BOHR RADIUS FOR CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX ANALYTIC AND HARMONIC MAPPINGS

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Abstract. We say that a class $B$ of analytic functions $f$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ satisfies a Bohr phenomenon if for the largest radius $R_f < 1$, the following inequality
\[
\sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial f(D))
\]
holds for $|z| = r \leq R_f$ and for all functions $f \in B$ in $D$. The largest radius $R_f$ is called Bohr radius for the class $B$. In this article, we obtain Bohr radius for certain subclasses of close-to-convex analytic functions as well as close-to-convex harmonic mappings. We establish the Bohr inequality for certain analytic classes $S^*_\phi(\varphi), C_\phi(\varphi), C^*_\phi(\varphi), K_\phi(\varphi)$ and for harmonic class $M(\alpha, \beta)$. Using Bohr phenomenon for subordination classes [14, Lemma 1], we obtain some radius $R_f$ such that Bohr phenomenon for these classes holds for $|z| = r \leq R_f$. Generally, in this case $R_f$ need not be sharp, but we show that under some additional conditions on $\varphi$, the radius $R_f$ becomes sharp bound. As a consequence of these results, we obtain some interesting corollaries on Bohr phenomenon for these classes.

1. Introduction and Preliminaries

The classical Bohr inequality says that if $f$ is an analytic function in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ of the form
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
and $|f(z)| < 1$ for all $z \in D$, then the majorant series
\[
M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1
\]
holds for $z \in D$ with $|z| = r \leq 1/3$ and the constant $1/3$, referred to as the Bohr radius, cannot be improved. The inequality (1.2) was introduced by Bohr [17] in 1914. Bohr proved that the inequality (1.2) holds for $|z| = r \leq 1/6$. Later, the value $1/6$ was sharpened to $1/3$ independently by Wiener, Riesz and Schur. Other proofs of this result can also be found in [28, 35, 36]. The idea of Bohr’s theorem has been extended to several complex variables and thus, a variety of results on

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Bohr’s inequality in higher dimension has been obtained. For Bohr radius and Bohr phenomenon, we suggest the reader to glance through the articles [5, 6, 12, 15, 16, 28] and the references therein.

The inequality (1.2) can also be written in the following form

\[(1.3)\]

\[\sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0| = d(f(0), \partial f(D))\]

for \(|z| = r \leq 1/3\), where \(d\) is the Euclidean distance. It is worth noting that the existence of the radius \(1/3\) in (1.3) is independent of the coefficients of the power series (1.1). Analytic functions of the form (1.1) with modulus less than 1 satisfying the inequality (1.3), are sometimes said to satisfy the classical Bohr phenomenon. Therefore we conclude that Bohr phenomenon occurs in the class of analytic self-maps of the unit disk \(D\). The notion of Bohr phenomenon has been extended to the class of analytic functions from \(D\) into a given domain \(D \subseteq \mathbb{C}\). Let \(G\) be the class of analytic functions of the form (1.1) which map \(D\) into a given domain \(D\) such that \(f(D) \subseteq D\). Suppose there exists the largest radius \(r_D > 0\) such that

\[(1.4)\]

\[\sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial f(D)) \text{ in } |z| \leq r_D,\]

for all functions \(f \in G\). In this case, we say that \(G\) satisfies a Bohr phenomenon. In [7], it has been proved that the largest radius \(r_D\) for convex domain \(D\) coincides with the classical Bohr radius \(1/3\) while Abu-Muhanna [1] has obtained \(r_D = 3 - 2\sqrt{2}\) for any proper simply connected domain \(D\). For more intriguing aspects of Bohr phenomenon, we refer the reader to the articles (see [2, 3, 8, 9]) and references therein.

Let \(A\) denote the class of normalized analytic functions in \(D\) of the form

\[(1.5)\]

\[f(z) = z + \sum_{n=2}^{\infty} a_n z^n\]

and \(S\) be its standard subclass made up of normalized univalent (i.e. one-to-one) functions in \(D\). A domain \(\Omega \subseteq \mathbb{C}\) is said to be starlike with respect to a point \(z_0 \in \Omega\) if the linear segment joining \(z_0\) to every other point \(z \in \Omega\) lies entirely in \(\Omega\). A domain \(\Omega\) is said to be starlike domain if it is starlike with respect to \(z = 0\). The domain \(\Omega\) is said to be convex if it is starlike with respect to each of its points. A starlike (respectively convex) function is one which maps the unit disk \(D\) onto a starlike (respectively convex) domain. Let \(S^*\) (respectively \(C\)) be the subclass of \(S\) consisting of starlike (respectively convex) functions in \(D\). It is well-known that \(f \in S^*\) (\(C\) respectively) if, and only if, \(\text{Re} (zf'(z)/f(z)) > 0\) for \(z \in D\) \((\text{Re} (1 + zf''(z)/f'(z)) > 0\) for \(z \in D\) respectively). Let \(S^*(\alpha)\) and \(C(\alpha)\) be the subclasses of \(S\) consisting of functions starlike of order \(\alpha\) \((0 \leq \alpha < 1)\) and convex functions of order \(\alpha\) \((0 \leq \alpha < 1)\) respectively, with the characterizations: \(f \in S^*(\alpha)\) (respectively \(C(\alpha)\)) if, and only if, \(\text{Re} (zf'(z)/f(z)) > \alpha\) for \(z \in D\)
(Re \((1 + zf''(z)/f'(z)) > \alpha\) for \(z \in \mathbb{D}\) respectively). Clearly, \(f \in \mathcal{C}(\alpha)\) if, and only, if \(zf' \in \mathcal{S}_*^{*(\alpha)}\). Note that the classes \(\mathcal{S}_*^{*} := \mathcal{S}_0^{*}(0)\) and \(\mathcal{C} := \mathcal{C}(0)\) are the family of starlike and convex functions in \(\mathbb{D}\) respectively.

An analytic function \(f\) in \(\mathbb{D}\) is said to be subordinate to an analytic function \(g\) in \(\mathbb{D}\), denoted by \(f \prec g\) (sometimes written \(f(z) \prec g(z)\)), if \(f(z) = g(\omega(z))\) for \(z \in \mathbb{D}\), where \(\omega : \mathbb{D} \to \mathbb{D}\) is the analytic function such that \(\omega(0) = 0\) and \(|\omega(z)| < 1\) in \(\mathbb{D}\). In particular, when \(g\) is univalent in \(\mathbb{D}\), then \(f \prec g\) if, and only if, \(f(0) = g(0)\) and \(f(\mathbb{D}) \subseteq g(\mathbb{D})\). Let \(\phi : \mathbb{D} \to \mathbb{C}\), called Ma-Minda function which is analytic and univalent in \(\mathbb{D}\) such that \(\phi(\mathbb{D})\) has positive real part, symmetric with respect to the real axis, starlike with respect to \(\phi(0) = 1\) and \(\phi'(0) > 0\). Such Ma-Minda functions have the Taylor series expansion of the form \(\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n\) \((B_1 > 0)\).

