Invited Article

Vortices in stably-stratified rapidly rotating Boussinesq convection

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Abstract

We study the Boussinesq approximation for rapidly rotating stably-stratified fluids in a three dimensional infinite layer with either stress-free or periodic boundary conditions in the vertical direction. For initial conditions satisfying a certain quasi-geostrophic smallness condition, we use dispersive estimates and the large rotation limit to prove global-in-time existence of solutions. We then use self-similar variable techniques to show that the barotropic vorticity converges to an Oseen vortex, while other components decay to zero. We finally use algebraically weighted spaces to determine leading order asymptotics. In particular we show that the barotropic vorticity approaches the Oseen vortex with algebraic rate while the barotropic vertical velocity and thermal fluctuations go to zero as Gaussians whose amplitudes oscillate in opposite phase of each other while decaying with an algebraic rate.

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1. Introduction

1.1. Background

The rotating Boussinesq equations have been widely used to study the effects of rotation and density stratification on flow dynamics in the Earth’s atmosphere and oceans, as well as in many other geophysical settings. These equations are an approximation of the compressible, rotating Navier–Stokes equations where one assumes weak density variations in only the equation of state and the buoyancy term. Furthermore, one assumes that the density varies
linearly with temperature, and, if one is studying ocean dynamics, salinity. In this work we focus on the case where less dense fluid is above more dense fluid, known as stable stratification. The opposite case, where the fluid is convective with less dense fluid lying below more dense fluid the system is said to be unstably-stratified; see section 1.4 for a brief discussion on this case.

After the aforementioned approximations and non-dimensionalization, these equations take the form of the incompressible Navier–Stokes equation posed in a non-inertial rotating frame coupled with an equation for temperature fluctuations in the stratified fluid,

\[
\begin{align*}
\partial_t u &= \nu \Delta u - u \cdot \nabla u - \Omega e_3 \times u - \nabla p + \Gamma \theta e_3, \\
\partial_t \theta &= \nu' \Delta \theta - u \cdot \nabla \theta - \Gamma u_3, \\
0 &= \text{div} \ u, \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3, \quad x \in \mathbb{R}^3, \quad t > 0.
\end{align*}
\]

(1.1)

Here \(u\) describes the velocity field of the fluid, \(\nu > 0\) a non-dimensional kinematic viscosity, and \(\nu' > 0\) the thermal diffusivity. The term \(\Omega e_3 \times u\) arises from the effect of the Coriolis force due to the rotating frame, with \(\Omega\) a non-dimensionalized frequency proportional to the inverse of the Rossby number, which measures the rate of rotation relative to the characteristic length scale of the fluid. While \(\Omega\) is in general \(x\)-dependent (say for spherical geometry in the case of the atmosphere of the Earth), it suffices in many situations to use the ‘\(f\)-plane’ approximation where \(\Omega\) is constant. We will assume this for the rest of the work. Next \(\theta\) and \(p\) denote scaled thermal and pressure fluctuations about a horizontally homogeneous mean state which varies linearly with \(x_3\) and has mean density and mean pressure in hydrostatic balance, where the pressure gradient is balanced by the force of gravity on the fluid. This approximation/balance arises from the fact that in typical applications of this model, such as the Earth’s oceans, the horizontal scale is much larger than than the vertical scale; see \[42\] for more explanation.

Finally, following \[3, 25, 29\], \(\Gamma\) is a non-dimensional parameter which combines the gravitational constant and the Brunt–Väisälä frequency for the neutral oscillation of a small ‘parcel’ of vertically unstable fluid feeling the effects of buoyancy and gravity inside a uniform background of stably-stratified fluid; see \[39,\] section 1.3] for more discussion on this.

For a full derivation of this system see \[22\]. For general discussion on these equations, the many interesting physical phenomenon they model, and its various approximating limits, see \[22, 42, 44\]. For a more mathematically focused introduction and review of the subject see \[15, 34, 36, 39, 40\].

1.2. Overview of mathematical literature

Before describing our own results, we discuss some of the other research on this system, focussing in particular on research that either appeared in Nonlinearity, or deals with themes and methods central to the focus of this journal. Due to the wealth of physical applications and the historical inaccessibility of various asymptotic parameter limits in numerical simulations, a sizeable body of mathematical research has been performed on this system and its many variants, with many interesting analytical tools brought to bear on the problem. In addition, the fact that dynamical systems theory has often proven a valuable tool in analyzing the behavior of these equations means that they have featured in Nonlinearity’s pages since its earliest days. For instance, papers \[35, 36\] both treat models very similar to that treated here, and discuss their applicability to questions of ocean and atmospheric dynamics.

In order to discuss some of the key issues in the analysis of this system we set \(\nu = \nu' = 1\); see remark 1.4 for discussion on the \(\nu \neq \nu'\) and \(\nu = \nu' \neq 1\) cases. As discussed in \[29\] (and reviewed below), the linear part of equation (1.1) has one physically relevant eigendirection,
which we call the ‘quasi-geostrophic mode’, which undergoes no dispersive smoothing. However, the other eigendirections correspond to rapidly oscillating waves known as ‘inertial’ or ‘Poincaré waves’. While these do not decay in $L^2$ they do decay in $L^p$ spaces with $p > 2$. This can be quantified with the aid of Strichartz estimates. One then attempts to control such oscillatory behavior and determine the long-time, quasi-geostrophic dynamics of the system. Such dynamics are briefly discussed below. See the book by Chemin et al [15] for more discussion on the role of inertial wave phenomenon in the context of a rotating incompressible Navier–Stokes system without density effects. There they also illustrate interesting connections of the $\Omega \to \infty$ limit with the incompressible limit (low Mach number) of compressible Navier–Stokes. Also see the text of Majda which discusses these topics in the context of inviscid shallow-water wave equations [39].

Global well-posedness and regularity has been one of the main focuses of mathematical research on this system. Here results date back to the works of Lions et al [35, 36] which proved global existence for weak solutions and estimates on the dimensions of global attractors, for a related model, known as the ‘primitive’ equations, which focused on ocean dynamics. This model includes an equation for water salinity and considered the system in a thin layer on a sphere, $S^2 \times (-H, 0)$ where the depth, $H$ was dependent on the spherical variables. The primitive equations are obtained from (1.1) under the hydrostatic approximation, where one uses the disparity between the vertical and horizontal length scales in the Earth’s ocean, to replace the equation for $u_3$ with the leading order balance $\frac{\partial u_3}{\partial z} = -\theta g$. Subsequent works (see [7, 24, 28, 51] and references therein) generalized these results, extending them to strong solutions (both small- and global-time existence) and to more general domains.

The works of Babin et al [2–4] considered systems of the form (1.1) with an additional forcing term, posed on fully periodic domains $x \in [0, 2\pi a_1] \times [0, 2\pi a_2] \times [0, 2\pi a_3]$ or horizontally periodic domains with ‘stress-free’ boundary conditions in the vertical direction:

$$u_3 = \theta = 0, \quad \partial_3 u_1 = \partial_3 u_2 = 0, \quad x = 0, 2\pi a_1.$$  \hspace{1cm} (1.2)

There, inertial waves cannot ‘escape to infinity’ due to the bounded domain, possibly inducing resonant three dimensional wave interactions. Small divisor techniques are required to prove global existence and regularity of strong solutions, along with the existence of attractors, for certain non-resonant domains in this setting. In these works, they also studied how, in the joint limit $\Omega = 1/\epsilon, N^2 = \bar{N}/\epsilon, \epsilon \to 0$ solutions of (1.1) can be uniformly approximated in time by solutions of the quasi-geostrophic equations, a classic and often-used model first derived by Charney [9] for the slow dynamics of the atmosphere.

There also has been an intense study of the full Boussinesq system (1.1) in the whole space $x \in \mathbb{R}^3$. In a series of works [11–14], Charve has shown convergence to the quasi-geostrophic system in the whole space $x \in \mathbb{R}^3$ for both weak and strong solutions of various regularity and initial condition size, in the same limit $\epsilon \to 0$ mentioned above. The idea underlying all these approaches is to decompose the solution as a part governed by the quasi-geostrophic system and a ‘remainder’ and to show that the oscillatory effects arising from the fast rotation allow one to prove existence and uniqueness of solutions in three-dimensions, even for large initial data.

Subsequent results such as [32] extend the use of dispersive estimates in $\mathbb{R}^3$ to other $L^p$ spaces. The most recent result [29] investigates more deeply the connections between harmonic analysis and the principal curvatures of the linear dispersion relation $p_\eta(k)$; see [23] for example. In particular, degeneracies of the Hessian, $(\partial_{k,k}^2 p_\eta)_\eta$, of the dispersion relation are used to study more general initial conditions, only requiring smallness in the quasi-geostrophic component. We remark that because of the smallness of the quasi-geostrophic initial
condition these last two results omit study of the quasi-geostrophic limit as in Charve’s work. See [29] for a more detailed discussion of these topics and review of the related literature.

1.3. The role of dynamical systems

The tools of dynamical systems have also been integral to the study of geophysical fluid dynamics. In fact, the study of such fluid models has been one of the main motivations for the development of many mathematical tools such as global attractors, inertial manifolds, and slow manifolds. Again, Nonlinearity was a leader in this work and already, in the second volume, dynamical systems methods were exploited to study the stability of stratified fluids [26].

1.3.1. Slow manifolds.

During the 1970s and 1980s many researchers, including Lorenz [37], posited the existence of a slow invariant manifold in the Boussinesq system which controlled and organized the dynamics. In an effort to understand the interplay between fast, oscillatory gravity-waves and the slow geostrophic dynamics, many works considered what is known as the quasi-geostrophic limit $\Omega, \Gamma \sim O(1/\epsilon)$ with small $\epsilon$. Here researchers worked to discover if there was a manifold in the (infinite-dimensional) phase space, which was invariant under the time evolution, devoid of fast waves, and contained the slowly evolving geostrophic dynamics which govern much of the long term behavior of the system. Such manifolds would then be of use as they could give information on the global attractor of the system. While the current consensus is that no such exact manifold exists [38], there has been rigorous work characterizing approximately invariant, or ‘fuzzy’, manifolds when the Boussinesq system undergoes certain types of forcing [43, 49, 50]. These works have shown that in finite-time the fast dynamics decay and stay $O(\epsilon)$-small, so that the dynamics stay near geostrophic balance. See also [52, 53] for studies which consider the additional limits $\Omega \sim O(1/\epsilon), \Gamma \sim O(1)$ and $\Omega \sim O(1), \Gamma \sim O(1/\epsilon)$. For nice reviews of this topic, the important work in the area, and some of the most pressing open questions, see [50, 52].

1.3.2. Scaling variables, invariant manifolds, and omega-limit sets.

In work that is the closest to our own in both technique and aims, Gallay and Roussier-Michon have used dynamical systems techniques to study long-time asymptotics of the rotating incompressible Navier–Stokes system

$$\partial_t u + (u \cdot \nabla) u + \Omega e_3 \times u - \Delta u = 0, \quad \text{div} u = 0. \quad (1.3)$$

In [18], they considered the above system posed on $\mathbb{R}^2 \times [0, 1]$ with periodic boundary conditions in the vertical direction. They showed, for any size initial condition and a correspondingly large $\Omega$, the existence of global, infinite energy solutions which converge to the $x_3$-independent diffusively decaying Oseen vortex solution. In other words, they showed for high rotation rates, solutions asymptotically behave like the non-rotating 2D Navier–Stokes equations. Using the barotropic-baroclinic decomposition described in [15], which splits the vector field into an $x_3$-independent component, $\bar{u}$, and an $x_3$-mean-zero component, $\tilde{u}$, one obtains a system of three equations, one each for the vertical vorticity $\tilde{\omega}_3 = \partial_1 \tilde{u}_2 - \partial_2 \tilde{u}_1 = \int_0^1 (\nabla \times u)_{3} \text{d}x_3$, the vertical barotropic velocity $\bar{u}_3$, and the baroclinic velocity $\tilde{u}_3$. They obtain exponential decay of the baroclinic component using Poincare’s inequality (due to vertical mean-zero and boundedness of vertical domain) and Strichartz estimates. After using linear convection-diffusion estimates of Carlen and Loss [8] to show the algebraic decay of $\nabla \tilde{u}_3$, they then study the barotropic vorticity which satisfies an equation very similar to the vorticity formulation of 2D Navier–Stokes. Taking the approach of Gallay and Wayne [20], they employ scaling variables...
and compactness arguments to determine the omega-limit set of the system, showing that \( \bar{\omega} \), and hence \( u \) itself, asymptotically converge to the Oseen vortex solution as \( t \to \infty \). However, they do not consider the effects of the coupling between the velocity field and the temperature, nor do they derive detailed asymptotics for a solution near a vortex, like those in theorem 1 below.

### 1.4. Unstably-stratified fluids

The unstably-stratified system, which models the case where a cold fluid lies atop a warmer one, has also been the focus of much recent study. This configuration is modeled by switching the sign on the last term of the second equation in the Boussinesq system (1.1). By scaling, one typically combines the effects of gravity, thermal expansion, and buoyancy, into the coefficient of the thermal forcing term—the last term of the first equation in (1.1). This coefficient, which we call \( R \), known as the Rayleigh number and its size roughly controls the stability and bifurcation structure of the system. Also, physical boundary conditions once again take the form (1.2). Such models typically arise when studying more extreme systems, such as rapidly-rotating convective atmospheric layers and are contained in the more complicated magneto-hydrodynamic models used to study stellar plasmas. See [6, 33] for a review of these applications.

The convective nature of the unstable stratification leads to a rich family of structures and dynamics for various parameter ranges. Experimental and numerical studies, with various geometries and boundary conditions, have exhibited a wealth of interesting behaviors, such as convection cells, localized plumes, and large-scale turbulent vortices as the Rayleigh number is varied. See the introduction of [17] for a nice review of the numerical and experimental literature and more discussion on this topic.

In the case of rapid rotation, \( \Omega \gg 1 \), the Coriolis term suppresses behaviors characteristic of 3D incompressible fluids, such as direct turbulence cascades. This causes the system to behave similarly to a forced 2D fluid, where an inverse cascade dominates turbulent dynamics; see once again [6, 33]. Assuming large enough Rayleigh number, the thermal forcing creates small-scale eddies which become vertically ‘aligned’ and coalesce into large domain-scale turbulent vortices.

