The topological susceptibility of QCD: from Minkowskian to Euclidean theory

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Abstract

We show how the topological susceptibility in the Minkowskian theory of QCD is related to the corresponding quantity in the Euclidean theory, which is measured on the lattice. We discuss both the zero-temperature case ($T = 0$) and the finite-temperature case ($T \neq 0$). It is shown that the two quantities are equal when $T = 0$, while the relation between them is much less trivial when $T \neq 0$. The possible existence of “Kogut–Susskind poles” in the matrix elements of the topological charge density between states with equal four-momenta turns out to invalidate the equality of these two quantities in a strict sense. However, an equality relation is recovered after one re-defines the Minkowskian topological susceptibility by using a proper infrared regularization.
1. Introduction

Since the pioneering works of Witten and Veneziano in 1979 for the solution of the “$U(1)$ problem” \[1,2\], a relevant role in the non–perturbative study of QCD has been played by the so–called “topological susceptibility”, defined as (in the theory at zero temperature, $T = 0$)

$$\chi \equiv -i \int d^4x \langle 0 | TQ(x)Q(0) | 0 \rangle ,$$

where $Q(x)$ is the topological charge density operator:

$$Q(x) \equiv \frac{g^2}{64\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_\mu^a F_\rho^a .$$

The topological susceptibility $\chi$ enters into the anomalous Ward identities of QCD \[2,3\] and its determination is of great relevance for understanding the role of the $U(1)$ axial symmetry in the spectrum and in the phase structure of the theory, both at zero and at finite temperature. The lattice is a unique tool to determine from first principles quantities like $\chi$, which have non–trivial dimensions in mass and therefore cannot be computed by perturbation theory. The first lattice determination of $\chi$ is that of Ref. \[4\], where it was evaluated for the pure–gauge theory. After that “seminal” work, a lot of groups have developed refined lattice techniques to obtain more realistic estimates of $\chi$ (see, for example, Refs. \[5,6\] and references therein).

Anyway, to be rigorous, what is normally evaluated on the lattice is not $\chi$ but the corresponding quantity $\chi_E$ defined in the Euclidean theory. In this paper we discuss how the topological susceptibility $\chi$ in the Minkowski theory is related to the corresponding quantity $\chi_E$ in the Euclidean theory. We consider both the zero–temperature case ($T = 0$) and the finite–temperature case ($T \neq 0$). In Sect. 2 we shall prove that the two quantities $\chi$ and $\chi_E$ are equal when $T = 0$. The case $T \neq 0$ is much less trivial and is discussed in detail in Sect. 3.

The finite temperature topological susceptibility $\chi(\beta)$ (where $\beta = 1/kT$, $k$ being the Boltzmann constant) is defined as

$$\chi(\beta) \equiv -i \int d^4x \langle TQ(x)Q(0) \rangle_\beta ,$$
where \( \langle \ldots \rangle_\beta \) is the usual quantum thermal average over the Gibbs ensemble [7, 8, 9]. Now the problem to be solved is the following. In Eq. (1.3) we have a finite-temperature Green function at different times, namely \( \langle TQ(x)Q(0) \rangle_\beta \), which is then integrated over the entire Minkowskian space-time. In order to do Monte Carlo simulations on the lattice to measure a certain Green function, one needs to represent such a Green function by a path-integral expression in the Euclidean theory. This is well known for Green functions at \( T = 0 \) (see Sect. 2) and also for finite-temperature (\( T \neq 0 \)) Green functions at equal times, i.e., of the kind \( \langle O(\vec{x}, 0)O(0, 0) \rangle_\beta \). However it is not so immediate to express a finite temperature Green function at different times, i.e., \( \langle TO(\vec{x}, t)O(\vec{0}, 0) \rangle_\beta \), in terms of a path-integral. As discussed in Sect. 3, the Euclidean topological susceptibility \( \chi_E(\beta) \) can be expressed as a path-integral over an Euclidean four-space limited in the time direction (so that it has the topology of \( R^3 \otimes T^1 \), where \( T^1 \) is a one-dimensional torus). However, the relation between \( \chi(\beta) \) and \( \chi_E(\beta) \) turns out to be much less trivial than in the \( T = 0 \) case. In particular, it comes out that the possible existence of “Kogut–Susskind poles” in the matrix elements of the topological charge density between states with equal four-momenta could invalidate the equality of these two quantities in a strict sense. Nevertheless, it will be shown in Sect. 3 that an equality relation is recovered after one re-defines the Minkowskian topological susceptibility by using a proper infrared regularization. Finally, in the Appendix we shall give a more accurate discussion about the correct definition of the topological susceptibility with the inclusion of the so-called “contact term” (see Refs. [1, 3]).

2. The zero-temperature case (\( T = 0 \))

Let us start with the most simple case: the zero-temperature one (\( T = 0 \)). The topological susceptibility at \( T = 0 \) is defined as

\[
\chi \equiv -i \int d^4x \langle 0|TQ(x)Q(0)|0 \rangle ,
\]

(2.1)

where \( Q(x) \) is the operator for the topological charge density and \( TQ(\vec{x}, t)Q(\vec{0}, 0) \) is the usual time-ordered product:

\[
TQ(\vec{x}, t)Q(\vec{0}, 0) = \theta(t)Q(\vec{x}, t)Q(\vec{0}, 0) + \theta(-t)Q(\vec{0}, 0)Q(\vec{x}, t) .
\]

(2.2)
[Actually, Eq. (2.1) is not the correct definition of the topological susceptibility: a certain equal–time commutator term (also called “contact term”) must be added to the right–hand side \[1, 3\]. A more careful discussion about the correct formula including the contact term is given in the Appendix. As it will be shown in the Appendix, all the results we find are not affected by the inclusion of the contact term. This last, however, solves some questions about positivity arising in connection with some of the formulas below.]

