On Dijkgraaf-Witten Type Invariants

Danny Birmingham

Universiteit van Amsterdam, Instituut voor Theoretische Fysica,
Valckenierstraat 65, 1018 XE Amsterdam,
The Netherlands

Mark Rakowski

School of Mathematics, Trinity College, Dublin 2, Ireland

Abstract

We explicitly construct a series of lattice models based upon the gauge group $\mathbb{Z}_p$ which have the property of subdivision invariance, when the coupling parameter is quantized and the field configurations are restricted to satisfy a type of mod-$p$ flatness condition. The simplest model of this type yields the Dijkgraaf-Witten invariant of a 3-manifold and is based upon a single link, or 1-simplex, field. Depending upon the manifold’s dimension, other models may have more than one species of field variable, and these may be based on higher dimensional simplices.

ITFA-94-07
February 1994

\footnote{Supported by Stichting voor Fundamenteel Onderzoek der Materie (FOM)}
Email: Dannyb@phys.uva.nl

\footnote{Email: Rakowski@maths.tcd.ie}
1 Introduction

An intriguing three dimensional lattice model was constructed by Dijkgraaf and Witten in [1]. By general considerations in gauge theory, it was shown that three dimensional Chern-Simons theories are classified by the cohomology classes in $H^4(BG, \mathbb{Z})$, where $BG$ is the universal classifying space for the group $G$. In the case of a finite group, they showed that the Boltzmann weight of such a theory was a 3-cocycle in $H^3(BG, R/Z)$; the cocycle condition being equivalent to the equation which guaranteed subdivision invariance of the lattice model. Subdivision invariance is, roughly speaking, the analogue of metric independence of a continuum theory.

In this paper, we will find a more concrete formulation for lattice models which have some features similar to the Dijkgraaf-Witten theory; their theory will appear as the simplest example. Extensions of that model to all odd dimensions, which was implicit in their formulation, appear as one series of models in our construction. The Chern-Simons type series just mentioned is based on dynamical variables associated only to links of the lattice, and is the closest to standard gauge theory. We also find other theories in our approach which have a superficial resemblance to the continuum $U(1)$ theory introduced by Schwarz [2], which was related to Ray-Singer, and equivalently, Franz-Reidemeister, torsion. These theories will also involve lattice variables associated to higher dimensional simplices. Additional models which do not really lie within either of these two categories will also be formulated. Generically, this construction falls outside of the scope of [1] which is rooted in link based gauge theory.

We work exclusively with the gauge group $\mathbb{Z}_p$. Subdivision invariance follows naturally in each model when the field configurations are restricted to satisfy a type of mod-$p$ flatness condition. While in three dimensions subdivision invariance of the partition function is sufficient to conclude that one has a topological invariant, the situation is more delicate in higher dimensions. There, subdivision invariance yields a combinatorial invariant of the piecewise linear structure. This situation is analogous to the continuum model phenomenon where metric independence allows one to conclude immediately that one has a diffeomorphism invariant, though further considerations may show that the theory is topological.
2 General Formalism

A lattice model is based on a simplicial complex which combinatorially encodes the topological structure of some manifold. Let us recall some of the essential ingredients that are required in such a formulation; we refer the reader to [3, 4, 5] for a more complete account.

Let $V = \{v_i\}$ denote a finite set of $N_0$ points which we will refer to as the vertices of a simplicial complex. An ordered $k$-simplex is an array of $k + 1$ distinct vertices which we denote by,

$$[v_0, \cdots, v_k].$$

(1)

It will usually be convenient to use simply the indices themselves to label a given vertex when no confusion will arise, so the above simplex is denoted more economically by $[0, \cdots, k]$. Pictorially, a $k$-simplex should be regarded as a point, line segment, triangle, or tetrahedron for $k$ equals zero through three respectively. A simplex which is spanned by any subset of the vertices is called a face of the original simplex. An orientation of a simplex is a choice of ordering of its vertices, where we identify orderings that differ by an even permutation, but for the models described here we will require an ordering of all vertices. One then checks that the invariant we compute is actually independent of the choice made in vertex ordering.

The boundary operator $\partial$ on the ordered simplex $\sigma = [v_0, \cdots, v_k]$ is defined by,

$$\partial \sigma = \sum_{i=0}^{k} (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_k],$$

(2)

where the ‘hat’ indicates a vertex which has been omitted. It is easy to show that the composition of boundary operators is zero; $\partial^2 = 0$.

