ON GROMOV’S POSITIVE SCALAR CURVATURE CONJECTURE FOR DUALITY GROUPS

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Abstract. We prove the inequality

\[ \dim_{mc} \tilde{M} \leq n - 2 \]

for the macroscopic dimension of the universal covers \( \tilde{M} \) of almost spin \( n \)-manifolds \( M \) with positive scalar curvature whose fundamental group \( \pi_1(M) \) is a virtual duality group that satisfies the coarse Baum-Connes conjecture.

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1. Introduction

In the book dedicated to Gelfand’s 80th anniversary [Gr1] Gromov introduced the notion of macroscopic dimension to formulate his observation that the universal cover $\tilde{M}$ of a closed $n$-manifold with a positive scalar curvature is dimensionally deficient on large scales:

1.1. Conjecture (Gromov [Gr1]). The macroscopic dimension of the universal cover $\tilde{M}$ of a closed $n$-manifold with a positive scalar curvature metric is at most $n - 2$:

$$\dim_{mc}\tilde{M}^n \leq n - 2.$$ 

This conjecture was proven by D. Bolotov for 3-manifolds [Bo]. Then it was proved by Bolotov and the author [BD] for spin $n$-manifolds, $n > 3$ whose fundamental group satisfies the Analytic Novikov Conjecture and the Rosenberg-Stolz condition [RS]: The natural transformation of the connective real $K$-theory of the group to periodic, $ko_*(\pi) \to KO_*(\pi)$ is a monomorphism. Perhaps the only known class of groups for which Rosenberg-Stolz condition is satisfied consists of the products of free groups.

In this paper we prove the conjecture for spin manifolds whose fundamental group is the virtual duality group that satisfies the Analytic Novikov Conjecture. Moreover, we prove it for almost spin manifolds, i.e. manifolds with spin universal cover, with the virtual duality fundamental group that satisfies the coarse Baum-Connes conjecture. It allows substantially enlarge the class of groups for which the Gromov conjecture holds true by adding to the (virtual) products of free groups, virtually nilpotent groups, arithmetic groups, knot groups, braid groups, mapping class groups, $Out(F_n)$, and their products.

A weak version of the Gromov conjecture states that

$$\dim_{mc}\tilde{M}^n \leq n - 1$$

for positive scalar curvature manifolds $M$. It first appeared in [Gr2] in a different language. Even the weak Gromov’s conjecture is extremely difficult, since it implies the famous Gromov-Lawson conjecture [GL] [R2]: A closed aspherical manifold cannot carry a metric with positive scalar curvature. In this paper we show that the condition $\dim_{mc}\tilde{M}^n \leq n - 1$ is the integral version of the notion of macroscopically small manifolds introduced by Gong and Yu [GY].

In [Dr2], [Dr3] we proved the weak Gromov conjecture for all rationally inessential orientable closed $n$-manifold $M$ whose fundamental group is a duality group. Note that in the case of a spin manifold with a
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positive scalar curvature metric the rational inessentiality follows from Rosenberg’s theorem and the KO-homology Chern-Dold character. In this paper we extend this result proving the weak Gromov conjecture for almost spin closed $n$-manifold $M$, $n \geq 5$ whose fundamental group satisfies virtually the following coarse version of the Rosenberg-Stolz condition: The natural transformation of the connective $K$-theory to periodic $ku_{*}^{f}(E\pi) \rightarrow K_{*}^{f}(E\pi)$ is a monomorphism where $E\pi$ is the universal cover of the classifying space $B\pi = K(\pi,1)$. We note that the duality groups satisfies this condition.

In the last section of the paper we reformulate the property $\dim_{mc}\tilde{M}^{n} \leq n - 1$ in terms of the Stone-Čech compactification $\beta\tilde{M}^{n}$ of $\tilde{M}^{n}$. Then we present the reduction of the weak Gromov conjecture to the rationality conjecture for certain group homology classes and the reduction of the later conjecture to a conjecture about the cohomology classes of $\beta\tilde{M}^{n}$.

2. GROMOV’S MACROSCOPIC DIMENSION

Here is the definition of the macroscopic dimension.

2.1. Definition. [Gr1] A metric space $X$ has the macroscopic dimension less or equal to $k$, $\dim_{mc}X \leq k$, if there is a continuous uniformly cobounded map $f : X \rightarrow N^{k}$ to a $k$-dimensional simplicial complex.

A map $f : X \rightarrow Y$ of a metric space is uniformly cobounded if there is a constant $C > 0$ such that $\text{diam}(f^{-1}(y)) < C$ for all $y \in Y$.

In [Dr1] this definition was modified to $\dim_{MC}X$ by asking $f$ to be Lipschitz for a uniform metric on $N^{k}$. This modification was useful for attacking another conjecture of Gromov. Note that the inequalities $\dim_{mc}X \leq \dim_{MC}X \leq \text{asdim}X$ follows from the definitions. Here asdim is Gromov’s asymptotic dimension [Gr3],[BeD]. As it follows from [Dr4] there is an example that makes both inequality strict. In particular, every essential $n$-manifold with the fundamental group $\mathbb{Z}^{m}$ for $m > n$ is such.

2.1. Inessential manifolds. Gromov calls an $n$-manifold $M$ inessential if a map $f : M \rightarrow B\pi$ that classifies its universal covering $p : \tilde{M} \rightarrow M$ can be deformed to a map into the $(n - 1)$-dimensional skeleton, $g : M \rightarrow B\pi^{(n-1)}$. Otherwise a manifold is called essential. Clearly, for every inessential $n$-manifold $M$ we obtain $\dim_{mc}\tilde{M} < n$. Indeed, a lift $\tilde{g} : \tilde{M} \rightarrow E\pi^{(n-1)}$ of $g$ defines a required map to $(n - 1)$-dimensional complex.
The following characterization of inessential manifolds can be found in [Ba] (see also [BD], Proposition 3.2).

2.2. **Theorem.** For a closed oriented $n$-manifold $M$ the following are equivalent:

1. $M$ is inessential;
2. $f_\ast([M]) = 0$ in $H_n(B\pi;\mathbb{Z})$ where $[M] \in H_n(M;\mathbb{Z})$ is the fundamental class and $f : M \to B\pi$ classifies the universal cover of $M$.

In the light of this theorem an oriented manifold is called **rational inessential** if $f_\ast([M]) = 0$ in $H_n(B\pi;\mathbb{Q})$ [Gr2].

We note that the proof of the inequality $\dim mc\tilde{M} \leq n - 2$ for the cases covered in [BD] consists of constructing a deformation of the classifying map $f : M \to B\pi$ to the $(n - 2)$-dimensional skeleton $B\pi^{(n-2)}$. In this paper we construct a bounded deformation of the lift $\tilde{f} : \tilde{M} \to E\pi$ to $E\pi^{(n-2)}$. The proof in [BD] was performed in the frame of the classical obstruction theory with twisted coefficients. The proof in this paper is based on the obstruction theory that suits the problem. The formulation of the corresponding obstruction theory is performed in the rest of this section and the next section. It is culminated by the following theorem analogous to Theorem 2.2.

2.3. **Theorem.** For a closed oriented $n$-manifold $M$ the following are equivalent:

1. $\dim mc\tilde{M} < n$;
2. $\tilde{f}_\ast([\tilde{M}]) = 0$ in $H_n^{lf}(E\pi;\mathbb{Z})$ where $[\tilde{M}] \in H_n^{lf}(\tilde{M};\mathbb{Z})$ is the fundamental class of $\tilde{M}$ and $\tilde{f} : \tilde{M} \to E\pi$ is a lift of the map $f : M \to B\pi$ classifying the universal cover of $M$.

Thus the condition $\dim mc\tilde{M} < n$ can be considered as a 'coarse inessentiality' of $M$ with its characterization given by Theorem 2.3 analogous to the characterization of the inessentiality in Theorem 2.2.

2.2. **Macroscopic dimension and coarse inessentiality.** We call a cellular map $f : X \to Y$ **bounded** if there is $k \in \mathbb{N}$ such that for every cell $e \subset X$ the image $f(e)$ intersects at most $k$ cells.

For a finitely presented group $\pi$ we can choose an Eilenberg-McLane complex $K(\pi, 1) = B\pi$ to be locally finite and hence metrizable. We consider a proper geodesic metric on $B\pi$ and lift it to the universal cover $E\pi$. We recall that every closed ball with respect to a proper metric is compact. In particular, every ball in $E\pi$ is contained in a finite subcomplex. The metric space $E\pi$ is **uniformly contractible**, i.e., there is a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that every set of diameter $\leq r$ can be contracted to a point in a ball of diameter $\rho(r)$. 
For a CW complex $X$ we consider the product CW complex structure on $X \times I$. Clearly, a cellular homotopy $H : X \times I \to E\pi$ is bounded if and only if if there is a uniform upper bound on the diameter of paths $H(x \times I)$, $x \in X$.

The following proposition shows that the inequality $\dim_{mc} \widetilde{M} < n$ for the universal cover of a $n$-manifold can be considered as a coarse version of inessentiality.

2.4. Proposition. For a finite complex $M$ the following conditions are equivalent:

(1) $\dim_{mc} \widetilde{M} < n$.

(2) A lift $\widetilde{f} : \widetilde{M} \to E\pi$ to the universal coverings of a map $f : M \to B\pi$ that induces an isomorphism of the fundamental groups can be deformed to the $(n - 1)$-dimensional skeleton $E\pi^{(n-1)}$ of $E\pi$ by a bounded homotopy.

The proof of Proposition 2.4 can be extracted from [Dr1], Proposition 3.1.

