MOST IRREDUCIBLE REPRESENTATIONS OF THE 3-STRING BRAID GROUP

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1. Introduction

With \textit{iss}_n B_3 we denote the affine variety of all isomorphism classes of semi-simple \(n\)-dimensional representations of the 3-string braid group

\[ B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \]

It is well-known, see for example [8], [4] and [5], that any irreducible components \(X_\sigma\) of \textit{iss}_n B_3 containing a Zariski open subset of irreducible representations is determined by a dimension-vector \(\sigma = (a, b; x, y, z)\) satisfying

\[ n = a + b = x + y + z \quad \text{and} \quad x = \max(x, y, z) \leq b = \min(a, b) \]

with \(\dim X_\sigma = n_\sigma = 1 + n^2 - (a^2 + b^2 + x^2 + y^2 + z^2)\). As \(B_3\) is of wild representation type one cannot expect a full classification of all its finite dimensional irreducible representations. In fact, such a classification is only known for \(n \leq 5\) by work of Imre Tuba and Hans Wenzl [7]. Still, one can aim to describe ‘most’ irreducible representations by constructing for each component \(X_\sigma\) an explicit minimal (étale) rational map

\[ f_\sigma : \mathbb{A}^n_\sigma \longrightarrow X_\sigma \longrightarrow \text{iss}_n B_3 \]

having a Zariski dense image. Such rational dense parametrizations were constructed in [4] for all components when \(n < 12\). The purpose of the present paper is to extend this to all finite dimensions \(n\).

2. Linear systems and some rational quiver settings

A linear control system \(\Sigma\) is determined by the system of linear differential equations

\[
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\]

where \(\Sigma = (A, B, C) \in M_n(\mathbb{C}) \times M_{m \times n}(\mathbb{C}) \times M_{p \times n}(\mathbb{C})\) and \(u(t) \in \mathbb{C}^m\) is the control at time \(t\), \(x(t) \in \mathbb{C}^n\) is the state of the system and \(y(t) \in \mathbb{C}^p\) its output. Equivalent control systems differ only by a base change in the state space, that is \(\Sigma' = (A', B', C')\) is equivalent to \(\Sigma\) if and only if there exists a \(g \in GL_n(\mathbb{C})\) such that

\[ A' = gAg^{-1}, \quad B' = gB \quad \text{and} \quad C' = Cg^{-1} \]

\(\Sigma\) is said to be \textit{canonical} if the matrices

\[ c_\Sigma = [B \ AB \ A^2B \ldots \ A^{n-1}B] \quad \text{and} \quad o_\Sigma = [C \ CA \ CA^2 \ldots \ CA^{n-1}] \]

are of maximal rank.
Michiel Hazewinkel proved in [1] that the moduli space $\text{sys}_{m,n,p}$ of all such canonical linear systems is a smooth rational quasi-affine variety of dimension $(m+p)n$. We will give another short proof of this result and draw some consequences from it (see also [6]).

Consider the quiver setting with $m$ arrows $\{b_1, \ldots, b_m\}$ from left to right and $p$ arrows $\{c_1, \ldots, c_p\}$ from right to left.

To a system $\Sigma = (A, B, C)$ we associate the quiver-representation $V_\Sigma$ by assigning to the arrow $b_i$ the $i$-th column $B_i$ of the matrix $B$, to the arrow $c_j$ the $j$-th row $C_j$ of $C$ and the matrix $A$ to the loop. As the base change group $\mathbb{C}^* \times GL_n$ acts on these quiver-representations by

$$(\lambda, g).V_\Sigma = (gAg^{-1}, gB_1\lambda^{-1}, \ldots, gB_m\lambda^{-1}, \lambda C_1 g^{-1}, \ldots, \lambda C_p g^{-1})$$

with the subgroup $\mathbb{C}^*(1, 1_n)$ acting trivially, there is a natural one-to-one correspondence between equivalence classes of linear systems $\Sigma$ and isomorphism classes of quiver-representations $V_\Sigma$. Under this correspondence it is easy to see that canonical systems correspond to simple quiver-representations, see [6, Lemma 1]. Hence, the moduli-space $\text{sys}_{m,n,p}$ is isomorphic to the Zariski-open subset of the affine quotient-variety classifying isomorphism classes of semi-simple quiver-representations, proving smoothness, quasi-affineness as well as determining the dimension by general results, see for example [3].

