A New Proof for the Correctness of F5 (F5-Like) Algorithm

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Abstract

The famous F5 algorithm for computing Gröbner basis was presented by Faugère in 2002 without complete proofs for its correctness. The current authors have simplified the original F5 algorithm into an F5 algorithm in Buchberger’s style (F5B algorithm), which is equivalent to original F5 algorithm and may deduce some F5-like versions. In this paper, the F5B algorithm is briefly revisited and a new complete proof for the correctness of F5B algorithm is proposed. This new proof is not limited to homogeneous systems and does not depend on the strategy of selecting critical pairs (i.e. the strategy deciding which critical pair is computed first) such that any strategy could be utilized in F5B (F5) algorithm. From this new proof, we find that the special reduction procedure (F5-reduction) is the key of F5 algorithm, so maintaining this special reduction, various variation algorithms become available. A natural variation of F5 algorithm, which transforms original F5 algorithm to a non-incremental algorithm, is presented and proved in this paper as well. This natural variation has been implemented over the Boolean ring. The two revised criteria in this natural variation are also able to reject almost all unnecessary computations and few polynomials reduce to 0 in most examples.

Keywords: Gröbner basis, F5 algorithm, proof of correctness, variation algorithm

1. Introduction

Solving systems of polynomial equations is a basic problem in computer algebra, through which many practical problems can be solved easily. Among all the methods for solving polynomial systems, the Gröbner basis method is one of the most efficient approaches. After the conception of Gröbner basis is proposed in 1965 (Buchberger, 1965), many algorithms have been presented for computing Gröbner basis, including (Lazard, 1983; Gebauer and Moller, 1986; Giovini et al., 1991; Mora et al., 1992; Faugère, 1999, 2002). Currently, F5 algorithm is one of the most efficient algorithms.

After the F5 algorithm is proposed, many researches have been done. For example, Bardet et al. study the complexity of this algorithm in (Bardet et al., 2004). Faugère and
Ars use the F5 algorithm to attack multivariable systems in (Faugère and Ars, 2003). Stegers revisits F5 algorithm in his master thesis (Stegers, 2005). Eder discusses the two criteria of F5 algorithm in (Eder, 2008) and proposes a variation of F5 algorithm (Eder and Perry, 2009). Ars and Hashemi present two variation of criteria in (Ars and Hashemi, 2009). Recently, Gao et al. give a new incremental algorithm in (Gao et al., 2010). The current authors discuss the F5 algorithm over boolean ring and present a branch F5 algorithm in (Sun and Wang, 2009a,b). We also discuss the F5 algorithm in Buchberger’s style in (Sun and Wang, 2010).

Currently, available proofs for the correctness of F5 algorithm can be found from (Faugère, 2002; Stegers, 2005; Eder, 2008; Eder and Perry, 2009). However, these proofs are somewhat not complete, particularly for non-homogeneous systems.

The main purpose of current paper is to present a new complete proof for the correctness of F5 (F5-like) algorithm. As we have shown in (Sun and Wang, 2010) that the F5 algorithm in Buchberger’s style (F5B algorithm) is equivalent to the original F5 algorithm in (Faugère, 2002) and may deduces various F5-like algorithms, therefore, we will focus on proving the correctness of F5B algorithm in this paper. The proposed new proof is not limited to homogeneous systems and does not depend on the strategy of selecting critical pairs (s-pairs), so the correctness of all versions of F5 algorithm mentioned in (Sun and Wang, 2010) can be proved at the same time. After a slight modification, the correctness of the variation of F5 algorithm in (Ars and Hashemi, 2009), which is quite similar as the natural variation in this paper, can also be proved.

Meanwhile, according to the new proposed proof, we find that the key of F5 (F5-like) algorithm is the special reduction procedure, which ensures the correctness of both criteria in F5 algorithm. Thus, maintaining this special reduction procedure, many variations of F5 algorithm become available. We propose and prove a natural variation of F5 algorithm after the main proofs. This variation algorithm avoids computing Gröbner basis incrementally such that the Gröbner bases for subsets of input polynomials are not necessarily computed. Besides, the two revised criteria in this variation are also able to reject almost all unnecessary reductions as shown in the experimental data.

This paper is organized as follows. We revisit the F5 algorithm in Buchberger’s style (F5B algorithm) in Section 3 after introducing basic notations in Section 2. The complete proof for the correctness of F5B algorithm is presented in Section 4. The key of F5 algorithm and the natural variation algorithm are discussed in Section 5. This paper is concluded in Section 6.

2. Basic Notations

Let $K$ be a field and $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring with coefficients in $\mathbb{K}$. Let $\mathbb{N}$ be the set of non-negative integers and $PP(X)$ the set of power products of $\{x_1, \ldots, x_n\}$, i.e. $PP(X) := \{x^\alpha \mid x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha_i \in \mathbb{N}, i = 1, \ldots, n\}$.

Let $\prec$ be an admissible order defined over $PP(X)$. Given $t = x^\alpha \in PP(X)$, the degree of $t$ is defined as $\deg(t) := |\alpha| = \sum_{i=1}^{n} \alpha_i$. For a polynomial $0 \neq f \in \mathbb{K}[x_1, \ldots, x_n]$, we have $f = \sum c_\alpha x^\alpha$. The degree of $f$ is defined as $\deg(f) := \max\{|\alpha|, c_\alpha \neq 0\}$ and the leading power
product of \( f \) is \( \text{lpp}(f) := \max_{\alpha} \{x^\alpha, c_\alpha \neq 0\} \). If \( \text{lpp}(f) = x^\alpha \), then the leading coefficient and leading monomial of \( f \) are defined to be \( \text{lc}(f) := c_\alpha \) and \( \text{lm}(f) := c_\alpha x^\alpha \) respectively.

3. The F5 Algorithm in Buchberger’s Style

In brief, F5 algorithm introduces a special reduction (F5-reduction) and provides two new criteria (Syzygy Criterion\(^2\) and Rewritten Criterion) to avoid unnecessary reductions.

In this section, we give the definitions of signatures and labeled polynomials first, and then describe the Syzygy Criterion and Rewritten Criterion as well as the special reduction, F5-reduction. At last, we present the F5 algorithm in Buchberger’s Style (F5B algorithm) as discussed in (Sun and Wang, 2010).

As a preparation for the main proofs, an important auxiliary concept is introduced. That is the numbers of labeled polynomials, which reflect the order of when labeled polynomials are generated. This auxiliary concept simplifies the description of the Rewritten Criterion and benefits for the main proofs. For more details about the F5B algorithm, please see (Sun and Wang, 2010).

3.1. Signature and Labeled Polynomial

Consider a polynomial system \( \{f_1, \ldots, f_m\} \subset K[X] \) and denote \( (f_1, \ldots, f_m) \) to be a polynomial \( m \)-tuple in \( (K[X])^m \). We call the \( f_i \)'s initial polynomials, as they are initial generators of ideal \( \langle f_1, \ldots, f_m \rangle \subset K[X] \).

Let \( e_i \) be the canonical \( i \)-th unit vector in \( (K[X])^m \), i.e. the \( i \)-th element of \( e_i \) is 1, while the others are 0. Consider the homomorphism map \( \sigma \) over the free module \( (K[X])^m \):

\[
\sigma : (K[X])^m \rightarrow \langle f_1, \ldots, f_m \rangle,
\]

\[
(g_1, \ldots, g_m) \mapsto g_1 f_1 + \cdots + g_m f_m.
\]

Then \( \sigma(e_i) = f_i \). More generally, if \( g = g_1 e_1 + \cdots + g_m e_m \), where \( g_i \in K[X] \) for \( 1 \leq i \leq m \), then \( \sigma(g) = g_1 f_1 + \cdots + g_m f_m \).

The admissible order \( \prec \) on \( PP(X) \) extends to the free module \( (K[X])^m \) naturally in a POT (position over term) fashion\(^3\):

\[
x^\alpha e_i \prec x^\beta e_j \text{ (or } x^\beta e_j \succ x^\alpha e_i) \text{ iff } \begin{cases} i > j, \\ \text{or} \\ i = j \text{ and } x^\alpha \prec x^\beta. \end{cases}
\]

Thus we have \( e_m \prec e_{m-1} \prec \cdots \prec e_1 \).

With the admissible order on \( (K[X])^m \), we can define the leading power product, leading coefficient and leading monomial of a \( m \)-tuple vector \( g \in (K[X])^m \) in a similarly way. For

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\(^2\)Also called F5 Criterion in some papers. To avoid confusion with the name of F5 algorithm, we call it Syzygy Criterion in this paper.

\(^3\)This order of signatures is imported from (Faugère 2002). We will introduce another order of signatures to deduce the natural variation of F5 algorithm after the main proofs.
example, let \( g = (2x^2 + y^2, 3xy) \in (\mathbb{Q}[x,y])^2 \) or equivalently \( g = (2x^2 + y^2)e_1 + 3xye_2 \). According to the Lex order \( \prec \) on \( PP(x,y) \) where \( x \succ y \), we have \( \text{lpp}(g) = x^2e_1 \), \( \text{lc}(g) = 2 \) and \( \text{lm}(g) = 2x^2e_1 \).

Now, we give the mathematical definition of signatures.

**Definition 3.1** (signature). Let \( g \in (\mathbb{K}[X])^m \) be an \( m \)-tuple vector. If polynomial \( g = \sigma(g) \in \langle f_1, \cdots, f_m \rangle \subset K[X] \), then the leading power product \( \text{lpp}(g) \) is defined to be a signature of \( g \).

Consider a simple system \( \{ f_1 = x^2 + 2y, f_2 = xy - z \} \subset \mathbb{Q}[x,y] \) with the Graded Reverse Lex Order \((x \succ y)\). The s-polynomial of \( f_1 \) and \( f_2 \) is \( yf_1 - xf_2 = 2y^2 + xz \). According to the above definition, \( ye_1 \) is a signature of the polynomial \( 2y^2 + xz \), as \( \sigma(ye_1 - xe_2) = 2y^2 + xz \) and \( \text{lpp}(ye_1 - xe_2) = ye_1 \).

Now we are able to assign a signature to each polynomial \( g \in \langle f_1, \cdots, f_m \rangle \). To tighten up the relation between a polynomial and its signature, we integrate them together and call it labeled polynomial.

**Definition 3.2** (labeled polynomial). Let \( g \in \langle f_1, \cdots, f_m \rangle \) be a polynomial. If \( x^\alpha e_i \) is a signature of \( g \), then \( G = (x^\alpha e_i, g, k) \) is defined to be a labeled polynomial of \( g \), where \( k \in \mathbb{N} \) reflects the order of when this labeled polynomial is generated.

For convenience, denote the signature, polynomial and number of the labeled polynomial \( G \) by \( \text{Sign}(G) := x^\alpha e_i \), \( \text{Poly}(G) := g \) and \( \text{Num}(G) := k \). Besides, the leading power product and leading monomial of \( G \) are denoted as: \( \text{lpp}(G) := \text{lpp}(g) \) and \( \text{lm}(G) := \text{lm}(g) \) respectively.

The number of labeled polynomial is an important auxiliary concept for the main proofs. It is designated by the algorithm and reflects the order of when the labeled polynomials are generated. The meaning of number will be much clearer after the F5B algorithm is presented.

Remark that a polynomial in the ideal \( \langle f_1, \cdots, f_m \rangle \) may have several different signatures, but during the computations, the signature and number of each polynomial are uniquely determined by the algorithm.

Therefore, in the above simple example \( \{ f_1 = x^2 + y, f_2 = xy - z \} \subset \mathbb{Q}[x,y] \), the labeled polynomials corresponding to \( f_1 \) and \( f_2 \) are \( (e_1, f_1, 1) \) and \( (e_2, f_2, 2) \) respectively. For the s-polynomial \( yf_1 - xf_2 = 2y^2 + xz \) of \( f_1 \) and \( f_2 \), its labeled polynomial is \( (ye_1, 2y^2 + xz, 3) \).

Then the signature, polynomial and number of \( G = (ye_1, 2y^2 + xz, 3) \) are \( \text{Sign}(G) = ye_1 \), \( \text{Poly}(G) = 2y^2 + xz \) and \( \text{Num}(G) = 3 \) respectively. We also have \( \text{lpp}(G) = y^2 \) and \( \text{lm}(G) = 2y^2 \). Notice that the numbers of labeled polynomials \( (e_1, f_1, 1) \) and \( (e_2, f_2, 2) \) are 1 and 2, both of which are smaller than the number \( \text{Num}(G) = 3 \). This indicates the labeled polynomial \( G = (ye_1, 2y^2 + xz, 3) \) is generated later than labeled polynomials \( (e_1, f_1, 1) \) and \( (e_2, f_2, 2) \).

