Categorical Extension of Dualities: From Stone to de Vries and Beyond, I

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Abstract

Propounding a general categorical framework for the extension of dualities, we present a new proof of the de Vries Duality Theorem for the category $\mathbf{K\text{Haus}}$ of compact Hausdorff spaces and their continuous maps, as an extension of a restricted Stone duality. Then, applying a dualization of the categorical framework to the de Vries duality, we give an alternative proof of the extension of the de Vries duality to the category $\mathbf{Tych}$ of Tychonoff spaces that was provided by Bezhanishvili, Morandi and Olberding. In the process of doing so, we obtain new duality theorems for both categories, $\mathbf{K\text{Haus}}$ and $\mathbf{Tych}$.

Keywords

Compact Hausdorff space · Tychonoff space · Stone space · Regular closed/open set · Irreducible map · Quasi-open map · Projective cover · (complete)Boolean algebra · (normal)Contact algebra · de Vries algebra · Ultrafilter · Cluster · Right/left lifting of a dual adjunction · Semi-right adjoint functor · Covering class · Stone duality · Tarski duality · de Vries duality · (universal)de Vries pair · Booleanization of a de Vries pair · (universal)Boolean de Vries extension

Dedicated to the memory of Professor Hendrik de Vries (November 5, 1932–May 13, 2021)

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1 Introduction

A restriction of the Stone duality renders the category of extremally disconnected compact Hausdorff spaces and their continuous maps as dually equivalent to the category of complete Boolean algebras and Boolean homomorphisms. Extending this duality, in 1982 de Vries [10] showed that the category $\text{KHaus}$ of all compact Hausdorff spaces is dually equivalent to the category $\text{deV}$, whose objects are complete Boolean algebras that come equipped with a so-called normal contact relation, originally called compingent algebras, but now known as de Vries algebras. The morphisms of $\text{deV}$ are maps satisfying certain compatibility conditions with the Boolean structure and the contact relation, but they are not necessarily Boolean homomorphisms; also, their categorical composition is generally not given by the ordinary map composition.

The de Vries duality is realized by the contravariant functors $\text{RC}$ and $\text{Clust}$. The functor $\text{RC}$ assigns to every space $X$ in $\text{KHaus}$ the complete Boolean algebra $\text{RC}(X)$ of its regular closed sets, provided with the relation that declares two sets to be in contact when they intersect. (Recall that for $X$ extremally disconnected, $\text{RC}(X)$ coincides with the algebra $\text{CO}(X)$ of closed and open sets in $X$, as used in the Stone duality.) The functor $\text{Clust}$ generalizes Stone’s formation of the space of ultrafilters in a (complete) Boolean algebra, assigning to a de Vries algebra its space of so-called clusters. Of course, $\text{RC}(X)$ may equivalently be replaced by the de Vries algebra $\text{RO}(X)$ of regular open sets in $X$ and, as we show in this paper, the functor $\text{Clust}$ is (just like the ultrafilter functor for Boolean algebras) represented by the two-chain 2 and may be replaced by the $\text{KHaus}$-valued contravariant hom-functor $\text{deV}(-,2)$.

In their 2019 paper [7], Bezhanishvili, Morandi and Olberding (BMO) showed that the de Vries duality may be extended from $\text{KHaus}$ to the category $\text{Tych}$ of all Tychonoff spaces when, on the algebraic side, one considers so-called maximal de Vries extensions. These are de Vries algebras that come equipped with an embedding into a complete atomic Boolean algebra satisfying a maximality condition. As this condition corresponds precisely to the universal property of the Stone–Čech compactification of a Tychonoff space, we call them universal Boolean de Vries extensions. The resulting category $\text{UBdeV}$ that is dually equivalent to $\text{Tych}$ contains (a copy of) $\text{deV}$ as a full subcategory and is itself a full subcategory of the arrow category of $\text{deV}$.

The primary goal of this paper is to show that both duality extensions (as displayed in the above diagram) may be obtained with the help of a general categorical technique (which we develop here in Theorem 14) for the extension of a given (bottom) dual equivalence when one of the two categories is fully embedded into a larger (top) category. This approach (which in [18] we applied to obtain an alternative proof of the Fedorchuk duality [23], using a special case of Theorem 14) then gives us a first equivalent description of the other (top) category.
in terms of the given data and, in fact, leads us to some alternative new dual equivalences
for both categories, \( \mathbf{KHaus} \) and \( \mathbf{Tych} \). Furthermore, the categorical viewpoint reveals
that the extension technique used for obtaining the BMO duality (on the right) is essentially dual
to that one used to obtain the de Vries duality (on the left). Here we must say “essentially
dual” since, roughly speaking, the role of the projective covers used to establish the de Vries
duality is taken by the Stone–Čech compactification in the case of the BMO duality, but
one knows that the compactification is functorial while the formation of the cover is not [2].
Nevertheless, the de Vries duality is obtained here with the help of Theorem 14, while the
BMO duality is proved using the dualization of the special case of Theorem 14 presented in
Corollary 15.1

The pursuit of the first goal naturally leads us to reaching our second goal, namely to
describe a category, \( \mathbf{deVBoo} / \sim \), which is equivalent to \( \mathbf{dev} \) and, in fact, has the same objects
as \( \mathbf{dev} \), but whose morphisms are easier to handle than those of \( \mathbf{dev} \). In fact, the definition of
de Vries morphism does not flow naturally out of the object definition, but rather appears to be
a peculiar mix of preservation conditions for some of the object structure, which justifies itself
only by the fact that, in the end, it “works”. By contrast, the morphisms in \( \mathbf{deVBoo} \) are simply
Boolean homomorphisms reflecting the contact relation, to be composed by ordinary map
composition, and also the equivalence relation defining the quotient category \( \mathbf{deVBoo} / \sim \) has an easy description, both categorically and in terms of the Boolean structure.2 In our
proof of the de Vries duality, \( \mathbf{deVBoo} / \sim \) acts as a “mediator” between \( \mathbf{KHaus} \) and \( \mathbf{dev} \).

Similarly, also in presenting an alternative proof of the BMO duality, we form a “mediating” category, \( \mathbf{Udev} \), whose definition flows naturally from our categorical extension
technique. As a consequence, this category of \emph{universal de Vries pairs} is easily seen to
be dually equivalent to \( \mathbf{Tych} \), and we then show that it is equivalent to the BMO category
\( \mathbf{UBdeV} \). In order for us to reach the BMO duality, in both of these categories that are dually
equivalent to \( \mathbf{Tych} \), we must stick with the de Vries notion of morphism. However, in the
sequel [19] to this paper, we use the morphisms of \( \mathbf{deVBoo} / \sim \) rather than those of \( \mathbf{dev} \) to
establish a category dually equivalent to \( \mathbf{Tych} \) and thereby avoid the inconveniences pertaining
to the notion of de Vries morphisms. In addition, [19] offers various other applications
of our categorical extension technique, to obtain alternative proofs or modifications of the
dualities established in [8,13] for the categories of locally compact Hausdorff spaces and
of normal Hausdorff spaces and their continuous maps, as well as some other new duality
theorems.

Here is an outline of the contents and the organization of the paper. For the reader’s
convenience, we recall in Sect. 2 the needed standard facts and fix the notation pertaining to
all, the Stone duality, the Tarski duality (of sets and complete atomic Boolean algebras), the
de Vries duality, and its Fedorchuk variation. In fact, following a communication by Richard
Garner to the third author, we present the Stone duality as obtainable from the \( \mathbf{Set} \)-valued
functor of Boolean algebras represented by 2, via the Eilenberg–Moore construction for its
induced monad, which is the ultrafilter monad. We do so since, as we show in Sect. 6, the
corresponding fact turns out to hold for de Vries algebras as well. At the end of the section
we summarize some key topological facts pertaining to projective covers, also known as
absolutes, including Gleason’s Theorem and Alexandroff’s Theorem.

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1 Using Corollary 15, one can obtain as well the recent extension of the Stone Duality Theorem to the category
of zero-dimensional Hausdorff spaces and continuous maps, as established in [17].

2 We note that in [16], as the restriction of a duality involving the category of all locally compact Hausdorff
spaces, another category dually equivalent to \( \mathbf{KHaus} \) is presented. While its composition law may be considered
to be more natural than that of the category \( \mathbf{dev} \), its morphisms, which are special multi-valued maps, may
not.
The categorical extension constructions for dual equivalences as given in Theorem 14 and Corollary 18 of Sect. 3 provide the categorical framework that we use to establish new proofs for, respectively, the de Vries duality and the BMO duality; moreover, in the process of doing so, we obtain new duality theorems for the categories $\mathbf{KHaus}$ and $\mathbf{Tych}$. Our notion of $\mathcal{X}$-covering class (Definition 11) is the common backbone for both constructions. But while Corollary 18 entails just the dualization of the construction that we used in [18] for an alternative proof of the Fedorchuk duality, Theorem 14 involves a quotient construction that is needed when we apply it towards the de Vries duality, in order to offset the non-functoriality of projective covers.

In Sect. 4 we give our alternative approach to the de Vries duality, via a two-step procedure. First, when applied to the restricted Stone duality, Theorem 14 leads us to a modified de Vries duality theorem (Theorem 27), whereby $\mathbf{KHaus}$ is dually equivalent to the category $\mathbf{devBoo} / \sim$, which we consider to be a viable contender to $\mathbf{deV}$ for a dual representation of $\mathbf{KHaus}$. The second step for completing our alternative proof of the de Vries duality theorem then lies in showing that the categories $\mathbf{deV}$ and $\mathbf{devBoo} / \sim$ are equivalent (Theorem 30).

Fedorchuk’s duality theorem [23] circumvents the inconvenience of the composition law of the category $\mathbf{deV}$ (caused by the non-functoriality of projective covers), by admitting only special morphisms in both categories of the de Vries duality (while keeping the objects), namely so-called quasi-open maps in $\mathbf{KHaus}$ and sup-preserving Boolean homomorphisms reflecting the contact relation in $\mathbf{deV}$. In Sect. 5 we briefly indicate how to obtain a proof of this duality theorem following the categorical framework but refer to [18] for all details.

As first proved in [15] and recorded here as Proposition 32, just like ultrafilters in a Boolean algebra $A$ may equivalently be described as 2-valued Boolean homomorphisms on $A$, clusters in a de Vries algebra $A$ may equivalently be presented by 2-valued de Vries morphisms on $A$. Having recognized the functor $\text{Clust} : \mathbf{deV}^{\text{op}} \to \text{Set}$ just like $\text{Ult} : \mathbf{Boo}^{\text{op}} \to \text{Set}$ as representable by 2, with $\text{Set}(-, 2)$ serving as a left adjoint in both cases, one easily sees that both adjunctions induce the ultrafilter monad on $\text{Set}$. In Theorem 34 we conclude that the de Vries dual equivalence as given by the lifted hom-functor

$$\mathbf{deV}(-, 2) \cong \text{Clust} : \mathbf{dev}^{\text{op}} \longrightarrow \mathbf{KHaus}$$

is therefore nothing but the comparison functor of $\mathbf{dev}^{\text{op}}$ into the Eilenberg–Moore category of the ultrafilter monad, i.e., into $\mathbf{KHaus}$ [28,29]. From a categorical perspective, this result makes the de Vries duality look very natural indeed.

Similarly to our proof of the de Vries duality, in Sect. 7 we give a two-step procedure for the establishment of the BMO duality. First, with Corollary 18 applied to the de Vries dual equivalence and the category $\mathbf{KHaus}$ as embedded into $\mathbf{Tych}$, we obtain the category $\mathbf{Udev}$ and its dual equivalence with the category $\mathbf{Tych}$ (Theorem 41). The $\mathbf{Udev}$-objects are de Vries algebras $A$ that come equipped with a subset $Y$ of (the compact Hausdorff space) $\mathbf{dev}(A, 2)$ which may then serve as the Stone–Čech compactification of its subspace $Y$. The second step towards establishing the BMO duality (see [7], and also [5,6,8]) then consists of employing the Tarski duality to encode the subset $Y$ by its power set, treated as a complete atomic Boolean algebra. Briefly, rather than subsets $\bar{Y}$ one may equivalently consider de Vries embeddings $A \to B$ into complete atomic Boolean algebras $B$ and arrive at the above-mentioned category $\mathbf{UBdeV}$, which turns out to be equivalent to $\mathbf{Udev}$ (Theorem 49).
2 Brief Review: Stone, Tarski, de Vries, Fedorchuk

Facts 1 (The Stone duality). We start by considering the basic dual adjunction

\[ \text{Boo}^{\text{op}} \rightleftharpoons \text{Set}_\text{op} \]

of contravariant hom-functors represented by 2, considered as both, a two-element Boolean algebra and a mere doubleton set. Under this adjunction, for any Boolean algebra \( A \) and any set \( X \), the Boolean homomorphisms \( \varphi : A \to \text{Set}(X, 2) \) correspond bijectively to the maps \( f : X \to \text{Boo}(A, 2) \), via \( f(x)(a) = \varphi(a)(x) \) for all \( a \in A, x \in X \). Of course, \( \text{Set}(-, 2) \) represents the contravariant powerset functor \( P \), and \( \text{Boo}(2, 2) \) may equivalently be described as the set \( \text{Ult}(A) \) of ultrafilters in \( A \), i.e., as the set of maximal, non-empty, down-directed, and upwards-closed proper subsets of \( A \); on morphisms, both contravariant functors operate by taking inverse images. The bijective correspondence between homomorphisms \( \varphi : A \to PX \) and maps \( f : X \to \text{Ult}(A) \) is now described by \( (a \in f(x) \iff x \in \varphi(a)) \) for all \( a \in A, x \in X \). The identity map on \( \text{Ult}(A) \) corresponds to the co-unit of the above adjunction, which is therefore described by

\[ \varepsilon_A : A \to P(\text{Ult}(A)), \quad a \mapsto \{ u \in \text{Ult}(A) \mid a \in u \}. \]

The adjunction induces the ultrafilter monad on \( \text{Set} \), with underlying endofunctor \( X \mapsto \beta X = \text{Ult}(P(X)) \). Its Eilenberg–Moore category is known to be (equivalent to) the category \( \text{KHaus} \) of compact Hausdorff spaces (see [28,29]). Therefore, the functor \( \text{Ult} \) can assume the role of the comparison functor from \( \text{Boo}^{\text{op}} \) into the E–M category of \( \beta \) and, as such, becomes \( \text{KHaus} \)-valued: one takes the sets \( \varepsilon_A(a), a \in A \), as the basic open (and closed) sets of a compact Hausdorff topology on the set \( \text{Ult}(A) \) that, having a base for clopen (= closed and open) sets, is zero-dimensional and, thus, makes \( \text{Ult}(A) \) a Stone space. The "lifted" functor \( \text{Ult} \) is easily seen to still have a left adjoint; it assigns to a compact Hausdorff space \( X \) the Boolean algebra \( CO(X) \) of its clopen subsets:

\[ \text{Boo}^{\text{op}} \rightleftharpoons \text{KHaus}_\text{op} \]

The unit of this adjunction at a compact Hausdorff space \( X \) is the map

\[ \eta_X : X \to \text{Ult}(CO(X)), \quad x \mapsto \{ M \in CO(X) \mid x \in M \}, \]

which is continuous since \( \eta_X^{-1}(\varepsilon_{CO(X)}(M)) = M \) for all \( M \in CO(X) \). By compactness of \( X \), it is also surjective, as a standard argument shows: if we had \( u \in \text{Ult}(CO(X)) \) outside the image of \( \eta_X \), every \( x \in X \) would have a clopen neighbourhood \( M_x \not= u \), i.e., \( X \setminus M_x \in u \), which would be possible only if the sets \( M_x \) cover \( X \); but finitely many of these cannot cover \( X \) since, as a member of \( u \), the intersection of their complements cannot be empty. Consequently, \( \eta_X \) is a homeomorphism if (and, trivially, only if,) \( \eta_X \) is injective, and since \( X \) is Hausdorff, this happens precisely when \( CO(X) \) is a base for the topology of \( X \), i.e., if \( X \) is a Stone space. Therefore, by restriction of the previous adjunction, one obtains the Stone dual equivalence ([37]; see, for example, [25–27])

\[ \text{Boo}^{\text{op}} \rightleftharpoons \text{Stone}. \]
with **Stone** denoting the obvious full subcategory of **KHaus**.

We note the well-known fact \([25,27]\) that, under the Stone duality, *complete* Boolean algebras correspond to compact Hausdorff spaces that are *extremally disconnected* (so that the closure of any open set is still open). Hence, in a self-explanatory notation one has the dual equivalence

\[
\text{CBoo}^{\text{op}} \cong EKH.
\]

It is interesting to note that the complete Boolean algebras are precisely the injective objects in **Boo** while the extremally disconnected spaces are the projective objects in **Stone**; see Facts 8 below.

**Facts 2** *(Morphism restrictions of the Stone duality).* For later use we list some useful restrictions of the Stone duality to special types of morphisms. Recall that a continuous map \(f : X \to Y\) of topological spaces is *quasi-open* \([30]\) if, for every open set \(U\) in \(X\), the interior \(\text{int}(f(U))\) of its image under \(f\) may be empty only if \(U\) itself is empty. As shown in \([14, \text{Corollary 3.2(c)}]\) and \([12, \text{Corollary 2.4(c)}]\), under the Stone duality the quasi-open maps of Stone spaces correspond precisely to those homomorphisms \(\varphi : A \to B\) of Boolean algebras which preserve all (existing) suprema; that is: for all \(D \subseteq A\), the existence of \(\bigvee D\) in \(A\) implies the existence of \(\bigvee \varphi(D)\) in \(B\) and the equality \(\varphi(\bigvee D) = \bigvee \varphi(D)\). Hence, in an obvious notation, the Stone duality restricts to the dual equivalence

\[
(\text{Boo}_{\text{sup}})^{\text{op}} \cong \text{Stone}_{\text{q-open}}.
\]

A “merger” of this duality with the last duality of Facts 1 gives us the dual equivalence

\[
(\text{CBoo}_{\text{sup}})^{\text{op}} \cong EKH_{\text{q-open}}.
\]

It is well known that a continuous mapping between extremally disconnected compact Hausdorff spaces is quasi-open if, and only if, it is open (for a more general statement see, e.g., \([11, \text{Lemma 2.7)}]\). Hence, the category \(EHK_{\text{q-open}}\) may be written more simply as \(EHK_{\text{open}}\).

**Facts 3** *(The Tarski duality).* Returning to the basic dual adjunction of Facts 1, one notes that the contravariant powerset functor \(P\) actually takes values in **CABA**, the category of complete *atomic* Boolean algebras (in which every element is the join of *atoms*, i.e., of minimal non-bottom elements) and their *complete* (= sup- and, hence, also inf-preserving) Boolean homomorphisms; this, of course, is a full subcategory of **CBoo**\(_{\text{sup}}\), as considered in Facts 2. Routine exercises in Boolean algebra (see, for example, \([38]\)) show that the map

\[
\theta_A : A \longrightarrow P(\text{At}(A)), \quad a \longmapsto \{ x \in \text{At}(A) \mid x \leq a \}
\]

becomes an isomorphism when the complete Boolean algebra \(A\) is atomic; here \(\text{At}(A)\) denotes the set of atoms in \(A\). In other words, **CABA**\(_{\text{op}}\) is actually the essential image of **Set** under the functor \(P : \text{Set} \to \text{Boo}^{\text{op}}\), and one has the dual equivalence

\[
\text{CABA}^{\text{op}} \cong \text{Set},
\]
also known as the Tarski duality. Explicitly, the functor $At$ sends a complete homomorphism $\varphi : A \to B$ to the map $At(B) \to At(A)$ which assigns to an atom $y \in B$ the atom (as one can show) $At(\varphi)(y) := \bigwedge\{a \in A \mid y \leq \varphi(a)\}$ in $A$. In other words, for $\varphi : A \to B$ in $\text{Boo}$ and all $a \in A$, $y \in At(B)$ one has

$$y \leq \varphi(a) \iff At(\varphi)(y) \leq a.$$  

In this way one sees easily that there is a natural bijection $\varkappa : At(A) \to \text{CABA}(A, 2)$ which maps $x \in At(A)$ to the characteristic function $\xi_x := \varkappa_A(x)$ of the set $\{a \in A \mid x \leq a\}$, and its inverse $\varkappa_A^{-1}$ maps $\xi : A \to 2$ to $\bigwedge\{a \in A \mid \xi(a) = 1\}$. Consequently,

$$\text{CABA}(−, 2) \cong \text{Set} \cong \text{CABA}^{\text{op}}(−, 2)$$

is an equivalent presentation of the Tarski duality.

