Intuition, iteration, induction
(draft)

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October 6, 2015

Abstract

In Mathematical Thought and Its Objects, Charles Parsons argues that our knowledge of the iterability of functions on the natural numbers and of the validity of complete induction is not intuitive knowledge; Brouwer disagrees on both counts. I will compare Parsons’ argument with Brouwer’s and defend the latter. I will not argue that Parsons is wrong once his own conception of intuition is granted, as I do not think that that is the case. But I will try to make two points: (1) Using elements from Husserl and from Brouwer, Brouwer’s claims can be justified in more detail than he has done; (2) There are certain elements in Parsons’ discussion that, when developed further, would lead to Brouwer’s notion thus analysed, or at least something relevantly similar to it. (This version contains a postscript of May, 2015.)

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1 Introduction

Consider the following two principles:

1. Iterability: Any total function $f : \mathbb{N} \to \mathbb{N}$ can be iterated arbitrarily many times;

2. Complete induction:

$$\begin{array}{c}
[A(a)] \\
\vdots \\
A(0) \quad A(Sa) \quad N(t) \\
\hline \\
A(t)
\end{array}$$

where $A$ is a placeholder for any well-defined predicate on the natural numbers, and $N$ is a natural number-predicate.

In his book *Mathematical Thought and Its Objects* [Parsons, 2008], Charles Parsons argues that our knowledge that these two general principles are valid is not intuitive knowledge. Brouwer, on the other hand, in his dissertation [Brouwer, 1907] claims that they are.

I will compare Parsons' position on iteration and induction with Brouwer's and defend Brouwer's view, in the following sense. Brouwer's and Parsons' conceptions of intuition are, of course, different. I will not argue that Parsons is wrong once his own conception of intuition is granted, as I do not think that that is the case. But I will try to make two points. The first is that, using elements from Husserl and from Brouwer, Brouwer's claims that the principle of induction and the iterability of functions are intuitive can be justified in more detail than he has done. The second is that there are certain elements in Parsons' discussion that, when developed further, would lead to Brouwer's notion thus analysed, or at least something similar to it.

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1Instead of a rule, induction may also be formulated as an axiom schema (using implication). In the Brouwerian context, however, the latter also has to be construed as a rule: As mathematical objects and proofs exist only as the result of the subject's activity, to hypothesise that a proof exists therefore must be to hypothesise that the subject has constructed a proof. Making an assumption towards proving an implication then has epistemic import. As pointed out in Sundholm and van Atten [2008], this means that in this respect Brouwerian logic differs from Gentzen's Natural Deduction, in which this is not the case.
2 Parsons

For Parsons, intuition plays two rôles in the foundations of mathematics. Intuition of objects falling under a given concept shows that the mathematical concept in question is not empty; and intuitive knowledge is more evident than knowledge of other types, in particular knowledge whose justification involves appeals to principles of reason [Parsons, 2008, pp. 113, 336]. But unlike Brouwer, Parsons is not a constructivist, and intuition has no overall legislative rôle for him. He does not generally require that, for a mathematical concept to be instantiated, it should be instantiated by an intuited object; this is only required for concepts at the bottom of our conceptual edifice. Other concepts then arise as combinations and idealisations of the lower ones, but there is no requirement that we have intuitions of objects that fall under these higher concepts. Naturally, an account is then needed of which idealizations are legitimate and which ones go too far. Parsons says this will depend on a theory of reason. He develops a number of ideas on this in the last chapter of Mathematical Thought and Its Objects, ‘Reason’, but my concern here will rather be with his views on the intuitive part of mathematics.

Parsons distinguishes intuition of objects from intuition that a proposition is true. For mathematical intuition specifically, Parsons takes the objects of intuition to be strings of strokes, e.g., |||||. He allows for the possibility to see specific inscriptions of such strings (tokens) as types, following Husserl here in holding that sometimes imagination of the token can found intuition of a type [Parsons, 2008, p. 173].

The first thing Parsons says in defining intuitive knowledge is that we have intuitive knowledge that p if p can be ‘seen’ (quotation marks Parsons’) to be true on the basis of intuiting objects that it is about [Parsons, 2008, p. 171]. He then generalises this by counting as intuition of an object not only an actual perception of one, but also an imagination of an arbitrary one. Otherwise, we could not justify the claim that it is intuitive knowledge that any such string can be extended by placing one more stroke at the right. An arbitrary string, Parsons says, may be imagined in the way we imagine a large crowd at a baseball game; it need not be part of the content of that imagination that the crowd consists of exactly n people [Parsons, 2008, pp. 173–174].

Intuitiveness of an operation is explained in terms of intuitiveness of a

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\[A \text{ remarkble conclusion from this undoubtedly correct observation is drawn in}\]

\[\text{Borges’ } \text{’argumentum ornithologicum’ } \cite{Borges, 1964, p. 29}\]
proposition: An operation will be intuitive if we have intuitive knowledge that it is well-defined.

As I mentioned, Parsons argues against the intuitiveness of the general principle of induction. His objection turns on its essentially higher-order character [Parsons, 1986, p. 227]: the principle does not speak directly about objects and operations on them that are intuitive in his sense, but about all predicates defined on these objects. On Parsons’ conception, intuition that is only possible where all types involved can be instantiated in (concrete) perception [Parsons, 2008, §28].

For the same reason, Parsons argues that the principle that every operation on numbers can be iterated is not intuitive. The question of its intuitiveness is fact prior to that for induction, because, as Parsons points out, induction is specific to reasoning about domains in which the objects are obtained from an initial object by arbitrary finite iteration of a given operation. Similarly, a simple way of defining an operation inductively is to define it as the iteration of another operation. For example, addition can be defined as iteration of the successor operation, multiplication as iteration of addition, and exponentiation as iteration of multiplication.

This does of course not exclude the possibility that certain specific operations that are usually defined inductively are intuitive after all; it is just that, in such a case, this intuitiveness would not have its ground in the fact that one way of defining that operation proceeds by appropriately instantiating the general schema of definition by induction.

An example is addition. Addition is usually understood according to an inductive definition like this:

\[
\begin{align*}
a + 0 &= a \\
a + Sb &= S(a + b)
\end{align*}
\]

Parsons accepts an argument proposed by Bernays that here an alternative understanding is available that does not depend on iteration [Parsons, 2008, p. 255]. If \(a\) is given in intuition (as a string of strokes) and also \(b\), then so can \(a + b\), because the strings representing \(a\) and \(b\) can be concatenated in one step, without iterating through them.

If one abstracts from questions of feasibility, as Bernays, Parsons, and Brouwer all do, it is, I think, unproblematic that we can juxtapose any two strings in one step. But I am not convinced that this means that we can come to accept addition as an intuitive operation without any appeal to iteration. We have to verify that the new object is not a mere juxtaposition of the two original ones, but itself a string of strokes – a concept that, Parsons
says, ‘involves iteration’ [Parsons, 2008, p. 175]. In other words, we have to know that the operation of concatenating strings is type-preserving. I do not think that we can know that the result of an act of concatenation can in turn be iterated through without having verified this in some particular cases. (Such cases then serve to found, in acts of what Husserl would call ‘eidetic variation’, the general judgement.) But I will leave this aside for the moment.

For multiplication one may propose an argument analogous to that for addition. Multiplication is usually understood in a way that depends on iteration:

\[
a \times 0 = 0 \\
\times Sb = a \times b + a
\]

However, given strings \(a\) and \(b\), we may obtain \(a \times b\) directly, by replacing each stroke in \(b\) by a copy of \(a\). In fact, we are then doing the same thing as when we constructed \(b\), except that we now take \(a\) as the unit instead of a single stroke.

Parsons notes that this way of understanding addition and multiplication does not easily generalise to exponentiation [Parsons, 2008, p. 256]. The way I would elaborate this is to say that, in the case of addition and multiplication, if we leave aside the objection mentioned above concerning type-preservation, we can indicate how to transform an image of the function arguments into an image of the function value without invoking any arithmetical concept, let alone iteration of arithmetical operations. Only direct manipulations of the image strings are required, such as copying, concatenating, and replacing one string by another. It seems that all such direct manipulations of strings can be understood in terms of part-whole relations. But the relation that exponentiation determines between one of its arguments, the exponent string, and the string representing the result of the operation, namely, the former indicates the number of iterations of multiplication needed to arrive at the latter, is not a part-whole relation.

For example, consider the exponentiation \(2^3\). The salient relation between 3 and 8 that is established by performing this operation is not that in a construction of 8 strokes, a string of 3 strokes enters as a part. Of course there is a part-whole relation between a string of 8 strokes and any 3 consecutive strokes in it; but that relation is not brought about by carrying out the operation of exponentiation. This seems to rule out that exponentiation can be understood in the same concrete sense as addition and multiplication.
Bernays once proposed an argument to the effect that, even if we accept that any understanding of exponentiation has to involve an appeal to iteration, it can still be understood as intuitive on different grounds. His argument, as translated by Parsons, runs as follows:

Consider the example of the number $10^{10^{10}}$. We can reach this number in a finitary way as follows: We start from the number 10, which, in accordance with one of the normalizations given earlier we represent by the expression

$$1111111111.$$  

Now let $z$ be any number which is represented by a corresponding expression. If we replace in the above expression each 1 by the expression $z$, then again a number expression arises, as we can make clear intuitively, which for communication we designate with “$10 \times z$”. Thus we obtain the process of multiplying a number by 10. From this we obtain the process of passing from a number $a$ to the number $10^a$, in that we let the number 10 correspond to the first 1 in $a$ and, to each attached 1, the process of multiplying by 10, and continue until we are at the end of the expression $a$. The number obtained by the final process of multiplying by 10 is designated by $10^a$.

