Rationally extended many-body truncated Calogero-Sutherland model

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Abstract

We construct a rational extension of the truncated Calogero-Sutherland model by Pittman et al. The exact solution of this rationally extended model is obtained analytically and it is shown that while the energy eigenvalues remain unchanged, however the eigenfunctions are completely different and written in terms of exceptional $X_1$ Laguerre orthogonal polynomials. The rational model is further extended to a more general, the $X_m$ case by introducing $m$ dependent interaction term. As expected, in the special case of $m = 0$, the extended model reduces to the conventional model of Pittman et al. In the two appropriate limits, we thereby obtain rational extensions of the celebrated Calogero-Sutherland as well as Jain-Khare models.

1 Introduction

The discovery of the two new orthogonal polynomials namely the exceptional $X_m$-Laguerre and exceptional $X_m$-Jacobi orthogonal polynomials \cite{1,2,3} has inspired the discovery of a number of new exactly solvable (ES) conventional one-body potentials through the rational extension of several conventional ES potentials. In most of these cases while the eigenvalues remain unchanged, the eigenfunctions are in terms of these newly discovered exceptional orthogonal polynomials (EOPs) \cite{4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20}. The next obvious question is how to construct rational extensions of many body problems as well as one body non-central but separable potentials. Kumari et al, \cite{21} took first step in that direction and considered the Calogero-Wolfe type three-body problems and constructed a class of corresponding rationally extended three-body systems and obtained their exact solutions in terms of $X_m$ exceptional Laguerre and $X_m$ exceptional Jacobi polynomials. In another paper \cite{22} they

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constructed the rational extension of a number of non-central but separable one-body potentials and again showed that while the energy-eigenvalue spectrum is unchanged, the eigenfunctions are now in terms of $X_m$ Laguerre and $X_m$ Jacobi polynomials. The obvious next step is to consider the rational extension of the many-body problems. One obvious candidate to consider would be the celebrated Calogero-Sutherland (CSM) \[23, 24\] $N$-body problem on a line with harmonic confinement. Rational extension of $N$-particle Calogero model with harmonic confining term and arbitrary interaction of the form $U(\sqrt{N}\rho)$ ($\rho$ being the radial coordinate), was carried by Basu-Mallick et.al \[25\]. Bound state eigenfunctions for specific angular part solution (hence QES solutions) were explicitly calculated using supersymmetric technique in terms of $X_m$ exceptional Laguerre polynomials. In this context, it is worth recalling that several years ago, Jain and Khare (JK) \[27\] considered a variant of CSM on the full line where there was only nearest and next-to-nearest neighbor interaction through two body and three body interactions and obtained its eigen spectrum. Recently, Pittman et al \[28\] generalized the JK model by considering $N$-body problem on a line with harmonic confinement in which the tunable inverse square as well as the three-body interaction extends over a finite number of neighbors and were able to obtain its eigenspectrum. One of the nice feature of this model is that in the appropriate limits its eigen values and eigenfunctions smoothly goes over to those of JK and CSM. One of the common feature in all the three cases is that a part of the eigenfunction is in terms of the celebrated classical Laguerre polynomials.

The purpose of this note is to consider the rational extension of the truncated Calogero-Sutherland (TCS) model of Pittman et al. \[28\] by introducing new interaction terms over and above the two-body and three-body terms and obtain the exact solutions of this model in terms of $X_1$ exceptional Laguerre polynomials. We further generalize it to the more general $X_m$ Laguerre case by introducing an $m$-dependent polynomial type interaction term. It must be mentioned here that the energy eigenvalue spectrum remains unchanged and is identical to the TCS model. As expected, in the special case of $m = 0$, the model reduces to the usual TCS model \[28\]. In the appropriate limits we thus obtain the rational extension of the celebrated classical Laguerre polynomials.

The plan of the manuscript is as follows: In section 2, we briefly recall the TCS model and briefly discuss its solutions. In section 3, we extend the TCS model by introducing a new rational term and obtain the exact solution in terms of $X_1$ exceptional Laguerre polynomials. The generalization to the $X_m$ Laguerre case is discussed in subsection 3.1. Finally we summarize our results in section 4.

