Zero Assignment, Pole Placement and Matrix Extension Problems: A Common Point of View

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Abstract

The paper studies a general inverse eigenvalue problem which contains as special cases many well studied pole placement and matrix extension problems. It is shown that the studied problem corresponds on the geometric side to a central projection from some projective variety. The degree for this variety is computed in the critical dimension.

1 Introduction and motivational examples

Let $\mathbb{K}$ be an arbitrary field and consider matrices of size $E, A$ of size $n \times n$ and matrices $B, H$ of size $n \times m$. These matrices define the discrete dynamical

\[ Ex_{t+1} + Ax_t + Hu_{t+1} + Bu_t = 0. \]  

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Consider the vector space $\text{Mat}_{m \times n}$ consisting of all $m \times n$ matrices defined over $\mathbb{K}$. Let $\mathcal{L} \subset \text{Mat}_{m \times n}$ be a linear subspace of dimension $d$. This paper will be devoted to the following ‘constrained’ pole placement question:

**Problem 1.1** Given an arbitrary monic polynomial $\varphi(s) \in \mathbb{K}[s]$ of degree $n$. Is there a feedback law of the form $u_t = F x_t$ having the following properties:

1. The closed loop system

\[
(E + HF)x_{t+1} + (A + BF)x_t = 0
\]

has characteristic polynomial $\varphi(s)$.

2. The feedback satisfies the constraint $F \in \mathcal{L}$.

By definition the characteristic polynomial of the system (1.2) is the unique monic polynomial which is a scalar multiple of $\det [s(E + HF) + (A + BF)]$.

Note that the set of monic polynomials of degree $n$ can be identified with the vector space $\mathbb{K}^n$. If Problem 1.1 has a positive answer for all monic polynomials $\varphi(s) \in \mathbb{K}^n$ of degree $n$, then we will say that the system (1.1) is arbitrarily pole assignable in the class of feedback compensators $\mathcal{L}$. If for a generic set of monic polynomials $\varphi(s) \in \mathbb{K}^n$ of degree $n$ Problem 1.1 has a positive answer, then we will say that system (1.1) is generically pole assignable in the class of feedback compensators $\mathcal{L}$.

A dimension argument immediately reveals that (1.1) is generically pole assignable only if $\dim \mathcal{L} \geq n$. Another natural necessary condition for generic pole assignability is the left primeness of the matrix pencil $[sE + A \ sH + B]$. This last condition is satisfied for a generic set of matrices $E, A, B, H$.

If the pencil $[sE + A \ sH + B]$ is left prime then the transfer function $(sE + A)^{-1}(sH + B)$ defines a system of McMillan degree $n$ which has the generic controllability indices and every $n \times m$ transfer function of McMillan degree $n$ with the generic controllability indices is of the form $(sE + A)^{-1}(sH + B)$. In terms of transfer functions the problem therefore asks: Given a $n \times m$ transfer function $(sE + A)^{-1}(sH + B)$ of McMillan degree $n$ which has the generic controllability indices and given a polynomial $\varphi(s) \in \mathbb{K}[s]$ of degree $n$ whose roots are disjoint from the roots of $\det(sE + A)$, is there a $m \times n$ matrix $F$ with $F \in \mathcal{L}$ and having the property that the zeroes of the transfer function $[I_n + (sE + A)^{-1}(sH + B)F]$ coincide with the roots of $\varphi(s)$?

The following set of examples show that Problem 1.1 is very general indeed and it contains in special cases many well studied pole placement and matrix extension problems.

**Example 1.2** Let $E = I_n$, $H = 0$ and let $\mathcal{L} = \text{Mat}_{m \times n}$. In this situation Problem 1.1 consists of the well known state feedback pole placement problem. In this case Problem 1.1 has a solution if and only if the matrices $A, B$ form a controllable pair. In other word the genericity condition $\text{rank }[B, AB, \ldots, A^{n-1}B] = n$ has to be satisfied. This last condition is equivalent with the left primeness of the matrix pencil $[sI_n + AB]$. 

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Example 1.3 Consider the static output feedback pole placement problem over the complex numbers $\mathbb{C}$:

\[
\dot{x} = Ax + Bu, \quad y = Cx, \quad x \in \mathbb{C}^n, u \in \mathbb{C}^m \text{ and } y \in \mathbb{C}^p.
\] (1.3)

The problem asks for a static feedback law $u = Ky$ such that the closed loop system

\[
\dot{x} = (A + BK) x, \quad y = Cx
\]

has some desired closed loop characteristic polynomial. One immediately verifies that Problem 1.1 covers this situation if one chooses $E = I_n$, $H = 0$ and $\mathcal{L} := \{KC \mid K \in \text{Mat}_{m \times p}\}$.

Over the complex numbers the main result in this area of research was given by Brockett and Byrnes [2]. It states:

**Theorem 1.4** If $n \leq mp = \dim \mathcal{L}$ then for a generic set of matrices $A, B, C$, the system (1.3) is arbitrarily pole assignable. Moreover if $n = mp$ then when counted with multiplicities there are exactly as many solutions as the degree of the complex Grassmannian variety $\text{Grass}(m, \mathbb{C}^{m+p})$ once embedded via the Plücker embedding.

Example 1.5 Let $m = n$, $E = I_n$, $H = 0$ and $B = I_n$. In this case Problem 1.1 asks for conditions which guarantee that the characteristic map

\[
\chi_A : \mathcal{L} \longrightarrow \mathbb{K}^n, \quad F \longmapsto \det(sI + A + F)
\] (1.4)

is surjective or at least ‘generically’ surjective. This general matrix extension problem contains itself many of the matrix completion problems as they were studied in [1, 4, 5, 7].

The main result in the situation of Example 1.5 has been derived in [11]. It states:

**Theorem 1.6** If the base field $\mathbb{K}$ is algebraically closed then for a generic set of matrices $A \in \text{Mat}_{n \times n}$ the characteristic map (1.4) is dominant (generically surjective) if and only if

1. $\dim \mathcal{L} \geq n$.

2. There must be at least one element $L \in \mathcal{L}$ whose trace $\text{tr}(L) \neq 0$, i.e. $\mathcal{L} \not\subset \text{sl}_n$.

The main result of this paper (Theorem 2.6) will show that over an algebraically closed field system (1.1) is generically pole assignable for a generic set of matrices $E, A, B, H$ if and only if $\dim \mathcal{L} \geq n$.

The paper is structured as follows: In Section 2 we will introduce a natural compactification of the linear space $\mathcal{L}$ which we will denote by $\hat{\mathcal{L}}$. In order to prove the main theorem we will show that one has a characteristic map $\chi$ defined on a Zariski open set of the variety $\hat{\mathcal{L}}$. Geometrically $\chi$ describes a central projection from the variety $\hat{\mathcal{L}}$ to the projective space $\mathbb{P}^n$. As a consequence the number of solutions in the critical dimension, i.e. in the situation where $\dim \hat{\mathcal{L}} = n$, is equal to $\deg \hat{\mathcal{L}}$ when counted with multiplicities and when taken into account some possible ‘infinite solutions’.
The degree of the variety $\mathcal{L}$ is of crucial importance for the understanding of the characteristic map $\chi$. In Section 3 we compute the degree of $\mathcal{L}$ in many special cases. As a corollary we will rediscover several matrix completion results as they were derived earlier in [3, 4, 5, 7].

