A POLYNOMIAL DEFINED BY THE $SL(2; \mathbb{C})$-REIDEMEISTER TORSION FOR A HOMOLOGY 3-SPHERE OBTAINED BY A DEHN SURGERY ALONG A $(2p, q)$-TORUS KNOT

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Abstract. Let $K$ be a $(2p, q)$-torus knot. Here $p$ and $q$ are coprime odd positive integers. $M_n$ is a 3-manifold obtained by $\frac{1}{n}$-Dehn surgery along $K$. We consider a polynomial $\sigma_{(2p,q,n)}(t)$ whose zeros are the inverses of the Reidemeister torsion of $M_n$ for $SL(2; \mathbb{C})$-irreducible representations under some normalization. Johnson gave a formula for the case of the $(2, 3)$-torus knot under another normalization. We generalize this formula for a $(2p, q)$-torus knot by using Tchebychev polynomials.

1. Introduction

Reidemeister torsion is a piecewise linear invariant for manifolds. It was originally defined by Reidemeister, Franz and de Rham in 1930’s. In 1980’s Johnson [1] developed a theory of the Reidemeister torsion from the viewpoint of relations to the Casson invariant. He also derived an explicit formula for the Reidemeister torsion of homology 3-spheres obtained by $\frac{1}{n}$-Dehn surgeries along a torus knot for $SL(2; \mathbb{C})$-irreducible representations.

Let $K$ be a $(2p, q)$-torus knot where $p, q$ are coprime, positive odd integers. Let $M_n$ be a closed 3-manifold obtained by a $\frac{1}{n}$-surgery along $K$. We consider the Reidemeister torsion $\tau_{n}(M_n)$ of $M_n$ for an irreducible representation $\rho : \pi_1(M_n) \rightarrow SL(2; \mathbb{C})$.

Johnson gave a formula for any non trivial values of $\tau_{n}(M_n)$. Furthermore in the case of the trefoil knot, he proposed to consider the polynomial whose zero set coincides with the set of all non trivial values $\{\frac{1}{\tau_{n}(M_n)}\}$, which is denoted by $\sigma_{(2,3,n)}(t)$. Under some normalization of $\sigma_{(2,3,n)}(t)$, he gave a 3-term relation among $\sigma_{(2,3,n+1)}(t), \sigma_{(2,3,n)}(t)$ and $\sigma_{(2,3,n-1)}(t)$ by using Tchebychev polynomials.

In this paper we consider one generalization of this polynomial for a $(2p, q)$-torus knot. Main results of this paper are Theorem 4.3 and Proposition 5.1.

Acknowledgements. This research was partially supported by JSPS KAKENHI 25400101.

2010 Mathematics Subject Classification. 57M27.
Key words and phrases. Reidemeister torsion, a torus knot, Brieskorn homology 3-sphere, $SL(2; \mathbb{C})$-representation.
2. Definition of Reidemeister torsion

First let us describe definitions and properties of the Reidemeister torsion for $SL(2; \mathbb{C})$-representations. See Johnson [11], Kitano [2, 3] and Milnor [5, 6, 7] for details. Let $W$ be a $d$-dimensional vector space over $\mathbb{C}$ and let $b = (b_1, \cdots, b_d)$ and $c = (c_1, \cdots, c_d)$ be two bases for $W$. Setting $b_i = \sum j p_{ij} c_j$, we obtain a nonsingular matrix $P = (p_{ij})$ with entries in $\mathbb{C}$. Let $[b/c]$ denote the determinant of $P$.

Suppose
\[ C_* : 0 \to C_k \to C_{k-1} \to \cdots \to C_1 \to C_0 \to 0 \]
is an acyclic chain complex of finite dimensional vector spaces over $\mathbb{C}$. We assume that a preferred basis $c_i$ for $C_i$ is given for each $i$. Choose some basis $b_i$ for $B_i = \text{Im}(\partial_{i-1})$ and take a lift of it in $C_{i+1}$, which we denote by $\tilde{b}_i$. Since $B_i = Z_i = \text{Ker}\partial_i$, the basis $b_i$ can serve as a basis for $Z_i$. Furthermore since the sequence
\[ 0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0 \]
is exact, the vectors $(b_i, \tilde{b}_{i-1})$ form a basis for $C_i$. Here $\tilde{b}_{i-1}$ is a lift of $b_{i-1}$ in $C_i$. It is easily shown that $[b_i, \tilde{b}_{i-1}/c_i]$ does not depend on the choice of a lift $\tilde{b}_{i-1}$. Hence we can simply denote it by $[b_i, b_{i-1}/c_i]$.

