Rotational solutions of prescribed period of Hamiltonian systems on $\mathbb{R}^{2n-k} \times T^k$

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Abstract

In this paper, we consider Hamiltonian systems on $\mathbb{R}^{2n-k} \times T^k$. Multiple rotational solutions are obtained.

1 Introduction and main results

In this paper we study the multiplicity of rotational solutions for Hamiltonian systems

$$\dot{z} = JH'(z), \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \quad \text{(HS)}$$

For $1 \leq k \leq 2n - 1$, let

$$z = (z_1, z_\Pi), \quad z_1 = (z_1, \cdots , z_{2n-k}), \quad z_\Pi = (z_{2n-k+1}, \cdots , z_{2n}), \quad \text{(1.1)}$$

and

$$H_{z_1} = (H_{z_1}, \cdots , H_{z_{2n-k}}), \quad H_{z_\Pi} = (H_{z_{2n-k+1}}, \cdots , z_{2n}). \quad \text{(1.2)}$$

We make the following basic hypothesis on the Hamiltonian

(H0) $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$.
(H1) $H(z_1, z_\Pi + v) = H(z_1, z_\Pi), \quad \forall z = (z_1, z_\Pi) \in \mathbb{R}^{2n}, \quad v \in \mathbb{Z}^k$.
(H2) There exist constants $r > 0, \mu > 1$ such that

$$0 < \mu H(z) \leq z_1 \cdot H_{z_1}(z), \quad \forall z = (z_1, \cdots , z_{2n}) \in \mathbb{R}^{2n}, \quad |z_1|^2 \geq r^2.$$

A periodic solution $z(t)$ of (HS) on $\mathbb{R}^{2n-k} \times T^k$ satisfies

$$z(T) - z(0) = (0, v) \quad \text{(1.3)}$$
for period $T > 0$ and vector $v \in \mathbb{Z}^k$. If $v = 0$, solutions are contractible. If $v \neq 0$, it is called rotational vector and corresponding solutions are called rotational solutions, which are non-contractible.

If $z(t)$ is a rotational solution of (HS), then $z(mt)$ is also such a solution with rotational vector $mv$. We restrict choices of rotational vectors to $Z^k_1$ defined as follows.

**Definition 1.** A rotational vector $v = (v_1, v_2, \cdots, v_k) \in \mathbb{Z}^k \setminus \{0\}$ is called prime, if one of its coordinates equals to 1 while others are zero, or two of them are relatively prime. Denotes by $Z^k_1$ the set of all prime rotational vectors in $\mathbb{Z}^k$.

In this paper, we consider the following boundary value problem

$$
\begin{cases}
  \dot{z}(t) = JH'(z(t)), \\
  z(T) = z(0) + (0, v),
\end{cases}
$$

(1.4)

For given $T > 0$ and $v \in Z^k_1$. Denote by $\mathcal{P}_H(T, v)$ the set of distinct solutions.

Main results in this paper are the following theorems.

**Theorem 1.** Assume that $k = n$ and $H$ satisfies (H0)–(H2). For every $T > 0$ and $v \in Z^k_1$, we have

$$
\# \mathcal{P}_H(T, v) \geq k.
$$

**Theorem 2.** Assume that $k > n$ and $H$ satisfies (H0)–(H2) and

(H3) There exist positive numbers $a, b$ and $s < \mu - \frac{1}{2}$ such that

$$
|H_z z(z)| \leq a|z_i|^s + b, \quad \forall z = (z_I, z_{\mathbb{R}}) \in \mathbb{R}^{2n-k} \times \mathbb{R}^k.
$$

For every $v \in Z^k_1 \cap \{0\} \times Z^{2n-k} \times \{0\}$ with $0 \in \mathbb{R}^{k-n}$ and $T \in I(s)$, we have

$$
\# \mathcal{P}_H(T, v) \geq k,
$$

where the interval $I(s)$ is defined as follows

$$
I(s) = \begin{cases}
  (0, +\infty), & s < \frac{\mu}{2}, \\
  (0, \delta), & \frac{\mu}{2} \leq s < \mu - \frac{1}{2},
\end{cases}
$$

and $\delta$ is sufficiently small.

### 2 Variational settings

Make substitution

$$
z(Tt) = x(t) + t(0, v),
$$

(2.1)
then solutions of \((1.4)\) are in one to one correspondence with 1-periodic solutions of the following problem:

\[
\begin{aligned}
\dot{x}(t) + (0, v) &= T J H'(x(t) + t(0, v)), \\
x(1) &= x(0).
\end{aligned}
\] (2.2)

We choose the Hilbert space \(E = W^{1,2}(S^1, \mathbb{R}^{2n})\) with its inner product and norm defined by

\[
\langle x, y \rangle = x_0 \cdot y_0 + \sum_{j \in \mathbb{Z}} 2\pi|j|x_j \cdot y_j, \quad \forall x, y \in E,
\] (2.3)

and

\[
||x|| = \langle x, x \rangle^{\frac{1}{2}},
\] (2.4)

where

\[
x(t) = \sum_{j \in \mathbb{Z}} e^{2\pijt} \xi_j, \quad \xi_j \in \mathbb{R}^{2n},
\]

is the Fourier series expansion of \(x\). For \(x \in E \cap C^\infty\) we define

\[
A(x) = -\frac{1}{2} \int_0^1 J \dot{x}(t) \cdot x(t) dt.
\] (2.5)

