Detecting Topology Variations in Dynamical Networks

Giorgio Battistelli and Pietro Tesi

Abstract—This paper considers the problem of detecting topology variations in dynamical networks. We consider a network whose behavior can be represented via a linear dynamical system. The problem of interest is then that of finding conditions under which it is possible to detect node or link disconnections from prior knowledge of the nominal network behavior and on-line measurements. The considered approach makes use of analysis tools from switching systems theory. A number of results are presented along with examples.

I. INTRODUCTION

Recent years have witnessed a growing interest towards networks of dynamical systems. There is in fact a trend to build modern infrastructures as large-scale networks, which are possibly geographically distributed [1]-[4]. Networks of dynamical systems also arise and play a fundamental role in vehicle formation, cooperative robotics, surveillance and environment monitoring, to name a few [5]-[8].

In many situations, the network behavior is determined or strictly related to its underlying topology. This is the case, for instance, in consensus, coordination and synchronization problems, where the dynamical system that describes the evolution of the network is related to the structure of the graph that models the interaction among the various networks components [9], [10]. On the other hand, also the problem of inferring the network topology from observations of the network behavior is of paramount importance, and it is the objective of this paper to explore such a topic.

Variations of the network topology can have a major impact on stability and/or performance. For example, in consensus-like networks a link disconnection may slow down convergence or even destroy agreement when the graph connectivity is lost [7], [9]. More importantly, variations of the topology may affect the network secure and reliable operation. In fact, the strong interdependency among the various elements of the network is such that a failure in one part of the communication infrastructure can rapidly create global cascading effects. This issue is amplified by the fact that failures in the communication infrastructure can be caused not only by equipment failures or human errors but also by intentional attacks [11]-[13].

This paper considers the problem of inferring variations of the network topology from observations of the network behavior. Specifically, we consider a network whose behavior can be represented via a linear dynamical system. The problem of interest is then that of finding conditions under which it is possible to detect a node or link disconnection from prior knowledge of the nominal network topology and measurements of the network state or a subset of it. Contributions to this topic have been recently proposed. In [14], [15], the authors address the problem of detecting single and multiple link failures in a multi-agent system under the agreement protocol. A notion of distinguishable flow graphs is introduced and sufficient conditions for achieving distinguishable dynamics are stated in terms of inter-nodal distances. In [16], the authors investigate the problem of detection and isolation of link failures by exploiting the presence of discontinuities in the derivatives of the output responses of a subset of nodes. It is worth noting that the problem of inferring variations of the network topology can also be addressed by means of topology identification algorithms [17]-[19]. However, identification algorithms do not assume prior knowledge of the nominal network topology, which is possible in many practical circumstances, and, as such, do not take full advantage of such extra information, which may be crucial for achieving early detection of stability and/or performance losses.

The approach taken in this paper makes use of analysis tools from switching systems theory. Specifically, networks with switching topology can be naturally modeled as a switching system, where the switching signal determines the current network configuration (operating mode). Thus, the problem of detecting a node/link disconnection can be naturally cast as the problem of determining under what conditions the operating mode of the system can be uniquely reconstructed from observations. In the relevant literature, this problem is known as the discernibility, distinguishability or mode-observability problem [20]-[26].

For linear systems, discernibility can be fully characterized through simple algebraic conditions. In fact, it is completely characterized by the eigenspace components related to the switching signal determines the current operating modes of the system. These conditions are generally difficult to refine because the dynamics related to the various operating modes of the system need not be related with one another. In the present case, however, the situation is different because the dynamics resulting from a node or link disconnection can be related with the nominal one via interlacing theorems [27]. Moreover, for several graphs of practical relevance, such as complete, ring, path and grid graphs, an explicit expression for the eigenspace components is available.

By exploiting these features, we provide necessary and
Appendix. remarks. For convenience, the proofs are reported in the
describe the framework of interest and formulate the detection
ment of sensor placement algorithms.
Section IV. Finally, Section V ends the paper with concluding
Connections with least-square identification are established in
A. Problem formulation
sufficient conditions for detecting topology variations for both
the cases of node and link disconnections. These conditions
are based on simple algebraic tests, which can be easily
checked numerically as well as analytically whenever an
explicit expression for the eigenspace components turns out
to be available. While the analysis is mainly oriented towards
a theoretical characterization of the detection problem, the
results also provide several insights on how detection can be
addressed in practice, as well as guidelines for the develop-
ment of sensor placement algorithms.
The remainder of this paper is as follows. In Section II, we
describe the framework of interest and formulate the detection
problem. In Section III, the main results of the paper are given.
Connections with least-square identification are established in
Section IV. Finally, Section V ends the paper with concluding
remarks. For convenience, the proofs are reported in the
Appendix.
II. FRAMEWORK AND PROBLEM FORMULATION
We consider a network of $n$ nodes, whose topological
structure is represented by an undirected graph $G := (V, E)$,
where $V := \{1, 2, \ldots, n\}$ denotes the node set and $E \subseteq V \times V$
denotes the edge set. We assume that the network behavior
can be represented via a linear dynamical system
\begin{equation}
\dot{x} = \Phi x
\end{equation}
where $x \in \mathbb{R}^n$ denotes the network state; $x_i \in \mathbb{R}^n$, $i \in V$,
denotes the state of the $i$-th network node; $\Phi \in \mathbb{R}^{n \times n}$ is the matrix that
determines the network behavior. We assume that $\Phi = \Phi'$ and that $\phi_{ij} \neq 0$ if and only if $(i, j) \in E$, where $\phi_{ij}$
denotes the $(i, j)$-th entry of $\Phi$.

As a relevant example, consider a classical agreement
problem in a network of continuous-time integrators with local
dynamics $\dot{x}_i = u_i$, which implement a linear consensus
protocol with unitary weights,
\begin{equation}
u_i = \sum_{j \in N_i} (x_j - x_i)
\end{equation}
where $N_i$ denotes the set of neighbors of node $i$. This gives
rise to the linear system $\dot{x} = -Lx$, where $L$ denotes the graph
Laplacian induced by $G$. The system is therefore in the same
form as \( \Phi = -L \), i.e., with $\phi_{ij} = 1$ for $j \neq i$ and $\phi_{ij} = -|N_i|$ for $j = i$.

