SEPARATION OF GATES IN QUANTUM PARALLEL PROGRAMMING

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Abstract. The number of qubits in current quantum computers is a major restriction on their wider application. To address this issue, Ying conceived of using two or more small-capacity quantum computers to produce a larger-capacity quantum computing system by quantum parallel programming ([M. S. Ying, Morgan-Kaufmann, 2016]). In doing so, the main obstacle is separating the quantum gates in the whole circuit to produce a tensor product of the local gates. In this study, we theoretically analyse the (sufficient and necessary) separability conditions of multipartite quantum gates in finite or infinite dimensional systems. We then conduct separation experiments with n-qubit quantum gates on IBM quantum computers using QSI software.

1. Introduction

With the development of quantum hardware, programming for quantum computers has become an urgent task [1, 2, 3, 4]. As reported in [1, 5, 6, 7], extensive research has been conducted on quantum programming over the last decade, and several quantum programming platforms have been developed over the last two decades. The first quantum programming environment was the ‘QCL’ project proposed by Ömer in 1998 [8, 9]. In 2003, Bettelli et al. defined a quantum language called Q language as a C++ library [2]. In recent years, more scalable and robust quantum programming platforms have emerged. In 2013, Green et al. proposed a scalable functional quantum programming language, called Quipper, using Haskell as the host language [10]. JavadiAbhari et al. defined Scaffold in 2014 [11], presenting its accompanying compilation system Scaffold [12]. Wecker and Svore from QuArc (the Microsoft Research Quantum Architecture and Computation team) developed LIQUi|⟩ as a modern tool-set embedded within F# [14]. At the end of 2017, QuARC announced a new programming language and simulator designed specifically for full-stack quantum computing, known as Q#, which represents a milestone in quantum programming. In the same year, Liu et al. released the quantum program Q|SI⟩ that supports a more complicated loop structure [13]. To date, the structures of programming languages and tools have mainly

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been sequential. However, beyond the constraints of quantum hardware, there remain several barriers to the development of practical applications for quantum computers. One of the most serious barriers is the number of physical qubits provided in physical machines. For example, IBMQ produces two five-qubits quantum computers [17] and one 16-qubit quantum computer [18], which are available to programmers through the cloud, but these are far fewer qubits than are required by practical quantum algorithms. Today, quantum hardware is in its infancy. As the number of available qubits is gradually increasing, many researchers are considering the possibility of combining various quantum hardware components to work as a single entity and thereby enable advances in the number of qubits [7]. To increase the number of accessible qubits in quantum hardware, one approach uses concurrent or parallel quantum programming. Although current quantum-specific environments are sequential in structure, some researchers are working to exploit the possibility of parallel or concurrent quantum programming on the general programming platform from different respects. Vizzotto and Costa applied mutually exclusive access to global variables to enable concurrent programming in Haskell [20]. Yu and Ying studied the termination of concurrent programs [19]. Researchers provide mathematics tools for process algebras to describe their interaction, communication and synchronization [21, 22, 23, 24]. Recently, Ying and Li defined and established operational (denotational) semantics and a series of proof rules for ensuring the correctness of parallel quantum programs [25]. Recall quantum gates are unitary operators on a system, which are fundamental and common ingredients of quantum circuits. Naturally, when implementing parallel quantum programs, the first challenge is to separate multipartite quantum gates into the tensor products of local gates. If separation is possible, a potential parallel execution is the natural result. Here, we provide the sufficient and necessary conditions for the separability of multipartite gates. It is showed that multipartite quantum gates that can be separated simply seldom exist. We then conduct separation experiments with n-qubit quantum gates on IBM quantum computers using QSI software.

