at the very large wavelength limit. Indeed, another way to characterize particle spatial distributions is via the static structure factor, $S(k) \equiv \langle |\hat{\rho}(k)|^2 \rangle / N$, which quantifies the amplitude of the density fluctuations in Fourier space. If for small wavevectors it follows a power law $S(k) \sim k^\alpha$, then $\beta = 2 - \alpha$ for $0 \leq \alpha < 1$ and $\beta = 1$ for $1 < \alpha$. Hence hyperuniformity is obtained when $\alpha > 0$, that is, when density fluctuations become smaller when reducing the wavenumber $k$ [11]. Here, we show that for finite systems the number variance is not the best observable to characterize hyperuniformity, but rather the structure factor, which tends to zero for small wavenumbers.

The experimental setup where we observe dynamical hyperuniformity is the same one reported previously [12, 13]. It consists of a rigid shallow box, where $N = 11704$ stainless steel monodisperse spheres are placed inside, with diameter $a = 1 \text{ mm}$. The box transverse dimensions are $L_x = L_y = 100a$, such that the filling fraction is $\phi = N \pi a^2 / 4 L^2 = 0.919$. The whole setup is forced sinusoidally with an electromagnetic shaker, with vertical displacement $z(t) = A \sin(\omega t)$ (Fig. 1). Top view images are obtained with a high-speed camera. The frame rate is either 10 or 500 fps, depending on the quantity to be measured. The control parameter is the dimensionless acceleration $\Gamma = A \omega^2 / g$, where $\omega = 2 \pi f$ is the angular frequency and $g$ the gravitational acceleration, which is varied in the range 1-6. For additional details, see the Supplementary Material [14].

The box height can be fixed between one and two particle diameters, allowing to have efficient vertical to horizontal collisional energy transfer mechanisms producing liquid-like states but, at the same time, is small enough to avoid particles to jump over another. This last feature allows to track all particles in the projected two-dimensional motion and therefore analyze simultaneously the microscopic and global dynamics. One of the most remarkable features of this geometry is the existence of a liquid-to-solid transition [15, 16]. When the vibration amplitude increases above a certain threshold, stable crystalline clusters are formed, coexisting with a liquid phase. Notably, this transition can be either continuous or discontinuous depending on the box height. For the continuous case, prior to the transition, patches of the solid phase form, with length scales and mean lifetimes that diverge critically at the transition point (Fig. 1d) [12, 13, 17].

Experiments are performed at fixed vibration frequency, $f = 80 \text{Hz}$, and for the box height where the transition is continuous, $L_z = 1.94a \pm 0.02a$. The vibration amplitude $A$ is varied in a range below the transition, where the system remains...
in a stable fluid phase. Some experiments are also performed above the solid-liquid transition. For these parameters, the critical acceleration is $\Gamma_c = 4.73 \pm 0.15$. With the particle positions obtained using a high-speed camera, it is direct to obtain the two-dimensional structure factor. For intermediate wavelengths $ka \sim 0.2$, a pre-peak is observed with a height that grows when approaching the transition (see Fig. 2b), although no critical divergence is observed \cite{11}. Notably, for smaller wavenumbers, the structure factor decreases and approaches zero when $k$ decreases as $S(k) \sim k^{\alpha}$, which is a clear signature of a hyperuniform state. In Fig. 2b we present the fitted exponents using the range of $ka$ below the pre-peak. There is a small $\Gamma$ dependence in the fitted exponents, although variations are within the adjustment errorbars. The average value and its error are $\alpha = 0.93 \pm 0.11$ \cite{14}.

Hyperuniformity in this system is closely linked to the existence of the intermittent solid patches. In these, particles are trapped in a state where they collide rapidly with the top and bottom walls, together with the neighboring particles. Friction with the walls is therefore enhanced and the whole patch behaves as a region with large friction, being therefore almost immobile. In the fluid phase, the particles move but this motion is limited by the presence of these regions. At the critical point, the crystalline patches are scale free and their degree of order. For each particle,

$$Q_4^i = \frac{1}{N_j} \sum_{s=1}^{N_j} e^{i4\theta s},$$

where $N_j$ is the number of nearest neighbors of particle $j$ and $\theta_s$ is the angle between the neighbor $s$ of particle $j$ and the $x$ axis. For a particle in a square lattice, $|Q_4^i| = 1$ and the complex phase measures the square lattice orientation respect to the $x$ axis. A map of $|Q_4^i|$ for an acceleration $\Gamma = 4.5 < \Gamma_c$ is shown in Fig. 1b. Below the transition, the structure factor of $Q_4$ presents an Ornstein-Zernike form, $S_4(k) = S_4(0) / [1 + (\xi_4 k)^2]$, where the amplitude $S_4(0)$ and correlation length $\xi_4$ diverge at the critical point. The friction with the top and bottom walls is always present and therefore, the lateral momentum is rapidly dissipated and, hence, it is not relevant for the slow large-scale dynamics \cite{18}.

