CRITICAL METRICS OF THE VOLUME FUNCTIONAL WITH PINCHED CURVATURE

H. BALTAZAR AND C. QUEIROZ

Abstract. In this paper, we prove that a critical metric of the volume functional with pinched Weyl curvature is isometric to a geodesic ball in $S^n$. Moreover, we provide a necessary and sufficient condition on the norm of the gradient of the potential function in order to classify such critical metrics.

1. Introduction

The investigation of critical metrics becomes a classical topic in geometric analysis. In this sense, much attention has been given to the critical metrics of the volume functional $V : M^m_n \rightarrow \mathbb{R}$, where $M^m_n$ denotes the space all structures on $M$ with constant scalar curvature and prescribed boundary metric $\gamma$. In [16], inspired by ideas developed in [13], Miao and Tam provided sufficient conditions for a metric $g \in M^m_n$ to be a critical point. In fact, they proved that $g$ is a critical point of the functional $V|_{M^m_n}$ if and only if there exist a smooth function $f$ on $M$ such that $f|_{\partial M} = 0$ and satisfies the following system of PDE’s

$$-(\Delta f)g + Hess f - f Ric = g,$$

provided that the first Dirichlet eigenvalue of $(n - 1)\Delta g + R$ is positive. Here, the terms $Ric$ and $Hess f$ stands for the Ricci tensor and Hessian form associated to $g$ on $M$, respectively.

Following the terminology used in [9, 16, 17] we recall the definition of such critical metrics investigated by Miao-Tam, which for simplicity will be called Miao-Tam critical metrics.

**Definition 1.** A Miao-Tam critical metric is a 3-tuple $(M^n, g, f)$, where $(M^n, g)$, $n \geq 3$, is a compact oriented Riemannian manifold with a smooth boundary $\partial M$ and $f : M^n \rightarrow \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ and satisfies (1.1).

There are several works about classification of Miao-Tam critical metrics involving the Weyl curvature tensor, see for instance, [3, 4, 5, 7, 9, 17]. Hence, motivated by [12] and based in techniques developed in [1, 2], we shall provide a classification of such critical metrics considering a pointwise pinching condition. More precisely, we have the following result.

2010 Mathematics Subject Classification. Primary 53C25, 53C20, 53C21; Secondary 53C65.

Key words and phrases. Volume Functional; critical metrics; Weyl tensor.
Theorem 1. Let \((M^n, g, f)\) be a Miao-Tam critical metric with nonnegative scalar curvature and satisfying
\[
W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \tilde{Ric} \otimes g \leq \frac{R}{\sqrt{2(n-1)(n-2)}}.
\]
Then \((M^n, g)\) is isometric to a geodesic ball in \(\mathbb{S}^n\).

The symbol \(\otimes\), in the above expression, denotes the Kulkarni-Nomizu product which is defined for any two symmetric \((0, 2)\)-tensor \(S\) and \(T\) as follows
\[
(S \otimes T)^{ijkl} = S^{ik}T^{jl} + S^{jl}T^{ik} - S^{il}T^{jk} - S^{jk}T^{il}.
\]

Since the Weyl tensor vanishes in dimension three, we obtain the following consequence:

Corollary 1. Let \((M^3, g, f)\) be a three dimensional Miao-Tam critical metric with nonnegative scalar curvature and satisfying
\[
|Ric| \leq \frac{R}{\sqrt{24}}.
\]
Then \((M^3, g)\) is isometric to a geodesic ball in \(\mathbb{S}^3\).

Remark 1. It is easy to verify that, the pinching \((1.3)\) becomes in this case
\[
|Ric| \leq \frac{R}{\sqrt{24}} \leq \frac{R}{\sqrt{6}}.
\]
Then it suffices to apply [7, Corollary 1.5] to conclude that \((M^3, g)\) is isometric to a geodesic ball in \(\mathbb{S}^3\).