2. Criteria for separation of quantum gates and IBMQ experiments

In this section, let \( \mathcal{H}_k \) be a separable complex Hilbert space of finite or infinite dimension, \( 1 \leq k \leq n \), and \( \bigotimes_{k=1}^{n} \mathcal{H}_k \) the tensor product of \( \mathcal{H}_k \)'s. Denote by \( \mathcal{B}(\bigotimes_{k=1}^{n} \mathcal{H}_k) \), \( \mathcal{U}(\bigotimes_{k=1}^{n} \mathcal{H}_k) \) and \( \mathcal{B}_s(\bigotimes_{k=1}^{n} \mathcal{H}_k) \) respectively the algebra of all bounded linear operators, the group of all unitary operators (quantum gates), and the space of all self-adjoint operators on the underline space \( \bigotimes_{k=1}^{n} \mathcal{H}_k \).
Let $U$ be a multipartite gate on the composite system $\otimes_{k=1}^n \mathcal{H}_k$. We call that $U$ is separable (local or decomposable) if there exist quantum gates $U_k$ on $\mathcal{H}_k$ such that

\begin{equation}
U = \otimes_{k=1}^n U_k.
\end{equation}

Considering each a unitary $U \in \mathcal{U}(\otimes_{k=1}^n \mathcal{H}_k)$, if $U = \exp[i\mathbf{H}]$ with $\mathbf{H} \in \mathcal{B}_n(\otimes_{k=1}^n \mathcal{H}_k)$, whether or not can we obtain separability criteria of $U$ from the structure of $\mathbf{H}$? Here, we first put forward the separation problem for multipartite gates in the arbitrary (finite or infinite) dimensional systems as follows.

**The Separation Problem:** Consider the multipartite system $\otimes_{k=1}^n \mathcal{H}_k$. If $U = \exp[i\mathbf{H}]$ with $\mathbf{H} = \sum_{i=1}^{NH} A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)}$ for a multipartite unitary gate $U$, determine whether there exist unitary operators $U_k$ on $\mathcal{H}_k$ such that $U = \otimes_{k=1}^n U_k$. Further, how does the structure of each $U_k$ depend on the exponents of $A_k^{(j)}$, $i = 1, 2, \ldots, n$?

**Remark 2.1.** Note that generally speaking, in the decomposition of $\mathbf{H} = \sum_{i=1}^{NH} A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)}$ in the above problem, there are many selections of the operator set $\{A_i^{(j)}\}_{ij}$ (even $A_i^{(j)}$ exist that may not be self-adjoint). However, by [29], for an arbitrary (self-adjoint or non-self-adjoint) decomposition $\mathbf{H} = \sum_{i=1}^{NH} B_i^{(1)} \otimes B_i^{(2)} \otimes \ldots \otimes B_i^{(n)}$, there exists a self-adjoint decomposition $\mathbf{H} = \sum_{i=1}^{NH} A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)}$ such that

$$\text{span}\{B_1^{(j)}, B_2^{(j)}, \ldots, B_n^{(j)}\} = \text{span}\{A_1^{(j)}, A_2^{(j)}, \ldots, A_n^{(j)}\}.$$

So we always assume that $\mathbf{H}$ takes its self-adjoint decomposition in the following.

To answer the separation question, we begin the discussion with a simple case: the length $NH$ is 1, i.e., $\mathbf{H} = A_1 \otimes A_2 \otimes \ldots \otimes A_n$. Let us first deal with the case $n = 2$.

**Theorem 2.2.** Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be a bipartite system of any dimension. For a quantum gate $U = \exp[i\mathbf{H}] \in \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\mathbf{H} = A \otimes B$, the following statements are equivalent:

(I) There exist unitary operators $C, D$ such that $U = C \otimes D$;

(II) One of $A, B$ belongs to $\mathbb{R}I$.

Furthermore, there exist real scalars $\alpha, \beta$ such that either $C = \exp[i(tA + \alpha I)]$, $D = I$ if $B = tI$, or $D = \exp[i(sB + \beta I)]$, $C = I$ if $A = sI$.

Before giving the proof of Theorem 2.2, we recall the following lemma concerning the separate vectors of operator algebras. Let $\mathcal{A}$ be a $C^*$-algebra on a Hilbert space $\mathcal{H}$. A vector $|x_0\rangle \in \mathcal{H}$ is called a separate vector of $\mathcal{A}$ if, for any $T \in \mathcal{A}$, $T(|x_0\rangle) = 0 \Rightarrow T = 0$. The following lemma is necessary to complete the proof of Theorem 2.2.

**Lemma 2.3.** [35] Every Abel $C^*$-algebra has separate vectors.
**Proof of Theorem 2.2.** (II)⇒ (I) is obvious. We only need to check (I) ⇒ (II).

