GOLDBACH’S PROBLEM IN PRIMES WITH BINARY EXPANSIONS OF A SPECIAL FORM

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Abstract. Let \( \mathbb{N}_0 \) be a class of natural numbers whose binary expansions contain even numbers of ones. Goldbach’s problem in numbers of class \( \mathbb{N}_0 \) is solved.

1. Introduction

Let \( n = e_0 + e_1 2 + \ldots + e_k 2^k \) be a binary expansion of a natural number \( n \), \( (e_j = 0, 1) \). Let \( \mathbb{N}_0 \) be a set of natural numbers whose binary expansions have an even number of ones, \( \mathbb{N}_1 = \mathbb{N} \setminus \mathbb{N}_0 \). Let

\[
\varepsilon(n) = \begin{cases} 
1, & \text{if } n \in \mathbb{N}_0; \\
-1, & \text{if } n \in \mathbb{N}_1.
\end{cases}
\]

In 1968, A.O. Gelfond [1] proved that numbers from the sets \( \mathbb{N}_0 \) and \( \mathbb{N}_1 \) are regularly distributed in arithmetical progressions.

In 1991, The author got [2] the asymptotical formula for the sum

\[
\sum_{n \leq x, n \in \mathbb{N}_0} \tau(n)
\]

and so solved Dirichlet divisors problem in the numbers of class \( \mathbb{N}_0 \).

In 2010, C. Mauduit and J. Rivat [3] proved in particular that the densities of sets of primes of the classes \( \mathbb{N}_0 \) and \( \mathbb{N}_1 \) are equal to each other. B. Green gave another proof of this fact [4]. These papers are based on estimates of exponential sums of a special type, which, by the force and by methods of proofs, are variants of estimate, derived by the author in 1991, of the integral of modulus of a trigonometric sum of the special type [2].

In this paper the ternary Goldbach problem in prime numbers of the set \( \mathbb{N}_0 \) is solved.

The main results are contained in the following theorems.

Theorem 1. Let \( \alpha \) be an arbitrary real number. There exists an absolute constant \( \kappa > 0 \) such that

\[
S = \sum_{n \leq X} \varepsilon(n)\Lambda(n)e^{2\pi i \alpha n} = O(X^{1 - \kappa}).
\]

The constant in sign \( O \) is absolute.

Theorem 2. Let \( J(N) \) be the number of representations of odd \( N \) by sum of three primes, and \( J_0(N) \) be the number of representations of odd \( N \) by sum of three primes from the set \( \mathbb{N}_0 \).

Then the equality

\[
J_0(N) = \frac{1}{8} J(N)(1 + O(N^{-\kappa} \ln N)),
\]

holds, where \( \kappa > 0 \) is a constant from theorem 1.

Key words and phrases. Goldbach’s problem, Gelfond’s problem, binary expansion, sequence of natural numbers, trigonometric sum, complex-valued function, inequality of the large sieve.
2. Auxiliary lemmas

Lemma 1. Let

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \ (a, q) = 1, \ q \geq 1, \ |\theta| \leq 1.$$ 

Then for any $\beta \in \mathbb{R}, \ U > 0, \ P \geq 1$ we have

$$\sum_{x=1}^{P} \min \left( U, \|\alpha x + \beta\|^{-1} \right) \leq 6 \left( \frac{P}{q} + 1 \right) (U + q \log q).$$

Proof see in [5, chapter 4].

Lemma 2. (A.O. Gelfond) Let $Q \in \mathbb{N}$. The inequality

$$\left| \prod_{r=0}^{2Q-1} \left( 1 - e^{i2\pi 2^r} \right) \right| \leq \frac{2}{\sqrt{3}} q^{2Q} \lambda,$$

holds, where $\lambda = \frac{\ln 3}{\ln 4} = 0, 7924812\ldots$

Proof see in [6].

Corollary 1. For any $\alpha \in \mathbb{R}$ the estimate

$$\left| \sum_{n \leq X} \varepsilon(n)e^{2\pi i\alpha n} \right| = O(X^\lambda \ln X).$$

holds.

