Cramér-Rao Bounds for Direction Estimation in the Presence of Circular and Noncircular Signals

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ABSTRACT In direction finding, the use of noncircularity of signals arriving at a sensor array can allow us to improve estimation performance. It will be more realistic, rather than the assumption of all noncircular sources, to consider some sources transmitting circular signals. In the presence of both circular and noncircular signals, this paper presents the deterministic Cramér-Rao bound (CRB) of direction estimation. The derivation of the CRB is quite general so that the bounds with unknown signal structures and with known waveforms can be readily obtained from it. Moreover, we show that the more information on signal structures is available the less the CRB is. And we present general conditions under which the CRBs for the direction parameters obtained from different Fisher information matrices are equal. Based on these basic theoretical results, the bounds with different information are analytically compared. Numerical results confirm the theoretical results.

INDEX TERMS Direction estimation, Cramér-Rao bound, circular and noncircular signals, signal structures.

I. INTRODUCTION
The noncircularity of signals arriving at a sensor array can be incorporated into the direction estimation. Unless the average of the squared complex envelope of a signal, as in binary phase shift keying (BPSK) and amplitude modulation (AM), is zero the noncircularity holds. Various high-resolution methods for direction estimation [1]–[8] have been developed that utilize the nonzero average of the squared envelope. With the utilization of the noncircular property, we can not only improve estimation accuracy but also resolve them even if the number of signals is larger than the number of sensors.

The Cramér-Rao bound (CRB) provides a lower bound on the variance of a parameter estimated by an unbiased estimator [9]. We can evaluate how well unbiased estimators work in terms of mean squared error (MSE) by comparing estimation results with the CRB. Furthermore, through the analysis of CRB under various signal scenarios, achievable performance can be readily obtained. Incident signals can be viewed by two different, deterministic and stochastic, models [9]–[11]. In the scenario of fully noncircular signals, the CRBs for the models are found in the literature of direction estimation. The deterministic CRB can be attained from a unified approach based on a general signal structure [12].

The bounds for multidimensional parameters that can include directions of departure and Doppler shifts in addition to directions of arrival have been derived in [13]. As for the stochastic approach, the CRB is addressed under the Gaussian distribution of incoming signals that are fully noncircular [14] and that are partially noncircular [15]. Hereafter, by “noncircular”, we mean “fully noncircular”.

It will be more realistic to consider some sources transmitting circular signals as well as noncircular ones. However, the CRBs in [12]–[14] have been obtained under the assumption that incident signals are all noncircular. In this paper, we derive the deterministic CRB for the direction estimation in the general case where both circular and noncircular signals are present. The CRB matrix is given by the inverse of the Fisher information matrix formed by the derivatives of a log-likelihood function with respect to unknown parameters in the estimation. This general case complicates the derivation of CRB as more parameter types should be handled. Representing some derivatives in complex form, we effectively derive it.

In [16], based on the Slepian-Bangs formula, a closed-form expression of the deterministic CRB for a mixture of circular and noncircular signals has been presented. However, the computation of CRB with the expression is very intensive, particularly when the number, \( N \), of snapshots is large, because it requires the inversion of a square matrix.
of size larger than KN where K is the number of incident signals. In contrast, the computation by the derived result, which is obtained through a systematic approach based on the repeated application of block matrix inversion, needs only the which is obtained through a systematic approach based on the repeated application of block matrix inversion, needs only the inversion of matrices of size equal to or less than K, regardless of N, which allows us to effectively find the bounds without the computational difficulty in matrix inversion even if N is very large.

The derivation of this paper is quite general so that other bounds with different information on signal structures can be attained from it. The bounds when no information is available [17] and when signal waveforms are known [18] are dealt with using the derived result. Although Fisher information matrices have different forms, the bounds for direction parameters can be the same. We present general conditions for the same bound. Moreover it is shown that CRB becomes smaller as more information is available. Using these basic theoretical results, we analytically compare the bounds with different information. Numerical calculation is performed, which confirms the theoretical results.

Superscripts T, *, and H designate transpose, complex conjugate, and complex conjugate transpose, respectively, and E[·] stands for expectation. Matrices and vectors are in bold type. Symbol I denotes an identity matrix of appropriate size and diag[·] stands for a diagonal matrix. For a complex matrix \( Z, P_Z \) represents a projection matrix onto the null space of \( Z \). If \( Z \) is positive definite (semidefinite), we denote it by \( Z > 0 \) (\( Z \geq 0 \)). The notations \( \bar{Z} = \text{Re}(Z) \) and \( \bar{Z} = \text{Im}(Z) \) where \( \text{Re} \) and \( \text{Im} \) represent real and imaginary parts of complex numbers.

II. PROBLEM FORMULATION

K narrowband signals with the complex envelopes \( s(t) = [s_1(t), \ldots, s_K(t)]^T \) at time \( t \) impinge on an array of \( M \) sensors from \( \theta = [\theta_1, \ldots, \theta_K]^T \) where \( s_k(t) \) and \( \theta_k \) denote, respectively, the complex envelope of the \( k \)th incident signal and its arrival angle. The array output vector \( x(t) \) can be written as

\[
x(t) = A(\theta)s(t) + n(t)
\]  

(1)

where \( n(t) \) is the noise vector and

\[
A(\theta) = [a(\theta_1), \ldots, a(\theta_K)]
\]  

(2)

with \( a(\theta) \) denoting the array response vector for a direction \( \theta \).

