Minor summation formula of hyperpfaffians and Selberg integrals

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\abstract

In a previous paper (J. Combin. Theory Ser. A, 120, 2013, 1263–1284) H. Tagawa and the two authors proposed an algebraic method to compute certain Pfaffians whose form resembles to Hankel determinants associated with moment sequences of the classical orthogonal polynomials. At the end of the paper they offered several conjectures. In this work we employ a completely different approach to evaluate this type of Pfaffians. The idea is to apply certain de Bruijn type formulas and to convert the evaluation of the Pfaffians to certain Selberg type integrals. This method works not only for Pfaffians but also for hyperpfaffians. Hence it enables us to establish much more generalized identities than those conjectured in the previous paper. We also investigate some Pfaffians related to classical $q$-orthogonal polynomials.

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1 Introduction

In [18] Tagawa and the two authors studied Pfaffians associated to moment sequences of certain classical orthogonal polynomials. In particular, using $LU$-type decomposition of skew-symmetric matrix, they proved the Pfaffian identity [18, Corollary 3.2]

$$\text{Pf} \left( (q^{i-1} - q^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)} q^{n(n-1)(4n+1)/3+n(n-1)r} \prod_{k=1}^{n} \frac{(aq; q)_{2k+r-1}(bq; q)_{2k-1}(q; q)_{2k-1}}{(abq^2; q)_{2(k+n)+r-3}},$$

(1.1)

where we use the standard notation for $q$-shifted factorial (see [2, 12]):

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}},$$

for any integer $n$. We frequently use the compact notation $(a_1, \ldots, a_r; q)_n = (a_1; q)_n \cdots (a_r; q)_n$, where $n$ is an integer or $\infty$.

Besides they formulated three conjectures [18, Conjecture 6.1, Conjecture 6.2 and Conjecture 6.3] concerning so-called Hankel Pfaffians. More precisely, Conjecture 6.1 is related to Hankel Pfaffians of moment sequence of the Al-Salam-Carlitz polynomials (see the first two identities of Theorem 6.1), Conjecture 6.2 is about Hankel Pfaffians of combinatorial numbers such as the Motzkin numbers, the central Delannoy numbers, the Schröder numbers and the Narayana numbers (see Corollary 4.2), and Conjecture 6.3 is on a Hankel Pfaffian involving a combinatorial sequence which comes from Tamm’s Hankel determinant [38] (see the first identity of Conjecture 7.1). A proof of Conjecture 6.2 was given in [17] by using Zeilberger’s Holonomic Ansatz and assisted by computer. In this paper we prove general hyperpfaffian identities of Narayana polynomials of Coxeter groups which include Conjecture 6.2 as special cases. The second proof of (1.1) was given in [15, Corollary 3.4], using a
quadratic formula for the basic hypergeometric series related to the Askey-Wilson polynomials. In this paper we give another proof of this identity by reducing the formula to the \( k = 2 \) case of the Askey-Habsieger-Kadell \( q \)-Selberg integral formula (5.18) via de Bruijn’s formula. We believe that our new proof gives a more simple method and essential insights to Pfaffians of Hankel type. We note that Conjecture 6.3 of [18] is still open.

In [8] de Bruijn presented two Pfaffian formulas, see [8, (4,7)] and [8, (7.3)]. Note that Luque and Thibon proved a hyperpfaffian version of de Bruijn’s second formula using a new minor summation formula [28, (87)] of hyperpfaffians, which generalizes the original formula of Pfaffians in [19]. Here we further generalize Luque and Thibon’s minor summation formula (see Theorem 2.7) to prove our hyperpfaffian generalizations, i.e., Theorem 3.1 and Theorem 3.2, of de Bruijn’s first and second Pfaffian formulas, where our second formula generalizes Luque and Thibon’s formula. In [29, Section 3] Luque and Thibon computed the hyperdeterminants of Catalan numbers and the central binomial coefficients, and in [30] they used hyperdeterminant calculations to prove the Selberg and Aomoto integrals. We shall give their Hankel Hyperpfaffian counterpart in this paper (see Section 4). In [33] Matsumoto studied Toeplitz hyperdeterminants and applied his theory to the Jack symmetric functions.

In this paper we work in general context of hyperpfaffians to maximize the power of the Selberg-Aomoto integrals. But for \( q \)-analogues we can work only on Pfaffian cases. Following Luque and Thibon [28, 29, 30], we say that an \( m \)-dimensional hypermatrix \( A = (A(i_1, \ldots, i_m))_{1 \leq i_1, \ldots, i_m, i_m \leq n} \) of size \( n \) is of Hankel type if \( A(i_1, \ldots, i_m) = f(i_1 + \cdots + i_m) \) holds for a certain function \( f \). Similarly we say that a skew-symmetric hypermatrix \( A \) is of Hankel type if \( A(i_1, \ldots, i_m) = \prod_{1 \leq k < l \leq m} (g(i_l) - g(i_k)) \cdot f(i_1 + \cdots + i_m) \) holds for certain functions \( f \) and \( g \). In most cases we take \( g(i) = i \) or \( q^i \). We call a hyperpfaffian of a skew-symmetric hypermatrix in such form Hankel hyperpfaffians.

Recall that Selberg’s beautiful integral formula [36] is

\[
S_n(\alpha, \beta, \gamma) = \int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1} (1 - t_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \, dt = \prod_{j=1}^{n} \frac{\Gamma(\alpha + (n-j)\gamma) \Gamma(\beta + (n-j)\gamma) \Gamma(j\gamma + 1)}{\Gamma(\alpha + \beta + (n+j-2)\gamma) \Gamma(\gamma + 1)},
\]

where \( dt = dt_1 \cdots dt_n \). A comprehensive account of the history and mathematics related to the Selberg integral is given in [11], and some discrete analogues of Selberg type integral with combinatorial applications are recently given in [7]. In [3] Aomoto proved a slightly more general integral formula:

\[
\int_{[0,1]^n} \left( \prod_{i=1}^{k} t_i \right) \prod_{i=1}^{n} t_i^{\alpha-1} (1 - t_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \, dt = S_n(\alpha, \beta, \gamma) \prod_{j=1}^{k} \frac{\alpha + (n-j)\gamma}{\alpha + \beta + (2n-j-1)\gamma},
\]

(1.3)
which implies

$$\int_{[0,1]^n} e_k(t) \prod_{i=1}^n t_i^{\alpha-1} (1-t_i)\beta^{-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \, dt = \binom{n}{k} S_n(\alpha, \beta, \gamma) \prod_{j=1}^k \frac{\alpha + (n-j)\gamma}{\alpha + \beta + (2n-j-1)\gamma},$$

(1.4)

where $e_k(t) = e_k(t_1, \ldots, t_n)$ stands for the elementary symmetric function which is defined by $\sum_{k=0}^n e_k(t) y^k = \prod_{i=1}^n (1 + t_i y)$.

This paper is composed as follows. In Section 2 we recall hyperdeterminants and give a slightly generalized definition of Barvinok’s original hyperpfaffian [6]. We prove a formula, Theorem 2.7, which is a hyperpfaffian-hyperdeterminant version of minor summation formula [19, 20]. In Section 3 we generalize two Pfaffian formulas of de Bruijn [8] to hyperpfaffians, i.e., Theorem 3.1 and Theorem 3.2, as an application of Theorem 2.7. In this paper Theorem 3.2 is more important, and we derive Corollary 3.4, which is used throughout this paper. Section 4 is devoted to the Hankel hyperpfaffians of the Narayana polynomials of the Coxeter groups as an application of Corollary 3.5 and the Selberg-Aomoto integrals, (1.2) and (1.3). We prove several remarkable identities which generalize the identities of Conjecture 6.2 in [18]. In Section 5 we prepare all the ingredients to evaluate $q$-Hankel Pfaffians, which will also be used in the next section, and give a new proof of (1.1), i.e., Theorem 5.1 ([18, Corollary 3.2], [15, Corollary 3.4]). In Section 6 we settle Conjecture 6.2 in [18] using the multivariate Al-Salam-Carlitz I and II polynomials [1, 9, 5]. In the last section we state more conjectures concerning Conjecture 6.3 in [18].

2 Minor summation for hyperpfaffians

The aim of this section is to give a hyperpfaffian-hyperdeterminant version of the minor summation formula [19, Theorem 1], [20, Theorem 3.2]. First we give a general form of hyperpfaffian-hyperdeterminant version in Theorem 2.7, and then derive corollaries which will be applied to obtain formulas linking the so-called Hankel hyperpfaffians to the Selberg type integrals in the next section. Some special cases of Theorem 2.7 are obtained in [28, (87)] and [32, Theorem 5.1]. The relation between the so-called Hankel Pfaffians and the Selberg integrals is fully understood if we generalize Pfaffian to hyperpfaffian. In fact the Selberg integral seems a very general formula and the application to evaluations of Pfaffians needs only a special case. If we pay attention to the hyperpfaffian, it will reveal a full view of the relation. However, more general arguments of hyperpfaffian may apparently look cumbersome. Hence this section will provide the complete proof of our theorems in a general form.

First of all, we fix some notation. Let $\{a_1, \ldots, a_k\} <$ denote the set $\{a_1, \ldots, a_k\} \subseteq \mathbb{R}$ where $a_1 < \cdots < a_k$. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ for any positive integer $n$. Let $\binom{S}{r}$ denote the set of all $r$-element subsets of $S \subseteq \mathbb{R}$. If $S = \{s_1, \ldots, s_n\} < \subseteq \mathbb{R}$ is an $n$-element set, then we use the symbol $S_I$ to denote the $r$-element subset $\{s_{i_1}, \ldots, s_{i_r}\} <$ for $I = \{i_1, \ldots, i_r\} \in \binom{[n]}{r}$. Hence we have $\binom{S}{r} = \left\{ S_I \mid I \in \binom{[n]}{r} \right\}$. More generally, let $l$ and $n$ be
positive integers and $S$ any $ln$-element set. Let $\binom{S}{I}$ ($l$ is repeated $n$ times in the lower line) denote the set of $n$-tuples $(S_1, \ldots, S_n)$ such that $S_j \in \binom{S}{i}$ for $1 \leq j \leq n$ and $S_1 \cup \cdots \cup S_n = S$ (disjoint union). In other word we have $\binom{S}{I} = \{(S_{I_1}, \ldots, S_{I_n}) \mid (I_1, \ldots, I_n) \in \binom{l^n}{n}\}.$

Let $\mathcal{S}_n$ be the set of permutations of $[n]$, which form the symmetric group of degree $n$. We use the one-line notation $\sigma = (\sigma(1), \ldots, \sigma(n))$ to denote an element $\sigma$ of $\mathcal{S}_n$. If $\sigma \in \mathcal{S}_n$, we split $\sigma$ into $n$ blocks of length $l$, i.e., $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_k = (\sigma((k-1)l + 1), \ldots, \sigma(kl))$ for $1 \leq k \leq n$. Let $\mathcal{S}_{ln}(l, \ldots, l)$ ($n$ times $l$) denote the subset of $\mathcal{S}_{ln}$ which is composed of the permutations $\sigma = (\sigma_1, \ldots, \sigma_n)$, in which the entries of each block are strictly increasing, i.e.,

$$\mathcal{S}_{ln}(l, \ldots, l) = \{ \sigma = (\sigma(1), \ldots, \sigma(ln)) \in \mathcal{S}_{ln} \mid \sigma((j-1)l + 1) < \cdots < \sigma(jl) \text{ for } 1 \leq j \leq n \}.$$

For example, we have

$$\mathcal{S}_{4}(2, 2) = \{(1, 2, 3, 4), (1, 3, 2, 4), (1, 4, 2, 3), (2, 3, 1, 4), (2, 4, 1, 3), (3, 4, 1, 2)\}.$$

Let $S \subseteq \mathbb{R}$ be an $ln$-element set. There is a natural bijection from $\mathcal{S}_{ln}(l, \ldots, l)$ to $\binom{S}{l, \ldots, l}$ defined by

$$\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{S}_{ln}(l, \ldots, l) \mapsto (S_{\sigma_1}, \ldots, S_{\sigma_1}) \in \binom{S}{l, \ldots, l}.$$

Hence we can identify an element $I \in \binom{S}{l, \ldots, l}$ with the corresponding permutation $\sigma \in \mathcal{S}_{ln}(l, \ldots, l)$, and use the symbol $\text{sgn} I$ to denote $\text{sgn} \sigma$. For example, if $l = n = 2$ and $S = \{1, 3, 5, 7\}$, then we have

$$\binom{S}{2, 2} = \{(\{1, 3\}, \{5, 7\}), (\{1, 5\}, \{3, 7\}), (\{1, 7\}, \{3, 5\}), (\{3, 5\}, \{1, 7\}), (\{3, 7\}, \{1, 5\}), (\{5, 7\}, \{1, 3\})\}.$$

The signs of those elements are $+, -, +, +, -$ and $+$, respectively.

We divide this section into three subsections. In Subsection 2.1 we recall the definition of hyperdeterminant, and prove some basic properties, Proposition 2.2, Proposition 2.3. Especially Lemma 2.4 is crucial to prove Theorem 2.7. Next, in Subsection 2.2, we introduce a new definition of hyperpfaffian, which is Barvinok’s original definition [6]. Lemma 2.6 is a hyperpfaffian generalization of a useful Pfaffian identity, which will be used to prove the main theorem of the next section, i.e., Theorem 3.2. Finally, in Subsection 2.3, we state a hyperpfaffian-hyperdeterminant version of the minor summation formula, i.e., Theorem 2.7, and two corollaries (Corollary 2.8 and Theorem 2.9).

### 2.1 Hyperdeterminant

Let $m$ be an even positive integer and $n$ be a positive integer. An $m$-dimensional tensor of order $n$ is a map $A : [n]^m \to F$, $(i_1, \ldots, i_m) \mapsto A(i_1, \ldots, i_m)$, where $F$ is a field
of characteristic 0. We express this tensor by \( A = (A(i_1, \ldots, i_m))_{1 \leq i_1, \ldots, i_m \leq n} \), which we call an \( m\text{-dimensional hypermatrix of size } n \). We also say the \((i_1, \ldots, i_{m-1})\)th row of \( A \) is \((A(i_1, \ldots, i_{m-1}, 1), \ldots, A(i_1, \ldots, i_{m-1}, n))\). The hyperdeterminant \( \det[m] A \) of \( A \) is defined to be

\[
\det[m] A = \frac{1}{n!} \sum_{\sigma_1, \ldots, \sigma_m \in S_n} \text{sgn}(\sigma_1 \cdots \sigma_m) \prod_{i=1}^n A(\sigma_1(i), \sigma_2(i), \ldots, \sigma_m(i))
= \sum_{\sigma_1, \ldots, \sigma_m \in S_n} \text{sgn}(\sigma_1 \cdots \sigma_m) \prod_{i=1}^n A(\sigma_1(i), \ldots, \sigma_{m-1}(i), i). \tag{2.1}
\]

Especially the \( m = 2 \) case corresponds to the ordinary determinant \( \det A \). For example, if \( m = 4 \) and \( n = 2 \), then

\[
\det[4] A = A(1, 1, 1, 1)A(2, 2, 2, 2) - A(1, 1, 1, 2)A(2, 2, 2, 1) - A(1, 1, 2, 1)A(2, 2, 1, 2) \\
+ A(1, 1, 2, 2)A(2, 2, 1, 1) - A(1, 2, 1, 1)A(2, 1, 2, 2) + A(1, 2, 1, 2)A(2, 1, 2, 1) \\
+ A(1, 2, 2, 1)A(2, 1, 1, 2) - A(1, 2, 2, 2)A(2, 1, 1, 1).
\]

**Proposition 2.2.** Let \( m \) and \( n \) be positive integers. If we put

\[
\eta_i = \sum_{i^{(1)}, \ldots, i^{(m-1)} \in [n]} A(i^{(1)}, \ldots, i^{(m-1)}, i) \xi_{i^{(1)}} \cdots \xi_{i^{(m-1)}} \tag{2.2}
\]
for \( i \in [n] \), then we have

\[
\eta_I = \eta_{i_1} \cdots \eta_{i_r} = \sum_{I^{(1)}, \ldots, I^{(m-1)} \in \binom{[n]}{r}} \det [m] A_{I^{(1)}, \ldots, I^{(m-1)}, I} \xi_{I^{(1)}} \oplus \cdots \oplus \xi_{I^{(m-1)}}
\] (2.3)

for any \( I = \{i_1, \ldots, i_r\} \subseteq \binom{[n]}{r} \), where \( \xi_{I^{(k)}} = \xi_{i_1}^{(k)} \cdots \xi_{i_r}^{(k)} \) for \( I^{(k)} = \{i_1^{(k)}, \ldots, i_r^{(k)}\} \subseteq \binom{[n]}{r} \). Especially, if \( r = n \) and \( I = [n] \), we obtain

\[
\eta_1 \cdots \eta_n = \det [m] (A) \cdot \xi \oplus \cdots \oplus \xi
\] (m-1) times

where \( \xi = \xi_1 \cdots \xi_n \).

**Proof.** By (2.2) we have

\[
\eta_{i_k} = \sum_{i_k^{(1)}, \ldots, i_k^{(m-1)} \in [n]} A(i_k^{(1)}, \ldots, i_k^{(m-1)}, i_k) \xi_{i_k^{(1)}} \oplus \cdots \oplus \xi_{i_k^{(m-1)}}.
\]

Multiplying this and using anti-commutativity, we obtain

\[
\eta_I = \eta_{i_1} \cdots \eta_{i_r} = \sum_{i_1^{(1)}, \ldots, i_r^{(m-1)} \in \binom{[n]}{r}} \sum_{i_1^{(1)}, \ldots, i_r^{(m-1)} \in \binom{[n]}{r}} A(i_1^{(1)}, \ldots, i_r^{(m-1)}, i_1) \cdots A(i_r^{(1)}, \ldots, i_r^{(m-1)}, i_r) \times \xi_{i_1^{(1)}} \cdots \xi_{i_r^{(1)}} \oplus \cdots \oplus \xi_{i_1^{(m-1)}} \cdots \xi_{i_r^{(m-1)}}
\]

\[
= \sum_{I^{(1)} = \{i_1^{(1)}, \ldots, i_r^{(1)}\} \subseteq \binom{[n]}{r}} \sum_{I^{(m-1)} = \{i_1^{(m-1)}, \ldots, i_r^{(m-1)}\} \subseteq \binom{[n]}{r}} \sum_{\sigma^{(1)}, \ldots, \sigma^{(m-1)} \in S_r} \text{sgn } \sigma^{(1)} \cdots \text{sgn } \sigma^{(m-1)} A(i_1^{(1)}, \ldots, i_1^{(m-1)}, i_1) \cdots A(i_r^{(1)}, \ldots, i_r^{(m-1)}, i_r) \xi_{I^{(1)}} \oplus \cdots \oplus \xi_{I^{(m-1)}}
\]

\[
= \sum_{I^{(1)}, \ldots, I^{(m-1)} \in \binom{[n]}{r}} \det [m] A_{I^{(1)}, \ldots, I^{(m-1)}, I} \xi_{I^{(1)}} \oplus \cdots \oplus \xi_{I^{(m-1)}},
\]

which is the desired identity.

Next, let \( I^{(1)}, \ldots, I^{(m)} \in \binom{[n]}{r} \). The \((I^{(1)}, \ldots, I^{(m)})\)-cofactor of \( A \) is defined to be

\[
\tilde{a}_{I^{(1)}, \ldots, I^{(m)}} = (-1)^{|I^{(1)}| + \cdots + |I^{(m)}|} \det [m] A_{I^{(1)}, \ldots, I^{(m)}}
\] (2.5)

where \( |I^{(k)}| = \sum_{i \in I^{(k)}} i \) and \( I^{(k)} = [n] \setminus I^{(k)} \) stands for the complement of \( I^{(k)} \) in \([n]\). In the following proposition we show that we can generalize Proposition 2.1 of [32] (early version of [33]) whereas we will not use this proposition in this paper.
Proposition 2.3. Let \( m \) and \( n \) be positive integers. For any \( m \)-dimensional hypermatrix \( A = (A(i_1, \ldots, i_m))_{1 \leq i_1, \ldots, i_m \leq n} \) of size \( n \), we have

\[
\det^{[m]} A = (-1)^{m\left(\frac{r+1}{2}\right)} \sum_{I^{(1)}, \ldots, I^{(m-1)} \in \binom{[n]}{r}} a_{I^{(1)}, \ldots, I^{(m-1)}, I} \tilde{a}_{I^{(1)}, \ldots, I^{(m-1)}, I} \tag{2.6}
\]

for any \( I \in \binom{[n]}{r} \). Here we use the symbol \( a_{I^{(1)}, \ldots, I^{(m-1)}, I} \) to denote the \( (I^{(1)}, \ldots, I^{(m-1)}, I) \)-minor of \( A \) for simplicity.