We model a closed $n$-dimensional manifold as a collection $K = \{\sigma_i\}$ of $n$-simplices constructed from the set of vertices $V$, subject to a few technical conditions. Most importantly, every $(n - 1)$-face of any given $n$-simplex appears as an $(n - 1)$-face of precisely two different $n$-simplices in the collection $K$. One thinks of the $n$-simplices then as glued together along $(n - 1)$-faces. There is an additional restriction on the “link” of a vertex for the the simplicial complex to represent a manifold, but this condition will not play a role in the sequel and we refer the reader to [3] for a more complete discussion.
The dynamical variables in the theories we construct will be objects which assign an element in the cyclic group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, which we represent as the set of integers,

$$\{0, \cdots, p-1\} ,$$

(3)
to ordered simplices of some specified dimension. We call these dynamical variables $k$-colours with coefficients in $\mathbb{Z}_p$, and denote the evaluation of some $k$-colour $B^{(k)}$ on the ordered $k$-simplex $[0, \cdots, k]$ by

$$< B^{(k)}, [0, \cdots, k] > = B_{0 \cdots k} \in \mathbb{Z}_p .$$

(4)
The superscript $(k)$ will usually be omitted when its value is clear from context. It is important to note that we are assigning a $\mathbb{Z}_p$ element in a way which depends on the ordering of vertices in the simplex; we do not have the rule $B^{(1)}_{01} = -B^{(1)}_{10}$, for example. Instead, we shall assume that,

$$B^{(1)}_{10} = -B^{(1)}_{01} \mod p ,$$

(5)
and similarly extend this to a $k$-colour for odd permutations of the vertices. The case closest to conventional lattice gauge theory is where a 1-colour variable is assigned to every 1-simplex in the complex.

The coboundary operator $\delta$ acts on the dynamical variables as follows. Given a $(k-1)$-colour, an application of the coboundary operator produces an integer in $\mathbb{Z}$, when evaluated on an ordered $k$-simplex, namely

$$< \delta B^{(k-1)}, [0, \cdots, k] > = < B, \partial [0, \cdots, k] > = B_{123 \cdots k} - B_{023 \cdots k} + B_{013 \cdots k} - \cdots .$$

(6)
We must emphasize that the above sum of integers is not taken with modular $p$ arithmetic; it is simply an element in $\mathbb{Z}$. In cases where we will need to take some combination mod-$p$, we will put those terms between square brackets, so for example,

$$[a + b] = a + b \mod p .$$

(7)
There is also a cup product operation on colours which takes a $k$-colour $B^{(k)}$ and a $l$-colour $C^{(l)}$ and gives an integer in $\mathbb{Z}$ when evaluated on a $(k+l)$-ordered simplex:

$$< B \cup C, [0, \cdots, k+l] > = B_{0\cdots k} \cdot C_{k\cdots k+l} .$$

(8)
Note once again that this product is in $Z$ and the value is not taken mod-$p$.

Let us now put these ingredients together and define our theories. First, we must be given some oriented simplicial complex $K$ which we take to represent a manifold of dimension $n$. One then has some collection of $n$-simplices defined up to orientation. Take the vertex set of this complex and give it an ordering. This is done arbitrarily and we will have to show that our construction is independent of this choice. Now we can write down an ordered collection of the $n$-simplices; each of the simplices is written in ascending order and a sign in front of that simplex indicates whether that ordering is positively or negatively oriented with respect to the orientation of the complex $K$. Let us denote this ordered set of $n$-simplices by $K^n$,

$$K^n = \sum_i \epsilon_i \sigma_i ,$$

where the index $i$ runs over the ordered $n$-simplices $\sigma_i$ and $\epsilon_i$ is a sign which indicates the orientation. We will assign a Boltzmann weight $W[K^n]$ to $K^n$ by taking a product of factors, one for every $n$-simplex,

$$W[K^n] = \prod_i W[\sigma_i]^{\epsilon_i} .$$

Each of the individual factors is a nonzero complex number and will be some function of the colours. The details of which colours we use and how the function is defined will depend on the particular model. Finally, the partition function, which we will require to be a combinatorial invariant, is defined to be a quantity which is proportional to the sum of the Boltzmann weights over all colourings,

$$Z = \frac{1}{|G|^N} \sum_{\text{colours}} W[K^n] .$$

Here $N$ is the total number of colour summations and $|G|$ is the order of the group where the colours take their values. In a theory based entirely on a single 1-colour field, for example, this number is equal to the number of 1-simplices in the complex. This factor simply serves to normalize all the group summations to have unit volume. Let us make all of this very explicit by defining some specific models.
3 The Dijkgraaf-Witten Invariant

