ON SOME $P$-$Q$ MIXED MODULAR EQUATIONS OF DEGREE 5

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ABSTRACT. In his second notebook, Ramanujan recorded total of 23 $P$-$Q$ modular equations involving theta-functions $\varphi(q)$, $\psi(q)$ and $f(-q)$. In this paper, modular equations analogous to those recorded by Ramanujan are obtained involving $f(-q)$. As a consequence, several values of quotients of theta-function are evaluated.

Dedicated to Prof. C. Adiga on the occasion of his 62$^{nd}$ Birthday

1. INTRODUCTION

For $|q| < 1$, let $(a; q)_\infty$ denote the infinite product $\prod_{n=0}^{\infty} (1 - aq^n)$, where $a, q$ are complex numbers and $f(a, b)$ be the Ramanujan theta-function:

$$f(a, b) := \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1,$$

The following definitions of theta-functions $\varphi$, $\psi$ and $f$ follows as special cases of $f(a, b)$:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad (1.1)$$
$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \quad (1.2)$$
$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \quad (1.3)$$

The ordinary or Gaussian hypergeometric function is defined by

$$2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad 0 \leq |z| < 1,$$

where $a, b, c$ are complex numbers, $c \neq 0, -1, -2, \ldots$, and

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) \text{ for any positive integer } n.$$
Let $K(k)$ be the complete elliptic integral of the first kind of modulus $k$. Recall that

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^2}{(n!)^2} k^{2n}, \quad (0 < k < 1), \quad (1.4)$$

and set $K' = K(k')$, where $k' = \sqrt{1 - k^2}$ is the so called complementary modulus of $k$. It is classical to set $q(k) = e^{-\pi K(k')/K(k)}$ so that $q$ is one-to-one increases from 0 to 1.

In the same manner introduce $L_1 = K(\ell_1)$, $L'_1 = K(\ell'_1)$ and suppose that the following equality

$$n_1 \frac{K'}{K} = \frac{L'}{L_1} \quad (1.5)$$

holds for some positive integer $n_1$. Then a modular equation of degree $n_1$ is a relation between the moduli $k$ and $\ell_1$ which is induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = \ell_1^2$. We say that $\beta$ is of degree $n_1$ over $\alpha$. The multiplier $m$, corresponding to the degree $n_1$, is defined by

$$m = \frac{K}{L_1} = \varphi^2(q)/\varphi^2(q^{n_1}). \quad (1.6)$$

for $q = e^{-\pi K(k')/K(k)}$.

Let $K$, $K'$, $L_1$, $L'_1$, $L_2$, $L'_2$, $L_3$ and $L'_3$ denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let $n_1$, $n_2$ and $n_3$ be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L'_1}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L'_2}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L'_3}{L_3}, \quad (1.7)$$

hold. Then a “mixed” modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.7). We say that $\beta$, $\gamma$ and $\delta$ are of degrees $n_1$, $n_2$ and $n_3$, respectively over $\alpha$. The multipliers $m = K/L_1$ and $m' = L_2/L_3$ are algebraic relation involving $\alpha$, $\beta$, $\gamma$ and $\delta$.

At scattered places of his second notebook [13], Ramanujan recorded a total of nine $P$–$Q$ “mixed” modular relations of degrees 1, 3, 5 and 15. These relations were proved by B. C. Berndt and L. -C. Zhang [5, 6] and the same has been reproduced in the book by Berndt [4, pp. 214-235]. In [7], S. Bhargava, C. Adiga and M. S. Mahadeva Naika have established several new $P$–$Q$ “mixed” modular relations with four moduli. For more information one can see [11] and [12]. Motivated by all these works, we establish some new modular equations of “mixed” degrees and as an application, we establish some new general formulas for the explicit evaluations of a remarkable product of theta function.
In Section 2, we collect some identities which are useful in proofs of our main results.

In Section 3, we establish several new modular equations of degree 5. In Section 4, we establish several new \(P-Q\) “mixed” modular equations akin to those recorded by Ramanujan in his notebooks.

Mahadeva Naika, M. C. Maheshkumar and Bairy [9], have defined a new remarkable product of theta-functions \(b_{s,t}\):

\[
b_{s,t} = \frac{te^{-(t-1)s} \sqrt{\pi} \phi^2 \left(-e^{-\pi \sqrt{st}}\right)}{\psi^2 \left(-e^{-\pi \sqrt{\pi}}\right) \phi^2 \left(-e^{-2\pi \sqrt{\pi}}\right)},
\]

where \(s, t\) are real numbers such that \(s > 0\) and \(t \geq 1\). They have established some new general formulas for the explicit evaluations of \(b_{s,t}\) and computed some particular values of \(b_{s,t}\). In Section 5, we establish some new modular relations connecting a remarkable product of theta-functions \(b_{s,5}\) with \(b_{r^2 s,5}\) for \(r = 2, 4\) and 6 and explicit values of \(b_{s,5}\) are deduced.

2. Preliminary results

In this section, we list some of the relevant identities which are useful in the proofs of our results.

