On the Cross-Correlation of a Ternary $m$-sequence of Period $3^{4k} - 1$ and Its Decimated Sequence by\[
\frac{(3^{2k}+1)^2}{20}.
\]

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Abstract—Let $d = \frac{(3^{2k}+1)^2}{20}$, where $k$ is an odd integer. We show that the magnitude of the cross-correlation values of a ternary $m$-sequence $\{s_t\}$ of period $3^{4k} - 1$ and its decimated sequence $\{s_{dt}\}$ is upper bounded by $5\sqrt{3^m} + 1$, where $n = 4k$.

I. INTRODUCTION

One problem of considerable interest has been to find a decimation value $d$ such that the cross-correlation between a $p$-ary $m$-sequence $\{s_t\}$ of period $p^n - 1$ and its decimation $\{s_{dt}\}$ is low. When $\gcd(d, p^n - 1) = 1$, the decimated sequence $\{s_{dt}\}$ is also a $m$-sequence of period $p^n - 1$. Basic results on the cross-correlation between two $m$-sequences can be found in [1-3].

When $\gcd(d, p^n - 1) \neq 1$, the sequence $\{s_{dt}\}$ has period $\frac{p^n - 1}{\gcd(d, p^n - 1)}$. For this case, the reader is referred to [4-6]. Recently, Choi, No, and Chung [7] investigated into the cross-correlation of a ternary $m$-sequence $\{s_t\}$ of period $3^{4k} - 1$ and its decimated sequence $\{s_{dt}\}$ by $d = \frac{(3^{2k}+1)^2}{20}$. They derived the upper bound as $2\sqrt{3^n} + 1$.

In this paper, we employ the method of [7] to investigate the cross-correlation of a ternary $m$-sequence $\{s_t\}$ of period $3^{4k} - 1$ and its decimated sequence $\{s_{dt}\}$ by $d = \frac{(3^{2k}+1)^2}{20}$, where $k$ is an odd integer. We show that the magnitude of the cross-correlation values is upper bounded by $5\sqrt{3^m} + 1$, where $n = 4k$.

II. PRELIMINARIES

Let $p$ be an odd prime, $\omega = e^{\frac{2\pi}{3n}}$ be a complex $p$th root of unity. Let $\{a_t\}$ and $\{b_t\}$ be two sequences of period $L$ over GF$(p)$. The cross-correlation between these two sequences at shift $\tau$ is defined by

$$C_{a,b}(\tau) = \sum_{t=0}^{L-1} \omega^{a_{t+\tau} - b_{t}},$$

where $0 \leq \tau < L$.

Let GF$(p^n)$ be the finite field with $p^n$ elements and GF$(p^n)^* = \text{GF}(p^n) \setminus \{0\}$. The trace function $\text{Tr}_m$ from the field GF$(p^n)$ onto the subfield GF$(p^m)$ is defined by

$$\text{Tr}_m^n(x) = x + x^{p^m} + x^{p^{2m}} + \cdots + x^{p^{(h-1)m}},$$

where $h = n/m$.

Let $\alpha$ be a primitive element of GF$(p^n)$. A $p$-ary $m$-sequence $\{s_t\}$ is given by

$$s_t = \text{Tr}_1^n(\alpha^t),$$

where Tr$_1^n$ is the trace function from GF$(p^n)$ onto GF$(p)$. The periodic cross correlation function $C_d(\tau)$ between $\{s_t\}$ and $\{s_{dt}\}$ is defined by

$$C_d(\tau) = \sum_{t=0}^{p^n-2} \omega^{s_{t+\tau} - s_{dt}},$$

where $0 \leq \tau \leq p^n - 2$.

We will use the following notation unless otherwise specified. Let $\alpha$ be a primitive element of GF$(3^n)$, $n = 4k$, $\omega = e^{\frac{2\pi}{3^{2k}+1}}$ and $d = \frac{(3^{2k}+1)^2}{20}$, where $k$ is an odd integer.

III. CROSS-CORRELATION AND THE RANK OF THE QUADRATIC FORM

In this section, we will give some results needed to prove our main result.

Lemma 1: Let the symbols be defined as above. Then the cross-correlation between $\{s_t\}$ and $\{s_{dt}\}$ is given by

$$C_d(\tau) = -1 + S_d(\tau).$$

Here,

$$2S_d(\tau) = \sum_{x \in \text{GF}(3^n)} \omega^{q_1(x)} + \sum_{x \in \text{GF}(3^n)} \omega^{q_2(x)},$$

where

$$q_1(x) = \text{tr}_1^n(ax^{3^{2(k+1)}+1} - x^{3^{2k}+1}),$$

$$q_2(x) = \text{tr}_1^n(ax^{3^{2(k+1)}+1} - r^ax^{3^{2k}+1}),$$

$a = \omega^7$ and $r$ is a nonsquare in GF$(3^4)$.

