EXISTENCE OF SOLUTION FOR A CLASS OF QUASILINEAR PROBLEM
IN ORLICZ-SOBOLEV SPACE WITHOUT $\Delta_2$-CONDITION

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ABSTRACT. In this paper we study existence of solution for a class of problem of the type
\[
\begin{cases}
-\Delta_\Phi u = f(u), & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying some conditions, and $\Phi : \mathbb{R} \to \mathbb{R}$ is a N-function which is not assumed to satisfy the well known $\Delta_2$-condition, then the Orlicz-Sobolev space $W^{1,\Phi}_0(\Omega)$ can be non reflexive. As main model we have the function $\Phi(t) = (e^t - 1)/2$. Here, we study some situations where it is possible to work with global minimization, local minimization and mountain pass theorem, however some estimates are not standard for this type of problem.

1. INTRODUCTION

In this paper we study existence of weak solution for a class of quasilinear problem of the type
\[
(P) \quad\begin{cases}
-\Delta_\Phi u = f(u), & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying some conditions which will be mentioned later on, and
\[
\Delta_\Phi u = \text{div}(\phi(|\nabla u|\nabla u))
\]
where $\Phi : \mathbb{R} \to \mathbb{R}$ is a N-function of the form
\[
\Phi(t) = \int_0^{|t|} s\phi(s) \, ds
\]
and $\phi : [0, +\infty) \to [0, +\infty)$ is a $C^1$ function verifying the following conditions
\[
(\phi_1) \quad t \mapsto t\phi(t); \quad t > 0 \quad \text{increasing and } t \mapsto t^2\phi(t) \quad \text{is convex in } \mathbb{R}.
\]
\[
(\phi_2) \quad \lim_{t \to 0} t\phi(t) = 0, \quad \lim_{t \to +\infty} t\phi(t) = +\infty.
\]
\[
(\phi_3) \quad t \mapsto \frac{t^2\phi(t)}{\Phi(t)}, \quad \text{is increasing for } t > 0 \quad \text{with } \frac{t^2\phi(t)}{\Phi(t)} \geq l > 1, \quad \forall t > 0.
\]
for some $l > 1$.
\[
(\phi_4) \quad \frac{t^2\phi(t)}{\Phi(t)} \leq 1 + \frac{t\phi'(t)}{\phi(t)} \leq \frac{2t^2\phi(t)}{\Phi(t)}, \quad \forall t > 0.
\]
If \( d \) is twice the diameter of \( \Omega \), then
\[
(\phi_5) \quad \limsup_{t \to 0^+} \frac{\Phi(t)}{\Phi(t/d)} < +\infty.
\]
For all \( A, B, q > 0 \) with \( A/B < q \), we have
\[
(\phi_6) \quad \lim_{t \to +\infty} \frac{(\Phi(Bl))^q}{\Phi(At)} = +\infty.
\]
\[
(\phi_7) \quad \liminf_{t \to +\infty} \frac{\Phi(t)}{t^p} > 0, \quad \text{for some} \quad p > N.
\]

The last assumption implies that the embedding
\[
W^{1,\Phi}_0(\Omega) \hookrightarrow W^{1,p}(\Omega) \quad \text{for some} \quad p > N
\]
is continuous. Hence, by Sobolev embedding, the embedding
\[
W^{1,\Phi}_0(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)
\]
is continuous for some \( \alpha \in (0,1) \) and
\[
W^{1,\Phi}(\Omega) \hookrightarrow C(\bar{\Omega})
\]
is compact. The condition \((\phi_7)\) also implies that there is \( C > 0 \) such that
\[
\|u\|_{W^{1,p}(\Omega)} \leq C \left( \int_{\Omega} \Phi(|\nabla u|) \, dx \right)^{\frac{1}{p}}, \quad \forall u \in W^{1,\Phi}_0(\Omega).
\]
From this,
\[
(1.3) \quad \|u\|_{C(\Omega)} \leq C \left( \int_{\Omega} \Phi(|\nabla u|) \, dx \right)^{\frac{1}{p}}, \quad \forall u \in W^{1,\Phi}_0(\Omega).
\]
for some \( C > 0 \).

Before continuing this section, we would like to point out that \( \Phi(t) = (e^{t^2} - 1)/2 \) and \( \Phi(t) = |t|^p/p \) with \( p > N \) satisfy \((\phi_1)-(\phi_7)\). Moreover, we would like to recall that \( u \in W^{1,\Phi}_0(\Omega) \) is a weak solution of \((P)\) if
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx = \int_{\Omega} f(u)v \, dx, \quad \forall v \in W^{1,\Phi}_0(\Omega).
\]

Quasilinear elliptic problem have been considered using different assumptions on the \( N \)-function \( \Phi \). Here we refer the reader to [4, 5, 12–15, 17, 19] and references therein. In these works was considered the \( \Delta_2 \)-condition which implies that the Orlicz-Sobolev space \( W^{1,\Phi}_0(\Omega) \) is a reflexive Banach space. This is used in order to get a nontrivial solution for elliptic problems taking into account the weak topology. In our work the main feature is to consider problem \((P)\) where the function \( \Phi \) is not assumed to verify the \( \Delta_2 \)-condition, then we cannot use that \( W^{1,\Phi}_0(\Omega) \) is reflexive which brings serious difficulty to apply variational methods. To overcome this difficulty, we apply the weak* topology recovering the compactness required in variational methods. We would like to recall that \( \Phi(t) = |t|^p/p \) for \( p > 1 \) satisfies the \( \Delta_2 \)-condition, while \( \Phi(t) = (e^{t^2} - 1)/2 \) does not verify the \( \Delta_2 \)-condition. For more details involving the \( \Delta_2 \)-condition see Section 2.

In [11], García-Huidobro, Khoi, Manásevich and K. Schmitt have studied the existence of solution for the following nonlinear eigenvalue problem
\[
(P_2) \quad \begin{cases}
-\Delta \Phi u = \lambda \Psi(u), & \text{in} \quad \Omega \\
 u = 0, & \text{on} \quad \partial \Omega,
\end{cases}
\]
where \( \Phi : \mathbb{R} \to \mathbb{R} \) is a N-function and \( \Psi : \mathbb{R} \to \mathbb{R} \) is a continuous function verifying some technical conditions. In that paper, the authors have considered the situation where the function \( \Phi \) does not satisfy the well known \( \Delta_2 \)-condition, for example, in the first part of this paper the authors considered the function

\[
\Phi(t) = e^{t^2} - 1, \quad \forall t \in \mathbb{R}.
\]

After in [3], Bocea and Mihăilescu made a careful study about the eigenvalues of the problem

\[
(P_3) \begin{cases}
-\text{div}(e^{\|\nabla u\|^2} \nabla u) - \Delta u = \lambda u, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Recently, da Silva, Gonçalves and Silva [8] have studied the existence of multiple solutions for \((P_3)\). In their paper the \( \Delta_2 \)-condition is not also assumed and the main tool used was the truncation of the nonlinearity and minimization of the energy functional associated to the quasilinear elliptic problem \((P)\).

