HOLOGRAPHIC WEYL ANOMALY FOR GJMS OPERATORS:
ONE LAPLACIAN TO RULE THEM ALL

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ABSTRACT. The holographic Weyl anomaly for GJMS operators (or conformal powers of the Laplacian) are obtained in four and six dimensions. In the context of AdS/CFT correspondence, free conformal scalars with higher-derivative kinetic operators are induced by an ordinary second-derivative massive bulk scalar. At one-loop quantum level, the duality dictionary for partition functions entails an equality between the functional determinants of the corresponding kinetic operators and, in particular, it provides a holographic route to their Weyl anomalies. The heat kernel of a single bulk massive scalar field encodes the Weyl anomaly (type-A and type-B) coefficients for the whole tower of GJMS operators whenever they exist, as in the case of Einstein manifolds where they factorize into product of Laplacians.

While a holographic derivation of the type-A Weyl anomaly was already worked out some years back, in this note we compute holographically (for the first time to the best of our knowledge) the type-B Weyl anomaly for the whole family of GJMS operators in four and six dimensions. There are two key ingredients that enable this novel holographic derivation that would be quite a daunting task otherwise: (i) a simple prescription for obtaining the holographic Weyl anomaly for higher-curvature gravities, previously found by the authors, that allows to read off directly the anomaly coefficients from the bulk action; and (ii) an implied WKB-exactness, after resummation, of the heat kernel for the massive scalar on a Poincaré-Einstein bulk metric with an Einstein metric on its conformal infinity.

The holographically computed Weyl anomaly coefficients are explicitly verified on the boundary by exploiting the factorization of GJMS operators on Einstein manifolds and working out the relevant heat kernel coefficient.

1. INTRODUCTION

Conformal powers of the Laplacian $P_{2k}$ (or GJMS operators for short [1]) are higher-derivative generalizations of the conformal Laplacian or Yamabe operator of the form

\[ P_{2k} = \Delta^k + \text{LOT} \]

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with principal part given by an integer power of the Laplacian and complemented by lower order (in derivative) terms (LOT) built up out of the Ricci tensor and covariant derivatives. They first arose within the general Fefferman-Graham program \cite{2} induced by the $k$-th power of the ambient Laplacian $\tilde{\Delta}^k$ and allowed Branson’s characterization of the Q-curvature in general even dimensions as given by their zeroth order term $\tilde{\Delta}^k$ \cite{5, 6}.

In the alternative Fefferman-Graham formulation where the ambient metric is traded by a Poincaré-Einstein metric in one dimension lower, the conformal structures are realized on the conformal boundary at infinity. This latter approach, that provides geometric roots for the celebrated AdS/CFT correspondence in physics \cite{7, 8, 9}, leads to a description of GJMS operators as residues of the scattering operator (aka two-point correlation function in CFT phraseology) as established by Graham and Zworski \cite{10}. The (critical) Q-curvature also arises in this context in connection with the volume asymptotics of the Poincaré-Einstein metric. When the dimensionality of the conformal boundary is odd, the renormalized volume is related to the bulk integral of the Q-curvature via the Chern-Gauss-Bonnet formula \cite{11, 12, 13}; when the dimensionality of the conformal boundary is even, in turn, the boundary integral of the Q-curvature is the volume anomaly or, equivalently, the renormalized volume is the conformal primitive of the Q-curvature \cite{10, 14, 17}.

Now, it was in the study of functional determinants of conformally invariant differential operators, such as the GJMS operators, where the Q-curvature made its first appearance \cite{18}. The infinitesimal variation of the determinant under a conformal (or Weyl) rescaling of the metric reveals the conformal (or Weyl or trace) anomaly; whereas the corresponding finite variation, i.e. its conformal primitive, leads to generalized Polyakov formulas \cite{19}. The Q-curvature arose in this context as a particular combination of local curvature invariants with a linear transformation law under conformal rescaling of the metric, playing the analog role of the Gaussian curvature on surfaces. Graham \cite{17} already noticed that the conformal invariance properties of the renormalized volume of a Poincaré-Einstein metric are reminiscent of those for the functional determinants of conformally invariant differential operators, e.g. conformal Laplacian and higher-order GJMS operators, being conformal invariant in odd dimensions but having an anomaly in even dimensions and, on the other hand, those for the volume anomaly are similar to those for the constant term in the expansion of the integrated heat kernel for the conformally invariant differential operator, which vanishes in odd dimensions but in even dimensions is a conformal invariant obtained by integrating a local expression in curvature, namely the conformal anomaly.