**Lemma 2.1.** [3] Ch. 17, Entry 12 (i) and (iii), p. 124] For \(0 < x < 1\), let

\[
f(e^{-y}) = \sqrt{2}s^{-1/3} \left\{x(1-x)e^y\right\}^{1/24},
\]

\[
f(e^{-2y}) = \sqrt{3}s^{-1/3} \left\{x(1-x)e^y\right\}^{1/12},
\]

where \(z := 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)\) and \(y := \pi 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)\).

**Lemma 2.2.** [3] Ch. 16, Entry 24 (ii) and (iv), p. 39] We have

\[
f^3(-q) = \phi^2(-q)\psi(q),
\]

\[
f^3(-q^2) = \phi(-q)\psi^2(q).
\]

**Lemma 2.3.** [3] Ch. 19, Entry 13 (xii) and (vii), pp. 281-282]

If \(\beta\) is of degree 5 over \(\alpha\), then

\[
\left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta (1-\beta)}{\alpha (1-\alpha)}\right)^{1/4} = m,
\]

\[
\left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha (1-\alpha)}{\beta (1-\beta)}\right)^{1/4} = \frac{5}{m},
\]
where $m$ is the multiplier for degree 5.

Lemma 2.4. [14] p. 55 If $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}$, then

$$XY + \frac{5}{XY} = \left(\frac{Y}{X}\right)^3 + \left(\frac{X}{Y}\right)^3.$$ \hspace{1cm} (2.7)

Lemma 2.5. [14] p. 55 If $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^3)}{q^{1/2}f(-q^{15})}$, then

$$(XY)^3 + \left(\frac{5}{XY}\right)^3 + \left[\left(\frac{X}{Y}\right)^6 - \left(\frac{Y}{X}\right)^6\right] + 9 \left[\left(\frac{X}{Y}\right)^3 + \left(\frac{Y}{X}\right)^3\right] = 0.$$ \hspace{1cm} (2.8)

Lemma 2.6. [14] p. 55 If $X := \frac{f(-q)}{q^{1/6}f(-q^5)}$ and $Y := \frac{f(-q^4)}{q^{2/3}f(-q^{20})}$, then

$$(XY)^3 + \left(\frac{5}{XY}\right)^3 = \left(\frac{X}{Y}\right)^5 + \left(\frac{Y}{X}\right)^5 - 8 \left\{\left(\frac{X}{Y}\right)^3 + \left(\frac{Y}{X}\right)^3\right\}$$

$$+ 4\left(\frac{X}{Y} + \frac{Y}{X}\right) + 4/\left(\frac{X}{Y} + \frac{Y}{X}\right).$$ \hspace{1cm} (2.9)

Lemma 2.7. [8] If $P := \frac{\varphi(-q)}{\varphi(-q^5)}$ and $Q := \frac{\varphi(-q^4)}{\varphi(-q^{20})}$, then

$$\frac{P^4}{Q^4} + \frac{Q^4}{P^4} + 24 \left[\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right] + 8 \left[P^2Q^2 + \frac{5^2}{P^2Q^2}\right] + 3 \left[Q^4 + \frac{5^2}{Q^4}\right] + 120$$

$$= 20 \left[P^2 + \frac{5}{P^2}\right] + 32 \left[Q^2 + \frac{5}{Q^2}\right] + \left[P^2Q^2 + \frac{5^4}{P^2Q^2}\right] + 3 \left[\frac{5P^2}{Q^2} + \frac{Q^4}{P^2}\right].$$ \hspace{1cm} (2.10)

Lemma 2.8. [1] Theorem 5.3] If $U := \frac{\varphi^2(q)}{\varphi^2(q^5)}$ and $V := \frac{\psi^2(-q)}{q\psi^2(-q^5)}$, then

$$U + UV = 5 + V.$$ \hspace{1cm} (2.11)

Lemma 2.9. [2] Theorem 2.17] If $U := \frac{\varphi(-q)}{\varphi(-q^5)}$ and $V := \frac{\varphi(-q^2)}{\varphi(-q^{10})}$, then

$$\frac{U^2}{V^2} + \frac{V^2}{U^2} + 4 = V^2 + \frac{5}{V^2}.$$ \hspace{1cm} (2.12)
3. **P-Q Modular Equations of Degree 5**

In this section, we establish some new modular equations of degree 5.