**Definition 2.1.** The torsion of the chain complex $C_*$ is given by the alternating product
\[ \prod_{i=0}^{k} [b_i, b_{i-1}/c_i]^{(-1)^i} \]
and we denote it by $\tau(C_*)$.

**Remark 2.2.** It is easy to see that $\tau(C_*)$ does not depend on the choices of the bases $\{b_0, \cdots, b_k\}$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let $X$ be a finite CW-complex and $\tilde{X}$ a universal covering of $X$. The fundamental group $\pi_1 X$ acts on $\tilde{X}$ from the right-hand side as deck transformations. Then the chain complex $C_*(\tilde{X}; \mathbb{Z})$ has the structure of a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.

Let $\rho : \pi_1 X \to SL(2; \mathbb{C})$ be a representation. We denote the 2-dimensional vector space $\mathbb{C}^2$ by $V$. Using the representation $\rho$, $V$ admits the structure of a $\mathbb{Z}[\pi_1 X]$-module and then we denote it by $V_\rho$. Define the chain complex $C_*(X; V_\rho)$ by $C_*(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis
\[ (\tilde{u}_1 \otimes e_1, \tilde{u}_1 \otimes e_2, \cdots, \tilde{u}_d \otimes e_1, \tilde{u}_d \otimes e_2) \]
of $C_*(X; V_\rho)$ where $\{e_1, e_2\}$ is a canonical basis of $V = \mathbb{C}^2$ and $u_1, \cdots, u_d$ are the $i$-cells giving a basis of $C_i(X; \mathbb{Z})$.

Now we suppose that $C_*(X; V_\rho)$ is acyclic, namely all homology groups $H_*(X; V_\rho)$ are vanishing. In this case we call $\rho$ an acyclic representation.
Definition 2.3. Let \( \rho : \pi_1(X) \to SL(2; \mathbb{C}) \) be an acyclic representation. Then the Reidemeister torsion \( \tau_\rho(X) \in \mathbb{C} \setminus \{0\} \) is defined by the torsion \( \tau(C_*(X; \rho)) \) of \( C_*(X; \rho) \).

Remark 2.4.

1. We define \( \tau_\rho(X) = 0 \) for a non-acyclic representation \( \rho \).
2. The definition of \( \tau_\rho(X) \) depends on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant for \( X \) with \( \rho \).

Now let \( M \) be a closed orientable 3-manifold with an acyclic representation \( \rho : \pi_1(M) \to SL(2; \mathbb{C}) \). Here we take a torus decomposition of \( M = A \cup \mathbb{T} \cup B \). For simplicity, we write the same symbol \( \rho \) for restricted representations to images of \( \pi_1(A), \pi_1(B), \pi_1(T^2) \) in \( \pi_1(M) \) by inclusions. By this decomposition, we have the following formula.

Proposition 2.5. Let \( \rho : \pi_1(M) \to SL(2; \mathbb{C}) \) a representation. Assume all homology groups \( H_*(T^2; V_\rho) = 0 \). Then all homology groups \( H_*(M; V_\rho) = 0 \) if and only if both of all homology groups \( H_*(A; V_\rho) = H_*(B; V_\rho) = 0 \). In this case, it holds

\[
\tau_\rho(M) = \tau_\rho(A)\tau_\rho(B).
\]

3. Johnson’s theory

We apply the above proposition to a 3-manifold obtained by Dehn-surgery along a knot. Now let \( K \subset S^3 \) be a \((2p, q)\)-torus knot with coprime odd integers \( p, q \). Further let \( N(K) \) be an open tubular neighborhood of \( K \) and \( E(K) \) its knot exterior \( S^3 \setminus N(K) \). We denote its closure of \( N(K) \) by \( \overline{N} \) which is homeomorphic to \( S^1 \times D^2 \). Now we write \( M_n \) to a closed orientable 3-manifold obtained by \( \frac{1}{n} \)-surgery along \( K \). Naturally there exists a torus decomposition \( M_n = E(K) \cup \overline{N} \) of \( M_n \).