Now we introduce two notations about signatures and labeled polynomials. Define \( S(X) \) to be the set of signatures, i.e. \( S(X) := \{ x^\alpha e_i \mid x^\alpha \in PP(X), 1 \leq i \leq m \} \subset \mathbb{R}^m \),
and \( L(X) \) to be the set of labeled polynomials, i.e. \( L(X) := \{ (x^\alpha e_i, g, k_g) \mid x^\alpha e_i \in S(X) \) is a signature of \( g \in K[X] \}. \) In the rest of current paper, we use the flourish, such as \( F, G, H \), to represent labeled polynomials, while the lowercase, such as \( f, g, h \), stand for polynomials in \( K[X] \). The boldface, \( f, g, h \), refer to the elements in free module \( (K[X])^m \).

In F5 algorithm, labeled polynomials are the basic elements in computation instead of polynomials in \( K[X] \). Suppose \( f, g \in (K[X])^m \) such that \( \sigma(f) = f \) and \( \sigma(g) = g \). Then \( F = (lpp(f), f, k_f) \), \( G = (lpp(g), g, k_g) \in L[X] \) are labeled polynomials. Assume \( cx^\gamma \) is a non-zero monomial. Then

- \( cx^\gamma F = (x^\gamma lpp(f), cx^\gamma f, k_f) \), as \( \sigma(cx^\gamma f) = cx^\gamma f \).
- \( F + G = (\max_{\alpha} \{ lpp(f), lpp(g) \}, f + g, k_{f+g}) \), as \( \sigma(f + g) = f + g \), where \( k_{f+g} = k_f \) or \( k_g \) corresponding to the maximal one of \( \{ lpp(f), lpp(g) \} \).

Unlike polynomials in \( K[X] \), labeled polynomials in \( L[X] \) can compare in following way:

\[
(x^\alpha e_i, f, k_f) \prec (x^\beta e_j, g, k_g) \quad \text{or} \quad (x^\beta e_j, g, k_g) \succ (x^\alpha e_i, f, k_f) \quad \text{iff} \quad \begin{cases} 
\begin{aligned}
& x^\alpha e_i < x^\beta e_j, \\
& or \\
& x^\alpha e_i = x^\beta e_j \text{ and } k_f > k_g.
\end{aligned}
\end{cases}
\]

Particularly, denote \( (x^\alpha e_i, f, k_f) \succ (x^\beta e_j, g, k_g) \), if \( x^\alpha e_i = x^\beta e_j \) and \( k_f = k_g \). Remark that in this case, the polynomial \( f \) may not equal to \( g \).

In the simple example \( \{ f_1 = x^2 + y, f_2 = xy - z \} \subset \mathbb{Q}[x, y] \). We have \( (e_2, f_2, 2) \prec (e_1, f_1, 1) \), since \( e_2 \prec e_1 \). For the s-polynomial \( y f_1 - x f_2 = 2y^2 + xz \) of \( f_1 \) and \( f_2 \), its labeled polynomial is \( (ye_1, 2y^2 + xz, 3) \). Notice that \( y(\alpha e_1, f_1, 1) = (ye_1, yf_1, 1) \). So we also have \( (ye_1, 2y^2 + xz, 3) \prec (ye_1, yf_1, 1) \) due to the numbers of these two labeled polynomials.

The critical pair (s-pair) of labeled polynomials is defined in a similar way as well. For labeled polynomials \( F, G \in L[X] \), we say \( [F, G] := (u, F, v, G) \) is the critical pair of \( F \) and \( G \), if \( u, v \) are monomials in \( X \) such that \( \text{ulm}(F) = \text{vlm}(G) = \text{lcm}(lpp(F), lpp(G)) \) and \( uF \triangleright vG \). Besides, the s-polynomial of \( [F, G] = (u, F, v, G) \) is denoted as \( \text{spoly}(F, G) = uF - vG \).

Remark that labeled polynomials in the critical pair \( [F, G] = (u, F, v, G) \) is ordered by \( uF \triangleright vG \). Moreover, critical pairs can compare with each other in the following way:

\[
(u, F, v, G) \prec (r, P, t, Q) \quad \text{or} \quad (r, P, t, Q) \succ (u, F, v, G) \quad \text{iff} \quad \begin{cases} 
\begin{aligned}
& uF \prec rP, \\
& or \\
& uF \succ rP \text{ and } vG \prec tQ.
\end{aligned}
\end{cases}
\]

### 3.2. Syzygy Criterion and Rewritten Criterion

First, we describe the Syzygy Criterion. We begin by the following definition.

**Definition 3.3 (Comparable).** Let \( F = (x^\alpha e_i, f, k_f) \in L[X] \) be a labeled polynomial, \( cx^\gamma \) a non-zero monomial in \( X \) and \( B \subset L[X] \) a set of labeled polynomials. The labeled polynomial \( cx^\gamma F = (x^{\gamma+\alpha} e_i, cx^\gamma f, k_f) \) is said to be comparable by \( B \), if there exists a labeled polynomial \( G = (x^\beta e_j, g, k_g) \in B \) such that:

1. \( \text{lpp}(g) \mid x^{\gamma+\alpha} \), and
Criterion 1 — Syzygy Criterion

Let \([F, G] := (u, F, v, G)\) be the critical pair of \(F\) and \(G\), where \(u, v\) are monomials in \(X\) such that \(u_{lm}(F) = v_{lm}(G) = \text{lcm}(\text{lpp}(F), \text{lpp}(G))\) and \(uF \triangleright vG\). And \(B \subset L[X]\) is a set of labeled polynomials. If either \(uF\) or \(vG\) is comparable by \(B\), then the critical pair \([F, G]\) meets the Syzygy Criterion.

2. \(e_i \succ e_j\), i.e. \(i < j\).

Then the Syzygy Criterion is described as follow.

Next, we describe the Rewritten Criterion. Again we start with a definition.

Definition 3.4 (Rewritable). Let \(F = (x^\alpha e_i, f, k_f) \in L[X]\) be a labeled polynomial, \(cx^\gamma\) a non-zero monomial in \(X\) and \(B \subset L[X]\) a set of labeled polynomials. The labeled polynomial \(cx^\gamma F = (x^\gamma \alpha e_i, cx^\gamma f, k_f)\) is said to be rewritable by \(B\), if there exists a labeled polynomial \(G = (x^\beta e_i, g, k_g) \in B\), such that:

1. \(x^\beta e_i | x^{\gamma + \alpha} e_i\), and
2. \(\text{Num}(F) < \text{Num}(G)\), i.e. \(k_f < k_g\).

The Rewritten Criterion is given as follow.

Criterion 2 — Rewritten Criterion

Let \([F, G] := (u, F, v, G)\) be the critical pair of \(F\) and \(G\), where \(u, v\) are monomials in \(X\) such that \(u_{lm}(F) = v_{lm}(G) = \text{lcm}(\text{lpp}(F), \text{lpp}(G))\) and \(uF \triangleright vG\). And \(B \subset L[X]\) is a set of labeled polynomials. If either \(uF\) or \(vG\) is rewritable by \(B\), then the critical pair \([F, G]\) meets the Rewritten Criterion.

In \(F5\) (\(F5B\)) algorithm, if a critical pair meets either Syzygy Criterion or Rewritten Criterion, then it is not necessary to reduce its corresponding s-polynomial.

3.3. \(F5\)-Reduction

The concept of signatures itself is not sufficient to ensure the correctness of two new criteria. It is the special reduction procedure that guarantees the critical pairs detected by criteria are really useless. The same is true for other \(F5\)-like algorithms.

Let us start with the definition of \(F5\)-reduction.

Definition 3.5 (\(F5\)-reduction). Let \(F = (x^\alpha f_i, f) \in L[X]\) be a labeled polynomial and \(B \subset L[X]\) a set of labeled polynomials. The labeled polynomial \(F\) is \(F5\)-reducible by \(B\), if there exists \(G = (x^\beta f_i, g) \in B\) such that:

\[\text{Deleting the conditions 3 and 4 does not affect the correctness of algorithm, but leads to redundant computations/reductions.}\]
1. \( \text{lpp}(g) \mid \text{lpp}(f) \), denote \( x^\gamma = \text{lpp}(f)/\text{lpp}(g) \) and \( c = \text{lc}(f)/\text{lc}(g) \),

2. \( \text{Sign}(F) \succ \text{Sign}(cx^\gamma G) \), i.e. \( x^\alpha e_i \succ x^{\gamma+\beta} e_j \),

3. \( x^\gamma G \) is not comparable by \( B \), and

4. \( x^\gamma G \) is not rewritable by \( B \).

If \( F \) is \( F5 \)-reducible by \( B \), let \( F' = F - cx^\gamma G \). Then this procedure: \( F \Rightarrow_B F' \) is called one step \( F5 \)-reduction. If \( F' \) is still \( F5 \)-reducible by \( B \), then repeat this step until \( F' \) is not \( F5 \)-reducible by \( B \). Suppose \( F^* \) is the final result that is not \( F5 \)-reducible by \( B \). We say \( F \) \( F5 \)-reduces to \( F^* \) by \( B \), and denote it as \( F \Rightarrow_B^* F^* \).

The key of \( F5 \)-reduction is the condition \( \text{Sign}(F) \succ \text{Sign}(cx^\gamma G) \), i.e. \( x^\alpha e_i \succ x^{\gamma+\beta} e_j \), which makes \( F5 \)-reduction much different from other general reductions. The major function of this condition is to preserve the signature of \( F \) during reductions. Thus a direct result is that, if labeled polynomial \( F \) \( F5 \)-reduces to \( F^* \) by \( B \) (i.e. \( F \Rightarrow_B^* F^* \)), then the signatures of \( F \) and \( F^* \) are identical, i.e.

\[
\text{Sign}(F) = \text{Sign}(F^*).
\]

This property plays a crucial role in the main proofs for the correctness of \( F5B \) algorithm. For convenience of reference, we describe this property by the following proposition.

**Proposition 3.6 (\( F5 \)-reduction property).** If labeled polynomial \( F \) \( F5 \)-reduce to \( F^* \) by set \( B \), i.e. \( F \Rightarrow_B^* F^* \), then there exist polynomials \( p_1, \cdots, p_s \in K[X] \) and labeled polynomials \( G_1, \cdots, G_s \subset B \), such that:

\[
F = F^* + p_1 G_1 + \cdots + p_s G_s,
\]

where leading power product \( \text{lpp}(F) \succeq \text{lpp}(p_i G_i) \) and signature \( \text{Sign}(F) \succ \text{Sign}(p_i G_i) \) for \( 1 \leq i \leq s \). Moreover, signature \( \text{Sign}(F) = \text{Sign}(F^*) \) and labeled polynomial \( F \triangleright \triangleright F^* \).

The proof of this proposition is trivial by the definition of \( F5 \)-reduction.

### 3.4. The \( F5 \) algorithm in Buchberger’s style

With the definitions of Syzygy Criterion, Rewritten Criterion and \( F5 \)-reduction, we can simplify the \( F5 \) algorithm in Buchberger’s style (\( F5B \) algorithm).

According to the above algorithm, the number \( \text{Num}(P) \) of labeled polynomial \( P \) is actually the order of when \( P \) is being added to the set \( B \). So the bigger \( \text{Num}(P) \) is, the later \( P \) is generated. Notice that the numbers of labeled polynomials in the set \( B \) are distinct from each other.

The strategy of selecting critical pairs is not specified in the \( F5B \) algorithm, instead we simply use

\[
\text{cp} \leftarrow \text{select a critical pair from } CP,
\]

since the new proof proposed in next section does not depend on the specifical strategies. Moveover, we have shown in (Sun and Wang, 2010) that the original \( F5 \) algorithm differs from \( F5B \) algorithm only by a strategy of selecting critical pairs, so the proof for the correctness of \( F5B \) algorithm can also prove the correctness of the original \( F5 \) (or \( F5 \)-like) algorithm. So next, we focus on proving the correctness of \( F5B \) algorithm.
Algorithm 1 — The F5 algorithm in Buchberger’s style (F5B algorithm)

| Input:   | a polynomial m-tuple: \((f_1, \cdots, f_m) \subset K[X]^m\), and an admissible order \(\prec\). |
|----------|------------------------------------------------------------------------------------------------------------------|
| Output:  | The Gröbner basis of the ideal \(\langle f_1, \cdots, f_m \rangle \subset K[X]\). |

begin
\(F_i \leftarrow (e_i, f_i, i)\) for \(i = 1, \cdots, m\)
\(k \leftarrow m\) \# to track the number of labeled polynomials
\(B \leftarrow \{F_i \mid i = 1, \cdots, m\}\)
\(CP \leftarrow \{\text{critical pair } [F_i, F_j] \mid 1 \leq i < j \leq m\}\)
while \(CP\) is not empty do
  \(cp \leftarrow\) select a critical pair from \(CP\)
  \(CP \leftarrow CP \setminus \{cp\}\)
  if \(cp\) meets neither Syzygy Criterion nor Rewritten Criterion, then
    \(SP \leftarrow \text{the s-polynomial of critical pair } cp\)
    \(P \leftarrow\) the F5-reduction result of \(SP\) by \(B\), i.e. \(SP \Rightarrow^* B P\)
    \(\text{Num}(P) \leftarrow k + 1\) \# update the number of \(P\)
    if the polynomial of \(P\) is not 0, i.e. \(\text{Poly}(P) \neq 0\), then
      \(CP \leftarrow CP \cup \{\text{critical pair } [P, Q] \mid Q \in B\}\)
    end if
  end if
  \(k \leftarrow k + 1\)
  \(B \leftarrow B \cup \{P\}\) \# no matter whether \(\text{Poly}(P) \neq 0\) or not
end while
return \{polynomial part of \(Q \mid Q \in B\}\)

end

4. A New Proof for the Correctness of F5B Algorithm

The main work of this section is to prove the correctness of F5B algorithm presented in last section, i.e. show that the outputs of F5B algorithm construct a Gröbner basis of the ideal \(\langle f_1, \cdots, f_m \rangle \subset K[X]\).

