DISPERSION RELATIONS FOR STEADY PERIODIC WATER WAVES OF FIXED MEAN-DEPTH WITH TWO ROTATIONAL LAYERS

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Abstract. The aim of this paper is to obtain the dispersion relation for small-amplitude periodic travelling water waves propagating over a flat bed with a specified mean depth under the presence of a discontinuous piecewise constant vorticity. An analysis of the dispersion relation for a model with two rotational layers each having a non-zero constant vorticity is presented. Moreover, we present a stability result for the bifurcation inducing laminar flow solutions.

1. Introduction. The present paper is devoted to the study of small-amplitude two-dimensional steady periodic water waves propagating over a flat bed located at a fixed mean-depth from the surface. Furthermore, the flow is assumed to have a discontinuous piecewise constant vorticity function, with two layers of non-vanishing vorticity. Regarding the consideration of a rotational flow, we note that this type of flows are suitable for modelling wave-current interactions and other complex phenomena [38, 55, 57]. On the other hand, the chosen vorticity distribution is suitable for modelling a flow whose surface layer is affected by a wind that generates rotational waves (cf. [54, 56]) while in its near-bed layer there is another current generated, for instance, by sediment transport. The presence of vorticity raises serious mathematical difficulties for a rigorous analytical study of water waves. Indeed, the existence of solutions to the water wave problem exhibiting vorticity has remained evasive until the seminal work of Constantin and Strauss [3] that

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handled the case of a regular vorticity, the case of a discontinuous vorticity being treated in [12] by the same authors.

Following the ground-breaking papers of Constantin and Strauss [3,12], the study of rotational water flows with a fixed mass-flux received significant attention—evidentiated by an abundance of works on issues like symmetry [1,2,5,33,52], stability [6], regularity [11,13,21,22,24,25], stratification [20,27,30,32,34,59,60], the presence of stagnation points and critical layers [13,17,19,20,48,53,58]—much less has been done in the scenario of the fixed mean-depth. The fixedness of the mean-depth is a perspective, of physical relevance, proposed by Henry, cf. [29,31]. Indeed, recent numerical investigations [41,42] show that the mean-depth varies throughout the continuum of solutions to the water wave problem, provided the mass-flux is fixed. Towards extending the class of analytical studies in the setting of the fixed mean-depth, we will be concerned here with the dispersion relation for the small-amplitude waves arising from a particular choice of the vorticity distribution in the setting of the fixed mean-depth. Namely, we consider here two-layered water flows in which each layer has a constant non-vanishing vorticity. The dispersion relation obtained in this paper generalizes those derived in [26,28] where one of the layers was considered to be irrotational.

At this point we stress that dispersion relations are important from a physical standpoint as they relate several parameters like the wavelength and wave speed, and—as it is the case here—others like the mean depth of the flow, the location of the jump in vorticity and the values of the vorticity itself. In addition, the dispersion relation in this case is also important from an analytical perspective, as it provides a condition for the bifurcation from the laminar flows to occur. An added bonus of dispersion relations comes as the possibility to perform resonance analysis of water waves [8,49].

Detailed analyses of dispersion relations for water flows exhibiting piecewise constant distribution of the vorticity were carried out recently, cf. [12,14,44–48,51] in the case of the fixed mass flux, and [26,28] for the fixed mean-depth case. Moreover, dispersion relations pertaining to water flows incorporating geophysical effects and exhibiting discontinuous vorticity were found and investigated in [50].

A further piecewise constant vorticity distribution in the fixed mean-depth setting was considered in [40]. We would also like to note that dispersion relations were analyzed in several rotational flows of continuous non-constant vorticity [39]. Recently, the dispersion relation was analyzed in [43] for certain parallel shear flows.

We conclude our paper with a stability analysis of the laminar flow solutions in the fixed mean-depth framework following the ideas in [6,47]. To perform the before mentioned stability analysis we employ and adapt to our setting (in which the mean-depth is kept fixed) a variational formulation of the water waves equations that was devised in [6]. We note that variational formulations were recently and relatively recently employed in the study of rotational water waves allowing for a constant or piecewise constant vorticity, cf. [7,15,16,37].

2. Governing equations. The fluid motion to be considered is described by the equations of mass conservation and conservation of momentum for two-dimensional gravity water waves, arising as the free surface of an inviscid and incompressible water flow. In particular, we consider steady periodic travelling surface waves propagating over a flow of fixed depth \( d \) (with \( d > 0 \)) allowing for piecewise constant vorticity and in a coordinate system with the \( X \) axis pointing in the direction of wave propagation, the \( Y \) axis pointing upwards.
The assumption of travelling water waves means that the velocity field \((u,v)\), the pressure \(P\) and the free surface, given as the graph of the (unknown) function \(X \rightarrow \eta(X,t)\), present a dependence in the horizontal direction of the type \(X - ct\), where \(c > 0\) denotes the constant wave speed of the travelling waves. The change of coordinates
\[
x := X - ct, \quad y := Y
\]
implies that the fluid motion is governed (in the moving frame) by the Euler’s equations
\[
\begin{align*}
(u-c)u_x + vu_y &= -P_x, \\
(u-c)v_x + vv_y &= -P_y - g
\end{align*}
\]
and the continuity equation for an incompressible fluid
\[
u_x + v_y = 0.
\]
The previous equations of motion are supplemented with the kinematic boundary conditions
\[
\begin{align*}
v &= (u-c)\eta_x \quad \text{on } y = \eta(x), \\
v &= 0 \quad \text{on } y = -d
\end{align*}
\]
and the dynamic boundary condition
\[
P = P_{atm} \quad \text{on } y = \eta(x),
\]
where \(P_{atm}\) is the constant atmospheric pressure. Additionally, it is assumed that the horizontal velocity satisfies the condition
\[
u(x,y) < c \quad \text{for all } (x,y) \in D_\eta = \{(x,y) : -L \leq x \leq L; -d \leq y \leq \eta(x)\},
\]
stating that there are no stagnation points within the flow. In addition, a scaling of the variables is performed assuming that we are dealing with water waves of wavelength \(L\). Hence, if \(\kappa = \frac{2\pi}{L}\) is the wavenumber, the new dimensionless variables are obtained by the following transformation
\[
(x,y,t,g,\eta,u,v,P,c) \mapsto (\kappa x, \kappa y, \kappa t, \frac{g}{\kappa}, \kappa \eta, u, v, P, c)
\]
Furthermore, the vorticity, understood as the only nonzero term of the three-dimensional vorticity vector \(\nabla \times (u,0,v)\), is given by
\[
\omega = u_y - v_x,
\]
where \(\omega/\kappa\) will be the scaled vorticity. We note that the scaled variables satisfy (1) replacing \(g\) by \(g/\kappa\). In what follows we will deal with dimensionless variables only.

