Dispersion and attenuation of bending waves propagating in a beam in the material there of damages accumulate during the operation

D M Brikkel1,*, V I Erofeev1,2 and A V Leonteva1,2
1National Research Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia
2Mechanical Engineering Research Institute of RAS, Nizhny Novgorod, Russia

* archive.94@mail.ru

Abstract. An approach is proposed that determines the new dependences of parameters of a linear bending wave on the degree of material damage.

A beam performing bending vibrations is studied herein. The technical theory of Euler-Bernoulli [1] supposes that:
1) The cross sections of a beam, which are flat and perpendicular to its axis, remain to be flat and perpendicular to the deformed axis thereof during bending;
2) The longitudinal sections of a beam resist the bending independently without affecting each other (i.e., the normal stresses at sites parallel to the axis are negligible small);
3) The beam rotation inertia during bending may be ignored.

It follows from the second assumption that from the stress and strain tensor components only $\sigma_{11}$ and $\varepsilon_{11}$ shall be recognized as significant, i.e. fibers either stretch or shrink and it results in the potential energy accumulation.

We suppose that the beam was subjected to static or cyclic tests and damage might accumulate in its material. To describe the damage measure, we introduce the Kachanov – Rabotnov function $\Psi(x, t)$ with its values to be equal to zero when there are no damages and to be close to unity when the material is destroyed [2,3].

We denote the displacement of particles in the mid-line of the beam during bending by $W(x, t)$. The dynamics of the beam including damage to its material is described by the following system of equations:

$$
\begin{align*}
\frac{\partial^2 W}{\partial t^2} + e^i \cdot r^j \cdot \frac{\partial^4 W}{\partial x^4} &= \beta_i \cdot \frac{\partial \Psi}{\partial x}, \\
\frac{\partial \Psi}{\partial t} + \alpha \Psi &= \beta_i \cdot E \cdot \frac{\partial W}{\partial x}.
\end{align*}
$$

(1)
The following notations are introduced herein for $c_s = \sqrt{\frac{E}{\rho}}, r_s = \sqrt{\frac{J_f}{F}}$, where $E$ is the Young's modulus, $\rho$ is the material density, $J_f$ is the axial moment of inertia, $F$ is the beam cross-sectional area, $\alpha, \beta_1, \beta_2$ are constants characterizing the material damage (among them $\alpha = \frac{1}{T}$, where $T$ is the relaxation time [4], the physical meaning of other coefficients is not so obvious; $\beta_1, \beta_2 < 0$).

System (1) is reduced to one equation with respect to the transverse displacement $W(x,t)$, which has the following form:

$$\frac{\partial^2 W}{\partial t^2} - \frac{\beta_1 \beta_2 E}{\alpha} \cdot \frac{\partial^3 W}{\partial x^3} + c_s^2 r_s^2 \cdot \frac{\partial^4 W}{\partial x^4} + \frac{1}{\alpha} \cdot \frac{\partial^3 W}{\partial t^3} + \frac{c_s^2 r_s^2}{\alpha} \cdot \frac{\partial^4 W}{\partial x^4 \partial t} = 0 \tag{2}$$

It is easily seen that the availability of damage to the material gives rise to the appearance of summands in equation (2), which are equivalent to those resulting from the compression of the beam by the longitudinal force (the second summand) and the internal friction available in the beam material (the third and fourth summands).

The introduction of dimensionless quantities for the transverse displacement, coordinates and time shall reduce equation (2) to the form:

$$\frac{\partial^2 U}{\partial \tau^2} - a \cdot \frac{\partial^2 U}{\partial z^2} + \frac{\partial^4 U}{\partial \tau^3} + \frac{\partial^4 U}{\partial z^4 \partial \tau} = 0 \tag{4}$$

where $a = \frac{-\beta_1 \beta_2 E}{\alpha^2 c_s^2 r_s^2} > 0$.

We study further the effect of the damage to material on the parameters of a bending wave propagating in a beam.

Seeking the solution of equation (4) in the form of a traveling harmonic wave $U = A \cdot e^{i(\omega t - k z)}$ (where $A$ is the complex amplitude, $\omega$ is the circular frequency, $k$ is the wave number), we come to the dispersion equation:

$$\omega^2 + ak^4 - k^2 + i\omega^3 - i\omega k^4 = 0 \tag{5}$$

it follows therefrom that the wave number should be complex:

$$k = k_1 + ik_2 \tag{6}$$

By substituting (6) into the complex dispersion equation (5) and distinguishing its real and imaginary parts we arrive at the following system of algebraic equations:

$$\begin{cases}
-\omega^2 - ak_1^4 + ak_2^4 + k_1^4 - 6k_1^2 k_2^2 + k_2^4 - 4\omega k_1 k_2 + 4\omega k_1 k_2 = 0; \\
2ak_1 k_2 + 4k_1^3 k_2 - 4k_1 k_2^3 - \omega^3 - 6\omega k_1^2 k_2^2 + \omega k_2^4 = 0.
\end{cases} \tag{7}$$

