SU(5) gravitating monopoles

Yves Brihaye and Theodora Ioannidou

1 Faculté des Sciences, Université de Mons, 7000 Mons, Belgium
2 Mathematics Division, School of Technology, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece

E-mail: yves.brihaye@umh.ac.be and ti3@auth.gr

Received 22 September 2004, in final form 21 January 2005
Published 28 February 2005
Online at stacks.iop.org/CQG/22/1077

Abstract
Spherically symmetric and purely magnetic (i.e. \( A_0 = 0 \)) solutions of the full second-order SU(5) Einstein–Yang–Mills–Higgs equations are constructed using the harmonic map ansatz [1, 2]. In this way the problem reduces to solving a set of ordinary differential equations for the appropriate profile functions.

PACS numbers: 04.40.Dg, 04.70.Bw, 14.80.Hv

1. Introduction

Magnetic monopoles are of diverse interest since they are predicted from grand unified theories (GUT) and embody a rich mathematical structure. They appear in non-perturbative field theories and provide a new perspective on particle physics phenomenology. Since the SU(5) gauge group plays a central role in GUT, it is natural to classify the corresponding magnetic monopoles [3]. In the last decade, the effects of gravitation on monopoles has been considered [4] which revealed a rich pattern of solutions (including the occurrence of black holes) in terms of the gravitational parameter: \( \alpha^2 \equiv 4\pi G v^2 \). (Here, \( G \) is the Newton constant and \( v \in \mathbb{R} \) is the vacuum expectation value of the Higgs field). Recently, new interest for SU(5) monopoles was stimulated by the discovery of a deep analogy between their magnetic charges and the electric charges of (one generation) elementary particles [5]. This gave rise to several papers on the topic (see, e.g., [6, 7] and references therein).

In this paper, the harmonic map ansatz [2], recently applied to the SU(3) gravitating monopole [8], is used to construct their SU(5) counterparts. Note that in [10], SU(5) gravitating solutions (including black holes) have been derived; however, they are embeddings of the SU(2) ones. Here, SU(5) gravitating solutions are constructed which are not embeddings of the SU(2) ones and correspond to monopole–antimonopole configurations. In fact, they are solutions of the full second-order Yang–Mills–Higgs equations coupled with gravity which are not solutions of the first-order Bogomolny equations. Note that in the SU(2)}
case, gravitating solutions representing static monopole–antimonopole pairs are at most axially symmetric [9] (consistent with the flat limit [2]).

The Einstein–Yang–Mills–Higgs action is given by

$$S = \int \left( \frac{R}{16\pi G} - \frac{1}{2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) - \text{tr}(D_\mu \Phi D^\mu \Phi) - V(\Phi) \right) \sqrt{-g} \, d^4x$$

(1)

where the $SU(5)$ potential [5] is of the form

$$V(\Phi) = -\lambda_1 \text{tr}(\Phi^2) + \lambda_2 (\text{tr}(\Phi^2))^2 + \lambda_3 \text{tr}(\Phi^4) - V_{\text{min}}.$$  

(2)

Here $g$ is the determinant of the metric and $\eta$ is a constant matrix defined as $\eta = i\nu I_3$ with $\nu \in R$ and $I_3$ being the five-dimensional unitary matrix. The field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the covariant derivative of the Higgs field reads $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$. The term $V_{\text{min}} = -15\lambda_1^2/(60\lambda_2 + 14\lambda_3)$ in the potential has been subtracted due to the finiteness of the energy.

