THE ORBIT STRUCTURE OF DYNKIN CURVES

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To the memory of Peter Slodowy

Abstract. Let $G$ be a simple algebraic group over an algebraically closed field $k$; assume that char $k$ is zero or good for $G$. Let $B$ be the variety of Borel subgroups of $G$ and let $e \in \text{Lie} G$ be nilpotent. There is a natural action of the centralizer $C_G(e)$ of $e$ in $G$ on the Springer fibre $B_e = \{ B' \in B \mid e \in \text{Lie} B' \}$ associated to $e$. In this paper we consider the case, where $e$ lies in the subregular nilpotent orbit; in this case $B_e$ is a Dynkin curve. We give a complete description of the $C_G(e)$-orbits in $B_e$. In particular, we classify the irreducible components of $B_e$ on which $C_G(e)$ acts with finitely many orbits. In an application we obtain a classification of all subregular orbital varieties admitting a finite number of $B$-orbits for $B$ a fixed Borel subgroup of $G$.

1. Introduction

Let $G$ be a simple algebraic group over an algebraically closed field $k$; assume that the characteristic of $k$ is zero or good for $G$. Let $e \in \mathfrak{g} = \text{Lie} G$ be nilpotent and write $\mathcal{O} = G \cdot e$ for the adjoint $G$-orbit of $e$. Let $B$ be a Borel subgroup of $G$ and let $u$ be the Lie algebra of the unipotent radical of $B$; we may assume that $e \in u$. We write $B$ for the variety of Borel subgroups in $G$ and let $B_e = \{ B' \in B \mid e \in u' \}$ be the Springer fibre associated to $e$. We note that $B_e$ is connected ([30, Prop. 1]), equidimensional ([26, Prop. 1.12]) and of dimension $\frac{1}{2}(\dim C_G(e) - \text{rank} G)$ ([7, 85]).

The varieties $B_e$ occur as the fibres in Springer’s resolution $\{(e', B') \in \mathcal{N} \times B \mid e' \in u'\} \to \mathcal{N}$ of the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. This desingularization, which is the projection map onto $\mathcal{N}$, was given by T. A. Springer in [27], see also [16, 6.10]. The Springer fibres have attracted much research interest due to their importance in the representation theory of the Weyl group $W$ of $G$; more specifically, Springer constructed all irreducible representations of $W$ on the top cohomology groups of $B_e$, see [28] and [20]. Subsequently there has been much interest in the geometric structure of the varieties $B_e$, see for example [8], [15] and [31].

Since all the Borel subgroups of $G$ are conjugate, we have $B = \{ gB \mid g \in G \}$ is the variety of all $G$-conjugates of $B$. There is a well-known connection between the Springer fibre $B_e$ and the variety $\mathcal{O} \cap u$ due to N. Spaltenstein, see [26, II 1.9]. Let $\pi_1 : G \to B$ be the map $g \mapsto s^{-1}B$ and let $\pi_2 : G \to \mathcal{O}$ be the orbit map $g \mapsto g \cdot e$. There is an action of $C_G(e) \times B$ on $Y = \pi_1^{-1}(B_e) = \pi_2^{-1}(\mathcal{O} \cap u)$ by $(g, b) \cdot y = byg^{-1}$. There is a bijection between the $C_G(e)$-orbits in $B_e$ and the $(C_G(e) \times B)$-orbits in $Y$, and a bijection between the $B$-orbits in $\mathcal{O} \cap u$ and the $(C_G(e) \times B)$-orbits in $Y$. Therefore, there is a bijection between the $C_G(e)$-orbits in $B_e$ and the $B$-orbits in $\mathcal{O} \cap u$; moreover this bijection preserves the closure order on orbits. Further,
the orbits of the component group $A(e) = C_G(e)/C_G(e)^o$ of $C_G(e)$ on the set of irreducible components of $B_e$ are in bijective correspondence with the irreducible components of $\mathcal{O} \cap u$.

Due to the importance of the Springer fibres alluded to above it is a natural problem to try to understand the orbits of $C_G(e)$ in $B_e$ under the conjugation action. This is formalized in the problem posed below.

**Problem 1.1.** Determine the orbit structure for the action of $C_G(e)$ on $B_e$. In particular, determine when $C_G(e)^o$ acts on a given irreducible component of $B_e$ with a finite number of orbits and if this is not the case, then determine whether there is still a dense $C_G(e)^o$-orbit.

It follows from the above discussion that Problem 1.1 is equivalent to Problem 1.2 below.

**Problem 1.2.** Determine the orbit structure for the action of $B$ on $\mathcal{O} \cap u$. In particular, determine when $B$ acts on an irreducible component of $\mathcal{O} \cap u$ with a finite number of orbits and if this is not the case, then determine whether there is still a dense $B$-orbit.

First we record two elementary cases for Problem 1.1. If $e$ is in the regular nilpotent orbit, then $B_e$ is a point (cf. [30, 3.7 Thm. 1]) and there is nothing to show. For $e = 0$, we have $B_e = B$ is an irreducible homogeneous variety for $C_G(0) = G$.