Mathematically, much less is known about these equations compared with the stably-stratified case. While there are some results on existence and bifurcation of finite dimensional attractors in fully periodic domains \( \mathbb{T}^3 \) [27, 34], little work has been done to rigorously characterize coherent structures which arise in these systems. It would be interesting to determine if an invariant or approximately invariant slow-manifold existed in the system. Given the previous literature and discussion above, a reasonable candidate for a slow variable is the the barotropic component of the full 3D velocity,

\[
\bar{v} = \int_0^1 v(x_0, x_3) \, dx_3.
\]

The boundary conditions on \( \theta \) and \( u_3 \) immediately give that \( \bar{u}_3 = \bar{\theta} = 0 \). One then obtains that \( \bar{u}_3 \) satisfies a 2D Navier–Stokes equation which is forced by the 3D baroclinic system. One would hope to show that in the high rotation limit, \( \Omega = \epsilon^{-1}, \ 0 < \epsilon \ll 1 \) (possibly also scaling the Rayleigh number \( R = \tilde{R}/\epsilon \) as well), the Coriolis force overpowers the unstable stratification term causing the baroclinic variables \( \bar{v} \) to decay to \( O(\epsilon) \) sizes after a finite-time and remain small for all subsequent times, as in [50]. Hence, the barotropic subspace \( \{ \bar{v} = 0 \} \) would form an approximately invariant manifold for the full dynamics. One would then hope
to characterize the dynamics of $\bar{u}_h$, either explicitly characterizing large-scale vortices or at the very least showing the existence of, and characterizing the attractor. As forced 2D turbulence is in general only understood statistically (see [6]) this approach seems to be an interesting line of future research. Also note the recent work [5] where a similar vertically averaged system is studied on a horizontally periodic domain, but with no rotation and with Dirichlet boundary conditions in the vertical direction. They characterize the effect of the baroclinic forcing terms on the vertically averaged velocity in order to understand convective turbulence.

The unstably-stratified system is in general difficult to simulate, with direct numerical simulations for large Rayleigh number only being performed in the last few years; see for example [17]. Julien and Knobloch with various collaborators have derived a formal asymptotic quasi-geostrophic model which eliminates fast inertial waves and certain boundary effects, known as Ekman layers, and is thus much more tractable numerically while still retaining much of the 3D dynamics (as opposed to a 2D Navier–Stokes system with arbitrary forcing). In a series of works [30, 46, 48], they have shown this model exhibits various coherent structures and similar statistical behaviors observed in experiments and recent simulations of the full Boussinesq system. It would be of great interest to rigorously study such coherent structures in this asymptotic model, and consequently to rigorously characterize the asymptotic convergence of the full Boussinesq system to this formal model in a way similar to that of the quasi-geostrophic approximation of the stably-stratified system discussed above.

### 1.5. Our results

This paper focuses on determining leading order asymptotics of (1.1). We consider the stably-stratified Boussinesq system (1.1) above, posed on $\mathbb{D} = \mathbb{R}^2 \times [0, 1]$ with either stress-free boundary conditions

$$\partial_t u_1 = \partial_t u_2 = u_3 = 0, \quad \theta = 0, \quad x_3 = 0, 1,$$

or periodic boundary conditions

$$u(x_h, x_3) = u(x_h, x_3 + 1), \quad \theta(x_h, x_3) = \theta(x_h, x_3 + 1).$$

While the latter boundary conditions are for the most part non-physical, they have been often studied as an idealized version of the system.

#### 1.5.1. Local dynamics near a barotropic Oseen vortex

The main result of this paper (proved in section 3), is to determine the leading order asymptotics in the case of periodic vertical boundary conditions. As in many previous works, we begin by splitting the evolution into its barotropic (i.e. vertically averaged) and baroclinic parts, $\bar{v} = Q \bar{v} := \int_0^1 v(x_h, x_3) dx_3$, and $\tilde{v} = (1 - Q) \bar{v}$. The purely barotropic evolution is obviously an invariant manifold within the phase space of the full system. We then identify a family of explicit vortex solutions in the barotropic system. These vortices correspond to an Oseen vortex for the 2D Navier–Stokes equation in the two horizontal components of the velocity, and a coupled pair of vortices in the vertical velocity and temperature fields which oscillate with the Brunt–Väisälä frequency. We then show that regardless of the vortex amplitude, these solutions are locally stable with respect to the full Boussinesq evolution. From a dynamical systems perspective, this shows that these solutions are in the barotropic manifold and are at least locally attractive. We also note that, for this local analysis, we need not assume the rotation rate is large, even to obtain the stability of large vortex solutions. Another way of thinking of this result is that it emphasizes the essentially 2D nature of these stably stratified fluid layers. The emergence of the
Oseen vortex is typical of 2D flows in which the vorticity concentrates in large coherent structures, and our results show that the temperature fluctuations do not destroy that effect.

As we explain below, the detailed analysis of the convergence towards these vortices requires somewhat localized initial data. Technically, we adapt and extend the aforementioned techniques of [20] to prove global existence in weighted spaces and derive leading order asymptotics by decomposing solutions using the leading order eigenspaces of the linear system. We enforce algebraic spatial decay with the weighted spaces

\[ L^2(m) = \{ v(\xi, x) \in L^2(\mathbb{D})^4 : \| v \|_{L^2(m)}^2 := \| b^m v \|_{L^2(\mathbb{D})} < \infty \} \]

\[ L^2_0(m) = \{ f \in L^2(\mathbb{R}^2) : \| f \|_{L^2_0(m)} := \| b^m f \|_{L^2(\mathbb{R}^2)} < \infty \} , \]

where \( b^m = (1 + |\xi|^2)^{m/2} \). (The reason for this precise choice of weighted space is discussed in section 3, but roughly speaking, we can make a change of variables such that in these function spaces, the linearized equations in the neighborhood of a barotropic vortex have a particularly simple form, with explicitly computable leading order asymptotics and decay rate.)

We also denote

\[ \varphi_0(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \quad \bar{u}_0^0(\xi) = \frac{1 - e^{-|\xi|^2/4}}{2\pi \xi^2} \left( -\frac{\xi_2}{\xi_1} \right), \]

where \( \bar{u}_0^0 \) is the incompressible, 2D velocity field with Gaussian vorticity distribution \( \varphi_0 \). (This is the Oseen vortex, which governs the long-time evolution of the 2D Navier–Stokes equations [20].) Then, setting \( \zeta := \nabla \times \bar{u} = (\partial_2 \bar{u}_3, -\partial_1 \bar{u}_3, \partial_3 \bar{u}_1 - \partial_1 \bar{u}_2) \), \( \bar{w}_h = (\omega_1, \omega_2)^T \), and \( \Theta := \nabla \bar{w} \cdot \bar{\theta} = (\partial_2 \bar{\theta}_1, -\partial_1 \bar{\theta})^T \), and defining the zeroth moments of the initial data

\[ A = \int_{\mathbb{R}^2} \bar{\omega}_{3,0}(x_0) dx_0, \quad B = (B_1, B_2)^T := \left( \int_{\mathbb{R}^2} \bar{u}_{3,0}(x_0) dx_0, \int_{\mathbb{R}^2} \bar{\theta}_0(x_0) dx_0 \right)^T, \]

our precise result is as follows

**Theorem 1.** Fix \( \mu \in (0, 1/2) \), and \( m > 3 \). There exists a constant \( r > 0 \) such that for all initial data \( (\bar{w}_0, \Theta_0) \in L^2_0(m)^5 \), \( \bar{v}_0 \in L^2(m)^4 \) with barotropic moments \( A, B \) of arbitrary size and

\[ \| \bar{w}_{3,0} - A \varphi_0 \|_{L^2(m)} + \| \bar{w}_{0,0} - B_1 \nabla \bar{w} \cdot \varphi_0 \|_{L^2(m)}^2 + \| \Theta_0 - B_2 \nabla \bar{w} \cdot \varphi_0 \|_{L^2(m)} + \| \bar{v}_0 \|_{L^2(m)} < r, \]

there exists a global solution of (1.1), for which the quantities \( (\bar{w}, \Theta, \bar{v}) \) satisfy \( (\bar{w}, \Theta, \bar{v}) \in C([0, \infty), L^2_0(m)^3 \times L^2_0(m)^3 \times L^2_0(m)^4) \). Furthermore,

\[ \left\| \bar{w}_3 (\cdot, t) - \frac{A}{1 + t} \bar{\varphi}_0 \left( \frac{\cdot}{\sqrt{1 + t}} \right) \right\|_{L^p(\mathbb{R}^2)} \leq \frac{C_p}{(1 + t)^{1 + \mu - 1/\rho}}, \] \n
\[ \left\| \bar{w}_3 (\cdot, t) - \frac{1}{(1 + t)^{3/2}} (B_1 \cos(\Gamma t) + B_2 \sin(\Gamma t)) \nabla \bar{w} \cdot \varphi_0 \left( \frac{\cdot}{\sqrt{1 + t}} \right) \right\|_{L^p(\mathbb{R}^2)} \leq \frac{C_p}{(1 + t)^{3/2 + \mu - 1/\rho}} \] \n
\[ \left\| \Theta (\cdot, t) - \frac{1}{(1 + t)^{3/2}} (-B_1 \cos(\Gamma t) + B_2 \sin(\Gamma t)) \nabla \bar{w} \cdot \varphi_0 \left( \frac{\cdot}{\sqrt{1 + t}} \right) \right\|_{L^p(\mathbb{R}^2)} \leq \frac{C_p}{(1 + t)^{3/2 + \mu - 1/\rho}} \]
for $t \geq 0$ and any $p \in [1, 2]$.

**Corollary 1.1.** One can use the Biot–Savart relationships in proposition 3.4 to readily conclude the following decay estimates for $\bar{v}$, $1/q = 1/p - 1/2$ with $p \in [1, 2]$ and $t > 0$,

$$\|\bar{u}_h(\cdot, t) - \frac{A}{\sqrt{1+t}} \bar{u}^0_h \left( \frac{\cdot}{\sqrt{1+t}} \right) \|_{L^p(\mathbb{R}^2)} \lesssim \frac{C_p}{(1 + t)^{1/2 + \mu - 1/q}},$$

$$\|\bar{u}_3(\cdot, t) - \frac{1}{1+t} (B_1 \cos(\Gamma t) + B_2 \sin(\Gamma t)) \varphi_0 \left( \frac{\cdot}{\sqrt{1+t}} \right) \|_{L^p(\mathbb{R}^2)} \lesssim \frac{1}{(1 + t)^{1/2 + \mu - 1/q}},$$

$$\|\bar{u}_1(\cdot, t) - \frac{1}{1+t} (-B_1 \sin(\Gamma t) + B_2 \cos(\Gamma t)) \varphi_0 \left( \frac{\cdot}{\sqrt{1+t}} \right) \|_{L^p(\mathbb{R}^2)} \lesssim \frac{C_p}{(1 + t)^{1/2 + \mu - 1/q}}.$$

**Remark 1.2.** Physically, this result shows the initial conditions for our fluid system correspond to a perturbation of an Oseen vortex, the horizontal velocity of the resulting solution converges toward that vortex, while the vertical velocity and the temperature fields oscillate with frequency $\Gamma$ which is determined by the density profile of the stratified fluid (for the exact definition of $\Gamma$, see the discussion just prior to (1.1)). Furthermore, the motion has an essentially 2D character since the baroclinic components of $\bar{v}(x, t)$, the velocity and temperature, converge to zero faster than the barotropic parts which depend only on the horizontal components of $x$.

### 15.2. Global existence and asymptotics

If we wish to study more general initial data, rather than just small perturbations of the barotropic vortices, we must impose additional restrictions. In this case, we will work in the high rotation number limit, $|\Omega| \gg 1$. That we must place some conditions on the initial data is not surprising—our system of equations contains the three dimensional Navier–Stokes equations and we cannot hope to prove existence and uniqueness of solutions with general initial data without some restrictions. In the previous section, we assumed that we are initially near one of the barotropic vortices. In the present section, we will assume that $\Omega$ is large, and that the initial conditions have small quasi-geostrophic component. In the high rotation number limit, we can use dispersive estimates and energy estimates to obtain global existence and asymptotics, albeit with less information on the rate of convergence. In particular, requiring smallness of only the quasi-geostrophic part of the initial condition, that is initial data $\tilde{v}_0$ satisfying $\int_{\mathbb{R}^3} |\tilde{v}(k, 0), a_0(k)\rangle_{C_0^1} |k|^4 \, dk \ll 1$, with $a_0(k)$ defined in (2.12), we prove global existence of mild solutions. Once again, the restriction on the size of the quasi-geostrophic part of the initial condition is not surprising from a physical point of view, since this mode is not damped by the dispersive effects which we use to control other parts of the solution.

As in the prior theorem, we can again show that in this regime, 2D effects dominate so that thermal fluctuations decay to zero and the velocity field converges to the two dimensional vortex solution

$$\left( \frac{A}{\sqrt{1+t}} \bar{u}^0_h \left( \frac{\cdot}{\sqrt{1+t}} \right), 0 \right)^T,$$
with $\bar{u}_0^i$ defined above. The less detailed asymptotics contained in theorem 2 just below, allow us to work in function spaces which do not require the sort of decay at infinity which we imposed on the initial data in theorem 1. Depending on the boundary conditions, we consider solutions in one of two the Banach spaces

$$X_d = \{ \bar{v} \in H^1_{\text{loc}}(\mathbb{D})^4 \mid \text{div} \bar{u} = \text{div} \bar{u}_0 = 0, \bar{\theta} \in H^1(\mathbb{R}^2), v \text{ satisfies } (1.4) \},$$

$$X_p = \{ \bar{v} \in H^1_{\text{loc}}(\mathbb{D})^4 \mid \text{div} \bar{u} = \text{div} \bar{u}_0 = 0, \bar{\theta} \in H^1(\mathbb{R}^2), v \text{ satisfies } (1.5) \},$$

with the norms

$$\| \bar{v} \|_{X_d} = \| \bar{\theta} \|_{H^1(\mathbb{D})} + \| \bar{\omega}_3 \|_{L^1(\mathbb{R}^2)} + \| \bar{\omega}_3 \|_{L^1(\mathbb{R}^2)},$$

$$\| \bar{v} \|_{X_p} = \| \bar{\theta} \|_{H^1(\mathbb{D})} + \| \bar{\omega}_3 \|_{L^1(\mathbb{R}^2)} + \| \bar{\omega}_3 \|_{L^1(\mathbb{R}^2)} + \| \bar{\omega}_3 \|_{L^1(\mathbb{R}^2)} + \| \bar{\theta} \|_{L^2(\mathbb{R}^2)} \quad (1.11)$$

where once again $\bar{v} = Q \bar{v} := \int_0^1 \bar{v}(x_3, x_1) dx_3$, and $\bar{\theta} = (1 - Q)\bar{\theta}$. Also, we define a projection operator $S$ on $L^2(\mathbb{D})$ as

$$Sf = F^{-1} \left[ \hat{f}(\cdot), a_3(\cdot) \right]_{C^0},$$

where $F$ is the Fourier transform, defined on $L^2(\mathbb{D})$. Our results for these two boundary conditions are as follows:

**Theorem 2.** For all $\Gamma \in \mathbb{R} \setminus \{0\}$, and initial conditions $v_0 \in X_\Gamma$ with either $i = p$ or $i = sf$, and $\|\delta u_0\|_{L^2}$ sufficiently small, there exists $\Omega_0 > 0$ such that, for all $|\Omega| \geq \Omega_0$, the system (1.1) with either periodic boundary conditions (1.5) or stress-free boundary conditions (1.4) respectively, has a mild solution $v \in C^0([0, \infty), X_\Gamma)$ satisfying $v(\cdot, 0) = v_0$. Furthermore, there exists a $C > 0$ such that $\|v(t)\|_{X_\Gamma} \leq C$ for all $t > 0$ and given $A = \int_{\Omega} (\text{curl} u_0) dx$, this solution satisfies

$$\left\| v(\cdot; t) - A \left( \frac{1}{\sqrt{1+t}} \log(1+t), \frac{1}{\sqrt{1+t}} \log(1+t), 0, 0 \right) \right\|_{X_\Gamma} \to 0, \quad \text{as } t \to +\infty,$$

where $u_0^i = (\bar{u}_0^i, \bar{u}_0^3)^T$ is defined in (1.6) above.