In Sect. 7 we will use a hybrid of these two presentations of the Tarski duality and, for every complete atomic Boolean algebra $A$ and every set $X$, consider the natural isomorphisms

$$\tilde{\theta}_A : A \to P(\text{CABA}(A, 2)), \quad a \mapsto \{\xi \in \text{CABA}(A, 2) \mid \xi(a) = 1\},$$

$$\chi_X : X \to \text{CABA}(P(X), 2), \quad x \mapsto [\chi^X_x : P(X) \to 2, (\chi^X_x(M) = 1 \iff x \in M)].$$

**Facts 4** (The algebraic side of the de Vries duality). A set $F$ in a topological space $X$ is regular closed (or a closed domain [22]) if it is the closure of its interior in $X$: $F = \text{cl}(\text{int}(F))$. The collection $\text{RC}(X)$ of all regular closed sets in $X$ becomes a Boolean algebra, with the Boolean operations $\lor, \land, *, 0, 1$ given by

$$F \lor G = F \cup G, \quad F \land G = \text{cl}(\text{int}(F \cap G)), \quad F^* = \text{cl}(X \setminus F), \quad 0 = \emptyset, \quad 1 = X.$$  

The Boolean algebra $\text{RC}(X)$ is actually complete, with the infinite joins and meets given by

$$\bigvee_{i \in I} F_i = \text{cl}\left(\bigcup_{i \in I} F_i\right) = \text{cl}\left(\bigcup_{i \in I} \text{int}(F_i)\right) = \text{cl}(\text{int}(\bigcup_{i \in I} F_i)) \quad \text{and} \quad \bigwedge_{i \in I} F_i = \text{cl}(\text{int}(\bigcap_{i \in I} F_i)).$$

We note in passing that, under the assignment $F \mapsto \text{int}(F)$, $\text{RC}(X)$ is isomorphic to the Boolean algebra $\text{RO}(X)$ of regular open sets $U$ in $X$, i.e., of those subsets $U$ of $X$ for which $U = \text{int}(\text{cl}(U))$. With the contact relation $\sim_X$ given by

$$F \sim_X G \iff F \cap G \neq \emptyset,$$

$\text{RC}(X)$ becomes a contact algebra (see [20]), that is: a Boolean algebra $A$ provided with a relation $\sim$ satisfying the conditions (C1–4) below; if $X$ is a normal Hausdorff space, then $\text{RC}(X)$ becomes even a normal contact algebra $A$ (see [10,23] where, however, different names have been used), defined to satisfy also conditions (C5–6):

1. $a \sim a$ whenever $a > 0$;
2. $a \sim b$ implies $a > 0$ and $b > 0$;
3. $a \sim b$ implies $b \sim a$;
4. $a \sim (b \lor c)$ if, and only if, $a \sim b$ or $a \sim c$;
5. if $a \not\sim b$, then $a \not\sim c$ and $b \not\sim c^*$ for some $c$;
6. if $a < 1$, then there exists $b > 0$ such that $b \not\sim a$.

With the non-tangential inclusion relation $\ll$ on $A$ defined by $(a \ll b \iff a \not\sim b^*)$, these conditions may equivalently be stated as
I1. \( a \ll b \) implies \( a \leq b \);
I2. \( a \leq b \ll c \leq d \) implies \( a \ll d \);
I3. \( 0 \ll 0 \), and \( a \ll b \) implies \( b^* \ll a^* \);
I4. \( a \ll c \) and \( b \ll c \) implies \( a \lor b \ll c \);
I5. if \( a \ll c \), then \( a \ll b \ll c \) for some \( b \);
I6. if \( a > 0 \), then \( b \ll a \) for some \( b > 0 \);

that is: (C1-4) \( \iff \) (I1-4) and (C1-6) \( \iff \) (I1-6). In terms of \( \ll \), the contact relation \( \bowtie \) takes the form \( (a \bowtie b \iff a \ll b^*) \). In \( (RC(X), \bowtie_X) \), the associated relation \( \ll_X \) reads as

\[
F \ll_X G \iff F \subseteq \text{int}_X(G).
\]

Note that if a contact algebra \((A, \bowtie)\) satisfies also condition C6, then every \( a \in A \) may be written as

\[
a = \bigvee \{ b \in A \mid b \ll a \}.
\]

Every Boolean algebra \( A \) may be endowed with a largest contact relation, \( \bowtie_\bullet \), and a least one, \( \bowtie_\circ \), respectively defined by

\[
a \bowtie_\bullet b \iff a \neq 0 \neq b \quad \text{and} \quad a \bowtie_\circ b \iff a \land b \neq 0
\]

for all \( a, b \in A \). The non-tangential inclusion relation \( \ll_\circ \) associated with \( \bowtie_\circ \), is just the order relation of the Boolean algebra \( A \), which makes \((A, \bowtie_\circ)\) even a normal contact algebra. We call this NCA structure on \( A \) discrete.

One is now ready to define the category \( \text{deV} \) of de Vries algebras: these are complete normal contact algebras, and a de Vries morphism \( \varphi : A \to B \) is a map of de Vries algebras satisfying the following conditions for all \( a, b \in A \):

V1. \( \varphi(0) = 0 \);
V2. \( \varphi(a \land b) = \varphi(a) \land \varphi(b) \);
V3. if \( a^* \ll b \) in \( A \), then \( \varphi(a)^* \ll \varphi(b) \) in \( B \);
V4. \( \varphi(a) = \bigvee \{ \varphi(b) \mid b \ll a \} \).

From V1-3 alone one obtains \( \varphi(1) = 1 \), \( \varphi(a^*) \leq \varphi(a)^* \) and \( (a \ll b \implies \varphi(a) \ll \varphi(b)) \). It is important to note that a de Vries morphism is not necessarily a Boolean homomorphism; conversely, a Boolean homomorphism with \((a \ll b \implies \varphi(a) \ll \varphi(b))\) for all \( a, b \in A \) satisfies V1-3, but not necessarily V4, unless it preserves suprema.

The composite \( \psi \circ \varphi : A \to C \) of \( \psi \) with \( \psi : B \to C \) in the category \( \text{deV} \) is given by

\[
(\psi \circ \varphi)(a) = \bigvee \{ (\psi \circ \varphi)(b) \mid b \ll a \}
\]

for all \( a \in A \). In general, this composition differs from the ordinary composition of maps, but it will coincide with it when \( \psi \) preserves suprema. The identity map on \( A \) maintains this role in the category \( \text{deV} \).

**Facts 5** (The de Vries duality). Extending the contact relation \( \bowtie \) on a Boolean algebra \( A \) to a relation on its ultrafilters (which, for brevity, will again be denoted by \( \bowtie \)) by

\[
u \bowtie v \iff \forall c \in u, \ d \in v : c \bowtie d,
\]

one shows [20, Lemma 3.5, p. 222] that the contact relation for elements \( a, b \in A \) is characterized by its ultrafilter extension, via

\[
a \bowtie b \iff \exists u, v \in \text{Ult}(A) : a \in u, \ b \in v, \ u \bowtie v.
\]
Furthermore, if \( A \) is normal, then \( \sim \) is an equivalence relation on \( \text{Ult}(A) \) [20,21]. In order to relate de Vries algebras with compact Hausdorff spaces, instead of considering ultrafilters directly, one uses the closely related concept of cluster in a normal contact algebra \( A \); this is a subset \( c \) of \( A \) satisfying the following conditions for all \( a, b \in A \):

cl 1. \( c \neq \emptyset \);
cl 2. \( a, b \in c \) implies \( a \sim b \);
cl 3. \( a \lor b \in c \) implies \( a \in c \) or \( b \in c \);
cl 4. if \( a \sim b \) for all \( b \in c \), then \( a \in c \).

As an easy consequence one has the property \( (a \in c, a \leq b \implies b \in c) \). Proceeding as in the proof of Theorem 5.8 of [32] one shows that every ultrafilter \( u \) in a normal contact algebra \( A \) gives the cluster

\[
\mathcal{c}_u = \{ a \in A \mid \forall b \in u : a \sim b \},
\]

and every cluster \( c \) in \( A \) comes about this way, that is: \( c = \mathcal{c}_u \), for some \( u \in \text{Ult}(A) \); actually, for every \( a \in c \) one has \( c = \mathcal{c}_u \), for some \( u \in \text{Ult}(A) \) with \( a \in u \). One concludes that \( \mathcal{c}_u \) is the unique cluster containing a given ultrafilter \( u \), and that any two clusters comparable by inclusion must actually be equal. Most importantly, the relation \( \sim \) for ultrafilters is characterized by

\[
u \sim v \iff \mathcal{c}_u = \mathcal{c}_v.
\]

With \( \text{Clust}(A) \) denoting the set of all clusters in a contact algebra \( A \), very similarly to the Stone duality, one can now establish the de Vries dual equivalence

\[
deV \overset{\sim}{\longrightarrow} \text{KHaus},
\]

as follows. We already saw in Facts 4 that, for a compact Hausdorff space \( X \), \( RC(X) \) becomes a de Vries algebra, and by assigning to a continuous map \( f : X \to Y \) the morphism

\[
RC(f) : RC(Y) \to RC(X), \ G \mapsto \text{cl}(f^{-1}(\text{int}(G)));
\]

one obtains the functor \( RC \). For a compact Hausdorff space \( X \) one has the natural map

\[
\sigma_X : X \to \text{Clust}(RC(X)), \ x \mapsto \{ F \in RC(X) \mid x \in F \},
\]

and for a de Vries algebra \( A \), one considers the map

\[
\tau_A : A \to RC(\text{Clust}(A)), \ a \mapsto \{ c \in \text{Clust}(A) \mid a \in c \}.
\]

With the the topology on \( \text{Clust}(A) \) to be taken to have the sets \( \tau_A(a), \ a \in A \), as its basic closed sets, one can then compute the interior \( \text{int}(\tau_A(a)) \) as the complement of \( \tau_A(a^*) \) in \( \text{Clust}(A) \) (see [10]). Now \( \text{Clust} \) becomes a contravariant functor when one assigns to a de Vries morphism \( \varphi : A \to B \) the map

\[
\text{Clust}(B) \to \text{Clust}(A), \ \varnothing \mapsto \{ a \in A \mid \forall b \in A \ (b \triangleleft a^* \implies \varphi(b)^* \in \varnothing) \}.
\]

We note that, should the \( \text{deV} \)-morphism \( \varphi \) be a Boolean homomorphism, the above-defined map \( \text{Clust}(B) \to \text{Clust}(A) \) is more succinctly described by \( \mathcal{c}_v \mapsto \mathcal{c}_{\varphi^{-1}(v)} \), for all \( v \in \text{Ult}(B) \). Apart from having worked with regular open sets, rather than with regular closed sets, as we do here, de Vries [10] showed that the natural maps \( \sigma_X \) and \( \tau_A \) become natural isomorphisms and thereby established his duality theorem.
As de Vries [10] also noted, his duality extends the restricted Stone duality between $\text{CBoo}$ and $\text{EKH}$ (see the end of Facts 1). Indeed, for any complete Boolean algebra $A$, the discrete de Vries algebra $(A, \circ)$ (see Facts 4) satisfies $\text{Clust}(A) = \text{Ult}(A)$. In this way one sees that the diagram

$$\text{deV}^{\text{op}} \xrightarrow{\text{Clust}} \text{KHaus}$$

$$\downarrow \quad \downarrow$$

$$\text{CBoo}^{\text{op}} \xrightarrow{\text{Ult}} \text{EKH}$$

commutes.

**Facts 6** (*The Fedorchuk duality*). The Fedorchuk duality may be obtained from the de Vries duality by keeping the objects in both categories under consideration, but restricting the admissible morphisms. On the algebraic side this is done very naturally, as follows.

We take as objects of the category $\text{Fed}$ all de Vries algebras (= complete normal contact algebras), but as morphisms $\varphi : A \to B$ only suprema-preserving Boolean homomorphisms reflecting the contact relation, so that $(\varphi(a) \circ \varphi(b) \implies a \circ b)$ or, equivalently, preserving the relation $\ll$, that is $(a \ll b \implies \varphi(a) \ll \varphi(b))$, for all $a, b \in A$. In this way, $\text{Fed}$ becomes a non-full subcategory of $\text{dev}$, and Fedorchuk [23] proved that, under the de Vries duality, $\text{Fed}$ becomes dually equivalent to the category $\text{KHaus}_{\text{q-open}}$ of compact Hausdorff spaces and their quasi-open continuous maps (as defined in Facts 2). In other words, one has the commutative diagram

$$\text{dev}^{\text{op}} \xrightarrow{\text{Clust}} \text{KHaus}$$

$$\downarrow \quad \downarrow$$

$$\text{Fed}^{\text{op}} \xrightarrow{\text{Clust}} \text{KHaus}_{\text{q-open}}$$

and obtains the dual equivalence

$$\text{Fed}^{\text{op}} \xrightarrow{\text{Clust}} \text{KHaus}_{\text{q-open}} \simeq \text{RC}$$

We remark that, unlike in $\text{dev}$, the morphism composition in $\text{Fed}$ coincides with the set-theoretic composition of maps.

**Facts 7** (*Summary and a look ahead*). The cuboid of the following commutative diagram summarizes how the de Vries and Fedorchuk dualities extend certain restricted Stone duali-
ties, and its upper part visualizes the BMO extension of the de Vries duality:

\[
\begin{align*}
\text{UB}_{\text{deV}}^{\text{op}} & \cong \text{Tych} \\
\text{deV}^{\text{op}} & \cong \text{KHaus} \\
\text{Fed}^{\text{op}} & \cong \text{KHaus}_{q\text{-open}} \\
\text{CBoo}^{\text{op}} & \cong \text{EKH} \\
\left(\text{CBoo}_{\text{sup}}^{\text{op}}\right) & \cong \text{EKH}_{\text{open}}
\end{align*}
\]

Our goal is now to provide a general categorical framework which will allow us to build equivalent substitutes for the categories $\text{deV}$ and $\text{UB}_{\text{deV}}$ (as well as for $\text{Fed}$), just based on the bottom and right panels of this diagram. In the case of the de Vries duality, this means that, using just the restricted Stone duality between $\text{CBoo}$ and $\text{EKH}$, with the latter category being fully embedded into $\text{KHaus}$, the framework should provide us with a category that, by construction, is dually equivalent to $\text{KHaus}$ and fairly easily seen to be also equivalent to $\text{deV}$; likewise for $\text{UB}_{\text{deV}}$. Our key tool to this end is an abstraction of projective covers, also called absolutes. For the reader’s convenience, we briefly recall them next.

**Facts 8** (*Projective covers and essential surjections*). Recall that an object $P$ in a category $\mathcal{C}$ is *projective* if the hom-functor $\mathcal{C}(P, -) : \mathcal{C} \to \text{Set}$ preserves epimorphisms; that is: for every epimorphism $f : X \to Y$ in $\mathcal{C}$, any morphism $g : P \to Y$ factors as $g = f \circ h$, for some morphism $h : P \to X$. Given an object $X$, a projective object $P$ together with an epimorphism $p : P \to X$ is a *projective cover* of $X$ if the equality $p \circ t = p$, where $t$ is an endomorphism of $P$, is possible only if $t$ is an isomorphism. One sees immediately that $X$ determines $P$ and $p$ up to isomorphism: if $q : Q \to X$ is also a projective cover of $X$, then $q \circ h = p$ for some isomorphism $h : P \to Q$.

Also easily shown is the fact that an epimorphism $p : P \to X$ with $P$ projective is a projective cover of $X$ if, and only if, the epimorphism $p$ is *essential*, that is: for every epimorphism $r : Z \to P$, if $p \circ r$ is an epimorphism, so is $r$. Indeed, assuming $p$ to be a projective cover and considering $r$ such that $p \circ r$ is epic, the projectivity of $P$ gives a morphism $s : P \to Z$ with $p \circ s = p$, so that $r \circ s$ is an isomorphism by hypothesis and, thus, $r$ is forced to be a (split) epimorphism. Conversely, assuming $p$ to be essential, any endomorphism $t$ with $p \circ t = p$ must be epic, so that the projectivity of $P$ gives a morphism $s$ with $t \circ s = 1_P$; but since $p \circ s = p \circ t \circ s = p$, so that also $s$ must be an epimorphism, $s$, and then also $t$, is actually an isomorphism.

We note that the argumentation remains valid if we relativize the notions of projective object and projective cover, by trading “epimorphism” everywhere for “$\mathcal{E}$-morphism”, where $\mathcal{E}$ may be any class of morphisms in $\mathcal{C}$ that contains all split epimorphisms (=retractions), but a split monomorphism (=section) may lie in $\mathcal{E}$ only if it is an isomorphism. We also
note the known fact (see [2]) that, when every object \( X \) in \( C \) admits an \( \mathcal{E} \)-projective cover \( \pi_X : EX \to X \), with \( \mathcal{E} \) a class of extremal epimorphisms in \( C \) (i.e., of those epimorphisms \( p \) that factor through a monomorphism \( m \) as \( p = m \circ g \) only when \( m \) is an isomorphism), then \( E \) may be made a functor \( C \to C \) and \( \pi \) a natural transformation \( E \to \text{Id}_C \) only if every object in \( C \) is already \( \mathcal{E} \)-projective—despite the fact that \( EX \) and \( \pi_X \) are determined by \( X \), up to isomorphism.

In \( C = \text{KHaus} \), the epimorphisms are extremal and coincide with the surjections. An essential epimorphism \( f : X \to Y \) is also called irreducible, as it is characterized by the following two properties: 1) \( f(X) = Y \), and 2) for all closed subsets \( Z \) of \( X \), \( f(Z) = Y \) is possible only when \( Z = X \).

By Gleason’s Theorem [24], \( P \in \text{KHaus} \) is projective if, and only if, \( P \) is extremally disconnected. Consequently, any irreducible map \( p : P \to X \) in \( \text{KHaus} \) with \( P \) extremally disconnected serves as a projective cover for a given \( X \). It is well known (see, for example, [40]) that \( P \) may be constructed as the Stone-dual of the complete Boolean algebra \( RC(X) \), with \( p = \pi_X \) the map \( EX = \text{Ult}(RC(X)) \to X \) that assigns to an ultrafilter \( u \) of regular closed sets of \( X \) the only point in \( \bigcap u \).

Under the name absolute, projective covers have been investigated intensively in categories of topological spaces larger than \( \text{KHaus} \), such as the category of regular Hausdorff spaces (see, for example, [4,34,35]), with the notion of projectivity relativized to perfect surjections. Of course, in \( \text{KHaus} \) every map is perfect, that is: a closed map with compact fibres. With the restriction to perfect maps, the characterization of projective covers via extremal disconnectedness and irreducibility remains valid when compact Hausdorff spaces are traded for regular Hausdorff spaces. For the definition and properties of absolutes of arbitrary topological spaces, see the survey paper [34]. A modern and very efficient presentation appeared in [36].

We will make essential use of Alexandroff’s Theorem [3, p. 346] (see also [35, Theorem (d) (3), p. 455]), which one derives easily from Ponomarev’s results [33] on irreducible maps: For \( p : X \to Y \) a closed irreducible map of topological spaces, the map

\[
\rho_p : RC(X) \to RC(Y), \quad H \mapsto p(H),
\]

is a Boolean isomorphism, with \( \rho_p^{-1}(K) = \text{cl}_X(p^{-1}(\text{int}_Y(K))) \), for all \( K \in RC(Y) \).

Finally, we will also use the well-known fact (see, e.g., [9], p.271, and, for a proof, [39]) that, when \( Y \) is a dense subspace of a topological space \( X \), then one has the Boolean isomorphisms \( r \) and \( e \) that are inverse to each other:

\[
r : RC(X) \to RC(Y), \quad F \mapsto F \cap Y, \quad \text{and} \quad e : RC(Y) \to RC(X), \quad G \mapsto \text{cl}_X(G).
\]

### 3 A General Framework for Extending Dualities

In fixing our notation, for a dual adjunction we use the symbolism

\[
\begin{array}{ccc}
\mathcal{A}^{\text{op}} & \overset{T}{\longrightarrow} & \mathcal{X} \\
\varepsilon \downarrow & & \eta \\
S & \overset{\eta}{\longrightarrow} & \mathcal{A}
\end{array}
\]

to indicate that \( T \) is right adjoint to \( S \) with adjunction units \( \eta_X : X \to TSX \) in \( \mathcal{X} \) and counits \( \varepsilon_A : A \to STA \) in \( \mathcal{A} \); they define the natural transformations \( \eta : \text{Id}_\mathcal{X} \to TS \) and \( \varepsilon : \text{Id}_\mathcal{A} \to ST \).
\(\varepsilon : \text{Id}_A \to ST\) satisfying the triangular identities\(^3\)

\[
T\varepsilon \circ \eta T = 1_T \quad \text{and} \quad S\eta \circ \varepsilon S = 1_S.
\]

When the dual adjunction is a dual equivalence, so that the units and counits are isomorphisms, we replace \(\top\) by \(\simeq\) in the symbolic notation, as we have already done so in the previous section.

