A. MATHEMATICAL NOTATION

In this short reference appendix, we report the classical mathematical notation and some common definitions that are used along the whole work.

Classic objects. We consider \( \mathbb{N} \) as the set of natural numbers and \( [m, n] \triangleq \{ k \in \mathbb{N} : m \leq k \leq n \} \), \( \langle m, n \rangle \triangleq \{ k \in \mathbb{N} : m < k < n \} \), \( |m, n| \triangleq \{ k \in \mathbb{N} : m < k < n \} \), and \( \{ k \in \mathbb{N} : m < k < n \} \) as its interval subsets, where \( \mathbb{N} \) is the set of natural numbers, \( \omega \) is the cardinality of \( \mathbb{N} \), i.e., the number of its elements, where \( \omega \) represents a more than countable cardinality, and by \( 2^\mathbb{X} \triangleq \{ y : y \subseteq X \} \) the powerset of \( X \), i.e., the set of all its subsets.

Relations. By \( R \subseteq X \times Y \) we denote a relation between the domain \( \text{dom}(R) \triangleq X \) and codomain \( \text{cod}(R) \triangleq Y \), whose range is indicated by \( \text{rng}(R) \triangleq \{ y \in Y : \exists x \in X . (x, y) \in R \} \). We use \( R^{-1} \triangleq \{ (y, x) \in Y \times X : (x, y) \in R \} \) to represent the inverse of \( R \) itself. Moreover, by \( S \circ R \), with \( R \subseteq X \times Y \) and \( S \subseteq Y \times Z \), we denote the composition of \( R \) with \( S \), i.e., the relation \( S \circ R \triangleq \{ (x, z) \in X \times Z : \exists y \in Y . (x, y) \in R \wedge (y, z) \in S \} \). We also use \( R^n \triangleq R^{n-1} \circ R \), with \( n \in [1, \omega] \), to indicate the n-iteration of \( R \subseteq X \times Y \), where \( Y \subseteq X \) and \( R^0 \triangleq \{ (y, y) : y \in Y \} \) is the identity on \( Y \). With \( R^+ \triangleq \bigcup_{n \in \omega} R^n \) and \( R^* \triangleq R^+ \cup R^0 \) we denote, respectively, the transitive and reflexive-transitive closure of \( R \). Finally, for an equivalence relation \( R \subseteq X \times X \), we represent \( (X/R) \triangleq \{ [x]_R : x \in X \} \), where \( [x]_R \triangleq \{ x' \in X : (x, x') \in R \} \), the quotient set of \( X \) w.r.t. \( R \), i.e., the set of all related equivalence classes \( [\_]_R \).

Functions. We use the symbol \( Y^X \subseteq 2^{X \times Y} \) to denote the set of total functions \( f \) from \( X \) to \( Y \), i.e., the relations \( f \subseteq X \times Y \) such that for all \( x \in \text{dom}(f) \) there is exactly one element \( y \in \text{cod}(f) \) such that \( (x, y) \in f \). Often, we write \( f : X \to Y \) and \( f : X \rightharpoonup Y \) to indicate, respectively, \( f \subseteq Y^X \) and \( f \subseteq \bigcup_{X \subseteq X} Y^{X'} \). Regarding the latter, note that we consider \( f \) as a partial function from \( X \) to \( Y \), where \( \text{dom}(f) \subseteq X \) contains all and only the elements for which \( f \) is defined. Given a set \( Z \), by \( f_Z \triangleq f \cap (Z \times X) \) we denote the restriction of \( f \) to the set \( X \cap Z \), i.e., the function \( f_Z : X \cap Z \to Y \) such that, for all \( x \in \text{dom}(f) \cap Z \), it holds that \( f_Z(x) = f(x) \). Moreover, with \( \emptyset \) we indicate a generic empty function, i.e., a function with empty domain. Note that \( X \cap Z = \emptyset \) implies \( f_Z = \emptyset \). Finally, for two partial functions \( f, g : X \to Y \), we use \( f \cup g \) and \( f \cap g \) to represent, respectively, the union and intersection of these functions defined as follows: \( \text{dom}(f \cup g) \triangleq \text{dom}(f) \cup \text{dom}(g) \setminus \{ x : \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x) \} \), \( \text{dom}(f \cap g) \triangleq \{ x \in \text{dom}(f) \cap \text{dom}(g) : f(x) = g(x) \} \), \( f \cup g)(x) = f(x) \) for \( x \in \text{dom}(f \cup g) \cap \text{dom}(f) \), \( f \cup g)(x) = g(x) \) for \( x \in \text{dom}(f \cup g) \cap \text{dom}(g) \), and \( f \cap g)(x) = f(x) \) for \( x \in \text{dom}(f \cap g) \).

