CLIFFORD ALGEBRAS AND
SHIMURA’S LIFT FOR THETA-SERIES

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ABSTRACT. Automorph class theory formalism is developed for the case of integral nonsingular quadratic forms in an odd number of variables. As an application, automorph class theory is used to construct a lifting of similitudes of quadratic $\mathbb{Z}$-modules of arbitrary nondegenerate ternary quadratic forms to morphisms between certain subrings of associated Clifford algebras. The construction explains and generalizes Shimura’s correspondence in the case of theta-series of positive definite ternary quadratic forms. Relations between associated zeta-functions are considered.

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Key words: Clifford algebras, Hecke operators, Shimura’s correspondence, quadratic forms, theta-series, zeta-functions.
Introduction

In his important paper [12] G. Shimura discovered a remarkable correspondence between modular forms of half-integral weight and forms of integral weight – the so-called “Shimura lift” (see [12, Main Theorem]). P. Ponomarev [10] determined the effect of Shimura’s lifting on the theta-series $\Theta(z, \mathbf{q})$ of certain positive definite ternary quadratic forms $\mathbf{q}$. Using purely arithmetical approach he expressed the lift of $\Theta(z, \mathbf{q})$ as an explicit linear combination of theta-series associated to a certain system of quaternary quadratic forms coming from a quaternion algebra (see [10], Theorem 1). Considering the Fourier coefficients of corresponding theta-series one can interpret these results as a link between the numbers of representations of integers by positive definite quadratic forms in three and in four variables. But according to the theory of automorph class rings (i.e. matrix Hecke rings of orthogonal groups) developed by A. Andrianov in [1] and [2], the majority of known multiplicative properties of the numbers of integral representations by quadratic forms turns out to be merely a reflection of certain relations between the representations themselves. Following ideas of A. Andrianov and P. Ponomarev one can try to define a direct algebraic correspondence between the representations of integers by quadratic forms in three and four variables which would underlie Shimura’s lift in case of theta-series. In the present paper we examine such a correspondence constructed by means of Clifford algebras. Our main result is contained in Theorems 6.5 and 6.7 together with Corollary 6.8 which can be summarized as follows (see notations in the end of this section):

**Proposition.** Let $\mathbf{q}$ be an integral nonsingular ternary quadratic form and let $\mathbf{n}$ be the integral quadratic form defined (up to integral equivalence) by the norm on the even subalgebra of the Clifford algebra $\mathcal{C}(\mathbf{q})$. Take $\{\mathbf{q}_1, \ldots, \mathbf{q}_H\}$ and $\{\mathbf{n}_1, \ldots, \mathbf{n}_H\}$ to be complete systems of representatives of different equivalence classes of $\mathbf{q}$ and of $\mathbf{n}$ respectively. Let $p$ be a prime number coprime to $\det \mathbf{q}$. Denote by

$$T_{\mathbf{q}}^*(p^t) = \bigcup_{i=1}^h E(\mathbf{q}_i) \setminus R^*(\mathbf{q}_i, p^t \mathbf{q})$$

and

$$T_{\mathbf{n}}^*(p^t) = \bigcup_{j=1}^H E(\mathbf{n}_j) \setminus R^*(\mathbf{n}_j, p^t \mathbf{n})$$

the sets of classes of (primitive) automorphs of $\mathbf{q}$ and of $\mathbf{n}$ respectively (with multiplier $p^t$). Then one can define an injection $\Psi : A \mapsto \Psi_A$, $T_{\mathbf{q}}^*(p^2) \rightarrow T_{\mathbf{n}}^*(p^2)$, which can be naturally continued to an injection $\Upsilon : A \mapsto \Upsilon_A$, $T_{\mathbf{q}}^*(p^2) \rightarrow T_{\mathbf{n}}^*(p)$, in such a way that $\Upsilon_A$ divides $\Psi_A$ from the right. Conversely, for any $\mathcal{M} \in T_{\mathbf{n}}^*(p)$ there exists a unique $A \in T_{\mathbf{q}}^*(p^2)$ such that $\mathcal{M}$ divides $\Psi_A$ from the right. This construction turns the set $\bigcup_j E(\mathbf{n}_j) \setminus R(\mathbf{n}_j, p \mathbf{n})$ of classes of quaternary automorphs into a 2-fold covering of the set $\bigcup_i E(\mathbf{q}_i) \setminus R^*(\mathbf{q}_i, p^2 \mathbf{q})$ of classes of ternary automorphs.

In view of Eichler’s commutation relations (2.5), (3.3) and (3.4) the construction corresponds to Shimura’s lift (4.1) for generic theta-series of ternary positive definite quadratic forms and generalizes the latter in a purely arithmetical way to the case of arbitrary indeterminate (nonsingular) ternary quadratic forms:

**Corollary.** With the notation and under the assumptions of the above Proposition, let in addition $\mathbf{q}$ be positive definite, then Shimura’s lift of the generic theta-series
\(\Theta_{\{q\}}(z)\) and the generic theta-series \(\Theta_{\{n\}}(z)\) of the norm \(n\) on the even Clifford subalgebra \(C_0(E, q)\) have the same eigenvalues \(p + 1\) for all Hecke operators \(T(p)\) with \(p \nmid \det q\).

(See also Theorem 6.3 and Corollary 6.4.)

Thus defined automorphic class lift (5.28) rises numerous questions which still await solution. Some of the possibilities, including those concerning relations between associated zeta-functions, will be discussed in Concluding Remarks (Section 7).

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Notations
As usual, the letters \(\mathbb{Z}, \mathbb{Q}\) and \(\mathbb{C}\) denote the ring of rational integers, the field of rational numbers and the field of complex numbers respectively. We put \(\mathbb{H} = \{z \in \mathbb{C} ; \text{Im } z > 0\}\) to be the upper half plane. \(\mathbb{A}_{m}^n\) is the set of all \((m \times n)\)– matrices with entries in a set \(\mathbb{A}, \mathbb{A}^m = \mathbb{A}^1_m, \mathbb{A}_n = \mathbb{A}^1_n\). We let \(\Lambda^m = GL_m(\mathbb{Z})\) denote the group of all integral invertible matrices of order \(m\) and \(1_m\) stand for the unit matrix of order \(m\). If \(M\) is a matrix, then \(tM\) denotes its transposed, \(\tilde{M}\) is its adjoint and \(|M|\) stands for the absolute value of its determinant (for square \(M\)). We write

\[L[M] = tMLM,\]

if the product of matrices on the right makes sense. We consider a quadratic form

\[
q(X) = \sum_{1 \leq i \leq j \leq m} q_{ij} x_i x_j, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix},
\]

in \(m\) variables with rational integral coefficients \(q_{ij}\). The symmetric matrix

\[
Q = (q_{ij}) + t(q_{ij})
\]

is called the matrix of the form \(q\) and its determinant \(\det Q = \det q\) is the determinant of \(q\). Quadratic form \(q\) is nonsingular if \(\det q \neq 0\). Clearly,

\[
2q(X) = Q[X].
\]

Note that \(Q\) is an even matrix (i.e. an integral symmetric matrix whose diagonal elements are even). The level of \(q\) (and of \(Q\)) is defined to be the least positive integer \(\ell\) such that \(\ell Q^{-1}\) is even. It should be noted that \(\ell\) and \(\det q\) have the same prime divisors. If \(A \in \mathbb{Q}_n^m\) with some \(m, n \in \mathbb{N}\), then \(q[A]\) denotes the quadratic form

\[
q'(Y) = \frac{1}{2} Q'[Y] = q(AY) = \frac{1}{2} Q[AY], \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},
\]
resulting from \( q \) by the linear change of variables \( X = AY \). The matrix \( A \) is called a representation (over \( \mathbb{Q} \)) of the form \( q' \) by the form \( q \). In case \( A \in \mathbb{Z}_n \), we call it an integral representation of \( q' \) by \( q \) and denote by

\[
R(q, q') = \{ A \in \mathbb{Z}_n^m ; q[A] = q' \} = \{ A \in \mathbb{Z}_n^m ; Q[A] = Q' \},
\]

the set of all such integral representations. In particular, if \( n = 1 \) and \( q' = ay^2 \) then the set \( R(q, q') \) coincides with the set \( R(q, a) \) of (integral) representations of the number \( a \) by \( q \):

\[
R(q, ay^2) = R(q, a) = \{ X \in \mathbb{Z}^m ; q(X) = a \}.
\]

In case \( q = q' \) we get \( E(q) = R(q, q) \cap \Lambda^m \) – the group of units of the form \( q \). Subset of \( R(q, q') \) consisting of matrices whose entries are coprime is denoted by \( R^*(q, q') \) and its elements are called primitive (integral) representations of \( q' \) by \( q \). We set

\[
\begin{align*}
r(q, q') &= |R(q, q')| , \quad r^*(q, q') = |R^*(q, q')| , \quad r(q, a) = |R(q, a)| , \\
e(q) &= |E(q)|
\end{align*}
\]

to be corresponding cardinalities. Finally, the abbreviation gcd stands for greatest common divisor.
1. Isotropic sums

If \( q \) and \( q' \) are integral quadratic forms in the same number of variables with equal determinants, we say that \( q' \) is similar to \( q \) if \( R(q, aq') \) is not empty for some \( a \) prime to \( \det q \).

The set of all quadratic forms similar to \( q \) is called the similarity class of \( q \). The Minkowski reduction theory of quadratic forms (see [3] for example) shows that the similarity class of a nonsingular integral quadratic form in \( m \) variables is a finite union

\[
\bigcup_{i=1}^{h} \{ q_i[U] ; U \in GL_m(\mathbb{Z}) \} ,
\]

of mutually disjoint classes of integrally equivalent quadratic forms. (One can compare the similarity class with the genus of \( q \). Although in general the two sets are different, but both consist of quadratic forms with arithmetical invariants equal to those of \( q \). Moreover, both sets are finite disjoint unions of classes modulo integral equivalence.)

For similar quadratic forms \( q \) and \( q' \) the elements of the set \( R(q, aq') \) are called automorphs (of \( q \) to \( q' \)) with multiplier \( a \).

**Lemma 1.1.** For any integral nonsingular quadratic forms \( q \) and \( q' \) and for any nonzero number \( a \in \mathbb{Z} \) the set \( R(q, aq') \) of automorphs with multiplier \( a \) is a finite union of left cosets modulo \( E(q) \).

**Proof.** Let \( m \) denote the rank of \( q \) and \( q' \) and let \( Q, Q' \) be their respective matrices. If \( M \in R(q, aq') \) then \( Q[M] = aQ' \). Considering corresponding determinants we have

\[
|M| = (a^m \det q / \det q')^{1/2} = |a|^{m/2} \in \mathbb{Z}
\]

which means that \( R(q, aq') \subset \Delta^m (|a|^{m/2}) \), where

\[
\Delta^m(d) = \{ D \in \mathbb{Z}_m^m ; |\det D| = d \}
\]

for a positive integer \( d \). We claim that \( \Delta^m(d) \) is a finite union of left cosets modulo \( \Lambda^m \) (see Notations). To establish this we note first that according to the theory of elementary divisors (see [3], Lemma 3.2.2) each double coset \( \Lambda^m D \Lambda^m \), \( D \in \Delta^m(d) \) has a unique representative of the form \( \text{ed}(D) = \text{diag}(d_1, \ldots, d_m) \) with \( d_i > 0 \), \( d_i | d_{i+1} \), and so the cardinality \( |\Lambda^m \setminus \Delta^m(d)/\Lambda^m| \) is bounded above by number of factorizations \( d = \prod_{i=1}^{m} d_i \), \( d_i > 0 \), \( d_i | d_{i+1} \), which means that \( \Delta^m(d) \) is a finite disjoint union of double cosets modulo \( \Lambda^m \). Next we note that each of these double cosets is a finite (disjoint) union of left cosets:

\[
\Lambda^m D \Lambda^m = \bigcup_{D' \in D^{-1} \Lambda^m D \cap \Lambda^m} \Lambda^m D' ,
\]

because \( \Lambda^m DU = \Lambda^m D \) with \( U \in \Lambda^m \) if and only if \( U \in D^{-1} \Lambda^m D \cap \Lambda^m \) and the latter subgroup has finite index in \( \Lambda^m \). To justify the last claim consider the principle congruence subgroup of level \( d \) of \( \Lambda^m \):

\[
\Lambda^m(d) = \{ M \in \Lambda^m ; M \equiv 1_m (\text{mod } d) \} .
\]
Clearly $\Lambda^m(d)$ has a finite index in $\Lambda^m$ as the kernel of reduction modulo $d : \Lambda^m \to GL_m(\mathbb{Z}/d\mathbb{Z})$. On the other hand $\Lambda^m(d) \subset D^{-1}\Lambda^m D$ since entry-wise $D\Lambda^m(d)\tilde{D} \equiv D\tilde{D} \equiv 0 \pmod{d}$ and so $D\Lambda^m(d)D^{-1} = (\pm d)^{-1} D\Lambda^m(d)\tilde{D} \subset \Lambda^m$, here $\tilde{D}$ is the matrix adjoint to $D$. Thus $D^{-1}\Lambda^m D \cap \Lambda^m$ contains subgroup $\Lambda^m(d)$ of finite index in $\Lambda^m$, which means that $D^{-1}\Lambda^m D \cap \Lambda^m$ is itself of finite index in $\Lambda^m$. Summing up, we see that $\Delta^m(d)$ is a finite (disjoint) union of double cosets each of which is a finite (disjoint) union of left cosets modulo $\Lambda^m$ and thus $\Delta^m(d)$ is indeed a finite union of left cosets modulo $\Lambda^m$.

It remains to note that $R(\mathbf{q}, a\mathbf{q}') \subset \Delta^m(\lfloor a^{\lfloor m/2 \rfloor} \rfloor)$ is therefore also a finite union of left cosets modulo $\Lambda^m$. But each left coset $\Lambda^m M \cap R(\mathbf{q}, a\mathbf{q}')$ consists of a single left coset $E(\mathbf{q})M$ (if not empty), because if $UM = M' \in R(\mathbf{q}, a\mathbf{q}')$ with $U \in \Lambda^m$ then $Q[U] = Q[M'M^{-1}] = aQ'[M^{-1}] = Q$, i.e. $U \in E(\mathbf{q})$. We conclude that $R(\mathbf{q}, a\mathbf{q}')$ is a finite union of left cosets modulo $E(\mathbf{q})$.

The above property of automorphs will allow us to construct certain Hecke algebras of orthogonal groups in section 3. Then the obvious inclusion

$$R(\mathbf{q}, a\mathbf{q}') \cdot R(\mathbf{q}', b) \subset R(\mathbf{q}, ab)$$

for any quadratic forms $\mathbf{q}, \mathbf{q}'$ and for any integers $a, b$ (the dot refers to the usual matrix multiplication) will help us to define an action of these Hecke algebras on representations of integers by quadratic forms. In order to prepare for later quantitative investigation of that action, we first need to address the question of inverse inclusion, i.e. we need to factor a given element $K \in R(\mathbf{q}, ab)$ into a product of an automorph $M \in R(\mathbf{q}, a\mathbf{q}')$ and a representation $L \in R(\mathbf{q}', b)$. We also want to find total number of such factorizations. For our purposes it is enough to restrict attention to the case when the automorph multiplier $a$ is a power of a fixed prime $p$ and quadratic forms $\mathbf{q}, \mathbf{q}'$ are similar. Moreover, it appears that consideration of just two automorph multipliers $a = p$ or $a = p^2$ is completely sufficient for all situations (see [2]). The case of quadratic forms in an even number of variables and automorph multiplier $p$ was treated in great detail in [1]. Our first goal is to conduct a similar investigation for the case of quadratic forms in an odd number of variables and automorph multiplier $p^2$ (it is easy to see that $p^2$ is the least power of an automorph multiplier possible for quadratic forms in an odd number of variables, see the proof of the Lemma 1.1).

Thus, given $K \in R(\mathbf{q}, p^2 b)$ we seek to find total number of factorizations $K = ML$ with $M \in R(\mathbf{q}, p^2 \mathbf{q}')$, $L \in R(\mathbf{q}', b)$, where $\mathbf{q}, \mathbf{q}'$ are similar integral quadratic forms in an odd number of variables $m = 2k + 1 \geq 3$ and $p$ is a prime coprime to $\det \mathbf{q}$. Following general method of investigation of questions of this type developed in [1], we will view a matrix $M \in R(\mathbf{q}, p^2 \mathbf{q}')$ as a solution of quadratic congruence

$$\mathbf{q}[M] \equiv 0 \pmod{p^2},$$

whose matrix of elementary divisors $D_p$ has a specific form.

**Lemma 1.2.** Let $\mathbf{q}, \mathbf{q}'$ be similar integral quadratic forms in an odd number of variables $m = 2k + 1 \geq 3$ and let $p$ be a prime coprime to $\det \mathbf{q}$. Then

$$R(\mathbf{q}, p^2 \mathbf{q}') = \begin{cases} R^*(\mathbf{q}, p^2 \mathbf{q}') & \text{if } \mathbf{q} \text{ and } \mathbf{q}' \text{ are not integrally equivalent,} \\ R^*(\mathbf{q}, p^2 \mathbf{q}') \cup pR(\mathbf{q}, \mathbf{q}') & \text{otherwise,} \end{cases}$$
where \( R^*(q, p^2 q') \) is the set of primitive automorphs, \( R(q, q') \subset \Lambda^m \), and the union on the right hand side is disjoint. Furthermore, the matrix of elementary divisors of an automorph from \( pR(q, q') \) is equal to \( p1_m \) and the matrix of elementary divisors of a primitive automorph has the form

\[
D_p = D_p(d) = \begin{pmatrix} 1_d & \mu_1 b & \ldots & \mu_k b \\ p1_b & & & \\ \ldots & & & \\ p^21_d & & & \\
\end{pmatrix}
\]  

for some \( d, 1 \leq d \leq k \), here \( k = (m - 1)/2 \) and \( b = m - 2d \).

**Proof.** Let \( M \in R(q, p^2 q') \) be an automorph and let \( \mu \) denote the greatest common divisor of its entries. Since \( q[M] = p^2 q' \) then \(|\det M| = |M| = p^m\), which implies that \( \mu|p \) and so either \( \mu = 1 \) and \( M \in R^*(q, p^2 q') \) or \( \mu = p \) and \( p^{-1}M \in R(q, q') \). We also note that any elementary divisor of \( M \) is a power (at most \( m^{th} \)) of \( p \). Next, since \( q \) and \( q' \) are similar, and in particular have equal determinants, then \( R(q, q') \subset \Lambda^m \) and therefore this set is empty if \( q \) and \( q' \) are not integrally equivalent. It is also clear that any double coset from \( \Lambda^m pR(q, q') \Lambda^m \subset p\Lambda^m \) coincides with \( \Lambda^m p1_m \Lambda^m \). Furthermore, if \( M \in R^*(q, p^2 q') \), then \( p^2M^{-1} = (p^2/\det M) \cdot \tilde{M} = (\det q)^{-1}\tilde{q}'^{-1}MQ \) is an integral matrix since \( \det M \) and \( \det q \) are coprime. Therefore the matrix \( D_p \) of elementary divisors of \( M \) has the form \( diag(1_s, p1_s, p^21_s) \) with \( s_0 + s_1 + s_2 = m \) and \( s_1 + 2s_2 = m \), which leaves us with \( s_1 = s_2 = d \) and \( 1 \leq d \leq (m - 1)/2 \).

Thus if \( M \in R^*(q, p^2 q') \), then \( q[M] \equiv 0 \pmod{p^2} \) and \( M \in \Lambda^m D_p\Lambda^m \) for a matrix \( D_p \) of the form (1.2). Next, since at the moment we do not want to distinguish a particular form \( q' \) in its equivalence class \( \{ q'[U] : U \in \Lambda^m \} \), we will be interested only in different cosets \( M\Lambda^m \in \Lambda^m D_p\Lambda^m/\Lambda^m \) (and thus only in cosets \( R^*(q, p^2 q')/E(q') \) after a choice of particular form \( q' \)). Finally, because our interest is in factorizations of a particular column \( K \in \mathbb{Z}^m \) such that \( q(K) \equiv 0 \pmod{p^2} \), we will consider only those \( M \) which divide from the left such a \( K \). To this end we introduce **isotropic sums** of the form

\[
S_{p^2}(q, D_p, K) = \sum_{M \in \Lambda^m D_p\Lambda^m/\Lambda^m \atop q[M] \equiv 0 \pmod{p^2}, M|K} 1.
\]  

We will use geometric methods to compute the above sum. The following notions and results will help us to reinterpret it in geometric terms.

**Lemma 1.3.** Let \( \delta \) be a positive integer and \( p \) be a rational prime. For a matrix \( M \in \mathbb{Z}_n^m \) with \( m > n \) the following statements are equivalent:

i) There exists a primitive matrix \( M' \in \mathbb{Z}_n^m \) such that \( M' \equiv M \pmod{p^\delta} \).

ii) Columns of \( M \) are linearly independent modulo \( p \).

iii) The matrix \( M \) can be complemented to a matrix \( (MM'') \in \mathbb{Z}_m^m \) such that \( (MM'') \equiv U \pmod{p^\delta} \) with \( U \in \Lambda^m \).
Proof. First of all, recall that an integer matrix is called \textit{primitive} if and only if the greatest common divisor of its principal minors is equal to 1. Note that if \( M' \) is primitive then not all of its principal minors are zero and so its \( \text{rank}_Z \) is equal to \( n \). Then \( \text{rank}_{\mathbb{Z}/p^\delta \mathbb{Z}} M = \text{rank}_{\mathbb{Z}/p^\delta \mathbb{Z}} M' = n \), i.e. columns of \( M \) are linearly independent modulo \( p^\delta \). It remains to note that for an arbitrary \( \delta \), linear independence modulo \( p^\delta \) is equivalent to linear independence modulo \( p \). Thus \( \text{ii)} \) implies \( \text{ii)} \). Conversely, suppose that columns of \( M \) are linearly independent modulo \( p \) (and thus modulo \( p^\delta \)). Then \( \text{rank}_{\mathbb{Z}/p^\delta \mathbb{Z}} M = n \) and using elementary transformations over \( \mathbb{Z}/p^\delta \mathbb{Z} \) one can find \( A \in \Lambda^m \) such that

\[
AM = \begin{pmatrix}
a_1 & \ldots & 0 \\
p^\delta & \ldots & p^\delta \\
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix} = L \pmod{p^\delta}
\]

with \( \prod a_i \not\equiv 0 \pmod{p} \). Moreover, choosing appropriate representatives modulo \( p^\delta \) one can assume that \( a_i \)'s are pairwise coprime. Clearly such \( L \) is primitive. Then \( M' = A^{-1}L \) is congruent to \( M \) modulo \( p^\delta \) and also primitive since multiplication by an invertible matrix preserves primitiveness. (Indeed, using Binet-Cauchy formula, one can see that

\[
\det(M') = \det(M) = 1,
\]

i.e. it is an integral linear combination of minors of order \( n \) of \( M' \). Therefore gcd of principal minors of \( M' \) divides gcd of those of \( L \) which is equal to 1 since \( L \) is primitive.) This establishes equivalence of \( \text{i)} \) and \( \text{ii)} \).