**Remark 1.3.** Note that due to the presence of the $L^1$-norm of $\omega_3$ in the definition of the $X_\Gamma$ norm, and the fact that the third component of the vorticity is to leading order $\varphi_3(\frac{\nabla \times \bar{u}}{1+t})$, this term has non-zero $X_\Gamma$ norm as $t \to \infty$. However, the remaining terms in the asymptotics in theorem 1 all vanish in the $X_\Gamma$-norm—they are ‘invisible’ in this theorem. Thus, while theorem 2 has the advantage of treating global initial data, it gives far less detailed information about the long-time behavior of solutions than theorem 1.

The proof of theorem 2 for each type of boundary condition follows the approach of [18] with modifications to account for the inclusion of temperature effects. Namely, we use a barotropic/baroclinic decomposition to split $\bar{v}$ into a $x_3$ independent part $\bar{v}$ and a $x_3$-mean zero part $\bar{v}$. We shall further decompose the barotropic vector $\bar{v}$ studying a system of equations for $\omega_3 = (\nabla \times \bar{u})_3, \bar{u}_3,$ and $\bar{\theta}$. Note stress-free boundary conditions force $\bar{u}_3 \equiv 0 \equiv \bar{\theta}$ identically.

We use dispersive estimates to show $\bar{v}$ decays exponentially fast and then use energy methods and Gronwall’s inequality to prove global existence. Diffusive estimates and dynamical systems techniques can then be used to determine the leading order asymptotics of $\bar{u}_0^i$, the barotropic part, hence determining the leading order dynamics of the system. Since the proofs
Remark 1.4. Qualitatively similar results should hold for the more general situation of differing viscosities \( \nu' \neq \nu \). In this situation, while the expressions for the eigenvalues and eigenvectors of the linear system in Fourier space are more complicated and the collection of eigenvectors are not orthogonal, three of these vectors are still orthogonal to the Fourier vector \( (k_1, k_2, k_3, 0)^T \). Thus, one should be able to push through the required dispersive estimates, as done in [10, section 4] for example, and obtain the same asymptotics for \( \tilde{v} \) up to an \( \mathcal{O}(\Omega^{-1}) \)-sized correction. The changes we expect if \( \nu = \nu' \neq 1 \) are slightly more complicated. On the whole space, one can change the length scale so that the viscosity is always equal to one. However, if we make such a rescaling in our situation, it would change the thickness of the fluid layer. A different value of \( \nu \) would probably result in a different decay rate for the baroclinic components of the solution, but we still expect on heuristic grounds that they would decay more rapidly than any inverse power of \( t \). For the barotropic components of the solution, we expect the decay rates would be the same as those in theorem 1, since these parts of the solution depend only on the unbounded variables \( (x_1, x_2) \).

While such results seem to have been implicitly known in the literature, (namely that for sufficiently small quasi-geostrophic part the fast-oscillatory component decays as \( t \to +\infty \), or in other words inertial waves escape to infinity, and the system asymptotes to solutions of 2D Navier–Stokes equations), to our knowledge it has never been rigorously stated or proven, nor has the detailed asymptotic behavior obtained in theorem 1 been derived, especially not for perturbations of large, barotropic vortices. Furthermore, we believe the viewpoint taken here could be useful in future studies, especially in the study of unstably-stratified convective rotating systems where a wealth of coherent structures arise; see section 1.4.

A natural next line of study extending from this work would be to investigate how much one can say about asymptotics for fully arbitrary initial conditions \( v_0 \in X \), where there is no restriction on the quasi-geostrophic component. As mentioned above, Charve has shown in \( \mathbb{R}^3 \) that general initial data evolves towards solutions of the quasi-geostrophic equation. Thus, a useful question to frame such an inquiry could be:

‘can dynamical systems techniques allow for a more refined characterization and classification of the asymptotic dynamics of the stably-stratified system with realistic boundary conditions?’

Finally, we believe it would of great interest to investigate whether dynamical systems techniques could be used to characterize the physical phenomenon observed in unstably-stratified rotating convection, described in section 1.4 above. This seems natural as such approaches have been very successful in describing coherent structures in other contexts (see for example [31]), so one expects that they could be successful for the cellular and plume like dynamics of the rotating system. Furthermore, it would be of interest whether the scaling variables/invariant manifold approach of [19, 20] could be used to describe the large-scale turbulent vortices which arise and grow out of small-scale turbulent eddies.

1.5.3. Outline of rest of work. The rest of the work is organized as follows. In section 2, we collect some facts about (1.1) with both periodic and stress-free boundary conditions and set-up our framework. In section 3, we prove our main result theorem 1, leaving proofs of several technical propositions to appendix A. In section 4.1 we collect estimates on the linear dispersive equation associated with our system. We then use this information in section 4.2 to
prove global existence in the stress-free case, while in section 4.3 we prove the leading order asymptotics under our assumptions, completing the proof of theorem 2.

2. Preliminaries

In this section we rewrite the basic system (1.1) in a way which allows us to more easily identify the primarily 2D parts of the motion (the ‘barotropic’ part of the solution) which ultimately dominate its long-time asymptotics. We also write the perturbation of the barotropic motion in a way which allows us to separate out the effects of rotation and stratification. Recall that (1.1) has the form:

\[ \partial_t u + (u \cdot \nabla) u + \Omega e_3 \times u + \nabla p = \Delta u + \Gamma \theta e_3, \]

(2.1)

\[ \partial_t \theta + (u \cdot \nabla) \theta = \Delta \theta - \Gamma u_3, \]

(2.2)

\[ \text{div } u = 0, \]

(2.3)

\[ u \in \mathbb{R}^3, (x_h, x_3) \in \mathbb{D} := \mathbb{R}^2 \times [0, 1]. \]

We denote \( v = (u_1, u_2, u_3, \theta)^T \) to be the combined vector of velocity and thermal fluctuations so that (2.1)–(2.3) takes the form

\[ \partial_t v + (u \cdot \tilde{\nabla}) v - \text{diag}(\Delta) v + J_{\Omega, \Gamma} v + \tilde{\nabla} p = 0, \]

div \( u = 0 \)

(2.4)

with \( \tilde{\nabla} = (\partial_1, \partial_2, \partial_3, 0)^T \), and

\[ J_{\Omega, \Gamma} = \begin{pmatrix} \Omega J & 0 \\ 0 & \Gamma J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

As it will ease computations, we also set \( J_\eta = \Gamma^{-1} J_{\Omega, \Gamma} \) with \( \eta = \Omega / \Gamma \). Applying the Helmholtz projection, denoted as \( \mathbb{P} \), onto divergence free vector fields to the velocity component of (2.4) we obtain

\[ \partial_t v + \text{diag}(\Delta) v + \Gamma \mathbb{P} J_\eta \mathbb{P} v + \mathbb{P}(v \cdot \tilde{\nabla}) v = 0, \]

div \( u = 0 \).

(2.5)

For periodic boundary conditions, \( \mathbb{P} \) is defined in Fourier space as

\[ \hat{\mathbb{P}} v(k) = P(k) \hat{v}(k), \quad P(k) = \begin{pmatrix} \delta_{ij} - \frac{k_i k_j}{k^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{R}^2 \times \{2\pi \mathbb{Z}\} \]

(2.6)

with \( \delta_{ij} \) the Kroenecker delta function and \( k = (k_1, k_2)^T \in \mathbb{R}^2, k_3 = 2\pi n, n \in \mathbb{Z} \). For stress-free boundary conditions, \( \mathbb{P} \) takes a similar form where one must consider sine and cosine series in the vertical direction.

2.1. Barotropic/baroclinic decomposition

2.1.1. Periodic case. In order to exhibit the leading order dynamics of the system, we decompose our vector field into a vertically independent “barotropic” part and a vertically mean-zero “baroclinic” part. For periodic boundary conditions (1.5) we set

\[ u = \bar{u} + \check{u}, \quad \theta = \bar{\theta} + \check{\theta}, \quad \bar{u} = Qu := \int_0^1 u(x_h, x_3) dx_3, \quad \bar{\theta} = Q\theta := \int_0^1 \theta(x_h, x_3) dx_3, \]
with $x_3$-independent barotropic variables $\tilde{u}, \tilde{\theta}$ and $x_3$-mean zero baroclinic variables $\tilde{u}, \tilde{\theta}$ which satisfy vertical periodic boundary conditions. With this decomposition, the $x_3$-dependent terms have the following Fourier decomposition,

$$
\tilde{v}(x) = \int_{\mathbb{R}^2} \sum_{n \in \mathbb{Z}_+} \hat{\omega}_n(k)e^{i(k_n x_n + 2\pi n x_3)}\,dk_n,
$$

(2.7)

$$
\tilde{\omega}(x) = \int_{\mathbb{R}^2} \sum_{n \in \mathbb{Z}_+} \hat{\omega}_n(k)e^{i(k_n x_n + 2\pi n x_3)}\,dk_n
$$

(2.8)

where $k_n = (k_1, k_2)^T$, $x_n = (x_1, x_2)^T$, $\mathbb{Z}_+ = \mathbb{Z} \setminus \{0\}$, and $\hat{\omega}$ and $\tilde{\omega}$ denote the Fourier transform of $\omega$ and $\tilde{\omega}$ respectively. From the incompressibility condition, the Biot–Savart law relating $\tilde{u}$ and $\tilde{\omega}$ is found to be

$$
\tilde{u}_n(k) = \frac{1}{|k|^2 + 4\pi^2 n^2} A_n(k) \hat{\omega}_n(k) := \frac{1}{|k|^2 + 4\pi^2 n^2} \begin{pmatrix} 0 & -2\pi n & ik_1 \\ 2\pi n & 0 & -ik_1 \\ -ik_1 & ik_1 & 0 \end{pmatrix} \hat{\omega}_n(k),
$$

(2.9)

while the skew-Hermitian term $\mathbb{P} J_{\eta} \mathbb{P}$ takes the form

$$
\mathbb{P} J_{\eta} \mathbb{P} = \frac{1}{|k|^2} \begin{pmatrix} 0 & 4\pi^2 n^2 \eta & -2\pi nk_2 \eta & -2\pi nk_1 \\ -4\pi^2 n^2 \eta & 0 & 2\pi nk_1 \eta & -2\pi nk_2 \\ 2\pi nk_2 \eta & -2\pi nk_1 \eta & 0 & |k|^2 \\ 2\pi nk_1 & 2\pi nk_2 & -|k|^2 & 0 \end{pmatrix},
$$

(2.10)

where $k = (k_1, k_2, 2\pi n)^T$, $|k|^2 := |k_1|^2 + (2\pi n)^2$. The spectral information of this matrix is computed in [29, section 2] and one finds it has eigenvalues 0, 0, $\pm ip_\eta(k)$ with corresponding eigenvectors

$$
a_\pm(k) = 1 \begin{pmatrix} ik_2 \\ -ik_1 \\ 0 \\ 2\pi n \eta \end{pmatrix}, \quad a_0(k) = 1 \begin{pmatrix} ik_1 \\ ik_2 \\ 0 \\ 2\pi n \eta \end{pmatrix},
$$

(2.11)

$$
a_+(k) = \frac{1}{\sqrt{2}|k||k||k_\eta|} \begin{pmatrix} 2\pi (k_2 \eta |k| + ik_1 |k_\eta|) \\ 2\pi (-ik_1 \eta |k| + k_2 |k_\eta|) \\ -i|k|^2 |k_\eta| \\ |k_\eta|^2 |k| \end{pmatrix}, \quad a_-(k) = a_+(k)
$$

(2.12)

where $k_\eta = (k_1, k_2, 2\pi n \eta)$. We can then decompose (2.4) into the following barotropic/baroclinic system,

$$
\dot{u}_t = \Delta u + \Gamma \dot{\theta} - Q[\bar{u} \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{u}] - \nabla p,
$$

$$
\dot{\theta} = \Delta \theta - \Gamma \dot{u} - Q[\bar{u} \cdot \nabla \bar{\theta} + \bar{\theta} \cdot \nabla \bar{u}],
$$

$$
\tilde{v}_t = \Delta \tilde{v} - J_{1,1} \tilde{v} - \nabla p - Q[\bar{u} \cdot \nabla \bar{v}] - (1 - Q)[(\bar{u} \cdot \nabla)\tilde{v} + (\bar{u} \cdot \nabla)\bar{v} + (\bar{\theta} \cdot \nabla)\bar{v}],
$$

$$
0 = \text{div} \tilde{u}.
$$

(2.13)

### 2.1.2. Stress-free case.

The form of the equations in the stress-free case is very similar to those in the periodic case. For completeness, we include them in appendix B.
3. Existence and asymptotics in algebraically weighted spaces

Our main result in this section concerns the existence and asymptotics of solutions of the Boussinesq equation (2.4) with periodic boundary conditions (1.5) for initial data that lie in algebraically weighted spaces. The existence result follows in a fairly standard fashion by rewriting the equation as an integral equation, coupled with estimates on the linear evolution. We note that the coupling of the vorticity formulation with temperature fluctuations in our system requires a more subtle analysis of the linear evolution; see appendix A. The analysis of the asymptotics makes use of scaling variables which have been very useful in previous studies of the asymptotics of the Navier–Stokes equations. We work with the baroclinic-barotropic decomposition (2.13) described in section 2.

