Cosmological Magnetic Field and Dark Energy as two sides of the same coin.

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It has been recently argued [1] that the de Sitter phase in cosmology might be naturally generated as a result of dynamics of the topologically nontrivial sectors in a strongly coupled QCD-like gauge theory in expanding universe. It is known that the de Sitter phase is realized in the history of our Universe twice: the first occurrence is coined as the inflation, while the second time (which is occurring now) is dubbed as the dark energy (DE). The crucial element of the proposal [1] is the presence of a nontrivial gauge holonomy which is the source of the vacuum energy leading to the de Sitter behaviour. It has been also argued that the anomalous coupling of the system with the Standard Model (SM) particles leads to the reheating epoch in case of the inflationary phase. A similar anomalous coupling of the system with the Maxwell E&M field during the DE epoch generates the cosmological magnetic field. The intensity of the field is estimated on the level of $10^{-10}$G while the corresponding correlation length reaches the scale of the visible Universe.

I. INTRODUCTION

This work is mostly motivated by recent proposal [1] where the vacuum energy (and accompanying it the de Sitter phase) is dynamically generated. The proposal [1] can be viewed as a synthesis of two naively unrelated ideas discussed previously in refs. [2–4] and [5–7] correspondingly. On the gravity side [2–4] the nontrivial element of the construction is represented by the Euclidean spacetime with a time compactified to a circle $S^1$. On the gauge field theory side [5–7] the same $S^1$ plays a crucial role when the gauge configurations may assume a nontrivial holonomy along $S^1$. Precisely the gauge configurations with the nontrivial holonomy along $S^1$ may serve as a source of vacuum energy density, which eventually leads to the de Sitter behaviour.

The focus of proposal [1] coined as the “holonomy inflation” was the study of the vacuum energy and the corresponding de Sitter behaviour in application to the inflationary Universe. It has been also suggested in that proposal that the holonomy inflation ends as a result of anomalous coupling of the system with massless Standard Model (SM) gauge fields with known coefficients.

The present work applies the same ideas on dynamical generation of the vacuum energy to the dark energy (DE) epoch when the corresponding strongly coupled gauge theory is well known, it is QCD characterized by a single dimensional parameter, $\Lambda_{\text{QCD}} \sim 0.1$ GeV. A similar anomalous coupling (which was the source of the reheating in the “holonomy inflation” in [1] when the vacuum energy is transferred to the massless gauge fields) generates the cosmological magnetic helical configurations with enormous correlation length reaching the size of the entire visible Universe during the present DE epoch. The focus of the present work is an analysis of the generation mechanism of such long ranged magnetic field.

Before we proceed with outline of this work we would like to make few remarks on conventional approaches to study the cosmological magnetic field. We refer to the classical review papers [8, 9] and more recent review [10] for details and references. It is normally assumed that magnetic fields in astronomical structures of different sizes, from stars $R \sim 10^{13}$ cm to galaxy clusters $R \sim 10^{24}$ cm are produced by amplification of pre-existing weaker “seed” magnetic fields via different types of dynamo. Two broad classes of models for the origin of the seed fields are discussed: 1. primordial magnetic field (seeds) is produced during different dramatic events in evolution of the Universe such as inflation, electroweak phase transition, QCD transition, i.e. during the epochs preceding the structure formation; 2. the process of generation of the seed magnetic fields accompanies the gravitational collapse leading to structure formation. We shall not comment on many problems related to this conventional picture referring to the reviews [8–10].

The unorthodox mechanism we are advocating in the present work is drastically distinct from previous conventional approaches. Essentially, the magnetic field in our framework is generated with enormous scale from the moment when it was born as the source of its energy is the DE occupying the entire Universe. Therefore, there is no need for amplification nor for different types of inverse cascades as the correlation length of the produced field is already characterized by the largest possible scale. The intensity of this correlated magnetic field is estimated on the level of $B \sim 10^{-10}$ G, and the intensity of the field $B^2$ is proportional to the DE density $\rho_{\text{DE}} \approx (2.3 \cdot 10^{-7}$eV$)^4$ with calculable (in principle) coefficient.

This intensity is very close to the upper limit, but not ruled out. In fact such fields can be studied by future UHECR telescopes, see Fig. 14 in ref. [10].

Our presentation is organized as follows. In next section II we overview the basic ideas and results on the nature of vacuum energy from ref. [1]. The nature of the DE plays a crucial role in our framework as it is the source of the cosmic magnetic field, which is the main subject of the present work. Therefore, we overview the basic ideas of [1] in context of the present work in great details for benefits of the readers. In Sect. III we explain how the DE couples to the EM field through the chiral anomaly. Precisely this coupling is responsible for the generation of the long ranged magnetic field, which is the subject of Section IV where we estimate its intensity.
We conclude in Section V with few comments on future development, and possible observational tests which may support or rule out this new paradigm when DE and cosmic magnetic field represent two sides of the same coin and are produced at the same epoch.

II. THE TOPOLOGY AS THE SOURCE OF THE GRATIVATING VACUUM ENERGY

The goal here is to overview the basic ideas advocated in [1], see also a number of precursor references therein.

In approach [1] the vacuum energy entering the Friedmann equation is defined as $\Delta \rho \equiv \rho_{\text{FRW}} - \rho_{\text{Mink}}$. This definition for the vacuum energy for the first time was advocated in 1967 by Zeldovich [11] who argued that $\rho_{\text{vac}} = \Delta \rho \sim Gm_p^6$ must be proportional to the gravitational constant with $m_p$ being the proton’s mass. Later on such definition for the relevant energy $\Delta \rho \equiv \rho_{\text{FRW}} - \rho_{\text{flat}}$ which enters the Einstein equations has been advocated from different perspectives in a number of papers written by the researchers from different fields, including particle physics, cosmology, condensed matter physics. This subtraction prescription is consistent with conventional subtraction procedure of the divergent ultra local bare cosmological constant because in the infinitely large flat space-time the corresponding contribution is proportional to the $\delta^4(x)$ function as explained in [1]. At the same time the nontrivial correction to $\Delta \rho$ as discussed below is a non-local function of the geometry and cannot be renormalized by any UV counter-terms.

In the present work we consider the geometry $\mathbb{R}^3 \times S^1$ instead of FRW geometry to simplify the arguments. The key element in this framework is the presence of a dimensional parameter $T^{-1}$ which plays the role of the Hubble constant $H$ in FRW geometry which distinguishes FRW geometry from flat infinite space-time geometry. In other words, we have a dimensional parameter $T$ which is assumed to be order $\sim H^{-1}$ and which parametrizes the difference between nontrivial and trivial (flat infinite space-time $\mathbb{R}^4$) geometries. In computations [1] parameter $T$ is the proper length of the $S^1$-period. As we already mentioned, this prescription (when $\Delta \rho \equiv [\rho_{\text{FRW}} - \rho_{\text{flat}}]$ is identified with physical energy, similar to the Casimir Effect) is consistent with the Einstein equations when the vacuum energy approaches zero, $\Delta \rho \to 0$ for the flat space-time which itself may be considered as a limiting case with $T \to \infty$.

