QUASI $f$-SIMPLICIAL COMPLEXES AND QUASI $f$-GRAPHS

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ABSTRACT. The notion of $f$-ideal is recent and has so far been studied in several papers. In [15], the idea of $f$-ideal is generalized to quasi $f$-ideals, which is much larger class than the class of $f$-ideals. In this paper, we introduce the concept of quasi $f$-simplicial complex and quasi $f$-graph. We give a characterization of quasi $f$-graphs on $n$ vertices. A complete solution of connectedness of quasi $f$-simplicial complexes is described. We have also shown a method of constructing Cohen-Macaulay quasi $f$-graphs.

Key words: $f$-ideal; $f$-graph; quasi $f$-ideal; facet ideal; Stanley-Reisner ideal

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1. INTRODUCTION

Commutative algebra supplies basic methods in the algebraic study of combinatorics on convex polytopes and simplicial complex. Richard Stanley was the first who used in a systematic way concepts and technique for commutative algebra to study simplicial complex by considering the Hilbert function of Stanley-Reisner rings, whose defining ideals are generated by square-free monomials. A square-free monomial ideal $I$ is an ideal of a polynomial ring $S = k[x_1, x_2, ..., x_n]$ in $n$ indeterminate over the field $K$ generated by the square-free monomials. Corresponding to every square-free monomial ideal $I$ of $S$, there are two natural simplicial complexes, namely, the facet complex of $I$, denoted by $\delta_F(I)$, and the non-face complex $\delta_N(I)$. The equality of $f$-vectors of these two complexes gives us $f$-ideals; whereas the quasi $f$-ideals shows the interconnections and relevance of the $f$-vectors of these two naturally associated complexes to $I$. The notion of $f$-ideals was introduced in 2012 in [1]. Later on, the idea of $f$-graphs was introduced in [12]. A simple finite graph $G$ on $n$ vertices is an $f$-graph if its edge ideal $I(G)$ is an $f$-ideal of degree 2. These notions have been studied for it various properties in the papers [1], [2], [9], [10], [11], [12], [13], [14], and [16]. In [15], the authors extended this concept to the notion of quasi $f$-ideal which is, in fact, a generalization of $f$-ideal. It turns out that every $f$-ideal is quasi $f$-ideal but not the converse. Moreover, the class of quasi

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f-ideals is much bigger class than the class of f-ideals. Various characterizations and construction, and the formula for computing Hilbert function and Hilbert series of the polynomial ring modulo quasi f-ideals of degree 2 can be found in [15].

This paper is set up in five sections. In second section, we recall some basic concepts, and introduce the term of quasi f-simplicial complex and quasi f-graph. The section focuses on primary study of f-graphs; Theorem 3.4 provides a characterization of quasi f-graphs. The fourth section of this paper is devoted to the connectedness of quasi f-simplicial complexes. We show that all the quasi f-simplicial complexes with dimension greater and equal to 2 are connected, Theorem 4.1. Moreover, Theorem 4.2 classify all connected quasi f-graphs. In the last section, we have given a construction of Cohen-Macaulay quasi f-graphs.

2. SOME FUNDAMENTAL CONCEPTS

Throughout this paper, the character k represents a field, R is a polynomial ring over k in n indeterminate \(x_1, x_2, \ldots, x_n\), and G will denote a finite simple graph on vertex set V with no isolated vertex. Let us recall some basic concepts to get familiar with simplicial complexes and square-free monomial ideals. Let V be a non-empty finite set and \(\Delta\) be a finite collection of subsets of V. Then \(\Delta\) is said to be a simplicial complex on V if

(i) \(\{v\} \in \Delta\) for all \(v \in V\) and,
(ii) For every subset \(E\) of \(F \in \Delta\) implies \(E \in \Delta\)

Here V we call the vertex set of the simplicial complex \(\Delta\). Each elements of \(\Delta\) is known as face and the maximal faces under \(\subseteq\) are known as facets. A subset \(F \subset V\) is said to be non-face of \(\Delta\) if \(F \notin \Delta\) and we denote by \(\mathcal{N}(\Delta)\), the set of minimal non-face of \(\Delta\). The dimension of a face \(F\) is defined as \(|F| - 1\), while the dimension of \(\Delta\) is the maximum of the dimensions of all faces of \(\Delta\). If \(F_1, F_2, \ldots, F_r\) are the facets of \(\Delta\), we write simplicial complex as

\[\Delta = \langle F_1, F_2, \ldots, F_r \rangle\]

to say that \(\Delta\) is generated by these \(F_i's\). A simplicial complex \(\Delta\) is said to be pure if all of its facets have the same dimension.

Remark 2.1. A finite simple graph is actually a 1-dimensional simplicial complex, it is usually denoted by G. We shall denote by \(E(G)\), is the set of all facets of 1-dimensional simplicial complex have dimension 1.