Lemma 1. A generic canonical system $\Sigma$ is equivalent to a triple $(A_n, B_{nm}, C_{pn})$ with

$$A_n = \begin{bmatrix} 0 & 0 & \ldots & x_n \\ 1 & 0 & \ldots & x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & 0 & \ldots & x_2 \\ 1 & \end{bmatrix}, \quad B_{nm} = \begin{bmatrix} 1 & b_{12} & \ldots & b_{1m} \\ 0 & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \ldots & b_{nm} \end{bmatrix}$$

that is, where $A_n$ is a companion $n \times n$-matrix, $B_{nm}$ is the generic $n \times m$-matrix with fixed first column and $C_{pn}$ a generic $p \times n$-matrix.

Proof. A generic representation of the quiver-setting

$$\xymatrix{ & v \ar[dl] \ar[dr]_{A_n} \\
1 & & A}$$

will have the property that $v$ is a cyclic-vector for the matrix $A$, that is, $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ are linearly independent. But then, performing a base-change we get a representation of the form

$$\xymatrix{ & [1 \ 0 \ \ldots \ 0]^t \ar[dl] \ar[dr]_{A_n} \\
1 & & A}$$

where $A_n$ is a companion matrix whose $n$-th column expresses the vector $-A^n v$ in the new basis. As the automorphism group of this representation is reduced to $\mathbb{C}^*(1, 1_n)$, any general representation $V_\Sigma$ is isomorphic to one with
\[ B_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T, \quad A = A_n \text{ and the other columns of } B \text{ and all rows of } C \text{ generic vectors.} \]

**Lemma 2.** The following representations give a rational parametrization of the isomorphism classes of simple representations of these quiver-settings

\[
R_k : \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T \xrightarrow{A_k} \begin{bmatrix} y_1 & y_2 & \ldots & y_k \end{bmatrix} \quad \text{and} \quad S_k : \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T \xrightarrow{A_k^\dagger} \begin{bmatrix} 0 \\ 1_{k-1} \end{bmatrix}
\]

where \( A_k \) (resp. \( A_k^\dagger \)) is the generic \( k \times k \) companion matrix (resp. the reduced \( k-1 \times k \) companion matrix)

\[
A_k = \begin{bmatrix}
0 & 0 & \ldots & x_k \\
1 & 0 & \ldots & x_{k-1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & x_2 \\
& & 0 & x_1 \\
\end{bmatrix} \quad \text{and} \quad A_k^\dagger = \begin{bmatrix}
1 & 0 & \ldots & x_{k-1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & x_2 \\
& & 1 & x_1 \\
\end{bmatrix}
\]

**Proof.** By invoking the first fundamental theorem of \( GL_n \)-invariants (see for example [2] Thm. II.4.1) we can in case \( R_k \) eliminate the base-change action in the right-most vertex, giving a natural one-to-one correspondence between isoclasses of representations

\[
\begin{array}{ccc}
\begin{array}{c}
1 \\
\uparrow v \\
\downarrow w^r \\
\end{array} & \xrightarrow{X} & \begin{array}{c}
2 \\
\uparrow \tau \\
\downarrow Y \\
\end{array} & \leftrightarrow & \begin{array}{c}
1 \\
\uparrow v \\
\downarrow w^r \\
\end{array} & \xrightarrow{Y.X} & \begin{array}{c}
2 \\
\uparrow \tau \\
\downarrow Y \\
\end{array}
\end{array}
\]

and hence the claim follows from the previous lemma. As for case \( S_k \) we can again apply the first fundamental theorem for \( GL_n \)-invariants, now with respect to the base-change action in the middle vertex, to obtain a natural one-to-one correspondence between isoclasses of representations

