A Geometric Approach for the Upper Bound Theorem for Minkowski Sums of Convex Polytopes

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Abstract We derive tight expressions for the maximum number of \(k\)-faces, \(0 \leq k \leq d - 1\), of the Minkowski sum, \(P_1 + \cdots + P_r\), of \(r\) convex \(d\)-polytopes \(P_1, \ldots, P_r\) in \(\mathbb{R}^d\), where \(d \geq 2\) and \(r < d\), as a (recursively defined) function on the number of vertices of the polytopes. Our results coincide with those recently proved by Adiprasito and Sanyal (Publ Math Inst Hautes Etudes Sci. doi: 10.1007/s10240-016-0083-7, 2016). In contrast to Adiprasito and Sanyal’s approach, which uses tools from Combinatorial Commutative Algebra, our approach is purely geometric and uses basic notions such as \(f\)- and \(h\)-vector calculus, stellar-subdivisions and shellings, and generalizes the methodology used in Karavelas and Tzanaki (Discrete Comput Geom 55:748–785, 2016) and Karavelas et al. (J Comput Geom 6(1):21–74, 2015) for proving upper bounds on the \(f\)-vector of the Minkowski sum of two and three convex polytopes, respectively. The key idea behind our approach is to express the Minkowski sum \(P_1 + \cdots + P_r\) as a section of the Cayley polytope \(\mathcal{C}\) of the summands; bounding the \(k\)-faces of \(P_1 + \cdots + P_r\) reduces to bounding the subset of the \((k + r - 1)\)-faces of \(\mathcal{C}\) that contain vertices from each of the \(r\) polytopes. We end our paper with an explicit construction that establishes the tightness of the upper bounds.
Keywords Discrete geometry · Combinatorial geometry · Combinatorial complexity · Minkowski sum · Convex polytopes · Upper bound

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1 Introduction

Given two sets $A$ and $B$ in $\mathbb{R}^d$, $d \geq 2$, their Minkowski sum $A + B$ is the set $\{a + b \mid a \in A, b \in B\}$. The Minkowski sum definition can be extended naturally to any number of summands: $A_1[\ldots] := A_1 + A_2 + \cdots + A_r = \{a_1 + a_2 + \cdots + a_r \mid a_i \in A_i, 1 \leq i \leq r\}$. Minkowski sums have a wide range of applications, including algebraic geometry, computational commutative algebra, collision detection, computer-aided design, graphics, robot motion planning and game theory, just to name a few (see also [1,14] and the references therein).

In this paper we focus on convex polytopes, and we are interested in computing the worst-case complexity of their Minkowski sum. More precisely, given $r$ $d$-polytopes $P_1, \ldots, P_r$ in $\mathbb{R}^d$, we seek tight bounds on the number of $k$-faces $f_k(P[\ldots])$, $0 \leq k \leq d - 1$, of their Minkowski sum $P[\ldots] := P_1 + P_2 + \cdots + P_r$. This problem, which can be seen as a generalization of the upper bound theorem (UBT) for polytopes [17], has a history of more than 20 years. Gritzmann and Sturmfels [10] were the first to consider the problem, and gave a complete answer to it, for any number of $d$-polytopes in $\mathbb{R}^d$, in terms of the number of non-parallel edges of the $r$ polytopes. More than 10 years later, Fukuda and Weibel [6] proved tight upper bounds on the number of $k$-faces of the Minkowski sum of two 3-polytopes, expressed either in terms of the number of vertices or number of facets of the summands. Fogel et al. [5] extended one of the results in [6], and expressed the number of facets of the Minkowski sum of $r$ 3-polytopes in terms of the number of facets of the summands. Quite recently Weibel [23] provided a relation for the number of $k$-faces of the Minkowski sum of $r \geq d$ summands in terms of the $k$-faces of the Minkowski sums of subsets of size $d - 1$ of these summands. This result should be viewed in conjunction with a result by Sanyal [19] stating that the number of vertices of the Minkowski sum of $r d$-polytopes, where $r \geq d$, is strictly less than the product of the vertices of the summands (whereas for $r \leq d - 1$ this is indeed possible). About 3 years ago, the authors of this paper proved the first tight upper bound on the number of $k$-faces for the Minkowski sum of two $d$-polytopes in $\mathbb{R}^d$, for any $d \geq 2$ and for all $0 \leq k \leq d - 1$ (cf. [13]), a result which was subsequently extended to three summands in collaboration with Konaxis (cf. [14]).

In a recent paper, Adiprasito and Sanyal [1] provide the complete resolution of the Upper Bound Theorem for Minkowski sums (UBTM). In particular, they show that there exists, what they call, a Minkowski-neighborly family of $r$ $d$-polytopes $N_1, \ldots, N_r$, with $f_0(N_i) = n_i$, $1 \leq i \leq r$, such that for any $r$ $d$-polytopes $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ with $f_0(P_i) = n_i$, $1 \leq i \leq r$, $f_k(P[\ldots])$ is bounded by above by $f_k(N[\ldots])$, for all $0 \leq k \leq d - 1$. The majority of the arguments in the UBTM proof by Adiprasito and Sanyal make use of powerful tools from Combinatorial Commutative Algebra. The
high-level layout of the proof is analogous to McMullen’s proof of the UBT, as well as the proofs of the UBTM in [13] and [14] for two and three summands, respectively:

1. Consider the Cayley polytope $C \subset \mathbb{R}^{d+r-1}$ of the $r$ polytopes $P_1, P_2, \ldots, P_r$, and identify their Minkowski sum as a section of $C$ with an appropriately defined $d$-flat $W$. Let $\mathcal{F} \subset \mathbb{R}^{d+r-1}$ be the faces of $C$ that intersect $W$, and let $\mathcal{K}$ be the closure of $\mathcal{F}$ under subface inclusion ($\mathcal{K}$ is a $(d + r - 1)$-polypetal complex). By the Cayley trick, there is a bijection between the faces of $\mathcal{F}$ and the faces of $P_r$; as a result, to bound the number of faces of $P_r$ it suffices to bound the number of faces of $\mathcal{F}$.

2. Define the $h$-vector $h(\mathcal{F})$ of $\mathcal{F}$, and prove the Dehn–Sommerville equations for $h(\mathcal{F})$, relating its elements to the elements of $h(\mathcal{K})$.

3. Prove a recurrence relation for the elements of $h(\mathcal{F})$.

4. Use the recurrence relation above to prove upper bounds for $h_k(\mathcal{F})$, for all $0 \leq k \leq \left\lfloor \frac{d + r - 1}{2} \right\rfloor$.

5. Prove upper bounds for $h_k(\mathcal{K})$, for all $0 \leq k \leq \left\lfloor \frac{d + r - 1}{2} \right\rfloor$. Due to the Dehn–Sommerville equations, these are bounds for $h_k(\mathcal{F})$ for all $k > \left\lfloor \frac{d + r - 1}{2} \right\rfloor$.

6. Compute upper bounds for the elements of $f(\mathcal{F})$ using the upper bounds for the elements of $h(\mathcal{F})$ and $h(\mathcal{K})$.

7. Provide necessary and sufficient conditions under which the elements of both $h(\mathcal{F})$ and $h(\mathcal{K})$ are maximized for all $k$. These conditions are conditions on the lower half of the $h$-vector of $\mathcal{F}$. Due to the relation between the $f$- and $h$-vectors of $\mathcal{F}$, these are also conditions for the maximality of the elements of $f(\mathcal{F})$.

8. Describe a family of polytopes for which the necessary and sufficient conditions hold; clearly, such a family establishes the tightness of the upper bounds.

In Adiprasito and Sanyal’s proof steps 2, 3 and 4 are proved by introducing a powerful new theory that they call the relative Stanley–Reisner theory for simplicial complexes. The focus of this theory is on relative simplicial complexes, and is able to reveal properties of such complexes not only under topological restrictions, but also account for their combinatorial and geometric structure. To apply their theory, Adiprasito and Sanyal consider the simplicial complex $\mathcal{K}$ and then define $\mathcal{F}$ as a relative simplicial complex (they call them the Cayley and relative Cayley complex, respectively). They then apply their relative Stanley–Reisner theory to $\mathcal{F}$ to establish the Dehn–Sommerville equations of step 2, the recurrence relation of step 3 and finally the upper bounds for $h(\mathcal{F})$ in 4. Steps 5, 6 and 7 are done by clever algebraic manipulation of the $h$-vectors of $\mathcal{F}$ and $\mathcal{K}$, by utilizing the Dehn–Sommerville equations, by exploiting the geometric properties of $\mathcal{K}$, and by making use of the recurrence relation in step 3. Step 8 is reduced to results by Matschke et al. [16] and Weibel [23].

Our contribution. In what follows, we provide a completely geometric proof of the UBTM, that generalizes the technique we used in [13] and [14] for two and three summands to the case of $r$ summands, when $r < d$. Our proof, in essence, differs from that of Adiprasito and Sanyal in steps 2, 3 and 4 of the layout above. Step 5 is done as in Adiprasito and Sanyal. Instead of relying on algebraic tools, we interpret the $h$-vectors mentioned above, using basic notions from combinatorial geometry and topology, such as stellar subdivisions and shellings.

In more detail, to prove the various intermediate results, towards the UBTM, we consider the Cayley polytope $C$ and we perform a series of stellar subdivisions to get
a simplicial polytope $Q$. From the analysis of the combinatorial structure of $Q$, we derive the Dehn–Sommerville equations of step 2 (see Sects. 3 and 4), as well as the recurrence relation of step 3 (see Sect. 5). This recurrence relation is then used for establishing the upper bounds for the elements of $h(\mathcal{F})$, $h(\mathcal{K})$ and $f(\mathcal{F})$ (see Sect. 6). We end with a construction similar to the one presented in [16, Thm. 2.6], that establishes the tightness of the upper bounds (see Sect. 7).

2 Preliminaries

Let $P$ be a $d$-dimensional polytope, or $d$-polytope for short. Its dimension is the dimension of its affine span. The faces of $P$ are $\emptyset$, $P$, and the intersections of $P$ with its supporting hyperplanes. The $\emptyset$ and $P$ faces are called improper, while the remaining faces are called proper. Each face of $P$ is itself a polytope, and a face of dimension $k$ is called a $k$-face. Faces of $P$ of dimension 0, 1, $d - 2$ and $d - 1$ are called vertices, edges, ridges, and facets, respectively.

A $d$-dimensional polytopal complex or, simply, $d$-complex, $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that (i) $\emptyset \in \mathcal{C}$, (ii) if $P \in \mathcal{C}$ then all the faces of $P$ are also in $\mathcal{C}$ and (iii) the intersection $P \cap Q$ for two polytopes $P$ and $Q$ in $\mathcal{C}$ is a face of both. The dimension $\dim(\mathcal{C})$ of $\mathcal{C}$ is the largest dimension of a polytope in $\mathcal{C}$. A polytopal complex is called pure if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the facets of $\mathcal{C}$. A polytopal complex is simplicial if all its faces are simplices. A polytopal complex $\mathcal{C}'$ is called a subcomplex of a polytopal complex $\mathcal{C}$ if all faces of $\mathcal{C}'$ are also faces of $\mathcal{C}$. For a polytopal complex $\mathcal{C}$, the star of $v$ in $\mathcal{C}$, denoted by $\text{star}(v, \mathcal{C})$, is the subcomplex of $\mathcal{C}$ consisting of all faces that contain $v$, and their faces. The link of $v$, denoted by $\mathcal{C}/v$, is the subcomplex of $\text{star}(v, \mathcal{C})$ consisting of all the faces of $\text{star}(v, \mathcal{C})$ that do not contain $v$.

A $d$-polytope $P$, together with all its faces, forms a $d$-complex, denoted by $\mathcal{C}(P)$. The polytope $P$ itself is the only maximal face of $\mathcal{C}(P)$, i.e., the only facet of $\mathcal{C}(P)$, and is called the trivial face of $\mathcal{C}(P)$. Moreover, all proper faces of $P$, along with the empty face, form a pure $(d - 1)$-complex, called the boundary complex $\mathcal{C}(\partial P)$, or simply $\partial P$, of $P$. The facets of $\partial P$ are just the facets of $P$.

For a $(d - 1)$-complex $\mathcal{C}$, its $f$-vector is defined as $f(\mathcal{C}) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where $f_k = f_k(\mathcal{C})$ denotes the number of $k$-faces of $P$ and $f_{-1}(\mathcal{C}) := 1$ corresponds to the empty face of $\mathcal{C}$. From the $f$-vector of $\mathcal{C}$ we define its $h$-vector as the vector $h(\mathcal{C}) = (h_0, h_1, \ldots, h_d)$, where $h_k = h_k(\mathcal{C}) := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\mathcal{C})$, $0 \leq k \leq d$.

Denote by $\mathcal{Y}$ a subset of faces of a polytopal complex $\mathcal{C}$, and define its dimension $\dim(\mathcal{Y})$ as the maximum of the dimensions of its faces. Let $\dim(\mathcal{Y}) = \delta - 1$; then we may define (if not already properly defined), the $h$-vector $h(\mathcal{Y})$ of $\mathcal{Y}$ as

$$h_k(\mathcal{Y}) = \sum_{i=0}^{\delta} (-1)^{k-i} \binom{\delta-i}{d-k} f_{i-1}(\mathcal{Y}). \tag{2.1}$$
We can further define the \( m \)-order \( g \)-vector of \( \mathcal{Y} \) according to the following recursive formula:

\[
g_k^{(m)}(\mathcal{Y}) = \begin{cases} h_k(\mathcal{Y}), & m = 0, \\ g_k^{(m-1)}(\mathcal{Y}) - g_k^{(m-1)}(\mathcal{Y}), & m > 0. \end{cases} \tag{2.2}
\]

Clearly, \( g^{(m)}(\mathcal{Y}) \) is nothing but the backward \( m \)-order finite difference of \( h(\mathcal{Y}) \); therefore:

\[
g_k^{(m)}(\mathcal{Y}) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} h_{k-i}(\mathcal{Y}), \quad k, m \geq 0. \tag{2.3}
\]

Observe that for \( m = 0 \) we get the \( h \)-vector of \( \mathcal{Y} \), while for \( m = 1 \) we get what is typically defined as the \( g \)-vector.

The relation between the \( f \)- and \( h \)-vector of \( \mathcal{Y} \) is better manipulated using generating functions. We define the \( f \)-polynomial and \( h \)-polynomial of \( \mathcal{Y} \) as follows:

\[
f(\mathcal{Y}; t) = \sum_{i=0}^{\delta} f_{i-1} t^{\delta-i} = f_{\delta-1} + f_{\delta-2} t + \cdots + f_{-1} t^{\delta},
\]

\[
h(\mathcal{Y}; t) = \sum_{i=0}^{\delta} h_{i} t^{\delta-i} = h_{\delta} + h_{\delta-1} t + \cdots + h_{0} t^{\delta},
\]

where we simplified \( f_i(\mathcal{Y}) \) and \( h_i(\mathcal{Y}) \) to \( f_i \) and \( h_i \). In this set-up, the relation between the \( f \)-vector and \( h \)-vector (cf. (2.1)) can be expressed as

\[
f(\mathcal{Y}; t) = h(\mathcal{Y}; t + 1), \quad \text{or, equivalently, as} \quad h(\mathcal{Y}; t) = f(\mathcal{Y}; t - 1). \tag{2.4}
\]

Based on (2.3) and (2.4), we may also define the \( g^{(m)} \)-polynomial of \( \mathcal{Y} \) as

\[
g^{(m)}(\mathcal{Y}; t) = \sum_{i=0}^{\delta} g_i^{(m)}(\mathcal{Y}) t^{\delta-i}, \tag{2.5}
\]

or, in terms of the \( h \)-polynomial of \( \mathcal{Y} \), as

\[
g^{(m)}(\mathcal{Y}; t) = (t - 1)^m h(\mathcal{Y}; t). \tag{2.6}
\]

### 2.1 The Cayley Embedding, the Cayley Polytope and the Cayley Trick

Let \( P_1, P_2, \ldots, P_r \) be \( d \)-polytopes with vertex sets \( \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r \), respectively. Let \( e_0, e_1, \ldots, e_{r-1} \) be an affine basis of \( \mathbb{R}^{r-1} \) and call \( \mu_i : \mathbb{R}^d \rightarrow \mathbb{R}^{r-1} \times \mathbb{R}^d \) the affine inclusion given by \( \mu_i(x) = (e_{i-1}, x), 1 \leq i \leq r \). The Cayley embedding \( \mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r) \) of the point sets \( \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r \) is defined as

\[
\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r) = \bigcup_{i=1}^{r} \mu_i(\mathcal{V}_i).
\]

The polytope corresponding to the convex hull \( \text{conv}(\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r)) \) is called the Cayley polytope.
of the Cayley embedding $C(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r)$ of $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r$ is typically referred to as the Cayley polytope of $P_1, P_2, \ldots, P_r$.

The following lemma, known as the Cayley trick for Minkowski sums, relates the Minkowski sum of the polytopes $P_1, P_2, \ldots, P_r$ with their Cayley polytope.

**Lemma 2.1** ([11, Lem. 3.2]) Let $P_1, P_2, \ldots, P_r$ be $d$-polytopes with vertex sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r \subseteq \mathbb{R}^d$. Moreover, let $\bar{W}$ be the $d$-flat defined as $\{ \frac{1}{r_1}e_0 + \cdots + \frac{1}{r_{r-1}}e_{r-1} \} \times \mathbb{R}^d \subseteq \mathbb{R}^{r-1} \times \mathbb{R}^d$. Then, the Minkowski sum $P_{[r]}$ has the following representation as a section of the Cayley embedding $C(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r)$ in $\mathbb{R}^{r-1} \times \mathbb{R}^d$:

$$P_{[r]} \cong \text{conv}(C(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r)) \cap \bar{W}$$

where $(e_{i-1}, v_i) \in C(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r), 1 \leq i \leq r$.

Moreover, $F$ is a facet of $P_{[r]}$ if and only if it is of the form $F = F' \cap \bar{W}$ for a facet $F'$ of $\text{conv}(C(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r))$ containing at least one point $(e_{i-1}, v_i)$ for all $1 \leq i \leq r$.

Let $C_{[r]}$ be the Cayley polytope of $P_1, P_2, \ldots, P_r$, and call $\mathcal{F}_{[r]}$ the set of faces of $C_{[r]}$ that have non-empty intersection with the $d$-flat $\bar{W}$. A direct consequence of Lemma 2.1 is a bijection between the $(k-1)$-faces of $P_{[r]}$ and the $(k-r)$-faces of $\mathcal{F}_{[r]}$, for $r \leq k \leq d + r - 1$. This further implies that

$$f_{k-1}(\mathcal{F}_{[r]}) = f_{k-r}(P_{[r]}), \quad \text{for all } r \leq k \leq d + r - 1. \quad (2.7)$$

In what follows, to keep the notation lean, we identify $V_i$ with its pre-image $\mathcal{V}_i$. For any $\emptyset \subseteq R \subseteq [r]$, we denote by $C_R$ the Cayley polytope of the polytopes $P_i$, where $i \in R$. In particular, if $R = \{ i \}$ for some $i \in [r]$, then $C_{\{i\}} \equiv P_i$. We shall assume below that $C_{[r]}$ is as simplicial as possible. This means that we consider all faces of $C_{[r]}$ to be simplicial, except possibly for the trivial faces $[C_R]^1, \emptyset \subset R \subseteq [r]$.

