Functional Analysis for Helmholtz Equation in the Framework of Domain Decomposition

Mikhael Balabane*

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Abstract

This paper gives a geometric description of functional spaces related to Domain Decomposition techniques for computing solutions of Laplace and Helmholtz equations. Understanding the geometric structure of these spaces leads to algorithms for solving the equations. It leads also to a new interpretation of classical algorithms, enhancing convergence. The algorithms are given and convergence is proved. This is done by building tools enabling geometric interpretations of the operators related to Domain Decomposition technique. The Despres operators, expressing conservation of energy for Helmholtz equation, are defined on the fictitious boundary and their spectral properties proved. It turns to be the key for proving convergence of the given algorithm for Helmholtz equation in a non-dissipating cavity. Using these tools, one can prove that the Domain Decomposition setting for the Helmholtz equation leads to an ill-posed problem. Nevertheless, one can prove that if a solution exists, it is unique. And that the algorithm do converge to the solution.

1 Introduction

In the framework of domain decomposition, given a bounded open set \( \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma \) where the two open sets \( \Omega_1 \) and \( \Omega_2 \) are not overlapping, and \( \Gamma \) a

*University Paris 13 - France - email: balabane@math.univ-paris13.fr
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common subset of their boundary (called the fictitious boundary), and given a (global) solution \( u \) of the Helmholtz equation

\[
\Delta u + k^2 u = f \in L^2(\Omega) \quad \text{and} \quad u \in H^1_0(\Omega)
\]

the aim of this paper is to understand the dynamics of the sequence \((v_1^n, v_2^n)_{n \in \mathbb{N}}\) solving separately the Helmholtz equations on \(\Omega_1\) and \(\Omega_2\), when equating the fluxes through \(\Gamma\): \((m = 1, 2 \text{ resp. } m' = 2, 1)\)

\[
\frac{\partial v_m^n}{\partial n_m} - i\gamma v_m^n = -\frac{\partial v_{m'}^{n-1}}{\partial n_{m'}} - i\gamma v_{m'}^{n-1} \quad \text{on } \Gamma
\]

Its ultimate aim is to prove convergence to \((u_{|\Omega_1}, u_{|\Omega_2})\) of the sequence \((u_1^n, u_2^n)_{n \in \mathbb{N}}\) solving the Helmholtz equations on \(\Omega_1\) and \(\Omega_2\) with a penalization on the boundary \(\Gamma\) is added, namely:

\[
\frac{\partial u_m^n}{\partial n_m} - i\gamma u_m^n = \theta[\frac{\partial u_{m-1}^{n-1}}{\partial n_m} - i\gamma u_{m-1}^{n-1}] - (1 - \theta)[\frac{\partial u_{m'}^{n-1}}{\partial n_{m'}} + i\gamma u_{m'}^{n-1}] \quad \text{on } \Gamma
\]

For this sake, the geometry of the set of solutions of the Helmholtz equation on \(\Omega_1 \times \Omega_2\) with equated energy fluxes is studied, through the study of the coupling operator defined on \(L^2(\Gamma) \times L^2(\Gamma)\) which intertwins the fluxes. It turns out that the key for understanding the convergence of the sequence \((u_1^n, u_2^n)_{n \in \mathbb{N}}\) is the analysis of the spectral properties of the intertwinning operator.

Using these tools, one can prove that the Domain Decomposition setting for the Helmholtz equation leads to an ill-posed problem. Nevertheless, one can prove that if a solution exists, it is unique. And that the algorithm do converge to the solution.

Convergence of the penalized algorithm is proven and numerical tests for solving the Helmholtz equation through this domain decomposition algorithm are given.

The geometric analysis given here provides the theoretical background for another numerical algorithm for computing the global solution \(u\), by a specific spectral method. A forthcoming paper describes and gives the numerical analysis of this algorithm.

This domain decomposition algorithm (in a dissipating cavity case, i.e. with a Sommerfeld-like radiation condition on part of the boundary), was first
initiated and studied by B.Despres in [D1] [D2] [BD], and computational results given by J.D.Benamou [B] [BD], F.Collino and P.Joly [CGJ].

In order to perform the geometric analysis of the set of solutions of the Helmholtz equation on $\Omega_1 \times \Omega_2$, one has first to make a complete description of the geometry of the set of solutions of the Laplace equation on $\Omega_1 \times \Omega_2$. Geometric properties of this set proven below makes it possible to revisit the classical penalized Dirichlet/Neumann domain decomposition algorithm (with penalization) for solving the Laplace equation. A new version of this algorithm is given here, and proved to converge to the global solution, enhancing the usual assumption on the penalization parameter.

This completes classical results by O.Widlund [PW], P.L.Lions [L], or A.Quarteroni and A.Valli [FMQT] [FQZ] [QV].

The paper is organized as follows: in section 2 basic facts are revisited, although classical, and completed in order to set the geometric framework needed. (It also makes the paper self contained). A precise study of duality, and the link with the Poincare-Steklov operators, is performed, which turns to be central for the remainder of the paper. In section 3 a new version of the Dirichlet/Neumann algorithm for the Laplace equation is given, and convergence is proved. In section 4 geometric tools for the Helmholtz equation, and related domain decomposition algorithm, are given. Despres operators are studied and their spectral properties investigated. As is the intertwinnig operator. In section 5, convergence of the domain decomposition algorithm for Helmholtz equation is proved. In section 6 numerical tests are given.

Throughout this paper, when dealing with the Helmholtz equation, the frequency $k$ is assumed to be non-resonnant for the Dirichlet boundary condition. More precisely we shall always make the following

Assumption (A) $-k^2$ is not an eigenvalue of the Laplace operator on $\Omega$ with Dirichlet boundary condition, i.e. the following problem is well posed for $f \in L^2(\Omega)$:

$$\Delta u + k^2u = f \quad \text{and} \quad u \in H^1_0(\Omega)$$

We shall also adopt the following

Notation (N) normal derivatives at the boundary of an open set are always meant as the derivative along the outward unit normal vector.
2 Basics

Let $\Omega \subset R^d$ be a bounded open set whose boundary $\partial \Omega$ is a $C^1$-submanifold of $R^d$. Let $\Gamma$ be an open $C^\infty$-submanifold of $R^d$, such that:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma, \quad \partial \Omega_1 = (\partial \Omega \cap \partial \Omega_1) \cup \Gamma, \quad \partial \Omega_2 = (\partial \Omega \cap \partial \Omega_2) \cup \Gamma$$

where $\Omega_1$ and $\Omega_2$ are open sets in $R^d$. We assume that $\Omega_1$ and $\Omega_2$ fulfill the strict cone property (see [Ag] for instance) and that $\Gamma$ is transverse to $\partial \Omega$ in the following sense: $\Gamma$ is a $C^1$-submanifold of $R^d$ with boundary, and there exists $a < 1$ such that for any $\sigma \in \partial \Omega \cap \Gamma$, we have:

$$-a \leq n_\Gamma(\sigma) \cdot n_{\partial \Omega}(\sigma) \leq a$$

(1)

where $n_\Gamma(\sigma) \in C^0(\Gamma)$ is a unit vector normal to $\Gamma$ at $\sigma$ and $n_{\partial \Omega}(\sigma) \in C^0(\partial \Omega)$ a unit vector normal to $\partial \Omega$ at $\sigma$.

2.1 Functional spaces associated to $\Gamma$

Let $H^1_0(\Omega)$ be endowed with the scalar product

$$(u, v)_{H^1_0(\Omega)} = \int_{\Omega} \nabla u \nabla v \, dx$$

For $m = 1, 2$ let $H^m = \{u \in H^1(\Omega_m); u|_{\partial \Omega \cap \partial \Omega_m} = 0\}$. Boundedness of the trace operators from $H^1_0(\Omega)$ to $H^{\frac{1}{2}}_0(\partial \Omega \cap \partial \Omega_m)$ imply that these are Hilbert spaces when endowed with the scalar products:

$$(u, v)_{H^m} = \int_{\Omega_m} \nabla u \nabla v \, dx$$

Let $\rho^\Gamma$ (resp. $\rho^\Gamma_m$ for $m = 1, 2$) be the trace operator on $\Gamma$, i.e. the bounded linear operator from $H^1_0(\Omega)$ (resp. $H^m$) to $H^{1/2}(\Gamma)$ which maps $u$ to $u|\Gamma$.

Let

$$\Lambda = \{u|\Gamma; u \in H^1_0(\Omega)\} = H^1_0(\Omega)/Ker \rho^\Gamma \simeq (Ker \rho^\Gamma)^\perp$$

and for $m = 1, 2$

$$\Lambda_m = \{u|\Gamma; u \in H^m\} = H^m/Ker \rho^\Gamma_m \simeq (Ker \rho^\Gamma_m)^\perp$$
Remark 1 Obviously $\text{Ker} \rho_m^\Gamma = H_0^1(\Omega_m)$, $\Lambda_m \subset H^{1/2}(\Gamma)$, $\Lambda \subset H^{1/2}(\Gamma)$

Because $\rho^\Gamma$ and $\rho_m^\Gamma$ are bounded, $\Lambda$ and $\Lambda_m$ are Hilbert spaces when endowed with the following norms:

\[ \forall \lambda \in \Lambda, \| \lambda \|_\Lambda = \inf_{\{u; u|_\Gamma = \lambda\}} \| u \|_{H^1_0(\Omega)} \quad \text{and} \quad \forall \lambda \in \Lambda_m, \| \lambda \|_{\Lambda_m} = \inf_{\{u; u|_\Gamma = \lambda\}} \| u \|_{H_m} \]

Remark 2 Obviously, for any $v \in H_0^1(\Omega)$ and $w \in H_m$

\[ \| \rho^\Gamma(v) \| \Lambda \leq \| v \|_{H_0^1(\Omega)} \quad \text{and} \quad \| \rho_m^\Gamma(w) \| \Lambda \leq \| w \|_{H_m} \]

Proposition 1 Let $m = 1, 2$. For any $\lambda \in \Lambda_m$, $\mu \in \Lambda_m$:

1- There exists a unique $u_\lambda^m \in H_m$ such that $\Delta u_\lambda^m = 0$ in $\Omega_m$ and $\rho_m^\Gamma(u_\lambda^m) = \lambda$

2- One has: $\| \lambda \|_{\Lambda_m} = \| u_\lambda^m \|_{H_m}$ and $(\lambda, \mu)_{\Lambda_m} = (u_\lambda^m, u_\mu^m)_{H_m}$

3- For any $\lambda \in \Lambda$, let $u_\lambda = u_\lambda^1$ on $\Omega_m$, $m = 1, 2$. Then

\[ \| \lambda \|^2_\Lambda = \| u_\lambda^1 \|^2_{H_1} + \| u_\lambda^2 \|^2_{H_2} \]

4- Using the previous notation, for any $\lambda \in \Lambda$, $\mu \in \Lambda$

\[ (\lambda, \mu)_\Lambda = (u_\lambda, u_\mu)_{H_0^1(\Omega)} \]

Proof:

1- Uniqueness follows well posedness of the Laplace problem in $H_0^1(\Omega_m)$. In order to prove existence, let $u \in H_m$ be such that $\lambda = \rho_m^\Gamma(u)$. Then $\Delta u \in H^{-1}(\Omega_m)$. Let $v$ be the unique solution in $H_0^1(\Omega_m)$ of $\Delta v = \Delta u$. Then $u_\lambda^m = u - v$ fulfills the property.

2- Because of remark 1, one has to show:

\[ w \in H_0^1(\Omega_m) \Rightarrow (u_\lambda^1, w)_{H_m} = 0 \]

which follows from the Green formula.

3- Because $\rho_1^\Gamma(u_1^1) = \rho_2^\Gamma(u_1^1) = \lambda$ one has $u_\lambda^1 \in H_0^1(\Omega)$. Obviously it is orthogonal to $\text{Ker} \rho^\Gamma$, so

\[ \| \lambda \|^2_\Lambda = \| u_\lambda^1 \|^2_{H^1(\Omega)} = \| u_1^1 \|^2_{H_1} + \| u_2^1 \|^2_{H_2} \]
2.2 $\Lambda = \Lambda_1 = \Lambda_2$

By symmetry, it is enough to prove $\Lambda = \Lambda_1$. In order to prove this algebraic and topological equality, two key tools are needed. The first tool is the Calderon extension theorem [Ag], which applies here because $\Omega_1$ has the strict cone property, by assumption, and which gives a bounded linear operator $E$ from $H^1(\Omega_1)$ to $H^1(R^d)$ such that:

$$\forall w \in H^1(\Omega_1), \quad Ew|_{\Omega_1} = w$$

The second key tool is:

**Theorem 1** There exists a bounded linear operator $\tau$ in $H^1(R^d)$ such that:

$$\forall v \in H^1(R^d), \quad v|_{\partial\Omega_1 \cap \partial\Omega} = 0 \Rightarrow (\tau v|_{\partial\Omega} = 0 \quad \text{and} \quad \tau v|_{\Omega_1} = v|_{\Omega_1})$$

**proof:**

Assumption (1) gives a finite open covering $(\omega_j)_j$ of $\Gamma \cap \partial\Omega$ and a change of variables $(a_j)_j$ such that $V_j = a_j(\omega_j)$ is a neighbourhood of zero in $R^d$ and:

$$a^j(\Gamma \cap \omega_j) = \{z^j \in V_j; z_1^j = 0, z_2^j > 0\} \quad \text{and} \quad a^j(\partial\Omega \cap \omega_j) = \{z^j \in V_j; z_2^j = 0\}$$

$$a^j(\Omega_1 \cap \omega_j) = \{z^j \in V_j; z_1^j < 0, z_2^j > 0\} \quad \text{and} \quad a^j(\Omega_2 \cap \omega_j) = \{z^j \in V_j; z_1^j > 0, z_2^j > 0\}$$

Regularity of the submanifold $\Gamma$ gives an open covering $(\omega'_k)_k$ of $\Gamma$ and change of variables $(b_k)_k$ such that $W_k = b^k(\omega'_k)$ is a neighbourhood of zero in $R^d$ and:

$$b^k(\Gamma \cap \omega'_k) = \{z^k \in W_k; z_1^k = 0\}$$

$$b^k(\Omega_1 \cap \omega'_k) = \{z^k \in W_k; z_1^k < 0\} \quad \text{and} \quad b^k(\Omega_2 \cap \omega'_k) = \{z^k \in W_k; z_1^k > 0\}$$

Compactness of $\Gamma$ enables to select a finite subcovering of $\Gamma$ still denoted by $(\omega_j)_j \cup (\omega'_k)_k$ having the previous properties.