From the equation of mass conservation (1b) we infer the existence of the stream function \(\psi\), which is defined (up to a constant) by means of
\[
\begin{align*}
\psi_y &= u - c, \\
\psi_x &= -v.
\end{align*}
\]
It follows from (1c) and (1d) that the stream function is constant on both boundaries of the fluid domain. Making the choice \(\psi(x,\eta(x)) = 0\), we then infer from (4) that \(\psi(x,-d) = -p_0\), where \(p_0\) is the relative mass flux
\[
p_0 = \int_{-d}^{0(\eta)} (u(x,y) - c) \, dy < 0,
\]
a quantity that does not depend on \(x\)-as can be derived from (1b), (1c) and (1d).
The governing equations (1) can be reformulated in terms of the stream function (cf. [3,10]) as
\[
\begin{cases}
\Delta \psi = \omega & \text{for } y \in (-d, \eta(x)), \\
|\nabla \psi|^2 + 2g(y + d) = Q & \text{on } y = \eta(x), \\
\psi = 0 & \text{on } y = \eta(x), \\
\psi = -p_0 & \text{on } y = -d,
\end{cases}
\]
where \(Q\) is a constant, known as the hydraulic head.

To shift our analysis to a problem in a fixed domain we make the following change of variables
\[
(q,p) = (q(x,y), p(x,y)) := (x, \frac{\psi(x,y)}{p_0}),
\]
(7)
which, together with the assumption (2), assures the existence of a diffeomorphism
\[(x,y) \in D \rightarrow (q,p) \in \mathbb{R} := [-L,L] \times [-1,0].\]

Let \(\phi\) be the function defined by
\[
\phi(y) = \frac{1}{p_0} \psi(x,y) \text{ for any given } x.
\]
(8)
This function is invertible (again by (2)) and we can define
\[
h(q,p) = \frac{\phi^{-1}(p)}{d} - p.
\]
(9)
The system of equations (6) is transformed by means of \(h\) (following the idea of the hodograph transformation in [18]) into
\[
\begin{cases}
\left(1 + \frac{d^2}{p^2}\right)h_{pp} - 2h_q(h_p + 1)h_{pq} + (h_p + 1)^2h_{qq} + \frac{\gamma(p)}{p_0}(h_p + 1)^3 = 0 & \text{in } R, \\
\left(1 + \frac{d^2}{p^2}\right) + \frac{(h_p + 1)^2}{p_0^2}\left[2gd(h+1) - Q\right] = 0 & \text{on } p = 0, \\
h = 0 & \text{on } p = -1,
\end{cases}
\]
(10)
where \(\omega = \gamma(p)\) is a function of \(p\) alone known as vorticity function (cf. [31]).
Moreover, the no stagnation assumption (2) is now equivalent to
\[
h_p + 1 > 0.
\]
(11)

We note that (10) represents an alternative reformulation of the governing equations for water waves exhibiting a fixed mean-depth, devised by Henry [29, 31] in the context of continuous vorticity and extended to the discontinuous vorticity case by Henry and Sastre-Gómez [35] and by Henry, Martin and Sastre-Gómez [36].

Furthermore, the system (10) consists of a uniformly elliptic quasilinear partial differential equation with oblique nonlinear boundary conditions that can be expressed, cf. [35], in the divergence form
\[
\begin{cases}
\frac{1 + d^2}{2d^2(1+h_p)} \left(\frac{\Gamma(p)}{2d^2}\right)_p - \left(\frac{h_q}{1+h_p}\right)_q = 0 & \text{for } p \in (-1,0), \\
\frac{1 + d^2}{2d^2(1+h_p)} + \frac{gd(h+1)}{p_0^2} = \frac{Q}{2p_0^2} & \text{on } p = 0, \\
h = 0 & \text{on } p = -1,
\end{cases}
\]
(12)
where
\[ \Gamma(p) = 2 \int_0^p \frac{d^2 \gamma(s)}{sp_0} \, ds \quad -1 \leq p \leq 0. \] (13)

The solutions of (12) are understood as functions \( h \in W^{2,r}_{\text{per}}(R) \subset C^{1,\alpha}_{\text{per}}(R) \) for \( r > 2/(1-\alpha) \) with \( \alpha \in (1/3, 1) \), allowing for a discontinuous vorticity function [35]. When considering solutions of (12) which are independent of \( q \), the resulting flows are laminar flows whose streamlines are horizontal. If \( \Gamma_{\text{min}} = \min_{p \in (-1,0)} \Gamma(p) \), a family of laminar flows for (12) parametrized by \( \lambda \in (-\Gamma_{\text{min}}, Q) \), is given by

\[ H(p; \lambda) = \int_0^p \frac{1}{\sqrt{\lambda + \Gamma(p)}} \, ds + \frac{1}{2q^d} \left[ Q - \frac{p_0^2}{d^2} \lambda \right] - (p + 1), \quad -1 < p \leq 0. \] (14)

Furthermore, the parameter \( \lambda \) is related with the velocity at the flat surface by

\[ \sqrt{\lambda} = \frac{1}{\sqrt{p_0} + 1} \left. \frac{d(u - c)}{p_0} \right|_{y=0}. \] (15)

It was proven in [35] that small amplitude waves which are perturbations of these laminar flows exist if and only if there exists a nontrivial solution \( M \in C^{1,\alpha}_{\text{per}}(-1,0) \) for the Sturm-Liouville problem

\[
\begin{aligned}
(a^3 M_p)_p = d^2 a M & \quad & \text{if} \quad -1 < p < 0, \\
a^3 M_p = \frac{q d^3}{p_0^2} M & \quad & \text{if} \quad p = 0, \\
M = 0 & \quad & \text{if} \quad p = -1,
\end{aligned}
\] (16)

where \( a(p; \lambda) = \frac{1}{\sqrt{p_0} + 1} = \sqrt{\lambda + \Gamma(p)} \in C^{2,\alpha}_{\text{per}}(-1,0). \) In the sequel we will use the Sturm-Liouville problem to obtain the dispersion relation for our two-layer rotational flow. In achieving the latter we will adapt to our setting (of two rotational layers) the method used by Constantin [14] and Constantin and Strauss [12] in the context of one rotational layer neighboring an irrotational one. We emphasize that our physical setting (in which the mean-depth is fixed) is essentially different from the case studied by Constantin and Strauss ( [12]) in which the mass flux is taken to be a constant throughout the set of solutions to the water wave equations.

3. The Sturm-Liouville problem. We are interested in a vorticity function that models two layers of constant, but different, vorticity. Let us consider the two layers separated by an intermediate depth \( y = -\nu \), where \( \gamma_1 \) will be the vorticity in the layer adjacent to the free surface and \( \gamma_2 \) the vorticity in the layer near the bottom,

\[ \omega(x, y) = \begin{cases} 
\gamma_1 & \text{if} \quad -\nu < y < 0, \\
\gamma_2 & \text{if} \quad -d < y < -\nu. 
\end{cases} \] (17)

In addition, by denoting

\[ p_1 = \frac{\psi(x, -\nu)}{p_0}, \] (18)

we obtain the same expression in terms of the variables \((q, p)\),

\[ \omega(q, p) = \begin{cases} 
\gamma_1 & \text{if} \quad p_1 < p < 0, \\
\gamma_2 & \text{if} \quad -1 < p < p_1. 
\end{cases} \] (19)
In particular, for laminar flows (whose vertical velocity vanishes) we have that $\omega = u_y$. Hence, by (15), it follows that

$$c - u(y) = \begin{cases} -\frac{p_0 \sqrt{\lambda}}{d} - \gamma_1 y & \text{if } -\nu < y < 0, \\ -\frac{p_0 \sqrt{\lambda}}{d} + \nu \gamma_1 - \gamma_2 (y + \nu) & \text{if } -d < y < -\nu. \end{cases}$$