2.3. Primary obstruction to a bounded extension. Here we recall some basic facts of the elementary obstruction theory. Let $f : X \to Y$ be a cellular map that induces an isomorphism of the fundamental groups. We want to deform the map $f$ to a map to the $(n - 1)$-skeleton $Y^{(n-1)}$. For that we consider the extension problem

$$X \supset X^{(n-1)} \xrightarrow{f} Y^{(n-1)},$$

i.e., the problem to extend $f : X^{(n-1)} \to Y^{(n-1)}$ continuously to a map $\widetilde{f} : X \to Y^{(n-1)}$. The primary obstruction for this problem $o_f$ is the obstruction to extend $f$ to the $n$-skeleton. It lies in the cohomology group $H^n(X; L)$ where $L = \pi_{n-1}(Y^{(n-1)})$ is the $(n - 1)$-dimensional homotopy group considered as a $\pi$-module for $\pi = \pi_1(Y) = \pi_1(X)$.

The class $o_f$ is generated by the cocycle $C_f : C_n(X) \to L$ defined by the obvious rule:

$$C_f(e) = [f \circ \phi_e : S^{n-1} \to Y^{(n-1)}] \in \pi_{n-1}(Y^{(n-1)})$$

where $\phi_e : \partial D^n \to X^{(n-1)}$ is the attaching map of an $n$-cell $e$. Clearly, $f$ can be extended if and only if $C_f = 0$.

The obstruction theory says that an extension $g : X \to Y^{(n-1)}$ that agrees with $f$ on the $(n - 2)$-skeleton $X^{(n-2)}$ exits if and only if $o_f = 0$. The primary obstruction is natural: If $g : Z \to X$ is a cellular map, then $o_{gf} = g^*(o_f)$. In particular, in our case $o_f = f^*(o_1)$ where $o_1 \in H^n(Y; L)$ is the primary obstruction to the retraction of $Y$ to the $(n - 1)$-skeleton.
Now in the above setting we consider the extension problem for the universal coverings
\[ \tilde{X} \supset \tilde{X}^{(n-1)} \xrightarrow{\tilde{f}} \tilde{Y}^{(n-1)}. \]
We are looking for an extension \( g : \tilde{X} \to \tilde{Y}^{(n-1)} \) which is on a bounded distance from \( \tilde{f} : \tilde{X} \to \tilde{Y} \). We call such extension \emph{bounded}. In view of Proposition 2.4 existence of such extension in the case when \( Y = B\pi \) means exactly the inequality \( \dim_{mc} \tilde{X} < n \). Clearly, the vanishing of the classical obstruction \( o_{\tilde{f}} \) would not guarantee a bounded extension. We need to restrict the freedom in the choice of cochains. Note that the obstruction cochain \( C_{\tilde{f}} \) is \( \pi \)-equivariant and its class in the equivariant cohomology \( H^n_{\pi}(\tilde{X}; L) = H^n(X; L) \) is exactly the class \( o_{\tilde{f}} \). Of course vanishing of \( o_{\tilde{f}} \) would imply a bounded extension but it is not a necessary condition. It turns out that the coarsely equivariant cochains do the required job.

2.5. \textbf{Definition.} The group of coarsely equivariant cochains \( C^n_{ce}(\tilde{X}, L) \) consists of homomorphisms \( \phi : C_n(\tilde{X}) \to L \) such that the set
\[ \{ \gamma^{-1} \phi(\gamma e) \mid \gamma \in \pi \} \subset L \]
is contained in a finitely generated subgroup of \( L \) for every \( n \)-cell \( e \).

2.6. \textbf{Proposition.} Let \( \tilde{f} : \tilde{X} \to \tilde{Y} \) be a lift to the universal coverings of a cellular map \( f : X \to Y \) of a finite complex to a locally finite that induces an isomorphism of the fundamental groups. Then
\begin{enumerate}
  \item\( \tilde{f} : \tilde{X}^{(n-1)} \to Y^{(n-1)} \) has bounded extension to \( X^{(n)} \) if and only if \( C_{\tilde{f}} = 0 \).
  \item\( \tilde{f} : \tilde{X}^{(n-2)} \to Y^{(n-1)} \) has bounded extension to \( X^{(n)} \) if and only if \( C_{\tilde{f}} = \delta \Psi \) where \( \Psi \) is a coarsely equivariant cochain.
\end{enumerate}
\textbf{Proof.} The proof goes along the lines of a similar statement from the classical obstruction theory.
\begin{enumerate}
  \item Let \( \pi = \pi_1(X) \). If \( C_{\tilde{f}} = 0 \) then \( \tilde{f} \) has a \( \pi \)-equivariant extension which is, clearly, bounded. In the other direction the statement is trivial.
  \item Let \( C_{\tilde{f}} = \delta \Psi \) where \( \Psi : C_{n-1}(\tilde{X}) \to L \), \( L = \pi_{n-1}(\tilde{Y}^{(n-1)}) \), is a coarsely equivariant homomorphism. For each \( (n-1) \)-cell \( e \) of \( X \) we fix a lift \( \tilde{e} \subset \tilde{X}^{(n-1)} \). Since \( \Psi \) is coarsely equivariant, the set
  \[ \{ \gamma^{-1} \Psi(\gamma \tilde{e}) \mid \gamma \in \pi \} \]
spans a finitely generated subgroup \( G \subset L \). Therefore, \( G \) is contained in the image of \( \pi_{n-1}(F_{\tilde{e}}) \) for some finite subcomplex \( F_{\tilde{e}} \subset \tilde{Y}^{(n-1)} \).
Like in the classical obstruction theory we define a map $g_\gamma : \gamma e \to \gamma F \subset Y^{(n-1)}$, $\gamma \in \pi$ on cells $\gamma e$ such that $g_\gamma$ agrees with $\tilde{f}$ outside a small $(n-1)$-ball $B_\gamma \subset \gamma e$ and the difference of the restriction of $\tilde{f}$ and $g_\gamma$ to $B_\gamma$ defines a map

$$d_{\tilde{f}, g_\gamma} : S^{n-1} = B_\gamma^+ \cup B_\gamma^- \to \gamma F \subset Y^{(n-1)}$$

that represents the class $-\Psi(\gamma e) \in \pi_{n-1}(\tilde{Y}^{(n-1)})$. The union $\cup_{\gamma \in \pi} g_\gamma$ defines a bounded map $g : \tilde{X}^{(n-1)} \to \tilde{Y}^{(n-1)}$.

The elementary obstruction theory implies that for every $n$-cell $\sigma \subset \tilde{X}^{(n)}$ there is an extension $\tilde{g}_\sigma : \sigma \to \tilde{Y}^{(n-1)}$ of $g|_{\partial \sigma}$. Since $\partial \sigma$ is contained in finitely many $(n-1)$-cells, there are $e_1, \ldots, e_k \subset X^{(n-1)}$ and $\gamma_1, \ldots, \gamma_k \in \pi$ such that

$$g(\partial \sigma) \subset \bigcup_{i=1}^k \gamma_i F_{\tilde{e}_i} = F.$$

Since $F$ is a finite complex, the group $\pi_{n-1}(F)$ is finitely generated over $\mathbb{Z} \pi_1(F)$. Hence there is a finite subcomplex $W \subset \tilde{Y}^{(n-1)}$ containing $F$ such that for the homotopy groups $\pi_{n-1}$ we have $\ker(j_W)_* = \ker j_*$ where $j_W : F \to W$ and $j : F \to Y^{(n-1)}$ are the inclusions. Therefore, we may assume that $\tilde{g}_\sigma(\sigma) \subset W$. Since $g(\partial \gamma \sigma) \subset \gamma W$, we may assume that $\tilde{g}_\gamma(\gamma \sigma) \subset \gamma W$ for all translates of $\sigma$, $\gamma \in \pi$. Thus, the resulting extension $\bar{g} : \tilde{X}^{(n)} \to \tilde{Y}^{(n-1)}$ is bounded.

In the other direction, if there is a $k$-bounded map $\bar{g} : \tilde{X} \to Y^{(n-1)}$ that coincides with $\tilde{f}$ on the $(n-2)$-dimensional skeleton, then the difference cochain $d_{\tilde{f}, \bar{g}}$ is coarsely equivariant. Indeed, the $k$-bounded maps preserving a base point $\phi : S^{n-1} \to Y^{(n-1)}$ define a finitely generated subgroup of $\pi_{n-1}(Y)$. Then the formula $\delta d_{\tilde{f}, \bar{g}} = C_{\bar{g}} - C_{\tilde{f}}$ and the fact that $o_{\bar{g}} = 0$ imply that $o_{\tilde{f}} = 0$. □

3. Coarsely equivariant cohomology

Let $X$ be a CW complex and let $E_n(X)$ denote the set of its $n$-dimensional cells. We recall that (co)homology groups of a CW complex $X$ with coefficients in a $\pi$-module $L$ where $\pi = \pi_1(X)$ are defined by by means of the chain complex $\text{Hom}_\pi(C_n(X), L)$ where $\tilde{X}$ is the universal cover of $X$ with the cellular structure induced from $X$. The chain complex defining the homology groups $H_*(X; L)$ is $\{C_n(\tilde{X}) \otimes_{\pi} L \}$. The resulting groups $H_*(X; L)$ and $H^*(X; L)$ do not depends on the CW structure on $X$. 