It can be extended to the whole space \(E\). Note that the space \(E\) can be orthogonally decomposed as

\[
E = E^+ \oplus E^- \oplus E^0,
\] (2.6)

where

\[
E^\pm = \left\{ \sum_{j \in \mathbb{Z} \setminus \{0\}} e^{2\pi j t} \xi_j \bigg| \pm j > 0 \right\}
\]

\[
= \text{span}_{\mathbb{R}}\left\{ \cos(2\pi j t)e_i \pm \sin(2\pi j t)e_{i+n}, \quad 1 \leq i \leq n, \quad j \in \mathbb{N}\right\},
\] (2.7)

\[
E^0 = \text{span}_{\mathbb{R}}\{e_1, \cdots, e_{2n}\} = \mathbb{R}^{2n}.
\] (2.8)

Denote by \(P^0\) and \(P^\pm\) the orthogonal projections from \(E\) onto \(E^0\) and \(E^\pm\) respectively. Let

\[
x^0 = P^0 x, \quad x^\pm = P^\pm x.
\] (2.9)

Define a linear, bounded and selfadjoint operator

\[
L = P^+ - P^- : E \rightarrow E.
\] (2.10)

Then the functional \(A : E \cap C^\infty \rightarrow \mathbb{R}\) defined by \((2.5)\) can be extended to \(E\) as follows

\[
A(x) = \frac{1}{2} (Lx, x) = \frac{1}{2} (||x^+||^2 - ||x^-||^2).
\] (2.11)

Consider a Hamiltonian \(H\) satisfying \((H0), (H1)\) and the growth condition

\[
|H(z)| \leq a|z_1|^2 + b, \quad \forall z = (z_1, z_2), \quad z_1 \in \mathbb{R}^{2n-k}, \quad z_2 \in \mathbb{R}^k,
\] (2.12)
for some constants $a, b, s > 0$. Then define a functional $B : E \rightarrow \mathbb{R}$ by
\[
B(x) = \int_0^1 \left[ H(x(t) + t(0, v)) + x(t) \cdot J(0, v) \right] dt.
\] (2.13)

This functional is well defined, of class $C^1$ and its derivative $B'$ is compact. Now we can define on $E$ the functional
\[
\Phi(x) = A(x) - B(x).
\] (2.14)

**Proposition 2.1.** $x \in E$ is a critical point of $\Phi$ iff $x$ is a solution of (2.2).

For proofs of main theorems, we need different decompositions instead of (2.6). Note that $E^0$ can be decomposed as $E^0 = E^0_1 \oplus E^0_\Pi$, where
\[
E^0_1 = \text{span}\{e_1, \ldots, e_{2n-k}\},
\]
\[
E^0_\Pi = \text{span}\{e_{2n-k+1}, \ldots, e_{2n}\}.
\] (2.15) (2.16)

We define subspaces $X, Y, \mathcal{H}$ of $E$ such that
\[
\mathcal{H} = X + Y = E^+ \oplus E^- \oplus E^0_1.
\] (2.17)

Then the space $E$ can be decomposed as
\[
E = \mathcal{H} \oplus E^0_\Pi.
\] (2.18)

Let
\[
E_1 = \left\{ (x_1, \ldots, x_{2n-k}, 0) \in E \left| \int_0^1 x_i(t) dt = 0 \right. \right\},
\] (2.19)
\[
E_\Pi = \left\{ (0, x_{2n-k+1}, \ldots, x_{2n}) \in E \left| \int_0^1 x_i(t) dt = 0 \right. \right\}.
\] (2.20)

Subspace $X, Y$ can be defined according to various values of $k$ as follows.

**Case 1.** $k = n$.
\[
X = E^- \oplus E^0_1, \quad Y = E_\Pi.
\] (2.21)

**Case 2.** $k > n$.
\[
x = (\ast, \ldots, \ast, \ast, \ldots, \ast, \ast, \ldots, \ast, \ast, \ldots, \ast, \ast, \ldots, \ast, \ast, \ldots, \ast).
\]

\[
X_1 = \left\{ (p_1, \ldots, p_{2n-k}, 0, q_1, \ldots, q_{2n-k}, 0) \in E \left| \int_0^1 p_i(t) dt = 0, \int_0^1 q_i(t) dt = 0 \right. \right\}.
\]
\[
X = (E^- \cap X_1) \oplus E^0_1, \quad Y = E_\Pi.
\] (2.22)

We consider symmetries involved in the problem.
Firstly, since $H$ satisfies (H1), the functional $\Phi$ defined by (2.14) is $Z^k$-invariant and can be defined on
$$E = E/Z^k = \mathcal{H} \times T^k.$$ (2.23)
Secondly, P. Felmer [6] points out that the following $S^1$-action is free, and the functional $\Phi$ is $S^1$-invariant.
$$\theta \cdot x(t) = x(t + \theta) + (0, \theta v), \ x \in \mathcal{E}. \quad (2.24)$$
where $\theta \in S^1 = [0, 1]/\{0, 1\}$.

