Abstract

We cast new light on the existing models of 1-way deterministic topological automata by introducing a new, convenient model, in which, as each input symbol is read, an interior system of an automaton, known as a configuration, continues to evolve in a topological space by applying continuous transition operators one by one. The acceptance and rejection of a given input are determined by observing the interior system after the input is completely processed. Such automata naturally generalize 1-way finite automata of various types, including deterministic, probabilistic, quantum, and pushdown automata. We examine the strengths and weaknesses of the power of this new automata model when recognizing formal languages. We investigate tantalizing effects of various topological features of our topological automata by analyzing their behaviors when different kinds of topological spaces and continuous maps, which are used respectively as configuration spaces and transition operators, are provided to the automata.

Keywords: topological automata, topological space, continuous map, compact, discrete topology, Kolmogorov separation axiom, quantum finite automata

1 Prelude: Background and Challenges

1.1 A Historical Account of Topological Automata

In the theory of computation, finite-state automata (finite automata, or even automata) are one of the simplest and most intuitive mathematical models to describe “mechanical procedures,” each of which depicts a finite number of “operations” in order to determine the membership of any given input word to a fixed language. Such procedures have resemblance of physical systems, which make discrete time evolution, contrary to continuous time evolution. Over decades of their study, these machines have found numerous applications in the fields of engineering, physics, biology, and even economy (see, e.g., [13]). In particular, a one-way finite automaton reads input symbols one by one and then processes them by changing a status of the automaton’s interior system step by step. This machinery models the computation of streamlined process, in which it receives streamlined input data and processes it piece by piece by applying operations predetermined for each of the input symbols. For such a machine, a computation is a description of a series of “evolutions” of the interior system.

To cope with numerous computational problems, various types of finite automata have been proposed as their appropriate computational models in the past literature. In the 1970s, many features of the existing 1-way finite automata were generalized into so-called “topological automata” (see [8] for early expositions and references therein). A topology is a mathematical concept of dealing mostly with open sets and continuous maps that preserve the openness of point sets. More general automata were also defined in terms of category in, e.g., [10]. Topological automata embody characteristic features of various types of finite automata and this fact has helped us take a unified approach toward the study of formal languages and automata theory. The analysis of topological features of the topological automata thus guides us to the better understandings of the theory itself.

Back in the 1970s, Brauer (see references in [8]) and Ehrig and Künnel [8] discussed topological automata as a topological generalization of Mealy machines, which behave as “transducers,” which simply produce outputs from inputs. In contrast, following a discussion of Bozapalidis [5] on a generalization of stochastic functions and quantum functions (see also [25]), Jeandel [15] studied another type of topological automata that behave as “acceptors” of inputs. Jeandel’s model naturally generalizes not only probabilistic finite automata [21] but also measure-once quantum finite automata [18]. The main motivation of Jeandel’s work,
nonetheless, was to study a nondeterministic variant of quantum finite automata and he thus used his topological automata to obtain an upper-bound of the language recognition power of nondeterministic quantum finite automata. Concerning the types of “inputs” fed into topological automata, in contrast, Ehrg and Kühnel [8] applied a quite general framework to inputs, which are taken from arbitrary compactly generated Hausdorff spaces, whereas Jeandel [15] used the standard framework with finite alphabets and languages generated over them. Jeandel further took “measures” (which assign real numbers to final configurations) to determine the acceptance or rejection of inputs. Since we are more concerned with the computational power of topological automata in comparison with the existing finite automata, we wish to make our model as simple and intuitive as possible by introducing, unlike the use of measures, sets of accepting and rejecting configurations, into which the machine’s interior system finally fall.

Given an input string over a fixed alphabet Σ, the evolution of an interior status of our topological automaton is described in the form of a series of configurations, which constitutes a computation of the machine. A list of transition operators thus serves as a “program”, which completely dictates the behaviors of the machine on each input. Since arbitrary topological spaces can be used as configuration spaces, topological automata are no longer “finite-state” machines; however, they evolve sequentially as they read input symbols one by one until they completely read the entire inputs and final configurations are observed once (referred to as an “observe once” feature). Moreover, our topological automata enjoy a “deterministic” nature in the sense that which transition operators are applied to the current configurations is completely determined by input symbols alone. This gives rise to “1-way deterministic topological automata” (or 1dta’s, in short). Although their tape heads move in one direction from the left to the right, 1dta’s turn out to be quite powerful in recognizing formal languages. By extending transition maps to “multi-valued” maps, it is possible to consider nondeterministic moves of topological automata [15].

1.2 A New Model of Topological Automata

All the aforementioned models of topological automata are based only on a relatively small range of appropriately defined topologies, such as compactly generated Hausdorff spaces. We instead wish to study all possible topologies with no initial restrictions other than discrete applications of transition operators.

This paper thus aims at shedding new light on the basic structures of topological automata and the acting roles of their transition operators that force configurations to evolve consecutively. For this purpose, we start our study with a suitable abstraction of 1-way finite automata using arbitrary topological spaces for configurations and arbitrary continuous maps for transitions. Such an abstraction serves as a skeleton to construct our topological automata. We call this skeleton an automata base. Since the essential behaviors of topological automata are strongly influenced by the choice of their automata bases, we are mostly concerned with the properties of these automata bases.

In general, the choice of topologies significantly affects the computational power of topological automata. As shown later, the trivial topology induces the language family composed only of Ø and Σ* (for each fixed alphabet Σ) whereas the discrete topology allows topological automata to recognize arbitrary complex languages. All topologies on a fixed space V form a complete lattice; thus, it is possible to classify the topologies according to the endowed power of associated topological automata.

Initially, a study on topological automata should be focused on achieving the following four key goals.

1. Understand how various choices of topological spaces and continuous maps affect the computational power of underlying machines by clarifying the strengths and weaknesses of the language recognition power of the machines.
2. Determine what kinds of topological features of topological automata nicely characterize the existing finite automata of various types by examining the descriptive power of such features.
3. Explore different types of topological automata to capture fundamental properties (such as closure properties) of formal languages and finite automata.
4. Find useful applications of topological automata to other fields of science.

Organizational of the Paper. In Section 2 we will formulate our basic model of 1dta’s. These automata are naturally induced from automata bases and they can express numerous types of the existing 1-way finite automata. Through Section 4 we will discuss basic properties of the 1dta’s, including closure properties and the elimination of two endmarkers. Following an exploration of such basic properties, we will compare different topologies in Section 5 by measuring how much computational power is endowed to underlying
topological automata. In Section 4 we will show that unique features of well-known topological concepts, such as compactness and equicontinuity, help us characterize 1-way deterministic finite automata (or 1dfa’s). We will lay out a necessary and sufficient condition on a topological space for which underlying machines are no more powerful than 1dfa’s. In Section 7, we will consider a nondeterministic variant of our topological automata (called Inta’s) by introducing multi-valued transition operators. It is known that, for weak machine models, nondeterministic machines can be easily simulated by deterministic ones. By formalizing this situation, we will argue what kind of topology makes Inta’s simulated by 1dfa’s.

We strongly hope that this work initiates a systematic study on the significant roles of topologies played by topological automata and also it leads to better understandings of ordinary finite automata in the end.

2 Basics of Topologies and Automata Bases

One-way deterministic topological automata can represent the existing one-way finite automata of numerous types. We begin our study of such powerful automata by describing their “basic” framework, which we intend to call an automata base. In what follows, we will provide the fundamental notion of automata bases, which are founded solely on topologies.

2.1 Numbers, Sets, and Languages

Let \(\mathbb{Z}, \mathbb{R},\) and \(\mathbb{C}\) respectively indicate the sets of all integers, of all real numbers, and of all complex numbers. Given a real number \(e \geq 0,\) let \(\mathbb{C}^{\leq e} = \{a \in \mathbb{C} | |a| \leq e\} .\) We denote by \(\mathbb{N}\) the set of all natural numbers (i.e., nonnegative integers) and set \(\mathbb{N}^+\) to be \(\mathbb{N} - \{0\} .\) For any two integers \(m\) and \(n\) with \(m \leq n,\) an integer interval \([m,n]\) expresses the set \([m,m+1,m+2,\ldots,n]\) in contrast with a real interval \([\alpha, \beta]\) for \(\alpha, \beta \in \mathbb{R} .\) We further abbreviate \([1,n]\) as \([n]\) for any number \(n \in \mathbb{N}^+.\)

An alphabet refers to a nonempty finite set \(\Sigma\) of “symbols” or “letters.” A string over an alphabet \(\Sigma\) is a finite sequence of symbols in \(\Sigma\) and the length \(|x|\) of a string \(x\) is the total number of symbols used to form \(x\). In particular, the empty string \(\lambda\) is a string of length 0 and is denoted by \(\lambda\). Given two strings \(x\) and \(y\) over the same alphabet, \(x\) is an initial segment of \(y\) if \(y = xz\) holds for a certain string \(z\). For each number \(n \in \mathbb{N}, \Sigma^n\) denotes the set of all strings of length exactly \(n;\) moreover, we set \(\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n\) and a language over \(\Sigma\) is a subset of \(\Sigma^*\). A language is called unary (or tally) if it is defined over a single-letter alphabet. Given a language \(L\) over \(\Sigma,\) we use the same symbol \(L\) to denote its characteristic function; that is, for any \(x \in \Sigma^*, L(x) = 1\) if \(x \in L\) and \(L(x) = 0\) otherwise. For two languages \(A\) and \(B\) over \(\Sigma,\) the notation \(AB\) denotes \(\{xy | x \in A, y \in B\}\). In particular, when \(A\) is a singleton \(\{s\},\) we write \(sB\) in place of \(\{s\}B\). Similarly, we write \(As\) for \(A\{s\}\). The reversal of a string \(x = x_1x_2\cdots x_n\) with \(x_i \in \Sigma\) for any \(i \in [n]\) is \(x_\pi x_{\pi -1} \cdots x_2x_1\) and is denoted by \(x^R\).

Given a set \(X,\) the notation \(\mathcal{P}(X)\) denotes the power set of \(X,\) i.e., the set of all subsets of \(X,\) and \(\mathcal{P}(X)^+\) expresses \(\mathcal{P}(X) - \{\emptyset\}\). A monoid \(C\) is a semigroup with an identity \(I\) in \(C\) and an associative binary operator \(\cdot\) on \(C.\)

2.2 Topologies and Related Notions

Let us review basic terminology in the theory of general topology (or point-set topology). Given a set \(V\) of points, a topology \(T_V\) on \(V\) is a collection of subsets of \(V,\) called open sets, such that \(T_V\) satisfies the following three axioms: (1) \(\emptyset, V \in T_V,\) (2) any (finite or infinite) union of sets in \(T_V\) is also in \(T_V,\) and (3) any finite intersection of sets in \(T_V\) belongs to \(T_V.\) Hence, \(T_V\) is a subset of \(\mathcal{P}(V).\) With respect to \(V,\) the complement of each open set \(V\) is called a closed set. Moreover, a closed set is a set that is both open and closed. Clearly, \(\emptyset\) and \(V\) are clopen with respect to \(V.\) A neighborhood of a point \(x \in V\) is a set in \(T_V\) that contains \(x.\) Often, we write \(N_x\) to indicate a neighborhood of \(x.\) A topological space is a pair \((V, T_V).\) When \(T_V\) is clear from the context, we omit \(T_V\) and simply call \(V\) a topological space. For a practical reason, we implicitly assume that \(V \neq \emptyset\) throughout this paper. For two topological spaces \((V_1, T_{V_1})\) and \((V_2, T_{V_2}),\) we say that \((V_2, T_{V_2})\) is finer than \((V_1, T_{V_1})\) (also \((V_1, T_{V_1})\) is coarser than \((V_2, T_{V_2})\)) if both \(V_1 \subseteq V_2\) and \(T_{V_1} \subseteq T_{V_2}.\) In such a case, we write \((V_1, T_{V_1}) \subseteq (V_2, T_{V_2}),\) or simply \(T_{V_1} \subseteq T_{V_2}\) if \(V_1\) and \(V_2\) are clear from the context. For a topological space \(V,\) a basis of its topology \(T_V\) is a set \(B\) of subsets of \(V\) such that every open set in \(T_V\) is expressed as a union of sets of \(B.\) In this case, we say that the basis \(B\) induces the topology \(T_V.\) Given two topological spaces \(V\) and \(W,\) the product topology (or Tychonoff topology) \(T_{V \times W}\) on the Cartesian product \(V \times W\) is the topology induced by the basis \(\{A \times B | A \in T_V, B \in T_W\}\). We further write
$T_V^+$ for $T_V - \{O\}$.

Take a point set $V$ and consider all possible topologies on $V$. Let $T(V)$ denote the collection of all topologies $T_V$ on $V$. This set $T(V)$ forms a complete lattice in which the meet and the join of a collection $A$ of topologies on $V$ correspond to the intersection of all elements in $A$ and the meet of the collection of all topologies on $V$ that contain every element of $A$.