## 2 The conventional TCS model

The $N$-body TCS model \[28\] is characterized by the Hamiltonian

$$
\hat{H} = \sum_{i=1}^{N} \left[ -\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \omega^2 x_i^2 \right] + V_{int},
$$

where

$$
V_{int} = \sum_{i<j}^{N} \frac{\lambda(\lambda - 1)}{|x_i - x_j|^2} + \sum_{i<j<k}^{N} \frac{\lambda^2 r_{ij} r_{jk}}{r_{ij}^2 r_{jk}^2}, \quad \lambda \neq 0,
$$

\[2\]
i.e. the particles are interacting through a pair wise two body potential as well as a three body term. The vector along $x$-axis is $\mathbf{r}_{ij} = (x_i - x_j)\hat{x}$. The above two body interactions are attractive for $0 < \lambda < 1$ and repulsive for $\lambda \geq 1$. It is worth pointing out that in the particular cases of $r = 1$ and $r = N - 1$, this Hamiltonian reduces to those of JK [27] and CSM [23, 24] respectively.

The solution of the above model is obtained in [28] and is given by

$$\Psi(x) = \phi(x)\xi(x); \quad x = (x_1, x_2, ..., x_N)e^{i\mathbb{R}^N},$$

(3)

where

$$\phi(x) = \prod_{i<j}(x_i - x_j)^{\lambda}$$

(4)

while the function $\xi$ satisfies the equation

$$-\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2 \xi}{\partial x_i^2} - \lambda \sum_{i<j} \frac{1}{x_i - x_j} \left( \frac{\partial \xi}{\partial x_i} - \frac{\partial \xi}{\partial x_j} \right) + \left( \frac{1}{2} \sum_{i} \omega^2 x_i^2 - E \right) \xi = 0.$$

(5)

To get exact solutions of the above equation, one assume $\xi$ as

$$\xi = \Phi(\rho)P_s(x); \quad \text{where} \quad \rho^2 = \sum_{i=1}^{N} x_i^2.$$

(6)

Substituting $\xi(x)$ in Eq. (5), one finds that $\Phi$ satisfies the differential equation

$$\Phi''(\rho) + \left( N + 2s - 1 + \lambda r(2N - r - 1) \right) \frac{1}{\rho} \Phi'(\rho) + 2\left( E - \frac{1}{2} \omega^2 \rho^2 \right) \Phi(\rho) = 0,$$

(7)

while the function $P_s(x)$ behaves as a homogeneous polynomial of degree $s(= 0, 1, 2, ...)$ and satisfies a generalized Laplace equation

$$\left[ \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\lambda \sum_{i<j} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \right] P_s(x) = 0.$$

(8)

The solutions of this Laplace equation are discussed in detail in Refs. [26, 27, 28] and the Eq. (7) is the well known equivalent radial equation for the oscillator potential in arbitrary dimensions. The solution of this radial equation is in terms of the classical Laguerre orthogonal polynomial $L_n^{(\alpha)}(\omega \rho^2)$ and is given by

$$\Phi(\rho) \simeq \exp\left(-\frac{\omega \rho^2}{2}\right)L_n^{(\alpha)}(\omega \rho^2); \quad n = 0, 1, 2, ...$$

(9)

while the corresponding energy eigenvalues are

$$E_n = \omega \left( 2n + s + \frac{N}{2} + \frac{\lambda r}{2}(2N - r - 1) \right),$$

(10)

where $\alpha = (s - 1 + \frac{N}{2} + \frac{\lambda r}{2}(2N - r - 1))$. As shown in [28], for $r = 1$ and $r = N - 1$, the results reduces to those of JK [27] and CSM [23, 24] respectively.
3 The extended truncated CS model with new interaction term

