Hereditary kernel identification method of nonlinear polymeric viscoelastic materials

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Abstract

This paper deals with a polymeric matrix composite material. The matrix behaviour is described by the modified Rabotnov’s nonlinear viscoelastic model assuming the material is nonlinear viscoelastic. The parameters of creep and stress-relaxation kernels of the model are determined. From the experimental data related to kernels approximated by spline functions and by means of the method of weighted residual, the formulas for the determination of viscoelastic parameters are derived.

1 Introduction

Most of polymeric matrix composite materials are characterized by a nonlinear viscoelastic behaviour, even at moderate loading levels. A long term behaviour modeling these materials requires the determination of viscoelastic characteristics. For the hereditary-type model the methods for determining viscoelastic characteristics play a role of great importance and reduce to the establishment and identification of creep and stress-relaxation kernels. The past few years, many authors studied the nonlinear viscoelastic matrix behaviour modelled as per Schapery’s constitutive law. See [1] and references therein. Approaches based on coupling Schapery model and viscoplastic model proposed by Zapas and Crissman are used to predict a long term material behaviour by homogenization [2]. In the phenomenon approach of Goldenblat-Kopnov [3], the hereditary model was developed for building long term strength phenomenon tests for anisotropic composite materials. This nonlinear model is based on the coupling of plastic potential of Mises-Hill for anisotropic materials and Iliuchin motion approach. In this case, taking into account the peculiarities of material mechanical properties, the viscoelastic parameters can be determined on the basis of long term strength tests of material.

In a series of papers [4-9], it was proposed the identification methods for Schapery model using the uniaxial tests of creep-recovery. This approach presents two disadvantages: first, it is too difficult to reproduce exactly the theoretical creep recovery tests, and secondly, one generally deals with the creep recovery tests separately.

In this paper, in order to avoid these difficulties, we propose a method of hereditary kernel identification for nonlinear viscoelastic materials as follows. The experimental data of the kernels are provided by achieving independent tests whose number is equal to the multiple of the number of kernel unknowns. The precision of parameter values of the viscoelastic model requires an efficient method for determining hereditary parameters of nonlinear viscoelastic materials whose matrix behaviour is modelled as per modified Rabotnov’s hereditary integral equations with creep and stress-relaxation kernels, taking into account the experimental parameter data for small time values. The experimental data of kernels are approximate by spline functions, and, by means of the method of weighted residual, we obtain formulas for the determination of viscoelastic parameters.
2 Mathematical model

For nonlinear viscoelastic materials, the matrix behaviour is modelled by the modified Rabotnov’s nonlinear hereditary-type integral equation \[10\]

\[
\varphi_0(\varepsilon(t)) = \sigma(t) + \lambda \int_0^t K(t - \tau)\sigma(\tau)\,d\tau
\]

\[1\]

\[
\sigma(t) = \varphi_0(\varepsilon(t)) - \lambda \int_0^t R(t - \tau)\varphi_0(\varepsilon(\tau))\,d\tau
\]

\[2\]

where \(K\) and \(R\) represent the hereditary creep and the stress-relaxation kernels defined, respectively, by

\[
K(t - \tau) = \frac{1}{(t - \tau)^{\alpha}} \sum_{n \geq 0} \frac{(-\beta)^n (t - \tau)^{(1-\alpha)n}}{\Gamma((1-\alpha)(1 + n))} \]

\[3\]

\[
R(t - \tau) = \frac{1}{(t - \tau)^{\alpha}} \sum_{n \geq 0} \frac{(-\lambda + \beta)^n (t - \tau)^{(1-\alpha)n}}{\Gamma((1-\alpha)(1 + n))}.
\]

\[4\]

\(\lambda\) is a parameter; \(\alpha\) and \(\beta\) stand for the kernel parameters, \(t\) is the time and \(\Gamma\) the function defined by

\[
\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\,dx.
\]

\[5\]

\(\varepsilon\) and \(\sigma\) are, respectively, the deformation and the stress-relaxation functions depending on \(t\). The scalar function \(\varphi_0\), depending nonlinearly on \(\varepsilon\), is well approximated by a power function as follows:

\[
\varphi_0(\varepsilon) = \frac{H}{q} \varepsilon^q \]

\[6\]

The coefficients \(H\) and \(q\) are provided by uniaxial traction experimental data.

