Backward energy flow in simple four-wave electromagnetic fields

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Abstract

Electromagnetic energy backflow is a phenomenon that occurs in regions where the direction of the Poynting vector is opposite to that of the propagation of the wave field. It is particularly remarkable in the nonparaxial regime and has been exhibited in the focal region of sharply-focused beams, for vector Bessel beams, and vector-valued spatiotemporally localized waves. A detailed study is undertaken of this phenomenon and the conditions for its appearance are examined in detail in the case of a superposition of four plane waves in free space, the simplest electromagnetic arrangement for the observation of negative energy flow, as well as its comprehensive and transparent physical interpretation. It is shown that the state of polarization of the constituent components of the electromagnetic plane wave quartet determines whether or not energy backflow takes place and what values the energy flow velocity assumes. Depending on the polarization angles, the latter can assume any value from \( c \) (the speed of light in vacuum) to \( -c \) in certain spatiotemporal regions of the field.

Keywords: Poynting vector, energy velocity, polarization, energy backflow

(Some figures may appear in colour only in the online journal)

1. Introduction

The energy backflow of a free-space electromagnetic (EM) field is an unusual effect exhibited in regions where the direction of the Poynting vector is opposite to the propagation direction of the wave field. In this sense, the terms ‘reversed’ or ‘negative’ Poynting vector, as well as ‘negative propagation of light’ and ‘optical retropropagation’, are also used.

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The Poynting vector is commonly introduced to physics students in a course of electricity and magnetism, and illustrated with a plane wave. In this simple case the Poynting vector has the same direction as that of the wave propagation and its quotient by the EM energy density is equal to \( c \). In more advanced courses dealing with left-handed (metamaterial) media the students become aware of situations where the Poynting and wave vectors have opposite directions. The fact, however, that such phenomena may occur even in free space, can be rarely found—if at all—in textbooks on electrodynamics or optics. The reason is probably that, typically, the conditions for the appearance of a negative Poynting vector and the energy backflow effect require complicated nonparaxial EM fields. So far, the effect has been studied in the focal region of sharply-focused optical beams [1–3], in superpositions of four plane waves directed under equal angles, \( \theta \), with respect to the propagation axis [2, 4, 5], for vector Bessel beams [6–10] and vector X-waves [11]. Note that Bessel beams are superpositions of infinitely many monochromatic plane waves making an angle \( \theta \) with respect to a specific axis (say \( z \)) which corresponds to the direction of propagation of the Bessel beam, and X-waves—representatives of the rich family of so-called localized waves—are just the same superpositions of ultrashort pulses [12–15]. Energy backflow has attracted much interest in connection with applications in microparticle manipulation like ‘tractor beams’, etc. (see reviews [16, 17]).

Our purpose is to study in detail the mechanism and conditions for the appearance of the reversed Poynting vector in a quartet of plane waves—the simplest of EM fields for observation of the effect. Distinct from earlier studies, we consider time-dependent EM fields instead of common cycle-averaged ones, and make use of the energy flow velocity. Although the latter is given by a well-known formula related to the Poynting vector and the energy density, it exhibits some surprising properties, e.g., a drastic discrepancy between the group velocity and energy flow velocity [18, 19]. We hope that such a study will result in a more comprehensible and transparent picture of the nature of the backward energy flow effect than can be deduced from cumbersome formulas describing the effect in the case of the vector Bessel beams and other more complicated fields.

In the next two sections we shall study a quartet of plane waves and show how the polarizations of the constituents of the quartet determine whether or not the energy backflow takes place, and what values the energy flow velocity assumes. Section 4 is devoted to studying the general dependence of the energy flow velocity on the polarization angles. In section 5 we discuss the results, touch on the problem of the physical interpretation of the Poynting vector, and some alternative definitions—noteworthy in the given context—of the energy flow velocity. Concluding remarks are provided in section 6.