A vector \((f_0, f_1, \ldots, f_d) \in \mathbb{Z}^{d+1}\) is said to be an f-vector of a d-dimensional simplicial complex \(\Delta\) if and only if \(f_i\) is a number of i-dimensional faces of \(\Delta\). It is usually denoted by \(f(\Delta)\).

A simplicial complex \(\Delta\) over V is said to be connected if for any two facets \(F\) and \(F'\) of \(\Delta\), there exists a sequence of facets \(F = F_0, F_1, \ldots, F_r = F'\) such that \(F_i \cap F_{i+1} \neq \varnothing\), where \(0 \leq i \leq r - 1\). A simplicial complex is said to be disconnected if its not connected.

In the following definitions we recall the relationship between the algebraic and combinatorial structures due to R. P. Stanley (see [6]) and S. Faridi [5].
Definition 2.2. (facet ideal and non-face ideal) The facet ideal $I_F(\Delta) \subset R$ of a simplicial complex $\Delta = \langle F_1, F_2, ..., F_r \rangle$, is a square-free monomial ideal generated by the square-free monomials $m_1, m_2, ..., m_r$ such that $m_i = \prod_{v_j \in F_i} x_j$, where $i$ is coming from $\{1, 2, ..., r\}$. A square-free monomial ideal of $R$ of a simplicial complex $\Delta$, denoted by $I_\Delta$ called non-face ideal (or Stanley-Reisner ideal) if it is generated by the square-free monomials $x_F = \prod_{v_j \in F} x_j$ where $F \in \mathcal{N}(\Delta)$ i.e. $I_\Delta = (x_F : F \in \mathcal{N}(\Delta))$.

Definition 2.3. (facet complex and non-face complex) Let $R = k[x_1, x_2, ..., x_n]$ be a polynomial ring over the field $k$ and $I$ be a square-free monomial ideal of $R$. We use $G(I)$ to denote the unique set of minimal generators of $I$. The facet complex of $I$ is a simplicial complex

$$\delta_F(I) = \{\{v_{i_1}, v_{i_2}, ..., v_{i_r}\} \subseteq V \mid x_{i_1}x_{i_2}...x_{i_r} \in G(I)\}$$

and the non-face complex of $I$ is a simplicial complex

$$\delta_N(I) = \{\{v_{i_1}, v_{i_2}, ..., v_{i_r}\} \subseteq V \mid x_{i_1}x_{i_2}...x_{i_r} \notin I\}$$

Now we recall the definition of $f$-ideal.

Definition 2.4. A square-free monomial ideal $I$ of the polynomial ring $R = k[x_1, x_2, ..., x_n]$ is said to be an $f$-ideal if $f(\delta_F(I)) = f(\delta_N(I))$. A simplicial complex $\Delta$ on $n$ vertices is said to be an $f$-simplicial complex if the facet ideal of $\Delta$ is an $f$-ideal of $R$. A 1-dimensional $f$-simplicial complex is termed as $f$-graphs.

We refer the readers to [1], [2], [9], [10], [11], [12], [13], [14], and [16] to know more about $f$-ideals, $f$-graphs and $f$-simplicial complexes. The notion of $f$-ideals was generalized to quasi $f$-ideals in [15]. The study of quasi $f$-ideals is the study of interconnection between the $f$-vectors of the facet complex and the non-face complex of the ideal. The idea is to read off one vector through the other (see [15] for more details). It is defined as follows.

Definition 2.5. Let $(a_1, a_2, ..., a_s) \in \mathbb{Z}^s$. A square-free monomial ideal $I$ in the polynomial ring $R = k[x_1, x_2, ..., x_n]$ over the field $k$ is said to be quasi $f$-ideal of type $(a_1, a_2, ..., a_s)$ if and only if $f(\delta_N(I)) - f(\delta_F(I)) = (a_1, a_2, ..., a_s)$.

Example 2.6. Let $I = (x_1x_2x_4, x_1x_2x_5, x_3x_4x_5, x_1x_4x_5)$ be a pure square-free monomial ideal of degree 3 in the polynomial ring $R[x_1, x_2, x_3, x_4, x_5]$. Then the primary decomposition of $I$ is $I = (x_1, x_3) \cap (x_1, x_4) \cap (x_1, x_5) \cap (x_2, x_4) \cap (x_2, x_5) \cap (x_4, x_5)$. The facet and the non-face complexes of $I$ are

$$\delta_F(I) = \langle\{1, 2, 4\}, \{1, 2, 5\}, \{3, 4, 5\}, \{1, 4, 5\}\rangle$$

and

$$\delta_N(I) = \langle\{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\rangle.$$  
Then $f(\delta_F(I)) = (5, 8, 4)$ and $f(\delta_N(I)) = (5, 10, 6)$. Thus $I$ is a quasi $f$-ideal with type $(0, 2, 2)$.