The above $N$-body truncated CS model can be extended by adding a new interaction term $V_{new}$ as

$$\hat{H}_{ext} = \hat{H} + V_{new},$$

where

$$V_{new} = \frac{(\alpha_1 + \alpha_2 \omega^2 \rho^2)}{(\beta_1 + \beta_2 \omega^2 \rho^2)^2},$$

where $\alpha_{1,2}$ and $\beta_{1,2}$ are unknown constants. Following the procedure adopted above in the case of conventional model, the solution of the Schrödinger equation

$$\hat{H}_{ext} \Psi_{ext} = E_{ext} \Psi_{ext}$$

corresponding to the extended Hamiltonian ($\hat{H}_{ext}$) is obtained by assuming the extended wavefunction

$$\Psi_{ext}(x) = \phi(x) \xi_{ext},$$

where $\phi(x)$ again is as given by Eq. (4) while $\xi_{ext}$ satisfies the equation

$$-\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2 \xi_{ext}}{\partial x_i^2} - \lambda \sum_{i<j} \frac{1}{x_i - x_j} \left( \frac{\partial \xi_{ext}}{\partial x_i} - \frac{\partial \xi_{ext}}{\partial x_j} \right) + \left( \frac{1}{2} \sum_{i} \omega^2 x_i^2 + V_{new} - E_{ext} \right) \xi_{ext} = 0.$$ (15)

As in the conventional case as discussed above, we redefine the function $\xi_{ext}$ as

$$\xi_{ext} = \Phi_{ext}(\rho) P_s(x).$$ (16)

In that case Eq. (15) reduces to the $\rho$ dependent equation

$$\Phi''_{ext}(\rho) + (N + 2s - 1 + \lambda r(2N - r - 1)) \frac{1}{\rho} \Phi'_{ext}(\rho) + 2(E - \left( \frac{1}{2} \omega^2 \rho^2 + V_{new} \right)) \Phi_{ext}(\rho) = 0,$$ (17)

with $P_s(x)$ satisfying the same generalized Laplace equation Eq. (8). Note that here a prime on $\Phi_{ext}(\rho)$ indicates derivative with respect to $\rho$.

To get the exact form of the defined new interaction term [12] and the solutions of the above equation, we assume

$$\Phi_{ext}(\rho) = f(\rho) \zeta(g(\rho)),$$ (18)

where $f(\rho)$ and $g(\rho)$ are two undermined functions and $\zeta(g)$ is a special function which satisfies a second-order differential equation

$$\zeta''(g(\rho)) + Q_1(g(\rho)) \zeta'(g(\rho)) + R_1(g(\rho)) \zeta(g(\rho)) = 0.$$ (19)

The functions $Q_1(g)$ and $R_1(g)$ are well defined for any special function $\zeta(g)$. Substituting Eq. (18) into Eq. (17), we get

$$\zeta''(g) + \left( \frac{2f'(\rho)}{f(\rho)g'(\rho)} + \frac{g''(\rho)}{g'(\rho)^2} + \frac{\tau}{\rho f'(\rho)} \right) \zeta'(g)$$

$$+ \frac{1}{g'(\rho)^2} \left( \frac{f''(\rho)}{f(\rho)} + \frac{\tau f'(\rho)}{\rho f(\rho)} + 2(E_{ext} - V_{ext}) \right) \zeta(g) = 0,$$ (20)
where \( V_{ext} = \frac{1}{2} \omega \rho^2 + V_{new} \) and \( \tau = (N + 2s - 1 + \lambda r(2N - r - 1)) \). On comparing Eq. (20) with Eq. (19), we get

\[
Q_1(g) = \frac{2f'(\rho)}{f(\rho)g'(\rho)} + \frac{g''(\rho)}{g'(\rho)^2} + \frac{\tau}{\rho f'(\rho)} \tag{21}
\]

and

\[
R_1(g) = \frac{1}{g'(\rho)^2} \left( \frac{f''(\rho)}{f(\rho)} + \frac{\tau f'(\rho)}{\rho f(\rho)} + 2(E_{ext} - V_{ext}) \right). \tag{22}
\]