Remark 1: Although this paper is only concerned with networks
whose topological structure is represented by an
undirected graph, most of the conclusions can be extended
to directed graphs as well.

A. Problem formulation
We regard the pair $(G, \Phi)$ as representative of the nominal
behavior of the network. The problem of interest is then that
of finding conditions under which it is possible to detect a
variation of the network topology via: i) knowledge of the
nominal matrix $\Phi$; and ii) measurements of
\begin{equation}
y = Mx, \quad M = \text{col}\{e_i, i \in \mathcal{M}\}
\end{equation}
where $e_i \in \mathbb{R}^n$ denotes the $i$-th versor, and $\mathcal{M} \subseteq V$ denotes
the set of nodes whose state is available for measurements.

A variation in the network topology is specified by means of a
pair $(G, \Phi)$, where $G$ describes the novel topological structure
and $\Phi$ describes the novel network behavior, i.e.,
\begin{equation}
\dot{x} = \Phi x
\end{equation}
In particular, we assume that $\bar{G}$ is an undirected graph defined
as $\bar{G} := (V, \bar{E})$, where $\bar{E} \subset E$: $\dot{\Phi} = \bar{\Phi}$ with $\phi_{ij} \neq 0$ if and
only if $(i, j) \in \bar{E}$, where $\phi_{ij}$ denotes the $(i, j)$-th entry of $\Phi$. As detailed hereafter, $(G, \Phi)$ captures several scenarios of
practical relevance.

1) Link disconnection without dynamics reconfiguration:
suppose that the link disconnection affects the nodes $i, j \in V$.
Then $\bar{G}$ is characterized by $\bar{E} = E \setminus \{(i, j), (j, i)\}$, and
\begin{equation}
\bar{\Phi} = \Phi - \phi_{ij} \left( e_i e_j' + e_j e_i' \right)
\end{equation}
In words, the network dynamics remains unchanged with the
exception of $\phi_{ij} = \bar{\phi}_{jj} = 0$.

2) Link disconnection with dynamics reconfiguration:
this scenario is the same as the previous one with the exception
that, in addition to having $\bar{\phi}_{ij} = \bar{\phi}_{ji} = 0$, a variation occurs
also in $\phi_{ii}$ and $\phi_{jj}$. For example, in the linear consensus
problem described above, one has $\bar{\phi}_{ii} = \phi_{ii} + 1$ and $\phi_{jj} + 1$, and
\begin{equation}
\bar{\Phi} = \Phi - \phi_{ij} \left( e_i e_j' + e_j e_i' \right)
\end{equation}

3) Node disconnection without dynamics reconfiguration:
Suppose that the node disconnection affects the node $i \in V$.
Then $\bar{G}$ is characterized by $\bar{E} = E \setminus \{(i, j), (j, i): j \in N_i\}$, while
\begin{equation}
\bar{\Phi} = \Phi - \sum_{j \in N_i} \phi_{ij} \left( e_i e_j' + e_j e_i' \right)
\end{equation}

4) Link disconnection with dynamics reconfiguration:
this scenario is the same as the previous one with the exception
that a variation occurs also in $\bar{\phi}_{ii}$. In the linear consensus
problem described above, one has $\bar{\phi}_{ii} = 0$, and
\begin{equation}
\bar{\Phi} = \Phi + \sum_{j \in N_i} \left( e_i e_j' + e_j e_i' - e_i e_j' - e_j e_i' \right)
\end{equation}

III. MAIN RESULTS

In this section, we first introduce a notion of discernible
networks. We then present the main results of this paper
and establish a number of connections with several graphs
of practical interest. To begin with, notice that the problem
of detecting a node or link disconnection from measurements
can be casted as the problem of finding conditions under
which $\Phi$ and $\bar{\Phi}$ do not give rise to the same dynamics.
Networks satisfying this property can be therefore referred to
as discernible.

We formalize these concepts.

Definition 1: A dynamical network is said to described by the
pair $(G, \Phi)$ if $G$ is the graph describing the network

\begin{align}
\dot{x} &= \Phi x \\
\dot{\Phi} &= \Phi - \phi_{ij} \left( e_i e_j' + e_j e_i' \right)
\end{align}

\begin{align}
\dot{\Phi} &= \bar{\Phi} - \sum_{j \in N_i} \phi_{ij} \left( e_i e_j' + e_j e_i' \right)
\end{align}
topology and the network behavior obeys \( \dot{x} = \Phi x \), where \( x \) is the network state.

**Definition 2:** Consider two dynamical networks described by \((\mathcal{G}, \Phi)\) and \((\bar{\mathcal{G}}, \bar{\Phi})\), respectively. The networks are said to be **indiscernible** with respect to the state \( x_0 \) if \( e^{\Phi t} x_0 = e^{\bar{\Phi} t} x_0 \) for all \( t \in \mathbb{R}_{\geq 0} \). Otherwise, they are said to be discernible. We denote by \( \mathcal{I} \) the set of states for which \((\mathcal{G}, \Phi)\) and \((\bar{\mathcal{G}}, \bar{\Phi})\) are indiscernible.

**Definition 3:** Given a matrix \( M \) as in \((3)\), the networks are said to be **\( M \)-indiscernible** with respect to the pair of states \((x_0, \bar{x}_0)\) if \( M e^{\Phi t} x_0 = M e^{\bar{\Phi} t} \bar{x}_0 \) for all \( t \in \mathbb{R}_{\geq 0} \). Otherwise, they are said to be \( M \)-discernible. We denote by \( \mathcal{I}(M) \) the set of pairs of states for which \((\mathcal{G}, \Phi)\) and \((\bar{\mathcal{G}}, \bar{\Phi})\) are \( M \)-indiscernible.