Now we want to include a natural notion relative to quasi $f$-ideals. There are quasi $f$-simplicial complexes and quasi $f$-graphs. They are given below:
Definition 2.7. Let \((a_1, a_2, \ldots, a_s) \in \mathbb{Z}^s\); let \(\Delta\) be a simplicial complex on the vertex set \(V = \{v_1, v_2, \ldots, v_n\}\). We say that \(\Delta\) is quasi \(f\)-simplicial complex of type \((a_1, a_2, \ldots, a_s)\) if the facet ideal of \(\Delta\) is quasi \(f\)-ideal of type \((a_1, a_2, \ldots, a_s)\) in the ring \(R = k[x_1, x_2, \ldots, x_n]\). It is natural to call 1-dimensional quasi \(f\)-simplicial complex as quasi \(f\)-graph. Indeed, the type of quasi \(f\)-graph would be some ordered pair of integers.

In the following Figure 1 we give the complete list of all non-isomorphic quasi \(f\)-graphs on \(n \leq 6\) vertices with type indicated.

![Quasi f-graphs up to n ≤ 6](image)

Figure 1. Quasi \(f\)-graphs up to \(n \leq 6\)

Remark 2.8. It is important to mention that all quasi \(f\)-graphs will be of the type \((0, b)\), for instance, the type of quasi \(f\)-graph \(G\) on vertex set \(V\) will be ordered pair \((a, b) \in \mathbb{Z}^2\). However, since \(G\) is a simple graph with no isolated vertex it means that the edge ideal \(I(G)\) of \(R = k[x_1, x_2, \ldots, x_n]\) is pure square-free monomial quasi \(f\)-ideal of degree 2 with type \((a, b) \in \mathbb{Z}^2\) and also \(\text{supp}(G(I(G))) = \{x_1, x_2, \ldots, x_n\}\). Therefore, both the facet complex and the non-face complex of \(I(G)\) will have the
same vertex set, this means that \( a \) must be zero in the ordered pair \((a, b)\). Thus any quasi \( f \)-graph must be of the type \((0, b)\).

**Example 2.9.** Every \( f \)-simplicial complex (\( f \)-graph) is a quasi \( f \)-simplicial (quasi \( f \)-graph) with type \( \mathbf{0} \)-vector.

**Example 2.10.** In Example 2.6 the facet complex of \( I \) is a quasi \( f \)-simplicial complex of type \((0, 2, 2)\).

**Example 2.11.** The simplicial complex \( \Delta = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_4, v_5\}\} \) on \( V = \{v_1, v_2, v_3, v_4, v_5\} \) is a non-pure quasi \( f \)-simplicial complex of type \((0, 1, 0)\).

**Example 2.12.** A graph \( G = \langle\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_1, v_5\}\rangle \) on \( V = \{v_1, v_2, v_3, v_4, v_5\} \) is not a quasi \( f \)-graph.

### 3. Quasi \( f \)-Graphs and its Characterization

The purpose of the present section is to give a complete characterization of quasi \( f \)-graphs. First of all, we would like to recall [11, Definition 2.1] of perfect sets of \( R \). Let \( sm(R) \) denote the set of all square-free monomials in \( R \); let \( sm(R)_d \) be the set of all square-free monomials of degree \( d \) in \( sm(R) \). For a subset \( U \subseteq sm(R) \), we set

\[
\cup(U) = \{gx_i \mid g \in U, x_i \text{ does not divide } g, 1 \leq i \leq n\} \subset sm(R)_{d+1}
\]

and

\[
\cap(U) = \{h \mid h = g/x_i \text{ for some } g \in U \text{ and some } x_i \text{ with } x_i|g\} \subset sm(R)_{d-1}
\]

The set \( U \) is then called upper perfect if \( \cup(U) = sm(R)_{d+1} \), and it is said to be lower perfect if \( \cap(U) = sm(R)_{d-1} \). The set \( U \) is called a perfect set if and only if it is both lower and upper perfect. In general, perfect sets can have different cardinalities; for example, every subset of \( sm(R)_d \) containing a perfect set is again a perfect set. The smallest number among the cardinalities of perfect sets of degree \( d \) is called the \((n, d)^{th}\) perfect number, and is denoted by \( N(n, d) \). By [11, Lemma 3.3], for a positive \( t \) and \( n \geq 4 \), we have the following equations:

\[
N(n, 2) = \begin{cases} 
t^2 - t, & \text{when } n = 2t; \\
t^2, & \text{when } n = 2t + 1.
\end{cases}
\]

The following lemma plays an important role in the characterization of quasi \( f \)-graphs.

**Lemma 3.1.** Let \( G \) be a simple graph on the set of vertices \( \{v_1, v_2, \ldots, v_n\} \). Then the complementary graph \( \overline{G} \) of \( G \) is triangle-free if \( G \) is quasi \( f \)-graph of type \((0, b)\).

