Higher variations, conservation laws, and the Jacobi equations for Yang–Mills Lagrangians on a Minkowskian background

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Abstract

We investigate higher order variations of Lagrangians in the framework of finite order variational sequences. In particular we obtain explicit expressions for second variations that are naturally related to the geometric structure of the problem. We recover the definition of the Jacobi morphism and of the Hessian at an arbitrary order, and show the relation between them. We investigate the relation between Jacobi fields, symmetries of higher order variations and conserved currents; we show that a pair given by a symmetry of the l-th variation of a Lagrangian and by a Jacobi field of the s-th variation of the same Lagrangian (with $s < l$) is associated with a (strongly) conserved current. Furthermore we prove that a pair of Jacobi fields always generates a (weakly) conserved current. The example of the Jacobi equation for a Yang-Mills theory on a Minkowskian background is worked out and the current associated with two Jacobi fields is obtained in this case.

Key words: variational sequence, higher order variation, Jacobi morphism, conservation law, Yang–Mills theory.
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1 Introduction

This paper deals with a geometric formulation of calculus of variations, focusing on the study of second and higher order variations of Lagrangians. We will be endowed with the framework of finite order jet prolongations of fibered manifolds, in which it is possible to formulate quite naturally the calculus of variation. Moreover, on jet prolongations we can define the finite order variational sequence, by which some classical problems of mathematical physics can be related to the cohomology of the configuration space. We will shortly review the construction of the variational sequence on finite order prolongations of fibered manifolds, then we will define the interior Euler operator in order to approach the so-called representation problem. By this machinery we obtain results on higher order variations of Lagrangians and conservation laws.

The geometric aspects of jet prolongations (or jet spaces) are quite interesting in themselves [14, 15, 17, 67]; here, we are interested in their use in the formulation of calculus of variation, discussed e.g. in [1, 2, 13, 32, 35, 36, 42, 54, 67, 69], in particular we shall refer in detail to [42, 43, 44, 45] and to the reviews [46, 47]. In fact, the geometric environment of jet prolongations allows to deal with differential operators and differential equations, Lagrangians, variations, Euler–Lagrange equations and Noether theorems, with a precise, geometric definition of conserved current. Undoubtedly the most important feature of jet spaces is the contact structure, that can be introduced by means of the affine structure of jet prolongations (see [17, 67]); the definition and the properties of the so-called contact ideal were discussed in [42, 43, 44, 45, 47, 48, 49, 50, 51] and constitute a fundamental point in our work. Essentially, the contact structure allows to introduce a decomposition of (pull-backs of) forms that plays a major rôle in the calculus of variations, e.g. to geometrically obtain variation formulae.

The finite order variational sequence, introduced by Krupka in [44], is obtained taking the quotient of the de Rham sequence of sheaves of forms by the contact structure. It turns out that it is a resolution (by soft sheaves of forms) of the constant sheaf; a fact implying that the cohomology of the complex of global sections is isomorphic to the standard de Rham cohomology of the configuration space (see e.g. [44, 17]). The idea behind the construction of this sequence can be traced back to Lepage: assigning to a Lagrangian the corresponding Euler–Lagrange equations is an operation that can be identified with the exterior derivative of forms up to a “congruence”, i.e. by
In order to give an actual proof of the fact the classes and arrows in the sequence are relevant objects for the calculus of variations, one need to solve the “representation problem”, i.e., to assign uniquely a global differential form to any class in the sequence (and then a map between forms to any arrow). This problem can be faced in various way; we follow Krbek and Musilová’s use of the so called interior Euler operator \[39, 40, 41\]. By such an operator, the canonical maps of the calculus of variations, such as the Euler–Lagrange mapping, which assigns to a Lagrangian its Euler–Lagrange equations, can be seen as quotient morphisms in the variational sequence; on the other hand, equivalence classes are interpreted as differential forms relevant for calculus of variation (Lagrangians, currents, source forms and so on).

Through the representation by the interior Euler operator we obtain several original results concerning higher variations, conservation laws and, in particular, the Jacobi morphism. Such results are presented in section 2, section 3, section 4 and section 5.

Applications to higher order variations of Lagrangians are presented and they constitute the first part of our original work. We first discuss contact symmetries and variational Lie derivatives, by which we get the first variation formula. In section 2 by means of an inductive process, we write higher order variations in terms of the interior Euler operator.

In section 3 the second variation case is analyzed in detail: by the interior Euler operator we define a Jacobi morphism and we recover the classical fact the the Jacobi operator is self adjoint along solutions (a property proved in \[32\] for first order field theories). Moreover, we define the Jacobi equation as the equation for the kernel of this morphism; its solutions are called Jacobi fields. An important characterization of higher order variations in terms of Jacobi morphisms is given in Remark 3.3. We also formulate the Hessian of the action functional in terms of the Jacobi morphism and characterize some important properties for higher order field theories.

The explicit example of the Jacobi equation for a Yang-Mills theory on a Minkowskian background is worked out in section 4. We remark that our result is comparable with the classical definition of a Jacobi operator for Yang–Mills Lagrangians, see e.g. \[3, 7\], adapted to our specific case.

Therefore we focus on conservation laws: after recalling the concepts of Lagrangian symmetry and of generalized symmetry, in section 5 we obtain relevant results which explicate the relation between Jacobi fields, symmetries of higher order variations and conserved currents. In particular, in subsection
the current associated with two Jacobi fields is obtained for a Yang-Mills theory on a Minkowskian background.

Finally, for the convenience of the reader and without any claim of completeness, an Appendix is added where we synthetically relate (higher) variations with Lie derivatives with respect to projectable vector fields, as well as we relate the interior Euler operator with the first order contact component of the differential of a Lepage equivalent. We recall some important facts concerning the calculus of variation on jet spaces, referred to through the paper. In particular, details can be found on the Takens representation of the variational sequence which is at the base of our main results.

1.1 Jet prolongations and the finite order contact structure

We briefly recall the geometric framework in which modern geometric calculus of variation is studied. Mainly we refer to [67] and to [47], [13], [36] for jet prolongations and their rôle in mathematical physics; to [48, 49, 50, 51, 67] for the contact structure.

We will denote by $X$ a differentiable manifold of dimension $n$ and by $Y$ a differentiable manifold of dimension $m+n$; we assume that it exists a fibered manifold structure $(Y, \pi, X)$ in which $X$ is the base space, $Y$ is the total space and $\pi$ is the projection. Only local fibered coordinates, i.e. adapted to the fibration, $(x^i, y^\sigma)$, with $i = 1 \ldots n$ and $\sigma = 1 \ldots m$, will be used. In what follows we shall denote $ds = dx^1 \wedge \ldots \wedge dx^n$ the local expression of a volume element on $X$; furthermore, we use the following notation

$$ds_i = \frac{\partial}{\partial x^i}|ds, \quad ds_{ij} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}|ds, \quad \ldots$$

and so on.

An equivalence relation identifies local sections of $\pi$ defined in a neighborhood of $x \in X$ such that they have the same values and derivatives, up to the order $k$, at $x$. The $k$-th jet space $J^k \pi$ is defined as the space of such equivalence classes which we denote by $j^k_x \phi$. We have, for $0 \leq h < k$, a map $\pi_{k,h} : J^k \pi \to J^h \pi$ such that $\pi_{k,h}(j^k_x \phi) = j^h_x \phi$, where we set $J^0 \pi = Y$ and $j^0_x \phi = \phi(x)$; moreover, maps $\pi_k : J^k \pi \to X$ such that $\pi_k(j^k_x \phi) = x$ are defined.
It turns out that $J^k\pi$ is a manifold. For $0 \leq h \leq k$, by setting
\[ y^\sigma_{j_1 \ldots j_h}(j^k_x \phi) = \left. \frac{\partial^h \phi^\sigma}{\partial x^{j_1} \ldots \partial x^{j_h}} \right|_x, \]
we have a set of functions $(x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_k})$ defined locally on $J^k\pi$; they are not a proper coordinate system because of the symmetry properties of partial derivatives, but restricting to $j_t \leq j_q$ if $t \leq q$ we can fix this problem. Anyway, it is easy to verify that $(J^k\pi, \pi_{k,h}, J^h\pi)$ and $(J^k\pi, \pi_k, X)$ are fibered manifolds for every $k$ and every $0 \leq h \leq k$. Moreover, $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$ is always an affine bundle.

Given a section of $\pi$, denoted by $\sigma$, we can always build canonically a section of $\pi_k$ called prolongation of $\sigma$ and denoted by $j^k\sigma$, setting
\[ y^\sigma_{j_1 \ldots j_h}(j^k \sigma) = \left. \frac{\partial^h \phi^\sigma}{\partial x^{j_1} \ldots \partial x^{j_h}} \right|_x. \]
Sections of $\pi_k$ that are not of this type are called non holonomic sections.

Finally, given a function $f$ defined on an open set $V$ of $J^k\pi$ and an index $1 \leq i \leq n$ the $i$-th total derivative is a function defined on $\pi_{k+1,k}^{-1}(V)$ with expression
\[ df_i = \frac{\partial f}{\partial x^i} + \sum_{t=1}^k \sum_{j_1 \leq \ldots \leq j_t} \frac{\partial f}{\partial y^\sigma_{j_1 \ldots j_t}} y^\sigma_{j_1 \ldots j_t i}. \]
We shall use the following convention on multi-indices:

- a multi-index will be an ordered $s$-uple $I = (i_1, \ldots, i_s)$;
- the length of $I$ is given by the number $s$;
- an expression of the kind $Ij$ denotes the multi-index given by the $(s+1)$-uple $(i_1, \ldots, i_s, j)$.

Therefore, we have
\[ \frac{\partial^{|I|}}{\partial x^l} = \frac{\partial^s}{\partial x^{i_1} \ldots \partial x^{i_s}}. \]
Note that, if we restrict to $1 \leq i_1 \leq \ldots \leq i_s \leq n$, then the system $(x^1, y^\sigma_{j_1})$, with
\[ y^\sigma_{j_1}(j^1 \gamma) = \left. \frac{\partial^s \gamma^\sigma}{\partial x^{i_1} \ldots \partial x^{i_s}} \right|_x, \]
for $I = (i_1 \ldots i_s)$ and $s \leq k$, is a system of coordinates on $J^rY$.

In the following we will often consider sums over multi-indices. We will always start a sum over multi-indices from the 0-length multi-index (i.e. the void multi-index), unless otherwise specified. The upper limit in such a sum will be usually given by multi-indices of length equal to the order of the jet prolongation under consideration. We restrict the sum to $s$-uple of indices such that $i_1 \leq \cdots \leq i_s$, because otherwise $y^\sigma_{ij}$ would not be a proper system of coordinates. However, multiplying partial derivatives $\frac{\partial}{\partial y^\sigma_{ij}}$ by a suitable numerical factor, we are allowed to sum over all multi-indices $I$ (see Appendix 2 in [47] for a detailed discussion). In the sequel, for convenience reasons, sometimes we will skip that restriction, indeed.

Let $\Omega_q(J^k\pi)$ denote the module of $q$-forms on $J^k\pi$. A major rôle in the calculus of variation is played by the so called contact structure induced by the affine bundle structure of $\pi_{k,k-1}$ (see [67], [47]).

A differential $q$-form $\alpha$ on $J^k\pi$ is called a contact form if, for every section $\gamma$ of $\pi$, we have

$$(j^k\gamma)^*(\alpha) = 0.$$ 

It is easy to see that forms $\omega$ locally given as

$$\omega^\sigma_{j_1\ldots j_h} = dy^\sigma_{j_1\ldots j_h} - y^\sigma_{j_1\ldots j_h} dx^i$$

for $0 \leq h < k$ are indeed contact 1-forms. In particular, it is easy to show that $(dx^i, \omega^\sigma, \omega^\sigma_{j_1\ldots j_h}, \ldots, \omega^\sigma_{j_1\ldots j_{k-1}}, dy^\sigma_{j_1\ldots j_k})$ is an alternative local basis for 1-forms on $J^k\pi$. It is important to notice that the ideal of the exterior algebra generated by contact forms on a fixed jet order prolongation is not closed under exterior derivation, while if $\alpha$ is contact so is $d\alpha$. For a detailed discussion of the structure of the contact ideal see [44, 45, 47].

For every form $\rho \in \Omega_q(J^k\pi)$, by the contact structure we obtain the canonical decomposition

$$\pi_{k+1,k}^*(\rho) = p_0 \rho + p_1 \rho + \cdots + p_q \rho$$

where $p_0 \rho$ is a form that is horizontal on $X$ (and so is often denoted by $h \rho$) while $p_i \rho$ is an $i$-contact $q$-form, that is a form generated by wedge products containing exactly $i$ contact 1-forms. Indeed a vector bundle homomorphism over $\pi_{k+1,k}$

$$h : TJ^{k+1} \pi \to TJ^k \pi$$
is defined by assigning to a tangent vector \( \Phi \) at \( j^k \psi \) the tangent vector 
\((T_x j^k \psi \circ T \pi_{k+1}) \cdot \Phi\). Then we set

\[ p \Phi = T \pi_{k+1,k} \cdot \Phi - h \Phi \]

and, for every \( q \)-form \( \rho \) on \( J^k \pi \) ([17]),

\[ p_l \rho \bigg|_{j^k \psi} (\Phi_1, \ldots, \Phi_q) = + \rho(p \Phi_1, p \Phi_2, \ldots, p \Phi_l, h \Phi_{l+1}, h \Phi_{l+2}, \ldots, h \Phi_q) \]
\[ - \rho(p \Phi_1, p \Phi_2, \ldots, p \Phi_l, h \Phi_{l+1}, h \Phi_{l+2}, \ldots, h \Phi_q) \]
\[ + \rho(p \Phi_1, \ldots, p \Phi_{l+1}, p \Phi_{l+2} h \Phi_{l-1}, h \Phi_{l}, \ldots, h \Phi_q) \]

where the sum is taken over all permutations \((i_1 \ldots i_q)\) such that \( i_1 < \cdots < i_l \)
and \( i_{l+1} < \cdots < i_q \) and taking into account the sign of the permutation. We remark that if \( q > n \) every \( q \)-form \( \rho \) is contact; then we call it strongly contact if \( p_q - n = 0 \).

The contact structure induces also the splitting of the exterior differential 
\( \pi^*_{k+1,k} d \rho = d_H \rho + d_V \rho \) in the so called horizontal and vertical differentials
respectively, given by 
\( d_H \rho = \sum_{l=0}^{q} p_l dp_l \rho \) and 
\( d_V \rho = \sum_{l=0}^{q} p_{l+1} dp_l \rho \).