3.1. Dynamics on barotropic invariant subspace

In this section we study the dynamics of the barotropic part of the velocity and temperature fields—i.e. the averaged velocity and temperature across the layer—ignoring for the moment the baroclinic part of the solution. We show that the horizontal part of the barotropic velocity field satisfies the 2D Navier–Stokes equation, and hence its motion is dominated by the Oseen vortex, an explicit solution. The vertical component of the barotropic velocity and the temperature oscillate and exchange energy with a frequency \( \Gamma \) determined by the stratification. (Recall that \( \Gamma^2 = -g(d\bar{\rho}/dx_3) \).) We begin by noting that the subspace on which the baroclinic velocity is zero, i.e. \( \bar{\nu} \equiv 0 \), is invariant under the evolution of the system of equation (2.13). In this subspace, the equations reduce to:

\[
\begin{align*}
(u_0)_t &= \Delta \bar{u}_0 - \bar{u}_0 \cdot \nabla \bar{u}_0 - Q \nabla \bar{p}_h, \\
(u_3)_t &= \Delta \bar{u}_3 + \Gamma \bar{\theta} - \bar{u} \cdot \nabla \bar{u}_3, \\
\bar{\theta}_t &= \Delta \bar{\theta} - \Gamma \bar{u}_3 - \bar{u} \cdot \nabla \bar{\theta},
\end{align*}
\]

where \( \bar{p}_h(x_1, x_2) = (p_1, p_2)(x_1, x_2) \). Note further that since \( \partial_3 \bar{u}_3 = 0 \), we have \( \nabla \cdot \bar{u}_h \equiv \partial_1 \bar{u}_1 + \partial_2 \bar{u}_2 = 0 \). Finally note that the equation for \( \bar{u}_h \) is independent of the evolution of \( (\bar{u}_3, \bar{\theta}) \) (and also of \( \bar{\Omega} \)—the horizontal components of the barotropic velocity are unaffected by the rotation of the system). Furthermore, this is just the 2D Navier–Stokes equation for which the long-time asymptotics are well understood [19, 20]. In particular, all solutions with integrable initial vorticity converge to an Oseen vortex, an explicit solution of the equation with Gaussian vorticity profile.

We will study dynamics using a vorticity formulation. Since velocity and vorticity have different spatial decay and regularity properties, we consider the vorticity \( \tilde{\omega}_h = \partial_2 \bar{u}_1 - \partial_1 \bar{u}_2 \) of the horizontal velocity components as well as the skew-gradients \( \tilde{\omega}_h = \nabla \times \bar{u}_h \) and \( \bar{\Omega} = \nabla \times \bar{\theta} \), with \( \nabla \times = (\partial_2, -\partial_1)^T \), of the vertical velocity and temperature respectively. With these quantities, one readily obtains from (3.1), the following system

\[
\begin{align*}
(\tilde{\omega}_h)_t &= \Delta \tilde{\omega}_h + \Gamma \bar{\Omega} - \bar{u}_h \cdot \nabla \tilde{\omega}_h + \tilde{\omega}_h \cdot \nabla \bar{u}_h, \\
(\tilde{\omega}_3)_t &= \Delta \tilde{\omega}_3 - \bar{u}_h \cdot \nabla \tilde{\omega}_3, \\
\bar{\Omega}_t &= \Delta \bar{\Omega} - \Gamma \tilde{\omega}_h - \bar{u}_h \cdot \nabla \bar{\Omega} + \bar{\Omega} \cdot \nabla \bar{u}_h, \\
\nabla \cdot \bar{\Omega} &= \nabla \cdot \tilde{\omega}_h = 0.
\end{align*}
\]

Here, \( \bar{u}_h, \bar{u}_3, \) and \( \bar{\theta} \) are obtained from \( \tilde{\omega}_3, \tilde{\omega}_h, \) and \( \bar{\Omega} \) respectively via the Biot Savart laws

\[
\nu_h(x_h) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x_h - y_h) \cdot \nu_{3h}}{|x_h - y_h|^2} \nu_{3h} dy_h
\]
\[ \mathfrak{u}_3(x_h) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x_h - y_h) \wedge \mathfrak{u}_h(y_h)}{|x_h - y_h|^2} dy_h, \quad (3.4) \]

\[ \bar{\theta}(x_h) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x_h - y_h) \wedge \bar{\Theta}(y_h)}{|x_h - y_h|^2} dy_h, \quad (3.5) \]

with \( x \wedge y = x_1 y_2 - x_2 y_1 \), and \( x^+ = (x_2, -x_1)^T \).

### 3.1.1 Rotating coordinates and vorticity formulation.

Note that by ignoring all terms with derivatives on the right hand side of the equations for \( \mathfrak{u}_h \) and \( \bar{\Theta} \) in (3.2), we obtain a linear oscillator with frequency \( \Gamma \). The next natural step would be to remove this oscillation by going to a rotating coordinate system. While this simplifies the linear dynamics, it complicates the baroclinic nonlinearity in the resulting system. Hence, we will use a rotating frame to study the linear system, and then use the stationary frame in the nonlinear system. With this in mind we introduce the following system of coordinates:

\[
\begin{pmatrix} \mathfrak{w} \\ \bar{\Theta} \end{pmatrix} := e^{T t I_l} \begin{pmatrix} \mathfrak{w}_h \\ \bar{\Theta}_h \end{pmatrix} = \begin{pmatrix} \cos(\Gamma t) I_2 & \sin(\Gamma t) I_2 \\ -\sin(\Gamma t) I_2 & \cos(\Gamma t) I_2 \end{pmatrix} \begin{pmatrix} \mathfrak{w}_h \\ \bar{\Theta}_h \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad (3.6)
\]

with \( I_2 \) the 2D identity matrix, so that (3.2) now takes the form

\[
\begin{align*}
(\mathfrak{w}_h)_t &= \Delta \mathfrak{w}_h - \mathfrak{u}_h \cdot \nabla \mathfrak{w}_h + \mathfrak{w}_h \cdot \nabla \mathfrak{u}_h \\
(\bar{\Theta}_h)_t &= \Delta \bar{\Theta}_h - \bar{\Theta}_h \cdot \nabla \bar{\Theta}_h + \bar{\Theta}_h \cdot \nabla \bar{\mathfrak{u}}_h \\
\nabla \cdot \bar{\Theta}_h &= 0.
\end{align*}
\]

Note that the equation for \( \mathfrak{w}_3 \) is exactly the 2D Navier–Stokes equation in the vorticity formulation and it is known that solutions of this equation tend to an Oseen vortex for long times [20]. The goal of the remainder of this section is to determine how that vortical solution affects the evolution of \( \mathfrak{w}_h \) and \( \bar{\Theta} \). Note that the Oseen vortex to which \( \mathfrak{w}_3 \) tends may not be small. Thus, it creates a large perturbation of the zero solution in the equations for \( \mathfrak{w}_h \) and \( \bar{\Theta} \). Nonetheless we find that due to the special form of the Oseen vortex velocity and vorticity fields we are able to compute explicitly the leading order terms in the evolution for \( \mathfrak{w}_h \) and \( \bar{\Theta} \).

To analyze the asymptotic behavior of the solutions of these equations it is convenient to introduce ‘scaling variables’, i.e. to rescale both the dependent and independent variables in the equations as:

\[
\begin{align*}
\tilde{\mathfrak{w}}(x_h, t) &= \frac{1}{1 + t} \mathfrak{w} \left( x_h \sqrt{1 + t}, \log(1 + t) \right), \\
\tilde{\bar{\Theta}}(x_h, t) &= \frac{1}{1 + t} \bar{\Theta} \left( x_h \sqrt{1 + t}, \log(1 + t) \right), \\
\tilde{\bar{\theta}}(x_h, x_3, t) &= \frac{1}{1 + t} \bar{\theta} \left( x_h \sqrt{1 + t}, \log(1 + t) \right), \\
\tilde{\mathfrak{u}}_h(x_h, t) &= \frac{1}{1 + t} \mathfrak{u}_h \left( x_h \sqrt{1 + t}, \log(1 + t) \right), \\
\tilde{\bar{\mathfrak{u}}}_3(x_h, t) &= \frac{1}{1 + t} \bar{\mathfrak{u}}_3 \left( x_h \sqrt{1 + t}, \log(1 + t) \right), \\
\xi &= x_h \sqrt{1 + t}, \quad \tau = \log(1 + \tau). \quad (3.8)
\end{align*}
\]
In terms of these variables (3.2) takes the form
\[
\begin{align*}
(\tilde{w}_h)_\tau &= \mathcal{L}w_h - \pi_h \cdot \nabla w_h + w_h \cdot \nabla \pi_h \\
(\tilde{w}_3)_\tau &= \mathcal{L}w_3 - \pi_h \cdot \nabla w_h \\
\Phi_\tau &= \mathcal{L}\Phi - \nabla \pi_h \cdot \nabla \Phi + \Phi \cdot \nabla \pi_h,
\end{align*}
\]
where
\[
\mathcal{L} := \Delta + \frac{1}{2} \xi \cdot \nabla \xi + 1
\]
and \(\pi_h\) is the velocity field associated to the vorticity \(\tilde{w}_3\) via the 2D Biot–Savart law (3.3) (which somewhat remarkably is unchanged by the introduction of the scaling variables (3.8) above). As noted in section 1.5, we study these equations in the weighted Hilbert spaces \(L^2(m)\) and \(L^2_{2D}(m)\). These function spaces admit an explicit characterization of the spectrum of \(\mathcal{L}\) in terms of the weight exponent \(m\). In particular, the algebraic weights push the continuous spectrum of \(\mathcal{L}\) away from the imaginary axis, and reveal point eigenvalues lying on the non-positive half-integers. We also need the associated weighted Sobolev spaces
\[
H^1(m) = \left\{ v \in L^2(m)^4 : \nabla v \in L^2(m)^4 \right\}, \quad H^1_{2D}(m) = \left\{ v \in L^2_{2D}(m) : \nabla v \in L^2_{2D}(m) \right\},
\]
with norm
\[
\|v\|_{H^1(m)}^2 := \int_\Omega \left( 1 + |x_h|^2 \right)^m \left( |f(x)|^2 + |\nabla f(x)|^2 \right) dx,
\]
and the weighted \(L^q\) spaces
\[
L^q(m) := \left\{ v \in L^q(\Omega)^4 : \|v\|_{L^q(m)} := \|b^m v\|_{L^q(\Omega)} < \infty \right\}, \quad W^{1,q}(m) := \left\{ v \in L^q(m) : \nabla v \in L^q(m)^4 \right\}.
\]
The operator \(\mathcal{L}\) then has the following spectral properties when posed on \(L^2_{2D}(m)\). For a more detailed account of these facts see [19, section A] or [45, section 4.2].

3.1.2. Spectral information.

(i) For any \(m \geq 0\), \(\mathcal{L}\) has eigenvalues \(\lambda_k = -k/2\) for all \(k \in \mathbb{N}\) with eigenfunctions
\[
\varphi_\alpha(\xi) = (\partial_\xi^\alpha \varphi_0)(\xi), \quad \varphi_0(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \quad \alpha \in \mathbb{N}^2, \quad |\alpha| = k.
\]
(ii) Each eigenvalue has a spectral projection \(P_n : L^2_{2D}(m) \to L^2_{2D}(m)\)
\[
(P_n f)(\xi) = \sum_{|\alpha| \leq n} \left( \int_{\mathbb{R}^2} H_\alpha(\xi') f(\xi') d\xi' \right)^{1/2} \varphi_\alpha(\xi),
\]
where \(H_\alpha(\xi) = \frac{2^{|\alpha|}}{\alpha!} e^{\xi^2/2} \partial_\xi^\alpha (e^{-|\xi|^2/2})\) are the Hermite polynomials.
(iii) \(\sigma(\mathcal{L}) = \left\{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq \frac{1-m}{2} \right\} \cup \{-n/2 : n \in \mathbb{N}\}.
\]
Hence \(P_0\) is the projection onto the 0-eigenspace and takes the form \(P_0 f = \left( \int_{\mathbb{R}} f(\xi) d\xi \right) \varphi_0\). We also denote the complementary projections
\[
Q_n : L^2_{2D}(m) \to L^2_{2D}(m), \quad n \geq 0
\]
\[
Q_n : f \mapsto (I - P_n) f,
\]
\[
R15
\]
as well as for completeness $Q_{-1} = Q$. We also denote

$$L_{2D_{m}}^2(m) := R_{g} L_{m}^2(m) \{ Q_{n} \}$$

to be the range of the projection $Q_{n}$. Next, take the weight $m > 3$ in the Hilbert spaces $L^2(m)$ and $L_{2D_{m}}^2(m)$ for which the spectrum of $L$ consists of a simple eigenvalue $0$, with eigenfunction the Gaussian $\varphi_0$ and all of the rest of the spectrum in the complex half-plane with real part less than or equal to $-1/2$. Thus, at least for small initial data, the vertical vorticity, $\omega_3(\tau)$, of (3.9) should converge toward some multiple of the Gaussian $\varphi_0$ with a rate $\sim e^{-\tau/2}$. We next turn to the equations of $\omega_3$ and $\Phi$ to see how $\Pi_{h}$ affects their evolution. Looking at the first equation in (3.9), and replacing $\Pi_{h}$ with its asymptotic limit $A \Pi_{h}$, where $\Pi_{h}^0(\xi) = \frac{1-e^{-|\xi|^2/4}}{2\pi|\xi|^4} (-\xi_2, \xi_1)^T$ is the velocity profile obtained from the Gaussian vorticity $\varphi_0$, we obtain the equation

$$(\omega_{3h})_{\tau} = L \omega_{3h} - A \Pi_{h}^0 \cdot \nabla \omega_{3h} + A \omega_{3h} \cdot \nabla \Pi_{h}^0.$$  (3.10)

While we are not able to explicitly compute the entire spectrum of the operator on the right hand side of this equation, a very similar operator arises in the study of the linearization of the 2D Navier–Stokes equation about an Oseen vortex (see [20, section 4]) and using the insights gained there, one can analyze the leading eigenvalue of this operator. One has

**Proposition 3.1.** For any $A \in \mathbb{R}$, the spectrum of $\Phi \mapsto L \Phi - A \Pi_{h}^0 \cdot \nabla \Phi + A \Phi \cdot \nabla \Pi_{h}^0$ on the weighted Hilbert space $L_{2D_{m}}^2(m) \cap \{ \nabla \cdot \Phi = 0 \}$ consists of the half-plane $\{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \leq -1 \}$, and a simple eigenvalue $-1/2$ whose eigenfunction is $\nabla_{h}^\perp \varphi_0$.