**Proof.** If \( G \) is a quasi \( f \)-graph, [Definition 3.1] implies \( I(G) \) is a quasi \( f \)-ideal. By using [15, Theorem 4.3] the minimal generating set \( G(I(G)) \) is upper perfect, this means that \( sm(S)_3 \subseteq I(G) \). Suppose that \( \overline{G} \) contains a triangle of edges \( \{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\} \text{ and } \{v_{i_3}, v_{i_1}\} \). Then it means that there exists a monomial \( x_{i_1}x_{i_2}x_{i_3} \notin \cup(G(I(G))) \) which is a contradiction. \( \square \)

It will be interesting to determine the bounds on the values of \( b \), which is given in the [Proposition 4.4] below. Here we recall a result from [15].
Proposition 3.2. Let $I$ be a quasi $f$-ideal of degree 2 and type $(0, b)$ in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$. Then the following holds true:

$$-\left(\frac{n}{2}\right) + 2 \leq b \leq \left(\frac{n}{2}\right) - 2N(n, 2)$$

Corollary 3.3. Let $G$ be a quasi $f$-graph of type $(0, b)$ on a vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Then the bounds of $b$ are follows:

$$-\left(\frac{n}{2}\right) + 2 \leq b \leq \left(\frac{n}{2}\right) - 2N(n, 2)$$

Proof. If $G$ is a quasi $f$-graph of type $(0, b)$ on a vertex set $V = \{v_1, v_2, \ldots, v_n\}$, then it means that $I(G)$ is a quasi $f$-ideal in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ and type $(0, b)$. Using Theorem 3.2 we have the desired inequality. □

Now we give a characterization of quasi $f$-Graphs below.

Theorem 3.4. Let $V = \{v_1, v_2, \ldots, v_n\}$, let $G$ be a simple graph on the vertex $V$ with no isolated vertices and $|E(G)| = \frac{1}{2}\binom{n}{2} - b$, where $|b| < \binom{n}{2}$. Then $G$ will be a quasi $f$-graph of type $(0, b)$ if and only if the complementary graph $\overline{G}$ of $G$ is triangle-free.

Proof. Suppose $G$ is a quasi $f$-graph with type $(0, b)$, then [Lemma 3.3] follows the desired result. For the converse of this theorem, suppose $\overline{G}$ is a triangle free graph. Therefore, $sm(S)_3 \subseteq I(G)$ and this implies $dim(\delta_N(I(G))) \leq 1$. By using the fact that $|b| < \binom{n}{2}$ yields that $dim(\delta_N(I(G))) = 1 = dim(\delta_F(I(G)))$. Since $G$ is a simple graph with no isolated vertices, $supp(I(G)) = \{x_1, x_2, \ldots, x_n\}$ and in view of [11, Remark 2.7] the both facet complex $(\delta_F(I(G)))$ and the non face complex $(\delta_N(I(G)))$ are on same number of vertices, which implies $f_0(\delta_N(I(G))) - f_0(\delta_F(I(G))) = 0$. Note that $(\delta_N(I(G))) = \overline{G}$. By [11, Lemma 3.2] we have $f_1(\delta_N(I(G))) = \binom{n}{2} - f_1(\delta_F(I(G)))$ and as given in above $f_1(\delta_F(I(G))) = |E(G)| = \frac{1}{2}\binom{n}{2} - b$ together implies $f_1(\delta_N(I(G))) - f_1(\delta_F(I(G))) = b$. The parity of $\binom{n}{2}$ is same as the parity of $b$ implies that $\binom{n}{2} \equiv 0 \mod 2$ (1 mod 2) if $b$ is even (odd). Hence $I(G)$ is a quasi $f$-ideal and using [Definition 3.1], $G$ is a quasi $f$-graph of type $(0, b)$ □

4. Connectedness of quasi $f$-simplicial complexes

We now concentrate on the problem of the connectedness of quasi $f$-simplicial complexes. In this section, we will classify connected and disconnected quasi $f$-simplicial complexes in terms of their dimensions.

Theorem 4.1. Let $V = \{v_1, v_2, \ldots, v_n\}$ be a vertex set and let $\Delta$ be a pure simplicial complex on $V$ of dimension $d$ with $d > 1$. If $\Delta$ is a quasi $f$-simplicial complex, then $\Delta$ will be connected.

Proof. Suppose $\Delta$ is disconnected quasi $f$-simplicial complex on a vertex set $V$. This means that there are two non-empty disjoint subsets say $V_1$ and $V_2$ of $V$ such that $V = V_1 \cup V_2$ with property that no any facet of $\Delta$ has vertices lie in both $V_1$ and
V_2. Therefore, We may choose a face \( F_1 \in P(V_1) \) and another face \( F_2 \in P(V_2) \) with 
\( \dim(F_1) = d - 1 \) and \( \dim(F_2) = 1 \) respectively. Then the square-free monomial 
\( x_{F_1 \cup F_2} \) of degree \( d + 2 \) does not belong to \( I_F(\Delta) \), which is contradiction to fact that 
\( G(I_F(\Delta)) \) is upper perfect. \( \square \)

The above theorem says that all quasi \( f \)-simplicial complexes of dimension greater or equal to 2 are connected. However, for the case of dimension 1, the situation is different. 1-dimensional quasi \( f \)-simplicial complexes may or may not be connected as shown in Figure 2. Now for any graph quasi \( f \)-graph \( G \), it is natural to ask the following questions:

(1) When is quasi \( f \)-graph connected?
(2) When is quasi \( f \)-graph disconnected?