2. Four-wave fields with regions of negative energy velocity

A quartet of plane waves symmetrically directed with respect to the propagation axis of the whole field constitutes the simplest EM field for which energy backflow is possible4. In order to study the superposition of four interfering plane waves of different polarizations, we start from a ‘seed’ plane wave propagating along the \( z \)-axis with the wavevector \( \mathbf{k}_0 = (0, 0, k) \) and polarization vector \( \mathbf{e}_0(\phi) = (\cos \phi, \sin \phi, 0) \), where \( \phi \) is an arbitrary angle of (linear) polarization with respect to the \( x \)-axis. By making use of well-known rotation \((3 \times 3)\)-matrices \( \mathbf{R}_x(\theta) \) and \( \mathbf{R}_y(\theta) \) which rotate vectors by an angle \( \theta \) about the \( x \)- and \( y \)-axes, respectively, we express

4 There is a claim in the literature [16] that a superposition of two plane waves is sufficient to introduce a negative Poynting vector. This is not wrong, but in this case the vector is not reversed with respect to the propagation direction of the resultant field.
the wave vectors of our quartet of plane waves (see figure 1) as follows:

\[
\begin{align*}
    \mathbf{k}_1 &= \mathbf{R}_y(\theta) \mathbf{k}_0, \\
    \mathbf{k}_2 &= \mathbf{R}_x(-\theta) \mathbf{k}_0, \\
    \mathbf{k}_3 &= \mathbf{R}_y(-\theta) \mathbf{k}_0, \\
    \mathbf{k}_4 &= \mathbf{R}_x(\theta) \mathbf{k}_0.
\end{align*}
\]

In this section we study the case where the ‘seed’ wave for all four waves is polarized along the x-axis, i.e., \( \phi = 0 \). Consequently, we obtain the following polarization vectors in the quartet:

\[
\begin{align*}
    \mathbf{e}_1 &= \mathbf{R}_y(\theta) \mathbf{e}_0(0) = (\cos \theta, 0, -\sin \theta), \\
    \mathbf{e}_2 &= \mathbf{R}_x(-\theta) \mathbf{e}_0(0) = (1, 0, 0), \\
    \mathbf{e}_3 &= \mathbf{R}_y(-\theta) \mathbf{e}_0(0) = (\cos \theta, 0, \sin \theta), \\
    \mathbf{e}_4 &= \mathbf{R}_x(\theta) \mathbf{e}_0(0) = (1, 0, 0),
\end{align*}
\]

which are also depicted in figure 1.

Our aim is to study not only the spatial but also the temporal dependence of energy flow, i.e., we cannot restrict ourselves to the calculation of a cycle-averaged Poynting vector. Therefore, instead of complex field expressions, we have to work with real ones. We assume a spatio-temporal dependence of the constituent plane waves of the form \( \cos(\mathbf{k} \mathbf{r} - \omega t) \), where \( k = \omega/c = 2\pi/\lambda \) and \( \mathbf{r} = (x, y, z) \). As a matter of fact, if the cosine is multiplied by a pulse envelope or replaced by any other suitable real function describing the pulse, some final results
Likewise, for the magnetic field we get

$$E(x, y, z, ct) = \sum_{j=1}^{4} \cos (k, r - kct) e_j$$

where unity amplitude of each plane wave has been assumed. Equation (3) indicates that the field propagates along the \(z\)-axis due to the symmetry of the pairs of plane waves in the quartet. Likewise, for the magnetic field we get

$$B(x, y, z, ct) = \sum_{j=1}^{4} \cos (k, r - kct) (k_j \times e_j)$$

where Heaviside–Lorentz units \((\varepsilon_0 = \mu_0 = 1)\) and normalized speed of light \(c = 1\) are assumed. In the given units the Poynting vector \(S\) and the energy density \(w\) are given by

$$S(x, y, z, ct) = E(x, y, z, ct) \times B(x, y, z, ct),$$

$$w(x, y, z, ct) = \frac{1}{2} \left[ E^2(x, y, z, ct) + B^2(x, y, z, ct) \right].$$

For the quartet the expression of \(S\) turns out to be not too cumbersome if one introduces the following rescaled and dimensionless coordinates: \(\tilde{x} \equiv kx \sin \theta, \tilde{y} \equiv ky \sin \theta, \tilde{z} \equiv kz \cos \theta, \tilde{t} \equiv kct.\) With these definitions we obtain

$$S(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \begin{pmatrix} -4 \cos (\tilde{z} - \tilde{t}) \sin (\tilde{z} - \tilde{t}) \sin \theta \sin \tilde{x} (\cos \tilde{x} + \cos \tilde{y} \cos \theta) \\ -4 \cos (\tilde{z} - \tilde{t}) \sin (\tilde{z} - \tilde{t}) \sin \theta \sin \tilde{y} (\cos \tilde{y} + \cos \tilde{x} \cos \theta) \\ 4 \cos^2 (\tilde{z} - \tilde{t}) (\cos \tilde{x} + \cos \tilde{y} \cos \theta) (\cos \tilde{y} + \cos \tilde{x} \cos \theta) \end{pmatrix}. \quad (6)$$