Assume I. Then, for any unit vectors \(|x\), \(|x'\) in the first system and \(|y\), \(|y'\) in the second system, one has

\[
U|xy⟩⟨x' y'| = \exp[iA ⊗ B]|xy⟩⟨x' y'|
= |xy⟩⟨x' y'| + iA ⊗ B|xy⟩⟨x' y'|
\]
(2.2)
\[
+ i^2 \frac{A^2 ⊗ B^2|xy⟩⟨x' y'|}{2!} + ... \\
+ i^k \frac{A^k ⊗ B^k|xy⟩⟨x' y'|}{k!} + ...
\]
and,

\[
(2.3)
U|xy⟩⟨x' y'| = C ⊗ D|xy⟩⟨x' y'|.
\]

Connecting Eq. 2.2 and 2.3 and taking a partial trace of the second (first) system respectively, we obtain that

\[
⟨y|D|y'⟩C|x⟩⟨x'| = ⟨y|y'⟩x⟩⟨x'⟩ + i⟨y|B|y'⟩A|x⟩⟨x'|
\]
\[
+ i^2 ⟨y|B^2|y'⟩ \frac{A^2}{2!}|x⟩⟨x'| + ... \\
+ i^k ⟨y|B^k|y'⟩ \frac{A^k}{k!}|x⟩⟨x'| + ...
\]
and

\[
⟨x|C|x'⟩D|y⟩⟨y'| = ⟨x|x'⟩y⟩⟨y'| + i⟨x|A|x'⟩B|y⟩⟨y'|
\]
\[
+ i^2 ⟨x|A^2|x'⟩ \frac{B^2}{2!}|y⟩⟨y'| + ... \\
+ i^k ⟨x|A^k|x'⟩ \frac{B^k}{k!}|y⟩⟨y'| + ...
\]

Then it follows from the arbitrariness of \(|x'\) and \(|y'\) that

\[
⟨y|D|y'⟩C|x⟩ = ⟨y|y'⟩I|x⟩ + i⟨y|B|y'⟩A|x⟩ + i^2 ⟨y|B^2|y'⟩ \frac{A^2}{2!}|x⟩ + ... \\
+ i^k ⟨y|B^k|y'⟩ \frac{A^k}{k!}|x⟩ + ...
\]
(2.4)
and

\[
⟨x|C|x'⟩D|y⟩ = ⟨x|x'⟩I|y⟩ + i⟨x|A|x'⟩B|y⟩ + i^2 ⟨x|A^2|x'⟩ \frac{B^2}{2!}|y⟩ + ... \\
+ i^k ⟨x|A^k|x'⟩ \frac{B^k}{k!}|y⟩ + ...
\]
(2.5)

Now there are the three cases that we should deal with.

**Case 1.** \(B = tI\). In this case, by taking \(y' = y\) in Eq. 2.4 we see that

\[
⟨y|D|y'⟩C|x⟩ = I|x⟩ + iA|x⟩ + i^2 t^2 \frac{A^2}{2!}|x⟩ + ... + i^k t^k \frac{A^k}{k!}|x⟩ + ...
= \exp[itA]|x⟩
\]
holds for all $|x\rangle$. Note that $C$ and $\exp[itA]$ are unitary, so there exists some $\alpha \in \mathbb{R}$ such that $C = \exp[i\alpha] \exp[itA] = \exp[i(tA + \alpha I)]$. It follows that $U = \exp[i(tA + \alpha I)] \otimes I$.

**Case 2.** $A = sI$. Similar to Case 1, in this case we have $D = \exp[i\beta] \exp[i sB] = \exp[i(sB + \beta I)]$ for some $\beta \in \mathbb{R}$. It follows that $U = I \otimes \exp[i(sB + \beta I)]$.

**Case 3.** $A, B \notin \mathbb{R}I$. In this case, a contradiction will be induced, so that Case 3 does not happen. Dividing the following proof to the two subcases.