Let $\omega^1 = R^d \setminus \overline{\Omega_1}$ and $\omega^2 = R^d \setminus \overline{\Omega_2}$, so:

$$R^d = \omega^1 \cup \omega^2 \cup (\cup_j \omega_j) \cup (\cup_k \omega'_k)$$

Let

$$(\alpha^1, \alpha^2, (\alpha_j)_j, (\alpha'_k)_k)$$

be a $C^\infty$-partition of unity associated with this open covering of $R^d$. 
Let $\varepsilon > 0$ be such that $\forall z \in (\cup_j V_j) \cup (\cup_k W_k)$, $0 < z^j_1 < \varepsilon \Rightarrow z \in \Omega_2$

Let $\psi(s) \in C^\infty(R)$ be equal to one for $s < 0$ and zero for $s > \varepsilon$.

Let $\varphi(\theta) \in C^\infty(R)$ be equal to zero for $\theta < 0$ and equal to one for $\theta > \frac{\pi}{2}$.

For any $v \in H^1(R^d)$,

$$v = \alpha^1 v + \alpha^2 v + \sum_j \alpha_j v + \sum_k \alpha'_k v$$

we define $\tau v$ as:

$$\tau(v) = \tau(\alpha^1 v) + \tau(\alpha^2 v) + \sum_j \tau(\alpha_j v) + \sum_k \tau(\alpha'_k v)$$

with:

$\tau(\alpha^1 v) \equiv 0$

$\tau(\alpha^2 v) = \alpha^2 v$

$\tau(\alpha'_k v)(z^k) = \psi(z^k)\alpha'_k(z^k) v(z^k)$ in the local coordinates.

These three quantities are multiplication of $v$ by $C^\infty$ functions, which are bounded as well as all their derivatives. It is linear and bounded in $H^1(R^d)$ with respect to $v \in H^1(R^d)$.

In order to define $\tau(\alpha_j v)$, we first write $\alpha_j v$ in the cylindrical coordinates as follows:

$$\alpha_j v(z^j_1, z^j_2, z^j_3, ..., z^j_d) = \overline{\alpha_j} v(r^j, \theta^j, z^j_3, ..., z^j_d) \quad \text{with} \quad z^j_1 = r^j \cos(\theta^j), \ z^j_2 = r^j \sin(\theta^j)$$

and define $\tau(\alpha_j v)$ in these coordinates as:

$$\tau(\overline{\alpha_j} v)(r^j, \theta^j, z^j_3, ..., z^j_d) = \varphi(\theta^j)\overline{\alpha_j} v(r^j, \theta^j, z^j_3, ..., z^j_d)$$

This quantity is linear with respect to $v$, and we prove its boundedness in $H^1(R^d)$ with respect to $v \in H^1(R^d)$ as follows (we omit the index $j$ and denote the measure $dz_3...dz_d$ by $d\overline{z}$):

$$||\tau(\alpha v)||^2_{H^1(R^d)} = \int_V |\overline{\tau(\alpha v)}|^2 r dr d\theta d\overline{z} + \int_V \left| \frac{\partial}{\partial r} \overline{\tau(\alpha v)} \right|^2 r dr d\theta d\overline{z}$$

$$+ \int_V \frac{1}{r^2} \left| \frac{\partial}{\partial \theta} \overline{\tau(\alpha v)} \right|^2 r dr d\theta d\overline{z} + \sum_{j=3}^{d} \int_V \left| \frac{\partial}{\partial z_j} \overline{\tau(\alpha v)} \right|^2 r dr d\theta d\overline{z}$$

$$= \int_V |\varphi(\theta)\overline{\alpha v}|^2 r dr d\theta d\overline{z} + \int_V |\varphi(\theta)\frac{\partial}{\partial r} \overline{\alpha v}|^2 r dr d\theta d\overline{z}$$
\begin{equation*}
+ \int_{V} \frac{1}{r^2} |\varphi(\theta) \frac{\partial}{\partial \theta} \tilde{\alpha} v + \varphi'(\theta) \tilde{\alpha} v|^2 r dr d\theta d\bar{z} + \sum_{j} \int_{V} |\varphi(\theta) \frac{\partial}{\partial z_j} \tilde{\alpha} v|^2 r dr d\theta d\bar{z}
\leq \sup \| \varphi \|^2 \left( \int_{V} |\tilde{\alpha} v|^2 r dr d\theta d\bar{z} + \int_{V} \left| \frac{\partial}{\partial r} \tilde{\alpha} v \right|^2 r dr d\theta d\bar{z} \right)
\end{equation*}

\begin{equation*}
\begin{split}
+ \int_{V} \frac{2}{r^2} \left| \frac{\partial}{\partial \theta} \tilde{\alpha} v \right|^2 r dr d\theta d\bar{z} + \sum_{j} \int_{V} \left| \frac{\partial}{\partial z_j} \tilde{\alpha} v \right|^2 r dr d\theta d\bar{z}
+ 2 \sup \| \varphi' \|^2 \int_{V} \frac{1}{r^2} |\tilde{\alpha} v|^2 r dr d\theta d\bar{z}
\leq 2 \sup \| \varphi \|^2 \| \alpha v \|^2_{H^1(R^d)} + 2 \sup \| \varphi' \|^2 \int_{V} \frac{1}{r^2} |\tilde{\alpha} v|^2 r dr d\theta d\bar{z}
\end{split}
\end{equation*}

In order to estimate this last quantity we use the assumption $v|_{\partial \Omega_1 \cup \partial \Omega} = 0$ to have:

$$
\tilde{\alpha} v(r, \theta, \bar{z}) = - \int_{\theta}^{\pi} \frac{\partial}{\partial \theta} \tilde{\alpha} v(r, s, \bar{z}) ds
$$

which gives (with $\varepsilon'$ the radius of the support of $\alpha$ in the $r$ variable, and $B$ a ball containing the support of $\alpha$ in the $\bar{z}$ variable):

\begin{equation*}
\begin{split}
\int_{V} \frac{1}{r^2} |\tilde{\alpha} v|^2 r dr d\theta d\bar{z} \leq \int_{B} \int_{0}^{\varepsilon'} \int_{-\pi}^{\pi} \frac{1}{r^2} \left| \int_{\theta}^{\pi} \frac{\partial}{\partial \theta} \tilde{\alpha} v(r, s, \bar{z}) ds \right|^2 r dr d\theta d\bar{z}
\leq 4 \pi^2 \int_{B} \int_{0}^{\varepsilon'} \int_{-\pi}^{\pi} \frac{1}{r^2} \left| \frac{\partial}{\partial \theta} \tilde{\alpha} v(r, s, \bar{z}) \right|^2 r dr d\theta d\bar{z} \leq 4 \pi^2 \| \alpha v \|^2_{H^1(R^d)}
\end{split}
\end{equation*}

we summarize to have:

$$
\| \tau(\alpha v) \|^2_{H^1(R^d)} \leq C \| \alpha v \|^2_{H^1(R^d)} \leq C' \| v \|^2_{H^1(R^d)}
$$

and this ends the proof of the boundedness of $\tau$ in $H^1(R^d)$.

We end the proof of theorem \[\square\] using the following obvious observations:

$\tau v|_{\Omega_1} = v|_{\Omega_1}$ because $\psi \equiv 1$ on $R_-$ and $\varphi \equiv 1$ for $\theta \geq \frac{\pi}{2}$

$\tau v|_{\partial \Omega_1 \cup \partial \Omega} = 0$ by assumption

$\tau v|_{\partial \Omega_2 \cup \partial \Omega} = 0$ because $\varphi(0) = 0$ and $\psi \equiv 0$ for $z_j > \varepsilon$.

**Corollary 1** For $m = 1, 2$, there exists a bounded linear map $E_m$ from $H_m$ to $H^1_0(\Omega)$ such that

$$
\forall u \in H_m \quad (E_m u)|_{\Omega_m} = u
$$

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proof: for $m = 1$ for instance let $E_1 u$ be the restriction to $\Omega$ of $\tau E u$. Boundedness follows from theorem [1].

**Corollary 2** $\Lambda_1 = \Lambda_2 = \Lambda$ and the three norms $\| \cdot \|_\Lambda, \| \cdot \|_{\Lambda_1}$ and $\| \cdot \|_{\Lambda_2}$ are equivalent.

proof: obviously $\Lambda \subset \Lambda_m$ and the previous corollary gives the converse inclusion. Moreover we have $\| \cdot \|_\Lambda \geq \| \cdot \|_{\Lambda_m}$ and the previous corollary gives the converse inequality.

**Corollary 3** $\mathcal{D}(\Gamma)$ is a dense subspace of $\Lambda$ for any of the three norms.

proof: $\mathcal{D}(\Gamma)$ is the set of traces on $\Gamma$ of functions in $\mathcal{D}(\Omega)$ because $\Gamma$ is a $C^\infty$-submanifold. Density of $\mathcal{D}(\Gamma)$ for the $\| \cdot \|_\Lambda$ norm follows density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$. Equivalence of the three norms ends the proof.

**Remark 3** If we denote as usual by $H^1_0(\Gamma)$ the closure of $\mathcal{D}(\Omega)$ in $H^{1/2}(\Gamma)$ (which exists because $\Gamma$ is $C^\infty$), then the previous corollary asserts that $\Lambda \subset H^1_0(\Gamma)$. Boundedness of the trace operators gives constants $C, C_m$ such that:

\[
\forall \lambda \in \Lambda, \| \lambda \|_{H^{1/2}(\Gamma)} \leq C \| \lambda \|_\Lambda \quad \text{and} \quad \forall \lambda \in \Lambda_m, \| \lambda \|_{H^{1/2}(\Gamma)} \leq C_m \| \lambda \|_{\Lambda_m}
\]

### 2.3 Well-posedness of the Laplace-Dirichlet problem in $H^{-1}(\Omega_m) \times \Lambda$

**Theorem 2** For $m = 1, 2$

1- for any $(f, \lambda) \in H^{-1}(\Omega_m) \times \Lambda$ there exists a unique $u \in H_1$ such that

\[
\Delta u = f \quad \text{in} \quad \Omega_m \quad \text{and} \quad \rho_m^\Gamma(u) = \lambda
\]

2- we have the estimate $\| u \|_{H_m} \leq \| f \|_{H^{-1}(\Omega_m)} + \| \lambda \|_{\Lambda_m}$

proof: Let $u^\lambda_m$ be given by Proposition [1]. Let $v = u - u^\lambda_m$. The problem is equivalent to

\[
v \in H^1_0(\Omega_m) \quad \text{and} \quad \Delta v = f
\]

This is a well-posed problem and we have, because the Riesz representation operator is isometric, $\| v \|_{H^1_0(\Omega_m)} = \| f \|_{H^{-1}(\Omega_m)}$. So

\[
\| u \|_{H_m} \leq \| u^\lambda_m \|_{H_m} + \| v \|_{H_m} \leq \| f \|_{H^{-1}(\Omega_m)} + \| \lambda \|_{\Lambda_m}
\]
2.4 Duality and the Poincare-Steklov operators

Let $\Lambda'$ denote the dual space to $\Lambda$, endowed with one of the three equivalent norms associated with the equivalent norms on $\Lambda$ defined previously.

We denote by $(.,.)_{\Lambda\Lambda'}$ the duality product, and by $(.,.)_{D'D'}$ the duality product in $D'(\Gamma)$.

Because of Corollary 3 and remark 3, we have the usual injections:

$$D \subset \Lambda \subset H^1_0(\Gamma) \subset L^2(\Gamma) \subset \Lambda' \subset D'$$

and for any $\lambda \in D(\Gamma)$ and $\nu \in \Lambda'$: $(\nu, \lambda)_{\Lambda\Lambda'} = (\nu, \lambda)_{DD'} = (\nu, \overline{\eta})_{L^2(\Gamma)}$ where $\overline{\eta}$ denotes the complex conjugate of functions or distributions $\eta$.

For any of the three scalar products on $\Lambda$ we have $(\lambda, \eta) = (\overline{\lambda}, \overline{\eta})$ so for all norms:

$$\forall \lambda \in \Lambda, \quad \|\lambda\| = \|\overline{\lambda}\| \quad \text{and} \quad \forall \nu \in \Lambda', \quad \|\nu\| = \|\overline{\nu}\|$$

**Notation 1** Let $\tilde{S}$ denote the antilinear Riesz representation operator for $\Lambda'$ in the $\Lambda$-scalar product, and $\tilde{S}_m$ $(m = 1, 2)$ this representation in the $\Lambda_m$-scalar product, i.e.

$$\forall \lambda \in \Lambda, \forall \nu \in \Lambda' \quad (\nu, \lambda)_{\Lambda\Lambda'} = (\lambda, \tilde{S}^{-1}\nu)_{\Lambda} = (\lambda, \tilde{S}_1^{-1}\nu)_{\Lambda_1} = (\lambda, \tilde{S}_2^{-1}\nu)_{\Lambda_2}$$

Let $S$ (resp. $S_m$) be the linear isometric bijections from $\Lambda$ to $\Lambda'$ defined by

$$\forall \lambda \in \Lambda \quad \tilde{S}\lambda = S\overline{\lambda}, \quad \tilde{S}_1\lambda = S_1\overline{\lambda}, \quad \tilde{S}_2\lambda = S_2\overline{\lambda}$$

We denote by $n_m$ the normal unit vector on $\Gamma$ pointing outward with respect to $\Omega_m$. We denote by $\frac{\partial \varphi}{\partial n_m}$ the normal derivative on $\Gamma$ of $\varphi \in D(\Omega)$, and by $\frac{\partial}{\partial n_m}$ bounded extensions of this operator to any functional space.

