ON THE RATIONALITY OF THE SPECTRUM

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Abstract. Let $\Omega \subset \mathbb{R}$ be a compact set with measure 1. If there exists a subset $\Lambda \subset \mathbb{R}$ such that the set of exponential functions $E_\Lambda := \{e_\lambda(x) = e^{2\pi i \lambda x}|_{\Omega} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\Omega)$, then $\Lambda$ is called a spectrum for the set $\Omega$. A set $\Omega$ is said to tile $\mathbb{R}$ if there exists a set $\mathcal{T}$ such that $\Omega + \mathcal{T} = \mathbb{R}$. A conjecture of Fuglede suggests that Spectra and Tiling sets are related. Lagarias and Wang [14] proved that Tiling sets are always periodic and are rational. That any spectrum is also a periodic set was proved in [3], [8]. In this paper, we give some partial results to support the rationality of the spectrum.

1. Introduction

In this paper we explore the rationality of the spectrum in $\mathbb{R}$.

Let $\Omega \subset \mathbb{R}^d$ be a (compact) set with Lebesgue measure $|\Omega| = 1$.

Definition 1. $\Omega$ is said to be a spectral set if there exists a subset $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E_\Lambda := \{e_\lambda(x) = e^{2\pi i \lambda x}|_{\Omega} : \lambda \in \Lambda\}$ is an orthonormal basis for the Hilbert space $L^2(\Omega)$. The set $\Lambda$ is said to be a spectrum for $\Omega$, and the pair $(\Omega, \Lambda)$ is called a spectral pair.

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It is easy to see that for a spectral set $\Omega$, the spectrum need not be unique, and conversely given a spectrum $\Lambda$, there may be many sets $\Omega$ such that $\Omega, \Lambda$ is a spectral pair.

Interest in the spectrum arose from a conjecture due to Fuglede relating spectral and tiling properties of sets. $\Omega$ is said to tile $\mathbb{R}^d$ if there exists a subset $T \subset \mathbb{R}^d$ such that $\Omega + T$ is a partition (a.e.) of $\mathbb{R}^d$. $(\Omega, T)$ is called a tiling pair and $T$ is called a tiling set.

Fuglede’s Conjecture. A set $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if $\Omega$ tiles $\mathbb{R}^d$ by translations.

The conjecture suggests that there could be a strong relationship between Spectra and Tiling sets for a given $\Omega$.

Fuglede’s conjecture is known to be false in dimensions $d \geq 3$ \cite{15, 11}. However interest in this conjecture is still alive. For $d = 2$ it has been proved for convex planar domains for $d = 2$ \cite{9} and for $d = 3$, Fuglede’s conjecture for Convex polytopes in $\mathbb{R}^3$ has been proved very recently by Greenfeld and Lev \cite{7}.

We will restrict to dimension $d = 1$. In this case Fuglede’s conjecture is known to hold when $\Omega$ is the union of two intervals \cite{12} and for the case of three intervals the authors proved that Tiling implies Spectral, and except for one situation, Spectral implies tiling too, \cite{1, 2}. In Lagarias and Wang \cite{14} studied the structure of tiling sets $T$, and proved that if $(\Omega, T)$ is a tiling pair for some $\Omega$, then the tiling set $T$ is periodic with an integer period and is rational, i.e. $T$ is of the form

$$T = \bigcup_{j=0}^{n-1} (t_j + n \mathbb{Z})$$

with $t_0 = 0$ and $t_j \in \mathbb{Q}$, $j = 0, 1, ..., n - 1$. 
Our aim is to study the structure of spectra $\Lambda$ for spectral pairs $(\Omega, \Lambda)$. In [3], the authors proved that if $\Omega$ is the union of finitely many intervals and $(\Omega, \Lambda)$ is a spectral pair, then $\Lambda$ is periodic with an integer period. In [8] this result was then proved for any compact set $\Omega$.

