On irregular links at infinity of algebraic plane curves

Walter D. Neumann and Le Van Thanh

Abstract. We give two proofs of a conjecture of Neumann [N1] that a reduced algebraic plane curve is regular at infinity if and only if its link at infinity is a regular toral link. This conjecture has also been proved by Ha H. V. [H] using Lojasiewicz numbers at infinity. Our first proof uses the polar invariant and the second proof uses linear systems of plane curve singularities. The second approach also proves a stronger conjecture of [N1] describing topologically the regular link at infinity associated with an irregular link at infinity.

1. Introduction

Let $P: \mathbb{C}^2 \rightarrow \mathbb{C}$ be a polynomial such that $V = P^{-1}(0)$ is a reduced curve. The intersection of $V$ with a sphere of sufficiently large radius centered at the origin is transverse and gives a link $\mathcal{L}(V, \infty) := (S^3, V \cap S^3)$, called the link at infinity of $V$. It is independent of the radius of $S^3$ (up to isotopy) and is a toral link. We shall abbreviate it $\mathcal{L} = \mathcal{L}(V, \infty) = (S^3, L)$.

One says that $V$ is regular at infinity if the defining polynomial $P$ gives a trivial fibration in a neighborhood of $V$ at infinity. Only finitely many fibers of $V$ are irregular at infinity.

We recall the construction in [N1] of the splice diagram $\Omega = \Omega(\mathcal{L})$ for the link at infinity. (See [N1] or [EN] for basics on splice diagrams)

By compactifying we can write $V = \overline{V} - (\overline{V} \cap \mathbb{P}^1) \subset \mathbb{P}^2 - \mathbb{P}^1 = \mathbb{C}^2$. $\overline{V}$ meets $\mathbb{P}^1$ in finitely many points $Y_1, \ldots, Y_n$, say. Let $D_0$ be a 2-disk in $\mathbb{P}^1$ which contains $\overline{V} \cap \mathbb{P}^1$ and $D$ a 4-disk neighborhood of $D_0$ in $\mathbb{P}^2$ whose boundary $S = \partial D$ meets $\overline{V} \cup \mathbb{P}^1$ transversally (see Fig. 1).

Key words: Link at infinity, regular at infinity, polar invariant, splice diagram

The authors thank the Max-Planck-Institut für Mathematik in Bonn, where this work was done, for its support. The first author also acknowledges partial support by the NSF.
Then $\mathcal{L}_0 = (S, (\mathbb{P}^1 \cup \mathbb{V}) \cap S)$ is a link which can be represented by a splice diagram $\Gamma$ as follows:

Here $K_0$ is the component $\mathbb{P}^1 \cap S$ and each $\leftarrow \Gamma_i$ is the diagram representing the local link of $\mathbb{P}^1 \cup \mathbb{V}$ at the point $Y_i$.

Let $N\mathbb{P}^1$ be a closed tubular neighborhood of $\mathbb{P}^1$ in $\mathbb{P}^2$ whose boundary $S^3 = \partial N\mathbb{P}^1$ is the sphere at infinity and $\mathcal{L}' = (S^3, S^3 \cap V)$ is, but for orientation, the link at infinity that interests us. We may assume that $N\mathbb{P}^1$ is obtained from $D$ by attaching a 2-handle along $K_0$, so $\mathcal{L}'$ is obtained from $\mathcal{L}_0$ by (+1)-Dehn surgery on $K_0 \subset S = \partial D$.

As in [N1], we call a weight in the splice diagram near or far according as it is on the near or far end of its edge, viewed from $K_0$. As described in [N1], the splice diagram for $\mathcal{L}'$ is

where $\Gamma'_i$ is obtained from $\Gamma_i$ by replacing each far weight $\beta_v$ by $\beta_v - \lambda_v^2 \alpha_v$ with $\alpha_v$ equal to the product of the near weights at vertex $v$ and $\lambda_v$ the product of the weights adjacent to, but not on, the simple path from $v$ to the vertex corresponding to $K_0$.

Finally, we must reverse orientation to consider $S^3$ as a large sphere in $\mathbb{C}^2$ rather than as $\partial N\mathbb{P}^1$. The effect is to reverse the signs of all near weights. We can then forget the leftmost vertex, which is redundant, to get a diagram $\Omega$ for $\mathcal{L} = \mathcal{L}(V, \infty)$ as follows.

The leftmost vertex of $\Omega$ is called the root vertex. $\Omega$ is an RPI splice diagram in the sense of [N1]. “RPI” stands for “reverse Puiseux inequalities” and refers to certain inequalities that the weights of $\Omega$ satisfy.

