Tight uniform continuity bound for a family of entropies

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Abstract

We prove a tight uniform continuity bound for a family of entropies which includes the von Neumann entropy, the Tsallis entropy and the \(\alpha\)-Rényi entropy, \(S_\alpha\), for \(\alpha \in (0, 1)\). We establish necessary and sufficient conditions for equality in the continuity bound and prove that these conditions are the same for every member of the family. Our result builds on recent work in which we constructed a state which was majorized by every state in a neighbourhood (\(\varepsilon\)-ball) of a given state, and thus was the minimal state in majorization order in the \(\varepsilon\)-ball. This minimal state satisfies a particular semigroup property, which we exploit to prove our bound.

1 Introduction

Entropies play a fundamental role in quantum information theory as characterizations of the optimal rates of information theoretic tasks, and as measures of uncertainty. The mathematical properties of entropic functions therefore have important physical implications. The von Neumann entropy \(S\), for instance, as a function of \(d\)-dimensional quantum states, is strictly concave, continuous, and is bounded by \(\log d\). As the von Neumann entropy characterizes the optimal rate of data compression for a memoryless quantum information source [Sch96], continuity of the von Neumann entropy, for example, implies that the quantum data compression limit is continuous in the source state. The \(\alpha\)-Rényi entropies \(S_\alpha\) are parametrized by \(\alpha \in (0, 1) \cup (1, \infty)\), and are a generalization of the von Neumann entropy in the sense that \(\lim_{\alpha \to 1} S_\alpha = S\). The \(\alpha\)-Rényi entropy has been used to bound the quantum communication complexity of distributed information-theoretic tasks [vH02], can be interpreted in terms of the free energy of a quantum or classical system [Bae11], and is the fundamental quantity defining the entanglement \(\alpha\)-Rényi entropy [Wan+16].

In fact, the \(\alpha\)-Rényi entropies are members of a large family of entropies called the \((h, \phi)\)-entropies, which are parametrized by two functions \(h, \phi\) on \(\mathbb{R}\) subject to certain constraints (see Section 2). This family includes the Tsallis entropies [Tsa88] and the unified entropies (considered by Rastegin in [Ras11]). Note that the \((h, \phi)\)-entropy of a quantum state is the classical \((h, \phi)\)-entropy of its eigenvalues, and therefore the results here apply equally well to probability distributions on finite sets.

Continuity is a useful property of entropic functions, particularly when cast in the form of a uniform continuity bound: given two \(d\)-dimensional states which are at a trace distance of at

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most $\varepsilon \in (0,1)$, this provides a bound on their entropy difference entirely in terms of $\varepsilon$ and $d$. Fannes first proved a uniform continuity bound for the von Neumann entropy [Fan73]. This bound was improved to a tight form by Audenaert [Aud07] and is often called the called the Audenaert-Fannes bound (see also [Pet05] Theorem 3.8). Rastegin proved similar continuity bounds for the unified entropies, which include the $\alpha$-Rényi entropies and Tsallis entropies, but the resulting bounds are not known to be tight [Ras11]. Recently, Chen et al proved continuity bounds for the $\alpha$-Rényi entropy for $\alpha \in (0,1) \cup (1,\infty)$ using techniques similar to Audenaert’s proof of the Audenaert-Fannes bound [Che+17], but the resulting bounds are known to be not tight [MF17].

In [HD17], we considered local continuity bounds. Given a $d$-dimensional quantum state $\sigma$, a local continuity bound of an entropic function $H$ at $\sigma$ is a bound on the entropy difference $|H(\omega) - H(\sigma)|$ for any $\omega$ in an $\varepsilon$-ball around $\sigma$, which depends not only on $\varepsilon$ and $d$ but also on the state $\sigma$ itself. These local bounds hence incorporate additional information about the state $\sigma$, for example, its spectrum, to yield a bound which is tighter than a uniform continuity bound.

By finding maximizers and minimizers of the majorization order on $d$-dimensional quantum states over the $\varepsilon$-ball around $\sigma$, local bounds were obtained for any $(h,\phi)$-entropy, in fact, for any Schur concave entropic function in [HD17].

Given a quantum state $\sigma$ and $\varepsilon \in (0,1]$, we denote the $\varepsilon$-ball in trace distance around $\sigma$ by $B_\varepsilon(\sigma)$ (defined by eq. (1) below). For a given $\sigma$ and $\varepsilon$, there exist two quantum states $\sigma_\varepsilon, \sigma_{\varepsilon,\varepsilon} \in B_\varepsilon(\sigma)$ such that for any $\omega \in B_\varepsilon(\sigma)$ centered at $\sigma$,

$$\sigma_\varepsilon \prec \omega \prec \sigma_{\varepsilon,\varepsilon}$$

where $\prec$ denotes the majorization order (defined in Section 2). In [HD17], this fact was proved by explicit construction of these states, using the notation $\rho_\varepsilon(\sigma)$ for $\sigma_\varepsilon$ and $\rho_{\varepsilon,\varepsilon}(\sigma)$ for $\sigma_{\varepsilon,\varepsilon}$. These states were also independently found by Horodecki, Oppenheim, and Sparaciari [HOS17], and considered in the context of thermal majorization [Mee16; vNW17]. In [HD17] we also established that the minimal state $\rho_\varepsilon(\sigma) \equiv \sigma^\ast$ in the majorization order, satisfied a semigroup property: $\rho_{\varepsilon_1 + \varepsilon_2}(\sigma) = \rho_{\varepsilon_1}(\rho_{\varepsilon_2}(\sigma))$. This property plays a key role in the proof of the main results of this paper.

In Section 2 we introduce the basic notation and definitions and in Section 3 we state our main results. The proof strategy is described in Section 4 and in Section 5 the construction of the minimal state (in the majorization order), $\sigma^\ast$, which we use in our proof, is formulated. Section 6 consists of a proof of the main technical result Theorem 4.1 and employs certain lemmas which are proved in Section 7. In Appendix A we recall an elementary property of concave functions.

2 Notation and definitions

Let $\mathcal{H}$ denote a finite-dimensional Hilbert space, with dim $\mathcal{H} = d$, $\mathcal{B}(\mathcal{H})$ the set of (bounded) linear operators on $\mathcal{H}$, and $\mathcal{B}_{sa}(\mathcal{H})$ the set of self-adjoint linear operators on $\mathcal{H}$. A quantum state (or density matrix) is a positive semidefinite element of $\mathcal{B}(\mathcal{H})$ with trace one. Let $\mathcal{D}(\mathcal{H})$ be the set quantum states on $\mathcal{H}$. We denote the completely mixed state by $\tau := \frac{1}{d}$. A pure state is a rank-1 density matrix; we denote the set of pure states by $\mathcal{D}_{pure}(\mathcal{H})$. For two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, the trace distance between them is given by

$$T(\rho, \sigma) = \frac{1}{2} ||\rho - \sigma||_1.$$  

We define the $\varepsilon$-ball around $\sigma \in \mathcal{D}(\mathcal{H})$ as the set

$$B_\varepsilon(\sigma) = \{ \omega \in \mathcal{D}(\mathcal{H}) : T(\omega, \sigma) \leq \varepsilon \}. \quad (1)$$
For any $A \in B_{sa}(\mathcal{H})$, let $\lambda_+(A)$ and $\lambda_-(A)$ denote the maximum and minimum eigenvalue of $A$, respectively, and $k_+(A)$ and $k_-(A)$ denote their multiplicities. Let $\lambda_j(A)$ denote the $j$th largest eigenvalue, counting multiplicity; that is, the $j$th element of the ordering

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_d(A).$$

We set $\tilde{\lambda}(A) := (\lambda_i(A))_{i=1}^d \in \mathbb{R}^d$ and denote the set of eigenvalues of $A \in B_{sa}(\mathcal{H})$ by \text{spec} $A \subset \mathbb{R}$.