According with [41] we define the formal derivative with respect to the \( i \)-th coordinate, \( i = 1, \ldots, n \), by an abuse of notation also denoted by \( d_i \), as
an operator acting on forms. It is defined by requiring that is commutes with the exterior derivative and that it satisfies the Leibnitz rule with respect to the wedge product. We see that 
\( d_H \rho = (-1)^q d_i \rho \wedge dx^i \) if \( \rho \) is a \( q \)-form. In particular, on functions this operator is simply the total derivative, and on the basis 1-forms we have

\[ d_i dx^j = 0, \quad d_i \omega_{j_1 \cdots j_r}^\sigma = \omega_{j_1 \cdots j_r i}^\sigma, \quad d_i dy^\sigma = dy_i^\sigma \]

By an abuse of notation, \( d_i \) will also indicate the vector field on \( J^k \pi \) along \( \pi^*_{k+1,k} \)

\[ d_i = \frac{\partial}{\partial x^i} + \sum_{t=1}^{k} \sum_{j_1 \leq \cdots \leq j_t} y_{j_1 \cdots j_t}^\sigma \frac{\partial}{\partial y_{j_1 \cdots j_t}^\sigma}. \]

Then for every vector projectable field \( \Phi \) on \( J^k \pi \) locally written as

\[ \Phi = \phi^i \frac{\partial}{\partial x^i} + \sum_{t=1}^{k} \sum_{j_1 \leq \cdots \leq j_t} \phi_{j_1 \cdots j_t}^\sigma \frac{\partial}{\partial y_{j_1 \cdots j_t}^\sigma}, \]
we have

$$\Phi \circ \pi_{k+1,k} = \Phi_H + \Phi_V,$$

where

$$\Phi_H = \phi_i d_i, \quad \Phi_V = \sum_{t=1}^{k} \sum_{j_1 \leq \cdots \leq j_t} (\phi_{j_1 \cdots j_t} - y_{j_1 \cdots j_t} \phi_i) \frac{\partial}{\partial y_{j_1 \cdots j_t}}.$$ 

In particular, taking $\Xi$ as a projectable vector field on $Y$ it is possible to define a prolongation $j^k \Xi$ that is a vector field on $J^k \pi$. For $\Xi = \xi^i (x) \frac{\partial}{\partial x^i} + \xi^\sigma (x, y) \frac{\partial}{\partial y^\sigma}$ we have (17)

$$j^k \Xi = \xi^i \frac{\partial}{\partial x^i} + \xi^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{t=1}^{k} \sum_{j_1 \cdots j_t} \Xi_{j_1 \cdots j_t} \frac{\partial}{\partial y_{j_1 \cdots j_t}},$$

defined by the recurrence formula

$$\Xi_{j_1 \cdots j_t} = d_{j_t} \Xi_{j_1 \cdots j_{t-1}} - y_{j_1 \cdots j_{k-1}} \xi^i \frac{\partial}{\partial x^i};$$

note that if $\Xi$ is vertical, then

$$\Xi_{j_1 \cdots j_t} = d_{j_1} \cdots d_{j_t} \Xi.$$

The prolongation vector field $j^k \Xi$ is, indeed, defined by the prolongation of the flow of $\Xi$ (see, e.g. [42, 67, 69]).

Concerned with the integration by parts procedure, we will use the fundamental formula $\omega_1^\sigma \wedge ds = -d \omega_1^\sigma \wedge ds_i$. We also recall the properties $d_{j_i} \omega^\sigma = \omega_{j_i}^\sigma$ and $\frac{\partial}{\partial y_{j_i}} \omega_{j_i}^\sigma = \delta^\sigma_{j_i} \delta_{j_i}^l$ (where the Kronecker symbol with multi-indices has the obvious meaning: it is 1 if the multi-indices coincide up to a rearrangement and 0 otherwise).

In the following, differential forms which are $\omega^\sigma$ generated $l$-contact $(n+l)$-forms will be called source forms.

### 1.2 The finite order variational sequence and its representation

The contact structure of jet prolongations enables to define a geometric object deeply related to the calculus of variations: the variational sequence. We refer
to [47], [44], [39], [41], [38] and to the review [59] for the construction and the representation of finite order variational sequences.

We assume that the reader be familiar with sheaves and their cohomology at least at an elementary level. The concept of a sheaf can be dated back to Leray [56]; a classical reference on this topic is e.g. [8]. We also refer to [47], where an elementary introduction can be found finalized to the construction of the variational sequence.

Let $\Omega^q$ denote the sheaf of differential $q$-forms on $J^k\pi$. It can be seen as a sheaf on $Y$; in fact we assign to an open set $W \subseteq Y$ a form defined on $\pi^{-1}(W)$. We set $\Omega^0 = \{0\}$ and denote by $\Omega^q$ the sheaf of contact $q$ forms, for $q \leq n$, or the sheaf of strongly contact $q$-forms if $q > n$. We define the sheaf

$$\Theta^k_q = \Omega^k + d\Omega^k_{-1,c},$$

where $d\Omega^k_{-1,c}$ is the presheaf image through $d$ of $\Omega^k_{-1,c}$ and $d\Omega^k_{-1,c}$ denotes the sheaf generated by $d\Omega^k_{-1,c}$. It can be shown ([47], [44]) that $\Theta^k_q = \{0\}$ if $q > M = m\left(\frac{n+k-1}{n}\right) + 2n - 1$

and that we get an exact sequence of soft sheaves

$$\{0\} \to \Theta^k_1 \to \Theta^k_2 \to \cdots \to \Theta^k_M \to \{0\}$$

that is a subsequence of the de Rham sequence

$$\{0\} \to \mathbb{R}_Y \to \Omega^1 \to \Omega^k \to \cdots \to \Omega^k_N \to \{0\}$$

where $N = \text{dim}(J^kY)$ and $\mathbb{R}_Y$ is the constant sheaf over $Y$. The quotient sequence

$$\{0\} \to \mathbb{R}_Y \to \Omega^k_0 \to \Omega^k_1/\Theta^k_1 \to \cdots \to \Omega^k_M/\Theta^k_M \to \Omega^k_{M+1} \to \cdots \to \Omega^k_N \to \{0\}$$

is called the Krupka’s variational sequence of order $k$. Quotient sheaves will be denoted by $\mathcal{V}^k_q$. We remark that here morphisms are quotients of the exterior derivative, namely $[\rho] \to [d\rho]$. They will be denoted by $\mathcal{E}_q : \mathcal{V}^k_q \to \mathcal{V}^k_{q+1}$. Notice that $\mathcal{E}_q([\rho]) = [d\rho]$ by definition of quotient map.

A fundamental theorem due to Krupka states that the variational sequence is a (soft sheaf) resolution of the constant sheaf $\mathbb{R}_Y$; see [44] and
for a recent review [47]. The sequence is exact as a sheaf sequence, i.e. the corresponding sequence of stalks is exact, and we recall that the global section functor is not exact on the right, meaning that the sequence is not exact as a sequence of presheaves. Indeed, the cohomology groups of the cochain complex of global section of the sequence are identified with the de Rham cohomology groups $H^q_{dR} Y$ of $Y$ by the abstract de Rham theorem. This important intrinsic aspect of the calculus of variations has been used to state cohomological obstructions to the existence of critical sections [65]; such obstructions appear relevant e.g. in Chern-Simons theories [10].

The variational sequence is a geometric description of the structure of the variational calculus on finite order jet prolongations of fibered manifold. Main aspects of such a geometric underlying structure is shortly recalled in the Appendix. The following facts are true (see e.g. [44], [47]).

1. The sheaf $\Omega^k_n/\Theta^k_n$ is isomorphic with a subsheaf of the sheaf of Lagrangians of order $k + 1$ (horizontal forms on $J^{k+1}\pi$) given by Lagrangians associated with $n$-horizontal forms on $J^k\pi$.

2. The sheaf $\Omega^k_{n+1}/\Theta^k_{n+1}$ is isomorphic with a subsheaf of the sheaf of $(n + 1)$-source forms.

3. The quotient mapping $\mathcal{E}_n : \Omega^k_n/\Theta^k_n \to \Omega^k_{n+1}/\Theta^k_{n+1}$ is the Euler–Lagrange mapping.

This means that we can interpret equivalence classes in the sheaves $\Omega^k_n/\Theta^k_n$ and $\Omega^k_{n+1}/\Theta^k_{n+1}$ as forms defined on higher order jets and that these forms are, respectively, Lagrangian and Euler–Lagrange forms. With this interpretation we can study variational problems concerning the cohomology of $Y$.

The so called representation problem, roughly speaking, consists in showing that classes of forms, i.e. elements of the quotient groups $\mathcal{V}_q^r$, can be associated with global differential forms.

Therefore, for $s \geq r$, we look for a suitable abelian group of forms of order $s$, denoted by $\Phi^s_q$, and an operator (called representation mapping) $T_q^r : \Omega^r_q \to \Phi^s_q$.
such that \( \ker \mathcal{I}_q = \Theta_q \). This gives us the following diagram

\[
\begin{array}{c}
\Theta_q \\
\downarrow \\
\Omega_q \\
\mathcal{V}_q \\
\cong \\
\mathcal{I}_q(\Omega_q)
\end{array}
\]

and an isomorphism \( \mathcal{V}_q \cong \mathcal{I}_q(\Omega_q) \). We present the solution of this problem based on the concept of Euler operator associated with a formal operator according to [39, 40, 41]; other approaches are possible, see e.g. [38, 47, 70].

### 1.3 Interior Euler operator

Let us introduce an operator which plays an essential rôle in the representation theory for the variational sequence. It was introduced to the calculus of variations within the variational bicomplex theory [2, 4] and adapted to the finite order situation of the variational sequence in [39, 40, 41, 52, 70]; see also the review in [47] and applications to the representation of variational Lie derivatives in [59]. This operator, called interior Euler operator, and denoted by \( \mathcal{I} \), reflects in an intrinsic way the procedure of getting a distinguished representative of a class \([\rho] \in \Omega^k_q/\Theta^k_q\) for \( q > n \) by applying to \( \rho \) the operator \( p_{q-n} \) and a factorization by \( \Theta^k_q \).

We will admit multi-indices with entries which are not in non-decreasing order. In this way, when we have a sum over a multi-index \( I \) of length \( k \), this sum is taken over all \( k \)-uple of integers from 1 to \( n \). Now, let

\[
J^r \Xi = \sum_{|J|=0}^r d_J \Xi^\sigma \frac{\partial}{\partial y^j},
\]

be the prolongation of a vertical vector field. A formal differential operator \( P \), locally given by \( P(\Xi) = \sum_{|J|=0}^r d_J \Xi^\sigma P^J_\sigma \), where \( P^J_\sigma \) are suitable \( q \)-forms on \( J^r Y \), can be written in coordinates as

\[
P(\Xi) = \sum_{|I|=0}^r d_I (\Xi^\sigma Q^I_\sigma),
\tag{1}
\]
with
\[ Q^J_\sigma = \sum_{|J|=0}^{r-|I|} \binom{|I| + |J|}{|I|} (-1)^{|J|} d_J P^{IJ}_\sigma ; \quad (2) \]
for \(|J| = r\), the terms \(Q^J_\sigma\) are the so-called symbol of the formal differential operator.

Let then \(W \subseteq Y\) be an open set and \(P : VW \to p_k\Omega^r_{n+k}W\) be a formal differential operator of order \(r\); there exists a unique formal differential operator, the Euler operator of \(P\), \(Q : VW \to p_k\Omega^{2r}_{n+k}W\) of order 0 such that \(P(\Xi) = Q(\Xi) + p_k dp_k R(\Xi)\), where the differential operator \(R\) is defined only locally. Moreover, in a system of coordinates \(Q(\Xi) = Q_\sigma \Xi^\sigma\).

A particular, and important, class of Euler operators is constituted by contraction Euler operators, i.e., Euler operators corresponding to the formal differential operator defined by the contraction
\[ J^{r+1}\Xi |p_k\rho, \]
where \(\rho\) is a (local) \((n+k)\)-form. Locally
\[ p_k \rho = \sum_{0 \leq |J_1|, \ldots, |J_k| \leq r} p^J\sigma_1 \cdots \sigma_k \omega^{\sigma_1}_{J_1} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge ds, \]
and then
\[ J^{r+1}\Xi |p_k\rho = \sum_{|J|=0}^r d_J \Xi^\sigma \left( \frac{\partial}{\partial y^J_f} |p_k\rho \right). \]
The corresponding contraction Euler operator \(I\) is given locally by \(I(\Xi) = \Xi^\sigma I_\sigma\), where
\[ I_\sigma = \sum_{|J|=0}^r (-1)^{|J|} d_J \left( \frac{\partial}{\partial y^J_f} |p_k\rho \right). \]
Define thus a map \(\mathcal{I} : \Omega^r_{n+k}W \to \Omega^{2r+1}_{n+k}W\) as
\[ \mathcal{I}(\rho) = \frac{1}{k} \omega^\sigma \wedge I_\sigma = \frac{1}{k} \omega^\sigma \wedge \sum_{|J|=0}^r (-1)^{|J|} d_I \left( \frac{\partial}{\partial y^J_f} |p_k\rho \right). \quad (3) \]
This map is called *interior Euler mapping* or *interior Euler operator*; it is a definition adapted to finite order jets from the one given by Anders, see [2]. It turns out that $\mathcal{I}(\rho)$ is a globally defined form.

Moreover, the operator $\mathcal{I}$ has the following important property.

**Lemma 1.1** For any $\eta \in \Omega_{n+k}^r W$, it holds $I \circ p_k \circ d \circ p_k \eta = 0$.

For a given $\rho$, $\mathcal{I}(\rho)$ is a source form of degree $n + k$. Moreover, it is by construction a $k$-contact form (we recall that $k$-contact means generated by wedge products of exactly $k$ terms $\omega_\sigma^J$); then, it is interesting to study the difference between $(\pi^{2r+1,r+1})(p_k \rho)$ and $\mathcal{I}(\rho)$.

By the duality between contact 1-forms and vertical vector fields, a local operator $\mathcal{R}$, called the *residual operator*, is defined by

$$(\pi^{2r+1,r+1})(p_k \rho) = \mathcal{I}(\rho) + p_k dp_k \mathcal{R}(\rho),$$

and $\mathcal{R}(\rho)$ is a local strongly contact $(n + k - 1)$-form. Consequently, for any $(n + k)$-form $\rho$ it holds

$$(\pi^{2r+1,r+1})(\rho) - \mathcal{I}(\rho) \in \Theta_{n+k}^{2r+1}$$

$\mathcal{I}$ behaves like a projector, *i.e.*

$\mathcal{I} \circ \mathcal{I} = (\pi^{4r+3,2r+1})^* \circ \mathcal{I} = \mathcal{I} \circ (\pi^{2r+1,r+1})^*.$

Moreover, by Lemma 1.1 $\Theta_{n+k}^r \subseteq \ker \mathcal{I}$. The opposite inclusion also holds: if $\mathcal{I}(\rho) = 0$, then by (4)

$$(\pi^{2r+1,r+1})^*(\rho) \in \Theta_{n+k}^{2r+1},$$

and this can be seen to imply $\rho \in \Theta_{n+k}^r$.