Noise is assumed to be a zero mean white Gaussian random process with a variance of \( \sigma^2 \) and to be uncorrelated from element to element. To handle various information on signal structures, we employ a general representation for the complex envelope vector as

\[
s(t) = By(t)
\]  

(3)

where \( B \), which can be constant or depend on time, is a diagonal matrix. The incident signals consists of \( \kappa_u \) noncircular and \( \kappa_c \) circular signals. Then \( s(t) \) can be generally expressed as

\[
s(t) = \begin{bmatrix}
s_u(t) \\
0 \\
s_c(t)
\end{bmatrix} =
\begin{bmatrix}
B_u & 0 \\
0 & B_c
\end{bmatrix}
\begin{bmatrix}
y_u(t) \\
y_c(t)
\end{bmatrix}
\]  

(4)

where \( s_u(t) \) and \( s_c(t) \) are the complex envelope vectors of noncircular and circular, respectively. Signals. In the representation of (4), some parameters can be contained in either \( B_u \) or \( y_u(t) \) and in either \( B_c \) or \( y_c(t) \). For example, in case the complex envelopes are expressed in terms of magnitudes and phases regardless of circularity, the elements of \( y_u(t) \) and \( y_c(t) \), representing magnitudes, are positive while the diagonal elements of \( B_u \) and \( B_c \) are given by complex exponential functions with unit magnitudes. They will be properly defined for the convenience of analysis on CRB taking account of signal structures known.

The initial phase vector for the noncircular signals is denoted by

\[
\phi_u = [\phi_1, \ldots, \phi_{\kappa_u}]^T
\]  

(5)

where \( \phi_k \) is the initial phase of the \( k \)th signal. In the derivation of CRB with the circularity known, we set \( B_u = \text{diag}[e^{i\phi_1}, \ldots, e^{i\phi_{\kappa_u}}] \) and \( B_c = I \). Accordingly \( y_u(t) \) becomes a real vector and \( y_c(t) \) is equal to \( s_c(t) \). Then (1) is expressed as

\[
x(t) = A(\theta_u)B_u y_u(t) + A(\theta_c)s_c(t) + n(t)
\]  

(6)

where

\[
\theta_u = [\theta_1, \ldots, \theta_{\kappa_u}]^T
\]  

(7a)

\[
\theta_c = [\theta_{\kappa_u+1}, \ldots, \theta_K]^T.
\]  

(7b)

Suppose that \( N \) observations \( x(1), \ldots, x(N) \) are available. The CRB depends on unknown parameters in the probability density function. Given the \( N \) observations, the log-likelihood function is expressed as

\[
L = c - MN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{t=1}^{N} \| x(t) - A(\theta)s(t) \|^2
\]  

(8)

where \( c \) is a constant independent of the parameters and \( \| \cdot \| \) denotes the Euclidean norm. The CRB matrix can be attained by inverting the Fisher information matrix, the elements of which are related to derivatives of \( L \) with respect to unknown parameters. For a complex vector \( v \), we use the notational convention \( \partial L/\partial v = \partial L/\partial \bar{v} + j \partial L/\partial \bar{v} \). Differentiating (8) with respect to \( \sigma^2, s_u(t), y_u(t), \phi_u(t), \theta_u \) and \( \theta \) yields

\[
\frac{\partial L}{\partial \sigma^2} = -\frac{MN}{\sigma^2} + \frac{1}{\sigma^4} \sum_{t=1}^{N} n^H(t)n(t)
\]  

(9a)

\[
\frac{\partial L}{\partial s_u(t)} = \frac{2}{\sigma^2} A^H(\theta_u)n(t)
\]  

(9b)

\[
\frac{\partial L}{\partial y_u(t)} = \frac{2}{\sigma^2} \text{Re}(B_u^H A^H(\theta_u)n(t))
\]  

(9c)

\[
\frac{\partial L}{\partial \phi_u} = \frac{2}{\sigma^2} \sum_{t=1}^{N} \text{Im}(S_u^H(t)A^H(\theta_u)n(t))
\]  

(9d)

\[
\frac{\partial L}{\partial \theta} = \frac{2}{\sigma^2} \sum_{t=1}^{N} \text{Re}(S_u^H(t)D^H(\theta_u)n(t))
\]  

(9e)
where
\[ S_u(t) = \text{diag}(s_1(t), \ldots, s_{K_u}(t)) \quad (10a) \]
\[ S(t) = \text{diag}(s_1(t), \ldots, s_K(t)) \quad (10b) \]
\[ D(\theta) = \left[ \frac{\partial a(\theta)}{\partial \theta_1}, \ldots, \frac{\partial a(\theta_K)}{\partial \theta_K} \right]. \quad (10c) \]

The information matrix can be formed using the derivatives. In the next section we derive the CRB matrix for unbiased estimates of \( \theta \).

### III. Derivation of Cramér-Rao Bounds

The unknown parameters are first divided into three groups, namely complex, real, and direction groups, in such a way that
\[
p = [s_c^T, \beta^T, \theta^T]^T
\]
where
\[ s_u = [s_u^T(1), s_u^T(N), \ldots, s_u^T(N), s_u^T(N)]^T, \quad \alpha = c \text{ or } u \quad (12a) \]
\[ \beta = [\gamma_u^T, \phi_u^T]^T \quad (12b) \]
with \( \gamma_u = [\gamma_u^T(1), \gamma_u^T(N)]^T \). The vector \( s_u \) is used later. The numbers of parameters contained in the real-valued vectors \( s_c, \beta, \) and \( \theta \) are \( 2K_c N, K_u(N+1), \) and \( K \), respectively, and the size of \( p \) is \( N(K + K_c) + K_u + K \). The Fisher information matrix is given by
\[ F = E(\mathbf{h}h^T) \quad (13) \]
where
\[ h = \frac{\partial L}{\partial \theta}. \quad (14) \]

For the sake of simplicity, we omitted the term \( \partial L/\partial \sigma^2 \) in (14) as its correlations with the other derivatives are zero, i.e.,
\[ E \left[ \frac{\partial L}{\partial \sigma^2} \left( \frac{\partial L}{\partial \sigma^2} \right)^T \right] = 0. \quad (15) \]

