Interacting light-cone gauge conformal fields

R.R. Metsaev

Department of Theoretical Physics, P.N. Lebedev Physical Institute,
Leninsky prospect 53, Moscow 119991, Russia

Abstract

Totally symmetric arbitrary spin conformal field propagating in flat space of dimension greater than or equal to four are studied. For such fields, we develop ordinary-derivative light-cone gauge formulation. We derive explicit form of restrictions imposed by conformal algebra kinematical symmetries on interaction vertices. Cubic interaction vertices for conformal scalar fields with arbitrary conformal dimensions and vector conformal fields in space time of arbitrary dimensions are considered in details.

* E-mail: metsaev@lpi.ru
1 Introduction

Light-cone formulation of relativistic dynamics [1]-[4] offers many interesting conceptual and technical simplifications of approaches to problems of superstring theories and modern quantum field theory. As the example of a powerful application of light-cone formalism we can mention the solution to a light-cone gauge superstring field theory [5] and the construction of various supersymmetric theories in terms of off-shell superfields (see, e.g., Refs.[6]-[11]). Light-cone formalism turned also to be helpful for the construction of interaction vertices of higher-spin fields [12]-[19] (for recent study of this theme, see Refs.[20]). Interesting applications of the light-cone formalism for studying gauge/gravity duality may be found in Refs.[21]. Light-cone gauge formulation of fields dynamics in AdS space and CFT may be found in Refs.[22, 23]. In addition to above-said, we also note that, as was demonstrated in Refs.[24], light-cone gauge formulation may sometimes be a good starting point for deriving new interesting Lorentz covariant formulations.

In this paper, we develop light-cone formalism for studying interaction vertices for conformal fields. We note that commonly used formulations of most conformal fields involve higher derivatives (for review, see Ref.[25]). In Ref.[26, 27], we developed an ordinary-derivative (second-derivative) Lagrangian gauge invariant formulation for free conformal fields. This is to say that our gauge invariant and Lorentz covariant Lagrangian formulation for free bosonic conformal fields does not involve higher than second order terms in derivatives. Using such Lorentz covariant formulation in Refs.[26,27], we developed ordinary-derivative light-cone gauge formulation for free conformal fields in Refs.[28]. Our light-cone gauge Lagrangian for free bosonic conformal fields in Ref.[28] does not involve higher than second order terms in derivatives with respect to transverse spacial coordinates. Our main aim in this paper is to generalize light-cone gauge approach for free conformal fields in Ref.[28] to the case of interacting conformal fields. In this paper, we develop method for constructing interaction vertices and use this method to find explicit expressions for cubic vertices for scalar and vector fields. We consider scalar fields having arbitrary conformal dimension and vector conformal field in flat space of arbitrary dimension.

Before proceeding to main theme of this paper let us briefly review various approaches to conformal fields which have been discussed in the literature. Interacting conformal fields by using conformal space method were studied in Ref.[29]. Weyl invariant densities for conformal gravity were discussed, e.g., in Refs.[30, 31, 32]. BRST approach to conformal gravity and arbitrary spin conformal fields was discussed in respective Ref.[33] and Ref.[34]. Conformal fields in AdS and curved backgrounds were studied in Refs.[35]-[37]. Mixed-symmetry conformal fields were investigated in Ref.[38]. Scattering amplitudes for conformal fields were studied in Refs.[39]. Study of interacting higher-spin conformal fields may be found in Refs.[40, 41].

This paper is organized as follows. In Section 2, we describe ordinary-derivative light-cone gauge formulation of free arbitrary spin conformal fields propagating in flat space \( R^{d-1,1} \). For arbitrary values of \( d \), we present \( so(d-2) \) covariant version of the formalism. For \( d = 4 \), we also present helicity basis formulation of light-cone gauge conformal fields in \( R^{3,1} \).

Section 3 is devoted to \( n \)-point interaction vertices of conformal fields. In this section, we find restrictions imposed by kinematical symmetries of the conformal \( so(d, 2) \) algebra on the interaction vertices.

In Section 4, we discuss light-cone dynamical principle and present complete list of equations which are required to determine cubic interaction vertices uniquely. We apply those equations for detailed study of cubic interaction vertices for scalar conformal fields having arbitrary conformal dimensions and vector conformal field in \( R^{d-1,1} \) with \( d \)-arbitrary.
2 Free light-cone gauge conformal fields

According to the method discussed in Ref.[1], a problem of finding a new dynamical system amounts to a problem of finding a new solution of commutation relations of a basic symmetry algebra. For the case of conformal fields propagating in flat space $R^{d-1,1}$, basic symmetries are governed by the conformal algebra $so(d,2)$. Therefore we start with a discussion of a realization of the conformal algebra symmetries on a space of conformal fields. In this section, we focus on free light-cone gauge conformal fields.

The conformal algebra $so(d,2)$ is spanned by the translation generators $P^\mu$, the dilatation operator $D$, the conformal boost generators $K^\mu$ and rotation generators $J^{\mu\nu}$ which are generators of the $so(d-1,1)$ Lorentz algebra. The conformal algebra commutators we use are normalized as

\[
\begin{align*}
[D, P^\mu] &= -P^\mu, & [P^\mu, J^{\nu\rho}] &= \eta^{\mu\nu}P^\rho - \eta^{\mu\rho}P^\nu, \\
[D, K^\mu] &= K^\mu, & [K^\mu, J^{\nu\rho}] &= \eta^{\mu\nu}K^\rho - \eta^{\mu\rho}K^\nu, \\
[P^\mu, K^\nu] &= \eta^{\mu\nu}D - J^{\mu\nu}, & [J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\mu\rho}J^{\nu\sigma} + 3 \text{ terms},
\end{align*}
\]

where $\eta^{\mu\nu}$ stands for the mostly positive flat metric tensor. The translation and conformal boost generators $P^\mu, K^\mu$ are chosen to be hermitian, while the dilatation generator $D$ and Lorentz algebra generators $J^{\mu\nu}$ are considered to be antihermitian. In order to discuss the light-cone gauge approach, we, in place of the Lorentz basis coordinates $x^\mu$, introduce the light-cone basis coordinates $x^\pm, x^i$ defined by the following relations:

\[
x^\pm \equiv \frac{1}{\sqrt{2}}(x^{d-1} \pm x^0), \quad x^i, \quad i = 1, \ldots, d - 2.
\]

Throughout this paper the $x^+$ is treated as an evolution parameter. In light-cone frame, the Lorentz basis vector $X^\mu$ is decomposed as $(X^+, X^-, X^i)$ and a scalar product of two vectors of the Lorentz algebra $so(d-1,1)$ is decomposed as

\[
\eta_{\mu\nu}X^\mu Y^\nu = X^+ Y^- + X^- Y^+ + X^i Y^i.
\]

We note also contravariant and covariant components of vectors are related as $X^+ = X_-, X^- = X_+, X^i = X_i$.

In the light-cone approach, the conformal algebra generators are separated into the following two groups of kinematical and dynamical generators:

\[
P^+, \quad P^i, \quad J^{+i}, \quad J^{+-}, \quad J^{ij}, \quad D, \quad K^i, \quad K^+, \quad \text{kinematical generators;} \tag{2.4}
\]

\[
P^-, \quad J^{-i}, \quad K^-, \quad \text{dynamical generators.} \tag{2.5}
\]

In the framework of light-cone gauge approach, one postulates then that, for $x^+ = 0$, the kinematical generators (2.4) in the field realization are quadratic in the physical fields. In general, the dynamical generators given in (2.5) receive higher-order interaction-dependent corrections.

Commutation relations of the conformal algebra in light-cone basis can be obtained from the ones in (2.1) by using the light-cone flat metric having the following non vanishing elements:

\[\eta^{ij} = \delta^{ij}, \quad \eta^{+-} = 1, \quad \eta^{-+} = 1.\]

---

1 $\mu, \nu, \rho, \sigma = 0, 1, \ldots, d - 1$ are $so(d-1,1)$ vector indices; ‘transverse’ indices $i, j, k = 1, \ldots, d - 2$ are $so(d-2)$ vector indices.

2 This is to say that, for $x^+ \neq 0$ kinematical generators (2.4) admit the representation $G = G_1 + x^+ G_2$, where a functional $G_1$ is quadratic in fields, while a functional $G_2$ contains higher order terms in fields.
In order to find a realization of the $so(d, 2)$ algebra generators on a space of conformal fields we use the light-cone gauge description of those fields. We discuss scalar, vector and arbitrary spin conformal fields in turn.