This procedure offers no difficulty from the intuitive point of view. [Parsons, 2008, p. 258]

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3This is how I would begin answering Tait’s criticism of Parsons’ account (in the earlier presentation of Parsons [1998]):

It isn’t that we are told [by Parsons] exactly how to reason about the dead symbols and what the limits of such reasoning are, and that what we are told does not support exponentiation. Aside from examples, we are told nothing about how to reason concerning them, except that it should be logic-free. It is true that a certain sketch is given of how to understand addition and multiplication finitistically, using the operations of concatenation and replacement (of each occurrence of $|$ in a word by a word), where there is no reasonable extension of this sketch to exponentiation. But these constructions are themselves just examples. [Tait, 2010, 101n.14]

4[Bernays writes $10^{10^{1000}}$.]

5[Footnote Bernays, omitted by Parsons: ‘Here we have a symbol “with content”’.]

6Auch hier bestehen Grenzen für die Vollziehbarkeit der Wiederholungen
Parsons objects that ‘processes are not clearly objects of intuition’ [Parsons, 2008, p. 258]; I would say that, from a Husserlian point of view, they are, and that, phenomenologically, Bernays’ account will turn out to be acceptable. I will come back to the intuitive givenness of processes below, and argue that a phenomenological account is available for induction more generally. All of this obviously requires a notion of intuition that goes beyond the immediate givenness of spatio-temporal configurations, however idealised. That is a notion that does not fit with Hilbert and Bernays’ descriptions of intuition elsewhere; and not with Parsons’ own notion either.

Parsons suggests that Bernays possibly means that ‘what the 1 is to be replaced by is the result of multiplying by 10 what one obtained at the previous stage’. Note that this reading of Bernays replaces his appeal to intuition of a process by an appeal to intuition of an object (of a type that Bernays accepts as intuitive). Parsons points out that, if it is really intuition of the result that is meant, the argument becomes circular. It then implicitly appeals to induction: the argument, on this reading, presupposes that, for any $n$, when we set out to form an intuitive representation of $x^{n+1}$, the result of raising $x$ to the power $n$, $x^n$, is intuitively given; and that assumption is in effect an induction hypothesis. The problem with this kind of circularity, Parsons notes, is not specific to the case of exponentiation; the problem would arise in any attempt to justify the intuitiveness of an operation by defining it as the iteration of another operation that has already been seen

sowohl im Sinne der wirklichen Vorstellbarkeit wie auch im Sinne der physikalischen Realisierung. Betrachten wir beispielsweise die Zahl $10^{10^{1000}}$. Zu dieser können wir auf finitem Wege folgendermaßen gelangen: Wir gehen aus von der Zahl 10, die wir gemäß der einen von unsern früher angegebenen Normierungen durch die Figur

repräsentieren. Sei nun $z$ irgendeine Zahl, die durch eine entsprechende Figur repräsentiert wird. Ersetzen wir in der vorigen Figur jede 1 durch die Figur $z$, so entsteht, wie wir uns anschaulich klarmachen können, wieder eine Zahlfigur, die zur Mitteilung mit “$10 \times z$” bezeichnet wird. [Footnote Bernays: ‘Hier handelt es sich um ein Zeichen “mit Bedeutung”’] Wir erhalten so den Prozeß der Verzehnfachung einer Zahl. Aus diesem gewinnen wir den Prozeß des Überganges von einer Zahl $a$ zu $10^a$, indem wir der ersten 1 in $a$ die Zahl 10 und jeweils jeder angehängten 1 den Prozeß der Verzehnfachung entsprechen lassen und hierin so weit gehen, bis wir mit der Figur $a$ am Ende sind. Die durch den letzten Prozeß der Verzehnfachung gewonnene Zahl bezeichnen wir mit $10^a$.

Dies Verfahren bietet für die anschauliche Einstellung grundsätzlich keinerlei Schwierigkeit. [Bernays, 1976, pp. 38–39]
to be intuitive.

3 Brouwer

According to Brouwer, all mathematical acts and the objects constructed in them are developed out of the ‘basic intuition’ or ‘Urintuition’. This is based on ‘the perception of the move of time’:

the perception of the move of time, i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.

[Brouwer, 1952, p. 141]

The ‘two distinct things’ that Brouwer speaks of are two phases of consciousness, that of the present and that of the immediate past, each with their full experiential content. Brouwer elsewhere specifies that the intuitive temporal continuum is ‘a measureless one-dimensional continuum in a single subject’ [Brouwer, 1975, p. 116], the experience of which exists independently of any outer experience [Brouwer, 1907, pp. 97-98, 118]. Brouwer’s intuitive time corresponds to Husserl’s inner time awareness; Brouwer distinguishes it from ‘scientific time’, which is not a priori but a posteriori and presupposes the existence of mathematics developed on the basis of intuitive time.
Brouwer, 1907, pp. 99n.]. He also emphasises that mathematics, conceived of as the activity of constructing mathematical objects on its basis of this intuition, is not of a linguistic nature Brouwer, 1907, pp. 169, 176–177. It is, to use his well-known later formulation, ‘essentially languageless’ Brouwer, 1952, p. 141).

For Brouwer, then, the objects of mathematical intuition are not, as in Parsons’ model, strings of strokes, but constructions out of inner time awareness. But the objects that are intuitive in Parsons’ sense can be mapped to objects that are intuitive in Brouwer’s sense, as successive strokes in strings can be mapped to successive intervals in time11. Intuitiveness of operations and of propositions are, in the abstract, defined in the same way as for Parsons. I claim, but will not argue for it at this point, that all arithmetical principles that are intuitive knowledge for Parsons are also intuitive knowledge for Brouwer. On the other hand, if a certain object or principle is not intuitive on Parsons’ account, it may still be on Brouwer’s.

Induction is a case in point. When, at the beginning of his dissertation, Brouwer gives construction methods for $9 \times 4$ and $4^5$, these are, in effect, instantiations of straightforward inductive definitions.

By $9 \times 4$ I mean: Count up to 4, write 1 on another line, add 4 on the first line (the operation ‘+4’ described above), write 2 on the second line, etc., till 9 has been written on the second line. By $9 \times 4$ I mean the last number on the first line. Brouwer, 1907, pp. 4–5/ Brouwer, 1975, p. 15

These are clearly meant to be examples of constructions that are intuitive, as is confirmed at the end of that chapter Brouwer, 1907, p. 77/ Brouwer, 1975, pp. 5–52]. Heyting, in his edition of Brouwer’s Collected Works, comments on this passage that

The definitions of the arithmetical operations by recursion and the derivation of their properties by induction are intuitionistically correct. Probably Brouwer intended to demonstrate that the notions of these operations are more primitive than the general notions of recursion and induction. Brouwer, 1975, p. 565]

Indeed, as we will see below, for Brouwer the general notion of induction and recursion belong to ‘mathematics of the second order’, whereas concrete

11Compare ‘It is of course natural also to view the generation of strings temporally. I believe that the structure that results, and the issues concerning it, are the same as in Brouwer’s case.’ Parsons, 2008, p. 174n.70

12The construction method he gives for $3+4$ is rather of the type of direct concatenation.
additions and multiplications belong to ‘mathematics of the first order’. But one never finds in Brouwer attempts at accounts for the intuitiveness of these operations of the kind proposed by Bernays or Parsons, and he did accept induction as an intuitive principle. In the list of ‘theses’ that go with his dissertation (stellingen), Brouwer states:

The admissibility of complete induction cannot only not be proved, but it ought neither to be considered as a separate axiom nor as a separately seen intuitive truth. Complete induction is an act of mathematical constructing, which is already justified by the basic intuition of mathematics. [Brouwer, 1975, p. 98, thesis II, trl. modified] 13

Induction ought not to be considered as a separate axiom because, in Brouwer’s view, it is neither an axiom nor separate. It is not an axiom but an act, and a principle only in the derived sense of being a correct description of that act, upon reflection on it. Induction as a principle would perhaps best be formulated as a rule under which mathematical construction is closed: If I have obtained a construction for $A(0)$ and if, by whatever mathematical (not necessarily merely logical) means, I can obtain a construction for $A(Sn)$ whenever I have obtained a construction for $A(n)$, then I have the mathematical means to obtain a construction for $\forall x A(x)$. 14 Correspondingly, an account of induction should be given in terms not of propositions and operations on them, but of acts and operations on them. This of course requires objectification of acts (see below), but objectified acts are still different from propositions.

And neither is induction ‘separate’, because, as I gloss that term, it is not evidentially independent. It is not an option to do any intuitionistic

13De geoorloofdheid der volledige inductie kan niet alleen niet worden bewezen, maar behoort ook geen plaats als afzonderlijk axioma of afzonderlijk ingeziene intuïtieve waarheid in te nemen. Volledige inductie is een daad van wiskundig bouwen, die in de oerintuïtie der wiskunde reeds haar rechtvaardiging heeft. [Brouwer, 1907, loose leaf, Stelling II] [van Dalen, 2001, p. 139].

