Background magnetic field and quantum correlations in the Schwinger effect

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In this work we consider two complex scalar fields distinguished by their masses coupled to constant background electric and magnetic fields in the (3 + 1)-dimensional Minkowski spacetime and subsequently investigate a few measures quantifying the quantum correlations between the created particle-antiparticle Schwinger pairs. Since the background magnetic field itself cannot cause the decay of the Minkowski vacuum, our chief motivation here is to investigate the interplay between the effects due to the electric and magnetic fields. We start by computing the entanglement entropy for the vacuum state of a single scalar field. Second, we consider some maximally entangled states for the two-scalar field system and compute the logarithmic negativity and the mutual information. Qualitative differences of these results pertaining to the charge content of the states are pointed out. Based upon our analyses, we make some speculations on the effect of a background magnetic field on the well known phenomenon of degradation of entanglement between states in an accelerated frame, for charged quantum fields.

I. INTRODUCTION

Correlation between the states or entanglement is one of the fundamental characteristics of quantum mechanics. There are several measures quantifying such correlations, studied in a wide range of theoretical researches, e.g. [1–10] and references therein. These correlations constitute the foundation of quantum information theory, see [11] and references therein.

A natural framework to study quantum entanglement is a system where pair creation can take place. This includes, most popularly, spacetimes endowed with non-extremal Killing horizons, e.g. [12–14] or the cosmological scenario, e.g. [15–22] (also references therein). We also refer our reader to e.g. [23–27] and references therein for discussions on quantum entanglement from the holographic perspective.

In this work, we wish to investigate some measures of quantum correlations (namely, the vacuum entanglement entropy, the logarithmic negativity and mutual information for entangled states)
the context of the Schwinger pair creation mechanism \cite{28, 29}. The entanglement entropy and some other correlation measures for pairwise modes for such a system with a background electric field was studied in \cite{30, 31}. See also \cite{32, 33, 34, 35} for subsequent developments.

It is well known that a magnetic field itself cannot give rise to pair creation but can affect its rate if a background electric field is also present. Thus it seems interesting to ask: what will be the effect of a background magnetic field on the quantum correlations between the particle-antiparticle pairs? We may intuitively expect \textit{a priori} that the magnetic field will oppose the effect of the electric field. However, how do these correlations explicitly depend upon the magnetic field strength, e.g., are they monotonic? How do these behaviour differ subject to the charge content of the state we choose? We wish to address these questions in this work for a complex scalar field in the Minkowski spacetime in (3 + 1)-dimensions.

The rest of the paper is organised as follows. We review very briefly the relevant information quantities in Section II for the convenience of reader and obtain the solution of the complex scalar’s mode functions with the background electromagnetic field in Section III. We compute the vacuum entanglement entropy for a single scalar field in Section IV and the logarithmic negativity and mutual information for maximally entangled states of the two scalar fields in Section V. We emphasise the qualitative differences of the results subject to the charge content of the states. Finally, we summarise and discuss our results and related issues in Section VI. In particular, we speculate that the well known degradation of the quantum entanglement in an accelerated frame e.g. \cite{14}, can perhaps be restored for a charged field, upon application of a ‘strong enough’ magnetic field.

We work with the mostly positive signature of the metric and set $\hbar = c = 1$ throughout. The logarithms are understood as $\log_2$ in our numerical calculations.

II. MEASURES OF CORRELATIONS – A QUICK LOOK

Following e.g. \cite{1}, let us consider a bipartite system constituted by subsystems, $A$ and $B$, so that the Hilbert space can be decomposed as $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho_{AB}$ be the density matrix of states on $\mathcal{H}_{AB}$. $\rho_{AB}$ is called a \textit{separable} (\textit{entangled}) when it can (cannot) be decomposed as $\sum_i p_i \rho_A^i \otimes \rho_B^i$ (e.g., in the trace norm), where $\{p_i\}$ is a probability distribution, and $\rho_A^i$ ($\rho_B^i$) is the density operator on $\mathcal{H}_A$ ($\mathcal{H}_B$).
The reduced density matrix of the subsystem $A$ is defined by

$$\rho_A = \text{Tr}_B \rho_{AB},$$  \hspace{1cm} (1)$$

where the partial traces $\text{Tr}_B$ is taken only over the Hilbert space $\mathcal{H}_B$. The reduced density matrix of $B$, $\rho_B$, is defined in parallel with $\rho_A$.

### A. Entanglement entropy

Entanglement entropy is well known as a good measure of entanglement for pure states. The entanglement entropy of $A$ is defined as the von Neumann entropy of $\rho_A$,

$$S(\rho_A) = -\text{Tr}_A (\rho_A \log \rho_A) = -\sum_{i=1}^{\text{dim}(\rho_A)} \lambda_i^A \log \lambda_i^A,$$ \hspace{1cm} (2)$$

where $\lambda_i^A$ is the $i$-th eigenvalue of $\rho_A$, and $\text{dim}(\rho_A)$ is the dimension of $\rho_A$. Similarly, $S(\rho_B) = -\text{Tr}_B (\rho_B \log \rho_B) = -\sum_i \lambda_i^B \log \lambda_i^B$, where $\lambda_i^B$ is the $i$-th eigenvalue of $\rho_B$.

