On Crossing Changes for Surface-Knots

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Abstract. In this paper, we discuss the crossing change operation along exchangeable double curves of a surface-knot diagram. We show that under certain condition, a finite sequence of Roseman moves preserves the property of those exchangeable double curves. As an application for this result, we also define a numerical invariant for a set of surface-knots called $du$-exchangeable set.

1. Introduction

A surface-knot $F$ is an orientable connected, closed surface smoothly embedded in the Euclidean 4-space $\mathbb{R}^4$. It is is called an unknotted or trivial if it is isotopic to the boundary of a handlebody embedded in $\mathbb{R}^3 \times \{0\}$. To describe a surface-knot $F$, we consider the image of the surface-knot under the orthogonal projection $p : \mathbb{R}^4 \to \mathbb{R}^3$ that is defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. We may slightly perturb $F$ by an isotopy so that its projection image in $\mathbb{R}^3$ is a generic surface [1]; that is, its singularity set consists of at most three types: double points, isolated triple points or isolated branch points. The surface-knot diagram, or simply the diagram of a surface-knot $F$, denoted by $D$, is the generic projection image of $F$ in 3-space with crossing information.

The crossing change in classical knot theory, that is defined by exchanging an upper arc and a lower arc at a crossing point in a knot diagram, can be generalized to theory of surface-knots. When the crossing change operation is defined, it is natural to consider the following problem.

Problem 1. Let $p(F)$ be the projection of a surface-knot $F$ in 3-space. Is there an unknotted surface-knot $F_0$ such that $p(F) = p(F_0)$?

In other words, can we transform any surface-knot diagram into a diagram of

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the trivial surface-knot by making suitable crossing changes? The crossing change
operation is called an unknotted operation for a surface-knot $F$ if an unknotted
surface-knot is obtained from $F$ by a finite sequence of crossing changes. Although
some projections of surface-knots are known to be projections of an unknotted
surface-knot [11], it is still an open problem whether all projections satisfy this
property; that is whether the crossing change is an unknotted operation for any
surface-knot. The answer for the problem is affirmative in the case of classical knot
theory and this fact is used to compute the invariants that are defined by skein
relations, e.g., [8]. Moreover an invariant, the unknotting number is defined.

When we explore into the literature related to Problem 1, we see that some
partial results were found. A special kind of crossing changes was defined [16],
but this kind can not be applied to any surface-knot diagram. A special surface
braid diagram is shown to be unknotted by crossing changes ([4], [6]). K. Tanaka
in [13] proved that any pseudo-ribbon surface-knot diagram can be deformed by
crossing changes into a diagram with knot group isomorphic to $Z$. K. Yoshida [17]
verified that the projection $p$ of an $S^n$-knot ($n \neq 3, 4$) is the projection of a trivial
$n$-knot provided that the singularity set of $p$ consists of only double points and is
homeomorphic to a disjoint union of $(n-1)$-spheres. E. Ogasa [9] proved that there
exists a projection of an $S^n$-knot which is not the projection of any trivial knot, for
$n \geq 3$.

In this paper, we define a set of surface-knots called the $du$-exchangeable set
that contains $du$-exchangeable surface-knots. We show that for a $du$-exchangeable
surface-knot, there exists a finite sequence of surface-knot diagrams each of which
can be unknotted by the crossing changes. As an application for this result, we also
define a numerical invariant for the $du$-exchangeable set.

The rest of the paper is organized as follows. In section 2, we give some basics
about surface-knots to facilitate later discussions. In section 3, we recall Roseman
moves. In section 4, we define a finite sequence for a surface-knot represented by its
surface-knot diagrams, called a $t$-descendent sequence. In section 5, we review the
crossing change operation and section 6 is devoted to stating and proving the main
result. Finally, section 7 introduces an invariant for a set of surface-knots called the
$du$-exchangeable set.