P. Felmer [5] essentially proves the following $S^1$-equivariant saddle point type theorem.

**Proposition 2.2. (Theorem 1.1 of [5])** Assume that $E$ can be splitted as $E = (X \cup Y) \times T^k$ with $X \neq \{0\}$. Let $I \in C^1(\mathcal{E}, \mathbb{R})$ be an $S^1$-invariant functional of the form
$$I(z) = \frac{1}{2} \langle Lz, z \rangle + B(z), \quad (2.25)$$
where
(1) $L : E \to E$ is a linear, bounded and selfadjoint operator; $X$ is an invariant subspace.
(2) $b \in C^1(\mathcal{E}, \mathbb{R})$ and $b' : \mathcal{E} \to \mathcal{H}$ is compact.
(3) $I$ satisfies (PS) property.
(4) There exist constants $\alpha < \beta$ and $\gamma$ such that
$$I|_{Q \times T^k} \leq \alpha, \ I|_{Y \times T^k} \geq \beta, \ I|_{Q \times T^k} \leq \gamma,$$
where $Q = \{x \in X \mid ||x|| \leq R\}$ and $\partial Q = \{x \in X \mid ||x|| = R\}$. (2.26)
Then $I$ possesses at least $k$ distinct critical points with critical values less than or equal to $\gamma$.

### 3 Proofs of main results

Since $H$ satisfies (H0)-(H2), then
$$H(z) \geq a_1|z_1|^\mu - a_2, \ \forall z = (z_1, z_2) \in \mathbb{R}^{2n-k} \times \mathbb{R}^k,$$ (3.1)
where
$$a_1 = \min_{|z_1| \geq r} \frac{H(z)}{|z_1|^\mu}, \ a_2 = \max_{|z_1| \leq r} |H(z)|. \quad (3.2)$$
Choose undetermined constants $K_2 > K_1 > r$ and a function $\chi \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that
$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq K_1, \\ 0, & t \geq K_2, \end{cases} \quad \text{and} \quad \chi'(t) < 0, \ K_1 < t < K_2.$$ We define
$$H_K(z) = \chi(|z_1|)H(z) + (1 - \chi(|z_1|))\rho|z_1|^\mu,$$ (3.3)
where

\[ \rho \geq \max_{K_1 \leq |z| \leq K_2} \frac{|H(z)|}{|z|^\mu} \quad (3.4) \]

The function \( H_K \) satisfies (H0)-(H1) and

\[ 0 < \mu H_K(z) \leq z_1 \cdot (H_K)_z, \quad |z_1| \geq r. \]

Then

\[ a_1 |z_1|^\mu - a_2 \leq H_K(z) \leq \rho |z_1|^\mu + a_2, \quad (3.5) \]

\[ \int_0^1 (z_1 \cdot (H_K)_z) - H_K dt \geq \int_0^1 ((\mu - 1)H - a_3) dt, \quad (3.6) \]

\[ H_K(z) < \min_{|z_1| < K_2} H_K(z), \quad |z_1| < K_2. \quad (3.7) \]

where

\[ a_3 = \max_{|z_1| \leq r} (\mu - 1)H(z) - \min_{|z_1| \leq r} z_1 \cdot H(z). \quad (3.8) \]

Let

\[ B_K(x) = \int_0^1 \left[ T H_K(x(t) + t(0,v)) + x(t) \cdot J(0,v) \right] dt, \quad (3.9) \]

and

\[ \Phi_K(x) = A(x) - B_K(x), \quad \forall x \in \delta. \quad (3.10) \]

Certainly functionals \( B_K \) and \( \Phi_K \) are well-defined and of \( C^1 \) class, and \( \Phi_K \) satisfies (I1) and (I2). We next prove that \( \Phi_K \) satisfies (I3) and (I4).

**Proposition 3.1.** \( \Phi_K \) satisfies (I3), i.e., (PS) condition.

**Proof.** Let us consider a sequence \( x^{(m)} = (w^{(m)}, \theta^{(m)}) \in \mathcal{H} \times \mathcal{T}^k \)

such that

\[ \Phi_K(w^{(m)}, \theta^{(m)}) \leq c \quad \text{and} \quad \Phi'_K(w^{(m)}, \theta^{(m)}) \to 0 \quad \text{as} \quad m \to \infty. \quad (3.11) \]

Certainly \( \{\theta^{(m)}\}_{m \in \mathbb{N}} \) has a convergent subsequence. We show that \( \{w^{(m)}\}_{m \in \mathbb{N}} \) is bounded in \( \mathcal{H} \).