To prove equivalence of \( \text{i)} \) and \( \text{iii)} \) it is enough to show that a matrix \( U' \in \mathbb{Z}_n^m \) is primitive if and only if it can be complemented to \( U = (U',U'') \in \Lambda^m \). This was done in a much more general setting (for matrices over arbitrary Dedekind rings) in Lemmas 2.3, 2.4 and 2.5 of [1]. Here we will present another explanation of this fact for our particular case of integral matrices. If \( U' \in \mathbb{Z}_n^m \) can be complemented to \( U = (U',U'') \in \Lambda^m \), then there exists \( U^{-1} = (A^tB) \) with \( A \in \mathbb{Z}^n, B \in \mathbb{Z}^{m-n} \). So we have:

\[
U^{-1} \cdot U = \begin{pmatrix} AU' & AU'' \\ BU' & BU'' \end{pmatrix} = 1_m ,
\]

in particular \( AU' = 1_n \). Applying Binet-Cauchy formula one has:

\[
\sum_{1 \leq \gamma_1 < \ldots < \gamma_n \leq m} |(A)_{\gamma_1,\ldots,\gamma_n}| \cdot |(U')_{\gamma_1,\ldots,\gamma_n}| = \det(A \cdot U') = 1 ,
\]

which means that an integral linear combination of principal minors of \( U' \) is equal to 1, i.e. their gcd is 1 and \( U' \) is primitive.
Conversely, assume first that \( n = 1 \) and \( U' \in \mathbb{Z}^m \) is a primitive column. Using induction on \( m \geq 2 \) we want to prove that \( U' \) can be complemented to an invertible matrix. In the base case \( m = 2 \) one has \( U' = t(u_1, u_2) \) with \( \gcd(u_1, u_2) = 1 \). Therefore \( xu_1 + yu_2 = 1 \) for some \( x, y \in \mathbb{Z} \) and then
\[
\begin{pmatrix}
  u_1 & -y \\
  u_2 & x
\end{pmatrix} \in \Lambda^2
\]
as intended. If \( m > 2 \) and \( U' = t(u_1, \ldots, u_m) \) is primitive, then either \( u_2 = \ldots = u_m = 0 \) and we can take
\[
U = \begin{pmatrix}
  u_1 & \cdots & 0 \\
  \vdots & \ddots & 1_m \\
  0 & \cdots & 1_m
\end{pmatrix} \in \Lambda^m
\]
since \( u_1 = \pm 1 \), or some of \( u_2, \ldots, u_m \) are not zero and by induction hypothesis we can complement the primitive column \( V' = t(u_2/d, \ldots, u_m/d) \in \mathbb{Z}^{m-1} \) to an invertible matrix \( (V'V'') \in \Lambda^m \), here \( d = \gcd(u_2, \ldots, u_m) \). Note that \( \gcd(u_1, d) = 1 \) as \( U' \) is primitive, so \( xu_1 + yd = 1 \) for some \( x, y \in \mathbb{Z} \) and then
\[
U = \begin{pmatrix}
  u_1 & 0 & y \\
  dV' & V'' & xV'
\end{pmatrix} \in \Lambda^m
\]
since \( \det U = xu_1(-1)^{m-1} + yd(-1)^{m+1} = \pm 1 \). Thus any primitive column can be complemented to an invertible matrix. Next assume that \( n > 1 \), \( U' \in \mathbb{Z}^m \) is primitive and \( t(u_1, \ldots, u_m) \) is the first column of \( U' \). For any selection \( 1 \leq i_1 < \ldots < i_n \leq m \), \( \gcd(u_{i_1}, \ldots, u_{i_n}) \) divides the minor \( |(U')_{i_1,\ldots,i_n}^1,\ldots,n| \) and so \( \gcd(u_1, \ldots, u_m) = 1 \) as \( \gcd \) of minors of order \( n \) of \( U' \) is equal to 1. Therefore \( \sum_{i=1}^{m} x_i u_i = 1 \) for some \( x_i \in \mathbb{Z} \). The column \( t(x_1, \ldots, x_m) \) is primitive, and as we already know it can be complemented to a matrix \( X \in \Lambda^m \). Then \( tXU' \) is primitive with \( (tXU')_{11} = 1 \). Applying elementary transformations to rows of \( tXU' \) we can find matrix \( Y \in \Lambda^m \), such that
\[
YU' = \begin{pmatrix}
  1 & \cdots & * \\
  \vdots & V' \\
  0 & \cdots & 1
\end{pmatrix} \in \mathbb{Z}_n^m
\]
is again a primitive matrix whose minors of order \( n \) are either 0 (in case the minor does not include the first row), or have the form \( |(V')_{i_1,\ldots,i_{n-1}}^1,\ldots,n-1|, 1 \leq i_1 < \ldots < i_{n-1} \leq m \). The latter implies that \( \gcd \) of principal minors of \( V' \) is 1, i.e. \( V' \in \mathbb{Z}_n^{m-1} \) is also primitive and we can recursively apply the above procedure to it. Continuing in this fashion one can find a matrix \( Z \in \Lambda^m \) such that
\[
ZU' = \begin{pmatrix}
  1 & \cdots & * \\
  \vdots & \ddots & \ddots \\
  0 & \cdots & 1 \\
  0 & \cdots & 0
\end{pmatrix} \in \mathbb{Z}_n^m ,
\]
then the matrix $W =\begin{pmatrix} ZU' & 0 \\ 1_{m-n} \end{pmatrix}$ is clearly invertible and we can take $U = (U', U'') = Z^{-1} \cdot W$ to complement $U'$ to an invertible matrix. \[\square\]

We will consider quadratic modules of the form

$$V_{p^\delta}(1_m) = ((\mathbb{Z}/p^\delta \mathbb{Z})^m, q \mod p^\delta)$$

(1.4)

obtained by reduction modulo $p^\delta$ of quadratic module $(\mathbb{Z}^m, q)$ whose (scalar product) matrix with respect to the standard basis of $\mathbb{Z}^m$ is $Q$, the matrix of $q$. For a matrix $M \in \mathbb{Z}_n^m$ we will denote by $V_{p^\delta}(M)$ the quadratic submodule of $((\mathbb{Z}/p^\delta \mathbb{Z})^m, q \mod p^\delta)$ spanned by the columns of $M$ modulo $p^\delta$.

**Lemma 1.4.** Let $D_p = D_p(d)$ be a matrix of the form (1.2) and let $M \in \Lambda^m D_p \Lambda^m$. Then the map $M \mapsto V_{p^\delta}(M)$ defines a bijection of the set of right cosets $M \Lambda^m \subset \Lambda^m D_p \Lambda^m / \Lambda^m$ to the set of all subgroups of $(\mathbb{Z}/p^2 \mathbb{Z})^m$ whose invariants are $(p^2, \ldots, p^d, p, \ldots, p)$, here $b = m - 2d$.

**Proof.** First of all we note that if $M' \in M \Lambda^m$ then clearly $V_{p^\delta}(M') = V_{p^\delta}(M)$ as generators of any of these two groups are integral linear combinations of generators of the other. Thus our map is defined coset-wise. Next, let $M = UD_p V$ with $U, V \in \Lambda^m$ and let $U_i$'s denote the columns of $U$. Then each $V_{p^\delta}(U_i), 1 \leq i \leq d$ is a cyclic group of order $p^2$ and each $V_{p^\delta}(pU_i), d \leq i \leq d + b$ is a cyclic group of order $p$. (To see this note that if $n$ is the order of a column $W \in (\mathbb{Z}/p^2 \mathbb{Z})^m$ then $nW \equiv 0 \pmod{p^2}$ and so $W \equiv 0 \pmod{p^2/n}$. But the gcd of elements of any of $U_i$'s is equal to 1 since $uin \Lambda^m$. Thus the order of $W = U_i$ is $p^2$ and the order of $W = pU_i$ is $p$.) Next we claim that

$$V_{p^\delta}(M) = V_{p^\delta}(U_1) \oplus \ldots \oplus V_{p^\delta}(U_d) \oplus V_{p^\delta}(pU_{d+1}) \oplus \ldots \oplus V_{p^\delta}(pU_{d+b})$$

i.e. $V_{p^\delta}(M)$ is a direct sum of $d$ cyclic subgroups of order $p^2$ and $b$ cyclic subgroups of order $p$. Indeed,

$$V_{p^\delta}(M) = V_{p^\delta}(UD_p) = V_{p^\delta}(U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b}) =$$

$$V_{p^\delta}(U_1) + \ldots + V_{p^\delta}(U_d) + V_{p^\delta}(pU_{d+1}) + \ldots + V_{p^\delta}(pU_{d+b})$$

and we just need to show that the above sum is direct. To see this consider the following: if

$$\sum_{i=1}^d \alpha_i U_i + p \sum_{j=d+1}^{d+b} \alpha_j U_j \equiv 0 \pmod{p^2}$$

then $U^{-1} \cdot (U_1, \ldots, U_{d+b}) \cdot t(\alpha_1, \ldots, \alpha_d, p\alpha_{d+1}, \ldots, p\alpha_{d+b}) \equiv 0 \pmod{p^2}$ and so $p^2|\alpha_i$ for $1 \leq i \leq d$ and $p|\alpha_j$ for $d \leq j \leq d + b$, $\alpha_i$ thus the zero of the group can be written only as the sum of zeros of the subgroups, i.e. the sum is indeed direct. Therefore the invariants of $V_{p^\delta}(M)$ are $(p^2, \ldots, p^d, p, \ldots, p)$.
Conversely, assume that \( G \subset (\mathbb{Z}/p^2\mathbb{Z})^m \) is a subgroup with this set of invariants, i.e.

\[
G = V_{p^2}(K_1) \oplus \ldots \oplus V_{p^2}(K_d) \oplus V_{p^2}(L_1) \oplus \ldots \oplus V_{p^2}(L_b),
\]

where the order of each \( K_i \) is \( p^2 \) and the order of each \( L_i \) is \( p \). Since \( pL_i \equiv 0 \pmod{p^2} \) then \( p|L_i \) for \( 1 \leq i \leq b \). The integral columns \( K_1, \ldots, K_d, p^{-1}L_1, \ldots, p^{-1}L_b \) are linearly independent modulo \( p \) (because the sum of the cyclic groups generated by \( K_i \)'s and \( L_j \)'s is direct). Therefore, according to the Lemma 1.3 there exists a primitive matrix \( V \) such that \( G \) is an isotropic submodule of quadratic module \( V \).

Thus any \( G \subset (\mathbb{Z}/p^2\mathbb{Z})^m \) with invariants \( (p^2, \ldots, p^2, p, \ldots, p) \) has the form \( V_{p^2}(A'D_p) \) for some \( A' \in \Lambda^m \).

The above Lemma allows us to replace summation over \( M \in \Lambda^m D_p \Lambda^m / \Lambda^m \) in (1.3) by summation over different submodules \( V_{p^2}(M) \subset (\mathbb{Z}/p^2\mathbb{Z})^m \) with \( M \in \Lambda^m D_p \Lambda^m \). Furthermore, conditions \( q[M] \equiv 0 \pmod{p^2} \), \( M / K \) in (1.3) mean exactly that \( V_{p^2}(M) \) should be an isotropic submodule of quadratic module \( V_{p^2}(1_m) = ((\mathbb{Z}/p^2\mathbb{Z})^m, q \pmod{p^2}) \) containing a fixed vector \( K \). In order to find the total number of such submodules (and thus to compute the isotropic sum (1.3)), we will first determine the total number of bases of certain kind which generate modules of type \( V_{p^2}(M) \), \( M \in \Lambda^m D_p \Lambda^m \) and then factor this number by the number if different bases generating the same module. We start with exhibiting the bases of \( V_{p^2}(M) \) we are interested in. We choose

\[
\{ U \cdot D_p : U \in \Lambda^m / (D_p \Lambda^m D_p^{-1} \cap \Lambda^m) \}
\]
as a complete system of representatives from classes \( M \Lambda^m \in \Lambda^m D_p \Lambda^m / \Lambda^m \) and let \( U_i, 1 \leq i \leq m \) denote the columns of matrix \( aU \). Then any isotropic module \( V_{p^2}(M) = V_{p^2}(UD_p) = V_{p^2}(U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b}), \) where \( b = m - 2d \).

Using Lemma 1.3 we replace condition \( U \in \Lambda^m \) with requirement that \( U_1, \ldots, U_{d+b} \) should be linearly independent modulo \( p \). Then the condition \( q[M] \equiv q[UD_p] \equiv 0 \pmod{p^2} \) should be replaced with

\[
\begin{align*}
\left\{ \begin{array}{l}
^tU_iQU_j \equiv 0 \pmod{p^2}, \quad \text{if } 1 \leq i \leq j \leq d, \\
^tU_iQU_j \equiv 0 \pmod{p}, \quad \text{if } 1 \leq i \leq d \text{ and } d + 1 \leq j \leq d + b.
\end{array} \right.
\end{align*}
\]

Conversely, it is easy to see that any (ordered) collection \( \{ U_1, \ldots, U_{d+b} \} \subset (\mathbb{Z}/p^2\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})^b \) of linearly independent modulo \( p \) vectors satisfying (1.5) defines a basis

\[
U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b}
\]
of an isotropic module of the form \( V_{p^2}(M), M \in \Lambda^m D_p \Lambda^m \). From now on, when referring to a basis of the form (1.6), we will always assume that vectors \( \{ U_1, \ldots, U_{d+b} \} \subset (\mathbb{Z}/p^2\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})^b \) of linearly independent modulo \( p \) vectors satisfying (1.5) defines a basis...
L^d \times (\mathbb{Z}/p\mathbb{Z})^b$ are linearly independent modulo $p$ and satisfy (1.5). Let us find the total number of such bases.

**Lemma 1.5.** Under the above notations there exist

$$p^{d(m+b-1)} \cdot |GL_b(\mathbb{F}_p)| \cdot \prod_{s=0}^{d-1} (p^{2(k-s)} - 1), \text{ where } k = (m-1)/2,$$

different (ordered) bases of the form (1.6).

**Proof.** We note first that $U_1, \ldots, U_d$ should form an isotropic system of vectors of the quadratic module $V^2_p(1_m)$, see (1.4), i.e. a system of linear independent modulo $p$ vectors spanning an isotropic submodule. In particular they should also form an isotropic system of $d$ vectors in $V^2_p(1_m)$. The number of isotropic systems in $V^2_p(1_m)$ consisting of $d$ vectors was computed in [3], Proposition A.2.14 and equals to

$$p^{d(d-1)/2}(p^{2k} - 1)(p^{2(k-1)} - 1) \cdots (p^{2(k-d+1)} - 1).$$

It remains to compute in how many ways such a system modulo $p$ can be lifted to an isotropic system in $V^2_p(1_m)$. Note that any vector $L \in (\mathbb{Z}/p\mathbb{Z})^m$ splits into $p^m$ different vectors of the form $L + pX$ in $(\mathbb{Z}/p\mathbb{Z})^m$. Therefore, if $\{L_1, \ldots, L_d\}$ is an isotropic system in $V^2_p(1_m)$, then $\{L_1 + pX_1, \ldots, L_d + pX_d\}$ would form an isotropic system in $V^2_p(1_m)$ if and only if

$$^t(L_i + pX_i)Q(L_j + pX_j) \equiv ^tL_iQL_j + p(^tL_iQX_j + ^tX_iQL_j) \equiv 0 \pmod{p^2}$$

for $1 \leq i \leq j \leq d$. (We do not worry about linear independence of the vectors because linear independence modulo $p^\delta$ is equivalent for any $\delta$ to linear independence modulo $p$, as we already noted in the proof of Lemma 1.3.) The above congruences form a system of $d(d+1)/2$ linear equations (over $\mathbb{F}_p$) with respect to $md$ unknowns – elements of columns $X_i$. Since columns $L_i$, $1 \leq i \leq d$ are linearly independent modulo $p$, the rank of the matrix of this linear system as well as the rank of its extended matrix is equal to $d(d+1)/2$ and therefore the system has

$$p^{dm - d(d+1)/2}$$

solutions in $\mathbb{F}_p$. (Note that this is exactly the number of different isotropic systems in $V^2_p(1_m)$ which lie above a fixed isotropic system in $V_p(1_m)$.) We conclude that there exist

$$p^{md - d(d+1)/2} \cdot p^{d(d-1)/2} \prod_{s=0}^{d-1} (p^{2(k-s)} - 1) = p^{d(m-1)} \cdot \prod_{s=0}^{d-1} (p^{2(k-s)} - 1)$$

different isotropic systems $\{U_1, \ldots, U_d\} \subset V^2_p(1_m)$. In how many ways such a system can be extended to an isotropic basis of the form (1.6)? From (1.5) one can see that each
\[ U_j, \ d \leq j \leq d + b \] should be orthogonal to the linear span of \( U_1, \ldots, U_d \), i.e. it should satisfy the following system of linear equations

\[
\begin{pmatrix}
\begin{bmatrix}
^t U_1 Q \\
\vdots \\
^t U_d Q
\end{bmatrix} \\
y_1 \\
y_m
\end{pmatrix} \equiv 0 \pmod{p}
\]

over \( \mathbb{F}_p \). The above system has \( m - d = d + b \) linear independent (modulo \( p \)) solutions. Besides, each of \( U_i, 1 \leq i \leq d \) is its solution. Therefore the remaining (ordered) set \( U_i, d + 1 \leq i \leq d + b \) can be chosen in

\[ p^{db} \cdot |GL_b(\mathbb{F}_p)| \]

different ways. Multiplying (1.8) by the number of different isotropic systems of \( d \) vectors in \( V_{p^2}(1_m) \) computed above, we deduce the result of the Lemma.

Next we need to determine which of the bases in question generate the same isotropic module. Assume that \( \{ U_1, \ldots, pU_{d+b} \} \) and \( \{ V_1, \ldots, pV_{d+b} \} \) are two (ordered) bases of the form (1.6) such that \( V_{p^2}(U_1, \ldots, pU_{d+b}) = V_{p^2}(V_1, \ldots, pV_{d+b}) \). Then

\[
(V_1, \ldots, V_d, pV_{d+1}, \ldots, pV_{d+b}) \equiv (U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b}) \cdot \begin{pmatrix} A & B \\ pC & D \end{pmatrix} \pmod{p^2},
\]

where \( A \in \mathbb{Z}^d_d, B \in \mathbb{Z}_b^d, C \in \mathbb{Z}_d^b, D \in \mathbb{Z}_d^d \). Because \( (U_1, \ldots, U_{d+b}) \cdot \begin{pmatrix} B \\ pD \end{pmatrix} \equiv 0 \pmod{p} \), then necessarily \( B \equiv 0 \pmod{p} \). Since vectors

\[
(V_1, \ldots, V_{d+b}) \equiv (U_1, \ldots, U_{d+b}) \cdot \begin{pmatrix} A & p^{-1}B \\ pC & D \end{pmatrix} \pmod{p^2}
\]

are linearly independent modulo \( p \), then the system of congruences

\[
\begin{pmatrix} A & p^{-1}B \\ pC & D \end{pmatrix} \cdot \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{d+b} \end{pmatrix} \equiv 0 \pmod{p}
\]

should have only trivial solution, for which it is necessary and sufficient that

\[
\begin{pmatrix} A & p^{-1}B \\ pC & D \end{pmatrix} \in GL_{d+b}(\mathbb{F}_p),
\]

i.e. \( A \in GL_d(\mathbb{F}_p) \) and \( D \in GL_b(\mathbb{F}_p) \). Therefore two sets of vectors \( \{ U_1, \ldots, pU_{d+b} \} \) and \( \{ V_1, \ldots, pV_{d+b} \} \) generate the same isotropic module only if they differ by a matrix from the group

\[
\Gamma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{d+b}(\mathbb{Z}/p^2\mathbb{Z}) \right\};
\]

\[
A \in GL_d(\mathbb{Z}/p^2\mathbb{Z}), \ D \in GL_b(\mathbb{Z}/p^2\mathbb{Z}), \ B \equiv 0 \pmod{p}
\].

(1.9)
Clearly the converse is also true. Thus $\Gamma$ acts transitively on the set of bases of the form (1.6) of an isotropic module of type $V_{p^2}(UD_p)$. Let $\Gamma'$ denote stabilizer of a basis $(U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b})$ under this action and let \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma' \), i.e.

\[
(U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b}) \cdot \begin{pmatrix} 1_d - A \\ C \\ \end{pmatrix} \begin{pmatrix} B \\ 1_b - D \end{pmatrix} \equiv 0 \pmod{p^2}.
\]

Because $U_i$'s are linearly independent over $\mathbb{F}_p$, the last congruence is equivalent to the set of conditions: $A \equiv 1_d \pmod{p^2}$, $B \equiv 0 \pmod{p^2}$, $C \equiv 0 \pmod{p}$, $D \equiv 1_b \pmod{p}$. Combining this with (1.9) one can see that the cardinality of $\Gamma$ modulo the stabilizer $\Gamma'$ is given by

\[
|\Gamma/\Gamma'| = p^{d(d+2b)} \cdot |GL_b(\mathbb{F}_p)| \cdot |GL_b(\mathbb{F}_p)|
\]

and this is the number of different bases of form (1.6) generating some fixed isotropic module of the type $V_{p^2}(UD_p), U \in \Lambda^m$. Using Lemma 1.5 we conclude that the number of different isotropic modules of the type we are interested in is equal to

\[
S_{p^2}(q, D, 0) = p^{d(d-1)} \cdot |GL_d(\mathbb{F}_p)|^{-1} \cdot \prod_{s=0}^{d-1} (p^{2(k-s)} - 1) ,
\]

where $k = (m - 1)/2$ and $0 \in \mathbb{Z}^m$ is the zero vector. We note also that since the zero vector belongs to any submodule of $V_{p^2}(1_m)$, then $S_{p^2}(q, D, K) = S_{p^2}(q, D, 0)$ in case $K \equiv 0 \pmod{p^2}$, see (1.3).

In order to finish our computation of isotropic sums (1.3), we still need to find the number of isotropic modules of the form $V_{p^2}(UD_p), U \in \Lambda^m$, which contain a fixed nonzero isotropic vector $K \in (\mathbb{Z}/p^2\mathbb{Z})^m$. It will be more convenient to consider two separate cases: either $K \not\equiv 0 \pmod{p}$, or $K \equiv 0 \pmod{p}$ but $K \not\equiv 0 \pmod{p^2}$.

**LEMMA 1.6.** Under the above notations, assume that $K \not\equiv 0 \pmod{p}$. Then there exist

\[
S_{p^2}(q, D, K) = (p^d - 1) \cdot p^{(d-1)^2} \cdot |GL_d(\mathbb{F}_p)|^{-1} \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1)
\]

different isotropic modules of the form $V_{p^2}(UD_p), U \in \Lambda^m$, which contain the vector $K$.

**Proof.** If $K \in V_{p^2}(UD_p) = V_{p^2}(U_1, \ldots, U_d, pU_{d+1}, \ldots, pU_{d+b})$ and $K \not\equiv 0 \pmod{p}$ then this vector can be complemented to a basis $\{V_1, \ldots, K, \ldots, V_d, pV_{d+1}, \ldots, pV_{d+b}\}$ of $V_{p^2}(UD_p)$. As we already know, such a basis differs from $\{U_1, \ldots, pU_{d+b}\}$ only by a matrix from $\Gamma$, see (1.9). Moreover, if

\[
K \equiv \sum_{i=1}^{d} c_i U_i + p \sum_{j=d+1}^{d+b} c_j U_j \pmod{p^2} ,
\]

(1.11)
then among the first $d$ columns of a matrix from $\Gamma$ which transforms $\{U_1, \ldots, pU_{d+b}\}$ into $\{V_1, \ldots, K, \ldots, V_d, pV_{d+1}, \ldots, pV_{d+b}\}$ there should be a column of the form

\[ t(\gamma_1, \ldots, \gamma_d, \gamma'_{d+1}, \ldots, \gamma'_{d+b}) \text{ with } \gamma'_j \equiv \gamma_j \pmod{p} \text{ for } d + 1 \leq j \leq d + b \]  

Since there exist exactly

\[ \frac{d}{p^d - 1} \cdot |GL_d(F_p)| \cdot |GL_b(F_p)| \cdot p^{d^2 + 2b^2 - 3db - d - b} \]

different matrices from $\Gamma$ with this property, then dividing by the order of the stabilizer $|\Gamma'|$ we deduce that a fixed isotropic module $V_{p^2}(UD_p)$ contains

\[ \frac{d}{p^d - 1} \cdot |GL_d(F_p)| \cdot |GL_b(F_p)| \cdot p^{d^2 + 2db - d - b} \]  

(1.12)
different bases of the form $\{V_1, \ldots, K, \ldots, V_d, pV_{d+1}, \ldots, pV_{d+b}\}$. Now we just need to find the total number of bases of the form (1.6) which contain $K$ among their first $d$ vectors.

We will proceed in a fashion similar to the proof of Lemma 1.5: first we compute the number of isotropic systems of $d$ vectors in $(\mathbb{Z}/p^2\mathbb{Z})^m$ containing the column $K$ and then multiply the result by the number of ways such an isotropic system can be extended to an isotropic basis of the form we are interested in (the latter was already computed in the course of proof of Lemma 1.5, see (1.8)).