Let $(\Omega, \Lambda)$ be a spectral pair. Since any translate of $\Lambda$ is again a spectrum for $\Omega$, we may assume that $\Lambda$ is of the form

$$\Lambda = \bigcup_{j=0}^{d-1}(\lambda_j + d\mathbb{Z}) = \Gamma + d\mathbb{Z},$$

with $\lambda_0 = 0$. Further, by the structure theorem proved in [3] we know that $\Lambda$ is also a spectrum for a set $\Omega_1$ which is a union of $d$ equal intervals, whose end points lie on the lattice $\mathbb{Z}/d$; i.e.,

$$\Omega_1 = \bigcup_{j=0}^{d-1}[a_j/d, a_j + 1/d]$$

$j = 0, 1, \ldots, d - 1$, with $a_0 = 0$. Thus to resolve the question of rationality of a spectrum, it is enough to assume that the associated spectral set $\Omega$ is of the above form (such sets are called clusters). After rescaling, we write

$$\Omega = A + [0, 1]$$

with $A \subset \mathbb{Z}_+$, $0 \in A$, $|A| = d$. Then $\Lambda = \bigcup_{j=1}^{d-1}(\lambda_j/d + Z) = \Gamma + Z$, and $E_\Lambda = \{1/\sqrt{d}e_\lambda(x), \ \lambda \in \Lambda\}$ is a complete orthonormal set.

All known spectra of sets in $\mathbb{R}$ are rational; however it is not known whether this must always be so. In [4] it is shown that if Fuglede’s conjecture is true in one dimension, then every spectral set of Lebesgue measure 1 has a rational spectrum.

The following result due to Laba [13] is the only result we are aware of in the literature, which addressed the problem of rationality of spectra for clusters:
Theorem 1 (Lab). Suppose that $\Omega = A + [0, 1]$, $A \subset \mathbb{Z}_+$, $|A| = d$ is a spectral set. If $\Omega \subset [0, M]$, where $M < \frac{5d}{2}$, then all spectra associated to $\Omega$ are rational.

In section 2, we deduce two interesting facts about a spectrum from known results. First we observe that elements of any spectrum are either rational or transcendental; next, we relate the rationality of the spectrum to integer zeros of exponential polynomials. In Section 3, we show that if for some $\Omega$ such that $(\Omega, \Lambda)$ is a spectral pair and $\Omega$ contains some patterns, then $\Lambda$ has to be rational. In section 4, we prove that if the set $\Omega - \Omega$ contains some rigid structures, then the spectrum is rational.

2.

2.1. We first observe that the elements of the spectrum are either rational or transcendental. We explain this below:

We have

$$\hat{\chi}_\Omega(\xi) = (1 - e^{2\pi i \xi})(1 + e^{2\pi i \lambda_1 \xi} + \ldots + e^{2\pi i \lambda_d - 1 \xi}),$$

and for every $\lambda_j$, $j = 0, 1, \ldots, d - 1$, $\hat{\chi}_\Omega(\lambda_j) = 0$. So each $e^{2\pi i \lambda}$ is an algebraic number, in fact, an algebraic integer. Recall the following famous result:

**Theorem 2** (Gelfond-Schneider). If $\alpha$ and $\beta$ are algebraic numbers with $\alpha \neq 0, 1$, and if $\beta$ is not a rational number, then any value of $\alpha^\beta = exp(\beta \log \alpha)$ is a transcendental number.

We apply this theorem to $\alpha = e^{\pi i} = -1$, and $\beta = 2\lambda_k$. Since $\alpha^\beta = e^{2\pi i \lambda_k}$ is an algebraic integer, $2\lambda_k$ is either rational or is not an
algebraic number. In other words, elements $\lambda_k$ of the spectrum are either rational or transcendental numbers.

2.2. **Zeros of Exponential Polynomials.** Consider the tempered distribution obtained as Dirac masses on points of $\Lambda$, i.e., the distribution

$$\delta_{\lambda} = \sum_{j=0}^{d-1} \sum_{n \in \mathbb{Z}} \delta_{n+\lambda_j/d} = \delta_{\Gamma} \ast \delta_Z$$

Then

$$\hat{\delta}_{\Gamma}(x) = \sum_{j=0}^{d-1} e^{2\pi i \lambda_j x/d}$$

Recall that with $\Omega$ and $\Lambda$ as above, where $|\Omega| = d$, $(\Omega, \Lambda)$ is a spectral pair iff $\frac{1}{d} |\widehat{\chi_\Omega}|^2 \ast \delta_{\lambda} \equiv d$ iff $|\widehat{\chi_\Omega}|^2 \ast \delta_{\Gamma} \ast \delta_Z \equiv d^2$ iff $(\Omega - \Omega)|_Z \subseteq \mathbb{Z}(\hat{\delta}_{\Gamma}) \cup \{0\}$, where $\mathbb{Z}(\hat{\delta}_{\Gamma}) = \{k \in \mathbb{Z} : \hat{\delta}_{\Gamma}(k) = 0\}$.