Conformal scalar field with arbitrary conformal dimension in $R^{d-1,1}$. To discuss ordinary-derivative light-cone gauge formulation of conformal scalar field having conformal dimension $\Delta = \frac{d-2}{2} - k$, $k$-arbitrary positive integer, we use $k + 1$ scalar fields of the algebra $so(d - 2)$,

$$\phi_{k'}, \quad k' \in [k]_2, \quad k - \text{ arbitrary positive integer}.$$

(2.6)

In (2.6) and below, the notation $\lambda \in [k]_2$ amounts to the following summation rule:

$$\lambda \in [k]_2 \iff \lambda = -k, -k + 2, -k + 4, \ldots, k - 4, k - 2, k.$$

(2.7)

Conformal dimensions of the scalar fields $\phi_{k'}$ (2.6) are given by

$$\Delta(\phi_{k'}) = \frac{d - 2}{2} + k'. $$

(2.8)

To streamline a discussion of light-cone gauge formulation we use creation operators $v^{\ominus}, v^{\oplus}$ and the respective annihilation operators $\bar{v}^{\ominus}, \bar{v}^{\oplus}$,

$$[\bar{v}^{\ominus}, v^{\ominus}] = 1, \quad [\bar{v}^{\ominus}, v^{\oplus}] = 1, \quad \bar{v}^{\ominus} = v^{\oplus\dagger}, \quad \bar{v}^{\oplus} = v^{\ominus\dagger},$$

(2.9)

$$\bar{v}^{\ominus}|0\rangle = 0, \quad \bar{v}^{\ominus}|0\rangle = 0.$$  

(2.10)

The creation and annihilation operators (2.9), (2.10) will be referred to as oscillators in this paper. Using the oscillators $v^{\ominus}, v^{\oplus}$, we collect all scalar fields (2.6) into a ket-vector $|\phi\rangle$ defined as

$$|\phi\rangle \equiv \sum_{k' \in [k]_2} \frac{1}{(k + k')!} (v^{\ominus})^{k+k'} (v^{\oplus})^{k+k'} \phi_{k'} |0\rangle.$$  

(2.11)

The ket-vector $|\phi\rangle$ (2.11) is a degree-$k$ homogeneous polynomial in the oscillators $v^{\ominus}, v^{\oplus}$,

$$(N_v - k)|\phi\rangle = 0, \quad N_v \equiv v^{\ominus\dagger} v^{\ominus} + v^{\oplus\dagger} v^{\oplus}.$$  

(2.12)

Ordinary-derivative Lagrangian for fields (2.6) may be found in Ref.[26]. In the literature, conformal scalar field with conformal dimension $\Delta = \frac{d-2}{2} - k$ with $k > 0$, is sometimes referred to as higher-order singleton. Study of various aspects of higher-order singleton and related higher-spin algebras may be found, e.g., in Refs.[43].

Vector conformal field in $R^{d-1,1}$. We now consider conformal vector field in $R^{d-1,1}$ having conformal dimension $\Delta = 1$. To discuss ordinary-derivative light-cone gauge formulation of spin-1 conformal field in $R^{d-1,1}$, $d \geq 4$, we use $k + 1$ vector fields $\phi_{k'}^i$ and $k$ scalar fields $\phi_{k'}$:

$$\phi_{k'}^i, \quad k' \in [k]_2; \quad \phi_{k'}, \quad k' \in [k - 1]_2; \quad k \equiv \frac{d - 4}{2}.$$  

(2.13)

The vector fields $\phi_{k'}^i$ and scalar fields $\phi_{k'}$ transform in the respective vector and scalar irreps of the algebra $so(d - 2)$. These fields have the following conformal dimensions

$$\Delta(\phi_{k'}^i) = \frac{d - 2}{2} + k', \quad \Delta(\phi_{k'}) = \frac{d - 2}{2} + k'.$$  

(2.14)
We note that scalar fields $\phi_{k'}$ (2.14) enter field content only when $d \geq 6$ (i.e., $k \geq 1$).

To streamline a discussion of light-cone gauge formulation we use additional creation operators $\alpha^i, \zeta$ and the respective annihilation operators $\bar{\alpha}^i, \bar{\zeta},$

\[
[\bar{\alpha}^i, \alpha^j] = \delta^{ij}, \quad [\bar{\zeta}, \zeta] = 1, \quad \bar{\alpha}^i = \alpha^{i\dagger}, \quad \bar{\zeta} = \zeta^{\dagger},
\]  
\[\bar{\alpha}^i|0\rangle = 0, \quad \bar{\zeta}|0\rangle = 0.
\]  

(2.15) (2.16)

The creation and annihilation operators (2.15), (2.16) will also be referred to as oscillators in this paper. Using the oscillators $\zeta, \bar{\zeta}$, we collect fields (2.13) into a ket-vector $|\phi\rangle$ defined as

\[
|\phi\rangle = |\phi_1\rangle + \zeta|\phi_0\rangle,
\]  
\[|\phi_1\rangle \equiv \sum_{k' \in [k]_2} \frac{1}{(\frac{k+k'}{2})!} \alpha^{i\dagger}(v^{\ominus})^{\frac{k+k'}{2}}(v^{\oplus})^{\frac{k-k'}{2}} \phi_{k'}^i|0\rangle,
\]  
\[|\phi_0\rangle \equiv \sum_{k' \in [k-1]_2} \frac{1}{(\frac{k-1+k'}{2})!} (v^{\ominus})^{\frac{k+k'+1}{2}}(v^{\oplus})^{\frac{k-k'+1}{2}} \phi_{k'}|0\rangle.
\]  

(2.17) (2.18) (2.19)

Using (2.17)-(2.19), we verify that the ket-vectors $|\phi\rangle, |\phi_1\rangle, |\phi_0\rangle$ obey the following constraints

\[
(N_\alpha + N_\zeta)|\phi\rangle = |\phi\rangle, \quad (N_\zeta + N_{\bar{\zeta}})|\phi\rangle = k|\phi\rangle,
\]  
\[N_\alpha|\phi_1\rangle = k|\phi_1\rangle, \quad N_\zeta|\phi_0\rangle = (k-1)|\phi_0\rangle,
\]  
\[N_\alpha \equiv \alpha^{i\dagger}\bar{\alpha}^i, \quad N_\zeta \equiv \zeta\bar{\zeta}.
\]  

(2.20) (2.21) (2.22)

From algebraic constraints (2.20), we learn that the ket-vector $|\phi\rangle$ is degree-1 homogeneous polynomial in the oscillators $\alpha^i, \zeta$, and degree-$k$ homogeneous polynomial in the oscillators $\zeta, v^{\ominus}, v^{\oplus}$. From algebraic constraints (2.21), we learn that the ket-vectors $|\phi_1\rangle$ and $|\phi_0\rangle$ are the respective degree-$k$ and $k-1$ homogeneous polynomials in the oscillators $v^{\ominus}, v^{\oplus}$.

**Arbitrary spin conformal field.** Totally symmetric spin-$s$ conformal field propagating in $\mathbb{R}^{d-1,1}$ has conformal dimension $\Delta = 2-s$. To develop light-cone gauge formulation for such field we use the following set of fields of the $so(d-2)$ algebra:

\[
\phi_{k'}^{i_1\cdots i_{s'}} s' = \begin{cases} 
0, 1, \ldots, s; & \text{for } d \geq 6; \\
1, 2, \ldots, s; & \text{for } d = 4; \\
\frac{k' + d - 6}{2}. & \end{cases}
\]

(2.23)

In the set of fields given in (2.23), the fields $\phi_{k'}$ and $\phi_{k'}^i$ are the respective scalar and vector fields of the $so(d-2)$ algebra, while the field $\phi_{k'}^{i_1\cdots i_{s'}} s' \geq 2$, $s' \geq 2$, is rank-$s'$ totally symmetric traceless tensor field of the $so(d-2)$ algebra,

\[
\phi_{k'}^{i_1\cdots i_{s'}} = 0, \quad s' \geq 2.
\]  

(2.24) (2.25)

In other words, the tensor field $\phi_{k'}^{i_1\cdots i_{s'}}$ transforms as irreps of the $so(d-2)$ algebra. From (2.23), we see that the scalar fields $\phi_{k'}$ enter the field content only when $d \geq 6$.

Alternatively, field content (2.23) can be represented as

\[
\phi_{k'}^{i_1\cdots i_s}, \quad k' \in [k_s]_2;
\]  

(2.26)
Among other things, we use the $so(d - 2)$ algebra vector oscillators $\alpha^i$. Use of twistor-like variables for a discussion of conformal fields may be found in Refs.\cite{44,45}. We note also that Lorentz algebra vector oscillators are popular in the framework of BRST approach to higher-spin fields (see, e.g., Refs.\cite{46}).

\[ \phi^{i_1 \ldots i_{s-1}}_{k'}, \quad k' \in [k_s - 1]_2; \]
\[ \ldots \quad \ldots \]
\[ \ldots \quad \ldots \]
\[ \phi^i_{k'}, \quad k' \in [k_s - s + 1]_2; \]
\[ \phi_{k'}, \quad k' \in [k_s - s]_2; \]
\[ k_s \equiv s + \frac{d - 6}{2}. \]

Namely, for $d \geq 6$, the field content is given in (2.26)-(2.29), while, for $d = 4$, the field content is given in (2.26)-(2.28).

