Split spin-squeezed Bose–Einstein condensates

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Abstract

We investigate and model a method for producing entanglement between two spatially separated Bose–Einstein condensates (BECs). In our approach, a spin-polarized BEC is squeezed using a \( (S^z)^2 \) interaction, then are split into two separate clouds. After the split, we consider that the particle number in each cloud collapses to a fixed number. We show that this procedure is equivalent to applying an interaction corresponding to squeezing each cloud individually plus an entangling operation. We analyze the system’s inter-well entanglement properties and show that it can be detected using correlation-based entanglement criteria. The nature of the states is illustrated by Wigner functions and have the form of a correlated squeezed state. The conditional Wigner function shows high degrees of non-classicality for dimensionless squeezing times beyond \( 1/\sqrt{N} \), where \( N \) is the number of particles per BEC.

1. Introduction

Entanglement has been traditionally considered to be a fragile quantum phenomenon which only happens in the microscopic world. As the state of the art improves, it has become increasingly realizable in the macroscopic world, and thereby accessible to future quantum technologies. One of the first demonstrations of entanglement between remote macroscopic objects was achieved between atomic ensembles containing \( 10^{12} \) atoms, eventually realizing quantum teleportation \([1–3]\). The weak nonlinearity and quantum correlations between fields of light have also been investigated to generate macroscopic entanglement \([4, 5]\). Such entanglement has been considered for being used in various quantum technologies such as metrology and quantum computing \([6–8]\). In terms of long distance entanglement, the current records for entanglement are using space-based distribution methods, between photons \([9, 10]\). Entanglement in the mesoscopic regime has been experimentally realized with superconductors, which is now considered to be one of the leading candidates for quantum computing \([11–14]\). Micromechanical resonators also exhibit entanglement in the mesoscopic regime, where micrometer-size structures display quantum mechanical behavior \([15–18]\).

For Bose–Einstein condensates (BECs), most studies until recently have been focused on creating entanglement within a single atomic cloud. Spin squeezing, as a means to reduce quantum noise and improve measurement precision beyond the standard quantum limit \([19, 20]\), can be also applied for the generation of entanglement in BECs \([21–24]\). The creation of many-particle entanglement in one BEC ensemble has been investigated thoroughly \([21, 23, 25–27]\), which is localized in a single spatial location. However, it is now known that entanglement can be furthermore subdivided into various categories, including steerable and non-local entanglement \([28]\).

These categories are in order of increasing quantum correlation, such that non-local entanglement implies both steerability and entanglement, but it is possible to have entangled states that are neither steerable nor...
2. Split spin-squeezed BECs

2.1. The model

We now construct a model for the splitting of a spin–squeezed BEC illustrated in figure 1. Our model has general similarities to that considered in [45], but with differing treatments of the splitting procedure. The basic assumptions are that each trap in figure 1 can be approximated by two bosonic modes. Thus before the BEC is

Figure 1. Schematic procedure for realizing split–squeezed Bose–Einstein condensates (BECs). (a) An initial spin coherent state in a single trap is spin–squeezed, producing correlations (dashed lines) between the bosons. (b) The bosons are then spatially separated into two ensembles via a beam splitting interaction. The atoms in the cloud remain correlated, both within and between the two ensembles.

non–local. Bell correlations within a single BEC have been observed, showing that the strength of the quantum correlations are of the strongest variety [29].

Recently, several experiments showing entanglement between spatially separated parts of a single BEC were reported [30–32]. In these works, the images of a single BEC are partitioned into regions, which define the subsystems. In this sense the partition is not physical but imposed in the process of entanglement detection: there are no pairs of BECs which could be individually addressed, as many quantum information tasks would require. To date, there has not been an experimental realization of entanglement between two completely spatially distinct BECs. Several theoretical studies have analyzed the entanglement created by a \( S^z S^z \) type interaction, pointing to a complex entanglement structure with fractal characteristics [33, 34]. Numerous proposal have been made for generating such entanglement, based on cold atomic collisions [35], atom–light interactions [36–40], and Rydberg excitations [41]. Several proposals for quantum information applications have been made based on such entanglement [38, 42, 43].

In this paper, we analyze an approach for creating entanglement between spatially distinct BECs. The situation that we consider is shown in figure 1. First a maximally \( S^z \)-polarized spin coherent state is squeezed by one–axis twisting [44]. The one–axis twisting produces multipartite entanglement that is present between all the atoms in the BEC. The BEC is then spatially split into two ensembles. The entanglement that is produced in the squeezing operation is inherited by the two BECs, and remain entangled across the wells after the split. In this work we study the quantum correlations between the two BECs obtained from this protocol, and discuss methods for detecting entanglement using correlations of observables. Recently, Oudot, Sangouard, and co–workers examined entanglement witnesses for such entangled BECs in the presence of local white noise [45]. Our physical model differs from this work and those experimental results in [30–32] as we consider the spatial separations to be large enough such that particle number superpositions on the left and right wells collapse to a fixed number. This is reasonable from the perspective that quantum tunneling should be highly suppressed for large splitting separations. In addition to the different treatment of the split–squeezed procedure, we also analyze the non–classicality and non–Gaussianity of the quantum states that are generated in this system by reconstructing the Wigner functions for the states at various squeezing times. This helps to visualize the nature of the states to a better degree. Furthermore, we compare several correlations based methods for detecting entanglement, and find that the Giovannetti et al [46] and covariance matrix [47] methods provide a powerful and efficient method for achieving this. We also show that our system allows for correlations stronger than entanglement as detected by a Einstein–Podolsky–Rosen (EPR) steering criterion.