14When in Notebook 1 Brouwer quotes the following passage from La science et l’hypothèse:

Poincaré : Les mathématiciens procèdent donc « par construction », ils « construisent » des combinaisons de plus en plus compliquées. Revenant ensuite par l’analyse de ces combinaisons, de ces ensembles, pour ainsi dire, à leurs éléments primitifs, ils aperçoivent les rapports de ces éléments et en déduisent les rapports des ensembles eux-mêmes” [Poincaré, 1902, p. 26].

he adds in the margin, next to ‘et en déduisent’, ‘liever: et essayent d’en construire’ [Brouwer. Archive, notebook I, p. 36].
mathematics without appealing to something from which, Brouwer claims, induction can be made evident as well – the ‘basic intuition of mathematics’. One may of course restrict oneself to doing intuitionistic mathematics without actually using induction, but there is no analogy between doing so and, for example, doing classical set theory without using the axiom of choice.

A rephrasing of the quoted ‘thesis’ on induction is known from Brouwer’s letter to Jan de Vries, of which the copy that remains in the Brouwer archive is undated, but which was clearly written around the time of the thesis defence. The letter comments on various ideas of the dissertation, and remarks on induction:

I replace the ‘axiom of complete induction’ with the ‘mathematical construction-act of complete induction’ and show how, given the intuition of time, this is nothing new. [van Dalen, 2001, p. 155]

But in spite of what he says here, no detailed attempt at showing this is found in the dissertation. One does however find an important indication of the form that such an account should take, in a footnote to this list of three examples of valid (general) synthetic a priori judgements:

1. the very possibility of mathematical synthesis, of thinking many-one-ness, and of the repetition thereof in a new many-one-ness.
2. the possibility of intercalation (namely that one can consider as a new element not only the totality of two already compounded, but also that which binds them: that which is not the totality and not the element)
3. the possibility of infinite continuation (axiom of complete induction)

15858–1940; professor of geometry in Utrecht from 1897 to 1928. No further contact between Brouwer and De Vries before or in the period of Brouwer’s dissertation is known. It is likely that the occasion for the exchange had been created by Brouwer’s thesis advisor Diederik Korteweg, who was a friend of De Vries and incidentally had been the thesis advisor of his younger brother Gustav, who received his PhD in 1894. See [Willink 2006].

16Brouwer to J. de Vries: ’Ik stel in plaats van het “axioma van de volledige inductie” de “wiskundige opbouw-handeling van volledige inductie”; en laat zien, hoe die na de tijdsintuïtie niets nieuws meer is.’

17[Perhaps in a hurry, Brouwer here writes ‘axiom of complete induction’ instead of ‘mathematical construction-act of complete induction’, which in the letter to De Vries he says is what he replaces the former with.]
The footnote to this list states:

One must however not try to base mathematics or experience on such judgements: they are the result of viewing the basic intuition mathematically, and hence presuppose the basic intuition in the viewing as well in what is viewed; they belong to what we shall call in the next chapter mathematics of the second order.

Mathematics of the second order had, in fact, already been defined a few pages earlier:

Strictly speaking the construction of intuitive mathematics in itself is an action and not a science; it only becomes a science, i.e. a totality of causal sequences, repeatable in time, in a mathematics of the second order, which consists of the mathematical consideration of mathematics or of the language of mathematics.

The latter type of mathematics of the second order is the better known one in the literature, no doubt because it is central to the intuitionistic conception of logic as the study of patterns in such descriptions. But in order to make the general principles of iterability and induction evident, it will have to be the first type that we engage in, because in this case we are concerned with patterns in intuitive acts, not in language. Although Brouwer points out that a judgement of second order-mathematics can play no rôle in founding
mathematics, second order-mathematics is all the same intuitive, so that judgments based on it still express intuitive knowledge.

In his 1911 review of Mannoury’s book *Methodologisches und philosophisches zur Elementar-Mathematik*, Brouwer speaks of ‘the intuition of complete induction’ [Brouwer, 1911, p. 200] and in 1912, in his inaugural lecture ‘Intuitionism and formalism’, he remarks that for finite numbers as understood intuitionistically, induction is ‘evident on the basis of their construction’ [Brouwer, 1975, p. 129–130, trl. modified] But this is in passing, without development either there or in later publications.

In the notebooks in which Brouwer drafted his dissertation between 1904 and 1907, induction is occasionally commented on, but never to question its status of an acceptable principle. One comment, in the last notebook, is of particular interest. It is occasioned by the following passage in Poincaré’s *La Science et l’Hypothèse*:

> Ce procédé est la démonstration par récurrence. On établit d’abord un théorème pour \( n = 1 \); on montre ensuite que s’il est vrai de \( n - 1 \), il est vrai de \( n \) et on en conclut qu’il est vrai pour tous les nombres entiers. [Poincaré, 1902, p. 19]

Brouwer comments:

> The principle of induction is not: ‘if the theorem is correct for 1, and for \( n + 1 \) if it is correct for \( n \), then it is correct for every number’, but the possibility to think the same thing repeated forever, so also buildings [i.e., construction acts], so also attempts at buildings in which at each number one gets smacked in the face by the principle of contradiction. [Brouwer, Archive, notebook VIII, p. 65]

It would have been more accurate for Brouwer to write here, as he does in the dissertation, that one is ‘smacked in the face’ not primarily by the (propositional) principle of contradiction, but by the fact that a certain construction

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21 ‘de intuïtie der volledige inductie’
22 ‘dit principe [i.e., volledige inductie], dat voor de eindige getallen van de intuitionist op grond hunner constructie evident is . . . ’ [Brouwer, 1912, pp. 15–16].
23 These notebooks have neither been translated nor published yet. But there are many quotations from it (with translations) in John Kuiper’s dissertation [Kuiper, 2004].
24 ‘Het principe der inductie is niet, dat: ‘als stelling[ing] geldt voor 1 en voor \( n + 1 \) als voor \( n \), dan voor elk getal’, maar de mogelijkheid, om zich een gelijksoortig ding altijd door herhaald te denken, dus ook gebouwen, dus ook een pogen om te bouwen, die bij elk getal opnieuw zijn neus stoot door den Satz vom Widerspruch.’
act ‘does not go through’ (‘niet verder gaat’ [Brouwer, 1907, p. 127]). Be that as it may, what is noteworthy in this quotation is the insistence on induction as an instance of iteration, which Brouwer does not make explicit elsewhere.

Brouwer’s preceding remarks on induction, which are those I have been able to find, may be summarised in five tenets:

B1 induction is primarily an act, not a proposition;
B2 the act of induction is an instance of iteration;
B3 the intuition of time is a condition of possibility of the act of induction;
B4 the judgement of the validity of the induction principle is a result of ‘second-order mathematics’;
B5 the induction principle is evident on the basis of the intuitionistic construction of the finite numbers.

Before turning to the details of an account based on these tenets, which is not to be found in Brouwer’s published or unpublished writings, I should like to show to what (varying) degree several published intuitionistic accounts of later authors do not show these tenets B1–B5.

I do not know how much of (the ideas in) Brouwer’s unpublished notes on induction were known to his foremost student, Arend Heyting. But Brouwer’s emphasis on acts and iteration over propositions and deduction is absent from the explicit justification of induction proposed by Heyting in his Intuitionism. An Introduction [Heyting, 1956]. Brouwer’s claim in the inaugural lecture that induction is ‘evident on the basis of the intuitionistic construction of the finite numbers’, however, is echoed there, although one suspects that Brouwer and Heyting had different ideas as to exactly how induction becomes evident on that basis.

Clearly the construction of a natural number \( n \) consists in building up successively certain natural numbers, called the numbers from 1 to \( n \), in signs: \( 1 \to p \). At any step in the construction we can pause to investigate whether the number reached at that step possesses a certain property or not. …

\[ m \neq n \text{ and } m > n \implies n < m. \]
complete induction admits of a proof of the same kind. Suppose $E(x)$ is a predicate of natural numbers such that $E(1)$ is true and that, for every natural number $n$, $E(n)$ implies $E(n')$, where $n'$ is the successor of $n$. Let $p$ be any natural number. Running over $1 \rightarrow p$ [the number 1 to $p$ in their natural order] we know that the property $E$, which is true for 1, will be preserved at every step in the construction of $p$; therefore $E(p)$ holds.

Justifications of the same type are given in Troelstra’s Principles of Intuitionism of 1969 [Troelstra, 1969, p. 12] and in Troelstra and Van Dalen’s Constructivism in Mathematics of 1988. To quote the latter:

The justification of induction is based on the mental picture we have of the natural numbers, obtained by successively adding ‘abstract units’; given $A(0), \forall x (A(x) \rightarrow A(Sx))$ we build parallel to the construction of $n \in \mathbb{N}$ a proof of $A(n)$:

$$
\begin{align*}
A(0) & \quad A(0) \rightarrow A(1) \\
A(1) & \quad A(1) \rightarrow A(2) \\
& \quad A(2) \ldots \\
& \quad \text{etc.}
\end{align*}
$$

[Troelstra and van Dalen, 1988, vol.1, p. 114]

While this does suffice to show that starting from the construction of a given number $n$, we can obtain a proof of $A(n)$, it does not suffice for demonstrating that we have one construction that works for all (infinitely many) $n$ at once, as would be required for a justification of complete induction. As Parsons notes in his discussion of this type of reasoning (without citing a particular occurrence), to claim that it does yield complete induction is fallacious:

In fact, for each $x$, we can construct a formal proof of $A(x)$ by beginning with $A(0)$ and building up by modus ponens, using $A(x) \rightarrow A(Sx)$. As a proof of induction, this is circular: the ‘construction’ of $x$ by a succession of steps is itself inductively defined, and it is by a corresponding induction that it is established that $A$ holds at each point in the construction. [Parsons, 2008, p. 266, original emphasis]27

Similarly, Yessenin-Volpin rejected both mathematical induction and the soundness principle that ‘If the axioms of a formal system are true and the rules of inference conserve the truth then each theorem is true’, for the reason that these are mutually dependent. ‘It is essentially on these grounds that I am not searching for any axiomatic theory in my program.’ [Yessenin-Volpin, 1970, p. 6]. Note that Brouwer’s grounding of induction, discussed below, is not axiomatic.
Dummett argues in *Elements of Intuitionism* that, although there is ‘no uniform proof skeleton’ for proofs of $A(n)$ from the induction basis and the induction step, except if one presupposes induction, we recognise all the same that the operation of chaining, at each $n$, $n$ applications of modus ponens, will yield a proof of $A(n)$ for each $n$. I do not see how that argument fares better.