Remembering that, in the Heisenberg representation of quantum operators, \(Q(\vec{x}, t) = e^{iHt}Q(\vec{x}, 0)e^{-iHt}\) (\(H\) being the total Hamiltonian operator), we may insert between the two \(Q\)’s a complete set of eigenstates \(\{|n\}\) which diagonalize the total four–momentum operator \(P^\mu\) (i.e., \(P^\mu|n\rangle = q^\mu_n|n\rangle\); remember that \([P^\mu, P^\nu] = 0\), with \(P^0 \equiv H\).) Then we may perform the integrations in the space coordinates as well as in the time coordinate, making use (in this last case) of the following relations:

\[
\begin{align*}
\int_0^{+\infty} e^{\alpha t} dt &= \frac{P \alpha}{\pi} + \pi \delta(\alpha) , \\
\int_{-\infty}^0 e^{\alpha t} dt &= -\frac{P \alpha}{\pi} + \pi \delta(\alpha).
\end{align*}
\] (2.3)

(“\(P\)” stands for “principal part”.) We thus find the following expression:

\[
\chi = -2(2\pi)^3 \sum_{n\neq 0} \frac{1}{q^0_n} |\langle 0|Q(0)|n\rangle|^2 \delta^{(3)}(\vec{q}_n) .
\] (2.4)

In the derivation of this formula we have also made use of the fact that \(Q(x)\) is an hermitean operator and that its vacuum expectation value is zero for parity invariance (\(\langle 0|Q(x)|0\rangle = 0\) since \(Q(x)\) is odd under parity transformations). We have simbolically indicated a “sum” over a complete set of eigenstates \(|n\rangle\) with four–momenta \(q^\mu_n\), but this clearly means that one has to sum over the discrete modes and integrate over the continuous distribution of eigenstates.

More generally, one can consider \(\chi\) as the value at zero four–momentum \((p = 0)\) of the Fourier transform of the two–point Green function \(I(x) \equiv -i\langle 0|TQ(x)Q(0)|0\rangle\); i.e., \(\chi = \tilde{I}(p = 0)\), where

\[
\tilde{I}(p) \equiv -i \int d^4xe^{ipx} \langle 0|TQ(x)Q(0)|0\rangle .
\] (2.5)

This quantity can be written in an elegant way in terms of the so–called “spectral density” \(\rho(p)\) of \(I(x)\):

\[
\rho(p) \equiv (2\pi)^3 \sum_n |\langle 0|Q(0)|n\rangle|^2 \delta^{(4)}(p - q_n) .
\] (2.6)
Thanks to the fact that $\langle 0|Q(x)|0 \rangle = 0$, the contribution from the vacuum states ($n = 0$) vanishes, that is:

$$\rho_0(p) \equiv (2\pi)^3 |\langle 0|Q(0)|0 \rangle|^2 \delta^{(4)}(p) = 0 ,$$

and we are left with

$$\bar{\rho}(p) \equiv (2\pi)^3 \sum_{n \neq 0} |\langle 0|Q(0)|n \rangle|^2 \delta^{(4)}(p - q_n) .$$

For covariance reasons, it is clear that

$$\bar{\rho}(p) = \bar{\sigma}(p^2) \bar{\theta}(p^0) ,$$

where $\bar{\theta}(p^0) = 1$ for $p^0 > 0$ and $\bar{\theta}(p^0) = 0$ for $p^0 \leq 0$. Because of the “natural” conditions which are assumed on the spectrum of $P^\mu$, $\bar{\sigma}(p^2)$ vanishes for $p^2 < 0$ and is real and positive semidefinite for $p^2 \geq 0$. One can then write $\tilde{I}(p)$ in the so-called “spectral representation”, firstly derived by Källen and Lehmann in Refs. [10, 11]:

$$\tilde{I}(p) = \int_{-\infty}^{+\infty} d\tau \int d^3 \vec{x} e^{i\mu \tau - i\vec{p} \cdot \vec{x}} \langle 0|TQ(\vec{x}, -i\tau)Q(\vec{0}, 0)|0 \rangle .$$

Therefore, an alternative expression for $\chi$ is also derived:

$$\chi = \tilde{I}(p = 0) = -\int_{0}^{+\infty} d(\mu^2) \bar{\sigma}(\mu^2) P \frac{1}{\mu^2} .$$

Let us consider, now, the Euclidean theory. We may define $Q(\vec{x}, \tau) = e^{iH\tau} Q(\vec{x}, 0)e^{-iH\tau}$ for every complex $\tau$. In particular, we shall consider the following quantity integrated over the imaginary axis from $+i\infty$ to $-i\infty$:

$$\tilde{G}(p) \equiv -\int_{-\infty}^{+\infty} d\tau \int d^3 \vec{x} e^{i\mu \tau - i\vec{p} \cdot \vec{x}} \langle 0|TQ(\vec{x}, -i\tau)Q(\vec{0}, 0)|0 \rangle .$$

The following prescription for the T–ordered product of a bosonic field $B$ in the imaginary domain is used:

$$TB(\tau_1) B(\tau_2) = B(\tau_1) B(\tau_2), \quad \text{if } i\tau_1 > i\tau_2,$$

$$TB(\tau_1) B(\tau_2) = B(\tau_2) B(\tau_1), \quad \text{if } i\tau_1 < i\tau_2 ,$$

(2.13)
In other words, $\theta(-it) \equiv \theta(t)$, for every real $t$: this prescription is used in order to keep the T–ordering unchanged when going from Minkowskian to Euclidean theory, $(x^0, \vec{x}) \rightarrow (-ix_{E4}, \vec{x}_E)$. Proceeding as for the derivation of Eq. (2.4), we can derive the following expression for $\tilde{G}(p)$:

$$\tilde{G}(p) = -2(2\pi)^3 \sum_{n \neq 0} \frac{q_0^n}{(q_0^n)^2 + (p^0)^2} |\langle 0 | Q(0) | n \rangle|^2 \delta^{(3)}(\vec{p} - \vec{q}_n) .$$  \hspace{1cm} (2.14)$$

Therefore, at zero four–momentum ($p = 0$) we find exactly the same expression reported in Eq. (2.4) for $\tilde{I}(p = 0)$:

$$\tilde{G}(p = 0) = -2(2\pi)^3 \sum_{n \neq 0} \frac{1}{q_0^n} |\langle 0 | Q(0) | n \rangle|^2 \delta^{(3)}(\vec{q}_n) = \tilde{I}(p = 0) .$$  \hspace{1cm} (2.15)$$

It is easy to see that $\tilde{G}(p = 0)$ is the topological susceptibility $\chi^E$ in the Euclidean theory. In fact:

$$\tilde{G}(p = 0) = - \int_{-\infty}^{+\infty} d\tau \int d^3 \vec{x} \langle 0 | TQ(\vec{x}, -i\tau)Q(\vec{0}, 0) | 0 \rangle$$

$$= \int d^4 x_E \langle Q_E(x_E)Q_E(0) \rangle_E \equiv \chi^E ,$$  \hspace{1cm} (2.16)$$

where

$$\langle Q_E(x, \tau)Q_E(\vec{0}, 0) \rangle_E \equiv -\langle 0 | TQ(\vec{x}, -i\tau)Q(\vec{0}, 0) | 0 \rangle$$  \hspace{1cm} (2.17)$$

