The interplay of the polar decomposition theorem and the Lorentz group

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November 2002

Abstract: It is shown that the polar decomposition theorem of operators in (real) Hilbert spaces gives rise to the known decomposition in boost and spatial rotation part of any matrix of the orthochronous proper Lorentz group \(SO(1,3)\). This result is not trivial because the polar decomposition theorem is referred to a positive defined scalar product while the Lorentz-group decomposition theorem deals with the indefinite Lorentz metric. A generalization to infinite dimensional spaces can be given. It is finally shown that the polar decomposition of \(SL(2,\mathbb{C})\) is preserved by the covering homomorphism of \(SL(2,\mathbb{C})\) onto \(SO(1,3)\).

I. Introduction and notation.

If \(H\) is a, either real or complex, Hilbert space, a bounded bijective operator \(T : H \to H\) can be uniquely decomposed as both \(T = UP\) and \(T = P'U'\) where \(U, U'\) are orthogonal/unitary operators and \(P, P'\) are bounded self-adjoint positive operators. These decompositions are called the polar decompositions of \(T\). Consider the special orthochronous Lorentz group \([1, 2, 3]\)

\[
SO(1,3) : = \{ \Lambda \in M(4, \mathbb{R}) \mid \Lambda \eta \Lambda^t = \eta, \text{ det } \Lambda = 1, \Lambda^0_0 > 0 \}, \tag{1}
\]

where \(M(n, \mathbb{R})\) denotes real vector space of real \(n \times n\) matrices, \(0\) in \(\Lambda^0_0\) is referred to the first element of the canonical basis of \(\mathbb{R}^4\), \(e_0, e_1, e_2, e_3\) and \(\eta = \text{diag}(-1, 1, 1, 1)\). If \(\Lambda \in SO(1,3)\) one may consider the polar decompositions \(\Lambda = \Omega P = P'\Omega'\) where \(\Omega, \Omega' \in O(4)\) and \(P, P'\) are non singular, symmetric, positive matrices in \(M(4, \mathbb{R})\). A priori those decompositions could be physically meaningless because \(\Omega\) and \(P\) could not to belong to \(SO(1,3)\); the notions of symmetry, positiveness, orthogonal group \(O(4)\) are refereed to the positive scalar product of \(\mathbb{R}^4\) instead of the indefinite Lorentz scalar product (similar comments can be made for the other polar decomposition). The main result presented in this work is that the polar decompositions of \(\Lambda \in SO(1,3)\) are in fact physically meaningful. Indeed, they coincides with the known physical decompositions of \(\Lambda\) in spatial-rotation and boost parts (this fact also assures the uniqueness of the physical decompositions). In part, the result can be generalized to infinite dimensional (real or complex) Hilbert spaces. As a subsequent issue, considering the universal covering
of $SO(1,3) \uparrow$, $SL(2, \mathbb{C}) [1, 2, 3]$, we show that the covering homomorphism $\Pi : SL(2, \mathbb{C}) \to SO(1,3) \uparrow$ preserves the polar decompositions of $SL(2, \mathbb{C})$ transforming them into the analogous decompositions in $SO(1,3) \uparrow$.

II. Square roots and polar decomposition.

A real Hilbert space $H$ is a vector space equipped with a symmetric scalar product $(\cdot | \cdot)$ and complete with respect to the induced norm topology. Henceforth we adopt the usual notation and definitions concerning adjoint, self-adjoint, unitary operators in Hilbert spaces (e.g, see [4]), using them either in complex or real Hilbert spaces $H$. Moreover $\mathcal{B}(H)$ denotes the space of all bounded operators $T : H \to H$. $T \in \mathcal{B}(H)$ is said positive ($T \geq 0$) if $(u | Tu) \geq 0$ for all $u \in H$. The lemma and the subsequent theorem below straightforwardly extend the polar decomposition theorem (Theorem 12.35 in [4]) encompassing both the real and the complex case. The proofs are supplied in the appendix. (In complex Hilbert spaces bounded positive operators are self-adjoint [4], in that case the self-adjointness property can be omitted in the hypotheses and the thesis of the lemma and the theorem and elsewhere in this work.)