Through the bond-orientational order parameter $Q_4$, particles can be classified to belong to the liquid or the solid phase. We have previously shown that the stationary distribution of $|Q_4^i|$ is bimodal, with a wide maximum around $|Q_4^i| = 0.3$, and a much narrower one at $|Q_4^i| \approx 0.95$ (details in the supplementary material of \cite{19}). The local minimum between the two distribution peaks is at $|Q_4^i| = 0.7$. In order to determine the diffusive nature of particles in each phase we track them for a given period of time $T_o = 0.4$ s, which is large compared to the fast time scale of energy injection and dissipation (vibration period $1/f = 12$ ms). To compute the mean-square displacement of particles in each phase, we impose that the time average $\langle|Q_4^i|\rangle_{T_o}$, computed for the observation time $T_o$, must be $\leq 0.4$ or $\geq 0.7$ to be classified as liquid or solid particles, respectively \cite{13}. Next, the mean-square displacement of the ensemble of particles that satisfy the previous conditions is computed using the tracked trajectories. Using this procedure we find that the behavior of the mean squared displacement (Fig. 3) is radically different for the liquid and solid

![Figure 1](image-url)

**FIG. 1.** a, Schematic of the experimental setup. (left) Top view of the quasi-2D cell, with $L_x = L_y = 100a$. (right) Side view of the setup. The vertical height in the cell is $L_z = (1.94 \pm 0.02)a$. The cell is illuminated from below with a 2D array of light emitting diodes, where light is diffused with a white acrylic sheet placed between the array and the cell. (1) camera, (2) quasi-2D cell, (3) electromechanical shaker, (4) accelerometer. b, Color map of the absolute value of the fourfold bond-orientational order parameter in real space, $|Q_4^i|$, for $\Gamma = 4.5$. c, Schematic representation of the density wave interactions with the crystalline patches shown in b.
phases. The former exhibits diffusion, while the later shows subdiffusive dynamics. This subdiffusion can be modeled by a very small diffusion coefficient, a property that is a result of caging and the enhanced friction.

With these elements, we propose a model, where density evolves by diffusion

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D \nabla \rho + \eta).$$

Here, $\eta$ is a fluctuating mass flux which is modeled as a white noise satisfying $\langle \eta(r,t) \rangle = 0$ and $\langle \eta_i(r,t) \eta_k(r',t') \rangle = C_1 \delta_{ij} \delta(t-t') \delta(r-r')$, where $i, k \in \{1, 2\}$ are the Cartesian coordinates. The diffusion coefficient $D$ depends on a local order field $\psi$, related to $Q_4$, which is described by the critical equation

$$\frac{\partial \psi}{\partial t} = \mu \nabla^b \psi - \nabla \psi + \xi,$$

where $\xi$ is a white noise satisfying $\langle \xi(r,t) \rangle = 0$ and $\langle \xi_i(r,t) \xi_k(r',t') \rangle = C_2 \delta(t-t') \delta(r-r')$. The constant $\mu$ accounts for the spatial coupling of $\psi$, while $\nu$ measures the distance to the critical point. The order parameter $\psi$ presents fluctuations that grow when approaching the transition and, hence, we identify large values of $\psi$ as being representative of the solid phase. Then, the diffusion coefficient is modeled as $D = D_0 e^{-\mu \psi}$, taking finite values in the liquid phase, while vanishing asymptotically in the solid phase. Other expressions for $D$ with the same asymptotic limits generate similar properties. To fix the exponent $b$, we require that the $\psi-\rho$ coupling becomes relevant, in the renormalization group sense, at large scales. We consider the scaling $r \to s r, t \to s^\nu t$ and $\psi \to s^\alpha \psi$, and Eqn. (3) reads

$$\frac{\partial \psi}{\partial t} = s^{\alpha-b} \mu \nabla^b \psi - s^\nu \nabla \psi + s^{-(d+z)/2} \nabla \cdot \xi,$$

after multiplying it by $s^{-z}$. Then, the different parameters scale as $\mu \to s^{\alpha-b} \mu$, $\nu \to s^\nu \nu$, and $C_2 \to s^{\alpha-d-2z} C_2$, where $d$ is the spatial dimensionality. Units are fixed by choosing that $\alpha = 1$.