We use Proposition 1 to have:

**Proposition 2**

$$\forall \lambda \in \Lambda \quad S_m\lambda = \frac{\partial u_m^\lambda}{\partial n_m} \quad \text{and} \quad S = S_1 + S_2$$

**proof:** by Green formula

$$\forall \lambda \in \Lambda, \eta \in \Lambda \quad (S_m\eta, \lambda)_{\Lambda_m\Lambda'} = (\lambda, \eta)_{\Lambda_m} =$$
\((u^\lambda_m, u^\eta_m)_{H^1(\Omega_m)} = \int_{\Omega_m} \nabla u^\lambda_m \nabla u^\eta_m dx = \int_{\Gamma} \lambda \frac{\partial u^\eta_m}{\partial n_m} d\sigma\)

this proves that the distribution \(\frac{\partial u^\eta_m}{\partial n_m}\) is bounded in the \(\Lambda_m\) norm, so \(\frac{\partial u^\eta_m}{\partial n_m} \in \Lambda\) and

\[ \tilde{S}_m \eta = \frac{\partial u^\eta_m}{\partial n_m} = \frac{\partial u^\eta_m}{\partial n_m} = \frac{\partial u^\eta_m}{\partial n_m} \]

**Corollary 4**

\(\bar{\Lambda} = S\Lambda \quad \bar{S_m \Lambda} = S_m \bar{\Lambda}\)

**Remark 4** For any \(v \in H_m\) such that \(\Delta v \in L^2(\Omega_m)\), we have \(\frac{\partial v}{\partial n_m} \in \Lambda^\prime\) and

\[ \| \frac{\partial v}{\partial n_m} \|_{\Lambda^\prime} \leq C (? v \|_{H_m} + \| \Delta v \|_{L^2(\Omega_m)}) \]

This is because for any \(\varphi \in D(\Omega)\) we have:

\[ (\frac{\partial v}{\partial n_m}, \varphi)_D = \int_{\Omega_m} \nabla \varphi \nabla v dx + \int_{\Omega_m} \varphi \Delta v dx \]

and this formula shows that the distribution \(\frac{\partial v}{\partial n_m}\) is bounded on \(\Lambda\).

### 2.5 Adjointes

**Notation 2** :
1. For a bounded linear operator \(T\) from \(\Lambda\) to \(\Lambda^\prime\), we denote by \(T^\prime\) its adjoint for the \((\Lambda, \Lambda^\prime)\) duality, i.e.

\[ \forall \lambda \in \Lambda, \forall \eta \in \Lambda \quad (T\eta, \lambda)_{\Lambda^\prime} = (T^\prime\lambda, \eta)_{\Lambda^\prime} \]

2. For a bounded linear operator \(T\) from \(\Lambda\) to \(\Lambda\), we denote by \(T^\ast\) the adjoint operator in \(\Lambda\), i.e.

\[ \forall \lambda \in \Lambda, \forall \eta \in \Lambda \quad (T\eta, \lambda)_\Lambda = (\eta, T^\ast \lambda)_\Lambda \]

**Proposition 3** For \(m = 1, 2\)

\[ S^\prime = S \quad S_m^\prime = S_m \]
proof: by definition of $S$ we have: $\forall \lambda \in \Lambda, \forall \eta \in \Lambda$

$$(S\lambda, \eta)_{\Lambda\Lambda'} = (S\eta, \lambda)_{\Lambda\Lambda'} = (\lambda, \eta)_{\Lambda} = (S\eta, \lambda)_{\Lambda\Lambda'} = (S\eta, \lambda)_{\Lambda\Lambda'}$$

**Theorem 3** For $m = 1, 2$ let $m' = 2, 1$. For all $\lambda \in \Lambda, \eta \in \Lambda$

1- $(S^{-1}_m S_{m'} \eta, \lambda)_{\Lambda_m} = (\eta, \lambda)_{\Lambda_{m'}}$
2- $(S^{-1}_m S_{m'} \eta, \lambda)_{\Lambda_{m'}} = (S_{m'} \eta, S_{m'} \lambda)_{\Lambda_{m'}}$
3- $S_{m'}$ is self adjoint in $\Lambda_m$ and in $\Lambda_{m'}$
4- $(S_1^{-1} S_2 + S_2^{-1} S_1)$ is selfadjoint in $\Lambda$

proof:
1- $(S^{-1}_m S_{m'} \eta, \lambda)_{\Lambda_m} = (\lambda, S^{-1}_m S_{m'} \eta)_{\Lambda_m} = (S_{m'} \eta, \lambda)_{\Lambda_{m'}} = (\lambda, \eta)_{\Lambda_{m'}}$
2- We use Proposition 3 to have

$$(S^{-1}_m S_{m'} \eta, \lambda)_{\Lambda_{m'}} = (S_{m'} \eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_{m'}} = (S_{m'} \eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_{m'}}$$

3- We use Proposition 3 to have

$$(S^{-1}_m S_{m'} \eta, \lambda)_{\Lambda_m} = (S_{m'} \eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_{m'}} = (\eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_m}$$

On the other hand

$$(S^{-1}_m S_{m'} \eta, \lambda)_{\Lambda_{m'}} = (S_{m'} \eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_{m'}} = (S_{m'} \eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_{m'}} = (\eta, S^{-1}_m S_{m'} \lambda)_{\Lambda_{m'}}$$

4- follows 3

**Corollary 5** Coerciveness: For $m = 1, 2$ let $m' = 2, 1$. There exists $C > 0$

such that for all $\lambda \in \Lambda$

1- $(S^{-1}_m S_{m'} \lambda, \lambda)_{\Lambda_m} = \|\lambda\|_{\Lambda_m}^2 \geq C\|\lambda\|_{\Lambda_m}^2$
2- $(S^{-1}_m S_{m'} \lambda, \lambda)_{\Lambda_{m'}} = \|S_{m'} \lambda\|_{\Lambda_{m'}}^2 \geq C\|\lambda\|_{\Lambda_{m'}}^2$
3- $(S^{-1}_m S_2 + S_2^{-1} S_1) \lambda, \lambda)_{\Lambda_m} \geq (1 + C)\|\lambda\|_{\Lambda}^2$

2.6 On the Neumann problem

**Proposition 4** For $m = 1, 2$ and for any $\nu \in \Lambda'$ there exists a unique $u_m \in H_m$ such that

$$\Delta u_m = 0 \text{ in } \Omega_m \text{ and } \frac{\partial u_m}{\partial n_m} = \nu \text{ on } \Gamma$$

Moreover

$$\|u_m\|_{H_m} = \|\nu\|_{\Lambda'}$$
proof: uniqueness is straightforward, and existence is provided by proposition \( \square \) and \( u_m = u S_m^{-1} \nu \)

Moreover isometry of the Riesz representation gives:

\[
\|u_m\|_{H_m} = \|S_m^{-1} \nu\|_\Lambda = \|\nu\|_{\Lambda'}
\]

**Proposition 5** For \( m = 1, 2 \) and for any \( f \in L^2(\Omega_m) \) there exists a unique \( u_m \in H_m \) such that

\[
\Delta u_m = f \quad \text{and} \quad \frac{\partial u_m}{\partial n_m} = 0 \quad \text{on } \Gamma
\]

Moreover

\[
\|u_m\|_{H_m} \leq C \|f\|_{L^2(\Omega_m)}
\]

**proof:** Uniqueness is straightforward. For existence let \( v_m \in H^1_0(\Omega_m) \) be the unique solution of \( \Delta v_m = f \). In remark \( \square \) we have shown that \( \nu = \frac{\partial v_m}{\partial n_m} \in \Lambda' \) and

\[
\|\nu\|_{\Lambda_m} \leq C (\|f\|_{L^2(\Omega_m)} + \|v_m\|_{H_m}) \leq C (\|f\|_{L^2(\Omega_m)} + \|f\|_{H^{-1}(\Omega_m)}) \leq C \|f\|_{L^2(\Omega_m)}
\]

The function \( u_m = v_m - u S_m^{-1} \nu \) solves the problem, and we have:

\[
\|u_m\|_{H_m} \leq \|v_m\|_{H_m} + \|u S_m^{-1} \nu\|_{H_m} \leq \|f\|_{L^2(\Omega_m)} + \|S_m^{-1} \nu\|_{\Lambda_m}
\]

\[
\leq \|f\|_{L^2(\Omega_m)} + \|\nu\|_{\Lambda_m} \leq C \|f\|_{L^2(\Omega_m)}
\]

3 A two-sided Dirichlet-Neumann domain decomposition algorithm for the Laplace operator

**Proposition 6** Let \( f \in L^2(\Omega) \) and \( u \in H^1_0(\Omega) \) be the unique solution of \( \Delta u = f \). For \( m = 1, 2 \) let \( f_m = f|_{\Omega_m} \) and \( g_m \in H^1_0(\Omega_m) \) be the unique solution of \( \Delta g_m = f_m \). Let \( \eta_m = g_m|_{\Gamma} \)

\[
\lambda = u|_{\Gamma} \iff (S_1 + S_2)\lambda = -(S_1 \eta_1 + S_2 \eta_2)
\]

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proof: the direct implication is stating continuity of \(u\) and its normal derivatives through \(\Gamma\). The converse implication states that taking \(u|_{\Omega_m} = u_m^\lambda\) solves the global problem.

We use the same notation as in the previous proposition to state:

**Theorem 4** Let \(0 < \theta < 1\) with \(\|S_1^{-1}S_2 + S_2^{-1}S_1\|_{\mathcal{L}(\Lambda)} < \frac{2(1-\theta)}{\theta}\). Any sequence \((\lambda_n)_n \subset \Lambda\) which fulfills

\[
\lambda_{n+1} = \left((1 - \theta)Id - \frac{\theta}{2}(S_1^{-1}S_2 + S_2^{-1}S_1)\right)\lambda_n - \frac{\theta}{2}(S_1^{-1} + S_2^{-1}))(S_1\eta_1 + S_2\eta_2)
\]

do converge in \(\Lambda\) (with geometric rate \(1 - \frac{3\theta}{2}\) at least) and its limit is \(u|_{\Gamma}\).

**proof:** theorem 3 states selfadjointness of \(S_1^{-1}S_2 + S_2^{-1}S_1\) in \(\Lambda\) and theorem 5 states coerciveness of \(S_1^{-1}S_2 + S_2^{-1}S_1\) so:

\[
\|((1 - \theta)Id - \frac{\theta}{2}(S_1^{-1}S_2 + S_2^{-1}S_1))\|_{\mathcal{L}(\Lambda)} =
\]

\[
\sup_{\Lambda} \frac{|(((1 - \theta)Id - \frac{\theta}{2}(S_1^{-1}S_2 + S_2^{-1}S_1))\lambda, \lambda)|_{\Lambda}}{\|\lambda\|_{\Lambda}^2} =
\]

\[
\sup_{\Lambda} \frac{(((1 - \theta)Id - \frac{\theta}{2}(S_1^{-1}S_2 + S_2^{-1}S_1))\lambda, \lambda)}{\|\lambda\|_{\Lambda}^2} \leq 1 - \theta - \frac{\theta}{2}(1+C) < 1 - \frac{3\theta}{2} < 1
\]

4 Tools for a Domain Decomposition algorithm for the Helmholtz equation

In the sequel we shall assume **connectedness** of the open sets \(\Omega_m, m = 1, 2\).

4.1 On the \(j\) operator

We define the linear bounded operator \(j\) from \(\Lambda\) to \(\Lambda'\) as the composition of the bounded linear injections

\[
\Lambda \subset H^{-\frac{1}{2}}_0(\Gamma) \subset L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset \Lambda'
\]

It will play a key role for Helmholtz equations. Its properties are summarized by:
Proposition 7.
1- \( j \) is a compact and one-to-one operator
2- For any \( \lambda \in \Lambda \), \( j(\lambda) = \overline{j(\lambda)} \)
3- For \( m = 1, 2 \)
\[ \forall \lambda \in \Lambda, \sigma \in \Lambda, (j(\lambda), \sigma)_{\Lambda \Lambda'} = (\sigma, S_m^{-1} j(\lambda))_{\Lambda_m} = (\sigma, \overline{\lambda})_{L^2(\Gamma)} \]
\[ \forall \lambda \in \Lambda, \quad (j(\lambda), \overline{\lambda})_{\Lambda \Lambda'} = \|\lambda\|_{L^2(\Gamma)}^2 \]
4- \( \lambda' = j \)
5- For \( m = 1, 2 \) the operator \( S_m^{-1} j \) is selfadjoint in \( \Lambda_m \)
6- For \( m = 1, 2 \) the operator \( j S_m^{-1} \) is selfadjoint in \( \Lambda'_m \)

proof: Item 1 comes from the Rellich compactness of the injection from \( H^1_0(\Gamma) \) to \( L^2(\Gamma) \).
Item 3 comes from:
\[ (\sigma, S_m^{-1} j(\overline{\lambda}))_{\Lambda_m} = \int_{\Omega_m} \nabla u^* \nabla u S_m^{-1} j(\lambda) = \int_{\Gamma} \sigma \frac{\partial u S_m^{-1} j(\lambda)}{\partial n_m} = \int_{\Gamma} \sigma \lambda = (\sigma, \overline{\lambda})_{L^2(\Omega_m)} \]
Item 4 follows item 3 because \( (j(\lambda), \sigma)_{\Lambda \Lambda'} = \int_{\Gamma} \sigma \lambda = (j(\lambda), \lambda)_{\Lambda \Lambda'} \)
Item 5 follows items 3 and 4 because
\[ (\sigma, S_m^{-1} j(\overline{\lambda}))_{\Lambda_m} = (j(\lambda), \sigma)_{\Lambda \Lambda'} = (j(\sigma), \lambda)_{\Lambda \Lambda'} = (S_m^{-1} j(\sigma), \overline{\lambda})_{\Lambda_m} \]
Item 6 follows item 5 because
\[ (j S_m^{-1} \mu, \nu)_{\Lambda'_m} = (S_m^{-1} j S_m^{-1} \mu, S_m^{-1} \nu)_{\Lambda_m} = (S_m^{-1} \mu, S_m^{-1} j S_m^{-1} \nu)_{\Lambda_m} = (\mu, j S_m^{-1} \nu)_{\Lambda'_m} \]

4.2 The spectrum of the local Helmholtz problems

This paragraph is devoted to the study of the operators

\[ m = 1, 2 \quad (S_m + i \gamma j) : \Lambda \longrightarrow \Lambda' \]
related to the Laplace equation, and to the like \( (S_m^{k} + i \gamma j) \) operators related to the Helmholtz equation. Let \( \gamma \) denote a real number.