For each vertex $v$ of $\Omega$ we denote by $\delta_v$ its valency (number of incident edges). We have three types of vertices: arrowheads (corresponding to components of $\mathcal{L}$), leaves (non-arrowheads of valency 1) and nodes (non-arrowheads of valency $> 2$). For each non-arrowhead, denote by $S_v$ a corresponding virtual component for $\mathcal{L}$ (see [N1]). In particular, $S_o = K_0$, where $o$ denotes the root vertex. The linking number $b_v := l(S_v, K_0)$ is called the braid index of $v$ and $l_v := l(S_v, L)$ (sum of linking numbers of $S_v$ with all components
of $\mathcal{L} = (S^3, L)$ is the \textit{(total) linking coefficient} at $v$ (called “multiplicity at $v$ in [EN]). The linking coefficient $l_v$ at the root vertex is the degree $d$ of the defining polynomial.

\textbf{Definition 1.1}. $\Omega$ is a regular RPI splice diagram if $l_v \geq 0$ for all non-arrowhead vertices.

As described in [N1], the diagram $\Omega$ depends on the particular compactification $\mathbb{C}^2 \subset \mathbb{P}^2$ of $\mathbb{C}^2$ that we choose, but if one RPI splice diagram for $\mathcal{L}$ is regular then every RPI splice diagram for $\mathcal{L}$ is. Thus the regularity or irregularity of $\Omega$ is a topological property of $\mathcal{L}$, and we say the toral link $\mathcal{L}$ is regular or irregular accordingly.

In [N1] it was shown that $\mathcal{L}(V, \infty)$ is a regular link if $V$ is regular at infinity, and the converse was conjectured. This conjecture has been proved by Ha H. V. in [H]. One purpose of this note is to give a new short proof of this using completely different methods, namely the polar invariant of $V$. We shall prove

\textbf{Theorem 1.2.} The following conditions are equivalent:

(1) $V = P^{-1}(0)$ is regular at infinity.

(2) $\mathcal{L}(V, \infty)$ is a regular toral link.

(3) For every component $\gamma$ of the polar curve for $V$, the natural valuation at infinity on holomorphic functions on $\gamma$ satisfies $v_\infty^\gamma(P|\gamma) \geq 0$.

(4) There exist $R > 0$ and $\delta > 0$ and a line field $\alpha \partial/\partial x + \beta \partial/\partial y$ which is transverse to $P^{-1}(t) \cap (\mathbb{C}^2 - B_R)$ for all $t < \delta$, where $B_R = \{(x, y) \in \mathbb{C}^2 : |(x, y)| < R\}$.

Using different methods we shall also describe the proof of a stronger conjecture from [N1], which computes the link at infinity of any regular fiber of the defining polynomial $P$ in terms of any RPI splice diagram $\Omega$ for an irregular link at infinity of $P$. Namely, let $\Omega^-$ be the full subgraph of $\Omega$ on all vertices $v$ with $l_v < 0$ and arrowheads adjacent to them, and let $\Omega^-_1, \ldots, \Omega^-_k$ be the connected components of $\Omega^-$. For each $j = 1, \ldots, k$, $\Omega^-_j$ is connected to the rest of $\Omega$ by a single edge. It is easy to see that there is a unique way of cutting at this edge and replacing $\Omega^-_j$ by a graph of the form

\begin{align*}
\text{(i) } l_w &= 0, \text{ where } w \text{ is the new node;} \\
\text{(ii) } l_v &= \text{unchanged for every non-arrowhead vertex of } \Omega - \Omega^-_j.
\end{align*}

Let $\Omega_0$ be the result of doing this for each $j = 1, \ldots, k$.

\textbf{Theorem 1.3.} $\Omega_0$ is an RPI splice diagram for the link at infinity of any regular fiber of $P$.

\textbf{2. Proof of Theorem 1.2}

That (1)$\Rightarrow$(2) was proved in [N1]. That (4)$\Rightarrow$(1) is trivial: if (4) holds then the vector field $(\alpha \partial/\partial x + \beta \partial/\partial y)/(\alpha \partial P/\partial x + \beta \partial P/\partial y)$ trivializes the neighborhood at infinity $P^{-1}(\Delta_\delta) \cap (\mathbb{C}^2 - B_R)$ of $V = P^{-1}(0)$, where $\Delta_\delta = \{t \in \mathbb{C} : |t| < \delta\}$ and $B_R = \{(x, y) \in \mathbb{C}^2 : |(x, y)| < R\}$.
Suppose (3) holds, that is, for all components $\gamma$ of the polar curve
\[ \Pi = \{(x, y) : \alpha \frac{\partial P}{\partial x} + \beta \frac{\partial P}{\partial y} = 0\} \quad \alpha \text{ and } \beta \text{ sufficiently general}, \]
one has $v^\infty_\gamma(P|\gamma) \geq 0$. This means that for all $\gamma \subset \Pi$, $P(x, y)|\gamma \not\to 0$ as $|(x, y)| \to \infty$, so there exists $\delta > 0$ such that for sufficiently large $R$ one has:
\[ P^{-1}(\Delta_\delta) \cap (\mathbb{C}^2 - B_R) \cap \Pi = \emptyset, \]
This is statement (4) of Theorem 1.2.