We write $z_j = e^{2\pi i \lambda_j / d}$, then

$$\hat{\delta}_{\Gamma}(k) = \sum_{j=0}^{d-1} z_j^k, \ k \in \mathbb{Z}$$

We are thus led to study the integer zeros of exponential polynomials; more specifically the integer zeros of exponential polynomials. An important result in this context is the Skolem-Mahler-Lech theorem, which says that the sets of integer zeros of exponential polynomials are of the form $X \cup F$, where $X$ is a union of finitely many complete arithmetic progressions, and $F$ is a finite set.

In [14] Lagarias and Wang considered the case of the exponential polynomial, $\hat{\delta}_{\Gamma}$, and gave more precise description of the set $X$, which we need to state.

With the above rescaling, we have, $\Gamma = \{\lambda_0 = 0, \lambda_1/d, ..., \lambda_{d-1}/d\}$. we write $\gamma_j = \lambda_j$, and define an equivalence relation on $\Gamma$ as $\gamma_i \sim \gamma_j$ iff
\( \gamma_i - \gamma_j \in \mathbb{Q} \), and we partition \( \Gamma \) into its rational equivalence classes,

\[
\Gamma = \bigcup_{j=1}^{k} \Gamma_j^*
\]

Then

\[
\hat{\delta}_\Gamma(\xi) = \hat{\delta}_{\Gamma_1^*}(\xi) + \cdots + \hat{\delta}_{\Gamma_k^*}(\xi)
\]

where,

\[
\Gamma_j^* = \{ \gamma_j, \gamma_j + \frac{l_j,2}{m_j}, \gamma_j + \frac{l_j,3}{m_j}, \ldots, \gamma_j + \frac{l_j,n_j}{m_j} \}
\]

Notice

\[
\hat{\delta}_{\Gamma_j^*}(\xi) = e^{2\pi i \gamma_j \xi} \left( 1 + e^{\frac{2\pi i l_j,2}{m_j} \xi} + \cdots + e^{\frac{2\pi i l_j,n_j}{m_j} \xi} \right)
\]

Hence if

\[
\hat{\delta}_{\Gamma_j^*}(\xi_0) = 0 \text{ then } \hat{\delta}_{\Gamma_j^*}(\xi_0 + m_j p) = 0, \forall p \in \mathbb{Z}
\]

Thus the zero set of \( \hat{\delta}_{\Gamma_j^*} \) is \( m_j \) periodic.

Let \( M := LCM \{ m_1, m_2, \ldots, m_k \} \). Let \( X \) be the common integer zero set of \( \{ \hat{\delta}_{\Gamma_j^*} \} \) i.e.

\[
X := \bigcap_{i=1}^{k} \mathbb{Z}(\hat{\delta}_{\Gamma_j^*})
\]

Then \( X \) is \( M \) periodic and \( X \subseteq \mathbb{Z}(\hat{\delta}_\Gamma) \), and \( F := \mathbb{Z}(\hat{\delta}_\Gamma) \setminus X \) is a finite set.

As a consequence, we prove:

**Theorem 3.** Let \((\Omega, \Lambda)\) be a spectral pair as above. Then \( \Lambda \) is rational if and only if \((A - A) \cap F = \phi \).

**Proof.** Clearly if \( \Lambda \) is rational, then \( F = \phi \). For the converse, we have

\[
\chi_\Omega \ast \chi_\Omega \cdot \hat{\delta}_\Gamma \cdot \delta_Z = d^2
\]
Consider the equivalence class $\Gamma_1^\ast$, for which 
\[ \chi_\Omega \ast \chi_\Omega \cdot \delta_{\Gamma_1} \cdot \delta_z = d|\Gamma_1| \leq d^2. \]
But 
\[ |\hat{\chi}_\Omega|^2 \ast \delta_{\Gamma_1} \ast \delta_z(0) = d^2 \]
Hence $|\Gamma_1^\ast| = d$, so that there is only one rational class. \qed

Remark 1.

(1) For the Skolem-Mahler-Lech theorem, the structure and cardinality of the finite set $F$ have been studied (see [5], and references therein), but "effective" results are largely unknown [5], [17].