Both discernibility and \( M \)-discernibility can be viewed as particular observability problems, which can be addressed by looking at the parallel interconnection of \( \dot{x} = \Phi x \) and \( \dot{\bar{x}} = \bar{\Phi} \bar{x} \). For instance, discernibility is equivalent to the observability of the pair \((\Delta, \Gamma)\), where \( \Delta = \text{diag}(\Phi, \bar{\Phi}) \) and \( \Gamma = [I - I] \), over the set \( \mathcal{X} = \{(w, \xi) \in \mathbb{R}^{2n} : w = \xi\} \). On the other hand, \( M \)-discernibility is equivalent to the (standard) observability of \((\Delta, \Gamma)\), where \( \Delta = \text{diag}(\Phi, \bar{\Phi}) \) and \( \Gamma = [I - I] \). The latter is the classical condition for reconstructing the active mode of a switching linear system from output measurements \([20, 21]\).

One sees that both discernibility and \( M \)-discernibility depend entirely on the eigenspaces of \( \Phi \) and \( \bar{\Phi} \), which are in general difficult to analyze. With respect to the case of switching systems, however, the analysis here considerably simplifies since \( \Phi \) and \( \bar{\Phi} \) comes with a symmetric structure. For clarity, we address the cases of discernibility and \( M \)-discernibility separately.

### A. Discernibility

We first consider the discernibility problem. Notice that since \( \Phi \) and \( \bar{\Phi} \) are symmetric, there exist orthonormal matrices \( S \) and \( \bar{S} \) such that

\[
\Phi = S \Lambda \bar{S}', \quad \bar{\Phi} = \bar{S} \bar{\Lambda} S'
\]

with \( \Lambda \) and \( \bar{\Lambda} \) diagonal matrices. Let \( \text{spec}(\Phi) \) denote the spectrum of \( \Phi \). Moreover, for any \( \lambda \in \text{spec}(\Phi) \), let \( \mu(\lambda) \) and \( V(\lambda) \) denote its multiplicity and eigenspace, respectively.

Finally, let \( S(\lambda) \) be the set of columns of \( S \) that generate \( V(\lambda) \), i.e., \( V(\lambda) = \text{span}(S(\lambda)) \), where span denotes the linear span. We then have

\[
e^{\Phi t} = \sum_{\lambda \in \text{spec}(\Phi)} e^{\lambda t} S(\lambda) S'(\lambda)
\]

\[
e^{\bar{\Phi} t} = \sum_{\lambda \in \text{spec}(\bar{\Phi})} e^{\lambda t} \bar{S}(\lambda) \bar{S}'(\lambda)
\]

From the above expressions, it is straightforward to draw the following conclusions:

i) If \( \text{spec}(\Phi) \cap \text{spec}(\bar{\Phi}) = \emptyset \), then \( \mathcal{I} = \{0\} \), i.e. the only indiscernible state is the zero state. This is obviously the smallest indiscernibility set that one may have.

ii) Nonzero indiscernible states exist if and only if there exists some eigenvalue \( \lambda \) common to \( \Phi \) and \( \bar{\Phi} \) such that \( V(\lambda) \cap V(\bar{\lambda}) \neq \emptyset \), or, equivalently, such that \( \text{rank}\{[S(\lambda) \bar{S}(\bar{\lambda})]\} < \mu(\lambda) + \mu(\bar{\lambda}) \).

Let \( \Psi(\lambda) \) be a matrix whose columns form an orthonormal basis of \( V(\lambda) \cap V(\bar{\lambda}) \). Hence, the set \( \mathcal{I} \) of states for which \((\mathcal{G}, \Phi)\) and \((\bar{\mathcal{G}}, \bar{\Phi})\) are indiscernible is given by

\[
\mathcal{I} = \{x : x \in \text{span}(\Psi(\lambda), \lambda \in \text{spec}(\Phi) \cap \text{spec}(\bar{\Phi}))\}
\]

In view of the above considerations, it is interesting to investigate under which circumstances the variation in the network topology may lead to a new system matrix \( \bar{\Phi} \) sharing eigenvalue/eigenvector pairs \((\lambda, x)\) with the original system matrix \( \Phi \). Clearly, this amounts to searching for necessary and sufficient conditions for the existence of pairs \((\lambda, x)\) such that

\[
\Phi x = \bar{\Phi} x = \lambda x
\]

With respect to the four scenarios of interest described in Section II-A, the following results can be stated, which show that indiscernible states can be readily inferred by inspection of the components of the eigenvalues of \( \Phi \).

**Theorem 1:** (Link disconnection without dynamics reconfiguration) Consider a disconnection of link \((i, j)\) in \(\mathcal{G}\). Then, the networks are indiscernible with respect to a state \( x \in V(\lambda) \), with \( \lambda \in \text{spec}(\Phi) \), if and only if \( x_i = x_j = 0 \).

**Theorem 2:** (Link disconnection with dynamics reconfiguration) Consider a disconnection of link \((i, j)\) in \(\mathcal{G}\). Then, the networks are indiscernible with respect to a state \( x \in V(\lambda) \), with \( \lambda \in \text{spec}(\Phi) \), if and only if \( x_i = x_j \).

**Theorem 3:** (Node disconnection without dynamics reconfiguration) Consider a disconnection of node \( i \) in \(\mathcal{G}\). Then, the networks are indiscernible with respect to a state \( x \in V(\lambda) \), with \( \lambda \in \text{spec}(\Phi) \), if and only if

\[
x_i \sum_{j \in N_i} \phi_{ij} = 0, \quad \sum_{j \in N_i} \phi_{ij} x_j = 0.
\]

If, in addition, we assume that \( \phi_{ij} \geq 0 \) for any \( i \neq j \), then condition \((13)\) becomes \( x_i = 0 \) and \( x_j = 0 \) for any \( j \in N_i \).

**Theorem 4:** (Node disconnection with dynamics reconfiguration) Consider a disconnection of node \( i \) in \(\mathcal{G}\). Then, the networks are indiscernible with respect to a state \( x \in V(\lambda) \), with \( \lambda \in \text{spec}(\Phi) \), if and only if

\[
x_j = x_i, \quad \forall j \in N_i.
\]

If, in addition, we assume that \( \Phi = -L \) (as in the linear consensus protocol \((2)\)) and we consider a non-null Laplacian eigenvalue \( \lambda \), then condition \((14)\) becomes \( x_i = 0 \) and \( x_j = 0 \) for any \( j \in N_i \).

