The opportunities of umbrella-type shells

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Abstract

Relevance. The necessity of division of umbrella surfaces and surfaces of umbrella type into two separated classes is explained in introduction. Earlier, umbrella surfaces and surfaces of umbrella type were in the same class of surfaces because they consist of the identical fragments lying on the surfaces of revolution. Umbrella surfaces are compound surfaces on the base surface of revolution, but umbrella-type surfaces are kinematic surfaces formed by continuous movement of a changing curve and that is why taking into account the methods of construction of these surfaces they were divided in two separate classes. The aim of the work is a collection of main publications on all areas of the investigation of umbrella-type shells. Methods. For the determination of principal results of investigation of umbrella-type shells, it is necessary to know differential geometry of surfaces, structural mechanics of thin shells, and approaches used in architecture of spatial structures. Results. In this article, the principal scientific papers on geometry, analysis, and offers of applications of thin-walled shells of umbrella type in building and of reflectors of umbrella type for space apparatuses. The accurate parametric equations of some determined surfaces are presented. The approximated computer models of middle surfaces of the real umbrella shells but in the form of umbrella-type surfaces are given. The examples of determination of stress-strain state of thin-walled shells of umbrella type without dividing of the whole shell in identical fragments are shown. New information and materials already known about shells of umbrella type give reasons to suppose that the shells of this type will be claimed by engineers and architects.

Keywords: umbrella shell, shell of umbrella type, waving dome, reflector of umbrella type for space apparatus, shell of radar installation, finite difference energy method

Introduction

At present, the movement in the direction of increasing of the interest for design of large-span structures begins to show. Interesting facts from the history of building and design of thin-walled spatial structures and perspectives of their application in future are presented in a review paper [1] where the authors insist that “there are signs, however, that shells are attracting interest among the new generation of architects and engineers”. The modern architects’ views at the place of shell structures in modern architecture of arbitrary forms and in building are set forth in papers [2; 3].

Consider the investigations on geometry, analysis, and application of umbrella-type shells and show up their advantages in comparison with umbrella shells.
The definitions of umbrella surfaces and umbrella-type surfaces are given in a monograph [4] or in an encyclopedia [5]. Adduce them almost word for word.

“A cyclic symmetrical spatial structure formed from several identical elements is called an umbrella dome (Figure 1). Curves obtained as a result of the intersection of their middle surfaces are the generatrix curves of any dome-shaped surface of revolution. A dome-shaped surface of revolution, on which the contour curves of the elements of a dome are placed, is called a contour surface. The contour curves of the element are the curves bounding the contour of the middle surface of the element of the dome.”

In scientific and technical literature, the shells put together from identical fragments of the hypar are also called umbrella domes. F. Candela [4; 8] designed and built the first umbrella shell in the form of four intersecting hypars that formed a covering of a restaurant in Xochomilco, Mexico, in 1957.

“The cyclic symmetrical surfaces consisting from several identical elements are called surfaces of umbrella type. But unlike an umbrella surface, the whole surface of umbrella type and all surfaces of the identical elements forming the whole surface are determined by one and the same explicit, implicit or parametric equations. In popular scientific literature, the umbrella-type shells are often called also waving or wave-shaped domes.”

Much useful information is presented in a manuscript [9] where 240 individual solutions of different domes are given and various classifications are presented. Some types of domes presented in this manuscript can be reckoned among umbrella shells, for example, wave-and-folded domes with folds of arched form and segment domes.

The results of investigation

A review of researches on geometry of umbrella-type surfaces

Firstly, umbrella-type surfaces were introduced into practice in papers [10; 11] where fourteen surfaces were proposed, their analytical formulas were given, and Gaussian quantities of the first and second orders in the theory of surfaces were obtained. Later, a part of these surfaces was investigated with the help of the MathCAD computer program in a work [12] where changes of the surface form were examined depending on constant parameters containing in analytical equations of surfaces of umbrella type.

Two surfaces from fourteen surfaces presented in works [10; 11] were offered by other authors. These are the Skidan’s ruled surface [13] and the crossed trough [14. P. 286]. V.N. Ivanov [15] offered a method of forming of the umbrella-type surface in the shape of Joachimsthal canal surface with a director circular sinusoid. This surface is formed by the rotation of the circle of variable radius with the common chord [16] (Figure 2). The generatrix circles of canal surfaces are lines of principal curvature. Cutting out from the common surface the fragment with radii \( r_1 \) and \( R \), one can obtain an umbrella-type shell with the opening at the vertex.