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These groups can be interpreted as the equivariant (co)homology:

\[ H_*(X; L) = H_{f, \pi}^*(\tilde{X}; L) \quad \text{and} \quad H^*(X; L) = H^*_{\pi}(\tilde{X}; L). \]

The last equality is obvious since the equivariant cohomology groups \( H^*_{\pi}(\tilde{X}; L) \) are defined by equivariant cochains \( C^n_{\pi}(\tilde{X}), L \), i.e., homomorphisms \( \phi : C_n(\tilde{X}) \to L \) such that the set

\[ S_{\phi,c} = \{ \gamma^{-1}\phi(\gamma c) \mid \gamma \in \pi \} \]

consists of one element for every \( c \in C_n(\tilde{X}) \). Here we consider the group of coarsely equivariant cochains \( C^n_{ce}(\tilde{X}, L) \). Thus we have a chain of inclusions of cochain complexes

\[ C^*_{\pi}(\tilde{X}, L) \subset C^*_{ce}(\tilde{X}, L) \subset C^*(\tilde{X}, L) \]

which induces a chain of homomorphisms of corresponding cohomology groups

\[ H^n(X; L) \xrightarrow{cc_X} H^n_{ce}(\tilde{X}; L) \to H^n(\tilde{X}; L) \]

where the first homomorphism is called the equivariant coarsening homomorphism. The cohomology groups \( H^*_{ce}(\tilde{X}; L) \) are called the coarsely equivariant cohomology of \( \tilde{X} \) with coefficients in a \( \pi \)-module \( L \).

3.1. Remark. In [Dr1] we defined the almost equivariant cohomology groups by means of almost equivariant cochains \( C^n_{ae}(\tilde{X}, L) \), i.e., homomorphisms \( \phi : C_n(\tilde{X}) \to L \) for which the set \( S_{\phi,c} \) is finite for every chain \( c \in C_n(\tilde{X}) \). The inclusions \( C^*_{\pi}(\tilde{X}, L) \subset C^*_{ae}(\tilde{X}, L) \subset C^*_a(\tilde{X}, L) \)

define the homomorphisms of corresponding cohomology groups

\[ H^n(X; L) \xrightarrow{pert^*_X} H^n_{ae}(\tilde{X}; L) \xrightarrow{\alpha} H^n_{ae}(\tilde{X}; L) \]

where \( ec_X = \alpha \circ pert^*_X \).

Note that the groups \( H^*_{ce}(\tilde{X}; L) \) do not depend on the choice of a CW complex structure on \( X \). Also note that in the case when \( L = \mathbb{Z} \) is a trivial module the cohomology theory \( H^*_{ce}(\tilde{X}; \mathbb{Z}) \) equals the standard (cellular) cohomology of the universal cover \( \tilde{X} \).

A proper cellular map \( f : X \to Y \) that induces an isomorphism of the fundamental groups lifts to a proper cellular map of the universal covering spaces \( \tilde{f} : \tilde{X} \to \tilde{Y} \). The lifting \( \tilde{f} \) defines a chain homomorphism \( \tilde{f}_* : C_n(\tilde{X}) \to C_n(\tilde{Y}) \) and the homomorphism of coarsely equivariant cochains \( \tilde{f}^* : Hom_{ce}(C_n(\tilde{Y}), L) \to Hom_{ce}(C_n(\tilde{X}), L) \). The latter defines a homomorphisms of the coarsely equivariant cohomology groups

\[ \tilde{f}_{ce}^* : H^*_{ce}(\tilde{Y}; L) \to H^*_{ce}(\tilde{X}; L). \]
Suppose that $\pi$ acts freely on CW complexes $\tilde{X}$ and $\tilde{Y}$ such that the actions preserve the CW complex structures. We call a cellular map $g: \tilde{X} \to \tilde{Y}$ coarsely equivariant if the set
\[
\bigcup_{\gamma \in \pi} \{\gamma^{-1}g(\gamma e)\}
\]
is contained in a finite subcomplex $K \subset \tilde{Y}$ for every cell $e$ in $\tilde{X}$.

3.2. Proposition. Let $f: X \to Y$ be a proper coarsely equivariant cellular map. Then the induced homomorphism on cochains takes the coarsely equivariant cochains to coarsely equivariant.

We omit the proof since it is similar to the proof of Proposition 2.3 in [Dr1]. Proposition 3.2 and the standard facts about cellular chain complexes imply the following.

3.3. Proposition. Let $X$ and $Y$ be complexes with free cellular actions of a group $\pi$.

(A) Then every coarsely equivariant cellular map $f: X \to Y$ induces an homomorphism of the coarsely equivariant cohomology groups
\[
f^*: H^*_ce(Y; L) \to H^*_ce(X; L).
\]

(B) If two coarsely equivariant maps $f_1, f_2: X \to Y$ are homotopic by means of a cellular coarsely equivariant homotopy $H: X \times [0, 1] \to Y$, then they induce the same homomorphism of the coarsely equivariant cohomology groups, $f_1^* = f_2^*$.

3.1. Coarsely equivariant homology. We recall that the equivariant locally finite homology groups are defined by the complex of infinite locally finite invariant chains
\[
C^n_\pi(\tilde{X}; L) = \{ \sum_{e \in E_n(\tilde{X})} \lambda_e e \mid \lambda_{ge} = g\lambda_e, \lambda_e \in L \}.
\]
The local finiteness condition on a chain requires that for every $x \in \tilde{X}$ there is a neighborhood such that the number of $n$-cells $e$ intersecting $U$ for which $\lambda_e \neq 0$ is finite. This condition is satisfied automatically when $X$ is a locally finite complex. It is known that $H_\pi(X; L) = H^*_{ce}(\tilde{X}; L)$.

Similarly one can define the coarsely equivariant homology groups on a locally finite CW complex by considering infinite coarsely equivariant chains. Let $X$ be a complex with the fundamental group $\pi$ and the universal cover $\tilde{X}$. We call an infinite chain $\sum_{e \in E_n(\tilde{X})} \lambda_e e$ coarsely equivariant if the group generated by the set $\{\gamma^{-1}\lambda_{\gamma e} \mid \gamma \in \pi\} \subset L$ is finitely generated for every cell $e$. As we already have mentioned, the
complex of equivariant locally finite chains defines equivariant locally finite homology $H_{lf,\pi}^*(\tilde{X}; L)$. The homology defined by the coarsely equivariant locally finite chain are called the coarsely equivariant locally finite homology. We use notation $H_{lf,ce}^*(\tilde{X}; L)$. Like in the case of cohomology this definition can be carried out for the singular homology and it gives the same groups. In particular the groups $H_{lf,ce}^*(\tilde{X}; L)$ do not depend on the choice of a CW complex structure on $X$. As in the case of cohomology for any complex $K$ there is an equivariant coarsening homomorphism

$$ec^K_\pi : H_*(K; L) = H_{lf,\pi}^*(\tilde{K}; L) \rightarrow H_{lf,ce}^*(\tilde{K}; L).$$

Also, there is an analog of Proposition 3.3 for the coarsely equivariant locally finite homology:

3.4. Proposition. Let $X$ and $Y$ be complexes with free cellular actions of a group $\pi$.

(A) Then every coarsely equivariant cellular map $f : X \rightarrow Y$ induces a homomorphism of the coarsely equivariant homology groups

$$f_* : H_{lf,ce}^*(X; L) \rightarrow H_{lf,ce}^*(Y; L).$$

(B) If two coarsely equivariant maps $f_1, f_2 : X \rightarrow Y$ are homotopic by means of a cellular coarsely equivariant homotopy, then they induce the same homomorphism of the coarsely equivariant cohomology groups, $(f_1)_* = (f_2)_*$.

Let $M$ be an oriented $n$-dimensional PL manifold with a fixed triangulation. Denote by $M^*$ the dual complex. There is a bijection between $k$-simplices $e$ and the dual $(n-k)$-cells $e^*$ which defines the Poincaré duality isomorphism. This bijection extends to a similar bijection on the universal cover $\tilde{M}$. Let $\pi = \pi_1(M)$. For any $\pi$-module $L$ the Poincaré duality on $M$ with coefficients in $L$ is given by the cochain-chain level by isomorphisms

$$Hom_\pi(C_k(\tilde{M}^*), L) \xrightarrow{PD_k} C_{n-k}^{lf,\pi}(\tilde{M}; L)$$

where $PD_k$ takes a cochain $\phi : C_k(\tilde{M}^*) \rightarrow L$ to the following chain $\sum_{e \in E_{n-k}(\tilde{M})} \phi(e^*) e$. The family $PD_*$ is a chain isomorphism which is also known as the cap product

$$PD_k(\phi) = \phi \cap [\tilde{M}]$$

with the fundamental class $[\tilde{M}] \in C_n^{lf,\pi}(\tilde{M})$, where $[\tilde{M}] = \sum_{e \in E_n(\tilde{M})} e$.

We note that the homomorphisms $PD_k$ and $PD_k^{-1}$ extend to the coarsely
equivariant chains and cochains:
\[ \text{Hom}_{ce}(C_k(\tilde{M}^*), L) \xrightarrow{PD} C_{n-k,ce}(\tilde{M}; L). \]
Thus, the homomorphisms \( PD_* \) define the Poincaré duality isomorphisms \( PD_{ce} \) between the coarsely equivariant cohomology and homology. We summarize this in the following

3.5. Proposition. For any closed oriented \( n \)-manifold \( M \) and any \( \pi_1(M) \)-module \( L \) the Poincaré duality forms the following commutative diagram:

\[
\begin{array}{ccc}
H^k(M; L) & \xrightarrow{\text{ec}^M} & H^k_{ce}(\tilde{M}; L) \\
\downarrow & & \downarrow \\
H_{n-k}(M; L) & \xrightarrow{\text{ec}^M} & H^{l,ce}_{n-k}(\tilde{M}; L).
\end{array}
\]

We note that the operation of the cap product for equivariant homology cohomology automatically extends on the chain-cochain level to the cap product on the coarsely equivariant homology and cohomology. Then the Poincaré Duality isomorphism \( PD_{ce} \) for \( \tilde{M} \) can be described as the cap product with the homology fundamental class \( [\tilde{M}] \in H^n_l(\tilde{M}). \)

3.2. Obstruction to the inequality \( \dim_{mc} \tilde{M}^n < n \). In view of Proposition 2.4 we can reformulate Proposition 2.6 as follows:

3.6. Theorem. Let \( X \) be a finite \( n \)-complex with \( \pi_1(X) = \pi \) and let \( f : X \to B\pi \) be a map that induces an isomorphism of the fundamental groups. Then \( \dim_{mc} \tilde{X} < n \) if and only if the obstruction \( o_f \in H^n_{ce}(\tilde{X}; \pi_{n-1}(Y^{(n-1)})) \) is trivial.