This section is organized as follows. First, we show the difficult point in the whole proofs by a toy example; second, we sketch the structure of proofs and prove the main theorem; at last, we provide the detail proofs for the lemmas and propositions used in the proof of main theorem.

4.1. The Thorny Problem

There exists a very interesting thing in F5B (or F5) algorithm. That is, when a critical pair is detected and discarded by the two criteria, this critical pair is usually not useless at that time (i.e. its s-polynomial cannot F5-reduce to 0 by the corresponding set \(B\)), but when the algorithm terminates, this detected critical pair becomes really redundant (i.e. its
s-polynomial F5-reduce to 0 by the final set $B$). This indicates that the two criteria of F5 algorithm can detect unnecessary computations/reductions in advance. This is so amazing and becomes a big thorny problem in the correctness proof of F5B algorithm.

This phenomenon happens frequently, particularly in non-homogeneous systems. Let us see a toy example first. In order to highlight this peculiar phenomenon, a special strategy of selecting critical pair is used.

**Example 4.1.** Compute the Gröbner basis of the following system in $\mathbb{Q}[x, y, z]$ with Graded Reverse Lex Order ($x \succ y \succ z$) by F5B algorithm:

\[
\begin{align*}
\{ f_1 &= y^2 + yz - x, \\
    f_2 &= y^2 - z^2 + z. 
\end{align*}
\]

The strategy of selecting critical pairs in this toy example is: first, find the minimal degree of critical pairs in the set $CP$ (the degree of critical pair $[F_i, F_j]$ refers to the degree of $\text{lcm}(\text{lpp}(F_i), \text{lpp}(F_j))$), and then select the maximal critical pair from the set $CP$ with the order $\prec$ at this minimal degree.

After initialization, the initial labeled polynomials are

\[
B^{(0)} = \{ F_1 = (e_1, y^2 + yz - x, 1), F_2 = (e_2, y^2 - z^2 + z, 2) \},
\]

and critical pairs are

\[
CP^{(0)} = \{ [F_1, F_2] \}.
\]

**LOOP 1:** Critical pair $[F_1, F_2]$ is selected from set $CP^{(0)}$. The s-polynomial of $[F_1, F_2]$ is $(e_1, yz + z^2 - x - z, 1)$ which is not F5-reducible by set $B^{(0)}$. Then after updating the number, labeled polynomial $F_3 = (e_1, yz + z^2 - x - z, 3)$ adds to the set $B^{(0)}$.

\[
B^{(1)} = \{ F_1, F_2, F_3 \} \text{ and } CP^{(1)} = \{ [F_3, F_1], [F_3, F_2] \}.
\]

**LOOP 2:** Critical pair $[F_3, F_1] = (y, F_3, z, F_1)$ is selected from set $CP^{(1)}$. But labeled polynomial $zF_1 = (ze_1, z(y^2 + yz - x), 1)$ is rewritable by set $B^{(1)}$, since there exists labeled polynomial $F_4 = (e_1, yz + z^2 - x - z, 3)$ in $B^{(1)}$ such that signature $e_1 \mid ze_1$ and number $3 > 1$. So critical pair $[F_3, F_1]$ is rejected by the Rewritten Criterion. Now

\[
B^{(2)} = \{ F_1, F_2, F_3 \} \text{ and } CP^{(2)} = \{ [F_3, F_2] \}.
\]

**LOOP 3:** Critical pair $[F_3, F_2]$ is selected from set $CP^{(2)}$. The s-polynomial of $[F_3, F_2]$ is $(ye_1, yz^2 + z^3 - xy - yz - z^2, 3)$ which F5-reduces to $(ye_1, -xy - yz + xz, 3)$ by set $B^{(2)}$. Then after updating the number, labeled polynomial $F_4 = (ye_1, -xy - yz + xz, 4)$ adds to the set $B^{(2)}$.

\[
B^{(3)} = \{ F_1, F_2, F_3, F_4 \} \text{ and } CP^{(3)} = \{ [F_4, F_1], [F_4, F_2], [F_4, F_3] \}.
\]

**LOOP 4:** Critical pair $[F_1, F_1] = (-y, F_4, x, F_1)$ is selected from set $CP^{(3)}$. But labeled polynomial $-yF_4 = (y^2e_1, -y(-xy - yz + xz), 4)$ is comparable by set $B^{(3)}$, since there
exists labeled polynomial \( F_2 = (e_2, y^2 - z^2 + z, 2) \) in \( B^{(3)} \) such that leading power product \( \text{lpp}(F_2) = y^2 \mid y^2 \) and \( e_1 \succ e_2 \). So critical pair \([F_4, F_1]\) is rejected by the Syzygy Criterion. Now

\[
B^{(4)} = \{F_1, F_2, F_3, F_4\} \text{ and } CP^{(4)} = \{[F_4, F_2], [F_4, F_3]\}.
\]

**LOOP 5:** Critical pair \([F_4, F_2]\) = \((-y, F_4, x, F_2)\) is selected from set \( CP^{(4)} \). But labeled polynomial \(-yF_4 = (y^2e_1, -y(-xy - yz + xz), 4)\) is comparable by set \( B^{(4)} \), since there exists labeled polynomial \( F_2 = (e_2, y^2 - z^2 + z, 2) \) in \( B^{(4)} \) such that leading power product \( \text{lpp}(F_2) = y^2 \mid y^2 \) and \( e_1 \succ e_2 \). So critical pair \([F_4, F_2]\) is rejected by the Syzygy Criterion. Now

\[
B^{(5)} = \{F_1, F_2, F_3, F_4\} \text{ and } CP^{(5)} = \{[F_4, F_3]\}.
\]

**LOOP 6:** Critical pair \([F_4, F_3]\) is selected from set \( CP^{(5)} \). The s-polynomial of \([F_4, F_3]\) is \((yze_1, -2xz^2 + yz^2 + x^2 + xz, 4)\) which is not \( F_5 \)-reducible by set \( B^{(5)} \). Then after updating the number, labeled polynomial \( F_5 = (yze_1, -2xz^2 + yz^2 + x^2 + xz, 5) \) adds to the set \( B^{(5)} \). Now

\[
B^{(6)} = \{F_1, F_2, F_3, F_4, F_5\} \text{ and } CP^{(6)} = \{[F_5, F_1], [F_5, F_2], [F_5, F_3], [F_5, F_4]\}.
\]

**LOOP 7:** Critical pair \([F_5, F_1]\) = \((-y/2, F_5, x^2, F_1)\) is selected from set \( CP^{(6)} \). But labeled polynomial \((-y/2)F_5 = (y^2ze_1, (y/2)(-2x^2z + yz^2 + x^2 + xz), 5)\) is comparable by set \( B^{(6)} \), since there exists labeled polynomial \( F_2 = (e_2, y^2 - z^2 + z, 2) \) in \( B^{(6)} \) such that leading power product \( \text{lpp}(F_2) = y^2 \mid y^2 \) and \( e_1 \succ e_2 \). So critical pair \([F_5, F_1]\) is rejected by the Syzygy Criterion. Now

\[
B^{(7)} = \{F_1, F_2, F_3, F_4, F_5\} \text{ and } CP^{(7)} = \{[F_5, F_1], [F_5, F_2], [F_5, F_3]\}.
\]

**LOOP 8:** Critical pair \([F_5, F_3]\) = \((-y/2, F_5, xz, F_3)\) is selected from set \( CP^{(7)} \). But labeled polynomial \((-y/2)F_5 = (y^2ze_1, (y/2)(-2x^2z + yz^2 + x^2 + xz), 5)\) is comparable by set \( B^{(7)} \), since there exists labeled polynomial \( F_2 = (e_2, y^2 - z^2 + z, 2) \) in \( B^{(7)} \) such that leading power product \( \text{lpp}(F_2) = y^2 \mid y^2 \) and \( e_1 \succ e_2 \). So critical pair \([F_5, F_3]\) is rejected by the Syzygy Criterion. Now

\[
B^{(8)} = \{F_1, F_2, F_3, F_4, F_5\} \text{ and } CP^{(8)} = \{[F_5, F_1], [F_5, F_2]\}.
\]

**LOOP 9:** Critical pair \([F_5, F_1]\) = \((-y^2/2, F_5, x^2, F_1)\) is selected from set \( CP^{(8)} \). But labeled polynomial \((-y^2/2)F_5 = (y^3ze_1, (-y^2/2)(-2x^2z^2 + yz^2 + x^2 + xz), 5)\) is comparable by set \( B^{(8)} \), since there exists labeled polynomial \( F_2 = (e_2, y^2 - z^2 + z, 2) \) in \( B^{(8)} \) such that leading power product \( \text{lpp}(F_2) = y^2 \mid y^3 \) and \( e_1 \succ e_2 \). So critical pair \([F_5, F_1]\) is rejected by the Syzygy Criterion. Now

\[
B^{(9)} = \{F_1, F_2, F_3, F_4, F_5\} \text{ and } CP^{(9)} = \{[F_5, F_2]\}.
\]

**LOOP 10:** Critical pair \([F_5, F_2]\) = \((-y^2/2, F_5, x^2, F_2)\) is selected from \( CP^{(9)} \). But labeled polynomial \((-y^2/2)F_5 = (y^3ze_1, (-y^2/2)(-2x^2z^2 + yz^2 + x^2 + xz), 5)\) is comparable by set \( B^{(9)} \), since there exists labeled polynomial \( F_2 = (e_2, y^2 - z^2 + z, 2) \) in \( B^{(9)} \) such that
leading power product \( \text{lpp}(F_3) = y^2 | y^3z \) and \( e_1 \gg e_2 \). So critical pair \([F_5, F_2]\) is rejected by the **Syzygy Criterion.** Now
\[
B^{(10)} = \{F_1, F_2, F_3, F_4, F_5\} \text{ and } CP^{(10)} = \emptyset.
\]

Since set \( CP^{(10)} \) is empty, F5B algorithm terminates and the final set \( B^{(10)} = \{F_1, F_2, F_3, F_4, F_5\} \). Then the polynomial set \( \{\text{Poly}(F_1), \text{Poly}(F_2), \text{Poly}(F_3), \text{Poly}(F_4), \text{Poly}(F_5)\} \) is a Gröbner basis of the ideal generated by \( \{f_1 = y^2 + yz - x, f_2 = y^2 - z^2 + z\} \).

At last, we check whether the critical pairs rejected by two criteria are really redundant. The labeled polynomial in round bracket is the \( s \)-polynomial of corresponding critical pair.

**LOOP 2:** \([F_3, F_4] = (y, F_3, z, F_1)\), then
\[
(yF_3 - zF_4) - F_4 = (ye_1, 0, 3).
\]

**LOOP 4:** \([F_4, F_1] = (-y, F_4, x, F_1)\), then
\[
(-yF_4 - xF_1) + 2xF_3 - zF_1 + F_5 = (y^2e_1, 0, 4).
\]

**LOOP 5:** \([F_4, F_2] = (-y, F_4, x, F_2)\), then
\[
(-yF_4 - xF_2) +xF_3 - zF_1 + F_5 = (y^2e_1, 0, 4).
\]

**LOOP 7:** \([F_5, F_4] = (-y/2, F_5, -z^2, F_4)\), then
\[
\left((-y/2)F_5 + z^2F_4\right) + \left(z^2/2\right)F_1 + \left(z/2\right)F_5 - \left(x/2\right)F_4 = (-y^2z e_1, 0, 5).
\]

**LOOP 8:** \([F_5, F_3] = (-y/2, F_5, xz, F_3)\), then
\[
\left((-y/2)F_5 - xzF_3\right) + \left(z^2/2\right)F_1 - \left(z/2\right)F_5 - \left(x/2\right)F_4 = (y^2ze_1, 0, 5).
\]

**LOOP 9:** \([F_5, F_1] = (-y^2/2, F_5, xz^2, F_1)\), then
\[
\left((-y^2/2)F_5 - xz^2F_1\right) + \left(yz^2/2 - z^3/2 + x^2/2 + xz/2\right)F_1 + \left(xz^2 - x^2/2\right)F_3 + \left(z^2/2\right)F_5 = (y^3ze_1, 0, 5).
\]

**LOOP 10:** \([F_5, F_2] = (-y^2/2, F_5, xz^2, F_2)\), then
\[
\left((-y^2/2)F_5 - xz^2F_2\right) + \left(yz^2/2 - z^3/2 + x^2/2 + xz/2\right)F_1 + \left(z^2/2\right)F_5 - \left(x^2/2\right)F_3 = (y^3ze_1, 0, 5).
\]

All these \( s \)-polynomials F5-reduces to 0 by \( B^{(10)} \), so both the criteria are correct.