The present paper was motivated by results found in [3] and [11] which can be applied for a class of quasilinear problems where the operator can be driven by N-function with exponential growth. Our first result uses the mountain pass theorem and we assume that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function satisfying the following conditions:

\[
(f_1) \quad \lim_{t \to 0} \frac{F(t)}{\Phi(t)} = 0,
\]

where \( F(t) = \int_0^t f(s)ds \).

There are \( \theta > 1, R > 0 \) in such way that

\[
(f_2) \quad 0 < \theta F(t) \leq h(t)f(t)t, \quad |t| \geq R
\]

holds true with \( h(t) = \frac{\Phi(t)}{t^\phi(t)} \).

The condition \((f_2)\) suggests that \( F \) is \( \Phi \)-superlinear, that is, the limit below holds

\[
(1.4) \quad \lim_{|t| \to +\infty} \frac{F(t)}{\Phi(t)} = +\infty.
\]

In fact, by fixing \( M > R > 0 \) and integrating the sentence

\[
\theta \frac{t \phi(t)}{\Phi(t)} \leq \frac{f(t)}{F(t)} \quad t \geq M > R
\]

we deduce that

\[
(1.5) \quad \frac{F(t)}{\Phi(t)} \geq \frac{F(M)}{\Phi(M)^q} \Phi(t)^{q-1} \to +\infty \quad \text{as } t \to +\infty.
\]

A similar argument works to prove that

\[
\frac{F(t)}{\Phi(t)} \to +\infty \quad \text{as } t \to -\infty.
\]

Here, we would like to point out that \( f(t) = \frac{d}{dt}(\Phi(t))^q \), for \( q > 1 \), satisfies the conditions \((f_1) - (f_2)\), because in this case

\[
F(t) = (\Phi(t))^q, \quad \forall t \in \mathbb{R}.
\]

Our first theorem is the following:

**Theorem 1.1.** Assume that \((\phi_1) - (\phi_7)\) and \((f_1) - (f_2)\) hold. Then, there is \( \theta^* > 0 \) such that if \( \theta \) as in \((f_2)\) verifies \( \theta > \theta^* \) the problem \((P)\) has a nontrivial solution.
To the best our knowledge the Theorem 1.1 is the first existence result for a class of quasilinear problem driven by a N-function with exponential growth by using the mountain pass theorem. Here, we have had serious difficulty in order to find a correct definition for the Ambrosetti-Rabinowitz condition for nonlinearity $f$, which makes the result interesting.

Our second result involves the existence of solution for a situation where the energy functional has a global minimum. For this case, we assume the following conditions on $f$:

\[(f_3) \quad 0 \leq F(t) \leq b_1(\Phi(t/d))^s, \quad \forall t \in \mathbb{R} \quad \text{and for some} \quad s \in (0, 1).\]

and

\[(f_4) \quad F(t) \geq c_1(\Phi(t))^{\gamma}, \quad \forall t \in (0, \delta) \quad \text{for some} \quad \gamma \in (0, 1) \quad \text{and} \quad \delta > 0.\]

Related to $\Phi$, we assume that for any $A, B > 0$

\[(\phi_8) \quad \lim_{t \to 0} \frac{(\Phi(Bt))^{\gamma}}{\Phi(At)} = +\infty.\]

where $\gamma$ is like in $(f_4)$.

The reader is invited to see that $\Phi(t) = (e^{t^2} - 1)/2$ and $\Phi(t) = |t|^p/p$ for $p > N$ also satisfy $(\phi_8)$.

Our second result has the following statement

**Theorem 1.2.** Assume $(f_3) - (f_4)$, $(\phi_1) - (\phi_2)$ and $(\phi_8)$. Then, problem (P) has a nontrivial solution.

Theorem 1.2 completes the study made in [8] and [11], in the sense that we have worked with a class of nonlinearity where the minimization arguments can be used, but it was not considered in the above references.

Our third result is associated with a concave-convex problem for the $\Phi$-Laplacian, which was introduced by Ambrosetti, Brézis and Cerami [2] for the Laplacian operator. For this result, we suppose that $f$ is continuous with primitive $F$ of the form

\[(f_5) \quad F(t) = \lambda \frac{(\Phi(t))^\alpha}{\alpha} + \frac{(\Phi(t))^q}{q}, \quad \forall t \in \mathbb{R},\]

where $\lambda > 0$, $\alpha \in (0, 1)$ and $q > 1$.

Our third result can be stated as below

**Theorem 1.3.** Assume $(f_5)$ and $(\phi_1) - (\phi_8)$. Then, problem (P) has two nontrivial solutions for $\lambda$ small enough.

In the proof of Theorem 1.3 we will use Ekeland’s Variational Principle and Mountain Pass Theorem. Theorem 1.3 completes the study made in [6], because in that paper the authors have considered the concave-convex case for a nonlinearity $f$ of the type

\[f(x, t) = \lambda a(x)|t|^{\alpha-2}t + b(x)|t|^{\gamma-2}t, \quad \forall t \in \mathbb{R},\]

where $a(x), b(x), \alpha$ and $q$ satisfy some technical conditions.

Before concluding this introduction we would like to point out that in the references above mentioned it was showed that if $0 \in \partial I(u)$ and $u$ is a minimum point of $I$, then $u$ is a weak solution of the problem, where $I$ denotes the functional energy associate with the problem and $\partial I(u)$ denotes the subdifferential of $I$ at $u$. Here, after a careful study we have improved this information, in the sense that we have proved that if $u$ is a critical point of $I$, which means $0 \in \partial I(u)$, then $u$ is a weak solution for problem. In our opinion this is a very important information for this class of problem, for more details see Proposition 3.4 and Corollary 3.5 in Section 3.
2. Basics on Orlicz-Sobolev spaces

In this section we recall some properties of Orlicz and Orlicz-Sobolev spaces, which can be found in [1, 18]. First of all, we recall that a continuous function \( \Phi : \mathbb{R} \to [0, +\infty) \) is a N-function if:

(i): \( \Phi \) is convex.

(ii): \( \Phi(t) = 0 \iff t = 0 \).

(iii): \( \lim_{t \to 0} \frac{\Phi(t)}{t} = 0 \) and \( \lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty \).

(iv): \( \Phi \) is even.