2. Preliminaries

Let $F$ be a surface-knot and let $h : \mathbb{R}^4 \to \mathbb{R}$ be the height function,
h$(x_1, x_2, x_3, x_4) = x_4$. Let $\text{cl}(S)$ stand for the closure of the set $S$.

The closure of the set

$$\{x \in F : \# p^{-1}(p(x)) \geq 2\}$$

can be regarded as the image of compact 1-dimensional manifold immersed into $F$.
It is divided into two families $S_a = \{s^a_1, \ldots, s^a_n\}$ and $S_b = \{s^b_1, \ldots, s^b_n\}$ of immersed
closed intervals or simple closed curves in $F$ such that $h(x) > h(x')$ holds for any
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\[ x \in s_i^a \text{ and } x' \in s_i^b \quad (i = 1, 2, \ldots, n) \]

Let \( S_a = \bigcup_{i=1}^{n} \text{cl}(s_i^a) \) and let \( S_b = \bigcup_{i=1}^{n} \text{cl}(s_i^b) \).

The union \( S_a \cup S_b \) is called the double decker set (see [1] and [2] for details).

Double points in a generic projection image of a surface-knot in 3-space form a disjoint union of 1-manifolds that may appear as open arcs or simple closed curves. We say that such an open arc is called a double edge. Let \( D \) be a surface-knot diagram of a surface-knot \( F \). Let \( e_1, \ldots, e_n, e_{n+1} = e_1 \) be double edges and let \( T_1, \ldots, T_n, T_{n+1} = T_1 \) be triple points of \( D \). For \( i = 1, \ldots, n \), assume that the boundary points of each of \( e_i \) are \( T_i \) and \( T_{i+1} \) and that \( e_i \) and \( e_{i+1} \) are in opposition to each other at \( T_{i+1} \). The closure of the union \( e_1 \cup e_2 \cup \ldots \cup e_n \) forms a circle component called a double point circle of the diagram.

Similarly, let \( e_1, \ldots, e_n \) be double edges, \( T_1, \ldots, T_{n-1} \) be triple points of \( D \) and suppose \( b_1 \) and \( b_n \) are branch points of \( D \). Assume the boundary points of each of \( e_1 \) are the triple point \( T_1 \) and the branch point \( b_1 \). Assume also that the double edge \( e_n \) is bounded by \( T_{n-1} \) and \( b_n \). For \( i = 2, \ldots, n - 1 \), the double edge \( e_i \) is bounded by \( T_{i-1} \) and \( T_i \). If \( e_i \) and \( e_{i+1} \) are in opposition to each other at \( T_i \) \( (1, 2, \ldots, n - 1) \), then the closure of the union \( e_1 \cup e_2 \cup \ldots \cup e_n \) forms an arc component called a double point interval of the diagram. By a double curve, we refer to a double point circle or a double point interval. Let \( T \) be a triple point of \( D \) and \( B(T) \) a 3-ball neighbourhood of \( T \). The intersection of \( B(T) \) and the double edges consists of six short arcs. We call them the branches of double edges at \( T \). A branch at \( T \) is called a b/m-, b/t- or m/t-branch if it is the intersection between bottom and middle or bottom and top or middle and top sheets, respectively.

3. Roseman Moves

D. Roseman introduced analogues of the Reidemeister moves as local moves to surface-knot diagrams. Let \( D \) and \( D' \) be surface-knot diagrams of \( F \) and \( F' \), respectively. It is known that \( F \) and \( F' \) are equivalent if and only if there exists a finite sequence of surface-knot diagrams \( D = D_0 \rightarrow D_1 \rightarrow \ldots \rightarrow D_n = D' \) such that for all \( i = 0, 1, \ldots, n - 1 \), \( D_i \) and \( D_{i+1} \) differ by one of seven Roseman moves [10]. We will write \( D \sim D' \) to indicate that \( D \) and \( D' \) present the same surface-knot.