Remarkably, a ‘holographic formula’ stemming from AdS/CFT heuristic\footnote{For recent results on recursive relations and explicit construction of GJMS operators and the associated Q-curvatures, we refer to the works \cite{3, 4} and references therein.} provided a direct link between the renormalized volume of the (d+1)-dimensional

\footnote{The AdS/CFT correspondence certainly predicted the matching of the volume anomaly with the combined conformal anomalies for the free scalars, spinors, and 1-form that enter the four-dimensional vector multiplet of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory at leading large $N$, as confirmed in \cite{12, 13}. But this connection is somewhat indirect, it relies on non-renormalization theorems of the supersymmetric boundary CFT. In fact, in six dimensions the matching for the free superconformal $\mathcal{N} = (2, 0)$ tensor multiplet is only achieved for the type-B content \cite{13} of the Q-curvature, the type-A central charge $a$ is not protected by the supersymmetry so that the combined anomalies do not add up to reproduce the Q-curvature.}
bulk Poincaré-Einstein metric and functional determinants on the d-dimensional conformal boundary

\[
\frac{\det_-[-\nabla^2 + m^2]}{\det_+[-\nabla^2 + m^2]} \bigg|_{\text{bulk}} = \det \langle O_\lambda O_\lambda \rangle \bigg|_{\text{bndry}}
\]

The bulk side contains the one-loop effective action for a massive scalar computed with the resolvent and spectral parameter \( \lambda_+ = \frac{d}{2} + \nu \) and its analytic continuation to \( \lambda_- = \frac{d}{2} - \nu \). The boundary counterpart contains the functional determinant of the two-point function of the dual boundary operator \( O_\lambda \), a nonlocal integral kernel corresponding to the scattering operator for the radial propagation in the bulk interior. The relation between bulk mass of the scalar field and boundary scaling dimension is, according to the AdS/CFT dictionary, given by

\[
m^2 = -\frac{d^2}{4} + \nu^2.
\]

The formula originated in an attempt to compute an \( O(1) \) correction to the partition function under the renormalization group (RG) flow triggered by a boundary double-trace deformation [20, 21, 22, 23]. The residues of the scattering operator at its poles become conformally invariant differential operators that in the case of the bulk massive scalar field \( (\nu \to k, \ k = 1, 2, 3, \ldots) \) correspond to the family of GJMS operators \( P_{2k} \)

\[
\frac{\det_-[-\nabla^2 - \frac{d^2}{4} + k^2]}{\det_+[-\nabla^2 - \frac{d^2}{4} + k^2]} \bigg|_{\text{bulk}} = \det P_{2k} \bigg|_{\text{bndry}}
\]

In the conformal class of round metrics on the spheres, the similarities noticed before get promoted to a full-fledged equality because on the bulk side the volume of Euclidean AdS (or hyperbolic space) factorizes in the effective action due to its homogeneity. In this way, for even \( d \), a Polyakov formula for the determinant of the GJMS operators was 'holographically' obtained [43] and, perhaps more importantly, the two chief roles of the Q-curvature were directly connected. In particular, a compact formula for the type-A Weyl anomaly coefficient was obtained from the bulk Green’s function (or resolvent) at coincident points.

A subsequent extension of this clean entry of the AdS/CFT dictionary beyond conformal flatness has remained stalled ever since. Two main obstacles become readily apparent. One is the absence of a viable holographic route to compute the type-B Weyl anomaly in higher-derivative gravities; this is to be contrasted with the simple prescription of evaluating the bulk action at the AdS background to obtain the type-A Weyl anomaly [45]. Second, powers of the Weyl tensor and its derivatives will appear in the heat kernel coefficients to all orders; this is again to be contrasted with the well-known WKB-exactness of the heat kernel in the AdS background [46, 47, 48] that leaves only the first few terms after resummation.

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3Further extensions of the holographic formula to fields other than the scalar and to quotients of AdS have been studied ever since [24]-[41].

4Quotients of AdS, like thermal AdS for example, allow explicit results in terms of Patterson-Selberg zeta functions. In odd dimensions, these examples were also reported in the conformal geometry literature [42].

5This holographically derived formula for the central charge \( a \) was verified later on by using the more standard zeta function regularization combined with Branson’s factorization of GJMS operators on the round spheres [43].
It is the aim of this note to show how these difficulties can be overcome and to present a holographic derivation of both type-A and type-B Weyl anomaly coefficients for the whole family of GJMS operators in four and six dimensions. We start in Section 2 by first going to a generic compact Einstein manifold on the boundary, exploiting the factorization of GJMS operators into Laplacians, and computing the constant term of their heat kernel expansion in four and six dimensions so as to have the Weyl anomaly beforehand. Section 3 is devoted to the main contribution of this paper, namely the holographic derivation of the Weyl anomaly by considering the heat kernel of the bulk scalar in the corresponding bulk Poincaré-Einstein metric and the resummation that must occur in order to meet the (by now expected) central charges. In the conclusion, Section 4, we summarize and discuss our results. In Appendix A we provide more details about the WKB-exactness and the resummation properties of the bulk scalar heat kernel on the relevant Poincaré-Einstein metric.