We already have noted above that any isotropic system modulo $p^2$ is also an isotropic system modulo $p$. Therefore we start with computation of the number of isotropic systems of $d$ vectors in $(\mathbb{Z}/p\mathbb{Z})^m$ containing $K \mod p$. We use induction: assume that $X_1, \ldots, X_{n-1} \in (\mathbb{Z}/p\mathbb{Z})^m$ form an isotropic system. In how many ways it can be complemented to an isotropic system of $n$ vectors? Since $X_1, \ldots, X_{n-1}$ form a basis of the isotropic subspace $V_p(X_1, \ldots, X_n)$ of nondegenerate quadratic space $V_p(1_m)$ then there exist vectors $X'_1, \ldots, X'_{n-1} \in V_p(1_m)$ such that each pair $X_i, X'_i$, $1 \leq i \leq n-1$, is hyperbolic, i.e. vectors $X_i, X'_i$ are isotropic, linear independent and $t(X_i, X'_i) \equiv 1 \pmod{p}$, see [3], proposition A.2.12 or [9], Satz 2.24. Then the spaces $V_p(X_i, X'_i)$, $1 \leq i \leq n - 1$ are (nondegenerate) hyperbolic planes and

\[ V_p(1_m) = V_p(X_1, X'_1) \oplus V_p(X_{n-1}, X'_{n-1}) \oplus V \]

is a direct sum of pairwise orthogonal subspaces. Therefore a vector $Y \in V_p(1_m)$ can be isotropic and orthogonal to each of $X_i$'s only if

\[ Y \equiv \sum_{i=1}^{n-1} \alpha_i X_i + v \pmod{p} \]

where $v$ is an isotropic vector of $V$. The vectors $X_1, \ldots, X_{n-1}, Y$ will be linearly independent only if $v \not\equiv 0 \pmod{p}$. Since $V$ is nondegenerate and $\dim V = m - 2(n - 1)$ then the number of nonzero isotropic vectors in it equals to $p^{m-2n+1} - 1$ (see [3], Proposition
A.2.14). Thus the vector $X_n$ complementing $\{X_1, \ldots, X_{n-1}\}$ to an isotropic system of $n$ vectors can be chosen in $p^{n-1}(p^{m-2n+1} - 1)$ ways. Using induction on $n$ we conclude that a fixed isotropic vector $K \not\equiv 0 \pmod{p}$ can be complemented to

$$p^{d(d-1)/2} \cdot \prod_{s=1}^{d-1} (p^{2(k-s)-1})$$

different (ordered) isotropic systems $\{K, X_2, \ldots, X_d\} \subset V_p(1, m)$. Clearly, the total number of isotropic systems of $d$ vectors in $V_p(1, m)$ containing $K$ is $d$ times that. Next we need to lift our solution to $V_{p^2}(1, m)$ keeping in mind that $K$ is fixed modulo $p^2$. Since any vector $X \in (\mathbb{Z}/p\mathbb{Z})^m$ splits in $(\mathbb{Z}/p^2\mathbb{Z})^m$ into $p^m$ different vectors $X + pY$, $Y \in (\mathbb{Z}/p\mathbb{Z})^m$, we need to look for isotropic systems of the form

$$\{X_1 + pY_1, \ldots, K = X_i, \ldots, X_d + pY_d\} \subset (\mathbb{Z}/p^2\mathbb{Z})^m .$$

Such a system will be isotropic modulo $p^2$ if and only if

$$^t KQY_j \equiv -\frac{1}{p} \cdot ^t X_jQK \pmod{p} ;$$

$$^t X_sQY_j + {^t X_jQY_s} \equiv -\frac{1}{p} \cdot ^t X_sQX_j \pmod{p} \text{ for } s, j \neq i .$$

This is a system of $(d-1)(d+2)/2$ linear nonhomogeneous equations in $m(d-1)$ variables (elements of columns $U_j, j \neq i$) over $\mathbb{F}_p$. Since the columns $X_1, \ldots, X_d$ are linearly independent modulo $p$, so are the columns $QX_1, \ldots, QX_d$, and therefore the rank of the matrix of the above system as well as the rank of its extended matrix is equal to $(d-1)(d+2)/2$, which implies that it has $p^{(d-1)(m-(d+2)/2)}$ solutions. We conclude that $V_{p^2}(1, m)$ contains

$$dp^{(m-1)(d-1)} \cdot \prod_{s=1}^{d-1} (p^{2(k-s)})$$
different isotropic systems of $d$ vectors one of which is $K$. According to (1.8), each of these systems can be complemented to a basis of the form (1.6) of some isotropic module $V_{p^2}(UD_p)$, $U \in \Lambda^m$ in $p^{db}|GL_b(\mathbb{F}_p)|$ different ways. Multiplying the last two numbers we get the total number of bases of the form (1.6) which contain $K$ among their first $d$ vectors.

Dividing the latter by (1.12) we finally deduce the result of the Lemma.

It remains to consider the isotropic sum $S_{p^2}(\mathbf{q}, D_p, K)$ in the case when $K \equiv 0 \pmod{p}$ but $K \not\equiv 0 \pmod{p^2}$. Let $K \in V_{p^2}(UD_p)$ for some $U = (U_1, \ldots, U_m) \in \Lambda^m$ and assume (1.11). Then $^t(\gamma_1, \ldots, \gamma_d) \equiv 0 \pmod{p}$ and we are faced with two possibilities:

$$(i) \quad (\gamma_{d+1}, \ldots, \gamma_{d+b}) \not\equiv 0 \pmod{p};$$

$$(ii) \quad (\gamma_{d+1}, \ldots, \gamma_{d+b}) \equiv 0 \pmod{p}, (\gamma_1, \ldots, \gamma_d) \not\equiv 0 \pmod{p^2} .$$

In the first case the column $^t(\gamma_1, \ldots, \gamma_{d+b})$ can be complemented to a matrix from $\Gamma$, see (1.9), and in the second case this is impossible. Note that the type of representation
(1.11) of the vector \( K \) does not depend on the choice of basis of \( V_{p^2}(UD_p) \). Therefore, if \( K \equiv 0 \pmod{p} \) but \( K \not\equiv 0 \pmod{p^2} \) then there exists two distinct types of isotropic modules which contain \( K \): in modules of the first type there exist a basis of the form \( \{V_1, \ldots, V_d, pV_{d+1}, \ldots, K, \ldots, pV_{d+b}\} \), but in the modules of the second type there are no such bases. In the two following Lemmas we compute the number of isotropic modules of each these two types.

**Lemma 1.7.** Under the above notations, assume that \( K \equiv 0 \pmod{p} \) but \( K \not\equiv 0 \pmod{p^2} \). Then the number of isotropic modules of the form \( V_{p^2}(UD_p) \), \( U \in \Lambda^m \) for which (1.11) implies case (i) above is equal to

\[
\begin{cases}
\eta_p \cdot p^{-d}(p^k - \chi_q(p)\varepsilon_p)(p^{k-d} + \chi_q(p)\varepsilon_p)\prod_{s=1}^{d-1}(p^{2(k-s)} - 1), & \text{if } q(p^{-1}K) \not\equiv 0 \pmod{p}, \\
\eta_p \cdot \prod_{s=1}^{d}(p^{2(k-s)} - 1), & \text{if } q(p^{-1}K) \equiv 0 \pmod{p}.
\end{cases}
\]

Here \( \eta_p = p^{d^2+b-1}(p^b - 1)|GL_{b-1}(\mathbb{F}_p)||GL_d(\mathbb{F}_p)|^{-1}|GL_b(\mathbb{F}_p)|^{-1} \), the character \( \chi_q(p) \) of the form \( q \) is defined via (2.4) below and \( \varepsilon_p = \varepsilon_p \left(q(p^{-1}K)\right) \) is given by the Legendre symbol

\[
\varepsilon_p(a) = \left(\frac{2a}{p}\right), \quad a \in \mathbb{N}.
\]

**Proof.** As we already noted above, if an isotropic module satisfies conditions of the Lemma, then it has a basis of the form \( \{V_1, \ldots, V_d, pV_{d+1}, \ldots, K, \ldots, pV_{d+b}\} \). Thus we need to compute the total number of bases of the form (1.6) which have \( K \) among the last \( b \) vectors, and then divide it by the number of such bases generating the same isotropic module. The latter is equal to the number of matrices in \( \Gamma \) (1.9) one of whose last \( b \) columns is fixed modulo \( (\mathbb{Z}/p^2\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})^b \) divided by the order of stabilizer \( |\Gamma'| \):

\[
b(p^b - 1)^{-1}|GL_d(F_p)| \cdot |GL_B(\mathbb{F}_p)|p^{d^2+2db-d}.
\]

Now we need to compute the total number of bases of the form (1.6) which have \( K \) among their last \( b \) vectors. Assume first that \( q(p^{-1}K) \not\equiv 0 \pmod{p} \), i.e. the column \( p^{-1}K \) is anisotropic. Then \( V_p(1_m) = V_p(p^{-1}K) \oplus V \), where \( V \) is an orthogonal complement of \( V_p(p^{-1}K) \) (see [9], Satz 1.15). The module \( V \) is nondegenerate and its dimension \( \dim V = m - 1 \) is even. Any vector orthogonal to \( p^{-1}K \) belongs to \( V \) and in particular is linear independent (modulo \( p \)) with \( p^{-1}K \). Therefore first \( d \) columns of a basis of the type in question should form an isotropic system in \( V \). According to Proposition A.2.14 of [3], the number of such systems is equal to

\[
p^{d(d-1)/2}(p^k - \chi_q(p)\varepsilon_p)(p^{k-d} + \chi_q(p)\varepsilon_p)\prod_{s=1}^{d-1}(p^{2(k-s)} - 1),
\]

where \( \varepsilon_p = \varepsilon_p \left(q(p^{-1}K)\right) \) is the Legendre symbol (1.13). Each of these systems splits into \( p^{dm-d(d+1)/2} \) different isotropic systems in \( V_{p^2}(1_m) \), see (1.7), any of which can be
chosen as first $d$ columns of a basis of the form (1.6). Now we need to find in how many ways an isotropic system $U_1, \ldots, U_d, K$ can be complemented by $b-1$ vectors from $(\mathbb{F}_p)^m$ to form a basis of the type (1.6). Orthogonality conditions imply that elements of each of the complementary columns should satisfy the following system of linear homogeneous equations:

$$
\begin{pmatrix}
^{t}U_1Q \\
\vdots \\
^{t}U_dQ
\end{pmatrix} \begin{pmatrix}
y_1 \\
\vdots \\
y_m
\end{pmatrix} \equiv 0 \pmod{p}
$$

This system of $d$ equations in $m$ variables over $\mathbb{F}_p$ has $d + b$ linear independent modulo $p$ solutions which can be chosen in the form $\vartheta = (U_1, \ldots, U_d, p^{-1}K, V_2, \ldots, V_b)$. Any other set of linear independent solutions has the form $\vartheta \cdot M$ with $M \in GL_{d+b}(\mathbb{F}_p)$. Among such matrices $M$ there are exactly $b|GL_{b-1}(\mathbb{F}_p)|p^{(b-1)(d-1)}$ matrices which do not change the first $d$ columns of the basis and leave $p^{-1}K$ intact (possibly changing its position). And this is exactly the number of different (modulo $p$) sets of vectors $pU_{d+1}, \ldots, K, \ldots, pU_{d+b}$ which complement a given isotropic system $U_1, \ldots, U_d$ to a basis of the form (1.6). Multiplying this number by (1.15) times (1.7) and dividing by (1.14) we deduce the statement of the Lemma in the case when $q(p^{-1}K) \not\equiv 0 \pmod{p}$.

We turn to the case of an isotropic vector $p^{-1}K \in \mathbb{Z}^m$. Assume that $q(p^{-1}K) \equiv 0 \pmod{p}$ and let $V \subset V_p(1_m)$ again denote its orthogonal complement. We see that $\dim V = m - 1$ again but now $V$ is degenerate: it has nontrivial radical $rad V = V_p(p^{-1}K)$. Nevertheless, $V = \text{rad } V + W$, where $W$ is a nondegenerate subspace orthogonal to $\text{rad } V$. The first $d$ vectors of a basis of the type we are interested in should form an isotropic system in $V$. Suppose that $U_1, \ldots, U_d$ is a set of elements of $V$. Each of these vectors can be uniquely represented in the form

$$U_i \equiv \alpha_i p^{-1}K + w_i \pmod{p}, \quad \text{where } \alpha_i \in \mathbb{F}_p, w_i \in W.$$  

Naturally, each of $U_i$’s is orthogonal to $p^{-1}K$. They will be isotropic an pairwise orthogonal if and only if $^{t}w_iQw_j \equiv 0 \pmod{p}$ for $1 \leq i \leq j \leq d$. Furthermore, the columns $U_1, \ldots, U_d, p^{-1}K$ will be linearly independent (modulo $p$) if and only if $w_1, \ldots, w_d \in W$ are linearly independent. We conclude that $U_1, \ldots, U_d$ will be an isotropic system if and only if $w_1, \ldots, w_d$ form an isotropic system. The number of isotropic systems of $d$ vectors in $W$ is equal to

$$p^{d(d-1)/2} \prod_{s=1}^{d} (p^{2(k-s)} - 1)$$

(see [3], Proposition A.2.14). Multiplying this number by $p^d$ we get the number of different isotropic systems in $V$ consisting of $d$ vectors linear independent with $p^{-1}K$. The rest of the computations completely coincide with the case of anisotropic $p^{-1}K$ and lead to the result stated in the Lemma. 

Let us turn to the second case.
Lemma 1.8. Under the above notations, assume that $K \equiv 0 \pmod{p^2}$ but $K \not\equiv 0 \pmod{p^2}$. Then the number of isotropic modules of the form $V_{p^2}(UD_p)$, $U \in \Lambda^m$ for which (1.11) implies case (ii) above is equal to

$$\begin{cases} 0, & \text{if } q(p^{-1}K) \not\equiv 0 \pmod{p}, \\ p^{(d-1)^2}|GL_{d-1}(\mathbb{F}_p)|^{-1} \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1), & \text{if } q(p^{-1}K) \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let $\{U_1, \ldots, pU_{d+b}\}$ be a basis of the form (1.6). Assume that (1.11) holds for some column $^t(\gamma_1, \ldots, \gamma_{d+b})$. Then the conditions $(\gamma_1, \ldots, \gamma_{d+b}) \equiv 0 \pmod{p}$ and $(\gamma_1, \ldots, \gamma_d) \not\equiv 0 \pmod{p^2}$ will hold if and only if $p^{-1}K \in V_p(U_1, \ldots, U_d)$. But the subspace $V_p(U_1, \ldots, U_d) \subset V_p(1_m)$ is isotropic, therefore the column $^t(\gamma_1, \ldots, \gamma_{d+b})$ satisfies (ii) if and only $q(p^{-1}K) \equiv 0 \pmod{p}$, otherwise there exist no isotropic modules which would satisfy the conditions of the Lemma. This proves its first case.

Suppose now that $q(p^{-1}K) \equiv 0 \pmod{p}$. In order to find the number of isotropic modules we are interested in, we need to divide the number of different bases of the form (1.6) for which $p^{-1}K \in V_p(U_1, \ldots, U_d)$ by the number of different bases of the form (1.6) which generate the same isotropic module, i.e. by $|\Gamma/\Gamma'|$ computed above, see discussion immediately following (1.9). Next, the number of bases of the form (1.6) with the property in question is equal to the product of the number of different isotropic systems $\{U_1, \ldots, U_d\} \in V_{p^2}(1_m)$ for which $p^{-1}K \in V_p(U_1, \ldots, U_d)$ multiplied by the number of ways an isotropic system of $d$ vectors can be complemented to a basis of the form (1.6), i.e. by $p^{db} \cdot |GL_b(\mathbb{F}_p)|$, see (1.8). According to (1.7), the number of isotropic systems $\{U_1, \ldots, U_d\} \in V_{p^2}(1_m)$ for which $p^{-1}K \in V_p(U_1, \ldots, U_d)$ is equal in turn to the number of isotropic systems $\{U_1, \ldots, U_d\} \in V_p(1_m)$ for which $p^{-1}K \in V_p(U_1, \ldots, U_d)$ times $p^{dn-d(d+1)/2}$. Thus it remains to find the number of isotropic systems $\{U_1, \ldots, U_d\} \in V_p(1_m)$ for which $p^{-1}K \in V_p(U_1, \ldots, U_d)$. Since $K \not\equiv 0 \pmod{p^2}$ then $p^{-1}K$ is a nonzero isotropic vector of $V_p(1_m)$, and so if $p^{-1}K \in V_p(U_1, \ldots, U_d)$ then $p^{-1}K$ can be complemented to a basis of $V_p(U_1, \ldots, U_d)$. Therefore the number of different isotropic systems in $V_p(1_m)$ with the property in question is equal to the number of isotropic systems of $d$ vectors in $V_p(1_m)$ containing the vector $p^{-1}K$ (which was computed in the proof of Lemma 1.6) divided by $dp^{d-1} \cdot |GL_{d-1}(\mathbb{F}_p)|$, i.e. by the number of such isotropic systems generating the same module, and multiplied by $|GL_d(\mathbb{F}_p)|$, i.e. by the number of different bases of a $d$-dimensional linear space over $\mathbb{F}_p$. This finishes the proof of the Lemma.

Combining the results of Lemmas 1.4 and 1.6–1.8 along with formulas (1.3) and (1.10), we deduce the following Theorem:

Theorem 1.9. Let $q$ be an arbitrary integral nonsingular quadratic form in an odd number of variables $m = 2k + 1 \geq 3$, and let $p$ be a rational prime not dividing the determinant $\det q$ of the form $q$. Then for each integral column $K \in \mathbb{Z}^m$ such that $q[K] \equiv 0 \pmod{p^2}$ the value of the isotropic sum (1.3) is given by

$$S_{p^2}(q, D_p(d), K) = \alpha_p(d) \cdot \left\{ 1 + (\varepsilon_p \chi_q(p)p^{k-1} + \kappa_p(d)) \cdot \delta_{(p^{-1}K)} + p^{m-2} \cdot \delta_{(p^{-2}K)} \right\}, \quad (1.16)$$
where \( D_p(d) \), \( 1 \leq d \leq k \) is a matrix of the form (1.2), \( \varepsilon_p = \varepsilon_p(\mathbf{q}(p^{-1}K)) \) is given by the Legendre symbol (1.13), the character \( \chi_q(p) \) of the form \( \mathbf{q} \) is defined via (2.4), \( \delta_X \) is the generalized Kronecker symbol (2.7) and 

\[
\alpha_p(d) = p^{1-d} \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1)/(1 - p^{-s}) ,
\]

\[
\kappa_p(d) = \frac{p^{m-2} - p^{d-1}}{p^d - 1} - 1 .
\]

**Proof.** It is well-known that for \( n \geq 1 \)

\[
|GL_n(\mathbb{F}_p)| = \prod_{s=0}^{n-1} (p^n - p^s) = p^n \cdot \prod_{s=1}^{n} (1 - p^{-s}) .
\]

We also put \(|GL_0(\mathbb{F}_p)| = 1\) and note that \(|GL_{n-1}(\mathbb{F}_p)|^{-1} \cdot |GL_n(\mathbb{F}_p)| = p^{n-1} \cdot (p - 1)\).

Assume that \( K \equiv 0 \pmod{p} \), then according to Lemma 1.6

\[
S_{p^2}(\mathbf{q}, D_p(d), K) = p^{(d-1)^2 - d^2} (p^d - 1) \cdot (1 - p^{-d})^{-1} \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1)/(1 - p^{-s}) = \alpha_p(d) ,
\]

which coincides with the right-hand side of (1.16) for such \( K \) since \( \delta_{(p^{-1}K)} = \delta_{(p^{-2}K)} = 0 \) in this case. Next, assume that \( K \equiv 0 \pmod{p} \) but \( K \not\equiv 0 \pmod{p^2} \). Combining results of Lemmas 1.7, 1.8 and employing a rather straightforward calculation one can see that \( S_{p^2}(\mathbf{q}, D_p(d), K) \) is equal to

\[
(p^d - 1)^{-1} (p^k - \varepsilon_p\chi_q(p)) (p^{k-d} + \varepsilon_p\chi_q(p)) \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1)/(1 - p^{-s}) ,
\]

if \( \mathbf{q}(p^{-1}K) \not\equiv 0 \pmod{p} \) and

\[
(p^d - 1)^{-1} (p^{a+b-1} - 1) \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1)/(1 - p^{-s}) ,
\]

if \( \mathbf{q}(p^{-1}K) \equiv 0 \pmod{p} \). Note that in the latter case the symbol \( \varepsilon_p = \varepsilon_p(\mathbf{q}(p^{-1}K)) \) in (1.13) is equal to zero and that otherwise \( \varepsilon_p = \pm 1 \). Therefore

\[
S_{p^2}(\mathbf{q}, D_p(d), K) = (p^d - 1)^{-1} . (p^{d+b-1} - 1 + \varepsilon_p\chi_q(p)p^k(1 - p^{-d})) \cdot \prod_{s=1}^{d-1} (p^{2(k-s)} - 1)/(1 - p^{-s}) = \alpha_p(d) \cdot \{ 1 + \varepsilon_p\chi_q(p)p^{k-1} + \kappa_p(d) \} .
\]
which coincides with the right-hand side of (1.16) for \( K \equiv 0 \pmod{p} \), \( K \not\equiv 0 \pmod{p^2} \).

Finally, if \( K \equiv 0 \pmod{p^2} \) then \( \varepsilon_p = 0 \) and, using (1.10) we have

\[
S_{p^2}(q, D_p(d), K) = \alpha_p(d) \cdot p^{d-1} \cdot (p^{2k} - 1)/(p^d - 1)
\]

which coincides with the right-hand side of (1.16) for such \( K \).

Summing up the equalities (1.16) over \( d \) ranging from 1 to \( k = (m - 1)/2 \) we immediately deduce the following

**Corollary 1.10.** Under the notations and assumptions of Theorem 1.8,

\[
\sum_{d=1}^k S_{p^2}(q, D_p(d), K) = c_p \cdot \left\{ 1 + (\varepsilon_p q(p)p^{h-1} + c_p^{-1}\beta_p) \cdot \delta_{p-1}K + p^{m-2} \cdot \delta_{p-2}K \right\}
\]

where \( c_p = c_p(q) \) and \( \beta_p = \beta_p(q) \) are given by (2.3).

We have finished computation of the isotropic sums of type (1.3). In order to apply the results of Theorem 1.9 and Corollary 1.10 to investigation of multiplicative arithmetic of integral representations by quadratic form \( q \), we need to rewrite the isotropic sums in a slightly different terms. Recall that the Minkowski reduction theory of quadratic forms (see [3] for example) shows that the similarity class of a nonsingular integral quadratic form in \( m \) variables is a finite union of mutually disjoint classes of integrally equivalent quadratic forms. Let us fix a complete system

\[ \{q_1, \ldots, q_h\} \]

of representatives of different equivalence classes contained in the similarity class of \( q \). Then the isotropic sum (1.3) can be rewritten as follows. The condition \( q[M] \equiv 0 \pmod{p^2} \) means that \( q[M] = p^2 q' \), where \( q' \) is an integral quadratic form in \( m \) variables. Since \( \det M = \det D_p = p^m \) then \( \det q' = \det q \) and thus quadratic form \( q' \) is similar to \( q \). Since \( M \) is defined only modulo right multiplication by \( \Lambda^m \), we can replace \( q' \) by any (integrally) equivalent form \( q'[U], U \in \Lambda^m \). In particular we can assume that \( q[M] = p^2 q_i \) for exactly one of \( q_i \)'s, \( 1 \leq i \leq h \) and \( M \in (R(q, p^2 q_i) \cap \Lambda^m D_p(d) \Lambda^m) / \Lambda^m \) for some \( d \). Furthermore, if \( M, MU \in R(q, p^2 q_i) \cap \Lambda^m D_p(d) \Lambda^m \) where \( U \in \Lambda^m \) then \( U \in R(q_i, q_i) = E(q_i) \) and so \( M \in R(q, p^2 q_i)/E(q_i) \cap \Lambda^m D_p(d) \Lambda^m \). Therefore

\[
S_{p^2}(q, D_p(d), K) = \sum_{i=1}^h \sum_{M \in \left( R(q, p^2 q_i) \cap \Lambda^m D_p(d) \Lambda^m \right) / E(q_i) \Lambda^m} 1
\]

Next we note that according to the Lemma 1.2 the set of primitive automorphs \( R^*(q, p^2 q_i) \) is a disjoint union

\[
R^*(q, p^2 q_i) = \bigcup_{1 \leq d \leq (m-1)/2} \left( R(q, p^2 q_i) \cap \Lambda^m D_p(d) \Lambda^m \right)
\]
Therefore summing both sides of (1.17) over \(d\) ranging from 1 to \((m - 1)/2\) and applying Corollary 1.10 we deduce the following Theorem:

**Theorem 1.11.** Let \(q\) be an arbitrary integral nonsingular quadratic form in an odd number of variables \(m = 2k + 1 \geq 3\), and let \(p\) be a rational prime not dividing the determinant \(\det q\) of the form \(q\). Let \(\{q_1, \ldots, q_h\}\) be a complete system of representatives of different equivalence classes contained in the similarity class of \(q\). Then for each integral column \(K \in \mathbb{Z}^m\) such that \(q[K] \equiv 0 \pmod{p^2}\) we have

\[
\sum_{i=1}^{h} \sum_{M \in R^* (q, p^2 q_1)/E(q_i) \cap M|K} 1 = c_p \cdot \left\{ 1 + (\varepsilon_p \chi(p) p^{k-1} + c_p^{-1} \beta_p) \cdot \delta(p^{-1}, K) + p^{m-2} \cdot \delta(p^{-2}, K) \right\}, \tag{1.18}
\]

where the symbol \(\varepsilon_p = \varepsilon_p (q(p^{-1}K))\) is defined in (1.13), \(\chi(p)\) is the character (2.4), \(\delta(X)\) is the generalized Kronecker symbol (2.7) and \(c_p = c_p(q), \beta_p = \beta_p(q)\) are given by (2.3).
2. Action of Hecke operators on theta-series

In case when $q$ is positive definite, the numbers $r(q, n)$ are finite and one can consider the theta-series

$$\Theta(z, q) = \sum_{X \in \mathbb{Z}^m} e^{2\pi i q(X)} = \sum_{n \geq 0} r(q, n) e^{2\pi i nz}, \quad z \in \mathbb{H}$$

and show that it is a (classical) modular form of weight $m/2$ and some character with respect to a certain congruence subgroup (see, for example [3], Theorem 2.2.2). Although theta-series generally are not eigenfunctions of Hecke operators, but the spaces spanned by theta-series which come from a fixed similitude class of quadratic forms are invariant under the action of Hecke operators. The fundamental discovery of M. Eichler [6] is that in this situation the corresponding eigenmatrices (“Anzahlmatrizen” in Eichler’s terminology) are purely arithmetical and can be defined without any reference to modular forms. (The same phenomenon occurs in case of Siegel modular forms of arbitrary degree, as was shown by E. Freitag [7] and A. Andrianov [3].)