Proof: By the definition of $C_d(\tau)$, we get

$$C_d(\tau) = \sum_{t=0}^{3^n-2} \omega^{s_{t+\tau} - s_{dt}}$$

$$= \sum_{t=0}^{3^n-2} \omega^{\text{tr}_1^n(a^{t+\tau} - a^{dt})}$$

$$= -1 + \sum_{x \in \text{GF}(3^n)} \omega^{\text{tr}_1^n(ax - x^d)}$$

$$= -1 + S_d(\tau).$$
where $a = a^T$.

Since
\[ \gcd(3^{2k+1} + 1, 3^{2k} + 1) = 2, \]
\[ \gcd(3^{2k+1} + 1, 3^{2k} - 1) = 2, \]
and
\[ 3^{2k+1} + 1 \equiv 2 \mod 4, \]
we have
\[ \gcd(3^{2k+1} + 1, 3^n - 1) = 2. \]

It follows that $x^{3^{2k+1}+1}$ runs twice through all the squares in $GF(3^n)$ when $x$ runs through the nonzero elements of $GF(3^n)$. Let $r$ be a nonsquare in $GF(3^4)$. Then $r$ is also a nonsquare in $GF(3^n)$, since $\frac{3}{3^k} = k$ is odd. Therefore, $rx^{3^{2k+1}+1}$ runs twice through all the nonsquares in $GF(3^n)$ when $x$ runs through the nonzero elements of $GF(3^n)$. Furthermore, since
\[ d(3^{2k+1}+1) = \frac{(3^{2k+1}+1-1)+20(3^{2k+1}+1)}{(3^{2k+1}+1-1)}, \]
we have
\[ 2S_d(r) = \sum_{x \in GF(3^n)} \omega q_1(x) + \sum_{x \in GF(3^n)} \omega q_2(x). \]

**Lemma 2.** Let $f(x) \in GF(p^n)[x]$ be a quadratic form in $GF(p^n)[x_1, x_2, \cdots, x_n]$. Furthermore, let
\[ Y = \{ y \in GF(p^n) : f(x+y) = f(x) \text{ for all } x \in GF(p^n) \}. \]
Then $\text{rank}(f) = \text{dim}(Y)$.

From Lemma 2, we know that we should find the number of solutions $y \in GF(3^n)$ such that $q_i(x+y) = q_i(x)$ for all $x \in GF(3^n)$ to determine the rank of $q_i(x)$, where $i = 1, 2$.

To this end, we have the following lemmas.

**Lemma 3.** The number of solutions $y \in GF(3^n)$ such that $q_1(x+y) = q_1(x)$ for all $x \in GF(3^n)$ equals the number of solutions $y \in GF(3^n)$ of
\[ a^{3^{2k+1}} y^{81} + y^9 + ay = 0, \]
and the number of solutions $y \in GF(3^n)$ such that $q_2(x+y) = q_2(x)$ for all $x \in GF(3^n)$ equals the number of solutions $y \in GF(3^n)$ of
\[ (ar)^{3^{2k+1}} y^{81} - (x^d + r^{q_4}y)^9 + ary = 0, \]
where $r$ is a nonsquare in $GF(3^4)$.

**Proof.** By the definition of $q_1(x)$, we have $q_1(x+y) = q_1(x)$ if and only if
\[ \text{tr}^n(a(x+y)^{3^{2k+1}+1} - (x+y)^{3^{2k+1}}) \]
which implies
\[ \text{tr}^n(a^{3^{2k+1}}(y^{81} + y^9 + ay) + (a^{3^{2k+1}} - y^{3^{2k+1}})) = 0. \]

Eq. (3) holds for all $x \in GF(3^n)$ if and only if
\[ a^{3^{2k+1}} y^{81} + y^9 + ay = 0, \]
and
\[ \text{tr}^n(a^{3^{2k+1}} - y^{3^{2k+1}}) = 0. \]

Next, we will show that Eq. (5) is a consequence of Eq. (4). From Eq. (4) we obtain
\[ -y^9 = a^{3^{2k+1}} y^{81} + ay. \]
Raising the $3^i$ power for Eq. (6) gives
\[ -y^{3^{2i+1}} = a^{3^{2(k+1)^i}} y^{3^{2i+1}} + a^{3^i} y^{3^i}. \]

By the definition and the properties of trace function, we have
\[ -\text{tr}^i(y^{3^{2k+1}}) = -\text{tr}^i((y^{3^k})^{3^{2k+1}}) = -\sum_{i=0}^{n-1} (y^{3^{2k+1}} + 3^i) = \sum_{i=0}^{n-1} (-y^{3^{2i+1}}). \]
Using Eq. (7), we have
\[ -\text{tr}_n^n (y^{3^{k+1}}) \]
\[ = \sum_{i=0}^{n-1} (ax^{2(2k+1)+i} + ay^{2i})y^{3^{2(k+1)+i}} \]
\[ = \text{tr}_n^n ((ay)^{2(2k+1)+1}y^{3^{2(k+1)+1}}) + \text{tr}_n^n (ay)^{2(2k+1)+1} \]
\[ = 2\text{tr}_n^n (ay)^{2(2k+1)+1} \]
\[ = -\text{tr}_n^n (y^{3^{2(k+1)+1}}), \]

i.e., Eq. (5) holds. Thus, we only need to determine the number of solutions of Eq. (1) to find the number of solutions of Eq. (3).