Words. By \( X^n \), with \( n \in \mathbb{N} \), we denote the set of all \( n \)-tuples of elements from \( X \), by \( X^* \triangleq \bigcup_{n=0}^{\omega} X^n \) the set of finite words on the alphabet \( X \), by \( X^+ \triangleq X^* \setminus \{ \varepsilon \} \) the set of non-empty words.
and by $X^\omega$ the set of infinite words, where, as usual, $\varepsilon \in X^*$ is the empty word. The length of a word $w \in X^\omega \triangleq X^* \cup X^\omega$ is represented with $|w| \in \overline{\mathbb{N}}$. By $(w)_i$, we indicate the $i$-th letter of the finite word $w \in X^+$, with $i \in [0, \ |w| ]$. Furthermore, by $\text{fst}(w) \triangleq (w)_0$ (resp., $\text{lst}(w) \triangleq (w)_{|w|-1}$), we denote the first (resp., last) letter of $w$. In addition, by $(w)_{\leq i}$ (resp., $(w)_{> i}$), we indicate the prefix up to (resp., suffix after) the letter of index $i$ of $w$, i.e., the finite word built by the first $i + 1$ (resp., last $|w| - i - 1$) letters $(w)_0, \ldots, (w)_i$ (resp., $(w)_{i+1}, \ldots, (w)_{|w|-1}$). We also set, $(w)_{< 0} \triangleq \varepsilon$, $(w)_{< i} \triangleq (w)_{< i-1}$, $(w)_{\geq 0} \triangleq w$, and $(w)_{> i} \triangleq (w)_{> i-1}$, for $i \in [1, \ |w| ]$. Mutatis mutandis, the notations of $i$-th letter, first, prefix, and suffix apply to infinite words too. Finally, by $\text{prefix}(w_1, w_2) \in X^\omega$ we denote the maximal common prefix of two different words $w_1, w_2 \in X^\omega$, i.e., the finite word $w \in X^*$ for which there are two words $w_1', w_2' \in X^\omega$ such that $w_1 = w \cdot w_1'$, $w_2 = w \cdot w_2'$, and $\text{fst}(w_1') \neq \text{fst}(w_2')$. By convention, we set $\text{prefix}(w, w) \triangleq w$.

Trees. For a set $\Delta$ of objects, called directions, a $\Delta$-tree is a set $T \subseteq \Delta^*$ closed under prefix, i.e., if $t \cdot d \in T$, with $d \in \Delta$, then also $t \in T$. We say that it is complete if it holds that $t \cdot d' \in T$ whenever $t \cdot d \in T$ for all $d' < d$, where $< \subseteq \Delta \times \Delta$ is an a priori fixed strict total order on the set of directions that is clear from the context. Moreover, it is full if $T = \Delta^*$. The elements of $T$ are called nodes and the empty word $\varepsilon$ is the root of $T$. For every $t \in T$ and $d \in \Delta$, the node $t \cdot d \in T$ is a successor of $t$ in $T$. The tree is $b$-bounded if the maximal number $b$ of its successor nodes is finite, i.e., $b = \max_{t \in T} \{|t \cdot d \in T : d \in \Delta|\} < \omega$. A branch of the tree is an infinite word $w \in \Delta^\omega$ such that $(w)_{\leq i} \in T$, for all $i \in \mathbb{N}$. For a finite set $\Sigma$ of objects, called symbols, a $\Sigma$-labeled $\Delta$-tree is a quadruple $(\Sigma, \Delta, T, \nu)$, where $T$ is a $\Delta$-tree and $\nu : T \rightarrow \Sigma$ is a labeling function. When $\Delta$ and $\Sigma$ are clear from the context, we call $(T, \nu)$ simply a (labeled) tree.

B. PROOFS OF SECTION 4

In this appendix, we report the proofs of lemmas needed to prove the behavorial of $\text{SL}[1\|]$. Before this, we describe two relevant properties that link together Skolem dependence functions of a given quantification prefix with those of the dual one. These properties report, in the Skolem dependence functions framework, what is known to hold, in an equivalent way, for first and second order logic. In particular, they result to be two key points towards a complete understanding of the strategy quantifications of our logic.

The first of these properties enlighten the fact that two arbitrary dual Skolem dependence functions $\theta$ and $\overline{\theta}$ always share a common valuation $\nu$. To better understand this concept, consider for instance the functions $\theta_1$ and $\overline{\theta}_6$ of the examples illustrated just after Definition 4.4 of Skolem dependence functions. Then, it is easy to see that the valuation $\nu \in \text{Val}_{D}(V)$ with $\nu(x) = \nu(y) = 1$ and $\nu(z) = 0$ resides in both the ranges of $\theta_1$ and $\overline{\theta}_6$, i.e., $\nu \in \text{rng}(\theta_1) \cap \text{rng}(\overline{\theta}_6)$.

**Lemma B.1 (Dependence Incidence).** Let $\varphi \in \text{Qnt}(V)$ be a quantification prefix over a set of variables $V \subseteq \text{Var}$ and $D$ a generic set. Moreover, let $\theta \in \text{SDF}_{D}(\varphi)$ and $\overline{\theta} \in \text{SDF}_{D}(\overline{\varphi})$ be two Skolem dependence functions. Then, there exists a valuation $\nu \in \text{Val}_{D}(V)$ such that $\nu = \theta(\nu_{[\varphi]} \overline{\nu}_{[\overline{\varphi}]} \nu_{[\overline{\varphi}]}$).

**Proof.** W.l.o.g., suppose that $\varphi$ starts with an existential quantifier. If this is not the case, the dual prefix $\overline{\varphi}$ necessarily satisfies the above requirement, so, we can simply shift our reasoning on it.