Proof. Define natural number $Q$ with inequalities

$$2^{2(Q-1)} < X + 1 \leq 2^Q.$$

Then

$$\left| \sum_{n \leq X} \varepsilon(n)e^{2\pi i\alpha n} \right| = \left| \sum_{n < 2^Q} \varepsilon(n)e^{2\pi i\alpha n} \sum_{n_1 \leq X} 1_{2^Q} \sum_{l=1}^{2^Q} e^{2\pi i(l-n_1)2^{-Q}} \right| \leq$$

$$\leq 2^{-2Q} \sum_{l=1}^{2^Q} \left| \sum_{n < 2^Q} \varepsilon(n)e^{2\pi i(l-2^{-Q})} \right| \left| \sum_{n_1 \leq X} e^{-2\pi in_1l2^{-Q}} \right|.$$

Furthermore, it follows from from inequality

$$\sum_{n < 2^Q} \varepsilon(n)e^{2\pi i(l-2^{-Q})} = \prod_{r=0}^{2Q-1} \left( 1 - e^{2\pi i(l+2^{-Q})} \right)^2$$

and lemma 2 that

$$\left| \sum_{n \leq X} \varepsilon(n)e^{2\pi i\alpha n} \right| \ll X^\lambda 2^{-2Q} \sum_{l=1}^{2^Q} \min(X, \|l2^{-Q}\|^{-1}).$$

From this and Lemma 1 we have Corollary 1.

Lemma 3. (Gallagher). Let $S(t)$ be a complex valued function with continuous first derivative on $[t_0, t_k]$ and, $t_0 < t_1 < \ldots < t_{k-1} < t_k.$

Then, assuming that $\delta = \min_{0 \leq r < k} (t_{r+1} - t_r), we have
\[
\sum_{r=1}^{k} |S(t_r)| \leq \frac{1}{\delta} \int_{t_0}^{t_k} |S(t)| \, dt + \frac{1}{2} \int_{t_0}^{t_k} |S'(t)| \, dt.
\]

Proof see in [6, chapter 1].

3. The main lemma and its corollaries

**Lemma 4.** Let \( Q \in \mathbb{N}, \)

\[
S_Q(a) = \prod_{r=0}^{2Q-1} \left( 1 - e^{2\pi i 2^r} \right).
\]

The inequality

\[
\int_{0}^{1} |S_Q(\alpha)| \, d\alpha \leq 2^{2Q} \theta_0,
\]

holds, where \( \theta_0 = \log_2 \sqrt{2 + \sqrt{2}} = 0,88577 \ldots \)

Proof see in [2].

**Corollary 2.** The inequality

\[
\int_{0}^{1} \left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i n \alpha} \right| \, d\alpha \leq X^{1/2} \ln X.
\]

Proof. Let \( Q \in \mathbb{N}, 2^{2(Q-1)} < X + 1 \leq 2^Q \)

Then

\[
\int_{0}^{1} \left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i n \alpha} \right| \, d\alpha \ll 2^{-2Q} \sum_{l=1}^{2Q} \min(X, \|l2^{-2Q}\|^{-1}) \int_{0}^{1} |S_Q(\alpha + l2^{-2Q})| \, d\alpha.
\]

Since \( S_Q(\alpha) \) is a periodic function of \( t \) with period 1,

\[
\int_{0}^{1} |S_Q(\alpha + l2^{-2Q})| \, d\alpha = \int_{0}^{1} |S_Q(\alpha)| \, d\alpha.
\]

By lemma 1,

\[
2^{-2Q} \sum_{l=1}^{2Q} \min(X, \|l2^{-2Q}\|^{-1}) \ll \ln X.
\]

The assertion of Corollary 2 follows from Lemma 4.

**Corollary 3.** \( k \in \mathbb{N}. \) The estimate

\[
2^{-k} \sum_{r=0}^{2k-1} \left| \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi i \frac{rx}{2k}} \right| \ll k 2^{\theta k}
\]

holds.