**Proof.** The fact that $\nabla_{h}^\perp \varphi_0$ is an eigenfunction with eigenvalue $-1/2$ follows by direct computation, while the fact that the remainder of the spectrum lies in the half plane with real part less than or equal to $-1$ is proven in lemma A.7.

Thus, the leading order asymptotics of the horizontal vorticity and temperature components, $\omega_{3h}$ and $\Phi_{h}$ in (3.9) are also given by derivatives of a Gaussian

$$\Phi_{h}(\xi, \tau) \approx e^{-\tau/2} B_{1} \nabla_{h}^\perp \varphi_0(\xi) + \mathcal{O}(e^{-\tau}), \quad \omega_{3h}(\xi, \tau) \approx e^{-\tau/2} B_{2} \nabla_{h}^\perp \varphi_0(\xi) + \mathcal{O}(e^{-\tau}),$$

where $B_{1} = \int \omega_{0}(x_{h}) dx_{h}$ and $B_{2} = \int \theta(\omega_{0}) dx_{h}$.

If we now revert to our original, unscaled variables, and reexpress the barotropic motion in terms of the velocity, rather than the vorticity, we find that solutions of (3.1), for small initial data, should behave as

$$\begin{align*}
\bar{u}_{3}(x_{h}, \tau) &\approx \frac{A}{\sqrt{1 + \iota}} \bar{u}_{h}^{0} \left( \frac{x_{h}}{\sqrt{1 + \iota}} \right) + \mathcal{O} \left( \frac{1}{\iota^{1/2}} \right), \\
\bar{u}_{h}(x_{h}, \tau) &\approx \frac{1}{\sqrt{1 + \iota}} \left( B_{1} \cos(\Gamma \tau) + B_{2} \sin(\Gamma \tau) \right) \varphi_{0} \left( \frac{x_{h}}{\sqrt{1 + \iota}} \right) + \mathcal{O} \left( \frac{1}{\iota^{3/2}} \right), \\
\bar{\theta}(x_{h}) &\approx \frac{1}{\sqrt{1 + \iota}} \left( -B_{1} \sin(\Gamma \tau) + B_{2} \cos(\Gamma \tau) \right) \varphi_{0} \left( \frac{x_{h}}{\sqrt{1 + \iota}} \right) + \mathcal{O} \left( \frac{1}{\iota^{3/2}} \right).
\end{align*}$$

(3.11)

We note a few somewhat surprising facts about these asymptotics:

**Remark 3.2.** The vertical component of the barotropic velocity, $\bar{u}_{3}$, and the barotropic temperature variations $\bar{\theta}$ are much more strongly localized than the horizontal components of the barotropic velocity $\bar{u}_{h}$. From the Biot–Savart law, one knows that $|\bar{u}_{h}^{0}(\xi)| \sim |\xi|^{-1}$, as $|\xi| \to \infty$ whereas $\bar{u}_{3}$ and $\bar{\theta}$ decay as Gaussians.
Remark 3.3. Note that from a dynamical systems point of view, we will show that, at least locally, any of the vortex solutions defined in the barotropic subspace \( \{ \Theta = 0 \} \) are at least locally attractive for the full equations, much as if we had a center-manifold in a finite dimensional dynamical system.

3.2. Full dynamics In this section we analyze the full equation (2.13). Regarding the barotropic motions as an invariant manifold within the full phase space of this system, we show that perturbations of the leading order solutions computed in the previous section are also stable with respect to perturbations in this larger space. In fact, we will prove (see proposition 3.6) that perturbations in the baroclinic direction decay faster than any inverse power of time, at least at the linear level. We begin by taking the curl of \( \bar{u} \) and \( \tilde{u} \) to eliminate the pressure terms. As mentioned previously, because \( \tilde{\theta} \) has different regularity and spatial decay properties than \( \tilde{\omega} \), we consider the evolution of its horizontal skew-gradient, \( \Theta = \nabla^h_+ \tilde{\theta} = (\partial_3 \tilde{\theta}, -\partial_1 \tilde{\theta})^T \). Thus we study a system in terms of the variables \( \tilde{\omega}, \Theta, \tilde{\omega}, \) and \( \tilde{\theta} \), obtaining

\[
(\tilde{\omega})_t = \Delta \tilde{\omega} + \left( \Gamma \tilde{\Theta} \right)_0 - N_{n}(v),
\]

\[
\Theta_t = \Delta \Theta - \Gamma \tilde{\omega} - N_{\Theta}(v) \tag{3.13}
\]

\[
\tilde{\omega}_t = \Delta \tilde{\omega} - \Omega \partial_3 \bar{u} + \Gamma \nabla^h_+ \tilde{\theta} - \tilde{N}_{\omega}(v) \tag{3.14}
\]

\[
\tilde{\theta}_t = \Delta \tilde{\theta} - \Gamma \tilde{\omega}_3 - \tilde{N}_{\theta}(v), \tag{3.15}
\]

\[
\text{div} \, \tilde{\omega} = 0, \quad \text{div} \, \bar{u} \tilde{\theta} = 0 \tag{3.16}
\]

where \( \tilde{\omega} = \nabla \times \bar{u} \), \( \tilde{\omega}_3 = \nabla \times \bar{u}_h, \nabla^h_+ = (\partial_2, -\partial_1)^T \), \( \text{div} \, u_h = \partial_1 u_1 + \partial_2 u_2 \), and the nonlinearities are defined as

\[
N_{\omega}(v) = (\bar{u}_h \cdot \nabla) \tilde{\omega} - (\tilde{\omega} \cdot \nabla) \bar{u} + Q \left[ (\bar{u} \cdot \nabla) \tilde{\omega} - (\tilde{\omega} \cdot \nabla) \bar{u} \right],
\]

\[
N_{\Theta}(v) = (\bar{u}_h \cdot \nabla \bar{u}) \Theta - (\tilde{\Theta} \cdot \nabla h) \bar{u}_h + Q \left[ (\bar{u} \cdot \nabla) \nabla^h_+ \tilde{\theta} - (\nabla^h_+ \tilde{\theta} \cdot \nabla h) \bar{u}_h + \partial_3 \tilde{\omega} \cdot (\nabla^h_+ \tilde{\theta}) - \partial_1 u_1 \cdot (\nabla^h_+ \tilde{\theta}) \right],
\]

\[
\tilde{N}_{\omega}(v) = (1 - Q) \left[ (\bar{u} \cdot \nabla) \tilde{\omega} - (\bar{u} \cdot \nabla) \bar{u} + (\bar{u} \cdot \nabla) \tilde{\omega} + (\bar{u} \cdot \nabla) \bar{u} - (\bar{u} \cdot \nabla) \bar{u} \right],
\]

\[
\tilde{N}_{\theta}(v) = (1 - Q) \left[ (\bar{u} \cdot \nabla) \tilde{\theta} + (\bar{u} \cdot \nabla) \tilde{\theta} + (\bar{u} \cdot \nabla) \tilde{\theta} \right]. \tag{3.17}
\]

Note here that \( \tilde{\omega} = (\partial_2 \bar{u}_3, -\partial_1 \bar{u}_3, \partial_2 u_1 - \partial_1 u_2)^T \) and the third component of \( \tilde{\omega} \cdot \nabla \bar{u} \) is zero due to the incompressibility condition (3.16).

Our approach for this nonlinear system is the following. We first introduce self-similar variables, and then use the properties of the approximate solution to derive a solution for the residual. We then derive an equivalent mild/integral formulation for this residual and use estimates on the linear evolution to show nonlinear existence and asymptotics via a fixed-point argument.

3.2.1. Scaling variables. We now convert the system (3.12)–(3.15) into scaling variables. In addition to the definition of the variables \( \tilde{\omega}, \Theta, \tilde{\omega}, \) and \( \tilde{\theta} \) introduced in (3.8), we also make the change of coordinates
\[ 
\tilde{\omega}(x_h, x_3, t) = \frac{1}{1 + t} \tilde{\omega} \left( \frac{x_h}{1 + t}, x_3, \log(1 + t) \right), \\
\tilde{u}(x_h, x_3, t) = \frac{1}{1 + t} \tilde{u} \left( \frac{x_h}{1 + t}, x_3, \log(1 + t) \right), \\
\tilde{\theta}(x_h, x_3, t) = \frac{1}{1 + t} \tilde{\theta} \left( \frac{x_h}{1 + t}, x_3, \log(1 + t) \right), 
\]

where as before \( \xi = \frac{x_h}{\sqrt{1 + t}} \), \( \tau = \log(1 + \tau) \). Note also that \( \nabla^2 \tilde{\varphi} = \tilde{\Phi} \). We then obtain from (3.12)–(3.15) the following system of equations
\[ \tilde{\omega}_t = \mathcal{L} \tilde{\omega} + e^\tau \left( \begin{array}{c} 1 \\ \tilde{\Phi} \\ 0 \end{array} \right) - \tilde{N}_\omega(\tilde{\omega}, \tilde{\varphi}, \tilde{\Psi}, \tau), \tag{3.18} \]
\[ \tilde{\Phi}_t = \mathcal{L} \tilde{\Phi} - e^\tau \Gamma \tilde{\omega}_t - \tilde{N}_\Phi(\tilde{\omega}, \tilde{\varphi}, \tilde{\Psi}, \tau), \tag{3.19} \]
\[ \tilde{\omega}_t - (\mathcal{L} + e^\tau \partial_3^2) \tilde{\omega} - \Omega e^{3\tau/2} \partial_3 \tilde{u} + \Gamma e^\tau \left( \nabla_3^2 \tilde{\phi} \right) = \tilde{N}_\omega(\tilde{\omega}, \tilde{\varphi}, \tilde{\psi}, \tilde{\Psi}, \tau), \tag{3.20} \]
\[ \tilde{\phi}_t = (\mathcal{L} + e^\tau \partial_3^2 - 1/2) \tilde{\phi} - \Gamma e^\tau \tilde{u}_3 - \tilde{N}_\phi(\tilde{\omega}, \tilde{\varphi}, \tilde{\psi}, \tilde{\Psi}, \tau), \tag{3.21} \]

with \( \nabla_3^2 \) and where the nonlinear terms are obtained from (3.17). Note because of the form of the scaling variables we must replace any 3D gradients \( \nabla \) with \( \nabla_3 = (\partial_2, -\partial_1) \). Also, here and throughout, \( \tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{\phi}) \) denote the combined vector of baroclinic vorticity and temperature in scaling variables, while \( \tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{\phi}, \tilde{\Psi}) \) will denote the corresponding collection of barotropic quantities. We let \( \tilde{N} = (\tilde{N}_\omega, \tilde{N}_\Phi)^T, \tilde{\Phi} = (\tilde{N}_\omega, \tilde{N}_\Phi)^T \) denote the full baroclinic and barotropic nonlinearities. Note also that the incompressibility conditions now take the form
\[ \nabla \cdot \tilde{w} = \nabla_h \cdot \tilde{w}_h = \nabla_h \cdot \tilde{\Phi} = 0, \]

with \( \tilde{w} \) and \( \tilde{\Phi} \) and \( \tilde{\phi} \) are still related via the Biot–Savart laws, (3.3)–(3.5). Also, \( \tilde{w} \) and \( \tilde{u} \) are related by a scaled version of the Biot Savart law (2.9) taking into account the \( e^{\tau/2} \) factor accompanying \( x_3 \)-derivatives. Furthermore, we shall need the following estimates which relate norms of the velocities in terms of their corresponding vorticities. These are derived in [45, section 4.2] (see also [19]):

**Proposition 3.4 (Biot–Savart).** If \( p \in (1, 2) \) and \( \tilde{w} \in L^p(\mathbb{R}^2) \) then \( \tilde{\Phi} \in L^{2p/(2-p)}(\mathbb{R}^2) \) and
\[ \| \tilde{\Phi} \|_{L^{2p/(2-p)}(\mathbb{R}^2)} \leq C \| \tilde{w} \|_{L^p(\mathbb{R}^2)}. \]  \( \tag{3.22} \)

If \( \tilde{w} \in L^2(\mathbb{D}) \) then \( \tilde{u} \in L^q(\mathbb{D}) \) for all \( q \in [2, 6] \) and there exists a \( C > 0 \) such that
\[ \| \tilde{u} \|_{L^q(\mathbb{D})} \leq C \tau^{3/2} \| \tilde{w} \|_{L^2(\mathbb{D})}. \]  \( \tag{3.23} \)

Furthermore, if \( \tilde{w} \in L^2(\mathbb{D}) \) then \( \tilde{u} \in H^1(\mathbb{D}) \) and there exists a \( C > 0 \) such that
\[ \| \tilde{u} \|_{H^1(\mathbb{D})} \leq C \| \tilde{w} \|_{L^2(\mathbb{D})}. \]  \( \tag{3.24} \)
We consider the existence, and asymptotic behavior of solutions of this equation in the space
\[
Z = \left\{ (\bar{u}, \Phi, \bar{\omega}, \bar{\phi}) \in L^2_{\text{D}}(m)^3 \times L^2_{\text{D}}(m)^3 \times H^1(m) \right\} \cap \left\{ \nabla_h \cdot \bar{u} = 0, \nabla_h \cdot \bar{\Phi} = 0, \nabla \cdot \bar{\omega} = 0 \right\},
\]
with the natural norm. We expect, from the form of the system (2.13), that the temperature should have the same regularity as the velocity. If the vorticity is in \(L^2\), then we expect (as in (3.24)) that the velocity would be in \(H^1\). For this reason we require increased regularity in the temperature component \(\phi\).