From the above results, it can be seen that, in general, node disconnections are easier to detect than link disconnections. Similar considerations can be made to conclude that a link/node disconnection without dynamics reconfiguration is easier to detect as compared to a disconnection with dynamic configuration.
As an example, consider again the case $\Phi = -L$, where $L$ is the graph Laplacian of $\mathcal{G}$. As well known, the all-ones vector $1$ is always an eigenvector of $L$ associated to the eigenvalue $0$. After a link/node disconnection with dynamic reconfiguration, the novel dynamic matrix $\tilde{\Phi}$ will coincide with $-L$, where $L$ is the Laplacian of the graph $\mathcal{G}$. Hence, the all ones vector $1$ will be an eigenvector associated to the $0$ eigenvalue also for $\tilde{\Phi} = -L$. Hence, in this case, the stationary state $x = 1$ turns out to be indiscernible for any link/node disconnection. This is consistent with the fact that $x = 1$ satisfies the conditions of Theorem 2 and 4 for any $i$ and any $j$, being all its components identical. On the contrary, when a link/node disconnection without dynamic reconfiguration occurs, $x = 1$ is no longer an eigenvector of $\tilde{\Phi}$ and, hence, is discernible (indeed $x = 1$ does not satisfy the conditions of Theorem 1 and 3). If we further assume that the graph $\mathcal{G}$ is connected, then $1$ turns out to be the unique eigenvector with eigenvalue $0$. Then, the second part of Theorem 4 can be exploited to conclude that, with the exception of $x = 1$, the two cases of node disconnection with or without dynamic reconfiguration give rise to the same indiscernible eigenvectors.

**B. M-Discernibility**

We now turn the attention to the problem of detecting a topology variation from observations of an output vector $y = Mx$. In this respect, notice preliminarily that a state $x \in \mathcal{I}$ for which $(\mathcal{G}, \Phi)$ and $(\mathcal{G}, \tilde{\Phi})$ are indiscernible does always generate indiscernible output trajectories. Then, if we define the set $\mathcal{I}_P = \{(x, x) : x \in \mathcal{I}\}$, we have that $\mathcal{I}_P \subseteq \mathcal{I}(M)$ irrespective of the choice of the output matrix $M$. On the other hand, by letting $M = I$ (i.e., by observing all the network nodes) we clearly have $\mathcal{I}_P = \mathcal{I}(I)$, where $I$ stands for the identity matrix of an appropriate dimension. Hence, a first important problem is how to choose the matrix $M$ so that $\mathcal{I}_P = \mathcal{I}(M)$. This amounts to asking where sensors nodes should be placed in order to guarantee that discernibility implies $M$-discernibility. When such a condition holds, we say that the matrix $M$ ensures output discernibility. Since we have

$$Me^{\Phi t} = \sum_{\lambda \in \text{spec}(\Phi)} e^{\lambda t} MS(\lambda)s'(\lambda)$$

$$Me^{\tilde{\Phi} t} = \sum_{\lambda \in \text{spec}(\tilde{\Phi})} e^{\lambda t} M\tilde{S}(\lambda)s'(\lambda)$$

the following result follows at once.

**Theorem 5**: Consider an observation vector $y$ as in (3). Then, condition $\mathcal{I}_P = \mathcal{I}(M)$ holds if and only if the following conditions are satisfied:

(i) $\text{rank} \{MS(\lambda)\} = \mu(\lambda)$ for any $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\tilde{\Phi})$;

(ii) $\text{rank} \{M\tilde{S}(\lambda)\} = \bar{\mu}(\lambda)$ for any $\lambda \in \text{spec}(\tilde{\Phi}) \setminus \text{spec}(\Phi)$;

(iii) $\text{rank} \{M[S(\lambda)\bar{S}(\lambda)]\} = \text{rank} \{[S(\lambda)\bar{S}(\lambda)]\}$ for any $\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})$.

Notice that condition (i) amounts to requiring that all the states belonging to $V(\lambda)$, with $\lambda$ eigenvalue of $\Phi$ but not of $\tilde{\Phi}$, are observable from the output $y = Mx$. The same property is required by condition (ii) for all the eigenvalues of $\tilde{\Phi}$, which are not eigenvalues of $\Phi$. Finally, condition (iii) amounts to requiring that, for any eigenvalue $\lambda$ that is shared by $\Phi$ and $\tilde{\Phi}$, and for any $(x, \bar{x}) \in V(\lambda) \times \bar{V}(\lambda)$, one has $Mx = M\bar{x}$ if and only if $x = \bar{x}$.

Building upon Theorem 5, an expression for $\mathcal{I}(M)$ can be given. To this end, for any $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\Phi)$, let $K(\lambda, M)$ be a matrix whose columns form a basis of the linear space $(x, 0) \in \mathbb{R}^{2n} : x \in V(\lambda)$ and $Mx = 0$. Then, condition (ii) is defined in a similar way with respect to $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\Phi)$. Finally, for any $\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})$, let $T(\lambda, M)$ be a matrix whose columns form a basis of the linear space $(x, \bar{x}) \in \mathbb{R}^{2n} : x \in V(\lambda), \bar{x} \in \bar{V}(\lambda)$, and $Mx = \bar{M}x$. Then, we have

$$\mathcal{I}(M) = \text{span} \{T(\lambda, M), \lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})\}$$

$$\cup \text{span} \{K(\lambda, M), \lambda \in \text{spec}(\Phi) \setminus \text{spec}(\tilde{\Phi})\}$$

$$\cup \text{span} \{\tilde{T}(\lambda, M), \lambda \in \text{spec}(\tilde{\Phi}) \setminus \text{spec}(\Phi)\}$$

Theorem 5 provides interesting insights on the number of sensors needed so as to have output discernibility. Consider, for example, the ideal situation in which $\Phi$ and $\tilde{\Phi}$ are discernible from all the states, i.e., $\mathcal{I} = \{0\}$. Then, condition (iii) becomes

$$\text{rank} \{M[S(\lambda)\bar{S}(\lambda)]\} = \mu(\lambda) + \bar{\mu}(\lambda)$$

for any $\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})$. Notice also that the rank in the left-hand side cannot exceed $\text{rank}(M)$, which, in turn, is equal to the number of measured nodes. Then, we can conclude that, in order to have output discernibility, one needs a number of sensors at least equal to the maximum among $\mu(\lambda)$ for $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\tilde{\Phi})$, $\bar{\mu}(\lambda)$ for $\lambda \in \text{spec}(\tilde{\Phi}) \setminus \text{spec}(\Phi)$, and $\mu(\lambda) + \bar{\mu}(\lambda)$ for $\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})$. Notice that this is just a lower bound, since Theorem 5 does not exclude that a larger number of sensors may be needed. Nevertheless, such considerations indicate that, similar to what happens when standard observability is addressed [28], the number of nodes that should be available for measurements increases with the multiplicity of the eigenvalues.