Thanks to the equivalence between these two problems we immediately obtain a partial answer of Problem 1.1 in the case of spherical nilpotent orbits. Suppose that $\mathcal{O}$ is a spherical nilpotent orbit; that is one on which $B$ admits a dense orbit. Thanks to a fundamental result, due to M. Brion [5] and E. B. Vinberg [32] in characteristic 0 and to F. Knop [19, 2.6] in arbitrary characteristic, $B$ acts on a spherical orbit $\mathcal{O}$ with a finite number of orbits. Therefore, it is clear that there are only finitely many $B$-orbits in each irreducible component of $\mathcal{O} \cap u$. Correspondingly, in the spherical case there are only finitely many $C_G(e)$-orbits in $B_e$. All spherical nilpotent orbits for char $k = 0$ have been classified by D. I. Panyushev in [21]. This classification has recently been shown to hold for positive good characteristic in work of R. Fowler and the third author [9].

The purpose of this paper is to give complete answers to Problems 1.1 and 1.2 when $e$ lies in the subregular nilpotent orbit, see Theorems 2.2 and 2.4. When $G$ is simply laced, this classification is nicely expressed in terms of coefficients of the highest root $\rho$ of $G$. As explained below, $B_e$ is a Dynkin curve. In case $G$ is simply laced, the irreducible components of $B_e$ are indexed by the simple roots of $G$, and Theorem 2.2 says that there is a finite number of $C_G(e)$-orbits in the irreducible component corresponding to the simple root $\alpha$ if and only if the coefficient of $\alpha$ in $\rho$ is 1. For $G$ non-simply laced, the classification requires the notion of associated simply laced root systems, see §2.2.

Let $\mathcal{O}$ be a nilpotent $G$-orbit. The irreducible components of the variety $\mathcal{O} \cap u$ are called *orbital varieties* and are of interest in the representation theory of the Weyl group of $G$, see for example [16, §9.13]. They are also of interest in the study of primitive ideals of the universal enveloping algebra $U(g)$ of $g$, see for example [3] and [17]. As an application of Theorem 2.4, we classify when there is a finite number of $B$-orbits in an irreducible component of $\mathcal{O} \cap u$, when $\mathcal{O}$ is the subregular nilpotent orbit, see Theorem 4.1.

From now on let $e$ be a representative of the subregular nilpotent orbit $\mathcal{O}$ in $g$. We recall from [30, §3.10] that in this case $B_e$ is a Dynkin curve. A Dynkin curve is a non-empty, connected union of certain projective lines determined by the root system of $G$. Each of the projective lines has a type $\alpha$, where $\alpha$ is a simple root of $G$; they are denoted by $C^\alpha_\alpha$, where
Dynkin curves are of interest as they arise in resolutions of Kleinian singularities. More specifically, a Dynkin curve occurs as the exceptional divisor in the minimal resolution of a Kleinian singularity. We refer the reader to Slodowy’s book \[25, \text{specifically, a Dynkin curve occurs as the exceptional divisor in the minimal resolution of a Kleinian singularity.} \]

As a consequence of our solution to Problem 1.1, we deduce that there are finitely many $C_G(e)$-orbits in $\mathcal{B}_e$ if and only if $G$ is of type $A_n$ or $B_{\ell}$; in both of these cases there are exactly $2r-1$ orbits. Further, we see that the action of $C_G(e)$ on $\mathcal{B}_e$ is trivial if and only if $G$ is of type $E_8$; see Corollary 2.3.

Since the irreducible components of $\mathcal{B}_e$ are projective lines $C_\alpha^i \cong \mathbb{P}^1$, if $C_G(e)^\circ$ acts on $C_\alpha^i$ with a dense orbit, then its complement in $C_\alpha^i$ is closed and so it is a finite set of points. This implies that in the subregular case, $C_G(e)^\circ$ admits a dense orbit in $C_\alpha^i$ precisely when there are finitely many orbits and so these two parts of Problem 1.1 are equivalent in this instance; correspondingly, this is also the case for Problem 1.2.

We remark that Problems 1.1 and 1.2 could be stated for connected reductive $G$. We choose not to work in this generality, as one can easily reduce to the case when $G$ is simple, so there is no loss in generality. Further, we could consider the analogous problems for unipotent $G$-conjugacy classes in place of nilpotent $G$-orbits. Under our assumption that char $k$ is zero or a good prime for $G$, there exists a Springer isomorphism, see [29, III, 3.12] and [1, Cor. 9.3.4], so that these problems are equivalent.

For general references on reductive algebraic groups and nilpotent orbits the reader is referred to Borel’s book [2] and Jantzen’s monograph [16].

2. Preliminaries and statements of results

2.1. Notation. Let $G$ be a simple algebraic group over an algebraically closed field $k$; we assume throughout that the characteristic of $k$ is zero or a good prime $p$ for $G$. Let $B$ be a Borel subgroup of $G$ with unipotent radical $U$ and let $T$ be a maximal torus of $G$ contained in $B$. We write $g$, $b$, $u$ and $t$ for the Lie algebras of $G$, $B$, $U$ and $T$, respectively. We write $\mathcal{B}$ for the variety of Borel subgroups of $G$. Since all Borel subgroups of $G$ are conjugate and $B$ is self-normalizing, we may identify $\mathcal{B}$ with $G/B$.

Let $\Phi = \Phi(G,T)$ be the root system of $G$ with respect to $T$. Let $\Phi^+$ be the system of positive roots determined by $B$ and $\Pi$ the corresponding base. We denote the Dynkin diagram of $\Phi$ by $\Delta$; we identify the nodes in $\Delta$ with the elements of $\Pi$. Let $\rho$ be the highest root of $\Phi$ with respect to $\Pi$. The Cartan matrix of $\Phi$ is written as $(\langle \alpha, \beta \rangle)_{\alpha, \beta \in \Pi}$. Let $\alpha_s$ denote a fixed root of short length (if there is only one root length in $\Phi$, then all roots are short). We label the simple roots in the Dynkin diagram $\Delta$ in accordance with [4, Planches I–IX].