(“$E$” stands for “Euclidean”) is just the Green function in the Euclidean theory. [A more accurate discussion about Eqs. (2.16) and (2.17) with the inclusion of the contact term is given in the Appendix.] We remind that the correspondence relationships from Minkowskian to Euclidean theory are:

$$(x^0, \vec{x}) \rightarrow (-ix_{E4}, \vec{x}_E) ;$$

$$A_0(x) \rightarrow iA_{E1}(x_E) ;$$

$$A_k(x) \rightarrow A_{Ek}(x_E) \quad (k = 1, 2, 3) ,$$  \hspace{1cm} (2.18)$$

with $x_E = (\vec{x}_E, x_{E4})$. And therefore:

$$Q(x) \rightarrow iQ_E(x_E) .$$  \hspace{1cm} (2.19)$$
From Eqs. (2.16), (2.15) and (2.4) we finally derive that

\[ \chi = \chi_E , \quad (2.20) \]

i.e., the topological susceptibility in the Minkowskian theory coincides with the topological susceptibility in the Euclidean theory. This last is the quantity which can be measured on the lattice [5, 6].

3. The finite–temperature case \((T \neq 0)\)

Let us address, now, the most difficult case: the finite–temperature case \((t \neq 0)\). The finite temperature topological susceptibility \(\chi(\beta)\) (where \(\beta = 1/kT\), \(k\) being the Boltzmann constant) is defined as

\[ \chi(\beta) \equiv -i \int d^4x \langle TQ(x)Q(0) \rangle_\beta . \quad (3.1) \]

[Actually, as in the \(T = 0\) case, Eq. (3.1) is not quite correct and a certain contact term must be added to the right–hand side. We refer again to the Appendix for a more accurate discussion about this point.]

The expectation value \(\langle \ldots \rangle_\beta\) is the usual quantum thermal average over the Gibbs ensemble [4, 8, 9]:

\[ \langle O \rangle_\beta \equiv \frac{\text{Tr}[e^{-\beta H}O]}{Z(\beta)} , \quad (3.2) \]

where

\[ Z(\beta) \equiv \text{Tr}[e^{-\beta H}] \quad (3.3) \]

is the partition function of the system. As before, \(\chi(\beta)\) can be seen as the value at zero four–momentum \((p = 0)\) of the Fourier transform of the two–point Green function at finite temperature \(I_\beta(x) = -i\langle TQ(x)Q(0) \rangle_\beta\); that is, \(\chi(\beta) = \tilde{I}_\beta(p = 0)\), where:

\[ \tilde{I}_\beta(p) \equiv -i \int d^4x e^{ipx} \langle TQ(x)Q(0) \rangle_\beta . \quad (3.4) \]
One can derive an expression for $\tilde{I}_\beta(p)$ in terms of the finite–temperature spectral density $\rho_\beta(p)$, defined as
\begin{equation}
\rho_\beta(p) \equiv \frac{(2\pi)^3}{Z(\beta)} \sum_{m,n} e^{-\beta q_m^0} |\langle m|Q(0)|n\rangle|^2 \delta^{(4)}(p + q_m - q_n) .
\end{equation}

(3.5)

It is easy to see, from covariance arguments and from the explicit expression (3.5), that the spectral density $\rho_\beta(p)$ has the following properties:
\begin{align*}
\rho_\beta(p) &= \rho_\beta(p^2, p^0) ; \\
\rho_\beta(p^2, -p^0) &= e^{-\beta p^0} \rho_\beta(p^2, p^0) .
\end{align*}

(3.6)

Analogously to the zero–temperature case, one can separate within $\rho_\beta(p)$ the contribution at $p^0 = 0$:
\begin{align*}
\rho_\beta(p^2, p^0) &= \rho_{\beta,0}(p^2, p^0) + \tilde{\rho}_\beta(p^2, p^0) ,
\end{align*}

(3.7)

where:
\begin{align*}
\rho_{\beta,0}(p^2, p^0) &\equiv \frac{(2\pi)^3}{Z(\beta)} \sum_{q_m^0 = q_n^0} e^{-\beta q_m^0} |\langle m|Q(0)|n\rangle|^2 \delta^{(4)}(p + q_m - q_n) ; \\
\tilde{\rho}_\beta(p^2, p^0) &\equiv \frac{(2\pi)^3}{Z(\beta)} \sum_{q_m^0 \neq q_n^0} e^{-\beta q_m^0} |\langle m|Q(0)|n\rangle|^2 \delta^{(4)}(p + q_m - q_n) .
\end{align*}

(3.8)

Therefore, by definition:
\begin{align*}
\rho_{\beta,0}(p^2, p^0) &= 0 , \quad \text{when } p^0 \neq 0 ; \\
\tilde{\rho}_\beta(p^2, p^0) &= 0 , \quad \text{when } p^0 = 0 .
\end{align*}

(3.9)

In terms of these two quantities, the following expression for $\tilde{I}_\beta(p)$ can be derived:
\begin{align*}
\tilde{I}_\beta(p) &= \int_0^\infty d(E^2) \rho_\beta(E^2 - p^2, E)(1 - e^{-\beta E}) \times \\
&\times \left[ \frac{1}{p^2_0 - E^2 + i\varepsilon} - i \frac{2\pi}{e^\beta E - 1} \delta(p_0^2 - E^2) \right] - 2\pi i \rho_{\beta,0}(p^2, p_0) .
\end{align*}

(3.10)

Therefore, at zero four–momentum $(p = 0)$:
\begin{align*}
\chi(\beta) &= \tilde{I}_\beta(p = 0) = -\int_0^\infty d(E^2) \rho_\beta(E^2, E)(1 - e^{-\beta E}) \frac{1}{E^2} - 2\pi i \rho_{\beta,0}(0, 0) .
\end{align*}

(3.11)
In other words, separating the real part and the imaginary part:

\[
\text{Re}[\chi(\beta)] = \text{Re}[\tilde{I}_\beta(p = 0)] = - \int_0^\infty d(E^2) \tilde{\rho}_\beta(E^2, E)(1 - e^{-\beta E}) P \frac{1}{E^2} = \frac{(2\pi)^3}{Z(\beta)} \sum_{q_m^0, q_n^0} \frac{e^{-\beta q_m^0} - e^{-\beta q_n^0}}{q_m^0 - q_n^0} |\langle m | Q(0) | n \rangle|^2 \delta(3)(q_m - q_n) . \tag{3.12}
\]

