Finite Size Spectrum of $SU(N)$ Principal Chiral Field from Discrete Hirota Dynamics

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Abstract: Using recently proposed method of discrete Hirota dynamics for integrable (1+1)D quantum field theories on a finite space circle of length $L$ we derive and test numerically a finite system of nonlinear integral equations for the exact spectrum of energies of $SU(N) \times SU(N)$ principal chiral field model as functions of $mL$, where $m$ is the mass scale. We propose a determinant solution of the underlying $Y$-system, or Hirota equation, in terms of determinants (Wronskians) of $N \times N$ matrices parameterized by $N-1$ functions of the spectral parameter $\theta$ with the known analytical properties at finite $L$. Although the method works in principle for any state, the explicit equations are written for states in the $U(1)$ sector only. For $N > 2$, we encounter and clarify a few subtleties in these equations related to the presence of bound states, absent in the previously considered $N = 2$ case. In particular, we solve these equations numerically for a few low-lying states for $N = 3$ in a wide range of $mL$.

Keywords: Hirota, QFT, Integrability.

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1. Introduction

Integrable 1 + 1 dimensional quantum field theories on a finite space circle were rather intensively studied in the last 20 years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. A great deal of success in the exact treatment of the finite size effects in various integrable QFT’s was due to the thermodynamic Bethe ansatz (TBA) approach resulting in a system (in most of the cases infinite) of non-linear integral equations. It was realized that the TBA equations could be rewritten in a functional, Y-system form [2].

Recently, a novel, and hopefully quite general approach to this problem was proposed in [11] based on the integrability of the Y-system. The Y-system is known to be a gauge invariant version of the famous Hirota bilinear equation, often called the T-system, in its discrete form [12]. The underlying discrete Hirota dynamics is integrable and general solutions with various boundary conditions, appropriate for various problems, from matrix models to quantum spin chains and quantum sigma-models, can be found in a determinant (Wronskian) form [13]. For finite rank symmetry groups the matrix of the Wronskian contains only a finite number of functions of the spectral parameter, drastically simplifying the problem: the infinite Y-system is reduced to a finite number of non-linear functional equations for these functions. Then the subtlest point comes: We should guess the analytic properties of these functions w.r.t. the spectral parameter. This is relatively easy to do for the spin chains where the polynomiality of transfer matrix leads to the final answer in terms of a set of Bethe ansatz equations. For the QFT’s at a finite volume $L$ (length of the space circle) the situation is much more complicated and the analyticity properties of the Y-functions are not so obvious. Nevertheless, it often appears to be possible to extract them, partially from physical considerations, partially from certain assumptions of the absence of unphysical singularities. It helps to transform the Y-system into a system of non-linear integral equations (NLIE), more tractable, and better suitable for the numerical studies. The resulting equations can remind the Destri-DeVega NLIE [14] or even coincide with them for the (rare) models where these DDV equations are known.

This program was performed in [11] for the $SU(2) \times SU(2)$ principal chiral field (PCF) for a general quantum state, and the numerical study of the finite size spectrum was successfully done for a variety of interesting states, from the vacuum and mass-gap to quite general states, in the so called $U(1)$ sector or even lying out of it (i.e. having excitations in left and right $SU(2)$ modes).

In this paper, we will construct within these lines the corresponding NLIE’s for $SU(N) \times SU(N)$ PCF at any $N$. We use the Wronskian solution of [13] for the underlying Hirota equation in terms of determinants of $N \times N$ matrices and guess the correct analytic form of the functions entering the Wronskian. For the vacuum state, the asymptotic Bethe ansatz (ABA) based on the scattering theory and, strictly speaking, valid only for sufficiently large length $L$, teaches us that there are no singularities on the physical strip of the rapidity plane, at least for not too small $L$’s.\footnote{The argument can’t exclude the possibility that at sufficiently small size, some extra singularities occur. However, our numerics give serious evidence that at least at $N = 3$ such extra singularities do not appear.}

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For excited states there are certain poles entering the physical strip, and their qualitative structure is guessed from the ABA. The explicit construction is done only for states in the $U(1)$ sector, but we sketch out the generalization to any state. We show how the exact S-matrix of the model (including the CDD factor) naturally emerges from this approach by simple analyticity assumptions.

The presence of additional singularities on the physical strip related to the bound states, absent for $N = 2$, leads in $N > 2$ case to significant modifications, already in the expression for the energies of excited states. We find from our NLIE’s the finite size (Lüscher) corrections which reveal the presence of the so called $\mu$-terms. We also test our NLIE’s analytically, comparing the results with the known analytic data in the ultraviolet (conformal) limit. Finally, we demonstrate the power of our approach by solving the resulting NLIE’s numerically, for the vacuum energy and the energies of some low lying excited states as functions of the size $mL$ for $N = 3$.

One of the principal motivations for our work is the hope to fulfill the same program in the case of recently constructed AdS/CFT Y-system [14, 15, 16] for the exact spectrum of anomalous dimensions in $N=4$ supersymmetric Yang-Mills theory. The PCF model, having $N - 1$ particles (including $N - 2$ bound states) in its asymptotic spectrum, bears many similarities with the AdS/CFT case where the number of bound states is infinite. The corresponding Wronskian solutions of AdS/CFT Y-system, or Hirota equation with the so called T-hook boundary conditions is already available [17, 18].

2. The principal chiral field model in the large volume

In this section we will give the definition of the PCF model, remind the reader the scattering theory for the physical particles and the ABA equations, and describe the finite size equations in terms of the Y-system.

2.1 The PCF model, its S-matrix and the large $L$ ABA

The $SU(N) \times SU(N)$ PCF model has the classical action

$$S_{PCF} = -\frac{1}{2e_0^2} \int dt \int_0^L dx \ tr(h^{-1} \partial_\alpha h)^2,$$  \hspace{1cm} h \in SU(N). \hspace{1cm} \hspace{1cm} \hspace{1cm} (2.1)$$

The spectrum of this asymptotically free theory in the infinite volume $L \to \infty$ consists of $N - 1$ physical particles with masses

$$m_a = m \frac{\sin \frac{\pi a}{N}}{\sin \frac{\pi}{N}}$$ \hspace{1cm} (2.2)$$

where the lowest mass scales with the bare charge $e_0$ according to the asymptotic freedom

$$m = \frac{A_0}{e_0} e^{-\frac{4\pi}{N_0}} \ (\Lambda \text{ is a cut-off}).$$

Its wave function transforms in the fundamental representation under each of the $SU(N)$ subgroups. The exact S-matrix for bi-fundamental particles, found from the conditions of factorizability, crossing, unitarity, analyticity and the bound
state structure \[13\], reads\[20, 21\]:\[
\hat{S}_{12}(\theta) = \chi_{CDD}(\theta) \cdot S_0(\theta) \frac{\hat{R}_{L,R}(\theta)}{\theta - i} \otimes S_0(\theta) \frac{\hat{R}_{L,R}(\theta)}{\theta - i}
\]
(2.3)
\[
S_0(\theta) = \frac{\Gamma\left(i \frac{\theta}{N}\right) \Gamma\left(\frac{1-i\theta}{N}\right)}{\Gamma\left(-i \frac{\theta}{N}\right) \Gamma\left(\frac{1+i\theta}{N}\right)}, \quad \chi_{CDD} = \frac{\sinh(\pi\theta/N + i\pi/N)}{\sinh(\pi\theta/N - i\pi/N)}
\]
(2.4)
where we introduced the standard $SU(N)$ R-matrix $\hat{R}_{L,R}(\theta) = \theta + i\hat{P}_{L,R}$ and $\hat{P}$ is the permutation operator exchanging the left/right spins of the scattering particles. In particular, crossing and unitarity lead to the following identity
\[
\prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} S_0(\theta + ik) = (-1)^N \frac{\theta - i\frac{N-1}{2}}{\theta + i\frac{N-1}{2}}
\]
(2.5)
on the scalar (dressing) factor.

We can use this S-matrix to study the spectrum of $N$ particles on a periodic space circle of a sufficiently big circumference $L \gg m^{-1}$ imposing the wave function periodicity
\[
\prod_{j=1}^{N} \hat{S}(\theta_k - \theta_j) \prod_{j=1}^{k-1} \hat{S}(\theta_k - \theta_j)|\Psi \rangle = e^{-imL \sinh(\pi\theta_k)}|\Psi \rangle,
\]
(2.6)
which quantizes the momenta of the physical particles. The asymptotic spectrum is then given by
\[
E \simeq \sum_{j=1}^{N} m \cosh\left(\frac{2\pi}{N} \theta_j\right) + O(e^{-mL})
\]
(2.7)
where $\theta_j$ are solutions to the system of nested Bethe equations following from the diagonalization of (2.6). This diagonalization can be performed by means of the algebraic Bethe ansatz and leads to the ABA equations \[4.20\] and \[4.9\] \[22, 23\].\[3\] We will derive these equations as a large $L$ limit of the Y-system of the model - a system of equations valid at any finite volume $L$ and presented in the next subsection.

The eq. \[4.20\] represents the diagonalized version of the periodicity condition (2.6). \[4.9\] is the set of $2(N-1)$ nested Bethe equations for the auxiliary right and left magnon roots $u_j^{(k)}$ and $v_j^{(k)}$ following from a regularity condition, as we will see in section 4.1.

Note that the ABA equations \[4.9\] remind the Bethe ansatz equations for two inhomogeneous $SU(N)$ spin chains with the inhomogeneity parameters $\theta_j$ given by the rapidities of physical particles. Their dynamics is defined by the periodicity equation \[4.20\]. So the large $L$ limit can be also called the “spin chain limit”.

\[2\]In the $N = 2$ case, these definitions give $\chi_{CDD} = -1$, which can also be expressed by multiplying $S_0$ by $i$.

\[3\]In what follows we will measure all dimensional quantities in the units of the mass $m$, so that we put everywhere $m = 1$. The only continuous parameter of the problem is now the volume $L$. 


2.2 TBA, Y-system and Hirota equation

The generalization of these ABA equations to any length $L$ is achieved by the TBA trick \cite{2}: the system is put on the space time torus, with a finite space period $L$ and a big euclidean “time” period $R \to \infty$. Then, using the relativistic invariance, we exchange the roles of time and space and rather solve the problem for the same system but with infinite space extent $R$ placed into a periodic time $L$ interpreted as the inverse temperature \cite{24}. The full energy spectrum of such an infinite system can be found from the nested BAE (2.4) and from (2.7) by means of the so called string hypothesis. The resulting equations for the densities of bound states are presented in \cite{25}, following the direct solution of the PCF given in \cite{26} (see also \cite{27}). The free energy calculation at a finite temperature for such an infinite volume system can be done thermodynamically, using the saddle point approximation due to \cite{28}. Then the resulting integral TBA equations can be rearranged into the Y-system

\begin{equation}
Y_{a,s}^+ Y_{a,s}^- = \frac{1 + Y_{a,s+1}}{1 + (Y_{a+1,s})^{-1}} \frac{1 + Y_{a,s-1}}{1 + (Y_{a-1,s})^{-1}} \\
a = 1, 2, \ldots, N - 1; \quad -\infty < s < \infty
\end{equation}

where, by definition, $Y_{0,s} = Y_{N,s} = \infty$ and we have the following boundary conditions at $\theta \to \pm \infty$:

\begin{equation}
Y_{a,s} \sim e^{-Lp_a(\theta)\delta_{s,0}} \times \text{const}_{a,s}, \quad p_a = \cosh\left(\frac{2\theta\pi}{N}\right) \frac{\sin\left(\frac{a\pi}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}
\end{equation}

This Y-system describing PCF is an infinite set of functional equations (2.3) with the functions $Y_{a,s}(\theta)$ of the spectral parameter $\theta$ defined in the nodes marked by black and white bullets in the interior of the infinite strip in $a, s$ lattice represented in fig. 2.1.