In order to justify our second result will be necessary to remember the classical simple connected examples of Miao-Tam critical metrics and a property involving its potential function. The first one is the Euclidean ball of radius \(r_0\) in \(\mathbb{R}^n\) with standard metric \(g_0\) and potential function
\[
f(x) = \frac{1}{2(n-1)}(r_0^2 - |x|^2).
\]
Now, since the scalar curvature is null and \(|\nabla f|_{\partial M} = \frac{r_0}{n-1}\) we can deduce
\[
|\nabla f|^2 + \frac{Rf^2}{n(n-1)} + \frac{2f}{n-1} = \frac{r_0^2}{(n-1)^2} = |\nabla f|^2_{\partial M}.
\]
At the same time, it is not hard to verify that \((\mathbb{S}^n, g_0)\), the standard sphere with canonical metric \(g_0\), is a Miao-Tam critical metric with potential function
\[
f(x) = \frac{1}{n-1} \left( \frac{\cos r}{\cos r_0} - 1 \right),
\]
where \(r\) is the geodesic distance from the point \((0, \ldots, 1)\) and radius \(r_0 \neq \frac{\pi}{2}\). Furthermore, with a straightforward computation we obtain the same identity,
\[
|\nabla f|^2 + \frac{Rf^2}{n(n-1)} + \frac{2f}{n-1} = \frac{\text{sen}^2 r_0}{(n-1)^2 \cos^2 r_0} = |\nabla f|^2_{\partial M}.
\]
Finally, the last interesting example is the hyperbolic space $\mathbb{H}^n$ embedded in the well-known Minkowski space $(\mathbb{R}^{n+1}, dx_1^2 + \ldots + dx_n^2 - dt^2)$,

$$\mathbb{H}^n = \{(x_1, \ldots, x_n, t) \mid x_1^2 + \ldots + x_n^2 - t^2 = -1, \ t > 0\}.$$  

The geodesic ball in $\mathbb{H}^n$ centred at $(0, \ldots, 0, 1)$ is a Miao-Tam critical metric with potential function

$$f(x) = \frac{1}{n-1} \left( 1 - \frac{\cosh r}{\cosh r_0} \right).$$

Similarly, the potential function must satisfy the following expression

$$|\nabla f|^2 + \frac{Rf^2}{n(n-1)} + \frac{2f}{n-1} = \frac{\text{sech}^2 r_0}{(n-1)^2 \cosh^2 r_0} = |\nabla f|_{\partial M}^2.$$

It is natural to ask whether these examples are the only Miao-Tam critical metrics that satisfy such a property. In this sense, inspired by ideas developed by Leandro [15] we provide a complete answer to this question. More precisely, we prove the following result.

**Theorem 2.** Let $(M^n, g, f)$ be a Miao-Tam critical metric satisfying

$$(1.4) \quad |\nabla f|^2 + \frac{Rf^2}{n(n-1)} + \frac{2f}{n-1} = |\nabla f|_{\partial M}^2.$$  

Then $M^n$ is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $S^n$.

**Remark 2.** Let us point out that, for equality $(1.4)$ to make sense, it is necessary to remember that a Miao-Tam critical metric has $|\nabla f|$ constant along $\partial M$, see [8, Section 3] for more details.

### 2. Preliminaries

In this section we need recall some special tensors which will be important for understanding the desired results. We start with Weyl tensor, given by the decomposition formula

$$(2.1) \quad W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (\text{Ric} \otimes g)_{ijkl} + \frac{R}{2(n-1)(n-2)} (g \otimes g)_{ijkl},$$

where $R_{ijkl}$ stands for the Riemann curvature tensor. In the sequel, we have the Cotton tensor $C$ which is given by

$$(2.2) \quad C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (\nabla_i R_{gjk} - \nabla_j R_{gik}).$$

It is easy to check that $C_{ijk}$ is skew-symmetric in the first two indices and trace-free in any two indices. These two tensors described above are related as follows

$$(2.3) \quad C_{ijk} = \frac{(n-2)}{(n-3)} \nabla_i W_{ijkl},$$

provided $n \geq 4$. 

Now, we note that the fundamental equation of a Miao-Tam critical metric can be rewritten in the tensorial language as follows

\begin{equation}
- \Delta f g_{ij} + \nabla_i \nabla_j f - f R_{ij} = g_{ij}.
\end{equation}

Tracing \(2.4\) we have

\begin{equation}
\Delta f = - \frac{R}{n-1} f - \frac{n}{n-1}.
\end{equation}

Furthermore, by using \(2.5\) it is not difficult to check that

\begin{equation}
f \tilde{Ric} = H \tilde{ess} f,
\end{equation}

where \(\tilde{T}\) stands for the traceless of \(T\).

Under this notation we get the following formula for a Miao-Tam critical metric

\begin{equation}
\nabla_j f R_{ik} - \nabla_i f R_{jk} = f C_{ijk} - R_{ijkl} \nabla_l f - \frac{R}{n-1} (\nabla_i f g_{jk} - \nabla_j f g_{ik}).
\end{equation}

Its proof can be found in \cite{9}.

3. Key lemmas

In this section we shall deduce a couple of integral identities, which allows us to obtain the classification of the Miao-Tam critical metrics under a curvature estimate. Both integrals are direct consequences of the Bochner type formulas obtained by first author and Ribeiro Jr. in \cite{7}, see Lemma 3.1 and Lemma 3.2 for more details.