This paper is structured as follows: in section 2, we give a detailed derivation how the process of squeezing and splitting is equivalent to an entangling interaction between two clouds. In section 3, we discuss the basic properties of the split spin–squeezed BEC, including the degree of entanglement and its non–Gaussian characteristics represented by the Wigner function. In section 4 we consider how to detect the entanglement in practice by using several different approaches: the widely used Duan–Giedke–Cirac–Zoller criterion [48] which is based on the variance of a pair of EPR-type operators, the covariance matrix formalism [47, 49, 50], and the criterion of Giovannetti et al [46]. We also show that EPR steering can be observed in the same system in section 5. The conclusions are given in section 6.
split, there are two bosonic modes, consisting of the spatial ground state for the two populated hyperfine states. After the BEC is split, there are four modes, corresponding to two from each well. Further details of this treatment can be found in [7, 8, 25].

First let us introduce some notation. We denote a general spin coherent state as

$$|\alpha, \beta\rangle = \frac{1}{\sqrt{N!}}(\alpha a^\dagger + \beta b^\dagger)^N|0\rangle,$$

where $a^\dagger$, $b^\dagger$ are bosonic creation operators of the two hyperfine spin states respectively, and $\alpha$, $\beta$ are arbitrary complex coefficients satisfying $|\alpha|^2 + |\beta|^2 = 1$. The Schwinger boson (total spin) operators are defined as

$$S^x = a^\dagger b + b^\dagger a, \quad S^y = i(a^\dagger b - b^\dagger a), \quad S^z = a^\dagger a - b^\dagger b,$$

which obey the commutation relations $[S^i, S^j] = i\epsilon_{ijk}S^k$, where $\epsilon_{ijk}$ is the Levi-Civita symbol.

Initially the BEC is prepared in a spin coherent state, where all $N$ atoms occupy the same spin state $a$. After that, we rotate it to a maximally $S^x$-polarized spin coherent state, according to

$$|\alpha, \beta\rangle \rightarrow e^{iS^x\pi/4}|\alpha, \beta\rangle = \frac{1}{\sqrt{N!}}(\alpha a^\dagger + \beta b^\dagger)^N|0\rangle.$$

This is then squeezed using the one-axis twisting evolution [44], corresponding to

$$e^{iS^x\pi t}|\alpha, \beta\rangle \rightarrow e^{i(S^x\pi/2)}|\alpha, \beta\rangle = \frac{1}{\sqrt{2^N N!}}\sum_{k=0}^N\binom{N}{k} e^{i(2k\pi/4)}|k\rangle,$$

where

$$|k\rangle = \frac{(a^\dagger)^k(b^\dagger)^{N-k}}{\sqrt{k!(N-k)!}}|0\rangle,$$

are the Fock states. Here $t$ is a dimensionless time of application of the squeezing operation. The atoms are then spatially separated into two parts. This is described by the transformation

$$a = \frac{1}{\sqrt{2}}(a_L + a_R), \quad b = \frac{1}{\sqrt{2}}(b_L + b_R),$$

where $a_{L,R}$ and $b_{L,R}$ are the left and right well modes for the two hyperfine states respectively. In the context of quantum optics, this is equivalent to a beam splitter operation which produces a coherent superposition of states in two outgoing modes [31]. In general a beam splitter can produce interference if the ingoing modes are occupied. In this case, the other input corresponds to the vacuum state with zero atoms. We then obtain the state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2^N N!}}\sum_{\text{all } N_L, N_R=0}^{N} \sum_{k=0}^{N-N_L} \sum_{k^*=0}^{N^*} \frac{\binom{N}{N_L} \binom{N-N_L}{k_L} \binom{N-N_L}{k_R}}{k_L! k_R! (N-L)! (N-R)!}$$

$$\times e^{i(2k_L + 2k_R - N^*)\pi/4} |k_L\rangle_{N_L} |k_R\rangle_{N-R},$$

where $N_L$ is the number of atoms in left ensemble. Here, the eigenstates of $S^x$ are

$$|k\rangle_{N_L} = \frac{(a^\dagger)^k(b^\dagger)^{N-L-k}}{\sqrt{k!(N-L-k)!}}|0\rangle,$$

where the subscripts take the values $i \in \{L, R \}$ and $N_i$ labels the number of atoms in the BEC.

The wavefunction (7) has a sum over $N_L$, the number of particles in the left well. This means that the particle number per well is not fixed, and exist in a superposition across both wells. This superposition arises due to the transformation (6) which produces coherence across the wells. However, for completely spatially separated BECs, it may be challenging to realize atoms delocalized over large distances. For example, there might be some inadvertent leakage of the particle number information which will collapse the superposition to a particular $N_L$ and $N_R$ on the left and right wells respectively. At the very least, there will always be a collapse during the measurement process, which will simultaneously measure particle number as well as spin. We thus assume that there is additionally some decoherence which can be taken into account by a
projection (number fixing measurement) onto a left-well particle number \(N_L\) performed by the operator

\[
P_{N_L} = \left( \sum_{k_L=0}^{N_L} |k_L\rangle_N \langle k_L| \right) \left( \sum_{k_R=0}^{N_R} |k_R\rangle_N \langle k_R| \right),
\]

where the number of atoms in the right well is

\[
N_R = N - N_L.
\]

Here the terms in the brackets give the identity operation in a fixed subspace of particle number for the left and right wells. After being projected, the normalized state can be expressed as

\[
|\Psi^{N_L}(t)\rangle = \frac{P_{N_L}|\Psi(t)\rangle}{\sqrt{p(N_L)}} = \frac{1}{\sqrt{2^N}} \sum_{k_L=0}^{N_L} \sum_{k_R=0}^{N_R} \left( \binom{N_L}{k_L} \binom{N_R}{k_R} \right) e^{i2k_L+2k_R-N^2t^2/2} |k_L\rangle_N |k_R\rangle_N,
\]

where the probability of obtaining the state with \(N_L\) atoms in the left well is given by

\[
p(N_L) = \langle \Psi(t)|P_{N_L}^2|\Psi(t)\rangle = \frac{1}{2^N} \binom{N}{N_L}.
\]