Otherwise, we can employ the so-called bottom-vertex triangulation [15, Sect. 6.5, pp. 160–161] to triangulate all proper faces of $C_{[r]}$ except for the trivial ones, i.e., $[C_R], \emptyset \subset R \subseteq [r]$. The resulting complex is polytopal (cf. [4]) with all of its faces being simplices, except possibly for the trivial ones. Moreover, it has the same number of vertices as $C_{[r]}$, while the number of its $k$-faces is never less than the number of $k$-faces of $C_{[r]}$.

For each $\emptyset \subset R \subseteq [r]$, we denote by $\mathcal{F}_R$ the set of faces of $\partial C_R$ having at least one vertex from each $V_i, i \in R$, and we call it the set of mixed faces of $C_R$. We trivially have that $\mathcal{F}_{\{i\}} \equiv \partial P_i$. We define the dimension of $\mathcal{F}_R$ to be the maximum dimension of the faces in $\mathcal{F}_R$, i.e., $\dim(\mathcal{F}_R) = \max_{F \in \mathcal{F}_R} \dim(F) = d + |R| - 2$. Under the as simplicial as possible assumption above, the faces in $\mathcal{F}_R$ are simplices. We denote by $\mathcal{K}_R$ the closure, under subface inclusion, of $\mathcal{F}_R$. By construction, $\mathcal{K}_R$ contains: (1) all faces in $\mathcal{F}_R$, (2) all faces that are subfaces of faces in $\mathcal{F}_R$, and (3) the empty set. It is easy to see that $\mathcal{K}_R$ does not contain any of the trivial faces $[C_S], \emptyset \subset S \subseteq R$.

$^1$ We denote by $[C_R]$ the polytope $C_R$ as a trivial face itself (without its non-trivial faces).
and thus, $\mathcal{K}_R$ is a pure simplicial $(d + |R| - 2)$-complex. It is also easy to verify that

$$f_k(\mathcal{K}_R) = \sum_{\emptyset \subseteq S \subseteq R} f_k(\mathcal{F}_S), \quad -1 \leq k \leq d + |R| - 2. \tag{2.8}$$

where in order for the above equation to hold for $k = -1$, we set $f_{-1}(\mathcal{F}_S) = (-1)^{|S|} - 1$ for all $\emptyset \subseteq S \subseteq R$. In what follows we use the convention that $f_k(\mathcal{F}_R) = 0$, for any $k < -1$ or $k > d + |R| - 2$.

A general form of the Inclusion–Exclusion Principle states that if $f$ and $g$ are two functions defined over the subsets of a finite set $A$, such that $f(A) = \sum_{\emptyset \subseteq B \subseteq A} g(B)$, then $g(A) = \sum_{\emptyset \subseteq B \subseteq A} (-1)^{|A|-|B|} f(B)$ [9, Thm. 12.1]. Applying this principle to (2.8), we deduce that

$$f_k(\mathcal{F}_R) = \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S), \quad -1 \leq k \leq d + |R| - 2. \tag{2.9}$$

In the majority of our proofs that involve evaluation of $f$- and $h$-vectors, we use generating functions as they significantly simplify calculations. The starting point is to evaluate $\mathcal{L}(\mathcal{K}_R; t)$ (resp., $\mathcal{L}(\mathcal{F}_R; t)$) in terms of the generating functions $\mathcal{L}(\mathcal{F}_S; t)$ (resp., $\mathcal{L}(\mathcal{K}_S; t)$), $\emptyset \subseteq S \subseteq R$, for each fixed choice of $\emptyset \subseteq R \subseteq [r]$. Then, using (2.4) we derive the analogous relations between their $h$-vectors.

Recalling that $\dim(\mathcal{K}_R) = d + |R| - 2$ and $\dim(\mathcal{F}_S) = d + |S| - 2$ we have

$$\mathcal{L}(\mathcal{K}_R; t) = \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{K}_R) t^{d+|R|-1-k} \overset{(2.8)}{=} \sum_{k=0}^{d+|R|-1} \sum_{\emptyset \subseteq S \subseteq R} f_{k-1}(\mathcal{F}_S) t^{d+|R|-1-k} \sum_{\emptyset \subseteq S \subseteq R} t^{|R|-|S|} \mathcal{L}(\mathcal{F}_S; t). \tag{2.10}$$

Rewriting the above relation as $t^{-|R|} \mathcal{L}(\mathcal{K}_R; t) = \sum_{\emptyset \subseteq S \subseteq R} t^{-|S|} \mathcal{L}(\mathcal{F}_S; t)$ and using Möbius inversion, we get

$$\mathcal{L}(\mathcal{F}_R; t) = \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} t^{|R|-|S|} \mathcal{L}(\mathcal{K}_S; t). \tag{2.11}$$

Setting $t := t - 1$ in (2.10) we have

$$\mathcal{L}(\mathcal{K}_R; t - 1) = \mathcal{L}(\mathcal{K}_R; t - 1) = \sum_{\emptyset \subseteq S \subseteq R} (t - 1)^{|R|-|S|} \mathcal{L}(\mathcal{F}_S; t - 1) \overset{(2.10)}{=} \sum_{\emptyset \subseteq S \subseteq R} (t - 1)^{|R|-|S|} \mathcal{L}(\mathcal{K}_S; t). \tag{2.12}$$
And similarly, from (2.11) we obtain
\[
h(F_R; t) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R| - |S|} g^{(|R| - |S|)}(K_S; t). \tag{2.13}
\]
Comparing coefficients in the above generating functions, we deduce that
\[
h_k(K_R) = \sum_{\emptyset \subset S \subseteq R} g_k^{(|R| - |S|)}(F_S), \quad \text{for all } 0 \leq k \leq d + |R| - 1, \text{ and}
\]
\[
h_k(F_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R| - |S|} g_k^{(|R| - |S|)}(K_S), \quad \text{for all } 0 \leq k \leq d + |R| - 1. \tag{2.15}
\]

3 The Construction of the Auxiliary Simplicial Polytope \(Q_{[r]}\)

The proper faces of the Cayley polytope \(C_{[r]}\) of \(P_1, \ldots, P_r\) are the faces in each \(F_R, \emptyset \subset R \subseteq [r]\), as well as all trivial faces \(\{C_R\}\) with \(\emptyset \subset R \subset [r]\). Since the latter are not necessarily simplices, the Cayley polytope \(C_{[r]}\) may not be simplicial. In order to exploit the combinatorial structure of \(C_{[r]}\), we add auxiliary points on \(C_{[r]}\) so that the resulting polytope, denoted by \(Q_{[r]}\), is simplicial.

The main tool for describing our construction is stellar subdivisions. Let \(P \subset \mathbb{R}^d\) be a \(d\)-polytope, and consider a point \(y_F\) in the relative interior of a face \(F \in \partial P\). The stellar subdivision \(st(y_F, \partial P)\) of \(\partial P\) over \(F\), replaces \(F\) by the set of faces \(\{y_F, F'\}\) where \(F'\) is a non-trivial face of \(F\). It is a well-known fact that stellar subdivisions preserve polytopality (cf. [3, pp. 70–73]), in the sense that the newly constructed complex is combinatorially equivalent to a polytope each facet of which lies on a distinct supporting hyperplane.

Our goal is to triangulate each (possibly) non-simplicial face of \(\partial C_{[r]}\) i.e., each lower dimensional Cayley polytope \(C_R\) with \(\emptyset \subset R \subset [r]\). We obtain this by considering a point \(y_R\) in the relative interior and performing a stellar subdivision over each such face (see Fig. 1 for the case of three polytopes). We thus arrive at the auxiliary simplicial polytope \(Q_{[r]}\). Since \(C_R\) consists of the faces in \(C_R \setminus \{i\}, i \in R\), and those in \(F_R\), the addition of the auxiliary point \(y_R\) and the iteration of the above procedure, produces faces corresponding to combinations of faces in \(F_R\) along with auxiliary vertices \(y_{S_1}, \ldots, y_{S_\ell}\) where \(S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\). One can think of this procedure as a variant of the barycentric subdivision on \(\hat{C}_{[r]}\), where the barycentric subdivision step is ignored for the faces which are already simplices (cf. [18, Sects. 9.1–9.4] for a nice overview on barycentric subdivisions).

We summarize all the above in the following lemma, where we formulate the statement for any arbitrary \(Q_R, \emptyset \subset R \subseteq [r]\) (instead of \(Q_{[r]}\)). Since the majority of our subsequent arguments are inductive, we work with \(Q_R\) until the main theorem (Theorem 6.8). Unless otherwise stated, all set unions are disjoint.
Fig. 1 The “simplicialization” of $C_3$, a The Cayley polytope $C_3$ of the three line segments $P_1$, $P_2$ and $P_3$ is the prism depicted above. The triple colored line segment is the intersection $W \cap C_3$ and corresponds to the Minkowski sum $P_1 + P_2 + P_3$ (scaled by a factor of 1/3). For $\{i, j\} \in \binom{3}{2}$, the Cayley polytope $C_{\{i, j\}}$ of $P_i$ and $P_j$ is the rectangle containing $P_i$ and $P_j$. The mixed faces in $F_3$ are the two triangular facets, while the mixed faces in $F_{\{i, j\}}$ are the double-colored line segments having both the color of $P_i$ and $P_j$, b $C_3$ after the addition of $y_i$, $i = 1, 2, 3$, c The simplicial polytope $Q_3$, i.e., the polytope we obtain after the addition of all auxiliary points.

**Lemma 3.1** For $\emptyset \subset R \subseteq [r]$, the non-trivial faces of the simplicial polytope $Q_R$ are:

$$\partial Q_R = \bigcup_{\emptyset \subset S \subseteq R} \mathcal{F}_S \bigcup_{\emptyset \subset S \subseteq R} \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_S\}$$ (3.1)

where $\{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_S\}$ is the set of faces formed by the vertices $y_{S_1}, \ldots, y_{S_\ell}$ and a face in $\mathcal{F}_S$, while $y_{\emptyset} = \mathcal{F}_\emptyset := \emptyset$.

The next lemma expresses the faces of $\partial Q_R$ in terms of the sets $\mathcal{K}_S$, $\emptyset \subset S \subseteq R$, and the auxiliary vertices added.

**Lemma 3.2** For $\emptyset \subset R \subseteq [r]$, the non-trivial faces of the simplicial polytope $Q_R$ are:

$$\partial Q_R = \mathcal{K}_R \bigcup_{\emptyset \subset S \subseteq R} \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{K}_S\}.$$ (3.2)

**Proof** Recall that the faces of $\mathcal{K}_R$ are all faces in $\bigcup_{\emptyset \subset S \subseteq R} \mathcal{F}_S$. We can therefore write the right-hand side of (3.2) as:

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\[
\bigcup_{\emptyset \subset S \subset R} \bigcup_{S = S_1 \subset \cdots \subset S_\ell \subset R} \{y_{S_1}, \ldots, y_{S_\ell}, \bigcup_{\emptyset \subset S' \subset R} \mathcal{F}_{S'}\}
\]

\[
= \bigcup_{\emptyset \subset S \subset R} \bigcup_{\emptyset \subset S' \subset R} \{y_{S_1}, \ldots, y_{S_\ell}, \mathcal{F}_{S'}\},
\]

which is precisely the quantity in (3.1) and thus equal to the set of faces of \( \partial Q_R \). \ \Box

The next lemma shows how the iterated stellar subdivisions performed above are captured in the enumerative structure of \( Q_R \). We remark that eq. (3.4) below, resembles that in [2, Lem. 1]. This, somehow reflects the fact that, geometrically, we have performed barycentric subdivisions over all faces contained in some \( K_S, \emptyset \subset S \subset R \).

**Lemma 3.3** For any \( \emptyset \subset R \subset [r] \) and \(-1 \leq k \leq d + |R| - 2\), we have

\[
f_k(\partial Q_R) = f_k(\mathcal{F}_R) + \sum_{\emptyset \subset S \subset R} \left( \sum_{i=0}^{\min{|R| - |S|, |R| - |S| = k - 1}} i! \binom{k}{i+1} f_{k-i}(\mathcal{F}_S) \right)
\]

\[
= f_k(\mathcal{K}_R) + \sum_{\emptyset \subset S \subset R} \left( \sum_{i=0}^{\min{|R| - |S| - 1, |R| - |S| = k - 1}} (i+1)! \binom{k}{i} f_{k-i-1}(\mathcal{K}_S) \right)
\]

where \( S_m^k \) are the Stirling numbers of the second kind [20]

\[
S_m^k = \frac{1}{k!} \sum_{i=0}^{\min{k, m}} (-1)^{k-i} \binom{k}{i} t^m, \quad \text{when } m \geq k \geq 0,
\]

and \( S_m^k = 0 \) otherwise.

**Proof** To prove (3.4), we count the \((k + 1)\)-element subsets of the set \( \partial Q_R \) in relation (3.2) of Lemma 3.2. This gives

\[
f_k(\partial Q_R) = f_k(\mathcal{K}_R) + \sum_{\emptyset \subset S \subset R} \sum_{i=1}^{\min{|R| - |S|, |R| - |S| = k - 1}} f_{k-i}(\mathcal{K}_S)
\]

\[
= f_k(\mathcal{K}_R) + \sum_{\emptyset \subset S \subset R} \sum_{i=1}^{\min{|R| - |S| - 1, |R| - |S| = k - 1}} |\{S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\}| f_{k-i}(\mathcal{K}_S).
\]

(3.5)

The cardinality of the set \( \{S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\} \) is equal to that of \( \{\emptyset \subset S_2 \setminus S \subset \cdots \subset S_\ell \setminus S \subset R \setminus S\} \) which, arguing as in the proof of [2, Lem. 2.1], is equal to \( i! S_m^i \). We therefore have
To prove (3.3), we count the \((k + 1)\)-element subsets of the set \(\partial QR\) in relation (3.1) of Lemma 3.1. This gives:

\[
f_k(\partial QR) = f_k(K_R) + \sum_{\emptyset \subset S \subset R} |R| - |S| \sum_{i=1}^{i! S_i^{i+1}} f_{k-i}(K_S)
\]

\[
= f_k(K_R) + \sum_{\emptyset \subset S \subset R} |R| - |S| - 1 \sum_{i=0}^{i!(i+1)! S_i^{i+1}} f_{k-i}(K_S).
\]

where the value \(i = k + 1\) in (3.6) combined with the fact that \(f_{-1}(F_S) = (-1)^{|S|-1}\), counts precisely the elements in \(\bigcup_{\emptyset \subset S_1 \subset S_2 \subset \cdots \subset S_l \subset R} \{y_{S_1}, y_{S_2}, \ldots, y_{S_l}\}\) in relation (3.1) via inclusion exclusion. The cardinality of the set \(\{S \subseteq S_1 \subset S_2 \subset \cdots \subset S_l \subset R\}\) is equal to \(|\{S = S_1 \subset S_2 \subset \cdots \subset S_l \subset R\}| + |\{S \subset S_1 \subset S_2 \subset \cdots \subset S_l \subset R\}|\) which, in turn, equals

\[
i! S_m^i + (i + 1)! S_m^{i+1} = i! (S_m^i + (i + 1) S_m^{i+1}) = i! S_m^{i+1},
\]

where \(m = |R| - |S|\), and, in the last equation, we used the recurrence \(S_m^{i+1} = S_m^i + (i + 1) S_m^{i+1}\) of the Stirling numbers. We therefore have

\[
f_k(\partial QR) = f_k(F_R) + \sum_{\emptyset \subset S \subset R} |R| - |S| \sum_{i=1}^{i! S_i^{i+1}} f_{k-i}(F_S)
\]

\[
= f_k(F_R) + \sum_{\emptyset \subset S \subset R} |R| - |S| - 1 \sum_{i=0}^{i!(i+1)! S_i^{i+1}} f_{k-i}(F_S).
\]

Restating relations (3.3) and (3.4) in terms of generating functions, we arrive at Lemma 3.4. These relations will be used to transform (3.3) and (3.4) in their \(h\)-vector equivalents.
Lemma 3.4 For all $\emptyset \subset R \subseteq [r]$ we have

$$
\mathcal{F}(\partial Q_R; t) = \mathcal{F}(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{\lfloor |R|-|S| \rfloor} i! S_{|R|-|S|+i}^{i+1} t^{|R|-|S|-i} \mathcal{F}(F_S; t), \tag{3.7}
$$

$$
\mathcal{F}(\partial Q_R; t) = \mathcal{F}(K_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} (i+1)! S_{|R|-|S|+1}^{i+1} t^{|R|-|S|-i} \mathcal{F}(K_S; t). \tag{3.8}
$$

Proof Using relation (3.3) and recalling that $\dim(\partial Q_R) = d + |R| - 2$, we have

$$
\mathcal{F}(\partial Q_R; t)
= \sum_{k=0}^{d+|R|-1} f_{k-1}(\partial Q_R) t^{d+|R|-1-k}
\leq \sum_{k=0}^{d+|R|-1} f_{k-1}(F_R) t^{d+|R|-1-k}
+ \sum_{k=0}^{d+|R|-1} \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|} i! S_{|R|-|S|+1}^{i+1} f_{k-1-i}(F_S) t^{d+|R|-1-k}
\leq \mathcal{F}(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|} i! S_{|R|-|S|+1}^{i+1} t^{|R|-|S|-i} \sum_{k=0}^{d+|R|-1} f_{k-1-i}(F_S) t^{d+|S|-1-k+i}
\leq \mathcal{F}(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} i! S_{|R|-|S|+1}^{i+1} t^{|R|-|S|-i} \sum_{k=i}^{d+|S|-1} f_{k-1-i}(F_S) t^{d+|S|-1-k+i}
\leq \mathcal{F}(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|} i! S_{|R|-|S|+1}^{i+1} t^{|R|-|S|-i} \sum_{k=0}^{d+|S|-1} f_{k-1}(F_S) t^{d+|S|-1-k}
\leq \mathcal{F}(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|} i! S_{|R|-|S|+1}^{i+1} t^{|R|-|S|-i} \mathcal{F}(F_S; t),
$$

where, to go from the third to the fourth line, we used the fact that $f_{k-i-1}(F_S) = 0$ for $k < i$ or $k > d + |S| - 1 + i$.

