Synchronization of Non-Linear Random Walk Dynamics in Complex Networks

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This work addresses synchronization in transient, non-linear stochastic dynamics corresponding to accesses performed by self-avoiding walks originating at each node of a complex network. More specifically, the synchronizability of accesses incoming from other nodes has been considered and quantified in terms of the entropy of the mean periods of access, being closely associated to the efficiency of access delivery to each node. The concept of synchronous support of a node i has also been suggested as corresponding to the nodes which contribute the most for the synchronization of the accesses to i. These concepts have been applied to the analysis of 6 networks of different types, leading to markedly smaller synchronizability being obtained for the Watts-Strogatz and a geographical models. The more uniform synchronizabilities were identified for the Watts-Strogatz and path-regular structures. Varying degrees of correlations were found between the synchronizability and the degree or outward accessibility of the nodes. The synchronous support of a node i has been found to present diverse structure, including nodes which may be near to i, or nodes which are scattered through the network and far away from i.

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‘That which is static and repetitive is boring. That which is dynamic and random is confusing. In between lies art.’ (J. A. Locke)

I. INTRODUCTION

One special kind of dynamics regards the synchronization of events, manifesting itself in many natural systems such as brain activity, heart cell oscillations, and animal movement and development. Synchronization is also closely related to the efficiency of several types of dynamics. For instance, the uniform rate at which components arrive at a factory is essential for avoiding delays or accumulation of parts, the rate in which molecules are produced and transformed into other molecules and tissues are critical for life, among other possibilities. Therefore, it is hardly surprising that growing attention has been focused on the study of synchronization in complex networks (e.g. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12). However, the majority of such investigations have addressed synchronization of linear dynamics.

The current work addresses the stochastic synchronization in transient non-linear dynamics defined by self-avoiding random walks (the extension to other types of walks is immediate). We start by estimating the transition probabilities of agents moving into a node after having initiated random walks at the other nodes h steps before. Several types of mean frequency and period of accesses are defined, and the synchronizability is defined in terms of the normalized entropy of the mean period of accesses to each node i by moving agents which initiated their walks at any of the H previous steps. The concept of synchronous support of each node i is introduced, corresponding to the set of nodes which contribute the most to the synchronizability of i. The potential of these concepts and methods for characterization of the transient, non-linear dynamics in complex networks is illustrated with respect to 6 networks of different types, leading to a series of interesting results regarding not only the intrinsic overall synchronizability of each type of network, but also concerning correlations between synchronizability and degree or outward accessibility. Of particular interest has been the identification of substantially lower synchronizabilities for the Watts-Strogatz and path-regular models and the possibility of relatively accurate estimation of the synchronizability of nodes in ER networks from the respective degree.

This article starts by summarizing the main concepts in complex networks and random walks and proceeds by defining synchronizability and synchronous support. These concepts are then applied for the characterization of transient non-linear dynamics in 6 networks of distinct kinds.

II. BASIC CONCEPTS

This section summarizes the main concepts in network representation, measurement, random walks, as well as the six network models assumed in the present article.

A. Complex Networks Basics and Models

A directed and weighted network Γ can be represented in terms of its weight matrix W, so that each connection with weight r from node i to j, i, j ∈ {1, 2, ..., N} and i ≠ j, implies W(j, i) = r. Observe that these weights can correspond to probabilities in some cases. The absence of a connection between those nodes is represented as W(j, i) = 0. W becomes symmetric in the case of undi-
rected networks. The topology of $\Gamma$ can be represented in terms of the respective adjacency matrix $K$, obtained as $K(i, j) = \delta(W(i, j), 1)$, where $\delta(a, b)$ is the Kronecker’s delta. Observe that $K$ is a binary matrix. The instrength of a node $i$ is given by adding the weights of all the respective incoming edges (an analogue definition holds for the outstrength). The indegree of a node $i$ in $K$ is the number of incoming edges (similar for the outdegree). If $K$ is symmetric, indegree = outdegree = degree.

