Research Article

Some Qualitative Properties of Traveling Wave Fronts of Nonlocal Diffusive Competition-Cooperation Systems of Three Species with Delays

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This paper is concerned with nonlocal diffusion systems of three species with delays. By modified version of Ikehara’s theorem, we prove that the traveling wave fronts of such system decay exponentially at negative infinity, and one component of such solutions also decays exponentially at positive infinity. In order to obtain more information of the asymptotic behavior of such solutions at positive infinity, for the special kernels, we discuss the asymptotic behavior of such solutions of such system without delays, via the stable manifold theorem. In addition, by using the sliding method, the strict monotonicity and uniqueness of traveling wave fronts are also obtained.

1. Introduction

In this paper, we are concerned with the following nonlocal diffusive competition-cooperation systems of three species with delays:

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1(J_1 \ast u_1)(x,t) - d_1u_1(x,t) + r_1u_1(x,t)[1 - u_1(x,t - \tau_{11}) - a_{12}u_2(x,t - \tau_{12}) - a_{13}u_3(x,t - \tau_{13})], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2(J_2 \ast u_2)(x,t) - d_2u_2(x,t) + r_2u_2(x,t)[1 + a_{21}u_1(x,t - \tau_{21}) - u_2(x,t - \tau_{22}) - a_{23}u_3(x,t - \tau_{23})], \\
\frac{\partial u_3(x,t)}{\partial t} &= d_3(J_3 \ast u_3)(x,t) - d_3u_3(x,t) + r_3u_3(x,t)[1 - a_{31}u_1(x,t - \tau_{31}) - a_{32}u_2(x,t - \tau_{32}) - u_3(x,t - \tau_{33})],
\end{align*}
\]

(1)

where \( (J_i \ast u_i)(x,t) = \int_{\mathbb{R}} J_i(x-y)u_i(y,t)dy \), \( i = 1, 2, 3 \), \( J_i \) denote the diffusive kernel functions of three species, respectively, and \( u_i(x,t) \) represent the density of three species. We assume that \( d_i, r_i > 0, 0 < a_{ij} < 1, i \neq j, i, j = 1, 2, 3 \) and \( \tau_{ij} \geq 0, i, j = 1, 2, 3 \).
By directly computing, (1) has eight equilibria as follows, which are also the equilibria of the corresponding ordinary differential system of (1):

\[ e_0 = (0, 0, 0), \]
\[ e_1 = (1, 0, 0), \]
\[ e_2 = (0, 1, 0), \]
\[ e_3 = (0, 0, 1), \]
\[ e_4 = \left( \frac{1 + a_{12}}{1 - a_{13}a_{21}}, \frac{1 + a_{23}}{1 - a_{13}a_{21}}, 0 \right) = (k_1, k_2, 0), \]
\[ e_5 = \left( \frac{1 - a_{13}}{1 - a_{13}a_{31}}, 0, \frac{1 - a_{31}}{1 - a_{13}a_{31}} \right), \]
\[ e_6 = \left( 0, \frac{1 - a_{23}}{1 - a_{23}a_{32}}, \frac{1 - a_{32}}{1 - a_{23}a_{32}} \right), \]
\[ e_7 = \left( \frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D} \right), \]

where \( D = \begin{vmatrix} 1 & -a_{12} & a_{13} \\ -a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{vmatrix}, \)

\[ D_1 = \begin{vmatrix} 1 & -a_{12} & a_{13} \\ 1 & 1 & a_{23} \\ 1 & a_{32} & 1 \end{vmatrix}, \]

\[ D_2 = \begin{vmatrix} 1 & 1 & a_{13} \\ -a_{23} & 1 & a_{23} \\ a_{31} & 1 & 1 \end{vmatrix}, \]

\[ D_3 = \begin{vmatrix} 1 & -a_{12} & 1 \\ -a_{21} & 1 & 1 \\ a_{31} & a_{32} & 1 \end{vmatrix}. \]

Moreover, if \( \phi_0(\xi), i = 1, 2, 3, \) are monotone in \( \xi \in \mathbb{R}, \) then we call it the traveling wave front.

If the diffusive kernel functions \( J_i(x) = \delta(x) - \delta''(x), \) \( i = 1, 2, 3, \) where \( \delta \) is the Dirac delta function and \( \delta'' \) is the second derivative of the Dirac delta function, then we call it the traveling wave front.

On the other hand, for (1) without diffusion and delays, by direct calculation, \( e_3 \) is unstable, and \( e_4 \) is stable, if

\[ 1 - a_{31}k_1 - a_{32}k_2 < 0. \]

Furthermore, if (5) and \( D > 0, \) then the third component of \( e_7 \) is negative, which implies that \( e_7 \) is not in the first octant.

The existence and other properties of traveling wave solutions are important research fields in diffusive equations including reaction diffusion equations and nonlocal diffusion equation. For (1), the vector value function

\[ (u_1(x, t), u_2(x, t), u_3(x, t)) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) = \phi(\xi), \quad \xi = x + ct, \]

is called a traveling wave solution connecting \( e_3 \) and \( e_4, \) if it satisfies

\[
\begin{align*}
    d_1(J_1 * \varphi_1)(\xi) - d_1\varphi_1(\xi) - c\varphi_1'(\xi) + r_1\varphi_1(\xi)[1 - \varphi_1(\xi - cr_{11}) + a_{12}\varphi_2(\xi - cr_{12}) - a_{13}\varphi_3(\xi - cr_{13})] &= 0, \\
    d_2(J_2 * \varphi_2)(\xi) - d_2\varphi_2(\xi) - c\varphi_2'(\xi) + r_2\varphi_2(\xi)[1 + a_{21}\varphi_1(\xi - cr_{21}) - \varphi_2(\xi - cr_{22}) - a_{23}\varphi_3(\xi - cr_{23})] &= 0, \\
    d_3(J_3 * \varphi_3)(\xi) - d_3\varphi_3(\xi) - c\varphi_3'(\xi) + r_3\varphi_3(\xi)[1 - a_{31}\varphi_1(\xi - cr_{31}) - a_{32}\varphi_2(\xi - cr_{32}) - \varphi_3(\xi - cr_{33})] &= 0,
\end{align*}
\]

\[ \lim_{\xi \to -\infty} \phi(\xi) = e_3, \]

\[ \lim_{\xi \to +\infty} \phi(\xi) = e_4. \]
et al. invested the nonlinear stability of such traveling wave fronts of (1) via the weighted energy method and comparison principle. In addition, Liu and Weng obtained the relevant conclusions of the asymptotic speeds of (1) by using the method of super-sub solutions and comparison principle in [12]. Very recently, in [21], Tian and Zhao proved the existence and the global stability of bistable traveling wave fronts of (1) with finite or infinite delays by using the theory of monotone semiflows, the squeezing technique, and the dynamical systems approach. When delays are taken into consideration and $D, D_1, D_2 > 0$ holds, in [10], Huang and Weng proved the existence of traveling wave fronts of (1) connecting $e_3$ with $e_r$ if (4) exists and connecting $e_3$ with $e_4$ if (5) exists.

Besides the aforementioned papers, for the research on the existence of traveling wave solutions, we can refer to [2–4, 7, 8, 13, 15, 17, 19–21, 23, 24]. Specially, Li et al. in [13] introduced the nonlocal diffusion and time delays into the classical Lotka–Volterra reaction diffusion system and proved the existence, asymptotic behavior, and uniqueness of the traveling wave front of this system connecting the equilibria on the two axes. From the above statements, as we know, there is no conclusion about the existence, asymptotic behavior, and other properties of the traveling wave solutions of (1), thus, inspired by [13], we mainly consider the asymptotic behavior, strict monotonicity, and uniqueness of the traveling wave fronts of (1), connecting $e_2$ and $e_4$. Firstly, we make a variable substitution $\tilde{u}_1 = u_1, \tilde{u}_2 = u_2, \tilde{u}_3 = 1 - u_3$. Thus, (1) is changed into the following cooperative system, by dropping the tildes for simplicity:

$$\frac{\partial u_i(x,t)}{\partial t} = d_i \left( f_i(u_1)(x,t) - d_1 u_i(x,t) + r_i u_i(x,t) \left[ 1 - a_{13} - u_1(x,t-\tau_11) + a_{12} u_2(x,t-\tau_12) + a_{13} u_3(x,t-\tau_13) \right] \right),$$

$$\frac{\partial u_1(x,t)}{\partial t} = d_1 \left( f_1 u_1(x,t) - d_1 u_1(x,t) + r_1 u_1(x,t) \left[ 1 - a_{23} + a_{21} u_1(x,t-\tau_21) - u_2(x,t-\tau_22) + a_{23} u_3(x,t-\tau_23) \right] \right),$$

$$\frac{\partial u_3(x,t)}{\partial t} = d_3 \left( f_3 u_3(x,t) - d_3 u_3(x,t) + r_3 \left[ 1 - t u_3 n(x,t) \right] \left[ a_{31} u_1(x,t-\tau_31) + a_{32} u_2(x,t-\tau_32) - u_3(x,t-\tau_33) \right] \right).$$

Correspondingly, system (9) also has eight corresponding equilibria as follows:

$$E_0 = (0, 0, 1),$$

$$E_1 = (1, 0, 1),$$

$$E_2 = (0, 1, 1),$$

$$E_3 = (0, 0, 0),$$

$$E_4 = \left( \frac{1 + a_{12}}{1 - a_{13} a_{21}}, \frac{1 + a_{21}}{1 - a_{12} a_{21}}, 1 \right),$$

$$E_5 = \left( \frac{1 - a_{13}}{1 - a_{13} a_{31}}, 0, \frac{a_{31} (1 - a_{13})}{1 - a_{13} a_{31}} \right),$$

$$E_6 = \left( 0, \frac{1 - a_{23}}{1 - a_{23} a_{32}}, \frac{a_{32} (1 - a_{23})}{1 - a_{23} a_{32}} \right),$$

$$E_7 = \left( \frac{D_1}{D}, \frac{D_2}{D}, 1 - \frac{D_3}{D} \right).$$

From the above statements, the traveling wave fronts of (1) connecting $e_3$ with $e_4$ are equivalent to the traveling wave fronts of (9) connecting $E_3$ with $E_4$. And, the boundary conditions (7) and (8) are correspondingly changed into
\[
\begin{align*}
\begin{cases}
    d_1 (f_1 \ast \varphi_1) (\xi) - d_1 \varphi_1 (\xi) - c \varphi'_1 (\xi) + r_1 \varphi_1 (\xi) f_1 (\varphi_1, \varphi_2, \varphi_3) = 0, \\
    d_2 (f_2 \ast \varphi_2) (\xi) - d_2 \varphi_2 (\xi) - c \varphi'_2 (\xi) + r_2 \varphi_2 (\xi) f_2 (\varphi_1, \varphi_2, \varphi_3) = 0, \\
    d_3 (f_3 \ast \varphi_3) (\xi) - d_3 \varphi_3 (\xi) - c \varphi'_3 (\xi) + r_3 [1 - \varphi_3 (\xi)] f_3 (\varphi_1, \varphi_2, \varphi_3) = 0,
\end{cases}
\end{align*}
\]
(11)

\[
\begin{align*}
\lim_{\xi \rightarrow -\infty} \varphi (\xi) = E_3, \\
\lim_{\xi \rightarrow +\infty} \varphi (\xi) = E_4,
\end{align*}
\]
(12)

where

\[
\begin{align*}
\begin{cases}
    f_1 (\varphi_1, \varphi_2, \varphi_3) = 1 - a_{13} - \varphi_1 (\xi - c r_{12}) + a_{12} \varphi_2 (\xi - c r_{12}) + a_{13} \varphi_3 (\xi - c r_{13}), \\
    f_2 (\varphi_1, \varphi_2, \varphi_3) = 1 - a_{23} + a_{21} \varphi_1 (\xi - c r_{22}) - \varphi_2 (\xi - c r_{22}) + a_{23} \varphi_3 (\xi - c r_{23}), \\
    f_3 (\varphi_1, \varphi_2, \varphi_3) = a_{31} \varphi_1 (\xi - c r_{31}) + a_{32} \varphi_2 (\xi - c r_{32}) - \varphi_3 (\xi - c r_{33}).
\end{cases}
\end{align*}
\]
(13)