By the similar argument, we can get the number of solutions \( y \in GF(3^n) \) such that \( q_2(x+y) = q_2(x) \) for all \( x \in GF(3^n) \) equals the number of solutions \( y \in GF(3^n) \) of
\[ (ar)^{3^{2(k+1)}} y^{2d} - (r^{9d} + r^{d \cdot 3^{2(k+1)}}) y^9 + ary = 0. \] 
(9)
Since \( r \) is a nonsquare in \( GF(3^4) \) and \( 4|2(k+1) \), we have \( r^{9 \cdot 3^{2(k+1)}} = r^d \). Hence Eq. (9) is equivalent to Eq. (2).

From Lemma 3, in order to calculate the rank of \( q_1(x) \) and \( q_2(x) \) we have to determine the number of solutions \( y \in GF(3^n) \) of Eqs. (1) and (2), respectively. Note that Eqs. (1) and (2) are both linearized forms, the possible number of solutions in \( GF(3^n) \) for both equations are 1, 9, or 81.

Lemma 4: The Eq. (2)
\[ (ar)^{3^{2(k+1)}} y^{2d} - (r^{d} + r^{9d}) y^9 + ary = 0 \]
has \( y = 0 \) as its only solution in \( GF(3^n) \), where \( r \) is a nonsquare in \( GF(3^4)^* \).

Proof: Note that \( r \) is a nonsquare in \( GF(3^4) \), and that both \( k \) and \( \frac{3k}{10} \) are odd, we have
\[ r^{3^{2(k+1)}} = r, \]
\[ r^{3^{k+1}} = r^{40} = -1 \]
and
\[ r^{40} (\frac{3^{2(k+1)}}{20})^2 = r^8 (\frac{3^{2(k+1)}}{20})^2 = r^{8d} = -1. \]
Therefore,
\[ (ar)^{3^{2(k+1)}} = ra^{3^{2(k+1)}} \]
and
\[ r^{9d} + r^{d} = r^{9d} + r^{d} = -r^{d} + r^{d} = 0. \]
Further, since \( ra \neq 0 \), we get Eq. (2) is equivalent to
\[ g(y^{a^{3^{2(k+1)} - 1} y^{80} + 1}) = 0. \]
Suppose \( a^{3^{2(k+1)} - 1} y^{80} = -1 \). Note that \( 3^{2(k+1)} - 1 \) can be divided by \( 3^4 - 1 = 80 \), we have
\[ a^{3^{2(k+1)} - 1} y^{80} = \beta^{80} = -1 \]
for some \( \beta \in GF(3^n) \), which implies that 80 divides some odd multiples of \( \frac{3^{2(k+1)} - 1}{2} \). This is a contradiction because 160 does not divide any odd multiples of \( 3^{4k} - 1 \).

From the above lemmas, we know that \( q_1(x) \) has the rank of \( n, n - 2, n - 4 \) and \( q_2(x) \) has the rank of \( n \).

IV. UPPER BOUND ON CROSS-CORRELATION MAGNITUDES
In this section, we will give the upper bound on the magnitude of the cross-correlation function \( C_d(\tau) \) of the ternary \( m \)-sequence \( \{s_t\} \) and its decimated sequence \( \{s_{dt}\} \) in Lemma 1. To this end, we will use the following definition and lemmas.

The quadratic character of \( GF(p^n) \) is defined as
\[ \eta(x) = \begin{cases} 1, & \text{if } x \text{ is a nonzero square in } GF(p^n) \\ -1, & \text{if } x \text{ is a nonsquare in } GF(p^n) \\ 0, & \text{if } x = 0. \end{cases} \]

Lemma 5: Let \( \eta \) be the quadratic character of \( GF(p) \). Suppose that \( f(x) \) is a nondegenerate quadratic form in \( t \) variables with determinant \( \Delta \). Then the number of solutions \( N(c) \) of \( f(x) = c \) is given as follows:
Case 1) \( t \) even;
\[ N(c) = \begin{cases} p^{t-1} - \epsilon p^{\frac{t-2}{2}}, & \text{if } c \neq 0 \\ p^{t-1} + \epsilon(p-1)p^{\frac{t-2}{2}}, & \text{if } c = 0. \end{cases} \]
where \( \epsilon = \eta((-1)^\frac{t-1}{2} \Delta) \).
Case 2) \( t \) odd;
\[ N(c) = \begin{cases} p^{t-1} - \epsilon p^{\frac{t-2}{2}}, & \text{if } c \neq 0 \\ p^{t-1} + \epsilon(p-1)p^{\frac{t-2}{2}}, & \text{if } c = 0. \end{cases} \]
where \( \epsilon = \eta((-1)^\frac{t-1}{2} \Delta) \).