FIG. 2. a, Experimental static structure factor, $S(k)$, in the large wavelength limit for different accelerations $\Gamma \in [2.0, 4.7]$. b, Exponent $\alpha$ obtained from the fits $S(k) = S_0 k^\alpha$ for different $\Gamma < \Gamma_c = 4.73 \pm 0.15$. The fits were performed in the range $k a = [0.0314, 0.08]$. The average value $\langle \alpha \rangle = 0.93$ is shown as dashed horizontal line. The individual errorbars are computed from the 95% confidence interval of each fit. The error associated to $\alpha$ is 0.11, computed from the maximum of the errorbars divided by $\sqrt{20}$, with 20 the number of values of $\alpha$. c, Experimental number particle variance $\sigma^2_N$ as a function of the box size $\ell$ for three different $\Gamma$. Note that in the available experimental range, $\sigma^2_N$ does not follow the classical exponent $\beta = 2$, neither the hyperuniform exponent $\beta = 1$. d, $S(k)$ obtained from the numerical solution of the model proposed in 1D and 2D for different values of the spatial coupling exponent $b$. The exponent obtained from the fit is $\alpha = 1.6$ in 1D ($b = 2$), $\alpha = 0.8$ in 2D ($b = 4$). e, Numerical solution of $S(k)$ when $b = 4$ in 2D, for three different values of $\nu$. The offset $\Delta S$ varies when the critical point is approached ($\nu \to 0$). f, Number particle variance $\sigma^2_N$ obtained from synthetic data for two different cell sizes (see Supplementary Material for details). It shows that when the system size is $100a \times 100a$, it is impossible to observe hyperuniformity, and it can only be observed for large enough systems. The dashed line shows the expected law for $\alpha = 1$. 

\[ S(k) \]

\[ \nabla^2 (1D) \]

\[ \nabla^4 (1D) \]

\[ \nabla^2 (2D) \]

\[ \nabla^4 (2D) \]
\(\mu\) and \(C_2\) are not modified when the spatial scale \(s\) is changed, resulting in \(z = b\) and \(\chi = (b - d)/2\). For the \(\psi\)-\(p\) coupling to become relevant, \(\psi\) must take large values close to the critical point at increasingly large spatial scales. This is obtained when \(\chi > 0\) or, equivalently, \(b > d\). That means that for \(d = 1\) and \(b = 2\) (normal spatial coupling for \(\psi\)) it is possible to observe large scale structures close to the critical point while for \(d = 2\), a larger exponent is needed, for example \(b = 4\) to keep analyticity.

It is possible to fix dimensions such that \(\mu = \lambda = C_1 = C_2 = 1\) (see Supplementary Material for details [14]). Thus, the free parameters of the problem are \(b, \nu\) and the system length \(L\). The characteristic time and lengths scales are \(t_c = 1/\nu\) and \(\ell_c = t_c^{1/b}\). Considering the spatial and temporal discretization \(\Delta t\) and \(\Delta x\), respectively, large structures will be well resolved if \(\Delta x \ll \ell_c \ll \ell\) and \(\Delta t \ll t_c \ll T\), where \(T\) is the total simulation time [14]. The numerical solution of the model under these conditions shows that the structure factor of \(\psi\) displays an Ornstein-Zernike form. The density field static structure factor indeed presents a notorious decrease when reducing the wavevector, which is well fitted as \(S(k) \sim \Delta S + S_0 k^\alpha\). In 1D we obtain \(\alpha = 1.6\), independent of the spatial coupling exponent \(b\); in 2D we get \(\alpha = 0.8\), only for \(b = 4\) (see Fig. [2]). The offset \(\Delta S\) vanishes when the critical point is approached \((\nu \to 0)\) and hyperuniformity is obtained (Fig. [2]). Both the experimental and theoretical normalized structure factors, \(S(k)/k^\alpha\), reach constant values at low wavenumbers (see Supplementary Material).