A direct but rather tedious derivation of this Y-system was performed in the Appendix A of \cite{11} for $N = 2$. The generalization of this calculation to $N > 2$ is rather straightforward, but the Y-system (2.8) is known from other considerations \cite{29} and is a very universal system of equations describing the integrable Hirota dynamics.

\footnote{To make many formulas less bulky, the shifts of the spectral parameter will be often denoted as follows $f^\pm = f(\theta \pm \frac{1}{2})$, $f^{\pm \pm} = f(\theta \pm i)$, and in general $f^{\pm k} = f(\theta \pm \frac{k}{4})$.}
As we will see later, the expression for the momentum $p_a(\theta)$ is the only one compatible with the Y-system and relativistic invariance, up to a normalization that can be absorbed into the definition of the size $L$ of the spin chain. As a result of (2.9) we see that the middle node Y-functions, $Y_{a,0}$, $a = 1, 2, \cdots, N - 1$, are exponentially suppressed at large $L$ or at large $|\theta|$.

Obviously, the Y-system (2.8) has many solutions and to specify a solution we have to describe its analytic properties. To have a qualitative idea of the analyticity we have to consider a certain limit for the solution where we know the corresponding Y-functions entirely as analytic functions of the spectral parameter $\theta$. The most convenient limit is $L \to \infty$ where we can solve the Y-system directly, with the appropriate physically natural analyticity assumptions, to obtain explicitly all Y-functions and make a link with the exact scattering matrix and the resulting ABA equations, as it was done for the $N = 2$ case in [11]. We will give this asymptotic solution in the next section. Then the Y-system, in the form of TBA equations, can be in principle solved numerically by iterations, starting at large $L$ and then adiabatically approaching finite, and even very small $L$'s corresponding to the ultraviolet CFT behavior. The method was successfully used for various integrable sigma models, including the $SU(2)$ PCF [11, 30, 31, 7]. It will be also the main method of this paper devoted to the $SU(N)$ PCF for $N > 2$. For the vacuum state, the information from ABA is trivial: it suggests that we don’t have any singularities in the physical strip $-iN/4 < \text{Im}(\theta) < iN/4$, at least for not too small $L$’s, since there are no Bethe roots.

The TBA procedure described above leads to the following expression for the vacuum energy

$$E_{\text{vacuum}}(L) = -\frac{m}{N} \sum_{a=1}^{N-1} \frac{\sin(\frac{\pi a}{N})}{\sin(\frac{\pi}{N})} \int_{-\infty}^{\infty} d\theta \cosh\left(\frac{2\pi}{N} \theta\right) \log (1 + Y_{a,0}(\theta)) \quad (2.10)$$

With certain modifications in the analytic properties of Y-functions, described in the next section, the equations (2.9-2.10) appear to be appropriate not only for the vacuum state, as it was originally derived from the string hypothesis, but also for the excited states [3, 4]. Y-functions for various excited states differ by their analytic properties which can be qualitatively inferred, as it was mentioned above, from the same states in the ABA. A naive heuristic proposal which worked well for $N = 2$ case is that the excited states correspond to the appearance of logarithmic poles in the integrand of (2.10) at the points $\theta_j$ where

$$Y_{1,0}(\theta_j + iN/4) + 1 = 0 \quad (2.11)$$

If the contour is deformed so that it encircles these singularities, the pole calculation will give a contribution $\sum_{j=1}^{N} m c \cosh(\frac{2\pi \theta_j}{N})$ which fits well the prediction of the ABA formula (2.7). However, the situation appears to be more complicated at $N \geq 3$, already because of the fact that unlike the $N = 2$ case of [11], the solutions $\theta_j$ of (2.11) are not necessarily real and this naive prescription should be slightly modified in order to get a real energy.

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5So that the length $L$ is actually measured in units of mass.

6It does not guarantee that we will not have some singularities entering the physical strip when $L$ becomes small enough. But our numerical result don’t suggest such a strange behavior.
This will be explained in detail in the section 5. One should admit that the right formulas for energies of excited states in the integrable sigma-models are still rather a matter of a natural guess then of a reliable derivation. More insight is needed into this issue.

To solve the $Y$-system equation (2.8) we will often use it in the form of the Hirota equation

$$T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$  \hspace{1cm} (2.12)

on a set $\{T_{a,s}\}$ of functions of the spectral parameter $\theta$ related to the original $Y$-functions as follows

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}.$$  \hspace{1cm} (2.13)

The Hirota equation (2.12) is invariant under the gauge transformation

$$T_{a,s} \rightarrow \chi_1^{[a+s]} \chi_2^{[a-s]} \chi_3^{[-a+s]} \chi_4^{[-a-s]} T_{a,s}$$  \hspace{1cm} (2.14)

so that $T$-functions are gauge dependent, whether as $Y$-functions (2.13) are gauge invariant. Another useful relation following from (2.12) is

$$1 + Y_{a,s} = \frac{T_{a,s}^+ T_{a,s}^-}{T_{a+1,s} T_{a-1,s}}$$  \hspace{1cm} (2.15)

### 3. Central node equations

The central node $Y$-functions $Y_{a,0}$ related to the black, momentum carrying nodes on the fig. 2.1 play a special role in the $Y$-system. It will be useful for the future to solve the corresponding $Y$-system equations for these functions entering the l.h.s. of (2.8) in terms of the r.h.s.

Let us rewrite the $Y$-system (2.8) in the form

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{(Y_{a+1,s})^{1-\delta_{a,N-1}} (Y_{a-1,s})^{1-\delta_{a,1}}} = \frac{1 + Y_{a,s+1}}{(1 + Y_{a+1,s})^{1-\delta_{a,N-1}}} \frac{1 + Y_{a,s-1}}{(1 + Y_{a-1,s})^{1-\delta_{a,1}}}$$  \hspace{1cm} (3.1)

At $s = 0$ it can be rewritten using (2.13) as follows:

$$Y_{a,0}^* \Delta = \frac{T_{a,1}^* (T^{(L)})_{a,-1}^*}{T_{a+1,0}^* T_{a-1,0}^*} \times \left( \frac{T_{N,0}^* T_{N,-1}^*}{T_{N,1}^* T_{N,-1}^*(L)} \right)^{\delta_{a,N-1}} \left( \frac{T_{0,1}^* T_{0,-1}^*}{T_{0,0}^* T_{0,-1}^*(L)} \right)^{\delta_{a,1}}$$  \hspace{1cm} (3.2)

where we introduced a discrete D'Alembert operator $\Delta$ on the interval $a \in [1, N-1]$ defined by the formula\footnote{The terming ”discrete D’Alembert operator” becomes clear if one takes the logarithm of the r.h.s. and the l.h.s. of (3.1).}

$$F_{a}^{* \Delta} := \frac{F_{a}^{+} F_{a}^{-}}{(F_{a+1}^{+})^{1-\delta_{a,N-1}} (F_{a-1}^{-})^{1-\delta_{a,1}}}$$  \hspace{1cm} (3.3)
we denoted a $T$-function in a gauge of a type ($L$) which can be different from the gauge of the other $T$- functions in that formula. We can do that because $1 + Y_{a,s} = \frac{(T_{a,s})^\Delta}{T_{0,a} T_{N,s}^{\Delta}}$ in the right hand side of (3.1) are gauge invariant, and we are allowed to write each $Y$-function in terms of $T$’s each taken in a different gauge. The meaning and the notation of the gauge ($L$) will be explained later.

We can act by $\Delta^{-1}$ on both sides of (3.2), to get

$$Y_{a,0} = e^{-Lp_a(\theta)} \frac{T_{a,1} T_{a,-1}^{(L)}}{T_{a+1,0} T_{a-1,0}} \left( \frac{\delta_{a,N-1}}{T_{N,0} T_{N,0}} + \frac{\delta_{a,1}}{T_{0,0} T_{0,0}} \right) \cdot \Delta^{-1}$$

where $p_a = \cosh(\frac{2\theta}{N}) \sin(\frac{a\pi}{N})$. The first factor is the only zero mode of $\Delta$ fitting the relativistic invariance and it gives the right large $L$ asymptotics (2.3). Furthermore, the action of the operator $\Delta^{-1}$ can be easily identified by the fourier transform in $\theta, a$ variables, so that the final expression is

$$Y_{a,0} = e^{-Lp_a(\theta)} \frac{T_{a,1} T_{a,-1}^{(L)}}{T_{a+1,0} T_{a-1,0}} \left( \Pi_{N-a} \left[ \frac{T_{N,0} T_{N,0}}{T_{0,0} T_{0,0}} \right] \Pi_a \left[ \frac{T_{N,0} T_{N,0}}{T_{0,0} T_{0,0}} \right] \right) K_N$$

where the “fusion” operator $\Pi_a$ is defined as the following product

$$\Pi_k[f](\theta) = \prod_{j=-(k-1)/2}^{(k-1)/2} f(\theta + i j)$$

and the kernel $K_N$ is the operator inverse to $\Pi_N$: $\forall f$ regular, $(\Pi_N[f]) K_N = f$. Its fourier transform is

$$\widehat{K}_N(\omega) = \frac{1}{\sum_{j=-N+1}^{N-1} e^{2\pi j \omega}}$$

Back in the $\theta$-space it takes the form

$$K_N = \tan \left( \frac{1}{2} \frac{\pi}{N} \left( \frac{1}{2} - \frac{2\theta}{N} \right) \right) + \tan \left( \frac{1}{2} \frac{\pi}{N} \left( \frac{1}{2} + \frac{2\theta}{N} \right) \right)$$

4. The large $L$, “spin chain” limit of $Y$-system and its relation to ABA

We will derive in this section the large $L$, ABA equations (1.20), (4.3) directly from the $Y$-system (2.8). Following the logic of (11) we use the fact that the $Y$-functions of the momentum carrying (black) nodes are exponentially small in this limit:

$$Y_{a,0} = \frac{T_{a,1} T_{a,-1}}{T_{a+1,0} T_{a-1,0}} \sim e^{-Lp_a(\theta)}$$

This implies that the two wings, left (for $s < 0$) and right (for $s > 0$), of the $Y$-system (2.8) are almost decoupled and can be treated separately.