Proof. Applying Lemma 3, putting it

\[
S(t) = \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi i tx}, \ t_r = \frac{r}{2k},
\]
Then
\[ S'(t) = 2\pi i \sum_{x=0}^{2^k-1} x\varepsilon(x)e^{2\pi i tx}, \]
\[ \delta = 2^{-k}. \]

Then
\[ 2^{-k} \sum_{r=0}^{2^k-1} \left| \sum_{x=0}^{2^k-1} \varepsilon(x)e^{2\pi i tx} \right| = 2^{-k} \sum_{r=0}^{2^k-1} |S(t_r)| \leq \]
\[ \leq \int_0^1 |S(t)| \, dt + 2^{-k} \int_0^1 |S'(t)| \, dt. \]

Apply Abel’s transform to \( S'(t) \):
\[ 2^{-k} \int_0^1 |S'(t)| \, dt \ll 2^{-k} \int_0^1 \left| \sum_{x=0}^{2^k-1} \varepsilon(x)e^{2\pi i tx} \right| \, dt + \]
\[ + 2^{-k} \int_0^1 \int_0^{2^k-1} \left( \left| \sum_{r \leq m} \varepsilon(x)e^{2\pi i tx} \right| \, du \right) \ll \int_0^1 \left| \sum_{x \leq u_0} \varepsilon(x)e^{2\pi i tx} \right| \, dt, \]

where \( u_0 \) is a number from the segment \([0, 2^k - 1]\) such that the last integral reaches its maximum. Now, Corollary 3 follows from Corollary 2.

**Corollary 4.** Let \( k \geq t \) be integers, \( 0 \leq t \leq k \). Suppose that the inequality \( m \in \mathbb{N} \) holds. Then
\[ \hat{\varepsilon}_m(r) = 2^{-m} \sum_{x=0}^{2^{m-1}} \varepsilon(m)e^{-2\pi i \frac{mx}{y}}, \]

Let \( a \) is any number of the segment \([0, 2^k - 1]\). Then
\[ \sum_{r \equiv a \pmod{2^k}} \left| \hat{\varepsilon}_k(r) \right| \ll 2^{(0,5-c)(k-t)} |\hat{\varepsilon}_t(a)| k, \]

where \( c = 1/2 - \theta_0/2, \theta_0 \) is a number from lemma 4.

Proof. By definition we have
\[ \sum_{r \equiv a \pmod{2^k}} \left| \hat{\varepsilon}_k(r) \right| = 2^{-k} \sum_{r \equiv a \pmod{2^k}} \left| \sum_{x=0}^{2^{k-1}} \varepsilon(x)e^{2\pi i \frac{mx}{y}} \right| = \]
\[ = 2^{-k} \sum_{r \equiv a \pmod{2^k}} \left| \sum_{y=0}^{2^{k-t}-1} \varepsilon(x+2^{k-t}y)e^{2\pi i \frac{mx+y}{2^t}} (x+2^{k-t}y) \right|. \]

Since \( \varepsilon(x+2^{k-t}y) = \varepsilon(x)\varepsilon(y) \), we have
\[ \sum_{r \equiv a \pmod{2^k}} \left| \hat{\varepsilon}_k(r) \right| = 2^{-(k-t)} \sum_{r \equiv a \pmod{2^k}} \left| \sum_{x=0}^{2^{k-t}-1} \varepsilon(x)e^{2\pi i \frac{mx+y}{2^t}} \right| 2^{-t} \sum_{y=0}^{2^{k-t}-1} \varepsilon(y)e^{2\pi iy/t} = \]
\[
2^{-(k-t)} \sum_{r(\mod 2^{k-t})} \left| \sum_{x=0}^{2^{k-t}-1} \varepsilon(x) e^{2\pi i \left( \frac{x}{2^{k-t-1}} + \frac{a}{x} \right)} \right| \varepsilon_t(a).
\]

The sum on the right side of this inequality is estimated in the same way as a similar amount of Corollary 3.