A typical landmark of hyperuniformity is the existence of a sublinear growth of the particle number variance with the box sizes. For this feature to be observable, it is necessary to have a full scale separation between the system size and the wavelength above which the structure factor displays the power law \(S(k) \sim k^\alpha\). In our experiments \(L = 100a\), with a structure factor peak located at \(ka \sim 0.2\), leaving a limited range to observe the sublinear behavior (see Fig. [2]). Furthermore, the variance must vanish for a box equal to the system size, limiting even more the range of box sizes where hyperuniformity can be observed. Indeed, although in the system under study the structure factor clearly evidences the presence of hyperuniformity, it is not possible to observe it with \(\sigma_N^2\) (Fig. [2]). Synthetic generated values of \(\sigma_N^2\) using the experimental values of \(S(k)\) show that the sublinear growth would only be observed for large system sizes \(L \geq 1000a\) (Fig. [2]). See the Supplementary Material for details on the number particle variance computation and the synthetic number particle variance generation [14].

We have shown that hyperuniformity can be dynamically generated in fluid-like states if friction is highly heterogeneous. In our case, we have built a model where density evolves by diffusion, with a diffusion coefficient that can become very small in fluctuating patches. In systems where momentum is conserved, a similar behavior should be obtained if the viscosities are heterogeneous, with patches of high friction.

In small systems, the finite size effects make it difficult to identify hyperuniformity with the particle number variance. On the other hand, in the analysis of density fluctuations the finite size effects and boundary inhomogeneities do appear in both \(\langle \tilde{\rho}_k^2 \rangle\) and \(\langle \tilde{\rho}_k \rangle\), but notably they cancel in the structure factor, which is remarkably isotropic [13]. This feature makes \(S(k)\) an ideal observable to identify hyperuniform states.

Hyperuniformity is characterized by positive values of the exponent \(\alpha\). In this article we report the values 0.93 ± 0.11 (experiment in 2D), 1.6 (simulation in 1D), and 0.8 (simulation in 2D). This diversity in values is consistent with a similar variability reported in the literature by measuring the structure factor: 0.45 in critical absorbing states [7], 0.5 in periodically driven emulsions [8], 1 in jammed packings of polydisperse spheres [1, 2], 1 in the photoreceptive cells of chicken eyes [6], or deducing it from the measurement of the number variance exponent: 0.8 in block-copolymer assemblies [4], 0.21 in jammed packings of soft spheres [3], 1 in two-phase random media [20], and 2 in quasicrystals [5]. In remains to be understood if this range of values is a signature a different universally classes or if \(\alpha\) is not an universal exponent at all.