Given $x \in \mathbb{R}^d$, write $x^{\downarrow} = (x_j^{\downarrow})_{j=1}^d$ for the permutation of $x$ such that $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_d^{\downarrow}$.

For $x, y \in \mathbb{R}^d$, we say $x$ majorizes $y$, written $x \succ y$, if

$$\sum_{j=1}^k x_j^{\downarrow} \geq \sum_{j=1}^k y_j^{\downarrow} \quad \forall k = 1, \ldots, d-1, \quad \text{and} \quad \sum_{j=1}^d x_j^{\downarrow} = \sum_{j=1}^d y_j^{\downarrow}. \tag{2}$$

Given two states $\rho, \sigma \in \mathcal{D}$, we say $\sigma$ majorizes $\rho$, written $\rho \prec \sigma$ if $\tilde{\lambda}(\rho) \prec \tilde{\lambda}(\sigma)$. We say that $\varphi : \mathcal{D} \to \mathbb{R}$ is Schur convex if $\varphi(\rho) \leq \varphi(\sigma)$ for any $\rho, \sigma \in \mathcal{D}$ with $\rho \prec \sigma$. If $\varphi(\rho) < \varphi(\sigma)$ for any $\rho, \sigma \in \mathcal{D}$ such that $\rho \prec \sigma$, and $\rho$ is not unitarily equivalent to $\sigma$, then $\varphi$ is strictly Schur convex. We say $\varphi$ is Schur concave (resp. strictly Schur concave) if $(-\varphi)$ is Schur convex (resp. strictly Schur convex).

Let $h : \mathbb{R} \to \mathbb{R}$ and $\phi : [0,1] \to \mathbb{R}$ with $\phi(0) = 0$ and $\phi(1) = 0$, such that either $h$ is strictly increasing and $\phi$ strictly concave, or $h$ strictly decreasing and $\phi$ strictly convex. Then the $(h, \phi)$-entropy, $H_{(h,\phi)}$, is defined by

$$H_{(h,\phi)}(\rho) := h(\text{Tr}[\phi(\rho)]) \tag{3}$$

where $\phi$ is defined on $\mathcal{D}(\mathcal{H})$ by functional calculus, i.e. given the eigen-decomposition $\rho = \sum_i \lambda_i(\rho) \pi_i$, we have $\phi(\rho) = \sum_i \phi(\lambda_i(\rho)) \pi_i$. Every $(h, \phi)$-entropy is strictly Schur concave and unitarily invariant; moreover, if $h$ is concave, then $H_{(h,\phi)}$ is concave \cite{2016arXiv160908592B}. Here, we are most interested in the following three examples of $(h, \phi)$ entropies:

- The von Neumann entropy

  $$S(\rho) = -\text{Tr}(\rho \log \rho).$$

  $S$ is the $(h, \phi)$ entropy with $h = \text{id}$, i.e., $h(x) = x$ for $x \in \mathbb{R}$, and with $\phi(x) = -x \log x$ for $x \in [0,1]$. The von Neumann entropy satisfies the following tight continuity bound known as the Audenaert-Fannes bound \cite{auden} (see also \cite[Theorem 3.8]{pet}). Given $\varepsilon \in (0,1]$ and $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ with $T(\rho, \sigma) \leq \varepsilon$,

  $$|S(\rho) - S(\sigma)| \leq \begin{cases} 
  \varepsilon \log(d-1) + h(\varepsilon) & \text{if } \varepsilon < 1 - \frac{1}{d}, \\
  \log d & \text{if } \varepsilon \geq 1 - \frac{1}{d} 
  \end{cases} \tag{4}$$

  where $h(\varepsilon) := -\varepsilon \log \varepsilon - (1-\varepsilon) \log(1-\varepsilon)$ denotes the binary entropy.

- The $q$-Tsallis entropy for $q \in (0,1) \cup (1, \infty)$,

  $$T_q(\rho) = \frac{1}{1-q} [\text{Tr}(\rho^q) - 1].$$

  $T_q$ can be written as the $(h, \phi)$-entropy with $h(x) = x - \frac{1}{1-q} x^q$ for $x \in \mathbb{R}$ and $\phi(x) = \frac{1-q}{q} x^q$. With these choices, $h$ is strictly increasing and affine (and therefore concave) and $\phi$ is strictly concave, for all $q \in (0,1) \cup (1, \infty)$.
The $\alpha$-Rényi entropy for $\alpha \in (0, 1) \cup (1, \infty)$, 

\[ S_\alpha(\rho) = \frac{1}{1-\alpha} \log (\text{Tr} \rho^\alpha). \]

$S_\alpha$ is the $(h, \phi)$-entropy with $h(x) = \frac{1}{1-\alpha} \log x$ for $x \in \mathbb{R}$ and $\phi(x) = x^\alpha$ for $x \in [0, 1]$. For $\alpha \in (0, 1)$, $h$ is concave and strictly increasing and $\phi$ is strictly concave. For $\alpha > 1$, $h$ is convex and strictly decreasing, and $\phi$ is strictly convex. It is known that $\lim_{\alpha \to 1} S_\alpha(\rho) = S(\rho)$.

In the above, all logarithms are taken to base 2.

3 Main results

**Theorem 3.1 (Uniform continuity bounds).** Let $H_{(h, \phi)}$ be an $(h, \phi)$-entropy, defined through (3) with $h$ concave and $\phi$ strictly concave. For $\varepsilon \in (0, 1]$ and any states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, we have

\[ |H_{(h, \phi)}(\rho) - H_{(h, \phi)}(\sigma)| \leq \begin{cases} h(\phi(1 - \varepsilon) + (d - 1)\phi(\varepsilon^{d-1})) & \varepsilon < 1 - \frac{1}{d} \\ h(d\phi(\frac{1}{d})) & \varepsilon \geq 1 - \frac{1}{d} \end{cases} \tag{5} \]

and in particular, for $\alpha \in (0, 1)$,

\[ |S_\alpha(\rho) - S_\alpha(\sigma)| \leq \begin{cases} \frac{1}{1-\alpha} \log((1 - \varepsilon)^\alpha + (d - 1)^{1-\alpha} \varepsilon^\alpha) & \varepsilon < 1 - \frac{1}{d} \\ \log d & \varepsilon \geq 1 - \frac{1}{d} \end{cases} \tag{6} \]

and for $q \in (0, 1) \cup (1, \infty)$,

\[ |T_q(\rho) - T_q(\sigma)| \leq \begin{cases} \frac{1}{d^{1-q} - 1} (1 - \varepsilon)^q + (d - 1)^{1-q} \varepsilon^q - 1 & \varepsilon < 1 - \frac{1}{d} \\ \frac{1}{d^{1-q} - 1} & \varepsilon \geq 1 - \frac{1}{d} \end{cases} \tag{7} \]

where $d = \dim \mathcal{H}$. Moreover, equality in (5), (6), or (7) occurs if and only if one of the two states (say, $\sigma$) is pure, and either

1. $\varepsilon < 1 - \frac{1}{d}$ and $\tilde{\lambda}(\rho) = (1 - \varepsilon, \varepsilon^{\frac{d}{d-1}}, \ldots, \varepsilon^{\frac{d}{d-1}})$, or
2. $\varepsilon \geq 1 - \frac{1}{d}$, and $\rho = \tau := \frac{1}{d}$.

