CONCENTRATION OF EIGENFUNCTIONS OF THE
LAPLACIAN ON A CLOSED RIEMANNIAN
MANIFOLD

KEI FUNANO AND YOHEI SAKURAI

Abstract. We study concentration phenomena of eigenfunctions of the Laplacian on closed Riemannian manifolds. We prove that the total measure of a closed manifold concentrates around nodal sets of eigenfunctions exponentially. Applying the method of Colding and Minicozzi we also prove restricted exponential concentration inequalities and restricted Sogge-type $L_p$ moment estimates of eigenfunctions.

1. Introduction

Eigenfunctions of the Laplacian naturally appeared as an important object in analysis and geometry ([3]). Their global behavior was vastly investigated in several literature. In this paper we study the global feature of eigenfunctions with focus on their concentration properties.

Let $(M,g)$ be a closed Riemannian manifold and $\varphi_\lambda$ be an eigenfunction corresponding to a closed eigenvalue $\lambda$ of the Laplacian on $M$. In [1] Brüning proved that there is some constant $C = C(M,g)$ depending only on $(M,g)$ such that the $(C/\sqrt{\lambda})$-neighborhood of the nodal set $\varphi_\lambda^{-1}(0)$ covers the whole manifold $M$ (see [3] Theorem 4.1). One might wonder how much the measure of $M$ concentrates around the $(r/\sqrt{\lambda})$-neighborhood of $\varphi_\lambda^{-1}(0)$ for any given $r > 0$. One of our main results answers this question. For $r > 0$ and $\Omega \subset M$, let $B_r(\Omega)$ denote the closed $r$-neighborhood of $\Omega$. We denote by $m_M$ the uniform volume measure on $M$ normalized as $m_M(M) = 1$.

Theorem 1.1. Let $M$ be a closed Riemannian manifold. Then for all $r > 0$ we have

$$m_M(M \setminus B_r(\varphi_\lambda^{-1}(0))) \leq \exp(1 - \sqrt{\lambda}r).$$
Let $\text{Ric}_M$ denote the infimum of the Ricci curvature over $M$. Under a lower Ricci curvature bound, we also consider how much the eigenfunction $\varphi_\lambda$ concentrates to zero and obtain the following exponential concentration inequality on large subsets:

**Theorem 1.2.** Let $M$ be an $n$-dimensional closed Riemannian manifold with $\text{Ric}_M \geq -(n-1)$. Then there exists a constant $C_n > 0$ depending only on $n$ such that the following holds: If the eigenvalue $\lambda$ is at least $C_n$, then for every $\xi \in (0, 1)$, there is a Borel subset $\Omega = \Omega_{\lambda, \xi} \subset M$ with $m_M(\Omega) \geq 1 - \xi$ such that

$$m_M(\Omega \cap \{|\varphi_\lambda| > r\}) \leq \exp \left(1 - \frac{C_n \sqrt{\xi}}{\|\varphi_\lambda\|_2} r\right)$$

for every $r > 0$, where $C_n > 0$ is a constant depending only on $n$, and $\|\cdot\|_2$ denotes the standard $L_2$ norm on $(M, m_M)$.

From the above theorem the standard argument yields the following.

**Corollary 1.3.** Under the same setting and the same notation of Theorem 1.2 we have the following: For any $p \geq 1$

$$\left(\int_{\Omega_{\lambda, \xi}} |\varphi_\lambda|^p \, dm_M\right)^{\frac{1}{p}} \leq \Gamma(p+1)^{\frac{1}{p}} \frac{\|\varphi_\lambda\|_2}{C_n \sqrt{\xi}},$$

where $\Gamma$ is the gamma function.

**Remark 1.1.** The Stirling formula tells us that the ratio of the right hand side of (1.1) to $p \|\varphi_\lambda\|_2 (C_n \sqrt{\xi})^{-1}$ tends to 1 as $p \to \infty$.