In fact, for any pair of integers $s \geq r$, the quotient map $Q_{q}^{s,r}$ in the following diagram

$$\begin{array}{ccc}
\Theta_q^s & \longrightarrow & \Omega_q^s \\
\downarrow (\pi^{s,r})^* & & \downarrow (\pi^{s,r})^* \\
\Theta_q^r & \longrightarrow & \Omega_q^r \\
\end{array}$$

is injective; therefore we have

$$\ker \mathcal{I} = \Theta_{n+k}^r.$$


Remark 1.2 Explicit coordinate expressions for the residual operator can be obtained; see [41] for details. Let

\[ p_k \rho = \sum_{|I|=0}^{r} \omega^\sigma_I \wedge \eta^I_\sigma , \]

with \( \eta^I_\sigma \) are \((k-1)\)-contact \((n+k-1)\)-forms. We have that

\[ \sum_{|I|=0}^{r} \omega^\sigma_I \wedge \eta^I_\sigma = \sum_{|I|=0}^{r} d_I (\omega^\sigma \wedge \zeta^I_\sigma) , \]

where

\[ \zeta^I_\sigma = \sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|I| + |J|}{|J|} d_J \eta^I_\sigma . \]

Moreover we can write \( \omega^\sigma \wedge \zeta^I_\sigma = \chi^I \wedge ds \) for suitable \( k \)-contact \( k \)-forms \( \chi^I \) on \( J^{2r} Y \). Since

\[ \sum_{|I|=1}^{r} d_I (\omega^\sigma \wedge \zeta^I_\sigma) = d_i \sum_{|I|=0}^{r-1} d_I \chi^I_i \wedge ds , \]

(the term \( \omega^\sigma \wedge \zeta_\sigma \) defines the interior Euler operator), after some straightforward manipulations, the residual operator is given by

\[ \mathcal{R}(\rho) = \sum_{|I|=0}^{r-1} (-1)^k d_I \chi^{I_j} \wedge ds_j . \]

In subsection 5.4 we shall apply the above expression in coordinates to Yang-Mills theories on a Minkowskian background.

2 Higher order variations by the interior Euler operator

By the interior Euler operator, we can obtain a sequence of sheaves of differential forms (rather than of classes of differential forms), such that both
the objects and the morphisms have a straightforward interpretation in the calculus of variations. Such a representation of the variational sequence, introduced by Krbek and Musilová (see [39, 40, 41]) basically concerns source forms (of all degrees), and in [59] has been called the Takens representation [68]. In the Appendix we summarize its peculiarities.

The aim of this section is to obtain formulas for higher order variations of a Lagrangian based on an iteration of the first variation formula expressed through the Takens representation, i.e. by means of the interior Euler operator.

### 2.1 First variation formula

The following version of the First Noether Theorem, concerned with variational Lie derivatives of classes of degree $n$, holds true (see in particular [59] and references therein).

**Theorem 2.1** For $q = n$ and for any $\pi$-projectable vector field $\Xi$ it holds, up to pull-backs by canonical projections,

$$
\tilde{R}_n(\mathcal{L}_{J^{r+1}}\Xi[p]) = L_{J^{r+1}}h\rho = 
\Xi_V | E_n(h\rho) + d_H(J^{r+1}\Xi_V | p_{d\psi h}\rho + \Xi_H | h\rho),
$$

(8)

where

$$
p_{d\psi h}\rho = -p_1 R(dh\rho).
$$

**Proof.** We need to take the pull-back of $L_{J^{r+1}}h\rho$ to $J^{2r+1}Y$. We use the well known property

$$(\pi^{2r+1,r+1})^*(L_{J^{r+1}}h\rho) = L_{J^{r+1}}(\pi^{2r+1,r+1})^*(h\rho).$$

This enables us to apply all the splittings provided by the contact structure. Moreover, being the left hand side dependent only on the jet prolongation of $\Xi$ up to the order $r + 1$, we can write that

$$(\pi^{2r+1,r+1})^*(L_{J^{r+1}}h\rho) = L_{J^{r+1}}(\pi^{2r+1,r+1})^*(h\rho),$$

where $J^{r+1}\Xi$ is regarded as a vector field along $\pi^{2r+1,r+1}$. By the Cartan formula

$$
L_{J^{r+1}}h\rho = J^{r+1}\Xi| dh\rho + d(J^{r+1}\Xi | \rho),
$$
and omitting the pull-back for simplicity, we have
\[
L_{J^r \Xi} h\rho = \left( J^{r+3} - J^r \Xi V \right) (d_H + d_{\mathcal{V}})(h\rho) + (d_H + d_{\mathcal{V}})((J^{r+4} - J^r \Xi V)(h\rho)).
\]
The right hand side is equal to
\[
J^{r+3} H \Xi d_{\mathcal{V}} h\rho + J^{r+3} V \Xi d_{\mathcal{V}} h\rho + d_H(J^{r+1} \Xi H)(h\rho) + d_{\mathcal{V}}(J^{r+1} \Xi H)(h\rho).
\]
It is simple to show that
\[
J^{r+3} H \Xi d_{\mathcal{V}} h\rho = -d_{\mathcal{V}}(J^{r+1} \Xi H)(h\rho).
\]
Then one gets
\[
L_{J^r \Xi} h\rho = J^{r+3} V \Xi d_{\mathcal{V}} h\rho + d_H(J^{r+1} \Xi H)(h\rho).
\]
Now, being \( h\rho \) horizontal, \( d_{\mathcal{V}} h\rho = p_1 d h\rho \) and we have, up to pull-backs,
\[
p_1 d h\rho = \mathcal{I}(d h\rho) + p_1 d p_1 \mathcal{R}(d h\rho).
\]
Using the representation sequence, we can write
\[
\mathcal{I}(d h\rho) = E_n(h\rho),
\]
while
\[
J^{r+3} V \Xi p_1 d p_1 \mathcal{R}(d h\rho) = -h d(J^{r+1} \Xi V \mathcal{R}(d h\rho)) = d_H(J^{r+1} \Xi V)(-p_1 \mathcal{R}(d h\rho)).
\]
By setting \( p_{d_{\mathcal{V}} h\rho} = -p_1 \mathcal{R}(d h\rho) \), formula (8) follows because \( h\rho \) is horizontal over \( X \) and \( E_n(h\rho) \) is a source \((n + 1)\)-form.

We stress that (8) can be regarded as the local first variation formula for Lagrangians \( h\rho \) (with a projectable variation vector field) \[42, 47] . The term under the horizontal differential is denoted by \( \epsilon \Xi h\rho \) and we call it \( Noether current for h\rho associated with \Xi \). The term \( p_{d_{\mathcal{V}} h\rho} = -p_1 d \mathcal{R}(d h\rho) \) is called \( local generalized momentum \). A generalization of formula (8) to class of degree greater or lower than \( n \) has been obtained \[9, 59] .

### 2.2 Second variation formula

Now we obtain a formula for the second variation, which will be further explored in section 3. We note that \( L_{J^r \Xi} h\rho = hL_{J^r \Xi} \rho \), and then apply a standard inductive reasoning. Of course, the iterated variation is pulled-back up to a suitable order, in order to suitably split the Lie derivatives.
Theorem 2.2 Let \( \rho \) be an \( n \)-form on \( J^rY \), \([\rho]\) its class and \( \lambda = h\rho \) the associated Lagrangian. For any pair of projectable vector fields \( \Xi_1 \) and \( \Xi_2 \), we have

\[
(\pi^{4r+3,r+1})^*(L_{J^{r+1}\Xi_2} L_{J^{r+1}\Xi_1} h\rho) = \Xi_2, V] E_n(\Xi_1, V] E_n(h\rho)) + d_H \epsilon_{\Xi_2}(\Xi_1, V] E_n(h\rho)) + d_H \epsilon_{\Xi_2}(d_H \epsilon_{\Xi_1}(h\rho)) \tag{9}
\]

where

\[
\epsilon_{\Xi_2}(\Xi_1, V] E_n(h\rho)) = \Xi_2, H] E_n(h\rho) + J^{r+1}\Xi_2, V] p_{d_H \Xi_1, V] E_n(h\rho)},
\]

\[
\epsilon_{\Xi_2}(d_H \epsilon_{\Xi_1}(h\rho)) = \Xi_2, H] d_H(J^{r+1}\Xi_1, V] p_{d_H \Xi_1, V] p_{d_H \Xi_1, V] E_n(h\rho)} + J^{r+1}\Xi_2, V] p_{d_H \Xi_1, V] p_{d_H \Xi_1, V] E_n(h\rho)}.
\]

Proof. We recall again that, taking a \( q \)-form \( \theta \) on \( J^kY \), a projectable vector field \( \Xi \) and a pair of integers \((s,k)\) with \( s > r \), then

\[
(\pi^{s,r})^*(L_{J^s} \Xi \theta) = L_{J^s} \Xi ((\pi^{s,k})^*(\theta)).
\]

Using this property we can write

\[
(\pi^{4r+3,r+1})^*(L_{J^{r+1}\Xi_2} L_{J^{r+1}\Xi_1} h\rho) = (\pi^{4r+3,2r+1})^*(L_{J^{2r+1}\Xi_2} (\pi^{2r+1,r+1})^*(L_{J^{r+1}\Xi_1} h\rho)).
\]

Now, applying Theorem 2.1 we have

\[
(\pi^{2k+1,k+1})^*(L_{J^{r+1}\Xi_1} h\rho) = \Xi_1, V] E_n(h\rho) + d_H(J^{r+1}\Xi_1, V] p_{d_H \Xi_1, V] p_{d_H \Xi_1, V] E_n(h\rho)}.
\]

Using linearity of the Lie derivative we can apply again Theorem 2.1 we obtain that

\[
(\pi^{4r+3,r+1})^*(L_{J^{r+1}\Xi_2} L_{J^{r+1}\Xi_1} h\rho)
\]
depends only on the components of the prolongations of the fields up to the order \( r + 1 \). Then we can conclude after some straightforward calculations.

Being the operators \( \mathcal{I} \) and \( \mathcal{R} \) directly related with the representation of the variational sequence, the expression (9) for the second variation seems to us the most natural one. From the above, interesting identities can be obtained.
Proposition 2.3  The following facts are true.

1. We have that the identity
\[ d_H(j^{k+1} \Xi_1, V^d V) + \Xi_1, H \rho) = (10) \]
\[ d_H(h(\Xi_2, V^d V) + \Xi_1, H \rho)) \]

2. For every pair of vertical vector fields \( \Xi_1 \) and \( \Xi_2 \) it holds
\[ \Xi_1, E_n(\Xi_2, E_n(h \rho)) + [\Xi_2, \Xi_1] E_n(h \rho) = (11) \]
\[ \Xi_2, E_n(\Xi_1) E_n(h \rho)) + d_H(\epsilon_{\Xi_2}(\Xi_1, E_n(h \rho))) \]

3. For every pair of vertical vector fields \( \Xi_1 \) and \( \Xi_2 \) it holds
\[ d_H(\epsilon_{\Xi_2}(\Xi_1, E_n(h \rho)) + d_H(\epsilon_{\Xi_1}(\Xi_2, E_n(h \rho))) = 0. (12) \]

Proof.

1. Up to pull-backs we have,
\[ L_{j^{k+1} \Xi_1, h \rho} = \Xi_1, E_n(h \rho) + d_H(\epsilon_{\Xi_1}(h \rho)) \]

We can certainly write, denoting by \( s \) and \( s' \) the orders of the terms on the right hand side,
\[ L_{j^{k+1} \Xi_2} L_{j^{k+1} \Xi_1, h \rho} = L_{j^{s} \Xi_2}(\Xi_1, V^d V) E_n(h \rho)) + L_{j^{s'} \Xi_2}(d_H(\epsilon_{\Xi_1}(h \rho))) \]

In fact higher order terms of \( j^{s} \Xi_2 \) and \( j^{s'} \Xi_2 \) play no rôle in the Lie derivative of the left hand side and so their contributions to Lie derivatives of the two terms in the right hand side have to cancel each other. We know that \( d_H \) commutes with \( L_{j^{s'} \Xi_2} \) (up to the order of the prolongation) so we write
\[ L_{j^{s'} \Xi_2}(d_H(\epsilon_{\Xi_1}(h \rho))) = d_H(L_{j^{s'-1} \Xi_2}(j^{k+1} \Xi_1, V^d V) p_{dv h \rho} + \Xi_1, H \rho)) \]

and, since the two forms on which we are applying Lie derivatives are \( (n - 1) \) horizontal forms, we get
\[ L_{j^{s'} \Xi_2}(d_H(\epsilon_{\Xi_1}(h \rho))) = d_H(h(\Xi_2, V^d V) d_V(j^{k+1} \Xi_1, V^d V) p_{dv h \rho} + \Xi_1, H \rho) + \Xi_2, H \rho) d_H(j^{k+1} \Xi_1, V^d V) p_{dv h \rho} + \Xi_1, H \rho)) \]

again we can forget about \( s' \) as before. Then we obtain an expression for \( L_{j^{k+1} \Xi_2} L_{j^{k+1} \Xi_1, h \rho} \); imposing the equality with (11) and having a \( d_H \)

closed \( (n - 1) \)-form, by the exactness of the representation sequence we can conclude.
2. Denote by $s$ the order of $\Xi_1|E_n(h\rho)$. The vector field being vertical and $E_n(h\rho)$ being a source $(n + 1)$-form horizontal over $Y$, we have $\Xi_1|E_n(h\rho) = J^s\Xi_1|E_n(h\rho)$. Now we use a well known property of the Lie derivative, namely

$$L_J \Xi_1 = J^s \Xi_1|E_n(h\rho) = \Xi_1|E_n(h\rho) = \Xi_1|E_n(\Xi_2|E_n(h\rho))$$

and the fact that $[J^s\Xi_2, J^s\Xi_1]|E_n(h\rho) = [\Xi_2, \Xi_1]|E_n(h\rho)$. By the exactness of the representation sequence, we have

$$J^s\Xi_1|L_J \Xi_2 E_n(h\rho) = \Xi_1|E_n(\Xi_2|E_n(h\rho))$$.

Finally, by linearity we see that

$$L_J E_n(h\rho) = \Xi_1|E_n(h\rho) + \Xi_2|E_n(h\rho) + dH \epsilon_2 E_n(h\rho).$$

Then, comparing this with the expression for second variation (9), we get the result.

3. Here we apply (11) and, simply changing the rôle of $\Xi_1$ and $\Xi_2$, we can conclude.

### 2.3 Higher order variation formula

Formulas for higher order variations of Lagrangians $h\rho$ can be obtained easily using what we have shown for the first and the second variation. Precisely, we state the following.