To obtain the light-cone gauge description of arbitrary spin conformal field in an easy–to–use form we use the oscillators $\alpha^i$, $\zeta$, $\upsilon^\theta$, $\upsilon^{\bar{\theta}}$, and collect fields (2.23) into ket-vector $|\phi\rangle$ defined by\footnote{\small Among other things, we use the $so(d - 2)$ algebra vector oscillators $\alpha^i$. Use of twistor-like variables for a discussion of conformal fields may be found in Refs.\cite{44,45}. We note also that Lorentz algebra vector oscillators are popular in the framework of BRST approach to higher-spin fields (see, e.g., Refs.\cite{46}).}

\[ |\phi\rangle \equiv \sum_{s'=0}^{s} \frac{\zeta^{s-s'}}{(s-s')!} |\phi^{s'}\rangle, \quad \text{for } d \geq 6, \]
\[ |\phi\rangle \equiv \sum_{s'=1}^{s} \frac{\zeta^{s-s'}}{(s-s')!} |\phi^{s'}\rangle, \quad \text{for } d = 4, \]
\[ |\phi^{s'}\rangle \equiv \sum_{k'=\lfloor k_s ceil_2} \frac{1}{s'!(k_{s'} + k')!} \alpha^{i_1} \ldots \alpha^{i_{s'}} (\upsilon^\theta)^{k_{s'}+k'} (\upsilon^{\bar{\theta}})^{k_{s'}+k'} |\phi^i_{k'} \cdots i_{s'}\rangle |0\rangle. \]

Using (2.31), (2.32), we verify that the ket-vectors $|\phi\rangle$, $|\phi^{s'}\rangle$ satisfy the following homogeneity relations

\[ (N_\alpha + N_\zeta - s)|\phi\rangle = 0, \quad (N_\zeta + N_\upsilon - k_s)|\phi\rangle = 0, \]
\[ (N_\alpha - s')|\phi^{s'}\rangle = 0, \quad (N_\upsilon - k_{s'})|\phi^{s'}\rangle = 0. \]

Definition of $k_{s'}$ is given in (2.24). From algebraic constraints (2.33), we learn that the ket-vector $|\phi\rangle$ is degree-$s$ homogeneous polynomial in the oscillators $\alpha^i$, $\zeta$ and degree-$k_s$ homogeneous polynomial in the oscillators $\zeta$, $\upsilon^\theta$, $\upsilon^{\bar{\theta}}$, while, from algebraic constraints (2.34), we learn that the ket-vector $|\phi^{s'}\rangle$ is degree-$s'$ homogeneous polynomial in the oscillators $\alpha^i$ and degree-$k_{s'}$ homogeneous polynomial in the oscillators $\upsilon^\theta$, $\upsilon^{\bar{\theta}}$. Also it is easy to see that, in terms of the ket-vector $|\phi\rangle$, the tracelessness condition (2.25) can be represented as

\[ \bar{\alpha}^2 |\phi\rangle = 0, \quad \bar{\alpha}^2 \equiv \bar{\alpha}^i \bar{\alpha}_i. \]

We now describe a realization of the conformal algebra symmetries on a space of conformal fields. A representation of the kinematical and dynamical generators in terms of differential operators acting on the ket-vector $|\phi(x^+, p)\rangle$ is given by

**kinematical generators:**

\[ P^i = p^i, \quad P^+ = \beta, \]
\[ (2.36) \]
\[ D = ix^+P^- - \partial_\beta \beta - \partial_{\rho_i} p^i + \frac{d - 2}{2} + M^{\oplus\oplus}, \]  
\[ J^+ = ix^+P^- + \partial_\beta \beta, \]  
\[ J^{+i} = ix^+p^i + \partial_{\rho_i} \beta, \]  
\[ J^{ij} = p^i \partial_{\rho^j} - p^j \partial_{\rho^i} + M^{ij}, \]  
\[ K^+ = \frac{1}{2}(2ix^+\partial_\beta - \partial_{\rho^i} \partial_{\rho^j} + M^{\oplus\oplus})\beta + ix^+D, \]  
\[ K^i = \frac{1}{2}(2ix^+\partial_\beta - \partial_{\rho^i} \partial_{\rho^j} + M^{\oplus\oplus})p^i - \partial_{\rho^i} D - M^{\oplus\oplus}p^i + M^{\oplus\oplus} + iM^{i-}x^+, \]  
\[ \text{dynamical generators:} \]  
\[ P^- = -\frac{p^i p^i + M^{\oplus\oplus}}{2\beta}, \]  
\[ J^{-i} = -\partial_\beta p^i + \partial_{\rho^i} P^- + M^{-i}, \]  
\[ K^- = \frac{1}{2}(2ix^+\partial_\beta - \partial_{\rho^i} \partial_{\rho^j} + M^{\oplus\oplus})P^- - \partial_\beta D - \partial_{\rho^i} M^{-i} - M^{\oplus\oplus}p^i + \frac{1}{\beta}B, \]  
where we use the following notation
\[ M^{\oplus\oplus} = u^{\oplus\oplus}, \]  
\[ M^{\oplus\ominus} = 4u^{\oplus\ominus}, \]  
\[ M^{\ominus\oplus} = u^{\ominus\oplus} - u^{\oplus\ominus}, \]  
\[ M^{\oplus\ominus} = e_1 \bar{\alpha}^i + A^i \bar{\alpha}_1, \]  
\[ M^{\ominus\oplus} = -r_{0,1} \bar{\alpha}^i + A^i \bar{r}_{0,1}, \]  
\[ M^{\ominus\ominus} = \alpha^i \bar{\alpha}^j - \alpha^j \bar{\alpha}^i, \]  
\[ M^{-i} = M^{ij} p^j + \frac{1}{\beta}M^{\ominus\ominus}, \]  
\[ B = -s - N_\zeta(2s + d - 4 - N_\zeta), \]  
\[ A^i = \alpha^i - \alpha^2 \frac{1}{2N_\alpha + d - 2} \bar{\alpha}^i, \]  
\[ e_1 = \zeta e_\zeta \bar{v}^{\ominus}, \quad \bar{e}_1 = -v^{\ominus} e_\zeta \bar{\zeta} \]  
\[ r_{0,1} = 2\zeta e_\zeta \bar{v}^{\ominus}, \quad \bar{r}_{0,1} = -2v^{\ominus} e_\zeta \bar{\zeta}, \]  
\[ e_\zeta = \left(\frac{2s + d - 4 - N_\zeta}{2s + d - 4 - 2N_\zeta}\right)^{1/2}, \]  
\[ \text{and the operators } N_\alpha, N_\zeta \text{ are defined in (2.22). Also we use the notation} \]
\[ \beta \equiv p^+ , \quad \partial_\beta \equiv \partial/\partial \beta , \quad \partial_{\rho^i} \equiv \partial/\partial p^i. \]  
For the reader convenience, we note the following helpful commutation relations for operators
\[ [M^{ij}, M^{kl}] = \delta^{jk} M^{il} + 3 \text{ terms}, \]
\[ [M^{\otimes i}, M^{\otimes j}] = -M^{\otimes j} M^{\otimes i}, \quad [M^{\otimes i}, M^{\otimes j}] = M^{\otimes j} M^{\otimes i}, \]
\[ [M^{\otimes i}, M^{\otimes j}] = 2M^{\otimes i} \]
\[ [M^{\otimes i}, M^{\otimes j}] = 2M^{\otimes i} \]
\[ [M^{\otimes i}, M^{\otimes j}] = 4M^{\otimes i} \]  
\[ (2.60) \]
\[ (2.61) \]
\[ (2.62) \]
\[ (2.63) \]

From the commutation relations given in (2.59), we see that a spin operator \( M^{ij} \) satisfies commutation relations of the \( so(d-2) \) algebra.