We say that a N-function \( \Phi \) verifies the \( \Delta_2 \)-condition, if

\[
\Phi(2t) \leq K \Phi(t), \quad \forall t \geq t_0,
\]

for some constants \( K, t_0 > 0 \). In what follows, fixed an open set \( \Omega \subset \mathbb{R}^N \) and a N-function \( \Phi \), we define the Orlicz space associated with \( \Phi \) as follows

\[ L^\Phi(\Omega) = \{ u \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} \Phi \left( \frac{|u|}{\lambda} \right) dx < +\infty \text{ for some } \lambda > 0 \} . \]

The space \( L^\Phi(\Omega) \) is a Banach space endowed with the Luxemburg norm given by

\[ \|u\|_{\Phi} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi \left( \frac{|u|}{\lambda} \right) dx \leq 1 \right\} . \]

The complementary function \( \Phi^* \) associated with \( \Phi \) is given by the Legendre's transformation, that is,

\[ \Phi^*(s) = \max_{t \geq 0} \{ st - \Phi(t) \}, \quad \text{for } s \geq 0. \]

The functions \( \Phi \) and \( \Phi^* \) are complementary each other. Moreover, we also have a Young type inequality given by

\[ st \leq \Phi(t) + \Phi^*(s), \quad \forall s, t \geq 0. \]

Using the above inequality, it is possible to prove a Hölder type inequality, that is,

\[ \left| \int_{\Omega} uv dx \right| \leq 2 \| u \|_{\Phi} \| v \|_{\Phi^*}, \quad \forall u \in L^\Phi(\Omega) \text{ and } \forall v \in L^{\Phi^*}(\Omega). \]

The corresponding Orlicz-Sobolev space is defined as follows

\[ W^{1,\Phi}(\Omega) = \{ u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega), \quad i = 1, \ldots, N \}, \]

endowed with the norm

\[ \|u\|_{1,\Phi} = \|\nabla u\|_{\Phi} + \|u\|_{\Phi}. \]

The space \( W^{1,\Phi}_0(\Omega) \) is defined as the closure of \( C_0^\infty(\Omega) \) with respect to Orlicz-Sobolev norm above.

The spaces \( L^\Phi(\Omega) \), \( W^{1,\Phi}(\Omega) \) and \( W^{1,\Phi}_0(\Omega) \) are separable and reflexive, when \( \Phi \) and \( \Phi^* \) satisfy the \( \Delta_2 \)-condition.

If \( E^\Phi(\Omega) \) denotes the closure of \( L^\infty(\Omega) \) in \( L^\Phi(\Omega) \) with respect to the norm \( \| \cdot \|_{\Phi} \), then \( L^\Phi(\Omega) \) is the dual space of \( E^{\Phi^*}(\Omega) \), while \( L^{\Phi^*}(\Omega) \) is the dual space of \( E^\Phi(\Omega) \). Moreover, \( E^\Phi(\Omega) \) and \( E^{\Phi^*}(\Omega) \) are separable spaces and any continuous linear functional \( M : E^\Phi(\Omega) \to \mathbb{R} \) is of the form

\[ M(v) = \int_{\Omega} v(x) g(x) dx \quad \text{for some } g \in L^{\Phi^*}(\Omega). \]

When \( \Phi \) verifies the \( \Delta_2 \)-condition, we have that \( E^\Phi(\Omega) = L^\Phi(\Omega) \).
Before concluding this section, we would like to state a lemma whose proof follows directly of a result found in Donaldson [7, Proposition 1.1].

**Lemma 2.1.** Assume that \( \Phi \) is a N-function and \( \Phi^* \) verifies the \( \Delta_2 \)-condition. If \( (u_n) \subset W_0^{1, \Phi}(\Omega) \) is a bounded sequence, then there are a subsequence of \( (u_n) \), still denoted by itself, and \( u \in W_0^{1, \Phi}(\Omega) \) such that

\[
\begin{align*}
  u_n & \rightharpoonup u \quad \text{in} \quad W_0^{1, \Phi}(\Omega) \\
  \int_{\Omega} u_n v \, dx & \rightarrow \int_{\Omega} u v \, dx, \\
  \int_{\Omega} \frac{\partial u_n}{\partial x_i} w \, dx & \rightarrow \int_{\Omega} \frac{\partial u}{\partial x_i} w \, dx, \quad \forall v, w \in E_{\Phi^*}(\Omega) = L_{\Phi^*}(\Omega).
\end{align*}
\]

The above lemma is crucial when we are working in a situation where the space \( W_0^{1, \Phi}(\Omega) \) is not reflexive, for example if \( \Phi(t) = (e^{t^2} - 1)/2 \). However, if \( \Phi(t) = |t|^p / p \) and \( p > 1 \), the above lemma is not necessary because \( \Phi \) satisfies the \( \Delta_2 \)-condition, and so, \( W_0^{1, \Phi}(\Omega) \) is reflexive. Here we would like to point out that the condition (f3) ensures that \( \Phi^* \) verifies the \( \Delta_2 \)-condition, for more details see Fukagai, Ito and Narukawa [10]. From this, we can apply the above lemma in the present paper.

3. Mountain pass

The main goal of this section is proving Theorem 1.1, then throughout this section we assume the assumptions of this theorem. We start by recalling that the conditions (φ1) - (φ4) do not imply that \( \Phi \) satisfies the \( \Delta_2 \)-condition, then \( W_0^{1, \Phi}(\Omega) \) can be non reflexive. In the case where we lose the \( \Delta_2 \)-condition, it is well known that there is \( u \in W_0^{1, \Phi}(\Omega) \) such that

\[
\int_{\Omega} \Phi(|\nabla u|) \, dx = +\infty.
\]

However, independent of \( \Delta_2 \)-condition, the condition (f1) always guarantees that

\[
\left| \int_{\Omega} F(u) \, dx \right| < +\infty, \quad \forall u \in W_0^{1, \Phi}(\Omega).
\]

Having this in mind, the energy functional \( I : W_0^{1, \Phi}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\} \) associated with (P) given by

\[
I(u) = \int_{\Omega} \Phi(|\nabla u|) \, dx - \int_{\Omega} F(u) \, dx,
\]

is well defined. Hereafter, we denote by \( D(I) \subset W_0^{1, \Phi}(\Omega) \) the set

\[
D(I) = \left\{ u \in W_0^{1, \Phi}(\Omega) : \int_{\Omega} \Phi(|\nabla u|) \, dx < +\infty \right\}.
\]

The reader must observe that \( D(I) = W_0^{1, \Phi}(\Omega) \) when \( \Phi \) satisfies the \( \Delta_2 \)-condition.

As an immediate consequence of the above remarks, we cannot guarantee that \( I \) belongs to \( C^1(W_0^{1, \Phi}(\Omega), \mathbb{R}) \). However, the functional \( J : W_0^{1, \Phi}(\Omega) \rightarrow \mathbb{R} \) given by

\[
J(u) = \int_{\Omega} F(u) \, dx
\]

belongs to \( C^1(W_0^{1, \Phi}(\Omega), \mathbb{R}) \) with

\[
J'(u)v = \int_{\Omega} f(u)v \, dx, \quad \forall u, v \in W_0^{1, \Phi}(\Omega).
\]
This can be done using Lebesgue Convergence Theorem and the fact that \( f \) is a continuous function. Related to the functional \( Q : W^1,\Phi_0(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\} \) given by
\[
Q(u) = \int_\Omega \Phi(|\nabla u|) \, dx
\]
we know that it is continuous, strictly convex and l.s.c. with respect to the weak* topology. Moreover, \( Q \in C^1(W^1,\Phi_0(\Omega), \mathbb{R}) \) when \( \Phi \) satisfies the \( \Delta_2 \)-condition.

