Effects of Kaluza-Klein theory on a position dependent mass system with uniform magnetic field in a magnetic cosmic string space-time

Faizuddin Ahmed
Ajmal College of Arts and Science, Dhubri-783324, Assam, India

Abstract

In this paper, we study the relativistic quantum dynamics of a position mass system on curved background within the Kaluza-Klein theory (KKT) with Cornell-type potential. We solve the Klein-Gordon equation in the magnetic cosmic string space-time background subject to a uniform magnetic field with a Cornell-type scalar potential and observe a relativistic analogue of Aharonov-Bohm effect for bound states. We show the energy levels get modify due to the presence of global parameters characterizing the space-time and break their degeneracy.

*keywords:* cosmic string, Relativistic wave equation, electromagnetic interactions, potential, energy spectrum, wave-functions, Aharonov-Bohm effect, special functions.

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1 Introduction

The relativistic wave-equations are of current research interest for theoretical physicists [1, 2] including in nuclear and high energy physics [3, 4]. In recent years, many studies have carried out to explore the relativistic energy eigenvalues and eigenfunctions on the curved background with the cosmic string (see, [5, 6, 7, 8, 9] and references therein).

\[^1\]faizuddinahmed15@gmail.com ; faiz4U.enter@rediffmail.com
Cosmic strings have been produced by phase transition in the early universe \cite{10, 11} as predicted in the string theory \cite{12} and particle physics \cite{13, 14}. These include domain walls, cosmic strings and monopoles. Among them, cosmic strings and monopoles are the best candidates to be observed. Cosmic strings \cite{15} and global monopoles \cite{16} are exotic topological objects that modify the space-time geometry, producing a planar and solid angle deficit, respectively. A series of cylindrically symmetric solutions of the Einstein and Einstein-Gauss-Bonnet equations in the context of Kaluza-Klein theory was investigated in \cite{17}.

In this paper, we study Klein-Gordon equation in the background of 5-D space-time geometry produced by topological defects in the context of Kaluza-Klein theory \cite{18, 19, 20}. Topological defects play an important role in condensed matter physics systems \cite{21, 22, 23, 24, 16} where, topological defects analogue to the cosmic strings appear in phase transitions in liquid crystals \cite{25, 26}. Geometric quantum phases \cite{27} describe the phase shifts acquire by wave-function of a quantum mechanical particle. A well-known quantum phase is the Aharonov-Bohm effect \cite{28, 29, 30} due to the presence of an internal magnetic flux. This effect has investigated in, Newtonian theory \cite{31}, bound states of massive fermions \cite{32}, scattering of dislocated wave-fronts \cite{33}, torsion effects with a position-dependent mass system \cite{34, 35, 36, 37, 38, 39, 40}, bound states solution of spin-0 particles \cite{41, 42, 43, 44, 45}. In the context of Kaluza-Klein theory, Aharonov-Bohm effect for bound states \cite{46, 47, 48, 49, 50, 51, 52, 53}, torsion effects \cite{54, 55, 56, 57, 58, 59, 60}, fermions \cite{57, 61, 62, 63}, Lorentz symmetry violation \cite{64, 65, 66}, and graphene layer \cite{67} have investigated in literature.
2 Position dependent mass system in a magnetic cosmic string space-time

In the context of Kaluza-Klein theory [18, 19], the metric with a magnetic quantum flux (Φ) passing along the symmetry axis of the string assumes the following form

\[ ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\phi^2 + dz^2 + [dx + K A_\mu(x) dx^\mu]^2, \]

(1)

where \( t \) is the time-coordinate, \( x \) is the coordinate associated with fifth additional dimension having ranges \( 0 < x < 2\pi a \) where, \( a \) is the radius of the compact dimension of \( x \), \( (r, \phi, z) \) are the cylindrical coordinates with the usual ranges, and \( K \) is the Kaluza constant [46]. The parameter \( \alpha = (1 - 4\mu) \) [11] characterizing the wedge parameter where, \( \mu \) is the linear mass density of the string.