**Theorem 2.4** Let $\rho$ be an $n$-form on $J^kY$. Consider the Lagrangian $h\rho$ and take $l$ variation vector fields $\Xi_1, \ldots, \Xi_l$. Define recursively a sequence $r_l$ by

$$r_l = 2r_{l-1} + 1, \quad r_0 = r$$

We have

$$(\pi^{r_l+1})^*(L_{J^r+1}\Xi_1 \ldots L_{J^r+1}\Xi_l h\rho) =$$

$$= \Xi_1|E_n(\Xi_2|E_n(\ldots \Xi_{2V}|E_n(\Xi_{1V}|E_n(h\rho)) \ldots)) +$$

$$+ dH \epsilon_2(\Xi_2|E_n(h\rho)) +$$

$$+ dH \epsilon_3(\Xi_3|E_n(h\rho)) +$$

$$+ \cdots$$

$$+ dH \epsilon_{r_l}(dH \epsilon_{r_l-1}(\Xi_{r_l-2V}|E_n(\ldots \Xi_{1V}|E_n(h\rho)) \ldots)) \tag{13}.$$
Proof. The proof is a straightforward induction using as base step the case \( l = 1 \) or \( l = 2 \). Taking into account the exactness of the representation sequence, the inductive step follows easily.

**Remark 2.5** Working recursively starting from (8) or (9), we can get more explicit expressions (similar to the ones in (9)) for the terms containing Noether currents in formula (13).

**Remark 2.6** Since the variation of any order of a Lagrangian \( h\rho \) is still an horizontal form, \( i.e. \) again a Lagrangian, we can study its variation by means of formula (8). On the other hand formula (13) gives us the possibility to investigate how to relate the so called *symmetries* of a variation of \( h\rho \) to \( h\rho \) itself; see sections 5.1 and 5 for the definition of symmetry and details about that.

## 3 The second variation and the Jacobi morphism

In this section we focus on the analysis of the particularly interesting case of the second variation. Within the framework of first order field theories, Goldschmidt and Sternberg worked out the second variations and introduced in that the definitions of *Hessian* and *Jacobi equation* associated with a Lagrangian [32]. The representation of second order variational derivatives in the variational sequence has been studied in [24, 25, 26] also for higher order field theories; also the definition and the rôle of the so called *generalized Jacobi morphism*, specifically for the relation between the Noether theorems and the second variation, and for applications to canonical conservation laws was investigated in [21, 22, 27, 60, 61, 62, 63, 64, 66]. The second variation in a (Poincaré-Cartan equivalent) Lagrangian context has been also studied by some authors in particular in relation with Finsler geometry, see \( e.g. \) [12] constituting a step towards a systematic intrinsic and global study of this area of the calculus of variations.

Here the definition of the Jacobi morphism will be given within the representation sequence, \( i.e. \) by the interior Euler operator. We show that the Jacobi morphism is self adjoint along extremals, finding also explicit coordinate expressions, and we introduce the Jacobi equation and Jacobi fields.
In this framework, we easily define the Hessian of the action, which turns out to be related to the Jacobi morphism. Our discussion is inspired by the classical paper [32] and by [26].

3.1 Self-adjoint operators associated with source \((n+2)\)-forms

We recall the definition of adjoint of a differential operator associated with a suitable \((n+2)\)-form, then we define the Jacobi morphism associated with a Lagrangian. The reader can consult, for example, [32, 6] for an introduction.

Consider a global \((n+2)\)-form on \(J^{r+1}Y\) with local coordinate expression given by
\[
\omega = \sum_{|J| = 0}^r A^J_{\tau\sigma} \omega^\tau_J \wedge \omega^\sigma \wedge ds;
\]
the local expressions for \(\mathcal{I}(\omega)\) are
\[
\mathcal{I}(\omega) = \sum_{|J| = 0}^k \frac{1}{2} \omega^\tau \wedge (-1)^{|J|} d_J(A^J_{\tau\rho} \omega^\sigma) \wedge ds - \sum_{|J| = 0}^k \frac{1}{2} \omega^\sigma \wedge A^J_{\tau\sigma} \omega^\tau_J \wedge ds.
\]
We can now introduce \(\tilde{\mathcal{I}}(\omega)\), associated with \(\mathcal{I}(\omega)\) and defined as
\[
\tilde{\mathcal{I}}(\omega) = - \sum_{|J| = 0}^r (-1)^{|J|} d_J(A^J_{\rho\sigma} \omega^\sigma) \otimes \omega^\rho \otimes ds + \sum_{|J| = 0}^r A^J_{\tau\sigma} \omega^\tau_J \otimes \omega^\sigma \otimes ds.
\]
Let us set
\[
\tilde{\omega} = \sum_{|J| = 0}^r A^J_{\tau\rho} \omega^\tau_J \otimes \omega^\sigma \otimes ds.
\]

We introduce some formal differential operators associated with \(\omega\). Define
\[
\nabla_\omega : X_Y(Y) \rightarrow C^1_0 \otimes \Omega^r_{n,Y}(J^{r+1}Y)
\]
\[
\Xi \rightarrow \tilde{\omega}(J^{r+1}\Xi, \bullet)
\]
where we have denoted by \(X_Y(Y)\) the space of vertical vector fields on \(Y\) and by \(C^1_0\) the space of contact 1-forms generated by \(\omega^\sigma\). In coordinates
\[
\nabla_\omega \left( \Xi^\sigma \frac{\partial}{\partial y^\sigma} \right) = \sum_{|J| = 0}^r A^J_{\tau\rho} d_J(\Xi^\tau) \omega^\rho \otimes ds
\]
Moreover we set
\[ \nabla^* \omega : X_V(Y) \to C^0_0 \otimes \Omega^r_{n,X}(J^r Y) \]
\[ \Xi \to (\hat{\omega} - \hat{T}(\omega))(J^{r+1} \Xi, \bullet) \tag{15} \]
that in coordinates is
\[ \nabla^*_\omega \left( \Xi^\sigma \frac{\partial}{\partial y^\sigma} \right) = \sum_{|J|=0}^r (-1)^{|J|} d_J(A^J_\rho \Xi^\sigma \omega^\tau \otimes ds. \tag{16} \]

The choice in the notation is motivated by the fact that \( \nabla^*_\omega \) can be seen as an adjoint operator for \( \nabla_\omega \).

### 3.2 The Jacobi morphism

Now we define the Jacobi morphism associated with a Lagrangian \( \lambda \) on \( J^r Y \); it will be denoted by \( \mathcal{J}(\lambda) \).

**Definition 3.1** The map
\[ \mathcal{J} : \Omega^r_{n,X}(J^r Y) \to X^*_V(J^{2r+1} Y) \otimes X^*_V(Y) \otimes \Omega^r_{n,X}(J^r Y) \]
\[ \lambda \to \bullet \int E_n(\bullet \int E_n(\lambda)) \tag{16} \]
is called the Jacobi morphism associated with \( \lambda \).

We will often use the notation \( \mathcal{J} \Xi_1(\lambda) \) to denote \( E_n(\Xi_1 \int E_n(\lambda)) \).

Roughly speaking, the Jacobi morphism is the second variation (generated by vertical vector fields) of the Lagrangian \( \lambda \) if we forget about horizontal differentials (see [26]); as a consequence, it is possible to associate this morphism with some horizontally closed \((n-1)\)-forms. We will discuss this fact later on.

Now we write down an explicit coordinate expression for \( \mathcal{J}(\lambda) \); doing that, we will discover its self adjointness along critical sections. An alternative proof of the following theorem can be found in [26].

**Theorem 3.2** Along critical sections it holds

1. For every vertical vector field on \( Y \) denoted by \( \Xi \),
\[ E_n(\Xi \int E_n(\lambda)) = \sum_{|J|=0}^{2r+1} (-1)^{|J|} d_J \left( \Xi^\rho \frac{\partial E^\rho_\lambda}{\partial y^\sigma_J} \right) \omega^\sigma \wedge ds. \tag{17} \]
2. For every vertical vector field on $Y$ denoted by $\Xi$,

$$E_n(\Xi | E_n(\lambda)) = \sum_{|J|=0}^{2r+1} d_J \Xi^\rho \frac{\partial E_\rho(\lambda)}{\partial y^J_\sigma} \omega^\sigma \wedge ds \cdot$$  \hspace{1cm} (18)

3. The Jacobi morphism is self adjoint.

**Proof.**

1. Simply writing down the coordinate expressions of the objects involved we have

$$E_n(\Xi | E_n(\lambda)) = \sum_{|J|=0}^{2r+1} (-1)^{|J|} d_J \left( \frac{\partial (E_\rho(\lambda) \Xi^\rho)}{\partial y^J_\sigma} \right) \omega^\sigma \wedge ds =$$

$$= \sum_{|J|=0}^{2r+1} (-1)^{|J|} d_J \left( \Xi^\rho \frac{\partial E_\rho(\lambda)}{\partial y^J_\sigma} \right) \omega^\sigma \wedge ds \cdot \left( \frac{\partial \Xi^\rho}{\partial y^J_\sigma} E_\rho(\lambda) \right) \omega^\sigma \wedge ds .$$

Now we note that along solutions of the Euler–Lagrange equations the terms of the form

$$\left( \frac{\partial \Xi^\rho}{\partial y^J_\sigma} E_\rho(\lambda) \right)$$

vanish.

2. We know that, up to pull-backs,

$$p_1 d\lambda = \mathcal{I}(d\lambda) + p_1 dp_1 \mathcal{R}(d\lambda) ,$$

and $p_1 d\lambda = d\lambda$; then we can write

$$d\mathcal{I}(d\lambda) = -dp_1 dp_1 \mathcal{R}(d\lambda) .$$

However, $p_1 dp_1 \mathcal{R}(d\lambda) \in \Theta^{2k+1}_{n+1}$; but then we have also $dp_1 dp_1 \mathcal{R}(d\lambda) \in \Theta^{2k+1}_{n+2}$ and $J^{2r+1} \Xi | dp_1 dp_1 \mathcal{R}(d\lambda) \in \Theta^{2k+1}_{n+1}$. This implies that

$$\mathcal{I}(J^{2r+1} \Xi | d\mathcal{I}(d\lambda)) = 0 ,$$
for any vertical vector field $\Xi$. Using the coordinate expression for the interior Euler operator we get

$$I(J^{2r+1}\Xi) = \sum_{|J|=0}^{2k+1} \frac{\partial E_\sigma(\lambda)}{\partial y_J^\rho} d_J^\rho \omega^\sigma \wedge ds +$$

$$- \sum_{|J|=0}^{2k+1} (-1)^{|J|} d_J \left( \frac{\partial E_\sigma(\lambda)}{\partial y_J^\rho} \Xi^\rho \wedge ds, \right),$$

and we can conclude.

3. By direct inspection we see that the expression (17) gives us precisely the adjoint of (18).

Note that this argument is equivalent to show that

$$J^{2r+1}\Xi_2 I(J^{2r+1}\Xi_1 d\lambda) = 0,$$

for any pair of vertical vector fields $\Xi_1, \Xi_2$.

**Remark 3.3** we can write formula (13) in terms of Jacobi morphisms:

$$(\pi^{r,r+1})^* (L_{J^{2r+1}\Xi_1} \ldots L_{J^{2r+1}\Xi_1} h\rho) =$$

$$= \Xi_{l,V} J_{\Xi_{l-1,V}} (\Xi_{l-2,V} J_{\Xi_{l-3,V}} \ldots (h\rho) \ldots) +$$

$$+ d_H \epsilon_{\Xi_{l-1,V}} (\Xi_{l-2,V} J_{\Xi_{l-3,V}} \ldots (h\rho) \ldots) +$$

$$+ d_H \epsilon_{\Xi_{l-1}} (d_H \epsilon_{\Xi_{l-1}} (\Xi_{l-2,V} J_{\Xi_{l-3,V}} \ldots (h\rho) \ldots) +$$

$$+ \cdots + d_H \epsilon_{\Xi_{l-1}} (d_H \epsilon_{\Xi_{l-1}} d_H \ldots d_H \epsilon_{\Xi_{l-1}} (h\rho) \ldots).$$

A quite important rôle is played by vector fields that are in the kernel of the Jacobi morphism.

**Definition 3.4** Let $\lambda$ be a Lagrangian of order $r$. A *Jacobi field* for the Lagrangian $\lambda$ is a vertical vector field $\Xi$ that belongs to the kernel of the Jacobi morphism, i.e.

$$J_{\Xi}(\lambda) = 0$$

The previous equation is called the *Jacobi equation* for the Lagrangian $\lambda$. 
The following proposition shows that the Jacobi equation can be correctly defined as an equation for vector fields along an extremal.

**Proposition 3.5** Consider the Jacobi equation

\[ J_\Xi(\lambda) = 0, \]

evaluated along an extremal \( \gamma \). The equation depends only on the values of the vector field \( \Xi \) along \( \gamma \).

**Proof.** The claim follows from the application of (18) together with the fact that \( d_j\Xi^\sigma \), when evaluated along \( \gamma \), depends only on the values of \( \Xi \) along \( \gamma \).

Taking into account the above proposition, we will speak of *Jacobi fields along an extremal \( \gamma \); Theorem 3.2 provides us with the coordinate expression of the equation for Jacobi fields along an extremal.*

### 3.3 The Jacobi morphism and the Hessian of the action functional

In the fundamental article [32] of Goldschmidt and Sternberg, the Jacobi equation was introduced and its relation to the Hessian of the action functional was explored in the case of first order field theories. We will reformulate their work in our formalism, generalizing it to arbitrary order; in particular by the above definition of a Jacobi morphism we recover the Hessian at any order.

In this section we will say often that a vector field on \( J^rY \), with \( r \geq 0 \), vanishes identically on a subset of \( X \) meaning that it vanishes along the fibers over that subset.

Take the action functional associated with a Lagrangian \( \lambda \) on \( J^rY \):

\[ A_D[\gamma] = \int_D (j^\gamma)^*(\lambda) \]

where \( D \) is an \( n \)-region and \( \gamma \) is a section. We can introduce variations that do not change \( \gamma \) on \( \partial D \); it is sufficient to generate it with vector fields that vanish on \( \partial D \). We will work only on \( D \) so we can require that these fields vanish also in \( X \setminus D \) (all vector fields compactly supported in an open proper subset of \( D \) are an example).
We denote by $\Gamma_D^\tau$ the space of sections defined on $D$ and equal to a fixed section $\tau$ on $\partial D$, by $X_{V,\gamma}(Y)$ the space of vertical vector fields defined along $\gamma \in \Gamma_D^\tau$ and by $T_\gamma \Gamma_D^\tau$ the subspace of $X_{V,\gamma}(Y)$ containing vector fields that are null on $\partial D$. The last notation is motivated by the fact that $T_\gamma \Gamma_D^\tau$ can be thought as the tangent space to $\Gamma_D^\tau$ at the point $\gamma$ (this is a quite standard fact from the theory of infinite dimensional manifolds). We then give the following definition (which is standard too).

**Definition 3.6** The differential of $A_D$ along a section $\gamma \in \Gamma_D^\tau$ is the map

$$dA_D[\gamma] : T_\gamma \Gamma_D^\tau \to \mathbb{R}$$

$$\nu \to \frac{d}{dt}A_D[\Gamma_{\Xi}(t)]\bigg|_{t=0}$$

(19)

where $\Xi$ is any extension of $\nu$ to the whole $X_{V}(Y)$ vanishing on $\partial D$ (and on $X/D$), while $\Gamma_{\Xi}$ is a one parameter variation of $\gamma$ generated by $\Xi$.