Let \( w = w^{(m)}, \theta = \theta^{(m)} \), and decompose \( w \) as

\[ w = w_0^0 + w_+^+ + w_+^- = w_0^0 + \tilde{w}_1 + \tilde{w}_\| = w_1 + \tilde{w}_\|, \quad (3.12) \]

where

\[ w_\pm \in E_\pm, \quad w_0^0 \in E_0^0, \quad \tilde{w}_1 \in E_1, \quad \tilde{w}_\| \in E_\|. \]

By (3.11) and for large \( m \) we have

\[ \Phi_K(w, \theta) - \langle \Phi'_K(w, \theta), w_1 \rangle \leq c + ||w_1||. \quad (3.13) \]
Case 1. \( k = n \).

\[
\omega_1 = \omega_1^0 + \tilde{\omega}_1 = (p, 0) \in E_1^0 \oplus E_1, \quad \tilde{\omega}_\Pi = (0, q) \in E_\Pi.
\]

Set \( \zeta(t) = x(t) + t(0, v) \), we have

\[
\Phi_K(x) = \int_0^1 [p(t) \cdot q(t) - TH_K(\zeta(t)) + p(t) \cdot v] \, dt,
\]

(3.14)

\[
\Phi'_K(x) = \left( \dot{q} + v - T(H_K)_{p, -} - T(H_K)_{q} \right).
\]

(3.13)

\[
\langle \Phi'_K(x), (p, 0) \rangle = \int_0^1 [p(t) \cdot q(t) + p(t) \cdot v - p(t) \cdot T(H_K)_p(\zeta(t))] \, dt,
\]

(3.15)

\[
\Phi_K(x) - \langle \Phi'_K(x), (p, 0) \rangle = \int_0^1 [p(t) \cdot (H_K)_p(\zeta(t)) - H_K(\zeta(t))] \, dt
\]

\[
\geq T \left( \int_0^1 (\mu - 1)H_K(\zeta(t)) \, dt - a_3 \right).
\]

By (3.13) and (3.15), we have

\[
c + ||w_1|| \geq T \left( \int_0^1 (\mu - 1)H_K(\zeta(t)) \, dt - a_3 \right)
\]

\[
\geq T \left( \int_0^1 (\mu - 1)(a_1|w_1|^\mu - a_2) \, dt - a_3 \right).
\]

Then

\[
||w_1||^\mu \leq b_1(1 + ||w_1||) \leq b_1(1 + ||w||).
\]

(3.16)

Here \( ||z||_A \) denotes the standard norm of \( z \in L^A \). We have

\[
|w_1^0| \leq b_2(1 + ||w||)^{1/\mu}.
\]

(3.17)

We next estimate \( \tilde{\omega}_1 + \tilde{\omega}_\Pi = w^+ + w^- \) with \( w^\pm \in E^\pm \). Note that

\[
\langle \Phi'_K(x), x^\pm \rangle = \int_0^1 \left( (x^+ - x^-) \cdot x^\pm - TH'_K(\zeta(t)) \cdot x^\pm - J(0, v) \cdot x^\pm \right) \, dt.
\]

(3.18)

By (3.11) we have

\[
||w^\pm||^2 \leq \int_0^1 |TH'_K(\zeta(t)) \cdot w^\pm| \, dt + ||w^\pm||.
\]

By definition of \( H_K \) we have

\[
|H'_K(x)| \leq b_3(|z_1|^{\mu - 1} + 1).
\]

Then

\[
\int_0^1 |TH'_K(\zeta(t)) \cdot w^\pm| \, dt \leq b_3 \left( \int_0^1 ||w_1||^{\mu - 1} + 1 ||w^\pm|| \, dt \leq b_3 \left( ||w_1||^{\mu - 1} + 1 \right) ||w^\pm||, \right)
\]

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and
\[ ||w^+|| \leq b_4 \left( 1 + ||w_1||^{\mu-1}_\mu \right) \leq b_5 \left( 1 + ||w||^\mu / \mu \right). \] (3.19)

By (3.17) we have
\[ ||w|| \leq ||w_1^0|| + ||w^+|| + ||w^-|| \leq b_6 \left( 1 + ||w||^\mu / \mu \right). \]

Since \( \mu > 1 \), we conclude that the sequence \( \{w^{(m)}\} \) is bounded in \( \mathcal{H} \).

**Case 2.** \( k > n \). Let
\[ \begin{cases} x = (p_1, p_\Pi, q_1, q_\Pi), \\ p_1 = (p_1, \cdots, p_{2n-k}), \\ p_\Pi = (p_{2n-k+1}, \cdots, p_n), \\ q_1 = (q_1, \cdots, q_{2n-k}), \\ q_\Pi = (q_{2n-k+1}, \cdots, q_n). \end{cases} \] (3.20)

Then
\[ w_1 = (p_1, 0, 0, 0), \quad w_\Pi = (0, p_\Pi, q_1, q_\Pi). \]

Each vector \( v \in \mathbb{Z}^k \) can be written as
\[ v = (v_\Pi^t, v_1, v_\Pi), \quad v_1 \in \mathbb{Z}^{2n-k}, \quad v_\Pi, v_\Pi^t \in \mathbb{Z}^{k-n}. \] (3.21)