From (6) the following conclusions can be drawn:

(a) The field of the Poynting vector propagates as a whole with velocity \(v = c/\cos \theta\) along the axis \(z\) and it is invariant because it depends on time only through the difference \(z \cos \theta - ct\) (we name it the propagation variable);

(b) If \(z = t = 0\) and generally on planes \(k(z \cos \theta - ct) = n\pi, n \in \mathbb{Z}\) the transverse components of \(S\) vanish;

(c) On planes \(k(z \cos \theta - ct) = (2n + 1)\pi/2, n \in \mathbb{Z}\) the field \(S\) vanishes;

(d) The longitudinal component \(S_z\) is invariant with respect to interchange of coordinates \(x\) and \(y\);

(e) The transverse components transform to each other if the coordinates \(x\) and \(y\) are interchanged;

(f) There are regions in any transverse plane where the longitudinal component \(S_z\) is negative, i.e., the energy flows backward and this phenomenon takes place irrespective of time instant or cycle averaging, although the latter affects the numerical value of negative \(S_z\). The periodical locations of such regions are indicated in figure 2.
Figure 2. Energy density and energy flow for the quartet of plane EM waves with polarization vectors given by (2) and \( \theta = \frac{2\pi}{5} = 72^\circ \). (a) Greyscale plot of \( w \) as a function of the transverse coordinate \( \tilde{x} \) at \( \tilde{y} = 0 \) (or \( \tilde{y} \) at \( \tilde{x} = 0 \)) and the propagation variable \( \tilde{z} - \tilde{t} \); (b) the same with decreased contrast and superimposed by a vector field plot of the energy flow velocity. Shown are projections of the velocity vectors onto the plane \((\tilde{z}, \tilde{x})\); (c) dependence on \( \tilde{x} \) (along vertical axis) of normalized longitudinal \((\tilde{x})\) component of the Poynting vector \( S_z(\tilde{x}, 0, 0, 0)/S_z(0, 0, 0, 0) \) showing the minimal values \(-0.28\). The range of the vertical axes is \([-2.5\pi, 2.5\pi]\) and of the propagation axes is \([-1.25\pi, 1.25\pi]\). In plot (c) the scale of the transversal coordinate \( \tilde{x} = kx \sin \theta \) along the vertical is shown in units of \( \pi \).

The energy density is given as follows in terms of the dimensionless coordinates:

\[
w(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = 2 \cos^2 (\tilde{z} - \tilde{t}) \left[ (1 + \cos^2 \theta) \cos^2 \tilde{x} + 4 \cos \theta \cos \tilde{x} \cos \tilde{y} + (1 + \cos^2 \theta) \cos^2 \tilde{y} \right] + 2 \sin^2 (\tilde{z} - \tilde{t}) (\sin^2 \tilde{x} + \sin^2 \tilde{y}) \sin^2 \theta.
\]

Similarly to \( S_z \), the energy density obeys the symmetry \( w(x, y, z, ct) = w(y, x, z, ct) \). This conclusion is confirmed by numerically calculated plots, an example of which is presented in figure 2.

The interference pattern in the energy density (figure 2(a)) remarkably differs from that of two plane waves: if the pair (waves \# 2, 4) were removed from the quartet, all maxima would become of equal intensity (e.g., five equally bright maxima along the central vertical \( \tilde{z} - \tilde{t} = 0 \) instead of three in figure 2(a)). Since the magnitude of the Poynting vector \( S \) is strong enough only in regions of local maxima and therefore a vector field plot of \( S \) would be of low distinctness, we present the vector field plot of the energy flow velocity \( V = S/w \) instead. Figure 2(b) clearly reveals that at locations of weak maxima of \( w \) the energy flows backward. While the strength of the component \( S_z \) at these locations comprises only 28% (see figure 2(c)) of that in locations of strong maxima of \( w \), the velocities \( V \) have exactly the same absolute value \( 1 \) (or \( c \) with dimensions) but the opposite signs. Numerical evaluation shows—and it is evident from figure 2(b) also—that wherever the velocity is purely longitudinal it equals the physically maximal possible value \( \pm c \) (small irregularities in the vector field at points of vanishing \( w \) are computational artifacts due to ‘zero divide by zero’). We point out the very noteworthy result
that energy backflow occurs also with velocity $c$. While according to (3) and (4) neither of the fields $E$ or $B$ is transverse, at locations where $V_z = \pm c$ the fields are locally TEM, mutually perpendicular and of equal magnitude (in the chosen units)—as they have to be for a null EM field.