When $\rho_{AB}$ corresponds to a pure state, one has $S(\rho_A) = S(\rho_B)$. Furthermore, when $\rho_{AB}$ is a pure and separable state, one has $S(\rho_A) = S(\rho_B) = 0$. For mixed states, the entanglement entropy does not vanish even when they are separable. The von Neumann entropies satisfy a subadditivity: $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$, where $S(\rho_{AB})$ is the Von Neumann entropy of $\rho_{AB}$. The equality holds if and only if $\rho_{AB} = \rho_A \otimes \rho_B$. More details on these properties can be found in e.g. [11].

### B. Quantum mutual information

The quantum mutual information is a measure of quantum as well as classical correlations between the subsystems $A$ and $B$. In the state $\rho_{AB}$, it is defined as,

$$I(A, B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$ \hspace{1cm} (3)$$

The lower bound of the mutual information, $I(A, B) \geq 0$, is immediately obtained by the subadditivity of the entanglement entropy, where the equality holds only if $\rho_{AB} = \rho_A \otimes \rho_B$. Further properties of the quantum mutual information can be found in e.g. [11].
C. Entanglement negativity and logarithmic negativity

Even for mixed states, there is a measure of the entanglement of bipartite states \([5, 7]\), called the entanglement negativity, defined as

\[
N(\rho_{AB}) = \frac{1}{2} \left( \|\rho_{AB}^T\|_1 - 1 \right),
\]

where \(\rho_{AB}^T\) is the partial transpose of \(\rho_{AB}\) with respect to the subspace of \(A\), i.e., \(\langle i \rangle_A |n⟩ \otimes \langle j \rangle_B(\ell)\)\(^T\) \(A := |n⟩_A (i) \otimes |j⟩_B(\ell)\). Here, \(\|\rho_{AB}^T\|\) is the trace norm, \(\|\rho_{AB}^T\|_1 = \sum_{i=1}^{\text{all}} |\mu_i|\), where \(\mu_i\) is the \(i\)-th eigenvalue of \(\rho_{AB}^T\).

The logarithm of \(\|\rho_{AB}^T\|_1\) is called the logarithmic negativity, which can be written as

\[
L_N(\rho_{AB}) = \log (1 + 2N(\rho_{AB})).
\]

The logarithmic negativity is the upper bound to the distillable entanglement \([1, 7]\). The negativity is convex, while the logarithmic negativity is not \([7, 8]\). These quantities are entanglement monotones which do not increase under local and classical communications.

These quantities measure violation of the positive partial transpose (PPT) in \(\rho_{AB}\). The PPT criterion can be stated as follows. If \(\rho_{AB}\) is separable, the eigenvalues of \(\rho_{AB}^T\) are non-negative. Hence, if \(N \neq 0\) (\(L_N \neq 0\)), \(\rho_{AB}\) is an entangled state. On the other hand, if \(N = 0\) (\(L_N = 0\)), we cannot judge the existence of the entanglement from this measure, since there exist PPT and entangled states in general. However, the logarithmic negativity can be useful since it is a calculable measure. Further discussions on it can be found in e.g. \([2]\).

III. COMPLEX SCALAR IN BACKGROUND ELECTROMAGNETIC FIELD

Let us now focus on the complex scalar field theory coupled to external or background electromagnetic fields in the four-dimensional Minkowski spacetime. In this system charged scalar particles and antiparticles are pair created by the external electric field. Our analysis in this section is in parallel with \([30, 36, 37]\).

The Klein-Gordon equation reads

\[
(D_\mu D^\mu - m^2) \phi(t, \vec{x}) = 0,
\]

where \(D_\mu = \partial_\mu - iqA_\mu\) is the gauge covariant derivative and \(q\) stands for the electric charge of the field. We consider the external gauge field as \(A_\mu = (Ez, -By, 0, 0)\), where the electric field \(E\) and the magnetic field \(B\) are constants.
We quantise the field as,

$$\phi(x) = \int \frac{dk^0 dk_x}{\sqrt{4\pi k^0}} \left[ a_k \phi^{(+)}_k + b^*_k \left( \phi^{(-)}_k \right)^* \right],$$

(7)

where $k^0$ is restricted to be positive, and $a_k$ ($b^*_k$) corresponds to the annihilation (creation) operator for the particle (antiparticle). The mode functions $\phi^{(\pm)}_k$ are given by

$$\phi^{(+)}_k = e^{-i(k^0 t - k_x x)} \phi^{(p)}_k (y, z), \quad \left( \phi^{(-)}_k \right)^* = e^{i(k^0 t - k_x x)} \left( \phi^{(a)}_k (y, z) \right)^*,$$

(8)

where $p(a)$ stands for particle (antiparticle). Eq. (9) gives,

$$\left[ (k^0 + qEz)^2 - (k^x + qBy)^2 + \partial_y^2 + \partial_z^2 - m^2 \right] \phi^{(\pm)}_k (y, z) = 0.$$