Proposition 8.
1- \( \forall \lambda \in \Lambda \quad ((S_m + i \gamma j) \lambda, \overline{\lambda})_{\Lambda \Lambda'} = \|\lambda\|_{\Lambda_m}^2 + i \gamma \|\lambda\|_{L^2(\Gamma)} \)
2- \( (S_m + i \gamma j) \) has a bounded inverse.
3- \( \forall \lambda \in \Lambda \quad (S_m + i \gamma j) \overline{\lambda} = (S_m - i \gamma j) \lambda \quad \text{and} \quad T_m \overline{\lambda} = T_m^{-\gamma}(\lambda) \)

proof:
1- is straightforward applying proposition \[7\] and notation \[2\]
2- item 1 shows that $\text{Ker}(S_m + i\gamma j) = \{0\}$ and $\text{Im}(S_m + i\gamma j)$ is closed in $\Lambda'$. It remains to show that $\text{Im}(S_m + i\gamma j)$ is everywhere dense in $\Lambda'$. By propositions 3 and 7

$$\forall \lambda \in \Lambda, ((S_m + i\gamma j)\lambda, \eta)_{\Lambda'\Lambda} = 0 \implies$$

$$\forall \lambda, ((S_m + i\gamma j)\eta, \lambda)_{\Lambda'\Lambda} = 0 \implies (S_m + i\gamma j)\eta = 0 \implies \eta = 0$$

because $(S_m + i\gamma j)$ is one to one.

3- is straightforward.

**Proposition 9** For $m = 1, 2$ and any $\nu \in \Lambda'$ there exists a unique $u \in H_m$ such that

$$\Delta u = 0 \text{ in } \Omega_m \text{ and } \frac{\partial u}{\partial n_m} + i\gamma j \rho_m^\Gamma u = \nu$$

In fact $u = u^{(S_m + i\gamma j)^{-1}} \nu$ and

$$\|u\|_{H_m} \leq C\|\nu\|_{\Lambda'}$$

**proof:** Uniqueness is straightforward by the Green formula. For existence we apply the previous proposition to get $\lambda \in \Lambda$ such that $(S_m + i\gamma j)\lambda = \nu$, and check that $u = u^\lambda$ solves the problem.

The following remark will be crucial to prove convergence of domain decomposition algorithms for the Helmholtz equation:

**Remark 5** with the notation of the preceeding proposition, if $\nu \in L^2(\Gamma)$ then

$$\frac{\partial u}{\partial n_m} \in L^2(\Gamma) \text{ and } \left\| \frac{\partial u}{\partial n_m} \right\|_{L^2(\Gamma)} \leq C\|\nu\|_{L^2(\Gamma)}$$

**proof:** $\frac{\partial u}{\partial n_m} = \nu - i\gamma j \rho_m^\Gamma u \in L^2(\Gamma) + \Lambda \subset L^2(\Gamma)$ and

$$\left\| \frac{\partial u}{\partial n_m} \right\|_{L^2(\Gamma)} \leq \|\nu\|_{L^2(\Gamma)} + |\gamma|\|\rho_m^\Gamma u\|_{L^2(\Gamma)} \leq \|\nu\|_{L^2(\Gamma)} + |\gamma|\|u\|_{H_m}$$

$$\leq \|\nu\|_{L^2(\Gamma)} + C|\gamma|\|\nu\|_{\Lambda'} \leq C\|\nu\|_{L^2(\Gamma)}$$
Proposition 10 For \( m = 1, 2 \) and any \( f \in L^2(\Omega_m) \) there exists a unique \( u \in H_m \) such that

\[
\Delta u = f \text{ in } \Omega_m \quad \text{and} \quad \frac{\partial u}{\partial n_m} + i\gamma j \rho_m^\Gamma u = 0
\]

and we have the estimate

\[
\|u\|_{H_m} \leq C\|f\|_{L^2(\Omega_m)}
\]

proof: let \( v \in H^1_0(\Omega_m) \) solve \( \Delta v = f \). Let \( w = u - v \). The function \( w \) fulfills \( \Delta w = 0 \) and \( \frac{\partial w}{\partial n_m} + i\gamma j \rho_m^\Gamma w = \frac{\partial v}{\partial n_m} \). Remark 4 shows that \( \frac{\partial v}{\partial n_m} \in \Lambda' \). The result follows from the previous proposition and the estimate follows estimates in remark 4 and proposition 9.

Remark 6 Using the notations of proposition 10 we have, as in remark 5, \( \frac{\partial u}{\partial n_m} \in L^2(\Gamma) \) and the estimate:

\[
\left\| \frac{\partial u}{\partial n_m} \right\|_{L^2(\Gamma)} \leq C\|f\|_{L^2(\Omega_m)}
\]

We can now proceed to compute the eigenfrequencies of the local Helmholtz problems involved in the Domain Decomposition algorithm. For that sake, we will use the following

Notation 3 For \( m = 1, 2 \) we denote by

\[
D_m^\gamma : L^2(\Omega_m) \rightarrow L^2(\Omega_m)
\]

\( f \rightarrow u \)

where \( u \) is given by proposition 10.

This map has the following properties:

Proposition 11 .
1- \( D_m^\gamma \) is a compact operator in \( L^2(\Omega_m) \)
2- the adjoint map of \( D_m^\gamma \) for the \( L^2(\Omega_m) \) scalar product is \( D_m^{-\gamma} \)
3- we have \( D_m^\gamma \overline{\eta} = D_m^{-\gamma} g \) and \( D_m^\gamma D_m^{-\gamma} \overline{\eta} = D_m^{-\gamma} D_m^\gamma \overline{\eta} \)
4- \( \text{Im} D_m^\gamma \subset H_m \)
5- \( \frac{\partial}{\partial n_m} D_m^\gamma f \in L^2(\Gamma) \)
proof:
1- translates Rellich compactness of the imbedding of \( H_m \) in \( L^2(\Omega_m) \)
2- If \( u = D_m^\gamma f \) and \( v = D_m^{-\gamma}g \) then Green formula gives
\[
\int_{\Omega_m} u \overline{v} dx - \int_{\Omega_m} v f dx = \int_{\Gamma} \frac{\partial u}{\partial n_m} \overline{v} ds - \int_{\Gamma} \frac{\partial v}{\partial n_m} u ds = -i \gamma \int_{\Gamma} u \overline{v} ds + i \gamma \int_{\Gamma} u v ds = 0
\]
3- is straightforward
4- follows proposition \([10]\)
5- follows remark \([6]\)

We collect the spectral properties of \( D_m^\gamma \) in the following

**Proposition 12** We denote by \( \sigma \) the spectrum of an operator and by \( \sigma_p \) the set of its eigenvalues. We have for \( m = 1, 2 \):

1- \( \sigma(D_m^\gamma) = \{0\} \cup \sigma_p(D_m^\gamma) \)
2- \( \mu \in \sigma(D_m^\gamma) \iff \overline{\mu} \in \sigma(D_m^{-\gamma}) \)
3- For any \( f \in L^2(\Omega_m) \), if \( u = D_m^\gamma f \) then
\[
(D_m^\gamma f, f)_{L^2(\Omega_m)} = -\int_{\Omega_m} |\nabla u|^2 dx + i \gamma \int_{\Gamma} |\rho_m^\Gamma u|^2 ds
\]
4- there exists a constant \( c \) such that if \( \mu \in \sigma(D_m^\gamma) \), \( \mu \neq 0 \), then
\[
\text{Re} \mu \leq 0, \gamma \text{Im} \mu \in \mathbb{R}_+, |\text{Im} \mu| \leq c|\gamma| |\text{Re} \mu|
\]
5- If \( \gamma \neq 0 \) then
\[
\sigma(D_m^\gamma) \cap \mathbb{R} = \{0\}
\]

proof:
1- follows compactness of \( D_m^\gamma \) asserted in proposition \([11]\)
2- is obvious by taking the complex conjugate of the eigenfunction associated with \( \mu \).

3-
\[
(D_m^\gamma f, f)_{L^2(\Omega_m)} = \int_{\Omega_m} u \overline{f} dx = \int_{\Omega_m} u \overline{\Delta u} dx =
\]
\[
-\int_{\Omega_m} |\nabla u|^2 dx + \int_{\Gamma} \rho_m^\Gamma u \frac{\partial u}{\partial n_m} \overline{d\sigma} = -\int_{\Omega_m} |\nabla u|^2 dx + i \gamma \int_{\Gamma} |\rho_m^\Gamma u|^2 ds
\]
4- if \( \mu \) is an eigenvalue of \( D_m^\gamma \) with associated eigenfunction \( f \), and \( u = D_m^\gamma f \), then
\[
\mu \int_{L^2(\Omega_m)} |f|^2 dx = -\int_{\Omega_m} |u|^2 dx + i \gamma \int_{\Gamma} |\rho_m^\Gamma u|^2 d\sigma
\]
and the result follows with $c$ the constant of continuity of the trace operator $\rho^\Gamma$ from $H$ to $L^2(\Gamma)$.

5- If $\mu \neq 0$ is a real eigenvalue of $D_m^\gamma$ with associated eigenfunction $f$, then the previous formula shows that $\rho^\Gamma_m u = 0$ and, because $u \in \text{Im}(D_m^\gamma)$, $\frac{\partial u}{\partial n_m} = -i\gamma \rho^\Gamma_m u = 0$. On the other hand, the equality $D_m^\gamma f = \mu f$ translates to $\Delta u = \frac{1}{\mu} u$. Because the Laplace operator is hyperbolic in the direction $n_m$, and both data on $\Gamma$ are zero, this implies $u = 0$ on a neighbourhood of $\Gamma$. Solutions of elliptic equations being analytic, and $\Omega_m$ being connected, this implies $u = 0$ on $\Omega_m$. Then $f = 0$, which contradicts the assumption on $f$ as an eigenfunction.

**Theorem 5** Let $k^2 \in \mathbb{R}$, $k^2 \neq 0$. Let $\gamma \in \mathbb{R}, \gamma \neq 0$. For $m = 1, 2$:  
1- for any $f$ in $L^2(\Omega_m)$ and $\nu \in \Lambda'$, there exits a unique $u \in H_m$ such that:  
$$\Delta u + k^2 u = f \quad \frac{\partial u}{\partial n_m} - i\gamma j \rho^\Gamma_m u = \nu$$  
2- we have the estimate  
$$\|u\|_{H_m} \leq C(\|f\|_{L^2(\Omega_m)} + \|\nu\|_{\Lambda'})$$  
3- if $\nu \in L^2(\Gamma)$ then $\frac{\partial u}{\partial n_m} \in L^2(\Gamma)$ and we have the estimate:  
$$\|\frac{\partial u}{\partial n_m}\|_{L^2(\Gamma)} \leq C(\|f\|_{L^2(\Omega_m)} + \|\nu\|_{L^2(\Gamma)})$$

**proof:**
1- Let $\lambda = (S_m - i\gamma j)^{-1} \nu$ and let $v = u - u^\lambda$. Then:  
$$\Delta u + k^2 u = f \quad \text{and} \quad \frac{\partial u}{\partial n_m} - i\gamma j \rho^\Gamma_m u = \nu \iff$$  
$$\Delta v = f - k^2 u^\lambda - k^2 v \quad \text{and} \quad \frac{\partial v}{\partial n_m} - i\gamma j \rho^\Gamma_m v = 0 \iff$$  
$$v = D_m^\gamma (f - k^2 u^\lambda - k^2 v) \iff D_m^\gamma v + k^{-2} v = k^{-2} D_m^\gamma (f - k^2 u^\lambda)$$

The previous proposition 12 shows that $-k^{-2} \notin \sigma(D_m^\gamma)$ so this problem is well-posed.

2- we have the estimate:  
$$\|v\|_{L^2(\Omega)} \leq C(k^{-2} \|D_m^\gamma f\|_{L^2(\Omega)} + \|D_m^\gamma u^\lambda\|_{L^2(\Omega)}) \leq C(k^{-2} \|f\|_{L^2(\Omega)} + \|u^\lambda\|_{L^2(\Omega)})$$
so
\[ \| u \|_{L^2(\Omega)} \leq C'(k^{-2}\| f \|_{L^2(\Omega)} + \| u^\lambda \|_{L^2(\Omega)}) \leq C'(k^{-2}\| f \|_{L^2(\Omega)} + \| u^\lambda \|_{H^m}) \]
\[ \leq C'(k^{-2}\| f \|_{L^2(\Omega)} + \| \lambda \|_{\Lambda}) \leq C'(k^{-2}\| f \|_{L^2(\Omega)} + \| \nu \|_{\Lambda'}) \]
We write
\[ \Delta u = f - k^2u \quad \frac{\partial u}{\partial n_m|\Gamma} - i\gamma j \rho^\gamma_m u = \nu \]
and the \textit{H}_m estimate follows proposition \ref{prop9} and proposition \ref{prop10}
3- If \( \nu \in L^2(\gamma) \) we write again
\[ \Delta u = f - k^2u \quad \frac{\partial u}{\partial n_m|\Gamma} - i\gamma j \rho^\gamma_m u = \nu \]
and the estimate follows remark \ref{rem5} and remark \ref{rem6}

4.3 Despres operators and the energy fluxes

We now define the building blocks of the intertwining operator on the fictitious boundary \( \Gamma \): the Despres operators.