(20)

In order to describe the Sturm-Liouville problem for the given vorticity, an expression for $\Gamma(p)$ is obtained,

$$\Gamma(p) = \begin{cases} 2d^2 \gamma_1 p & \text{if } p_1 < p < 0, \\ 2d^2 \left[ \gamma_2 p + p_1 (\gamma_1 - \gamma_2) \right] & \text{if } -1 < p < p_1. \end{cases}$$

(21)

which yields

$$a(p; \lambda) = \begin{cases} \sqrt{\lambda + 2d^2 \gamma_1 p} & \text{if } p_1 < p < 0, \\ \sqrt{\lambda + 2d^2 \left[ \gamma_2 p + p_1 (\gamma_1 - \gamma_2) \right]} & \text{if } -1 < p < p_1. \end{cases}$$

(22)

The Sturm-Liouville problem (16) for the vorticity function (19) is specified now by the previous expression for $a(p; \lambda)$. We will seek a solution of this problem by providing its expressions in both layers, corresponding to $[p_1, 0]$ and to $[-1, p_1]$. For the near-surface layer, we look for a solution of

$$\begin{cases} (a^3 m_p)_p = d^2 a m & \text{if } p_1 < p < 0, \\ a^3 m_p = \frac{d^2}{p_0} m & \text{if } p = 0, \end{cases}$$

(23)

where $a(p; \lambda)$ is given by (22) for $p \in (p_1, 0)$. We seek a solution of (22) in the form

$$m(p) = \frac{1}{a(p)} \left[ c_1 \sinh \left( \frac{p_0 a(p)}{d \gamma_1} \right) + c_2 \cosh \left( \frac{p_0 a(p)}{d \gamma_1} \right) \right],$$

for constants $c_1, c_2 \in \mathbb{R}$. For the second part, we have

$$\begin{cases} (a^3 \tilde{m}_p)_p = d^2 a \tilde{m} & \text{if } -1 < p < p_1, \\ \tilde{m} = 0 & \text{if } p = -1, \end{cases}$$

(24)

where $a(p; \lambda)$ is given by (22) for $p \in (-1, p_1)$, and the equation has a simpler boundary condition, which allows us to search for a solution of the form

$$\tilde{m}(p) = \frac{C}{a(p)} \sinh \left( \frac{p_0 a(p) - a(-1)}{d \gamma_2} \right) \quad \text{for } C := \frac{c_3}{\cosh \left( \frac{p_0 a(-1)}{d \gamma_2} \right)},$$

where $c_3$ is another constant. We then set

$$M(p) := \begin{cases} m(p) & \text{if } p \in [p_1, 0], \\ \tilde{m}(p) & \text{if } p \in [-1, p_1]. \end{cases}$$
In order to simplify the calculations, we adopt the following notation
\[
\begin{align*}
\theta_1 &= \frac{p_0 a(p_1)}{d\gamma_1}, \\
\theta_2 &= \frac{p_0[a(p_1) - a(-1)]}{d\gamma_2}.
\end{align*}
\tag{25}
\]
By the continuity of \( M \) at \( p = p_1 \), we obtain a first condition involving the unknown constants \( c_1, c_2 \), and \( C \):
\[
c_1 \sinh(\theta_1) + c_2 \cosh(\theta_1) = C \sinh(\theta_2).
\tag{26}
\]
Secondly, from the continuity of the first derivative of \( M \) at \( p = p_1 \), it follows that
\[
-\frac{C d\gamma_2}{p_0} \sinh(\theta_2) + Ca(p_1) \cosh(\theta_2) = -\frac{d\gamma_1}{p_0} \left[ c_1 \sinh(\theta_1) + c_2 \cosh(\theta_1) \right] + a(p_1) \left[ c_1 \cosh(\theta_1) + c_2 \sinh(\theta_1) \right].
\tag{27}
\]

**Remark 1.** At this point we note that the matching conditions (26) and (27)—once satisfied—ensure that \( M \in C^1([-1, 0]) \). By utilizing Schauder type estimates (cf. [23]) we can argue along the lines of [36] (see [12] in the setting of the fixed mass flux) and obtain that, in fact, \( M \in C^{1,\alpha}([-1, 0]) \) as required for the existence of small-amplitude water waves that are perturbations of the laminar flow solutions (14).

Making now use of (26),
\[
c_1 \cosh(\theta_1) + c_2 \sinh(\theta_1) = \frac{1}{a(p_1)} \frac{dC}{p_0} (\gamma_1 - \gamma_2) \sinh(\theta_2) + C \cosh(\theta_2).
\tag{28}
\]
We write now (26) and (27) as the system of equations
\[
\begin{align*}
c_1 \sinh(\theta_1) + c_2 \cosh(\theta_1) &= C \sinh(\theta_2), \\
c_1 \cosh(\theta_1) + c_2 \sinh(\theta_1) &= \frac{1}{a(p_1)} \frac{dC}{p_0} (\gamma_1 - \gamma_2) \sinh(\theta_2) + C \cosh(\theta_2),
\end{align*}
\tag{29}
\]
that will be solved for \( c_1 \) and \( c_2 \) in terms of \( C \). Utilizing now (29) and the condition
\[
a^3(0)m_{p|p=0} = \frac{gd^3}{p_0^3} m(0),
\]
it follows that the existence of a non-trivial solution of the Sturm-Liouville problem (16) is ensured if and only if the algebraic equation
\[
\begin{align*}
p_0^2 \left[ \cosh \left( \theta_2 - \theta_1 + \frac{p_0 \sqrt{\lambda}}{d\gamma_1} \right) + \frac{d}{p_0 a(p_1)} (\gamma_1 - \gamma_2) \sinh(\theta_2) \cosh \left( \frac{p_0 \sqrt{\lambda}}{d\gamma_1} - \theta_1 \right) \right] \lambda \\
- \sinh \left( \theta_2 - \theta_1 + \frac{p_0 \sqrt{\lambda}}{d\gamma_1} \right) + \frac{d}{p_0 a(p_1)} (\gamma_1 - \gamma_2) \sinh(\theta_2) \sinh \left( \frac{p_0 \sqrt{\lambda}}{d\gamma_1} - \theta_1 \right) \right] p_0 d\gamma_1 \sqrt{\lambda} \\
= \left[ \sinh \left( \theta_2 - \theta_1 + \frac{p_0 \sqrt{\lambda}}{d\gamma_1} \right) + \frac{d}{p_0 a(p_1)} (\gamma_1 - \gamma_2) \sinh(\theta_2) \sinh \left( \frac{p_0 \sqrt{\lambda}}{d\gamma_1} - \theta_1 \right) \right] gd^2
\end{align*}
\tag{30}
\]
has a unique positive solution, \( \lambda^* \).
4. The algebraic equation. The problem of existence of small amplitude water waves has been transformed, through the Sturm–Liouville problem (16), into the analysis of the algebraic equation (30). In order to study the latter, we will find suitable relations for the terms involved. Let us begin by considering the mass flux \( p_0 \). By integrating the velocity expression (20) we obtain
\[
p_0(\sqrt{\lambda} - 1) = \frac{\gamma_2 - \gamma_1}{2} (d - \nu)^2 + \frac{\gamma_1}{2} d^2. \tag{31}
\]
Furthermore, by (18), the assumption of \( \psi(0) = 0 \) and the definition of the stream function, we have
\[
-p_1 p_0 = \psi(0) - \psi(-\nu) = \int_{-\nu}^{0} (u - c) \, dy = \left( \frac{p_0 \sqrt{\lambda}}{d} \nu - \frac{\gamma_1}{2} \nu^2 \right).
\]
Thus, we obtain
\[
\frac{\gamma_1}{2p_0} \nu^2 - \frac{\sqrt{\lambda}}{d} \nu - p_1 = 0, \tag{32}
\]
which is a polynomial in \( \nu \) with roots
\[
\nu = \frac{\sqrt{\lambda} \pm \sqrt{\lambda + \frac{2 \gamma_1 d^2 p_1}{p_0}}}{d \gamma_1 / p_0}. \tag{33}
\]
We claim that only a minus sign is possible in (33). Indeed, if the choice of a positive sign is made, then any positive value of \( \gamma_1 \) implies that \( \nu \) is negative due to the fact that \( p_0 \) is negative. However, \( \nu \) must be positive. On the contrary, the expression (33) with the choice of a minus sign is positive regardless the sign of \( \gamma_1 \).

