Singularity Formation for the General Poiseuille Flow of Nematic Liquid Crystals

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Abstract
We consider the Poiseuille flow of nematic liquid crystals via the full Ericksen-Leslie model. The model is described by a coupled system consisting of a heat equation and a quasilinear wave equation. In this paper, we will construct an example with a finite time cusp singularity due to the quasilinearity of the wave equation, extended from an earlier result on a special case.

Keywords Liquid crystals · Cusp singularity · Quasilinear wave equation

Mathematics Subject Classification 76A15 · 35M31 · 35L52 · 35L67

1 Introduction
The state of a nematic liquid crystal is characterized by its velocity field $u$ for the flow and its director field $n \in S^2$ for the alignment of the rod-like feature. These two characteristics interact with each other so that any distortion of the director $n$ causes a motion $u$ and, likewise, any flow $u$ affects the alignment $n$. One famous model on nematic liquid crystal is the Ericksen-Leslie model for nematics that was first proposed by Ericksen [11] and Leslie [15] in the 1960s.

In this paper, we consider the Poiseuille flow via the full Ericksen-Leslie model, when $u$ and $n$ take the form

$$u(x, t) = (0, 0, u(x, t))^T \quad \text{and} \quad n(x, t) = (\sin \theta(x, t), 0, \cos \theta(x, t))^T,$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and the motion $u$ is along the $z$-axis and the director $n$ lies in the $(x, z)$-plane with angle $\theta$ made from the $z$-axis. Then, the Ericksen-Leslie system can be written as

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\[ u_t = \left( g(\theta)u_x + h(\theta)\theta_t \right)_x, \quad (1) \]
\[ \theta_{tt} + \gamma_1 \theta_t = c(\theta) \left( c(\theta) \theta_x \right)_x - h(\theta)u_x, \quad (2) \]

where the \( C^\infty \) functions \( c, g, \) and \( h \) are explicitly given by

\[
\left\{ \begin{array}{l}
g(\theta) := \alpha_1 \sin^2 \theta \cos^2 \theta + \frac{\alpha_5 - \alpha_2}{2} \sin^2 \theta + \frac{\alpha_3 + \alpha_6}{2} \cos^2 \theta + \frac{\alpha_4}{2}, \\
h(\theta) := \alpha_3 \cos^2 \theta - \alpha_5 \sin^2 \theta = \frac{\gamma_1 + \gamma_2 \cos(2\theta)}{2}, \\
\kappa^2(\theta) := K_1 \cos^2 \theta + K_3 \sin^2 \theta.
\end{array} \right. \quad (3)
\]

Here, \( K_1, K_3 \) are positive elastic constants in the Oseen-Frank energy. The material coefficients \( \gamma_1 \) and \( \gamma_2 \) reflect the molecular shape and the slippery part between fluid and particles. The coefficients \( \alpha_i \)'s and coefficients \( \gamma_1, \gamma_2 \) satisfy the following physical relations:

\[
\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \quad \alpha_2 + \alpha_3 = \alpha_6 - \alpha_5. \quad (4)
\]

The first two relations are compatibility conditions, while the third relation is called Parodi's relation, derived from Onsager reciprocal relations expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium (cf. [17]). The coefficients also satisfy the following empirical relations (p.13, [16]):

\[
\left\{ \begin{array}{l}
\alpha_4 > 0, \quad 2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 > 0, \quad \gamma_1 = \alpha_3 - \alpha_2 > 0, \\
2\alpha_4 + \alpha_5 + \alpha_6 > 0, \quad 4\gamma_1 (2\alpha_4 + \alpha_5 + \alpha_6) > (\alpha_2 + \alpha_3 + \gamma_2)^2.
\end{array} \right. \quad (5)
\]

A detailed derivation of the Ericksen-Leslie system for Poiseulle flows can be found in [7], where without loss of generality, we choose the density \( \rho \) to be 1, and the inertial coefficient \( \kappa \) of the director \( n \) to be 1. Furthermore, for simplicity, the constant \( \alpha \) in the model in [7], which is the gradient of pressure along the flow direction, is set to be zero.

Due to the quasilinearity of the wave equation on \( \theta \), finite time gradient blow-up might happen even when the initial data are smooth. In [7], Chen et al. established an example showing such kind of singularity formation phenomena, for a special case of (1) and (2) when \( g = h = 1 \) and \( \gamma_1 = 2 \).

The construction of the singularity formation example relies on the framework in [13] by Glassy-Hunter-Zheng on the variational wave equation

\[ \theta_{tt} = c(\theta)(c(\theta) \theta_x)_x. \quad (6) \]

The global well-posedness theories of Hölder continuous solutions for variational wave equations and systems related to nematic liquid crystals have been intensively studied in the last two decades [1–6, 9, 10, 14, 18–20].

The proof of finite time singularity formation in [7] for the special case of (1) and (2) also relies on a crucial estimate on the \( L^\infty \) bound of a new function \( J \), where in the general case,

\[ J = u_x + \frac{h}{g} \theta_t. \quad (7) \]
We note that functions $u_x$ and $\theta_t$ both blow up when the singularity forms; however, their combination $J$ will be proved uniformly bounded. This estimate, first obtained in [7] for the special case (when $g = h = 1$ and $\gamma_1 = 2$), is crucial for both the singularity formation and the global existence of the Hölder continuous solution for the system (1) and (2).

Using (7), the wave equation (2) can be written as

$$
\theta_{tt} + \left[ \gamma_1 - \frac{h^2(\theta)}{g(\theta)} \right] \theta_t = c(\theta)(c(\theta)\theta_x)_x - h(\theta)J.
$$

From (3)–(5), we have the following bounds for $g$, $h$, and $c$:

$$
\begin{align*}
&g_L \leq g(\theta) \leq g_U, \\
&h_L \leq h(\theta) \leq h_U, \\
&C_L \leq c(\theta) \leq C_U,
\end{align*}
$$

where $g_L, g_U, h_L, h_U, C_L, \text{ and } C_U$ are constants such that $g_L, g_U, h_U, C_L, \text{ and } C_U$ are strictly positive. Furthermore, physical laws in (4) and (5) give that

$$
\min \left\{ \gamma_1 - \frac{h^2(\theta)}{g(\theta)}, g(\theta) \right\} > C
$$

for some positive constant $C$, where the proof can be found in [7].