2. Weyl anomaly for GJMS: take I

Let us start by examining the GJMS operators on an even d-dimensional compact manifold where the very existence of the “supercritical” ones, i.e. \( P_{2k} \) with \( k > d/2 \), is not granted in general. Even if they exist, as in the case of Einstein manifolds, their higher-derivative nature precludes the use of standard heat kernel methods. In the conformal class of round spheres, nevertheless, Branson’s factorization of GJMS operators into product of Laplacians [6] comes to rescue and the type-A Weyl anomaly coefficient can be worked out either by adding the constant terms of the heat expansion for the individual Laplacians or by zeta function regularization [44].

In going beyond the conformally flat class of round metrics on the spheres, as required to access the type-B Weyl anomaly, the leap forward we need is facilitated by Gover’s remarkable extension of the factorization of GJMS operators to the more general case of Einstein manifolds [49].

\[
P_{2k} = \prod_{i=0}^{k-1} \left[ -\nabla^2 + \frac{(d+2i)(d-2i-2)}{4(d-1)} R \right]
\]

starting \((i = 0)\) with the conformal Laplacian or Yamabe operator

\[
Y = -\nabla^2 + \frac{d-2}{4(d-1)} R
\]

The contribution of each Laplacian to the functional determinant, and to the anomaly, can then be computed with standard heat kernel techniques. In addition, as it has already been noticed and successfully put into use [50] [51] [52], although the Einstein condition brings in many simplifications, the curvature invariants that enter the type-B Weyl anomaly remain independent and their coefficients can be efficiently obtained by this shortcut route.
2.1. **Factorization and heat kernel at 4D: two birds, one stone.**

As explained before, a direct way to work out the Weyl anomaly for the GJMS operators is to exploit their factorization on a generic compact Einstein manifold, look for the relevant heat kernel coefficient for each individual factor and then add them all. We will need then the $b_4$ heat coefficient for each of the “shifted Laplacians” in the product

\[ P_{2k} = \prod_{i=0}^{k-1} \left( -\nabla^2 + \frac{(2+i)(1-i)}{12} R \right) \]

Each shifted Laplacian has the form $-\nabla^2 - E$, where $E$ is an endomorphism (see e.g. [53] for details) and it is straightforward to get the heat coefficient restricted to the Einstein metric

\[ b_4^{(i)} = \left( \frac{i^2(i+1)^2}{288} - \frac{1}{2160} \right) R^2 + \frac{1}{180} W^2 \]

Now we simply have to add up the contributions of the individual Laplacians to get the Weyl anomaly for the 4D GJMS operators

\[ \mathcal{A}_4[P_{2k}] = \sum_{i=0}^{k-1} b_4^{(i)} = \left( \frac{k^5}{240} - \frac{k^3}{144} \right) \frac{R^2}{6} + \frac{k}{180} W^2 \]

Then, regarding the Weyl anomaly basis in 4D, one can trade the Euler density $E_4$ by the Q-curvature $Q_4$ (type-A) and maintain the Weyl tensor squared $W^2 \equiv W_{abcd} W^{abcd}$ which is the obvious independent Weyl-invariant local curvature combination (type-B). The full information on $a$ and $c$ can be gained at one go\footnote{This is a slightly more efficient way than the usual trick (see, e.g. [54]) that restricts first to the round sphere for computing $a$ and then to a Ricci-flat manifold for computing $c-a$.} by considering the generic Einstein metric $g_E$, since then the Q-curvature reduces to a multiple of the Ricci scalar squared, $Q_4 = R^2/24$, and the Weyl tensor-squared remains unchanged; therefore we have the following rewriting [50]

\[ \mathcal{A}_4 = -a E_4 + c W^2 \]

\[ = -4a Q_4 + (c-a) W^2 \]

\[ = -a R^2/6 + (c-a) W^2 \]

Comparing the above relation with the accumulated heat coefficient of the “shifted Laplacians”, we finally obtain the Weyl anomaly coefficients for the whole GJMS family in 4D
Two remarks are worth mentioning here. First, the quintic polynomial $a_k$ follows as well from the generic expression found in [43] and corroborated by explicit zeta regularization in [44]. Second, only the shifted type-B anomaly coefficient turns out to be linear in $k$ and, in consequence, meets the holographic expectation of [56, 57] on Ricci-flat backgrounds.