\[
\begin{array}{ccc}
\begin{array}{c}
1 \\
\uparrow v \\
\downarrow w^r \\
\end{array} & \xrightarrow{X} & \begin{array}{c}
3 \\
\uparrow \tau \\
\downarrow Y \\
\end{array} & \leftrightarrow & \begin{array}{c}
1 \\
\uparrow v \\
\downarrow w^r \\
\end{array} & \xrightarrow{X.Y} & \begin{array}{c}
2 \\
\uparrow \tau \\
\downarrow Y \\
\end{array}
\end{array}
\]

and again the claim follows from the previous lemma, taking into account the extra free loop in the left-most vertex, which corresponds to \( y_1 \). \qed
Lemma 3. The following representations give a rational parametrization for the isomorphism classes of simple representations of the quiver-setting

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
y_1 & y_2 & \cdots & y_k
\end{pmatrix} \rightarrow \begin{pmatrix}
A^i_k \\
0 & 1_{k-1}
\end{pmatrix}
\]

where \( B \) is a generic \( k-1 \times k-1 \) matrix and, as before, \( A^i_k \) is a reduced generic companion matrix.

Proof. Forgetting the end-vertices (and maps to and from them) we are in the situation of the previous lemma. For general values these are simple quiver-representations and hence the automorphism group is reduced to \( \mathbb{C}^\ast(1, 1_k, 1_k^{-1}) \). If we now add the end vertices we can use base-change in them to force one of the two arrows to be the identity map, leaving the remaining map generic. Alternatively, we can use the first fundamental theorem of \( GL_n \)-invariants as before, to obtain the claimed result. \( \square \)

3. Luna slices and the action map

We quickly recall the basic strategy of \([4]\). As the central generator \( c = (\sigma_1 \sigma_2)^3 = (\sigma_1 \sigma_2 \sigma_1)^2 \) of \( B_3 \) acts via a scalar \( \lambda \in \mathbb{C}^\ast \) on any irreducible \( B_3 \)-representation it suffices to study irreducible representations of the quotient group \( B_3/\langle c \rangle \simeq C_2 \ast C_3 = \langle s, t \mid s^2 = e = t^3 \rangle \) where \( s \) is the class of \( \sigma_1 \sigma_2 \sigma_1 \) and \( t \) that of \( \sigma_1 \sigma_2 \). Note that this quotient-group is isomorphic to the modular group \( \text{PSL}_2(\mathbb{Z}) \). The action of \( s \) and \( t \) on a finite dimensional \( C_2 \ast C_3 \)-representation \( V \) induce two decompositions of \( V \) into eigen-spaces

\[ V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2} \]

where \( \rho \) is a primitive 3-nd root of unity. Hence \( V \) is fully determined by a base-change matrix \( B = (B_{ij})_{1 \leq i, j \leq 2} \) from a fixed basis compatible with the first decomposition to a fixed basis compatible with the second, that is by a representation of the quiver-setting

Bruce Westbury observed in \([8]\) that under this correspondence isoclasses of \( C_2 \ast C_3 \)-representations coincide with isoclasses of quiver-representations, and that irreducible group-representations correspond to stable quiver-representations wrt. the stability structure \( \theta = (-1, -1; 1, 1, 1) \). It then follows from this stability condition that the dimension-vectors \( \sigma = (a, b; x, y, z) \) containing a Zariski open subset of irreducible \( n \)-dimensional \( C_2 \ast C_3 \)-representations must satisfy \( a + b = n = x + y + z \) as well as \( \max(x, y, z) \leq \min(a, b) \).
Working backwards, we obtain for each \( \lambda \in \mathbb{C}^* \) an irreducible \( B_3 \)-representation determined by the above base-change matrix \( B \) via

\[
\begin{align*}
\sigma_1 & \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^{2_1} y & 0 \\ 0 & 0 & \rho^{1_z} \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 0 & \rho^{1_z} \end{bmatrix} \\
\sigma_2 & \mapsto \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^{2_1} y & 0 \\ 0 & 0 & \rho^{1_z} \end{bmatrix} B
\end{align*}
\]