Analogously, converting (3.4) into its generating function equivalent, we get:

$$
\mathcal{F}(\partial Q_R; t)
= \sum_{k=0}^{d+|R|-1} f_{k-1}(K_R) t^{d+|R|-1-k}
\leq \sum_{k=0}^{d+|R|-1} f_{k-1}(K_R) t^{d+|R|-1-k}
+ \sum_{k=0}^{d+|R|-1} \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{|R|-|S|-1} (i+1)! S_{|R|-|S|}^{i+1} f_{k-1-i}(K_S) t^{d+|R|-1-k}
,$$
\[
\begin{align*}
&= f(K_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} (i+1)! S^{i+1}_{\left| R \right| - \left| S \right|} t^{\left| R \right| - \left| S \right| - i} \\
&\quad \times \sum_{k=0}^{d+\left| R \right| - 1} f_{k-1}(K_S) t^{d+\left| S \right| - 1 - k + i} \\
&= f(K_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} (i+1)! S^{i+1}_{\left| R \right| - \left| S \right|} t^{\left| R \right| - \left| S \right| - i} \sum_{k=0}^{d+\left| R \right| - 1} f_{k-1}(K_S) t^{d+\left| S \right| - 1 - k} \\
&= f(K_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} (i+1)! S^{i+1}_{\left| R \right| - \left| S \right|} t^{\left| R \right| - \left| S \right| - i} f(K_S; t),
\end{align*}
\]

where, in order to go from the third to the fourth line, we changed variables (in the last sum) and we used the fact that \( f_{k-1}(K_S) = 0 \) for \( k > d + \left| S \right| - 1 \).

The \( h \)-vector relations stemming from the \( f \)-vector relations above are the subject of the following lemma.

**Lemma 3.5** For all \( \emptyset \subset R \subseteq \left[ r \right] \) we have

\[
\begin{align*}
\h(\partial Q_R; t) &= \h(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{\left| R \right| - \left| S \right| - 1} E^j_{\left| R \right| - \left| S \right|} t^{j+1} \h(F_S; t), \\
\h(\partial Q_R; t) &= \h(K_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{\left| R \right| - \left| S \right| - 1} E^j_{\left| R \right| - \left| S \right|} t^{j} \h(K_S; t),
\end{align*}
\]

where \( E^k_m \) are the Eulerian numbers \([8, 22]\):

\[
E^k_m = \sum_{i=0}^{k} (-1)^i \binom{m+1}{i} (k + 1 - i)^m, \quad \text{when } m \geq k + 1 > 0,
\]

and \( E^m_m = 0 \) in any other case.

**Proof** Using (2.4) and (3.7) we get

\[
\begin{align*}
\h(\partial Q_R; t) &= f(\partial Q_R; t - 1) \\
&= f(F_R; t - 1) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{\left| R \right| - \left| S \right|} i! S^{i+1}_{\left| R \right| - \left| S \right| + 1} (t - 1)^{\left| R \right| - \left| S \right| - i} f(F_S; t - 1) \\
&\quad \underbrace{\quad \text{cf. (8.2)} \quad}_{\text{(3.5)}} \\
&= \h(F_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{\left| R \right| - \left| S \right| - 1} E^j_{\left| R \right| - \left| S \right|} t^{j+1} \h(F_S; t).
\end{align*}
\]

\( \Box \)

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Analogously, using (2.4), (3.8) we deduce that:

\[
\begin{align*}
\mathcal{h}(\partial Q_R; t) &= f(\partial Q_R; t - 1) \\
&= \mathcal{h}(\mathcal{K}_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{i=0}^{(|R| - |S|) - 1} (i + 1)! S^{i+1}_{|R| - |S|} (t - 1)^{i} \mathcal{h}(\mathcal{K}_S; t) \\
&= \mathcal{h}(\mathcal{K}_R; t) + \sum_{\emptyset \subset S \subset R} E^i_{|R| - |S|} t^i \mathcal{h}(\mathcal{K}_S; t).
\end{align*}
\]

\qed

4 The Dehn–Sommerville equations

A very important structural property of the Cayley polytope \( C_R \) is, what we call, the Dehn–Sommerville equations. For a single polytope these equations reduce to the well-known Dehn–Sommerville equations, whereas for two or more summands they relate the \( h \)-vectors of the sets \( \mathcal{F}_R \) and \( \mathcal{K}_R \). The Dehn–Sommerville equations for \( C_R \) are one of the major key ingredients for establishing our upper bounds, as they permit us to reason for the maximality of the elements of \( \mathcal{h}(\mathcal{F}_R) \) and \( \mathcal{h}(\mathcal{K}_R) \) by considering only the lower halves of these vectors.

**Theorem 4.1** (Dehn–Sommerville equations) Let \( C_R \) be the Cayley polytope of the \( d \)-polytopes \( P_i, i \in R \). Then, the following relation holds:

\[
t^{d + |R| - 1} \mathcal{h}(\mathcal{F}_R; \frac{1}{t}) = \mathcal{h}(\mathcal{K}_R; t)
\]

or, equivalently,

\[
h_{d + |R| - 1 - k}(\mathcal{F}_R) = h_k(\mathcal{K}_R), \quad 0 \leq k \leq d + |R| - 1.
\]

**Proof** We prove our claim by induction on the size of \( R \), the case \( |R| = 1 \) being the Dehn–Sommerville equations for a \( d \)-polytope. We next assume that our claim holds for all \( \emptyset \subset S \subset R \) and prove it for \( R \). The ordinary Dehn–Sommerville relations, written in generating function form, for the (simplicial) \( (d + |R| - 1) \)-polytope \( Q_R \) imply that

\[
\mathcal{h}(\partial Q_R; t) = t^{d + |R| - 1} \mathcal{h}(\partial Q_R; \frac{1}{t}).
\]

In view of relation (3.9) of Lemma 3.5, the right-hand side of (4.3) becomes

\[
t^{d + |R| - 1} \mathcal{h}(\mathcal{F}_R; \frac{1}{t}) + t^{d + |R| - 1} \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{(|R| - |S|) - 1} E^j_{|R| - |S|} t^{-j-1} \mathcal{h}(\mathcal{F}_S; \frac{1}{t}).
\]
Using relation (3.10), along with the induction hypothesis, the left-hand side of (4.3) becomes
\[ h(K_R, t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{\left| R \right| - |S| - 1} E^j_{|R| - |S|} t^j h(K_S, t) \]
(4.5)

\[ = h(K_R, t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{\left| R \right| - |S| - 1} E^j_{|R| - |S|} t^{|R| - |S| - j - 1} h(K_S, t) \]
(4.6)

\[ = h(K_R, t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{\left| R \right| - |S| - 1} E^j_{|R| - |S|} t^{|R| - |S| - j - 1} t^d + |S| - 1 h(F_S; \frac{1}{t}) \]
(4.7)

where to go from (4.5) to (4.6) we changed variables and used the symmetry of the Eulerian numbers.

Now, substituting (4.4) and (4.7) in (4.3), we deduce that
\[ t^d + |R| - 1 h(F_R; \frac{1}{t}) = h(K_R; t), \]
which is, coefficient-wise, equivalent to (4.2).

\[ \square \]

5 The Recurrence Relation for \( h(F_R) \)

The subject of this section is the generalization, for the \( h \)-vector of \( F_R, \emptyset \subset R \subseteq [r] \), of the recurrence relation:
\[ (k + 1) h_{k+1}(\partial P) + (d - k) h_k(\partial P) \leq n h_k(\partial P), \quad 0 \leq k \leq d - 1, \]
(5.1)

that holds true for any simplicial \( d \)-polytope \( P \subset \mathbb{R}^d \). This is the content of the next theorem. Its proof is postponed until Sect. 5.6. In the next five subsections we build upon the necessary intermediate results for proving this theorem.

**Theorem 5.1 (Recurrence inequality)** For any \( \emptyset \subset R \subseteq [r] \) we have
\[ h_{k+1}(F_R) \leq \frac{n_R - d - |R| + 1 + k}{k + 1} h_k(F_R) + \sum_{i \in R} \frac{n_i}{k + 1} g_k(F_{R \setminus \{i\}}), \]
(5.2)

where: (1) \( n_R = \sum_{i \in R} n_i \) and, (2) \( g_k(F_{\emptyset}) = g_k(\emptyset) = 0, \) for all \( k \).

5.1 Relating the \( h \)-Vector of \( Q_R/v \) with the \( h \)-Vectors of \( F_R/v \) and \( K_R/v \)

For any \( \emptyset \subset R \subseteq [r] \), let \( V_R := \bigcup_{i \in R} V_i \). We define the link of a vertex \( v \in V_R \) in \( F_R \) as the intersection of the link \( K_R/v \) with \( F_R \). The following lemma relates the \( h \)-vector of \( \partial Q_R/v \) with the \( h \)-vectors of \( F_R/v \) and \( K_R/v \).
Lemma 5.2 For any \( v \in V_j, j \in R \), we have
\[
\h(\partial Q_R/v; t) = \h(F_R/v; t) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} E^i_{\left| R \right| - \left| S \right| - 1} t^i \h(F_S/v; t), \tag{5.3}
\]
and
\[
\h(\partial K_R/v; t) = \h(K_R/v; t) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} E^i_{\left| R \right| - \left| S \right| - 1} t^i \h(K_S/v; t). \tag{5.4}
\]

Proof Let us fix some \( v \in V_j, j \in R \). In view of relation (3.1) in Lemma 3.1 we can write:
\[
\partial Q_R/v = \bigcup_{\emptyset \subseteq S \subseteq R} \bigcup_{\sum_{i=0}^{\left| R \right| - \left| S \right| - 1} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} E^i_{\left| R \right| - \left| S \right| - 1} t^i \h(K_S/v; t). \tag{5.5}
\]
where it is understood that both \( F_S/v \) and \( \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, F_S/v\} \) are empty if \( v \notin V_S \). Taking this into account, we simplify (5.5) as follows:
\[
\partial Q_R/v = \bigcup_{\{j\} \subseteq S \subseteq R} \bigcup_{\sum_{i=0}^{\left| R \right| - \left| S \right| - 1} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} E^i_{\left| R \right| - \left| S \right| - 1} t^i \h(K_S/v; t). \tag{5.6}
\]
Since each auxiliary point in \( \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, F_S/v\} \) increases the dimension of a face in \( F_S/v \) by one, (5.6) implies:
\[
f_k(\partial Q_R/v) = \sum_{\{j\} \subseteq S \subseteq R} f_k(F_S/v) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=1}^{\left| R \right| - \left| S \right|} \sum_{S \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_\ell \subseteq R} f_{k-i}(F_S/v).
\]
The equation above resembles (3.6), from which we immediately deduce that
\[
f_k(\partial Q_R/v) = f_k(F_R/v) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} i! E^i_{\left| R \right| - \left| S \right| - 1} t^i f_{k-i}(F_S/v).
\]
Recalling that \( \dim(F_S/v) = d + \left| S \right| - 3 \) and converting the above relation into its generating function equivalent, we get
\[
\ell(\partial Q_R/v; t) = \ell(F_R/v; t) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\left| R \right| - \left| S \right| - 1} i! E^i_{\left| R \right| - \left| S \right| - 1} t^i \ell(F_S/v; t), \tag{5.7}
\]
which converts to
\[
\h(\partial Q_R/v; t) = \h(F_R/v; t) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\frac{|R|-|S|-1}{2}} E^j_{[R]-[S]} t^{i+1} \h(F_S/v; t),
\]

Let us now turn our attention to relation (5.4). In view of (3.2) of Lemma 3.2 we have
\[
\partial Q_R/v = K_{\partial Q_R/v} \cup \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, K_S/v\},
\]
which in turn gives
\[
f_k(\partial Q_R/v) = f_k(K_{\partial Q_R/v}) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=1}^{\frac{|R|-|S|-1}{2}} f_{k-i}(K_S/v) = f_k(K_{\partial Q_R/v}) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=1}^{\frac{|R|-|S|-1}{2}} i! S^i_{[R]-[S]} f_{k-i}(K_S/v) = f_k(K_{\partial Q_R/v}) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\frac{|R|-|S|-1}{2}} (i+1)! S^i_{[R]-[S]} f_{k-i-1}(K_S/v).
\]

Recalling that \(\dim(K_S/v) = d + |S| - 3\) and converting the above relation into its generating function form, we get
\[
f_k(\partial Q_R/v; t) = f_k(K_{\partial Q_R/v}; t) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\frac{|R|-|S|-1}{2}} (i+1)! S^i_{[R]-[S]} t^{i} f_{k-i-1}(K_S/v; t),
\]
which further implies that
\[
h(\partial Q_R/v; t) = h(K_{\partial Q_R/v}; t) + \sum_{\{j\} \subseteq S \subseteq R} \sum_{i=0}^{\frac{|R|-|S|-1}{2}} E^j_{[R]-[S]} t^{i} h(K_S/v; t).
\]

\(\square\)

5.2 The Link of \(y_S\) in \(\partial Q_R\)

Our next goal is to find an expression analogous to those of Lemma 5.2, but now involving links of type \(\partial Q_R/y_S\), where \(\emptyset \subset S \subseteq R\). To do this, we first need to
express \( f_k(\partial Q_R/\partial S) \) in terms of sums of \( f_i(\mathcal{F}_X) \) with \( i \leq k \) and \( X \subseteq S \). This is the content of the next Lemma. In order to state it we need to introduce a new set. Let \( X \subseteq T \subset R \) and \( \ell \) be a positive integer. We define the set
\[
D(R, T, X, \ell) := \{ (S_1, \ldots, S_\ell) : X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \leq i \leq \ell \},
\]
and denote by \( D(R, T, X, \ell) \) its cardinality.

**Lemma 5.3** For every \( \emptyset \subset S \subset R \) we have:
\[
\ell(\partial Q_R/\partial S; t) = \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} D(R, S, X, \ell) t^{|R| - |X| - \ell} \ell(\mathcal{F}_X; t).
\]

**Proof** First of all, notice that, in view of relation (3.1), if we denote by \( y_S * \partial Q_R \) the set of all faces in \( \partial Q_R \) containing \( y_S \), we have
\[
y_S * \partial Q_R = \bigcup_{\emptyset \subset X \subseteq S \quad X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \quad S_i = S \text{ for some } 1 \leq i \leq \ell} \{ y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_X \}.
\]
Then clearly,
\[
f_k(\partial Q_R/\partial S) = f_{k+1}(y_S * \partial Q_R) = \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} f_{k-\ell+1}(\mathcal{F}_X)
\]
\[
= \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} D(R, S, X, \ell) f_{k-\ell+1}(\mathcal{F}_X).
\]
Using the fact that \( \dim(\partial Q_R/\partial S) = d + |R| - 3 \), and rewriting in terms of generating functions, the above becomes:
\[
\ell(\partial Q_R/\partial S; t)
\]
\[
= \sum_{k=0}^{d+|R|-2} \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} D(R, S, X, \ell) f_{k-\ell+1}(\mathcal{F}_X) t^{d + |R| - 2 - k}
\]
\[
= \sum_{k=0}^{d+|R|-2} \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} t^{|R| - |X| - \ell} D(R, S, X, \ell) f_{k-\ell+1}(\mathcal{F}_X) t^{d + |X| - 1 - (k-\ell+1)}
\]
\[
= \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} D(R, S, X, \ell) t^{|R| - |X| - \ell} \sum_{k=0}^{d+|R|-2} f_{k-\ell+1}(\mathcal{F}_X) t^{d + |X| - 1 - (k-\ell+1)}
\]
\[
= \sum_{\emptyset \subset X \subseteq S} \sum_{\ell = 1}^{|R| - |X|} D(R, S, X, \ell) t^{|R| - |X| - \ell} \sum_{k=0}^{d+|R|-2} f_{k-\ell+1}(\mathcal{F}_X) t^{d + |X| - 1 - (k-\ell+1)}
\]
where to go from (5.11) to (5.12) we used the fact that the non-vanishing range of 
\[ f_{k-\ell+1}(F_X) \] is \[ \ell - 2 \leq k \leq d + |X| - 2 + \ell \], which, combined with the inequality 
\[ d + |X| - 2 + \ell \leq d + |X| - 2 + |R| - |X| = d + |R| - 2 \], justifies the range of \( k \) in
the third sum of (5.12). \( \Box \)

The following lemma expresses the sum of the \( h \)-vectors of the links \( \partial Q_R/yS \) in terms of the \( h \)-vectors of the sets \( F_X, X \subseteq S \).

**Lemma 5.4** For every \( \emptyset \subset R \subseteq [r] \) we have

\[
\sum_{\emptyset \subset S \subset R} h(\partial Q_R/yS; t) = \sum_{\emptyset \subset X \subset R} \sum_{j=1}^{d+|X|} \left( E_j^{r-|X|+1} - E_j^{r-|X|} \right) t^{d+|X|-j} h(F_X; t). 
\] (5.13)

**Proof** Converting relation (5.10) of the above lemma to its \( h \)-vector equivalent we get

\[
h(\partial Q_R/yS; t) = \ell(\partial Q_R/yS; t-1) \\
= \sum_{\emptyset \subset X \subseteq S} \sum_{\ell=1}^{d+|X|} D(R, S, X, \ell)(t-1)^{d+|X|-\ell} h(F_X; t). \] (5.14)

By means of relation (5.14), the sum \( \sum_{\emptyset \subset S \subset R} h(\partial Q_R/yS; t) \) is equal to

\[
\sum_{\emptyset \subset X \subseteq S \subset R} \sum_{\ell=1}^{d+|X|} D(R, S, X, \ell)(t-1)^{d+|X|-\ell} h(F_X; t) \\
= \sum_{\ell=1}^{d+|X|} \sum_{\emptyset \subset X \subset R} \sum_{X \subseteq S \subset R} D(R, S, X, \ell)(t-1)^{d+|X|-\ell} h(F_X; t). \] (5.15)
\[ |R| - |X| \sum_{\ell=0}^{\ell} \ell! S_{|R| - |X| + 1}^{\ell+1} (t - 1)^{|R| - |X| - \ell} \mathcal{h}(F_{X}; t) \] (5.16)

where, to go from (5.15) to (5.16), we used Lemma 9.1.