**Lemma 1.** (Existence and uniqueness of (positive) square roots in Hilbert spaces). Let $T \in \mathcal{B}(H)$ be a self-adjoint positive operator where $H$ is a, either real or complex, Hilbert space. There exists exactly one operator $\sqrt{T} \in \mathcal{B}(H)$ such that $\sqrt{T}^* = \sqrt{T} \geq 0$ and $\sqrt{T}^2 = T$. If $T$ is bijective, $\sqrt{T}$ is so. $\sqrt{T}$ is said the (positive) square root of $T$.

**Theorem 1.** (Polar Decomposition in either Real or Complex Hilbert spaces). If $T \in \mathcal{B}(H)$ is a bijective operator where $H$ is a, either real or complex, Hilbert space:

1. there is a unique decomposition $T = UP$, where $U$ is unitary, and $P$ is bounded, bijective, self-adjoint and positive. In particular $P = \sqrt{T^*T}$ and $U = T(\sqrt{T^*T})^{-1}$;
2. there is a unique decomposition $T = P'U'$, where $U'$ is unitary and $P'$ is bounded, bijective, self-adjoint and positive. In particular $U' = U$ and $P' = UPU^*$.

III. Lorentz group and polar decomposition.

Let us come to the main point by focusing attention on the real Hilbert space $H = \mathbb{R}^4$ endowed with the usual positive scalar product. In that case $\mathcal{B}(H) = M(4, \mathbb{R})$. Unitary operators are orthogonal matrices, i.e., elements of of $O(4)$ and, if $A \in \mathcal{B}(H)$ the adjoint $A^*$ coincides with the transposed matrix $A^t$, therefore self-adjoint operators are symmetric matrices. The Lie algebra of $SO(1,4) \uparrow$ (1) admits a well-known basis made of **boost generators** $K_1, K_2, K_3$ and **spatial rotation generators** $S_1, S_2, S_3$:

\[
K_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad K_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]
Theorem 2. The next theorem clarifies the interplay of boosts, spatial rotations and polar decomposition. Let \( \Lambda \) be the relative velocity of the two inertial frames with Minkowski coordinate systems related by \( \Lambda \) and rotate to a trivial faithful representation of \( \Lambda \) and finally, the pairs \((\pi, \mathbf{n})\) and \((\pi, -\mathbf{n})\) individuates the same rotation.) The elements of one-parameter subgroups of \( \text{SO}(3) \), \( \Lambda = e^{\theta \mathbf{n} \cdot \mathbf{T}} \), with \( \theta \in \mathbb{R} \) and \( \mathbf{n} \) versor in \( \mathbb{R}^3 \), do not affect the time coordinate of the two Minkowski coordinate systems related by \( \Lambda \) and rotate the spatial axes by \( e^{\theta \mathbf{n} \cdot \mathbf{T}} \). These elements are called spatial proper rotations. They give rise to a trivial faithful representation of \( \text{SO}(3) \) in \( \text{SO}(1,3)^+ \). Conversely, the (Lorentz) boosts are the elements of one-parameter subgroups of \( \text{SO}(1,3)^+ \), \( \Lambda = e^{m \mathbf{K}} \), with \( m \in \mathbb{R} \) and \( \mathbf{m} \) versor in \( \mathbb{R}^3 \). (The correspondence between boosts and pairs \((\chi, \mathbf{n})\) is one-to-one with the following exceptions: \( \chi = 0 \) individuates the trivial boost \( I \) not depending on \( \mathbf{n} \), \((\chi, \mathbf{m})\) and \((-\chi, -\mathbf{m})\) define the same boost.) The vector \( \mathbf{v} := c \sinh \chi \mathbf{m} \) \( (c > 0 \text{ being the velocity of light}) \) has the components of the relative velocity of the two inertial frames with Minkowski coordinate systems related by \( \Lambda \). The next theorem clarifies the interplay of boosts, spatial rotations and polar decomposition.