Expressions given in (2.36)-(2.45) provide a realization of the conformal algebra \( so(d,2) \) in terms of differential operators acting on the physical field \( |\phi \rangle \). To discuss light-cone gauge interaction vertices we use a field theoretical realization of the conformal algebra in terms of the physical field \( |\phi \rangle \) propagating in \( R^{d-1,1} \). As we have already said, the kinematical generators \( G^{\text{kin}} \) are realized quadratically in the field \( |\phi \rangle \), while the dynamical generators \( G^{\text{dyn}} \) are realized non-linearly in the \( |\phi \rangle \). To quadratic order in the field \( |\phi \rangle \), both \( G^{\text{kin}} \) and \( G^{\text{dyn}} \) can be presented as
\[ G_{[2]} = \int \beta d^{d-1}p \left\langle \phi(p) | G_{\text{diff}} | \phi(p) \right\rangle, \quad d^{d-1}p \equiv d\beta d^{d-2}p, \]
\[ (2.64) \]
where \( G_{\text{diff}} \) in (2.64) stands for realization of conformal algebra generators in terms of the differential operators given in (2.36)-(2.45). Throughout this paper the bra-vectors are defined according the rule
\[ \langle \phi(p) | \equiv (|\phi(p)\rangle)^\dagger. \]
\[ (2.65) \]
The field \( |\phi \rangle \) satisfies the Poisson-Dirac commutator
\[ [[|\phi(p, \alpha)\rangle, |\phi(p', \alpha')\rangle]]_{\text{equal \( x^+ = \)}} = \frac{1}{2\beta} \delta(\beta + \beta')\delta^{d-2}(p + p')|\rangle|', \]
\[ (2.66) \]
where \(|\rangle|'\rangle\) stands for a projector that respects the algebraic constraints (2.33),(2.34). Using these relations, we get the standard commutation relation
\[ [[|\phi\rangle, G_{[2]} | = G_{\text{diff}} |\phi \rangle. \]
\[ (2.67) \]
As noted in the literature (see, e.g., Refs.[5]), in the framework light-cone gauge approach, a Lagrangian for light-cone gauge interacting fields takes the standard form
\[ S = \int dx^+ d^{d-1}p \left\langle \phi(p) | i\beta \partial^- | \phi(p) \right\rangle + \int dx^+ P^-, \]
\[ (2.68) \]
where \( P^- \) is the light-cone Hamiltonian.

Incorporation of an internal symmetry into theory of interacting conformal fields can be done via the Chan–Paton method used in string theory [42] (for example, see Ref.[17]).

**Helicity basis for light-cone gauge conformal fields.** To discuss conformal fields in \( R^{d,1} \), we can use a helicity basis. To this end we introduce a frame of complex coordinates \( x^R, x^L \) defined by the relations
\[ x^R \equiv \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad x^L \equiv \frac{1}{\sqrt{2}}(x^1 - ix^2). \]
\[ (2.69) \]
In such frame, a vector of the \( so(2) \) algebra \( X^i \) is decomposed as \( X^i = X^R, X^L \), while a scalar product of two vectors \( X^i, Y^j \) is represented as \( X^i Y^j = X^R Y^L + X^L Y^R \).
In the frame of the complex coordinates, we decompose the oscillators \( \alpha^i = \alpha^R, \alpha^L \), and, in place of the ket-vector \(|\phi\rangle\) given in (2.31), (2.32), we use a ket-vector of fields defined by

\[
|\phi\rangle \equiv \sum_{s'=1}^{s} \frac{\zeta^{s-s'}}{\sqrt{(s - s')!}} |\phi^{s'}\rangle,
\]

\[
|\phi^{s'}\rangle \equiv \sum_{k'\in[k_s]} \frac{1}{\sqrt{s'! \left(\frac{k_{s}+k'}{2}\right)! \left(\frac{k_{s}-k'}{2}\right)!}} \alpha^{s'}_L \left(\psi^{s'}\right) \left(\psi^{s'}\right)^{\dagger} \phi^{s'}_k |0\rangle,
\]

(2.70)

where \( \alpha^L = \alpha^R \) and \( \phi^{s'}_k \) are fields having positive helicities \( s', s' = 1, 2, \ldots, s \).

We note also that, in the frame of the complex coordinates, the operators \( M \oplus i, M \ominus i \) are decomposed as \( M \oplus i = M \oplus R, M \oplus L, M \ominus i = M \ominus R, M \ominus L \), while the \( \text{so}(2) \) algebra generator \( M^{ij} = -M^{ji} \) is represented as \( M^{RL} \). For generators of the \( \text{so}(4, 2) \) algebra in (2.36)-(2.45), a realization on space of \(|\phi\rangle \) (2.70) can easily be read from (2.46)-(2.57),

\[
M^{\oplus\oplus} = v^\oplus \bar{v}^\oplus,
\]

(2.71)

\[
M^{\ominus\ominus} = 4v^\ominus \bar{v}^\ominus,
\]

(2.72)

\[
M^{\ominus\oplus} = v^\ominus \bar{v}^\oplus - v^\oplus \bar{v}^\ominus,
\]

(2.73)

\[
M^{\oplus R} = \alpha^R e_1,
\]

(2.74)

\[
M^{\oplus L} = e_1 \bar{\alpha}^L,
\]

(2.75)

\[
M^{RL} = \alpha^R \bar{\alpha}^L,
\]

(2.76)

\[
M^{\ominus R} = \alpha^R \bar{r}_{0,1},
\]

(2.77)

\[
M^{\ominus L} = -r_{0,1} \bar{\alpha}^L,
\]

(2.78)

\[
M^{-R} = M^{RL} \frac{p^R}{\beta} + \frac{1}{\beta} M^{\oplus R},
\]

(2.79)

\[
M^{-L} = -M^{RL} \frac{p^L}{\beta} + \frac{1}{\beta} M^{\ominus L},
\]

(2.80)

\[
e_1 = \zeta e_\zeta \bar{v}^\oplus, \quad \bar{e}_1 = -v^\oplus e_\zeta \bar{\zeta},
\]

(2.81)

\[
r_{0,1} = 2\zeta e_\zeta \bar{v}^\ominus, \quad \bar{r}_{0,1} = -2v^\ominus e_\zeta \bar{\zeta},
\]

(2.82)

\[
e_\zeta = \left(\frac{2s - N_\zeta}{2s - 2N_\zeta}\right)^{1/2},
\]

(2.83)

\[
B = -s - N_\zeta (2s - N_\zeta).
\]

(2.84)

Operators in (2.71)-(2.84) are simply obtained by restricting the operators in (2.46)-(2.57) on space of \(|\phi\rangle\) given in (2.70). The oscillators satisfy the commutators

\[
[\bar{\alpha}^L, \alpha^R] = 1, \quad \bar{\alpha}^L = \alpha^R\dagger.
\]

(2.85)

To quadratic order in fields, a field theoretical representation for generators of the \( \text{so}(4, 2) \) algebra takes the form

\[
G^{[2]} = \int \beta d^2p \langle \phi(p) | G_{\text{diff}} | \phi(p) \rangle, \quad \langle \phi(p) | \equiv | \phi(p) \rangle\dagger,
\]

(2.86)
where \( G_{\text{diff}} \) in (2.86) are obtained by restricting \( G_{\text{diff}} \) (2.36)-(2.45) on space \(|\phi\rangle\) (2.70). The Poisson-Dirac commutator for fields entering ket-vector (2.70) takes the form

\[
[\phi_{k'}(p'), \phi_{k''}(p'')^\dagger]|_{x^+} = \frac{1}{\beta'} \delta(\beta' - \beta'') \delta^2(p' - p'') \delta_{s's''} \delta_{k'k''}. \tag{2.87}
\]

### 3 Interaction vertices. Restrictions imposed by kinematical symmetries.

In this Section, we discuss the general structure of the conformal algebra dynamical generators (2.5). This is to say that we find restrictions imposed by kinematical symmetries on the dynamical generators of the \( so(d,2) \) algebra. Namely, we find equations obtained from commutation relations between the kinematical generators (2.4) and dynamical generators (2.5).

In general, in theories of interacting conformal fields, the dynamical generators receive corrections involving higher powers of physical fields. The dynamical generators can be expanded as

\[
G_{\text{dyn}} = \sum_{n=2}^{\infty} G_{[n]}^{\text{dyn}}, \tag{3.1}
\]

where \( G_{[n]}^{\text{dyn}} \) appearing in (3.1) stands for the functional that has \( n \) powers of physical fields \(|\phi\rangle\).