In next part of this section, we have addressed these questions. However, we need to set some notations and terminologies. Let \( m \) and \( n \) be two positive integers. A graph \( G \) is said to be a \( [m : n] \)-graph if the complementary graph \( \overline{G} \) of \( G \) is a complete bipartite graph on \( m + n \) vertices. i.e. \( \overline{G} = K_{m,n} \). Note that \( [m : n] \)-graph \( G \) is a disconnected graph having two components \( K_m \) and \( K_n \), and we can write it as \( G = K_m \coprod K_n = K_n \coprod K_m \).

![Figure 2. Connected and Disconnected quasi \( f \)-graphs](image)

**Theorem 4.2.** A graph \( G \) will be a disconnected quasi \( f \)-graph of type \((0, b)\) if and only if \( G \) is \([m : n] \)-graph such that \((m - n)^2 = m + n - 2b\).

**Proof.** If \( G \) is a disconnected quasi \( f \)-graph of type \((0, b)\), then \( G \) would have connected components (say) \( G_1 \) and \( G_2 \). Let \( m \) and \( n \) be positive integers and let \( |V(G_1)| = m \) and \( |V(G_2)| = n \). Obviously, \( m, n > 1 \) since \( G \) is a simple graph with on isolated vertices. In order to prove \( G \) is a \([m : n] \)-graph, it is sufficient to show that \( G_1 = K_m \) and \( G_2 = K_n \). If \( G_1 \) is not a complete graph on \( m \) vertices, then this means \( |E(G_1)| < \binom{m}{2} \), which implies that there is at least one edge exists in the complementary graph of \( G_1 \) (say) \( e \) with vertices \( v_i \) and \( v_j \). In particular, \( e \in E(\overline{G}) \). If \( v \) is any vertex in \( G_2 \), then the edges \( \{v, v_i\}, \{v_i, v_j\} \) and \( \{v_j, v\} \) forms a cycle of length three must contained in \( E(\overline{G}) \), which contradict to [Lemma 3.1]. Therefore, \( G_1 = K_m \). Similarly, \( G_2 = K_n \). Next, we want prove that \((m - n)^2 = m + n - 2b\).
holds. As we have proved that \( G = K_{m,n} \), this means that \( |E(G)| = \binom{m}{2} + \binom{n}{2} \) and as the number of edges of \( G \) is \( \frac{1}{2}(\binom{m+n}{2} - b) \) since \( G \) is a quasi \( f \)-graph of type \((0,b)\) on \( m + n \) vertices, we have the following equation

\[
\frac{1}{2}\left(\binom{m+n}{2} - b\right) = \binom{m}{2} + \binom{n}{2}
\]

It is easy to verify that \( E(G) = \binom{m+n}{2} - E(G) = \frac{1}{2}(\binom{m+n}{2} + b) \). As \( G = K_{m,n} \), this means that \( E(G) = mn \) therefore, we have

\[
\frac{1}{2}\left(\binom{m+n}{2} + b\right) = mn
\]

\[
\Rightarrow \frac{1}{2}\left(\binom{m+n}{2} - b\right) = mn - b
\]

Equation (2) and equation (4), together implies

\[
\Rightarrow \binom{m}{2} + \binom{n}{2} = mn - b
\]

\[
\Rightarrow \frac{m(m-1)}{2} + \frac{n(n-1)}{2} = mn - b
\]

\[
\Rightarrow m^2 - m + n^2 - n = 2mn - 2b
\]

\[
\Rightarrow m^2 + n^2 - 2mn = m + n - 2b
\]

\[
\Rightarrow (m - n)^2 = m + n - 2b
\]

Conversely, suppose \( G \) is \([m : n]\)-graph on \( m + n \) vertices such that \((m - n)^2 = m + n - 2b\) holds. Obviously, \( G \) is disconnected since \( G \) is \([m : n]\)-graph. Now we need to prove that \( G \) is a quasi \( f \)-graph. Since \( G \) is \([m : n]\)-graph, so \( G = K_{m,n} \) this means \( G \) is a triangle-free graph, because a complete bipartite graph \( K_{m,n} \) contains no cycle of odd length. Next, we show that the parity of \( \binom{m+n}{2} \) is same as the parity of \( b \) and \( |E(G)| = \frac{1}{2}(\binom{m+n}{2} - b) \). It is noted that if \( G \) is \([m : n]\)-graph, then \( |E(G)| = \binom{m}{2} + \binom{n}{2} \) and \( |E(G)| = mn \). From relation \((m - n)^2 = m + n - 2b\), we have

\[
m^2 + n^2 - 2mn = m + n - 2b
\]

\[
\Rightarrow m^2 - m + n^2 - n = 2mn - 2b
\]

\[
\Rightarrow \frac{m(m-1)}{2} + \frac{n(n-1)}{2} = mn - b
\]

\[
\Rightarrow \binom{m}{2} + \binom{n}{2} = mn - b
\]