T. Yashiro [14] showed that the Roseman’s seven moves can be described by six moves depicted in Figure 1 (see also [7]). We call these moves also Roseman moves. Each move from left to right is denoted by \( R-X^+ \) and right to left by \( R-X^- \) except \( R-6 \). The move \( R-X^- \) is called the reverse of \( R-X^+ \). Note that the information on height has not been specified in Figure 1.
Here we describe the $R$-6 move. Type $R$-6 move consists of deformations of two disks $P_1$ and $P_2$ in a surface-knot diagram $D$. The disk $P_1$ forms a saddle point and the other disk $P_2$ passing through the saddle point of $P_1$ in a direction perpendicular to the tangent plane at the saddle point. The two disks $P_1$ and $P_2$ intersect at two double edges, say $e_1$ from $a_1$ to $a_2$ and $e_2$ from $b_1$ to $b_2$, where $a_i$ and $b_i$ ($i = 1, 2$) are boundary points of $P_2$. We give an order to these boundary points such that the four points are ordered as $\{a_1, a_2, b_1, b_2\}$ with respect to the orientation of the boundary. As $P_2$ passes through the saddle point of $P_1$, the two double edges $e_1$ and $e_2$ get closer and join at the middle point of each double edge. As a result, the new double edges $e'_1$ from $a_1$ to $b_2$ and $e'_2$ from $a_2$ to $b_1$ appear.

Assume that the $R$-6 move is applied to the pair of disks $P_1$ and $P_2$. In the notation above, let $a$ be the middle point of the double segment $e_1$ and let $b$ be the middle point of the double segment $e_2$. Then we can find a disk $P$, in a closure of one of complementary open regions of $D$, satisfying the following properties:

1. $\partial P = l_1 \cup l_2$, where $l_1$ and $l_2$ are two simple arcs in $D$, each of which is
terminated by $a$ and $b$;

(2) One of $l_i$ ($i = 1, 2$) is on $P_1$ and the other one is on $P_2$;

(3) The pre-images of $l_i$ ($i = 1, 2$) do not meet with $S_a \cup S_b$ other than their end points.

(4) The two endpoints of one of the pre-images $l_i$ ($i = 1, 2$) are on $S_a$ and both endpoints of the other one are on $S_b$.

The disk $P$ is called a descendent disk of $D$ [3]. Conversely, if a descendent disk exists, then $R$-$6$ can be applied. In this sense, the two descendent disks involved in the left and right of $R$-$6$ are said to be dual to each other.

4. $t$-descendent Sequences

Let $D = D_0 \to D_1 \to \ldots \to D_n = D'$ be a finite sequence of surface-knot diagrams satisfying the following condition:

(*) A transition from $D_i$ to $D_{i+1}$ ($i = 0, \ldots, n - 1$) is done by one of Roseman moves which can be realized by an isotopy of the surface-knot in $\mathbb{R}^4$ without creating a triple point in the projection.

In other words, the above condition says that the finite sequence of Roseman moves that connects $D$ and $D'$ does not include the Roseman moves $R_i^+$ ($i = 2, 3, 5$). Note that the reverse moves of these prohibited moves are allowed.

**Definition 4.1.** A finite sequence of surface-knot diagrams satisfying the condition (*) is called a $t$-descendent sequence.

The terminology "t-descendent" is used to clarify that the number of triple points will not increase under the sequence.

**Remark 4.2.** We point out here that M. Jabonowski published a paper [5] in which he provided an example of two equivalent surface-knot diagrams which can not be connected by a $t$-descendent sequence. Both diagrams in the example are with singularity set consisting of only closed 1-manifolds.

5. The Crossing Change Operation

A crossing change operation is a local operation for a diagram of a surface-knot which has a natural analogy to the crossing changes of classical knots.