**Dynamical generators for arbitrary \( n \geq 3 \).** Using commutation relations of the dynamical generators with \( P_i \) and \( P^+ \), we learn that, for \( n \geq 3 \), the dynamical generators \( G_{[n]}^{\text{dyn}} \) can be presented as

\[
P_{[-n]} = \int d\Gamma_n \langle \Phi_{[n]} | p_{[-n]}^-- \rangle, \tag{3.2}
\]

\[
J_{[-i]} = \int d\Gamma_n \langle \Phi_{[n]} | j_{[-i]}^- \rangle + (X^i \langle \Phi_{[n]} | p_{[-n]}^- \rangle, \tag{3.3}
\]

\[
K_{[-i]} = \int d\Gamma_n \langle \Phi_{[n]} | k_{[-i]}^- \rangle - (X^i \langle \Phi_{[n]} | j_{[-i]}^- \rangle - \frac{1}{2} (X^i X^i \langle \Phi_{[n]} | p_{[-n]}^- \rangle, \tag{3.4}
\]

where we use the notation

\[
\langle \Phi_{[n]} | \equiv \prod_{a=1}^{n} \langle \phi(p_a, \alpha_a) |, \tag{3.5}
\]

\[
d\Gamma_n \equiv (2\pi)^{d-1} \delta^{d-1} \left( \sum_{a=1}^{n} p_a \right) \prod_{a=1}^{n} \frac{q^{d-1} p_a}{(2\pi)(d-1)/2}, \tag{3.6}
\]

\[
X^i \equiv -\frac{1}{n} \sum_{a=1}^{n} \partial p_a^i, \tag{3.7}
\]

while the ket-vectors of densities \(|p_{[-n]}^-\rangle\), \(|j_{[-i]}^-\rangle\), and \(|k_{[-i]}^-\rangle\) appearing in (3.2)-(3.4) can be presented as

\[
|p_{[-n]}^-\rangle = p_{[-n]}(p_a, \beta_a; \alpha_a)|0\rangle, \tag{3.8}
\]

\[
|j_{[-i]}^-\rangle = j_{[-i]}(p_a, \beta_a; \alpha_a)|0\rangle, \tag{3.9}
\]
In (3.5)-(3.10) and below, the indices \( p \) momentum for the case of conformal fields in (2.31) the shortcut (3.10) depend on the momenta \( \beta_a \) and \( \alpha \) appearing on r.h.s in (3.8)-(3.10) depend on the momenta \( p_a \), \( \beta_a \), and spin variables denoted by \( \alpha_a \) in this paper. Note that for the case of conformal fields in (2.31), the shortcut \( \alpha_a \) stands for the oscillators \( \alpha_a, \zeta_a, \upsilon_a, \upsilon_a^\oplus \). Note also that the density \( p_a \) will often be referred to as an \( n \)-point interaction vertex (or cubic interaction vertex when \( n = 3 \)).

\[ J^{+-} \text{-symmetry.} \] Commutation relations of the dynamical generators (2.5) with the kinematical generator \( J^{+-} \) gives the relations

\[
\sum_{a=1}^{n} \beta_a \partial_{\beta_a} |p_a^-\rangle = 0 ,
\]

\[
\sum_{a=1}^{n} \beta_a \partial_{\beta_a} |\tilde{J}_a^i\rangle = 0 ,
\]

\[
\sum_{a=1}^{n} \beta_a \partial_{\beta_a} |k_a^-\rangle = 0 .
\]

\[ D\text{-symmetry.} \] Commutation relations of the dynamical generators (2.5) with the kinematical generator \( D \) gives the relations

\[
\sum_{a=1}^{n} (\beta_a \partial_{\beta_a} + p_a^i \partial_{p_a^i}) |p_a^-\rangle = \left( 2 + \frac{d-2}{2} (2-n) + \sum_{a=1}^{n} M_{a}^{\oplus} \right) |p_a^-\rangle ,
\]

\[
\sum_{a=1}^{n} (\beta_a \partial_{\beta_a} + p_a^i \partial_{p_a^i}) |\tilde{J}_a^i\rangle = \left( 1 + \frac{d-2}{2} (2-n) + \sum_{a=1}^{n} M_{a}^{\oplus} \right) |\tilde{J}_a^i\rangle ,
\]

\[
\sum_{a=1}^{n} (\beta_a \partial_{\beta_a} + p_a^i \partial_{p_a^i}) |k_a^-\rangle = \left( \frac{d-2}{2} (2-n) + \sum_{a=1}^{n} M_{a}^{\oplus} \right) |k_a^-\rangle .
\]

\[ J^{ij} \text{-symmetries.} \] Commutation relations of the dynamical generators (2.5) with the kinematical generators \( J^{ij} \) lead to the relations

\[
\sum_{a=1}^{n} (p_a^i \partial_{p_a^i} - p_a^j \partial_{p_a^j} + M_a^{ij}) |p_a^-\rangle = 0 ,
\]

\[
\sum_{a=1}^{n} (p_a^i \partial_{p_a^i} - p_a^j \partial_{p_a^j} + M_a^{ij}) |\tilde{J}_a^{ij}\rangle = \delta^{ik} |\tilde{J}_a^i\rangle - \delta^{jk} |\tilde{J}_a^j\rangle ,
\]

\[
\sum_{a=1}^{n} (p_a^i \partial_{p_a^i} - p_a^j \partial_{p_a^j} + M_a^{ij}) |k_a^-\rangle = 0 .
\]

\[ K^i \text{-symmetries.} \] Commutation relations of the dynamical generators (2.5) with the kinematical generators \( K^i \) gives the relations

\[
|\tilde{J}_a^{ij}\rangle - K^{ij} |p_a^-\rangle - X^{ij} |p_a^-\rangle = 0 ,
\]
where we use the notation

\[ K^i \equiv \sum_{a=1}^{n} K^i_a , \]

\[ K^i_a = \frac{1}{2} \rho_a ( - \partial_{p_a} \partial_{p_a} + M_a^{\circ \circ} \right) + D^i_a \partial_{p_a} - M^{ij} \partial_{p_a} + M^{\circ i} , \]

\[ D^i_a = \beta_a \partial_{\beta_a} + p_a \partial_{p_a} + \frac{d-2}{2} + M_a^{\circ \circ} . \]

Using (3.7), (3.23), we note the following helpful relation

\[ [K^i, X^j] = \frac{1}{2n} \delta^{ij} \sum_{a=1}^{n} M_a^{\circ \circ} + \frac{1}{n} \sum_{a=1}^{n} (\partial_{p_a} \partial_{p_a} - \frac{1}{2} \delta^{ij} \partial_{p_a} \partial_{p_a}) . \]

**\( K^+ \)-symmetry.** Commutations relations of the dynamical generators (2.5) with the kinematical generator \( K^+ \) tell us that the densities \( p^i_{[n]} \), \( j^i_{-i} \), and \( k^i_{-i} \) should satisfy the following equations

\[ \sum_{a=1}^{n} \beta_a (M_a^{\circ \circ} - \partial_{p_a} \partial_{p_a}) |p^i_{[n]} \rangle = 0 , \]

\[ \sum_{a=1}^{n} \beta_a (M_a^{\circ \circ} - \partial_{p_a} \partial_{p_a}) |j^i_{-i} \rangle = 0 , \]

\[ \sum_{a=1}^{n} \beta_a (M_a^{\circ \circ} - \partial_{p_a} \partial_{p_a}) |k^i_{-i} \rangle = 0 . \]

**\( J^+i \)-symmetries.** Commutations relations of the dynamical generators (2.5) with the kinematical generators \( J^+i \) tell us that the densities \( p^i_{[n]} \), \( j^i_{-i} \), and \( k^i_{-i} \), appearing on the r.h.s in (3.8)-(3.10) depend on the momenta \( p^i_{[n]} \) through the new momentum variables \( \mathbb{P}^{i}_{ab} \) defined by

\[ \mathbb{P}^{i}_{ab} \equiv p^i_{a} \beta_b - p^i_{b} \beta_a . \]

In other words, the densities \( p^i_{[n]} \), \( j^i_{-i} \), \( k^i_{-i} \) (3.8)-(3.10) turn out to be functions of \( \mathbb{P}^{i}_{ab} \) in place of \( p^i_{[n]} \),

\[ p^i_{-i} = p^i_{-i} (\mathbb{P}^{i}_{ab}, \beta_a ; \alpha_a) , \]

\[ j^i_{-i} = j^i_{-i} (\mathbb{P}^{i}_{ab}, \beta_a ; \alpha_a) , \]

\[ k^i_{-i} = k^i_{-i} (\mathbb{P}^{i}_{ab}, \beta_a ; \alpha_a) . \]

We summarize our study of kinematical symmetries for dynamical generators (2.5) by the following two remarks.
i) The commutation relations between the dynamical generators \(G_{[3]}^{\text{dyn}}\) and the kinematical generators \(J^{+}, D, J^{ij}, K^{i}, K^{+}\) lead to equations (3.11)-(3.29) for the densities \(p_{[n]}, j_{[n]}, k_{[n]}\) (3.8)-(3.10).

ii) The commutation relations between the dynamical generators \(G_{[3]}^{\text{dyn}}\) and the kinematical generators \(J^{+}\) tell us that the densities \(p_{[n]}, j_{[n]}^{i}, k_{[n]}\) (3.8)-(3.10) turn out to be functions of \(P^{i}_{ab}\) in place of \(p_{a}^{i}\) (3.31)-(3.33).