For such \(\phi\), Ma-Minda \cite{27} considered the more general classes \(\mathcal{S}_*^{*}(\phi)\) and \(\mathcal{C}(\phi)\), called Ma-Minda type starlike and Ma-Minda type convex classes associated with \(\phi\) respectively, where \(\mathcal{S}_*^{*}(\phi)\) and \(\mathcal{C}(\phi)\) are the subclasses of functions in \(\mathcal{S}\) with the following characterization:

\[
\frac{zf'(z)}{f(z)} \prec \phi(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)
\]

respectively. Clearly, \(f \in \mathcal{C}(\phi)\) if, and only if, \(zf' \in \mathcal{S}_*^{*}(\phi)\). It is important to note that for every such \(\phi\) described in above, \(\mathcal{S}_*^{*}(\phi)\) and \(\mathcal{C}(\phi)\) are always subclasses of the classes \(\mathcal{S}_*^{*}\) and \(\mathcal{C}\) respectively by taking \(\phi(z) = (1 + z)/(1 - z)\). For various \(\phi\), the classes \(\mathcal{S}_*^{*}(\phi)\) and \(\mathcal{C}(\phi)\) yield various important subclasses of starlike and convex functions, respectively. When \(\phi(z) = (1 + (1 - 2\alpha))/(1 - z)\), we obtain the classes \(\mathcal{S}_*^{*}(\alpha)\) and \(\mathcal{C}(\alpha)\). By taking \(\phi(z) = (1 + Az)/(1 + Bz)\), \(\mathcal{S}_*^{*}(\phi)\) and \(\mathcal{C}(\phi)\) reduce to the Janowski starlike class \(\mathcal{S}_*[A, B]\) and Janowski convex class \(\mathcal{C}[A, B]\) respectively. By taking \(\phi(z) = ((1 + z)/(1 - z))^{\alpha}\) for \(0 < \alpha \leq 1\), we obtain the classes of strongly convex and strongly starlike functions of order \(\alpha\). The extremal functions \(k\) and \(h\) respectively for the classes \(\mathcal{C}(\alpha)\) and \(\mathcal{S}_*^{*}(\alpha)\) as follows:

\[
1 + \frac{zk''(z)}{k'(z)} = \phi(z) \quad \text{and} \quad \frac{zh'(z)}{h(z)} = \phi(z)
\]

with the normalizations \(k(0) = k'(0) - 1 = 0\) and \(h(0) = h'(0) - 1 = 0\). Obviously the functions \(k\) and \(h\) belong to the classes \(\mathcal{C}(\alpha)\) and \(\mathcal{S}_*^{*}(\alpha)\) and play the role of Koebe functions in the respective classes. Ma and Minda \cite{27} have obtained the following subordination theorems and growth estimates for the classes \(\mathcal{S}_*^{*}(\phi)\) and \(\mathcal{C}(\phi)\).

**Lemma 1.7.** \cite{27} Let \(f \in \mathcal{S}_*^{*}(\phi)\). Then \(zf'(z)/f(z) \prec zh'(z)/h(z)\) and \(f(z)/z \prec h(z)/z\).

**Lemma 1.8.** \cite{27} Assume \(f \in \mathcal{S}_*^{*}(\phi)\) and \(|z| = r < 1\). Then

\[
-h(-r) \leq |f(z)| \leq h(r).
\]

Equality holds for some \(z \neq 0\) if, and only, if \(f\) is a rotation of \(h\).

**Lemma 1.10.** \cite{27} Let \(f \in \mathcal{C}(\phi)\). Then \(zf''(z)/f'(z) \prec zk''(z)/k'(z)\) and \(f'(z) \prec k'(z)\).
Lemma 1.11. [27] Assume \( f \in \mathcal{C}(\phi) \) and \( |z| = r < 1 \). Then
\[
- k(-r) \leq |f(z)| \leq k(r).
\]
Equality holds for some \( z \neq 0 \) if, and only, if \( f \) is a rotation of \( k \).

Ma-Minda functions \( \phi \) have been considered with the condition \( \phi'(0) > 0 \). Motivated by this, recently, Kumar and Banga [25] have introduced the function \( \Phi \), called non-Ma-Minda function, with the condition \( \Phi'(0) < 0 \) and the other conditions are same as that of \( \phi \). Note that \( \Phi \) is obtained from \( \phi \) by a rotation, namely, \( z \) by \(-z\). By going a similar manner as the definition of \( S^*(\phi) \) and \( C(\phi) \) [27], Kumar and Banga have considered the classes \( S^*(\Phi) \) and \( C(\Phi) \) and also studied the growth estimates and some other properties of these classes.

Let \( \mathcal{K} \) and \( \mathcal{C}^* \) respectively denote the classes of close-to-convex and quasi-convex functions in \( \mathbb{D} \) which are defined as:
\[
\mathcal{K} = \left\{ f : f \in \mathcal{A}, g \in \mathcal{S}^*, \quad \text{and} \quad \text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D} \right\}
\]
and
\[
\mathcal{C}^* = \left\{ f : f \in \mathcal{A}, g \in \mathcal{C}, \quad \text{and} \quad \text{Re} \left( \frac{(zf'(z))'}{g'(z)} \right) > 0, \quad z \in \mathbb{D} \right\}.
\]
In 1959, Sakaguchi [24] introduced the subclass \( S^*_s \) of functions starlike with respect to symmetric points, which consists of functions \( f \in \mathcal{S} \) satisfying the condition
\[
\text{Re} \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0
\]
for \( z \in \mathbb{D} \). Motivated by \( S^*_s \), Wang et.al. [38] have considered \( \mathcal{C}_s \), i.e. a function \( f \in \mathcal{C}_s \) if \( f \) satisfies the following inequality
\[
\text{Re} \left( \frac{(zf'(z))'}{(f(z) - f(-z))'} \right) > 0, \quad \text{for} \quad z \in \mathbb{D}
\]
A function \( f \in \mathcal{A} \) is starlike with respect to conjugate points and convex with respect to conjugate points in \( \mathbb{D} \) if \( f \) satisfies the conditions
\[
\text{Re} \left( \frac{zf'(z)}{f(z) + f(\bar{z})} \right) > 0, \quad \text{and} \quad \text{Re} \left( \frac{(zf'(z))'}{(f(z) + f(\bar{z}))'} \right) > 0, \quad z \in \mathbb{D}
\]
respectively. A function \( f \in \mathcal{A} \) is starlike with respect to symmetric conjugate points in \( \mathbb{D} \) if it satisfies the inequality
\[
\text{Re} \left( \frac{zf'(z)}{f(z) - f(\bar{z})} \right) > 0, \quad z \in \mathbb{D}.
\]
In more general, Ravichandran [30] has defined the classes \( S^*_s(\phi) \) and \( \mathcal{C}_s(\phi) \).
Definition 1.1. [30] A function \( f \in \mathcal{A} \) is in the class \( \mathcal{S}_s^*(\phi) \) if
\[
\frac{2zf'(z)}{f(z) - f(-z)} < \phi(z), \quad z \in \mathbb{D}
\]
and is in the class \( \mathcal{C}_s(\phi) \) if
\[
\frac{2(zf'(z))'}{f'(z) + f'(-z)} < \phi(z), \quad z \in \mathbb{D}.
\]

Similarly, let \( \mathcal{S}_{sc}^*(\phi) \) and \( \mathcal{S}_{sc}^*(\phi) \) be the corresponding classes of starlike functions with respect to conjugate points and symmetric conjugate points respectively. Let \( \mathcal{C}_s(\phi) \) and \( \mathcal{C}_{sc}(\phi) \) be the corresponding classes of convex functions with respect to conjugate points and symmetric conjugate points respectively. The following lemmas are required to prove some of our results.

Lemma 1.13. [30] Let \( \min_{|z|=r} |\phi(z)| = \phi(-r), \max_{|z|=r} |\phi(z)| = \phi(r), |z| = r \). If \( f \in \mathcal{C}_s(\phi) \), then
\[
\frac{1}{r} \int_0^r \phi(-t)[k'(-t^2)]^{1/2} dt \leq |f'(z)| \leq \frac{1}{r} \int_0^r \phi(t)[k'(t^2)]^{1/2} dt.
\]

From [38], for \( f \in \mathcal{C}_s(\phi) \), we have
\[
\int_0^r \frac{1}{s} \int_0^s \phi(-t)[k'(-t^2)]^{1/2} dt ds \leq |f(z)| \leq \int_0^r \frac{1}{s} \int_0^s \phi(t)[k'(t^2)]^{1/2} dt ds
\]
and the results are sharp for the function
\[
f(z) = \int_0^z \frac{1}{\xi} \int_0^\xi \phi(-\eta)[k'(-\eta^2)]^{1/2} d\eta d\xi \in \mathcal{C}_s(\phi),
\]

since it has real coefficients and is in \( \mathcal{C}(\phi) \).