In this paper, we will first find a bound on $J$ in terms of the initial energy. Therefore, (8) is a damped variational wave equation adding a uniform bounded source term $hJ$. Then, we can prove the singularity formation of cusp singularity using the methods in [7, 10, 13]. Note the source term $hux$ in the original equation (2) may be unbounded.

In another companion paper [8], we will show that $J$ is bounded under the norm of $L^2 \cap L^\infty \cap C^\alpha$ for some positive constant $\alpha$, even for weak solutions including singularities. This is one of the key estimates for the global existence proof.

Except showing one example forming the cusp singularity, another motivation of this paper is to introduce the major mathematical idea why we can get better regularity on $J$ than $u_x$ or $\theta_t$, in the general case. For smooth solutions, we can explain the idea in a relatively easier manner than for weak solutions.

Now, let us explain the main difficulty in controlling $J$ for the general case comparing to the special case. For the special case considered in [7] (when $g = h = 1$ and $\gamma_1 = 2$), (1) is

$$
u_t = u_{xx} + \theta_x,
$$

thus $u$ can be solved directly by $\theta_x$. However, this is not true for the general case. The main difficulty we need to overcome is the varying coefficient $g(\theta)$ in the heat equation (1). Although $g(\theta)$ is strictly positive and uniformly bounded, it is only Hölder continuous on $x$ and $t$ at the blow-up. Therefore, the derivatives of $g(\theta)$ on $x$ and $t$ blow-up when the singularity forms. This creates a lot of essential problems for finding a uniform bound on $J$, since we need to use both heat and wave equations to bound $J$. One of our key ideas is to consider the potential $A$ of $J$, with $A_x = J$.

Before we state the main result we define the following function $\phi(x)$ with $x \in \mathbb{R}$, that is used to design the initial data. Take $\phi \in C^1$ with

$$
\phi(0) = 0 \quad \text{and} \quad \phi(a) = 0 \quad \text{for} \quad a \notin (-1, 1),
$$

(11)
\[-\phi'(0) > \max \left\{ 16C_u \| \gamma_1 - \frac{h^2}{g} \|_{L^\infty}, \exp \left( \frac{\| \gamma_1 - \frac{h^2}{g} \|_{L^\infty}}{C_L} \right) \right\} \text{ and } |\phi'(x)| \leq C_2, \quad (12)\]

and
\[\int_{-1}^{1} (\phi')^2(a) \, da < k_0, \quad (13)\]

where \( \theta^* \) is a constant such that \( c'(\theta^*) > 0 \) and \( C_2, k_0 \) are some positive constants.

Here is the main singularity formation result.

**Theorem 1**  
*Let the initial data be*

\[
\theta(x, 0) = \theta_0(x) := \theta^* + \varepsilon \phi \left( \frac{x}{\varepsilon} \right), \quad \theta_t(x, 0) = \theta_1(x) := (-c(\theta_0(x)) + \varepsilon)\theta_0'(x), \quad (14)
\]

*and*

\[
u(x, 0) = \nu_0(x) := \begin{cases} 0, & x \in (-\infty, -\varepsilon), \\ \int_{-\varepsilon}^{x} \frac{h c(\theta_0(a))}{g} \theta_0'(a) \, da, & x \in [-\varepsilon, \varepsilon], \\ \chi(x), & x \in (\varepsilon, \varepsilon + 2), \\ 0, & x \in (\varepsilon + 2, \infty), \end{cases}
\]

*where \( \theta^* \) is a constant such that \( c'(\theta^*) > 0 \), and \( \chi(x) \) is a \( C^1 \) function satisfying*

\[|\chi'(x)| \leq \frac{3}{2} \left\| \frac{h}{g} \right\|_{L^\infty} C_u C_2 \varepsilon, \quad (15)\]

*and such that \( \nu_0(x) \) is a \( C^1 \) function. Then, there exists a sufficiently small positive choice of the parameter \( \varepsilon \) such that the solution \((\theta, \nu)\) of \((1)\) and \((2)\) is \( C^1 \) only up to a finite time. More precisely, the solution is continuously differentiable up to some time before*

\[T = \min \left\{ \frac{2 \ln 2}{\| \gamma_1 - \frac{h^2}{g} \|_{L^\infty}}, 1 \right\}. \]

In the proof of the theorem, we will show that the singularity happens in the following form: there exists a time \( 0 < t_0 < T \) such that

\[\theta_t \to \infty, \quad \theta_x \to -\infty \quad (16)\]

as \( t \to t_0^- \) along the characteristic having the blow-up phenomena. In this example, the initial energy is small, and the initial data are smooth, but the finite time singularity still forms. The singularity is a cusp type of singularity, combining with the existence result in [8], also see [7, 13].

The rest of the paper is divided into 3 sections as follows. In Sect. 2 we show the decay of the energy associated with a smooth solution. Section 3 contains the main
estimate on the quantity $J$: we show the uniform bound of $J$ over a fixed time interval. In Sect. 4, we prove the singularity formation result in Theorem 1.

## 2 The Energy of the System

The energy of the system (1) and (2) is defined as

$$\mathcal{E}(t) := \frac{1}{2} \int_{\mathbb{R}} (\theta^2_t + c(\theta)^2 \theta_x^2 + u^2) \, dx.$$  \hspace{1cm} (17)

In this section, we show the energy decay for smooth solutions.

**Proposition 1** For any smooth solution $(\theta(x, t), u(x, t))$ of the system (1), (2), the energy $\mathcal{E}(t)$ decays with the following rate:

$$\frac{d}{dt} \mathcal{E}(t) = - \int_{\mathbb{R}} \left( b(\theta) u_x^2 + \gamma_1 \left( \theta_t + \frac{h(\theta)}{\gamma_1} u_x \right)^2 \right) \, dx,$$ \hspace{1cm} (18)

where $b(\theta) := g(\theta) - \frac{h^2(\theta)}{\gamma_1} > 0$.