**Lemma 1.** Let \((M^n, g, f)\) be a Miao-Tam critical metric. Then we have

\[
\int_M |\tilde{Ric}|^2 |\nabla f|^2 dM_g = \frac{n-3}{2(n-1)} \int_M f^2 |C_{ijk}|^2 dM_g + \int_M f^2 |\nabla \tilde{Ric}|^2 dM_g \\
+ \frac{n}{n-1} \int_M f |\tilde{Ric}|^2 dM_g + \frac{2}{n-1} \int_M f^2 |\nabla \tilde{Ric}|^2 dM_g \\
+ \int_M f^2 \left( \frac{n}{n-2} tr(\tilde{Ric}^3) - W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} \right) dM_g \\
- \frac{n-2}{n-1} \int_M f C_{ijk} W_{ijkl} \nabla_l f dM_g,
\]

where \(\tilde{Ric}^3\) is the 2-tensor defined by \((\tilde{Ric}^3)_{ij} = \hat{R}_{ik} \hat{R}_{kl} \hat{R}_{ij}\).

**Proof.** Using \cite{7} Lemma 3.1 it is immediate to obtain the following integral identity

\[
- \int_M \langle \nabla f^2, \nabla |\tilde{Ric}|^2 \rangle dM_g = \int_M f^2 |C_{ijk}|^2 dM_g + 2 \int_M f^2 |\nabla \tilde{Ric}|^2 dM_g \\
+ \frac{2n}{n-2} \int_M f^2 tr(\tilde{Ric}^3) dM_g + \frac{2}{n-1} \int_M f^2 |\nabla \tilde{Ric}|^2 dM_g \\
+ 4 \int_M f C_{ijk} \nabla_j f R_{ik} dM_g - 2 \int_M f^2 W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dM_g,
\]

\[(3.1)\]
where we have used that
\[ \text{tr}(\text{Ric}^3) = \text{tr}(\hat{\text{Ric}}^3) + \frac{3}{n} R|\text{Ric}|^2 + \frac{1}{n^2} R^3. \]

On the other hand, using \[(2.7)\] and \[(2.1)\], it is not difficult to verify that
\[ (3.2) \quad \int_M f C_{ijk} \nabla_j f R_{ik} dM_g = \frac{n-2}{2(n-1)} \int_M |f^2| C_{ijk}^2 dM_g - f C_{ijk} W_{ijkl} \nabla_l f dM_g. \]

Moreover, since \( M^n \) has constant scalar curvature we may use \[(2.5)\] to deduce
\[ \text{div}(|\hat{\text{Ric}}|) \nabla f^2 - \langle \nabla f^2, |\text{Ric}|^2 \rangle = 2 f \Delta f |\hat{\text{Ric}}|^2 + 2 |\hat{\text{Ric}}|^2 |\nabla f|^2. \]

As consequence, upon integrating \[(3.3)\] over \( M \) we apply the divergence theorem to arrive at
\[ (3.4) \quad - \int_M \langle \nabla f^2, |\text{Ric}|^2 \rangle dM_g = - \frac{2}{n-1} \int_M f^2 |\text{Ric}|^2 dM_g - \frac{2n}{n-1} \int_M f |\hat{\text{Ric}}|^2 dM_g + \frac{n}{n-1} \int_M f |\hat{\text{Ric}}|^2 dM_g. \]

Finally, just replace \[(3.2)\] and \[(3.4)\] into \[(3.1)\] to get the desired result.

Now, we use \[ [7, \text{Lemma 3.2}] \], to get another formula for \( \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 dM_g \).

**Lemma 2.** Let \((M^n, g, f)\) be a Miao-Tam critical metric. Then we have
\[ \frac{3}{2} \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 dM_g = - \frac{1}{n-1} \int_M f^2 C_{ijk}^2 dM_g + \int_M f^2 |\nabla \text{Ric}|^2 dM_g \]
\[ + \frac{n}{2(n-1)} \int_M f |\hat{\text{Ric}}|^2 dM_g + \frac{3}{2(n-1)} \int_M f^2 R|\hat{\text{Ric}}|^2 dM_g \]
\[ - \frac{n-2}{n-1} \int_M f C_{ijk} W_{ijkl} \nabla_l f dM_g. \]

**Proof.** By direct computation using \[ [7, \text{Lemma 3.2}] \], we achieve
\[ - \frac{3}{4} \int_M \langle \nabla f^2, |\text{Ric}|^2 \rangle dM_g = - \int_M f^2 C_{ijk}^2 dM_g + \int_M f^2 |\nabla \text{Ric}|^2 dM_g \]
\[ - \frac{n}{n-1} \int_M f |\hat{\text{Ric}}|^2 dM_g + 2 \int_M f C_{ijk} \nabla_j f R_{ik} dM_g. \]

To conclude, just consider the same argument which was used in the last part of the proof of Lemma \[ [1] \].