Proof. If \( \dim_{mc} \tilde{X} < n \), then by Proposition 2.4 there is a a bounded cellular homotopy of \( \tilde{f} : \tilde{X} \to E\pi \) to a map \( g : \tilde{X} \to E\pi^{(n-1)}. \) By Corollary 3.7 the map \( g \) is coarsely equivariant. Then by Proposition 3.3 \( o_f = \tilde{f}^*(o_1) = g^*i^*(o_1) = 0 \) where \( i : E\pi^{(n-1)} \subset E\pi \) is the inclusion and \( o_1 \) is the primary obstruction to bounded retraction of \( E\pi \) onto \( E\pi^{(n-1)}. \)

We assume that \( B\pi \) is a locally finite simplicial complex. If \( o_f = 0 \), then by Proposition 2.6 there is a bounded cellular map \( g : \tilde{X} \to E\pi^{(n-1)} \) which agrees with \( \tilde{f} \) on the \( (n-2) \)-skeleton \( X^{(n-2)}. \) The uniform contractibility of \( E\pi \) implies that \( g \) is homotopic to \( \tilde{f} \) by means of a bounded cellular homotopy. Then Proposition 2.4 implies the inequality \( \dim_{mc} \tilde{X} < n. \) \qed
3.7. **Proposition.** Let $X$ and $Y$ be universal covers of finite and locally finite complexes respectively with the same fundamental group $\pi$. Then a bounded cellular homotopy $\Phi : X \times [0, 1] \to Y$ of a coarsely equivariant map is coarsely equivariant.

We refer to [Dr1] for the proof.

3.3. **Proof of Theorem 2.3.**

1. $\dim_{\text{mc}} \tilde{M} < n \Rightarrow \tilde{f}_*(\tilde{M}) = 0$. We may assume that $f : M \to B\pi$ is cellular and Lipschitz for some metric CW complex structure on $B\pi$. If $\dim_{\text{mc}} \tilde{M} < n$, then by Proposition 2.4 there is a coarsely Lipschitz cellular homotopy of $\tilde{f} : \tilde{X} \to E\pi$ to a map $g : \tilde{X} \to E\pi^{(n-1)}$ with a compact projection to $B\pi$. By Proposition 3.7 it is coarsely equivariant. Then by Proposition 3.3 it follows that $\tilde{f}_*(\tilde{M}) = 0$.

2. $\tilde{f}_*(\tilde{M}) = 0 \Rightarrow \dim_{\text{mc}} \tilde{M} < n$. Let $o_{\tilde{f}}$ be the primary obstruction for a bounded deformation of $\tilde{M}$ to $E\pi^{(n-1)}$. Note that $o_{\tilde{f}} = \tilde{f}_*(o_1)$ where $o_1$ is the primary obstruction for bounded retraction of $E\pi^{(n)}$ to $E\pi^{(n-1)}$. Then $\tilde{f}_*(\tilde{M}) \cap o_{\tilde{f}} = \tilde{f}_*(\tilde{M}) \cap o_1 = 0$. Since the induced homomorphism

$$\tilde{f}_* : H_0^{lf,ce}(\tilde{M}; L) \to H_0^{lf,ce}(E\pi; L)$$

is an isomorphism for every $\pi$-module $L$, we obtain $[\tilde{M}] \cap o_{\tilde{f}} = 0$. By the Poincaré Duality (Proposition 3.5), $o_{\tilde{f}} = 0$. We apply Theorem 3.6 to obtain the inequality $\dim_{\text{mc}} \tilde{M} < n$.

The following statement is in spirit of Brunnbauer-Hanke results [BH].

3.8. **Corollary.** For each $n$ there is a subgroup $H_n^{sm}(B\pi) \subset H_n(B\pi)$ of small classes such that for an orientable $n$-manifold $M$ its universal covering satisfies $\dim_{\text{mc}} \tilde{M} < n$ if and only if $f_*([M]) \in H_n^{sm}(B\pi)$ where $f : M \to B\pi$ is a classifying map for $M$.

**Proof.** Define $H_n^{sm}(B\pi) = \ker\{ec_\pi^* : H_n(B\pi) \to H_n^{lf,ce}(E\pi)\}$. Since $\tilde{f}_* \circ ec_\pi^* = ec_\pi^* \circ f_*$, we have that

$$\dim_{\text{mc}} \tilde{M} < n \Leftrightarrow [\tilde{M}] \in \ker \tilde{f}_* \Leftrightarrow f_*([M]) \in H_n^{sm}(B\pi).$$

\[ \square \]

4. **Coarse homology**

The coarse homology groups $HX_*(Y)$ of a metric space $Y$ were defined by Roe [Ro1], [Ro3] by means an anti-Čech approximation of $X$. 
An anti-Čech approximation of a metric space $Y$ is a sequence of uniformly bounded locally finite open covers of $Y$ with finite multiplicity

$$\mathcal{U}_1 \prec \mathcal{U}_2 \prec \mathcal{U}_3 \prec \ldots$$

such that for every $i$ the diameter of elements of $\mathcal{U}_i$ less than the Lebesgue number of $\mathcal{U}_{i+1}$,

$$\text{mesh}(\mathcal{U}_i) < \text{Leb}(\mathcal{U}_{i+1}).$$

Then the refinement $\mathcal{U}_i \prec \mathcal{U}_{i+1}$ defines the simplicial map $p_i : N(\mathcal{U}_i) \to N(\mathcal{U}_{i+1})$ of the nerves. The coarse homology of $Y$ are defined as the direct limit of homology groups of the nerves:

$$H_{\text{X}_k}(Y) = \lim_{\to} \{H_{\text{lf}}^k(N(\mathcal{U}_i)), (p_i)_*\}.$$ 

Let $q^i : Y \to N(\mathcal{U}_i)$ be the projection to the nerve defined by a partition of unity on $Y$ subordinated to $\mathcal{U}_i$. For a metric CW complex $Y$ it induces the homomorphism $q^i_1 : H_{\text{lf}}^k(Y) \to H_{\text{lf}}^k(N(\mathcal{U}_i))$ which defines a natural homomorphism $c_Y : H_{\text{lf}}^k(Y) \to H_{\text{X}_*}(Y)$ called coarsening (see [HR]). The coarse homology groups are invariant under the coarse equivalence and, in particular, under quasi-isometries. Thus, for a closed manifold $M$ with $\pi_1(M) = \pi$ there is a natural isomorphism $H_{\text{X}_*}(\pi) \to H_{\text{X}_*}(\tilde{M}).$

4.1. **Macrosopically small manifolds.** The following is an integral version of Gong-Yu’s definition of macroscopically large manifolds [GY],[NY].

4.1. **Definition.** The universal covering $\tilde{M}$ of a closed manifold $M$ with the lifted metric is called macroscopically large if $c_{\tilde{M}}(\tilde{M}) \neq 0$ in $H_{\text{X}_*}(\pi)$ for integral coefficients. Otherwise we call $\tilde{M}$ macroscopically small.

In the original Gong-Yu definition [GY],[NY] the coefficients were not mentioned though they were assumed to be rational. Latter Brunnbauer and Hanke [BH] proved that Gong-Yu’s concept of the rational macroscopically large manifolds coincides with Gromov’s notion of a manifold with the infinite rational filling radius [Gr2]. In the following theorem we prove a similar statement integrally.

4.2. **Theorem.** For a closed $n$-manifold $M$ its universal cover $\tilde{M}$ is macroscopically small if and only if $M$, $\dim_{\text{mc}} \tilde{M} < n$.

**Proof.** Suppose that an $n$-manifold $M$, $\dim_{\text{mc}} \tilde{M} < n$. By Proposition 2.4 there is a uniformly cobounded bounded cellular map $g : \tilde{M} \to L \subset E\pi^{(n-1)}$ to an $(n-1)$-dimensional complex $L$. Moreover, we may
assume that $g$ is a quasi-isometry to $L$. Therefore, $c_{\bar{M}} = c_L \circ g_*$. Since $L$ is $(n-1)$-dimensional, $g_*([\bar{M}]) = 0$. Thus, $\bar{M}$ is macroscopically small.

Suppose that $c_{\bar{M}}([\bar{M}]) = 0$. Then $q_i^*[\bar{M}] = 0$ for some $i$ where $q_i^*: \bar{M} \to N(U_i)$ is the projection to the nerve in an anti-Čech approximation of $\bar{M}$. The uniform contractibility of $E\pi$ and finite-dimensionality of $N(U_i)$ imply that there is a map $s: N(U_i) \to E\pi$ such that the map $s \circ q_i$ is proper homotopic to $\bar{f}$. Therefore, $f_*([\bar{M}]) = 0$ and by Theorem 2.3 $\dim_{mc} \bar{M} < n$. □

We note that in [Dr4] we called manifolds $M$ with $\dim_{mc} \bar{M} < n$ md-small where md stands for macroscopic dimension. In this paper in view of Theorem 4.2 we stick to Gong-Yu’s terminology calling $\bar{M}$ macroscopically small.