There are two typical topologies on $V$: the trivial topology $T_{\text{trivial}}(V) = \{O, V\}$ and the discrete topology $T_{\text{discrete}}(V) = P(V)$. Notice that any topology $T_V$ on $V$ is located between $T_{\text{trivial}}(V)$ and $T_{\text{discrete}}(V)$ in the lattice $T(V)$.

A map $B$ from a topological space $V$ to another topological space $W$ is said to be continuous if, for any $v \in V$ and any neighborhood $N$ of $B(v)$, there exists a neighborhood $N'$ of $v$ satisfying $B(N') \subseteq N$, where $B(N') = \{B(w) \mid w \in N'\}$. Given a set $B$ of continuous maps, the notation $C_B(V)$ denotes the set of all continuous maps in $B$ on $V$ (i.e., from $V$ to itself) together with a certain given topology, expressed as $T_{C_B(V)}$. When $B$ is the set of all continuous maps on $V$, we often omit subscript $B$ from $C_B(V)$ and $T_{C_B(V)}$.

2.3 Automata Bases

In 1970s, topological automata were sought to take inputs from arbitrary topological spaces (e.g., [8]). Although such a general treatment of topological automata provides a bird-eye view of a topological landscape of a standard setting of formal languages and automata theory, as noted in Section 1, we wish to limit our interest within fixed discrete alphabets because our intention is to compare the language recognition power of topological automata with the existing finite automata that deal only with languages over small discrete alphabets.

To discuss structures of topological automata, we first introduce a fundamental notion of “automata base,” which is a skeleton of various topological automata introduced in Section 3. We are now ready to introduce a fundamental concept of automata base used as a foundation to our model of topological automata. A left act $(V, \bullet)$ over $C_B(V)$ with a continuous map $\circ$ on $C_B(V)$ satisfies that $(B_1 \circ B_2) \bullet v = B_1 \bullet (B_2 \bullet v)$ and $I \bullet v = v$ for any $v \in V$ and any $B_1, B_2 \in C_B(V)$.

Automata Bases. A triplet $(V, B, O)$ is called an automata base if $V, B,$ and $O$ satisfy the following three conditions.

1. $V$ is a set of certain topological spaces (which are called configuration spaces).
2. $B$ is a set of continuous maps (called transition operators) from any space $V$ in $V$ to itself for which
   (i) $(C_B(V), \circ)$ is a monoid with a multiplication operator $\circ$,
   (ii) $\circ$ is a continuous map on $C_B(V)$,
   (iii) $(V, \bullet)$ is a left act over $C_B(V)$ with $\circ$, and
   (iv) $\bullet$ must be a continuous map on $V$.
3. $O$ is a set of observable pairs $(E_{\text{acc}}, E_{\text{rej}})$, both of which are clopen sets in a certain space $V$ in $V$ (where $E_{\text{acc}}$ and $E_{\text{rej}}$ are respectively called by an accepting space and a rejecting space).

We often say that an automata base $(V, B, O)$ is reasonable if $V, B,$ and $O$ are all nonempty. In the rest of this paper, we will deal only with reasonable automata bases.

It is often convenient to deal with $(V, B)$ without $O$. A pair $(V, B)$ is conveniently called a sub-automata base. For operators $A, B$ in $B$ and a point $v$ of $V$, we simply write $A(v)$ or even $Av$ for $A \bullet v$ and we also abbreviate $A \circ B$ as $AB$. Note that $AB(v) = (A \circ B)(v) = A(B(v))$ for every $v \in V$. Given an operator $B$, we say that $O$ is closed under $B$ if $(B(E_1), B(E_2)) \in O$ holds for any pair $(E_1, E_2) \in O$. Given a “property” $P$ associated with topological spaces, we say that $V$ satisfies $P$ if all topological spaces in $V$ satisfy $P$.

3 One-Way Deterministic Topological Automata

We formally describe our model of one-way deterministic topological automata (or 1dta’s, for short), which are based on the choice of automata bases. To demonstrate the expressiveness of 1dta’s, we show how the existing finite automata of various types can be redefined in terms of our topological automata.

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4In this paper, we demand the clopenness of $E_{\text{acc}}$ and $E_{\text{rej}}$. It is, however, possible to require only the openness.

5This informal term “property” is used in a general sense throughout this paper, not limited to “topological properties,” which usually means the “invariance under homeomorphisms.”
3.1 Basic Models of \((V,B,O)\)-1dta's

Formally, let us introduce our model of topological automata, each of which reads input symbols taken from a fixed alphabet, modifies configurations in a deterministic manner, and finally observes the final configurations to determine the acceptance or rejection of the given inputs. Concerning the last step of observation, a similar situation appears in quantum computation as “measurement.” Our model is “observe once” because we observe only the final configuration after a computation terminates instead of observing configurations at every move of the automaton.

Hereafter, let \((V,B,O)\) denote an arbitrary reasonable automata base. Here, we use two endmarkers \(\xi\) (left endmarker) and \(\$\) (right endmarker) that surround each input string \(x\) as \(\xi x\$\).

**Framework of \((V,B,O)\)-1dta's.** Assuming an arbitrary input alphabet \(\Sigma\) with \(\xi,\$\notin\Sigma\), let us define a basic model of our topological automata. An \(I\)-way (observe-many\(^6\)) deterministic \((V,B,O)\)-topological automaton with the endmarkers (succinctly called a \((V,B,O)\)-1dta) \(M\) is a tuple \((\Sigma,\{\xi,\$\},V,\{B_\sigma\}_{\sigma\in\Sigma};v_0,E_{\text{acc}},E_{\text{rej}})\), where \(\Sigma = \Sigma \cup \{\xi,\$\}\) is an extended alphabet, \(V\) is a configuration space in \(V\) with a certain topology \(TV\) on \(V\), \(v_0\) is the initial configuration in \(V\), each \(B_\sigma\) is a transition operator in \(B\) acting on \(V\), and \((E_{\text{acc}},E_{\text{rej}})\) is an observable pair in \(O\) for \(V\) satisfying the exclusion principle: \(E_{\text{acc}}\) and \(E_{\text{rej}}\) are disjoint (i.e., \(E_{\text{acc}}\cap E_{\text{rej}} = \emptyset\)). For convenience, let \(E_{\text{non}} = V - (E_{\text{acc}}\cup E_{\text{rej}})\).

Notice that the use of the endmarkers helps us avoid an introduction of a special transition operator associated with \(\lambda\) (the empty string). Without endmarker, by contrast, 1dta's must process an input string with no knowledge of the end of the string.

Our definition of 1dta's is different from the existing ones stated in Section\(^4\) in the following ways. Ehrig and Künnel\(^8\) took compactly generated Hausdorff spaces for our \(\Sigma\) and \(V\). Jeandel\(^13\) took a metric space for \(V\) and also used a measure, which maps \(V\) to \(\mathbb{R}\) instead of our observable pair \((E_{\text{acc}},E_{\text{rej}})\).

Notice that the use of the endmarkers helps us avoid an introduction of a special transition operator associated with \(\lambda\) (the empty string). Without endmarker, by contrast, 1dta's must process an input string with no knowledge of the end of the string.

**Configurations and Computation.** Let \(x = x_1 x_2 \cdots x_n\) be an input string of length \(n\) in \(\Sigma^*\). We set \(\tilde{x} = x_0 x_1 \cdots x_{n+1}\) to be an endmarked input string, which includes \(x_0 = \xi\) (left endmarker) and \(x_{n+1} = \$\) (right endmarker). This \(\tilde{x}\) can be considered as a string over \(\Sigma\).

The machine \(M\) works as follows. A configuration of \(M\) on \(x\) is a point of \(V\). A configuration in \(E_{\text{acc}}\) (resp., \(E_{\text{rej}}\)) is called an accepting configuration (resp., a rejecting configuration). Both accepting and rejecting configurations are collectively called halting configurations. We begin with the initial configuration \(v_0\) in \(V\), which is the 0-th configuration of \(M\) on \(x\). At the 1st step, we apply \(B_1\) and obtain \(v_1 = B_1(v_0)\). For an index \(i \in [n]\), we assume that \(v_i\) is the \(i\)th configuration of \(M\) on \(x\). At Step \(i + 1\) (\(0 \leq i \leq n\)), the \(i+1\)-th configuration \(v_{i+1}\) is obtained from \(v_i\) by applying the operator \(B_{\sigma_i}\) corresponding to \(x_i\); that is, \(v_{i+1} = B_{\sigma_i}(v_i)\). For any series \(\sigma_1,\sigma_2,\ldots,\sigma_{i-1},\sigma_i \in \Sigma\), we abbreviate as \(B_{\sigma_1\sigma_2\cdots\sigma_i}\) the multiplication \(B_{\sigma_i}B_{\sigma_{i-1}}\cdots B_{\sigma_2}B_{\sigma_1}\). Notice that, since \(B\) is a monoid with the multiplication, \(B_{\sigma_1\sigma_2\cdots\sigma_i}\) also belongs to \(B\). The final configuration \(v_{n+1}\) is obtained from \(v_{n+1}\) by \(v_{n+1} = B_\emptyset(v_{n+1})\) and it coincides with \(B_\emptyset\emptyset\emptyset(v_0)\). The obtained series of configurations, \((v_0,v_1,\ldots,v_{n+1})\), is called a computation of \(M\) on the input \(x\). When a 1dta has no endmarker, by contrast, a computation \((v_0,v_1,\ldots,v_{n+1})\) is simply generated by \(v_i = B_{\sigma_i}(v_{i-1})\) for each \(i \in [n]\), and the final configuration \(v_{n+1}\) coincides with \(B_\emptyset(v_0)\).

**Acceptance and Rejection.** Finally, we determine whether a 1dta accepts and rejects each input string by checking whether the final configuration \(v_{n+2}\) falls into \(E_{\text{acc}}\) and \(E_{\text{rej}}\), respectively. To be precise, we say that \(M\) accepts (resp., rejects) \(x\) if \(v_{n+2} \in E_{\text{acc}}\) (resp., \(v_{n+2} \in E_{\text{rej}}\)). We say that \(M\) recognizes \(L\) if, for every string \(x \in \Sigma^*\), the following two conditions are met: (1) if \(x \in L\), then \(M\) accepts \(x\) and (2) if \(x \notin L\), then \(M\) rejects \(x\). We use the notation \(L(M)\) to denote the language that is recognized by \(M\).

We define \((V,B,O)\)-1DITA to be the family of all languages, each of which is defined over a certain alphabet \(\Sigma\) and is recognized by a certain \((V,B,O)\)-1dta.

Two 1dta's \(M_1\) and \(M_2\) having the common \(\Sigma\) and \(V\) are said to be (computationally) equivalent if \(L(M_1) = L(M_2)\). This relation satisfies reflexivity, symmetry, and transitivity.

For two topological spaces \(V_1\) and \(V_2\), \(V_1\) is homeomorphic to \(V_2\) by a map \(f\) if (i) \(f\) is a bijection, (ii) \(f\) is continuous, and (iii) the inverse function \(f^{-1}\) is continuous. This function \(f\) is called a homeomorphism. Let \((V,B,O)\) be any reasonable automata base. For each index \(i \in \{1,2\}\), let \(M_i = (\Sigma,\{\xi,\$\},V_i,\{B_{\sigma,i}\}_{\sigma\in\Sigma};v_{i0},E_{i,\text{acc}},E_{i,\text{rej}})\) denote an arbitrary \((V,B,O)\)-1dta. We say that \(M_1\) and

\(^6\)It is possible to consider an observe-many model of 1dta in which, at each step, the 1dta checks if, at each step, the current configuration falls into \(E_{\text{acc}}\cup E_{\text{rej}}\). For a further discussion, refer to Section\(^8\)
$M_2$ are homeomorphic if (i) $V_1$ is homeomorphic to $V_2$ via a certain homeomorphism $f$ with $f(v_{1,0}) = v_{2,0}$, (ii) $B_{1,\tau}$ and $B_{2,\tau}$ satisfy that $B_{1,\tau}(v) = v$ implies $B_{2,\tau}(f(v)) = f(w)$, and (iii) for each $\tau \in \{ \text{acc, rej} \}$, $E_{1,\tau}$ is homeomorphic to $E_{2,\tau}$ via $f|_{E_{1,\tau}}$ (i.e., $f$ restricted to $E_{1,\tau}$).

As shown below, two homeomorphic 1dta’s must recognize exactly the same language.  

**Lemma 3.1** Let $(V, B, O)$ be any reasonable automata base. Let $M_i$ be any $(V, B, O)$-1dta for each $i \in \{1, 2\}$. If $M_1$ is homeomorphic to $M_2$, then $M_1$ and $M_2$ are computationally equivalent.  