Finally in Section 4 we will compute the degree of $\mathcal{L}$ for a generic subspace $\mathcal{L} \subset \text{Mat}_{n \times n}$.

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2 Compactification of the problem

The inverse eigenvalue problem formulated in Problem 1.1 describes an intersection problem in the linear variety $\mathcal{L}$. In order to invoke results from intersection theory [6] it is important to understand the intersection at the ‘boundary’ of $\mathcal{L}$. What is needed is a good compactification of $\mathcal{L}$. It turns out that Problem 1.1 induces in a natural way a compactification and we will explain this in the sequel.

The closed loop characteristic polynomial can be written as

$$
\det [s(E + HF) + (A + BF)] = \det [sE + A | sH + B] \left[ \begin{array}{c} I_n \\ F \end{array} \right] = \det \left[ \begin{array}{cc} I_m & F \\ -sH - B & sE + A \end{array} \right].
$$

(2.1)

Following an idea introduced by Brockett and Byrnes [2] for the static output pole placement problem we will identify rowsp $[I_m F]$ with an element of Grass($m, \mathbb{K}^{m+n}$). In this way we have natural embeddings

$$
\mathcal{L} \longrightarrow \text{Grass}(m, \mathbb{K}^{m+n}) \longrightarrow \mathbb{P} \left( \wedge^m \mathbb{K}^{m+n} \right) =: \mathbb{P}^N.
$$

Definition 2.1 Let $\mathcal{L}$ be the projective closure of $\mathcal{L}$.

By definition $\bar{\mathcal{L}}$ is a projective variety of dimension dim $\bar{\mathcal{L}} = \dim \mathcal{L}$. The remainder of the paper will be devoted to a large extend in the study of this variety. In order to have a general idea of how the projective closure of $\mathcal{L}$ is defined, we start with an illustrative example.

Example 2.2 Let $\mathcal{L} \in \text{Mat}_{3 \times 3}$ be defined by

$$
\mathcal{L} = \left\{ \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix} \right\}
$$

(2.2)
where \( a, b, c \in \mathbb{K} \) are arbitrary elements. Then for fixed \( a, b, c \)
\[
\begin{bmatrix}
1 & 0 & 0 & a & b & 0 \\
0 & 1 & 0 & c & a & b \\
0 & 0 & 1 & 0 & c & a
\end{bmatrix}
\]  \tag{2.3}

is a point in \( \text{Grass}(3, \mathbb{K}^6) \). Let \( z_{ijk} \) be the full size minor of (2.3) consisting of the \( i \)th, \( j \)th, \( k \)th columns. Then \( \{z_{ijk}\} \) are the Plücker coordinates of \( \text{Grass}(3, \mathbb{K}^6) \) in \( \mathbb{P}^{19} \). \( \mathcal{L} \) is defined by 6 linear equations of its entries. In terms of the Plücker coordinates, they become
\[
\begin{align*}
z_{234} &= -z_{135}, \\
z_{234} &= z_{126}, \\
z_{235} &= -z_{136}, \\
z_{125} &= -z_{134}, \\
z_{125} &= 0, \\
z_{236} &= 0.
\end{align*}
\]  \tag{2.4}

\( \mathcal{L} \) has 9 minors of size \( 2 \times 2 \), but there are only 6 monomials of degree 2 of \( a, b, c \):
\[
a^2, b^2, c^2, ab, ac, bc.
\]

So there are 3 linear relations among the \( 2 \times 2 \) minors. In terms of the Plücker coordinates, they are
\[
\begin{align*}
z_{146} &= -z_{245}, \\
z_{345} &= z_{156}, \\
z_{346} &= -z_{256}.
\end{align*}
\]  \tag{2.5}

The monomials \( a^2, b^2, c^2, ab, ac, bc \) are not algebraically independent, they satisfy the relation
\[
\text{rank } \det \begin{bmatrix}
a^2 & ab & ac \\
ab & b^2 & bc \\
ac & bc & c^2
\end{bmatrix} \leq 1
\]  \tag{2.6}

i.e. all the \( 2 \times 2 \) minors of (2.6) are zero, which induce 6 quadratic relations among the \( 2 \times 2 \) minors of \( \mathcal{L} \):
\[
\begin{align*}
z_{346}^2 + z_{246}z_{356} &= 0, \\
(z_{246} + z_{345})^2 - z_{356}z_{145} &= 0, \\
z_{346}(z_{246} + z_{345}) + z_{356}z_{146} &= 0.
\end{align*}
\]  \tag{2.7}

It is not hard to show that \( \bar{\mathcal{L}} \) is defined by (2.4), (2.5), and (2.7) in \( \text{Grass}(3, \mathbb{K}^6) \subset \mathbb{P}^{19} \).

Note that every element in \( \bar{\mathcal{L}} \) can be simply represented by a subspace of the form rowsp \([F_1 \ F_2]\), where the \( m \times m \) matrix \( F_1 \) is not necessarily invertible. Row span \([F_1 \ F_2]\) describes an element of \( \mathcal{L} \) if and only if \( F_1 \) is invertible. Note that a characteristic equation is even defined if \( F_1 \) is singular unless the polynomial in (2.1) is the zero polynomial.

Let \( f_i, i = 0, \ldots, N \) be the Plücker coordinates of rowsp \([F_1 \ F_2]\). In terms of the Plücker coordinates the characteristic equation can then be written up to a constant factor as:
\[
\det \begin{bmatrix}
F_1 & F_2 \\
-sH-B & sE+A
\end{bmatrix} = \sum_{i=0}^{N} f_i p_i(s),
\]  \tag{2.8}
where the \( p_i(s) \) represent up to sign and order the full size minors of \([-sH - B \ sE + A]\).

Let \( Z \subset \mathbb{P}^N \) be the linear subspace defined by

\[
Z = \{ z \in \mathbb{P}^N | \sum_{i=0}^{N} p_i(s)z_i = 0 \}. \tag{2.9}
\]

Identify a closed loop characteristic polynomial \( \varphi(s) \) with a point in \( \mathbb{P}^n \). In analogy to the situation of the static pole placement problem considered in [2, 17] (compare also with [15, Section 5]) one has a well defined characteristic map

\[
\chi : \bar{\mathcal{L}} - Z \rightarrow \mathbb{P}^n \quad \text{rowsp} [F_1 \ F_2] \rightarrow \sum_{i=0}^{N} f_i p_i(s). \tag{2.10}
\]

It will turn out that surjectiveness of the map \( \chi \) will imply the generic pole assignability of system (1.1) in the class of compensators \( \mathcal{L} \). The geometric properties of the map \( \chi \) are as follows:

**Theorem 2.3** The map \( \chi \) defines a central projection. In particular if \( Z \cap \bar{\mathcal{L}} = \emptyset \) and \( \dim \mathcal{L} = n \) then \( \chi \) is surjective of mapping degree equal to the degree of the variety \( \bar{\mathcal{L}} \).