And also:

\[
\text{Im}[\chi(\beta)] = \text{Im}[\tilde{I}_\beta(p = 0)] = -2\pi \rho_{\beta,0}(0, 0) = \frac{(2\pi)^4}{Z(\beta)} \sum_{m,n} e^{-\beta q_m^0} |\langle m | Q(0) | n \rangle|^2 \delta(4)(q_m - q_n) . \tag{3.13}
\]

In order to compare Eq. (3.11) with the corresponding quantity in the Euclidean theory, we shall proceed as in the previous Section and consider the following quantity, integrated over the imaginary axis from 0 to \(-i\beta\):

\[
\tilde{G}_\beta(p) \equiv - \int_0^\beta d\tau \int d^3\vec{x} e^{ip_0 \tau - i\vec{p} \cdot \vec{x}} \langle TQ(\vec{x}, -i\tau)Q(\vec{0}, 0) \rangle_\beta , \tag{3.14}
\]

where the prescription for the T-ordered product is given by Eq. (2.13). We remind that the correlation function \(\langle B(t) B(0) \rangle_\beta\) is analytical for \(-\beta < \text{Im}(t) < 0\), so that the integral in Eq. (3.14) is taken over the analyticity domain of \(\langle Q(\vec{x}, -i\tau)Q(\vec{0}, 0) \rangle_\beta\). The following expression for \(\tilde{G}_\beta(p)\) can be derived:

\[
\tilde{G}_\beta(p) = \frac{(2\pi)^3}{Z(\beta)} \sum_{m,n} e^{-\beta q_m^0} - e^{-\beta q_n^0 + i\beta p^0} |\langle m | Q(0) | n \rangle|^2 \delta(3)(\vec{p} + \vec{q}_m - \vec{q}_n) . \tag{3.15}
\]

In particular, at \(p = 0\):

\[
\tilde{G}_\beta(p = 0) = \frac{(2\pi)^3}{Z(\beta)} \sum_{q_m^0 \neq q_n^0} e^{-\beta q_m^0} \left[ \frac{1 - e^{\beta (q_m^0 - q_n^0)}}{q_m^0 - q_n^0} \right] |\langle m | Q(0) | n \rangle|^2 \delta(3)(\vec{q}_m - \vec{q}_n)
- \beta \frac{(2\pi)^3}{Z(\beta)} \sum_{q_m^0 = q_n^0} e^{-\beta q_m^0} |\langle m | Q(0) | n \rangle|^2 \delta(3)(\vec{q}_m - \vec{q}_n) . \tag{3.16}
\]
Therefore, from Eqs. (3.16), (3.12) and (3.13):
\[
\tilde{G}_\beta(p = 0) = \text{Re}[\chi(\beta)] - \beta \left(\frac{2\pi}{Z(\beta)}\right)^3 \sum_{q_m^0 = q_n^0} e^{-\beta q_m^0} |\langle m|Q(0)|n\rangle|^2 \delta^{(3)}(\vec{q}_m - \vec{q}_n);
\]
\[
\text{Im}[\chi(\beta)] = -\frac{(2\pi)^4}{Z(\beta)} \sum_{m,n} e^{-\beta q_m^0} |\langle m|Q(0)|n\rangle|^2 \delta^{(4)}(q_m - q_n).
\]  

(3.17)

An alternative way to derive these equations starts from considering some basic properties of finite-temperature Green functions. In particular, one uses the fact that the correlation function \(\langle B(t)B(0)\rangle_\beta\) is analytical for \(-\beta < \text{Im}(t) < 0\), while \(\langle B(0)B(t)\rangle_\beta\) is analytical for \(0 < \text{Im}(t) < \beta\). Moreover, one finds the following boundary conditions in the analyticity domains (if \(B\) is a bosonic field):
\[
\langle B(0)B(t)\rangle_\beta = \langle B(t-i\beta)B(0)\rangle_\beta,
\]
\[
\langle B(t)B(0)\rangle_\beta = \langle B(0)B(t+i\beta)\rangle_\beta.
\]  

(3.18)

valid for every real \(t\). By virtue of all these results, one may apply the Cauchy theorem when integrating \(\langle Q(\vec{x},\tau)Q(\vec{0},0)\rangle_\beta\) along the path represented in Fig. 1. One thus finds the same result (3.17), using also the fact that:
\[
\lim_{T \to +\infty} \int_{-\beta}^{\beta} d\tau \int d^3\vec{x} \langle Q(\vec{x},\tau)Q(\vec{0},0)\rangle_\beta = -i\beta \left(\frac{2\pi}{Z(\beta)}\right)^3 \sum_{q_m^0 = q_n^0} e^{-\beta q_m^0} |\langle m|Q(\vec{0},0)|n\rangle|^2 \delta^{(3)}(\vec{q}_m - \vec{q}_n).
\]  

(3.19)

Now let us observe that, by virtue of the correspondence relationships (2.18) and (2.19) from Minkowskian to Euclidean theory:
\[
-\langle TQ(\vec{x},-i\tau)Q(\vec{0},0)\rangle_\beta \equiv \langle Q_E(\vec{x},\tau)Q_E(\vec{0},0)\rangle_{E,\beta} \equiv M^{(E)}_\beta(\vec{x},\tau).
\]  

(3.20)

This is just the Matsubara finite-temperature Green function, defined in the Euclidean theory, with \(0 \leq \tau \leq \beta\). Therefore \(\tilde{G}_\beta(p = 0)\) is the Euclidean topological susceptibility at finite temperature, which we shall indicate with \(\chi_E(\beta)\):
\[
\tilde{G}_\beta(p = 0) = -\int_{-\beta}^{\beta} d\tau \int d^3\vec{x} \langle TQ(\vec{x},-i\tau)Q(\vec{0},0)\rangle_\beta = \int_{-\beta}^{\beta} d\tau \int d^3\vec{x} \langle Q_E(\vec{x},\tau)Q_E(\vec{0},0)\rangle_{E,\beta} \equiv \chi_E(\beta).
\]  

(3.21)
[A more accurate discussion about Eqs. (3.20) and (3.21) with the inclusion of the contact term is given in the Appendix.] This is the quantity which can be measured on the lattice, thanks to the fact that a Matsubara Green function, such as $M_\beta^{(E)}(\vec{x}, \tau)$, admits a representation in a path–integral form as follows (see, for example, Ref. [12]):

$$M_\beta^{(E)}(\vec{x}, \tau) = \int_{\text{periodic}} dA_E \int_{\text{anti–p.}} [d\psi_E][d\bar{\psi}_E] Q_E(\vec{x}, \tau)Q_E(0,0)e^{-\int_0^\beta d\tau' \int d^3\vec{x}L_E} ,$$

where $L_E$ is the Euclidean lagrangian density of QCD. The partition function itself may be written as a path–integral over all the Euclidean gauge–field configurations which are periodic, with period $\beta$, in the Euclidean time, and over all the fermion configurations which are anti–periodic, with period $\beta$, in the Euclidean time [12]:

$$Z(\beta) \equiv \text{Tr}[e^{-\beta H}] = \int_{\text{periodic}} dA_E \int_{\text{anti–p.}} [d\psi_E][d\bar{\psi}_E]e^{-\int_0^\beta d\tau' \int d^3\vec{x}L_E} .$$