From the above commentaries, in the present paper we will use a minimax method developed by Szulkin [20]. In this sense, we will say that \( u \in D(I) \) is a critical point for \( I \) if \( 0 \in \partial I(u) \), where
\[
\partial I(u) = \left\{ \chi \in (W^1,\Phi_0(\Omega))' : Q(v) - Q(u) - J'(u)(v - u) \, dx \geq \chi(v - u), \ \forall v \in W^1,\Phi_0(\Omega) \right\}
\]
We recall that \( \partial I(u) \) is the subdifferential of \( I \) at \( u \). Thereby, \( u \in D(I) \) is a critical point for \( I \) if
\[
Q(v) - Q(u) \geq J'(u)(v - u), \ \forall v \in W^1,\Phi_0(\Omega),
\]
or equivalently
\[
\int_\Omega \Phi(|\nabla v|) \, dx - \int_\Omega \Phi(|\nabla u|) \, dx \geq \int_\Omega f(u)(v - u) \, dx, \ \forall v \in W^1,\Phi_0(\Omega).
\]

If \( \Phi \) satisfies the \( \Delta_2 \)-condition, the functional \( I \in C^1(W^1,\Phi_0(\Omega), \mathbb{R}) \) and the last inequality is equivalent to
\[
I'(u)v = 0, \ \forall v \in W^1,\Phi_0(\Omega),
\]
or yet
\[
\int_\Omega \phi(|\nabla u|) \nabla u \nabla v \, dx = \int_\Omega f(u)v \, dx, \ \forall v \in W^1,\Phi_0(\Omega),
\]
showing that \( u \) is a weak solution of \((P)\). However, when \( \Phi \) does not satisfy the \( \Delta_2 \)-condition the above conclusion is not immediate and a careful analysis must be done, for more details see Lemma 3.2 below.

Hereafter, we denote by \( \| \cdot \| \) the usual norm in \( W^1,\Phi_0(\Omega) \) given by
\[
\|u\| = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left( \frac{|\nabla u|}{\lambda} \right) \, dx \leq 1 \right\}.
\]
Since \( \Phi \) is not assumed to satisfy the \( \Delta_2 \)-condition, we cannot claim that \( \| \cdot \| \) is an equivalent norm to induced norm by \( W^1,\Phi_0(\Omega) \). However, it is very important to point out that we have a Poincaré type inequality which can be stated of the form
\[
\int_\Omega \Phi(|u|/d) \leq \int_\Omega \Phi(|\nabla u|) \, dx \quad \forall u \in W^1,\Phi_0(\Omega),
\]
where \( d = 2 \text{diam}(\Omega) \). For more details see [11, Lemma 2.1].

Hereafter, we will denote by \( \text{dom}(\phi(t)t) \subset W^1,\Phi_0(\Omega) \) the following set
\[
\text{dom}(\phi(t)t) = \left\{ u \in W^1,\Phi_0(\Omega) : \int_\Omega \Phi^*(\phi(|\nabla u|)|\nabla u|) \, dx < \infty \right\}.
\]
As \( \Phi^* \) verifies \( \Delta_2 \)-condition, the above set can be written of the form
\[
\text{dom}(\phi(t)t) = \left\{ u \in W^1,\Phi_0(\Omega) : \phi(|\nabla u|)|\nabla u|) \in L^{\Phi^*}(\Omega) \right\}.
\]
The set \( \text{dom}(\phi(t)t) \) is not empty, because it is easy to see that \( C^\infty_0(\Omega) \subset \text{dom}(\phi(t)t) \).
Lemma 3.1. For each \( u \in D(I) \), there is a sequence \( (u_n) \subset \text{dom}(\phi(t)) \) such that

\[
\int_\Omega \Phi(|\nabla u_n|) \, dx \leq \int_\Omega \Phi(|\nabla u|) \, dx \quad \text{and} \quad \|u - u_n\| \leq 1/n.
\]

Proof. For each \( \epsilon \in (0, 1] \), we know by a result found in [16, Lemma 4.1] that \( v_\epsilon = (1 - \epsilon)u \in \text{dom}(\phi(t)) \). By convexity of \( \Phi \), it follows that

\[
\int_\Omega \Phi(|\nabla v_\epsilon|) \, dx \leq \int_\Omega \Phi(|\nabla u|) \, dx, \quad \forall \epsilon \in (0, 1].
\]

On the other hand, we claim that

\[(3.5)\]

\( v_\epsilon \to u \quad \text{in} \quad W^{1,\Phi}_0(\Omega) \quad \text{as} \quad \epsilon \to 0. \)

Indeed, fixed \( \delta > 0 \), for all \( \epsilon \in (0, \delta) \) we have

\[
\frac{\Phi(|\nabla u - \nabla v_\epsilon|)}{\delta} = \frac{\Phi(|\epsilon \nabla u|)}{\delta} \leq \frac{\epsilon}{\delta} \Phi(|\nabla u|) \leq \Phi(|\nabla u|) \in L^1(\Omega).
\]

Applying the Lebesgue’s Theorem, we get

\[
\int_\Omega \frac{\Phi(|\nabla u - \nabla v_\epsilon|)}{\delta} \, dx \to 0 \quad \text{as} \quad \epsilon \to 0.
\]

Then

\[
\|u - v_\epsilon\| < \delta
\]

for \( \epsilon \) small enough, showing the desired result.

Our next lemma establishes that a critical point \( u \) in the sense \((3.2)\) is a weak solution for \((P)\) if \( u \in \text{dom}(\phi(t)) \).

Lemma 3.2. Let \( u \in D(I) \) be a critical point of \( I \). If \( u \in \text{dom}(\phi(t)) \), then it is a weak solution for \((P)\), that is,

\[
\int_\Omega \phi(|\nabla u|) \nabla u \nabla w \, dx = \int_\Omega f(u)w \, dx, \quad \forall w \in W^{1,\Phi}_0(\Omega).
\]

Proof. By following the arguments found in García-Huidobro, Khoi, Manásevich and Schmitt [11], the directional derivative \( \frac{\partial Q(u)}{\partial v} \) given by

\[
\frac{\partial Q(u)}{\partial v} = \lim_{t \to 0} \frac{Q(u + tv) - Q(u)}{t}
\]

exists for all \( v \in D(I) \cap \text{dom}(\phi(t)) \) with

\[
\frac{\partial Q(u)}{\partial v} = \int_\Omega \phi(|\nabla u|) \nabla u \nabla v \, dx.
\]

Since \( J \in C^1(W^{1,\Phi}_0(\Omega), \mathbb{R}) \), we must have

\[
\frac{\partial J(u)}{\partial v} = \int_\Omega f(u)v \, dx, \quad \forall v \in D(I) \cap \text{dom}(\phi(t)).
\]