Let us mention the result due to Sogge. In [7] (see also [8, Theorem 9.2]) Sogge obtained a global $L_p$ moment estimate for eigenfunctions:

$$\|\varphi_\lambda\|_p \leq O(\delta(n, p)) \|\varphi_\lambda\|_2,$$

where $\delta(n, p)$ is a some function of $n$ and $p$. He also mentioned that his inequality is sharp for certain $n$ and $p$. The crucial point of Corollary 1.3 is that once we restrict the eigenfunction $\varphi_\lambda$ on a some large subset $\Omega$ then we do not need the $\lambda$-term of the Sogge inequality to bound the $L_p$ moment of $\varphi_\lambda 1_\Omega$ in terms of $\|\varphi_\lambda\|_2$.

**2. Dirichlet eigenvalues**

Throughout this section, let $M$ be a connected compact Riemannian manifold with boundary. We denote by $\partial M$ its boundary and by $d_M$ the Riemannian distance.
2.1. Key estimates. The key ingredient of the proof of our main results is the following concentration inequality around the boundary in terms of the first Dirichlet eigenvalue of the Laplacian. In the proof we closely follows the argument of Gromov and Milman ([3, 5, Theorem 3.1]). Their context was the first nontrivial closed eigenvalue and Neumann eigenvalue.

**Proposition 2.1.** For every $r > 0$ we have
\begin{equation}
(2.1) \quad m_M(M \setminus B_r(\partial M)) \leq \exp\left(1 - \sqrt{\lambda_1^D(M)} r\right),
\end{equation}
where $\lambda_1^D(M)$ is the first Dirichlet eigenvalue of the Laplacian on $M$.

**Proof.** First, we show that for all $\epsilon, r > 0$ we have
\begin{equation}
(2.2) \quad (1 + \epsilon^2 \lambda_1^D(M)) m_M(M \setminus B_{r+\epsilon}(\partial M)) \leq m_M(M \setminus B_r(\partial M)).
\end{equation}
We set $\Omega_1 := B_r(\partial M)$, $\Omega_2 := M \setminus B_{r+\epsilon}(\partial M)$, and put $v_\alpha := m_M(\Omega_\alpha)$ for each $\alpha = 1, 2$. Let us define a Lipschitz function $\varphi : M \to \mathbb{R}$ by
\[
\varphi(x) := \min \left\{ \frac{1}{\epsilon} d_M(x, \Omega_1), 1 \right\}.
\]
The function $\varphi$ satisfies the following properties:

1. $\varphi \equiv 0$ on $\Omega_1$, and $\varphi \equiv 1$ on $\Omega_2$;
2. $\|\nabla \varphi\| \leq \epsilon^{-1} m_M$-almost everywhere on $M$,
where $\nabla \varphi$ denotes the gradient of $\varphi$, and $\| \cdot \|$ denotes the canonical norm induced from the Riemannian metric on $M$. It follows that
\[
\int_M \varphi^2 \ d m_M \geq v_2, \quad \int_M \|\nabla \varphi\|^2 \ d m_M \leq \frac{1}{\epsilon^2} (1 - v_1 - v_2).
\]
The min-max principle leads us to
\[
\lambda_1^D(M) \leq \frac{\int_M \|\nabla \varphi\|^2 \ d m_M}{\int_M \varphi^2 \ d m_M} \leq \frac{1}{\epsilon^2 v_2} (1 - v_1 - v_2).
\]
This implies (2.2).