2.2. Factorization and heat kernel at 6D: four birds, one stone.

In 6D, we follow the same procedure as in 4D. The factorization of the GJMS operators in terms of “shifted Laplacians” is now given by

$$P_{2k} = \prod_{i=0}^{k-1} \left( -\nabla^2 + \frac{(3+i)(2-i)}{30} R \right)$$

The endomorphism term is $E = -\frac{(3+i)(2-i)}{30} R$ and we denote $d_i = \frac{(3+i)(2-i)}{30}$. The relevant heat-kernel coefficient of the individual Laplacians can be worked out (see e.g. [53]) and the raw result on a 6D Einstein metric, modulo trivial total derivatives, reads

$$b_6^{(i)} = \frac{d_i^3}{6} R^3 + \frac{d_i^2}{12} R^3 - d_i \left( \frac{1}{180} R_{\text{Riem}}^2 - \frac{1}{180} R_{\text{Ric}}^2 + \frac{1}{72} R^3 \right)$$

$$+ \frac{1}{7!} \left( -3|\nabla R_{\text{Riem}}|^2 + \frac{44}{9} R_{\text{Riem}}^3 - \frac{80}{9} R_{\text{Ric}}^3 + \frac{16}{3} R_{\text{Ric}} R_{\text{Riem}}^2 \right.$$

$$+ \frac{14}{3} R R_{\text{Riem}}^2 - \frac{8}{3} R_{\text{Riem}} R_{\text{Ric}} + \frac{8}{9} R_{\text{Ric}}^3 - \frac{14}{3} R R_{\text{Ric}}^2 + \frac{35}{9} R^3 \bigg)$$

On the Einstein metric there is a lot of simplifications: the Cotton tensor, the Bach tensor and the traceless part of the Ricci tensor all vanish. Nonetheless, the type-A and the three type-B terms remain independent [50]. We keep a generic 6D Einstein boundary metric $g_E$ so that the Einstein condition reduces the Q-curvature to a multiple of the Ricci scalar cubed, $Q_6 = R^3/225$; the two cubic contractions of the Weyl tensor, denoted by $I_1 = W_{\text{Riem}}^2$ and $I_2 = W_{\text{Ric}}^3$, remain unchanged; while the third Weyl invariant reduces to $I_3 = W \nabla^2 W - \frac{8}{15} R W^2$ modulo the trivial

\footnote{For notation and conventions we refer to [50].}
total derivative $\frac{3}{2} \nabla^2 W^2$ (see e.g. [64]) that we omit in what follows. The 6D Weyl anomaly can then be casted in the following convenient form

\begin{equation}
\mathcal{A}_6 = -a E_6 + c_1 I_1 + c_2 I_2 + c_3 I_3
\end{equation}

\begin{equation}
= -48 a Q_6 + (c_1 - 96a)I_1 + (c_2 - 24a)I_2 + (c_3 + 8a)I_3
\end{equation}

\begin{equation}
= -16 a R^3/75 + (c_1 - 96a)I_1 + (c_2 - 24a)I_2 + (c_3 + 8a)I_3
\end{equation}

| Curvature invariant | $Q_6 = R^3/225$ | $I_1$ | $I_2$ | $I_3$ |
|---------------------|-----------------|------|------|------|
| $A_{10}$            | $R^3$           | 225  | -    | -    |
| $A_{11}$            | $RRic^2$        | 75/2 | -    | -    |
| $A_{12}$            | $RRiem^2$       | 15   | 20   | -5   | -5   |
| $A_{13}$            | $Riem^3$        | 25/4 | -    | -    |
| $A_{14}$            | $Riem Ric^2$    | 25/4 | -    | -    |
| $A_{15}$            | $Ric Riem^2$    | 5/2  | 10/3 | -5/6 | -5/6 |
| $A_{16}$            | $Riem^4$        | 1    | 4    | 0    | -1   |
| $A_{17}$            | $-Riem^3$       | 1    | -2   | 1/4  | 1/4  |
| $A_5$               | $|\nabla Riem|^2$| -    | -32/3| 8/3  | 5/3  |

Making use of the table above to go to the standard anomaly basis and adding up the heat coefficients of the individual Laplacians (tedious but straightforward) we end up with

\begin{equation}
7! A_6[P_{2k}] = 7! \sum_{i=0}^{k-1} b^{(i)}_6
\end{equation}

\begin{equation}
= -\frac{16}{75} \left( -3k^7 + 21k^5 - 28k^3 \right) R^3
\end{equation}

\begin{equation}
+ \frac{14(k^3 - k)}{9} (4I_1 - I_2 - I_3) - \frac{k}{9} (24I_1 - 30I_2 - 13I_3)
\end{equation}