(9)

We consider a particle that is incoming in the $z$-direction at $|z| = \infty$. The independent solutions of (9) with this boundary condition is derived as

$$\phi^{(p)\text{in}}_k (y, z) = N^{-1} e^{-y^2/2} H_{n_L}(y+) D_\nu(\zeta_+),$$

$$\left[ \phi^{(a)\text{in}}_k (y, z) \right]^* = N^{-1} e^{-y^2/2} H_{n_L}(y-) \left[ D_\nu(\zeta_-) \right]^*,$$

(10)

where $H_{n_L}(y_\pm)$ is the Hermite polynomial, and $D_\nu(\zeta_\pm)$ is the parabolic cylinder function. The variables $y_\pm$ and $\zeta_\pm$ are defined by

$$y_\pm = \sqrt{|qB|} \left( y \pm \frac{k_x}{qB} \right), \quad \zeta_\pm = e^{i\pi/4} \sqrt{2 |qE|} \left( z \pm \frac{k^0}{qE} \right).$$

(11)

Also, $\nu = -(1 + i\mu)/2$, with the parameter $\mu$ given by

$$\mu = \frac{m^2 + |qB| (2n_L + 1)}{|qE|},$$

(12)

for the Landau levels $n_L = 0, 1, 2, \cdots$.

The incoming modes, $\phi^{(\pm)\text{in}}_k (x)$, satisfy the orthonormality conditions, defined via the Klein-Gordon inner product, $\langle \phi_1, \phi_2 \rangle = i \int d^3x (\phi_1^* \partial_t \phi_2 - \phi_2^* \partial_t \phi_1^*)$. Using the properties of the parabolic cylinder functions [38], it is easy to check that

$$\langle \phi^{(\pm)\text{in}}_k (x), \phi^{(\pm)\text{in}}_{k'} (x) \rangle = - \left\{ \left( \phi^{(-)\text{in}}_k (x) \right)^*, \left( \phi^{(-)\text{in}}_{k'} (x) \right)^* \right\} = \delta(k^0 - k'^0) \delta(k^x - k'^x) \delta_{nn'},$$

$$\langle \phi^{(+)\text{in}}_k (x), \left( \phi^{(-)\text{in}}_{k'} (x) \right)^* \rangle = 0.$$  

(13)

Similarly, we find the orthonormal outgoing modes for particles $\phi^{(p)\text{out}}_k \propto e^{-y^2/2} H_{n_L}(y+) [D_\nu(-\zeta_+)]^*$ and for antiparticles $\left[ \phi^{(a)\text{out}}_k \right]^* \propto e^{-y^2/2} H_{n_L}(y-) D_\nu(-\zeta_-)$. These modes also satisfy the orthonormality conditions in the same way as [13].
The incoming and the outgoing modes furnish two independent quantisation of the scalar field. These modes are related via the Bogoliubov transformation,
\[
\phi_{k}^{(+)} = \alpha_{k} \phi_{k}^{(+)} + \beta_{k} \left( \phi_{-k}^{(-)} \right)^{\ast},
\]
where \( \alpha_{k} \) and \( \beta_{k} \) are the Bogoliubov coefficients. The relation \((14)\) yields,
\[
D_{\nu}(\zeta_{+}) = \alpha_{k} \left[ D_{\nu}(\zeta_{+}) \right]^{\ast} + \beta_{k} D_{\nu}(\zeta_{+}),
\]
where \( [D_{\nu}(\zeta)]^{\ast} = D_{-\nu-1}(i\zeta) \). Using the relation \((38)\),
\[
D_{\nu}(\zeta) = e^{-i\pi \nu} D_{\nu}(-\zeta) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-\frac{i\pi(\nu+1)}{2}} D_{-\nu-1}(i\zeta),
\]
we obtain
\[
\alpha_{k} = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-\frac{i\pi(\nu+1)}{2}}, \quad \beta_{k} = e^{-i\pi \nu},
\]
which satisfy \( |\alpha_{k}|^{2} - |\beta_{k}|^{2} = 1 \). Employing the orthonormality conditions, we derive the transformations for the creation and annihilation operator as
\[
a_{k}^{\text{in}} = \alpha_{k} a_{k}^{\text{out}} - \beta_{k} b_{-k}^{\ast}, \quad b_{k}^{\text{in}} = -\beta_{k} a_{k}^{\text{out}} + \alpha_{k} b_{-k}. \tag{18}
\]