**Definition 5.1.** Let $D$ be a surface-knot diagram of a surface-knot. Let $\Gamma = \bigcup_{j=1}^{r} \gamma_j$ be a union of double curves in $D$. $\Gamma$ is exchangeable if a surface-knot diagram is obtained from $D$ by changing the upper/lower information along the double curves of $\Gamma$ simultaneously. This operation is called the crossing change operation along $\Gamma$, and we denote by $D(\Gamma)$ the surface-knot diagram obtained from $D$ by the operation.
We do not assume that $D$ and $D(\Gamma)$ present distinct surface-knots.

Let $T$ be a triple point of $D$. The figure below shows all possible cases of changing the crossing information around the triple point $T$, where the branches at $T$ contained in $\Gamma$ are bold lines for each possible case.

![Figure 2: Changing the crossing information around a triple point](image)

**Remark 5.2.** Let $D$ be a surface-knot diagram and $\gamma$ be a double curve in $D$. Assume $\gamma \subset \Gamma$ where $\Gamma$ is exchangeable. Suppose $\gamma$ has an edge which contains a $b/t$-branch at a triple point. Then, the $b/m$-branches or $m/t$-branches at $T$ are subsets of a double curve in $D$ that is contained in $\Gamma$.

**Definition 5.3.** Let $\Gamma$ be an exchangeable union of double curves of a surface-knot diagram $D$. We say that $\Gamma$ satisfies the *descendent disk condition* for $D$ if exchanging the crossing information along the double curves of $\Gamma$ preserves all descendent disks of $D$. We say that $\Gamma$ satisfies the *unknotting condition* for $D$ if the surface-knot diagram $D(\Gamma)$ presents an unknotted surface-knot.

**Definition 5.4.** A surface-knot diagram $D$ is called *du-exchangeable* if it presents an unknotted surface-knot or it has an exchangeable union of double curves $\Gamma$ satisfying both the descendent disk condition and the unknotting condition for $D$.

**Definition 5.5.** A surface-knot $F$ is *du-exchangeable* if there is a surface-knot diagram $D$ presenting $F$ such that $D$ is du-exchangeable.

**Remark 5.6.** It is not difficult to see that any surface-knot diagram of a surface-knot has exchangeable double curves satisfying the descendent disk condition. For example, the union of all double curves of a surface-knot diagram is exchangeable and it satisfies the descendent disk condition.

### 6. The Main Result

In this section we prove the main theorem in this paper (Theorem 6.4.). In particular, the proof is divided into three lemmas.

Let $D$ be a surface-knot diagram of a surface-knot $F$. Let $\Gamma = \bigcup_{j=1}^{r} \gamma_j$ be an exchangeable union of double curves in $D$. The surface-knot diagram obtained
after cross-change operation along $\Gamma$ applied is denoted by $D(\Gamma)$. Throughout the proof of the three lemmas in this section, $\Gamma^{(s)}$ denotes the union of double curves $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_{s-1} \cup \hat{\gamma}_s \cup \gamma_{s+1} \cup \ldots \cup \gamma_r$ that is obtained from $\Gamma$ by deleting the double curve $\gamma_s$. Similarly, we define $\Gamma^{(s,w)}$ to be the union of double curves that is obtained from $\Gamma$ by deleting the double curves $\gamma_s$ and $\gamma_w$.

**Lemma 6.1.** Suppose that $D$ is transformed into $D'$ by one of the Roseman moves of $R-i^- (i = 2, 3, 5)$. For any exchangeable union $\Gamma$ of double curves of $D$, there is an exchangeable union $\Gamma'$ of double curves of $D'$ such that $D(\Gamma) \sim D'(\Gamma')$. Moreover, if $\Gamma$ satisfies the descendent disk condition, then we may assume that $\Gamma'$ also satisfies it.

**Proof.** Let $\Gamma = \cup_{j=1}^r \gamma_j$ be an exchangeable union of double curves in $D$ satisfying the descendent disk condition for $D$. We need to define an exchangeable union of double curves $\Gamma'$ in $D'$ that satisfies the assertion of the lemma for each Roseman move $R-i^- (i = 2, 3, 5)$.