The whole proof proceeds by induction on the alternation number $\text{alt}(\varphi)$ of $\varphi$. As base case, if $\text{alt}(\varphi) = 0$, we define $\nu \triangleq \theta(\varphi)$, since $[\varphi] = \emptyset$. Obviously, it holds that $\nu = \theta(\nu_{[\varphi]} \overline{\nu}_{[\overline{\varphi}]}$. due to the fact that $\nu_{[\varphi]} = \emptyset$ and $\nu_{[\overline{\varphi}]} = \nu$. Now, as inductive case, suppose that the statement is true for all prefixes $\varphi' \in \text{Qnt}(V')$ with $\text{alt}(\varphi') = n$, where $V' \subseteq V$. Then, we prove that it is true for all prefixes $\varphi \in \text{Qnt}(V)$ with $\text{alt}(\varphi) = n + 1$ too. To do this, we have to uniquely split $\varphi = \varphi' \cdot \varphi''$ into the two prefixes $\varphi' \in \text{Qnt}(V')$ and $\varphi'' \in \text{Qnt}(V \setminus V')$ such that $\text{alt}(\varphi') = n$ and $\text{alt}(\varphi'') = 0$. At this point, the following two cases can arise.
— If \( n \) is even, it is immediate to see that \( \langle v' \rangle = \emptyset \). So, consider the Skolem dependence functions \( \theta' \in \text{SDF}_D(\langle v' \rangle) \) and \( \overline{\theta} \in \text{SDF}_D(\langle v \rangle) \) such that \( \theta'(v_{1|v'}) = \theta(v_{1|v'}) \) and \( \overline{\theta}(v) = \overline{\theta}(v_{1|v'}) \), for all valuations \( v \in \text{Val}_D(\langle v \rangle) \) and \( \overline{v} \in \text{Val}_D(\langle v \rangle) = \text{Val}_D(\langle v \rangle) \). By the inductive hypothesis, there exists a valuation \( v' \in \text{Val}_D(V') \) such that \( v' = \theta'(v_{1|v'}) = \overline{\theta}(\langle v \rangle_{1|v}) \). So, set \( v \triangleq \overline{\theta}(\langle v \rangle_{1|v}) \).

— If \( n \) is odd, it is immediate to see that \( \langle v' \rangle = \emptyset \). So, consider the Skolem dependence functions \( \theta' \in \text{SDF}_D(\langle v' \rangle) \) and \( \overline{\theta} \in \text{SDF}_D(\langle v \rangle) \) such that \( \theta'(v) = \theta(v_{1|v'}) \) and \( \overline{\theta}(\langle v \rangle_{1|v}) = \overline{\theta}(\langle v \rangle_{1|v}) \), for all valuations \( v \in \text{Val}_D(\langle v \rangle) \) = \text{Val}_D(\langle v \rangle) \). By the inductive hypothesis, there exists a valuation \( v' \in \text{Val}_D(V') \) such that \( v' = \theta'(v_{1|v'}) = \overline{\theta}(\langle v \rangle_{1|v}) \). So, set \( v \triangleq \theta(v_{1|v'}) \).

Now, it is easy to see that in both cases the valuation \( v \) satisfies the thesis, i.e., \( v = \theta(v_{1|v'}) = \overline{\theta}(\langle v \rangle_{1|v}) \). □

The second property we are going to prove describes the fact that, if all Skolem dependence functions \( \theta \) of a given prefix \( \varphi \), for a dependent specific universal valuation \( v \), share a given property then there is a dual Skolem dependence functions \( \overline{\theta} \) that has the same property, for all universal valuations \( \overline{v} \). To have a better understanding of this idea, consider again the examples reported just after Definition 4.4 and let \( P \triangleq \{ (0, 0, 1), (0, 1, 0) \} \subset \text{Val}_D(V) \), where the triple \( (l, m, n) \) stands for the valuation that assigns \( l \) to \( x \), \( m \) to \( y \), and \( n \) to \( z \). Then, it is easy to see that all ranges of the Skolem dependence functions \( \theta_i \) for \( \varphi \) intersect \( P \), i.e., for all \( i \in [0, 3] \), there is \( v \in \text{Val}_D(\langle v \rangle) \) such that \( \theta_i(v) \in P \). Moreover, consider the dual Skolem dependence functions \( \overline{\theta}_2 \) for \( \overline{v} \). Then, it is not hard to see that \( \overline{\theta}_2(\overline{v}) \in P \), for all \( \overline{v} \in \text{Val}_D(\langle \overline{v} \rangle) \).

**Lemma B.2 (Dependence Dualization).** Let \( \varphi \in \text{Qnt}(V) \) be a quantification prefix over a set of variables \( V \subseteq \text{Var}, D \) a generic set, and \( P \subseteq \text{Val}_D(V) \) a set of valuations of \( V \) over \( D \). Moreover, suppose that, for all Skolem dependence functions \( \theta \in \text{SDF}_D(\varphi) \), there is a valuation \( v \in \text{Val}_D(\langle \varphi \rangle) \) such that \( \theta(v) \in P \). Then, there exists a Skolem dependence function \( \overline{\theta} \in \text{SDF}_D(\langle \varphi \rangle) \) such that, for all valuations \( \overline{v} \in \text{Val}_D(\langle \overline{\varphi} \rangle) \), it holds that \( \overline{\theta}(\overline{v}) \in P \).

**Proof.** The proof easily proceeds by induction on the length of the prefix \( \varphi \). As base case, when \( |\varphi| = 0 \), we have that \( \text{SDF}_D(\varphi) = \text{SDF}_D(\overline{\varphi}) = \{ \emptyset \} \), i.e., the only possible Skolem dependence functions is the empty function, which means that the statement is vacuously verified. As inductive case, we have to distinguish between two cases, as follows.

\[ \varphi = \langle \langle x \rangle \rangle \cdot \varphi' \]

As first thing, note that \( \langle \varphi \rangle = \langle \varphi' \rangle \) and, for all elements \( e \in D \), consider the projection \( P_e \triangleq \{ v' \in \text{Val}_D(V(\varphi')) : v'[x \mapsto e] \in P \} \) of \( P \) on the variable \( x \) with value \( e \).