Observe now that

\[ |R| - |X| \sum_{\ell=0}^{\ell} \ell! S_{|R| - |X| + 1}^{\ell+1} (t - 1)^{|R| - |X| - \ell} \]

\[ = \sum_{\ell=0}^{\ell} (\ell + 1)! S_{|R| - |X| + 1}^{\ell+1} (t - 1)^{|R| - |X| - \ell} \]

\[ - \sum_{\ell=0}^{\ell} \ell! S_{|R| - |X| + 1}^{\ell+1} (t - 1)^{|R| - |X| - \ell} \] (5.17)

\[ = \sum_{j=0}^{j} E_{j}^{j = |R| - |X| + 1} (t - 1)^{|R| - |X| - j} - \sum_{j=0}^{j} E_{j}^{j = |R| - |X|} (t - 1)^{|R| - |X| - j} \] (5.18)

\[ = \sum_{j=1}^{j} (E_{j}^{j = |R| - |X| + 1} - E_{j}^{j = |R| - |X|}) (t - 1)^{|R| - |X| - j}, \] (5.19)

where, to go from (5.17) to (5.18), we used Lemma 8.1, while to obtain (5.19) we exploited the fact that \( E_{m}^{0} = 1 \) for all \( m > 1 \).

\[ \square \]

5.3 Links and Non-links

McMullen [17], in his original proof of the Upper Bound Theorem for polytopes, proved that for any simplicial \( d \)-polytope \( P \), the following relation holds:

\[(k + 1)h_{k+1}(\partial P) + (d - k)h_{k}(\partial P) = \sum_{v \in \text{vert}(\partial P)} h_{k}(\partial P / v), \quad 0 \leq k \leq d - 1.\] (5.20)

From [13, Lem. 3] we have that, in terms of generating functions, (5.20) becomes

\[ d \mathcal{h}(\partial P; t) + (1 - t)\mathcal{h}'(\partial P; t) = \sum_{v \in \text{vert}(\partial P)} \mathcal{h}(\partial P / v; t). \] (5.21)

The following theorem generalizes (5.21) in the context of Cayley polytopes.
Theorem 5.5 For any $\emptyset \subset R \subseteq [r]$, 

$$(d + |\emptyset| - 1)\mathbb{h}(\mathcal{F}_R; t) + (1 - t)\mathbb{h}'(\partial \mathcal{F}_R; t) = \sum_{v \in \mathcal{V}_R} \mathbb{h}(\mathcal{F}_R/v; t),$$  \hspace{1cm} (5.22) 

where $\mathcal{V}_R = \bigcup_{i \in R} V_i$.

Proof We proceed by induction on the size of $R$. The case $|\emptyset| = 1$ is precisely equation (5.21). Assume now that (5.22) holds for all $\emptyset \subset S \subset R$. By applying relation (5.21) to the simplicial polytope $Q_R$ we have

$$(d + |\emptyset| - 1)\mathbb{h}(\partial Q_R; t) + (1 - t)\mathbb{h}'(\partial Q_R; t) = \sum_{v \in \text{vert}(\partial Q_R)} \mathbb{h}(\partial Q_R/v; t).$$  \hspace{1cm} (5.23) 

Recall from Lemma 3.5 that

$$\mathbb{h}(\partial Q_R; t) = \mathbb{h}(\mathcal{F}_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{|\emptyset| - |S| - 1} E_j^{|\emptyset| - |S| - j} \mathbb{h}(\mathcal{F}_S; t).$$  \hspace{1cm} (5.24) 

Multiplying both sides of (5.24) by $d + |\emptyset| - 1$ we get

$$(d + |\emptyset| - 1)\mathbb{h}(\partial Q_R; t) = (d + |\emptyset| - 1)\mathbb{h}(\mathcal{F}_R; t) + (d + |\emptyset| - 1)$$

$$\times \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{|\emptyset| - |S| - 1} E_j^{|\emptyset| - |S| - j} \mathbb{h}(\mathcal{F}_S; t).$$

Differentiating both sides of (5.24) and multiplying by $(1 - t)$ we get

$$(1 - t)\mathbb{h}'(\partial Q_R; t) = (1 - t)\mathbb{h}'(\mathcal{F}_R; t)$$

$$+ (1 - t) \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{|\emptyset| - |S| - 1} (|\emptyset| - |S| - j) E_j^{|\emptyset| - |S| - j - 1} \mathbb{h}(\mathcal{F}_S; t)$$

$$+ (1 - t) \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{|\emptyset| - |S| - 1} E_j^{|\emptyset| - |S| - j} \mathbb{h}'(\mathcal{F}_S; t).$$

Summing up the above two relations and using relation (5.21) for the $(d + |\emptyset| - 1)$-polytope $Q_R$, we conclude that the right-hand side of (5.23) is equal to

$$(d + |\emptyset| - 1)\mathbb{h}(\partial Q_R; t) + (1 - t)\mathbb{h}'(\partial Q_R; t)$$

$$= (d + |\emptyset| - 1)\mathbb{h}(\mathcal{F}_R; t) + (d + |\emptyset| - 1) \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{|\emptyset| - |S| - 1} E_j^{|\emptyset| - |S| - j} \mathbb{h}(\mathcal{F}_S; t)$$

$$+ (1 - t)\mathbb{h}'(\mathcal{F}_R; t)$$
where  

$$A = (d + |R| - 1)h(F_R; t) + (1 - t)h'(F_R; t).$$

In order to use our induction hypothesis, we regroup the terms of the above expression as follows:

$$A + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{[|R| - |S| - 1]} |R| - |S| - j E^j_{[|R| - |S|]} t^{[|R| - |S| - j - 1]} h(F_S; t)$$

$$+ \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{[|R| - |S| - 1]} (d + |S| - 1) E^j_{[|R| - |S|]} t^{[|R| - |S| - j]} h(F_S; t)$$

$$+ \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{[|R| - |S| - 1]} (|R| - |S| - j) E^j_{[|R| - |S|]} t^{[|R| - |S| - j - 1]} h(F_S; t)$$

$$+ (1 - t) \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{[|R| - |S| - 1]} E^j_{[|R| - |S|]} t^{[|R| - |S| - j]} h'(F_S; t),$$

Using the well known recurrence relation for the Eulerian numbers (cf. [8]):

$$E^j_m = (m - i) E^i_{m - 1} + (i + 1) E^i_{m - 1},$$

and the induction hypothesis, the above expression simplifies to

$$A + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{[|R| - |S| - 1]} E^j_{[|R| - |S|]} t^{[|R| - |S| - j]} h(F_S/v; t)$$

$$+ \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{[|R| - |S| - 1]} (E^{j+1}_{[|R| - |S| + 1]} - E^{j+1}_{[|R| - |S|]}) t^{[|R| - |S| - j - 1]} h(F_S; t).$$

(5.25)
Since the vertices of $Q_R$ are either vertices of some polytope $P_i$, $i \in R$, or auxiliary points $y_S$, $\emptyset \subset S \subset R$, we split the sum in the right-hand side of (5.23) as follows:

$$
\sum_{v \in \text{vert}(\partial Q_R)} h(\partial Q_R/v; t) = \sum_{v \in V_R} h(\partial Q_R/v; t) + \sum_{\emptyset \subset S \subset R} h(\partial Q_R/y_S; t).
$$

Using relations (5.13) and (5.3), the right-hand side of the above equation is equal to

$$
\sum_{v \in V_R} h(\partial \mathcal{F}_R/v; t) + \sum_{v \in V_S} \sum_{i \in R} \sum_{j=0}^{\lfloor |R|-|S|-1 \rfloor} E_j^{i_{|R|-|S|-1}} t^{j+1} h(\mathcal{F}_S/v; t)
$$

$$
+ \sum_{\emptyset \subset S \subset R} \sum_{\ell=1}^{\lfloor |R|-|X| \rfloor} (E_{|R|-|S|+1}^{\ell} - E_{|R|-|S|}^{\ell}) t^{\lfloor |R|-|X|+1 \rfloor} h(\mathcal{F}_X; t)
$$

$$
= B + \sum_{\emptyset \subset S \subset R} \sum_{v \in V_S} \sum_{j=0}^{\lfloor |R|-|S|-1 \rfloor} E_j^{i_{|R|-|S|-1}} t^{j+1} h(\mathcal{F}_S/v; t)
$$

$$
+ \sum_{\emptyset \subset S \subset R} \sum_{\ell=1}^{\lfloor |R|-|X| \rfloor} (E_{|R|-|S|+1}^{\ell} - E_{|R|-|S|}^{\ell}) t^{\lfloor |R|-|X|+1 \rfloor} h(\mathcal{F}_X; t)
$$

$$
= B + \sum_{\emptyset \subset S \subset R} \sum_{v \in V_S} \sum_{j=0}^{\lfloor |R|-|S|-1 \rfloor} E_j^{i_{|R|-|S|-1}} t^{\lfloor |R|-|S|-1 \rfloor} h(\mathcal{F}_S/v; t)
$$

$$
+ \sum_{\emptyset \subset S \subset R} \sum_{\ell=0}^{\lfloor |R|-|X| \rfloor} (E_{|R|-|S|+1}^{\ell} - E_{|R|-|S|}^{\ell}) t^{\lfloor |R|-|X|-1 \rfloor} h(\mathcal{F}_X; t). \tag{5.26}
$$

Equating (5.25) and (5.26) we conclude that $A = B$, which is precisely relation (5.22).

Comparing coefficients in (5.22) we conclude the following:

**Corollary 5.6** For any $\emptyset \subset R \subset [r]$ and all $0 \leq k \leq d + |R| - 2$ we have

$$
(k + 1)h_{k+1}(\mathcal{F}_R) + (d + |R| - 1 - k)h_k(\mathcal{F}_R) = \sum_{v \in V_R} h_k(\mathcal{F}_R/v), \tag{5.27}
$$

or equivalently

$$
(k + 1)h_{k+1}(\mathcal{F}_R) + (d + |R| - 1 - k)h_k(\mathcal{F}_R)
$$

$$
= \sum_{\emptyset \subset S \subset R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(K_S/v), \tag{5.28}
$$

where $V_S = \bigcup_{i \in S} V_i$. 

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Proof Relation (5.27) is immediate from (5.22): it suffices to compare the coefficients of the generating functions of left- and right-hand sides of (5.22).

To go from (5.27) to (5.28) we use the Inclusion–Exclusion principle (as in (2.15)), and notice that $K_{S/v}$ is the empty set for $v \not\in K_{S}$:

$$
\sum_{v \in V_R} h_k(F_R/v) = \sum_{v \in V_R} \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k(|R|-|S|)(K_{S/v}) = \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_R} g_k(|R|-|S|)(K_{S/v}) = \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k(|R|-|S|)(K_{S/v}).
$$

$$
\square
$$

5.4 Using Shellings to Bound the $g$-Vectors of Links

The main result of this subsection is Theorem 5.12, which is essential for proving the recursive relation in Theorem 5.1. Before proving it, a couple of preparatory lemmas are in order. The first (Lemma 5.10) concerns inequalities of $h$-vectors, which are proved using their interpretation as in-degrees of the dual graph of simplicial polytopes (cf. [12]). The second (Lemma 5.11) shows that there exists a particular shelling of the polytope $\partial Q_R$, for which the previous lemma is applicable.

We start with some definitions.

Definition 5.7 Let $C$ be a pure $d$-dimensional complex. A shelling of $C$ is a linear ordering $F_1, \ldots, F_s$ of its facets such that either $C$ is 0-dimensional, or it satisfies the following conditions:

(a) the boundary complex $\partial F_1$ of the first facet has a shelling,
(b) for $1 < j \leq s$ the intersection of the facet $F_j$ with the previous facets is nonempty and is a beginning segment of a shelling of the $(d-1)$-dimensional boundary complex $\partial F_j$, i.e., $F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \cdots \cup G_r$ for some shelling $G_1, \ldots, G_r, \ldots, G_r$ of $\partial F_j$.

A complex is shellable if it is pure and has a shelling.

As was pointed out in [24, Rem. 8.3], in the special case where $C$ is a pure $d$-dimensional simplicial complex, the definition above simplifies as follows: a shelling of $C$ is a linear ordering $F_1, \ldots, F_s$ of its facets such that either $C$ is 0-dimensional, or for each $1 < j \leq s$ the intersection of the facet $F_j$ with the previous facets is nonempty and pure $(d-1)$-dimensional.

In the case where $C$ is the boundary complex of a simplicial polytope, one can define the (oriented) dual graph $\mathcal{V}(\partial C)$ of $C$. This is the content of the next definition.

Definition 5.8 The dual graph $\mathcal{V}(\partial C)$ of a simplicial polytope $C$ is the graph whose nodes are the maximal simplices (i.e., facets) and whose edges correspond to pairs of adjacent facets, i.e., facets which intersect in codimension 1.
Fig. 2. Let $P$ be the 3-dimensional cross polytope, depicted on the left. For any shelling order of $P$ starting with its four top facets and finishing with its four bottom ones, its oriented dual graph $\mathcal{V}^\Delta(P)$ is depicted on the right. The $h$-vector of $P$ is $h(P) = (1, 3, 3, 1)$, which indeed counts the nodes of $\mathcal{V}^\Delta(P)$ according to their in-degree. If $A$ is the beginning segment of the shelling of $P$ consisting of the four top facets, then $h(A) = (1, 2, 1, 0)$ which, again, counts the nodes of $\mathcal{V}^\Delta(A)$ (blue subgraph) according to their in-degree. To verify Lemma 5.10, notice that the difference $h(P) - h(A) = (0, 1, 2, 1)$ counts the nodes in $\mathcal{V}^\Delta(P) \setminus \mathcal{V}^\Delta(A)$ (i.e., facets $F_6, \ldots, F_8$ on the graph) according to their in-degree in $\mathcal{V}^\Delta(P)$.

If, in addition, we consider a linear ordering $F_1, \ldots, F_\ell$ of the facets of $C$, we can impose an orientation on the graph $\mathcal{V}^\Delta(C)$ as follows: an edge connecting two nodes, i.e., two facets $F_i, F_j$, is oriented from $F_i$ to $F_j$ if $F_i$ precedes $F_j$ in the above order.

In the case where $C$ is a simplicial polytope, its $h$-vector encodes information about the in-degrees of its dual graph $\mathcal{V}^\Delta(C)$. This is the content of the next theorem.

Theorem 5.9 ([12],[24, Exer. 8.10]) Let $C$ be a simplicial $d$-polytope with dual graph $\mathcal{V}^\Delta(C)$, oriented according to a shelling order of its facets. Then, $h_k(C), 0 \leq k \leq d$, counts the number of vertices of the dual graph of $C$ with in-degree $k$ (and is independent of the shelling chosen).

It is also implicit in [24, Chap. 8] that, the dual graph $\mathcal{V}^\Delta(C)$ is well defined when $C$ is the beginning segment of a shelling of $P$, in which case Theorem 5.9 is also applicable. The reader is invited to verify Theorem 5.9 in Fig. 2.

Let $S$ be a simplicial polytope $P$, or the beginning segment of a shelling of $P$ and assume that $F_1, \ldots, F_\ell, F_{\ell+1}, \ldots, F_s$ is a shelling order of its facets. Let $A$ be the subcomplex of $S$ whose facets are $F_1, \ldots, F_\ell$. Clearly, $A$ is shellable as an initial segment of a shelling of $S$. Consider now the set $B$ containing all faces in $S \setminus A$. Notice that $B$ has no complex structure since it contains the facets $F_{\ell+1}, \ldots, F_s$ but not all their subfaces. We can, however, naturally define its $f$-vector and, since all its maximal faces are facets of $S$, make the convention that $\dim(B) = \dim(S)$. Moreover, as the following lemma suggests, the $h$-vector of $B$ admits a combinatorial interpretation.

Lemma 5.10 $h_k(B)$ counts the number of nodes in the subgraph $\mathcal{V}^\Delta(S) \setminus \mathcal{V}^\Delta(A)$ of $\mathcal{V}^\Delta(S)$ of in-degree $k$ in $\mathcal{V}^\Delta(S)$.

Proof In view of Theorem 5.9 we have that (refer also to Fig. 2): (i) $h_k(S)$ counts the number of nodes of the dual graph $\mathcal{V}^\Delta(S)$ of $S$ with in-degree $k$ and (ii) $h_k(A)$ counts

---

2 This can be deduced by exploiting the correspondence of the size of the restriction of a facet on a shelling order of $P$ [24, Sect. 8.3] with its in-degree of the dual graph [24, Exer. 8.10]. Since this correspondence holds also during a shelling of $P$, Theorem 5.9 is applicable for any beginning segment of a shelling of $P$. Springer
the number of nodes of the dual graph $\gamma(\Delta)(A)$ of $A$ with in-degree $k$. However, since the facets in $A$ are an initial segment of a shelling of $S$, their in-degree in $\gamma(\Delta)(A)$ and $\gamma(\Delta)(S)$ is the same (the out-degrees of nodes in $\gamma(\Delta)(A)$ might be greater when seen as nodes in $\gamma(\Delta)(S)$). Thus, the difference $h_k(B) = h_k(S) - h_k(A)$ counts the number of nodes in $\gamma(\Delta)(S) \setminus \gamma(\Delta)(A)$ with in-degree $k$ in $\gamma(\Delta)(S)$.

Using the machinery developed above, we may now show that $\partial Q_R$ admits a particular shelling, as stated in the following lemma.

**Lemma 5.11** There exists a shelling of $\partial Q_R$ starting from facets in $\bigcup_{j \in R \setminus \{1\}} \star(y_R \setminus \{j\}, \partial Q_R)$, and finishing with facets in $\star(y_R \setminus \{1\}, \partial Q_R)$.

**Proof** Let us start with some definitions: we denote by $Z$ the $(d + |R| - 1)$-complex we get by performing the construction of Sect. 3 until the last but one step, i.e., after having added all the auxiliary vertices $y_S$ with $|S| \leq |R| - 2$. Clearly, the facets of $Z$ are the $(d + |R| - 2)$-polytopes $Q_{R \setminus \{i\}}, i \in R$, as well as all facets in $F_R$. Since $Z$ is polytopal, each line in general position induces a shelling order of its facets (cf. [24, Sect. 8.2]). We will chose a line in such a way, so that the induced line shelling of $Z$ leads us (after adding the remaining auxiliary vertices $y_R \setminus \{i\}$) to the sought-for shelling of $\partial Q_R$.

Notice that, by the definition of the Cayley embedding, there exists a hyperplane in $\mathbb{R}^{d + |R| - 1}$ containing $P_{1}$ and being parallel to $C_{R \setminus \{1\}}$ (and thus to $Q_{R \setminus \{1\}}$). We can therefore choose a line $\ell$ beyond $y_1$ in $Z$ and intersecting $Q_{R \setminus \{1\}}$ in its interior. This line $\ell$ yields a shelling $S(Z)$ of $Z$ starting from facets in $\star(y_1, Z)$ and finishing with $Q_{R \setminus \{1\}}$. Since the facets in $\star(y_1, Z)$ are nothing but the polytopes $Q_{R \setminus \{i\}}, i \in R \setminus \{1\}$, the shelling $S(Z)$ starts with all $Q_{R \setminus \{i\}}, i \in R \setminus \{1\}$ (continues with the facets in $F_R$) and ends with $Q_{R \setminus \{1\}}$. Our next goal is to replace each facet $Q_{R \setminus \{i\}}$ in $S(Z)$ by all facets in $\star(y_R \setminus \{i\}, \partial Q_R \setminus \{i\})$, ordered so that the conditions in Definition 5.7 are satisfied.