We have
\[ \Phi_K(x) = \int_0^1 \left( p_1 \cdot q_1 + p_\Pi \cdot q_\Pi - TH_K(\xi(t)) + p_1 \cdot v_1 + p_\Pi \cdot v_\Pi - q_\Pi \cdot v_\Pi^t \right) dt, \] (3.22)
\[ \Phi'_K(x) = \left( q_1 + v_1 - T(H_K)p_1, q_\Pi + v_\Pi - T(H_K)p_\Pi, -p_1 - T(H_K)q_1, -p_\Pi - v_\Pi - T(H_K)q_\Pi \right), \] (3.23)
\[ \langle \Phi'_K(x), (p_1, 0) \rangle = \int_0^1 \left( p_1 \cdot q_1 + p_1 \cdot v_1 - p_1 \cdot T(H_K)p_1(\xi(t)) \right) dt, \]
\[ \Phi(x) - \langle \Phi'(x), (p_1, 0) \rangle = \int_0^1 \left( p_1 \cdot T(H_K)p_1(\xi(t)) - TH_K(\xi(t)) + p_\Pi \cdot (q_\Pi + v_\Pi^t) - q_\Pi \cdot v_\Pi^t \right) dt. \] (3.24)

By the definition of \( H_K \), we have
\[ ||(H_K)_{x_2}|| \leq b_7. \] (3.25)

By (3.23), (3.24) and for large \( m \) we have
\[ ||p_\Pi + v_\Pi^t + T(H_K)q_\Pi(\xi(t))||_2 \leq 1, \]
\[ ||q_\Pi + v_\Pi - T(H_K)p_\Pi(\xi(t))||_2 \leq 1. \] (3.26)

Since \( w_\Pi = (0, p_\Pi, q_1, q_\Pi) \in E_\Pi \), by Wirtinger’s inequality we have
\[ ||\bar{p}_\Pi||_2 \leq \frac{1}{2\pi} ||\bar{p}_\Pi||_2, \quad ||q_\Pi||_{L^2} \leq \frac{1}{2\pi} ||q_\Pi||_2. \]
By (3.25) and (3.26) we have
\[
\left| \int_0^1 (p_\Pi \cdot (q_\Pi + v'_\Pi) - q_\Pi \cdot v'_\Pi) dt \right| \leq b_8. \tag{3.27}
\]

By (3.24), (3.13), (H2) and the above inequality, we have
\[
c + ||w|| \geq T \left( \int_0^1 (\mu - 1)H_K(\xi(t))dt - a_3 \right)
\geq T \left( \int_0^1 (\mu - 1)(a_1|w|^\mu - a_2)dt - a_3 \right).
\]

The rest part of proof is similar with Case 1.

**Proposition 3.2.** \( \Phi_K \) satisfies (I4).

**Proof.**

**Case 1.** \( k = n \). By (2.21), each \( x = (w, \theta) \in Y \times T^n \) satisfies \( w = (0, q) \), then
\[
\Phi_K \bigg|_{Y \times T^n} (x) = \int_0^1 -TH_K(0, q(t) + \theta + tv)dt \geq \beta.
\]

Each \( x = (w, \theta) \in X \times T^n \) can be written as
\[
w = w^- + w^0, \quad w^- = (p^-, q^-), \quad w^0 = (p^0, 0).
\]

\[
\Phi_K \bigg|_{X \times T^n} (x) = -||w^-||^2 + p^0 \cdot v - \int_0^1 TH_K(p^-, p^0, q^- + tv + \theta)dt
\leq -||w^-||^2 + |w^0| \cdot |v| - \int_0^1 T(a_1|p^- + p^0|^\mu - a_2)dt. \tag{3.28}
\]

**Subcase 1.1.** \( 1 < \mu < 2 \). Let \( f(t) = \frac{(1 + t)^\mu}{1 + t^\mu}, \quad t \geq 0 \). We have
\[
f'(t) = \frac{\mu(1 + t)^{\mu-1}(1 + t^\mu) - (1 + t)^\mu \mu^{\mu-1}}{(1 + t^\mu)^2}
= \frac{\mu(1 + t)^{\mu-1}}{(1 + t^\mu)^2}(1 - t^{\mu-1}) > 0 \text{ iff } t < 1.
\]

Then \( \max_{t \geq 0} f(t) = f(1) = 2^{\mu-1} \) and
\[
|u_2|^{\mu} = |u_1 + u_2 - u_1|^{\mu} \leq (|u_1 + u_2| + |u_1|)^{\mu} \leq 2^{\mu-1}(|u_1 + u_2|^{\mu} + |u_1|^{\mu}), \quad \forall u_1, u_2 \in \mathbb{R}^n.
\]

Choose \( u_1 = p^- \) and \( u_2 = p^0 \), we have
\[
|w^0| = |p^0|^{\mu} \leq 2^{\mu-1}(|p^- + p^0|^{\mu} + |p^-|^{\mu}) \leq 2^{\mu-1}(|p^- + p^0|^{\mu} + |w^-|^{\mu}).
\]
By (3.28) we have
\[
\Phi_{K|X \times T^k}(x) \\
\leq -||w^-||^2 + ||w^0|| \cdot |v| - Ta_1 \left[ 1 \int_0^1 (21^{-\mu} |w^0|^\mu - |w^-|^\mu) \, dt + Ta_2 \right] \\
= -||w^-||^2 + Ta_1 ||w^-||^\mu + ||w^0|| \cdot |v| - 21^{-\mu} Ta_1 |w^0|^\mu + Ta_2 \\
\leq -||w^-||^2 + Ta_1 ||w^-||^\mu + ||w^0|| \cdot |v| - 21^{-\mu} Ta_1 |w^0|^\mu + Ta_2. 
\tag{3.29}
\]