**Theorem 3.1.** If \( M := \frac{\varphi(-q)}{\varphi(-q^5)} \) and \( N := \frac{\varphi(-q^6)}{\varphi(-q^{30})} \), then

\[
\begin{multline*}
\left[ \frac{M^8}{N^8} + \frac{N^8}{M^8} \right] + 96 \left[ \frac{M^6}{N^6} + \frac{N^6}{M^6} \right] + 1146 \left[ \frac{M^4}{N^4} + \frac{N^4}{M^4} \right] + 2868 \left[ \frac{M^2}{N^2} + \frac{N^2}{M^2} \right] \\
+ \left[ N^8 + \frac{5^4}{N^8} \right] - 16 \left[ N^6 + \frac{5^3}{N^6} \right] + 188 \left[ N^4 + \frac{5^2}{N^4} \right] - 1696 \left[ N^2 + \frac{5}{N^2} \right] \\
- 54 \left[ M^6 + \frac{5^3}{M^6} \right] + 498 \left[ M^4 + \frac{5^2}{M^4} \right] - 2106 \left[ M^2 + \frac{5}{M^2} \right] - 4 \left[ N^8 + \frac{5^6}{N^8} \right] \\
+ 6 \left[ \frac{N^8}{M^4} + \frac{5^2 M^4}{N^8} \right] - 4 \left[ \frac{N^8}{M^4} + \frac{5^3 M^2}{N^8} \right] - 144 \left[ \frac{N^6}{M^4} + \frac{5 M^4}{N^6} \right] + 64 \left[ \frac{N^6}{M^2} + \frac{5^2 M^2}{N^6} \right] \\
- 479 \left[ \frac{N^4}{M^2} + \frac{5 M^2}{N^4} \right] - 165 \left[ \frac{M^6}{N^4} + \frac{5 N^4}{M^6} \right] + 124 \left[ \frac{M^6}{N^2} + \frac{5^2 N^2}{M^6} \right] - 936 \left[ \frac{M^4}{N^2} + \frac{5 N^2}{M^4} \right] \\
+ 10 \left[ M^4 N^4 + \frac{5^4}{M^4 N^4} \right] + 516 \left[ M^2 N^2 + \frac{5^2}{M^2 N^2} \right] - 39 \left[ M^2 N^4 + \frac{5^3}{M^2 N^4} \right] \\
- 120 \left[ M^4 N^2 + \frac{5^3}{M^4 N^2} \right] + 12 \left[ M^6 N^2 + \frac{5^4}{M^6 N^2} \right] - \left[ M^6 N^4 + \frac{5^5}{M^6 N^4} \right] + 6748 = 0.
\end{multline*}
\]

(3.1)

**Proof.** Using the equation \((2.8)\) after changing \( q \) to \( q^2 \), we get

\[
X^9 Y^9 + 125 X^3 Y^3 + X^{12} - Y^{12} + 9 X^9 Y^3 + 9 Y^9 X^3 = 0.
\]

(3.2)

where

\[
X := \frac{f(-q^2)}{q^{1/3} f(-q^{10})} \quad \text{and} \quad Y := \frac{f(-q^6)}{q f(-q^{30})}.
\]

Cubing the equation \((3.2)\) and using the equations \((2.3)\) and \((2.4)\), we deduce

\[
T M^3 R^3 N^6 T^2_2 + 125 M R N^2 T + M^4 R^4 - N^8 T^2 + 9 M^3 R^3 N^2 T + 9 T N^6 T^2 M R = 0.
\]

(3.3)

where

\[
R := \frac{\psi^2(q)}{q \psi^2(q^5)} \quad T := \frac{\psi(q^5)}{q^3 \psi(q^{30})} \quad \text{and} \quad T_2 := \frac{\psi^2(q^6)}{q^6 \psi^2(q^{30})}.
\]

Using the equation \((2.11)\) after changing \( q \) to \(-q\), we have

\[
R := \frac{M^2 - 5}{M^2 - 1} \quad \text{and} \quad T_2 := \frac{N^2 - 5}{N^2 - 1}.
\]

(3.4)

Collecting the terms containing \( T \) on one side of the equation \((3.3)\) and using the equation \((3.4)\), we get

\[
A(M, N) B(M N) = 0,
\]

(3.5)
where
\[ A(M, N) := (625 - 500M^2 - 20N^6 - 500N^2 + 150M^4 + N^8 + 150N^4 \]
\[ + M^8 - 20M^6 + 16M^6N^6 + 400M^2N^2 + 120M^4N^2 + 24M^6N^4 + 24M^4N^6 \]
\[ - 300M^4N^4 - 16M^2N^6 - 16M^6N^2 - 4M^8N^2 + N^8M^8 - 4N^8M^6 \]
\[ + 6N^8M^4 - 4N^8M^2 - 4M^8N^6 + 120M^2N^4 + 6M^8N^4 \]
and
\[ B(M, N) := (625M^8 + N^{16} - 825N^{12}M^2 + 150M^{12} - 500M^{10} + M^{16} \]
\[ + 12900M^6N^6 - 4875M^6N^4 - 15000M^4N^6 + 6250M^4N^4 + 7500M^2N^6 \]
\[ - 2000M^8N^2 + 1146M^{12}N^4 - 720M^{12}N^2 - 2395M^{10}N^4 + 1600M^{10}N^2 \]
\[ + 188N^{12}M^8 - 479N^{12}M^6 + 1146N^{12}M^4 - 1696N^{10}M^8 + 2868N^{10}M^6 \]
\[ - 4680N^{10}M^4 + 3100N^{10}M^2 + 6748N^8M^8 - 10530N^8M^6 + 12450N^8M^4 \]
\[ - 6750N^8M^2 + 10M^{12}N^4 - 120M^{12}N^2 - 39M^{10}N^4 + 516M^{10}N^2 \]
\[ - 2106M^{10}N^2 + 2868M^{10}N^6 - 8480M^8N^6 + 498M^{12}N^8 - 936M^{12}N^6 \]
\[ - 165M^{14}N^4 + 96M^{14}N^2 - M^{14}N^2 + 12M^{14}N^{10} - 144M^{14}N^2 - 80M^{14} \]
\[ + 124M^{14}N^6 + 96N^{14}M^2 - 16N^{14}M^8 + 64N^{14}M^6 - 14N^{14}M^4 \]
\[ - 4N^{16}M^2 + N^{16}M^8 - 4N^{16}M^6 + 6N^{16}M^4 - 3125M^2N^4 + 4700M^8N^4 \).