For a simple root $\alpha \in \Pi$, we let $e_\alpha$ be a generator of the root subspace of $g$ corresponding to $\alpha$. We write $P_\alpha$ for the minimal parabolic subgroup of $G$ corresponding to $\alpha$, i.e. $P_\alpha$ is the parabolic subgroup of $G$ generated by $B$ and the root subgroup corresponding to $-\alpha$. We write $u_\alpha$ for the Lie algebra of the unipotent radical of $P_\alpha$.

For the rest of this section we let $e$ be a representative of the subregular nilpotent orbit $O = G \cdot e$; we assume that $e \in u$. We write $C_G(e)$ for the centralizer of $e$ in $G$ and $A(e) = C_G(e)/C_G(e)^\circ$ for the component group of $C_G(e)$. We denote the Springer fibre associated to $e$ by $\mathcal{B}_e = \{ B' \in \mathcal{B} \mid e \in u' \}$.

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2.2. Dynkin curves. We recall, from [30, §3.10] (see also [25, §6.3]), that the Springer fibre $B$ is the Dynkin curve determined by the root system $\Phi$. This Dynkin curve $\mathcal{C}$ is a non-empty union of projective lines $\mathcal{C}_i^\prime$, where $\alpha \in \Pi$ is the type of $\mathcal{C}_i^\prime$ and $i = 1, \ldots, |\alpha|^2/|\alpha_s|^2$. A line of type $\alpha$ meets exactly $-\langle \alpha, \beta \rangle$ lines of type $\beta \neq \alpha$ and they intersect in a single point; lines of the same type do not meet. We write $\mathcal{C}_e$ for $\mathcal{C}_1^1$, $\mathcal{C}_e^\prime$ for $\mathcal{C}_2^\prime$ and $\mathcal{C}_e^\prime\prime$ for $\mathcal{C}_3^\prime\prime$. Under the identification of $B$ with $G/B$ lines of type $\alpha$ are of the form $xP_\alpha/B$, for some $x \in G$, see [25, §6.3].

It is straightforward to explicitly describe the Dynkin curve in case $G$ is simply laced: there is a single projective line $\mathcal{C}_\alpha$ for each $\alpha \in \Pi$; and $\mathcal{C}_\alpha$ intersects $\mathcal{C}_\beta$ if and only if $\alpha$ and $\beta$ are adjacent in $\Delta$. In other words, the Dynkin curve $\mathcal{C}$ is the diagram dual to the Dynkin diagram $\Delta$, see [25, §6.3].

In order to give an explicit description of $\mathcal{C}$ in case $G$ is non-simply laced we need to recall some further notation from [25, §6.2]. Suppose $\Delta$ is non-simply laced, then we define the associated simply laced diagram $\hat{\Delta}$ of $\Delta$ and the associated symmetry group $\Gamma(\Delta)$ by Table 1 below; in this table $S_n$ denotes the symmetric group of degree $n$.

| $\Delta$ | $B_r$ | $C_r$ | $F_4$ | $G_2$ |
|----------|-------|-------|-------|-------|
| $\hat{\Delta}$ | $A_{2r-1}$ | $D_{r+1}$ | $E_6$ | $D_4$ |
| $\Gamma(\Delta)$ | $S_2$ | $S_2$ | $S_2$ | $S_3$ |

Table 1. The associated Dynkin diagrams

We write $\hat{\Phi}$ for the root system and $\hat{\Pi}$ for the base of $\hat{\Phi}$ corresponding to $\hat{\Delta}$, and $\hat{\rho}$ for the highest root of $\hat{\Phi}$ with respect to $\hat{\Pi}$. As explained in [25, §6.2], there is a unique faithful action of $\Gamma(\Delta)$ on $\hat{\Delta}$ and we can regard $\Delta$ as the quotient of $\hat{\Delta}$ by this action; we refer the reader to [25, §6.2] for a description of the $\Gamma(\Delta)$-orbit in $\hat{\Pi}$ corresponding to a simple root in $\Pi$.

Suppose $G$ is non-simply laced. Then the Dynkin curve $\mathcal{C}$ can be explicitly described as follows. Let $\tilde{\mathcal{C}}$ be the Dynkin curve determined by the simply laced root system $\hat{\Phi}$. As a variety $\mathcal{C}$ is the same as $\tilde{\mathcal{C}}$. A projective line in $\mathcal{C}$ is of type $\alpha$ if its type in $\tilde{\mathcal{C}}$ is in the $\Gamma(\Delta)$-orbit corresponding to $\alpha$.

The action of $G$ on $B$ restricts to an action of $C_G(e)$ on $B_e$. This in turn induces an action of the component group $A(e)$ of $C_G(e)$ on the set of lines of type $\alpha$ in $B_e$. The group $\Gamma(\Delta)$ is precisely the component group $A(e)$ of $C_G(e)$; further, the action of $\Gamma(\Delta)$ on $\hat{\Delta}$ corresponds in a natural way to the action of $A(e)$ on the lines in $\mathcal{C}$, see [25, Prop. 7.5]. More precisely, there is an isomorphism $\phi: \Gamma(\hat{\Delta}) \to A(e)$ such that for $\beta \in \hat{\Phi}$ and $\mathcal{C}_\beta$ the line of type $\beta$ in $\tilde{\mathcal{C}}$, and $g \in \Gamma(\hat{\Delta})$, we have $\phi(g) \cdot \mathcal{C}_\beta = \mathcal{C}_{\hat{\rho} \beta}$. In particular, the action of $A(e)$ on lines of type $\alpha$ is transitive.