For each \( w = w^- + w^0 \in Q \), we have \( ||w^-||^2 + ||w^0||^2 = R^2 \). Then
\[
||w^-|| \geq \frac{1}{\sqrt{2}} R \text{ or } ||w^0|| \geq \frac{1}{\sqrt{2}} R.
\]

By (3.29), since \( 1 < \mu < 2 \), for large \( R \), we have
\[
\Phi_{K|\partial Q \times T^k}(x) \leq \beta - 1. 
\tag{3.30}
\]

**Subcase 1.2.** \( \mu \geq 2 \). Similarly to computations in subcase 1.1, we have
\[
\Phi_{K|X \times T^k}(x) \leq -||w^-||^2 + ||w^0|| \cdot |v| - T \left( a_1 \left[ 1 \int_0^1 |p^- + p^0|^\mu \, dt - a_2 \right] \right) \\
\leq -||w^-||^2 + ||w^0|| \cdot |v| - T \left( a_1 \left[ 1 \int_0^1 |p^-|^\mu \, dt \right] - a_2 \right) \\
= -||w^-||^2 + ||w^0|| \cdot |v| - T \left( a_1 \left[ 1 \int_0^1 |p^-|^2 + |p^0|^2 \, dt \right] - a_2 \right) \\
\leq ||w^0|| \cdot |v| - Ta_1 |w^0|^\mu + Ta_2.
\]

Then
\[
\Phi_{K|\partial Q \times T^k}(x) \leq ||w^0|| \cdot |v| - Ta_1 |w^0|^\mu + Ta_2 \leq \beta - 1
\]
for sufficiently large \( R \).

**Case 2.** \( k > n \). By (2.22),
\[
X = (E^- \cap X_1) \oplus E^0_1, \ Y = E^0_2.
\]

Similarly to Case 1, we have
\[
\Phi_{K|Y \times T^k}(x) \geq \beta, \ \forall x \in Y \times T^k,
\]
and
\[
\Phi_{K|\partial Q \times T^k}(x) \leq \beta - 1, \ \forall x \in \partial Q \times T^k.
\]

By Propositions 3.1, 3.2 and 2.2, the functional \( \Phi_K \) possess at least \( k \) distinct critical points. Rest proofs for Theorem 1 and Theorem 2 are preliminary estimates for these critical points.
3.1 Proof of Theorem 1

Let

\[ \gamma = \begin{cases} 
T^{\frac{2}{2(2-\mu)}} \left[ \frac{\mu a_1}{2} \right]^{2(2-\mu)} + T^{-\frac{1}{\mu}} \cdot 2 \left[ \frac{\mu^2}{\mu a_1} \right]^{1(1-\mu)} \left( 1 - \frac{1}{\mu} \right) + Ta_2, & 1 < \mu < 2, \\
T^{-\frac{1}{\mu}} \cdot \left[ \frac{\mu^2}{\mu a_1} \right]^{1(1-\mu)} \left( 1 - \frac{1}{\mu} \right) + Ta_2, & \mu \geq 2. 
\end{cases} \] (3.31)

Proposition 3.3. If \( x(t) = (p(t), q(t)) \) is a critical point of \( \Phi_K \) with \( \Phi_K(x) \leq \gamma \), then

\[ |p(t)| \leq K, \quad H_K(x(t) + t(0, v)) = H(x(t) + t(0, v)), \quad \forall t \in \mathbb{R}, \]

and \( x \) is a solution of (2.2).

Proof. By (3.15),

\[ \gamma \geq \Phi(x) = \Phi(x) - \langle \Phi'(x), (p, 0) \rangle \]

\[ \geq T \left( \int_0^1 (\mu - 1)H_K(p, q + tv)dt - a_3 \right) \]

\[ = T(\mu - 1)H_K(p, q + tv) - a_3 \geq T(\mu - 1)a_1 |p(t)|^\mu - a_2 - a_3. \]

We have \( |p(t)| \leq K. \]

3.2 Proof of Theorem 2

Proposition 3.4. If \( x(t) = (x_1(t), x_2(t)) \) is a critical point of \( \Phi_K \) with \( \Phi_K(x) \leq \gamma \), then

\[ |x_1(t)| \leq K_1, \quad H_K(x_1(t) + t(0, v)) = H(x_1(t) + t(0, v)), \quad \forall t \in \mathbb{R}, \]

and \( x \) is a solution of (2.2).

Proof. Set \( \xi(t) = x(t) + t(0, v) \).