It is illuminating to write (11) in the form

\[
|\Psi^{N_L}(t)\rangle = e^{i(S_z^L + S_z^R)t} \left| \frac{1}{\sqrt{\binom{N}{N_L}}} \right\rangle_{N_L} \left| \frac{1}{\sqrt{\binom{N}{N_L}}} \right\rangle_{N_R} = \frac{1}{\sqrt{\binom{N}{N_L}}} \binom{N}{N_L} e^{i2k_L+2k_R-N^2t^2/2} |k_L\rangle_N |k_R\rangle_N,
\]

which shows that the final state can be viewed to have a fixed particle number in each well acted on by the Hamiltonian

\[
H_{\text{eff}} = (S_z^L + S_z^R)^2 = (S_z^L)^2 + (S_z^R)^2 + 2S_z^L S_z^R.
\]

This makes it clear that the split squeezing operation is equivalent to a combination of squeezing each of the split BECs individually, as well as applying the \(S_z^L S_z^R\) operation, which creates entanglement between the two BECs [33, 34].

The state (11) corresponds to the conditional state for a particular collapse of the atom number, to \(N_L\) atoms in the left well. Experimentally, this would correspond to post-selecting the measurement outcome with \(N_L\) atoms in the left well. After many runs of the experiment, the state that is measured is a probabilistic mixture of various outcomes. The mixed state that is obtained is

\[
\rho = \sum_{N_L=0}^{N} p(N_L)|\Psi^{N_L}(t)\rangle \langle \Psi^{N_L}(t)|.
\]

We will examine both the conditional state (11) and the average ensemble state (16) to examine their properties in section 3.

### 2.2. Equivalence of split-squeezing with squeeze-splitting

In (14) we found that the procedure of splitting a squeezed BEC was equivalent to splitting a condensate and then applying a squeezing on the total spin of the system (15). On first glance this is puzzling because the spin operators \(S_x^L, S_x^R\) do not commute with the splitting operation (6). To show that in fact the order of these operations do not matter, let us first define the splitting operators

\[
\Sigma^x_a = a_L^+ a_R + a_R^+ a_L,
\]

\[
\Sigma^y_a = -i(a_L^+ a_R + a_R^+ a_L),
\]

\[
\Sigma^z_b = b_L^+ b_R + b_R^+ b_L,
\]

\[
\Sigma^z_b = -i(b_L^+ b_R + i b_R^+ b_L).
\]

The operation of (6) then amounts to the transformation

\[
e^{i\Sigma^z_b \pi/4} b_R e^{-i\Sigma^z_b \pi/4} = \frac{1}{\sqrt{2}} (a_L + a_R)
\]

\[
e^{i\Sigma^z_b \pi/4} b_R e^{-i\Sigma^z_b \pi/4} = \frac{1}{\sqrt{2}} (b_L + b_R).
\]

Thus in this formulation we identify the original modes as right-well operators \(a = a_R, b = b_R\) and (6) is equivalent to (18). The state (11) can be thus written equivalently as
\[ |\Psi^{NL}(t)\rangle = \frac{P_{NL}e^{i\Sigma_{L}z/4}e^{i\Sigma_{R}z/4}e^{i\Sigma_{L}r/4}e^{i\Sigma_{R}r/4}|1 - \frac{1}{\sqrt{2}}i\rangle_R}{\sqrt{p(N_L)}}, \]

where the initial state is written in terms of the left-well operators \( a_L, b_L \).

The question here is whether the squeezing operation can be put after the splitting operation. Simply inverting the \( S^z_L \) and \( \Sigma^y_{a,b} \) operations is clearly not allowed since they do not commute:

\[ [S^z_L, \Sigma^y_a] = i\Sigma^x_a, \quad [S^z_L, \Sigma^y_b] = -i\Sigma^x_b. \]

The key point to notice is that since

\[ [S^z_L, \Sigma^y_a] = -i\Sigma^x_a, \quad [S^z_L, \Sigma^y_b] = i\Sigma^x_b \]

the combination of \( S^z_L + S^z_R \) does commute with the splitting operators

\[ [S^z_L + S^z_R, \Sigma^y_{a,b}] = 0. \]

Using this we can directly show the desired relation. Since there are no right-well operators in the state (19), we can introduce a additional factor of \( e^{i(S^z_L + 2S^z_R)t} \) just before the state, giving

\[ |\Psi^{NL}\rangle = \frac{P_{NL}e^{i\Sigma_{L}z/4}e^{i\Sigma_{R}z/4}e^{i\Sigma_{L}r/4}e^{i\Sigma_{R}r/4}|1 - \frac{1}{\sqrt{2}}i\rangle_R}{\sqrt{p(N_L)}}, \]

Now using (22), we can commute the squeezing operation to the end, giving

\[ |\Psi^{NL}\rangle = \frac{e^{i(S^z_L + S^z_R)t}P_{NL}e^{i\Sigma_{L}z/4}e^{i\Sigma_{R}z/4}e^{i\Sigma_{L}r/4}e^{i\Sigma_{R}r/4}|1 - \frac{1}{\sqrt{2}}i\rangle_R}{\sqrt{p(N_L)}}, \]

where we used the fact that

\[ [P_{NL}, S^x_L] = [P_{NL}, S^x_R] = 0 \]

to commute the squeezing operation to the end. This shows the desired relation.

\section*{3. Basic properties of the state}

We now examine the basic properties of the conditional state (11) and the averaged state (16). These can be efficiently characterized by entanglement between the collective spin in the left and right wells of the BECs, and the Wigner functions representing the states. These are examined in the following sections.