**Remark.**

- When (5) is applied to the von Neumann entropy $S$, one recovers the Audenaert-Fannes bound (4), with equality conditions. The sufficiency of these equality conditions were shown in [Aud07], and their necessity was recently derived in [HD17] by an analysis of the proof of the bound presented in [Pet08] Thm. 3.8 and [Win16], which involves a coupling argument. We establish that these necessary and sufficient conditions are the same for every $(h, \phi)$-entropy satisfying the conditions of the theorem.
- The inequality (6) reduces to the Audenaert-Fannes bound (4) when the limit $\alpha \to 1$ is taken on both sides of it.
- The bound (7) appeared in [Che+17] as Lemma 1.2, and was derived with a different method. However, the equality conditions were not established.
- See Figure 1 for a comparison of our uniform continuity bound for the $\alpha$-Rényi entropy, (6), for $\alpha = \frac{1}{2}$, with those obtained in [Ras11] and [Che+17].

\[ \diamond \]
4 Proof strategy

Given a state $\sigma \in \mathcal{D}(\mathcal{H})$ and $\varepsilon \in (0, 1]$, one can construct two states $\sigma^*_\varepsilon, \sigma_{*\varepsilon} \in B_\varepsilon(\sigma)$ such that

$$\sigma^*_\varepsilon \prec \omega \prec \sigma_{*\varepsilon}$$

for any $\omega \in B_\varepsilon(\sigma)$. This was done in [HD17], with the notation $\rho^*_\varepsilon(\sigma)$ (resp. $\rho_{*\varepsilon}(\sigma)$) to denote $\sigma^*_\varepsilon$ (resp. $\sigma_{*\varepsilon}$). These states were also independently found in [HOS17], and considered in the context of thermal majorization in [Mee16; vNW17]. The proof of our main result relies on the form of $\sigma^*_\varepsilon$ and its properties. An explicit construction of $\sigma^*_\varepsilon$ is given in Section 5, and its properties are described in Proposition 5.1.

Consider an $(h, \phi)$ entropy $H_{(h,\phi)}$, and let $\varepsilon \in (0, 1]$, and $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ with $T(\rho, \sigma) \leq \varepsilon$. If $H_{(h,\phi)}(\rho) \geq H_{(h,\phi)}(\sigma)$, then since $\rho \in B_\varepsilon(\sigma)$,

$$|H(\rho) - H(\sigma)| = H(\rho) - H(\sigma) \leq \max_{\omega \in B_\varepsilon(\sigma)} H(\omega) - H(\sigma) = H(\sigma^*_\varepsilon) - H(\sigma)$$

where the last equality follows from the first majorization relation in eq. (8) and the strict Schur concavity of $H_{(h,\phi)}$. Similarly, if $H_{(h,\phi)}(\sigma) \geq H_{(h,\phi)}(\rho)$, eq. (9) holds with $\sigma$ (resp. $\sigma^*_\varepsilon$) replaced by $\rho$ (resp. $\rho^*_\varepsilon$). Hence, in general,

$$|H_{(h,\phi)}(\rho) - H_{(h,\phi)}(\sigma)| \leq \max\{\Delta_\varepsilon(\rho), \Delta_\varepsilon(\sigma)\} \leq \max_{\omega \in \mathcal{D}(\mathcal{H})} \Delta_\varepsilon(\omega),$$

where

$$\Delta_\varepsilon : \mathcal{D}(\mathcal{H}) \to \mathbb{R}_{\geq 0}, \quad \omega \mapsto H_{(h,\phi)} \circ M(\omega) - H_{(h,\phi)}(\omega),$$
and $\mathcal{M}_\varepsilon$ is the majorization-minimizer map,

$$\mathcal{M}_\varepsilon : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$$

$$\omega \mapsto \omega^*_\varepsilon.$$

This map is defined explicitly by eq. (15) in Section 5. Note that $\Delta_\varepsilon(\omega) > 0$ for $\omega \in \mathcal{D}(\mathcal{H})$ follows from the Schur concavity of the $(h, \phi)$-entropy. To prove Theorem 4.1, it remains to maximize $\Delta_\varepsilon$ over $\mathcal{D}(\mathcal{H})$.

We show that for $(h, \phi)$-entropies for which $h$ is concave and $\phi$ (strictly) convex, $\Delta_\varepsilon$ is a Schur convex function on $\mathcal{D}(\mathcal{H})$, which is our main technical result. We defer its proof to Section 6.

**Theorem 4.1.** Assume $h$ is concave and $\phi$ is strictly concave. Let $\varepsilon \in (0, 1]$. Then $\Delta_\varepsilon : \mathcal{D}(\mathcal{H}) \rightarrow \mathbb{R}_{\geq 0}$ is Schur convex. That is, if $\rho < \sigma$,

$$\Delta_\varepsilon(\rho) \leq \Delta_\varepsilon(\sigma).$$

Moreover, $\Delta_\varepsilon(\rho) = \Delta_\varepsilon(\sigma)$ implies $\lambda_+(\rho) = \lambda_+(\sigma)$. Lastly, if $h$ is strictly concave, then $\Delta_\varepsilon$ is strictly Schur convex.

Note that if $h$ is not strictly concave, $\Delta_\varepsilon$ need not be strictly Schur convex. In fact, for the von Neumann entropy we can find a counterexample to strict Schur convexity of $\Delta_\varepsilon$. Setting $\rho = \text{diag}(0.1, 0.2, 0.2, 0.5)$ and $\sigma = \text{diag}(0.1, 0.15, 0.25, 0.5)$ yields $\rho < \sigma$ and that $\rho$ and $\sigma$ are not unitarily equivalent. However, for $\varepsilon \leq 0.05$, we have $\Delta_\varepsilon(\rho) = \Delta_\varepsilon(\sigma)$.

**Corollary 4.2.** If $h$ is concave, $\phi$ strictly concave, and $\varepsilon \in (0, 1]$, then $\Delta_\varepsilon$ achieves a maximum on $\mathcal{D}(\mathcal{H})$, and moreover $\arg\max \Delta_\varepsilon = \mathcal{D}_{\text{pure}}(\mathcal{H})$.

**Proof.** Since any pure state $\psi$ satisfies $\rho < \psi$ for every $\rho \in \mathcal{D}(\mathcal{H})$, we have $\Delta_\varepsilon(\psi) \geq \Delta_\varepsilon(\rho)$ for every $\rho \in \mathcal{D}(\mathcal{H})$. Therefore, $\mathcal{D}_{\text{pure}}(\mathcal{H}) \subset \arg\max \Delta_\varepsilon$. On the other hand, if $\omega \in \mathcal{D}(\mathcal{H})$ has $\omega \in \arg\max \Delta_\varepsilon$, then

$$\Delta_\varepsilon(\omega) = \Delta_\varepsilon(\psi)$$

for a pure state $\psi$. Therefore, $\lambda_+(\omega) = \lambda_+(\psi) = 1$, and $\omega$ must be a pure state. $\square$

Using these results, the proof of Theorem 3.1 is completed as follows. Let $\psi$ be any pure state, $\psi \in \mathcal{D}_{\text{pure}}(\mathcal{H})$. Then for any $\omega \in \mathcal{D}(\mathcal{H})$, we have $\omega < \psi$. Therefore, by Theorem 4.1 we have $\Delta_\varepsilon(\omega) \leq \Delta_\varepsilon(\psi)$, for any $\omega \in \mathcal{D}(\mathcal{H})$, and in particular for $\omega \in \{\rho, \sigma\}$. Therefore, by (10) we have

$$|H_{(h, \phi)}(\rho) - H_{(h, \phi)}(\sigma)| \leq \Delta_\varepsilon(\psi).$$

By computing $\Delta_\varepsilon(\psi)$ using the form given in Proposition 5.1(g) we obtain the right-hand side of eq. (5).