Lemma 1.16. [21] Let \( f(z) = z + a_{l+1}z^{l+1} + \cdots \in \mathcal{C}(\phi) \), then we have
\[
[k'(-r^2)]^{1/2} \leq |f'(z)| \leq [k'(r^2)]^{1/2}.
\]

In particular for \( l = 2 \) we can obtain the bounds of \( |f'(z)| \) for odd convex functions. From Lemma 1.16 the following can be easily obtained for \( l = 2 \)
\[
\int_0^r [k'(-t^2)]^{1/2} dt \leq |f(z)| \leq \int_0^r [k'(t^2)]^{1/2} dt.
\]
The result is sharp for the function \( K(z) := \int_0^z [k'(\xi^2)]^{1/2} d\xi \). It is easy to see that \( K \) is odd convex function belongs to \( \mathcal{C}(\phi) \). From [21], the function \( H(z) := [h(z^2)]^{1/2} \) is a Koebe type function for the odd starlike class \( \mathcal{S}_s^*(\phi) \), where the function \( K \) defined by
\[
zK'(z) = H(z),
\]
is a Koebe type function for odd convex class in \( \mathcal{C}(\phi) \).
Lemma 1.18. [20] Let \( \min_{|z|=r} |\phi(z)| = \phi(-r) \), \( \max_{|z|=r} |\phi(z)| = \phi(r) \), \( |z| = r \). If \( f \in S^*_c(\phi) \), then

(i) \( h'(-r) \leq |f'(z)| \leq h'(r) \)
(ii) \( -h(-r) \leq |f(z)| \leq h(r) \)
(iii) \( f(D) \supseteq \{ w : |w| \leq h(-1) \} \).

The results are sharp.

Lemma 1.19. [20] Let \( \min_{|z|=r} |\phi(z)| = \phi(-r) \), \( \max_{|z|=r} |\phi(z)| = \phi(r) \), \( |z| = r \). If \( f \in C_c(\phi) \), then

(i) \( k'(-r) \leq |f'(z)| \leq k'(r) \)
(ii) \( -k(-r) \leq |f(z)| \leq k(r) \)
(iii) \( f(D) \supseteq \{ w : |w| \leq k(-1) \} \).

The results are sharp.

Motivated by the class \( S^*_c \), Gao and Zhou [20] have studied the class \( K_s \) of close-to-convex univalent functions, where \( K_s \) is the class of all functions \( f \in S \) satisfying the condition

\[
\text{Re} \left( \frac{z^2f'(z)}{g(z)g(-z)} \right) < 0, \quad z \in \mathbb{D}.
\]

A more general class \( K_s(\phi) \) has been studied extensively by Cho et al. [19] and Wang et al. [37]. For the brevity, we write the definition.

Definition 1.2. [37] For a function \( \phi \) with positive real part, the class \( K_s(\phi) \) consists of functions \( f \in A \) satisfying

\[
-\frac{z^2f'(z)}{g(z)g(-z)} < \phi(z) \quad (z \in \mathbb{D})
\]

for some function \( g \in S^*(1/2) \).

In particular, for \( \phi(z) = (1 + (1 - 2\gamma)z)/(1 - z) \) with \( 0 \leq \gamma < 1 \), the class \( K_s(\phi) \) reduces to \( K_s(\gamma) \) which was recently investigated by Kowalczyk and Les-Bomba [29]. When \( \gamma = 0 \), we can obtain \( K_s \), the subclass of close-to-convex functions which has been defined by Gao and Zhou [20]. When \( \phi(z) = (1 + \beta z)/(1 - \alpha \beta z) \), where \( 0 \leq \alpha \leq 1 \) and \( 0 < \beta \leq 1 \), the class \( K_s(\phi) \) reduces to \( K_s(\alpha, \beta) \) defined in [37]. Now let \( q(z) = \sum_{n=1}^{\infty} q_n z^n \) be analytic in \( \mathbb{D} \). Then for fixed \( f \in K_s(\phi) \), we define

\[
S^K_f(\phi) := \left\{ q(z) = \sum_{n=1}^{\infty} q_n z^n : q < f \right\}
\]

The distortion and growth theorems for the class \( K_s(\phi) \) have been obtained in [19]. Let \( \phi \) be a Ma-Minda function.

Lemma 1.21. [19] Let \( \min_{|z|=r} |\phi(z)| = \phi(-r) \), \( \max_{|z|=r} |\phi(z)| = \phi(r) \), \( |z| = r \). If \( f \in K_s(\phi) \), then the following sharp inequalities hold:

(i) \( \frac{\phi(-r)}{1 + r^2} \leq |f'(z)| \leq \frac{\phi(r)}{1 - r^2} \) \( (|z| = r < 1) \)
(ii) \( \int_0^r \frac{\phi(t)}{1 + t^2} \, dt \leq |f(z)| \leq \int_0^r \frac{\phi(t)}{1 - t^2} \, dt \) \( (|z| = r < 1) \).
Let \( \mathcal{H} \) be the class of all complex-valued harmonic functions \( f = h + \overline{g} \) defined on \( \mathbb{D} \) normalized by the conditions \( h(0) = h'(0) - 1 = 0 \) and \( g(0) = 0 \) of the form

\[
(1.22) \quad f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n
\]

and \( S_{\mathcal{H}} \) be the subclass of sense-preserving harmonic mappings of the form (1.22) in \( \mathbb{D} \). A harmonic mappings in \( \mathbb{D} \) is sense-preserving if, and only, if \( |h'(z)| > |g'(z)| \) for all \( z \in \mathbb{D} \). Set \( S^0_{\mathcal{H}} = S_{\mathcal{H}} \cap \mathcal{H} \). In 2016, Sun et al. defined the class \( \mathcal{M}(\alpha, \beta) \) of close-to-convex harmonic mappings.

**Definition 1.3.** For \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \) and \( -1/2 \leq \beta < 1 \), let \( \mathcal{M}(\alpha, \beta) \) denote the class of harmonic mappings \( f \) of the form (1.22), with \( h'(0) \neq 0 \), which satisfies

\[
g'(z) = \alpha z h'(z) \quad \text{and} \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta.
\]

For \( \alpha = 1 \) and \( \beta = -1/2 \), \( \mathcal{M}(\alpha, \beta) \) reduces to \( \mathcal{M}(1, -1/2) \), which has been studied extensively by Bshouty [18] and references therein. The class \( \mathcal{M}(1, -1/2) \) with \( |\alpha| = 1 \), has been extended to \( \mathcal{M}(\alpha, -1/2) \) in [13]. It is worth to point out that when the co-analytic part \( g \equiv 0 \), then \( \mathcal{M}(\alpha, \beta) \) coincides with the well-known analytic convex class \( \mathcal{C}(\beta) \). Coefficient bounds and growth theorem for the class \( \mathcal{M}(\alpha, \beta) \) have also been obtained in [26].

**Lemma 1.23.** [26] Let \( f \in \mathcal{M}(\alpha, \beta) \) be of the form (1.22). Then

(i) \( |a_n| \leq \frac{1}{n!} \prod_{j=0}^{n} (j - 2\beta) \quad (n=2, 3 \cdots) \),

(ii) \( |b_2| = \frac{\alpha}{2} \) and \( |b_n| \leq \frac{(n-1)|\alpha|}{n!} \prod_{j=0}^{n} (j - 2\beta) \quad (n=3, 4 \cdots) \)

Moreover these bounds are sharp with the extremal functions

\[
f_{\alpha, \beta}(z) = \int_{0}^{z} \frac{dt}{(1 - \gamma t)^{2-2\beta}} + \int_{0}^{z} \frac{\alpha dt}{(1 - \gamma t)^{2-2\beta}} \quad (|\gamma| = 1; \quad z \in \mathbb{D}).
\]

**Lemma 1.25.** [26] Let \( f \in \mathcal{M}(\alpha, \beta) \) with \( 0 \leq \beta < 1 \). Then \( f \) satisfies the following inequalities

\[
L(r, \alpha, \beta) \leq |f(z)| \leq R(r, \alpha, \beta),
\]

where

\[
L(r, \alpha, \beta) = \begin{cases} 
\frac{(1 + |\alpha|)r}{1 + r} - |\alpha| \log(1 + r), & \beta = 0 \\
-|\alpha| r + (1 + |\alpha|) \log(1 + r), & \beta = 1/2 \\
- (|\alpha| + 2\beta)(1 + r) + (1 + r)^{2\beta} (|\alpha| + 2\beta - (2\beta - 1)|\alpha|r), & \beta \neq 0, 1/2 \\
2\beta(2\beta - 1)(1 + r) & \end{cases}
\]

\[
R(r, \alpha, \beta) = \begin{cases} 
\frac{|\alpha| r}{1 + r}, & \beta = 0 \\
|\alpha| r + (1 + |\alpha|) \log(1 + r), & \beta = 1/2 \\
(\frac{|\alpha| + 2\beta}{2\beta(2\beta - 1)}(1 + r)^{2\beta} (|\alpha| + 2\beta - (2\beta - 1)|\alpha|r), & \beta \neq 0, 1/2
\end{cases}
\]
and
\[ R(r, \alpha, \beta) = \begin{cases} 
\frac{(1 + |\alpha|)r}{1 - r} + |\alpha|\log(1 - r), & \beta = 0 \\
-|\alpha|r - (1 + |\alpha|)\log(1 - r), & \beta = 1/2 \\
(\alpha + 2\beta)(1 - r) - (1 - r)^{2\beta}(|\alpha| + 2\beta + (2\beta - 1)|\alpha|r) & \beta \neq 0, 1/2.
\end{cases} \]