**Proof.** For two given $(V, B, O)$-1dta’s $M_1$ and $M_2$, let $f$ be a homeomorphism from $M_1$ to $M_2$. We intend to show that $L(M_1) = L(M_2)$. Let $x$ be any input string. By induction, it is possible to prove that, for any $k \in [0, n+1]_Z$ and any $v \in V$, $B_{1,\tau x_1 \ldots x_k}(v_{1,0}) = v_{1,0}$ if $B_{2,\tau x_1 \ldots x_k}(f(v_{1,0})) = f(v_{1,0})$, provided that $x_0 = \xi$ and $x_{n+1} = \xi$. Thus, if $x \in L(M_1)$, then $B_{1,\tau x}\left(v_{1,0}\right) = v_{acc} \in E_{1,\tau \text{acc}}$ and thus $B_{1,\tau}(v) = v_{acc}$ follows, where $v_{1,0} = B_{1,\xi}(v_{1,0})$. This implies $f(v) = B_{2,\tau x}\left(f(v_{1,0})\right)$ and $B_{2,\tau}(f(v)) = f(v_{acc})$. Therefore, we obtain $B_{2,\tau x}(f(v_{1,0})) = B_{2,\tau}(f(v)) = f(v_{acc})$. Since $E_{1,\text{acc}}$ is homeomorphic to $E_{2,\text{acc}}$ via $f|_{E_{1,\text{acc}}}$, it follows that $f(v_{acc}) \in E_{2,\text{acc}}$. This yields the conclusion that $x \in L(M_2)$.

By a similar argument, we also conclude that $x \notin L(M_1)$ implies $x \notin L(M_2)$. $\square$

Hence, we can freely identify all 1dta’s that are homeomorphic to each other.

### 3.2 Conventional Finite Automata are 1dta’s

Our topological-automata framework naturally extends the existing 1-way finite automata of various types. To see this fact, let us demonstrate that typical models of 1-way finite automata can be nicely fit into our framework. This demonstrates the usefulness of our computational model.

As concrete examples, we here consider only the following types of well-known finite automata used in the past literature. To relate to the definition of topological automata, all the existing finite automata discussed below are assumed to equip with two endmarkers $\xi$ and $\xi$.

**(i) Deterministic Finite Automata.** A 1-way deterministic finite automaton (or a 1dfa) with two endmarkers can be viewed as a special case of $(V, B, O)$-1dta’s $(\{\xi, \xi\}, V, \{B_1\}_{\xi \in \Sigma}^\Sigma, v_0, E_{\text{acc}}, E_{\text{rej}})$ when $V$ equals $\{\{k\} \mid k \in \mathbb{N}^+\}$ with the discrete topology, $B$ contains all maps from $\{k\}$ to $\{k\}$ for each $k \in \mathbb{N}^+$, and $O$ contains all nonempty partitions $(E_{\text{acc}}, E_{\text{rej}})$ of $\{k\}$ for each $k \in \mathbb{N}^+$. Languages recognized by 1dfa’s are called and REG denotes the set of all regular languages.

**(ii) Probabilistic Finite Automata [21].** A stochastic matrix is a nonnegative-real matrix in which every column sums up to exactly 1. A bounded-error 1-way probabilistic finite automaton (or 1pfa) is a special case of $(V, B, O)$-1dta, where $V = \{\{0, 1\}^k \mid k \in \mathbb{N}^+\}$ (in which each point of $\{0, 1\}^k$ is expressed as a column vector), $B$ is composed of all $k \times k$ stochastic matrices, and $O$ is the set of all pairs $(E_{\text{acc}}, E_{\text{rej}})$, each of which is defined as the inverse images of projections onto unit basis vectors in $\{0, 1\}^k$. The notation 1BPFA denotes the set of all languages recognized by 1pfa’s with bounded-error probability. When we consider unbounded-error probability, we write SL to denote the set of all stochastic languages (i.e., languages that are recognized by 1pfa’s with unbounded-error probability). It is known that 1BPFA = REG [21] and REG $\subseteq$ SL since $L_{\text{sl}} = \{a^m b^n \mid m, n \in \mathbb{N}, m < n\}$ is in SL.

**(iii) Measure-Once Quantum Finite Automata [18].** A bounded-error measure-once 1-way quantum finite automaton (or mo-1qfa, in short) is a quantum extension of bounded-error 1pfa, which is allowed to measure the inner state of the mo-1qfa only once after reading off all input symbols. exactly a $(V, B, O)$-1dta, where $V$ is a set of spaces $V = \{\mathbb{C}^{\leq 1}\}^k$ for $k \in \mathbb{N}^+$, $B$ consists of $k \times k$ unitary matrices, $O$ contains all pairs $(E_{\text{acc}}, E_{\text{rej}})$ such that $E_{\text{acc}} = \{v \in V \mid \|\Pi_{\text{acc}}v\|^2 > 1 - \varepsilon\}$ and $E_{\text{rej}} = \{v \in V \mid \|\Pi_{\text{rej}}v\|^2 > 1 - \varepsilon\}$ for two projections $\Pi_{\text{acc}}, \Pi_{\text{rej}}$ and a constant $\varepsilon \in [0, 1/2]$, where $\|\cdot\|$ denotes the $\ell^2$-norm. Write MO-1QFA to denote the collection of all languages recognized by mo-1qfa’s with bounded-error probability. Note that MO-1QFA coincides with 1PAUT [3].

**(iv) Measure-Many Quantum Finite Automata [18].** A bounded-error measure-many 1-way quantum finite automaton (or mm-1qfa) is a variant of mo-1qfa, which makes a measurement every time the mm-1qfa reads an input symbol. Each mm-1qfa can be described as a $(V, B, O)$-1dta, where $V$ contains all sets $V$ of the form $(\mathbb{C}^{\leq 1})^k \otimes \{0, 1\} \otimes \{0, 1\}$ and $B$ consists of all maps $T$ defined in [25] as

$$T(v, \gamma_1, \gamma_2) = \left(\Pi_{\text{acc}}Bv, \text{sgn}(\gamma_1)\sqrt{\gamma_1^2 + \|\Pi_{\text{acc}}Bv\|^2}, \text{sgn}(\gamma_2)\sqrt{\gamma_2^2 + \|\Pi_{\text{rej}}Bv\|^2}\right),$$

Unlike the standard definition, in accordance with our topological automata, we apply each stochastic matrix to column vectors from the left, not from the right in the early literature.
where $\text{sgn}(\gamma) = +1$ if $\gamma \geq 0$ and $-1$ if $\gamma < 0$, for a certain $k \times k$ unitary matrix $B$ and 3 projections $\Pi_{\text{acc}}, \Pi_{\text{rej}}$, and $\Pi_{\text{on}}$ onto the spaces spanned by different sets of basis vectors. Concerning bounded-error 1qfa’s, we set $E_{\text{acc}} = \{(v, \gamma_1, \gamma_2) \in V \mid \gamma_1 \geq 1 - \epsilon\}$ and $E_{\text{rej}} = \{(v, \gamma_1, \gamma_2) \in V \mid \gamma_2 \geq 1 - \epsilon\}$ for each constant $\epsilon \in [0, 1/2)$. Let $O$ express the set of all such pairs $(E_{\text{acc}}, E_{\text{rej}})$. For basic properties of $T$, see Appendix. We write MM-1QFA to denote the collection of all languages recognized by 1qfa’s with bounded-error probability. It is known that MM-1QFA $\subseteq$ REG [16].

(v) Quantum Finite Automata with Mixed States and Superoperators [11, 9, 23] (see also a survey [11]). A 1-way quantum finite automaton with mixed states and superoperators (or simply, a 1qfa) extends mo-qfa’s and mm-qfa’s in computational power. Let $V$ denote the set of $k \times k$ complex matrices, let $v_0 = \text{diag}(1, 0, \ldots, 0)$, let $B_\sigma(v) = \sum_{j=1}^{m} A_{\sigma,j}vA_{\sigma,j}^\dagger$ for a set $\{A_{\sigma,j}\}_{\sigma \in \Sigma, j \in [m]} \subseteq V$ satisfying $\sum_{j=1}^{m} A_{\sigma,j}A_{\sigma,j}^\dagger = I$ (the identity matrix). Let $\Pi_{\text{acc}}$ and $\Pi_{\text{rej}}$ be projections onto the spaces spanned by disjoint sets of basis vectors. Each pair $(E_{\text{acc}}, E_{\text{rej}})$ is defined as $E_{\text{acc}} = \{v \in V \mid \text{tr}(\Pi_{\text{acc}}v) \geq 1 - \epsilon\}$ and $E_{\text{rej}} = \{v \in V \mid \text{tr}(\Pi_{\text{rej}}v) \geq 1 - \epsilon\}$ for any constant $\epsilon \in [0, 1/2)$, where $\text{tr}(D)$ is the trace of a square matrix $D$.

(vi) Deterministic Pushdown Automata. A 1-way deterministic pushdown automaton (or a 1dpda) $M$ can be seen as a $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta when $(\mathcal{V}, \mathcal{B}, \mathcal{O})$ satisfies the following properties. Let $\mathcal{V} = \{[k] \times \pm \Gamma^* \mid k \in \mathbb{N}^+\}$, alphabet, where $\pm$ is a distinguished bottom marker not in $\Gamma$. Let $\mathcal{B}$ be composed of all maps of the form $B(q, z) = (\mu_1(q, z), \mu_2(q, z), \ldots, \mu_l(q, z))$ for 2 functions $\mu_1 : [k] \times \Gamma \rightarrow [k]$ and $\mu_2 : [k] \times \Gamma \rightarrow \Gamma^\leq$, where $z = z_1z_2 \cdots z_n$ and $l \in \mathbb{N}^+$. Intuitively, $B$ simulates a series of moves in which $M$ reads one symbol and then makes a non-$\lambda$-move followed by a certain number of $\lambda$-moves. Let $\mathcal{O}$ consist of all pairs $(E_{\text{acc}}, E_{\text{rej}})$ with $E_{\text{acc}} = Q_1 \times \pm \Gamma^*$ and $E_{\text{rej}} = Q_2 \times \pm \Gamma^*$, where $Q_1$ and $Q_2$ are partitions of $[k]$. We write DCFL for the class of these languages.

4 Basic Properties of $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta’s

We have introduced the model of $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta’s in Section 3 and this computational model has an ability to characterize the existing finite automata of various types, as shown in Section 3.2. Next, we will explore basic properties of those $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta’s and their language families $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1DTA.

4.1 Elimination of Endmarkers

In many cases, it is possible to eliminate two endmarkers from a 1dta $M$ without changing the language $L(M)$ recognized by $M$. A simple way to do so is to modify the initial configuration, say, $v_0$ of $M$ to a new initial configuration $B_0(v_0)$ using an operator $B_0$ of $M$. Even if we stick to the same $v_0$, a slight modification of all operators $B_{\sigma}$ of $M$ plays the same effect as shown in Lemma 1.1.

We say that a set $\mathcal{B}$ of operators is continuously invertible if every operator $B$ in $\mathcal{B}$ is invertible and its inverse $B^{-1}$ is in $\mathcal{B}$ and continuous.

Lemma 4.1 Let $(\mathcal{V}, \mathcal{B}, \mathcal{O})$ be any reasonable automata base. Assume that $\mathcal{B}$ is continuously invertible. For every $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta $M = (\Sigma, \{\$, \$\}, \mathcal{V}, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}})$, there exists another equivalent $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta $N$ with the same $\Sigma$, $\mathcal{V}$, $v_0$, $E_{\text{acc}}$, and $E_{\text{rej}}$ but no left-endmarker.

Proof. We define a new set of operators $B_\sigma'$ for each symbol $\sigma \in \Sigma \cup \{\$\}$. Define $B_\sigma' = B^{-1}_{B_\sigma}B_\sigma B_{B_\sigma}$ for any $\sigma \in \Sigma$ and let $B'_\$ = $B_\$B_\$. The desired $N$ reads an input $x\$ and behaves exactly as $M$ does by applying $\{B_\sigma'\}_{\sigma \in \Sigma \cup \{\$\}}$. $\square$

We can eliminate $\$ as well by slightly changing observable $M$ as stated in Lemma 1.2. The set $\mathcal{O}$ is closed under inverse image of operations in $\mathcal{B}$ if, for any operator $B \in \mathcal{B}$ and for any pair $(E_1, E_2) \in \mathcal{O}$, the pair $(B^{-1}(E_1), B^{-1}(E_2))$ also belongs to $\mathcal{O}$, where $B^{-1}(A) = \{v \in \mathcal{V} \mid B(v) \in A\}$.

Lemma 4.2 Let $(\mathcal{V}, \mathcal{B}, \mathcal{O})$ be any reasonable automata base. Assume that $\mathcal{O}$ is closed under inverse images of operators in $\mathcal{B}$. For every $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta $M$, there exists its equivalent $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta $N$ with the same $\Sigma$, $\mathcal{V}$, $B_\sigma$, and $v_0$ but no right-endmarker.

Proof. Let $M$ be any $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta $M$ in the premise of the lemma. We define a new observable pair $(E_{\text{acc}}', E_{\text{rej}}')$ of $N$ by setting $E_{\text{acc}}' = \{v \in \mathcal{V} \mid B_\$<br>
\( B^{-1}_h(E_{acc}) = \{ B^{-1}_h(v) \mid v \in E_{acc} \} \) and \( B^{-1}_h(E_{rej}) = \{ B^{-1}_h(v) \mid v \in E_{rej} \} \). Since \( \mathcal{O} \) is closed under inverse images of operators in \( \mathcal{B} \), it follows that \((B^{-1}_h(E_1), B^{-1}_h(E_2)) \in \mathcal{O}\) for any \((E_1, E_2) \in \mathcal{O}\). In particular, since \((E_{acc}, E_{rej}) \in \mathcal{O}\), we conclude that \((E'_{acc}, E'_{rej}) \in \mathcal{O}\).