**Subcase 3.1.** Both $A$ and $B$ have two distinct eigenvalues. It follows that there exist two real numbers $t_1, t_2$ with $t_1 \neq t_2$ such that $A|x_1\rangle = t_1|x_1\rangle$ and $A|x_2\rangle = t_2|x_2\rangle$, and $s_1, s_2$ with $s_1 \neq s_2$ such that $B|y_1\rangle = s_1|y_1\rangle$ and $B|y_2\rangle = s_2|y_2\rangle$. Taking $|x\rangle = |x'\rangle = |x_1\rangle$ and $|x\rangle = |x'\rangle = |x_2\rangle$ in Eq. 2.5 respectively, and $|y\rangle = |y'\rangle = |y_1\rangle$ and $|y\rangle = |y'\rangle = |y_2\rangle$ in Eq. 2.4 respectively, we have that

$$\langle x_1|C|x_1\rangle D = \exp[t_1B], \quad \langle x_2|C|x_2\rangle D = \exp[t_2B],$$

and

$$\langle y_1|D|y_1\rangle C = \exp[s_1A], \quad \langle y_2|D|y_2\rangle C = \exp[s_2A].$$

It follows that

$$\frac{\langle y_1|D|y_1\rangle \exp[t_1B]}{\exp[s_1t_1]} = D = \frac{\langle y_1|D|y_1\rangle \exp[t_2B]}{\exp[s_1t_2]}.$$  

Taking the inner product for $|y_2\rangle$ on both sides of the above equation, we have

$$\frac{\exp[t_1s_2]}{\exp[t_1s_1]} = \frac{\exp[t_2s_2]}{\exp[t_2s_1]}.$$  

It follows that $\exp[t_1s_2 - t_1s_1] = \exp[t_2s_2 - t_2s_1]$, which leads to $t_1 = t_2$ as $s_1 - s_2 \neq 0$. This is a contradiction.

**Subcase 3.2.** At least one of $A$ and $B$ has no distinct eigenvalues.

In this case, we must have $\dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \infty$ and at least one of $\sigma(A)$ and $\sigma(B)$, respectively the spectrum of $A$ and $B$, is an infinite closed subset of $\mathbb{R}$. With no loss of generality, say $\sigma(A)$ has infinite many points. Let $\mathcal{A} = \text{cl span}\{I, A, A^2, \ldots, A^n, \ldots\}$, then $\mathcal{A}$ is an Abelian $C^*$-algebra. By Lemma 2.3, $\mathcal{A}$ has a separate vector $|x_0\rangle$. Replacing $|x\rangle$ with $|x_0\rangle$ and taking vectors the $|y\rangle, |y'\rangle$ satisfying $\langle y|D|y'\rangle = 0$ in Eq. 2.4 we see that

$$0 = \langle y|D|y'\rangle C|x_0\rangle$$

$$= \langle y|y'\rangle I|x_0\rangle + \langle y|B|y'\rangle A|x_0\rangle - \langle y|B^2|y'\rangle \frac{A^2}{2} |x_0\rangle - \ldots$$

$$+ i^k \langle y|B^k|y'\rangle \frac{A^k}{k!} |x_0\rangle + \ldots$$

$$= (\sum_k \lambda_k A^k)|x_0\rangle,$$

where

$$\lambda_k = \langle y|y'\rangle I + \langle y|B|y'\rangle A + \langle y|B^2|y'\rangle \frac{A^2}{2} + \ldots + i^k \langle y|B^k|y'\rangle \frac{A^k}{k!}.$$
where $\lambda_k = \frac{i^k \langle y | B^k | y' \rangle}{k!}$. As $|x_0\rangle$ is a separate vector, we must have $\sum_k \lambda_k A^k = 0$.

We claim that each $\lambda_k = 0$. For any fixed $|y\rangle, |y'\rangle$, note that the function $f(z) = \sum_k \lambda_k z^k$ is analytic. Since $f(A) = 0$, the spectrum $\sigma(f(A))$ of $f(A)$ contains the unique element 0. So, by the spectrum mapping theorem, we have

$$\{0\} = \sigma(f(A)) = \{f(\lambda) | \lambda \in \sigma(A)\}.$$ 

Note that, by the assumption of this subcase, $\sigma(A)$ is an infinite set and has at most one isolated point. So the analytic function $f(z)$ must by zero. Then each $\lambda_k = 0$. It follows that, for each $k = 0, 1, 2, ..., n, ...$,