If \( \gamma_1 > 0 \), the numerator \( \sqrt{\lambda} - \sqrt{\lambda + 2 \gamma_1 d^2 p_1} / p_0 \), is negative as well as the denominator \( d \gamma_1 / p_0 \). Hence, \( \nu > 0 \). On the other hand, if \( \gamma_1 < 0 \) then the numerator in (33) is positive as well as \( d \gamma_1 / p_0 \). Thus, so \( \nu > 0 \). Therefore, we have that
\[
\nu = \frac{\sqrt{\lambda} - \sqrt{\lambda + \frac{2 \gamma_1 d^2 p_1}{p_0}}}{d \gamma_1 / p_0} = \frac{a(0) - a(p_1)}{d \gamma_1 / p_0},
\]
and from (25),
\[
\theta_1 = \frac{p_0 \sqrt{\lambda}}{d \gamma_1} - \nu. \tag{34}
\]
In order to obtain an expression of the same kind for \( \theta_2 \), we start by considering (31) as a polynomial in \( d \), written in the following form
\[
\gamma_2 d^2 + 2 \nu (\gamma_1 - \gamma_2) d - (\gamma_1 - \gamma_2) \nu^2 - 2 p_0 (\sqrt{\lambda} - 1) = 0.
\]
Multiplying by \( \gamma_2 \) and rearranging the terms we obtain
\[
\left( \gamma_2 d + \nu (\gamma_1 - \gamma_2) \right)^2 = \nu^2 \gamma_1 (\gamma_1 - \gamma_2) + 2 p_0 \gamma_2 (\sqrt{\lambda} - 1),
\]
which after multiplying by \( d^2 / p_0^2 \), yields
\[
\frac{d^2}{p_0} \left( \gamma_2 d + \nu (\gamma_1 - \gamma_2) \right)^2 = \frac{2 d^2}{p_0} \gamma_2 \sqrt{\lambda} + \frac{2 d^2}{p_0} \gamma_2 = \frac{\nu^2 d^2}{p_0^2} \gamma_1 (\gamma_1 - \gamma_2) \tag{35}
\]
On the other hand, by (22) and (32)

\[ a^2(-1; \lambda) = \lambda - \frac{2d^2}{\gamma_0} \gamma_2 + \frac{\nu^2 d^2}{\gamma_0} \gamma_1(\gamma_1 - \gamma_2) - \frac{2 \nu d}{\gamma_0} (\gamma_1 - \gamma_2) \sqrt{\lambda}. \]

From (35), the last expression becomes

\[ a^2(-1; \lambda) = \left\{ \frac{d}{\gamma_0} \left( \frac{\gamma_2 d + \nu(\gamma_1 - \gamma_2)}{d} \right) - \sqrt{\lambda} \right\}^2 \]  

(36)

In order to choose the correct sign, we note that \( c - u(y) \) is positive for all \( y \) in \([-d, 0]\). In particular, for \( y = -d \),

\[ c - u(-d) = -\frac{p_0 \sqrt{\lambda}}{d} + \nu(\gamma_1 - \gamma_2) + \gamma_2 d > 0, \]

which holds if and only if

\[ \frac{d}{\gamma_0} \left( \frac{\gamma_2 d + \nu(\gamma_1 - \gamma_2)}{d} \right) - \sqrt{\lambda} < 0. \]

Thus, from (36) and the previous argument,

\[ a(-1; \lambda) = \sqrt{\lambda} - \frac{d}{\gamma_0} \left( \frac{\gamma_2 d + \nu(\gamma_1 - \gamma_2)}{d} \right). \]

(37)

Hence, we obtain

\[ \theta_2 = \frac{p_0}{d \gamma_2} \left( a(p_1) - a(-1) \right) = \frac{p_0}{d \gamma_2} \left( a(p_1) - a(0) \right) + \frac{p_0}{d \gamma_2} \left( a(0) - a(-1) \right) = d - \nu. \]

Multiplying now (30) by \( p_0 a(p_1) \) and rewriting it by means of the identities

\[
\begin{align*}
\theta_1 &= \frac{p_0 \sqrt{\lambda}}{d \gamma_1} - \nu, \\
\theta_2 &= d - \nu, \\
\theta_2 - \theta_1 + \frac{p_0 \sqrt{\lambda}}{d \gamma_1} &= d,
\end{align*}
\]

(38)

we obtain

\[
\begin{align*}
p_0^2 &\left\{ p_0 a(p_1) \cosh(d) + d(\gamma_1 - \gamma_2) \sinh(d - \nu) \cosh(\nu) \right\} \lambda \\
&- \left\{ p_0 a(p_1) \sinh(d) + d(\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) \right\} p_0 d \gamma_1 \sqrt{\lambda} \\
&- \left\{ p_0 a(p_1) \sinh(d) + d(\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) \right\} g d^3 = 0.
\end{align*}
\]

(39)

Then, (25) together with (38) yields

\[ p_0 a(p_1) = p_0 \sqrt{\lambda} - d \nu \gamma_1. \]

Finally, we obtain the polynomial equation

\[
p_0 \cosh(d) \lambda^{3/2} \\
+ p_0^2 \left\{ -d \nu \gamma_1 \cosh(d) + d(\gamma_1 - \gamma_2) \sinh(d - \nu) \cosh(\nu) - d \gamma_1 \sinh(d) \right\} \lambda \\
+ p_0 \left\{ d^2 \nu \gamma_1^2 \sinh(d) + d^2 \gamma_1(\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) - g d^2 \sinh(d) \right\} \lambda^{1/2} \\
+ \left\{ g d^3 \nu \gamma_1 \sinh(d) - g d^3 (\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) \right\} = 0.
\]

(40)

Provided (40) has a unique positive solution, it gives the (relative) speed at the free surface of the bifurcation inducing laminar solution. The formula giving the unique positive solution of (40) is called the dispersion relation.
Remark 2. By setting $\gamma_1 = 0$ and $\gamma_2 = \gamma$ in the previous expression, the polynomial (3.11) in [26] is recovered. In the same way, taking $\gamma_2 = 0$, $\gamma_1 = \gamma$ and making the appropriate change of variables, the polynomial (14) in [28] is also recovered.