The key element of the framework [1] is that the vacuum energy receives the linear correction $T^{-1}$ at large $T$ in contrast with naively expected quadratic corrections $T^{-2}$ such that the vacuum energy entering the Friedmann equation assumes the form

$$\rho_{\text{DE}} \equiv \left( E_{\text{vac}}[\mathbb{R}^3 \times S^1] - E_{\text{vac}}[\mathbb{R}^3 \times \mathbb{R}^4] \right) = \Lambda_{\text{QCD}}^3 \frac{\tilde{c}_T}{T},$$

where the vacuum energy can be represented as follows

$$E_{\text{vac}}[\mathbb{R}^3 \times S^1] \simeq - \frac{32 \pi^2}{9 \pi^4} \Lambda_{\text{QCD}}^4 \left( 1 - \frac{c_T}{T \Lambda_{\text{QCD}}} \right)$$

$$\simeq - \frac{32 \pi^2}{9 \pi^4} \Lambda_{\text{QCD}}^4 + \Lambda_{\text{QCD}}^3 \frac{\tilde{c}_T}{T} + O(\frac{1}{T^2}).$$

In this expression we redefined $\tilde{c}_T \equiv \frac{32 \pi^2}{9 \pi^4} c_T$. The coefficient $c_T \sim 1$ is, in principle, a calculable parameter, expected to be order of one. The linear dependence $T^{-1}$ of the relevant portion of the vacuum energy (1) on external parameter $T^{-1} \sim H$ suggests that $\rho_{\text{DE}}$ numerically is very close to the observed value today, i.e.

$$\rho_{\text{DE}} \simeq \Lambda_{\text{QCD}}^3 \frac{\tilde{c}_T}{T} \sim \Lambda_{\text{QCD}}^3 H \sim (10^{-3} \text{eV})^4.$$ (3)

One should also mention that this numerical coincidence in estimate (3) was the main motivation to advocate the proposal [13, 14] that the driving force for the dark energy is a nontrivial dynamics of the topological sectors in strongly coupled QCD (admittedly, without much deep understanding behind the formula at that time).

Few important comments regarding formulae (1) and (2) are in order.

1. All computations leading to (2) are performed in the Euclidean space-time where the relevant gauge configurations describing the tunnelling processes are defined. Using this technique one can compute the energy density $\rho$ and the pressure $P$ in the Euclidean space. As usual, we assume that there is analytical continuation to Lorentizan space-time where the physical energy density has the same form. In our context it means that the parameters $P, \rho$ and equation of state (EoS) as given by (7) below are interpreted as the corresponding parameters in physical Lorentizan space-time.

2. The same arguments also suggest that the parameter $T$ entering (2) is a constant parameter of the system (not to be confused with observed Hubble $H_{\text{obs}}(t)$ which is time dependent in FRW Universe). The cosmological evolution in the Lorentizan space-time is determined by the analytic continuation as discussed in ref. [1].

3. What is the interpretation of the parameter $T$ in physical Lorentizan space-time? In the system with Euclidian signature the parameter $T$ is determined by the size of $S^1$, which is normally can be interpreted as the inverse temperature of the system in Lorentizan space-time. We think it is a proper interpretation even though there is no any thermodynamical processes which are occurring and characterized by extremely low temperature $T^{-1} \sim H \sim 10^{-33} \text{eV}$.

4. The vacuum energy $E_{\text{vac}}$ is defined in conventional way in terms of the path integral. It has a “non-dispersive” nature, which implies that the corresponding vacuum energy cannot be expressed in terms

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1 It can be in principle computed in strongly coupled QCD using the lattice Monte Carlo simulations, similar to studies [12].
of conventional propagating degrees of freedom (absorptive part) using the dispersion relations to compute the dispersive part. Furthermore, all effects represented by eq. (2) are obviously non-analytical in coupling constant \( \sim \exp(-1/g^2) \) and can not be seen in perturbation theory.\(^2\)

5. One can view the relevant topological Euclidean configurations which saturate (2) as the 3d magnetic monopoles wrapping around \( S^2 \) direction. These configurations are characterized by the non-vanishing holonomy, which eventually generates the linear (rather than quadratic) correction \( \sim 1/T \) to the vacuum energy density. For the specific geometry (leading to the de Sitter feature of non-locality implies that the relevant energy configuration with nontrivial holonomy. Precisely this is highly non-local effect as it is saturated by the gauge parameter is a constant\(^3\).

6. In the cosmological context such configurations are highly unusual objects: they obviously describe the non-local physics as the holonomy is a non-local object. Indeed, the holonomy defines the dynamics along the entire history of evolution of the system. This entire gauge configuration is a mere saddle point in Euclidean path integral computation which describes the instantaneous tunnelling event, rather than propagation of a physical degree of freedom.

7. The generation of the “non-dispersive” energy \( E_{\text{vac}} \) is highly non-local effect as it is saturated by the gauge configurations with nontrivial holonomy. Precisely this feature of non-locality implies that the relevant energy \( \Delta \rho \) which enters the Friedmann equation (3), cannot be expressed in terms of a gradient expansion in any effective local field theory.

8. The basic idea of the framework [1] on dynamical generation of the vacuum energy leading to the de Sitter behaviour is that there is a linear correction (with respect to the inverse size of the system) to the energy

\[
E_{\text{vac}}[\mathbb{R}^3 \times S^2] - E_{\text{vac}}[\mathbb{R}^3 \times \mathbb{R}] \sim \left( 1 - \frac{cT}{T\Lambda_{\text{QCD}}} \right).
\]

This correction \( \sim T^{-1} \) is generated in spite of the fact that the system has a gap \( \Lambda_{\text{QCD}} \) which naively implies that the system must not be sensitive to the size \( T \) of the system at all. We already mentioned that the correction \( T^{-1} \) is nevertheless generated because the vacuum energy (1), (3) has a “non-dispersive” nature, not associated with any propagating massive degrees of freedom, but rather is related to instantaneous tunnelling events (expressed in terms of the Veneziano ghost, mentioned in footnote 2, as the presence of the topologically protected pole). Explicit computations in hyperbolic space \( S^1 \times \mathbb{H}^3 \) \(^7\) and simplified “deformed QCD” model \(^18\), along with the lattice simulations \(^12\) support this claim.