Then, by hypothesis, we can derive that, for all \( e \in D \) and \( \theta' \in \text{SDF}_D(\langle v' \rangle) \), there exists \( v' \in \text{Val}_D(\langle \varphi' \rangle) \) such that \( \theta'(v') \in P_e \). Indeed, let \( e \in \text{Val}_D(\langle v' \rangle) \) and build the function \( \theta : \text{Val}_D(\langle \varphi \rangle) \rightarrow \text{Val}_D(V) \) given by \( \theta(v) \triangleq \theta'(v'[x \mapsto e]) \), for all \( v \in \text{Val}_D(\langle \varphi \rangle) = \text{Val}_D(\langle \varphi \rangle) \). It is immediate to see that \( \theta \in \text{SDF}_D(\varphi) \). So, by the hypothesis, there is \( v' \in \text{Val}_D(\langle \varphi' \rangle) \) such that \( \theta(v') \in P_e \), which implies \( \theta'(v'[x \mapsto e]) \in P_e \), and so, \( \theta'(v') \in P_e \).

Now, by the inductive hypothesis, for all elements \( e \in D \), there exists \( \overline{\theta}_e \in \text{SDF}_D(\langle \overline{\varphi} \rangle) \) such that, for all \( \overline{v} \in \text{Val}_D(\langle \overline{\varphi} \rangle) \), it holds that \( \overline{\theta}_e(\overline{v}) \in P_e \), i.e., \( \overline{\theta}_e(\overline{v}[x \mapsto e]) \in P \).

At this point, consider the function \( \overline{\theta}_e : \text{Val}_D(\langle \overline{\varphi} \rangle) \rightarrow \text{Val}_D(V) \) given by \( \overline{\theta}(v) \triangleq \overline{\theta}_e(\overline{v}[x \mapsto \overline{v}[x]]) \), for all \( \overline{v} \in \text{Val}_D(\langle \overline{\varphi} \rangle) \). Then, it is possible to verify that \( \overline{\theta} \in \text{SDF}_D(\varphi) \).

Indeed, for each \( y \in \langle \varphi \rangle \) and \( \overline{v} \in \text{Val}_D(\langle \overline{\varphi} \rangle) \), we have that \( \overline{\theta}(\overline{v}[x \mapsto \overline{v}[x] = \overline{\theta}_e(\overline{v}[x \mapsto \overline{v}[x]]) \). Now, if \( \overline{v} = x \) then \( \overline{\theta}(\overline{v}[x] = \overline{v}(y) \). Otherwise, since \( \overline{\theta}_e(\overline{v}[x]) \) is a Skolem dependence function, it holds that \( \overline{\theta}(\overline{v}[x]) = \overline{\theta}_e(\overline{v}[x]) = \overline{v}(y) \). So, Item 1 of Definition 4.4 of Skolem dependence functions is verified. It only remains to prove Item 2. Let
\( y \in \langle \langle \tau \rangle \rangle \) and \( \tau_1, \tau_2 \in \text{Val}_D(\langle \langle \tau \rangle \rangle) \), with \( \tau_1|_{\text{Dep}(\tau,y)} = \tau_2|_{\text{Dep}(\tau,y)} \). It is immediate to see that \( x \in \text{Dep}(\tau,y) \), so, \( \tau_1(x) = \tau_2(x) \), which implies that \( \bar{\theta}_{\tau(x)} = \bar{\theta}_{\tau_2(x)} \). At this point, again for the fact that \( \bar{\theta}_{\tau(x)} \) is a Skolem dependence function, for each \( \tau \in \text{Val}_D(\langle \langle \tau \rangle \rangle) \), we have that \( \bar{\theta}_{\tau(x)}(\tau_1|_{\langle \langle \tau \rangle \rangle})(y) = \bar{\theta}_{\tau(x)}(\tau_2|_{\langle \langle \tau \rangle \rangle})(y) \). Thus, it holds that \( \bar{\theta}(\tau_1)(y) = \bar{\theta}(\tau)(\tau_1|_{\langle \langle \tau \rangle \rangle})(y) \). Finally, it is enough to observe that, by construction, \( \bar{\theta}(\tau) \in P \), for all \( \tau \in \text{Val}_D(\langle \langle \tau \rangle \rangle) \), since \( \bar{\theta}_{\tau(x)}(\tau_1|_{\langle \langle \tau \rangle \rangle})(y) \in P_{\tau(x)} \). Thus, the thesis holds for this case.

\[ \neg \varphi = \langle \langle x \rangle \rangle \cdot \varphi' \].

We first show that there exists \( e \in D \) such that, for all \( \varphi' \in \text{SDF}_D(\langle \langle \varphi \rangle \rangle) \), there is \( \varphi' \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \) for which \( \theta'(\varphi') \in P_e \) holds, where the set \( P_e \) is defined as above.

To do this, suppose by contradiction that, for all \( e \in D \), there is a \( \theta'_e \in \text{SDF}_D(\langle \langle \varphi \rangle \rangle) \), such that, for all \( \varphi' \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), it holds that \( \theta'_e(\varphi') \not\in P_e \). Also, consider the function \( \theta : \text{Val}_D(\langle \langle \varphi \rangle \rangle) \rightarrow \text{Val}_D(V) \) given by \( \theta(\varphi) \triangleq \theta'_e(\varphi) \), for all \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \). Then, is possible to verify that \( \theta(\varphi) \in \text{SDF}_D(\langle \langle \varphi \rangle \rangle) \). Indeed, for each \( y \in \langle \langle \varphi \rangle \rangle \) and \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), we have that \( \theta(\varphi)(y) = \theta'_e(\varphi)(y) \), for all \( y \in \langle \langle \varphi \rangle \rangle \) and \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \). Otherwise, since \( \theta'_e(\varphi) \) is a Skolem dependence function, it holds that \( \theta(\varphi)(y) = \theta'_e(\varphi)(y) \). So, Item 1 of Definition 4.4 of Skolem dependence functions is verified. It only remains to prove Item 2. Let \( y \in \langle \langle \varphi \rangle \rangle \) and \( v_1, v_2 \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), with \( v_1|_{\text{Dep}(\varphi,y)} = v_2|_{\text{Dep}(\varphi,y)} \). It is immediate to see that \( x \in \text{Dep}(\varphi,y) \), so, \( v_1(x) = v_2(x) \), which implies that \( \theta'_e(v_1(x)) = \theta'_e(v_2(x)) \). This point, again for the fact that \( \theta'_e(\varphi) \) is a Skolem dependence function, for each \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), we have that

\[ \theta'(v_1)|_{\langle \langle \varphi \rangle \rangle}(y) = \theta'(v_2)|_{\langle \langle \varphi \rangle \rangle}(y) \].