It turns out that the endmarker elimination is quite costly by placing heavy restrictions on an automata base \((\mathcal{V}, \mathcal{B}, \mathcal{O})\). It is not clear, however, under what “minimal” condition we can eliminate the endmarkers.

### 4.2 Closure Properties

Let us discuss closure properties of a family \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA. First, we start with the closure property under inverse homomorphism, which turns out to be satisfied by every family \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA. Given two alphabets \(\Sigma\) and \(\Gamma\), a homomorphism \(h\) is a function from \(\Sigma\) to \(\Gamma^*\) with its extension satisfying that \(h(\lambda) = \lambda\) and \(h(xa) = h(x)h(a)\) for any \(x \in \Sigma^*\) and any \(a \in \Sigma\). We say that a language family \(\mathcal{C}\) is closed under inverse homomorphism if, for any language \(L \in \mathcal{C}\) and any homomorphism \(h\), the set \(h^{-1}(L) = \{ x \mid h(x) \in L \}\) also belongs to \(\mathcal{C}\).

**Lemma 4.3** For any reasonable automata base \((\mathcal{V}, \mathcal{B}, \mathcal{O})\), \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA is closed under inverse homomorphism.

**Proof.** Let \(\Sigma\) and \(\Gamma\) denote two alphabets and consider a homomorphism \(h : \Sigma \rightarrow \Gamma^*\) and its extension to \(\Sigma^*\). For any given automata base \((\mathcal{V}, \mathcal{B}, \mathcal{O})\) stated in the premise of the lemma, take a \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA \(M = (\Gamma, \{\ell, sp\}, \mathcal{V}, \{B_\ell\}_{\ell \in \mathcal{B}}, v_0, E_{acc}, E_{rej})\) that recognizes a language \(L\).

Let us define a new \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA \(N = (\Sigma, \{\ell, sp\}, \mathcal{V}, \{B'_\sigma\}_{\sigma \in \Sigma}, v_0, E_{acc}, E_{rej})\), which intends to recognize \(h^{-1}(L)\). Initially, we set \(B'_\ell = B_\ell\) and \(B'_v = B_v\). For each symbol \(\sigma \in \Sigma\), we define \(B'_\sigma = B_h(\sigma)\). The definition of \(\{B'_\sigma\}_{\sigma \in \Sigma}\) implies \(B'_{x\sigma} = B_h(x)\) for any string \(x \in \Sigma^*\). It then follows that, for each input \(x \in \Sigma^*\), \(B'_{x\sigma}(v_0) = B'_\sigma(B'_{x}(B'_{x}(v_0))) = B'_\sigma(B_h(x)(B_h(v_0))) = B_h(x)\). We thus conclude that \(x \in L(N)\) iff \(h(x) \in L\). From this equivalence, \(L(N) = h^{-1}(L)\) follows.

Boolean closures are quite fundamental properties. To state our claims regarding those closure properties, we start with new terminology to use. We say that \(\mathcal{O}\) is symmetric if, for any pair \((A, B) \in \mathcal{O}\), \((B, A)\) also belongs to \(\mathcal{O}\). We consider the “product” of two topological spaces \(V_1\) and \(V_2\) by setting \(V = V_1 \times V_2\) and by taking the associated product topology \(T_V = T_{V_1 \times V_2}\). An automata base \((\mathcal{V}, \mathcal{B})\) is said to be closed under product if, for any \(V_1, V_2 \in \mathcal{V}\) and any \(B_1, B_2 \in \mathcal{B}\), \(V_1 \times V_2\) and \(B_1 \times B_2\) are homeomorphic to certain \(V'\) in \(\mathcal{V}\) and \(B'\) in \(\mathcal{B}\), respectively.

Next, let \((\mathcal{V}, \mathcal{B}, \mathcal{O})\) be any reasonable automata base.

**Lemma 4.4** Let \((\mathcal{V}, \mathcal{B}, \mathcal{O})\) be any reasonable automata base.

1. If \(\mathcal{O}\) is symmetric, then \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA is closed under complementation.

2. If \((\mathcal{V}, \mathcal{B})\) is closed under product and \((\mathcal{V}, \mathcal{O})\) is closed under accept-union product, then \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA is closed under union.

3. If \((\mathcal{V}, \mathcal{B})\) is closed under product and \((\mathcal{V}, \mathcal{O})\) is closed under reject-union product, then \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA is closed under intersection.

The language family DCFL is closed under neither union nor intersection. Similarly, MM-1QFA is not closed under union \(\mathcal{B}^3\). Since, as shown in Section 3.2, 1DTA’s can characterize those language families, the premises of Lemma 4.4 may not be removed.

**Proof of Lemma 4.4** (1) The closure property of \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA under complementation can be obtained simply by exchanging between \(E_{acc}\) and \(E_{rej}\) since \(\mathcal{O}\) is symmetric.

(2) For each \(i \in \{1, 2\}\), we take a language \(L_i\) over \(\Sigma\) recognized by a certain \((\mathcal{V}, \mathcal{B}, \mathcal{O})\)-1DTA \(M_i = (\Sigma, \{\ell, sp\}, \mathcal{V}, \{B_\ell\}_{\ell \in \mathcal{B}}, v_0, E_{acc}, E_{rej})\). Let \(L = L_1 \cup L_2\). For \((V_1, B_1, \sigma)\) and \((V_2, B_2, \sigma)\), we consider \((V, B_\sigma)\) defined by \(V = V_1 \times V_2\) and \(B_\sigma = B_1, \sigma \times B_2, \sigma\). Moreover, set \(v_0 = (v_{1,0}, v_{2,0})\), \(E_{acc} = (V_1 \times E_{acc}) \cup (E_{acc} \times V_2)\), and \(E_{rej} = E_{1, rej} \times E_{2, rej}\). For any initial segment \(z\) of \(x\), \(B_{\ell,z}(v_0) = (B_{\ell, z}(v_{1,0}), B_{\ell, z}(v_{2,0}))\). It thus follows that (i) \(B_{z\ell, z}(v_0) \in E_{acc}\) iff either \(B_{1, z, \ell}(v_{1,0}) \in E_{1, acc}\) or \(B_{2, z, \ell}(v_{2,0}) \in E_{2, acc}\) and (ii)
must be desirable to leave out all points that are not visited by \( M \) from the rest of the points. Therefore, when we discuss the true power of topologies used to define 1dta’s, it does not seem to represent the actual behavior of \( M \), because the points that cannot be visited by \( M \) from \( V \) do not cover all points in \( V \). In such a case, the topological feature of \( V \) does not seem to represent the actual behavior of \( M \), because the points that cannot be visited by \( M \) may satisfy a completely different property from the rest of the points. Therefore, when we discuss the true power of topologies used to define 1dta’s, it must be desirable to leave out unreachable points from the set of all points that \( M \) can reach.

\[ B_{2,d}(v_0) \in E_{rej} \text{ iff both } B_{1,d}(v_1,0) \in E_{1, rej} \text{ and } B_{2,d}(v_2,0) \in E_{2, rej}. \] Finally, we define the desired \( N \) as \((\Sigma, \{ \varepsilon, \}, V, \{ B_\sigma \}_{\sigma \in \Sigma}, v_0, E_{acc}, E_{rej})\).

(3) Similar to (1) in principle, but we need to exchange the roles of “acc” and “rej”.

\[
\begin{equation}
\end{equation}
\]

4.3 Computational Power Endowed by the Trivial and Discrete Topologies

We briefly discuss the language recognition power endowed to 1dta’s by the trivial topology as well as the discrete topology. In fact, while the trivial topology makes 1dta’s recognize only trivial languages, the discrete topology makes them powerful enough to recognize all languages. This latter fact, in particular, assures us to be able to characterize any language family by an appropriate choice of topologies for 1dta’s.

**Proposition 4.5** Let \((V, B, O)\) be a reasonable automata base has the trivial topology \( T_{\text{trival}}(V) \) for each \( V \in V \). For any \((V, B, O)-1dta M\) with an alphabet \( \Sigma \), \( L(M) \) is either \( \emptyset \) or \( \Sigma^* \).

**Proof.** Given an automata base \((V, B, O)\) in the lemma, let us consider any \((V, B, O)-1dta M = (\Sigma, \{ \varepsilon, \}, V, \{ B_\sigma \}_{\sigma \in \Sigma}, v_0, E_{acc}, E_{rej})\). Since \( E_{acc} \) is clopen with respect to \( T_{\text{trival}}(V) \), it must be either \( \emptyset \) or \( V \). The same holds for \( E_{rej} \). Hence, \( M \) either accepts all strings or rejects all strings. Thus, \( L(M) \) is either \( \Sigma^* \) or \( \emptyset \).

The trivial topology provides little power to 1dta’s. In contrast, the discrete topology gives underlying automata enormous computational power, as shown below, to recognize all languages.

**Proposition 4.6** There is a reasonable automata base \((V, B, O)\) with the discrete topology for each \( V \in V \) such that, for any language \( L \), there is a \((V, B, O)-1dta \) that recognizes \( L \). This is true for the 1dta model with or without endmarkers.

**Proof.** Let \( V \) be composed of all \( \Sigma^* \)'s for any alphabet \( \Sigma \) and let \( T_V \) be the discrete topology on \( V \). Moreover, let \( B \) consist of all continuous operators \( B_\sigma \) on \( V \) in \( V \). Moreover, let \( O = \{(L, \Sigma^* - L), (\{s_0\}, \{s_1\}) | L \subseteq \Sigma^*, s_0, s_1 \in \Sigma^+, s_0 \neq s_1, \Sigma: \text{alphabet}\} \). Since \( T_V \) is discrete topology on \( V \), \( O \) is a set of valid observable pairs.

First, we consider the case where 1dta’s use no endmarkers. First, we set \( V = \Sigma^* \), \( v_0 = \lambda \), \( B_\sigma v = \sigma v \), where \( \sigma v \) is the concatenation of \( v \) and \( \sigma \), \( E_{acc} = L \), and \( E_{rej} = \Sigma^* - L \). These definitions imply that \( B_\sigma x = x \in L \) if \( x \in E_{acc} \).

Next, we consider the case where 1dta’s use two endmarkers. Let \( L \) be any language over alphabet \( \Sigma \). We fix two distinguished distinct points \( s_0, s_1 \in \Sigma^+ \). We define a 1dta \( M \) as follows. Define \( V = \Sigma^* \), \( v_0 = \lambda \), \( B_\varepsilon = I \), \( B_\sigma v = \sigma v \), \( B_\varepsilon v = s_{L(v)} \), \( E_{acc} = \{s_1\} \), and \( E_{rej} = \{s_0\} \). For each \( x \in \Sigma^* \), we obtain \( B_{\varepsilon} x = x \) and \( B_{\varepsilon} x v_0 = B_{\varepsilon} x s_{L(x)} \). Hence, \( x \in L \) implies \( B_{\varepsilon} x v_0 \in E_{acc} \), and \( x \not\in L \) implies \( B_{\varepsilon} x v_0 \in E_{rej} \).

Based upon Proposition 4.6 hereafter, we will shift our attention to topologies strictly between the trivial topology and the discrete topology.

4.4 Slender Topological Automata

To design finite automata, it is sometimes imperative to make them “small” enough. Such a requirement often gives rise to a notion of “minimal” finite automata. For instance, Ehrig and Kühnel \[\text{[8]}\] earlier discussed the minimality of topological automata when they have compactly generated Hausdorff metric spaces, where a compactly generated space is a topological space \( V \) such that every subset \( A \) of \( V \) is open iff \( A \cap C \) is open for any compact subspace \( C \subseteq V \). From a different viewpoint, Jeandel \[\text{[15]}\] considered “small” topological automata under the term of “purge” by excluding all points of a configuration space that cannot be reached (or visited) along any computation. We wish to take a similar approach to leave out unreachable points from each topological space.

Let us consider a \((V, B, O)-1dta M = (\Sigma, \{ \varepsilon, \}, V, \{ B_\sigma \}_{\sigma \in \Sigma}, v_0, E_{acc}, E_{rej})\) for a given automata base \((V, B, O)\). There is often a case where all configurations generated (or visited) by \( M \) starting with \( v_0 \) do not cover all points in \( V \). In such a case, the topological feature of \( V \) does not seem to represent the actual behavior of \( M \), because the points that cannot be visited by \( M \) may satisfy a completely different property from the rest of the points. Therefore, when we discuss the true power of topologies used to define 1dta’s, it must be desirable to leave out all points that are not visited by \( M \) and to be focused on the set of all points that \( M \) can reach.
As a simple example, let us consider two topological spaces: $V_1 = \{v_0, v_1\}$ with $T_{V_1} = \{O, \{v_0\}, \{v_1\}, V_1\}$ and $V_2 = \{v_0, v_1, v_2\}$ with $T_{V_2} = \{O, \{v_0\}, \{v_1\}, \{v_0, v_1\}, V_2\}$. Note that $(V_1, T_{V_1})$ is the discrete topology but $(V_2, T_{V_2})$ is not. Let us consider the operator $B_\mathcal{V}$ defined as $B_\mathcal{V}(v_0) = v_0$, $B_\mathcal{V}(v_0) = v_1$, $B_\mathcal{V}(v_1) = v_0$, $B_\mathcal{V}(v_1) = v$ for any $v$. Any 1dta works in the same way on $V_1$ and $V_2$ using $\{B_{\mathcal{V}} \}_{\sigma \in \{a, b, s\}}$ since $v_2$ is not reached from $v_0$.