5. Analysis of the dispersion relation. The unique positive solution of (40) gives the relative speed at the surface of the laminar flows in terms of several parameters. In our case, those parameters are the fixed mean depth of the flow, the location of the jump in vorticity, the vorticity distribution itself and the wavelength (which is not explicit but that can be recovered by means of the dimension variables prior to the scaling (3)). Furthermore, it was shown that non-laminar small-amplitude waves due to bifurcation exist if and only if (40) holds, i.e., if and only if there is a unique root $\sqrt{\lambda}$ of (40) compatible with (2). If such a root exists, the velocity at the surface is recovered via (15). This inspires the following change of variables,

$$x := c - u(0) = -\frac{p_0 \sqrt{\lambda}}{d} \quad (> 0) \quad (41)$$

which, after dividing by $-d^3$, transforms (40) into

$$p(x) := \cosh(d) x^3 + \left\{ \nu \gamma_1 \cosh(d) - (\gamma_1 - \gamma_2) \sinh(d - \nu) \cosh(\nu) + \gamma_1 \sinh(d) \right\} x^2$$

$$+ \left\{ \nu \gamma_1^2 \sinh(d) - \gamma_1 (\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) - g \sinh(d) \right\} x$$

$$+ g \left\{ -\nu \gamma_1 \sinh(d) + (\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) \right\} = 0 \quad (42)$$

Note that the above equation coincides with the one derived in [47], in the fixed mass-flux setting. This aspect is indicative of the fact that the differences between the fixed mass flux formulation and the fixed mean-depth formulation are perceptible in waves of large amplitude, cf. [29,36].

To decide upon whether equation (40) has a unique positive solution, we will make use of the following remark.

Remark 3 (Viète’s formulae for third-order polynomials). Let $x_1, x_2, x_3$ denote the roots of the polynomial $P(x) = Ax^3 + Bx^2 + Bx + D$. It holds that

$$x_1 x_2 x_3 = -\frac{D}{A}, \quad (43a)$$

$$x_1 + x_2 + x_3 = -\frac{B}{A}. \quad (43b)$$

The next result follows along the lines of [47]. For the sake of self-containedness, we include here a sketch of the proof.

Lemma 5.1. If $\gamma_1$ and $\gamma_2$ are both positive, then there exists a unique positive root of (42). Therefore, local bifurcation always occurs.

Proof. Let $p(x)$ be the polynomial given by (42). It follows from the properties of the function $\sigma(\nu) = \sinh(d) \nu - \sinh(d - \nu) \sinh(\nu)$ that

$$p(0) = -g \left[ \nu \gamma_1 \sinh(d) - (\gamma_1 - \gamma_2) \sinh(d - \nu) \sinh(\nu) \right] < 0.$$

Thus, there exists at least one positive root of the polynomial because $p$ approaches infinity as $x$ goes to infinity. This root is unique and the bifurcation occurs. Indeed,
if we assume the existence of another positive root, then (43a) and the fact that
\[- \frac{p(0)}{\cosh(d)} > 0\]
yields that the third root will be real and positive too. However, this situation is impossible because the coefficient of the second-order term in (42) is positive and, therefore, the sum of the three positives roots will then be negative.

We consider further the instance when \(\gamma_1 \cdot \gamma_2 < 0\).

**Lemma 5.2.** Let \(\gamma_1\) be negative and \(\gamma_2\) be positive. There exists a unique positive root of the dispersion relation (42) satisfying the non-stagnation condition (2) if and only if
\[
\frac{g}{\gamma_1^2} > \nu^2 \frac{\cosh(\nu)}{\sinh(\nu)} - \nu.
\]  

**Proof.** The non-stagnation condition for \(\gamma_2 > 0\) is now equivalent to
\[-p_0 \frac{\sqrt{\lambda}}{d} + \gamma_1 \nu = x + \gamma_1 \nu > 0.\]

This suggests the change of variables
\[
\tilde{x} = -\frac{1}{\gamma_1} x - \nu.
\]

Thus, the polynomial (42) in the new variable \(\tilde{x}\) is given by
\[
q(\tilde{x}) = A \tilde{x}^3 + B \tilde{x}^2 + C \tilde{x} + D,
\]  

where
\[
A = \cosh(d),
B = \left\{2\nu \cosh(d) - \sinh(\nu) \cosh(d - \nu) - \frac{\gamma_2}{\gamma_1} \sinh(d - \nu) \cosh(\mu)\right\},
C = \left\{\nu^2 \cosh(d) - \sinh(d - \nu) \sinh(\nu) + \nu \left[\sinh(d - \nu) \cosh(\nu) - \sinh(\nu) \cosh(d - \nu)\right] + \frac{\gamma_2}{\gamma_1} \sinh(d - \nu) \right\} - \frac{\gamma_2}{\gamma_1} \sinh(\nu)\right\},
D = \sinh(d - \nu) \left\{\frac{\gamma_2}{\gamma_1} \sinh(\nu) - \frac{g}{\gamma_1^2} \sinh(\nu) + \left(\nu^2 \cosh(\nu) - \nu \sinh(\nu)\right) \left(1 - \frac{\gamma_2}{\gamma_1}\right)\right\}.
\]

We note here that \(x\) is a positive root of (42) that satisfies the non-stagnation condition (2) if and only if \(\tilde{x}\) is a positive root of \(q(\tilde{x})\). Since \(1 - \frac{\gamma_2}{\gamma_1} > 0\), it is easy to see that \(q(0) < 0\) if and only if (44) is satisfied. A similar argument to the one presented in the previous lemma concludes that there is a unique positive root of (45) (as \(B > 0\)).

In addition, the condition (44) is necessary for the bifurcation to occur. Indeed, if we assume that
\[
\frac{g}{\gamma_1^2} \leq \nu^2 \frac{\cosh(\nu)}{\sinh(\nu)} - \nu,
\]  

then \(q(\tilde{x})\) has no positive roots and the bifurcation does not occur. To see that we note that (46) implies that \(q(0) > 0\). In addition, \(q'(\tilde{x}) > 0\) for all \(\tilde{x} > 0\) because
\[
q'(\tilde{x}) = 3A \tilde{x}^2 + 2B \tilde{x} + C,
\]
where all the coefficients are positive as it was proven in [47].

The final case, when both layers possess negative vorticity, is analyzed in this paper.
Lemma 5.3. Assume that $\gamma_1 < \gamma_2 < 0$. Then, if (44) holds, local bifurcation occurs if and only if

$$\frac{\gamma_2 - \gamma_1}{\gamma_2(d - \nu)} \left[ \frac{g - \gamma_1 x_c}{x^2} \sinh(\nu) - \cosh(\nu) \right] \sinh(d - \nu) < \frac{g - \gamma_1 x_c}{x^2} \sinh(d) - \cosh(d),$$

where $x_c = -\gamma_2(d - \nu) - \gamma_1 \nu$.