Now, by the contradiction hypothesis, we have that \( \theta(\varphi) \not\in P \), for all \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), since \( \theta'(v_1)|_{\langle \langle \varphi \rangle \rangle} \not\in P_{\varphi(x)} \), which is in evident contradiction with the hypothesis.

At this point, by the inductive hypothesis, there exists \( \overline{\theta} \in \text{SDF}_D(\langle \langle \varphi \rangle \rangle) \) such that, for all \( \tau \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), it holds that \( \overline{\theta}(\tau) \in P \), i.e., \( \overline{\theta}(\langle \langle \varphi \rangle \rangle)|_{\langle \langle \tau \rangle \rangle} \in P \).

Finally, build the function \( \bar{\theta} : \text{Val}_D(\langle \langle \varphi \rangle \rangle) \rightarrow \text{Val}_D(V) \) given by \( \bar{\theta}(\varphi) \triangleq \overline{\theta}(\langle \langle \varphi \rangle \rangle)|_{\langle \langle \tau \rangle \rangle} \), for all \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \) = \( \text{Val}_D(\langle \langle \varphi \rangle \rangle) \). It is immediate to see that \( \bar{\theta} \in \text{SDF}_D(\langle \langle \varphi \rangle \rangle) \). Moreover, for all valuations \( \varphi \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \), it holds that \( \bar{\theta}(\varphi) \in P \). Thus, the thesis holds for this case too.

Hence, we have done with the proof of the lemma. \( \square \)

At this point, we are able to give the proofs of Lemma 4.8 of adjoint Skolem dependence functions, Lemma 4.17 of dependence-vs-valuation duality, and Lemma 4.19 of enascertainment characterization.

**Lemma B.3 (Adjoint Skolem Dependence Functions).** Let \( \varphi \in \text{Qnt}(V) \) be a quantification prefix over a set of variables \( V \subseteq \text{Var} \), and \( D \) and \( T \) two generic sets, and \( \theta : \text{Val}_{T \rightarrow D}(\langle \langle \varphi \rangle \rangle) \rightarrow \text{Val}_{T \rightarrow D}(V) \) and \( \bar{\theta} : T \rightarrow (\text{Val}_{T \rightarrow D}(\langle \langle \varphi \rangle \rangle) \rightarrow \text{Val}_{T \rightarrow D}(V)) \) two functions such that \( \bar{\theta} \) is the adjoint of \( \theta \). Then, \( \theta \in \text{SDF}_{T \rightarrow D}(\varphi) \) iff, for all \( t \in T \), it holds that \( \theta(t) \in \text{SDF}_D(\varphi) \).

**Proof.** To prove the statement, it is enough to show, separately, that Items 1 and 2 of Definition 4.4 of Skolem dependence functions hold for \( \theta \) if \( \bar{\theta}(t) \) satisfies the same items, for all \( t \in T \), and vice versa.

**[Item 1, iff]** Assume that \( \bar{\theta}(t) \) satisfies Item 1, for each \( t \in T \), i.e., \( \bar{\theta}(t)(\langle \langle \varphi \rangle \rangle)|_{\langle \langle t \rangle \rangle} = v \), for all \( v \in \text{Val}_D(\langle \langle \varphi \rangle \rangle) \). Then, we have that \( \bar{\theta}(t)(\langle \langle \varphi \rangle \rangle)|_{\langle \langle t \rangle \rangle} = \bar{\theta}(t)(\langle \langle \varphi \rangle \rangle)|_{\langle \langle t \rangle \rangle} = v \), for all \( g \in \text{Val}_{T \rightarrow D}(\langle \langle \varphi \rangle \rangle) \) and \( x \in \langle \langle \varphi \rangle \rangle \). By hypothesis, we have that \( \bar{\theta}(g)(x)(t) = \bar{\theta}(t)(\langle \langle \varphi \rangle \rangle)|_{\langle \langle t \rangle \rangle} = v \), for all \( g \in \text{Val}_{T \rightarrow D}(\langle \langle \varphi \rangle \rangle) \).

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At this point, let $\theta$ satisfies Item 1, i.e., $\theta(g)_{\|\|\|v\|} = g$, for all $g \in ValT_{D}(\|v\|)$.

Then, we have that $\theta(g)(x)(t) = g(x)(t)$, for all $x \in \|v\|$ and $t \in T$. By hypothesis, we have that $\theta(t)(g(x))(t) = \theta(t)(g)(x)(t)$, so $\theta(t)(g(x))(t) = g(x)(t) = g(t)(x)$, which means that $\theta(t)(g(x))(t)_{\|\|\|v\|} = g(t)(x)_{\|\|\|v\|}$.

Now, since for each $v \in ValD(\|v\|)$, there is an $g \in ValT_{D}(\|v\|)$ such that $\theta(t) = v$, we obtain that $\theta(t)(v)_{\|\|\|v\|} = v$, for all $v \in ValD(\|v\|)$ and $t \in T$.