We illustrate the above discussion with an example, cf. [25, 6.3].

**Example 2.1.** Consider the case when $G$ is of type $G_2$. Let $\alpha = \alpha_1$ and $\beta = \alpha_2$ be the short and long simple roots of $\Pi$, respectively. The Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. Therefore, $\mathcal{C}$ is of the form $\mathcal{C} = \mathcal{C}_\alpha \cup \mathcal{C}_\beta \cup \mathcal{C}_\beta^\prime \cup \mathcal{C}_\beta^\prime\prime$ and the intersection pattern is illustrated below.
As follows from the discussion above, as an algebraic variety, the Dynkin curve for $G_2$ is the same as that for $C_3$ and $D_4$; although the types of the lines differ.

For fixed $\alpha \in \Pi$, let $\Upsilon_\alpha$ be the partition of $C_\alpha$ afforded by the points in the intersections $C_\alpha \cap C_\beta$, $C_\alpha \cap C'_\beta$, $C_\alpha \cap C''_\beta$ for all $\beta \in \Pi \setminus \{\alpha\}$, and the complement in $C_\alpha$ of the union of all these points; $\Upsilon'_\alpha$ and $\Upsilon''_\alpha$ are defined analogously. We observe that the partition of all of $B_e$ that we obtain by taking the union of all the $\Upsilon_\alpha$, $\Upsilon'_\alpha$ and $\Upsilon''_\alpha$ for all $\alpha \in \Pi$ is related to the component configuration of $B_e$ as defined in [6, 1.4].

We can now state the main theorem of this article; it is proved in the next section. For $\alpha \in \Pi$, we let $\hat{\alpha} \in \hat{\Pi}$ be a representative of the $\Gamma(\Delta)$-orbit corresponding to $\alpha$.

**Theorem 2.2.** Let $e$ be a representative of the subregular nilpotent $G$-orbit in $\mathfrak{g}$. Then the following hold:

(a) There is a finite number of $C_G(e)^o$-orbits in $C_\alpha$ if and only if
   (i) the coefficient of $\alpha$ in $\rho$ is 1, for $G$ simply laced;
   (ii) the coefficient of $\hat{\alpha}$ in $\hat{\rho}$ is 1, for $G$ non-simply laced.

(b) If $C_G(e)^o$ acts on $C_\alpha$ with a finite number of orbits, then each component of the partition $\Upsilon_\alpha$ of $C_\alpha$ is a single $C_G(e)^o$-orbit. Otherwise the action of $C_G(e)^o$ on $C_\alpha$ is trivial.

The following is immediate from Theorem 2.2.

**Corollary 2.3.** Let $e$ be a representative of the subregular nilpotent $G$-orbit in $\mathfrak{g}$. Then the following hold:

(i) There are finitely many $C_G(e)$-orbits in $B_e$ if and only if $G$ is of type $A_r$ or $B_r$. Moreover, in both of these cases the number of $C_G(e)$-orbits is $2r - 1$.

(ii) The action of $C_G(e)$ on $B_e$ is trivial if and only if $G$ is of type $E_8$.

2.3. The varieties $\mathcal{O} \cap \mathfrak{u}$. We now discuss the varieties $\mathcal{O} \cap \mathfrak{u}$ for $\mathcal{O} = G \cdot e$ the subregular nilpotent orbit.

The irreducible components of $\mathcal{O} \cap \mathfrak{u}$ are the intersections $\mathcal{O} \cap u_\alpha$ for $\alpha \in \Pi$. Under the correspondence between the $C_G(e)$-orbits in $B_e$ and the $B$-orbits in $\mathcal{O} \cap \mathfrak{u}$, the $C_G(e)$-orbits in lines of type $\alpha$ correspond to the $B$-orbits in $\mathcal{O} \cap u_\alpha$, i.e. on the level of correspondence of irreducible components: the $A(e)$-orbit of lines of type $\alpha$ corresponds to $\mathcal{O} \cap u_\alpha$. This allows one to derive from the knowledge of the intersection pattern of projective lines in the Dynkin curve $\mathcal{C}$ that the intersection of $\mathcal{O}$ with $u_\alpha \cap \mathfrak{u}_\beta$ is either empty or a single $B$-orbit. More precisely, $\mathcal{O} \cap (u_\alpha \cap \mathfrak{u}_\beta) \neq \emptyset$, if and only if $\langle \alpha, \beta \rangle \neq 0$, and in this case this intersection
is a single $B$-orbit. This fact can also be deduced from Richardson's dense orbit theorem ([23]), as explained in the next paragraph.