Observe that in lifting irreducibles from \( C_2 \times C_3 \) to \( B_3 \) we get an action by multiplication of 6-th roots of unity on the components which contain irreducibles, which accounts for the fact that the irreducible components \( X_\sigma \) containing irreducible \( B_3 \)-representations are classified by the dimension vectors \( \sigma = (a, b; x, y, z) \) as above with the extra condition that \( b = \min(a, b) \) and \( x = \max(x, y, z) \). We will now construct special semi-simple \( C_2 \times C_3 \)-representations \( M_0 \) in every component, with all its irreducible factors being 1- or 2-dimensional.

There are 6 one-dimensional irreducible \( C_2 \times C_3 \)-representations, corresponding to the quiver-representations \( S_i \) for \( 1 \leq i \leq 6 \):

and three one-parameter families of two-dimensional irreducibles corresponding to the quiver-representations \( T_i(q) \) for \( q \neq 0, 1 \) and \( 1 \leq i \leq 3 \)

The semi-simple representation

\[
M_0 = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6} \oplus T_1(q)^{\oplus b_0} \oplus T_2(q)^{\oplus b_3} \oplus T_3(q)^{\oplus b_2}
\]

clearly belongs to the component \( X_\sigma \) with dimension vector \( \sigma = (a, b; x, y, z) \) where

\[
\begin{align*}
a &= a_1 + a_3 + a_5 + b_0 + b_3 \\
b &= a_2 + a_4 + a_6 + b_0 + b_3 \\
x &= a_1 + a_4 + b_0 + b_3 \\
y &= a_2 + a_5 + b_0 + b_3 \\
z &= a_3 + a_6 + b_0 + b_3
\end{align*}
\]
and is fully determined by the base-change matrix $B_0$ with block-form as above

\[
\begin{bmatrix}
1_{a_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_{1b_a} & 0 & 0 \\
0 & 0 & 0 & 0 & q_{1b_d} & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{b_d} \\
0 & 0 & 0 & 0 & 0 & 1_{b_d}
\end{bmatrix}
\begin{bmatrix}
1_{a_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1_{a_5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{b_a} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{b_a}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

We will now determine the structure of the base-change matrices $B$ of isoclasses of $C_2 \ast C_3$-representations $M$ in a Zariski open neighborhood of $[M_0]$ in $\text{iss}_\sigma C_2 \ast C_3$.

As $M_0$ is semi-simple, its isomorphism class forms a Zariski closed orbit $\text{O}(M_0)$ in the smooth irreducible component $\text{rep}_\sigma C_2 \ast C_3$ under the action of $GL(\sigma) = GL_a \times GL_b \times GL_c \times GL_d$. The stabilizer subgroup $\text{Stab}(M_0)$ is the automorphism group and is the subgroup of $GL(\sigma)$ we will denote by $GL(\tau) = GL_{a_1} \times GL_{a_2} \times GL_{a_3} \times GL_{a_4} \times GL_{a_5} \times GL_{a_6} \times GL_{b_a} \times GL_{b_b} \times GL_{b_c} \times GL_{b_d}$. The normal space to the orbit $\text{O}(M_0)$ can be identified as $GL(\tau)$-representation with the vectorspace of self-extensions $\text{Ext}^1_{C_2 \ast C_3}(M_0, M_0)$, see for example [2] II.2.7. The Luna slice theorem, see for example [2] §4.2, asserts that the action map

\[
GL(\sigma) \times^{GL(\tau)} \text{Ext}^1_{C_2 \ast C_3}(M_0, M_0) \longrightarrow \text{rep}_\sigma C_2 \ast C_3
\]

sending the class of $(\gamma, \vec{n})$ in the associated fibre bundle to the $C_2 \ast C_3$-representation $g.(M + \vec{n})$ is a $GL(\sigma)$-equivariant étale map with a Zariski dense image. Taking $GL(\sigma)$-quotients on both sides we obtain an étale map