Variation of (1) with respect to the metric $g^{\mu\nu}$ leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

(3)

with the stress–energy tensor $T_{\mu\nu} = g_{\mu\nu} \mathcal{L} - 2 \frac{\partial_\mu \Phi \partial_\nu \Phi}{\eta}$ given by

$$T_{\mu\nu} = \text{tr}(2D_\mu \Phi D_\nu \Phi - g_{\mu\nu} D_\alpha \Phi D^\alpha \Phi) + 2 \text{tr}(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}) - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - g_{\mu\nu} V(\Phi).$$

(4)

The boundary conditions are that the energy is finite and that the Higgs field at infinity in a chosen direction (say $x_3$-axis) is a constant matrix $\Phi(0,0,\infty) = i\Phi_0$ where

$$\Phi_0 = \text{diag}(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5).$$

(5)

Note that, since $\Phi \in su(5)$ $\sum_i \kappa_i = 0$. In addition, (along this axis) the asymptotic value $G_0$ of the magnetic charge $G = (1 + |z|^2)^2 F_{zz}$ is

$$G_0 = G(0,0,\infty) = \text{diag}(n_1, n_2 - n_1, n_3 - n_2, n_4 - n_3, -n_4).$$

(6)

where $n_j$ are the magnetic charges [11] which characterize topologically the monopoles. In fact, the difference $m_j = \kappa_j - \kappa_{j+1}$ determines the mass of the monopole of type $j$, of which there are $n_j$ in total. In flat space, the corresponding energy is given by $E \geq \sum_j m_j n_j$ and the equality is valid in the Bogomolny limit. If the Higgs eigenvalues are not distinct, say $\kappa_j = \kappa_{j+1}$ then the monopole of type $j$ is massless and the integer $n_j$ is no longer a topological quantity (for more details, see [2]).

2. The harmonic map ansatz

For simplicity, we use the coordinates $r, z, \bar{z}$ on $\mathbb{R}^3$ where in terms of the usual spherical coordinates $r, \theta, \phi$ the Riemann sphere variable $z$ is given by $z = e^{i\theta} \tan(\theta/2)$. Then the Schwarzschild-like metric becomes

$$ds^2 = -A^2(r) B(r) \, dr^2 + \frac{1}{B(r)} \, dr^2 + \frac{4r^2}{(1 + |z|^2)^2} \, dz \, d\bar{z}, \quad B(r) = 1 - \frac{2m(r)}{r},$$

(7)

where $A(r)$ and $B(r)$ are real functions and depend only on the radial coordinate $r$, and $m(r)$ is the mass function. The (dimensionful) mass of the solution is $m_\infty \equiv m(\infty)$ and the square root of the determinant is

$$\sqrt{-g} = i A(r) \frac{2r^2}{(1 + |z|^2)^2}. $$

(8)
In this system of coordinates, action (1) becomes
\[
S = \int \left\{ \frac{B(1 + \vert z \vert^2)^2}{r^2} \text{tr}(|F_{r}z|^2) + \frac{(1 + \vert z \vert^2)^4}{4r^4} \text{tr}(F_{z}^2) - B \text{tr}((D_{r} \Phi)^2) - \frac{(1 + \vert z \vert^2)^2}{r^2} \text{tr}((D_{r} \Phi)^2) - V(\Phi) \right\} \sqrt{-g} r^2 dr,
\]
and its variation with respect to matter fields gives the matter equations.

Also, the Einstein equations (3) simplify to
\[
\frac{2}{r^2} m' = 8\pi G T_{00}^0, \quad \frac{2}{r} A' - \frac{2}{r} A = 8\pi G (T_{00}^0 - T'_{00})
\]
where prime denotes the derivative with respect to \( r \) and
\[
T_{00}^0 = \frac{(1 + \vert z \vert^2)^4}{4r^4} \text{tr}(F_{z}^2) - \frac{B(1 + \vert z \vert^2)^2}{r^2} \text{tr}(|F_{r}z|^2) - B \text{tr}((D_{r} \Phi)^2) - \frac{(1 + \vert z \vert^2)^2}{r^2} \text{tr}((D_{r} \Phi)^2) - V(\Phi)
\]
\[
T_{00}^0 - T'_{00} = -\frac{2B(1 + \vert z \vert^2)^2}{r^2} \text{tr}(|F_{r}z|^2) - 2B \text{tr}((D_{r} \Phi)^2).
\]