Note that the velocity components $V_z$ and $V_x$ on the plane ($y = 0$) constitute the vector field plot in figure 2(b). If instead we plotted figure 2(b) with $V_z$ and $V_y$ in the same plane, we would obtain another projection of the energy flow velocity, where all arrows would be directed with respect to the $z$-axis, either parallel (in the regions of stronger maxima of $w$) or antiparallel (in the regions of weaker maxima of $w$). This is understandable because according to (6) $S_y$ vanishes on the plane ($y = 0$).

Naturally, the smaller the angle $\theta$, the less remarkable will be the backflow effect: peak negative values of the normalized $S_z$ decrease to 11% if $\theta = \pi/3$ and to 3% if $\theta = \pi/4$. Of course, in the case of paraxial geometry the energy backflow vanishes. However, the abovementioned characteristic features of the velocity vector field remain unchanged with decreasing $\theta$, except transverse narrowing of the backflow channels.

Similar narrowing of the backflow channels is observable also if one moves away from the planes $z - \tau = n\pi$, $n \in \mathbb{Z}$. Curves in figure 3 show the dependence of the longitudinal component of the energy flow velocity in comparison with that of the energy density along the central vertical in figure 2(b), i.e., where $z - \tau = 0$. We see in figure 3(a) that transverse widths of the regions where $V_z = \pm c$ are approximately equal to FWHM of the maxima of $w$. But on the plane $z - \tau = 0.4\pi$, which is close to the plane $z - \tau = \pi/2$ where $V = 0$, the backflow regions narrow substantially and their FWHM become about two times less than those of the minima or maxima of the energy density. By introducing the separate secondary scale for $\tilde{w}(x, 0, 0.4\pi, 0) = w(x, 0, 0.4\pi, 0)/w(0, 0, 0, 0)$ on the right border of the plot, it is clearly seen in figure 3(b) that the negative minima of $V_z$ are encompassed by the (positive) minima of $\tilde{w}$.

In the discussion so far it has been assumed that the seed polarization angle $\phi = 0$. However, it turns out in general that neither $S$ nor $V$ depend on the angle of polarization $\phi$ (if its value is the same for all four waves). This is understandable because, despite the fact that the polarization vectors of the quartet do depend on $\phi$, according to (2), the combination of their mutual relative directions, which determines $S$ and $w$, is invariant.
3. Four-wave fields without regions of negative energy velocity

A closer study of the effect of the angle $\phi$ on the energy flow reveals that what is essential is the difference between the angle $\phi_{13}$ for the pair $(1,3)$ and the angle $\phi_{24}$ for the pair $(2,4)$ in the quartet of waves. Thus, if $\phi_{13} - \phi_{24} = 0$, we return to the results of the previous section. Of special interest are the cases $\phi_{13} - \phi_{24} = \pm \pi/2$. Since the polarization vectors, together with fields $E$ and $B$, depend on both $\phi_{13}$ and $\phi_{24}$, we prescribe values to them separately and for simplicity make one of them equal to zero. So, first we consider the case $\phi_{13} = \pi/2$ and $\phi_{24} = 0$. Then $\mathbf{e}_1 = \mathbf{e}_3 = (0,1,0)$ and $\mathbf{e}_2 = \mathbf{e}_4 = (1,0,0)$, i.e., the pair $(1,3)$ is nothing but a replica of the pair $(2,4)$ rotated clockwise about the $z$-axis by $\pi/2$. Analogously to (3) and introducing the dimensionless arguments, we obtain

$$
E(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \begin{pmatrix}
2 \cos (\tilde{z} - \tilde{r}) \cos \tilde{y} \\
2 \cos (\tilde{z} - \tilde{r}) \cos \tilde{x} \\
0
\end{pmatrix}
$$

and likewise for the magnetic field

$$
B(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \begin{pmatrix}
-2 \cos (\tilde{z} - \tilde{r}) \cos \tilde{x} \cos \theta \\
2 \cos (\tilde{z} - \tilde{r}) \cos \tilde{y} \cos \theta \\
-2 \sin (\tilde{z} - \tilde{r}) (\sin \tilde{x} - \sin \tilde{y}) \sin \theta
\end{pmatrix}
$$