\[
\text{Ext}^1_{C_2 \ast C_3}(M_0, M_0)/GL(\tau) \longrightarrow \text{iss}_\sigma C_2 \ast C_3
\]

with a Zariski dense image. The crucial observation to make is that it follows from the theory of local quivers, [2] §4.2], that as a $GL(\tau)$-representation $\text{Ext}^1_{C_2 \ast C_3}(M_0, M_0)$ is isomorphic to $\text{rep}_\tau Q$ for the quiver $Q$ having 9 vertices (one for each of the distinct simple factors of $M_0$) and having as many directed arrows from the vertex corresponding to the simple factor $S$ to that of the simple factor $T$ as is the dimension of the space $\text{Ext}^1_{C_2 \ast C_3}(S, T)$. This then allows to identify the quotient variety $\text{Ext}^1_{C_2 \ast C_3}(M_0, M_0)/GL(\tau)$ with the affine variety $\text{iss}_\tau Q$ whose points are the isoclasses of semi-simple representations of $Q$ of dimension-vector $\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_a, b_b, b_c)$, and the action map induces an étale map with dense image

\[
\text{iss}_\tau Q \longrightarrow \text{iss}_\sigma C_2 \ast C_3
\]

Computing the normal space to the orbit $\text{O}(M_0)$ as in the proof of [4] Thm. 4] but for the more complicated representation $M_0$ one obtains that the sub quiver of $Q$ on the 6 vertices corresponding to the 1-dimensional simple components $S_1, \ldots, S_6$
coincides with that of [H], that is corresponds to the quiver-setting

The additional quiver-setting depending on the 3 vertices corresponding to the 2-dimensional simple factors $T_1(q)$, $T_2(q)$ and $T_3(q)$ can be verified to be

which concludes the proof of the following:

**Theorem 1.** The étale action map $GL(\sigma) \times GL(\tau) \rightarrow \text{rep}_\sigma Q \rightarrow \text{rep}_\tau C_2 \ast C_3$ sends a $\tau$-dimensional $Q$-representation to the $C_2 \ast C_3$-representation determined by the base-change matrix $B$

| 1_{a_1} 0 0 0 | 0 0 | 0 0 1_{a_4} 0 0 0 | 0 0 |
|----------------|-----|-----------------|-----|
| 0 C_{34} C_{34} 0 0 D_{a_4} | q_{1_{b_{1_\alpha}}} + E_{a_1} q_{1_{b_{1_\beta}}} F_{a_{1_\beta}} | 0 | 0 |
| 0 D_{3_{b_1}} 0 0 0 | q_{1_{b_{1_\alpha}}} + E_{a_1} q_{1_{b_{1_\beta}}} F_{a_{1_\beta}} | 0 0 |
| C_{1_{2_\beta}} 0 0 0 | D_{a_2} 0 | 0 0 1_{a_4} 0 0 0 | 0 0 |
| 0 0 1_{a_{1_\beta}} 0 0 0 | F_{a_{1_\beta}} q_{1_{b_{1_\alpha}}} + E_{a_1} q_{1_{b_{1_\beta}}} F_{a_{1_\beta}} | 0 | 0 |
| 0 0 0 0 0 | F_{a_{1_\gamma}} q_{1_{b_{1_\alpha}}} + E_{a_1} q_{1_{b_{1_\beta}}} F_{a_{1_\beta}} | 0 | 0 |
| 0 1_{a_3} 0 0 0 | D_{a_3} 0 | 0 0 1_{a_4} 0 0 0 | 0 0 |
| 0 0 D_{a_3} 0 0 0 | 1_{b_{1_\alpha}} + E_{a_1} 0 | 0 | 0 |
| 0 0 0 0 0 | F_{a_{1_\gamma}} 0 1_{b_{1_\gamma}} | 0 | 0 |
| 0 0 0 0 0 | F_{a_{1_\gamma}} 0 1_{b_{1_\gamma}} | 0 | 0 |