The immediate neighbors of a node $i$ are those nodes which can be reached from $i$ through a single edge. A walk can be defined as a sequence of adjacent edges. A path is a walk which never repeats an edge or node. The length of a walk (or path) is equal to the number of edges along it. A random walk is defined by a moving agent as it performs a walk along the network. An emphself-avoiding random walk or path-walk is defined by a moving agent as it performs a walk without repeating nodes or edges.

Six models of complex networks are considered in the present article: Erdős-Rényi (ER), Barabási-Albert (BA), Watts-Strogatz (WS), a geographical model (GG), as well as two knitted networks (PN and PA) [13, 14, 15, 16, 17, 18, 19]. The ER network (see also [20]) is obtained by establishing connections between pairs of nodes with fixed probability. The traditional preferential attachment rule (e.g. [13, 14, 15, 16]) is used here in order to obtain BA networks. The geographical network adopted in this article is obtained by distributing $N$ nodes along a two-dimensional space and connecting every pair of node whose nodes are closer than a given threshold. Path-regular network (PN) can be obtained by performing several path-walks along the $N$ nodes [17, 18, 19], and path-transformed BA networks are obtained by converting a BA networks through the star-path transformation of network connectivity [17, 15]. Therefore, this type of network includes paths with several lengths (power law). All networks considered in this work, except the virtual networks established by the transition probabilities, are undirected and have similar node degree ($k$) $\approx 6$ and number of nodes $N$. Only the largest connected component, which contains most nodes because of the relatively high adopted value of $\langle k \rangle$, of each networks is used herein.

### B. Synchronization in Random Walk Models

Let $\Gamma$ be a complex network in which moving agents perform self-avoiding walks after having started at a specific node. Let the transition probabilities of the moving agent from node $i$ to node $j$ after $h$ steps along its self-avoiding walk to be $P_h(i, j)$. Observe that, except for $h = 1$, non-zero values of $P_h(i, j)$ do not necessarily mean the presence of a physical edge extending from node $i$ to $j$, being here understood as virtual edges cite-Costa:2004. The set of nodes $j$ which connect to node $i$ through paths of length $h$ are henceforth represented as $\Phi_h(i)$. Observe that such a formulation can be immediately extended to other types of Markovian walks. The transition probabilities $P_h(i, j)$ can be estimated computationally by performing several walks of length $H$ starting at all the network nodes and accumulating the number of visits to each node (the probabilities are estimated by the relative frequencies). Figure 1(b) illustrates the probabilities $P_2(q, 5)$ considering the simple network in (a), $q \in \Phi_2(5) = \{1, 4, 6, 8\}$. The nodes in gray correspond to the nodes in $\Phi_2(5)$. Having started the walk at node 1, the moving agent has probability $P_1(a, 2) = 1/2$ of going to node 2 and then $P_1(2, 5) = 1/3$ of reaching node 5. As this is the only path from 1 to 5, we have $P_2(1, 5) = P_1(1, 2)P_1(2, 5) = 1/6$ because of the Markovianity of such a dynamic stochastic system.

![Figure 1: A simple network containing $N = 7$ nodes and the respective transition probabilities $P_2(q, 5)$, with $q \in \Phi_2(5) = \{1, 4, 6, 8\}$.](image)

Observe that the transition probability matrix $P_h$ can be understood as the transpose of a weight matrix describing virtual edges between the original nodes. After estimating $P_h(i, j)$ for $i, j \in \{1, 2, \ldots, N\}$, it becomes possible to define the accessibility to each node [21]. The inward accessibility of a node $i$ after $h$ steps has been understood [21] as being equal to $IA_h(i) = \sum_{v=1}^{N} P_h(v, i)/(N - 1)$. The outward accessibility of node $i$ after $h$ steps has been defined as $OA_h(i) = \exp(E_h(i))/(N - 1)$, where $E_h(i)$ is the diversity entropy [21] of node $i$, given as $E_h(i) = -\sum_{v=1}^{N} P_h(i, v)\log(P_h(i, v))$.

It is assumed henceforth that self-avoiding walks are initiated from all nodes at each time step. The mean frequency of accesses to node $i$ received from node $j$ after $h$ steps is immediately given as $f_{h}(j, i) = P_h(j, i)$. The respective mean period of accesses is $T_{h}(j, i) = 1/f_{h}(j, i)$. Observe that $T_{h}(i, j) \in \{1, 2, \ldots, N - 1\}$ for any $i$ and $j$. Table II includes other important mean frequencies of accesses. Observe that the respective mean periods of accesses can be immediately obtained by taking the reciprocal of the respective mean frequencies.