It is well-known that the subregular class $O$ is the Richardson class of $P_\alpha$ for each $\alpha \in \Pi$, i.e. $O$ meets each $u_\alpha$ in an open dense subvariety and $O \cap u_\alpha$ is a single $P_\alpha$-orbit. Observe that if $\langle \alpha, \beta \rangle = 0$, then $O \cap \langle u_\alpha, u_\beta \rangle = \emptyset$, as in this case $u_\alpha \cap u_\beta$ is the Lie algebra of the unipotent radical of a larger parabolic subgroup of $G$, which has a different Richardson class. It follows from Lemma 3.4 later in this paper that $O \cap \langle u_\alpha, u_\beta \rangle$ cannot be at most one $B$-orbit; and in case $\langle \alpha, \beta \rangle \neq 0$ one can show $O \cap \langle u_\alpha, u_\beta \rangle \neq \emptyset$ by using the root subgroups corresponding to $\beta$ and $-\beta$ to find a representative of the Richardson orbit in $u_\beta$ for which the coefficient of $e_\alpha$ is zero.

Let $\Xi_\alpha$ be the partition of $O \cap u_\alpha$ afforded by all the subvarieties $O \cap \langle u_\alpha, u_\beta \rangle$ for all $\beta \in \Pi$ with $\langle \alpha, \beta \rangle \neq 0$ along with the complement in $O \cap u_\alpha$ of the union of all these.

Now we state the counterpart to Theorem 2.2 in the context of the action of $B$ on $O \cap u$; due to the discussion in the introduction, these two theorems are equivalent. In the next section we prove both theorems by proving complementary parts of each.

**Theorem 2.4.** Let $O$ be the subregular nilpotent $G$-orbit in $g$. Then the following hold:

(a) There is a finite number of $B$-orbits in $O \cap u_\alpha$ if and only if

(i) the coefficient of $\alpha$ in $\rho$ is 1, for $G$ simply laced;
(ii) the coefficient of $\check{\alpha}$ in $\check{\rho}$ is 1, for $G$ non-simply laced.

(b) If $B$ acts on $O \cap u_\alpha$ with a finite number of orbits, then each component of the partition $\Xi_\alpha$ of $O \cap u_\alpha$ is a single $B$-orbit.

We may now state an analogue of Corollary 2.3(i) in the setting of $B$ acting on $O \cap u$.

**Corollary 2.5.** Let $O$ be the subregular nilpotent $G$-orbit in $g$. Then there are finitely many $B$-orbits in $O \cap u$ if and only if $G$ is of type $A_r$ or $B_r$. Moreover, in both of these cases the number of $B$-orbits is $2r - 1$.

### 3. Proofs of Theorems 2.2 and 2.4

Before we begin the proof of Theorems 2.2 and 2.4, we state the following well-known lemma about the action of tori and unipotent groups on projective lines. It can be deduced easily from [2, Prop. 10.8], which says that any action of an algebraic group $H$ on $\mathbb{P}^1$ is given by a homomorphism $H \to \text{PGL}_2(k)$.

**Lemma 3.1.** Let $H$ be a connected algebraic group acting non-trivially on $\mathbb{P}^1$.

(i) Suppose $H$ is unipotent. Then $H$ has two orbits in $\mathbb{P}^1$: one is a fixed point and the other is the complement of the fixed point.

(ii) Suppose $H$ is a torus. Then $H$ has three orbits in $\mathbb{P}^1$: two are fixed points and the other is their complement.

We recall that a nilpotent element $e \in g$ is called distinguished if $C_g(e)^0$ is unipotent, see for example [16, §4.1]. Our next lemma follows directly from [25, §7.5 Lem. 4].

**Lemma 3.2.** The subregular nilpotent class $O = G \cdot e$ of $G$ is distinguished unless $G$ is of type $A_r$ or $B_r$, in which case the Levi factor of $C_g(e)^0$ is a one-dimensional torus.

Armed with Lemmas 3.1 and 3.2, we can deduce that the action of $C_g(e)^0$ on $C_\alpha$ ($\alpha \in \Pi$) is trivial in many cases. These are the instances when $e$ is distinguished and $C_\alpha$ meets at
least two other lines. Since the connected centralizer $C_G(e)^\circ$ fixes these intersection points and is unipotent, it has to act trivially on all of $C_\alpha$, by Lemma 3.1(i).

This leaves us to consider the following cases: $G$ is of type $A_r$ and $B_r$ and $\alpha$ is any simple root; or $G$ is of any type and $\alpha$ is an end-node of $\Delta$.

First, we use computations explained in [10] to complete the proof of Theorem 2.4 (equivalently Theorem 2.2) for $G$ of exceptional type. It follows from the results of these computations that $B$ acts on $O \cap u_\alpha$ with a dense orbit precisely in the cases stated in Theorem 2.4. This in turn implies that $C_G(e)^\circ$ admits a dense orbit in $C_\alpha$ in the stated cases. Finally, we use the fact that $C_G(e)^\circ$ is unipotent and Lemma 3.1(i) to deduce that $C_G(e)^\circ$ acts on $C_\alpha$ with two orbits if it acts with a dense orbit. This completes the discussion of the exceptional cases.

Next we complete the proof of Theorems 2.2 and 2.4 for $G$ of classical type. First, we explain why we can reduce to considering groups of matrices.

Let $\phi : G \to \tilde{G}$ be an isogeny. Then it is clear that Theorems 2.2 and 2.4 hold for $G$ if and only if they hold for $\tilde{G}$. Therefore, we may assume $G$ is one of $SL_n(k)$, $SO_{2n+1}(k)$, $Sp_{2n}(k)$ or $SO_{2n}(k)$. In these cases the nilpotent $G$-orbits are parameterized by the partitions given by the Jordan normal form (with some exceptions in case $G = SO_{2n}(k)$ and $n$ is even, which are not relevant for our purposes here), see for example [16, Chapter 1]. We require the following well-known lemma, giving the partitions corresponding to subregular elements, see for example [13, 4.5.6, Cor. 2] and [29, IV 2.33].