Following after V.N. Ivanov’s method [16], his postgraduate student Nasr Younis A. Abboushi [17] made several models of Joachimsthal canal surfaces defining three methods of their formation:

1) a surface is formed by the rotation of the circle of variable radius \( R(u) \) so that a distance from the axis of rotation till the point of contact with the generatrix circle \( c = (r^2 - R^2)^{1/2} \) remains constant;
2) a surface is formed by the rotation of the circle of variable radius \( R(u) \) around the common chord;
3) a surface is formed by the rotation of the circle of variable radius \( R(u) \) around the common tangent, i.e. \( c = 0, r = R \). Here, \( r(u) \) is a distance from the axis of rotation till the center of the generatrix circle.

The methods 2 \((r < R)\) and 3 \((r = R)\) give the opportunity to obtain umbrella-type surfaces.

A computer program in the AutoCAD system for tracing of umbrella-type surfaces with parabolic generatrices and with a circular opening at the vertex is presented in a paper [18]. It gives the possibility to consider a process of forming umbrella-type surfaces in dynamic conditions by means of making mini-film. Analogous program for another surface of umbrella type is given in a paper [19] (Figure 3).

![Figure 3](image)

**Figure 3.** A forming surface with radial sinusoidal waves damping in the central point and formed by the cubic parabolas

Umbrella-type surfaces with elliptic contour surfaces and with the circular sinusoidal equator were taken as a base of research of waving domes in a paper [20]. A vector equation of the waving ellipsoidal surface may taken in the form

\[
\begin{align*}
   r(u,v) &= a\{[1 + \mu\cos(pu)]\cos\nu h(u) + \gamma\sin\nu k\}, \\
   h(u) &= i\cos u + jsin u, \\
   \gamma &= b/a,
\end{align*}
\]

where \( a, b \) are the semi-axes of the ellipsoid of revolution; \( u, v \) are the parameters at the interval \([0, \pi]\); \( i, j, k \) are the unit vectors; \( \mu \) is a ratio of the amplitude of the sinusoid to the radius \( a \) of the circle; \( p \) is a number of waves of the sinusoid. The waving ellipsoidal surfaces were modeled under different parameters \( \gamma, \mu, p \).

**Umbrella shells and umbrella-type shells in architecture and technics in the 20th and 21st centuries**

It should be noted, that umbrella shells are presented in architecture of the 20th century very widely [4; 21]. Real erections in the form of umbrella-type surfaces were not found. But outward examination of some shells showed that their middle surfaces can be quite given by analytical formulae, i.e. these shells can be number among umbrella-type shells. For example, a covering of a restaurant attached to a hotel “La Concha Hotel”, San Juan, Puerto Rico [22], arch. Jose R. Marchand, 2009 [Available from: http://www.architecturaldigest.com/homes/hotels/2009], can be approximated by a paraboloid of revolution with four radial waves [5. P. 388] or by a surface with astroidal level lines that is generated by biquadratic parabolas [5. P. 392].

A wave-shaped surface of the Marché Royan, France, consisting of 13 reinforced sinusoidal parabolic fragments is well simulated by a paraboloid of revolution with radial waves [5. P. 388], that has a parametrical form of definition:

\[
\begin{align*}
   x &= x(u,v) = cu\cos v, \\
   y &= y(u,v) = cu\sin v, \\
   z &= z(u,v) = [asin(nv) + b]\nu^2,
\end{align*}
\]

where \( \nu \) is the angle read from the axis \( Ox \) in the direction to the axis \( Oy \); \( a = \text{const} \) is an amplitude of a wave; \( n \) is a number of the vertexes of the waves; \( b \) is a constant parameter of the datum paraboloid of revolution, \( c = \text{const} \).

When \( a = 0 \), a paraboloid of revolution with radial waves degenerates into a paraboloid of revolution. The real shell has a 100 mm thickness and a 50 m span. It supports at 13 points along the outward perimeter (Figure 4). A market was built in 1956 and now it is architectural and historical possessions of the town.