\subsection*{3.1. Inter-well entanglement}

As a measure of the entanglement between the left and right wells of the BECs, we use the logarithmic negativity \cite{Ollivier99, Horodecki01} defined as

\[ E(\rho) = \log_2\|\rho^{\text{tr}}\|, \]

where \( \rho^{\text{tr}} \) is the partial transpose of (16) with respect to left well and \( \|X\| = \text{Tr}[X] = \text{Tr}\sqrt{X^\dagger X} \) is the trace norm of an operator \( X \). For two BECs each with two components with particle numbers \( N_L \) and \( N_R \), a maximally entangled state is

\[ |\Psi_{\text{max}}\rangle = \sum_{k=0}^{\min(N_L, N_R)} |k\rangle_{N_L} |k\rangle_{N_R}. \]

This has a logarithmic negativity of

\[ E(|\Psi_{\text{max}}\rangle \langle \Psi_{\text{max}}|) = \log_2[\min(N_L, N_R) + 1]. \]

We note that (27) is only one out of \( (N + 1)^2 \) mutually orthogonal maximally entangled states. For example, there are 4 orthogonal Bell states in the case of two qubits which are all maximally entangled. The remaining states can be generated by local unitary transformations.

Figure 2 shows the inter-well entanglement as a function of the dimensionless squeezing time for both the conditional state (11) and the averaged state (16). The first thing that is evident is that all the curves have a similar form to the ‘devil’s crevasse’ entanglement as seen for a \( S^z_L S^z_R \) interaction \cite{Jing2018, Jing2019}. The entanglement increases monotonically until the characteristic time \( t = 1/2\sqrt{N} \), with a overall periodicity of \( T = \pi/4 \) and dips occurring at time that are a fractional multiple of this time, in agreement to a pure \( S^z_L S^z_R \) interaction. This is the expected behavior considering that the final interaction can be written in the form (13) which is a combination of this interaction and local squeezing on each well. Since inter-well entanglement is invariant under local
unitary transformations, the effect of the local squeezing \((S_z^1)^2 + (S_z^2)^2\) is not visible in the curves shown in figure 2. Thus the generation of entanglement between two wells is only due to the \(S_z^1 S_z^2\) interaction, which commutes with the local squeezing terms, explaining why the entanglement is equivalent to that seen in \([33,34]\).

The dominant effect of the different partitions of the particle number between the wells is to change the amplitude of the inter-well entanglement. This can be understood from the relation (28) which shows that the amount of entanglement is related to the dimensionality of the minimum of the particle number between the left and right well. The curves in figure 2 never reach the maximal value of entanglement (\(\log_2(N/2+1) \approx 2.6\)) here, as the type of entangled states that are generated from a \(S_z^1 S_z^2\) interaction do not given the maximum value (28) \([33]\). As expected, the amplitude of the entanglement for the case \(N_L = N_R = 5\) is the largest with respect to all other \(N\) partitions, and is symmetric around this midpoint. The entanglement is well-preserved despite the mixing between all possible partition of \(N\) in the state (16). This agrees with the general arguments of section 2.2, where the entangling operation commutes with the splitting and localization operations.

### 3.2. Wigner functions

The results of the previous section show that the split squeezed states are entangled, but do not clearly show the nature of the states, including what kind of quantum correlations are present between the two BECs. To visualize the state it is beneficial to calculate the Wigner functions \([34–56]\), defined for a two mode BEC. The Wigner function may give a representation of the quantum state as a quasiprobability distribution on the Bloch sphere. The Wigner function is defined as

\[
W(\theta, \varphi) = \sum_{k=0}^{2j} \sum_{q=-k}^{k} \rho_{kq} Y_{kq}(\theta, \varphi),
\]

where \(Y_{kq}(\theta, \varphi)\) are the spherical harmonics. Here, \(\rho_{kq}\) is defined as

\[
\rho_{kq} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} (-1)^{j-m} \sqrt{2k+1} \left( \begin{array}{ccc} j & k & j \\ m & q & m' \end{array} \right) \langle jm|\rho|m'\rangle,
\]

where \(\left( \begin{array}{ccc} j & k & j \\ m & q & m' \end{array} \right)\) is the Wigner 3j symbol. In the above, we have used notation conventional to the Dicke representation in terms of angular momentum eigenstates. This can be related to our previous notation according to the correspondence

\[
|jm\rangle = |k = j + m\rangle_{N=2j},
\]

where the left hand side are the Dicke state and the right hand side is defined according to (8).

The above representation allows us to represent any state of a two component BEC with a fixed particle number as a Wigner distribution. In our case, we have two BECs, each with two components, which allows for several possibilities of plotting a Wigner function. The most illustrative quantities are the marginal and conditional Wigner functions \([37]\). For the marginal Wigner function, the density matrix for one of the BECs is traced over, and the Wigner function is computed from the remaining density matrix. The reduced density matrix is a mixture of all the states that is present on a single BEC. For the conditional Wigner function, a projection onto a specific Fock state \(|k\rangle\) is made on one of the BECs, and normalized accordingly. Qualitatively, the difference between the two types distribution is that the marginal Wigner function gives an average state of one of the BECs disregarding the state of the other BEC. The conditional Wigner function gives the correlations that are present given a particular measurement \(|k\rangle\) (i.e. post-selection) on the other BEC. A similar argument was used to construct a visualization for the states in \([33]\). We note that it is also possible to plot the Wigner function for the two BEC system considering it as a single angular momentum state. Due to the equivalence of

![Figure 2. Inter-well entanglement in split squeezed Bose–Einstein condensates, as quantified by the logarithmic negativity. Solid line corresponds to the state (16), the dashed line and dotted lines represent the conditional state (13) with \(N_L = 5\) and \(N_L = 3\) respectively. The total number of atoms is \(N = 10\).](Image)
split-squeezing to squeeze-splitting, in this case we reproduce the well-known Wigner function for one-axis twisting \[25, 44\].