**Remark 4.2.** Notice that the \( s \)-polynomial of \([F_3, F_1]\) F5-reduces to 0 by the labeled polynomial \( F_4 \), which is generated in **LOOP 3.** However, the critical pair \([F_3, F_1]\) is rejected in **LOOP 2**, which implies that when this critical pair is being discarded, its \( s \)-polynomial \( yF_3 - zF_1 \) cannot F5-reduce to 0 by the set \( B^{(1)} = \{F_1, F_2, F_3\} \). Similar cases also happen to critical pairs \([F_4, F_1]\) and \([F_4, F_2]\). These facts illustrate the thorny problem mentioned earlier.

This thorny problem is a big handicap for the correctness proof of F5B (or F5) algorithm, and as we know, it is not well handled in other existing proofs for F5 algorithm.

The new proof presented in this paper averts this thorny problem subtly. Instead of proving the critical pairs are useless when they are being detected, we concentrate on showing that these critical pairs are redundant after the algorithm terminates. This is detailed in next subsection.
4.2. Main Theorem

In order to show the detected critical pairs are redundant after the algorithm terminates, we need to save these critical pairs and discuss them afterwards. Thus, we modify F5B algorithm slightly.

Algorithm 2 — The F5B algorithm modified by a subtle trick (F5M algorithm)

**Input:** a polynomial $m$-tuple: $(f_1,\cdots,f_m) \subset K[X]^m$, and an admissible order $\prec$.

**Output:** The Gröbner basis of the ideal $\langle f_1,\cdots,f_m \rangle \subset K[X]$.

```
begin
  $F_i \leftarrow (e_i, f_i, i)$ for $i = 1,\cdots,m$
  $k \leftarrow m$  # to track the number of labled polynomials
  $B \leftarrow \{F_i \mid i = 1,\cdots,m\}$
  $D \leftarrow \emptyset$
  $CP \leftarrow \{\text{critical pair } [F_i, F_j] \mid 1 \leq i < j \leq m\}$
  while $CP$ is not empty do
    $cp \leftarrow$ select a critical pair from $CP$
    $CP \leftarrow CP \setminus \{cp\}$
    if $cp$ meets neither Syzygy Criterion nor Rewritten Criterion, then
      $SP \leftarrow$ the s-polynomial of critical pair $cp$
      $P \leftarrow$ the F5-reduction result of $SP$ by $B$, i.e. $SP \Rightarrow_B^* P$
      $\text{Num}(P) \leftarrow k + 1$  # update the number of $P$
      if the polynomial of $P$ is not 0, i.e. $\text{Poly}(P) \neq 0$,
        then
          $CP \leftarrow CP \cup \{\text{critical pair } [P, Q] \mid Q \in B\}$
        end if
      $k \leftarrow k + 1$
    else
      $D \leftarrow D \cup \{cp\}$  # save the detected critical pairs
    end if
  end while
  return $\{\text{polynomial part of } Q \mid Q \in B\}$
end
```

The only difference between the F5B algorithm and F5M algorithm is: the detected critical pairs are all saved in set $D$. For convenience, we use the notations $B_{\text{end}}$ and $D_{\text{end}}$ to express the corresponding sets $B$ and $D$ when the F5M algorithm terminates.

Since initial polynomial set $\{f_1,\cdots,f_m\} = \{\text{Poly}(Q) \mid Q \in B_0\}$ and $B_0 \subset B_{\text{end}}$ by the F5M algorithm, our main purpose of this paper is to prove the following correctness theorem.

**Theorem 4.3** (Correctness Theorem). The set $\{\text{Poly}(Q) \mid Q \in B_{\text{end}}\} \subset K[X]$ itself is a Gröbner basis.
To prove this theorem, we need a powerful tool: \( t \)-representation for labeled polynomials.

**Definition 4.4 (\( t \)-representation).** Let \( \mathcal{F} \in L[X] \) be a labeled polynomial, \( B \subset L[X] \) a set of labeled polynomials and \( t \in PP(X) \) a power product. We say labeled polynomial \( \mathcal{F} \) has a \( t \)-representation w.r.t. set \( B \), if there exist polynomials \( p_1, \cdots, p_s \in K[X] \) and labeled polynomials \( \mathcal{G}_1, \cdots, \mathcal{G}_s \in B \), such that:

\[
\text{Poly}(\mathcal{F}) = p_1 \text{Poly}(\mathcal{G}_1) + \cdots + p_s \text{Poly}(\mathcal{G}_s),
\]

where labeled polynomial \( \mathcal{F} \) \( \succeq \) \( p_i \mathcal{G}_i \) and power product \( t \succeq \text{lpp}(p_i \mathcal{G}_i) \) for \( i = 1, \cdots, s \).

Compared with the definition of \( t \)-representation in polynomial version, the \( t \)-representation for labeled polynomials has an extra condition \( \mathcal{F} \) \( \succeq \) \( p_i \mathcal{G}_i \) on the signatures and numbers.

For convenience, we say the critical pair \([\mathcal{F}, \mathcal{G}]\) = \((u, \mathcal{F}, v, \mathcal{G})\) has a \( t \)-representation w.r.t. set \( B \), if the s-polynomial of \([\mathcal{F}, \mathcal{G}]\) has a \( t \)-representation w.r.t. set \( B \) where \( t \prec \text{lcm} (\text{lpp}(\mathcal{F}), \text{lpp}(\mathcal{G})) \).

The following theorem is the main result on \( t \)-representation for labeled polynomials. Its proof is straight from its polynomial version, so we omit the detail proof here. For interesting readers, please see [Becker et al., 1993].

**Theorem 4.5 (\( t \)-representation).** Let \( B \subset L[X] \) be a set of labeled polynomials. If for all labeled polynomials \( \mathcal{F}, \mathcal{G} \in B \), critical pair \([\mathcal{F}, \mathcal{G}]\) always has a \( t \)-representation w.r.t. set \( B \), then the polynomial set \( \{\text{Poly}(\mathcal{P}) \mid \mathcal{P} \in B\} \subset K[X] \) itself is a Gröbner basis.

So far, in order to prove the Correctness Theorem 4.3, it suffices to show that for any labeled polynomials \( \mathcal{F}, \mathcal{G} \in B_{end} \), the critical pair \([\mathcal{F}, \mathcal{G}]\) always has a \( t \)-representation w.r.t. set \( B_{end} \). In fact, if we examine all these critical pairs in detail, there are only two kinds of critical pairs generated by set \( B_{end} \):

1. The ones that have been operated during the loops, i.e. their s-polynomials have been calculated and then \( F5 \)-reduced. These \( F5 \)-reduction results have added to set \( B_{end} \).
2. The ones detected by either Syzygy Criterion or Rewritten Criterion. In \( F5M \) algorithm, all these critical pairs have been collected into set \( D_{end} \).

For the first kind of critical pairs, the following proposition, which is proved in next subsection, ensures that these critical pairs have \( t \)-representations w.r.t. set \( B_{end} \).

**Proposition 4.6 (first kind).** If a critical pair is operated during the loops, i.e. it is not detected by the two criteria, then it has a \( t \)-representation w.r.t. set \( B_{end} \).

For the second kind of critical pairs, the proof that they have \( t \)-representations w.r.t. set \( B_{end} \) is a bit complicated. In fact, we cannot show this directly, since an extra condition is necessary.

Let \( \mathcal{F}, \mathcal{G} \in L[X] \) be two labeled polynomials and \( B \subset L[X] \) a set of labeled polynomials. We say all the lower critical pairs of \([\mathcal{F}, \mathcal{G}]\) have \( t \)-representations w.r.t. set \( B \), if for any critical pair \([\mathcal{P}, \mathcal{Q}]\) such that \([\mathcal{P}, \mathcal{Q}] \prec [\mathcal{F}, \mathcal{G}]\) where \( \mathcal{P}, \mathcal{Q} \in B \), the critical pair \([\mathcal{P}, \mathcal{Q}]\) always has a \( t \)-representation w.r.t. set \( B \).

The following theorem shows the second kind of critical pairs have \( t \)-representations w.r.t. set \( B_{end} \) with an extra condition.
Theorem 4.7 (second kind). Let $[\mathcal{F}, \mathcal{G}] = (u, \mathcal{F}, v, \mathcal{G})$ be a critical pair, where $\mathcal{F}, \mathcal{G} \in B_{\text{end}}$ and $u, v$ are monomials in $X$ such that $\text{ulm}(\mathcal{F}) = \text{ulm}(\mathcal{G}) = \text{lcm}(\text{lpp}(\mathcal{F}), \text{lpp}(\mathcal{G}))$. Then the critical pair $[\mathcal{F}, \mathcal{G}]$ has a $t$-representation w.r.t. set $B_{\text{end}}$, if

1. labeled polynomial $u\mathcal{F}$ (or $v\mathcal{G}$) is either comparable or rewritable by $B_{\text{end}}$, and
2. all the lower critical pairs of $[\mathcal{F}, \mathcal{G}]$ have $t$-representations w.r.t. set $B_{\text{end}}$.

With Proposition 4.6 (first kind) and Theorem 4.7 (second kind), we are now able to prove the Correctness Theorem 4.3. The extra condition in Theorem 4.7 is satisfied subtly.

Theorem 4.3 (Correctness Theorem). The set $\{\text{Poly}(Q) \mid Q \in B_{\text{end}}\} \subset K[X]$ itself is a Gröbner basis.

Proof. Let $CP_{\text{all}}$ be the set of all critical pairs generated by set $B_{\text{end}}$. Then all the critical pairs in $CP_{\text{all}} \setminus D_{\text{end}}$ have $t$-representations w.r.t. $B_{\text{end}}$ by Proposition 4.6 (first kind). Next, it only remains to show that critical pair $cp$ has a $t$-representation w.r.t. set $B_{\text{end}}$ for all $cp \in D_{\text{end}}$.

The strategy of the proof is as follows.

1. Select the minimal critical pair, say $cp_{\text{min}}$, from set $D_{\text{end}}$ w.r.t. the order $\succ$. 
2. Show the critical pair $cp_{\text{min}}$ has a $t$-representation w.r.t. set $B_{\text{end}}$.
3. Remove the critical pair $cp_{\text{min}}$ from set $D_{\text{end}}$.

If set $D_{\text{end}}$ is not empty, then repeat the steps (1), (2) and (3). Since the cardinality of set $D_{\text{end}}$ is finite, this procedure terminates after finite steps. If all the critical pairs in set $D_{\text{end}}$ are proved in this way, the theorem is proved.

The steps (1) and (3) are trivial, so it only needs to show how the step (2) is done. Since critical pair $cp_{\text{min}}$ is the minimal one in set $D_{\text{end}}$, then all the critical pairs which are lower than $cp_{\text{min}}$ should be contained in the set $CP_{\text{all}} \setminus D_{\text{end}}$ and hence have $t$-representations w.r.t. set $B_{\text{end}}$ (because set $D_{\text{end}}$ contains all the unproved critical pairs). Critical pair $cp_{\text{min}} \in D_{\text{end}}$ also means $cp_{\text{min}}$ meets either Syzygy Criterion or Rewritten Criterion, so the critical pair $cp_{\text{min}}$ has a $t$-representation w.r.t. set $B_{\text{end}}$ by Theorem 4.7 (second kind).

After all, the critical pairs in $CP_{\text{all}}$ all have $t$-representations w.r.t. set $B_{\text{end}}$. Then the polynomial set $\{\text{Poly}(P) \mid P \in B_{\text{end}}\}$ itself is a Gröbner basis by Theorem 4.5 ($t$-representation).

The proof of Proposition 4.6 (first kind) for the first kind of critical pairs is simple. However, the proof of Theorem 4.7 (second kind) for the second kind of critical pairs is quite complicated. Next, we sketch the idea of this proof. All the following lemmas and propositions are proved in next subsection. We begin by an important definition.

Definition 4.8 (strictly lower representation). Let $\mathcal{F} \in L[X]$ be a labeled polynomial and $B \subset L[X]$ a set of labeled polynomials. We say labeled polynomial $\mathcal{F}$ has a strictly lower representation w.r.t. set $B$, if there exist polynomials $p_1, \ldots, p_s \in K[X]$ and labeled polynomials $\mathcal{G}_1, \ldots, \mathcal{G}_s \in B$, such that:

$$\text{Poly}(\mathcal{F}) = p_1 \text{Poly}(\mathcal{G}_1) + \cdots + p_s \text{Poly}(\mathcal{G}_s),$$
where labeled polynomial $F \triangleright p_i G_i$ for $i = 1, \ldots, s$.

Compared with the $t$-representation defined earlier, the strictly lower representation does not need the constraints on the leading power products $\text{lpp}(p_i G_i)$. Besides, the relation "$\geq$" in Definition 4.4 (t-representation) becomes "$\triangleright$" here, which is why we name it as strictly lower representation.

By the above definition, we first have two propositions on comparable and rewritable.