The circularity that arises can also be analysed in terms of the BHK explanation, which construes an implication as the existence of a function from proof objects of the antecedent to proof objects of the consequent. For each $n$, the instantiation of the premise $\forall x(Ax \rightarrow ASx)$ required for the application of modus ponens yields a different function (one with as domain proofs of $A(n)$ and as range proofs of $A(Sn)$). These infinitely many different functions cannot be used directly in a finite proof; one has to use induction to reason about the application of all of them, and this introduces the circularity.

As we saw, Brouwer’s idea of the relation between the evidence for induction and the construction of the natural numbers was different from its construal in this circular argument; for him the central idea was iteration. In 1960 a constructive account was proposed that did not, as its originator might have put it, (cl)aim to reconstruct Brouwer’s thought, yet in effect agreed with Brouwer on this point. It was part of Kreisel’s ‘theory of constructions’, and runs as follows ($*$ is the concatenation operator on constructions):

Supposons qu’une propriété $P$ (portant sur les nombres naturels) soit déterminée par la construction $\rho_P(b, a) = 0$ si $b$ est une preuve que $a$ satisfait $P$, $= 1$ dans le cas contraire) et que les résultats

$$P(0), P(a) \rightarrow P(a * 1)$$

soient établis; autrement dit, on a deux constructions $b_0$ et $\rho$ telles que $\rho_P(b_0, 0) = 0$ et que $\{\rho_P(b, a) = 0$ implique $\rho_P[\rho(b, a), a* 1] = 0\}$.

Alors, étant donnée une construction $a$ faite à partir de (i) et (ii), il suffit de suivre cette construction pour obtenir une $b_a$ telle que $\rho_P(b_a, a) = 0$ : à chaque application de (ii) dans la construction de $a$ correspond une application de $\rho$.

---

28 In the proof tree given by Troelstra and Van Dalen, shown above, these steps of instantiation are left implicit.

29 [Defined on p. 389 as (i) $Z(0)$ and (ii) $Z(a) \rightarrow Z(a*1)$ (which formulations incidentally go back to Hilbert)]

16
Ce raisonnement s'exprime dans le cadre de la théorie des constructions abstraites, qui a pour base certains axiomes existentiels assez élémentaires. [Kreisel, 1960, p. 390]

This account is of a different type than Heyting's. It in effect avoids circularity by appealing to one operation, \( \rho \), instead of infinitely many. It achieves this by taking not just a natural number as argument, but also an abstract construction, in such a way that the operation can be iterated. These ‘abstract constructions’ are, in Kreisel’s framework, themselves objects of which only their formal properties are taken into account; it is a ‘formal semantic foundation’ the value of which, qua formal theory, Kreisel acknowledges to be ‘primarily technical’ [Kreisel, 1962, p. 199]. The questions if and how his notion of abstract construction may be related to Brouwer’s notion of construction objects as resulting from construction acts out of a ‘basic intuition’ thus fall out of the scope of Kreisel’s paper.

An approach that likewise leads to a construal of induction as iteration, but now in explicitly Brouwerian terms and (therefore) not in the context of formal semantics, was given by Van Dalen in 2008:

Given \( A(1) \) and \( \forall n (A(n) \rightarrow A(n+1)) \), we want to show \( \forall n (A(n)) \).

‘Show’ means for a constructivist ‘present a proof’, where we have to keep in mind that already in 1907 Brouwer was aware that proofs are constructions; he spoke of ‘erecting mathematical buildings’ and ‘fitting buildings into other buildings’. In modern terms this would be read as ‘constructing mathematical structures’ and ‘constructing a structure on the basis of (out of) another structure’. It is quite clear that he knew how proof-constructions for implication, universally and existentially quantified statements were to be made. The cases of conjunction and disjunction were tacitly understood. So – returning to the matter of induction – we may assume that there is a proof \( a_1 \) of \( A(1) \); notation – \( a_1 : A(1) \). Now a proof for \( \forall n (A(n) \rightarrow A(n+1)) \) is a construction \( c \) that for any given \( n \) and proof \( a : A(n) \) yields

---

30 Also, in a slightly different form, in [Kreisel 1962, p. 209].
31 In the 1962 paper mentioned to in the previous note, Kreisel does refer to Heyting 1956, for its informal explanation of the meaning of the intuitionistic logical constants [Heyting 1956, p. 98]; but Kreisel does not comment on Heyting’s argument for induction elsewhere in that book.
32 Kreisel acknowledges that he indulges in ‘the “mixing” of mathematics and metamathematics stressed in the informal writings of intuitionists’ [Kreisel 1962, p. 202n.9] – stressed, one may add, as something that goes against the foundational order [Brouwer, 1907, pp. 169–178].
a proof \( c(n, a) : A(n + 1) \). So \( a_2 = c(1, a_1) \) and \( a_2 : A(2) \), and 
\( a_3 = c(2, a_2) \) and \( a_3 : A(3) \), \ldots Hence parallel to the con-
struction of the natural numbers we obtain the (potentially infi-
nite) sequence of proofs \( a_1, a_2, a_3, \ldots \), i.e., a proof for \( \forall n(A(n)) \).

In a footnote, Van Dalen adds that ‘In systems with an explici-
t recursor, one can often write down a term for the proof-construction given by the sequence’ [van Dalen, 2008, p. 11n.6]. The procedure sketched may be thought of in terms of a primitive recursive function \( f \) such that \( f(n) \) is a proof \( a_n \) of \( A(n) \):

\[
\begin{align*}
  f(1) & = a_1 \\
  f(Sn) & = c(n, f(n))
\end{align*}
\]

In a type theory with a recursor \( R \) satisfying the conversion relations

\[
\begin{align*}
  Ru v 1 & \leadsto u \\
  Ru v(Sn) & \leadsto vn(Ruvn)
\end{align*}
\]

we can put \( a_1 \) for \( u \), and for \( v \) an operation corresponding to \( c \). In (slides for) a lecture in Groningen in 2009, Van Dalen claims that ‘The Ur-intuition also yields the recursor!’ [van Dalen, 2008, slide 25]

I understand that claim as follows. Brouwer in 1907 of course did not have a theory of recursive functions [34] but he did, as we have noted, have a solid idea of iteration and its relation to induction. Primitive recursion can be reduced to iteration; applied to the recursive function above, this can be done as follows [35] In a type theory with an iterator \( I \)

\[
\begin{align*}
  Iuv1 & \leadsto u \\
  Iuv(Sn) & \leadsto v(Iuvn)
\end{align*}
\]

and with pairing, we can set

\[ u = \langle 1, a_1 \rangle \]

[33] ‘De herhaalbaarheid van de successor-operatie is een onderdeel van de oer-intuïtie, de z.g. “zelfontvouwing”. De oer-intuïtie levert ook de recursor!’ [van Dalen, 2009, slide 25]

[34] In a letter of July 17, 1928, Brouwer suggested that Heyting add to the latter’s formalisation of intuitionistic logic and analysis a formalisation of the notion of law (in the sense of a so-called spread law) [van Dalen, 2011, p. 334]; Heyting did not take this up. A lawlike sequence, or to be precise a spread with a lawlike sequence as its single element, is given as a limiting case; and a recursive sequence is intuitionistically lawlike. (Whether the converse is also true is a different question.)

[35] In the literature, the reduction of recursion to iteration goes back to Kleene’s iterative rendering of the recursive predecessor function [Kleene, 1938]. For a systematical discussion, see Robinson [1947].
\begin{align*}
v(Iuvn) = \langle S\pi_1(Iuvn), c(\pi_1(Iuvn), \pi_2(Iuvn)) \rangle
\end{align*}

where \(\pi_1\) and \(\pi_2\) are the left and right projection operators. Note that the operations of pairing and projection are readily understood in terms of an invocation of Brouwer’s two-ity. Ordered pairing is acceptable as a general intuitive operation because it consists in an order-preserving mapping of the two parts of the empty two-ity onto whatever the elements of the pair will be and projection in separating one element out of a two-ity.

In the following, I should like to argue in some detail that Brouwer, in holding to the tenets B1-B5 (p. 14 above), indeed had in mind a development that leads from time awareness to intuitiveness of the iterator, and, from there, of induction.

Picking up from the two-ity again, we first turn to the natural numbers. Brouwer identifies the natural number 2 with the empty two-ity. Once I have created an empty two-ity as an object, time moves on again, a created two-ity sinks into the past and this, when I decide to turn my attention to it, will then become one component of a new two-ity. Brouwer identifies this new, nested two-ity with the natural number 3. Because time keeps moves on, and I can keep turning my attention to it, I thus obtain an iterative structure, and thereby the natural numbers. There is an intrinsic ordering of the natural numbers as constructed intuitionistically in the sense that the construction object of \(n + 1\) includes that of \(n\) as a proper part.