All these bounds are sharp, the extremal function is \( f_{\alpha, \beta} \) or its rotations, where
\[
f_{\alpha, \beta}(z) = \begin{cases} 
\frac{1}{1 - z} + \frac{\alpha(1 - z)}{1 - z} + \log(1 - z), & \beta = 0 \\
-\log(1 - z) - \frac{\alpha(z + \log(1 - z))}{1 - z}, & \beta = 1/2 \\
\frac{1 - (1 - z)^{2\beta - 1}}{2\beta - 1} + \frac{\alpha}{2\beta} [1 - (1 - z)^{2\beta - 1} (1 + (2\beta - 1)z)], & \beta \neq 0, 1/2.
\end{cases}
\]

In 2018, Bhowmik and Das [14] proved an interesting result for subordination classes. Let \( f \) and \( g \) be two analytic functions in \( \mathbb{D} \) such that \( g < f \). Let
\[
g(z) = \sum_{n=0}^{\infty} b_n z^n.
\]

Lemma 1.28. [14] Let \( f \) and \( g \) be analytic in \( \mathbb{D} \) with Taylor expansions (1.1) and (1.27) respectively and \( g < f \), then
\[
\sum_{n=0}^{\infty} |b_n| r^n \leq \sum_{n=0}^{\infty} |a_n| r^n
\]
for \( |z| = r \leq 1/3 \).

2. Main Results

Before going to state our main theorems we prove an elementary result which is required to prove some of our results.

Lemma 2.1. (i) Let \( f \) and \( g \) be analytic in \( \mathbb{D} \) with series representation \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) and (1.27) respectively such that
\[
f(z) = \int_0^r g(z) \, dt \quad \text{for} \quad |z| = r < 1.
\]
Here \( M_f(r) \) and \( M_g(r) \) are respectively the majorant series associated with \( f \) and \( g \).

(ii) Let \( f \) and \( g \) be analytic in \( \mathbb{D} \) with Taylor expansions (1.1) and (1.27) respectively and \( g < f \), then \( M_G(r) \leq M_F(r) \) for \( |z| = r \leq 1/3 \), where
\[
G(z) = \int_0^r g(z) \, dz \quad \text{and} \quad F(z) = \int_0^r f(z) \, dz \quad \text{for} \quad z \in \mathbb{D}.
\]

Now let \( \min_{|z| = r} |\phi(z)| = \phi(-r), \max_{|z| = r} |\phi(z)| = \phi(r), \) \( |z| = r \), and we assume these through the articles. Here \( \phi \) is the Ma-Minda function.
Theorem 2.2. Let $f \in \mathcal{K}_s(\phi)$ be of the form (1.5). Then

\begin{equation}
|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(D))
\end{equation}

for $|z| = r \leq R_f$, where $R_f = \min\{1/3, r_f\}$ and $r_f$ is the smallest positive root of $R(r) = L(1)$ in $(0, 1)$. Here $R(r) := \int_0^r (M_\phi(t))/(1 - t^2) \, dt$, $L(r) := \int_0^r (\phi(-t))/(1 + t^2) \, dt$ and $M_\phi$ is the associated majorant series of $\phi$.

Remark 2.1. (i) Assume that the coefficients of $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ in the above theorem are all positive i.e. $B_n > 0$ for $n \geq 1$. Then the majorant series $M_\phi(r) = \phi(r)$, $0 < r < 1$. Then $R(r) := \int_0^r (\phi(t))/(1 - t^2) \, dt$.

(ii) [Bohr phenomenon for corresponding class $\mathcal{K}_s(\Phi)$ associated with non-Ma-Minda functions]

Let $\Phi$ be the corresponding non-Ma-Minda function of $\phi$, which is actually a rotation by mere replacing $z$ by $-z$. Therefore the image of the unit disk $D$ under the functions $\Phi$ and $\phi$ are identical. Thus we conclude that $\mathcal{K}_s(\Phi) = \mathcal{K}_s(\phi)$ and the above Bohr phenomenon (2.3) holds for the class $\mathcal{K}_s(\Phi)$ for same $R_f$.

Some applications:

Lemma 2.4. [Bohr phenomenon for the corresponding subordination class]

Let $q(z) = \sum_{n=1}^{\infty} q_n z^n \in S^f_*(\phi)$ as defined in (1.20). Then

\begin{equation}
\sum_{n=1}^{\infty} |q_n||z|^n \leq d(f(0), \partial f(D))
\end{equation}

for $|z| = r \leq R_f$, where $R_f$ is defined as in the Theorem 2.2.

Corollary 2.5. (i) [Bohr phenomenon for the class $\mathcal{K}_s(\gamma)$]

When $\phi(z) = (1 + (1 - 2\gamma) z)/(1 - z)$, the class $\mathcal{K}_s(\phi)$ reduces to $\mathcal{K}_s(\gamma)$. Then any $f \in \mathcal{K}_s(\gamma)$ with $0 \leq \gamma < 0.259056404$ satisfies the inequality (2.3) for $|z| = r \leq r_f$, where $r_f$ is the root of the equation

\begin{equation}
\frac{\gamma}{2} \ln \left( \frac{1 + r}{1 - r} \right) + (1 - \gamma) \frac{r}{1 - r} = \frac{1 - \gamma}{2} \ln 2 + \frac{\gamma \pi}{4} \quad \text{in} \quad (0, 1/3).
\end{equation}

(ii) In particular, for $\gamma = 0$, $\mathcal{K}_s(\phi)$ reduces to $\mathcal{K}_s$. Each function $f \in \mathcal{K}_s$ satisfies the Bohr inequality (2.3) for $|z| = r \leq r_f$, where $r_f = \frac{\ln 2}{2 + \ln 2} \approx 0.257374415$.

Corollary 2.7. (i) When $\phi(z) = (1 + \beta z)/(1 - \alpha \beta z)$, where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, the class $\mathcal{K}_s(\phi)$ reduces to $\mathcal{K}_s(\alpha, \beta)$. Then $\mathcal{K}_s(\alpha, \beta)$ satisfies the Bohr phenomenon (2.3) for $|z| = r \leq r_f = \min\{1/3, r_f\}$, where $r_f$ is the smallest root of the equation

\begin{equation}
\int_0^r \frac{1 + \beta t}{(1 - \alpha \beta t)(1 - t^2)} \, dt = \int_0^1 \frac{1 - \beta t}{(1 + \alpha \beta t)(1 + t^2)} \, dt \quad \text{in} \quad (0, 1).
\end{equation}

In particular, for $\alpha = \beta = 1$, then $\mathcal{K}_s(\alpha, \beta)$ coincides with $\mathcal{K}_s$ and we can easily obtain $r_f$ from (2.8).
Theorem 2.9. Let \( f \in S_c^*(\phi) \) be of the form (1.5). Then

\[
|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\mathbb{D}))
\]

for \(|z| = r \leq \min\{1/3, r_f\}\) and \(r_f\) is the smallest positive root of \( P(r) + h(-1) = 0 \) in \((0, 1)\), where \( P(r) := \int_0^r ((M_h(t)M_\phi(t))/t) \, dt \). Here \( M_h(t) \) and \( M_\phi(t) \) are respectively the majorant series of \( h \) and \( \phi \).