The vector \( h \), which is the differentiation of the log-likelihood function with respect to the parameter vector, is termed the information vector. The information matrix can be represented using the following complex matrices that are associated with derivatives with respect to the parameters:
\[ C_{s_c \theta} = A_h^H A_c \quad (16a) \]
\[ C_{s_c \gamma} = A_h^H A_u B_u \quad (16b) \]
\[ C_{s_c \phi} = j A_h^H A_u S_u(n) \quad (16c) \]
\[ C_{s_c \theta} = A_h^H D(\theta) S(n) \quad (16d) \]
\[ C_{s_c \gamma} = B_u^H A_c^H A_u B_u \quad (16e) \]
\[ C_{s_c \phi} = j B_u^H A_c^H A_u S_u(n) \quad (16f) \]
\[ C_{\gamma u \theta} = B_u^H A_c^H D(\theta) S(n) \quad (16g) \]
\[ C_{\gamma u \phi} = S_u^H(n) A_u^H A_c^H A_u S_u(n) \quad (16h) \]
\[ C_{\phi u \gamma} = -j S_u^H(n) A_u^H D(\theta) S(n) \quad (16i) \]
\[ C_{\phi u \phi} = S_u^H(n) D(\theta)^2 D(\theta) S(n) \quad (16j) \]

where \( A_u = A(\theta_u) \) and \( A_c = A(\theta_c) \). The CRB matrix is the inverse of \( F \), which can be obtained through the inversion of partitioned matrices. Under known circularity (KC), the CRB matrix for \( \theta \) is represented as
\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left[ \Gamma - \Omega^T \Psi^{-1} \Omega \right]^{-1} \quad (17) \]

where
\[
\Gamma = \sum_{n=1}^{N} (\mathbf{R}_{\theta \theta, n} - \mathbf{R}_{\gamma_u \gamma_u})^{-1} \quad (18a) \]
\[
\Omega = \sum_{n=1}^{N} (\mathbf{R}_{\gamma_u \gamma_u, n} - \mathbf{R}_{\theta \theta, n}^{-1}) \quad (18b) \]
\[
\Psi = \sum_{n=1}^{N} (\mathbf{R}_{\gamma_u \gamma_u, n} - \mathbf{R}_{\theta \theta, n}^{-1}) \quad (18c) \]

For derivation, see Appendix A. The bound is linearly proportional to \( \sigma^2 \). The matrices \( \mathbf{R}_{\gamma_u \gamma_u}, \mathbf{R}_{\gamma_u \phi}, \) and \( \mathbf{R}_{\theta \theta} \) are symmetric. Clearly \( \text{CRB}(\theta) \) is symmetric.

The deterministic CRB for a mixture of circular and noncircular signals has been presented in [16]. When calculating the CRB according to (17) or the result of [16], the most noticeable difference consists in matrix inversion. To compute \( \text{CRB}(\theta) \) by (17) needs the inversion of the square matrices \( C_{s_c \gamma}, C_{s_c \phi}, \Psi, \) and \( \text{CRB}^{-1}(\theta) \), which are of size \( K_c, K_u, K_u, \) and \( K \), respectively. To invert the matrices requires \( O(K^3) \) multiplications. As discussed in Appendix A, the computation of CRB by the formula in [16] requires the inversion of a matrix of size \( N_d \), which costs \( O(N_d^3) \) multiplications, where
\[ N_d = N(K + K_c) + K_u. \quad (20) \]

The computational load depends on \( N \) so that it can be extremely heavy when \( N \) is large. Given \( N \) and \( K \), the minimum of \( N_d \) is \( K(N + 1) \) when \( K_u = K \). Even the lowest
The CRB matrix is given by

\[ C \]  

the removal of the terms with the other circular ones has 2 unknown parameters are known or are not related to a given signal. In (11), the noncircularity of the other signals except the first one is unknown.

The derivation of the CRB, which has been made under a signal model of (6), is quite general so that it can allow us to find bounds in different signal scenarios. To this end, it is necessary to use the symbols \( \kappa_u \) and \( \kappa_s \) flexibly regardless of circularity. In (11), \( s_c \) and \( \beta \) have 2N and N + 1, respectively, unknown parameters per signal. We use \( \kappa_u \) and \( \kappa_s \) to denote the numbers of signals with the N + 1 unknown parameters and with the 2N unknown parameters, respectively. In (19), each \( C \) matrix is associated with the derivatives of \( L \) with respect to the parameters which subscripts indicate. If some parameters are known or are not related to a given signal scenario, the rows and columns of the Fisher information matrix corresponding to those should be eliminated, which results in removing the terms in (19) with the \( C \) matrices associated with at least one of known or unrelated parameters.

In a case where no information on signal structures is available, which is referred to as unknown information (UI), each complex envelope of the incident signals including noncircular ones has 2N unknown parameters. Hence, the CRB can be found from (17) by setting \( \kappa_c = K \), which results in the removal of the terms with the other \( C \) matrices except \( C_{s_c\kappa_c} \) and \( C_{s_u\kappa_s} \) in (19). And \( P_A^{\perp} \) becomes \( P_A^{\perp}(\theta) \). Then \( \Omega \) and \( \Psi \) are totally removed and \( \Gamma \) is reduced to \( \sum_{n=1}^{N} R_{\theta,n} \). The CRB matrix is given by

\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left[ \Gamma_{kw} - \text{Re}(\Omega_{kw}^{H} \Psi_{kw}^{-1} \Omega_{kw}) \right]^{-1} 
\]  

(21)

Equation (21) corresponds to the result derived in [17]. If all the incident signals are circular, i.e., actually \( \kappa_c = K \), the CRB has the same form as (21). On the contrary, if they are all noncircular so that \( K = \kappa_u \), the terms with subscript \( s_c \) should be eliminated, which gives rise to the replacement of \( P_A^{\perp} \) in (19) with an identity matrix. The CRB matrix is the same as (17) with the replacement, which corresponds to the CRB for AM in [12].