**IV. ENTANGLEMENT ENTROPY FOR THE VACUUM**

We consider first the vacuum state of incoming modes. The Hilbert space \( \mathcal{H} \) is constructed by the tensor product, \( \mathcal{H} = \prod_{k} \mathcal{H}_{k} \otimes \mathcal{H}_{-k} \), where \( \mathcal{H}_{k} \) and \( \mathcal{H}_{-k} \) are the Hilbert spaces of the modes of the particle and the antiparticle, respectively. The full ‘in’ vacuum state \( |0\rangle_{\text{in}} \) is described by
\[
|0\rangle_{\text{in}} = \prod_{k,-k} |0_{k}\rangle_{\text{in}} \otimes |0_{k}^{\ast}\rangle_{\text{in}} \equiv \prod_{k,-k} |0_{k}0_{-k}\rangle_{\text{in}},
\]
where
\[
a_{k}^{\text{in}} |0_{k}\rangle_{\text{in}} = b_{-k}^{\ast} |0_{k}\rangle_{\text{in}} = 0, \tag{20}
\]
and likewise for the ‘out’ states. The state \( |0_{k}0_{-k}\rangle_{\text{in}} \) can be expanded in terms of the ‘out’ states as
\[
|0_{k}0_{-k}\rangle_{\text{in}} = \sum_{n=0}^{\infty} C_{n_{k}}^{\ast} |n_{k}n_{-k}\rangle_{\text{out}}, \tag{21}
\]
by using the Schmidt decomposition. The normalisation, \( \sum_{n=0}^{\infty} |C_{n_k}^0|^2 = 1 \), yields

\[
\sum_{n=0}^{\infty} |C_{n_k}^0|^2 = 1.
\]

The properties of \( C_{n_k}^0 \) and the Bogoliubov transformation \( (18) \) yield the recurrence relation \( C_{n_k}^0 = (\beta_k/\alpha_k)C_{(n-1)k}^0 \), giving,

\[
C_{n_k}^0 = \left( \frac{\beta_k}{\alpha_k} \right)^n C_{0_k}^0,
\]  
(22)

as discussed in [30]. Using now \( (17), (22) \), we obtain

\[
|C_{0_k}^0| = \frac{1}{|\alpha_k|} = \frac{1}{\sqrt{2\pi}} \left| \Gamma(-\nu) e^{i\pi(1+\nu)/2} \right|.
\]  
(23)

Then \( (22) \) can be rewritten as

\[
C_{n_k}^0 = \sqrt{1 - |\gamma|^2 e^{i\theta_0}},
\]  
(24)

where

\[
\gamma = \frac{1}{\sqrt{1 + e^{i2\pi}}} \exp \left[ i \left( \frac{3\pi}{4} + \arg \Gamma(-\nu) \right) \right], \quad \theta_0 = \frac{\pi}{4} + \arg \Gamma(-\nu) + \phi_c,
\]  
(25)

where \( \phi_c \) is a constant. Note that \( |\gamma| < 1/\sqrt{2} \) when \( \mu > 0 \), and hence \( C_{n_k}^0 \) approaches 0 as the label \( n \) increases.

Let us comment on other features of \( C_{n_k}^0 \). First, \( C_{n_k}^0 \) depends on only the variable \( \mu \), Eq. \( (12) \). Thus \( C_{n_k}^0 \) reflects the charge and the mass but not the momentum \( k^0 \) and \( k^x \) as the feature of the (anti)particle. Second, when \( \mu \to \infty \), \( |C_{n_k}^0| \to \delta_{n0} \) since \( |\gamma| \) approaches 0, and hence \( (21) \) becomes \( |0_{0}0_{-k}\rangle_{in} \to C_{0_k}^0 |0_{0}0_{-k}\rangle_{out} \), where the difference between the left- and right-hand side is just the phase factor. Third, when \( \mu \to 0 \), \( |\gamma| \) approaches \( 1/\sqrt{2} \), and hence \( |C_{n_k}^0| \to 2^{-(n+1)/2} \).

The density matrix for the ‘in’ vacuum state \( |0_{0}0_{-k}\rangle_{in} \) is given by \( \rho^{(v)} = |0_{0}0_{-k}\rangle_{in}(0_{0}0_{-k}| \), which is a pure state. Employing \( (21) \), we obtain the reduced density matrix for the particle as

\[
\rho_k = \text{Tr}_{-k} \rho^{(v)} = \sum_{n=0}^{\infty} |C_{n_k}^0|^2 |n_k\rangle_{out} \langle n_k|, \text{ and hence the entanglement entropy, defined by (2), is given by}
\]

\[
S_k = -\text{Tr}_k \rho_k \log \rho_k = -|\beta_k|^2 \log |\beta_k|^2 + \left( 1 + |\beta_k|^2 \right) \log \left( 1 + |\beta_k|^2 \right).
\]  
(26)

Since we are dealing with a pure state, the entanglement entropies for the particle and antiparticle sectors satisfy \( S_k = S_{-k} \).
FIG. 1. The entanglement entropy $S_k$ of the state $|0_k0_{-k}\rangle$ vs. the parameter $\mu$ given by Eq. (12). Considering $m^2$ and $E$ to be fixed, $S_k$ is maximum in the small $\mu$ ($|B|, n_L \ll |E|$), where the number of outgoing particle is unity, and vanishing for large $\mu$ ($E \ll n_L, |B|$), where the number of outgoing particle is zero. See the text for discussions.