It remains to check under which conditions equality occurs in (5). Assume without loss of generality that $H_{(h, \phi)}(\rho) \geq H_{(h, \phi)}(\sigma)$. Equality in (10) is equivalent to $\sigma \in \mathcal{D}_{\text{pure}}(\mathcal{H})$ by Corollary 4.2. Next, since the $(h, \phi)$-entropy is strictly Schur concave and $\sigma^*_\varepsilon < \rho$, equality in (9) is equivalent to the fact that $\rho$ is unitarily equivalent to $\sigma^*_\varepsilon$. The expression for $\sigma^*_\varepsilon$ when $\sigma \in \mathcal{D}_{\text{pure}}(\mathcal{H})$ is given in Proposition 5.1(g). This completes the proof. $\square$

Theorem 4.1 does not extend to the $\alpha$-Rényi entropy for $\alpha > 1$, in which case $h$ is convex and $\phi$ strictly convex. This is discussed in the remark following Lemma 6.1 and is illustrated in Figure 2.
Figure 2: In dimensions $d = 3$, we parametrize $\sigma = \text{diag}(x, y, 1 - x - y)$, and plot $(x, y) \mapsto \Delta_\varepsilon(\sigma)$ for $\varepsilon = 0.1$, with $H(\rho, \phi) = S_\alpha$, the Rényi entropy. That is, above each $(x, y)$ in the $xy$-plane, the value of $\Delta_\varepsilon(\text{diag}(x, y, 1 - x - y))$ is plotted. Top row, left: $\alpha = \frac{1}{2}$, right: $\alpha = 1$. Bottom row, left: $\alpha = 1.5$, right: $\alpha = 2$. The three points $(0, 0), (0, 1), (1, 0)$ in the $xy$-plane correspond to the pure states $\text{diag}(0, 0, 1), \text{diag}(0, 1, 0), \text{diag}(1, 0, 0)$, respectively. The central point $(\frac{1}{3}, \frac{1}{3})$ corresponds to the completely mixed state $\tau = \frac{1}{3} \mathbb{1}$. We observe for $\alpha = \frac{1}{2}$ and $\alpha = 1$ the maximum of $\Delta_\varepsilon$ appears to occur at the pure states. On the other hand, for $\alpha = 1.5$, the maximum is along the boundary (i.e. for a state $\sigma$ with exactly one zero eigenvalue), and for $\alpha = 2$, the maximum occurs at states without any zero eigenvalues.

5 The majorization-minimizer map $\mathcal{M}_\varepsilon$

In order to prove Theorem 3.1, we need to use properties of the majorization-minimizer map $\mathcal{M}_\varepsilon$ introduced in (12). Let $\sigma \in \mathcal{D}(\mathcal{H})$ and $\varepsilon \in (0, 1]$. We formulate the definition of $\mathcal{M}_\varepsilon$ by constructing $\sigma^*_\varepsilon$. Note that the following is a reformulation of Lemma 4.1 of [HD17]. For notational simplicity, we often suppress dependence on $\sigma$ and $\varepsilon$ in this section, and write $\lambda_j = \lambda_j(\sigma)$ so that the eigenvalues of $\sigma$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$.

We first define a quantity $\gamma_+^{(m)} \equiv \gamma_+^{(m)}(\sigma, \varepsilon)$, for $m \in \{0, 1, \ldots, d - 1\}$, as follows

$$\gamma_+^{(m)} := \begin{cases} \frac{1}{m} \left( \sum_{i=1}^{m} \lambda_i - \varepsilon \right) & \text{if } T(\sigma, \tau) > \varepsilon \text{ and } m \neq 0 \\ \frac{1}{d} & \text{else.} \end{cases}$$

Similarly, a quantity $\gamma_-^{(m)} \equiv \gamma_-^{(m)}(\sigma, \varepsilon)$ is defined by

$$\gamma_-^{(m)} := \begin{cases} \frac{1}{d} \left( \sum_{i=d-m+1}^{d} \lambda_i + \varepsilon \right) & \text{if } T(\sigma, \tau) > \varepsilon \text{ and } m \neq 0 \\ \frac{1}{d} & \text{else.} \end{cases}$$
Then for $\sigma \neq \tau$, we define $m_+ = m_+(\sigma, \varepsilon)$ as the unique solution to the following inequalities:

$$
\lambda_{m+1} \leq \gamma_+^{(m)} < \lambda_m, \quad m \in \{1, \ldots, d - 1\}
$$

and we set $m_+(\tau, \varepsilon) = 0$. Similarly, for $\sigma \neq \tau$, we define $m_- = m_-(\sigma, \varepsilon)$ as the unique solution to the inequalities:

$$
\lambda_{d-m+1} < \gamma_-^{(m)} \leq \lambda_{d-m}, \quad m \in \{1, \ldots, d - 1\}
$$

and set $m_-(\tau, \varepsilon) = 0$. Finally, we set $\gamma_+ = \gamma_+(\sigma, \varepsilon) := \gamma_+^{(m_+)}$ and $\gamma_- = \gamma_-(\sigma, \varepsilon) := \gamma_-^{(m_-)}$.

Given the eigen-decomposition $\sigma = \sum_{i=1}^d \lambda_i |i\rangle\langle i|$, we define

$$
\mathcal{M}_\varepsilon(\sigma) := \sigma_\varepsilon^* := \sum_{i=1}^{m_+} \gamma_+ |i\rangle\langle i| + \sum_{i=m_+ + 1}^{d-m_-} \lambda_i |i\rangle\langle i| + \sum_{i=d-m_- + 1}^d \gamma_- |i\rangle\langle i|.
$$

To summarize, we construct $\sigma_\varepsilon^*$ as follows: we decrease the $m_+$ largest eigenvalues of $\sigma$ by setting them to $\gamma_+$ (where $m_+$ and $\gamma_+$ are related by eq. $13$), increase the $m_-$ smallest eigenvalues of $\sigma$ by setting them to $\gamma_-$ (where $m_-$ and $\gamma_-$ are related by eq. $14$), and we keep the other eigenvalues of $\sigma$ unchanged. This is illustrated in Figure 3 for a state $\sigma \in \mathcal{D}(\mathcal{H})$ with $\varepsilon = 0.07$ and $d = 12$.

![Figure 3: We choose $d = 12$, a state $\sigma \in \mathcal{D}(\mathcal{H})$, and $\varepsilon = 0.07$, for which $m_+ = 2$ and $m_- = 4$. Left: the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ of $\sigma$ are plotted. Center: the smallest four eigenvalues of $\sigma$ are increased to $\gamma_- = \frac{1}{4}[\lambda_1 + \cdots + \lambda_4 + \varepsilon]$, and the largest two eigenvalues of $\sigma$ decreased to $\gamma_+ = \frac{1}{2}[\lambda_1 + \lambda_2 - \varepsilon]$. Right: the eigenvalues of $\sigma_\varepsilon^*$ are $\gamma_+$ with multiplicity two, $\lambda_3, \lambda_4, \ldots, \lambda_{d-4}$, and $\gamma_-$ with multiplicity four.

Considered as a map on $\mathcal{D}(\mathcal{H})$, $\mathcal{M}_\varepsilon$ has several useful properties which are presented in the following proposition. It should be noted, however, that $\mathcal{M}_\varepsilon$ is not a linear map.