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[1] Aleksandar Donev, Frank H. Stillinger, and Salvatore Torquato, “Unexpected density fluctuations in jammed disordered sphere packings,” Phys. Rev. Lett. 95, 090604 (2005).
[2] Ludovic Berthier, Pinaki Chaudhuri, Corentin Coulais, Olivier Dauchot, and Peter Sollich, “Suppressed compressibility at large scale in jammed packings of size-disperse spheres,” Phys. Rev. Lett. 106, 120601 (2011).
[3] Remi Dreyfus, Ye Xu, Tim Still, L. A. Hough, A. G. Yodh, and Salvatore Torquato, “Diagnosing hyperuniformity in two-dimensional, disordered, jammed packings of soft spheres,” Phys. Rev. E 91, 012302 (2015).
[4] Gianluigi Zito, Giulia Rusciano, Giuseppe Pesce, Anna Malafronte, Rocco Di Girolamo, Giovanni Ausanio, Antonio Vecchione, and Antonio Sasso, “Nanoscale engineering of two-dimensional disordered hyperuniform block-copolymer assemblies,” Phys. Rev. E 92, 050601 (2015).
[5] Erdal C. Öğuz, Joshua E. S. Socolar, Paul J. Steinhardt, and Salvatore Torquato, “Hyperuniformity of quasicrystals,” Phys. Rev. B 95, 054119 (2017).
[6] Yang Jiao, Timothy Lau, Haralampos Hatzikirou, Michael Meyer-Hermann, Joseph C. Corbo, and Salvatore Torquato, “Avian photoreceptor patterns represent a disordered hyperuniform solution to a multiscale packing problem,” Phys. Rev. E 89, 022721 (2014).
[7] Daniel Hexner and Dov Levine, “Hyperuniformity of critical absorbing states,” Phys. Rev. Lett. 114, 110602 (2015).
[8] Joost H. Weijs, Raphaël Jeanneret, Rémi Dreyfus, and Denis Bartolo, “Emergent hyperuniformity in periodically driven emulsions,” Phys. Rev. Lett. 115, 108301 (2015).
[9] Vijay Narayan, Siriram Ramaswamy, and Narayanan Menon, “Long-lived giant number fluctuations in a swarming granular nematic,” Science 317, 105–108 (2007).
[10] H. P. Zhang, Avraham Be’er, E.-L. Florin, and Harry L. Swinney, “Collective motion and density fluctuations in bacterial colonies,” PNAS 107, 13626–13630 (2010).
[11] Salvatore Torquato, “Hyperuniformity and its generalizations,” Phys. Rev. E 94, 022122 (2016).
[12] Gustavo Castillo, Nicolás Mujica, and Rodrigo Soto, “Fluctuations and Criticality of a Granular Solid-Liquid-Like Phase Transition,” Phys. Rev. Lett. 109, 095701 (2012).
[13] Gustavo Castillo, Nicolás Mujica, and Rodrigo Soto, “Universality and criticality of a second-order granular solid-liquid-like phase transition,” Phys. Rev. E 91, 012141 (2015).
[14] See supplementary material for some details on the experimental setup, the normalized structure factor, the classification of solid and liquid particles for the computation of dynamic quantities, the dimensionless model and its numerical solution, the number particle variance computation and the synthetic number particle variance generation.
[15] Alexis Prevost, Paul Melby, David A. Egolf, and Jeffrey S. Urbach, “Nonequilibrium two-phase coexistence in a confined granular layer,” Phys. Rev. E 70, 050301 (2004).
[16] Paul Melby, Francisco Vega Reyes, Alexis Prevost, R Robertson, P Kumar, D A. Egolf, and Jeffrey S. Urbach, “The dynamics of thin vibrated granular layers,” Journal of Physics: Condensed Matter 17, 2689–S2704 (2005).
[17] Marcelo Guzmán and Rodrigo Soto, “Critical phenomena in quasi-two-dimensional vibrated granular systems,” Phys. Rev. E 97, 012907 (2018).
[18] Marcel G. Clerc, Patricio Cordero, Jocelyn Dunstan, Kathryn D. Huff, Nicolás Mujica, Dino Risso, and Germán Varas, “Liquid-solid-like transition in quasi-one-dimensional driven granular media,” Nature Physics 4, 249–254 (2008).
[19] Li-Hua Luu, Gustavo Castillo, Nicolás Mujica, and Rodrigo Soto, “Capillarylike fluctuations of a solid-liquid interface in a noncohesive granular system,” Phys. Rev. E 87, 040202 (2013).
[20] Chase E Zachary and Salvatore Torquato, “Hyperuniformity in point patterns and two-phase random heterogeneous media,” Journal of Statistical Mechanics: Theory and Experiment 2009, P12015 (2009).
Supplementary Material for
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Gustavo Castillo¹, Nicolás Mujica², Néstor Sepúlveda², Juan Carlos Sobarzo², and Rodrigo Soto²

¹Instituto de Ciencias de la Ingeniería, Universidad de O’Higgins, Rancagua, Chile and
²Departamento de Física, FCFM, Universidad de Chile, Santiago, Chile

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In this supplementary paper we present some details on the experimental setup, the normalized structure factor, the classification of solid and liquid particles for the computation of dynamic quantities, the dimensionless model and its numerical solution, the number particle variance computation and the synthetic number particle variance generation.

System setup

The box consists of two 10-mm-thick glass plates separated by a square metallic frame. Each inner glass surface has an indium tin oxide (ITO) coating, which dissipates electrostatic charges generated by collisions of particles with the walls. A piezoelectric accelerometer is fixed to the base, allowing the measurement of the imposed forcing acceleration with a resolution of 0.01 g. For each Γ, three videos of 3000 images were acquired, two at 10 fps for the computation of $S(k)$ and one at 500 fps to obtain the diffusion coefficients. The acquired images have a resolution of $1600 \times 1600$ pix². Particle positions are determined at sub-pixel accuracy. The particle detection is done by using a modified open source Matlab code, which uses a least-square algorithm [1]. Our modified version in C++ and CUDA allows faster computation for a large number of particles [2, 3]. The algorithm allows us to detect both layers of particles in a dense solid cluster, where the top-layer particles are placed in the valleys that the bottom particles form.