**Lemma 3.3.** The partition corresponding to the subregular nilpotent class is:

(i) $(n - 1, 1)$ for $G = SL_n(k)$;

(ii) $(2n - 1, 1, 1)$ for $G = SO_{2n+1}(k)$;

(iii) $(2n - 2, 2)$ for $G = Sp_{2n}(k)$;

(iv) $(2n - 3, 3)$ for $G = SO_{2n}(k)$.

The following lemma, which holds for an arbitrary simple algebraic group $G$, is used extensively to prove existence of a dense $B$-orbit in $O \cap u_\alpha$ in the classical cases. It can be proved using a simple dimension argument using the two equalities: $\dim P_\alpha \cdot e = \dim u_\alpha$ and $\dim B = \dim P_\alpha - 1$; we omit the details.

**Lemma 3.4.** Let $\alpha \in \Pi$ and suppose $e \in u_\alpha$ is subregular. Then $\dim B \cdot e = \dim u_\alpha$ or $\dim u_\alpha - 1$. Thus if $\dim C_B(e) < \dim C_{P_\alpha}(e)$, then $B \cdot e$ is dense in $u_\alpha$.

We now prove that there is a dense $B$-orbit in $u_\alpha$ ($\alpha \in \Pi$) for the cases stated in Theorem 2.4.

First we consider the case $G = SL_n(k)$. We note that in this case the existence of a dense $B$-orbit in $u_\alpha$ can be deduced directly from the main theorem in [12], but we give a more elementary proof here. We take $T$ to be the maximal torus of diagonal matrices in $G$ and $B$ to be the Borel subgroup of upper triangular matrices in $G$. We write $e_{i,j}$ for the elementary matrix with $(i, j)$th entry 1 and all other entries 0. Given a root $\alpha \in \Phi$, we let $x_\alpha : k \to G$ be a parametrization of the corresponding root subgroup of $G$ and we write $e_\alpha$ for a generator of the root subspace, so $e_\alpha = e_{i,j}$ for some $i, j$. 


Suppose \( \alpha = \alpha_i \), where \( i \neq 1, n - 1 \). Consider
\[
e = \left( \sum_{j=1, j \neq i}^{n-1} e_{j,j+1} \right) + e_{i,i+2} = \left( \sum_{j=1, j \neq i}^{n-1} e_{\alpha_j} \right) + e_{\alpha_i + \alpha_{i+1}}.
\]
One can check that \( e^{n-2} \neq 0 \) and \( e^{n-1} = 0 \). So, using Lemma 3.3, we deduce that \( e \) lies in the subregular nilpotent orbit. Consider \( x_{-\alpha}(s) \cdot e \). One can see that there exists \( t_s \in T \) such that 
\[
t_s x_{-\alpha}(s) \cdot e = e
\]
for all but one value of \( s \). This implies that \( \dim C_B(e) < \dim C_{P_\alpha}(e) \).
So, by Lemma 3.4, we have that \( B \cdot e \) is dense in \( u_\alpha \). From the equivalence of Theorems 2.2 and 2.4, we deduce there is a dense \( C_B(e) \)-orbit in \( C_\alpha \). It now follows from Lemma 3.1 that there are precisely three \( C_B(e) \)-orbits. This completes the proof of Theorem 2.2, and

Therefore, we have
\[
u_\alpha \setminus (B \cdot e) = \bigcup_{\alpha \neq \beta} u_\alpha \cap u_\beta,
\]
where the union is taken over all simple roots \( \beta \neq \alpha \).

If \( \beta \neq \alpha_2 \), then \( (G \cdot e) \cap (u_\alpha \cap u_\beta) = \emptyset \), as in this case \( u_\alpha \cap u_\beta \) is the Lie algebra of the unipotent radical of a parabolic subgroup bigger than \( P_\alpha \). Using Lemma 3.4, we see that \( O \cap (u_\alpha \cap u_\beta) \) is a single \( B \)-orbit when \( \beta = \alpha_2 \). So there are precisely two \( B \)-orbits in \( O \cap u_\alpha \).

This completes the proof of Theorem 2.4 in this case.

We now consider the other classical groups; the proofs in these cases are similar to the one for \( SL_n(k) \). We just give representatives of the dense \( B \)-orbit in \( u_\alpha \) in the cases stated in Theorem 2.4; one can prove that the representatives do indeed give a dense orbit and that there is the right number of \( B \)-orbits in \( O \cap u_\alpha \) using arguments similar to those for the \( SL_n(k) \)-case, so we omit the details.

Let \( G \) be one of \( SO_{2n+1}(k) \), \( Sp_{2n}(k) \) or \( SO_{2n}(k) \). Let \( V \) be the natural \( G \)-module with standard (ordered) basis \( v_1, \ldots , v_n, v_0, v_{-n}, \ldots , v_{-1} \) and \( G \)-invariant symmetric or alternating bilinear form \( (, ) \) defined by \( (v_0, v_i) = (v_0, v_i) = 0, (v_0, v_0) = 1, (v_i, v_j) = (v_{-i}, v_{-j}) = 0 \) and \( \delta_{i,j} = \delta_{i,j} \) for \( i,j = 1, \ldots , n \), where we omit \( v_0 \) everywhere if \( G \) is \( Sp_{2n}(k) \) or \( SO_{2n}(k) \).