Brouwer calls the successive construction of iterated two-ities the ‘self-unfolding’ of the empty two-ity. The idea of this self-unfolding includes the ideas that time is potentially infinite, and that in particular every life moment will fall apart into a two-ity, one part of which is a previous two-ity. These ideas obviously cannot be taken to be empirical observations, but should be seen as a priori insights into the structure of time awareness.

In some of Husserl’s writings, the relation between iteration and the continuum of time awareness is made explicit. Central to his analyses of inner time awareness are the notions of retentional and protentional intentionality – the intentional directedness of the living present towards the phase of the stream of consciousness that has just elapsed and towards the phase that is about to come, respectively [Husserl, 1969, p. 297, p. 333]. In a text of 1916, published as Appendix I to the 1905 lectures on time awareness [Husserl, 1907, p. 179n1]/[Brouwer, 1907, p. 97n1]. See also Parsons’ remarks on this point [Parsons, 2008, pp. 177–178].
wrote:

It is inherent in the essence of every linear continuum that, starting from any point whatsoever, we can think of every other point as continuously produced from it; and every continuous production is a production by means of continuous iteration. We can indeed divide each interval in infinitum and, in the case of each division, think of the later point of the division as produced mediatly through the earlier points . . . Now this is also true in the case of temporal modification – or rather, while the use of the word ‘production’ is a metaphor in the case of other continua, here it is used authentically. The time-constituting continuum is a flow of continuous production of modifications of modifications. The modifications in the sense of iterations proceed from the actually present now, the actual primal impression i . . . [Husserl, 1991, p. 106] . . .

Likewise, in a manuscript of 1931, Husserl speaks of the ‘the future horizon as flowing continuity of the implicit iteration of coming realisations of always new presents’ and ‘this flowing continuity of pasts, an endless horizon of iteratively nested pasts’ [Husserl, 2006, p. 405].

Thus, the iterative structure of nested two-ities arises as an abstraction from a structure present in inner time awareness: In the perception of the move of time, just as the present phase of consciousness is modified into a retention, prior retentions are modified into retentions of themselves, so that, for example, a simple retention now becomes a retention of a retention. In the constitution of a two-ity out of two phases of consciousness, the form of the retentional relation between these two phases is retained, while abstraction is made from the intermediate retentional relations along the temporal

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39‘Im Wesen jedes linearen Kontinuums liegt es, daß wir, von einem beliebigen Punkt ausgehend, jeden anderen Punkt aus ihm stetig erzeugt denken können, und jede stetige Erzeugung durch stetige Iterierung. Jeden Abstand können wir ja in infinitum teilen und bei jeder Teilung den späteren Teilungspunkt mittelbar durch die früheren erzeugt denken . . . So ist es nun auch bei der zeitlichen Modifikation, oder vielmehr, während sonst, bei anderen Kontinu, die Rede von der Erzeugung ein Bild ist, ist sie hier eine eigentliche Rede. Das zeitkonstituierende Kontinuum ist ein Fluß stetiger Erzeugung von Modifikationen von Modifikationen. Vom aktuellen Jetzt aus, der jeweiligen Urimpression u, gehen die Modifikationen im Sinn von Iterationen . . . ’ [Husserl, 1969, p. 100]

40‘[der] Zukunftshorizont als strömender Kontinuität der impliziten Iteration kommen der Verwirklichungen von immer neuen Gegenwarten’ and ‘diese strömende Kontinuität der Vergangenheiten, [ein] endloser Horizont wiederum iterativ ineinander geschachtelter Vergangenheiten’.
continuum in between. Protention, in turn, is an intention directed towards the phase of consciousness that is just about to come. It plays a rôle in the intuition that the sequence of numbers constructed so far can always be extended. I read the statement in Brouwer’s notebooks that ‘The sequence \( \omega \) can only be constructed on the continuous intuition of time’ as, in effect, a recognition of this dependence of the construction of nested two-ities on the retentional and protentional intentionalities that characterise the awareness of inner time.

We obtain intuitive knowledge of the structures of time awareness is through acts of reflection – reflection in the phenomenological sense of turning our attention to an earlier episode in our flow of consciousness. The structure of inner time awareness is not given passively altogether, for we have to engage in the appropriate kind of reflection, which is an activity. But that activity consists in objectifying the flow of time and its parts and phases; it does not form that which is reflected on. There is no circularity, then, in saying that iterative structures are constructed in intuition by projecting from the iterative structure of time awareness, as we can obtain intuitive knowledge of whatever structure inner time awareness exhibits without first having had to engage in an activity of constructing it. Note that this means that the form of inner time is not categorial form in Husserl’s sense, as he holds that categorial form is constituted in \textit{active formation} by the ego. Indeed, although Husserl for some time between 1907 and 1909 did believe that the form of time is categorial, he gave up that idea when he around 1909 discovered the absolute flow of time which constitutes itself as a flow – one is tempted to say, which unfolds itself.

The question may be raised how we know that the construction of the natural numbers based on these ideas of Brouwer and Husserl results in the standard numbers. After all, there are constructive proofs of the existence of non-standard models of the theory of the successor and, more importantly, of Heyting Arithmetic (HA). Dedekind already devised a non-standard model for the theory of successor [Dedekind, 1890, p. 100]. As for non-standard models of arithmetic, the existence proofs by Skolem and Gödel are of course not constructive, and McCarty has proved constructively that, under the assumption of Markov’s Principle and a weak form (consequence) of Church’s

\[41\] For a phenomenological constitution analysis of potentially infinite sequences (with choice sequences as a special case), see van Atten [2007, section 6.2].

\[42\] ‘De reeks \( \omega \) is alleen op te bouwen op de continue tijdsintuïtie.’

\[43\] Husserl critically discusses various skeptical claims regarding reflection in section 79 of Ideen I [Husserl, 1976]. See also Hopkins [1989].

\[44\] For more on this, see van Atten [2015b].
Thesis, HA has no non-standard models [McCarty, 1988]. The latter result does not settle the matter however as, in a Brouwerian setting including the theory of the creating subject, even the weak form of Church’s Thesis involved is false, and there is a weak counterexample to Markov’s Principle. Indeed, De Swart’s proof of the compactness theorem for intuitionistic predicate logic [de Swart, 1977, section 3] does give rise to non-standard models (which have not been studied so far) acceptable from a Brouwerian point of view.

However, non-standard models, even if constructive, pose no threat to the intuitionistic account of the natural numbers and their arithmetic. Dummett has written, in a passage that Parsons [2008, p. 279] draws attention to:

Within any framework which makes it possible to speak coherently about models for a system of number theory, it will indeed be correct to say that there is just one standard model, and many non-standard ones; but since such a framework within which a model for the natural numbers can be described will itself involve either the notion of ‘natural number’ or some equivalent or stronger notion such as ‘set’, the notion of a model, when legitimately used, cannot serve to explain what it is to know the meaning of the expression ‘natural number’. [Dummett, 1978, p. 193]

Within the specifically intuitionistic context, the point can be strengthened by observing that the natural numbers are privileged not only conceptually,

45That weak form of Church’s Thesis is

$$\forall n(P(n) \lor \neg P(n)) \to \neg\exists e\{e\}$$

is $P$’s characteristic function.

This is inconsistent with Kripke’s example of a function that the creating subject is able to compute yet cannot be assumed to be recursive, on pain of contradiction [van Atten, 2008]. Markov’s Principle is used in this form:

$$\forall n(P(n) \lor \neg P(n)) \to (\neg\exists n P(n) \to \exists n P(n))$$

As pointed out in Troelstra and Van Dalen [Troelstra and van Dalen, 1988, I:pp. 205–206 and 237], this is equivalent to

$$\forall x \in \mathbb{R}(x \neq 0 \to \exists k(|x| > 2^{-k}))$$

to which Brouwer presented a weak counterexample in ‘Essentially negative properties’ [Brouwer, 1948].

46A constructive non-standard model that, unlike De Swart’s construction, does not depend on choice sequences, and which has given rise to further work, was later built by Moerdijk [1995].
but also genetically. The construction of any model of the theory of successor or of arithmetic, whether standard or non-standard, depends, like all mathematical activity, on the unlimited self-unfolding of the empty two-ity, and hence on the intuition ‘and so on’. But if one acknowledges that intuition, one obtains, by thematising a structure that is already present in time awareness, the natural numbers directly from it.

Parsons notes that

the point of Dummett’s observation that the notion of natural number must be used in the construction of models of arithmetic is that, in the end, we have to come down to mathematical language as used, and this cannot be made to depend on semantic reflection on that same language. [Parsons, 2008, pp. 287-288]

Phenomenologically, one would take certain properties of the mind to be (partly) explanatory of the constraints on the use of language that Dummett takes as the point of departure for his meaning-theoretical considerations. For example, that we have the capacity to iterate the construction of two-ities is an intuitive truth, as is the fact that, with our kind of mind, we cannot complete infinitely many iterations; Dummett’s observation that

Even if we can give no formal characterisation which will definitely exclude all such [non-standard] elements, it is evident that there is not in fact any possibility of anyone’s taking any object, not described (directly or indirectly) as attainable from 0 by iteration of the successor operation, to be a natural number. [Dummett, 1978, p. 193]

depends on these facts. A vagueness remains in the circumstance that, although at any given moment in the self-unfolding of the two-ity, in a sense only finitely many unfoldings have been made, this sense of ‘finite’ cannot be replaced by a definition in numerical terms, as then the account would become circular. So this particular notion of finiteness has to be taken as understood prior to the notion of natural number.