**Proof** Multiplying (1) by $u$ and (2) by $\theta_t$, and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int u^2 \, dx = - \int g(\theta) u_x^2 \, dx - \int h(\theta) \theta_t u_x \, dx,$$ \hspace{1cm} (19)

and

$$\frac{1}{2} \frac{d}{dt} \int (\theta^2_t + c^2(\theta) \theta_x^2) \, dx = - \int \gamma_1 \theta_t^2 \, dx - \int h(\theta) u_x \theta_t \, dx.$$ \hspace{1cm} (20)

Sum up (19) and (20) to get

$$\frac{1}{2} \frac{d}{dt} \int \left( \theta^2_t + c^2(\theta) \theta_x^2 + u^2 \right) \, dx = - \int \left( \gamma_1 \theta_t^2 + 2 h(\theta) \theta_t u_x + g(\theta) u_x^2 \right) \, dx$$

$$= - \int_{\mathbb{R}} \left( b(\theta) u_x^2 + \gamma_1 \left( \theta_t + \frac{h(\theta)}{\gamma_1} u_x \right)^2 \right) \, dx.$$  \hspace{1cm} (21)

Therefore, considering only smooth solutions, for any $T > 0$, we have

$$\max_{0 \leq t \leq T} \mathcal{E}(t) \leq \mathcal{E}(0).$$

## 3 Uniform Bound on $J$ in Finite Time

For convenience, we just fix $T_0 = 1$ and give a uniform upper bound on $J$, defined in (7), when $0 \leq t \leq T_0$, when the solution is smooth. In fact, one can choose $T_0$ to be any positive constant, and still prove that $J$ is bounded.
Before our calculation, we recall the bounds in (9) and (10), and also note that \(|h'|\) and \(|g'|\) are bounded above by a constant because of (3).

### 3.1 An Integral Relation on \(J\) and Its Estimate

To get the uniform \(L^\infty\) estimate on \(J\), we first find an integral relation on \(J\) using both (1) and (2).

We study the potential of \(J\), defined as

\[
A := \int_{-\infty}^{x} \left( u_x + \frac{h}{g} \theta_t \right) \, dz = \int_{-\infty}^{x} J \, dx.
\]

Thus

\[
A_x = J.
\]

In terms of the quantity \(J\), we can write the system as

\[
\begin{align*}
  u_t &= \left( g(\theta)u_x + h(\theta)\theta_t \right)_x, \quad (22) \\
  \theta_t + \left( \gamma_1 - \frac{h^2}{g} \right) \theta_x &= c(\theta) \left( c(\theta) \theta_x \right)_x - hJ. \quad (23)
\end{align*}
\]

And we derive the following equation for \(A\):

\[
\begin{align*}
A_t &= \int_{-\infty}^{x} \frac{(gJ)_t}{g} \, dz - \int_{-\infty}^{x} \frac{g'\theta_t}{g} J \, dz \\
&= (gJ)_x + \int_{-\infty}^{x} \frac{1}{g} \left( (g'\theta_t - \gamma_1 g) J + \left( h' - \frac{g'h}{g} \right) \theta^2 + (\gamma_1 g - h^2) u_x + h c(c(\theta) \theta_x) \right) \, dz \\
&\quad - \int_{-\infty}^{x} \frac{g'\theta_t}{g} J \, dz \\
&= (g(\theta)J)_x + \int_{-\infty}^{x} \frac{1}{g} \left( (\gamma_1 - \frac{h^2}{g}) J + \left( h' - \frac{g'h}{g} \right) \theta^2 + (\gamma_1 g - h^2) u_x + h c(c(\theta) \theta_x) \right) \, dz.
\end{align*}
\]

Integrating it by parts, we get

\[
\begin{align*}
A_t &= g(\theta)A_{xx} - \gamma_1 A + g t \theta_x J \\
&\quad + \int_{-\infty}^{x} \left( \left( \frac{h^2}{g} - \frac{g'h}{g^2} \right) \theta^2 - \left( \gamma_1 - \frac{h^2}{g} \right)^' u_x - \left( \frac{h(\theta)c(\theta)}{g} \right)^' c(\theta) \theta_x^2 \right) \, dz \\
&\quad + \left( \gamma_1 - \frac{h^2}{g} \right) u + \frac{h(\theta)c^2(\theta)}{g} \theta_x.
\end{align*}
\]

(24)

In summary, we consider the following Cauchy problem:

\[
A_t - g(\theta)A_{xx} + \gamma_1 A = g' \theta_x J + F_1(\theta, u) + F_2(\theta, u), \quad (25)
\]

where
\[
F_1 = \int_{-\infty}^{x} \left( \left( \frac{h'}{g} - \frac{g'h}{g^2} \right) \theta_i^2 - \left( \gamma_1 - \frac{h^2}{g} \right) \theta_u - \left( \frac{h(\theta)c(\theta)}{g} \right) \theta_z^2 \right) d\xi,
\]
\[
F_2 = \left( \gamma_1 - \frac{h^2}{g} \right) u + \frac{h(\theta)c^2(\theta)}{g} \theta_x
\]

with

\[
A(x, 0) = \int_{\mathbb{R}} \left( u_0'(x) + \frac{h(\theta_0)}{g(\theta_0)} \theta_1 \right) dx : = A_0(x).
\]

It is easy to verify that there exists some positive constant \( C \), such that

\[
\|F_1(\theta, u)\|_{L^2(\mathbb{R})} \leq C E(t) \leq C E(0), \quad \|F_2(\theta, u)\|_{L^2(\mathbb{R})}^2 \leq C E(t) \leq C E(0).
\]

In this section, without confusion, we always use \( C \) to denote different positive constants for different estimates.