*We use the notation $f K_N$ for $e^{\log f} K_N$ where $*$ stands for convolution.
4.1 Expressions for T-functions in the spin chain limit

Eq. (4.1) suggests that either $T_{a,1} \sim e^{-Lp_a(\theta)}$ or $T_{a,-1} \sim e^{-Lp_a(\theta)}$. Which one does so (whereas another one is finite) is a matter of choice of a gauge for T-functions.

We will work with two different gauges $(R)$ and $(L)$, such that in the large $L$ limit we have

$$
\begin{align*}
T_{a,-1}^{(R)} &\ll 1 \quad , \
T_{a,1}^{(L)} &\ll 1 \quad , \
1 \leq a \leq N-1 \\
T_{a,s \geq 0}^{(R)} &\sim 1 \quad , \
T_{a,s \leq 0}^{(L)} &\sim 1
\end{align*}
(4.2)
$$

$(R)$ will be called the “right-wing-gauge” and $(L)$ the “left-wing-gauge”, and when this superscript will be omitted it will be implicitly assumed that we are working in the $(R)$ gauge.

In the large $L$ limit, the T-functions of the left $(L)$ and right $(R)$ gauge both describe the same $Y$ functions but (up to exponential corrections) they satisfy Hirota equation restricted to the wings $s \geq 0$ (resp $s \leq 0$). Moreover, these $T$-functions are in this limit analytic on the whole complex plane, and therefore polynomial.

Such a solution of Hirota equation is well known in applications to the fusion procedure in similar spin chain systems, bosonic or even supersymmetric. First we parameterize $T_{1,s}$ in terms of $N$ functions $X_{(j)}^{(W)}(\theta), \quad j = 1, \ldots, N$ by means of the following generating functional

$$
\hat{W}^{(W)} = \left(1 - X^{(W)}_{(N)}(\theta) e^{i\theta b}\right)^{-1} \left(1 - X^{(W)}_{(N-1)}(\theta) e^{i\theta b}\right)^{-1} \cdots \left(1 - X^{(W)}_{(1)}(\theta) e^{i\theta b}\right)^{-1}
(4.3)
$$

These functions can be further expressed as follows

$$
X^{(W)}_{(k)} = \frac{Q^{(W)}_{k-1}[N/2-k-1]}{Q^{(W)}_{k-1}[N/2-k+1]} \frac{Q^{(W)}_{k}[N/2-k+2]}{Q^{(W)}_{k}[N/2-k]} , \quad k = 1, 2, \ldots, N
(4.4)
$$
in terms of some $Q^{(W)}$-functions, where the superscript $W = R, L$ indicates the wing that we study (either right or left). These $Q$ functions in the corresponding gauge are polynomials characterizing different solutions of Hirota equation in the large $L$ limit, their roots are the Bethe roots describing various excited states - solutions of the $Y$-system:

$$
Q^{(R)}_k(\theta) = \prod_{j=1}^{j_k^{(R)}} \left(\theta - u_j^{(k)}\right) , \quad Q^{(L)}_k(\theta) = \prod_{j=1}^{j_k^{(L)}} \left(\theta - u_j^{(k)}\right) , \quad (k = 1, \ldots, N-1) \quad (4.5)
$$

$$
Q^{(R,L)}_N(\theta) \equiv \varphi(\theta) = \prod_{j=1}^{N} (\theta - \theta_j) , \quad Q^{(R,L)}_0(\theta) \equiv 1
(4.6)
$$

---

9We assume here that there exists a gauge such that the large $L$ limit is described polynomial $Q^{(W)}_k$ functions. Although it needs a better understanding from the point of view of $Y$-system, it is the case if we start treating the large $L$ limit from the S-matrix by the ABA approach.
In particular, we have from (4.13)

$$T_{11}^{(W)}(\theta) = \varphi(\theta - iN/4) \sum_{k=1}^{N} X_{(k)}^{(W)}(\theta)$$  \hspace{1cm} (4.8)

$T_{1,1}$ should be free of poles, i.e. polynomial. But for each Bethe root $w_j = u_j^{(k)}$ or $w_j = v_j^{(k)}$, the two functions $X_{(k)}^{(W)}$ and $X_{(k-1)}^{(W)}$ have a pole at the same position $w_j - iN/2 (\frac{N}{2} - k)$. By requiring their cancellation in the sum (4.8), we get a constraint on the position of $w_j$, which we will call the auxiliary Bethe equation:

$$1 = \frac{Q_{k-1}^{(R/L)}(w_j - i/2) Q_k^{(R/L)}(w_j + i) Q_{k+1}^{(R/L)}(w_j - i/2)}{Q_k^{(R/L)}(w_j + i/2) Q_{k-1}^{(R/L)}(w_j - i) Q_{k+1}^{(R/L)}(w_j + i/2)}, \quad \begin{cases} k = 1, 2, \ldots, N-1 \\ w_j = u_j^{(k)} \text{ resp. } v_j^{(k)} \end{cases}$$  \hspace{1cm} (4.9)

The rest of the $T$-functions in the right wing can be expressed through the Bazhanov-Reshetikhin determinant [34]

$$T_{a,s} = \frac{\det_{1 \leq j,k \leq a} T_{1,s+k-j} \left( \theta + \frac{iN}{2} (a+1-k-j) \right)}{\Pi_{a-1} \left( \varphi^{-s-N/2} \right)}$$  \hspace{1cm} (4.10)

and they are also automatically polynomial in virtue of (4.9).

Among these $Q$-functions, the polynomial function $Q_N = \varphi$, encoding, as its roots, the rapidities of all physical particles, will be of a particular importance, and the vanishing of $T_{a,-1}$ or $T_{a,1}$ due to (4.11) implies the following asymptotics\(^{10}\)

$$\varphi(\theta) = \lim_{L \to \infty} T_{a,0}^{(R/L)}(\theta + iN - 2a/4)$$  \hspace{1cm} (4.11)

$$= \lim_{L \to \infty} T_{0,s>0}^{(R)}(\theta + iN + 2s/4) = \lim_{L \to \infty} T_{0,s<0}^{(L)}(\theta - iN - 2s/4)$$  \hspace{1cm} (4.12)

$$= \lim_{L \to \infty} T_{N,s>0}^{(R)}(\theta - iN + 2s/4) = \lim_{L \to \infty} T_{N,s<0}^{(L)}(\theta - iN - 2s/4)$$  \hspace{1cm} (4.13)

These relations translate all the zeroes $\theta_j$ of $Y_{1,0}(\theta + iN/4) + 1$, according to the Bethe equation (2.11), into the zeroes of $T$-functions.

### 4.2 Asymptotic Bethe ansatz (ABA)

Now we will reproduce from the Y-system in large volume limit $L \to \infty$ the ABA equations (4.20), (4.19) for the spectrum of energies. In this spin chain limit, (4.13) can be employed to compute $Y_{a,0}$ to the leading order, by using the asymptotic behaviors (4.12).

At this point, it is interesting to notice that the crossing relation implies that up to a zero mode of $K_N$

$$\left( \frac{\varphi^+}{\varphi^-} \right)^{K_N} = \left( \frac{\varphi^{[2-N]}_{[-2N+1]}}{\varphi^{[-N]}_{[-2N+1]}} \right)^{K_N} = \left( \frac{\varphi^{[-N]}_{[-2N]} S^{[N]}_{[-N]} S^{[2N]}}{\varphi^{[-N]}_{[-2N+1]}} \right)^{K_N} = \left( \frac{\varphi^{[-N]}_{[-2N]} S^{[N]}_{[-N]} S^{[2N]}}{\varphi^{[-N]}_{[-2N+1]}} \right)^{K_N}$$  \hspace{1cm} (4.14)

\(^{10}\)The claim in (4.11) is a bit too strong and we will see further that it actually only holds inside some strips on the complex plane.
There exists no full proof procedure to generalize the formula (2.10) to the excited states.

5. Expressions for the energy of excited states

Logarithmic poles (different for each state) appear under the integral in (2.10). From (4.11) general state in our model. The procedure of [4, 8] claims that for the excited states a set of the poles and double zeroes of which gives for instance

\[
\lim_{L \to \infty} \prod_{N-a} \left[ \frac{T_{0,0}^+ T_{0,0}^-}{T_{N-1,0}^L T_{N-1,0}^L} \right]^{K_N} = \prod_{N-a} \left[ \frac{\varphi^{-N/2+1}}{\varphi^{-N/2-1}} \right]^{K_N}
\]

which gives for instance

\[
= \prod_{N-a} \left[ \frac{\varphi^{-N/2+1}}{\varphi^{-N/2-1}} \right]^{K_N}
\]

As a consequence, the large \( L \) limit of equation (3.5) is

\[
Y_{a,0}(\theta) \sim e^{-LP_a} \frac{T_{a,1} T_{a-1}^{(L)}}{T_{a+1,0} T_{a-1,0}} \varphi^{-N/2-1} \varphi^{-N/2-1} \frac{1}{\chi_{CDD}^{-N/2}}
\]

where the factor \( \chi_{CDD} \) was added as another zero mode, necessary to transform the double poles and double zeroes of \( S^2 \) into the simple ones [21]. We will also see in section 7.3.2 that this factor arises in our Y-system formalism in a natural way.

In particular, at \( a = 1 \), we get the ABA equation (periodicity condition):

\[
-1 = e^{-iL \sinh \frac{2\pi}{N} \theta} \frac{1}{\chi_{CDD}(\theta)} \frac{Q_N^{(L)}(\theta - i/2) Q_N^{(R)}(\theta - i/2)}{Q_N^{(L)}(\theta + i/2) Q_N^{(R)}(\theta + i/2)}
\]

which expresses the fact that \( Y_{1,0}(\theta + iN/4) + 1 = 0 \) (here \( T_{1,1} \) was replaced by the single surviving, last term of (4.3)).

In conclusion, we have shown here that the \( Y \) system implies the familiar ABA equations [23, 21]. In the next sections we will see how these ABA equations for the spectrum of PCF can be generalized to any finite size \( L \).