In the sequel, the traveling wave front of (9) as we mention is the nondecreasing solution to (11) and (12). We give the basic assumptions of this paper to end this section. Firstly, we always assume that (5) holds. Secondly, the kernels \( J_i, i = 1, 2, 3 \), satisfy

\[
\begin{align*}
\text{(P1)} & \quad J_i \in C(\mathbb{R}), J_i (x) = J_i (-x) \geq 0, \quad \int_{\mathbb{R}} J_i (x) dx = 1, \\
\text{(P2)} & \quad \text{For every } \lambda > 0, \quad \int_{\mathbb{R}} |x|^{1/2} J_i (x) e^{-\lambda x} dx < +\infty, \quad j = 0, 1, 2.
\end{align*}
\]

In Section 2, we introduce some notations and the main results. In Section 3, we discuss the asymptotic behavior of the nondecreasing solutions to (11) and (12). In Section 4, we give the strict monotonicity and uniqueness of the nondecreasing solutions to (11) and (12).

\section{2. Preliminaries and Main Results}

In this section, we will discuss the eigenvalue problems and introduce the main results in this paper. First of all, let

\[
\begin{align*}
\Delta_1 (\lambda, c) &= d_1 \int_{\mathbb{R}} J_1 (y) e^{-cy} dy - d_1 - c \lambda + r_1 (1 - a_{13}), \\
\Delta_2 (\lambda, c) &= d_2 \int_{\mathbb{R}} J_2 (y) e^{-cy} dy - d_2 - c \lambda + r_2 (1 - a_{23}), \\
\Delta_3 (\lambda, c, \tau_{33}) &= d_3 \int_{\mathbb{R}} J_3 (y) e^{-cy} dy - d_3 - c \lambda - r_3 e^{-\lambda \tau_{33}}.
\end{align*}
\]
(14)

By some simple computations, we have the following lemma.

\textbf{Lemma 1.} Assume that (P1) and (P2) hold. Then, there exist \( \lambda_i^*, c_i^* \), such that \( \Delta_i (\lambda_i^*, c_i^*) = 0 \) and \( \partial \Delta_i (\lambda, c_i^*) / \partial \lambda \big|_{\lambda = \lambda_i^*} = 0, \quad i = 1, 2 \). For \( c < c_1^* \), the equation \( \Delta_1 (\lambda, c) = 0 \) has two positive roots \( \lambda_1, \lambda_2 \) with \( 0 < \lambda_1 < \lambda_2 < \lambda_2 \), while for \( 0 < c < c_2^* \), \( \Delta_1 (\lambda, c) > 0 \) for \( \lambda \in \mathbb{R} \). Similarly, for \( c > c_2^* \), the equation \( \Delta_2 (\lambda, c) = 0 \) has two positive roots \( \lambda_3, \lambda_4 \) with \( 0 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_4 \), while for \( 0 < c < c_2^* \), \( \Delta_2 (\lambda, c) > 0 \) for \( \lambda \in \mathbb{R} \).

Note that

\[
\begin{align*}
    f' (\lambda) &= -d_1 \int y J_1 (y) e^{-cy} dy - c, \\
    f'' (\lambda) &= d_3 \int y^2 J_3 (y) e^{-cy} dy > 0,
\end{align*}
\]
(16)

and \( f (0) = 0 \). Thus, \( f' (\lambda) \) is increasing in \([0, +\infty)\). Then, we remark that there are \( y_1 \in (-\infty, 0) \) and \( \delta > 0 \), such that \( y_1 + \delta < 0 \) and \( J_3 (y) > 0 \) on \([y_1 - \delta, y_1 + \delta]\); otherwise, \( J_3 (y) \equiv 0 \) by the symmetry of \( J_3 \), which contradicts (P1). Therefore, by (P2),

\[
\lim_{\lambda \rightarrow +\infty} f' (\lambda) \leq \lim_{\lambda \rightarrow +\infty} \left( -d_3 \int_{\delta}^{\infty} y J_3 (y) e^{-cy} dy - d_3 \int_{y_1 - \delta}^{y_1 + \delta} y J_3 (y) e^{-cy} dy - c \right) = +\infty,
\]
(17)

which implies \( f' (\lambda) = 0 \) has a unique zero. Hence, in \([0, +\infty)\), there is only \( \lambda_0 \) such that
Case 1. \( f(\lambda_0) = \inf_{\lambda \in (0, +\infty)} f(\lambda) < 0 \), and \( f'(\lambda) > 0 \) for \( \lambda \in (\lambda_0, +\infty) \). Thus, by further noting \( f(+\infty) = +\infty \), \( g'(\lambda) < 0 \) and \( g(+\infty) = 0 \), there is only \( \lambda_3(\tau_{33}) > \lambda_0 > 0 \) such that \( f(\lambda) = g(\lambda) \).

If \( \tau_{33} = 0 \), then let \( \lambda_5(0) \) be the corresponding root, which in fact satisfies \( f(\lambda) = r_3 \). Suppose \( \lambda_5(0) \neq \lambda_3(\tau_{33}) \), for \( \tau_{33} \in (0, \tau_0) \). Since \( \lambda_5(\tau_{33}) \), \( \lambda_3(\tau_{33}) \in (\lambda_0, +\infty) \), then \( f(\lambda_5(0)) < f(\lambda_5(\tau_{33})) \). Hence, we have

\[
r_3 = f(\lambda_5(0)) < f(\lambda_5(\tau_{33})) = r_3 e^{-c_{31} \lambda_5(\tau_0)},
\]

which is a contradiction. Thus, \( \lambda_5(\tau_{33}) \leq \lambda_2(0) \). The rest of the proof is similar to the proof of Lemma 2.8 in [13], so we omit it here.

Let \( c^* = \max\{c_1^*, c_2^*\} \), \( A_1 \in \{\lambda_1, \lambda_2\} \) and \( A_2 \in \{\lambda_3, \lambda_4\} \). Now, we give the following the exponential decay rates of \( \varphi(\xi) \) at the minus infinity.

**Theorem 1.** Assume that (P1), (P2), and (5) hold. For \( c > c^* \), assume that \( \varphi(\xi) \) is any nondecreasing solution of (11) and (12) for either \( \tau_i = 0 \) or sufficiently small positive \( \tau_i \), \( i = 1, 2, 3 \). Then, (i) There exist \( \theta_i = \theta_i(\varphi_1, \varphi_2, \varphi_3) \), \( (1 \leq i \leq 8, i \in \mathbb{Z}) \) such that

\[
\lim_{\xi \to -\infty} \frac{\varphi_1(\xi + \theta_1)}{e^{\lambda_1 \xi}} = 1, \quad \lim_{\xi \to -\infty} \frac{\varphi_1(\xi + \theta_2)}{e^{\lambda_2 \xi}} = 1, \quad \text{when} \ c > c^*,
\]

\[
\lim_{\xi \to -\infty} \frac{\varphi_1(\xi + \theta_3)}{e^{\lambda_3 \xi}} = 1, \quad \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_4)}{e^{\lambda_4 \xi}} = 1, \quad \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_5)}{e^{\lambda_5 \xi}} = 1, \quad \text{when} \ c = c^* = c_1^* > c_2^*,
\]

\[
\lim_{\xi \to -\infty} \frac{\varphi_1(\xi + \theta_6)}{e^{\lambda_6 \xi}} = 1, \quad \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_7)}{e^{\lambda_7 \xi}} = 1, \quad \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_8)}{e^{\lambda_8 \xi}} = 1, \quad \text{when} \ c = c^* = c_1^* > c_2^*.
\]

Case 1. When \( \lambda_5(\tau_{33}) \leq \min\{A_1, A_2\} \) and \( c > c^* \),

Case 2. When \( \lambda_5(\tau_{33}) > \max\{A_1, A_2\} \).

If \( c > c^* \), then

\[
\lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_10)}{e^{\lambda_1 \xi}} = 1, \quad \text{when} \ \lambda_4 > \lambda_3 > \lambda_2 > \lambda_1,
\]

\[
\lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_11)}{e^{\lambda_1 \xi}} = 1, \quad \text{when} \ \lambda_2 > \lambda_1 \geq \lambda_4 > \lambda_3,
\]

\[
\lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_12)}{e^{\lambda_1 \xi}} = 1, \quad \text{when} \ \lambda_1 = \lambda_3,
\]

\[
\lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_13)}{e^{\lambda_1 \xi}} = 1, \quad \text{when} \ \lambda_1 \neq \lambda_3, \ \lambda_2 = \lambda_4,
\]

where \( \lambda_\pm = \lambda_1 = \lambda_3, \lambda_\pm = \min\{\lambda_1, \lambda_3\} \).

If \( c = c^* = c_1^* > c_2^* \), then
Assume that (P1), (P2), and (5) hold. \( \lambda_1 = \lambda_2 = \lambda_3 \).

\[
\begin{aligned}
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_3)}{e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_3 < \lambda_4 \leq \lambda_1 = \lambda_2 = \lambda_3 \leq \lambda_4, \\
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_5)}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_1 = \lambda_2 < \lambda_3 = \lambda_4, \\
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_6)}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_1 = \lambda_2 = \lambda_3 < \lambda_4,
\end{aligned}
\]

where \( \lambda_3 = \lambda_1 = \lambda_2 = \lambda_3 \).

If \( c^* = c_3^* > c_1^* \), then

\[
\begin{aligned}
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_7)}{e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_1 < \lambda_2 \leq \lambda_3 = \lambda_4 \text{ or } \lambda_1 < \lambda_3 = \lambda_4 \leq \lambda_2, \\
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_8)}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_3 = \lambda_4 < \lambda_1 < \lambda_2, \\
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_9)}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_1 = \lambda_2 = \lambda_3 < \lambda_4.
\end{aligned}
\]

where \( \lambda_3 = \lambda_1 = \lambda_3 = \lambda_4 \).