The simplest model of the type we are describing will lead to the Dijkgraaf-Witten invariant of 3-manifolds \([1]\). Further analysis of this model has been presented in \([3]-[8]\). So, let us be given a simplicial complex of dimension 3 and an ordering of vertices as described above. This model will be constructed out of a single 1-colour (with values in \(\mathbb{Z}_p\)) denoted by \(A\). The weight assigned to some ordered 3-simplex \([0, 1, 2, 3]\) is:

\[
W[[0, 1, 2, 3]] = \exp\{\beta < A \cup \delta A, [0, 1, 2, 3]>\} = \exp\{\beta A_{01} (A_{12} + A_{23} - A_{13})\} .
\] (12)

Here \(\beta\) is a complex number which at this stage is unrestricted. Clearly, our motivation for taking this particular structure is to try and mimic the action of a continuum Chern-Simons theory. We will now see that the requirement of subdivision invariance will quantize this coupling parameter.

Consider the subdivision of a specific ordered 3-simplex \([0, 1, 2, 3]\) obtained by installing a new vertex \(c\) at the center and linking it to the other 4 vertices; symbolically,

\[
[0, 1, 2, 3] \to [c, 1, 2, 3] - [c, 0, 2, 3] + [c, 0, 1, 3] - [c, 0, 1, 2] .
\] (13)

Let us declare this new vertex to be the first in the total ordering of all vertices. It is a simple exercise to show that,

\[
W[[0, 1, 2, 3]] \exp\{-\beta < \delta A \cup \delta A, [c, 0, 1, 2, 3]>\} = W[[c, 1, 2, 3]] W[[c, 0, 2, 3]]^{-1} W[[c, 0, 1, 3]] W[[c, 0, 1, 2]]^{-1} .
\] (14)

Thus, we see that our Boltzmann weight is not generally invariant under the replacement of the original Boltzmann factor of \(W[[0, 1, 2, 3]]\) by the 4 factors on the right hand side of (14); there is this added “insertion” which somehow must be trivialized. While one might imagine other more complicated suggestions, the conditions that lead to the Dijkgraaf-Witten invariant are to impose a restriction on the sum over colourings and on the parameter \(\beta\). Those conditions are to take \(s = e^\beta\) to be a \(p^2\) root of unity \((s^{p^2} = 1)\) and to restrict the sum over colourings to those which satisfy

\[
\delta A = 0 \mod p ,
\] (15)
for all 2-simplices in the complex $K$. This restriction shall be termed a “flatness” condition; for example on the 2-simplex $[0, 1, 2]$, we have the restriction
\[ [A_{01} + A_{12} - A_{02}] = 0. \]  
(16)
We remind the reader that the brackets denote a sum which is to be taken mod-$p$, so this particular equation can also be written as
\[ [A_{01} + A_{12}] = A_{02}. \]  
(17)
With only these flat field configurations, the product $\delta A \cup \delta A$ is clearly a multiple of $p^2$ and the above insertion becomes unity. The resulting identity (14) shall be referred to as the $5W$ identity. It should be remarked that subdivision invariance is achieved without the necessity of summing over the additional colour fields attached to the vertex $c$, and this will be a general feature of the models presented here. Notice also that the Boltzmann weight of $[0, 1, 2, 3]$ becomes
\[ \exp\left\{ \frac{2\pi ik}{p^2} A_{01}(A_{12} + A_{23} - [A_{12} + A_{23}]) \right\}, \]  
(18)
with $k \in \{0, \cdots, p-1\}$. This is precisely the well known representation of a 3-cocycle for the group cohomology of $Z_p$ with coefficients in $Z_p$ (or $U(1)$).

As discussed in [1], one can now check that the Boltzmann weight is gauge invariant for a closed manifold. This property, together with a verification that the partition function is independent of the chosen vertex ordering, follows immediately from the $5W$ identity.