**Proposition 5.1 (Properties of $\mathcal{M}_\varepsilon$).** Let $\sigma \in \mathcal{D}(\mathcal{H})$. We have the following properties of $\mathcal{M}_\varepsilon$, for any $\varepsilon \in (0, 1]$.

a. Maps states to states: $\mathcal{M}_\varepsilon : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$.

b. Minimal in majorization order: $\mathcal{M}_\varepsilon(\sigma) \in B_\varepsilon(\sigma)$ and for any $\omega \in B_\varepsilon(\sigma)$, we have $\mathcal{M}_\varepsilon(\sigma) \prec \omega$.

c. Semi-group property: if $\varepsilon_1, \varepsilon_2 \in (0, 1]$ with $\varepsilon_1 + \varepsilon_2 \leq 1$, we have $\mathcal{M}_{\varepsilon_1 + \varepsilon_2}(\sigma) = \mathcal{M}_{\varepsilon_1} \circ \mathcal{M}_{\varepsilon_2}(\sigma)$.

d. Majorization-preserving: let $\rho \in \mathcal{D}(\mathcal{H})$ such that $\rho \prec \sigma$. Then $\mathcal{M}_\varepsilon(\rho) \prec \mathcal{M}_\varepsilon(\sigma)$. 

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e. \( \tau = \frac{1}{d} \) is the unique fixed point of \( \mathcal{M}_\varepsilon \), i.e. the unique solution to \( \sigma = \mathcal{M}_\varepsilon(\sigma) \) for \( \sigma \in \mathcal{D}(\mathcal{H}) \).

Moreover, for any \( \sigma \neq \tau \), \( \mathcal{M}_\varepsilon(\sigma) \) is not unitarily equivalent to \( \sigma \).

f. For any state \( \sigma \in \mathcal{B}_\varepsilon(\tau) \), we have \( \mathcal{M}_\varepsilon(\sigma) = \tau \).

g. For any pure state \( \psi \in \mathcal{D}_{\text{pure}}(\mathcal{H}) \), the state \( \mathcal{M}_\varepsilon(\psi) \) has the form

\[
\mathcal{M}_\varepsilon(\psi) = \begin{cases} 
\text{diag}(1 - \varepsilon, \frac{\varepsilon}{d-1}, \ldots, \frac{\varepsilon}{d-1}) & \varepsilon < 1 - \frac{1}{d} \\
\tau := \frac{1}{d} & \varepsilon \geq 1 - \frac{1}{d}.
\end{cases}
\] (16)

The proof of properties [a] and [b] can be found in [HD17; HOS17]; the property [c] was proved in [HD17], property [d] can be found in Lemma 2 of [HOS17]. The property [e] can be shown as follows. \( \mathcal{M}_\varepsilon(\rho) \) is not unitarily equivalent to \( \rho \) for \( \rho \neq \tau \) follows from the construction presented above, in particular, the fact that the eigenvalues of \( \mathcal{M}_\varepsilon(\rho) \) differ from \( \rho \). One immediately has that \( \tau \) is a fixed point of \( \mathcal{M}_\varepsilon \), and uniqueness follows from the fact that \( \mathcal{M}_\varepsilon(\sigma) \) is not unitarily equivalent to \( \sigma \) for \( \sigma \neq \tau \). Lastly, the properties [f] and [g] follow from the construction given above.

6 Proof of Theorem 4.1

6.1 Reducing to \( h = \text{id} \)

Our first task is to reduce to the case when \( h = \text{id} \), i.e. \( h(x) = x \) for all \( x \in \mathbb{R} \). Fix \( \varepsilon \in (0, 1] \) and \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) such that \( \rho \prec \sigma \) and \( \rho \) and \( \sigma \) are not unitarily equivalent. Let us define four variables

\[
a := H_{(\text{id}, \phi)} \circ \mathcal{M}_\varepsilon(\rho), \quad b := H_{(\text{id}, \phi)}(\rho), \quad c := H_{(\text{id}, \phi)} \circ \mathcal{M}_\varepsilon(\sigma), \quad d := H_{(\text{id}, \phi)}(\sigma)
\]

which are non-negative real numbers. Theorem 4.1 is the statement that

\[
h(a) - h(b) \leq h(c) - h(d).
\] (17)

Lemma 6.1. Let \( h \) be concave, and \( \phi \) strictly concave. If \( a - b \leq c - d \), then (17) holds. Moreover, if \( h \) is strictly concave, then (17) holds with strict inequality.

Proof. By the strict Schur concavity of the \((\text{id}, \phi)\)-entropy, we have \( b < a \) and \( d < c \), and by Proposition 5.1[b] we have \( b > d \) and \( a > c \). Therefore, since \( h \) is concave, we apply Proposition A.1 to obtain

\[
\frac{h(b) - h(d)}{b - d} \geq \frac{h(a) - h(c)}{a - c}.
\]

That is,

\[
[h(b) - h(d)] \frac{a - c}{b - d} \geq h(a) - h(c)
\]

Since we have \( a - c \leq b - d \) using the assumption, then \( \frac{a - c}{b - d} \leq 1 \), and therefore

\[
h(b) - h(d) \geq h(a) - h(c)
\]

and adding \( h(c) - h(b) \) to each side yields (17). \( \square \)

Therefore, it remains to establish \( a - b \leq c - d \), which is Theorem 4.1 when \( h = \text{id} \).
Remark. An extension of Theorem 3.1 to treat the α-Rényi entropy for α > 1 would need to address the case in which h is convex and strictly decreasing, and φ is strictly convex. In this case, ρ ↦ Tr φ(ρ) is Schur convex, and we have a < b, c < d, b < d, and a < c. The analog to Lemma 6.1 would be to show that a − b ≥ c − d implies (17). However, repeating the proof of Lemma 6.1 in this case yields e.g.

\[ |h(b) - h(d)| \frac{c - a}{d - b} \leq h(a) - h(c) \]

which is inconclusive in showing (17) when a − b ≥ c − d. This is the technical reason this proof does not extend to the α-Rényi entropy for α > 1.

In fact, the associated quantity \( \Delta_\varepsilon \) for an α-Rényi entropy with α > 1 is not Schur convex. For the example stated after Theorem 4.1, it can be shown that choosing \( H_{(h,\phi)} = S_\alpha \) for α > 1 yields \( \Delta_\varepsilon(\rho) > \Delta_\varepsilon(\sigma) \).

6.2 The case \( h = \text{id} \)

We prove Theorem 4.1 in several steps. First, we use the semigroup property of \( M_\varepsilon \) to decompose \( \Delta_\varepsilon \) for \( \varepsilon = \varepsilon_1 + \varepsilon_2 \) in terms of \( \varepsilon_1 \) and \( \varepsilon_2 \) in Lemma 6.2. Then we define a quantity \( \delta(\rho, \sigma) \) in Definition 6.3 such that for \( \varepsilon \leq \delta(\rho, \sigma) \), we can show that \( \Delta_\varepsilon(\rho) \leq \Delta_\varepsilon(\sigma) \) if \( \rho \prec \sigma \) (Lemma 6.5), using properties of \( \delta(\rho, \sigma) \) presented in Lemma 6.4. Finally, we show that for arbitrary \( \varepsilon \in (0, 1] \), we can use Lemma 6.2 finitely many times to prove Theorem 4.1. We state the lemmas here but defer their proofs to Section 7.