3.2.1. Marginal Wigner function

The reduced density matrix on the left well is

\[ \rho_L = \text{Tr}_R (|\Psi_N(t)\rangle\langle\Psi_N(t)|) . \]  

Inserting this state into (29), we obtain the marginal Wigner function

\[ W(\theta, \varphi) = \frac{1}{2\pi} \sum_{k=0}^{N_L} \sum_{q=-k}^{k} Y_{aq}(\theta, \varphi) \sum_{m', m'' = -N_L/2}^{N_L/2} \sqrt{2k + 1} \times \left( \begin{array}{c} N_L \\ N_L/2 + m' \\
 N_L/2 + m'' \end{array} \right) \left( \begin{array}{c} k \\ N_L/2 \\
 -m' \end{array} \right) \times e^{i(\nu_m m'' + m'' - N_L/2)(1 + e^{i(n_m m' - m' + \nu_m m' - m' - N_L/2)})} , \]

where the total spin on the left well was taken to be a fixed value \( j_L = \frac{N_L}{2} \).

In figure 3, we see the behavior of marginal Wigner function for different times. Initially, at \( t = 0 \), the state is a spin coherent state located at \( \theta = \pi/2, \varphi = 0 \). With increasing time, the coherent state (figure 3(a)) evolves into a squeezed state (figure 3(b)), which is expected due to the single site squeezing terms of the effective Hamiltonian in (15). For example, for the left BEC, the effective Hamiltonian contains a squeezing term \( S_L^2 \) which produces the characteristic diagonal distribution. For longer times this becomes elongated even more (figure 3(c)). The width along the squeezing direction is however broader than what would be obtained from a genuine one-axis twisting dynamics, due to the averaging obtained by tracing over the BEC in the right well. After the characteristic time \( t \sim 1/\sqrt{N} \), the state transforms into a non-Gaussian state, where the elongation starts to wrap around the whole Bloch sphere. At \( t = \pi/8 \), the state evolves into two coherent states located at \( \varphi = 0 \) and \( \varphi = \pi \) respectively, characteristic of a Schrödinger cat state (figure 3(d)). At this point the state can be written

\[ |\Psi_N(t = \pi/8)\rangle = \frac{e^{i\pi/4}}{\sqrt{2}} \left( \left| \frac{1}{\sqrt{2}} \right. , \frac{1}{\sqrt{2}} \right)_L \left| \frac{1}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right)_R - i(-1)^{N_L + N_R}/2 \left| \frac{-1}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right)_L \left| \frac{-1}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right)_R \right) . \]  

Figure 3. Marginal Wigner function for the split spin–squeezed state at various interaction times. (a) \( t = 0 \), (b) \( t = 1/N \), (c) \( t = 1/2\sqrt{N} \), (d) \( t = \pi/8 \). In all plots, the total number of atoms is \( N = 20 \), and the numbers of atoms in each ensemble are \( N_L = N_R = 10 \).
which is only valid for even \( N_{L}, N_{R} \). Tracing out the BEC in the right well thus gives the state

\[
\rho_{L}(t = \pi/8) = \frac{1}{2} \left( \left| \frac{1}{\sqrt{2}} \right\rangle \left\langle \frac{1}{\sqrt{2}} \right|_{L} + \left| -\frac{1}{\sqrt{2}} \right\rangle \left\langle -\frac{1}{\sqrt{2}} \right|_{L} \right),
\]

which is a mixture of the states at opposite ends of the Bloch sphere, matching with the distribution in figure 3(d).

### 3.2.2. Conditional Wigner function

The conditional Wigner function is obtained by projecting the final state (11) onto different \( |k_{R}\rangle \) states.

The state after projection is written as

\[
P_{k_{R}}|W_{N}(t)\rangle = \frac{1}{\sqrt{2^{N}}}|k_{R}\rangle e^{i(S_{z}^{L})^{T}t} e^{i(S_{z}^{R})^{T}t} \times \left| \begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right\rangle_{L} \left| \begin{array}{c} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{array} \right\rangle_{L} \right|_{N_{L}} e^{i(2k_{R} e^{-N_{R} t})/\sqrt{2}} - e^{-i(2k_{R} e^{-N_{R} t})/\sqrt{2}},
\]

where \( P_{k_{R}} = |k_{R}\rangle_{N_{L}} \langle k_{R}|_{N_{L}} \) is the projector onto the Fock states of the right well and we used (14). Together with (36) and (29), we thus obtain

\[
W_{k_{R}}(\theta, \varphi) = \frac{1}{\tilde{N}} \sum_{k=0}^{N_{L}} \sum_{q=-k}^{k} Y_{qk}(\theta, \varphi) \sum_{m_{L}, m_{L}'}^{N_{L}/2} \sum_{m_{R}, m_{R}'}^{N_{R}/2} \sqrt{2k + 1} \times \left| \begin{array}{c} N_{L} \\ N_{L}/2 + m_{L} \end{array} \right\rangle_{L} \left| \begin{array}{c} N_{L}/2 \end{array} \right\rangle_{L} \times \left| \begin{array}{c} N_{R} \\ k_{R} \end{array} \right\rangle_{R} e^{i(4m_{L} - m_{R})/2} \times \left| \begin{array}{c} -m_{L} \\ q \end{array} \right\rangle_{L} \left| \begin{array}{c} m_{L}' \end{array} \right\rangle_{L} \times \left| \begin{array}{c} N_{L}/2 \\ N_{L}/2 + m_{L}' \end{array} \right\rangle_{L} \left| \begin{array}{c} -N_{L}/2 - m_{L}' \end{array} \right\rangle_{L},
\]

where \( \tilde{N} \) is the normalization factor and again we took \( j_{L} = \frac{N_{L}}{2} \).