4.3. Problem. Suppose that the universal covering $\bar{M}$ of a closed manifold is macroscopically large. Does it follow that $\bar{M}$ is macroscopically large rationally?

4.2. Coarse Baum-Connes conjecture. We recall that for a generalized homology theory $h_*$ the locally finite homology groups $h_{lf}^i(X)$ of a locally compact space are defined as the Steenrod $h_*$-homology of the one-point compactification $\alpha X$.

Roe introduced certain $C^*$-algebra $C^*_\text{Roe}(X)$ of an open Riemannian manifold $X$ and defined a coarse index map [Ro1], [Ro2], [R2]

$$A_*: K_{lf}^*(X) \to K_*(C^*_\text{Roe}(X)).$$

The construction of $C^*_\text{Roe}(X)$ and $A$ can be extended to any metric space $X$ [HR]. In particular, $C^*_\text{Roe}(X)$ is a coarse invariant. Note that the nerve of a uniformly bounded cover $U$ can be given a metric such that the projection $q: X \to N(U)$ is a quasi-isometry. Then using an anti-Čech approximation $\{U_i\}$ of $X$ and taking the direct limit of the coarse index maps $K_{lf}^*(N(U_i)) \to K_*(C^*_\text{Roe}(N(U_i))) \cong K_*(C^*_\text{Roe}(X))$ one can defined a homomorphism

$$A_\infty: KX_*(X) \to K_*(C^*_\text{Roe}(X))$$

called the coarse assembly map.

4.4. Conjecture (Coarse Baum-Connes conjecture [Ro1], [HR]). For a proper metric space $X$ the coarse assembly map is an isomorphism.

We say that a finitely generated group $\pi$ satisfies the coarse Baum-Connes conjecture if $\pi$ satisfies the coarse Baum-Connes conjecture as a metric space with respect to the word metric for a finite set of
generators. For the definition of the coarse $K$-homology in the case of discrete space one can take a sequence of Rips complexes

$$\pi \subset R_1(\pi) \subset R_2(\pi) \subset \cdots \subset R_m(\pi) \subset \cdots$$

instead of an anti-Čech system. Then

$$KX_*(\pi) = \lim_{\to} K_{lf}^*(R_m(\pi)).$$

We recall that the Rips complex $R_m(\pi)$ has $\pi$ as the set of vertices and a finite subset $\{\gamma_0, \ldots, \gamma_k\} \subset \pi$ spans a $k$-simplex in $R_m(\pi)$ if and only if $d(\gamma_i, \gamma_j) \leq m$ for all $0 \leq i, j \leq k$.

4.5. Proposition. For every closed manifold $M$ with the fundamental group $\pi$ there is a homomorphism $b$ that makes a commutative diagram

$$\begin{array}{ccc}
K_{lf}^*(\tilde{M}) & \xrightarrow{c_{\tilde{M}}} & KX_*(\tilde{M}) \\
\downarrow & & \downarrow \cong \\
K_{lf}^*(E\pi) & \xleftarrow{b} & KX_*(\pi).
\end{array}$$

The homomorphism $b$ is an isomorphism for geometrically finite groups $\pi$.

Proof. We construct a sequence of proper maps $b_i : R_i(\pi) \to E\pi$ such that $b_{i+1}|_{R_i(\pi)} = b_i$ for all $i \in \mathbb{N}$ where $b_0 : \pi \to E\pi$ is an orbit of a base point, $b_0(\gamma) = \gamma(x_0)$.

\[\square\]

5. Gromov’s scalar curvature conjecture

We recall that the Gromov conjecture states that $\dim_{mc} \tilde{M} \leq n - 2$ for the universal cover of a closed $n$-manifold with positive scalar curvature. Clearly, the conjecture has two stages where the first stage is to prove the inequality $\dim_{mc} \tilde{M} \leq n - 1$. We call it the weak Gromov conjecture. In view of Theorem 4.2 the weak Gromov conjecture can be reformulated as following: The universal covering of a manifold with positive scalar curvature is macroscopically small.

We call manifolds with spin universal cover almost spin. The spin structure on $\tilde{M}$ defines a $K$-theory orientation. Thus, there exists the $K$-theory fundamental class $[\tilde{M}]_K \in K_{lf}^n(\tilde{M})$. We denote by $\text{ku}_*$ the connective $K$-theory. Then there is the fundamental class $[\tilde{M}]_{\text{ku}}$ which is taken to $[\tilde{M}]_K$ under the transformation $\text{ku}_* \to KU_* = K_*$. 
5.1. **Deformation to the** \((n - 1)\)-**skeleton.** Our deformation of \(\bar{f} : \tilde{M} \to E\pi\) to \(E\pi^{(n-1)}\) is parallel to the deformation of \(f : M \to B\pi\) to \(B\pi^{(n-1)}\) performed in [BD]. In [BD] the main ingredients were the Analytic Novikov conjecture and the vanishing theorem or Rosenberg [R1]. Here we use the coarse Baum-Connes conjecture and the vanishing theorem of Roe [Ro1],[HR].

5.1. **Theorem** (Roe). *Suppose that a closed almost spin manifold \(M\) has positive scalar curvature. Then the coarse index map

\[
A_\ast : K^\lf_\ast(\tilde{M}) \to K_\ast(C^\ast_{\text{Roe}}(\pi))
\]

takes the \(K\)-theory fundamental class \([\tilde{M}]_K\) to zero.*

Another parallel with [BD] is the usage of the connective \(K\)-theory.

5.2. **Theorem.** *Suppose that the fundamental group \(\pi\) of a closed almost spin \(n\)-manifold \(M\), \(n \geq 5\), with a positive scalar curvature metric satisfies the coarse Baum-Connes conjecture and the natural transformation \(ku^\lf_n(E\pi) \to K^\lf_n(E\pi)\) is injective. Then the weak Gromov conjecture holds true: \(\dim_{mc}\tilde{M} < n\).*

**Proof.** It suffices to show that \(\tilde{f}_\ast([\tilde{M}])\) is zero in \(H^\lf_\ast(E\pi)\).

Note that the coarse index map for \(\tilde{M}\) is the composition

\[
\mathcal{A} = A_\infty \circ i_\ast \circ c_{\tilde{M}}
\]
in the following commutative diagram

\[
\begin{array}{ccc}
K^\lf_\ast(\tilde{M}) & \xrightarrow{c_{\tilde{M}}} & KX_\ast(\tilde{M}) \\
\tilde{f}_\ast \downarrow & & \downarrow i_\ast \cong \\
K^\lf_\ast(E\pi) & \xleftarrow{b} & KX_\ast(\pi) \xrightarrow{A_\infty} K_\ast(C^\ast_{\text{Roe}}(\pi)).
\end{array}
\]

Since the coarse Baum-Connes conjecture holds for \(\pi\), the coarse assembly map \(A_\infty\) is an isomorphism. Then in view of Proposition 4.5 and Theorem 5.1 \(\tilde{f}_\ast([\tilde{M}]_K) = 0\) in \(K^\lf_\ast(E\pi)\). By the condition of the theorem we obtain \(\tilde{f}_\ast([\tilde{M}]_{ku}) = 0\) in \(ku^\lf_\ast(E\pi)\).

We note that the natural transformation of the connected spectrum to the Eilenberg-McLane spectrum \(ku \to H(\mathbb{Z})\) takes the fundamental class \([\tilde{M}]_{ku}\) to the fundamental class \([\tilde{M}]\). Hence \(\tilde{f}_\ast([\tilde{M}]) = 0\). By Theorem 2.3 \(\dim_{mc}\tilde{M} < n\). \(\square\)
5.2. Deformation to the $(n - 2)$-skeleton.

5.3. Lemma. Let $\dim_{mc} \widetilde{M} < n$ for an oriented closed $n$-manifold $M$ with a classifying map $f : M \to B\pi$. Then its lifting $\tilde{f} : \tilde{M} \to E\pi$ admits a bounded deformation to a map $g : \tilde{M} \to E\pi^{(n-1)}$ such that $g(\tilde{M}^{(n-1)}) \subset E\pi^{(n-2)}$.

Proof. First we show that the map $\tilde{g} = f|_{\tilde{M}^{(n-1)}} : \tilde{M}^{(n-1)} \to E\pi^{(n-1)}$ admits a bounded deformation to a map $q : \tilde{M}^{(n-1)} \to E\pi^{(n-2)}$ that agrees with $\tilde{f}$ on $\tilde{M}^{(n-3)}$. In view of Proposition 2.6 and Theorem 3.6 it suffices to show that $o\tilde{g} = 0$ in $H^{n-1}_{ce}(M; \pi_{n-2}(E\pi^{(n-2)}))$. Since $o\tilde{g} = \tilde{f}^*(o_1)$ and $\tilde{f}_*([\tilde{M}]) = 0$, we obtain

$$\tilde{f}_*(\tilde{M}) \cap o\tilde{g} = \tilde{f}_*([\tilde{M}]) \cap o_1 = 0.$$ 

Since $\tilde{f}_*$ is an isomorphism in dimension 1, $[\tilde{M}] \cap o\tilde{g} = 0$. By the Poincaré duality, $o\tilde{g} = 0$.