Lemma 6.2. Let \( \rho \in \mathcal{D}(\mathcal{H}) \), and \( \varepsilon_1, \varepsilon_2 \in (0, 1] \) with \( \varepsilon_1 + \varepsilon_2 \leq 1 \). Then

\[ \Delta_{\varepsilon_1+\varepsilon_2}(\rho) = \Delta_{\varepsilon_1} \circ M_{\varepsilon_2}(\rho) + \Delta_{\varepsilon_2}(\rho). \]

Definition 6.3 (\( \delta(\rho, \sigma) \)). Let \( \rho \in \mathcal{D}(\mathcal{H}) \) for \( \rho \neq \tau \). Let \( \mu_1 > \mu_2 > \cdots > \mu_\ell \) denote the distinct ordered eigenvalues of \( \rho \), and define

\[ \delta(\rho) = \min\{k_+(\rho)(\mu_1 - \mu_2), k_-(\rho)(\mu_\ell - \mu_\ell-1)\}. \]  

(18)

For \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) with \( \rho \neq \tau \neq \sigma \), define

\[ \delta(\rho, \sigma) = \min\{\delta(\rho), \delta(\sigma)\}. \]  

(19)

For any \( \varepsilon \leq \delta(\rho, \sigma) \), the map \( M_\varepsilon \) only “moves” the largest and smallest eigenvalue of \( \rho \) and of \( \sigma \), as shown by the following result and illustrated through an example in Figure 4.

Lemma 6.4. Let \( \rho \neq \tau \). For any \( \varepsilon \leq \delta(\rho) \), we have

\[ m_+(\rho, \varepsilon) = k_+(\rho), \quad \text{and} \quad m_-(\rho, \varepsilon) = k_-(\rho). \]

Moreover, if \( \varepsilon = \delta(\rho) \) then either \( k_+(M_\varepsilon(\rho)) > k_+(\rho) \) or \( k_-(M_\varepsilon(\rho)) > k_-(\rho) \).

Using this result, we can prove the Schur convexity of \( \Delta_\varepsilon \) for \( \varepsilon \) small enough (depending on \( \rho \) and \( \sigma \)).

Lemma 6.5. Let \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) with \( \rho \prec \sigma \). Let \( \varepsilon \leq \delta(\rho, \sigma) \), defined by (19). Then

\[ \Delta_\varepsilon(\rho) \leq \Delta_\varepsilon(\sigma). \]  

(20)

Moreover, equality in (20) implies that \( \lambda_\pm(\rho) = \lambda_\pm(\sigma) \).

We can iterate this result using Lemma 6.2 to prove Theorem 4.1 in general.
Step 1. Set \( \varepsilon = 0 \) and \( \sigma_1 = \sigma \). If \( \varepsilon \leq \delta_1 := \delta(\rho_1, \sigma_1) \), we conclude via Lemma 6.5. Otherwise, set \( \varepsilon_1 = \varepsilon - \delta_1 \). Then by Lemma 6.2
\[
\Delta_{\varepsilon_1 + \delta_1}(\rho_1) = \Delta_{\varepsilon_1} \circ M_{\delta_1}(\rho_1) + \Delta_{\delta_1}(\rho_1)
\]
and
\[
\Delta_{\varepsilon_1 + \delta_1}(\sigma_1) = \Delta_{\varepsilon_1} \circ M_{\delta_1}(\sigma_1) + \Delta_{\delta_1}(\sigma_1).
\]
We invoke Lemma 6.5 to find \( \Delta_{\delta_1}(\rho_1) \leq \Delta_{\delta_1}(\sigma_1) \); it remains to show
\[
\Delta_{\varepsilon_1}(M_{\delta_1}(\rho_1)) \leq \Delta_{\varepsilon_1}(M_{\delta_1}(\sigma_1)).
\]

Step 2. Set \( \rho_2 = M_{\delta_1}(\rho_1) \) and \( \sigma_2 = M_{\delta_1}(\sigma_1) \). If either \( \rho_2 = \tau \) or \( \sigma_2 = \tau \) we conclude by the argument presented at the start of the proof. Otherwise, we set \( \delta_2 := \delta(\rho_2, \sigma_2) \) and proceed.

If \( \varepsilon_1 \leq \delta_2 \), we conclude by Lemma 6.5. Otherwise, set \( \varepsilon_2 = \varepsilon_1 - \delta_2 \), expand e.g.
\[
\Delta_{\varepsilon_2 + \delta_2}(\rho_2) = \Delta_{\varepsilon_2} \circ M_{\delta_2}(\rho_2) + \Delta_{\delta_2}(\rho_2), \quad \Delta_{\varepsilon_2 + \delta_2}(\sigma_2) = \Delta_{\varepsilon_2} \circ M_{\delta_2}(\sigma_2) + \Delta_{\delta_2}(\sigma_2),
\]
and conclude \( \Delta_{\delta_2}(\rho_2) \leq \Delta_{\delta_2}(\sigma_2) \) by Lemma 6.5. It remains to show
\[
\Delta_{\varepsilon_2} \circ M_{\delta_2}(\rho_2) \leq \Delta_{\varepsilon_2} \circ M_{\delta_2}(\sigma_2).
\]
Step $k$. We continue recursively: for $k \geq 3$, we define $\rho_k = M_{\delta_k-1}(\rho_{k-1})$, $\sigma_k = M_{\delta_k-1}(\sigma_{k-1})$. If either $\rho_k = \tau$ or $\sigma_k = \tau$, we conclude as before; otherwise, set $\delta_k = \delta(\rho_k, \sigma_k)$. If $\varepsilon_{k-1} \leq \delta_k$, we conclude by Lemma 6.5; otherwise, define $\varepsilon_k = \varepsilon_{k-1} - \delta_k$, expand by Lemma 6.2 to find

$$\Delta_{\varepsilon_k + \delta_k}(\rho_k) = \Delta_{\varepsilon_k} \circ M_{\delta_k}(\rho_k) + \Delta_{\delta_k}(\rho_k), \quad \Delta_{\varepsilon_k + \delta_k}(\sigma_k) = \Delta_{\varepsilon_k} \circ M_{\delta_k}(\sigma_k) + \Delta_{\delta_k}(\sigma_k),$$

and conclude $\Delta_{\delta_k}(\rho_k) \leq \Delta_{\delta_k}(\sigma_k)$ by Lemma 6.5. At the end of step $k$, it remains to show that

$$\Delta_{\varepsilon_k} \circ M_{\delta_k}(\rho_k) \leq \Delta_{\varepsilon_k} \circ M_{\delta_k}(\sigma_k).$$

This process must terminate in less than $4d$ steps, as follows. At each step $k$ for which the process does not conclude, we have either $\delta(\rho_k, \sigma_k) = \delta(\rho_k)$ and therefore $k_+(\rho_k) > k_+(\rho_{k-1})$ or $k_-(\rho_k) > k_-(\rho_{k-1})$ or else $\delta(\rho_k, \sigma_k) = \delta(\sigma_k)$ and therefore $k_+(\sigma_k) > k_+(\sigma_{k-1})$ or $k_-(\sigma_k) > k_-(\sigma_{k-1})$, by Lemma 6.4. Since $k_{\pm}(\omega) \leq d$ for $\omega \in \mathcal{D}(\mathcal{H})$ and one of each of the four integers $k_{\pm}(\rho), k_{\pm}(\sigma)$ increases at each step, there cannot be more than $4d$ steps in total. Note that $\Delta_{\varepsilon}(\rho) = \Delta_{\varepsilon}(\sigma)$ implies equality in the use of Lemma 6.5 in Step 1, which requires $\lambda_+(\rho) = \lambda_+(\sigma)$.

\[\square\]

7 Proof of lemmas

**Proof of Lemma 6.2** We expand

$$\Delta_{\varepsilon_1 + \varepsilon_2}(\rho) = H_{(h,\phi)} \circ M_{\varepsilon_1 + \varepsilon_2}(\rho) - H_{(h,\phi)}(\rho).$$

By Proposition 5.4(c), we have $M_{\varepsilon_1 + \varepsilon_2}(\rho) = M_{\varepsilon_1} \circ M_{\varepsilon_2}(\rho)$. Thus,

$$\Delta_{\varepsilon_1 + \varepsilon_2}(\rho) = H_{(h,\phi)} \circ M_{\varepsilon_1}(M_{\varepsilon_2}(\rho)) - H_{(h,\phi)}(M_{\varepsilon_2}(\rho)) + H_{(h,\phi)}(M_{\varepsilon_2}(\rho)) - H_{(h,\phi)}(\rho) = \Delta_{\varepsilon_1}(M_{\varepsilon_2}(\rho)) + \Delta_{\varepsilon_2}(\rho).$$

\[\square\]

**Proof of Lemma 6.4** We use the notation of Definition 6.3. We check that the choice $m = k_+(\rho)$ satisfies the definition of $m_+(\rho, \varepsilon)$, namely that the choice $m = k_+(\rho)$ solves (13).