In the flat case [2], solutions of the full second-order Yang–Mills–Higgs equations can be obtained by assuming that the Higgs and gauge fields (for a real gauge, i.e. \( A_{z} = -A_{r} \)) are of the form
\[
\Phi = i \sum_{j=0}^{3} h_{j} \left( P_{j} - \frac{1}{N} \right), \quad A_{z} = \sum_{j=0}^{3} g_{j}[P_{j}, \partial_{z} P_{j}], \quad A_{r} = 0
\]
where \( h_{j}(r), g_{j}(r) \) are the radial dependent matter profile functions and \( P(z, \bar{z}) \) are 5 \times 5 Hermitian projectors: \( P_{j} = P_{j}^{\dagger} = P_{j}^{2} \), independent of \( r \). The projectors are orthogonal to each other since \( P_{j} P_{j} = 0 \) for \( i \neq j \) and defined by
\[
P_{k} = \frac{(\Delta f)^{k}}{|\Delta f|^2}, \quad k = 0, \ldots, 4
\]
where \( \Delta f = \partial_{z} f = \frac{\lambda_{1}(f^{z} f)}{f_{z}} \) (for details see [12]).

For spherical symmetric solutions the matter fields can be obtained from (12) by applying the orthogonalization procedure to the initial holomorphic vector
\[
f = (1, 2z, \sqrt{6} z^{2}, 2z^{3}, z^{4}).
\]

Using (12), the energy–momentum tensor \( T_{00}^0 \) can be evaluated explicitly:
\[
T_{00}^0 = \frac{4B}{r^2} \left[ c_{0}^{2} + \frac{3}{2} (c_{1}^{2} + c_{2}^{2}) + c_{3}^{2} \right] + \frac{4}{r^4} \left[ c_{0}^{2}b_{0}^{2} + \frac{3}{2} (c_{1}^{2}b_{1}^{2} + c_{2}^{2}b_{2}^{2}) + c_{3}^{2}b_{3}^{2} \right]
\]
\[
+ \frac{4B}{5} \left[ b_{0}^{2} + \frac{3}{2} (b_{1}^{2} + b_{2}^{2}) + b_{3}^{2} + b_{0}^{2} \left( b_{2} + \frac{3b_{1}}{2} + \frac{b_{1}^{2}}{2} \right) + b_{1}^{2} \left( b_{1}^{2} + \frac{3b_{2}}{2} \right) + 2b_{1}b_{2}^{2} \right] + \frac{1}{r^4} \left[ 8c_{1}^{4} + 18(c_{1}^{2} + c_{2}^{2}) + 8c_{3}^{4} - 6(c_{1}^{2} + c_{2}^{2}) - 4c_{3}^{2} \right]
\]
\[
- \frac{1}{r^4} \left[ 8c_{1}^{4} + 18(c_{1}^{2} + c_{2}^{2}) + 8c_{3}^{4} + 6(c_{1}^{2} + c_{2}^{2}) - 4c_{3}^{2} \right] - V(\Phi)
\]
where
\[
V(\Phi) = \frac{4\lambda_{1}}{5} \left[ b_{0}^{2} + \frac{3}{2} (b_{1}^{2} + b_{2}^{2}) + b_{3}^{2} + b_{0}^{2} \left( b_{2} + \frac{3b_{1}}{2} + \frac{b_{1}^{2}}{2} \right) + b_{1}^{2} \left( b_{1}^{2} + \frac{3b_{2}}{2} \right) + 2b_{1}b_{2}^{2} \right]
\]
\[
\begin{align*}
&+ \frac{16\lambda_2}{25} \left[ b_0^3 + \frac{3}{2} (b_1^2 + b_2^2) + b_3^3 + b_0 \left( b_2 + \frac{3b_1}{2} + \frac{b_3}{2} \right) + b_3 \left( b_1 + \frac{3b_2}{2} \right) \right] \\
&+ 2b_1b_2 \right]^2 + \frac{\lambda_3}{125} \left[ 52 \left[ b_0^3 \left( b_3 + 3b_1 + 2b_2 \right) + b_3^3 \left( b_0 + 3b_2 + 2b_1 \right) \right] \\
&+ 56 \left[ b_1^3 \left( b_3 + \frac{3b_0}{2} + 2b_2 \right) + b_2^3 \left( b_0 + \frac{3b_1}{2} + 2b_1 \right) \right] \\
&+ 42 \left( b_1^2 + b_2^2 \right) + 52 \left( b_0^2 + b_2^2 \right) + 108 \left[ b_0^2 b_2 \left( b_2 + b_3 \right) + b_1^2 b_1 \left( b_1 + b_0 \right) \right] \\
&+ 192 b_1 b_2 \left( b_2 + b_3 \right) \left( b_0 + b_1 \right) + 42 b_0^2 b_2^2 + 198 \left( b_1^2 b_2^2 + b_0^2 b_1^2 \right) \\
&+ 12 \left[ b_0 b_1 (11b_0 + 7b_1) (b_3 + 2b_2) + b_2 b_3 (11b_1 + 7b_2) (b_0 + 2b_1) \right] \right] - V_{\text{min}},
\end{align*}
\]