Proof. Let \( \eta \) be as in (2.2). Then, for any \( I \in \binom{[n]}{r} \), we have, from (2.3),

\[
\eta = \eta_1 \cdots \eta_n = (-1)^{|I|-\left(\frac{r+1}{2}\right)} I \| \xi\|
\]

\[
= (-1)^{|I|-\left(\frac{r+1}{2}\right)} \sum_{I^{(1)}, \ldots, I^{(m-1)} \in \binom{[n]}{r}} \sum_{J^{(1)}, \ldots, J^{(m-1)} \in \binom{[n]}{r}} a_{I^{(1)}, \ldots, I^{(m-1)}, I} a_{J^{(1)}, \ldots, J^{(m-1)}, J} \xi_{I^{(1)}} \xi_{J^{(1)}} \oplus \cdots \oplus \xi_{I^{(m-1)}} \xi_{J^{(m-1)}}.
\]

Note that

\[
\xi_{I^{(k)}} \xi_{J^{(k)}} = \begin{cases} (-1)^{|I^{(k)}|-\left(\frac{r+1}{2}\right)} \xi & \text{if } I^{(k)} \cap J^{(k)} = \emptyset, \\
0 & \text{otherwise.}
\end{cases}
\]

Hence we obtain

\[
\eta = (-1)^{m\left(\frac{r+1}{2}\right)} \sum_{I_1, \ldots, I_{m-1} \in \binom{[n]}{r}} a_{I^{(1)}, \ldots, I^{(m-1)}, I} \tilde{a}_{I^{(1)}, \ldots, I^{(m-1)}, I} \xi \oplus \cdots \oplus \xi.
\]

Meanwhile, we have already shown that \( \eta = \det^{[m]} A \cdot \xi \oplus \cdots \oplus \xi \). Hence we obtain the desired identity. Note that \( (-1)^{m\left(\frac{r+1}{2}\right)} = 1 \) if \( m \) is even. \( \square \)

The following lemma gives the Laplace expansion of a hyperdeterminant along the last index and one can state it in more general form. But this form is enough to use in the proof of Theorem 2.7.

Lemma 2.4. Let \( l, m, n \) and \( N \) be positive integers such that \( ln \leq N \). Let \( A = (A(i_1, \ldots, i_m))_{1 \leq i_1, \ldots, i_m \leq N} \) be an \( m \)-dimensional hypermatrix of size \( N \). Let \( I = \{i_1, \ldots, i_l\} < \binom{[N]}{l} \) and \( I = (I_1, \ldots, I_n) \) be the \( l \)-block notation of \( I \), i.e., \( I_j = \{i_{(j-1)l+1}, \ldots, i_{jl}\} < \binom{[N]}{l} \) for \( 1 \leq j \leq n \). Then we have

\[
\det^{[m]} A_{[l_1, \ldots, [l_n], I} = \sum_{I^{(1)}, \ldots, I^{(m-1)} \in \binom{[n]}{l}} \sgn I^{(1)} \cdots \sgn I^{(m-1)} \prod_{j=1}^{n} \det^{[m]} A_{I_j^{(1)}, \ldots, I_j^{(m-1)}, I_j} \tag{2.7}
\]

where \( I^{(k)} = (I_1^{(k)}, \ldots, I_n^{(k)}) \) with \( I_j^{(k)} \in \binom{[l]}{l} \) for \( 1 \leq j \leq n \) and \( 1 \leq k \leq m-1 \).
Proof. Let

$$\eta_l = \sum_{i^{(1)}, \ldots, i^{(m-1)} \in [ln]} A(i^{(1)}, \ldots, i^{(m-1)}, i) \xi_{i^{(1)}} \oplus \cdots \oplus \xi_{i^{(m-1)}}$$

as in (2.2). Then, since $$\eta_l = \eta_{i_1} \cdots \eta_{i_n} = \eta_{i_1} \cdots \eta_{i_n}$$, we have, by (2.3),

$$\eta_l = \sum_{i^{(1)}, \ldots, i^{(m-1)} \in [ln]} \cdots \sum_{l^{(1)}, \ldots, l^{(m-1)} \in [ln]} \det[m] A_{i^{(1)}}^{(1)}, \ldots, i^{(m-1)}_{i_n}, l^{(1)}, \ldots, l^{(m-1)}_{l_n} \times \xi_{i^{(1)}} \cdots \xi_{i^{(m-1)}} \oplus \cdots \oplus \xi_{l^{(1)}} \cdots \xi_{l^{(m-1)}}.$$

Note that $$\xi_{i^{(1)}} \cdots \xi_{i^{(m)}}$$ vanishes unless $$[ln]$$ is disjoint union of $$\{i^{(k)}_1, \ldots, i^{(k)}_{l_n}\}$$ for $$1 \leq k \leq m-1$$. If we put $$I^{(k)} = (I^{(k)}_1, \ldots, I^{(k)}_{l_n})$$ for $$1 \leq k \leq m-1$$, then $$\xi_{i^{(k)}_1} \cdots \xi_{i^{(k)}_{l_n}} = \text{sgn} I^{(k)} \xi_{[ln]}$$ implies

$$\eta_l = \sum_{l^{(1)}, \ldots, l^{(m-1)} \in [ln]} \text{sgn} l^{(1)} \cdots \text{sgn} l^{(m-1)} \det[m] A_{i^{(1)}}^{(1)}, \ldots, i^{(m-1)}_{i_n}, l^{(1)}, \ldots, l^{(m-1)}_{l_n} \xi_{[ln]} \oplus \cdots \oplus \xi_{[ln]}.$$

Meanwhile, by (2.4), we have $$\eta_l = \det[m] A_{[ln], \ldots, [ln]} \xi_{[ln]} \oplus \cdots \oplus \xi_{[ln]}$$, which proves the desired identity.

It is clear that we can obtain a similar expansion formula along the $$k$$th index.

### 2.2 Hyperpfaffian

Barvinok [6] gave the first definition of hyperpfaffian, and Matsumoto used a variant of hyperpfaffian in [33]. Here we give a slightly generalized definition, which unifies the original definitions in [6, 28, 33, 37]. Ordinary Pfaffian is defined for a skew-symmetric matrix, that is a matrix $$B = (B(i, j))_{i, j \in [2n]}$$ satisfying $$B(j, i) = -B(i, j)$$. Since this condition implies $$B(i, i) = 0$$, a skew-symmetric matrix is regarded as a map $$\mathcal{B} : \{ (2n) \} \to F, \{ i, j \} \mapsto B(i, j)$$. For a positive even integer $$l$$, Barvinok considered an $$l$$-alternating tensor $$B : [ln]^l \to F, i = (i_1, \ldots, i_l) \mapsto B(i) = B(i_1, \ldots, i_l)$$ of order $$ln$$ which satisfies $$B(\tau(i)) = \text{sgn} \tau \cdot B(i)$$ for $$\tau \in \mathfrak{S}_l$$, where $$\tau$$ stands for $$(i_{\tau(1)}, \ldots, i_{\tau(l)})$$. The reader can easily see that this $$l$$-alternating tensor $$B$$ of order $$ln$$ is equivalent to giving a map $$\mathcal{B} : \{ i_1, \ldots, i_l \} \to F$$ defined by $$\mathcal{B}(\{ i_1, \ldots, i_l \}) = B(i_1, \ldots, i_l)$$. In this paper we consider more general situation.

Let $$l, m$$ and $$n$$ be positive integers, and let $$B : [ln]^l \times \cdots \times [ln]^l \to F, (i^{(1)}, \ldots, i^{(m)}) \mapsto B(i^{(1)}, \ldots, i^{(m)})$$ be an $$ln$$-dimensional tensor of order $$ln$$ with $$i^{(1)}, \ldots, i^{(m)} \in [ln]^l$$. We say that this tensor $$B = (B(i_1, \ldots, i_m))_{i_1, \ldots, i_m \in [ln]^l}$$ is $$l$$-alternating if it satisfies

$$B(\tau_1(i^{(1)}), \ldots, \tau_m(i^{(m)})) = \text{sgn}(\tau_1) \cdots \text{sgn}(\tau_m) B(i^{(1)}, \ldots, i^{(m)})$$
for any \((\tau_1, \ldots, \tau_m) \in (\mathcal{S}_l)^m\). This means that, if we permute each index, then the sign of the permutation is multiplied. We can associate a map \(\mathcal{B} : \left(\binom{[n]}{l}\right) \times \cdots \times \left(\binom{[n]}{l}\right) \to F\) with \(B\) by setting its value \(\mathcal{B}(I^{(1)}, \ldots, I^{(m)}) = B(i^{(1)}, \ldots, i^{(m)})\), where each \(l\)-tuple \(i^{(k)} = (i_1^{(k)}, \ldots, i_l^{(k)})\) has the same entries as \(I^{(k)} = \{i_1^{(k)}, \ldots, i_l^{(k)}\}\) \((1 \leq k \leq m)\). This latter map \(\mathcal{B} : \left(\binom{[n]}{l}\right) \times \cdots \times \left(\binom{[n]}{l}\right) \to F\) is denoted by

\[
\mathcal{B} = \left(\mathcal{B}(I^{(1)}, \ldots, I^{(m)})\right)_{I^{(1)}, \ldots, I^{(m)} \in \binom{[n]}{l}},
\]

and called a \(m\)-dimensional \(l\)-block array of size \(ln\). An \(lm\)-dimensional \(l\)-alternating tensor \(B\) of order \(ln\) and an \(m\)-dimensional \(l\)-block array \(\mathcal{B}\) of size \(ln\) are equivalent since we can recover \(B\) from \(\mathcal{B}\) by the \(l\)-alternating property.

The hyperpfaffian \(\text{Pf}^{[l,m]}(\mathcal{B})\) of an \(m\)-dimensional \(l\)-block array \(\mathcal{B}\) of size \(ln\) is defined to be

\[
\text{Pf}^{[l,m]}(\mathcal{B}) = \frac{1}{n!} \sum_{I^{(1)}, \ldots, I^{(m)} \in \binom{[n]}{l}} \text{sgn}(I^{(1)}) \cdots \text{sgn}(I^{(m)}) \prod_{j=1}^{n} \mathcal{B}(I_j^{(1)}, \ldots, I_j^{(m)}),
\]  

(2.8)

where \(I^{(k)} = (I_1^{(k)}, \ldots, I_n^{(k)})\) with \(I_j^{(k)} \in \binom{[n]}{l}\) for \(1 \leq j \leq n\) and \(1 \leq k \leq m\). We shall write \(\text{Pf}^{[l,m]}(\mathcal{B}(I_1, \ldots, I_m))_{I_1, \ldots, I_m \in \binom{[n]}{l}}\) for \(\text{Pf}^{[l,m]}(\mathcal{B})\). By removing the sign, we define hyperhafnian \(\text{Hf}^{[l,m]}(\mathcal{B})\) of \(\mathcal{B}\) as

\[
\text{Hf}^{[l,m]}(\mathcal{B}) = \frac{1}{n!} \sum_{I^{(1)}, \ldots, I^{(m)} \in \binom{[n]}{l}} \prod_{j=1}^{n} \mathcal{B}(I_j^{(1)}, \ldots, I_j^{(m)}),
\]  

(2.9)

which reduces to the ordinary Hafnian when \(l = 2\) and \(m = 1\). We use it only in Corollary 2.9. Note that, if \(l\) and \(m\) are both odd then \(\text{Pf}^{[l,m]}(\mathcal{B})\) is always zero. Hence we assume \(l\) is always even hereafter. The \(m = 1\) case reduces to \([6, (3.3.1)], [28, (76)], [37, Sec.1.1]\) and the \(l = 2\) case reduces to \([33, (2.2)]\). Especially the \((l, m) = (2, 1)\) case reduces to the ordinary Pfaffian

\[
\text{Pf}^{[2,1]}(B) = \frac{1}{n!} \sum_{\sigma = (\sigma_1(1), \sigma_1(2), \ldots, \sigma_n(1), \sigma_n(2)) \in \Sigma_2n, \sigma_1(1) < \sigma_2(1), \ldots, \sigma_n(1) < \sigma_n(2)} \text{sgn } \sigma \cdot B(\sigma_1(1), \sigma_1(2)) \cdots B(\sigma_n(1), \sigma_n(2)),
\]

which we simply write \(\text{Pf}(B)\). But, in fact, our new definition of hyperpfaffian \(\text{Pf}^{[l,m]}(\mathcal{B})\) does not essentially generalize Barvinok’s original hyperpfaffian \(\text{Pf}^{[1,1]}(\mathcal{B})\) since one can write \(\text{Pf}^{[l,m]}(\mathcal{B})\) in the form of \(\text{Pf}^{[lm,1]}(\tilde{\mathcal{B}})\) by taking a suitable \(\tilde{\mathcal{B}}\) in a similar way as Matsumoto stated in \([33, \text{Proposition A1}]\). We will prove it at the end of this section, i.e., Proposition 2.10. The reader may skip it since it is not used in this paper.
Proposition 2.5. Given an $m$-dimensional $l$-block array $\mathcal{B} = \{ \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \}_{I^{(1)}, \ldots, I^{(m)} \in \binom{[ln]}{l}}$ of size $ln$, we put

$$\zeta = \sum_{I^{(1)}, \ldots, I^{(m)} \in \binom{[ln]}{l}} \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \xi_{I^{(1)}} \oplus \cdots \oplus \xi_{I^{(m)}}. \quad (2.10)$$

Then we have

$$\zeta^n = n! \text{Pf}^{[l,m]}(\mathcal{B}) \underbrace{\xi \oplus \cdots \oplus \xi}_{m \text{ times}}, \quad (2.11)$$

where $\xi = \xi_1 \cdots \xi_{ln}$.

Proof. From the definition (2.10), we obtain

$$\zeta^n = \sum_{I^{(1)}, \ldots, I^{(m)} \in \binom{[ln]}{l}^{m}} \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \cdots \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \times \xi_{I^{(1)}} \cdots \xi_{I^{(m)}} \oplus \cdots \oplus \xi_{I^{(m)}} \cdots \xi_{I^{(m)}}.$$

If we set $I^{(k)} = (I^{(k)}_1, \ldots, I^{(k)}_n)$ for $1 \leq k \leq m$, then we have

$$\zeta^n = \sum_{I^{(1)}, \ldots, I^{(m)} \in \binom{[ln]}{l}^{m}} \text{sgn} I^{(1)} \cdots \text{sgn} I^{(m)} \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \cdots \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \times \xi \oplus \cdots \oplus \xi,$$

because $\xi_{I^{(1)}} \cdots \xi_{I^{(k)}} = \text{sgn} I^{(k)} \xi$ if $I^{(k)} = [ln]$ (disjoint union), or 0 otherwise for $1 \leq k \leq m$. Hence we obtain the desired identity. 

Let $l$, $r$ and $n$ be positive integers such that $l$ is even and $r \leq n$. Let $\mathcal{B} = \{ \mathcal{B}(I_1, \ldots, I_m) \}_{I_1, \ldots, I_m \in \binom{[ln]}{l}}$ be an $m$-dimensional array of size $ln$. When $(S^{(1)}, \ldots, S^{(m)})$ is an $m$-tuple such that $S^{(k)} \in \binom{[ln]}{l}$ for $1 \leq k \leq m$, we let $\mathcal{B}(S^{(1)}, \ldots, S^{(m)})$ denote the $m$-dimensional $l$-block array

$$\{ \mathcal{B}(I_1, \ldots, I_m) \}_{I_1 \in \binom{S^{(1)}}{l}, \ldots, I_m \in \binom{S^{(m)}}{l}},$$

of size $ln$. We call the hyperpfaffian

$$\text{Pf}^{[l,m]}(\mathcal{B}(S^{(1)}, \ldots, S^{(m)})) = \frac{1}{r!} \sum_{I^{(1)} \in \binom{S^{(1)}}{l}, \ldots, I^{(m)} \in \binom{S^{(m)}}{l}} \prod_{k=1}^m \text{sgn}(I^{(k)}) \prod_{j=1}^r \mathcal{B}(I^{(1)}_j, \ldots, I^{(m)}_j). \quad (2.12)$$

the subhyperpfaffian of $\mathcal{B}$, where we write $I^{(k)} = (I^{(k)}_1, \ldots, I^{(k)}_r)$.

We need the following lemma for later use. Let $D_l(\lambda) = \{ l(\lambda - 1) + 1, l(\lambda - 1) + 2, \ldots, l(\lambda - 1) + l \}$ for a positive integer $\lambda$, and let $D_l(\lambda_1, \ldots, \lambda_n) = D_l(\lambda_1) \cup \cdots \cup D_l(\lambda_n)$ for $1 \leq \lambda_1 < \cdots < \lambda_n \leq N$. We may write $D_l(\lambda)$ in short for $\lambda = \{ \lambda_1, \ldots, \lambda_n \} \in \binom{[N]}{n}$. The following lemma is an easy consequence of (2.12) and we omit the proof.
Lemma 2.6. Let \( l, m, n \) and \( N \) be positive integers such that \( l \) is even and \( n \leq N \). Let \( \mathcal{B} = (\mathcal{B}(I^{(1)}, \ldots, I^{(m)}))_{I^{(1)}, \ldots, I^{(m)} \in \binom{[N]}{l}} \) be an \( m \)-dimensional \( l \)-block array of size \( lN \) defined by

\[
\mathcal{B}(I^{(1)}, \ldots, I^{(m)}) = \begin{cases} 
1 & \text{if } I^{(k)} = D_l(\lambda^{(k)}) \text{ for some } \lambda^{(k)} \in [N] (1 \leq k \leq m), \\
0 & \text{otherwise.}
\end{cases}
\tag{2.13}
\]

Let \( S^{(1)}, \ldots, S^{(m)} \in \binom{[N]}{l} \). Then we have

\[
\text{Pf}^{[l,m]}(\mathcal{B}_{S^{(1)}, \ldots, S^{(m)}}) = \begin{cases} 
1 & \text{if } S^{(k)} = D_l(\lambda^{(k)}) \text{ for some } \lambda^{(k)} \in \binom{[N]}{n} (1 \leq k \leq m), \\
0 & \text{otherwise.}
\end{cases}
\tag{2.14}
\]

For example, if \( l = 2 \) and \( m = 1 \), then \( \mathcal{B} \) is the array whose entry is

\[
\mathcal{B}(I) = \begin{cases} 
1 & \text{if } I = \{2l - 1, 2\lambda\} \text{ for } 1 \leq \lambda \leq N, \\
0 & \text{otherwise.}
\end{cases}
\]

Assume \( S \in \binom{[2N]}{2n} \). Then \( \text{Pf}(\mathcal{B}_S) \) equals 1 if \( S \) is of the form \( S = \{2\lambda_1 - 1, 2\lambda_1\} \cup \cdots \cup \{2\lambda_n - 1, 2\lambda_n\} \) for some \( 1 \leq \lambda_1 < \cdots < \lambda_n \leq N \), or 0 otherwise.