What is an intuitive way to understand the effect? Imagine that we study the Aharonov-Casher effect. We insert an external charge into a superconductor when the electric field \( E \) is screened, i.e. \( E \sim Q \exp(-r/\lambda) \) with \( \lambda \) being the penetration depth. Nevertheless, a neutral magnetic fluxon will be still sensitive to an inserted external charge \( Q \) at arbitrary large distances in spite of the screening of the physical field. This genuine quantum effect is purely topological and non-local in nature and can be explained in terms of the dynamics of the gauge sectors which are responsible for the long range dynamics. Imagine now that we study the same effect but in a time dependent background. The corresponding topological sectors which saturate the vacuum energy will be modified due to the external background. However, this modification can not be described in terms of any local dynamical fields, as there are no any propagating long range fields in the system since the physical electric field is screened. The effect is obviously non-local in nature as the Aharonov-Casher effect itself is a non-local phenomenon, and cannot be expressed in terms of the local operator \( F_{\mu\nu} \), but rather is expressed in terms of the gauge invariant, but non-local operator, the holonomy \( \sim \exp(iQ \oint A_\mu dx^\mu) \).

We conclude this short overview on generation of the dynamical vacuum energy (as a result of dynamics of the topological sectors) with comment that this type of energy behaves in all respects as a cosmological constant if anomalous coupling with other gauge fields is switched off. Indeed, one can use conventional thermodynamical relation

\[
dF = TdS - PdV, \quad P = -\frac{\partial F}{\partial V}
\]

to convince yourself that the correction \( \sim T^{-1} \) does not modify the equation of state. In fact, it behaves exactly in the same way as the cosmological constant does, i.e.

\[
\rho = \frac{F}{V} = -\frac{32\pi^2}{g^4} \Lambda_{\text{QCD}}^4 \left( 1 - \frac{cT}{T\Lambda_{\text{QCD}}} \right).
\]

The equation (6) implies that the corresponding equation of state assumes the form

\[
w = \frac{\Delta \rho}{\Delta \rho} = -1, \quad a(t) \sim \exp(Ht), \quad H \sim T^{-1}.
\]

\(^2\) This non-dispersive nature of the vacuum energy is well known to the QCD community: it appears in computation of the topological susceptibility (which is expressed as the second derivative of the vacuum energy with respect to \( \theta \)). The corresponding non-dispersive contact term was postulated by Witten in [15], while the same term with a “wrong sign” in the correlation function was saturated by the Veneziano ghost in \([16, 17]\), see Appendix A1 in [1] for references and details.

\(^3\) A nonzero holonomy for the vacuum configurations saturating the vacuum energy represents a technical explanation why the conventional argument (that the correction in (2) must be quadratic in \( H^2 \) in gravitational background rather than linear in \( H \)) fails. The point is that the holonomy is independent gauge invariant non-local characteristic of the system, similar to the Polyakov’s line, which cannot be expressed in terms of the local curvature \( R \), which is indeed quadratic in \( H \) as \( R \sim H^2 \). Explicit computations in Hyperbolic space support this claim, see item 8.
where $\Delta P$ and $\Delta \rho$ are defined by subtracting the constant value computed in infinitely large flat space time, as explained above and expressed by (1), (3).

The regime described by (7) would be the final destination of our Universe if the interaction of the QCD gauge configurations (saturating the vacuum energy) with massless EM photons were always switched off. When the coupling of the QCD vacuum fields with EM field is switched back on, the end of de Sitter behaviour is triggered precisely by this interaction which itself is unambiguously fixed by the triangle anomaly as we discuss in next section III.

The corresponding physics of the energy transfer from the vacuum energy given by (1), (3) to the cosmic magnetic energy is very similar in all respects to the physics of the reheating epoch at the end of inflation when the vacuum energy is transferred to the light gauge SM fields as discussed in [1]. The technical (very challenging) problems which need to be resolved to address these questions are also very similar in spirit as we discuss in next section.

III. COUPLING OF THE VACUUM ENERGY TO PHOTONS

This section is separated in two parts. In the first subsection III A we explain the formal procedure (based on the Euclidean path integral formulation) which in principle allows to compute the desired rate and other characteristics of the energy transfer. While the corresponding procedure is well defined it is not technically feasible yet. Therefore, in subsection III B we introduce an alternative technique in terms of the auxiliary topological auxiliary fields to attack the problem.

A. Formulation of the problem in terms of the tunnelling transitions

The vacuum energy (1), (3) in our framework is expressed in terms of the tunnelling transitions which are normally computed in terms of the Euclidean path integral and the corresponding (Euclidean) field configurations which describe the interpolation between distinct topologically $|k\rangle$ sectors. In conventional QFT computations the corresponding procedure selects a specific superposition of the $|k\rangle$ states which generates the $|\theta\rangle$ state with energy $E_{\text{vac}}(\theta)$. In the context of DE, when the background assumes a non-trivial FRW geometry (in contrast with conventional case described by $\mathbb{R}^4$) the corresponding computations become profoundly more complicated, though the corresponding procedure is well defined in principle:

1. One should describe the relevant Euclidean configurations satisfying the proper boundary conditions for a nontrivial geometry (similar to calorons with nontrivial holonomy, reviewed in Appendix A2 in [1]) represented by parameter $H \sim T^{-1}$;

2. One should compute the corresponding path integral which includes all possible positions and orientations of the relevant gauge configurations interpolating between different topological $|k\rangle$ sectors and physically describing the tunnelling transitions between them;

3. The corresponding computations for the vacuum energy $\rho$ and pressure $P$ must be done with all massless fields which couple to QCD. In our case the only massless particles to be considered are the photons as production of all massive particles is exponentially suppressed. Precisely this coupling of the QCD gauge configurations with EM field is responsible for transferring the vacuum energy to the magnetic energy;

4. As the last step, one should subtract the corresponding expression (computed on $\mathbb{R}^4$) as explained in previous section II. Precisely this remaining portion of the vacuum energy is interpreted as the relevant energy which enters the Friedmann equation, and which cannot be removed by any subtraction procedure and cannot be renormalized by any UV counter terms. The corresponding formulae for $\Delta \rho, \Delta P$ will depend, in general, on properties of the manifold (parametrized by $H$), the relevant coupling constant $\alpha$ with EM field, and the environment where the magnetic field is generated. This procedure will unambiguously predict the magnetic energy of the produced field along with its basic features (such as the correlation length, helical features etc).