Using definition in (3.30), we verify that by virtue of momentum conservation laws not all momenta \(P^{i}_{ab}\) are independent. Namely, we verify that the \(n\)-point vertex involves \(n-2\) independent momenta \(P^{i}_{ab}\). This implies that, for \(n = 3\), there is only one independent \(P^{i}_{ab}\). This simplifies the kinematical symmetry equations for the dynamical generators above-discussed. We discuss now kinematical symmetry equations for cubic densities \(p_{[3]}, j_{[3]}^{i}, k_{[3]}\).

**Kinematical symmetry equations of dynamical generators \(G_{[3]}^{\text{dyn}}\).** Using the momentum conservation laws

\[
p_{1}^{i} + p_{2}^{i} + p_{3}^{i} = 0, \quad \beta_{1} + \beta_{2} + \beta_{3} = 0, \quad (3.34)
\]

we verify that momenta \(P^{i}_{12}, P^{i}_{23}, P^{i}_{31}\) can be expressed in terms of a new momentum \(P^{i}\) as

\[
P^{i}_{12} = P^{i}_{23} = P^{i}_{31} = P^{i}, \quad (3.35)
\]

where the new momentum \(P^{i}\) is defined by the relations

\[
P^{i} = \frac{1}{3} \sum_{a=1}^{3} \beta_{a} P^{i}_{a}, \quad P^{i}_{a} = \beta_{a+1} - \beta_{a+2}, \quad \beta_{a} \equiv \beta_{a+3}. \quad (3.36)
\]

We prefer to use the momentum \(P^{i}\) (3.36) because this momentum is manifestly invariant under cyclic permutations of the external line indices 1, 2, 3. Therefore the densities \(p_{[3]}^{i}, j_{[3]}^{i}, k_{[3]}\) are eventually a functions of \(P^{i}, \beta_{a}\) and \(\alpha_{a}\):

\[
p_{[3]}^{i} = p_{[3]}^{i}(P, \beta_{a}; \alpha_{a}), \quad j_{[3]}^{i} = j_{[3]}^{i}(P, \beta_{a}; \alpha_{a}), \quad k_{[3]} = k_{[3]}(P, \beta_{a}; \alpha_{a}). \quad (3.37)
\]

We now represent the kinematical symmetry equations (3.11)-(3.29) in terms of the densities given in (3.37).

**\(J^{+}\)-symmetry equations:**

\[
(P^{i}_{j} \partial_{p}^{j} + \sum_{a=1}^{3} \beta_{a} \partial_{\beta_{a}}) |p_{[3]}^{-}\rangle = 0, \quad (3.38)
\]

\[
(P^{i}_{j} \partial_{p}^{j} + \sum_{a=1}^{3} \beta_{a} \partial_{\beta_{a}}) |j_{[3]}^{-i}\rangle = 0, \quad (3.39)
\]

\[
(P^{i}_{j} \partial_{p}^{j} + \sum_{a=1}^{3} \beta_{a} \partial_{\beta_{a}}) |k_{[3]}^{-}\rangle = 0. \quad (3.40)
\]

**\(D\)-symmetry equations:**

\[
P^{i}_{j} \partial_{p}^{j} |p_{[3]}^{-}\rangle = \left(\frac{6 - d}{2} + M^{\oplus_{0}}\right) |p_{[3]}^{-}\rangle, \quad (3.41)
\]

\[
P^{i}_{j} \partial_{p}^{j} |j_{[3]}^{-i}\rangle = \left(\frac{4 - d}{2} + M^{\oplus_{0}}\right) |j_{[3]}^{-i}\rangle. \quad (3.42)
\]
\[ \mathbb{P}^j \partial_{\mathbb{P}^j} |k_{[a]}^-\rangle = \left( \frac{2 - d}{2} + M^{\oplus\oplus} \right) |k_{[a]}^-\rangle , \]  
\[ M^{\oplus\oplus} \equiv \sum_{a=1}^{3} M^{\oplus\oplus}_a . \]  
(3.43)

Note that for derivation of equations (3.41)-(3.43) we use equations (3.38)-(3.40).

**J^{ij}**-symmetry equations:

\[ J^{ij} |p_{[a]}^-\rangle = 0 , \]  
(3.45)
\[ J^{ij} |j_{[a]}^-k_{[a]}\rangle = \delta^{jk} |j_{[a]}^-i_{[a]}\rangle - \delta^{ik} |j_{[a]}^-j_{[a]}\rangle , \]  
(3.46)
\[ J^{ij} |k_{[a]}^-\rangle = 0 , \]  
(3.47)

where we use the notation

\[ J^{ij} \equiv L^{ij}(\mathbb{P}) + M^{ij} , \]  
(3.48)
\[ L^{ij}(\mathbb{P}) \equiv \mathbb{P}^i \partial_{\mathbb{P}^j} - \mathbb{P}^j \partial_{\mathbb{P}^i} , \quad M^{ij} \equiv \sum_{a=1}^{3} M^{ij}_a . \]  
(3.49)

**K^{i}**-symmetry equations:

\[ |j_{[a]}^-\rangle - K^{\dagger i} |p_{[a]}^-\rangle = 0 , \]  
(3.50)
\[ K^{\dagger i} |j_{[a]}^-j_{[a]}\rangle - [K^{\dagger i}, X^j] |p_{[a]}^-\rangle + \delta^{ij} |k_{[a]}^-\rangle = 0 , \]  
(3.51)
\[ K^{\dagger i} |k_{[a]}^-\rangle + [K^{\dagger i}, X^i] |j_{[a]}^-\rangle = 0 , \]  
(3.52)

where in (3.50)-(3.52) we use the realization of operators \( K^{\dagger i} \) and \([K^{\dagger i}, X^j] \) on space of densities depending on \( \mathbb{P}^i, \beta_a, \) and \( \alpha_a (3.37) \):

\[ K^{\dagger i} = (\mathbb{D}_\beta - M^{\oplus\oplus}) \partial_{\mathbb{P}^i} - M^{ij} \partial_{\mathbb{P}^j} + M^{\ominus i} - \frac{\mathbb{P}^i}{6 \beta} \sum_{a=1}^{3} \beta_a \delta_a M^{\oplus\oplus}_a , \]  
(3.53)
\[ [K^{\dagger i}, X^j] = \frac{1}{6} \delta^{ij} M^{\oplus\oplus} + \frac{\Delta \beta}{9} (\partial_{\mathbb{P}^i} \partial_{\mathbb{P}^j} - \frac{1}{2} \delta^{ij} \partial_{\mathbb{P}^i} \partial_{\mathbb{P}^j}) , \]  
(3.54)
\[ M^{\oplus\oplus} \equiv \frac{1}{3} \sum_{a=1}^{3} \delta_a M^{\oplus\oplus}_a , \quad M^{\dagger i} \equiv \frac{1}{3} \sum_{a=1}^{3} \delta_a M^{\dagger i}_a , \]  
(3.55)
\[ M^{\ominus i} \equiv \sum_{a=1}^{3} M^{\ominus i}_a , \quad M^{\ominus i} \equiv \sum_{a=1}^{3} M^{\ominus i}_a , \]  
(3.56)
\[ \mathbb{D}_\beta \equiv \frac{1}{3} \sum_{a=1}^{3} \beta_a \delta_a \partial_{\beta_a} , \quad \Delta \beta \equiv \sum_{a=1}^{3} \beta^2_a , \]  
(3.57)
\[ \beta \equiv \beta_1 \beta_2 \beta_3 . \]  
(3.58)

Note also that for derivation of Eqs. (3.50)-(3.52) we use Eqs. (3.20)-(3.22) and relation \([X^i, \mathbb{P}^j] = 0 \).
Kinematical symmetry equations:

\[ (\beta \partial_{\mathcal{P}} \partial_{\mathcal{P}} + \sum_{a=1}^{3} \beta_a M_a) |p_{[i]}\rangle = 0 , \]  
\[ (\beta \partial_{\mathcal{P}} \partial_{\mathcal{P}} + \sum_{a=1}^{3} \beta_a M_a) |j^{-i}_{[i]}\rangle = 0 , \]  
\[ (\beta \partial_{\mathcal{P}} \partial_{\mathcal{P}} + \sum_{a=1}^{3} \beta_a M_a) |k^{-i}_{[i]}\rangle = 0 , \]

where \( \beta \) is defined in (3.58).

Kinematical restrictions by themselves are not sufficient to determine the dynamical generators \( G^\text{dyn}_{[i]} \) uniquely. These generators can be fixed by studying additional requirement which we refer to as light-cone dynamical principle.