We see that in the given case the EM field as a whole is transverse electric (TE). Equation (5) gives for the Poynting vector and the energy density the expressions

$$
S(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \begin{pmatrix}
-4 \cos (\tilde{z} - \tilde{r}) \sin (\tilde{z} - \tilde{r}) \sin \theta \cos \tilde{x} (\sin \tilde{x} - \sin \tilde{y}) \\
4 \cos (\tilde{z} - \tilde{r}) \sin (\tilde{z} - \tilde{r}) \sin \theta \cos \tilde{y} (\sin \tilde{x} - \sin \tilde{y}) \\
4 \cos^2 (\tilde{z} - \tilde{r}) \cos \theta (\cos^2 \tilde{x} + \cos^2 \tilde{y})
\end{pmatrix}
$$

and

$$
w(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = 2 \cos^2 (\tilde{z} - \tilde{r}) (1 + \cos^2 \theta) (\cos^2 \tilde{x} + \cos^2 \tilde{y})$$

$$+ 2 \sin^2 (\tilde{z} - \tilde{r}) \sin^2 \theta (\sin^2 \tilde{x} - \sin^2 \tilde{y}) .$$

The conclusions from (9) are the same as those derived from (6) in the previous section, but with one essential exception: the sixth conclusion now is that the longitudinal component $S$, cannot take negative values, i.e., the energy flows only forward (of course, it is assumed that $|\theta| \leq \pi/2$ since otherwise the constituent plane waves would propagate backward). These conclusions, including the absence of energy backward flow, are confirmed by numerically calculated plots, see figure 4.

As seen in figure 4, all maxima of the energy density are of equal intensity, the energy velocity vector field is the same at all maxima and contains no backward flow. But what is very noteworthy is that everywhere the energy velocity is essentially subluminal (less than $c$) and wherever the energy flows along the propagation $z$-axis, its velocity is given by the expression

$$
\frac{|V|}{c} = \frac{V_z}{c} = \frac{2\beta}{\beta^2 + 1} = \frac{2\beta^{-1}}{\beta^{-2} + 1},
$$

where $\beta = \cos \theta$. In figure 4, we obtain for $\theta = 2\pi/5$ the value $V_z = 0.564c$. Precisely the same value comes out from (10). For comparison, as mentioned in the previous section, all fields $E, B, S$, and $w$ propagate with velocity $v = c/\cos \theta = \beta^{-1} c = 3.24 c$, i.e., superluminally. Equation (10) was first found in [18] where it was shown that various fields—crossing plane
waves, Bessel beams, Bessel-X pulses and other propagation-invariant wave fields, obey this relation between the axial phase, or group velocity and energy flow velocity.

Next, we consider the case $\phi_{13} = 0$ and $\phi_{24} = \pi/2$. Then the polarization vectors in the quartet turn out to be

\[
\begin{align*}
    \mathbf{e}_1 &= (\cos \theta, 0, -\sin \theta), \\
    \mathbf{e}_2 &= (0, \cos \theta, -\sin \theta), \\
    \mathbf{e}_3 &= (\cos \theta, 0, \sin \theta), \\
    \mathbf{e}_4 &= (0, \cos \theta, \sin \theta),
\end{align*}
\]

i.e., the pair (2, 4) is nothing but a replica of the pair (1, 3) rotated clockwise about the $z$-axis by $\pi/2$. In contradistinction to the previous case, the EM field now is transverse magnetic (TM) and the magnetic field turns out to be equal to (7) but with a minus sign in front of the first component. The expression for the Poynting vector turns out to be the same as in (9) if in the latter the difference of sines is replaced by their sum, and the factor four in the second component by $-4$. As a result, the quartet with polarization vectors of (11) exhibits the same subluminal velocity of the energy flow as in the previous case, and figure 4 illustrates equally well the present case. In [18] we have shown that (i) a TM field of a symmetrical pair of plane waves—the propagating direction of the first one lies on the $(x, z)$ plane and is inclined by angle $+\theta$ with respect to the $z$-axis, and the second one by angle $-\theta$ on the same plane—exhibit subluminal energy velocity along the $z$-axis according to (10); (ii) any superpositions of such pairs rotated about the $z$-axis have the same property. The quartet of the present case is just a particular example of such superposition where the two pairs are rotated clockwise by $\pi/2$.