It can be easily shown that $1 < T(i) \leq N - 1$. The probability that all the moving agents which started their walks at the nodes in $\Phi_h(i)$ reach node $i$ after $h$ steps is given as $S_h(i) = \prod_{v \in \Phi_h(i)} P_h(v, i)$. Consequently, the
 maximal surge of accesses to a node $i$ by nodes which started their walks at $h$ steps before will occur with period $R_h(i) = 1/S_h(i)$. The surge frequency considering all time steps is $S_h(i) = \sum_{v \in \Phi_h(i)} P_h(v, i)$.

Figure 2 illustrates the above concepts with respect to the network in Figure 1. In Fig. 2(a) are shown the visits to node 5 along the time $t$ received by moving agents which started their walks at nodes 1, 4, 6, and 8 (each line of the diagram). Figure 2(b) illustrates the total of number of received visits at each time $t$ by agents which started their walks $h = 2$ steps before. For this case, we have that $f_2(1, 5) = 1/6$, $f_2(4, 5) = 5/12$, $f_2(6, 5) = 1/3$, $f_2(8, 5) = 1/4$, and $f_2(i) = 7/6$.

![FIG. 2: The accesses to node 5 along time $t$ by agents which left nodes 1, 4, 6 and 8 $h = 2$ steps before (a), and the respective total of such moving agents arriving at node 5 at each time $t$.](image)

Though the agents arriving at node 5 after 2 steps from the start of their respective walks at the nodes $v \in \Phi_2(5)$ will present a well-defined mean frequency of arrivals $f_2(v, 5)$, because of the statistical fluctuations it makes little sense speaking of phase or phase synchronization between such time series. Still, we consider in this work the synchronization between the agents arriving at a node $i$ in terms of the similarity between their mean frequencies of arrival at $i$. In order to do so, let us assume that each node $i$ needs to receive one visit from each of all other nodes (the involving subset of these nodes are immediate) before being able to treat them (e.g. to produce a respective molecule of perform some computation). Let us also suppose that, having received such visits, node $i$ takes a period of time $\tau(i)$ to process them. The ideal situation regarding the processing speed and occupation of node $i$ takes place if and only $\tau(i) = T(i)$. Such a matching of frequencies is what we understand by synchronization in the present work. It becomes interesting to devise an objective quantification of the frequency synchronizability of each node in the network.

In this work, the synchronizability is simply expressed in terms of the normalized entropy of the effective periods of accesses to each node per unit of time $T(i)$, i.e.

$$
\epsilon(i) = - \sum_{v=1}^{N} p(T(v, i) \log(p(T(v, i)))
$$

$$
\sigma(i) = \frac{\log(N - 1) - \epsilon(i)}{\log(N - 1)}
$$

where $T(v, i) = 1/f(v, i)$ (see Table I), $\epsilon(i)$ is the entropy of $T(v, i)$ and $\log(N - 1)$ is the maximum entropy possibly achieved in any network. It can be verified that $0 < \sigma(i) \leq 1$. Thus, in case a node $i$ receives accesses with identical frequencies $F(i)$, we will have $\sigma(i) = 1$, indicating maximum synchronizability. The synchronizability of node $i$, and therefore the efficiency in the use of the access received at that node, will decrease when the probabilities $p(T(i))$ tend to concentrate on a few nodes. Figure 3 illustrates that when a node $i$ (in this case, node 2) has immediate neighbors with different degrees, different rates of accesses to node $i$ (i.e. $P_1(2, 1) = 1/3$, $P_1(3, 1) = 1$ and $P_1(4, 1) = 1/5$) will be obtained, implying high entropy of mean periods of accesses to $i$. In

| Mean frequency of accesses to node $j$ by moving agents which, $h$ steps ago, left from node $i$: | $f_{h}(i, j) = P_{h}(i, j)$ |
| Mean frequency of accesses to node $i$ by moving agents which, $h$ steps ago, left from all other nodes: | $f_{h}(i) = \sum_{v=1}^{N} P_{h}(v, i)$ |
| Mean frequency of accesses to node $j$ by moving agents which, from 1 to $H$ steps ago, left from node $i$: | $f(i, j) = \sum_{h=1}^{H} P_{h}(i, j)$ |
| Mean frequency of accesses to node $i$ by moving agents which, from 1 to $H$ steps ago, left from all other nodes: | $f(i) = \sum_{h=1}^{H} \sum_{v=1}^{N} P_{h}(i, j) = \sum_{v=1}^{N} f(v, i)$ |