Remark 2.2. (i) Bohr radius for \( S_c^*(\phi) \) when \( \phi \) has positive coefficients

Let \( \phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \). It is worth to point out that if we impose one further condition on \( \phi \) that the coefficients \( B_n \)'s are positive, then the majorant series \( M_\phi(r) = \phi(r) \). From the definition of \( h \) in (1.6), we have

\[
h(z) = z \exp \left( \int_0^z \frac{\phi(t) - 1}{t} \, dt \right) = \exp \left( \sum_{n=1}^{\infty} \frac{B_n}{n} z^n \right).
\]

Thus from (2.11), it is easy to see that \( M_h(r) = h(r) \). Then \( P(r) = \int_0^r ((h(t)\phi(t))/t) \, dt = h(r) \). Then each \( f \in S_c^*(\phi) \) satisfies the inequality (2.10) for \(|z| \leq \min\{1/3, r_f\}\), where \( r_f \) is the root of the equation \( h(r) + h(-1) = 0 \). In particular, when \( r_f \leq 1/3 \), the radius \( r_f \) is the best possible for the function \( f(z) = h(z) \in S_c^*(\phi) \), since it has real coefficients and is in \( S^*(\phi) \). Indeed, for \(|z| = r_f \), \( M_h(r_f) = h_{r_f} = -h(-1) = d(h(0), \partial h(\mathbb{D})) \), which shows that \( r_f \) is best possible.

(ii) Bohr phenomenon for corresponding class \( S_c^*(\Phi) \) associated with non-Ma-Minda functions

Let \( \Phi \) be the corresponding non-Ma-Minda function of \( \phi \). Since \( \Phi \) is actually obtained from \( \phi \) by a rotation \( z \) by \(-z\), the image of the unit disk \( \mathbb{D} \) under the functions \( \Phi \) and \( \phi \) are identical. Thus we conclude that \( S_c^*(\Phi) = S_c^*(\phi) \) and the Bohr radius for the class \( S_c^*(\Phi) \) is same as that of \( S_c^*(\phi) \).

Let \( S_{cf}^*(\phi) \) denote the class of analytic functions \( g \) subordinate to a fixed function \( f \in S_c^*(\phi) \).

Lemma 2.12. [ Bohr phenomenon for the corresponding subordination class \( S_{cf}^*(\phi) \) ]

Let \( g \in S_{cf}^*(\phi) \) be of the form \( g(z) = \sum_{n=1}^{\infty} g_n z^n \). Then

\[
\sum_{n=1}^{\infty} |g_n||z|^n \leq d(f(0), \partial f(\mathbb{D}))
\]

for \(|z| = r \leq \min\{1/3, r_f\}\), where \( r_f \) is as in the Theorem 2.9.

Similar results on the Bohr phenomenon of the class \( S_c^*(\phi) \) holds also for the class \( S_{cf}^*(\phi) \). Now from the above Remark 2.2 and Lemma 2.12 in particular, we obtain the following interesting corollaries.

Corollary 2.14. Let \( \phi(z) = (1 + sz)^2 \) with \( 0.444981 < s \leq 1/\sqrt{2} \), then \( S_c^*(\phi) \) reduces to the class \( S_c^*((1 + sz)^2) \). Then the class \( S_c^*((1 + sz)^2) \) (and \( S_{cf}^*((1 + sz)^2) \)) satisfies the Bohr inequality (2.10) for \(|z| = r \leq r_f \), where \( 0 < r_f < 1/3 \) and \( r_f \) is the
root of the equation

\[(2.15) \quad r \exp \left( s \left( 2r + \frac{sr^2}{2} \right) \right) = \exp \left( s \left( -2 + \frac{s}{2} \right) \right).\]

The radius \( r_f \) is the best possible.

Table 1

| \( s \) | \( r_f \) | \( s \) | \( r_f \) |
|---|---|---|---|
| 0.1 | 0.71184 | 0.45 | 0.330472 |
| 0.15 | 0.619461 | 0.5 | 0.3040402 |
| 0.2 | 0.546344 | 0.55 | 0.28091732 |
| 0.25 | 0.486934 | 0.6 | 0.2605657 |
| 0.3 | 0.437693 | 0.65 | 0.24256 |
| 0.35 | 0.39624 | 0.7 | 0.226558 |
| 0.4 | 0.360903 | 1/\sqrt{2} | 0.22443096 |

From Table 1, it is easy to see that when \( s < 0.444981 \), \( r_f > 1/3 \), hence Bohr phenomenon holds for \( r \leq 1/3 \) and when \( 0.444981 < s \leq 1/\sqrt{2} \), \( r_f < 1/3 \), hence the radius \( r_f \) is best possible.

**Corollary 2.16.** For \( \phi(z) = \alpha + (1 - \alpha)\exp z \) with \( 0 \leq \alpha < 0.05284 \), the class \( \mathcal{S}_c^b(\phi) \) satisfies the Bohr phenomenon \((2.10)\) for \(|z| = r \leq r_f\), where \( 0 < r_f < 1/3 \). The radius \( r_f \) is the best possible.

Table 2

Existence of sharp radius \( r_f \) in \((0, 1/3)\) for different \( \alpha \in [0, 0.05284) \)

| \( \alpha \) | \( h(1/3) \) | \( h(-1) \) | Sign of \( D_2(0) \) | Sign of \( D_2(1/3) \) |
|---|---|---|---|---|
| 0.0 | 0.47935 | 0.4508594 | – | + |
| 0.01 | 0.477619 | 0.454465 | – | + |
| 0.02 | 0.476887697 | 0.458100015 | – | + |
| 0.03 | 0.47416191 | 0.4617638 | – | + |
| 0.04 | 0.47244238 | 0.465456 | – | + |
| 0.05 | 0.470729 | 0.469179 | – | + |
| 0.06 | 0.469022 | 0.47293 | – | – |
| 0.07 | 0.46732112 | 0.4767143 | – | – |

From Table 2, it is clear that when \( 0 \leq \alpha < 0.05284 \), \( r_f \) lies in \((0, 1/3)\) and hence \( r_f \) is best possible. On the other hand for \( \alpha > 0.05284 \), \( r_f > 1/3 \) and corresponding Bohr phenomenon holds for \( r \leq 1/3 \).

**Corollary 2.17.** Let \( \phi(z) = ((1 + z)/(1 - z))^\alpha \) with \( 0 < \alpha \leq 1 \). Also assume \( h(1/3) > -h(-1) \), where

\[ h(r) = r \exp \left( \int_0^r \frac{(1+t)\alpha}{t} - 1 \, dt \right) \]
and
\[-h(-1) = \exp \left( \int_0^{-1} \frac{(1+t)^\alpha - 1}{t} \, dt \right) \].

Then the class $S_c^* (\phi)$ satisfies the Bohr phenomenon (2.10) for $|z| = r \leq r_f$, where $r_f$ is the smallest root of the equation $D_3(r) := h(r) + h(-1) = 0$.

| $\alpha$ | $h(1/3)$ | $-h(-1)$ | Sign of $D_3(0)$ | Sign of $D_3(1/3)$ |
|--------|----------|----------|-----------------|-----------------|
| 0.2    | 0.38335  | 0.65515  | -               | -               |
| 0.4    | 0.4453711| 0.475453 | -               | -               |
| 0.45   | 0.4631699| 0.443795 | -               | +               |
| 0.5    | 0.482023 | 0.415759 | -               | +               |
| 0.6    | 0.523214 | 0.368431 | -               | +               |
| 0.7    | 0.569663 | 0.330139 | -               | +               |
| 0.8    | 0.62222  | 0.298621 | -               | +               |
| 0.9    | 0.681928 | 0.272286 | -               | +               |

From the above table it is easy to see that for different values of $\alpha$, the constant $r_f$ sometimes not lies in $(0, 1/3)$. But when $r_f$ lies in $(0, 1/3)$, then corresponding $r_f$ is the best possible and Bohr phenomenon for the class $S_c^* (\phi)$ holds for $r \leq r_f$.

**Corollary 2.18.** Let $\phi(z) = (1 + (1 - 2\gamma)z)/(1 - z)$ with $0 \leq \gamma < 1/2$. Then each $f \in S_c^* ((1 + (1 - 2\gamma)z)/(1 - z))$ satisfies the inequality (2.10) for $|z| = r \leq r_f$, where $0 < r < 1/3$ and $r_f$ is the root of the equation
\[(2.19) \quad r + 2r^{1/(2(1-\gamma))} - 1 = 0.\]

The radius $r_f$ is the best possible.