**Theorem 2.** If \( UP = P'U = \Lambda \) (with \( P' = UPU^t \)) are polar decompositions of \( \Lambda \in \text{SO}(1,3)^+ \):

1. \( P, P', U \in \text{SO}(1,3)^+ \), more precisely \( P, P' \) are boosts and \( U \) a spatial proper rotation;
2. there are no other decompositions of \( \Lambda \) as a product of a Lorentz boost and a spatial proper rotation different from the two polar decompositions above.

**Proof.** If \( P \in M(4, \mathbb{R}) \) we shall use the representation:

\[
P = \begin{bmatrix}
g & B^t \\
C & A
\end{bmatrix},
\]

where \( g \in \mathbb{R}, B, C \in \mathbb{R}^3 \) and \( A \in M(3, \mathbb{R}) \).

(1) We start by showing that \( P, U \in \text{O}(1,3) \). As \( P = P^t \), \( \Lambda \eta \Lambda^t = \Lambda \) entails \( UP \eta PU^t = \eta \). As \( U^t = U^{-1} \) and \( \eta^{-1} = \eta \), the obtained identity is equivalent to \( UP^{-1} \eta P^{-1} U^t = \eta \) which, together with \( UP \eta PU^t = \eta \), implies \( P \eta P = P^{-1} \eta P^{-1} \), namely \( \eta P^2 \eta = P^{-2} \), where we have used \( \eta = \eta^{-1} \) once again. Both sides are symmetric (notice that \( \eta = \eta^t \)) and positive by construction, by Lemma 1 they admit unique square roots which must coincide. The square root of \( P^{-2} \) is \( P^{-1} \) while the square root of \( \eta P^2 \eta \) is \( \eta P \eta \) since \( \eta P \eta \) is symmetric positive and \( \eta P \eta P \eta = \eta PP \eta = \eta P^2 \eta \). We conclude that \( P^{-1} = \eta P \eta \) and thus \( \eta = P \eta P \) because
\( \eta = \eta^{-1} \). Since \( P = P^t \) we have found that \( P \in O(1, 3) \) and thus \( U = \Delta P^{-1} \in O(1, 3) \). Let us prove that \( P, U \in SO(1, 3) \). \( \eta = P \eta P^t \) entails \( \det P = \pm 1 \), on the other hand \( P = P^t \) is positive and thus \( \det P \geq 0 \) and \( P^0 0 \geq 0 \). As a consequence \( \det P = 1 \) and \( P^0 0 \geq 0 \). We have found that \( P \in SO(1, 3) \). Let us determine the form of \( P \) using (4). \( P = P^t \), \( P \geq 0 \) and \( P \eta P = \eta \) give rise to the following equations: \( C = B, 0 < g = \sqrt{1 + B^2}, AB = gB, A = A^*, A \geq 0 \) and \( A^2 = I + BB^t \). Since \( I + BB^t \) is positive, the solution of the last equation \( A = \sqrt{A^2} = I + BB^t/(1 + g) \geq 0 \) is the unique solution by Lemma 1. We have found that a matrix \( P \in O(1, 3) \) with \( P \geq 0 \), \( P = P^* \) must have the form

\[
P = \begin{bmatrix}
\cosh \chi & (\sinh \chi) n^t \\
(\sinh \chi) n & I - (1 - \cosh \chi) nn^t
\end{bmatrix},
\]

where we have used the parameterization \( B = (\sinh \chi)n, n \) being any versor in \( \mathbb{R}^3 \) and \( \chi \in \mathbb{R} \). If \( n' := e^{\theta m^t} n \) (which is a versor since \( e^{\theta m^t} n \in SO(3) \)), by direct computation it arises that:

\[
e^{\theta m} s P(e^{\theta m} s)^t = \begin{bmatrix}
\cosh \chi & \sinh \chi (n')^t \\
(\sinh \chi) n' & I - (1 - \cosh \chi) nn'^t
\end{bmatrix},
\]