As for the sensor placement, one can see that in order to satisfy condition (iii) the sensors $i \in M$ must be positioned so that the rows of the matrix $[S(\lambda)\bar{S}(\lambda)]$ corresponding to the indices $i \in M$ contain at least one non-zero minor of order $\text{rank} \{[S(\lambda)\bar{S}(\lambda)]\}$. Analogous considerations can be given for conditions (i) and (ii).

In particular, condition (ii) becomes tricky when, starting from a connected graph $\mathcal{G}$, a topology variation gives rise to multiple connected components in the graph $\mathcal{G}$ (notice that this always happens in the case of node disconnection). Specifically, let $\mathcal{G}$ consist of $N$ mutually disjoints components $\mathcal{G}^1, \ldots, \mathcal{G}^N$, and let $\mathcal{N}^k$ be the set of nodes belonging to $\mathcal{G}^k$ (clearly $\sum_{k=1}^{N} |\mathcal{N}^k| = n$). Then, as well-known,

$$\text{spec}(\mathcal{G}) = \bigcup_{k=1}^{N} \text{spec}(\mathcal{G}^k)$$
and, in addition, for any $\lambda \in \text{spec}(\Omega^k)$ there exist eigenvectors $x \in V(\lambda)$ such that $x_i \neq 0$ if and only if $i \in N^k$. As a consequence, it is immediate to verify that condition (ii) can be satisfied only by placing sensors in each one of the mutually disjoints components $\Omega^1, \ldots, \Omega^N$.

Clearly, this latter requirement can be quite restrictive in practice. For instance, this implies that one can have output discernibility with respect to any possible node disconnection only by placing a sensor in each network node. Hence, instead of requiring complete output discernibility, in many situations it may be of interest to restrict the attention only to some of the connected components of the graph $\tilde{G}$. This can be done in a straightforward way by considering in condition (ii) only the eigenvalues and eigenvectors pertaining to the connected components of interest. For example, in the case of disconnection of node $i$, one can restrict the attention to the component with node set $N \setminus \{i\}$ by excluding from condition (ii) the eigenvector $e_i$ pertaining to the trivial component $\{i\}$.

IV. A LEAST-SQUARES CRITERION FOR DETECTION OF TOPOLOGY VARIATIONS

In the previous section, we have provided conditions under which it is theoretically possible to detect a topology variation by observing the evolution of the state $x_i$ in a subset $M$ of the network nodes $N$. From a practical point of view, this can be done by resorting to a least-squares criterion, as detailed hereafter.

Suppose that, starting from time $t_0$, a certain number, say $N$, of samples of the output vector $y$ are collected at the time instants $t = t_0 + kT$ for $k = 0, \ldots, N - 1$, where $T \in \mathbb{R}_{> 0}$. In particular, to account for the possible presence of a measurement noise, let each sample $z_k$ be of the form

$$z_k = y(t_0 + kT) + v_k$$

with $v_k$ an unknown but bounded discrete-time noise signal. We assume that an upper bound $E_v$ on the energy of the sequence $\{v_k\}$ is known, i.e., $(\sum_{k=0}^{N-1} \|v_k\|^2)^{1/2} \leq E_v$ where $\|\cdot\|$ stands for Euclidean norm. Hereafter, the vector of all the collected samples will be denoted by

$$Z_N = \text{col}(z_0, \ldots, z_{N-1})$$

Remark 2: It is worth noting that (15) amounts to making use of synchronous measurements. While this hypothesis may be restrictive in some cases, there are many applications where the measurement devices are equipped with global positioning system (GPS) units. This is the case, for instance, in many smart grid applications where Phasor Measurement Units are sampled from widely dispersed locations and synchronized via a common GPS reference [2].

Let $O_N$ and $\hat{O}_N$ denote the sampled-data observability matrices associated with $G$ and $\tilde{G}$, respectively. Clearly, we have

$$O_N := \begin{pmatrix} M & M e^{\Phi T} & \vdots & M e^{\Phi (N-1)T} \\ M e^{\Phi T} & \vdots & \vdots & M e^{\Phi (N-1)T} \end{pmatrix}, \quad \hat{O}_N := \begin{pmatrix} M & M e^{\tilde{\Phi} T} & \vdots & M e^{\tilde{\Phi} (N-1)T} \\ M e^{\tilde{\Phi} T} & \vdots & \vdots & M e^{\tilde{\Phi} (N-1)T} \end{pmatrix}$$

(16)

Notice now that, when the state evolution is generated by the nominal network $(G, \Phi)$, the sampled outputs are of the form $Z_N = O_N x(t_0) + V_N$ where $V_N = \text{col}(v_0, \ldots, v_{N-1})$ and $x(t_0)$ is the (unknown) state at time $t_0$. Then, the least-squares cost function

$$\pi(Z_N) = \min_{x \in \mathbb{R}^n} \|Z_N - O_N x\|$$

provides a quantitative measure of how close the observed output behavior is to the nominal ones. In fact, whenever the output samples $Z_N$ arises from the nominal network, we have $\pi(Z_N) \leq E_v$.

Similarly, any output behavior generated by $(\tilde{G}, \tilde{\Phi})$ leads to sampled outputs of the form $Z_N = \hat{O}_N x(t_0) + V_N$ and, hence, the least-squares cost function

$$\tilde{\pi}(Z_N) = \min_{x \in \mathbb{R}^n} \|Z_N - \hat{O}_N x\|$$

provides a quantitative measure of the distance between the observed outputs and the set of behaviors associated with the modified topology.