**Definition 9** For given functors \(I : A \to B\) and \(J : X \to Y\), a dual adjunction

\[
\begin{array}{ccc}
\mathcal{B}^{\text{op}} & \xrightarrow{T} & \mathcal{Y} \\
\varepsilon \top \eta & \Downarrow & \\
\mathcal{S} & \xleftarrow{\epsilon \top \eta} & \mathcal{X}
\end{array}
\]

is a a right lifting of \(\mathcal{A}^{\text{op}} \xrightarrow{T} \mathcal{X}\) (along \(I\) and \(J\)) if \(I\) and \(J\) commute with the right adjoints, so that \(\tilde{T}I = JT\) holds:

\[
\begin{array}{ccc}
\mathcal{B}^{\text{op}} & \xrightarrow{T} & \mathcal{Y} \\
I & \Downarrow & \\
\mathcal{A}^{\text{op}} & \xleftarrow{\epsilon \top \eta} & \mathcal{X}
\end{array}
\]

it is a left lifting if \(IS = \tilde{S}J\) holds.

**Proposition 10** (1) For a right lifting as in Definition 9, there is a natural transformation \(\gamma : IS \to \tilde{S}J\) satisfying the conditions \(\tilde{T}\gamma \circ \tilde{\eta}J = J\eta \circ \gamma T \circ I\varepsilon = \tilde{\varepsilon}I\), and being uniquely determined by each of them; \(\gamma\) is an isomorphism if the given dual adjunctions are both dual equivalences.

(2) For a left lifting as in Definition 9, there is a natural transformation \(\delta : JT \to \tilde{T}I\) satisfying the conditions \(\tilde{S}\delta \circ \tilde{\varepsilon}I = I\varepsilon\); and \(\delta S \circ J\eta = \tilde{\eta}J\) and being uniquely determined by each of them; \(\delta\) is an isomorphism if the given dual adjunctions are both dual equivalences.

**Proof** (1) Since for all \(B \in |\mathcal{B}|, Y \in |\mathcal{Y}|\) one has the natural bijections \(\mathcal{B}(B, \tilde{S}Y) \cong \mathcal{Y}(Y, \tilde{T}B)\), the transformation \(\gamma : IS \to \tilde{S}J\) corresponds by adjunction to the transformation \(J\eta : J \to JT S = \tilde{T}IS\); explicitly, one has \(\gamma = \tilde{S}J\eta \circ \tilde{\varepsilon}IS\). For dual equivalences, \(\eta\) and \(\tilde{\varepsilon}\) are both isomorphisms, so that then also \(\gamma\) is an isomorphism.

(2) follows dually from (1); the explicit formula for \(\delta\) is now \(\delta = \tilde{T}I\varepsilon \circ \tilde{\eta}JT\).

Given a dual equivalence \(\mathcal{A}^{\text{op}} \xrightarrow{T} \mathcal{X}\) and an embedding \(J\) of \(\mathcal{X}\) as a full subcategory of a category \(\mathcal{Y}\) that, without loss of generality, is assumed to be an inclusion functor, we wish to give a natural construction for a category \(\mathcal{B}\) into which \(\mathcal{A}\) may be fully embedded via \(I\) and which allows for a right lifting along \(I\) and \(J\) that renders \(\mathcal{B}\) as dually equivalent to \(\mathcal{Y}\). Our construction depends on a given class \(\mathcal{P}\) of morphisms in \(\mathcal{Y}\), the role model for which is the class of essential surjections of projective covers of compact Hausdorff spaces, as described in Facts 8.

\(^3\) Of course, formally we should have written \(\varepsilon^{\text{op}} : ST \to \text{Id}_A^{\text{op}}\) for the counit of the dual adjunction and listed the triangular equalities as \(T\varepsilon^{\text{op}} \circ \eta T = 1_T\), \(\varepsilon^{\text{op}} S \circ S\eta = 1_S\). But in this paper we will generally suppress the explicit use of the op-formalism for functors and natural transformations, in order to keep the notation simple and maintain the symmetric presentation of the units in \(\mathcal{X}\) and the counits in \(A\).
Definition 11 For a full subcategory $\mathcal{X}$ of $\mathcal{Y}$, we call a class $\mathcal{P}$ of morphisms in $\mathcal{Y}$ an $\mathcal{X}$-covering class in $\mathcal{Y}$ if it satisfies the following conditions:

P1. $\forall X \in |\mathcal{X}|: 1_X \in \mathcal{P}$;
P2. every $\mathcal{X}$-object $X$ is $\mathcal{P}$-projective in $\mathcal{Y}$, that is:
\[ \forall (p: Y \to Y') \in \mathcal{P}, \ f: X \to Y' \exists g: X \to Y : p \circ g = f; \]
P3. $\mathcal{Y}$ has enough $\mathcal{P}$-projectives in $\mathcal{X}$, that is:
\[ \forall Y \in |\mathcal{Y}| \exists (p: X \to Y) \in \mathcal{P} : X \in |\mathcal{X}|. \]

Without loss of generality, one may assume (and we will often do so) that the domain of every morphism in $\mathcal{P}$ lies in $\mathcal{X}$, and that $\mathcal{P}$ is closed under precomposition with isomorphisms in $\mathcal{X}$ and under post-composition with isomorphisms in $\mathcal{Y}$; indeed, with $\mathcal{P}$ also the class $\mathcal{P} := \{ p: X \to Y | X \in |\mathcal{X}|, \ p \in \text{Iso}(\mathcal{Y}) \circ \mathcal{P} \circ \text{Iso}(\mathcal{X}) \}$ satisfies P1-3.

Examples 12 (1) If $\mathcal{X}$ is a full coreflective subcategory of $\mathcal{Y}$, one always has an $\mathcal{X}$-covering class $\mathcal{P}$ of morphisms in $\mathcal{Y}$. Just take for $\mathcal{P}$ the class of coreflections of $\mathcal{Y}$-objects into $\mathcal{X}$, that is: of $\mathcal{Y}$-morphisms $p: X \to Y$ with $X \in |\mathcal{X}|$ such that every $\mathcal{Y}$-morphism $X' \to Y$ with $X' \in |\mathcal{X}|$ factors uniquely through $p$.

(2) More generally than in (1), assume that the full embedding $J: \mathcal{X} \to \mathcal{Y}$ admits a functor $E: \mathcal{Y} \to \mathcal{X}$ and a natural transformation $\pi: JE \to \text{Id}_{\mathcal{Y}}$, such that $E J \cong \text{Id}_{\mathcal{X}}$ and $\pi J$ is an isomorphism. Then the class
\[ \mathcal{P}_\pi = \{ p: X \to Y | X \in |\mathcal{X}|, \ \beta \circ p = \pi Y \text{ for some isomorphism } \beta: E Y \to X \} \]
is an $\mathcal{X}$-covering class in $\mathcal{Y}$.

Construction 13 As a precursor to the construction of a category $\mathcal{B}$ as envisaged before Definition 11, for any given functor $T: A^{\text{op}} \to \mathcal{X}$ and an $\mathcal{X}$-covering class $\mathcal{P}$ in $\mathcal{Y}$, we consider the category
\[ C(\mathcal{A}, \mathcal{P}, \mathcal{X}), \]
defined as follows:

- objects in $C(\mathcal{A}, \mathcal{P}, \mathcal{X})$ are pairs $(A, p)$ with $A \in |\mathcal{A}|$ and $p: TA \to Y$ in the class $\mathcal{P}$;
- morphisms $(\varphi, f): (A, p) \to (A', p')$ in $C(\mathcal{A}, \mathcal{P}, \mathcal{X})$ are given by morphisms $\varphi: A \to A'$ in $\mathcal{A}$ and $f: Y' \to Y$ in $\mathcal{Y}$, such that $p \circ T \varphi = f \circ p'$:

\[
\begin{array}{ccc}
TA & \xrightarrow{T \varphi} & TA' \\
p \downarrow & & \downarrow p' \\
Y & \leftarrow & Y'
\end{array}
\]

- composition is as in $\mathcal{A}$ and $\mathcal{Y}$, so that $(\varphi, f)$ as above gets composed with $(\varphi', f') : (A', p') \to (A'', p'')$ by the horizontal pasting of diagrams, that is,
\[(\varphi', f') \circ (\varphi, f) = (\varphi' \circ \varphi, f \circ f').\]

- the identity morphism of an object $(A, p)$ in the category $C(\mathcal{A}, \mathcal{P}, \mathcal{X})$ is the $C(\mathcal{A}, \mathcal{P}, \mathcal{X})$-morphism $(1_A, 1_{\text{cod}(p)})$.

On the hom-sets of $C(\mathcal{A}, \mathcal{P}, \mathcal{X})$ we define a compatible equivalence relation by
\[(\varphi, f) \sim (\psi, g) \iff f = g.\]
for all \((\varphi, f), (\psi, g) : (A, p) \rightarrow (A', p')\). We denote the equivalence class of \((\varphi, f)\) by \([\varphi, f]\), and now let \(\mathcal{B}\) be the quotient category
\[
\mathcal{B} = C(A, \mathcal{P}, \mathcal{X})/\sim,
\]
i.e., \(|\mathcal{B}| = |C(A, \mathcal{P}, \mathcal{X})|, \text{Mor}(\mathcal{B}) = \{[\varphi, f] \mid (\varphi, f) \in \text{Mor}(C(A, \mathcal{P}, \mathcal{X}))\}\) and
\[
[\varphi', f'] \circ [\varphi, f] = [\varphi' \circ \varphi, f \circ f']/.
\]
Thanks to condition P1 one has the functor \(I : \mathcal{A} \rightarrow \mathcal{B}\), defined by
\[
(\varphi : A \rightarrow A') \mapsto (I\varphi = [\varphi, T\varphi] : (A, 1_{TA}) \rightarrow (A', 1_{TA'})),
\]
which, when \(T\) is faithful, is easily seen to be a full embedding, so that \(\mathcal{A}\) may be considered as a full subcategory of \(\mathcal{B}\). When \(T\) is also full, by P2, morphisms \([\varphi, f]\) in \(\mathcal{B}\) are fully determined by \(f\), and when \(T\) is a dual equivalence, as it will be the case henceforth, by P3 \(\mathcal{B}\) becomes dually equivalent to \(\mathcal{Y}\).

In the following theorem, in which we keep the quotient notation for the morphisms of \(\mathcal{B}\), since it facilitates a more intuitive and convenient description of its applications, we give a detailed account of the dual equivalence \(\mathcal{B}^{op} \simeq \mathcal{Y}\) as a right lifting of the given dual equivalence \(\mathcal{A}^{op} \simeq \mathcal{X}\).

**Theorem 14** Let \(\mathcal{X}\) be a full subcategory of \(\mathcal{Y}\) and \(\mathcal{P}\) an \(\mathcal{X}\)-covering class in \(\mathcal{Y}\). Then
\[
\mathcal{A}^{op} \xrightarrow{\varepsilon \simeq \eta} \mathcal{X} \quad \text{has a right lifting} \quad \mathcal{B}^{op} \xrightarrow{\tilde{\varepsilon} \simeq \tilde{\eta}} \mathcal{Y} \quad \text{along} \quad I \quad \text{and} \quad J,
\]
with \(J : \mathcal{X} \hookrightarrow \mathcal{Y}\) the inclusion and \(\mathcal{B} := C(A, \mathcal{P}, \mathcal{X})/\sim\), as well as the full embedding \(I : \mathcal{A} \rightarrow \mathcal{B}\), being defined as above. The lifted dual equivalence may be chosen to satisfy
\[
\tilde{T}\tilde{S} = \text{Id}_\mathcal{Y}, \quad \tilde{\eta} = 1_{\text{Id}_\mathcal{Y}}, \quad \tilde{T}\tilde{\varepsilon} = 1_{\tilde{T}}, \quad \tilde{\varepsilon}\tilde{S} = 1_{\tilde{S}},
\]
and the canonical isomorphism \(\gamma : IS \rightarrow \tilde{S}J\) then satisfies \(\tilde{T}\gamma = J\eta\) and \(\gamma T \circ I\varepsilon = \varepsilon I\).

**Proof** \(\tilde{T}\) is given by the projection \([\varphi, f] \mapsto f\); this trivially gives a faithful functor. With P2 we see that \(\tilde{T}\) is full: given \((A, p : TA \rightarrow Y), (A', p' : TA' \rightarrow Y') \in |\mathcal{B}|\) and \(f : Y' \rightarrow Y\) in \(\mathcal{Y}\), with P2 one obtains \(g : TA' \rightarrow TA\) with \(p \circ g = f \circ p'\), and then \(g\) may be written as \(T\varphi\) with \(\varphi : A \rightarrow A'\) in \(\mathcal{A}\) since \(T\) is full. To define \(\tilde{S}\) on objects, with P3 one chooses for every \(Y \in |\mathcal{Y}|\) a morphism \(\pi_Y : EY \rightarrow Y\) in \(\mathcal{P}\), with \(\pi_X = 1_X\) for all \(X \in |\mathcal{X}|\) (according to P1), and then puts \(\tilde{S}Y = (SEY, \pi_Y \circ \eta_{EY}^{-1})\). For a morphism \(f : Y' \rightarrow Y\) in \(\mathcal{Y}\), again, P2 and the fullness of \(T\) allow one to choose a morphism \(\varphi_f : SEY \rightarrow SEY'\) in \(\mathcal{A}\) with \(\pi_Y \circ \eta_{EY}^{-1} \circ T\varphi_f = f \circ \pi_Y \circ \eta_{EY}^{-1}\); we then put \(\tilde{S}f = [\varphi_f, f]\). Checking the functoriality of \(\tilde{S}\) and the identity \(\tilde{T}\tilde{S} = \text{Id}_\mathcal{Y}\) is straightforward.

For \((A, p : TA \rightarrow Y) \in |\mathcal{B}|\) one puts \(\tilde{\varepsilon}_{(A,p)} = [\varphi_{(A,p)}, 1_Y]\), with any \(\mathcal{A}\)-morphism \(\varphi_{(A,p)} : A \rightarrow SEY\) satisfying \(p \circ T\varphi_{(A,p)} = \pi_Y \circ \eta_{EY}^{-1}\). Clearly, \(\tilde{\varepsilon}\) is, like \(\tilde{\eta} = 1_{\text{Id}_\mathcal{Y}}\), a natural isomorphism satisfying the claimed identities. By Proposition 10(1), the canonical
Let us now look at the special case of $\mathcal{X}$ being coreflective in $\mathcal{Y}$, so that we have an adjunction $J \dashv E : \mathcal{Y} \to \mathcal{X}$ with counit $\pi : JE \to \text{Id}_\mathcal{Y}$. For simplicity, and without loss of generality, we may assume that the coreflector $E$ and the counit $\pi$ have been chosen to satisfy $EJ = \text{Id}_\mathcal{X}$ and $\pi J = 1_J$. As observed in Examples 12, we can then take the class $\mathcal{P} = \mathcal{P}_\pi$ to be given by the coreflections $\pi_Y : J E Y \to Y$, $Y \in \mathcal{Y}$, precomposed by any $\mathcal{X}$-isomorphisms, and apply Construction 13. The equivalence relation $\sim$ on $C(\mathcal{A}, \mathcal{P}_\pi, \mathcal{X})$ considered there becomes discrete in this case, so that one may simply consider $\mathcal{B} = C(\mathcal{A}, \mathcal{P}_\pi, \mathcal{X})$ and the full embedding

$$I : \mathcal{A} \to \mathcal{B}, \ (\varphi : A \to A') \mapsto (I \varphi = (\varphi, T \varphi) : (A, 1_{T A}) \to (A', 1_{T A'})),$$

when applying Theorem 14 in this situation. We now obtain the following corollary:

**Corollary 15** Let $J : \mathcal{X} \leftarrow \mathcal{Y}$ be coreflective, with $J \dashv E$ and counit $\pi : JE \to \text{Id}_\mathcal{Y}$. Then

$$\mathcal{A}^{\text{op}} \xleftarrow{\varepsilon \simeq \eta} \mathcal{X} \xrightarrow{\tilde{T}} \mathcal{Y}$$

has a right lifting $\mathcal{B}^{\text{op}} \xleftarrow{\tilde{\varepsilon} \simeq \tilde{\eta}} \mathcal{Y}$ along $I$ and $J$, satisfying

$$\tilde{T} \tilde{S} = \text{Id}_\mathcal{Y}, \quad \tilde{\eta} = 1_{\text{Id}_\mathcal{Y}}, \quad \tilde{\varepsilon} = 1_{\tilde{T}}, \quad \tilde{\varepsilon} \tilde{S} = 1_{\tilde{T}};$$

here the full embedding $I : \mathcal{A} \to \mathcal{B} = C(\mathcal{A}, \mathcal{P}_\pi, \mathcal{X})$ has a reflector $D \dashv I$ with unit $\rho : \text{Id}_\mathcal{B} \to ID$, satisfying $DI = \text{Id}_\mathcal{A}$, $\rho I = 1_I$ and making the diagram

\[
\begin{array}{ccc}
\mathcal{B}^{\text{op}} & \xrightarrow{\tilde{S}} & \mathcal{Y} \\
\downarrow D & & \downarrow E \\
\mathcal{A}^{\text{op}} & \xleftarrow{\tilde{T}} & \mathcal{X}
\end{array}
\]

commute, so that $\mathcal{A}^{\text{op}} \xleftarrow{\varepsilon \simeq \eta} \mathcal{X}$ becomes a left lifting of

\[
\begin{array}{ccc}
\mathcal{B}^{\text{op}} & \xrightarrow{\tilde{\varepsilon} \simeq \tilde{\eta}} & \mathcal{Y} \\
\end{array}
\]

along $D$ and $E$. In addition to the canonical isomorphism $\gamma : IS \to \tilde{SJ}$, there is therefore the canonical isomorphism $\beta : ET \to TD$, determined by each of the conditions $\beta \tilde{S} = \eta E$ and $S \beta \circ \varepsilon D = D \tilde{\varepsilon}$; furthermore, $\beta$ and $\gamma$ connect $\pi$ and $\rho$ via

$$\tilde{T} \rho \circ J \beta = \pi \tilde{T}, \quad \gamma E \circ \rho \tilde{S} = \tilde{S} \pi.$$

**Proof** Assuming, without loss of generality, that $EX = X$ and $\pi_X = 1_X$ holds for all $X \in \mathcal{X}$, and writing $(\varphi, f)$ instead of $[\varphi, f]$ everywhere in the proof of Theorem 14, one may proceed verbatim as in that proof, in order to define the lifted dual equivalence $\tilde{S} \dashv \tilde{T}$. (Note that, in the definitions of $\tilde{S}$ and $\tilde{\varepsilon}$, the morphisms $\varphi f$ and $\varphi(A, p)$ are now uniquely determined.) For $D$ one simply takes the projection

$$D : \mathcal{B} \to \mathcal{A}, \quad ((\varphi, f) : (A, p) \to (A', p')) \mapsto (\varphi : A \to A'),$$

\(\blacklozenge\) Springer
so that $DI = \text{Id}_A$ and $D\tilde{S} = SE$ hold trivially. By the last identity, just as $\gamma : IS \to SI$ corresponds by adjunction to $J\eta$, one obtains with Proposition 10 the natural isomorphism $\beta : ET \to TD$ corresponding to $D\tilde{S}$ by adjunction and, thus, satisfying the stated identities $\beta\tilde{S} = \eta E$ and $S\beta \circ \varepsilon D = D\tilde{e}$. Explicitly, for $(A, p : TA \to Y) \in |\mathcal{B}|$, the (iso)morphism $\beta_{(A, p)} : EY \to TA$ is the only one with $p \circ \beta_{(A, p)} = \pi_Y$. One easily confirms that $\rho_{(A, p)} = (1_A, p) : (A, p) \to (A, 1_TA) = ID(A, p)$ is an $I$-universal arrow for $(A, p)$ and therefore serves as a unit $\rho : \text{Id}_{\mathcal{B}} \to ID$ for the adjunction $D \dashv I$. Now the identity $p \circ \beta_{(A, p)} = \pi_Y$ means $T\rho \circ J\beta = \pi T\rho$. It is also straightforward to confirm the remaining identities $\beta\tilde{S} = \eta E$ and $\gamma E \circ \rho\tilde{S} = \tilde{S}\pi$. \hfill $\Box$

**Remark 16** As shown in Theorem 3.4 of [18], our Corollary 15 remains intact in the more general context of Examples 12(2), that is, when the full subcategory $\mathcal{X}$ of $\mathcal{Y}$ admits a pseudo-retraction $E : \mathcal{Y} \to \mathcal{X}$ and a natural transformation $\pi : JE \to \text{Id}_{\mathcal{Y}}$ such that $\pi J$ is an isomorphism, provided that $\pi$ has the additional property that $\pi_Y \circ \alpha = \pi_Y$ for an automorphism $\alpha : EY \to EY$ is possible only for $\alpha = 1_{EY}$. (In this case, $E$ is just semi-right adjoint to $J$ in the sense of [31], that is: generally, only one of the two triangular identities required for an adjunction is assumed to hold.) Under this generalization of Corollary 15, one has to return to the construction of the category $\mathcal{B}$ as given in the proof of Theorem 14, but has the advantage that $E$ and $\pi$ facilitate (not a unique, but still) a functorial choice of a representative in all equivalence classes $[\varphi, f]$ that form the morphisms of $\mathcal{B}$, and it then suffices to operate with these representatives. But as we are not aware of any relevant example that would benefit from this generalization of Corollary 15, we skip any further elaboration of it in this paper.