4 Light-cone dynamical principle

Our general strategy consists of the following three steps which we refer to as the light-cone dynamical principle:

i) First, we find restrictions imposed by the conformal algebra commutation relations between the dynamical generators (2.5). In other words, at this step we consider the following commutation relations

\[ [P^-, J^{-i}] = 0 , \]  
\[ [P^-, K^-] = 0 , \quad [J^{-i}, J^{-j}] = 0 , \quad [J^{-i}, K^-] = 0 . \]

ii) Second, we require a vertex \( p_{[i]}^- \) and the corresponding densities \( j^{-i}_{[i]} \), \( k^{-i}_{[i]} \) (3.37) be polynomials in \( \mathbb{P}^i \). This requirement will be referred to as the light-cone locality condition.

iii) Third, we find a vertex \( p_{[i]}^- \) that cannot be removed by using field redefinitions.

We now use the light-cone dynamical principle for studying a vertex \( p_{[i]}^- \) and the corresponding densities \( j^{-i}_{[i]} \), \( k^{-i}_{[i]} \). To this end, we note the following important feature of light-cone gauge formulation of conformal fields. Kinematical \( K^i \)-symmetry equations given in (3.50)-(3.52) are obtained from the commutation relations

\[ [P^-, K^i] = -J^{-i} , \quad [J^{-i}, K^j] = \delta^{ij} K^- , \quad [K^-, K^i] = 0 . \]

Using Jacoby identities, we verify that if commutators (4.1) and (4.3) are satisfied, then the commutators (4.2) are satisfied automatically. Thus, if we respect \( K^i \)-symmetry equations (3.50)-(3.52), then we can restrict ourselves to the study of commutators in (4.1). To cubic order in fields, the commutators (4.1) take the form

\[ [P_{[i]}^-, J_{[i]}^{-i}] + [P_{[i]}^-, J_{[i]}^{-i}] = 0 . \]

Equations (4.4) amount to the following equations for the densities \( p_{[i]}^- (\mathbb{P}, \beta_a ; \alpha_a) \) and \( j_{[i]}^{-i} (\mathbb{P}, \beta_a ; \alpha_a) \),

\[ J_{[i]}^{-i} |p_{[i]}^-\rangle + P^- |j_{[i]}^{-i}\rangle = 0 , \]  
\[ 15 \]
where we use the notation
\[
P^{-} \equiv \sum_{a=1}^{3} P_{a}^{-} , \quad J^{-i}_{a} \equiv \sum_{a=1}^{3} J_{a}^{-i} , \quad (4.6)
\]
\[
P_{a}^{-} \equiv -p_{a}^{i} p_{a}^{i} + M_{a}^{\oplus} / 2 \beta_{a} , \quad (4.7)
\]
\[
J_{a}^{-i} \equiv p_{a}^{i} \partial_{\beta_{a}} - p_{a}^{-} \partial_{p_{a}^{i}} - 1 / \beta_{a} (M_{a}^{ij} p_{a}^{j} + M_{a}^{ai}) . \quad (4.8)
\]

Taking into account that \(p_{[3]}^{-}\) and \(j_{[3]}^{-i}\) appearing in (4.5) depend on the generic momenta \(p_{i}^{a}\) through the momenta \(P_{i}^{a}\), we represent \(P^{-}\) and \(J^{-i}\) as follows
\[
P^{-} = \frac{P_{i}^{a} P_{i}^{a}}{2 \beta} - \sum_{a=1}^{3} M_{a}^{\oplus} / 2 \beta_{a} , \quad (4.9)
\]
\[
J^{-i} = - \frac{P_{i}^{a} \partial_{\beta_{a}}}{\beta} + \frac{M_{a}^{ij} P_{j}^{a} + \sum_{a=1}^{3} 6 \beta_{a} M_{a}^{\oplus} \partial_{p_{a}^{i}} - 1 / \beta_{a} M_{a}^{ai}}{6 \beta_{a} M_{a}^{\oplus}} . \quad (4.10)
\]

where we use the notation as in (3.55)-(3.58).

We now are ready to demonstrate the attractive feature of equations for cubic vertex \(p_{[3]}^{-}\) of conformal fields. Namely, we note that Eq.(3.50) allows us to express \(j_{[3]}^{-i}\) in terms of \(p_{[3]}^{-}\). Doing so and using Eq.(4.5), we obtain closed equations for the cubic vertex \(p_{[3]}^{-}\),
\[
(J^{-i} + P^{-} K^{i})|p_{[3]}^{-}⟩ = 0 . \quad (4.11)
\]

Thus, to cubic order in fields, we exhaust all commutation relations of the conformal algebra \(so(d, 2)\). Equations (4.11) together with kinematical symmetry equations given in (3.38)-(3.61) provide the complete list of restrictions imposed by commutation relations of the conformal algebra on the densities \(p_{[3]}^{-}\), \(j_{[3]}^{-i}\), and \(k_{[3]}^{-}\). To select physically relevant densities \(p_{[3]}^{-}\), \(j_{[3]}^{-i}\), \(k_{[3]}^{-}\), i.e. to determine them uniquely, we use the light-cone locality condition. In other words we require the densities \(p_{[3]}^{-}\), \(j_{[3]}^{-i}\), and \(k_{[3]}^{-}\), to be polynomials in \(P_{a}^{a}\). We now consider various interaction vertices for scalar and vector fields in turn.

**Cubic vertex for conformal scalar fields in \(R^{d-1,1}\).** We start with a discussion of cubic interaction vertex for three conformal scalar fields having conformal dimensions given by
\[
\Delta_{a} = \frac{d-2}{2} - k_{a} , \quad k_{a} \geq 0 , \quad k_{a} \text{ integers} . \quad (4.12)
\]

Our result for a cubic vertex \(p_{[3]}^{-}\) and the corresponding densities \(j_{[3]}^{-i}\), \(k_{[3]}^{-}\) is given by
\[
p_{[3]}^{-} = v^{n} \prod_{a=1}^{3} \upsilon_{a}^{\oplus n_{a}} , \quad (4.13)
\]
\[
j_{[3]}^{-i} = 0 , \quad (4.14)
\]
\[
k_{[3]}^{-} = \frac{1}{6} M^{\oplus} p_{[3]}^{-} , \quad (4.15)
\]
where we use the notation

\[ n = \frac{1}{2} \left( \frac{6 - d}{2} + \sum_{a=1}^{3} k_a \right), \tag{4.16} \]

\[ n_a = \frac{1}{2} \left( \frac{d - 6}{2} + k_a - k_{a+1} - k_{a+2} \right), \tag{4.17} \]

\[ v = v_1^\oplus v_2^\oplus v_3^\oplus + v_2^\oplus v_3^\oplus v_1^\oplus + v_3^\oplus v_1^\oplus v_2^\oplus, \tag{4.18} \]

and \( M^{\ominus\ominus} \) is defined as in (3.56).

We now discuss the restrictions to be imposed on the values of \( k_a \) and space-time dimension \( d \). To this we note that the powers of the form \( v \) and oscillators \( v_a^\oplus \) in (4.13) must be non-negative integers,

\[ n \geq 0, \quad n_a \geq 0 \quad \text{for all } a, \tag{4.19} \]

where \( n \) and \( n_a \) are defined in (4.16), (4.17). Using (4.16), (4.17), we note that inequalities (4.19) amount to the following restrictions:

\[ k - 2k_{\min} \leq \frac{d - 6}{2} \leq k, \quad k \equiv \sum_{a=1}^{3} k_a, \quad k_{\min} = \min_{a=1,2,3} k_a. \tag{4.20} \]

Restrictions (4.20) lead to a surprisingly simple result for allowed values of space-time dimensions. Indeed, taking into account that \( n \) and \( n_a \) should be integer, we obtain that given values \( k_1, k_2, k_3 \), the space-time dimension \( d \) takes the values

\[ d = 2k + 6, 2k + 2, 2k - 2, \ldots, 2k + 6 - 4k_{\min}. \tag{4.21} \]

Relation (4.21) implies that given values \( k_1, k_2, k_3 \), the number of space-times which admit conformal invariant cubic vertices for scalar fields having conformal dimensions as in (4.12) is given by

\[ k_{\min} + 1. \tag{4.22} \]

We note a interesting similarity of relations (4.20), (4.21), and (4.22) in this paper with the respective relations (5.13), (5.15), and (5.16) in Ref. [18].

For the illustration purposes, we note that considering a cubic vertex for conformal scalar fields having canonical dimensions, i.e. the case \( k_a = 0, a = 1, 2, 3 \), we see that (4.22) implies that there is only one space-time admitting a cubic interaction vertex for such conformal scalar fields, while, from (4.21), we learn that dimension of that space-time is equal to \( d = 6 \).