**Corollary 2.20.** If $\phi(z) = (1 + Az)/(1 + Bz)$ with $-1 \leq B < A \leq 1$, then

(i) When $B = 0$, every function $f \in S_c^* ((1 + Az)/(1 + Bz))$ satisfies the inequality (2.10) for $|z| = r \leq r_f$, where $0 < r_f < 1/3$ and $r_f$ is the unique root of the equation
\[(2.21) \quad re^{Ar} = e^{-A},\]

provided $A \geq (3/4) \ln 3$. The radius $r_f$ is the best possible.

(ii) When $B \neq 0$, every function $f \in S_c^* ((1 + Az)/(1 + Bz))$ satisfies the inequality (2.10) for $|z| = r \leq r_f$, where $0 < r_f < 1/3$ and $r_f$ is the unique root of the equation
\[(2.22) \quad r (1 + Br)^{\frac{A-B}{B}} = (1 - B)^{\frac{A-B}{B}},\]

provided $\frac{1}{3} (1 + B/3)^{\frac{A-B}{B}} \geq (1 - B)^{\frac{A-B}{B}}$. The radius $r_f$ is the best possible.
Table 4
The radius \( r_f \) for different \( B \) when \( A = 1 \) and \( A = 1/2 \)

| \( B \)  | \( r_f \)  | \( B \)   | \( r_f \)   |
|-------|---------|-------|---------|
| -0.1  | 0.261789| -0.1  | 0.432852|
| -0.2  | 0.247088| -0.2  | 0.395824|
| -0.3  | 0.23402 | -0.3  | 0.364714|
| -0.4  | 0.222323| -0.4  | 0.338205|
| (A = 1) -0.5 | 0.21179 | (A = 1/2) -0.5 | 0.31534|
| -0.6  | 0.202239| -0.6  | 0.295418|
| -0.7  | 0.193548| -0.7  | 0.277899|
| -0.8  | 0.185599| -0.8  | 0.262372|
| -0.9  | 0.1783  | -0.9  | 0.248514|
| -1.0  | 0.17157 | -1.0  | 0.236068|

From the Table 4, we see that for different values of \( A \) and \( B \), sometimes radius \( r_f < 1/3 = 0.33333 \) and in that case \( r_f \) is the best possible. When \( r_f > 1/3 \), Bohr phenomenon for class \( S_C^* \) holds for \( r \leq 1/3 \).

Theorem 2.23. Let \( f \in C_c(\phi) \) be of the form (1.5). Then

\[
|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\Omega))
\]

for \( |z| = r \leq \min\{1/3, r_f\} \) and \( r_f \) is the smallest positive root of \( T(r) = -k(-1) \) in \((0, 1)\) and \( T(r) := \int_0^r \frac{1}{s} \int_0^s M_K(t)M_\phi(t) \, dt \, ds \).

The other results for this class, for particular \( \phi \), may be obtained easily and hence omitted.

Theorem 2.25. Let \( f \in C_s(\phi) \) be of the form (1.5). Then

\[
|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\Omega))
\]

for \( |z| = r \leq \min\{1/3, r_f\} \) and \( r_f \) is the smallest positive root of \( R_s(r) = L_s(1) \) in \((0, 1)\), where

\[
R_s(r) := \int_0^r \frac{1}{s} \int_0^s M_K(t)M_\phi(t) \, dt \, ds \quad \text{and} \quad L_s(r) := \int_0^r \frac{1}{s} \int_0^s [k(-t^2)]^{1/2} \phi(-t) \, dt \, ds
\]

and \( K'(r) = [k'(t^2)]^{1/2} \).

Remark 2.3. (i) Let \( \Phi \) be corresponding non-Ma-Minda class of \( \phi \). Then Bohr radius for the class \( C_s(\Phi) \) is same as that of \( C_s(\phi) \).
(ii) Let $S^*_{sf}(\phi)$ be the class of analytic functions $g$ of the form $g(z) = \sum_{n=1}^{\infty} g_n z^n$ in $D$ subordinate to a fixed function $f \in C_s(\Phi)$, then

$$\sum_{n=1}^{\infty} |g_n||z|^n \leq d(f(0), \partial f(D))$$

for $|z| = r \leq \min\{1/3, r_f\}$ and $r_f$ is explained in 2.25.

**Theorem 2.27.** Let $f \in \mathcal{M}(\alpha, \beta)$ be of the form (1.22) with $|\alpha| \leq 1, 0 \leq \beta < 1$. Then

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(D))$$

for $|z| = r \leq r_f$, where $r_f$ is the smallest root of $R(r, \alpha, \beta) = L(1, \alpha, \beta)$. The radius $r_f$ is sharp.

From the above theorem we obtain the following interesting results. Ali et.al. [9] obtained the Bohr radius for the class of convex functions of order $\beta$ for $-1/2 \leq \beta < 1$. Here we showed that this result can be obtained for $0 \leq \beta < 1$ as an application of the Theorem 2.27.

**Corollary 2.28 (Bohr radius for convex functions of order $\beta$).** Let $f = h + \overline{g} \in \mathcal{M}(\alpha, \beta)$. If the co-analytic part $g \equiv 0$, then $\mathcal{M}(\alpha, \beta)$ reduces to the analytic class $\mathcal{C}(\beta)$. If $0 \leq \beta < 1$, then $\mathcal{C}(\beta)$ satisfies the Bohr phenomenon

$$|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(D))$$

for $|z| = r \leq r_f$, where $r_f$ is the unique root of $h_{\beta}(r) + h_{\beta}(-1) = 0$ in $(0, 1)$, where

$$h_{\beta}(z) = \begin{cases} \frac{1-(1-z)2^{\beta-1}}{2^{\beta-1}}, & \beta \neq 1/2 \\ -\log(1-z), & \beta = 1/2. \end{cases}$$

The radius $r_f$ is sharp.

3. **Proof of the main results**

**Proof of Lemma 2.1.**

(i) The relation $f(z) = \int_0^x g(z) \, dz$ gives

$$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \frac{b_n-1}{n} z^n.$$

Therefore

$$M_f(r) = \sum_{n=1}^{\infty} \frac{|b_n-1|}{n} r^n = \int_0^r \sum_{n=0}^{\infty} |b_n| t^n \, dt = \int_0^r M_g(t) \, dt, \quad r < 1.$$
(ii) From Lemma 1.28, we have \( M_g(r) \leq M_f(r) \) for \( r \leq 1/3 \). Integrating this we obtain
\[
\int_0^r M_g(t) \, dt \leq \int_0^r M_f(t) \, dt \quad \text{for} \quad r \leq 1/3.
\]

Hence the result follows from the first part of this Lemma.

\[\square\]

**Proof of Theorem 2.2** Let \( f \in K_s(\phi) \), then from Lemma 1.21, the Euclidean distance between \( f(0) \) and the boundary of \( f(D) \) is
\[
d(f(0), \partial f(D)) = \liminf_{|z| \to 1} |f(z) - f(0)| \geq \int_0^1 \frac{\phi(-t)}{1 + t^2} \, dt.
\]

By subordination principle, there exists analytic function \( \omega : D \to D \) such that
\[
-\frac{z^2 f'(z)}{g(z)g(-z)} = \phi(\omega(z)).
\]

Let \( G(z) := -\frac{g(z)g(-z)}{z} \). Clearly, \( G \) is odd starlike function in \( D \). Let \( G(z) = z + \sum_{n=2}^{\infty} g_{2n-1} z^{2n-1} \). It is well-known that \( |g_{2n-1}| \leq 1 \) for \( n \geq 2 \). Therefore
\[
M_G(r) \leq r + \sum_{n=2}^{\infty} r^{2n-1} = \frac{r}{1 - r^2}, \quad 0 < r < 1.
\]

From (3.2), we have \( zf'(z) = G(z)\phi(\omega(z)) \), which immediately follows that
\[
f(z) = \int_0^z \frac{G(\xi)\phi(\omega(\xi))}{\xi} \, d\xi.
\]