Finally we have the following path–integral expression for the Euclidean topological susceptibility at finite temperature $\chi_E(\beta)$:

$$\chi_E(\beta) = \int_0^\beta d\tau \int d^3\vec{x} \int_{\text{periodic}} dA_E \int_{\text{anti–p.}} [d\psi_E][d\bar{\psi}_E] Q_E(\vec{x}, \tau)Q_E(0,0)e^{-\int_0^\beta d\tau' \int d^3\vec{x}L_E} \int_{\text{periodic}} dA_E \int_{\text{anti–p.}} [d\psi_E][d\bar{\psi}_E]e^{-\int_0^\beta d\tau' \int d^3\vec{x}L_E} .$$

This expression indicates that, in the lattice formulation of the theory, one must use an asymmetrical lattice of the form $N_\sigma^3 \times N_\tau$, with a number of time sites $N_\tau$ (much) smaller than the number of sites $N_\sigma$ along a space coordinate axis. In this way, $a \cdot N_\tau = \beta$ and $(a \cdot N_\sigma)^3 = V \to \infty$, where $a$ is the lattice spacing and $V$ is the space volume, which must be sent to infinity in the thermodynamical limit (see Ref. [13] and references therein).

Let us compare, now, $\chi_E(\beta)$ with the “physical” quantity $\chi(\beta)$ in the Minkowskian theory. At first sight, the two quantities $\chi(\beta) = I_\beta(p = 0)$, given by Eq. (3.11), and $\chi_E(\beta) = \bar{G}_\beta(p = 0)$, given by Eq. (3.16), are not equal. From Eq. (3.17) it is clear that a crucial role, in the relation between $\chi(\beta)$ and $\chi_E(\beta)$, is played by those terms which contain the expectation values $\langle m|Q(0)|n \rangle$ between states with equal four–momenta, $q_m = q_n$. It is known, however, that $Q(x)$ is the four–divergence of a four–current, i.e., $Q(x) = \partial^\mu K_\mu(x)$, with

$$K_\mu = \frac{g^2}{16\pi^2} \varepsilon_{\mu\alpha\beta\gamma} A_\alpha^\gamma (\partial^\beta A^\gamma_a - \frac{1}{3}gf_{abc} A_\beta^b A_\gamma^c) ,$$

(3.25)
where $\mu, \alpha, \beta, \gamma$ are Lorentz indices, $a, b, c$ are colour indices and $f_{abc}$ are the structure constants of the colour group. Therefore we deduce that:

$$\langle m|Q(0)|n \rangle = i(q_m - q_n)^\mu \langle m|K_\mu(0)|n \rangle.$$  

(3.26)

Normally, if $J_\mu(x)$ is some gauge–invariant (observable) four–current and if the physical spectrum is free of massless particles, then $(q_m - q_n)^\mu \langle m|J_\mu(0)|n \rangle$ must tend to zero as $q_m - q_n \to 0$. However, $K_\mu(x)$ is gauge–noninvariant and $\langle m|K_\mu(0)|n \rangle$ can contain massless singularities of the form:

$$\langle m|K_\mu(0)|n \rangle \simeq k_{mn} \frac{(q_m - q_n)^\mu}{(q_m - q_n)^2} \quad (k_{mn} = \text{constant}).$$  

(3.27)

This behaviour is known in the literature as a “Kogut–Susskind (KS) pole”, since it was first conjectured by Kogut and Susskind in Ref. [14] (for the matrix element $\langle \pi\pi|K_\mu(0)|\eta \rangle$) in the context of a mechanism for explaining the large (unsuppressed) decay amplitude of $\eta \to 3\pi$. If there were no KS poles, from Eqs. (3.17) we would be allowed to conclude that $\chi(\beta)$ is real and equal to the corresponding Euclidean quantity $\chi_E(\beta)$. Instead, this conclusion can be invalidated by the possible existence of KS poles.

Let us go back to Eqs. (3.17) and try to explore more carefully the relationship between $\chi(\beta)$ and $\chi_E(\beta)$. Since

$$\lim_{q_m^0 \to q_n^0} \frac{1 - e^{\beta(q_m^0 - q_n^0)}}{q_m^0 - q_n^0} = -\beta$$  

(3.28)

is a finite quantity, the last term in Eq. (3.16) vanishes assuming that the (possible) KS poles lie within a region of continuous distribution of energy eigenvalues $(q_m^0, q_n^0)$, in which case the set $q_m^0 = q_n^0$ has null measure in the plane $(q_m^0, q_n^0)$, and also assuming that the first term in Eq. (3.16) is finite (which seems to be confirmed a posteriori by lattice measurements). We are then left with

$$\tilde{G}_\beta(p = 0) = \frac{(2\pi)^3}{Z(\beta)} \sum_{q_m^0 \neq q_n^0} e^{-\beta q_m^0} - e^{-\beta q_n^0} \langle m|Q(0)|n \rangle^2 \delta(3)(q_m^0 - q_n^0).$$  

(3.29)

That is, comparing with Eq. (3.12):

$$\tilde{G}_\beta(p = 0) = \text{Re}[\tilde{I}_\beta(p = 0)].$$  

(3.30)
On the other hand, from Eq. (3.10) we also have that:

\[
\begin{align*}
\text{Re}[\tilde{I}_\beta(p)] &= \int_0^\infty d(E^2)\tilde{\rho}_\beta(E^2, E)(1 - e^{-\beta E})P \frac{1}{p_0^2 - E^2} ; \\
\text{Im}[\tilde{I}_\beta(p)] &= -\pi(1 - e^{-\beta|p_0|})\tilde{\rho}_\beta(p^2, |p_0|) - 2\pi \rho_{\beta,0}(p^2, p_0) .
\end{align*}
\]

(3.31)