As quoted above, Brouwer in his dissertation claims that the a priori judgement that mathematical synthesis can be repeated is a result of ‘mathematics of the second order’, the result of viewing mathematics mathematically. In that same work, Brouwer defines mathematical viewing as ‘seeing

47This is also pointed out by Dedekind, in section 6 of his letter to Keferstein [Dedekind, 1890]. In effect, Dedekind himself relies on such a pre-numerical understanding of ‘finite’ in his understanding of proofs as finite objects.
in 1929 he analyses it into the two phases of assuming the ‘temporal attitude’ – accepting inner time awareness, which is a necessary condition for mathematics – and then the ‘causal attitude’.\footnote{\textsc{Brouwer, 1907}, pp. 81, 105-106.}

Nunmehr besteht die \textit{kausale Einstellung} im Willensakt der ‘Identifizierung’ verschiedener sich über Vergangenheit und Zukunft erstreckender zeitlicher Erscheinungsfolgen. Dabei entsteht ein als \textit{kausale Folge} zu bezeichnendes gemeinsames Substrat dieser identifizierten Folgen. \cite[Brouwer, 1929, p. 153]{Brouwer1929}

If we count the givenness of an objectified act in reflection as an ‘Erscheinung’, then it is clear that viewing our mathematical activity mathematically allows for the total or partial identification of temporally distinct acts (or series of acts) of mathematical construction; this is a way of ‘seeing repetition’. Such identification is a form of applied mathematics, first, because taking two (or more) objectified acts together in one awareness depends on the formation of two-ities of them, and second, because isolating a mathematical structure that is common to two acts consists in the construction and then successful projection of the same mathematical structure onto both objectified acts. As construction and projection themselves take place in intuition, the determination of a common structure leads to intuition of act types and, correspondingly, of object types, namely the type of object constructed in acts of a given type. Thus, instead of having to keep, e.g., empty two-ities constructed at different times ontologically distinct, we can identify them and consider these as repeated constructions of the same empty two-ity. It is also in reflection that the ‘processes’ in Bernays’ account of exponentiation (quoted on p. \textit{3}) can be given in intuition as individual objects and, founded on that, as types.

Partial identification is the foundation for abstraction. For example, in reflection on acts in which we perform $5 + 2$ and $7 + 2$, we may come to identify the act type ‘adding 2 to a natural number’. This is the example that came up at Brouwer’s thesis defence, where the question of the possibility of identifying act types intuitively was raised. Gerrit Mannoury there objected that in ‘and so on’, the ‘so’ is not primitive (and hence neither is ‘and so on’), but consists in a relation between relations \cite[van Dalen, 2001, p. 151]{vanDalen2001}.

\footnote{\textsc{wiskundig bekijken, het zien van herhalingen van volgreeksen Brouwer, 1907, pp. 81, 105-106.}}

\footnote{\textsc{zeitliche Einstellung’}; ‘kausale Einstellung’}

\footnote{See also Brouwer’s remark in a letter to Korteweg of January 16, 1907, on a letter in which Mannoury, as was the custom, had informed Brouwer of his planned objection. Mannoury’s letter unfortunately seems not to have been preserved.}
Mannoury’s idea seems to have been that the relation between 5 and 7 and the relation between 7 and 9 are, in their full intuitive concreteness, different; but a similarity relation holds between them. Brouwer replied:

What you say at the end, namely that the ‘so’ in ‘and so on’ is just a relation between relations and not itself a relation, can, it seems to me, not be upheld either. Mathematics could not exist, if I cannot repeatedly think the same thing again, e.g., first Jan, then Piet, then the same Jan again. Likewise I can think the same relation, and so between five and seven there exists after all the same relation as between seven and nine, namely, ‘+2 =’.  

Unfortunately, this is where Brouwer’s reply ends. But this insight naturally leads from the generation of the natural numbers, that is, from the iteration of the successor operation, to the general principle of iteration. First observe that, on Brouwer’s conception of intuition, it is not just the natural numbers as individual objects that are constructed in intuition, but so is the (growing) object that is the potentially infinite sequence of them. He speaks of

the intuitive truth that mathematically we cannot construct but finite sequences, and also, on the basis of the clearly conceived ‘and so on’, the order type $\omega$, but only consisting of equal elements. [Brouwer, 1975, p. 80, trl. modified]

A footnote elucidates the ‘and so on’:

The expression ‘and so on’ means the indefinite repetition of one and the same object or operation, even if that object or that

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51 Wat U tenslotte zegt, dat enzovoort in zo slechts een relatie tussen relaties en niet een relatie zelf ziet, is geloof ik evenmin vol te houden. Wiskunde zou niet kunnen bestaan, als ik niet meermalen weer hetzelfde ding kon denken, b.v. eerst Jan, dan Piet, dan weer diezelfde Jan. Zo kan ik ook meermalen dezelfde relatie denken, en zo bestaat wel degelijk tussen vijf en zeven dezelfde relatie als tussen zeven en negen nl. ‘+2 =’. [van Dalen, 2001, p. 151]

52 Compare Tait in ‘Finitism’: ‘we understand $n + 2$ not via understanding each of the infinitely many instances, $0 + 2, 1 + 2$, and so on. Rather, we understand these via our understanding of what it is for one sequence to be a two-element extension of another.’ Tait, 1981, p. 530

53 De intuitieve waarheid, dat wij wiskundig niet anders kunnen scheppen, dan eindige rijen, verder op grond van het duidelijk gedachte ‘en zoo voort’ het orndtype $\omega$, doch alleen bestaande uit gelijke elementen’ Brouwer, 1907, p. 142–143. 

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operation is defined in a rather complex way. \[\text{Brouwer, 1975, p. 80n1, original emphasis}\]

Now suppose we have a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) (or, more generally, \( f : A \rightarrow B \) such that \( B \subseteq A \)). We now construct, in intuition, two potentially infinite sequences in parallel, one being that of the natural numbers starting at 1, and the other the sequence of (the results of) the operation of applying \( f \), beginning with \( f(a) \) for some given \( a \):

\[
\begin{align*}
1 & \quad f(a) \\
2 & \quad f(f(a)) \\
3 & \quad f(f(f(a))) \\
\vdots & \quad \vdots \\
\end{align*}
\]

The object constructed in these acts in parallel with \( n \) is \( f^n(a) \). The justification for this claim is that, on the hypothesis that we have a construction method for a given natural number \( n \), we also know that the series of applications of \( f \) of length \( n \) admits of composition, because each time we apply one and the same operation \( f \) whose range is included in its domain. Moreover, by an appeal to the facts that the sequence construction of natural numbers is given as such in intuition, and that the construction of \( f^n(a) \) proceeds in parallel with that of \( n \), we also know that we have an intuitive construction of the potentially infinite sequence of the \( n \)-fold iterations.

As the insight that operations \( \mathbb{N} \rightarrow \mathbb{N} \) can be iterated depends immediately on the insight that the construction of the natural numbers is iterative, the former is hardly more reducible than the latter. At the end of the dissertation, Brouwer emphasises:

there are elements of mathematical construction that in the system of definitions which must remain irreducible, and which therefore, when communicated, must be understood from a single word, sound, or symbol; they are the elements of construction that are immediately read off from the Urintuition or intuition of the continuum; notions such as continuous, unity, once more, and so on are irreducible.\[55\]

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\[54\] Waar men zegt: "en zoo voort", bedoelt men het onbepaald herhalen van eenzelfde ding of operatie ook al is dat ding of die operatie tamelijk complex gedefinieerd. \[\text{Brouwer, 1907, p. 180}\]

\[55\] Er zijn elementen van wiskundige bouwing, die om het systeem der definities onherleidbaar moeten blijven, dus bij mededeling door een enkel woord, klank of teken, weerklank moeten vinden; het zijn de uit de oer-intuïtie of de continuum-intuïtie afgelezen bouwelementen; begrippen als continu, eenheid, nog eens, enzovoort zijn onherleidbaar. \[1907:180\]
Brouwer’s view of induction as an act in mathematical intuition depends on his particular understanding of a mathematical property not in terms of definitions of predicates but in terms of mathematical construction acts, which for him are languageless:

Often it is very simple to introduce inside such a system, independently of the way it came into being, new buildings, as the elements of which we take elements of the old one or systems thereof, in a new arrangement, but bearing in mind the arrangement in the old building. What are called the ‘properties’ of a given system amount to the possibility of building such new systems in a certain connection with a priorly given system.

And it is exactly this fitting in of new systems in a given system that plays an important part in building up mathematics, often in the form of an inquiry into the possibility or impossibility of a fitting-in satisfying certain conditions, and in the case of possibility into the various ways in which it is possible. [Brouwer, 1975, p. 51–52, trl. modified, original emphasis]

Brouwer calls the whole of the old building, the new one, and the correspondences between them, a ‘fitting-in’ (Dutch: inpassing). There is a familiar act-object ambiguity here. A fitting-in, as an object, is given by two buildings and a correspondence between elements of these buildings. A fitting-in as an act is the act of building up a fitting-in as an object. In either sense, the concept of property in Brouwer’s sense stands in contrast to a primarily logical (and hence linguistic) concept of property.