4 Another Model in Three Dimensions

Having illuminated the general formalism, which in the case of a single 1-colour yields the Dijkgraaf-Witten model, we can immediately consider generalizations. In three dimensions, we have the obvious choice of a theory with two independent 1-colour fields. Let us now treat this theory in some detail. The Boltzmann weight of an ordered 3-simplex $[0, 1, 2, 3]$ is defined as:
\[ W[[0, 1, 2, 3]] = s^{<B \cup \delta A, [0,1,2,3]>} = s^{B_{01} (A_{12} + A_{23} - A_{13})}, \]  
(19)
where the two independent 1-colour fields are denoted by \( B \) and \( A \).

Our first duty is to consider the behaviour of the theory under the subdivision of eqn. (13), and we find

\[
W([[0, 1, 2, 3]]) s^{-\delta B \cup \delta A, [c, 0, 1, 2, 3]} = W([[c, 1, 2, 3]]) W([[c, 0, 2, 3]]) W([[c, 0, 1, 3]]) W([[c, 0, 1, 2]])^{-1}.
\]  

(20)

In this case, we see that invariance under subdivision can be achieved by again quantizing the coupling scale \( s \) to be a \( p^2 \) root of unity, and restricting the sum over colourings to those which satisfy the “flatness” conditions:

\[
[\delta B] = [\delta A] = 0.
\]  

(21)

The Boltzmann weight of a single ordered 3-simplex then assumes the form

\[
W([[0, 1, 2, 3]]) = \exp\left\{ \frac{2\pi ik}{p^2} B_{01} (A_{12} + A_{23} - [A_{12} + A_{23}]) \right\},
\]  

(22)

where \( k \in \{0, \ldots, p - 1\} \) as before.

Let us now address the issue of gauge invariance on closed manifolds. We wish to show that the Boltzmann weight (22) is invariant under independent gauge transformations of the \( A \) and \( B \) colour fields. Consider the \( A \) and \( B \) colour fields defined on the ordered 1-simplex \([0, 1]\); then, the gauge transformations of those fields are defined as

\[
A'_{01} = [A_{01} + k_{01} - k], \quad B'_{01} = [B_{01} + l_0 - l_1].
\]  

(23)

Here, the \( k \) and \( l \) fields are 0-colours defined on the vertices of the complex. It suffices to consider one example, so let us treat the case of a gauge transformation of \( A \) and \( B \) at the 1-vertex. The transformed Boltzmann weight, by definition, is given by,

\[
W'([[0, 1, 2, 3]]) = s^{B_{01} - l_1} ([k_1 + A_{12}] + A_{23} - [k_1 + A_{12} + A_{23}]) W([[0, 1, 2, 3]]) s^{B_{01} ([k_1 + A_{12}] - A_{12} - [k_1 + A_{12} + A_{23}] + [A_{12} + A_{23}])}
\]

(24)

In order to prove that the Boltzmann weight of a simplicial complex for a closed manifold is indeed gauge invariant, we need to show that the additional
terms generated on the right hand side of eqn. (24) are cancelled by other factors in the Boltzmann weight of the complex. To see this, one makes use of the fact that on a closed complex, each 2-simplex is common to precisely two 3-simplices. It is then a simple matter to check for the required cancellation.

As we have noted, the Boltzmann weight is defined with an arbitrary choice of ordering of the vertex set $V$. In order to establish independence of our results on the choice of ordering, we require a few identities. Given the Boltzmann weight on an ordered 3-simplex $[0, 1, 2, 3]$, it suffices to examine the behaviour under orientation reversal of the $0-1$, $1-2$, and $2-3$, vertices. Again, we shall establish that the value of the partition function is indeed independent of the vertex ordering, on a closed complex.

Under reversal of the $0-1$ vertices, for example, $W[[0, 1, 2, 3]]$ is replaced by $W[[1, 0, 2, 3]]^{-1}$, and we have the result

$$W[[1, 0, 2, 3]]^{-1} = W[[0, 1, 2, 3]] s^{B_{01} (|A_{01} + A_{12}| - A_{12} - |A_{01} + A_{12} + A_{23}| + |A_{12} + A_{23}|)} .$$

Similarly, reversal of the $1-2$ vertices, yields

$$W[[0, 2, 1, 3]]^{-1} = W[[0, 1, 2, 3]] s^{-B_{12} (A_{21} + |A_{12} + A_{23}| - A_{23})} ,$$

and finally, $2-3$ reversal gives

$$W[[0, 1, 3, 2]]^{-1} = W[[0, 1, 2, 3]] s^{-B_{01} (A_{23} + A_{32})} .$$

To actually establish order independence of the Boltzmann weight for a closed manifold, we again take recourse to the fact that each 2-simplex is common to precisely two 3-simplices. Again, we find that the required cancellations occur.