We now indicate how to dualize Theorem 14 and Corollary 15, giving an explicit formulation only in the case of Corollary 15, as it will be used in Sect. 7. In our symbolism, noting that we put the “op” sign only for categories, the dualization of a dual adjunction $A^{op} \xrightarrow{T} \mathcal{X}$ gives us $A \xleftarrow{\eta \perp \varepsilon} S \mathcal{X}^{op}$. Hence, $T$ is now considered as left adjoint to $S$, and the roles of $\eta$ and $\varepsilon$ as unit and counit have been interchanged. A dual equivalence may then be written as

$$A \xleftarrow{\eta \simeq \varepsilon} S \mathcal{X}^{op}.$$

Under this dualization, a right lifting along $I, J$ (as defined in Definition 9) becomes a left lifting along $I, J$, and conversely.

Instead of a full coreflective embedding $J : \mathcal{X} \hookrightarrow \mathcal{Y}$, we now have to assume that $\mathcal{X}$ be reflective in $\mathcal{Y}$, with $E : \mathcal{Y} \to \mathcal{X}$ being left adjoint to $J$. Since in our primary example the reflection morphisms will be embeddings, we denote the unit of the adjunction $E \dashv J$ by $\iota : \text{Id}_{\mathcal{Y}} \to JE$, instead of $\pi$ as used in the dual situation. Likewise, in dualizing the class $\mathcal{P}_\pi$ and, thus, switching from a notion of projectivity to a notion of injectivity, we now use the notation $\partial_\iota$ for the class of all reflection morphisms $\iota_Y : Y \to EY$ ($Y \in |\mathcal{Y}|$), post-composed by any isomorphisms in $\mathcal{X}$. 

\begin{figure}
\centering
\includegraphics{diagram.png}
\caption{Diagram for dual equivalence}
\end{figure}
Construction 17 From the class $\mathcal{J}_i$ as above one builds the category

$$D(A, \mathcal{J}_i, \mathcal{X}),$$

dually to $C(A, \mathcal{P}_\pi, \mathcal{X})$, as follows:

- objects are pairs $(A, j)$ with $A \in |A|$ and $j : Y \to TA$ in the class $\mathcal{J}_i$;
- morphisms $(\varphi, f) : (A, j) \to (A', j')$ are given by morphisms $\varphi : A \to A'$ in $A$ and $f : Y' \to Y$ in $\mathcal{Y}$ with $T\varphi \circ j' = j \circ f$:

\[
\begin{array}{ccc}
T A & \xrightarrow{T \varphi} & T A' \\
\downarrow j & & \uparrow j' \\
Y & \xleftarrow{f} & Y'
\end{array}
\]

- composition in $D(A, \mathcal{J}_i, \mathcal{X})$ proceeds by the horizontal pasting of diagrams;
- the identity morphism of a $D(A, \mathcal{J}_i, \mathcal{X})$-object $(A, j)$ is $(1_A, 1_{\text{dom}(j)})$.

As in the dual situation, there is a full embedding

$I : A \to B = D(A, \mathcal{J}_\pi, \mathcal{X}), (\varphi : A \to A') \mapsto (I\varphi = (\varphi, T\varphi) : (A, 1_{TA}) \to (A', 1_{TA'}))$.

The dualization of Corollary 15 now reads as follows:

Corollary 18 Let $J : \mathcal{X} \hookrightarrow \mathcal{Y}$ be reflective, with $E \dashv J$ and unit $\iota : \text{Id}_Y \to JE$. Then

$$\begin{array}{ccc}
A & \xleftarrow{T} & \mathcal{X}^{\text{op}} \\
\downarrow \eta \simeq \varepsilon & & \downarrow S \\
S & \xrightarrow{\tilde{T}} & \mathcal{Y}^{\text{op}}
\end{array}$$

has a left lifting $B \xleftarrow{\tilde{S}} \mathcal{Y}^{\text{op}}$ along $I$ and $J$ satisfying

$$\tilde{T}\tilde{S} = \text{Id}_Y, \quad \tilde{\eta} = 1_{\text{Id}_Y}, \quad \tilde{T}\tilde{\varepsilon} = 1_{\tilde{T}}, \quad \tilde{\varepsilon}\tilde{S} = 1_S;$$

here the full embedding $I : A \to B = D(A, \mathcal{J}_\pi, \mathcal{X})$ has a coreflector $D$ with counit $\rho : ID \to \text{Id}_B$, satisfying $DI = \text{Id}_A$, $\rho I = 1_I$ and making the diagram

$$\begin{array}{ccc}
B & \xleftarrow{\tilde{S}} & \mathcal{Y}^{\text{op}} \\
\downarrow D & & \downarrow E \\
A & \xleftarrow{S} & \mathcal{X}^{\text{op}}
\end{array}$$

commute, so that $A \xleftarrow{T} \mathcal{X}^{\text{op}}$ is a right lifting of $B \xleftarrow{\tilde{T}} \mathcal{Y}^{\text{op}}$ along $D$ and $E$. In addition to the canonical isomorphism $\gamma : IS \to \tilde{S}J$, there is therefore the canonical isomorphism $\beta : ET \to TD$, determined by each of the conditions $\beta\tilde{S} = \eta E$ and $S\beta \circ \varepsilon D = D\tilde{\varepsilon}$; furthermore, $\beta$ and $\gamma$ connect $\iota$ and $\rho$ via

$$J\beta \circ \iota \tilde{T} = \tilde{T}\rho, \quad \rho\tilde{S} = \tilde{S}I \circ \gamma E.$$
For completeness, we list the “pointwise” definitions of \( \tilde{T} \), \( D \), \( \tilde{S} \), \( \tilde{\epsilon} \), \( \rho \), \( \gamma \), \( \beta \) appearing in Corollary 18, as derived by dual adjustment of those given in Corollary 15:

- \( \tilde{T} : ((\varphi, f) : (A, j : Y \to TA) \to (A', j' : Y' \to TA')) \mapsto (f : Y' \to Y) \);
- \( D : ((\varphi, f) : (A, j) \to (A', j')) \mapsto (\varphi : A \to A') \);
- \( \tilde{S} : Y \mapsto (SEY, \eta_{EY} \circ tY), (f : Y' \to Y) \mapsto (\varphi_f, f), \) with \( T\varphi_f \circ \eta_{EY} \circ tY = \eta_{EY} \circ tY \circ f \);
- \( \tilde{\epsilon}_{(A, j : Y \to TA)} = (\varphi_{(A, j)}, 1_Y) \), with \( \varphi_{(A, j)} : A \to SEY \) satisfying \( T\varphi_{(A, j)} \circ \eta_{EY} \circ tY = j \);
- \( \rho(A, j) = (A, 1_{TA}); \gamma_X = (1_X, \eta_X); \beta_{(A, j : Y \to TA)} \) is determined by \( \beta_{(A, j)} \circ j = t_Y \).

4 A New Approach to the de Vries Duality

As mentioned in Facts 8, with Gleason’s Theorem one sees easily that the class \( \mathcal{P} \) of irreducible continuous maps of compact Hausdorff spaces with extremally disconnected domain is an \( EKH \)-covering class in \( KHaus \) (as given in Definition 11). An application of Theorem 14 to the restricted Stone duality

\[
\mathcal{A}^{\text{op}} = \mathcal{CBoo}^{\text{op}} \xrightarrow{T = UlT} EKH = \mathcal{X}
\]

(see Facts 1) produces the following dual representation of \( KHaus \):

\[
\text{Proposition 19} \quad \mathcal{B}^{\text{op}} := (C(A, \mathcal{P}, \mathcal{X})/\sim)^{\text{op}} \xrightarrow{\tilde{T}} KHaus =: \mathcal{Y}.
\]

Here the objects of the category \( C(A, \mathcal{P}, \mathcal{X}) \) are pairs \((A, p : UlT(A) \to Y)\) with a complete Boolean algebra \( A \) and \( p \in \mathcal{P}; \) morphisms \((\varphi, f) : (A, p) \to (A', p')\) are given by Boolean homomorphisms \( \varphi : A \to A' \) and maps \( f \) that make the diagram

\[
\begin{array}{ccc}
UlT(A) & \xrightarrow{UlT(\varphi)} & UlT(A') \\
p \downarrow & & \downarrow p' \\
Y & \xleftarrow{f} & Y'
\end{array}
\]

commute (see Construction 13). We note that such a map \( f \) is necessarily continuous and uniquely determined by \( \varphi \) since, as a continuous surjection of compact Hausdorff spaces, the map \( p' \) in \( \mathcal{P} \) provides its codomain with the quotient topology of its domain. The projection functor \( C(A, \mathcal{P}, \mathcal{X})^{\text{op}} \to KHaus \), \((\varphi, f) \mapsto f\), induces the compatible relation \( \sim \) on \( C(A, \mathcal{P}, \mathcal{X}) \), so that \((\varphi, f) \sim (\psi, g) \iff f = g\) for \((\varphi, f), (\psi, g) : (A, p) \to (A', p')\).

We obtain the quotient category \( \mathcal{B} \), with the same objects as in \( C(A, \mathcal{P}, \mathcal{X}) \). The contravariant functor \( \tilde{T} \) is induced by the projection functor; that is:

\[
\tilde{T} : \mathcal{B}^{\text{op}} \to KHaus, \quad ([\varphi, f] : (A, p) \to (A', p')) \mapsto f.
\]

With \( \pi_Y : EY = UlT(RC(Y)) \to Y \) denoting the Gleason cover of a compact Hausdorff space \( Y \) (see Facts 8), the adjoint \( \tilde{S} \) of \( \tilde{T} \) as constructed in Theorem 14 formally assigns to \( Y \) the \( \mathcal{B} \)-object \((CO(EY), \pi_Y \circ \eta_{EY}^{-1})\) which, however, is naturally isomorphic to \((RC(Y), \pi_Y);\)
here \( \eta \) and \( \varepsilon \) are as in Facts 1:

\[
E_Y = \text{Ult}(RC(Y)) \xleftarrow{\text{Ult}(\phi_{RC(Y)})} \text{Ult}(CO(E_Y))
\]

\[
\begin{array}{c|c}
\pi_Y & 1_Y \\
\hline
Y & Y
\end{array}
\]

We may therefore assume \( \tilde{S}(Y) = (RC(Y), \pi_Y) \) and \( \tilde{S}(f) = [\varphi_f, f] \) for \( f : Y' \to Y \) in \textbf{Khaus}, where \( \varphi_f : RC(Y) \to RC(Y') \) is a Boolean homomorphism such that \( \pi_Y \circ \text{Ult}(\varphi_f) = f \circ \pi_{Y'} \). This leaves all assertions of Theorem 14 in tact. In fact, it simplifies them since we now have that the natural isomorphism \( \gamma \) of Theorem 14 is actually an identity transformation: \( IS = \tilde{S}J \).

In order to show that \textbf{Khaus} is dually equivalent to the category \textbf{deV} (see Facts 5), we just need to exhibit \textbf{deV} as equivalent to the category \( \mathcal{B} \) of Proposition 19. We start by proving a technical lemma which generalizes \cite[Proposition 1.5.4]{10}.

**Definition 20** For complete normal contact (= de Vries) algebras \( A \) and \( A' \) and a monotone function \( \varphi : A \to A' \), we call the function \( \varphi^\vee : A \to A' \), defined by

\[
\varphi^\vee(a) := \bigvee \{ \varphi(b) \mid b \ll a \}
\]

for every \( a \in A \), the de Vries transform of \( \varphi \).

With the pointwise order of functions of ordered sets, one notes immediately that (1) \( \varphi^\vee \) is monotone; (2) \( \varphi^\vee \leq \varphi \); (3) if \( \varphi \leq \tilde{\varphi} \), then \( \varphi^\vee \leq \tilde{\varphi}^\vee \). Less trivially one has:

**Lemma 21** Let \( A, A', A'' \) be de Vries algebras. If the monotone function \( \varphi : A \to A' \) preserves the relation \( \ll \), then one has the implication

\[
b \ll a \implies \varphi(b) \ll \varphi^\vee(a)
\]

for all \( a, b \in A \); furthermore, for every monotone function \( \psi : A' \to A'' \) one obtains

\[
(\psi \circ \varphi)^\vee = (\psi^\vee \circ \varphi^\vee)^\vee.
\]

**Proof** For the first assertion, given \( b \ll a \) in \( A \), one picks \( c \in A \) with \( b \ll c \ll a \) and obtains \( \varphi(a) \ll \varphi(c) \leq \varphi^\vee(b) \), by the \( \ll \)-preservation of \( \varphi \) and the definition of \( \varphi^\vee(b) \).

For the second assertion, exploiting the properties (1-3) above, one first observes \( \psi^\vee \circ \varphi^\vee \leq \psi \circ \varphi \) and concludes \( (\psi^\vee \circ \varphi^\vee)^\vee \leq (\psi \circ \varphi)^\vee \). To prove “\( \geq \)”, considering any \( b \ll a \) in \( A \), we again pick \( c \in A \) with \( b \ll c \ll a \) and obtain with the first assertion \( \varphi(b) \ll \varphi^\vee(c) \) which, by definition of \( \psi^\vee(\varphi^\vee(c)) \) and \( (\psi^\vee \circ \varphi^\vee)^\vee(a) \), gives us

\[
(\psi \circ \varphi)(b) = \psi(\varphi(b)) \leq \psi^\vee(\varphi^\vee(c)) = (\psi^\vee \circ \varphi^\vee)(c) \leq (\psi^\vee \circ \varphi^\vee)^\vee(a).
\]

Thus, taking the join over all \( b \ll a \), we see \( (\psi \circ \varphi)^\vee(a) \leq (\psi^\vee \circ \varphi^\vee)^\vee(a) \), as desired.

One routinely verifies that the composition in the category \textbf{deV} of de Vries algebras, given by (see Facts 4)

\[
\psi \circ \varphi = (\psi \circ \varphi)^\vee,
\]
stays in the range of de Vries morphisms. Now we form the auxiliary category

\textbf{deV Boo}

whose objects are the same as those of deV, i.e., are complete normal contact algebras; a morphism \( \varphi : A \to A' \) in deV Boo is a Boolean homomorphism reflecting the contact relation or, equivalently, preserving the associated relation \( \ll \) (see Facts 4). Unlike \( \diamond \) in deV, the composition of deV Boo-morphisms proceeds by ordinary map composition. With the map \( \varepsilon_A : A \to CO(Ult(A)) \) defined as in Facts 1, we first show:

**Proposition 22** There are functors

\[
C(A, \mathcal{P}, X) \xrightarrow{U} \text{deV Boo} \xrightarrow{V} \text{deV},
\]

where \( V \) maps objects identically and \( U \) provides the complete Boolean algebra \( A \) belonging to an object \((A, p)\) in \( C(A, \mathcal{P}, X) \) with the normal contact relation

\[
a \ll b \iff p(\varepsilon_A(a)) \cap p(\varepsilon_A(b)) \neq \emptyset \iff \exists u, v \in Ult(A) : a \in u, b \in v, p(u) = p(v).
\]

**Proof** First we confirm that \( \ll_P \) as defined makes \( A \) a complete normal contact algebra, for \((A, p : Ult(A) \to Y) \in |C(A, \mathcal{P}, X)|\). Indeed, by Alexandroff’s Theorem (see Facts 8), the map \( \rho_P : RC(Ult(A)) \to RC(Y), H \mapsto p(H), \) is a Boolean isomorphism, and so is \( \varepsilon_A : A \to CO(Ult(A)) = RC(Ult(A)), \) by the Stone duality. Hence, \( A \) is, like \( RC(Y), \) a complete normal contact algebra, since the contact relation \( \ll_P \) has been obtained by transferring the contact relation of \( RC(Y) \) along the (inverse of the) isomorphism \( \rho_P \circ \varepsilon_A. \)

Defining \( U \) on morphisms by \([(\varphi, f) : (A, p) \to (A', p')] \mapsto \varphi, \) we must just show that \( \varphi : A \to A' \) reflects the contact relations imposed by \( U. \) That, however, is obvious: if \( \varphi(a) \ll \varphi(b) \) in \( A', \) then \( p'(u') = p'(v') \) for some \( u', v' \in Ult(A') \) with \( \varphi(a) \in u', \varphi(b) \in v'; \) consequently, \( p(\varphi^{-1}(u')) = f(p'(u')) = f(p'(v')) = p(\varphi^{-1}(v')), \) which implies \( a \ll b. \)

We define \( V \) on a morphism \( \varphi : A \to B \) in deV Boo by \( V\varphi := \varphi \ll \) and must first confirm that the map \( \varphi \ll : A \to B \) satisfies conditions (V1-4) of Facts 4. It is straightforward to show that, since \( \varphi \) satisfies (V2), the map \( \varphi \ll \) satisfies (V2) and (V4). Condition (V1) holds trivially for \( \varphi \ll \). Hence, we are left with having to confirm that \( \varphi \ll \) satisfies (V3). Given \( a, b \in A \) and \( a^* \ll b, \) we find \( c, d \in A \) such that \( a^* \ll c \ll d \ll b. \) Since \( \varphi \) preserves the Boolean negation and the relation \( \ll, \) we have

\[
(\varphi \ll(a))^* = (\bigvee \{\varphi(e) \mid e \in A, e \ll a\})^* = \bigwedge \{\varphi(e)^* \mid e \in A, e \ll a\} = \bigwedge \{\varphi(e^*) \mid e \in A, e \ll a\} = \bigwedge \{\varphi(e) \mid e \in A, a^* \ll e\} \leq \varphi(c) \ll \varphi(d) \leq \varphi \ll(b).
\]

One has \( a = \bigvee \{b \in A \mid b \ll a\} \) for all \( a \in A, \) so that \( V \) preserves the identity map on \( A. \) The preservation by \( V \) of the composition follows from Lemma 21. \( \square \)

**Remark 23** Note that if, \((A, p : Ult(A) \to Y) \in |C(A, \mathcal{P}, X)|, \) then for all \( u, v \in Ult(A), \)

\[
u \ll_P v \iff p(u) = p(v).
\]

Indeed, the implication “\( \iff \)” follows easily from the Hausdorffness of \( Y \) and the fact that \( \{\varepsilon_A(a) \mid a \in A\} \) is an open base for \( Ult(A), \) while the converse implication is obvious.
Proposition 24 The functor $U$ has a right inverse $W : \text{devBoo} \to C(A, \mathcal{P}, \mathcal{X})$ with $W \circ U \cong \text{Id}$. In particular, $U$ is an equivalence of categories.