Let us deal with a case of known waveform (KW). To this end, we express \( B \) and \( \gamma(t) \) in (3) as

\[
B = \text{diag}[e^{j\phi_1}, \ldots, e^{j\phi_K}]
\]

\[
\gamma(t) = [\alpha_1 \gamma'_1(t), \ldots, \alpha_K \gamma'_K(t)]^T
\]  

(22)

(23)

where \( \alpha_k \) is a real constant and \( \gamma'_k(t) \), which is complex, represents a waveform of the \( k \)th signal. In (22) all initial phases of incident signals are contained in \( B \), whereas in (6) the initial phases of circular signals are included in \( \gamma_c(t) \). When \( \gamma_c(t) = [\gamma'_1(t), \ldots, \gamma'_K(t)]^T \) is known, the CRB is given by [18]

\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left[ \Gamma_{kw} - \text{Re}(\Omega_{kw}^{H} \Psi_{kw}^{-1} \Omega_{kw}) \right]^{-1} 
\]  

(24)

where

\[
\Gamma_{kw} = \sum_{n=1}^{N} \text{Re}(S_n^{H}(n)D^{H}(\theta)D(\theta)S(n)) 
\]  

(25a)

\[
\Omega_{kw} = \sum_{n=1}^{N} S_n^{H}(n)A^{H}(\theta)A(\theta)S(n) 
\]  

(25b)

\[
\Psi_{kw} = \sum_{n=1}^{N} S_n^{H}(n)A^{H}(\theta)A(\theta)S(n). 
\]  

(25c)

Suppose that \( \alpha = [\alpha_1, \ldots, \alpha_K]^T \) is also known. In the case that every \( \gamma'_k(t), k = 1, \ldots, K, \) is known, which is termed the known waveform plus (KW+), the rows and columns of the Fisher information matrix associated with the derivatives with respect to \( \gamma_k(t) \) should be removed. The CRB for KW+ can be obtained from (17) through such removal. Putting \( \kappa_u = K \) and removing the terms with subscript \( u \) in (19), we have

\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left[ \Gamma_{kw+} - \Omega_{kw+}^{H} \Psi_{kw+}^{-1} \Omega_{kw+} \right]^{-1} 
\]  

(26)

where \( \Gamma_{kw+} = \Gamma_{kw}, \Omega_{kw+} = \text{Im}(\Omega_{kw}), \) and \( \Psi_{kw+} = \text{Re}(\Psi_{kw}) \). The same bound as (26) is found in [18]. As \( \kappa_u \) is set at \( K \), every \( \gamma'_k(t) \) is regarded as real. Even if they are complex, the derivation of (26) from (17) is valid, which is justified below (27) in Section IV.

IV. COMPARISON OF CRAMÉR-RAO BOUNDS

Here we compare CRBs for the cases of UI, KC, KW, and KW+. Their parameter vectors can be written as

\[
p_{ui} = [s_T, \theta_T]^T 
\]  

(27a)

\[
p_{kc} = [s_c, \gamma'_u, \Phi_u, \theta_T]^T 
\]  

(27b)

\[
p_{kw} = [\alpha_T, \phi_T, \theta_T]^T 
\]  

(27c)

\[
p_{kw+} = [\Phi_T, \Theta_T]^T 
\]  

(27d)

where

\[
s = [s_T, s_c_T]^T 
\]  

(28a)

\[
\phi = [\phi_{k+1}, \ldots, \phi_k]^T 
\]  

(28b)

with \( \phi_k = [\phi_{k+1}, \ldots, \phi_k]^T \) and \( s_n \) defined in (12a). Even if \( \gamma(t) \) is complex the CRB for KW+ is obtained as (26) although it has been derived from (17) on the premise that \( \gamma_u(t) \) is real. To show it, suppose for a moment that \( \gamma_u(t) \) is complex. It is easy to see that

\[
\frac{\partial L}{\partial \gamma_u(t)} = \frac{2}{\sigma^2} P_u^{H} A^{H}(\theta_u)n(t) 
\]  

(29)
Note that \( \partial L / \partial \tilde{y}_n(t) \) is identical to (9c). The derivatives \( \partial L / \partial \phi_u \) and \( \partial L / \partial \theta \) are still equal to (9d) and (9e) even though \( y_u(t) \) is a complex vector. Once the derivatives of (9) are given, the Fisher information matrix of (13) has the same form irrespective of whether \( y_u(t) \) is a real vector or a complex vector. Hence, when the parameter vector is \( p = [\phi_u^T, \theta^T]^T \), the CRB matrix for direction estimates is given by (17) with \( \kappa_u = K \). The parameter vector \( p_{kw+} \) is the same as the \( p \) with \( \tilde{y}_n \) removed. Therefore, for complex \( y(t) \), the CRB for KW+ can be found from (17) by putting \( \kappa_u = K \) and removing the terms with subscript \( y_{an} \) in (19).

In (6), \( s(t) \) is formulated in accordance with the circularity known in advance. Considering the case where the information on circularity is not available, we differently define \( B \) in (4), which is dependent on time so that \( s_n(t) \) is represented as

\[
s_n(t) = B_u(t)y_u(t). \tag{30}
\]

For comparison of CRBs, we need another form of the parameter vector for UI. In the case of UI, \( B_u(t) \) can be written as

\[
B_u(t) = \text{diag}[e^{i\phi_1(t)}, \ldots, e^{i\phi_{\kappa_u}(t)}] \equiv B_{\kappa_u}(t) \tag{31}
\]

where \( \phi_k(t) \) is the phase of the \( k \)th signal. Let \( y_u(t) \) be \( y_{\kappa_u}(t) \) when \( B_u(t) = B_{\kappa_u}(t) \). The elements of \( y_{\kappa_u}(t) \), which denote the magnitudes of the complex envelopes, are all positive and are related to those of \( y_u(t) \) in (6) by

\[
y_{\kappa_u,k}(t) = |y_k(t)|, \quad k = 1, \ldots, \kappa_u. \tag{32}
\]