- If $T(\rho, \tau) > \varepsilon$, then $\gamma_+^{(m)}(\rho, \varepsilon) = \frac{1}{m} \left( \sum_{i=1}^{m} \lambda_i(\rho) - \varepsilon \right)$. And indeed, taking $m = k_+(\rho)$ we find

$$\lambda_{k_++1}(\rho) = \mu_2 \leq \frac{1}{k_+} \left( \sum_{i=1}^{k_+} \lambda_i(\rho) - \varepsilon \right) = \mu_1 - \varepsilon$$

since $\frac{\varepsilon}{k_+} \leq \frac{1}{k_+} \delta(\rho, \sigma) \leq \mu_1 - \mu_2$. Additionally, $\mu_1 - \frac{\varepsilon}{k_+} < \mu_1 = \lambda_{k_+}(\rho)$. Therefore, $m = k_+(\rho)$ solves (13), hence $m_+(\rho, \varepsilon) = k_+(\rho)$.

- In the case $0 < T(\rho, \tau) \leq \varepsilon$. Then $\gamma_+(\rho, \varepsilon) = \frac{1}{\delta}$. Since $\rho \neq \tau$, we have $\lambda_{k_+(\rho)}(\rho) = \mu_1 > \frac{1}{\delta}$. Moreover,

$$k_+(\rho)(\mu_1 - \frac{1}{\delta}) \leq \text{Tr}(\rho - \tau) = T(\rho, \tau) \leq \varepsilon \leq k_+(\rho)(\mu_1 - \mu_2)$$

and therefore $\mu_1 - \frac{1}{\delta} \leq \mu_1 - \mu_2$, yielding $\mu_2 \leq \frac{1}{\delta}$. Thus, $m_+(\rho, \varepsilon) = k_+(\rho)$.
Proving that \( m_-(\rho, \varepsilon) = k_- (\rho) \) is analogous.

Next, consider \( \varepsilon = \delta (\rho) \). If \( 0 < T (\rho, \tau) \leq \varepsilon \), then \( \mathcal{M}_\varepsilon (\rho) = \tau \) (by Proposition \[5.1\]) and \( d = k_+ (\mathcal{M}_\varepsilon (\rho)) > k_+ (\rho) \), by the assumption that \( \rho \neq \tau \). Otherwise, without loss of generality, assume \( \delta (\rho, \sigma) = k_+ (\rho) (\mu_1 - \mu_2) \). We show that \( k_+ (\mathcal{M}_\varepsilon (\rho)) > k_+ (\rho) \). By the above, \( m_+ (\rho, \varepsilon) = k_+ (\rho) \), and therefore

\[
\gamma_+ (\rho, \varepsilon) = \mu_1 + \frac{\varepsilon}{k_+ (\rho)} = \mu_1 + (\mu_1 - \mu_2) = \mu_2.
\]

As

\[
\mathcal{M}_\varepsilon (\rho) = \sum_{i=1}^{m_+ (\rho, \varepsilon)} \gamma_+ (\rho, \varepsilon) |i\rangle \langle i| + \sum_{i=m+1}^{d-m_+ (\rho, \varepsilon)} \gamma_- (\rho, \varepsilon) |i\rangle \langle i| + \sum_{i=d-m_+ (\rho, \varepsilon)+1}^{d} \gamma_- (\rho, \varepsilon) |i\rangle \langle i|
\]

by eq. \([15] \) and \( \lambda_{m_+ (\rho, \varepsilon)+1} (\rho) = \mu_2 \), we have that \( k_+ (\mathcal{M}_\varepsilon (\rho)) \), the multiplicity of \( \mu_2 \) for \( \mathcal{M}_\varepsilon (\rho) \), is strictly larger than \( k_+ (\rho) = m_+ (\rho, \varepsilon) \).

**Proof of Lemma 6.5** As in the proof of Theorem \[4.1 \] if \( \sigma = \tau \), then \( \rho \prec \sigma \) implies \( \rho = \tau \), and hence \( \Delta_\varepsilon (\rho) = 0 = \Delta_\varepsilon (\sigma) \). If \( \rho = \tau \neq \sigma \), then \( \Delta_\varepsilon (\rho) = 0 < \Delta_\varepsilon (\sigma) \) by the strict Schur concavity of the \( H_{\text{id}, \phi} \) entropy. Now, assume \( \rho \neq \tau \neq \sigma \). We aim to show

\[
H_{\text{id}, \phi} \circ \mathcal{M}_\varepsilon (\rho) - H_{\text{id}, \phi} (\rho) \leq H_{\text{id}, \phi} \circ \mathcal{M}_\varepsilon (\sigma) - H_{\text{id}, \phi} (\sigma).
\]

By two applications of Lemma \[6.4 \] we have \( m_+ (\rho, \varepsilon) = k_+ (\rho) \), \( m_- (\rho, \varepsilon) = k_- (\rho) \), \( m_+ (\sigma, \varepsilon) = k_+ (\sigma) \), and \( m_- (\sigma, \varepsilon) = k_- (\sigma) \). Therefore, by \([3] \) and \([15] \),

\[
H_{\text{id}, \phi} (\mathcal{M}_\varepsilon (\rho)) = k_+ (\rho) \phi (\gamma_+ (\rho)) + \sum_{i=k_+ (\rho)+1}^{d-k_- (\rho)} \phi (\lambda_i (\rho)) + k_- (\rho) \phi (\gamma_- (\rho))
\]

since \( h = \text{id} \). The \( \phi (\lambda_i (\rho)) \) terms for \( i = k_+ (\rho) + 1, \ldots, d - k_- (\rho) \) therefore cancel in \( \Delta_\varepsilon (\rho) \) yielding

\[
H_{\text{id}, \phi} \circ \mathcal{M}_\varepsilon (\rho) - H_{\text{id}, \phi} (\rho) = k_+ (\rho) [\phi (\gamma_+ (\rho, \varepsilon)) - \phi (\lambda_+ (\rho))] + k_- (\rho) [\phi (\gamma_- (\rho, \varepsilon)) - \phi (\lambda_- (\rho))]
\]

and similarly

\[
H_{\text{id}, \phi} \circ \mathcal{M}_\varepsilon (\sigma) - H_{\text{id}, \phi} (\sigma) = k_+ (\sigma) [\phi (\gamma_+ (\sigma, \varepsilon)) - \phi (\lambda_+ (\sigma))] + k_- (\sigma) [\phi (\gamma_- (\sigma, \varepsilon)) - \phi (\lambda_- (\sigma))].
\]

We conclude by invoking Lemma \[7.1 \] below, which bounds the first term (resp. second term) of \([22] \) by the first term (resp. second term) of \([23] \).