While these steps are well defined in principle, it is not feasible to perform the corresponding computations because even the first step in this direction, a finding the relevant Euclidean configurations satisfying the proper boundary conditions for a nontrivial geometry, is yet unknown. Nevertheless, this procedure, in principle, shows that the de Sitter behaviour (7) in this framework emerges without any local field $\Phi(x)$ as explained in previous section II because the physics leading to (7) is not associated with any scalar fields, but related to the tunnelling events. This procedure, in principle, also shows how the vacuum may transfer its energy to the magnetic field in a time dependent background.

In many respects this energy transfer is very similar to the so-called Dynamical Casimir Effect (DCE) when the photons are radiated from the vacuum in a time dependent background. The difference with conventional DCE is that the photons are emitted in our case not from conventional virtual fluctuating particles which are always present in the system. The key difference with DCE is that the photons in our system are emitted from vacuum configurations which describe the tunnelling processes between different topological sectors $|k\rangle$.

This difference (in comparison with DCE) in nature of emission explicitly displays a hard challenging technical problem in computation of the corresponding emission rate. Indeed, our topological configurations interpolating between different topological sectors $|k\rangle$ are formulated in terms of the Euclidean path integral, while the emission of real particles on mass shell represents an inherent Minkowski process. At present time the con-
ventional technical tools developed for Euclidean versus Minkowski descriptions are very different and designed for different purposes and different problems. For example, conventional lattice QCD MC simulations are not designed to compute physical processes such as on shell scattering amplitudes, but perfectly adapted to compute the Euclidean correlation functions such as topological susceptibility which assumes a nonzero value exclusively due to the tunnelling events between different topological sectors.

B. Formulation of the problem in terms of the auxiliary topological fields

Fortunately, the key ingredients which are relevant for our future studies can be understood in alternative way, in terms of the auxiliary topological non-propagating fields $b(x, H)$ which effectively describes the relevant infrared physics (IR) representing the key elements of the steps 1-4 highlighted in section III A. Parameter $H$ here represents the deviation of the manifold under consideration (for example $1/T$) from trivial $\mathbb{R}^4$.

The basic idea is to construct the effective Lagrangian for the auxiliary topological field $b(x, H)$ using the Euclidean conventional formulation. As the next step one can utilize the standard formulae to rewrite the corresponding action in Minkowski spacetime. Finally, one can study the emission of real particles and generation of new magnetic field using the obtained effective Lagrangian written in Minkowski space. This procedure effectively resolves the fundamental technical problem formulated at the end of section III A and originated from the differences in descriptions in Euclidean versus Minkowski spacetimes.

The formal technique we are about to overview is widely used in particle physics and condensed matter (CM) communities. We refer to Appendix B in ref.[1] for the highlights of the main ideas and results of this approach within context of the present work. In particular, this approach is extremely useful in description of the topologically ordered phases when the IR physics is formulated in terms of the topological Chern-Simons (CS) like Lagrangian. One should emphasize that the corresponding physics, such as calculation of the braiding phases between quasiparticles, computation of the degeneracy etc, can be computed (and in fact originally had been computed) without Chern-Simons Lagrangian and without auxiliary fields. Nevertheless, the discussions of the IR physics in terms of CS like effective action is proven to be very useful, beautiful and beneficial. In our case, it is not simply a matter of convenience, but in fact matter of necessity because we cannot proceed with explicit computations along the lines 1-4 as explained in section III A.

In context of the present work the auxiliary topological non-propagating fields $b(x, H)$ is introduced in conventional way as Lagrange multiplier in the course of inserting the corresponding $\delta$- functional into the path integral which effectively constraints the relevant degrees of freedom, see Appendix B in ref.[1] for references and technical details.\footnote{The computations have been performed in a simplified version of QCD, the so-called weakly coupled “deformed QCD” model [19] which preserves all relevant features of the strongly coupled QCD such as confinement, nontrivial $\theta$ dependence, generation of the “non-dispersive” vacuum energy, etc [20]. The corresponding results have been reproduced in [21] using the technique of the auxiliary topological fields $b(x)$ exploited in the present work. It is expected that similar description in terms of the auxiliary topological field also holds in strongly coupled QCD. In fact, the Veneziano ghost postulated in refs. [16, 17] can be identified with the auxiliary topological fields [21].}

As a consequence of this fundamental feature the topological auxiliary field $b(x, H)$ field is in fact an angular topological variable and it has the same $2\pi$ periodic properties as the original $\theta$ parameter.\footnote{As it is known the $\theta$ parameter can be promoted to the dynamical axion field $\theta(x)$ by adding the canonical kinetic term $[\partial_\mu \theta(x)]^2$ to the effective Lagrangian. The difference of the $b(x, H)$ field with the dynamical axion field $\theta(x)$ field is that the auxiliary topological field $b(x, H)$ does not have a canonical axion kinetic term.} In other words, the desired coupling of $b(x, H)$ field with $F_{\mu\nu}$ photons is

$$ L_{b\gamma\gamma}(x) = \frac{\alpha}{4\pi} N \sum_{q=1}^{Q^2} \frac{N_f}{N} \left[ \theta + b(x, H) \right] \cdot F_{\mu\nu} F^{\mu\nu}(x), \quad (8) $$

where $\alpha$ is the fine-structure constant, $Q_i$ are the electric charges of $N_f$ light quarks and $N = 3$ is the number of colours of the strongly coupled QCD and everything is written already in Minkowski metric. As we already mentioned, the coupling (8) is unambiguously fixed because the auxiliary $b(x, H)$ field always accompanies the $\theta$ parameter in the specific combination $[\theta + b(x, H)]$ and describes the anomalous interaction of the topological auxiliary $b(x, H)$ field with $E&M$ photons.

The next question we want to address is as follows: what are the typical fluctuation scales of the auxiliary quantum $b(x, H)$ field? The answer is quite obvious: the typical fluctuations are of order $\Delta_{\text{QCD}}$ as the $b(x, H)$ effectively describes the tunnelling events, and in particular, saturates the topological susceptibility (which can be explicitly computed in weakly coupled “deformed QCD”
as studied in [21] where all computations are under complete theoretical control).

What happens when the same system is defined on a nontrivial manifold characterized by some dimensional parameters such as $H \sim T^{-1} \ll \Lambda_{\text{QCD}}$? In this case the field $b(x, H)$ will continue to fluctuate with typical frequencies $\Lambda_{\text{QCD}}$. However, the relevant correlation functions should demonstrate the emergence of the linear corrections with respect to these small parameters $\sim T^{-1}$. In particular, the topological susceptibility (expressed as the second derivative of the vacuum energy with respect to $\theta$) should be of order $\Lambda_{\text{QCD}}^4$ with corrections of order $(\Lambda_{\text{QCD}} T)^{-1}$ as expressions (1), (2) suggest.