We obtain the $\mu$-dependence of $S_k$ as shown in Fig. 1. Thus $S_k$ decreases as $|B|$ or $n_L$ increases (assuming fixed $|E|$ and $m$ values). The entanglement entropy is maximum, $S_k = 2$, in the limit $\mu \to 0$ ($m^2, |B|, n_L \ll |E|$). The entanglement decreases as $\mu$ increases, and it vanishes in the large limit of $\mu$ ($|E| \ll m^2$ or $|E| \ll |B|$), where the reduced density matrix $\rho_k$ returns to the incoming pure state. This corresponds to the suppression of pair creation due to the stabilisation of the vacuum with increasing $B$ and or the Landau level.

V. MUTUAL INFORMATION AND LOGARITHMIC NEGATIVITY IN SYSTEMS OF TWO SCALAR FIELDS

Let us now consider systems which are constructed by two complex scalar fields. There are two species of (anti)particles, which do not interact with each other. The total Hilbert space $\mathcal{H}$ is given by, $\mathcal{H} = \prod_{s,k} \mathcal{H}_s \otimes \mathcal{H}_{-s} \otimes \mathcal{H}_k \otimes \mathcal{H}_{-k}$, where $s$ and $k$ stand for the two species of scalar fields. We assume that these two scalar fields have the same charge, but different masses are allowed.

We shall focus on the maximally entangled states for the incoming states of the (anti)particles. Now, the gauge transformation properties of the wavefunctional of a charged field in quantum electrodynamics puts a constraint on how one can prepare those states, as follows. The wavefunctionals corresponding to two states with different charge content will have different transformation properties under the local gauge transformation. Hence if we add two or more states to construct an entangled state, we must ensure that the charge content of each of these states are the same, so that the wavefunctional for the full state has a definite transformation property. This will be
reflected in the states \((27)\) and \((32)\) we work with.

### A. Single-charge state

Based upon the above argument, we consider a maximally entangled single-charge state, \(\rho^{(1)} = |\psi^{(1)}_{sk}\rangle\langle \psi^{(1)}_{sk}|\), which is a pure state, with

\[
|\psi^{(1)}_{sk}\rangle = \frac{|0_s0_{-s}; 1_k0_{-k}\rangle_{\text{in}} + |1_s0_{-s}; 0_k0_{-k}\rangle_{\text{in}}}{\sqrt{2}} \quad \tag{27}
\]

In our notation, the first (second) pair of entries appearing in the kets stands for the first (second) scalar. For a specific pair, the first (second) entry represents particle (antiparticle).

Using the expansion of the incoming vacuum, \((21)\), we rewrite \(|1_k0_{-k}\rangle_{\text{in}} = a^{\text{in}*}_k |0_k0_{-k}\rangle_{\text{in}}\) by the outgoing states as

\[
|1_k0_{-k}\rangle_{\text{in}} = \sum_{n=0}^{\infty} C^1_{nk} |(n + 1)k(n-k)\rangle_{\text{out}}, \quad \tag{28}
\]

where the coefficient \(C^1_{nk}\) is given by

\[
C^1_{nk} = \frac{\sqrt{n+1}}{\alpha_k} C^0_{nk} = \left(1 - |\gamma_k|^2\right) \gamma_k^n e^{i\theta_\nu}, \quad \theta_\nu = 2 \left(\frac{\pi}{4} + \arg(\Gamma(-\nu))\right) + \phi_c \quad \tag{29}
\]

Here, we write the label \(k\) for \(\gamma_k\), since it depends on the mass and the charge of the particle which has the momentum \(k\). The features of \(C^1_{nk}\) are given in parallel with that of \(C^0_{nk}\) in the preceding section.

The coefficients \(C^1_{nk}\) depend on only the variable \(\mu\), and hence \(C^1_{nk}\) reflects the charge and the mass but not the momentum \(k^0\) and \(k^x\). When \(\mu \to \infty\), we obtain \(|C^1_{nk}| \to \delta_{n0}\), and hence \((28)\) becomes \(|1_k0_{-k}\rangle_{\text{in}} \to C^1_{nk} |1_k0_{-k}\rangle_{\text{out}}\), where the difference between the left- and right-hand side is just a phase factor. When \(\mu \to 0\), we have \(|C^1_{nk}| \to \sqrt{n+1}/\sqrt{2(n+2)}\).

Using the relations \((21)\) and \((28)\), the single-particle ‘in’ state \(|\psi^{(1)}_{sk}\rangle\) can be written in terms of the ‘out’ states.

#### 1. Quantum Mutual information

Here we compute the quantum mutual information defined by \((3)\), corresponding to the state in Eq. \((27)\). We shall focus on two reduced density matrices that characterise the particle-particle and also the particle-antiparticle correlations between the two scalar fields.
FIG. 2. The quantum mutual information of $\rho_{s,k}^{(1)}$ vs. $\mu(k)$ (i.e., the particle-particle sector) for each value of $\Delta$, corresponding to the single charge state in Eq. (27). All lines approach $S(\rho_s^{(1)}) + S(\rho_k^{(1)}) = 2$ as $\mu(k)$ increases.