Remark 3.1. This manifold \( M_n \) is diffeomorphic to a Brieskorn homology 3-sphere \( \Sigma(2p, q, N) \) where \( N = [2pqn + 1] \).

Here the fundamental group of \( E(K) \) has a presentation as follows.

\[
\pi_1(E(K)) = \pi_1(S^3 \setminus K) = \langle x, y \mid x^{2p} = y^q \rangle
\]

Furthermore the fundamental group \( \pi_1(M_n) \) admits the presentation as follows;

\[
\pi_1(M_n) = \langle x, y \mid x^{2p} = y^q, ml^n = 1 \rangle
\]

where \( m = x^{-r}y^s \) \( (r, s \in \mathbb{Z}, 2ps - qr = 1) \) is a meridian of \( K \) and \( l = x^{-2p}m^2pq = y^{-q}m^2pq \) is similarly a longitude.

Let \( \rho : \pi_1(E(K)) = \pi_1(S^3 \setminus K) \to SL(2; \mathbb{C}) \) a representation. It is easy to see a given representation \( \rho \) can be extended to \( \pi_1(M_n) \to SL(2; \mathbb{C}) \) as a representation if and only if \( \rho(ml^n) = E \). Here \( E \) is the identity matrix in \( SL(2; \mathbb{C}) \). In this case by applying Proposition 2.5,

\[
\tau_\rho(M_n) = \tau_\rho(E(K))\tau_\rho(N)
\]
for any acyclic representation \( \rho : \pi_1(M_n) \to SL(2; \mathbb{C}) \).

Now we consider only irreducible representations of \( \pi_1(M_n) \), which is extended from the one on \( \pi_1(E(K)) \). It is seen that the set of the conjugacy classes of the \( SL(2; \mathbb{C}) \)-irreducible representations is finite. Any conjugacy class can be represented by \( \rho_{(a,b,k)} \) for some \( (a,b,k) \) such that

1. \( 0 < a < 2p, 0 < b < q, a \equiv b \mod 2 \),
2. \( 0 < k < N = |2pqn + 1|, k \equiv na \mod 2 \),
3. \( \operatorname{tr}(\rho_{(a,b,k)}(x)) = 2 \cos \frac{a\pi}{2p} \),
4. \( \operatorname{tr}(\rho_{(a,b,k)}(y)) = 2 \cos \frac{b\pi}{q} \),
5. \( \operatorname{tr}(\rho_{(a,b,k)}(m)) = 2 \cos \frac{a\pi}{N} \).

Johnson computed \( \tau_{\rho_{(a,b,k)}}(M_n) \) as follows.

**Theorem 3.2 (Johnson).**

1. A representation \( \rho_{(a,b,k)} \) is acyclic if and only if \( a \equiv b \equiv 1, k \equiv n \mod 2 \).
2. For any acyclic representation \( \rho_{(a,b,k)} \) with \( a \equiv b \equiv 1, k \equiv n \mod 2 \),
   \[ \tau_{\rho_{(a,b,k)}}(M_n) = \frac{1}{2(1 - \cos \frac{a\pi}{2p})(1 - \cos \frac{b\pi}{q})(1 + \cos \frac{a\pi}{N})}. \]

**Remark 3.3.**

- In fact Johnson proved this theorem for any torus knot, not only for a \((2p,q)\)-torus knot.
- Johnson’s result was generalized for any Seifert fiber manifold in [2]. Please see [2] as a reference.
- In general, it is not true that the set of \( \{ \tau_{\rho}(M_n) \} \) is finite. There exists a manifold whose Reidemister torsion can be variable continuously. Please see [3].