Now, let us prove (2.1). We put $\epsilon_0 := \lambda_1^D(M)^{-\frac{1}{2}}$. We first consider the case where $r \in (0, \epsilon_0)$. Let $l \geq 1$ denote the integer determined by $\epsilon_0 r^{-1} \in [(l + 1)^{-1}, l^{-1})$. Using the inequality (2.2) $l$ times, we arrive at
\[
m_M(M \setminus B_r(\partial M)) \leq m_M(M \setminus B_{l\epsilon_0}(\partial M)) \leq 2^{-l} m_M(M \setminus B_{(l-1)\epsilon_0}(\partial M)) \leq 2^{-l} \leq 2^{1 - \frac{r}{\epsilon_0}}.
\]
This proves (2.1). In the case where $r \in [\epsilon_0, \infty)$, it holds that
\[
m_M(M \setminus B_r(\partial M)) \leq \exp\left(1 - \frac{r}{\epsilon_0}\right)
\]
since the right hand side is at least 1. Therefore, we conclude (2.1). □

2.2. Boundary separation distances. We call \( X = (X, d_X, \mu_X) \) a metric measure space with boundary when \( X \) is a connected complete Riemannian manifold with boundary, \( d_X \) is the Riemannian distance, and \( \mu_X \) is a Borel probability measure on \( X \). Let \( X = (X, d_X, \mu_X) \) be a metric measure space with boundary, and let \( k \geq 1 \) be an integer. For positive numbers \( \eta_1, \ldots, \eta_k > 0 \), we denote by \( S_X(\eta_1, \ldots, \eta_k) \) the set of all sequences \( \{\Omega_{\alpha}\}_{\alpha=1}^k \) of Borel subsets \( \Omega_{\alpha} \) with \( \mu_X(\Omega_{\alpha}) \geq \eta_{\alpha} \).

For a sequence \( \{\Omega_{\alpha}\}_{\alpha=1}^k \in S_X(\eta_1, \ldots, \eta_k) \), we define

\[
D_X(\{\Omega_{\alpha}\}_{\alpha=1}^k) := \min \left\{ \min_{\alpha \neq \beta} d_X(\Omega_{\alpha}, \Omega_{\beta}), \min_{\alpha} d_X(\Omega_{\alpha}, \partial X) \right\}.
\]

The author [6] has introduced the \((\eta_1, \ldots, \eta_k)\)-boundary separation distance \( BSep(X; \eta_1, \ldots, \eta_k) \) of \( X \) as follows (see Definition 3.2 in [6]): If \( S_X(\eta_1, \ldots, \eta_k) \neq \emptyset \), then

\[
BSep(X; \eta_1, \ldots, \eta_k) := \sup D_X(\{\Omega_{\alpha}\}_{\alpha=1}^k),
\]

where the supremum is taken over all \( \{\Omega_{\alpha}\}_{\alpha=1}^k \in S_X(\eta_1, \ldots, \eta_k) \); otherwise, \( BSep(X; \eta_1, \ldots, \eta_k) := 0 \). The second author [6] has presented the following relation with the Dirichlet eigenvalue:

**Lemma 2.2** ([6, Lemma 4.1]). For all \( \eta_1, \ldots, \eta_k > 0 \), we have

\[
BSep((M, d_M, m_M); \eta_1, \ldots, \eta_k) \leq \frac{2}{\sqrt{\lambda_k^D(M)}} \frac{\log e}{\eta},
\]

where \( \lambda_k^D(M) \) is the \( k \)-th Dirichlet eigenvalue of the Laplacian on \( M \).

By applying Proposition 2.1 to our setting, we obtain the following refined estimate in the case where \( k = 1 \):

**Theorem 2.3.** For every \( \eta > 0 \), we have

\[
BSep((M, d_M, m_M); \eta) \leq \frac{1}{\sqrt{\lambda_1^D(M)}} \log e \frac{\eta}{\eta}.
\]

**Proof.** We may assume that the left hand side is positive. Fix a Borel subset \( \Omega \subset M \) with \( m_M(\Omega) \geq \eta \). From Proposition 2.1 for every \( r > 0 \) with \( r > \lambda_1^D(M)^{-\frac{1}{2}}(1 - \log \eta) \), we derive \( m_M(B_r(\partial M)) > 1 - \eta \); in particular, \( B_r(\partial M) \cap \Omega \neq \emptyset \) and \( d_M(\Omega, \partial M) \leq r \). By letting \( r \to \lambda_1^D(M)^{-\frac{1}{2}}(1 - \log \eta) \), we obtain \( d_M(\Omega, \partial M) \leq \lambda_1^D(M)^{-\frac{1}{2}}(1 - \log \eta) \). Since \( \Omega \) is arbitrary, we complete the proof. □
3. Proof of the main results