It is useful to treat $\langle b(x, H) \rangle$ as the axial chemical potential\(^6\), i.e.

$$\mu_5 \equiv \langle b(x, H) \rangle, \quad (9)$$

which can be easily understood by performing the the $U(1)_A$ chiral time-dependent transformation in the path integral to rotate away the coupling (8). The corresponding interaction reapers in the form of a singlet non-vanishing axial chemical potential $\mu_5$ for light $N_f$ flavours as stated in (9).

Few comments are in order. In formula (9) we use notation for the expectation value $\langle b(x, H) \rangle$ to emphasize that we treat $b(x, H)$ entering (8) as the external parameter ignoring a complicated quantum dynamics of the $b(x, H)$ field itself (which would require to proceed with steps 1-4 as formulated in Sect. III A). In what follows we also neglect the back-reaction of $F_{\mu\nu}$ field on $b(x, H)$. In other words, we approximate the dynamics of the $b(x, H)$ by taking its expectation value $\langle b(x, H) \rangle$, and treat it as an (almost constant) external thermodynamical parameter of the system. One should emphasize that $\mu_5$ is not a genuine thermodynamical parameter. Furthermore, $\mu_5$ does not satisfy any classical equation of motion as there is no a canonical kinetic term in the Lagrangian for $b(x, H)$ field itself. Instead, the $b(x, H)$ field was introduced as Lagrange multiplier to account for complicated dynamics of the tunnelling events.

Our next comment is related to estimation of the expectation value $\langle b(x, H) \rangle$. As we discussed in the previous section II the dimensional parameters entering our framework must be computed by subtracting the corresponding expectation values computed on $\mathbb{R}^4$. This procedure unambiguously implies\(^7\) that $\langle b(x, H) \rangle \sim H$ as it must vanish at $H = 0$ and it must be linear in $H$ as discussed in Sect. II.

Therefore, our problem is now reduced to the study of the magnetic field generation determined by coupling (8) with a source which can be parametrized as follows

$$\mu_5 \equiv \langle b(x, H) \rangle = c_1 H, \quad (10)$$

where numerical coefficient $c_1 \sim 1$ is order of one, similar to $c_T$ from eq. (2) and it can be, in principle, computed from the first principles by following the steps 1-4 as highlighted in Sect. III A. The dimensional parameter $H$ in eq. (10) should not be confused\(^8\) with the observed (time dependent) Hubble constant $H_{\text{obs}}(t)$. Instead, $H$ should be treated as a parameter of the system which behaves as a cosmological constant (7) saturating the DE today (1).

Numerically, these parameters are the same order of magnitude today, i.e. $H \sim H_{\text{obs}}(t) \sim 10^{-33}$ eV.

IV. GENERATION OF THE MAGNETIC FIELD THROUGH THE CHIRAL ANOMALY

A. Basic equations

The coupling of the $E\&M$ fields with auxiliary topological field (8) parametrized by (10) generates an additional source term in the Maxwell equations

$$\vec{\nabla} \times \vec{B} = \sigma \vec{E} + \frac{\alpha}{2\pi} N \sum Q^2_{N_f} \cdot (\mu_5 \vec{B}), \quad (11)$$

where $\sigma$ is the conductivity to be estimated below, and term $\sim \langle \vec{b}(x, H) \rangle \times \vec{E}$ was neglected as a result of spatial isotropy of the tunnelling events. The extra induced non-dissipating current $\vec{j} \sim \vec{B}$ has been a very active area of research for many years in a number of different fields, including heavy ion physics, see reviews [22, 23], the axion searches, see reviews [24-32], earlier studies in condensed matter physics [33, 34], more recent studies in condensed matter physics [35] to name just a few.

There are also numerous applications of this anomalous term $\sim \mu_5 \vec{B}$ to cosmology related to the topic of the present work, and we want to mention just few papers [36-39] relevant for our future discussions. The drastic difference with most previous studies is that the source (10) in our case is not a dynamical field, but rather, an auxiliary field accounting for the tunnelling transitions in a time dependent background generating the vacuum dark energy (1), (3) as discussed in Section II. Nevertheless, for our purposes we can use some technical tools from previous studies treating $\mu_5$ as almost constant thermodynamical parameter.

With these comments in mind we consider the following simple ansatz for magnetic field [38, 39]

$$\vec{B} = B(t) \begin{bmatrix} \sin(kz), & \cos(kz), & 0 \end{bmatrix}, \quad (12)$$

\(^6\) There is a close analogy with heavy ion physics when a large domain with induced $\theta_{\text{ind}} \neq 0$ can be formed resulting in generation of the axial chemical potential $\mu_5 = \theta_{\text{ind}}$ in this $\theta_{\text{ind}}$-domain. It may produce a number of interesting $P$ odd phenomena, see [22] for review and references.

\(^7\) For this specific case $\langle b(x, H = 0) \rangle = 0$. Therefore, the subtraction in this case is a triviality.

\(^8\) As we mentioned in Section II for a specific geometry studied in [1] the parameters $H$ and $\mathcal{T}$ are related: $H \sim \pi/\mathcal{T}$ and describe the de Sitter behaviour for constant $H$. 
while the Bianchi identity \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \) implies that the corresponding electric field assumes the form

\[
\vec{E} = -\frac{1}{k} \vec{B}(t) \left[ \sin(kz), \cos(kz), 0 \right] = -\frac{1}{k} \vec{E}.
\]

(13)

The configuration (12) is a special case of the force-free field which satisfies

\[
\nabla \times \vec{B} = k \vec{B},
\]

(14)

see [38, 39] for references and details generalizing the ansatz (12). Substituting (12), (13) and (14) into (11) we arrive to the following equation for \( \vec{B}(t) \):

\[
k \vec{B}(t) = -\frac{\sigma}{k} \vec{B}(t) + \frac{\alpha}{\pi} \vec{e} H B(t), \quad \vec{e} \equiv c_1 \frac{N \sum_i Q_i^2}{2N_f}.
\]

(15)

where we introduced \( \vec{e} \) replacing the previously defined numerical coefficient \( c_1 \) as given by (10).

We are looking for a solution in the form

\[
\vec{B}(t) = B_0 \exp(\gamma t),
\]

(16)

which returns the following formula for the exponent \( \gamma \)

\[
\gamma = \frac{k}{\sigma} \left[ \frac{\alpha}{\pi} \vec{e} H - k \right].
\]

(17)

The exponential growth of the magnetic field occurs for very long waves,

\[
\gamma > 0 \quad \Rightarrow \quad k < \frac{\alpha}{\pi} \vec{e} H.
\]

(18)

The instability with respect to generation of the magnetic field \( B(t) \sim \exp(\gamma t) \) due to the coupling (8) is well known phenomenon and was discussed previously in the literature, including the cosmological applications [36–39] and heavy ion collisions [40]. In context of the inflationary scenario the same type of coupling could be responsible for the reheating epoch as discussed in [1].