where we have set \( h_j = \sum_{k=j}^3 b_k \) and \( c_j = 1 - g_j - g_{j+1} \) for \( j = 0, \ldots, 3 \) and \( g_3 = 0 \). The properties of a given solution can be read off by computing the values of the Higgs field and magnetic charge at \( x = (0, 0, \infty) \) which corresponds to the direction \( z = 0 \):

\[
\Phi_0 = \frac{1}{2} \text{diag}(b_3 + 3b_1 + 2b_2 + 4b_0, b_3 + 3b_1 + 2b_2 - b_0, b_3 - b_0 - 2b_1 + 2b_2),
\]

\[
b_3 - 3b_2 - b_0, -4b_3 - 3b_2 - b_0)
\]

\[
G_0 = \text{diag} (4 (1 - c_0^2), 2 (1 + 2c_0^2 - 3c_1^2), 6 (c_1^2 - c_2^2), -2(1 + 2c_3^2 - 3c_2^2), -4 (1 - c_3^2))
\]

while the monopole masses are related to the Higgs profile functions since \( m_j = b_j \).

Variation of (9) imply that the equations of motions of the gauge profile functions are

\[
\begin{align*}
\frac{1}{A} (ABc'_0) &= b_0^2 c_0 + \frac{1}{r^2} c_0 \left( 4c_0^2 - 3c_1^2 - 1 \right), \\
\frac{1}{A} (ABC'_0) &= b_0^2 c_0 + \frac{1}{r^2} c_0 \left( 6c_1^2 - 2c_0^2 - 3c_2^2 - 1 \right)
\end{align*}
\]

while the Higgs fields profile functions satisfy

\[
\begin{align*}
\frac{2AAb'_0}{r^2} &= \frac{1}{r^2} \left( 4b_0 c_0^2 - 3b_1 c_1^2 \right) - \frac{1}{2} \lambda_1 b_0 - \frac{4\lambda_2}{5} b_0 \left[ b_0^2 + \frac{3}{2} \left( b_1^2 + b_2^2 \right) \right] \\
&+ b_0^2 + b_0 \left( b_2 + \frac{3b_1}{2} + \frac{b_3}{2} \right) + b_3 \left( b_1 + \frac{3b_2}{2} \right) + 2b_1 b_2 - \frac{\lambda_3}{25} b_0 \left[ 13b_0^2 + 27b_1^2 \right] \\
&+ 12b_1^2 + 18b_0 \left( b_2 - \frac{3b_1}{2} + \frac{b_3}{2} \right) + 18b_1 (b_3 + 2b_2) + 12b_2 b_3
\end{align*}
\]