5. Expressions for the energy of excited states

There exists no full proof procedure to generalize the formula (2.10) to the excited states. The analytic continuation of (4) with respect to the mass is difficult, if possible at all for a general state in our model. The procedure of [4, 8] claims that for the excited states a set of logarithmic poles (different for each state) appears under the integral in (2.10). From (4.11) and the ABA we know that at very large \( L \), \( T_{a,0}(\theta) \approx \varphi(\theta - iN/4 + ia/2) \), where \( \varphi(\theta) = \prod_j (\theta - \theta_j) \) is a polynomial encoding all real roots. For finite \( L \) the roots \( \theta_j \) will be shifted and in general become complex. These exact Bethe roots, as opposed to the approximate

\[\text{We consider here for simplicity only the situation when the Bethe roots } \theta_j \text{ are real in the asymptotic limit. The case when they occur in complex conjugated pairs should not be very different but at the moment we did not try to do it.}\]
ones given by (2.120), are defined by the exact Bethe equations $T_{a,0}(\theta^{(a)}_j) - iN/4 + ia/2 = 0$. There is a whole family of such roots when $a \in [0, N]$, because even though the two functions $T_{a,0}(\theta)$ and $T_{a+1,0}(\theta + i/2)$ have the same limit at large $L$, they do not necessarily have the same roots at finite size. Each of these roots also gives rise to two zeroes and two poles in the $Y$-functions, namely, as we see from (2.13), $1 + Y_{a,0}(\theta^{(a)}_j) - iN/4 + ia/2 \pm i/2 = 0$ and $1 + Y_{a\pm1,0}(\theta^{(a)}_j) - iN/4 + ia/2 = \infty$. Among these families of finite size Bethe roots, we will actually restrict ourselves to the roots $\theta^{(2)}_j$ for even $N$, and $\theta^{(N\pm1)}_j$ for odd $N$. We will argue that only those ones will contribute as poles caught by an integration contour.

Separating the logarithmic poles (where $1 + Y_a$ cancels) in the contour integral (2.10) should give a familiar contribution $\sum_j \cosh(\frac{2\pi}{N}\theta_j)$ to the energy of a finite $L$ state. This appears to be the right, though not completely well understood and justified, answer for some models, including the PCF at $N = 2$ [1].

For PCF at $N > 2$ this procedure encounters another difficulty: the zeros under the logarithm in (2.10) appear to correspond in general to complex Bethe roots $\theta_j$. We have to decide what is the right integration contour in (2.10) when this formula is applied to an excited state. We are not aware of any well justified procedure for fixing the contour but we shall try to guess it on the basis of our numerical observations and the symmetry considerations.

In the rest of this section, we consider the formula for excited states of the $U(1)$ sector - the one which corresponds to the wave function $|\Psi\rangle$ having the the maximal value of total spin $S_L = S_R = N/2$ w.r.t. the $SU(N)_R$ and $SU(N)_L$ symmetries. In this case $J_k^{(L,R)} = 0$ and there are no auxiliary roots in the ABA $Q$-functions (4.3) (all of them are equal to 1 except $Q^{(R,L)}_N = \varphi$, see [1.6][4.7]. In what follows, we shall distinguish even and odd $N$'s.

### 5.1 Energy of state in the $U(1)$ sector at odd $N$'s

It is believed that the energy of an excited state can be obtained from (2.10) by an analytic continuation in the parameter $L$. This continuation has the effect of appearance of new singularities of the integrand in the physical strip in (2.10) and a certain choice of the integration contour, enclosing some singularities of the integrand [3.4]. How it happens...
in each particular model or state is usually a rather complicated question. It implies the
analysis of positions of these singularities at a finite \( L \) but the large \( L \) asymptotics often
serves as an important guiding principle.

Here we propose a formula for the energies of excited states in the \( U(1) \) sector which
seems to work well for any odd \( N \). It is based on our numerical and analytic observations,
in particular for the \( N = 3 \) case. It reads as follows

\[
E(L) = - \frac{m}{N} \sum_{a=1}^{N-1} \frac{\sin(\frac{\pi a}{N})}{\sin(\frac{\pi}{N})} \int_{-\infty}^{\infty} \frac{d\theta}{-N^2 + a^2} \cosh \left( \frac{2\pi}{N} \theta \right) \log \left( 1 + Y_{a,0}(\theta) \right)
\] (5.1)

so that we have the straight integration contours parallel to the real axis and shifted by
\(-N^2 + a^2\).\(^{12}\)

Let us explain the reason for such a choice of contours. First, let us note that in order
to have a real energy from (5.1) we should impose the following property of \( Y \)-functions
under complex-conjugation: \( Y_{a,s}(\theta) = Y_{N-a,s}(\bar{\theta}) \). We will restrict ourselves to the gauges
where this property is a consequence of the relation

\[
T_{a,s}(\theta) = T_{N-a,s}(\bar{\theta})
\] (5.2)

For finite \( L \), we will focus on the roots \( \theta_j \) defined\(^{13}\) by \( T_{N-1,0}(\theta_j + i/4) = 0 \).

Due to the very definition of \( 1 + Y_{a,0} = \frac{T_{a,0} T_{a+1,0}^*}{T_{a+1,0} T_{a,0}^*} \) each \( \theta_j \) gives rise to two zeros and
poles. In particular, \( 1 + Y_{N-1/2,0}(\theta) \) has a zero and a pole\(^{14}\) at respective positions \( \theta_j - i/4 \) and \( \overline{\theta_j} - i/4 \) because \( T_{N-1/2,0}(\theta_j - i/4) = 0 \) and \( T_{N-1/2,0}(\overline{\theta_j} - i/4) = 0 \). In the large \( L \) limit
these zero and pole almost coincide since \( \theta_j \) is almost real. By complex conjugation, we
can also say that \( 1 + Y_{N+1/2,0}(\theta) \) has a zero and a pole at respective positions \( \overline{\theta_j} + i/4 \) and
\( \theta_j + i/4 \).

This structure is illustrated for \( N = 3 \) in figure 2. From the Lüscher corrections\(^{15}\), we
can say that the pole occurs below the zero for \( 1 + Y_{1,0} \) and vice versa for \( 1 + Y_{2,0} \), at least
for roots with even momentum numbers\(^{16}\). This is important to ensure the right answer if
we want the contours to be straight.

In (5.1) we chose the integration contour to pass, for the \( N-1/2 \) and \( N+1/2 \) th term in the
sum, between those zero and pole. Deforming the contour to the real axis and computing

\(^{12} \)One of the advantages of this straight contour is that it can be easily implemented in numerics. We
will see indeed that the \( Y \) functions can be most easily computed on exactly these lines. We will also see
that the statement holds only for roots with even momentum number, but for odd momentum number, the
(slightly modified) contour stays very close to this straight line.

\(^{13} \)In this section, we will denote \( \theta_j \) for \( \theta_j^{(N/2)} \), because the other types of finite size roots don’t contribute.

\(^{14} \)In addition to this zero and pole, \( 1 + Y_{N-1/2,0}(\theta) \) has another zero at \( \theta_j + 3i/4 \) and a pole at each root
of \( T_{N-1,0} \), but this will not have any consequence in the contour argument.

\(^{15} \)In section 8.1 we detail how this is proved in the asymptotic limit. Our numerics suggests that it is
still true at finite size, and even in the conformal limit.

\(^{16} \)So that the contour will actually have to be slightly modified for roots having odd momentum number.
This will be done in such a manner that (5.3) will stay true.
the contributions of the logarithmic poles enclosed by the contour during that deformation, one gets the following formula

\[ E(L) = -\frac{m}{N} \sum_{a=1}^{N-1} \frac{\sin\left(\frac{\pi a}{N}\right)}{\sin\left(\frac{\pi}{N}\right)} \int_{-\infty}^{\infty} d\theta \cosh\left(\frac{2\pi}{N}\theta\right) \log\left(1 + Y_{a,0}(\theta)\right) \]  

\[ + i \sum_{j} m \frac{\cos\frac{\pi}{N}}{\sin\frac{\pi}{N}} \left[ \sinh\left(\frac{2\pi}{N}(\theta_j - i/4)\right) - \sinh\left(\frac{2\pi}{N}(\theta_j + i/4)\right) \right] \]  

In the thermodynamic limit \((L \gg 1)\), the Bethe roots \(\theta_j\) become real and the second line of (5.3) reduces to the asymptotic result (2.7), whereas the term in the first line appears to be \(O(e^{-mL})\).

5.2 Energy of state in the \(U(1)\) sector at even \(N\)’s

When \(N\) is even, the corresponding contour cannot be chosen as a straight line. We will conjecture here the analogue of (5.3) to be simply

\[ E(L) = -\frac{m}{N} \sum_{a=1}^{N-1} \int_{-\infty}^{\infty} \frac{p_a(\theta)}{N} \log\left(1 + Y_{a,0}(\theta)\right) d\theta + \sum_{j} \cosh(\theta_j) \]  

where the roots \(\theta_j\) are defined by \(T_{N/2,0}(\theta_j) = 0\), so that the second term in (5.4) is real due to the reality of \(T_{N/2,0}\). The corresponding contour is shown in figure 3.

We should admit here that this formula for the masses at even \(N\) has a status of a natural conjecture. We have not enough of numerical, or analytic evidence to be 100% sure in it. It would be good to verify it at least for the mass gap at \(N = 4\), numerically and by means of the Lüscher corrections at large \(L\).

6. Hirota form of \(Y\)-system and wronskian solution

For the principal chiral field, \(T_{a,s}\) is defined for \(a = 0, 1, \ldots, N\), while \(Y_{a,s}\) is defined for \(a = 1, 2, \ldots, N - 1\). We can solve the Hirota finite difference equation (2.13) (and the
corresponding Y-system) with the appropriate boundary conditions using its integrability. Any solution of (2.12) is gauge equivalent to a solution where \( T_{0,s}(\theta) = T_{0,0}(\theta - s\frac{i}{2}) \) and \( T_{N,s}(\theta) = T_{N,0}(\theta + s\frac{i}{2}) \). The most general solution can thus be expressed \(^{13}\) as an \( N \times N \) determinant, in terms of \( 2N \) unknown functions \( q_j \) and \( \overline{q}_j \):

\[
T_{a,s}(\theta) = -(-1)^a \text{Det}(c_{j,k})_{1 \leq j,k \leq N} \tag{6.1}
\]

where

\[
c_{j,k} = \overline{q}_j \left( \theta + \frac{i}{2} \left( s + a + 1 + \frac{N}{2} - 2k \right) \right) \quad \text{if } k \leq a
\]

and

\[
c_{j,k} = q_j \left( \theta + \frac{i}{2} \left( -s + a + 1 + \frac{N}{2} - 2k \right) \right) \quad \text{if } k > a
\]

At this point, \( q_j \) is not necessarily the complex-conjugate of \( q_j \) and the gauge freedom reduces to two independent functions \( g \) and \( \overline{g} \):

\[
q_j(\theta) \rightarrow g(\theta) \cdot q_j(\theta) \quad \tag{6.2}
\]

\[
\overline{q}_j(\theta) \rightarrow \overline{g}(\theta) \cdot \overline{q}_j(\theta) \quad \tag{6.3}
\]

As an example of this determinant solution, the large \( L \), spin chain limit solution corresponding to the \( U(1) \) sector state with roots \( \theta_i \) is easily identified, in the \( (R) \) (or \( (L) \)) gauge, by plugging the following definitions for \( q_j \) into (6.1)

\[
q_j(\theta) = \overline{q}_j(\theta) = \frac{\theta^{j-1}}{(j-1)!} \tag{6.4}
\]

\[
q_N(\theta) = \overline{q}_N(\theta) = P_\infty(\theta) \tag{6.5}
\]

where

\[
(e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta})^{N-1} P_\infty = \varphi = \prod_k \left( \theta - \theta_k \right) \tag{6.6}
\]

To see that this is the correct parameterization of the \( U(1) \) solution, first, we can convince ourselves that \( T_{a,-1} = 0 \), \( T_{a,0} = \varphi(\theta - i\frac{N-2a}{4}) \), and second, that it reproduces the \( T_{1,s} \) generated by \((1.3)(4.5)\) where all \( Q_j(\theta) \) \( j < N \) are set to 1. For the vacuum state \( P_\infty(\theta) = \frac{e^{N-1}}{(N-1)!} \).