### 4. Proof of Theorem 1

Using Vaughan’s identity (see, e.g. \[5\] chapter 3, problem 9), with \(u = X^{0.1}\):

\[
S = \sum_{n \leq X} \Lambda(n) \varepsilon(n) = W_1 - W_2 - W_3 + O(u \ln u),
\]

where

\[
W_1 = \sum_{d \leq u} \mu(d) \sum_{n \leq X} \varepsilon(dn) e^{2\pi i \alpha dn} \ln n,
\]

\[
W_2 = \sum_{d \leq u} \mu(d) \sum_{n \leq u} \Lambda(n) \sum_{dn \leq X} \varepsilon(dn) e^{2\pi i \alpha dn},
\]

\[
W_3 = \sum_{u < m \leq X} \alpha_m \sum_{u < n \leq u \cdot m^{-1}} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn},
\]

\[
\alpha_m = \sum_{d|m,d \leq u} \mu(d).
\]

Sums \(W_1\) and \(W_2\) are estimated in the same way. Estimate \(W_1\).

Fix \(d \leq u\). Apply to the inner sum, which we denote \(S_1(d)\), the Abel transform, we obtain:

\[
|S_1(d)| \ll \left| \sum_{dn \leq u_0} \varepsilon(dn) e^{2\pi i \alpha dn} \right| \log X,
\]

where \(u_0\) is a number not exceeding \(X\).

Furthermore,

\[
\sum_{dn \leq u_0} \varepsilon(dn) e^{2\pi i \alpha dn} = \sum_{m \leq u_0} \varepsilon(m) e^{2\pi i \alpha m} \frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i \frac{bm}{d}}.
\]

The sum over \(m\) estimate by Corollary 1:

\[
\left| \sum_{m \leq u_0} \varepsilon(m) e^{2\pi i \left( \alpha + \frac{b}{d} \right) m} \right| \ll X^\lambda \ln X,
\]

where \(\lambda = 0.792\ldots\)

Thus, for any \(d \leq u\) we have

\[
|S_1(d)| \ll X^\lambda \ln X,
\]

therefore,

\[
|W_1| \ll u X^\lambda \ln X.
\]
Similarly, we arrive at the estimate

$$|W_2| \ll u^2 X^3 \ln X.$$

The parameter $u$ is chosen so that

$$|W_1| \ll X^{1-\varepsilon}, |W_2| \ll X^{1-\varepsilon},$$

where $\varepsilon > 0$ is an absolute constant.

We now estimate $W_3$. Divide the interval of summation over $m$ in $O(\ln X)$ intervals of the form $(\frac{M}{2}, M]$, where $u < \frac{M}{2} < M \leq \frac{X}{u}$; one of these intervals may be incomplete. So, we have:

$$W_3 = \ln X \sum_M W_3(M),$$

where

$$W_3(M) = \sum_{M/2 < m \leq M_1} a_m \sum_{u < n \leq X/m} \Lambda(n) \varepsilon(mn)e^{2\pi i \alpha mn},$$

where $M/2 < M_1 \leq M$.

Furthermore,

$$W_3(M) = \sum_{M/2 < m \leq M_1} a_m \sum_{u < n \leq X/M_1} \Lambda(n) \varepsilon(mn)e^{2\pi i \alpha mn} +$$

$$+ \sum_{M/2 < m \leq M_1} a_m \sum_{X/M_1 < n \leq X/m} \Lambda(n) \varepsilon(mn)e^{2\pi i \alpha mn}.$$

Splitting the interval of summation over $n$ in $O(\log X)$ intervals of the form $(\frac{N}{2}, N_1]$, where $\frac{N}{2} < N_1 \leq N$, $u < N \leq \frac{N}{u}$, we arrive at the inequality

$$|W_3| \ll |W_3(M, N)| \ln^2 X,$$

where

$$W_3(M, N) = \sum_{M/2 < m \leq M_1} a_m \sum_{N/2 < n \leq N_1} \Lambda(n) \varepsilon(mn)e^{2\pi i \alpha mn},$$

where $u < M/2 < M_1 \leq M \leq Xu^{-1}$, $u < N/2 < N_1 \leq N \leq Xu^{-1}$; it may be that $N_1 = X/m$.