Based on [46, 48, 53, 52], we introduce a uniform magnetic field \( B_0 \) and magnetic quantum flux \( \Phi \) through the line-element of the cosmic string space-time (1) in the following form

\[ ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\phi^2 + dz^2 + \left[ dx + \left( -\frac{1}{2} \alpha B_0 r^2 + \frac{\Phi^2}{2\pi} \right) d\phi \right]^2, \]

(2)

where the gauge field given by

\[ A_\phi = K^{-1} \left( -\frac{1}{2} \alpha B_0 r^2 + \frac{\Phi}{2\pi} \right) \]

(3)

gives rise to a uniform magnetic field \( \vec{B} = \vec{\nabla} \times \vec{A} = -K^{-1} B_0 \hat{z} \) [68], \( \hat{z} \) is the unitary vector in the \( z \)-direction. Here \( \Phi = \text{const.} \) is the magnetic quantum flux [30, 68] through the core of the topological defects [69].

The relativistic quantum dynamics of a position dependent mass system is described by [34, 35, 52]:

\[ \left[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - (m + S)^2 \right] \Psi = 0, \]

(4)
with \( g \) is the determinant of metric tensor with \( g^{\mu\nu} \) its inverse. For the metric \( g^{\mu\nu} \)

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\alpha^2 r^2} & 0 & -\frac{K A_\phi}{\alpha^2 r^2} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{K A_\phi}{\alpha^2 r^2} & 0 & 1 + \frac{K^2 A^2_\phi}{\alpha^2 r^2}
\end{pmatrix}.
\]

(5)

By considering the line-element (2) into the Eq. (4), we obtain the following differential equation:

\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{\alpha^2 r^2} \left( \frac{\partial}{\partial \phi} - K A_\phi \frac{\partial}{\partial x} \right)^2 - \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} - (m + S)^2 \right] \Psi(t, r, \phi, z) = 0.
\]

(6)

Since the line-element (2) is independent of \( t, \phi, z, x \). One can choose the following ansatz for the function \( \Psi \) as:

\[
\Psi(t, r, \phi, z, x) = e^{i(-Et + l \phi + k z + q x)} \psi(r),
\]

(7)

where \( E \) is the total energy of the particle, \( l = 0, \pm 1, \pm 2, .. \in \mathbb{Z} \), and \( k, q \) are constants.

Substituting the ansatz (7) into the Eq. (4), we obtain the following equation:

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + E^2 - k^2 - q^2 - \frac{(l - K q A_\phi)^2}{\alpha^2 r^2} - (m + S)^2 \right] \psi(r) = 0.
\]

(8)

Case A : Interactions with Cornell-type potential

Cornell-type potential consists of linear plus Coulomb-like term is a particular case of the quark-antiquark interaction [70, 71]. The Coulomb potential is responsible at small distances or short range interactions and linear potential leads to confinement of quark. This type of potential is given by [72, 83, 85]

\[
S(r) = \frac{\eta_c}{r} + \eta_L r
\]

(9)
where \( \eta_c, \eta_L \) are the potential parameters.

Substituting the (3) and (9) into the Eq. (8), we obtain the following equation:

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \lambda - \frac{j^2}{r^2} - \Omega^2 r^2 - \frac{a}{r} - b r \right] \psi(r) = 0, \quad (10)
\]

where

\[
\lambda = E^2 - k^2 - q^2 - m^2 - 2 \eta_c \eta_L - 2 m \omega \frac{(l - \frac{q \Phi}{2 \pi})}{\alpha},
\]

\[
\Omega = \sqrt{m^2 \omega^2 + \eta_L^2},
\]

\[
j = \sqrt{\frac{(l - \frac{q \Phi}{2 \pi})^2}{\alpha^2} + \eta_c^2},
\]

\[
\omega = \frac{q B_0}{2 m},
\]

\[
a = 2 m \eta_c,
\]

\[
b = 2 m \eta_L. \quad (11)
\]

Introducing a new variable \( \rho = \sqrt{\Omega} r \), Eq. (10) becomes

\[
\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \zeta - \frac{j^2}{\rho^2} - \rho^2 - \frac{\eta}{\rho} - \theta \rho \right] \psi(\rho) = 0, \quad (12)
\]

where

\[
\zeta = \frac{\lambda}{\Omega}, \quad \eta = \frac{a}{\sqrt{\Omega}}, \quad \theta = \frac{b}{\Omega^2}. \quad (13)
\]

Suppose the possible solution to Eq. (12) is

\[
\psi(\rho) = \rho^j e^{-\frac{1}{2} (\rho + \theta) \rho} H(\rho). \quad (14)
\]

Substituting the solution Eq. (14) into the Eq. (12), we obtain

\[
H''(\rho) + \left[ \frac{\gamma}{\rho} - \theta - 2 \rho \right] H'(\rho) + \left[ -\frac{\beta}{\rho} + \Theta \right] H(\rho) = 0, \quad (15)
\]
where
\[\gamma = 1 + 2j,\]
\[\Theta = \zeta + \frac{\theta^2}{4} - 2(1 + j),\]
\[\beta = \eta + \frac{\theta}{2}(1 + 2j).\]  \hfill (16)

Equation (15) is the biconfluent Heun’s differential equation \cite{34, 35, 41, 51, 52, 73, 74} and \(H(\rho)\) is the Heun polynomials.