The proof for this theorem is identical to the one given in [17]. In the algebraic geometry literature (see e.g. [4, 13]) \( \chi \) is sometimes referred to as a *projection of \( \mathcal{L} \) from the center \( Z \) to \( \mathbb{P}^n \) and \( Z \cap \bar{\mathcal{L}} \) is sometimes referred to as the *base locus*. Of course the interesting part of the theorem occurs when \( Z \cap \bar{\mathcal{L}} = \emptyset \) since in this situation very specific information on the number of solutions is provided. If \( Z \cap \bar{\mathcal{L}} = \emptyset \) and \( \dim \mathcal{L} = n \) then one says that \( \chi \) describes a *finite morphism* from the projective variety \( \bar{\mathcal{L}} \) onto the projective space \( \mathbb{P}^N \).

In analogy to the situation of the static pole placement problem [4, 17] and the dynamic pole placement problem [13] we introduce a definition for this important situation:

**Definition 2.4** A particular system \( E, A, B, H \) is called \( \mathcal{L} \)-nondegenerate if \( Z \cap \bar{\mathcal{L}} = \emptyset \). A system which is not \( \mathcal{L} \)-nondegenerate will be called \( \mathcal{L} \)-degenerate.

In general it will always happen that certain systems \( E, A, B, H \) are \( \mathcal{L} \)-degenerate. The next theorem shows that if the dimension of \( \mathcal{L} \) is not too large then the set of matrices \( E, A, B, H \) which are \( \mathcal{L} \)-degenerate are contained in a proper algebraic subset when viewed as a subset in the vector space \( \mathbb{K}^{2m(m+n)} \).

**Lemma 2.5** Assume the base field \( \mathbb{K} \) is algebraically closed. If \( \dim \mathcal{L} > n \) then every system \( E, A, B, H \) is \( \mathcal{L} \)-degenerate. If \( \dim \mathcal{L} \leq n \) then a generic set of systems \( E, A, B, H \) is \( \mathcal{L} \)-nondegenerate.

**Proof:** If \( \dim \mathcal{L} > n \) then \( Z \cap \bar{\mathcal{L}} \) is nonempty by the (projective) dimension theorem (see e.g. [13]) and the fact that \( \dim Z \geq N - n - 1 \).
Assume now that $\dim \mathcal{L} \leq n$. Identify the set of systems $E, A, B, H$ with the vector space $\mathbb{K}^{2m(m+n)}$. In analogy to the proof of [13, Lemma 5.3] we compute the dimension of the coincidence set

$$\mathcal{S} := \left\{ (F_1, F_2; E, A, B, H) \in \bar{\mathcal{L}} \times \mathbb{K}^{2m(m+n)} \mid \det \begin{bmatrix} F_1 & F_2 \\ -sH-B & sE+A \end{bmatrix} = 0 \right\}. \tag{2.11}$$

Using the same arguments as in [13] one computes

$$\dim \mathcal{S} = \dim \bar{\mathcal{L}} + 2m(m+n) - n - 1.$$ 

Since $\bar{\mathcal{L}}$ is projective the projection onto the second factor (namely $\mathbb{K}^{2m(m+n)}$) is an algebraic set by the main theorem of elimination theory (see e.g. [13]). This projection can result in an algebraic set of dimension at most $\dim \mathcal{S}$. The claim therefore follows. 

We are now in a position to state one of the main theorems of this paper:

**Theorem 2.6** Assume the base field $\mathbb{K}$ is algebraically closed. Let $\mathcal{L} \subset \text{Mat}_{n \times n}$ be a fixed subspace. Then the map $\chi$ introduced in (2.10) is surjective for a generic set of matrices $E, A, B, H$ if and only if $\dim \mathcal{L} \geq n$. If $\dim \mathcal{L} = n$ then for a generic set of matrices $E, A, B, H$ the intersection $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$ and the characteristic map $\chi$ describes a finite morphism of mapping degree which is equal to the degree of the variety $\bar{\mathcal{L}}$.

*Proof:* (Compare with [14, Theorem 2.14]). If $\dim \mathcal{L} < n$ then a simple dimension argument shows that $\chi$ cannot be surjective. We therefore will assume that $\dim \mathcal{L} := d \geq n$.

Consider once more the coincidence set $\mathcal{S}$ introduced in (2.11) and consider the projection onto $\mathbb{K}^{2m(m+n)}$. For a generic point inside $\mathbb{K}^{2m(m+n)}$ the fiber of the projection has dimension equal to $d - n - 1$. Let $E, A, B, H$ be a system whose fiber has this dimension and let $\mathcal{Z}$ be the corresponding center as defined in (2.9). By construction we have that

$$\dim \mathcal{Z} \cap \bar{\mathcal{L}} = \dim \mathcal{L} - n - 1 = d - n - 1.$$ 

In particular if $\dim \mathcal{L} = n$ then $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$ and the characteristic map $\chi$ is a finite morphism.

If $d > n$ choose a subspace $H \subset \mathbb{P}^{N}$ having codimension $d - n$ inside $\mathbb{P}^{N}$ and having the property that

$$\bar{\mathcal{L}} \cap \mathcal{Z} \cap H = \emptyset. \tag{2.12}$$

Such a subspace $H$ exists by [13, Corollary (2.29)]. Let $\pi_1 : \bar{\mathcal{L}} \to \mathbb{P}^{d}$ be the central projection with center $\mathcal{Z} \cap H$ and let $\pi_2 : \mathbb{P}^{d} - \pi_1(\mathcal{Z}) \to \mathbb{P}^{n}$ be the central projection with center $\pi_1(\mathcal{Z})$. Then $\pi_1$ is a finite morphism which is surjective over $\mathbb{K}$. $\pi_2$ is a linear map, it is surjective as well and

$$\chi = \pi_2 \circ \pi_1.$$ 

It follows that $\chi$ is surjective as soon as $\dim \mathcal{L} \geq n$. 

□
Theorem 2.6 assumes that the field is algebraically closed. For general fields it is often possible to deduce some results by considering the corresponding question over the algebraic closure. The following results is of this sort:

**Corollary 2.7** If the degree of the variety $\mathcal{L}$ defined over the complex numbers $\mathbb{C}$ is odd and if $\dim \mathcal{L} \geq n$ then $\chi$ is also surjective over the real numbers $\mathbb{R}$ for a generic set of real matrices $E, A, B, H$.