From this,

\[
\frac{\partial I(u)}{\partial v} = \frac{\partial Q(u)}{\partial v} - \frac{\partial J(u)}{\partial v}, \quad \forall v \in D(I) \cap \text{dom}(\phi(t))
\]

and so,

\[
\frac{\partial I(u)}{\partial v} = \int_\Omega \phi(|\nabla u|) \nabla u \nabla v \, dx - \int_\Omega f(u)v \, dx, \quad \forall u, v \in D(I) \cap \text{dom}(\phi(t)).
\]
On the other hand, by (3.2),
\[
\int_{\Omega} \Phi(|\nabla u + tv|) \, dx - \int_{\Omega} \Phi(|\nabla u|) \, dx \geq t \int_{\Omega} f(u)v \, dx, \quad \forall v \in D(I) \cap \text{dom}(\phi(t)t) \quad \text{and} \quad t \in \mathbb{R},
\]
which leads to
\[
\frac{\partial Q(u)}{\partial v} = \lim_{t \to 0^+} \frac{\int_{\Omega} \Phi(|\nabla u + tv|) \, dx - \int_{\Omega} \Phi(|\nabla u|) \, dx}{t} \geq \int_{\Omega} f(u)v \, dx = \frac{\partial J(u)}{\partial v},
\]
or equivalently,
\[
\frac{\partial I(u)}{\partial v} \geq 0, \quad \forall v \in D(I) \cap \text{dom}(\phi(t)t).
\]
Since \(v\) is arbitrary and \(-v \in D(I) \cap \text{dom}(\phi(t)t)\), the last inequality gives
\[
\frac{\partial I(u)}{\partial v} = 0, \quad \forall v \in D(I) \cap \text{dom}(\phi(t)t),
\]
and so,
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx = \int_{\Omega} f(u)v \, dx, \quad \forall v \in D(I) \cap \text{dom}(\phi(t)t).
\]
In particular,
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx = \int_{\Omega} f(u)v \, dx, \quad \forall v \in C_0^\infty(\Omega).
\]
Now the result follows by using the density of \(C_0^\infty(\Omega)\) in \(W_0^{1,\Phi}(\Omega)\) together with the fact that \(\phi(|\nabla u|)|\nabla u| \in L^{\Phi^*}(\Omega)\).

The next result shows that \(I\) possesses the mountain pass geometry.

**Lemma 3.3.** The functional \(I\) satisfies the mountain pass geometry, that is,

(a) There are \(r, \rho > 0\) such that
\[
I(u) \geq \rho \quad \text{for} \quad \int_{\Omega} \Phi(|\nabla u|) \, dx = r.
\]

(b) There is \(\epsilon \in W_0^{1,\Phi}(\Omega)\) with \(\int_{\Omega} \Phi(|\nabla \epsilon|) \, dx > r\) and \(I(\epsilon) < 0\).

**Proof.** We begin recalling that by \((f_1)\), given \(\epsilon > 0\) there is \(r > 0\) such that
\[
F(t) \leq \epsilon \Phi(t) \quad \text{for} \quad |t| \leq r.
\]
Combining the last inequality with (1.2), it follows that
\[
I(u) \geq \int_{\Omega} \Phi(|\nabla u|) \, dx - \epsilon \int_{\Omega} \Phi(u) \, dx, \quad \text{for} \quad \int_{\Omega} \Phi(|\nabla u|) \, dx = r.
\]
Now, by \((\phi_5)\) and (1.3) there exists \(C > 0\) such that
\[
\Phi(u) \leq C \Phi(d^{-1} u) \quad \text{for} \quad \int_{\Omega} \Phi(|\nabla u|) \, dx = r.
\]
Using the last inequality together with Poincaré inequality (3.4), we get
\[
I(u) \geq (1 - \epsilon C) \int_{\Omega} \Phi(|\nabla u|) \, dx = \rho, \quad \text{for} \quad \int_{\Omega} \Phi(|\nabla u|) \, dx = r \quad \text{and} \quad \rho = (1 - \epsilon C)r > 0,
\]
showing (a). Now, we will prove (b). To this end, we set \(\Psi \in C^\infty(\overline{\Omega}) \cap W_0^{1,\Phi}(\Omega)\) with
\[
\Psi(x) > 0 \quad \forall x \in \Omega, \quad \Psi(x) = 0 \quad \forall x \in \partial \Omega,
\]
and
\[ A = |\nabla \Psi|_{\infty, \Omega} \quad \text{and} \quad B = \inf_{x \in \Omega_0} \Psi(x) \quad (\Omega_0 \subset \subset \Omega). \]
By (1.5), there are \( A_0, B_0 > 0 \) such that
\[ F(t) \geq A_0 \Phi(t)^\theta - B_0, \quad \forall t \in \mathbb{R}. \]
Hence, for any \( t > 0 \), we mention that
\[ I(t\Psi) \leq \int_\Omega \Phi(t|\nabla |) \, dx - A_0 \int_\Omega \Phi(t\Psi)^\theta \, dx + B_0|\Omega|, \]
\[ \leq \int_\Omega \Phi(t|\nabla |) \, dx - A_0 \int_\Omega \Phi(t\Psi)^\theta \, dx + B_0|\Omega|, \]
\[ \leq C_1 \Phi(At) - C_2(\Phi(Bt))^\theta + B_0|\Omega|. \]
Now, fixing \( \theta^* > 0 \) such that \( \frac{A}{\theta} < \theta^* \) and \( \theta > \theta^* \), the condition \( (\phi_0) \) leads to
\[ I(t\Psi) \to -\infty \quad \text{as} \quad t \to +\infty, \]
showing (b).

\[ \square \]

Remark 1. In the proof of the last lemma we have used the condition \( (\phi_0) \), but the reader is invited to observe that it is not necessary when \( \Omega \) contains a ball \( B_r(x_0) \) with \( r > 1 \), because in this case it is easy to build a function \( \Psi \in C^\infty(\Omega) \cap W^{1,\Phi}_0(\Omega) \) verifying
\[ \Psi(x) > 0 \quad \forall x \in \Omega, \quad \Psi(x) = 0 \quad \forall x \in \partial \Omega \quad \text{and} \quad A = |\nabla \Psi|_{\infty, \Omega} < B = \inf_{x \in \Omega_0} \Psi(x) \quad (\Omega_0 \subset \subset \Omega). \]
Using this information together with the fact that \( \Phi \) is increasing for \( t \geq 0 \), we get
\[ I(t\Psi) \leq C_1 \Phi(Bt) - C_2(\Phi(Bt))^\theta + B_0|\Omega| \to -\infty \quad \text{as} \quad t \to +\infty. \]

The next result establishes that any \((PS)\) sequence of \( I \) is bounded. We recall that \((u_n) \subset W^{1,\Phi}_0(\Omega) \) is a \((PS)\) sequence at level \( c \in \mathbb{R} \), if there is \( \tau_n \to 0 \) such that
\[ (3.6) \quad I(u_n) \to c \quad \text{as} \quad n \to +\infty \]
and
\[ (3.7) \quad \int_\Omega \Phi(|\nabla v|) \, dx - \int_\Omega \Phi(|\nabla u_n|) \, dx \geq \int_\Omega f(u_n)(v-u_n) \, dx - \tau_n\|v-u_n\|, \quad \forall v \in W^{1,\Phi}_0(\Omega) \quad \text{and} \quad n \in \mathbb{N}. \]
In the sequel we say that \( I \) satisfies the \((PS)\) condition, if any \((PS)\) sequence possesses a convergent subsequence in \( W^{1,\Phi}_0(\Omega) \) in the strong topology. However, we would like point out that by (3.6), if \((u_n)\) is a \((PS)\) sequence for \( I \), then \((u_n) \subset D(I)\).