2.3 Minor summation formulas of hyperpfaffians

The following theorem and its corollaries are generalizations of the minor summation formulas of Pfaffians (See [19, 20, 32, 33]). The original formula corresponds to the case where \( (l, m, r) = (2, 2, 1) \) and \( n \) is a positive integer. That is to say, if \( \mathcal{B} = (\mathcal{B}(I))_{I \in \binom{[N]}{2}} = (\mathcal{B}([i,j])_{j<k \in \binom{[N]}{2}} \) is a 2-block array of size \( N \) and \( H = (H(i,j))_{1 \leq i \leq 2n, 1 \leq j \leq N} \) a \( 2n \times N \) matrix, then we have

\[
\sum_{J \in \binom{[N]}{2n}} \text{Pf}(\mathcal{B}_J) \det H_{[2n],J} = \text{Pf}(Q)
\]

where

\[
Q(I) = Q([i,j]) = \sum_{K=\{k,l\} \in \binom{[N]}{2}} \mathcal{B}([k,l]) \det \begin{pmatrix}
H(i,k) & H(i,l) \\
H(j,k) & H(j,l)
\end{pmatrix}
\]

for \( I = [i,j] \in \binom{[2n]}{2} \) (see [19, Theorem 1], [20, Lemma 2.1]).

Theorem 2.7. Let \( l, m, n \) and \( r \) be positive integers such that \( l \) is even and \( ln \leq N \). Let \( H(\nu) = (H(\nu)(i_1, \ldots, i_m))_{1 \leq i_1, \ldots, i_m \leq n, 1 \leq i_m \leq N} \) be an \( m \)-dimensional hypermatrix of size \( (ln, \ldots, ln, N) \) for \( 1 \leq \nu \leq r \), and let \( \mathcal{B} = (\mathcal{B}(I^{(1)}, \ldots, I^{(m)}))_{I^{(1)}, \ldots, I^{(m)} \in \binom{[N]}{l}} \) be an \( r \)-dimensional \( l \)-block array of size \( N \). Then we have

\[
\sum_{S^{(1)}, \ldots, S^{(r)} \in \binom{[N]}{l}} \text{Pf}^{[l,r]}(\mathcal{B}_{S^{(1)}, \ldots, S^{(r)}}) \prod_{\nu=1}^{r} \det^m [H(\nu)]_{[ln], \ldots, [ln], S^{(\nu)}} = \text{Pf}^{[l, m-1, r]}(Q),
\tag{2.15}
\]
Proof. From the definition (2.8) of the hyperpfaffian, we have

\[
\begin{align*}
&\text{where } Q = (Q(I^{(1,1)}, \ldots, I^{(1,m-1)}, \ldots, I^{(r,1)}, \ldots, I^{(r,m-1)}))_{I^{(1,1)}, \ldots, I^{(r,m-1)} \in (\mathbb{I}^n)^r} \text{ is the } (m-1)r\text{-dimensional } l\text{-block array of size } ln \text{ defined by}

\begin{align*}
&Q(I^{(1,1)}, \ldots, I^{(1,m-1)}, \ldots, I^{(r,1)}, \ldots, I^{(r,m-1)}) \\
&= \sum_{K^{(1)}, \ldots, K^{(r)} \in (\mathbb{N})^r} \mathcal{B}(K^{(1)}, \ldots, K^{(r)}) \prod_{\nu=1}^{r} \det^{[m]}(H(\nu)_{I^{(1,1)}, \ldots, I^{(r,m-1)}, K^{(\nu)}}).
\end{align*}
\]

(2.16)

By formula (2.7) in Lemma 2.4, it reduces to

\[
\begin{align*}
&\prod_{j=1}^{n} Q(I^{(1,1)}_j, \ldots, I^{(1,m-1)}_j, \ldots, I^{(r,1)}_j, \ldots, I^{(r,m-1)}_j),
\end{align*}
\]

where \( I^{(\nu,k)} = (I^{(\nu,k)}_1, \ldots, I^{(\nu,k)}_n) \) with \( I^{(\nu,k)}_j \in (\mathbb{I}^n) \) for \( 1 \leq j \leq n \), \( 1 \leq \nu \leq r \) and \( 1 \leq k \leq m-1 \). Hence, by (2.16), \( n! \operatorname{Pr}^{[(m-1)r]}(Q) \) is equal to

\[
\begin{align*}
&\sum_{I^{(1,1)}, \ldots, I^{(1,m-1)} \in (\mathbb{I}^n)} \cdots \sum_{I^{(r,1)}, \ldots, I^{(r,m-1)} \in (\mathbb{I}^n)} \prod_{k=1}^{m-1} \prod_{\nu=1}^{r} \text{sgn}(I^{(\nu,k)}) \\
&\times \sum_{K^{(1)}_1, \ldots, K^{(r)}_1 \in (\mathbb{N})^r} \cdots \sum_{K^{(1)}_n, \ldots, K^{(r)}_n \in (\mathbb{N})^r} \prod_{j=1}^{r} \mathcal{B}(K^{(1)}_j, \ldots, K^{(r)}_j) \prod_{\nu=1}^{r} \det^{[m]}(H(\nu)_{I^{(1,1)}_j, \ldots, I^{(r,m-1)}, K^{(\nu)}_j}) \\
&= \prod_{j=1}^{n} \mathcal{B}(K^{(1)}_1, \ldots, K^{(r)}_1) \cdots \mathcal{B}(K^{(1)}_n, \ldots, K^{(r)}_n) \\
&\times \prod_{k=1}^{m-1} \prod_{j=1}^{n} \text{sgn}(I^{(1,k)}) \prod_{j=1}^{n} \det^{[m]}(H(1)_{I^{(1,1)}_j, \ldots, I^{(r,m-1)}, K^{(1)}_j}) \\
&\times \cdots \times \sum_{I^{(r,1)}, \ldots, I^{(r,m-1)} \in (\mathbb{I}^n)} \prod_{k=1}^{m-1} \prod_{j=1}^{n} \text{sgn}(I^{(r,k)}) \prod_{j=1}^{n} \det^{[m]}(H(r)_{I^{(r,1)}_j, \ldots, I^{(r,m-1)}, K^{(r)}_j}).
\end{align*}
\]

By formula (2.7) in Lemma 2.4, it reduces to

\[
\begin{align*}
&\sum_{K^{(1)}_1, \ldots, K^{(r)}_1 \in (\mathbb{N})^r} \cdots \sum_{K^{(1)}_n, \ldots, K^{(r)}_n \in (\mathbb{N})^r} \mathcal{B}(K^{(1)}_1, \ldots, K^{(r)}_1) \cdots \mathcal{B}(K^{(1)}_n, \ldots, K^{(r)}_n) \\
&\times \det^{[m]}H(1)_{\mathbb{I}^n, \ldots, \mathbb{I}^n, K^{(1)}_1} \cdots \det^{[m]}H(r)_{\mathbb{I}^n, \ldots, \mathbb{I}^n, K^{(r)}_1},
\end{align*}
\]

where \( K^{(\nu)} = (K^{(\nu)}_1, \ldots, K^{(\nu)}_n) \) for \( 1 \leq \nu \leq r \). Since the hyperdeterminant vanishes unless \( K^{(\nu)}_1, \ldots, K^{(\nu)}_n \) are mutually disjoint for each \( \nu \), we can set \( S^{(\nu)} = K^{(\nu)}_1 \cup \cdots \cup K^{(\nu)}_n \in (\mathbb{N})^r \).
Hence we have \( K^{(\nu)} = (K_1^{(\nu)}, \ldots, K_n^{(\nu)}) \in \binom{S^{(\nu)}}_{i=1} \) for each \( \nu \). In this case, the hyperdeterminant becomes \( \det[m] H(s)_{[ln], \ldots, [ln], K^{(\nu)}} = \text{sgn } K^{(\nu)} \det[m] H(s)_{[ln], \ldots, [ln], S^{(\nu)}} \) and the above sum becomes

\[
\sum_{S^{(1)}, \ldots, S^{(\nu)} \in \binom{[N]}{ln}} \cdots \sum_{K^{(\nu)} \in \binom{S^{(\nu)}}_{i=1}} \prod_{\nu=1}^r \text{sgn } K^{(\nu)} 
\times \prod_{j=1}^n \text{Pr}_{[\nu]} (K_j^{(1)}, \ldots, K_j^{(r)}) \prod_{\nu=1}^r \det[m] H(\nu)_{[ln], \ldots, [ln], S^{(\nu)}}. \tag{2.17}
\]

By (2.12), this reduces to

\[
n! \sum_{S^{(1)}, \ldots, S^{(\nu)} \in \binom{[N]}{ln}} \text{Pr}_{[\nu]} (\mathcal{B}_{S^{(1)}, \ldots, S^{(\nu)}}) \prod_{\nu=1}^r \det[m] H(\nu)_{[ln], \ldots, [ln], S^{(\nu)}}.
\]

This proves the theorem. \( \square \)

If we put \( m = 2 \) in Theorem 2.7, then the hyperdeterminant becomes the ordinary determinant and we obtain the following corollary:

**Corollary 2.8.** Let \( l, n, N \) and \( r \) be positive integers such that \( l \) is even and \( ln \leq N \). Let \( H(\nu) = (h_{ij}(\nu))_{1 \leq i, j \leq N} \) be \( ln \times N \) rectangular matrices for \( 1 \leq \nu \leq r \), and let \( \mathcal{B} = (\mathcal{B}(I^{(1)}, \ldots, I^{(r)}))_{I^{(1)}, \ldots, I^{(r)} \in \binom{[N]}{ln}} \) be an \( r \)-dimensional \( l \)-block array of size \( N \). Then we have

\[
\sum_{S^{(1)}, \ldots, S^{(\nu)} \in \binom{[N]}{ln}} \text{Pr}_{[\nu]} (\mathcal{B}_{S^{(1)}, \ldots, S^{(\nu)}}) \prod_{\nu=1}^r \det H(\nu)_{[ln], S^{(\nu)}} = \text{Pr}_{[\nu]} (Q), \tag{2.18}
\]

where \( Q = (Q(I^{(1)}, \ldots, I^{(r)}))_{I^{(1)}, \ldots, I^{(r)} \in \binom{[ln]}{1}} \) is the \( r \)-dimensional \( l \)-block array of size \( ln \) defined by

\[
Q(I^{(1)}, \ldots, I^{(r)}) = \sum_{K^{(1)}, \ldots, K^{(r)} \in \binom{[N]}{l}} \mathcal{B}(K^{(1)}, \ldots, K^{(r)}) \prod_{\nu=1}^r \det (H(\nu)_{I^{(\nu)}, K^{(\nu)}}). \tag{2.19}
\]

Actually the following theorem is not a direct corollary of Theorem 2.7, but a consequence of the proof of Theorem 2.7.

**Theorem 2.9.** Let \( l, n, N \) and \( r \) be positive integers such that \( l \) is even and \( ln \leq N \). Let \( H(\nu) = (h_{ij}(\nu))_{1 \leq i, j \leq N} \) be an \( ln \times N \) rectangular matrices for \( 1 \leq \nu \leq r \), and \( \mathcal{A} = (\mathcal{A}(I))_{I \in \binom{[N]}{l}} \) a 1-dimensional \( l \)-block array of size \( N \). If we define the array \( Q = (Q(I^{(1)}, \ldots, I^{(r)}))_{I^{(1)}, \ldots, I^{(r)} \in \binom{[ln]}{1}} \) by

\[
Q(I^{(1)}, \ldots, I^{(r)}) = \sum_{K \in \binom{[N]}{l}} \mathcal{A}(K) \prod_{\nu=1}^r \det (H(\nu)_{I^{(\nu)}, K}), \tag{2.20}
\]

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then we obtain the following identity:

\[
\text{Pf}^{[l,r]}(Q) = \begin{cases} 
\sum_{S \in \binom{[n]}{l}} \text{Pf}^{[l,1]}(S) \prod_{\nu=1}^r \det (H(\nu)_{[m],S}) & \text{if } r \text{ is odd}, \\
\sum_{S \in \binom{[n]}{l}} \text{Hf}^{[l,1]}(S) \prod_{\nu=1}^r \det (H(\nu)_{[m],S}) & \text{if } r \text{ is even}.
\end{cases}
\] (2.21)

**Proof.** Recall the proof of Theorem 2.7 when \( m = 2 \). We put

\[ \mathcal{B}(I^{(1)}, \ldots, I^{(r)}) = \mathcal{A}(I^{(1)}) \delta_{I^{(1)}, I^{(2)}} \cdots \delta_{I^{(1)}, I^{(r)}} \]

for the given \( \mathcal{A} : \binom{[n]}{l} \rightarrow F \). Then (2.17) becomes

\[ \sum_{S \in \binom{[n]}{l}} \sum_{K \in \binom{[s]}{m}} (\text{sgn } K)^r \prod_{j=1}^n \mathcal{A}(K_j) \prod_{\nu=1}^r \det H(\nu)_{[m],S^{(\nu)}}. \]

Note that \( (\text{sgn } K)^r = \text{sgn } K \) if \( r \) is odd, or 1 if \( r \) is even. Hence we obtain the desired result.

At the end of this section, we briefly sketch the proof of the following fact:

**Proposition 2.10.** Given an \( m \)-dimensional \( l \)-block array \( \mathcal{B} = \left( \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) \right)_{I^{(1)}, \ldots, I^{(m)} \in \binom{[n]}{l}} \) of size \( ln \), we define the 1-dimensional \( lm \)-block array \( \mathcal{A} = (\mathcal{A}(J))_{J \in \binom{[mn]}{lm}} \) of size \( lm \) by

\[ \mathcal{A}(J) = \begin{cases} \mathcal{B}(I^{(1)}, \ldots, I^{(m)}) & \text{if } J = \bigcup_{k=1}^m (I^{(k)} + ln(k-1)), \\
0 & \text{otherwise}, \end{cases} \]

where \( I^{(k)} + ln(k-1) \) stands for the subset \( \{i + ln(k-1) \mid i \in I^{(k)} \} \in \binom{[mn]}{lm} \) for \( 1 \leq k \leq m \). Then we have

\[ \text{Pf}^{[l,m]}(\mathcal{B}) = \text{Pf}^{[lm,1]}(\mathcal{A}). \] (2.22)

**Proof.** The proof is similar to that of [33, Proposition A1]. Given an \( m \)-tuple \( (I^{(1)}, \ldots, I^{(m)}) \) such that \( I^{(k)} = (I^{(k)}_1, \ldots, I^{(k)}_n) \in \binom{[n]}{l} \) for \( 1 \leq k \leq m \), we associate \( J = (J_1, \ldots, J_n) \in \binom{[mn]}{lm} \) with \( J_j = \bigcup_{k=1}^n (I^{(k)}_j + ln(k-1)) \) for \( 1 \leq j \leq n \). It is easy to check that \( (I^{(1)}, \ldots, I^{(m)}) \mapsto J \) gives a injection \( \binom{[n]}{l}^m \rightarrow \binom{[mn]}{lm} \), and \( \text{sgn } J = \text{sgn } I^{(1)} \cdots \text{sgn } I^{(m)} \). Further, from the definition of \( \mathcal{A} \), we have \( \mathcal{A}(J_1) \cdots \mathcal{A}(J_n) = \prod_{j=1}^n \mathcal{B}(I^{(1)}_j, \ldots, I^{(m)}_j) \) if \( J \) is in the image of this injection, and 0 otherwise. This proves the identity (2.22).

3 De Bruijn’s formula and Hankel hyperpfaffians

De Bruijn’s original paper contains two major Pfaffian formulas [8, (4.7)] and [8, (7.3)], both of which are called by his name. In this section we establish certain hyperpfaffian analogues of both formulas, see Theorem 3.1 and Theorem 3.2. Especially the latter implies Corollary 3.5, which gives a relation between the Hankel hyperpfaffians and Selberg integrals.
However we are not completely satisfied with Corollary 3.4 since it is hard to evaluate the integral in the right-hand side of (3.10) unless \( l = 2 \). The main idea of the proof of those two theorems is to use Theorem 2.7. The aim of this section is to reduce certain (hyper)pfaffians and their \( q \)-analogues, which we call Hankel (hyper)pfaffians, to Selberg type integrals. Note that \( \text{Pf}^{[l,1]} \) stands for Barvinok’s original hyperpfaffian [6], where \( l \) is a positive even integer.

### 3.1 Jackson integral

First we recall the definition of the Jackson integral. Let \( f(x) \) be a function defined on an interval \([\alpha, \beta]\). The Jackson integral from \( \alpha \) to \( \beta \) is defined by

\[
\int_{\alpha}^{\beta} f(x) \, dq_x = \int_{0}^{\beta} f(x) \, dq_x - \int_{0}^{\alpha} f(x) \, dq_x
\]

where

\[
\int_{0}^{\alpha} f(x) \, dq_x = (1 - q)\alpha \sum_{n=0}^{\infty} f(\alpha q^n)q^n
\]

[12, (1.11.2), (1.11.3)]. Here we assume \( 0 < q < 1 \) unless otherwise stated. If \( w \) is the weight function of a measure \( dq_\omega \), we have

\[
\int_{0}^{\alpha} f(x) \, dq_\omega(x) = \int_{0}^{\alpha} f(x) \, w(x) dq_x.
\]  

(3.1)

More generally, we define the multiple \( q \)-integral by

\[
\int_{0 \leq x_1 < \cdots < x_n \leq \alpha} f(x) dq_\omega(x) = (1 - q)^n\alpha^n \sum_{0 \leq i_1 < \cdots < i_n} f(\alpha q^{i_1}, \ldots, \alpha q^{i_n}) \prod_{k=1}^{n} w(\alpha q^{i_k})q^{i_1 + \cdots + i_n}.
\]  

(3.2)

Let \( l \) and \( m \) be positive integers, and let \( f : ([0, \alpha]^l)^m \to \mathbb{C} \) be a function. We say \( f \) is \( l \)-alternating if it satisfies \( f(\sigma^{(1)}(y^{(1)}), \ldots, \sigma^{(m)}(y^{(m)})) = \text{sgn} \sigma^{(1)} \cdots \text{sgn} \sigma^{(m)} f(y^{(1)}, \ldots, y^{(m)}) \), where \( y^{(k)} = (y^{(k)}_1, \ldots, y^{(k)}_l) \in [0, \alpha]^l \) and \( \sigma^{(1)}, \ldots, \sigma^{(m)} \in S_1 \) for \( 1 \leq k \leq m \). This is equivalent to giving a function \( ([0,\alpha]^l)^m \to \mathbb{C} \) such that \( \beta(Y^{(1)}, \ldots, Y^{(m)}) = f(y^{(1)}, \ldots, y^{(m)}) \) where \( Y^{(k)} = \{y^{(k)}_1, \ldots, y^{(k)}_l\} \) for \( 1 \leq k \leq m \). This means that it is enough to use only the values of an \( l \)-alternating function \( f(y^{(1)}, \ldots, y^{(m)}) \) in the domain \( y^{(k)}_1 < \cdots < y^{(k)}_l \) \( (1 \leq k \leq m) \).