Expanding in powers of \( q \), the first and second factor of the equation (3.5), one gets respectively,
\[ A(M, N) = (256 - 1536q^8 - 512q^{10} + 1152q^{11} + 3840q^{11} + 4736q^{12} + \cdots) \]
and
\[ B(M, N) = q^8 \left( 8448 + 33792q + 33792q^2 - 54528q^3 - 194208q^4 - 268608q^5 + \cdots \right) . \]

As \( q \to 0 \), the factor \( B(M, N) \) of the equation (3.5) vanishes whereas the other factor \( A(M, N) \) do not vanish. Hence, we arrive at the equation (3.4) for \( q \in (0, 1) \). By analytic continuation the equation (3.4) is true for \( |q| < 1 \). \( \square \)

**Remark 1.** The modular relation connecting
\[ \frac{\psi(q)}{q^{1/2}\psi(q^2)} \] and \[ \frac{\psi(q^6)}{q^3\psi(q^{10})}, \]
can be obtained by eliminating \( M \) and \( N \) from the equation (3.3).
ON SOME $P$–$Q$ MIXED MODULAR EQUATIONS OF DEGREE 5

In this section, we establish several new $P$–$Q$ “mixed” modular equations with four moduli. Throughout this section, we set

$$A := \frac{-f(q)f(q^2)}{q^{1/2}f(-q^3)f(-q^{10})}, \quad B_n := \frac{-f(q^n)f(q^{2n})}{q^{n/2}f(-q^{5n})f(-q^{10n})},$$

and

$$C_n := \frac{q^{n/6}f(-q^n)f(-q^{10n})}{f(-q^{2n})f(-q^{3n})}.$$

Theorem 4.1. For $|q| < 1$,

$$\frac{f^2(q)f^2(-q^2)}{q^4f^2(-q^5)f^2(-q^{10})} = \frac{U(U - 5)}{U - 1}, \quad U > 1,$$

$$\frac{f^2(q)f^2(-q^2)}{q^4f^2(-q^5)f^2(-q^{10})} = \frac{V(V - 5)}{V - 1}, \quad V > 1,$$

where $U := \phi^2(q)/\phi^2(-q^5)$ and $V := \psi^2(q)/\psi^2(q^5)$.

Proof of (4.2). The equations (2.5) and (2.6) can be rewritten as

$$m \left\{ \alpha(1 - \alpha) \right\} \frac{1}{4} + 1 = \frac{5}{m} + \left\{ \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right\} \frac{1}{4}.$$ \hfill (4.4)

Employing the equation (2.1) after changing $q$ to $-q$ and equation (2.2) in the equation (4.4), we arrive at the equation (4.2).

Proof of (4.3). Using the equation (2.11) in the equation (4.2), we arrive at the equation (4.3).

Theorem 4.2. For $|q| < 1$,

$$\frac{qf^6(q)f^6(-q^{10})}{f^6(-q^5)f^6(-q^{10})} = \frac{U(U - 1)}{U - 5},$$

$$\frac{qf^6(q)f^6(-q^{10})}{f^6(-q^5)f^6(-q^{10})} = \frac{(V - 5)}{V - 1},$$

where $U := \phi^2(q)/\phi^2(-q^5)$ and $V := \psi^2(q)/\psi^2(q^5)$.

Proof. The proof of equations (4.5) and (4.6) are similar to the proof of (4.2) and (4.3). Hence, we omit the details.

Theorem 4.3. If $P := AB_2$ and $Q := \frac{A}{B_2}$, then

$$\left(Q^4 + \frac{1}{Q^4}\right) - 3 \left(Q^2 + \frac{1}{Q^2}\right) - \left(P + \frac{5^2}{P}\right) \left(Q + \frac{1}{Q}\right) - 12 = 0.$$ \hfill (4.7)
Proof. Taking the cube of both sides of the equation (2.7), we deduce

\[
X^3Y^3 + \frac{125}{X^3Y^3} + 15\left(\frac{X^3}{Y^3} + \frac{Y^3}{X^3}\right) = \frac{X^9}{Y^9} + \frac{Y^9}{X^9},
\]

(4.8)

where \( X := \frac{f(-q)}{q^{1/6}f(-q^5)} \) and \( Y := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \).