In our hypothesis the derivation passes under the integral and we have

$$dA_D[\gamma](\nu) = \int_D (j^\tau \gamma)^*(\delta_\Xi \lambda).$$

We need to show that the definition does not depend on $\Xi$, but thanks to this last remarks this means that we need to prove

$$\int_D (j^\tau \gamma)^*(L_{j^\tau \xi} \lambda) = \int_D (j^k \gamma)^*(L_{j^k \xi'} \lambda),$$

where $\Xi'$ is an alternative choice of the field. Here we simply apply (9) and note that the horizontal differential plays no rôle thanks to Stokes theorem and the fact that the fields vanish on $\partial D$; then we use the fact that, because of the pull-back by $j^\tau \gamma$, everything depends only on the values of the fields along $\gamma$.

So we have a notion of differential of the action; an extremal point is exactly a solution of Euler–Lagrange equations. In an analogous manner, we introduce an Hessian for the action, but we need the section $\gamma$ to be an extremal.
Definition 3.7 The Hessian of $A_D$ along an extremal section $\gamma \in \Gamma_D^\tau$ is the map
\[ H(A_D)[\gamma] : T_\gamma \Gamma^\tau_D \times T_\gamma \Gamma^\tau_D \rightarrow \mathbb{R} \]
\[ (\nu, \kappa) \mapsto \frac{\partial^2}{\partial t_1 \partial t_2} A_D[\Gamma_{\Xi_1, \Xi_2}(t_1, t_2)] \bigg|_{t_1=t_2=0} \tag{20} \]
where $\Xi_1$ and $\Xi_2$ are extensions in $X_V(Y)$ of $\nu$ and $\kappa$ respectively and vanish on $\partial D \cup X \setminus D$, while $\Gamma_{\Xi_1, \Xi_2}(t_1, t_2)$ is a two parameter variation of $\gamma$ generated by $\Xi_1$ and $\Xi_2$.

Deriving again under the integral sign
\[ H(A_D)[\gamma](\nu, \kappa) = \int_D (j^r \gamma)^* (L_{j^r \Xi_1} L_{j^r \Xi_2} \lambda) . \]

In order to show that the Hessian is well defined, we need to prove that it does not depend on the extensions $\Xi_1$ and $\Xi_2$ chosen; moreover, we want to show that it is symmetric. Here we need the hypothesis that the section is an extremal; in fact this implies, by applying (11), the Stokes theorem and the fact that the fields vanish on $\partial D$,
\[ \int_D (j^r \gamma)^* (L_{j^r \Xi_1} L_{j^r \Xi_2} \lambda) = \int_D (j^r \gamma)^* (L_{j^r \Xi_2} L_{j^r \Xi_1} \lambda) . \]

Now, using (9), we see that the definition can be restated as
\[ H(I_A)[\sigma](\nu, \kappa) = \int_D (j^{2+1} \gamma)^* (\Xi_1] E_n(\Xi_2] E_n(\lambda)) , \]
that obviously does not depend on the extension $\Xi_1$. However, by symmetry it cannot depend on $\Xi_2$ too. Then the Hessian is a well defined symmetric bilinear map.

Now we can easily recover the properties stated at the first order by Goldschmidt and Sternberg in [32]: we need only to apply our intrinsic results.

Proposition 3.8 If $\gamma$ is a local minimum, the Hessian along it is positive semi definite.
Proof. We consider $\Gamma_\Xi$, a one parameter variation of $\gamma$ generated by $\Xi$. We have by hypothesis
\[
\int_D (j^r T_{\Xi}(t))^*(\lambda) \geq \int_D (j^r \gamma)^*(\lambda),
\]
for all $t$ in a neighborhood of 0. Consequently
\[
\frac{d^2}{dt^2} \int_D (j^r T_{\Xi}(t))^*(\lambda) \geq 0.
\]
But, by definition, the left hand side is the Hessian along $\gamma$ calculated on the pair $(\Xi \circ \gamma, \Xi \circ \gamma)$. Thanks to the arbitrariness of $\Xi$ we can conclude that the quadratic form associated with the Hessian along $\gamma$ has only values $\geq 0$, that is our claim.

Our construction gives us immediately also the relation between the Hessian and the Jacobi morphism. In fact it is clear from the previous discussion that
\[
\mathcal{H}(A_D)[\gamma](\nu, \kappa) = \int_D (j^{2r+1}\gamma)^*(\Xi_1 \mid J_{\Xi_2}(\lambda)).
\]
Finally, considering that our definitions of Hessian and Jacobi field do not depend on the order, we can prove the following result applying the same argument used in [32] for the first order case.

**Proposition 3.9** A vector field $\nu \in T_\gamma \Gamma^r_D$, where $\gamma$ is an extremal, belongs to the null space of the Hessian along $\gamma$ if and only if it is a Jacobi field along $\gamma$.

Proof. If we have a Jacobi field, then it is immediate to see that it belongs to the null space of the Hessian, because of the relation of the Hessian with the Jacobi morphism.

Conversely, if $\nu \in T_\gamma \Gamma^r_D$ is in the nullspace of the Hessian, then
\[
\int_D (j^{2r+1}\gamma)^*(\Xi_1 \mid J_{\Xi_2}(\lambda)) = 0,
\]
for any $\Xi_1$ and any $\Xi_2$ that extends $\nu$. The arbitrariness of $\Xi_1$ enables us to conclude.
4 Yang-Mills theories on a Minkowskian background

In this section we discuss the Jacobi equation for Yang-Mills theories \[71\] on a Minkowskian background. Note that in this example, the (configuration) fibered manifolds have the structure of bundles.

In the sequel, by assuming a Minkowskian background we mean that the spacetime manifold, that is the base space \( M \) of the configuration bundle for the theory, is equipped with a fixed Minkowskian metric. This means that \( M \) is a Lorentzian manifold such that we can choose a system of coordinates in which the metric is expressed in the diagonal form \( \eta_{\mu\nu} \); there are two possible conventions on the signature but, since the signature will play no rôle in our discussion, the choice is left to the reader.

Consider a principal bundle with a semi-simple structure group \( G \), denoted by \((P,p,M,G)\); this bundle is the structure bundle of the theory. We consider the bundle \((C_P,\pi,M)\) of principal connections on \( P \) (the total space can be seen as \( J^1 P/G \); see \[67\]). In this section lower Greek indices will denote space time indices, while capital Latin indices label the Lie algebra \( g \) of \( G \). Then, on the bundle \( C_P \), we introduce coordinates \((x^\mu,\omega^A_\sigma)\).

We define a Yang-Mills Lagrangian as follows. We consider the Cartan-Killing metric \( \delta \) on the Lie algebra \( g \), and choose a \( \delta \)-orthonormal basis \( T_A \) in \( g \); the components of \( \delta \) will be denoted \( \delta_{AB} \). We will use \( \delta_{AB} \) to raise and lower Latin indices. On a spacetime manifold with a generic (not necessarily Minkowskian) fixed background \( g_{\mu\nu} \) the Yang-Mills Lagrangian is defined by

\[
\lambda_{YM} = -\frac{1}{4} F^A_{\mu\nu} g^{\rho\sigma} F^B_{\rho\sigma} \delta_{AB} \sqrt{g} ds ,
\]

where

- \( g \) stands for the absolute value of the determinant of the metric \( g_{\mu\nu} \)
- we set \( \omega^A_\mu = d_\nu \omega^A_\mu \)
- we denote by \( c^A_{BC} \) the structure constants of \( g \);
- \( F^A_{\mu\nu} = \omega^A_{\nu,\mu} - \omega^A_{\mu,\nu} + c^A_{BC} \omega^B_\mu \omega^C_\nu \) is the so called field strength.

The well known variation procedure for this Lagrangian gives the equations

\[
E^\nu_B = d_\mu (\sqrt{g} F^\mu_B) + \sqrt{g} F^\mu_A c^A_{BC} \omega^C_\mu = 0 .
\]
From now on, we will assume the metric to be Minkowskian and we will always work in coordinates such that it is expressed in diagonal form.

We start our discussion from Maxwell theory; it is well known that we can regard it as an abelian Yang-Mills theory where the structure group is $U(1)$. Being the group one dimensional we can drop Latin indices; moreover, the structure constants vanish identically. We have then the well known equations:

$$E^\nu = d_\mu (F^{\mu \nu}) = 0.$$ 

We calculate the expression of the Jacobi equation along solutions using the explicit formula (18).

A vertical vector field $\Xi = \Xi_\sigma \frac{\partial}{\partial \omega_\sigma}$ satisfies the Jacobi equation along solutions if and only if

$$\Xi_\sigma \frac{\partial E^\nu}{\partial \omega_\sigma} + d_\rho \Xi_\sigma \frac{\partial E^\nu}{\partial \omega_{\sigma, \rho}} + d_\sigma d_\rho \Xi_\sigma \frac{\partial E^\nu}{\partial \omega_{\sigma, \rho}} = 0.$$ 

Clearly, in this particular case,

$$\frac{\partial E^\nu}{\partial \omega_\sigma} = 0 \quad \text{and} \quad \frac{\partial E^\nu}{\partial \omega_{\sigma, \rho}} = 0,$$

so that we need only to calculate the last term. Note that

$$d_\mu (F^{\mu \nu}) = \eta^{\alpha \mu} \eta^{\beta \nu} (\omega_{\beta, \alpha \mu} - \omega_{\alpha, \beta \mu})$$

and so, by commutativity of total derivatives we get easily that our system of equations is equivalent to have, for every $\nu$,

$$\eta^{\sigma \nu} \eta^{\tau \kappa} d_\kappa (d_\tau \Xi_\sigma - d_\sigma \Xi_\tau) = 0.$$ 

(21)

If we consider any semi-simple group $G$ instead of $U(1)$, a vertical vector field has the form

$$\Xi = \Xi_\sigma \frac{\partial}{\partial \omega_\sigma};$$

in the particular case of investigation, since

$$d_\mu (F_B^{\mu \nu}) = \delta_B A \eta^{\lambda \mu} \eta^{\sigma \nu} (\omega_{\sigma, \lambda \mu} - \omega_{\lambda, \sigma \mu} + c^{A} C D \omega_{\lambda, \rho} \omega^{D} C + c^{A} C D \omega_{\lambda} \omega^{D} C \sigma, \mu),$$
The Euler–Lagrange expressions for Yang–Mills are given by

\[ E^\nu_B = \delta_{BA} \eta^{\mu \nu} \eta^{\sigma \nu} (\omega^A_{\sigma, \mu} - \omega^A_{\lambda, \sigma} + c^D_{CD} \omega^C_{\mu} \omega^D_{\sigma, \mu} + c^D_{CD} \omega^C_{\lambda, \sigma} \omega^D_{\mu, \sigma}) + \eta^{\lambda \mu \nu} \eta^{\sigma \nu} \delta_{BA} (\omega^D_{\sigma, \lambda} + \omega^D_{\lambda, \sigma} + c^D_{EF} \omega^E_{\lambda} \omega^F_{\sigma, \lambda}) c^{A \nu}_{BC} \omega^C_{\mu} \]

The Jacobi equation now becomes

\[ \Xi^\alpha_a \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha}} + d_\beta \Xi^\alpha_a \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha, \beta}} + d_\gamma d_\beta \Xi^\alpha_a \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha, \beta \gamma}} = 0 . \]

Clearly now the terms

\[ \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha}} \quad \text{and} \quad \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha, \beta}} \]

do not vanish identically; indeed we have

\[ \Xi^\alpha_a \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha}} = \Xi^\alpha_a \left[ \delta_{BA} C^A_{DZ} \omega^C_{\mu} \eta^{\lambda \mu \nu} \eta^{\sigma \nu} + \eta^{\alpha \mu} \eta^{\sigma \nu} c^D_{CZ} \omega^C_{\mu} \delta_{DA} + \eta^{\lambda \mu} \eta^{\sigma \nu} C^A_{BC} \omega^C_{\mu} \delta_{DA} + \eta^{\lambda \mu} \eta^{\sigma \nu} F^D_{\lambda \mu} c_{BZ} \delta_{DA} \right] , \]

and

\[ d_\beta \Xi^\alpha_a \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha, \beta}} = d_\beta \Xi^\alpha_a \left[ \delta_{BA} \eta^{\lambda \mu \nu} \eta^{\alpha \nu} \delta_{BA} (c^D_{DZ} \delta^a_{\beta} \delta^\sigma_{\mu} \omega^D_{\sigma} + c^A_{CZ} \delta^a_{\beta} \delta^\sigma_{\mu} \omega^C_{\sigma}) + \eta^{\alpha \mu} \eta^{\sigma \nu} \delta_{Z A} C^A_{BC} \omega^C_{\mu} - \eta^{\lambda \mu} \eta^{\alpha \nu} \delta_{Z A} C^A_{BC} \omega^C_{\mu} \right] . \]

The third term is analogous to the one in Maxwell case:

\[ d_\gamma d_\beta \Xi^\alpha_a \frac{\partial E^\nu_B}{\partial \omega^a_{\alpha, \beta \gamma}} = \delta_{BZ} \eta^{\alpha \nu} \eta^{\gamma \tau} d_\gamma (d_\tau \Xi^\alpha_a - d_\sigma \Xi^\alpha_a) . \]

Summing up these terms and doing some straightforward calculations, we have, for any pair \((\nu, B)\),

\[ \eta^{\nu \alpha} \eta^{\alpha \beta} \left\{ d_\beta \left[ (d_\alpha \Xi^A_a + c^A_{CZ} \Xi^Z_a \omega^C_{\alpha}) \delta_{BA} \right] + \left[ (d_\alpha \Xi^D_a + \Xi^Z_a c^D_{CZ} \omega^F_{\alpha}) \delta_{AD} \right] c^A_{BC} \omega^C_{\beta} + \right. \]

\[ \left. - d_\beta \left[ (d_\alpha \Xi^A_a + c^A_{CZ} \Xi^Z_a \omega^C_{\alpha}) \delta_{BA} \right] - \left[ (d_\alpha \Xi^D_a + \Xi^Z_a c^D_{CZ} \omega^F_{\alpha}) \delta_{AD} \right] c^A_{BC} \omega^C_{\beta} + + F^D_{\beta \alpha} c^A_{BZ} \Xi^Z_a \delta_{AD} \right\} = 0 . \]
It is now noteworthy that we can further simplify the expression introducing a suitable induced connection.