With these notations, alternatively the parameter vector of UI can be expressed as

\[
p_{\kappa_u1} = [s_c^T, y_{\kappa_u1}^T, \phi_{\kappa_u1}^T, \theta^T]^T \tag{33}
\]

where

\[
y_{\kappa_u1} = [y_{\kappa_u1}(1), \ldots, y_{\kappa_u1}(N)]^T \tag{34a}
\]

\[
\phi_{\kappa_u1} = [\phi_{\kappa_u1}(1), \ldots, \phi_{\kappa_u1}(N)]^T \tag{34b}
\]

with \( \phi_{\kappa_u1}(t) = [\phi_1(t), \ldots, \phi_{\kappa_u}(t)]^T \). The complex envelopes of the noncircular signals in \( p_{\kappa_u1} \) are represented in polar form. The information vector corresponding to a parameter vector is denoted with the same subscript, and CRB(v|h) designates the CRB matrix of \( v \) when the information vector is \( h \). Accordingly CRB(\( \theta | h_{\kappa_u} \)), CRB(\( \theta | h_{\kappa_u1} \)), and CRB(\( \theta | h_{\kappa_u+} \)) are identical to (17), (21), (24), and (26), respectively. Before the bounds are compared, we first introduce basic theorems that are useful for the analysis of CRB.

In the theorems, the parameters of interest are \( \theta \).

**Theorem 1:** Matrices \( Z_1 \) and \( Z_2 \) are positive definite Hermitian. \( Z_1 > Z_2 \) \((Z_1 \geq Z_2)\) if and only if \( Z_1^{-1} < Z_2^{-1} \) \((Z_1^{-1} \leq Z_2^{-1})\).

**Theorem 2:** Let \( p_i = [v_i^T, \theta_i^T]^T \), \( i = 1, 2 \), where \( v_1 \) and \( v_2 \) are related by \( v_2 = [v_0^T, v_1^T]^T \). Then,

\[
\text{CRB}(\theta | h_1) \leq \text{CRB}(\theta | h_2). \tag{35a}
\]

In addition, if the matrix \( E[(\partial L/\partial v_0)(\partial L/\partial p_1)^T] \) is of full column rank,

\[
\text{CRB}(\theta | h_1) < \text{CRB}(\theta | h_2). \tag{35b}
\]

**Theorem 3:** Two parameter vectors \( p_i = [v_i^T, \theta_i^T]^T \), \( i = 1, 2 \), have the same size. If \( \partial L / \partial v_1 = Q \partial L / \partial v_2 \) and the matrix \( Q \) is nonsingular and independent of the expectation operation, the CRB for \( \theta \) is irrelevant to \( Q \) so that

\[
\text{CRB}(\theta | h_2) = \text{CRB}(\theta | h_1). \tag{36}
\]

The proof of Theorem 1 is given in Appendix B. Equations (35) and (36) are derived in Appendix C. In (35) CRB(\( \theta \)) is a matrix. We are concerned with its diagonal entries, and denote the \( k \)th diagonal by CRB(\( \theta_k | h_1 \)). Even if the matrix in Theorem 2 is not of full column rank, the inequality of CRB(\( \theta_k | h_1 \)) \(<\) CRB(\( \theta_k | h_2 \)) would hold in most cases because very specific conditions must be satisfied in order that CRB(\( \theta_k | h_1 \)) \(<\) CRB(\( \theta_k | h_2 \)). As more information on signal structures is known the size of the parameter vector becomes smaller. Theorem 2 implies that the more information is available the less the CRB is.

When \( \kappa_{uk} \) smaller than the actual number \( \kappa_u \) in (17) is used, the corresponding parameter vector \( p_{k+1} \) has a larger size than \( p_{uc} \). It can be written as \( p_{k+1} = [v^T, p_{k+1}^T]^T \) where \( v \) consists of \((\kappa_u - \kappa_{uk})\) parameters. According to (35a)

\[
\text{CRB}(\theta | h_{k+1}) \leq \text{CRB}(\theta | h_{k+1}). \tag{37}
\]

Equation (37) implies that the CRB increases as \( \kappa_{uk} \) decreases.

The complex envelopes of noncircular signals in (33) are represented in polar form while those in (27a) are in rectangular form. The CRB(\( \theta \)) would be irrelevant to the representation of complex envelopes and thus

\[
\text{CRB}(\theta | h_{ui}) = \text{CRB}(\theta | h_{ui}). \tag{38}
\]

Let us show it. Replacing \( s_n(t) \) in (4) by \( B_{u1}(t)y_{u1}(t) \) and taking the derivative of (8) with respect to \( \phi_{u1}(t) \), we have

\[
\frac{\partial L}{\partial \phi_{u1}(t)} = \frac{2}{\sigma^2} \text{Im}(S_u^H(t)A^H(\theta)m(t)). \tag{39}
\]

The derivatives with respect to \( s_n(t) \) and with respect to \( y_{u1}(t) \) are the same, respectively, as (9b) with the change from subscript \( c \) to \( u \) and as the right side of (9c) except that \( B_u \) is replaced by \( B_{u1}(t) \). It is straightforward to obtain

\[
\begin{bmatrix}
\frac{\partial L/\partial y_{u1}(t)}{\partial L/\partial \phi_{u1}(t)}
\end{bmatrix} = T_u(t) \begin{bmatrix}
\frac{\partial L/\partial s_n(t)}{\partial L/\partial \phi_{u1}(t)}
\end{bmatrix}\tag{40}
\]

where

\[
T_u(t) = \begin{bmatrix}
\bar{B}_{u1}(t) & \bar{B}_{a1}(t) \\
-S_{n1}(t) & S_u(t)
\end{bmatrix}. \tag{41}
\]