After simplifying \( Q_1(g) \), one finds that

\[
f(\rho) \simeq (g'(\rho))^{-\frac{1}{2}} \rho^{-\frac{\alpha}{2}} \exp \left( \frac{1}{2} \int Q_1(g) dg \right). \tag{23}
\]

Using \( f(\rho) \) in the expression of \( R_1(g) \) we get

\[
E_{ext} - V_{ext} = \frac{1}{2} \left[ \frac{g''(\rho)}{2g'(\rho)} - \frac{3}{4} \frac{g''(\rho)^2}{g(\rho)^2} + \frac{\tau/(2(\tau/2 - 1)}{\rho^2} + g'(\rho)^2 \left( R_1(g) - \frac{Q_1^2(g)}{2} - \frac{Q_2^2(g)}{4} \right) \right]. \tag{24}
\]

Thus, once we choose \( Q_1(g) \) and \( R_1(g) \) corresponding to the given special function \( \zeta(g) \) the extended potential \( V_{ext} \) and the corresponding energy \( E_{ext} \) can be obtained for given \( g(\rho) \) as defined in the case of conventional model.

Let us consider the special function \( \zeta(g) \) in the form of \( X_1 \) Laguerre polynomial \( \hat{L}_{n}^{(\alpha)}(g) \) satisfying the differential equation

\[
\hat{L}_n^{(\alpha)}(g(\rho)) + Q(g)\hat{L}_n^{(\alpha)}(g(\rho)) + R(g)\hat{L}_n^{(\alpha)}(g(\rho)) = 0; \quad n \geq 1, \tag{25}
\]

with

\[
Q_1(g) = \frac{-(g - \alpha)(g + \alpha + 1)}{g(g + \alpha)} \quad \text{and} \quad R_1(g) = \frac{1}{g} \left( \frac{g - \alpha}{g + \alpha} + n - 1 \right). \tag{26}
\]

Using above equations in Eqs. (23) and (24) and by defining

\[
g(\rho) = \omega \rho^2; \quad \alpha = \frac{\tau}{2} - \frac{1}{2} \tag{27}
\]

and replacing \( n \to n + 1 \), we get

\[
V_{ext} = \frac{1}{2} \omega^2 \rho^2 + \frac{4\omega}{(2\omega \rho^2 + \tau - 1)} - \frac{8\omega(\tau - 1)}{(2\omega \rho^2 + \tau - 1)^2}, \tag{28}
\]

and the energy eigenvalues \( E_{ext} \) turn out to be the same as that of the conventional model as discussed in Sec. II and are given by Eq. (10). Note however that the corresponding eigenfunction \( \Phi_{ext}(\rho) \) is completely different. Using \( f(\rho) \) and replacing \( \zeta(g) \to \hat{L}_{n+1}^{(\alpha)}(g) \) in Eq. (18), the expression for the energy eigenfunctions is obtained in terms of \( X_1 \) exceptional orthogonal Laguerre polynomials \( (\hat{L}_n^{(\alpha)}(g)) \) as

\[
\Phi_{ext}(\rho) \simeq \frac{\exp\left(-\frac{\omega \rho^2}{2}\right)}{(2\omega \rho^2 + \alpha)} \hat{L}_{n+1}^{(\alpha)}(\omega \rho^2); \quad n = 0, 1, 2, \ldots. \tag{29}
\]
Note that the $X_1$ Laguerre polynomial ($\tilde{L}^{(a)}_{n+1}(g)$) is related to the classical Laguerre polynomials by
\begin{equation}
\tilde{L}^{(a)}_{n+1}(g) = -(g + \alpha + 1)L^{(a)}_{n}(g) + L^{(a)}_{n-1}(g).
\end{equation}
The constant parameters $\alpha, \beta_1, \beta_2$ for which the Hamiltonian (11) is ES can easily be determined by comparing Eqs. (11) and (28) and one finds that
\begin{align}
\alpha_1 &= -4\omega(\tau - 1); \quad \alpha_2 = 8, \\
\beta_1 &= \tau - 1; \quad \text{and} \quad \beta_2 = 2/\omega.
\end{align}
In the special cases of $r = 1$ and $r = N - 1$ we then obtain the rational extension of the JK and the CSM respectively.