Item 2, if. Assume that $\theta(t)$ satisfies Item 2, for each $t \in T$, i.e., $\theta(t)(v_{1})(x) = \theta(t)(v_{2})(x)$, for all $v_{1}, v_{2} \in ValD(\|v\|)$ and $x \in \|v\|$ such that $v_{1}, v_{2} \in ValD(\|v\|)$ and $x \in \|v\|$. Then, we have that $\theta(t)(g_{1})(x)(t) = \theta(t)(g_{2})(x)(t)$, for all $g_{1}, g_{2} \in ValT_{D}(\|v\|)$ such that $g_{1}, g_{2} \in ValT_{D}(\|v\|)$ such that $g_{1} = g_{2}$.

By hypothesis, we have that $\theta(t)(g_{1})(x)(t) = \theta(t)(g_{1})(x)(t)$ and $\theta(t)(g_{2})(x)(t) = \theta(t)(g_{2})(x)(t)$, thus $\theta(t)(g_{1})(x)(t) = \theta(t)(g_{2})(x)(t)$.

Hence, $\theta(t)(g_{1})(x)(t) = \theta(t)(g_{2})(x)(t)$, for all $g_{1}, g_{2} \in ValT_{D}(\|v\|)$ and $x \in \|v\|$ such that $g_{1}, g_{2} \in ValT_{D}(\|v\|)$.

Item 2, only if. Assume that $\theta(t)$ satisfies Item 2, i.e., $\theta(g_{1})(x) = \theta(g_{2})(x)$, for all $g_{1}, g_{2} \in ValT_{D}(\|v\|)$ and $x \in \|v\|$ such that $g_{1}, g_{2} \in ValT_{D}(\|v\|)$ and $x \in \|v\|$. Then, we have that $\theta(g_{1})(x)(t) = \theta(g_{2})(x)(t)$, for all $t \in T$. By hypothesis, we have that $\theta(t)(g_{1})(x)(t) = \theta(t)(g_{1})(x)(t)$ and $\theta(t)(g_{2})(x)(t) = \theta(t)(g_{2})(x)(t)$, hence $\theta(t)(g_{1})(x)(t) = \theta(t)(g_{2})(x)(t)$.

Now, since for each $v_{1}, v_{2} \in ValD(\|v\|)$, with $v_{1}, v_{2} \in ValD(\|v\|)$, there are $g_{1}, g_{2} \in ValT_{D}(\|v\|)$ such that $g_{1}(t) = v_{1}$ and $g_{2}(t) = v_{2}$, with $g_{1} = g_{2}$, we obtain that $\theta(t)(v_{1})(x) = \theta(t)(v_{2})(x)$, for all $v_{1}, v_{2} \in ValD(\|v\|)$ and $x \in \|v\|$ such that $v_{1}, v_{2} \in ValD(\|v\|)$.
functions such that $s_{\rho}((\wp)_{i+1}) = ((\wp)_{i}, \theta)$ and $s_{\rho}((\wp)_{i}) = ((\wp)_{i}, \theta)$. Consequently, by substituting the values of the even schemes $s_{\rho}$ and $s_{\rho}$, it holds that $((\wp)_{i+1}) = s_{\rho}((\wp)_{i} \cdot ((\wp)_{i}, \theta))$ and $((\wp)_{i+1}) = s_{\rho}((\wp)_{i} \cdot ((\wp)_{i}, \theta))$. Furthermore, by the definition of the odd schemes $s_{\rho}$ and $s_{\rho}$, it follows that $s_{\rho}((\wp)_{i} \cdot ((\wp)_{i}, \theta)) = \tau((\wp)_{i}, \theta \circ \zeta_{i}) = s_{\rho}((\wp)_{i} \cdot ((\wp)_{i}, \theta))$. Thus, we have that $((\wp)_{i+1}) = ((\wp)_{i+1})$, which implies $(\wp)_{i+1} = ((\wp)_{i+1})$.

If $\rho$: Suppose that player odd wins the dual TPG $H(\mathcal{G}, s, P_{th}(s) \setminus \wp)$, then there exists an odd scheme $s_{\wp} \in Sch_{\mathcal{G}}$ such that, for all even schemes $s_{\wp} \in Sch_{\mathcal{G}}$, it holds that $mtc_{\mathcal{G}}(s_{\wp}, s_{\wp}) \in P$.

Now, to prove that $\rho$ wins the TPG $H(\mathcal{G}, s, P, \wp, b)$, we have to show that there exists an even scheme $s_{\wp} \in Sch_{\mathcal{G}}$ such that, for all odd schemes $s_{\wp} \in Sch_{\mathcal{G}}$, it holds that $mtc_{\mathcal{G}}(s_{\wp}, s_{\wp}) \in P$.

To do this, let us first consider the two functions $g : Tr(k(s)) \rightarrow 2^{Val_{\mathcal{A}}(V)}$ and $h : Tr(k(s)) \rightarrow 2^{s_{\wp}}$ such that $g(\rho) = \{\bar{\theta}(\wp) : \bar{\theta} \in SDF_{\mathcal{A}}(\wp) \land \wp \in Val_{\mathcal{A}}(\wp)\}$ and $h(\rho) = \{\bar{\theta}(\rho \cdot (\operatorname{lst}(\rho), \bar{\theta})) : \bar{\theta} \in SDF_{\mathcal{A}}(\wp)\}$, for all $\rho \in Tr(k(s))$. Now, it is easy to see that, for each $\rho \in Tr(k(s))$ and $\bar{\theta} \in SDF_{\mathcal{A}}(\wp)$, there is $\wp \in Val_{\mathcal{A}}(\wp)$ such that $\bar{\theta}(\wp) \in g(\rho)$. Consequently, by Lemma B.2 on dependence dualization, for all $\rho \in Tr(k(s))$, there is $\theta_{\rho} \in SDF_{\mathcal{A}}(\wp)$ such that, for each $\wp \in Val_{\mathcal{A}}(\wp)$, it holds that $\theta_{\rho}(\wp) \in g(\rho)$, and so, $\tau(\operatorname{lst}(\rho), \theta_{\rho}(\wp) \circ \zeta_{i}) \in h(\rho)$.