\[
\begin{align*}
\frac{2AAb'_1}{r^2} &= \frac{1}{r^2} \left( 6b_1 c_1^2 - 3b_2 c_2^2 - 2b_0 c_0^2 \right) - \frac{1}{2} \lambda_1 b_1 \\
&- \frac{4\lambda_2}{5} b_1 \left[ b_0^2 + \frac{3}{2} \left( b_1^2 + b_2^2 \right) + b_3^2 + b_3 \left( b_1 + \frac{3b_2}{2} + \frac{b_0}{2} \right) \right] \\
&+ b_0 \left( b_2 + \frac{3b_1}{2} \right) + 2b_1 b_2 - \frac{\lambda_3}{25} b_1 \left[ 3b_1^2 + 12b_2^2 + 7b_3^2 + 3b_0^2 \right] \\
&+ 6b_1 \left( 2b_0 + b_1 \right) - 3b_0 (4b_2 + b_1) + 6b_1 b_2
\end{align*}
\]

The other equations can be obtained from the permutation \( 0 \leftrightarrow 3, 1 \leftrightarrow 2 \).
Finally, the Einstein equations (10) become

\[
\frac{2}{r^2} m' = 8\pi G T_0^0,
\]

\[
\frac{1}{r} \frac{A'}{A} = 8\pi G \left\{ \frac{4}{r^2} \left[ c_0^2 + \frac{3}{2} (c_1^2 + c_2^2) + c_3^2 \right] + 4 \left[ b_0^2 + \frac{3}{2} (b_1^2 + b_2^2) + b_3^2 + b'_0 \left( b'_2 + \frac{3 b'_1 b'_2 + b'_3 + 2 b'_1 b'_2}{2} \right) \right] \right\}
\]

(22)

where \( m(r) \) and \( T_0^0 \) are given by (7) and (15) and (16), respectively.

The above system of equations has to be solved with specific boundary conditions which ensure the regularity of the solutions and the finiteness of the ADM mass: \( M_{\text{ADM}} = m(\infty)/a^2 \).

(i) The Einstein equations impose boundary conditions for the metric functions: \( m(0) = 0 \) and \( A(\infty) = 1 \). The latter condition fixes the invariance of the equations under the arbitrary scale \( A(r) \to k A(r) \) (for \( k \) constant) and implies that spacetime is asymptotically flat.

(ii) The regularity of the matter fields at the origin requires that \( c_j(0) = 1 \) and \( b_j(0) = 0 \) and the finiteness of the ADM mass implies that \( b_j(\infty)c_j(\infty) = 0 \). However, the specific choice of the boundary conditions on \( b_j(\infty) \) and \( c_j(\infty) \) is determined by the type of the solution (e.g. maximal or minimal symmetry breaking) we are interested in.

In the absence of potential, the ‘length’ of the Higgs fields is not fixed since when \( \Phi \to \lambda \Phi \) and \( r \to r/\lambda \) the ADM mass scales accordingly i.e.

\[
M_{\text{ADM}}(\lambda, \Phi) = \lambda M_{\text{ADM}}(\Phi),
\]

(23)

which is true also in the flat limit, where the ADM mass is interpreted as the classical energy of the solution.

The projectors used in the harmonic map ansatz have a sigma model interpretation in terms of instanton and anti-instanton configurations as discussed in [12]. Thus, as shown in the following section, monopole and monopole–antimonopole configurations can be constructed for appropriate boundary conditions of the matter field profile functions.

In the flat limit (and in absence of Higgs potential), first-order equations (called the Bogomolny equations) implying the second-order equations can be obtained [2]. The solutions of these equations, called self-dual (SD) solutions, are characterized by the fact that the asymptotic values of the \( b_j \) are all of the same sign (positive for monopoles and negatives for antimonopoles). In contrast, solutions of the full second-order Yang–Mills–Higgs equations which do not obey the Bogomolny equations exist when the values of the \( b_j \) have opposite signs [2]; these solutions, which are named naturally non-self-dual (NSD), can be interpreted as monopole–antimonopole solutions.