**Proof:** Let $E, A, B, H$ be a set of real matrices whose fiber has dimension equal to $d - n - 1$. (Since the set of real matrices inside $\mathbb{C}^{2m(m+n)}$ is not contained in an algebraic set, such real matrices exist.) Let $Z$ be the induced center. If the degree of $\mathcal{L}$ is odd then the finite morphism $\pi_1 : \mathcal{L} \to \mathbb{P}^d$ is surjective over the real numbers. Indeed over the complex numbers the inverse image $\pi^{-1}_1(y) \subset \mathcal{L}$ represents a finite set of complex conjugate points for every real point $y \in \mathbb{P}^d$. But then also $\pi_2$ and $\chi$ are surjective over the reals. \qed

3 The degree of $\mathcal{L}$ in some special situations

In this and in the next section we will assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. We will show in a moment that the compactification $\mathcal{L}$ is in many cases isomorphic to the product of some Schubert varieties. This will allow us to compute the degree of $\mathcal{L} \subset \mathbb{P}^N$ in these cases.

For the convenience of the reader we summarize the basic notions. More details can be found in [12, 16] and [6, Chapter 14].

Consider a flag

$$F : \{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_{m+n} = \mathbb{K}^{m+n}$$

where we assume that $\dim V_q = q$ for $q = 1, \ldots, m + n$. Let $\nu = (\nu_1, \ldots, \nu_m)$ be an ordered index set satisfying

$$1 \leq \nu_1 < \ldots < \nu_m \leq m + n.$$

With respect to the flag $F$ one defines the Schubert variety

$$S(\nu_1, \ldots, \nu_m) := \{ W \in \text{Grass}(m, \mathbb{K}^{m+n}) \mid \dim (W \bigcap V_{\nu_k}) \geq k \text{ for } k = 1, \ldots, m \}$$

and the Schubert cell

$$C(\nu_1, \ldots, \nu_m) := \{ W \in S(\nu_1, \ldots, \nu_m) \mid \dim (W \bigcap V_{\nu_{k-1}}) = k - 1; \text{ for } k = 1, \ldots, m \}.$$

The closure of the Schubert cell $C(\nu_1, \ldots, \nu_m)$ inside the variety $\text{Grass}(m, \mathbb{K}^{m+n}) \subset \mathbb{P}^N$ is equal to the Schubert variety $S(\nu_1, \ldots, \nu_m)$. By definition, $S(\nu_1, \ldots, \nu_m)$ is a projective variety. There is a well known formula for the degree of a Schubert variety [12, Chapter XIV, §6, (7)]:

$$\deg S(\nu_1, \ldots, \nu_k) = (\sum_i (\nu_i - i))! \prod_{j>i} (\nu_j - \nu_i) \prod_i (\nu_i - 1)!.$$
Let $\mathcal{B} := \{v_1, \ldots, v_{m+n}\} \subset \mathbb{K}^{m+n}$ be a basis which is compatible with the flag $\mathcal{F}$. In other words this basis has the property that $V_i = \text{span}(v_1, \ldots, v_i)$. With respect to the basis $\mathcal{B}$ one can represent the Schubert cell $C(v_1, \ldots, v_m)$ as the set of all $m$-dimensional subspaces in $\mathbb{K}^{m+n}$ which are the rowspaces of a matrix of the form:

$$
\begin{bmatrix}
* & \cdots & * & 1 & 0 & \cdots & 0 & \cdots & 0 \\
* & \cdots & * & 0 & \cdots & \ast & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & \cdots & 0 & \cdots & * & 0 & \cdots & \ast & 1 \\
\end{bmatrix}
\tag{3.1}
$$

where the 1’s are in the columns $v_1, \ldots, v_m$.

The cell $C(v_1, \ldots, v_m)$ is isomorphic to $\mathbb{K}^d$, where $d = \sum_{i=0}^m (\nu_i - i)$. In particular the cell $C(v_1, \ldots, v_m)$ is isomorphic to every subspace $\mathcal{L} \subset \text{Mat}_{m \times n}$ having dimension $\dim \mathcal{L} = d$. In general it is not true that the closures $S(v_1, \ldots, v_m) \subset \mathbb{P}^N$ and $\mathcal{L} \subset \mathbb{P}^N$ are isomorphic. This happens however in the following situation:

Let $E_{i,j}$ be the $m \times n$ matrix whose $i, j$-entry is 1 and all the other entries are 0. Let $\mu = (\mu_1, \ldots, \mu_m)$ be an ordered index sets satisfying

$$0 \leq \mu_1 \leq \ldots \leq \mu_m \leq n.$$

**Definition 3.1** $\mathcal{L} \subset \text{Mat}_{m \times n}$ is called a lower left filled linear space of type $\mu$ if $\mathcal{L}$ is spanned by the matrices

$$E_{i,j} \quad \text{for} \quad j \leq \mu_i, \ i = 1, \ldots, m.$$

**Lemma 3.2** If $\mathcal{L} \subset \text{Mat}_{m \times n}$ is a lower left filled linear space of type $\mu$ then $\bar{\mathcal{L}}$ is isomorphic to the Schubert variety $S(\mu_1 + 1, \mu_2 + 2, \ldots, \mu_m + m)$.

**Proof:** Let $\nu_i := \mu_i + 1, \ i = 1, \ldots, m$. There is a fixed $(m+n) \times (m+n)$ permutation matrix $P$ such that the set

$$\{[I_m|F] P \mid F \in \mathcal{L}\} \subset \text{Mat}_{m \times (m+n)}$$

is equal to the cell $C(\nu_1, \ldots, \nu_m)$ described in (3.1). The linear transformation $P \in \text{Gl}_{m+n}$ extends to a linear transformation in $\mathbb{P}(\wedge^m \mathbb{K}^{m+n}) = \mathbb{P}^N$ and this linear transformation maps $\bar{\mathcal{L}}$ isomorphically onto $S(\nu_1, \ldots, \nu_m)$.

The proof of the lemma shows in particular that permutations of the columns inside $\text{Mat}_{m \times n}$ result in isomorphic compactifications. The following lemma shows that a broader range of transformations do not change the topological properties of the compactification.

**Lemma 3.3** Assume there are subspaces $\mathcal{L}_1, \mathcal{L}_2 \subset \text{Mat}_{n \times n}$. If there are linear transformations $S \in \text{Gl}_n$ and $T \in \text{Gl}_n$ such that $\mathcal{L}_2 = S \mathcal{L}_1 T^{-1}$ then there exists an automorphism of $\mathbb{P}^N$ which maps the compactification $\bar{\mathcal{L}}_1$ isomorphically onto the compactification $\bar{\mathcal{L}}_2$. 

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Proof: 
\[ [I_m \mid SL_1T^{-1}] = S[I_m \mid L_1] \begin{bmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \].