Proof. Let us consider the monic polynomial

$$p(x) = x^3 + \frac{1}{\cosh(d)} \left[ \nu_1 \gamma_1 \cosh(d) + (\gamma_2 - \gamma_1) \sinh(d - \nu) \cosh(\nu) + \gamma_1 \sinh(d) \right] x^2$$

$$+ \frac{1}{\cosh(d)} \left[ \nu_1^2 \sinh(d) + \nu_1^2 (\gamma_2 - \gamma_1) \sinh(d - \nu) \sinh(\nu) - g \sinh(d) \right] x$$

$$+ \frac{g}{\cosh(d)} \left[ - \nu_1 \sinh(d) - (\gamma_2 - \gamma_1) \sinh(d - \nu) \sinh(\nu) \right],$$

which, of course, has the same roots as (42). We show that under the condition (44), $p(x)$ has the form described in Figure 1. Certainly, the value of the polynomial at the origin is given by

$$p(0) = \frac{g}{\cosh(d)} \left[ - \gamma_1 \nu \sinh(d) + \gamma_1 \sinh(d - \nu) \sinh(\nu) - \gamma_2 \sinh(d - \nu) \sinh(\nu) \right]$$

$$> \frac{g}{\cosh(d)} \left[ - \gamma_1 \nu \sinh(d) + \gamma_1 \sinh(d - \nu) \sinh(\nu) \right]$$

$$= -\gamma_1 \frac{g}{\cosh(d)} \left[ \nu \sinh(d) - \sinh(d - \nu) \sinh(\nu) \right].$$

Note now that the function

$$\nu \to \sigma(\nu) = \sinh(d) \nu - \sinh(d - \nu) \sinh(\nu), \quad \text{for} \quad \nu > 0$$

FIGURE 1. Sketch of the graph of the polynomial providing the dispersion relation for two negative vorticities under the condition (44). The root $x_{01}$ does not satisfy the non-stagnation condition while the root $x_{02}$ verifies it if and only if (47) is assumed.
is a strictly increasing function for \( \nu > 0 \) and satisfies \( \sigma(0) = 0 \). Thus, \( \sigma(\nu) > 0 \) for any positive \( \nu \). In addition, taking into account the negative value of \( \gamma_1 \), it follows that (49) is strictly positive. Therefore by Viète’s formula (see remark (3)), the polynomial \( p \) has either one or three negative roots. Let us consider now the point \( x_0 = -\nu \gamma_1 \), for which the polynomial takes the value

\[
p(x_0) = \frac{\gamma_1^2 (\gamma_1 - \gamma_2)}{\cosh(d)} \left[ -\nu^2 \cosh(\nu) + \nu \sinh(\nu) + \frac{g}{\gamma_1^2} \sinh(\nu) \right] \sinh(d - \nu).
\]

Since \( \gamma_1 - \gamma_2 < 0 \), we have that \( p(x_0) < 0 \) if and only if

\[
-\nu^2 \cosh(\nu) + \nu \sinh(\nu) + \frac{g}{\gamma_1^2} \sinh(\nu) > 0,
\]

which is equivalent to the condition (44). Hence, provided (44) holds, the polynomial has at least one positive root because there exists \( x_0 > 0 \) such that \( p(x_0) < 0 \). Furthermore, the only possibility is that \( p \) has two positive roots and one negative root, being of the form showed in Figure 1. Let \( x_{01} < x_{02} \) the two positive roots of \( p \). We claim that the first positive root does not satisfy the non-stagnation condition (2). In order to see this, we note that the first positive root \( x_{01} \) is such that \( x_{01} < x_0 \). Indeed, the function given in (20) is increasing in \( [-d, 0] \) for \( \gamma_1 \) and \( \gamma_2 \) negative. This means that the non-stagnation condition is satisfied for all \( y \) in \( [-d, 0] \) if and only if \( c - u(-d) > 0 \). Therefore, by (41), any root \( x_\lambda = -\frac{m \sqrt{\Delta}}{d} \) of \( p \) must satisfy

\[
x_\lambda > (\gamma_2 - \gamma_1) \nu - \gamma_2 d =: x_c.
\]

However, \( x_{01} < x_0 \) and

\[
x_0 \leq (\gamma_2 - \gamma_1) \nu - \gamma_2 d, \quad \text{since} \quad \gamma_2 < 0
\]

and the only admissible positive root agreeing with the non-stagnation condition is \( x_{02} \). Thus, once (44) holds, bifurcation occurs if and only if the root \( x_{02} \) satisfies (50), which due to the properties of the polynomial (as illustrated in Figure 1), is equivalent to

\[
p(x_c) < 0.
\]

Evaluating now the polynomial at \( x_c \) it follows (after some algebraic manipulations) that

\[
p(x_c) = -\gamma_2 (d - \nu) x_c^2 + (\gamma_2 - \gamma_1) x_c^2 \frac{\sinh(d - \nu) \cosh(\nu)}{\cosh(d)} +
\]

\[
+ \gamma_2 (d - \nu) [g - \gamma_1 x_c] \tanh(d) - (\gamma_2 - \gamma_1) \left[ g - \gamma_1 x_c \right] \frac{\sinh(d - \nu) \sinh(\nu)}{\cosh(d)}.
\]

Furthermore,

\[
-\frac{\cosh(d)}{\gamma_2 (d - \nu) x_c^2} > 0
\]

and (51) is now equivalent to

\[
\cosh(d) - \frac{\gamma_2 - \gamma_1}{\gamma_2 (d - \nu)} \sinh(d - \nu) \cosh(\nu) - \frac{g - \gamma_1 x_c}{x_c^2} \sinh(d) +
\]

\[
+ \frac{\gamma_2 - \gamma_1}{\gamma_2 (d - \nu) x_c^2} \left[ g - \gamma_1 x_c \right] \sinh(d - \nu) \sinh(\nu) < 0
\]

which, by rearranging the terms, is equivalent to condition (47). \( \square \)
Remark 4. The solutions of (42) can be obtained by Cardano’s formula. However, it can be more instructive to give an estimate of those solutions. This can be done similarly as in [47]. We perform this task in the next two lemmas.

Lemma 5.4. Let be \( \gamma_1 < \gamma_2 \) and \( \nu < d \). Given the equations
\[
x^2 + \gamma_1 \tanh(d) x - g \tanh(d) = 0 \tag{52}
\]
and
\[
x^2 + \gamma_1 \tanh(\nu) x - g \tanh(\nu) = 0, \tag{53}
\]
the positive solution, \( x \), of (42) satisfies
\[
x^+_\nu < x < x^+_d, \tag{54}
\]
where \( x^+_d \) is the positive solution of (52) and \( x^+_\nu \) is the positive solution of (53).