If \( c^* = c_1^* = c_2^* \), then

\[
\begin{aligned}
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_{20})}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4, \\
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_{21})}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_1 = \lambda_2 < \lambda_3 = \lambda_4, \\
&\lim_{{\xi \to -\infty}} \frac{\varphi_3(\xi + \theta_{22})}{|\xi|^\mu e^{\lambda_3 \xi}} = 1, \quad \text{when } \lambda_3 = \lambda_4 < \lambda_1 < \lambda_2,
\end{aligned}
\]

where \( \lambda_3 = \lambda_1 = \lambda_2 = \lambda_3 \).

\[\mu = \begin{cases} 0, & \text{when } Q_i(\lambda_i) = 0 \text{ or } \Lambda_i \text{ is a simple root of } \Delta_i(\lambda, c) = 0, \\ 1, & \text{when } Q_i(\lambda_i) \neq 0 \text{ and } \Lambda_i \text{ is a double root of } \Delta_i(\lambda, c) = 0, \end{cases} \quad i = 1, 2,\]

\[
\begin{aligned}
Q_1(\lambda) &= \int_{-\infty}^{\infty} \varphi_1(\xi) \varphi_3(\xi - c\tau_{11}) - a_{12} \varphi_2(\xi - c\tau_{12}) - a_{13} \varphi_3(\xi - c\tau_{13}) e^{-\lambda \xi} d\xi, \\
Q_2(\lambda) &= \int_{-\infty}^{\infty} \varphi_2(\xi) [-a_{21} \varphi_1(\xi - c\tau_{21}) + \varphi_2(\xi - c\tau_{22}) - a_{23} \varphi_3(\xi - c\tau_{23})] e^{-\lambda \xi} d\xi.
\end{aligned}
\]
\[
\frac{\partial^2 \Delta_4(\lambda, c)}{\partial \lambda^2} = d_3 \int \gamma^2 J_3(y) e^{-\lambda y} dy > 0, \quad \Delta_4(-\infty, c) = +\infty, \tag{29}
\]
and \(\Delta_4(0, c) = -r_j(a_{1j}k_1 + a_{2j}k_2 -1) < 0\) from (5), then obviously \(\Delta_4(\lambda, c) = 0\) has a unique negative root \(\lambda_6 < 0\).

Then, we introduce the following theorem. \(\square\)

**Theorem 2.** Assume that (P1), (P2), and (5) hold. For \(c \geq c^*\), assume that \(\varphi(\xi)\) is any nondecreasing solution of (11) and (12). Then, there exists \(\theta_{23} = \theta_{23}(\varphi_1, \varphi_2, \varphi_3)\) such that
\[
\lim_{\xi \to +\infty} \frac{\varphi_3'(\xi)}{1 - \varphi_3(\xi + \theta_{23})} = -\lambda_6. \tag{30}
\]

We cannot use the method offered in [13] to obtain the asymptotic behavior similar to Theorem 1. In order to obtain more detailed results of \(\varphi(\xi)\) at the plus infinity, we consider a special case, \(\tau_{ij} = 0, i, j = 1, 2, 3\), and \(J_i(x) = \delta(x) - \delta''(x), i = 1, 2, 3\). Moreover, from Lemma 8 in Section 3, we conclude that \(\lambda_6 > \gamma_9\) are the negative roots of
\[
(d_1\lambda^2 - c\lambda - r_1k_1)(d_2\lambda^2 - c\lambda - r_2k_2) - a_{12}a_{21}r_1r_2k_1k_2 = 0. \tag{31}
\]

Then, with the methods in [5, 18, 22] and the stable manifold theorem, we give the following asymptotic behavior of \(\varphi(\xi)\) at the plus infinity.

**Theorem 3.** Assume that (P1), (P2), and (5) hold. For \(c \geq c^*\), we assume that \(\varphi(\xi)\) is any nondecreasing solution of (11) and (12), \(\tau_{ij} = 0, i, j = 1, 2, 3\), and
\[
J_i(x) = \delta(x) - \delta''(x), \quad i = 1, 2, 3, \tag{32}
\]
where \(\delta\) is the Dirac delta function. In addition, let
\[
\lambda_7 = \frac{c - \sqrt{c^2 + 4d_1r_3(a_{13}k_1 + a_{23}k_2 - 1)}}{2d_3}. \tag{33}
\]

Then, when \(\xi \to +\infty\), \(\varphi(\xi)\) has the following asymptotic behavior:
(i) When \(\lambda_7 > \lambda_8 > \lambda_9\),
\[
\begin{align*}
\varphi_1(\xi) &= k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) &= k_2 - \alpha_1 s_2 e^{\lambda_7 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) &= 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{align*}
\tag{34}
\]
where \(\alpha_1 > 0\).
\[
s_1 = \frac{a_{13}r_1k_1(-d_1\lambda_7^2 + c\lambda_7 + r_1k_2) + a_{12}a_{23}r_1r_2k_1k_2}{(d_1\lambda_7^2 - c\lambda_7 - r_1k_1)(d_2\lambda_7^2 - c\lambda_7 - r_2k_2) - a_{12}a_{21}r_1r_2k_1k_2},
\]
\[
s_2 = \frac{a_{23}r_2k_2(-d_1\lambda_7^2 + c\lambda_7 + r_2k_1) + a_{12}a_{13}r_1r_3k_1k_2}{(d_1\lambda_7^2 - c\lambda_7 - r_1k_1)(d_2\lambda_7^2 - c\lambda_7 - r_2k_2) - a_{12}a_{21}r_1r_2k_1k_2}. \tag{35}
\]

(ii) When \(\lambda_8 > \lambda_9 > \lambda_7\),
\[
\begin{align*}
\varphi_1(\xi) &= k_1 - \alpha_2 s_1 e^{\lambda_8 \xi} - \alpha_3 e^{\lambda_7 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) &= k_2 - \alpha_2 s_2 e^{\lambda_8 \xi} - \alpha_3 e^{\lambda_7 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) &= 1 - \alpha_2 e^{\lambda_8 \xi} + \text{h.o.t.},
\end{align*}
\tag{36}
\]
where \(\alpha_1, \alpha_2 > 0\), \(\tau_1 = (-d_1\lambda_8^2 + c\lambda_8 + r_1k_1)/a_{12}r_1k_1 > 0\).
(iii) When \(\lambda_8 > \lambda_7 > \lambda_9\),
\[
\begin{align*}
\varphi_1(\xi) &= k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} - \alpha_2 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) &= k_2 - \alpha_1 s_2 e^{\lambda_7 \xi} - \alpha_2 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) &= 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{align*}
\tag{37}
\]
where \(\alpha_1, \alpha_2 > 0\).
Moreover, suppose
\[
\begin{align*}
\frac{c - d_2}{d_3^2} &+ \left( \frac{c - d_2}{d_3} \right)^2 + 4 \frac{r_2k_2}{d_3^2} \left( 1 + \frac{a_{12}a_{21}}{a_{13}} \right), \\
\# &- \frac{c - d_2}{d_3} + \left( \frac{c - d_2}{d_3} \right)^2 + 4 \frac{r_3}{d_3} (a_{31}k_1 + a_{32}k_2 - 1),
\end{align*}
\tag{38}
\]
(iv) When \(\lambda_7 = \lambda_8 > \lambda_9\),
\[
\begin{align*}
\varphi_1(\xi) &= k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} - \alpha_3 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) &= k_2 - \alpha_1 s_2 e^{\lambda_7 \xi} - \alpha_3 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) &= 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{align*}
\tag{39}
\]
where \(\alpha_1 > 0, \alpha_2 > 0\).
\[
\tau_1 = \frac{(a_{23}r_2k_2/a_{13}r_1k_1)(c - 2d_2\lambda_8) - (c - 2d_2\lambda_8)r_1}{-d_1\lambda_8^2 + c\lambda_8 + r_1k_2 + (a_{12}a_{23}/a_{13})r_2k_2}, \tag{40}
\]
\[
\bar{\mu}_1 = \frac{c - 2d_1\lambda_8 - a_{12}r_1k_1}{a_{13}r_1k_1} > 0.
\]
(v) When \(\lambda_8 > \lambda_7 = \lambda_9\),
\[
\begin{align*}
\varphi_1(\xi) &= k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} - \alpha_2 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) &= k_2 - \alpha_1 s_2 e^{\lambda_7 \xi} - \alpha_2 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) &= 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{align*}
\tag{41}
\]
where \(\alpha_1 > 0, \alpha_3 > 0 (\bar{\mu}_2 > 0), \alpha_4 < 0 (\bar{\mu}_2 < 0)\), and
\[
\tau_3 = \frac{(a_{23}r_2k_2/a_{13}r_1k_1)(c - 2d_2\lambda_9) - (c - 2d_2\lambda_9)r_2}{-d_1\lambda_9^2 + c\lambda_9 + r_2k_2 + (a_{12}a_{23}/a_{13})r_2k_2}, \tag{42}
\]
\[
\bar{\mu}_2 = \frac{c - 2d_1\lambda_9 - a_{12}r_1k_1}{a_{13}r_1k_1}.
\]
We give a remark at once to show (38) is reasonable.

Remark 1. Choose $d_2 = r_2 = r_3 = 1$, $d_3 = 2$, $a_{12} = a_{21} = a_{13} = a_{23} = a_{32} = 1/2$; then, $k_1 = k_2 = 2$. Thus, the left side of the equation (38) is equal to $-c + \sqrt{c^2 + 2}$, while the right side of the equation (38) is $(c/2) + \sqrt{(c/2)^2 + 2}$, which are not equal for $c > 0$.

Finally, we introduce conclusions on the strict monotonicity and the uniqueness.

Theorem 4. Assume that (P1), (P2), and (5) hold, and $\tau_{ii} = 0$, $i = 1, 2, 3$. For $c \geq c^\ast$, if $\varphi(\xi)$ is any nondecreasing solution of (11) and (12), then it is strictly monotone.

Theorem 5. Under the assumptions of Theorem 3, let $\varphi_1(\xi)$ and $\varphi_2(\xi)$ be two nondecreasing solutions of (11) and (12), with the same speed $c \geq c^\ast$ and the same decay rate at $\xi = \pm \infty$. Then, they are unique (up to a translation).

Remark 2. When $\tau_{ii} = 0$, $i, j = 1, 2, 3$, and $J_i(x) = \delta(x) - \delta_i(x)$, $i = 1, 2, 3$, the existence of nondecreasing solutions of (11) and (12) had been discussed in [9]. From Theorems 1–5, we further investigate the asymptotic behavior, strict monotonicity, and uniqueness of such solutions in this paper.

Remark 3. The proof of existence of traveling wave fronts is standard. For example, suppose $D > 0$, and (P1), (P2), (5) hold. Since system (11) is the quasimonotone system ($\tau_{ii} = 0$, $i = 1, 2, 3$) or weak quasimonotone system ($\tau_{ii} > 0$, $i = 1, 2, 3$), with the methods in the [11–13, 17, 19, 23] and references therein, the existence of traveling wave fronts to (9) can be proved.

3. Asymptotic Behavior of Traveling Wave Solutions

In this section and next section, we always assume $\tau_{ii} = 0$ or $\tau_{ii}, i = 1, 2, 3$, small enough, and assume (P1), (P2), and (5) hold; then, we investigate the asymptotic behavior, strict monotonicity, and uniqueness of nondecreasing solutions to (11) and (12).

To study the asymptotic behavior, we introduce the following result given in [24].