This conclusion makes us introduce a new notion of slender Idta’s, which can visit all points in $V$. Formally, we say that the 1dta $M$ is slender if, for every $v \in V$, there exists a string $x \in \Sigma^*$ for which either $B_{\mathcal{V}}(v_0) = v$ or $B_{\mathcal{V}}(v_0) = v$ holds. This notion will become important in a later section. Notice that, for any slender 1dta, since $\Sigma^*$ is countable, its associated topological space is also countable.

We show how to build, for any given 1dta, its equivalent slender 1dta. The normalization of $M$ is defined to be a $(\hat{V}_M, \hat{B}_M, \hat{O}_M)$-1dta, denoted by $M_{\text{norm}}$, which is obtained from $M$ in the following way. First, we define $B'$ to be the closure of $\{B_x \mid x \in \Sigma^*\}$ under functional composition. We define $V_M = \{v_0, B_k(v_0), B_kB_k(v_0), B_kB_kB_k(v_0) \mid B \in B'\}$ with its subspace topology $T_{V_M}$ induced by $T_V$ (namely, $T_{V_M} = \{A \cap V_M \mid A \in T_V\}$). Notice that $(V_M, T_{V_M})$ and $(V, T_V)$ may be quite different in nature. We further set $V_M = \{V_M\}$. To define $\hat{B}_M$, we need to restrict the domain of each operator $B \in B' \cup \{B_k, B_k\}$ onto $V_M$.

We write the obtained map as $\hat{B}$. The desired $\hat{B}_M$ is set to be the family of all operators $\hat{B}$ induced from operators $B$ in $B' \cup \{B_k, B_k\}$. Finally, we set $\hat{O}_M$ to be $\{(E_{\text{acc}} \cap V_M, E_{\text{rej}} \cap V_M)\}$. For each $\tau \in \{\text{acc}, \text{rej}\}$, since $E_\tau$ is a clopen set, $E_{\text{acc}} \cap V_M$ is also clopen with respect to $T_{V_M}$. It thus follows that all points of $V_M$ are visited by $M$ while reading certain input strings over the alphabet $\Sigma$.

**Lemma 4.7** 1. $M_{\text{norm}}$ is slender.

2. $M_{\text{norm}}$ is computationally equivalent to $M$.

*Proof.* (1) We first claim that $M_{\text{norm}}$ is slender. Let $v \in V_M$. Then we obtain $v = v_0, v = B_kB_k(v_0), v = B_kB_k(v_0)$ for a certain $B \in B'$. By the definition of $B'$, there is an $x \in \Sigma^*$ such that $B = B_x$ for a certain $x \in \Sigma^*$. Then we conclude that $v = v_0, v = B_k(v_0), v = B_kB_k(v_0)$.

(2) Next, we want to show by induction on $n \in \mathbb{N}$ that $\hat{B}_{k^n}(v_0) = B_{k^n}(v_0)$ and $\hat{B}_{k^n}(v_0) = B_{k^n}(v_0)$ for any $x \in \Sigma^n$. This yields the computational equivalence between $M_{\text{norm}}$ and $M$. Let $x \in \Sigma^n$ and consider $B_{k^n}(v_0)$. Note that $v_0 \in V_M$. Assume that $\hat{B}_{k^n}(v_0) = B_{k^n}(v_0) \in V_M$. Take any $\sigma \in \Sigma \cup \{\$\}$. Since $\hat{B}_{k^n}(w) = B_{k^n}(w)$ for any $w \in V_M$, it follows by the induction hypothesis that $\hat{B}_{k^n}(v_0) = B_{k^n}(v_0) = B_{k^n}(v_0) = B_{k^n}(v_0)$. In particular, we obtain $\hat{B}_{k^n}(v_0) = B_{k^n}(v_0)$.

**Lemma 4.8** If $(V, B, O)$-1dta $M = (\Sigma, \{q, \$,\}, V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}})$ is slender and $B$ contains $B' \cup \{B_k, B_k\}$ (which is defined earlier), then $M_{\text{norm}}$ is also a $(V, B, O)$-1dta.

*Proof.* Since $M$ is slender, we obtain $V_M = V$ and thus $T_{V_M} = T_V$. For any $\hat{B} \in \hat{B}_M$, since $B$ contains $B' \cup \{B_k, B_k\}$, there is another operator $B' \in B'$ such that $\hat{B}$ equals $B'$ restricted to $V_M$. Since $V_M = V$, we obtain $\hat{B} = B'$ and $\hat{O}_M = \{(E_{\text{acc}} \cap V_M, E_{\text{rej}} \cap V_M)\} \subseteq O$.

For slender 1dta’s, we can eliminate the right endmarker.

**Lemma 4.9** Let $(V, B, O)$ be any automata base such that $O$ is closed under inverse images of operators in $B$. For any slender $(V, B, O)$-1dta $M$, there exists a slender $(V, B, O)$-1dta $N$ with no right endmarker such that $L(M) = L(N)$.

*Proof.* We first note the following: for any slender $(V, B, O)$-1dta $M = (\Sigma, \{q, \$,\}, V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}})$, it follows that $B_\$^{-1}(E_{\text{acc}}) \cup B_\$^{-1}(E_{\text{rej}}) = V_M$ and $B_\$^{-1}(E_{\text{acc}}) \cap B_\$^{-1}(E_{\text{rej}}) = \emptyset$. To eliminate the right endmarker $\$, we redefine new halting sets $E_{\text{acc}} \cup E_{\text{rej}}$ as $E = B_\$^{-1}(E')$ for each $\tau \in \{\text{acc}, \text{rej}\}$. Since $\{E_{\text{acc}}, E_{\text{rej}}\}$ partitions the entire space $V$, $E_{\text{acc}}$ and $E_{\text{rej}}$ are also clopen sets. Since $O$ is closed under inverse images of $B_\$, $(E_{\text{acc}}, E_{\text{rej}})$ must belong to $O$.

The notion of slender topological automata will play an important role in Sections

5-6.
5 Computational Strengths of Properties on Topological Spaces

Since topological spaces give fundamental grounds to topological automata, we want to compare the strengths of two different “properties” of the topological spaces by comparing the computational power of the corresponding topological automata. For instance, the Hausdorff separation axiom is one of those properties.

Let us recall the notion of slender 1dta’s from Section 4. Hereafter, we pay our attention to the slender 1dta’s. Given a property \( P \), we say that an automata base \((V, B, \mathcal{O})\) meets \( P \) if every slender \((V, B, \mathcal{O})\)-1dta \( M \) satisfies \( P \). Let \( P_1 \) and \( P_2 \) be two properties on point sets. We say that \( P_2 \) is at least as computationally strong as \( P_1 \), denoted by \( P_1 \preceq \text{comp} \ P_2 \), if, for any reasonable automata base \((V_1, B_1, \mathcal{O}_1)\) that meets \( P_1 \), there exists another reasonable automata base \((V_2, B_2, \mathcal{O}_2)\) meeting \( P_2 \) such that every slender \((V_1, B_1, \mathcal{O}_1)\)-1dta has a computationally equivalent slender \((V_2, B_2, \mathcal{O}_2)\)-1dta. Notice that \( P_1 \) is always at least as computationally strong as \( P_1 \) itself. Moreover, \( P_2 \) is said to be computationally stronger than \( P_1 \) if \( P_1 \preceq \text{comp} \ P_2 \) and \( P_2 \not\preceq \text{comp} \ P_1 \). In this case, we succinctly express \( P_1 \prec \text{comp} \ P_2 \).

For two properties \( P_1 \) and \( P_2 \), we say that \( P_2 \) supersedes \( P_1 \), denoted by \( P_1 \preceq P_2 \), exactly when, for every automata base \((V, B, \mathcal{O})\), if it meets \( P_1 \), then it also meets \( P_2 \).

**Lemma 5.1** For two properties \( P_1 \) and \( P_2 \), if \( P_1 \preceq P_2 \), then \( P_1 \preceq \text{comp} \ P_2 \).

**Proof.** Assume that \( P_1 \preceq P_2 \). Consider any reasonable automata base \((V, B, \mathcal{O})\) meeting \( P_1 \). Since \( P_1 \preceq P_2 \), \((V, B, \mathcal{O})\) also satisfies \( P_2 \). Thus, \( P_1 \preceq \text{comp} \ P_2 \) follows. \( \Box \)

In what follows, we present a result concerning topological indistinguishability. Let \((V, T_V)\) be any topological space. Two points \( x \) and \( y \) of \( V \) are topologically indistinguishable if there exists an open set \( N \in T_V \) such that either (i) \( x \in N \) and \( y \notin N \) or (ii) \( x \notin N \) and \( y \in N \). Otherwise, they are topologically indistinguishable. The Kolmogorov separation axiom (or simply, the Kolmogorov condition) dictates that any pair of distinct points of \( V \) are topologically indistinguishable. Any space that satisfies the Kolmogorov condition is called a Kolmogorov space. For example, let us consider \((V, T_V)\) with \( V = \{1, 2, 3\} \) and \( T_V = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\} \). This \((V, T_V)\) is a Kolmogorov space but it does not have the discrete topology on \( V \). While the discrete topology always satisfies the Kolmogorov condition, the trivial topology violates the Kolmogorov condition.

The next theorem exhibits a clear difference between the trivial topology and any topology that violates the Kolmogorov condition.

**Theorem 5.2** There is a topology not satisfying the Kolmogorov separation axiom and it is computationally stronger than the trivial topology.

**Proof.** Let us consider the language \( \text{ZERO} = \{0^n \mid n \in \mathbb{N}\} \) over the binary alphabet \( \Sigma = \{0, 1\} \). By Proposition 4.4, \( \text{ZERO} \) cannot be recognized by any \((V, B, \mathcal{O})\)-1dta’s having the binary alphabet for any reasonable automata base \((V, B, \mathcal{O})\) with every \( V \) in \( V \) having the trivial topology. Here, we want to prove the existence of an automata base \((V, B, \mathcal{O})\) and a \((V, B, \mathcal{O})\)-1dta \( M = \langle \Sigma, \{\emptyset, \$\}, V, \{B_\sigma\}_{\sigma \in \Sigma}, \nu, E_{\text{acc}}, E_{\text{ rej}} \rangle \) satisfying that (i) \( V \) consists of finite topological spaces violating the Kolmogorov condition and (ii) \( M \) recognizes \( \text{ZERO} \). Since the trivial topology provides only \( \{\emptyset, \Sigma^*\} \), the topology on \( V \) is computationally stronger.

Let us define the desired \((V, B, \mathcal{O})\)-1dta \( M \) as follows. Let \( V = \{0, 1, 2\} \) and \( T_V = \{\emptyset, V, \{0\}, \{1, 2\}\} \) so that \((V, T_V)\) cannot satisfy the Kolmogorov condition. We define \( B_0 = B_1 = I \) and \( B_0(n) = n \) and \( B_1(n) = \min\{n + 1, 2\} \) for any \( n \in V \). Clearly, \( B_\sigma \) is continuous for each \( \sigma \in \Sigma \). Moreover, we set \( \nu_0 = 0 \), \( E_{\text{acc}} = \{0\} \), and \( E_{\text{ rej}} = \{1, 2\} \). It is immediate that \( M \) accepts all strings of the form \( 0^n \) for \( n \in \mathbb{N} \) and rejects all the strings containing \( 1 \). Therefore, \( M \) recognizes \( \text{ZERO} \). Finally, we set \( V = \{V\}, B = \{B_\sigma \mid \sigma \in \Sigma\}, \) and \( \mathcal{O} = \{(E_{\text{acc}}, E_{\text{ rej}})\} \). \( \Box \)

**Proposition 5.3** The discrete topology is computationally stronger than any topology not satisfying the Kolmogorov separation axiom.

**Proof.** Let us consider any language \( L \) over the binary alphabet \( \Sigma = \{0, 1\} \) satisfying the following condition: for any distinct strings \( x_1 \) and \( x_2 \), there exists a string \( y \) for which \( L(x_1 y) \neq L(x_2 y) \). We assert that any slender \((V, B, \mathcal{O})\)-1dta violating the Kolmogorov condition cannot recognize \( L \). Take any slender \((V, B, \mathcal{O})\)-1dta \( M \) whose topological space \((V, T_V)\) does not satisfy the Kolmogorov condition; that is, there exists a pair of distinct points \( v_1, v_2 \in V \) that are topologically indistinguishable.
First, we consider the case where $B_{\Sigma^i}(v_0) = v_1$ and $B_{\Sigma^j}(v) = v_2$. In this case, since $v_1$ and $v_2$ both belong to either $E_{\text{acc}}$ or $E_{\text{rej}}$, we conclude that $L(x_1) = L(x_2)$, leading to a contradiction against the definition of $L$.