The Roseman move $R-2^-$ can be viewed that it takes the paraboloid away from the double edge so that the paraboloid does not meet with the two intersecting disks (see $R-2^-$ in Figure 1). Let $\gamma_s$ be a double point curve in $D$ containing branches formed by the two disks. By applying $R-2^-$, $\gamma_s$ is restricted to a double curve $\gamma_s'$ of $D'$ that has less triple points by two. The exchangeable union of double curves $\Gamma'$ of $D'$ is defined as follows.

$$
\Gamma' = \begin{cases} 
\Gamma & \text{if } \gamma_s \not\subseteq \Gamma \\
\Gamma^{(s)} \cup \gamma_s' & \text{if } \gamma_s \subseteq \Gamma.
\end{cases}
$$

We see that the diagram $D'(\Gamma')$ differs from $D(\Gamma)$ by the move $R-2^-$ and thus they are equivalent.

The move $R-3^-$ can be viewed that it takes the paraboloid away from the triple point $T$ so that it does not meet with the three intersecting disks (see $R-3^\pm$ in Figure 1). This leads to elimination of six triple points and three double point circles. Assume that the $b/m$- branches at $T$ are subsets of the double curve $\gamma_s$ of $D$, the $m/l$- branches at $T$ are subsets of the double curve $\gamma_w$ of $D$ and the $b/t$- branches at $T$ are subsets of the double curve $\gamma_k$ of $D$. By applying $R-3^-$, $\gamma_s$ is transformed to a double curve $\gamma_s'$ of $D'$ such that $\gamma_s'$ has less double edges than $\gamma_s$ by two. Similarly, $\gamma_w$ and $\gamma_k$ of $D$ are transformed to $\gamma_w'$ and $\gamma_k'$ in $D'$, respectively. We define $\Gamma'$ by

$$
\Gamma' = \begin{cases} 
\Gamma & \text{if } \gamma_s, \gamma_w, \gamma_k \not\subseteq \Gamma \\
\Gamma^{(s)} \cup \gamma_s' & \text{if } \gamma_s \subseteq \Gamma, \gamma_w, \gamma_k \not\subseteq \Gamma \\
\Gamma^{(w)} \cup \gamma_w' & \text{if } \gamma_w \subseteq \Gamma, \gamma_s, \gamma_k \not\subseteq \Gamma \\
\Gamma^{(k,s)} \cup \gamma_s' \cup \gamma_k' & \text{if } \gamma_s \text{ and } \gamma_k \subseteq \Gamma, \gamma_w \not\subseteq \Gamma \\
\Gamma^{(k,w)} \cup \gamma_w' \cup \gamma_k' & \text{if } \gamma_w \text{ and } \gamma_k \subseteq \Gamma, \gamma_s \not\subseteq \Gamma \\
\Gamma^{(k,s,w)} \cup \gamma_s' \cup \gamma_w' \cup \gamma_k' & \text{if } \gamma_s, \gamma_w, \gamma_k \subseteq \Gamma.
\end{cases}
$$
From the definition of $\Gamma'$, we obtain that the diagram $D'(\Gamma')$ differs from $D(\Gamma)$ by the move $R-3^-$. It follows that they are equivalent.

The move $R-5^-$ can be described as moving a disk with a branch point away from the second disk (see $R-5^\pm$ in Figure 1) and thus a triple point $T$ is cancelled. Let the branch of $T$ whose other endpoint is a branch point be a subset of a double curve $\gamma_s$ of $D$. By applying $R-5^-$ to $D$, $\gamma_s$ is restricted to the double curve $\gamma'_s$ of $D'$ that has less double edges by one. Define

$$
\Gamma' = \begin{cases} 
\Gamma & \text{if } \gamma_s \not\subseteq \Gamma \\
\Gamma^{(s)} \cup \gamma'_s & \text{if } \gamma_s \subset \Gamma 
\end{cases}
$$

It follows that the transition from $D(\Gamma)$ to $D'(\Gamma')$ is done by $R-5^-$. We obtain that $D(\Gamma) \sim D'(\Gamma')$.