At this point, we have shown that to achieve subdivision invariance, we must restrict the sum over colourings to those which satisfy the “flatness” conditions on each 2-simplex in the simplicial complex. Recall that we began with a partition function, (11), which was defined with respect to a sum over all colourings, and with a normalization factor of $|G|$ for each independent colour field summation, i.e., $|G|^{-2N_1}$, in the theory under study, where $N_1$ is the number of 1-simplices in the complex. However, the subdivision invariant
Boltzmann weight is one which contains an insertion of delta functions which impose these flatness restrictions. It only remains to check the overall scale of the partition function.

This can be obtained quite easily by noting that the product of delta functions before and after subdivision are equal, up to the scale factor $|G|^3$. Since the number of 3-simplices increases by 3 under this move, the subdivision invariant partition function contains the normalization factor, $|G|^{2(N_3-N_1)}$. One can rewrite this by noting that for the case of a closed 3-manifold, the Euler number is zero: $N_3 - N_2 + N_1 - N_0 = 0$. Furthermore, for the case of a closed complex, we have the restriction that $N_2 = 2N_3$, and hence the subdivision invariant partition function can be written as:

$$Z = \frac{1}{|G|^{2N_0}} \sum_{flat} W[K^n], \quad (28)$$

where the sum is now over all flat colourings.

Since the Boltzmann weight and the delta function restrictions are gauge invariant objects, one has the freedom to gauge fix arbitrarily the values of a certain number of the colour configurations. In the case of a 1-colour field, the maximal allowable gauge fixing is called a maximal tree. A simple argument shows that a maximal tree is specified by the requirement that it should contain no closed 2-simplices. Given the vertex set of $N_0$ elements, it is clear that an ordering exists such that the maximal tree contains $N_0 - 1$ links. In this way, the partition function can be reduced to a sum over all gauge inequivalent flat colourings (denoted as $flat'$), with a normalization as follows:

$$Z = \frac{1}{|G|^{2N_0}} \sum_{flat'} W[K^n]. \quad (29)$$

Therefore, we note that the normalization coincides with that used in the definition of the Dijkgraaf-Witten theory, where the partition function is defined as a sum over all inequivalent flat connections, $\text{Hom}(\pi_1(K), G)$.

From a practical point of view, the freedom to perform this gauge fixing facilitates the evaluation of the partition function, to which we now turn. For the case of the 3-sphere, $S^3$, a suitable simplicial complex is provided by the boundary of a single 4-simplex. An easy calculation then shows that there is only a single gauge inequivalent flat colouring, for both the $A$ and $B$
field. The subdivision invariant value of the partition function is therefore:

\[ Z(S^3) = \frac{1}{|G|^2} , \]

(30)

for all groups \( G = Z_p \), and all roots of unity \( s \).

An equally simple calculation establishes the result,

\[ Z(S^2 \times S^1) = 1 , \]

(31)

for all \( Z_p \), and all roots of unity \( s \). Both these results yield the square of the value obtained in the \( Z_p \) Dijkgraaf-Witten theory; this will not be the case in the next example.

An interesting case to consider is provided by the real projective 3-space, \( \mathbb{R}P^3 \), and we shall deal here with the gauge group \( Z_2 \). We refer to [9], where a convenient simplicial complex in terms of a small number of vertices is provided. One should bear in mind, however, that attention must be paid to the relative orientation of the simplices in the triangulation of ref. [9], so that the boundary of the complex is zero. The relevant flatness conditions can then be solved, and one finds that each of the independent 1-colour fields \( A \) and \( B \) has 2 gauge inequivalent flat solutions. When a non-trivial 4-th root of unity is taken for \( s \), only one of the 4 total field configurations has a Boltzmann weight different from 1, and the result is,

\[ Z(\mathbb{R}P^3) = \frac{1}{4} (1 + 1 + 1 - 1) = \frac{1}{2} . \]

(32)

The point to note here is that this value differs from the calculation in the \( Z_2 \) Dijkgraaf-Witten theory, where a value of zero is obtained. It is more meaningful, however, to compare the \( B\delta A \) model with the \( Z_2 \times Z_2 \) Dijkgraaf-Witten theory. One nontrivial way to represent a group cocycle in that case is to take the action to be a sum of two independent Chern-Simons type terms,

\[ A \cup \delta A + B \cup \delta B . \]