It remains to prove that (2) $\Rightarrow$ (3). Recall that $\Gamma_i$ is the splice diagram for the local link $L_i = (S^3(Y_i), S^3(Y_i) \cap \bar{V})$ of $V \subset \mathbb{P}^2$ at the point $Y_i$. If $v$ is a non-arrowhead vertex of $\Gamma_i$, let $S_v$ be the corresponding virtual component of $L_i$. Write $L_i = (S^3, L_i)$, and denote the “local braid index” and “local linking coefficient” at $v$ by
\[ b^\text{loc}_v := l_{\mathcal{L}_i}(S_v, K_0), \]
\[ l^\text{loc}_v := l_{\mathcal{L}_i}(S_v, L_i). \]
(Note that this local braid index is the braid index with respect to the special line $z = 0$ rather than with respect to a general line, which is how the term is usually used.) Since we can consider $v$ as a vertex of $\Omega$, the (global) braid index $b_v$ and linking coefficient $l_v$ are also defined.

**Lemma 2.1.**
(i) $b_v = b^\text{loc}_v$
(ii) $l_v = b_v d - l^\text{loc}_v$.

**Proof.** Recall (e.g., Lemma 3.2 of [N1]) that the linking number of any two components (virtual or genuine) of a toral link is the product of all weights adjacent to but not on the simple path joining the corresponding vertices of a splice diagram for the link. Given the relationship between the weights of $\Gamma$ and $\Omega$ described in Section 1, (i) is then immediate and (ii) is a simple calculation.

Both (i) and (ii) are also easy to see topologically. For (ii) one uses the fact that $(1/d)$-Dehn surgery on a knot $K$ in $S^3$ replaces the linking number $l(C_1, C_2)$ of any two disjoint 1-cycles which are disjoint from $K$ by $l(C_1, C_2) - dl(C_1, K)l(C_2, K)$. Applying this with $C_1 = S_v, C_2 = L$, we see, in the notation of Section 1, $l_v = l_{\mathcal{L}_i}(S_v, L) = l_{\mathcal{L}_i}(S_v, K_0) = -l_v - l_{\mathcal{L}_i}(S_v, L) = -(l^\text{loc}_v - b_v d)$, since $l_{\mathcal{L}_0}(S_v, L) = l_{\mathcal{L}_0}(S_v, L_i) = l_{\mathcal{L}_i}(S_v, L_i) = l^\text{loc}_v$.

Let us assume part (3) of Theorem 1.2 is not true. We may assume coordinates are chosen such that the polar curve is given as
\[ \Pi = \{(x, y) \in \mathbb{C}^2 : \frac{\partial P(x, y)}{\partial y} = 0\} \]
and, in addition, the component $\gamma \subset \Pi$ contravening (3) has Puiseux expansion at infinity
\[ \gamma = \{(x, y) : y = y_\gamma(x)\} \quad \text{with} \quad y_\gamma(x) = \sum_{\alpha \in 1+\mathbb{Q}^-} a_\alpha x^\alpha, \]
where $\mathbb{Q}^- = \{ \alpha \in \mathbb{Q} : \alpha < 0 \}$ and $P_y(x, y_\gamma(x)) \equiv 0$. In particular, the point at infinity in question is $y = 0$.

One has
\[ v_\gamma^\infty(P|\gamma) = v_x^\infty(P(x, y_\gamma(x))). \]

On the other hand, if $\tilde{P}(z, y) = z^d P(\frac{1}{z}, \frac{y}{z})$ is the equation for $P^{-1}(0)$ in local coordinates at infinity, then $\tilde{P}_y(z, zy_\gamma(\frac{1}{z})) \equiv 0$, which means that
\[ \tilde{\gamma} := \{(z, y) : y = zy_\gamma(\frac{1}{z}) \} = y_\tilde{\gamma}(z) \]
is the compactification of $\gamma$. One has $v_z(\tilde{P}(z, y_\tilde{\gamma}(z))) = d - v_x^\infty(P(x, y_\gamma(x)))$, so, by assumption,
\[ v_z(\tilde{P}(z, y_\tilde{\gamma}(z))) > d. \]