Now using the conclusion from Chapter 1, Theorem 12 in [12], we can formally solve \( A \) by (25) as

\[
A(x, t) = \int_{\mathbb{R}} \Gamma(x, t; \xi, 0) A_0(\xi) d\xi \\
+ \int_0^t \int_{\mathbb{R}} \Gamma(x, t; \xi, \tau) (g' \theta_J + F_1(\theta, u) + F_2(\theta, u)) (\xi, \tau) d\xi d\tau,
\]

where the kernel \( \Gamma \) can be written in terms of the heat kernel as follows:

\[
\Gamma(x, t; \xi, \tau) = H^{\xi, \tau}(x - \xi, t - \tau) + \int_0^t \int_{\mathbb{R}} H^{\xi, s}(x - y, t - s) \Phi(y, s; \xi, \tau) dy ds,
\]

where

\[
H^{\xi, \tau}(x - \xi, t - \tau) = \frac{\sqrt{g(\Phi(\xi, \tau))}}{2\sqrt{\pi}} e^{-\frac{g(\Phi(\xi, \tau))^2}{4(t - \tau)}}.
\]

The function \( \Phi \) is determined by the condition

\[\mathcal{L} \Gamma = 0,\]

where

\[\mathcal{L} := \partial_t - g(\theta) \partial_{xx} + \gamma_1.\]

It can be shown that such function exists and

\[
|\Phi(y, s; \xi, \tau)| \leq \frac{C}{(s - \tau)^{5/4}} e^{\frac{d_1-s^2}{4(s-\tau)}},
\]

where \( d \) is a constant depending on \( g \). Moreover, we have the following bounds for \( \Gamma \) and \( \Gamma_x \):
\[ |F(x, t; \xi, \tau)| \leq \frac{C}{\sqrt{t - \tau}} e^{-\frac{\|x-x_0\|^2}{4(t-\tau)}} , \quad (32) \]

\[ |\Gamma_3(x, t; \xi, \tau)| \leq \frac{C}{t - \tau} e^{-\frac{\|x-x_0\|^2}{4(t-\tau)}} . \quad (33) \]

The reader can find the proof of the above estimates in Chapter 1, Theorem 11 in [12].

Now, differentiating (28) w.r.t. \( x \) we obtain that \( J \) satisfies the following relation:

\[ J(x, t) = \int_{\mathbb{R}} \Gamma_3(x, t; \xi, 0) A_0(\xi) \, d\xi \]

\[ + \int_0^t \int_{\mathbb{R}} \Gamma_3(x, t; \xi, \tau) F_1(\theta, u)(\xi, \tau) \, d\xi \, d\tau \]

\[ + \int_0^t \int_{\mathbb{R}} \Gamma_3(x, t; \xi, \tau) F_2(\theta, u)(\xi, \tau) \, d\xi \, d\tau \]

\[ + \int_0^t \int_{\mathbb{R}} \Gamma_3(x, t; \xi, \tau) (g^\prime \theta^\prime J)(\xi, \tau) \, d\xi \, d\tau \]

\[ := L_0(x, t) + L_1(x, t) + L_2(x, t) + \int_0^t \int_{\mathbb{R}} \Gamma_3(x, t; \xi, \tau) (g^\prime \theta^\prime J)(\xi, \tau) \, d\xi \, d\tau. \quad (34) \]

Then, we use this expression to find the uniform upper bound on \( J \). First, we give uniform bounds on \( L_0 \), \( L_1 \), and \( L_2 \) in terms of \( J_0 \), \( A_0 \) and the initial energy.

### 3.2 \( L^\infty \) Estimates on \( L_1 \) and \( L_2 \)

Recall the estimates (33) and (27). First,

\[ |L_1| \leq \|F_1\|_{L^\infty} \int_0^t \int_{\mathbb{R}} \frac{C}{t - \tau} e^{-\frac{\|x-x_0\|^2}{4(t-\tau)}} \, d\xi \, d\tau \leq C t^{1/2} \|F_1\|_{L^\infty(\Omega)} \leq t^{1/2} CE(0). \quad (35) \]

Second,

\[ |L_2| \leq \left[ \int_0^t \int_{\mathbb{R}} \frac{1}{|t - \tau|^{3/2}} e^{-\frac{\|x-x_0\|^2}{4(t-\tau)}} \, d\xi \, d\tau \right]^{1/2} \left( \int_0^t \int_{\mathbb{R}} \frac{1}{|t - \tau|^{2r}} F_2^2 \, d\xi \, d\tau \right)^{1/2} \quad (36) \]

For \( r = \frac{3}{8} \) we obtain

\[ |L_2| \leq t^{1/2} C \|F^2\|_{L^\infty((0,t),L^2(\mathbb{R}))} \leq t^{1/4} CE^{3/2}(0). \quad (37) \]

### 3.3 \( L^\infty \) Estimates on \( L_0 \)

**Lemma 1** Let \( A_0(x) \) be such that \( A_0(x), A_0(x) \in L^\infty(\mathbb{R}) \), we have
The integral related to the second term of (42) can be easily estimated as

\[ |\int_\mathbb{R} \Gamma_x(x, t; \xi, 0) A_0(\xi) \, d\xi| \leq C \|J_0(x)\|_{L^\infty(\mathbb{R})} + C_1 \|\theta'_0(x)\|_{L^\infty(\mathbb{R})} \|A_0(x)\|_{L^\infty(\mathbb{R})} + C_2 \|A_0(x)\|_{L^\infty(\mathbb{R})} \]  

(39)

for some constants \(C, C_1,\) and \(C_2.\)

**Proof** Recall

\[
\Gamma_x(x, t; \xi, 0) = H_x^{\xi,0}(x - \xi, t) + \int_0^t \int_\mathbb{R} H_x^{\xi,s}(x - y, t - s) \Phi(y, s; \xi, 0) \, dy \, ds
\]

and

\[
H_x^{\xi,s}(x - \xi, t - \tau) = \frac{\sqrt{g(\theta(\xi, \tau))}}{2\sqrt{\pi}} e^{-\frac{g(\theta(\xi, \tau))}{4(t - \tau)}}.
\]