Proof. Let us consider the function \( d \mapsto \chi(d) = x^+_d \).
\[
\chi(d) = \frac{-\gamma_1 \tanh(d) + \sqrt{\gamma_1^2 \tanh^2(d) + 4g \tanh(d)}}{2}.
\]
It is easy to see that \( \chi'(d) \) is positive and therefore \( \chi \) is strictly increasing, proving that \( 0 < x^+_\nu < x^+_d \). On the other hand, (42) can be rewritten as
\[
\frac{1}{\cosh(\nu)} \left[ x^2 + \gamma_1 \tanh(d) x - g \tanh(d) \right] (x + \gamma_1 \nu) = \left( \gamma_1 - \gamma_2 \right) \frac{\sinh(d - \nu)}{\cosh(d)} \left[ x^2 + \gamma_1 \tanh(\nu) x - g \tanh(\nu) \right]. \tag{55}
\]
Therefore, if we assume that a root, \( x \), of (42) is such that \( x \geq x^+_d \), then
\[
x^2 + \gamma_1 \tanh(d) x - g \tanh(d) \geq 0. \tag{56}
\]
In addition, the non-stagnation condition (20) implies that
\[
x + \gamma_1 \nu = -\frac{p_0 \sqrt{\lambda}}{d} + \gamma_1 \nu > 0.
\]
Thus,
\[
x^2 + \gamma_1 \tanh(\nu) x - g \tanh(\nu) \leq 0
\]
and \( 0 < x \leq x^+_\nu \), which is a contradiction with the assumption \( x \geq x^+_d \). Finally, we obtain that \( x < x^+_d \) and therefore,
\[
x^2 + \gamma_1 \tanh(d) x - g \tanh(d) < 0,
\]
which by (55) and (20) again, yields \( x > x^+_\nu \).

Lemma 5.5. If \( \gamma_1 > \gamma_2 \) then
\[
0 < x < x^+_\nu \quad \text{or} \quad x > x^+_d,
\]
where \( x \) denotes the positive solution of (42) and \( x^+_\nu, x^+_d \) are as in Lemma 5.4.

Proof. The proof is similar to the one in Lemma 5.4.
6. **Stability of the laminar solutions.** This section comprises a stability analysis of the laminar flow solutions (15) following the point of view presented in [6, 47] in the scenario of the fixed mass flux. The essential difference, if compared with [6, 47], is that we work in the physical setting of the fixed mean-depth. We introduce first some preliminary notation. Let \( \mathcal{F} \) be the functional given by

\[
\mathcal{F}(\psi) = \frac{1}{2dp_0^2} \int_D \left[ |\nabla \psi(x, y)|^2 + Q + \frac{p_0^2}{2} \Gamma \left( \frac{\psi(x, y)}{p_0} \right) \right] dx \, dy \\
- \int_D g_d \frac{1}{p_0^2} (1 + h(x, y)) h_y \, dx \, dy.
\]

(57)

The change of variables (7) transforming the free-surface dependent domain \( D \) into \( R \), yields the following functional

\[
\mathcal{L}(h) = \int_R \left[ \frac{1 + d^2h_0^2}{2d^2(1 + h_p)^2} + \frac{Q}{2p_0} + \frac{\Gamma(p)}{2d^2} (1 + h_p) \right] (1 + h_p) \, dq \, dp - \int_R g_d \frac{1}{p_0^2} (1 + h) h_p \, dq \, dp.
\]

(58)

The jump in vorticity (17) induces a loss in regularity of solutions \( h \) of (10) along the line \( p = p_1 \), (see for [46] details). Furthermore, it can be shown as in [46] that, given \( \alpha \in (0, 1) \), there exists a real-analytic curve in \( C^{1, \alpha}(R) \) consisting of solutions

\[
h(q, p) = \begin{cases} 
H(q, p) & \text{for} \ (q, p) \in [0, L] \times [p_1, 0], \\
h(q, p) & \text{for} \ (q, p) \in [0, L] \times [-1, p_1]
\end{cases}
\]

(59)

to the problem (12) such that

\[
H \in C^{2, \alpha}([0, L] \times [p_1, 0]) \cap C^\infty([0, L] \times (p_1, 0)), \\
h \in C^{2, \alpha}([0, L] \times [-1, p_1]) \cap C^\infty([0, L] \times (-1, p_1))
\]

(60)

and \( h \) is continuously differentiable with respect \( p \) along \([0, L] \times p_1 \). In order to characterize the critical points of the functional (58) we compute the first variation of \( \mathcal{L} \). Let \( t \) be the \( L \)-periodic and even-in-\( q \) functions

\[
t(q, p) = \begin{cases} 
T(q, p) & \text{for} \ (q, p) \in [0, L] \times [p_1, 0], \\
t(q, p) & \text{for} \ (q, p) \in [0, L] \times [-1, p_1]
\end{cases}
\]

(61)

which are continuously differentiable with respect \( p \) along \([0, L] \times p_1 \) and such that \( T \in C^{2, \alpha}([0, L] \times [p_1, 0]), t \in C^{2, \alpha}([0, L] \times [-1, p_1]) \) and

\[
\int_0^L T(q, 0) \, dq = 0, \\
T(q, -1) = 0.
\]

(62)

Taking \( t \) as test functions, we obtain

\[
\langle \delta \mathcal{L}(h), t \rangle = \lim_{\epsilon \to 0} \frac{\mathcal{L}(h + \epsilon t) - \mathcal{L}(h)}{\epsilon} \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_R \left[ \frac{1 + d^2(h_0 + \epsilon q)^2}{2d^2(1 + h_p + \epsilon t_p)^2} - \frac{1 + d^2h_0^2}{2d^2(1 + h_p)^2} \right] t_p \, dq \, dp \\
+ \int_R \left[ \frac{Q}{2p_0} + \frac{\Gamma(p)}{2d^2} \right] t_p \, dq \, dp - \lim_{\epsilon \to 0} \int_R g_d \frac{1}{p_0^2} (1 + h) t_p + \epsilon t \, dq \, dp
\]
Hence, the first variation of the functional (58) is given by
\[ i.e, \text{critical points of } L \]
Therefore, any critical point of the functional

**Definition 6.1.** The function \( h \) is said to be formally stable if \( \langle \delta^2 \mathcal{L}(h) t, t \rangle \geq 0 \) for all the functions \( t \) given in (61).

The second variation of \( \mathcal{L} \) at \( h \) in the direction of \( t \) and \( s \) is defined by

\[
\langle \delta^2 \mathcal{L}(h)s, t \rangle = \lim_{\epsilon \to 0} \frac{\langle \delta \mathcal{L}(h + \epsilon t), s \rangle - \langle \delta \mathcal{L}(h), s \rangle}{\epsilon}.
\]
Let \( h \) be a critical point of \( \mathcal{L} \). Then, it follows from (63) and (66) that
\[
\langle \delta^2 \mathcal{L}(h), t, t \rangle = -\frac{1}{2} \int_0^L \mathcal{G}_\eta(H)[S](1 + H_p)^{-2}T \bigg|_{p=0} dq - \int_R \mathcal{G}_\eta(h)[s](1 + h_p)^{-3} t dq dp,
\]
where
\[
\mathcal{G}_\eta(H)[S] = \lim_{\epsilon \to 0} \frac{\mathcal{G}_\eta(H + \epsilon S) - \mathcal{G}_\eta(H)}{\epsilon}
\]
and
\[
\mathcal{G}_\eta(h)[s] = \lim_{\epsilon \to 0} \frac{\mathcal{G}(h + \epsilon s) - \mathcal{G}(h)}{\epsilon}
\]
are the derivatives of the correspondent functionals (64) and (65). We are interested in the stability of laminar solutions
\[
\bar{h}(p) = \begin{cases} \bar{H}(p) & \text{for } p \in [p_1, 0], \\ \bar{h}(p) & \text{for } p \in [-1, p_1]. \end{cases}
\]
Let \( \lambda^\ast \) be the critical value for which the bifurcation from the laminar solutions occurs and which is related to the velocity on the surface by (15). Then we have the following result.