![Figure 4](image)

**Figure 4.** Marché Royan, France, and its possible computer model

The three-tiered many-waved umbrella church in the suburb of St. Louis (USA) designed by G. Obata with participation of P.L. Nervi apparently was arranged from fragments of cylindrical parabolic shells [22; 23]. Middle surfaces of these three tiers can be easily approximated by a paraboloid of revolution with radial waves (Figure 5), i.e. by a surface of umbrella type (2) \( a = b \).
Umbrella glass-fibre plastic dome of radar installation consist of fragments of the same type united between themselves by bolted joints (Figure 6, a) [24]. The joints are sealed by glass fabric on elastic special mastic. Airports Domodedovo, Vnukovo, Sheremetyevo, and other airports of Russia exploit the shells of this type. The form of these domes is rather like surface of a sphere with external cycloidal crimps (Figure 6, b) which may be given by parametrical equations:

\[ x = x(u, \varphi) = (R + r) \cos \varphi - r \cos(n + 1)\varphi \cos u, \]
\[ y = y(u, \varphi) = (R + r) \sin \varphi - r \sin(n + 1)\varphi \cos u, \]
\[ z = z(u) = R \sin u, \]

where \( u \) is the angle read from the plane \( xOy \) in the direction of the axis \( Oz \); \( 0 \leq z \leq R; \ 0 \leq \varphi \leq 2\pi; \ 0 \leq u \leq \pi / 2 \). In the cross section of the surface in question by the planes \( z = \text{const} \), i.e., when \( u = u_0 = \text{const} \), we have the epicycloids:

\[ x = x(\varphi) = ((R + r) \cos \varphi - r \cos(n + 1)\varphi) \cos u_0, \]
\[ y = y(\varphi) = ((R + r) \sin \varphi - r \sin(n + 1)\varphi) \cos u_0, \]

with \( n = \text{const} \), \( n \) is a number of the vertexes of the epicycloid on the circular plane; \( n = R/r, 2r \) is the maximum amplitude of the crimps at the base of the surface; \( R \) is a radius of the equator of the sphere; \( \varphi \) is the angle read from the axis \( Ox \) in the direction of the axis \( Oy \).

In case of need, domes of radar installation can be designed in the form of corrugated paraboloid of revolution with external crimps (Figure 6, c) with a circular waving curve at the foot with the vertexes directed out of the center of the circular base:

\[ x = (R + a \cos n\varphi) \cos \varphi, \]
\[ y = (R + a \cos n\varphi) \sin \varphi, \ z = 0, \]

where \( n \) is a number of the vertexes of the sinusoid on a circular plan; \( a \) is an amplitude of the crimps at the foot of the surface; \( R \) is a radius of the base circle of the corrugated paraboloid in the foot relative to which, the circular sinusoid is constructed.

Parametrical of the definition of the corrugated paraboloid of revolution with the external crimps:

\[ x = x(r, \varphi) = r \left(1 + \frac{ar \cos n\varphi}{R^2}\right) \cos \varphi, \]
\[ y = y(r, \varphi) = r \left(1 + \frac{ar \cos n\varphi}{R^2}\right) \sin \varphi, \]
\[ z = z(r) = h \left(1 - \frac{r^2}{R^2}\right), \]

where \( h \) is the height of a corrugated paraboloid of revolution, \( 0 \leq z \leq h; \ 0 \leq \varphi \leq 2\pi; \ 0 \leq r \leq R \).
One can find several works devoted to design of parabolic reflectors of antennas of space apparatus of umbrella type. In a paper [25], a review of results of modeling of parabolic reflectors the structure of which consist of parabolic spokes and a form of the reflecting surface is created by metal mesh cloth. The initial form of the spokes coincides with contour curves on a theoretical paraboloid. Having assumed an edge of a reflector in the form of a hypocycloid one may create a surface of umbrella type with the parabolic generatrixes and the round opening at the vertex [5. P. 389] (Figure 7). In any cross section of the surface by a plane passing through the central axis, the parabola would be placed. Suitableness of the proposed surface for the reflectors with taking into account distortion of the reflecting surface one can corroborate only by experimental measurements.

Designers offer enormous choice of tent umbrella structures for sheltering from the sun and rain. For these purposes, it is possible to use existing surfaces of umbrella type. For example, architectural street umbrellas “Edelveis” designed by specialists of the firm “TeniRadi” are unique durable umbrella structures for using at the grounds with large wind loads (Figure 8, a). They have built-in drainage system. In Figure 8, b, two possible forms for approximation of a surface of an umbrella shown in Figure 8, a are presented. The upper surface is a surface of umbrella type on the cycloidal plan formed by semi-cubic parabolas, the lower surface is a waving surface formed by semi-cubic parabollas with the waving foot line [10; 11].