Using (2.3), (2.4) and (4.1), we deduce

\[
A^4V_1^4B_2^4V_2^4 + 125A^2V_1^2B_2^3V_2^2 + 12A^4V_1^4B_2^2V_2^3 + 12B_2^4V_2^4A^2V_1^2
\]

\[= A^6V_1^6 + B_2^6V_2^6, \quad (4.9)\]

where \( V_1 := \frac{\psi(q)}{q^{1/6}\psi(q^5)} \) and \( V_2 := \frac{\psi(q^2)}{q^{1/3}\psi(q^{10})} \).

Using the equation (4.3) in the equation (4.9), we deduce

\[
12B_2^5A^2vu + A^6B_2^6uv + 5A^6B_2^6uv + 5A^4B_2^6uv + 12A^6B_2^2uv + 625A^2B_2^2v
\]

\[+ 625A^2B_2^2u - 18A^8u - 18B_2^8v + 185A^4B_2^4v + 185A^4B_2^4u + 300A^4B_2^4u
\]

\[+ 60A^4B_2^4u + 300B_2^4A^2v + 60B_2^4A^2vu + A^8B_2^8v - 250A^6 - 250B_2^6
\]

\[+ 1350A^4B_2^4 + 2125A^4B_2^4 + A^6B_2^6v + 37A^4B_2^4v + 40A^6B_2^4v + 8A^6B_2^6v
\]

\[+ 40A^6B_2^6u + 37A^4B_2^4u + 5A^4B_2^4u + 8A^6B_2^6u + 60A^6B_2^6u + 60B_2^6A^2v
\]

\[+ 425A^4B_2^4v - 50A^6u - 50B_2^6v + 12A^2B_2^4u + 5A^8B_1^4v + 425A^2B_2^4u \quad (4.10)
\]

\[+ 12A^8B_2^2v + 125A^2B_2^2uv + 64A^6B_2^6v + 96A^6B_2^6v + 96B_2^6A^2u + 25A^4B_2^4uv
\]

\[- 2B_2^3v - 2A^14u + 2125A^2B_2^4 + 3125A^2B_2^4 + A^8B_2^8 + 8A^8B_2^8 + 37A^8B_2^8
\]

\[+ 8A^8B_2^8 + 296A^4B_2^4 + 37A^4B_2^4 + 296A^4B_2^4 + 60A^8B_2^8 + 480A^6B_2^8
\]

\[+ 60A^2B_2^8 + 480A^2B_2^8 - 120B_2^8 - 120A^8 - 24B_2^14 - 24A^{14} - 2A^{12}
\]

\[- 2B_2^{12} = 0.\]

where \( u := \pm \sqrt{A^4 + 6A^2 + 25} \) and \( v := \pm \sqrt{B_2^4 + 6B_2^2 + 25}. \)

Eliminating \( u \) and \( v \) from the equation (4.10), we find and squaring both sides, we deduce

\[
(3125 - 390625A^4B_2^4 - 12A^4B_2^4 - 3A^4B_2^4 - 12A^4B_2^4 - 25A^4B_2^4 - 3A^2B_2^6 - 25A^2B_2^4
\]

\[+ B_2^8) (A^{16} + B_2^{16} - 187500A^8B_2^8 - 187500A^8B_2^8 - 26900A^8B_2^8
\]

\[\quad + 93125A^6B_2^6 - 57500A^4B_2^8 - 15625A^2B_2^8 - 9600B_2^{10}A^4
\]

\[+ 87512A^2 - 7676A^8B_2^6 + 57500A^8B_2^6 + 15625A^8B_2^6 - 15625A^8B_2^6
\]

\[+ B_2^{10} - 1076A^8B_2^{10} - 92A^8B_2^{10} - 92A^8B_2^{10} - 35A^14B_2^4 - 384A^{12}B_2^6
\]
Proof.

Expanding in powers of \( q \), the first and second factors of (4.11) one gets respectively

\[-4q^{11} (4 + 8q - 27q^{2} - 68q^{3} + 40q^{4} + 278q^{5} + 62q^{6} - 723q^{7} + \cdots)\]

and

\[-(-2 + 2q^{2} - 10q^{3} + 552q^{5} - 2016q^{6} + 1038q^{7} + 15620q^{8} + \cdots).\]

As \( q \) tends to 0 the first factor of (4.11) vanishes whereas the second factor does not vanish. Hence we arrive at (4.7) for \( q \in (0, 1) \). By analytic continuation (4.7) is true for \( |q| < 1 \). \( \square \)