Proposition 1. Let $c > 0$ be a constant and $B(\cdot)$ be a continuous function having finite limits at infinity $B(\pm \infty) = \lim_{x \to \pm \infty} B(x)$. Let $z(\cdot)$ be a measurable function satisfying

$$
\int_{\mathbb{R}} J(y)e^{\int_{x}^{y}z(s)ds}dy - cz(x) + B(x) = 0, \quad x \in \mathbb{R}.\quad (43)
$$

Then, $z$ is uniformly continuous and bounded. In addition, $\mu = \lim_{x \to \pm \infty} z(x)$ exist and are real roots of the characteristic equation

$$
\int_{\mathbb{R}} J(y)e^{\int_{x}^{y}z(s)ds}dy - cz + B(\pm \infty) = 0.\quad (44)
$$

Firstly, we give some properties of the solutions $\varphi(\xi)$ to (11) and (12).

Lemma 4. Suppose (P1), (P2), and (5) hold, and for $c \geq c^\ast$, $\varphi(\xi)$ is any nondecreasing solution of (11)-(12). Then,

$$
\varphi_1(\xi), \varphi_2(\xi) > 0, \varphi_3(\xi) < 1, \quad \xi \in \mathbb{R}.\quad (45)
$$

Moreover, if $\tau_{ii} = 0$, $i = 1, 2, 3$, then

$$
\varphi_1(\xi) < k_1, \varphi_2(\xi) < k_2, \varphi_3(\xi) > 0, \quad \xi \in \mathbb{R}.\quad (46)
$$

Proof. From the assumptions, we remark that $\varphi_1$ is nondecreasing and $\lim_{x \to -\infty} \varphi_1(\xi) = k_1$. Thus, let $\xi_1 \in \mathbb{R}$ be the right-most point of $\varphi_1(\xi) = 0$; then, $\varphi_1(\xi)$ attains the minimum at $\xi_1$, implying $\varphi_1(\xi_1) = 0$. It follows from the first equation of (11) that $(f_1(\cdot) \ast \varphi_1)(\xi_1) = 0$. By (P1), there exist $y_2 > 0$, $\delta_2 > 0$ such that $J_1(y) \neq 0$ for $y \in [y_2 - \delta_2, y_2 + \delta_2]$. Since

$$
0 = (J_1 \ast \varphi_1)(\xi_1) = \int_{\mathbb{R}} J_1(\xi_1 - y)\varphi_1(y)dy
$$

$$
= \int_{\mathbb{R}} J_1(y)\varphi_1(\xi_1 - y)dy
$$

$$
= \int_{\mathbb{R}} J_1(y)\varphi_1(\xi_1 + y)dy,
$$

following from $J_1(-y) = J_1(y)$, $y \in \mathbb{R}$. Then, we obtain $\varphi_1(\xi_1 \pm y) = 0$ for $y \in [y_2 - \delta_2, y_2 + \delta_2]$, which contradicts the choice of $\xi_1$. Thus, $\varphi_1(\xi) > 0$ in $\xi \in \mathbb{R}$. Similarly, we can prove $\varphi_2(\xi) > 0, \varphi_3(\xi) < 1, \xi \in \mathbb{R}$.

When $\tau_{ii} = 0$, similarly, suppose there exists a point $\xi_2 \in \mathbb{R}$ such that $\varphi_1(\xi_2) = k_1$, implying $\varphi_1(\xi_2) = 0$. By recalling $\varphi_3(\xi) < 1$ and $\varphi_3(\xi) \leq k_2$, $\xi \in \mathbb{R}$, we deduce

$$
0 = d_i (J_1(\ast \varphi_1)(\xi_2) - d_i \varphi_1(\xi_2) + r_1 \varphi_1(\xi_2)
$$

$$
\cdot [1 - a_{13} - \varphi_1(\xi_2) + a_{12}\varphi_2(\xi_2 - c\tau_{12}) + a_{13}\varphi_3(\xi_2 - c\tau_{13})]
$$

$$
< d_i (J_1(\ast \varphi_1)(\xi_2) - d_i k_1 + r_1 k_1 [-a_{13} k_2 + a_{12} \varphi_2(\xi_2 - c\tau_{12})]
$$

$$
\leq d_i (J_1(\ast \varphi_1)(\xi_2) - d_i k_1.
$$

Thus, $(J_1 \ast \varphi_1(\xi_2)) > k_1$. On the other hand,

$$
(J_1 \ast \varphi_1)(\xi_2) = \int_{\mathbb{R}} J_1(\xi_2 - y)\varphi_1(y)dy
$$

$$
= \int_{\mathbb{R}} J_1(y)\varphi_1(\xi_2 - y)dy
$$

$$
\leq k_1 \int_{\mathbb{R}} J_1(y)dy = k_1,
$$

which is a contradiction. Thus, $\varphi_1(\xi) < k_1$. Similarly, when $\tau_{ii} = 0$, $\varphi_2(\xi) < k_2$. Complexity

$$
\int_{\mathbb{R}} J(y)e^{\int_{x}^{y}}z(s)ds\,dy - cz + B(\pm \infty) = 0.\quad (44)
$$
When $r_{33} = 0$, we also assume there exists a point $\xi_3 \in \mathbb{R}$ such that $\varphi_3(\xi_3) = 0$, and thus $\varphi_1(\xi_3) = 0$. Also by noting $\varphi_1(\xi), \varphi_2(\xi) > 0, \xi \in \mathbb{R}$, we have

$$0 = d_3 (J_\ast \varphi_3)(\xi_3) - d_3 \varphi_3(\xi_3) - r_3 \sigma_3(1 - \varphi_3(\xi_3))$$

$$+ [a_{31} \varphi_1(\xi_3 - c r_{31}) + a_{32} \varphi_2(\xi_3 - c r_{32}) - \varphi_3(\xi_3)]$$

$$= d_3 (J_\ast \varphi_3)(\xi_3) + r_3 [a_{31} \varphi_1(\xi_3 - c r_{31}) + a_{32} \varphi_2(\xi_3 - c r_{32})].$$

(50)

Therefore,

$$0 \leq d_3 (J_\ast \varphi_3)(\xi_3) = -r_3 [a_{31} \varphi_1(\xi_3 - c r_{31}) + a_{32} \varphi_2(\xi_3 - c r_{32})] < 0,$$

(51)

which is a contradiction, and thus $\varphi_3(\xi) > 0$.

By Lemmas 1 and 4 and Proposition 1, it is easy to verify the following theorem.

**Theorem 6.** Suppose (P1), (P2), (5) hold, and for $c \geq c^*$, $\varphi(\xi)$ is any nondecreasing solution of (11) and (12). Then,

$$\lim_{\xi \to -\infty} \varphi_1(\xi) = \Lambda_1 \in \{\Lambda_1, \Lambda_2\},$$

$$\lim_{\xi \to +\infty} \varphi_1(\xi) = \Lambda_2 \in \{\Lambda_3, \Lambda_4\},$$

(52)

where $\lambda_i, i = 1, 2, 3, 4$, are described as Lemma 1.

Define the bilateral Laplace transform

$$L(\lambda, \varphi) = \int_{-\infty}^{\infty} \varphi(\xi) e^{-\lambda \xi} d\xi,$$

(53)

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function. The following lemma is derived from (12) and Theorem 6.

**Lemma 5.** Suppose (P1) and (P2) hold, and for $c \geq c^*$, $\varphi(\xi)$ is any nondecreasing solution of (11) and (12). Then,

$$L(\lambda, \varphi_1) < +\infty, \quad \lambda \in (0, \Lambda_1),$$

$$L(\lambda, \varphi_1) = +\infty, \quad \lambda \in \mathbb{R} \setminus (0, \Lambda_1),$$

$$L(\lambda, \varphi_2) < +\infty, \quad \lambda \in (0, \Lambda_2),$$

$$L(\lambda, \varphi_2) = +\infty, \quad \lambda \in \mathbb{R} \setminus (0, \Lambda_2).$$

(54)

Then, we discuss the convergence of $L(\lambda, \varphi_3)$ via the following lemma.

**Lemma 6.** Suppose (P1), (P2), and (5) hold, and for $c \geq c^*$, suppose $\varphi(\xi)$ is any nondecreasing solution of (11) and (12). Then,

$$L(\lambda, \varphi_3) < +\infty, \quad \lambda \in (0, \gamma(r_{33})),$$

$$L(\lambda, \varphi_3) = +\infty, \quad \lambda \in \mathbb{R} \setminus (0, \gamma(r_{33})).$$

(55)

where $\gamma(r_{33}) = \min\{\Lambda_1, \Lambda_2, \Lambda_2, \gamma(r_{33})\}$.

**Proof.** We first search $\sigma$ such that $L(\lambda, \varphi_3) < -\infty, \lambda \in (0, \sigma)$. From Lemma 2, we can conclude that $d_3 \int R J_\ast (y + \chi \varphi_3) dy - d_3 - c \lambda = 0$ has two roots 0 and $\Lambda^* (0 < \Lambda^* < \Lambda)$ for $c > 0$, and

$$d_3 \int R J_\ast (y + \chi \varphi_3) dy - d_3 - c \lambda < 0,$$

(56)

for $\lambda \in (0, \Lambda^*)$. Since (P2) implies that

$$-\infty < \int_{-\infty}^{0} y J_\ast (y + \chi \varphi_3) dy < 0,$$

(57)

for $\lambda > 0$, then there exists a small $\Lambda > 0$ such that

$$d_3 \int_{-\infty}^{0} y J_\ast (y + \chi \varphi_3) dy - c + \gamma(r_{33}) > 0,$$

(58)

for $\lambda \in (0, \Lambda)$.

By Lemma 5, we take $z_0 < 0$ such that

$$\varphi_3(\xi) \leq 1/2$$

for $\xi \leq z_0 + c r_{33} + (1/\lambda)$.