It remains to prove that $\Gamma'$ defined above for each Roseman move $R-i^-$ ($i = 2, 3, 5$) satisfies the descendent disk condition. Note that none of the three moves $R-i^-$ ($i = 2, 3, 5$) create new double edges and thus no new descendent disks are involved in $D'$. Since $D$ satisfies the descendent disk condition, we can assume that $D'$ also does.

**Lemma 6.2.** Suppose that $D$ is transformed into $D'$ by one of the Roseman moves of $R-i^-$ and $R-i^+$ ($i = 1, 4$). For any exchangeable union $\Gamma$ of double curves of $D$, there is an exchangeable union $\Gamma'$ of double curves of $D'$ such that $D(\Gamma) \sim D'(\Gamma')$. Moreover, if $\Gamma$ satisfies the descendent disk condition, then we may assume that $\Gamma'$ also satisfies it.

**Proof.** We prove the moves $R-1^+$ and $R-1^-$. The moves $R-4^+$ and $R-4^-$ are similarly proved. The Roseman move $R-1^+$ has the affect of adding a simple double point circle, denoted by $\gamma'$, that is independent from the other double curves of the diagram. The resulting double curve might be involved in a descendent disk where the other involved double edge is in $D$. Assume that the new descendent disk created is $P$ and that each of $\gamma'$ and $\gamma_s$ contains a boundary point of $P$, where $\gamma_s$ is a double curve of $D$. We define $\Gamma'$ in this case such that

$$
\Gamma' = \begin{cases} 
\Gamma & \text{if } \gamma_s \not\subseteq \Gamma \\
\Gamma^{(w)} \cup \gamma' & \text{if } \gamma_s \subset \Gamma 
\end{cases}
$$

If there is no such a descendent disk, we can assume that $\Gamma' = \Gamma$. It is not hard to see that $D(\Gamma) \sim D'(\Gamma')$ in both cases and that $\Gamma'$ satisfies the descendent disk condition by the definition. The move $R-1^-$ is the reverse of $R-1^+$. The double edge $\gamma'$ will be eliminated as a result. Denote $\gamma'$ by $\gamma_w$ for this move. Then, $\Gamma'$ of $D'$ that satisfy the assertion of the lemma can be defined such that

$$
\Gamma' = \begin{cases} 
\Gamma & \text{if } \gamma_w \not\subseteq \Gamma \\
\Gamma^{(w)} & \text{if } \gamma_w \subset \Gamma 
\end{cases}
$$

The lemma follows. \qed
Lemma 6.3. Suppose that $D$ is transformed into $D'$ by Roseman move of R-6. Let $\Gamma$ be an exchangeable union of double curves of $D$ satisfying the descendent disk condition. Then there is an exchangeable union $\Gamma'$ of double curves of $D'$ such that $D(\Gamma) \sim D'(\Gamma')$ and that $\Gamma'$ also satisfies the descendent disk condition.