The above equation (15) can be solved by the Frobenius method. We consider the power series solution \cite{75}
\[H(\rho) = \sum_{i=0}^{\infty} c_i \rho^i\]  \hfill (17)

Substituting the above power series solution into the Eq. (15), we obtain the following recurrence relation for the coefficients:
\[c_{n+2} = \frac{1}{(n+2)(n+2+2j)} \left[ \{\beta + \theta(n+1)\} c_{n+1} - (\Theta - 2n) c_n \right].\]  \hfill (18)

And the various coefficients are
\[c_1 = \left(\frac{\eta}{\gamma} + \frac{\theta}{2}\right) c_0,\]
\[c_2 = \frac{1}{4(1+j)} \left[ (\beta + \theta) c_1 - \Theta c_0 \right].\]  \hfill (19)

We must truncate the power series by imposing the following two conditions \cite{34, 35, 41, 48, 49, 50, 51, 52}:
\[\Theta = 2n, \quad (n = 1, 2, \ldots)\]
\[c_{n+1} = 0.\]  \hfill (20)

By analyzing the condition \(\Theta = 2n\), we get the following second degree
expression of the energy eigenvalues $E_{n,l}$:

$$\frac{\lambda}{\Omega} + \frac{\theta^2}{4} - 2 (1 + j) = 2 n$$

$$\Rightarrow \quad E_{n,l} = \pm \left\{ k^2 + q^2 + m^2 + 2 \Omega \left( n + 1 + \sqrt{\frac{(l - \frac{q_\Phi^2}{2\pi})^2}{\alpha^2} + \eta_c^2} \right) \right.$$

$$+ 2 \eta_c \eta_L + 2 m \omega \frac{(l - \frac{q_\Phi^2}{2\pi})}{\alpha} - \frac{m^2 \eta_L^2}{\Omega^2} \}^{\frac{1}{2}}.$$

(21)

For $\alpha \to 1$, the relativistic energy eigenvalue (21) is consistent with those result in [49].

Now, we impose additional recurrence condition $c_{n+1} = 0$ to find the individual energy levels and wave-functions one by one as done in [34, 35, 51, 52, 41]. For $n = 1$, we have $\Theta = 2$ and $c_2 = 0$ which implies from Eq. (19)

$$c_1 = \frac{2}{\beta + \theta} c_0 \Rightarrow \left( \frac{\eta}{1 + 2 j} + \frac{\theta}{2} \right) = \frac{2}{\beta + \theta}$$

$$\Omega_{1,l}^3 - \frac{a^2}{2 (1 + 2 j)} \Omega_{1,l}^2 - ab \left( \frac{1 + j}{1 + 2 j} \right) \Omega_{1,l} - \frac{b^2}{8} (3 + 2 j) = 0 \quad (22)$$

a constraint on the parameter $\Omega_{1,l}$. The magnetic field $B_{0,l}^{1,l}$ is so adjusted that Eq. (22) can be satisfied and we have simplified by labelling:

$$\omega_{1,l} = \frac{1}{m} \sqrt{\Omega_{1,l}^2 - \eta_L^2} \leftrightarrow B_{0,l}^{1,l} = \frac{2}{q} \sqrt{\Omega_{1,l}^2 - \eta_L^2}. \quad (23)$$

Therefore, the ground state energy level for $n = 1$ is given by

$$E_{1,l} = \pm \left\{ k^2 + q^2 + m^2 + 2 \Omega_{1,l} \left( 2 + \sqrt{\frac{(l - \frac{q_\Phi^2}{2\pi})^2}{\alpha^2} + \eta_c^2} \right) \right.$$

$$+ 2 \eta_c \eta_L + 2 m \omega_{1,l} \frac{(l - \frac{q_\Phi^2}{2\pi})}{\alpha} - \frac{m^2 \eta_L^2}{\Omega_{1,l}^2} \}^{\frac{1}{2}}.$$