As a side remark we note that, in terms of \( k_a \), the restrictions \( n_a \geq 0, a = 1, 2, 3, (4.17) \) can be represented as

\[ k_1 + k_2 - \frac{d - 6}{2} \leq k_3 \leq \frac{d - 6}{2} - |k_1 - k_2|. \tag{4.23} \]

Note that the restriction \( d \leq 2k + 6 \) has a simple explanation. Namely, in the framework of higher-derivative approach, the scalar conformal field having conformal dimension \( \Delta = \frac{d-2}{2} - k \) is described by a field \( \phi_{-k} \) entering (2.6) with \( k' = -k \). Writing symbolically a higher-derivative cubic Lagrangian for three scalar fields \( \phi_{-k_a} \) with conformal dimensions as in (4.12)

\[ L_{[3]}^{\text{high-deriv}} = \partial^1 \phi_{-k_1} \partial^2 \phi_{-k_2} \partial^3 \phi_{-k_3}, \tag{4.24} \]

17
where \( \mathcal{D} \) stands for \( l \) derivatives, we note that requiring the dilatation symmetry gives the relation

\[
d = 2k + 6 - \sum_{a=1}^{3} l_a.
\]

(4.25)

Taking into account that \( l_a \geq 0 \), we see from (4.25) that \( d \leq 2k + 6 \).

Note that our vertex (4.13) does not involve momenta \( P^i \). For the case of three scalar fields, a general cubic vertex can depend on \( P^i P^i \). However one can make sure that using field redefinitions dependence of the vertex on \( P^i P^i \) can be removed.

Cubic vertices for two conformal scalar fields and one conformal vector field in \( R^{d-1,1} \). We now discuss cubic vertex for two scalar fields having the same conformal dimensions \( \Delta = \frac{d-2}{2} - k \), where \( k \) is integer, \( k \geq 0 \), and one vector conformal field with conformal dimension \( \Delta = 1 \). Let us use the notation \((s, \Delta)\) to label conformal field having spin \( s \) and conformal dimension \( \Delta \). Using such notation we note that we consider a vertex for three fields with the following values of \( s \) and \( \Delta \):

\[
(s_1 = 1, \Delta_1 = 1), \quad (s_2 = 0, \Delta_2 = \frac{d-2}{2} - k), \quad (s_3 = 0, \Delta_3 = \frac{d-2}{2} - k),
\]

(4.26)

i.e. the conformal vector field carries external line index \( a = 1 \), while the two conformal scalar fields carry external line indices \( a = 2, 3 \). We are interested in the vertex which does not involve higher than first order terms in the momenta \( P^i \). Our result for such vertex \( p_{[3]}^- \) and the corresponding densities \( j_{[3]}^- \), \( k_{[3]}^- \) is given by

\[
p_{[3]}^- = A_1 v^m v_1^{\ominus n_1},
\]

\[
+ \frac{m}{\beta_1} \zeta_1 v_2 v_3 v_1 v_1^{\ominus n_1} - \frac{n_1 \beta_1}{2 \beta_1} \zeta_1 v^m v_1^{\ominus (n_1 - 1)},
\]

(4.27)

\[
\dot{j}_{[3]}^- = \frac{2 \beta_1}{3} \alpha_1 v^m v_1^{\ominus n_1},
\]

(4.28)

\[
k_{[3]}^- = \frac{4m \beta_1}{3 \beta_1} \zeta_1 v_2 v_3 v_3 v_1 v_1^{\ominus n_1} + \frac{1}{6} M^{\ominus \ominus} p_{[3]}^-,
\]

(4.29)

\[
m \equiv k, \quad n_1 \equiv \frac{d-4}{2} - k,
\]

(4.30)

\[
A_a \equiv \frac{P^i \alpha_a}{\beta_a}, \quad \beta_a \equiv \beta_{a+1} - \beta_{a+2},
\]

(4.31)

\[
v \equiv v_1^{\ominus} v_2^{\ominus} v_3^{\ominus} + v_2^{\ominus} v_3^{\ominus} v_1^{\ominus} + v_3^{\ominus} v_1^{\ominus} v_2^{\ominus},
\]

(4.32)

\[
v_{ab} \equiv v_a^{\ominus} v_b^{\ominus} \beta_b - v_b^{\ominus} v_a^{\ominus} \beta_a,
\]

(4.33)

\[
\bar{\beta}_1 = \beta_2 - \beta_3,
\]

(4.34)

where \( M^{\ominus \ominus} \) is defined in (3.56).

From (4.30), we see that \( m = 0, n_1 = 0 \) when \( d = 4 \) and \( k = 0 \). For this case, our vertex (4.37) describes a standard cubic interaction vertex of one Yang-Mills field and two massless scalar fields in \( R^{3,1} \).
We now discuss the restrictions to be imposed on the values \( k \) and space-time dimension \( d \). To this end we note that the powers of the form \( v \) and the oscillator \( v^\ominus_1 \) in (4.27) must be non-negative integers. Using (4.30), we find

\[
0 \leq k \leq \frac{d - 4}{2}.
\]

We note that the restriction \( k \geq 0 \) is satisfied by assumption from the very beginning.

**Cubic vertex for three conformal vector fields in \( R^{d-1,1} \).** We now discuss cubic vertex for self-interacting vector conformal field in \( R^{d-1,1} \) which has a conformal dimension \( \Delta = 1 \). As before we use the notation \((s, \Delta)\) to label conformal field having spin \( s \) and conformal dimension \( \Delta \). Using such notation, we consider a vertex for three fields with the following values of \( s \) and \( \Delta \):

\[
(s_1 = 1, \Delta_1 = 1), \quad (s_2 = 1, \Delta_2 = 1), \quad (s_3 = 1, \Delta_3 = 1).
\]

We are interested in a vertex that does not involve higher than first order terms in the momenta \( \mathbb{P}^i \). Our result for such vertex \( p^\ominus_{[3]} \) and the corresponding densities \( j^\ominus_{[3]} \), \( k^\ominus_{[3]} \) is given by

\[
p^\ominus_{[3]} = \sum_{a=1}^{3} A_a (v \alpha_{a+1} \zeta_{a+2} v^\ominus_a) v^{m-1}
\]

\[
+ \sum_{a=1}^{3} m \zeta_a (v \alpha_{a+1} \zeta_{a+2} v^\ominus_a) v_{a+1} v_{a+2} v^{m-2}
\]

\[
- \frac{m \tilde{\beta}}{2\beta} \zeta_1 \zeta_2 \zeta_3 v^{m-1},
\]

\[
j^\ominus_{[3]} = -3 \sum_{a=1}^{3} 2\tilde{\beta}_a \alpha_a (v \alpha_{a+1} \zeta_{a+2} v^\ominus_a) v^{m-1},
\]

\[
k^\ominus_{[3]} = -3 \sum_{a=1}^{3} \frac{4m \tilde{\beta}_a \zeta_a}{\beta_a} v^\ominus_{a+1} v^\ominus_{a+2} \left(v \alpha_{a+1} \zeta_{a+2} v^\ominus_a - (m - 1) \zeta_{a+1} \zeta_{a+2} v^\ominus_a\right) v^{m-2}
\]

\[
+ \frac{1}{6} \mathbb{M}^\ominus p^\ominus_{[3]},
\]

\[
m \equiv \frac{d - 4}{2},
\]

\[
A_a \equiv \frac{\mathbb{P}^i \alpha_a}{\beta_a}, \quad \alpha_{ab} \equiv \alpha_a \alpha_b,
\]

\[
v \equiv v^\oplus_1 v^\ominus_2 v^\ominus_3 + v^\oplus_2 v^\ominus_3 v^\ominus_1 + v^\oplus_3 v^\ominus_1 v^\ominus_2,
\]

\[
v_{ab} \equiv v^\ominus_a v^\ominus_b \beta_a - v^\ominus_b v^\ominus_a \beta_b
\]

\[
\beta \equiv \beta_1 \beta_2 \beta_3, \quad \tilde{\beta} \equiv \tilde{\beta}_1 \tilde{\beta}_2 \tilde{\beta}_3, \quad \tilde{\beta}_a \equiv \beta_{a+1} - \beta_{a+2},
\]

where \( \mathbb{M}^\ominus \) is defined as in (3.56).

From (4.40), we see that \( m = 0 \) when \( d = 4 \). For this case our vertex (4.37) describes a standard cubic interaction vertex of the Yang-Mills fields in \( R^{3,1} \).

We note also that requiring the power of the form \( v \) (4.39) to be non-negative integer, \( m \geq 0 \), leads to the restriction (see (4.40))

\[
d \geq 4.
\]

19
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