Figures 4 and 5(a) shows the conditional Wigner functions for various projected states \( |k_{R}\rangle \) for the right well. We see that for all the projected values the Wigner functions take the form of one-axis squeezing \((S_{z}^{R})^{2}\), with the characteristic diagonal distribution. Unlike the marginal Wigner function, the width of the distributions are narrowed, displaying genuine squeezing. This can be attributed to the \((S_{z}^{R})^{2}\) term in (36) which produces the squeezing effect. In figure 4, we show that for various projected states \( |k_{R}\rangle \), the distributions are offset by various positions around the Bloch sphere, as can be seen by the arguments of the spin coherent state in (36). This arises because of the \( S_{z}^{L} S_{z}^{R} \) interaction produce a type of correlation where if one projects on an \( S_{z}^{L} \) eigenstate, the other BEC will have correlated rotations around the equator of the Bloch sphere as has been discussed in detail in [33]. This is exactly what is seen here, where the squeezed state distribution is displaced around the equator. Thus the integrated density corresponds to the probability of a particular measurement result \( k_{R} \) is obtained. The probability distribution is

\[
P(k_{R}) = Tr(\rho_{N} P_{k_{R}}) = \frac{1}{2^{N}} \sum_{k_{R}}^{N_{R}} \left| \begin{array}{c} N_{L} \\ k_{R} \end{array} \right\rangle \left| \begin{array}{c} N_{R} \\ k_{R} \end{array} \right\rangle e^{i(4m_{L} - m_{R})/2} \times \left| \begin{array}{c} -m_{L} \\ q \end{array} \right\rangle_{L} \left| \begin{array}{c} m_{L}' \end{array} \right\rangle_{L} \times \left| \begin{array}{c} N_{L}/2 \\ N_{L}/2 + m_{L}' \end{array} \right\rangle_{L} \left| \begin{array}{c} -N_{L}/2 - m_{L}' \end{array} \right\rangle_{L},
\]

In figure 5, we show the conditional Wigner distributions for the state (36) for various evolution times projected at \( k_{R} = N_{R}/2 \) with time evolution. The \( k_{R} = N_{R}/2 \) projection is chosen because it is the most...
The probable result according to (38). The behavior of the conditional Wigner function is consistent with that of marginal Wigner function. For early times it shows the characteristic squeezed distribution (figure 5(a)). For longer times we see greater amounts of squeezing, but also larger regions with a negative Wigner function (figures 5(b), (c)), indicating non-classical behavior [57]. This is in contrast with the marginal Wigner function, which is always positive as was seen in figure 3. According to Hudson’s theorem, only pure quantum states with non-negative Wigner functions shows Gaussianity [58, 59]. Therefore the positivity of Wigner function in this case does not necessarily mean the quantum state is always Gaussian. Finally, at \( t = \pi/8 \) we see the characteristic Schrödinger cat distribution, where there are two peaks at opposite sides of the Bloch sphere, as well as the stripes with alternating positive and negative regions. Similar distributions are seen in quantum optical states where there is a superposition of coherent states. The conditional Wigner function, corresponding to post-selecting certain measurement outcomes, exhibits more non-Gaussianity and higher degree of quantumness.

In summary, for short times, the Wigner function analysis give consistent results with what is expected from a Hamiltonian of the form in (15). The states show local squeezing on each BEC, and show the correlations that are expected of the type of entanglement discussed. For longer times, the Wigner functions show a more complex behavior which is not as easily understood. However, at \( t = \pi/8 \) the system shows the distributions characteristic of a Schrödinger cat state, in agreement to the expression (34).

4. Detection of inter-well entanglement

The type of the entangled state presented between the BEC is a rather complex many-body state. Thus although we have shown already in figure 2 that the inter-well entanglement between the two wells of the split squeezed BEC is always present except for times \( t = 0 \) modulo \( T = \pi/4 \), detecting it in an experimentally feasible manner is a non-trivial task. While the logarithmic negativity unambiguously shows the presence of inter-well entanglement, it requires tomography of full density matrix, which for a BEC is rather difficult due to the large number of degrees of freedom. For example, in [30] the total number of atoms was \( N \approx 500 \). Typically in an experiment, only low order expectation values of the total spins are observable. Thus the use of correlation-based entanglement criteria [46–50, 60], where a small number of observables are measured is desirable. In this section, we compare several types to see which gives the most sensitive detection of inter-well entanglement.

![Figure 5](image.png)
4.1. DGCZ entanglement criterion

We first calculate a DGCZ criterion for the split squeezed BEC. Since the spins are initially polarized in the $S^x$-direction, we may use a Holstein–Primakoff transformation to treat the other spins as approximate position and momentum operators

$$X_i \approx \frac{S_i^x}{\sqrt{2N_i}},$$
$$P_i \approx \frac{S_i^y}{\sqrt{2N_i}},$$

where $i \in \{L, R\}$ and $S_i^z \approx N_i$ is treated classically. As derived in [33], these variables can be used to create EPR-like variances which we expect to have suppressed noise fluctuations. Any separable state then obeys the inequality

$$\mathcal{E}_D = \frac{\langle \Delta^2[S_L^x - S_R^x]\rangle + \langle \Delta^2[S_L^y - S_R^y]\rangle}{2\langle S_L^2 \rangle + 2\langle S_R^2 \rangle} \geq 1,$$

where the variance of an operator $A$ is $\langle \Delta^2 A \rangle = \langle A^2 \rangle - \langle A \rangle^2$. A violation of (40) signals the presence of entanglement.