It is simply proven that the matrix in the right hand side of (6) coincides with \( e^{\chi n'} K \) if \( n' = e_3 \) and this happens for a suitable choice of parameters \( m_\theta, \theta_\theta \). Therefore we have the decomposition \( P = e^{\theta m' s} e^{\chi e_3 K}(e^{\theta m' s})^t \) for \( m' = -m_\theta \). On the other hand, from the commutation relations \( [S_i, K_j] = \sum_{k=1}^3 \epsilon_{ijk} K_k \) it is simply proven that, for all versors \( m, n \) and \( \theta \in \mathbb{R} \):

\[
e^{\theta m} s n \cdot K (e^{\theta m} s)^t = (e^{\theta m} s n) \cdot K
\]

(the proof is based on the fact the functions of \( \chi \) in both sides satisfy the same differential equation with the same initial condition). As a consequence,

\[
e^{\theta m} s e^{\chi n} K (e^{\theta m} s)^t = e^{\chi(e^{\theta m} s n)} K.
\]

Specializing to the case \( n = e_3, \theta = \theta_\theta \) and \( m = m' \), we have found that every matrix \( P \in O(1, 3) \) with \( P \geq 0 \) and \( P = P^t \) can be written as \( P = e^{\theta p} K \) for some \( \chi \in \mathbb{R} \) and some \( p \) versor of \( \mathbb{R}^3 \).

In other words \( P \) is a Lorentz boost. (The same proofs apply to \( P^t \).)

Let us pass to consider \( U \). Since \( \Lambda, P \in SO(1, 3) \), from \( \Lambda P^{-1} = U \) we conclude that \( U \in SO(1, 3) \). \( U \eta = \eta(U^t)^{-1} \) (i.e. \( U \in O(1, 3) \)) and \( U^t = U^{-1} \) (i.e. \( U \in O(3) \)) entail that \( U \eta = \eta U \) and thus the eigenspaces of \( \eta \), \( E_\lambda \) (with eigenvalue \( \lambda \)), are invariant under the action of \( U \). In those spaces \( U \) acts as an element of \( O(\dim(E_\lambda)) \) and the whole matrix \( U \) has a block-diagonal form. \( E_{\lambda=1} \) is generated by \( e_0 \) and thus \( U \) reduces to \( \pm I \) therein. The sign must be + because of the requirement \( U^0 0 > 0 \). The eigenspace \( E_{\lambda=1} \) is generated by \( e_1, e_2, e_3 \) and therein \( U \) reduces to an element of \( R \in O(3) \). Actually the requirement \( \det U = 1 \) (together with \( U^0 0 = 1 \)) implies that \( R \in SO(3) \) and thus \( R = e^{\theta m} s \) for some versor \( m \) and some real \( \theta \).
Using the found block-diagonal structure of the matrix Ω and the definition of the matrices S in functions of the matrices T, it is straightforwardly proven that Ω = e^{θmS}.

(2) If ΩB = Λ ∈ SO(1, 3)↑ where B is a pure boost and Ω is a spatial proper rotation. B = e^{χnK} is symmetric by construction since $K_i = K_i^t$. As a consequence of (7) we find $e^{χnK} = (e^{θmS})^t e^{χK3} e^{θmS}$ (m is orthogonal to e3 and n and θ is the rotation angle around m of the rotation which transforms e3 into n). By direct inspection one see that $e^{χK3}$ is positive and thus B is so. On the other hand if Ω = e^{θnS}, $Ω^t = e^{θnS^t} = e^{-θnS} = Ω^{-1}$ and thus Ω is orthogonal. We conclude that Λ = ΩB is one of the two polar decompositions (using the uniqueness property in Theorem 1). The proof for the other case Λ = B′Ω′ is strongly analogous. □

The result can be partially generalized into the following theorem. The proof is part of the proof of the statement (1) of Theorem 2 with $\mathbb{R}^4, η, τ$ replaced by $H, E, τ$ respectively.