**Definition 1**: Let \( k \in R, \gamma \neq 0, \gamma \in R \). For any \( \nu \in \Lambda' \), let \( u \in H_m \) be the unique solution, given by Theorem \ref{thm5} of the following equation on \( \Omega_m, m = 1, 2 \):
\[ \Delta u + k^2 u = 0, \quad \frac{\partial u}{\partial n_m|\Gamma} - i\gamma j \rho^\gamma_m u = \nu \quad \text{on} \quad \Gamma \]
Let \( \tilde{P}^\gamma_m \) be the linear bounded operator in \( \Lambda' \) defined by:
\[ \tilde{P}^\gamma_m \nu = \frac{\partial u}{\partial n_m} + i\gamma j \rho^\gamma_m u \quad \text{on} \quad \Gamma \]

**Remark 7** Boundedness of \( \tilde{P}^\gamma_m \) follows Remark \ref{rem3} and Proposition \ref{prop7}

**Proposition 13**: We obviously have \( \tilde{P}^\gamma_m \tilde{P}^{-\gamma}_m = \tilde{P}^{-\gamma}_m \tilde{P}^\gamma_m = Id_{\Lambda'} \)
Notation 4 Let \( \tilde{A}^\gamma \) denote the linear bounded operator in \( \Lambda' \times \Lambda' \) given by:

\[
\tilde{A}^\gamma = \begin{pmatrix}
0 & -\tilde{P}^\gamma \\
-\tilde{P}_2 & 0
\end{pmatrix}
\]

Remark 8: The inverse of \( \tilde{A}^\gamma \) in \( \Lambda' \times \Lambda' \) is the bounded linear operator given by:

\[
(\tilde{A}^\gamma)^{-1} = \begin{pmatrix}
0 & -\tilde{P}^{-\gamma} \\
-\tilde{P}_1^{-\gamma} & 0
\end{pmatrix}
\]

In order to use conservation of energy, and to gain compactness, we use Theorem 5 to introduce:

Notation 5 For \( \gamma \neq 0 \) and \( k \in \mathbb{R} \), let the bounded operator in \( L^2(\Gamma) \) denoted by \( P^\gamma_m \) be the restriction of \( \tilde{P}^\gamma_m \) to \( L^2(\Gamma) \). Let the bounded operator in \( L^2(\Gamma) \times L^2(\Gamma) \) denoted by \( A^\gamma \) be the restriction of \( \tilde{A}^\gamma \) to \( L^2(\Gamma) \times L^2(\Gamma) \).

Conservation of energy fluxes through \( \Gamma \) reads:

**Proposition 14** Let \( \gamma \neq 0 \) and \( m = 1, 2 \).

(i) \( P^\gamma_m \) is an isometry in \( L^2(\Gamma) \):

\[
\forall \nu \in L^2(\Gamma), \quad \|\nu\|_{L^2(\Gamma)} = \|P^\gamma_m \nu\|_{L^2(\Gamma)}
\]

(ii) \( A^\gamma \) is an isometry in \( L^2(\Gamma) \times L^2(\Gamma) \):

\[
\forall (\nu, \eta) \in L^2(\Gamma) \times L^2(\Gamma), \quad \|(\nu, \eta)\|_{L^2(\Gamma) \times L^2(\Gamma)} = \|A^\gamma(\nu, \eta)\|_{L^2(\Gamma) \times L^2(\Gamma)}
\]

**proof:** For \( \nu \in L^2(\Gamma) \) let \( u \in H_m \) solve by Theorem 5 the following equation:

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_m, \quad \frac{\partial u}{\partial n_m} - i\gamma j \rho^\gamma_m u = \nu \quad \Gamma
\]

Multiplying by \( \overline{u} \) the equation fulfilled by \( u \) and integrating on \( \Omega_m \) gives

\[
\int_{\Omega_m} |\nabla u|^2 \, dx - k^2 \int_{\Omega_m} |u|^2 \, dx = \int_{\Gamma} \frac{\partial u}{\partial n_m} \overline{u} \, d\sigma
\]

Taking the imaginary part gives

\[
\Im \int_{\Gamma} \frac{\partial u}{\partial n_m} \overline{u} \, d\sigma = 0
\]
The result follows integration of the following identity on $\Gamma$:

$$\left| \frac{\partial u}{\partial n_m} + i\gamma j \rho_m^\gamma u \right|^2 - \left| \frac{\partial u}{\partial n_m} - i\gamma j \rho_m^\gamma u \right|^2 = 4i\gamma \mathbb{I} m \, \pi \, \frac{\partial u}{\partial n_m}$$

An important consequence of this property will be crucial in the next section:

**Corollary 6:** For $m = 1, 2$ and $\gamma \neq 0$,

(i) $P^\gamma_m$ is a normal operator in $L^2(\Gamma)$

(ii) $A^\gamma$ is a normal operator in $L^2(\Gamma) \times L^2(\Gamma)$

### 4.4 Spectral properties of the Despres operators

In the preceding section we proved that the Despres operator $P^\gamma_m$, ($m = 1, 2$ and $\gamma \neq 0$) is a bijective isometry in $L^2(\Gamma)$, and consequently a normal operator in $L^2(\Gamma)$. It follows that its spectrum is a subset of the unit circle in the complex plane. We now investigate this spectrum more accurately.

**Definition 2** Let $\gamma \neq 0$ and $m = 1, 2$. Let $C^\gamma_m$ be the operator in $L^2(\Gamma)$ given by:

$$\forall \nu \in L^2(\Gamma), \quad C^\gamma_m \nu = j \rho_m^\gamma u$$

where $u \in \Lambda$ is the solution given by Theorem 5 of the equation:

$$(\Delta + k^2)u = 0 \text{ in } \Omega_m, \quad \frac{\partial u}{\partial n_m} - i\gamma \rho_m^\gamma u = \nu \text{ on } \Gamma$$

Compactness of the injection $j$ from $\Lambda$ to $L^2(\gamma)$ gives:

**Proposition 15**:

(i) For $m = 1, 2$ and $\gamma \neq 0$, $P^\gamma_m = I + 2i\gamma C^\gamma_m$

(ii) $C^\gamma_m$ is a normal and compact operator in $L^2(\gamma)$

**Notation 6** Let

(i) $\Sigma^\gamma_m$ denote the spectrum of $P^\gamma_m$

(ii) $\Sigma^{\text{Dir}}_m$ denote the sequence of eigenvalues of the Laplace operator on $\Omega_m$ with Dirichlet boundary condition on $\partial\Omega_m$, i.e.

$-k^2 \notin \Sigma^{\text{Dir}}_m$ if and only if the following problem is well posed:

$$(\Delta + k^2)u = f \in L^2(\Omega_m), \quad u \in H^1_0(\Omega_m)$$
We have:

**Proposition 16**: Let $\gamma \neq 0$, and $m = 1, 2$.

(i) $1$ belongs to $\Sigma^m_\gamma$

(ii) $1$ is an eigenvalue of $P^\gamma_m$ if and only if $-k^2 \in \Sigma^\text{Dir}_m$

**proof:**

(i) Because $C^\gamma_m$ is a compact normal operator (Proposition 15) in $L^2(\Omega_m)$, its spectrum is a sequence of eigenvalues and its limit zero. This implies that $1$ is in the closure of $\Sigma^\gamma_m$, which is closed.

(ii) If $-k^2 \in \Sigma^\text{Dir}_m$, then there exist an eigenfunction $\varphi_k$ of the Laplace operator such that

$$
\Delta \varphi_k + k^2 \varphi_k = 0 \quad \text{and} \quad \varphi_k|_{\Gamma} = 0
$$

Let $\nu_k = \frac{\partial \varphi_k}{\partial n_m}|_{\Gamma}$. We have $\nu_k \neq 0$ or else $\varphi_k$ solving an elliptic equation, condition $\varphi_k|_{\Gamma} = 0$ and connectedness of $\Omega_m$ would imply $\varphi_k = 0$ on $\Omega_m$, which contradicts the fact that $\varphi_k$ is an eigenfunction. Obviously $\nu_k$ is an eigenvector of $P^\gamma_m$ for the eigenvalue $1$.

Conversely, if $1$ is an eigenvalue of $P^\gamma_m$, with eigenvector $\nu \neq 0$, then there exist $u$ in $H_m$ such that

$$
\Delta u + k^2 u = 0 \quad \text{and} \quad \nu = \frac{\partial u}{\partial n_m}|_{\Gamma} - i\gamma j \rho^\Gamma_m u = \frac{\partial u}{\partial n_m}|_{\Gamma} + i\gamma j \rho^\Gamma_m u
$$

This implies $\rho^\Gamma_m u = 0$, which added to the fact that $u \in H_m$ implies $u \in H^1_0(\Omega_m)$. So $u$ is an eigenfunction of the Laplace operator for the eigenvalue $-k^2$, provided $u$ is not identically zero. And this is ruled out because $\frac{\partial u}{\partial n_m}|_{\Gamma} = \nu \neq 0$

The following Theorem gives a complete spectral description of the Despres operators:

**Theorem 6**: Assume that $-k^2 \notin \Sigma^\text{Dir}_m$. For $\gamma \neq 0$ and $m = 1, 2$:

(i) $\Sigma^\gamma_m = \{1\} \cup (e^{i\sigma^m_n})_{n \in \mathbb{N}}$ where $(\sigma^m_n)_{n \in \mathbb{N}}$ is a sequence of numbers with $\sigma^m_n \in R$, $\sigma^m_n \neq 0$, and $\sigma^m_n \longrightarrow 0$ when $n \longrightarrow \infty$.

(ii) For each $n \in \mathbb{N}$, $e^{i\sigma^m_n}$ is an eigenvalue of $P^\gamma_m$, with finite multiplicity.

(iii) $L^2(\Gamma)$ is the Hilbert direct sum of the eigenspaces associated with the eigenvalues $e^{i\sigma^m_n}$.
Proof: $C_m^\gamma$ is a normal and compact operator in $L^2(\Gamma)$ (proposition 15). Its kernel is trivial (proposition 16). The diagonalization theorem gives a sequence $(\lambda_m^n)_{n\in\mathbb{N}}$, $(\lambda_m^n \neq 0)$, for its eigenvalues (they have finite multiplicity), with limit zero, and $L^2(\Gamma)$ is the Hilbert direct sum of the associated eigenspaces. The diagonalization of $P_m^\gamma = I + 2i\gamma C_m^\gamma$ follows, with eigenvalues $(1 + 2i\gamma \lambda_m^n)_{n\in\mathbb{N}}, (1 + 2i\gamma \lambda_m^n \neq 1)$. Proposition [14] implies that these eigenvalues have modulus one: we set $1 + 2i\gamma \lambda_m^n = e^{i\sigma_m^n}$.

### 4.5 Spectral Properties of $P_1^\gamma P_2^\gamma$ and $P_2^\gamma P_1^\gamma$

Properties of the intertwining operator $A^\gamma$ rely heavily on the spectral properties of $P_1^\gamma P_2^\gamma$ and $P_2^\gamma P_1^\gamma$ that we investigate now.

We first list obvious properties which follow from the previous section:

**Proposition 17**:
(i) $P_1^\gamma P_2^\gamma$ and $P_2^\gamma P_1^\gamma$ are isometric bijections in $L^2(\gamma)$.
(ii) $P_1^\gamma P_2^\gamma$ and $P_2^\gamma P_1^\gamma$ are normal operators in $L^2(\gamma)$.
(iii) $P_1^\gamma P_2^\gamma - I$ and $P_2^\gamma P_1^\gamma - I$ are compact operators in $L^2(\gamma)$.

**Proof:**
(i) follows the fact that $P_1^\gamma$ and $P_2^\gamma$ are isometric bijections in $L^2(\Gamma)$.
(ii) follows (i)
(iii) follows proposition [15] through:

$$P_1^\gamma P_2^\gamma = (I + 2i\gamma C_1^\gamma)(I + 2i\gamma C_2^\gamma) = I + 2i\gamma C_1^\gamma + 2i\gamma C_2^\gamma - 4\gamma^2 C_1^\gamma C_2^\gamma$$

An important property that we shall need is the spectral status of 1:

**Proposition 18**: 1 is not an eigenvalue of $P_1^\gamma P_2^\gamma$ or $P_2^\gamma P_1^\gamma$ in $L^2(\Gamma)$.

**Proof**: By symmetry, it is enough to prove it for $P_1^\gamma P_2^\gamma$. Let $\nu \in L^2(\Gamma)$ be such that $P_1^\gamma P_2^\gamma \nu = \nu$. This translates to the existence of $u_1 \in H_1$ and $u_2 \in H_2$ satisfying:

$$\Delta u_2 + k^2 u_2 = 0 \text{ in } \Omega_2; \quad \frac{\partial u_2}{\partial n_2} - ik u_2 = \nu, \quad \frac{\partial u_2}{\partial n_2} + ik u_2 = P_2^\gamma \nu \quad \text{on } \Gamma$$
\[ \Delta u_1 + k^2 u_1 = 0 \text{ in } \Omega_1; \quad \frac{\partial u_1}{\partial n_1} - iku_1 = P_2^\gamma \nu, \quad \frac{\partial u_1}{\partial n_1} + iku_1 = \nu \text{ on } \Gamma. \]

This implies that on \( \Gamma \) these functions fulfill:

\[ \frac{\partial u_1}{\partial n_1} - iku_1 - \frac{\partial u_2}{\partial n_2} - iku_2 = 0; \quad \frac{\partial u_1}{\partial n_1} + iku_1 - \frac{\partial u_2}{\partial n_2} + iku_2 = 0 \]

Adding and subtracting gives:

\[ u_1 = -u_2 \quad \text{and} \quad \frac{\partial u_1}{\partial n_1} = \frac{\partial u_2}{\partial n_2} \]

We define \( u \) on \( \Omega \) as \( u_{|\Omega_1} = u_1 \) and \( u_{|\Omega_2} = -u_2 \). It solves the Helmholtz equation on \( \Omega_1 \) and \( \Omega_2 \), its has no jump accross \( \Gamma \), neither has its normal derivative. So it solves Helmholtz equation on \( \Omega \). Moreover \( u_{|\partial\Omega} = 0 \). Assumption (A) gives \( u = 0 \), so \( u_1 = 0 \) and \( u_2 = 0 \); and \( \nu = 0 \) follows.