Making the limits \(p^0 \to 0\) and \(\vec{p} \to \vec{0}\) in these equations, we get:

\[
\lim_{p^0 \to 0} \lim_{\vec{p} \to \vec{0}} \text{Re}[\tilde{I}_\beta(p)] = -\int_0^\infty d(E^2)\tilde{\rho}_\beta(E^2, E)(1 - e^{-\beta E})P \frac{1}{E^2} = \text{Re}[\tilde{I}_\beta(p = 0)] .
\]

(3.32)

And also, using Eqs. (3.9):

\[
\lim_{p^0 \to 0} \lim_{\vec{p} \to \vec{0}} \text{Im}[\tilde{I}_\beta(p)] = 0 ,
\]

(3.33)

which in general may be different from

\[
\text{Im}[\tilde{I}_\beta(p = 0)] = -2\pi \rho_{\beta,0}(0, 0) .
\]

(3.34)

This means that, while the real part of \(\tilde{I}_\beta(p)\) is continuous at \(p = 0\), the imaginary part may (in general) have a discontinuity at \(p^0 = 0\), due to the existence of KS poles. In other words, \(\tilde{I}_\beta(p)\) may be discontinuous at \(p^0 = 0\). From Eqs. (3.32) and (3.33) one has that:

\[
\lim_{p^0 \to 0} \lim_{\vec{p} \to \vec{0}} \tilde{I}_\beta(p) = \text{Re}[\tilde{I}_\beta(p = 0)] .
\]

(3.35)

On the contrary, from Eqs. (3.15) and (3.16) one deduces that \(\tilde{G}_\beta(p)\) is continuous at \(p = 0\): \(\lim_{p^0 \to 0} \tilde{G}_\beta(p) = \tilde{G}_\beta(p = 0)\).

Comparing Eq. (3.35) with Eq. (3.30), one concludes that:

\[
\tilde{G}_\beta(p = 0) = \lim_{p^0 \to 0} \lim_{\vec{p} \to \vec{0}} \tilde{I}_\beta(p) .
\]

(3.36)

That is, by the definitions (3.21) and (3.4):

\[
\chi_E(\beta) = \lim_{p^0 \to 0} \lim_{\vec{p} \to \vec{0}} \left( -i \int d^4xe^{ipx}\langle TQ(x)Q(0) \rangle_\beta \right) = \chi^{(\text{reg})}(\beta) .
\]

(3.37)
In conclusion, the Euclidean topological susceptibility at finite temperature $\chi_E(\beta)$ is not, in a strict sense, equal to the Minkowskian topological susceptibility $\chi(\beta)$, but it is equal to the limit for $p^0 \to 0$ and for $\vec{p} \to \vec{0}$ of the Fourier transform of $-i\langle TQ(x)Q(0) \rangle_{\beta}$: we have called this quantity $\chi^{(\text{reg})}(\beta)$, meaning that it can be seen as a re-definition of the Minkowskian topological susceptibility by introducing a proper infrared regularization. Of course, in the zero-temperature limit $T \to 0$ (i.e., $\beta \to \infty$) one recovers the result $\chi_E = \chi$, obtained at the end of the previous Section. In fact, in this limit the discontinuity (3.34) disappears:

$$\lim_{\beta \to \infty} \rho_{\beta,0}(0,0) = 0,$$

since $\langle 0|Q(0)|0 \rangle = 0$, so that $\chi = \chi(\beta \to \infty)$ becomes real and equal to $\chi_E = \chi_E(\beta \to \infty) = \chi^{(\text{reg})}(\beta \to \infty)$, in agreement with Eq. (2.20). Instead, at finite temperature ($T \neq 0$), Eq. (3.37) is not trivial and furnishes the correct equality relation between the Euclidean topological susceptibility and the “infrared–regularized” Minkowskian topological susceptibility.

We want to observe, also, that the “infrared–regularized” topological susceptibility $\chi^{(\text{reg})}$, if considered in the “quenched” limit $N_c \to \infty$ ($N_c$ being the number of colours), is exactly the quantity which plays a fundamental role in the Witten–Veneziano mechanism for the solution of the $U(1)$ problem: it provides the $\eta'$ meson with a large “gluonic” mass [1, 2]. In fact, in the approach of Witten and Veneziano one first sums all diagrams in $\tilde{I}(p)$ of a given order in $1/N_c$ and then considers the limit $p \to 0$. In other words, one first writes $\tilde{I}(p)$ as $\tilde{I}(p) = \tilde{I}_0(p) + \tilde{I}_1(p) + \tilde{I}_2(p) + \ldots$, where $\tilde{I}_n(p)$ is the sum of all diagrams with $n$ quark loops (each quark loop is suppressed by a factor of $1/N_c$). The limit $p \to 0$ is taken at the end. The Witten–Veneziano mechanism was originally formulated for the $T = 0$ case [1, 2]. However, it can be generalized to the $T \neq 0$ case, as was done in Ref. [13]. We hope to return to a more detailed discussion about this point in the near future.

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Appendix: The contact term.

Some subtleties arise in connection with the proper definition of the topological susceptibility $\chi$ and the use of the $T$-ordered product in Eqs. (1.1), (1.3), (2.1) and (3.1). The problem is that the two-point Green function $\langle TQ(x)Q(0) \rangle$ is not well defined as $x \to 0$: one must give a prescription for treating the product of the two (composite) operators $Q(x)$ and $Q(0)$ when approaching the same space–time point $x \to 0$. Actually, this ambiguity is eliminated in the “correct” definition of the topological susceptibility, which can be found in Ref. [1]:

$$\chi \equiv \frac{1}{VT} \left. \frac{i}{2} \frac{d^2 Z[\theta]}{d\theta^2} \right|_{\theta=0} .$$  \hspace{1cm} (A.1)

Here $VT$ is an infinite four–volume which must be factorized [in a symbolic notation: $VT = \int d^4 x = \int d^4 x e^{ipx} \big|_{p=0} = (2\pi)^4 \delta(4)(0)$] and $Z[\theta]$ is the partition function of the theory (in the path–integral formalism) with the addition of a $\theta$–term to the usual action:

$$Z[\theta] \equiv \int [dA][d\psi][d\bar{\psi}] e^{i(S+\theta q[A])} .$$  \hspace{1cm} (A.2)