3.2.2. Approximate solutions and residual equation. We now make rigorous the discussion in section 3.1 by writing the solution of the full Boussinesq system as the sum of an approximate solution given by the preceding heuristic considerations, plus a remainder, deriving equations satisfied by the remainder, and then proving that solutions of the remainder equations tend to zero more rapidly than the leading order terms in the approximate solution. We take initial data, \((\bar{w}_0, \bar{\Phi}_0, \bar{\omega}_0, \bar{\phi}_0) \in Z\). (Note that due to the way in which the scaling variables are defined, the initial values for \(\bar{w}\) are the same as that for \(\bar{\omega}\) and similarly for the remaining variables.) Translating our formal approximate solution above, we then define

\[
(\bar{u}_h)_{\text{app}}(\xi) = A\bar{u}_h(\xi),
\]
\[
(\bar{w})_{\text{app}}(\xi, \tau) = e^{-\tau/2} (B_1 c(\tau) + B_2 s(\tau)) \varphi_0(\xi),
\]
\[
\bar{\phi}_{\text{app}}(\xi, \tau) = e^{-\tau/2} (-B_1 s(\tau) + B_2 c(\tau)) \bar{\phi}_0(\xi),
\]
\[
\bar{w}_{\text{app}}(\xi, \tau) = (0, 0, A\varphi_0(\xi))^T + e^{-\tau/2} (B_1 c(\tau) + B_2 s(\tau)) (\partial_\tau \varphi_0(\xi), -\partial_\tau \varphi_0(\xi), 0)^T
\]
\[
\bar{\Phi}_{\text{app}}(\xi, \tau) = e^{-\tau/2} (-B_1 s(\tau) + B_2 c(\tau)) \nabla_h^\perp \varphi_0(\xi)
\]

where \(c(\tau) := \cos(\Gamma(e^\tau - 1)), s(\tau) := \sin(\Gamma(e^\tau - 1)), (\bar{u}_h)_{\text{app}}\) is the velocity profile associated with the Gaussian \(\varphi_0\) from the Biot–Savart law (3.3), and the constants are determined by the initial data

\[
A = \int_{\mathbb{R}^2} (\bar{w}_3)_{\text{D}}(\xi) d\xi, \quad B_1 = \int_{\mathbb{R}^2} (\bar{u}_3)_{\text{D}}(\xi) d\xi, \quad B_2 = \int_{\mathbb{R}^2} \bar{\theta}_0(\xi) d\xi.
\]

Direct computation then readily gives that the solution \(\bar{w}_\text{app} = (\bar{w}_{\text{app}}, \bar{\Phi}_{\text{app}})^T\) solves the barotropic subsystem (3.18) and (3.19) of the full nonlinear system with \((\bar{\omega}, \bar{\phi}) \equiv 0\). One also readily finds that the subspace \(L^2_{\text{D}}(m)^2 \times L^2_{\text{D}}(m) \times L^2_{\text{D}}(m)^2\), which is the projection off the leading order modes, is invariant under the evolution of this subsystem. We can thus make the following decomposition, which splits off the leading order approximate dynamics

\[
\bar{w}_h = \bar{w}_{\text{app}} + \bar{w}_{R,h}, \quad \bar{w}_3 = \bar{w}_{\text{app},3} + \bar{w}_{R,3}, \quad \bar{\Phi} = \bar{\Phi}_{\text{app}} + \bar{\Phi}_R
\]

where the residuals satisfy

\[
\bar{w}_{R,h}, \bar{\Phi}_R \in \text{Rg}\{Q_1 \oplus Q_1\} = L^2_{\text{D}}(m)^2, \quad \bar{w}_{R,3} \in \text{Rg}\{Q_0\} = L^2_{\text{D}}(m).
\]
Here $\text{Rg}\{Q_1 \oplus Q_1\}$ denotes the range of the projection $Q_1 \oplus Q_1 : L^2(m)^2 \rightarrow L^2_{2D}(m)^2$. Note the scaled velocity and thermal components, $\bar{\mathbf{w}}$ and $\bar{\varphi}$, can also be obtained from $\bar{\mathbf{w}}_R$ and $\bar{\Phi}_R$ respectively using the 2D Biot–Savart laws (3.3) and (3.4) above. We thus consider the residual on the space $Y \times L^2(m)^3 \times H^1(m)$ with

$$Y = Y_{h,1} \times Y_{3,0} \times Y_{r,1},$$

$$Y_{h,n} = \{ \bar{\mathbf{w}}_h \in L^2_{2D,n}(m)^2 \mid \text{div}_h \bar{\mathbf{w}}_h = 0 \}, \quad Y_{3,n} = L^2_{2D,n}(m), \quad Y_{r,n} = \{ \bar{\Phi} \in L^2_{2D,n}(m)^2 \mid \text{div}_h \bar{\Phi} = 0 \},$$

and the norms

$$\|(\bar{\mathbf{w}}, \bar{\Phi})\|_m := \|\bar{\mathbf{w}}\|_m + \|\bar{\Phi}\|_m, \quad \|(\bar{\mathbf{w}}, \bar{\varphi})\|_{s,m} := \|\bar{\mathbf{w}}\|_m + \|\bar{\varphi}\|_{H^1(m)}.$$  

In order to simplify the notation, we also define $\bar{\mathbf{W}}_R = (\bar{\mathbf{w}}_R, \bar{\Phi}_R)^T$ and $\bar{\mathbf{W}} = (\bar{\mathbf{w}}, \bar{\Phi})^T$. Inserting this decomposition into the full system (3.18)–(3.21), and using the aforementioned facts of the approximate solution, we find that the barotropic and baroclinic residuals $\bar{\mathbf{W}}_R$ and $\bar{\mathbf{W}}$ satisfy the following system

$$(\bar{\mathbf{W}}_R)_\tau = (L(\tau) + \bar{\Lambda}) \bar{\mathbf{W}}_R + \bar{\mathbf{N}}(\bar{\mathbf{W}}_R, \bar{\mathbf{W}}), \quad \bar{\mathbf{W}}_R = (L(\tau) + \bar{\Lambda}) \bar{\mathbf{W}}_R + \bar{\mathbf{N}}(\bar{\mathbf{W}}_R, \bar{\mathbf{W}}),$$

with the linear operators

$$\bar{L}(\tau) \bar{\mathbf{W}}_R := \bar{L} \begin{pmatrix} \bar{\mathbf{w}}_R \\ \bar{\Phi}_R \end{pmatrix} + e^\tau \begin{pmatrix} \bar{T}_R \\ 0 \\ -\bar{\mathbf{W}}_{R,h} \end{pmatrix},$$

$$\bar{L}(\tau) \bar{\mathbf{W}} := \bar{L} + e^\tau \begin{pmatrix} \bar{\mathbf{w}} \\ \bar{\Phi} \end{pmatrix} - \begin{pmatrix} 0 \\ \bar{\varphi} / 2 \end{pmatrix} + \Gamma e^\tau \begin{pmatrix} \bar{\nabla}_h \bar{\varphi} \\ 0 \\ -\bar{U}_h \end{pmatrix} - \begin{pmatrix} \Omega e^{\tau/2} \partial_3 \bar{\mathbf{u}} \\ 0 \end{pmatrix},$$

$$\bar{\Lambda} \bar{\mathbf{W}}_R := \bar{\mathbf{N}}(\bar{\mathbf{W}}_{app} + \bar{\mathbf{W}}_R, \bar{\mathbf{W}}) - (\bar{\mathbf{N}}(0, \bar{\mathbf{W}}) + \bar{\mathbf{N}}(\bar{\mathbf{W}}_{app}, 0) + \bar{\mathbf{N}}(\bar{\mathbf{W}}_R, 0)),$$

$$= \bar{\mathbf{w}}_{app} \cdot \bar{\nabla} \bar{\mathbf{W}}_R + \bar{\Phi}_{R,h} \cdot \bar{\nabla} \bar{\mathbf{W}}_{app} - \begin{pmatrix} (\bar{\mathbf{w}}_{app} \cdot \bar{\nabla}) \bar{\Phi}_{R,h} \\ 0 \\ (\bar{\mathbf{w}}_{app} \cdot \bar{\nabla}) \bar{\Phi}_{app,h} \end{pmatrix} - \begin{pmatrix} (\bar{\Phi}_R \cdot \bar{\nabla}) \bar{\Phi}_{app,h} \\ 0 \end{pmatrix},$$

$$\bar{\Lambda} \bar{\mathbf{W}} := \bar{\mathbf{N}}(\bar{\mathbf{W}}_{app}, \bar{\mathbf{W}}) - \bar{\mathbf{N}}(0, \bar{\mathbf{W}}),$$

$$= \bar{\mathbf{w}}_{app} \cdot \bar{\nabla} \bar{\mathbf{W}} - \bar{\Phi}_{app} \cdot \bar{\nabla} \begin{pmatrix} \bar{\Phi}_{app} \\ 0 \end{pmatrix} + \bar{\mathbf{u}} \cdot \bar{\nabla} \begin{pmatrix} \bar{\mathbf{w}}_{app} \\ \bar{\Phi}_{app} \end{pmatrix} - \bar{\mathbf{w}}_{app} \cdot \bar{\nabla} \tau \begin{pmatrix} \bar{\mathbf{u}} \\ 0 \end{pmatrix},$$

and with the nonlinear components defined as
\[ \nabla (\dot{W} + \dot{\bar{W}}) := \bar{Q} \left[ \frac{\nabla (W_{app} + \dot{W}), \dot{\bar{W}}}{-\dot{\bar{W}}(W_{app}, 0) - \dot{\bar{W}}(W_{app}, 0)} \right] \]
\[ \nabla \bar{W}(\dot{W}, \dot{\bar{W}}) := \dot{Q} \left[ \frac{\bar{N} (W_{app} + \dot{W}, \dot{W}) - \bar{N} (W_{app}, 0)}{\bar{N}(\dot{W}, 0) - \bar{N}(W_{app}, 0)} \right] \]
\[ = (\bar{\mu} R \cdot \nabla) \bar{\mu} R - (\bar{\mu} R \cdot \nabla) \bar{\mu} R + \bar{Q} \left[ (\bar{u} \cdot \nabla) \bar{u} - (\bar{u} \cdot \nabla) \bar{u} \right] \]
\[ \nabla \bar{W}(\dot{W}, \dot{\bar{W}}) := \dot{Q} \left[ \frac{\bar{N} (W_{app} + \dot{W}, \dot{W}) - \bar{N} (W_{app}, 0)}{\bar{N}(\dot{W}, 0) - \bar{N}(W_{app}, 0)} \right] \]
\[ = (\bar{\mu} R \cdot \nabla) \bar{\mu} R - (\bar{\mu} R \cdot \nabla) \bar{\mu} R + \bar{Q} \left[ (\bar{u} \cdot \nabla) \bar{u} - (\bar{u} \cdot \nabla) \bar{u} \right] \]

Here, we set \( \bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3)^T \), \( \bar{N}_p = (\bar{N}_1, \bar{N}_2, \bar{N}_3)^T \) for the barotropic nonlinearity \( N = (N_1, N_2, N_3, N_4, N_5)^T \). The projection \( \bar{Q} := \dot{Q}_1 \oplus \dot{Q}_2 \oplus \dot{Q}_2 \oplus \dot{Q}_1 \) projects \( L^2_{2d}(m)^5 \) onto \( Y \).

Before considering the full nonlinear equations (3.28) and (3.29), we list some results on the barotropic and baroclinic components of the linear evolution. Since the linear operators are time-dependent, we must characterize the linear evolution in terms of evolutionary families of operators in the sense of Pazy [41, chapter 5]. We find that these evolutionary families inherit nearly the same temporal decay properties as the unperturbed linear evolutions, despite the fact that the perturbations \( N \) and \( \Lambda \) may be large in the operator norm. In particular, we find only a slight reduction in the decay rate. We leave the details of their proof to the appendix. We define \( a(\tau) = 1 - e^{-\tau} \), the standard basis vectors \( e_i \in \mathbb{R}^2 \), and the projections \( e_i^+ = 1 - e_i^T \).

**Proposition 3.5 (Barotropic evolution).** Let \( m > 3, 0 < \mu < 1/2, \alpha \in \mathbb{N}^2 \) with \( |\alpha| \leq 1 \), and \( W_0 \in Y \). Then the operator \( L(\tau) + \Lambda \) generates an evolutionary family of operators, \( \hat{S}(\tau, \sigma) \), with \( \tau \geq \sigma \geq 0 \), on \( Y \). If also \( b^m W_0 \in L^q(\mathbb{R}^2)^5 \), then it satisfies the following decay estimates for any \( q \in [1, 2] \),

\[ \|e_3^+ (\partial^\alpha G(\tau, \sigma) W_0)\|_{L^q} \leq \frac{C}{a(\tau - \sigma)} \|b^m W_0\|_{L^q}, \]
\[ \|e_3^+ (\partial^\alpha G(\tau, \sigma) W_0)\|_{L^q} \leq \frac{C}{a(\tau - \sigma)^{1/2} \|b^m W_0\|_{L^q}} \]

for some constant \( C > 0 \).

**Proof.** See appendix A.1. \( \square \)

**Proposition 3.6 (Baroclinic evolution).** The linear operator \( \bar{L}(\tau) + \bar{\Lambda} \) generates an evolutionary family of operators \( \bar{S}(\tau, \sigma) \), for \( \tau \geq \sigma \), on \( L^2(m)^3 \times H^1(m) \) which, for \( m > 1, \alpha \in \mathbb{N}^3 \) with \( |\alpha| \leq 1, q \in [2, 2] \), \( 0 < \delta < 4\pi, f \in L^2(m)^3 \times H^1(m) \) and with \( b^m f \in L^q(\mathbb{R}) \), satisfies the following estimate
\[ \|\partial^\sigma \tilde{S}(\tau, \sigma)f\|_{\infty,m} \leq \frac{e^{-(4\gamma^2-\delta)(e^\tau-e^\sigma)}}{a(\tau-\sigma)^{1/2+1/2+\delta/2}a(e^\tau-e^\sigma)^{(1/2+1/2+\delta/2)}} \left(\|\partial^\sigma e^\tau f\|_{L^2(\mathbb{R})} + \|\partial^\sigma e^\sigma f\|_{W^{1,2}(\mathbb{R})}\right) \]

for \( \tau > \sigma \geq 0 \), and some constant \( C > 0 \).

**Proof.** See appendix A.2. \( \square \)

### 3.2.3. **Fixed point operator.** Having collected information about the linear system in \( W_R := (W_R, W)^T \), we now study the nonlinear system (3.28) and (3.29). In particular we study mild solutions of this system via the equivalent integral formulation

\[ W_R(\tau) = \mathbf{F}(W_R) := \tilde{S}(\tau, \sigma) W_{R,0} + \int_0^\tau \tilde{S}(\tau, \sigma) \mathbf{N}(W_R(\sigma))d\sigma, \tag{3.36} \]

\[ \tilde{W}(\tau) = \mathbf{F}(\tilde{W}) := \tilde{S}(\tau, \sigma) \tilde{W}_0 + \int_0^\tau \tilde{S}(\tau, \sigma) \tilde{N}(W_R(\sigma))d\sigma. \tag{3.37} \]

We note here that neither \( \tilde{S} \) nor \( \tilde{\mathbf{S}} \) act diagonally on vectors in \( \mathbb{R}^5 \) and \( \mathbb{R}^4 \) respectively, so the mild formulation cannot be broken down into components as in [45]. Using this formulation we can then prove the following result about the asymptotic behavior of small solutions of (3.21).