Without loss of generality, we assume that $M \leq N$.

Using the fact that $|a_m| \leq \tau(m) \ll m$ we apply the Cauchy inequality:

$$|W_3(M, N)|^2 \ll M^{1+\varepsilon} \sum_{M/2 < m \leq M_1} \sum_{N/2 < n \leq N_1} \Lambda(n) \varepsilon(mn)e^{2\pi i \alpha mn}.$$ 

Let $H = [X^\rho]$, where $0 < \rho < 10^{-3}$ - a small parameter to be chosen later. We apply van der Corput’ inequality (see, eg, [4], [1]):

$$\left| \sum_{N/2 < n \leq N_1} \Lambda(n) \varepsilon(mn)e^{2\pi i \alpha mn} \right|^2 \ll \frac{N}{H} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H}\right) \times$$

$$\times \sum_{N/2 < n \leq N_1, N/2 < n + h \leq N_1} \Lambda(n)\Lambda(n + h) \varepsilon(mn)\varepsilon(mn + mh)e^{2\pi i \alpha mn}e^{-2\pi i \alpha (n + h)}.$$
The contribution of $h = 0$ is estimated as $O\left(\frac{N^2}{H}\right)$, so

$$|W_3(M, N)|^2 \ll \frac{X^{1+\varepsilon}}{H} \sum_{h=1}^{H} \sum_{N/2 < n < N} \left| \sum_{M/2 < m < M_1} \varepsilon(mn)\varepsilon(mn + mh)e^{-2\pi i \alpha m(n + h)} \right| + \frac{X^{2+\varepsilon}}{H}.$$  

Fix $h \in [1, H]$. Now it is sufficient to prove that

$$W_4(M, N) = \sum_{N/2 < n < N} \left| \sum_{M/2 < m < M_1} \varepsilon(mn)\varepsilon(mn + mh)e^{-2\pi i \alpha m(n + h)} \right| \ll X^{1-\varepsilon}.$$  

Choose a positive integer $k$ from the inequalities

$$2^{k-1} < M X^{2\rho} \leq 2^k.$$  

Introduce the symbol $\varepsilon_k(n)$:

$$\varepsilon_k(n) = \begin{cases} 1, & \text{if the sum of the first } k \text{ binary digits of } n \text{ is even;} \\ -1, & \text{otherwise.} \end{cases}$$  

Prove that

$$|W_4(M, N)| \ll \sum_{N/2 < n < N} \left| \sum_{M/2 < m < M_1} \varepsilon_k(mn)\varepsilon_k(mn + mh)e^{-2\pi i \alpha m(n + h)} \right| + X^{1-\rho+\varepsilon}.$$  

Divide $mn$ by $2^k$ with remainder: $mn = 2^k q + r$, $0 \leq r < 2^k$. If $r < 2^k - 2MH$, then $mn + mh = 2^k q + r + mh$, $0 < r + mh < 2^k$.

For such $mn$, we have

$$\varepsilon(mn)\varepsilon(mn + mh) = \varepsilon(r)\varepsilon(r + mh) = \varepsilon_k(mn)\varepsilon_k(mn + mh).$$  

The number of pairs $(m, n)$ such that the remainder $r$ lies between $2^k - 2MH$ and $2^k - 1$ is $O(X^{2-k}MHX^{\varepsilon}) = O(X^{1-\rho+\varepsilon})$.