Multiplying the third equation of (11) by $e^{-\lambda \xi}$ with $\lambda \in (0, \min\{\Lambda_1, \Lambda_2, \Lambda^*, (1/c r_{33}), \Lambda\})$ and integrating from $z \leq z_0$ to $+\infty$ gives

$$r_3 a_{31} \int_{z}^{\infty} [1 - \varphi_3(\xi)] \varphi_1(\xi - c r_{31}) e^{-\lambda \xi} d\xi$$

$$+ r_3 a_{32} \int_{z}^{\infty} [1 - \varphi_3(\xi)] \varphi_2(\xi - c r_{32}) e^{-\lambda \xi} d\xi$$

$$= -d_3 \int_{z}^{\infty} \int R J_\ast (\xi - y) \varphi_3(y) e^{-\lambda \xi} dy d\xi$$

$$+ d_3 \int_{z}^{\infty} \varphi_3(\xi) e^{-\lambda \xi} d\xi + c \int_{z}^{\infty} \varphi_3(\xi) e^{-\lambda \xi} d\xi$$

$$+ r_3 \int_{z}^{\infty} [1 - \varphi_3(\xi)] \varphi_3(\xi - c r_{33}) e^{-\lambda \xi} d\xi.$$

(59)

For the left side of (59), obviously,

$$r_3 a_{31} \int_{z}^{\infty} [1 - \varphi_3(\xi)] \varphi_1(\xi - c r_{31}) e^{-\lambda \xi} d\xi \leq r_3 a_{31} L(\lambda, \varphi_1),$$

(60)

$$r_3 a_{32} \int_{z}^{\infty} [1 - \varphi_3(\xi)] \varphi_2(\xi - c r_{32}) e^{-\lambda \xi} d\xi \leq r_3 a_{32} L(\lambda, \varphi_2).$$

(61)

By recalling the convergence in Lemma 5, from (56)–(61), similar to [13], we deduce

$$r_3 a_{31} L(\lambda, \varphi_1) + r_3 a_{32} L(\lambda, \varphi_2)$$

$$\geq \left( d_3 \int_{-\infty}^{0} y J_\ast (y + \chi \varphi_3) dy - c + r_{33} \right) \varphi_3(z) e^{-\lambda \xi},$$

(62)

which implies, for any $\lambda \in (0, \min\{\Lambda_1, \Lambda_2, \Lambda^*, (1/c r_{33}), \Lambda\}),$

$$0 < \sup_{z \in [0]} \varphi_3(z) e^{-\lambda \xi} < +\infty.$$

(63)
Furthermore, if \( \gamma \) is analytic in the strip \( 0 < \Re \lambda < \Lambda \), \( i = 1, 2 \), then

\[
\Delta_i(\lambda, c, \tau_{33})L(\lambda, \phi_2) = -r_3 \int_{-\infty}^{\infty} [1 - \phi_3(\xi)][a_{31}\phi_1(\xi - \tau_{33}) + a_{32}\phi_2(\xi - \tau_{33})]e^{-\lambda \xi}d\xi.
\]

(64)

We claim that \( \gamma(\tau_{33}) \leq \lambda_5(\tau_{33}) \). Otherwise, if \( \gamma(\tau_{33}) > \lambda_5(\tau_{33}) \), then \( L(\lambda, \phi_2) < +\infty \). Taking \( \lambda = \lambda_5(\tau_{33}) \) in (64), the left side of (64) equals 0 by \( \Delta_3(\lambda_5(\tau_{33}), c, \tau_{33}) = 0 \), while the right side of (64) is always negative by \( \phi_3(\xi) \leq 1 \), which is a contradiction. It also follows easily from (64) that \( \gamma(\tau_{33}) = \min(\lambda_1, \lambda_2) \) if \( \lambda_1, \lambda_2 > \lambda_5(\tau_{33}) \) or \( \gamma(\tau_{33}) = \lambda_5(\tau_{33}) \) if \( \lambda_1, \lambda_2 < \lambda_5(\tau_{33}) \). Furthermore, if \( \lambda_1, \lambda_2 > \lambda_5(\tau_{33}) \), it also follows that \( \lim_{\xi \to \infty} \Phi_3(\xi, \tau_{33})L(\lambda, \phi_2) = 1 \).

In order to study the asymptotic behavior, we introduce the following modified version of Ikehara’s theorem [1], which has been applied early in [6].

**Lemma 7** (Ikehara’s theorem). Let \( \phi \) be a positive nondecreasing function on \( \mathbb{R} \) and define \( F(\lambda) = \int_{-\infty}^{\infty} \phi(\xi)e^{-\lambda \xi}d\xi \). Assume that \( F(\lambda) \) can be written as \( F(\lambda) = H(\lambda)/(\lambda - \alpha)^{\nu+1} \), where \( \nu > -1, \alpha > 0, \) and \( H \) is analytic in the strip \( 0 < \Re \lambda \leq \alpha \); then,

\[
\lim_{\xi \to -\infty} \frac{\phi(\xi)}{[\xi]^{\nu+2}} = \frac{H(\alpha)}{\Gamma(\nu + 1)}.
\]

(65)

With the aid of the above lemmas, we will finish the proof of Theorem 1.

**Proof of Theorem 1.** The proof is motivated by [13–15]. From Lemmas 5 and 6, \( L(\lambda, \phi) \) is well defined for \( \lambda \in \mathbb{C} \) with \( \Re \lambda \in (0, \Lambda) \), \( i = 1, 2 \), and \( L(\lambda, \phi) \) is well defined for \( \lambda \in \mathbb{C} \) with \( \Re \lambda \in (0, \gamma(\tau_{33})) \), respectively. It follows from (11) that

\[
\Delta_i(\lambda, c) \int_{-\infty}^{\infty} \phi_1(\xi)e^{-\lambda \xi}d\xi = r_iQ_i(\lambda),
\]

(66)

for \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda < \Lambda_i \), \( i = 1, 2 \), and

\[
\Delta_3(\lambda, c, \tau_{33}) \int_{-\infty}^{\infty} \phi_3(\xi)e^{-\lambda \xi}d\xi = -r_3a_{31}e^{-\lambda \tau_{33}}r_iQ_i(\lambda)\Delta_i(\lambda, c) + r_3Q_3(\lambda),
\]

(67)

for \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda < \gamma(\tau_{33}) \), where \( Q_1, Q_2 \) are defined in Theorem 1 and

\[
Q_3(\lambda) = \int_{-\infty}^{\infty} \phi_3(\xi)[a_{31}\phi_1(\xi - \tau_{33}) + a_{32}\phi_2(\xi - \tau_{33})]e^{-\lambda \xi}d\xi.
\]

(68)

By Lemmas 2, 5, and 6 and directly calculating, we conclude the following facts.

1. \( \lambda = \Lambda \) is a unique root with \( \Re \Lambda = \Lambda_i \) of \( \Delta_i(\lambda, c) = 0 \), \( i = 1, 2 \), and \( \lambda = \lambda_5(\tau_{33}) \) is a unique root with \( \Re \lambda = \lambda_5(\tau_{33}) \) of \( \Delta_3(\lambda, c, \tau_{33}) = 0 \).

2. \( Q_i(\lambda) \) is analytic in the strip \( 0 < \Re \lambda < \Lambda_i + \gamma(\tau_{33}) \), \( i = 1, 2 \), and \( Q_3(\lambda) \) is analytic in the strip \( 0 < \Re \lambda < 2\gamma(\tau_{33}) \).

From (66), we define

\[
F_i(\lambda) = \int_{-\infty}^{0} \phi_1(\xi)e^{-\lambda \xi}d\xi - \int_{0}^{\infty} \phi_1(\xi)e^{-\lambda \xi}d\xi, \quad i = 1, 2,
\]

(69)

\[
H_i(\lambda) = \frac{r_iQ_i(\lambda)}{\Delta_i(\lambda, c)/(\Lambda_i - \lambda)^{\nu+1}} - (\Lambda_i - \lambda)^{\nu+1} \int_{0}^{\infty} \phi_1(\xi)e^{-\lambda \xi}d\xi, \quad i = 1, 2,
\]

(70)

where \( \nu = 0 \) when \( c > c^* \) and \( \nu = \mu \) when \( c = c^* \).

Then, from (1) and (2), we deduce that \( H_i(\lambda) \) is analytic in the strip \( 0 < \Re \lambda < \Lambda_i \), \( i = 1, 2 \). Thus, from Lemma 7,

\[
\lim_{\xi \to -\infty} \frac{\phi_1(\xi)}{[\xi]^\nu e^{\lambda \xi}} = \frac{H_i(\Lambda_i)}{\Gamma(\nu + 1)}, \quad i = 1, 2.
\]

(72)

which implies (i) in Theorem 1 holds if \( H_i(\Lambda_i) \neq 0 \). In the following, we will prove that \( H_i(\Lambda_i) \neq 0 \), \( i = 1, 2 \), in two cases.

For \( c > c^* \), \( i = 1, 2 \); since \( \nu = 0 \) and \( \Lambda_i \) is a simple root of \( \Delta_i(\lambda, c) = 0 \), then we have \( \Delta_i(\lambda, c)/(\Lambda_i - \lambda) \neq 0 \) at \( \lambda = \Lambda_i \). For \( i = 1, 2 \), if we suppose \( Q_i(\Lambda_i) = 0 \), then from (69), \( L(\Lambda_i, \phi) \)
exist, which contradicts Lemma 5. Thus, $Q_i(\Lambda_i) \neq 0$, $i = 1, 2$. Therefore, from (70), we arrive at

$$H_i(\Lambda_i) = \frac{r_i Q_i(\lambda)}{\Delta_i(\lambda, c)/(\Lambda_i - \lambda)} \neq 0, \quad i = 1, 2. \quad (73)$$

For $c = c^*$, in this case, there are three subcases, $c = c^* = c_1^* > c_2^*$, $c = c^* = c_2^* > c_1^*$, and $c = c^* = c_1^* = c_2^*$. The first two subcases are equivalent by changing the roles of $\phi_1$ and $\phi_2$. Thus, without loss of generality, we consider $c = c^* = c_1^* > c_2^*$. In this subcase, $\Lambda_2$ is a simple root of $\Delta_2(\lambda, c) = 0$; repeating the above proofs of $H_i(\Lambda_i) \neq 0$ with $c > c^*$, $i = 1, 2$, take $\nu = \mu = 0$, and we directly have

$$H_2(\Lambda_2) \neq 0. \quad (74)$$

Since $\Lambda_1 = \lambda_1 = \lambda_2$ is a double root of $\Delta_1(\lambda, c) = 0$, then $\Delta_1(\lambda, c)/(\lambda_1 - \lambda)^2 \neq 0$ at $\lambda = \lambda_1$. When $Q_1(\lambda_i) \neq 0$, we can take $\nu = \mu = 1$ such that

$$H_1(\lambda_i) = \frac{r_1 Q_1(\lambda)}{\Delta_1(\lambda, c)/(\lambda_1 - \lambda)^2} \neq 0. \quad (75)$$

While $Q_1(\lambda_1) = 0$, $\lambda_1$ must be a simple root of $Q_1(\lambda) = 0$; otherwise, $L(\lambda, \phi_i)$ exists by (69), which contradicts Lemma 5. Thus, we can take $\nu = \mu = 0$ such that

$$H_1(\lambda_i) = \frac{r_1 Q_1(\lambda)}{\Delta_1(\lambda, c)/(\lambda_1 - \lambda)} \neq 0. \quad (76)$$

When $c = c^* = c_1^* = c_2^*$, repeating the above proofs gives $H_i(\Lambda_i) \neq 0$, $i = 1, 2$, where $\Lambda_1 = \lambda_1 = \lambda_2$, $\Lambda_2 = \lambda_3 = \lambda_4$.