In my view, a fitting-in, as an object, is what is elsewhere known as a state of affairs (Sachverhalt). This brings me to Parsons’ observation that

Binnen zulk een opgebouwd systeem zijn dikwijls, geheel buiten zijn wijze van ontstaan om, nieuwe gebouwen zeer eenvoudig aan te brengen, als elementen waarvan de elementen van het oude of systemen daarvan worden genomen, in nieuwe rangschikking, waarbij men de rangschikking in het oude gebouw voor oogen behoudt. Op de mogelijkheid van zulk bouwen van nieuwe systemen in bepaalde samenhang met een vooraf gegeven systeem, komt neer, wat men noemt de ‘eigenschappen’ van het gegeven systeem.

En een belangrijke rol speelt bij den opbouw der wiskunde juist dat inpassen in een gegeven systeem van nieuwe systemen, dikwijls in den vorm van een onderzoek naar de mogelijkheid of onmogelijkheid van een inpassing, die aan bepaalde voorwaarden voldoet, en in geval van mogelijkheid naar de verschillende wijzen waarop. [Brouwer, 1907, pp. 77–78, original emphasis]
Brouwer is not as clear as he might be about the distinction between intuition of and intuition that. Writers about Brouwer tend to be even less so. [Parsons, 2008, p. 175n71]

I confess that I, as a writer about Brouwer, have never stated my view on this clearly; but it has always been that intuition that is a special case of intuition of, namely, of a state of affairs. In this I follow Husserl, who had learned the term ‘Sachverhalt’, in this particular use, from Carl Stumpf, his teacher in Halle; and the concept itself had been used by the teacher of both, Franz Brentano [Smith, 1989]. The fact that Husserl saw things this way is remarked on by Parsons:

The basic notion for Husserl is intuition of; nonetheless intuition that distinguishes different ways of entertaining a proposition from actively knowing it. But it seems to me that Husserl reduces intuition that to a form of intuition of, where the object is not what we would call a proposition but rather a state of affairs (Sachverhalt). [Parsons, 2008, 143n10]

and

Husserl seems to regard intuition that as a species of intuition of: Evidence of a judgement is a situation in which the state of affairs that obtains if it is true is ‘itself given’. Because, typically, a proposition involves reference to objects, evidence will involve intuition of those objects, but they play the rôle of constituents of a state of affairs that is also intuitively present, at least in the ideal case. [Parsons, 2008, 146]

In a footnote, Parsons adds:

In his discussion of truth, Husserl talks about the ‘ideal of final fulfillment’ (LU VI [Husserl [1984]] §§37–39). Later, he concedes that this is in interesting cases not achieved or even achievable, so that final fulfillment is a kind of Kantian idea. [Parsons, 2008, 146n21]

But it can be argued that, while this is true in general, Husserl’s conception of a particular class of objects, the purely categorial objects, does not in fact allow the use of Kantian ideas to conceive of the fulfilment of intentions directed towards them. That class includes the objects of pure mathematics; for further discussion, I refer to van Atten [2010, pp. 78–79].

28
Let me now return to induction and state the assertion-condition for \( \forall x (A(x) \rightarrow A(Sx)) \) as follows: I should have a construction method \( f \) that, given a constructed number object \( x \), yields a construction method \( g = f(x) \) to construct a fitting-in \( A(Sx) \) whenever I am given a fitting-in \( A(x) \), so that \( g(A(x)) = A(Sx) \). But if that condition is fulfilled, I can combine these two methods \( f \) and \( g \), together with the device of ordered pairing, into one method \( h \) that, given an ordered pair of construction objects \( x \) and the fitting-in \( A(x) \), yields the ordered pair of construction objects \( Sx \) and the fitting-in \( A(Sx) \): Define

\[
h(\langle x, A(x) \rangle) = \langle Sx, g(A(x)) \rangle
\]

Then

\[
h(\langle x, A(x) \rangle) = \langle Sx, A(Sx) \rangle
\]

As \( h \) is uniform in its operation on the two components of the pair, by our earlier consideration we have intuitive knowledge that we can iterate it. Given a construction for 0 and, by hypothesis, a construction for \( A(0) \), we construct in intuition the ordered pair \( \langle 0, A(0) \rangle \) and iterate \( h \):

\[
\begin{align*}
\langle 0, A(0) \rangle \\
h(\langle 0, A(0) \rangle) & = \langle 1, A(1) \rangle \\
h(\langle 1, A(1) \rangle) & = \langle 2, A(2) \rangle \\
& \vdots
\end{align*}
\]

Thus we have a uniform construction of fitting-ins of \( n \) into \( A(n) \), justifying the conclusion of the induction principle. Again, on Brouwer’s account not just the individual members of this potentially infinite sequence are given in intuition, but also that sequence as such.

The generality of the account, and hence of the validity of induction, with respect to the predicate \( A \) follows because it imposes no conditions on the predicate other than that it be total and that it be constructive in Brouwer’s sense. The generality claim is not grounded on a prior overview over the domain of all constructible predicates, but on general knowledge I have, through reflection on acts, of the genesis of fitting-ins. Casting a mathematical view on acts of mathematical construction that proceed iteratively, one identifies their common structure and thereby obtains iteration as an act type, itself given in intuition. It is in this sense that the judgement that the induction principle is valid is the result of second-order mathematics.

Parsons writes that

\[57\] Note that here ‘\( A(Sx) \)’ stands for a mathematical construction in intuition, not for a proposition. The ambiguity of the notation will be resolved by the context.
What [Brouwer] calls the ‘original intuition of mathematics’ is not an intuition of iteration or of the natural numbers. I think one can regard Brouwer as holding that any natural number can be given in intuition; iteration and the structure of the natural numbers arise through the ‘self-unfolding’ of the intuition, but there is no reason to suppose that either is an object of intuition. The phrase ‘intuition of iteration’ does not, so far as I know, occur in Brouwer’s writings. [Parsons, 1986, p. 214]

However, as we have seen, a phrase that does occur in Brouwer’s writings is that ‘and so on’ is ‘immediately read off from the Urintuition’ [1907:180]; and, in a handwritten note to that passage, that ‘and so on’ is among the ‘polarizations of the Urintuition’ [van Dalen, 2001, p. 136n108]. These formulations imply that ‘and so on’ is given as part of the Urintuition, understood in its self-unfolding, as ‘and so on’ involves more than one act; specifically, as what Husserl would call a dependent part, a part that cannot be given independently. If this is combined with Brouwer’s understanding of ‘and so on’ as ‘the indefinite repetition of one and the same object or operation, even if that object or operation is defined in a rather complex way’, which implies generality, then it seems clear that Brouwer does think of iteration as an object of intuition. As reconstructed here, it is given as an act type, the intuition of which is founded on objectified iterative acts, given in intuition by reflection. Thus, Brouwer disagrees with Parsons here, and also with Tait, who both hold that the general idea of iteration is ‘not found in intuition’ [Tait, 1981, p. 539], [Parsons, 1986, p. 225].

The quotation continues: ‘it was used by Hermann Weyl, who said that on the basis of the intuition of iteration we are convinced that the concept of natural number is “extensionally definite” (umfangsdefinit; Weyl [1913], p. 85), that is, that the natural numbers are a domain over which classical quantification is valid. In my opinion, such a view at best presupposes a different conception of intuition [than Kant’s or Brouwer’s, it seems, given the following sentence] and is at worst confused. In fact, Weyl’s conception of intuition seems to derive not from Kant or Brouwer but from Husserl.’

Tait’s rejection of intuition goes even further:

However and in whatever sense one can represent the operation of successor, to understand Number one must understand the idea of iterating this operation. But to have this idea, itself not found in intuition, is to have the idea of Number independent of any sort of representation in intuition. The same objection applied to Brouwer’s analysis of number in terms of consciousness of succession in time (two-ity). Again, the essence of number is in the iteration of that operation, and the idea of iteration is not founded on time consciousness. [Tait, 1981, p. 539–540]

I agree with Parsons’ comment on that passage:
4 Some questions to Parsons

The account in the previous section suggests various reflections on Parsons’ arguments.

As Parsons notes,

Although the concept of a string of strokes involves iteration, the proposition that every such string can be extended is not an inductive conclusion. A proof by induction would be circular. (Parsons, 2008, p. 175)

In the Brouwerian account, the circularity was, in effect, blocked by arguing that there is one iterative form that is given to us without our having to construct it first – the structure of inner time awareness. Parsons, in his own setting, proposes a different type of solution, which is to say that, just to know that a string of strokes can be extended, we do not have to think of the string we are extending as having been obtained by iterated application of adding one more string. We can make do, Parsons says (2008, p. 175), with a ‘proto-conception’ of string, in which we so to speak willingly forget about those iterations and then add one more stroke to it. The idea is that the proto-conception is rich enough to make us see extendability of every string, but too poor to set a circularity in motion. But here one could, I think, ask a question about type-preservation again. How do we know that the result is a string? Parsons explains that

...to see the possibility of adding one more, it is only the general structure that we use, and not the specific fact that what we have before us was obtained by iterated additions of one more. This is shown by the fact that, in the same sense in which a new stroke can be added to any string of strokes, a new stroke can be added to any bounded geometric configuration. (Parsons, 2008, p. 175)

But precisely because of that generality – the independence of the type of the given geometric configuration – it is not clear to me that this argument gives us a purchase on the type of the particular resulting configuration.