$$\langle y | B^k | y' \rangle = 0$$

holds for any vectors $|y\rangle, |y'\rangle$ satisfying $\langle y | D | y' \rangle = 0$. Particularly, for the case $k = 0$, we have that, for any vectors $|y\rangle, |y'\rangle$, $\langle y | D | y' \rangle = 0 \Rightarrow \langle y | y' \rangle = 0$. This ensures that $D \in \mathbb{R}I$. Now consider the case $k = 1$, one obtains that, for any vectors $|y\rangle, |y'\rangle$, $\langle y | D | y' \rangle = 0 \Rightarrow \langle y | B | y' \rangle = 0$. This implies that $B$ is linearly dependent to $D$. So we get $B \in \mathbb{R}I$, which is a contradiction.

This completes the proof. $\square$

Next, we extend Theorem 2.2 to the multipartite systems. Before stating the result, let us give some notations.

Let $A_i$s be self-adjoint operators on $\mathcal{H}_i, i = 1, 2, ..., n$ such that $\mathcal{H} = A_1 \otimes A_2 \otimes ... \otimes A_n$. If there exists at most one element in the set $\{A_1, A_2, ..., A_n\}$ that does not belong to the set $\mathbb{R}I$, we can define a scalar

$$\delta(A_j) = \begin{cases} 
\prod_{k \neq j} \lambda_k, & \text{if } A_j \notin \mathbb{R}I; \\
0, & \text{if } A_j \in \mathbb{R}I
\end{cases}$$

(2.7)

where $A_k = \lambda_k I$ if $A_k \in \mathbb{R}I$.

Based on Theorem 2.2, we reach the following conclusion in the multipartite case.

**Theorem 2.4.** Let $\otimes_{i=1}^n \mathcal{H}_i$ be a multipartite system of any dimension. For a multipartite quantum gate $U = \exp[i \mathcal{H}] \in \mathcal{U}(\otimes_{i=1}^n \mathcal{H}_i)$ with $\mathcal{H} = A_1 \otimes A_2 \otimes ... \otimes A_n$, the following statements are equivalent:

(I) There exist unitary operators $C_i \in \mathcal{U}(\mathcal{H}_i)$ $(i = 1, 2, ..., n)$ such that $U = \otimes_{i=1}^n C_i$;

(II) At most one element in $\{A_i\}_{i=1}^n$ does not belong to $\mathbb{R}I$.

Furthermore, there is a unit-model number $\lambda$ such that

$$U = \lambda \otimes_{j=1}^n \exp[i \delta(A_j) A_j],$$

(2.8)

where $\delta(A_j)$s are as that defined in Eq. 2.7.
Proof. (II) $\Rightarrow$ (I) is straightforward. To prove (I) $\Rightarrow$ (II), we use induction on $n$.

According to Theorem 2.2, (I) $\Rightarrow$ (II) is true for $n = 2$. Assume that the implication is true for $n = k$. Now let $n = k + 1$. We have that

\[
\exp[iA_1 \otimes A_2 \otimes \cdots \otimes A_{k+1}] = \exp[iH]
\]

\[
= C_1 \otimes C_2 \otimes \cdots \otimes C_k \otimes C_{k+1}
\]

\[
= T \otimes C_{k+1}.
\]

It follows from Theorem 2.2 that either $A_{k+1} \in \mathbb{R}I$ or $A_1 \otimes A_2 \otimes \cdots \otimes A_k \in \mathbb{R}I$. If $A_1 \otimes A_2 \otimes \cdots \otimes A_k \in \mathbb{R}I$, then each $A_i$ belongs to $\mathbb{R}I$. According to the induction assumption, (II) holds true. If $A_{k+1} \in \mathbb{R}I$, assume that $A_{k+1} = wI$, then

\[
\exp[iH] = \exp[iwA_1 \otimes A_2 \otimes \cdots \otimes A_k] \otimes I = C_1 \otimes C_2 \otimes \cdots \otimes C_k \otimes I.
\]

It follows from the induction assumption that (II) holds true. Eq. (2.8) is obtained by repeating to use (II) in Theorem 2.2. We complete the proof. \(\Box\)