Let \((\phi^a)\) be a set of coordinates on the group \(G\). Introducing right invariant vector fields \(\rho_A\), we have \(\rho_A = R^A_a(\phi) \partial_a\), where \(\partial_a\) denotes \(\frac{\partial}{\partial \phi^a}\), the standard local system of generators of vector fields on \(G\). We will denote by \(R^{-1}_a(\phi)\) the inverse matrix of \(R^A_a(\phi)\); in an analogue way, using left invariant vector fields \(\lambda^A_{\nu}\), we introduce the matrix \(L^A_{a\nu}(\phi)\) and its inverse \(L^{-1}_{a\nu}(\phi)\). Moreover, we introduce \(\text{Ad}_{B}^{A}(\phi) = R^{-1}_B(\phi) L^A_{a\nu}(\phi)\), that is the adjoint representation of \(G\) on \(\mathfrak{g}\). If we chose another system of fibered coordinates on \(P\), \((x'_{\nu}, \phi'_{b})\), we recall that \(\omega'_{\nu B} = J^{\mu}_{\nu \lambda}(\text{Ad}^{B}_{A}(\phi)\omega_{A \mu} - R^{B}_{a}(\phi)\phi_{\mu}^{a})\), where \(J^{\mu}_{\nu \lambda}\) denotes the inverse of the Jacobian matrix of the change of coordinates in the base space.

Then the components of a vertical vector field satisfy the transformation rule \(\Xi'_{B \nu} = \text{Ad}^{B}_{A}(\phi)\Xi_{A \mu}J^{\mu}_{\nu}\).

Following a standard approach we can see \(\Xi\) as a section of a suitable bundle. Indeed, consider the fibered product \(P \times_M L(M)\) where \(L(M)\) is the frame bundle of \(M\); \(P \times_M L(M)\) is clearly a principal bundle with structure group \(G \times GL(n)\), where \(GL(n)\) is the general linear group of degree \(n = \text{dim}(M)\). We introduce the vector space \(V = \mathfrak{g} \otimes \mathbb{R}^n\) and the representation

\[
\lambda : G \times GL(n) \times V \to V \\
(\phi, J, \Xi_{\nu}) \to \Xi'^{B}_{\nu} = \text{Ad}^{B}_{A}(\phi)\Xi_{A \mu}J^{\mu}_{\nu},
\]

by which we construct the bundle \(B = (P \times_M L(M)) \times_{\lambda} V\), which turns out to be associated with \(P \times_M L(M)\). As well known, its sections are in one to one correspondence with vertical vector fields over \(C_P\).

Now we consider that a principal connection on \(P \times_M L(M)\) is induced by any pair \((\omega, \Gamma)\), where \(\omega\) is a principal connection on \(P\) (for example, an extremal of the Yang-Mills Lagrangian) while \(\Gamma\) is a principal connection on \(L(M)\) (see [16, 33, 34] and, for gauge-natural theories, [18, 19]). In coordinates, if \(\rho_{A}^{\mu}\) are right invariant vector fields on \(L(M)\),

\[
\Omega = dx^{\mu} \otimes (\partial_{\mu} - \omega_{A \mu} \rho_{A} - \Gamma_{\mu \nu}^{A} \rho_{A}^{\nu})
\]

is a principal connection on \(P \times_M L(M)\). However, since we are considering a manifold \(M\) that admits a global Minkowskian metric, we get a connection on \(L(M)\) with coefficients vanishing in a whole class of system of coordinates (the ones in which the metric is written as \(\eta_{\mu \nu}\)). We are already working in
these coordinates, because we have required the metric to be expressed in
diagonal form; then we can assume $\Gamma^\lambda_{\nu\mu} = 0$.

We thus induce a connection on any bundle associated with $P \times_M L(M)$;
in particular, we have a connection on $B$ given by

$$\tilde{\Omega} = dx^\mu \otimes (\partial_\mu - \omega_B^{\lambda \mu}(x, \phi) \partial_\lambda).$$

Now, taking into account that the coefficients of $\Gamma$ are assumed to vanish,

$$\omega^{\lambda \mu}_B(x, \phi) = T^a_\lambda \partial_a \lambda^B(e, \Xi) \omega^A_\mu(x),$$

where

- $e$ denotes the identity element of $G \times GL(n)$
- $\lambda^B_\sigma$ denotes the “components” of the representation $\lambda$
- $T^a_\lambda = T^a_\lambda \partial_a$ is a fixed basis of $\mathfrak{g}$ $\cong T_{id}G$ ($id$ is the identity element of $G$).

Working out this expression in local coordinates we get

$$\omega^{\lambda \mu}_B(x, \phi) = -c^{B \sigma}_A \Xi^{D \sigma} \omega^A_\mu.$$

Therefore, by some careful manipulations, we rewrite the Jacobi equation for
the Yang-Mills Lagrangian on that specific background as

$$\eta^{\nu \sigma} \eta^{\beta \alpha} \left\{ \nabla_\beta \left[ (\nabla_\alpha \Xi^A - \nabla_\sigma \Xi^A) \delta_{BA} \right] + F^{D \sigma}_B \Xi^{Z \sigma} \delta_{AD} \right\} = 0,$$

for any pair $(\nu, B)$.

We note that the above is comparable with the classical definition of a
Jacobi operator [3, 7]. It can be easily checked by writing down in our case
the expression corresponding to $L_A = d^*_A d_A + *[*F,]$ for the Jacobi operator
given in [3] page 553 (we stress that the expression for the second variation
in [3, 7] is reproduced by our approach by taking the twice iterated variation
by the same variation field, see also in particular [32]).

The solution of the Jacobi equation defines the kernel $\mathfrak{k}$ of the Jacobi
operator $\mathcal{J}$, which, in particular, is characterized by Proposition 3.8 and Proposition 3.9. Note that we have fixed an orthonormal basis for the Cartan-Killing metric; working with a specific group $G$ of course the equation can be further specialized writing down the structure constants.
5 Conservation laws and higher order variations

The compatibility of the Lie derivative by jet prolongations of vector fields with the contact structure allows to discuss in an elegant and geometrical way the Noether theorem and the theory of conserved currents. Moreover, using the approach focused on iterated variational Lie derivatives, we can investigate the existence of conservation laws associated with the Jacobi equation or to symmetries of higher order variations. In this section we shall explore these topics.

We refer to the Appendix for a geometric approach to conservation laws based on Lepage equivalents and their specific relation with the interior Euler operator; we mainly followed [47]. The original articles of Noether and Bessel-Hagen are respectively [58] and [5] (see also the historical review [37]); a fundamental reference is [69]. As for generalized symmetries, see also in particular [9, 30].

The results presented here on the relation between Jacobi fields and conservation laws are original; the reader can consult [61, 62, 64] for some related topics. Further aspects concerning the relation between second variational Lie derivative and conserved currents are discussed in [28, 29].

We deal with Jacobi fields and symmetries of higher order variations. We prove that a pair of Jacobi fields is associated with a conserved current. Moreover, by means of our higher order variation formulas, we show that also a pair given by a symmetry of the \( l \)-th variation of a Lagrangian and a Jacobi field of the \( s \)-th variation of the same Lagrangian (with \( s < l \)) is associated with a (strongly) conserved current.

5.1 Lagrangian and generalized symmetries

**Definition 5.1** Let \( \lambda \) be a Lagrangian on \( J^rY \) (or on an open subset \( V^r \subseteq J^rY \) where \( V \) is open in \( Y \)). A *symmetry of \( \lambda \)* is an automorphism \( f \) of \( Y \) such that \( J^rf \) is an invariance transformation of \( \lambda \).

By abuse of notation, we will use the term *symmetries* for infinitesimal generators of symmetries too. The following collects some important properties of symmetries, see *e.g.* [47, 69].
1. A projectable vector field $\Xi$ is a symmetry of $\lambda$ if and only if
   \[ L_{J^r\Xi}\lambda = 0. \]

2. Symmetries of a Lagrangian constitute a subalgebra of the algebra of vector fields on $J^rY$.

3. Given a symmetry $\Xi$ of a Lagrangian $\lambda$, for any Lepage equivalent $\Theta$ of order $s$ and any section $\gamma$,
   \[ h(J^s\Xi|E_\lambda) + h(dJ^s\Xi|\Theta) = 0, \]
   or, equivalently,
   \[ (j^s\gamma)^*(J^s\Xi|E_\lambda) + d(j^s\gamma)^*(J^s\Xi|\Theta) = 0. \]

4. A projectable vector field $\Xi$ is a generator of invariance transformations for a source form $\omega \in \Omega^r_{n+1,YV}$ if and only if
   \[ L_{J^r\Xi}\omega = 0. \]

5. Generators of invariance transformations constitute a subalgebra of the algebra of projectable vector field on $J^rY$.

6. An invariance transformation of $\lambda$ is an invariance transformation of $E_\lambda$; furthermore given an invariance transformation $f$ of $E_\lambda$, $\lambda - (J^rf)^*(\lambda)$ is a trivial Lagrangian.

**Definition 5.2** Given a section $\gamma \in \Gamma_{loc}(\pi)$ and an open set $W$ in $J^rY$, an $(n-1)$-form $\epsilon \in \Omega_{n-1,Y}W$ such that
   \[ d(j^r\gamma)^*(\epsilon) = (j^{r+1}\gamma)^*d_H\epsilon = 0, \]
   is called *conserved current* along $\gamma$. The previous equality is a *weak conservation law* along $\gamma$.

The term weak is related to the fact that the form $\epsilon$ is closed (equivalently, horizontally closed) only along the section $\gamma$. When a current is horizontally closed everywhere, one speaks of a strongly conserved current.

**Definition 5.3** An invariance transformation of $E_\lambda$ is called *generalized symmetry* of $\lambda$. A generator of invariance transformations of $E_\lambda$ is an *infinitesimal generator of generalized symmetries* of $\lambda$; we will call it simply *generalized symmetry*. 
5.2 Conserved currents and Jacobi fields

Our aim is now to investigate the possibility of associating conservation laws to Jacobi fields and symmetries of variations. Our results concerning the second variation and Jacobi fields can be applied as follows.

Theorem 5.4 Let $\rho$ be an $n$-form on $J^{r-1}Y$ and $h\rho$ the associated Lagrangian on $J^rY$. Consider two vertical vector fields $\Xi_1$ and $\Xi_2$ on $Y$.

1. Suppose that $\Xi_2$ is a symmetry of the first variation of $\lambda$ generated by $\Xi_1$ and that $\Xi_1$ and $\Xi_2$ satisfy

$$\Xi_2| J_{\Xi_1}(h\rho) = 0,$$

then

$$d_H\varepsilon_{\Xi_2}(L_{\Xi_1}h\rho) = 0. \tag{22}$$

2. Suppose that $\Xi_1$ and $\Xi_2$ are Jacobi fields, i.e.

$$J_{\Xi_i}(h\rho) = 0,$$

then, along solutions,

$$d_H\varepsilon_{\Xi_2}(\Xi_1|E_n(h\rho)) = 0. \tag{23}$$

Proof.

1. It follows immediately from (9).

2. We simply apply (11) and, using the conditions imposed on the fields, we get

$$d_H\varepsilon_{\Xi_2}(\Xi_1|E_n(h\rho)) = [\Xi_2, \Xi_1]|E_n(h\rho),$$

thus we conclude.

Remark 5.5 We stress that (22) can be interpreted as a strong conservation law. On the other hand (23) can be seen as a weak conservation law associated with Jacobi fields. We can conclude that, taking two Jacobi fields $\Xi_1$ and $\Xi_2$ and working along solutions, $d_H\varepsilon_{\Xi_2}(\Xi_1|E_n(h\rho))$ and $d_H\varepsilon_{\Xi_2}(d_H\varepsilon_{\Xi_1}(h\rho))$ vanish separately.

Remark 5.6 We note that if the hypothesis of Theorem 5.4 hold along solutions, then (22) and (23) hold along solutions too, as we can see in a completely analogous manner.
5.3 Symmetries and Jacobi fields of higher order variations

We observe that the Jacobi equation for variations of $h\rho$ can be expressed in terms of $h\rho$. In fact, just using the exactness of the representation sequence, we have

$$E_n(\Xi|E_n(\text{L}_{J^{r+1}}\Xi_s\ldots L_{J^{r+1}}h\rho)) = E_n(\Xi|E_n(\text{E}_{s+1} E_n(\ldots \Xi_1|E_n(h\rho)\ldots)))\,.$$

The application of Theorem 5.4 to an iterated variation of a Lagrangian gives results that are relevant for the Lagrangian itself; in fact, using (13) we can relate the Noether current of the $s$-th variation with Noether currents of lower order variations. More precisely, we can state the following important result.

**Theorem 5.7** If we take a symmetry of an $(l-1)$-th variation of $h\rho$ and we suppose that the $s$-th variation ($s < l$) is taken with respect to a Jacobi field of the $(s-1)$-th variation, then

$$d_H\epsilon_{\Xi_l}\ldots d_H\epsilon_{\Xi_{s+1}}(L_{J^{r+1}}\Xi_s\ldots L_{J^{r+1}}h\rho) = 0\,.$$

**Proof.** Actually we have

$$d_H\epsilon_{\Xi_l}(L_{J^{r+1}}\Xi_{l-1}\ldots L_{J^{r+1}}h\rho) = 0\,,$$

with some terms that vanish separately. In fact, applying the definition of Jacobi field, we get

$$\Xi_{l-1}|E_n(\Xi_{l-2}|E_n(\ldots \Xi_2|E_n(\Xi_1|h\rho)\ldots)) = 0\,,$$

$$d_H\epsilon_{\Xi_{l-1}}(L_{J^{r+1}}\Xi_{l-2}\ldots L_{J^{r+1}}h\rho) = 0\,,$$

...  

$$d_H\epsilon_{\Xi_{l}}\ldots d_H\epsilon_{\Xi_{s+2}}(\Xi_{s+1}|E_n(\ldots (\Xi_1|h\rho)\ldots)) = 0\,.$$

Then the result stated follows easily using (13) (Theorem 2.4).