Here assume \( K = T(2, 3) \) is the trefoil knot. By considering the set of non trivial values \( \tau_{\rho}(M_n) \) for irreducible representation \( \rho : \pi_1(M_n) \to SL(2; \mathbb{C}) \), Johnson defined the polynomial \( \tilde{\sigma}_{(2,3,n)}(t) \) of one variable \( t \) whose zeros are the set of \( \{ \frac{1}{\tau_{\rho}(M_n)} \} \), which is well defined up to multiplications of nonzero constants.

**Theorem 3.4 (Johnson).** Under normalization by \( \tilde{\sigma}_{(2,3,n)}(0) = (-1)^n \), there exists the 3-term relation s.t.

\[ \tilde{\sigma}_{(2,3,n+1)}(t) = (t^3 - 6t^2 + 9t - 2)t_{(2,3,n)}(t) - \tilde{\sigma}_{(2,3,n-1)}(t). \]

**Remark 3.5.** The polynomial \( t^3 - 6t^2 + 9t - 2 \) is given by \( 2T_6 \left( \frac{1}{2} \sqrt{7} \right) \). Here \( T_6(x) \) is the sixth Tchebychev polynomial.

Recall the \( n \)-th Tchebychev polynomial \( T_n(x) \) of the first kind can be defined by expressing \( \cos n\theta \) as a polynomial in \( \cos \theta \). We give a summary of these polynomials.

**Proposition 3.6.** The Tchebychev polynomials have following properties.
(1) $T_0(x) = 1$, $T_1(x) = x$.

(2) $T_n(x) = T_n(x)$.

(3) $T_n(1) = 1$, $T_n(-1) = (-1)^n$.

(4) $T_n(0) = \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
(-1)^{\frac{n}{2}} & \text{if } n \text{ is even}.
\end{cases}$

(5) $T_{n+1}(x) = 2xT_n - T_{n-1}(x)$.

(6) The degree of $T_n(x)$ is $n$.

(7) $2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x)$.

He we put a short list of $T_n(x)$.

- $T_0(x) = 1$,
- $T_1(x) = x$,
- $T_2(x) = 2x^2 - 1$,
- $T_3(x) = 4x^3 - 3x$,
- $T_4(x) = 8x^4 - 8x^2 + 1$,
- $T_5(x) = 16x^5 - 20x^3 + 5x$,
- $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$.

**Example 3.7.** Put $p = 1$, $q = 3$ and $n = -1$. Then $N = |2 \cdot 3 \cdot (-1) + 1| = 5$ and $M_{-1} = \Sigma(2, 3, 5)$. In this case, it is easy to see that $a = b = 1$ and $k = 1, 3$. By the above formula, we obtain

\[ \frac{1}{\frac{1}{2}\tau_{(1,3)}(M_{-1})} = \frac{1}{2(1 - \cos \frac{2}{3})(1 - \cos \frac{\pi}{3})(1 + \cos \frac{6\pi}{5})} = \frac{1}{1 + \cos \frac{6\pi}{5}} = 3 + \sqrt{5}. \]

Hence we have two non trivial values of $\frac{1}{\frac{1}{2}\tau_{(1,3)}(M_{-1})}$ as

\[ \frac{1}{\frac{1}{2}\tau_{p}(M_{-1})} = \frac{1}{3 + \sqrt{5}} = \frac{2}{3 \pm \sqrt{5}} = \frac{3 \pm \sqrt{5}}{2}. \]

Therefore we have

\[ \left( t - \left( \frac{3 - \sqrt{5}}{2} \right) \right) \left( t - \left( \frac{3 + \sqrt{5}}{2} \right) \right) = t^2 - 3t + 1. \]