### 4. Miao Tam critical metrics with pinched curvature

In this section we will prove the Theorem \[ [1] \] announced in the introduction.
4.1. **Proof of Theorem 1**. First of all, we take the difference between the expressions obtained in Lemma 1 and Lemma 2 to infer

\[ 0 = \int_M |\tilde{\text{Ric}}|^2 |\nabla f|^2 dM_g + \int_M f^2 |C_{ijk}|^2 dM_g + \frac{n}{n-1} \int_M f |\tilde{\text{Ric}}|^2 dM_g \]

(4.1) 

\[ + \frac{1}{n-1} \int_M f^2 R |\tilde{\text{Ric}}|^2 dM_g + 2 \int_M f^2 \left( \frac{n}{n-2} \text{tr}(\tilde{\text{Ric}})^3 - W_{ijkl} \tilde{R}_{ik} \tilde{R}_{jl} \right) dM_g. \]

Before proceeding it is important to remember the following inequality

\[ \left| \frac{n}{n-2} \tilde{R}_{ij} \tilde{R}_{jk} \tilde{R}_{ik} - W_{ijkl} \tilde{R}_{ik} \tilde{R}_{jl} \right| \leq \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{2n}{n-2} |\tilde{\text{Ric}}|^2 \right)^{1/2} |\tilde{\text{Ric}}|^2, \]

which is true for an arbitrary \( n \)-dimensional Riemannian manifold. Its proof can be found in [1] or [14].

Hence, using the above inequality and our curvature estimate (1.2), it is immediate to check that

\[ \frac{1}{n-1} R |\tilde{\text{Ric}}|^2 + 2 \left( \frac{n}{n-2} \text{tr}(\tilde{\text{Ric}})^3 - W_{ijkl} \tilde{R}_{ik} \tilde{R}_{jl} \right) \]

\[ \geq \left\{ \frac{1}{n-1} R - 2 \sqrt{\frac{n-2}{2(n-1)}} |W| + \sqrt{n} \sqrt{2} (\tilde{\text{Ric}} \otimes g) \right\} |\tilde{\text{Ric}}|^2 \geq 0. \]

As consequence, returning to Identity (4.1), we force \( M^n \) to be Einstein manifold.

Then, we are in position to use Theorem 1.1 of [17] to conclude that \( (M^n, g) \) is isometric to a geodesic ball in \( S^n \). So, the proof is completed.

5. **Proof of Theorem 2**

5.1. **Proof of Theorem 2**. To start with, taking into account that \( M \) has constant scalar curvature, we derive our hypothesis on both side to infer

\[ \nabla_i \nabla_j f \nabla_j f + \frac{R f}{n(n-1)} \nabla_i f + \frac{1}{n-1} \nabla_i f = 0. \]

Hence, by (2.23), we have

\[ f R_{ij} \nabla_j f + \left( \Delta f + 1 + \frac{R f}{n(n-1)} + \frac{1}{n-1} \right) \nabla_j f = 0, \]

which can be written using (2.25) as

\[ (5.1) \quad \text{Ric}(\nabla f) = \frac{R}{n} \nabla f. \]

Here, we have used that \( f \) is positive in the interior of \( M \) and the equality immediately follows by continuity.
On the other hand, using again our hypothesis we may conclude the following expression of the laplacian on the norm of the gradient of the potential function \( f \),
\[
\frac{1}{2} \Delta |\nabla f|^2 = -\frac{R}{2n(n-1)} \Delta f^2 - \frac{1}{n-1} \Delta f
\]
\[
= -\frac{Rf + n}{n(n-1)} \Delta f - \frac{R}{n(n-1)} |\nabla f|^2
\]
\[
= \frac{1}{n} (\Delta f)^2 - \frac{R}{n(n-1)} |\nabla f|^2.
\]

(5.2)