Proof With the contact relation $\prec$ of an object $A$ in $\text{devBoo}$ extended to an equivalence relation on $\text{Ult}(A)$ as in Facts 5, one puts  

$$WA = (A, \pi_A : \text{Ult}(A) \to \text{Ult}(A)/\prec),$$

where $\pi_A$ is the canonical projection. That $\pi_A$ is an irreducible continuous map in $\text{KHauss}$ with extremely disconnected domain and, thus, belongs to $\mathcal{P}$, has been confirmed in Lemma 4.4 of [18]. Hence, $WA$ is indeed a $(C(A, \mathcal{P}, \mathcal{X})$-object; moreover, as $\pi_A$ induces the original contact relation on $A$ (see Proposition 22 and Facts 5), one has $U(WA) = A$.

For a morphism $\varphi : A \to A'$ in $\text{devBoo}$, reflection of $\prec$ by $\varphi$ implies preservation of $\prec$ by $\text{Ult}(\varphi)$; indeed, as shown in Proposition 4.6 of [18], this follows quite easily from the Hausdorffness of $Y = \text{Ult}(A)/\prec$. Hence, with $Y' = \text{Ult}(A')/\prec$ one obtains a uniquely determined continuous map $f_\varphi$ making the diagram  

$$
\begin{array}{ccc}
\text{Ult}(A) & \xrightarrow{\text{Ult}(\varphi)} & \text{Ult}(A') \\
\pi_A \downarrow & & \downarrow \pi_{A'} \\
Y & \xleftarrow{f_\varphi} & Y'
\end{array}
$$

commute; explicitly, $f_\varphi(\pi_A(u')) = \pi_A(\varphi^{-1}(u'))$, for all $u' \in \text{Ult}(A')$. Obviously then, with $W\varphi = (\varphi, f_\varphi)$, we obtain a well-defined functor $W$. Since, trivially, $U(W\varphi) = \varphi$ for all $\varphi$, one has $U \circ W = \text{Id}_{\text{devBoo}}$.

Finally, for $(A, p) \in |C(A, \mathcal{P}, \mathcal{X})|$, $U$ provides $A$ with the contact relation $\prec_p$, and then $W$ provides $A$ with the projection $\pi_A$. We may define  

$$\lambda_{(A, p)} : W(U(A, p)) = (A, \pi_A) \to (A, p), \quad \lambda_{(A, p)} = (1_A, \ell_p),$$

with $\ell_p$ determined by the commutative diagram  

$$
\begin{array}{ccc}
\text{Ult}(A) & \xrightarrow{\text{Ult}(1_A)} & \text{Ult}(A) \\
\pi_A \downarrow & & \downarrow \pi_A \\
\text{Ult}(A)/\prec_p & \xleftarrow{\ell_p} & Y
\end{array}
$$

Quite trivially, $\ell_p$ is a homeomorphism, making $\lambda_{(A, p)}$ an isomorphism in $C(A, \mathcal{P}, \mathcal{X})$, which is obviously natural in $(A, p)$. \hfill $\square$

Unlike $U$, the functor $V$ of Proposition 22 is not an equivalence of categories. But it induces an equivalence relation on $\text{devBoo}$ that is compatible with the equivalence relation $\sim$ on $C(A, \mathcal{P}, \mathcal{X})$, as considered in Proposition 19. Concretely:

Lemma 25 For all morphisms $(\varphi, f), (\psi, g) : (A, p) \to (A', p')$ in $C(A, \mathcal{P}, \mathcal{X})$, one has $f = g$ if, and only if, $V\varphi = V\psi$; that is:  

$$(f = g \iff \varphi^\vee = \psi^\vee).$$

Proof Under the hypothesis $f = g$ one has $p \circ \text{Ult}(\varphi) = f \circ p' = g \circ p' = p \circ \text{Ult}(\psi)$, which means $\varphi^{-1}(u') \prec_p \psi^{-1}(u')$ for all $u' \in \text{Ult}(A')$. To show $\varphi^\vee(a) \leq \psi^\vee(a)$ for $a \in A$, it suffices to prove that for all $b \ll a$ there is some $c \ll a$ with $\varphi(b) \leq \psi(c)$. Assuming the opposite and, for the then given $b \ll a$, picking some $c \in A$ with $b \ll c \ll a$, one has
ϕ(b) \not\leq ψ(c). This implies \( d := ϕ(b) \land ψ(c) > 0 \), so that we find \( u' \in \text{Ult}(A') \) containing \( d \). Consequently, \( b \in ϕ^{-1}(u') \) and \( c^* \in \psi^{-1}(u') \). But then \( b \not\sim_p c^* \), which means \( b \not\ll c \), a contradiction. Since \( ϕ^-(a) \leq ϕ^-(a) \) follows by symmetry, the identity \( ϕ^\lor = \psi^\lor \) is thereby confirmed.

Conversely, having \( ϕ^\lor = \psi^\lor \), since \( p' \) is surjective, it suffices to show \( ϕ^{-1}(u') \not\sim_p \psi^{-1}(u') \) for all \( u' \in \text{Ult}(A') \) to conclude \( f = g \) (see Remark 23). Assuming \( ϕ^{-1}(u') \not\sim_p \psi^{-1}(u') \) for some \( u' \in \text{Ult}(A') \) we obtain \( b \in ϕ^{-1}(u') \) and \( a \in A \) with \( a^* \in \psi^{-1}(u') \) and \( b \not\sim_p a^* \), which means \( ϕ(b), \psi(a^*) \in u' \) and \( b \ll a \). Consequently, with the monotonicity of \( ψ \) one has

\[ 0 < ϕ(b) \land \psi(a)^* \leq ϕ(b) \land (\bigvee_{c \ll a} ψ(c))^* \]

which implies \( ϕ(b) \not\leq \bigvee_{c \ll a} ψ(c) = (Vψ)(a) \). But since \( ϕ(b) \leq ϕ^-(a) \), this contradicts the hypothesis \( ϕ^\lor = \psi^\lor \).

\[ \square \]

**Construction 26** With the compatible equivalence relation \( \sim \) on \( C(A, \mathcal{P}, \mathcal{X}) \) as defined in Proposition 19, and with \( \sim \) denoting the equivalence relation on \( \text{deVBoo} \) induced by \( V \) (so that \( ϕ \sim ψ \iff ϕ^\lor = ψ^\lor \)), the assertion of Lemma 25 reads as \( (\langle ϕ, f \rangle \sim (\langle ψ, g \rangle \iff ϕ \sim ψ \)). Denoting by \( \langle ϕ \rangle \) the \( \sim \)-equivalence class of a morphism \( ϕ \) in \( \text{deVBoo} \), the functors \( U \) and \( V \) therefore induce faithful functors

\[ \mathcal{B} = C(A, \mathcal{P}, \mathcal{X})/\sim \xrightarrow{U} \text{deVBoo}/\sim \xrightarrow{V} \text{deV} \]

mapping objects like \( U \) and \( V \), and morphisms by \( [\varphi, f] \mapsto \langle ϕ \rangle \) and \( \langle ϕ \rangle \mapsto ϕ^\lor \), respectively.

To obtain the de Vries duality, it now suffices to prove that the functors \( U \) and \( V \) are equivalences of categories. The first of these two assertions comes almost for free now, and it actually produces a new duality for the category \( \text{K Haus} \) that is of independent interest:

**Theorem 27** (Modifed de Vries Duality Theorem) The functor \( U \) is an equivalence of categories. As a consequence, \( \text{K Haus} \) is dually equivalent to the category \( \text{deVBoo}/\sim \), whose objects are complete normal contact algebras and whose morphisms are equivalence classes of Boolean morphisms reflecting the contact relations, to be composed by ordinary map composition of their representatives.

**Proof** Just as \( U \) induces the functor \( U \), the functor \( W \) of Proposition 24 induces the functor

\[ \overline{W} : \text{deVBoo}/\sim \longrightarrow C(A, \mathcal{P}, \mathcal{X}), \quad \langle ϕ \rangle \mapsto [ϕ, f_ϕ]. \]

It maps objects as \( W \) does and still satisfies \( U \circ \overline{W} = \text{Id} \) and \( \overline{W} \circ U \cong \text{Id} \). Hence, \( U \) is an equivalence of categories, with quasi-inverse \( \overline{W} \). The claimed duality now follows with Proposition 19.

\[ \square \]

Since \( \overline{V} \) maps objects identically and is trivially faithful, it suffices to show that \( \overline{V} \) is full, in order for us to conclude that it is an isomorphism of categories. To this end, we first confirm the following proposition in which the assertions (2) and (3) are due to de Vries [10], but our proofs are new. Here \( ε_A \) and \( τ_A \) are defined as in Facts 1 and Facts 5, and \( π_A : \text{Ult}(A) \rightarrow \text{Ult}(A)\sim \) is the irreducible projection of Proposition 24.

**Proposition 28** (1) For every de Vries algebra \((A, \sim)\), one has the homeomorphism

\[ γ_A : \text{Ult}(A)\sim \longrightarrow \text{Clust}(A), \quad [u] \mapsto ε_u = \{ a \in A \mid ∀b \in u : a \sim b \} \]
(making $\text{Clust}(A)$ a compact Hausdorff space), and the Boolean isomorphism

$$\rho_{\gamma_A \circ \pi_A} : CO(Ult(A)) \longrightarrow RC(\text{Clust}(A)), \ H \mapsto \gamma_A(\pi_A(H)),$$

which fits into the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\tau_A} & CO(Ult(A)) \\
\downarrow{\rho_{\gamma_A \circ \pi_A}} & & \downarrow{\rho_{\gamma_A \circ \pi_A}} \\
\tau_A & & \tau_A \\
RC(\text{Clust}(A)) & \xrightarrow{RC(\gamma_A)} & RC(\text{Clust}(A)).
\end{array}$$

(2) $\tau_A$ is a Boolean isomorphism which preserves the relation $\ll$ and, thus, is a deV-isomorphism.

(3) For every morphism $\alpha : A \rightarrow A'$ in deV, one has the continuous map

$$\hat{\alpha} : \text{Clust}(A') \rightarrow \text{Clust}(A), \ c' \mapsto \{ a \in A \mid \forall b \in A \ (b \ll a^* \Rightarrow \alpha(b)^* \in c') \},$$

which fits into the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow{\tau_A} & & \downarrow{\tau_A'} \\
RC(\text{Clust}(A)) & \xrightarrow{RC(\hat{\alpha})} & RC(\text{Clust}(A')).
\end{array}$$

**Proof** (1) We first note that the family $\{ \tau_A(a) \mid a \in A \}$ can indeed be taken as a closed base for the space $\text{Clust}(A)$ since $\tau_A(0) = \emptyset$, and for all $a, b \in A$ one has

$$\tau_A(a \vee b) = \{ c \in \text{Clust}(A) \mid a \vee b \in c \} = \{ c \in \text{Clust}(A) \mid a \in c \text{ or } b \in c \} = \tau_A(a) \cup \tau_A(b).$$

From the assertions of Facts 5 we know that $\gamma_A$ is a well-defined bijective map. Also, by applying Alexandroff’s Theorem (see Facts 8) to $\pi_A : Ult(A) \rightarrow Y = Ult(A)/\sim$, we obtain the Boolean isomorphism

$$\rho_{\pi_A} : CO(Ult(A)) = RC(Ult(A)) \rightarrow RC(Y), \ H \mapsto \pi_A(H).$$

That $\gamma_A$ is actually a homeomorphism follows with

$$\gamma_A(\pi_A(\varepsilon_A(a))) = \{ c_u \mid a \in u \in Ult(A) \} = \{ c \mid a \in c \in \text{Clust}(A) \} = \tau_A(a),$$

for all $a \in A$. Consequently,

$$\rho_{\gamma_A} : RC(Y) \rightarrow RC(\text{Clust}(A)), \ K \mapsto \gamma_A(K),$$

and then also $\rho_{\gamma_A \circ \pi_A} = \rho_{\gamma_A} \circ \rho_{\pi_A}$, are Boolean isomorphisms, and the triangle of (1) commutes. Finally, since $Ult(A)/\sim$ is a compact Hausdorff space (as shown in [18, Lemma 4.4]) and $\gamma_A$ is a homeomorphism, $\text{Clust}(A)$ is also a compact Hausdorff space.

(2) Clearly, by (1), $\tau_A$ is a Boolean isomorphism. With the assertions of Facts 5 (see also [20, Corollary 3.4, p.222]), for all $a, b \in A$ one has

$$a \sim b \iff \exists u, v \in Ult(A) : a \in u, b \in v, u \sim v$$

$$\iff \exists u, v \in Ult(A) : a \in u, b \in v, c_u = c_v$$

$$\iff \exists c \in \text{Clust}(A) : a, b \in c$$

$$\iff \tau_A(a) \sim_Z \tau_A(b).$$
where $Z := Clust(A)$. Consequently, $\tau_A$ preserves the relation $\ll$.

(3) We must first confirm that $\hat{\alpha}'$ is a cluster of $A$, for every $c' \in Clust(A')$. Obviously, $1 \in \hat{\alpha}'(c')$, so that (cl 1) is satisfied. For (cl 2), we consider $a, b \in \hat{\alpha}'(c')$ and assume that we had $a \not< b$, that is: $a \ll b^*$. Choosing $c, d \in A$ such that $a \ll c \ll d^* \ll b^*$ or, equivalently, $b \ll d \ll c^* \ll a^*$, the definition of $\hat{\alpha}'(c')$ gives $(\alpha(c^*))^* \in c'$ and $(\alpha(d^*))^* \in c'$, so that $(\alpha(c^*))^* \wedge (\alpha(d^*))^* \in c'$, which implies $(\alpha(d^*))^* \not< (\alpha(c^*))^*$, a contradiction.

To confirm (cl 3), we consider $a, b \in A$ with $a \vee b \in \alpha'(c')$ and assume we had $a \notin \hat{\alpha}'(c')$ and $b \notin \hat{\alpha}'(c')$. Then we obtain $c, d \in A$ such that $c \ll a^*, d \ll b^*$ and $(\alpha(c))^* \notin c'$, $(\alpha(d))^* \notin c'$. From $c \wedge d \ll a^* \wedge b^* = (a \vee b)^*$ we conclude
\[(\alpha(c) \wedge d)^* = (\alpha(c) \wedge (a(d))^*) = (\alpha(c))^* \vee (\alpha(d))^* \in c',\]
which implies $(\alpha(c))^* \in c'$ or $(\alpha(d))^* \in c'$, a contradiction.

Finally, for (cl 4), we consider $a \in A$ with $a \wedge b$ for every $b \in \hat{\alpha}'(c')$ and assume $a \notin \hat{\alpha}'(c')$, so that, for some $c \in A$, one has $c \ll a^*$, but $(\alpha(c))^* \notin c'$. Then $(\alpha(c))^* \notin c'$. We claim that $c$ lies in $\hat{\alpha}'(c')$. Indeed, if $d \in A$ satisfies $d \ll c^*$, then, as $\alpha$ preserves $\ll$, we have $\alpha(d) \ll \alpha(c^*)$; therefore, $\alpha(c) \leq (\alpha(c^*))^* \ll (\alpha(d))^*$ which, with $\alpha(c) \in c'$, implies $(\alpha(d))^* \in c'$; thus $c \in \hat{\alpha}'(c')$. But, since $c \ll a$, we have $a \not< c$, despite $c \in \hat{\alpha}'(c')$—a contradiction.

To show the continuity of $\hat{\alpha}$, we first observe that, since $\{\tau_A(a) \mid a \in A\} = RC(Z)$ is a base for closed sets in $Z = Clust(A)$, the set
\[RO(Z) = \{\text{int}(F) \mid F \in RC(Z)\} = \{\text{int}(\tau_A(a)) \mid a \in A\}\]
is a base for open sets in $Z$. We must therefore show that $\hat{\alpha}^{-1}(\text{int}(\tau_A(a)))$ is open in $Z' = Clust(A')$, for every $a \in A$. Indeed, denoting the Boolean operations of $RC(Z)$ as in Facts 4, with $F = \tau_A(a)$ we have
\[\text{int}(F) = Z \setminus \text{cl}(Z \setminus F) = Z \setminus F^* = Z \setminus \tau_A(a^*) = \{c \in Z \mid a^* \not< c\}.\]
Hence, for every $c' \in Z'$, one obtains
\[c' \in \hat{\alpha}^{-1}(\text{int}(F)) \iff a^* \not< \hat{\alpha}'(c') \iff \exists b \in A \text{ (} b \ll a, (\alpha(b))^* \not< c'\).\]
Since $(\alpha(b))^* \not< c'$ means equivalently $c' \notin \tau_{A'}((\alpha(b))^*)$, we conclude that the set
\[\hat{\alpha}^{-1}(\text{int}(F)) = \bigcup_{b \ll a} Z' \setminus \tau_{A'}((\alpha(b))^*) = \bigcup_{b \ll a} \text{int}(\tau_{A'}((\alpha(b)^*)))\]
is indeed open in $Z'$. Furthermore, continuing to take advantage of the Boolean isomorphism $\tau_A$ and the de Vries morphism $\alpha$, we see that
\[\text{cl}(\hat{\alpha}^{-1}(\text{int}(F))) = \text{cl}(\bigcup_{b \ll a} \text{int}(\tau_{A'}((\alpha(b))^*))) = \bigvee_{b \ll a} \tau_{A'}((\alpha(b)) = \tau_{A'}(\bigvee_{b \ll a} \alpha(b)) = \tau_{A'}(\alpha(a))\]
holds in $Z'$. But this proves precisely the claimed identity $RC(\hat{\alpha}) \circ \tau_A = \tau_{A'} \circ \alpha$. □

As a final step in completing the proof of the fullness of the functor $\overline{V}$ we prove the following lemma. Here we understand $T_3$-space (resp., $T_4$-space) to mean a regular (resp., normal) Hausdorff space.

**Lemma 29** Let $p : X \rightarrow Y$, $p' : X' \rightarrow Y'$, $f : X' \rightarrow X$, $g : Y' \rightarrow Y$ be continuous maps of topological spaces with $p \circ f = g \circ p'$. If $p$ and $p'$ are closed irreducible and $Y$ a $T_4$-space, then every $G \in RC(X)$ satisfies the identity
\[\text{cl}_{Y'}(g^{-1}(\text{int}_Y(p(G)))) = \bigcup \{p'\text{cl}_{X'}(f^{-1}(\text{int}_X(H))) \mid H \in RC(X), p(H) \subseteq \text{int}_Y(p(G))\}.\]
Proof Alexandreff’s Theorem (see Facts 8) gives us the Boolean isomorphism
\[ \rho_p : RC(X) \rightarrow RC(Y), \quad H \mapsto p(H), \]
whose inverse maps \( K \in RC(Y) \) to \( cl_X(p^{-1}(\text{int}_Y(K))) \). Since \( Y \) is a \( T_3 \)-space, for every \( G \in RC(X) \) the theorem implies
\[ \text{int}(p(G)) = \bigcup \{ p(H) \mid H \in RC(X) \text{ and } p(H) \subseteq \text{int}(p(G)) \}. \]
In describing the inverse image under \( g \) of this union, we first note that, since \( p' \) is surjective, we have \( g^{-1}(p(H)) = p'(p^{-1}(g^{-1}(p(H)))) = p'(f^{-1}(p^{-1}(p(H)))) \) for every \( H \in RC(X) \), which gives us
\[ g^{-1}(\text{int}(p(G))) = \bigcup \{ p'(f^{-1}(p^{-1}(p(H)))) \mid H \in RC(X) \text{ and } p(H) \subseteq \text{int}(p(G)) \}. \]
Since \( Y \) is a \( T_3 \)-space, Alexandreff’s Theorem gives us for every \( H \) with \( p(H) \subseteq \text{int}(p(G)) \) contributing to this union a set \( H' \in RC(X) \) with \( p(H) \subseteq \text{int}(p(H')) \subseteq p(H') \subseteq \text{int}(p(G)) \); moreover, \( H' = p^{-1}_p(p(H')) = cl(p^{-1}(\text{int}(p(H')))) \). Consequently,
\[ H \subseteq p^{-1}(p(H)) \subseteq p^{-1}(\text{int}(p(H'))) \subseteq \text{int}(\text{cl}(p^{-1}(\text{int}(p(H'))))) = \text{int}(H') \subseteq H'. \]
As a result,
\[ g^{-1}(\text{int}(p(G))) = \bigcup \{ p'(\text{cl}(f^{-1}(\text{int}(H)))) \mid H \in RC(X) \text{ and } p(H) \subseteq \text{int}(p(G)) \}. \]
Since \( f^{-1}(H) \subseteq f^{-1}(\text{int}(H')) \subseteq \text{cl}(f^{-1}(\text{int}(H'))) \subseteq f^{-1}(H') \), we obtain that
\[ g^{-1}(\text{int}(p(G))) = \bigcup \{ p'(\text{cl}(f^{-1}(\text{int}(H)))) \mid H \in RC(X) \text{ and } p(H) \subseteq \text{int}(p(G)) \}. \]
Finally, using Facts 4, we conclude the claimed formula. \( \square \)

We note that, for obtaining the last formula in the proof of Lemma 29, we used only the surjectivity of the map \( p' \). The stronger requirement that \( p' \) be closed irreducible is needed only for establishing the formula claimed in Lemma 29. Also, note that in this formula, the requirement \( p(H) \subseteq \text{int}(p(G)) \) is equivalent to requiring \( p^{-1}(p(H)) \subseteq \text{int}(G) \). Indeed, the implication from left to right was already shown in the proof of Lemma 29. So, assuming conversely \( p^{-1}(p(H)) \subseteq \text{int}(G) \) and using two lemmas for closed irreducible mappings of Ponomarev’s paper [33], we obtain \( p(G) = p(\text{cl}(\text{int}(G))) = \text{cl}(p^\#(\text{int}(G))) \), where \( p^\#(\text{int}(G)) = \{ y \in Y \mid p^{-1}(y) \subseteq \text{int}(G) \} \) is open in \( Y \). Hence, \( p(H) \subseteq p^\#(\text{int}(G)) \subseteq \text{int}(p(G)) \) follows.