Under Johnson's normalization $\bar{\sigma}_{(2,3,-1)}(0) = -1$,

\[ \bar{\sigma}_{(2,3,-1)}(t) = -\frac{t^2 + 3t - 1}{5}. \]
Next put $n = 1$. In this case

$$\tau_{p(1,1)}(M_1) = \frac{1}{2 \left(1 - \cos \frac{\pi}{2} \right) \left(1 - \cos \frac{\pi}{3} \right) \left(1 + \cos \frac{6\pi}{5} \right)}$$

$$= \frac{1}{1 + \cos \frac{6\pi}{5}}.$$ 

We can see as

$$\left( t - 2 \left(1 + \cos \frac{6\pi}{5} \right) \right) \left( t - 2 \left(1 + \cos \frac{6 \cdot 3\pi}{5} \right) \right) \left( t - 2 \left(1 + \cos \frac{6 \cdot 5\pi}{5} \right) \right)$$

$$= t^3 - 5t^2 + 6t - 1$$

$$= \bar{\sigma}(2,3,1)(t).$$

On the other hand, by using Johnson’s formula

$$(t^3 - 6t^2 + 9t - 2) \bar{\sigma}_{(2,3,0)}(t) - \bar{\sigma}_{(2,3,-1)}(t) = (t^3 - 6t^2 + 9t - 2) \cdot 1 - (-t^2 + 3t - 1)$$

$$= t^3 - 5t^2 + 6t - 1,$$

we obtain the same polynomial.

4. Main theorem

From this section, we consider the generalization for a $(2p, q)$-torus knot. Here $p, q$ are coprime odd integers. In this section we give a formula of the torsion polynomial $\sigma_{(2p,q,n)}(t)$ for $M_n = \Sigma(2p, q, N)$ obtained by a $\frac{n}{p}$-Dehn surgery along $K$. Although Johnson considered the inverses of the half of $\tau_p(M_n)$, we simply treat torsion polynomials as follows.

**Definition 4.1.** A one variable polynomial $\sigma_{(2p,q,n)}(t)$ is called the torsion polynomial of $M_n$ if the zero set coincides with the set of all non trivial values \{ $\frac{1}{\bar{\tau}_p(M_n)}$ \} and it satisfies the following normalization condition as $\sigma_{(2p,q,n)}(0) = (-1)^{\frac{2pqk-1}{M_n}}$.

**Remark 4.2.**

(1) If $n = 0$, then clearly $M_n = S^3$ with the trivial fundamental group. Hence we define the torsion polynomial to be trivial.

From here assume $n \neq 0$. Recall Johnson’s formula

$$\frac{1}{\tau_{p,a,k}(M_n)} = 2 \left(1 - \cos \frac{ap}{2p} \right) \left(1 - \cos \frac{b\pi}{q} \right) \left(1 + \cos \frac{2pqk\pi}{N} \right)$$

where $0 < a < 2p$, $0 < b < q$, $a \equiv b \equiv 1 \ mod \ 2$, $k \equiv n \ mod \ 2$. Here we put

$$C_{(2p,q,a,b)} = \left(1 - \cos \frac{ap}{2p} \right) \left(1 - \cos \frac{b\pi}{q} \right)$$

and we have

$$\frac{1}{\tau_{p,a,b,k}(M_n)} = 4C_{(2p,q,a,b)} \cdot \frac{1}{2} \left(1 + \cos \frac{2pqk\pi}{N}\right).$$
Main result is the following.

**Theorem 4.3.** The torsion polynomial of $M_n$ is given by

$$\tau_{(2p,q,n)}(t) = \prod_{(a,b)} Y_{(a,b)}(t)$$

where

$$Y_{(a,b)}(t) = \begin{cases} 
2C_{(2p,q,a,b)} \left( \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} \right) - T_{N-1} \left( \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} \right) & (n > 0) \\
-2C_{(2p,q,a,b)} \left( \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} \right) - T_{N-1} \left( \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} \right) & (n < 0).
\end{cases}$$

Here $N = |2pqn + 1|$ and a pair of integers $(a, b)$ is satisfying the following conditions:

- $0 < a < 2p, 0 < b < q$,
- $a \equiv b \equiv 1 \mod 2$,
- $0 < k < N, k \equiv 2n \mod 2$.

**Proof.**

Case 1: $n > 0$

We modify one factor $(1 + \cos \frac{2pqk\pi}{N})$ of $\frac{1}{\tau_p(M_n)}$ as follows.