Now we prove (ii) in Theorem 1. From (67), we define

$$F_3(\lambda) = \int_{-\infty}^{0} \phi_3(\xi)e^{-\lambda\xi}d\xi$$

$$= -\int_{-\infty}^{0} \phi_3(\xi)e^{-\lambda\xi}d\xi - r_3 a_{31}e^{-\lambda r_3 Q_1(\lambda)} \frac{\Delta_1(\lambda, c)\Delta_3(\lambda, c)}{\Delta_2(\lambda, c)\Delta_3(\lambda, c) + r_3 Q_3(\lambda)} + r_3 Q_3(\lambda)$$

$$H_3(\lambda) = (y(\tau_{33}) - \lambda)^{-1} F_3(\lambda)$$

$$= -\frac{r_3 a_{31}e^{-\lambda r_3 Q_1(\lambda)}}{\Delta_1(\lambda, c)\Delta_3(\lambda, c)/(y(\tau_{33}) - \lambda)^{-1}}$$

$$- \frac{r_3 a_{32}e^{-\lambda r_3 Q_2(\lambda)}}{\Delta_2(\lambda, c)\Delta_3(\lambda, c)/(y(\tau_{33}) - \lambda)^{-1}}$$

$$+ \frac{r_3 Q_3(\lambda)}{\Delta_3(\lambda, c)/(y(\tau_{33}) - \lambda)^{-1}}$$

$$- [y(\tau_{33}) - \lambda]^{-1} \int_{0}^{\infty} \phi_3(\xi)e^{-\lambda\xi}d\xi, \quad (77)$$

where $\nu$ is the same as above. By using a similar argument as (i), $H_3(\lambda)$ is analytic in the strip $0 < \text{Re}\lambda \leq y(\tau_{33})$. Hence, also from Lemma 7,

$$\lim_{\xi \to -\infty} \frac{\phi_3(\xi)}{[\xi]^m e^{\xi^m + \xi}} = H_3(\gamma(\tau_{33})) \Gamma(\gamma(\tau_{33}) + 1). \quad (79)$$

which implies (ii) in Theorem 1 holds if $H_3(\gamma(\tau_{33})) \neq 0$. In the following, we will prove that $H_3(\gamma(\tau_{33})) \neq 0$ in three cases.

**Case 1.** $\lambda_5(\tau_{33}) \leq \min\{\lambda_1, \lambda_2\}$. In this case, $\gamma(\tau_{33}) = \lambda_5(\tau_{33})$ and

$$\text{Case 2.}$$
\[
H_3(\lambda) = -r_3 \left\{ \int_{-\infty}^{\infty} [\varphi_3(\xi) \varphi_3(\xi - cr_{33}) + a_{31} [1 - \varphi_3(\xi)] \varphi_1(\xi - cr_{33}) + a_{32} [1 - \varphi_3(\xi)] \varphi_2(\xi - cr_{33})] e^{-\lambda \xi} d\xi \right\} \\
\cdot \left( \lambda_5 (r_{33}) - \lambda \right) (\Delta_3 (\lambda, c))^{-1} - [\lambda_5 (r_{33}) - \lambda] \int_{0}^{\infty} \varphi_3(\xi) e^{-\lambda \xi} d\xi.
\]

If \( H_3(\lambda_5 (r_{33})) = 0 \), then

\[
\int_{-\infty}^{\infty} [\varphi_3(\xi) \varphi_3(\xi - cr_{33}) + a_{31} [1 - \varphi_3(\xi)] \varphi_1(\xi - cr_{33}) + a_{32} [1 - \varphi_3(\xi)] \varphi_2(\xi - cr_{33})] e^{-\lambda_5 (r_{33}) \xi} d\xi = 0,
\]

implying \( \varphi_1(\xi) \equiv \varphi_2(\xi) \equiv \varphi_3(\xi) \equiv 0 \) in \( \mathbb{R} \), which contradicts the results in Lemma 4.

First of all, when \( c > c^* \), in this case \( \nu = 0 \), and there are four subcases:

(i) \( \lambda_4 > \lambda_3 \geq \lambda_2 > \lambda_1 \),
(ii) \( \lambda_2 > \lambda_4 > \lambda_3 > \lambda_1 \),
(iii) \( \lambda_1 = \lambda_3 \),
(iv) \( \lambda_1 \neq \lambda_3, \lambda_2 = \lambda_4 \).

For subcase (i), \( \nu(r_{33}) = \lambda_4 \), and \( \lambda_1 \) is a simple root of \( \Delta_1 (\lambda, c) = 0 \) and \( \Delta_2 (\lambda, c) > 0 \); then, by repeating the process to prove \( H_4(\lambda_4) \neq 0 \) with \( c > c^* \), we easily obtain

\[
H_3(\lambda_4) = -r_3 a_{31} e^{-\lambda \gamma} r_1 Q_1 (\lambda) \Delta_2 (\lambda, c) - r_3 a_{32} e^{-\lambda \gamma} r_2 Q_2 (\lambda) \Delta_1 (\lambda, c) \\
\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c) / (\nu(r_{33}) - \lambda)^{\gamma + 1}
\]

\[ \neq 0. \]  

(82)

For subcase (ii), \( \nu(r_{33}) = \lambda_4 \), and \( \lambda_1 \) is a simple root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(83)

For subcase (iii), \( \nu(r_{33}) = \lambda_3 \), and \( \lambda_1 \) is a simple root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(84)

For subcase (iv), \( \nu(r_{33}) = \lambda_2 \), and \( \lambda_1 \) is a simple root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(85)

For subcase (ii), \( \nu(r_{33}) = \lambda_4 \), and \( \lambda_1 \) is a double root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(86)

For subcase (iii), \( \nu(r_{33}) = \lambda_3 \), and \( \lambda_1 \) is a double root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(87)

For subcase (iv), \( \nu(r_{33}) = \lambda_2 \), and \( \lambda_1 \) is a double root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(88)

For subcase (ii), \( \nu(r_{33}) = \lambda_4 \), and \( \lambda_1 \) is a double root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(89)

For subcase (iii), \( \nu(r_{33}) = \lambda_3 \), and \( \lambda_1 \) is a simple root of \( \Delta_1 (\lambda, c) = 0 \), and thus

\[
\frac{\Delta_1 (\lambda, c) \Delta_2 (\lambda, c) \Delta_3 (\lambda, c)}{(\lambda_1 - \lambda)^{\gamma}} \neq 0,
\]

\[ \lambda = \lambda_1. \]  

(90)
For subcase (iv), $\gamma(\tau_{33}) = \lambda_1 = \lambda_2 = \lambda_3 = \tilde{\lambda}$ is a double root of $\Delta_1(\lambda, c) = 0$, while it is a simple root of $\Delta_2(\lambda, c) = 0$. Since $Q_2(\tilde{\lambda}) \neq 0$, we can take $\nu = \mu = 1$ such that

$$H_3(\hat{\lambda}) = -r_3 \alpha_{31} e^{-\lambda c_{13} \tau_3} r_1 Q_1(\lambda) \Delta_2(\lambda, c) \Delta_3(\lambda, c)/(\hat{\lambda} - \lambda)^2 \bigg|_{\hat{\lambda} = \lambda} \neq 0. \quad (91)$$

When $c = c^* = c_t^*$, there are three subcases:

(i) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$,

(ii) $\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4$,

(iii) $\lambda_3 = \lambda_4 < \lambda_1 = \lambda_2$.

Repeating the above proofs, one can easily obtain $H_3(\gamma(\tau_{33})) \neq 0$ in each subcase.

Case 3. $\min[A_1, A_2] \leq \lambda_3(\tau_{33}) \leq \max[A_1, A_2]$. In this case, the proof is similar.

$$d_1(f_1*\varphi_1)(\xi) - d_1\varphi_1(\xi) - \tilde{c}\varphi_1'(\xi) + r_1[k_1 - \varphi_1(\xi)] \tilde{f}_1(\varphi_{1, r_1}, \varphi_{2, r_2}, \varphi_{3, r_3}) = 0,$$

$$d_2(f_2*\varphi_2)(\xi) - d_2\varphi_2(\xi) - \tilde{c}\varphi_2'(\xi) + r_2[k_2 - \varphi_2(\xi)] \tilde{f}_2(\varphi_{1, r_1}, \varphi_{2, r_2}, \varphi_{3, r_3}) = 0,$$

$$d_3(f_3*\varphi_3)(\xi) - d_3\varphi_3(\xi) - \tilde{c}\varphi_3'(\xi) + r_3\varphi_3(\xi) \tilde{f}_3(\varphi_{1, r_1}, \varphi_{2, r_2}, \varphi_{3, r_3}) = 0,$$ \quad (94)

The following corollary is obvious by Theorem 1.

**Corollary 1.** If $\varphi(\xi)$ is described as Theorem 1, then

$$\lim_{\xi \to -\infty} (\varphi_1'(\xi)/\varphi_3(\xi)) = \nu.$$

Now, we investigate the exponential decay rates of $\varphi(\xi)$ at the plus infinity. For convenience, let $\bar{\varphi}_1 = \varphi_1 - \varphi_1$, $\bar{\varphi}_2 = \varphi_2 - \varphi_2$, and $\bar{\varphi}_3 = 1 - \varphi_3$; substituting $\bar{\varphi}_1$, $\bar{\varphi}_2$, $\bar{\varphi}_3$ into (11), we have

$$\bar{\varphi}_1 = a_{12}\varphi_1(\xi - ct_{11}) + a_{13}\varphi_3(\xi - ct_{13}),$$

$$\bar{\varphi}_2 = a_{21}\varphi_1(\xi - ct_{21}) - \varphi_2(\xi - ct_{22}) + a_{23}\varphi_3(\xi - ct_{23}),$$

$$\bar{\varphi}_3 = -a_{31}k_1 - a_{32}k_2 + 1 + a_{31}\varphi_1(\xi - ct_{31}) + a_{32}\varphi_2(\xi - ct_{32}) - \varphi_3(\xi - ct_{33}).$$ \quad (96)

Let $\tilde{z}(\xi) = \varphi_1(\xi)/\varphi_3(\xi)$. Since $\varphi_3(\xi) < 1$ by Lemma 4, Theorem 2 is based on $\lim_{\xi \to -\infty} \tilde{z}(\xi) = 0$ and Proposition 1.