(33)

The partition function simply factorizes and one merely has to square the \( Z_p \) result. Once again a value of zero is obtained for \( \mathbb{R}P^3 \) when a nontrivial 4-th
root of unity is taken for \( s \). However, the \( B\delta A \) model we have been discussing has a Boltzmann weight which can be regarded as a function from \( G \times G \times G \) to \( \mathbb{Z}_p \) (where \( G = \mathbb{Z}_p \times \mathbb{Z}_p \)) which satisfies the equation for subdivision invariance. This follows from associating one copy of \( \mathbb{Z}_p \) to each of the \( A \) and \( B \) variables. Since this equation is equivalent to the group cocycle condition, this \( B\delta A \) theory is presumably a representation of a different inequivalent 3-cocycle in the \( \mathbb{Z}_p \times \mathbb{Z}_p \) Dijkgraaf-Witten model. This is interesting since normally in gauge theory the only possibility when writing down an action for a model based on a direct product group is to take a sum of terms, one for each factor, as in (33).

5 DW Models in Higher Dimensions

An immediate question at this point is whether the higher dimensional extensions of the Dijkgraaf-Witten model can also be interpreted within the formalism we have been discussing. In \( n = 2m \) dimensions, the action one would take, based on a single 1-colour field, is clearly a \( \cup \)-product of \( m \) copies of \( \delta A \). In terms of the Boltzmann weight, one has

\[
W[\sigma] = \exp\{\beta < \delta A \cup \cdots \cup \delta A, \sigma >\}.
\]  

(34)

Since this structure is a “total derivative”, the Boltzmann weight is always 1 on a closed \( 2m \)-manifold, and no interesting phases can result. While the group cohomology of \( \mathbb{Z}_p \) with \( U(1) \) coefficients is trivial in even dimensions, this is not so with \( \mathbb{Z}_p \) coefficients. In fact, a simple application of the universal coefficient theorem [1],

\[
H^n(X, G) = H^n(X, Z) \otimes G \oplus Tor(H^{n+1}(X, Z), G)
\]  

(35)

to the result \( H^{\text{even}}(B\mathbb{Z}_p, \mathbb{Z}) = \mathbb{Z}_p \) and \( H^{\text{odd}}(B\mathbb{Z}_p, \mathbb{Z}) = 0 \), shows that

\[
H^n(B\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p
\]  

(36)

for all nonnegative \( n \). In particular for \( n = 4 \), when the flatness condition is imposed and \( s^p = 1 \), eqn. (34) provides a representation of the 4-cocycle. In this particular model, the trouble is that when one multiplies together all the \( W \) factors for a closed complex, the total Boltzmann weight is 1. Since
the Boltzmann weights are actually $Z_p$ valued, it would be fascinating if they could be realized in some other lattice model in even dimension.

For $2m+1$ dimensions, one easily writes the higher dimensional analogue of the 3d Chern-Simons term. One takes the Boltzmann weight,

$$W[\sigma] = \exp\{\beta < A \cup \delta A \cdots \cup \delta A, \sigma >\} ,$$

(37)

where one has $m$ factors of $\delta A$ in the action. The same analysis that we have given earlier goes through without difficulty, and one finds a subdivision invariant model when the factor $s = e^{\beta}$ is a $p^{m+1}$-root of unity. These would be concrete realizations of the more abstract models implicit in [1].

We also remark that, as in three dimensions, we have the freedom to consider the $2m+1$ dimensional model, with an array of different 1-colour fields. For example, in five dimensions, we obviously can define models with the following Boltzmann weights,

$$W[\sigma] = \exp\{\beta < B^{(2)} \cup \delta A^{(1)} \cup \delta A^{(1)}, \sigma >\} ,$$

$$W[\sigma] = \exp\{\beta < B^{(1)} \cup \delta B^{(1)} \cup \delta A^{(1)}, \sigma >\} ,$$

$$W[\sigma] = \exp\{\beta < A^{(1)} \cup \delta B^{(1)} \cup \delta C^{(1)}, \sigma >\} .$$

(38)

The expectation would be that such models are related in some way to the single 1-colour model for product groups.