Consequently, combining the classical Bochner’s formula [11],
\[
\frac{1}{2} \Delta |\nabla f|^2 = \text{Ric}(\nabla f, \nabla f) + \langle \nabla \Delta f, \nabla f \rangle + |\text{Hess} f|^2,
\]
Equalities (5.2) and (2.5), we arrive at
\[
\frac{1}{n} (\Delta f)^2 - \frac{R}{n(n-1)} |\nabla f|^2 = \text{Ric}(\nabla f, \nabla f) + \langle \nabla \left(-\frac{Rf + n}{n-1}\right), \nabla f \rangle + |\text{Hess} f|^2
\]
\[
= \text{Ric}(\nabla f, \nabla f) - \frac{R}{n-1} |\nabla f|^2 + |\text{Hess} f|^2 + \frac{1}{n} (\Delta f)^2,
\]
that is,
\[
|\text{Hess} f|^2 + \text{Ric}(\nabla f, \nabla f) = 0.
\]

Finally, using (5.1), the last equality allow us to conclude that the \( \text{Hess} f \) is identically zero in \( M^n \). This implies, from (2.6), that the \( M \) is an Einstein manifold and we are in position to use again Theorem 1.1 of [17] to conclude that \( (M^n, g) \) is isometric to a geodesic ball in a simply connected space form \( \mathbb{R}^n, \mathbb{H}^n \) or \( \mathbb{S}^n \).

Acknowledgement. The first author was partially supported by PPP/FAPEPI/MCT/CNPq, Brazil [Grant: 007/2018] and CNPq/Brazil [Grant: 422900/2021-4].

References

[1] Baltazar, H.: On critical point equation of compact manifolds with zero radial Weyl curvature. Geom. Dedicata 202, 337–355 (2019)
[2] Baltazar, H.: Besse conjecture for compact manifolds with pinched curvature. Arch. Math. 115, no. 2, 229–239 (2020).
[3] Baltazar, H., Barros, A., Batista, R. and Viana, E.: On static manifolds and related critical spaces with zero radial Weyl curvature. Monatsh. Math. 191, no. 3, 449–463 (2020).
[4] Baltazar, H., Batista, R., Bezerra, K.: On the volume functional of compact manifolds with boundary with harmonic Weyl tensor. arXiv:1710.06217v1 [math.DG] (2017)
[5] Baltazar, H., Diógenes, R. and Ribeiro Jr., E.: Volume functional of compact manifolds with prescribed boundary metric, J. Geom. Anal. 31, 4703–4720 (2021).
[6] Baltazar, H. and Ribeiro Jr., E.: Critical metrics of the volume functional on manifolds with boundary. Proc. of the Amer. Math. Soc. 145, 3513–3523 (2017).
[7] Baltazar, H. and Ribeiro Jr., E. Remarks on critical metrics of the scalar curvature and volume functionals on compact manifolds with boundary. Pacific J. Math. 1, 29–45 (2018).
[8] Batista, R., Diógenes, R., Ranieri, M. and Ribeiro Jr., E.: Critical metrics of the volume functional on compact three-manifolds with smooth boundary. J. Geom. Anal. 27 1530–1547 (2017).
[9] Barros, A., Diógenes, R. and Ribeiro Jr., E.: Bach-Flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. J. Geom. Anal. 25, 2698–2715 (2015).
[10] Barros, A. and da Silva, A.: Rigidity for critical metrics of the volume functional. Math. Nachr. 292, 709–719 (2019).
[11] Bochner, S.: Vector fields and Ricci curvature, Bull. Am. Math. Soc. 52, 776–797 (1946).
[12] Catino, G.: Complete gradient shrinking Ricci solitons with pinched curvature. Math. Ann. 355, 629-635 (2013).
[13] Fan, X.-Q., Shi, Y.-G., Tam, L.-F.: Large-sphere and small-sphere limits of the Brown-York mass. Commun. Anal. Geom. 17, 37–72 (2009).
[14] Fu, H.P.: On compact manifolds with harmonic curvature and positive scalar curvature. J. Geom. Anal. 27, 3120–3139 (2017).
[15] Leandro, B.: A note on critical point metrics of the total scalar curvature functional. J. Math. Anal. Appl. 424, 1544–1548 (2015).
[16] Miao, P., Tam, L.-F.: On the volume functional of compact manifolds with boundary with constant scalar curvature. Calc. Var. PDE, 36, 141–171 (2009)
[17] Miao, P. and Tam, L.-F.: Einstein and conformally at critical metrics of the volume functional. Trans. Amer. Math. Soc. 363, 2907–2937 (2011).

(H. Baltazar) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO PIAUÍ, 64049-550 TERESINA, PIAUÍ, BRAZIL.
Email address: halyson@ufpi.edu.br

(C. Queiroz) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO PIAUÍ, 64049-550 TERESINA, PIAUÍ, BRAZIL.
Email address: chrisqueiroz@ufpi.edu.br