Finally, from Lemma 6, the convergence of $L(\lambda, \varphi_3)$ is determined both by $L(\lambda, \varphi_1)$ and $L(\lambda, \varphi_2)$, while for $\xi = +\infty$, we cannot obtain the convergence of $L(\lambda, \varphi_1)$ and $L(\lambda, \varphi_2)$ only by $L(\lambda, \varphi_3)$. Thus, in order to investigate the asymptotic behavior of the nondecreasing solution $\varphi(\xi)$, we enforce the assumptions and adopt the methods in [5, 18, 22]. Now, we suppose $\tau_{ij} = 0$, $i, j = 1, 2, 3$, and $J_i(x) = \delta(x) - \delta''(x)$, $i = 1, 2, 3$, where $\delta$ is the Dirac delta function. With these assumptions, system (11) is transformed into the following system:

$$d_1\varphi_1'' - c\varphi_1' + r_1\varphi_1(1 - a_{13} - \varphi_1 + a_{12}\varphi_2 + a_{13}\varphi_3) = 0,$$

$$d_2\varphi_2'' - c\varphi_2' + r_2\varphi_2(1 - a_{23} - a_{23}\varphi_1 - a_{23}\varphi_2 + a_{23}\varphi_3) = 0,$$

$$d_3\varphi_3'' - c\varphi_3' + r_3(1 - \varphi_3)(a_{31}\varphi_1 + a_{32}\varphi_2 - \varphi_3) = 0,$$ \quad (97)

which is equivalent to (by taking $\varphi_1 = \varphi_1', \varphi_2 = \varphi_2', \varphi_3 = \varphi_3'$)
\[
\begin{align*}
\phi_1' &= \phi_1,
\phi_1' &= \frac{1}{d_1} \left[ c\phi_1 - r_1\phi_1 (1 - a_{13} - \phi_1 + a_{12}\phi_2 + a_{13}\phi_3) \right],
\phi_2' &= \phi_2,
\phi_2' &= \frac{1}{d_2} \left[ c\phi_2 - r_2\phi_2 (1 - a_{23} + a_{21}\phi_1 - \phi_2 + a_{23}\phi_3) \right],
\phi_3' &= \phi_3,
\phi_3' &= \frac{1}{d_3} \left[ c\phi_3 - r_3 (1 - \phi_3) (a_{31}\phi_1 + a_{32}\phi_2 - \phi_3) \right].
\end{align*}
\]

Linearizing the above system at the equilibrium point \((k_1, 0, k_2, 0, 1, 0)\) gives
\[
\Phi_1' = A\Phi_1,
\]
where
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
r_1k_1 & c & -a_{12}r_1k_1 & 0 & -a_{13}r_1k_1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{-a_{21}r_2k_2}{d_2} & 0 & \frac{r_2k_2}{d_2} & c & \frac{-a_{23}r_2k_2}{d_2} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \frac{r_3(a_{31}k_1 + a_{32}k_2 - 1)}{d_3} & \frac{c}{d_3}
\end{pmatrix},
\]
and \(\phi_1 = \phi_1 - k_1, \phi_2 = \phi_2 - k_2, \phi_3 = \phi_3 - 1.\) Let
\[
B(\lambda) = A - \lambda I,
\]
and then the characteristic equation of matrix \(A\) is
has the following four zero points:

\[\lambda_1 = 0, \quad \lambda_2 = \frac{c + \sqrt{c^2 + 4d_1 r_1 k_1}}{2d_1} > 0, \quad \lambda_3 = \frac{c - \sqrt{c^2 + 4d_1 r_1 k_1}}{2d_1} < 0, \quad \lambda_4 = \frac{c - \sqrt{c^2 + 4d_2 r_2 k_2}}{2d_2} < 0.\]

When \(\mu_1 \neq \mu_3\) and \(\mu_2 \neq \mu_4\), then \(h_0(\lambda) = 0\) has four different roots, two positive and two negative. Moreover, we notice that the image of \(h_1(\lambda)\) is strictly decreasing as \(\zeta\) is increasing, and \(\forall \zeta \in [0, 1], \quad h_1(0) = (1 - \zeta c_{a_{12}a_{21}}) r_1 r_2 k_1 k_2 > 0, h_1(\pm \infty) = \pm \infty.\) Thus, we deduce that \(h_1(\lambda) = 0\) has two different positive roots and two different negative roots.

For \(\mu_1 = \mu_3\) and \(\mu_2 = \mu_4\), \(h_0(\lambda) = 0\) has one positive root (the algebraic multiplicity is 2) and two different negative roots. Repeating the above process, we can prove that \(h_1(\lambda) = 0\) has two different positive roots and two different negative roots. When \(\mu_1 \neq \mu_3\) and \(\mu_2 = \mu_4\) or \(\mu_1 = \mu_3\) and \(\mu_2 = \mu_4\), the same result still holds. \(\square\)

From Lemma 8, (104) has two different negative roots, denoted by \(\lambda_8\) and \(\lambda_9\) separately. Without loss of generality, we assume \(\lambda_8 < \lambda_9 < 0.\)

Now, we give the proof of Theorem 3 to finish this section.

**Proof of Theorem 3.** We firstly suppose

\[(d_1^2 \lambda_7^2 - c \lambda_7 - r_1 k_1)(d_2^2 \lambda_7^2 - c \lambda_7 - r_2 k_2) - a_{12} a_{21} r_1 r_2 k_1 k_2 \neq 0,\]

(110)
which is equivalent to \( \lambda_7, \lambda_8, \) and \( \lambda_9 \) which are three different negative real roots of \(|B(\lambda)| = 0\). Then, there are three corresponding eigenfunctions:

\[
\begin{align*}
\mathbf{r}_1^1 &= (s_1, s_1\lambda_7, s_2, s_2\lambda_7, 1, \lambda_7)^T, \\
\mathbf{r}_2^1 &= (1, \lambda_8, r_1, r_1\lambda_8, 0, 0)^T, \\
\mathbf{r}_3^1 &= (1, \lambda_9, r_2, r_2\lambda_9, 0, 0)^T,
\end{align*}
\]

(111)

where

\[
\begin{align*}
\tau_1 &= \frac{-d_1^2 \lambda_7^2 + c \lambda_7 + r_1 k_1}{a_2 r_1 k_1}, \\
\tau_2 &= \frac{-d_1^2 \lambda_8^2 + c \lambda_8 + r_1 k_1}{a_2 r_1 k_1}, \\
s_1 &= \frac{a_1 r_1 k_1 (d_1^2 \lambda_7^2 - c \lambda_7 - r_1 k_1) (d_2^2 \lambda_7^2 - c \lambda_7 - r_2 k_2) - a_1 a_2 r_1 r_2 k_1 k_2}{(d_1^2 \lambda_7^2 - c \lambda_7 - r_1 k_1) (d_2^2 \lambda_7^2 - c \lambda_7 - r_2 k_2) - a_1 a_2 r_1 r_2 k_1 k_2}, \\
s_2 &= \frac{a_2 r_2 k_2 (d_1^2 \lambda_8^2 - c \lambda_8 + r_1 k_1) (d_2^2 \lambda_8^2 - c \lambda_8 + r_1 k_1) - a_2 a_3 r_1 r_2 k_1 k_2}{(d_1^2 \lambda_8^2 - c \lambda_8 + r_1 k_1) (d_2^2 \lambda_8^2 - c \lambda_8 + r_1 k_1) - a_2 a_3 r_1 r_2 k_1 k_2}.
\end{align*}
\]

(112)

From the proofs of Lemma 8, it is easy to infer

\[
\lambda_8 > \max\{|\mu_2|, |\mu_3|\} > \lambda_9,
\]

(113)

and thus \( \tau_1 > 0, \tau_2 < 0 \). Therefore, when \( \xi \rightarrow +\infty \), (99) has the following solutions:

\[
\Phi_1\xi = C_1 r_1^1 e^{\lambda_7 \xi} + C_2 r_2^1 e^{\lambda_8 \xi} + C_3 r_3^1 e^{\lambda_9 \xi},
\]

(114)

where \( C_i, i = 1, 2, 3 \), are arbitrary constants. Thus, by the stable manifold theorem, we have

\[
\begin{cases}
\varphi_1(\xi) = k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} - \alpha_2 s_2 e^{\lambda_8 \xi} - \alpha_3 s_3 e^{\lambda_9 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) = k_2 - \alpha_1 s_1 e^{\lambda_7 \xi} - \alpha_2 s_2 e^{\lambda_8 \xi} - \alpha_3 s_3 e^{\lambda_9 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) = 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{cases}
\]

(115)

Similar to [18], we finally determine the positive or negative of \( \alpha_i, i = 1, 2, 3 \). Since \( \varphi_1 \leq 1 \), then \( \alpha_1 > 0 \).

When \( \lambda_8 < \lambda_7 < 0 \), we deduce that \( \lambda_7 > \max\{|\mu_2|, |\mu_3|\} \).

Thus,

\[
d_2^2 \lambda_7^2 + c \lambda_7 + r_1 k_1 > 0, \quad i = 1, 2, \quad \xi \rightarrow \infty,
\]

(116)

which implies \( s_1, s_2 > 0 \). Moreover, by noting \( \lim_{\xi \rightarrow \infty} (e^{\lambda_7 \xi} / e^{\lambda_8 \xi}) = +\infty, i = 8, 9 \), (115) can be simplified into

\[
\begin{cases}
\varphi_1(\xi) = k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) = k_2 - \alpha_1 s_2 e^{\lambda_8 \xi} + \alpha_3 s_3 e^{\lambda_9 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) = 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{cases}
\]

(117)

When \( \lambda_7 < \lambda_8 < \lambda_9 \), the denominators of \( s_j \) are positive, \( i = 1, 2 \), and \( \lim_{\xi \rightarrow \infty} (e^{\lambda_7 \xi} / e^{\lambda_8 \xi}) = 0, i = 8, 9 \). Since the signs of molecules of \( s_j \) may not be positive, \( i = 1, 2, \alpha_1, \alpha_3 \geq 0 \) and \( (\alpha_1, \alpha_3) \neq (0, 0) \). If \( \alpha_1 = 0 \), we suppose \( \alpha_3 > 0 \); then, \( -\alpha_1 s_2 e^{\lambda_8 \xi} > 0 \) for large \( \xi, \varphi_2 > \varphi_3 \). We suppose \( \alpha_3 < 0 \); similarly, \( \varphi_1 > \varphi_3 \) for large \( \xi \). Thus, \( \alpha_3 = 0 \), which is a contradiction. Hence, \( \alpha_3 > 0 \). Thus, (115) can be simplified into

\[
\begin{cases}
\varphi_1(\xi) = k_1 - \alpha_1 s_1 e^{\lambda_7 \xi} - \alpha_2 s_2 e^{\lambda_8 \xi} + \text{h.o.t.}, \\
\varphi_2(\xi) = k_2 - \alpha_1 s_2 e^{\lambda_8 \xi} + \alpha_3 s_3 e^{\lambda_9 \xi} + \text{h.o.t.}, \\
\varphi_3(\xi) = 1 - \alpha_1 e^{\lambda_7 \xi} + \text{h.o.t.},
\end{cases}
\]

(118)

When \( \lambda_7 < \lambda_8 < \lambda_9 \), the proof is similar.

In the following, we assume that (101) does not hold, namely, \( \lambda_7 = \lambda_8 \) or \( \lambda_7 = \lambda_9 \), and we want to find two independent vectors of \( B^2(\lambda) \mathbf{r} = 0 \) at first. By directly computing, \( \mathbf{r}_1^2 = (1, \lambda, \tau, r_1, 0, 0)^T \) is a vector of \( B^2(\lambda) \mathbf{r} = 0 \); when \( \lambda = \lambda_7 = \lambda_8 \), \( \mathbf{r}_1^2 = \mathbf{r}_1^1 \), with \( \tau = \tau_1 > 0 \), and when \( \lambda = \lambda_7 = \lambda_9 \), \( \mathbf{r}_1^2 = \mathbf{r}_1^3 \), with \( \tau = \tau_2 < 0 \).