This means that \( E(G) = mn - b \). As we know \( E(G) + E(G) = \binom{m+n}{2} \), so we have the following equation

\[
2mn - b = \binom{m+n}{2}
\]
\[ 2mn - 2b = \left( \frac{m + n}{2} \right) - b \]
\[ \Rightarrow E(G) = mn - b = \frac{1}{2} \left( \frac{m + n}{2} \right) - b \]

The equation (5) shows that the parity of \( \left( \frac{m + n}{2} \right) \) is same as the parity of \( b \).

**Corollary 4.3.** A quasi \( f \)-graph \( G \) on a vertex set \( V \) of type \( (0, b) \) is a connected if \( G \) is not \([m : n] \)-graph

*Proof.* If a quasi \( f \)-graph \( G \) is not \([m : n] \)-graph, then obviously it is connected. \( \square \)

**Corollary 4.4.** Let \( n \) and \( r \) be two positive integers and let \( 1 < r < n \). Then for \( n \geq 4 \), \([n : n-r] \)-graph \( G \) is disconnected quasi \( f \)-graph of type \((0, \frac{1}{2}(2n-r-r^2))\)

*Proof.* We need to show that \([n : n-r] \)-graph \( G \) is disconnected quasi \( f \)-graph of type \((0, \frac{1}{2}(2n-r-r^2))\). Let \( b = \frac{1}{2}(2n-r-r^2) \) and let \( m = n-r \). Using [Theorem 4.2] it is sufficient to show that the relation \((n-m)^2 = m + n - 2b \) holds. Let’s see \( m+n-2b = n-r+n-2\frac{1}{2}(2n-r-r^2) = 2n-r-2n+r^2 = r^2 = (n-m)^2 \). \( \square \)

**Figure 3.** Disconnected quasi \( f \)-graphs on 9 and 10 vertices
5. Construction of Cohen-Macaulay quasi $f$-simplicial complexes

In this section, we will give a construction of quasi $f$-graphs which are Cohen-Macaulay. Let us first recall the definition of Cohen-Macaulay Graphs.

**Definition 5.1.** The ring $R$ is called Cohen-Macaulay if its depth is equal to its dimension.

**Definition 5.2.** A graph $G$ on the vertex set $V = \{x_1, x_2, \ldots, x_n\}$ is said to be Cohen-Macaulay over the field $k$ if $k[x_1, x_2, \ldots, x_n]/I(G)$ is a Cohen-Macaulay ring.

**Theorem 5.3.** Let $b$ is an integer such that $|b| < \left\lfloor \frac{n}{2} \right\rfloor$ and $G$ be a graph on $n$ vertices which is constructed by following cases:

1. If $n = 4k, b = 2b'$, $G$ consists of two components $G_1$ and $G_2$ joined with $k - b'$ edges, where both $G_1$ and $G_2$ are complete graphs on $2k$ vertices.
2. If $n = 4k + 1, b = 2b'$, $G$ consists of two components $G_1$ and $G_2$ joined with $k - b'$ edges, where $G_1$ and $G_2$ are complete graphs on $2k + 1$ vertices and $2k$ vertices, respectively.
3. If $n = 4k + 2, b = 2b' + 1$, $G$ consists of two components $G_1$ and $G_2$ joined with $k - b'$ edges, where both $G_1$ and $G_2$ are complete graphs on $2k + 1$ vertices.
4. If $n = 4k + 3, b = 2b' + 1$, $G$ consists of two components $G_1$ and $G_2$ joined with $k - b'$ edges, where $G_1$ and $G_2$ are complete graphs on $2k + 2$ vertices and $2k + 1$ vertices, respectively.

Then $G$ is a Cohen-Macaulay quasi $f$-graph of type $(0, b)$.

**Proof.** The condition $|b| < \left\lfloor \frac{n}{2} \right\rfloor$ ensures that in each case, $b' < k$, so there are always a positive number of edges joining $G_1$ and $G_2$.

First, we check that the number of edges of $G$ as constructed is $\frac{1}{2}\binom{n}{2} - b$. In fact, for the case (1), the number of edges of $G$ is

$$2 \left( \binom{2k}{2} \right) + k - b' = 4k^2 - k - b' = \frac{1}{2} \left( \binom{4k}{2} - b \right).$$

Similarly, we can check number of edges to be $\frac{1}{2}\left( \binom{n}{2} - b \right)$ for the cases (2), (3) and (4). Thus, it is easy to see from the above construction that $G$ is a quasi $f$-graph of type $(0, b)$ - since the complement of $G$ is a bipartite graph, which does not contain any triangle and it has $\frac{1}{2}\left( \binom{n}{2} - b \right)$ edges.