3.1 The extended TCS model associated with $X_m$-exceptional Laguerre polynomials

The above model Eq. (11) can easily be generalized to any positive integer values of $m$ by replacing $V_{new}$ with an $m$ dependent polynomial type interaction term $V_{m,new}$ i.e.,
\begin{equation}
\hat{H}_{m,ext} = \hat{H} + V_{m,new}.
\end{equation}
Unlike the $X_1$ case, it is not easy to define the exact form of $V_{m,new}$ in the general $X_m$ case. We shall obtain the interaction terms by assuming the solution of the Schrödinger equation
\begin{equation}
\hat{H}_{m,ext}\Psi_{ext}(x) = E_{m,ext}\Psi_{m,ext}(x)
\end{equation}
as
\begin{equation}
\Psi_{m,ext}(x) = \phi(x)\xi_{m,ext}(x),
\end{equation}
Similar to the $X_1$ case, we redefine Eqs. (16) and (18) by replacing $\Phi_{ext}(\rho) \to \Phi_{m,ext}(\rho)$ and $f(\rho) \to f_m(\rho), \zeta(g) \to \zeta_m(g)$ respectively. In this way the differential Eq. (17) will be also $m$ dependent i.e.,
\begin{equation}
\Phi''_{m,ext}(\rho) + (N + 2s - 1 + \lambda r(2N + r - 1))\frac{1}{\rho} \Phi'_{m,ext}(\rho) + 2\left(E - \frac{1}{2}\omega^2\rho^2 + V_{m,new}\right)\Phi_{m,ext}(\rho) = 0.
\end{equation}
and $\zeta_m(g)$ satisfies an equivalent second-order differential equation
\begin{equation}
\zeta''_m(g) + Q_m(g)\zeta'_m(g) + R_m(g)\zeta_m(g) = 0.
\end{equation}
Now using $\Phi_{m,ext}(\rho)$ into Eq. (35), we get
\begin{align}
\zeta''_m(g) &= \left(\frac{2f''_m(\rho)}{f_m(\rho)g'(\rho)} + \frac{g''(\rho)}{g'(\rho)^2} + \frac{\tau}{\rho g'(\rho)}\right)\zeta'_m(g) \\
&\quad + \frac{1}{g'(\rho)^2}\left(\frac{f''_m(\rho)}{f_m(\rho)} + \frac{\tau f'(\rho)}{\rho f_m(\rho)} + 2(E_{m,ext} - V_{m,ext})\right)\zeta_m(g) = 0,
\end{align}
where $V_{m,ext} = \frac{1}{2}\omega \rho^2 + V_{m,new}$. The functions $Q_m(g)$ and $R_m(g)$ become
\begin{align}
Q_m(g) &= \frac{2f''_m(\rho)}{f_m(\rho)g'(\rho)} + \frac{g''(\rho)}{g'(\rho)^2} + \frac{\tau}{\rho g'(\rho)}
\end{align}
and
\begin{align}
R_m(g) &= \frac{1}{g'(\rho)^2}\left(\frac{f''_m(\rho)}{f_m(\rho)} + \frac{\tau f'(\rho)}{\rho f_m(\rho)} + 2(E_{m,ext} - V_{m,ext})\right).
\end{align}
In terms of $Q_m(g)$, the function $f_m(\rho)$ is given by

$$f_m(\rho) \simeq (g'(\rho))^{-\frac{\tau}{2}} \rho^{-\frac{\tau}{4}} \exp\left(\frac{1}{2} \int_{\rho}^{g} Q_m(g) dg\right). \quad (40)$$

Using $f_m(\rho)$ back in the expression of $R_m(g)$ and get

$$E_{m,\text{ext}} - V_{m,\text{ext}} = \frac{1}{2} \left[ \frac{g''(\rho)}{g'(\rho)^2} \cdot \frac{3 g'(\rho)^2}{4 g(\rho)^2} \cdot \frac{\tau/2(\tau/2 - 1)}{\rho^2} \cdot \rho^2 + g'(\rho) \cdot \left( R_m(g) - \frac{Q_m'(g)}{2} - \frac{Q_m^2(g)}{4} \right) \right]. \quad (41)$$