**Theorem 6.2.** A laminar solution \( \bar{h}(\cdot, \lambda) \) is formally stable if and only if \( \lambda \geq \lambda^\ast \).

**Proof.** In order to obtain the second variation of \( \mathcal{L} \) at \( \bar{h} \) we will consider the test functions \( t \) and \( s \) of the form (61). The independence of \( \bar{h} \) on \( q \), yields
\[
\mathcal{G}_\eta(\bar{H})[S] = \frac{2gd}{p_0^2}(1 + \bar{H}_p)^2S + \frac{2}{p_0}(1 + \bar{H}_p)[2gd(\bar{H} + 1) - Q]S_p
\]
and
\[
\mathcal{G}_\eta(\bar{h})[s] = \frac{1}{d^2} s_{pp} + (1 + \bar{h}_p)^2 s_{qq} + 3 \frac{\gamma(p)}{p_0} (1 + \bar{h}_p)^2 s_p
\]
Thus, from (68)
\[
\langle \delta^2 \mathcal{L}(\bar{h}), t, t \rangle = -\frac{1}{2} \int_0^L \left[ \frac{2gd}{p_0^2}T^2 + \frac{2}{p_0^2}(1 + \bar{H}_p)[2gd(\bar{H} + 1) - Q]T_p \right] \bigg|_{p=0} dq
- \int_R \left[ \frac{1}{d^2(1 + \bar{h}_p)^3} t_{pp} + \frac{1}{1 + \bar{h}_p} t_{pq} + 3 \frac{\gamma(p)}{p_0(1 + \bar{h}_p)} t_p \right] dq dp.
\]
Taking into account the regularity of the test functions \( t \) and \( \bar{H} \) along \( p = p_1 \) and the assumption \( T(q, -1) = 0 \), we can see that
\[
\int_{-1}^0 \frac{1}{d^2(1 + \bar{h}_p)^3} t_{pp} dp = \frac{T T_p}{d^2(1 + \bar{h}_p)^3} \bigg|_{p=0} - \int_{-1}^0 \frac{1}{d^2(1 + \bar{h}_p)^3} t_{pp}^2 dp + \int_{-1}^0 \frac{3 \bar{h}_{pp}}{d^2(1 + \bar{h}_p)^3} \frac{t_p}{(1 + \bar{h}_p)} dp.
\]
Using that the laminar solutions satisfy the relation
\[
\frac{\bar{h}_{pp}}{(1 + \bar{h}_p)^3} = -\frac{d^2 \gamma(p)}{p_0},
\]
we integrate with respect to \( q \) in (69) and obtain
\[
\int_R \frac{1}{d^2(1 + \bar{h}_p)^3} t_{pp} dq dp = \int_0^L \frac{T T_p}{d^2(1 + \bar{h}_p)^3} \bigg|_{p=0} dq - \int_R \frac{1}{d^2(1 + \bar{h}_p)^3} t_{pp}^2 dq dp
\]
\[- \int \int_R \frac{3 \gamma(p)}{p_0(1 + \mathcal{H}_p)} t \ t_p \ dq dp. \]  

(70)

In addition, by (14) and (15)

\[ 2gd(h(0) + 1) - Q = - \frac{p_0^2}{d^2(1 + \mathcal{H}_p(0))^2}. \]  

(71)

Finally, by (70) and (71) we obtain

\[ \langle \delta^2 \mathcal{L}(\mathcal{H}) t, t \rangle = - \int \int_R \left[ \frac{tt_{qq}}{(1 + \mathcal{H}_p)} - \frac{t_p^2}{d^2(1 + \mathcal{H}_p)^3} \right] dq dp. \]  

(72)

Due to the requirements of the water wave problem, we have taken continuously differentiable, \(-L\)-periodic and even-in-\(q\) test functions. Therefore, we can expand \(t\) in Fourier series

\[ t(q, p) = \sum_{n=0}^{\infty} t_n(p) \cos \left( \frac{2\pi n q}{L} \right). \]  

(73)

where

\[ t_n(p) = \begin{cases} T_n(p) = \frac{2}{L} \int_0^L T(q, p) \cos \left( \frac{2\pi n q}{L} \right) dq & \text{for } p \in [p_1, 0], \\ t_n(p) = \frac{2}{L} \int_0^L t(q, p) \cos \left( \frac{2\pi n q}{L} \right) dq & \text{for } n \geq 1 \\ t_n(p) = \frac{2}{L} \int_0^L t(q, p) dq & \text{for } p \in [-1, p_1] \end{cases} \]

and

\[ t_0(p) = \begin{cases} T_0(p) = \frac{1}{L} \int_0^L T(q, p) dq & \text{for } p \in [p_1, 0], \\ t_0(p) = \frac{1}{L} \int_0^L t(q, p) dq & \text{for } p \in [-1, p_1] \end{cases} \]

are both continuously differentiable along \(p = p_1\) and such that \(t_0(0) = T_0(0) = 0\).

Now (72) takes the form

\[ \langle \delta^2 \mathcal{L}(\mathcal{H}) t, t \rangle = L \left[ - \frac{gd}{p_0^2} t_0^2(0) + \int_{-1}^{0} \left( \frac{t_n^2(p)}{d^2(1 + \mathcal{H}_p)^3} \right) dp \right] + \]

\[ + \frac{L}{2} \sum_{n=1}^{\infty} \left\{ - \frac{gd}{p_0^2} t_n^2(0) + \int_{-1}^{0} \left[ \frac{(t_n'(p))^2}{d^2(1 + \mathcal{H}_p)^3} + \left( \frac{2\pi n}{L} \right)^2 \frac{t_n^2(p)}{(1 + \mathcal{H}_p)} \right] dp \right\} \]

\[ = \frac{L}{2} \sum_{n=0}^{\infty} \left\{ - \frac{gd}{p_0^2} t_n^2(0) + \int_{-1}^{0} \left[ \frac{(t_n'(p))^2}{d^2(1 + \mathcal{H}_p)^3} + \left( \frac{2\pi n}{L} \right)^2 \frac{t_n^2(p)}{(1 + \mathcal{H}_p)} \right] dp \right\}. \]  

(74)

It was shown in section 3 of [35] that if \( \hat{k} = \frac{2\pi n}{L} \) in (73) then

\[ \inf_t \frac{- \frac{gd t^2(0)}{p_0^2} + \int_{-1}^{0} \frac{(t')^2}{d^2(1 + \mathcal{H}_p)^3} dp}{\int_{-1}^{0} \frac{t^2(p)}{d^2(1 + \mathcal{H}_p)} dp} \geq -\hat{k}^2 \quad \text{if and only if} \quad \lambda \geq \lambda^*. \]
Therefore, by the last inequality and (74), it follows that \( \langle \delta^2 \mathcal{L}(\bar{h})t, t \rangle \geq 0 \) if and only if \( \lambda \geq \lambda^* \).

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