Normalized structure factor

Figure 1 presents the normalized structure factors $S(k)/ka$. The exponents $\alpha$ are obtained from fits performed in the small wavenumber regime. Both the experimental and theoretical normalized structure factors reach constant values at low wavenumbers, consistent with hyperuniformity.

Classification of particles in liquid and solid phases for computing the mean square displacement

In Fig. 2a we present two typical time series of $|Q|$ for a particle in the solid phase and one in the liquid phase for a complete time duration of 0.4 s. The former has a high average $|Q|$, whereas the later has a lower average; both realizations show fluctuations. In Fig. 2b we present the standard deviation of $|Q|$, $\sigma_{|Q|}$, versus $|Q|$ for $\Gamma = \{2.5, 4, 6\}$. At lower and intermediate forcing most particles are in the liquid phase, with some particles that occasionally condense, temporally, in small ordered solid clusters. Well above the transition there are many particles that stay in the solid phase during the complete time series. At intermediate and high accelerations a large fraction of particles fluctuate between both

FIG. 1. (a) Experimental $S(k)$ normalized by $ka$ for values of $\Gamma$ in the range $[2.0, 4.7]$. The shaded region shows where the fits were performed. (b) Normalized static structure factor $S(k)/ka$ obtained from the model, for both 1D and 2D and two different values of the spatial coupling exponent $b$. Transition
then we chose to make it dimensionless. As the product $\lambda \psi^2$ must be also dimensionless, we can take $\lambda = 1$. Considering Eq. (3), the dimensions of the parameters are $[\mu] = \mu \cdot t^{-1}$ and $[C_2] = \mu \cdot t$. It is then possible to fix length and time units such that $\mu = C_2 = 1$. By Eq. (2) of the paper, $[\psi] = \rho \cdot \xi^{1/2}$. As the units of $\rho$ are arbitrary, it is possible to chose the units of $\psi$ in such a way that $C_1 = 1$. With these elections, the dimensionless model has only $b$, $v$, and $L$ as free parameters.

### Numerical solution of the model

The integration of Eq. (3) of the paper was performed using the Crank–Nicolson method [4], where the spatial part is computed with spectral methods using the FFTW library [5]. Eq. (2) was integrated with the Alternating-Direction Implicit method [4]. For both equations we used a time step $\delta t = 0.05$ (in 1D and 2D) and spatial grid $\delta x = 0.25$ (in 1D), $\delta x = 0.65$ (in 2D), and the number of nodes for the discretization in 1D was $N_x = 4096$, and in 2D in both directions was $N_x = N_y = 512$. Simulations were first relaxed for a time $T_1 = 5 \times 10^6$ (1D) and $T_1 = 2.5 \times 10^9$ (2D) and later configurations were recorded every $\Delta T = 250$ (1D) and $\Delta T = 50$ (2D), for a total simulation time $T_2 = 1 \times 10^7$ (1D) and $T_2 = 5 \times 10^9$ (2D).

#### Experimental number particle variance

In order to quantify the spatial correlations, the average number of particles $\langle N \rangle$ and its variance $\sigma^2 = \langle N^2 \rangle - \langle N \rangle^2$ in boxes of different sizes are measured. From each image we compute the number of particles in subsystems of different size, defined by square windows ranging in size from $a \times a$ to $80a \times 80a$. Due to the presence of some spatial heterogeneities, each square window of size $\ell$ is displaced throughout the entire image (excluding 10a at each border), which gives us a spatial average of $\langle N \rangle$. Then, for the set of images at a fixed $\Gamma$ we determine, for each subsystem size, the average $\langle N \rangle$ over all images and thus, its standard deviation, $\sigma_N(\ell)$. In Fig.4a of the paper we shown the behavior of $\sigma^2(\ell)$ for different accelerations $\Gamma$.