Then we extend the map $q$ to a map $g : \tilde{M} \to E\pi^{(n-1)}$ which is in a finite distance to $\tilde{f}$. We may assume that $M$ has one $n$-cell $e$. Let $\tilde{e}$ be any lift of $e$ to $\tilde{M}$. Since $q$ is coarsely equivariant, there is a finite complex $K \subset E\pi$ such that

$$\bigcup_{\gamma \in \pi} \gamma^{-1}q(\partial\gamma \tilde{e}) \subset K.$$ 

Since $E\pi$ is contractible, there is a finite complex $F$ containing $K$ such that the inclusion $K \to F$ is nullhomotopic. Then the inclusion $K^{(n-2)} \subset F^{(n-1)}$ is nullhomotopic. Let $\psi_\gamma : \gamma \tilde{e} \to F^{(n-1)}$ be an extension of $\gamma^{-1}q$ restricted to the boundary $\partial\gamma \tilde{e}$ of the $n$-cell $\gamma \tilde{e} \subset \tilde{M}$, $\gamma \in \pi$. Then the union of maps $\cup_{\gamma \in \pi} \gamma \psi_\gamma$ and $q$ defines the required map $g$. 

We note that the Atiyah-Hirzebruch spectral sequence converges for the Steenrod homology for a finite dimensional compact metric spaces [EH]. Thus, for finite dimensional $X$ for any generalized homology theory $H_*$ there is the Atiyah-Hirzebruch spectral sequence for the locally finite homology which has the $E^2$-term $E^2_{p,q} = H^f_p(X; h_q(pt))$ and converges to $h^f_*(X)$.

5.4. Lemma. Suppose that $g : (\tilde{M}, \tilde{M}^{(n-1)}) \to (E\pi^{(n-1)}, E\pi^{(n-2)})$ is a coarsely equivariant bounded cellular map of the universal cover of an almost spin manifold with the fundamental group $\pi$. Assume that $H^i_f(E\pi) = 0$ for $0 < i < n - 1$. Then $\dim_{mc} \tilde{M} \leq n - 2$.

Proof. First we show that for every proper map $g : \tilde{M} \to E\pi^{(n-1)}$ with $g(\tilde{M}^{(n-1)}) \subset E\pi^{(n-2)}$ takes the ko-fundamental class to zero. Since
Therefore, the image \( g_\ast([\widetilde{M}]_{ko}) \) lives in the image of the \( E^2 \)-terms \( \text{im}\{g_\ast : E^2_{n-1,0}(\widetilde{M}) \to E^2_{n-1,0}(E\pi^{(n-1)})\} \). We show that the induced homomorphism

\[
g_\ast : H^i_{n-1}(\widetilde{M}; ko_0(pt)) \to H^i_{n-1}(E\pi^{(n-1)}; ko_0(pt))
\]

is zero, then it would follow that \( g_\ast([\widetilde{M}]_{ko}) = 0 \).

Note that for the one-point compactifications the quotient \( \alpha \widetilde{M}/\alpha \widetilde{M}^{(n-1)} \) is homeomorphic to the one-point compactification of an infinite wedge of \( n \)-spheres. Then \( H_{n-1}(\alpha \widetilde{M}/\alpha \widetilde{M}^{(n-1)}) = 0 \). From the Steenrod homology exact sequence of the pair \( (\alpha \widetilde{M}, \alpha \widetilde{M}^{(n-1)}) \) it follows that the inclusion homomorphism \( i_\ast : H^i_{n-1}(\widetilde{M}^{(n-1)}) \to H^i_{n-1}(\widetilde{M}) \) is an epimorphism. Since \( H^i_{n-1}(E\pi^{(n-2)}) = 0 \), the commutative diagram

\[
\begin{align*}
H^i_{n-1}(\widetilde{M}) & \xrightarrow{g_\ast} H^i_{n-1}(E\pi^{(n-1)}) \\
i_\ast & \\
H^i_{n-1}(\widetilde{M}^{(n-1)}) & \longrightarrow H^i_{n-1}(E\pi^{(n-2)})
\end{align*}
\]

implies that \( g_\ast \) is zero homomorphism.

We construct a CW complex \( L \) by replacing every \((n-1)\)-dimensional cell \( \sigma \subset E\pi^{(n-1)} \) by an \( n \)-cell \( D_\sigma \) attached by the composition \( \phi_\sigma \circ \beta : S^{n-1} \to E\pi^{(n-2)} \) where \( \beta : S^{n-1} \to S^{n-2} \) generates \( \pi_{n-1}(S^{n-2}) \). Let

\[
\psi : L \to E\pi^{(n-1)}
\]

denote the map which is obtained by extending of the identity map on \( E\pi^{(n-2)} \) by the cones of the attaching maps \( \phi_\sigma \circ \beta \).

Since the inclusion \( E\pi^{(n-2)} \subset E\pi^{(n-1)} \) is nullhomotopic, the image of \( \pi_n(E\pi^{(n-1)}, E\pi^{(n-2)}) \) under the boundary homomorphism generates \( \pi_{n-1}(E\pi^{(n-2)}) \). Thus the group \( \pi_{n-1}(E\pi^{(n-2)}) \) is generated by the boundaries of \((n-1)\)-simplices, \( \pi_{n-1}(\partial \sigma) \). Therefore any map \( \phi : S^{n-1} \to E\pi^{(n-2)} \) is nullhomotopic in \( L \). Hence the map \( g : \widetilde{M}^{(n-1)} \to E\pi^{(n-2)} \) extends to a coarsely equivariant bounded cellular map \( f : \widetilde{M} \to L \).

For every \( n \)-cell \( D_\sigma \) we may assume that there is a regular value of \( f \), i.e. there is a closed \( n \)-disk \( D'_\sigma \subset D_\sigma \), such that \( f^{-1}(D) = \bigsqcup B_i \) and the restriction of \( f \) to each \( B_i \) is a diffeomorphism between \( B_i \) and \( D'_\sigma \) for all \( i \). We join all \( B_i \) by tubes in \( \widetilde{M} \) in one \( n \)-ball \( B_\sigma \). We can do it in such a way that all \( B_\sigma \) are disjoint and uniformly bounded.

We use the notation \( \overset{\circ}{B} \) for the interior of \( B \). Let \( r : L \to L \) denote a map that deforms \( D_\sigma \setminus D'_\sigma \) to \( E\pi^{(n-2)} \) and defines a homeomorphism
of $D^\sigma_\sigma$ to $D^\sigma_\sigma$ for all $\sigma$. Thus the restriction of $r$ to $L \setminus (\cup \sigma D^\sigma_\sigma)$ is a retraction onto $E^{\pi(n-2)}$.

If the degree $d_\sigma = \sum_i \deg(f|_{B_i})$ is even, then in view of the fact that $\pi_{n-1}(S^{n-2}) = \mathbb{Z}_2$ for $n \geq 5$, the map $r \circ f|_{\partial B_\sigma}$ is nulhomotopic. Thus, if $d_\sigma$ is even for all $\sigma$, there is a bounded deformation of $g$ into $E^{\pi(n-2)}$.

Consider the commutative diagram generated by exact sequences of pairs and the maps $r \circ f$ and $\psi$:

$$
\begin{array}{cccc}
ko_n^f(\tilde{M}) & \xrightarrow{r \circ f} & ko_n^f(L) & \xrightarrow{\psi^*} & ko_n^f(E^{\pi(n-1)}) \\
\downarrow j^1 & & \downarrow j^2 & & \downarrow j^3 \\
ko_n^f(\tilde{M}, \tilde{M} \setminus B_\sigma) & \xrightarrow{r \circ f} & ko_n^f(L, L \setminus D_\sigma) & \xrightarrow{\psi^*} & ko_n^f(E^{\pi(n-1)}, E^{\pi(n-1)} \setminus \tilde{\sigma}) \\
\cong q_1^1 & & \cong \downarrow q_2^1 & & \cong q_3^1 \\
ko_n(B_\sigma/\partial B_\sigma) & \xrightarrow{d_\sigma} & ko_n(D_\sigma/\partial D_\sigma) & \xrightarrow{\Sigma_\beta^*} & ko_n(\sigma/\partial \sigma).
\end{array}
$$

As it already has been proved, the conditions of the lemma imply that $\psi_r f_* ([\tilde{M}]_{ko}) = 0$. Therefore, $\Sigma_\beta^* \circ d_\sigma \circ q_1^1 \circ j^1_* ([\tilde{M}]_{ko}) = 0$. Note that the suspension $\Sigma_\beta : S^n \to S^{n-1}$ induces the epimorphism $\Sigma_\beta_* : \mathbb{Z} \to \mathbb{Z}_2$ and $q_*^1 \circ j_*^1 ([\tilde{M}]_{ko})$ is a generator of $ko_n(B_\sigma/\partial B_\sigma) = \mathbb{Z}$. Therefore $d_\sigma$ cannot be odd.

Thus, $d_\sigma$ is even for all $\sigma$, the maps $r \circ f|_{\partial B_\sigma}$ are nulhomotopic, and there is a bounded deformation of $g$ in $E^{\pi}$ to a map $\tilde{f} : \tilde{M} \to E^{\pi(n-2)}$. By Proposition 2.4 $\dim_{mc} \tilde{M} \leq n - 2$. \qed

### 5.3. Duality groups. We recall that a group $\pi$ is called a duality group [Br] if there is a $\pi$-module $D$ such that

$$H^i(\pi, M) \cong H_{m-i}(\pi, M \otimes D)$$

for all $\pi$-modules $M$ and all $i$ where $m = cd(\pi)$ is the cohomological dimension of $\pi$. The groups that admit finite $B\pi$ are called geometrically finite or of the type FL.