Now we will explain how to generalize this large \( L \) solution to any finite \( L \).

7. Solution of the Y-system for PCF at a finite size

This section describes how to solve the Y-system by reducing it to a finite number of non-linear integral equations (NLIEs), that can be solved in its turn by iterative numerical methods.

We will focus on \( U(1) \) sector states, although the method is in principle applicable to any excited state (see the discussion in subsection \[3\]).

\(^{17}\)The general solution doesn’t assume that \( q \)'s and \( \overline{q} \)'s are complex-conjugated. Nonetheless, our numerics has shown that at least for the states in the \( U(1) \) sector, it is sufficient to restrict ourselves to the solutions where \( q \)'s and \( \overline{q} \)'s are complex conjugated.
7.1 Definition of the jump densities

We propose here an ansatz for the finite size $L$ solution by adding to the large $L$ polynomial expressions (6.4-6.6) for $q$’s certain finite $L$ terms decreasing for $\theta \to \pm \infty$ and exponentially small for $L \to \infty$ or $\theta \to \infty$: The finite $L$ $q$’s take thus the form

$$q_j(\theta) = \frac{\theta^{j-1}}{j-1} + F_j(\theta) \quad \text{When } j < N \text{ and } Im(\theta) \leq 0$$ \hspace{1cm} (7.1)

$$\overline{q}_j(\theta) = \frac{\theta^{j-1}}{j-1} + \overline{F}_j(\theta) \quad \text{When } j < N \text{ and } Im(\theta) \geq 0$$ \hspace{1cm} (7.2)

$$q_N(\theta) = P(\theta) + F_N(\theta) \quad \text{When } Im(\theta) \leq 0$$ \hspace{1cm} (7.3)

$$\overline{q}_N(\theta) = P(\theta) + \overline{F}_N(\theta) \quad \text{When } Im(\theta) \geq 0$$ \hspace{1cm} (7.4)

Where

$$F_j(\theta) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{f_j(\eta)}{\theta - \eta} d\eta \quad \text{When } Im(\theta) < 0$$ \hspace{1cm} (7.5)

$$\overline{F}_j(\theta) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\overline{f}_j(\eta)}{\theta - \eta} d\eta \quad \text{When } Im(\theta) > 0$$ \hspace{1cm} (7.6)

and the polynomial $P$ has the same degree as $P_\infty = \lim_{L \to \infty} P$ given by (6.6)\(^{18}\). As a consequence of these definitions, we have

$$q_j^{[+0]} - q_j^{[-0]} = \lim_{\epsilon \to 0} q_j^{[+\epsilon]} - q_j^{[-\epsilon]} = -f_j$$ \hspace{1cm} (7.7)

so that $f_j$ is actually the jump between the functions $q_j$ and $\overline{q}_j$ on the real axis.

Eqs.(7.1-7.4) define $q_j$ only below the real axis and $\overline{q}_j$ above the real axis, so that the determinant only allows to compute $T_{a,s}$ inside the strip $\text{Im}(\theta) \in [-\frac{N-2a}{4} - \frac{3a+1}{2}, -\frac{N-2a}{4} + \frac{a+1}{2}]$. We can already see that these strips are the minimal strips to compute the $Y$ functions on the integration contour of equation (5.1), and we will see that it enables to compute the exact\(^{19}\) energy of states in the $U(1)$ sector, at any length $L$.

The jump densities $f_j(\eta)$ are well defined on the real axis where they take only imaginary values, which follows from (7.7) and they are exponentially suppressed at large $L$ or large $\cosh(\frac{2\pi}{N}\eta)$, as can be inferred from (2.9).

7.2 Relation to the analyticity of $T$ functions

From the ABA (4.19) and the finite size equation (3.3), we can see that

$$1 + Y_{a,0} \xrightarrow{\theta \to \infty} 1 \quad \text{When } |Im(\theta)| < \frac{N}{4}$$ \hspace{1cm} (7.8)

which means that $1 + Y_{a,0} = \frac{T_{a,0}}{T_{a+1,0}T_{a-1,0}}$ has a proper behavior in this strip, being a meromorphic function regular at infinity. On the other hand, when $|Im(\theta)| = \frac{N}{4}$, $1 + Y_{a,0}$

\(^{18}\) The way we fix this polynomial will be explained in section 7.4, where the finite size Bethe equations are discussed.

\(^{19}\) up to the precision of our numerical procedure solving these NLIE’s.
oscillates at $\text{Re}(\theta) \to \infty$, and it diverges when, e.g., $|\text{Im}(\theta)| \in \left[\frac{N}{4}, \frac{3N}{4}\right]$. By that reason we conclude that the analyticity strip of $1 + Y_{a,0} = \frac{T_{a,0}^+ T_{t,0}^-}{T_{a+1,0}^+ T_{a-1,0}^-}$ is $\{\theta, |\text{Im}(\theta)| < \frac{N}{4}\}$. From this we can identify the strips where the asymptotics (4.11) hold for $T_{a,s}$:

$$
T_{0,0} \xrightarrow{L \cosh(\frac{2\pi N}{N}) \to \infty} \varphi^{[-N/2]} \quad \text{when } |\text{Im}(\theta)| < \frac{N}{4} \quad (7.9)
$$

$$
T_{a,0} \big|_{0 < a < N} \xrightarrow{L \cosh(\frac{2\pi N}{N}) \to \infty} \varphi^{[a-N/2]} \quad \text{when } |\text{Im}(\theta)| < \frac{N}{4} + \frac{1}{2} \quad (7.10)
$$

$$
T_{N,0} \xrightarrow{L \cosh(\frac{2\pi N}{N}) \to \infty} \varphi^{[+N/2]} \quad \text{when } |\text{Im}(\theta)| > -\frac{N}{4} \quad (7.11)
$$

These conditions ensure the proper analyticity of $1 + Y_{a,0} = \frac{T_{a,0}^+ T_{t,0}^-}{T_{a+1,0}^+ T_{a-1,0}^-}$, and the boundaries of the analyticity strips of each $1 + Y_{a,0}$ are given by the boundaries of the analyticity strips of the corresponding $T$ functions\textsuperscript{20}.

Now, since we know that the $T$ functions are described by wronskian determinants, these analyticity strips suggest that

$$
q_j \quad \text{is analytic when } \quad \text{Im}(\theta) < 1/2 \quad (7.12)
$$

$$
\overline{q_j} \quad \text{is analytic when } \quad \text{Im}(\theta) > -1/2 \quad (7.13)
$$

The fact that the analyticity strip for $q_j$ ends up at $\text{Im}(\theta) = 1/2$ is reflected for instance in the fact that $T_{N-1,0}$ is not analytic when $\text{Im}(\theta) > N/4 + 1/2$, which explains that $Y_{N-1,0}$ isn’t analytic when $\text{Im}(\theta) > N/4$.

The equations (7.12,7.13) teach us that the analyticity domain is a bit bigger than what is necessary for (7.1,7.4). It tells us that in the definitions (7.3,7.6), the contour can be shifted up to $\pm i/2$. In other words the functions $f_j(\eta)$ are analytic on the strip $\text{Im}(\eta) < 1/2$.

It is noteworthy that even with these contour deformations, the determinant expressions (7.1,7.4) describe the function $T_{a,s}(\theta)$ inside the strip $\text{Im}(\theta) \in \left[-\frac{N-2a}{4} - \frac{N-2a}{4} + \frac{2d}{2} + 1\right]$, which is narrower than in equations (7.9,7.11). But we will show that the relatively narrow strips given by this ansatz are sufficient to solve the $Y$-system and compute the energies.

### 7.3 Closed system of NLIEs

The gauge freedom under the transformation (6.2,6.3) can be used to impose $F_1(\theta) = T_{t,0}^+ T_{t,0}^- = 0$, which leaves only $N - 1$ independent densities to compute. $N - 1$ equations on this densities can be obtained by asking that the state is symmetric, i.e. that $Y_{a,-s} = Y_{a+1,0}$ and inserting this requirement into the middle node $Y$-system equation (6.11)\textsuperscript{21}, in the same manner as (6.11).

\textsuperscript{20}The analyticity strips for $T_{0,0}$ and $T_{N,0}$ can be chosen as half plane thanks to an appropriate gauge.

\textsuperscript{21}As a consequence, we are solving the $Y$-system under the two following constraints : $Y_{a,s} \sim e^{-L_{p,s}(\theta)s} \times \text{const}_{a,s}$ on the one hand, and $T_{a,0}^+ T_{t,0}^- = T_{a,0}^1$ on the other hand. This second constraint is specific to symmetric states (which includes the states in the $U(1)$ sector), such that $Y_{a,-s} = Y_{a,s}$.
Equation (3.3) is then reduced to:

\[ Y_{a,0} = e^{-Lp_{a}} \frac{(T_{a,1})^2}{T_{a-1,0}T_{a+1,0}} \left( \frac{T_{0,0}^{[a-N]}T_{N,0}^{-[a]}}{T_{0,0}^{[a-N]}T_{N,0}^{-[a]}} \right)^{\star K_N} \]  

(7.14)

On the other hand, we can write the determinant expression (5.1) for \( T_{\alpha-1} (\theta - i \frac{N-2a}{4}) \), and notice that \( c_{a,j} = q_j^{[+0]} \) while \( c_{a+1,j} = q_j^{[-0]} \). After subtracting one column from another in the determinant, there is a full column of \( q_j^{[+0]} - q_j^{[-0]} = -f_j \) which is exponentially small and that explains the exponential suppression of \( T_{\alpha-1} \). Expanding the determinant w.r.t. these columns\(^{22}\) gives the following linear system relating \( f_j \)'s to \( T_{\alpha-1} \)’s:

\[
T_{\alpha-1} (\theta - i \frac{N-2a}{4}) = \sum_j d_{a,j}(\theta) f_j(\theta) \\
\text{Where } d_{a,j} = (-1)^j \frac{\det(c_{k,l})_{k \neq j, l \neq a} + \det(c_{k,l})_{k \neq j, l \neq a+1}}{2} 
\]

(7.15)

(7.16)

These \( c_{k,l} \) are the coefficients of the determinant (5.1) defining \( T_{\alpha-1} (\theta - i \frac{N-2a}{4}) \), and finally equations (7.14,17) can be recast into

\[
\sum_j d_{a,j}(\theta) f_j(\theta) = e^{-Lp_{a}(\theta - i \frac{N-2a}{4})} T_{a,1}^{[a-N/2]} \left( \frac{T_{0,0}^{+[N/2]}T_{N,0}^{-[N/2]}}{T_{0,0}^{[2a-3N/2]}T_{N,0}^{-[N/2+2a]}} \right)^{\star K_N} 
\]

(7.17)

This is a closed system of equations on \( \{f_j(\theta)\}_{\theta \in \mathbb{R}} \) because all coefficients \( d_{a,j} \), and all \( T \)'s can be computed out of \( f_j \)'s through several convolutions.