More specifically, let $\{q_1, \ldots, q_h\}$ be a full system of representatives of different equivalence classes of the similarity class of $q$ and let $p$ be a prime number which does not divide $\det q$, then

$$\Theta(z, q) \mid_{T(p)} T(p) = c_p^{-1} \cdot \sum_{1 \leq j \leq h} r^*(q, p q_j) e(q_j) \Theta(z, q_j), \quad (2.1)$$

if $m$ (the number of variables of $q$) is even and

$$\Theta(z, q) \mid_{T(p^2)} T(p^2) = c_p^{-1} \left( \sum_{1 \leq j \leq h} r^*(q, p^2 q_j) e(q_j) \Theta(z, q_j) - \beta_p \Theta(z, q) \right), \quad (2.2)$$

if $m$ is odd. Here

$$c_p = c_p(q) = \begin{cases} 1 & \text{if } m = 2, \\ \prod_{i=0}^{k-2} (1 + \chi_q(p) p^i) & \text{if } m = 2k > 2, \\ \sum_{a=1}^{k} \prod_{i=1}^{a-1} \frac{p^{2(k-i)} - 1}{1 - p^{-i}} \cdot p^{1-a} & \text{if } m = 2k + 1 \geq 3, \end{cases} \quad (2.3)$$

$$\beta_p = \beta_p(q) = \sum_{a=1}^{k} \left( \prod_{i=1}^{a-1} \frac{p^{2(k-i)} - 1}{1 - p^{-i}} \cdot \frac{p^{m-2} - p^{a-1}(p + 1) + 1}{p^{a-1}(p^a - 1)} \right),$$

and the quadratic character $\chi_q(p)$ is given by the Legendre symbol

$$\chi_q(p) = \left( \frac{-1}{p} \right)^k \det q. \quad (2.4)$$

For a detailed proof of formula (2.1) for the action of Hecke operators $T(p)$ on theta-series of integral weight see Chapter 5 of [3] and specifically Theorem 5.2.5 there. The second
formula (2.2) for the action of Hecke operators $T(p^2)$ on theta-series of half-integral weight easily follows from combination of Theorem 1.7 of [12] and Theorem 1.11 above:

**Proof of Formula (2.2).** Assume that $\mathbf{q}$ is a positive definite (integral) quadratic form in an odd number of variables $m = 2k + 1 \geq 3$. Then using Theorem 1.7 of [12] we have

$$
\Theta(z, \mathbf{q}) \big|_{\frac{m}{2}} T(p^2) = \sum_{n \geq 0} \left\{ r(\mathbf{q}, p^2n) + \chi(p) \left( \frac{2n}{p} \right) p^{k-1} r(\mathbf{q}, n) + p^{m-2} r(\mathbf{q}, n/p^2) \right\} \cdot e^{2\pi i nz},
$$

where we understand that $r(\mathbf{q}, n/p^2) = 0$ if $p^2 \nmid n$. On the other hand, summing up both sides of (1.18) over $K \in R^*(\mathbf{q}, p^2n)$ for some $n \in \mathbb{N} \cup \{0\}$ we conclude that the left-hand side

$$
\sum_{i=1}^{h} \sum_{K \in R(\mathbf{q}, p^2n)} \sum_{M \in R^*(\mathbf{q}, p^2\mathbf{q}_i) / E(\mathbf{q}_i)} 1 = \sum_{i=1}^{h} \sum_{M \in R^*(\mathbf{q}, p^2\mathbf{q}_i) \setminus L \in R(\mathbf{q}, n)} \frac{1}{e(\mathbf{q}_i)} = \sum_{i=1}^{h} \frac{r^*(\mathbf{q}, p^2\mathbf{q}_i)}{e(\mathbf{q}_i)} \cdot r(\mathbf{q}_i, n)
$$

is equal to the right-hand side

$$
\sum_{K \in R(\mathbf{q}, p^2n)} c_p \cdot \left\{ 1 + (\varepsilon_p(n)\chi(p)p^{k-1} + c_p^{-1}\beta_p) \cdot \delta_{(p-1)K} + p^{m-2} \cdot \delta_{(p-2)K} \right\} = c_p \cdot \left\{ r(\mathbf{q}, p^2n) + (\varepsilon_p(n)\chi(p)p^{k-1} + c_p^{-1}\beta_p) \cdot r(\mathbf{q}, n) + p^{m-2} r(\mathbf{q}, n/p^2) \right\}.
$$

Thus

$$
c_p^{-1} \cdot \left( \sum_{i=1}^{h} \frac{r^*(\mathbf{q}, p^2\mathbf{q}_i)}{e(\mathbf{q}_i)} r(\mathbf{q}_i, n) - \beta_p r(\mathbf{q}, n) \right)
$$

coincides with the $n^{th}$ Fourier coefficient of $\Theta(z, \mathbf{q}) \big|_{\frac{m}{2}} T(p^2)$, and on the other hand it is equal to the $n^{th}$ Fourier coefficient of the series on the right-hand side of (2.2). This finishes the proof of (2.2) because of the uniqueness of Fourier decomposition. 

One can easily see that in case of odd number of variables the formulas are not as nice as when $m$ is even. (Unfortunately this is always the case.) Since we can substitute for $\mathbf{q}$ any of $\mathbf{q}_i$, $1 \leq i \leq h$, the above formulas lead to the so-called Eichler’s commutation relation:

$$
\left( \frac{\Theta(z, \mathbf{q}_i)}{e(\mathbf{q}_i)} \right) \big|_{\frac{m}{2}} T(p') = c_p^{-1} \cdot \left( t'(p') - \beta_p \delta_{(\frac{m}{2}+1)1} h \right) \cdot \left( \frac{\Theta(z, \mathbf{q}_i)}{e(\mathbf{q}_i)} \right), \quad (2.5)
$$

where

$$
l = \begin{cases} 
1 & \text{if } m = 2k \text{ is even,} \\
2 & \text{if } m = 2k + 1 \text{ is odd,}
\end{cases}
$$
the square matrix of order $h$

$$t^*(p') = t^*_q(p') = \left( \frac{r_q(p', q_j)}{e(q_j)} \right)$$  \hspace{1cm} (2.6)

is the Eichler’s “Anzahlmatrix” (with multiplier $p'$) and $\delta$ is generalized Kronecker symbol defined by

$$\delta_X = \begin{cases} 
1 & \text{if } X \text{ is an integral matrix,} \\
0 & \text{if matrix } X \text{ is not integral.} 
\end{cases}$$  \hspace{1cm} (2.7)

Finally, following C. L. Siegel one can consider the \textit{generic} theta-series

$$\Theta_{\{q\}}(z) = \sum_{j=1}^h e^{-1}(q_j) \Theta(z, q_j)$$

and show that for $p \nmid \det q$ it is an eigenfunction of corresponding Hecke operators $T(p')$ with eigenvalues $c^{-1}_p \left( \sum_j \frac{r^*_q(p, p'q_j)}{e(q_j)} - (\iota - 1) \beta_p \right)$, $\iota$ is either 1 or 2 depending on the parity of $m$. In case $m$ is even and $\det q = 1$ the generic theta-series is proportional to the Eisenstein series of weight $\frac{m}{2}$. (This implies the famous Siegel theorem on mean numbers of integral representations in the case of quadratic forms of determinant 1, see for example Exercise 5.1.18 in [3].)

The above formulas together with remarkable multiplicative properties of corresponding Hecke algebras (see [3]) reveal multiplicative relations between the Fourier coefficients of the theta-series which are best expressed in terms of associated zeta-functions:

$$\sum_{(n, \det q) = 1} \frac{\bar{r}(n a)}{n^s} = \prod_{p \nmid \det q} \left( \frac{1}{1 - c^{-1}_p t^*(p)p^{-s} + \chi_q(p)p^{k-1-2s}} \right) \cdot \bar{r}(a)$$  \hspace{1cm} (2.8)

in case $m = 2k$ is even and $a$ is any non-zero integer, or

$$\sum_{(n, \det q) = 1} \frac{\bar{r}(n^2 a)}{n^s} = \prod_{p \nmid \det q} \left( \frac{1 - \chi_q(p)(\frac{2a}{p})p^{k-1-s}}{1 - c^{-1}_p (t^*(p^2) - \beta_p)p^{-s} + p^{m-2-2s}} \right) \cdot \bar{r}(a)$$  \hspace{1cm} (2.9)

in case $m = 2k + 1 \geq 3$ is odd and $a$ is a square-free integer.

For $\bar{r}(n)$ we can substitute either the column-vector $t(\ldots, r_q(n, q_j), \ldots)$ or the mean number $\sum_{j=1}^h \frac{r_q(n, q_j)}{e(q_j)}$. The first case corresponds to Fourier coefficients of the vector-valued modular form $t(\ldots, \Theta(z, q_j), \ldots)$ the second — to Fourier coefficients of the generic theta-series $\Theta_{\{q\}}(z)$. Then $t^*(p')$ stands for either Eichler’s matrix (2.6) or the average $\sum_{j=1}^h \frac{r_q(p', q_j)}{e(q_j)}$ respectively. In any case the formulas (2.8) and (2.9) give us explicit expressions of multiplicative relations between the numbers $r_q(n, n)$ of integral representations of integers by quadratic forms. The well-known formula (2.8) easily follows from (2.5) and Exercise 4.3.6 of [3] for example. A proof of the formula (2.9) follows below.
Proof of Formula (2.9). Let \( \bar{t}(n) = t(\ldots, r_{q_j, n}^{\chi_q(p)}(p)\ldots) \) and let \( \bar{t}^*(p^2) \) be Eichler’s matrix (2.6). Substituting quadratic form \( f. \) andrianov in other words

\[
\in \mathbb{K}
\]

Then formally

The last equality implies the following identities for the following formal power series in

\[
\{ r(q_j, p^2n) + (\varepsilon_p(n)\chi_q(p)p^{k-1} + c_p^{-1}\beta_p) r(q_j, n) + p^{m-2}r(q_j, n/p^2) \}
\]

for any \( j, 1 \leq j \leq h \) and any \( n \in \mathbb{N} \cup \{0\} \). In matrix notations:

\[
c_p^{-1} \cdot \bar{t}^*(p^2) \cdot \bar{r}(n) = \bar{r}(p^2n) + (\varepsilon_p(n)\chi_q(p)p^{k-1} + c_p^{-1}\beta_p) \cdot \bar{r}(n) + p^{m-2} \cdot \bar{r}(n/p^2)
\]

The last equality implies the following identities for the following formal power series in \( t \) with coefficients in \( \mathbb{Q}^h \):

\[
(1_h - c_p^{-1}(\bar{t}^*(p^2) - \beta_p)t + p^{m-2}t^2) \cdot \sum_{\delta \geq 0} \bar{r}(p^{2\delta}a)t^\delta = \bar{r}(a) + (\bar{r}(p^2a) - c_p^{-1}(\bar{t}^*(p^2) - \beta_p)\bar{r}(a)) + \sum_{\delta \geq 2} \left\{ \bar{r}(p^{2\delta}a) - c_p^{-1}(\bar{t}^*(p^2) - \beta_p)\bar{r}(p^{2(\delta-1)}a) + p^{m-2}\bar{r}(p^{2(\delta-2)}a) \right\} t^\delta = (1_h - \varepsilon_p(a)\chi_q(p)p^{k-1}t) \cdot \bar{r}(a),
\]

where \( a \) is an arbitrary integer not divisible by \( p^2 \). Define formal power series in \( t \) with coefficients in \( \mathbb{Z}^h \) by

\[
\sum_{\delta \geq 0} \hat{t}_a^*(p^{2\delta})t^\delta = \frac{1_h - \varepsilon_p(a)\chi_q(p)p^{k-1}t}{1_h - c_p^{-1}(\bar{t}^*(p^2) - \beta_p)t + p^{m-2}t^2},
\]

in other words

\[
\begin{align*}
\hat{t}_a^*(1) &= 1_h, \\
\hat{t}_a^*(p^2) &= (c_p^{-1}(\bar{t}^*(p^2) - \beta_p) - \varepsilon_p(a)\chi_q(p)p^{k-1}) \cdot 1_h, \\
\hat{t}_a^*(p^{2\delta}) &= c_p^{-1}(\bar{t}^*(p^2) - \beta_p)\hat{t}_a^*((p^{2(\delta-1)}) - p^{m-2}\hat{t}_a^*(p^{2(\delta-2)}) \text{, for } \delta \geq 2.
\end{align*}
\]

Then formally

\[
\sum_{\delta \geq 0} \bar{r}(p^{2\delta}a)t^\delta = \sum_{\delta \geq 0} \hat{t}_a^*(p^{2\delta})t^\delta \cdot \bar{r}(a)
\]

and so \( \bar{r}(q_j a) = \hat{t}_a^*(p^{2\delta}) \cdot \bar{r}(a) \) for \( \delta \geq 0 \) and \( a \) not divisible by \( p^2 \). Define

\[
\hat{t}_a^*(n^2) = \prod_{s=1}^l \hat{t}_a^*(p_s^{2\delta}) \text{, if } n = p_1^{\delta_1} \ldots p_l^{\delta_l},
\]
here \( p_j \)'s are distinct prime factors of \( n \) and \( \delta_j \)'s are their corresponding orders. Note that the definition does not depend on the order of the factors. To see this it is enough to note that according to Lemmas 3.3–3.6 of [4], Eichler matrices \( \hat{t}^*(p^2) \), \( \hat{t}^*(u^2) \) commute for different primes \( p, u \) not dividing \( \det q \). Next we claim that

\[
\tilde{v}(n^2a)\hat{t}_a^*(n^2) \cdot \tilde{v}(a)
\]

for any integer \( n \) coprime to \( \det q \), and any integer \( a \) not divisible by the square of any prime factor of \( n \). Indeed, let \( n = p_1^{\delta_1} \cdots p_l^{\delta_l} \) be coprime to \( \det q \). We will use induction on \( l \) to justify our claim. If \( l = 1 \) then the statement is obvious because of the definition of \( \hat{t}_a^*(n^2) = \hat{t}_a^*(p_1^{2\delta_1}) \). Assume now that if \( b \) is a product of powers of \( p_1, \ldots, p_{l-1} \) then \( \tilde{v}(b^2a)\hat{t}_a^*(b^2) \cdot \tilde{v}(a) \) for any \( a \) not divisible by \( p_i^2 \), \( i \) ranging from 1 to \( l-1 \). Fix an integer \( a \) not divisible by any of \( p_i^2 \), \( 1 \leq i \leq l \), then

\[
\tilde{v}(n^2a) = \tilde{v}\left(\prod_{i=1}^{l-1} p_i^{2\delta_i} \cdot p_l^{2\delta_l} a\right) = \hat{t}_a^*(p_1^{2\delta_1}) \cdot \tilde{v}(p_l^{2\delta_l} a) = \hat{t}_a^* \left( \prod_{i=1}^{l-1} p_i^{2\delta_i} \right) \cdot \hat{t}_a^*(p_l^{2\delta_l}) \cdot \tilde{v}(a) = \hat{t}_a^*(n^2) \cdot \tilde{v}(a)
\]

since \( \hat{t}_a^*(p^2) = \hat{t}_a^*(p^2) \) if \( u \) and \( p \) are coprime. By induction on \( l \) we conclude that \( \tilde{v}(n^2a)\hat{t}_a^*(n^2) \cdot \tilde{v}(a) \) for any integer \( n \) coprime to \( \det q \) and any integer \( a \) not divisible by the square of a prime factor of \( n \) (in particular for any square-free integer \( a \)). Combining the above equalities we deduce the following identities for the formal zeta-functions:

\[
\sum_{n, \det q = 1} \tilde{v}(n^2a) = \sum_{n, \det q = 1} \hat{t}_a^*(n^2) \cdot \tilde{v}(a) = \prod_{p / \det q} \left( \sum_{\delta \geq 0} \hat{t}_a^*(p^{2\delta}) \cdot \tilde{v}(a) \right) = \prod_{p / \det q} \left( \frac{1 - \chi_q(p)(\frac{2a}{p})p^{k-1-s}}{1 - c_p^{-1}(\hat{t}^*(p^2) - \beta_p)p^{-s} + p^{m-2-2s}} \cdot \tilde{v}(a) \right)
\]

for any square-free integer \( a \) which finishes the proof of (2.9) for the case of column-vectors \( \tilde{v}(n) = t^{(\ldots, n_{(\text{det } q_i)}, \ldots)} \). Proof of (2.9) for the case of the mean numbers \( \sum_j r(q_j, n)/e(q_j) \) proceeds in a similar fashion. One just needs to multiply both sides of the matrix equalities above by the row \( (1, \ldots, 1) \) of length \( h \) and observe that

\[
(1, \ldots, 1) \cdot \left( \frac{r^*(q_j, p^j q_j)}{e(q_j)} \right) = (c_p(1 + p^{m-2}) + \beta_p) \cdot (1, \ldots, 1)
\]

according to comparison of the 0th Fourier coefficients in (2.2).

Let us note also that the zeta-functions on the right hand side of (2.8) are closely related to Epstein’s zeta-functions

\[
\zeta(s, q_j) = \sum_{n \geq 1} \frac{r(q_j, n)}{n^s}
\]

which are Mellin transforms of corresponding theta-series \( \Theta(z, q_j) \) and therefore have good analytical properties (i.e. meromorphic continuation on the whole \( s \)-plane and standard functional equation, for more details see [8], [12] or exercises in Section 4.3.1 of [3]).
3. Matrix Hecke rings of orthogonal groups

For a long time it was somewhat of a mystery that the numbers \( r(q, n) \) of integral solutions of certain quadratic equations have general multiplicative properties even though no similar relations between the solutions themselves were known. (The sole exception to this rule was the case of so-called composition of quadratic forms. Then the integral representations can be interpreted as elements of certain arithmetical rings and multiplicative structure of these rings is reflected in relations between representations. A classical example of this phenomenon is Gauss theory of binary quadratic forms. Unfortunately, according to a theorem of Hurwitz, there is no composition unless the number of variables equals to 1, 2, 4 or 8.)

In [1] A. Andrianov developed Shimura’s construction of abstract matrix Hecke algebras (see [11]) specifically for the case of orthogonal groups and introduced so-called automorph class rings. The latter play the role of Hecke operators acting directly on the sets of solutions of quadratic Diophantine equations. The basic idea behind this action is that

\[
R(q, aq') \cdot R(q', bq'') \subset R(q, abq'')
\]

for any integral quadratic forms \( q, q', q'' \) and for any integers \( a, b \) (the dot refers to usual matrix multiplication). This purely algebraic approach revealed existence of general multiplicative relations between automorphs \( R(q, nq') \) and representations \( R(q, n) \) of integers by quadratic forms. Remarkably, these relations can be expressed in terms of associated zeta-functions in the form almost identical to (2.8) and (2.9). In fact, the latter formulas for numbers of representations appear to be immediate consequences of the former relations between representations themselves! This suggests existence of some correlation between rings of Hecke operators for symplectic and for orthogonal groups.

Let us examine the construction in some detail. For an integral nonsingular quadratic form \( q \) in \( m \) variables we fix a complete system \( \{ q_1, \ldots, q_h \} \) of representatives of equivalence classes contained in the similarity class of \( q \). Consider free \( \mathbb{Z} \)-modules \( A_{ij} \), \( 1 \leq i, j \leq h \) generated by double cosets

\[
\{ E(q_i)AE(q_j) \mid \text{where } A \in E(q_i) \setminus \bigcup_{a \in \mathbb{N}} R(q_i, aq_j)/E(q_j) \}.
\]

Since any double coset of the above type is a finite disjoint union of left cosets modulo \( E(q_i) \), we will write elements of \( A_{ij} \) as finite linear combinations with integral coefficients of (formal) symbols \( < A > \) which bijectively correspond to left cosets \( E(q_i)A \), \( A \in E(q_i) \setminus \bigcup_{a \in \mathbb{N}} R(q_i, aq_j) \). Note that elements of \( A_{ij} \) are invariant under the right multiplication by matrices from \( E(q_j) \). This and the obvious inclusion

\[
R(q_i, aq_j) \cdot R(q_j, bq_k) \subset R(q_i, abq_k)
\]

allows us to define natural multiplication

\[
A_{ij} \times A_{jk} \rightarrow A_{ik},
\]

\[
\sum_{\alpha} a_\alpha < A_\alpha > \cdot \sum_{\beta} b_\beta < B_\beta > = \sum_{\alpha, \beta} a_\alpha b_\beta < A_\alpha B_\beta > .
\]
Finally, we introduce the ring of \( h \times h \) matrices

\[
\mathbb{A} = \mathbb{A}_{(q)} = \{ (A_{ij}) ; A_{ij} \in \mathbb{A}_{ij}, 1 \leq i, j \leq h \}
\]

which is the automorph class ring (over \( \mathbb{Z} \)) of the form \( q \). Sometimes it is more convenient to consider \( \mathbb{A} \otimes \mathbb{Q} \) — the automorph class ring over \( \mathbb{Q} \).

The automorph counterparts of the classical Hecke operators \( T(p') \) (\( i \) is 1 or 2 depending on the parity of \( m \)) are elements of the form

\[
\mathfrak{T}^*(p') = \mathfrak{T}^*_{ij}(p') = \left( \mathfrak{T}^*_{ij}(p') \right), \quad \text{where} \quad \mathfrak{T}^*_{ij}(p') = \sum_{D \in E(q_{ij}) \setminus \mathbb{R}^*(q_{ij}, p') \mathbb{Q}_{ij}} <D> . \tag{3.1}
\]

Note that these matrices are close relatives of Eichler’s “Anzahlmatrizen” (2.6).

Next we should define an action of \( \mathbb{A} \) on the integral representations by the forms \( q_1, \ldots, q_h \). The idea is based on the natural inclusion: \( R(q_i, nq_j) \cdot R(q_j, a) \subset R(q_i, na) \).

In order to suite our formalism we introduce column-vectors of length \( h \):

\[
\mathfrak{R}(a) = \mathfrak{R}_q(a) = \left( \begin{array}{c} \vdots \\ \mathfrak{R}_j(a) \\ \vdots \end{array} \right), \quad \text{where} \quad \mathfrak{R}_j(a) = \sum_{L \in E(q_i) \setminus \mathbb{R}(q_i, a)} \mu_{q_i}^{-1}(L) \cdot <L> , \tag{3.2}
\]

which encode all orbits modulo \( E(q_j) \) of integral representations of \( a \) by forms \( q_j \). Here \( \mu_{q_i}(L) \) is a suitably normalized measure of the stabilizer \( E_L(q_j) = \{ U \in E(q_j) ; UL = L \} \).

The vectors of type (3.2) can be thought of as elements of \( \mathbb{Q} \)-module \( \mathbb{D} = \mathbb{D}_q = \mathbb{D}_1 \times \ldots \times \mathbb{D}_h \) whose \( j^{th} \) component \( \mathbb{D}_j \) is a free \( \mathbb{Q} \)-module generated by (formal) symbols \( <L> \) which are in one-to-one correspondence with orbits \( E(q_j) \) in \( E(q_j) \setminus \mathbb{Z}^m \).

In this notations an automorph analog of Eichler’s commutation relation (2.5) take the form:

\[
\mathfrak{R}(pa) + \chi_{q}(p)p^{k-1}[p]\mathfrak{R}(a/p) = c_p^{-1} \mathfrak{T}^*(p)\mathfrak{R}(a) , \tag{3.3}
\]

if \( m = 2k \), or

\[
\mathfrak{R}(p^2a) + \chi_{q}(p)p^{k-1}\left( \frac{2a}{p} \right)[p]\mathfrak{R}(a) + p^{m-2}[p^2]\mathfrak{R}(a/p^2) = c_p^{-1} \left( \mathfrak{T}^*(p^2) - \beta p_1 \right)\mathfrak{R}(a) \tag{3.4}
\]

if \( m = 2k + 1 \geq 3 \). Here \( p \) is a prime coprime to \( \det q, a \) is any positive integer and \( [p'] = \text{diag}(<p'^1>, \ldots, <p'^1>) \in \mathbb{A} \) are simple elements of the automorph class ring responsible for nonprimitive automorphs. The automorph class theory formalism was initially developed in [1], where one can find the original proof if the formula (3.3) above as formula (1.18) of Section 1. The formula (3.4) was proved in [4], Theorem 3.1 (for the case of positive-definite quadratic forms) and in (2.8) of [5] for the case of arbitrary nonsingular forms.

The above formulas relating sets of representations \( R(q, p'a) \) and \( R(q, a) \) immediately imply analogous relations between numbers \( r(q, p'a) \) and \( r(q, a) \) of representations. Namely, let us consider two “coefficient” homomorphisms \( \pi \):

\[
\mathbb{A} \longrightarrow \mathbb{M}_h(\mathbb{Z}) \quad \text{or} \quad \mathbb{D} \longrightarrow \mathbb{Z}^h
\]
defined entry-wise by
\[ \pi : \sum_\alpha a_\alpha < A_\alpha > \mapsto \sum_\alpha a_\alpha \]
for a corresponding finite formal linear combination of “orbits” \( < \cdot > \). For example,
\[ \pi \left( \mathcal{I}_q^*(p^i) \right) = t_q^*(p^i) = \left( \left| E(q_i) \backslash R^*(q_i, p^i q_j) \right| \right) \]
is the Eichler’s “Anzahlmatrix” (2.6). In case \( q \) is positive definite all sets of representations under consideration are finite, in particular
\[ \pi \left( \mathcal{R}_q(n) \right) = r_q(n) = \begin{pmatrix} \vdots \\ r(q_j, n) \\ \vdots \end{pmatrix} \]
coincides with the \( n^{th} \) Fourier coefficient of the vector modular form \( t(\ldots, \theta(z, q_j), \ldots) \).