#### Synthetic number particle variance

For a finite 2D system of size $L \times L$, the density field, its Fourier transform, and inverse transform are

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i), \quad \tilde{\rho}_k = \sum_{i=1}^{N} e^{ik \cdot \mathbf{r}_i}, \quad \rho(\mathbf{r}) = L^{-2} \sum_k \tilde{\rho}_k e^{-ik \cdot \mathbf{r}}.$$  

(1)

Here $\mathbf{r}_i$ are the positions of the particles and the wavevectors are $\mathbf{k} = (2\pi n_x/L, 2\pi n_y/L)$, with $n_{x,y} \in \mathbb{Z}$. In a subvolume $\ell \times \ell \times \ell$.
the number of particles is

\[ N(\ell) = \int_{x,t} p(x) d^2r. \quad (2) \]

Using the expression for the inverse transform and noting that for a statistically homogeneous system, \( \langle p_k \rangle = N N_0 \), the variance in the number of particles in the subvolume can be computed in terms of the structure factor as \[ \sigma^2_n(\ell) = \frac{\rho^2}{N} \sum_{k} S(k) |w(k,\ell)|^2, \quad (3) \]
where

\[ w(k,\ell) = \int_{x,t} e^{-ik \cdot r} d^2r \quad (4) \]
is the Fourier transform of the subvolume indicatrix and the prime indicates that \( k = 0 \) should be excluded from the sum. For the square subvolumes considered here,

\[ |w(k,\ell)|^2 = \left[ \frac{2 \sin(kx\ell/2)}{kx} \right]^2 \left[ \frac{2 \sin(ky\ell/2)}{ky} \right]^2. \quad (5) \]

It can be easily verified that the correct limiting values are obtained: \( \lim_{\ell \to 0} \sigma^2_n(\ell) = \lim_{\ell \to \infty} \sigma^2_n(\ell) = 0 \). For large box sizes, the sum in Eq. (3) of this supplementary document can be casted into an integral when \( r = x,y \) are continuous variables. If the structure factor follows a power law, \( S(k) = S_0 \frac{ka}{1 + S_1(ka)^{\alpha}} \), \( \alpha \in [0,1) \).

\[ \sigma^2_n(\ell) \sim \ell^{2-\alpha}. \quad \text{For} \quad \alpha = 1, \quad \text{the lower limit of the integral must be considered, in which case} \quad \sigma^2_n(\ell) \sim \ell \log \ell \quad [7]. \]

In order to determine the effects of finite experimental sizes, we fit the measured structure factor in the range \( ka < 0.6 \) to the function

\[ S(k) = \frac{S_0 a}{1 + S_1(ka)^\alpha}, \quad (7) \]
with \( S_0 = 9.49, S_1 = 72.7, n = 2.4 \), which captures the growth and the maximum at \( ka \approx 0.2 \). This is done for \( \Gamma = 4.40 < \Gamma_c \). Note that for simplicity we use \( \alpha = 1 \), although almost identical results are obtained for \( \alpha = 0.93 \). With this function, the variance (3) is directly computed for different system sizes (Fig. 2f of the paper). In practice, the sum is done for \( n_{\max} \leq n_{\max} \) and the sum converges using \( n_{\max} = 100 \) for \( L = 100a \) and using \( n_{\max} = 600 \) for \( L = 1000a \).

\[ \begin{align*}
[1] & \text{Mark D. Shattuck and Scott V. Franklin, \textit{Handbook of Granular Materials} (CRC Press, 2015).} \\
[2] & \text{Juan Silva, "Optimización de proceso de detección de partículas a partir de imágenes de video mediante paralelización," Computer Science Engineering thesis, Universidad de Chile (2012).} \\
[3] & \text{Mauricio Cerda, Cristóbal A. Navarro, Juan Silva, Scott R. Waitukaitis, Nicolás Mujica, and Nancy Hitschfeld-Kahler, "A high-speed tracking algorithm for dense granular media," Computer Physics Communications 227, 8–16 (2018).} \\
[4] & \text{William H Press, \textit{Numerical recipes 3rd edition: The art of scientific computing} (Cambridge university press, 2007).} \\
[5] & \text{Matteo Frigo and Steven G Johnson, "The Design and Implementation of FFTW3," Proceedings of the IEEE 93, 216–231 (2005).} \\
[6] & \text{Salvatore Torquato and Frank H. Stillinger, \"Local density fluctuations, hyperuniformity, and order metrics," Phys. Rev. E 68, 041113 (2003).} \\
[7] & \text{Chase E Zachary and Salvatore Torquato, \"Hyperuniformity in point patterns and two-phase random heterogeneous media," Journal of Statistical Mechanics: Theory and Experiment \textit{2009}, P12015 (2009).} 
\end{align*} \]