Under this map, simple $Q$-representations are mapped to irreducible $C_2 \ast C_3$-representations, and if the coefficients of the block-matrices $C_{ij}, D_{ij}, E_i$ and $F_{ij}$ occurring in $B$ give a parametrization of a Zariski open subset of the quotient variety $\text{iss}_\tau Q$, then the $n$-dimensional representations of the 3-string braid group $B_3$
given by

\[
\begin{align*}
\sigma_1 &\mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\
\sigma_2 &\mapsto \lambda^{1/6} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B
\end{align*}
\]

contain a Zariski dense set of irreducible $B_3$-representations in the component $X_\sigma$ of $\text{iss}_n B_3$.

4. The main result

In view of the previous section it remains to find for each $\sigma = (a, b; x, y, z)$ satisfying

\[a + b = n = x + y + z \quad \text{and} \quad x = \text{max}(x, y, z) \leq b = \text{min}(a, b)
\]
a judiciously chosen dimension-vector $\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma)$ of type $\sigma$ together with an explicit rational parametrization of $\text{iss}_\tau Q$. We will separate this investigation in two cases, sharing the same underlying strategy. First we choose $a_1, a_2, a_3, a_4, a_5, a_6$ such that $\sigma_1 = (a_1 + a_3 + a_5, a_2 + a_4 + a_6; a_1 + a_4, a_2 + a_5, a_3 + a_6)$ is a component containing simples and such that we have an explicit rational parametrization of the isoclasses of the quiver-setting

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array}
\]

The upshot being that for a general representation the stabilizer subgroup reduces to $\mathbb{C}^*(1_{a_1} \times \ldots \times 1_{a_6})$. But then, the additional arrows $D_{ij}$ and $E_{ij}$, that is the quiver setting

\[
\begin{array}{c}
7 \\
8 \\
9 \\
10 \\
11 \\
12
\end{array}
\]

give three settings corresponding to quiver settings of canonical linear systems with $m = p = a_i + a_{i+3}$ and the results of section 2 give a rational parametrization of the isoclasses and further reduces the stabilizer subgroup to $\mathbb{C}^*(1_{a_1} \times \ldots \times 1_{a_6} \times$
1_{b_a} \times 1_{b_b} \times 1_{b_c})$. This then leaves the trivial action on the remaining arrows $F_{ij}$ and hence these generic matrices conclude the desired rational parametrization.

4.1. **Case 1: $a > b$.** Define 

\[d = a - b, \quad e = d - 1, \quad f = b - z, \quad g = b - y, \quad h = b - x,\]

then the dimension-vector

\[\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (d, e, e, 0, 1, 1, f, g, h)\]

is of type $\sigma$. If we denote by

\[
\begin{bmatrix}
* & \text{a generic matrix} \\
\text{the column vector} & \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
\text{T}_n & \text{the } n + 1 \times n \text{ matrix} \\
\end{bmatrix}
\]

and the (reduced) companion matrices as in lemma \[a\] then using lemma \[b\] a rational parametrization of the first stage is given by the representations

By lemma \[c\] a rational parametrization of the second stage is then given by the representations

This concludes the proof of
Theorem 2. A Zariski dense rational parametrization of the component $X_\sigma$ of $\text{iss}_n B_3$ where $\sigma = (a,b;x,y,z)$ with $a > b$ is given by the representations

$$
\begin{align*}
\sigma_1 \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix}
1_x & 0 & 0 \\
0 & \rho^2 y & 0 \\
0 & 0 & \rho z
\end{bmatrix} B \begin{bmatrix}
1_a & 0 \\
0 & -1_b
\end{bmatrix}
\end{align*}
$$

and

$$
\begin{align*}
\sigma_2 \mapsto \lambda^{1/6} \begin{bmatrix}
1_x & 0 \\
0 & -1_b
\end{bmatrix} B^{-1} \begin{bmatrix}
1_x & 0 \\
0 & \rho^2 y & 0 \\
0 & 0 & \rho z
\end{bmatrix} B
\end{align*}
$$