**TABLE I:** The definition of the several mean frequency of accesses by moving agents.
networks where a node with high degree has an inherent change of being attached to nodes with diverse degrees, a negative correlation between the synchronizability and the node degree is expected. Such a relationship, however, is not guaranteed because the synchronizability involves more global connectivity around each node.

![Diagram](Image)

FIG. 3: A simple situation illustrating how the entropy of mean periods of accesses to a node (1) increases when the immediate neighbors of that node have different degrees.

It is also interesting to identify for each node what are the other nodes in the network which contribute the most for the synchronization of accesses to i. This can be done by identifying the highest value of \( p(T(v,i)) \), henceforth represented as \( p_{\text{max}}(i) \), and finding out which nodes \( s \) have \( T(s,i) \) nearly equal to the respective mean period. Such more highly synchronized nodes are henceforth called the synchronous support of node i, henceforth represented as \( S(i) \). Finally, observe that the probabilities \( p(T(v,i)) \) can be easily estimated from the histogram of relative frequencies of \( T(i) \), derived from the transition matrices \( P_h \).

### III. RESULTS AND DISCUSSION

In order to illustrate the concepts described in this article, we considered 6 theoretical complex networks — namely Erdős-Rényi (ER), Barabási-Albert (BA), Watts-Strogatz (WS), a geographical model (GG) as well as two knitted networks (PN and PA), all with \( N = 100 \) (except GG, for which \( N = 91 \)) and \( \langle k \rangle \approx 6 \). In all cases, a total of 2000 self-avoiding walks were initiated at each of the nodes in order to estimate the transition probabilities \( P_h(i,j) \) for all nodes and for \( h = 1 \) to \( H = 10 \).

Figure 4 shows the histograms of synchronizabilities obtained considering each nodes for each network and \( H = 10 \) (the mean and standard deviations of the synchronizabilities are shown in the respective titles). The ER, BA, PN and PA networks yielded similar and relatively high synchronizabilities, while the WS and GG were characterized by substantially lower values (see Figs. 4c and d)). The smallest dispersions of synchronizability were obtained for WS and PN. Additional experiments performed indicated that the mean synchronizability tends to decrease substantially with smaller values of \( H \).

Figure 5 shows the scatterplots of the synchronizabilities \( \sigma \) of each node against its respective degree \( k \). The Pearson correlation coefficients are shown in the title of each plot. Such results show that the correlations between \( \sigma \) and \( k \) vary markedly for each type of network. ER showed the highest correlation, suggesting that the synchronizability of its nodes can be reasonably predicted from the respective degree. In the case of the BA structure, \( \sigma \) exhibited an almost quadratic increase with \( k \), also leading to a well-defined relationship between these two features. Moderate positive correlations were also found for the PN and PA network. However, small correlations were obtained for the WS and GG structures, with negative weak correlation being observed for the WS network. This implies that it is virtually to infer the synchronizability of a node in the GG structure from its degree.

Figure 6 shows the geographical network considered in this article, with the synchronizabilities of each node identified by the color legend. Node 90 presented the highest synchronizability value (\( \sigma(90) = 0.45 \)). Observe that the values of \( \sigma \) tend to be similar among neighboring nodes (assortativeness of synchronizability). The more central nodes, identified by larger outward accessibility, resulted with the lowest synchronizabilities. Also, the more isolated group of nodes at the upper left-hand side of the figure presented the highest values of synchronizability. Because such nodes are known [21] to have low outward accessibility \( O_A \), it is interesting to consider possible correlations between this measurement and synchronizability.