One can notice that while in the present section the polarizations of the ‘seed’ wave for the pair (1, 3) and (2, 4) are mutually orthogonal, they—on the contrary—were collinear in section 2, resulting in the energy backflow effect. Further study showed that almost the same...
4. Dependence of energy velocity on polarization angles: general case

Since the energy flow velocity field so drastically depends on whether $\phi_{13} = \pi/2$ or $\phi_{13} = 0$, it is of interest to study the dependence over the full range of angles $\phi_{13} = \phi \in [0, \pi]$ (keeping $\phi_{24} = 0$). Let us focus, in particular, on regions where $\mathbf{V}$ is parallel or antiparallel to the $z$-axis and has magnitude $|\mathbf{V}|/c = 1$ or $V_z/c = \pm 1$, see figure 2. For the sake of definiteness, we consider two points on the $x$-axis: first, the point where $\bar{x} = \bar{y} = \bar{z} - \bar{t} = 0$ and, second, where $\bar{x} = \pi, \bar{y} = \bar{z} - \bar{t} = 0$. For the first point we obtain

$$\frac{V_z}{c} = \frac{2 \cos \theta + \cos \phi + \cos \phi \cos^2 \theta}{\cos^2 \theta + 1 + 2 \cos \phi \cos \theta}, \tag{12}$$

which for $\phi = 0$ reduces to $V_z/c = 1$ as it has to (see figure 2) and for $\phi = \pi/2$ reduces to (10) as it also has to (see figure 4). For the second point, a slightly modified version of (12) comes out: only a replacement $\cos \phi \rightarrow -\cos \phi$ must be done. Again, the modified expression for $\phi = 0$ reduces to $V_z/c = -1$ and for $\phi = \pi/2$ to (10), as expected. The behavior of $V_z/c$ over the full range of angles is depicted in figure 5.

We see that if $\phi = \pi/2$ the velocity is given by (10) in accord with figure 4. At values $\phi$ close to $\pi$ the regions of forward and backward flow of energy interchange. Finally, if $\phi = \pi$
the backflow with \( V_z/c = -1 \) takes place along the optical axis\(^5\). Equating the nominator of (12) to zero allows one to answer the question: what are the values of \( \phi \) for which the energy backflow disappears, i.e., both curves in figure 5 become positive? The answer is: \( \phi \) must obey the condition \( \cos \phi = \pm 2 \cos \theta / (\cos^2 \theta + 1) \), i.e., the right hand side of (10). Whether such a surprising coincidence has a deeper physical meaning needs further study.

Similarly to the preceding section, we also studied the behavior of the velocity if the values of angles \( \phi_{13} \) and \( \phi_{24} \) are interchanged, i.e. the velocity dependence on \( \phi_{24} \) over the full range of angle \( \phi_{24} = \phi \in [0, \pi] \) (keeping \( \phi_{13} = 0 \)). Although the interchange of the angles alters the fields \( \mathbf{E} \) and \( \mathbf{B} \) (interchange their components and signs), the Poynting vector \( \mathbf{S} \), as well as the energy density \( w \) and therefore also \( \mathbf{V} \), remain unchanged. Hence, the results given above hold in this case as well.

5. Discussion

Since the variable along the horizontal axis in figures 2(a) and 2(b) and 4(a) and 4(b) is \( z \cos \theta - ct \), these plots can be viewed either as \( z \)-dependence or time-dependence of the fields. The latter viewpoint allows one to read out from these plots what would be the temporally cycle-averaged quantities commonly evaluated from complex-valued EM fields. We see that the direction of energy flow does not flip in time (and/or along the \( z \)-axis) and, in particular, where it flows backward in figure 2(b), it does so for all time. Also, the transverse components of the velocity vanish as a result of such averaging. This follows already from symmetry considerations and from the observation that the product \( \cos (\tilde{z} - \tilde{t}) \sin (\tilde{z} - \tilde{t}) \) entering the transverse components in (6) and (9) vanishes after cycle averaging. In [5] the cycle-averaged \( z \)-component of the Poynting vector has been evaluated for the same polarization vectors as in our (2) and plotted on the plane \( (z = 0) \). Our more detailed results are in accord with corresponding results in [5]. An example of a wave quartet which does not reveal negative \( P_z \) has been also studied by the authors of the cited article, but they did it with another geometry of the polarization vectors than in our two cases of section 3. In their quartet one pair of the polarization vectors is also a replica of the other rotated clockwise about the \( z \)-axis by \( \pi/2 \); the field is also TM and the cycle-averaged \( z \)-component of the Poynting vector is positive everywhere. We studied the cycle nonaveraged vector field of energy velocity in such a quartet and found that, in contradistinction to figure 4(b), the velocity vanishes at the maxima of the energy density \( w \), whereas its longitudinal component at the minima of \( w \) is given by (10).