The matrix to the right, an element of $GL_{m+n}$, induces a linear transformation on the projective space $\mathbb{P}(\wedge^m \mathbb{K}^{m+n}) = \mathbb{P}^N$ which maps $\overline{L_1}$ onto $\overline{L_2}$. \qed

**Theorem 3.4** Assume there are linear transformations $S \in Gl_m$ and $T \in Gl_n$ such that 
\[ S L T^{-1} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & L_k \end{pmatrix}, \]

where each $L_i$, $l = 1, \ldots, k$ is the space of $m_l \times n_l$ lower left filled matrices of type $\mu^l$:

\[ 0 \leq \mu^l_1 \leq \cdots \leq \mu^l_{m_l} \leq n_l. \]

Then $\overline{L}$ is isomorphic to the product of Schubert varieties 
\[ \overline{S}((\mu^1_1 + 1, \mu^2_1 + 2, \ldots, \mu^1_{m_1} + m_1) \times \cdots \times S(\mu^k_1 + 1, \mu^k_2 + 2, \ldots, \mu^k_{m_k} + m_k) \]

and

\[ \deg \overline{L} = \left( \sum \mu^l_i \right) \frac{\prod_{i \neq l} (\mu^l_i + l - \mu^l_i)}{(\prod_i (\mu^1_i + l - 1))}. \]

**Proof:** The closure of $[L_i, I_{m_l}]$ in the Grassmannian variety $\text{Grass}(m_l, \mathbb{K}^{m_l+n_l})$ is the Schubert variety $S(\mu^1_1 + 1, \ldots, \mu^l_{m_l} + m_l)$, and $\overline{L}$ is a product of Schubert varieties.

The degree formula of a product of projective varieties under the Segre embedding [18, Proposition 2.1] is given by

\[ \deg Z_1 \times \cdots \times Z_k = \left( \sum_i \dim Z_i \right)! \frac{\prod_i \dim Z_i!}{\prod_i (\dim Z_i)!} \prod_i \deg Z_i. \]

Combining these formulas gives the result. \qed

**Corollary 3.5** When $\mu^1_1 = \cdots = \mu^1_{m_1} = n_1$ and $\mu^l_i = 0$ for $l > 1$, then the compactification $\overline{L} = \text{Grass}(m_1, \mathbb{K}^{m_1+n_1})$ and its degree is

\[ \frac{(m_1 n_1)! 2! \cdots (m_1 - 1)!}{n_1! (n_1 + 1)! \cdots (n_1 + m_1 - 1)!} \].
Using Lemma 3.3 and Corollary 3.5 we can deduce Theorem 1.4, the result of Brockett and Byrnes. For this assume that \( \mathcal{L} = \{ BFC \mid F \in \text{Mat}_{m \times p} \} \). Without loss of generality we can assume that \( B, C \) have full rank, \( \text{rank} \, B = m \) and \( \text{rank} \, C = p \). (Theorem 1.4 assumes genericity!) There are invertible matrices \( S, T \) such that 
\[
SB = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \quad \text{and} \quad CT^{-1} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}.
\]
It follows that 
\[
SLT^{-1} = \left\{ \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \in \text{Mat}_{n \times n} \mid F \in \text{Mat}_{m \times p} \right\}.
\]

According to Lemma 3.3 and Corollary 3.5 the compactification is isomorphic to the Grassmannian \( \text{Grass}(m, K^{m+p}) \) as predicted by Theorem 1.4. In order to fully prove Theorem 1.4 it remains to be shown that for a generic set of matrices \( A \in \text{Mat}_{n \times n} \) the system is \( \mathcal{L} \)-nondegenerate as soon as \( n = mp \).

**Corollary 3.6** When \( m_l = n_l = 1 \) and \( \mu_1^l = 1 \) for all \( l \), then \( \mathcal{L} = \prod_{l=1}^n \mathbb{P}^1 = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \) and its degree is 
\[
n!.
\]

Corollary 3.6 covers a result first studied by Friedland [4, 5]. Indeed the subspace \( \mathcal{L} \subset \text{Mat}_{n \times n} \) corresponds in this case exactly to the set of diagonal matrices. By Theorem 2.6 we know that for a generic set of matrices \( E, A, B, H \) the characteristic map \( \chi \) is a finite morphism of mapping degree \( m! \). Friedland [4, 5] and Byrnes and Wang [3] did show that the set of all matrices of the form \( I_n, A, I_n, 0 \) belongs to this generic set. We therefore have the result:

**Theorem 3.7 ([3, 4, 5])** Let \( \mathcal{L} \subset \text{Mat}_{n \times n} \) be the set of all diagonal matrices defined over an algebraically closed field \( K \). If \( A \in \text{Mat}_{n \times n} \) is an arbitrary matrix and \( \varphi \in K[s] \) is an arbitrary monic polynomial of degree \( n \) then there are exactly \( n! \) diagonal matrices \( F \in \mathcal{L} \) (when counted with multiplicity) such that the matrix \( A + F \) has characteristic polynomial \( \varphi(s) \).

## 4 The degree of \( \mathcal{L} \) in the generic situation

In the previous section we computed the degree of the variety \( \mathcal{L} \) in many special cases. The set of all subspace \( \mathcal{L} \subset \text{Mat}_{n \times n} \) having the property that \( \text{dim} \, \mathcal{L} = d \) can be identified with the Grassmannian variety \( \text{Grass}(d, \mathbb{K}^{mn}) \). The degree attains its maximal value on a Zariski open subset of \( \text{Grass}(d, \mathbb{K}^{mn}) \). This largest possible degree is sometimes referred to as the generic degree. In this section we will determine this generic degree if \( d = m = n \). The result is as follows:

**Theorem 4.1** Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero. There is a generic subset \( U \subset \text{Grass}(n, \mathbb{K}^{n^2}) \) such that the compactification \( \mathcal{L} \subset \mathbb{P} \left( \wedge^n \mathbb{K}^{2n} \right) = \mathbb{P}^N \) of every element \( \mathcal{L} \in U \) has degree \( n(n-1)^{n-1} \). This is also equal to the maximal possible degree among all varieties \( \mathcal{L} \) with \( \text{dim} \, \mathcal{L} = n \) and \( \mathcal{L} \subset \text{Mat}_{n \times n} \).
The proof of this theorem will require a fair amount of algebraic geometry. In particular, our proof involves a “blowing-up method.” This is an important method in algebraic geometry and is the main tool in the resolution of singularities of algebraic varieties, or in the elimination of the points of indeterminacy of rational maps in consideration. (A rational map is a morphism which is only defined on some open and (Zariski-) dense subset.) This blowing-up construction enables us to compute the degree of the rational map. The interested reader may want to consult [8, 10] for the notation and basic facts on blowing-up.

In order to make the proof more understandable we will explain it first in the specific examples \( n = 3 \) and \( n = 4 \). Thereafter we will give the general proof.