Now we estimate

$$|W_5(M, N)| = \sum_{N/2 < n < N} \left| \sum_{M/2 < m < M_1} \varepsilon_k(mn)\varepsilon_k(mn + mh)e^{-2\pi i \alpha hm} \right|.$$  

Introduce the discrete Fourier transform for the character $\varepsilon_k(r)$:

$$\hat{\varepsilon}_k(r) = 2^{-k} \sum_{l=0}^{2^k-1} \varepsilon_k(l)e^{-2\pi i \frac{rl}{2^k}}.$$  

From this definition it follows that

$$\varepsilon_k(mn) = \sum_{r=0}^{2^k-1} \hat{\varepsilon}_k(r) \exp\left\{ \frac{2\pi i rmn}{2^k} \right\}, \quad \varepsilon_k(mn + mh) = \sum_{s=0}^{2^k-1} \hat{\varepsilon}_k(s) \exp\left\{ \frac{2\pi is(mn + mh)}{2^k} \right\}.$$  

Summing linear sums over $m$, we obtain:

$$|W_5(M, N)| \leq \sum_{r=0}^{2^k-1} \sum_{s=0}^{2^k-1} |\hat{\varepsilon}_k(r)||\hat{\varepsilon}_k(s)| \sum_{N/2 < n < N} \min\left( M, \left\| \frac{r + s}{2^k} n + \frac{hs}{2^k} - h\alpha \right\|^{-1} \right).$$  

From now on we will assume that $\rho = \frac{1}{\theta_0}$, where $\theta = \frac{1-h\theta_0}{2}$, $\theta_0$ is a constant from lemma 4. Let $t$ – non-negative integer such that $2^t \|(r + s)$.
Suppose first $0 \leq t \leq k - \frac{2\rho}{c} \log_2 X$. Note that from inequality $2^k > M \geq X^{1/10}$, it follows that $k > \frac{4}{t} \log_2 X$; this and $\frac{2\rho}{c} = \frac{1}{100}$ implies the inequality $k - \frac{2\rho}{c} \log_2 X \geq \frac{1}{5} k$.

We apply Lemma 1 with $q = 2^{k-t}$, $\alpha = \frac{8}{2k}$, $\beta = \frac{sh}{2k} - \alpha h$ to the sum
\[
\sum_{N/2 < n \leq N} \min \left( \frac{N}{2^{k-t} + 1} (M + k2^{k-t}) \right).
\]

Simplify the right-hand side of this inequality.
We have:
\[
\frac{N}{2^{k-t}} + 1 < \frac{NX^{2\rho}}{2^{k-t}} + 1 \leq 2 \frac{NX^{2\rho}}{2^{k-t}};
\]
and so,
\[
\left( \frac{N}{2^{k-t}} + 1 \right) (M + k2^{k-t}) < 4MX^{2\rho} + k2^{k-t} < 2MX^{2\rho};
\]

Now estimate the sum
\[
\sum_{2^k-1}^{2^k} \left| \hat{\varepsilon}_k(r) \right| \left| \hat{\varepsilon}_k(s) \right| \leq \sum_{r=0}^{2^{k-1}} \sum_{s=0}^{2^{k-1}} \left| \hat{\varepsilon}_k(r) \right| \left| \hat{\varepsilon}_k(s) \right| \sum_{r=0}^{2^{k-1}} \sum_{s=0}^{2^{k-1}} \left| \hat{\varepsilon}_k(r) \right| \left| \hat{\varepsilon}_k(s) \right|.
\]
Use the corollary 4:
\[
\sum_{r=0}^{2^{k-1}} \sum_{s=0}^{2^{k-1}} \left| \hat{\varepsilon}_k(r) \right| \left| \hat{\varepsilon}_k(s) \right| \ll 2^{(1-2\rho)(k-t)} \log^2 X \sum_{a=0}^{2^{k-1}} \left| \hat{\varepsilon}_k(a) \right|^2 = 2^{(1-2\rho)(k-t)} n^2 X.
\]
In this case, the estimate is achieved
\[
|W_5(M, N)| \ll X^{1+6\rho}2^{-2\rho(k-t)} n^2 X.
\]
Recall that
\[
k - t \geq \frac{4}{5} k, \quad 2^k > M \geq X^{1/10},
\]
it follows that
\[
|W_5(M, N)| \ll X^{1+6\rho} - \frac{6\rho}{5c};
\]
finally, from the inequality $\rho \leq \frac{c}{100}$ we have $6\rho - \frac{8}{50} c < -\rho$, 
\[
|W_5(M, N)| \ll X^{1-\rho}.
\]
It remains to consider the case
\[
k - \frac{2\rho}{c} \log_2 X < t \leq k.
\]
We have:

\[ |W_5(M, N)| \leq \sum_{r=0}^{2k-1} \sum_{s=0}^{2k-1} |\hat{\varepsilon}(r)||\hat{\varepsilon}(s)| \min\left(M, \left\| \frac{r+s}{2^k} n + \frac{sh}{2^k} - \alpha h \right\|^{-1}\right) = \]

\[ = \sum_{N/2 < n \leq N} \sum_{s=0}^{2k-1} \sum_{r=0}^{2k-1} |\hat{\varepsilon}(r_1 + 2^t r_2)||\hat{\varepsilon}(s_1 + 2^t s_2)| \times \]

\[ \times \min\left(M, \left\| \frac{r_1 + s_1 + 2^t(r_2 + s_2)}{2^k} n + \frac{(s_1 + 2^t s_2)h}{2^k} - \alpha h \right\|^{-1}\right). \]

It follows from the inequalities $0 \leq s_1; r_1 < 2^t$ and congruence $r_1 + s_1 \equiv 0 \pmod{2^t}$ that either $r_1 = s_1 = 0$ or $r_1 + s_1 = 2^t$.

From this and Lemma 2 we get

\[ |W_5(M, N)| \ll 2^{-2k(1-\lambda)} \sum_{N/2 < n \leq N} \sum_{s=0}^{2k-1} \sum_{r=0}^{2k-1} \sum_{s_1=0}^{2^k-1} \min\left(M, \left\| \frac{h s_1}{2^k} + \beta \right\|^{-1}\right) + \]

\[ + X 2^{-2k(1-\lambda)} 2^{2k(t-t)} \tag{1} \]

where $\beta = \frac{1 + r + s_1}{2^k} n + \frac{h s_2}{2^k} - \alpha h$.

Let $\frac{h s_1}{2^k} = \frac{h s_2}{2^k}$, where $(h s_1, 2) = 1$. From Lemma 1 and inequalities $2^{-k_1} \leq 2^{-k} X^{\rho}$, $M + 2^{k_1} k_1 \ll M X^{3\rho}$ it follows that

\[ \sum_{s_1=0}^{2^k-1} \min\left(M, \left\| \frac{h s_1}{2^k} + \beta \right\|^{-1}\right) \ll M X^{5\rho}. \]

Substituting this inequality in (1):

\[ |W_5(M, N)| \ll X^{1+5\rho} 2^{2k(t-t)} 2^{-2k(1-\lambda)}. \]

Now use the fact that

\[ 2^{k-t} \leq X^{4\rho/c}, \quad \frac{\rho}{c} \leq \frac{1}{200}, \quad c < 0.06, \quad 2^{k} > M \geq X^{0.1}. \]

We got:

\[ |W_5(M, N)| \ll X^{1-0.01}. \]

Theorem 1 is proved.

5. Proof of Theorem 2

Define sums $S(\alpha)$ and $S_0(\alpha)$:

\[ S(\alpha) = \sum_{p \leq N} e^{2\pi i \alpha p}, \quad S_0(\alpha) = \sum_{p \leq N} \varepsilon(p) e^{2\pi i \alpha p}. \]

Then

\[ J_0(N) = \frac{1}{8} \int_0^1 (S(\alpha) + S_0(\alpha))^3 e^{-2\pi i \alpha N} d\alpha. \]
Expanding the brackets and using Theorem 1 and Cauchy’s inequality, we obtain

\[ J_0(N) = \frac{1}{8} \int_0^1 S^3(\alpha)e^{-2\pi i \alpha N} d\alpha + O(\pi(N)N^{1-\varepsilon}). \]

Since

\[ J(N) = \int_0^1 S^3(\alpha)e^{-2\pi i \alpha N} d\alpha, \quad J(N) \gg N^2(\ln N)^{-3} \]

(for sufficiently large odd \( N \)), theorem 2 is proved.

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