Although there is (to our knowledge) no counterpart of the Bogomolny equations in the gravitating case, we will use in the following sections the terminology ‘self-dual’ (SD) and ‘non-self-dual’ to distinguish the gravitating deformations of the corresponding solutions in the flat case.

3. Numerical results

For definiteness, we take the Higgs potential to be zero in our numerical study and expect that the results do not change significantly if the parameters \( \lambda_j \) are small but nonzero (similar to the \( SU(2) \) case).
3.1. Maximal symmetry breaking

First, we discuss solutions with maximal symmetry breaking which occurs when all the eigenvalues of $Φ_0$ (or any permutation) are different. Since there are many possibilities (in fact, 120 possible permutations exist) we limit ourselves to few generic cases.

In the flat limit (i.e. $α = 0$), the simplest case corresponds to the self-dual solution where $Φ_0 = \text{diag}(2, 1, 0, -1, -2)$ with monopole masses equal to unity. Since $b_j(∞) = 1$ for $j = 0, \ldots, 3$ the gauge functions $c_j(r)$ have to vanish asymptotically and therefore $G_0 = \text{diag}(4, 2, 0, -2, -4)$ implying that $(n_1, n_2, n_3, n_4) = (4, 6, 6, 4)$. The corresponding mass is equal to $(M_{\text{ADM}})^{\text{SD}}_{\text{max}} = \sum_{j=1}^{4} b_j n_j = 20$ (as expected).

Another choice would be $b_0(∞) = b_3(∞) = 3, b_1(∞) = b_2(∞) = -2$, which corresponds to a non-self-dual solution. This time the eigenvalues of $Φ_0$ are not ordered while the entries of the magnetic charge must also be permuted and this results in $G_0 = \text{diag}(2, -4, 0, 4, -2)$. Therefore, the charges are $(n_1, n_2, n_3, n_4) = (2, -2, -2, 2)$. This solution cannot be constructed analytically since the corresponding equations are not integrable. However, it can be obtained numerically and its mass is evaluated to be equal to $(M_{\text{ADM}})^{\text{NSD}}_{\text{max}} = 27$. This suggests that we should think of the configuration as the composite of monopole, antimonopole and monopole–antimonopole pairs.

In order to appreciate qualitatively the differences between these solutions, we superposed their energy densities on figure 1 (curves $α = 0$). It can be observed that the energy density of the self-dual (line SD, MAX) solution has a maximum at $r = 0$ while the energy density of the non-self-dual (line NSD, MAX) one is more extended in space and presents an additional local maximum around $r = 2$ (indicating bound states of monopole–antimonopole pairs).
When these solutions are coupled to gravity (i.e. $\alpha \neq 0$), numerical simulations indicate that they deformed to gravitating configurations and their presence progressively deforms spacetime. In particular, figure 1 shows that the energy density $T^{0}_{0}(r)$ is higher and more concentrated in the region of the origin for gravitating solitons than the flat ones. In addition, figure 2 indicates that the metric functions $A(0)$ and $B_{m}$ decrease as $\alpha$ increases for both solutions. In particular, the function $B(r)$ develops a minimum $B_{m}$ at some intermediate value of $r = r_{h}$; while $A(r)$ takes its minimum value at the origin and tends to unity at infinity. Similarly, in figure 3 the $\alpha$ dependence of the product $\alpha M_{\text{ADM}}$ is plotted (using the same conventions for the various lines) and shows that the ADM mass and the product $\alpha M_{\text{ADM}}$ decreases and increases (respectively) as $\alpha$ increases. Due to the peculiar normalization of the Higgs field the energies are not directly comparable; therefore, the ratio $\alpha M_{\text{ADM}}/|\Phi_{0}|$ should be considered instead. Both branches stop at some maximal value of $\alpha$: (i) the self-dual solution can be deformed by gravity up to $\alpha_{m} \approx 0.63$ while (ii) the non-self-dual one exists up to $\alpha_{m} \approx 0.45$. As in the $SU(3)$ case [8], the interval of the parameter $\alpha$ where gravitating monopoles exist gets smaller when the mass of the flat solution gets larger. Moreover, the main branches of gravitating solutions are completed by secondary branches which exist on a rather small interval of $\alpha$, similar to $SU(2)$ [4] and $SU(3)$ [8] models. Indeed, the secondary branch exists in the intervals:

$$\begin{align*}
\text{SD, MAX:} & \quad \alpha_{cr} \approx 0.621, \quad \alpha_{m} \approx 0.627 \\
\text{NSD, MAX:} & \quad \alpha_{cr} \approx 0.424, \quad \alpha_{m} \approx 0.448.
\end{align*}$$

(24)

Note that, when $\alpha \to \alpha_{cr}$ the minimum of $B(r)$ tends to zero which means that (in this limit) the solution develops a horizon (see figure 2). In fact, $B_{m} \to 0$ faster than $A(0)$ in terms of
the critical value of $\alpha$ which means that the $SU(5)$ gravitating monopole bifurcates into an extremal Reissner–Nordstrom black hole. This configuration corresponds to solutions of the Abelian Einstein–Maxwell equations and can be embedded into the non-Abelian ones. Its mass is equal to

$$m_{RN} = m_{\infty, RN} - \frac{\alpha^2 Q^2}{2r}$$

where $Q$ is the charge of the black hole and can be read off from the energy–momentum tensor, i.e.

$$\frac{Q^2}{2} = \left[ 8(c_0^4 + c_1^4) + 18(c_0^2 + c_1^2) - 4(c_0^2 + c_1^2) - 6(c_1^2 + c_2^2) - 12(c_1^2 c_0^2 + c_1^2 c_2^2) - 18c_1^2 c_2^2 + 10 \right]_{r=\infty}. \tag{26}$$

Both our solutions have charge equal to $Q = \sqrt{20}$ in consistence with the numerical simulations (see figure 3).

3.2. Minimal symmetry-breaking solutions

In accordance with the maximal symmetry breaking, there are many minimal breaking patterns producing $SU(5)$ solutions with non-Abelian stability group. In what follows, we present two types of such solutions which are invariant under the $SU(3) \times SU(2)/U(1)$ group.

First, we investigate the self-dual solution in the flat space where $\Phi_0 = \text{diag}(3, 3, -2, -2, -2)$ i.e. for $b_k(\infty) = 0$ for $k = 0, 2, 3$ and $b_1(\infty) = 5$ and choose $c_j$ to satisfy the following asymptotic values: $c_0 = \frac{1}{2}, c_1 = 0, c_2 = \frac{1}{\sqrt{3}}$ and $c_3 = \frac{1}{\sqrt{2}}$. Note that the function $b_1$ approaches its asymptotic value as $|b_1 - 5| \sim O(1/r)$. The corresponding solution has energy equal to $E = 30$ with mass lower compared to all types of solutions we investigated—when
normalized appropriately. Moreover, when coupled with gravity it exists up to $\alpha_m \approx 0.38$ and (on the main branch) bifurcates into a Reissner–Nordstrom solution with charge $Q = \sqrt{15}$. This was confirmed by the numerical simulations which indicate the lack of a second branch. Figure 4 illustrates the way the matter functions approach their constant values outside the horizon at $r = \alpha_c$ of the Reissner–Nordstrom solution in the flat, $\alpha = 0.1$, and critical, $\alpha = 0.3797$, limit.