### 3.2 The first de Bruijn type formula

De Bruijn presented two types of formulas in his paper [8]. We establish hyperpfaffian versions of both. We deduce the following theorem from Theorem 2.7, which is a generalization of de Bruijn’s formula [8, (4,7)].
Theorem 3.1. Let $l$, $m$, $n$ and $r$ be positive integers such that $l$ is even. Let $\phi_{i_1,\ldots,i_{m-1}}^{(\nu)}(y)$ be a function on $[0,\alpha]$ for $i_1,\ldots,i_{m-1} \in [ln]$ and $1 \leq \nu \leq r$. Let $f(y^{(1)},\ldots,y^{(r)})$ be an $l$-alternation function, where $y^{(\nu)} = (y_1^{(\nu)},\ldots,y_l^{(\nu)}) \in [0,\alpha]^l$ for $1 \leq \nu \leq r$. Then we have

$$\int_{0 \leq x_1^{(1)} < \cdots < x_{ln}^{(1)} \leq \alpha} \cdots \int_{0 \leq x_1^{(r)} < \cdots < x_{ln}^{(r)} \leq \alpha} \Pr^{[l,r]}(f(x_1^{(1)},\ldots,x_r^{(r)}))_{I^{(1)},\ldots,I^{(r)} \in \{\ln\}^l} \times \prod_{\nu=1}^r \det[m](\phi^{(\nu)}_{j_1^{(\nu)},\ldots,j_{r-m}^{(\nu)}}(x_j^{(\nu)}))_{1 \leq j_1^{(\nu)},\ldots,j_{r-m}^{(\nu)} \leq ln} d\omega(x^{(s)}) = \Pr^{[l,(m-1)r]}(Q),$$

(3.3)

where

$$Q(I^{(1)},\ldots,I^{(r-1)},I^{(r,m-1)},\ldots,I^{(r,m-1)},I^{(r-1)},\ldots,I^{(r,m-1)}) = \int_{0 \leq x_1^{(1)} < \cdots < x_{ln}^{(1)} \leq \alpha} \cdots \int_{0 \leq x_1^{(r)} < \cdots < x_{ln}^{(r)} \leq \alpha} f(x^{(1)},\ldots,x^{(r)}) \prod_{\nu=1}^r \det[m](\phi^{(\nu)}_{i_1^{(\nu)},\ldots,i_{r-m}^{(\nu)}}(x_i^{(\nu)}))_{I^{(\nu)} \in \{\ln\}^l} d\omega(x^{(s)})$$

(3.4)

for $I^{(1)},\ldots,I^{(r-1)},I^{(r,m-1)},\ldots,I^{(r,m-1)},I^{(r-1)},\ldots,I^{(r,m-1)} \in \{\ln\}$. Here we use the notation $x^{(\nu)} = (x_1^{(\nu)},\ldots,x_l^{(\nu)})$ and $x^{(\nu)}_{I^{(\nu)}} = (x_i^{(\nu)}_{i_1^{(\nu)}},\ldots,x_i^{(\nu)}_{i_{r-m}^{(\nu)}})$ for $I^{(\nu)} = \{i_1^{(\nu)},\ldots,i_{r-m}^{(\nu)}\} \in \{\ln\}$ $(1 \leq \nu \leq r)$ in (3.3). Meanwhile $x^{(\nu)}$ stands for $(x_1^{(\nu)},\ldots,x_l^{(\nu)})$ in (3.4).

Proof. Let $N$ be an integer such that $N \geq ln$. In this proof we use the notation $\alpha q^{I-1} = \{\alpha q^{i-1},\ldots,\alpha q^{i-1}\}$ for any $l$-element set $I = \{i_1,\ldots,i_l\} \in \{\ln\}$. We take

$$\mathcal{B}(I^{(1)},\ldots,I^{(r)}) = f(\alpha q^{I^{(1)-1}},\ldots,\alpha q^{I^{(r)-1}})$$

for $I^{(1)},\ldots,I^{(r)} \in \{\ln\}$, and

$$H(\nu)(i_1,\ldots,i_{m-1},i_m) = (1-q)\alpha q^{m-1}\phi_{i_1,\ldots,i_{m-1}}^{(\nu)}(\alpha q^{m-1})\omega(\alpha q^{m-1})$$

in Theorem 2.7. Then, by (2.12), the hyperpfaffian in the left-hand side of (2.15) becomes

$$\Pr^{[l,r]}(\mathcal{B}_{s^{(1)}},\ldots,s^{(r)}) = \frac{1}{n!} \sum_{I^{(1)} \in \{s^{(1)}\},\ldots,I^{(r)} \in \{s^{(r)}\}} \prod_{\nu=1}^r \operatorname{sgn}(I^{(\nu)}) \prod_{j=1}^n f(\alpha q^{j-1},\ldots,\alpha q^{j-1}),$$

where $I^{(\nu)}$ is in the form $I^{(\nu)} = (I_1^{(\nu)},\ldots,I_n^{(\nu)})$ with $I_j^{(\nu)} \in \{s^{(\nu)}\}$ $(1 \leq j \leq n, 1 \leq \nu \leq r)$. Meanwhile, by the definition (2.1), the hyperdeterminant in the left-hand side of (2.15) equals

$$\det[m]H(\nu)_{[ln],\ldots,[ln],s^{(\nu)}}$$

$$= \sum_{\sigma_1,\ldots,\sigma_{m-1} \in \mathcal{S}_m} \operatorname{sgn}(\sigma_1 \cdots \sigma_{m-1}) \prod_{j=1}^n H(\nu)(\sigma_1(j),\sigma_2(j),\ldots,\sigma_{m-1}(j),s_j^{(\nu)})$$

$$= \sum_{\sigma_1,\ldots,\sigma_{m-1} \in \mathcal{S}_m} \operatorname{sgn}(\sigma_1 \cdots \sigma_{m-1}) \prod_{j=1}^n (1-q)\alpha q^{j-1}\phi_{\sigma_1(j),\ldots,\sigma_{m-1}(j)}^{(\nu)}(\alpha q^{j-1})\omega(\alpha q^{j-1}),$$

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where $S^{(\nu)} = \{ s^{(\nu)}_1, \ldots, s^{(\nu)}_n \}$. If $N$ tends to $\infty$, then it is not hard to see that the left-hand side of (2.15) becomes the left-hand side of (3.3) because of (3.2). The right-hand side of (3.3) is more easy to see by putting $N \to \infty$ in (2.16) under this assumption. This completes the proof of the theorem. 

\[ \square \]

### 3.3 The second de Bruijn type formula and its corollaries

The following theorem is a hyperpfaffian analogue of another de Bruijn’s formula [8, (7.3)], which will play an important role in this paper. A special case, i.e., $r = 1$, is obtained in [28, (96)].

**Theorem 3.2.** Let $l$, $m$, $n$ and $r$ be positive integers such that $l$ is even. Let $\phi^{(\nu,k)}_{i'}(x) = \phi^{(\nu,k)}_{i_1,\ldots,i_{m-1}}(x)$ be a function on $[0, \alpha]$ for $i' = (i_1, \ldots, i_{m-1}) \in [ln]^{m-1}$, $\nu \in [r]$ and $k \in [l]$. Then we have

\[
\int_{0 \leq x_1^{(1)} < \cdots < x_n^{(1)} \leq \alpha} \cdots \int_{0 \leq x_1^{(r)} < \cdots < x_n^{(r)} \leq \alpha} \prod_{\nu=1}^{r} \det^{[m]}(\phi^{(\nu,1)}_{i'}(x_j^{(\nu)})) \cdots \det^{[m]}(\phi^{(\nu,l)}_{i'}(x_j^{(\nu)}))_{i' \in [ln]^{m-1}, j \in [n]} \\
\times \prod_{\nu=1}^{r} d_q\omega(x^{(\nu)}) = P[l,(m-1)r](Q),
\]

(3.5)

where $d_q\omega(x^{(\nu)}) = d_q\omega(x^{(\nu)}_1) \cdots d_q\omega(x^{(\nu)}_n)$ and $Q$ is the $(m-1)r$-dimensional $l$-block array of size $ln$ defined by

\[
Q(I^{(1)}, \ldots, I^{(1,m-1)}, \ldots, I^{(r,1)}, \ldots, I^{(r,m-1)})
\]

\[
= \int_{[0,\alpha]^{r}} \prod_{\nu=1}^{r} \det^{[m]}(\phi^{(\nu,k)}_{i^{(\nu,1)},\ldots,i^{(\nu,m-1)}}(x^{(\nu)}))_{i^{(\nu,1)} \in I^{(\nu,1)}, \ldots, i^{(\nu,m-1)} \in I^{(\nu,m-1)}, k \in [l]} d_q\omega(x^{(\nu)})
\]

(3.6)

for $I^{(1)}, \ldots, I^{(r,m-1)} \in [ln]$. Here $(\phi^{(\nu,1)}_{i'}(y_1)) \cdots (\phi^{(\nu,l)}_{i'}(y_j))_{i' \in [ln]^{m-1}, j \in [n]}$ stands for the $m$-dimensional hypermatrix of size $ln$ whose $i'$th row is given by

\[
(\phi^{(\nu,1)}_{i'}(y_1), \ldots, \phi^{(\nu,l)}_{i'}(y_1), \ldots, \phi^{(\nu,1)}_{i'}(y_n), \ldots, \phi^{(\nu,l)}_{i'}(y_n)).
\]

**Proof.** Let $N$ be a positive integer such that $N \geq n$. We replace $N$ by $lN$ in Theorem 2.7, and take the $r$-dimensional $l$-block array $B = (B(I^{(1)}, \ldots, I^{(r)}))_{I^{(1)},\ldots,I^{(r)}} \in [ln]^{r}$ of size $ln$ defined by (2.13) in Lemma 2.6. We take the $m$-dimensional hypermatrix $H(\nu) = (H(\nu)(y^{(\nu)}, i_m))_{y^{(\nu)} \in [ln]^{m-1}, i_m \in [ln]}$ whose entries are given by

\[
H(\nu)(y^{(\nu)}, i_m) = \begin{cases} 
\phi^{(\nu,k)}_{i_m} (\alpha q^{-1}) \omega (\alpha q^{-1}) (1 - q) \alpha q^{-1} & \text{if } \kappa = 1, \\
\phi^{(\nu,k)}_{i_m} (\alpha q^{-1}) & \text{if } 2 \leq \kappa \leq l,
\end{cases}
\]

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where \( \iota \) and \( \kappa \) are the uniquely determined integers which satisfy \( i_m = \iota(t-1) + \kappa, 1 \leq \iota \leq N \) and \( 1 \leq \kappa \leq l \). Then (2.16) reads

\[
Q(I^{(1)}, \ldots, I^{(1,m-1)}, \ldots, I^{(r,1)}, \ldots, I^{(r,m-1)})
= \sum_{\lambda^{(1)}, \ldots, \lambda^{(r)} \in [N]} \prod_{\nu=1}^r \det[m] (H(\nu)_{I^{(\nu)},\ldots,I^{(\nu,m-1)},D_l(\lambda^{(\nu)})})
= \sum_{\lambda^{(1)}, \ldots, \lambda^{(r)} \in [N]} \prod_{\nu=1}^r \det[m] \left( \phi_{\nu}^{(\nu,1)}(\alpha q^{\lambda^{(\nu)}-1}) \cdots \phi_{\nu}^{(\nu,l)}(\alpha q^{\lambda^{(\nu)}-1}) \right)_{i^{(\nu)},j \in [n]} \times \omega(\alpha q^{\lambda^{(\nu)}-1}) \times (1-q)^{\alpha q^{\lambda^{(\nu)}-1}}
\]

for \( I^{(1)}, \ldots, I^{(1,m-1)}, \ldots, I^{(r,1)}, \ldots, I^{(r,m-1)} \in \binom{[N]}{l} \). If we put \( N \to \infty \) in this identity, we obtain (3.6) from (3.1). On the other hand, in the left-hand side of (2.15), the hyperpaffian vanishes by (2.14) unless each \( S^{(\nu)} = D_l(\lambda^{(\nu)}) \) for some \( \lambda^{(\nu)} = (\lambda_1^{(\nu)}, \ldots, \lambda_n^{(\nu)}) \in \binom{[N]}{n} \) \((1 \leq \nu \leq r)\). In this case we have

\[
\det[m] H(\nu)_{i^{(\nu)} \in [n], j \in [n], D_l(\lambda^{(\nu)})} = \det[m] \left( \phi_{\nu}^{(\nu,1)}(\alpha q^{\lambda^{(\nu)}-1}) \cdots \phi_{\nu}^{(\nu,l)}(\alpha q^{\lambda^{(\nu)}-1}) \right)_{i^{(\nu)},j \in [n]} \times \omega(\alpha q^{\lambda^{(\nu)}-1}) \times (1-q)^{\alpha q^{\lambda^{(\nu)}-1}}.
\]

Hence we can obtain the left-hand side of (3.5) from (3.2) by putting \( N \to \infty \). Here we use the fact that \( l \) is even to rearrange the size of the \( i' \)th row as it fits (3.2). \( \square \)

For example, if \( l = 2, m = 2 \) and \( r = 1 \), then the above formula (3.5) reads

\[
\int_{0 \leq x_1 < \cdots < x_n \leq \alpha} \det(\phi_i(x_j) | \psi_i(x_j))_{i \in [2n], j \in [n]} d_{q}\omega(x) = \text{Pf} \left( \int_0^\alpha \phi_i(x_j) \psi_i(x_j) d_{q}\omega(x) \right)_{i_1, i_2 \in [2n]},
\]

where \( (\phi_i(x_j) | \psi_i(x_j))_{i \in [2n], j \in [n]} \) stands for the matrix whose \( i \)th row equals

\[
(\phi_i(x_1) \quad \psi_i(x_1) \quad \ldots \quad \phi_i(x_n) \quad \psi_i(x_n))
\]

for \( i \in [2n] \). One can prove the following corollary directly from Theorem 2.9 by similar arguments. The \( r = 1 \) case of (3.7) agrees with equation (96) in [28].

**Corollary 3.3.** Let \( l, n \) and \( r \) be positive integers such that \( l \) is even. Let \( d_{q}\omega(x) = w(x) d_{q}x \) be a measure on \([0, \alpha]\), and let \( \phi_i(x) \) be a function on \([0, \alpha]\) for \( i \in [l] \) and \( t \in [l] \). Then we have

\[
\int_{0 \leq x_1 < \cdots < x_n \leq \alpha} \prod_{\nu=1}^r \det \left( \phi_i^{(\nu,1)}(x_j) \cdots \phi_i^{(\nu,t)}(x_j) \right)_{i \in [l], j \in [n]} d_{q}\omega(x) = \begin{cases} 
\text{Pf}^{[l]}(Q(I^{(1)}, \ldots, I^{(r)}))_{I^{(1)}, \ldots, I^{(r)} \in \binom{[l]}{r}}, & \text{if } r \text{ is odd,} \\
\text{Hf}^{[l]}(Q(I^{(1)}, \ldots, I^{(r)}))_{I^{(1)}, \ldots, I^{(r)} \in \binom{[l]}{r}}, & \text{if } r \text{ is even.}
\end{cases}
\]

(3.7)
where
\[ Q(I^{(1)}, \ldots, I^{(r)}) = \int_0^\alpha \prod_{k=1}^r \det(\phi^{(\nu,\mu)}(x))_{1 \leq \lambda, \nu \leq l} \, d_q \omega(x) \]  
(3.8)
for \( I^{(\nu)} = \{i^{(\nu)}_1, \ldots, i^{(\nu)}_l\} \in \binom{[n]}{l} \) \( \nu \in [r] \).

Note that equation (3.7) generalizes Luque and Thibon’s generalization [28, (96)] of de Bruijn’s second formula. From here we study how to apply this second de Bruijn type formula. As in [16] we use the symbol
\[ \Delta_k^l(x_1, \ldots, x_n) = \prod_{i,j=1}^{k-1} (x_j - q^\nu x_i) (x_j - q^{-\nu} x_i) \]  
(3.9)
and \((q)_k = (q; q)_k\) in short. Recall that \( Pf^{[l]} \) stands for Barvinok’s original hyperpfaffian in the following corollary.

**Corollary 3.4.** Let \( l \) and \( n \) be positive integers such that \( l \) is even. Let \( t \) be an integer, and let \( \mu_i = \int_0^\alpha x^t d_q \omega(x) \) denote the \( i \)th moment of the measure \( \omega \). Then we have
\[ Pf^{[l]} \left( \prod_{1 \leq j < k \leq l} (q^{i_j-1} - q^{i_k-1}) \cdot \mu_{\sum_{k=1}^t i_k+t-l} \right)_{1 \leq i_1 < i_2 < \cdots < i_l \leq n} = q^n \frac{(l)}{2} (q)_k^{-n} \int_{0 \leq x_1 < \cdots < x_n \leq \alpha} \prod_{1 \leq i \leq n} (x_j - x_i)^{-1} \prod_{k=1}^l \Delta_k^l(x) \, d_q \omega(x), \]  
(3.10)
where \( d_q \omega(x) = d_q \omega(x_1) \cdots d_q \omega(x_n) \).

**Proof.** The proof mainly appeals to the Vandermonde determinant
\[ \Delta_N(X) = \det(X_{j-1}^{i-1})_{1 \leq i,j \leq N} = \prod_{1 \leq i < j \leq N} (X_j - X_i). \]  
(3.11)
We put \( m = 2 \) and \( r = 1 \) in Theorem 3.2 (or \( r = 1 \) in Corollary 3.3). Here we can write \( \phi_i^{(k)} \) for \( \phi_i^{(1,k)} \) and \( I \) for \( I^{(1)} \) in short in (3.5) and (3.6). We take \( \phi_i^{(k)}(x) = \begin{cases} q^{(l-k)(i-1)x^t+1+t} & \text{if } k = 1, \\ q^{(l-k)(i-1)x^{-1}} & \text{if } 2 \leq k \leq l, \end{cases} \)
in (3.6), then we obtain \( \det(\phi_i^{(k)}(x)))_{j,k \in [l]} = \det((q^{i-1}x^{-l-k})\, j,k \in [l] \, x^{|l-t|+t}, \) which implies
\[ Q(I) = \prod_{1 \leq j < k \leq l} (q^{i_j-1} - q^{i_k-1}) \int_0^\alpha x^{|l-t|+t} d_q \omega(x) = \prod_{1 \leq j < k \leq l} (q^{i_j-1} - q^{i_k-1}) \mu_{|l|^{-t+l+t}} \]
from (3.11) with \( N = l \). On the other hand, if we perform the same substitution as before in the left-hand side of (3.5), we find that
\[ \det \left( \phi_i^{(1)}(x_j) \right)_{i \in [n], j \in [n]} = \prod_{j=1}^n x_j^t \cdot \det \left( q^{(l-1)(i-1)x_j^{-1}} \right)_{i \in [n], j \in [n]}, \]
where the \(i\)th row of the matrix equals
\[
(q^{(l-1)(i-1)}x_1^{i-1}, \ldots, q^{i-1}x_1^{i-1}, x_1^{i-1}, \ldots, q^{(l-1)(i-1)}x_n^{i-1}, \ldots, q^{i-1}x_n^{i-1})
\]
which can be realized as the Vandermonde determinant (3.11) with \(N = ln\) as follows. Each integer \(i \in [ln]\) can be written in a unique way as \(i = l(j-1) + r\) for \(j \in [n]\) and \(r \in [l]\). If we choose \(X_i = q^{l-r}x_j\) for \(i \in [ln]\) with \(i = l(j-1) + r\) \((j \in [n]\) and \(r \in [l]\)), then we can write \(\Delta_{ln}(X) = \prod_{1 \leq i < i' \leq ln} (X_{i'} - X_i)\) as
\[
\prod_{j=1}^{n} \prod_{1 \leq r < r' \leq l} (X_{l(j-1)+r'} - X_{l(j-1)+r}) \prod_{1 \leq j < j' \leq n} \prod_{r,r' \in [l]} (X_{l(j'-1)+r'} - X_{l(j-1)+r})
\]
\[
= \prod_{j=1}^{n} x_j^{(l)} \prod_{1 \leq r < r' \leq l} (q^{l-r} - q^{l-r'})^{n-r'} \prod_{1 \leq j < j' \leq n} \prod_{r,r' \in [l]} q^{l-r'} \prod_{1 \leq j < j' \leq n} \prod_{r,r' \in [l]} (x_{j'} - q^{r-r}x_j)
\]
\[
= q^{(l^2)_j} (\prod_{j=1}^{n} x_j^{(l)})^{n-1} \prod_{k=1}^{l} (1 - q^k)^{n-k} \prod_{1 \leq j < j' \leq n} \prod_{k=1}^{l} (x_{j'} - q^kx_j)^{l-k} \prod_{k=1}^{l} (x_{j'} - q^{-k}x_j)^{l-k}
\]
by direct computation. Hence we obtain
\[
Pr^{[l,1]} \left( \prod_{1 \leq \lambda < \mu \leq l} (q^{k_{\lambda-1}} - q^{k_{\mu-1}}) \cdot \mu_{\sum_{\nu=1}^{l} \nu-l+u} \right)_{1 \leq i_1 < \cdots < i_l \leq ln}
\]
\[
= q^{(l^2)_j} \prod_{k=0}^{l-1} (q)_k^n \int_{0 \leq x_1 < \cdots < x_n \leq a} \prod_{i \leq j} (x_j - x_i)^l \prod_{k=1}^{l} (x_j - q^kx_i)^{l-k} \prod_{k=1}^{l} (x_j - q^{-k}x_i)^{l-k} \times \prod_{i=1}^{n} x_i^{u+\binom{l}{2}} d\omega(x_i).
\]
This immediately implies the desired identity.