In this section, we will prove the main results. In what follows, let $M$ be an $n$-dimensional closed Riemannian manifold and $\varphi_\lambda$ be an eigenfunction corresponding to a closed eigenvalue $\lambda$ of the Laplacian on $M$. For a Borel subset $\Omega \subset M$, let $m_\Omega$ stand for the normalized volume measure on $\Omega$ defined as

$$m_\Omega := \frac{1}{v_M(\Omega)} v_M|_\Omega,$$  \hspace{1cm} (3.1)

where $v_M$ is the volume measure of $M$.

3.1. Proof of Theorem 1.1. Let us prove Theorem 1.1.

Proof of Theorem 1.1. We fix a nodal domain $\Omega$ of $\varphi_\lambda$, i.e., a connected component of $M \setminus \varphi_\lambda^{-1}(0)$. Applying the same argument in the proof of Proposition 2.1 to $\Omega$, we see

$$m_M(\Omega \setminus B_r(\partial \Omega)) = m_\Omega (\Omega \setminus B_r(\partial \Omega)) \leq \exp(1 - \sqrt{\lambda D_1(\Omega)} r),$$

where $\partial \Omega$ is the boundary of $\Omega$, and $\lambda D_1(\Omega)$ is the first Dirichlet eigenvalue of the Laplacian on $\Omega$. It is well-known that $\lambda D_1(\Omega)$ is equal to $\lambda$ ([2, Lemma 1 in Chapter 1]). Hence,

$$m_M(\Omega \setminus B_r(\partial \Omega)) \leq \exp (1 - \sqrt{\lambda} r) m_M(\Omega). \hspace{1cm} (3.2)$$

We now decompose the set $M \setminus \varphi_\lambda^{-1}(0)$ into the nodal domains $\Omega_\alpha$ of $\varphi_\lambda$ as $M \setminus \varphi_\lambda^{-1}(0) = \sqcup_\alpha \Omega_\alpha$. From (3.2) we derive

$$m_M(M \setminus B_r(\varphi_\lambda^{-1}(0))) = \sum_\alpha m_M(\Omega_\alpha \setminus B_r(\partial \Omega_\alpha)) \leq \exp (1 - \sqrt{\lambda} r) \sum_\alpha m_M(\Omega_\alpha).$$

By $\sum_\alpha m_M(\Omega_\alpha) \leq 1$, we obtain the desired inequality. \hfill \Box

3.2. Proof of Theorem 1.2. In the present subsection, we will give a proof of Theorem 1.2 by applying Theorem 1.1. In order to apply the theorem we need to control the gradient of an eigenfunction on large subsets on $M$. To do so we shall follow the argument of Colding and Minicozzi in [3].

We first recall the following (see the proof of [3, Theorem 1.1]):

Lemma 3.1 ([3]). If $\text{Ric}_M \geq -(n-1)$, then there is a constant $C_n > 0$ depending only on $n$ such that $M = B_R(\varphi_\lambda^{-1}(0))$, where $R := C_n \lambda^{-\frac{1}{2}}$. 

In the proof of [3, Theorem 1.1] Colding and Minicozzi have also obtained the following fact by combining the mean value inequality with the Bochner formula:

**Lemma 3.2** ([3]). If \( \text{Ric}_M \geq -(n - 1) \), then there exists a constant \( C_n > 0 \) depending only on \( n \) such that for all \( x \in M \) and \( r \in (0, 1) \), the supremum of \( \| \nabla \varphi_\lambda \|^2 \) over \( B_r(x) \) is at most

\[
\exp \left( C_n \left( 1 + r \sqrt{2(\lambda + n - 1)} \right) \right) \int_{B_{2r}(x)} \| \nabla \varphi_\lambda \|^2 \, dm_{B_{2r}(x)},
\]

where \( m_{B_{2r}(x)} \) is the normalized measure on \( B_{2r}(x) \) defined as (3.1).