Then application of homomorphisms \( \pi \) to (3.3) and (3.4) leads directly to Eichler’s identities (2.5) written in terms of Fourier coefficients of corresponding theta-series. Furthermore, automorph class ring \( \mathcal{A} \) itself has good multiplicative properties (as one would expect from a Hecke algebra). Combining this properties with (3.3) or (3.4) one deduces Euler product expansions of (formal) zeta-functions
\[ \sum_{(n, \det q) = 1} \frac{\mathcal{R}(n)}{n^s} \quad \text{or} \quad \sum_{(n, \det q) = 1} \frac{\mathcal{R}(n^2 a)}{n^s} \]
very similar to Euler products (2.8) or (2.9) respectively (for exact formulas and their detailed proofs see formula (1.19) of [1] and formula (1.13) with Theorem 1.1 of [5]). This method also provides an alternative proof of identities (2.8), (2.9) themselves via appropriate application of the “coefficient” homomorphisms \( \pi \). One should stress here once again that all of the above considerations are purely algebraic and remain valid for any nondegenerate quadratic form over a Dedekind domain. (For an extensive account of multiplicative properties of general automorph class rings and of Euler products of associated formal zeta-functions the reader is referred to [1],[5] and especially [2].)
4. Shimura’s lift for theta-series

In [12] G. Shimura showed that if a cusp form $f(z) = \sum_{n \geq 1} c(n) \exp(2\pi inz)$ of weight $m/2$ (where $m = 2k + 1 \geq 3$) is a common eigenfunction for all Hecke operators $T(p^2)$, then zeta-function $\zeta(z, \chi_m)$ of square-free positive integer $m$ defined by the Euler product of denominators of local factors $\zeta(z, \chi_m)$ has Euler product decomposition of type (2.9) for every square-free positive integer $a$ (one should just replace $r(\cdot)$ by $c(\cdot)$ and $c_p^{-1}(\bar{r}^*(p^2) - \beta_p)$ by corresponding eigenvalues of Hecke operators). Comparing denominators of local factors in (2.8) and (2.9) one can note their striking similarity. And indeed, Shimura used the Weil criterion to prove that the zeta-function

$$\zeta(z, \chi_m) = \sum_{n \geq 1} \frac{c(n^2a)/n^s}{n^s}$$

defined by the Euler product of denominators of local factors of $\sum_{n \geq 1} c(n^2a)/n^s$ is a Mellin transform of a modular form $f^{(a)}(z)$ of integral weight $m - 1 = 2k$ (see [12], Main theorem). The form $c^{-1}(a) \cdot f^{(a)}(z)$ is independent of $a$ and is called the Shimura’s lift of $f(z)$. In what follows we will use this name for $f^{(a)}(z)$ as well.

P. Ponomarev in [10] investigated the lift for theta-series $\Theta(z, \mathbf{q})$ of certain ternary positive definite quadratic forms. The theta-series are neither cusp forms nor Hecke eigenforms in general, but one can still consider lift $\Theta^{(a)}(z, \mathbf{q})$ defined for an integer $a$ via product of type (4.1). As the reader already expects, in order to get a complete picture one should work not with an individual quadratic form $\mathbf{q}$ but rather with a complete system

$$\{\mathbf{q}_1, \ldots, \mathbf{q}_h\}$$

of representatives of different equivalence classes of the similarity class of $\mathbf{q}$. Employing particular case $m = 3$ of Eichler’s commutation relation (2.5) P. Ponomarev showed that

$$\Theta^{(a)}(z, \mathbf{q}_k) = \sum_{j=1}^{h} \left(r(\mathbf{q}_j, a) \cdot \sum_{n \geq 0} \pi_{kj}(n^2) e^{2\pi inz} \right), \quad 1 \leq k \leq h,$$

where each $\sum_{n \geq 0} \pi_{kj}(n^2) \exp(2\pi inz)$ is a finite linear combination of theta-series of quaternary quadratic forms associated with certain lattices in a quaternion algebra (see [10], Theorem 1). The quaternion algebra is chosen so that the reduced norm on its pure part is similar over $\mathbb{Q}$ to the form $\mathbf{q}$. In particular then $(\pi_{kj}(p^2)) = (r^*(\mathbf{q}_k, p^2\mathbf{q}_j)/e(\mathbf{q}_j))$ is nothing else but transposed of Eichler’s matrix (2.6).

One of the goals of the present paper is to find an automorph analog of Shimura’s lift for theta-series of ternary quadratic forms. Let us start with a brief reexamination of P. Ponomarev’s main result from a somewhat different point of view. For simplicity assume that $\mathbf{q}$ is an integral ternary positive definite quadratic form of class number $h$. Fix the system (4.2) of representatives from the similitude class. Then the generic theta-series $\Theta_{\mathbf{q}_1}(z)$ is an eigenform of all Hecke operators $T(p^2)$ with $p \nmid \det \mathbf{q}$ and the equality (2.9) holds for associated zeta-functions. From (2.3) it follows that for ternary quadratic forms the local constants in (2.9) are given by: $c_p^{-1} = 1$, $\beta_p = 0$. Define coefficients $b_n$ by

$$\sum_n \frac{b_n}{n^s} = \prod_p (1 - \bar{r}^*(p^2)p^{-s} + p^{1-2s})^{-1},$$
where
\[ \bar{t}^*(p^2) = \sum_{j=1}^{h} \frac{r^*(q_j, p^2 q_j)}{e(q_j)}, \]
and put
\[ B(z) = \sum_n b_n e^{2\pi i n z}. \]
Note in particular that \( b_1 = 1, b_p = \bar{t}^*(p^2) \). We expect \( B(z) \) to be equal to a finite linear combination
\[ B(z) = \sum_{j=1}^{H} x_j \Theta(z, n_j) = \sum_n \left( \sum_{j=1}^{H} x_j r(n_j, n) \right) e^{2\pi i n z} \]
of theta-series associated to some quaternary (integral) quadratic forms \( n_j \). We also expect \( B(z) \) to satisfy
\[ B(z) \mid_{2T(p)} = \bar{t}^*_q(p^2) B(z) \]
for any prime \( p, p \not| \det q \). A plausible candidate would be
\[ B(z) = \left( \sum_{i=1}^{H} \frac{r(n_i, 1)}{e(n_i)} \right)^{-1} \cdot \sum_{j=1}^{H} e^{-1}(n_j) \Theta(z, n_j) \] (4.3)
proportional to generic theta-series \( \Theta(n)(z) \) of a quaternary form \( n \) of class number \( H \). In this case we would have
\[ \bar{t}^*_q(p^2) = b_p = \left( \sum_{i=1}^{H} \frac{r(n_i, 1)}{e(n_i)} \right)^{-1} \cdot \sum_{j=1}^{H} \frac{r(n_j, p)}{e(n_j)}. \] (4.4)
On the other hand from Eichler’s commutation relations (3.3) or (2.5) we know that
\[ \left( \begin{array}{c} r(n_i, p) \\ e(n_i) \end{array} \right) = c^{-1}_p(n) \left( \begin{array}{c} r^*(n_i, pn_j) \\ e(n_i) \end{array} \right) \left( \begin{array}{c} r(n_i, 1) \\ e(n_i) \end{array} \right), \]
which implies that
\[ \sum_{i=1}^{H} \frac{r(n_i, p)}{e(n_i)} = c^{-1}_p(n) \sum_{i=1}^{H} \left( \sum_{j=1}^{H} \frac{r^*(n_i, pn_j)}{e(n_i)} \right) \frac{r(n_j, 1)}{e(n_j)}. \]
But the sum \( (\sum_i r(n_i, pn_j)/e(n_i)) \) does not depend on \( j \). (In fact, the sum coincides with \( c_p(n) \) times zero Fourier coefficient of \( \Theta(z, n_j) \mid_{2T(p)} \) which is equal to 1 + \( p \) for any \( j \)). Therefore, together with hypothetical (4.4), the last equality would imply that
\[ \sum_{i=1}^{h} \frac{r^*(q_i, p^2 q_i)}{e(q_i)} = \bar{t}^*_q(p^2) = c^{-1}_p(n) \sum_{i=1}^{H} \frac{r^*(n_i, pn)}{e(n_i)}. \]
where $c_p(n) = (1 + \chi_n(p))$ according to (2.3). The above formula can be rewritten in a slightly more general form

$$\sum_{A \in \bigcup_{i=1}^{b} R^*(q, p^2 q_i) / E(q_i)} 1 = (1 + \chi_n(p))^{-1} \sum_{M \in \bigcup_{i=1}^{H} R(n, p n_i) / E(n_i)} 1$$

which make sense for any nonsingular (integral) ternary form $q_i$. Thus according to our heuristic argument, one can expect that an automorph $A \in R(q, p^2 q')$ of quadratic form $q$ can be “lifted” to an automorph $M \in R(n, p n')$ of some quaternary form $n$ associated to $q$. Moreover, the equality (4.5) should appear as a corollary of the expected lifting of automorphs. In the next sections we will prove that this is exactly the case by providing an explicit construction for the “lift” (see (5.30), Theorems 6.3, 6.7 and Corollary 6.8).
5. Clifford algebras and automorph lift

Since our construction of the automorph lift is based on Clifford algebras, we start with a brief recall of their properties. In the course we also introduce some convenient notations. Let

\[ q(X) = q_1 x_1^2 + b_{12} x_1 x_2 + b_{13} x_1 x_3 + q_2 x_2^2 + b_{23} x_2 x_3 + q_3 x_3^2 \]  \hspace{1cm} (5.1)

be an integral nonsingular (ternary) quadratic form. We set

\[ Q_0 = \begin{pmatrix} q_1 & b_{12} & b_{13} \\ 0 & q_2 & b_{23} \\ 0 & 0 & q_3 \end{pmatrix}, \quad B = \begin{pmatrix} -b_{23} \\ b_{13} \\ -b_{12} \end{pmatrix}. \]  \hspace{1cm} (5.2)

Then the matrix \( Q \) of the form \( q \) and its determinant \( \det q \) are given by

\[ Q = t Q_0 + Q_0, \quad 2\Delta = \det q = 2 \cdot (4 \det Q_0 - Q_0 B), \]  \hspace{1cm} (5.3)

where \( \det Q_0 = q_1 q_2 q_3 \) and \( \Delta = \det q / 2 \). The matrices adjoint to \( Q_0 \) and \( Q \) are

\[ \tilde{Q}_0 = \begin{pmatrix} q_2 q_3 & -q_3 b_{12} & b_{12} b_{23} - q_2 b_{13} \\ 0 & q_1 q_3 & -q_1 b_{23} \\ 0 & 0 & q_1 q_2 \end{pmatrix} \]  \hspace{1cm} (5.4)

and

\[ \tilde{Q} = 2(t \tilde{Q}_0 + \tilde{Q}_0) - B \cdot t B. \]  \hspace{1cm} (5.5)

We associate to \( q \) quadratic \( \mathbb{Z} \)-module \((E, q) = (\mathbb{Z}^3, q)\) with corresponding symmetric bilinear form

\[ b(x, y) = q(x + y) - q(x) - q(y), \quad x, y \in E \]  \hspace{1cm} (5.6)

whose matrix \((b(e_i, e_j))\) relative to the standard basis \(e_1, e_2, e_3\) of \(\mathbb{Z}^3\) is \(Q\). The Clifford algebra \(C(E) = C(E, q)\) of \((E, q)\) is a free \(\mathbb{Z}\)-algebra with monomorphism \(\iota : E \hookrightarrow C(E)\) such that \(\iota(x)^2 = q(x) \cdot 1_{C(E)}\) for any \(x \in E\). It is uniquely characterized by the following universal property: for any homomorphism \(\eta : E \to D\) into an algebra \(D\) with \(\eta(x)^2 = q(x) \cdot 1_D\), there exists unique homomorphism of algebras \(\tau : C(E) \to D\) such that the diagram

\[ \begin{array}{ccc}
E & \xleftarrow{\text{id}} & E \\
\downarrow \eta & & \downarrow \iota \\
D & \xleftarrow{\text{\tau}} & C(E)
\end{array} \]  \hspace{1cm} (5.7)

commutes (i.e. \(\eta = \tau \circ \iota\)). The algebra \(C(E)\) can be realized as a free \(\mathbb{Z}\)-module of rank 8 generated by products of \(e_1, e_2, e_3\) (see [9] for details). We will denote the product \(e_i e_j\)
in \( C(E) \) by \( e_{ij} \) and \( e_{ij}e_k \) by \( e_{ijk} \). The unit element of \( C(E) \) is denoted by \( e_0 \) and we identify \( \mathbb{Z} \) with \( \mathbb{Z}e_0 \). We also identify \( E \) with \( \iota(E) \subset C(E) \). The even subalgebra

\[
C_0(E) = \mathbb{Z}e_0 \oplus \mathbb{Z}e_{23} \oplus \mathbb{Z}(-e_{13}) \oplus \mathbb{Z}e_{12}
\]  

(5.7)

is a free \( \mathbb{Z} \)-module of rank 4 generated by products of even number of vectors of \( E \). (Note the order of the elements of the basis in (5.7)! In spite of first strange impression, this is the natural order. We will refer to this basis of \( C_0(E) \) as the natural basis). Setting further \( C_1(E) = \mathbb{Z}(-e_{123}) \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \) one has \( C(E) = C_0(E) \oplus C_1(E) \). Sometimes it is also useful to consider \( C(E) \otimes \mathbb{Q} \) or \( C_0(E) \otimes \mathbb{Q} – \) the Clifford algebra of \( (E, q) \) over \( \mathbb{Q} \) and its even subalgebra, in which \( C(E) \) and \( C_0(E) \) are orders. The algebra \( C(E) \) has an antiautomorphism \( x \mapsto \bar{x} \) defined on the generators by

\[
\bar{e}_j = -e_j , \quad j = 1, 2, 3.
\]  

(5.8)

The restriction of the antiautomorphism \( x \mapsto \bar{x} \) to \( C_0(E) \) can be written in coordinates as

\[
x_0e_0 + x_1e_{23} - x_2e_{13} + x_3e_{12} \mapsto (x_0 + x_3b_{12} - x_2b_{13} + x_1b_{23})e_0 - x_1e_{23} + x_2e_{13} - x_3e_{12}
\]  

(5.9)

and gives us the standard involution on \( C_0(E) \). That is to say,

\[
s(x) = x + \bar{x} \in \mathbb{Z} , \quad n(x) = x \cdot \bar{x} \in \mathbb{Z}
\]  

(5.10)

for any \( x \in C_0(E) \). In particular each element \( x \) of \( C_0(E) \) satisfies the equation \( (z - x)(z - \bar{x}) = z^2 - s(x)z + n(x) = 0 \) which is unique in case \( x \) and \( e_0 \) are linearly independent. Integers \( s(x) \) and \( n(x) \) are called respectively the trace and the norm of \( x \in C_0(E) \). The norm \( n \) turns the even subalgebra \( C_0 \) into quadratic \( \mathbb{Z} \)-module \( (C_0(E), n) \) of rank 4 with associated bilinear form

\[
s(xy) = n(x + y) - n(x) - n(y) , \quad x, y \in C_0(E).
\]  

(5.11)

The matrix \( N \) of the form \( n \) with respect to the natural basis (5.7) is

\[
N = \begin{pmatrix} 2 & -tB \\ -B & (t\bar{Q}_0 + \bar{Q}_0) \end{pmatrix}
\]  

(5.12)

One easily computes the determinant

\[
\det n = \Delta^2 = ((\det q)/2)^2
\]  

(5.13)

and the inverse matrix

\[
N^{-1} = \Delta^{-1} \cdot \begin{pmatrix} 2 \cdot \det Q_0 & t(Q_0B) \\ (Q_0B) & Q \end{pmatrix},
\]  

(5.14)
where $Q_0$ and $B$ are given by (5.2). From the above formulas one can see that the quadratic modules $(E, q)$ and $(C_0(E), \mathbf{n})$ are closely related. In fact $(C_0(E), \mathbf{n})$ contains a certain quadratic submodule of rank 3 which is “almost” $(E, q)$. We will present the construction shortly, but first we need to introduce the special element

$$
t = e_{123} + e_{123} = e_{123} - e_{321} \in C(E)
$$

(5.15)

used by M. Kneser in [9]. Straightforward (but rather tiresome) calculations show that $t$ belongs to the center of $C(E)$, $t = \bar{t}$ and

$$
t = (-e_{123}, e_1, e_2, e_3) \begin{pmatrix}
-2 \\
B
\end{pmatrix}, \quad t^2 = -\Delta .
$$

(5.16)

Consider now $\mathbb{Z}$-submodule $Et$ of $C_0(E)$. Using (5.16) we see that $Et$ has rank 3 and

$$
(e_1t, e_2t, e_3t) = (e_0, e_{23}, -e_{13}, e_{12}) \cdot T
$$

form its $\mathbb{Z}$-basis, where

$$
T = \begin{pmatrix}
\langle t(Q_0B) \rangle \\
Q
\end{pmatrix} \in \mathbb{Z}_3^4 .
$$

(5.17)

Let us compute the matrix of the restriction of the norm-form $\mathbf{n}$ onto $Et$ with respect to the basis $e_1t, e_2t, e_3t$. We have

$$
N \cdot T = \begin{pmatrix}
2 \cdot \langle t(Q_0B) \rangle - tB \cdot Q \\
\langle (t\hat{Q} + \hat{Q})(tQ_0 + Q_0) - B \cdot tB \cdot tQ_0 \rangle
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \Delta & 1
\end{pmatrix}
$$

and so

$$
N[T] = \Delta \cdot Q .
$$

(5.18)

Recall that $\Delta = \det q/2$ is an integer (see (5.3)), thus $T \in R(\mathbf{n}, \Delta \mathbf{q})$ and $(Et, \Delta \mathbf{q}) \hookrightarrow (C_0(E), \mathbf{n})$ as we stated. Finally we also note that

$$
-e_{123} \cdot t = (e_0, e_{23}, -e_{13}, e_{12}) \begin{pmatrix}
2 \det Q_0 \\
Q_0B
\end{pmatrix},
$$

(5.19)

which extends multiplication of $E$ by $t$ to injection $C_1(E) \cdot t \hookrightarrow C_0(E)$, whose matrix with respect to our choice of bases is $\Delta \cdot N^{-1}$.

Let $\mathbf{q}''$ be another integral ternary quadratic form. We associate with it a quadratic module $(E'', \mathbf{q}'')$ and Clifford algebra $C(E'')$ exactly as above. Let $A \in R(\mathbf{q}, \mathbf{q}'')$ be an integral automorph. From geometric point of view $A$ defines an isometry $\alpha = \alpha_A$ from quadratic module $(E'', \mathbf{q}'')$ into $(E, \mathbf{q})$:

$$
(\alpha_A(e_1''), \alpha_A(e_2''), \alpha_A(e_3'')) = (e_1, e_2, e_3) \cdot A ,
$$
where $e_1''', e_2''', e_3'''$ is the standard basis of $E''' = \mathbb{Z}^3$. Let $\iota$ and $\iota''$ be natural inclusions of $(E, q)$ and $(E'', q''')$ into corresponding Clifford algebras. Since $(\iota \circ \alpha(x''))^2 = q(\alpha(x'')) \cdot e_0 = q''(x'') \cdot e_0$ for any $x'' \in E''$, there exists a unique homomorphism of algebras $\varphi = \varphi_A$ which makes the diagram

$$
(E, q) \xleftarrow{\iota} (E', q') \xrightarrow{\alpha} (E'', q'') \xrightarrow{\iota''} C(E) \xrightarrow{\varphi_A} C(E'')
$$

commutative. (This follows immediately from the universal property (5.6) of Clifford algebras.) Clearly the homomorphism $\varphi_A$ is compatible with the involutions (5.8) on $C(E)$ and $C(E'')$:

$$
\varphi_A(x'') = \overline{\varphi_A(x'')} \text{ for all } x'' \in C(E'').
$$

Therefore the restriction of $\varphi_A$ to the even subalgebra $C_0(E'')$ is an isometry of quadratic modules $\varphi_A : (C_0(E''), n''') \rightarrow (C_0(E), n)$ whose matrix with respect to our choice of bases (5.7) is

$$
\Phi_A = \begin{pmatrix}
1 & tA_3Q_0A_2 & -tA_3Q_0A_1 & tA_2Q_0A_1 \\
0 & 0 & 0 & t\bar{A}
\end{pmatrix},
$$

so that

$$
(\varphi_A(e_0'''), \varphi_A(e_2'''), \varphi_A(-e_3'''), \varphi_A(e_1''')) = (e_0, e_2, -e_3, e_1) \cdot \Phi_A.
$$

Here $A_j$ denotes the $j^{th}$ column of the original automorph $A \in R(q, q''')$ which gave rise to $\Phi_A \in R(n, n''')$. (By $n, n'''$ we mean the integral quaternary quadratic forms which are specializations of the corresponding norm-forms at the natural bases (5.7) of $C_0(E)$ and $C_0(E'')$ respectively.) To complete the picture we also need a description of action of $\varphi_A$ on the special element $t'' \in C(E'')$ of the form (5.15):

**Lemma 5.1.** Let $A = (a_{ij}) \in R(q, q''')$ be an integral automorph. Then

$$
\varphi_A(t'') = (\det A) \cdot t
$$

in the notations introduced above.

**Proof.** Note that elements of $E$ are multiplied in $C(E)$ according to the following rule:

$$
(e_1, e_2, e_3) \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} \cdot (e_1, e_2, e_3) \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = (e_0, e_2, -e_3, e_1) \begin{pmatrix}
tYQ_0X \\
x_2y_3 - x_3y_2 \\
x_3y_1 - x_1y_3 \\
x_1y_2 - x_2y_1
\end{pmatrix},
$$
Therefore
\[
\varphi_A(-e''_{123}) = -\varphi_A(e''_1)\varphi_A(e''_2)\varphi_A(e''_3) = -\alpha_A(e''_1)\alpha_A(e''_2)\alpha_A(e''_3) = \\
-1 \cdot (e_1, e_2, e_3) A_1 \cdot (e_1, e_2, e_3) A_2 \cdot (e_1, e_2, e_3) A_3 = (-e_{123}, e_1, e_2, e_3) \begin{pmatrix} \delta \\ Z \end{pmatrix} ,
\]
where \( A = (A_1 A_2 A_3) \), \( \delta = \det A \) and components \( z_1, z_2, z_3 \) of the column \( Z \) are given by
\[
z_1 = -(t A_2 Q_0 A_1) a_{13} + b_{23} a_{13} \hat{a}_{23} + b_{23} a_{13} \hat{a}_{23} - b_{12} a_{13} \hat{a}_{33} - q_2 a_{23} \hat{a}_{33} + q_3 a_{33} \hat{a}_{23} ,
\]
\[
z_2 = -(t A_2 Q_0 A_1) a_{23} - b_{13} a_{13} \hat{a}_{23} - b_{23} a_{23} \hat{a}_{23} + q_1 a_{13} \hat{a}_{33} - q_3 a_{33} \hat{a}_{13} ,
\]
\[
z_3 = -(t A_2 Q_0 A_1) a_{33} + b_{12} a_{13} \hat{a}_{13} + q_2 a_{23} \hat{a}_{13} - q_1 a_{13} \hat{a}_{23} ,
\]
and \( \hat{a}_{ij} \) is the algebraic cofactor of \( a_{ij} \) in \( A \). Using (5.16) applied to \( t'' \) we have
\[
\varphi_A(t'') = (-e_{123}, e_1, e_2, e_3) \begin{pmatrix} \delta & 0 & 0 \\ Z & A \end{pmatrix} \begin{pmatrix} -2 \\ B'' \end{pmatrix} .
\]
On the other hand, from (5.16) we also know the coordinates of \( t \) in this basis. Therefore, in order to prove the Lemma it remains to check that
\[
A \cdot \begin{pmatrix} -b''_{23} \\ b''_{13} \\ -b''_{12} \end{pmatrix} - 2 \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \delta \cdot \begin{pmatrix} -b_{23} \\ b_{13} \\ -b_{12} \end{pmatrix} \quad (5.23)
\]
Before entering into computations let us make several remarks. First, since \( t A_2 Q_0 A_1 = t A_2 Q A_1 = b''_{12} \), then \( 2 \cdot t A_2 Q_0 A_1 = t A_2 (Q_0 - t Q_0) A_1 = b_{12} \hat{a}_{33} + b_{13} \hat{a}_{23} - b_{23} \hat{a}_{13} \). Second, \( Q[A] = Q'' \) implies that \( \delta^{-1} t A Q'' = Q A \). Third, \( (a_{ij}) = A = (A^{-1})^{-1} = \delta^{-1} A = \delta^{-1} (\hat{a}_{ij}) \), where \( A \) is the double adjoint of \( A \). Now we proceed to calculation of entries of the column-vector on the left hand side of (5.23).