(24)

And the radial wave-functions is

$$\psi_{1,l} = \rho \sqrt{\frac{a^2 - \frac{q_\Phi^2}{2\pi}}{\alpha^2} + \eta_c^2} e^{-\frac{1}{2} \left( \frac{2 m \eta_L}{a_{1,l}^2} + \rho \right) \rho} (c_0 + c_1 \rho), \quad (25)$$
where
\[
c_1 = \left( \frac{2 m \eta_c}{\sqrt{\Omega_{1,l}^2 (1 + 2 \sqrt{(\frac{l-a}{a})^2 + \eta_L^2})}} + \frac{m \eta_L}{\Omega_{1,l}^{\frac{3}{2}}} \right) c_0. \tag{26}
\]

**Case B : Interactions with Coulomb-type potential**

We consider \( \eta_L \to 0 \) in the scalar potential \( S \). Thus the Coulomb potential is given by
\[
S(r) = \frac{\eta_c}{r}, \tag{27}
\]
This kind of potential has used to study position-dependent mass systems \([52, 49, 78, 79]\) in the relativistic quantum mechanics.

The radial wave-equations Eq. (10) becomes
\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \tilde{\lambda} - \frac{j^2}{r^2} - m^2 \omega^2 r^2 - \frac{a}{r} \right] \psi(r) = 0, \tag{28}
\]
where \( \tilde{\lambda} = E^2 - k^2 - q^2 - m^2 - 2 m \omega \left( \frac{l-a}{a} \right) \).

Introduce a new variable \( \rho = \sqrt{m \omega} r \), Eq. (28) becomes
\[
\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \tilde{\lambda} - \frac{j^2}{\rho^2} - \rho^2 - \tilde{\eta} \right] \psi(\rho) = 0, \tag{29}
\]
where \( \tilde{\eta} = \frac{a}{\sqrt{m \omega}} \).

Suppose the possible solution to Eq. (29) is
\[
\psi(\rho) = \rho^j e^{-\frac{\rho^2}{2}} H(\rho). \tag{30}
\]
Substituting the solution Eq. (14) into the Eq. (12), we obtain
\[
H''(\rho) + \left[ \frac{1 + 2 j}{\rho} - 2 \rho \right] H'(\rho) + \left[ -\frac{\tilde{\eta}}{\rho} + \tilde{\Theta} \right] H(\rho) = 0, \tag{31}
\]
where \( \tilde{\Theta} = \frac{\tilde{\lambda}}{m \omega} - 2 (1 + j) \).
Equation (31) is the biconfluent Heun’s differential equation \[34, 35, 41, 51, 52, 73, 74\] and \( H(\rho) \) is the Heun polynomials. Substituting the power series solution (17) into the Eq. (31), we obtain the following recurrence relation for the coefficients:

\[
    c_{n+2} = \frac{1}{(n+2)(n+2+2j)} \left[ \tilde{\eta} c_{n+1} - (\tilde{\Theta} - 2n) c_n \right].
\]  
(32)

And the various coefficients are

\[
    c_1 = \frac{\tilde{\eta}}{1+2j} c_0, \quad c_2 = \frac{1}{4(1+j)} [\tilde{\eta} c_1 - \Theta c_0].
\]  
(33)

The power series expansion (17) becomes a polynomial of degree \( n \) by imposing two conditions \[34, 35, 41, 48, 49, 50, 51, 52\]:

\[
    c_{n+1} = 0, \quad \tilde{\Theta} = 2n \quad (n = 1, 2, \ldots)
\]  
(34)

By analyzing the condition \( \tilde{\Theta} = 2n \), we get the following energy eigenvalues \( E_{n,l} \):

\[
    E_{n,l} = \pm \sqrt{k^2 + q^2 + m^2 + 2m \omega \left( n + 1 + \sqrt{(l - \frac{q \Phi}{2\pi})^2 + \eta_c^2 + \left( l - \frac{q \Phi}{2\pi} \right)} \right)}.
\]  
(35)

For the radial mode \( n = 1 \), we have \( \tilde{\Theta} = 2 \) and \( c_2 = 0 \) which implies

\[
    \omega_{1,l} = \frac{2m \eta_c^2}{\left( 1 + 2 \sqrt{(l - \frac{q \Phi}{2\pi})^2 + \eta_c^2} \right)} \leftrightarrow B_{0,1,l}^{1,l} = \frac{4m^2 \eta_c^2}{q \left( 1 + 2 \sqrt{(l - \frac{q \Phi}{2\pi})^2 + \eta_c^2} \right)}.
\]  
(36)

a constraint on the parameter \( \omega_{1,l} \) or the magnetic field \( B_{0,1,l}^{1,l} \).