Similar to the $X_1$ case, the special function $\zeta_m(g)$ satisfies the $X_m$ exceptional Laguerre differential

$$\hat{L}_{n,m}^{(\alpha)}(g(\rho)) + Q_m(g)\hat{L}_{n,m}^{(\alpha)}(g(\rho)) + R_m(g)\hat{L}_{n,m}^{(\alpha)}(g(\rho)) = 0, \quad (42)$$

with

$$Q_m(g) = \frac{1}{g} \left[ (\alpha + 1 - g) - 2g \frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} \right]$$

and

$$R_m(g) = \frac{1}{g} \left[ n - 2\alpha \frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha)}(-g)} \right]. \quad (43)$$

Using above $Q_m(g)$ and $R_m(g)$ in Eqs. (40) and (41) and replacing $n \to n + m$, we get

$$V_{m,\text{new}} = -2\omega^2 \rho^2 \frac{L_{m-2}^{(\alpha+1)}(-g)}{L_m^{(\alpha-1)}(-g)} + 2\omega(\alpha + \omega \rho^2 - 1) \frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} + 4\omega^2 \rho^2 \left( \frac{L_{m-1}^{(\alpha)}(-g)}{L_m^{(\alpha-1)}(-g)} \right)^2 - 2m\omega, \quad (44)$$

while the energy eigenvalues $E_{m,\text{ext}}$ are again unchanged and are same as that of the $X_1$ or conventional cases i.e, $E_{m,\text{ext}} = E_{\text{ext}} = E$. The energy eigen functions $\Phi_{m,\text{ext}}(\rho)$ are however different and are given by

$$\Phi_{m,\text{ext}}(\rho) \simeq \exp\left(\frac{-\omega \rho^2}{2}\right) \frac{L_{m}^{(\alpha)}(-\omega \rho^2)}{L_{m}^{(\alpha-1)}(-\omega \rho^2)} \hat{L}_{n+m}^{(\alpha)}(\omega \rho^2); \quad n, m = 0, 1, 2, \ldots, \quad (45)$$

where the $X_m$ Laguerre polynomial $\hat{L}_{n+m}^{(\alpha)}(g)$ is related to the classical Laguerre polynomials by

$$\hat{L}_{n+m}^{(\alpha)}(g) = L_{n}^{(\alpha)}(-g) L_{m}^{(\alpha-1)}(-g) + L_{m}^{(\alpha-1)}(-g) L_{n-1}^{(\alpha)}(-g). \quad (46)$$

As expected, for $m = 1$, the above results reduce to the corresponding $X_1$-case while for the $m = 0$ case one gets back the conventional TCS model.

4 Results and discussion

In this paper we have constructed an extended truncated Calogero-Sutherland model by introducing new interaction terms. The exact solutions of this extended model are in terms of the
newly discovered special function, the $X_1$-exceptional Laguerre Polynomials while the energy eigenvalues remain unchanged and are same as those of TCS model. The model is further extended to the $X_m$ case and the corresponding $m$-dependent interaction term is obtained. In the particular case of $m = 0$ and $r = 1$ or $r = N – 1$, it can be easily shown that the Hamiltonian and the corresponding eigenvalues and eigenfunctions reduce to that of JK model or CSM respectively. Thus for $r = 1$ or $r = N – 1$, one obtains extended JK model or CSM corresponding to the $X_m$-case simply by putting $r = 1$ or $r = N – 1$ in Eq. (32).

This paper raises some obvious possibilities. What we have done in this paper is basically obtained rational extension of $A_N$ JK or TCS models. The obvious question is can one extend these results to the other cases like $B_N$, $C_N$, $BC_N$, $D_N$ or even to the exceptional groups? Further, are there other N-body problems where such rational extensions are possible? We hope to address some of these issues in the near future.

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