In order to prove Theorem 1.2, we show the following assertion based on the above two lemmas. The idea goes back to Colding and Minicozzi ([3, Theorem 1.1]).

**Lemma 3.3.** We assume \( \text{Ric}_M \geq -(n - 1) \). Then there exists a constant \( C_n > 0 \) depending only on \( n \) such that the following holds: If \( \lambda \geq C_n \), then for every \( \xi \in (0, 1) \), there exists a Borel subset \( \Omega \subset M \) with \( m_M(\Omega) \geq 1 - \xi \) such that

\[
(3.3) \quad \Omega \cap B_r(\varphi_\lambda^{-1}(0)) \subset \Omega \cap \left\{ |\varphi_\lambda| \leq C_n \sqrt{\frac{\lambda}{\xi}} \| \varphi_\lambda \|_2 \right\}
\]

for every \( r > 0 \), where \( C_n > 0 \) is a constant depending only on \( n \).

**Proof.** By Lemma 3.1, there exists a constant \( C_{1,n} > 0 \) depending only on \( n \) such that \( M = B_{10R}(\varphi_\lambda^{-1}(0)) \), where \( R := C_{1,n} \lambda^{-\frac{1}{2}} \). We define \( C_n := \max\{(10 C_{1,n})^2, n - 1\} \), and suppose \( \lambda \geq C_n \). Then \( 10R \in (0, 1] \), and hence Lemma 3.2 implies that there is a constant \( C_{2,n} > 0 \) depending only on \( n \) such that for every \( x \in M \), the supremum of \( \| \nabla \varphi_\lambda \|^2 \) over \( B_{10R}(x) \) is smaller than or equal to

\[
\exp \left( C_{2,n} \left( 1 + 10R \sqrt{2(\lambda + n - 1)} \right) \right) \int_{B_{20R}(x)} \| \nabla \varphi_\lambda \|^2 \, dm_{B_{20R}(x)}.
\]

From \( \lambda \geq n - 1 \), it follows that for every \( x \in M \) we have

\[
(3.4) \quad \sup_{B_{10R}(x)} \| \nabla \varphi_\lambda \|^2 \leq C_{3,n} \int_{B_{20R}(x)} \| \nabla \varphi_\lambda \|^2 \, dm_{B_{20R}(x)},
\]

where \( C_{3,n} := \exp\left(C_{2,n} \left( 1 + 20C_{1,n} \right) \right) \).

Let us take a maximal family \( \{ B_R(x_i) \}_{i \in I} \) of disjoint balls centered at \( \varphi_\lambda^{-1}(0) \). The maximality leads to \( \varphi_\lambda^{-1}(0) \subset \bigcup_{i \in I} B_{2R}(x_i) \); in particular,

\[
(3.5) \quad M = B_R(\varphi_\lambda^{-1}(0)) = \bigcup_{i \in I} B_{3R}(x_i).
\]
By the Bishop-Gromov volume comparison, there is a constant $C_{4,n} > 0$ depending only on $n$ such that the multiplicity of $\{B_{20R}(x_i)\}_{i \in I}$ is at most $C_{4,n}$. Furthermore, the standard covering argument tells us that

\begin{equation}
\sum_{i \in I} \int_{B_{20R}(x_i)} \|\nabla \varphi_{\lambda}\|^2 \, dv_M \leq C_{4,n} \int_M \|\nabla \varphi_{\lambda}\|^2 \, dv_M
= C_{4,n} \lambda \int_M \varphi_{\lambda}^2 \, dv_M.
\end{equation}