Clearly \( T_u(t) \) is nonsingular. Equation (40) brings about the transformation matrix \( Q \) that is nonsingular and independent of the expectation operation. Theorem 3 leads to (38). In the case of \( N = 1 \), the explicit expression of \( Q \) is given in Appendix D.
When only a single snapshot $x(1)$ is available, information on signal structures known in advance as in the cases of KC and KW cannot be exploited for direction estimation. For example, even though $y'(t)$ is known at every sampling time, the known waveforms cannot be incorporated into direction estimation if only one snapshot is available. Therefore, it is expected that the CRBs with $N = 1$ for UI, KC, and KW will be the same. When $N = 1$, the number of unknown parameters, excluding the noise variance, in the case of KW is $3K$ and is the same as that in UI. However, the size of $p_{kw}$ is $2K$. According to Theorems 2 and 3, we have

$$
\text{CRB}_1(\theta|\theta_{kw+}) \leq \text{CRB}_1(\theta|\theta_{kw}) = \text{CRB}_1(\theta|\theta_{kc})
$$

where $\text{CRB}_1(\theta)$ designates the CRB when $N = 1$ and the single snapshot is $x(n)$. It is easy to see by comparing (21) and (24) that $\text{CRB}_1(\theta|\theta_{kw}) = \text{CRB}_1(\theta|\theta_{uk})$. The equality $\text{CRB}_1(\theta|\theta_{uk}) = \text{CRB}_1(\theta|\theta_{kc})$ is clearly shown using (36) in Appendix D. The relationships of (42) are valid at arbitrary single snapshot $x(n)$ as well as at $x(1)$ and so $\text{CRB}_1(\theta|\theta_{kw}) = \text{CRB}_1(\theta|\theta_{uk})$. Using (17) and (18) together with the equality, we can have a different form of (21) as

$$
\text{CRB}(\theta|\theta_{uk}) = \frac{\sigma^2}{2} \left[ \Gamma_n - \Omega_n \Psi_n^{-1} \Omega_n \right]^{-1}
$$

where

$$
\Gamma_n = \mathcal{R}_{\theta,u,n} - \mathcal{R}_{\phi,u,n} \mathcal{R}^{-1}_{\phi,u,n} \mathcal{R}_{\phi,u,n}, \quad \Omega_n = \mathcal{R}_{\phi,u,n} - \mathcal{R}_{\phi,u,n} \mathcal{R}^{-1}_{\phi,u,n} \mathcal{R}_{\phi,u,n}, \quad \text{and} \quad \Psi_n = \mathcal{R}_{\phi,u,n} \mathcal{R}^{-1}_{\phi,u,n} \mathcal{R}_{\phi,u,n}.
$$

In the inequality of (37), $\text{CRB}(\theta|\theta_{kc})$ becomes equal to $\text{CRB}(\theta|\theta_{kc})$ when $N = 1$. If $N = 1$, the sizes of $p_{kc}$ and $p_u$ are identical to Theorem 3 the bounds become the same. It is interesting that (43) holds even if $\kappa_u$ is replaced by $\kappa_{ak}$.

The entries of $\mathbf{y}_u$ and $\mathbf{y}_u$ can differ only in sign. Consider the parameter vector $\mathbf{p}_{au} = [s^T, \mathbf{y}_u^T, \phi^T, \mathbf{u}^T]$ where $\phi^T$ is equal to $\phi$ in (33) except that their entries can differ only in sign. It is obvious according to (36) that $\text{CRB}(\theta|\theta_{uk}) = \text{CRB}(\theta|\theta_{uk})$. As a result, $\text{CRB}(\theta|\theta_{uk}) = \text{CRB}(\theta|\theta_{uk})$. The phase $\phi_{u,2}(1)$ is identical to $\phi_u$ in (27b). By (35a), $\text{CRB}(\theta|\theta_{uk}) \leq \text{CRB}(\theta|\theta_{uk})$. Consequently,

$$
\text{CRB}(\theta|\theta_{uk}) \leq \text{CRB}(\theta|\theta_{uk}) \leq \text{CRB}(\theta|\theta_{uk}) \leq \text{CRB}(\theta|\theta_{uk}).
$$

The sizes of $p_{kw}$ and $p_{kw}$ are constant regardless of $N$. But the sizes of $p_{uk}$ and $p_{uk}$ increase with an increase in $N$. If $N$ is a little large so that the full rank condition in Theorem 2 is satisfied, it is guaranteed that $\text{CRB}(\theta|\theta_{kw}) = \text{CRB}(\theta|\theta_{kw})$ irrespective of complex envelopes $s(1), \ldots, s(N)$ and directions $\theta_1, \ldots, \theta_K$.

As seen in (19a), $\mathcal{R}_{\phi,u,n}$ is independent of the sampling time $n$. For simplicity it is denoted as $\mathcal{R}_u$. For sufficiently large $N$, (21) is written as [17]

$$
\text{CRB}(\theta) = \frac{\sigma^2}{2N} \left[ \text{Re}(\mathcal{D}^H(\theta) \mathcal{P}^{-1}_{A_\theta}(\mathcal{D}(\theta))) \cdot \mathbf{R}_u \right]^{-1}
$$

where $\cdot$ denotes the Hadamard product and

$$
\mathbf{R}_u = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{v}(n) \mathbf{v}^H(n).
$$