Next, we assume that two distinct strings $x_1$ and $x_2$ satisfy that $B_{\Sigma^i}(v_0) = v_1$ and $B_{\Sigma^j}(v_0) = v_2$. Note that there is no case where $B_{\Sigma^i}(v_0) = v_1$ and $B_{\Sigma^j}(v_0) = v_2$ or the other way round. Since no open set topologically distinguishes between $v_1$ and $v_2$, for any string $y$, $B_{\Sigma^i}(v_1)$ and $B_{\Sigma^j}(v_2)$ cannot be topologically distinguishable and they must fall into the same set $E_\tau$ for a certain $\tau \in \{\text{acc, rej}\}$. Therefore, we conclude that $L(x_1y) = L(x_2y)$, a contradiction.

Since $L$ is recognized by a certain 1dta with the discrete topology, as shown in Proposition 4.6, the lemma immediately follows. \hfill $\Box$

In the proof of Theorem 5.2, we have used finite topologies, each of which is composed of a finite number of open sets. As shown in Section 5.2, any 1dta can be simulated by a certain 1dta with a finite discrete topology. Conversely, we argue in Theorem 5.4 that any finite topology provides topological automata with no more recognition power than 1dta’s. Hence, 1dta’s with finite topologies can characterize REG.

**Theorem 5.4.** Let $(V, B, O)$ be any reasonable automata base with finite topologies. It then follows that $\text{REG} \subseteq (V, B, O)\text{-}1\text{DTA}$.

**Proof.** The inclusion $\text{REG} \subseteq (V, B, O)\text{-}1\text{DTA}$ follows from Section 5.2, in which we describe 1dta’s in terms of finite topological spaces. Hereafter, we intend to prove that $(V, B, O)\text{-}1\text{DTA} \subseteq \text{REG}$.

Assuming a reasonable automata base $(V, B, O)$ with finite topologies, we take a $(V, B, O)\text{-}1\text{DTA} M = (\Sigma, \{\#, \}, V, \{B_v\}_{v \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}})$. We want to convert $M$ into another equivalent 1dta $N$. For any two points $v, w \in V$, we define a binary relation $\equiv$ as: $v \equiv w$ iff $v$ and $w$ are topologically indistinguishable. We first show that this relation $\equiv$ is an equivalence relation on $V$. Clearly, $v \equiv v$ holds. If $v \equiv w$, then $w \equiv v$ holds. Assume that $v \equiv w$ and $w \equiv z$. If $v \neq z$, then there is an open set $A \in TV$ such that either ($v \in A$ and $z \notin A$) or ($v \notin A$ and $z \in A$). Without loss of generality, we assume that $v \in A$ and $z \notin A$. Since $v \equiv w$, we obtain $w \in A$. This means that $z \neq w$, a contradiction.

Let us consider a set $V/\equiv$ of all equivalence classes. Given a point $v \in V$, let $[v] = \{w \in V \mid v \equiv w\}$. It then follows that $V/\equiv = \{[v] \mid v \in V\}$. Here, we want to show that $V/\equiv$ is finite. If $V/\equiv$ is an infinite set, then we can take an infinite subset $S$ of $V$ such that any two distinct points are topologically distinguishable. There must be an infinite number of open sets in $TV$. This contradicts the finiteness of $TV$. Thus, $|V/\equiv|$ must be finite. We set $m = |V/\equiv|$.

We choose $m$ points $v_0, v_1, \ldots, v_{m-1} \in V$ such that $[v_i] \neq [v_j]$ for any distinct pair $i, j \in \{0, m - 1\}$. We define a new 1dta $N = (Q, \Sigma, \{\#, \}, \delta, v_0, Q_{\text{acc}}, Q_{\text{rej}})$ as follows. Let $Q = \{v_0, v_1, \ldots, v_{m-1}\}$ and set $Q_{\text{acc}} = \{v_i \mid [v_i] \cap E_{\text{acc}} \neq \emptyset\}$ and $Q_{\text{rej}} = \{v_i \mid [v_i] \cap E_{\text{rej}} \neq \emptyset\}$. The transition function $\delta : Q \times \Sigma \rightarrow Q$ is defined as: $\delta(v_i, \sigma) = v_j$ iff there are points $w_1, w_2 \in V$ such that $[v_i] = [w_1], [v_j] = [w_2]$, and $B_v(w_1) = w_j$.

For any initial segment $z$ of $x$, it follows that $B_z(v_0) \in E_{\text{acc}}$ iff $\delta^*(v_0, \varepsilon_z) \in Q_{\text{acc}}$, where $\delta^*(q, w)$ denotes an inner state obtained just after reading $w$, starting in state $q$.

Next, we assert that $[v] = [w]$ implies $[B_z(v)] = [B_z(w)]$. Assume that $[B_z(v)] \neq [B_z(w)]$. Take a neighborhood $N$ of $B_z(v)$ satisfying $B_z(w) \notin N$. Choose another neighborhood $N'$ of $v$ for which $B_z(N') \subseteq N$. Since $w \in N'$, we obtain the desired contradiction.

Finally, we show that there is no index $i \in [0, m - 1]_Z$ such that $[v_i] \cap E_{\text{acc}} \neq \emptyset$ and $[v_i] \cap E_{\text{rej}} \neq \emptyset$. This is because, otherwise, there are two distinct points $w_1, w_2 \in [v_i]$ satisfying that $w_1 \in E_{\text{acc}}$ and $w_2 \in E_{\text{rej}}$, and thus $v_1 \neq v_2$, follows a contradiction.

Since $N$ can simulate $M$, we conclude that $L(M) = L(N)$. \hfill $\Box$

### 6 Compactness, Equicontinuity, and Regularity

In general topology, the notion of compactness of topological spaces plays an important role. This notion also makes a significant effect on the computational complexity of 1dta’s. For a metric space $V$ and a topological automaton $M$, Jeandel claimed in [15, Theorem 3] that, using our notation, the compactness of $V_M$ and $\hat{B}_M$ yields the regularity of the language recognized by $M$. In contrast, our topological automata use arbitrary topologies, not limited to metric space; therefore, we need to show a more general statement, which gives a necessary and sufficient condition for the regularity of languages.

With an appropriate index set $I$, a collection $\{W_i\}_{i \in I}$ of open subsets of $V$ is called a covering if
A subcovering of \( \{ W_i \}_{j \in I} \) is a collection \( \{ W_j \}_{j \in B} \) for a certain subset \( J \) of \( I \) satisfying that \( V \subseteq \bigcup_{j \in B} W_j \). When \( J \) is a finite set, the subcovering is said to be finite. A topological space \((V, T_V)\) is called compact if every open covering of \( V \) has a finite subcovering. Recall that the set \( C_B(V) \) is assumed to have a topology, denoted by \( T_{C_B(V)} \). We say that a sub-automata base \((V, B)\) is compact if, for any \( V \in V \), \( V \) is compact and \( C_B(V) \) is also compact.

A uniform structure on \( V \) is a collection \( \Phi \) of subsets of \( V \times V \) satisfying that, for any \( U \in \Phi \) and any \( W \subseteq V \times V \), (i) \( \{ (v, v) \mid v \in V \} \subseteq U \), (ii) \( U \subseteq W \) implies \( W \in \Phi \), (iii) \( W \in \Phi \) implies \( U \cap W \in \Phi \), (iv) there exists a set \( X \in \Phi \) for which \( X \circ X \subseteq U \), where \( X \circ X = \{(v, w) \in V^2 \mid \exists z \in V \mid [v, z], (z, w) \in X \} \), and (v) \( U^{-1} = \{(w, v) \in V^2 \mid \{v, w\} \in U \} \). For the set \( C(V) \) of all continuous maps on \( V \), a subset \( F \) of \( C(V) \) is uniformly topologically equicontinuous if, for any element \( A \) of a uniform structure on \( V \), the set \( \{ (u, v) \in V^2 \mid \forall f \in F \mid ([f(u), f(v)] \subseteq A) \} \) is also an element of the uniform structure. A uniform structure \( \Phi \) on \( V \) is said to be compatible with the topology \( T_V \) if, for every set \( A \subseteq V \), \( A \in T_V \) holds exactly when, for any \( x \neq A \), there is a set \( U \in \Phi \) satisfying that \( U \subseteq A \), where \( U \subseteq \{ y \in V \mid (x, y) \in A \} \). A topological space \((V, T_V)\) is uniformizable if there exists a uniform structure compatible with the topology \( T_V \). We say that a sub-automata base \((V, B)\) is uniformly topologically equicontinuous if, for any \( V \in V \), \( C_B(V) \) is uniformly topologically equicontinuous.

Next, we show one of our main theorems, which gives a necessary and sufficient condition on \((V, B, O)\) that ensures \((V, B, O)\)-1dta = REG. This is a complete characterization of regular languages in terms of topological automata.

**Theorem 6.1** For any language \( L \), the following two statements are logically equivalent.

1. \( L \) is regular.

2. There is a reasonable automata base \((V, B, O)\) such that every element in \( V \) is uniformizable, \((V, B)\) is compact and uniformly topologically equicontinuous, and \( L \) is recognized by a certain slender \((V, B, O)\)-1dta.

To prove Theorem 6.1 we need the following technical lemma.

**Lemma 6.2** Let \((V, B, O)\) be any reasonable automata base such that \( V \)'s elements are uniformizable and \( O \) is closed under inverse images of operators in \( B \). If \((V, B)\) is compact and uniformly topologically equicontinuous, then any language recognized by a slender \((V, B, O)\)-1dta is a regular language.

**Proof.** Given an automata base \((V, B, O)\), we assume that \((V, B)\) is compact and uniformly topologically equicontinuous and that \( V \)'s elements are uniformizable. Let \( L \) be any language over an alphabet \( \Sigma \) recognized by a certain slender \((V, B, O)\)-1dta \( M = (\Sigma, \{ \xi, \emptyset \}, V, \{ B_T \}_{x \in \Sigma^*}, v_0, E_{acc}, E_{rej}) \). Hereafter, we want to show that \( L \) is a regular language by converting \( M \) into an equivalent 1dfa \( N \).

To simplify our proof, we eliminate from \( M \) the right endmarker \( \$ \). For this purpose, we define \( E^s_{acc} = B^{-1}_{g}(E_{acc}) \) and \( E^s_{rej} = B^{-1}(E_{rej}) \). As in the proof of Lemma 6.1, we note that \( E^s_{acc} \) and \( E^s_{rej} \) are clopen and partition the whole space \( V \). By the uniformizability of \( V \), there exists a uniform structure \( \Phi_M \) of \( V \) that is compatible with the topology \( T_V \). For convenience, we set \( \Phi_M = \{ A \cap (E^s_{acc} \times E^s_{rej}) \mid \tau \in \{ acc, rej \}, A \in \Phi_M \} \).

Next, we partition \( \Sigma^* \) into equivalence classes as follows. Given two strings \( x, y \in \Sigma^* \), we write \( x \approx y \) if \( L(xz) = L(yz) \) holds for all strings \( z \in \Sigma^* \). Since \( \approx \) is an equivalence relation, we consider the collection \( \Sigma^*/\approx \) of all equivalence classes. If \( | \Sigma^*/\approx | = 1 \), then either \( \Sigma = \emptyset \) or \( \Sigma = \emptyset \) holds, and thus \( L \) is regular. Hereafter, we assume that \( | \Sigma^*/\approx | > 1 \). Let us choose two strings \( x, y \in \Sigma^* \) from different equivalence classes, that is, \( x \neq y \). In what follows, for simplicity, we write \( v_x \) for \( B_{gz}(v_0) \). We define \( C_{x,y} = \{ a \in \Phi_M \mid \exists z \in \Sigma^* \mid (B_z(v_x), B_z(v_y)) \notin A \} \) and consider a “maximal” set \( D_{x,y} \in C_{x,y} \) in such a sense that, for any \( a \in C_{x,y}, D_{x,y} \subseteq A \) implies \( D_{x,y} = A \). From the set \( D_{x,y} \), we define \( F_{x,y} = \{ (v_1, v_2) \in V^2 \mid \forall z \in \Sigma^* \mid (B_z(v_1), B_z(v_2)) \in D_{x,y} \} \). Obviously, we obtain \( (v_x, v_y) \notin F_{x,y} \) and \( (v, v_x) \in F_{x,y} \) for any \( v \in V \). Since \( C_B(V) \) is uniformly topologically equicontinuous, \( F_{x,y} \) falls into \( \Phi_M \).

Finally, we set \( P = \{ (u, F_{x,y}) \mid x \in \Sigma^*, x \neq y, 3w([u, w] \in F_{x,y}) \} \). Here, we claim that \( V = \bigcup_{(u, F) \in P} F[u] \). Since \( F[u] \subseteq V \), it suffices to show that \( V \subseteq \bigcup_{(u, F) \in P} F[u] \). Choose \( x, y \) with \( x \neq y \).