3 Hereditary kernel identification

The choice of efficient method for the kernel parameters determination depends essentially on the physical context. So, with respect to stress-relaxation kernel \(R\) of model (1,2), from the condition \(\varepsilon(t) = \varepsilon(0) = \text{const}\), the time derivative of equation (2) gives

\[
R(t) = -\frac{1}{\lambda} \frac{\varepsilon^q}{H} \frac{d\sigma(t)}{dt}
\]

\[7\]

Since the above expression of the kernel \(R\) is proportional to the stress-relaxation velocity \(\frac{d\sigma(t)}{dt}\) with the coefficient of proportionality remaining constant for \(\varepsilon(t) = \text{const}\), we can determine the kernel parameters of the model described by (1) and (2) by the derivation of the experimental curves of the kernel \(R\).

With respect to the creep kernel \(K\) of this model (1) and (2), the physics imposes to satisfy the condition \(\sigma(t) = \sigma(0) = \text{const}\) that provides the time derivative of equation (1) as follows:

\[
K(t) = \frac{1}{\lambda} \frac{H(\varepsilon(t))^{q-1}}{\sigma} \frac{d\varepsilon(t)}{dt}
\]

\[8\]

As this expression of the kernel \(K\) is proportional to the creep velocity \(\frac{d\varepsilon(t)}{dt}\), with the coefficient of proportionality remaining variable for \(\sigma(t) = \text{const}\), the above approach is no longer applicable, i.e., we cannot determine the kernel parameters by derivation of the experimental curves of kernel. In this case, the kernel parameters are provided by using the similarity condition between the stress-relaxation isochronic curves and the creep curve of the model described by (1) and (2): \[10\]

\[
\varphi_0(\varepsilon, 0) = (1 + G(t))\varphi_1(\varepsilon, t)
\]

\[9\]
where $1 + G(t)$ is the similarity function defined as
\[ 1 + G(t) = 1 + \lambda \int_0^t K(\tau) d\tau. \quad (10) \]

Here, $\varphi_0(.)$ is the function represented by the deformation curve and $\varphi_t(.)$, the function represented by the creep isochronic curves at every instant $t$.

Setting $S(t) = 1 + G(t)$, the condition (10) can be differentiated with respect to $t$, yielding the deformation velocity as follows:
\[ \frac{d\varepsilon(t)}{dt} = \sigma \left\{ \frac{d\varphi_0^{-1}[S(t)\sigma]}{d[S(t)\sigma]} \right\} \frac{dS(t)}{dt}, \quad \sigma = \text{const} \quad (11) \]

which, taking into account the relation (8), allows to write
\[ K(t) = \frac{1}{\lambda} \frac{dS(t)}{dt} \quad (12) \]

4 Creep data approximation and method of weighted residual

In this section, we use spline functions to approximate the creep kernel experimental data. Then, by method of weighted residual, we obtain formulas for the viscoelastic parameter value computation.

4.1 Approximation by spline functions

The similarity of curves representing functions $\varphi_t(.)$ and $\varphi_0(.)$ is considered in the plane $(\varepsilon, \varphi)$ for each fixed deformation level $\varepsilon_i, i = 1 \cdots l$ and for the parameter $t_j, j = 1, \cdots, n$. Then data $\varphi_0(\varepsilon_i, 0) = \frac{H_i}{q_i}$ can be approximated by the quantity $\varphi_t(\varepsilon_i, t_j)$. For this purpose we consider the functional $\Pi$ defined as
\[ \Pi(\overline{S}_j(t)) = \sum_{i=1}^l \left[ \varphi_0(\varepsilon_i, 0) - \overline{S}_j(t) \varphi_t(\varepsilon_i, t_j) \right]^2, \quad (13) \]

where $\overline{S}_j(t)$ is the similarity mean function. Then the functional minimum is reached for
\[ \overline{S}_j(t) = \frac{\sum_{i=1}^l \varphi_0(\varepsilon_i, 0) \varphi_t(\varepsilon_i, t_j)}{\sum_{i=1}^l \left[ \varphi_t(\varepsilon_i, t_j) \right]^2}. \quad (14) \]

In order to obtain the best approximation of the discrete data of the similarity mean function, we choose the spline functions defined as (11)
\[ \overline{S}_j(t) = \overline{A}_j + \overline{B}_j(t - t_j) + \overline{C}_j(t - t_j)^2 + \overline{D}_j(t - t_j)^3 \]

where $\overline{A}_j, \overline{B}_j, \overline{C}_j, \overline{D}_j$ are the coefficients, $j = 1, \ldots, n$. By substituting the equation (15) into equation (12), the experimental data $K(t_j)$ can be estimated by the functions $K_j(t, q)$ defined on $[t_j, t_{j+1}]$ as
\[ K_j(t, q) = B_j + 2C_j(t - t_j) + 3D_j(t - t_j)^2, \quad j = 1, \cdots, n \quad (16) \]

with the coefficients
\begin{align*}
B_j &= K(t_j), \quad (17) \\
2C_j &= 2 \frac{t_j[K(t_j) - K(t_{j-1})]}{h_{j-1}(2t_j - h_{j-1})}, \quad C_1 = 0, \quad (18) \\
3D_j &= \frac{K(t_j) - K(t_{j-1})}{h_{j-1}(2t_j - h_{j-1})}, \quad D_1 = 0, \quad (19) \\
h_j &= t_{j+1} - t_j. \quad (20)
\end{align*}