The general conclusion from the preceding section is in accord with what has been found for the Bessel beams and X-waves [7, 11]: without superposition of TE and TM fields, there is no reversed Poynting vector. This is understandable, since the quartet is an elementary example of superpositions that construct these much more complicated waves.

Although we considered here only monochromatic waves, the results obtained hold also for ultrashort propagation-invariant pulses, more exactly—for apex regions of their X-like or double conical field profile. This can be readily proved by applying the approach used in [18, 19] for the evaluation of the Poynting vector of few-cycle and single-cycle light sheets. Keeping in mind that the velocity \( v = c / \cos \theta \) for such pulses is simultaneously phase and

\(^5\) Negative values of the cycle-averaged Poynting vector in the region of the optical axis have also been derived in [2] for the polarization geometry of the quartet, which corresponds to the case \( \phi = \pi \) in our designations.
group velocity\(^6\), we encounter a striking conclusion: according to (10), the energy flows subluminally while the pulse itself propagates superluminally! Definitely the statement ‘if an energy density is associated with the magnitude of the wave...the transport of energy occurs with the group velocity, since that is the rate of which the pulse travels along’ (in [20], section 7.8) cannot hold if the group velocity exceeds \(c\). A possibility to resolve this contradiction is offered by the notion of reactive energy [19].

It is well known that the Poynting vector is not defined uniquely by the Poynting theorem. Could it be that, consequently, the energy-flow velocity we have dealt with throughout this paper is also not defined uniquely and this is the reason for the energy backflow effect and nonequality of its velocity to the group velocity? The discussion of whether the Poynting vector gives the local energy flow or not has a century-long history, see, e.g., a review [22]. In particular, an alternative definition of the Poynting vector has been proposed which is more in harmony with the notion of group velocity [23]. However, we are not going to dig into the problem here and simply refer to [20] (section 6.7), where it is stated that if one takes into account that the Poynting vector defines also the momentum density of the field and the definition must not violate the theory of relativity, the common expression for the Poynting vector is unique.

An intriguing alternative definition of the energy velocity is proposed in [24]. It springs forth from an idea that a correct velocity in a given location must be equal to the velocity of an inertial frame in which energy flux turns out to be zero at the given location. Such an approach leads to the following relation between the magnitude \(V \equiv |V| = |S|/w\) of the commonly defined energy velocity and its alternative version \(V_a\)

\[
\frac{V}{c} = \frac{2 (V_a/c)}{1 + (V_a/c)^2}.
\] (13)

This relation, which is (11) from [24] rewritten in our designations, surprisingly coincides with (10) if we equate \(V_a/c\) to the normalized group velocity \(\beta^{-1}\) or to its reciprocal. Indeed, in the reference frame moving with \(V_a = \beta\) along the z-axis the angle \(\theta\) transforms to \(\pi/2\) not only for the wave quartet but also for Bessel beams and X-type pulsed localized waves [25], resulting, according to (9), in the vanishing of \(S\). However, further analysis of the alternative definition of the energy flow velocity is outside the scope of this study.

6The phase velocity is given as the reciprocal of the quotient of the gradient of the spatial phase and the frequency, while the group velocity is given as the reciprocal of the derivative of the gradient of the spatial phase with respect to the frequency (see e.g., [20, 21]). As one can see from (3), (4), (7) and (8), for the given type of fields, the z-component of the gradient of the spatial phase depends linearly on the frequency (or on the wavenumber as \(\cos \theta\)), in which case the quotient and the derivative coincide and both velocities are superluminal (since \(1/\cos \theta > 1\), see also conclusion (a) after equation (6)). These circumstances are typical for so-called X-type waves with propagation-invariant fields, see [12–15].

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The generalization of the results for interfering few-cycle ultrashort pulses is straightforward. They may also be useful in studies of quantum backflow and optical analogues of this intriguing effect in particle physics [26–28].

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