Proof. Let $P$ be a descendent disk of $D$ and suppose R-6 move is applied along $P$ to obtain the surface-knot diagram $D'$. Let $e_1$ and $e_2$ be double edges of $D$ each of which contains a boundary point of $P$. Assume that $e_1 \subset \gamma_s$ and $e_2 \subset \gamma_w$, where $\gamma_s$ and $\gamma_w$ are double curves of $D$. Since $\Gamma$ satisfies the descendent disk condition, either the upper/lower information of both $\gamma_s$ and $\gamma_w$ are exchanged or neither. In the latter case, the result follows by letting $\Gamma' = \Gamma$. On the other hand, let $\gamma_s$ and $\gamma_w$ be subsets of $\Gamma$. Apply R-6 to obtain the surface-knot diagram $D'$. The connection between the double edges $e_1$ and $e_2$ is changed so that we obtain new double edges, say $e_1'$ and $e_2'$ in $D'$. Assume that $e_1'$ and $e_2'$ are subsets of double curves $\gamma'_s$ and $\gamma'_w$ of $D'$, respectively. Exchanging the upper/lower information of $\Gamma' = \Gamma^{(s,w)} \cup \gamma'_s \cup \gamma'_w$ in $D'$ gives a surface-knot diagram equivalent to $D(\Gamma)$. \qed

Theorem 6.4. Let $F$ be a du-exchangeable surface-knot and $D$ be a du-exchangeable surface-knot diagram of $F$. There exists a finite $t$-descendent sequence $D = D_0 \rightarrow D_1 \rightarrow \ldots \rightarrow D_n = D'$ such that for each $i = 1, \ldots, n$, $D_i$ is du-exchangeable.

Proof. Without loss of generality, we can assume that $D'$ is obtained from $D$ by applying a single Roseman move of one of the possible types in a $t$-descendent sequence. The theorem then follows from Lemma 6.1., Lemma 6.2. and Lemma 6.3. \qed

Example 6.5. The triple point number of a surface-knot, denoted by $t(F)$ is defined to be the minimal number of triple points over all possible diagrams of the surface-knot. Let $K$ be a knot in $\mathbb{R}^3$ such that $K$ has a knot diagram $D_K$ with $c$ crossings in which there is a full twisted trivial tangle with 2-strings. S. Satoh and A. Shima estimated in [12] the upper bound of triple point number for the $m$-twist spun of the knot $K$, $\tau^m(K)$. In fact, they showed that $t(\tau^m(K)) \leq 2(c - 2)m$. T. Yashiro in [15] gave an improved upper bound for some family of twist-spun knots including the $m$-twist spun of the knot $K$ described above. Both upper bounds can be obtained by deforming surface-knot diagrams that are obtained by following Satoh’s construction [11]. In particular, the deformations applied in both cases are done by $t$-descendent sequences.

7. The du-exchange Index

Let $F$ be a du-exchangeable surface-knot. The du-exchange index $du(F)$ is defined as follows.

Definition 7.1. The du-exchange index $du(F)$ of a du-exchangeable surface-knot
$F$ is the minimum number of double curves required, taken over all du-exchangeable diagrams representing $F$, to convert $F$ into a trivial surface-knot.

The du-exchange index is an invariant for du-exchangeable surface-knots.

**Example 7.2.** S. Satoh and A. Shima in [12] gave an estimate for a lower bound of the triple point number for tri-colourable surface-knots. Using this estimate, they showed that the 2-twist spun trefoil has the triple point number four. Let $K$ be the trefoil knot and let $D_K$ be a knot diagram of $K$ with 3 crossings. Let $T(K)$ be the one string tangle obtained from $D_K$ by removing a small 1-dim trivial neighbourhood of a point on the diagram $D_K$. By following Satoh’s construction of diagrams of twist-spun knots [11], we obtain a surface-knot diagram of the 2-twist spun trefoil with twelve triple points. In particular, Satoh’s diagram can be deformed into one with four triple points by a $t$-descendent sequence involving Roseman moves $R-2^-, R-5^-$ and $R-6$. Figure 3 depicts a schematic picture of the double curves of a t-minimal diagram of the 2-twist spun trefoil showing the types of double branches at each triple point.

![Figure 3: Schematic picture of the double curves of a t-minimal diagram of the 2-twist spun trefoil](image)

From the figure, we see that there are three double curves and two of them are double point intervals. Note that exchanging the crossing information of the double point circle gives a trivial 2-knot diagram and this shows that the 2-twist spun trefoil has the du-exchange index equal to one.

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