The solution of the \( Y \)-system is therefore achieved by solving this system of \( N-1 \) equations on \( N-1 \) densities. The simple inversion of the linear system (7.17) brings (7.17) into the form

\[
f_j(\theta) = H_j(\{f_k(\eta)\}_{k=2,\ldots,N-1}) \quad \eta \in \mathbb{R} 
\]

(7.18)

This \( H_j \) is a contraction mapping in some vicinity of \( f_j = 0 \) when \( L \) is sufficiently large since it leads to an exponentially small \( f_j(\theta) \). This implies that in some vicinity of \( L = \infty \), the mapping \( H_j \) has a fixed point that can be numerically found through repeated iterations of \( H \).

The way we solve \( Y \)-system is therefore simply the iteration of (7.18) and a good news is that, at least for \( N = 3 \), even at very small \( L \), this procedure seems to converge to a fix point of \( H_j \), giving a complete solution of (7.18) and thus of the \( Y \)-system.

\(^{22}\)The two terms in (7.16) correspond to the fact that before expanding the determinant, we’ve added and subtracted columns to get one full column of \( q_j^{[+0]} - q_j^{[-0]} = -f_j \) and the other one of \( q_j^{[+0]} q_j^{[-0]} \) (which corresponds to a principal value in (7.3,7.6)).
### 7.3.1 Numerically workable form for the NLIE’s

One difficulty of this process is that in order to compute the right hand side of (7.17), it is necessary to compute \( \left( \frac{T_{0,0}^{\left[+N/2\right]} T_{N,0}^{[-N/2]}}{T_{0,0}^{\left[-N/2\right]} T_{N,0}^{[+N/2]} \delta_{N-2a+1}} \right)^{K_N} \). But we have already seen that the description it terms of \( q_j \)'s having cuts on the real axis allows to compute \( T_{0,0}(\theta) \) only when \( \text{Im}(\theta) < -\frac{N-1}{2} \). So, for instance, \( T_{0,0}(\theta + i\frac{N}{2}) \) cannot be computed when the spectral parameter \( \theta \) is real. The denominator can nonetheless be computed if in the convolutions we shift appropriately both the argument of the kernel and of the T-functions.

\[
\left( \frac{1}{T_{0,0}^{[2a-3N/2]} T_{N,0}^{[-N/2+2a]}} \right)^{K_N} = \left( \frac{1}{T_{0,0}^{[-N/2-N+a+1]} T_{N,0}^{[+N/2-a]} \delta_{N-a-1}} \right)^{K_N} \quad (7.19)
\]

since \( K_N(\theta) \) is regular when \( \text{Im}(\theta) \in [-\frac{N-1}{2}, \frac{N-1}{2}] \) and because \( a \in [1, N-1] \). Eq.(7.19) simply reflects the fact that if \( k(\theta) \) is analytic for \( \text{Im}(\theta) \in [0, b/2] \), then for real \( \theta \), \( (f^{[b]})^{*k} = f^{*(k)[b]} \), \( \forall f \).

The same idea, applied to the numerator, would give \( (T_{0,0}(\theta + N\frac{i}{4}))^{*K_N} = (T_{0,0}(\theta - N\frac{i}{4}))^{*K_N^{[-N]}} \). But the equality fails because \( K_N \) has a pole at \( i\frac{N-1}{2} \).

Instead, one can use the following relation:

\[
\left( T_{0,0}(\theta + N\frac{i}{4}) \right)^{*K_N} = \frac{T_{0,0}(\theta - N\frac{i}{4} + \frac{i}{2})}{T_{0,0}(\theta - N\frac{i}{4} - \frac{i}{2})} \left( \frac{T_{0,0}^{[+N/2]} T_{N,0}^{[-N/2+2]} \delta_{N-a}}{T_{0,0}^{[-N/2-N+a+1]} T_{N,0}^{[+N/2-a]} \delta_{N-a-1}} \right)^{K_N} \quad (7.20)
\]

which simply uses the fact that for \( \forall f \) regular, \( (\Pi_{N}[f])^{*K_N} = f \). This is true only up to a zero mode of \( K_N \), discussed in the section \( 7.3.2 \).

Finally, the last factor in the eq.(7.17) can be put into a numerically workable form by rewriting

\[
\left( \frac{T_{0,0}^{[+N/2]} T_{N,0}^{[-N/2]}}{T_{0,0}^{\left[-N/2\right]} T_{N,0}^{[+N/2]} \delta_{N-2a+1}} \right)^{K_N} = \left( \frac{T_{0,0}^{[-N/2-1]} T_{N,0}^{[N/2-1]} \delta_{N+a-1}}{T_{0,0}^{[-N/2]} T_{N,0}^{[N/2]} \delta_{N-1}} \right)^{K_N^{[a-1]}} \left( \frac{T_{0,0}^{[3N/2-N+a+1]} T_{N,0}^{[-3N/2+2]} \delta_{N-a-1}}{T_{0,0}^{[-3N/2]} T_{N,0}^{[3N/2-2+a]} \delta_{N-a+1}} \right)^{K_N^{[-N+a+1]}} \quad (7.22)
\]

This will help us to transforms the eq.(7.17), once the appropriate zero mode is added, into a really closed system of NLIEs where the right hand side can indeed be computed by knowing the functions \( f_j \) only on the real axis.
7.3.2 $\chi_{CDD}$ factor

It is clear from the derivation of eq. (7.17), as well as of the eqs. (7.20, 7.22) that they are fixed only up to a zero mode of the operator $K_N$. A zero mode $Z$ therefore has to be added to (7.17), to get

$$\sum_j d_{a,j}(\theta) f_j(\theta) = Z e^{-L \cosh(\frac{2\pi}{N}(\theta - i \frac{N - 2a}{4}))} \frac{\sin(\frac{a\pi}{N})}{\sin(\pi N)} T_{a,1}$$

Such zero mode can include for instance the factors $e^{-L \cosh(\frac{2\pi}{N} \theta)}$ and $\chi_{CDD}$ from equation (4.19).

In the asymptotic limit ($L \to \infty$), the zero modes in the equation (7.14) can be obtained by comparison with (1.19). In the same manner, the zero modes implicitly present in equation (7.22) can be computed in the asymptotic limit, by replacing $T_{a,0}$’s by their asymptotic values in terms of $\varphi$, see eq. (4.11), so that we can directly compute the zero mode $Z$ when $L \to \infty$. We notice that, although both (7.17) and (7.22) are true up to a non-trivial zero mode, the zero mode in (7.23) happens to be

$$Z = 1,$$

at least in the asymptotic limit. In the numerical solution of the $Y$-system we therefore assume that $Z = 1$ holds even at a finite size, i.e. that the analyticity structure of the zero modes is the same at finite $L$ as at $L \to \infty$. We explicitly see that at finite size, we obtain $Y$ functions having simple poles only.

7.4 Finite size Bethe equations

Bethe equations emerge in this procedure as a regularity requirement on the jump densities $f_j$’s. Let us illustrate it for a general $U(1)$ state in the $SU(3)$ case, and also show why these finite $L$ analogues of Bethe equations are equivalent, at large $L$, to the ABA Bethe equations on the roots of $\varphi$.

For such a state, the linear system (7.15) can be written as

$$\begin{pmatrix} A & B \\ -\overline{A} & -\overline{B} \end{pmatrix} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} T_{1,-1}(\theta - i/4) \\ T_{2,-1}(\theta + i/4) \end{pmatrix}$$

---

$^{23}$Actually the right hand side of (7.23) is itself defined up to a factor of $e^{i \frac{2k\pi}{N}}$, because any $f^* K = e^{K \log f}$ is defined up to a $e^{2i\pi f K}$ corresponding to the choice of the branch of the log. As a consequence, a more exact statement for (7.24) is $Z = e^{i \frac{2k\pi}{N}}$. The choice of $k$ is done to reproduce (1.19), where the phase in (8.3) is chosen in such a manner that one particle at rest ($\theta_1 = 0$) is a solution of the Bethe equation (1.20).

For states with zero momentum (like the mass gap and the vacuum), this extra phase can also be obtained by requiring a $\theta \to -\theta$ symmetry, which these states should exhibit.
where \( A = -\frac{1}{2}(q_3 + \overline{q}_3) + \overline{q}_3^- \) and \( B = \frac{1}{2}(q_2 + \overline{q}_2) - q_2^- \).

Inverting the matrix \( \begin{pmatrix} A & B \\ -\overline{A} - \overline{B} \end{pmatrix} \), some singularity could occur at the zeroes of the function \( \overline{A}B - A\overline{B} \), i.e., when the determinant is zero. If we want \( f_j \)'s to be regular, we need the numerator to vanish at the same \( \theta \) to cancel this pole. This gives the following finite size Bethe equation:

\[
\text{If } \quad (\overline{A}B - A\overline{B})|_{\theta_j} = 0 \quad \text{then} \quad \begin{cases} T_{1,-1}(\tilde{\theta}_j - i/4)\overline{A}(\tilde{\theta}_j) = T_{2,-1}(\tilde{\theta}_j + i/4)A(\tilde{\theta}_j) \\ T_{1,-1}(\tilde{\theta}_j - i/4)\overline{B}(\tilde{\theta}_j) = T_{2,-1}(\tilde{\theta}_j + i/4)B(\tilde{\theta}_j) \end{cases} \quad (7.26)
\]

One can notice that at such \( \tilde{\theta}_j \) the two conditions in the r.h.s. are equivalent.

The claim that \( \tilde{\theta}_j \) are a finite size analogue of the Bethe roots is supported by the fact that at large \( L \), the roots of \( \overline{A}B - A\overline{B} \) are precisely the Bethe roots. Indeed, at large \( L \), \( B \simeq i \) and \( A \simeq P^{++} - P \), giving \( \overline{A}B - A\overline{B} \simeq iP^{--} - P + P^{++} - P \) = \( i\varphi \). Moreover, the second relation in the r.h.s. of (7.26) reduces then to the reality condition \( \frac{T_{1,-1}(\theta_j - i/4)}{T_{1,-1}(\theta_j - i/4)} = -1 \).