**Example 4.2** Let

\[
\mathcal{L} = \begin{bmatrix}
z_1 & z_2 & z_3 \\ 0 & z_1 & z_2 \\ z_3 & 0 & z_1
\end{bmatrix},
\]

be a 3-dimensional linear subspace in \( \text{Mat}_{3 \times 3} \). The full size minors of \([I, \mathcal{L}]\) give the following 20 coordinates:

\[
(1, z_1, z_2, z_3, z_1^2, z_2z_3, -z_1z_3, z_3^2 - z_1^2, z_2^2 - z_1z_3, \ldots, z_1^3 + z_2^2z_3 - z_1z_3^2).
\]

By adding another variable \( z_0 \) to compactify \( \mathcal{L} \) and homogenize the coordinates, we get

\[
(z_0^3, z_0^2z_1, z_0^2z_2, z_0^2z_3, z_0z_1^2, z_0z_2z_3, \ldots, z_1^3 + z_2^2z_3 - z_1z_3^2).
\]

Let

\[
\phi : \mathbb{P}^3 \longrightarrow \mathbb{P}^{19}
\]

be the rational map defined by the above set of degree 3 homogeneous polynomials on \( \mathbb{P}^3 \), say \( \mathcal{D} \), which is a sublinear system of the complete linear system \( |\mathcal{O}_{\mathbb{P}^3}(3)| \). In general, \( |\mathcal{O}_{\mathbb{P}^n}(d)| \) determines a morphism, which is called the \( d \)-uple embedding, from \( \mathbb{P}^n \) to \( \mathbb{P}^N \) where \( N = \binom{n+d}{n} - 1 \), defined by the algebraically independent homogeneous polynomials of degree \( d \) in \( n + 1 \) variables.

Note that \( \phi \) is not defined on the cubic curve \( B := \{ z \in \mathbb{P}^3 | z_0 = 0, z_1^3 + z_2^2z_3 - z_1z_3^2 = 0 \} \) in \( \mathbb{P}^3 \). The curve \( B \) coincides with the indeterminacy locus of the rational map \( \phi \) (also scheme theoretically to be precise). We denote \( \phi \) by \( |\mathcal{D} - B| \). Note also that \( B \) is nonsingular and irreducible. Let

\[
\pi : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3
\]

be the blowing-up of \( I_B \), the ideal of \( B \) in \( \mathbb{P}^3 \). Since we blew-up the smooth curve \( B \), \( \mathbb{P}^3 \) is a projective manifold containing the smooth exceptional divisor \( \widetilde{B} := \pi^{-1}(B) \). Let \( \widetilde{\mathcal{D}} := \pi^*\mathcal{D} \) be the pulled-back sublinear system on \( \widetilde{\mathbb{P}^3} \). \( \widetilde{B} \) is isomorphic to \( \mathbb{P}(N_B^{\vee}/\mathbb{P}^3) \), the projective space bundle of hyperplanes in the conormal bundle \( N_B^{\vee}/\mathbb{P}^3 \) of rank equal to 2, codimension of \( B \) in \( \mathbb{P}^3 \). There is the natural projection morphism \( p : \mathbb{P}(N_B^{\vee}/\mathbb{P}^3) \rightarrow B \), and the tautological line bundle \( \xi \) on \( \mathbb{P}(N_B^{\vee}/\mathbb{P}^3) \) whose restriction to the fiber \( \mathbb{P}^1 \) of \( p \) is isomorphic
to $O_{\mathbb{P}^1}(1)$. By the argument as in [10, II, Example 7.17.3], we have a well-defined morphism 
$\tilde{\phi} = |\tilde{D} - \tilde{B}| : \mathbb{P}^3 \to \mathbb{P}^{19}$ factoring through $\phi$:

\[
\begin{array}{c}
\mathbb{P}^3 \\
\downarrow \phi \\
\mathbb{P}^3 \\
\downarrow \pi \\
\mathbb{P}^3 \\
\end{array}
\xrightarrow{\tilde{\phi}}
\begin{array}{c}
\mathbb{P}^{19}
\end{array}
\]

Since $\mathcal{L}$ can be identified as $\{ z = (z_0, z_1, z_2, z_3) \in \mathbb{P}^3| z_0 \neq 0 \}$, $\mathcal{L}$ lies in $\mathbb{P}^3 - B$ on which $\pi$ is an isomorphism. So the degree of the closure $\phi(\mathcal{L})$ of $\phi(\mathcal{L})$ is equal to the degree of $\tilde{\phi}(\mathbb{P}^3)$ which is equal to the self-intersection number

$$(\tilde{D} - \tilde{B})^3 = \tilde{D}^3 - 3\tilde{D}^2 \cdot \tilde{B} + 3\tilde{D} \cdot \tilde{B}^2 - \tilde{B}^3.$$ (i) $\tilde{D}^3 = 3^3$ since $\mathbb{P}^3$ is the intersection number of three hypersurfaces of degree 3 on $\mathbb{P}^3$ which are generic elements in $\mathcal{D}$.

(ii) $\tilde{D} \cdot \tilde{B} = 0$ since $\mathbb{P}^3$ and generic enough not to meet the cubic curve $B$.

Let $E$ be a vector bundle of rank $r$ on a nonsingular projective variety $X$. In the next computation, we use the following ‘Chern relation’ in the cohomology ring of the projective space bundle $p : \mathbb{P}(E) \to X$ which gives a relation between the Chern classes of $E$ and the tautological line bundle $\xi$ on $\mathbb{P}(E)$:

$$\sum_{i=0}^{r} (-1)^i p^* c_i(E) \xi^{r-i} = 0$$

in $H^{2r}(\mathbb{P}(E))$. We will suppress the pull-back $p^*$.

(iii) $\tilde{B}^3 = \tilde{B}|_{\tilde{B}} \cdot \tilde{B}|_{\tilde{B}} = (-\xi)^2$. By the Chern relation for $\mathbb{P}(N^\vee_{B/\mathbb{P}^3})$

$$\xi^2 - c_1(N^\vee_{B/\mathbb{P}^3}) \xi + c_2(N^\vee_{B/\mathbb{P}^3}) = 0,$$

and by the fact that $c_2(N^\vee_{B/\mathbb{P}^3})$ restricted to $B$ is automatically trivial since $\dim B < 2$, we get $\xi^2 = c_1(N^\vee_{B/\mathbb{P}^3}) \xi = c_1(N^\vee_{B/\mathbb{P}^3})$. Now observe that $N_{B/\mathbb{P}^3} \cong O_{\mathbb{P}^3}(1) \oplus O_{\mathbb{P}^3}(3)$ since $B$ is a complete intersection of two hypersurfaces of degree 1 and 3. Therefore

$$\xi^2 = \deg (\wedge^2 N^\vee_{B/\mathbb{P}^3})|_B = -\deg O_B(4) = -12,$$

which yields $\tilde{B}^3 = -12$.