**Proposition 4.9** (comparable). Let $F \in B_{\text{end}}$ be a labeled polynomial and $cx^\gamma$ a non-zero monomial in $X$. If labeled polynomial $cx^\gamma F$ is comparable by $B_{\text{end}}$, then $cx^\gamma F$ has a strictly lower representation w.r.t. set $B_{\text{end}}$.

**Proposition 4.10** (rewritable). Let $F \in B_{\text{end}}$ be a labeled polynomial and $cx^\gamma$ a non-zero monomial in $X$. If labeled polynomial $cx^\gamma F$ is rewritable by $B_{\text{end}}$, then $cx^\gamma F$ has a strictly lower representation w.r.t. set $B_{\text{end}}$.

Next, the key lemma connect the strictly lower representation and $t$-representation. We say all the lower critical pairs of $F$ have $t$-representations w.r.t. set $B$, where $F$ is a labeled polynomial and $B$ is a set of labeled polynomials, if for all critical pairs $[P, Q] = (r, P, t, Q)$ such that $r P < \ll F$ where $P, Q \in B$, the critical pair $[P, Q]$ always has a $t$-representation w.r.t. set $B$.

**Lemma 4.11** (key lemma). Let $F \in L[X]$ be a labeled polynomial. If
1. labeled polynomial $F$ has a strictly lower representation w.r.t. set $B_{\text{end}}$, and
2. all the lower critical pairs of $F$ have $t$-representations w.r.t. set $B_{\text{end}}$.

Then labeled polynomial $F$ has a $t$-representation w.r.t. set $B_{\text{end}}$ where $t = \text{lpp}(F)$. Furthermore, there exists a labeled polynomial $H \in B_{\text{end}}$ such that: $\text{lpp}(H) \mid \text{lpp}(F)$ and $F \triangleright x^\gamma H$ where $x^\gamma = \text{lpp}(F)/\text{lpp}(H)$.

Based on Lemma 4.11 (key lemma), it is easy to obtain the following two propositions. Please pay attention to the position of the labeled polynomial $F$ in the critical pair of each proposition.

**Proposition 4.12** (left). Let $[F, G] = (u, F, v, G)$ be a critical pair, where $F, G \in B_{\text{end}}$ are labeled polynomials and $u, v$ are monomials in $X$ such that $\text{ulm}(F) = \text{ulm}(G) = \text{lcm}(\text{lpp}(F), \text{lpp}(G))$. Then the critical pair $[F, G]$ has a $t$-representation w.r.t. set $B_{\text{end}}$, if
1. labeled polynomial $u F$ has a strictly lower representation w.r.t. set $B_{\text{end}}$, and
2. all the lower critical pairs of $[F, G]$ have $t$-representations w.r.t. set $B_{\text{end}}$.

**Proposition 4.13** (right). Let $[G, F] = (v, G, u, F)$ be a critical pair, where $G, F \in B_{\text{end}}$ are labeled polynomials and $v, u$ are monomials in $X$ such that $\text{vlm}(G) = \text{vlm}(F) = \text{lcm}(\text{lpp}(G), \text{lpp}(F))$. Then the critical pair $[G, F]$ has a $t$-representation w.r.t. set $B_{\text{end}}$, if
1. labeled polynomial $u F$ has a strictly lower representation w.r.t. set $B_{\text{end}}$, and
2. all the lower critical pairs of $[G, F]$ have $t$-representations w.r.t. set $B_{\text{end}}$.

Now, combined with Propositions 4.9 (comparable), 4.10 (rewritable), 4.12 (left) and 4.13 (right), Theorem 4.7 (second kind) is proved.
4.3. Proofs of Lemmas and Propositions

In this subsection, we list the detail proofs for the lemmas and propositions appearing in last subsection.

Proposition 4.6 (first kind). If a critical pair is operated during the loops, i.e. it is not detected by the two criteria, then it has a \( t \)-representation w.r.t. set \( B_{\text{end}} \).

Proof. Let \( cp = [F, G] \) be a critical pair which is not rejected by the two criteria. Assume \( cp \) is being selected in the \( l \)th loop (from set \( CP^{(l-1)} \)) and \( B^{(l-1)} \) is the labeled polynomial set before the \( l \)th loop begins.

Since critical pair \( cp \) is not rejected by two criteria, its s-polynomial is calculated and F5-reduces by set \( B^{(l-1)} \) to a new labeled polynomial \( P \), i.e. \( \text{spoly}(F, G) \Rightarrow_{B^{(l-1)}} * B \). Next, only two possibilities may happen to the labeled polynomial \( P \).

1. If \( \text{Poly}(P) = 0 \), it is easy to check that the s-polynomial \( \text{spoly}(F, G) \) of \([F, G]\) has a \( t \)-representation w.r.t. set \( B^{(l-1)} \) where \( t = \text{lpp}(\text{spoly}(F, G)) \) by the definition of F5-reduction and hence \( t \prec \text{lcm}(\text{lpp}(F), \text{lpp}(G)) \).

2. If \( \text{Poly}(P) \neq 0 \), then the number of \( P \) is updated and denote this new labeled polynomial as \( P' \). Since signature \( \text{Sign}(\text{spoly}(F, G)) = \text{Sign}(P) = \text{Sign}(P') \) and the number \( \text{Num}(\text{spoly}(F, G)) = \text{Num}(P) < \text{Num}(P') \), then labeled polynomial \( \text{spoly}(F, G) \triangleright P' \) by the definition of “\( \triangleright \)”. Therefore, the s-polynomial \( \text{spoly}(F, G) \) has a \( t \)-representation w.r.t. set \( B^{(l-1)} \cup \{P'\} \) where \( t = \text{lpp}(\text{spoly}(F, G)) < \text{lcm}(\text{lpp}(F), \text{lpp}(G)) \). Notice that set \( B^{(l)} = B^{(l-1)} \cup \{P'\} \) by the algorithm and both \( B^{(l-1)}, B^{(l)} \subset B_{\text{end}} \).

Thus in either of the above cases, the critical pair \([F, G]\) has a \( t \)-representation w.r.t. set \( B_{\text{end}} \).

Next, we begin the proofs for Theorem 4.7 (second kind). The following lemma reveals the meanings of signatures and it is also used in the proof of Proposition 4.9 (comparable) and 4.10 (rewritable).

Lemma 4.14 (signature). If labeled polynomial \( F = (x^\alpha e_j, f, k) \in B_{\text{end}} \), then

\[
f = cx^\alpha f_j + p_1 \text{Poly}(G_1) + \cdots + p_s \text{Poly}(G_s),
\]

where \( c \) is a non-zero constant in \( K \), \( p_i \in K[X] \) and \( G_i \in B_{\text{end}} \) such that either \( p_i = 0 \) or signature \( \text{Sign}(F) \triangleright \text{Sign}(p_i G_i) \) for \( i = 1, \cdots, s \).

Proof. We prove this proposition by induction of the loop \( l \). Let \( B^{(l-1)} \) be the labeled polynomial set before the \( l \)th loop begins and \( B^{(l)} \) the labeled polynomial set when the \( l \)th loop is over.

First, when \( l = 0 \), consider the set \( B^{(0)} = \{(e_i, f_i, i) \mid i = 1, \cdots, m\} \) where \( f_i \)'s are initial polynomials. Clearly,

\[
f_i = f_i,
\]

which shows the proposition holds for the set \( B^{(0)} \).
Second, suppose the proposition holds for the set $B^{(l-1)}$. Then the next goal is to show the proposition holds for the set $B^{(l)}$. Denote the critical pair that is selected (from set $CP^{(l-1)}$) in the $l$th loop as $cp = \{Q_1, Q_2\} = (u_1, Q_1, u_2, Q_2)$, where $Q_1, Q_2 \in B^{(l-1)}$ and $u_1, u_2$ are monomials in $X$ such that $u_1 \cdot \text{lcm}(Q_1) = u_2 \cdot \text{lcm}(Q_2) = \text{lcm(lpp}(Q_1), \text{lpp}(Q_2))$.

If critical pair $cp$ meets either of criteria, then this critical pair is discarded and no labeled polynomial adds to set $B^{(l-1)}$, which means $B^{(l)} = B^{(l-1)}$. Then the proposition holds for set $B^{(l)}$.

It remains to show that when the critical pair $cp$ does not meet either of criteria, the proposition still holds for set $B^{(l)}$. In this case, the s-polynomial $\text{spoly}(Q_1, Q_2)$ is calculated and F5-reduces to a new labeled polynomial $P$ by the set $B^{(l-1)}$, i.e. $\text{spoly}(Q_1, Q_2) \Rightarrow \text{spoly}(P)$.

Then the number of $P$ is updated and denote this new labeled polynomial as $P'$. Clearly, signature $\text{Sign}(P) = \text{Sign}(P')$ and polynomial $\text{Poly}(P) = \text{Poly}(P')$. Next, $B^{(l)} = B^{(l-1)} \cup \{P'\}$ by the algorithm. Therefore, it suffices to prove that the proposition holds for $P'$.

By Proposition 3.3 (F5-reduction property), as s-polynomial $\text{spoly}(Q_1, Q_2) \Rightarrow \text{spoly}(P)$, there exist polynomials $p_1, \ldots, p_s \in K[X]$ and labeled polynomials $G_1, \ldots, G_s \in B^{(l-1)}$, such that $P = \text{spoly}(Q_1, Q_2) + p_1G_1 + \cdots + p_sG_s$, where signature $\text{Sign}(P') = \text{Sign}(\text{spoly}(Q_1, Q_2)) \Rightarrow \text{Sign}(p_iG_i)$ for $i = 1, \ldots, s$. Notice that s-polynomial $\text{spoly}(Q_1, Q_2) = u_1Q_1 - u_2Q_2$. The above equation equals to

$$P = u_1Q_1 - u_2Q_2 + p_1G_1 + \cdots + p_sG_s. \quad (1)$$

The definition of critical pair $[Q_1, Q_2]$ shows $u_1Q_1 \succ u_2Q_2$. As labeled polynomial $u_1Q_1$ is not rewritable by $B^{(l-1)}$, then signature $\text{Sign}(u_1Q_1) \Rightarrow \text{Sign}(u_2Q_2)$ holds; otherwise $u_1Q_1$ is rewritable by $\{Q_2\} \subset B^{(l-1)}$. Therefore, according to the addition of labeled polynomials, signature $\text{Sign}(P') = \text{Sign}(P) = \text{Sign}(u_1Q_1) = \text{Sign}(\text{spoly}(Q_1, Q_2)) \Rightarrow \text{Sign}(p_iG_i)$ for $i = 1, \ldots, s$ and $\text{Sign}(P') = \text{Sign}(P) = \text{Sign}(u_1Q_1) \Rightarrow \text{Sign}(u_2Q_2)$.

Now consider the polynomial part of equation (1):

$$\text{Poly}(P') = \text{Poly}(P) = u_1\text{Poly}(Q_1) - u_2\text{Poly}(Q_2) + p_1\text{Poly}(G_1) + \cdots + p_s\text{Poly}(G_s). \quad (2)$$

Since labeled polynomial $Q_1 \in B^{(l-1)}$, assume $Q_1 = (x^e j, q, k')$, by the induction hypothesis,

$$\text{Poly}(Q_1) = cx^e j + q_1\text{Poly}(H_1) + \cdots + q_r\text{Poly}(H_r),$$

where $c$ is a non-zero constant in $K$, $q_i \in K[X]$ and $H_i \in B^{(l-1)}$ such that either $q_i = 0$ or signature $\text{Sign}(Q_1) \Rightarrow \text{Sign}(q_iH_i)$ for $i = 1, \ldots, r$. Since $u_1$ is a non-zero monomial in $X$, signature $\text{Sign}(P') = \text{Sign}(u_1Q_1) = \text{lpp}(u_1)x^e j$. Substitute the above expression of $\text{Poly}(Q_1)$ back into equation (2), then a new representation of $\text{Poly}(P')$ is obtained, which shows that the proposition holds for set $B^{(l)}$. Then the proposition is proved.

The above lemma explains the implications of the signatures, i.e. for any labeled polynomial $f = (x^e j, f, k) \in B_{\text{ad}}$, its polynomial $f$ is F5-reduced from the polynomial $x^e f_j$, where $f_j$ is an initial polynomial. In fact, this lemma holds more generally.
Corollary 4.15 (signature). Let \( \mathcal{F} = (x^\alpha e_j, f, k) \in B_{\text{end}} \) be a labeled polynomial and \( cx^\gamma \) a non-zero monomial in \( X \). For the labeled polynomial \( cx^\gamma \mathcal{F} = (x^{\gamma + \alpha} e_j, cx^\gamma f, k) \), then

\[
x^{\gamma} f = \bar{c} x^{\gamma + \alpha} f_j + p_1 \text{Poly}(G_1) + \cdots + p_s \text{Poly}(G_s),
\]

where \( \bar{c} \) is a non-zero constant in \( K \), \( p_i \in K[X] \) and \( G_i \in B_{\text{end}} \) such that either \( p_i = 0 \) or signature \( \text{Sign}(cx^\gamma \mathcal{F}) \supset \text{Sign}(p_i G_i) \) for \( i = 1, \ldots, s \).