**Lemma 7.1.** For \( \rho \prec \sigma \) with \( \rho \neq \tau \neq \sigma \) and \( 0 < \varepsilon \leq \delta (\rho, \sigma) \), we have

\[
k_\pm (\rho) [\phi (\gamma_\pm (\rho, \varepsilon)) - \phi (\lambda_\pm (\rho))] \leq k_\pm (\sigma) [\phi (\gamma_\pm (\sigma, \varepsilon)) - \phi (\lambda_\pm (\sigma))]
\]

and that equality in \([24] \) implies \( \lambda_\pm (\rho) = \lambda_\pm (\sigma) \).

To prove this result, we first recall a simple consequence of the majorization order \( \rho \prec \sigma \).

**Lemma 7.2.** If \( \rho \prec \sigma \), then \( T (\rho, \tau) \leq T (\sigma, \tau) \).

**Proof.** If \( \rho \prec \sigma \), then by Theorem 2-2 (b) of \[AUS2 \], we have \( \rho = \Phi (\sigma) \) for a map \( \Phi (\cdot) = \sum_i p_i U_i \cdot U_i^* \) where \( p_i \) is a finite probability distribution and each \( U_i \) is unitary. \( \Phi \) is completely positive and trace-preserving (CPTP) as well as unital. Since \( \Phi (\tau) = \tau \),

\[
T (\rho, \tau) = T (\rho, \Phi (\tau)) = T (\Phi (\sigma), \Phi (\tau)) \leq T (\sigma, \tau)
\]

where the inequality follows from the monotonicity of the trace distance under CPTP maps.
Proof of Lemma 7.1 We prove the case + in eq. (24); the case − is proved analogously. First, we have $\lambda_+(\rho, \varepsilon) \leq \lambda_+(\sigma, \varepsilon)$ and $\lambda_+(\rho) \leq \lambda_+(\sigma)$, using that $\rho < \sigma$ and $\mathcal{M}_\rho < \mathcal{M}_\sigma$ by Proposition 5.1. Moreover, by definition, $\gamma_+(\rho, \varepsilon) < \lambda_+(\rho)$ and $\gamma_+(\sigma, \varepsilon) < \lambda_+(\sigma)$. Therefore, by applying Proposition A.1 and multiplying by minus one, we have

$$\frac{\phi(\gamma_+(\rho, \varepsilon)) - \phi(\lambda_+(\rho))}{\lambda_+(\rho) - \gamma_+(\rho, \varepsilon)} \leq \frac{\phi(\gamma_+(\sigma, \varepsilon)) - \phi(\lambda_+(\sigma))}{\lambda_+(\sigma) - \gamma_+(\sigma, \varepsilon)}.$$  \hspace{1cm} (25)

and that equality requires $\lambda_+(\rho) = \lambda_+(\sigma)$.

Now, we complete the proof of (24) in three cases.

Case 1: $T(\sigma, \tau) \leq \varepsilon$. Then $T(\rho, \tau) \leq \varepsilon$ as well, and

$$\lambda_+(\rho) - \gamma_+(\rho, \varepsilon) = \lambda_+(\rho) - \frac{1}{d}.$$  

As shown in the proof of Lemma 6.4, the second-largest eigenvalue of $\rho$ is less or equal to $\frac{1}{d}$. Therefore,

$$T(\rho, \tau) = \text{Tr}[(\rho - \tau)_+] = k_+((\rho_\tau) = \frac{1}{k_+((\rho_\tau)},$$

and hence,

$$\lambda_+(\rho) - \gamma_+(\rho, \varepsilon) = \frac{1}{k_+((\rho_\tau)}.$$  \hspace{1cm} (26)

As $T(\sigma, \tau) \leq \varepsilon$, we likewise have $\lambda_+(\sigma) - \gamma_+(\sigma, \varepsilon) = \frac{1}{k_+(\sigma)} T(\sigma, \tau)$. Then (25) yields

$$\frac{k_+((\rho_\tau)[\phi(\gamma_+(\rho, \varepsilon)) - \phi(\lambda_+(\rho))]}{T(\rho, \tau)} \leq \frac{k_+((\sigma_\tau)[\phi(\gamma_+(\sigma, \varepsilon)) - \phi(\lambda_+(\sigma))]}{T(\sigma, \tau)}.$$  

Since $T(\sigma, \tau) \geq T(\rho, \tau)$, we may bound the right-hand side by $\frac{k_+((\sigma_\tau)[\phi(\gamma_+(\sigma, \varepsilon)) - \phi(\lambda_+(\sigma))]}{T(\rho, \tau)}$. Then multiplying by $T(\rho, \tau)$ yields (24).

Case 2: $T(\rho, \tau) \leq \varepsilon < T(\sigma, \tau)$. In this case, (26) holds, and $\gamma_+(\sigma, \varepsilon) = \lambda_+(\sigma) - \frac{\varepsilon}{k_+(\sigma)}$. Therefore, (25) yields

$$\frac{k_+((\rho_\tau)[\phi(\gamma_+(\rho, \varepsilon)) - \phi(\lambda_+(\rho))]}{T(\rho, \tau)} \leq \frac{k_+((\sigma_\tau)[\phi(\gamma_+(\sigma, \varepsilon)) - \phi(\lambda_+(\sigma))]}{\varepsilon}.$$  

Similarly to the previous case, the inequality $\varepsilon \geq T(\rho, \tau)$ bounds the right-hand side by $\frac{k_+((\sigma_\tau)[\phi(\gamma_+(\sigma, \varepsilon)) - \phi(\lambda_+(\sigma))]}{T(\rho, \tau)}$, and multiplying by $T(\rho, \tau)$ yields (24).

Case 3: $T(\rho, \tau) > \varepsilon$. Then $\gamma_+(\rho, \varepsilon) = \lambda_+(\rho) - \frac{\varepsilon}{k_+(\rho)}$, and $\gamma_+(\sigma, \varepsilon) = \lambda_+(\sigma) - \frac{\varepsilon}{k_+(\sigma)}$. Therefore, (25) yields

$$\frac{k_+((\rho_\tau)[\phi(\gamma_+(\rho, \varepsilon)) - \phi(\lambda_+(\rho))]}{\varepsilon} \leq \frac{k_+((\sigma_\tau)[\phi(\gamma_+(\sigma, \varepsilon)) - \phi(\lambda_+(\sigma))]}{\varepsilon}.$$  

and multiplying by $\varepsilon$ yields (24).

Note that in each case, equality in (24) requires equality in (25). \hfill \Box
A An elementary property of concave functions

Given a function \( \phi : I \to \mathbb{R} \) defined on an interval \( I \subset \mathbb{R} \), we define the “slope function,”

\[
s(x_1, x_2) = \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}
\]

for \( x_1, x_2 \in I \) with \( x_1 \neq x_2 \). Note that \( s \) is symmetric in its arguments. It can be shown that \( \phi \) is concave (resp. strictly concave) if and only if \( s \) is monotone decreasing (resp. strictly decreasing) in each argument.

**Proposition A.1.** Let \( I \subset \mathbb{R} \) be an interval and \( \phi : I \to \mathbb{R} \) be concave. For any \( x_1, x_2, y_1, y_2 \in I \) such that \( x_1 \neq x_2, y_1 \neq y_2, x_1 \leq y_1 \) and \( x_2 \leq y_2 \) we have

\[
s(x_1, x_2) \geq s(y_1, y_2).
\]

If \( \phi \) is strictly concave, then equality is achieved if and only if \( x_1 = y_1 \) and \( x_2 = y_2 \).

**Proof.** For \( \phi \) concave, we have \( s(x_1, x_2) \geq s(y_1, x_2) \geq s(y_1, y_2) \). Next, assume \( \phi \) is strictly concave. Then equality holds in the first inequality if and only if \( x_2 = y_2 \), and in the second if and only if \( x_1 = y_1 \), completing the proof. \( \square \)

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