Lemma 6: Let \( \eta \) be the quadratic character of \( GF(3) \) (i.e., \( \eta(0) = 0, \eta(1) = 1, \eta(2) = -1 \). Let \( f(x) \) be a nondegenerate quadratic form in \( t \) variables with determinant \( \Delta \). Then
\[ S = \sum_{x \in GF(3^t)} \omega_f(x) \]
is given by
\[ S = \begin{cases} \epsilon 3^t, & \text{if } t \text{ is even} \\ \epsilon i 3^t, & \text{if } t \text{ is odd} \end{cases} \]
where \( \epsilon = \eta((-1)^\frac{t-1}{2} \Delta) \) for even \( t \), \( \epsilon = \eta((-1)^\frac{t-1}{2} \Delta) \) for odd \( t \).

Using Lemma 6, we can derive the upper bound of the magnitude of \( C_d(\tau) \) in Lemma 1.

Theorem 1: The magnitude of the cross-correlation function \( C_d(\tau) \) in Lemma 1 is upper bounded by
\[ |C_d(\tau)| \leq 5 \cdot 3^t + 1. \]

Proof: From Lemma 1, we know that
\[ C_d(\tau) = -1 + S_d(\tau) \]
and
\[ 2S_d(\tau) = \sum_{x \in GF(3^t)} \omega q_1(x) + \sum_{x \in GF(3^t)} \omega q_2(x), \]
where
\[ q_1(x) = \text{tr}_n^n (ax^{2(2k+1)+1} + x^{3^{2k}+1}), \]
Using Lemma 6, we get
\[ q_2(x) = tr^n_1(\alpha^x^{3^{2(k+1)+1} - r^d} x^{3^{2k+1}}), \]
a = \alpha^x and \( r \) is a nonsquare in \( GF(3^4) \). Let \( \epsilon_1 \) and \( \epsilon_2 \) denote the values given in Lemma 6 corresponding to \( q_1(x) \) and \( q_2(x) \), respectively. From [8] we know that when the rank \( t \) of a quadratic form is less than \( n \), the corresponding exponential sum should be multiplied by \( 3^{n-t} \).

From Lemma 3, the possible rank combinations of \( q_1(x) \) and \( q_2(x) \) are \( (n, n) \), \( (n-2, n) \), \( (n-4, n) \). Therefore, we should consider the following three cases to find the values of \( S_d(\tau) \).

Case 1) Rank\( (q_1) = n \) and rank\( (q_2) = n \);
Using Lemma 6, we get
\[ 2S_d(\tau) = \sum_{x \in GF(3^n)} \omega^{q_1(x)} + \sum_{x \in GF(3^n)} \omega^{q_2(x)} \]
\[ = (\epsilon_1 + \epsilon_2)3^{\frac{n}{2}}. \quad (10) \]
Hence, we have \( |C_d(\tau)| = | -1 + S_d(\tau)| \leq 3^{\frac{n}{2}} + 1. \)

Case 2) Rank\( (q_1) = n - 2 \) and rank\( (q_2) = n \);
Using Lemmas 5 and 6, we have
\[ 2S_d(\tau) = \sum_{x \in GF(3^n)} \omega^{q_1(x)} + \sum_{x \in GF(3^n)} \omega^{q_2(x)} \]
\[ = (3\epsilon_1 + \epsilon_2)3^{\frac{n}{2}}. \quad (11) \]
Therefore, we obtain \( |C_d(\tau)| = | -1 + S_d(\tau)| \leq 2 \cdot 3^{\frac{n}{2}} + 1. \)

Case 3) Rank\( (q_1) = n - 4 \) and rank\( (q_2) = n \);
Using Lemmas 5 and 6, we get
\[ 2S_d(\tau) = \sum_{x \in GF(3^n)} \omega^{q_1(x)} + \sum_{x \in GF(3^n)} \omega^{q_2(x)} \]
\[ = (9\epsilon_1 + \epsilon_2)3^{\frac{n}{2}}. \quad (12) \]
Hence, we obtain \( |C_d(\tau)| = | -1 + S_d(\tau)| \leq 5 \cdot 3^{\frac{n}{2}} + 1. \)
As a conclusion, we have \( |C_d(\tau)| \leq 5\sqrt{3^n} + 1. \)

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