**Theorem 3.** Let H be a, either real or complex, Hilbert space and $G_E$ the group of all of operators $Λ ∈ B(H)$ such that $ΛEΛ^* = E$, for a fixed $E ∈ B(H)$ which is not necessarily positive and satisfies $E = E^{-1} = E^*$. The polar decompositions of $Λ ∈ G_E$, $Λ = PU = UP'$ (where U is the unitary operator) are such that $P, P', U ∈ G_E$ and the eigenspaces of E are invariant for U.

Notice that in the hypotheses above for $E, σ(E) ⊂ \{-1, +1\}$.

Let us come to the last result. As is well known, the simply connected Lie group $SL(2, \mathbb{C})$ is the universal covering of $SO(1, 3)↑ [1, 2, 3]$. Hence there is a surjective Lie-group homomorphism $Π : SL(2, \mathbb{C}) → SO(1, 3)↑$ which is a local Lie-group isomorphism about each $L ∈ SL(2, \mathbb{C})$.

**Theorem 4.** Let $σ$ denote the vector whose components are the well-known Pauli’s matrices

$$σ_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad σ_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad σ_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (8)$$

If $L ∈ SL(2, \mathbb{C})$ and $L = PU = UP'$ are its polar decompositions:

(1) $P, P', U ∈ SL(2, \mathbb{C})$, in particular $P = e^{χnσ/2}, U = e^{-θm-iσ/2}$ for some $n, m$ versors in $\mathbb{R}^3$ and $χ, θ ∈ \mathbb{R}$.

(2) $Π(e^{χnσ/2}) = e^{χnK}$ and $Π(e^{-θm-iσ/2}) = e^{-θmS}$ and thus Π maps the polar decompositions of any $L ∈ SL(2, \mathbb{C})$ into the corresponding polar decompositions of $Π(L) ∈ SO(1, 3)↑$.

**Proof.** (1) We deal with the decomposition PU only the other case being analogous. As $0 ≤ P = P^*$, P can be reduced in diagonal form with positive eigenvalues so that $det P ≥ 0$. As a consequence $1 = det L = det P det U$ entails that det $U > 0$. In turn, the condition $U^{-1} = U^*$ implies $|det U|^2 = 1$ and thus $det U = 1$. We have proven that $U ∈ SL(2, \mathbb{C})$ and also that $P = LU^{-1} ∈ SL(2, \mathbb{C})$. From the spectral theorem (see theorem 12.37 in [4]) there are two bounded self-adjoint operators S, Q (i.e. Hermitean matrices of $M(2, \mathbb{C})$) such that $P = e^S$ and $U = e^{iQ}$. Since the matrices $σ_1 := I, σ_1, σ_2, σ_3$ are a basis of the real vector space of $2 × 2$
Hermitean matrices, \( S = aI + \chi n \cdot \sigma \) and \( Q = bI + \theta m \cdot \sigma \) for some versors \( n, m \in \mathbb{R}^3 \) and reals \( a, b, \chi, \theta \). Using \( \det e^X = e^{\text{tr} X} \) and the fact that Pauli matrices are traceless, the constraint \( \det P = \det U = 1 \) implies \( a = b = 1 \). This completes the proof of (1).

(2) By definition \( \Pi \) maps a one-parameter subgroup with initial tangent vector \( d\Pi_{t}X \) into a one-parameter subgroup with initial tangent vector \( d\Pi_{t}X \). Since \( \Pi(L_j) = 1, j = 0, 1, 2, 3 \), it holds \( d\Pi_{t} : -i n \cdot \sigma / 2 \rightarrow n \cdot S \) and similarly \( d\Pi_{t} : n \cdot \sigma / 2 \rightarrow n \cdot K \) for \( i = 1, 2, 3 \). Hence the one parameters groups \( \theta \mapsto e^{-i \theta m \cdot \sigma / 2} \) and \( \chi \mapsto e^{i \chi n \cdot \sigma / 2} \) are respectively mapped into \( \theta \mapsto e^{i m \cdot S} \) and \( \chi \mapsto e^{i n \cdot K} \).

\[\Box\]

A1. Proofs of some propositions.