(iv) Finally,

$$\tilde{D} \cdot \tilde{B}^2 = \tilde{D}|_{\tilde{B}} \cdot \tilde{B}|_{\tilde{B}}$$

$$= (9 - \text{fibers of } \pi|_{\tilde{B}}) \cdot (-\xi)$$

$$= -9$$

13
Therefore \((\widetilde{D} - \widetilde{B})^3 = 12 = 3 \cdot 2^2\).

**Example 4.3** Let \(\mathcal{L} = \{M(z_1, \ldots, z_4)\} \subset \text{Mat}_{4 \times 4}\) be a 4-dimensional subspace where \(M(z_1, \ldots, z_4)\) represents a \(4 \times 4\) matrix whose entries are given by linear polynomials in \(z_1, \ldots, z_4\). Then the indeterminacy locus

\[
B = \{ z = (z_0, z_1, z_2, z_3, z_4) \in \mathbb{P}^4 | z_0 = 0, f(z_1, \ldots, z_4) = 0 \}
\]

where \(f(z_1, \ldots, z_4) = \det M(z_1, \ldots, z_4)\) is a homogeneous polynomial of degree 4. We may assume that \(B\) is nonsingular and irreducible for the moment; the reason for it will be given in the proof of Theorem 4.1. Such an assumption is necessary in order to carry out the computation using the normal bundle \(N_{\mathcal{L}/\mathbb{P}^4}\) of \(B\) in \(\mathbb{P}^4\) and to use the Chern relations as in the previous case. Let \(\phi : \mathbb{P}^4 \rightarrow \mathbb{P}^4(1)\) be the rational map defined by the sublinear system \(\mathcal{D}\) of the complete linear system \(|\mathcal{O}_{\mathbb{P}^4}(4)|\), which is obtained by taking the Plücker coordinates for \(\mathcal{L}\). Now we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}^4 & \xrightarrow{\sim} & \mathbb{P}^4(4) \\
\pi \downarrow & & \downarrow \phi \\
\mathbb{P}^4 & \xrightarrow{\phi} & \mathbb{P}^4(1)
\end{array}
\]

where \(\pi\) is the blowing-up of \(I_B\), \(\phi = |\mathcal{D} - B|\) and \(\widetilde{\phi} = |\widetilde{D} - \widetilde{B}|\) as in the case \(n = 3\). Now we want to calculate the self-intersection number

\[
(\widetilde{D} - \widetilde{B})^4 = \widetilde{D}^4 - \left(\begin{array}{c} 4 \\ 1 \end{array}\right) \widetilde{D}^3 \cdot \widetilde{B} + \left(\begin{array}{c} 4 \\ 2 \end{array}\right) \widetilde{D}^2 \cdot \widetilde{B}^2 - \left(\begin{array}{c} 4 \\ 3 \end{array}\right) \widetilde{D} \cdot \widetilde{B}^3 + \widetilde{B}^4.
\]

(i) \(\widetilde{D}^4 = 4^4\).

(ii) \(\widetilde{D}^3 \cdot \widetilde{B} = 0\) since \(\widetilde{D}^3\) is a curve in \(\mathbb{P}^4\) and generic enough not to meet the quadric surface \(B\).

(iii) \(\widetilde{D} \cdot \widetilde{B}^3 = 4 \pi^* H \cdot \widetilde{B}^3\) where \(H \in |\mathcal{O}_{\mathbb{P}^4}(1)|\). \(\pi^* H \cdot \widetilde{B}^3 = \widetilde{B}^3|_{\pi^* H}\), i.e. \(\widetilde{B}^3\) restricted to the pull-back of the curve \(C := H \cap B\) of degree 4 in \(H \cong \mathbb{P}^3\). Let us still denote it by \(\widetilde{B}^3\).

Note that by Bertini’s theorem [10, II, Theorem 8.18], for generic \(H\), \(C\) is nonsingular and irreducible. Then on \(\pi^{-1}(C) \cong \mathbb{P}(N_{\mathcal{L}/\mathbb{P}^3}^\vee)\),

\[
\widetilde{B}^3 = \widetilde{B} |_{\pi^* H} \cdot \widetilde{B} |_{\pi^* H} = (-\eta)^2
\]

where \(\eta\) is the tautological line bundle on \(\mathbb{P}(N_{\mathcal{L}/\mathbb{P}^3}^\vee)\). Observe that \(N_{\mathcal{L}/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(4)\). Now we use the Chern relation for \(\mathbb{P}(N_{\mathcal{L}/\mathbb{P}^3}^\vee)\):

\[\eta^2 - c_1(N_{\mathcal{L}/\mathbb{P}^3}^\vee) \eta = 0\]

as in the previous case. Therefore \(\eta^2 = c_1(N_{\mathcal{L}/\mathbb{P}^3}^\vee) = -5 \cdot 4 = -20\) and \(\widetilde{D} \cdot \widetilde{B}^3 = -4^2 \cdot 5\).
Proof of Theorem 4.1: Let \( \mathcal{L} \) be the trivial flat family of \( \mathbb{P} \) over \( \text{Mat}_{n \times n} \) be an \( n \times n \) matrix whose entries are given by linear polynomials in \( z_1, \ldots, z_n \). The Plücker coordinates of \( [I, M(z_1, \ldots, z_n)] \) define a polynomial map from \( \mathbb{A}^2_\mathbb{K} \) to \( \mathbb{A}^{(2n)-1}_\mathbb{K} \subset \mathbb{P}^{(2n)-1} \). Homogenizing the map, we have a map \( \phi : \mathbb{P}^n \longrightarrow \mathbb{P}^{(2n)-1} \) defined by the sublinear system \( \mathbb{D} \) of \( |\mathcal{O}_{\mathbb{P}^n}(n)| \). The restriction of \( \phi \) to \( \mathcal{L} \) is the Plücker embedding which is described in the beginning of Section 2. However, the map \( \phi \) is not well-defined on a subvariety of codimension 2, i.e. \( \phi \) is not a “morphism,” but a “rational map” with non-empty indeterminacy locus. To eliminate this indeterminacy locus we will construct a “blow-up.”