The functions $K_j(t, q)$ obtained by approximation are presented in Table 1.
At the first stage we determine experimentally the initial values \( \lambda_0 \) and \( q_0 \) of parameters \( \lambda \) and \( q \) in the equation (22) ;

At the second stage, the values \( \lambda_0 \) and \( q_0 \) are used, yielding

\[
\bar{w}_j(t) = \left\{ 1 + \left| \frac{K(t_j) - \lambda_0 K_j(t_j, q_0)}{K(t_*) - \lambda_0 K_j(t_*, q_0)} \right|^m \right\}^{-1}.
\]

Then the weighting functions \( w_j \) can be provided on the basis of the minimal value \( \delta_{\text{min}} \) of residual

\[
\delta = \sum_{j=1}^{n} \left\{ \bar{w}_j(t) [K(t_j) - \lambda_0 K_j(t_j, q_0)] \right\}^2
\]

where the sum decreases as \( m \) increases.

### 4.2 Method of weighted residual

The test of best approximation of data \( \{K(t_j), \ j = 1, \ldots, n\} \) by the functions \( \{K_j(t, q), \ j = 1, \ldots, n\} \) remains the method of weighted residual seeking to minimize the residual (error) as

\[
\Omega[\lambda, q] = \sum_{j=1}^{n} \left\{ w_j(t) [K(t_j) - \lambda K_j(t_j, q)] \right\}^2,
\]

for a finite set of weighting functions \( w_j \), \( j = 1, \ldots, n \) defined as

\[
w_j(t) = \left\{ 1 + \left| \frac{K(t_j) - \lambda K_j(t_j, q)}{K(t_*) - \lambda K_j(t_*, q)} \right|^m \right\}^{-1}
\]

satisfying the following conditions

\[
K_j(t, q) \to \infty, \quad w_j(t) \to 0,
\]

\[
K(t_j) = \lambda K_j(t_j, q), \quad w_j(t) = 1.
\]

\( n \) is the number of discret values of creep kernel for \( t \) from 0 to \( t_* = 1050 \) and \( m \), the order of difference moments, \( m = 2, 3, 4, \ldots \).

The parameter \( \lambda \) and the creep kernel parameter \( q \) of the equation (22) are defined in two steps:

- At the first stage we determine experimentally the initial values \( \lambda_0 \) and \( q_0 \) of parameters \( \lambda \) and \( q \) in the equation (22) ;
- At the second stage, the values \( \lambda_0 \) and \( q_0 \) are used, yielding

| \( j \) | \( t_j \) | \( B_j \) | \( 2C_j \) | \( 3D_j \) | \( K_j(t, q) \) |
|---|---|---|---|---|---|
| 1 | 0 | 3750 | 0 | 0 | 3750 |
| 2 | 5 | 3500 | -100 | -10 | 3500 - 100(t - 5) - 10(t - 5)^2 |
| 3 | 7 | 3250 | -149 | -10.42 | 3250 - 149(t - 7) - 10.42(t - 7)^2 |
| 4 | 10 | 2900 | -137 | -6.86 | 2900 - 137(t - 10) - 6.86(t - 10)^2 |
| 5 | 12 | 2600 | -167 | -6.82 | 2600 - 167(t - 12) - 6.82(t - 12)^2 |
| 6 | 15 | 2250 | -130 | -4.32 | 2250 - 130(t - 15) - 4.32(t - 15)^2 |
| 7 | 17 | 1900 | -186 | -5.47 | 1900 - 186(t - 17) - 5.47(t - 17)^2 |
| 8 | 30 | 1500 | -39.3 | -0.65 | 1500 - 39.3(t - 30) - 0.65(t - 30)^2 |
| 9 | 70 | 1150 | -12.25 | -0.09 | 1150 - 12.25(t - 70) - 0.09(t - 70)^2 |
| 10 | 80 | 900 | -27 | -0.17 | 900 - 27(t - 80) - 0.17(t - 80)^2 |
| 11 | 100 | 750 | -8.3 | -0.04 | 750 - 8.3(t - 100) - 0.04(t - 100)^2 |
| 12 | 150 | 500 | -6 | -0.02 | 500 - 6(t - 150) - 0.02(t - 150)^2 |
| 13 | 250 | 300 | -2.5 | -0.005 | 300 - 2.5(t - 250) - 0.005(t - 250)^2 |
| 14 | 350 | 250 | -0.6 | -0.0008 | 250 - 0.6(t - 350) - 0.0008(t - 350)^2 |
| 15 | 750 | 150 | -0.34 | -0.0002 | 150 - 0.34(t - 750) - 0.0002(t - 750)^2 |
| 16 | 1050 | 100 | -0.2 | -0.0001 | 100 - 0.2(t - 1050) - 0.0001(t - 1050)^2 |

