Glueballs on $S^3$

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For SU(2) gauge theory on the three-sphere we study the dynamics of the low-energy modes. By explicitly integrating out the high-energy modes, the one-loop correction to the Hamiltonian for this problem is obtained. After imposing the $\theta$ dependence through boundary conditions in configuration space, we obtain the glueball spectrum of the effective theory with a variational method.

1. INTRODUCTION

It is our goal to study the influence of the structure of the configuration space on the SU(2) glueball spectrum. The method we use consists of putting the theory in a finite spatial volume and reducing the full dynamics to an effective theory of a finite number of degrees of freedom. Asymptotic freedom implies that a small volume corresponds to a small coupling constant. Hence changing the volume from small to intermediate values allows us to monitor the onset of non-perturbative phenomena.

The Yang-Mills configuration space is the space of gauge orbits $A/G$ ($A$ is the collection of gauge fields or connections, $G$ the group of local gauge transformations). We know from Singer that the topology of this configuration space is highly non-trivial when $G$ is non-abelian. At increasing coupling, the wave functional will start to spread out and will become sensitive to the non-trivial topology, like non-contractable closed loops, of the configuration space.

This spreading out will occur first in those directions in configuration space, where the potential rises the slowest, like in the direction of the low-energy modes of the gauge field, or in the direction of the sphalerons associated to tunnelling from one Gribov copy of the vacuum to another. As long as the global features of the configuration space only affects this finite number of modes, one can capture these non-perturbative phenomena in an effective model of the low-energy modes. One uses the Born-Oppenheimer approximation to account for the influence of all the other (high-energy) modes, that is, they are assumed to behave perturbatively and are integrated out from the path integral.

We are interested in the influence of the multiple vacuum structure of the theory on the glueball spectrum; in particular we would like to see the dependence of the energies on the $\theta$ angle. The $\theta$ angle shows up when one implements gauge invariance under large gauge transformations. If $n[g]$ is the winding number of the gauge transformation $g$, gauge invariance in the Hamiltonian formulation is implemented by

$$\Psi[^nA] = e^{in[g]\theta} \Psi[A].$$

We impose gauge invariance by restricting the theory to a so-called fundamental domain. Here the $\theta$ angle shows up in the boundary conditions that have to be imposed at the boundary of the fundamental domain.

To compare results with lattice calculations it would be most natural to take the finite (spatial) volume to be a 3-torus $T^3$. On the torus, however, the instantons, which are the gauge field configurations that describe tunnelling between different vacua, are only known numerically. To circumvent this problem we take our space to be the three-sphere $S^3$, where the instantons are exactly known. The sphaleron degrees of freedom for this geometry happen to be embedded in the space of the low-energy modes.
2. THE EFFECTIVE THEORY

To isolate the low-energy modes, we examine the potential energy and the quadratic fluctuation operator $\mathcal{M}$ defined by

$$V(A) = -\frac{1}{2\pi^2} \int_{S^3} \frac{1}{2} \text{tr}(F_{ij}^2)$$

$$= -\frac{1}{2\pi^2} \int_{S^3} \text{tr}(A_i M_{ij} A_j) + \mathcal{O}(A^3).$$

The space of low-energy modes is the lowest eigenspace of $\mathcal{M}$. It is 18 dimensional and can be parametrized by

$$A_\mu(c,d) = (c^a_i e_\mu^i + d^a_i \bar{e}_\mu^i) \frac{\sigma_a}{2},$$

with $\sigma_a = i\tau_a$, $\tau_a$ the Pauli matrices, and $e_\mu^i$ and $\bar{e}_\mu^i$ certain framings on $S^3$. We now have to restrict ourselves to the cross section of this space with the fundamental domain $\Lambda$ and then solve for the lowest eigenfunctions of the hamiltonian. To make this problem well-defined we have to specify boundary conditions. To keep things transparent we now focus on a subspace of the 18-dimensional space. This two-dimensional space of field configurations (see fig. 1) contains three copies of the vacuum (large dots), which are related by large gauge transformations. This space is important because of all the tunnelling paths connecting the vacua, it contains those paths that have the lowest energy barrier. These barrier configurations are saddle points of $V$ and are called sphalerons (small dots); they are also gauge copies of each other. At the regime we start to see the non-perturbative effects, that is, where energies become of the order of the sphaleron energy, the only relevant boundary conditions are those at the sphalerons:

$$\Psi(A(1,0)) = e^{i\theta} \Psi(A(0,1)).$$

The lowest order (or truncated) hamiltonian becomes:

$$-\frac{f}{2} \left( \frac{\partial^2}{\partial c_\mu^a \partial c_\mu^a} + \frac{\partial^2}{\partial d_\mu^a \partial d_\mu^a} \right) + \frac{1}{f} \frac{V(c,d)}{2\pi^2},$$

with $f = \frac{g(R)^2}{2\pi^2}$ the renormalized coupling constant.

The potential $V(c,d)$ for the truncated hamiltonian is obtained directly from eq. (2). The one-loop correction to the hamiltonian is obtained by explicitly integrating out the high-energy modes, that is, the non-$(c,d)$ modes in the path integral. Using background gauge fixing, $D(B(c,d))_\mu Q_\mu = 0$, we find for the effective euclidean action:

$$-S_{\text{Eff}}^E[B] = -S_{\text{Eff}}^E[B] - S_{\text{Eff}}^E(B),$$

with the one-loop contribution $-S_{\text{Eff}}^E(B)$ given by

$$\ln \int D\psi D\bar{\psi} DQ_\mu \exp \left[ \int d\tau \int_{S^3} d\vec{x} \times \right.$$}

$$\left. \text{tr} \left( \bar{\psi}(-D_\mu^2(B))\psi + Q_\mu W_{\mu\nu}(B)Q_\nu \right) \right] =$$

$$\ln \left[ \det(-D_\mu^2(B))/\det\left(\frac{1}{2}(W_{\mu\nu}(B)) \right) \right].$$

Figure 1. The $(u,v)$ plane: $c_\mu^a = -u\delta_\mu^a$, $d_\mu^a = -v\delta_\mu^a$. Location of the classical vacua, sphalerons, lines of equal potential. The boundary of the fundamental domain lies between the Gribov horizon (fat sections) and the lower bound $\tilde{\Lambda}$ (drawn parabola): $\tilde{\Lambda} \subset \Lambda \subset \Omega$, with $\Omega$ the Gribov region.
Here we introduced ghost fields $\psi$ and $\bar{\psi}$, and $W(B)$ denotes the inverse field propagator in the background field $B$.

3. THE RAYLEIGH-RITZ ANALYSIS

We approximate the excitation energies of the effective model, which are the masses of the various glueballs, with a variational method. We introduce the radial coordinates

\[ r_c = \left[c_i^a c_i^a \right]^{\frac{1}{2}}, \quad r_d = \left[d_i^a d_i^a \right]^{\frac{1}{2}}. \tag{8} \]

We rewrite the Hamiltonian of eq. (5) in radial and angular coordinates. Remembering the remarks above on the relevance of the boundary conditions, we can implement them through (see [9] for details)

\[ \psi(Sph, 0) = e^{i\theta} \psi(0, Sph) \]
\[ \frac{\partial (r_c \psi)}{\partial r_c} (Sph, 0) = -e^{i\theta} \frac{\partial (r_d \psi)}{\partial r_d} (0, Sph) \tag{9} \]

The trial wave functions we use for the Rayleigh-Ritz method [11] are essentially the eigenfunctions of the kinetic part of the Hamiltonian. The rotational symmetry is used to classify the various glueball states.

4. RESULTS

Results for the mass of the first scalar and tensor glueball are shown in fig. 2. For small coupling these masses can be calculated perturbatively, but at $f \approx 0.2$ we see the onset of the influence of the boundary conditions, i.e. of the instantons. Beyond $f \approx 0.3$, our model will no longer be valid, because the wave function will no longer be sensitive to just the boundary conditions at the sphaleron, as can be checked explicitly using plots of the wave function. More results, including the dependence on the $\theta$ angle, can be found in [9,10].

It is important to emphasize one should not expect our results for the spectrum to be accurate for large volumes, but it has been the main aim of this study to demonstrate that instanton effects on the low-lying spectrum are large, but calculable as long as energies remain close to the sphaleron energy, where nevertheless semiclassical techniques will completely fail.

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