**Remark 5.8** The previous result is a strong conservation law: the conserved current is the $(l-s-1)$-th variation of the horizontal differential of the Noether current for the $s$-th variation of $h\rho$. The result is not trivial because we are not assuming that $\Xi_{s+1}$ is a symmetry of the $s$-th variation.
5.4 Yang-Mills theories on a Minkowskian background: weak conservation law associated with Jacobi fields

As an example of application of the arguments discussed in this section, we calculate the current \( \epsilon_\Xi(\Xi|E_n(\lambda_{YM})) \) for two given Jacobi fields \( \Xi \) and \( \tilde{\Xi} \) along an extremal of the Yang-Mills Lagrangian \( \lambda_{YM} \) on a Minkowskian background (see the previous section). Being the vector fields vertical, the current has the form

\[
\epsilon_\Xi(\Xi|E_n(\lambda_{YM})) = -J_3^3 \tilde{\Xi}| p_1 R \left( d(\Xi|E_n(\lambda_{YM})) \right).
\]

Having denoted the components of the principal connection by \( \omega^A_\mu \), in order to avoid confusion we will use in this example the notation \( \theta^A_\mu, \theta^A_{\mu,\nu}, \theta^A_{\mu,\nu,\rho}, \ldots \) to indicate generators of contact forms. We apply Remark 1.2 and formula (7) to the form

\[
d(\Xi|E_n(\lambda_{YM})) = \sum_{|J|=0}^{|I|=0} \theta^A_\mu,J \wedge \eta^A_{\mu,I} + \sum_{|J|=1}^{|I|=0} \theta^A_\mu,J \wedge ds + \sum_{|J|=2}^{|I|=0} \theta^A_\mu,J \wedge ds,
\]

where we recall that \( E^\nu_B \) denotes the coordinate expression of the Euler–Lagrange form. We need to rewrite this form as \( \sum_{|I|=0}^{|I|=2} d_I(\omega^A_\mu \wedge \zeta_A^{\mu,I}) \) with

\[
\zeta_A^{\mu,I} = \sum_{|J|=0}^{|I|=2} (-1)^{|J|} \binom{|I|+|J|}{|J|} d_J \eta_A^{\mu,J}_I.
\]

Actually, we are interested only in \( \zeta_A^{\mu,I} \) for \( |I|=1 \) or \( |I|=2 \); the case \( |I|=0 \) gives no contribution to the residual operator. We have, for \( |I|=1 \),

\[
\zeta_A^{\mu,I} = \eta_A^{\mu,I} - 2d_\tau \eta_A^{\mu,\tau I},
\]

and, for \( |I|=2 \)

\[
\zeta_A^{\mu,I} = \eta_A^{\mu,I}.
\]
Then
\[
\sum_{|I|=1}^2 d_I (\theta^A \wedge \zeta^{\mu,I}_A) = d_\xi \left[ \theta^Z_\rho \wedge \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi} - 2d_\tau \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) \right) ds \right] + \\
+ d_\tau d_\xi \left[ \theta^Z_\rho \wedge \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) ds \right],
\]
that can be rewritten as
\[
d_\xi \left\{ \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi} - d_\tau \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) \right\} \theta^Z_\rho + \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) \theta^Z_{\rho,\tau} \wedge ds,
\]
Consequently
\[
\mathcal{R} (d (\Xi \mid E_n (\lambda_{YM}))) = - \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi} - d_\tau \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) \right) \theta^Z_\rho \wedge ds_\xi + \\
- \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) \theta^Z_{\rho,\tau} \wedge ds_\xi,
\]
and the current is
\[
\epsilon_\Xi (\Xi \mid E_n (\lambda_{YM})) = \left[ \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi} \Xi^Z_\rho - d_\tau \left( \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} \right) \Xi^Z_\rho + \Xi^B_\nu \frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} d_\tau \Xi^Z_\rho \right] ds_\xi,
\]
where
\[
\frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi} = [\delta_{BA}c_{ZD}^{A} \omega^{D}_\sigma (\eta^{\rho \xi} \eta^{\sigma \rho} - \eta^{\sigma \xi} \eta^{\rho \rho}) + \delta_{ZAC}^{B} c_{BC}^{A} \omega^{C}_\sigma (\eta^{\rho \rho} \eta^{\sigma \eta} - \eta^{\rho \xi} \eta^{\xi \eta})],
\]
\[
\frac{\partial E^\nu_B}{\partial \omega^{Z}_\rho,\xi,\tau} = \delta_{BZ} (\eta^{\xi \tau} \eta^{\rho \omega} - \eta^{\rho (\tau} \eta^{\xi) \omega}).
\]
(the brackets on the superscripts denote symmetrization). Substituting we can write, for the coefficients of the current,
\[
\left( \Xi^B_\nu \Xi^Z_\rho - d_\sigma \Xi^B_\nu \Xi^Z_\rho \right) (\eta^{\xi \tau} \eta^{\rho \omega} - \eta^{\rho (\tau} \eta^{\xi) \omega}) \delta_{BZ} + \\
+ \Xi^B_\nu \Xi^Z_\rho \delta_{BA}c_{ZD}^{A} \omega^{D}_\sigma (\eta^{\sigma \rho} \eta^{\xi \rho} - \eta^{\sigma \xi} \eta^{\rho \rho}) + \Xi^B_\nu \Xi^Z_\rho \delta_{ZAC}^{B} c_{BC}^{A} \omega^{C}_\sigma (\eta^{\rho \rho} \eta^{\sigma \eta} - \eta^{\rho \xi} \eta^{\xi \eta}).
\]
Denoting with square brackets the anti-symmetrization, we can formulate this as
\[ \eta^{[\rho \sigma]} \delta_{BA} c^A_{ZD} \omega^D_\sigma \left( \Xi^B \tilde{\Xi} Z^\rho \delta_{BZ} + d_{\sigma} \Xi^B \tilde{\Xi} Z^\rho \delta_{BZ} + \Xi^B \tilde{\Xi} Z^\rho \delta_{Z \sigma} c^A_{BZ} \omega^D_\sigma - \Xi^B \tilde{\Xi} Z^\rho \delta_{BA} c^A_{ZD} \omega^D_\sigma \right). \]

In conclusion, the current is
\[ \epsilon_{\Xi} (\Xi \big| E_n (\lambda_{YM})) = \left[ \eta^{[\rho \sigma]} \delta_{BA} c^A_{ZD} \omega^D_\sigma \left( \Xi^B \tilde{\Xi} Z^\rho - \Xi^Z \tilde{\Xi} B^\rho \right) + (\eta^{[\rho \sigma]} \delta_{BA} c^A_{ZD} \omega^D_\sigma \left( \Xi^B \tilde{\Xi} Z^\rho \delta_{Z \sigma} c^A_{BZ} \omega^D_\sigma - \Xi^B \tilde{\Xi} Z^\rho \delta_{BA} c^A_{ZD} \omega^D_\sigma \right) \right] ds_\xi. \] (24)

Notice that if \( \Xi = \tilde{\Xi} \) the current vanishes identically.

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A Appendix

A.1 (Higher) variations and Lie derivatives with respect to projectable vector fields

In the framework of jet prolongations of a fibered manifold \((Y, \pi, X)\) is given a geometric notion of Lagrangian either as a bundle morphism or, equivalently, as an horizontal \(n\)-form.

- **Definition** Let \((Y, \pi, X)\) be a fibered manifold. A Lagrangian of order \(r\) on \(Y\) is a fibered morphism over the identity

\[ J^r Y \xrightarrow{\lambda} \Lambda^n (T^* X) \]

\[ X \]

\[ \]
Notice that the previous definition is equivalent to ask $\lambda$ to be an horizontal $n$-form (called a Lagrangian form) over $X$ defined on $J^rY$. In a system of coordinates, we have by definition $\lambda = \mathfrak{L}(x^i, y^\sigma, y^\sigma_I)\,ds$ where $ds$ is the local volume form on $X$ and $\mathfrak{L}$ is called Lagrangian density.

**Definition** An $n$-region $D$ is a compact embedded $n$-dimensional submanifold with boundary $\partial D$ that is an embedded compact $(n-1)$-dimensional submanifold of $X$. Consider a section $\gamma$ of $\pi$ (a configuration). The action of $\gamma$ over $D$ associated with $\lambda$ is denoted by $A_D[\gamma]$ and defined as

$$A_D[\gamma] = \int_D \lambda \circ j^r\gamma,$$

where $\lambda \circ j^r\gamma = (j^r\gamma)^*(\lambda) = (\mathfrak{L} \circ j^r\gamma)\,ds$.

Varying the section $\gamma$, we get a functional called the action functional over $D$ associated with $\lambda$. If we choose an open set $V \subseteq Y$ and an arbitrary $n$-form $\rho \in \Omega^n_r(V)$, thanks to the decomposition $(\pi^{r+1,r})^*(\rho) = h\rho + p_1\rho + \cdots + p_q\rho$, we can set

$$A_D[\gamma] = \int_D (j^r\gamma)^*(\rho) = \int_D (j^{r+1}\gamma)^*((\pi^{r+1,r})^*(\rho)) = \int_D (j^{r+1}\gamma)^*((h\rho)).$$

We introduce a suitable notion of variation by which we handle different aspects of the theory in a unified way; see e.g. [24, 32, 47]. First, we shall define variations of sections.

**Definition** Consider an open subset $U \subseteq X$ and a section $\gamma$ of $\pi$ defined on $U$. Take an integer $s \geq 0$. An $s$-parameters variation of $\gamma$ is a map $\Gamma : I^s \times X \to Y$, where $I^s$ denotes the $s$-cube with side $][-1,1[$, such that

- if $i_X$ denote the inclusion $X \to I^s \times X$, then $\Gamma \circ i_X|_U : U \to Y$ is a section;
- we have $\Gamma(0, \ldots, 0, x) = \gamma(x)$ for any $x \in U$. 


Let now $\Xi_1, \ldots, \Xi_s$ be vertical vector fields on $Y$ such that

\[
\frac{\partial \Gamma(t_1, 0, \ldots, 0)}{\partial t_1} \bigg|_{t_1=0} = \Xi_1 \circ \gamma \\
\frac{\partial \Gamma(t_1, t_2, 0, \ldots, 0)}{\partial t_2} \bigg|_{t_2=0} = \Xi_2 \circ \Gamma(t_1, 0 \ldots 0, x) \\
\vdots
\frac{\partial \Gamma(t_1, \ldots, t_s)}{\partial t_s} \bigg|_{t_s=0} = \Xi_s \circ \Gamma(t_1, t_2, \ldots, t_{s-1}, 0, x).
\]

We say that $\Gamma$ is generated by the variation vector fields $\Xi_1, \ldots, \Xi_s$ and write $\Gamma_{\Xi_1, \ldots, \Xi_s}$. This definition holds true also for projectable vector fields; see [47].

This enables us to suitably define variations of forms along sections.

**Definition** Let $\rho \in \Omega^k_q W$ be a local $q$-form. Consider an $s$-parameters variation $\Gamma$ of a section $\gamma$. The $s$-variation of $\rho$ along $\gamma$ associated with $\Gamma$, denoted by $\delta^s_\Gamma \rho$, is

\[
\delta^s_\Gamma \rho|_{j^r \gamma(x)} = \left. \frac{\partial^s (\rho \circ j^r \Gamma(t_1, \ldots, t_s, x))}{\partial t^1 \cdots \partial t^s} \right|_{t_1=\cdots=t_s=0}.
\]

Variations of forms are Lie derivatives; indeed the following holds true.

**Proposition** Let $\rho \in \Omega^k_q W$ be a local $q$-form and consider an $s$-variation of $\rho$ along $\gamma$ associated with $\Gamma$, where $\Gamma$ is generated by $\Xi_1, \ldots, \Xi_s$. Then

\[
\delta^s_\Gamma \rho|_{\gamma(x)} = L_{j^r \Xi_1} \cdots L_{j^r \Xi_s} \rho \circ j^r \gamma(x).
\]

Moreover, formal variations generated by $\Xi_1, \ldots, \Xi_s$, for short variations, are defined as

\[
\delta_{\Xi_1, \ldots, \Xi_s} \rho = L_{j^r \Xi_1} \cdots L_{j^r \Xi_s} \rho.
\]

**Definition** Let $\Gamma$ be a one parameter variation of a section $\gamma$ and let $\Xi$ be the variation vector field; $t$ will denote the parameter of the flow of $\Xi$. The variation of the action induced by $\Xi$ and evaluated at $\gamma$ is defined as

\[
\delta_\Xi A_D[\gamma] = \frac{d}{dt} \bigg|_{t=0} A_D[\psi_t \circ \gamma] = \frac{d}{dt} \bigg|_{t=0} \int_D \lambda \circ j^r \psi_t \circ j^r \gamma.
\]
The variation of the action can be expressed as

$$\int_D (j^r \gamma)^* (L_{j^r \Xi} \lambda) = \int_D L_{j^r \Xi} \lambda \circ j^r \gamma.$$

Note that the above formula holds true also for projectable vector fields (see, e.g. [47]). We can easily generalize it to iterated variations.

The $s$-th variation of the action generated by $\Xi_1, \ldots, \Xi_s$ at $\gamma$ is given by

$$\delta_{\Xi_1, \ldots, \Xi_s} A_D[\gamma] = \int_D L_{j^r \Xi_1} \ldots L_{j^r \Xi_s} \lambda \circ j^r \gamma.$$

Finally we can give the definition of extremal. If $\Xi$ is a vector field along a local section $\gamma$ defined on $U$, $\Xi$ denote the set $\text{cl}\{x \in U \text{ s.t. } \Xi|_x \neq 0\}$.

where $\text{cl}$ means closure. Recall that it is possible to extend $\Xi$ to a vector field $\tilde{\Xi}$ defined in a neighborhood of $\gamma(U)$.

- Definition A section $\gamma$ defined on $U$ is called extremal of the action functional (or of the action) if, for every $\pi$-vertical vector fields $\Xi$ such that $\Xi \circ \gamma \subseteq D$, it holds

$$\delta_\Xi A_D(\gamma) = \int_D L_{j^r \Xi} \lambda \circ j^r \gamma = 0.$$

A.2 Takens representation of the variational sequence

- If $q = 0$ we define $R_0$ be the identity mapping.

- If $0 < q \leq n$ then for every $q$-form $\rho$ of order $k$, the contact decomposition takes the form

$$\pi_{k+1,k}^* \rho = h\rho + p_1 \rho + \cdots + p_q \rho,$$

and $\Theta^k_q = \Omega^k_{q,c} + (d\Omega^k_{q-1,c})$, where $\Omega^k_{q,c}$ are sheaves of contact forms.

Summarizing, for $q \leq n$ the operator of horizontalization,

$$h : \Omega^k_q \rightarrow \Omega^k_{q,X} \subset \Omega^{k+1}_q, \quad \rho ightarrow h\rho,$$
induces a representation mapping,

\[ R_q : \mathcal{V}_q^k \to \Omega_q^{k+1}, \quad R_q([\rho]) = h\rho. \]

- If \( q = n + l \) for \( l > 1 \) then for every \( q \)-form \( \rho \) of order \( k \), the contact decomposition takes the form

\[ \pi_{k+1, k}^* \rho = p_l \rho + p_{l+1} \rho + \cdots + p_q \rho, \]

Denoted by \( \mathcal{I} \Omega_{n+l}^k \) the image of \( \Omega_{n+l}^k \) by \( \mathcal{I} \), the interior Euler operator

\[ \mathcal{I} : \Omega_{n+l}^k \to \mathcal{I} \Omega_{n+l}^k \subset \Omega_{n+l}^{2k+1}, \quad \rho \to \mathcal{I}(\rho), \]

induces a representation mapping,

\[ R_{n+l} : \mathcal{V}_{n+l}^k \to \Omega_{n+l}^{2k+1}, \quad R_{n+l}([\rho]) = \mathcal{I}(\rho). \]

Every class \([\rho] \in \mathcal{V}_{n+l}^k\) is completely determined by a unique canonical source form, i.e. a form such that \( \rho = \mathcal{I}(\rho) \).