**Proposition 19:**

(i) The spectrum of \( P_1^\gamma P_2^\gamma \) in \( L^2(\Gamma) \) is \( \{1\} \cup (e^{i\tau_{12}^n})_{n \in \mathbb{N}} \), where \( (\tau_{12}^n)_{n \in \mathbb{N}} \) is an infinite sequence of real numbers, \( \tau_{12}^n \neq 0 \), and \( \tau_{12}^n \to 0 \) when \( n \to \infty \).

(ii) \( (e^{i\tau_{12}^n})_{n \in \mathbb{N}} \) is the set of eigenvalues of \( P_1^\gamma P_2^\gamma \). They have finite multiplicity. If we denote by \( E_{12}^n \) the eigenspace associated with \( e^{i\tau_{12}^n} \), then \( L^2(\Gamma) \) is the Hilbert direct sum of the subspaces \( (E_{12}^n)_{n \in \mathbb{N}} \).

(iii) \( P_2^\gamma P_1^\gamma \) has the same properties, and we set the obvious notations: \( (e^{i\tau_{21}^n})_{n \in \mathbb{N}} \) for eigenvalues and \( (E_{21}^k)_{n \in \mathbb{N}} \) for eigenspaces.

**Proof:** The operator \( P_1^\gamma P_2^\gamma - I \) is a normal compact operator (proposition 17). So by the diagonalization theorem its spectrum is the union of \{1\} and an infinite sequence of eigenvalues with finite multiplicity \( (t^n)_{n \in \mathbb{N}}, \) \( t^n \neq 0 \). Zero is not an eigenvalue of \( P_1^\gamma P_2^\gamma - I \) (proposition 18). So the whole set of eigenvalues is \( (t^n)_{n \in \mathbb{N}} \). If \( E_{12}^n \) denotes the eigenspace associated with \( t^n \), then \( L^2(\Gamma) \) is the Hilbert direct sum of \( (E_{12}^n)_{n \in \mathbb{N}} \). By proposition 17 we know that \( P_1^\gamma P_2^\gamma \) is an isometry in \( L^2(\Gamma) \), so \(|1 + t^n| = 1\), and we write it: \( t^n = e^{i\tau_{12}^n} \).

The theorem translates proven properties of \( t_n \) into properties of \( \tau_{12}^n \).

In order to study the relationship between \( (\tau_{12}^n)_{n \in \mathbb{N}} \) and \( (\tau_{21}^n)_{n \in \mathbb{N}} \), and between \( (E_{12}^n)_{n \in \mathbb{N}} \) and \( (E_{21}^n)_{n \in \mathbb{N}} \), we prove the following lemmi:
Lemma 1 Let $\gamma \neq 0$. For $m = 1, 2$ and any $\nu \in L^2(\Gamma)$: $P_m\nu = P_m^{-\gamma}\overline{\nu}$

Proof: Let $m = 1$ or 2. By definition of $P_m^\gamma$ there exists $u^m \in H_m$ with:

$$\Delta u^m + k^2 u^m = 0; \quad \frac{\partial u^m}{\partial n_m |\Gamma} - i\gamma j \rho_m u^m = \nu; \quad \frac{\partial u^m}{\partial n_m |\Gamma} + i\gamma j \rho_m u^m = P_m^\gamma \nu$$

Taking the complex conjugate of these equalities gives $\overline{u^m} \in H_m$ such that

$$\Delta \overline{u^m} + k^2 \overline{u^m} = 0; \quad \frac{\partial \overline{u^m}}{\partial n_m |\Gamma} - i\gamma j \rho_m \overline{u^m} = \overline{\nu}; \quad \frac{\partial \overline{u^m}}{\partial n_m |\Gamma} + i\gamma j \rho_m \overline{u^m} = \overline{P_m^\gamma \nu}$$

which by definition of $P_m^\gamma$ writes

$$P_m^{-\gamma} \overline{\nu} = \overline{P_m^\gamma \nu}$$

Lemma 2 Let $\gamma \neq 0$

(i) If $\lambda$ is an eigenvalue of $P_1^\gamma P_2^\gamma$ (resp $P_2^\gamma P_1^\gamma$) for the eigenvector $\nu$ then it is an eigenvalue of $P_2^\gamma P_1^\gamma$ (resp $P_1^\gamma P_2^\gamma$) with associated eigenvector $\overline{\nu}$

(ii) For all $n \in N$, $\tau_1^n = \tau_2^n \pmod{2\pi}$; we denote it by $\tau_n$

(iii) If we denote by $C$ the set of complex conjugates of distributions in a set $C$, then, for any $n \in N$,

$$E_{21}^n = E_{12}^n \quad \text{and} \quad E_{12}^n = E_{21}^n$$

Proof:

(i) If $P_1^\gamma P_2^\gamma \nu = \lambda \nu$ then $P_2^\gamma P_1^\gamma \overline{\nu} = \overline{\lambda \nu}$ which by lemma 1 writes $P_1^\gamma P_2^\gamma \overline{\nu} = \overline{\nu}$, which implies, by proposition 13 $P_2^\gamma P_1^\gamma \overline{\nu} = \overline{\nu} = \overline{\lambda \nu}$ because $\frac{1}{\lambda} = \lambda$ by proposition 19.

(ii) follows (i) and a renumbering.

(iii) follows (i) because it gives:

$$E_{21}^n \subset E_{12}^n \quad \text{and} \quad E_{12}^n \subset E_{21}^n$$

but then

$$E_{21}^n \subset E_{12}^n \subset E_{21}^k \quad \text{and} \quad E_{12}^n \subset E_{21}^n \subset E_{12}^n$$

which gives

$$E_{21}^n = E_{12}^n \quad \text{and} \quad E_{12}^n = E_{21}^n$$

Lemma 3: Let $\gamma \neq 0$. For any $n \in N$, $P_1^\gamma E_{21}^n = E_{12}^n$ and $P_2^\gamma E_{12}^n = E_{21}^n$
Proof: Let \( \nu \neq 0 \in E_{12}^n \) then \( P_1 \gamma P_2 \gamma \nu = \mu_{21}\nu \) so \( P_1 \gamma P_2 \gamma \nu = \epsilon_{21}\nu \) which proves that \( P_2 \gamma \nu \in E_{21}^n \) \((P_2 \gamma \nu \neq 0 \) because \( P_2 \gamma \) is bijective \( \) (proposition \[13\]).

This writes \( P_2 \gamma E_{12}^n \subset E_{21}^n \). Proposition \[13\] lemma \[2\] and invertibility of \( P_2 \gamma \) (proposition \[13\]) give \( \dim E_{21}^n = \dim E_{12}^n = \dim P_2 \gamma E_{12}^n \) so \( P_2 \gamma E_{12}^n = E_{21}^n \).

The following algebraic property and its consequences on the eigenprojectors (next theorem) are a key for understanding the geometric properties of the intertwining operator:

**Lemma 4**: Let \( \gamma \neq 0 \). For any \( \mu \notin \{1\} \cup (e^{i\tau})_{\tau \in \mathbb{R}} \):

\[
(P_1 \gamma P_2 \gamma - \mu I)^{-1} P_1 \gamma = P_1 \gamma (P_2 \gamma P_1 \gamma - \mu I)^{-1}
\]

\[
(P_2 \gamma P_1 \gamma - \mu I)^{-1} P_2 \gamma = P_2 \gamma (P_1 \gamma P_2 \gamma - \mu I)^{-1}
\]

Proof: we prove the second assertion, using resolvant identity:

\[
(P_2 \gamma P_1 \gamma - \mu I)^{-1} P_2 \gamma = (P_2 \gamma P_1 \gamma - \mu I)^{-1} P_2 \gamma (P_1 \gamma P_2 \gamma - \mu I)(P_1 \gamma P_2 \gamma - \mu I)^{-1} =
\]

\[
[[I + \mu(P_2 \gamma P_1 \gamma - \mu I)^{-1}]P_2 \gamma - \mu (P_2 \gamma P_1 \gamma - \mu I)^{-1} P_2 \gamma](P_1 \gamma P_2 \gamma - \mu I)^{-1} = P_2 \gamma (P_1 \gamma P_2 \gamma - \mu I)^{-1}
\]

**Lemma 5**: Let \( \gamma \neq 0 \). For any \( n \in \mathbb{N} \), if \( \Pi_1 \gamma \) (resp. \( \Pi_2 \gamma \)) denotes the spectral projector of the operator \( P_1 \gamma P_2 \gamma \) (resp. \( P_2 \gamma P_1 \gamma \)) on the eigenspace \( E_{12}^n \) (resp. \( E_{21}^n \)) then we have

\[
P_1 \gamma \Pi_2 \gamma = \Pi_1 \gamma P_2 \gamma \quad \text{and} \quad P_2 \gamma \Pi_1 \gamma = \Pi_2 \gamma P_1 \gamma
\]

Proof: By symmetry it is enough to prove the first formula. Let \( C_\gamma \) denote a positively oriented curve in the complex plane, which winds one time around the eigenvalue \( e^{i\tau} \), and none around any other eigenvalue, then the Dunford integral representation formula gives:

\[
\Pi_1 \gamma = \frac{-1}{2i\pi} \int_{C_\gamma} (P_1 \gamma P_2 \gamma - \mu I)^{-1} d\mu \quad \text{and} \quad \Pi_2 \gamma = \frac{-1}{2i\pi} \int_{C_\gamma} (P_2 \gamma P_1 \gamma - \mu I)^{-1} d\mu
\]

The previous lemma gives the following:

\[
P_2 \gamma \Pi_1 \gamma = \frac{-1}{2i\pi} \int_{C_\gamma} P_2 \gamma (P_1 \gamma P_2 \gamma - \mu I)^{-1} d\mu = \frac{-1}{2i\pi} \int_{C_\gamma} (P_2 \gamma P_1 \gamma - \mu I)^{-1} P_2 \gamma d\mu = \Pi_2 \gamma P_1 \gamma
\]
4.6 Spectral Properties of $A^\gamma$

We recall that

$$A^\gamma = \begin{pmatrix} 0 & -P_1^\gamma \\ -P_2^\gamma & 0 \end{pmatrix}$$

and that this operator is a bijective isometry in $L^2(\Gamma) \times L^2(\Gamma)$

**Theorem 7**: Let $\gamma \neq 0$.

(i) If $\lambda \notin \{\pm 1\} \cup \{\pm e^{i\frac{2\pi}{n}}\}_{n \in \mathbb{N}}$ then $\lambda$ belongs to the resolvant set of $A^\gamma$ and

$$(A^\gamma - \lambda I)^{-1} = \begin{pmatrix} \lambda(P_1^\gamma P_2^\gamma - \lambda^2 I)^{-1} & -(P_1^\gamma P_2^\gamma - \lambda^2 I)^{-1}P_1^\gamma \\ -(P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1}P_2^\gamma & \lambda(P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1} \end{pmatrix}$$

(ii) For any $n \in \mathbb{N}$, $\pm e^{i\frac{2\pi}{n}}$ is an eigenvalue of $A^\gamma$ with associated eigenspace:

$$F^\pm_n = \{(\mu, \mp e^{-i\frac{2\pi}{2n}}P_2^\gamma \mu); \quad \mu \in E_{12}^n\}$$

and associated eigenprojector:

$$P^\pm_n = \begin{pmatrix} \frac{1}{2}\Pi^n_{12} & \frac{1}{2}e^{-i\frac{2\pi}{2n}}P_1^\gamma \Pi^n_{21} \\ \mp \frac{1}{2}e^{-i\frac{2\pi}{2n}}P_2^\gamma \Pi^n_{12} & \frac{1}{2}\Pi^n_{21} \end{pmatrix}$$

(iii) $\{\pm 1\}$ belong to the spectrum of $A^\gamma$ and are not eigenvalues of $A^\gamma$

(iv) $(F^\pm_n)_{n \in \mathbb{N}: \pm}$ is an orthogonal family of subspaces and we have the Hilbert decomposition

$$L^2(\Gamma) \times L^2(\Gamma) = (\oplus_0^\infty F^+_n) \oplus (\oplus_0^\infty F^-_n)$$

(v) The following series are strongly convergent in $\mathcal{L}(L^2(\Gamma) \times L^2(\Gamma))$:

$$I = \sum_0^\infty P^+_n + \sum_0^\infty P^-_n$$

$$A^\gamma = \sum_0^\infty e^{i\frac{2\pi}{2n}} P^+_n - \sum_0^\infty e^{i\frac{2\pi}{2n}} P^-_n$$

**Proof**: 

(i) let $\lambda \notin \{\pm 1\} \cup \{\pm e^{i\frac{2\pi}{n}}\}_{n \in \mathbb{N}}$

$(A^\gamma - \lambda I)$ is injective: let $(\varphi, \psi) \in L^2(\Gamma) \times L^2(\Gamma)$ be such that

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
this writes
\[ P_1^\gamma \psi + \lambda \varphi = 0 \quad \text{and} \quad P_2^\gamma \varphi + \lambda \psi = 0 \]
which implies
\[ P_2^\gamma P_1^\gamma \psi - \lambda^2 \psi = 0 \quad \text{and} \quad P_1^\gamma P_2^\gamma \varphi - \lambda^2 \varphi = 0 \]
This implies \( \varphi = \psi = 0 \) by proposition 19 and lemma 2.
\((A^\gamma - \lambda I)\) is surjective: For any \((\xi, \eta) \in L^2(\Gamma) \times L^2(\Gamma)\) let:
\[
\varphi = (P_1^\gamma P_2^\gamma - \lambda^2 I)^{-1}(\lambda \xi - P_1^\gamma \eta) \quad \text{and} \quad \psi = (P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1}(\lambda \eta - P_2^\gamma \xi)
\]
We have, by lemma 3
\[
P_2^\gamma \varphi + \lambda \psi = P_2^\gamma (P_1^\gamma P_2^\gamma - \lambda^2 I)^{-1}(\lambda \xi - P_1^\gamma \eta) + \lambda(P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1}(\lambda \eta - P_2^\gamma \xi)
\]
\[
= (P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1}(\lambda P_2^\gamma \xi - P_2^\gamma P_1^\gamma \eta) + \lambda(P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1}(\lambda \eta - P_2^\gamma \xi)
\]
\[
= (P_2^\gamma P_1^\gamma - \lambda^2 I)^{-1}(-P_2^\gamma P_1^\gamma \eta + \lambda^2 \eta) = -\eta
\]
Similarly
\[ P_1^\gamma \psi + \lambda \varphi = -\xi \]
These two equalities write
\[
(A^\gamma - \lambda I) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}
\]
So surjectivity is proven. These expressions for \((\varphi, \psi)\) give the formula for the resolvent of \(A\).