First entry:
\[
-a_{11} b''_{13} + a_{12} b''_{12} - a_{13} b''_{12} - 2 z_1 = \\
-b_{23} (a_{13} \hat{a}_{13} + a_{23} \hat{a}_{23}) - b_{13} a_{13} \hat{a}_{23} + b_{12} a_{13} \hat{a}_{33} - b_{23} a_{23} \hat{a}_{23} + 2 q_2 a_{23} \hat{a}_{33} - 2 q_3 a_{33} \hat{a}_{23} \\
-(a_{11} b''_{23} - a_{12} b''_{13}) = \\
-\delta b_{23} + \hat{a}_{33} (b_{12} a_{13} + 2 q_2 a_{23} + b_{23} a_{33}) - \hat{a}_{23} (b_{13} a_{13} + b_{23} a_{23} + 2 q_3 a_{33}) \\
-(a_{11} b''_{23} - a_{12} b''_{13}) = \\
\delta^{-1} \hat{a}_{33} (\hat{a}_{21} b''_{13} + \hat{a}_{22} b''_{12} + \hat{a}_{23} 2 q''_3) - \delta^{-1} \hat{a}_{23} (\hat{a}_{31} b''_{13} + \hat{a}_{32} b''_{12} + \hat{a}_{33} 2 q''_3) - \delta b_{23} = \\
\delta^{-1} b''_{13} (\hat{a}_{33} \hat{a}_{21} - \hat{a}_{23} a_{31}) + \delta^{-1} b''_{23} (\hat{a}_{33} \hat{a}_{22} - \hat{a}_{23} a_{32}) - (a_{11} b''_{23} - a_{12} b''_{13}) = \\
\delta^{-1} b''_{13} (-\hat{a}_{12}) + \delta^{-1} b''_{23} (\hat{a}_{11}) - (a_{11} b''_{23} - a_{12} b''_{13}) = \delta b_{23} = - \det A \cdot b_{23} .
\]

Second entry:
\[
-a_{21} b''_{23} + a_{22} b''_{22} - a_{23} b''_{22} - 2 z_2 = \\
b_{13} (a_{13} \hat{a}_{13} + a_{23} \hat{a}_{23}) + \hat{a}_{13} (b_{13} a_{13} + b_{23} a_{23} + 2 q_3 a_{33}) - \hat{a}_{33} (2 q_1 a_{13} + b_{12} a_{23}) \\
-(a_{21} b''_{23} - a_{22} b''_{13}) = \\
\delta^{-1} \hat{a}_{13} (\hat{a}_{31} b''_{13} + \hat{a}_{32} b''_{22} - \hat{a}_{33} 2 q''_3) - \delta^{-1} \hat{a}_{33} (\hat{a}_{11} b''_{13} + \hat{a}_{12} b''_{23}) - (a_{21} b''_{23} - a_{22} b''_{13}) + \delta b_{13} = \det A \cdot b_{13} .
\]
Third entry:
\[-a_{31}b_{12}' + a_{32}b_{13}' - a_{33}b_{12}' = -2z_3 = \]
\[-b_{12}(a_{13}a_{13} + a_{33}a_{33}) + \hat{a}_{23}(2q_1a_{13} + b_{13}a_{33}) - \hat{a}_{11}(b_{12}a_{13} + 2q_2a_{23} + b_{23}a_{33}) \]
\[-a_{31}b_{23}' - a_{32}b_{13}' = \]
\[\delta^{-1}\hat{a}_{23}(\hat{a}_{11}b_{13}' + \hat{a}_{12}b_{23}') - \delta^{-1}\hat{a}_{13}(\hat{a}_{21}b_{13}' + \hat{a}_{22}b_{23}') - (a_{31}b_{23}' - a_{32}b_{13}') - \delta b_{12} = - \det A \cdot b_{12}. \]

}\]

The above lemma along with (5.16) leads us to the following criterion:

**Theorem 5.2.** Let \( \varphi : C_0(E'') \to C_0(E) \) be a homomorphism of even subalgebras of Clifford algebras associated to ternary quadratic \( \mathbb{Z} \)-modules \((E'', \mathfrak{q}'')\) and \((E, \mathfrak{q})\). Set \( \Delta = \det \mathfrak{q}/2, \Delta'' = \det \mathfrak{q}'/2 \) and let \( t \in C(E), t'' \in C(E'') \) be special elements of the form (5.15). Then \( \varphi \) coincides with the lift (5.20) of an isometry \( \alpha : (E'', \mathfrak{q}'') \to (E, \mathfrak{q}) \) if and only if
\[
\varphi(E'' \cdot t'') \subset \sqrt{\Delta''/\Delta} \cdot (E \cdot t). \tag{5.24}
\]
In this case the isometry \( \alpha \) is unique (up to sign) and is given by
\[
\alpha(x'') = \frac{\pm \varphi(x'' t'') \cdot t}{\sqrt{\Delta''/\Delta}} \quad \text{for } x'' \in E'', \tag{5.25}
\]
where one can fix either “+” or “−”.

**Proof.** Assume (5.24) and consider \( \alpha \) as defined in (5.25). For an \( x'' \in E'' \) we have:
\[
\alpha(x'') \subset (\sqrt{\Delta''})^{-1} \cdot \varphi(E''t'') \cdot t \subset (\sqrt{\Delta''})^{-1} \sqrt{\Delta''/\Delta} \cdot E \cdot t^2 \subset -|\Delta|^{-1} \Delta \cdot E \subset E
\]
Furthermore,
\[
\mathfrak{q} \left( \alpha(x'') \right) = \alpha(x'') \cdot \alpha(x'') = (\Delta\Delta'')^{-1} \cdot \varphi(x'' t') \cdot t \cdot \varphi(x'' t') \cdot t = (\Delta\Delta'')^{-1} \cdot \varphi(x'' t' x'' t') \cdot t^2 =
\]
\[
(\Delta\Delta'')^{-1} \cdot \varphi(-\Delta''(x''))^2 \cdot (-\Delta) = \varphi(\mathfrak{q}'(x'')) = \mathfrak{q}''(x''),
\]
thus \( \alpha : (E'', \mathfrak{q}'') \to (E, \mathfrak{q}) \) is indeed an isometry. According to (5.20) it can be lifted to an algebraic homomorphism \( \varphi_\alpha : C(E'') \to C(E) \) the restriction of which on \( C_0(E'') \) is homomorphism into \( C(E) \). Then
\[
\varphi_\alpha(\mathfrak{e}_{ij}'') = \alpha(\mathfrak{e}_{ij}'') \cdot \alpha(\mathfrak{e}_{ij}') = (\Delta\Delta'')^{-1} \cdot \varphi(\mathfrak{e}_{ij}'') \cdot \mathfrak{e}_{ij}' \cdot (\Delta\Delta'')^{-1} \cdot \varphi(\mathfrak{e}_{ij}') \cdot \mathfrak{e}_{ij}' = \varphi(\mathfrak{e}_{ij}')
\]
so \( \varphi_\alpha \) and \( \varphi \) coincide on \( C_0(E'') \).

Conversely, if \( \varphi = \varphi_\alpha \big|_{C_0} \) for some lift \( \varphi_\alpha \) of an isometry \( \alpha : (E'', \mathfrak{q}'') \to (E, \mathfrak{q}) \), then
\[
\varphi(x'' t'') = \varphi_\alpha(x'' t'') = \varphi_\alpha(x'') \cdot \det \alpha \cdot t = \alpha(x'') \cdot \det \alpha \cdot t \subset \det \alpha \cdot E \cdot t \subset \sqrt{\Delta''/\Delta} \cdot (E \cdot t),
\]
so (5.24) is satisfied. The equality (5.25) is also satisfied for one of the choices of signs. □
To summarize what we have established so far, the universal property (5.6) of Clifford algebras allows us to lift an integral automorph $A \in R(q, q')$ of ternary forms to an integral automorph $\Phi_A \in R(n, n'')$ of quaternary forms associated with norms on the even subalgebras of Clifford algebras. Conversely, the original automorph $A \in R(q, q'')$ can be recovered from the lift $\Phi_A \in R(n, n'')$ with the help of the Theorem 5.2 above. (We note again that quaternary quadratic forms $n$, $n''$ are specializations of norm-forms (5.10) at the natural bases (5.7).)

Recall that the quadratic forms we are working with are not necessarily positive definite and so the sets $R(q, q''), R(n, n'')$ . . . of automorphs or the sets $R(q, a), R(n, a)$ . . . of representations of a number $a$ can be infinite in general. But quotients modulo corresponding groups of units $(E(q), E(n) . . .)$ are always finite for nonsingular forms. Because of this it is more convenient sometimes to consider the orbits modulo groups of units rather than individual automorphs or representations (compare to section 3). The following lemma shows that this transition is compatible with lift (5.22):

**Lemma 5.3.** Let $A, D \in R(q, q'')$ be integral automorphs. Then $A \in E(q)D$ if and only if $\Phi_A \in E(n)\Phi_D$.

**Proof.** Let $E \in E(q)$ be such that $A = ED$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
(E, q) & \xleftarrow{\alpha_E} & (E, q) \\
\downarrow & & \downarrow \\
C(E) & \xleftarrow{\varphi_E} & C(E)
\end{array}
\quad
\begin{array}{ccc}
(E, q) & \xrightarrow{\alpha_D} & (E'', q'') \\
\downarrow & & \downarrow \\
C(E) & \xrightarrow{\varphi_D} & C(E'')
\end{array}
\]

where the isometries $\alpha_E, \alpha_D$ and algebraic homomorphisms $\varphi_E, \varphi_D$ are defined analogously to (5.20). Since $\alpha_E \circ \alpha_D = \alpha_A$, we know that there exists unique algebraic homomorphism $\varphi_A = \varphi_{ED} : C(E'') \to C(E)$ such that $\varphi_A \circ \alpha_A = \varphi_A \circ \varphi_D \circ \alpha_D = \varphi_{E} \circ \alpha_E \circ \alpha_D$ as in (5.20). But $\varphi_A = \varphi_{ED} = \varphi_E \circ \varphi_D$ and thus $\Phi_A = \Phi_E \Phi_D$ (see (5.22)). Note that since $E \in E(q)$ then $\Phi_E \in E(n)$.

Conversely, assume that $\Phi_A \in E(n)\Phi_D$. Then $\Phi_A \Phi_D^{-1} \in E(n)$ and in particular

\[
\begin{pmatrix}
1 & * & * & * \\
0 & 0 & \tilde{A} \\
0 & 0 & 0 & t \tilde{D}^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & * & * & * \\
0 & 0 & \tilde{D}^{-1} \\
0 & 0 & 0 & \frac{1}{\det \tilde{D}} t \tilde{A} \cdot t \tilde{D}
\end{pmatrix}
\]

is an integral matrix. But $\det D = \det A$, so $t \tilde{A}^{-1} \cdot t \tilde{D}$ is also an integral matrix. Then $E = DA^{-1}$ is also integral and $Q[E] = Q[DA^{-1}] = Q''[A^{-1}] = Q$, i.e. $E \in E(q)$ and $A \in E(q)D$. \qed
Therefore (5.14) implies \((5.14)\). By our assumption the forms \(q \in N \cap \Lambda\). Further, since \(\phi\) which maps \(\{q\} \) gives rise to an isomorphism \(\phi\) whose norm is also 1. Then the right multiplication \(x \mapsto x \cdot \phi(e_0)\) is a (linear) isometric isomorphism of quadratic modules: thus \((5.12)) \) are integrally equivalent.

Remark. An obvious modification of the above proof shows that

\[ A \in D \cdot E(q'') \iff \Phi_A \in \Phi_D \cdot E(n'') \]

We will also need the following technical result:

Lemma 5.4. Assume that (ternary) quadratic forms \(q\) and \(q''\) belong to the same similarity class. Then \(q\) and \(q''\) are integrally equivalent if and only if corresponding quadratic forms \(n\) and \(n''\) (see (5.12)) are integrally equivalent.

Proof. Suppose \(q\) and \(q''\) belong to the same equivalence class, i.e. there exists a matrix \(U \in R(q, q'') \cap \Lambda^3\). Then \(\alpha_U : (E'', q'') \to (E, q)\) is an isometric isomorphism of quadratic modules which gives rise to an isomorphism \(\varphi_U : C(E'') \to C(E)\) of Clifford algebras exactly as in (5.20). The restriction of \(\varphi_U\) on \(C_0(E'')\) is an isomorphism of even subalgebras whose matrix with respect to the natural bases (5.7) is \(\Phi_U \in R(n, n'') \cap \Lambda^4\) (see (5.22)). Thus \(n\) and \(n''\) are integrally equivalent.

Conversely, suppose that there exists \(U \in R(n, n'') \cap \Lambda^4\). The matrix \(U\) defines a (linear) isometric isomorphism of quadratic modules:

\[
\begin{align*}
\nu & : (C_0(E''), n'') \to (C_0(E), n), \\
(v(e_0''), v(e_2''), v(-e_{13}''), v(e_{12}'')) &= (e_0, e_{23}, -e_{13}, e_{12}) \cdot U.
\end{align*}
\]

Since \(n(v(e_0'')) = n''(e_0'') = 1\), the element \(v(e_0'')\) has an inverse \(v(e_0'')^{-1} = \overline{v(e_0'')} \in C_0(E)\) whose norm is also 1. Then the right multiplication \(x \mapsto x \cdot \overline{v(e_0'')}\) is a (linear) isometric automorphism of \(C_0(E)\). The composition of these two isometries

\[
\omega : x'' \mapsto \nu(x'') \cdot \overline{v(e_0'')} \quad x'' \in C_0(E'')
\]

gives a (linear) isometric isomorphism of quadratic modules \((C_0(E''), n'') \to (C_0(E), n)\) which maps \(e_0''\) to \(e_0\). Therefore its matrix with respect to natural bases (5.7) has the form

\[
W = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & W \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}
\]

with \(W \in \Lambda^3\).

Further, since \(N[W] = N''\), then \(N^{-1}[tW^{-1}] = (N'')^{-1}\) with \(N^{-1}, (N'')^{-1}\) given by (5.14). By our assumption the forms \(q\) and \(q''\) are similar, in particular \(\Delta = \det q/2 = \Delta''\). Therefore (5.14) implies

\[
\begin{pmatrix} 2 \det Q_0 & t(Q_0B) \\ (Q_0B) & Q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & tW^{-1} & * \\ * & * & tW^{-1} \end{pmatrix} = \begin{pmatrix} 2 \det Q_0'' & t(Q_0''B'') \\ (Q_0''B'') & Q'' \end{pmatrix}.
\]
It follows that $Q[tW^{-1}] = Q''$ with $W \in \Lambda^3$, i.e. the quadratic forms $q$ and $q''$ are (integ rall) equivalent. \hfill \Box

We now proceed directly to an automorph analog of Shimura’s lift. Let $q'$ be a quadratic form similar to $q$ (see section 1) and let $p$ be a prime coprime to $\det q = \det q'$. Given an integral automorph $A \in \mathcal{R}(q, p^2q')$ we can set $q'' = p^2q'$ and apply the construction (5.20) to get the homomorphism $\varphi_A$ of Clifford algebras:

\[
(E, q) \quad \xleftarrow{\alpha_A} \quad (E'', p^2q')
\]

\[
\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad (5.26)
\]

\[
C(E, q) \quad \xleftarrow{\varphi_A} \quad C(E'', p^2q')
\]

The restriction of $\varphi_A$ to the even subalgebra $C_0(E'', p^2q')$ will give us an automorph $\Phi_A \in \mathcal{R}(n, n'') = \mathcal{R}(N, N'')$ given by (5.22). Here $N$ is the matrix (5.12) and

\[
N'' = \begin{pmatrix}
2 & -p^2 \cdot iB' \\
-p^2B' & p^4(t\tilde{Q}_0' + \tilde{Q}_0')
\end{pmatrix} = N' \begin{pmatrix}
1 \\
p^213
\end{pmatrix}
\]

(5.27)

is the matrix (5.12) of the norm-form (5.10) of the even subalgebra $C_0(E'', p^2q')$ with respect to its natural basis (5.7). Note that $C_0(E'', p^2q')$ can be identified with the subring of $C_0(E', q')$ spanned by $e_0', p^2e_{23}', -p^2e_{13}', p^2e_{12}'$. (Indeed, the natural isometry of quadratic modules $E'' \cong pE'$ defined by $e_{i}' \mapsto pe_i'$ gives rise to an algebraic monomorphism $C(E'', p^2q') \hookrightarrow C(E', q')$, the restriction of which to the even subalgebra is the desired ring homomorphism.) In particular, $N'$ on the right hand side of (5.27) is exactly the matrix (5.12) of the norm-form $n'$ on the subalgebra $C_0(E', q')$. Combining the last equality with $N[\Phi_A] = N''$ we conclude that if $A \in \mathcal{R}(q, p^2q')$ then $N[\Psi_A] = N'[(p \cdot 14)] = p^2N'$, where we set

\[
\Psi_A = \Phi_A \cdot \begin{pmatrix}
p & p^{-1} \\
p^{-1}13
\end{pmatrix} = \begin{pmatrix}
p & p^{-1}(tA_3Q_0A_2 - tA_3Q_0A_1) \\
0 & p^2 \cdot tA^{-1}
\end{pmatrix}.
\]

(5.28)

**Lemma 5.5.** In the above notations $\Psi_A \in \mathcal{R}(n, p^2n')$, in particular the matrix $\Psi_A$ is integral.

**Proof.** We already know that $N[\Phi_A] = p^2N'$ therefore to prove the lemma it remains to show that $\Psi_A$ is integral. Since $A \in \mathcal{R}(q, p^2q')$ then $Q[A] = p^2Q'$ and so $p^2A^{-1} =
\((\det q)^{-1}\tilde{Q}'^tAQ\) which implies that \(\det q \cdot p^2A^{-1}\) is an integral matrix. But \(\det A = p^3\) is coprime to \(\det q\), thus \(p^2A^{-1}\) is integral. Next we need to deal with the first row of \(\Psi_A\) in (5.28). Denote

\[ tZ_A = \left( tA_3Q_0A_2, -tA_3Q_0A_1, tA_2Q_0A_1 \right). \] (5.29)

Then

\[
N[\Phi_A] = \begin{pmatrix}
2 & 2 \cdot tZ_A - p^3 \cdot t(A^{-1}B) \\
* & 2^{-1}(p^4\tilde{Q}' + (2Z_A - p^3A^{-1}B)^t(2Z_A - p^3A^{-1}B))
\end{pmatrix} = N'\left( \begin{pmatrix} 1 \\ p^21_3 \end{pmatrix} \right)
\]

which implies that \(2Z_A - p^3A^{-1}B = -p^2B' \equiv 0 \pmod{p^2}\). But \(p^2A^{-1}\) is integral, so \(p^3A^{-1}B \equiv 0 \pmod{p}\) and therefore \(Z_A \equiv 0 \pmod{p}\). Thus \(\Psi_A\) is indeed an integral matrix.

We conclude that the correspondence \(A \mapsto \Psi_A\) gives us an injection \(R(q,p^2q') \hookrightarrow R(n,p^2n')\). Because of the Lemma 5.3 we can also view this map as injection

\[
\Psi : E(q)\backslash R(q,p^2q') \longrightarrow E(n)\backslash R(n,p^2n') , \quad p \nmid \det q \quad . \] (5.30)

**Remark 5.6.** The above construction does not depend on primality of \(p\) and works equally well for any number \(a\) coprime to \(\det q\) resulting in injection \(R(q,a^2q') \hookrightarrow R(n,a^2n')\). In particular, if \(q\) is similar to \(q'\) then \(n\) is similar to \(n'\).

**Remark 5.7.** The lift \(\Psi\) is also compatible with the bijection \(R(q,p^2q') \longrightarrow R(q',p^2q)\) given by \(A \mapsto p^2A^{-1}\) in the sense that \(\Psi_{p^2A^{-1}} = p^2\Psi_A^{-1} \in R(n',p^2n)\). Indeed, the fact that \(\Psi_{p^2A^{-1}}\) and \(p^2\Psi_A^{-1}\) both belong to \(R(n',p^2n)\) follows directly from definition of \(\Psi\). Then using the equality \(N'[\Psi_{p^2A^{-1}}] = N'[p^2\Psi_A^{-1}]\) together with (5.12) and (5.28) one can see that \(\Psi_{p^2A^{-1}} = p^2\Psi_A^{-1}\).
6. Factorization of automorph lifts

Thus, we can lift an automorph $A \in R(q, p^2q')$ of ternary quadratic forms to an automorph $\Psi_A \in R(n, p^2n')$ of quaternary forms (see 5.28). But with quaternary quadratic forms we are no longer limited to multipliers $p^2$ and can use automorphs with multipliers $p$ (compare to existence of Hecke operators $T(p)$ rather than $T(p^2)$ for corresponding theta-series).

According to the Theorem 1.3 of [2] each integral automorphism $A \in \text{M} \subseteq \text{L}$ is a product $\mathcal{M}\mathcal{L}$ of automorphs $\mathcal{M} \in R(n, pf)$ and $\mathcal{L} \in R(f, p\mathbf{n}')$ for some quadratic form $f$ similar to $n$. Our next goal is to describe explicitly such factorizations for the constructed lift $\Psi_A \in R(n, p^2n')$. Since the general strategy behind Clifford algebras is to replace questions about representations by quadratic forms with questions concerning multiplicative arithmetic of these algebras, we will try to use for our purposes certain properties of ideals of the algebras. First of all let us look more closely at the algebraic multiplicative arithmetic of these algebras, we will try to use for our purposes certain properties of ideals of the algebras. Moreover, the norm of any element of $I$ includes $\phi$. Then from (5.28) it follows that the submodule $(e_0, e_23, -e_13, e_12) = (e_0, e_23, -e_13, e_12) \cdot \Phi_A$.

defined on the corresponding order of $C_0(E', q')$. One can easily extend this homomorphism to an isomorphism of $Q$-algebras:

$$\varphi_A : C_0(E', q') \otimes Q \longrightarrow C_0(E, q) \otimes Q,$$

whose matrix with respect to the natural bases is $\Phi_A \cdot \text{diag}(1, p^{-2}, p^{-2}, p^{-2}) = p^{-1}\Psi_A$. Using Lemma 5.5 one can see that the matrix $\Phi_A \cdot \text{diag}(1, p^{-1}, p^{-1}, p^{-1})$ is integral, which implies that the image under $\varphi_A$ of the order of $C_0(E')$ generated by $e_0, pe_23, -pe_13, pe_12$ belongs to $C_0(E)$. Then from (5.28) it follows that the submodule $(e_0, e_23, -e_13, e_12)\Psi_A \cdot Z^4 \subset C_0(E)$ is nothing else but the image $\varphi_A(pC_0(E'))$ of the principle ideal $pC_0(E') \subset C_0(E')$ and therefore is itself ideal of the order $C_0(E') \cap C_0(E)$. Consider the left $C_0(E)$-ideal

$$J_A = C_0(E) \cdot \varphi_A(pC_0(E')) ,$$

which extends $\varphi_A(pC_0(E'))$. We claim the following.

**Lemma 6.1.** In the above notations $J_A$ is a proper (left) $C_0(E)$-ideal which properly includes $\varphi_A(pC_0(E'))$:

$$\varphi_A(pC_0(E')) \subset J_A \subset C_0(E).$$

Moreover, the norm of any element of $J_A$ is divisible by $p$.

**Proof.** Validity of the above inclusions is clear and we only need to prove that both of them are proper. To show that $\varphi_A(pC_0(E')) \not\subseteq J_A$, assume the opposite. Since $J_A$ is a left $C_0(E)$-ideal, we would have then for any $c \in C_0(E)$ an inclusion $c \cdot \varphi_A(pC_0(E')) \subset \varphi_A(pC_0(E'))$. Recall that as an isomorphism of $Q$-algebras $\varphi_A$ is invertible, the matrix of
\( \varphi_A^{-1} \) with respect to the natural bases is \( p\Psi_A^{-1} = p^{-1}\Psi_{p^2 A^{-1}} \), in particular \( \varphi_A^{-1}(C_0(E)) \subset p^{-1}C_0(E') \). Thus, our assumption would imply that \( \varphi_A(\varphi_A^{-1}(c) \cdot p C_0(E')) \subset \varphi_A(p C_0(E')) \) and so \( \varphi_A^{-1}(c) \cdot p C_0(E') \subset p C_0(E') \) for any \( c \in C_0(E) \). The last inclusion would be possible only if \( \varphi_A^{-1}(c) \in C_0(E') \) for any \( c \in C_0(E) \), which is definitely not the case (for example for \( c = e_{23} \)). Therefore \( \varphi_A(p C_0(E')) \) is a proper subset of \( I_A \). Next, to prove that \( I_A \neq C_0(E) \) we will show that the norm of any element of \( I_A \) is divisible by \( p \). Indeed, any element of \( \bar{I}_A = C_0(E) \cdot \varphi_A(p C_0(E')) \) is a sum of elements of the form \( c \cdot \varphi_A(pc') \) with \( c \in C_0(E) \) and \( c' \in C_0(E') \). Observe that \( \mathbf{n}(\varphi_A(pc')) \equiv 0 \pmod{p^2} \) for any \( c' \in C_0(E') \), according to Lemma 5.5. It remains to note that if \( b, c \in C_0(E) \) and \( b', c' \in C_0(E') \) then

\[
\mathbf{n}(b \varphi_A(pb') + c \varphi_A(pc')) = \mathbf{n}(b \varphi_A(pb')) + \mathbf{n}(c \varphi_A(pc')) + \mathbf{s}(b \varphi_A(pb') \cdot c \varphi_A(pc'))
\]

\[
\mathbf{s}(b \varphi_A(pb') \cdot c \varphi_A(pc')) 
\]

\[
\equiv p \cdot \mathbf{s}(b \varphi_A(pb' c') \cdot c') 
\equiv 0 \pmod{p}
\]

therefore \( \mathbf{n} | \mathcal{J}_A \equiv 0 \pmod{p} \) as claimed and so \( \mathcal{J}_A \) is a proper subset of \( C_0(E) \).