But, in the local situation, one knows that
\[ v_z(\tilde{P}(z, y_\tilde{\gamma}(z))) = \frac{\tilde{\gamma} \cdot \tilde{P}^{-1}(0)}{\tilde{\gamma} \cdot \{z = 0\}}. \]

If $\{z = 0\}$ is a generic line, then the main result of [LMW] can be expressed that for some vertex $v$ of the splice diagram $\Gamma$ (which is the diagram for the link at 0 of $z\tilde{P}(z, y) = 0$, rather than $\tilde{P}(z, y) = 0$ as in [LMW]),
\[ \frac{\tilde{\gamma} \cdot \tilde{P}^{-1}(0)}{\tilde{\gamma} \cdot \{z = 0\}} = \frac{t_v^{loc}}{b_v^{loc}}. \]

Expressed in this form, the proof in [LMW] applies also if $\{z = 0\}$ is not a generic line (cf. [L]). Thus $t^{loc}/b^{loc} > d$, so $L$ is not regular by Lemma 2.1 (ii).

3. Proof of Theorem 1.3

Let $f(y, z)$ be a polynomial with $f(0, 0) = 0$ and $a$ and $b$ non-negative integers and consider the linear family of polynomials $f_t(y, z) = f(y, z) - ty^az^b$ with $t \in \mathbb{C}$. Let $V_t = f_t^{-1}(0)$. A special case of the main theorem of [N2] describes the local link at 0 of a generic member $V_t$ of this family in terms of the link at 0 of $V_0$. Namely, we can assume without loss of generality that $f(y, z)$ is not divisible by $y$ or $z$. Let $\Gamma$ be the splice diagram for the link of $yzf(y, z) = 0$ at 0 and let $K_y$ and $K_z$ be the components of this link given by $y = 0$ and $z = 0$ and $K$ the union of the remaining components (that is, the link of $f = 0$). For each vertex $v$ of $\Gamma$ consider the number $s_v := al(S_v, K_y) + bl(S_v, K_z) - l_v$.

Let $\Gamma^-$ be the full subgraph of $\Gamma$ on all vertices $v$ with $s_v < 0$ and arrowheads adjacent to them, and let $\Gamma_1^-, \ldots, \Gamma_k^-$ be the connected components of $\Gamma^-$. Suppose:

(*) For each $j = 1, \ldots, k$, $\Gamma_j^-$ is connected to the rest of $\Gamma$ by a single edge.
In [N2] it is shown that here is a unique way of cutting at this edge and replacing $\Gamma_j^-$ by a graph of the form

in such a way that in the new $\Gamma$:

(i) $s_w = 0$, where $w$ is the new node;

(ii) $s_v$ is unchanged for every non-arrowhead vertex of $\Gamma - \Gamma_j^-$. Moreover, the result $\Gamma_0$ of doing this for each $j = 1, \ldots, k$ is the splice diagram for the link at 0 of $yzf_t(y, z) = 0$ for generic $t$. (If (*) fails then $\Gamma^{-}$ is connected and meets the rest of $\Gamma$ in two edges and $\Gamma_0$ is computed similarly in [N2].)

Now let $\tilde{P}(z, y) = 0$ be the polynomial at a point at infinity for a curve $P(x, y) = 0$, as described in the previous section. Put $f(y, z) = \tilde{P}(z, y)$, $a = 0$, $b = \deg(P) = d$. Then $f_t(y, z) = 0$ is the equation of the point at infinity for $P(x, y) = t$. By Lemma 2.1(ii), and the relationship between the splice diagrams $\Gamma$ and $\Omega$ described in section 2, the above result of [N2] translates directly to give Theorem 1.3.

Bibliography

[EN] D. Eisenbud and W. D. Neumann, *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities*, Annals of Math. Studies 110 (Princeton Univ. Press 1985).

[H] Ha H.V., On the irregular at infinity algebraic plane curve, (Preprint 91/4, Institute of Math., National Center for Scientific Research of Vietnam).

[L] L. V. Than, Affine polar quotients of algebraic plane curves, (in preparation).

[LMW] Le D.T., F. Michel, C. Weber, Courbes polaires et topologie des courbes planes, Ann. Sc. Ec. Norm. Sup. 4e Series, 24 (1991), 141–169.

[N1] W. D. Neumann, Complex plane curves via their links at infinity, Invent. Math. 98 (1989), 445–489.

[N2] W. D. Neumann, Linear systems of plane curve singularities and irregular links at infinity, (in preparation).

Walter D. Neumann  
Department of Mathematics  
Ohio State University  
Columbus, OH 43210, USA  
neumann@mps.ohio-state.edu

Le Van Than  
Institute of Mathematics  
P.O. Box 631-10000  
Hanoi, Vietnam