Now, define the even scheme $s_{\wp} \in Sch_{\mathcal{A}}$ in $\mathcal{A}$ as follows: $s_{\wp}(\rho) = \theta_{\rho}(\wp)$, for all $\rho \in Tr(k(s))$. Moreover, let $s_{\wp} \in Sch_{\mathcal{A}}$ be a generic odd scheme in $\mathcal{A}$ and consider the derived even scheme $s_{\wp} \in Sch_{\mathcal{G}}$ in $\mathcal{G}$ defined as follows: $s_{\wp}(\rho) = \theta_{\rho}(\wp)$, for all $\rho \in Tr(k(s))$, where $\theta_{\rho} \in SDF_{\mathcal{A}}(\wp)$ is such that $s_{\wp}(\rho \cdot (\operatorname{lst}(\rho), \theta_{\rho})) = \theta_{\rho}(\wp \cdot (\theta_{\rho}(\omega), \bar{\theta}(\wp))).$ The existence of such a Skolem dependence function is ensured by the previous membership of the successor of $\operatorname{lst}(\rho)$ in $h(\rho)$.

At this point, it remains only to prove that $\wp = \wp$, where $\wp \in Sch_{\mathcal{G}}(s_{\wp}, s_{\wp})$ and $s_{\wp} \in mtc_{\mathcal{G}}(s_{\wp}, s_{\wp})$. To do this, we proceed by induction on the prefixes of the matches, i.e., we show that $(\wp)_{i} = (\wp)_{i}$, for all $i \in N$. The base case is immediate by definition of match, since we have that $(\wp)_{0} = (\wp)_{0}$. Now, as inductive case, suppose that $(\wp)_{i} = (\wp)_{i}$, for all $i \in N$. By the definition of match, we have that $(\wp)_{i+1} = s_{\rho}((\wp)_{i} \cdot s_{\rho}((\wp)_{i}))$ and $(\wp)_{i+1} = s_{\rho}((\wp)_{i} \cdot s_{\rho}((\wp)_{i}))$. Moreover, by the inductive hypothesis, it follows that $s_{\rho}((\wp)_{i} \cdot s_{\rho}((\wp)_{i})) = s_{\rho}((\wp)_{i} \cdot s_{\rho}((\wp)_{i}))$. Now, by substituting the values of the even schemes $s_{\wp}$ and $s_{\wp}$, we have that $(\wp)_{i+1} = s_{\rho}((\wp)_{i} \cdot (\wp)_{i})$ and $(\wp)_{i+1} = s_{\rho}((\wp)_{i} \cdot (\wp)_{i})$. This point, due to the choice of the Skolem dependence function $\theta_{\wp,i}$, it holds that $s_{\rho}((\wp)_{i} \cdot (\wp)_{i}) = s_{\rho}((\wp)_{i} \cdot (\wp)_{i})$. Thus, we have that $(\wp)_{i+1} = (\wp)_{i+1}$, which implies $(\wp)_{i+1} = (\wp)_{i+1}$.

Lemma B.5 (Encasement Characterization). Let $\mathcal{G}$ be a CGS, $s \in St$ one of its states, $P \subseteq P_{th}(s)$ a set of paths, $\wp \in Quat(\mathcal{V})$ a quantification prefix over a set of variables $V \subseteq \operatorname{Var}$, and $b \in Bnd(V)$ a binding. Then, the following hold:

(i) player even wins $H(\mathcal{G}, s, P, \wp, b)$ iff $P$ is an encasement w.r.t. $\wp$ and $b$;
(ii) if player odd wins $H(\mathcal{G}, s, P, \wp, b)$ then $P$ is not an encasement w.r.t. $\wp$ and $b$;
(iii) if $P$ is a Borelian set and it is not an encasement w.r.t. $\wp$ and $b$ then player odd wins $H(\mathcal{G}, s, P, \wp, b)$.

Proof. [Item i, only if]. Suppose that player even wins the TPG $H(\mathcal{G}, s, P, \wp, b)$. Then, there exists an even scheme $s_{\wp} \in Sch_{\mathcal{G}}$ such that, for all odd schemes $s_{\wp} \in Sch_{\mathcal{G}}$, it holds that $mtc(s_{\wp}, s_{\wp}) \in P$. Now, to prove the statement, we have to show that there exists a behavioral Skolem dependence function $\theta \in SDF_{\mathcal{St}(s)}(\wp)$ such that, for all assignments $\chi \in Asg(\wp), s$, it holds that $play(\theta(\chi) \circ \zeta_{i}, s) \in P$.

To do this, consider the function $w : Tr(k(s)) \rightarrow SDF_{\mathcal{A}}(\wp)$ constituting the projection of $s_{\wp}$ on the second component of its codomain, i.e., for all $\rho \in Tr(k(s))$, it holds that $s_{\wp}(\rho) = (\operatorname{lst}(\rho), w(\rho))$. By Lemma 4.8 on adjoint Skolem dependence functions, there exists a behavioral Skolem dependence function $\theta \in SDF_{\mathcal{St}(s)}(\wp)$ for which $w$ is the adjoint, i.e., $w = \theta$. Moreover, let $\chi \in Asg(\wp), s$.