Next we deal with the general case of $H$: $1 < N_H < \infty$. Assume that a multipartite quantum gate $U = \exp[-iH]$ with $H = \sum_{i=1}^{N_H} T_i$ and $T_i = A^{(i)}_1 \otimes A^{(i)}_2 \otimes \cdots \otimes A^{(i)}_n$. If at most one element in each set $\{A^{(i)}_1, A^{(i)}_2, \ldots, A^{(i)}_n\}$ does not belong to the set $\mathbb{R}I$, we define a function:

\[
\delta(A^{(i)}_k) = \begin{cases} 
\prod_{k \neq i} \lambda_j^{(k)}, & \text{if } A^{(i)}_j \notin \mathbb{R}I; \\
0, & \text{if } A^{(i)}_j \in \mathbb{R}I,
\end{cases}
\]

where we denote $A^{(k)}_j = \lambda_j^{(k)}I$ if $A^{(k)}_j \in \mathbb{R}I$. In the following theorem, we grasp a class of separable multipartite gates.

Before the theorem, we recall the Zassenhaus formula states that

\[
\exp[A + B] = \exp[A] \exp[B] \mathcal{P}_z(A, B),
\]

where $\mathcal{P}_z(A, B) = \prod_{i=2}^{\infty} \exp[C_i(A, B)]$ and each term $C_i(A, B)$ is a homogeneous Lie polynomial in variables $A, B$, i.e., $C_i(A, B)$ is a linear combination (with rational coefficients) of commutators of the form $[V_1, \ldots, [V_2, \ldots, [V_{m-1}, V_m] \ldots]]$ with $V_i \in \{A, B\}$ \cite{32, 34}. Especially, $C_2(A, B) = -\frac{1}{2}[A, B]$ and $C_3(A, B) = \frac{1}{3}[B, [A, B]] + \frac{1}{6}[A, [A, B]]$. As it is seen, if $\prod_{i=2}^{\infty} \exp[C_i(A, B)]$ is a multiple of the identity, then $\exp[A] \exp[B] = \lambda \exp[A + B]$ for some scalar $\lambda$. Particularly, if $AB = BA$, then $\prod_{i=2}^{\infty} \exp[C_i(A, B)] \in \mathbb{C}I$. 
Furthermore, for the multi-variable case, we have

\[
\exp\left[\sum_{i=1}^{N} A_i\right] = \prod_{i=1}^{N} \exp[A_i] P_z(A_{N-1}, A_N) \\
\cdot P_z(A_{N-2}, A_{N-1} + A_N) \cdots P_z(A_1, \sum_{j=2}^{N} A_j) \\
= \prod_{i=1}^{N} \exp[A_i] \prod_{k=1}^{N} P_z(A_k, \sum_{j=k+1}^{N} A_j).
\]

(2.11)

**Theorem 2.5.** For a multipartite quantum gate \( U \in U(\otimes_{k=1}^{n} \mathcal{H}_k) \), if \( U = \exp[-it\mathbf{H}] \) with \( \mathbf{H} = \sum_{i=1}^{N_H} T_i \) and \( T_i = A^{(1)}_i \otimes A^{(2)}_i \otimes \ldots \otimes A^{(n)}_i \) with \([T_k, T_l] = 0\) for each pair \( k, l \), and at most one element in each set \( \{A^{(1)}_i, A^{(2)}_i, \ldots, A^{(n)}_i\} \) does not belong to the set \( \mathbb{R} I \), then up to a unit modular scalar,

\[
U = U^{(1)} \otimes U^{(2)} \otimes \ldots \otimes U^{(n)},
\]

where \( U^{(i)} \) is the local quantum gate on \( \mathcal{H}_i \),

\[
U^{(i)} = \prod_{k=1}^{N_H} \exp[it\delta_k(A^{(i)}_k)A^{(i)}_k],
\]

where \( \delta_k^{(i)} \) is defined by Eq. 2.9.