Corollary 3.4 plays an important role in what follows. The idea of the proof relies on a judicious choice of the function \(\phi_i^{(k)}\) in Corollary 3.3 and we utilize the Vandermonde determinant. In fact we are not satisfied with this choice of the function since we are able to compute the integral in the right-hand side of (3.10) only in the case \(q \to 1\) or \(l = 2\). It is an interesting open problem to find another nice function \(\phi_i^{(k)}\) which will lead to define the right 
"Hankel \(q\)-hyperpfaffian". Meanwhile, if we take \(q \to 1\) in Corollary 3.4, we immediately obtain the following formula which is satisfactory for us since, in many cases, we can compute the hyperpfaffian for general \(l\) by appealing to the Aomoto-Selberg type integrals. We give an example of applications of this corollary in the next section.

**Corollary 3.5.** Let \(l\) and \(n\) be positive integers such that \(l\) is even, and \(t\) an integer. Let \(d\psi(x) = \psi'(x)dx\) be a measure on an interval \([0, \alpha]\) and let \(\mu_i = \int_0^{\alpha} x^i d\psi(x)\) denote the \(i\)th
moment of the measure $\psi$. Then we have

$$\text{Pf}_{1,1}^{[1,1]} \left( \prod_{1 \leq i < j \leq t} (i_k - i_j) \cdot \mu \sum_{i=1}^{t} i_k - l_i \right)_{1 \leq i_1 < \cdots < i_t \leq n}$$

$$= \prod_{k=1}^{t} \left( \frac{(k - 1)!}{n!} \right)^n \int_{[0, \alpha]^n} \prod_{i=1}^{t} x_{i+1} \prod_{i<j} (x_j - x_i)^2 \, d\psi(x).$$

(3.12)

**Proof.** If we take $q \to 1$ in (3.10) then it is easy to see that

$$\prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1} \prod_{k=1}^{t} \Delta_k(x) \to \prod_{i<j} (x_j - x_i)^2.$$ Since the integrand is a symmetric function in this case, we can change the range of the integration to $[0, \alpha]^n$ and divide by $n!$.

4 Hankel hyperpfaffians for Narayana polynomials

This section deals with an application of Corollary 3.5 to concrete hyperpfaffian computations. In [18, Conjecture 6.2] we presented a conjecture on Pfaffian identities involving Motzkin, Delannoy, Schröder numbers and Narayana polynomials. We define the Narayana polynomials of type $A$, $B$ and $D$, which provide us a unified treatment of these combinatorial numbers. The master theorem of this section is Theorem 4.1. The proof of this theorem shows when we can apply (1.2) or (1.3) to the hyperpfaffians involving the Narayana polynomials of type $A$, $B$ and $D$. As corollaries we derive remarkable Pfaffian identities from the master theorem.

4.1 Definitions and main results

For a nonnegative integer $n$, we introduce *Narayana numbers* of type $A$, $B$ and $D$ by

$$N_k(A_n) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}, \quad N_k(B_n) = \binom{n}{k}^2, \quad N_k(D_n) = \binom{n}{k} \left\{ \binom{n-1}{k} + \binom{n-2}{k-2} \right\}.$$ The Narayana polynomials $N(X_n, a) (n \geq 0)$ are defined by

$$N(X_n, a) = \sum_{k=0}^{n} N_k(X_n) a^k$$

(4.1)

for $X = A, B$ or $D$, where we use the convention that $N(A_0, a) = N(D_0, a) = 1, N(D_1, a) = \frac{a+1}{2}$ (see [34, pp. 277–278]).

For convenience we introduce the notation

$$\Phi_n(r, s, m) = \prod_{j=1}^{n} \frac{2m(j-1)+r+2s}{m(j-1)+r} \frac{2m(j-1)+2s}{m(j-1)+s} \prod_{k=2}^{n} \frac{mk}{m(2k-3)+r+s}$$

(4.2)

for nonnegative integers $r, s, m$ and $n$. This is related to the value of Selberg integral (see Lemma 4.7). In this paper we take $\sqrt{a} = \sqrt{r}e^{i\theta/2}$ for a complex number $a = re^{i\theta} (r \geq 0$ and $-\pi < \theta \leq \pi)$. The following is the master theorem of this section, and all the succeeding Pfaffian identities in this section follow from this theorem.
**Theorem 4.1.** Let $l$ and $n$ be positive integers such that $l$ is even. Let $r$ be an integer such that $r \geq -\binom{l}{2}$ (hence the entries of each hyperpfaffian are well-defined), and let $a \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Let $\tau_a = \frac{\left(\frac{\sqrt{a}}{\sqrt{\pi}}\right)^2}{\sqrt{a}}$ and $H_{l,n} = \frac{1}{n!} \prod_{k=1}^{l} \{ (k-1)! \}^n$. Then the following identities hold:

$$
\text{Pf}^{[l,1]} \left( \prod_{1 \leq j < k \leq l} (i_k - i_j) \cdot N \left( A_{|I|+r-l}, a \right) \right)_{I \in \binom{[l]}{r}}
$$

$$
= \begin{cases} 
2^{-n} H_{l,n} \cdot \Phi_n \left( r + \binom{l}{2}, 1, \frac{\omega}{2} \right) & \text{if } (a,r) = (1,r), \\
2^{-n} a^{n+\frac{r}{2}} \left( \frac{\omega}{2} \right) H_{l,n} \Phi_n \left( 1, 1, \frac{\omega}{2} \right) & \text{if } (a,r) = (a,1-\binom{l}{2}), \\
2^n a^{\frac{r}{2} + \frac{r}{2}} \left( \frac{\omega}{2} \right) H_{l,n} \Phi_n \left( 0, 0, \frac{\omega}{2} \right) \sum_{k=0}^{n} \left( \binom{n}{k} \right) \tau_a^{n-k} \prod_{j=1}^{k} \frac{3+2(n-j)}{2+2^{r}(2n-j-1)} & \text{if } (a,r) = (a,2-\binom{l}{2}).
\end{cases}
\quad (4.3)
$$

$$
\text{Pf}^{[l,1]} \left( \prod_{1 \leq j < k \leq l} (i_k - i_j) \cdot N \left( B_{|I|+r-l}, a \right) \right)_{I \in \binom{[l]}{r}}
$$

$$
= \begin{cases} 
H_{l,n} \Phi_n \left( r + \binom{l}{2}, 0, \frac{\omega}{2} \right) & \text{if } (a,r) = (1,r), \\
a^{\frac{r}{2}} \left( \frac{\omega}{2} \right) H_{l,n} \Phi_n \left( 0, 0, \frac{\omega}{2} \right) & \text{if } (a,r) = (a,-\binom{l}{2}), \\
2^{n} a^{\frac{r}{2} + \frac{r}{2}} \left( \frac{\omega}{2} \right) H_{l,n} \Phi_n \left( 0, 0, \frac{\omega}{2} \right) \sum_{k=0}^{n} \left( \binom{n}{k} \right) \tau_a^{n-k} \prod_{j=1}^{k} \frac{1+2(n-j)}{2+2^{r}(2n-j-1)} & \text{if } (a,r) = (a,1-\binom{l}{2}).
\end{cases}
\quad (4.4)
$$

$$
\text{Pf}^{[l,1]} \left( \prod_{1 \leq j < k \leq l} (i_k - i_j) \cdot N \left( D_{|I|+r-l}, a \right) \right)_{I \in \binom{[l]}{r}}
$$

$$
= \begin{cases} 
2^{n} H_{l,n} \Phi_n \left( r + \binom{l}{2} - 1, 0, \frac{\omega}{2} \right) \sum_{k=0}^{n} \left( \binom{n}{k} \right) \left( -\frac{1}{2} \right)^{n-k} \prod_{j=1}^{k} \frac{2r-l-1+2(n-j+1)}{2r-l+2(2n-j)} & \text{if } (a,r) = (1,r), \\
2^{n} \omega^{n+\binom{r}{2}} H_{l,n} \Phi_n \left( 1, 0, \frac{\omega}{2} \right) \sum_{k=0}^{n} \left( \binom{n}{k} \right) \left( -\frac{\omega}{8} \right)^{n-k} \prod_{j=1}^{k} \frac{3+2(n-j)}{4+2^{r}(2n-j-1)} & \text{if } (a,r) = (\omega,2-\binom{l}{2}).
\end{cases}
\quad (4.5)
$$

The sums in the right-hand side come from Aomoto integral (1.4). We first note that most of the famous combinatorial numbers are related to the Narayana polynomials, as detailed below.

Let $n$ be a nonnegative integer, and let $\omega = \frac{-1 + \sqrt{3}}{2}$, which is a primitive cube root of unity. By specializing $a$ in (4.1), we obtain several classical numbers: the Catalan numbers $\text{Cat}(n) = N(A_n, 1) = \frac{1}{2n+1} \binom{2n+1}{n}$, the large Schröder numbers $\text{Sch}(n) = N(A_n, 2) = \sum_{k=0}^{n} \binom{n}{2k} \text{Cat}(k)$, the central binomial coefficients $\text{CBC}(n) = N(B_n, 1) = \binom{2n}{n}$, the central Delannoy numbers $\text{Del}(n) = N(B_n, 2) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$. We call the sequence $\text{Cat}^D(n) =$
\( N(D_n, 1) = (3n - 2) \text{Cat}(n - 1) \) \((n \geq 1)\) the Catalan numbers of type \( D \), which have several combinatorial meanings (see [34, 12.3], A051924). Further we have the Motzkin numbers \( \text{Mot}(n) = (-1)^n \omega^{n+2} N(A_{n+1}, \omega) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \begin{array}{c} n \\ 2k \end{array} \right) \text{Cat}(k) \) for \( n \geq 0 \), and the central trinomial coefficients \( \text{CTC}(n) = (-1)^n \omega^n N(B_n, \omega) \) for \( n \geq 0 \), which is the coefficient of \( x^n \) in the expansion of \((1 + x + x^2)^n\). Let \( _2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n \) denote the hypergeometric series with Pochhammer symbol \((x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\). Finally, the Motzkin numbers of type \( D \) are defined by

\[
\text{Mot}^D(n) = (-1)^n \omega^n N(D_n, \omega) = _2F_1 \left( \frac{1-n}{2}, 1 - \frac{n}{2}; 1; 4 \right) + (n-2)_2F_1 \left( 1 - \frac{n}{2}, \frac{3-n}{2}; 2; 4 \right)
\]

for \( n \geq 2 \) (A298300). These specializations are roughly summarized in Table 1.

| Type | \( A \) | \( B \) | \( D \) |
|------|------|------|------|
| \( a = 1 \) | \text{Cat}(n) | \text{CBC}(n) | \text{Cat}^D(n) |
| \( a = 2 \) | \text{Sch}(n) | \text{Del}(n) | |
| \( a = \omega \) | \text{Mot}(n) | \text{CTC}(n) | \text{Mot}^D(n) |

**Table 1:** Specializations of Narayana polynomials

By putting \( l = 2 \) and substituting appropriate values into \( a \) and \( r \) in Theorem 4.1, we obtain the following identities conjectured in [18, Conjecture 6.2] for ordinary Pfaffians.

**Corollary 4.2.** Let \( n \) be a positive integer. Then the following identities hold:

\[
Pf \left( (j-i) \text{Mot}(i + j - 3) \right)_{1 \leq i, j \leq 2n} = \prod_{k=0}^{n-1} (4k + 1), \quad (4.6)
\]
\[
Pf \left( (j-i) \text{Del}(i + j - 3) \right)_{1 \leq i, j \leq 2n} = 2^{n^2 - 1} (2n - 1) \prod_{k=1}^{n-1} (4k - 1), \quad (4.7)
\]
\[
Pf \left( (j-i) \text{Sch}(i + j - 2) \right)_{1 \leq i, j \leq 2n} = 2^{n^2} \prod_{k=0}^{n-1} (4k + 1), \quad (4.8)
\]
\[
Pf \left( (j-i) N(A_{i+j-2}, a) \right)_{1 \leq i, j \leq 2n} = a^{n^2} \prod_{k=0}^{n-1} (4k + 1). \quad (4.9)
\]

In fact, by substituting other values into \( a \) and \( r \) of Theorem 4.1, we obtain several more remarkable Pfaffian identities. In Table 2 the reader can see how to specialize the parameters to obtain the identities in the corollaries above and below.
Table 2: Derivation of the Pfaffian identities

| Identity | Type | Master Identity | l | a | \( r \in \mathbb{Z} \) |
|----------|------|----------------|---|---|------------------|
| (4.6)    | A    | (4.3)          | 2 | \( \omega \) | 0               |
| (4.7)    | B    | (4.4)          | 2 | 2 | -1              |
| (4.8)    | A    | (4.3)          | 2 | 2 | 0               |
| (4.10)   | A    | (4.3)          | 2 | \( \omega \) | 1               |
| (4.11)   | B    | (4.4)          | 2 | 2 | 0               |
| (4.12)   | A    | (4.3)          | 2 | 2 | 1               |
| (4.13)   | D    | (4.5)          | 2 | \( \omega \) | 1               |
| (4.14)   | A    | (4.3)          | 2 | 1 | \( r \geq -1 \) |
| (4.15)   | B    | (4.4)          | 2 | 1 | \( r \geq -1 \) |
| (4.16)   | D    | (4.5)          | 2 | 1 | \( r \geq 0 \)  |

**Corollary 4.3.** Let \( n \) be a positive integer. Then

\[
Pf\left( (j - i) \text{Mot}(i + j - 2) \right)_{1 \leq i, j \leq 2n} = 2^{2n} \prod_{k=0}^{n-1} (4k + 1) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( -\frac{1}{4} \right)^{n-k} \prod_{j=1}^{k} \frac{3 + 4(n-j)}{2 + 4(2n-j)}. \tag{4.10}
\]

\[
Pf\left( (j - i) \text{Del}(i + j - 2) \right)_{1 \leq i, j \leq 2n} = 2^{n+\frac{n}{4}} \prod_{k=1}^{n-1} (4k - 1) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{3\sqrt{2} - 4}{8} \right)^{n-k} \prod_{j=1}^{k} \frac{1 + 4(n-j)}{4(2n-j) - 2}. \tag{4.11}
\]

\[
Pf\left( (j - i) \text{Sch}(i + j - 1) \right)_{1 \leq i, j \leq 2n} = 2^{n+\frac{n}{4}} \prod_{k=0}^{n-1} (4k + 1) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{3\sqrt{2} - 4}{8} \right)^{n-k} \prod_{j=1}^{k} \frac{3 + 4(n-j)}{2 + 4(2n-j)}. \tag{4.12}
\]

\[
Pf\left( (j - i) \text{Mot}(i + j - 1) \right)_{1 \leq i, j \leq 2n} = 2^{2n} \prod_{k=0}^{n-1} \frac{(4k + 2)!(4k)!}{(2k)!^2(2k + 2n - 1)!} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( -\frac{5}{8} \right)^{n-k} \prod_{j=1}^{k} \frac{3 + 4(n-j)}{4(2n-j)}. \tag{4.13}
\]

Further, the case of \( a = 1 \) in Theorem 4.1 gives us the following formulas:

**Corollary 4.4.** Let \( n \) be a positive integer, and \( r \geq -1 \) be an integer. Then

\[
Pf\left( (j - i) \text{Cat}(i + j + r - 2) \right)_{1 \leq i, j \leq 2n} = 2^{-n} \prod_{k=0}^{n-1} \frac{(4k + 2r + 2)!(4k + 2)!}{(2k + r + 1)!(2k + 2n + r)!}, \tag{4.14}
\]

\[
Pf\left( (j - i) \text{CBC}(i + j + r - 2) \right)_{1 \leq i, j \leq 2n} = \prod_{k=0}^{n-1} \frac{(4k)!(4k + 2r + 2)!(2k + 1)}{(2k + r + 1)!(2k + 2n + r - 1)!}, \tag{4.15}
\]

\[
Pf\left( (j - i) \text{Cat}(i + j + r - 2) \right)_{1 \leq i, j \leq 2n}
\]

\[
= \frac{2^n}{n!} \prod_{k=0}^{n-1} \frac{(4k + 2r)!}{(2k + r)!(2k)!(2n + 2k + r - 2)!} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( -\frac{1}{4} \right)^{n-k} \prod_{j=1}^{k} \frac{2r + 1 + 4(n-j)}{2r - 2 + 4(2n-j)}. \tag{4.16}
\]
Here we assume \( r \geq 0 \) in (4.16) (see the definition of \( \text{Cat}^D \)).

The *Hankel Pfaffian transform* of a sequence \( \{a_n\}_{n \geq 0} \) is defined to be the sequence \( \{\mathcal{P}_n\}_{n \geq 0} \) of Pfaffians \( \mathcal{P}_n = \text{Pf} ((j - i)a_{i+j-r})_{1 \leq i, j \leq 2n} \). Corollary 4.4 gives the Hankel Pfaffian transforms of the sequences \( \{\text{Cat}(n + r)\}_{n \geq 0} \) and \( \{\text{CBC}(n + r)\}_{n \geq 0} \) for a fixed \( r \). We summarize some identified sequences with OEIS references of (4.14) and 4.15 in Table 3.

| Identity | \( r = -1 \) | \( r = 0 \) | \( r = 1 \) | \( r = 2 \) |
|----------|--------------|--------------|--------------|--------------|
| (4.14)   | A000108      | A007696      | A001813      | A007696      |
| (4.15)   | A147626      |              |              |              |

Table 3: Hankel Pfaffian transforms of Catalan numbers and CBCs

As pointed in [18], we can also derive (4.14) and (4.15) from (1.1). It is interesting to note that (4.14) is a Pfaffian analogue of Desainte-Catherine and Viennot’s determinant identity:

\[
\det(\text{Cat}(i + j + r - 1))_{1 \leq i, j \leq n} = \prod_{1 \leq i \leq j \leq r} \frac{i + j + 2n}{i + j},
\]

(4.17)

see [10, 38]. The reader may think that the product in the right-hand side of (4.14) is not so simple as that of (4.17). But, if one takes the ratio of the products between \( r \) and \( r + 1 \), it looks very similar to the ratio of (4.17). As shown in [10], the determinant of (4.17) is the number of Young tableaux with entries from \([r]\) satisfying: there are at most \( 2n \) rows; the rows are strictly increasing; the columns are non-decreasing; every column has an even number of cells. It would be a challenging problem to find combinatorial objects which count some of the above Pfaffians.

### 4.2 Proof of Theorem 4.1

Now we proceed to prove Theorem 4.1. First we need to find the generating function

\[
G(X, a, z) = \sum_{n=0}^{\infty} N(X_n, a) z^n
\]

of the Narayana polynomials (4.1) for \( X = A, B \) or \( D \).

**Lemma 4.5.** We have

\[
G(A, a, z) = \frac{1 - (a - 1)z - \sqrt{(a - 1)^2z^2 - 2(a + 1)z + 1}}{2z},
\]

(4.18)

\[
G(B, a, z) = \frac{1}{\sqrt{(a - 1)^2z^2 - 2(a + 1)z + 1}},
\]

(4.19)

\[
G(D, a, z) = \frac{1}{2} \left\{ \sqrt{(a - 1)^2z^2 - 2(a + 1)z + 1} + \frac{1 + (a + 1)z}{\sqrt{(a - 1)^2z^2 - 2(a + 1)z + 1}} \right\}.
\]

(4.20)
Proof. Let $y$ be the function at the right-hand side of (4.18) multiplied by $z$. Then

$$y^2 + \{(a - 1)z - 1\}y + z = 0$$

and $\lim_{z \to 0} \frac{y}{z} = 1$. This equation can be rewritten as $y = z\left(1 + \frac{az}{1-y}\right)$. By Lagrange inversion formula with $\phi(x) = 1 + \frac{ax}{1-x}$ (see [2, Appendix E]), we have

$$[z^{n+1}]y = \frac{1}{n+1}[y^n]\left(1 + \frac{ay}{1-y}\right)^{n+1} = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} a^k [y^n] y^k (1-y)^{-k}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} a^k \frac{(k)_{n-k}}{(n-k)!} = \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{n}{k-1} a^k.$$

This establishes (4.18).