The above lemma immediately leads to

**Lemma 6.2.** Let \( \mathcal{I}_A \) be a generating matrix of the extension ideal \( \mathcal{I}_A \) (6.1) with respect to the natural basis, so that \( \mathcal{I}_A = (e_0, e_{23}, -e_{13}, e_{12}) \cdot \mathcal{I}_A \mathbb{Z}^4 \). Then \( \Psi_A \in \mathcal{I}_A \mathbb{Z}^4 \) and \( \mathcal{I}_A \in \mathbb{R} (\mathbf{n}, p\mathbf{f}) \) with integral quadratic form

\[
\mathbf{f}_{\mathcal{J}_A} = \frac{\mathbf{n} | \mathcal{J}_A}{\sqrt{|\mathcal{J}_A|}}
\]

similar to \( \mathbf{n} \). Here we have set \( |\mathcal{J}_A| = |\mathcal{I}_A| = |C_0(E)/\mathcal{J}_A| \), the index of \( \mathcal{J}_A \) in \( C_0(E) \).

**Proof.** Indeed, since \( \mathcal{J}_A \) is a proper (left) ideal of \( C_0(E) \), we know that \( |\mathcal{I}_A| \neq 1 \). Moreover, for a fixed element \( i \in \mathcal{J}_A \) we have \( C_0(E) \cdot i \subset \mathcal{J}_A \) and thus \( C_0(E) \cdot i = (e_0i, e_{23}i, -e_{13}i, e_{12}i) \mathbb{Z}^4 = (e_0, e_{23}, -e_{13}, e_{12}) \cdot \mathcal{I}_A \mathbb{Y} \mathbb{Z}^4 \) with some \( \mathbb{Y} \in \mathbb{Z}^4 \). Then, looking at the matrix of the restriction of the norm-form \( \mathbf{n} \) onto the submodule \( C_0(E) \cdot i \subset C_0(E) \) we see that \( \mathbf{n}(i)N = N[\mathcal{I}_A \mathbb{Y}] \) which implies that \( |\mathcal{J}_A| \) divides \( \mathbf{n}^2(i) \). Denote for a moment \( |\mathcal{I}_A| = a^2b \) with a square-free positive integer \( b \). Clearly then \( ab \) divides \( \mathbf{n}(i) \), but \( i \in \mathcal{J}_A \) was arbitrary and therefore the form \( \frac{1}{ab} \mathbf{n} | \mathcal{J}_A \) is integral as well as its matrix \( (ab)^{-1}N[\mathcal{I}_A] \) with respect to the natural basis. In particular \( \det ((ab)^{-1}N[\mathcal{I}_A]) = (|\mathcal{J}_A|^2 |\mathbf{n}|/(ab)^4) = (\det \mathbf{n})/b^2 = (\Delta/b)^2 \) is integral and so \( b \) divides \( \Delta = \det \mathbf{q}/2 \), see (5.13). On the other hand, using Lemma 6.1 we also see that \( \varphi_A(p C_0(E')) \) is a proper subset of \( \mathcal{J}_A \) which in terms of corresponding matrices means that \( \Psi_A = \mathcal{I}_A \mathbb{X} \) for some \( \mathbb{X} \in \mathbb{Z}^4 \), \( |\mathbb{X}| \neq 1 \) and in particular \( |\mathcal{J}_A| \) is a proper divisor of \( |\Psi_A| = p^4 \). Therefore \( b \mid \gcd(\Delta, p^4) \), but according to our assumption prime \( p \) is coprime to \( \det \mathbf{q} = 2\Delta \) thus \( b = 1 \) and \( |\mathcal{J}_A| = p^2 \). We conclude that the quadratic form

\[
\mathbf{f}_{\mathcal{J}_A}(x) = \frac{\mathbf{n} | \mathcal{J}_A(x)}{\sqrt{|\mathcal{J}_A|}} = p^{-1} \cdot \mathbf{n} | \mathcal{J}_A(x)
\]

(6.2)
that $\mathfrak{f}_{3A}$ with respect to the natural basis $(e_0, e_{23}, -e_{13}, e_{12}) \cdot \mathcal{I}_A$ is $F_{3A} = p^{-1} N[\mathcal{I}_A] \in \mathbb{Z}_p^4$ and its determinant $\det \mathfrak{f}_{3A}$ is equal to $\det \mathfrak{n}$. Summing up we see that $\mathfrak{f}_{3A}$ is similar to $\mathfrak{n}$ and that $\mathcal{I}_A \in R(\mathfrak{n}, p \mathfrak{f}_{3A})$.

Lemmas 6.1 and 6.2 in essence allow us to write a lift $\Psi_A \in R(\mathfrak{n}, p^2 \mathfrak{n}')$ as a product of automorphs with multipliers $p$ of similar quadratic forms $\mathfrak{n}, \mathfrak{n}'$ and $\mathfrak{f}$. Note that the quadratic form $\mathfrak{f} = \mathfrak{f}_{3A}$ in the above factorization is associated with a class of equivalent (left) ideals of $C_0(E, \mathfrak{q})$ and is defined only up to integral equivalence, see (6.2). (Two left $C_0(E)$-ideals $\mathfrak{I}, \mathfrak{I}' \subset C_0(E)$ are called equivalent if $\mathfrak{I}\alpha = \mathfrak{I}\beta$ for some $\alpha, \beta \in C_0(E)$.) An analogous factorization of $\Psi_A$ can be obtained if one considers the extension of $\varphi_A(p C_0(E'))$ to the right ideal $\varphi_A(p C_0(E')) \cdot C_0(E)$ instead of the left ideal as in (6.1). We now seek to find the total number of such factorizations (different modulo corresponding groups of units) for a given $\Psi_A$. Following a general method of investigation of questions of this type developed in [1], we will view a matrix $\mathcal{M} \in R(\mathfrak{n}, p \mathfrak{f})$ for a quadratic form $\mathfrak{f}$ similar to $\mathfrak{n}$ as a solution of the quadratic congruence

$$n[\mathcal{M}] \equiv 0 \pmod{p},$$

whose matrix of elementary divisors is equal to $\mathcal{D}_p = \text{diag}(1, 1, p, p)$. That is to say, $\mathcal{M}$ belongs to the double coset $\Lambda^4 \mathcal{D}_p \Lambda^4$ which is a necessary condition for $\mathcal{M}$ to define an automorph with multiplier $p$ of similar quaternary quadratic forms. (Indeed, we should have $|\mathcal{M}| = p^2$ and $p \mathcal{M}^{-1} \in \mathbb{Z}_p^4$, which leaves us with $\mathcal{D}_p$ as the only possible matrix of elementary divisors.) Since at the moment we do not want to distinguish a particular form $\mathfrak{f}$ in its equivalent class $\{\mathfrak{f}[U] ; U \in \Lambda^4\}$, we will next be counting only different left cosets $\mathcal{M} \Lambda^4 \subset \Lambda^4 \mathcal{D}_p \Lambda^4 / \Lambda^4$. Finally, because of our interest in factorizations of automorphs with multipliers $p^2$, we restrict our search to only those automorphs $\mathcal{M}$, which divide (from the left) a particular $\mathcal{A} \in R(\mathfrak{n}, p^2 \mathfrak{n}')$. Thus, following [1], in order to find the total number of factorizations of $\mathcal{A}$ one introduces isotropic sums

$$S_p(n, \mathcal{D}_p, \mathcal{A}) = \sum_{\mathcal{M} \in \Lambda^4 \mathcal{D}_p \Lambda^4 / \Lambda^4 \atop n[\mathcal{M}] \equiv 0 \pmod{p}, \mathcal{M}|\mathcal{A}} 1.$$  

(6.3)

General isotropic sums of the above type were computed in [1], Theorem 5.1 (see also [2] and [4]). The idea behind the computation is to associate to a matrix $\mathcal{L} \in \mathbb{Z}_p^4$ the subspace $V_p(\mathcal{L})$ of the quadratic space $((\mathbb{Z}/p\mathbb{Z})^4, n \mod p)$ spanned by the columns of $\mathcal{L}$ modulo $p$. Then the summation conditions on $\mathcal{M}$ in (6.3) mean exactly that $V_p(\mathcal{M})$ is a 2-dimensional isotropic subspace of $((\mathbb{Z}/p\mathbb{Z})^4, n)$ which contains fixed subspace $V_p(\mathcal{A})$. The number of such subspaces can be found with help of standard methods of Geometric Algebra over finite fields (see [9]). Specializing results of [1], Theorem 5.1 to our case we find that

$$S_p(n, \mathcal{D}_p, \mathcal{A}) = \begin{cases} 1 & \text{if } \text{rank}_{\mathbb{F}_p} \mathcal{A} = 2, \\ 1 + \chi_n(p) & \text{if } \text{rank}_{\mathbb{F}_p} \mathcal{A} = 1, \\ (1 + \chi_n(p))(p+1) & \text{if } \text{rank}_{\mathbb{F}_p} \mathcal{A} = 0, \end{cases}$$

(6.4)

where $p \nmid \det \mathfrak{q}$ and $\chi_n(p)$ is the character (2.4) of the form $\mathfrak{n}$. (Note that $\chi_n(p) = 1$ for $p \nmid \det \mathfrak{q}$ because of (5.13).) The above sum can be rewritten as follows. The condition
N[M] \equiv 0 \pmod{p} means that N[M] = pF for some even matrix F of order 4. Since 
\det M = \det D_p = p^2, then \det F = \det N and thus quadratic form f(X) = 2^{-1}F[X]

is similar to n. Since M is defined only modulo right multiplication by \Lambda^4, we can replace F by any (integri-

cally) equivalent form F[U], U \in \Lambda^4. Let us fix a complete system

\{n_1, \ldots, n_H\} of representatives of different equivalence classes of the similarity class of

n (analogous to (4.2)). Note that h \leq H since we can choose n_i with 1 \leq i \leq h to be

quadratic forms defined by the norms on the even Clifford subalgebras C_0(\mathbb{Z}^3, q_i), 1 \leq i \leq

h. (Indeed, according to Lemmas 5.4 and 5.5 these n_i’s 1 \leq i \leq h belong to the different

equivalence classes of the same similarity class.) Then for each M \in \Lambda^4D_p\Lambda^4/\Lambda^4 with

N[M] \equiv 0 \pmod{p} there exists a unique number i, 1 \leq i \leq H such that n_i[MU] = p n_i

for some U \in \Lambda^4. Next we note that according to the theory of elementary divisors (see

[3], Lemma 3.2.2) R(n, p n_i) \subset \Lambda^4D_p\Lambda^4, therefore

\[ S_p(n, D_p, A) = \sum_{i=1}^{H} \sum_{M \in R(n, p n_i) / E(n_i)} 1 \]  

(6.5)

which already resembles the right hand side of hypothetical (4.5) we seek to establish. Similar considerations of another isotropic sum allow us to compute also the left hand side of (4.5). Indeed, let \{q_1, \ldots, q_h\} be a complete system (4.2). According to the theory of elementary divisors the set of primitive automorphs R^+(q, p^2q_i) coincides with

R(q, p^2q_i) \cap \Lambda^3D_p\Lambda^3, where D_p = \text{diag}(1, p, p^2), see [4] Lemma 3.1. Conversely, any A \in \Lambda^3D_p\Lambda^3/\Lambda^3 such that Q[A] \equiv 0 \pmod{p^2} defines a quadratic form p^{-2}q[A] which is similar to q and therefore is integrally equivalent to exactly one of the forms q_i so that

q[AU] = p^2q_i for a unique number i, 1 \leq i \leq h and some U \in \Lambda^3. We conclude that

\[ \sum_{i=1}^{h} A \in R^+(q, p^2q_i) / E(q_i) 1 = S_{p^2}(q, D_p, O_3) = \sum_{A \in \Lambda^3D_p\Lambda^3/\Lambda^3 \atop Q[A] \equiv 0 \pmod{p^2}} 1 \]  

(6.6)

where O_3 is the zero matrix of order 3. Furthermore, using Theorem 2.2 of [4] we can provide an explicit value of the isotropic sum (6.6):

\[ S_{p^2}(q, D_p, O_3) = p + 1 \]  

(6.7)

Combining formulas (6.3)–(6.7) we finally establish (4.5) and deduce the following Theorem, which expresses total number of ternary automorphs of q with multiplier p^2 in terms of quaternary automorphs with multiplier p and can be viewed as a generalization of the Shimura’s correspondence for theta-series of ternary positive definite quadratic forms to the case of indeterminate forms:

**Theorem 6.3.** Let q be an integral nonsingular ternary quadratic form (5.1) with
determinant \det q = 2\Delta, and let p be a prime not dividing 2\Delta. Fix a complete system
\{q_1, \ldots, q_h\} of representatives of different equivalence classes of the similarity class of \(q\). Then

\[
\sum_{A \in \bigcup_{i=1}^h R^*(q, p^2q_i)/E(q_i)} 1 = (1 + \chi_n(p))^{-1} \sum_{M \in \bigcup_{i=1}^H R(n, p n_i)/E(n_i)} 1,
\]

where \(n\) is integral quaternary quadratic form defined (with respect to some basis) by the norm (5.10) on the even subalgebra \(C_0(E)\) of the Clifford Algebra \(C(E, q)\), \(\chi_n(p) = \left(\frac{(-1)^2 \Delta^2}{p}\right)\) is the character (2.4) of \(n\), and \(\{n_1, \ldots, n_H\}\) is a complete system of representatives of equivalent classes in the similarity class of \(n\).

Proof. Indeed, taking \(A\) to be the zero matrix of order 4 in (6.4) and using (6.7) we see that

\[
S_{p^2}(q, D_p, O_3) = (1 + \chi_n(p))^{-1} S_p(n, D_p, O_4)
\]

Because of (6.5) and (6.6), this relation between the isotropic sums is equivalent to the above relation between numbers of automorphs, which also proves (4.5).

Tracing back considerations of section 4, one can see that the above theorem immediately implies

**Corollary 6.4.** With the notation and under the assumptions of Theorem 6.3, let in addition \(q\) be positive definite. Then Shimura’s lift of the generic theta-series \(\Theta_{\{q\}}(z)\) and the generic theta-series \(\Theta_{\{n\}}(z)\) of the norm \(n\) on the even Clifford subalgebra \(C_0(E, q)\) have the same eigenvalues \(p + 1\) for all Hecke operators \(T(p)\) with \(p \nmid \det q\).

Unfortunately up to the present the isotropic sums of type (6.3) or (6.6) have not been computed in the case of prime \(p\) dividing the determinant of corresponding quadratic form. This prevents us from comparing directly eigenvalues of \(\Theta_{\{n\}}(z)\) and of Shimura’s lift of \(\Theta_{\{q\}}(z)\) on singular Hecke operators \(T(p)\) with \(p\) – divisor of level. Thus, the question if Shimura’s lift of \(\Theta_{\{q\}}(z)\) for arbitrary ternary form \(q\) is always proportional to \(\Theta_{\{n\}}(z)\) remains open.

Nevertheless automorph class lifting \(A \mapsto \Psi_A(5.30)\) constructed in section 5 allows us to make progress in another direction and exhibit algebraic origins of the equality (4.5) and therefore of Shimura’s correspondence for generic theta-series. Indeed, because of Eichler’s commutation relation (2.5), linear combinations of theta-series invariant under Hecke operators (such as generic theta-series, for example) are determined by corresponding sets of automorphs with multipliers \(p\) or \(p^2\) (according to the weight of the theta-series). Thus, algebraic relations between various sets of automorphs can in principle be responsible for correspondences between certain linear combinations of theta-series. (Moreover, they can provide a natural generalization of such correspondences to the case of indeterminate forms.) In particular, in the case of Shimura’s lift of generic theta-series of ternary forms we seek relations between two sets of automorphs:

\[
\bigcup_{i=1}^h E(q_i)\backslash R^*(q_i, p^2q_i) \quad \text{and} \quad \bigcup_{j=1}^H E(n_j)\backslash R(n_j, pn) .
\]
of ternary and quaternary quadratic forms respectively. Recall that by factoring \( \Psi_A \) in Lemmas 6.1 and 6.2, we extended our automorph lift \( A \mapsto \Psi_A \) to an inclusion \( A \mapsto I_A \) of \( \cup_i E(q_i) \backslash R^+_{\mathbb{Q}}(q_i, p^2 \mathbf{q}) \) into \( \cup_j E(n_j) \backslash R(n_j, pn) \). Now we claim that this map is in fact an injection whose inverse can be naturally extended turning the set of quaternary automorphs with multiplier \( p \) into a 2-fold covering of the set of ternary automorphs with multipliers \( p^2 \). More precisely we have the following

**Theorem 6.5.** Let \( \mathbf{q} \) be an integral nonsingular ternary quadratic form with matrix \( Q \) given by (5.3) and let \( N \) be the matrix \((5.12)\). If \( p \) is a prime number coprime to \( \det \mathbf{q} \), then for any \( M \in \Lambda^4 D_p \Lambda^4 / \Lambda^4 \) such that \( N[M] \equiv 0 \pmod{p} \) there exists (a unique) \( A \in \Lambda^3 D_p \Lambda^3 / \Lambda^3 \) such that \( Q[A] \equiv 0 \pmod{p^2} \) and \( M|\Psi_A \) (see (5.28)).

**Proof.** First of all, if \( M|\Psi_A \), then \( M^{-1} \Psi_A \) is an integral automorph (of quaternary forms) with multiplier \( p \), and thus \( p \Psi_A^{-1} M \) is an integral matrix. Therefore, using the Remark 5.7 we see that

\[
M|\Psi_A \iff \Psi_{p^2 A^{-1}} \cdot M \equiv 0 \pmod{p} \quad (6.8)
\]

Next, let \( M = UD_p W \) with \( U, W \in \Lambda^4 \). Since \( N[M] \equiv 0 \pmod{p} \) if and only if \( N[M W^{-1}] \equiv 0 \pmod{p} \), \( \Psi_{p^2 A^{-1}} M \equiv 0 \pmod{p} \) if and only if \( \Psi_{p^2 A^{-1}} M W^{-1} \equiv 0 \pmod{p} \) and \( M \) is defined only up to right multiplication by \( \Lambda^4 \), it is enough to consider \( W = 1_4 \) and \( M = UD_p \).

Similarly, let \( A = VD_p W \) with \( V, W \in \Lambda^3 \). Clearly \( Q[A] \equiv 0 \pmod{p^2} \) if and only if \( Q[A W^{-1}] \equiv 0 \pmod{p^2} \). We also claim that \( \Psi_{p^2 W A^{-1}} M \equiv 0 \pmod{p} \) if and only if \( \Psi_{p^2 W A^{-1}} M \equiv 0 \pmod{p} \). Indeed, consider the following commutative diagram:

\[
\begin{array}{ccc}
(\mathbb{Z}^3, \mathbf{q}[p^{-1} A W^{-1}]) & \xleftarrow{\alpha_W} & (\mathbb{Z}^3, \mathbf{q}[p^{-1} A]) & \xleftarrow{\alpha_{p^2 A^{-1}}} & (E, p^2 \mathbf{q}) \\
\downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} \\
C(\mathbb{Z}^3, \mathbf{q}[p^{-1} A W^{-1}]) & \xleftarrow{\varphi_W} & C(\mathbb{Z}^3, \mathbf{q}[p^{-1} A]) & \xleftarrow{\varphi_{p^2 A^{-1}}} & C(E, p^2 \mathbf{q})
\end{array}
\]

Using (5.20),(5.22) and (5.28) one can see that \( \Psi_{p^2 W A^{-1}} = \Phi_W \cdot \Psi_{p^2 A^{-1}} \in \Lambda^4 \Psi_{p^2 A^{-1}} \) since \( \varphi_W \) is an isomorphism of Clifford algebras. Thus we can take \( W = 1_3 \) and \( A = VD_p \).

Next we set \( V = (V_1, V_2, V_3) \) and

\[
U = (U_1, U_2, U_3, U_4) = (u_1, u_2, u_3, u_4).
\]

Thus given \( M = UD_p = (U_1, U_2, p U_3, p U_4) \) with \( i U_i N U_j \equiv 0 \pmod{p} \) for \( i = 1, 2 \) we want to find \( A = VD_p = (V_1, p V_2, p^2 V_3) \) such that \( V \in \Lambda^3 \) with \( Q[V_1] \equiv 0 \pmod{p^2} \), \( V_2 Q V_1 \equiv
0 (mod \( p \)) and
\[
\Psi_{p^2 A^{-1}} \cdot \mathcal{M} = \Psi_{p^2 D_p^{-1} V^{-1}} \cdot UD_p = \begin{pmatrix}
p & tZ \\
0 & tV_1 \\
0 & p^t V_2 \\
0 & p^2 tV_3
\end{pmatrix} \cdot \begin{pmatrix}
u_1 & u_2 & p u_3 & p u_4 \\
U_1 & U_2 & p U_3 & p U_4
\end{pmatrix} \equiv 0 \pmod{p},
\]
where \( Z = p^{-1} Z_{(p^2 A^{-1})} \) depends on \( V \), see (5.28). The last condition is equivalent to the system of congruences \( t Z U_i \equiv 0 \pmod{p} \), \( t V_i U_i \equiv 0 \pmod{p} \) for \( i = 1, 2 \). Note that all of the above conditions on columns of matrix \( V \in \Lambda^3 \) are modulo \( p \) (or \( p^2 \)) and therefore the choice of particular representatives \( \pmod{p^2} \) is irrelevant as long as \( V \in \Lambda^3 \). Recall that columns \( V_1, V_2 \in \mathbb{Z}^3 \) can be complemented to a matrix \( (V_1, V_2, V_3) \in \Lambda^3 \) if and only if the greatest common divisor of principal minors of \( (V_1, V_2) \in \mathbb{Z}_2^3 \) is 1. Using the above remarks and Lemma 1.3 above we can replace the condition \( V \in \Lambda^3 \) by linear independence of \( V_1 \) and \( V_2 \) modulo \( p \). Thus, combining all the conditions, we need to find \( V_1, V_2 \in \mathbb{Z}^3 \) such that \( V_1, V_2 \) are linearly independent modulo \( p \) and
\[
\begin{cases}
Q[V_1] \equiv 0 \pmod{p^2} \\
t V_2 Q V_1 \equiv 0 \pmod{p} \\
t V_1 \cdot U_i \equiv 0 \pmod{p} \quad \text{for } i = 1, 2 \\
t Z \cdot U_i \equiv 0 \pmod{p} \quad \text{for } i = 1, 2
\end{cases}
\]
Denote
\[
\mathcal{W} = \mathcal{U} = (\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4) = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 \\
\tilde{W}_1 & \tilde{W}_2 & \tilde{W}_3 & \tilde{W}_4
\end{pmatrix} \in \Lambda^4,
\]
then \( t \mathcal{W} \cdot \mathcal{U} = (t \mathcal{W}_i \cdot U_j) = 1_4 \) which implies that \( w_i \cdot u_j + t \tilde{W}_i \cdot U_j \equiv 0 \pmod{p} \) if \( i \neq j \). Next, since \( N[UD_p] \equiv 0 \pmod{p} \), then using (5.14) we have
\[
\tilde{N}[t \mathcal{U}^{-1}] = \Delta \cdot \begin{pmatrix}
2 \det Q_0 & t(Q_0 B) \\
(Q_0 B) & Q
\end{pmatrix} \begin{pmatrix}
w_i & \ldots \\
\tilde{W}_i & \ldots
\end{pmatrix} = \begin{pmatrix}
n & * & * & * \\
* & n & * & * \\
* & * & p & p \\
* & * & p & p
\end{pmatrix}
\]
and therefore \( t \mathcal{W}_i \tilde{N} \mathcal{W}_j \equiv 0 \pmod{p} \) for \( i, j = 3, 4 \). Because \( \mathcal{W} \in \Lambda^4 \), the columns \( \mathcal{W}_3, \mathcal{W}_4 \) are linearly independent modulo \( p \) and thus their linear span (over \( \mathbb{F}_p \)) contains the column \( X \not\equiv 0 \pmod{p} \) whose first component is zero modulo \( p \):
\[
a \mathcal{W}_3 + b \mathcal{W}_4 = X \equiv \begin{pmatrix}
0 \\
X
\end{pmatrix} \pmod{p}, \quad X \not\equiv 0 \pmod{p}
\]
Using properties of \( \mathcal{W}_3, \mathcal{W}_4 \) exhibited above we see that \( t X U_i \equiv t X U_i \equiv 0 \pmod{p} \) for \( i = 1, 2 \) and \( Q[X] \equiv \Delta^{-1} \tilde{N}[X] \equiv 0 \pmod{p} \). According to Lemma 2.3 of [4] there exists \( p^2 \) distinct (modulo \( p^2 \)) columns \( X \in \mathbb{Z}^3 \) such that \( X \equiv \tilde{X} \pmod{p} \) and \( Q[X] \equiv 0 \pmod{p^2} \). For any of these columns we can take \( V_1 \equiv X \pmod{p^2} \).
Next, consider the congruence \( tV_1QY \equiv 0 \pmod p \) as a linear equation with 3 variables over \( \mathbb{F}_p \). Since \( Q \in GL_3(\mathbb{F}_p) \) and \( V_1 \not\equiv 0 \pmod p \) then \( tV_1Q \not\equiv 0 \pmod p \) and therefore the space of solutions of the above equation is two dimensional. One nontrivial solution is \( V_1 \). Thus there exists \( V_2 \in \mathbb{Z}^3 \) such that \( tV_1QV_2 \equiv 0 \pmod p \) and \( V_1, V_2 \) are linearly independent modulo \( p \). Using Lemma 1.3 we can change \( V_1 \) and \( V_2 \) modulo \( p^2 \) in such a way that the matrix \( (V_1, V_2) \) is primitive and therefore complementable to an invertible matrix \( V = (V_1, V_2, V_3) \in \Lambda^3 \). Thus given \( \mathcal{U} \in \Lambda^4 \) such that \( N[\mathcal{U}D_p] \equiv 0 \pmod p \) we can find \( V \in \Lambda^3 \) such that \( Q[VD_p] \equiv 0 \pmod {p^2} \) and \( tV_1\mathcal{U}_i \equiv 0 \pmod p \) for \( i = 1, 2 \). To prove the Theorem we still need to show that \( tZ\mathcal{U}_i \equiv 0 \pmod p \) for \( i = 1, 2 \) (\( Z \) depends on \( V \), see above). Luckily, this condition is met automatically with our choice of \( V \). Indeed, set \( A = VD_p \) and \( q' = p^{-2} \cdot q[A] \), then \( A \in R(q, p^2q') \) and so \( p^2\Psi_A = \Psi_{p^2A-1} \in R(n', p^2n) \), in other words \( N'[^2\Psi_{p^2A-1}] = p^2N \) (see (5.12), (5.27) and Remark 5.7). Then \( N'[\Psi_{p^2A-1} \cdot \mathcal{U}_i] = p^2N[\mathcal{U}_i] \equiv 0 \pmod p \) for \( i = 1, 2 \). But

\[
\Psi_{p^2A-1} \cdot \mathcal{U}_i = \begin{pmatrix}
    p & tZ \\
    0 & tV_1 \\
    0 & p^tV_2 \\
    0 & p^{2t}V_3
\end{pmatrix} \cdot \begin{pmatrix}
    u_i \\
    \mathcal{U}_i
\end{pmatrix} \equiv \begin{pmatrix}
    tZ\mathcal{U}_i \\
    0 \\
    0 \\
    0
\end{pmatrix} \pmod p \quad \text{for} \quad i = 1, 2
\]

by the above choice of \( V \), therefore using (5.12) we have \( N'[\Psi_{p^2A-1} \cdot \mathcal{U}_i] \equiv 2(tZ\mathcal{U}_i)^2 \equiv 0 \pmod p \) and so \( tZ\mathcal{U}_i \equiv 0 \pmod p \) for \( i = 1, 2 \) as stated (recall that \( p \neq 2 \)) since \( 2 \mid \det q \).