Table 1: Approximated function \( K_j(t, q) \)
Furthermore, the minimum of residual

$$\lambda \mapsto \Omega[\lambda, q] = \sum_{j=1}^{n} \left\{ \bar{w}_j(t) \left[ K(t_j) - \lambda K_j(t, q) \right] \right\}^2$$

(27)

is reached for

$$\bar{\lambda} = \frac{\sum_{j=1}^{n} \bar{w}_j^2(t) K(t_j) K_j(t, q)}{\sum_{j=1}^{n} \bar{w}_j^2(t) K_j^2(t, q)}.$$  

(28)

Taking into account the equations (28) and (30), we obtain the formula for the parameter \(\bar{\lambda}\) value computation

$$\bar{\lambda} = \left[ \sum_{j=1}^{n} \frac{K(t_j) K_j(t, q)}{|K(t_j) - \lambda_0 K_j(t, q_0)|^2} \right]^{-1} \left[ \sum_{j=1}^{n} \frac{K_j^2(t, q)}{|K(t_j) - \lambda_0 K_j(t, q_0)|^2} \right],$$

(31)

$$t \in [t_j, t_{j+1}], j = 1, \ldots, n + 1.$$  

(32)

For instance, when \(t = t_j\), the formula (31) reduces to

$$\bar{\lambda} = \left[ \sum_{j=1}^{n} \frac{K_j^2(t_j)}{|(1 - \lambda_0) K(t_j)|^2} \right]^{-1} \left[ \sum_{j=1}^{n} \frac{K_j^2(t_j)}{|(1 - \lambda_0) K(t_j)|^2} \right] = 1.$$  

(33)

For each fixed deformation level \(\varepsilon_i, i = 1, \ldots, l\) and for the parameter \(t_j, j = 1, \ldots, n\), the equation (1) reduces to

$$\frac{H}{q_1} \varepsilon_i^q = \sigma \left[ 1 + \bar{\lambda} \int_{0}^{t_j} K_j(\tau, q) d\tau \right] \iff \varepsilon_i^q - \eta_j q = 0,$$

(34)

$$\eta_j = \frac{\sigma}{H} \left\{ 1 + \bar{\lambda} [B_j t_j - C_j t_j^2 + D_j t_j^3] \right\}, \sigma = \text{const}$$

(35)

which admits the solution \(\bar{q} \in [0, \bar{q}]\), where \(\bar{q}\) must satisfy the inequality

$$\varepsilon_i^q < \eta_j \bar{q}.$$  

(36)

Therefore we obtain the formula for the parameter \(\bar{q}\) value computation as follows:

$$\varepsilon_i^q - \eta_j \bar{q} = 0, \ i = 1 \cdots, l, \ j = 1, \cdots, 16.$$  

(37)

Finally, we arrive at the following formulas for the computation of the parameters \(\lambda\) and \(q\):

$$\bar{\lambda} = \left[ \sum_{j=1}^{n} \frac{K(t_j) K_j(t, q)}{|K(t_j) - \lambda_0 K_j(t, q_0)|^2} \right]^{-1} \left[ \sum_{j=1}^{n} \frac{K_j^2(t, q)}{|K(t_j) - \lambda_0 K_j(t, q_0)|^2} \right],$$

(38)

$$t \in [t_j, t_{j+1}], j = 1, \cdots, 15,$$

(39)

$$\varepsilon_i^q - \eta_j \bar{q} = 0, \ i = 1 \cdots, l, \ j = 1, \cdots, 16.$$  

(40)
5 Concluding remarks

The most important element for the kernel identification method of nonlinear viscoelastic models reveals to be the experimental data \{K(t_j), j = 1, \ldots, n\} related to the kernels which we applied the approximation test to. Among all known standard tests, the cubic spline method gave the best approximation and using the method of weighted residual to minimize the residual error, we obtained the formulas needed to compute the parameters \(\lambda\) and \(q\).

Thus, this study allowed us to identify the creep and stress-relaxation kernels of the nonlinear viscoelastic materials. Using the spline functions we approximated the experimental data related to the kernel that permitted, with the help of the method of weighted residual, to obtain formulas useful for the parameters computation of the nonlinear model.

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