### 5.5. Proposition. Let $\pi$ be a FL duality group. Then $H_i^{lf}(E\pi; \mathbb{Z}) = 0$ for all $i \neq cd(\pi)$.

**Proof.** From Theorem 10.1 of [Br] it follows that $H^i(\pi, \mathbb{Z}_\pi) = 0$ for $i \neq m = cd(\pi)$ and $H^m(\pi, \mathbb{Z}_\pi)$ is a free abelian group. In view of the equality $H_i^*(\pi, \mathbb{Z}_\pi) = H_i^*(E\pi; \mathbb{Z})$ for geometrically finite groups (see [Br] Theorem 7.5) and the short exact sequence for the Steenrod
homology of a compact metric space
\[ 0 \to \text{Ext}(H^{i+1}(X), \mathbb{Z}) \to H^i_\ell(X; \mathbb{Z}) \to \text{Hom}(H^i(X), \mathbb{Z}) \to 0 \]

applied to the one point compactification \( \alpha(E\pi) \) of \( E\pi \) we obtain that \( H^i_\ell(\alpha(E\pi); \mathbb{Z}) = 0 \) for \( i < m \). The equality \( H^i_\ell(E\pi; \mathbb{Z}) = H^i_\ell(\alpha(E\pi); \mathbb{Z}) \) completes the proof. \( \square \)

We note that every duality group \( \pi \) has type \( FP \) \([Br]\), i.e., \( B\pi \) is dominated by a finite complex. It is still an open problem whether \( FP = FL \) \([Br]\). A group \( \pi \) is called virtually \( FL \) if it contains a finite index subgroup \( \pi' \) which is \( FL \). We note that all classes of virtual duality groups listed in the introduction are virtually \( FL \).

5.6. Theorem. Suppose that the fundamental group \( \pi \) of a closed almost spin \( n \)-manifold \( M \) with positive scalar curvature, \( n \geq 5 \), is a virtual duality \( FL \) group that satisfies the coarse Baum-Connes conjecture. Then the Gromov Conjecture holds for \( M \), \( \dim_{mc} \tilde{M} \leq n - 2 \).

Proof. Let \( \pi' \) be a finite index subgroup of \( \pi \) which is a \( FL \) duality group. Since \( \pi' \) is quasi-isometric to \( \pi \), the coarse Baum-Connes conjecture holds for \( \pi' \). Let \( M' \to M \) be a covering that corresponds to \( \pi' \). Note that the metric on \( M' \) lifted from \( M \) has positive scalar curvature and \( \tilde{M}' = \tilde{M} \).

If \( n < \dim \pi' \), then Proposition 5.5 implies that \( H^i_\ell(E\pi') = 0 \) for \( 0 < i \leq n \). Thus, the condition of Theorem 5.2 is satisfied. Then \( \dim_{mc} \tilde{M}' < n \). We apply Lemma 5.3 and Lemma 5.4 to obtain the inequality \( \dim_{mc} \tilde{M}' \leq n - 2 \).

If \( n > \dim \pi + 1 \), the inequality \( \dim_{mc} \tilde{M}' \leq n - 2 \) holds automatically.

If \( n = \dim \pi + 1 \) we apply Lemma 5.3

Consider the case \( n = \dim \pi \). Let \( \tilde{f} : \tilde{M}' \to E\pi \) be a lift of the classifying map. As in the proof of Theorem 5.2 we obtain that \( \tilde{f}_*(\tilde{[M']}_{K}) = 0 \). From the Atiyah-Hirzebruch spectral sequence it follows that \( \tilde{f}_*([\tilde{M}']) = 0 \) for the integral locally finite homology. Then by Theorem 2.3

\[ \dim_{mc} \tilde{M}' < n. \]

Then we apply Lemma 5.3 and Lemma 5.4 to complete the proof. \( \square \)

It is known that the coarse Baum-Connes conjecture implies the Analytic Novikov conjecture. For spin manifolds one can replace the coarse Baum-Connes conjecture by the Analytic Novikov conjecture. We say that a group \( \pi \) virtually satisfies the Analytic Novikov conjecture if it contains a finite index subgroup \( \pi' \) that satisfies the Analytic Novikov conjecture.
5.7. Theorem. Suppose that the fundamental group $\pi$ of a closed spin $n$-manifold $M$, $n \geq 5$, with positive scalar curvature is a virtual FL duality group that virtually satisfies the Analytic Novikov conjecture. Then $\dim_{mc} \tilde{M} \leq n - 2$.

Proof. As in the proof of Theorem 5.6 we may assume that $\pi$ is a FL duality group that satisfies the Analytic Novikov conjecture. We note that the K-theory fundamental class $[M]_K$ goes under the transfer map $\text{trf}_{\tilde{M}} : K_*(M) \to K_!^f(\tilde{M})$ to the fundamental class $[\tilde{M}]_K$. By the Analytic Novikov conjecture and Rosenberg’s vanishing theorem [R1] it follows that $f_*([M]_K) = 0$ where $f : M \to BK$ is a classifying map. Therefore, $\tilde{f}_*([\tilde{M}]_K) = f_! \text{trf}_M ([M]_K) = \text{trf}_{B\pi} f_*([M]_K) = 0$ for a lift $\tilde{f}$ of $f$. The rest of the proof is the same as in Theorem 5.6. □

5.8. Remark. It is natural to expect an extension of Theorem 5.2 to the following: If $\pi$ satisfies the coarse Baum-Connes conjecture and $ko! (E\pi) \to KO! (E\pi)$ is injective then the Gromov conjecture holds true for almost spin manifolds with the fundamental group $\pi$. The weak Gromov conjecture follows in this case similarly to the proof of Theorem 5.2. The necessary tools to deal with the real K-theory are given in [Ro4]. Still, the proof of vanishing of the second obstruction looks like a very technical task due to the nature of the Steenrod generalized homology. Since it is unclear if the above injectivity condition brings new classes of groups, this goal is not pursued in the paper.

6. Rationality conjecture

Corollary 3.8 defines a subgroup of integral homology group $H_*(B\pi)$ of the macroscopically small homology classes $H^{sm}_*(B\pi)$ in spirit as Brunbauer-Hanke defined corresponding subgroups for several other classes of manifolds [BH]. A major difference is that Brunbauer and Hanke considered the rational homology. The following is closely related to the Gromov conjecture.

The Rationality Conjecture. Macroscopically small homology classes of a group $\pi$ are rational.

The precise meaning of this conjecture is that there are subgroups $H^{sm}_n(B\pi; \mathbb{Q}) \subset H_n(B\pi; \mathbb{Q})$ such that for an orientable $n$ manifold $M$ the universal covering $\tilde{M}$ is macroscopically small if and only if $f_*([M]) \in H^{sm}_n(B\pi; \mathbb{Q})$. Clearly, $H^{sm}_n(B\pi; \mathbb{Q}) = H_n^{sm}(B\pi) \otimes \mathbb{Q}$. Therefore, the Rationality Conjecture states that if $f_*([M])$ is a torsion in $H_n(B\pi)$ then $\dim_{mc} \tilde{M} < n$.

Note that an affirmative answer to Problem 4.3 implies the Rationality Conjecture.
6.1. **Theorem.** The Rationality Conjecture implies the weak Gromov conjecture for spin manifolds whose fundamental group satisfies the Novikov Conjecture.

*Proof.* It follows from Rosenberg’s theorem and the Chern character isomorphism for homology that $f_*(\{M\}) = 0$ in $H_n(B\pi; \mathbb{Q})$. Since $0 \in H_n^{sm}(B\pi)$, by the Rationality Conjecture $\tilde{M}$ might be macroscopically small. \hfill \Box

6.1. **The Stone-Čech compactification.** Let $M$ be a closed $n$-manifold and let $f : M \to B\pi$ be a classifying map of its universal cover. Since $M$ is compact, the universal covering map $p : \tilde{M} \to M$ can be extended to the continuous map of the Stone-Čech compactification $\bar{p} : \beta(\tilde{M}) \to B\pi$.

6.2. **Theorem.** For a closed $n$-manifold $M$ the following conditions are equivalent:

1. $\dim \pi_{\text{mc}}(\tilde{M}) < n$ for the universal cover $\tilde{M}$ of $M$.
2. The map $f \circ \bar{p} : \beta(\tilde{M}) \to B\pi$ of the Stone-Čech compactification can be deformed to the $(n-1)$-dimensional skeleton $B\pi^{n-1}$ of $B\pi$.

*Proof.* (1) $\Rightarrow$ (2). Let $\dim \pi_{\text{mc}}(\tilde{M}) < n$. By Proposition 2.4 a lift $\tilde{f} : \tilde{M} \to E\pi$ of a classifying map $f : M \to B\pi$ to the universal covering can be deformed to the $(n-1)$-dimensional skeleton $E\pi^{n-1}$ by a bounded homotopy. Using the uniform contractibility of $E\pi$ and the nature of the metric on it (see the proof of Proposition 3.1 in [Dr1]) we can construct a homotopy $H : \tilde{M} \times I \to E\pi$ joining $\tilde{f}$ with a map $\tilde{g} : \tilde{M} \to E\pi^{n-1}$ such that for some fixed $\lambda > 0$ the restrictions $H|_{x \times I}$ are $\lambda$-Lipschitz for all $x \in X$. Since the metric on $B\pi$ is proper, it follows that $p \circ H(\tilde{M} \times I)$ lies in a compact subcomplex $L$. By the Ascoli-Arzelà theorem the space $\text{Map}_\lambda(I, L)$ of all $\lambda$-Lipschitz maps $\phi : I \to L$ is compact. The map $p \circ H$ induces a continuous map $h : \tilde{M} \to \text{Map}_\lambda(I, L)$ to a compact space. Therefore it admits a continuous extension $\tilde{h} : \beta(\tilde{M}) \to \text{Map}_\lambda(I, L)$ to the Stone-Čech compactification. The map $\tilde{h}$ defines a homotopy $H : \beta(\tilde{M}) \times I \to L$ that joins $f \circ \bar{p}$ with a map to $B\pi^{n-1}$.