Figure 7 shows the scatterplots obtained considering the synchronizabilities and outward accessibilities (corresponding to the mean value of the accessibilities along the \( H \) steps) of each node for all the networks in this article. The Pearson correlation coefficients are also shown respectively to each plot. The results show that varying correlations are verified between \( \sigma \) and \( O_A \) for each type of network. The ER and PA structures showed particularly strong positive correlations (Fig. 7a and f), while the WS model exhibited the weakest correlation (Fig. 7c). Interestingly, the synchronizability resulted negatively correlated for the GG network (Fig. 7b).

The nodes belonging to the synchronous support of node 50, which has an intermediate synchronizability of \( \sigma(50) = 0.26 \), are shown in Figure 6 with wider borders. The respective histogram of relative frequencies of mean periods of accesses to this node, i.e. \( T(v,50) \), \( v = 1,2, \ldots , N \), is depicted in Figure 8. Observe that the synchronous support of node 50 comprises most of the nodes in the small community to which it belongs, except only for nodes 22, 24 and 25. Similar relationship with communities has been verified for nodes 61 and 52. However, the synchronous support of node 1 was found to include nodes 2, 15, 25, 36, 48, 63 and 71, which are all far away and scattered through the right-hand side of the network. This suggests that the synchronous supports of nodes in a GG network do not seem to follow a typical structural pattern.
IV. CONCLUDING REMARKS

We have considered the situation where, after leaving continuously from each of the nodes in a complex networks, moving agents perform self-avoiding walks of fixed and finite length along complex networks. After estimating the probabilities of accesses received by each node by agents leaving from each of the other nodes a number of steps behind, it is possible to consider synchronization between the respective mean frequencies. An objective measurement of the synchronizability of each node has been reported which takes into account the entropies of the mean periods of accesses from all other nodes by moving agents which initiated their walks at any of the previous steps. It has been shown that such a kind of synchronization optimizes the delivery to the reference node of agents originating at each of the other nodes at a constant rate and well-defined, avoiding accumulation or lack of accesses along the critical period. Such a result also allows the identification of optimal processing times for each node. Modifications of the reported framework to consider different rates of random walks emanating from each node or accesses from subsets of the network nodes, instead of from all nodes, are straightforward and can be useful for more general applications. The concept of synchronous support of a node has also been suggested in order to identify the nodes which most contribute to the synchronizability of each node. Several interesting results have been obtained, including the identification of distinct overall synchronizability exhibited among 6 types of networks (with the WS and GG being less synchronizable), as well as the identification, in the case of the GG, of assortativeness of synchronizability and the fact that the synchronous support of the nodes tend not to exhibit a typical pattern. The correlations between the synchronizability and degree or outward accessibility have also been considered and found to present diverse patterns. For instance, while the synchronizability of the nodes in an ER structure can be reasonably inferred from their respective nodes with relatively high accuracy, it is virtually impossible to perform such predictions in the case of the GG structure. Therefore, interesting patterns of relationships have been identified between a topological feature (node degree) and the dynamic property of synchronizability in the transient non-linear dynamics defined by self-avoiding random walks.

Future developments include the search for additional possible topological features which can explain or predict synchronizability, studies of scaling with the average degree and network size, and the investigation of synchronizabilities implied by different types of walks. Also interesting would be to search further for possible rela-
FIG. 5: Scatterplots of synchronizability against the node degree for the six considered networks.

relationships between the synchronizability and community structure. In addition, the concepts described in this article have good potential for applications in real-world problems, including biological systems, transportation and communications networks. Of special interest is the study of non-linear transient synchronizability in neuronal or cortical networks, especially by considering frontwave random walks, i.e. the progression of accesses along the successive neighborhoods of each node (e.g. [22]).

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FIG. 6: The values of the synchronizability ($H = 10$) of each node in the GG network considered in this article are shown accordingly with the color scale (increasing from bottom upwards). The nodes belonging to the synchronous support of node 50 are shown with wider borders.

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FIG. 7: Scatterplots of the synchronizability $\sigma$ against the respective node degrees $k$ obtained for each of the 6 networks.

FIG. 8: Histogram of the relative frequencies of mean periods $s$ of accesses to node 50 from all other nodes in the network.