(ii) By definition of \(E_{12}^n\) we have for any \(\mu \in E_{12}^n:\)
\[ A^\gamma(\mu , \mp e^{-i\frac{\tau}{2}} P_2^\gamma \mu) = (\pm e^{-i\frac{\tau}{2}} P_1^\gamma P_2^\gamma \mu , -P_2^\gamma \mu) =
\]
\[
(\pm e^{i\frac{\tau}{2}} \mu , -P_2^\gamma \mu) = e^{i\frac{\tau}{2}} (\mu , \mp e^{i\frac{\tau}{2}} P_2^\gamma \mu)
\]
Because \(E_{12}^n \neq \{0\}\), this proves that \(\pm e^{i\frac{\tau}{2}}\) is an eigenvalue of \(A^\gamma\). This proves moreover that \(F_\pm^n\) is a subset of the eigenspace of \(A^\gamma\) associated with the eigenvalue \(\pm e^{i\frac{\tau}{2}}\).
On the other hand, if \((\xi, \eta)\) is an eigenvector of \(A^\gamma\) for the eigenvalue \(\pm e^{i\frac{\tau}{2}}\) then
\[
-P_1^\gamma \eta = \pm e^{i\frac{\tau}{2}} \xi \quad \text{and} \quad -P_2^\gamma \xi = \pm e^{i\frac{\tau}{2}} \eta
\]
This implies
\[ P_1^\gamma P_2^\gamma \xi = \mp e^{i\frac{2\pi}{n}} P_1^\gamma \eta = e^{i\tau_n} \xi \quad \text{so} \quad \xi \in E_{12}^n \]
and
\[ \eta = \mp e^{-i\frac{2\pi}{n}} P_2^\gamma \xi \]
This completes the characterisation of the eigenspace.

We compute now the eigenprojector: for this sake, we make a choice of a branch for \( \sqrt{z} \). We take a positively oriented curve \( C_n^\pm \) in the complex plane which winds one time around \( \pm e^{i\frac{2\pi}{n}} \) and not around \( \mp e^{i\frac{2\pi}{n}} \) for \( n' \neq n \). Let \( D_n \) be the image of \( C_n^\pm \) by the function \( z \rightarrow z^2 \). \( D_n \) winds one time around \( e^{i\tau_n} \) and does not wind around \( e^{i\tau_{n'}} \) for \( n' \neq n \). Let \( D'_n \) winds one time around \( e^{i\tau_n} \), lying in the interior set delimited by \( D_n \).

The eigenprojector is given by the Dunford formula:
\[ P_n^\pm = -\frac{1}{2i\pi} \int_{C_n^\pm} (A^\gamma - \lambda I)^{-1} d\lambda \]
Using the representation formula given by (i) for \((A^\gamma - \lambda I)^{-1}\) leads to compute integrals of two different types:
For the first type it is straightforward and gives:
\[ -\frac{1}{2i\pi} \int_{C_n^\pm} \lambda (P_1^\gamma P_2^\gamma - \lambda^2 I)^{-1} d\lambda = -\frac{1}{4i\pi} \int_{D_n} (P_1^\gamma P_2^\gamma - \lambda I)^{-1} d\lambda = \frac{1}{2} \Pi_{12}^n \]
For the second type, we first use the resolvent identity to have:
\[ -\frac{1}{2i\pi} \int_{D_n} (P_1^\gamma P_2^\gamma - \lambda I)^{-1} \Pi_{12}^n \frac{d\lambda}{2\sqrt{\lambda}} = \]
\[ -\frac{1}{2i\pi} \int_{D_n} (P_1^\gamma P_2^\gamma - \lambda I)^{-1} \frac{d\lambda}{2\sqrt{\lambda}} = -\frac{1}{2i\pi} \int_{D'_n} (P_1^\gamma P_2^\gamma - \mu I)^{-1} d\mu = \]
\[ -\frac{1}{2i\pi} \frac{1}{2i\pi} \int_{D_n} \int_{D'_n} (P_1^\gamma P_2^\gamma - \lambda I)^{-1} (P_1^\gamma P_2^\gamma - \mu I)^{-1} d\lambda d\mu = \]
\[ -\frac{1}{2i\pi} \frac{1}{2i\pi} \int_{D_n} \int_{D'_n} \left( \frac{d\mu}{\lambda - \mu} \right) \frac{d\lambda}{2\sqrt{\lambda}} = \]
\[ -\frac{1}{2i\pi} \frac{1}{2i\pi} \int_{D_n} \left( \int_{D'_n} \frac{d\mu}{\lambda - \mu} \right) \frac{d\lambda}{2\sqrt{\lambda}} = \]
\[ -\frac{1}{2i\pi} \frac{1}{2i\pi} \int_{D'_n} \left( \int_{D_n} \frac{d\mu}{2\sqrt{\lambda}(\lambda - \mu)} \right) d\mu = \]
We compute now the second type of integral using this equality, properties of $\Pi_{n12}$, and lemma 5 to have:

\[
-\frac{1}{2i\pi} \int_{C_{n12}^+} (P_{1}^\gamma P_{2}^\gamma - \lambda^2 I)^{-1} P_1^\gamma d\lambda = -\left( \frac{1}{2i\pi} \int_{D_n} (P_{1}^\gamma P_{2}^\gamma - \lambda I)^{-1} \frac{d\lambda}{2\sqrt{\lambda}} \right) P_1^\gamma = \\
- \left( \frac{1}{2i\pi} \int_{D_n} (P_{1}^\gamma P_{2}^\gamma - \lambda I)^{-1} \Pi_{n12} \frac{d\lambda}{2\sqrt{\lambda}} \right) P_1^\gamma = - \left( \frac{1}{2i\pi} \int_{D_n} \frac{1}{(e^{i\tau n} - \lambda I)\Pi_{n12}^2} \frac{d\lambda}{2\sqrt{\lambda}} \right) P_1^\gamma \\
= \pm \frac{1}{2} e^{-i\frac{\tau n}{2}} \Pi_{n12}^2 P_1^\gamma = \pm \frac{1}{2} e^{-i\frac{\tau n}{2}} P_1^\gamma \Pi_{n21}^2
\]

(iii) $\pm 1$ are limits of the sequence of eigenvalues $(\pm e^{-i\frac{\tau n}{2}})_{n \in \mathbb{N}}$ so they belong to the spectrum of $A$. These values are not eigenvalues, or else 1 is an eigenvalue of $P_{2}^\gamma P_{1}^\gamma$ and $P_{1}^\gamma P_{2}^\gamma$, which is ruled out by proposition 18.

(iv) and (v) Assertions (i), (ii) and (iii) prove that the spectrum of $A^\gamma$ is $\{\pm 1\} \cup (\pm e^{-i\frac{\tau n}{2}})_{n \in \mathbb{N}}$. Normality of $A^\gamma$ (corollary 6) implies orthogonality of the family $(P_{n}^\pm)_{n \in \mathbb{N}}$, and gives the decomposition of $I$ and $A^\gamma$ as series of the eigenprojectors $(P_{n}^\pm)$.

**Remark 9**: Notice that the expression of $F_{n}^\pm$ in (ii) of the previous proposition is symmetric: in fact we have

\[
\{(\mu , \mp e^{-i\frac{\tau n}{2}} P_{1}^\gamma \mu'); \quad \mu \in E_{121}^n\} = \{(-\mp e^{-i\frac{\tau n}{2}} P_{1}^\gamma \mu', \mu'); \quad \mu' \in E_{211}^n\}
\]

this is because $P_{1}^\gamma E_{211}^n = E_{121}^n$ following lemma 8, so if we set $\mu = \mp e^{-i\frac{\tau n}{2}} P_{1}^\gamma \mu'$ it ensures $\mu \in E_{121}^n$ if $\mu' \in E_{211}^n$. Moreover, by definition of $\mu'$ we have:

\[
\mp e^{-i\frac{\tau n}{2}} P_{2}^\gamma \mu = e^{-i\tau n} P_{2}^\gamma P_{1}^\gamma \mu' = \mu'
\]
Domain Decomposition algorithm for the Helmholtz equation

5.1 The Domain Decomposition framework for the Helmholtz equation

**Proposition 20**: Let \( k \in \mathbb{R}, k \neq 0 \). For any \( f \in L^2(\Omega) \), let \( u \in H^1_0(\Omega) \) be the unique solution of the Helmholtz equation

\[
\Delta u + k^2 u = f
\]

Let \( \gamma \in \mathbb{R}, \gamma \neq 0 \). For \( m = 1, 2 \), let \( v_m \in H_m \) solve the equation

\[
\Delta v_m + k^2 v_m = f_{|\Omega_m} \quad \text{and} \quad \frac{\partial v_m}{\partial n_m} - i\gamma j \rho^\Gamma v_m = 0 \quad \text{on} \quad \Gamma
\]

Let

\[
\nu_m = 2i\gamma j \rho^\Gamma v_m \quad \text{and} \quad \eta = (P_1^\gamma \nu_2, P_2^\gamma \nu_1)
\]

Then the equation in \( L^2(\Gamma) \times L^2(\Gamma) \)

\[
(A^\gamma - Id)\pi = \eta
\]

has a unique solution:

\[
\pi = (\pi_1, \pi_2) \quad \text{with} \quad \pi_m = \frac{\partial u}{\partial n_m} + i\gamma j \rho^\Gamma u - 2i\gamma j \rho^\Gamma v_m
\]

**proof**: First notice that the assumption \( f \in L^2(\Omega) \) and assumption (A) on \( k \) imply \( u \in L^2(\Omega) \), so \( \Delta u \in L^2(\Omega) \). Regularity of \( \partial \Omega \) enables the use of classical regularity results ([Ag]) for solutions of elliptic boundary problems to have \( u \in H^2(\Omega) \), hence \( \frac{\partial u}{\partial n_m} \in H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \). This proves that \( \pi_m \in L^2(\Gamma) \).

Let \( w_m = u_{|\Omega_m} - v_m \). Then

\[
\pi_m = \frac{\partial u}{\partial n_m} + i\gamma j \rho^\Gamma u - 2i\gamma j \rho^\Gamma v_m = \frac{\partial w_m}{\partial n_m} + i\gamma j \rho^\Gamma w_m
\]

Because the function \( w_m \) fulfills

\[
\Delta w_m + k^2 w_m = 0, \quad w_m \in H_m, \quad \frac{\partial w_m}{\partial n_m} - i\gamma j \rho^\Gamma w_m = \frac{\partial u}{\partial n_m} - i\gamma j \rho^\Gamma u \quad \text{on} \quad \Gamma
\]
one has

\[ P_m^\gamma \left( \frac{\partial u}{\partial n_m} - i\gamma j\rho^\Gamma u \right) = \pi_m \]

so, if \( m' = 2, 1 \) for \( m = 1, 2 \)

\[ \pi_m = -P_m^\gamma \left( \frac{\partial u}{\partial n_{m'}} + i\gamma j\rho^\Gamma u \right) = -P_m^\gamma (\pi_{m'} + \nu_m') \]

This writes

\[ \pi = A^\gamma \pi - \eta \]

Uniqueness follows from theorem \( \text{[?] (iii).} \)

**Proposition 21**: Let \( k \in \mathbb{R}, k \neq 0, \gamma \in \mathbb{R}, \gamma \neq 0 \). For \( f \in L^2(\Omega) \) and for \( m = 1, 2 \), let \( v_m \in H_m \) solve the equation

\[ \Delta v_m + k^2 v_m = f|_{\Omega_m} \quad \text{and} \quad \frac{\partial v_m}{\partial n_m} - i\gamma j\rho^\Gamma v_m = 0 \] on \( \Gamma \)

Let

\[ \nu_m = 2i\gamma j\rho^\Gamma v_m \quad \text{and} \quad \eta = (P_1^\gamma \nu_2, P_2^\gamma \nu_1) \]

Let \( \pi = (\pi_1, \pi_2) \in L^2(\Gamma) \times L^2(\Gamma) \) solve the equation:

\[ (A^\gamma - \text{Id}) \pi = \eta \]

and let \( u_m \in H_m \) solve the equation

\[ \Delta u_m + k^2 u_m = f|_{\Omega_m} \quad \text{and} \quad \frac{\partial u_m}{\partial n_m} + i\gamma j\rho^\Gamma u_m = \pi_m + \nu_m \] on \( \Gamma \)

Then \( u \) given by \( u|_{\Omega_m} = u_m \) solve the Helmholtz equation

\[ \Delta u + k^2 u = f \quad u \in H^1_0(\Omega) \]

**proof**: By definition of \( u_m \) and \( v_m \) one has:

\[ \Delta (u_m - v_m) + k^2 (u_m - v_m) = 0, \quad u_m - v_m \in H_m \]

and

\[ \frac{\partial (u_m - v_m)}{\partial n_m} + i\gamma j\rho^\Gamma (u_m - v_m) = \pi_m \quad \text{on} \quad \Gamma \]
This implies through proposition 13:

\[(P_m^\gamma)^{-1}\pi_m = P_m^{-1}\pi_m = \frac{\partial(u_m - v_m)}{\partial n_m} - i\gamma j \rho \Gamma (u_m - v_m) = \frac{\partial u_m}{\partial n_m} - i\gamma j \rho \Gamma u_m\]

Because \(\pi = A^\gamma \pi - \eta\) this implies (with \(m' = 2, 1\) for \(m = 1, 2\))

\[\frac{\partial u_m}{\partial n_m} - i\gamma j \rho \Gamma u_m = -\pi_{m'} - \nu_{m'} = -\frac{\partial u_{m'}}{\partial n_{m'}} - i\gamma j \rho \Gamma u_{m'}\]

Adding and subtracting these equalities gives:

\[\frac{\partial u_m}{\partial n_m} = -\frac{\partial u_{m'}}{\partial n_{m'}}\quad \text{and} \quad \rho \Gamma u_m = \rho \Gamma u_{m'}\]

These jump conditions through \(\Gamma\) imply that \(\Delta u + k^2 u = f\) on \(\Omega\), and \(u\) fulfills the Dirichlet boundary condition on \(\partial\Omega\) because \(u_m \in H_m\).