The ground state energy eigenvalues for \( n = 1 \) is

\[
    E_{1,l} = \pm \sqrt{k^2 + q^2 + m^2 + 2m \omega_{1,l} \left( 2 + \sqrt{(l - \frac{q \Phi}{2\pi})^2 + \eta_c^2 + \left( l - \frac{q \Phi}{2\pi} \right)} \right)},
\]  
(37)
where $\omega_{1,l}$ is given by Eq. (36).

Equation (37) with (36) corresponds to the allowed values of energy levels for the radial mode $n = 1$ of a position-dependent mass particle subject to a Coulomb-type scalar potential in the context of Kaluza-Klein theory. For $\alpha \to 1$, the energy eigenvalues is consistent with those result in [49].

**Special case**

In this special case, we choose zero magnetic field, $B_0 \to 0$. The radial wave-equations from Eq. (28) becomes

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{E^2 - k^2 - q^2 - m^2 - \frac{j^2}{r^2} - \frac{a}{r}}{r^2} \right] \psi(r) = 0. \quad (38)$$

The above can now be expressed as [41, 43, 81]

$$\psi''(r) + \frac{1}{r} \psi'(r) + \frac{1}{r^2} \left(-\xi_1 r^2 + \xi_2 r - \xi_3\right) \psi(r) = 0. \quad (39)$$

where

$$\xi_1 = k^2 + q^2 + m^2 - E^2, \quad \xi_2 = -a, \quad \xi_3 = j^2. \quad (40)$$

The energy eigenvalues is given by

$$E_{n,l} = \pm m \sqrt{1 - \frac{\eta_c^2}{\left(n + \frac{1}{2} + \sqrt{\frac{(\xi_2 - \xi_3)^2}{a^2} + \eta_c^2}\right)^2 + \frac{k^2}{m^2} + \frac{q^2}{m^2}}}, \quad (41)$$

where $n = 0, 1, 2, \ldots$.

Equation (41) is the relativistic energy eigenvalues of a scalar charged particles in the magnetic cosmic string background in the Kaluza-Klein theory with a Coulomb-type scalar potential. For $\alpha \to 1$, the energy eigenvalues Eq. (41) is consistent with those result in [48].

The corresponding radial wave functions is given by

$$\psi_{n,l}(r) = |N|_{n,l} r^j e^{-\sqrt{k^2 + q^2 + m^2 - E_{n,l}^2} \cdot r} L_n^{(2j)}(r), \quad (42)$$
where $\langle N |_{n,l} = 2^{2j} \left( k^2 + q^2 + m^2 - E_{n,l}^2 \right)^{j + \frac{1}{2}} \left( \frac{n!}{(n + 2j)!} \right)^\frac{1}{2}$ is the normalization constant and $L_{n}^{(2j)}(r)$ is the generalized Laguerre polynomials. The polynomials $L_{n}^{(j)}(r)$ are orthogonal over $[0, \infty)$ with respect to the measure with weighting function $r^j e^{-r}$ as

$$\int_0^\infty r^j e^{-r} L_{n}^{(j)}(r) L_{m'}^{(j)}(r) \, dr = \frac{(n+j)!}{n!} \delta_{nm'}.$$  \hfill (43)

**Case C : Interactions with Linear potential**

We consider $\eta_c \to 0$. Thus the linear scalar potential is given by

$$S(r) = \eta_L r,$$  \hfill (44)

The linear potential have studied by many authors in the relativistic quantum mechanics [9, 78, 79, 80, 82, 83].

The radial wave-equations from Eq. (10) becomes

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \tilde{\lambda} - \frac{l_0^2}{r^2} - \Omega^2 r^2 - b r \right] \psi(r) = 0.$$  \hfill (45)

Introduce a new variable $\rho = \sqrt{\Omega}r$, then the Eq. (45) becomes

$$\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \tilde{\lambda} - \frac{l_0^2}{\rho^2} - \rho^2 - \theta \rho \right] \psi(\rho) = 0.$$  \hfill (46)

Let the possible solution to Eq. (46) is

$$\psi = \rho^{l_0} e^{-\frac{1}{2}(\theta + \rho)\rho} H(\rho)$$  \hfill (47)

Substituting Eq. (47) into the Eq. (46), we obtain

$$H''(\rho) + \left[ \frac{(1 + 2 |l_0|)}{\rho} - \theta - 2 \rho \right] H'(\rho) + \left[ -\frac{\theta}{2} (1 + 2 |l_0|) \rho \right] + \Theta_0 \right] \psi(\rho) = 0,$$  \hfill (48)
where $\Theta_0 = \frac{\dot{\lambda}}{\Omega} - 2(1 + |l_0|) + \frac{\theta^2}{4}$.