**Lemma 4.4.** The set $\{\cos \frac{2pqk\pi}{N} | 0 < k < n \equiv n \mod 2\}$ is equal to the set $\{\cos \frac{2pqk\pi}{N} | 0 < k < \frac{N}{2}\}$.

**Proof.** Now $N = 2pqn + 1$ is always an odd integer.

For any $k > \frac{N}{2}$, then clearly $N - k < \frac{N}{2}$. Then

$$\cos \frac{2pq(N - k)\pi}{N} = \cos \left( 2pq\pi - \frac{2pqk\pi}{N} \right) = (-1)^{2pq} \cos \left( \frac{2pqk\pi}{N} \right) = \cos \left( \frac{2pqk\pi}{N} \right).$$

Here if $k$ is even (resp. odd), then $N - k$ is odd (resp. even). Hence it is seen

$$\{\cos \frac{2pqk\pi}{N} | 0 < k < n \equiv n \mod 2\} = \{\cos \frac{2pqk\pi}{N} | 0 < k < \frac{N}{2}\}.$$

For any $k < \frac{N}{2}$, there exists uniquely $l$ such that $-\frac{N}{2} < l < \frac{N}{2}$ and $l \equiv l \mod N$. Further there exists uniquely $l$ such that $0 < l < \frac{N}{2}$ and $l \equiv \pm qk \mod N$. Here $\cos \frac{2pq\pi}{N} = \cos \frac{2pl\pi}{N}$ if and only if $2pqk \equiv \pm 2pl \mod N$.

Therefore it is seen that the set

$$\left\{\cos \frac{2pqk\pi}{N} | 0 < k < \frac{N}{2}\right\} = \left\{\cos \frac{2pqk\pi}{N} | 0 < k < \frac{N}{2}\right\}.$$ 

$\Box$
Now we can modify

\[ \frac{1}{2} \left( 1 + \cos \frac{2pk\pi}{N} \right) = \frac{1}{2} \cdot 2 \cos^2 \frac{2pk\pi}{2N} = \cos^2 \frac{pk\pi}{N}. \]

We put

\[ z_k = \cos \frac{pk\pi}{N} \quad (0 < k < N) \]

and substitute \( x = z_k \) to \( T_{N+1}(x) \). Then it holds

\[ T_{N+1}(z_k) = \cos \left( \frac{(N + 1)(pk\pi)}{N} \right) = \cos \left( pk\pi + \frac{pk\pi}{N} \right) = (-1)^{pk} z_k. \]

Similarly it is seen

\[ T_{N-1}(z_k) = \cos \left( \frac{(N - 1)(pk\pi)}{N} \right) = \cos \left( pk\pi - \frac{pk\pi}{N} \right) = (-1)^{pk} z_k. \]

Hence it holds

\[ T_{N+1}(z_k) - T_{N-1}(z_k) = 0. \]

By properties of Tchebyshev polynomials, it is seen that

- \( T_{N+1}(1) - T_{N-1}(1) = 0, \)
- \( T_{N+1}(-1) - T_{N-1}(-1) = 0. \)

Therefore we consider the following:

\[ X_n(x) = \begin{cases} \frac{T_{N+1}(x) - T_{N-1}(x)}{2(x^2 - 1)} & (n > 0) \\ \frac{T_{N+1}(1) - T_{N-1}(1)}{2(x^2 - 1)} & (n < 0). \end{cases} \]

We mention that the degree of \( X_n(x) \) is \( N - 1. \)
By the above computation, \( z_1, \ldots, z_{N-1} \) are the zeros of \( X_n(x) \). Further we can see

\[
z_{N-k} = \cos \frac{p(N - k)\pi}{N} = \cos(p\pi - \frac{pk\pi}{N}) = (-1)^p \cos\left(-\frac{pk\pi}{N}\right) = (-1)^p \cos\left(\frac{pk\pi}{N}\right) = -z_k.
\]

This means \( N - 1 \) roots \( z_1, \ldots, z_{N-1} \) of \( X_n(x) = 0 \) occur in a pairs. Because \( T_{N+1}(x) \), \( T_{N-1}(x) \) are even functions, they are functions of \( x^2 \). Hence \( X_n(x) \) is also an even function.