Similarly the asymptotic CRB for KC can be obtained from (17). Using $\text{Re}(\mathbf{Z}_1(\mathbf{Z}_2)) = \mathbf{Z}_1(\mathbf{Z}_2) - \mathbf{Z}_1(\mathbf{Z}_2)$ and $\text{Im}(\mathbf{Z}_1(\mathbf{Z}_2)) = \mathbf{Z}_1(\mathbf{Z}_2), we have

$$
\text{CRB}(\theta) = \frac{\sigma^2}{2N} \left[ \Gamma - \Omega^T \Omega^{-1} \Gamma \right]^{-1}
$$

where

$$
\Gamma' = \text{Re}(\mathcal{D}^H(\theta) \mathcal{P}^{-1}_{A_\theta}(\mathcal{D}(\theta))) \cdot \mathbf{R}_u - (\mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1) \cdot \mathbf{R}_u - \mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1 \cdot \mathbf{R}_u + (\mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1) \cdot \mathbf{R}_u^T
$$

$$
\Omega' = \text{Im}(\mathcal{D}^H(\theta) \mathcal{P}^{-1}_{A_\theta}(\mathcal{D}(\theta))) \cdot \mathbf{R}_u - (\mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1) \cdot \mathbf{R}_u - \mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1 \cdot \mathbf{R}_u + (\mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1) \cdot \mathbf{R}_u^T
$$

$$
\Psi' = \text{Re}(\mathcal{D}^H(\theta) \mathcal{P}^{-1}_{A_\theta}(\mathcal{D}(\theta))) \cdot \mathbf{R}_u - (\mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1) \cdot \mathbf{R}_u - \mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1 \cdot \mathbf{R}_u + (\mathcal{C}_1^T \mathcal{R}_u^{-1} \mathcal{C}_1) \cdot \mathbf{R}_u^T
$$

with

$$
\mathbf{C}_1 = \mathcal{B}_u^H \mathcal{B}_u \mathcal{P}^{-1}_{A_\theta}(\mathcal{D}(\theta))
$$

$$
\mathbf{C}_2 = \mathcal{B}_u^H \mathcal{B}_u \mathcal{P}^{-1}_{A_\theta}(\mathcal{D}(\theta))
$$

$$
\mathbf{R}_{\mathbf{v},\mathbf{v}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{v}(n) \mathbf{v}^H(n).
$$

The CRB is inversely proportional to $N$.

V. NUMERICAL EXAMPLES

Four signals are incident into a uniform linear array from $\theta_1 = -40^\circ$, $\theta_2 = 0^\circ$, $\theta_3 = -15^\circ$, and $\theta_4 = 20^\circ$ off broadside to the array. The array is composed of six sensors with an interelement spacing of half a wavelength. Every incident signal has the same signal-to-noise ratio (SNR). Employing BPSK for noncircular signals and QPSK for circular signals, we present the average of CRBs, which is computed as $\sum_{k=1}^{K} \text{CRB}(\theta_k)/4$. The unit of degree is used for the measurement of angles.

In Figs. 1 and 2, the incident signals are all noncircular. But the bounds are obtained as if the first $\kappa_u$ signals only are noncircular. Accordingly they are calculated by (17) with $\kappa_u = \kappa_{ak}$ and $\kappa_u = 4 - \kappa_{ak}$. In case $\kappa_u = 4$, the CRB becomes equal to (17) with $\kappa_u = 4$ and $\mathcal{P}^{-1}_{A_\theta}$ replaced by an identity matrix. The bounds for different $\kappa_u$ are shown against SNR at $N = 50$ in Fig. 1, and against $N$ at SNR = 15dB in Fig. 2. It is easy to show from (17) that the CRB for each arrival angle when every signal has the same SNR is inversely proportional to the SNR so is the average, which is seen in Fig. 1.
From Fig. 2, one can see that the bounds are virtually proportional to $N^{-1}$ unless $N$ is small. When $N = 1$, the parameter vectors have the same size regardless of $\kappa_{uk}$. As expected from the theoretical result (36), the CRBs at $N = 1$ are the same. Moreover, Figs. 1 and 2 show that for $N > 1$, the bound decreases with an increase in $\kappa_{uk}$, which is in agreement with (37).

In Figs. 3 and 4, the first two signals are noncircular ($\kappa_u = 2$) and the others are circular. The CRBs for the cases of KC, UI, KW, and KW+ are compared as functions of SNR at $N = 100$ in Fig. 3, and as functions of $N$ at SNR = 20dB in Fig. 4. It is seen from Fig. 3 that all the CRBs are inversely proportional to SNR. Fig. 4 shows that unless $N$ is small the CRBs are essentially proportional to $N^{-1}$, which are consistent with (45) and (47). We note that when $N = 1$, the bounds for KC, UI, and KW are identical and are larger than that for KW+, which confirms (42). Furthermore, the results of Figs. 3 and 4 agree with (44), demonstrating that the more information on signal structures is known the less the bound.