Then, by computing the quantities $\pi(Z_N)$ and $\tilde{\pi}(Z_N)$, the following conclusions can be readily drawn:

(a) when $\pi(Z_N) > E_v$, the output samples are not consistent with the nominal network; hence we can conclude that a variation from the nominal behavior has occurred.

(b) when $\pi(Z_N) \leq E_v$ and $\tilde{\pi}(Z_N) > E_v$, the output samples are consistent only with nominal behavior; hence we can exclude the variation associated with $(G, \Phi)$.

(c) when both $\pi(Z_N) \leq E_v$ and $\tilde{\pi}(Z_N) \leq E_v$, we cannot conclude since the sampled outputs are consistent with both the nominal and the modified behavior.

Of course, the considered framework can be easily extended so as to account for different possible topological variations. In particular, the detection of a topological variation as in case (a) requires only the computation of the cost function associated with the nominal behavior. On the other hand, a “validation” test as in case (b) requires, in general, the computation of one cost function for each possible variation. Further, also in case (a), computation of the other cost functions can be useful in order to possibly identify the topological variation.

As for case (c), this corresponds to the situation in which the information contained in sampled outputs is not sufficient to conclude on the underlying topology. Clearly, this case may arise when the measurement noise is sufficiently large so as to mask the data information. However, case (c) is also inherently connected to the concept of $M$-indiscernible states as previously defined. In fact, let $\Pi(M)$ and $\tilde{\Pi}(M)$ be the projections on the first and, respectively, last $n$ components of
the set \( \mathcal{I}(M) \), i.e.,

\[
\mathcal{I}(M) = \{ x \in \mathbb{R}^n : \exists \tilde{x} \in \mathbb{R}^n \text{ such that } (x, \tilde{x}) \in \mathcal{I}(M) \}
\]

\[
\bar{\mathcal{I}}(M) = \{ \tilde{x} \in \mathbb{R}^n : \exists x \in \mathbb{R}^n \text{ such that } (x, \tilde{x}) \in \mathcal{I}(M) \}
\]

Then, it is an easy matter to see that when the output behavior is generated by the modified network starting from state \( x(t_0) \in \bar{\mathcal{I}}(M) \), one has \( \pi(Z_N) \leq E_0 \) and \( \pi(Z_N) \leq E_0 \) since both \( \pi(\bar{O}_N x(t_0)) = 0 \) and \( \bar{\pi}(\bar{O}_N x(t_0)) = 0 \). In this respect, it is worth noting that although \( \bar{\mathcal{I}}(M) \) is defined with respect to an ideal situation (i.e., assuming to measure the noise-free continuous-time evolution of \( y = Mx \)), it turns out that such a set plays a fundamental role also in the practical situation of sampled outputs affected by measurement noises, provided that the sampling is non-pathological [29].

**Lemma 1:** Let the state trajectory be generated by the modified network \((\bar{G}, \bar{\Phi})\). Furthermore, suppose that for any \( \lambda, \bar{\lambda} \in \text{spec}(\Phi) \cup \text{spec}(\bar{\Phi}) \) with \( \lambda \neq \bar{\lambda} \) the following condition holds

\[
\text{Im}(\lambda - \bar{\lambda}) \neq \frac{2\pi h}{T} \quad \text{for } h \in \mathbb{Z} \setminus \{0\} \quad \text{whenever } \text{Re}(\lambda - \bar{\lambda}) = 0 .
\]

(17)

Finally, let \( N \geq 2n \). Then, case (c) can occur only if

\[
d(x(t_0), \bar{\mathcal{I}}(M)) \leq \alpha E_v \quad \text{(18)}
\]

where \( \alpha \) is a suitable positive constant and \( d(\cdot, \cdot) \) stands for point-set distance.

In view of Lemma 1, it can be seen that, when the state \( x(t_0) \) is far enough from the set of \( M \)-indiscernible states (in the sense that condition (18) does not hold) and the sampling is non-pathological, then the sampled outputs provide sufficient information for detecting that a topological variation has occurred. Bounds on the constant \( \alpha \) can be found, as in [25], in terms of the cosine of the smallest non-null angle between the linear subspaces \( \text{span}(\bar{O}_N) \) and \( \text{span}(O_N) \). Notice finally that condition (17) is nothing but the well-known Kalman-Bertram criterion for the observability of sampled-data systems applied to the pair \((\Gamma, \Delta)\).

Notice that a result analogous to Lemma 1 can be derived also in the case of output behaviors generated by the nominal network \((G, \Phi)\), with the set \( \mathcal{I}(M) \) replaced by \( \bar{\mathcal{I}}(M) \).

### V. An Example

In order to illustrate some of the previous results in an easy manner, we consider the simple, yet non-trivial, case of a linear consensus protocol with unitary weights over the \( n \)-dimensional path graph \( G = P_n \) with dynamics reconfiguration, which is standard in consensus-like algorithms. We focus on the discernibility problem. Recall that the Laplacian \( L \) of \( P_n \) has eigenvalues \( \lambda[k] = 2 - 2\cos(\pi k/n) \) and eigenvectors

\[
x_i[k] = \cos(\pi k i/n - \pi k/2n)
\]

\[
k \in \{0, 1, \ldots, n-1\}, i \in \{1, 2, \ldots, n\}
\]

where \( \lambda[k] \) denotes the \( k \)-th eigenvalue and \( x_i[k] \) denotes the \( i \)-th component of the \( k \)-th eigenvector. This describes the nominal network \((G, -L)\).

We consider first the case of disconnection of a link \((i, j)\). Notice that we can restrict the attention to the situation where \( j = i + 1 \) since the case \( j = i - 1 \) is specular. In accordance with Theorem 2, the link disconnection is not detectable if and only if

\[
\cos(\pi k i/n - \pi k/2n) = \cos(\pi k j/n - \pi k/2n) = \cos(\pi k i/n + \pi k/2n)
\]

for some \( k \in \{0, 1, \ldots, n-1\} \) and \( i \in \{1, 2, \ldots, n\} \).