Here by replacing \( x^2 \) by \( \frac{t}{4C(2p,q,a,b)} \), namely \( x \) by \( \frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}} \), we put

\[
Y_{(n,a,b)}(t) = X_n\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right)
\]

\[
= \frac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right)}{2\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right)^2 - 1}
\]

\[
= \frac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right)}{2\left(\frac{t}{4C(2p,q,a,b)} - 1\right)}
\]

\[
= 2C(2p,q,a,b) \frac{T_{N+1}\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right) - T_{N-1}\left(\frac{\sqrt{t}}{2\sqrt{C(2p,q,a,b)}}\right)}{t - 2C(2p,q,a,b)}
\]

Here it holds that its degree of \( Y_{(n,a,b)}(s) \) is \( \frac{N-1}{2} \), and the roots of \( Y_{(n,a,b)}(t) \) are \( 4C(2p,q,a,b)z_k^2 = 4C(2p,q,a,b) \cos^2 \frac{pk}{2pqn+1}, \) \( 0 < k < \frac{N-1}{2} \), which are all non trivial values of \( \frac{1}{\tau_{(n,a,b)}(M_n)} \). Therefore we obtain the formula.

Case 2: \( n < 0 \)

In this case we modify \( N = |2pqn + 1| = 2pqn - 1 \). By the same arguments, it is easy to see the claim of the theorem can be proved. Therefore the proof completes. \( \square \)

**Remark 4.5.** By defining as \( X_0(t) = 1 \), it implies \( Y_{(0,a,b)}(t) = 1 \). Then the above statement is true for \( n = 0 \).

**Corollary 4.6.** The degree of \( \sigma_{(2p,q,n)}(t) \) is given by \( \frac{(N-1)p(q-1)}{4} \).
Proof. The number of the pairs \((a, b)\) is given by \(\frac{p(q-1)}{2}\). As the degree of \(Y_{(n,a,b)}(t)\) is \(\frac{N-1}{2}\), then the degree of \(\sigma_{(2p,q,n)}(t)\) is given by \(\frac{(N-1)p(q-1)}{4}\). \(\square\)

5. 3-TERM RELATIONS

Finally we prove 3-term realtions for each factor \(Y_{(n,a,b)}(t)\) as follows.

**Proposition 5.1.** For any \(n\), it holds that

\[
Y_{(n+1,a,b)}(t) = D(t)Y_{(n,a,b)}(t) - Y_{(n-1,a,b)}(t)
\]

where \(D(t) = 2T_{2pq} \left( \frac{\sqrt{7}}{2 \sqrt{C_{2p,q,a,b}}} \right) \).

**Proof.** Recall Prop. 3.2 (7);

\[
2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x).
\]

Then if \(n > 0\) we have

\[
2T_{2pq}(x)X_n(x) = 2T_{2pq}(x)\left( \frac{T_{2pq+2}(x) - T_{2pq}(x)}{2(x^2 - 1)} \right)
\]

\[
= \frac{(T_{2pq+2pq+2}(x) + T_{2pq+2-2pq}(x)) - (T_{2pq+2pq}(x) + T_{2pq-2pq}(x))}{2(x^2 - 1)}
\]

\[
= \frac{T_{2pq(n+1)+2}(x) - T_{2pq(n+1)}(x) + T_{2pq(n-1)+2}(x) - T_{2pq(n-1)}(x)}{2(x^2 - 1)}
\]

\[
= X_{n+1}(x) + X_{n-1}(x).
\]

Therefore it can be seen that

\[
X_{n+1}(x) = 2T_{2pq}(x)X_n(x) - X_{n-1}(x)
\]

and

\[
Y_{(n+1,a,b)}(t) = 2T_{2pq} \left( \frac{\sqrt{7}}{2 \sqrt{C_{2p,q,a,b}}} \right) Y_{(n,a,b)}(t) - Y_{(n-1,a,b)}(t).
\]

If \(n=0\), 3-term relation is

\[
Y_{(1,a,b)}(t) = D(t)Y_{(0,a,b)}(t) - Y_{(-1,a,b)}(t).
\]

It can be seen by direct computation

\[
2T_{2pq}(x)X_0(x) - X_{-1}(x) = 2T_{2pq}(x) - X_{-1}(x)
\]

\[
= X_1(x).
\]

If \(n < 0\), it can be also proved. \(\square\)

We show some examples. First we treat \((2, 3)\)-torus knot again.