Then, we suppose another linearly independent vector is \( \mathbf{r}_2^2 = (x_1, x_2, x_3, x_4, x_5, x_6)^T \), which satisfies \( B^2(\lambda) \mathbf{r}_2^2 = 0 \) and \( B(\lambda) \mathbf{r}_2^2 = \mathbf{r}_1^2 \), since \( B(\lambda) \mathbf{r}_1^1 = 0 \). We can solve \( B(\lambda) \mathbf{r}_2^2 = \mathbf{r}_1^1 \) to find \( \mathbf{r}_2^2 \). Thus, in fact, \( \mathbf{r}_2^2 \) satisfies
\[
\begin{align*}
-\lambda x_1 + x_2 &= 1, \\
r_1 k_1 x_1 + \left( \frac{c}{d_1} - \lambda \right) x_2 - \frac{a_{12} r_1 k_1}{d_1} x_3 - \frac{a_{13} r_1 k_1}{d_1} x_2 &= \lambda, \\
-\lambda x_3 + x_4 &= \tau, \\
\frac{a_{21} r_2 k_2}{d_2} x_1 + \frac{r_2 k_2}{d_2} x_2 + \left( \frac{c}{d_2} - \lambda \right) x_4 - \frac{a_{23} r_2 k_2}{d_2} x_3 &= \tau \lambda, \\
-\lambda x_5 + x_6 &= 0, \\
\frac{r_3 (a_{31} k_1 + a_{32} k_2 - 1)}{d_3} x_5 + \left( \frac{c}{d_3} - \lambda \right) x_6 &= 0.
\end{align*}
\] (120)

Take $x_6 = \lambda x_5$ into the last formula above, and we get
\[
\left[ -d_3 \lambda^2 + c \lambda + r_3 (a_{31} k_1 + a_{32} k_2 - 1) \right] x_5 = 0. \quad (121)
\]

From (103), for any $x_5$, the coefficient of $x_5$ equals zero. If $x_5 = \bar{\mu}$, then $x_6 = \lambda \bar{\mu}$, for some $\bar{\mu} \neq 0$. Thus, we can deduce that
\[
\begin{align*}
-\lambda x_1 + x_2 &= 1, \\
r_1 k_1 x_1 + (c - d_1 \lambda) x_2 - a_{12} r_1 k_1 x_3 - a_{13} r_1 k_1 \bar{\mu} &= d_1 \lambda, \\
-\lambda x_3 + x_4 &= \tau, \\
-a_{21} r_2 k_2 x_1 + r_2 k_2 x_2 + (c - d_2 \lambda) x_4 - a_{23} r_2 k_2 \bar{\mu} &= d_2 \tau \lambda.
\end{align*}
\] (122)

Taking $x_1 = 0$, $x_2 = 1$ gives
\[
\begin{align*}
-\lambda x_3 + x_4 &= \tau, \\
r_2 k_2 x_3 + (c - d_2 \lambda) x_4 - a_{23} r_2 k_2 \bar{\mu} &= d_2 \tau \lambda.
\end{align*}
\] (123)

From the first equation, we get
\[
\bar{\mu} = \frac{c - 2d_2 \lambda - a_{12} r_1 k_1 x_3}{a_{13} r_1 k_1}. \quad (124)
\]

Substituting it into the last one gives
\[
\begin{align*}
-\lambda x_3 + x_4 &= \tau, \\
r_2 k_2 x_3 + (c - d_2 \lambda) x_4 - a_{23} r_2 k_2 \frac{c - 2d_2 \lambda - a_{12} r_1 k_1 x_3}{a_{13} r_1 k_1} &= 0.
\end{align*}
\]

Thus,
\[
x_3 = \frac{a_{23} r_2 k_2 / a_{13} r_1 k_1 \left( c - 2d_2 \lambda \right) - (c - 2d_2 \lambda) r_1}{-d_2 \lambda - c \lambda_5 + r_2 k_2 + (a_{12} a_{23} / a_{13}) r_2 k_2}. \quad (125)
\]

Thus,
\[
x_3 = \frac{(a_{23} r_2 k_2 / a_{13} r_1 k_1) \left( c - 2d_2 \lambda \right) - (c - 2d_2 \lambda) r_1}{-d_2 \lambda - c \lambda_5 + r_2 k_2 + (a_{12} a_{23} / a_{13}) r_2 k_2}. \quad (125)
\]

Then, we will prove $\bar{\mu} > 0$, which is equivalent to
\[
c - 2d_2 \lambda - a_{12} a_{23} r_2 k_2 (c - 2d_2 \lambda) - a_{13} r_1 k_1 (c - 2d_2 \lambda) r_1 \quad \frac{a_{13} r_1 k_1}{a_{12} a_{23} / a_{13}} r_2 k_2.
\] (130)

Thus,
\[
c - 2d_2 \lambda - a_{12} a_{23} r_2 k_2 (c - 2d_2 \lambda) - a_{13} r_1 k_1 (c - 2d_2 \lambda) r_1 \quad \frac{a_{13} r_1 k_1}{a_{12} a_{23} / a_{13}} r_2 k_2.
\] (131)
Since \( \max[\mu_2, \mu_4] < \lambda_8 < 0, -d_2 \lambda_8^2 + c \lambda_8 + r_2 k_2 > 0, r_1 > 0 \), then
\[
a_{13}(c - 2d_1 \lambda_9) \left( -d_2 \lambda_8^2 + c \lambda_8 + r_2 k_2 + \frac{a_{12}a_{23}}{a_{13}} r_2 k_2 \right) > a_{12}a_{23} r_2 k_2 (c - 2d_1 \lambda_9) - a_{13} r_1 k_1 (c - 2d_2 \lambda_9),
\]
which is equivalent to (131).

When \( \lambda = \lambda_9 \), from (38),
\[
-d_2 \lambda_8^2 + c \lambda_8 + r_2 k_2 + \frac{a_{12}a_{23}}{a_{13}} r_2 k_2 \neq 0.
\]

Since \( \tau_2 < 0, \lambda < 0 \), then
\[
x_3 = \frac{(a_{23} r_2 k_2/a_{13} r_1 k_1) (c - 2d_1 \lambda_9) - (c - 2d_2 \lambda_9) \tau_2}{-d_2 \lambda_8^2 + c \lambda_8 + r_2 k_2 + (a_{12}a_{23}/a_{13}) r_2 k_2} \neq 0.
\]

If \( x_3 < 0 \), then \( \pi > 0 \). If \( x_3 > 0 \), then from \( \tau_2 < 0, \lambda < 0 \), we have
\[
\frac{a_{23} r_2 k_2}{a_{13} r_1 k_1} (c - 2d_1 \lambda_9) - (c - 2d_2 \lambda_9) \tau_2 > 0,
\]
which implies
\[
-d_2 \lambda_8^2 + c \lambda_8 + r_2 k_2 + \frac{a_{12}a_{23}}{a_{13}} r_2 k_2 > 0.
\]

Suppose \( \pi \geq 0 \); by recalling \( \lambda < \min[\mu_2, \mu_4] \), from the above proof, we deduce that
\[
0 > -d_2 \lambda_8^2 + c \lambda_8 + r_2 k_2 \geq \frac{c - 2d_2 \lambda_9}{c - 2d_1 \lambda_9} a_{13} r_1 k_1 \tau_2 > 0,
\]
which is a contradiction, and thus \( \pi < 0 \).

Therefore, when \( \lambda = \lambda_9 > \lambda_9, \xi \rightarrow -\infty \), (99) has the following solutions:
\[
\Phi_{1,i}^* = C_i r_1^i e^{\lambda_i \xi} + C_2 r_2^i e^{\lambda_i \xi} + C_3 r_3^i e^{\lambda_i \xi},
\]
where \( C_i, i = 1, 2, 3 \), are arbitrary constants and \( r_2^i = (0, 1, r_1, \lambda_8 r_8, \mu_1, \lambda_8 \mu_1)^T \).

By the stable manifold theorem, we have
\[
\left\{ \begin{array}{l}
\Phi_1(\xi) = k_1 - a_1 e^{\lambda_1 \xi} - a_2 e^{\lambda_2 \xi} + h.o.t., \\
\Phi_2(\xi) = k_2 - a_1 r_1 e^{\lambda_1 \xi} - a_2 r_2 e^{\lambda_2 \xi} + h.o.t., \\
\Phi_3(\xi) = 1 - a_3 e^{\lambda_3 \xi} + h.o.t.,
\end{array} \right.
\]
for some \( a_i > 0, a_i > 0 \).

Moreover, when \( \lambda = \lambda_9 \), \( \xi \rightarrow -\infty \), (99) has the following solutions:
\[
\Phi_{1,i}^* = C_1 r_1^i e^{\lambda_1 \xi} + C_2 r_2^i e^{\lambda_2 \xi} + C_3 (r_3^i + r_3^i) e^{\lambda_i \xi},
\]
where \( C_i, i = 1, 2, 3 \), are arbitrary constants and \( r_2^i = (0, 1, r_1, \lambda_8 r_8, \mu_1, \lambda_8 \mu_1)^T \).

By the stable manifold theorem, we have
\[
\left\{ \begin{array}{l}
\Phi_1(\xi) = k_1 - a_1 e^{\lambda_1 \xi} - a_2 e^{\lambda_2 \xi} + h.o.t., \\
\Phi_2(\xi) = k_2 - a_1 r_1 e^{\lambda_1 \xi} - a_2 r_2 e^{\lambda_2 \xi} + h.o.t., \\
\Phi_3(\xi) = 1 - a_3 e^{\lambda_3 \xi} + h.o.t.,
\end{array} \right.
\]
for some \( a_i > 0, a_i > 0 \).

4. Other Properties

In this section, we adopt the sliding method to prove the strict monotonicity and the uniqueness. Similar to [13], we first give the strong comparison principle.

Lemma 9. Assume that (P1), (P2), and (S) hold. For \( c \geq c^* \), let \( \Phi_1(\xi) \) and \( \Phi_2(\xi) \) be two nondecreasing solutions of (11) and (12). Then, either \( \Phi_{11} < \Phi_{12}, \Phi_{21} < \Phi_{22}, \Phi_{31} < \Phi_{32} \) in \( \mathbb{R} \) or \( \Phi_{11} \equiv \Phi_{12}, \Phi_{21} \equiv \Phi_{22}, \Phi_{31} \equiv \Phi_{32} \) in \( \mathbb{R} \).

Finally, we finish the proof of Theorem 4.

Proof of Theorem 4. Let \( \Phi(\xi) \) be a nondecreasing solution of (11) and (12); then, \( \Phi_i(\xi) \geq 0, i = 1, 2, 3 \). By Theorem 6 and Corollary 1, for a large enough \( N > 0 \), we have \( \Phi_i(\xi) > 0, \xi \in (-\infty, -N], i = 1, 2, 3 \). We need to prove \( \Phi_i(\xi) > 0 \) for \( \xi \in [-N, +\infty), i = 1, 2, 3 \). Let \( \xi_i \) be the left-most point such that \( \Phi_i(\xi_i) = 0 \); then, \( \Phi_i(\xi_i) \) is the minimum of \( \Phi_i(\xi) \), implying \( \Phi_i(\xi_i) = 0 \). Differentiating the first equation of (11) gives
\[
0 = d_i (J_1 \Phi_i(\xi_i)) + r_i \Phi_i(\xi_i) [J_{a2} \Phi_i(\xi_i - c r_{12}) + a_{13} \Phi_i(\xi_i - c r_{13})] \geq 0.
\]

Thus, we conclude \( \Phi_i(\xi_i - y) = 0 \), \( y \in \mathbb{R} \), which contradicts the definition of \( \xi_i \). Therefore, \( \Phi_i(\xi_i) > 0 \). With similar process, \( \Phi_i(\xi_i) > 0 \) in \( \mathbb{R} \).

In the end, the proof of Theorem 5 is completely similar to [14, 15], so we omit it here.
Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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