We can now sum up and return to the functor
\[ \overline{V} : \text{deVBoo}/\sim \rightarrow \text{deV} \]
of Construction 26 and complete our alternative proof of the de Vries dual equivalence.

Theorem 30 (The de Vries Duality Theorem) The functor \( \overline{V} \) is an isomorphism of categories. Consequently, the category \( \text{KHaus} \) is dually equivalent to the category \( \text{deV} \) of de Vries algebras and their morphisms.

Proof In order to show that \( \overline{V} \) is full, given \( \alpha : A \rightarrow A' \) in \( \text{deV} \), we must show \( \alpha = V \varphi \), for some \( \varphi : A \rightarrow A' \) in \( \text{deVBoo} \). Proposition 28 produces the continuous map \( \hat{\alpha} = \text{Clust}(\alpha) : \text{Clust}(A') \rightarrow \text{Clust}(A) \), as well as the homeomorphism \( \gamma_A : \text{Ult}(A)/\sim \rightarrow \text{Clust}(A), \ [u] \mapsto c_u \). With \( \hat{f} = \gamma_A^{-1} \circ \hat{\alpha} \circ \gamma_A' \), the Gleason Theorem (see Facts 8) together
with the fact that the map \( \pi_A : \Ult(A) \to \Ult(A)/\sim \) is a surjection give us a continuous map \( f \) making the diagram

\[
\begin{array}{ccc}
\Ult(A) & \xleftarrow{f} & \Ult(A') \\
\pi_A \downarrow & & \pi_{A'} \downarrow \\
\Ult(A)/\sim & \xleftarrow{\tilde{f}} & \Ult(A')/\sim \\
\gamma_A \downarrow & & \gamma_{A'} \downarrow \\
\Clust(A) & \xleftarrow{\tilde{\alpha}} & \Clust(A')
\end{array}
\]

commute. By the Stone duality, \( f = \Ult(\varphi) \), for a Boolean homomorphism \( \varphi : A \to A' \).

Obviously, the commutativity of the above diagram implies that \( \Ult(\varphi) \) preserves the contact relation for ultrafilters, so that (by Proposition 4.6 of [18]) \( \varphi \) itself must reflect the contact relation and therefore be a morphism in \text{deVBoo}.

Applying Lemma 29 to the outer rectangle of the above diagram, thus putting \( p := \gamma_A \circ \pi_A, \ p' := \gamma_{A'} \circ \pi_{A'} \) and \( g := \tilde{\alpha} \), we obtain

\[ RC(\tilde{\alpha}) = (\rho_{p'} \circ CO(\Ult(\varphi)) \circ \rho_{p}^{-1})^\vee. \]

Now, Proposition 28(3) implies \( \tau_{A'} \circ \alpha \circ \tau_A^{-1} = (\rho_{p'} \circ CO(\Ult(\varphi)) \circ \rho_{p}^{-1})^\vee \) and thus

\[ \alpha = \tau_A^{-1} \circ \zeta^\vee \circ \tau_A, \quad \text{where} \quad \zeta := \rho_{p'} \circ CO(\Ult(\varphi)) \circ \rho_{p}^{-1}. \]

With Lemma 21 and Proposition 28(2) we then obtain

\[
\begin{align*}
\alpha &= \alpha^\vee = ((\tau_A^{-1} \circ \zeta^\vee) \circ \tau_A)^\vee = (((\tau_A^{-1})^\vee \circ \zeta^\vee \circ (\tau_A)^\vee)^\vee \\
&= ((\tau_{A'}^{-1} \circ \zeta)^\vee \circ (\tau_A)^\vee)^\vee = (\tau_{A'}^{-1} \circ \zeta \circ \tau_A)^\vee \\
&= (\tau_{A'}^{-1} \circ \rho_{p'} \circ CO(\Ult(\varphi)) \circ \rho_{p}^{-1} \circ \tau_A)^\vee.
\end{align*}
\]

Finally, applying Proposition 28(1) and the Stone Duality, we conclude

\[ \alpha = (\varepsilon_{A'}^{-1} \circ CO(\Ult(\varphi)) \circ \varepsilon_A)^\vee = \varphi^\vee = V(\varphi), \]

as desired. \( \square \)

We also confirm that our constructions leading up to Theorem 30 give, up to natural isomorphisms, the functors \( \Clust \) and \( RC \) furnishing the classical de Vries dual equivalence, with unit \( \sigma \) and counit \( \tau \) as described in Facts 5; that is:

\[ \textbf{Corollary 31} \quad \text{The diagram}
\]

![Diagram](image)

commutes in an obvious sense, up to natural isomorphism.
Proof It suffices to confirm that the functor $\tilde{T} \circ \tilde{W} \circ V^{-1}$ is naturally isomorphic to $Clust$ since then its left adjoint, $\tilde{V} \circ \tilde{U} \circ \tilde{S}$, must be naturally isomorphic to $RC$. Indeed, for $(A, \sim) \in |\text{deV}|$, by Proposition 28 one has the homeomorphism

$$\tilde{T}(\tilde{W}(V^{-1}(A, \sim))) = \tilde{T}(A, \pi_A) = \text{Ult}(A)/\sim \to Clust(A).$$

Checking that $\gamma_A$ is natural in $A$ involves going back to the morphism definitions of the functors involved, but that is a routine matter. \hfill \Box

5 Two Pathways to the Fedorchuk Duality

Sections 3 and 4 lead to two novel, but distinct methods of establishing the Fedorchuk duality, one using the de Vries duality, the other not using it. Under the former method, having established the de Vries duality with the help of the categorical extension technique as in Sect. 4, one may simply follow Fedorchuk’s original approach [23] and obtain his duality by showing that the de Vries duality may be restricted accordingly (as visualized by the top panel of the diagram of Facts 7).

Under the latter method, similarly to how we established the de Vries duality in this paper, one applies the categorical construction to the restricted Stone duality

$$\mathcal{A}^{op} = (\text{CBoo}, \sup)^{op} \cong \text{EKH}_{\text{open}} = \mathcal{X}.$$ 

As this approach has been outlined in [18], here we give only a brief summary of it.

Step 1. One first shows that the category $\mathcal{X}$ as above is coreflective in the category of compact Hausdorff spaces and quasi-open maps, $\mathcal{Y} = \text{KHaus}_{q-\text{open}}$; see [36] or Proposition 4.1 of [18].

Step 2. Now one can choose $\mathcal{P}$ to be the class of coreflections (as in Examples 12(1)) and apply Corollary 15 to obtain the dual equivalence

$$\mathcal{B}^{op} := C(\mathcal{A}, \mathcal{P}, \mathcal{X})^{op} \cong \text{KHaus}_{q-\text{open}} = \mathcal{Y}.$$ 

Step 3. One establishes an equivalence

$$\text{Fed} \cong C(\mathcal{A}, \mathcal{P}, \mathcal{X}),$$

of categories, with the functors $F$ and $G$ defined on objects like the functors $W$ and $U$ of Proposition 24 and Proposition 22, respectively.

Step 4. The composition of the dual equivalence of Step 2 with the dualization of the equivalence of Step 3 produces the Fedorchuk duality, as in

$$\text{Fed}^{op} \cong C(\mathcal{A}, \mathcal{P}, \mathcal{X})^{op} \cong \text{KHaus}_{q-\text{open}}.$$
6 The Hom Representation of the de Vries Duality

In Facts 1 we emphasized that the Stone duality arises naturally from a dual adjunction represented by the Boolean algebra 2 inducing the ultrafilter monad on \( \text{Set} \), as a restriction of the comparison functor into the Eilenberg–Moore category of the monad, \( \text{KHaus} \). In this section we show that the de Vries duality may be obtained in exactly the same manner; actually, more succinctly so, as no restriction of the comparison functor is necessary. The key observation, proved first in [15], is that, as a contravariant \( \text{Set} \)-valued functor, \( \text{Clust} \) is represented by 2, now considered as a discrete de Vries algebra.

In more detail, for every de Vries algebra \( A \), we consider the map

\[
\omega_A : \text{deV}(A, 2) \longrightarrow \text{Clust}(A), \quad \varphi \mapsto \{a \in A \mid \varphi(a^*) = 0\}.
\]

**Proposition 32** The maps \( \omega_A \) \((A \in [\text{deV}])\) define a natural isomorphism \( \omega : \text{deV}(\_ , 2) \longrightarrow \text{Clust} \) of functors \( \text{deV}^{\text{op}} \longrightarrow \text{Set} \).

**Proof** For every de Vries morphism \( \varphi : A \rightarrow 2 \), we should first confirm that \( \omega_A(\varphi) \) is indeed a cluster in \( A \).

(c1) Since \( \varphi(1^*) = \varphi(0) = 0 \), one trivially has \( 1 \in \omega_A(\varphi) \neq \emptyset \).

(c2) For \( a, b \in \omega_A(\varphi) \), let’s suppose we had \( a \not\sim b \), that is: \( a \ll b^* \). Then (V3) implies \((\varphi(a^*))^* \ll \varphi(b^*)\), i.e., \( 1 \leq 0 \), a contradiction.

(c3) In the presence of (V2), from \( a \lor b \in \omega_A(\varphi) \) one obtains \( 0 = \varphi((a \lor b)^*) = \varphi(a^* \land b^*) = \varphi(a^*) \land \varphi(b^*) \), which gives \( a \in \omega_A(\varphi) \) or \( b \in \omega_A(\varphi) \).

(c4) For \( a \in A \) with \( a \not\sim b \) for all \( b \in \omega_A(\varphi) \), let’s suppose we had \( a \notin \omega_A(\varphi) \), so that \( \varphi(a^*) = 1 \). Then, by (V4), there must be some \( b \in A \) with \( b \ll a^* \) and \( \varphi(b) = 1 \). Since \( a \not\sim b \), by hypothesis on \( a \) we must have \( b \notin \omega_A(\varphi) \), which means \( \varphi(b^*) = 1 \). But this contradicts the trivial fact \( 0 = \varphi(b \land b^*) = \varphi(b) \land \varphi(b^*) \).

Since, by definition, the cluster \( c = \omega_A(\varphi) \) satisfies

\[ \varphi(a) = 0 \iff a^* \in c \]

for all \( a \in A \), one can now take this equivalence to, conversely, define a map \( \varphi = \varphi_c : A \rightarrow 2 \) for any given cluster \( c \) in \( A \). Once we have shown that \( \varphi_c \) is a de Vries morphism, it is then clear that \( c \mapsto \varphi_c \) is inverse to \( \omega_A \), so that \( \omega_A \) is bijective.

(V1): From \( 0^* = 1 \in c \neq \emptyset \) one obtains \( \varphi_c(0) = 0 \).

(V2): Since the cluster \( c \) is upwards closed, the map \( \varphi_c \) is monotone, so that \( \varphi_c(a \land b) \leq \varphi_c(a) \land \varphi_c(b) \) follows for all \( a, b \in A \). Assuming now \( \varphi_c(a \land b) = 1 \), so that \( (a \land b)^* \notin c \), we have \( a^* \lor b^* \notin c \). This means that neither \( a^* \) nor \( b^* \) can lie in the upwards closed set \( c \). Consequently, \( \varphi_c(a) = 1 = \varphi_c(b) \). We conclude that \( \varphi_c(a \land b) = \varphi_c(a) \land \varphi_c(b) \) holds always.

(V3): Given \( a, b \in A \) with \( a^* \ll b \), assume first \( \varphi_c(a) = 0 \), so that \( a^* \in \sigma \). Since \( a^* \not\sim b^* \), we then obtain that \( b^* \notin c \), or \( \varphi_c(b) = 1 \). We conclude that \( (\varphi_c(a))^* \ll \varphi_c(b) \) holds, trivially so when \( \varphi_c(a) = 1 \).

(V4): Since \( \varphi_c \) is monotone (see (V2)), we certainly have for all \( a \in A \) that \( \varphi_c(a) \geq \sqrt{\varphi_c(b)} \mid b \in A, b \ll a \). To show equality, assume \( \varphi_c(a) = 1 \), that is, \( a^* \notin c \). Then, by (c14), there exists \( c \in c \) with \( a^* \not\sim c \), or \( c \ll a \). We may then pick some \( b \in A \) with \( c \ll b \ll a \). From \( c \not\sim b^* \) we now obtain \( b^* \notin c \), which means \( \varphi_c(b) = 1 \). As \( b \ll a \), this confirms (V4).
We must finally confirm the naturality of $\omega$, that is: for every de Vries morphism $\alpha : A \to A'$, we have to show the commutativity of the diagram

\[
\begin{array}{c}
\text{dev}(A, 2) \\
\downarrow \omega_A
\end{array}
\begin{array}{c}
\text{dev}(\alpha, 2) \\
\downarrow \omega_{A'}
\end{array}

\begin{array}{c}
\text{Clust}(A) \\
\downarrow \text{Clust}(\alpha)
\end{array}
\begin{array}{c}
\text{Clust}(A')
\end{array}
\]

Here the map $\omega_A \circ \text{dev}(\alpha, 2)$ sends every de Vries morphism $\varphi : A' \to 2$ to the cluster $\omega_A(\varphi \circ \alpha) = \{a \in A \mid (\varphi \circ \alpha)(a^*) = 0\}$,

while the map $\text{Clust}(\alpha) \circ \omega_{A'}$ sends $\varphi$ to the cluster $\{a \in A \mid \forall b (b \ll a^* \Rightarrow (\alpha(b))^* \in \omega_{A'}(\varphi))\} = \{a \in A \mid \forall b (b \ll a^* \Rightarrow (\varphi \circ \alpha)(b) = 0)\}$.

But since $(\varphi \circ \alpha)(a^*) = \bigvee \{(\varphi \circ \alpha)(b) \mid b \in A, b \ll a^*\}$, the two clusters coincide. $\square$

In strong analogy to the fundamental adjunction of Facts 1 underlying the Stone duality, we will now set up a dual adjunction, replacing Boolean by de Vries algebras, and then show that it may be used to build up the de Vries duality in a categorical manner, via the comparison functor of $\text{dev}^{\text{op}}$ into the Eilenberg–Moore category of its induced $\text{Set}$-monad which, just as in the case of the Stone duality, turns out to be (up to isomorphism) the category $\text{K Haus}$. To this end, in what follows we regard the power set $P(X)$ of a set $X$ as a discrete de Vries algebra, by taking $\ll$ to be the inclusion order; that is, $\text{Set}(X, 2)$ is regarded as a discrete de Vries algebra with the pointwise order inherited from the two-chain 2. In order to see that $X \mapsto \text{Set}(X, 2)$ is left adjoint to the representable functor $\text{dev}(\cdot, 2)$, it suffices to confirm that $\text{Set}(X, 2)$ serves as a direct product of $X$-many copies of 2 in $\text{dev}$. The proof of the following proposition confirms this fact, by describing the natural adjunction bijection explicitly.

**Proposition 33** There is an adjunction

\[
\begin{array}{c}
\text{dev}^{\text{op}} \\
\downarrow \text{Set}
\end{array}
\begin{array}{c}
\text{Set}
\end{array}
\]

whose induced monad on $\text{Set}$ is (isomorphic to) the ultrafilter monad.

**Proof** It suffices to establish, for every set $X$, a bijective correspondence between the de Vries morphisms $\varphi : A \to \text{Set}(X, 2)$ and the maps $f : X \to \text{dev}(A, 2)$, naturally so for every de Vries algebra $A$. Such a correspondence may be facilitated by the constraints $f(x)(a) = \varphi(a)(x)$ for all $x \in X, a \in A$, i.e., $f(x)(\cdot) = \varphi(\cdot)(x)$ for all $x \in X$. Indeed, given a de Vries morphism $\varphi$, the values of the map $f$ defined by these constraints are obviously de Vries morphisms, and conversely. The naturality of this correspondence is easily established, similarly to the naturality proof of $\omega$, which completes the proof for the claimed adjunction.
To compute the induced monad, by Proposition 32 one has, for every set $X$, the natural isomorphisms
\[
deV(\text{Set}(X, 2), 2) \cong \deV(P(X), 2) \cong Clust(P(X)) = \text{Ult}(P(X)).
\]
That the last identity holds more generally for all complete Boolean algebras was already mentioned at the end of Facts 5. In the case of the Boolean algebra $P(X)$, it is easy to see that a cluster $c$ in $P(X)$ is indeed an ultrafilter, since $c$ may be written as $c_u = \{F \subseteq X \mid \forall G \in u (F \cap G \neq \emptyset)\}$, for some ultrafilter $u$ on $X$ (see Facts 5). But every set $F \in c_u$ must actually lie in $u$, since otherwise we would have its complement lying in $u$ which, by definition of $c_u$, is impossible. Hence, $c = c_u = u$ is an ultrafilter on $X$. Proving that the resulting identity $Clust(P(X)) = \text{Ult}(P(X))$ extends to set maps involves only another routine check.

Since the bijective correspondence underlying the adjunction is the same as the correspondence of the fundamental adjunction of Facts 1 leading to the Stone duality, the pointwise definitions of the units and counits of the current adjunction are the same as the corresponding definitions for the fundamental adjunction. Consequently, the current adjunction induces a monad isomorphic to the monad induced by the fundamental adjunction, i.e., the ultrafilter monad.

Since the category of Eilenberg–Moore algebras of the ultrafilter monad on $\text{Set}$ is (isomorphic to) the category $\text{KHaus}$ (see [29], [28], [1]), the $\text{Set}$-valued functor $\deV(\cdot, 2)$ “lifts” to become the comparison functor $\deV^{\text{op}} : \deV^{\text{op}} \to \text{KHaus}$, with the topology on $\deV(A, 2)$ to be induced by the counit $\bar{\tau}_A$ of the adjunction, for every de Vries algebra $A$. But since $\deV(\cdot, 2) \cong Clust$ by Proposition 32, this counit must arise as a composite of the map $\tau_A$ of Facts 5 with the natural bijection induced by $\omega_A$ of Proposition 32, as in
\[
\bar{\tau}_A = [ A \xrightarrow{\tau_A} RC(Clust(A)) \xrightarrow{RC(\omega_A)} RC(\deV(A, 2)) ].
\]
As a consequence of the de Vries duality theorem one obtains the following theorem:

**Theorem 34** The de Vries dual equivalence may be presented as the comparison functor into the Eilenberg–Moore category of the monad induced by the adjunction of Proposition 33.

**Remarks 35** (1) We note that here we derived Theorem 34 by using the de Vries duality theorem in a significant way. But our presentation shows that this may be avoided. Indeed, since the category $\deV$ has equalizers, the general theory of monads describes how to obtain a left adjoint to the comparison functor $\deV(\cdot, 2) : \deV^{\text{op}} \to \text{KHaus}$ induced by the adjunction of Proposition 33, providing also an explicit description of the units and counits of the “lifted” adjunction. The core of the proof of the duality theorem then rests on the need to establish that these units and counits are all isomorphisms, for which one may use some of the key tools provided in Sect. 4.

(2) To the benefit of readers not familiar with basic monad theory and the Eilenberg–Moore construction, here is a more elaborate description of the functors and natural isomorphisms appearing in Theorem 34, as they emerge from the categorical setting.