For convenience, let us define \( A := \pi k i/n - \pi k/2n \) and \( B := \pi k i/n + \pi k/2n \). By looking at the cosine function (cf. Figure 1), one sees that discernibility is violated when \( A \) and \( B \) take the form \( A = \pm m\pi - \Delta/2 \) and \( B = A + \Delta \), where \( m \in \mathbb{N} \) and \( \Delta \in [0, \pi) \).

The reason for constraining \( \Delta \) to be less than \( \pi \) comes from the fact that \( B = A + \pi/k/n < A + \pi \) since \( k < n \). This in particular excludes indiscernible points located at the zeros of the cosine function as well as indiscernible points of periodicity of \( 2\pi \) or higher. Combining the previous expressions, we then have \( \Delta = \pi/k/n \) Thus discernibility is violated whenever exist \( k, i, n \) and \( m \) such that \( A = \pi k i/n - \pi k/2n = \pm m\pi - \pi k/2n \) or, equivalently,

\[
ki = nm \quad (19)
\]

where \( n > 1 \). The following conclusions can be drawn: i) A trivial solution to (19) is given by \( k = m = 0 \). As previously noted, this corresponds to the fact that the consensus state \( x = 1 \) turns out to be indiscernible for any link/node disconnection; ii) Nontrivial solutions to (19) also exist. Simple examples are \((k, i, n, m) = (2, 4, 8, 1) \) and \((k, i, n, m) = (8, 5, 10, 4) \); iii) Apart from the case where \( k = m = 0 \), there is no solution to (19) when \( i \in \{1, n - 1\} \). In fact, if \( i = n \) we obtain \( k = nm \). This has no solutions when \( k, m > 0 \) since \( k - m = k/n \). This has no solutions when \( k, m > 0 \) since \( k - m \) is integer, whereas \( k/n \) cannot be integer since \( k < n \).
Point iii) is interesting since it indicates that, apart from the stationary state $x = 1$, a link disconnection can always be detected if it involves one of the endpoints of the graph. The latter situation can also be viewed as a node disconnection, and, in fact, it is straightforward to verify that, apart from the stationary case, node disconnections are always detectable. To see this, notice that, in accordance with Theorem 4, a node disconnection is not detectable if and only if there exist an index $i$ such that

$$\cos(\pi ki/n - \pi k/2n) = \cos(\pi ki/n + \pi k/2n) = \cos(\pi ki/n - 3\pi k/2n)$$

By the point iii) above, one can restrict the attention to the case where $i \in \{3, 4, \ldots, n - 2\}$ since the cases $i \in \{1, 2\}$ and $i \in \{n - 1, n\}$ do involve the endpoints of the graph. Let $A$ and $B$ be as before, and let $C := \pi ki/n - 3\pi k/2n$. Thus, there must exist three points, namely $A, B$ and $C$ where the cosine function takes on the same value. Apart from the case $k = 0$, this is not possible since $B = C + 2\pi k/n < C + 2\pi$. Hence, we conclude that, apart from the stationary case, a node disconnection is always detectable.

It is worth mentioning that a very similar analysis could be carried out with respect to the grid graph since its eigenspace is completely determined by the eigenspace of the path graph [28].

VI. CONCLUSIONS

In this paper, we have addressed the problem of detecting topology variations in dynamical networks, considering both the cases of node and link disconnections. The results show that the detection problem can be characterized through simple algebraic conditions, which depend on the eigenspace components related to the nominal and faulty operating mode of the network. While the analysis is mainly oriented towards a theoretical characterization of the detection problem, the results also provide several insights on how detection can be addressed in practice, as well as guidelines for the development of sensor placement algorithms.

APPENDIX

Proof of Theorems 1-4. Let $x$ be an eigenvector of $\Phi$ with eigenvalue $\lambda$, i.e., $x$ is such that $\Phi x = \lambda x$. Then, Theorem 1 readily follows from the fact that, when a link disconnection without dynamics reconfiguration occurs, we have

$$\tilde{\Phi} x = \Phi x - \phi_{ij} (e_i e_j' + e_j e_i') x = \lambda x - \phi_{ij} (x_j e_i + x_i e_j).$$

Similarly, in the case of a link disconnection with dynamics reconfiguration we have

$$\tilde{\Phi} x = \Phi x + \phi_{ij} (e_i e_j' + e_j e_i') x = \lambda x + (x_i - x_j) e_i + (x_j - x_i) e_j$$

From which Theorem 2 follows. As for the case of a node disconnection without dynamics reconfiguration, we have

$$\tilde{\Phi} x = \Phi x - \sum_{j \in N_i} \phi_{ij} (e_i e_j' + e_j e_i') x$$

$$= \lambda x - \left( \sum_{j \in N_i} \phi_{ij} x_j \right) e_i - x_i \sum_{j \in N_i} \phi_{ij} e_j$$

which leads to the statement of Theorem 3. Finally, if we consider the case of a node disconnection with dynamics reconfiguration, we have

$$\tilde{\Phi} x = \Phi x + \sum_{j \in N_i} (e_i e_j' + e_j e_i') x$$

$$= \lambda x - \left( \sum_{j \in N_i} (x_i - x_j) \right) e_i + \left( \sum_{j \in N_i} (x_j - x_i) e_j \right)$$

and, hence, condition (15). Concerning the second part of Theorem 4, notice that, in accordance with Theorem 4, a node disconnection is always detectable.