The Legendre polynomials $P_n(x)$ have the explicit formula (see [35, p. 162]):

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k.$$

Substituting $a = \frac{x+1}{x-1}$, we obtain

$$\sum_{k=0}^{n} \binom{n}{k}^2 a^k = (a - 1)^n P_n \left(\frac{a + 1}{a - 1}\right).$$

Equation (4.19) then follows from the following generating function (see [35, Chapter 10])

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

By definition we have

$$N_k(D_n) = \left(\binom{n}{k}\right)^2 - \binom{n}{k} \left(\binom{n-2}{k-1}\right) = N_k(B_n) - nN_k(A_{n-1})$$

for $n \geq 2$, $N(A_0, a) = N(B_0, a) = N(D_0, a) = 1$, $N(B_1, a) = a + 1$ and $N(D_1, a) = \frac{a+1}{2}$.

Since

$$(zG(A, a, z))' = \sum_{n \geq 0} (n+1) N(A_n, a) z^n = \sum_{n \geq 1} n N(A_{n-1}, a) z^{n-1},$$

we have $G(D, a, z) = G(B, a, z) - (zG(A, a, z))' \cdot z - \frac{a-1}{2} z$, which immediately implies (4.20) by using (4.18) and (4.19).

Given sequence $\{(\mu_n)_{n \geq 0}\}$ of real numbers, a necessary and sufficient condition for the existence of a measure $\psi$ such that $\int_{-\infty}^{\infty} x^n d\psi(x) = \mu_n$ is that the Hankel determinants
\[ \det (\mu_{i+j-2})_{1 \leq i, j \leq n} \text{ are all nonzero (} n \geq 1 \text{). Let } G(z) = \sum_{n=0}^{\infty} \mu_n z^n. \text{ Then the Stieltjes transform of the measure } \psi \text{ is defined by} \]

\[ g(z) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{z - x} = \frac{1}{z} G \left( \frac{1}{z} \right). \]

The distribution function \( \psi \) can be recovered from \( g(z) \) by means of the Stieltjes inversion formula:

\[ \psi(t) - \psi(t_0) = -\frac{1}{\pi} \lim_{y \to +0} \int_{t_0}^{t} \text{Im} \ g(x + iy) \, dx, \]

where \( \text{Im} \ z \) stands for the imaginary part of \( z \). Namely

\[ \psi'(x) = \lim_{y \to +0} \frac{g(x - iy) - g(x + iy)}{2\pi i}, \quad (4.21) \]

see [9, Chapter 3].

**Lemma 4.6.** Let \( a \) be a positive real number. Let \( \psi(X, a, x) \) denote the distribution function for the moment generating function \( G(X, a, z) \) of type \( X = A, B \) or \( D \). Then

\[ \psi'(A, a, x) = \begin{cases} \frac{\sqrt{4a - (x-a-1)^2}}{2\pi x} & \text{if } (\sqrt{a} - 1)^2 \leq x \leq (\sqrt{a} + 1)^2, \\ 0 & \text{otherwise,} \end{cases} \]

\[ \psi'(B, a, x) = \begin{cases} \frac{1}{2\sqrt{4a - (x-a-1)^2}} & \text{if } (\sqrt{a} - 1)^2 \leq x \leq (\sqrt{a} + 1)^2, \\ 0 & \text{otherwise,} \end{cases} \]

\[ \psi'(D, a, x) = \begin{cases} \frac{2x^2 - 2(a+1)x + (a-1)^2}{2\pi x^2 \sqrt{4a - (x-a-1)^2}} & \text{if } (\sqrt{a} - 1)^2 \leq x \leq (\sqrt{a} + 1)^2, \\ 0 & \text{otherwise,} \end{cases} \]

**Proof.** To calculate \( \psi'(A, a, x) \) from (4.21), we use

\[ g(A, a, z) = \frac{1}{z} \left( I_{A, a, \frac{1}{z}} \right) = \frac{z - (a - 1) - \sqrt{z^2 - 2(a+1)z + (a-1)^2}}{2z}. \]

Since \( \frac{z - (a - 1)}{2z} \) is a rational function, this part does not contribute to \( \psi'(A, a, x) \). Note that

\( (x + iy)^2 - 2(a+1)(x + iy) + (a - 1)^2 = x^2 - y^2 - 2(a+1)x + (a-1)^2 \mp 2i\{x - (a + 1)\}y. \)

Hence, we conclude \( \psi'(A, a, x) = 0 \) if \( x^2 - 2(a+1)x + (a-1)^2 \geq 0 \). Assume \( x^2 - 2(a+1)x + (a-1)^2 < 0 \). Then we put \( y \to +0 \) if \( x - (a + 1) \geq 0 \), and \( y \to -0 \) if \( x - (a + 1) < 0 \). Thus we obtain (4.22). The other identities (4.23) and (4.24) are obtained by similar arguments.

\[ \square \]

The following lemma simplifies the values of the Selberg integrals in Theorem 4.1.

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Lemma 4.7. Let $r, s, m$ and $n$ be nonnegative integers. Then
\[ S_n \left( r + \frac{1}{2}, s + \frac{1}{2}, m \right) = \frac{\pi^n}{2^{2n(mn-1)+r+s}} \cdot \Phi_n(r, s, m) \quad (4.25) \]

Proof. We use $\Gamma(x + n) = \Gamma(x)(x)_n$ for nonnegative integer $n$. Then the Gamma function formulas $\Gamma \left( r + m(j - 1) + \frac{1}{2} \right) = \left( \frac{i}{2} (\frac{i}{2})_{m(j-1)+r} \right)$ and $\Gamma \left( \frac{i}{2} \right) = \sqrt{\pi}$ lead to the desired identity by direct computation. \qed

Proof of Theorem 4.1. Let $\mathcal{A} = \mathcal{A}(a, r)$, $\mathcal{B} = \mathcal{B}(a, r)$ and $\mathcal{D} = (a, r)$ be the left-hand side of (4.3), (4.4) and (4.5), respectively. Using (3.12) with (4.22), (4.23) and (4.24), we obtain

\[ \mathcal{A}(a, r) = \frac{H_{1,n}}{(2\pi)^n} \int_{I_n} \prod_{i=1}^{n} x_i^{r+i-1} \frac{x_i}{\sqrt{4a - (x_i - a - 1)^2}} \prod_{i<j} (x_j - x_i)^2 \, dx, \]

\[ \mathcal{B}(a, r) = \frac{H_{1,n}}{(2\pi)^n} \int_{I_n} \prod_{i=1}^{n} \frac{x_i^{r+i-1}}{\sqrt{4a - (x_i - a - 1)^2}} \prod_{i<j} (x_j - x_i)^2 \, dx, \]

\[ \mathcal{D}(a, r) = \frac{H_{1,n}}{(2\pi)^n} \int_{I_n} \prod_{i=1}^{n} \frac{x_i^{r+i-2} \left\{ 2x_i^2 - (a+1)x_i + (a-1)^2 \right\}}{\sqrt{4a - (x_i - a - 1)^2}} \prod_{i<j} (x_j - x_i)^2 \, dx, \]

where $I_n = [(\sqrt{a} - 1)^2, (\sqrt{a} + 1)^2]$. If we set $x_i = 4\sqrt{a} t_i + (\sqrt{a} - 1)^2$ in the above integrals, then a straightforward computation leads to the following identities:

\[ \mathcal{A}(a, r) = C^A_{n,r} \int_{[0,1]^n} \prod_{i=1}^{n} \left\{ t_i + \frac{(\sqrt{a} - 1)^2}{4\sqrt{a}} \right\}^{r+i-1} \frac{t_i}{t_i(1-t_i)} \prod_{i<j} (t_j - t_i)^2 \, dt, \quad (4.26) \]

\[ \mathcal{B}(a, r) = C^B_{n,r} \int_{[0,1]^n} \prod_{i=1}^{n} \left\{ t_i + \frac{(\sqrt{a} - 1)^2}{4\sqrt{a}} \right\}^{r+i-1} \frac{t_i}{t_i(1-t_i)} \prod_{i<j} (t_j - t_i)^2 \, dt, \quad (4.27) \]

\[ \mathcal{D}(a, r) = C^D_{n,r} \int_{[0,1]^n} \prod_{i=1}^{n} \left\{ t_i + \frac{(\sqrt{a} - 1)^2}{4\sqrt{a}} \right\}^{r+i-2} \left( t_i - \eta^+ \right) \left( t_i - \eta^- \right) \prod_{i<j} (t_j - t_i)^2 \, dt, \quad (4.28) \]

where $C^A_{n,r} = \frac{H_{1,n}}{(2\pi)^n} \left\{ \begin{array}{c} r+i-1 \end{array} \right\}^n \pi^2 \left( \begin{array}{c} n \end{array} \right)$, $C^B_{n,r} = C^D_{n,r} = \frac{H_{1,n}}{(2\pi)^n} \left\{ \begin{array}{c} r+i \end{array} \right\}^n \pi^2 \left( \begin{array}{c} n \end{array} \right)$ and $\eta^\pm = \frac{-3(a+1) + 8\sqrt{a} \pm \sqrt{4a^2 - 7(a-1)^2}}{16a}.$

Note that (4.26), (4.27) and (4.28) are proven under the assumption that $a > 0$ is a real number. However, in these identities the hyperpfaffians in the left-hand sides are polynomials of $a$ and the right-hand sides are rational functions of $\sqrt{a}$. Using the identity theorem [27, Chapter 3, Theorem 1.2], we see that the left-hand and the right-hand side agree as far as the square root is defined by analytic continuation.

Now we determine when these integrals are as in the form of the Selberg integral (1.2) or (1.4).
First, assume \((a, r) = (1, r)\). Then we obtain
\[
\mathcal{J}_A(1, r) = C_{n,r}^A \int_{[0,1]^n} \prod_{i=1}^n t_i^{r\left(\frac{l}{2}\right)-1/2}(1-t_i)^{1/2} \prod_{i<j} (t_j-t_i)^2 \, dt
\]
\[
= C_{n,r}^A \cdot S_n \left( r + \left( \frac{l}{2} \right) + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{l^2}{2} \right),
\]
\[
\mathcal{J}_B(1, r) = C_{n,r}^B \int_{[0,1]^n} \prod_{i=1}^n t_i^{r\left(\frac{l}{2}\right)-1/2}(1-t_i)^{-1/2} \prod_{i<j} (t_j-t_i)^2 \, dt
\]
\[
= C_{n,r}^B \cdot S_n \left( r + \left( \frac{l}{2} \right) + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{l^2}{2} \right),
\]
\[
\mathcal{J}_D(1, r) = C_{n,r}^D \int_{[0,1]^n} \prod_{i=1}^n (t_i - \frac{1}{4}) t_i^{r\left(\frac{l}{2}\right)-3/2} (1-t_i)^{-1/2} \prod_{i<j} (t_j-t_i)^2 \, dt
\]
\[
= C_{n,r}^D \cdot S_n \left( r + \left( \frac{l}{2} \right) - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{l^2}{2} \right) \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \left( -\frac{1}{4} \right)^{n-k} \prod_{j=1}^k \frac{r + \left( \frac{l}{2} \right) - \frac{1}{2} + \frac{l^2}{2} (n-j)}{r + \left( \frac{l}{2} \right) + \frac{l^2}{2} (2n-j-1)}
\]
from (1.2) and (1.4). Hence, using (4.25), we obtain the desired identities, i.e., (4.3), (4.4) and (4.5), when \((a, r) = (1, r)\).

Hereafter we may assume \(a \neq 1\). If \(r = 1 - \left( \frac{l}{2} \right)\) then (4.26) reduces to (1.2). Hence we obtain
\[
\mathcal{J}_A \left( a, 1 - \left( \frac{l}{2} \right) \right) = C_{n,r}^A \cdot S_n \left( \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{l^2}{2} \right) = 2^{-n} H_{i,n} a^{n+\frac{l^2}{2}} \Phi_n \left( 1, 1, \frac{l^2}{2} \right),
\]
which is equal to the second case of (4.3). If we put \(r = 2 - \left( \frac{l}{2} \right)\) in (4.26) then we obtain
\[
\prod_{i=1}^n (t_i + \tau_a)^{r\left(\frac{l}{2}\right)-1} = \sum_{k=0}^n e_k(t) \tau_{a}^{n-k}. \] Hence, by (1.4) we obtain
\[
\mathcal{J}_A \left( a, 2 - \left( \frac{l}{2} \right) \right) = C_{n,r}^A \cdot S_n \left( \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{l^2}{2} \right) \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \tau_{a}^{n-k} \prod_{j=1}^k \frac{\frac{3}{3} + \frac{l^2}{2} (n-j)}{3 + \frac{l^2}{2} (2n-j-1)},
\]
which proves the third case of (4.3).

Next, for \(X = B\), if we put \(r = -\left( \frac{l}{2} \right)\) in (4.27) then we obtain
\[
\mathcal{J}_B \left( a, -\left( \frac{l}{2} \right) \right) = C_{n,r}^B \cdot S_n \left( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{l^2}{2} \right) = H_{i,n} a^{\frac{l^2}{2}} \Phi_n \left( 0, 0, \frac{l^2}{2} \right),
\]
which proves the second case of (4.4). If we put \(r = 1 - \left( \frac{l}{2} \right)\) in (4.27), then we have
\[
\mathcal{J}_B \left( a, 1 - \left( \frac{l}{2} \right) \right) = C_{n,r}^B \cdot S_n \left( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{l^2}{2} \right) \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \tau_{a}^{n-k} \prod_{j=1}^k \frac{\frac{3}{3} + \frac{l^2}{2} (n-j)}{3 + \frac{l^2}{2} (2n-j-1)},
\]
which agrees with the third case of (4.4).

Finally, we consider the $X = D$ case. When $a \neq 1$, (4.28) can be in the form of (1.2) or (1.4) only if $r + \binom{l}{2} - 2 = 0$ and one of $\eta_a^+$ or $\eta_a^-$ equals 0 or 1. This actually happens if and only if $a = \omega$ or $a = \omega^{-1}$ in which case we have $\eta_a^- = 0$ and $\eta_a^+ = \frac{5}{8}$. Hence we obtain

$$J_D \left( \omega^{\pm 1}, 2 - \binom{l}{2} \right) = C_{n,r} S_n \left( \frac{3}{2}, \frac{1}{2}, \frac{l^2}{2} \right) \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{5}{8} \right)^{n-k} \prod_{j=1}^{k} \frac{3}{2} + \frac{l^2}{2} (n-j-1)$$

$$= \frac{1}{2^{2n}} \omega^{n} \Phi_n \left( \frac{\omega^{1/2}}{2}, \frac{\omega^{1/2}}{2} \right) \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{5}{8} \right)^{n-k} \prod_{j=1}^{k} \frac{3}{2} + \frac{l^2}{2} (2n-j-1)$$

by (1.4). This completes the proof of the theorem.

**Remark 4.8.** If one writes the product in (4.26), (4.27) or (4.28) as a linear combination of the Jack polynomials and uses [22, Theorem 1, 2] or [39, Corollary 1.2], then he/she can obtain more formulas of the hyperpfaffians for other values of $r$.

In this section we studied only the Narayana polynomials of type $A$, $B$ and $D$ since they cover the most of famous combinatorial numbers as we noted. The common feature of the combinatorial numbers of this section is that the generating functions of the sequences are always solutions of quadratic equations as we saw in Lemma 4.5. This makes us easier to find the distribution function and evaluate the related Selberg-Aomoto integral. Of course, we can choose other moment sequences. However it is not always easy to find the distribution function. Conjecture 7.1 in the last section is such an example.

## 5 Hankel Pfaffians and little $q$-Jacobi polynomials

The aim of this section is to give a new proof of (1.1), i.e., Theorem 5.1, (5.2). This $q$-identity first appeared in [18, Corollary 3.2], and also proved in [14], [15, Corollary 3.4] by using a quadratic formula for the basic hypergeometric series related to the Askey-Wilson polynomials. Here we use a different approach from the above two proofs, that is, we establish a key identity (5.12) which enable us to compute $q$-Pfaffians of this type. This key identity will be used to derive Theorem 5.1, and also used to derive Theorem 6.1 in the next section.

As mentioned just before Corollary 3.5, another case that we can evaluate (3.10) is when $l = 2$. Then we call the left-hand side of (3.10) $q$-Hankel Pfaffian. We give examples of $q$-Hankel Pfaffians in this section and the next one. In this section we reduce the evaluation of the $q$-Hankel Pfaffian (5.2) to the $k = 2$ case of the Askey-Habsieger-Kadell integral (5.18).

This section is composed as follows. In [16, 21] the $q$-Selberg integral has several expressions which include one of the $q$-difference products $\Delta_k(x)$, $\Delta^0_k(x)$, $\Delta^1_k(x)$ and $\Delta^2_k(x)$ (to be defined later). To prove Theorem 5.1, we first state the relations between the integrals which include the $q$-difference products (see (5.5), (5.9) and (5.10)). These identities hold for any $q$-measure $\omega$ and any $l = 2k$. Then we set $l = 2$ to obtain the Pfaffian identity (5.12), which will be an appropriate form to apply the Askey-Habsieger-Kadell formula (5.18). Finally we
take the measure which gives the little $q$-Jacobi polynomials, and perform a straightforward computation to prove Theorem 5.1.

If we put $l = 2$ in (3.10), we obtain

$$
\text{Pf} \left( (q^{l-1} - q^{l-1})_{i+j+r-2} \right)_{1 \leq i < j \leq 2n} = \frac{q^{n(n-1)}(1-q)^n}{n!} \int_{[0,a]^n} \prod_{i=1}^{2n} x_i^{r+1} \cdot \Delta_2^I(x) \, d_q \omega(x). \quad (5.1)
$$

In this section we use the $q$-analogue of the Selberg integral formula (5.18) to evaluate this Pfaffian when $\mu_n = \frac{(aqq)_n}{(abq^2;q)_n}$ is the $n$th moment of the little $q$-Jacobi polynomials. Our main result of this section is a new proof of the following theorem, which was first proved in [18].

**Theorem 5.1.** For integers $n \geq 1$ and $r \geq 0$, we have

$$
\text{Pf} \left( (q^{l-1} - q^{l-1})_{i+j+r-2} \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)} q^{n(4n+1)/3}(n-1)^r \prod_{k=1}^{n} \frac{(aq; q)_{2k+1}(bq; q)_{2(k-1)}(q; q)_{2k-1}}{(abq^2; q)_{2(k+n)+r-3}}. \quad (5.2)
$$

Let us use the notation (see [16, p.1476])

$$
\Delta_0^I(x) = \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_k (q; q)_k, \quad (5.3)
$$

$$
\Delta_k(x) = \frac{1}{n!} \sum_{\sigma \in S_n} \Delta_0^I(\sigma x). \quad (5.4)
$$

The $q$-gamma function is defined on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ by

$$
\Gamma_q(x) = (1-q)^{1-x} \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}}, \quad 0 < q < 1.
$$

The following lemma appeared in [39, Lemma 3.2] as a constant term identity.