Since \( (v, v) \in F_{x,y} \) for any \( v \in V \), we obtain \( (v_x, v_y) \in P ) \) and \( v \in F_{x,y} \). As a consequence, \( \{ F[u] \mid (u, F) \in P \} \) is a covering of \( V \).

By the compactness of \( V \), we can choose from \( \{ F[u] \mid (u, F) \in P \} \) a finite subcovering \( \{ P_i \}_{i \in [t]} \) of \( V \) for a certain number \( t \in \mathbb{N}^+ \). We consider all possible intersections of any number of sets in \( \{ P_i \}_{i \in [t]} \) and \( P \) denotes the set of all nonempty intersections. Next, we claim that, for any \( A \in P \) and for any \( v_x, v_y \in A \), both \( x \) and \( y \) belong to the same equivalence class; that is, \( x \approx y \). To show this claim, we assume that \( x \neq y \).
From this assumption, there is a string $z$ for which $(B_z(v_x), B_z(v_y)) \in (E_{\text{acc}}^S \times E_{\text{rej}}^S) \cup (E_{\text{rej}}^S \times E_{\text{acc}}^S)$. The definition of $D_X,y$ implies that $(B_z(v_x), B_z(v_y)) \notin D_X,y$. It then follows that $(v_x, v_y) \notin F_{X,y}$. For the set $A$, we take a pair $(u, F) \in P$ satisfying $A \subseteq F[u]$. Since $v_x, v_y \in A$, we obtain $(v_x, u), (u, v_y) \in F$. Since $F \notin \Phi_M$ and $\Phi_M$ is a uniform structure, there exists a set $X \in \Phi_M$ such that $(v_x, u), (u, v_y) \in X$ implies $(v_x, v_y) \in F$. This is a contradiction against $(v_x, v_y) \notin F_{X,y}$ because $F \subseteq F_{X,y}$. Let $P = \{P'_i\}_{i \in [m]}$ for a certain number $m \in \mathbb{N}^+$. For each index $i \in [m]$, we choose a point, say, $v_i$ that represents $P_i$. For convenience, we write $x \equiv y$ if $v_x$ and $v_y$ are in the same set $P_i$ for a certain $i \in [m]$.

Finally, we define the desired 1dfa $N = (Q, \Sigma, \{\delta, \nu\}, Q_{\text{acc}}, Q_{\text{rej}})$ with $Q = \{v_0, v_1, \ldots, v_{m-1}\}$ as follows. Let $Q_{\text{acc}} = \{v_i \in Q \mid v_i \in E_{\text{acc}}\}$ and $Q_{\text{rej}} = \{v_i \in Q \mid v_i \in E_{\text{rej}}\}$. Moreover, we define $\delta$ as follows: $\delta(v_i, \sigma) = v_j$ if there exists a point $w_j$ in $M_X$ such that $v_i \equiv w_j$ and $B_\sigma(v_i) = w_j$. In what follows, we claim that $\delta$ is a well-defined function from $Q \times \Sigma$ to $Q$. Assume that $\delta(v_k, \sigma) = v_i$ and $\delta(v_k, \sigma) = v_j$. There are two points $w_i$ and $w_j$ such that $v_i \equiv w_i$, $v_j \equiv w_j$, $B_\sigma(w_i) = w_i$, and $B_\sigma(w_j) = w_j$. Since $B_\sigma$ is a function, we obtain $w_i = w_j$. This implies that $v_i \equiv v_j$. By the choice of $v_i$ and $v_j$, we conclude that $v_i = v_j$.

By the definition of $N$, we conclude that $M$ accepts (resp., rejects) $x$ iff $N$ accepts (resp., rejects) $x$. Therefore, $L(M) = L(N)$ follows. This completes the proof of the lemma.

Let us return to the proof of Theorem 6.1 which is now an easy consequence of Lemma 6.2.

**Proof of Theorem 6.1** Any 1dfa can be viewed as a $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta of the particular form described in Section 3.2. By the description of this 1dta, it follows that $\mathcal{V}$’s points are uniformizable and $(\mathcal{V}, \mathcal{B})$ is compact and uniformly topologically equicontinuous. Combining this fact with Lemma 6.2, we immediately obtain the desired characterization of regular languages in terms of $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta’s.

The condition of compactness in Theorem 6.1 is needed because, without it, certain 1dta’s can recognize non-regular languages. The next lemma exemplifies such a situation.

**Lemma 6.3** Let $\mathcal{V} = \{\{\mathbb{Z}, \mathcal{P}(\mathbb{Z})\}\}, \mathcal{B} = \{K_1, K_2, B_n, B_k\}$ with $\Sigma = \{a, b\}$, $K_b = B_k = I$, $B_n(n) = n+1$, and $B_n(n) = n - 1$ for all $n \in \mathbb{Z}$, and $\mathcal{O} = \{(E_{\text{acc}}, E_{\text{rej}})\}$ with $E_{\text{acc}} = \{\emptyset\}$ and $E_{\text{rej}} = \mathbb{Z} - \{\emptyset\}$. The sub-automata base $(\mathcal{V}, \mathcal{B}, \mathcal{O})$ is uniformly topologically equicontinuous but not compact. There exists a $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta that recognizes the language $\text{Equal} = \{w \in \{a, b\}^* \mid \#_a(w) = \#_b(w)\}$, where $\#_a(w)$ indicates the total number of occurrences of a symbol $a$ in a string $w$.

**Proof.** Take $\mathcal{V}$, $\mathcal{B}$, and $\mathcal{O}$ defined in the premise of the lemma. First, we note that $\mathcal{O}(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$ is not compact because the set $W = \{\{n\} \mid n \in \mathbb{Z}\}$ is a covering of $\mathbb{Z}$ but there is no finite subcovering of $W$. We define $\Phi$ to be a collection of all sets $\{\{m, n\} \in \mathbb{Z}^2 \mid |n - m| \leq k\}$ for all numbers $k \in \mathbb{Z}$ and their super sets. It is not difficult to show that (1) $\Phi$ is a uniform structure on $\mathbb{Z}$, (2) $\Phi$ is compatible with $T_{\mathbb{Z}}$, and (3) $C_{\Phi}(\mathbb{Z})$ is uniformly topologically equicontinuous. As a consequence, we conclude that $(\mathcal{V}, \mathcal{B})$ is uniformly topologically equicontinuous.

Let us consider a $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta $M = (\Sigma, \{\mathbb{Z}\}, \mathcal{B}, \mathcal{O}, E_{\text{acc}}, E_{\text{rej}})$ with $v_0 = 0$, where $\Sigma$ and $B_n$’s are given in the premise of the lemma. Let $x = x_1, x_2, \ldots, x_n$ be any input string on $\Sigma$. By the definition of $B_n$’s, it is not difficult to show that $B_{\#_a}(v_0) = \#_a(x) - \#_a(x)$. Hence, it follows that $B_{\#_b}(v_0) \in E_{\text{acc}}$ iff $\#_a(x) = \#_b(x)$. Therefore, $M$ recognizes $\text{Equal}$. 

7 Multi-Valued Operators and Nondeterminism

Nondeterminism is a ubiquitous feature, which appears in many fields of computer science. To study nondeterministic quantum finite automata, Jeandel [15] considered such a feature for his model of topological automata. In a similar vein, we wish to define a nondeterministic version of our $(\mathcal{V}, \mathcal{B}, \mathcal{O})$-1dta’s, called 1-way nondeterministic topological automata (or 1nfa’s), each of which nondeterministically chooses one next state out of a predetermined set of possible states at every step.

7.1 Multi-Valued Operators and 1nfa’s

Unlike the previous sections, we deal with multi-valued operators, which map one element to “multiple” elements. To be more precise, a multi-valued operator is a map from each point $x$ of a given topological
space \((V_1, T_{V_1})\) to a certain number of points of a topological space \((V_2, T_{V_2})\). Although this operator can be viewed simply as an “ordinary” map from \(V_1\) to \(P(V_2)\), we customarily express such a multi-valued map as \(B: V_1 \rightarrow V_2\) as long as the multi-valuedness of \(B\) is clear from the context.

For example, let us consider \(V = \mathbb{R}\) and the discrete topology \(T_V\) on \(\mathbb{R}\). The function \(F\) defined by \(f_\varepsilon(v) = \{w \in V \mid |w - v| \leq \varepsilon\}\) is a multi-valued operator on \(V\). Another example is the inverse map \(B^{-1}\) of any operator \(B: V \rightarrow V\), where \(B^{-1}\) is defined as \(B^{-1}(v) = \{w \in V \mid B(w) = v\}\) for any \(v \in V\). This map \(B^{-1}\) is clearly a multi-valued operator. In comparison, any standard map is often referred to as a single-valued map. Notice that every single-valued operator is also a multi-valued operator.

A multi-valued operator \(B: V_1 \rightarrow V_2\) is said to be \(\text{continuous}\) if, for any \(x \in V_1\) and for any neighborhood \(N\) of \(B(x) \in V_2\), there exists a neighborhood \(N'\) of \(x\) satisfying \(B(N') \subseteq N\). To emphasize the multi-valuedness of operators \(B\) and \(B'\), we express their “composition” as \(B \circ B'\) defined by \(B \circ B'(v) = \bigcup_{w \in B'(v)} B(w)\) for any \(v\).

Let us define an \(\text{extended automata base}\) by expanding the notion of automata bases in the following way.

**Extended Automata Base.** An \(\text{extended automata base}\) is, similar to an automata base, a tuple \((V, B, O)\) in which \(V\) is a set of topological spaces, \(O\) is a set of observable pairs, and \(B\) consists of continuous multi-valued operators \(B: V \rightarrow V\) for each set \(V \in V\).

Since single-valued maps are also multi-valued, any automata base can be treated as a special case of extended automata bases.

**Definition of 1-inta’s.** Given an extended automata base \((V, B, O)\), a \(1\)-\(\text{way nondeterministic} (V, B, O)\)-\(\text{topological automaton}\) (or a \((V, B, O)\)-1nta, for short) \(M\) is a tuple \((\Sigma, \{\#, \}$\), \(V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}}\)\) similar to a 1dta except that, for each \(B_\sigma\), we apply a family of all languages recognized by certain \((V, B_\sigma)\)-1nta’s in a natural way.

We present a simple observation on the closure property under reversal. A language family \(L\) is expressed as \((\Sigma \times \{\#, \}$\), \(v_0, E_{\text{acc}}, E_{\text{rej}}\)\), where \(V\) is of the form \([k]\) for a certain constant \(k \in \mathbb{N}^+\), \(B_\sigma(v)\) is a subset of \(V\) for each \(\sigma \in \Sigma\), and \(E_{\text{acc}}\) and \(E_{\text{rej}}\) are disjoint nonempty subsets of \(V\).

\[(i) \text{ \textit{Nondeterministic Finite Automata.}}\quad \text{A \textit{one-way nondeterministic finite automaton} (or a 1nfa) is described as} (\Sigma, \{\#, \}$\), \(V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}}\), \text{where} V \text{ is of the form} \([k]\) \text{ for a certain constant} k \in \mathbb{N}^+, B_\sigma(v) \text{ is a subset of} V \text{ for each} \sigma \in \Sigma, \text{and} E_{\text{acc}}\text{ and} E_{\text{rej}} \text{ are disjoint nonempty subsets of} V.\]

\[(ii) \text{ \textit{Nondeterministic Pushdown Automata.}}\quad \text{A \textit{one-way nondeterministic pushdown automaton} (or a 1npda) is expressed as} (\Sigma, \{\#, \}$\), \(V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}}\), \text{where} V = \[k] \times \mathbb{L}^* \text{ for a constant} k \in \mathbb{N}^+\text{ and a fixed alphabet} \Gamma.\text{ We set} B_\sigma(q, \#) = \{z \in \[k] \times \{\#z_1z_2\cdots z_{n-1}w\mid w \in \Gamma^*\} | z = z_1z_2\cdots z_n\text{ and} l \in \mathbb{N}^+. \text{ Let} E_{\text{acc}} = Q_1 \times \mathbb{L}^* \text{ and} E_{\text{rej}} = Q_2 \times \mathbb{L}^* \text{ with a partition} (Q_1, Q_2) \text{ of} [k].\]

\[(iii) \text{ \textit{Interactive Proof Systems with Finite Quantum Automata}} [19, 20].\quad \text{An\ \textit{interactive proof system} with a 1qfa \textit{verifier} can be described by nondeterministically simulating interactions between a prover and a verifier for a \((V, B, O)\)-Inta that satisfies the following conditions. Let} V \text{ contain} V_1 \times V_2, \text{ where} V_1 = (\mathbb{C}^\leq)^{k_1} \text{ and} V_2 = (\mathbb{C}^\leq)^{k_2} \text{ for certain constants} k_1, k_2 \in \mathbb{N}^+. \text{ Define} B \text{ as a collection of multi-valued operators} B: V_1 \times V_2 \rightarrow V_1 \times V_2 \text{ for certain} V_1 \times V_2 \in V \text{ for which there are functions} B_1: V_1 \rightarrow V_1 \text{ and} B_2: V_1 \times V_2 \rightarrow V_1 \times V_2 \text{ satisfying} B(a, v) = B_2(B_1(a), v) \text{ for any} (a, v) \in V_1 \times V_2.\]

We present a simple observation on the closure property under reversal. A language family \(C\) is said to be \textit{closed under reversal} if, for any language \(L \in C\), its reversal \(L^R = \{x \mid x^R \in L\}\) also belongs to \(C\).