Transporting, for every $A \in \deV$, the topological structure of $Clust(A)$ onto $\deV(A, 2)$, via the bijection $\omega_A$, one obtains the family $\{\omega_A^{-1}(\tau_A(a)) \mid a \in A\}$ as a base for closed sets
for the topology on \( \text{deV}(A, 2) \) under which the map \( \omega_A \) becomes a homeomorphism. Then, for every \( \alpha \in \text{deV}(A, A') \), the map

\[
\text{deV}(\alpha, 2) : \text{deV}(A', 2) \longrightarrow \text{deV}(A, 2), \quad \varphi' \longmapsto \varphi' \circ \alpha,
\]
is continuous, by the commutativity of the diagram in the proof of Proposition 32. This defines the functor

\[
\text{deV}(-, 2) : \text{deV}^{\text{op}} \longrightarrow \text{KHaus},
\]

which, qua definition, is isomorphic to the de Vries equivalence \( \text{Clust} \) and, thus, produces the hom version of the de Vries duality. The counits

\[
\tilde{\tau}_A = R_C(\omega_A) \circ \tau_A : A \longrightarrow R_C(\text{deV}(A, 2))
\]
of the dual adjunction may (as defined above) be computed as

\[
\tilde{\tau}_A(a) = \omega_A^{-1}((c \in \text{Clust}(A) \mid a \in c)) = \{ \varphi \in \text{deV}(A, 2) \mid \varphi(a^*) = 0 \},
\]
for all \( a \in A \). Their complements in \( \text{deV}(A, 2) \) are given by \( \text{deV}(A, 2) \setminus \tilde{\tau}_A(a) = \{ \varphi \in \text{deV}(A, 2) \mid \varphi(a^*) = 1 \} \), and their interiors by

\[
\text{int}_{\text{deV}(A, 2)}(\tilde{\tau}_A(a)) = \{ \varphi \in \text{deV}(A, 2) \mid \varphi(a) = 1 \},
\]
for all \( a \in A \); they form a base for open sets of \( \text{deV}(A, 2) \). In the hom-version, the adjunction units, for every compact Hausdorff space \( X \) (see Facts 5), are (by necessity) given by

\[
\tilde{\sigma}_X = [ X \xrightarrow{\sigma_X} \text{Clust}(R_C(X)) \xrightarrow{\omega_{R_C(X)}^{-1}} \text{deV}(R_C(X), 2) ],
\]
so that, for all \( x \in X \) and \( F \in R_C(X) \), one has

\[
\tilde{\sigma}_X(x)(F) = 1 \iff x \in \text{int}_X(F).
\]

### 7 Extending de Vries’ Duality to Tychonoff Spaces

Having presented the de Vries duality in the form

\[
\mathcal{A}^{\text{op}} = \text{deV}^{\text{op}} \xrightarrow{T = \text{deV}(-, 2)} \text{KHaus} = \mathcal{X}
\]
(see Sect. 6), we now wish to extend it to the category \( \mathcal{J} = \text{Tych} \) of all Tychonoff spaces and continuous maps, through an application of Corollary 18. As \( \text{Tych} \) is reflective in \( \text{KHaus} \), to this end one considers the class \( \mathcal{J} \) of the Stone–\v{C}ech-compactifications of Tychonoff spaces, that is: \( \mathcal{J} \) contains precisely all continuous maps \( j : Y \to X \) with \( Y \in |\text{Tych}| \) and \( X \in |\text{KHaus}| \), satisfying the universal property that every continuous map \( f : Y \to Z \) with \( Z \in |\text{KHaus}| \) factors through \( j \), by a uniquely determined continuous map \( X \to Z \); equivalently, the maps \( j : Y \to X \) in \( \mathcal{J} \) are dense embeddings, with \( Y \in |\text{Tych}| \) and \( X \in |\text{KHaus}| \) such that \( j(Y) \) is \( C^* \)-embedded in \( X \).

According to Construction 17, we now form the category \( \mathcal{D}(\mathcal{A}, \mathcal{J}, \mathcal{X}) \) whose

- objects are pairs \( (A, j) \) with a de Vries algebra \( A \) and \( j : Y \to \text{deV}(A, 2) \) in \( \mathcal{J} \);
- morphisms \((\alpha, f) : (A, j) \rightarrow (A', j')\) are given by de Vries morphisms \(\alpha : A \rightarrow A'\) and continuous maps \(f : Y' \rightarrow Y\) in \(\text{Tych}\) with \(T \alpha \circ j' = j \circ f\):

\[
\begin{array}{ccc}
TA & \xleftarrow{T \alpha} & TA' \\
\downarrow{j} & & \downarrow{j'} \\
Y & \xleftarrow{f} & Y'
\end{array}
\]

- composition in \(D(A, B, X)\) proceeds by the horizontal pasting of diagrams;
- the identity morphism of a \(D(A, B, X)\)-object \((A, j)\) is \((1_A, 1_{\text{dom}(j)})\).

The category \(\text{dev}\) is fully and coreflectively embedded into \(\mathcal{B} = D(A, B, X)\) via

\[I : \mathcal{A} \rightarrow \mathcal{B}, \ (\alpha : A \rightarrow A') \longmapsto (I \alpha = (\alpha, T \alpha) : (A, 1_{T A}) \rightarrow (A', 1_{T A'})).\]

From Corollary 18 we obtain a first extension statement for the de Vries duality:

**Proposition 36** The de Vries duality extends to the duality

\[D(A, B, X)^{\text{op}} \xleftarrow{\tilde{T}} \mathcal{B} \xrightarrow{\tilde{S}} \text{Tych},\]

with object assignments \(\tilde{T} : (A, j : Y \rightarrow TA) \longmapsto Y\) and (see Theorem 34 for notation)

\[\tilde{S} : Y \longmapsto (RC(\beta Y), [\ Y \xrightarrow{\beta Y} Y \xrightarrow{\tilde{\delta}_{\beta Y}} \text{dev}(RC(\beta Y), 2)]).\]

**Remark 37** The direct reference to the the Stone–Čech compactification in the definition of the functor \(\tilde{S}\) may be avoided, as follows. Let us first note that, while the Boolean algebras \(RC(Y)\) and \(RC(\beta Y)\) are isomorphic for any Tychonoff space \(Y\) (see Facts 8), generally the contact algebras \((RC(Y), \sim_Y)\) and \((RC(\beta Y), \sim_{\beta Y})\) are not as such, unless \(Y\) is a normal Hausdorff space (thanks to Urysohn’s Lemma). However, under the relation \(\sim_{\beta Y}\) defined for all \(F, G \in RC(Y)\) by

\[F \sim_{\beta Y} G \iff F\text{ and }G\text{ are not completely separated}
\]

\[\iff \exists f : Y \rightarrow [0, 1] \text{ continuous with } f(F) = 0 \text{ and } f(G) = 1,
\]

the contact algebras \((RC(Y), \sim_{\beta Y})\) and \((RC(\beta Y), \sim_{\beta Y})\) become isomorphic. Consequently, the object assignment of the functor \(\tilde{S}\) is equivalently described by

\[\tilde{S} : Y \longmapsto ((RC(Y), \sim_{\beta Y}), [\ Y \xrightarrow{\tilde{\delta}_{\beta Y}} \text{dev}((RC(Y), \sim_{\beta Y}), 2)]).\]

We now embark on describing the objects \((A, j : Y \rightarrow \text{dev}(A, 2))\) of \(D(A, B, X)\) in a more algebraic fashion, without direct reference to the categories \(X = \text{KHaus}\) or \(\text{Tych}\). First of all, replacing \(Y\) by the homeomorphic subspace \(j(Y)\) of \(\text{dev}(A, 2)\) one observes that \((A, j)\) is isomorphic to \((A, j(Y) \hookrightarrow \text{dev}(A, 2))\) in \(D(A, B, X)\). Hence, without loss of generality, we may assume \(Y\) to be a subspace of the compact Hausdorff space \(\text{dev}(A, 2)\). The task is then to express, in algebraic terms, the status of \(\text{dev}(A, 2)\) as a largest compactification of \(Y\), so that (1) \(Y\) is dense in \(\text{dev}(A, 2)\), and (2) any dense embedding of \(Y\) into a compact

\[\begin{split}
{\sigma_{\beta_{\text{Y}}}}
\end{split}\]
A Hausdorff space may be extended to a continuous map on \textbf{deV}(A, 2). To this end we use for all \(a \in A\) the abbreviation
\[
Y_a = Y \cap \tilde{\tau}(a) = \{\varphi \in Y \mid \varphi(a^*) = 0\}
\]
and employ the following terminology, which will be justified by Proposition 39.

**Definition 38** (1) For a de Vries algebra \(A\) and a subset \(Y \subseteq \text{deV}(A, 2)\) we call \((A, Y)\) a de Vries pair if for every element \(a > 0\) in \(A\) one has some \(\varphi \in Y\) with \(\varphi(a) = 1\).

(2) A de Vries pair \((A, Y)\) is called universal if for any other de Vries pair \((A', Y')\) and any bijection \(h : Y \to Y'\) with \(h^{-1}(Y'_a) \mid a' \in A' = \{Y_a \mid a \in A\}\) there is a de Vries morphism \(\alpha : A' \to A\) with \(h(\varphi) = \varphi \circ \alpha\) for all \(\varphi \in Y\).

**Proposition 39** (1) For a de Vries algebra \(A\) and a subset \(Y\) of \(\text{deV}(A, 2)\), the pair \((A, Y)\) is a de Vries pair if, and only if, \(Y\) is a dense subset of the space \(\text{deV}(A, 2)\).

(2) A de Vries pair \((A, Y)\) is universal if, and only if, the space \(\text{deV}(A, 2)\) serves as a Stone–Čech compactification of its subspace \(Y\).

**Proof** (1) Since the sets \(\{\varphi \in \text{deV}(A, 2) \mid \varphi(a) = 1\} (a \in A)\) form a base for open sets in \(\text{deV}(A, 2)\), the statement is obvious.

(2) Assuming first the universality of the de Vries pair \((A, Y)\) and considering any compactification \(c : Y \to X\), one has a homeomorphism \(g : X \to \text{deV}(A', 2)\) with a de Vries algebra \(A'\). The map \(g \circ c\) restricts to a homeomorphism \(h : Y \to Y' := g(c(Y))\), which implies \(h^{-1}(F') \mid F' \in \text{RC}(Y')) = \text{RC}(Y)\). With Facts 8, this equality reads as \(h^{-1}(Y'_{a'}) | a' \in A' = \{Y_a | a \in A\}\). Therefore, the defining property of a universal de Vries pair gives a de Vries morphism \(\alpha : A' \to A\) which makes \(h\) a restriction of the continuous map \(\text{deV}(\alpha, 2) : \text{deV}(A, 2) \to \text{deV}(A', 2)\), \(\varphi \mapsto \varphi \circ \alpha\). Equivalently, the map \(g^{-1} \circ \text{deV}(\alpha, 2) : \text{deV}(A, 2) \to X\) is a continuous extension of \(c\), as desired.

Conversely, let \(\text{deV}(A, 2)\) be a Stone–Čech compactification of its subspace \(Y\) and consider a de Vries pair \((A', Y')\), along with a bijection \(h : Y \to Y'\) satisfying the equality \(h^{-1}(Y'_{a'}) | a' \in A' = \{Y_a | a \in A\}\). This equality says precisely that \(h\) and \(h^{-1}\) map the chosen bases of closed sets in \(Y\) and \(Y'\) given by the regular closed sets onto each other, so that \(h\) must be a homeomorphism. Hence, enlarging its codomain to \(\text{deV}(A', 2)\), we obtain another compactification of \(Y\) which, by hypothesis, must factor through a (uniquely determined) continuous map \(f : \text{deV}(A, 2) \to \text{deV}(A', 2)\). But by the de Vries duality, we may write \(f\) as \(f = \text{deV}(\alpha, 2)\) for a (uniquely determined) de Vries morphism \(\alpha : A' \to A\), so that \(h\) becomes a restriction of \(\text{deV}(\alpha, 2)\), as needed. \(\square\)

**Remark 40** The proof above shows that, in the defining property of a universal de Vries pair \((A, Y)\), the de Vries morphism \(\alpha\) is uniquely determined by the bijection \(h\).

Definition 38 gives us the objects of the category \(\text{UdeV}\) of universal de Vries pairs whose morphisms \(\alpha : (A, Y) \to (A', Y')\) are de Vries morphisms \(\alpha : A \to A'\) with \(\varphi' \circ \alpha \in Y\) for all \(\varphi' \in Y'\); they get composed as in \(\text{deV}\). With
\[
f_\alpha : Y' \to Y
\]
denoting the restriction of the map \( \text{deV}^\alpha_2 \), by Proposition 39 one then has the \( D(A, \mathcal{J}, \mathcal{X}) \) morphism \( (\alpha, f_\alpha) : (A, j_Y) \to (A', j_{Y'}) \), with inclusion maps \( j_Y, j_{Y'} \), as visualized by

\[
\begin{array}{c}
\text{deV}(A, 2) \\
\downarrow j_Y \\
Y \\
\downarrow f_\alpha \\
\text{deV}(A', 2) \\
\uparrow j_{Y'} \\
\end{array}
\]

This defines the full embedding

\[
\Psi : \text{UdeV} \longrightarrow D(A, \mathcal{J}, \mathcal{X}), \quad (A, Y) \longmapsto (A, j_Y).
\]

Proposition 39 allows us to define also the functor

\[
\Phi : D(A, \mathcal{J}, \mathcal{X}) \longrightarrow \text{UdeV}, \quad (A, j : Y \to \text{deV}(A, 2)) \longmapsto (A, j(Y)),
\]

with its obvious definition on morphisms. Clearly then, \( \Phi \circ \Psi = \text{Id}_{\text{deV}} \) and \( \Psi \circ \Phi \cong \text{Id}_{D(A, \mathcal{J}, \mathcal{X})} \). Together with Proposition 36, this proves the following duality theorem for the category of Tychonoff spaces.

**Theorem 41** The category \( \text{UdeV} \) is an equivalent retract of the category \( D(A, \mathcal{J}, \mathcal{X}) \) and therefore dually equivalent to the category \( \text{Tych} \).

The composite dual equivalence

\[
\begin{array}{c}
\text{UdeV}^{\text{op}} \\
\downarrow \Psi \\
D(A, \mathcal{J}, \mathcal{X})^{\text{op}} \\
\downarrow \Phi \\
\text{Tych} \\
\end{array}
\]

and its natural isomorphisms \( \tilde{\tau}, \tilde{\sigma} \) arising from \( \tilde{\tau}, \tilde{\sigma} \) are easily described by

- \( \tilde{\tau} \circ \Psi : [\alpha : (A, Y) \to (A', Y')] \longmapsto [f_\alpha : Y' \to Y] \);
- \( \Phi \circ \tilde{\sigma} : [f : Y' \to Y] \longmapsto [RC(\beta f) : (RC(\beta Y), \tilde{\sigma}_{\beta Y}(Y)) \to (RC(\beta Y'), \tilde{\sigma}_{\beta Y'}(Y'))] \);
- \( \tilde{\tau}_{A,Y} : (A, Y) \longrightarrow \Phi \tilde{\tau} \Psi (A, Y) = (RC(\text{deV}(A, 2)), \{ \varphi \circ \tilde{\tau}_A^{-1} \mid \varphi \in Y \}) \), where, for simplicity, one assumes \( \beta Y = \text{deV}(A, 2) \);
- \( \tilde{\sigma}_Y : Y \longrightarrow \tilde{\Psi} \Phi \tilde{\sigma}(Y) = \tilde{\sigma}_{\beta Y}(Y) \).

In order to obtain the Bezhanishvili-Morandi-Olberding Duality Theorem [7], we now undertake a further “algebraization” of the objects \( (A, Y) \) of \( \text{UdeV} \). Following the key idea from [7], we encode the subset inclusion \( Y \hookrightarrow \text{deV}(A, 2) \) of a de Vries pair \( (A, Y) \) by a de Vries morphism \( A \to P(Y) \), with the complete atomic Boolean algebra \( P(Y) \) regarded as a discrete de Vries algebra (so that \( \llap{\leq} = \llap{\subseteq} ; \) see Facts 4). As \( Y \) is recovered from \( PY \) as its set of atoms, no loss of information occurs in the process, as we make precise in what follows.

**Definition 42** Given a de Vries pair \( (A, Y) \), maintaining the notation of Theorem 34, we call the composite map

\[
\gamma(A, Y) = [ A \xrightarrow{\tilde{T}_A} RC(\text{deV}(A, 2)) \xrightarrow{\text{int}} RO(\text{deV}(A, 2)) \xrightarrow{Y \cap (-)} P(Y) ]
\]

the **Booleanization** of \( (A, Y) \). By definition, \( \gamma(A, Y) \) is for every \( a \in A \) described by

\[
\gamma(A, Y)(a) = Y \cap \text{int}(\tilde{T}(a)) = Y \cap \{ \varphi \in \text{deV}(A, 2) \mid \varphi(a) = 1 \} = \{ \varphi \in Y \mid \varphi(a) = 1 \}.
\]
A proposition similar to the next one was established in [7], but in another setting.

**Proposition 43** Let \((A, Y)\) be a de Vries pair.

1. With the complete atomic Boolean algebra \(P(Y)\) regarded as a discrete de Vries algebra, the Booleanization \(\gamma_{(A, Y)} : A \to P(Y)\) of \((A, Y)\) is an injective de Vries morphism, and every atom in \(P(Y)\) may be presented as a meet of elements in the image of \(\gamma_{(A, Y)}\).

2. For any other de Vries pair \((A', Y')\) and any bijection \(h : Y \to Y'\) satisfying (in the notation of Definition 38) \([h^{-1}(Y'_a) \mid a' \in A'] = \{Y_a \mid a \in A\}\),

(a) the de Vries morphism \(P(h) \circ \gamma_{(A', Y')}\) has the same image in \(P(Y)\) as \(\gamma_{(A, Y)}\), and

(b) if the de Vries pair \((A, Y)\) is universal, then there exists a unique de Vries morphism \(\alpha : A' \to A\) rendering the diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{\alpha} & A \\
\downarrow{\gamma_{(A', Y')}} & & \downarrow{\gamma_{(A, Y)}} \\
P(Y') & \xrightarrow{P(h)} & P(Y)
\end{array}
\]

commutative in the category \(\mathbf{deV}\), that is: \(P(h) \circ \gamma_{(A', Y')} = \gamma_{(A, Y)} \circ \alpha\).

**Proof** (1) To see that \(\gamma := \gamma_{(A, Y)}\) is injective, with \(X := \mathbf{deV}(A, 2)\) it suffices to show that the map \(Y \cap (-) : RO(X) \to P(Y)\) is injective. But this map is just a codomain enlargement of \(Y \cap (-) : RO(X) \to RO(Y)\), which is the composite of the Boolean isomorphisms \(c_X : RO(X) \to RC(X), r = Y \cap (-) : RC(X) \to RC(Y)\) (see Facts 8), and \(\text{int}_Y : RC(Y) \to RO(Y)\).

Furthermore, since the sets \(\text{int}_1(t_2(a)) \ (a \in A)\) form a base for open sets in \(X\), the sets \(\gamma(a) \ (a \in A)\) form a base for open sets in \(Y\). Every singleton set \([\varphi]\) in this \(T_1\)-space, i.e. every atom of \(P(Y)\), is an intersection of its basic open neighbourhoods in \(Y\), and these lie all in the image of \(\gamma\).

We verify that \(\gamma\) is a de Vries morphism. Indeed, as for every de Vries morphism \(\varphi : A \to 2\) one has \(\varphi(0) = 0, \) trivially \(\gamma(0) = 0\) follows, i.e., \(\gamma\) satisfies (V1). Similarly, one derives the satisfaction of (V2-4) for \(\gamma\) from the corresponding properties of the de Vries morphisms \(\varphi\), as follows.

(V2): Since \(1 = \varphi(a \land b) = \varphi(a) \land \varphi(b)\) means equivalently \(\varphi(a) = 1 = \varphi(b)\) for all \(\varphi\), the equality \(\gamma(a \land b) = \gamma(a) \land \gamma(b)\) follows.

(V3): Since \(a^* \leq b \) in \(A\) implies \(\varphi(a)^* \leq \varphi(b)\) for all \(\varphi\), so that when \(\varphi(a) = 0\) one must have \(\varphi(a)^* = 1\) and then also \(\varphi(b) = 1\), we conclude \(\gamma(a)^* = \{\varphi \in Y \mid \varphi(a) = 0\} \leq \gamma(b)\).