Summing up, there exists \( A \) such that \( \Psi_{p^2A-1} \cdot \mathcal{U}D_p \equiv 0 \pmod p \), i.e. \( \mathcal{M}[\Psi_A] \). Finally, it is easy to see that for a given \( \mathcal{M} \in \cup_i R(n_i, p^2n_i) \) such an \( A \in \cup_j R(q_j, p^2q_j)/E(q_j) \) is unique. Indeed, first note that since \( A \in \Lambda^3D_p\Lambda^3 \) then the double coset \( \Lambda^4\Psi_A\Lambda^4 \) contains matrix of the form

\[
\begin{pmatrix}
    p & * & * & * \\
    0 & & & \\
    0 & & D_p & \\
    0 & & & \\
\end{pmatrix}
\]

and so \( \text{rank} \, \Psi_A \) is either 1 or 2. Then (6.4) implies that each \( \Psi_A \in R(n, p^2n') \) is divisible from the left by at most \( 2 = 1 + \chi_n(p) \) different cosets \( \mathcal{M} \in \cup_{i=1}^HR(n_i, pn_i)/E(n_i) \). (The same is true for any \( A \in \cup_i R^*(q_i, p^2q_i) \)). If the same \( \mathcal{M} \) would divide \( \Psi_A \) for different classes of \( A \)'s, then the union over \( A \)

\[
\bigcup_{A \in \cup_{i=1}^HR^*(q_i, p^2q_i)/E(q_i)} \{ \mathcal{M} \in \cup_{i=1}^HR(n_i, pn_i)/E(n_i) : \mathcal{M}[\Psi_A] \}
\]

would not be disjoint and therefore its cardinality would be less than

\[
(1 + \chi_n(p)) \cdot \sum_{i=1}^h r^*(q_i, p^2q_i)/e(q_i) = (1 + \chi_n(p)) \cdot (p + 1).
\]

On the other hand, we have just proved that any \( \mathcal{M} \in \cup_j R(n_j, pn_j)/E(n_j) \) divides from the left some \( \Psi_A \) and thus belongs to the union (6.9), whose cardinality in view of this
should be at least \((1 + \chi_n(p)) \cdot (p + 1)\), see (6.4) and (6.5). We conclude that for any \(\mathcal{M}\) there exists exactly one \(A \in \Lambda^3 D \Lambda^3 / \Lambda^3\) such that \(Q[A] \equiv 0 \pmod{p^2}\) and \(\mathcal{M} | \Psi_A\). Incidentally we have also proved that rank\(_{\mathbb{F}_p}\) \(\Psi_A = 1\) for any such \(A\). \(\square\)

**Remark 6.6.** The above result means that for any class \(\mathcal{M} \mathcal{E}(n_i) \in R(n, p n_i) / E(n_i)\) there exists a unique class \(A \mathcal{E}(q_j) \in R^*(q_i, p^2 q_j) / E(q_j)\) such that any \(\mathcal{M}' \in \mathcal{M} \mathcal{E}(n_i)\) divides from the left \(\Psi_A\) for any \(\mathcal{M}' \in A \mathcal{E}(q_j)\). Thus the established correspondence is indeed between classes modulo groups of units rather than between individual automorphs. (To see this let \(\mathcal{M}' = A \mathcal{E}\) and \(\mathcal{M}' = \mathcal{M} \mathcal{E}\) with \(E \in E(q_j), \mathcal{E} \in E(n_i)\). Then using (5.28), (6.8) and Lemma 5.3 we have: \(\mathcal{M}' | \Psi_A' \iff \Psi_{p^2 A - 1} | \mathcal{M} \mathcal{E} \equiv 0 \pmod{p} \iff \Phi_{E - 1} \Psi_{p^2 A - 1} | \mathcal{M} \equiv 0 \pmod{p} \iff \mathcal{M} | \Psi_A\) since \(\Phi_{E - 1} \in E(n_j)\) and so it is invertible.

In order to obtain a generalization of Shimura’s lift in terms of automorph class theory, we need to consider correspondences between left (rather than right) classes of automorphs. To this end we extend the automorph class lift \(\Psi\) (5.28) and define the map \(\Upsilon : \bigcup_i E(q_i) \backslash R^*(q_i, p^2 q) \to \bigcup_j E(n_j) \backslash R(n_j, p n)\) via the following chain elements of which are already familiar (see Lemmas 5.5 and 6.2):

\[
\Upsilon : A \mapsto \Psi_A \mapsto p^2(\Psi_A)^{-1} = \Psi_{p^2 A - 1} \mapsto \mathcal{I}_{p^2 A - 1} \mapsto p(\mathcal{I}_{p^2 A - 1})^{-1} = \Upsilon_A \quad (6.10)
\]

Summing up our knowledge about the above maps we arrive to the following

**Theorem 6.7.** Let \(q\) be an integral nonsingular ternary quadratic form and let \(n\) be the integral quadratic form defined (up to integral equivalence) by the norm on the even subalgebra of the Clifford algebra \(C(q)\). Take \(\{q_1, \ldots, q_h\}\) and \(\{n_1, \ldots, n_H\}\) to be complete systems of representatives of different equivalence classes of the similarity classes of \(q\) and of \(n\) respectively. Let \(p\) be a prime number coprime to \(\det q\). Then map of classes of automorphs \(A \mapsto \Upsilon_A\) defined via (6.10), (5.28) and Theorem 6.5:

\[
\Upsilon : \bigcup_{i=1}^h E(q_i) \backslash R^*(q_i, p^2 q) \hookrightarrow \bigcup_{j=1}^H E(n_j) \backslash R(n_j, p n)
\]

is an injection such that \(\Upsilon_A\) divides \(\Psi_A\) from the right for any \(A \in \bigcup_i E(q_i) \backslash R^*(q_i, p^2 q)\).

**Proof.** Indeed, the map \(\Psi : E(q_i) \backslash R^*(q_i, p^2 q) \to E(n_i) \backslash R^*(n_i, p^2 n)\) is an injection according to (5.30) and Lemmas 5.3, 5.5. The second element of our chain is the bijection \(\mathcal{M} \mapsto p^2 \mathcal{M}^{-1}\) of classes of primitive automorphs \(E(n_i) \backslash R^*(n_i, p^2 n)\) restricted to the image of \(\Psi\). The equality \(p^2(\Psi_A)^{-1} = \Psi_{p^2 A - 1}\) was established in Remark 5.7. The map \(\mathcal{I}_{p^2 A - 1} \mapsto \mathcal{I}_{p^2 A - 1}\), \(R^*(n, p^2 n_i) / E(n_i) \to R(n, p^2 n_i) / E(n_i)\) for some \(j\) is defined via extension of ideals of corresponding Clifford algebras in Lemmas 6.1 and 6.2. It provides us with \(\mathcal{I}_{p^2 A - 1} \in \Lambda^4 D \Lambda^4 / \Lambda^4\) such that \(\mathcal{I}_{p^2 A - 1}\) divides \(\Psi_{p^2 A - 1}\) from the left and therefore is an injection according to the Theorem 6.5 and Remark 6.6. The last step in the chain (6.10) is the bijection \(\mathcal{M} \mapsto p \mathcal{M}^{-1}\), \(R(n, p n_j) / E(n_j) \to E(n_j) \backslash R(n_j, p n)\). Thus, being a composition of injections, \(\Upsilon\) is certainly an injection itself. Moreover, since \(\mathcal{I}_{p^2 A - 1}\) divides \(p^2(\Psi_A)^{-1}\) from the left, then \(\Upsilon_A = p(\mathcal{I}_{p^2 A - 1})^{-1}\) divides \(\Psi_A\) from the right. \(\square\)
From the above theorem it follows easily that one can consider the set of classes of quaternary automorphs of \( n \) as a 2-fold covering of the set of classes of ternary automorphs of \( q \):

**Corollary 6.8.** With the notation and under the assumptions of Theorem 6.7,

\[
\bigcup_{j=1}^{H} E(n_j) \backslash R(n_j, pn) = \bigcup_{A \in \cup_{i=1}^{h} E(q_i) \backslash R^*(q_i, p^2q)} \{ M \in \cup_j E(n_j) \backslash R(n_j, pn) ; \Psi_A M^{-1} \in \mathbb{Z}_4^2 \}, \tag{6.11}
\]

where for a fixed \( A \) the set \( \{ M \in \cup_j E(n_j) \backslash R(n_j, pn) ; \Psi_A M^{-1} \in \mathbb{Z}_4^2 \} \) consists of exactly 2 classes and the union over \( A \) is disjoint.

**Proof.** According to the Remark 6.6, for each \( M \in \cup_j E(n_j) \backslash R(n_j, pn) \) there exists a unique \( A \in \cup_i E(q_i) \backslash R^*(q_i, p^2q) \) such that \((pM)^{-1} \) divides \( \Psi_{p^2A^{-1}} \) from the left, which immediately implies that \( M \) divides \( \Psi_A = p^2(\Psi_{p^2A^{-1}})^{-1} \) from the right. This establishes the proposed equality (6.11) and shows that cardinality of the union over \( A \) in the right hand side of (6.11) is equal to \((1 + \chi_n(p)) \cdot (p + 1) \), see formulas (6.3),(6.4) and (6.5). On the other hand, because of (6.6) and (6.7) we see that this cardinality can be achieved only if the union over \( A \) in (6.11) is disjoint (recall that in the proof of Theorem 6.5 we already have established that \( \text{rank}_{p} \Psi_A = 1 \) and so the number of elements in the set \( \{ M \in \cup_j E(n_j) \backslash R(n_j, pn) ; \Psi_A M^{-1} \in \mathbb{Z}_4^2 \} \) is equal to \((1 + \chi_n(p)) = 2 \) according to (6.4)).

In other words, Corollary 6.8 states that each class in \( \cup_j E(n_j) \backslash R(n_j, pn) \) is a right divisor of a unique class in \( \Psi \left( \cup_i E(q_i) \backslash R^*(q_i, p^2q) \right) \). And conversely, each class in the image \( \Psi \left( \cup_i E(q_i) \backslash R^*(q_i, p^2q) \right) \) is divisible from the right by exactly \((1 + \chi_n(p)) = 2 \) classes of automorphs in \( \cup_j E(n_j) \backslash R(n_j, pn) \).
7. Concluding remarks

Automorph class lift and its factorizations studied in preceding sections sheds a new light on relations between representations of integers by ternary and quaternary quadratic forms. In particular case of positive definite ternary forms one can deduce from its construction relations between corresponding theta-series similar to Shimura’s lift as defined by P.Ponomarev in [10]. Still the theory is far from being complete and we would like to discuss some open questions.

First of all it is necessary to address the issue of singular primes, i.e. those $p$ which divide the determinant $\det q$. As we already noted in Corollary 6.4, the isotropic sums of types (6.3) and (6.6) have not been computed in the case of such primes. This prevents us from stating a stronger versions of Theorem 6.3 and Corollary 6.4 which would satisfactorily resolve the question of whether or not Shimura’s lift of $\Theta_{\{q\}}(z)$ is always equal to a normalized $\Theta_{\{n\}}(z)$. Computation of singular isotropic sums would also have other important applications, such as a complete Euler product decomposition of general Eichler and Epstein zeta-functions as well as investigation of their analytical properties (see [2]).

Another unclear issue concerning singular primes is proper construction of the automorph class lift (5.28) $\Psi : \cup_i E(q_i) \setminus R^*(q_i, p^2 q_i) \hookrightarrow \cup_j E(n_j) \setminus R^*(n_j, p^2 n_j)$ for such $p$ and its further factorization into a product of quaternary automorphs with prime multipliers (similar to results of Lemmas 6.1 and 6.2). All our proofs use the primality of $p$ to $\det q$ and it would be very interesting to see to what extent the same construction works in the singular case and what modifications (if any) should be made.

The next interesting possibility is connected to investigation of effects of the automorph class lift on individual quadratic forms. (In case of positive definite quadratic forms this could lead to correspondences between certain spaces of modular forms spanned by theta-series of different weights which are invariant under Hecke operators.) For this one needs to describe explicit factorizations of an individual image $\Psi(E(q_1) \setminus R^*(q_1, p^2 q_1))$ into products of quaternary automorphs with multiplier $p$. (Recall that we provided such a description only for the whole $\Psi(\cup_i E(q_i) \setminus R^*(q_i, p^2 q_i))$ which corresponds to Shimura’s lift of the generic theta-series $\Theta_{\{q\}}(z)$). Inspired by results of P.Ponomarev [10], here is a sketch of what one might expect.

The subspace of modular forms of half-integral weight $3/2$ spanned by $h$ theta-series of ternary quadratic forms (4.2) is invariant under Hecke operators $\frac{|S|}{T}(p^2)$ and its eigenmatrix with respect to $\Theta(z, q_i)/e(q_i), 1 \leq i \leq h$ is $t^q_i(p^2)$ given by (2.6). Let $F_i(z)$ denote Shimura’s lift $\Theta^{(a)}(z, q_i)$ for some fixed $a$ and let $F_q(z) = t^i(..., F_i(z), ...)$ be the associated vector-valued modular form of integral weight 2. Then we expect the space spanned by $F_i(z), 1 \leq i \leq h$ to be invariant under Hecke operators $\frac{|S|}{T}(p)$ whose eigenmatrix with respect to $F_i$’s is again given by $t^q_i(p^2)$. Furthermore, let $\Theta_n(z) = t^i(..., \Theta(z, n_j)/e(n_j), ...)$ be the the vector-valued theta-series of weight 2 associated to a complete system $\{n_1, ..., n_H\}$ of quaternary quadratic forms representing different equivalent classes of the similarity class of quadratic form defined by the norm on the even subalgebra of Clifford algebra $C(\mathbb{Z}^3, q_i)$ for some $i$. (We note again that we can take $n_i$ to be the norm on $C_0(\mathbb{Z}^3, q_i), 1 \leq i \leq h$ and so $h \leq H$, see Lemmas 5.4 and 5.5). We expect the space spanned by Shimura’s lifts of $\Theta(z, q_i), 1 \leq i \leq h$ to be a subspace of the space spanned by the normalized theta-series $\Theta(z, n_j)/e(n_j), 1 \leq j \leq H$, which is invariant un-
der Hecke operators $|_2 T(p)$. In other words, we hope that there exists a matrix $X \in \mathbb{Q}_H^h$ (of arithmetical nature) such that $F_q(z) = X \cdot \Theta_n(z)$. This would imply that

$$t_q^*(p^2) \cdot X \Theta_n(z) = F_q(z) |_2 T(p) = X \Theta_n(z) |_2 T(p) = (1 + \chi_n(p))^{-1} X t_n^*(p) \cdot \Theta_n(z)$$

Building up expectations, one hopes that $t_q^*(p^2) \cdot X = (1 + \chi_n(p))^{-1} X \cdot t_n^*(p)$ for some arithmetical matrix $X \in \mathbb{Q}_H^h$, in other words

$$\left( |E(q)| R^*(q_i, p^2 q_j)| \right)_{h \times h} \cdot X = (1 + \chi_n(p))^{-1} X \cdot \left( |E(n)| R^*(n_k, pn_l)| \right)_{h \times H} \ (7.1)$$

One can see that such an $X$ is not totally impossible using Theorem 6.3 and taking for example $h = 1$ and $X = (\sum_j r(n_j, 1)/e(n_j))^{-1} \cdot (1, \ldots, 1)$, the particular scalar multiple is chosen in order to make the first Fourier coefficient of $X \cdot \Theta_n(z)$ equal to 1. Another, rather trivial example of matrix $X$ with property (7.1) is

$$X = \begin{pmatrix} e^{-1}(q_1) & \ldots & e^{-1}(q_1) \\ \vdots & \ddots & \vdots \\ e^{-1}(q_h) & \ldots & e^{-1}(q_h) \end{pmatrix}_{h \times H}$$

for which we have $t_q^*(p^2) \cdot X = (p+1) \cdot X = (1 + \chi_n(p))^{-1} X \cdot t_n^*(p)$ according to the Theorem 6.3. Both examples produce the generic theta-series $\Theta_{\{n\}}(z)$ as a Hecke eigenform of weight 2 whose eigenvalue is equal to $t_q^*(p^2)$. It would be extremely interesting to find nontrivial (of rank $> 1$) examples of matrices with property (7.1), which could lead to construction of other subspaces invariant under Hecke operators of the space spanned by the theta-series $\Theta(z, n_j), 1 \leq j \leq H$ with eigenmatrix $t_q^*(p^2)$. Furthermore, and perhaps more interesting, equality of type (7.1) would provide a direct link between various zeta-functions associated to ternary and quaternary quadratic forms, which could lead to insights concerning their analytical properties (such as meromorphic continuation and functional equation). Let us illustrate this with a simple example.

If we start with $q(x) = x_1^2 + x_2^2 + x_3^2$ - the sum of 3 squares, then corresponding quaternary form (5.12) on $C_0(E, q)$ is the the sum of 4 squares $n(y) = y_1^2 + y_2^2 + y_3^2 + y_4^2$. Both forms are positive definite and both have the class number equal to 1 (i.e. $h = 1$ and $H = 1$), so their theta-series $\Theta(z, q) = e(q) \cdot \Theta_{\{q\}}(z)$ and $\Theta(z, n) = e(n) \cdot \Theta_{\{n\}}(z)$ are eigenforms of Hecke operators $|_2 T(p^2)$ or $|_2 T(p)$ with respective eigenvalues $t_q^*(p^2) = r^*(q, p^2 q)/e(q)$ or $t_q^*(p) = r(n, pn)/e(n)$, see (2.6). The only singular prime dividing the determinants in our example is $p = 2$. By the Corollary 6.4 odd-numbered Fourier coefficients of Shimura’s lift of $\Theta(z, q)$ coincide with those of $\frac{1}{8} \Theta(z, n)$, we normalize with $r(n, 1) = 8$. Note that according to the Theorem 6.3 the equality (7.1) actually holds in this situation with $X = 1$: $t_q^*(p^2) = \frac{1}{8} t_q^*(p)$ for $p \neq 2$. Next we set

$$v_k(m) = R\left( \sum_{i=1}^k x_i^2, m \right) / \left| E\left( \sum_{i=1}^k x_i^2 \right) \right| ,$$
Then, using (2.8) and (2.9) we have the following chain of equalities for any square-free positive integer \( a \) and any positive integer \( b \):

\[
\sum_{m \text{- odd}} \frac{r_4(mb)}{m^s} = \prod_{p \text{- odd}} \left( 1 - \frac{1}{4} t_4^*(p) p^{-s} + p^{1-2s} \right)^{-1} \cdot r_4(b) = \prod_{p \text{- odd}} \left( 1 - t_3^*(p^2) p^{-s} + p^{1-2s} \right)^{-1} \cdot r_4(b) = \sum_{m \text{- odd}} \frac{\chi_q(m)(2a_m)}{m^s} \prod_{p \text{- odd}} \left( 1 - \chi_q(p) \left( \frac{2a}{p} \right) p^{-s} \right) \cdot \prod_{p \text{- odd}} \left( 1 - t_3^*(p^2) p^{-s} + p^{1-2s} \right)^{-1} \cdot r_4(b) = L \left( s, \left( \frac{-a}{.} \right) \right) \cdot \sum_{m \text{- odd}} \frac{r_3(m^2a)}{m^s} \cdot \frac{r_4(b)}{r_3(a)}.
\]

Taking \( b = 1 \) and setting \( L \left( s, \left( \frac{-a}{.} \right) \right) = L_a(s) \), where \( \left( \frac{-a}{.} \right) \) is the Legendre symbol, we conclude that

\[
\sum_{m \text{- odd}} \frac{r_4(m^2a)}{m^s} = L_a^{-1}(s) \cdot \sum_{m \text{- odd}} \frac{r_3(m)}{m^s} \cdot r_3(a) \quad .
\]

(7.2)

The particular zeta-function on the left-hand side of (7.2) has Euler product (2.9) and is associated to theta-series \( \Theta(z, x_1^2 + x_2^2 + x_3^2) \) of half-integral weight. Now one can also deduce analytical properties (meromorphic continuation and functional equation) of this zeta-function from known properties of the Epstein zeta-function and the \( L \)-series on the right-hand side of (7.2).

Finally one can try to give an interpretation (and generalization) of equality (7.1) in terms of the automorh class theory which would provide a direct link between \( \Sigma_q^*(p^2) \) and \( \Sigma_n^*(p) \), see (3.1), i.e. between individual classes of ternary automorphs \( E(q_i) \backslash R^*(q_i, p^2 q_j) \) and quaternion automorphs \( E(n_k) \backslash R^*(n_k, p m_l) \) with fixed \( i, j, k, l \), for an arbitrary integral nonsingular ternary quadratic form \( q \). This would also automatically furnish a natural proof for the above conjectures on relations between numbers of representations and, as a consequence, between corresponding Eichler or Epstein zeta-functions. Such an interpretation seems to require a better understanding of arithmetic of even Clifford subalgebras \( C_0(E, q) \) in order to characterize explicitly factorizations of individual lifts \( \Psi_A \subset E(n_i) \backslash R^*(n_i, p^2 n_j) \) in terms of automorph classes in \( E(n_k) \backslash R^*(n_k, p m_l) \) for specific \( k \) and \( l \). To this day attempts to find such an interpretation were unsuccessful.

At last we mention the most intriguing mystery concerning Shimura’s lift for theta-series – the question of theta-series of quadratic forms in more than 3 variables. The apparatus of Clifford algebras extensively used in the present paper seems to be of no use in that situation since on one hand dimensions of corresponding algebras do not match the expected weights of the lifting and on the other hand the standard norm on an even Clifford subalgebra of dimension greater than 4 does not define a quadratic form in general. To author’s knowledge there is no result describing effects of Shimura’s lift on such theta-series. The mountain is still in clouds!
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