**Remark.** From Theorem 2.5, we can grasp a subclass of separable multipartite gates. Each element in the subclass is of the form \( U = \exp[-it\sum_{i=1}^{N_H} T_i] \in U(\otimes_{k=1}^{n} \mathcal{H}_k) \) with \( T_i = A^{(1)}_i \otimes A^{(2)}_i \otimes \ldots \otimes A^{(n)}_i \) satisfying \([T_k, T_l] = 0\) for each pair \( k, l \), and at most one element in each set \( \{A^{(1)}_i, A^{(2)}_i, \ldots, A^{(n)}_i\} \) does not belong to \( \mathbb{R} I \).

**Proof of Theorem 2.5** Let us first observe that for any real number \( r \), \( \exp[rT] = (\exp[T])^r \). Furthermore, \( \exp[rT \otimes S] = \exp[rT] \otimes \exp[S] \) if \( \exp[T \otimes S] = \exp[T] \otimes \exp[S] \). Indeed, for arbitrary positive integer \( N \), it follows from Baker formula that \( \exp[NT] = (\exp[T])^N \). In addition, \( \exp[T] = \exp[\frac{T}{N} \cdot M] \) gives \( \exp[\frac{T}{N}] = (\exp[T])^{\frac{1}{N}} \). So, for any rational number \( a \), we have \( \exp[aT] = (\exp[T])^a \). As \( \phi(a) = \exp[aT] \) is continuous in \( a \in [0, \infty) \) and \( \exp[-T] = (\exp[T])^{-1} \), one sees that \( \exp[aT] = (\exp[T])^a \) holds for any real number \( a \).
Now according to the assumption and the definition of $\delta^{(i)}_j$, write
\[
\prod_{k=1}^{N} P_z(T_k, \sum_{j=k+1}^{N} T_j) = \lambda I
\]
since $[T_k, T_l] = 0$ for each pair $k, l$, it follows from Theorem 2.4 and Eq. 2.11 that
\[
U = \exp[itH] = \exp[it(\sum_{i=1}^{N} T_i)]
\]
\[
= \prod_{i=1}^{N} \exp[itT_i] \prod_{k=1}^{N} P_z(T_k, \sum_{j=k+1}^{N} T_j)
\]
\[
= \lambda \prod_{i=1}^{N} \exp[itT_i]
\]
\[
= \lambda \prod_{i=1}^{N} \exp[itA^{(1)}_i \otimes A^{(2)}_i \otimes \ldots \otimes A^{(n)}_i]
\]
\[
= \lambda \prod_{k=1}^{N} \exp[it\delta^{(1)}_k A^{(1)}_k] \prod_{k=1}^{N} \exp[it\delta^{(2)}_k A^{(2)}_k]
\]
\[
\otimes \ldots \otimes \prod_{k=1}^{N} \exp[it\delta^{(n)}_k A^{(n)}_k].
\]
Absorbing the unit modular scalar $\lambda$ and letting $U^{(i)} = \prod_{k=1}^{N} \exp[it\delta^{(i)}_k A^{(i)}_k]$, we complete the proof. \(\square\)

**Example 2.6** Here, we show some simple separable two-qubit gates. We assume that the Planck constant equals to one and denote by $\sigma_X, \sigma_Y$ and $\sigma_Z$ the Pauli matrices
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

In the two-qubit composite spin-$\frac{1}{2}$ system, the total spin operator $S^2$ is defined by
\[
S^2 = S_X^2 + S_Y^2 + S_Z^2,
\]
where $S_X = \sigma_X \otimes I + I \otimes \sigma_X$, $S_Y = \sigma_Y \otimes I + I \otimes \sigma_Y$, $S_Z = \sigma_Z \otimes I + I \otimes \sigma_Z$. The three operators $S_X, S_Y, S_Z$ assign $X, Y, Z$ components of spin to the composite system respectively. The $X$-spin quantum gate $U_X = \exp[-itH_X]$ with the Hamiltonian $H_X = I \otimes \sigma_X + \sigma_X \otimes I$. According to Theorem 2.2, $U_X$ is separable and
\[
U_X = \exp[-itH_X] = \exp[-it\sigma_X] \otimes \exp[-it\sigma_X].
\]
Similarly, $U_Y$ and $U_Z$ can be defined analogously, and
\[
U_Y = \exp[-itH_Y] = \exp[-it\sigma_Y] \otimes \exp[-it\sigma_Y].
\]
\[
U_Z = \exp[-itH_Z] = \exp[-it\sigma_Z] \otimes \exp[-it\sigma_Z].
\]
Furthermore, let us consider the so-called special 7-parameter Hamiltonian introduced in [31], where