**Theorem 3.** There exists \( K_0 > 0 \) such that for \( 0 < \mu < 1/2, m > 3 \), initial data \( W_{R,0} = (W_{R,0}, \tilde{W}_0) \in Y \times L^2(m)^3 \times H^1(m) \), with \( \nabla \cdot \tilde{w}_0 = 0 \), and \( \|W_{R,0}\|_m + \|\tilde{W}_0\|_{*,m} < K_0 \), there exists a unique solution of (3.36) and (3.37) in \( C^0([0, \infty), Y \times L^2(m)^3 \times H^1(m)) \) which satisfies the asymptotic estimates:

\[
\lim_{\tau \to \infty} e^{\gamma \tau} \left( \|\tilde{w}(\cdot, \tau)\|_{L^2(\mathbb{R})} + \|\tilde{\phi}(\cdot, \tau)\|_{H^1(\mathbb{R})} \right) = 0 \tag{3.38}
\]

for any \( \gamma > 3/4 \).

The solution of the residual equation for sufficiently small initial residual data then readily implies existence and asymptotics in the full solution \( W = \tilde{W}_{\text{app}} + W_R \).

**Corollary 3.7.** There exists \( K_0 > 0 \) such that for \( 0 < \mu < 1/2 \), and initial data \( W_0 \in Z \) with \( \|W_0(0) - \tilde{W}_{\text{app}}(0)\|_m + \|W_0\|_{*,m} < K_0 \), there exists a unique solution of (3.18)–(3.21) in \( C^0([0, \infty), Z) \) which satisfies the following asymptotics

\[
\lim_{\tau \to \infty} e^{\gamma \tau} \left( \|\tilde{w}(\cdot, \tau)\|_{L^2(\mathbb{R})} + \|\tilde{\phi}(\cdot, \tau)\|_{H^1(\mathbb{R})} \right) = 0 \tag{3.39}
\]

for any \( \gamma > 3/4 \).
Remark 3.8. Note that if we rewrite these estimates in terms of our original variables, it says that the asymptotics of (3.11) are correct, (at least if we replace the $O(\frac{1}{t})$ error terms with $O(\frac{1}{t^2})$) and similarly for the $O(\frac{1}{t})$ terms and that the baroclinic components of the velocity and temperature decay at least like $O(t^{-3/2})$ that for the linearized evolution the baroclinic velocity actually decays exponentially fast.

Proof. To begin, we define a Banach space

$$X_{\mu,\gamma} = \{ W_R, \nabla W_R \in C^4([0, \infty); Y \times L^2(m)^3 \times H^1(m)) \mid \nabla \cdot \vec{v}(\tau) = \nabla h \cdot \vec{u}_h(\tau) = \nabla h \cdot \vec{u}_h(\tau) = 0 \},$$

with norm

$$\| W \|_{X_{\mu,\gamma}} = \sup_{\tau \geq 0} e^{\tau/2} \left( \| \tilde{\epsilon}_W \|_{\infty} + e^{\tau/2} \| \tilde{\epsilon}_W \|_{\infty} + a(\tau)^{1/2} (\| \tilde{\epsilon}_W \|_{\infty} + e^{\tau/2} \| \tilde{\epsilon}_W \|_{\infty}) \right) + e^{\tau/2} \left( \| \tilde{\epsilon}_W \|_{\infty} + a(\tau)^{1/2} (\| \tilde{\epsilon}_W \|_{\infty} + e^{\tau/2} \| \tilde{\epsilon}_W \|_{\infty}) \right).$$

(3.41)

where we recall that $e^{\frac{1}{4}} = 1 - e^{\frac{1}{2}}$ with $e^{\frac{1}{2}} = (0, 0, 1, 0)^T$. We let $S(\tau, \sigma)$ be the direct sum of the barotropic and baroclinic evolution operators $\mathcal{S}(\tau, \sigma)$ and $\tilde{S}(\tau, \sigma)$, and $F = (\hat{F}, \hat{F})^T$ where $F$ and $\hat{F}$ are the integral terms in (3.36) and (3.37) respectively. The existence of a global solution then follows by finding a fixed point of the mapping,

$$F(W_R) := \left( \hat{F}(W_R), \hat{F}(W_R) \right) = S(\tau, \sigma)W_{R0} + F(W_R)$$

defined on $X_{\mu,\gamma}$. Paired with the linear estimates on $\mathcal{S}$ and $\tilde{S}$, this follows in a standard fashion from the estimates

$$\| F(W_R) \|_{X_{\mu,\gamma}} \leq C_1 \| W_R \|_{X_{\mu,\gamma}}^2,$$

(3.42)

$$\| F(W_R) - F(W_R') \|_{X_{\mu,\gamma}} \leq C_2 \left( \sup_{\tau \geq 0} \| W_R(\tau) \|_2 + \sup_{\tau \geq 0} \| W_R'(\tau) \|_2 \right) \| W_R - W_R' \|_{X_{\mu,\gamma}}.$$

(3.43)

The estimate for (3.43) follows in a similar way to (3.42). For (3.42), since the evolutionary operators $\mathcal{S}$ and $\tilde{S}$ do not act diagonally we cannot consider the nonlinearities in $F$ component-wise as in [45]. We thus use proposition 3.5 to estimate $\hat{F}$ and proposition 3.6 to estimate $\hat{F}$, both with $q = 3/2$.

3.2.4. Estimates on barotropic nonlinearity $F$. We first find

$$\| \tilde{\epsilon}_W \|_{L^1(\mu)} \leq C \int_0^\tau e^{-(1/2 + \rho)(\tau - \sigma)/\mu} \left[ \| b^h \nabla \vec{u}_h \cdot \nabla \vec{w}_h \|_{L^2(\mu)} + \| b^h \vec{w}_h \cdot \nabla \mu \|_{L^2(\mu)} + \| b^h \nabla \phi \cdot \nabla \vec{w}_h \|_{L^2(\mu)} \right] d\sigma \leq C \int_0^\tau e^{-(1/2 + \rho)(\tau - \sigma)/\mu} \left[ \| b^h \nabla \vec{u}_h \cdot \nabla \vec{w}_h \|_{L^2(\mu)} + \| b^h \vec{w}_h \cdot \nabla \mu \|_{L^2(\mu)} + \| b^h \nabla \phi \cdot \nabla \vec{w}_h \|_{L^2(\mu)} \right] d\sigma + e^{\tau/2} \left( \| b^h \nabla \phi \cdot \nabla \vec{w}_h \|_{L^1(\mu)} + e^{\tau/2} \| b^h \nabla \phi \cdot \nabla \vec{w}_h \|_{L^1(\mu)} \right).$$

(3.44)
\[ \leq C \int_0^\tau \frac{e^{-(1/2 + \rho)(\tau - \sigma)}}{a(\tau - \sigma)^{1/\beta}} \left[ \| \tilde{w}_{R,3} \|_m \| \nabla \tilde{W} \|_m + \| \tilde{w}_{R,3} \|_m + \| \tilde{w}_{R,3} \|_{H^{1/2}(m)} \right] d\sigma + e^{\sigma/2} \| \tilde{w} \|_{\| \tilde{w} \|_{H^{1/2}(m)}} \right) d\sigma \tag{3.45} \]

where \( C > 0 \) is a constant which may change from line to line. To obtain the inequality (3.45), we go term-by-term. The 2D Biot–Savart law in proposition 3.4, the embedding \( L^q(m) \hookrightarrow L^r(D) \) for all \( q \in [2/(m + 1), 2] \), and Hölder’s inequality give

\[ \| b^m \tilde{w}_{R,3} \cdot \nabla \tilde{W} \|_{L^{1/2}} \leq \| \tilde{w}_{R,3} \|_{L^1(D)} \| \nabla \tilde{W} \|_m \leq C \| \tilde{w}_{R,3} \|_{L^1(D)} \| \nabla \tilde{W} \|_m \leq C \| \tilde{w}_{R,3} \|_{L^1(D)} \| \nabla \tilde{W} \|_m. \tag{3.46} \]

The estimate \( \| \nabla \tilde{u} \|_{L^p} \leq \| \tilde{W} \|_{L^p} \), which can be derived from [19, lemma 2.1], along with the Gagliardo–Nirenberg inequality gives

\[ \| b^m \tilde{w}_{R,3} \cdot \nabla \tilde{u}_{R,3} \|_{L^{1/2}} \leq \| \tilde{w}_{R,3} \|_{L^1(D)} \| \nabla \tilde{u}_{R,3} \|_{L^1(D)} \leq C \| \tilde{w}_{R,3} \|_{L^1(D)} \| \nabla \tilde{u}_{R,3} \|_{L^1(D)} \leq C \| \tilde{w}_{R,3} \|_{L^1(D)} \| \nabla \tilde{u}_{R,3} \|_{L^1(D)}. \tag{3.47} \]

A similar approach gives

\[ \| b^m \tilde{u} \cdot \nabla \tilde{W} \|_{L^{1/2}} \leq C \| \tilde{u} \|_{L^1(D)} \| \nabla \tilde{W} \|_{L^1(D)} \leq C \| \tilde{u} \|_{L^1(D)} \| \nabla \tilde{W} \|_{L^1(D)}. \tag{3.48} \]

For the baroclinic contributions, the 3D Biot–Savart law in proposition 3.4, the embedding \( L^q(m) \hookrightarrow L^r(D) \) for all \( q \in [1, 2] \), and finally the estimate \( \| \nabla \tilde{u} \|_{L^p} \leq \| \tilde{w} \|_{H^{1/2}(m)} \) from [45, lemma 2.4.2] give

\[ \| b^m \tilde{u} \cdot \nabla \tilde{u} \|_{L^{1/2}} \leq \| \tilde{u} \|_{L^1(D)} \| \nabla \tilde{u} \|_{L^1(D)} \leq C \| \tilde{u} \|_{L^1(D)} \| \nabla \tilde{u} \|_{L^1(D)}. \tag{3.49} \]

and

\[ \| b^m \tilde{u} \cdot \nabla \tilde{u} \|_{L^{1/2}} \leq C \| \tilde{u} \|_{L^1(D)} \| \nabla \tilde{u} \|_{L^1(D)}. \tag{3.50} \]

We can estimate the last three terms in (3.44) as

\[ \| b^m \nabla^2 \tilde{u} \cdot \nabla \tilde{u} \|_{L^{1/2}} \leq \| b^m \nabla^2 \tilde{u} \cdot \nabla \tilde{u} \|_{L^{1/2}} \leq C \| \tilde{u} \|_{L^1(D)} \| \tilde{u} \|_{L^1(D)} \leq C \| \tilde{u} \|_{H^{1/2}(m)}. \tag{3.51} \]

We can then conclude from (3.44)

\[ e^{(1/2 + \rho)\tau} \| \tilde{u}(t) \|_{L^1(D)} \leq C \int_0^\tau \frac{e^{-(1/2 + \rho)(\tau - \sigma)}}{a(\tau - \sigma)^{1/\beta}} \left[ \| \tilde{w}_{R,3} \|_m \| \nabla \tilde{W} \|_m + \| \tilde{w}_{R,3} \|_m + \| \tilde{w}_{R,3} \|_{H^{1/2}(m)} \right] d\sigma + e^{\sigma/2} \| \tilde{w} \|_{\| \tilde{w} \|_{H^{1/2}(m)}} \right) d\sigma \leq C \| \tilde{w}(t) \|_{H^{1/2}(m)}. \tag{3.52} \]
and in a similar manner
\[
e^{\tau/2} \| e^\tau \hat{F}(W_R) \|_{m} \leq C \int_0^\tau e^{\tau/2} \left[ \| \hat{w}_{R,3} \|_m \| \nabla \hat{w}_{R,3} \|_m + e^{\tau/2} \left( \| \hat{w} \|_m \| \nabla \hat{w} \|_m + \| \hat{w} \|_m \| \hat{W} \|_{H^{(m)}} \right) \right] d\sigma
\]
\[
\leq C \| W_R \|_{X_{\mu,\gamma}}. \quad (3.53)
\]
Here we make a slight abuse of notation, letting \( \| W_R \|_{X_{\mu,\gamma}} \) and \( \| \hat{W} \|_{X_{\mu,\gamma}} \) denote the \( X_{\mu,\gamma} \) norm of \((W_R, 0)\) and \((0, \hat{W})\) respectively. Continuing in this way one readily obtains estimates on the gradient terms
\[
a(\tau)^{1/2} e^{\tau/2} \left( \| e^\tau \nabla \hat{F}(W_R) \|_{m} + e^{\tau/2} \| e^\tau \nabla \hat{F}(W_R) \|_{m} \right) \leq C \| W_R \|_{X_{\mu,\gamma}}^2. \quad (3.54)
\]
Combining (3.52)–(3.54) we then obtain
\[
\| \hat{F}(W_R) \|_{X_{\mu,\gamma}} \leq C \| W_R \|_{X_{\mu,\gamma}}^2.
\]