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be a generic assignment and consider the derived odd scheme $s_o \in \text{Sch}_o$ defined a\d follows: $s_o(\rho \cdot \text{lst}(\rho), \theta')) = \tau(\text{lst}(\rho), \theta'(\chi(\rho)) \circ \zeta)$, for all $\rho \in \text{Trk}(s)$ and $\theta' \in \text{SDF}_{Ac}(\psi)$.

At this point, it remains only to prove that $\pi = \varpi$, where $\pi \models \text{play}(\theta(\chi) \circ \zeta, s)$ and $\varpi \models \text{mtrc}(s_o, s_o)$. To do this, we proceed by induction on the prefixes of both the play and the match, i.e., we show that $(\pi)_{\leq i} = (\varpi)_{\leq i}$, for all $i \in \mathbb{N}$. The base case is immediate by definition, since we have that $(\pi)_{\leq 0} = s = (\varpi)_{\leq 0}$. Now, as inductive case, suppose that $(\pi)_{\leq i} = (\varpi)_{\leq i}$, for $i \in \mathbb{N}$. On one hand, by the definition of match, we have that $(\text{var})_{i+1} = s_o((\varpi)_{\leq i} \cdot s_o((\varpi)_{\leq i}))$, from which, by substituting the value of the even scheme $s_e$, we derive $(\text{var})_{i+1} = s_o((\text{var})_{\leq i} \cdot ((\varpi)_{\leq i}) \circ \zeta))$. On the other hand, by the definition of play, we have that $(\pi)_{i+1} = \tau((\pi)_{i}, \theta((\pi)_{\leq i}) \circ \zeta)$, from which, by using the definition of the odd scheme $s_o$, we derive $(\pi)_{i+1} = s_o((\pi)_{\leq i} \cdot ((\pi)_{i}, \theta((\pi)_{\leq i})))$. Then, by the inductive hypothesis, we have that $(\text{var})_{i+1} = s_o((\text{var})_{\leq i} \cdot ((\varpi)_{\leq i}) \circ \zeta)) = s_o((\pi)_{\leq i} \cdot ((\pi)_{i}, \theta((\pi)_{\leq i})))$, which implies $(\pi)_{\leq i+1} = (\pi)_{\leq i+1}$.

[Item i, if]. Suppose that $P$ is an encasement w.r.t. $\varphi$ and $b$. Then, there exists a behavioral Skolem dependence function $\theta \in \text{BSDF}_{\text{Str}}(\psi)$ such that, for all assignments $\chi \in \text{Asg}([\psi], s)$, it holds that $\text{play}(\theta(\chi) \circ \zeta, s) \in P$. Now, to prove the statement, we have to show that there exists an even scheme $s_e \in \text{Sch}_e$ such that, for all odd schemes $s_o \in \text{Sch}_o$, it holds that $\text{mtrc}(s_e, s_o) \in P$.

To do this, consider the even scheme $s_e \in \text{Sch}_e$ defined as follows: $s_e(\rho) \equiv \text{lst}(\rho), \theta(\rho))$, for all $\rho \in \text{Trk}(s)$. Observe that, by Lemma 4.8 on adjoint Skolem dependence functions, the definition is well-formed. Moreover, let $s_o \in \text{Sch}_o$ be a generic odd scheme and consider a derived assignment $\chi \in \text{Asg}([\psi], s)$ satisfying the following property: $\chi(\rho) \in \{ v \in \text{Val}_{Ac}([\psi]) : s_o(\rho \cdot \text{lst}(\rho), \theta(\rho))) = \tau(\text{lst}(\rho), \theta(v) \circ \zeta) \}$, for all $\rho \in \text{Trk}(s)$.

At this point, it remains only to prove that $\pi = \varpi$, where $\pi \models \text{play}(\theta(\chi) \circ \zeta, s)$ and $\varpi \models \text{mtrc}(s_o, s_o)$. To do this, we proceed by induction on the prefixes of both the play and the match, i.e., we show that $(\pi)_{\leq i} = (\varpi)_{\leq i}$, for all $i \in \mathbb{N}$. The base case is immediate by definition, since we have that $(\pi)_{\leq 0} = s = (\varpi)_{\leq 0}$. Now, as inductive case, suppose that $(\pi)_{\leq i} = (\varpi)_{\leq i}$, for $i \in \mathbb{N}$. On one hand, by the definition of match, we have that $(\text{var})_{i+1} = s_o((\text{var})_{\leq i} \cdot s_o((\text{var})_{\leq i}))$, from which, by the definition of the even scheme $s_e$, we derive $(\text{var})_{i+1} = s_o((\text{var})_{\leq i} \cdot ((\varpi)_{\leq i}) \circ \zeta))$, from which, by the choice of the assignment $\chi$, we derive $(\pi)_{i+1} = s_o((\pi)_{\leq i} \cdot ((\pi)_{i}, \theta((\pi)_{\leq i})))$. Then, by the inductive hypothesis, we have that $(\text{var})_{i+1} = s_o((\text{var})_{\leq i} \cdot ((\varpi)_{\leq i}) \circ \zeta)) = s_o((\pi)_{\leq i} \cdot ((\pi)_{i}, \theta((\pi)_{\leq i})))$, which implies $(\pi)_{\leq i+1} = (\pi)_{\leq i+1}$.

[Item iii]. If player odd wins the TPG $H(\mathcal{G}, s, P, \varphi, b)$, we have that player even does not win the same game. Consequently, by Item i, it holds that $P$ is not an encasement w.r.t. $\varphi$ and $b$.

[Item iii]. If $P$ is not an encasement w.r.t. $\varphi$ and $b$, by Item i, we have that player even does not win the TPG $H(\mathcal{G}, s, P, \varphi, b)$. Now, since $P$ is Borelian, by the determinacy theorem [Martin 1975; 1985], it holds that player odd wins the same game. □

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