**Remark 10** Theorem 7 shows that the problem

\[(\bar{A} - Id)\pi = \eta\]

is ill-posed for \(\eta \in L^2(\Gamma) \times L^2(\Gamma)\). Proposition 20 shows that if \(\eta\) has the specific form given through the domain decomposition setting for the Helmholtz equation, the equation \((\bar{A} - Id)\pi = \eta\) do have a solution, (and this solution is unique by Theorem 7). Proposition 21 shows that this solution provides the solution of the Helmholtz equation.

### 5.2 The domain decomposition \(\theta\)-algorithm for Helmholtz equation

Let \(f \in L^2(\Omega)\). Let \(u \in H_0^1\) fulfill the non-dissipating Helmholtz equation \(\Delta u + k^2 u = f\) in \(\Omega\). The classical algorithm used, (for dissipating cavities with Sommerfeld-like boundary condition), to solve by a domain decomposition technique the Helmholtz equation ([B], [BD], [D1], [D2], [CGJ]) writes, in the non-dissipating case that discussed here, as follows: for any \(\pi^0 = (\pi_1^0, \pi_2^0)\) given in \(L^2(\Gamma) \times L^2(\Gamma)\) let

\[\pi^{p+1} = \theta \pi^p + (1 - \theta)A^\gamma \pi^p - (1 - \theta)\eta\]
where $\eta = (P_1^\gamma \nu_2, P_2^\gamma \nu_1)$ with $\nu_m = 2i\gamma j \rho^\Gamma v_m$ for $v_m \in H_m$ solving

$$\Delta v_m + k^2 v_m = f_{|\Omega_m} \quad \text{and} \quad \frac{\partial v_m}{\partial n_m} - i\gamma j \rho^\Gamma v_m = 0 \quad \text{on} \quad \Gamma$$

It is straightforward to translate the $\theta$-algorithm in a PDE setting: one use theorem 5 to get the (unique) function $w^m_n \in H_m$ such that

$$\Delta w^m + k^2 w^m = 0 \quad \text{and} \quad \pi^m = \frac{\partial w^m}{\partial n_m} + i\gamma j \rho^\Gamma w^m$$

In these $w^p = (w^p_1, w^p_2)$ variables the $\theta$-algorithm becomes: for $m = 1$ and 2 (resp. $m' = 2$ and 1)

$$\Delta w^{p+1}_m + k^2 w^{p+1}_m = 0, \quad w^{p+1}_m \in H_m$$

$$\frac{\partial w^{p+1}_m}{\partial n_m} - i\gamma j \rho^\Gamma w^{p+1}_m = \theta \left[ \frac{\partial w^{p}_m}{\partial n_m} - i\gamma j \rho^\Gamma w^{p}_m \right] - (1-\theta) \left[ \frac{\partial w^{p}_m}{\partial n_{m'}} + i\gamma j \rho^\Gamma w^{p}_m + \nu_m \right] \quad \text{on} \quad \Gamma$$

For practical use in computing codes, one writes this algorithm in the $w^n = (u^n_1, u^n_2)$ variables with $u^p_m = w^p_m + v_m$ and gets:

$$\Delta u^{p+1}_m + k^2 u^{p+1}_m = f_{|\Omega_m}, \quad u^{p+1}_m \in H_m$$

$$\frac{\partial u^{p+1}_m}{\partial n_m} - i\gamma j \rho^\Gamma u^{p+1}_m = \theta \left[ \frac{\partial u^{p}_m}{\partial n_m} - i\gamma j \rho^\Gamma u^{p}_m \right] - (1-\theta) \left[ \frac{\partial u^{p}_m}{\partial n_{m'}} + i\gamma j \rho^\Gamma u^{p}_m \right] \quad \text{on} \quad \Gamma$$

5.3 Convergence results for the $\theta$-algorithm

Notice that if the sequence $(\pi^p)_{p \in \mathbb{N}}$ has a limit $\pi^\infty$ in $L^2(\Gamma) \times L^2(\Gamma)$ then continuity of $A^\gamma$ gives:

$$(A^\gamma - Id)\pi^\infty = \eta$$

and $\pi^\infty$ provides the solution of the Helmholtz equation on $\Omega$ as stated in proposition 21. Alternatively, a way to solve the Helmholtz equation on $\Omega$ through solving Helmholtz equations on $\Omega_m$, $(m = 1, 2)$, is to notice that convergence of $(\pi^p)_{p \in \mathbb{N}}$ in $L^2(\Gamma) \times L^2(\Gamma)$ implies convergence in $\Lambda' \times \Lambda'$, and theorem 5 shows that the sequence $(w^p)_{p \in \mathbb{N}} = ((w^p_1, w^p_2))_{p \in \mathbb{N}}$ has a limit in $H_1 \times H_2$, which implies convergence of the sequence $(w^p)_{p \in \mathbb{N}} = ((w^p_1, w^p_2))_{p \in \mathbb{N}}$ in $H_1 \times H_2$. 35
Proposition 21 shows that its limit $u^\infty = (u_1^\infty, u_2^\infty)$ provides the solution of the Helmholtz equation, through:

$$(u_1^\infty, u_2^\infty) = (u_{|\Omega_1}, u_{|\Omega_2})$$

Here are the convergence results for the $\theta$-algorithm. We begin with a negative result:

**Proposition 22** If $\theta = 0$ then the sequence $(\pi^p)_{p\in\mathbb{N}}$ has no limit in $L^2(\Gamma) \times L^2(\Gamma)$ unless if its initial value fulfills $(A^\gamma - Id)\pi^0 = \eta$. Written for the $(u^p)_{p\in\mathbb{N}}$ sequence, this translates to $u_1^0 = u_{|\Omega_1}$ and $u_2^0 = u_{|\Omega_2}$ where $u \in H^1_0(\Omega)$ solves $\Delta u + k^2 u = f$ in $\Omega$.

**proof:** for $\theta = 0$ one has $\pi^p - \pi^{p-1} = A^\gamma(\pi^{p-1} - \pi^{p-2})$ and proposition 14 gives

$$\forall p \quad \|\pi^p - \pi^{p-1}\|_{L^2(\Gamma)\times L^2(\Gamma)} = \|\pi^{p-1} - \pi^{p-2}\|_{L^2(\Gamma)\times L^2(\Gamma)}$$

This prevents convergence unless if $\pi^1 = \pi^0$, i.e. $\pi^0$ fulfills

$$\pi^0 = A^\gamma \pi^0 - \eta$$

i.e. unless $u_1^0 = u_{|\Omega_1}$ and $u_2^0 = u_{|\Omega_2}$ by proposition 21.

**Remark 11** If $\theta = 0$ the sequence $(u^p)_{p\in\mathbb{N}}$ may have a limit in $H_1 \times H_2$ even if $u_1^0 \neq u_{|\Omega_1}$ or $u_2^0 \neq u_{|\Omega_2}$. This is because convergence of $(u^p)_{p\in\mathbb{N}}$ in $H_1 \times H_2$ implies convergence of $(\frac{\partial u^p}{\partial n_1} + i\gamma j \rho^\Gamma u_1^p, \frac{\partial u^p}{\partial n_2} + i\gamma j \rho^\Gamma u_2^p)$ in $\Gamma' \times \Gamma'$, i.e. convergence of $(\frac{\partial u^p}{\partial n_1} + i\gamma j \rho^\Gamma u_1^p, \frac{\partial u^p}{\partial n_2} + i\gamma j \rho^\Gamma u_2^p)$ in $\Gamma' \times \Gamma'$, i.e. convergence of $(\pi^p)_{p\in\mathbb{N}}$ in $\Gamma' \times \Gamma'$, which do not contradict divergence in $L^2(\Gamma) \times L^2(\Gamma)$.

We now turn to the main result:

**Theorem 8** : For $f \in L^2(\Omega)$ and $k \in R$ let $u \in H^1_0(\Omega)$ solve $\Delta u + k^2 u = f$. Let $\gamma \neq 0, \gamma \in R$. Then for any $0 < \theta < 1$:

(i) the sequence $(\pi^p)_{p\in\mathbb{N}}$ given by the $\theta$-algorithm converge in $L^2(\Gamma) \times L^2(\Gamma)$ to $\pi^u = (\pi^u_1, \pi^u_2)$ with:

$$\pi^u_1 = \frac{\partial u}{\partial n_1} + i\gamma j \rho^\Gamma u \quad \text{and} \quad \pi^u_2 = \frac{\partial u}{\partial n_2} + i\gamma j \rho^\Gamma u$$

(ii) the sequence $(u^p)_{p\in\mathbb{N}}$ given by the $\theta$-algorithm converge in $H_1 \times H_2$ to $(u_{|\Omega_1}, u_{|\Omega_2})$

(iii) There is no uniform geometric rate of convergence.
Proof:
(i) let \( \pi^u = (\pi^u_1, \pi^u_2) \) with:

\[
\pi^u_1 = \frac{\partial u}{\partial n^1} + i \gamma j \rho^\Gamma u \quad \text{and} \quad \pi^u_2 = \frac{\partial u}{\partial n^2} + i \gamma j \rho^\Gamma u
\]

Nullity of jumps of \( u \) and its normal derivatives through \( \Gamma \) gives

\[\pi^u = A\pi^u + \eta\]

This implies

\[\pi^p - \pi^u = \theta[\pi^{p-1} - \pi^u] + (1 - \theta)A[\pi^{p-1} - \pi^u]\]

We use eigenprojectors of \( A^\gamma \) given by theorem 7 and denote by:

\[\delta^p_{n, \pm} = P_{n}^\pm (\pi^p - \pi^u)\]

Completeness of the set of orthogonal eigenprojectors \( (P^\pm_n)_{n, \pm} \) proved in theorem 7 gives:

\[\pi^p - \pi^u = \sum_n \delta^p_{n, +} + \sum_n \delta^p_{n, -}\]

Decomposition of \( L^2(\Gamma) \times L^2(\Gamma) \) by eigenspaces of \( A^\gamma \) writes for successive terms of the \( \theta \)-algorithm sequence as follows:

\[\delta^p_{n, \pm} = [\theta \pm (1 - \theta)e^{i\frac{\pi}{2}}\delta^{p-1}_{n, \pm}\]

This implies:

\[\|\delta^p_{n, \pm}\|_{L^2(\gamma) \times L^2(\gamma)} = [1 - 2\theta(1 - \theta)(1 \mp \cos \frac{\tau_n}{2})]^{\frac{p}{2}}\|\delta^0_{n, \pm}\|_{L^2(\gamma) \times L^2(\gamma)\}
\]

and orthogonality of the eigenprojectors writes:

\[\|\pi^p - \pi^u\|_{L^2(\gamma) \times L^2(\gamma)}^2 = \sum_{n, \pm}(1 - 2\theta(1 - \theta)(1 \mp \cos \frac{\tau_n}{2}))^{2p}\|\delta^0_{n, \pm}\|^2_{L^2(\gamma) \times L^2(\gamma)}\]

Assumption \( 0 < \theta < 1 \) and theorem 7 (with proposition 18 asserting \( \tau_n \neq 0 \mod 2\pi \)) imply

\[0 < 1 - 2\theta(1 - \theta)(1 \mp \cos \frac{\tau_n}{2}) < 1\]

and Lebesgue convergence theorem gives

\[\pi^p \xrightarrow{L^2(\Gamma) \times L^2(\Gamma)} \pi^u\]

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This proves assertion (i).

Assertion (ii) is straightforward: convergence of $\pi^p$ to $\pi^n$ in $L^2(\Gamma) \times L^2(\Gamma)$ implies its convergence in $\Lambda' \times \Lambda'$ which implies convergence of the related sequence $u^p$ in $H_1 \times H_2$, and accordingly convergence of $u^p$ to $(u_{|\Omega_1}, u_{|\Omega_2})$.

Assertion (iii) is obvious by taking initial data for $\pi^p$ in the $n$–th eigenspace of $A^\gamma$ and notice that $\tau_n \longrightarrow 0$ (proposition [19]).

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