We do this by induction. Assuming that $Q_{R \setminus \{2\}}$ is the first facet in the shelling order $S(Z)$, we replace it by the facets in $\star(y_R \setminus \{2\}, \partial Q_R \setminus \{2\})$, in any order “inherited” from a shelling of $\partial Q_{R \setminus \{2\}}$. Next, without loss of generality, we assume that the facets $Q_{R \setminus \{j\}}$ with $2 \leq j < i$ are those preceding $Q_{R \setminus \{i\}}$ in the shelling order $S(Z)$. By our induction hypothesis, we have replaced all $Q_{R \setminus \{j\}}$ by star($y_{R \setminus \{j\}}, \partial Q_{R \setminus \{j\}}$) in a way that the conditions of our claim are satisfied; we want to do the same for $j = i$.

To this end, notice that the intersection of $Q_{R \setminus \{i\}}$ with the union of the previous facets, is the union of all $Q_{R \setminus \{i, j\}}$ with $2 \leq j < i$, whether we consider “previous” in the shelling $S(Z)$ or in the sought-for shelling until the current inductive step (i.e., until when each $Q_{R \setminus \{j\}}$ with $2 \leq j < i$ is stellarly subdivided). As a result, the second condition of Definition 5.7, namely that that there exists a shelling order of the facets of $\partial Q_{R \setminus \{i\}}$ starting with all facets of $\bigcup_{2 \leq j < i} \partial Q_{R \setminus \{i, j\}}$, holds. It suffices to choose a shelling order of star($y_{R \setminus \{i\}}, \partial Q_{R \setminus \{i\}}$) that respects the common shelling order with $\bigcup_{2 \leq j < i} \partial Q_{R \setminus \{i, j\}}$. Using the existence of the above mentioned shelling, we may replace the facet $Q_{R \setminus \{i\}}$ by those in star($y_{R \setminus \{i\}}, \partial Q_{R \setminus \{i\}}$) (the shelling orders of each star($y_S, \partial Q_S$) are inherited from those for $\partial Q_S$) and arrive at a shelling order of $\partial Q_R$ with the desired properties. The last facet $Q_{R \setminus \{1\}}$ can be replaced by star($y_{R \setminus \{1\}}, \partial Q_{R \setminus \{1\}}$) without any further concern, since the shelling conditions are already fulfilled from the shelling $S(Z)$. 

\[\square\]
The interested reader may refer to Fig. 3, visualizing the arguments in the proof of the above lemma for the case of three polytopes. More precisely, $Z$ is the polytope in Fig. 1b, for which we deduce that there exists a shelling starting with $Q_{1,2}$ and $Q_{1,3}$, i.e., the two facets containing $y_1$, and ending with the bottom facet $Q_1$. When we perform the stellar subdivisions $st(y_{1,i}, Q_{1,i})$, $i = 2, 3$, we exploit the fact that the intersection $Q_{1,2} \cap Q_{1,3}$ is a beginning segment of a shelling of $\partial Q_{1,i}$ for both $i = 2, 3$. This induces a shelling for each $st(y_{1,i}, Q_{1,i})$, starting with the facets containing $y_1$ (the shaded ones in Fig. 3). Thus, replacing each $Q_{1,i}$ in $S(Z)$ by the facets in $st(y_{1,i}, Q_{1,i})$ in the above mentioned order, we obtain an ordering of the facets of $\partial Q_1$ which respects the shelling of $Z$ and thus the requirements of Definition 5.7.

Exploiting Lemmas 5.10 and 5.11 we arrive at the following theorem, where we bound the right-hand side of (5.28) by an expression that does not involve the links $K_S/v$.

**Theorem 5.12** For all $v \in V_R$ and all $k \geq 0$ we have

$$
\sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(K_S/v) \leq \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(K_S),
$$

(5.29)

where $V_S = \bigcup_{i \in S} V_i$.

**Proof** Let us first observe that, by rearranging terms, we can rewrite relation (5.29) as

$$
\sum_{i \in R} \sum_{v \in V_i} \sum_{|i| \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(K_S/v) \\
\leq \sum_{i \in R} \sum_{v \in V_i} \sum_{|i| \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(K_S).
$$

(5.30)

Clearly, to show that relation (5.30) holds, it suffices to prove that

$$
\sum_{|i| \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(K_S/v) \leq \sum_{|i| \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(K_S),
$$

(5.31)

for any arbitrary fixed $i \in R$ and any $v \in V_i$. 
Without loss of generality we may assume that \( i = 1 \). Define \( G_1 = \mathcal{F}_R \cup \mathcal{F}_{R \setminus \{1\}} \). Since \( \mathcal{F}_R \) and \( \mathcal{F}_{R \setminus \{1\}} \) are disjoint, we can write:

\[
f_k(G_1) = f_k(\mathcal{F}_R) + f_k(\mathcal{F}_{R \setminus \{1\}})
= \sum_{S \subseteq R} (-1)^{|R| - |S|} f_k(\mathcal{K}_S) + \sum_{S \subseteq R \setminus \{1\}} (-1)^{|R| - 1 - |S|} f_k(\mathcal{K}_S)
= \sum_{S \subseteq R} (-1)^{|R| - |S|} f_k(\mathcal{K}_S) - \sum_{S \subseteq R \setminus \{1\}} (-1)^{|R| - |S|} f_k(\mathcal{K}_S)
= \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R| - |S|} f_k(\mathcal{K}_S).
\]

(5.32)

Similarly, for all \( v \in V_1 \):

\[
\sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R| - |S|} f_k(\mathcal{K}_S/v).
\]

(5.33)

Converting the above relations into \( h \)-vector relations (using generating functions and comparing coefficients), we deduce that

\[
h_k(G_1) = \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R| - |S|} g_k^{(|R| - |S|)}(\mathcal{K}_S),
\]

(5.34)

and

\[
h_k(G_1/v) = \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R| - |S|} g_k^{(|R| - |S|)}(\mathcal{K}_S/v).
\]

(5.35)

Thus, in view of (5.34) and (5.35), proving (5.31) reduces to showing that \( h_k(G_1/v) \leq h_k(G_1) \). Define \( \partial Q' \) to be the polytopal \((d + |R| - 1)\)-complex whose facets are the facets of \( \partial Q_R \) not incident to \( y_{R \setminus \{1\}} \). To understand the face structure of \( \partial Q' \), we use Lemma 3.1 to rewrite \( \partial Q_R \) as the union

\[
\mathcal{F}_R \cup \bigcup_{i \in R} \text{star}(y_{R \setminus \{i\}}, \partial Q_R)
\]

of, not necessarily disjoint, faces. After removing all faces of \( \partial Q_R \) incident to \( y_{R \setminus \{1\}} \) we are left with the following set of faces

\[
\partial Q'_R = \bigcup_{i \in R \setminus \{1\}} \text{star}(y_{R \setminus \{i\}}, \partial Q_R) \cup \mathcal{G}_1 \cup \mathcal{F}_R \cup \mathcal{F}_{R \setminus \{1\}}.
\]

(5.36)

Although the face sets in the above union are not disjoint, the face sets \( \mathcal{G} \) and \( \mathcal{G}_1 \) are. This further implies that the facets of \( \partial Q'_R \) are the facets in \( \mathcal{G} \) and those in \( \mathcal{G}_1 \) (the latter being the facets in \( \mathcal{F}_R \cup \mathcal{F}_{R \setminus \{1\}} \)). We next claim that \( \partial Q'_R \) is shellable and that there exists a shelling of \( \partial Q'_R \) in which all facets in \( \mathcal{G} \) come first.
Indeed, according to Lemma 5.11, there exists a shelling of $\partial Q'_R$ starting from facets in $\bigcup_{i \in R \setminus \{1\}}$ star$(y_{R \setminus \{1\}}, \partial Q'_R)$, continuing with those in $F_R$ and ending with facets in star$(y_{R \setminus \{1\}}, \partial Q_R)$. Discarding the facets in star$(y_{R \setminus \{1\}}, \partial Q_R)$ we obtain a shelling of $\partial Q'_R$ starting from facets in $\bigcup_{i \in R \setminus \{1\}}$ star$(y_{R \setminus \{1\}}, \partial Q_R)$ and ending with facets in $F_R$.

We finally apply\(^3\) Lemma 5.10 with $S := \partial Q'_R$ and $A := G$ and we deduce that $h_k(G_1) = h_k(\partial Q'_R) - h_k(G)$ is the number of nodes in $\mathcal{V}(\partial Q'_R) \setminus \mathcal{V}(G)$ with in-degree $k$ (in the graph $\mathcal{V}(\partial Q'_R)$). In view of [24, Lem. 8.7], the restriction of the shelling order (5.36) of $\partial Q'_R$ on the link of $v$ is a shelling order of $\partial Q'_R/v$, beginning with $G/v$.

We exploit this in two ways:

(i) We, once again, apply Lemma 5.10 with $S := \partial Q'_R/v$ and $A := G/v$ and we deduce that $h_k(G_1/v) = h_k(\partial Q'_R/v) - h_k(G/v)$ is the number of nodes in $\mathcal{V}(\partial Q'_R/v) \setminus \mathcal{V}(G/v)$ with in-degree $k$ (in the graph $\mathcal{V}(\partial Q'_R/v)$).

(ii) Since the shelling order of $\partial Q'_R/v$ is induced from that in $\partial Q'_R$, arguing as in the proof of [24, Lem. 8.26], we deduce that the graph $\mathcal{V}(\partial Q'_R/v)$ is a subgraph of $\mathcal{V}(\partial Q'_R)$, whose nodes have the same in-degree, both in $\mathcal{V}(\partial Q'_R/v)$ and $\mathcal{V}(\partial Q'_R)$.

Since the nodes in $\mathcal{V}(\partial Q'_R/v) \setminus \mathcal{V}(G/v)$ are contained in the nodes of $\mathcal{V}(\partial Q'_R) \setminus \mathcal{V}(G)$, (i) and (ii) allow us to compare them, along with their in-degrees. More precisely, we conclude that the number of nodes in $\mathcal{V}(\partial Q'_R/v) \setminus \mathcal{V}(G/v)$ of in-degree $k$, cannot exceed the number of nodes $\mathcal{V}(\partial Q'_R) \setminus \mathcal{V}(G)$ of in-degree $k$. This immediately implies that $h_k(G_1/v) \leq h_k(G_1)$ and completes our proof. □

5.5 The Last Step Towards the Recurrence Relation

The last step for proving Theorem 5.1, is the following lemma that involves calculations which simplify the right-hand side of (5.29).

**Lemma 5.13** Let $\emptyset \subset R \subseteq [r]$, and $V_S = \bigcup_{i \in S} V_i$, for all $\emptyset \subset S \subseteq R$. Then, for all $k \geq 0$ we have

$$\sum_{\emptyset \subset S \subseteq R} (-1)^{|R| - |S|} \sum_{v \in V_S} g_k^{(|R| - |S|)}(K_S) = n_R h_k(F_R) + \sum_{i \in R} n_i g_k(F_{R \setminus \{i\}}), \quad (5.37)$$

where $n_R = \sum_{i \in R} n_i$.

**Proof** From relation (2.14) and the definition of the $m$-order $g$-vector (cf. (2.2)), we can easily show that, for any $\emptyset \subset R \subseteq [r]$,

$$g_k^{(m)}(K_R) = \sum_{\emptyset \subset S \subseteq R} g_k^{(|R| - |S| + m)}(F_S).$$

\(^3\) The arguments in this part of the proof are identical with their analogues for the Minkowski sum of two and three polytopes, (see [13, Lem. 11] and [14, Lem. 7], respectively).
Hence, for all $0 \leq k \leq d + |R| - 1$, we get
\[
g_k^{(|R| − |S|)}(K_S) = \sum_{\emptyset \subset X \subseteq S} g_k^{(|S| − |X| + (|R| − |S|))}(F_X) = \sum_{\emptyset \subset X \subseteq S} g_k^{(|R| − |X|)}(F_X).
\]

Thus, the left-hand side of (5.37) becomes
\[
\sum_{\emptyset \subset X \subseteq S} (-1)^{|R| − |S|} \sum_{v \in V_S} g_k^{(|R| − |S|)}(K_S) = \sum_{\emptyset \subset X \subseteq S} (-1)^{|R| − |S|} n_S g_k^{(|R| − |S|)}(K_S) = \sum_{\emptyset \subset X \subseteq S} \left( \sum_{X \subseteq S} (-1)^{|R| − |S|} n_S \right) g_k^{(|R| − |X|)}(F_X).
\]

We next evaluate the coefficient of $g_k^{(|R| − |X|)}(F_X)$ in (5.38), i.e., the quantity
\[
\sum_{X \subseteq S \subseteq R} (-1)^{|R| − |S|} n_S.
\]

We separate cases:

(a) If $X = R$ the sum in (5.39) simplifies to $n_R$.
(b) If $|X| = |R| − 1$, then $X = R \setminus \{i\}$ for some $i \in R$ and the sum in (5.39) simplifies to
\[
\sum_{X \subseteq S \subseteq R} (-1)^{|R| − |S|} n_S = (-1)^{|R| − |R| − 1} n_{R \setminus \{i\}} + (-1)^{|R| − |R|} n_R = n_R - n_{R \setminus \{i\}} = n_i.
\]

(c) If $|X| < |R| − 1$ then for every $i \in R \setminus X$ and every $0 \leq j \leq |R| − |X| − 1$ there exist $(|R| − |X| − 1)$ sets of size $|X| + j + 1$ containing $i$. We therefore have
\[
\sum_{X \subseteq S \subseteq R} (-1)^{|R| − |S|} n_S = \sum_{i \in R \setminus X} \sum_{j=0}^{|R| − |X| − 1} (-1)^{|R| − |X| − 1} \binom{|R| − |X| − 1}{j} n_i.
\]

From (a)–(c) we deduce that the only non-zero coefficients of $g_k^{(|R| − |X|)}(F_X)$ in (5.38) are those for which $|X| = |R|$ or $|R| − 1$. Thus, the sum in (5.38) simplifies to
\[
n_R h_k(F_R) + \sum_{i \in R} n_i g_k(F_{R \setminus \{i\}}),
\]

which is precisely the right-hand side of (5.37). \hfill \Box
5.6 The Proof of Theorem 5.1

Proof of Theorem 5.1

To prove the inequality in the statement of the theorem, we generalize McMullen’s steps in the proof of his Upper Bound theorem [17].

Our starting point is relation (5.1) applied to the simplicial $(d + |R| - 1)$-polytope $Q_R$, expressed in terms of generating functions:

\[(d + |R| - 1) h(\partial Q_R; t) + (1 - t) h'(\partial Q_R; t) = \sum_{v \in \text{vert}(\partial Q_R)} h(\partial Q_R/v; t). \quad (5.40)\]

Exploiting the combinatorial structure of $Q_R$ in order to express: (1) $h(\partial Q_R)$ in terms of $h(F_S, \emptyset \subset S \subset R)$, and (2) $h(\partial Q_R/v)$ in terms of $h(F_R/v)$ and $h(F_S)$, $\emptyset \subset S \subset R$, relation (5.40) yields (see Sects. 5.1–5.3):

\[(d + |R| - 1) h(F_R; t) + (1 - t) h'(F_R; t) = \sum_{v \in V_R} h(F_R/v; t),\]

the element-wise form of which is

\[(k + 1) h_{k+1}(F_R) + (d + |R| - 1 - k) h_k(F_R) = \sum_{v \in V_R} h_k(F_R/v), \quad 0 \leq k \leq d + |R| - 2.\]

Noticing that $h_k(F_R/v)$ is equal to $\sum_{\emptyset \subset S \subset R} (-1)^{|R| - |S|} g_k^{(|R| - |S|)}(K_S/v)$ (by the Inclusion–Exclusion Principle), and using relation (5.29) from Theorem 5.12 we have

\[\sum_{v \in V_R} h_k(F_R/v) = \sum_{\emptyset \subset S \subset R} (-1)^{|R| - |S|} \sum_{v \in V_S} g_k^{(|R| - |S|)}(K_S/v) \leq \sum_{\emptyset \subset S \subset R} (-1)^{|R| - |S|} g_k^{(|R| - |S|)}(K_S).\]

From Lemma 5.13, the right-hand side above simplifies to $n_R h_k(F_R) + \sum_{i \in R} n_i g_k(F_R \setminus \{i\})$, which in turn suggests the following inequality:

\[(k + 1) h_{k+1}(F_R) + (d + |R| - 1 - k) h_k(F_R) \leq n_R h_k(F_R) + \sum_{i \in R} n_i g_k(F_R \setminus \{i\}), \quad (5.41)\]

that holds true for all $0 \leq k \leq d + |R| - 2$. Solving in terms of $h_{k+1}(F_R)$ results in (5.2).  

\[\square\]

6 Upper Bounds

In this section we derive the upper bounds for the $h$- and $f$-vector of $F_R$, $\emptyset \subset R \subseteq [r]$, which leads to upper bounds for the Minkowski sum of $P_{[r]}$. The approach, reasoning and technique is essentially the same as that in Adiprasito and Sanyal [1], and is
presented here for completeness and consistency of presentation and notation, as well as due to the fact that our definition of \( h \)-vectors slightly differs from that of in [1].

Let \( S_1, \ldots, S_r \) be a partition of a set \( S \) into \( r \) subsets. We say that \( A \subseteq \bigcup_{1 \leq i \leq r} S_i \) is a spanning subset of \( S \) if \( A \cap S_i \neq \emptyset \) for all \( 1 \leq i \leq r \).

**Definition 6.1** Let \( P_i, i \in R, \) be \( d \)-polytopes with vertex sets \( V_i, i \in R \). We say that their Cayley polytope \( C_R \) is \( R \)-neighborly if every spanning subset of \( \bigcup_{i \in R} V_i \) of size \( |R| \leq \ell \leq \lfloor \frac{d+|R|-1}{2} \rfloor \) is a face of \( C_R \) (or, equivalently, a face of \( F_R \)). We say that the Cayley polytope \( C_R \) is Minkowski-neighborly if, for every \( \emptyset \subset S \subseteq R \), the Cayley polytope \( C_S \) is \( S \)-neighborly.

The notion of Minkowski-neighborly polytopes was introduced in [1, Proposition 5.3]. The following lemma characterizes \( R \)-neighborly Cayley polytopes in terms of the \( f \)- and \( h \)-vector of \( F_R \).

**Lemma 6.2** The following are equivalent:

1. \( C_R \) is \( R \)-neighborly,
2. \( f_{\ell-1}(F_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S}{\ell} \), for all \( 0 \leq \ell \leq \lfloor \frac{d+|R|-1}{2} \rfloor \),
3. \( h_\ell(F_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S-d-|R|+\ell}{\ell} \), for all \( 0 \leq \ell \leq \lfloor \frac{d+|R|-1}{2} \rfloor \),

where \( n_i \) is the number of vertices of \( P_i \) and \( n_S = \sum_{i \in S} n_i \).

**Proof** To show the equivalence between (i) and (ii), notice, from the definition of spanning subsets, that every spanning subset of \( V_R = \bigcup_{i \in R} V_i \) of size \( \ell \geq |R| \) has

\[
\sum_{1 \leq k_i \leq n_i} \prod_{i \in R} \binom{n_i}{k_i}
\]

elements. Using induction on the size of \( R \), one can check that the above sum of products is equal to the expression on the right-hand side of (ii). Moreover, in the case where \( \ell < |R| \), the expression on the right-hand side of (ii) is 0. This agrees with the fact that there do not exist any spanning subsets of \( \bigcup_{i \in R} V_i \) of size \( \ell < |R| \).