From this expression for the accumulated heat coefficients for the shifted Laplacians we finally read off the 6D Weyl anomaly for the whole GJMS tower

\begin{equation}
7! a_k = -\frac{3k^7 - 21k^5 + 28k^3}{144}
\end{equation}

\begin{equation}
7!(c_{1,k} - 96a_k) = \frac{8}{9} k(7k^2 - 10)
\end{equation}
3. Weyl anomaly for GJMS: take II

Let us now turn to our main thrust and try to elucidate the way in which the information on the Weyl anomaly is encoded in the “hologram”, namely the bulk massive scalar. We proceed in two steps. First, we consider the holographic formula for a bulk Poincaré-Einstein metric with the Einstein metric of before on the boundary conformal class, following the prescription put forward in [50] that allows to read off the Weyl anomaly coefficient in higher-curvature gravities.

\[ \hat{g}_{PE} = dx^2 + (1 - \lambda x^2)^2 g_E \]

with \( \lambda = \frac{R}{2(d-1)} \) proportional to the boundary Ricci scalar.

At first sight this seems to be of little help because the heat kernel coefficients, in particular those depending on the nonvanishing Weyl tensor, will be present to all orders so that there will be infinitely many higher-curvature terms in the bulk one-loop effective action.

In a second step, and despite the above caveat, we compute the Weyl content of the first few heat coefficients. With this partial information at hand and under the crucial assumption of WKB-exactness after resummation, we are able to correctly reproduce the Weyl anomaly coefficients for the whole tower of GJMS in four and in six dimensions, as explained in what follows.

3.1. Holographic derivation from 5 to 4 dims.

We consider therefore the holographic formula in the above Poincaré-Einstein metric on the bulk and the corresponding generic compact Einstein metric on the boundary.

\[ \left. \frac{Z_{\text{MS}}^{(-)}}{Z_{\text{MS}}^{(+)}} \right|_{PE} = \left. Z_{\text{GJMS}} \right|_E \]
with the bulk one-loop effective action given by the functional determinants of the massive scalar field:

\[
Z_{\text{MS}}^{(+)} \bigg|_{P,E} = \left[ \det \left\{ -\hat{\nabla}^2 + m_k^2 \right\} \right]^{-1/2}
\]

(21)

We first recall the WKB-exact heat expansion in $\text{AdS}_5$ [46, 47, 48]. Although there are infinitely many heat kernel coefficients, after factorization of the exponential factor $e^{-k^2 t}$ only the first two remain in five dimensions

\[
\text{tr} e^{(\Psi^2 - k^2 + 4)t} \bigg|_{\text{AdS}_5} = \frac{1 + \frac{2t}{3} e^{-k^2 t}}{(4\pi t)^{5/2}}
\]

(22)

massive scalar $m_k^2 = k^2 - 4$:

\[
\text{We need now to depart from } \text{AdS}_5 \text{ and determine the pure-Weyl content of the heat kernel on the Poincaré-Einstein metric. The first contribution arises with } \hat{b}_4
\]

\[
\hat{b}_4 \sim \frac{1}{180} \hat{W}^2
\]

(23)

The relevant terms in the next heat coefficient $\hat{b}_6$ are the following

\[
\hat{b}_6 \sim \frac{1}{7!} \left( -3|\hat{\nabla}\hat{\text{Riem}}|^2 + \frac{44}{9} \hat{\text{Riem}}^3 - \frac{80}{9} \hat{\text{Riem}}^3 - \frac{16}{3} \hat{\text{Ric}}\hat{\text{Riem}}^2 
\]

\[
+ \frac{14}{3} \hat{\text{Riem}}^2 - \frac{8}{3} \hat{\text{Riem}}\hat{\text{Ric}}^2 + \frac{8}{9} \hat{\text{Ric}}^3 - \frac{14}{3} \hat{\text{Ric}}^2 + \frac{35}{9} \hat{\Phi}^3 \right)
\]

(24)

We now follow the prescription of [50] and go to the particular basis of Weyl invariants given by two independent cubic contractions, $\hat{W}^3$ and $\hat{W}^{r3}$, and the third one given by the 5D Fefferman-Graham invariant $\hat{\Phi}_5 = |\nabla \hat{W}|^2 - 8\hat{W}^2$

\[
\hat{b}_6 \sim \frac{1}{45} \hat{W}^2 - \frac{1}{7!} \left( \frac{80}{9} \hat{W}^{r3} - \frac{44}{9} \hat{W}^3 + 3 \hat{\Phi}_5 \right)
\]

(25)

We tabulate the dictionary below for convenience.