Also note that due to the dependence of \(H_x^{\xi,0}(x - \xi, t)\) on \(g(\theta(\xi, 0))\), we have the following relation:

\[
H_x^{\xi,0}(x - \xi, t) = -H_x^{\xi,0}(x - \xi, t) + \frac{g'\theta_0'}{4\sqrt{\pi} \sqrt{g}} e^{-\frac{g(\theta(\xi, t))}{4t}}/4t
\]

\[
- \frac{\sqrt{g} g'\theta_0'}{2\sqrt{\pi}} e^{-\frac{g(\theta(\xi, t))}{4t}}/4t
\]

(42)

Starting with the first term of \(\Gamma_x\) and using the above relation, there are three integrals to estimate. The integral related to the first term in (42) is

\[
\int_\mathbb{R} -H_x^{\xi,0}(x - \xi, t) A_0(\xi) \, d\xi = \int_\mathbb{R} H_x^{\xi,0}(x - \xi, t) A_{0,\xi}(\xi) \, d\xi.
\]

(43)

We have

\[
\left| \int_\mathbb{R} -H_x^{\xi,0}(x - \xi, t) A_0(\xi) \, d\xi \right| \leq \int_\mathbb{R} H_x^{\xi,0}(x - \xi, t) |A_{0,\xi}(\xi)| \, d\xi
\]

\[
\leq C \|A_{0,\xi}(x)\|_{L^\infty(\mathbb{R})}
\]

(44)

The integral related to the second term of (42) can be easily estimated as
\[ \left| \int_{\mathbb{R}} \frac{g'(\theta_0')}{2\sqrt{\pi}} H^{2,0}_x(x, \xi, t) A_0(\xi) \, d\xi \right| \leq C_1 \| \theta_0'(x) \|_{L^\infty(\mathbb{R})} \| A_0(x) \|_{L^\infty(\mathbb{R})}. \tag{45} \]

The integral related to the third term of (42) has a similar bound as the second term, using the change of variable
\[
u = \frac{x - \xi}{\sqrt{t}}.
\]
Combing above estimates, we obtain the following estimate:
\[
\left| \int_{\mathbb{R}} H^{2,0}_x(x, t; \xi, 0) A_0(\xi) \, d\xi \right| \leq C \| J_0(x) \|_{L^\infty(\mathbb{R})} + C_1 \| \theta_0'(x) \|_{L^\infty(\mathbb{R})} \| A_0(x) \|_{L^\infty(\mathbb{R})}. \tag{46} \]

For the term with \( \Phi \) (the second term in (40)), we have
\[
\left| \int_{\mathbb{R}} \int_{0}^{t} \int_{\mathbb{R}} H^{2,0}_x(x - y, t - s; \xi, 0) A_0(\xi) \, dy \, ds \, d\xi \right| \\
\leq \| A_0(\xi) \|_{L^\infty} \int_{0}^{t} \int_{\mathbb{R}} \frac{|x - y|}{(t - s)^{3/2}} e^{-\frac{g_1(x, y)}{4(t - s)}} \frac{1}{s^{5/4}} e^{-\frac{|x - s|^2}{4s}} \, dy \, ds \, d\xi \\
\leq C \| A_0(\xi) \|_{L^\infty} \int_{0}^{t} \frac{1}{(t - s)^{3/2}} e^{-\frac{g_1(x, y)}{4(t - s)}} \frac{1}{s^{5/4}} \, dy \, ds \\
\leq C \| A_0(\xi) \|_{L^\infty} \int_{0}^{t} \frac{1}{s^{3/4}} \, ds \leq C_2 \| A_0(\xi) \|_{L^\infty(\mathbb{R})}.
\]
Combining this estimate and (46), we prove the lemma.

### 3.4 Uniform Bound on \( J \)

Similarly as in (36), we have
\[
\left| \int_{0}^{t} \int_{\mathbb{R}} \Gamma_n(x, t; \xi, \tau) (g'(\theta_\xi) J)(\xi, \tau) \, d\xi \, d\tau \right| \\
\leq C_1^{1/4} \| J \|_{L^\infty((0, t), L^2(\mathbb{R}))} \| \theta_\xi \|_{L^\infty((0, t), L^2(\mathbb{R}))} \\
\leq C_1^{1/4} \| J \|_{L^\infty((0, t), L^2(\mathbb{R}))} C_1\frac{1}{2}(0), \tag{47} \]
where we use the fact that \( g' \) is uniformly bounded, (17), (9), and (21).

Suppose that
\[
CE_1\frac{1}{2}(0) \leq \frac{1}{2} \tag{48} \]
for the constant \( C \) in (47), then we have
\[
\left| \int_{0}^{t} \int_{\mathbb{R}} \Gamma_n(x, t; \xi, \tau) (g'(\theta_\xi) J)(\xi, \tau) \, d\xi \, d\tau \right| \leq \frac{1}{2} \| J \|_{L^\infty((0, 1), L^\infty(\mathbb{R}))} \tag{49} \]
for any $t \in [0, T_0]$ with $T_0 = 1$.

Hence by (34), (35), (37), (49), and Lemma 1, we have

$$
\|J\|_{L^\infty((0, 1), L^\infty(\mathbb{R}))} \leq 2 \left[ C \mathcal{E}(0) + C \mathcal{E}^1(0) + C \|J_0(x)\|_{L^\infty(\mathbb{R})} + C_1 \|\theta'_0(x)\|_{L^\infty(\mathbb{R})} + \|A_0(x)\|_{L^\infty(\mathbb{R})} + C_2 \|A_0(x)\|_{L^\infty(\mathbb{R})} \right].
$$

(50)

\section{Singularity Formation for Classical Solutions}

In this section, we prove Theorem 1. For reader’s convenience, we recall the system

$$
u_t = \left( g(\theta)u_x + h(\theta)\theta_t \right)_x,
$$

(51)

$$
\theta_{\tau} + \gamma_1 \theta_t = c(\theta)(c(\theta) \theta_x) - h(\theta)u_x,
$$

(52)

and the $C^1$ initial data

$$
\theta(x, 0) = \theta_0(x) = \theta^* + \varepsilon \phi \left( \frac{x}{\varepsilon} \right),
$$