Proposition 3.4. (Main Proposition) If \((u_n) \subset W^{1,\Phi}_0(\Omega) \) is a \((PS)\) sequence for \( I \), then \((u_n)\) is bounded and there exists \( u \in D(I) \cap \text{dom}(\phi(t)) \) such that for some subsequence, still denoted by itself, we have
\[ \int_\Omega f(u_n)v \, dx \to \int_\Omega f(u)v \, dx \quad \forall v \in W^{1,\Phi}_0(\Omega), \]
\[ \int_\Omega F(u_n) \, dx \to \int_\Omega F(u) \, dx \]
Our first step is showing that any (PS) sequence \((u_n)\) is bounded. To this end, consider the sequence

\[ v_n(x) = \frac{\Phi(u_n)(x)}{u_n(x)\phi(u_n(x))}, \quad x \in \Omega. \]

A direct computation leads to

\[ \nabla v_n = \left[ 1 - \frac{\Phi(u_n)}{u_n^2 \phi(u_n)} \left[ 1 + \frac{u_n \phi'(u_n)}{\phi(u_n)} \right] \right] \nabla u_n, \]

then by \((\phi_4)\),

\[ |\nabla v_n| \leq |\nabla u_n| \quad \forall n \in \mathbb{N}. \tag{3.8} \]

On the other hand, \((\phi_3)\) also gives

\[ |v_n(x)| \leq \frac{1}{I} |u_n(x)| \quad \forall x \in \Omega. \tag{3.9} \]

From (3.8)-(3.9), \(v_n \in D(I)\) with

\[ \int_{\Omega} \Phi(|\nabla v_n|) \, dx \leq \int_{\Omega} \Phi(|\nabla u_n|) \, dx, \quad \forall n \in \mathbb{N}. \]

Applying (3.7) with \(v = u_n + tv_n\) and taking the limit as \(t \to 0^+\) we get

\[ \int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla v_n \geq \int_{\Omega} f(u_n)v_n - \tau_n \|v_n\| \geq \int_{\Omega} f(u_n)v_n - |\tau_n|\|u_n\|, \quad \forall n \in \mathbb{N}, \]

that is,

\[ \frac{\partial I(u_n)}{\partial v_n} \geq -|\tau_n|\|u_n\|, \quad \forall n \in \mathbb{N}. \]

Combining the above informations, we obtain

\[ c + 1 \geq I(u_n) - \frac{1}{\theta} \frac{\partial I(u_n)}{\partial v_n} - \frac{1}{\theta} |\tau_n|\|u_n\|, \quad \forall n \in \mathbb{N}, \]

from where it follows that

\[ c + 1 \geq \int_{\Omega} \Phi(|\nabla u_n|) \, dx - \frac{1}{\theta} \int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^2 S(u_n) \, dx + \frac{1}{\theta} \int_{\Omega} (f(u_n)u_n h(u_n) - \theta F(u_n)) \, dx - \frac{1}{\theta} |\tau_n|\|u_n\|, \]

where

\[ h(t) = \frac{\Phi(t)}{t^2 \phi(t)} \quad \text{and} \quad S(t) = 1 - \frac{\Phi(t)}{t^2 \phi(t)} \left[ 1 + \frac{t \phi'(t)}{\phi(t)} \right]. \]

From \((f_2)\) and \((\phi_4)\), \(S(t) \leq 0\) for all \(t \in \mathbb{R}\), and so

\[ c + 1 \geq \int_{\Omega} \Phi(|\nabla u_n|) \, dx - K - \frac{1}{\theta} |\tau_n|\|u_n\|, \forall n \in \mathbb{N}, \]

for some \(K > 0\). Supposing by contradiction that \((u_n)\) possesses a subsequence, still denoted by itself, satisfying

\[ \|u_n\| \to +\infty \quad \text{as} \quad n \to +\infty, \]
we must have for $n$ large enough
\[
\int_\Omega \Phi(|\nabla u_n|) \, dx \geq \| u_n \|.
\]
Hence, for $n$ large enough
\[
c + 1 \geq \left( 1 - \frac{1}{\theta} |\tau_n| \right) \| u_n \| - K \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty,
\]
which is a contradiction. The above analysis shows that $(u_n)$ is a bounded sequence in $W_0^{1,\Phi}(\Omega)$.