$S$ is the usual action for the full theory and $q[A] \equiv \int d^4 x Q(x)$ is the (total) topological charge. As shown in the Appendix of Ref. [1], a certain equal–time commutator term (also called “contact term”) must be added to the right–hand side of Eq. (2.1) to get the correct formula (A.1) for the topological susceptibility:

$$\chi = -i \int d^4 x \langle TQ(x)Q(0) \rangle + \chi^{(1P)} ,$$  \hspace{1cm} (A.3)

where the contact term $\chi^{(1P)}$ is given by (in the temporal gauge $A^0 = 0$):

$$\chi^{(1P)} = 8 \left( \frac{g^2}{16\pi^2} \right)^2 \langle \text{Tr}[\vec{B}^2] \rangle ,$$  \hspace{1cm} (A.4)

and $B^a_i = -\frac{1}{2} \varepsilon_{ijk} F^a_{jk}$ is the non–Abelian magnetic field. [Here and in the following we shall adopt the suffix “1P” (which stands for “one–point”) to indicate the contact term, while the first term at the right–hand side of Eq. (A.3) will be denoted as $\chi^{(2P)}$, since the two–point function of $Q(x)$ appears in the integral.]
An alternative definition of \(\chi\) can be found in Ref. [3], where the T–ordering ambiguity is eliminated using the following prescription:

\[
\chi \equiv -i \int d^4(x - y) \partial_\mu \partial_\nu \langle TK_\mu(x)K_\nu(y)\rangle .
\] (A.5)

Here \(K_\mu(x)\) is the usual four–current defined in Eq. (3.25), whose divergence gives the topological charge density, \(Q(x) = \partial^\mu K_\mu(x)\). Eq. (A.5) can also be put in the following equivalent form:

\[
\chi \equiv -i \int d^4 x \partial_\mu \langle TK_\mu(x)Q(0)\rangle .
\] (A.6)

Both (A.5) and (A.6) give rise to the following (more explicit) expression for \(\chi\) [actually, the two integrand expressions in (A.5) and (A.6) differ by a quasi–local operator \(\Delta(x)\), which anyway vanishes after space–time integration]:

\[
\chi = -i \int d^4 x \langle TQ(x)Q(0)\rangle - i \int d^3 x \langle [K_0(x), Q(0)]_{x^0=0} \rangle .
\] (A.7)

The equal–time commutator term at the right–hand side of Eq. (A.7) is just the contact term, which is needed to make the definition of the topological susceptibility unambiguous. In fact, any T–ordered product ambiguity in (A.6) is necessarily a derivative \(\partial_x\) of \(\delta^{(4)}(x - y)\); but such terms are annihilated after integration in \(\int d^4 x\). The quantity \(\chi\), defined in (A.5) \(\div\) (A.7), enters into the anomalous Ward identities of QCD, which were derived in Ref. [3].

Using the same procedure outlined in the Appendix of Ref. [1] (canonical quantization in the temporal gauge \(A^0 = 0\)), one can easily verify that the contact term in Eq. (A.7) is exactly equal to the contact term (A.4), derived by Witten from the definition (A.1) of \(\chi\):

\[
- i \int d^3 x \langle [K_0(x), Q(0)]_{x^0=0} \rangle = \chi^{(1P)} = 8 \left( \frac{g^2}{16\pi^2} \right)^2 \langle \text{Tr}[\vec{B}^2] \rangle .
\] (A.8)

The same expression is also derived for the finite–temperature contact term \(\chi^{(1P)}_\beta\), except that one must now intend \(\langle \ldots \rangle\) not as the simple vacuum expectation value \(\langle 0|\ldots|0\rangle\), but as the usual quantum thermal average over the Gibbs ensemble \(\langle \ldots \rangle_\beta\), defined in Eqs. (3.2) and (3.3). Therefore one has that:

\[
\chi^{(1P)}_\beta \equiv -i \int d^3 x \langle [K_0(x), Q(0)]_{x^0=0} \rangle_\beta = 8 \left( \frac{g^2}{16\pi^2} \right)^2 \langle \text{Tr}[\vec{B}^2] \rangle_\beta .
\] (A.9)
The reason for this simple result is that in the explicit evaluation of the integral in Eq. (A.9) one simply uses canonical (equal–time) commutation relations, which are fundamental and do not depend on the temperature of the system. For these reasons, all the results obtained in this Appendix are valid both at zero temperature \((T = 0, \text{ in which case: } \langle \ldots \rangle = \langle 0| \ldots |0 \rangle)\) and at finite temperature \((T \neq 0, \text{ in which case: } \langle \ldots \rangle = \langle \ldots \rangle_\beta)\). We observe, in addition, that at \(T = 0\) the contact term can put in the Lorentz– and gauge–invariant form \(\chi^{(1P)} = (g^2/64\pi^2)G_2\), where \(G_2 \equiv (\alpha_s/\pi)\langle F^a_{\mu\nu}F^a_{\mu\nu} \rangle \) (with \(\alpha_s \equiv g^2/4\pi\) is the so–called gluon condensate: one uses the relation \(F^a_{\mu\nu}F^a_{\mu\nu} = 2\text{Tr}[F_{\mu\nu}F^{\mu\nu}] = 4\text{Tr}[\vec{B}^2 - \vec{E}^2]\) and the fact that, at zero temperature \((T = 0)\), the electric part of the gluon condensate \(G_2^{(el)} \equiv -(g^2/\pi^2)\langle \text{Tr}[\vec{E}^2] \rangle\) is equal to the magnetic part of the gluon condensate \(G_2^{(mag)} \equiv (g^2/\pi^2)\langle \text{Tr}[\vec{B}^2] \rangle\), because of Lorentz– and parity–invariance. Instead, at finite temperature \((T \neq 0)\), Lorentz invariance is broken down to \(O(3)\) rotational invariance: therefore \(G^{(mag)} \neq G^{(el)}\) and one must keep the expression (A.9) for the contact term, i.e., \(\chi^{(1P)} = (g^2/32\pi^2)G_2^{mag}\).

All the above refers to the physical, i.e., Minkowskian, space–time. Let us see, now, what happens in the Euclidean four–space. The Euclidean topological susceptibility \(\chi_E\) is defined by continuing the definition (A.1) or, equivalently, (A.5) ÷ (A.7), to the Euclidean world, by using the correspondence relationships (2.18) and (2.19) from Minkowskian to Euclidean theory:

\[
\chi_E \equiv -\frac{1}{V_ET_E} \frac{1}{Z_E[\theta]} \frac{d^2Z_E[\theta]}{d\theta^2} \bigg|_{\theta = 0} . \tag{A.10}
\]

\(Z_E[\theta]\) is the Euclidean partition function (in the path–integral formalism) with the addition of a \(\theta\)–term to the usual Euclidean action:

\[
Z_E[\theta] \equiv \int [dA_E][d\psi_E][d\bar{\psi}_E]e^{-S_E + i\theta q_E[A_E]} . \tag{A.11}
\]