(53)

$$
\theta_1(x, 0) = \theta_1(x) = (-c(\theta_0(x)) + \varepsilon \phi) \left( \frac{x}{\varepsilon} \right),
$$

(54)

$$
u(x, 0) = u_0(x) = \begin{cases} 0, & x \in (-\infty, -\varepsilon), \\ \int_{-\varepsilon}^{x} \frac{k}{g} c(\theta_0(a)) \theta'_0(a) \, da, & x \in [-\varepsilon, \varepsilon], \\ \chi(x), & x \in (\varepsilon, \varepsilon + 2), \\ 0, & x \in (\varepsilon + 2, \infty), \end{cases}
$$

where $\chi(x)$ is a $C^1$ function satisfying

$$
|\chi'(x)| \leq \frac{3}{2} \frac{h}{g} C_\gamma C_2 \varepsilon,
$$

(55)

and the $C^1$ function $\phi$ satisfies

$$
\phi(0) = 0 \quad \text{and} \quad \phi(a) = 0 \quad \text{for} \quad a \notin (-1, 1),
$$

(56)

$$
-\phi'(0) > \max \left\{ \frac{16 C_\gamma \|\gamma_1 - \frac{h^2}{g}\|_{L^\infty} \exp \left( \|\gamma_1 - \frac{h^2}{g}\|_{L^\infty} \right) }{c'(\theta^*) C_L \ln 2}, \frac{C_L}{C_\gamma} \right\} \quad \text{and} \quad |\phi'(x)| \leq C_2,
$$

(57)

and

$$
\int_{-1}^{1} (\phi')^2(a) \, da < k_0.
$$

(58)
Here, $\theta^*$ is a constant such that $c'(\theta^*) > 0$ and $k_0$ is some constant.

Remark 1 A choice of the function $\chi(x)$ can be a cubic polynomial constructed by satisfying the following constraints:

$\chi(\epsilon) = \int_{-\epsilon}^{\epsilon} \frac{h}{g} c(\theta_0(a))\theta_0'(a) \, da,$

$\chi'(\epsilon) = \frac{h}{g} c(\theta_0(\epsilon))\theta_0'(\epsilon) = \frac{h}{g} c(\theta_0(\epsilon))\phi'(1) = 0,$

$\chi(\epsilon + 2) = 0,$

$\chi'(\epsilon + 2) = 0.$

Some calculations lead to

$\chi(x) = \left( \frac{1}{4} \int_{-\epsilon}^{\epsilon} \frac{h}{g} c \phi' \, da \right) x^3 - \left( \frac{3}{4} (\epsilon + 1) \int_{-\epsilon}^{\epsilon} \frac{h}{g} c \phi' \, da \right) x^2 + \left( \frac{3}{4} \epsilon (\epsilon + 2) \int_{-\epsilon}^{\epsilon} \frac{h}{g} c \phi' \, da \right) x$

$- \left( \frac{1}{4} (\epsilon - 1)(\epsilon + 2)^2 \int_{-\epsilon}^{\epsilon} \frac{h}{g} c \phi' \, da \right).$

The derivative $\chi'(x)$ is a quadratic polynomial that vanishes at the end points $x = \epsilon$ and $x = \epsilon + 2$. The parabola is either concave or convex depending on the sign of the integral $\int_{-\epsilon}^{\epsilon} c \phi' \, da$. In any case, the critical point of the derivative is

$$\left( \epsilon + 1, \chi'(\epsilon + 1) \right) = \left( \epsilon + 1, -\frac{3}{4} \int_{-\epsilon}^{\epsilon} \frac{h}{g} c \phi' \, da \right).$$

This means

$$|\chi'(x)| \leq \frac{3}{2} \left\| \frac{h}{g} \right\|_{L^\infty} C_\psi C_2 \epsilon.$$

Now, we define the following gradient variables representing the rate of change of $\theta$ along the forward and backward characteristics:

$$S := \theta_t - c(\theta)\theta_x,$$

$$R := \theta_t + c(\theta)\theta_x.$$

Direct calculations give

$$S_t + c(\theta)S_x = \frac{c'}{4c}(S^2 - R^2) + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) (R + S) - hJ, \quad (59)$$

$$R_t - c(\theta)R_x = \frac{-c'}{4c}(S^2 - R^2) + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) (R + S) - hJ, \quad (60)$$

and the following balance laws:

$$(S^2)_x + (c(\theta)S^2)_x = \frac{c'}{2c}(S^2 R - SR^2) + \left( \frac{h^2}{g} - \gamma_1 \right) S(R + S) - 2hSJ, \quad (61)$$
Therefore, we can get the following equation:

\[
(R^2)_t - (c(\theta)R^2)_x = \frac{-c'}{2c} (S^2 R - SR^2) + \left( \frac{h^2}{g} - \gamma_1 \right) R(R + S) - 2hRJ. \tag{62}
\]

Therefore, we can get the following equation:

\[
(S^2 + R^2)_t + (c(\theta)(S^2 - R^2))_x = \left( \frac{h^2}{g} - \gamma_1 \right) (R + S)^2 - 2hJ(R + S) \tag{63}
\]

with the initial conditions

\[
R(x, 0) = \epsilon \phi'(\frac{x}{\epsilon}), \quad S(x, 0) = (-2c(\theta_0(x)) + \epsilon) \phi'(\frac{x}{\epsilon}). \tag{64}
\]

By (57), we have

\[
S(0, 0) = (-2c(\theta^*) + \epsilon) \phi'(0) = (2c(\theta^*) - \epsilon)(-\phi'(0))
\]

\[
> \max \left\{ 16CU \| \gamma_1 - \frac{h^2}{g} \|_{L^\infty}, \exp \left( \| \gamma_1 - \frac{h^2}{g} \|_{L^\infty} \right) \right\}. \tag{65}
\]