In context of the present work the equations (16), (18) unambiguously imply that the magnetic field will be generated on enormous scales of the entire visible Universe as a result of anomalous coupling of the DE with the Maxwell field (8). The generation of the magnetic field obviously implies that there will be the energy transfer from the vacuum to the magnetic field as a result of evolution of the Universe.

One should emphasize that a sample configuration (12), (13) considered above is oversimplified example. We, of course, do not expect the magnetic field to be uniformed running along z direction through the entire Universe. Instead, we expect the field to be twisted as it is highly helical (which is normally associated with linking and twisting of the magnetic fluxes). Indeed, the magnetic helicity is defined as

\[
\mathcal{H} \equiv \int \vec{A} \cdot d\vec{B} d^3x.
\]

(19)

The time evolution of the magnetic helicity is determined precisely by \( \dot{\vec{E}} \cdot \vec{B} \) entering eq. (8), i.e.

\[
\frac{d\mathcal{H}}{dt} = -2 \int \vec{E} \cdot \vec{B} \, d^3x.
\]

(20)

For our configuration (12), (13) considered above the magnetic helicity per unit volume \( \mathcal{H}/V \) is directly related to the magnetic energy density, i.e.

\[
\frac{\mathcal{H}(t)}{V} \approx \frac{B^2(t)}{k}.
\]

(21)

Furthermore, the time evolution of both observables is also the same as eq. (21) states.

One should comment here that the magnetic field with enormous correlation length is known to be present in our Universe, see original paper [41] and review [10]. The mechanism suggested in the present work automatically generates the fields with such large correlation lengths. On other hand, it is very hard, if at all possible, to generate such enormous correlation length within conventional approaches, see [10] for review.

The generation of the magnetic field from \( \mu_5 \) is not very new idea, and was previously discussed in the literature for different systems. Furthermore, it has been known for sometime that the generation of the helical magnetic field is normally accompanied by decreasing of \( \mu_5 \) which is the source of the produced field. In particular, such behaviour is shown to occur in heavy ion systems [40] and also in the systems relevant for cosmology [38, 39].

What is the efficiency of this energy transfer from the DE to the magnetic energy in our case? What is the typical time scale for this energy transfer? What is the intensity of the magnetic field generated by this mechanism? We have to estimate \( \sigma \) and other related parameters in order to address these and many other related questions, which is the topic of the next subsection.

B. Numerical estimates

This subsection is much more speculative in comparison with our previous discussions in subsection IV A which is entirely based on the Maxwell equations in the presence of additional axion term. Nevertheless, we want to proceed with our speculations here to argue that all conventional cosmological assumptions about the environment leads to the estimates for the magnetic field which is perfectly consistent with presently available observations. Future studies as discussed in [10] are capable to discover these long ranged fields.

We start with electric conductivity \( \sigma \) entering the expression (17) for \( \gamma \). It is normally estimated as follows

\[
\sigma = \frac{4\pi n_e \alpha \tau}{m_e},
\]

(22)

where \( \tau \) is the time scale when a free electron is loosing its coherence. This time scale for low density environment is normally estimated as a result of interaction of electrons with CMB photons through the Thomson scattering

\[
\tau^{-1} = n_e \sigma \tau, \quad \sigma \tau = \frac{8\pi \alpha}{3m_e^2}.
\]

(23)
where in conventional circumstances \( n_e = n_{\text{CMB}} \sim T^3 \), see e.g. [9]. However, as we estimate below in our framework the number density of the E\&M configurations characterized by \( \vec{B} \) and \( \vec{E} \) fields and given by (12) and (13) correspondingly is much higher than \( n_{\text{CMB}} \). Precisely these long-wave lengths configurations with very low \( k \) as given by (18) will be dominating the electron resistivity in low density environment.

The estimation for the electron density \( n_e \) entering (22) strongly depends on the scale under consideration. For example, if residual free electrons (after recombination \( p + e \leftrightarrow H + \gamma \)) dominate the physics their density is estimated as [9]

\[
n_e \approx 2 \cdot 10^{-10}(1 + z)^3 \text{cm}^{-3}
\]  

(24)

At the same time if one assumes that the IGM (intergalactic medium) is mostly ionized then \( n_e \) is about the average baryon density [10]

\[
n_e \approx \frac{\rho_B}{m_p} \approx 2 \cdot 10^{-7}(1 + z)^3 \text{cm}^{-3}
\]  

(25)

where \( \rho_B \) is the baryon density.

To proceed with our task we have to estimate \( n_e \) entering (23). We define the corresponding density \( n_\gamma \) as follows

\[
h \omega_k \cdot n_\gamma(t) \equiv \frac{B^2(t)}{2}
\]  

(26)

For convenience of the estimates we also introduce dimensionless suppression factor \( \xi(t) < 1 \) which relates the magnetic energy density in comparison with the DE density, i.e.

\[
\frac{B^2(t)}{2} \approx \xi(t) \cdot \rho_{\text{DE}}(t),
\]  

(27)

where \( \rho_{\text{DE}}(t) \) is the source of the magnetic energy and it is defined in our framework by eqs. (1), (3). Our goal is to estimate \( \xi(t) \), and therefore, the strength of the magnetic field \( B(t) \).

To achieve this goal we estimate the ratio \( k/\sigma \) entering the expression for \( \gamma \) in terms of the observable parameters as follows

\[
\frac{k}{\sigma} = \frac{m_e k n_\gamma \sigma_T}{4 \pi n_e \sigma} \approx \frac{\alpha}{3} \left( \frac{B^2}{n_e m_e} \right) \approx \frac{2 \alpha}{3} \left( \frac{\rho_{\text{DE}}}{n_e m_e} \right)
\]  

(28)

where we used previously defined relations (22), (23), (26), (27) and for estimates we take \( \omega_k \approx k \). Numerically, one has

\[
\frac{k}{\sigma} \approx 2 \cdot 10^2 \xi \left( \frac{2 \cdot 10^{-7} \text{cm}^3}{n_e} \right).
\]  

(29)

This is precisely the dimensionless parameter which enters expression (17) for \( \gamma \). It measures, according to (16), the typical time scale (in the Hubble units \( H^{-1} \)) when magnetic field is generated. In other words, the energy transfer from DE to the magnetic energy becomes highly efficient when as \( (\gamma \tau_{\text{form}}) \sim 1 \).