**Theorem 4.4.** If \( P = AB_4 \) and \( Q = \frac{A}{B_4} \) then

\[
\left( Q^8 + \frac{1}{Q^8} \right) - 52 \left( Q^6 + \frac{1}{Q^6} \right) - 1024 \left( Q^4 + \frac{1}{Q^4} \right) - 3987 \left( Q^2 + \frac{1}{Q^2} \right)
\]

\[
- \left( P + \frac{5^2}{P} \right) \left[ 1076 \left( Q + \frac{1}{Q} \right) + 384 \left( Q^3 + \frac{1}{Q^3} \right) + 35 \left( Q^5 + \frac{1}{Q^5} \right) \right]
\]

\[
- \left( P^2 + \frac{5^4}{P^2} \right) \left[ 92 \left( Q^2 + \frac{1}{Q^2} \right) + 8 \left( Q^4 + \frac{1}{Q^4} \right) + 149 \right] - \left( P^3 + \frac{5^6}{P^3} \right)
\]

\[
\times \left[ 12 \left( Q + \frac{1}{Q} \right) + \left( Q^3 + \frac{1}{Q^3} \right) \right] - \left( P^4 + \frac{5^8}{P^4} \right) - 7676 = 0.
\]

**Proof.** Using the equation (2.10) in the equation (4.2), we deduce

\[
625 - 125u + 125v - A^4uB_4^6v - 5A^4uB_4^4v - 90A^2uB_4^2v - 6A^2uB_4^6v
\]

\[
- 38A^2uB_4^4v + 40A^2u + 2A^6u - 1215B_4^2v - 63B_4^6v - A^6v - 56A^6B_4^2
\]

\[
- 36A^6B_4^4 - 8A^6B_4^6 - 6A^6B_4^8 + 3A^4v - 592A^4B_4^2 - 360A^4B_4^4 - 80A^4B_4^6
\]

\[
- 9A^4B_4^8 - 15A^2v - 3080A^2B_4^2v - 1732A^2B_4^4 - 376A^2B_4^6 - 39A^2B_4^8
\]

\[
- 25uv - 1360uB_4^2v - 688uB_4^4v - 144uB_4^6 - 13uB_4^8 - 125A^2 - 6000B_4^2
\]

\[
+ 29A^4 - 3A^6 + 2A^8 - 3216B_4^2 - 688B_4^4 - 63B_4^6 - 9A^4u - 13A^6B_4^2v
\]

\[
- A^6B_4^6v - 5A^6B_4^4v - 129A^4B_4^2v - 9A^4B_4^4v - 53A^4B_4^6v - A^4uv
\]
Theorem 4.6. If now (3.1) is used in place of (2.10).

Proof. Let $u := \pm \sqrt{A^4 + 6A^2 + 25}$ and $v := \pm \sqrt{B_4^2 + 6B_4^2 + 25}$. Eliminating $u$ and $v$ from the equation (4.15), we arrive at (4.12). \qed

Theorem 4.5. If $P = AB_6$ and $Q = \frac{A}{B_6}$, then

$$Q^{16} - 363Q^{14} - 30882Q^{12} - 698682Q^{10} - 6183702Q^8 - 16140317Q^6 + 37225608Q^4 + 231497788Q^2 + P \{ 6013380Q + 21753498Q^3 - 1148442Q^5 - 2210604Q^7 - 406488Q^9 - 26740Q^{11} - 519Q^{13} \} + P^3 \{ 6287236Q^2 + 858465Q^4 - 462222Q^6 - 150099Q^8 - 12840Q^{10} - 267Q^{12} + 10229305 \} + P^3 \{ 1132002Q + 362832Q^3 - 42462Q^5 - 37066Q^7 - 4323Q^9 - 78Q^{11} \} + P^4 \{ 74418Q^2 + 4471Q^4 - 5955Q^6 - 1026Q^8 - 12Q^{10} + 130902 \} + P^5 \{ 9171Q + 2028Q^3 - 588Q^5 - 171Q^7 - Q^9 \} + P^6 \{ 300Q^2 - 27Q^4 - 18Q^6 + 679 \} + P^7 \{ 24Q - Q^5 \} + P^8 + 36965548 = 0, \tag{4.14}$$

where $P^n = \left( P^n + \frac{52n}{P^n} \right)$ and $Q^n = \left( Q^n + \frac{1}{Q^n} \right)$.

Proof. The proof of the equation (4.14) is similar to the proof of the equation (4.12). Notice that now (3.1) is used in place of (2.10). \qed

Theorem 4.6. If $P = C_1C_2$ and $Q = \frac{C_1}{C_2}$, then

$$\left( P + \frac{1}{P} \right) \left( Q^3 + \frac{1}{Q^3} \right) + 2 = \left( P^2 + \frac{1}{P^2} \right). \tag{4.15}$$

Proof. Using the equation (2.12) in the equation (4.5), we deduce

$$10v - 10u - 2vC_6^6C_1^6 - 2vuC_2^6 + 6vu + 4uC_6^6 + 2vC_6^6 - 2C_2^{12}C_1^6 - 6 + 24C_2^6C_1^6 - 2uC_2^{12} + 24uC_6^6 + 6vC_6^6 - 46C_6^6 - 8C_2^6 + 4C_1^{12} + 2C_2^{12} = 0, \tag{4.16}$$

where $u := \pm \sqrt{C_1^{12} - 18C_1^6 + 1}$ and $v := \pm \sqrt{C_2^{12} - 18C_2^6 + 1}$. \qed
Eliminating \( u \) and \( v \) from the equation (4.16) leads to

\[
\left( C_1^6 - C_2^6 C_1^6 + C_2^6 - C_2^2 C_1^2 + C_2^2 C_1^8 + 2C_2^4 C_1^4 + C_2^8 C_2^2 \right) (C_2^4 C_1^{16} + \ldots)
\]