The criterion (40) only detects entanglement in a limited time range. The range can be estimated by evaluating the Heisenberg equations of motion

$$S_L^x(t) \approx S_L^x(0) + 4N_L t (S_L^y(0) + S_R^y(0)),$$
$$S_L^y(t) \approx S_L^y(0) + 4N_L t (S_L^x(0) + S_R^x(0)),$$
$$S_R^x(t) = S_L^x(0),$$
$$S_R^y(t) = S_L^y(0).$$

Substituting this into (40) for the initial state of completely polarized spins in the $S^x$-direction, we obtain

$$\langle \Delta^2[S_L^x - S_R^x]\rangle + \langle \Delta^2[S_L^y - S_R^y]\rangle \approx 2N(4N^2t^2 - 2Nt + 1).$$

Here, we have approximated that $N_L = N_R = N/2$, which is the most likely outcome of (12). Similarly, we have for the denominator of (40):

$$2\langle S_L^x \rangle + 2\langle S_R^x \rangle = 2N.$$

Thus we expect the entangled states to be detected in the region $0 < t < 1/2N$.

Evaluation of the criterion (40) is shown in figure 6. We see that inter-well entanglement can be detected up to times $t \approx 1/2N$ as expected. We note that the plot for figure 6 is calculated using the exact expressions for the quantities in (42) and (43), rather than the Holstein–Primakoff approximation as above. We attribute the failure of the criterion (40) beyond times $t > 1/2N$ to the break down of the Holstein–Primakoff approximation, since the total spin is no longer polarized in the $S^z$ direction after this time, as also can be observed from figure 3.
4.2. Covariance matrix entanglement criterion

Another correlation-based approach that can be used to detect entanglement is the covariance matrix formalism [61]. Using Holstein–Primakoff approximated variables, one can construct a $4 \times 4$ covariance matrix in operators $(X_L, P_L, X_R, P_R)$ which can be used to construct an entanglement criterion. In order to detect entanglement in a wider time range it is however desirable to not explicitly rely upon the Holstein–Primakoff approximation which we expect to break down for times $t > 1/N$. Recently, the covariance matrix procedure was generalized such that an arbitrary set of operators $\xi_n$ could be used to construct a covariance matrix [47]. In the procedure given in [47], the covariance matrix is constructed for a particular state

$$V_{nm} \equiv \frac{1}{2} \langle \{\Delta \xi_n, \Delta \xi_m\} \rangle,$$

where $\Delta \xi_n = \xi_n - \langle \xi_n \rangle$. Then combining this with the commutation matrix

$$\Omega_{nm} \equiv -i \langle [\xi_n, \xi_m] \rangle,$$

for any separable state we have

$$PT(V) + \frac{1}{2} PT(\Omega) \geq 0,$$

where $PT$ denotes a partial transposition operation. As (46) is a matrix equation, the meaning of the inequality is in the semi-positive definite nature of the matrix on the left hand side. This means that any negative eigenvalue of $PT(V) + \frac{1}{2} PT(\Omega)$ signals the violation of the separability condition.

In our case, the set of observables that we will use are

$$\xi = (S_L^x, S_L^y, S_L^z, S_R^x, S_R^y, S_R^z).$$

In figure 6 we plot the quantity

$$\xi_{CM} = \text{min}_N \lambda_i,$$

where $\lambda_i$ are the eigenvalues of the left hand side of (46). We see that this criterion also detects entanglement in the split squeezed BEC state, but for a wider range of times than using the DGCZ criterion. The region of applicability of the covariance matrix is found to be up to times $t \approx 1/\sqrt{12N}$ which we find empirically using several values of $N$. In this sense, we find that the covariance matrix formalism is more powerful than the DGCZ criterion as it can detect inter-well entanglement in a wider range. This is natural since more information is contained in the covariance matrix, since (47) giving all the $6 \times 6$ correlators, rather than the two types in (40).

4.3. Giovannetti et al entanglement criterion

Another entanglement criterion we consider is the one proposed in [46] by Giovannetti et al. This is given by

$$\xi_G = \sqrt{\frac{\langle \Delta^2 [g_S S'_L - S'_R] \rangle \langle \Delta^2 [g_S S'_R - S'_L] \rangle}{\langle g'_S S'_R \rangle \langle S'_R \rangle + \langle S'_L \rangle}} \geq 1,$$

where

$$S'_L = S_L^x \sin \theta + S_L^z \cos \theta$$

$$S'_R = S_R^x \cos \theta - S_R^z \sin \theta$$

are spin operators that are chosen to minimize the variance of $S'_L$ and $S'_R$ for $i \in \{L, R\}$. The squeezing angle is given by

$$\tan 2\theta = \frac{4 \sin(4t) \cos^{N-2}(4t)}{1 - \cos^{N-2}(8t)}.$$

The parameters $g_S, g'_S$ are free real parameters that are chosen to minimize $\xi_G$. As for the case with (40), the inequality (49) is valid for separable states. A violation of (49) signals the presence of entanglement. The criterion (49) was successfully adopted in [30] to detect entanglement between two regions of an expanded spin-squeezed BEC with $N \approx 600$.

In figure 6 we plot the criteria (49). It is evident that the criterion (49) allows to detect inter-well entanglement for a wider range of $t$ than the DGCZ criterion and the same range as the covariance matrix approach. We have verified that the covariance matrix and Giovannetti criterion give the same range of entanglement detection for any value of $N$ chosen, hence can be considered equivalent methods of detection for split squeezed BECs.