Let us start with the particle-particle correlation. The reduced density matrix is given by $\rho_{s,k}^{(1)} = \text{Tr}_{-s,-k}\rho^{(1)}$ and is written in terms of the ‘out’ states as

$$
\rho_{s,k}^{(1)} = \frac{1}{2} \sum_{n, \ell=0}^{\infty} \left( C_{\ell_s}^0 C_{n_k}^1 |\ell_s(n+1)_k\rangle_{\text{out}} + C_{\ell_s}^1 C_{n_k}^0 |(\ell + 1)_s n_k\rangle_{\text{out}} \right) \times (\text{h.c.}),
$$

where (h.c.) stands for the Hermitian conjugate of the first parenthesis. Eq. (30) is symmetric in $s$ and $k$. Even though (30) is no longer a pure state, it becomes pure and maximally entangled again in the limit of large $\mu(k)$ and $\mu(s)$, and the von Neumann entropy $S(\rho_{s,k}^{(1)})$ becomes vanishing.

The quantum mutual information is defined by $I(\rho_{s,k}^{(1)}) = S(\rho_s^{(1)}) + S(\rho_k^{(1)}) - S(\rho_{s,k}^{(1)})$, where $\rho_s^{(1)} = \text{Tr}_k \rho_{s,k}^{(1)}$ and $\rho_k^{(1)} = \text{Tr}_s \rho_{s,k}^{(1)}$. The summation in (30) converges rapidly and hence for numerical purpose, we replace the infinity with a finite but large $n$- and $\ell$-value. We thus obtain the $\mu$-dependence of $I(\rho_{s,k}^{(1)})$, shown in Fig. 2. Here we have defined

$$
\Delta \equiv \mu(s) - \mu(k),
$$

reflecting the mass difference between the fields.

Fig. 2 shows that $I(\rho_{s,k}^{(1)})$ approaches its maximum value, $S(\rho_s^{(1)}) + S(\rho_k^{(1)}) = 2$, as $\mu(k)$ increases, showing the correlation of the particle-particle sector is maximum for the large $\mu$ limit of (30).

When $\mu$ is small, the lines for the different values of $\Delta$ split, e.g., the mass difference of the two scalar fields can be estimated with fixed $E$ and $B$ in that region.

Next, we consider the particle-antiparticle correlation. We write the reduced density matrix, $\rho_{s,-k}^{(1)} = \text{Tr}_{-s,k}\rho^{(1)}$, in terms of the outgoing modes as

$$
\rho_{s,-k}^{(1)} = \frac{1}{2} \sum_{n, \ell=0}^{\infty} \left( C_{\ell_s}^0 C_{(n-1)_k}^1 |\ell_s(n-1)_{-k}\rangle_{\text{out}} + C_{\ell_s}^1 C_{n_k}^0 |(\ell + 1)_s n_{-k}\rangle_{\text{out}} \right) \times (\text{h.c.}),
$$
FIG. 3. The quantum mutual informations of $\rho_{s,-k}^{(1)}$ vs. $\mu(k)$ for each value of $\Delta$ (i.e., the particle-antiparticle sector), corresponding to the single charge state in Eq. (27). All lines approach zero, since $S(\rho_{s,-k}^{(1)}) = S(\rho_s^{(1)}) + S(\rho_{-k}^{(1)})$ in the limit of large $\mu(k)$. Note its qualitative difference from Fig. 2.

with the requirement $C^1_{(-1)k} = 0$. We also define $\rho_{-k}^{(1)} = \text{Tr}_s \rho_{s,-k}$. Unlike the case of $\rho_{s,k}^{(1)}$, Eq. (31) is not symmetric in $s$ and $-k$. Note that $\rho_{s,-k}^{(1)}$ becomes a product state $\rho_s^{(1)} \otimes \rho_{-k}^{(1)}$ in the limit of large $\mu(k)$ and $\mu(s)$, and consequently the mutual information becomes zero, as discussed in Section II B.

Fig. 3 shows that the mutual information of $\rho_{s,-k}^{(1)}$ approaches zero as $\mu(k)$ increases. This corresponds to the fact that for large $\mu(k)$ values, the Bogoliubov transformation becomes trivial, and the ‘out’ and ‘in’ states coincide modulo some trivial phase factors, as discussed in Section V A. However, (27) has no antiparticle content in it, resulting in a vanishing mutual information between the particle-antiparticle sector in this limit. On the other hand, for smaller $\mu$ values, the lines split as Fig. 2. However, we note the qualitative differences between Fig. 2 and Fig. 3.

2. Logarithmic negativity

Let us now compute the logarithmic negativity defined by (5), first for the particle-particle sector, $\rho_{s,k}^{(1)}$. The $\mu(k)$-dependence of the logarithmic negativity of $\rho_{s,k}^{(1)}$ is shown in Fig. 4 for different values of $\Delta$. The logarithmic negativity increases as $\mu(k)$ increases and for large $\mu(k)$-values, all the lines converge to unity. This is because $\rho_{s,k}^{(1)}$ approaches the maximally entangled pure state for large $\mu(k)$, as discussed below (30). Accordingly, it has the same eigenvalues as that of incoming modes, so that $\mathcal{L}_N \rightarrow \log_2 (4 \times 1/2) = 1$. For the particle-antiparticle sector however, we find that the logarithmic negativity is vanishingly small, $\mathcal{L}_N(\rho_{s,k}^{(1)}) \lesssim \mathcal{O}(10^{-15})$, for all $\mu(k)$ values, showing once again its qualitative difference with the above case.