Since

\[
R^2(x, 0) + S^2(x, 0) = (-c(\theta_0) + \epsilon)^2 \phi'(\frac{x}{\epsilon})^2 + c^2(\theta_0) \phi'(\frac{x}{\epsilon})^2 \leq C \phi'(\frac{x}{\epsilon})^2
\]

for some constant C, it is easy to get that

\[
\int_{-\infty}^{\infty} (R^2(x, 0) + S^2(x, 0)) \, dx = O(\epsilon).
\]

Similarly,

\[
\int_{-\infty}^{\infty} u^2(x, 0) \, dx = O(\epsilon^2).
\]

Under the help of energy decay, we obtain

\[
\mathcal{E}(t) \leq \mathcal{E}(0) = \frac{1}{2} \int_{-\infty}^{\infty} (R^2 + S^2 + 2u^2)(x, 0) \, dx = O(\epsilon). \tag{66}
\]

Hence, by (50), it is easy to get that

\[
\| J \|_{L^\infty((0,1), L^\infty(\mathbb{R}))} = O(\epsilon^{\frac{1}{2}}), \tag{67}
\]

where we use (50), (66), and

\[
| A_0 | \leq \int_{\mathbb{R}} | J_0 | \, dx = \int_{-\epsilon}^{\epsilon} \frac{h}{\epsilon} \phi'(x/\epsilon) \, dx + \int_{\epsilon}^{\epsilon+2} \chi' \, dx \approx \epsilon^2 + \epsilon = O(\epsilon).
\]

Furthermore, \( \epsilon \) is small enough such that (48) is satisfied.

Now, we consider the two characteristic curves \( x_{\pm}(t) \) given by
with $x_1 := x_+(0)$ and $x_2 := x_-(0)$. See Fig. 1. Since the wave speed $c$ has a positive lower bound, two characteristics intersect at some point, say $(x_0, t_0)$, we have $x_+(t_0) = x_-(t_0) = x_0$ and

$$|x_2 - x_1| \leq |x_2 - x_0| + |x_0 - x_1| \leq 2C_U t_0.$$  \hfill (68)

We assume that $t_0 \leq 1$. We will verify it later by showing that blow-up will happen before $t = 1$.

Integrating (63) over the triangle $\Omega$ (see Fig. 1), and applying the divergence theorem, then we have

$$
\int_{x_0}^{x_1} 2R^2(x, t_+(x)) \, dx + \int_{x_0}^{x_2} 2S^2(x, t_-(x)) \, dx - \int_{x_1}^{x_0} (R^2(x, 0) + S^2(x, 0)) \, dx
\\= \int \int_{\Omega} \left( \frac{h^2}{g} - \gamma_1 \right) (S + R)^2 \, dx \, dt - \int \int_{\Omega} \frac{h}{g} J (S + R) \, dx \, dt.
$$

Rearranging it, we have

$$
\int_{x_1}^{x_0} R^2(x, t_+(x)) \, dx + \int_{x_0}^{x_2} S^2(x, t_-(x)) \, dx
\\\leq \frac{1}{2} \int_{x_1}^{x_0} (R^2 + S^2)(x, 0) \, dx + \int \int_{\Omega} \left( -\frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) (S + R)^2 + h |J| (S + |R|) \right) \, dx \, dt.
$$

By the decay of energy and $t_0 \leq 1$, we obtain

$$
\int \int_{\Omega} -\frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) (S + R)^2 \, dx \, dt \leq t_0 \left\| \gamma_1 - \frac{h^2}{g} \right\|_{L^\infty(\Omega)} E(0) = O(\varepsilon). \hfill (70)
$$

Similarly, by the decay of energy, $t_0 \leq 1$, and also using (67) and (68), we have

$$
\int \int_{\Omega} h |J| (|S| + |R|) \, dx \, dt \leq O(\varepsilon).
$$

Therefore, we have

Fig. 1 The triangle $\Omega$
Next, we consider a forward characteristic that we denote by

$$x = \xi(t)$$

for

$$t \in \left[0, \min\left\{1, \frac{2 \ln 2}{\|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}\right\}\right]$$

such that

$$\frac{d\xi(t)}{dt} = c(\theta(\xi(t), t)), \quad \xi(0) = 0.$$ Integrating the equation

$$\frac{d\theta(\xi(t), t)}{dt} = R(\xi(t), t),$$

we obtain

$$|\theta(\xi(t), t) - \theta(\xi(0), 0)| = \left| \int_0^t R(\xi(s), s) \, ds \right| \leq \sqrt{t} \left( \int_0^t R^2(\xi(s), s) \, dt \right)^{1/2} = O(\sqrt{\epsilon}).$$

Using the smoothness of $c$, when $\epsilon$ is small enough,

$$c'(\theta(\xi(t), t)) > \frac{c'(\theta(\xi(0), 0))}{2} = \frac{c'(\theta^*)}{2} > 0.$$ Next, we claim that for smooth solutions we have $S(\xi(t), t) > 1$ as long as

$$t \in \left[0, \min\left\{1, \frac{2 \ln 2}{\|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}\right\}\right]$$

To prove it by an contradiction argument, we assume that $S(\xi(t), t) \leq 1$ for some time in the interval. Define

$$t^* := \inf \left\{ t \in \left(0, \min\left\{1, \frac{2 \ln 2}{\|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}\right\}\right] : S(t, \xi(t)) \leq 1 \right\}.$$ By the continuity of $S$, we have

$$S(\xi(t^*), t^*) = 1.$$ Define

$$\tilde{S} = e^{\mu(t,x)} S(x, t) := \exp \left( \int_0^t \frac{1}{2} \left( \gamma_1 - \frac{h^2}{g} \right)(x, s) \, ds \right) S(x, t).$$ Some calculations give
\[ \ddot{S} + c(\theta) \dot{S} = p_x(t, x) \ddot{S} + e^{p(t, x)} S_x + c(\theta) p_x(t, x) \dot{S} + c(\theta) e^{p(t, x)} S_x = (p_x + c p_x) \ddot{S} + e^{p(t, x)} (S_t + c S_x) \]

\[ = (p_x + c p_x) \ddot{S} + e^{p(t, x)} \left[ \frac{c'}{4c} (S^2 - R^2) + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) (R + S) - hJ \right] \]

\[ = (p_x + c p_x) \ddot{S} + \frac{c'}{4c} e^{-p(t, x)} \ddot{S}^2 + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) \ddot{S} - \frac{c'}{4c} e^{p(t, x)} R^2 \]

\[ + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) e^{p(t, x)} R - e^{p(t, x)} hJ \]