for all $n \times n$ matrices $B$ of the form

| $1_d$ | $0$ | $0$ | $0$ | $0$ | $\overline{1_e}$ | $0$ | $0$ | $*$ |
|-------|-----|-----|-----|-----|-----------------|-----|-----|-----|
| $0$   | $*$ | $q_{1f} + A_f$ | $0$ | $0$ | $0$             | $*$ | $0$ | $1_g$ |
| $0$   | $0$ | $0$ | $q_{1g}$ | $*$ | $1_f + A_f$     | $0$ | $*$ | $*$ |
| $*$   | $0$ | $0$ | $0$ | $* $ | $q_{1h} + A_h$ | $0$ | $0$ | $1_h$ |

where $d = a - b$, $e = d - 1$, $f = b - z$, $g = b - y$ and $h = b - x$.

4.2. Case $2: a = b$. Define $c = x + y + 1 - a$, $g = a - y - 1$ and $h = a - x$, which corresponds to the decomposition

If $c$ is odd, define $c = 2d + 1$, $e = d + 1$ and $f = d - 1$, then the dimension vector

$$
\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (c, e, 1, d, f, 0, 0, g, h)
$$

is of type $\sigma$. Then, using lemma 2, a rational parametrization for the first stage is given by the representations

$$
\begin{align*}
\sigma_1 \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix}
1_x & 0 & 0 \\
0 & \rho^2 y & 0 \\
0 & 0 & \rho z
\end{bmatrix} B \begin{bmatrix}
1_a & 0 \\
0 & -1_b
\end{bmatrix}
\end{align*}
$$

and

$$
\begin{align*}
\sigma_2 \mapsto \lambda^{1/6} \begin{bmatrix}
1_x & 0 \\
0 & -1_b
\end{bmatrix} B^{-1} \begin{bmatrix}
1_x & 0 \\
0 & \rho^2 y & 0 \\
0 & 0 & \rho z
\end{bmatrix} B
\end{align*}
$$

for all $n \times n$ matrices $B$ of the form
Using lemma 1 we then get that a rational parametrization of the second stage is given by the following representations

If \( c \) is even, we can define \( c = 2e \) and \( f = e - 1 \) in which case the dimension vector

\[
\tau = (a_1, a_2, a_3, a_4, a_5, b_a, b_6, b_7) = (e, e, 1, e, f, 0, 0, g, h)
\]

is of type \( \sigma \) and exactly the same representations give a rational parametrization of both stages if we replace all occurrences of \( d \) by \( e \). This then concludes the proof of

**Theorem 3.** A Zariski dense rational parametrization of the component \( X_{\sigma} \) of \( \text{iss}_nB_3 \) where \( \sigma = (a, b, x, y, z) \) with \( a = b \) is given by the representations

\[
\left\{
\begin{align*}
\sigma_1 & \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho_1 z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 1 \end{bmatrix} \\
\sigma_2 & \mapsto \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho_1 z \end{bmatrix} B
\end{align*}
\right.
\]

for all \( n \times n \) matrices \( B \) of the form

\[
\begin{array}{c|cccc}
1_e & 0 & 0 & 0 & 0 \\
0 & T_f & 0 & * & 0 \\
0 & 0 & q_{1g} & * & 0 \\
1_e & 0 & * & 0 & 0 \\
0 & 1_f & 0 & 0 & 0 \\
| & 0 & 0 & q_{1h} & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1_g + A_g & 0 & |* & 0 & 1_g & 0 \\
0 & 0 & 0 & 1_h & 0 & 0 & 0 & 1_h
\end{array}
\]

where \( g = a - y - 1, h = a - x \) and if \( c = x + y + 1 - a \) is odd we take \( c = 2d + 1, \)
\( e = d + 1 \) and \( f = d - 1 \) whereas if \( c = x + y + 1 - a \) is even we take \( c = 2e \) and \( f = e - 1 \) and we replace all occurrences of \( d \) in the matrix to \( e \).
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