With a little care, the representations in Lemma 4.14 (signature) and Corollary 4.15 (signature) only constrain the signatures of \( \mathcal{F} \) and \( p_i G_i \), and do not limit the leading power products \( \text{lpp}(\mathcal{F}) \) and \( \text{lpp}(p_i G_i) \).

Remark that Lemma 4.14 (signature) itself is not sufficient to provide a strictly lower representation for the labeled polynomial \( \mathcal{F} \), since signature \( \text{Sign}(\mathcal{F}) = x^\alpha e_j = \text{Sign}(x^\alpha \mathcal{F}_j) \) but the number \( \text{Num}(\mathcal{F}) \geq \text{Num}(x^\alpha \mathcal{F}_j) \), which means labeled polynomial \( \mathcal{F} \leq x^\alpha \mathcal{F}_j \), where \( \mathcal{F}_j \) is the labeled polynomial of initial polynomial \( f_j \).

The following two propositions show that if a labeled polynomial is either comparable or rewritable by \( B_{\text{end}} \), then this labeled polynomial has a strictly lower representation w.r.t. set \( B_{\text{end}} \).

Proposition 4.9 (comparable). Let \( \mathcal{F} = (x^\alpha e_j, f, k_f) \in B_{\text{end}} \) be a labeled polynomial and \( cx^\gamma \) a non-zero monomial in \( X \). If labeled polynomial \( cx^\gamma \mathcal{F} \) is comparable by \( B_{\text{end}} \), then \( cx^\gamma \mathcal{F} \) has a strictly lower representation w.r.t. set \( B_{\text{end}} \).

Proof. Since \( cx^\gamma \mathcal{F} = (x^{\gamma + \alpha} e_j, cx^\gamma f, k_f) \) is comparable by \( B_{\text{end}} \), there exists labeled polynomial \( \mathcal{G} = (x^\alpha e_g, g, k_g) \in B_{\text{end}} \) such that (1) \( \text{lpp}(g) \mid x^{\gamma + \alpha} \) and (2) \( e_j > e_l \). Denote \( x^\lambda = x^{\gamma + \alpha}/\text{lpp}(g) \), then \( x^{\gamma + \alpha} = x^\lambda \text{lpp}(g) \). Let \( \mathcal{F}_j = (e_j, f_j, j) \in B_{\text{end}} \) be the labeled polynomial of initial polynomial \( f_j \). Then the polynomial 2-tuple \((g, -f_j)\) is a principle syzygy of the 2-tuple vector \((f_j, g)\) in free module \( (K[X])^2 \). That is

\[
g f_j - f_j g = 0 \quad \text{and} \quad \text{lm}(g)f_j = f_j g - (g - \text{lm}(g)) f_j.
\]

As \( x^{\gamma + \alpha} = x^\lambda \text{lpp}(g) \), then

\[
x^{\gamma + \alpha} f_j = x^\lambda \text{lpp}(g) f_j = \frac{x^\lambda}{\text{lc}(g)} (f_j g - (g - \text{lm}(g)) f_j) = \frac{x^\lambda}{\text{lc}(g)} f_j g - \frac{x^\lambda}{\text{lc}(g)} (g - \text{lm}(g)) f_j = q_1 g + q_2 f_j = q_1 \text{Poly}(\mathcal{G}) + q_2 \text{Poly}(\mathcal{F}_j),
\]

where \( q_1 = \frac{x^\lambda}{\text{lc}(g)} f_j \) and \( q_2 = -\frac{x^\lambda}{\text{lc}(g)} (g - \text{lm}(g)) \).

As \( e_j > e_l \) holds by hypothesis, then labeled polynomial \( cx^\gamma \mathcal{F} \triangleright q_1 \mathcal{G} \). Also labeled polynomial \( cx^\gamma \mathcal{F} \triangleright q_2 \mathcal{F}_j \), as the signature \( \text{Sign}(cx^\gamma \mathcal{F}) = x^{\gamma + \alpha} e_j = x^\lambda \text{lpp}(g) e_j > x^\lambda \text{lpp}(g - \text{lm}(g)) e_j = \text{lpp}(q_2) e_j = \text{Sign}(q_2 \mathcal{F}_j) \).

Since labeled polynomial \( \mathcal{F} \in B_{\text{end}} \) and \( cx^\gamma \) is a non-zero monomial, Corollary 4.15 (signature) shows

\[
\text{Poly}(cx^\gamma \mathcal{F}) = cx^\gamma f = \bar{c} x^{\gamma + \alpha} f_j + p_1 \text{Poly}(\mathcal{H}_1) + \cdots + p_s \text{Poly}(\mathcal{H}_s),
\]
where \( \bar{c} \) is a non-zero constant in \( K \), \( p_i \in K[X] \) and \( \mathcal{H}_i \in B_{end} \) such that either \( p_i = 0 \) or signature \( \text{Sign}(cx^\gamma \mathcal{F}) \triangleright \text{Sign}(p_i \mathcal{H}_i) \) and hence labeled polynomial \( cx^\gamma \mathcal{F} \triangleright p_i \mathcal{H}_i \) for \( i = 1, \cdots, s \).

Substitute the expression of polynomial \( x^{\gamma+\alpha} f_j \) in equation (3) into (4). Then

\[
\text{Poly}(cx^\gamma \mathcal{F}) = \bar{c}_q \text{Poly}(\mathcal{G}) + \bar{c}_q \text{Poly}(\mathcal{F}_j) + p_l \text{Poly}(H_1) + \cdots + p_s \text{Poly}(H_s),
\]

where labeled polynomial \( cx^\gamma \mathcal{F} \triangleright \bar{c}_q \mathcal{G} \), \( cx^\gamma \mathcal{F} \triangleright \bar{c}_q \mathcal{F}_j \) and \( cx^\gamma \mathcal{F} \triangleright p_i \mathcal{H}_i \) for \( i = 1, \cdots, s \). This is already a strictly lower representation of the labeled polynomial \( cx^\gamma \mathcal{F} \) w.r.t. set \( B_{end} \).}

**Proposition 4.10 (rewritable).** Let \( \mathcal{F} = (x^\alpha \mathcal{e}_j, f, k_f) \in B_{end} \) be a labeled polynomial and \( cx^\gamma \) a non-zero monomial in \( X \). If labeled polynomial \( cx^\gamma \mathcal{F} \) is rewritable by \( B_{end} \), then \( cx^\gamma \mathcal{F} \) has a strictly lower representation w.r.t. set \( B_{end} \).

**Proof.** Since \( cx^\gamma \mathcal{F} = (x^{\gamma+\alpha} \mathcal{e}_j, cx^\gamma f, k_f) \) is rewritable by \( B_{end} \), there exists labeled polynomial \( \mathcal{G} = (x^\beta \mathcal{e}_j, g, k_g) \in B_{end} \) such that (1) \( x^\beta \mathcal{e}_j \mid x^{\gamma+\alpha} \mathcal{e}_j \) and (2) \( k_f < k_g \). Denote \( x^\lambda = x^{\gamma+\alpha-\beta} \).

On one hand, for labeled polynomial \( x^\lambda \mathcal{G} \), since \( \mathcal{G} \in B_{end} \), according to Corollary 4.15 (signature),

\[
\text{Poly}(x^\lambda \mathcal{G}) = x^\lambda g = c_1 x^{\lambda+\beta} f_j + q_1 \text{Poly}(R_1) + \cdots + q_l \text{Poly}(R_l),
\]

where \( c_1 \) is a non-zero constant in \( K \), \( q_i \in K[X] \) and \( R_i \in B_{end} \) such that either \( q_i = 0 \) or signature \( \text{Sign}(x^\lambda \mathcal{G}) \triangleright \text{Sign}(q_i R_i) \) for \( i = 1, \cdots, l \). As signature \( \text{Sign}(cx^\gamma \mathcal{F}) = x^{\gamma+\alpha} \mathcal{e}_j = x^{\lambda+\beta} \mathcal{e}_j = \text{Sign}(x^\lambda \mathcal{G}) \), then signature \( \text{Sign}(cx^\gamma \mathcal{F}) \triangleright \text{Sign}(q_i R_i) \) and hence labeled polynomial \( cx^\gamma \mathcal{F} \triangleright q_i R_i \) for \( i = 1, \cdots, l \).

On the other hand, since labeled polynomial \( \mathcal{F} \in B_{end} \) and \( cx^\gamma \) is a non-zero monomial, the Corollary 4.15 (signature) shows

\[
\text{Poly}(cx^\gamma \mathcal{F}) = cx^\gamma f = c_2 x^{\gamma+\alpha} f_j + p_1 \text{Poly}(H_1) + \cdots + p_s \text{Poly}(H_s),
\]

where \( c_2 \) is a non-zero constant in \( K \), \( p_i \in K[X] \) and \( H_i \in B_{end} \) such that either \( p_i = 0 \) or signature \( \text{Sign}(cx^\gamma \mathcal{F}) \triangleright \text{Sign}(p_i H_i) \) and hence labeled polynomial \( cx^\gamma \mathcal{F} \triangleright p_i H_i \) for \( i = 1, \cdots, s \).

Since \( x^{\lambda+\beta} = x^{\gamma+\alpha} \), substitute the expression of polynomial \( x^{\lambda+\beta} f_j \) in equation (5) into (6). Then

\[
\text{Poly}(cx^\gamma \mathcal{F}) = \frac{c_2}{c_1} (\text{Poly}(x^\lambda \mathcal{G}) - q_1 \text{Poly}(R_1) - \cdots - q_l \text{Poly}(R_l)) + p_1 \text{Poly}(H_1) + \cdots + p_s \text{Poly}(H_s),
\]

where \( c_1, c_2 \) are non-zero constants in \( K \), labeled polynomial \( cx^\gamma \mathcal{F} \triangleright q_i R_i \) for \( i = 1, \cdots, l \) and labeled polynomial \( cx^\gamma \mathcal{F} \triangleright p_i H_i \) for \( i = 1, \cdots, s \). Also notice that labeled polynomial \( cx^\gamma \mathcal{F} \triangleright x^\lambda \mathcal{G} \), since signature \( \text{Sign}(cx^\gamma \mathcal{F}) = x^{\gamma+\alpha} \mathcal{e}_j = x^{\lambda+\beta} \mathcal{e}_j = \text{Sign}(x^\lambda \mathcal{G}) \) and number \( \text{Num}(cx^\gamma \mathcal{F}) = k_f < k_g = \text{Num}(x^\lambda \mathcal{G}) \). Thus (7) is a strictly lower representation of the labeled polynomial \( cx^\gamma \mathcal{F} \) w.r.t. set \( B_{end} \).
The following lemma is the key lemma of the whole proofs, which shows when a labeled polynomial, who has a strictly lower representation, has a t-representation.

Lemma 4.11 (key lemma). Let \( \mathcal{F} \in L[X] \) be a labeled polynomial. If

1. labeled polynomial \( \mathcal{F} \) has a strictly lower representation w.r.t. set \( B_{\text{end}} \), and
2. all the lower critical pairs of \( \mathcal{F} \) have t-representations w.r.t. set \( B_{\text{end}} \).

Then the labeled polynomial \( \mathcal{F} \) has a t-representation w.r.t. set \( B_{\text{end}} \) where \( t = \text{lpp}(\mathcal{F}) \). Furthermore, there exists a labeled polynomial \( \mathcal{H} \in B_{\text{end}} \) such that: \( \text{lpp}(\mathcal{H}) | \text{lpp}(\mathcal{F}) \) and \( \mathcal{F} \triangleright x^\lambda \mathcal{H} \) where \( x^\lambda = \text{lpp}(\mathcal{F})/\text{lpp}(\mathcal{H}) \).

Proof. Since labeled polynomial \( \mathcal{F} \) has a strictly lower representation w.r.t. set \( B_{\text{end}} \), by definition of strictly lower representation, there exist polynomials \( p_1, \ldots, p_s \in K[X] \) and labeled polynomials \( \mathcal{G}_1, \ldots, \mathcal{G}_s \in B_{\text{end}} \), such that: \( \text{Poly}(\mathcal{F}) = p_1 \text{Poly}(\mathcal{G}_1) + \cdots + p_s \text{Poly}(\mathcal{G}_s) \), where labeled polynomial \( \mathcal{F} \triangleright p_i \mathcal{G}_i \) for \( i = 1, \ldots, s \).