Expanding in powers of \( q \), the first and second factors of (4.17), one gets respectively

\[
q^{11} \left( 8 - 32q - 8q^2 + 168q^3 - 220q^4 - 760q^5 + 1748q^7 + \cdots \right)
\]

and

\[
(3 - 24q + 117q^2 - 456q^3 + 1356q^4 - 3192q^5 + 7242q^6 - 17304q^7 + \cdots).
\]

As \( q \) tends to 0 the first factor of (4.17) vanishes whereas the second factor does not vanish. Hence we arrive at (4.15) for \( q \in (0, 1) \). By analytic continuation (4.15) is true for \(|q| < 1\). \( \square \)

**Theorem 4.7.** If \( P = C_1 C_4 \) and \( Q = \frac{C_1}{C_4} \), then

\[
\left( P^3 + \frac{1}{P^3} \right) \left( Q^3 + \frac{1}{Q^3} \right) = \frac{1}{Q^3} + \frac{1}{Q^6} + 13 \left( Q^4 + \frac{1}{Q^4} \right) + 52 \left( Q^2 + \frac{1}{Q^2} \right) + 82 = \left( P^6 + \frac{1}{P^6} \right).
\]

**Proof.** The proof of the equation (4.18) is similar to the proof of (4.15): Notice that now (2.10) is used in place of (2.12). \( \square \)

### 5. Remarkable Product of Theta-Functions

In this section, we establish several new modular identities connecting the remarkable product of theta-functions \( b_{s,5} \) with \( b_{r,s,5} \) for \( r = 2, 4, \) and 6.

**Lemma 5.1.** \( [9] \) If \( s \) and \( t \) are any positive rational, then

\[
b_{2s,t} b_{2s,t} = 1.
\]

**Lemma 5.2.** \( [10] \) 0 < \( b_{s,t} \leq 1 \) for all \( s \geq 2 \) and \( t \) positive integer greater than 1.
Theorem 5.1. If \( X = \sqrt{b_{s,5}b_{4s,5}} \) and \( Y = \sqrt{\frac{b_{s,5}}{b_{4s,5}}} \), then

\[
\left( Y^4 + \frac{1}{Y^4} \right) - 5 \left( Y^2 + \frac{1}{Y^2} \right) - 5 \left( \frac{1}{X} + X \right) \left( Y + \frac{1}{Y} \right) - 12 = 0.
\]

(5.2)

Proof. Using the equation (1.8) in the equation (4.7) we arrive at the equation (5.2). \(\square\)

Corollary 5.1.

\[
b_{4,5} = \sqrt{\frac{2 + 2\sqrt{5} - 2\sqrt{2} + 2\sqrt{5}}{2}},
\]

(5.3)

\[
b_{1,5} = \sqrt{\frac{2 + 2\sqrt{5} + 2\sqrt{2} + 2\sqrt{5}}{2}}.
\]

(5.4)

Proof. Putting \( s = \frac{1}{2} \), in (5.2) and using the fact that \( b_{1,5}b_{4,5} = 1 \), we deduce

\[
h^8 - 2h^6 - 2h^4 - 2h^2 + 1)(h^2 + h + 1)(h^2 - h + 1) = 0,
\]

(5.5)

where \( h := b_{4,5} \).

We observe that the first factor of (5.5) vanishes for specific value of \( q := e^{-\pi\sqrt{4/5}} \), whereas the other factors does not vanish. Hence, we have

\[
t^2 - 2t - 4 = 0,
\]

(5.6)

where \( t := h^2 + \frac{1}{h^2} \).

On solving the equation (5.6) for \( h \) and \( t > 0 \), we deduce

\[
h^2 + \frac{1}{h^2} = 1 + \sqrt{5}.
\]

(5.7)

On solving the equation (5.7) for \( h \) and \( 0 < h < 1 \), we arrive at (5.3) and (5.4). \(\square\)

Theorem 5.2. If \( X = \sqrt{b_{s,5}b_{16s,5}} \) and \( Y = \sqrt{\frac{b_{s,5}}{b_{16s,5}}} \), then

\[
\left( Y^8 + \frac{1}{Y^8} \right) - 52 \left( Y^6 + \frac{1}{Y^6} \right) - 1024 \left( Y^4 + \frac{1}{Y^4} \right) - 3987 \left( Y^2 + \frac{1}{Y^2} \right) - 5 \left( \frac{1}{X} + X \right) \left( Y + \frac{1}{Y} \right) - 1076 \left( Y^2 + \frac{1}{Y^2} \right) + 384 \left( Y^4 + \frac{1}{Y^4} \right) + 35 \left( Y^3 + \frac{1}{Y^3} \right) - 5^2 \left( X^2 + \frac{1}{X^2} \right) \left[ 92 \left( Y^2 + \frac{1}{Y^2} \right) + 8 \left( Y^4 + \frac{1}{Y^4} \right) + 149 \right] - 5^3 \left( X^3 + \frac{1}{X^3} \right) - 12 \left( Y + \frac{1}{Y} \right) + 5^4 \left( X^4 + \frac{1}{X^4} \right) - 7676 = 0.
\]