According to (3.20), we have

\[ \gamma \geq \Phi_K(x) = \int_0^1 \left[ p_1(t) \cdot q_1(t) + p_2(t) \cdot q_2(t) - TH_K(\xi(t)) + p_1(t) \cdot v_1 + p_2(t) \cdot 0 - q_2(t) \cdot 0 \right] dt \]

\[ = \int_0^1 \left[ p_1(t)(q_1(t) + v_1) + p_2(t) \cdot q_2(t) - TH_K(\xi(t)) \right] dt \]

\[ = \int_0^1 \left[ p_1(t)(TH_K)p_1(\xi(t)) - TH_K(\xi(t)) + p_2(t) \cdot q_2(t) \right] dt \]

\[ \geq \int_0^1 \left[ T(\mu - 1)H_K(\xi(t)) - a_3 \right] + p_2(t) \cdot (TH_K)p_2(\xi(t)) \right] dt. \] (3.32)
For arbitrary $t, t' \in [0, 1]$, we have
\[
\tilde{p}_\Pi(t) = \int_0^1 \left( \tilde{p}_\Pi(t) - \tilde{p}_\Pi(t') \right) dt' = \int_0^1 \left( \int_{t'}^t \tilde{p}_\Pi(t'') dt'' \right) dt'.
\]
Then
\[
|\tilde{p}_\Pi(t)| = \left| \int_0^1 \left( \int_{t'}^t |\tilde{p}_\Pi(t'')| dt'' \right) dt' \right| \leq \int_0^1 \left( \int_{t'}^t |\tilde{p}_\Pi(t'')| dt'' \right) dt' \leq \int_0^1 |\tilde{p}_\Pi(t'')| dt'' = \int_0^1 |\tilde{p}_\Pi(t)| dt = \int_0^1 \| (TH_K)_{qH}(\xi(t)) \| dt.
\]
We have
\[
\left| \int_0^1 \tilde{p}_\Pi \cdot (TH_K)_{pH}(\xi(t)) dt \right| \leq \int_0^1 |\tilde{p}_\Pi| \| (TH_K)_{pH}(\xi(t)) \| dt \leq \int_0^1 \| (TH_K)_{qH}(\xi(t)) \| dt \int_0^1 \| (TH_K)_{pH}(\xi(t)) \| dt \leq \left( \int_0^1 \| (TH_K)_{qH}(\xi(t)) \| dt \right)^2.
\]
By definition of $H_K$ and (H3), we have
\[
\| (H_K)_{qH} \| = \| \chi(|p_1|)H_{qH} \| \leq a|p_1|^r + b.
\]
Then
\[
\left| \int_0^1 \tilde{p}_\Pi \cdot (TH_K)_{pH}(\xi(t)) dt \right| \leq T^2 \left( \int_0^1 (a|p_1|^r + b) dt \right)^2 \leq T^2 \cdot 2(a^2\|p_1\|_{qH}^2 + b^2) dt.
\]
By (3.35), we have
\[
\begin{align*}
g & \geq \int_0^1 \left[ T((\mu - 1)H_K(\xi(t)) - a_3) \right] dt - T^2 \cdot 2 \left( a^2\|p_1\|_{qH}^2 + b^2 \right) \\
& \geq \int_0^1 T((\mu - 1)(a_1|p_1|^{\mu} - a_2) - a_3) dt - T^2 \cdot 2 \left( a^2\|p_1\|_{qH}^2 + b^2 \right) \\
& = T(\mu - 1)a_1\|p_1\|_{qH}^{\mu} - T^2 \cdot 2a^2\|p_1\|_{qH}^2 - (\mu - 1)a_2 - Ta_3 - T^2b^2 \\
& \geq T(\mu - 1)a_1\|p_1\|_{qH}^{\mu} - T^2 \cdot 2a^2\|p_1\|_{qH}^2 - (\mu - 1)a_2 - Ta_3 - T^2b^2. \quad (3.34)
\end{align*}
\]
Let
\[
C \overset{\text{def}}{=} g + T(\mu - 1)a_2 + Ta_3 + T^2b^2. \quad (3.35)
\]
Then (3.34) is
\[
C \geq T(\mu - 1)a_1\|p_1\|_{qH}^{\mu} - T^2 \cdot 2a^2\|p_1\|_{qH}^2. \quad (3.36)
\]
We claim that there exists a constant $R_1 > 0$ such that
\[
\|p_1\|_{qH} \leq R_1. \quad (3.37)
\]

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By (3.33), we have
\[
\gamma + Ta_3 + T^2 b^2 + T^2 a^2 R_1^2 \geq T(\mu - 1) \int_0^1 H_K(\xi(t))dt \equiv T(\mu - 1)H_K(\xi(t)) \geq T(\mu - 1)|p_1(t)|^\mu - a_2, \forall t.
\]

Then
\[
|p_1(t)| \leq \left( \frac{C + T^2 a^2 R_1^2}{T(\mu - 1)a_1} \right)^{1/\mu} = K_1.
\]

**Case 1.** $2s < \mu$. By (3.34), we have $||p_1||_s \leq R_1$.

**Case 2.** $2s = \mu$. For $T \in \left(0, \frac{(\mu - 1)a_1}{2a^2}\right)$ and
\[
R_1 = \left( \frac{\gamma + T(\mu - 1)a_2 + Ta_3 + T^2 b^2}{T(\mu - 1)a_1 - T^2 a^2} \right)^{1/\mu},
\]
we have $||p_1||_s \leq R_1$.