(2) In the case when each of the equivalence classes $\Gamma_j^\ast$ are singletons, i.e. $\lambda_j - \lambda_k \notin \mathbb{Q}$ for all $j \neq k$, then it can be easily seen that in fact $Z(\hat{\delta}_\Gamma) = F$. (See Corollary 1.20 in [5]).

3.

In this section we show that the existence of some specific structures or patterns (which we call flags) in the zero set $Z(\hat{\delta}_\Gamma)$ guarantees the rationality of $\Gamma$.

We begin with a result which follows from a result of Jager [10]. We state it in our setting and give a proof for the sake completeness. (Jager’s paper is difficult to find).

Definition 2. For fixed integers $m$, $r$ and $N > r$, let $S_0 = \{m+1, m+2, ... m+r\}$ and for an $N \geq r$ let $S_n = S_{n-1} + N$, $n = 1, 2, ..., s - 1$. These $S_n$’s are called strips and the the set $F = \bigcup_{0}^{s-1} S_n$ is called an $r \times s$-flag. We will think of a flag as an array:
Observe that the $S_n$'s are all disjoint, since $N > r$. One can think of a flag as a rectangular array of $s$ points $m+1, m+N+1, \ldots, m+(s-1)N+1$ on a vertical pole, and with $s$ horizontal strips $m+nN+1, m+nN+2, \ldots, m+nN+r$, $n = 0, 1, \ldots, (s-1)$.

Let $\Gamma = \{0, \lambda_1, \ldots, \lambda_{d-1}\}$. It is easy to see that if $d > 1$, then a $d \times 1$ flag cannot be contained in $\mathbb{Z}(\hat{\delta}_\Gamma) \cup \{0\}$. (by a simple Vandermonde argument). However, if $S_0$ is a strip of shorter length, we will consider several such strips in a flag configuration as above, and prove:

**Theorem 4.** Fix two integers $m, r$ with $\left\lceil \frac{d}{2} \right\rceil \leq r < d$. Suppose an $r \times d$ flag $F \subset \mathbb{Z}(\hat{\delta}_\Gamma)$. Then $\Gamma$ is rational.

**Proof.** Let $z_j = e^{2\pi i \lambda_j}$. The hypothesis implies that for each $k = 1, 2, \ldots, r$, we have the following system of equations:

\[
\sum_{j=0}^{d-1} z_j^{m+k} = 0
\]

\[
\sum_{j=0}^{d-1} z_j^{m+k+N} = 0
\]

\[
\vdots
\]

\[
\sum_{j=0}^{d-1} z_j^{m+k+(d-1)N} = 0
\]
Equivalently, for every $k = 1, 2, \ldots, r$,

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & z_1^N & \cdots & z_{d-1}^N \\
\vdots & \vdots & \ddots & \vdots \\
1 & z_1^{(d-1)N} & \cdots & z_{d-1}^{(d-1)N}
\end{pmatrix}
\begin{pmatrix}
1 \\
1^{m+k} \\
\vdots \\
1^{m+k}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

Thus we can conclude that the above Vandermonde matrix is singular. Hence $z_i^N = z_j^N$ for some $i \neq j$. We define an equivalence relation $z_k \sim z_l \iff z_k^N = z_l^N$, and we write $\rho_j$ as a representative of each equivalence class so obtained, $j = 0, 1, \ldots, t$. Also let $[\rho_j] = \{z_{j1}, z_{j2}, \ldots, z_{lj}\}$ be the set of elements ($l_j$ in number) in the $j$th equivalence class. We can now extract a subsystem of the above system of equations:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & \rho_1^N & \cdots & \rho_t^N \\
\vdots & \vdots & \ddots & \vdots \\
1 & \rho_1^{(t-1)N} & \cdots & \rho_t^{(t-1)N}
\end{pmatrix}
\begin{pmatrix}
\sum_{s=0}^{l_0} z_0^{m+k} \\
\sum_{s=1}^{l_1} z_1^{m+k} \\
\vdots \\
\sum_{s=t-1}^{l_{t-1}} z_{(t-1)s}^{m+k}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

Now the $\rho_j$’s are all distinct, and hence the Vandermonde matrix on the left is non-singular. Hence, for each $k = 1, 2, \ldots, r$ and $j = 0, 1, \ldots, t$, we have

\[
\sum_{s=1}^{l_j} z_{j_s}^{m+k} = 0.
\]