Using the leading-order large \( L \) expression of \( Y_{a,0} \) in terms of \( S \), eq.(1.19), we get at large \( L \)

\[
T_{1,-1}(\theta - i/4) \simeq \varphi - \varphi + 2\varphi^{--}\varphi^{++} - L\cosh(\frac{\varphi}{3}(\theta - i/4))
\]

where \( S(\theta) = \prod_j S_j^2(\theta - \theta_j)\chi_{CD}^{\theta_j}(\theta - \theta_j) \) \quad (7.27)

Using the fact that \( \varphi(\theta_i) = 0 \) at all Bethe roots \( \theta_i \), and dividing by the complex conjugate, the large \( L \) regularity requirement becomes

\[
\left. \frac{(\varphi^{--})^2}{\varphi^{++}} \frac{\varphi}{(\varphi^{++})^2 S^{++}} \right|_{\theta = \theta_j} e^{iL\sinh(\frac{\varphi}{3}\theta_j)} = -1 \quad (7.29)
\]

Using the crossing relation, the left hand side becomes simply \( S(\theta_j)e^{iL\sinh(\frac{\varphi}{3}\theta_j)} \), so that the finite size regularity condition stated above is equivalent at large \( L \) to the asymptotic Bethe equations (1.20).

As a consequence, the iterative solution of the closed, finite size equations (7.23), should start from the expression (7.4) where \( P = P_\infty \) is given in terms of the asymptotic Bethe roots by (6.6), and then at each iteration, this polynomial is updated in order to incorporate this regularity condition.

\( N > 3 \) case: The same construction leads to finite size Bethe equations for any odd \( N \). Like in equation (7.26) the number of regularity constraints at each zero of the determinant is apparently \( N - 1 \) but reduces to only one constraint: namely the cancellation of the projection of \( \begin{pmatrix} T_{1,-1}(\theta - i(N - 2)/4) \\ \vdots \\ T_{N-1,-1}(\theta + i(N - 2)/4) \end{pmatrix} \) to the kernel of the matrix \( d_{i,j} \) defining the linear system (7.13).
This procedure for finite-size Bethe equations was described here for odd $N$ and for states having real Bethe roots in the asymptotic limit. The subtlety which arises when $N$ is even, or for the states having, in the asymptotic limit, complex-conjugated complexes of Bethe roots, is that the zeroes of the determinant do not lie on the real axis but approximately on $\mathbb{R} \pm i/2$. The above procedure can in principle be applied anyway, but its interpretation is less clear because the regularity condition is imposed at the very boundary of the analyticity strip.

7.5 Numerical results

As seen in the figure 4, this method allows to compute the energies of $U(1)$ sector excited states for a wide range of lengths $L$. We can see that in the IR, $L \to \infty$ limit, the energies of individual states behave mainly according to the number of “particles” forming the state - the number of the Bethe roots $\theta_j$: The vacuum energy tends to 0, while the energies of the states $\theta_0, \theta_1$ and $\theta_2$ tend to 1, and the energy of $\theta_{00}$ tends to 2). In the conformal limit $L \to 0$ we will see that the behavior is defined by the “particle’s” mode numbers: The energy goes to $\frac{2\pi}{L}(-\frac{N^2-1}{12} + n)$ where $n$ is the total momentum mode number.

Numerical restrictions As the length $L$ is decreasing, the algorithm looks worse and worse converging, and the densities become more and more peaked around the endpoints of the distribution. By choosing a small enough interpolation step (the densities $f_j$ are numerically defined by polynomial interpolation from a finite number of values), it was nevertheless possible to make the algorithm reasonably convergent for the considered states and lengths $L$, when $N = 3$. Decreasing further the interpolation step means increasing the computation time, and the necessary amount of memory, which puts a limit to our precision and the minimal length$^{24}$.

$^{24}$We are aware of the fact that our precision for the energy levels is not sufficiently good. Our numerical results are still quite preliminary, we will try to improve them in the nearest future by using a better interpolation procedure and using more powerful computers. We definitely believe that our procedure can give a much better precision than these first results in the $N = 3$ case, and we hope also to tackle the $N \geq 4$ case.
Table 1: Numerical energies for several U(1) sector states at $N = 3$

| $L/\sin\left(\frac{\pi}{3}\right)$ | $E_{\text{vacuum}}$ | $E_{\theta_0}$ | $E_{\theta_1}$ | $E_{\theta_2}$ | $E_{\theta_{1,0}}$ |
|------------------------------------|---------------------|----------------|----------------|----------------|----------------|
| 60                                 | $-7 \times 10^{-24}$| $1.00000(0)$  | $1.00(7)$      | $1.28(6)$      | $2.00(3)$      |
| 20                                 | $-5.1(6) \times 10^{-8}$| $1.00000(0)$  | $1.06(3)$      | $1.23(5)$      | $2.02(6)$      |
| 10                                 | $-4.2(0) \times 10^{-4}$| $0.99(8)$      | $1.23(3)$      | $1.76(1)$      | $2.08(2)$      |
| 8                                  | $-2.80(1) \times 10^{-3}$| $0.98(9)$      | $1.3(4)$       | $2.0(6)$       | $2.1(0)$       |
| 4                                  | $-0.119(7)$          | $0.87(7)$      | $2.0(2)$       | $3.7(0)$       | $2.1(8)$       |
| 2                                  | $-0.782(3)$          | $0.45(2)$      | $3.4(2)$       | $6.8(2)$       | $2.4(3)$       |
| 1                                  | $-2.43(2)$           | $-0.59(2)$     | $6.2(0)$       | $13.0$         | $2.7(7)$       |
| 0.8                                | $-3.29(1)$           | $-1.15(7)$     | $7.5(8)$       | $16.0$         | $2.9(0)$       |
| 0.4                                | $-7.72(1)$           | $-4.21(0)$     | $9.8$          | $3.1 \times 10^1$ | $3.2(7)$     |
| 0.2                                | $-16.9(0)$           | $-10.9(1)$     | $2.8 \times 10^1$ | $6.1 \times 10^1$ | $3.2(6)$     |
| 0.1                                | $-35.8(8)$           | $-25.3$        | $5.7 \times 10^1$ | $1.2 \times 10^2$ | $2.0$       |
| 0.04                               | $-94.3(3)$           | $-71.7$        | $1.4 \times 10^2$ | $3.0 \times 10^2$ | $-5.4$      |
| 0.01                               | $-3.9(6) \times 10^2$| $-3.2(2) \times 10^2$ | $5.6 \times 10^2$ | $1.1 \times 10^3$ | $-6.1 \times 10^1$ |
| 0.004                              | $-1.0(1) \times 10^3$ | $-8.4(7) \times 10^2$ | $(1) \times 10^3$ | $(3) \times 10^3$ | $-2 \times 10^2$ |
| 0.001                              | $-4.1(6) \times 10^3$ | $-3.5(8) \times 10^3$ | $(5) \times 10^3$ | $(1) \times 10^4$ | $-1 \times 10^3$ |

Unfortunately, at $N \geq 4$ the calculations become heavier and with the size of interpolation steps we can afford our algorithm becomes instable already for $L$ of order $\sim 1$ (which means we cannot really check the conformal limit for instance). At the moment we cannot say whether this has a physical meaning (like some symmetry breaking down, or some new type of singularity appearing) or whether it is just a numerical artifact, due to a poor numeric accuracy, or to the choice of the equations. For instance, it could be that the function we iterate over stops to be a contraction mapping but still has a fixed point, and maybe even that by rewriting slightly the functions it could become a contraction again, and extracting its fixed point would be possible by iterations.

8. IR and UV limits

8.1 Leading order results at large $L$

The approach of this paper allows to compute the first exponential Lüschler type correction, the so called Lüschler correction, to the energy at large $L$, as we will show now on a few examples.

8.1.1 Vacuum

The large $L$ behavior of vacuum is given by the condition that

$$Y_{a,0}|_{L \rightarrow \infty}^{\text{Lüschler}} = (T_{a,1})^2 e^{-L p_a}$$

(8.1)
where $T_{a,1}$ is equal to the binomial coefficient $\binom{N}{a}$. This is obtained from (1.19) where $\varphi = 1$, and can be plugged directly into (2.10) to get the energy to the leading order. By construction, the resulting energy fits well our numeric results when $L$ is large enough.

8.1.2 Mass Gap at $N = 3$

When $N = 3$ it suffices to compute $Y_{1,0}$ to get the energy, because $Y_{2,0} = \overline{Y_{1,0}}$.

Moreover the previous analysis shows that

$$Y_{1,0} = e^{-L\cosh(\frac{2\pi}{3})} \frac{(T_{1,1})^2}{T_{0,0}T_{2,0}} \frac{1}{S_0(\theta - 3\frac{i}{4})} \frac{\varphi(\theta - 3\frac{i}{4})}{\varphi(\theta + \frac{i}{4})} \frac{1}{\chi_{CDD}(\theta - 3\frac{i}{4})}$$

(8.2)

$$= e^{-L\cosh(\frac{2\pi}{3})} \frac{(3\theta - 5\frac{i}{4})^2}{(\theta + \frac{i}{4})^2} \frac{1}{S_0(\theta - 3\frac{i}{4})} \frac{1}{\chi_{CDD}(\theta - 3\frac{i}{4})}$$

(8.3)

that enables to compute at large $L$ the leading order value of the integral term in (5.3).

The analogous analysis for the second term in (5.3) is a bit more tricky, as it involves the position of the Bethe root. This position can be estimated by computing the densities to the leading order, to deduce the first correction to $T_{1,0}$ in order to solve the equation $T_{1,0}(\theta_0 + i/4) = 0$.

For the mass gap, this root should be at the origin, up to exponential corrections in $L$. Moreover one can show that $T_{1,0}(0 + i/4) \sim \frac{i}{6} f_2(0) + i f_3(0) = O(e^{-L\sqrt{3}/2})$, while $T'_{1,0}(0 + i/4) \sim i$, so that $T_{1,0}(\theta_0 + i/4) = 0$ gives $\theta_0 \sim -\frac{i}{3} f_2(0) - f_3(0)$. Using the asymptotic expression for $f_j$'s (which can be extracted by keeping only the leading order in $T_{a,s}$ and in $d_{a,j}$ in the formula (7.11)), one gets $\theta_0 \sim \frac{i e^{-\sqrt{3}L/2} \Gamma(\frac{4}{3})^2}{\sqrt{3} \pi \Gamma(\frac{4}{3})^2}$, so that the second term in (5.3), which is $\sinh \left( \frac{2\pi}{3} (\theta_0 - i/4) \right) - \sinh \left( \frac{2\pi}{3} (\theta_0 + i/4) \right)$, can be computed at leading order.

That gives

$$E^{massgap}_{L \to \infty} \sim 1 - \left( \frac{32 e^{-\sqrt{3}L/2} \pi^3}{\Gamma(\frac{4}{3})^6} \right)$$

(8.4)

which is in very good agreement the numerical results, as can be seen in fig.

Moreover, this expression (8.4) coincides exactly with the so-called $\mu$-term [35, 36], which is known to dominate the finite-size corrections in the presence of the bound states.

