The Connectivity Order of Links

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Abstract — We associate at each link a connectivity space which describes its splittability properties. Then, the notion of order for finite connectivity spaces results in the definition of a new numerical invariant for links, their connectivity order. A section of this short paper presents a theorem which asserts that every finite connectivity structure can be realized by a link: the Brunn-Debrunner-Kanenobu Theorem.

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1 Order of Finite Connectivity Spaces

Let us recall the definition of a connectivity space [1, 4].

Definition 1 (Connectivity spaces) A connectivity space is a couple \(X = (X, K)\) where \(X\) is a set and \(K\) is a set of nonempty subsets of \(X\) such that

\[
\forall I \in \mathcal{P}(K), \quad \bigcap_{K \in I} K \neq \emptyset \implies \bigcup_{K \in I} K \in K.
\]

The set \(X\) is called the support of the space \(X\), the set \(K\) is its connectivity structure. The elements of \(K\) are called the connected subsets of \(X\). The morphisms between two connectivity spaces are the functions which transform connected subsets into connected subsets. A connectivity space is called integral if every singleton subset is connected. A connectivity space is called finite if the number of its points is finite.

Definition 2 (Irreducible connected subsets) Let \(X = (X, K)\) be a connectivity space. A connected subset \(K \in K\) is called reducible if there are two connected subsets \(A \subsetneq K\) and \(B \subsetneq K\) such that

\[K = A \cup B\] and \(A \cap B \neq \emptyset\).

A connected subset is said to be irreducible if it is not reducible.
In the sequel, all connectivity spaces will be supposed to be integral and finite.

**Definition 3 (Order of irreducible connected subsets)** Let \( X = (X, K) \) be a (finite and integral) connectivity space. We define by induction the order of each irreducible subset. Singleton subsets are said to be of order 0. The order \( \omega(L) \) of an irreducible subset \( L \) which has more than one point is defined by

\[
\omega(L) = 1 + \max_{K \in S(L)} \omega(K)
\]

where \( S(L) \) is the set of irreducible connected subsets which are strictly included in \( L \).

**Definition 4 (Order of a connectivity space)** The order \( \omega(X) \) of a (finite and integral) nonempty connectivity space \( X = (X, K) \) is the maximum of the order of its irreducible connected subsets.

Instead of \( \omega(X) \), we generally write just \( \omega(X) \).

**Remark.** It is useful to represent the structure of a (finite and integral) connectivity space \( X \) by a graph \( G \) whose vertices are the irreducible connected subsets of \( X \) and edges express the inclusion relations: if \( a \) and \( b \) are irreducible connected subsets of \( X \), \( (a, b) \) is an edge of \( G \) iff \( a \subseteq b \) and there is no irreducible connected subset \( c \) such that \( a \subsetneq c \subsetneq b \). In [5], I called this graph the *generic graph* of the connectivity space \( X \); the vertices of this graph, i.e. the irreducible connected subsets of \( X \), are the *generic points* of \( X \). The order of a space is then the maximal length of paths in the generic graph.

**Proposition 1** The order \( \omega(X) \) of a nonempty finite integral connectivity space is always less than the number \( |X| \) of its points: \( \omega(X) \leq |X| - 1 \). The order \( \omega(X) \) is zero iff the space is totally disconnected, that is the only connected subsets are the singletons.

**Example.** Let \( n \geq 1 \) an integer, and \( (A_n, A_n) \) the connectivity space defined by

\[
A_n = \{1, \cdots, n\} \quad \text{and} \quad A_n = \{\{1\}, \{2\}, \cdots, \{n\}\} \cup \{\{1, 2\}, \{1, 2, 3\}, \cdots, \{1, 2, \cdots, n\}\}.
\]

Then \( \omega(A_n) = n - 1 \). Indeed, in this space each connected subset is irreducible, and its order is, by induction, equal to its cardinal minus one. It is, up to isomorphism, the only connectivity space with \( n \) points which is of order \( n - 1 \).

## 2 The Brunn-Debrunner-Kanenobu Theorem

At each (tame) link \( L \), we can associate a connectivity space \( X_L \) taking the components of the link \( L \) as points of \( X_L \) and nonsplittable sublinks of \( L \) (one considers knots, i.e. sublinks with only one component, as nonsplittable links) defining the connected nonempty subsets of \( X_L \).

**Definition 5** The connectivity structure of the connectivity space \( X_L \) associated to a link \( L \) is called the splittability structure of \( L \).
Figure 1: A Borromean ring of borromean rings.

Example. The splittability structure of the Borromean ring is (isomorphic to) \( \{ \{1\}, \{2\}, \{3\}, \{1, 2, 3\} \} \).

In [4, 5], I asked whether every finite connectivity space can be represented by a link, that is whether exists (at least) one link whose connectivity structure is (isomorphic to) the one given. It appears that in 1892, Brunn [2] first asked this question, without formalizing the concept of a connectivity space. His answer was positive, and he gave the idea of a proof based on a construction using some of the links now called “brunnian”. In 1964, Debrunner [3], rejecting the Brunnian “proof”, gave another construction, proving it but only for \( n \)-dimensional links with \( n \geq 2 \). In 1985, Kanenobu [6, 7] seems to be the first to give a proof of the possibility to represent every finite connectivity structure by a classical link, a result which has not been, up to now, very wellknown. The key idea of those different constructions is already in the Brunn’s article: for him, what we call “Brunnian links” are not so interesting in themselves, but for the constructions they allow to make, that is the representation of all finite connectivity structures by links.

Theorem 2 (Brunn-Debrunner-Kanenobu) Every finite connectivity structure is the splittability structure of a link in \( \mathbb{R}^3 \).

3 The Connectivity Order of Links

At each link \( L \), we associate its connectivity order \( \omega(L) \), i.e. the order of the connectivity space associated to \( L \).

Examples. The link pictured on the figure 1 has a connectivity order 2. The one of the figure 2 has a connectivity order 8, which is the maximal order for a link with 9 components. The way this very asymmetrical link splits when a component is erased or cut highly depends on the position of this component in the link, as shows its connectivity structure, which is the one we called \( A_9 \).

Remark. The connectivity order is not a Vassiliev finite type invariant for links. For example, it is easy to check that the connectivity order of the singular link...
with two components, a circle and another component crossing this circle at $2n$ double-points, is greater than $2^n$.

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