Proof of Theorem 5. Clearly, conditions (i) and (ii) are necessary because, otherwise, we would have non-null initial states leading to zero output trajectories and hence indiscernible from the output. Further, condition (i) ensures that for any state $x$ such that $S'(\lambda)x \neq 0$ with $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\tilde{\Phi})$ the term $e^\lambda M S(\lambda)S'(\lambda)$ related to the output evolution of $(G, \Phi)$ is not null. Since this term is never present in the output evolution of $(\tilde{G}, \Phi)$, we can conclude that $(\tilde{G}, \Phi)$ is $M$-indiscernible from $(\tilde{G}, \Phi)$ in such a state. Similarly, condition (ii) ensures that $(\tilde{G}, \Phi)$ is $M$-indiscernible from $(\tilde{G}, \Phi)$ in all the states $x$ such that $S'(\lambda)x \neq 0$ for at least one $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\tilde{\Phi})$. Hence, under conditions (i) and (ii), a necessary condition for a pair of states $(x, \bar{x})$ to be $M$-indiscernible is $S'(\lambda)x = 0$ and $S'(\lambda)\bar{x} = 0$ for any $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\Phi)$ and $\lambda \in \text{spec}(\Phi) \setminus \text{spec}(\tilde{\Phi})$, or, equivalently, that $x$ belongs to $\bigcup_{\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})} \mathcal{V}(\lambda)$ and $\bar{x}$ belongs to $\bigcup_{\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})} \tilde{\mathcal{V}}(\lambda)$. Consider now two such states $x$ and $\bar{x}$. Condition (iii) ensures that, for any $\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})$, one has $MS(\lambda)S'(\lambda)x = MS(\lambda)S'(\lambda)\bar{x}$ if and only if $S(\lambda)S'(\lambda)x = S(\lambda)S'(\lambda)\bar{x}$. This, in turn, implies that $(x, \bar{x})$ are $M$-indiscernible if and only if $x = \bar{x} \in \mathcal{I}$, which proves the sufficiency of conditions (i)-(iii). Finally, the necessity of condition (iii) readily follows from the fact that, when such a condition is not satisfied for some $\lambda \in \text{spec}(\Phi) \cap \text{spec}(\tilde{\Phi})$, there exist pairs of states $(x, \bar{x}) \notin \mathcal{I}$ with $x \in V(\lambda)$, $\bar{x} \in \tilde{V}(\lambda)$ such that $MS(\lambda)S'(\lambda)x = MS(\lambda)S'(\lambda)\bar{x}$.
observability of sampled-data systems and conclude that, when conditions (17) and $N \geq 2n$ are satisfied, the null space of $[\mathcal{O}_N - \mathcal{O}_N^*]$ coincides with the set of unobservable states of the pair $(\mathcal{I}, \Delta)$, which is precisely the set $\mathcal{I}(M)$. Hence, we have $\mathcal{O}_N \mathcal{X} = \mathcal{O}_N \mathcal{X}'$ if and only if $(x, \bar{x}) \in \mathcal{I}(M)$. Then, the statement of Lemma 1 can be proven by proceeding as in the proof of Theorem 2 of [25], to which the reader is referred for details.

REFERENCES

[1] X. Zhang, A. Papachristodoulou, A real-time control framework for smart power networks with star topology, in: Proc. of the 2013 American Control Conference, Washington, DC, USA, 2013.

[2] A. Chakrabortty, P. Khargonekar, Introduction to wide-area control of power systems, in: Proc. of the American Control Conference, Washington, DC, USA, 2013.

[3] F. Dörfler, M. Chertkov, F. Bullo, Synchronization in complex oscillator networks and smart grids, Proc. of the National Academy of Sciences 110 (2013) 2005–2010.

[4] C. D. Persis, T. Jensen, R. Ortega, R. Wisniewski, Output regulation of large-scale hydraulic networks, IEEE Transactions on Control Systems Technology 22 (2014) 238–245.

[5] P. Ögren, E. Fiorelli, N. Leonard, Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed network, IEEE Transactions on Automatic Control 49 (2004) 1292–1302.

[6] R. Beard, T. McLain, D. Nelson, D. Kingston, D. Johansson, Decentralized cooperative aerial surveillance using fixed-wing miniature UAVs, Proceedings of the IEEE 94 (2006) 1306–1324.

[7] M. Arcak, Passivity as a design tool for group coordination, IEEE Transactions on Automatic Control 52 (2007) 1380–1390.

[8] C. Nowzari, J. Cortés, Self-triggered coordination of robotic networks for optimal deployment, Automatica 48 (2012) 1077–1087.

[9] R. Olfati-Saber, R. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Transactions on Automatic Control 49 (2004) 1520–1533.

[10] L. Scardovi, R. Sepulchre, Synchronization in networks of identical linear systems, Automatica 45 (2009) 2557–2562.

[11] S. Amin, A. Cárdenas, S. Sastry, Safe and secure networked control systems under denial-of-service attacks, in: Hybrid systems: Computation and Control, 2009, pp. 31–45.

[12] H. Sandberg, A. Teixeira, K. Johansson, On security indices for state estimators in power networks, in: First Workshop on Secure Control Systems, CPSWEEK, Stockholm, Sweden, 2010.

[13] C. D. Persis, P. Tesi, Resilient control under denial-of-service, in: The 19th IFAC World Congress, Cape Town, South Africa, 2014.

[14] M. Rahimian, A. Ajorlou, A. Aghdam, Characterization of link failures in multi-agent systems under the agreement protocol, in: Proc. of the American Control Conference, Montréal, Canada, 2012.

[15] M. Rahimian, A. Ajorlou, A. Aghdam, Detectability of multiple link failures in multi-agent systems under the agreement protocol, in: Proc. of the IEEE Conference on Decision and Control, Maui, Hawaii, 2012.

[16] M. Rahimian, V. Preciado, Detection and isolation of failures in linear multi-agent networks, in: arXiv:1309.5540, 2013.

[17] M. Franceschelli, A. Gasparri, A. Giua, C. Seatzu, Decentralized laplacian eigenvalues estimation for networked multi-agent systems, in: 48th IEEE Conference on Decision and Control, Shanghai, P.R. China, 2009.

[18] B. Sanandaji, T. Vincent, M. Wakin, Exact topology identification of large-scale interconnected dynamical systems from compressive observations, in: Proc. of the American Control Conference, San Francisco, CA, USA, 2011.

[19] A. Kibangou, Commault, Decentralized laplacian eigenvalues estimation and collaborative network topology identification, in: 3rd IFAC Workshop on Distributed Estimation and Control in Networked Systems, NeCsys 12, Santa Barbara, CA, USA, 2012.

[20] R. Vidal, A. Chiuso, S. Soatto, Observability and identifiability of jump linear systems, in: Proc. of the 41st IEEE Conference on Decision and Control, Las Vegas, Nevada, 2002, pp. 3614–3619.

[21] M. Babaali, M. Egerstedt, Observability of switched linear systems, in: R. Alur, G. J. Pappas (Eds.), Hybrid Systems: Computation and Control, Lecture Notes in Computer Science, Springer, 2004, pp. 48–63.