Equation (48) is the biconfluent Heun’s differential equation [34, 35, 41, 51, 52, 73, 74] and $H(\rho)$ is the Heun polynomials.

Substituting the power series solution (17) into the Eq. (48), we obtain the following recurrence relation for the coefficients:

$$c_{n+2} = \frac{1}{(n+2)(n+2+2l_0)} \left[ \frac{\theta}{2} (2n+3+2|l_0|) c_{n+1} - (\Theta_0 - 2n) c_n \right].$$

(49)

And the various coefficients are

$$c_1 = \frac{\theta}{2} c_0,$$

$$c_2 = \frac{1}{4(1+j)} \left[ \frac{\theta}{2} (3+2|l_0|) c_1 - \Theta_0 c_0 \right].$$

(50)

The power series expansion (17) becomes a polynomial of degree $n$ by imposing two conditions [34, 35, 41, 48, 49, 50, 51, 52]:

$$c_{n+1} = 0, \quad \Theta_0 = 2n \quad (n = 1, 2, 3, 4, \ldots)$$

(51)

By analyzing the condition $\Theta_0 = 2n$, we get the following energy eigenvalues $E_{n,l}$:

$$E_{n,l} = \pm \sqrt{k^2 + q^2 + m^2 + 2m\omega l_0 + 2 \Omega \left( n + 1 + \frac{|l - \frac{q\phi}{2\pi}|}{\alpha} \right) - \frac{m^2 \eta_L^2}{\Omega^2}},$$

(52)

where $l_0 = \frac{1}{\alpha} (l - \frac{q\Phi}{2\pi})$.

Equation (52) is the relativistic energy eigenvalues of a scalar charged particles in the magnetic cosmic string background in the Kaluza-Klein theory with a linear confining potential. For $\alpha \to 1$, the energy eigenvalues Eq. (52) is consistent with those result in [49].

For the radial mode $n = 1$, $c_2 = 0$ which implies

$$\Omega_{1,l} = \left[ \frac{m^2 \eta_L^2}{2} (3+2|l_0|) \right]^{\frac{1}{3}}.$$

(53)
a constraint on the parameter $\Omega_{1,l}$. Therefore, the magnetic field is given by

$$
\omega_{1,l} = \frac{1}{m} \sqrt{\Omega_{1,l}^2 - \eta_L^2}
$$

$$
\Rightarrow B_{0,1,l} = \frac{2}{q} \sqrt{\Omega_{1,l}^2 - \eta_L^2}
$$

$$
\Rightarrow = \frac{2}{q} \sqrt{\left[ \frac{m^2 \eta_L^2}{2} (3 + 2 |l_0|) \right]^{\frac{1}{2}}} - \eta_L^2. \quad (54)
$$

Therefore the ground state energy levels

$$
E_{1,l} = \pm \left( k^2 + q^2 + m^2 + 2m \omega_{1,l} \frac{(l - \frac{q \Phi}{2\pi})}{\alpha} + 2 \Omega_{1,l} \left( n + 1 + \frac{|l - \frac{q \Phi}{2\pi}|}{\alpha} \right) - \frac{m^2 \eta_L^2}{\Omega_{1,l}^2} \right)^{\frac{1}{2}}. \quad (55)
$$

Equation (55) with (54) corresponds to the allowed values of relativistic energy levels for the radial mode $n = 1$ of a position-dependent mass particle subject to a linear confining potential in a possible scenario described by a KKT.