Let us recall from [17] that if $G$ is a graph on $n$ vertices such that $ht(I(G)) = n - 2$ then $G$ is Cohen-Macaulay if and only if $\delta_N(I(G))$ is connected. Thus, it suffices to show that the complement of $G$, which is $\delta_N(I(G))$ (since it has no triangle), is connected. In fact, the main idea is the following: since the number of edges joining $G_1$ and $G_2$ is small compared to the maximal number of possible edges between $G_1$ and $G_2$, so when we take the complement, the number of edges matching the vertex sets of $G_1$ and $G_2$ is large enough to make it connected. We will give the calculation case by case and we will see further that we can take $|b| \leq \left\lfloor \frac{n}{2} \right\rfloor$ in the assumption.
of the theorem with some special exceptions (see remark below).

Let $G$ be the graph constructed above. The number of edges of the $G$ is $\frac{1}{2}\binom{n}{2} + b$. Suppose that $G$ is not connected, i.e., there exists the sets $V_1$ with $x$ vertices from $G_1$ and $V_2$ with $y$ vertices from $G_2$ such that all edges of $G$ are edges joining vertices from $V_1$ to $V_2$ and vertices from $V(G_1) - V_1$ to $V(G_2) - V_2$.

**Case 1:** The number of edges of $G$ is at most $xy + (2k - x)(2k - y)$. Without loss of generality, assume that $x \leq k$,

1. If $x = 0$, then $y \geq 1$. This means that the number of edges of $G$ is at most $2k(2k - y)$. Since $-2ky \leq -2k < -k + \frac{b}{2}$, we have
   
   $$2k(2k - y) = 4k^2 - 2ky < 4k^2 - k + \frac{b}{2} = \frac{1}{2}\binom{n}{2} + b$$
   
   which is a contradiction. Note that if $|b| = \lfloor \frac{n}{2} \rfloor$ then the inequality above becomes equality if and only if $x = 0, y = 1$ and $k = -\frac{b}{2}$.

2. If $x \geq 1$, then since $2xy \leq 2ky$ and $-2kx < -k + \frac{b}{2}$ it holds that
   
   $$xy + (2k - x)(2k - y) = 4k^2 + 2xy - 2ky - 2kx < 4k^2 - k + \frac{b}{2} = \frac{1}{2}\binom{n}{2} + b$$
   
   which is a contradiction. Note that if $|b| = \lfloor \frac{n}{2} \rfloor$, then the inequality above becomes equality if and only if $x = y = k = 1$ and $b = -2$ or $x = 1, y = 0$ and $k = -\frac{b}{2}$.

**Case 2:** The number of edges of $G$ is at most $xy + (2k + 1 - x)(2k - y)$. Without loss of generality, assume that $x \leq k$,

1. If $y = 0$, then $x \geq 1$. This means that the number of edges of $G$ is at most $2k(2k + 1 - x)$. Since $-2kx < -k + \frac{b}{2}$, we have
   
   $$2k(2k + 1 - x) = 4k^2 + 2k - 2kx < 4k^2 + k + \frac{b}{2} = \frac{1}{2}\binom{n}{2} + b$$
   
   which is a contradiction. Note that if $|b| = \lfloor \frac{n}{2} \rfloor$ then the inequality above becomes equality if and only if $y = 0, x = 1$ and $k = -\frac{b}{2}$.

2. If $y \geq 1$, then since $2xy \leq 2ky$, $-2kx - y < -k + \frac{b}{2}$ (even when $|b| = \lfloor \frac{n}{2} \rfloor$) we have
   
   $$xy + (2k + 1 - x)(2k - y) < 4k^2 + 2k + \frac{b}{2} = \frac{1}{2}\binom{n}{2} + b$$
   
   which is a contradiction.

**Case 3:** The number of edges of $G$ is at most $xy + (2k + 1 - x)(2k + 1 - y)$. Without loss of generality, assume that $x \leq k$,

1. If $x = 0$ then $y \geq 1$. Since $-2ky \leq -2k < -k + \frac{1}{2} + \frac{b}{2}$, we have
   
   $$(2k + 1)(2k + 1 - y) < 4k^2 + 3k + \frac{1}{2} + \frac{b}{2} = \frac{1}{2}\binom{n}{2} + b$$

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which is a contradiction. Also, if \( |b| = \left\lfloor \frac{n}{2} \right\rfloor \) then the inequality above becomes equality if and only if \( x = 0, y = 1 \) and \( b = -(2k + 1) \).