VI. CONCLUSION

The deterministic CRB in the presence of both circular and noncircular signals is given by (17). As (17) is derived in the general signal structure of (6), other bounds such as $\text{CRB}(\theta | h_{ub})$ and $\text{CRB}(\theta | h_{ku})$ can be easily obtained from it by removing the terms related to unnecessary parameters. Even if the supposed number $\kappa_{uk}$ of noncircular signals is less than the actual $\kappa_u$, (17) can still be applicable with the replacement of $K_u$ by $\kappa_{uk}$. As shown in Theorem 2, the more information is available the less the CRB is. Moreover, as shown in Theorem 3, the CRB for $\theta$ is the same for different parameter vectors with the same size if the transformation matrix between the information vectors is nonsingular and independent of the expectation operation. Based on the basic theorems, theoretical analyses have been made for the cases of KC, UI, KW, and KW+. Particularly when $N = 1$, the bounds for KC, UI, and KW become equal, but are larger than for KW+. For sufficiently large samples, they are inversely proportional to the number of snapshots. Numerical results validate the theoretical analyses.
APPENDIX A
DERIVATION OF (17)
From (11) and (13), the Fisher information matrix is represented as
\[
F = \begin{bmatrix}
\Delta_{s,s} & \Delta_{s,\beta} & \Delta_{s,\theta} \\
\Delta_{s,\beta}^T & \Delta_{\beta,\beta} & \Delta_{\beta,\theta} \\
\Delta_{s,\theta}^T & \Delta_{\beta,\theta} & \Delta_{\theta,\theta}
\end{bmatrix}
\]  
(50)
where
\[
\Delta_{s,v} = E\left[\frac{\partial L}{\partial v_1} \frac{\partial L}{\partial v_2}\right].
\]  
(51)
It is convenient, for a complex matrix \(Z\), to introduce the following notations
\[
U(Z) = \begin{bmatrix} Z & -\bar{Z} \\ \bar{Z} & Z \end{bmatrix},
\]  
(52)
\[
V(Z) = \begin{bmatrix} Z^T & -Z \end{bmatrix},
\]  
(53)
The elements of the information matrix are obtained, by substituting (9) into (51), as
\[
\Delta_{s,s} = \frac{2}{\sigma^2} \left[ U(C_{k_1s_{11}}) \quad 0 \quad \cdots \quad 0 \right],
\]  
(54a)
\[
\Delta_{s,\beta} = \frac{2}{\sigma^2} \left[ V(C_{k_1s_1}) \quad 0 \quad \cdots \quad V(C_{k_{11}s_{1N}}) \right],
\]  
(54b)
\[
\Delta_{s,\theta} = \frac{2}{\sigma^2} \left[ V(C_{k_1s_1\theta}) \quad \cdots \quad V(C_{k_{11}s_{1N}\theta}) \right],
\]  
(54c)
\[
\Delta_{\beta,\beta} = \frac{2}{\sigma^2} \text{Re} \left[ C_{\gamma_1s_1,\gamma_1s_1} \quad 0 \quad \cdots \quad 0 \right],
\]  
(54d)
\[
\Delta_{\beta,\theta} = \frac{1}{\sigma^2} \text{Re} \left[ C_{\gamma_1s_1,\gamma_1s_1} \quad \cdots \quad C_{\gamma_1s_1,\gamma_1s_1} \right],
\]  
\[
\Delta_{\theta,\theta} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} C_{\theta,\theta,n}.
\]  
(54f)
Using the inverse of the partitioned matrix, we have
\[
\text{CRB}^{-1}(\beta, \theta) = \left[\begin{array}{cc}
\Delta_{\beta,\beta} & \Delta_{\beta,\theta} \\
\Delta_{\beta,\theta}^T & \Delta_{\theta,\theta}
\end{array}\right]^{-1} = \frac{1}{\Delta_{ss}^2} \Delta_{s,s}^{-1} \Delta_{s,\beta} \Delta_{s,\theta}
\]  
(55)
It is straightforward to see that for complex matrices \(Z_1\) and \(Z_2\),
\[
U(Z_1)U(Z_2) = V(Z_1)Z_2 \quad \text{and} \quad U(Z_1)^TV(Z_2) = \text{Re}(Z_1^TZ_2).
\]  
(56)
and that for a nonsingular matrix \(Z\), the inverse of \(U(Z)\) is given by \(U(Z)^{-1}\). Inserting (54) in (55) together with the use of (56) and (57) yields
\[
Y_{11} = \begin{bmatrix} \mathfrak{R}_{\gamma_1s_1,\gamma_1s_1} & 0 & \mathfrak{R}_{\gamma_1s_1,\theta} \\ \cdot & \cdot & \cdot \\ \mathfrak{R}_{\gamma_1s_1,\theta} & \mathfrak{R}_{\gamma_1s_1,\gamma_1s_1} & \sum_{n=1}^{N} \mathfrak{R}_{\theta,\theta,n}, \mathfrak{R}_{\gamma_1s_1,\gamma_1s_1} \end{bmatrix}
\]  
(58a)
\[
Y_{12} = \begin{bmatrix} \mathfrak{R}_{\gamma_1s_1,\theta} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{n=1}^{N} \mathfrak{R}_{\theta,\theta,n} \end{bmatrix}
\]  
(58b)
\[
Y_{22} = \sum_{n=1}^{N} \mathfrak{R}_{\theta,\theta,n}.
\]  
(58c)
By the inversion of the partitioned matrix
\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} (Y_{11} - Y_{12}Y_{11}^{-1}Y_{12})^{-1}.
\]  
(59)
The matrices \(Y_{11}\) and \(Y_{12}\) can be partitioned into four and two, respectively, blocks. The inverse of \(Y_{11}\) can also be found through the inversion of the partitioned matrix. Putting the resultant inverse, the partitioned \(Y_{12}\), and (58c) in (59) leads to (17).

Let us add, in terms of computational complexity, brief discussion of the CRB in [16]. Equivalently it can be directly found by applying the partitioned matrix inversion to the matrix \(F\) of (50) once, which gives
\[
\text{CRB}(\theta) = (D - B^TA^{-1}B)^{-1}
\]  
(60)
where
\[
F = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}, \quad D = \Delta_{\theta,\theta}.
\]  

The major computational load in the calculation of (60) is the inversion of \(A\). The size of \(A\) is \(N^2_d\), which is given by (20). The inversion requires \(O(N^3_d)\) multiplications. With \(A^{-1}\) obtained, the matrix multiplication of \(B^TA^{-1}B\) costs \(O(KN^3_d)\). The overall complexity for the computation of (60) is \(O(N^3_d)\). It is lowest when all incident signals are noncircular, i.e., \(K = \kappa_u\). Then the complexity is \(O(N^3(K + 1)^3)\).
Recall that $Z_1 \geq Z_2$ if and only if $Z_1^{-1} \leq Z_2^{-1}$. Similarly that $Z_1 > Z_2$ if and only if $Z_1^{-1} < Z_2^{-1}$ can be proven. Theorem 7.7.3(a) of [19] says that $M_1 > M_2$ if and only if $\rho(M_1^{-1}M_2) < 1$, where $M_1$ and $M_2$ are positive definite and positive semidefinite, respectively, Hermitian matrices and $\rho(M)$ is the maximum eigenvalue of a matrix $M$ that has nonnegative eigenvalues. Accordingly, $Z_1 \geq Z_2$ and only if $\rho(Z_1^{-1}Z_2) = \rho(Z_2Z_1^{-1}) < 1$ if and only if $Z_1^{-1} < Z_2^{-1}$.

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