Finally, a non-self-dual solution with the same unbroken group can be constructed for $b_0(\infty) = b_2(\infty) = 1$, $b_1(\infty) = b_3(\infty) = -1$ and $c_1(\infty) = 0$. Then, $\Phi_0 = \text{diag}(2, -3, 2, -3, 2)$ and, in the flat limit, the mass of the configuration is equal to $(M_{\text{ADM}})_{\text{NSD}} = 48$. In this case, $b_0(r)$ increases monotonically while $b_1(r)$ presents a node at $r \approx 0.85$. Once more, the aforementioned solutions can be considered in the presence of gravitating fields and our numerical routines indicate that their gravitating analogues exist up to a maximal value of the coupling constant equal to $\alpha_m \approx 0.249$. In addition, a secondary branch exists which terminates at $\alpha \approx 0.229$ into an extremal Reissner–Nordstrom black hole of charge $Q = \sqrt{20}$. The corresponding results (line SD, MIN) and (line NSD, MIN) are presented in figures 2 and 3.

4. Conclusions

Four types of $SU(5)$ self-dual and non-self-dual monopoles have been constructed. One of our non-self-dual solutions with maximally symmetry breaking has a total magnetic charge equal to zero, and its energy density presents two maxima at $r = 0$ and at some finite value of $r$. This feature is, to our knowledge, not present for $SU(N)$ groups with $N < 5$.  

![Figure 4. The matter profile functions of the self-dual minimal symmetry-breaking solution for two different values of $\alpha$.](image.png)
The various solutions can be deformed by gravity forming branches of solutions labelled by the gravitational coupling constant $\alpha$. Numerical investigation of the $SU(5)$ Einstein–Yang–Mills–Higgs equations reveals (for three of the four cases studied) that for each branch a second one exists on the interval $\alpha \in [\alpha_c, \alpha_m]$, consistent with the results obtained in smaller gauge groups such as $SU(2)$ and $SU(3)$. In fact, the solution on the second branch has a higher mass than that with the same $\alpha$ on the main branch while in the limit $\alpha \to \alpha_c$, the minimum of the function $B(r)$ becomes deeper and deeper and approaches zero at some intermediate value $r_h$. Accordingly, the regular solution does not exist for $\alpha = \alpha_c$ and the metric fields approach that of an extremal Reissner–Nordstrom black hole on the interval $[r_h, \infty]$ and all matter fields tend to their asymptotic values. In the case of minimal symmetry breaking, the self-dual solution’s branch bifurcates directly into a Reissner–Nordstrom black hole.

Acknowledgments

We gratefully acknowledge Betti Hartmann for numerous interesting discussions which turned out to be at the basis of the present work. YB acknowledges the Belgian FNRS for financial support.

References

[1] Houghton C J, Manton N S and Sutcliffe P M 1998 Nucl. Phys. B 150 507
[2] Ioannidou T and Sutcliffe P M 1999 Phys. Rev. D 60 105009
[3] Dokos C P and Tomaras T N 1980 Phys. Rev. D 21 2940
[4] Breitenlohner P, Forgacs P and Maison D 1992 Nucl. Phys. B 383 357
  Breitenlohner P, Forgacs P and Maison D 1995 Nucl. Phys. B 442 126
[5] Vachaspati T 1996 Phys. Rev. Lett. 76 188
[6] Lepora N 2000 Monopoles and dyons in SU(5) gauge unification Preprint hep-th/0008322
[7] Pogosian L, Steer D A and Vachaspati T 2003 Phys. Rev. Lett. 90 061801
[8] Brihaye Y, Hartmann B, Ioannidou T and Zakrzewski W 2004 Class. Quantum. Grav. 21 517
[9] Kleihaus B and Kunz J 2000 Phys. Rev. Lett. 85 2430
[10] Brihaye Y and Hartmann B 2003 Phys. Rev. D 67 044001
[11] Goddard P, Nuyts J and Olive D 1977 Nucl. Phys. B 125 1
[12] Zakrzewski W J 1989 Low Dimensional Sigma Models (Bristol: Institute of Physics Publishing)