\[ H = \sum_{i=1}^{4} (a_i \sigma_i \otimes I + I \otimes b_i \sigma_i), \]

with \( a_0 + b_0 = \text{tr}(H) \) (so with seven not eight parameters). Rewrite \( H = (\sum_{i=1}^{4} a_i \sigma_i) \otimes I + I \otimes (\sum_{i=1}^{4} b_i \sigma_i). \)

\[ U = \exp[-itH] = \exp[-it(\sum_{i=1}^{4} a_i \sigma_i)] \otimes \exp[-it(\sum_{i=1}^{4} b_i \sigma_i)]. \]

\[ \square \]

In the following we devote to designing an algorithm to check whether or not a multipartite gate is separable in \( n \)-qubit case (see Algorithm 2.1). We perform the experiments on the IBM quantum processor \( \text{ibmqx4} \), while generate the circuits by \( Q|SI \rangle \) (the key code segments can be obtained in https://github.com/klinus9542).

### 3. Conclusion and discussion

We established a number of evaluation criteria for the separability of multipartite gates. These criteria demonstrate that almost all \( A \in \{A_i\}_{i=1}^{n} \) should belong to \( \mathbb{R}I \) for a separable multipartite gate \( U = \exp[iH], \) where \( H = A_1 \otimes A_2 \otimes \ldots \otimes A_n. \) Most of random multipartite gates cannot fundamentally satisfy the separability condition in Theorem 2.4. So we will put forward and discuss preliminarily the approximate separation question, which has the more practical meaning. Roughly speaking, the multipartite unitary is closed to some local unitary when their Hamiltonians are close to each other.

This work reveals that there are very few quantum computational tasks (quantum circuits) that can be automatically parallelized. Concurrent quantum programming and parallel quantum programming still needs to be researched for a greater understanding of quantum specific features concerning the separability of quantum states, local operations and classical communication and even quantum networks.

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Algorithm 2.1 Check whether a unitary is separable or not

Require: $U$

Ensure: $Status, NonIndentiIndex$

1: function [$Status, NonIndentiIndex]=CheckSeparable($U$) ▷ If separable, it can
tell the status; otherwise it will answer nothing about the status
2: $H \leftarrow$ Hermitian value of $U$
3: for $index=1$:Number of System do
4: if $PosChecker(H,index)$ then
5: return $Status \leftarrow$ Separable
6: return $NonIndentiIndex \leftarrow index$
7: function $Status=PosChecker(H,index)$ ▷ Recurse solve this problem
8: if $index == 1$ then
9: $Status \leftarrow$ CheckPosLastDimN($H$)
10: else
11: \[
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
= H
\]
where $dim(C_{11}) = dim(C_{12}) = dim(C_{21}) = dim(C_{22}) = \frac{1}{2} \times dim(H)$
12: if $C_{12}$ and $C_{21}$ is NOT all 0 matrix then ▷ Counter-diagonal matrix is all 0
13: $Status \leftarrow 0$
14: else if $C_{11}$ is NOT equal to $C_{22}$ then ▷ Ensure $C_{11}$ is a repeat of $C_{22}$
15: $Status \leftarrow 0$
16: else
17: if $PosChecker(C_{11}, index-1)$ then ▷ Recursion process sub-matrix
18: return $Status \leftarrow 0$
19: else
20: return $Status \leftarrow 1$
21: function $Status=CheckPosLastDimN($H$)$ ▷ If the dimension of input matrix
great or equal to 4, conduct this process; otherwise return true
23: \[
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
= H
\]
where $dim(C_{11}) = dim(C_{12}) = dim(C_{21}) = dim(C_{22}) = \frac{1}{2} \times dim(H)$
24: if $C_{11}, C_{12}, C_{21}$ and $C_{22}$ are all diagonal matrix with only 1 element then
25: $Status \leftarrow 0$;
26: else
27: $Status \leftarrow 1$;
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