**Lemma 5.2.** We have

$$
\int_{[0,a]^n} \Delta_1^I(x) \prod_{i=1}^{n} x_i^{r+1} \cdot d_q \omega(x) = \frac{n!}{\Gamma_q(n+1)} \int_{[0,a]^n} \Delta_2^I(x) \prod_{i=1}^{n} x_i^{r+1} \cdot d_q \omega(x), \quad (5.5)
$$

where $\Delta_1^I(x)$ is as defined in (3.9) and $\Delta_2^I(x)$ is defined by

$$
\Delta_2^I(x) = \prod_{1 \leq i < j \leq n} x_i^{2k} (q^{1-k}x_i/x_j; q)_{2k} = \prod_{1 \leq i < j \leq n} x_i^{2k} (q^{1-k}x_i/x_j)_{2k} \quad (5.6)
$$

(see [5, (3.30)]).
Proof. If we use
\[
\Delta_k(x) = \frac{\Gamma_{q^k}(n+1)}{n!} \prod_{i \neq j} (x_i/x_j; q)_k
\]  
(5.7)
which is proved in [16, (2.8)], then a direct computation shows
\[
\Delta_k(x) = (-1)^k \binom{n}{k} q^{k(n-1)/2} \Gamma_{q^k}(n+1) \prod_{i=1}^n x_i^{-k(n-1)} \cdot \Delta^1_k(x)
\]  
(5.8)
(see [16, (3.1)]). This implies that
\[
\int_{[0,a]^n} \prod_{i=1}^n x_i^{r+1} \cdot \Delta^1_k(x) \, dq \omega(x) = \left( \frac{(-1)^k \binom{n}{k} q^{k(n-1)/2} n!}{\Gamma_{q^k}(n+1)} \right) \int_{[0,a]^n} \prod_{i=1}^n x_i^{r+1+k(n-1)} \cdot \Delta_k(x) \, dq \omega(x).
\]  
(5.9)
Since \( \int_{[0,a]^n} f(\sigma x) \, d_q \sigma = \int_{[0,a]^n} f(x) \, d_q x \) for any \( \sigma \in \mathfrak{S}_n \), we obtain
\[
\int_{[0,a]^n} \prod_{i=1}^n x_i^{r+1} \cdot \Delta^1_k(x) \, dq \omega(x) = \left( \frac{(-1)^k \binom{n}{k} q^{k(n-1)/2} n!}{\Gamma_{q^k}(n+1)} \right) \int_{[0,a]^n} \prod_{i=1}^n x_i^{r+1+k(n-1)} \cdot \Delta^0_k(x) \, dq \omega(x).
\]  
(5.10)
A direct computation shows
\[
x_i^{2k} \left( q^{1-k} x_j/x_i; q \right)_{2k} = (-1)^k q^{-k} \binom{n}{k} q^{k(n-1)/2} (x_i x_j)^k (x_i/x_j; q)_k (qx_j/x_i; q)_k,
\]  
which implies
\[
\Delta^0_k(x) = (-1)^k \binom{n}{k} q^{k(n-1)/2} \prod_{i=1}^n x_i^{-k(n-1)} \prod_{1 \leq i < j \leq n} x_i^{2k} \left( q^{1-k} x_j/x_i; q \right)_{2k}.
\]  
(5.11)
Hence (5.10) and (5.11) immediately imply (5.5). \qed

From (5.1) and (5.5), we derive the key identity
\[
Pf(\left(q^{i-1} - q^{j-1}\right) \mu_{i+j+r-2})_{1 \leq i < j \leq 2n} = \frac{q^{n(n-1)}(1-q)^n}{\Gamma_{q^2}(n+1)} \int_{[0,a]^n} \Delta^2_k(x) \prod_{i=1}^n x_i^{r+1} \cdot dq \omega(x)
\]  
(5.12)
to compute our \( q \)-Pfaffians.

Next we begin our proof by recalling the notation of the little \( q \)-Jacobi polynomials. Let
\[
\phi_1^{(a,b; c; q; z)} = \sum_{n=0}^{\infty} \frac{(a,b; q)_n}{(c; q)_n} z^n
\]  
denote the basic hypergeometric series. The little \( q \)-Jacobi polynomials [12, 24] are defined by
\[
p_n(x; a, b; q) = \frac{(aq; q)_n}{(abq^{n+1}; q)_n} \left( q^{-n}; abq^{n+1}; aq; q \right)_{2\phi_1}^{\left[ q^{-n}; abq^{n+1}; aq; xq \right]}.
\]  
(5.13)
which are orthogonal with respect to the inner product defined as
\[
\int_0^1 f(x)g(x) \, dq(x) = \frac{(aq; q)_\infty}{(abq^2; q)_\infty} \sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (aq)^k f(q^k) \, g(q^k).
\]

Hence the measure is given by the weight function
\[
w(x) = \frac{1}{1-q} \cdot \frac{(aq, bq; q)_\infty}{(q, abq^2; q)_\infty} \cdot \frac{(qx; q)_\infty}{(q^{\beta+1}x; q)_\infty} x^\alpha,
\]
where \(a = q^\alpha\) and \(b = q^\beta\). Theorem 5.1 is equivalent to a polynomial identity in \(a\) and \(b\) if we multiply both sides by the denominator of the right-side of (5.2). So it is sufficient to prove it for \(a = q^\alpha\) and \(b = q^\beta\). By the \(q\)-binomial formula, the \(n\)th moment of the little \(q\)-Jacobi polynomials is
\[
\mu_n = \int_0^1 x^n w(x) \, dq(x) = \frac{(aq; q)_n}{(abq^2; q)_n} (n = 0, 1, 2, \ldots).
\]

Let
\[
A_n(x, y, k; q) = \prod_{j=1}^{n} \frac{\Gamma_q(x + (j - 1)k) \Gamma_q(y + (j - 1)k) \Gamma_q(jk + 1)}{\Gamma_q(x + y + (n + j - 2)k) \Gamma_q(k + 1)}.
\]

Askey proposed several \(q\)-analogues of the Selberg integral in [4]. The following is the well-known Askey-Habsieger-Kadell integral, conjectured by Askey [4, Conjecture 1], and proved independently by Habsieger [16, (3.2), (3.3)] and Kadell [21, Theorem 2; \(l = m = 0\)],
\[
\int_{[0,1]^n} \prod_{i<j} t_i^{2k} (q^{1-k} t_j / t_i; q)_{2k} \prod_{i=1}^{n} x_i^{n-1} \frac{(t_i q; q)_\infty}{(t_i q^{\beta}; q)_\infty} \, dq_i t = q^{kx(\binom{n}{2})+2k^2(\binom{n}{3})} A_n(x, y, k; q).
\]

Recently Kim and Stanton [23] gave a combinatorial interpretation of a \(q\)-Selberg integral, which is equivalent to (5.18).

**Lemma 5.3.** We have
\[
\int_{[0,1]^n} \Delta_k^2(t) \prod_{i=1}^{n} t_i^{r+1} w(t_i) \, dq_i t = \left\{ \frac{(aq, bq; q)_\infty}{(q, abq^2; q)_\infty} \right\}^n q^{k(\alpha+r+2)\binom{n}{2}+2k^2\binom{n}{3}} A_n(\alpha + r + 2, \beta + 1, k; q) \frac{1}{(1-q)^n},
\]
where \(w\) is the weight function (5.15).

**Proof.** From (5.15) and (5.6), we obtain
\[
I = \int_{[0,1]^n} \Delta_k^2(t) \prod_{i=1}^{n} t_i^{r+1} w(t_i) \, dq_i t
= \frac{1}{(1-q)^n} \left\{ \frac{(aq, bq; q)_\infty}{(q, abq^2; q)_\infty} \right\}^n \int_{[0,1]^n} \prod_{i<j} t_i^{2k} (q^{1-k} t_j / t_i; q)_{2k} \prod_{i=1}^{n} t_i^{\alpha+r+1} \frac{(t_i q; q)_\infty}{(t_i q^{\beta+1}; q)_\infty} \, dq_i t.
\]
It is easy to see that this equals the desired identity (5.19) from (5.18). 
\[\square\]
Proof of Theorem 5.1. If we put $k = 2$, we obtain the following identity from (5.12) and (5.19):

$$
\text{Pf}\left( (q^{i-1} - q^{j-1})\mu_{i+j+r-2} \right)_{1 \leq i < j \leq 2n}
= \left\{ \frac{(aq, bq; q)_{\infty}}{(q, abq^2; q)_{\infty}} \right\}^n \frac{q^{n(n-1)+2(\alpha+r+2)(\beta)+8(n)}}{\Gamma_q^2(n+1)} A_n(\alpha + r + 2, \beta + 1, 2; q).
$$

(5.20)

From (5.17) we have

$$
A_n(\alpha + r + 2, \beta + 1, 2; q) = \prod_{j=1}^n \frac{\Gamma_q(\alpha + r + 2 + 2(j - 1))\Gamma_q(\beta + 1 + 2(j - 1))\Gamma_q(2j + 1)}{\Gamma_q(\alpha + \beta + r + 3 + 2(n + j - 2))\Gamma_q(3)}.
$$

Using $\Gamma_q(x + k) = \Gamma_q(x)\frac{[q^{x+k}]_q}{(1-q)^x}$ when $k$ is nonnegative integer, we see that this equals

$$
\left\{ \frac{\Gamma_q(\alpha + 1)\Gamma_q(\beta + 1)}{\Gamma_q(\alpha + \beta + 2)} \right\}^n \prod_{j=1}^n \frac{\Gamma^{a+1}(q^{\alpha+1}; q)_{r+1+2(j-1)}(q^{\beta+1}; q)_{2(j-1)}(q; q)_{2j}}{\Gamma(q^{\alpha+\beta+2}; q)_{r+1+2(n+j-2)}(q; q)_{2j}}
= \left\{ \frac{(abq^2, q; q)_{\infty}}{(1-q^2)(aq, bq; q)_{\infty}} \right\}^n \prod_{j=1}^n \frac{(aq; q)_{r+1+2(j-1)}(bg; q)_{2(j-1)}(q; q)_{2j}}{(abq^2; q)_{r+1+2(n+j-2)}}.
$$

If we substitute $A_n(\alpha + r + 2, \beta + 1, 2; q)$ into (5.20) and use $(1-q^2)^n\Gamma_q^2(n+1) = \prod_{j=1}^n (1-q^{2j})$, then we immediately obtain the desired identity (5.2) by a straightforward computation. □

6 Hankel Pfaffians for Rogers-Szegő polynomials

The aim of this section is to prove Theorem 6.1 in which the first two identities are conjectured in [18, Conjecture 6.1]. In [18] we treated the Al-Salam-Carlitz I polynomials only, and only the product formulas are conjectured. But here we extend the conjectured identities to the Al-Salam-Carlitz I and II polynomials, and two more additive identities, i.e., (6.9) and (6.12), are added. The key tool for the proof is (5.12) as well, and we apply the results in [5].

This section is composed as follows. First we recall the Al-Salam-Carlitz I, II polynomials and their weight functions. We state our main result in Theorem 6.1, then we define the multivariate Al-Salam-Carlitz polynomials $\{U^{(a)}_\lambda(x; q, t)\}$ and $\{V^{(a)}_\lambda(x; q, t)\}$, which are both basis of the vector space of the symmetric functions in the variables $x = (x_1, \ldots, x_n)$ over $\mathbb{C}(q, t)$. The orthogonality [5, (4.27), (4.29), (4.34), (4.35)] of $\{U^{(a)}_\lambda(x; q, t)\}$ and $\{V^{(a)}_\lambda(x; q, t)\}$ plays an important role to prove the theorem. We use a result [5, (A4), (A5), (A6)] which tells us how to express the elementary symmetric function $e_r(x)$ by means of $\{U^{(a)}_\lambda(x; q, t)\}$ or $\{V^{(a)}_\lambda(x; q, t)\}$ and apply the above orthogonality.

In this section we use the notation:

$$
e_q(x) = \sum_{n=0}^\infty \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad E_q(x) = \sum_{n=0}^\infty \frac{q^{a(n+1)}x^n}{(q; q)_n} = (-x; q)_\infty.
$$

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These functions are the classical $q$-analogues of the exponential function. Al-Salam and Carlitz have introduced two families $\{U_n^{(a)}(x; q)\}_{n=0}^{\infty}$ and $\{V_n^{(a)}(x; q)\}_{n=0}^{\infty}$ of polynomials by means of the generating functions

$$\rho_a(x; q)e_q(xy) = \sum_{n=0}^{\infty} U_n^{(a)}(y; q) \frac{x^n}{(q; q)_n},$$  \hspace{1cm} (6.1)

$$\frac{1}{\rho_a(x; q)} E_q(-xy) = \sum_{n=0}^{\infty} V_n^{(a)}(y; q) \frac{(-1)^n q^{\frac{n(n-1)}{2}} x^n}{(q; q)_n},$$  \hspace{1cm} (6.2)

where $\rho_a(x; q) = (x; q)_\infty (ax; q)_\infty = E_q(-x)E_q(-ax)$. The families $\{U_n^{(a)}(x; q)\}_{n=0}^{\infty}$ and $\{V_n^{(a)}(x; q)\}_{n=0}^{\infty}$ are called the Al-Salam-Carlitz I polynomials and Al-Salam-Carlitz II polynomials, respectively. Al-Salam and Carlitz obtain the explicit orthogonal relations (see [1, 9])

$$\int_{1}^{\infty} U_n^{(a)}(x; q) U_n^{(a)}(x; q) w_U^{(a)}(x; q) d_q x = (1 - q)((-a^n q^{\frac{n(n-1)}{2}})(q; q)_n \delta_{m,n},$$  \hspace{1cm} (6.3)

$$\int_{1}^{\infty} V_n^{(a)}(x; q) V_n^{(a)}(x; q) w_V^{(a)}(x; q) d_q x = (1 - q)a^n q^{-n} (q; q)_n \delta_{m,n},$$  \hspace{1cm} (6.4)

where

$$w_U^{(a)}(x; q) = \frac{(q;x; q)_\infty (\frac{q}{a}; q)_\infty}{(q; q)_\infty (aq; q)_\infty (\frac{q}{a}; q)_\infty}, \hspace{1cm} w_V^{(a)}(x; q) = \frac{(q; q)_\infty (aq; q)_\infty (\frac{q}{a}; q)_\infty}{(x; q)_\infty (\frac{q}{a}; q)_\infty}.$$

Here $(x; q)_\infty'$ stands for the product except zeros, and the above Jackson integrals are defined by

$$\int_{1}^{\infty} f(x) d_q x = (1 - q) \left\{ \sum_{n=0}^{\infty} f(q^n) q^n - a \sum_{n=0}^{\infty} f(a q^n) q^n \right\},$$

$$\int_{1}^{\infty} f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} f(q^{-n}) q^{-n}.$$

Recall that $\{U_n^{(a)}(x; q)\}_{n=0}^{\infty}$ is positive-definite for $a < 0$ and $0 < q < 1$, and $\{V_n^{(a)}(x; q)\}_{n=0}^{\infty}$ is positive-definite for $a > 0$ and $0 < q < 1$. Then the moments $F_n(a; q)$ and $G_n(a; q)$ are defined by

$$\int_{1}^{\infty} x^n w_U^{(a)}(x; q) d_q x = (1 - q) F_n(a; q), \hspace{1cm} \int_{1}^{\infty} x^n w_V^{(a)}(x; q) d_q x = (1 - q) G_n(a; q)$$  \hspace{1cm} (6.5)

and have the expressions [12, 24]

$$F_n(a; q) = \sum_{k=0}^{n} \left[ \begin{array}{l} n \\ k \end{array} \right]_q a^k, \hspace{1cm} G_n(a; q) = \sum_{k=0}^{n} \left[ \begin{array}{l} n \\ k \end{array} \right]_q a^k q^{k(k-n)},$$  \hspace{1cm} (6.6)

where $\left[ \begin{array}{l} n \\ k \end{array} \right]_q = \frac{(aq)_n}{(q)_n (q; q)_{n-k}}$. We call $F_n(a; q)$ the Rogers-Szegö polynomials [12, Ex. 7.40]. The main result of this section is the following:
Theorem 6.1. Let $F_n(a; q)$ and $G_n(a; q)$ be as above. Then the following identities hold:

\[
\text{Pf}\left( (q^{j-1} - q^{j-1})F_{i+j-3}(a; q) \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)}q^{\frac{1}{n}(n-1)(4n-5)}\prod_{k=1}^{n}(q; q)_{2k-1}, \tag{6.7}
\]

\[
\text{Pf}\left( (q^{i-1} - q^{j-1})F_{i+j-2}(a; q) \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)}q^{\frac{1}{n}(n-1)(4n+1)}\prod_{k=1}^{n}(q; q)_{2k-1} \cdot G_n(a; q^2), \tag{6.8}
\]

\[
\text{Pf}\left( (q^{i-1} - q^{j-1})F_{i+j-1}(a; q) \right)_{1 \leq i < j \leq 2n} = \left(-a\right)^{n(n-1)}q^{n(n-1)(4n+7)/6}\prod_{i=1}^{n}(q; q)_{2i-1}
\times \sum_{i=0}^{n}(-a)^iq^{i(i+1)}/2n(q; q^2)_{\frac{n-i}{q^2}}G_{n-i}(a; t^2), \tag{6.9}
\]

\[
\text{Pf}\left( (q^{i-1} - q^{j-1})G_{i+j-3}(a; q) \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)}q^{-n(n-1)(4n-5)/3}\prod_{k=1}^{n}(q; q)_{2k-1}, \tag{6.10}
\]

\[
\text{Pf}\left( (q^{i-1} - q^{j-1})G_{i+j-2}(a; q) \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)}q^{-2n(n-1)(2n-1)/3}\prod_{k=1}^{n}(q; q)_{2k-1} \cdot F_n(a; q^2), \tag{6.11}
\]

\[
\text{Pf}\left( (q^{i-1} - q^{j-1})G_{i+j-1}(a; q) \right)_{1 \leq i < j \leq 2n} = a^{n(n-1)}q^{-n(n-1)(4n+1)/3}\prod_{i=1}^{n}(q; q)_{2i-1}
\times \sum_{i=0}^{n}a^iq^{-i(q; q^2)_{\frac{n-i}{q^2}}}F_{n-i}(a; q^2)^2. \tag{6.12}
\]

Remark 6.2. The first (resp. the second) identity in [18, Conjecture 6.1] obviously corresponds to (6.7) (resp. (6.8)). The powers of $q$ in conjectured identities look complicated and different. However (6.7) is equivalent to the conjectured identity and (6.8) corrects the mistaken conjectured formula.