Now, we will show that $(u_n)$ has a subsequence strongly convergent in $W_0^{1,\Phi}(\Omega)$. In order to do that, taking into account (1.2), there exists $u \in D(I) \cap C(\overline{\Omega})$ and a subsequence of $(u_n)$, still denoted by itself, such that
\[
u_n \rightarrow u \quad \text{in} \quad C(\overline{\Omega}).
\]
The last limit permits to conclude that
\[
\int_\Omega f(u_n)v \, dx \rightarrow \int_\Omega f(u)v \, dx, \quad \forall v \in W_0^{1,\Phi}(\Omega),
\]
and
\[
(3.10) \quad \int_\Omega F(u_n) \, dx \rightarrow \int_\Omega F(u) \, dx.
\]
Since $(I(u_n))$ is bounded, we will suppose that for some subsequence the sequence
\[
\left( \int_\Omega \Phi(|\nabla u_n|) \, dx \right)
\]
has limit which will be denoted by $L$, that is,
\[
\lim_{n \rightarrow +\infty} \int_\Omega \Phi(|\nabla u_n|) \, dx = L.
\]
As the functional $Q$ given in (3.1) is l.s.c. with respect to the weak* topology we obtain
\[
(3.11) \quad \int_\Omega \Phi(|\nabla u|) \, dx \leq \lim_{n \rightarrow +\infty} \int_\Omega \Phi(|\nabla u_n|) \, dx = L.
\]
From (3.7), we know that
\[
\int_\Omega \Phi(|\nabla v|) \, dx - \int_\Omega \Phi(|\nabla u_n|) \, dx \geq \int_\Omega f(u_n)(v-u_n) \, dx - \tau_n v - \tau_n u_n, \quad \forall v \in W_0^{1,\Phi}(\Omega) \quad \text{and} \quad n \in \mathbb{N},
\]
from where it follows that
\[
\int_\Omega \Phi(|\nabla v|) \, dx - \int_\Omega \Phi(|\nabla u|) \, dx \geq \int_\Omega f(u)(v-u) \, dx, \quad \forall v \in W_0^{1,\Phi}(\Omega).
\]
From this, $u \in D(I)$ and it is a critical point of $I$. Moreover, we also have
\[
\int_\Omega \Phi(|\nabla u|) \, dx - \int_\Omega \Phi(|\nabla u_n|) \, dx \geq \int_\Omega f(u_n)(u-u_n) \, dx - \tau_n u - \tau_n u_n, \quad n \in \mathbb{N}.
\]
Therefore,
\[
(3.12) \quad \int_\Omega \Phi(|\nabla u|) \, dx \geq \lim_{n \rightarrow +\infty} \int_\Omega \Phi(|\nabla u_n|) \, dx.
\]
Combining (3.11) with (3.12) we get
\[
(3.13) \quad \lim_{n \rightarrow +\infty} \int_\Omega \Phi(|\nabla u_n|) \, dx = \int_\Omega \Phi(|\nabla u|) \, dx.
\]
From (3.10) and (3.13),
\[
\lim_{n \rightarrow +\infty} I(u_n) = I(u).
\]
In the sequel, we will show that \( u \in \text{dom}(\phi(t)t) \). By Lemma 3.1, there is \( (v_n) \subset \text{dom}(\phi(t)t) \) such that
\[
\|v_n - u_n\| \leq 1/n \quad \text{and} \quad \int_{\Omega} \phi(|\nabla v_n|) \, dx \leq \int_{\Omega} \phi(|\nabla u_n|) \, dx, \quad \forall n \in \mathbb{N}.
\]
Consequently,
\[
v_n \rightharpoonup u \quad \text{in} \quad C(\overline{\Omega})
\]
and
\[
\int_{\Omega} \phi(|\nabla v|) \, dx - \int_{\Omega} \phi(|\nabla v_n|) \, dx \geq \int_{\Omega} f(u_n)(v-u_n) \, dx - |\tau_n||v_n-u_n|, \quad \forall v \in W^1_0(\Omega) \quad \text{and} \quad \forall n \in \mathbb{N}.
\]
Setting \( v = v_n - \frac{1}{n}v_n \), we get
\[
\int_{\Omega} \phi(|\nabla v_n|) \, dx - \int_{\Omega} \phi(|\nabla v_n|) \, dx \geq \int_{\Omega} f(u_n)(v_n - \frac{1}{n}v_n - u_n) \, dx - |\tau_n||v_n - \frac{1}{n}v_n - u_n|,
\]
or equivalently
\[
\int_{\Omega} \left( \frac{\phi(|\nabla v_n|)}{n} - \frac{\phi(|\nabla v_n|)}{n} \right) \, dx \leq -n \int_{\Omega} f(u_n)(v_n - u_n) \, dx + \int_{\Omega} f(u_n)v_n \, dx + n|\tau_n||v_n-u_n| + |\tau_n||v_n|.
\]
As \( (u_n) \) is bounded in \( W^1_0(\Omega) \), \( f(u_n) \) is bounded in \( L^\infty(\Omega) \), \( (\tau_n) \) is bounded in \( \mathbb{R} \) and \( \|v_n - u_n\| \leq \frac{1}{n} \), it follows that the right side of the above inequality is bounded. Therefore, there is \( M > 0 \) such that
\[
\int_{\Omega} \left( \frac{\phi(|\nabla v_n|)}{n} - \frac{\phi(|\nabla v_n|)}{n} \right) \, dx \leq M, \quad \forall n \in \mathbb{N}.
\]
Since \( \Phi \) is \( C^1 \), there is \( \theta_n(x) \in [0,1] \) verifying
\[
\frac{\Phi(|\nabla v_n|)}{n} - \frac{\Phi(|\nabla v_n|)}{n} = \phi((1 - \theta_n(x)/n)\nabla v_n)|1 - \theta_n(x)/n|\nabla v_n|^2.
\]
Recalling that \( 0 < 1 - \theta_n(x)/n \leq 1 \), we know that
\[
1 - \theta_n(x)/n \geq (1 - \theta_n(x)/n)^2,
\]
which leads to
\[
\int_{\Omega} \phi((1 - \theta_n(x)/n)\nabla v_n)|(1 - \theta_n(x)/n)^2|\nabla v_n|^2 \, dx \leq M, \quad \forall n \in \mathbb{N}.
\]
As \( u_n \rightharpoonup u \) in \( W^1_0(\Omega) \), we also have \( (1 - \theta_n(x)/n)v_n \rightharpoonup u \) in \( W^1_0(\Omega) \). Then, by using the fact that \( \phi(t)t^2 \) is convex, we can apply [11, Lemma 3.2] to get
\[
\liminf_{n \to +\infty} \int_{\Omega} \phi((1 - \theta_n(x)/n)\nabla v_n)|(1 - \theta_n(x)/n)^2|\nabla v_n|^2 \, dx \geq \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \, dx
\]
and so,
\[
\int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \, dx \leq M.
\]
Recalling that
\[
\phi(t)t^2 = \Phi(t) + \Phi^*(\phi(t)t), \quad \forall t \in \mathbb{R}
\]
we have
\[
\phi(|\nabla u|)|\nabla u|^2 = \Phi(|\nabla u|) + \Phi^*(\phi(|\nabla u|)|\nabla u|)
\]
which leads to
\[
\int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \, dx = \int_{\Omega} \Phi(|\nabla u|) \, dx + \int_{\Omega} \Phi^*(\phi(|\nabla u|)|\nabla u|) \, dx.
\]
Since \( \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \, dx \) and \( \int_{\Omega} \Phi(|\nabla u|) \, dx \) are finite, we see that \( \int_{\Omega} \Phi^* (\phi(|\nabla u|)|\nabla u|) \, dx \) is also finite, showing that \( u \in \text{dom}(\phi(t)t) \), finishing the proof. \( \square \)

As an immediate consequence of the last proposition we have

**Corollary 3.5.** Let \( u \in W_0^{1,\Phi}(\Omega) \) be a critical point of \( I \), that is, \( 0 \in \partial I(u) \). Then, \( u \) is a weak solution of \((P)\).

**Proof.** It is enough to apply the Proposition 3.4 with \( u_n = u \) for all \( n \in \mathbb{N} \). \( \square \)

### 3.1. Proof of Theorem 1.1.

**Proof.** From Lemmas 3.3 and 3.4, \( I \) verifies the assumptions of the mountain pass theorem due to Szulkin \cite{20}. Then the mountain pass level \( \beta \) of \( I \) is a critical level, that is, there is \( u \in W_0^{1,\Phi}(\Omega) \) such that

\[
I(u) = \beta > 0 \quad \text{and} \quad \int_{\Omega} \Phi(|\nabla u|) \, dx - \int_{\Omega} \Phi(|\nabla u|) \, dx \geq \int_{\Omega} f(u)(v - u) \, dx, \quad \forall v \in W_0^{1,\Phi}(\Omega).
\]

Thus, by Corollary 3.5 \( u \) is a nontrivial solution of \((P)\). \( \square \)

### 4. Global Minimization

In this section, we intend to prove Theorem 1.2 by showing that \( I \) has a critical point which can be obtained by global minimization.