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8From now on we denote bulk quantities with a hat to distinguish from the corresponding boundary ones.

9The merit of our special basis of curvature invariants is to unveil the direct relation between bulk and boundary Weyl invariants, but of course the contribution of each term of the $A$-basis has been worked out by other routes in the literature, see e.g. [60, 61, 62] and references therein.
After the dust has settled, we realize then that the $-1/45 \hat{W}^2$ in $\hat{b}_6$ can be absorbed by the $e^{-4t}$ factor that makes the resummation of the pure-Ricci terms and results in the well-known WKB-exactness of the heat kernel expansion in odd-dimensional hyperbolic space. The remaining Weyl invariant terms in $\hat{b}_6$ do not contribute to the holographic anomaly. Assuming that this WKB-exactness extends to the $\hat{W}^2$ term, the contribution of the one-loop effective Lagrangian of the massive bulk scalar to the holographic Weyl anomaly comes exclusively from the following combination of pure-Ricci (numbers since we set the radius of the asymptotic hyperbolic metric to unity) and pure-Weyl pieces

$$\int_{0}^{\infty} \frac{dt}{t^{7/2}} e^{-k^2 t} \left\{ 1 + \frac{2}{3} t + \frac{1}{180} \hat{W}^2 t^2 + ... \right\}$$

(26)

where the ellipsis stands for higher curvature pure-Weyl invariants that do not contribute to the 4D holographic Weyl anomaly. After proper time integration we obtain for the one-loop effective Lagrangian (modulo an overall normalization factor that can be easily worked out)

$$L_{1\text{-loop}}^{(\text{GJMS})} = \frac{4}{3} \left( \frac{k^5}{5} - \frac{k^3}{3} \right) \cdot 1 + \frac{k}{180} \hat{W}^2 + ...$$

(27)

The holographic recipe [50] tells us then how to read the anomaly: the volume part (pure-Ricci) $1$ ‘descends’ to the 4D Q-curvature and the pure-Weyl quadratic contraction of the 5D Weyl tensor ‘descends’ to the analog contraction of the 4D Weyl tensor. In all, the holographic Weyl anomaly one reads off is simply given by

$$A_4[P_{2k}] = -4 \left( \frac{k^3}{144} - \frac{k^5}{240} \right) Q_4 + \frac{k}{180} \hat{W}^2$$

(28)

in perfect and remarkable agreement with the boundary computation (eqn 8).

#### 3.2. Holographic derivation from 7 to 6.

We move on now to seven dimensions. The WKB-exact heat expansion in $AdS_7$ [46, 47, 48] requires factorization of the exponential factor $e^{-9t}$ so that only the first three terms remain in seven dimensions
massive scalar \( m_k^2 = k^2 - 9 \):

\[
\left. \frac{\text{tr} \, e^{(\Phi^2 - k^2 + 9)t}}{AdS_7} \right|_{t} = \frac{1 + 2t + \frac{16}{15} t^2}{(4\pi t)^{5/2}} e^{-k^2 t}
\]

To depart from \( AdS_7 \) and the conformally flat class of bulk and boundary metrics, we consider the pure-Weyl content of the heat kernel on the bulk Poincaré-Einstein metric. The first nontrivial contribution arises again with \( \hat{b}_4 \)

\[
\hat{b}_4 \sim \frac{1}{180} \hat{W}^2
\]

The next contribution comes form the next heat coefficient \( \hat{b}_6 \)

\[
\hat{b}_6 \sim \frac{1}{7!} \left( -3|\nabla \hat{\text{Riem}}|^2 + \frac{44}{9} \hat{\text{Riem}}^3 - \frac{80}{9} \hat{\text{Riem}}^3 - \frac{16}{3} \hat{\text{Ric}} \hat{\text{Riem}}^2 
+ \frac{14}{3} \hat{\text{Riem}}^2 - \frac{8}{3} \hat{\text{Riem}} \hat{\text{Ric}}^2 + \frac{8}{9} \hat{\text{Ric}}^3 - \frac{14}{3} \hat{\text{Ric}}^2 + \frac{35}{9} \hat{R}^3 \right)
\]