If \( H \) is a real Hilbert space \( H + iH \) denotes the complex Hilbert space obtained by defining on \( H \times H \): (i) the product \( (a + ib)(u + iv) := au - bv + i(bu + av) \) where \( a + ib \in \mathbb{C} \) and we have defined \( u + iv := (u, v) \in H \times H \), (ii) the sum of \( u + iv \) and \( x + iy \) in \( H \times H \): \( (u + iv) + (x + iy) := (u + x) + i(v + y) \), and (iii) the, anti-linear in the former entry, Hermitean scalar product \( \langle u + iv | w + ix \rangle := (u|v) + (v|x) + i(u|x) - i(v|w) \). Let us introduce a pair of useful operators. The complex conjugation \( J : u + iv \mapsto u - iv \) turns out to be an anti linear operator with \( \langle J(u + iv) | J(w + ix) \rangle = \langle w + ix | u + iv \rangle \) and \( JJ = I \). The unitary flip operator \( C : u + iv \mapsto v - iu \) satisfies \( C = C^* = C^{-1} \). A bounded operator \( A : H + iH \rightarrow H + iH \) is said to be real if \( JA = AJ \). It is simply proven that, (1) \( A \) is real if and only if there is a (uniquely determined) pair of bounded operators \( A_j : H \rightarrow H, j = 1, 2 \), such that \( A(u + iv) = A_1 u + iA_2 v \) for all \( u + iv \in H + iH \); (2) \( A \) is real and \( AC = CA \), if and only if there is a (uniquely determined) bounded operator \( A_0 : H \rightarrow H \), such that \( A(u + iv) = A_0 u + iA_0 v \) for all \( u + iv \in H + iH \).

**Proof of Lemma 1.** The proof in the complex case is that of Theorem 12.33 in [4]. Let us consider the case of a real Hilbert space \( H \). If \( T \in \mathcal{B}(H) \) is positive and self-adjoint, the operator on \( H + iH, A : u + iv \mapsto Tu + iTv \) is bounded, positive and self-adjoint. By Theorem 12.33 in [4] there is only one \( B \in \mathcal{B}(H + iH) \) with \( 0 \leq B (= B^*) \) and \( B^2 = A \), that is the square root of \( A \) which we indicate by \( \sqrt{A} \). Since \( A \) commutes with both \( J \) and \( C \), all of the real polynomials in \( A \) do so. If \( \Omega \subset \sigma(A) \) is a Borel set and \( P_\Omega \) is the associated orthogonal projector in the spectral measure of \( A \), there is a sequence of real polynomials in \( A \) which tends to \( P_\Omega \) in the strong operator topology (use Stone-Weierstrass’ theorem and the fact that the space of continuous functions is dense in any \( L^2(\mathbb{R}, \mu) \) if \( \mu \) is borel with respect to the topology of \( \mathbb{R} \)). Therefore every projector \( P_\Omega \) commutes with both \( J \) and \( C \) and, in turn, every real Borel function of \( A \) does so, \( \sqrt{A} \) in particular. We conclude that \( \sqrt{A} \) is real with the form \( \sqrt{A} : u + iv \mapsto Ru + iRv \). The operator \( \sqrt{T} := R \) fulfills all of requirements it being bounded, self-adjoint and positive because \( \sqrt{A} \) is so and \( R^2 = T \) since \( (\sqrt{A})^2 = A : u + iv \mapsto Tu + iTv \). If \( T \) is bijective, \( A \) is so by construction. Then, by Theorem 12.33 in [4], \( \sqrt{A} \) turns out to be bijective and, in turn, \( R \) is bijective too by construction. Let us consider the uniqueness of the found square root. If \( R' \) is another bounded positive self-adjoint square root of \( T \), \( B : u + iv \mapsto R'u + iR'v \) is a bounded-
adjoint positive square root of \( A \) and thus it must coincide with \( \sqrt{A} \). This implies that \( R = R' \). \( \square \)