(V4): Since \(\varphi(a) = \bigvee\{\varphi(b) \mid b \leq a\}\), for all \(\varphi\) one has \(\varphi(a) = 1\) precisely when \(\varphi(b) = 1\) for some \(b \leq a\) in \(A\). Hence, \(\gamma(a) = \bigcup\{\gamma(b) \mid b \leq a\}\).

(2)(a) Let us note first that \(P(h) : P(Y') \to P(Y)\) is a (sup-preserving) Boolean isomorphism and, hence, a de Vries morphism satisfying \(P(h) \circ \gamma' = P(h) \circ \gamma\), where \(\gamma' = \gamma_{(A', Y')}\). Since \(\gamma(A) = RO(Y)\) (see the proof of (1)) and, likewise, \(\gamma'(A') = RO(Y')\), and since the hypotheses make \(h : Y \to Y'\) a homeomorphism of the respective subspaces of \(deV(A, 2)\) and \(deV(A', 2)\), we conclude

\[
(P(h) \circ \gamma')(A') = P(h)(RO(Y')) = RO(Y) = \gamma(A).
\]

(b) The universality of the de Vries pair \((A, Y)\) gives us a uniquely determined de Vries morphism \(\alpha : A' \to A\) with \(h(\varphi) = \varphi \circ \alpha\) for all \(\varphi \in Y\). We claim that this last equality implies the commutativity of the diagram above, i.e., that \(h^{-1}(\gamma'(a')) = (\gamma \circ \alpha)(a')\) holds for all \(a' \in A'\), and that this implication is in fact reversible.
Indeed, for the necessity of the condition we have that, for all \(a' \in A'\) and \(\varphi \in Y\),
\[
\varphi \in h^{-1}(\gamma'(a')) \iff h(\varphi) \in \gamma(a') \\
\iff h(\varphi)(a') = 1 \\
\iff (\varphi \circ \alpha)(a') = 1 \quad \text{(by hypothesis)} \\
\iff \exists b' \ll a' \in A' : \varphi(\alpha(b')) = 1 \\
\iff \exists b' \ll a' \in A' : \varphi \in \gamma(\alpha(b')) \\
\iff \varphi \in (\gamma \circ \alpha)(a').
\]
Conversely, assuming the commutativity of the diagram and reshuffling the above equivalences, one obtains for all \(\varphi \in Y\) and \(a' \in A'\) the equivalence
\[
h(\varphi)(a') = 1 \iff (\varphi \circ \alpha)(a') = 1,
\]
which means \(h(\varphi) = \varphi \circ \alpha\) for all \(\varphi \in Y\). In conclusion, while the necessity of the condition shows the existence of the desired de Vries morphism \(\alpha\), the sufficiency implies the uniqueness of \(\alpha\), by Remark 40.

\[\square\]

**Remark 44** The de Vries duality associates the de Vries algebra \(2\) with the singleton space \(1 \cong \text{deV}(2, 2)\) which, in categorical terms, is a regular generator (also called separator) in the category \(\text{KHaus}\) (since every compact Hausdorff space is a quotient of the Stone–Čech compactification of its underlying set, considered as a discrete space). Consequently, \(2\) is a regular cogenerator (or coseparator) in the category \(\text{deV}\). This makes every de Vries algebra a subalgebra of some power of \(2\). The injectivity of the map \(\gamma(A, Y)\) of Proposition 43 may be seen as a consequence of this important categorical role of the de Vries algebra \(2\).

Moreover, \(2\) is regular injective in \(\text{deV}\) since, trivially, the singleton space \(1\) is regular projective in \(\text{KHaus}\), i.e., projective with respect to regular epimorphisms in \(\text{KHaus}\), which are the surjective continuous maps.

Proposition 43 motivates the following definition.

**Definition 45** (1) An injective de Vries morphism \(\gamma : A \to B\) into a complete atomic Boolean algebra \(B\) (regarded as a discrete de Vries algebra) is called a **Boolean de Vries extension** (or just a **de Vries extension**, [7]) of the de Vries algebra \(A\) if every atom in \(B\) is a meet of elements in the image of \(\gamma\).

(2) A Boolean de Vries extension \(\gamma : A \to B\) is called **universal** (or **maximal**, [7]), if every Boolean de Vries extension \(\gamma' : A' \to B\) into the same Boolean algebra \(B\) with \(\gamma(A) = \gamma'(A')\) factors as \(\gamma' = \gamma \circ \alpha\), for some de Vries morphism \(\alpha : A' \to A\).

We should clarify immediately how to regard a Boolean de Vries extension as a de Vries pair. Considering the fact that the complete Boolean homomorphisms \(\xi : B \to 2\) of the complete atomic Boolean algebra \(B\) correspond bijectively to the atoms \(x\) in \(B\), so that the map
\[
\gamma_B : \text{At}(B) \to \text{CABA}(B, 2), \quad x \mapsto \xi_x,
\]
where \((\xi_x(b) = 1 \iff x \leq b)\) for all \(b \in B\), is a natural bijection (see Facts 3), we define:

**Definition 46** For a Boolean de Vries extension \(\gamma : A \to B\), we consider the hom map
\[
\text{deV}(\gamma, 2) : \text{deV}(B, 2) \to \text{deV}(A, 2), \quad \xi \mapsto \xi \circ \gamma,
\]
and then the image $Y'$ of $\text{CABA}(B, 2) \subseteq \text{dev}(B, 2)$ under this map:

$$Y' = \{\xi \circ \gamma \mid \xi \in \text{CABA}(B, 2)\} = \{\xi \circ \gamma \mid \xi \in \text{CABA}(B, 2)\}.$$ 

Since, for every $a > 0$ in $A$, the injectivity of $\gamma$ guarantees $\gamma(a) > 0$ and, thus, $\gamma(a) > x$ for some $x \in At(B)$, we obtain $(\xi_x \circ \gamma)(a) = \xi_x(\gamma(a)) = 1$ with $\xi_x \circ \gamma \in Y'$. Hence, $(A, Y')$ is a de Vries pair, and we call it induced by $\gamma$ and $Y'$ the trace of $\gamma$.

With $\gamma_{at}$ denoting the (“atomic”) restriction of $\text{dev}(\gamma, 2)$ we have the trivially commuting diagram

$$\begin{array}{ccc}
\text{dev}(A, 2) & \xrightarrow{\text{dev}(\gamma, 2)} & \text{dev}(B, 2) \\
\downarrow & & \downarrow \\
Y' & \xleftarrow{\gamma_{at}} & \text{CABA}(B, 2).
\end{array}$$ 

We regard $Y'$ as a subspace of the compact Hausdorff space $\text{dev}(A, 2)$ and, likewise, the set $\text{CABA}(B, 2)$ as a subspace of the Stone space $\text{dev}(B, 2) \cong \text{Ult}(B)$, and first prove:

**Proposition 47** (1) For every Boolean de Vries extension $\gamma: A \rightarrow B$, the map

$$\gamma_{at}: \text{CABA}(B, 2) \rightarrow Y', \quad \xi \mapsto \xi \circ \gamma,$$

is a continuous bijection.

(2) The Booleanization $\gamma_{(A, Y)}$ of a universal de Vries pair $(A, Y)$ is a universal Boolean de Vries extension.

**Proof** (1) As $\gamma_{at}$ is a surjective subspace restriction of the continuous map $\text{dev}(\gamma, 2)$, it suffices to prove that $\gamma_{at}$ is injective. Every atom $x$ in $B$ may be written as $x = \bigwedge_{i \in I} \gamma(a_i)$. Hence, with the complete Boolean homomorphism $\xi_x: B \rightarrow 2$ representing $x$, one has $1 = \xi_x(x) = \bigwedge_{i \in I} \xi_x(\gamma(a_i))$. Assuming $\xi_x \circ \gamma = \xi_z \circ \gamma$ with $\xi_z$ representing $z \in At(B)$, we then obtain $\xi_z(x) = \bigwedge_{i \in I} \xi_z(\gamma(a_i)) = \bigwedge_{i \in I} \xi_x(\gamma(a_i)) = 1$, or $z \leq x$. Hence $z = x$ and thus $\xi_x = \xi_z$.

(2) By Proposition 43(1), $\gamma = \gamma_{(A, Y)}: A \rightarrow P(Y) = B$ is a Boolean de Vries extension. To confirm its universality, let $\gamma': A' \rightarrow B$ be an injective de Vries morphism with $\gamma'(A') = \gamma(A)$. With (1) we have that the composite map

$$h := (Y \xrightarrow{\gamma'_{(A, Y)}} \text{CABA}(B, 2) \xrightarrow{\gamma_{at}} Y' =: Y'), \quad \varphi \mapsto \xi_{\varphi} \circ \gamma',$$

is a bijection. Moreover, the Boolean isomorphism $P(h)$ makes the Boolean de Vries extension $\gamma': A' \rightarrow B$ factor through the Booleanization of the de Vries pair $(A', Y')$, induced by $\gamma'$, as in

$$\begin{array}{ccc}
A' & \xrightarrow{\gamma'} & P(Y) \\
\downarrow_{\gamma_{(A', Y')}} & & \downarrow_{P(h)} \\
\text{CABA}(B, 2) & \xrightarrow{\gamma_{at}} & Y'.
\end{array}$$
Indeed, for all $a' \in A'$ and $\varphi \in Y$ one has

$$\varphi \in (P(h) \circ \gamma_{(A', Y')})(a') \iff h(\varphi) \in \gamma_{(A', Y')}(a')$$

$$\iff h(\varphi)(a') = 1$$

$$\iff \xi_\varphi(\gamma'(a')) = 1$$

$$\iff \varphi \in \gamma'(a').$$

Consequently,

$$P(h)(RO(Y')) = P(h)(\gamma_{(A', Y')}(A')) = \gamma'(A') = \gamma(A) = RO(Y);$$

equivalently, $P(h)(RC(Y')) = RC(Y)$, or, in the notation of Proposition 43(2),

$$\{h^{-1}(Y'_a) \mid a' \in A'\} = \{Y_a \mid a \in A\},$$

which then gives the existence of a (unique) de Vries morphism $\alpha : A' \to A$ with $\gamma \circ \alpha = \gamma'$.

The universal Boolean de Vries extensions are the objects of the category

$$\text{UBdeV}$$

whose morphisms $(\alpha, \delta) : \gamma \to \gamma'$ are given by commutative squares

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow{\gamma} & & \downarrow{\gamma'} \\
B & \xrightarrow{\delta} & B'
\end{array}$$

in $\text{deV}$, with a de Vries morphism $\alpha$ and a complete Boolean homomorphism $\delta$, so that $\delta \circ \gamma = \gamma' \circ \alpha$; note that $\delta \circ \gamma$ is simply the map composite $\delta \circ \gamma$ since $\delta$ preserves suprema. The composition in $\text{UBdeV}$ proceeds as in the arrow category of $\text{deV}$, i.e., by horizontal pasting of diagrams, so that $\text{UBdeV}$ is in fact a full subcategory of $\text{deV}^2$.

**Proposition 48** (1) Assigning to every universal de Vries pair $(A, Y)$ its Booleanization $\gamma_{(A, Y)} : A \to P(Y)$ (as in Definition 42) describes the object map of a functor

$$\Gamma : \text{UdeV} \longrightarrow \text{UBdeV}.$$ (2) Assigning to every universal Boolean de Vries extension $\gamma : A \to B$ its induced de Vries pair $(A, Y')$ (as in Definition 46) describes the object map of a functor

$$\Delta : \text{UBdeV} \longrightarrow \text{UdeV}.$$  

**Proof** (1) By Proposition 47(1), $\Gamma$ is well-defined on objects. Further, for a morphism $\alpha : (A, Y) \to (A', Y')$ in $\text{UdeV}$, one lets $\Gamma(\alpha) = (\alpha, P(f_\alpha)) : \gamma_{(A, Y)} \to \gamma_{(A', Y')}$ and must confirm the commutativity of the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow{\gamma_{(A, Y)}} & & \downarrow{\gamma_{(A', Y')}} \\
P(Y) & \xrightarrow{P(f_\alpha)} & P(Y')
\end{array}.$$
But this may be be seen similarly as at the end of the proof of Proposition 43(2). Indeed, for all \(a \in A\) and \(\varphi' \in \gamma\)' one has
\[
\varphi' \in P(f_a)(\gamma(A,Y)(a)) \iff f_a(\varphi') \in \gamma(A,Y)(a) \\
\iff (\varphi' \circ \alpha)(a) = 1 \\
\iff \exists b \ll a : \varphi'(\alpha(b)) = 1 \\
\iff \varphi' \in \bigcup\{\gamma(A',Y')(\alpha(b)) \mid b \ll a\} \\
\iff \varphi' \in (\gamma(A',Y') \circ \alpha)(a) .
\]

The functoriality of \(\Gamma\) is now trivial.

We must verify that the de Vries pair \((A, Y')\) is universal when the Boolean de Vries extension \(\gamma : A \rightarrow B\) is universal. To this end, using the notation of Facts 3 and Definition 42 and Definition 46, one easily confirms that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B \\
\gamma(A,Y) \downarrow & & \downarrow \bar{\delta}_B \\
P(Y) & \xrightarrow{P(\gamma_a)} & P(CABA(B, 2))
\end{array}
\]
commutes. Hence, with the Boolean isomorphism \(\nu_B := \bar{\delta}_B^{-1} \circ P(\gamma_a)\) and with \(Y := Y'\), the map \(\nu_B \circ \gamma(A,Y)\) has the same image as \(\gamma\) in \(B\). Now, given any de Vries pair \((A', Y')\) and a bijection \(h : Y \rightarrow Y'\) as in Definition 38(2), from Proposition 43(2)(a) we know that \(P(h) \circ \gamma(A',Y')\) has the same image as \(\gamma(A,Y)\) in \(P(Y)\). Consequently, the Boolean de Vries extension \(\nu_B \circ P(h) \circ \gamma(A',Y')\) has the same image in \(B\) as \(\gamma\) and must therefore factor through \(\gamma\) in \(\text{dev}\), as shown in the diagram
\[
\begin{array}{ccc}
A' & \xrightarrow{\alpha} & A \\
\gamma(A',Y) \downarrow & & \downarrow \gamma \\
P(Y') & \xrightarrow{P(h)} & P(Y) \\
\end{array}
\]
As at the end of the proof of Proposition 43(2)(b), from \(P(h) \circ \gamma(A',Y') = \gamma(A,Y) \circ \alpha\) one concludes that \(h(\varphi) = \varphi \circ \alpha\) holds for every \(\varphi \in Y\). This confirms the universality of the de Vries pair \((A, Y')\).

For a morphism \((\alpha, \delta) : (\gamma : A \rightarrow B) \rightarrow (\gamma' : A' \rightarrow B')\) in \(\text{UBdev}\), since
\[
(\xi' \circ \gamma') \circ \alpha = \xi' \circ (\gamma' \circ \alpha) = (\xi' \circ \delta \circ \gamma) = (\xi' \circ \delta) \circ \gamma
\]
for all \(\xi' \in \text{CABA}(B', 2)\), the hom map \(\text{dev}(\alpha, 2) : \text{dev}(A', 2) \rightarrow \text{dev}(A, 2)\) maps \(Y'\) into \(Y\). Hence, \(\Delta(\alpha, \delta) := \alpha : (A, Y') \rightarrow (A', Y')\) is a morphism in \(\text{Udev}\), and the functoriality of \(\Delta\) is trivial. \(\square\)

We are now ready to prove the main theorem of this section.

**Theorem 49** The functors \(\Gamma\) and \(\Delta\) of Proposition 48 satisfy
\[
\Delta \circ \Gamma = \text{Id}_{\text{Udev}} \quad \text{and} \quad \Gamma \circ \Delta \cong \text{Id}_{\text{UBdev}}
\]
and therefore exhibit the category \(\text{Udev}\) as an equivalent retract of the category \(\text{UBdev}\).  

\(\square\) Springer
Proof For every universal de Vries pair \((A, Y)\), one has

\[
\Delta(\Gamma(A, Y)) = \Delta(\gamma(A,Y)) = (A, Y^{\gamma(A,Y)}),
\]

where

\[
Y^{\gamma(A,Y)} = \{ \xi \circ \gamma(A,Y) \mid \xi \in \text{CABA}(P(Y), 2) \} \leftrightarrow \text{dev}(A, 2).
\]

Hence, it suffices to show \(Y^{\gamma(A,Y)} = Y\) to be able to deduce \(\Delta(\Gamma(A, Y)) = (A, Y)\). But under the natural bijection \(\chi_Y : Y \rightarrow \text{CABA}(P(Y), 2)\) of Facts 3, we can write \(\xi \in \text{CABA}(P(Y), 2)\) equivalently as \(\chi_Y^\psi\) with \(\varphi \in Y\), and for all \(a \in A\) one has

\[
(\chi_Y^\psi \circ \gamma(A,Y))(a) = 1 \iff \chi_Y^\psi(\{\psi \in Y \mid \psi(a) = 1\}) = 1 \iff \varphi(a) = 1,
\]

so that \(\chi_Y^\psi \circ \gamma(A,Y) = \varphi\). Hence, \(Y^{\gamma(A,Y)} = Y\) follows. That \(\Delta \circ \Gamma\) maps morphisms identically as well is now obvious.

For every universal Boolean de Vries extension \(\gamma : A \rightarrow B\), we compute

\[
\Gamma(\Delta(\gamma)) = \Gamma(\gamma, Y^{\gamma}) = \{ \xi \circ \gamma \mid \xi \in \text{CABA}(B, 2) \} = (\gamma, Y^{\gamma}) : A \rightarrow P(Y^{\gamma}),
\]

with \(\gamma, Y^{\gamma}(a) = \{ \xi \circ \gamma \mid \xi \in \text{CABA}(B, 2), \xi(\gamma(a)) = 1 \}\), for all \(a \in A\). But with the Tarski isomorphism \(\theta_B\) (see Facts 3) and the bijection \(\gamma_{\text{at}}\) of Proposition 47(1) we have the natural composite Boolean isomorphism

\[
B \xrightarrow{\theta_B} P(\text{CABA}(B, 2)) \xrightarrow{P(y_{\text{at}}^{-1})} P(Y^{\gamma}), \quad b \mapsto \{ \xi : \xi(b) = 1 \} \mapsto \{ \xi \circ \gamma \mid \xi(b) = 1 \}.
\]

So, for every \(a \in A\), this isomorphism maps \(\gamma(a)\) precisely to the set \(\gamma, Y^{\gamma}(a)\), and we obtain the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow\gamma & & \downarrow\gamma, Y^{\gamma} \\
B & \xrightarrow{P(y_{\text{at}}^{-1}) \circ \theta_B} & P(Y^{\gamma})
\end{array}
\]

which represents an isomorphism \(\gamma \rightarrow \gamma, Y^{\gamma}\) in \(\text{UBdev}\). The confirmation of its naturality with respect to \(\gamma\) involves only a routine check.

\[\square\]

Corollary 50 The Bezhanishvili-Morandi-Olberding duality [7] may be obtained as the composite of the equivalences of Proposition 36 and Theorem 41 and Theorem 49:

\[
\begin{array}{ccc}
\text{UBdev}^{\text{op}} & \xrightarrow{\Delta} & \text{Udev}^{\text{op}} \\
\downarrow\simeq & & \downarrow\simeq \\
\Gamma & \xrightarrow{\psi} & D(A, J, X)^{\text{op}} \\
\downarrow\phi & & \downarrow\tilde{\xi} \\
\text{Tych} & \xrightarrow{\simeq} &
\end{array}
\]

Here the composite dual equivalence \(\tilde{T} \circ \Psi \circ \Delta\) assigns to a universal Boolean de Vries extension \(\gamma : A \rightarrow B\) its trace \(Y^{\gamma} = \{ \xi \circ \gamma \mid \xi \in \text{CABA}(B, 2) \}\) in \(\text{dev}(A, 2)\), and \(\Gamma \circ \Phi \circ S\) first embeds a Tychonoff space \(Y\) into the compact space \(\text{deV}(A, 2)\) with \(A = RC(\beta Y)\) (or, more simply, \(A = RC(Y)\), at the expense of having to use a less simple contact relation for the de Vries structure—see Remark 37) and then sends it to the Booleanization \(\gamma(A,Y) : A \rightarrow P(Y)\) of the universal de Vries pair \((A, Y)\), where we have identified \(Y\) with its image under the embedding into \(\text{deV}(A, 2)\).
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