**Example 5.2.** Put \(p = 1, q = 3\). In this case \(a = b = 1\). Then we see

\[
C_{2,3,1,1} = \left(1 - \cos \frac{\pi}{10}\right)\left(1 - \cos \frac{\pi}{3}\right) = \frac{1}{2}
\]
By applying the theorem 4.3 and the proposition 5.1,

\[
\sigma_{(2,3,-1)}(t) = \frac{T_6 \left( \frac{\sqrt{t}}{\sqrt{2}} \right) - T_4 \left( \frac{\sqrt{t}}{\sqrt{2}} \right)}{2(1 - \left( \frac{\sqrt{t}}{\sqrt{2}} \right)^2)}
= -4t^2 + 6t - 1.
\]

\[
\sigma_{(2,3,0)}(t) = 1.
\]

\[
\sigma_{(2,3,1)}(t) = 8t^3 - 20t^2 + 12t - 1.
\]

We show one more example.

**Example 5.3.** Here put \((2p,q) = (2, 5)\). In this case \((a,b) = (1, 1)\) or \((1, 3)\) and the constants \(C_{(2,5,1,1)}, C_{(2,5,1,3)}\) are given as follows:

\[
C_{(2,5,1,1)} = (1 - \cos \frac{\pi}{2})(1 - \cos \frac{\pi}{5})
= 1 - \cos \frac{\pi}{5}
= \frac{1}{4}(3 - \sqrt{5}).
\]

\[
C_{(2,5,1,3)} = (1 - \cos \frac{\pi}{2})(1 - \cos \frac{3\pi}{5})
= 1 - \cos \frac{3\pi}{5}
= \frac{1}{4}(3 + \sqrt{5}).
\]

First we put \(n = -1\). By Theorem 4.3,

\[
\sigma_{(2,5,-1)}(t) = Y_{(-1,1,1)}(t)Y_{(-1,1,3)}(t)
= X_{-1} \left( \frac{\sqrt{t}}{\sqrt{2}C_{2,5,1,1}} \right) X_{-1} \left( \frac{\sqrt{t}}{\sqrt{2}C_{2,5,1,3}} \right)
= 4C_{(2,5,1,1)}C_{(2,5,1,3)} \frac{T_{10} \left( \frac{\sqrt{t}}{2\sqrt{2}C_{2,5,1,1}} \right) - T_8 \left( \frac{\sqrt{t}}{2\sqrt{2}C_{2,5,1,1}} \right) + T_{10} \left( \frac{\sqrt{t}}{2\sqrt{2}C_{2,5,1,3}} \right) - T_8 \left( \frac{\sqrt{t}}{2\sqrt{2}C_{2,5,1,3}} \right)}{t - 2C_{(2,5,1,1)}}
= 64t^{10} + 384t^9 - 2880t^8 + 5952t^7 + 2336t^6
- 14856t^5 + 12192t^4 - 4608t^3 + 820t^2 - 60t + 1.
\]

By the definition,

\[
\sigma_{(2p,q,0)}(t) = 1.
\]

By applying the 3-term realtion

\[
Y_{(1,a,b)}(t) = 2T_{10} \left( \frac{\sqrt{t}}{2C_{(2,5,a,b)}} \right) Y_{(0,2p,q)}(t) - Y_{(-1,a,b)}(t),
\]
we obtain
\[
\sigma_{(2,5,1)}(t) = 256t^{12} + 384t^{11} - 16064t^{10} + 61056t^9 - 72000t^8
\]
\[- 57888t^7 + 197424t^6 - 172824t^5 + 273408t^4
\]
\[- 16632t^3 + 1880t^2 - 90t + 1.
\]

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