**Proof of Theorem 1.** (1) Consider the bijective operator \( T : H \to H \) where \( H \) is either real or complex. \( T^*T \) is bounded, self-adjoint, positive and bijective by construction. Define \( P := \sqrt{T^*T} \), which exists and is bounded, self-adjoint, positive and bijective by Lemma 1, and \( U := TP^{-1} \). \( U \) is unitary because \( UP = P^2P^{-1} = I \), where we have used \( P^* = P \). This proves that a polar decomposition of \( T \) exists because \( UP = T \) by construction. Let us pass to prove the uniqueness of the decomposition. If \( T = U_1P_1 \) is a other polar decomposition, \( T^*T = P_1U_1^*U_1P_1 = PU^*UP \). That is \( P_1^2 = P^2 \). Lemma 1 implies that \( P = P_1 \) and \( U = T^{-1}P = T^{-1}P_1 = U_1 \).

(2) \( P' := UPU^* \) is bounded, self-adjoint, positive and bijective since \( U^* \) is unitary and \( P'U' = UPU^*U = UP = T \). The uniqueness of the decomposition in (2) is equivalent to the uniqueness of the polar decomposition \( U^*P'^* = T^* \) of \( T^* \) which holds true by (1) replacing \( T \) by \( T^* \). \( \square \)

**Proof of the fact that \( SO(3) \) is made by all of the matrices \( e^{\theta n^\cdot T} \).** If \( R \in SO(3) \), the induced operator in \( \mathbb{R} + i\mathbb{R} \) is unitary and thus it admits a base of eigenvectors with eigenvalues \( \lambda_i \) with \( |\lambda_i| = 1 \), \( i = 1, 2, 3 \). As the characteristic polynomial of \( R \) is real, an eigenvalue must be real, the remaining pair of eigenvalues being either real or complex and conjugates. Since \( \det R = \lambda_1\lambda_2\lambda_3 = 1 \), 1 is one of the eigenvalues. We conclude that \( R \) has a real normalized eigenvector \( n \) with eigenvalue 1. By direct inspection one finds that \( R \) is represented by the matrix \( e^{\theta T_3} \) for some \( \theta \in [0, 2\pi] \) in any orthonormal base \( n_1 := n, n_2, n_3 \). In other words \( R = R'e^{\theta T_3}R'^t \) for some \( R' \in SO(3) \). On the other hand \( (T_3)_{jk} = -\epsilon_{ijk} \) entails that \( \sum_{i,j,k} U_{pi}U_{qj}U_{rk}\epsilon_{ijk} = \epsilon_{pqr} \) for all \( U \in SL(3, \mathbb{R}) \). That identity can be re-written as \( n^\cdot U T U^t = (U n^\cdot T) \) for every \( U \in SL(3, \mathbb{R}) \).

By consequence, if \( U \in SO(3) \) it also holds \( U e^{\theta n^\cdot T} U^t = e^{\theta(U n)^\cdot T} \). Therefore, the identity found above for any \( R \in SO(3) \), \( R = R'e^{\theta T_3}R'^t \) with \( R' \in SO(3) \), can equivalently be written as \( R = e^{\theta n^\cdot T} \) for some versor \( n = R'e_3 \). Finally, every matrix \( e^{\theta n^\cdot T} \) belongs to \( SO(3) \) because \( (e^{\theta n^\cdot T})^t = e^{\theta n^\cdot T^t} = e^{-\theta n^\cdot T} = (e^{\theta n^\cdot T})^{-1} \) and \( \det e^{\theta n^\cdot T} = e^{\theta trn T} = e^0 = 1 \).

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