**Special case**

In this special case, we choose zero magnetic field, $B_0 \rightarrow 0$. The radial wave-equations from Eq. (45) becomes

$$
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \tilde{\lambda} - \frac{l_0^2}{r^2} - \eta_L^2 r^2 - b r \right] \psi(r) = 0. \quad (56)
$$

Transforming $\rho = \sqrt{\eta_L} r$ into the Eq. (45), we have

$$
\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \tilde{\lambda} - \frac{l_0^2}{\rho^2} - \rho^2 - b \frac{\rho^3}{\eta_L^3} \right] \psi(r) = 0. \quad (57)
$$

Let us now discuss the asymptotic behavior of the possible solutions to Eq. (57), that is, we hope that $\psi(\rho) \rightarrow 0$ at $\rho \rightarrow 0$ and $\rho \rightarrow \infty$. Suppose the
possible solution to Eq. (57) is
\[ \psi(\rho) = \rho^{l_0} e^{-\frac{1}{2}(\rho + \frac{b}{\eta_L})^2} H(\rho). \] (58)

Substituting the solution Eq. (58) into the Eq. (57), we obtain
\[
H''(\rho) + \left[ \frac{1 + 2|l_0|}{\rho} - 2\rho - \frac{b}{\eta_L^2} \right] H'(\rho)
+ \left[ \frac{\tilde{\lambda}}{\eta_L} + \frac{b^2}{4\eta_L^2} - 2(1 + |l_0|) - \frac{b^2}{2\eta_L^2} \left(1 + 2|l_0|\right) \right] H(\rho) = 0. \] (59)

Substituting the power series solution Eq. (14) into the above equation, we get
\[
c_{n+2} = \frac{1}{(n+2)(n+2 + 2|l_0|)} \left[ \frac{b}{\eta_L^2} \left( n + \frac{3}{2} + |l_0| \right) c_{n+1} - \left\{ \frac{\tilde{\lambda}}{\eta_L} + \frac{b^2}{4\eta_L^2} - 2(1 + |l_0|) - 2n \right\} c_n \right], \] (60)

where few coefficients are
\[
c_1 = \frac{b}{2\eta_L^2} c_0, \quad c_2 = \frac{1}{4(1 + |l_0|)} \left[ \frac{b}{2\eta_L^2} \left( 3 + 2|l_0| \right) c_1 - \left( \frac{\tilde{\lambda}}{\eta_L} + \frac{b^2}{4\eta_L^2} - 2 - 2|l_0| \right) c_0 \right]. \] (61)

The power series solution becomes a polynomial of degree \( n \). For this, we must have \[ 34, 35, 41, 48, 49, 50, 51, 52 \]
\[
\frac{\tilde{\lambda}}{\eta_L} + \frac{b^2}{4\eta_L^2} - 2(1 + |l_0|) = 2n \quad (n = 1, 2, \ldots), \quad c_{n+1} = 0. \] (62)

For \( n = 1 \), we have \( c_2 = 0 \) which implies from (61)
\[
\eta_{11L} = \frac{m^2}{2} (3 + 2|l_0|). \] (63)
a constraint on the potential parameter $\eta_{1,l}$.

By analysing the condition $\frac{\lambda}{\eta L} + \frac{k^2}{4\eta L} - 2(1 + |l_0|) = 2n$, we get

$$E_{n,l} = \pm \sqrt{k^2 + q^2 + 2\eta L (n + 1 + |l_0|)}.$$  \hspace{1cm} (64)

Therefore, the ground state energy eigenvalue is given by

$$E_{1,l} = \pm \sqrt{k^2 + q^2 + 2\eta_{1,l} L (2 + |l_0|)}$$

$$= \pm m \sqrt{\frac{k^2}{m^2} + \frac{q^2}{m^2} + (3 + 2 \frac{|l - \frac{q\Phi}{2\pi}|}{\alpha})(2 + \frac{|l - \frac{q\Phi}{2\pi}|}{\alpha})}. \hspace{1cm} (65)$$

Equation (65) represents energy levels associated with the radial mode $n = 1$ of a Klein-Gordon particle subject to a linear central potential in a background governed by the Kaluza-Klein theory. For $\alpha \to 1$, the energy eigenvalue is consistent with those result in [50].