We next show the equivalence between (ii) and (iii). Taking the \((d-k)\)-th derivative of relation (2.4) for \( F_R \), it suffices to show that the values for \( f_{\ell-1}(F_R) \) and \( h_\ell(F_R) \), \( 0 \leq \ell \leq k \), in the statement of the theorem satisfy

\[
\sum_{i=0}^{k} f_{\ell-1}(F_R) \frac{(d-i)!}{(k-i)!} t^{k-i} = \sum_{i=0}^{k} \frac{(d-i)!}{(k-i)!} h_i(F_R) (t+1)^{k-i}.
\] (6.1)

Indeed, we have

\[
\sum_{i=0}^{k} h_i(F_R) \frac{(d+|R|-1-i)!}{(k-i)!} (t+1)^{k-i}
\]

\[
= \sum_{i=0}^{k} \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S-d-|R|+i}{i} \frac{(d+|R|-1-i)!}{(k-i)!} (t+1)^{k-i}
\]
From the recurrence relation in Theorem 5.1 we arrive at the following theorem.

**Theorem 6.3** For any \( \emptyset \subset R \subseteq \lfloor r \rfloor \) and \( 0 \leq k \leq d + |R| - 1 \), we have

\[
g_k(F_R) \leq \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - d - |R| + i}{i} \frac{(d+|R|-1-k)!}{(k-i)!} \sum_{j=0}^{k-i} \binom{k-i}{j} t_j \tag{6.2}
\]

\[
= \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{i=0}^{k} \binom{n_S - d - |R| + i}{i} \frac{(d+|R|-1-i)!}{(k-i)!} \sum_{j=0}^{k-i} \binom{k-i}{j} t_j \tag{6.3}
\]

\[
= \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \frac{(d+|R|-1-k+j)!}{j!} \binom{n_S - d - |R| + i}{i} \frac{(d+|R|-1-i)!}{(d+|R|-1-k+j)!} t_j \tag{6.4}
\]

\[
= \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \frac{(d+|R|-1-k+j)!}{j!} \binom{n_S}{k-j} \tag{6.5}
\]

where to go

- from (6.2) to (6.3) we used the relation:
  \[
  \frac{(d+|R|-1-i)!}{(k-i)!} \binom{k-i}{j} = \frac{(d+|R|-1-k+j)!}{j!} \binom{d+|R|-1-i}{d+|R|-1-k+j},
  \]
  and

- from (6.4) to (6.5) we used Relation 5.26 from [8]
  \[
  \sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1},
  \]
  holding for all non negative integers \( l, m, n \geq q \). \( \Box \)

### 6.1 Upper Bounds for the Lower Half of \( h(F_R) \)

From the recurrence relation in Theorem 5.1 we arrive at the following theorem.

**Theorem 6.3** For any \( \emptyset \subset R \subseteq \lfloor r \rfloor \) and \( 0 \leq k \leq d + |R| - 1 \), we have

\[
g_k(F_R) \leq \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - d - |R| + i}{i} \frac{(d+|R|-1-k)!}{(k-i)!} \sum_{j=0}^{k-i} \binom{k-i}{j} t_j \tag{6.6}
\]

\[
h_k(F_R) \leq \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - d - |R| + i}{i} \frac{(d+|R|-1-k)!}{(k-i)!} \sum_{j=0}^{k-i} \binom{k-i}{j} t_j \tag{6.7}
\]

where \( n_S = \sum_{i \in S} n_i \). Equalities hold for all \( 0 \leq k \leq \lfloor \frac{d+|R|-1}{2} \rfloor \) if and only if the Cayley polytope \( C_R \) is \( R \)-neighborly.
Proof We are going to show the wanted bounds by induction on $|R|$ and $k$. Clearly the bounds hold for $|R| = 1$ and for any $0 \leq k \leq d$ (this is the case of one $d$-polytope and the bounds of the lemma refer to the well-known bounds on the elements of the $h$- and $g$-vector of a polytope).

Suppose now that the bounds for $g_k(F_R)$ and $h_k(F_R)$ hold for all $|R| < m$ and for all $0 \leq k \leq d + |R| - 1$. Consider an $R$ with $|R| = m$. Then, for $k = 0$ we have

$$h_0(F_R) = f_{-1}(F_R) = (-1)^{|R|-1} = (-1)^{|R|}$$

$$= \sum_{i=1}^{|R|} (-1)^{|R|-i} \binom{|R|}{i} = \sum_{i=1}^{|R|} (-1)^{|R|-i} \sum_{\emptyset \subseteq S \subseteq R} 1$$

$$= \sum_{i=1}^{|R|} \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-i} = \sum_{\emptyset \subseteq S \subseteq R} \sum_{|S| = i} (-1)^{|R|-|S|} n_{|S|, |R|, k}$$

$$= \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} (n_{|S|-d-|R|, 0}^{d-|R|, 0})$$

and

$$g_0(F_R) = h_0(F_R) - h_{-1}(F_R) = \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} (n_{|S|-d-|R|, 0}^{d-|R|, 0})$$

$$= \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} (n_{|S|-d-|R|, 0}^{d-|R|, 0}).$$

For $k \geq 1$ we have

$$g_k(F_R) = h_k(F_R) - h_{k-1}(F_R)$$

$$\leq n_{R-d-|R|, k}^{k+1} h_{k-1}(F_R) + \sum_{i \in R} n_i^k g_{k-1}(F_R \setminus \{i\}) - h_{k-1}(F_R)$$

$$= n_{R-d-|R|, k}^{k+1} h_{k-1}(F_R) + \sum_{i \in R} n_i^k g_{k-1}(F_R \setminus \{i\}). \quad (6.8)$$

By our inductive hypotheses, we have

$$h_{k-1}(F_R) \leq \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} (n_{|S|-d-|R|+k-1, 0}^{d-|R|+k-1}) \quad (6.9)$$

and also, for all $i \in R$

$$g_{k-1}(F_R \setminus \{i\}) \leq \sum_{\emptyset \subseteq S \subseteq R \setminus \{i\}} (-1)^{|R\setminus \{i\}| - |S|} (n_{|S|-d-|R\setminus \{i\}|+1+k-1, 0}^{d-|R\setminus \{i\}|+1+k-1})$$

$$= - \sum_{\emptyset \subseteq S \subseteq R \setminus \{i\}} (-1)^{|R|-|S|} (n_{|S|-d-|R|, 0}^{d-|R|, 0}). \quad (6.10)$$
Substituting (6.9) and (6.10) in (6.8) we get

\[ g_k(F_R) \leq \frac{n_R-d-|R|}{k} \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} \left( n_S-d-|R|+k \right)_{k-1} \]
\[ - \sum_{i \in R} \left( \frac{n_i}{k} \sum_{\emptyset \subseteq S \subseteq R \setminus \{i\}} (-1)^{|R|-|S|} \left( n_S-d-|R|-1+k \right)_{k-1} \right) \]

(6.11)

Consider the sum \( \sum_{i \in R} \frac{n_i}{k} \sum_{\emptyset \subseteq S \subseteq R \setminus \{i\}} (-1)^{|R|-|S|} \left( n_S-d-|R|-1+k \right)_{k-1} \); observe that for any given \( \emptyset \subset S \subset R \) we get a contribution of \( \frac{n_i}{k} \) for \( (n_s-d-|R|-1+k) \), for any \( i \notin S \). In other words, we have the equality

\[ \sum_{i \in R} \frac{n_i}{k} \sum_{\emptyset \subseteq S \subseteq R \setminus \{i\}} (-1)^{|R|-|S|} \left( n_S-d-|R|-1+k \right)_{k-1} = \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} \frac{n_{R\setminus S}}{k} \left( n_S-d-|R|-1+k \right)_{k-1} \]

(6.12)

In view of (6.12) the inequality in (6.11) becomes

\[ g_k(F_R) \leq \frac{n_R-d-|R|}{k} \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} \left( n_S-d-|R|+k \right)_{k-1} \]
\[ - \sum_{\emptyset \subseteq S \subseteq R} (-1)^{|R|-|S|} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]
\[ + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ \leq \frac{n_R-d-|R|}{k} \left( n_R-d-|R|-1+k \right)_{k-1} \]
\[ + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ = \left( \frac{n_R-d-|R|}{k} + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

For any \( i \), \( (n_i-d-|R|)_{k-1} \) can be written as \( (n_i-d-|R|+k)_{k-1} - (n_i-d-|R|-1+k) \), so

\[ \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ = \left( \frac{n_R-d-|R|}{k} + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ \leq \left( \frac{n_R-d-|R|}{k} + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]

\[ + \sum_{\emptyset \subseteq S \subseteq R} \left( \frac{n_{R\setminus S}}{k} \right) \left( n_S-d-|R|-1+k \right)_{k-1} \]
Before proceeding with proving upper bounds for the relation (5.2) and the upper bound for \( h_k \), we define the following functions (see also Sect. 5.6 in [1], where the same arguments and steps are followed in that context).

Finally, the equality claim is immediate from Lemma 6.2.

6.2 Upper Bounds for \( h_k(\mathcal{F}_R) \) and \( h_k(\mathcal{K}_R) \) for all \( k \)

Before proceeding with proving upper bounds for the \( h \)-vectors of \( \mathcal{F}_R \) and \( \mathcal{K}_R \) we need to define the following functions (see also Sect. 5.6 in [1], where the same arguments and steps are followed in that context).
Definition 6.4 Let $d \geq 2$, $\emptyset \subset R \subseteq [r]$, $m \geq 0$, $0 \leq k \leq d + |R| - 1$, and $n_i \in \mathbb{N}$, $i \in R$, with $n_i \geq d + 1$. We define the functions $\Phi^{(m)}_{k,d}(n_R)$ and $\Psi_{k,d}(n_R)$ via the following conditions

1. $\Phi^{(0)}_{k,d}(n_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - d - |R| + k}{k}, 0 \leq k \leq \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor$, 
2. $\Phi^{(m)}_{k,d}(n_R) = \Phi^{(m-1)}_{k,d}(n_R) - \Phi^{(m-1)}_{k-1,d}(n_R), m > 0$, 
3. $\Psi_{k,d}(n_R) = \sum_{\emptyset \subset S \subseteq R} \Phi^{(|R|-|S|)}_{k,d}(n_S)$, 
4. $\Phi^{(0)}_{k,d}(n_R) = \Psi_{d+|R|-1-k,d}(n_R)$.

where $n_R$ stands for the $|R|$-dimensional vector whose elements are the values $n_i$, $i \in R$.

Notice that $\Phi^{(0)}_{k,d}(n_R)$ and $\Psi_{k,d}(n_R)$ are well defined, though in a recursive manner (in the size of $R$), since for any $k > \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor$, we have

$$\Phi^{(0)}_{k,d}(n_R) = \Psi_{d+|R|-1-k,d}(n_R) = \sum_{\emptyset \subset S \subseteq R} \Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(n_S)$$

$$= \Phi^{(0)}_{d+|R|-1-k,d}(n_R) + \sum_{\emptyset \subset S \subseteq R} \Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(n_S)$$

$$= \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S - k - 1}{d+|R|-1-k} + \sum_{\emptyset \subset S \subseteq R} \Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(n_S),$$

where the first summand of (6.13) is replaced by its explicit expression from Definition 6.4, due to the fact that if $k > \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor$ then $d + |R| - 1 - k \leq \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor$. Also, the second sum in (6.14) is to be understood as 0 when $|R| = 1$. In other words, $\Phi^{(0)}_{k,d}(n_R)$, and, thus, also $\Phi^{(m)}_{k,d}(n_R)$ for any $m > 0$, is fully defined for some $R$ and any $k$, once we know the values $\Phi^{(c)}_{k,d}(n_S)$ for all $\emptyset \subset S \subseteq R$, for all $0 \leq k \leq d + |S| - 1$, and for all $1 \leq l \leq |R| - 1$. Moreover, it is easy to verify that $\Phi^{(0)}_{k,d}(n_R)$ satisfies the following recurrence relation

$$\Phi^{(0)}_{k+1,d}(n_R) = \frac{n_R - d - |R| + k + 1}{k+1} \Phi^{(0)}_{k,d}(n_R)$$

$$+ \sum_{i \in R} \frac{n_i}{k+1} \Phi^{(1)}_{k,d}(n_{R\setminus\{i\}}), \quad 0 \leq k < \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor.$$

Lemma 6.5 For any $\emptyset \subset R \subseteq [r]$, any $k$ with $0 \leq k \leq \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor$, and any $\alpha$ with $0 \leq \alpha \leq \frac{d+1}{d-1}$, we have

$$h_k(F_R) - \alpha \sum_{i \in R} h_{k-1}(F_{R\setminus\{i\}}) \leq \Phi^{(0)}_{k,d}(n_R) - \alpha \sum_{i \in R} \Phi^{(0)}_{k-1,d}(n_{R\setminus\{i\}}).$$

To prove Lemma 6.5 we need the following intermediate result.

Lemma 6.6 For any $\emptyset \subset R \subseteq [r]$, any $k$ with $0 \leq k \leq \left\lfloor \frac{d + |R| - 1}{2} \right\rfloor$, and any $\alpha$ with $0 \leq \alpha \leq \frac{d+1}{d-1}$, we have

$$g_k(F_R) - \alpha \sum_{i \in R} g_{k-1}(F_{R\setminus\{i\}}) \leq \Phi^{(1)}_{k,d}(n_R) - \alpha \sum_{i \in R} \Phi^{(1)}_{k-1,d}(n_{R\setminus\{i\}}).$$
Proof Let us recall the recurrence relation from Theorem 5.1
\[
  h_k(F_R) \leq \frac{n_R - d - |R| + k}{k} h_{k-1}(F_R) + \sum_{i \in R} \frac{n_i}{k} g_{k-1}(F_{R \setminus \{i\}}).
\]
Subtracting \( h_{k-1}(F_R) + \alpha \sum_{i \in R} g_{k-1}(F_{R \setminus \{i\}}) \) from both sides of the inequality we get
\[
g_k(F_R) - \alpha \sum_{i \in R} g_{k-1}(F_{R \setminus \{i\}}) \leq \frac{n_R - d - |R|}{k} h_{k-1}(F_R) + \sum_{i \in R} \left( \frac{n_i}{k} - \alpha \right) g_{k-1}(F_{R \setminus \{i\}}). \tag{6.17}
\]
Observe that the coefficient of \( h_{k-1}(F_R) \) in (6.17) is non-negative
\[
n_R - d - |R| \geq |R|(d + 1) - d - |R| = d|R| + |R| - d - |R| = d(|R| - 1) \geq 0.
\]
The same holds for the coefficient of \( g_{k-1}(F_{R \setminus \{i\}}) \) in (6.17), since
\[
  \frac{n_i}{k} \geq \frac{d + 1}{\frac{d + |R| - 1}{2}} \geq \frac{d + 1}{\frac{d + |R| - 1}{2}} = \frac{2d + 2}{d + |R| - 1} \geq \frac{2d + 2}{d + (d - 1) - 1} = \frac{2d + 2}{2d - 2} \geq \alpha,
\] where we used the fact that \( |R| \leq r \leq d - 1 \). Hence, we can bound (6.17) from above by substituting \( g_{k-1}(F_R) \) and \( g_{k-1}(F_{R \setminus \{i\}}) \), \( i \in R \), by \( \Phi_{k-1,d}^{(i)}(n_R) \) and \( \Phi_{k-1,d}^{(i)}(n_{R \setminus \{i\}}) \), \( i \in R \), respectively. This gives
\[
g_k(F_R) - \alpha \sum_{i \in R} g_{k-1}(F_{R \setminus \{i\}}) \leq \frac{n_R - d - |R|}{k} \Phi_{k-1,d}^{(i)}(n_R) + \sum_{i \in R} \left( \frac{n_i}{k} - \alpha \right) \Phi_{k-1,d}^{(i)}(n_{R \setminus \{i\}})
\]
\[
= \frac{n_R - d - |R|}{k} \Phi_{k-1,d}^{(i)}(n_R) + \sum_{i \in R} \frac{n_i}{k} \Phi_{k-1,d}^{(i)}(n_{R \setminus \{i\}})
\]
\[
- \alpha \sum_{i \in R} \Phi_{k-1,d}^{(i)}(n_{R \setminus \{i\}})
\]
\[
= \Phi_{k,d}^{(i)}(n_R) - \alpha \sum_{i \in R} \Phi_{k-1,d}^{(i)}(n_{R \setminus \{i\}}).
\]

Having established Lemma 6.6, it is now straightforward to prove Lemma 6.5.

Proof of Lemma 6.5 First observe that \( h_i(F_X) \) may be written as a telescopic sum as follows
\[
h_i(F_X) = h_0(F_X) + \sum_{\ell=0}^{i-1} g_{i-\ell}(F_X). \tag{6.19}
\]
Since \( h_0(\mathcal{F}_X) = g_0(\mathcal{F}_X) \), the above expansion may be written in the more concise form
\[
h_i(\mathcal{F}_X) = \sum_{\ell=0}^{i} g_{i-\ell}(\mathcal{F}_X).
\] (6.20)

Using relations (6.19) and (6.20), and applying Lemma 6.6, we get
\[
h_k(\mathcal{F}_R) - \alpha \sum_{i \in R} h_{k-1}(\mathcal{F}_{R\setminus\{i\}})
\]
\[
= h_0(\mathcal{F}_R) + \sum_{\ell=0}^{k-1} g_{k-\ell}(\mathcal{F}_R) - \alpha \sum_{i \in R} \sum_{\ell=0}^{k-1} g_{k-\ell-1}(\mathcal{F}_{R\setminus\{i\}})
\]
\[
= h_0(\mathcal{F}_R) + \sum_{\ell=0}^{k-1} \left( g_{k-\ell}(\mathcal{F}_R) - \alpha \sum_{i \in R} g_{k-\ell-1}(\mathcal{F}_{R\setminus\{i\}}) \right)
\]
\[
\leq \Phi^{(0)}(\mathcal{F}_R) + \sum_{\ell=0}^{k-1} \left( \Phi^{(1)}_{k-\ell,d}(\mathcal{F}_R) - \alpha \sum_{i \in R} \Phi^{(1)}_{k-\ell-1,d}(\mathcal{F}_{R\setminus\{i\}}) \right)
\]
\[
= \Phi^{(0)}(\mathcal{F}_R) + \sum_{\ell=0}^{k-1} \Phi^{(1)}_{k-\ell,d}(\mathcal{F}_R) - \alpha \sum_{i \in R} \Phi^{(1)}_{k-\ell-1,d}(\mathcal{F}_{R\setminus\{i\}})
\]
\[
= \Phi^{(0)}(\mathcal{F}_R) + \left( \Phi^{(0)}_{k,d}(\mathcal{F}_R) - \Phi^{(0)}_{0,d}(\mathcal{F}_R) \right)
\]
\[
- \alpha \sum_{i \in R} \left( \Phi^{(0)}_{k-1,d}(\mathcal{F}_{R\setminus\{i\}}) - \Phi^{(0)}_{1,d}(\mathcal{F}_{R\setminus\{i\}}) \right)
\]
\[
= \Phi^{(0)}_{k,d}(\mathcal{F}_R) - \alpha \sum_{i \in R} \Phi^{(0)}_{k-1,d}(\mathcal{F}_{R\setminus\{i\}}),
\]

where we used the fact that \( \Phi^{(0)}_{-1,d}(\mathcal{F}_{R\setminus\{i\}}) = 0 \) for all \( \emptyset \subset R \subseteq [r] \) and \( i \in R \).