We take \( T \) to be the maximal torus of diagonal matrices in \( G \) and \( B \) to be the Borel subgroup of upper triangular matrices in \( G \).

First we consider \( G = SO_{2n+1}(k) \) and the case where \( \alpha = \alpha_1 \). Then
\[
e = \left( \sum_{j=2}^{n-1} e_{j,j+1} - e_{-(j+1),-j} \right) + (e_{n,0} - e_{0,-n}) = \sum_{j=2}^{n} e_{\alpha_j}.
\]
is a representative of the dense $B$-orbit in $u_\alpha$. For $\alpha = \alpha_i$ for $i = 2, \ldots, n - 1$,
\[
e = \left( \sum_{j=1, j \neq i}^{n-1} e_{j,j+1} - e_{-(j+1),-j} \right) + (e_{n,0} - e_{0,-n}) + (e_{i,i+2} - e_{-(i+2),-i})
\]
\[
= \left( \sum_{j=1, j \neq i}^{n} e_{\alpha_j} \right) + e_{\alpha_i + \alpha_i + 1}
\]
is a representative of the dense $B$-orbit in $u_\alpha$. In case $\alpha = \alpha_n$, a representative of the dense $B$-orbit in $u_\alpha$ is
\[
e = \left( \sum_{j=1}^{n-1} e_{j,j+1} - e_{-(j+1),-j} \right) + (e_{n-1,-n} - e_{n,-n-1}) = \left( \sum_{j=1}^{n-1} e_{\alpha_j} \right) + e_{\alpha_{n-1} + 2\alpha_n}.
\]

Now consider $G = \text{Sp}_{2n}(k)$. For $\alpha = \alpha_1$, we take
\[
e = \left( \sum_{j=2}^{n-1} e_{j,j+1} - e_{-(j+1),-j} \right) + e_{n,-n} + e_{1,-1} = \left( \sum_{j=2}^{n} e_{\alpha_j} \right) + e_{\rho};
\]
and for $\alpha = \alpha_n$ we take
\[
e = \left( \sum_{j=1}^{n-1} e_{j,j+1} - e_{-(j+1),-j} \right) + e_{n-1,-(n-1)} = \left( \sum_{j=1}^{n} e_{\alpha_j} \right) + e_{2\alpha_{n-1} + \alpha_n}.
\]

Finally, we consider the case $G = \text{SO}_{2n}(k)$. For $\alpha = \alpha_1$,
\[
e = \left( \sum_{j=2}^{n-1} e_{j,j+1} - e_{-(j+1),-j} \right) + (e_{n-1,-n} - e_{n,-(n-1)}) + (e_{1,-n} - e_{n,-1})
\]
\[
= \left( \sum_{j=2}^{n} e_{\alpha_j} \right) + e_{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2}) + \alpha_n}
\]
is a representative of the dense $B$-orbit. We finish with the case $\alpha = \alpha_{n-1}$; the case $\alpha = \alpha_n$ is equivalent, by symmetry. A representative of the dense $B$-orbit in $u_\alpha$ is
\[
e = \left( \sum_{j=1}^{n-2} e_{j,j+1} - e_{-(j+1),-j} \right) + (e_{n-1,-n} - e_{n,-(n-1)}) + (e_{n-2,-(n-1)} - e_{n-1,-(n-2)})
\]
\[
= \left( \sum_{j=1}^{n-2} e_{\alpha_j} \right) + e_{\alpha_n} + e_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n}.
\]

This completes the proof of Theorem 2.4 and so of Theorem 2.2.

4. Subregular Orbital Varieties

In this section we apply Theorem 2.4 to classify all subregular orbital varieties admitting a finite number of $B$-orbits. Let $O$ be the subregular nilpotent $G$-orbit in $g$. Since $O$ is the Richardson class of all semisimple rank 1 parabolic subgroups $P_\alpha$ of $G$, it follows that the orbital varieties of $O \cap u$ coincide with the nilradicals $u_\alpha$ of the $P_\alpha$ ($\alpha \in \Pi$). Observe that Theorem 2.4 gives a classification of all instances when $B$ acts on $u_\alpha$ with a dense orbit: for,
this is equivalent to $B$ admitting a dense orbit in $\mathcal{O} \cap u_\alpha$ which in turn is equivalent to $B$ acting on $\mathcal{O} \cap u_\alpha$ with a finite number of orbits, as explained at the end of the introduction. However, in contrast to Problems 1.1 and 1.2 in the subregular case, the questions of finiteness for the number of orbits versus that of the existence of a dense orbit are not equivalent for the action of $B$ on the orbital varieties $u_\alpha$ in $\mathcal{O} \cap u$.

**Theorem 4.1.** There is a finite number of $B$-orbits in $u_\alpha$ if and only if

(i) $G$ is of type $A_r$ for $r \leq 4$; or $G$ is of type $A_5$ and $\alpha \in \{\alpha_1, \alpha_3, \alpha_5\}$;
(ii) $G$ is of type $B_2$; or $G$ is of type $B_3$ and $\alpha = \alpha_2$;
(iii) $G$ is of type $C_2$; or $G$ is of type $C_3$ and $\alpha \in \{\alpha_1, \alpha_3\}$;
(iv) $G$ is of type $G_2$ and $\alpha = \alpha_2$.