By the representation mappings \( R_q \) we get the representation sequence

\[ 0 \to \mathcal{I} \mathcal{Y} \to R(\mathcal{V}_*^k) \to \mathcal{V}_*^k. \]

We denote by

\[ E_q : R_q(\mathcal{V}_q^k) \to R_{q+1}(\mathcal{V}_{q+1}^k), \quad q \geq 1, \]

the morphisms in the representation sequence and their definition follows by the commutativity of the diagrams

\[
\begin{array}{ccccccccc}
\cdots & \mathcal{E}_{q-1} & \mathcal{V}_q^k & \mathcal{E}_q & \mathcal{V}_{q+1}^k & \mathcal{E}_{q+1} & \cdots \\
R_q & R_q & R_q & R_q & R_q & \cdots \\
\cdots & E_{q-1} & R_q(\mathcal{V}_q^k) & E_q & R_{q+1}(\mathcal{V}_{q+1}^k) & E_{q+1} & \cdots \\
\end{array}
\]

(26)

The representation sequence \( 0 \to \mathcal{I} \mathcal{Y} \to R(\mathcal{V}_*^k) \) is an exact sheaf subsequence of the de Rham sequence [59]. The diagram (26) can be written in the explicit form

\[
\begin{array}{ccccccccc}
\cdots & \mathcal{E}_{n-1} & \Omega_n^k/\Theta_n^k & \mathcal{E}_n & \Omega_{n+1}^k/\Theta_{n+1}^k & \mathcal{E}_{n+1} & \cdots \\
R_n & R_{n+1} & R_{n+1} & R_{n+1} & R_{n+1} & \cdots \\
\cdots & E_{n-1} & \mathcal{I} \Omega_{n+1}^k & E_n & \mathcal{I} \Omega_{n+2}^k & E_{n+1} & \cdots \\
\end{array}
\]

(27)
where \( R_q \), \( 1 \leq q \leq n \), has the meaning of horizontalization \( h \) on classes of order \( k \), and \( R_{n+1} \) is the operator \( \mathcal{I} \) acting on \( p_1 \rho \) where \( \rho \) is of order \( k \). Then since elements of the sheaves \( R_q(\mathcal{V}_q^k) \) are functions for \( q = 0 \), horizontal forms for \( 1 \leq q \leq n \), and canonical source forms for \( q \geq n + 1 \), we have

\[
E_0(f) = hdf, \quad E_q(h\rho) = hdh\rho, \quad 1 \leq q \leq n - 1 \\
E_n(h\rho) = \mathcal{I}(dh\rho), \quad E_{n+1}(\mathcal{I}(\rho)) = \mathcal{I}(d\mathcal{I}(\rho)), \quad l \geq 1
\]

In particular, for \( q = n \) we can write in coordinates \( h\rho = Lds \), and then we get the Euler–Lagrange form by means of the interior Euler operator as an integrability condition for the inverse problem.

Hence, \( E_n : R_n(\mathcal{V}_n^k) \to R_{n+1}(\mathcal{V}_{n+1}^k) \) is the Euler–Lagrange mapping, assigning to every Lagrangian \( \lambda = h\rho \) its Euler–Lagrange form \( E_\lambda = \mathcal{I}(d\lambda) \).

Note that the morphism \( E_{n-1} : R_{n-1}(\mathcal{V}_{n-1}^k) \to R_n(\mathcal{V}_n^k) \) assigns to every horizontal \((n-1)\)-form \( \varphi \) (resp., if \( \dim X = n = 1 \), to a function \( f \)) a Lagrangian \( \lambda = hd\varphi \) (resp. \( \lambda = hdf \)). This is a so-called null-Lagrangian (also called variationally trivial Lagrangian).

In general, for \( q \geq 1 \), elements of \( R_q(\mathcal{V}_q^k) \) belonging to the kernel of the variational morphism \( E_q \) are called variationally trivial.
A.3 Variational derivatives and variations

The Lie derivative of differential forms with respect to prolongations of projectable vector fields preserves the contact structure; this fact allows to define a Lie derivative of classes of forms, a variational Lie derivative, as the equivalence class of the standard Lie derivative. The splitting provided by the interior Euler operator can then be used to get higher order variation formulas. This approach allows to study the structure of higher order variations with a certain detail; this section is devoted to this. A careful study of first order variational Lie derivatives can be found in [9, 59]; we refer also to the classical article [32] and to [25, 24, 26, 27, 61, 62] for topics related to second order variations.

The definition of the formal variation of a form, recalled in the Appendix, justifies our interest in Lie derivatives with respect to prolongations of projectable vector fields. We are then able to introduce variational Lie derivatives and to present an alternative derivation of the first variation formula. Our references are [9] and [59]; for the notion of contact symmetry the reader can see [48, 49, 50].

We explore the relation between Lie derivatives with respect to prolongations of vector fields and the contact structure. This will result in an alternative approach to the first variation formula that can be easily used for the study of higher order variations. The splitting of an \((n + k)\)-form provided by the interior Euler operator plays an important rôle. We recall that, apart form being interpretable as variations (see the Appendix), Lie derivatives are related to invariance transformations. Roughly speaking, a contact symmetry is a vector field that “preserves” the contact structure. In other words, a contact symmetry is a symmetry of the contact ideal. The following lemma characterizes projectable contact symmetries.

- **Lemma** Let \( \Phi \) be a vector field on \( J^rY \) projectable over \( X \). It holds \( \Phi = J^r\Xi \) for a certain projectable vector field \( \Xi \) if and only if for any local system of coordinates \( (x^i, y^\sigma) \) and any multi-index \( J \) such that \( 0 \leq |J| \leq r - 1 \), \( L_{\Phi}\omega_J^\sigma \) is contact.

We know that any jet prolongation of a projectable vector field \( \Xi \) is a contact symmetry; the previous lemma guarantees that a contact symmetry projectable over \( X \) is the prolongation of a projectable vector field \( \Xi \) on \( Y \).

Consider a contact symmetry \( \Phi \) and a \( q \)-form \( \rho \) whose order of contactness is greater than \( k \) (with \( k \leq q \)). It is obvious that \( L_{\Phi}\rho \) has still order of
contactness greater than \( k \); this implies that Lie derivatives with respect to contact symmetries preserve the sheaves \( \Theta^r_q \). Consequently, we can give the following definition.

- **Definition** The *variational Lie derivative* of a class \([\rho] \in \mathcal{V}^r_q\) with respect to a contact symmetry \( \Phi \) is denoted by \( \mathcal{L}_\Phi[\rho] \) and defined as

\[
\mathcal{L}_\Phi[\rho] = [L_\Phi \rho].
\]

Being \( \Phi \) a contact symmetry, the definition is well given. The variational Lie derivative is a natural transformation. Indeed for any \( q \geq 0 \), any \([\rho] \in \mathcal{V}^r_q\) and any contact symmetry \( \Phi \) on \( J^r Y \)

\[
\mathcal{L}_\Phi \mathcal{E}_q([\rho]) = \mathcal{E}_q([\Phi] d\rho).
\]

Note that \( \mathcal{E}_q([\Phi] d\rho) = \mathcal{E}_q(\mathcal{L}_\Phi[\rho]) \), because the variational sequence is a complex.

From now on we will consider only contact symmetries projectable on \( X \), *i.e.* jet prolongations of projectable vector fields on \( Y \). We use the representation of the variational sequence by the interior Euler operator in order to explore the structure of Lie derivatives of representatives of \( n \)-forms, *i.e.* Lagrangians; this means performing a careful investigation of the relation between the interior Euler operator and the Cartan formula. Our discussion follows mainly the papers [9, 59].

We define the contraction of a class \([\rho] \in \mathcal{V}^r_q\) with a prolongation \( J^r \Xi \) as

\[
J^r \Xi [\rho] = [J^s \Xi R_q \rho],
\]

where \( s \) depends on \( R_q \), *i.e.* on the degree \( q \).

We want to assign a differential form to each variational Lie derivative. We could use the standard representation of classes; however, there is a different choice that provides more interesting results. The idea is to follow the definition of the contraction of a prolongation with a class: we introduce an operator \( \tilde{R}_q \) requiring that

\[
\tilde{R}_q \mathcal{L}_J \Xi [\rho] = \tilde{R}_q [L_J \Xi \rho] = L_J \Xi R_q \rho,
\]

where, as before, \( s \) depends on the degree \( q \). More precisely, \( \tilde{R}_q \mathcal{L}_J \Xi [\rho] \) is equal to
• $L_{J^s} \Xi h \rho$, with $s = r + 1$, if $0 \leq q \leq n$
• $L_{J^s} \Xi (\rho)$, with $s = 2r + 1$, if $0 \leq q \leq n$
• $L_{J^s} \Xi \rho$, with $s = r$, if $q \geq M$

It can be shown that the prescriptions

$$\tilde{R}_{q-1} J^s \Xi [\rho] = J^s \Xi R_q [\rho],$$

and

$$\tilde{R}_{q+1} \circ \mathcal{E}_q = d \circ \tilde{R}_q,$$

are sufficient to define such an operator $\tilde{R}_q$.

### A.4 Infinitesimal first variation formula, source forms and Lepage equivalents

With the expression *first variation formula* one usually means a formula for the first variation of the action that leads to the well known Euler–Lagrange equations; getting this formula usually means splitting the integral in a suitable way. Here we will follow mainly the geometric approach of [47]. By using the standard *Cartan formula for the Lie derivative* we shall handle suitably the variation of a Lagrangian: we obtain then an *infinitesimal first variation formula*, concerned with variations of forms rather than of integrals.

Let $V$ be a chart domain in $Y$. A *Lepage equivalent* for a Lagrangian $\lambda \in \Omega^r_{n,x} V$, where $\Omega^r_{n,x} V$ is the space of horizontal forms defined on $V^r$, is a certain $\rho \in \Omega^s_n V$ with order of contactness $\leq 1$ and such that

- $h \rho = \lambda$ up to pull-backs;
- $p_1 d \rho$ is a source form (of order $r + 1$).

In particular, if a Lagrangian in a fibered chart $(V, \phi)$ is written as

$$\lambda = \mathcal{L} ds$$

then

$$\Theta_{\Xi} = \mathcal{L} ds + \sum_{k=0}^{r-1} \left( \sum_{l=0}^{r-1-k} (-1)^l d_{p_1} \ldots d_{p_l} \frac{\partial \Sigma}{\partial y_{j_1 \ldots i_k p_1 \ldots p_l i}} \right) \omega_{j_1 \ldots j_k}^\sigma \wedge ds_i$$
is called **principal Lepage equivalent** of \( \lambda \) in the chosen chart.

An important example, well known in the literature, is the Poincaré–Cartan form; Lepage equivalents are a generalization of the Poincaré–Cartan form in mechanics (see e.g. [51, 53]).

A generic Lepage equivalent \( \Theta \) of \( \lambda \) is given in coordinates

\[
\Theta = \mathfrak{L}ds + \sum_{k=0}^{r-1} \sum_{s=0}^{r-k-1} \left( (-1)^s d_{i_1...i_s} \frac{\partial \mathfrak{L}}{\partial y_{j_1...j_k i_1...i_s}} \right) \omega_{j_1...j_k}^s \wedge ds_i + \nu
\]

where \( \nu \) is any contact form such that \( p_1 d\nu = 0 \). It can be built a Lepage equivalent which is global; see also [43].

The form \( p_1 d\Theta \), where \( \Theta \) is a Lepage equivalent of a Lagrangian \( \lambda \), is called **Euler–Lagrange form of** \( \lambda \) and denoted by \( E_\lambda \). Its coefficients are called **Euler–Lagrange expressions**. In a chart we have

\[
E_\lambda = \mathcal{E}_\sigma(\mathfrak{L}) \omega^\sigma \wedge ds,
\]

where

\[
\mathcal{E}_\sigma(\mathfrak{L}) = \sum_{|\sigma|=0}^r (-1)^{|\sigma|} d_{\sigma} \frac{\partial \mathfrak{L}}{\partial y_{\sigma}}, \quad (28)
\]

Since a Lepage equivalent can be chosen to be global, the Euler–Lagrange form is global too. Thus, a map \( \lambda \rightarrow E_\lambda \), called **Euler–Lagrange mapping**, is defined.

We stress that, if \( \rho \) is an \( n \)-form with associated Lagrangian \( h\rho \) and taken a Lepage equivalent \( \Theta \) of \( h\rho \) (also called Lepage equivalent of \( \rho \) for short), then \( E_{h\rho} = p_1 d\Theta \).

The definition of the Euler–Lagrange mapping by a Lepage equivalent shows its relationship with the exterior derivative, that plays a major rôle for the study of calculus of variation. Considering the decomposition of \( p_1 d\Theta \) by the Interior Euler operator and the residual operator, we see that a Lepage equivalent is such that the part of the splitting containing the residual operator of its exterior differential vanishes, i.e. \( p_1 dp_1 R(d\Theta) = 0 \).

Take a Lepage equivalent \( \Theta \) of \( \lambda \) of order \( s \). We have (up to pull-backs)

\[
hL_{J^r \in} \Theta = L_{J^r \in} h\Theta = L_{J^r \in} \lambda \text{ and } p_1 dL_{J^r \in} \Theta = L_{J^r \in} p_1 d\Theta \text{ because Lie derivatives with respect to prolongations preserve the contact structure. This implies that } p_1 dL_{J^r \in} \Theta \text{ is a source form. Then } L_{J^r \in} \Theta \text{ is a Lepage equivalent of } L_{J^r \in} \lambda. \text{ We therefore conclude that } E_{L_{J^r \in} \lambda} = p_1 dL_{J^r \in} \Theta = L_{J^r \in} E_\lambda.
\]

**Theorem (Noether I.)** Let \( \lambda \in \Omega^r_{n,X} \) be a Lagrangian, \( \Theta \) a Lepage equivalent of order \( s \) and \( \gamma \) an extremal. Any symmetry of \( \lambda \) can be
associated with a weak conservation law along the extremal, namely
\[ d((J^s \gamma)^*(J^s \Xi | \Theta)) = 0. \]

- **Theorem** Let \( \lambda \) be a Lagrangian of order \( r \), \( \gamma \) an extremal and \( \Xi \) a generalized symmetry. Around any point in \( Y \) exist a fibered chart \( (V, \phi) \) and a \((n - 1)\)-form \( \beta \) defined in \( V^{r-1} \) such that, on \( \pi(V) \) (up to pull-backs)
\[ d(J^{2r-1} \gamma)^*(J^{2r-1} \Xi | \Theta _\lambda - \beta) = 0, \]
where \( \Theta _\lambda \) is the local principal Lepage equivalent in the chosen chart.

The above theorem provides us with an equivalent definition of generalized symmetry for \( \lambda \): it can be characterized as a vector field such that (up to pull-backs) \( L_{\mu \Xi \lambda} = d_H \beta \), for some horizontal \((n - 1)\)-form \( \beta \). The weakly conserved current of the previous theorem, associated with a generalized symmetry \( \Xi \), is called a *Noether-Bessel-Hagen current* \[9, 23, 30].

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