This means along the curve \( \xi(t) \) for \( t \in [0, t^*] \)

\[ \frac{d}{dt} S(\xi(t), t) = \left( \frac{d}{dt} p(t, \xi(t)) + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) \right) \ddot{S} + \frac{c'}{4c} e^{-p(t, \xi(t))} \ddot{S}^2 - \frac{c'}{4c} e^{p(t, \xi(t))} R^2 \]

\[ + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) e^{p(t, \xi(t))} R - e^{p(t, \xi(t))} hJ \]

\[ \quad = \frac{c'}{4c} e^{-p(t, \xi(t))} \ddot{S}^2 - \frac{c'}{4c} e^{p(t, \xi(t))} R^2 + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) e^{p(t, \xi(t))} R - e^{p(t, \xi(t))} hJ. \] (71)

Dividing it by \( \ddot{S}^2 \), then integrating it over \( [0, t^*] \), we have

\[ \frac{1}{\ddot{S}(0)} - \frac{1}{\ddot{S}(t^*)} \geq \int_0^{t^*} \frac{c'}{4c} e^{-p(t, \xi)} \, dt \]

\[ + \int_0^{t^*} \frac{1}{\ddot{S}^2} \left[ - \frac{c'}{4c} e^{p(t, \xi)} R^2 + \frac{1}{2} \left( \frac{h^2}{g} - \gamma_1 \right) e^{p(t, \xi)} |R| - e^{p(t, \xi)} hJ \right] \, dt. \]

Recalling \( 0 < t^* \leq \frac{2 \ln 2}{\| \gamma_1 - \frac{h^2}{g} \|_{L^\infty}} \), we have

\[ \frac{1}{\ddot{S}(t^*)} \leq \frac{1}{\ddot{S}(0)} - \int_0^{t^*} \frac{c'}{4c} e^{-p(t, \xi(t))} \, dt \]

\[ + \int_0^{t^*} \frac{1}{\ddot{S}^2} \left[ \frac{c'}{4c} e^{p(t, \xi(t))} R^2 + \frac{1}{2} \left( - \frac{h^2}{g} + \gamma_1 \right) e^{p(t, \xi(t))} |R| + e^{p(t, \xi(t))} hJ \right] \, dt \]

\[ \leq \min \left\{ \frac{c' \langle \theta^* \rangle \ln 2}{16 C_U \| \gamma_1 - \frac{h^2}{g} \|_{L^\infty}}, \exp \left( - \| \gamma_1 - \frac{h^2}{g} \|_{L^\infty} \right), \frac{c' \langle \theta^* \rangle}{32 C_U} \right\} + \frac{c' \langle \theta^* \rangle}{16 C_U} t^* + D \sqrt{\epsilon} \]

\[ \leq \exp \left( - \| \gamma_1 - \frac{h^2}{g} \|_{L^\infty} \right) + D \sqrt{\epsilon} \] (72)

for some positive constant \( D \).

For \( \epsilon \) small enough,
\[
\frac{1}{\bar{S}(t^*)} < \exp \left( -\frac{1}{2} \left\| \gamma_1 - \frac{h^2}{g} \right\|_{L^\infty} \right).
\]
then
\[
\bar{S}(t^*) > \exp \left( \frac{1}{2} \left\| \gamma_1 - \frac{h^2}{g} \right\|_{L^\infty} \right).
\]
Hence
\[
S(t^*) > \exp \left( \frac{1}{2} \left\| \gamma_1 - \frac{h^2}{g} \right\|_{L^\infty} \right) \exp \left( \int_0^{t^*} -\frac{1}{2} \left( \gamma_1 - \frac{h^2}{g} \right) \, ds \right) \geq \exp \left( \frac{1}{2} \left\| \gamma_1 - \frac{h^2}{g} \right\|_{L^\infty} (1 - t^*) \right),
\]
which gives \( S(t^*) > 1 \). This is a contradiction with the definition of \( t^* \).

This proves that \( S(\xi(t), t) > 1 \) as long as \( t \in \left[ 0, \min\{1, \frac{2 \ln 2}{\|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}\} \right] \).

Now using same calculations, we obtain
\[
\frac{1}{\bar{S}(t)} \leq \min \left\{ \frac{c'(\theta^*) \ln 2}{16C_U \|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}, \frac{c'(\theta^*)}{32C_U} \right\} - \frac{c'(\theta^*)}{16C_U} t + D \sqrt{\varepsilon}.
\]
As a consequence, \( \bar{S}(t) \) blows up before
\[
t = \min \left\{ \frac{\ln 2}{\|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}, \frac{1}{2} \right\} + \frac{16C_U D}{c'(\theta^*)} \sqrt{\varepsilon}.
\]
Now choose \( \varepsilon \) small enough, we know the solution will blow up before
\[
T = \min \left\{ \frac{2 \ln 2}{\|\gamma_1 - \frac{h^2}{g}\|_{L^\infty}}, 1 \right\}.
\]
More precisely, there exists a time \( t_0 < T \) such that
\[
S(t, \xi(t)) \to \infty
\]
as \( t \to t_0^- \). This shows that \( \theta_t \to \infty \) or \( \theta_x \to -\infty \) as \( t \to t_0^- \).

On the other hand, because of the smallness of \( R \) initially, i.e., \( R(x, 0) \) is of order \( O(\varepsilon) \), and (60), we know that \( R(x, t) \) remains uniformly bounded before the blow-up of \( S(x, t) \).
This shows that both
\[
\theta_t \to \infty \quad \text{and} \quad \theta_x \to -\infty
\]
simultaneously as \( t \to t_0^- \), at the blow-up point.

This completes the proof of Theorem 1.

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