Let

\[ B = \{ z = (z_0, z_1, \ldots, z_n) \in \mathbb{P}^n | z_0 = 0, f(z_1, \ldots, z_n) = 0 \} \]

be the indeterminacy locus of pure codimension 2 where \( f(z_1, \ldots, z_n) = \det M(z_1, \ldots, z_n) \) is homogeneous of degree \( n \). We may assume that \( B \) is nonsingular and irreducible. If not, consider

\[ p : \mathbb{P}^n \times \mathbb{A}^1_\mathbb{K} \rightarrow \mathbb{A}^1_\mathbb{K}, \]

the trivial flat family of \( \mathbb{P}^n \) over \( \mathbb{A}^1_\mathbb{K} \) (or the projection onto \( \mathbb{A}^1_\mathbb{K} \)) where \( \mathbb{A}^1_\mathbb{K} \) is the affine line over \( \mathbb{K} \). Let \( \mathbb{P}^n_t \equiv \mathbb{P}^n \) be the fiber over \( t \in \mathbb{A}^1_\mathbb{K} \), and consider the subvariety \( B_t \) of \( \mathbb{P}^n_t \) given by \((z_0 = 0, f + tg = 0)\) where \( g \in \mathbb{K}[z_1, \ldots, z_n] \) is homogeneous of degree \( n \). Such \( \{B_t\} \)
forms a flat family over $\mathbb{A}^1_k$ by varying $t \in \mathbb{A}^1_k$ with $B_0 = B$. Moreover, we can choose an appropriate $g$ so that the generic $B_t$ is nonsingular and irreducible. Let $I_{B_t}$ be the ideal of $B_t$ in $\mathbb{P}^n_t$ for $t \in \mathbb{A}^1_k$. Let $\mathcal{D}_t$ be the sublinear system of $|O_{\mathbb{P}^n}(n)|$ consisting of the same polynomials of degree $n$ as for $\mathcal{D}$, except that $f$ is replaced by $f + tg$. For each $t \in \mathbb{A}^1_k$, we take the blowing-up $\pi_t$ of $I_{B_t}$ with the exceptional divisor $\tilde{B}_t$, satisfying the following diagram:

$$
\begin{array}{c}
\mathbb{P}^n_t \\
\downarrow \pi_t \\
\mathbb{P}^n_t \\
\phi_t \\
\downarrow \\
\mathbb{P}^n(2^n) - 1
\end{array}
$$

with $\phi_t = |\mathcal{D}_t - B_t|$ and $\tilde{\phi}_t = |\tilde{\mathcal{D}}_t - \tilde{B}_t|$ where $\tilde{\mathcal{D}}_t := \pi_t^* \mathcal{D}_t$ is the pulled-back sublinear system on $\tilde{\mathbb{P}}^n_t$. On the other hand, we take the blowing-up of $I$, the ideal of the subscheme $\{B_t\}_{t \in \mathbb{A}^1_k}$ in $\mathbb{P}^n \times \mathbb{A}^1_k$:

$$\pi : \mathbb{P}^n \times \mathbb{A}^1_k \to \mathbb{P}^n \times \mathbb{A}^1_k$$

with the exceptional divisor $E$. Since $\mathbb{A}^1_k$ is a smooth curve, the composition

$$p \circ \pi : \mathbb{P}^n \times \mathbb{A}^1_k \to \mathbb{A}^1_k$$

is flat (cf. [4, Appendix B.6.7]). The fiber of $p \circ \pi$ over $t = 0$ is $\mathbb{P}^n$, i.e. the blowing up of $I_0$, the ideal of $B$ in $\mathbb{P}^n$. Note that the restriction of the exceptional divisor $E$ to a fiber, $E|_{(p \circ \pi)^{-1}(t)}$, is $\tilde{B}_t$. The flatness of $p \circ \pi$ assures that certain numerical invariants remain constant in the family (see [10, III, Theorem 9.9]). In particular, the self-intersection $(\tilde{\mathcal{D}}_t - \tilde{B}_t)^n$ is independent of $t \in \mathbb{A}^1_k$. This is why we can assume that $B$ is irreducible and nonsingular.

Let us proceed with the proof of the theorem. We write $\phi = |\mathcal{D} - B|$. We have the following diagram:

$$
\begin{array}{c}
\mathbb{P}^n \\
\downarrow \pi \\
\mathbb{P}^n \\
\phi \\
\downarrow \\
\mathbb{P}^n(2^n) - 1
\end{array}
$$

where $\pi$ is the blowing-up of $I_B$. $O_{\mathbb{P}^n}(1) \oplus O_{\mathbb{P}^n}(n)$. By using the Chern relation

$$
\xi^{n-1} - c_1(N_{B/\mathbb{P}^n}) \xi^{n-2} + \cdots + (-1)^{n-2} c_{n-2}(N_{B/\mathbb{P}^n}) \xi = 0
$$

we have

$$
\xi^{n-1} - c_1(N_{B/\mathbb{P}^n}) \xi^{n-2} + \cdots + (-1)^{n-2} c_{n-2}(N_{B/\mathbb{P}^n}) \xi = 0
$$
inductively as in the previous cases, we get the following:

\[
\begin{align*}
\tilde{D}^n &= n^n \\
\tilde{D}^{n-1} \cdot \tilde{B} &= 0 \\
\tilde{D}^{n-2} \cdot \tilde{B}^2 &= -n^{n-1} \\
\tilde{D}^{n-3} \cdot \tilde{B}^3 &= -n^{n-2}(n + 1) \\
\tilde{D}^{n-4} \cdot \tilde{B}^4 &= -n^{n-3}(n^2 + n + 1) \\
\tilde{D}^{n-5} \cdot \tilde{B}^5 &= -n^{n-4}(n^3 + n^2 + n + 1) \\
&\vdots \\
\tilde{D}^0 \cdot \tilde{B}^n &= -n(n^{n-2} + n^{n-3} + \cdots + n + 1),
\end{align*}
\]

which yields

\[
(\tilde{D} - \tilde{B})^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \tilde{D}^{n-k} \cdot \tilde{B}^k
\]

\[
= n^n + \sum_{k=2}^{n} (-1)^{k+1} \binom{n}{k} n^{n-k+1} \left( \sum_{i=0}^{k-2} n^i \right)
\]

\[
= n(n-1)^{n-1}.
\]

\[\square\]

Theorem 4.1 will allow us to answer Problem 1.1 in the “generic situation” if \(m = n = d\):

**Theorem 4.4** Assume \(\mathbb{K}\) is an algebraically closed field of characteristic zero. Let \(\mathcal{L} \subset \text{Mat}_{n \times n}\) be a “generic subspace”, let \(E, A, B, H\) be a “generic set of matrices” and let \(\varphi(s) \in \mathbb{K}^n\) be a “generic monic polynomial” of degree \(n\). Then there exist exactly \(n(n-1)^{n-1}\) different feedback laws \(F \in \mathcal{L}\) such that \(\det \left[ s(E + HF) + (A + BF) \right] = \varphi(s)\). In particular the system (1.1) is generically pole assignable in the class of feedback compensators \(\mathcal{L}\).

**Proof:** Consider the characteristic map \(\chi\) introduced in (2.10). According to Theorem 2.6 \(\chi\) is a finite morphism of degree \(\tilde{\mathcal{L}}\). By Theorem 4.1 the degree of \(\tilde{\mathcal{L}}\) is \(n(n-1)^{n-1}\). For a generic set of polynomials \(\varphi(s) \in \mathbb{K}^n \subset \mathbb{P}^n\) the inverse image \(\chi^{-1}(\varphi(s))\) contains \(\deg(\tilde{\mathcal{L}})\) different solutions and all these solutions are contained in \(\mathcal{L} \subset \tilde{\mathcal{L}}\).

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