(5.8)
Proof. Using the equation (1.8) in the equation (4.12) we arrive at the equation (5.8). \(\square\)

**Corollary 5.2.**

\[ b_{8,5} = \sqrt{(\sqrt{2} - 1)(\sqrt{5} - 2)}, \]  
(5.9)

\[ b_{1/2,5} = \sqrt{(\sqrt{2} + 1)(\sqrt{5} + 2)}. \]  
(5.10)

**Proof.** Putting \( s = 1/4 \), in (5.8) and using the fact that \( b_{1/2,5}b_{8,5} = 1 \), we deduce

\[
(h^8 - 8h^6 - 22h^4 - 8h^2 + 1) (h^4 + 3h^2 + 1) (h^4 - h^3 + h^2 + h + 1)
\]
(5.11)

where \( h := b_{8,5} \).

We observe that the first factor of (5.11) vanishes for specific value of \( q := e^{-\pi\sqrt{8/5}} \), whereas the other factors does not vanish. Hence, we have

\[
t^2 - 8t - 24 = 0,
\]  
(5.12)

where \( t := h^2 + \frac{1}{h^2} \).

On solving the equation (5.12) for \( h > 0 \), we deduce

\[
h^2 + \frac{1}{h^2} = 4 + 2\sqrt{10}.
\]  
(5.13)

On solving the equation (5.13) for \( h > 0 < h < 1 \), we arrive at (5.9) and (5.10). \(\square\)

**Theorem 5.3.** If \( X = \sqrt{b_{8,5}b_{36,8,5}} \) and \( Y = \sqrt{b_{8,5}b_{36,8,5}} \), then

\[
Y^{16} - 363Y^{14} - 30882Y^{12} - 698682Y^{10} - 6183702Y^8 - 16140317Y^6
\]
+ 37225608Y^4 + 231497788Y^2 + 5X \{ 60133800Y + 21753498Y^3
- 1148442Y^5 - 2210604Y^7 - 406488Y^9 - 26740Y^{11} - 519Y^{13} \}
+ 5^2X^2 \{ 6287236Y^2 + 858465Y^4 - 462222Y^6 - 150099Y^8 - 12840Y^{10}
- 267Y^{12} + 1029305 \} + 5^3X^3 \{ 1132002Y + 362832Y^3 - 42462Y^5
- 37066Y^7 - 4323Y^9 - 78Y^{11} \} + 5^4X^4 \{ 74418Y^2 + 4417Y^4 - 5955Y^6
- 1026Y^8 - 12Y^{10} + 130902 \} + 5^5X^5 \{ 9171Y + 2028Y^3 - 588Y^5 - 171Y^7
- Y^9 \} + 5^6X^6 \{ 300Y^2 - 27Y^4 - 18Y^6 + 679 \} + 5^7X^7 \{ 24Y - Y^5 \} + 5^8X^8
+ 36965548 = 0,
\]  
(5.14)
where $X^n = \left( X + \frac{1}{X} \right)$ and $Y^n = \left( Y + \frac{1}{Y} \right)$.

**Proof.** Using the equation (1.8) in the equation (4.14) we arrive at the equation (5.14). □

**Corollary 5.3.**

\[ b_{12,5} = \sqrt{\frac{(2 - \sqrt{3})(7 - 3\sqrt{5})}{2}} \]  

(5.15)

\[ b_{1/3,5} = \sqrt{\frac{(2 + \sqrt{3})(7 + 3\sqrt{5})}{2}} \]  

(5.16)

**Proof.** Putting $s = 1/6$, in (5.14) and using the fact that $b_{1/3,5}b_{12,5} = 1$, we deduce,

\[
(h^8 - 28h^6 + 63h^4 - 28h^2 + 1) \left( h^{12} + 10h^{10} + 15h^8 + 28h^6 + 15h^4 + 10h^2 + 1 \right) \\
(h^8 - 2h^7 + 4h^6 - h^5 + 7h^4 + 3h^3 + 4h^2 + 2h + 1) \left( h^8 + 2h^7 + 4h^6 + h^5 + 7h^4 - h^3 + 4h^2 - 2h + 1 \right) = 0,
\]

(5.17)

where $h := b_{12,5}$.

We observe that the first factor of (5.17) vanishes for specific value of $q := e^{-\pi \sqrt{12/5}}$, whereas the other factors does not vanish. Hence, we have

\[ t^2 - 28t + 61 = 0, \]  

(5.18)

where $t := h^2 + \frac{1}{h^2}$.

On solving the equation (5.18) for $h$ and $t > 0$, we deduce

\[ h^2 + \frac{1}{h^2} = 14 + 3\sqrt{15}. \]  

(5.19)

On solving the equation (5.19) for $h$ and $0 < h < 1$, we arrive at (5.15) and (5.16). □

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