Let $P_\lambda(x; q, t)$ denote Macdonald’s $P$-function with respect to a partition $\lambda$ in $n$-tuple $x = (x_1, \ldots, x_n)$ of variables (see [31, Chap. IV, (4.7)]). In what follows, we use the notation

\[
a_\lambda(s) = \lambda_i - j, \quad a'_\lambda(s) = j - 1, \\
\lambda_\lambda(s) = \lambda'_i - j, \quad \lambda'^{\prime}_\lambda(s) = i - 1 \tag{1.13}
\]

for each cell $s = (i, j) \in \lambda$ ([31, VI, (6.14)]), where $\lambda'$ denote the conjugate of $\lambda$. We also use the notation

\[
h_\lambda(q, t) = \prod_{s \in \lambda}(1 - q^{a_\lambda(s)}\theta^{a_\lambda(s)+1}), \quad h'_\lambda(q, t) = \prod_{s \in \lambda}(1 - q^{a_\lambda(s)+1}\theta^{a_\lambda(s)}) \tag{1.14}
\]

(see [5, (1.3), (1.4)]). The hypergeometric functions $0\mathcal{F}_0(x; q, t)$ and $0\mathcal{F}_0(x; y; q, t)$ are
defined by

\[0_{\mathcal{F}_0}(x; y; q, t) = \sum_{\lambda} \frac{t^{n(\lambda)}}{h_{\lambda}(q, t)P_{\lambda}(t^\delta; q, t)} P_{\lambda}(x; q, t)P_{\lambda}(y; q, t),\]  

(6.15)

\[0_{\varphi_0}(x; y; q, t) = \sum_{\lambda} \frac{(-1)^{|\lambda|}q^{n(\lambda')}}{h_{\lambda}(q, t)P_{\lambda}(t^\delta; q, t)} P_{\lambda}(x; q, t)P_{\lambda}(y; q, t),\]  

(6.16)

where \(n(\lambda) = \sum_{i \geq 1}(i - 1)\lambda_i\) ([5, (1.3), (1.4)], [31, I, (1.5)]) and \(t^\delta = (1, t, \ldots, t^{n-1})\). In [31, VI, (6.11')] (also see [5, (1.4)]) an explicit description of \(P_{\lambda}(t^\delta; q, t)\) is given that

\[P_{\lambda}(t^\delta; q, t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{\lambda_s(s)}t^{n_s(s)}}{1 - q^{\lambda_s(s)}t^{n_s(s)+1}} = t^{n(\lambda)} \prod_{(i, j) \in \lambda} (1 - q^{j-i-1}) \frac{1}{h_{\lambda}(q, t)}.\]  

(6.17)

Two families \(\{U^{(a)}_{\lambda}(x; q, t)\}\) and \(\{V^{(a)}_{\lambda}(x; q, t)\}\) of multivariate polynomials are introduced in [5, (2.32), (2.33)] using the following generating functions:

\[\prod_{i=1}^{n} \rho_n(y_i; q) \cdot 0_{\mathcal{F}_0}(x; y; q, t) = \sum_{\lambda} \frac{t^{n(\lambda)}P_{\lambda}(y; q, t)}{h_{\lambda}(q, t)P_{\lambda}(t^\delta; q, t)} U^{(a)}_{\lambda}(x; q, t),\]  

(6.18)

\[\frac{1}{\prod_{i=1}^{n} \rho_n(t^{-(n-1)}y_i; q)} \cdot 0_{\varphi_0}(x; y; q, t) = \sum_{\lambda} \frac{(-1)^{|\lambda|}q^{n(\lambda')}P_{\lambda}(y; q, t)}{h_{\lambda}(q, t)P_{\lambda}(t^\delta; q, t)} V^{(a)}_{\lambda}(x; q, t).\]  

(6.19)

It is shown in [5, (2.34)] that the following relation holds:

\[V^{(a)}_{\lambda}(x; q, t) = U^{(a)}_{\lambda}(x; q^{-1}, t^{-1}).\]  

(6.20)

Hereafter we write \(d_{\mu}^{U}(x) = \prod_{i=1}^{n} w^{(a)}_{U}(x_i; q) d_{x} x\) and \(d_{\mu}^{V}(x) = \prod_{i=1}^{n} w^{(a)}_{V}(x_i; q) d_{x} x\). Baker and Forrester proved the following identities:

\[\mathcal{N}_{\lambda}^{(U)}(a; q, t) = \int_{[1,\infty]^n} \Delta_{k}^{2}(x) d_{\mu}^{U}(x) = (1 - q)^{n} (-a)^{kn(n-1)/2} t^{k(3) - k(2)} \prod_{i=1}^{n} \frac{(q; q)_{ki}}{(q; q)_{k}},\]  

(6.21)

\[\mathcal{N}_{\lambda}^{(V)}(a; q, t) = \int_{[1,\infty]^n} \Delta_{k}^{2}(x) d_{\mu}^{V}(x) = (1 - q)^{n} (-a)^{kn(n-1)/2} t^{-2k(3)} \prod_{i=1}^{n} \frac{(q; q)_{ki}}{(q; q)_{k}},\]  

(6.22)

for \(t = q^k\) with any nonnegative integer \(k\), where \(\Delta_{k}^{2}(x)\) is as in (5.6) (see [5, (4.27), (4.29)]).

They also prove the orthogonality

\[\int_{[1,\infty]^n} U^{(a)}_{\lambda}(x; q, t)U^{(a)}_{\mu}(x; q, t) \Delta_{k}^{2}(x) d_{\mu}^{U}(x) = \mathcal{N}_{\lambda}^{(U)}(a; q, t) \delta_{\lambda\mu},\]  

(6.23)

\[\int_{[1,\infty]^n} V^{(a)}_{\lambda}(x; q, t)V^{(a)}_{\mu}(x; q, t) \Delta_{k}^{2}(x) d_{\mu}^{V}(x) = \mathcal{N}_{\lambda}^{(V)}(a; q, t) \delta_{\lambda\mu}.\]  

(6.24)
of \( \{U^{(a)}_\lambda(x; q, t)\} \) and \( \{V^{(a)}_\lambda(x; q, t)\} \) ([5, (3.29), (3.37)]) and obtain the normalization integrals
\[
\mathcal{N}^{(U)}_\lambda(a; q, t) = (-at^{n-1})^{|\lambda|} q^{n(|\lambda|)} t^{-2n(|\lambda|)} h'_\lambda(q, t) P_\lambda(t; q, t),
\]
\[
\mathcal{N}^{(V)}_\lambda(a; q, t) = (aq^{-1} t^{-2(n-1)})^{|\lambda|} q^{n(|\lambda|)} t^{n(|\lambda|)} h'_\lambda(q, t) P_\lambda(t; q, t).
\]
(6.25) (6.26)
for any \( \lambda \) ([5, (4.34), (4.35)]) when \( t = q^k, k = 0, 1, 2, \ldots \).

**Proof of Theorem 6.1.** In this proof we always assume \( t = q^k \), where \( k \) is a nonnegative integer. However, we basically use only the case when \( k = 2 \) to prove Theorem 6.1. Substituting (6.5), (6.21) and (6.22) into (5.12) with \( r = -1 \) and \( k = 2 \), we obtain
\[
\text{Pf}(q^{-1} - q^{j-1}) F_{i+j-3}(a; q) = \frac{q^{n(n-1)}(1-q)^n}{\Gamma q^n(n+1)} a^{n(n-1) q^{4(3)-4(2)}} \prod_{j=1}^{n} (q; q)_{2i}.
\]
(6.27) which proves (6.7) and (6.10) by direct computation.

Next we need to expand the \( r \)th elementary symmetric function \( \prod_{i=1}^{n} x_i = e_r(x) \) in terms of \( U^{(a)}_\lambda(x; q, t) \) (resp. \( V^{(a)}_\lambda(x; q, t) \)) to prove (6.8) (resp. 6.11)). Then we use the orthogonality (6.23) or (6.24) to calculate the right-hand side of (5.12) when \( r = 0 \). First we evaluate the integrals
\[
\int_{[a, 1]^n} \prod_{i=1}^{n} x_i \cdot \Delta_k^2(x) d_q \mu^{(U)}(x) \quad \text{and} \quad \int_{[1, \infty]^n} \prod_{i=1}^{n} x_i \cdot \Delta_k^2(x) d_q \mu^{(V)}(x)
\]
(6.29)
for any nonnegative integer \( k \). Baker and Forrester [5, (A.4), (A.5), (A.6)] show the \( r \)th elementary symmetric function \( e_r(x) \) can be expanded in \( U^{(a)}_\lambda(x; q, t) \) as
\[
e_r(x) = \sum_{i=0}^{r} \tilde{f}_{r-i}(a) \left[ \frac{n-i}{r-i} \right] U^{(a)}_{(i;r)}(x; q, t),
\]
(6.28)
where \( \tilde{f}_i(a) \) is defined by the initial condition \( \tilde{f}_0(a) = 1, \tilde{f}_1(a) = 1 + a \) and the recurrence equation
\[
\tilde{f}_i(a) = (1 + a)^{t^i-1} \tilde{f}_{i-1}(a) + at^{i-2}(1 - t^{i-1}) \tilde{f}_{i-2}(a).
\]
Using this recurrence, we can show that
\[
\tilde{f}_i(a) = \sum_{j=0}^{i} \left[ \begin{array}{c} i \\ j \end{array} \right] t^{(i-j) \frac{(j-1)}{2} + (i-j) \frac{(i-j-1)}{2}} a^j = t^i G_i(a; t)
\]
by induction. Since we have \( U^{(a)}_0(x; q, t) = 1 \) from (6.18), the first integral of (6.27) is written as
\[
\int_{[a, 1]^n} \prod_{i=1}^{n} x_i \cdot \Delta_k^2(x) d_q \mu^{(U)}(x) = \int_{[a, 1]^n} e_n(x) U^{(a)}_0(x; q, t) \Delta_k^2(x) d_q \mu^{(U)}(x)
\]
(6.29)
Substituting (6.28) with \( r = n \) into (6.29), we see this becomes
\[
\sum_{i=0}^{n} \tilde{f}_{n-i}(a) \int_{\text{[1,1]}} U^{(a)}_{(1')}(x; q, t)U^{(a)}_{0}(x; q, t) \Delta^2_k(x) d_{q\mu^{(U)}}(x)
\]
which equals
\[
\tilde{f}_{n}(a) \int_{\text{[1,1]}} \Delta^2_k(x) d_{q\mu^{(U)}}(x)
\]
from the orthogonality (6.23). Hence, by (6.21), we conclude that (6.29) equals
\[
\tilde{f}_{n}(a)(1 - q)^n(-a)^{kn(n-1)/2}q^{k^2(\gamma) - k(k-1)(\gamma)} \prod_{i=1}^{n} \frac{(q; q)_{ki}}{(q; q)_k},
\]
Finally, by substituting \( k = 2 \) into this result and using the identity (5.12) with \( r = 0 \), we obtain
\[
\text{Pf}\left( (q^{i-1} - q^{j-1})F_{i+j-2}(a; q) \right)_{1 \leq i < j \leq 2n} = \frac{q^{n(n-1)}(1 - q)^n}{\Gamma q^2(n + 1)} \tilde{g}_{n}(a) a^{n(n-1)}q^{3(\gamma) - 4(\gamma)} \prod_{i=1}^{n} \frac{(q; q)_{2i}}{(q; q)_2}.
\]
One can prove (6.8) from this identity by straight computation.

Next we use (6.20) to prove (6.11). The identities (6.28) and (6.20) imply
\[
e_r(x) = \sum_{i=0}^{r} \tilde{g}_{r-i}(a) \binom{n-i}{r-i} V^{(a)}_{(1')}((x; q, t), \quad (6.30)
\]
where \( \tilde{g}_{i}(a) \) is given by \( \tilde{g}_{i}(a) = t^{-\binom{i}{2}} \sum_{j=0}^{i} \binom{i}{j} a^j = t^{-\binom{i}{2}} F_{i}(a; t) \). A similar argument as above shows that we can rewrite the second integral of (6.27) as
\[
\int_{\text{[1,1]}} \prod_{i=1}^{n} x_i \cdot \Delta^2_k(x) d_{q\mu^{(V)}}(x) = \tilde{g}_{n}(a)(1 - q)^n a^{kn(n-1)/2} q^{-2k^2(\gamma) - k^2(\gamma)} \prod_{i=1}^{n} \frac{(q; q)_{ki}}{(q; q)_k}
\]
by (6.22), (6.24) and (6.30). Again the \( k = 2 \) case of this identity and (5.12) with \( r = 0 \) lead to
\[
\text{Pf}\left( (q^{i-1} - q^{j-1})G_{i+j-2}(a; q) \right)_{1 \leq i < j \leq 2n} = \frac{q^{n(n-1)}(1 - q)^n}{\Gamma q^2(n + 1)} \tilde{g}_{n}(a) a^{n(n-1)}q^{-8(\gamma) - 8(\gamma)} \prod_{i=1}^{n} \frac{(q; q)_{2i}}{(q; q)_2},
\]
which proves (6.11) immediately.

Next we can use (6.28), (6.30) and the orthogonality (6.23), (6.24) to obtain
\[
\int_{\text{[1,1]}} e_n(x)^2 \Delta^2_k(x) d_{q\mu^{(U)}}(x) = \sum_{i=0}^{n} \tilde{f}_{n-i}(a)^2 \mathcal{A}^{(U)}_{(1')} (a; q, t), \quad (6.31)
\]
\[
\int_{\text{[1,\infty]}} e_n(x)^2 \Delta^2_k(x) d_{q\mu^{(V)}}(x) = \sum_{i=0}^{n} \tilde{g}_{n-i}(a)^2 \mathcal{A}^{(V)}_{(1')} (a; q, t). \quad (6.32)
\]
Substituting $|\lambda| = i$, $n(\lambda) = \binom{i}{2}$, $n(\lambda') = 0$, $h_i(q, t) = (q, t)$, and $P_\lambda(t; q, t) = \frac{t^i}{(q, t)^i}$, we obtain

$$A_\lambda^{(U)}(a; q, t) = (-a)^{i(n-1)} \frac{(q, t)_i(t^{n-i+1}; q)_i}{(t, t)_i} A_\lambda^{(U)}(a; q, t), \quad (6.33)$$

$$A_\lambda^{(V)}(a; q, t) = a^i q^{-i} t^{-2(n-i)+2(\frac{i}{2})} \frac{(q, t)_i(t^{n-i+1}; q)_i}{(t, t)_i} A_\lambda^{(V)}(a; q, t). \quad (6.34)$$

Substituting (6.33) (resp. (6.34)) into (6.31) (resp. (6.32)), we obtain

$$\int_{[1, \infty]^n} e_n(x)^2 \Delta_2^2(x) d_q \mu^{(U)}(x) = (1 - q)^{n(1-a)} \sum_{i=0}^{n} \frac{(-a)^{i(n-1)}(q, t)_i(t^{n-i+1}; q)_i}{(t, t)_i} G_{n-i}(a; t)^2, \quad (6.35)$$

$$\int_{[a, \infty]^n} e_n(x)^2 \Delta_2^2(x) d_q \mu^{(V)}(x) = (1 - q)^{n(1-a)} a^{-k(n-1)/2} \sum_{i=0}^{n} \frac{(q, t)_i(t^{n-i+1}; q)_i}{(t, t)_i} F_{n-i}(a; t)^2. \quad (6.36)$$

By replacing $t$ by $q^k$, substituting $k = 2$ into (6.35) and (6.36), and using the identity (5.12) with $r = 1$, we obtain (6.9) and (6.12). This completes the proof of our theorem. □

The reader may notice that (6.10) (resp. (6.11), (6.12)) can be directly derived from (6.7) (resp. (6.8), (6.9)) by replacing $q$ by $q^{-1}$. Here we give an explicit form (6.30), which tells us how to express $e_r(x)$ by means of $\{V_\lambda^{(a)}(x; q, t)\}$. We would like to add the following remark at the end of this section.

**Remark 6.3.** We can evaluate the Pfaffian $\text{Pf}\left( (q^{i-1} - q^{j-1}) F_{i+j-r-2}(a; q) \right)_{1 \leq i, j \leq 2n}$ for a nonnegative integer $r$ if we can find a formula which expands the elementary symmetric function $e_{n+r+1}(x) = e_n(x_1, \ldots, x_n)^{r+1}$ by means of $U_\lambda^{(a)}(x; q, t)$.

### 7 Open Problems

In [18, Conjecture 6.3] we presented another conjecture. This conjecture is still open, but instead we add more conjectures of this type in this last section. Let us take $a_n = \frac{1}{2n+1} \binom{3n}{n} = \frac{1}{3n+1} \binom{3n+1}{n}$. This number is famous because Tamn [38] proved that the Hankel determinant $\det(a_{i+j-1})_{1 \leq i, j \leq n}$ equals the number of $(2n + 1) \times (2n + 1)$ alternating sign matrices that are invariant under vertical reflection. He evaluated the determinant by showing that the generating function of the sequence $\{a_n\}_{n \geq 0}$ is a ratio of hypergeometric series, i.e.,

$$g = \sum_{n=0}^{\infty} a_n x^n = 2F_1 \left( \frac{2}{3}, \frac{4}{3}, \frac{3}{2}, \frac{27}{4}; x \right) / 2F_1 \left( \frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{27}{4}; x \right),$$
and by using the continued fraction expansion of \( g \). In [13, Section 3] Gessel and Xin made a systematic application of this continued fraction method to a number of similar Hankel determinants. They observed empirically that there are five pairs of generating functions of the form

\[
\sum_{n=0}^{\infty} p_n x^n = 2 F_1 \left( a, b + 1; c + 1; \frac{27}{4}x \right) / 2 F_1 \left( a, b; c; \frac{27}{4}x \right),
\]

that can be expressed as polynomials in \( g \). Let \( \{a_n^{(i)}\}_{n \geq 0} \) (\( i = 1, \ldots, 5 \)) be Gessel and Xin’s five sequences (see below). Gessel and Xin studied the Hankel determinants of these sequences. By experiments we observe that the Hankel Pfaffian transforms of these sequences have mysteriously nice product formulas as in the following.

**Conjecture 7.1.** Let \( n \) be a nonnegative integer. The following identities would hold.

If \( a_n^{(1)} = \frac{1}{3n+1} \binom{3n+1}{n} \) then

\[
\text{Pf} \left( (j - i) a_{i+j-1}^{(1)} \right)_{1 \leq i, j \leq 2n} = 2^{-n} \prod_{k=0}^{n-1} \frac{(12k + 6)!(4k + 1)!(3k + 2)!}{(8k + 2)!(8k + 5)!(3k + 1)!}. \tag{7.1}
\]

If \( a_n^{(2)} = \frac{1}{3n+2} \binom{3n+2}{n+1} \) then

\[
\text{Pf} \left( (j - i) a_{i+j-2}^{(2)} \right)_{1 \leq i, j \leq 2n} = 12^{-n} \prod_{k=0}^{n-1} \frac{(12k + 10)!(4k + 2)!(4k + 1)}{(8k + 3)!(8k + 7)!(3k + 2)(12k + 5)}. \tag{7.2}
\]

If \( a_n^{(3)} = \frac{2}{3n+1} \binom{3n+1}{n+1} \) then

\[
\text{Pf} \left( (j - i) a_{i+j-2}^{(3)} \right)_{1 \leq i, j \leq 2n} = \frac{4}{3} \prod_{k=0}^{n-1} \frac{(12k + 15)!(4k + 5)!(2k + 1)}{(8k + 8)!(8k + 11)!(12k + 13)}. \tag{7.3}
\]

If \( a_n^{(4)} = \frac{2}{(3n+1)(3n+2)} \binom{3n+2}{n+1} \) then

\[
\text{Pf} \left( (j - i) a_{i+j-1}^{(4)} \right)_{1 \leq i, j \leq 2n} = \left( \frac{2}{3} \right)^n (6n + 1)! \prod_{k=0}^{n-1} \frac{(12k + 6)!(4k + 5)!(4k + 3)}{(8k + 5)!(8k + 10)!(k + 1)(3k + 1)}. \tag{7.4}
\]

If \( a_n^{(5)} = \frac{9n+5}{(3n+1)(3n+2)} \binom{3n+2}{n+1} \) then

\[
\text{Pf} \left( (j - i) a_{i+j-2}^{(5)} \right)_{1 \leq i, j \leq 2n} = 3^{-n} \prod_{k=0}^{n-1} \frac{(6k + 6)!(2k)!}{(4k + 1)!(4k + 4)!(3k + 2)}. \tag{7.5}
\]

The above four sequences \( \{a_n^{(i)}\}_{n \geq 0} \) (\( i = 1, \ldots, 5 \)) are identified in OEIS except the last one.
Remark 7.2. Further we identified the Hankel Pfaffian transform of \( \{a_n^{(5)}\}_{n \geq 0} \) as A059489 in OEIS. Namely we observe that

\[
Pf \left( (j - i) a_{i+j-2}^{(5)} \right)_{1 \leq i,j \leq 2n} = A^{(2)}_{UU}(4n; 1, 1, 1) \tag{7.6}
\]

would hold for \( n \geq 0 \). Kuperberg [26, Theorem 4] showed that the generating function \( A_{UU}(4n; x, y, z) \) of UUASMs factors as

\[
A_{UU}(4n; x, y, z) = A_{V}(2n + 1; z) A^{(2)}_{UU}(4n; x, y, z),
\]

which characterizes \( A^{(2)}_{UU}(4n; x, y, z) \). Recall that a UUASM is an alternating sign matrix with U-turn boundary on the right and on the top. Here we define the \( x \)-weight to be \( x^k \) if \( k \) is the number of \(-1\)s, \( y \)-weight to be \( y^k \) if \( k \) of the U-turns on the right are oriented upward, and \( z \)-weight to be \( z^k \) if \( k \) of the U-turns on the top are oriented to the right. Further \( A_{V}(2n + 1; x) \) denotes the generating function of \( (2n + 1) \times (2n + 1) \) vertically symmetric alternating sign matrices, where we define the \( x \)-weight be the number of \(-1\)s, as well. It is also a challenging problem to find the reason which explains \( A^{(2)}_{UU}(4n; 1, 1, 1) \) appears as the Hankel Pfaffian sequence of \( \{a_n^{(5)}\}_{n \geq 0} \).

The conjectured identity (7.1) first appeared in [18, Conjecture 6.3], but the others are new. We note that the generating functions of these sequences are roots of certain cubic equations, and so the evaluation of the Pfaffians would be more difficult than the Pfaffians in Section 4. We also note that we can find more Hankel determinants of this type in [25, Theorem 31]. We hope to address this topic in a forthcoming paper.

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