**Case 3.** $\mu < 2s < 2\mu - 1$.

**Subcase 3.1.** $1 < \mu < 2$. Let $\lambda = ||p_1||_s$ and
\[
C \overset{\text{def}}{=} \gamma + T(\mu - 1)a_2 + Ta_3 + T^2 b^2
\]
\[
= T^{2/\mu} \left( \frac{\mu a_1}{2} \right)^{2/\mu} + T^{1/\mu} \cdot 2 \left( \left| \frac{\mu}{\mu a_1} \right| \frac{1}{\mu - 1} \right)
\]
\[+ T \mu a_2 + Ta_3 + T^2 b^2. \quad (3.38)
\]

Set
\[
\phi(\lambda) = A \lambda^\alpha - B \lambda^\beta, \quad a < \beta.
\]

Then
\[
\phi'(\lambda) = A \lambda^{\alpha - 1} - B \beta \lambda^{\beta - 1} = 0 \Rightarrow \lambda_0 = \left( \frac{A \alpha}{B \beta} \right)^{1/(\beta - a)}.
\]

max $\phi(\lambda) = \phi(\lambda_0) = B \lambda_0^{\beta - 1} = B \left( \frac{A \alpha}{B \beta} \right)^{\frac{\beta - 1}{\alpha}}$.

Choose
\[
A = T(\mu - 1)a_1, \quad B = T^2 a^2, \quad \lambda_0 = \left( \frac{T(\mu - 1)a_1}{T^2 a^2 2s} \right)^{1/\mu},
\]
we have
\[
\max \phi(\lambda) = T(\mu - 1)a_1 \lambda_0^\mu - T^2 a^2 \lambda_0^{2s}
\]
\[
= T^2 a^2 \left( \frac{T(\mu - 1)a_1}{T^2 a^2 2s} \right)^{2/\mu} \left( \frac{2s}{\mu} - 1 \right)
\]
\[
= T^2 \left( \frac{T}{T^2} \right)^{2/\mu} \cdot 2a^2 \left( \frac{\mu - 1}{2a^2 2s} \right)^{2s/\mu} \left( \frac{2s}{\mu} - 1 \right)
\]

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where
\[ D_0 = 2a^2 \left( \frac{(\mu - 1)\alpha_1\mu}{2a^22s} \right)^{\frac{2s}{\mu}} \left( \frac{2s}{\mu} - 1 \right) \]
\[ = (2a^2)^{\frac{2s}{\mu}} \left( \frac{(\mu - 1)\alpha_1\mu}{2s} \right)^{\frac{2s}{\mu}} \left( \frac{2s}{\mu} - 1 \right). \]  
(3.41)

Let
\[ D_1 = \left( \frac{\mu_1}{\mu} \right)^{\frac{2s}{\mu}}, D_2 = 2 \left( \frac{|v|}{\mu a_1} \right)^{\frac{1}{\mu}} \left( 1 - \frac{1}{\mu} \right), D_3 = \mu a_2 + a_3, D_4 = 2b^2. \]
(3.42)

There exists \( \delta > 0 \) such that for \( T \in (0, \delta) \),
\[ T^{\frac{2s}{\mu}} \cdot D_1 + T^\frac{1}{\mu - 1} \cdot D_2 + T \cdot D_3 + T^2 \cdot D_4 < T^{\frac{2s}{\mu}} \cdot D_5, \]  
(3.43)
i.e.,
\[ T^{\frac{2s}{\mu}} \cdot D_1 + D_2 + T^{1 + \frac{1}{\mu - 1}} \cdot D_3 + T^{2 + \frac{1}{\mu - 1}} \cdot D_4 < T^{\frac{\mu(2s - 2\mu + 1)}{2s - \mu(\mu - 3)}} \cdot D_5. \]

In fact, since \( \mu < 2s < 2\mu - 1 \),
\[ \lim_{T \to 0^+} \text{Left} = D_2, \lim_{T \to 0^+} \text{Right} = +\infty. \]

By (3.38), (3.39), (3.40) and (3.43)
\[ C < T(\mu - 1)\alpha_1\lambda_0^\mu - T^22a^2\lambda_0^{2s} \]
\[ = \max_{\lambda \in \mathbb{R}} \left( T(\mu - 1)\alpha_1\lambda^\mu - T^22a^2\lambda^{2s} \right). \]  
(3.44)

Then the inequality (3.36) implies that
\[ ||p_I||_s \leq R_1 \text{ or } ||p_I||_s \geq R_2, \quad R_1 < \lambda_0 < R_2. \]  
(3.45)

By (3.7), we have \( ||p_I||_s \leq R_1 \).

**Remark 1.** In subcase 3.1, we must show that
\[ K_1 = \left( \frac{C + T^22a^2R_1^{2s}}{T(\mu - 1)\alpha_1} \right)^{\frac{1}{\mu}} < K_2 = R_2. \]

In fact, by (3.44) and (3.45), we have
\[ R_1 < \lambda_0 < R_2, \]
\[ C + T^22a^2\lambda_0^{2s} < T(\mu - 1)\alpha_1\lambda_0^\mu, \]
\[ K_1 < \left( \frac{C + T^22a^2\lambda_0^{2s}}{T(\mu - 1)\alpha_1} \right)^{\frac{1}{\mu}} < \lambda_0 < R_2 = K_2. \]
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