It is known that for two analytic functions \( f \) and \( g \) in \( D \), \( M_{fg}(r) \leq M_f(r)M_g(r) \), where \( M_f(r) \), \( M_g(r) \) and \( M_{fg}(r) \) are associated majorant series with \( f \), \( g \) and the product \( fg \). Then \( M_{G(\phi \circ \omega)}(r) \leq M_G(r)M_{\phi \circ \omega}(r) \). Since \( \phi \circ \omega \prec \phi \) then by Lemma 1.28, we have
\[
M_{\phi \circ \omega}(r) \leq M_\phi(r) \quad \text{for} \quad |z| = r \leq 1/3.
\]

Using Lemma 2.1 from (3.3), (3.4) and (3.5), we obtain
\[
M_f(r) \leq \int_0^r \frac{M_G(t)M_{\phi \circ \omega}(t)}{t} \, dt \leq \int_0^r \frac{M_\phi(t)}{1 - t^2} \, dt = R(r)
\]

for \( |z| = r \leq 1/3 \). Note that \( R(r) \) is less than or equals to \( L(1) \) whenever \( r \leq r_f \), where \( r_f \) is the smallest positive root of the equation \( R(r) = L(1) \) in \( (0, 1) \). Let
\[ H_1(r) = R(r) - L(1) \] and see \( H_1 \) is continuous in \( r \). Note that
\[
H_1(0) = L(1) = -\int_0^1 \frac{\phi(-t)}{1 + t^2} \, dt < 0
\]
and
\[
H_1(1) = R(1) - L(1) = \int_0^1 \frac{M_\phi(t)}{1 - t^2} \, dt - \int_0^1 \frac{\phi(-t)}{1 + t^2} \, dt > 0,
\]
since \( R(1) > L(1) \) and \( M_\phi(t) \geq |\phi(t)| \). Thus \( H_1 \) has a root in \((0, 1)\). Let \( r_f \) be the smallest root of \( H_1 \) in \((0, 1)\). Thus \( R(r) \leq L(1) \) for \( r \leq r_f \). Therefore using (3.1) and (3.6), we conclude that
\[
M_f(r) \leq \int_0^1 \frac{\phi(-t)}{1 + t^2} \, dt \leq d(f(0), \partial f(\mathbb{D}))
\]
for \(|z| = r \leq \min\{1/3, r_f\} = R_f \) \( \square \)

**Proof of Lemma 2.4** From the definition of \( S^K_f(\phi) \), we have \( q \prec f \). Then by Lemma 1.28 we obtain \( M_q(r) \leq M_f(r) \) for \(|z| = r \leq 1/3\). Hence the result follows from the inequality (2.3) \( \square \)

**Proof of Corollary 2.5**

(i) Let \( f \in \mathcal{K}_s(\gamma) \). Then a little computation shows that
\[
R(r) = \frac{\gamma}{2} \ln \left( \frac{1 + r}{1 - r} \right) + (1 - \gamma) \frac{r}{1 - r}
\]
and
\[
L(r) = (1 - \gamma) \ln \left( \frac{1 + r}{\sqrt{1 + r^2}} \right) + \gamma \arctan r.
\]
See \( L(1) = \left( \frac{1 - \gamma}{2} \right) \ln 2 + \frac{\pi \gamma}{4} \). Here \( H_1(r) := R(r) - L(1) \). Then \( H_1 \) is continuous in \( r \). Note that \( H_1(0) < 0 \) and \( H_1(1/3) > 0 \) if \( 0 \leq \gamma < 0.259056404 \). Thus \( H \) has a root in \((0, 1/3)\) and choose smallest root to be \( r_f \) in \((0, 1/3)\). Thus the inequality (2.3) holds for \(|z| = r \leq r_f \).

(ii) Putting \( \gamma = 0 \) in (2.6), we obtain \( r_f = \ln 2/(2 + \ln 2) \). \( \square \)

**Proof of Theorem 2.9** Let \( f \in \mathcal{S}^*_c(\phi) \), then using the Lemma (1.18) we obtain the Euclidean distance between \( f(0) \) and the boundary of \( f(\mathbb{D}) \) is
\[
d(f(0), \partial f(\mathbb{D})) = \lim inf_{|z| \to 1} |f(z) - f(0)| \geq -h(-1).
\]
Since \( f \in S^*_c(\phi) \) and \( \phi \) is starlike and symmetric with respect to real-axis, it follows that \( g(z) := (f(z) + f(\overline{z}))/2 \) is in \( S^*(\phi) \). Since \( g \in S^*(\phi) \), from Lemma 1.7 we have \( g(z)/z < h(z)/z \). Therefore from Lemma 1.28 we obtain

\[
(3.8) \quad M_g(r) \leq M_h(r) \quad \text{for} \quad |z| = r \leq 1/3.
\]

From the definition of \( S^*_c(\phi) \), we have

\[
(3.9) \quad zf'(z) = g(z)\phi(\omega(z)),
\]

where \( \omega \) is analytic in \( \mathbb{D} \) and \( \omega(0) = 0, |\omega(z)| < 1 \) in \( \mathbb{D} \). Since \( \phi \circ \omega \prec \omega \), from Lemma 1.28

\[
(3.10) \quad M_{\phi \circ \omega}(r) \leq M_\phi(r) \quad \text{for} \quad |z| = r \leq 1/3.
\]

Simplification of (3.9) gives

\[
(3.11) \quad f(z) = \int_0^z g(\xi)\phi(\omega(\xi)) \frac{d\xi}{\xi}.
\]

Now, by making use of the Lemma 2.1, (3.8) and (3.10), from (3.11) we obtain

\[
(3.12) \quad |z| + \sum_{n=2}^{\infty} |a_n||z|^n = M_f(r)
\]

\[
\leq \int_0^r \frac{M_g(t)M_{\phi \circ \omega}(t)}{t} dt
\]

\[
\leq \int_0^r \frac{M_h(t)M_\phi(t)}{t} dt
\]

\[
= P(r)
\]

for \( |z| = r \leq 1/3 \). Note that \( P(r) \leq -h(-1) \), whenever \( r \leq r_f \), where \( r_f \) is the smallest positive root of \( P(r) = -h(-1) \) in \( (0,1) \). Going by the similar line of argument as in the proof of the Theorem 2.2, the existence of the root \( r_f \) is ensured by the inequalities \( M_h(t) \geq |h(t)|, M_h(1) \geq |h(1)| \geq -h(-1) \) and \( M_h(0) < -h(-1) \). Thus, combining the inequalities (3.12) and (3.7) with the fact \( P(r) \leq -h(-1) \) for \( r \leq r_f \), we conclude that

\[
|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\mathbb{D}))
\]

for \( |z| = r \leq \min\{1/3, r_f\} \).

**Proof of Lemma 2.12.** From the definition of \( S^*_c(\phi) \), we have \( g \prec f \). Then by Lemma 1.28 we obtain \( M_g(r) \leq M_f(r) \) for \( |z| = r \leq 1/3 \). Hence the result follows from the inequality (2.10). \( \square \)
Proof of Corollary 2.14. Here the coefficients of $\phi(z) = (1 + sz)^2$ with $0 < s \leq 1/\sqrt{2}$ are all positive. Thus from the Remark 2.2, we obtain

$$P(r) = h(r) = r \exp \left( s \left( 2r + \frac{sr^2}{2} \right) \right).$$

Let $D_1(r) = h(r) + h(-1)$. Clearly $D$ is continuous in $r$. Observe that $D(0) < 0$ and

$$D_1 \left( \frac{1}{3} \right) = \frac{1}{3} \exp \left( s \left( \frac{s + 12}{18} \right) \right) - \exp \left( s \left( -2 + \frac{s}{2} \right) \right) > 0,$$

whenever $0.444981 < s \leq 1/\sqrt{2}$. Thus, $D_1$ has a real root in $(0, 1/3)$ and choose it to be $r_f$. Therefore from the Remark 2.2, the radius $r_f$ is the best possible. □