**Proof.** It was shown by Kashin in [18] that $B$ acts on $u$ with a finite number of orbits if and only if $G$ is of type $A_r$ for $r \leq 4$ or $B_2$. So clearly, $B$ acts on each $u_\alpha$ with a finite number of orbits in these cases. This classification was extended in [22, Cor. 1.4] to the case of minimal parabolic subgroups $P_\alpha$ of $G$. The results in [22] are stated and proved assuming that char $k = 0$; they are also valid provided char $k$ is a good prime for $G$, cf. [24]. Accordingly, $P_\alpha$ acts on $u_\alpha$ with a finite number of orbits if and only if $G$ is of type $A_r$ for $r \leq 5$, $B_r$ for $r \leq 3$, $C_r$ for $r \leq 3$, $D_4$, or $G_2$. Obviously, if $P_\alpha$ already acts on $u_\alpha$ with infinitely many orbits, so does $B$. Clearly, if $B$ acts on $u_\alpha$ with a finite number of orbits, then $B$ has only finitely many orbits in $\mathcal{O} \cap u_\alpha$. We thus infer from Theorem 2.4(a) and the classification results from [18] and [22] that, given $B$ acts on all of $u$ with infinitely many orbits, the only cases that need consideration for $B$ acting on $u_\alpha$ are as follows: each simple root for $A_5$ and $B_3$: $\alpha_1$ and $\alpha_3$ for $C_3$; each end-node simple root for $D_4$, and $\alpha_2$ for $G_2$.

In order to show that $B$ acts with an infinite number of orbits in several of these cases, we employ a method already used in [18] and [22] which we recall now for convenience. Let $n$ be a $B$-submodule of $u$, i.e. an ideal of $b$ in $u$. Let $N$ be the connected unipotent normal subgroup of $B$ with Lie algebra $n$. The action of $B$ on $n$ induces an action of $B$ on the quotient $n/\left[n, n\right]$ of $n$ by its commutator subalgebra, and this latter action of $B$ factors through $B/N$. Thus if $\dim B/N < \dim n/\left[n, n\right]$, then $B$ acts on $n$ with an infinite number of orbits. The idea in all the cases we consider below is to exhibit a suitable $B$-submodule $n$ of $u$ which satisfies this inequality. We collect the relevant information in Table 2 below where we list $n$ by means of the simple roots $\beta$ so that $n$ is generated by the root spaces $g_{\beta}$ as a $B$-module. We omit the details.

| Type of $G$ | $n$     | $\dim B/N$ | $\dim n/\left[n, n\right]$ |
|------------|---------|-------------|----------------------------|
| $A_5$      | $\alpha_1, \alpha_3, \alpha_5$ | 7            | 8                          |
| $B_3$      | $\alpha_2$         | 5            | 6                          |
| $C_3$      | $\alpha_1, \alpha_3$ | 4            | 5                          |
| $D_4$      | $\alpha_2$         | 7            | 8                          |
| $G_2$      | $\alpha_2$         | 3            | 4                          |

**Table 2.** Some ideals $n$ in $b$

Note that if $\alpha$ is a simple root of $G$ not among the roots of the generating root spaces for $n$ from Table 2, then $n \subseteq u_\alpha$. It follows from the discussion above and the information
given in Table 2 that then \( B \) acts on \( u_\alpha \) with an infinite number of orbits. This immediately rules out the remaining cases for \( D_4 \), as well as the cases \( \alpha_2, \alpha_4 \) for \( A_5 \) and \( \alpha_1, \alpha_3 \) for \( B_3 \). Consequently, it remains to be checked that each of the cases for \( A_5, B_3, C_3 \) and \( G_2 \) listed in the statement is indeed an instance when \( B \) acts on \( u_\alpha \) with a finite number of orbits. It turns out that for \( G \) of type \( A_5, B_3, C_3, G_2 \) there is a unique one-parameter family of \( B \)-orbits in \( u \) and moreover that this family is dense precisely in the ideal \( n \) given in Table 2; see Tables 1 and 2 in [6], see also the proof of [14, Prop. 4.8] for type \( A_5 \). It thus follows that \( B \) acts with a finite number of orbits in \( u_\alpha \) for the remaining instances listed in the statement, as then \( u_\alpha \) does not meet this infinite family of orbits in \( n \).

\[ \square \]

**Remark 4.2.** In each of the finite instances of Theorem 4.1 the number of \( B \)-orbits in \( u_\alpha \) can be determined from the computation of a list of representatives of the \( B \)-orbits in all of \( u \) by means of the algorithm from [6]. Apart from the \( A_5 \) case, the list of \( B \)-orbits is given in [6, Table 2]; the list for \( G \) of type \( A_5 \) was made available to us by W. Hesselink. If \( G \) is of type \( B_3 \), then there are 23 \( B \)-orbits in \( u_{\alpha_2} \), if \( G \) is of type \( C_3 \), there are 24 \( B \)-orbits in \( u_{\alpha_1} \) and 21 \( B \)-orbits in \( u_{\alpha_3} \), while if \( G \) is of type \( G_2 \), there are 8 \( B \)-orbits in \( u_{\alpha_2} \). For \( G \) of type \( A_5 \), there are 185 \( B \)-orbits on \( u_{\alpha_1} \) and 200 \( B \)-orbits on \( u_{\alpha_3} \).

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