The heat coefficients for the scalar Laplacian are universal in the sense that the number in front of each curvature invariant is independent of the dimensionality of the manifold. However, when following the prescription of [50] and going to the particular basis of Weyl invariants (see table below) given by two independent cubic contractions, \( \hat{W}^3 \) and \( \hat{W}^3' \), and the third one given now by the 7D Fefferman-Graham invariant \( \hat{\Phi}_7 = |\nabla \hat{W}|^2 - 8 \hat{W}^2 \), we obtain a different result

\[
\hat{b}_6 \sim \frac{1}{7!} \left( -\frac{1916}{9} \hat{W}^3' + \frac{503}{9} \hat{W}^3 - 54 \hat{\Phi}_7 \right)
\]

| Curvature invariant | \( \hat{W}^3 \) | \( \hat{W}^3' \) | \( \hat{\Phi}_7 \) |
|---------------------|----------------|----------------|----------------|
| \( A_{10} \) | \( \hat{R}^3 \) | - | - |
| \( A_{11} \) | \( \hat{R} \hat{\text{Ric}}^2 \) | - | - |
| \( A_{12} \) | \( \hat{R} \hat{\text{Riem}}^2 \) | -42 | 21/2 | -21/2 |
| \( A_{13} \) | \( \hat{\text{Ric}}^3 \) | - | - |
| \( A_{14} \) | \( \hat{\text{Riem}} \hat{\text{Ric}}^2 \) | - | - |
| \( A_{15} \) | \( \hat{\text{Riem}} \hat{\text{Riem}}^2 \) | -6 | 3/2 | -3/2 |
| \( A_{16} \) | \( \hat{\text{Riem}}^3 \) | -6 | 5/2 | -3/2 |
| \( A_{17} \) | \( -\hat{\text{Riem}}^3 \) | 1/2 | -3/8 | 3/8 |
| \( A_5 \) | \( |\nabla \hat{\text{Riem}}|^2 \) | 8 | -2 | 3 |

We now assume WKB-exactness after factorization of the \( e^{-9t} \) factor. The convolution with the exponential must absorb a \( -1/20 \hat{W}^2 \) contribution to \( \hat{b}_6 \), that in the 7D case can be rewritten in the Weyl basis \( \left[ \hat{W}^3, \hat{W}^3', \hat{\Phi}_7 \right] \). In fact, modulo a trivial total derivative \( \hat{W}^2 = \hat{W}^3' - \frac{1}{4} \hat{W}^3 + \frac{1}{4} \hat{\Phi}_7 \) on the Poincaré-Einstein metric.
So that we obtain, under the assumption of WKB-exactness, the following one-loop effective Lagrangian

\[
\int_0^\infty \frac{dt}{t^{9/2}} e^{-k^2 t} \left\{ 1 + \frac{2}{15} t^2 + \frac{1}{180} \hat{W}^2 t^2 + \frac{1}{7!} \left( \frac{352}{9} \hat{W}'^3 - \frac{64}{9} \hat{W}'^3 + 9 \hat{\Phi}_7 \right) t^3 + \ldots \right\}
\]

where again the ellipsis stands for higher-curvature terms in the Weyl tensor that do not contribute to the 6D holographic Weyl anomaly. After proper time integration we obtain for the one-loop effective Lagrangian (modulo an overall normalization factor)

\[
\mathcal{L}^{(\text{GJMS})}_{1\text{-loop}} = \frac{8}{315} \left( -3k^7 + 21k^5 - 28k^3 \right) \hat{1} - \frac{14k^3}{3 \cdot 7!} \left( 4 \hat{W}'^3 - \hat{W}'^3 + \hat{\Phi}_7 \right) + \frac{k}{9 \cdot 7!} \left( 352 \hat{W}'^3 - 64 \hat{W}'^3 + 81 \hat{\Phi}_7 \right) + \ldots
\]

Now, according to the holographic recipe [50], the holographic Weyl anomaly one reads off from the bulk effective Lagrangian is simply

\[
7! \mathcal{A}_6[P_{2k}] = -48 \frac{-3k^7 + 21k^5 - 28k^3}{144} Q_6 - \frac{14k^3}{3} (4I_1 - I_2 + \Phi_6) + \frac{k}{9} (352I_1 - 64I_2 + 81\Phi_6)
\]

We finally go to the standard basis of 6D Weyl invariants \([I_1, I_2, I_3]\) by use of the dictionary \(3\Phi_6 = I_3 - 16I_1 + 4I_2\)

\[
7! \mathcal{A}_6[P_{2k}] = -48 \frac{-3k^7 + 21k^5 - 28k^3}{144} Q_6 + \frac{14k^3}{9} (4I_1 - I_2 - I_3) - \frac{k}{9} (80I_1 - 44I_2 - 27I_3)
\]

and get perfect agreement with the outcome of the boundary computation (eqn. [15]).
4. Conclusion

We have shown the way one bulk Laplacian rules the whole family of boundary GJMS operators and, in particular, the way the conformal anomaly is encoded in the bulk heat kernel. Clearly, the alleged WKB-exactness of the bulk scalar heat kernel on the Poincaré-Einstein metric deserves further analysis and an independent confirmation thereof would be desirable. The boundary computation of the anomaly was facilitated by the factorization of the GJMS operator on a generic Einstein manifold and by the fact that the Einstein condition, besides the many simplifications, does not spoil the independence of the curvature invariants that enter the type-A and type-B Weyl anomaly.