Suppose $t > 1$, i.e. there are more than one equivalence classes $[\rho_j]$, then we can choose one equivalence class, say $\rho_{j_0}$ which has less than or equal to $[\frac{d}{2}] + 1 \leq r$ elements. For this $j_0$, we consider the first $l_{j_0}$ equations from the above set of $r$ equations. We have,
This is a contradiction since the $z_{j0}$'s are all distinct. Hence there is only one equivalence class. \[\square\]

**Remark 2.** If $\Lambda = \Gamma + \mathbb{Z}$ is a spectrum for a set $\Omega = A + [0,1]$, with $|A| = d$, we know that $A - A \subset \mathbb{Z}(\delta_\Gamma)$. Since $|A - A| \leq \frac{d(d-1)}{2}$, an $r \times d$ flag with $[\frac{d}{2}] + 1 \leq r$ cannot be contained in $A - A$. In the next theorem we will improve Jager’s's result and show that rationality follows from the existence of a 'shorter' flag in the integer zero set $\mathbb{Z}(\delta_\Gamma)$. Hence if the set $A - A$ itself has enough structure to contain these shorter flags, then we can conclude the rationality of the spectrum.

We will now extend a result due to Tijdeman [18], which in our notation can be stated as

**Theorem 5.** [18] Let $r = d - 1$ and suppose that a $(d - 1) \times 2$ flag is contained in $\mathbb{Z}(\delta_\Gamma)$, then the extended $(d - 1) \times d$ flag is also contained in $\mathbb{Z}(\delta_\Gamma)$.

Our extension of this theorem is for smaller values of $r$.

**Theorem 6.** Suppose that an $r \times (d - r + 1)$ flag is contained in $\mathbb{Z}(\delta_\Gamma)$, then the extended $r \times d$ flag is also contained in $\mathbb{Z}(\delta_\Gamma)$.

Let $z_0 = 1, z_1, ..., z_{d-1}$ be distinct complex numbers and

$$f_k = \sum_{0}^{d-1} z_j^k$$
Then the $f_k$s satisfy a $d$-term recurrence relation given by the Newton-Girard Formulae:

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_0 & c_1 & c_2 & \cdots & c_{d-1}
\end{pmatrix}
\begin{pmatrix}
f_k \\
f_{k+1} \\
\vdots \\
f_{d+k-1}
\end{pmatrix}
= \begin{pmatrix}
f_{k+1} \\
f_{k+2} \\
\vdots \\
f_{k+d+1}
\end{pmatrix}
\]

In brief, we will write these equations as:

\[U\nu_k = \nu_{k+1}\]

**Proof of Theorem 6.** Let $F$ be the $r \times (d-r+1)$ flag contained in $A-A\setminus\{0\} \subset \hat{Z}(\delta_T)$. Then, each of the vectors $\nu_{m+1}, \nu_{m+N+1}, \ldots, \nu_{m+(d-r)N+1}$ will be of the form given below:

\[
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
* \\
\vdots \\
*
\end{pmatrix}
\]

where the first $r$ entries are 0. For ease of notation we will write $\mu_j = \nu_{m+jN+1}$, $j = 0, 1, \ldots, (d-r)$. Clearly these $d-r+1$ vectors are linearly dependent, so there exist constants $\alpha_0, \alpha_1, \ldots, \alpha_{d-r}$ such that

\[\mu_{d-r+1} = \alpha_0 \mu_0 + \alpha_1 \mu_1 + \ldots + \alpha_{d-r} \mu_{d-r}\]

Now we apply the Newton-Girard matrix $U$ $N$ times to get

\[U^N \mu_j = \mu_{j+1}\]
Repeating this process, we see that all the vectors $\nu_{m+jN+1}$ are of the same form. But this means that the extended $r \times d$ flag is also contained in $\hat{\mathbb{Z}}(\delta_{\Gamma})$.

Combined with Theorem 4, we get

**Theorem 7.** Let $\Lambda = \Gamma + \mathbb{Z}$ be a spectrum for a set $\Omega = A + [0, 1]$, with $A \subset \mathbb{Z}$, $|A| = d$. Suppose $r \geq \left\lceil \frac{d}{2} \right\rceil$, and an $r \times (d - r + 1)$ flag is contained in $A - A$, then $\Lambda$ is rational.

We have found that the existence of some variations of the flag patterns in $A - A$ again imply rationality of the spectrum.

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