1 Due to similar reason, we expect the mutual information for the antiparticle-antiparticle sector to rapidly vanish as we increase $\mu(k)$. For this qualitative similarity, we do not investigate this case explicitly here.
We shall consider another example of entangled state below and will see its qualitative differences with that of (27).

**B. Zero-charge state**

Keeping in mind the discussion made at the end of Section V, we now consider an entangled state that has zero total charge, \( \rho^{(0)} = |\psi_{sk}^{(0)}\rangle \langle \psi_{sk}^{(0)}| \), which is a pure state, with

\[
|\psi_{sk}^{(0)}\rangle = \frac{1}{\sqrt{3}} \left( |0_s0_s;0_k0_k\rangle_{\text{in}} + |1_s0_s;0_k1_k\rangle_{\text{in}} + |0_s1_s;1_k0_k\rangle_{\text{in}} \right)
\]

(32)

Using Eq. (21), we rewrite \( |0_k1_{-k}\rangle_{\text{in}} = b_{-k}^{\dagger} |0_k0_{-k}\rangle_{\text{in}} \) as

\[
|0_k1_{-k}\rangle_{\text{in}} = \sum_{n=0}^{\infty} C^1_{n_k} |n_k(n+1)_{-k}\rangle_{\text{out}} ,
\]

(33)

in parallel with (28). Employing now (28) and (33), we can rewrite \( |\psi_{sk}^{(0)}\rangle \) in terms of the ‘out’ states.

**1. Quantum Mutual information**

For the particle-particle correlations, the reduced density matrix is given by

\[
\rho^{(0)}_{s,k} = \frac{1}{3} \sum_{n,\ell=0}^{\infty} \left( C^0_{\ell_s} C^0_{n_k} |\ell_s n_k\rangle_{\text{out}} + C^1_{(\ell-1)_s} C^1_{n_k} |(\ell - 1)_{s}(n+1)_{k}\rangle_{\text{out}} + C^1_{\ell_s} C^1_{(n-1)_k} |(\ell + 1)_{s}(n - 1)_{k}\rangle_{\text{out}} \right) \times (\text{h.c.}) ,
\]

(34)
FIG. 5. The quantum mutual informations of $\rho_{s,k}^{(0)}$ vs. $\mu(k)$ (i.e., the particle-particle sector) for each value of $\Delta$, corresponding to the state carrying zero net charge, Eq. (32). The all lines approach 0.252 in the limit of large $\mu(k)$. When $\mu$ is smaller, the smaller $\Delta$ gives the larger value of the mutual information from $\Delta = 0$ to $\Delta = 2$; however, the hierarchy of the curves is reversed between $\Delta = 2$ and $\Delta = 3$.

which is symmetric in $s$ and $k$. This originates from the equality in number of particles and antiparticles in each of the states appearing on the right hand side of Eq. (32). Unlike the case of (27), the state (34) becomes classically correlated and mixed for large $\mu(k)$ and $\mu(s)$.

The $\mu(k)$-dependence of the quantum mutual information corresponding to Eq. (34) for different values of $\Delta$ is depicted in Fig. 5. The mutual information of $\rho_{s,k}^{(0)}$ approach $-\frac{4}{3} \log 2 + \log 3 \approx 0.252$, as $\mu(k)$ increases. Also, the mutual information are maximum in the limit of small $\mu$’s and they also have minima. Also, when $\mu$ is smaller, the smaller $\Delta$ gives the larger value of the mutual information from $\Delta = 0$ to $\Delta = 2$; however, the hierarchy of the curves is reversed between $\Delta = 2$ and $\Delta = 3$, manifesting the non-linearity of the system. Clearly, such characteristics are thoroughly different from that of the single charge state, Eq. (27).

Since the zero charge state (32) has a symmetry in the number of particles and antiparticles, we expect the quantum mutual information for the antiparticle-antiparticle sector to behave qualitatively similarly to that of the particle-particle sector. Hence we shall not pursue the antiparticle-antiparticle sector explicitly here.

For the particle-antiparticle correlation between the two scalar fields, the reduced density matrix is given by

$$
\rho_{s,-k}^{(0)} = \frac{1}{3} \sum_{n,\ell=0}^{\infty} \left( C_{s,n-k}^{(0)} C_{s}^{(0)} |_{\ell s,n-k}^{\text{out}} + C_{(\ell-1)s}^{(1)} C_{(n-1)k}^{(1)} |_{(\ell-1)s,n-1-k}^{\text{out}} \\
+ C_{(\ell+1)s}^{(1)} C_{n-k}^{(1)} |_{(\ell+1)s,n+1-k}^{\text{out}} \right) \times \langle \text{h.c.} \rangle
$$

(35)

Fig. 6 shows that the mutual informations of $\rho_{s,-k}^{(0)}$ approach a specific value as $\mu(k)$ increases.
FIG. 6. The quantum mutual information of $\rho^{(0)}_{s,-k}$ vs. $\mu(k)$ (i.e., the particle-antiparticle sector) for each value of $\Delta$, corresponding to the state in Eq. (32). For large $\mu(k)$, the logarithmic negativities approach a finite value $\sim 1.29$. The curves for $\Delta = 0$ is less than $\Delta = 0.5$ in the almost all range of $\mu(k)$. However in the limit of small $\mu$‘s, the hierarchy of these curves is reversed.