To proceed with the estimates we need to make one more assumption which is formulated as follows. It is usually assumed (in strongly coupled systems) that in order to form a configuration characterized by a typical energy \( \omega \) one needs a time scale of order \( (2\pi)/\omega \), which is essentially a trivial manifestation of the uncertainty relation. In our (weakly coupled case) there is an additional fine structure coupling constant \( \alpha/(2\pi) \) entering (8) which suggests that the time scale \( \tau_{\text{form}} \) required to form the magnetic configuration with wave length \( k^{-1} \) from the DE source is \( (2\pi/\alpha)^2 \) much larger. In addition, the time which is available for the present Universe is \( H_{\text{obs}}^{-1} \) which is much shorter than \( \omega_k \) according to (18). This implies that \( \gamma \tau_{\text{form}} \) still cannot reach a magnitude of order one at present time; instead, it assumes only a fraction of its value \( \gamma \tau_{\text{form}} \sim (\omega_k/\alpha \bar{c}) \gamma \tau_{\text{form}} \) because \( H_{\text{obs}}^{-1} \sim 10^9 \) at present time when \( H \sim H_{\text{obs}} \). Collecting all these factors together we arrive to our final estimate

\[
\frac{\gamma}{\alpha \bar{c}} \sim \frac{k}{\sigma} \left[ \frac{\alpha}{3} \bar{c} \bar{H} - \bar{k} \right] \left( \frac{2 \pi}{\omega_k} \right) \left( \frac{2 \pi}{\alpha} \right)^2
\]  

(30)

\[
\sim 2 \cdot 10^2 \xi \left( \frac{2 \cdot 10^{-7} \text{cm}^3}{n_e} \right) \left( \frac{\omega_k}{H_{\text{obs}}} \right) \approx \frac{\alpha \bar{c}}{\pi}
\]  

where we approximated \( \omega_k \approx k \) and used the estimate (29) for \( k/\sigma \).

The numerical value for \( \xi \) which follows from relation (30) can be written as follows

\[
\xi \sim 10^{-2} \frac{\alpha \bar{c}}{(2\pi)^4} \left( \frac{n_e}{2 \cdot 10^{-7} \text{cm}^3} \right) \sim 10^{-12} \bar{c},
\]  

(31)

which implies that the intensity of the magnetic field at present time assumes the following value

\[
B \sim \sqrt{\xi \rho_{\text{DE}}} \sim 10^{-6} \cdot (2.3 \cdot 10^{-3} \text{eV})^2 \sim 2.6 \cdot 10^{-10} G.
\]  

(32)

where we expressed eV\(^2\) units in terms of conventional Gauss using the following relation: \( 1 G \approx 2 \cdot 10^{-2} \text{eV}^2 \).

This is of course, an order of magnitude estimate. However, the important point here is not just that a relatively strong magnetic field can be generated by this mechanism. Much more important element of the proposed mechanism is that the source of this field is the vacuum dark energy \( \rho_{\text{DE}} \) such that these two (naively unrelated) cosmological puzzles (the nature of the dark energy and magnetic coherent field) are intimately related because they are both originated from the same physics governed by the dynamics of the QCD topological sectors as reviewed in Sect. II.

One may wonder if the generation of the magnetic field at earlier times could produce a larger intensity field in comparison with estimate (32). The answer is “no”, and the reason for that is as follows. The wave length \( k \) is determined by eq. (18) where parameter \( H \) is defined in (3). At earlier times the right hand side of equation (30) will have an additional suppression factor \( H/H_{\text{obs}} \ll 1 \) as the time formation in physical units is getting shorter for the same frequency \( \omega_k \). This implies that parameter \( \xi \), and therefore, intensity of the generated magnetic field, will
receive an additional suppression. This argument implies that the strongest field is generated the last. It should be contrasted with conventional mechanisms which could produce very strong field at the moment of formation but become very weak due to the Hubble expansion.

V. CONCLUSION AND FUTURE DIRECTIONS

The main claim of this work is that the tunnelling transitions in QCD in expanding Universe will generate the coupling (8) due to the chiral anomaly. This interaction unambiguously implies that the Maxwell equations will be modified according to eq. (11). This additional non-dissipating current \( j \sim \vec{B} \) in the Maxwell system implies that there will be energy transfer from the vacuum DE to the magnetic energy. The correlation length of the produced magnetic field is determined by DE correlation length as \( B^2 \sim \rho_{DE} \) in this framework. The intensity of the field generated by this mechanism is estimated on the level of \( B \sim 10^{-10} \text{G} \) according to (32).

Can we test some of these ideas in tabletop experiments, at least, as a matter of principle? We want to argue that the ultimate answer is “yes”. Therefore, we claim that we are dealing with a real physics phenomenon, rather than with juggling of formal equations (such as insertions of the Lagrange multipliers, introductions of the auxiliary fields, subtractions of the UV counter terms and other formal elements which may look very suspicious for some readers).

The basic idea for a tabletop experiment goes as follows. The fundamentally new type of energy discussed in Section II can be, in principle, studied by measuring some specific corrections to the Casimir vacuum energy in the Maxwell theory as suggested in [42–46]. This fundamentally new contribution to the Casimir pressure emerges as a result of tunnelling processes, rather than due to the conventional fluctuations of the propagating photons with two physical transverse polarizations. Therefore, it was coined as the Topological Casimir Effect (TCE). The extra energy computed in [42–46] is the direct analog of the QCD non-dispersive vacuum energy (1), (2) which is the key player of the present work as it explicitly enters the EoS (7). In fact, an extra contribution to the Casimir pressure emerges in this system as a result of nontrivial holonomy for the Maxwell field. The nontrivial holonomy in \( E\&M \) system is enforced by the nontrivial boundary conditions imposed in refs [42–46], and related to the nontrivial mapping \( \pi_1(U(1)) = \mathbb{Z} \) relevant for the Maxwell abelian gauge theory\(^9\).

Furthermore, the emission of real physical photons from the Euclidean vacuum configurations describing the tunnelling events in the abelian Maxwell system (representing the direct analog of the non-abelian system discussed in Section IIIA) can also be studied in the Maxwell theory as argued in [45].

In fact, the same obstacle (related to the formulation of the tunnelling transitions in terms of the Euclidean path integral, while the emission of real particles on mass shell represents inherent Minkowski processes) can be also resolved by introducing the auxiliary topological fields in Maxwell system, similar to the discussions in Section III B, see [45] for the details.

To recapitulate the main point: the long range magnetic field with intensity of order \( B \sim 10^{-10} \text{G} \) can be generated as a result of variation of the QCD tunnelling transition rate in the time dependent background related to the Universe expansion. The two, naively distinct phenomena, are in fact closely related as the DE is the source for the magnetic energy in this framework, \( B^2 \sim \rho_{DE} \). This novel effect can be, in principle, tested in tabletop experiment, and in many respects is similar to the Dynamical Casimir Effect. What is more important is that such fields (correlated on the enormous scale of the visible Universe) can be studied by future UHECR telescopes, see Fig. 14 in ref. [10]. We finish this work on this optimistic note.