(2) If \( x \geq 1 \), then since \( 2xy - 2ky \leq 0 \), \( 1 - x - y \leq 0 \) and \( -2kx < -k + \frac{1}{2} + \frac{b}{2} \) we have

\[
x y + (2k + 1 - x)(2k + 1 - y) < 4k^2 + 3k + \frac{1}{2} + \frac{b}{2} = \frac{1}{2} \binom{n}{2} + b)
\]

(contradiction). Also, if \( |b| = \left\lfloor \frac{n}{2} \right\rfloor \), then the inequality above becomes equality if and only if \( x = 1, y = 0 \) and \( b = -(2k + 1) \).

**Case 4:** The number of edges of \( G \) is at most \( xy + (2k + 2 - x)(2k + 1 - y) \).

Without loss of generality, assume that \( x \leq k + 1 \),

(1) If \( y = 0 \), then \( x \geq 1 \). Since \( -2kx < -k + \frac{1}{2} + \frac{b}{2} \), we have

\[
(2k + 2 - x)(2k + 1) < 4k^2 + 3k + \frac{3}{2} + \frac{b}{2} = \frac{1}{2} \binom{n}{2} + b
\]

(contradiction). Note that if \( |b| = \left\lfloor \frac{n}{2} \right\rfloor \) then the inequality above becomes equality if and only if \( y = 0, x = 1 \) and \( b = -(2k + 1) \).

(2) If \( y \geq 1 \), then since \( 2xy - (2k + 2)y \leq 0 \) and \( -2kx < -k + \frac{1}{2} + \frac{b}{2} \) we have

\[
(2k + 2 - x)(2k + 1 - y) < 4k^2 + 3k + \frac{3}{2} + \frac{b}{2} = \frac{1}{2} \binom{n}{2} + b
\]

which is a contradiction.

**Remark 5.4.** As in the proof above, if the assumption of the theorem was \( |b| \leq \left\lfloor \frac{n}{2} \right\rfloor \) then the construction still gives us Cohen-Macaulay graphs except the following cases when \( |b| = \left\lfloor \frac{n}{2} \right\rfloor \):

(1) The graph \( C_4 \) is of type \((0, -2)\). (When \( x = y = k = 1 \) and \( b = -2 \))

(2) The graph \( K_{2k} \biguplus K_{2k} \) or \( K_{2k+1} \biguplus K_{2k} \) with \( 2k \) edges joining 1 vertex from the first component to all vertices of the second component. These are of type \((0, -2k)\).

(3) The graph \( K_{2k+1} \biguplus K_{2k+1} \) or \( K_{2k+2} \biguplus K_{2k+1} \) with \( 2k+1 \) edges joining 1 vertex from the first component to all vertices of the second component. These are of type \((0, -2k - 1)\).

**Example 5.5.** At extreme case when \( |b| = \left\lfloor \frac{n}{2} \right\rfloor \), the graph \( K_{2} \biguplus K_{2} \) is Cohen-Macaulay quasi \( f \)-graph of type \((0, 2)\) whereas the graph \( C_4 \) is non Cohen-Macaulay of type \((0, -2)\).

**Example 5.6.** (a) Take \( n = 7 \) with \( k = 1 \). Let us take \( b = 1 \) this means that \( b' = 0 \). The quasi \( f \)-graph will be obtained by joining the graphs \( G_1 = K_3 \) with \( G_2 = K_4 \) by 1 edge as shown in figure.

(b) Take \( n = 7 \) with \( k = 1 \). Let us take \( b = -1 \) this means that \( b' = -1 \). The quasi \( f \)-graph will be obtained by joining the graphs \( G_1 = K_3 \) with \( G_2 = K_4 \) by 2 edge as shown in figure.

(c) Take \( n = 8 \) with \( k = 2 \). Let us take \( b = 2 \) this means that \( b' = 1 \). The quasi \( f \)-graph will be obtained by joining the graphs \( G_1 = K_4 \) with \( G_2 = K_4 \) by 1 edge as
shown in figure.

(d) Take \( n = 8 \) with \( k = 2 \). Let us take \( b = -2 \) this means that \( b' = -1 \). The quasi \( f \)-graph will be obtained by joining the graphs \( G_1 = K_4 \) with \( G_2 = K_4 \) by 3 edge as shown in figure.

\begin{align*}
&\text{(0,1)} &\text{(0,-1)} &\text{(0,2)} &\text{(0,-2)}
\end{align*}

Figure 4. Cohen-Macaulay Quasi \( f \)-graphs

Unlike the fact that all \( f \)-graphs are Cohen-Macaulay, we have lot of examples of non-Cohen-Macaulay \( f \)-graphs. Some simple examples are described in remark above. In particular, among 5 graphs in 4 vertices of quasi \( f \)-graphs as in figure 1, two of them are Cohen-Macaulay and three of them are not. Also, there are a lot more Cohen-Macaulay quasi \( f \)-graphs even in small case that is not constructed by argument above, for example \( K_4 \prod K_2 \) (of type \((0, 1)\)). It would be interesting to characterize all Cohen-Macaulay quasi \( f \)-graphs in particular types.

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