\(S_E\) is the usual Euclidean action for the full theory and \(q_E[A_E] \equiv \int d^4x E_\mu(x)E^\mu(x)\) is the (total) Euclidean topological charge. Eq. (A.10) provides us with a rigorous definition of the Euclidean Green function \(\langle Q_E(x)Q_E(y) \rangle_E\), which appears in the expression:

\[
\chi_E \equiv \frac{1}{V_ET_E} \int d^4y_E \int d^4x_E \langle Q_E(x)Q_E(y) \rangle_E = \int d^4x_E \langle Q_E(x)Q_E(0) \rangle_E , \tag{A.12}
\]

in terms of an Euclidean path–integral:

\[
\langle Q_E(x)Q_E(0) \rangle_E \equiv \frac{1}{Z_E} \int [dA_E][d\psi_E][d\bar{\psi}_E]Q_E(x)Q_E(0)e^{-S_E} . \tag{A.13}
\]
In other words, the correct expression for the Euclidean Green function \( \langle Q_E(x_E)Q_E(y_E) \rangle_E \), in the sense given by Eqs. (A.10) \( \div \) (A.13), is not simply obtained by the analytic continuation of \( -\langle TQ(x)Q(y) \rangle \), as assumed in Eqs. (2.16) and (3.20): rigorously speaking, Eqs. (2.16) and (3.20) are only true when \( \tau \neq 0 \). In the general case, one must also include the analytic continuation of the contact term, since, by virtue of Eqs. (A.10) \( \div \) (A.13) [to be compared with Eqs. (A.1), (A.2) and (A.5), (A.6)], \( \langle Q_E(x_E)Q_E(y_E) \rangle_E \) is the analytic continuation of \( -\partial_x^\mu \partial_y^\nu \langle TK^\mu(x)K^\nu(y) \rangle \); that is:

\[
\langle Q_E(\vec{x}, \tau)Q_E(\vec{y}, \sigma) \rangle_E = -\partial_x^\mu \partial_y^\nu \langle TK^\mu(x)K^\nu(y) \rangle \big|_{(x^0, y^0) \to (-i\tau, -i\sigma)} .
\] (A.14)

Explicitly, this means that:

\[
\langle Q_E(\vec{x}, \tau)Q_E(\vec{0}, 0) \rangle_E = -\langle TQ(\vec{x}, -i\tau)Q(\vec{0}, 0) \rangle - i\delta(\tau)\langle [K^0(\vec{x}, -i\tau), Q(\vec{0}, 0)]_{\tau=0} \rangle + \Delta(\vec{x}, -i\tau) ,
\] (A.15)

where \( \Delta(x) \) is a quasi–local operator which vanishes after space–time integration [see Eq. (A.7)]. Therefore, after integration in \( \int d^3\vec{x} \int d\tau \), we obtain:

\[
\chi_E \equiv \int d^3\vec{x} \int d\tau \langle Q_E(\vec{x}, \tau)Q_E(\vec{0}, 0) \rangle_E = \chi^{(2P)}_E + \chi^{(1P)}_E ,
\] (A.16)

where

\[
\chi^{(2P)}_E \equiv -\int d^3\vec{x} \int d\tau \langle TQ(\vec{x}, -i\tau)Q(\vec{0}, 0) \rangle
\] (A.17)

is the contribution to \( \chi_E \) coming from the two–point function

\[
\langle Q_E(\vec{x}, \tau)Q_E(\vec{0}, 0) \rangle^{(2P)}_E \equiv -\langle TQ(\vec{x}, -i\tau)Q(\vec{0}, 0) \rangle ,
\] (A.18)

while

\[
\chi^{(1P)}_E = \chi^{(1P)} = -i\int d^3\vec{x}\langle [K^0(\vec{x}, 0), Q(\vec{0}, 0)] \rangle
\] (A.19)

is the contribution to \( \chi_E \) from the contact (equal–time commutator) term. We observe that \( \chi^{(1P)}_E \) turns out to be trivially equal to the Minkowskian contact term \( \chi^{(1P)} \). This is due to the fact that \( \chi^{(1P)} \) is an equal-time commutator term, so that the continuation to imaginary times \( x^0 \to -i\tau \) is trivial in this case. Let us observe also that, by virtue of the correspondence relationships (2.18) and (2.19) from Minkowskian to Euclidean theory, we have:

\[
K^0(x) \to K_{E4}(x_E) , \quad K^i(x) \to iK_{Ei}(x_E) .
\] (A.20)
so that \( Q(x) \rightarrow iQ_{E}(x_{E}) \), with \( Q_{E} = \partial_{E\alpha}K_{E\alpha} \). Therefore, we can re-write Eq. (A.14) in the following way:

\[
\langle Q_{E}(x_{E})Q_{E}(y_{E}) \rangle_{E} = \frac{\partial}{\partial x_{E\alpha}} \frac{\partial}{\partial y_{E\beta}} \langle TK_{E\alpha}(x_{E})K_{E\beta}(y_{E}) \rangle_{E} ,
\]

(A.21)
in agreement with (A.13). In other words:

\[
\chi^{(1P)}_{E} = \int d^{3}\vec{x}\langle[K_{E4}(\vec{x}, \tau), Q_{E}(\vec{0}, 0)]_{\tau=0} \rangle_{E}
\]

\[
= -i \int d^{3}\vec{x}\langle[K^{0}(\vec{x}, -i\tau), Q(\vec{0}, 0)]_{\tau=0} \rangle
\]

\[
= -i \int d^{3}\vec{x}\langle[K^{0}(\vec{x}, 0), Q(\vec{0}, 0)] \rangle = \chi^{(1P)} . \tag{A.22}
\]

In conclusion, the contact term for the topological susceptibility in the Euclidean world is exactly (and trivially) equal to the contact term in the Minkowski world. This is true both at zero temperature and at finite temperature. Therefore, the inclusion of the contact (equal–time commutator) term does not affect the results derived in the text (where we proved that \( \chi^{(2P)} = \chi^{(2P)}_{E} \), with the notation introduced above), except that it solves some questions about positivity arising with some formulas. As a rule, in all formulas where \( \chi \) is mentioned in the text, one must intend that it is actually \( \chi^{(2P)} \): then, to obtain the proper value of \( \chi \), consistently with the definitions (A.1) [(A.5), (A.6)] and (A.10), one must add the contact term \( \chi^{(1P)} \), in the way explained in this Appendix.
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FIGURE CAPTIONS

Fig. 1. The integration path for $\langle Q(\vec{x}, \tau)Q(\vec{0}, 0)\rangle_{\beta}$ in the complex $\tau$–plane.