The next theorem provides upper bounds for the \( h \)-vectors of \( \mathcal{F}_R \) and \( \mathcal{K}_R \), as well as necessary and sufficient conditions for these upper bounds to be attained (see also [1, Thm. 5.19]).

**Theorem 6.7** For all \( 0 \leq k \leq d + |R| - 1 \), we have

(i) \( h_k(\mathcal{F}_R) \leq \Phi^{(0)}_{k,d}(\mathcal{F}_R) \),

(ii) \( h_k(\mathcal{K}_R) \leq \Psi^{(0)}_{k,d}(\mathcal{F}_R) \).

**Proof** To prove the upper bounds we use recursion on the size of \( |R| \). For \( |R| = 1 \), the result for both \( h_k(\mathcal{F}_R) \) and \( h_k(\mathcal{K}_R) \) comes from the UBT for \( d \)-polytopes. For \( |R| > 1 \), we assume that the bounds hold for all \( S \) with \( \emptyset \subset S \subset R \), and for all \( k \) with \( 0 \leq k \leq d + |S| - 1 \). Furthermore, the upper bound for \( h_k(\mathcal{F}_R) \) for \( k \leq \left\lfloor \frac{d+|R|-1}{2} \right\rfloor \) is immediate from Theorem 6.3. To prove the upper bound for \( h_k(\mathcal{K}_R) \), \( 0 \leq k \leq \left\lfloor \frac{d+|R|-1}{2} \right\rfloor \), we use the following expansion for \( h_k(\mathcal{K}_R) \) (cf. [1, Lem. 5.14]):

\( \circ \) Springer
\( h_k(K_R) \)

\[
= \sum_{j=0}^{\lfloor \frac{|R|}{2} \rfloor} \sum_{s=c-2j+1}^{\lfloor \frac{|R|}{2j} \rfloor} \sum_{S \subseteq R, |S| = s} \binom{|R|-s}{2j} (h_{k-2j}(F_S) - \frac{1}{2j+1} \sum_{i \in S} h_{k-2j-1}(F_{S \setminus \{i\}}))
\]

(6.21)

\[
+ \sum_{j=0}^{\lfloor \frac{|R|}{2} \rfloor} \sum_{S \subseteq R, |S| = c-2j+1} \binom{|R|-|S|}{2j} (h_{k-2j}(F_S) - \frac{1}{2j+1} \sum_{i \in S} h_{k-2j-1}(F_{S \setminus \{i\}})),
\]

where \( c \) depends on \( k, d \) and \( |R| \). Under the assumption that \( r < d \), it is easy to show that (see Lemma 6.5 above)

\[
h_{k-2j}(F_S) - \frac{1}{2j+1} \sum_{i \in S} h_{k-2j-1}(F_{S \setminus \{i\}}) \leq \Phi_{k-2j,d}(n_S) - \frac{1}{2j+1} \sum_{i \in S} \Phi_{k-2j-1,d}(n_{S \setminus \{i\}}).
\]

(6.22)

Substituting the upper bound from (6.22) in (6.21), and reversing the derivation logic for (6.21), we deduce that \( h_k(K_R) \leq \Psi_{k,d}(n_R) \).

For \( k > \lfloor \frac{d+|R|-1}{2} \rfloor \) we have

\[
h_k(F_R) = h_{d+|R|-1-k}(K_R) \leq \Psi_{d+|R|-1-k,d}(n_R) = \Phi_{k,d}(n_R), \quad \text{and},
\]

\[
h_k(K_R) = h_{d+|R|-1-k}(F_R) \leq \Phi_{d+|R|-1-k,d}(n_R) = \Psi_{k,d}(n_R).
\]

The necessary and sufficient conditions are easy consequences of the equality claim in Theorem 6.3. \( \square \)

For any \( d \geq 2, \emptyset \subset R \subseteq [r], 0 \leq k \leq d + |R| - 1, \) and \( n_i \in \mathbb{N}, i \in R, \) with \( n_i \geq d+1 \), let

\[
\Xi_{k,d}(n_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} f_k(C_{d+|R|-1}(n_S))
\]

\[
+ \sum_{i=0}^{\lfloor \frac{d+|R|-1}{2} \rfloor} \binom{d+|R|-1}{i} \sum_{\emptyset \subset S \subseteq R} \Phi_{i,d}(n_S),
\]

where \( C_\delta(n) \) stands for the cyclic \( \delta \)-polytope with \( n \) vertices. It is straightforward to verify that for \( 0 \leq k \leq \lfloor \frac{d+|R|-1}{2} \rfloor \), \( \Xi_{k,d}(n_R) \) simplifies to \( \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} (n_S) \).

We are finally ready to state and prove the main result of the paper (which is also the analogue of Theorem 5.4 in [1]).

**Theorem 6.8** Let \( P_1, \ldots, P_r \) be \( r \)-polytopes, \( r < d \), with \( n_1, \ldots, n_r \) vertices respectively. Then, for all \( 1 \leq k \leq d \), we have

\[
f_{k-1}(P_{[r]}) \leq \Xi_{k+r,d}(n_{[r]}).
\]
Equality holds for all $0 \leq k \leq d$ if and only if the Cayley polytope $C_{[r]}$ of $P_1, \ldots, P_r$ is Minkowski-neighborly.

**Proof** We start by recalling that

$$f_{k-1}(\mathcal{F}_{[r]}) = \sum_{i=0}^{d+r-1} \binom{d+r-1}{k-i} h_i(\mathcal{F}_{[r]}).$$

In view of Theorem 6.7, the above expression is bounded from above by

$$\sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(n_{[r]}) + \sum_{i=\lfloor \frac{d+r-1}{2} \rfloor+1}^{d+r-1} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(n_{[r]})$$

(6.23)

$$= \sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(n_{[r]}) + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{d+r-i}{k-d-r+1+i} \Phi_{i,d+r-1-i,d}^{(0)}(n_{[r]})$$

(6.24)

$$= \sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(n_{[r]}) + \sum_{i=\lfloor \frac{d+r-2}{2} \rfloor}^{\lfloor \frac{d+r-1}{2} \rfloor+1} \binom{d+r-i}{k-d-r+1+i} \Psi_{i,d}^{(0)}(n_{[r]})$$

(6.25)

$$= \sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(n_{[r]}) + \sum_{i=\lfloor \frac{d+r-2}{2} \rfloor}^{\lfloor \frac{d+r-1}{2} \rfloor+1} \sum_{\emptyset \subset R \subseteq [r]} \Phi_{i,d}^{(r-|R|)}(n_{[r]})$$

(6.26)

$$= \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{d+r-1-i}{k-i} + \binom{d+r-i}{k-d-r+1+i} \sum_{\emptyset \subset R \subseteq [r]} (-1)^{|R|} \binom{n_{[r]}-d-r+i}{i}$$

$$+ \sum_{i=\lfloor \frac{d+r-2}{2} \rfloor}^{\lfloor \frac{d+r-1}{2} \rfloor+1} \sum_{\emptyset \subset R \subseteq [r]} \Phi_{i,d}^{(r-|R|)}(n_{[r]})$$

(6.27)

$$= \sum_{\emptyset \subset R \subseteq [r]} \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} (-1)^{|R|} f_k(C_{d+r-1}(n_{[r]}))$$

$$+ \sum_{\emptyset \subset R \subseteq [r]} \sum_{i=\lfloor \frac{d+r-2}{2} \rfloor}^{\lfloor \frac{d+r-1}{2} \rfloor+1} \Phi_{i,d}^{(r-|R|)}(n_{[r]})$$

(6.28)

where to go:

- from (6.23) to (6.24) we changed the variable of the second sum from $i$ to $d+r-1-i$,
- from (6.24) to (6.25) we used condition 3 in Definition 6.4,
- from (6.25) to (6.26) we used condition 4 in Definition 6.4,
- from (6.26) to (6.27) we wrote the explicit expression of $\Phi_{i,d}^{(0)}(n_{[r]})$ from relation (6.14),
from (6.27) to (6.28) we used that the number of \((k - 1)\)-faces of a cyclic \(\delta\)-polytope with \(n\) vertices is \(\sum_{i=0}^{\lfloor \frac{\delta}{2} \rfloor} \binom{\delta - i}{k - \delta + i} + \binom{i}{k - 1 + i} \), where \(\sum_{i=0}^{\lfloor \frac{\delta}{2} \rfloor} T_i\) denotes the sum of the elements \(T_0, T_1, \ldots, T_{\lfloor \frac{\delta}{2} \rfloor}\) where the last term is halved if \(\delta\) is even.

Finally, observing that the expression in (6.28) is nothing but \(\Xi_{k,d}(n_{\lfloor \tau \rfloor})\), and recalling that \(f_{k-1}(F_{\lfloor \tau \rfloor}) = f_{k-\tau}(P_{\lfloor \tau \rfloor})\), we arrive at the upper bound in the statement of the theorem. The equality claim is immediate from Theorem 6.7.

7 Tight Bound Construction

In this section we show that the bounds in Theorem 6.8 are tight. Before getting into the technical details, we outline our approach. We start by considering the \((d - r + 1)\)-dimensional moment curve, which we embed in \(r\) distinct subspaces of \(\mathbb{R}^d\). We consider the \(r\) copies of the \((d - r + 1)\)-dimensional moment curve as different curves, and we perturb them appropriately, so that they become \(d\)-dimensional moment-like curves. The perturbation is controlled via a non-negative parameter \(\zeta\), which will be chosen appropriately. We then choose points on these \(r\) moment-like curves, all parameterized by a positive parameter \(\tau\), which will again be chosen appropriately. These points are the vertices of \(r\) \(d\)-polytopes \(P_1, P_2, \ldots, P_r\), and we show that, for all \(\emptyset \subset R \subseteq \{r\}\), the number of \((k - 1)\)-faces of \(F_R\), where \(|R| \leq k \leq \lfloor \frac{d + |R| - 1}{2} \rfloor\), becomes equal to \(\Xi_{k,d}(\delta_{\{R\}})\) for small enough positive values of \(\zeta\) and \(\tau\). Our construction produces projected prod-simplicial neighborly polytopes (cf. [16]). For \(\zeta = 0\) our polytopes are essentially the same as those in [16, Thm. 2.6], while for \(\zeta > 0\) we get deformed versions of those polytopes. The positivity of \(\zeta\) allows us to ensure the tightness of the upper bound on \(f_k(P_{\lfloor \tau \rfloor})\), not only for small, but also for large values of \(k\).

At a more technical level (cf. Sect. 1), the proof that \(f_{k-1}(F_R) = \Xi_{k,d}(\delta_{\{R\}})\), for all \(|R| \leq k \leq \lfloor \frac{d + |R| - 1}{2} \rfloor\), is performed in two steps. We first consider the cyclic \((d - r + 1)\)-polytopes \(\hat{P}_1, \ldots, \hat{P}_r\), embedded in appropriate subspaces of \(\mathbb{R}^d\). The \(\hat{P}_i\)'s are the unperturbed, with respect to \(\zeta\), versions of the \(d\)-polytopes \(P_1, P_2, \ldots, P_r\) (i.e., the polytope \(\hat{P}_i\) is the polytope we get from \(P_i\), when we set \(\zeta\) equal to zero). For each \(\emptyset \subset R \subseteq \{r\}\) we denote by \(\hat{C}_R\) the Cayley polytope of \(\hat{P}_i, i \in R\), seen as a polytope in \(\mathbb{R}^d\), and we focus on the set \(\mathcal{W}_R\) of its mixed faces. Recall that the polytopes \(P_i, i \in R\), are parameterized by the parameter \(\tau\); we show that there exists a sufficiently small positive value \(\tau^*\) for \(\tau\), for which the number of \((k - 1)\)-faces of \(\mathcal{W}_R\) is equal to \(\Xi_{k,d}(\delta_{\{R\}})\) for all \(|R| \leq k \leq \lfloor \frac{d + |R| - 1}{2} \rfloor\). For \(\tau\) equal to \(\tau^*\), we consider the polytopes \(P_1, P_2, \ldots, P_r\) (with \(\tau\) set to \(\tau^*\)), and show that for sufficiently small \(\zeta\) (denoted by \(\zeta^*\)), \(f_{k-1}(F_R)\) is equal to \(\Xi_{k,d}(\delta_{\{R\}})\).

In the remainder of this section we describe our construction in detail. For each \(1 \leq i \leq r\), we define the \(d\)-dimensional moment-like curve\(^4\):

\[^4\] The curve \(\gamma(t; \zeta), \zeta > 0\), is the image under an invertible linear transformation, of the curve \(\gamma_1(t) = (t, t^2, \ldots, t^{d-r+i}, t^{d-r+i+2}, \ldots, t^{d+1})\). Polytopes whose vertices are \(n\) distinct points on this curve are combinatorially equivalent to the cyclic \(d\)-polytope with \(n\) vertices.
There exists a sufficiently small positive value towards our construction. We write each spanning subset $U$ of the polytopes $\hat{C}_R$ and the $d$-polytope
\[
P_i := \text{conv}(\{y_i(y_{i,1}; \zeta), \ldots, y_i(y_{i,n_i}; \zeta)\}),
\]
where the parameters $y_{i,j}$ belong to the sets $Y_i = \{y_{i,1}, \ldots, y_{i,n_i}\}$, $1 \leq i \leq r$, whose elements are determined as follows. Choose

- $n_{[r]} + d + r$ arbitrary real numbers $x_{i,j}$ and auxiliary numbers $M_i'$, such that:
  - $0 < x_{i,1} < x_{i,1} + \epsilon < x_{i,2} < x_{i,2} + \epsilon < \cdots < x_{i,n_i} + \epsilon$, for $1 \leq i \leq r - 1$,
  - $0 < x_{r,1} < x_{r,1} + \epsilon < x_{r,2} < x_{r,2} + \epsilon < \cdots < x_{r,n_r} + \epsilon < M_1' < \cdots < M_{d+r}'$
where $\epsilon > 0$ is sufficiently small and $x_{i,n_i} < x_{i+1,1}$ for all $i$, and

- $r$ non-negative integers $\beta_1, \beta_2, \ldots, \beta_r$, such that $\beta_1 > \beta_2 > \cdots > \beta_{r-1} > \beta_r \geq 0$.

We then set $y_{i,j} := x_{i,j} \tau^{\beta_i}, \tilde{y}_{i,j} := (x_{i,j} + \epsilon) \tau^{\beta_i}$ and $M_i := M_i' \tau^{\beta_i}$, where $\tau$ is a positive parameter.

The $y_{i,j}$’s and $\tilde{y}_{i,j}$’s are later used to define determinants whose value is positive for a small enough value of $\tau$ (see also Lemma 10.2 in the Appendix). The positivity of these determinants is crucial in defining supporting hyperplanes for the Cayley polytopes $\hat{C}_R$ and $C_R$ in Lemmas 7.1 and 7.2 below.

Next, for each $1 \leq i \leq r$, we define $\hat{P}_i := \lim_{\tau \to 0^+} P_i$. Clearly, each $\hat{P}_i$ is a cyclic $(d + r + 1)$-polytope embedded in the $(d + r + 1)$-flat $F_i$ of $\mathbb{R}^d$, where $F_i = \{x_j = 0 | 1 \leq j \leq r$ and $j \neq i\}$. The following lemma establishes the first step towards our construction.

**Lemma 7.1** There exists a sufficiently small positive value $\tau^*$ for $\tau$, such that, for any $\emptyset \subset U \subseteq [r]$, the set of mixed faces $W_R$ of the Cayley polytope of the polytopes $\hat{P}_1, \ldots, \hat{P}_r$ constructed above, has
\[
f_{k-1}(W_R) = \Xi_{k,d}(n_R), \quad |R| \leq k \leq \left\lfloor \frac{d+|R|-1}{2} \right\rfloor.
\]

**Proof** Let $\mathcal{V}_i$ be the set of vertices of $\hat{P}_i$ for $1 \leq i \leq r$ and set $\mathcal{V} := \bigcup_{i=1}^{r} \mathcal{V}_i$. The objective in the proof is, for each $\emptyset \subset U \subseteq [r]$ and each spanning subset $U$ of the partition $U = \bigcup_{i \in R} \mathcal{V}_i$, to exhibit a supporting hyperplane of the $(d + |R| - 1)$-dimensional Cayley polytope $\hat{C}_R$, containing exactly the vertices in $U$. In that respect, our approach is similar in spirit to the proof showing, by defining supporting hyperplanes constructed from Vandermonde determinants, that the cyclic $n$-vertex $d$-polytope $C_d(n)$ is neighborly (see, e.g., [24, Cor. 0.8]).

In our proof we need to involve the parameter $\zeta$ before taking the limit $\zeta \to 0^+$. This is due to the fact that, when $\emptyset \subset U \subseteq [r]$, the information of the relative position of the polytopes $\hat{P}_i, i \in R$, is lost if we set $\zeta = 0$ from the very first step. To describe our construction, we write each spanning subset $U$ of $U$ as the disjoint union of non-empty sets $U_i, i \in R$, where $U_i = U \cap \mathcal{V}_i$ and $|U_i| = \kappa_i \leq n_i$. For this particular $U$, we define the linear equation

\[\cdots\]
\[ H_U(x) = \lim_{\xi \to 0^+} \left( -1 \right) \frac{|R|(|R| - 1)}{2}^{ \sigma(R) \xi |R|-r } D_U(x; \xi), \]  

where \( x = (x_1, x_2, \ldots, x_{d+|R|-1}) \), and \( D_U(x; \xi) \) is the \((d + |R|) \times (d + |R|)\) determinant:

- whose first column is \((1, x)^T\),
- the next \( \kappa_i, i \in R \), pairs of columns are \((1, e_{i-1}, \gamma_i(y_{i,j}; \xi))^T\) and \((1, e_{i-1}, y_i(\gamma_i; \xi))^T\) where \( e_0, \ldots, e_{|R|-1} \) is the standard affine basis of \( \mathbb{R}^{|R|-1} \) \( y_{i,j} \in \{ y \in Y_i | \gamma_i(y); 0 \in U_i \} \), and
- the last \( s := d + |R| - 1 - \sum_{i \in R} \kappa_i \) columns are \((1, e_{|R|-1}, \gamma_{|R|-1}(M_i; \xi))^T\), \( 1 \leq i \leq s \); these columns exist only if \( s > 0 \).