Let \( x^{\delta} = \max_{\mathcal{H}} \{ \text{lpp}(p_1 \mathcal{G}_1), \ldots, \text{lpp}(p_s \mathcal{G}_s) \} \), so \( \text{lpp}(\mathcal{F}) \preceq x^{\delta} \) always holds. Now consider all possible strictly lower representations of \( \mathcal{F} \) w.r.t. set \( B_{\text{end}} \). For each such expression, we get a possibly different \( x^{\delta} \). Since a term order is well-ordering, we can select a strictly lower representation of \( \mathcal{F} \) w.r.t. set \( B_{\text{end}} \) such that power product \( x^{\delta} \) is minimal. Assume this strictly lower representation is

\[
\text{Poly}(\mathcal{F}) = q_1 \text{Poly}(\mathcal{H}_1) + \cdots + q_l \text{Poly}(\mathcal{H}_l), \tag{8}
\]

where \( q_i \in K[X], \mathcal{H}_i \in B_{\text{end}} \) and labeled polynomial \( \mathcal{F} \triangleright q_i \mathcal{H}_i \) for \( i = 1, \ldots, l \). We will show that once this minimal \( x^{\delta} \) is chosen, we have \( \text{lpp}(\mathcal{F}) = x^{\delta} \) and hence the lemma is proved. We prove this by contradiction.

Equality fails only when leading power product \( \text{lpp}(\mathcal{F}) \prec x^{\delta} \). Denote \( m(i) = \text{lpp}(q_i \mathcal{H}_i) \), and then we can rewrite polynomial \( \text{Poly}(\mathcal{F}) \) in following form:

\[
\text{Poly}(\mathcal{F}) = \sum_{m(i) = x^{\delta}} q_i \text{Poly}(\mathcal{H}_i) + \sum_{m(i) \prec x^{\delta}} q_i \text{Poly}(\mathcal{H}_i)
\]

\[
= \sum_{m(i) = x^{\delta}} \text{lm}(q_i) \text{Poly}(\mathcal{H}_i) + \sum_{m(i) = x^{\delta}} (q_i - \text{lm}(q_i)) \text{Poly}(\mathcal{H}_i) + \sum_{m(i) \prec x^{\delta}} q_i \text{Poly}(\mathcal{H}_i). \tag{9}
\]

The power products appearing in the second and third sums on the second line all \( \prec x^{\delta} \). Thus, the assumption \( \text{lpp}(\mathcal{F}) \prec x^{\delta} \) means that power products in the first sum also \( \prec x^{\delta} \). So the first sum must be a linear combination of s-polynomials, i.e.

\[
\sum_{m(i) = x^{\delta}} \text{lm}(q_i) \text{Poly}(\mathcal{H}_i) = \sum_{j,k} w_{jk} \text{spoly}(\mathcal{H}_j, \mathcal{H}_k). \tag{10}
\]

where \( w_{jk} \)'s are monomials in \( X \). For each s-polynomial \( \text{spoly}(\mathcal{H}_j, \mathcal{H}_k) = u_{jk} \mathcal{H}_j - v_{jk} \mathcal{H}_k \) in equation \((10)\), we have \( \mathcal{F} \triangleright w_{jk} u_{jk} \mathcal{H}_j \), because expression \((8)\) is a strictly lower representation of \( \mathcal{F} \).
The next step is to use the hypothesis that all the lower critical pairs of \( \mathcal{F} \) have \( t \)-representations w.r.t. \( B_{end} \). Therefore, for each \( s \)-polynomial \( \text{spoly}(\mathcal{H}_j, \mathcal{H}_k) \) in equation (10), there exist polynomials \( g_1, \ldots, g_r \in K[X] \) and labeled polynomials \( \mathcal{R}_1, \ldots, \mathcal{R}_r \in B_{end} \), such that

\[
\text{spoly}(\mathcal{H}_j, \mathcal{H}_k) = g_1 \text{Poly}(\mathcal{R}_1) + \cdots + g_r \text{Poly}(\mathcal{R}_r),
\]

where \( s \)-polynomial \( \text{spoly}(\mathcal{H}_j, \mathcal{H}_k) \geq g_i \mathcal{R}_i \) and \( \text{lcm}(\text{lpp}(\mathcal{H}_j), \text{lpp}(\mathcal{H}_k)) \succ \text{lpp}(g_i \mathcal{R}_i) \) for \( i = 1, \cdots, r \).

Substitute the above representations back into the equation (9) and hence into the equation (10). The power products in the new expression of (10) will all \( \prec x^\delta \). Then a new strictly lower representation of \( \mathcal{F} \) w.r.t. \( B_{end} \) appears with all power products \( \prec x^\delta \), which contradicts with the minimality of \( x^\delta \). So we must have \( \text{lpp}(\mathcal{F}) = x^\delta \).

Thus, there exist polynomials \( q_1, \ldots, q_l \in K[X] \) and labeled polynomials \( \mathcal{H}_1, \ldots, \mathcal{H}_l \in B_{end} \), such that:

\[
\text{Poly}(\mathcal{F}) = q_1 \text{Poly}(\mathcal{H}_1) + \cdots + q_l \text{Poly}(\mathcal{H}_l),
\]

where \( \mathcal{F} \triangleright q_i \mathcal{H}_i \) and leading power product \( \text{lpp}(\mathcal{F}) \geq \text{lpp}(q_i \mathcal{H}_i) \) for \( i = 1, \cdots, l \). And this is already a \( t \)-representation of \( \mathcal{F} \) w.r.t. \( B_{end} \) where \( t = \text{lpp}(\mathcal{F}) \). Furthermore, since the equality holds in equation (11), there exists an integer \( j \) where \( 1 \leq j \leq l \), such that \( \text{lpp}(\mathcal{F}) = \text{lpp}(q_j \mathcal{H}_j) \). The lemma is proved.

The next two propositions provide sufficient conditions when a critical pair has a \( t \)-representation. Please pay more attention to the position of \( \mathcal{F} \) in the critical pair of each proposition.

**Proposition 4.12 (left).** Let \([\mathcal{F}, \mathcal{G}] = (u, \mathcal{F}, v, \mathcal{G})\) be a critical pair, where \( \mathcal{F}, \mathcal{G} \in B_{end} \) are labeled polynomials and \( u, v \) are monomials in \( X \) such that \( \text{ulm}(\mathcal{F}) = \text{vlm}(\mathcal{G}) = \text{lcm}(\text{lpp}(\mathcal{F}), \text{lpp}(\mathcal{G})) \). Then the critical pair \([\mathcal{F}, \mathcal{G}]\) has a \( t \)-representation w.r.t. \( B_{end} \), if

1. labeled polynomial \( u \mathcal{F} \) has a strictly lower representation w.r.t. \( B_{end} \), and
2. all the lower critical pairs of \([\mathcal{F}, \mathcal{G}]\) have \( t \)-representations w.r.t. \( B_{end} \).

**Proof.** Since labeled polynomial \( u \mathcal{F} \) has a strictly lower representation w.r.t. \( B_{end} \), then there exist polynomials \( p_1, \cdots, p_s \in K[X] \) and labeled polynomials \( \mathcal{H}_1, \cdots, \mathcal{H}_s \in B_{end} \), such that

\[
\text{Poly}(u \mathcal{F}) = p_1 \text{Poly}(\mathcal{H}_1) + \cdots + p_s \text{Poly}(\mathcal{H}_s),
\]

where labeled polynomial \( u \mathcal{F} \triangleright p_i \mathcal{H}_i \) for \( i = 1, \cdots, s \). By the definition of critical pairs, labeled polynomial \( u \mathcal{F} \triangleright v \mathcal{G} \). Then the following equation holds:

\[
\text{Poly}(\text{spoly}(\mathcal{F}, \mathcal{G})) = \text{Poly}(u \mathcal{F} - v \mathcal{G}) = \text{Poly}(u \mathcal{F}) - \text{Poly}(v \mathcal{G})
= p_1 \text{Poly}(\mathcal{H}_1) + \cdots + p_s \text{Poly}(\mathcal{H}_s) - v \text{Poly}(\mathcal{G}).
\]

Denote \( p_{s+1} = -v \) and \( \mathcal{H}_{s+1} = \mathcal{G} \in B_{end} \). Then

\[
\text{Poly}(\text{spoly}(\mathcal{F}, \mathcal{G})) = p_1 \text{Poly}(\mathcal{H}_1) + \cdots + p_s \text{Poly}(\mathcal{H}_s) + p_{s+1} \text{Poly}(\mathcal{H}_{s+1}),
\]

21
where s-polynomial \( \text{spoly}(F, G) \) \( \triangleleft \) \( u \), for \( i = 1, \ldots, s + 1 \). Then this is a strictly lower representation of \( \text{spoly}(F, G) \) w.r.t. set \( B_{\text{end}} \). Combined with the hypothesis that all the lower critical pairs of \([F, G]\) have \( t \)-representations w.r.t. set \( B_{\text{end}} \), Lemma 4.11 (key lemma) shows the s-polynomial \( \text{spoly}(F, G) \) has a \( t \)-representation w.r.t. set \( B_{\text{end}} \) where \( t = \text{lpp}(\text{spoly}(F, G)) \) \( \prec \) \( \text{lcm}(\text{lpp}(F), \text{lpp}(G)) \).

**Proposition 4.13 (right).** Let \([G, F] = (v, u, F)\) be a critical pair, where \( G, F \in B_{\text{end}} \) are labeled polynomials and \( v, u \) are monomials in \( X \) such that \( \text{vlm}(G) = \text{ulm}(F) = \text{lcm}(\text{lpp}(G), \text{lpp}(F)) \). Then the critical pair \([G, F]\) has a \( t \)-representation w.r.t. set \( B_{\text{end}} \), if

1. labeled polynomial \( u \) \( F \) has a strictly lower representation w.r.t. set \( B_{\text{end}} \), and
2. all the lower critical pairs of \([G, F]\) have \( t \)-representations w.r.t. set \( B_{\text{end}} \).

**Proof.** Since labeled polynomial \( u \) \( F \) has a strictly lower representation w.r.t. \( B_{\text{end}} \) and all the lower critical pairs of \([G, F]\) have \( t \)-representations w.r.t. set \( B_{\text{end}} \), Lemma 4.11 (key lemma) shows that there exists a labeled polynomial \( H \in B_{\text{end}} \) such that \( \text{lpp}(H) \) \( \mid \) \( \text{lpp}(u \) \( F \) \( ) \) and \( u \) \( F \) \( \triangleleft w \) \( H \) where \( w = \text{lm}(u \) \( F \) \( ) / \text{lm}(H) \).

Notice that \( \text{lpp}(v \) \( G \) \( ) = \text{lpp}(u \) \( F \) \( ) = \text{lpp}(w \) \( H \) \( ) \) and \( v \) \( G \) \( \triangleright u \) \( F \) \( \triangleright w \) \( H \), then

\[
\text{spoly}(G, F) = v \) \( G - u \) \( F = (v \) \( G - w \) \( H) - (u \) \( F - w \) \( H) = \gcd(v, w)\) \( \text{spoly}(G, H) - \gcd(u, w)\) \( \text{spoly}(F, H).\)

Since critical pair \([G, F] \triangleright \) \([G, H]\) and \([G, F] \triangleright \) \([F, H]\) and all the lower critical pairs of \([G, F]\) have \( t \)-representations w.r.t. set \( B_{\text{end}} \), then the s-polynomial \( \text{spoly}(G, H) \) has a \( t \)-representation w.r.t. set \( B_{\text{end}} \) where \( t < \text{lcm}(\text{lpp}(G), \text{lpp}(H)) \), and similarly the s-polynomial \( \text{spoly}(F, H) \) also has a \( t \)-representation w.r.t. set \( B_{\text{end}} \) where \( t < \text{lcm}(\text{lpp}(F), \text{lpp}(H)) \).

Combined with the fact that \( \text{lcm}(\text{lpp}(G), \text{lpp}(F)) = \gcd(v, w)\) \( \text{lcm}(\text{lpp}(G), \text{lpp}(H)) = \gcd(u, w)\) \( \text{lcm}(\text{lpp}(F), \text{lpp}(H)), \) thus the s-polynomial \( \text{spoly}(G, F) \) has a \( t \)-representation w.r.t. set \( B_{\text{end}} \) where \( t < \text{lcm}(\text{lpp}(G), \text{lpp}(H)) \).

5. Available Variation of F5 Algorithm

5.1. Available Variations

Briefly, the F5 (F5B) algorithm introduces a special reduction (F5-reduction) and provides two new criteria (Syzygy Criterion and Rewritten Criterion) to avoid unnecessary computations/reductions.

From the proofs in last section, Lemma 4.11 (key lemma) plays a crucial role in the whole proofs, and the base of this key Lemma is the property of F5-reduction (Proposition 3.6). So the F5-reduction is the key of whole F5 (F5B) algorithm, and it ensures the correctness of the whole algorithm.

Therefore, various variations of F5 algorithm become available if we maintain the F5-reduction. For example,
1. use various strategies of selecting critical pairs, such as incremental F5 algorithm in (Faugère, 2002) and the F5 algorithm (reported by Faugère in INSCRYPT 2008);
2. use matrix technique when doing reduction, such as matrix-F5 algorithm mentioned in (Bardet et al., 2004);
3. add new initial polynomials during computation, such as branch Gröbner basis algorithm over boolean ring (Sun and Wang, 2009a,b);
4. change the order of signatures, such as Gröbner basis algorithms in (Ars and Hashemi, 2009; Sun and Wang, 2009a,b).

Next, we introduce a natural variation of F5 algorithm by change the order of signatures. This natural variation has been reported in (Sun and Wang, 2009a,b), and it is also quite similar as the variation in (Ars and Hashemi, 2009).