We define $J \subset I$ by the set of all $i \in I$ satisfying

\begin{equation}
\int_{B_{20R}(x_i)} \|\nabla \varphi_{\lambda}\|^2 \, dm_{B_{20R}(x_i)} \leq C_{4,n} \frac{\lambda}{\xi} \int_M \varphi_k^2 \, dm_M,
\end{equation}

and $J' := I \setminus J$. Note that if $i \in J$, then we deduce

\begin{equation}
\sup_{B_{10R}(x_i)} \|\nabla \varphi_{\lambda}\|^2 \leq C_{5,n} \Lambda
\end{equation}

from (3.4) and (3.7), where $C_{5,n} := (C_{3,n} C_{4,n})^{\frac{1}{2}}$ and $\Lambda := (\lambda \xi^{-1})^{\frac{1}{2}} \|\varphi_{\lambda}\|_2$. We set

\[ \Omega := \bigcup_{i \in J} B_{3R}(x_i), \quad \Omega' := \bigcup_{i \notin J} B_{3R}(x_i). \]

We will verify that $\Omega$ is a desired Borel subset. By the definition of $J'$,

\[ C_{4,n} \frac{\lambda}{\xi} \int_M \varphi_{\lambda}^2 \, dv_M \sum_{i \in J'} m_M(B_{20R}(x_i)) \leq \sum_{i \in J'} \int_{B_{20R}(x_i)} \|\nabla \varphi_{\lambda}\|^2 \, dv_M. \]

This together with (3.6) implies $\sum_{i \in J'} m_M(B_{20R}(x_i)) \leq \xi$, and hence $m_M(\Omega') \leq \xi$. On the other hand, $M = \Omega \cup \Omega'$ since (3.5). Therefore, one can conclude $m_M(\Omega) \geq 1 - \xi$.

Now, we check the inclusion (3.3) for $C_n = C_{5,n}$. For $r > 0$, we fix $x \in B_r(\varphi_{\lambda}^{-1}(0)) \cap \Omega$. Then the following hold:

1. there exists $x_0 \in M$ with $\varphi_{\lambda}(x_0) = 0$ such that $d_M(x, x_0) \leq r$;
2. there is $i_0 \in J$ with $\varphi_{\lambda}(x_{i_0}) = 0$ such that $d_M(x, x_{i_0}) \leq 3R$.

Let us consider the case where $r \in (0, 3R]$. We take a minimal geodesic $\gamma : [0, d_M(x, x_0)] \to M$ from $x$ to $x_0$. Using the triangle inequality and $d_M(x, x_0) \leq 3R$, we see that $\gamma$ lies in $B_{10R}(x_{i_0})$. By $\varphi_{\lambda}(x_0) = 0$, the Cauchy-Schwarz inequality, $d_M(x, x_0) \leq r$ and (3.8), we obtain

\[ |\varphi_{\lambda}(x)| \leq \int_0^{d_M(x, x_0)} \|\nabla \varphi_{\lambda}\|(\gamma(t)) \, dt \leq r \sup_{B_{10R}(x_{i_0})} \|\nabla \varphi_{\lambda}\| \leq C_{5,n} \Lambda r, \]

and this proves (3.3). When $r \in (3R, \infty)$, one can prove (3.3) by taking a minimal geodesic from $x$ to $x_{i_0}$, and by a similar argument to that in the case where $r \in (0, 3R]$. We complete the proof. □
We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** We assume \( \text{Ric}_M \geq -(n - 1) \). By Lemma 3.3, there is a constant \( C_n > 0 \) depending only on \( n \) such that the following holds: If we have \( \lambda \geq C_n \), then for every \( \xi \in (0, 1) \), there exists a Borel subset \( \Omega \subset M \) with \( m_M(\Omega) \geq 1 - \xi \) such that for every \( r > 0 \)

\[
\Omega \cap B_r(\varphi^{-1}(0)) \subset \Omega \cap \left\{ |\varphi_\lambda| \leq C_n \sqrt{\frac{\lambda}{\xi}} \|\varphi_\lambda\|_2 r \right\},
\]

where \( C_n > 0 \) is a constant depending only on \( n \).