ACKNOWLEDGEMENTS

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

[1] A. O. Barvinsky and A. R. Zhitnitsky, Phys. Rev. D 98, no. 4, 045008 (2018) [arXiv:1709.09671 [hep-th]].
[2] A. O. Barvinsky and A. Y. Kamenshchik, JCAP 0609, 014 (2006) [hep-th/0605132].

\(^9\) A similar new type of energy can be, in principle, also studied in superfluid He-II system which also shows a number of striking similarities with non-abelian QCD as argued in [47]. For the superfluid He-II system the crucial role plays the vortices which are classified by \( \pi_1(U(1)) = \mathbb{Z} \) similar to the abelian quantum fluxes studied in the Maxwell system.

[3] A. O. Barvinsky and A. Y. Kamenshchik, Phys. Rev. D 74, 121502 (2006) [hep-th/0611206].
[4] A. O. Barvinsky, Phys. Rev. Lett. 99 (2007) 071301 [arXiv:0704.0083 [hep-th]].
[5] A. R. Zhitnitsky, Phys. Rev. D 89, no. 6, 063529 (2014) [arXiv:1310.2258 [hep-th]].
[6] A. R. Zhitnitsky, Phys. Rev. D 90, no. 4, 043504 (2014) [arXiv:1404.5965 [hep-ph]].
[7] A. R. Zhitnitsky, Phys. Rev. D 92, no. 4, 043512 (2015) [arXiv:1505.05151 [hep-ph]].
[8] P. P. Kronberg, Rept. Prog. Phys. 57, 325 (1994).
[9] D. Grasso and H. R. Rubinstein, Phys. Rept. 348, 163 (2001) [arXiv:astro-ph/0009061].
[10] R. Durrer and A. Neronov, Astron. Astrophys. Rev. 21, 62 (2013), [arXiv:1303.7121 [astro-ph.CO]].
[11] Y. B. Zeldovich, JETP Lett. 6, 316 (1967) [Pisma Zh. Eksp. Teor. Fiz. 6, 883 (1967)].
[12] A. Yamamoto, Phys. Rev. D 90, no. 5, 054510 (2014) [arXiv:1405.6665 [hep-lat]].
[13] F. R. Urban and A. R. Zhitnitsky, Phys. Lett. B 688, 9 (2010) [arXiv:0906.2162 [gr-qc]].
[14] F. R. Urban and A. R. Zhitnitsky, Nucl. Phys. B 835, 135 (2010) [arXiv:0906.2162 [gr-qc]].
[15] E. Witten, Nucl. Phys. B 156, 269 (1979).
[16] G. Veneziano, Nucl. Phys. B 159, 213 (1979).
[17] P. Di Vecchia and G. Veneziano, Nucl. Phys. B 171, 253 (1980).
[18] E. Thomas and A. R. Zhitnitsky, Phys. Rev. D 85, 044039 (2012) [arXiv:1109.2608 [hep-th]].
[19] M. ¨Unsal and L. G. Yaffe, Phys. Rev. D 78, 065035 (2008), [arXiv:0803.0344 [hep-th]].
[20] E. Thomas and A. R. Zhitnitsky, Phys. Rev. D 86, 065029 (2012) doi:10.1103/PhysRevD.86.065029 [arXiv:1203.6073 [hep-ph]].
[21] A. R. Zhitnitsky, Annals Phys. 336, 462 (2013) [arXiv:1301.7072 [hep-ph]].
[22] D. E. Kharzeev, Annals Phys. 325, 205 (2010) [arXiv:0911.3715 [hep-ph]].
[23] D. E. Kharzeev, J. Liao, S. A. Voloshin and G. Wang, Prog. Part. Nucl. Phys. 88, 1 (2016) [arXiv:1511.04505 [hep-ph]].
[24] K. van Bibber and L. J. Rosenberg, Phys. Today 59N8, 30 (2006); 60N9, 52 (2007).
[25] S. J. Asztalos, L. J. Rosenberg, K. van Bibber, P. Sikivie, K. Zioutas, Ann. Rev. Nucl. Part. Sci. 56, 293-326 (2006).
[26] Pierre Sikivie, Lect. Notes Phys. 741, 19 (2008) arXiv:0610440v2 [astro-ph].
[27] G. G. Raffelt, Lect. Notes Phys. 741, 51 (2008) [hep-ph/0611350].
[28] P. Sikivie, Int. J. Mod. Phys. A 25, 554 (2010) [arXiv:0909.0949 [hep-ph]].
[29] L. J. Rosenberg, Proc. Nat. Acad. Sci. 112, 12278 (2015).
[30] P. W. Graham, I. G. Irastorza, S. A. Voloshin and G. Wang, Prog. Part. Nucl. Phys. 88, 1 (2016) [arXiv:1511.04050 [hep-ph]].
[31] A. Ringwald, PoS NOW 2016, 081 (2016) [arXiv:1612.08933 [hep-ph]].
[32] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. Lett. 108, 031301 (2012) [arXiv:1109.3350 [hep-th]].
[33] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 92, 043004 (2015) [arXiv:1504.04854 [hep-th]].
[34] Y. Akamatsu and N. Yamamoto, Phys. Rev. Lett. 111, 052002 (2013) [arXiv:1302.2125 [nucl-th]].
[35] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 87, 105012 (2013) [arXiv:1301.1796 [hep-th]].
[36] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 92, 043004 (2015) [arXiv:1504.04854 [hep-th]].
[37] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 91, 035017 (2015) [arXiv:1507.06703 [hep-th]].
[38] C. Cao, M. van Caspel and A. R. Zhitnitsky, Phys. Rev. D 87, no. 10, 105012 (2013) [arXiv:1301.1796 [hep-th]].
[39] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 92, 043004 (2015) [arXiv:1504.04854 [hep-th]].
[40] Y. Akamatsu and N. Yamamoto, Phys. Rev. Lett. 88, 181601 (2002) [arXiv:1302.2125 [nucl-th]].
[41] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 87, 105012 (2013) [arXiv:1301.1706 [hep-th]].
[42] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 94, 065049 (2016) [arXiv:1512.00470 [hep-th]].
[43] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 95, 065018 (2017) [arXiv:1605.01411 [hep-th]].
[44] A. Boyarsky, J. Frohlich and O. Ruchayskiy, Phys. Rev. D 96, 015013 (2017) [arXiv:1702.00012 [hep-ph]].
[45] A. Zhitnitsky, Nucl. Phys. B 916, 510 (2017) [arXiv:1609.08619 [cond-mat.stat-mech]].