5.2. A Natural Variation

In fact, the original F5 algorithm is always an incremental algorithm no matter which strategy of selecting critical pair is used. Specifically, the outputs of F5 algorithm not only contain the Gröbner basis of the ideal \( \langle f_1, \ldots, f_m \rangle \), but also include the Gröbner bases of the ideals \( \langle f_i, \ldots, f_m \rangle \) for \( 1 < i < m \).

However, there are three disadvantages of incremental algorithms.

1. Generally, the ideals \( \langle f_i, \ldots, f_m \rangle \) for \( 1 < i < m \) usually have higher dimensions than the ideal \( \langle f_1, \ldots, f_m \rangle \), so their Gröbner bases may be expensive to compute.
2. The Gröbner bases of ideals \( \langle f_i, \ldots, f_m \rangle \) for \( 1 < i < m \) are not necessary, since the Gröbner of ideal \( \langle f_1, \ldots, f_m \rangle \) is what we really need.
3. The order of initial polynomials influences the efficiency of algorithm significantly.

If we dig it deeper, we will find that it is the order of signatures that makes F5 algorithm incremental. Original F5 algorithm uses a POT (position over term) order of signatures defined on free module \((K[X])^m\). Thus, a nature idea is to change the POT order to the TOP (term over position) order. When using a TOP order of signatures, F5 algorithm will not be an incremental algorithm.

We extend the admissible order \( \prec \) on \( PP(X) \) to free module \((K[X])^m\) in the TOP (term over position) fashion:

\[
x^\alpha e_i \prec' x^\beta e_j \text{ (or } x^\beta e_j \succ' x^\alpha e_i \text{) } \text{ iff } \begin{cases} x^\alpha \text{lpp}(f_i) \prec x^\beta \text{lpp}(f_j), \\ \text{or} \\ x^\alpha \text{lpp}(f_i) = x^\beta \text{lpp}(f_j) \text{ and } i > j. \end{cases}
\]

Similarly, labeled polynomials are compared in the following way:

\[
(x^\alpha e_i, f, k_f) \prec' (x^\beta e_j, g, k_g) \text{ (or } x^\beta e_j, g, k_g) \succ' (x^\alpha e_i, f, k_f)) \text{ iff } \begin{cases} x^\alpha e_i \prec x^\beta e_j, \\ \text{or} \\ x^\alpha e_i = x^\beta e_j \text{ and } k_f > k_g. \end{cases}
\]

Particularly, denote \((x^\alpha e_i, f, k_f) \succ' (x^\beta e_j, g, k_g)\), if \(x^\alpha e_i = x^\beta e_j\) and \(k_f = k_g\).
There is no need to modify the definition of rewritable, as well as the descriptions of F5-
reduction, Syzygy Criterion and Rewritten Criterion. However, the definition of comparable
needs a bit adaption to fit the new order.

**Definition 5.1** (new-comparable). Let \( \mathcal{F} = (x^\alpha e_i, f, k_f) \in L[X] \) be a labeled polynomial,
\( cx^\gamma \) a non-zero monomial in \( X \) and \( B \subset L[X] \) a set of labeled polynomials. The labeled
polynomial \( cx^\gamma \mathcal{F} = (x^{\gamma+\alpha} e_i, cx^\gamma f, k_f) \) is said to be new-comparable by \( B \), if there exists a
labeled polynomial \( \mathcal{G} = (x^\beta e_j, g, k_g) \in B \) such that:

1. \( \text{lpp}(g) \mid x^{\gamma+\alpha} \), and
2. \( cx^\gamma \mathcal{F} \triangleright' x^\lambda \text{lpp}(f_i) \mathcal{G} \), where \( x^\lambda = x^{\gamma+\alpha}/\text{lpp}(g) \).

With this definition, the following proposition implies the new Syzygy Criterion is still
correct.

**Proposition 5.2** (new-comparable). Let \( \mathcal{F} = (x^\alpha e_j, f, k_f) \in B_{\text{end}} \) be a labeled polynomial
and \( cx^\gamma \) a non-zero monomial in \( X \). If labeled polynomial \( cx^\gamma \mathcal{F} \) is new-comparable by
\( B_{\text{end}} \), then \( cx^\gamma \mathcal{F} \) has a strictly lower representation w.r.t. set \( B_{\text{end}} \).

**Proof.** As \( cx^\gamma \mathcal{F} = (x^{\gamma+\alpha} e_j, cx^\gamma f, k_f) \) is new-comparable by \( B_{\text{end}} \), there exists labeled polyno-

mial \( \mathcal{G} = (x^\beta e_j, g, k_g) \in B_{\text{end}} \) such that (1) \( \text{lpp}(g) \mid x^{\gamma+\alpha} \) and (2) \( cx^\gamma \mathcal{F} \triangleright' x^\lambda \text{lpp}(f_i) \mathcal{G} \),
where \( x^\lambda = x^{\gamma+\alpha}/\text{lpp}(g) \). Let \( \mathcal{F}_j = (e_j, f_j, j) \in B_{\text{end}} \) be the labeled polynomial of the initial

polynomial \( f_j \). Then the polynomial 2-tuple \( (g, -f_j) \) is still a principle syzygy of the 2-tuple vector
\( (f_j, g) \) in free module \( (K[X])^2 \). So

\[
\begin{align*}
g f_j - f_j g & = 0 \quad \text{and} \quad \text{lmp}(g) f_j = f_j g - (g - \text{lmp}(g)) f_j. \\
\end{align*}
\]

Since \( x^{\gamma+\alpha} = x^\lambda \text{lpp}(g) \), then

\[
\begin{align*}
x^{\gamma+\alpha} f_j = x^\lambda \text{lpp}(g) f_j & = \frac{x^\lambda}{\text{lmp}(g)} (f_j g - (g - \text{lmp}(g)) f_j) = \frac{x^\lambda}{\text{lmp}(g)} f_j g - \frac{x^\lambda}{\text{lmp}(g)} (g - \text{lmp}(g)) f_j \\
& = q_1 g + q_2 f_j = q_1 \text{Poly}(\mathcal{G}) + q_2 \text{Poly}(\mathcal{F}_j), \quad (12)
\end{align*}
\]

where \( q_1 = \frac{x^\lambda}{\text{lmp}(g)} f_j \) and \( q_2 = -\frac{x^\lambda}{\text{lmp}(g)} (g - \text{lmp}(g)) \).

By the definition of new-comparable, labeled polynomial \( cx^\gamma \mathcal{F} \triangleright' x^\lambda \text{lpp}(f_i) \mathcal{G} = \text{lpp}(q_1) \mathcal{G} \).
Since \( x^{\gamma+\alpha} \text{lpp}(f_j) = x^\lambda \text{lpp}(g) \text{lpp}(f_j) \triangleright' x^\lambda \text{lpp}(g - \text{lmp}(g)) \text{lpp}(f_j) = \text{lpp}(q_2) \text{lpp}(f_j) \), then
\( cx^\gamma \mathcal{F} \triangleright' q_2 \mathcal{F}_j \) holds.

Since labeled polynomial \( \mathcal{F} \in B_{\text{end}} \) and \( cx^\gamma \) is a non-zero monomial, Corollary \( 4.15 \)
(signature) shows

\[
\text{Poly}(cx^\gamma \mathcal{F}) = cx^\gamma f = cx^{\gamma+\alpha} f_j + p_i \text{Poly}(\mathcal{H}_i) + \cdots + p_s \text{Poly}(\mathcal{H}_s), \quad (13)
\]

where \( \tilde{c} \) is a non-zero constant in \( K \), \( p_i \in K[X] \) and \( \mathcal{H}_i \in B_{\text{end}} \) such that either \( p_i = 0 \)
or signature \( \text{Sign}(cx^\gamma \mathcal{F}) \triangleright' \text{Sign}(p_i \mathcal{H}_i) \) and hence labeled polynomial \( cx^\gamma \mathcal{F} \triangleright' p_i \mathcal{H}_i \) for \( i = 1, \cdots, s \).
Substitute the expression of polynomial $x^{\gamma+\alpha} f_j$ in equation (12) into (13). Then
\[
\text{Poly}(cx^{\gamma} F) = \bar{c}q_1 \text{Poly}(G) + \bar{c}q_2 \text{Poly}(F_j) + p_1 \text{Poly}(H_1) + \cdots + p_s \text{Poly}(H_s),
\]
where labeled polynomial $cx^{\gamma} F \triangleright \bar{c}q_1 G$, $cx^{\gamma} F \triangleright \bar{c}q_2 F_j$ and $cx^{\gamma} F \triangleright p_i H_i$ for $i = 1, \ldots, s$. This is already a strictly lower representation of the labeled polynomial $cx^{\gamma} F$ w.r.t. set $B_{\text{end}}$.

**Remark 5.3.** For the labeled polynomials $F = (x^a e_i, f, k_f) \in L[X]$ and $G = (x^b e_j, g, k_g) \in B$ in Definition 5.1 (new-comparable). The second condition “$cx^{\gamma} F \triangleright x^l \text{pp}(f) G$” is in fact equivalent to the condition “signature Sign($G$) = $e_j$ and $e_i \succ e_j$, i.e. $i < j$”. So this new Syzygy Criterion only utilizes the principle syzygies of initial polynomials, which is the same as the criteria in (Ars and Hashemi, 2009). The technique “adding new initial polynomials during computation” introduced in (Sun and Wang, 2009a,b) will enhance this new Syzygy Criterion. Specifically, when a labeled polynomial $P = (x^a e_i, p, k_p)$ is generated during the computation, simply adding the labeled polynomial $P' = (e_i, p, k_p)$ into computation and updating critical pairs correspondingly do not affect the correctness of algorithm, where we prefer $l' > l$ and $k'_p > k_p$ such that $P \triangleright P'$.

The Syzygy Criterion in Ars and Hashema’s paper (Ars and Hashemi, 2009) can also be proved in a similar way as above.

### 5.3. Criteria of the Natural Variation

Although only the principle syzygies of initial polynomials are used, the new Syzygy Criterion also performs pretty good in experiments. We have implemented this natural variation of F5 algorithm over boolean ring (Sun and Wang, 2009a,b). The data structure ZDD (Zero-suppressed Binary Decision Diagrams) is used to express boolean polynomials, and the “adding new initial polynomials during computation” technique is also used to enhance the new Syzygy Criterion. Also matrix technique is used when F5-reducing labeled polynomials, but this procedure is not fully optimized yet, as only general Gaussian elimination is used.

The data about the two revised criteria in following table are obtained from the above implementation. Examples are randomly generated quadratic boolean polynomials, and the number of initial polynomials $m$ equals to the number of variables $n$. The timings are obtained from a computer (OS Linux, CPU Xion 4*3.0GHz, 16.0GB RAM). In the table, **comparable**, **F2-comparable**, and **rewritable** refer to the times of corresponding conditions being met in the computation. Remark that these numbers are not the numbers of rejected critical pairs, as F5-reduction also needs to check the **comparable**, **F2-comparable** and **rewritable**. Besides, **useful cp’s** is the number of critical pairs that are really operated during computation (i.e. not rejected by two criteria). **0-polys** is the number of labeled polynomials that F5-reduce to 0.

---

6F2-comparable is a special comparable which results from the characteristic of boolean ring, since for each boolean polynomial $f$, we always have $f^2 = f$ in boolean ring. For more details, please see (Sun and Wang, 2009a,b).
Table 1: The Revised Criteria

|   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| comparable | 0   | 36  | 566 | 898 | 72189 | 68337 | 99058 | 136404 |
| F2-comparable | 2   | 5   | 87  | 114 | 7770 | 6763 | 9374 | 11749 |
| re writable | 2   | 20  | 74  | 136 | 6908 | 4786 | 6293 | 8536 |
| useful cp’s | 21  | 77  | 225 | 305 | 841 | 3480 | 4469 | 5672 |
| 0-polys | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| Time(sec.) | 0.001 | 0.005 | 0.034 | 0.107 | 0.778 | 14.586 | 77.197 | 344.875 |

From the data in table 1, most of redundant computations/reductions are rejected by the revised *new-comparable* (Syzygy Criterion), particularly in large examples and no labeled polynomials F5-reduce to 0 in these examples. Therefore, the revised criteria in the natural variation of F5 algorithm are very effective and they are able to reject almost all unnecessary computations/reductions.

6. Conclusion

In this paper, a complete proof for the correctness of F5 (F5-like) algorithm is presented. As F5B algorithm is equivalent to the original F5 algorithm as well as some F5-like algorithms, we concentrate on the proof for the correctness of F5B algorithm. This new proposed proof is not limited to homogeneous systems and does not depend on the strategies of selecting critical pairs, so it can easily extends to other variations of F5 algorithm. From the new proof, we find that the F5-reduction is the key of the whole algorithm and it ensures the correctness of two criteria. With these insights, various variations of F5 algorithm become available by maintaining the F5-reduction. We present and prove a natural variation of F5 algorithm which is not incremental. We hope to study other variations of F5 algorithm in the future.

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