### 6 General Models

Let us now attend to the description of some potentially interesting new models in higher dimensions. In particular, we begin by considering a four-dimensional theory, which involves the new feature of a 2-colour field. The Boltzmann weight of an ordered 4-simplex $[0,1,2,3,4]$ is defined by:

$$W[[0,1,2,3,4]] = \exp\{\beta < B^{(2)} \cup \delta A^{(1)}, [0,1,2,3,4] >\} ,$$

(39)

where $B^{(2)}$ and $A^{(1)}$ are 2- and 1-colour fields, respectively.

To analyze the subdivision properties of this model, we consider the introduction of a new vertex $c$ at the centre of the simplex $[0,1,2,3,4]$, which
is then joined to all the vertices, namely

\[ [0, 1, 2, 3, 4] \to [c, 1, 2, 3, 4] - [c, 0, 2, 3, 4] + [c, 0, 1, 3, 4] \]
\[ - [c, 0, 1, 2, 4] + [c, 0, 1, 2, 3] . \] \hspace{1cm} (40)

As before, we declare this new vertex to be the first in the total ordering of all vertices. It follows immediately that

\[ W[[0, 1, 2, 3, 4]] s^{-<\delta B \cup \delta A, [c, 0, 1, 2, 3, 4]>} = W[[c, 1, 2, 3, 4]] \] \hspace{1cm} (41)
\[ W[[c, 0, 2, 3, 4]]^{-1} W[[c, 0, 1, 3, 4]] W[[c, 0, 1, 2, 4]]^{-1} W[[c, 0, 1, 2, 3]] . \]

Subdivision invariance of this four dimensional theory is now guaranteed by imposing quantization of the coupling \( s^p = 1 \), as well as a restriction of the colourings to those satisfying the flatness conditions

\[ [\delta B^{(2)}] = [\delta A^{(1)}] = 0 . \] \hspace{1cm} (42)

The subdivision invariant Boltzmann weight is then:

\[ W[[0, 1, 2, 3, 4]] = \exp \left\{ \frac{2\pi ik}{p^2} B_{012} (A_{23} + A_{34} - [A_{23} + A_{34}]) \right\} . \] \hspace{1cm} (43)

The above subdivision move is known as an Alexander move of type 4 \([11]\), or equivalently a \((1, 5)\) move \([11]\). In order to complete the proof of the combinatorial invariance of this four dimensional theory, we also need to check invariance under a remaining set of subdivision moves. These are conveniently represented by a set of \((k, l)\) moves, where \((k + l = 6, k = 1, \ldots, 5)\) \([11]\). However, it is straightforward to check invariance under the remaining moves, by an analysis similar to the above.

We mention also that the 2-colour field enjoys a gauge invariance of the form:

\[ B'_{012} = [B_{012} - L_{01} - L_{12} + L_{02}] , \] \hspace{1cm} (44)

where \( L \) is a 1-colour field. As in the previous models, the Boltzmann weight is gauge invariant for closed complexes.

In this four dimensional example, there is hope of finding a nontrivial phase in the Boltzmann weight when one has solutions to \([12]\) which do not reduce to solutions in the strong sense, when the mod-p brackets are removed.
Experience in the 3d DW theory suggests that one find a 4d example where torsion is present in both $H_1(K,\mathbb{Z})$ and $H_2(K,\mathbb{Z})$. The manifold $RP^3 \times S^1$ fills the bill, and it will be interesting to do an explicit computation of that partition function.

Moving on, we can now identify a series of models in $n$ dimensions with Boltzmann weight given by:

$$W[\sigma] = \exp\{\beta < B^{(r)} \cup \delta A^{(n-r-1)}, \sigma >\} \quad (45)$$

where $\sigma = [0,1,\cdots,n]$ is an $n$-simplex. In this case, the colour degrees are $r$ and $(n-r-1)$ respectively, and again subdivision invariance requires that $s = e^{\beta}$ is a $p^2$ root of unity, with field configurations being restricted by the flatness conditions:

$$[\delta B^{(r)}] = [\delta A^{(n-r-1)}] = 0 \quad (46)$$

At this point, it is worth remarking that non-trivial solutions to these flat conditions will generically exist, and these are enumerated by the relevant cohomology groups, $H^r(K,\mathbb{Z}_p)$ and $H^{(n-r-1)}(K,\mathbb{Z}_p)$, of the complex $K$.