[This] seems to imply either that the ‘idea of number’ is a concept of an abstract structure that does not depend on any manner in which an instance of the structure might be given, or that an instance is given in an essentially non-intuitive way. Tait does not argue for either of these positions, and I am inclined to reject both. (Parsons, 1986, p. 225n16)

A good discussion and reply to earlier criticisms of the idea that, on the conception
Parsons acknowledges that the limits of intuitive knowledge that he arrives at are rather narrow, and that many will hold that this is due to the very restricted character of the conception of intuition he develops [Parsons, 2008, p. 316]. One particularly strong constraint Parsons works with is Kantian in the broad sense that intuition is intuition of spatio-temporal objects. However, Parsons explicitly leaves open that there might be different models of intuition, on which there would be intuition also of other types of objects, and he mentions Husserl and Gödel. I think that material in the Gödel archive, reading notes and work for the revision of the Dialectica Interpretation, shows that he was much more committed to Husserl’s notion than he was willing to let on in either his publications or his conversations with people without too much interest in phenomenology [van Atten, 2015a, section 1.2 and chs 4, 6]. Be that as it may, the following two questions to Parsons are directly concerned with Husserl’s notion.

As we saw in section 2, Parsons says that we have intuitive knowledge that $p$ if $p$ can be ‘seen’ to be true on the basis of intuiting the objects that $p$ is about. But he does not say much more about how we see this. Does the ‘seeing’ involve any intuitive component other than that of the objects? We would arrive at something like Husserl’s ‘categorial intuition’, but I am not sure Parsons would be willing to embrace that. But if, alternatively, the component that the seeing involves beyond intuition of the objects is itself non-intuitive, what would justify calling the resulting knowledge ‘intuitive knowledge’?

The following question is motivated by the fact that, although Parsons mentions Husserl in Mathematical Thought and Its Objects, he does not mention the aspect of Husserl that brings Husserl closest to Brouwer, inner time awareness. It would seem to be a natural question, however, whether Parsons would not be willing to extend his notion of intuition, such as he explicitly describes it, with a Brouwerian or Husserlian intuition of inner time. It is

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61In ‘Mathematical intuition’, Parsons remarks that ‘Husserl does undertake to show that in categorial intuition there is something analogous to sensations in sense perception. In my view, he lapses into obscurity in explaining this (LU, VI [Husserl 1983], §56). I am not sure to what extent this can be cleared up’ [Parsons, 1980, p. 166n8]. In that section, Husserl outlines a theory of the ‘categorial representation’ (‘kategoriale Räpresentant’), which, however, he later dropped [Husserl, 1984, p. 535], [Lohmar, 1990]. Note that Parsons’ critical remark about it has not been included in Mathematical Thought and Its Objects. An alternative interpretation of categorial intuition is proposed in van Atten (2015).
not clear to me whether Parsons has a principled reason not to. One may of course observe that, given his long-standing engagement with both Brouwer’s and Husserl’s thought, if he had wanted to exploit their notion of intuition of inner time, he would have done so by now.

Yet in *Mathematical Thought and Its Objects*, Parsons appeals to Brouwer’s intuition of two-ity twice.

First, as one way of seeing that every string of strokes can be extended:

Let us return to the proposition that any string can be extended. The idea that this rests on a capability of the mind is a very natural one and in certain respects acceptable. I have proposed two different ways of seeing this, one resting on the figure-ground structure of perception and one (Brouwer’s) resting on temporal experience. . . . We experience the world as temporal, and have the conviction that we can continue into a proximate future, in which the immediate past is retained. [Parsons, 2008, p. 177] 62

The second appeal is made in a comment on Bernays. Bernays had written:

We are conscious of the freedom we have to advance from one position arrived at in the process of counting to the next one. 63

Parsons comments:

It would take some argument to show that there is no appeal here to the temporal character of experience, such as we find in Brouwer. [Parsons, 2008, p. 337]

As we just saw, Parsons is willing to appeal to Brouwer’s temporal experience as one way of upholding the claim that every string of strokes can be extended; and the present comment on Bernays is made by Parsons in

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62 Also in ‘Intuition in constructive mathematics’:

The ‘original intellectual phenomenon of the falling apart of a life moment into two qualitatively distinct things’ [Brouwer, 1929, p. 153], is in a certain way iterable: since we can divide our experience into past and present/future, independently of its objects, we can continue to repeat that division, so that there ‘arises by self-unfolding of the original intellectual phenomenon the temporal series of appearances of arbitrary multiplicity’ (ibid.) [Parsons, 1986, pp. 212–213]

63 [Zunächst sind wir uns der Freiheit bewusst, von einer erreichten Stelle im Zählprozess jeweils noch um Eins forzuschreiten [Bernays, 1955, p. 469]]
the context of defending his own claim that purely rational evidence cannot replace appeals to intuition completely. I therefore take also this comment to show that Parsons indeed wishes to accept Brouwer’s temporal intuition.

It is, of course, characteristic of Parsons’ general approach in the philosophy of mathematics that, to convince us that we have the capability to extend any string, he refers to two types of intuition – perception and temporal experience. However, in his description and use of Brouwer’s two-ity, it is the discrete elements in the two-ity and their order that he exploits, never quite mentioning the continuum in between the two things. Intuitionists subscribe to the view that the intuitions of the discrete and the continuous cannot be accepted independently from one another [Brouwer, 1907, p. 8]/[Brouwer, 1975, p. 17]. A question to Parsons, then, is whether he means to do just that, and if so, whether that is possible.

I should like to close with a remark on impredicativity.

Parsons [2008, pp. 293–294] accepts the following argument, proposed by Dummett, to the effect that the notion of natural number is impredicative:

The totality of natural numbers is characterised as one for which induction is valid with respect to any well-defined property, where by a ‘well-defined property’ is understood one which is well-defined relative to the totality of natural numbers. In the formal system, this characterisation is of course weakened to ‘any property definable within the formal language’; but the impredicativity remains, since the definitions of the properties may contain quantifiers whose variables range over the totality characterised. [Dummett, 1978, p. 199]

It is curious that Dummett, who raised the issue in his paper on Gödel’s incompleteness theorem of 1963, does not discuss it in his later book Elements of intuitionism [Dummett, 1977, 2000]. Be that as it may, John Myhill has argued that the constructivist can avoid this problem by saying that to the constructivist, the notion ‘finite’ or some equivalent idea such as ‘natural number’ or ‘ancestral’ is clear whereas impredicative definitions are not [Myhill, 1974, p. 27]. Parsons answers to Myhill that that reply depends on ‘a dogmatic view of the clarity of the notion of natural number and the evidence of mathematical induction’. He adds that ‘such a dogmatic view could plausibly be attributed to Poincaré and possibly also Brouwer’ [Parsons, 2008, p. 294n44].

I have tried to show that the view is, in Brouwer’s setting, less dogmatic than it may seem. Moreover, as we saw above (p. 27), for Brouwer a (mathematical) property is primordially not a logically defined predicate, but a
fitting of one mathematical building into another. As the notion of mathematical building depends on that of the two-ity and its unfolding, this means that the notion of property presupposes an intuition that by itself suffices to give the natural numbers. This contradicts Parsons’ conclusion that ‘the concept of natural number cannot determine what counts as a well-defined predicate’ [Parsons, 2008, p. 267]; but Brouwer and Parsons are speaking from different backgrounds.

Acknowledgement. I express my profound gratitude to Charles Parsons, for his generous instruction and friendship over the years.

This is a revised and expanded version of a paper presented in Jerusalem on December 4, 2013, at the conference on the work of Parsons, ‘Intuition and Reason’, held in Tel Aviv and Jerusalem, Dec 2–5, 2013. Earlier versions were presented at ‘Vuillemin, lecteur de Kant’ at the Archives Poincaré in Nancy (Dec 15, 2012); at the seminar ‘Mathématiques et Philosophie, 19e et 20e siècles’ at SPHERE, Paris (Feb 21, 2013); at the seminar ‘Les usages de la phénoménologie (I) : temps vécu et temps cosmique’ in Lille (Mar 28, 2013); and at the joint PHILMATH seminar of SND and IHPST, Paris (Oct 28, 2013). I thank the organisers for their invitations, and the audiences and later readers for their questions and comments; in particular, in addition to Parsons: Dirk van Dalen, Gerhard Heinzmann, Wilfried Sieg, Göran Sundholm, Bill Tait, Joseph Vidal-Rosset, Albert Visser, and Michael Wright.

Postscript, May 2015 When this paper had almost been completed, Charles Parsons shared with me his very recent manuscript ‘Intuition revisited’. Among other things, it contains a revision of the conception of intuition developed in chapter 5 of Mathematical Thought and Its Objects, in the light of criticism by Felix Mühlhölzer (Mühlhölzer 2010). Parsons revises that conception by replacing spatial intuition with temporal intuition as the foundation of intuitive knowledge in arithmetic, with reference to Brouwer. The question whether this revision has a bearing on the evidence of induction and recursion is explicitly left aside. A remaining difference between Parsons and Brouwer is that in Parsons’ exposition, external objects still have a role, so as to avoid charges of subjectivism or solipsism. A Brouwerian reply, based on a Husserlian reading of Brouwer’s writings and correspondence, would be to say that the structure of inner time awareness is identical across different minds because it is a structure of (not empirical but) transcenden-

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64 Even if one conceives of properties in terms of logically defined predicates, the intuition of ‘and so on’ is presupposed, namely in our grasp of the syntax.
tal subjectivity (Van Atten 2004, chapter 6). But Parsons does not want to commit himself to a transcendental view of consciousness.

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