Proof of Corollary 2.16. Let $\phi(z) = \alpha + (1 - \alpha)e^z$ then the coefficients of $\phi(z)$ are positive for $0 \leq \alpha < 1$. Consider $D_2(r) = h(r) + h(-1)$ where

$$h(r) = r \exp \left( (1 - \alpha) \int_0^r \left( \frac{1 + e^t}{t} \right) dt \right).$$

Note that

$$h \left( \frac{1}{3} \right) = \frac{1}{3} \exp \left( (1 - \alpha) \int_0^{1/3} \left( \frac{1 + e^t}{t} \right) dt \right) \approx \frac{1}{3} (1.43807)^{1-\alpha}$$

and

$$h(-1) = - \exp \left( (1 - \alpha) \int_0^{-1} \left( \frac{1 + e^t}{t} \right) dt \right) \approx -(0.450859463)^{1-\alpha}.$$

A little computation using Mathematica shows that $D_2(1/3) = h(1/3) + h(-1) > 0$ if, and only if, $0 \leq \alpha < 0.05284$. Clearly, $D_2(0) = h(-1) < 0$. Thus $D_2$ has a root in $(0, 1)$ and choose it to be $r_f$. By Remark 2.2, $r_f$ is the best possible. □

Proof of Corollary 2.17. Let $\phi(z) = ((1 + z)/(1 - z))^\alpha$ with $0 < \alpha \leq 1$. From [4], it is guaranted that the coefficients of $\phi$ are positive. Here

$$h(r) = r \exp \left( \int_0^r \left( \frac{1+t}{1-t} \right)^\alpha - 1 \ dt \right).$$

Then $D_3(r) := h(r) + h(-1)$ is continuous in $r$ and $D_3(0) < 0$ and $D_3(1/3) = h(1/3) + h(-1) > 0$. Thus $D_3$ has a root in $(0, 1)$ and choose it to be $r_f$. Hence from Remark 2.2, $r_f$ is the best possible. □
(3.14)\begin{equation}
f(z) = \int_{0}^{1} \int_{0}^{\xi} g'(\eta)\phi(\omega(\eta))\,d\eta\,d\xi.
\end{equation}

Since $g \in C(\phi)$, from Lemma 1.10 we have $g' \prec k'$ and hence by Lemma 1.28 we obtain
\begin{equation}
M_g(r) \leq M_{k'}(r) \quad \text{for} \quad r \leq 1/3.
\end{equation}
Using Lemma 2.1 from (3.14) and (3.15), we obtain

\[(3.16) \quad M_f(r) \leq \int_0^r \frac{1}{s} \int_0^s M_k'(t) M_\phi(t) \, dt \, ds = T(r) \quad \text{for} \quad r \leq 1/3.\]

From Lemma 1.19, the Euclidean distance between \(f(0)\) and the boundary of \(f(D)\) is

\[(3.17) \quad d(f(0), \partial f(D)) = \liminf_{|z| \to 1} |f(z) - f(0)| \geq -k(-1).\]

Note that \(T(r) \leq -k(-1)\), whenever \(r \leq r_f\), where \(r_f\) is the smallest positive root of \(T(r) = -k(-1)\) in \((0, 1)\). Going by the similar line of argument as in the proof of the Theorem 2.9, the existence of the root \(r_f\) is ensured by the inequalities \(M_k(r) \geq |k(r)|, M_k(1) \geq |k(1)| \geq -k(-1)\) and \(M_k(0) < -k(-1)\). Therefore from \((3.16)\) and \((3.17)\), we obtain

\[|z| + \sum_{n=0}^{\infty} |a_n||z|^n = M_f(r) \leq d(f(0), \partial f(D))\]

for \(|z| = r \leq \min\{1/3, r_f\}\). □

**Proof of Theorem 2.25.** Let \(f \in C_s(\phi)\), then it is evident that the Euclidean distance between \(f(0)\) and the boundary of \(f(D)\) is

\[(3.18) \quad d(f(0), \partial f(D)) = \liminf_{|z| \to 1} |f(z) - f(0)| \geq L_s(1).\]

Since \(f \in C_s(\phi)\) and \(\phi\) is starlike and symmetric with respect to real axis, then it follows that

\[(3.19) \quad g(z) := \frac{f(z) - f(-z)}{2} = z + \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1} \in C(\phi).\]

Here \(g\) is odd convex function. Note that the function \(K(z) = \int_0^r [k'(t^2)]^{1/2} \, dt\) defined in (1.17) is odd function in \(C(\phi)\). By Lemma 1.10 we have \(g' \prec K'\). Therefore from 1.28, we obtain

\[(3.20) \quad M_{g'}(r) \leq M_{K'}(r) \quad \text{for} \quad |z| = r \leq 1/3.\]

Now from the definition of \(C_s(\phi)\), we have

\[(3.21) \quad (z f'(z))' = g'(z)\phi(\omega(z)).\]

Simplication of (3.21) gives

\[(3.22) \quad f(z) = \int_0^r \frac{1}{\xi} \int_0^\xi g'(\eta)\phi(\omega(\eta)) \, d\eta \, d\xi.\]
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By making use of Lemmas 1.18 and 2.1 from (3.20) and (3.22), we obtain

\[
|z| + \sum_{n=2}^{\infty} |a_n| |z|^n = M_f(r) \leq \int_0^r \frac{1}{s} \int_0^s M_f(t) M_\phi(t), dt \, ds
\]

\[
\leq \int_0^r \frac{1}{s} \int_0^s M_K(t) M_\phi(t), dt \, ds
\]

\[
= R_s(r),
\]

for \( |z| = r \leq 1/3 \). Now \( R_s(r) \leq L_s(1) \) for \( r \leq r_f \), where \( r_f \) is the smallest root of \( R_s(r) = L_s(1) \) in \((0, 1)\). The existence of the root is ensured by the relation \( M_K(t) \geq |K'(t)|, R_s(1) \geq L_s(1) \) and \( R_s(0) \leq L_s(1) \) from growth inequality (1.14).

Let \( r_f \) be the smallest root. Using (3.23) and (3.18), we obtain

\[
|z| + \sum_{n=2}^{\infty} |a_n| |z|^n \leq d(f(0), \partial f(\mathbb{D})) \quad \text{for} \quad |z| = r \leq r_f.
\]

This completes the proof. \( \square \)

**Proof of Theorem 2.27.** From the Lemma 1.23, it is evident that the Euclidean distance between \( f(0) \) and the boundary of \( f(\mathbb{D}) \) is

\[
d(f(0), \partial f(\mathbb{D})) = \liminf_{|z| \to 1} |f(z) - f(0)| \geq L(1, \alpha, \beta).
\]

Note that \( r_f \) is the root of the equation \( R(r, \alpha, \beta) = L(1, \alpha, \beta) \) in \((0, 1)\). The existence of the root is ensured by the relation \( R(1, \alpha, \beta) > L(1, \alpha, \beta) \) from the growth inequality (1.26). Then for \( 0 < r \leq r_f \), it is easily seen that \( R(r, \alpha, \beta) \leq L(1, \alpha, \beta) \).

From the Lemma 1.23 and (3.24), for \( |z| = r \leq r_f \), we obtain

\[
|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \leq r_f + (|a_2| + |b_2|)r_f^2 + \sum_{n=3}^{\infty} (|a_n| + |b_n|)r_f^n
\]

\[
= R(r_f, \alpha, \beta) \leq L(1, \alpha, \beta) \leq d(f(0), \partial f(\mathbb{D})).
\]

To show the sharpness of the radius \( r_f \), we consider the function \( f = f_{\alpha, \beta} \), which is defined in Lemma 1.23 and clearly belongs to \( M(\alpha, \beta) \). Since the left side growth inequality in Lemma 1.25 holds for \( f = f_{\alpha, \beta} \) or its rotations, then \( d(f(0), \partial f(\mathbb{D})) = L(1, \alpha, \beta) \). Therefore the function \( f = f_{\alpha, \beta} \) for \( |z| = r_f \) gives

\[
|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n = r_f + (|a_2| + |b_2|)r_f^2 + \sum_{n=3}^{\infty} (|a_n| + |b_n|)r_f^n
\]

\[
= R(r_f, \alpha, \beta) = L(1, \alpha, \beta) = d(f(0), \partial f(\mathbb{D})),
\]

which shows that the radius \( r_f \) is the best possible. This completes the proof. \( \square \)

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