It would be interesting to explore the connection between the one-loop information encoded in the present holographic formula and one-loop Witten diagrams (see e.g. [63]). For example, one- and two-point correlators of the boundary stress tensor computed from graphs with one and two graviton legs, respectively, with the bulk scalar running in the loop ought to render the \( a \) and the \( c_T \) central charges.\(^{10}\)

One subtle feature of the present computation that we leave as a future direction to look into consists in the following. There is an ambiguity in the construction of GJMS operators given by the addition of terms containing the Weyl tensor. For example, one can add to the Paneitz \( P_4 \) operator a constant times \( W^2 \) without changing its conformal properties. In the case of \( P_6 \) in 6D, besides any of the three Weyl invariants \( I_1, I_2 \) and \( I_3 \), there is also the freedom to add another term quadratic in the Weyl tensor and in covariant derivatives (see e.g. [64, 65]). These additional Weyl terms will certainly modify the conformal anomaly of the differential operators. The choice implied by the factorization on Einstein manifolds that we have made use of clearly distinguishes pure-Ricci GJMS with no additional term containing the Weyl tensor. It remains then to be elucidated the way in which the possible additional Weyl terms find their way into the holographic picture.

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\(^{10}\)The coefficient of the two-point function of the stress tensor \( c_T \) in 4D is proportional to the \( c \) central charge and in 6D, to \( c_3 \). In 6D one would need additional (three-point) correlators to disentangle the remaining \( (c_1 \text{ and } c_2) \) type-B Weyl anomaly coefficients.
Appendix A. WKB-exactness of the scalar Laplacian

In this appendix, we explicitly compute the first heat coefficients and illustrate the way they get rearranged after factorization of the exponential factor.

5D PE/E.

\[ \text{tr} e^{(\hat{\nabla}^2) t} \big|_{PE} = \frac{1}{(4\pi t)^{5/2}} \left\{ 1 - \frac{10}{3} t + \frac{16}{3} t^2 + \frac{1}{180} \hat{W}^2 t^2 ight. \]
\[ \left. - \frac{16}{3} t^3 - \frac{1}{45} \hat{W}^2 t^3 - \frac{1}{7!} \left( \frac{80}{9} \hat{W}^3 - \frac{44}{9} \hat{W}^3 + 3 \hat{\Phi}_5 \right) t^3 + \mathcal{O}(t^4) \right\} \]
\[ = \frac{e^{-4t}}{(4\pi t)^{5/2}} \left\{ 1 + \frac{2}{3} t + \frac{1}{180} \hat{W}^2 t^2 \right. \]
\[ \left. - \frac{1}{7!} \left( \frac{80}{9} \hat{W}^3 - \frac{44}{9} \hat{W}^3 + 3 \hat{\Phi}_5 \right) t^3 + \mathcal{O}(t^4) \right\} \]

7D PE/E.

\[ \text{tr} e^{(\hat{\nabla}^2) t} \big|_{PE} = \frac{1}{(4\pi t)^{7/2}} \left\{ 1 - 7 t + \frac{707}{30} t^2 + \frac{1}{180} \hat{W}^2 t^2 \right. \]
\[ \left. - \frac{501}{10} t^3 - \frac{1}{7!} \left( \frac{1916}{9} \hat{W}^3 - \frac{503}{9} \hat{W}^3 + 54 \hat{\Phi}_7 \right) t^3 + \mathcal{O}(t^4) \right\} \]
\[ = \frac{1}{(4\pi t)^{7/2}} \left\{ 1 - 7 t + \frac{707}{30} t^2 + \frac{1}{180} \hat{W}^2 t^2 \right. \]
\[ \left. - \frac{501}{10} t^3 - \frac{1}{20} \hat{W}^2 t^3 + \frac{1}{7!} \left( \frac{352}{9} \hat{W}^3 - \frac{64}{9} \hat{W}^3 + 9 \hat{\Phi}_7 \right) t^3 + \mathcal{O}(t^4) \right\} \]
\[ = \frac{e^{-9t}}{(4\pi t)^{7/2}} \left\{ 1 + \frac{2}{5} t + \frac{16}{15} t^2 + \frac{1}{180} \hat{W}^2 t^2 \right. \]
\[ \left. + \frac{1}{7!} \left( \frac{352}{9} \hat{W}^3 - \frac{64}{9} \hat{W}^3 + 9 \hat{\Phi}_7 \right) t^3 + \mathcal{O}(t^4) \right\} \]
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