(2) $\Rightarrow$ (1). We define a function $\Psi : B\pi^I \to \mathbb{R}$ on the space of all paths $B\pi^I$ as follows: for every path $\phi : I \to B\pi$ we set $\Psi(\phi)$ to be the diameter of $\phi'(I)$ for a lift $\phi'$ of $\phi$. Since the metric on $E\pi$ is $\pi$-invariant, $\Psi(\phi)$ does not depend on the choice of $\phi'$. We leave to the reader to show that $\Psi$ is continuous.

A deformation $H : \beta(\tilde{M}) \to B\pi$ of $f \circ \bar{p}$ to a map $g : \beta(\tilde{M}) \to B\pi^{n-1}$ defines a continuous map $h : \beta(\tilde{M}) \to B\pi^I$. In view of compactness...
of $\beta(\tilde{M})$ the function $\Psi \circ h$ is bounded. Then a lift of the homotopy $H|_{\tilde{M} \times I} : \tilde{M} \times I \to B\pi$ defines a bounded homotopy of $\tilde{f}$ to the $(n-1)$-skeleton $E\pi^{(n-1)}$. Proposition 2.4 completes the proof. □

6.3. Conjecture. For every simply connected open $n$-manifold $N$ and any locally constant sheaf $S$ on the Stone-Čech compactification $\beta(N)$ with a free abelian group as the stalk, the cohomology group $H^n(\beta(N), S)$ is torsion free.

Since $H^n(N) = 0$, it follows from [CS] that $H^n(\beta(N), S) = 0$ in the case when $S$ is a constant sheaf.

6.4. Proposition. Conjecture 6.3 implies the Rationality Conjecture.

Proof. Suppose that $f_\ast([M]) = 0$ in $H_n(B\pi; \mathbb{Q})$. Thus, $f_\ast([M])$ is a torsion in $H_n(B\pi)$. Since $\dim \beta(\tilde{M}) = n$, The primary obstruction $\tilde{o}$ to deform $f \circ \tilde{p} : \beta(\tilde{M}) \to B\pi$ to $B\pi^{(n-1)}$ is the only obstruction and it lives in the $n$-dimensional cohomology group with locally constant coefficient system $(f \circ \tilde{p})^*(S)$ where $S$ is the system on $B\pi$ that corresponds to the $\pi$-module $\pi_{n-1}(B\pi^{(n-1)})$. Note that the stalk of $C$ is a free abelian group. Note that $\tilde{\partial} = \tilde{p}^*(o_f)$ where $o_f$ is the primary obstruction to deform $f$ into $B\pi^{(n-1)}$. Since $f_\ast([M])$ is a torsion, it follows that $o_f$ has finite order. Therefore $\tilde{o}$ has finite order. Then Conjecture 6.3 implies that $\tilde{\partial} = 0$. Then by Theorem 6.2 $\dim_{mc} \tilde{M} < n$. □

6.2. The Berstein-Schwarz cohomology class. We recall that the Berstein-Schwarz class $[Sw], [Be]$ $b = b_\pi \in H^1(\pi, I(\pi))$ of a group $\pi$ is defined by the cochain on the Cayley graph $\phi : G \to I(\pi)$ which takes an ordered edge $[g, g']$ to $g' - g$. Here $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$. We use notation $I(\pi)^k$ for the $k$-times tensor product $I(\pi) \otimes \cdots \otimes I(\pi)$ over $\mathbb{Z}$. Then the cup product $b^k = b \smile \cdots \smile b$ is defined as an element of $H^k(\pi, I(\pi)^k)$.

The Berstein-Schwarz class is universal in the following sense.

6.5. Theorem ([Sw], [DR]). For every $\pi$-module $L$ and every element $\alpha \in H^k(\pi, L)$ there is a $\pi$-homomorphism $\xi : I(\pi)^k \to L$ such that $\xi^*(b^k) = \alpha$ where

$$\xi^* : H^k(\pi, I(\pi)^k) \to H^k(\pi, L)$$

is the coefficient homomorphism.

By bringing in the Berstein-Schwarz class we can extend Theorem 2.3 to the following.
6.6. **Theorem.** For a closed oriented $n$-manifold $M$ with the classifying map $f : M \to B\pi$ and its lift to the universal covers $\tilde{f} : \tilde{M} \to E\pi$ the following are equivalent:

1. $\dim_{mc}\tilde{M} < n$;
2. $\tilde{f}_*(\lbrack\tilde{M}\rbrack) = 0$ in $H^i_{nf}(E\pi;\mathbb{Z})$ where $\lbrack\tilde{M}\rbrack \in H^i_{nf}(\tilde{M};\mathbb{Z})$ is the fundamental class of $\tilde{M}$;
3. $f_*(\lbrack M\rbrack) \in \ker(ec^n_\pi)$ where $\lbrack M\rbrack$ is the fundamental class of $M$;
4. $f^*(b^n) \in \ker(ec^*_M)$ where $b$ is the Berstein-Schwarz class of $\pi$.

**Proof.**

1. $\Rightarrow$ 2. We may assume that $f : M \to B\pi$ is cellular and Lipschitz for some metric CW complex structure on $B\pi$. If $\dim_{mc}\tilde{M} < n$, then by Proposition 2.4 there is a bounded cellular homotopy of $\tilde{f} : \tilde{X} \to E\pi$ to a map $g : \tilde{X} \to E\pi^{(n-1)}$ with a compact projection to $B\pi$. By Proposition 3.7, it is coarsely equivariant. Then by Proposition 3.3 it follows that $f_*(\lbrack\tilde{M}\rbrack) = 0$.

2. $\Rightarrow$ 3. $ec^n_\pi(f_*(\lbrack M\rbrack)) = \tilde{f}_*(\text{Pert}^M_\pi(\lbrack M\rbrack)) = 0$ and hence, $f_*(\lbrack M\rbrack) \in \ker(ec^n_\pi)$.

3. $\Rightarrow$ 4. If $f_*(\lbrack M\rbrack) \in \ker(\text{Pert}^\pi_\pi)$, then $ec^n_\pi(f_*(\lbrack M\rbrack) \cap b^k) = 0$. Since the commutative diagram

$$
\begin{array}{ccc}
H^i_{nf}(\tilde{M};I(\pi)^n) & \xrightarrow{\tilde{f}_*} & H^i_{nf}(E\pi;I(\pi)^n) \\
ec^*_M \uparrow & & \ec^n \uparrow \\
H_0(M;I(\pi)^n) & \xrightarrow{f_*} & H_0(B\pi;I(\pi)^n)
\end{array}
$$

has isomorphisms for horizontal arrows, $ec^*_M(\lbrack M\rbrack \cap (f^*(b^n))) = 0$. Thus, $ec^*_M(\lbrack M\rbrack) \cap ec^*_M(f^*(b^n)) = 0$. By the Poincare Duality, $ec^*_M(b^n) = 0$.

4. $\Rightarrow$ 1. We show that the obstruction $o_f$ to the inequality $\dim_{mc}\tilde{M} < n$ is zero and apply Theorem 3.6. By Theorem 6.5, there is a $\pi$-homomorphism $\xi : I(\pi)^n \to L = \pi_{n-1}(B\pi^{(n-1)})$ such that $\xi^*(b^n) = \kappa_1$ and $\xi^*(f^*(b^n)) = o_f$ where $\kappa_1$ is the primary obstruction to retract $B\pi$ onto $B\pi^{(n-1)}$ and $o_f$ is the primary obstruction do deform $f$ into $B\pi^{(n-1)}$. Then $o_\tilde{f} = ec^*_M(o_f) = \xi^*ec^*_M(b^n) = 0$. \qed

Every manifold $M$ with the fundamental group $\pi$ carries a local coefficients system $\mathcal{T}^n$ generated by the $\pi$-module $I(\pi)^n$.

6.7. **Proposition.** Suppose that Conjecture 6.3 holds true for the universal cover $\tilde{M}$ of a closed rationally inessential $n$-manifold $M$ with $S = \overline{p^*\mathcal{T}^n}$. Then $\dim_{mc}\tilde{M} < n$.

**Proof.** In view of Theorem 6.2 it suffices to show that the primary obstruction $\bar{o}$ to deform $f \circ \overline{\rho} : \beta(\tilde{M}) \to B\pi$ to $B\pi^{(n-1)}$ is zero. Note that
\( \tilde{o} = \bar{p}^* f^*(o_1) \) where \( o_1 \in H^n(\pi(B); S) \) is the primary obstruction to retraction of \( B\pi \) to \( B\pi^{(\alpha-1)} \). By the universality of the Berstein-Schwarz class there is a morphism of local coefficient systems \( I^\alpha \to S \) over \( B\pi \) such that the induced homomorphism of \( n \)th cohomology takes \( b^n \) to \( o_1 \). Then \( \bar{p}^* f^*(b^n) \) is taken to \( \tilde{o} \) by the corresponding homomorphism. Since \( f_*([M]) \) is a torsion, it follows that \( f^*(b^n) \) has finite order. Therefore \( \tilde{o} \) has finite order. Then Conjecture 6.3 for the sheaf \( \bar{p}^* I^\alpha \) implies that \( \tilde{o} = 0 \). \( \square \)

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