In the large limit of $\mu(k)$, the eigenvalues of $\rho^{(0)}_{s,k}$ are given by $(1 \pm \sqrt{5}/3)/2$. In this limit, the quantum mutual information equals 1.29. As depicted, only for $\Delta = 0$, we have a minimum in Fig 6. In addition, the curve for $\Delta = 0$ is less than that of $\Delta = 0.5$ in the almost all range of $\mu(k)$; however in the limit of small $\mu$, the hierarchy of these curves is reversed.

2. Logarithmic negativity

Finally, we come to the logarithmic negativity corresponding to the state in Eq. (32). For the particle-particle sector, Fig. 7 shows that the logarithmic negativities have maxima, whose magnitude decreases as $\Delta$ increases and moves to left. The curves approach zero in the large $\mu$ limit. This is because in this limit the reduced density matrix, as we stated earlier, becomes classically correlated, which implies $(\rho^{(0)}_{s,k})^{T_s} \rightarrow \rho^{(0)}_{s,k}$ and yields a vanishing logarithmic negativity.
FIG. 8. The logarithmic negativities of $\rho_{s,-k}^{(0)}$ vs. $\mu(k)$ (i.e., the particle-antiparticle sector) for each value of $\Delta$, corresponding to the zero net charge state in Eq. (32). In the limit of large $\mu(k)$, the logarithmic negativities approach $\log 5/3 \approx 0.74$.

Fig. 8 shows the variation of the logarithmic negativity of $\rho_{s,-k}^{(0)}$ (the particle-antiparticle sector) with respect to $\mu(k)$. They approach a finite value as $\mu(k)$ increases. In the large limit of $\mu(k)$, the summation of all eigenvalues of $(\rho_{s,-k}^{(0)})^T$ is $5/3$, and accordingly the logarithmic negativity approach $\log 5/3 \approx 0.74$. Note also that for $\Delta = 0$, the derivative of logarithmic negativity with respect to $\mu(k)$ shows a gap around $\mu(k) = 0.08$ and $\mu(k) = 0.25$.

From the symmetric structure of the number of particles and antiparticles of Eq. (32), we expect the antiparticle-antiparticle sector to behave qualitatively similarly to that of the particle-particle sector, Fig. 6 and Fig. 8, and hence we have not pursued this sector explicitly here. Interestingly, we also note from Fig. 5, Fig. 6, Fig. 7, and Fig. 8 that for given $\Delta$ and $\mu(k)$ values, the numerical values of the correlation quantities for the particle-particle sector are always greater than that of the particle-antiparticle sector. On the other hand, in Fig. 2, Fig. 3, and Fig. 4 corresponding to the single charged state Eq. (27), the particle-particle sector’s correlations are always higher compared to the particle-antiparticle sector.

VI. SUMMARY AND OUTLOOK

We now summarise our results. The chief motivation of this work was to quantify the effect of a background magnetic field on the quantum correlations between the Schwinger pairs. We have studied the vacuum entanglement entropy (Section IV), the quantum mutual information and logarithmic negativity for states with single and zero electric charges respectively in Section V A, Section V B. We also have emphasised the qualitative differences in the behaviour of the information quantities corresponding to these states. Extension of these results to the Rindler and the inflationary backgrounds would be interesting.
Finally, we note that in all the plots the various information quantities converge to some specific points for sufficiently large $\mu(k)$ values. Assuming the electric field to be constant, a large $\mu(k)$ corresponds to large values of the magnetic field or the Landau level, Eq. (12). At this limit the Bogoliubov transformation becomes trivial and an ‘out’ state becomes coincident with the ‘in’ state, modulo some trivial phase factor (cf., the discussion below Eq. (25)). Intuitively, then it seems possible that the degraded quantum correlation between two entangled states in an accelerated frame, e.g. [14], might be restored (for charged fields) via the application of a background magnetic field, as follows. The magnetic Lorentz force, $q\vec{v} \times \vec{B}$, acts in the same direction for the particle and antiparticle initially moving in the opposite direction after the pair creation. An electric field or background spacetime curvature/acceleration do the opposite effect by moving the created pairs away, e.g. [39]. Thus to the best of our understanding, it seems logical to expect that the particle-antiparticle pair creation causing the entanglement degradation in Rindler frame, will get diminished in the presence of a ‘sufficiently strong’ magnetic field. This effect can in particular be relevant for a black hole endowed with a strong magnetic field in its exterior. We hope to come back to this issue in our future work.

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