The quantity \( \sigma(R) \) above is a non-negative integer counting the total number of row swaps required to shift, for all \( j \in [r] \setminus R \), the \((|R| + j)\)-th row of \( D_U(x; \xi) \) to the bottom of the determinant, so that the powers of \( y_{i,j} \) in each column are in increasing order (notice that if \( R \equiv [r] \) no such row swaps are required). Moreover, \( \sigma(R) \) depends only on \( R \) and not on the choice of the spanning subset \( U \) of \( U \).

The equation \( H_U(x) = 0 \) is the equation of a hyperplane in \( \mathbb{R}^{d+|R|-1} \) that passes through the points in \( U \). We claim that, for any choice of \( U \), and for all vertices \( u \) in \( \mathcal{V} \setminus U \), we have \( H_U(u) > 0 \). To prove our claim, notice first that, for each \( j \in [r] \setminus R \), the \((|R| + j)\)-th row of the determinant \( D_U(u; \xi) \) will contain the parameters \( y_{i,j}, \gamma_{d-r+1+j} \) multiplied by \( \xi \). After extracting \( \xi \) from each of these rows and shifting them to their proper position (i.e., the position where the powers along each column increase), we will have a term \( \xi^{r-|R|} \) and a sign \((-1)^{\sigma(R)}\) (induced from the \( \sigma(R) \) row swaps required altogether). These terms cancel out with the term \((-1)^{\sigma(R)} \xi^{r-|R|} \) in (7.2). We can, therefore, transform \( H_U(u) \) in the form of the determinant \( D_N(Z; \alpha_1, \ldots, \alpha_m) \), \( Z = \{z_{i,j} \mid 1 \leq i \leq \rho, 1 \leq j \leq v_i \} \), \( N = \{v_1, v_2, \ldots, v_m\}, 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m \), shown below:

\[
D_N(Z; \alpha_1, \ldots, \alpha_m) := (-1)^{\frac{\rho(\rho-1)}{2}} |
\begin{array}{cccccc}
\xi_{1,1} & \cdots & \xi_{1,v_1} & 0 & 0 & 0 \\
0 & \xi_{2,1} & \cdots & \xi_{2,v_2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \xi_{\rho,1} & \cdots & \xi_{\rho,v_{\rho}} \\
0 & 0 & \cdots & 0 & \xi_{\rho+1,1} & \cdots \\
\xi_{1,1} & \cdots & \xi_{1,v_1} & 0 & 0 & 0 \\
0 & \xi_{2,1} & \cdots & \xi_{2,v_2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \xi_{\rho,1} & \cdots & \xi_{\rho,v_{\rho}} \\
\xi_{1,1} & \cdots & \xi_{1,v_1} & 0 & 0 & 0 \\
\end{array}|
\]

by means of the following determinant transformations:

(i) By subtracting rows 2 to \( |R| \) of \( H_U(u) \) from its first row.
(ii) By shifting the first column of $H_U(u)$ to the right, so that all columns of $H_U(u)$ are arranged in increasing order with respect to their parameters $z_{i,j}$. Clearly, this can be done with an even number of column swaps.

The determinant $D_K(Y; \mu_1, \ldots, \mu_m)$ is strictly positive for all $\tau$ between 0 and some value $\hat{\tau}(R, U, u)$, that, depends (only) on the choice of $R$, $U$ and $u$. Since there is a finite number of possible such determinants, the value $\hat{\tau}^* := \min_{R, U, u} \hat{\tau}(R, U, u)$ is necessarily positive. Choosing some $\tau^* \in (0, \hat{\tau}^*)$ makes all these determinants simultaneously positive; this completes our proof. \hfill \Box

The following lemma establishes the second (and last) step of our construction.

Lemma 7.2 There exists a sufficiently small positive value $\zeta^\diamond$ for $\zeta$, such that, for any $\emptyset \subset R \subseteq [r]$, the set $\mathcal{F}_R$ of mixed faces of the Cayley polytope $C_R$ of the polytopes $P_1, \ldots, P_r$ in (7.1) has

$$f_{k-1}(\mathcal{F}_R) = \Xi_{k,d}(n_R), \quad \text{for all } |R| \leq k \leq \lfloor \frac{d+|R|-1}{2} \rfloor.$$

Proof Briefly speaking, the value $\zeta^\diamond$ is determined by replacing the limit $\zeta \to 0^+$ in the previous proof, by a specific value of $\zeta$ for which the determinants we consider are positive.

More precisely, let $\mathcal{U}_i$ be the set of vertices of $P_i$, $1 \leq i \leq r$, and set $\mathcal{U} := \bigcup_{i=1}^r \mathcal{U}_i$. Our goal is, for each $\emptyset \subset R \subseteq [r]$ and each spanning subset $U$ of the partition $U = \bigcup_{i \in R} \mathcal{U}_i$, to exhibit a supporting hyperplane of the Cayley polytope $C_R$, containing exactly the vertices in $U$.

To this end, we define the linear equation $\widetilde{H}_U(x; \zeta) = 0, x = (x_1, x_2, \ldots, x_{d+|R|-1})$, with

$$\widetilde{H}_U(x; \zeta) = (-1)^\frac{|R|(|R|-1)}{2} + \sigma(R)\zeta^{-|R|-r}D_U(x; \zeta), \quad \zeta > 0, \quad (7.3)$$

where $D_U(x; \zeta)$ is the determinant in the proof of Lemma 7.1, where we have set $\tau$ to $\tau^*$. Clearly, for each $u \in \mathcal{U} \setminus U$, we have $\lim_{\zeta \to 0^+} \widetilde{H}_U(u; \zeta) = H_U(u) > 0$. This immediately implies that for each combination of $U$ and $u$ there exists a value $\hat{\zeta}(U, u)$ such that, for all $\zeta \in (0, \hat{\zeta}(U, u))$, $\widetilde{H}_U(u; \zeta) > 0$, which, due to the positivity of $\zeta$, yields that $\zeta r^{-|R|} \widetilde{H}_U(u; \zeta) > 0$. Since the number of possible combinations for $U$ and $u$ is finite, the minimum $\hat{\zeta}^\diamond := \min_{U, u} \hat{\zeta}(U, u)$ is well defined and positive. Taking $\zeta^\diamond$ to be any value in $(0, \hat{\zeta}^\diamond)$, satisfies our demands. \hfill \Box

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Appendix 1: Special Sets Related to the Derivation of the Dehn–Sommerville Equations

The following combinatorial identities are used in the proof of Lemma 3.5.

**Lemma 8.1** For any \( m \geq 1 \), we have:

\[
\sum_{i=0}^{m-1} (i + 1)! S_{m}^{i+1} (t - 1)^{m-i-1} = \sum_{j=0}^{m-1} E_{m}^{j} t^{j}.
\]

(8.1)

and

\[
\sum_{i=0}^{m} i! S_{m+1}^{i+1} (t - 1)^{m-i} = \sum_{j=0}^{m} E_{m}^{j} t^{j+1},
\]

(8.2)

**Proof** The remark following [8, Relat. (6.39)], states that \( \sum_{i=0}^{m} i! S_{m}^{i} (t - 1)^{m-i} = \sum_{j=0}^{m-1} E_{m}^{j} t^{j} \). Making use of this, we have:

\[
\sum_{i=0}^{m-1} (i + 1)! S_{m}^{i+1} (t - 1)^{m-i-1} = \sum_{j=0}^{m} j! S_{m}^{j} (t - 1)^{m-j} = \sum_{j=0}^{m} E_{m}^{j} t^{j}.
\]

Now, using the recurrence relation \( S_{m}^{i} = S_{m-1}^{i-1} + i S_{m-1}^{i} \) of the Stirling numbers, we have:

\[
\sum_{i=0}^{m} i! S_{m+1}^{i+1} (t - 1)^{m-i} = \sum_{i=0}^{m} i! S_{m}^{i} (t - 1)^{m-i} + \sum_{i=0}^{m} (i + 1)! S_{m}^{i+1} (t - 1)^{m-i} = \sum_{j=0}^{m-1} E_{m}^{j} t^{j} + (t - 1) \sum_{j=0}^{m-1} E_{m}^{j} t^{j} = \sum_{j=0}^{m-1} E_{m}^{j} t^{j+1}.
\]

\( \square \)

Appendix 2: Relations Appearing in the Derivation of the Recurrence Relation for the \( h \)-Vector of \( \mathcal{F}_{R} \)

Recall that \( \mathcal{D}(R, T, X, \ell) \) denotes the cardinality of the set:

\[
\mathcal{D}(R, T, X, \ell) := \{(S_{1}, \ldots, S_{\ell}) : X \subseteq S_{1} \subseteq S_{2} \subseteq \cdots \subseteq S_{\ell} \subseteq R \\
\text{and } S_{i} = T \text{ for some } 1 \leq i \leq \ell \}.
\]
The following lemma expresses the sum over all \( T \) with \( X \subseteq T \subset R \), of the cardinalities \( D(R, T, X, \ell) \), in terms of the Stirling numbers of the second kind.

**Lemma 9.1** For any \( \ell \in \mathbb{N} \), and \( X, R \) with \( \emptyset \subseteq X \subset R \), we have:

\[
\sum_{X \subseteq T \subset R} D(R, T, X, \ell) = \ell \ell! S_{|R|-|X|+1}^{\ell+1} + 1.
\]

**Proof** The left-hand side of (9.1) is the cardinality of the set:

\[
\mathcal{Y} = \{(S_1, \ldots, S_\ell) : X \subseteq T \subset R, X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \leq i \leq \ell\},
\]

which is nothing but \( \ell \) copies of the set:

\[
\mathcal{Z} = \{(S_1, S_2, \ldots, S_\ell) | X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\}.
\]

Indeed,

\[
\mathcal{Y} = \{(S_1, \ldots, S_\ell) : X \subseteq T \subset R, X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \leq i \leq \ell\}
\]

\[
= \{i : 1 \leq i \leq \ell\}
\]

\[
\times \{(S_1, \ldots, S_\ell) : X \subseteq T \subset R, X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T\}
\]

\[
= \{i : 1 \leq i \leq \ell\} \times \{(S_1, \ldots, S_\ell) : X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\}
\]

\[
= \{i : 1 \leq i \leq \ell\} \times \mathcal{Z}.
\]

As we have shown in the second part of the proof of Lemma 3.3, the cardinality of the set \( \mathcal{Z} \) is \( \ell! S_{|R|-|X|+1}^{\ell+1} + 1 \) and this completes our proof. \( \square \)

**Appendix 3: Determinants Used in the Tightness Construction**

**Definition 10.1** Let \( Y_i = \{y_{i,1}, \ldots, y_{i,\kappa_i}\} \), \( 1 \leq i \leq n \), be non-empty disjoint sets of real numbers. Set \( K := \kappa_1 + \kappa_2 + \cdots + \kappa_n \), \( m := K - 2n - 2 \) and let \( \mu_1 < \mu_2 < \cdots < \mu_m \) be non-negative integers. We denote by \( Y \) the partition \( Y_1 \cup \cdots \cup Y_n \) and we define the \( K \times K \) matrix \( \Delta_K(Y; \mu_1, \ldots, \mu_m) \) as follows:
We denote by $D_K(Y; \mu_1, \ldots, \mu_m)$ the signed determinant $(-1)^{\frac{n(n-1)}{2}} |\Delta_K(Y; \mu_1, \ldots, \mu_m)|$.

We, now, parameterize all $y_{i,j}$’s as follows: for each $1 \leq i \leq n$ we choose arbitrary real numbers $0 < x_{i,1} < x_{i,2} < \cdots < x_{i,K_i}$ and non-negative integers $0 \leq \beta_n < \beta_{n-1} < \cdots < \beta_1$. Then, we set $y_{i,j} := x_{i,j} \tau^{\beta_j}$ where $\tau$ is a positive parameter, and consider $D_K(Y; \mu_1, \ldots, \mu_m)$ as a polynomial in $\tau$. In the next lemma we essentially show that, for sufficiently small $\tau$, the determinant $D_K(Y; \mu_1, \ldots, \mu_m)$ is strictly positive.

**Lemma 10.2** If the elements of the sets $Y_i$, $1 \leq i \leq r$, are parameterized as above and $Y = Y_1 \cup \cdots \cup Y_n$, then $D_K(Y; \mu_1, \ldots, \mu_m) = A\tau^\alpha + O(\tau^{\alpha+1})$, where $A > 0$ and $\alpha$ is a positive integer.

**Proof** To prove our claim we use the Binet–Cauchy theorem [21]. To apply the Binet–Cauchy theorem in our case, notice that the matrix $\Delta_K(Y; \mu_1, \ldots, \mu_m)$ can be factorized into a product of a $K \times n(m+1)$ matrix $L$ and an $n(m+1) \times K$ matrix $R$ as shown in Figs. 4 and 5. Let $J$ be a subset of $\{1, 2, \ldots, n(m+1)\}$ of size $K$. We denote by $L_{[K],J}$ the $K \times K$ matrix whose columns are the columns of $L$ at indices from $J$ and by $R_{J,[K]}$ the $K \times K$ matrix whose rows are the rows of $R$ at indices from $J$. The Binet–Cauchy theorem states that:

$$\det(LR) = \sum_J \det(L_{[K],J}) \det(R_{J,[K]}),$$

(10.1)

where the sum is taken over all subsets $J$ of $\{1, 2, \ldots, n(m+1)\}$ of size $K$.

Recall that $y_{i,j} = x_{i,j} \tau^{\beta_j}$. It is not hard to see that, for each $J \subseteq \{1, \ldots, n(m+1)\}$ with $|J| = K$, the sub-matrix $L_{[K],J}$ is independent of $\tau$ while $R_{J,[K]}$ is a block-diagonal matrix whose blocks are generalized Vandermonde determinants (cf. [7]) from which we can extract powers of $\tau$. More precisely, we set $k^{(i)} := k + (i-1)(m+1)$ and we write each index set $J$ of the Binet–Cauchy expansion as $J_1 \cup J_2 \cup \cdots \cup J_n$, where $J_i \subseteq \{1^{(i)}, \ldots, (m+1)^{(i)}\}$. We then have:
The matrix $L$ from the factorization $\Delta K(\mathcal{Y}; \mu_1, \ldots, \mu_m) = LR$. The numbers over and sideways of $L$ indicate the column and row numbers, respectively.

$$
\det(R_{J,[K]}) = \tau^{\alpha(J)} \prod_{i=1}^{n} GVD(J_i), \quad (10.2)
$$

where

$$
\alpha(J) = \beta_1 \sum_{j^{(1)} \in J_1} \mu_j + \beta_2 \sum_{j^{(2)} \in J_2} \mu_j + \cdots + \beta_n \sum_{j^{(n)} \in J_n} \mu_j,
$$

and $GVD(J_i)$ is a positive generalized Vandermonde determinant,\(^5\) independent of $\tau$, depending on the $x_i,j$'s with $j^{(i)} \in J_i$. Thus, combining (10.1) and (10.2), we deduce that $D_K(\mathcal{Y}; \mu_1, \ldots, \mu_m)$ is a polynomial in $\tau$. To prove our claim it suffices to find the subset $J$ for which $\alpha(J)$ is minimal and, for this $J$, evaluate the sign of the coefficient of $\tau^{\alpha(J)}$.

Notice that a term $\det(L_{[K],J}) \det(R_{J,[K]})$ in the Binet–Cauchy expansion of $D_K(\mathcal{Y}; \mu_1, \mu_2, \ldots, \mu_m)$ vanishes in the following two cases:

(i) if $k^{(i)}, k^{(j)} \in J$ for some $3 \leq k \leq m+1$; in this case the $k^{(i)}$-th and $k^{(j)}$-th columns of $L_{[K],J}$ are identical, and thus $\det(L_{[K],J}) = 0$.

\(^5\) It is a well-known fact that, if the parameters in the columns of the generalized Vandermonde determinant are in strictly increasing order, then the Vandermonde determinant is itself strictly positive (see [7] for a proof of this fact).
(ii) if $|J_i| \neq k_i$ for at least some $1 \leq i \leq n$; in this case $R_{J_i[K]}$ is a block-diagonal square matrix with non-square non-zero blocks. The determinant of such a matrix is always zero.\footnote{To see this, consider the Laplace expansion of the matrix with respect to the columns of its top-left block.}

Among all possible index sets $J = J_1 \cup \cdots \cup J_n$ for which the product $\det(L_{[K],[J]}) \det(R_{J,[K]})$ does not vanish, we have to find the one for which the exponent $\alpha(J)$ in (10.2) is the minimum possible. To do this, we combine condition (i) above with the fact that $\beta_1 > \cdots > \beta_n$ and we deduce that the minimum exponent for $\alpha(J)$ is attained if, for all $1 \leq i \leq r$:

- $1^{(i)}, 2^{(i)} \in J_i$, and
- $\kappa^{(i)} \in J_i$ and $\lambda^{(i+1)} \in J_{i+1}$ for some $\kappa, \lambda > 2$, implies $\kappa < \lambda$.

Moreover, since from condition (ii) we have $|J_i| = k_i$, we conclude that:

- $J_1 = J_1^* := \{1^{(1)}, 2^{(1)}, 3^{(1)}, \ldots, k_1^{(1)}\} = \{1, \ldots, k_1\}$,
- $J_2 = J_2^* := \{1^{(2)}, 2^{(2)}, (k_1 + 1)^{(2)}, \ldots, (k_1 + k_2 - 2)^{(2)}\}$,
- $J_3 = J_3^* := \{1^{(3)}, 2^{(3)}, (k_1 + k_2 - 1)^{(3)}, \ldots, (k_1 + k_2 + k_3 - 4)^{(3)}\}$
- etc.
For the above choice of \( J^* = J_1^* \cup \cdots \cup J_n^* \), the matrix \( L_{[K], J^*} \) is:

\[
L_{[K], J^*} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]

Thus, in order to find the sign of our original determinant, we have to evaluate \( \det(L_{[K], J^*}) \). To do this, we perform the appropriate row and column swaps so that \( L_{[K], J^*} \) becomes the identity matrix. More precisely,

- we perform \( n - 1 + (n - 2) + (n - 3) + \cdots + 1 = \frac{n(n-1)}{2} \) row swaps so that, for all \( 1 \leq i \leq n \), row \( n + i \) is shifted upwards and paired with row \( i \), to become a \( 2 \times 2 \) identity matrix,
- we then perform an even number of column swaps to shift each \( I_{k_i-2} \) to its “proper” position to the right.

We therefore conclude that the sign of the dominant term of the expansion of the determinant of the matrix \( \Delta_K(Y; \mu_1, \ldots, \mu_m) \) as a polynomial in \( \tau \), is \((-1)^{\frac{n(n-1)}{2}}\). This completes our proof. \( \square \)

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