**Case D : Interactions without potential**

In this case, we consider zero scalar potential, $S = 0$. Therefore, the radial wave-equations Eq. (8) becomes

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \lambda_0 - \frac{l_0^2}{r^2} - m^2 \omega^2 r^2 \right] \psi(r) = 0,$$  \hspace{1cm} (66)

where

$$\lambda_0 = E^2 - k^2 - q^2 - m^2 - 2m \omega \frac{(l - \frac{q\Phi}{2\pi})}{\alpha},$$

$$\omega = \frac{q B_0}{2m}.$$  \hspace{1cm} (67)

Transforming to a new variable $\rho = m \omega r^2$ into the Eq. (66), we obtain [41][43][81]

$$\psi''(\rho) + \frac{1}{\rho} \psi'(\rho) + \frac{1}{\rho^2} \left( -\xi_1 \rho^2 + \xi_2 \rho - \xi_3 \right) \psi(\rho) = 0,$$  \hspace{1cm} (68)
where
\[ \xi_1 = \frac{1}{4}, \quad \xi_2 = \frac{\lambda_0}{4m\omega}, \quad \xi_3 = \frac{\ell_0^2}{4}. \] (69)

Therefore, the energy eigenvalues is given by
\[ E_{n,l} = \pm \sqrt{k^2 + q^2 + m^2 + qB_0 \left( 2n + 1 + \frac{(l - q\Phi)}{2\pi} + \frac{|l - q\Phi|}{2\pi} \right)}, \] (70)

where \( n = 0, 1, 2, ... \).

Equation (70) is the relativistic energy eigenvalues of a scalar charged particle subject to a uniform magnetic field including a magnetic quantum flux in cosmic string space-time within the Kaluza-Klein theory. For zero magnetic quantum flux, \( \Phi \to 0 \), the energy eigenvalues Eq. (70) is consistent with those result obtained in [46]. Thus we can see that the energy eigenvalues Eq. (70) get modify in comparison to those in [46] due to the presence of a magnetic quantum flux \( \Phi \).

The wave-functions is given by
\[ \psi_{n,l}(\rho) = |N|_{n,l} \rho^{\frac{l-q\Phi}{2\alpha}} e^{-\frac{\rho}{\alpha}} L_n^{(\frac{l-q\Phi}{2\alpha})}(\rho), \] (71)

where \( |N|_{n,l} = \left( \frac{n!}{2(n + \frac{q\Phi}{2\pi})!} \right)^{\frac{1}{2}} \) is the normalization constant and \( L_n^{(\frac{l-q\Phi}{2\alpha})}(\rho) \) is the generalized Laguerre polynomials.

We have observed in all cases that the angular momentum number \( l \) is shifted, \( l \to l_0 = \frac{1}{\alpha} (l - \frac{q\Phi}{2\pi}) \), an effective angular quantum number. Therefore, all the relativistic energy eigenvalues obtained here depend on the geometric quantum phase [30, 68]. Thus, we have that, \( E_{n,l}(\Phi + \Phi_0) = E_{n,l \pm \tau}(\Phi) \), where \( \Phi_0 = \pm \frac{2\pi}{q} \tau \) with \( \tau = 0, 1, 2, ... \). This dependence of the relativistic energy levels on the geometric quantum phase gives rise to a relativistic analogue of the Aharonov-Bohm effect for bound states [33, 35, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52].
3 Conclusions

In this work, we have investigated the quantum dynamics of massive charged particles with a uniform magnetic field in the cosmic string space-time in the context of Kaluza-Klein theory with various potential forms. In Case A, we have considered a Cornell-type scalar potential and obtained the energy eigenvalues (21). In Case B, we have considered a Coulomb-type scalar potential and obtained the energy eigenvalues (35). Furthermore, we have discussed a special case corresponds to zero external magnetic field, $B_0 \rightarrow 0$, and obtained the energy eigenvalues (41). In Case C, we have considered linear confining potential and obtained the energy eigenvalues (52). Furthermore, we have discussed a special case corresponds to zero external magnetic field, $B_0 \rightarrow 0$, and obtained the energy eigenvalues (64). We have observed that for $\alpha \rightarrow 1$, the energy eigenvalues reduces to those results obtained in [48, 49, 50]. Thus the presence of the topological defect parameter, $\alpha$, modify the energy spectrum of the quantum system and shifted the energy levels. In Case D, we have solved the Klein-Gordon equation subject to a uniform magnetic field including a magnetic quantum flux in cosmic string space-time in the context of Kaluza-Klein theory without potential. We have obtained the energy eigenvalues (70) and seen that for zero magnetic quantum flux, $\Phi \rightarrow 0$, this energy eigenvalues is consistent in [46]. Thus the energy eigenvalues (70) get modify in comparison to those result in [46] due to the presence of magnetic quantum flux $\Phi$ in the quantum systems.

Data Availability

No data has been used to prepare this paper.
Conflict of Interest

Author declares that there is no conflict of interest regarding publication this paper.

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