First, we make a quick preparation for our observation stated as Lemma 7.1. Given a 1nta with \(V = (E_{\text{acc}}, E_{\text{rej}})\), we choose two points \(v_{\text{acc}} \in E_{\text{acc}}\) and \(v_{\text{rej}} \in E_{\text{rej}}\), and we then define an operator \(D[v_{\text{acc}}, v_{\text{rej}}]: V \rightarrow V\) as \(D[v_{\text{acc}}, v_{\text{rej}}](v) = v_{\text{acc}}\text{ if} v \in E_{\text{acc}}; v_{\text{rej}}\text{ if} v \in E_{\text{rej}};\text{ and otherwise. It then follows that} D[v_{\text{acc}}, v_{\text{rej}}]\text{ is a continuous operator because} E_{\text{acc}} \cap E_{\text{rej}} = 0.\)

**Lemma 7.1** Let \((V, B, O)\) be any \(\text{reasonable automata base}\) such that \(B\) is \textit{closed under inverse}, \(B\) contains \(D[v_{\text{acc}}, v_{\text{rej}}]\) and all operators of the form \(D[v_{\text{acc}}, v_{\text{rej}}] \circ B\) for any \(B \in \mathcal{B}\) and any \(v_{\text{acc}}, v_{\text{rej}} \in E_{\text{acc}} \times E_{\text{rej}}\). Given a \((V, B, O)\)-Inta \(M\) with \(v_0, V,\) and \((E_{\text{acc}}, E_{\text{rej}})\) recognizing a language \(L\), if there is a point \(v_{\text{acc}} \in E_{\text{acc}}\) satisfying \(\{v_0\}, \{v_{\text{acc}}\} \in TV\), then there exists a \((V, B, O)\)-Inta \(N\) that recognizes the reverse \(L^R\) of \(L\).

**Proof.** Take a \(\text{reasonable automata base} (V, B, O)\) satisfying the premise of the lemma. Let \(M = \)
\((\Sigma, \{\epsilon, \$\}, V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}})\) be any \((V, B, O)\)-Inta that recognizes \(L\). We choose two points \(v_{\text{acc}} \in E_{\text{acc}}\) and \(v_{\text{rej}} \in E_{\text{rej}}\) such that \(\{v_{\text{acc}}\}\) is an open set. We define \(v_0 = v_{\text{acc}}\).

Let us consider another Inta \(N = (\Sigma, \{\epsilon, \$\}, V, \{B'_\sigma\}_{\sigma \in \Sigma}, v'_0, E'_{\text{acc}}, E'_{\text{rej}})\) defined in the following way. Recall that, for any operator \(B\) on \(V\), \(B^{-1}(v)\) denotes the set \(\{w \in V \mid v \in B(w)\}\). Given a symbol \(\sigma \in \Sigma \cup \{\$\}\), we define \(B'_\sigma = B_\sigma^{-1}\) and set \(B'_X = (D[v_{\text{acc}}, v_{\text{rej}}] \circ B_\delta)^{-1}\). Finally, we define \(E'_{\text{acc}} = \{v_0\}\) and \(E'_{\text{rej}} = V - \{v_0, v'_0\}\). By the premise of the lemma, \(E_{\text{acc}}\) and \(E_{\text{rej}}\) are both clopen sets. Our goal is to show that \(L(N) = L^R\). First, we claim that \((v)\) \(B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \supseteq B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_{\text{rej}}) \neq \emptyset\) for any \(k \in [0, n]\), where \(\epsilon = \text{acc} \text{ if } x \in L(M)\), and \(\epsilon = \text{rej} \text{ if } x \notin L(M)\). If \(x^R \in L\), then \(x \in L^R\) and thus \(B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \cap E_{\text{acc}} \neq \emptyset\). This implies that \((B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \cap E_{\text{acc}}) \neq \emptyset\) by Statement \((*)\).

Hereafter, we show Statement \((*)\) by downward induction. In the base case of \(k = n\), since \(B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \supseteq \{v_{\text{rej}}\}\), it follows that \((D[v_{\text{acc}}, v_{\text{rej}}] \circ B_\delta) \circ B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \supseteq \{v_{\text{rej}}\}\). Hence, we obtain \(B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \supseteq B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_{\text{rej}}) \neq \emptyset\). By the induction hypothesis, assume that \((B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k+1}}(v_0) \supseteq B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0) \neq \emptyset\). Since \(B_{\epsilon_{k+1}} \circ (B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0)) = B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k+1}}(v_0) \supseteq B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k}}(v_0)\), it follows that \(B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k+1}}(v_{\text{rej}}) \subseteq B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k+1}}(v_0)\). Thus, we obtain \(B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k+1}}(v_0) \subseteq B^{0}_{\epsilon_{1}\epsilon_{2}\cdots \epsilon_{k+1}}(v_{\text{rej}})\), as requested. \(\square\)

### 7.2 Relationships between 1Inta’s and Inta’s

Nondeterminism seems more powerful than determinism; however, it is known that Inta’s can be simulated by appropriate 1Inta’s using exponentially more inner states than the Inta’s. Here, we seek a similar simulation of 1Inta’s by certain Inta’s. In the following proposition, for a given topological space \((V, T^+_V)\), we expand \(V\) to \(T^+_V\) so that \((T^+_V, T^o(T^+_V))\) forms a topological space for an appropriately chosen topology \(T^o(T^+_V)\). Following Michael [17], we here take \(T^o(T^+_V)\) as the topology that is generated by the bases \(\{[A]^+, [A]^-[A \in T]\}\), where \([A]^+ = \{x \in T^+_V \mid x \subset A\}\) and \([A]^− = \{x \in T^+_V \mid x \cap A \neq \emptyset\}\). This topology is known as the Victoris topology, adapted to \(T^+_V\).

### Proposition 7.2

Let \((V, B, O)\) be any extended automata base. There exists an automata base \((V', B', O')\) with \(V' = \{T^+_V, T^o(T^+_V)\} \mid V \in V\) such that, for any \((V, B, O)\)-Inta \(M\) with \(v_0\) and \(V\), there is an equivalent \((V', B', O')\)-Inta \(N\), provided that \(\{v_0\} \in T^+_V\).

**Proof.** From a given extended automata base \((V, B, O)\), we define \(B'\) and \(O'\) as follows. Notationally, for each space \(V \subset V\), consider the set \(T^+_V\). We write \(B'(W)\) for the set \(\bigcup_{W \in V} B(w)\) for any map \(B: T^+_V \to T^+_V\) and any element \(W \in T^+_V\). Let \(B' = \{B': T^+_V \to T^+_V \mid V \in V, B \in B\}\) and let \(O'\) consist of all pairs \((E_1, E_2) \in T^+_V \times T^+_V\) for any \(V \in V\) for which (i) \(E_1, E_2 \in T^o(T^+_V) \cap \text{co-T}^o(T^+_V)\) with \(E_1 \cap E_2 = \emptyset\) and (ii) there exists a pair \((E_1, E_2) \in O\) satisfying that \(A_1 \cap E_1 \neq \emptyset\) and \(A_2 \subseteq E_2\) for any \(A_1 \in E_1'\) and \(A_2 \in E_2\).

Next, we argue that \((V', B', O')\) forms a proper automata base. Let \(B\) be any multi-valued continuous operator in \(B\) and its corresponding operator \(B'\) in \(B'\). We want to show that \(B'\) is continuous on \((T^+_V, T^o(T^+_V))\). Let \(B'(W) = U\) for arbitrary elements \(U, W \in T^+_V\) and consider any open set \(S\) in \(T^o(T^+_V)\) containing \(U\). Without loss of generality, we assume that \(S\) is either \([U]^+\) or \([U]^−\) because \(U\) is an open set in \(T^+_V\). If \(S = [U]^−\), then we take \(R = [W]^−\). For any \(Y \in R\), it follows that \(B'(Y) \subseteq U\), and thus \(B'(Y) \in S\). In contrast, if \(S = [U]^+\), then we take \(R = [W]^+\). For any \(Y \in R\), since \(W \cap Y \neq \emptyset\), we obtain \(U \cap B'(Y) \neq \emptyset\); hence, \(B'(Y) \in S\) follows.

Let \(M = (\Sigma, \{\epsilon, \$\}, V, \{B_\sigma\}_{\sigma \in \Sigma}, v_0, E_{\text{acc}}, E_{\text{rej}})\) be any given \((V, B, O)\)-Inta with \(\{v_0\} \in T^+_V\). Associated with \(M\), we take \(N = (\Sigma, \{\epsilon, \$\}, T^+_V, \{B'_\sigma\}_{\sigma \in \Sigma}, v'_0, E'_{\text{acc}}, E'_{\text{rej}})\) with \(v'_0 = \{v_0\}\).

By induction on the length of any input string \(w \in \{\epsilon, \$\}^* \cup \Sigma^* \cup \{\epsilon, \$\}^*\), we want to show that \(B'_w(v_0) = B'_w(v_0)\). In the base case, since \(B'_w(v_0) = B'_w(v_0)\), it follows that \(B'_w(v_0) = B'_w(v_0)\). In an induction step, assuming that \(B'_w(v_0) = B'_w(v_0)\), let us consider an input string of the form \(x^a\) with \(a \in \Sigma\) and define \(U_x\) to be \(B'_w(v_0)\). It then follows that \(B'_w(x^a)(v_0) = B'_w(x^a)(v_0)\). Since \(B'_w(x^a)(v_0) = B'_w(x^a)(v_0)\), it follows that \(B'_w(x^a)(v_0) = B'_w(x^a)(v_0)\). Therefore, \(x\) is accepted by \(M\) iff \(x\) is accepted by \(N\). Similarly, for the rejection of \(x\), we define \(E'_{\text{rej}} = \{E' \in T^o(T^+_V) \cap \text{co-T}^o(T^+_V) \mid \forall A \in E' [A \cap E_{\text{rej}} \neq \emptyset]\}\). \(\square\)
8 A Brief Discussion on Future Challenges

In the past literature, several mathematical models of topological automata were proposed and then studied on platforms quite different from ours. In order to categorize formal languages of various complexities, this paper has proposed new computational models of one-way deterministic and nondeterministic topological automata. The fundamental machinery of our new models is based on various choices of topologies ranging from the trivial topology to the discrete topology. Such topological automata are descriptionally powerful enough to represent the existing finite automata of numerous types, including quantum finite automata, pushdown automata, and interactive proof systems.

It turns out that topology and its associated concepts are quite powerful to describe language families. In Section 1.2, we have listed four key goals of the study of topological automata. Our study conducted in this paper is merely the initial step toward the full understandings of the topological features that characterize various language families. Toward a future study, we provide a short list of challenging open questions.

1. The family REG of all regular languages is one of the most basic language families. We have given a few characterizations of REG in terms of topological automata, e.g., in Theorem 6.1. Find a more “natural” automata base \((V, B, O)\) that fulfills the equality of \((V, B, O)\)-1DTA = REG.

2. Complementing the first question, find “natural” automata bases \((V, B, O)\) and \((V', B', O')\) for which \((V, B, O)\)-1DTA \(\not\subseteq\) REG and REG \(\not\subseteq\) \((V', B', O')\)-1DTA.

3. In Proposition 7.2 we have shown how to simulate each 1nta by a computationally-equivalent 1dta. Find a more “succinct” description of \((V', B', O')\)-1dta that is computationally equivalent to any given \((V, B, O)\)-1nta.

4. The complexity classes DCFL and MM-1QFA are not closed under intersection. Find a necessary and sufficient condition of \((V, B, O)\) such that \((V, B, O)\)-1DTA is not closed under intersection. This contrasts Lemma 4.4(3).

5. We have discussed only the case where any computation evolves in linear fashion. If we further expand the basic models using nonlinear evolutions, how do the corresponding finite automata look like?

6. Given an automata base \((V, B, O)\) with “natural” topologies, characterize the language family \((V, B, O)\)-1DTA in terms of standard automata.

7. In Section 4.4 we have discussed a type of “minimal” topological automata. Find a “natural” notion of minimality for our models of topological automata and give an exact condition on \((V, B, O)\)-1dta’s.

8. Neither vector spaces nor metric spaces have been discussed in this paper although certain finite automata require those spaces. For example, quantum finite automata are founded on Hilbert spaces with the \(\ell_2\)-norm. Develop a theory of topological automata using vector spaces or metric spaces.

9. We have discussed the Kolmogorov separation axiom in Section 5. Determine the computational complexity of \((V, B, O)\)-1DTA when \((V, B, O)\) violates the Kolmogorov separation axiom.

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