**The works on strength analysis of umbrella-type shells**

At the middle of the 20th century, some principles of forming surfaces of umbrella and umbrella-type domes and methods of their strength analysis were absent and that is why architectural and technical conceptions were bound [26]. But attempts of building of umbrella domes were undertaken by different architects and builders. Simplified methods of strength analysis were offered. For example, every fragment of the covering of the market in Royan (Figure 4) was considered as a beam on two hinges: at the foot and at the vertex of the dome. Taking into account that the covering works already 60 years, an analysis circuit was chosen successfully and can be recommended for preliminary calculation of umbrella-type shells at present time.

As it was noted before, middle surfaces of umbrella-type shells are defined by one analytical equation and it gives the opportunity to use the standard computer programs and to derive author’s programs for static and dynamic analysis of considered shells on diverse types of loading.

For example, a thin shell shown in Figure 2 was examined on action of dead load with the help of finite difference energy method [15]. This method was used also for strength analysis of two types of umbrella-type shells with middle surfaces in the form of Joachimsthal canal surfaces. They were subjected to action of their own weight [27].

**Figure 7.** A form of a parabolic reflector of umbrella type

**Figure 8.** The tent umbrella structure and its proposed computer models

**Figure 9.** A corrugated sphere

A finite element analysis of stress-strain state of a reinforced concrete shell of umbrella type with an el-
Критические поверхности (1) были реализованы в работе [20]. Анализ был выполнен на основе собственной массы при \( \gamma = 1 \) и замыкании тросов. Программный комплекс LIRA SAPR 2013 был применен. Он был основан на примере подвесной панели, когда контур поверхности (рис. 9) быстрым способом не наблюдается. Мы можем увеличить жесткость оболочки в сравнении с гладкой поверхностью. Наоборот, при увеличении жесткости оболочки, торсионные моменты появляются. Причинение значительно будет зависеть от формы конической поверхности (рис. 9) при волнистых оболочках. Конструкция должна быть основана на архитектуре и дизайне.

**Заключение**

Строители начали воздвигать оболочечные каменные купола несколько веков назад, но тонкосстенные оболочки применялись только в середине 20-го века. В это время, геометрическая наука не могла дать строгого определения поверхности. Оболочка была разделена на идентичные сегменты и анализ был выполнен для одного фрагмента, но результаты приложимы ко всем оболочкам.

В начале 21-го века, аналитические оболочечные поверхности стали использоваться. Давление собственной массы вызывает волнообразные поверхности. Существенно более двум из новых поверхностей были введены в практику. Научные работы посвящены определению напряженно-деформированного состояния оболочки и ее сегментов. На их применение начали поступать. В настоящее время, численные методы анализа востребованы.

По мнению автора, аналитического оболочечных форм не используют. Но автор согласуется с В.А. Лебедевом [26], который говорит о возвращении архитектурной формы. Аналитические оболочечные поверхности и их коэффициенты основных форм. Structural Mechanics of Engineering Constructions and Buildings, 2005;1(1):11–17. (In Russ.)

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Возможности оболочек зонтичного типа

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Аннотация
В статье объясняется актуальность разделения зонтичных поверхностей и поверхностей зонтичного типа на два разных класса. Раньше зонтичные поверхности и поверхности зонтичного типа входили в один класс поверхностей, так как состоят из тождественных фрагментов, лежащих на поверхности вращения. Учитывая способ построения этих поверхностей, а именно то, что зонтичные поверхности – составные поверхности на базовой поверхности вращения, а поверхности зонтичного типа – кинематические поверхности, образованные непрерывным движением изменяющейся кривой, они были разделены на два отдельных класса. Цель статьи – собрать воедино основные публикации по всем разделам исследований оболочек зонтичного типа.

Методы. Для получения основных результатов исследований оболочек зонтичного типа были изучены сведения о дифференциальной геометрии поверхностей, строительной механике оболочек и подходах, используемых в архитектуре пространственных структур. Результаты. Представлено основные научные работы по геометрии, расчету и предложениям по применению тонкостенных оболочек зонтичного типа в строительстве и для рефлекторов зонтичного типа космических аппаратов. Приведены уточненные параметрические уравнения некоторых рассматриваемых поверхностей. Проанализированы аппроксимационные компьютерные модели срединных поверхностей существующих зонтичных оболочек, но в форме поверхностей зонтичного типа. Показаны примеры определения напряженно-деформированного состояния тонкостенных оболочек зонтичного типа без разбиения целой оболочки на тождественные фрагменты. Полученные новые и уже известные сведения об оболочках зонтичного типа дают основания предполагать, что оболочки этого типа будут востребованы инженерами и архитекторами.

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