8.2 Conformal limit at $L \to 0$

Let us start from the vacuum. At very small $L$, the effective coupling constant becomes very small $g_0^2(L) \simeq \frac{2\pi}{\log L}$ and we can linearize the field on the group manifold in the vicinity of $g(\sigma, \tau) = I$ as $g^{-1} \partial_\mu g \simeq \partial_\mu A$, where $A(\sigma, \tau)$ is a Hermitian $N \times N$ traceless matrix field. The $SU(N)$ PCF model should become a 2d CFT of $N^2 - 1$ massless bosons: $R(L)$ is very big, the action will be

\[ \text{\textsuperscript{25}The precision is over 10\% when } L \geq 4. \]

\[ \text{\textsuperscript{26}These large } L \text{ expressions are obtained by neglecting integral terms in the determinant expression of } T_{1,0}. \]
The vacuum, and the states \( \theta \)

We only recall that for the mass gap, in the logarithmic approximation,

\[
[37] \text{and was compared with the numerical results following from the TBA approach in \[6\].}
\]

The energies of excited states are

\[
L \frac{2\pi}{2\pi} E_{\vec{n}_1, \vec{n}_2 \vec{n}_3 \ldots} (L) \approx -\frac{\pi(N^2 - 1)}{12} + \sum_{k=1}^{N} \sum_{\alpha=1}^{N^2-1} | n_k^{(\alpha)} |
\]

(8.6)

where \( \vec{n}_k = (n_k^{(1)}, n_k^{(2)}, \ldots, n_k^{(N^2-1)}) \) are the momentum numbers of particles constituting the state. We see that the small \( L \) asymptotics of our plots are well described by this formula: The vacuum, and the states \( \theta_0, \theta_{0,0}, \ldots \) have total momentum zero, and their energy satisfies \( L \frac{2\pi}{2\pi} E(L) \approx -\frac{\pi(N^2-1)}{12} \). On the other hand, a state like \( \theta_1 \) has momentum number equal to 1 and \( L \frac{2\pi}{2\pi} E(L) \approx -\frac{\pi(N^2-1)}{12} + 1, \ldots \)

The qualitative behavior of the states \( \theta_0, \theta_{0,0}, \ldots \), at very small \( L \)'s can be explained by the fact that the quantum fields are dominated by their zero modes. Since the momentum modes are not excited the field \( \vec{Y}(\sigma, \tau) \) does not depend on \( \sigma \). The action and the Hamiltonian become:

\[
S_\sigma \approx \frac{NL}{\ell_0^2(L)} \int dt \, \text{tr}(g^{-1} \partial_x g)^2, \quad \hat{H} = \frac{\ell_0^2(L)}{4NL} \text{tr} \hat{J}^2
\]

(8.7)

where the \( g(\tau) \) represents the coordinate of a material point (a top) on the group manifold, and \( \hat{J} \) is the corresponding angular momentum on the group. The quantum mechanical spectrum of this system is well known: the quantum states are classified according to the irreducible representations of \( su(N) \) characterized by highest weight with components \( (m_1 \geq m_2 \geq \cdots \geq m_N) \) usually represented by a Young tableaux \( \lambda \) with \( N \) rows with the lengths \( m_j, j = 1, \cdots, N \). The operator \( \text{tr} \hat{J}^2 \) is nothing but the second Casimir operator with the well known eigenvalues, so that

\[
L \frac{2\pi}{2\pi} (E_\lambda - E_0) \approx 1 \frac{\ell_0^2(L)}{8N\pi} \text{tr} \hat{J}^2 = \sum_{k=1}^{N} r_k (r_k - 2k + N + 1) \frac{N}{N \log(1/L)}
\]

(8.8)

where \( r_k = m_k - \frac{1}{N} \sum_{j=1}^{N} m_j \). For instance, for a state with only \( M \) real roots in the asymptotic limit (and without self-conjugated complexes of roots), we have \( m_1 = M, m_{k \geq 2} = 0 \), and

\[
L \frac{2\pi}{2\pi} (E_{\theta_0_{\{0,0,\ldots,0\}}^{M \text{ times}}} - E_0) \approx \frac{N - 1 M (M + N)}{N^2 \log(1/L)}
\]

(8.9)

This formula explains well the fact that the corresponding plots on fig.4 converge slowly, as inverse logarithm of \( L \), to \(-(N^2 - 1)/12\).

The perturbative calculation of the mass gap \([E_{\theta_0} (L) - E_0 (L)]\) for \( L \ll 1 \) was done in \[37\] and was compared with the numerical results following from the TBA approach in \[6\]. We only recall that for the mass gap, in the logarithmic approximation,
Figure 5: Numeric energy of the mass Gap (blue dots), compared to the analytic expression of $E_{\text{massgap}}^{L \to \infty}$ [green dotted line] and to the conformal limit $E_{\theta_0} = E_{\text{vac}} + \frac{8 \pi}{9} \log(1/L) + \frac{1}{3} \log(\log(1/L))$ [red dashed line], where $c = 12.3$ is chosen to fit the data.

\[
\frac{L}{2\pi} [E_{\theta_0}(L) - E_0(L)] \approx \frac{N^2 - 1}{N^2} \frac{1}{\log \frac{c}{L} + \frac{1}{2} \log \log \frac{c}{L}}, \quad (L \ll 1) \tag{8.10}
\]

which is in the perfect agreement with (8.8), as well as with our numerics.

The formula (8.8) also gives a prediction for state $\theta_{0\theta}$ with two zero-momentum-particles, namely $\frac{E_{\theta_{0\theta}}(L) - E_0(L)}{E_{\theta_0}(L) - E_0(L)} \approx \frac{10}{3}$. This result is consistent with our numerics when $L$ is between approximately $10^{-2}$ and 1 (see the fig. 3). For smaller $L$, although the algorithm seems to converge, the numeric precision is too poor because the integrand of (5.1) is very peaked, and the compatibility with $L \to 0$ predictions cannot be checked very accurately in this regime.

Although our approach is based on adding some terms to the finite size solution, it reproduces correctly these conformal expressions, which shows that this description is not only accurate in some vicinity of $L = \infty$, but even in the conformal limit where $L$ is very small. This confirms that these terms where added in a sufficiently general manner to describe the relevant solutions of the $Y$-system.

9. Discussion

We have presented here, on the example of the $SU(N)$ principal chiral field model, a potentially powerful and rather general approach to the study of the finite volume spectrum of various integrable 1+1 dimensional sigma-models. The approach continues the ideas of [11] where the method was proposed on the example of the $SU(2)$ PCF, but for $N > 2$ the method has to be seriously reconsidered due to many new physical features w.r.t. the $N = 2$ case. In particular, the presence of the bound state particles and the non-reality of the Bethe roots at finite $L$ show a few qualitatively new features within our approach.

For virtually all integrable sigma models, the TBA-like approach of Al.Zamolodchikov can be summarized in a very universal system of functional equations, the $Y$-system. The
Y-system is equivalent to the famous Hirota equation - the Master equation of integrability describing in this case the integrable discrete dynamics with respect to a pair of “representational” variables, $a, s$ and the spectral parameter (rapidity) $\theta$. The boundary conditions for $a, s$ are defined by the symmetry algebra of the model, whether as the analytic structure w.r.t. the $\theta$ variable is in general the most complicated issue, really defining the dynamics of the model. It is plausible that in fact even the possible analyticity structures are greatly constrained by Hirota dynamics and by the symmetry algebra. It would be interesting to classify possible types of analyticity stemming from Hirota dynamics and some simple physical arguments (relativistic invariance, crossing, absence of certain singularities, etc.) related to the finite volume sigma models, similarly to the S-matrix bootstrap theory of Al.&A.Zamolodchikov valid only at infinite volume. This could lead to an interesting classification of sigma models themselves and eventually to the discovery of new integrable models.

In this paper, we managed to transform one such relativistic $\sigma$-model, the $SU(N) \times SU(N)$ principal chiral field into a finite set of NLIEs, by solving its finite $L$ Y-system in terms of wronskian determinants of a finite number of Q-functions and parameterizing these Q-functions by $N-1$ densities correcting their large $L$ asymptotics to any finite $L$.

Generalizes the analytic and numerical results of [11] to $N \geq 2$, and we could numerically check, at least when $N = 2, 3$, that this procedures solves the Y-system, and enables to compute energies for a wide range of lengths $L$, compatible with the UV conformal limit and the IR finite size (Lüscher) corrections. On the way, we conjectured the natural generalization of the energy formula for excited states for the $U(1)$ at finite $L$, to $N \geq 2$. This generalization appears to be unexpectedly non-trivial and looks different for even and odd $N$. The question of definition of the energy formula for excited states deserves a better understanding and hopefully the eventual derivation.

The analysis was done for $U(1)$ sector states, and it certainly can and should be generalized to any excited state, as was done in [11] for $N = 2$. To do this, one will have to understand the asymptotic terms, and the structure of zeroes. In particular some extra zeroes would appear, which might affect the way the energy is computed by contour manipulation. Apart from that, the main difference with $U(1)$ sector would be that for non-symmetric states (i.e. when $Y_{a,s} \neq Y_{a,-s}$), it would be necessary to introduce $N-1$ densities for the right wing and $N-1$ densities for the left wing. One would have to write (3.3) as $2(N-1)$ different equations, by writing the left hand side either as $\frac{T_{11}^{(R)} T_{11}^{(R)}}{T_{2,0}^{(R)} T_{0,0}^{(R)}}$ or as $\frac{T_{11}^{(L)} T_{11}^{(L)}}{T_{1,0}^{(L)} T_{0,0}^{(L)}}$. But a similar approach based on the wronskian solution of Y-system should a priori still enable us to compute the energies of these states.

An interesting problem which our approach might help to solve is the planar $N \to \infty$ limit in PCF. This model has a rich history of its comparison to QCD and it might provide an important example of the exactly solvable 2+1 dimensional bosonic string theory, similarly to the matrix quantum mechanical model of the 1+1 dimensional, $c = 1$ non-critical string theory proposed and solved in [11]. An exact and explicit solution for it was given in the case of infinite volume $L$ but in the presence of specific ”magnetic” fields [41].
As concerns the numerics, our algorithm converges very well for any length when \( N \leq 3 \), but for \( N \geq 4 \) it is very unstable for small enough length \( L \), already at \( L \lesssim 1 \) (which means for instance that we cannot really check the convergence to the conformal UV limit). Hopefully this instability has no direct physical meaning and is just a numerical artifact, due to a poor numeric accuracy, or to a bad choice of the iteration procedure for our NLIEs.

We believe that this method of deriving a finite system of NLIEs for integrable sigma models is general and powerful enough to work for much more complicated cases of AdS/CFT correspondence, such as the superstring on the AdS\(_5 \times S^5\) background dual to N=4 SYM theory, and the so called ABJM model where the Y-system was already discovered \([14, 38, 39]\). The wronskian quasiclassical character \([17]\) and even the full quantum solution of the Hirota dynamics for AdS\(_5\)/CFT\(_4\) \([38]\) are already available. It is left there to understand the very rich and complicated analyticity structure of Q-functions for short operators to complete the derivation of the AdS/CFT NLIE.

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