By (3.9), for every \( r > 0 \) we see

\[
\Omega \cap B(C_n \Lambda^{-1}, r(\varphi^{-1}_\lambda(0))) \subset \Omega \cap \{ |\varphi_\lambda| \leq r \},
\]

where \( \Lambda := (\lambda \xi^{-1})^{\frac{1}{2}} \|\varphi_\lambda\|_2 \). It follows that

\[
m_M(\Omega \cap \{ |\varphi_\lambda| > r \}) \leq m_M\left( \Omega \setminus B(C_n \Lambda^{-1}, r(\varphi^{-1}_\lambda(0))) \right) \leq m_M\left( M \setminus B(C_n \Lambda^{-1}, r(\varphi^{-1}_\lambda(0))) \right).
\]

Due to Theorem 1.1, we arrive at the desired inequality

\[
m_M(\Omega \cap \{ |\varphi_\lambda| > r \}) \leq \exp\left( 1 - \sqrt{\Lambda} (C_n \Lambda)^{-1} r \right)
= \exp\left( 1 - \frac{C_n^{-1} \sqrt{\xi}}{\|\varphi_\lambda\|_2} r \right).
\]

This completes the proof of Theorem 1.2. \( \square \)

3.3. **Proof of Corollary 1.3.** We finally prove Corollary 1.3.

**Proof of Corollary 1.3.** Let \( p \in [1, \infty) \). We assume \( \text{Ric}_M \geq -(n - 1) \). By Theorem 1.1, there exists a constant \( C_n > 0 \) depending only on \( n \) such that the following holds: If \( \lambda \geq C_n \), then for every \( \xi \in (0, 1) \), there exists a Borel subset \( \Omega \subset M \) with \( m_M(\Omega) \geq 1 - \xi \) such that

\[
m_M(\Omega \cap \{ |\varphi_\lambda| > r \}) \leq \exp\left( 1 - \frac{C_n \sqrt{\xi}}{\|\varphi_\lambda\|_2} r \right)
\]

for every \( r > 0 \), where \( C_n > 0 \) is a constant depending only on \( n \).

The Cavalieri principle yields

\[
\int_{\Omega} |\varphi_\lambda|^p d m_M = \int_0^\infty m_M(\Omega \cap \{ |\varphi_\lambda|^p > t \}) dt.
\]

By letting \( r := t^{\frac{1}{p}} \), and by change of variables, we have

\[
\int_{\Omega} |\varphi_\lambda|^p d m_M = p \int_0^{\infty} m_M(\Omega \cap \{ |\varphi_\lambda| > r \}) r^{p-1} dr.
\]
Using (3.10), and change of variables again, we arrive at
\[
\int_{\Omega} |\varphi_{\lambda}|^p \, dm_M \leq e^p \int_0^\infty \exp \left(-\frac{C_n \sqrt{\xi}}{\|\varphi_{\lambda}\|_2} r\right) r^{p-1} \, dr
\]
\[= e^p \left(\frac{\|\varphi_{\lambda}\|_2^2}{C_n \sqrt{\xi}}\right)^p \int_0^\infty \exp (-s) \, s^{p-1} \, ds,
\]
where \( s := (C_n \sqrt{\xi} \|\varphi_{\lambda}\|_2^{-1}) \). Hence we have
\[
\left(\int_{\Omega} |\varphi_{\lambda}|^p \, dm_M \right)^{\frac{1}{p}} \leq e^\frac{1}{p} \left( p \Gamma(p) \right)^{\frac{1}{p}} \frac{\|\varphi_{\lambda}\|_2}{C_n \sqrt{\xi}} \leq e \Gamma(p + 1)^{\frac{1}{p}} \frac{\|\varphi_{\lambda}\|_2}{C_n \sqrt{\xi}}.
\]
Thus, we complete the proof of Corollary 1.3. \( \square \)

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Division of Mathematics & Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, 6-3-09 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980-8579, Japan
E-mail address: kfunano@tohoku.ac.jp

Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
E-mail address: sakurai@iam.uni-bonn.de