As the analytical study of the system (7) is quite difficult, to analyze it we apply the numerical simulation and obtain as a result the frequency dependences of the real $k_1(\omega)$ and imaginary $k_2(\omega)$ of the wave number (Figure 1).
If in the classical Euler-Bernoulli beam for bending waves there is one dispersion branch \( k_1 = \sqrt{\omega} \), \( k_2 = 0 \) for any value of \( \omega \) (dotted line in Figure 1), then for waves in the beam described by the equation (4) in the entire frequency range there are two dispersion branches for \( k_1 \) characterizing the wave propagation and two dispersion branches for \( k_2 \) characterizing its frequency-dependent attenuation.

![Figure 1](image)

**Figure 1.** Dependences of the real \( k_1(\omega) \) (green), imaginary \( k_2(\omega) \) (blue) parts of the wave number and the attenuation coefficient \( \gamma(\omega) \) (red) on the frequency.

An important factor characterizing the bending wave propagation process in this case is the frequency dependence of the parameter \( \gamma = \frac{k_2}{k_1} \) - the ratio of the coefficient characterizing the wave attenuation \( k_2 \) to its propagation constant \( (k_1) \).

The implicit equation for the attenuation coefficient dependence \( \gamma = \frac{k_2}{k_1} \) on the frequency shall have the form:

\[
a^2 \omega^2 \gamma^b - 4m \gamma^b - 2(a^2 \omega^2 + 8a^2 - 4m)\gamma^4 - 4m \gamma^2 + a^2 \omega^2 = 0
\]  

(8)

where the notation \( m = a^2 + 4\omega^2(1 + \omega^2)^2 \) is introduced, to abbreviate the record. Dependence \( \gamma(\omega) \) is shown in Figure 1 (red line).

It is seen from the Figure that for the lower \( k_1 \) dispersion branch \( \gamma > 1 \) within the entire frequency range, therefore, the attenuation here prevails and there are no traveling bending waves. For the upper \( k_1 \) dispersion branch \( \gamma < 1 \) within the entire frequency range \( \gamma < 1 \), therefore, there are here traveling damped bending waves.

The curve dynamics for different parameter values is shown in Figure 2, where asymptote \( \gamma = 1 \) is marked by a dotted line with spaces. The coordinates for the maximum point of the lower branch of the curve depend on the parameter and are determined from the solution of the system of equations. The derivative \( \gamma'(\omega) \) becomes zero on a curve.
\[ a^2 \gamma^4 + 2(a^2 - 8(3\omega^2 + 1)(\omega^2 + 1))\gamma^2 + a^2 = 0 \] \hspace{1cm} (9)

marked in blue in Figure 2. The simultaneous solution of equations (8) and (9) enables to find the coordinates of the maximum point \((\omega_{\text{max}}, \gamma_{\text{max}})\) for a fixed value of \(a\). As the parameter \(a\) increases, the curve approaches the asymptote and the maximum point shifts in the direction of increasing frequency and the attenuation coefficient (Figure 3).

Figure 2. Dependence of the attenuation coefficient \(\gamma(\omega)\) (red) on the frequency at different values of the parameter \(a\).

Figure 3. Dependencies \(\gamma_{\text{max}}(a)\) (upper red branch), \(\gamma_{\text{min}}(a)\) (lower red branch), \(\omega_{\text{max}}(a) = \omega_{\text{min}}(a)\) (blue).
Figure 4 shows the dependences of the phase \( v_{ph} = \frac{\omega}{k_1} \) and group \( v_{gr} = \frac{\partial \omega}{\partial k_1} \) velocities of a bending wave on the propagation constant \( k_1 \). The dotted line in the Figure shows the linear growth of the phase (bottom line) and group (top line) velocities of a bending wave propagating in the classical Euler-Bernoulli beam depending on \( k_1 \).

Figure 5 shows the dependence of the phase velocity on frequency \( v_{ph}(\omega) \) at a fixed value of parameter \( a \).

Figure 4. Dependences of phase \( v_{ph}(k_1) \) (green) and group \( v_{gr}(k_1) \) (blue) velocities on the wave number at a fixed value of the parameter \( a \).

Figure 5. The dependence of the phase velocity on frequency \( v_{ph}(\omega) \) at a fixed value of parameter \( a \).
The suggested approach has enabled us to formulate the frequency dependences of the wave number $k$ as well as the dependences of the group and phase velocities on the wave number $k$ of the beam in the material thereof damages accumulate.

To determine experimentally the relevant constants characterizing the damage to the material, the calculated dependencies may be used for diagnosing by acoustic methods long-term operated structures.

This paper complements the series of articles [4–9] handling problems of wave dynamics of damaged materials and structural elements which have been published by the authors in recent years.

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