**Proof.** By using the definition of \( I \) and \((f_3)\), we get

\[
I(u) \geq \int_{\Omega} \Phi(|\nabla u|) \, dx - b_1 \int_{\Omega} (\Phi(u/d))^s \, dx, \quad \forall u \in W_0^{1,\Phi}(\Omega).
\]

By Hölder’s inequality and \((3.4)\),

\[
I(u) \geq \int_{\Omega} \Phi(|\nabla u|) \, dx - C \left( \int_{\Omega} \Phi(|\nabla u|) \, dx \right)^s, \quad \forall u \in W_0^{1,\Phi}(\Omega).
\]

Now, as \( s \in (0,1) \) and

\[
\|u\| \to +\infty \Rightarrow \int_{\Omega} \Phi(|\nabla u|) \, dx \to +\infty,
\]

we derive

\[
I(u) \to +\infty \quad \text{as} \quad \|u\| \to +\infty,
\]

showing that \( I \) is coercive. This fact combined with the definition of \( I \) gives that \( I \) is bounded from below in \( W_0^{1,\Phi}(\Omega) \). Thereby, there is \( (u_n) \subset W_0^{1,\Phi}(\Omega) \) such that

\[
I(u_n) \to I_\infty = \inf_{u \in W_0^{1,\Phi}(\Omega)} I(u) \quad \text{as} \quad n \to +\infty.
\]

Consequently, by coercivity of \( I \), \( (u_n) \) is bounded in \( W_0^{1,\Phi}(\Omega) \). Thus, by Lemma 2.1, for some subsequence,

\[
u_n \overset{*}{\rightharpoonup} u \quad \text{in} \quad W_0^{1,\Phi}(\Omega).
\]

Now, applying \cite[Lemma 3.2]{11} and \cite{9}, the functional \( I \) is weak* lower semicontinuous, and so,

\[
\liminf_{n \to +\infty} I(u_n) \geq I(u),
\]

implying that

\[
I(u) = I_\infty.
\]
Therefore \( u \in D(I) \) and \( 0 \in \partial I(u) \), from where it follows that \( u \) is weak solution of \((P)\). Now, we will prove that \( u \neq 0 \). To this end, it is enough to show that \( I_\infty < 0 \). Fix \( v \in C^\infty_0(\Omega) \) with \( v \neq 0 \), and note that by \((f_4)\), if \( t > 0 \) is small enough,

\[
F(tv(x)) \geq c_1 \Phi(tv(x)), \quad \forall x \in \overline{\Omega}.
\]

Thereby,

\[
I(tv) \leq \int_\Omega \Phi(t|\nabla v|) \, dx - c_1 \int_\Omega (\Phi(tv))^\gamma \, dx
\]

\[
\leq A_1 \Phi(tA_2) - B_1 (\Phi(tB_2))^\gamma.
\]

From \((\psi_8)\), we see that \( I(tv) < 0 \) for \( t \) small enough. As \( I_\infty \leq I(tv) \), it follows that \( I_\infty < 0 \), finishing the proof.

\[\Box\]

5. The concave and convex case

In this section, our intention is showing the Theorem 1.3. Before proving this result, we recall that in this section the energy functional \( I : W^{1,\Phi}_0(\Omega) \rightarrow \mathbb{R} \) is given by

\[
I(u) = \int_\Omega \Phi(|\nabla u|) \, dx - \frac{\lambda}{\alpha} \int_\Omega (\Phi(|u|))^{\alpha} \, dx - \frac{1}{q} \int_\Omega (\Phi(|u|))^q \, dx.
\]

5.1. First solution.

**Proof.** By using Hölder and Poincaré inequalities,

\[
I(u) \geq \int_\Omega \Phi(|\nabla u|) \, dx - \frac{\lambda}{\alpha} \left( \int_\Omega \Phi(|\nabla u|) \, dx \right)^\alpha - \frac{1}{q} \left( \int_\Omega \Phi(|\nabla u|) \, dx \right)^q.
\]

From the above inequality, there are positive numbers \( \lambda^*, r \) and \( \rho > 0 \) such that

\[
(5.1) \quad I(u) > \rho \quad \text{for} \quad \int_\Omega \Phi(|\nabla u|) \, dx = r, \quad \text{and} \quad 0 < \lambda \leq \lambda^*.
\]

Hereafter, we denote by \( X \subset W^{1,\Phi}_0(\Omega) \) the following closed set

\[
X = \left\{ u \in W^{1,\Phi}_0(\Omega) : \int_\Omega \Phi(|\nabla u|) \, dx \leq r \right\},
\]

and by \( I_\infty \in [0, +\infty) \) the number

\[
I_\infty = \inf_{u \in X} I(u).
\]

Arguing as in Section 3, it is possible to ensure that there exists \( w \in \text{int}(X) \) with \( I(w) < 0 \). This information implies that

\[
(5.2) \quad \inf_{u \in X} I(u) < \inf_{u \in \partial X} I(u).
\]

By Using the Ekeland’s variational principle, we find a sequence \((u_n) \subset X\) verifying

\[
(5.3) \quad I(u_n) \rightarrow I_\infty \quad \text{and} \quad I(v) - I(u_n) \geq -\frac{1}{n} \|v - u_n\|, \quad \forall v \in X \setminus \{u_n\}.
\]

Since the functionals \( J \) is Gateaux differentiable at \( u_n \) and \( Q \) is convex, we derive that there exists \( \tau_n \rightarrow 0 \) verifying

\[
\int_\Omega \Phi(|\nabla v|) \, dx - \int_\Omega \Phi(|\nabla u_n|) \, dx \geq \int_\Omega f(u_n)(v - u_n) \, dx - \tau_n \|v - u_n\|, \quad \forall v \in W^{1,\Phi}_0(\Omega) \quad \text{and} \quad n \in \mathbb{N}.
\]

The above analysis gives that \((u_n)\) is a \((PS)\) sequence for \( I \).
A simple computation shows that \((f_5)\) leads to
\[
\lim_{t \to +\infty} \frac{F(t)}{h(t)tf(t)} = \frac{1}{q} < 1,
\]
from where it follows that condition \((f_2)\) is verified. Thereby, arguing as in Proposition 3.4 of Section 3, functional \(I\) verifies the \((PS)\) condition, and thus, there is \(u \in X\) such that
\[
I(u) = I_\infty < 0 \quad \text{and} \quad 0 \in \partial I(u).
\]
Therefore, \(u\) is our first nontrivial weak solution.

5.2. Second solution.

Proof. By above arguments, we know that \(f\) satisfies \((f_2)\) and \((5.1)\) guarantees \(I\) verifies the mountain pass geometry. Thereby, the same arguments explored in Section 3 work to show that \(I\) possesses a critical point \(w \in W^{1,\Phi}_0(\Omega)\) at the mountain pass level \(\beta\) of \(I\), that is,
\[
I(w) = \beta > 0 \quad \text{and} \quad 0 \in \partial I(w).
\]
Thus, \(w\) is a nontrivial solution. Moreover, \(w\) is not equal to first solution \(u\), because \(I(u) < 0 < I(w)\). Therefore, \(w\) is our second nontrivial weak solution.

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