In $2m+1$ dimensions, we can construct models with Boltzmann weight

$$W[\sigma] = \exp\{\beta < B^{(m)} \cup \delta B^{(m)}, \sigma >\} \quad (47)$$

or

$$W[\sigma] = \exp\{\beta < B^{(m)} \cup \delta A^{(m)}, \sigma >\} \quad (48)$$

where $\sigma = [0,1,\cdots,2m+1]$, and $B^{(m)}$ and $A^{(m)}$ are independent $m$-colour fields, which, as usual, will be restricted by the relevant flatness condition. The important point to note here is that these models have a structure distinct from the higher-dimensional Chern-Simons type theories of the previous section, which were based only on 1-colour fields.

It is also possible to consider extensions of these models in which the $B$ and $A$ fields take values in different groups, $\mathbb{Z}_p$ and $\mathbb{Z}_q$, say, and with the scale parameter being chosen as $s^{pq} = 1$.

To conclude this survey of models, we remark that theories which include a 0-colour field lead to a trivial Boltzmann weight. The reason for this can
be seen most easily in two dimensions. Taking the Boltzmann weight of the simplex $[0, 1, 2]$ to be:

$$W[[0, 1, 2]] = \exp\{\beta < B^{(0)} \cup \delta A^{(1)}, [0, 1, 2] > \} \quad , \quad (49)$$

we one finds that the relevant flatness conditions are

$$[\delta B^{(0)}] = 0 \quad , \quad [\delta A^{(1)}] = 0 \quad . \quad (50)$$

However, the inequivalent solutions to the 0-colour flatness condition impose the constraint that the $B^{(0)}$ field is constant on the vertices. One then sees that the Boltzmann weight is a “total derivative”, and will always be 1 on a closed 2-manifold.

7 Concluding Remarks

It is clear that when the scale parameter $s = 1$ the theories described above reduce simply to a sum over all gauge inequivalent solutions to the flatness conditions. Such an invariant is itself non-trivial, and thus the even dimensional models presented above certainly differ from the Franz-Reidemeister torsion, which is trivial in those dimensions. Our main interest, of course, is in obtaining more subtle behaviour at the non-trivial roots of unity. One should note that in all the theories described, the central identity obtained involves a product of $(n + 2)$ factors of the Boltzmann weight. In [12], a variation of the cup product was used to define a subdivision invariant lattice model in four dimensions. In that case, a similar identity involving six factors of the Boltzmann weight allowed one to establish triviality of the invariant. The reason for this is that the model was defined with an assignment of arbitrary group elements to each link, without the imposition of flatness restrictions. Perhaps, it is worth mentioning the possibility that expectation values of gauge invariant observables, beyond the partition function, may also yield some interesting structures, but we leave that for the future.

Acknowledgements

D.B. would like to thank R. Dijkgraaf, and M. de Wild Propitius, for discussions.
References

[1] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, Commun. Math. Phys. 129 (1990) 393.

[2] A.S. Schwarz, *The Partition Function of a Degenerate Quadratic Functional and the Ray-Singer Invariants*, Lett. Math. Phys. 2 (1978) 247.

[3] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, 1984.

[4] J. Rotman, *An Introduction to Algebraic Topology*, Springer-Verlag, New York, 1988.

[5] J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer-Verlag, New York, 1980.

[6] D. Altschuler and A. Coste, *Quasi-Quantum Groups, Knots, Three-Manifolds, and Topological Field Theory*, Commun. Math. Phys. 150 (1992) 83.

[7] D. Altschuler and A. Coste, *Invariants of Three-Manifolds from Finite Group Cohomology*, J. Geom. Phys. 11 (1993) 191.

[8] D.S. Freed and F. Quinn, *Chern-Simons Theory with Finite Gauge Group*, Commun. Math. Phys. 156 (1993) 435.

[9] W. Kuhnel, *Triangulations of Manifolds with Few Vertices*, in *Advances in Differential Geometry and Topology*, ed. F. Tricerri, World Scientific, 1990, pg. 59.

[10] J.W. Alexander, *The Combinatorial Theory of Complexes*, Ann. Math. 31 (1930) 292.

[11] M. Gross and S. Varsted, *Elementary Moves and Ergodicity in d-Dimensional Simplicial Quantum Gravity*, Nucl. Phys. B378 (1992) 367.

[12] D. Birmingham and M. Rakowski, *Combinatorial Invariants from Four Dimensional Lattice Models*, Int. J. Mod. Phys. A., to appear.