We observe that the range of entanglement that can be detected by the Giovannetti and covariance matrix methods is equivalently given by the total squeezing in the combined system. The entanglement in a single BEC
can be witnessed from the Wineland squeezing parameter \[19\]
\[
\xi^2 = \frac{N \Delta^2[S_{tot}^z]}{|S_{tot}^z|^2},
\]
where
\[
S_{tot}^z = S_{L}^z + S_{R}^z
\]
\[
S_{tot}^x = S_{L}^x + S_{R}^x
\]
which quantifies the metrological usefulness of a state for Ramsey interferometry. For a single BEC, \(\xi^2 \geq 1\) for all separable states, \(\xi^2 < 1\) reveals that the state is entangled. We find that the region that the Giovannetti and covariance matrix criteria detect entanglement coincides with all states with \(\xi^2 < 1\), as can be seen in figure 6. This is natural in view of the fact that the inter-well entanglement between the BECs originates from multipartite entanglement that is present originally in the form of squeezing on a single BEC.

4.4. Comparison of detection schemes
In all three schemes, the inter-well entanglement can only be detected for a small range of times, in contrast to figure 2 which showed that inter-well entanglement is always present except for times \(t = 0\) modulo \(T = \pi/4\). This is the price to be paid for not performing full tomography, since not all information of the density matrix is present using only correlations of spin operators. However, since experimentally squeezing can be only performed for relatively small times, fortunately the detected range of \(t\) is most probably sufficient for the next generation of experiments. Due to the complex nature of the entangled state and the limited region experimental measurement capability, it is not completely trivial to construct entanglement detectors that are effective. Furthermore, it is important to know which detectors are most effective to with regard to this task.

We note that in comparison with the criterion proposed in \[45\], the criteria for entanglement based on variances adopted in our work should be experimentally more practical. This is because the use of the second order moments of the collective spin operator along \(y\) and \(z\) directions would require the first order ones along these two axis to be (at least very close to) zero, which is experimentally very challenging to achieve.

5. EPR steering
We now show that the quantum correlations in the system are strong enough to allow for EPR steering \[62, 63\]. EPR steerable states are a subclass of entangled states where the quantum correlations are strong enough such that one part of the system can steer the state of another part of the system \[28\]. The criterion we consider is the one proposed in \[63\], and experimentally adopted in \[30\], according which the left BEC steers the right BEC if there is a violation of the inequality
\[
\mathcal{E}^{L \rightarrow R} = \frac{\sqrt{\langle (\Delta^2[g^z S_L^z - S_R^z]) \rangle \langle (\Delta^2[g^z S_L^z - S_R^z]) \rangle}}{\langle S_R^z \rangle} \geq 1,
\]
where \(g^z\) and \(g^r\) are free real parameters that are chosen to minimize \(\mathcal{E}^{L \rightarrow R}\). The criterion has an obvious similarity to the entanglement criterion \(49\) with a small modification of the denominator. Thus a measurement of the same quantities can give information both about entanglement and EPR steerability of the state.

In figure 7 we plot the criterion \(54\). It is evident that the criterion can be violated for a wide range of \(t\), and it is therefore suited to detect EPR steering between the two parts of a split spin-squeezed BEC. The range is smaller than for entanglement since steering is a type of correlation that is stronger than entanglement, and for large times the type of correlation between the BECs cannot be described in terms of the steering variables that \(54\) is designed to be a witness of. The criterion \(54\) was successfully adopted in \[30\] to detect steering between two regions of an expanded spin-squeezed BEC with \(N \approx 600\). In the case that the BEC is physically split along the same regions as considered in \[30\], we expect a similar detection of EPR steerable states.

We mention here that unlike entanglement or Bell correlations, steering is an intrinsically asymmetric concept: while one system can steer the other and vice-versa (two-way steering), there exists the possibility of having only one of the two systems able to steer the other (one-way steering) \[64\]. While in this theoretical study the state is symmetric, and therefore one-way steering always implies two-way steering, experimental implementation of our protocol might result in one-way steering only, because of asymmetric noise \[30\].
6. Summary and conclusions

We have modeled and analyzed an experimentally viable method for producing entanglement between two spatially separated BECs. In our approach, one spin polarized BEC is squeezed and split into two spatially separated parts. After the BECs are split, the particle number in each well collapses to a particular number state. The combination of the squeezing, splitting, and collapse is found to have the same effect as applying a Hamiltonian of the form

$$H_{\text{eff}} = \alpha (S_L^x)^2 + \beta (S_R^z)^2,$$

where $\alpha$ and $\beta$ are constants. This corresponds to a non-local entangling Hamiltonian between the two wells, combined with local squeezing. The situation found here is analogous to the generation of a two-mode squeezed state in optical systems. Such a state is well-known to be equivalent to applying a beam splitter operation on single mode squeezed states \[65\]. In the same way that photonic entanglement can be prepared by a beam splitter operation, here the entanglement between BECs is produced by splitting a single squeezed BEC into two spatially separated BECs.

The inter-well entanglement between the BECs was found to have the characteristic ‘devil’s crevasse’ form which arises from a $S_L^x S_R^z$ interaction, giving a non-zero entanglement for all time except the states equivalent to the initial product state. The marginal Wigner function, corresponding to the averaged quantum state over one BEC, evolves to a non-Gaussian state for evolution times exceeding $t > 1/\sqrt{N}$. The conditional Wigner function, corresponding to a (post-selected) particle number collapse across the wells, shows a high degree of non-classicality, as illustrated by negativities appearing in the Wigner function. We also provided several approaches for the detection of entanglement using correlations of spin variables: the DGCZ, Giovannetti, and a generalized covariance matrix criteria. The Giovannetti and covariance matrix approaches were found to give the largest range of entanglement detection and appear to be best suited to detecting entanglement.

Using a small modification of the Giovannetti entanglement criterion, we showed that one may also detect EPR steering in the system. An interesting question is whether Bell correlations between the two BECs could be detected by measuring low-order correlators, in a similar way. While there are known Bell inequalities that can be violated, these require always parity measurements which are experimentally challenging. Therefore, it is an intriguing theoretical task to find new Bell correlation witnesses that are more experimentally accessible for mesoscopic and macroscopic systems.

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