3.2.5. Estimates on baroclinic nonlinearity \( \hat{F} \). For \( \hat{F} \) we use similar estimates and (3.35) of proposition 3.6 with \( q = 3/2 \) to find
\[
e^{\tau} \| \hat{F}(W_R) \|_{m} \leq C \int_0^\tau \left[ \| \hat{w}_R \|_m \| \nabla \hat{w}_R \|_m + \| \hat{u} \|_m \| \nabla \hat{u} \|_m \right] d\sigma
\]
\[
\leq C \int_0^\tau \frac{e^{-(4\tau/3 - \delta)(e^{\tau} - e^{-\tau})}}{a(\tau - \sigma)^{1/2}(e^{\tau} - e^{-\tau})} \left[ \| \hat{w}_R \|_m \| \nabla \hat{w}_R \|_m \right. \\
\left. + \| \hat{u} \|_m \| \nabla \hat{u} \|_m \right] d\sigma
\]
\[
\leq C \int_0^\tau \frac{e^{-(4\tau/3 - \delta)(e^{\tau} - e^{-\tau})}}{a(\tau - \sigma)^{1/2}(e^{\tau} - e^{-\tau})} \left[ \| \hat{w}_R \|_m \| \nabla \hat{w}_R \|_m + \| \hat{u} \|_m \| \nabla \hat{u} \|_m \right. \\
\left. + \| \hat{w}_R \|_m \| \nabla \hat{w}_R \|_m + \| \hat{u} \|_m \| \nabla \hat{u} \|_m \right] d\sigma
\]
\[
\leq C \int_0^\tau \frac{e^{-(4\tau/3 - \delta)(e^{\tau} - e^{-\tau})}}{a(\tau - \sigma)^{1/2}(e^{\tau} - e^{-\tau})} \left[ \| \hat{w}_R \|_m \| \nabla \hat{w}_R \|_m + \| \hat{u} \|_m \| \nabla \hat{u} \|_m \right. \\
\left. + \| \hat{w}_R \|_m \| \nabla \hat{w}_R \|_m + \| \hat{u} \|_m \| \nabla \hat{u} \|_m \right] d\sigma \quad (3.55)
\]
Here we have used estimates similar to (3.46)–(3.51) for the bound on \( \hat{F} \) above, as well as the following
\[
\| b^{\mu} \hat{u} \cdot \nabla \phi \|_{L^{2}/2} \leq C \left( \| b^{\mu} \hat{u} \cdot \nabla \phi \|_{L^{2}/2} + \| b^{\mu} \nabla \left( \hat{u} \cdot \nabla \phi \right) \|_{L^{2}/2} \right)
\]
\[
\leq C \left( \| \hat{u} \|_{L^{2}} \| b^{\mu} \nabla \phi \|_{L^{2}} + \| \nabla \hat{u} \|_{L^{2}} \| b^{\mu} \nabla \phi \|_{L^{2}} + \| \hat{u} \|_{L^{2}} \| b^{\mu} \nabla \phi \|_{H^{(m)}} \right)
\]
\[
\leq C \left( \| \hat{u} \|_{L^{2}} \| b^{\mu} \nabla \phi \|_{L^{2}} + \| \nabla \hat{u} \|_{L^{2}} \| b^{\mu} \nabla \phi \|_{L^{2}} + \| \hat{u} \|_{L^{2}} \| \nabla \phi \|_{H^{(m)}} \right)
\]
\[
\leq C e^{\tau/2} \left( \| \hat{w} \|_{m} \| \nabla \phi \|_{m} + \| \hat{w} \|_{H^{(m)}} \| \nabla \phi \|_{m} + \| \hat{w} \|_{m} \| \nabla \phi \|_{H^{(m)}} \right). \quad (3.56)
\]
\[ \| \mathbf{b}^a \mathbf{\hat{u}} \cdot \nabla \mathbf{\bar{\phi}} \|_{W^{1/2,2}} \leq C \left( \| \mathbf{b}^a \mathbf{\hat{u}} \cdot \nabla \mathbf{\bar{\phi}} \|_{L^{3/2}} + \| \mathbf{b}^a \nabla (\mathbf{\hat{u}} \cdot \nabla \mathbf{\bar{\phi}}) \|_{L^{3/2}} \right) \]
\[ \leq C \left( \| \mathbf{\hat{u}} \|_{L^3} \| \mathbf{b}^a \nabla \mathbf{\bar{\phi}} \|_{L^3} + \| \nabla \mathbf{\hat{u}} \|_{L^2} \| \mathbf{b}^a \nabla \mathbf{\bar{\phi}} \|_{L^2} + \| \mathbf{\hat{u}} \|_{L^2} \| \nabla \mathbf{\bar{\phi}} \|_{L^6} \right) \]
\[ \leq C \left( \| \mathbf{\hat{u}} \|_m \| \Phi R \|_m + \| \mathbf{\hat{w}} \|_{H^1(m)} \| \mathbf{\bar{\phi}} R \|_m + \| \mathbf{\hat{w}} \|_m \| \mathbf{\bar{\phi}} R \|_{H^1(m)} \right) \]
(3.57)

and other estimates which follow in a similar way. Note here that the first term of (3.56) is obtained using the 3D Biot–Savart estimate in proposition 3.4 while the second term is obtained by once again using the estimate \( \| \nabla \mathbf{\hat{u}} \|_{L^3} \leq \| \mathbf{\hat{w}} \|_{H^1(m)} \) from [45, lemma 2.4.2]. The estimate (3.57) is obtained using nearly identical arguments paired with the estimate \( \| \mathbf{b}^a \nabla \mathbf{\bar{\phi}} \|_{L^2} \leq C \| \mathbf{\bar{\phi}} R \|_m \). Then, along with similar estimates for \( a(\tau)^{1/2} \| \nabla \mathbf{\hat{F}} \|_m \), we obtain
\[ \| \mathbf{\hat{F}}(W_R) \|_{X_{\mu,\gamma}} \leq C \| W_R \|_{X_{\mu,\gamma}}^2, \]
(3.58)

We thus obtain the quadratic estimate on \( F \) in (3.42) and conclude the existence of a fixed point for sufficiently small \( W_0 \in Z \). This implies global existence of solutions and the structure of the \( X_{\mu,\gamma} \) norm implies the temporal decay prescribed in (3.39).

\[ \square \]

4. Global existence and asymptotics

4.1. Dispersive estimates

We now turn to the proof of theorem 2. The basic conclusion of this theorem is similar to that of the preceding one. Namely, we prove that horizontal components of the velocity of solutions of the Boussinesq system tend, as time tends toward infinity, toward an Oseen vortex, emphasizing the predominately 2D nature of the flow. However, in this case, we do not restrict to small perturbations of the barotropic manifold, but allow the baroclinic part of the solution to initially be large. This has two consequences: first, we must assume that the rotation rate is large—how large, depends on how large the initial baroclinic part of the solutions is. Second, we obtain less sharp asymptotic estimates on the solution—while we can show that the solution tends to an Oseen vortex, we do not obtain the additional terms in the asymptotic approximation contained in theorem 1.

Theorem 2 treats two different choices of boundary conditions, periodic and stress-free. The proofs for the two cases are very similar, so we only provide the details of the proof for the latter. Hence for the remaining sections we will consider the boundary conditions (1.4).

Indeed, the proofs of these results follow very closely the strategy used by Roussier-Michon and Gallay in [18] so we will mainly just highlight the differences in the proof necessitated by the additional temperature dependence, \( \text{vis-\'a-\'vis} \) the purely rotational problem considered in that reference. In contrast to the preceding section, the dispersive nature of the linearized problem is paired with large-rotation rate, \( |\Omega| \gg 1 \), to obtain global existence for initial data in which the smallness assumptions are imposed only on the quasi-geostrophic part of the initial data.

To characterize such dispersive effects, it will be important to consider the linear Rossby-type equation
\[ \partial_t \mathbf{\hat{v}} + \Gamma \mathbf{P} \mathbf{\hat{J}} \mathbf{P} \mathbf{\hat{v}} = \Delta \mathbf{\hat{v}}, \quad \text{div} \, \mathbf{\hat{u}} = 0, \]
(4.1)
where the operator \( \mathbf{P} \mathbf{\hat{J}} \mathbf{P} \) arising from rotation and stratification is defined in (B.7) below and \( \eta = \Omega / \Gamma \). We note that since the spatial domain \( \mathcal{D} = \mathbb{R}^2 \times [0,1] \) is bounded in the vertical direction, and since this equation only acts on the baroclinic part of the solution (i.e. the part with non-trivial \( x_3 \)-dependence), all eigenvalues of the linear operator will have a negative real
part \( \sim -4\pi n^2 \), where \( n \) is the Fourier index in the \( x_3 \) direction. This immediately leads to the fact that for all \( s \geq 0 \) and \( \tilde{v}_0 \in (1 - Q)H^s(\mathbb{D})^4 \) with \( \text{div} \, \tilde{v}_0 = 0 \), a solution \( \tilde{v}(t) \) to (4.1) satisfies

\[
\| \tilde{v}(t) \|_{H^s(\mathbb{D})} \leq \| \tilde{v}_0 \|_{H^s(\mathbb{D})} e^{-4\pi^2 t}, \quad t \geq 0.
\]  

(4.2)

In addition, recalling that \( \mathbb{P}_J, \mathbb{P}_S \) is anti-symmetric, we see that

\[
\frac{1}{2} \partial_t \| \nabla \tilde{v} \|_{L^2(\mathbb{D})}^2 = -\| \Delta \tilde{v} \|_{L^2(\mathbb{D})}^2.
\]  

(4.3)

From this, we immediately conclude

**Lemma 4.1.** If \( \tilde{v}_0 \in H^s(\mathbb{D})^4 \), then for any \( T > 0 \), solutions of (4.1) satisfy

\[
\int_0^T \| \Delta \tilde{v}(\cdot,t) \|_{L^2(\mathbb{D})}^2 \, dt \leq \frac{1}{2} \| \nabla \tilde{v}_0 \|_{L^2(\mathbb{D})}^2.
\]  

(4.4)

**Remark 4.2.** Due to the form of the eigenvalues of the linearized equation, solutions \( \hat{v}(k,t) = \hat{v}(k,n,t) \) with \( |k| = \sqrt{k_1^2 + k_2^2 + 4\pi^2 n^2} \sim |k_1| + |k_2| + |n| \geq R \), decay like \( \sim e^{-R t} \), so in order to understand the dispersive properties of the solutions, it suffices to study the part of the solution localized in a neighborhood of zero in Fourier space.

With this in mind, let \( B_R = \{ k = (k_n, n) \in \mathbb{R}^2 \times \mathbb{Z} \mid |k| \leq R \} \) and \( S \) be the projection onto the quasi-geostrophic eigenspace, defined as above

\[
\tilde{S} \tilde{v}(k) = \langle a_g(k), \tilde{v}(k) \rangle_{\mathbb{C}^4} a_g(k).
\]

Because the quasi-geostrophic eigenvalue \(-|k|^2\) corresponding to \( a_g \) is independent of \( \Omega \) and \( \Gamma \), we cannot expect any dispersive smoothing in this mode. However for the other modes, we can prove the following estimates on solutions of the linear Rossby equation (4.1) for initial data compactly supported in Fourier space perpendicular to the quasi-geostrophic mode.

**Proposition 4.3.** For any \( R > 0 \), there exists \( C_R > 0 \) such that, for all \( \tilde{v}_0 \in (1 - Q)L^2(\mathbb{D})^4 \) with \( \text{div} \, \tilde{u}_0 = 0 \) and supp \( \tilde{v}_0 \subset B_R \), the solution \( \tilde{v} \) of (4.1) satisfies

\[
\| \tilde{v} \|_{L^1(\mathbb{R}^+; L^\infty(\mathbb{D}))} \leq C_R \left( |\eta|^{-1/4} \| (1 - \tilde{S}) \tilde{v}_0 \|_{L^2(\mathbb{D})} + \| \tilde{S} \tilde{v}_0 \|_{L^2(\mathbb{D})} \right).
\]  

(4.5)

**Proof.** In Fourier space, equation (4.1) takes the form

\[
\left( \partial_t + |k|^2 I_4 + 1 \mathbb{P}_J, \mathbb{P}_S \right) \hat{v}(k,n,t) = 0, \quad n \in \mathbb{Z} \setminus \{0\}, k_n \in \mathbb{R}^2.
\]  

(4.6)

We decompose \( \tilde{v} = \tilde{v}^+ + \tilde{v}^- + \tilde{v}^\| \), where in Fourier space

\[
\begin{align*}
\tilde{v}^+ (k,t) &= e^{-|k|^2 t/\mathbb{P}_S(k)} \langle (\tilde{v}_0)(k), a_{\pm}(k) \rangle_{\mathbb{C}^4} a_{\pm}(k) \\
\tilde{v}^- (k,t) &= e^{-|k|^2 t/\mathbb{P}_S(k)} \langle (\tilde{v}_0)(k), a_{\pm}(k) \rangle_{\mathbb{C}^4} a_{\pm}(k).
\end{align*}
\]  

(4.7)

We recall that the explicit expressions for the eigenvectors \( a_{\pm} \) are given in (2.12). Note here that the incompressibility condition gives \( \langle (\tilde{v}_0)(k), a_{\pm}(k) \rangle_{\mathbb{C}^4} = 0 \). Also note that the forms of the modes \( \tilde{v}^\| \) are almost identical to those of the dispersive modes in [18, equation (87)]. (see also [10, section 4.2] for a discussion of similar dispersive estimates, albeit in
an unbounded domain.) Following the arguments of ([18]; appendix B) more-or-less line for line, one obtains
\[
\|\tilde{v}^\pm\|_{L^1(\mathbb{R}, L^\infty(\mathbb{D}))} \leq C_R \left( |\eta|^{-1/4} \| (1 - S) \tilde{v}_0\|_{L^2(\mathbb{D})} \right).
\] (4.8)

Thus, we need only estimate \( \tilde{v}^\| \). However this term can be estimated directly without recourse to the duality methods used in [18]. Note that
\[
\tilde{v}^\| (x, t) = \sum_{n \neq 0} \int_{\mathbb{R}^2} e^{i k x} e^{2\pi i n y} e^{-|k|^2 t} e^{-4\pi^2 |x|^2 t} \left( \langle \tilde{v}_0 \rangle (k), a_g (k) \right)_I a_g (k) dk.
\] (4.9)

Thus, using the fact that all terms have \( n \neq 0 \), and the support condition on \( \tilde{v}_0 \) we find
\[
|\tilde{v}^\| (x, t)| \leq \sum_{n \neq 0} \int_{\mathbb{R}^2} e^{-|k|^2 t} e^{-4\pi^2 |x|^2 t} \left( \langle \tilde{v}_0 \rangle (k), a_g (k) \right)_I a_g (k) dk \\
\leq \sum_{n \neq 0} \int_{|k| \leq R} e^{-|k|^2 t} e^{-4\pi^2 |x|^2 t} \left( \langle \tilde{v}_0 \rangle (k), a_g (k) \right)_I a_g (k) dk \leq C_R e^{-4\pi^2 t}.
\] (4.10)

Integrating this estimate with respect to \( t \) and combining it with the estimates on \( \tilde{v}^\pm \) immediately yields the result.

Note that the restriction on the support of \( \tilde{v} \) in proposition 4.3 means that the solution of (4.1) lies in any Sobolev space \( H^p \) with \( s \geq 0 \). From this, we immediately obtain the following:

**Corollary 4.4.** With the assumptions of proposition 4.3, for any \( 1 \leq p \leq \infty \) and \( 2 \leq q \leq \infty \), with \( \frac{1}{p} + \frac{2}{q} \leq 1 \), the solution of (4.1) obeys:
\[
\| \tilde{v} \|_{L^p(\mathbb{R}, L^q(\mathbb{D}))} \leq C_R \left( |\eta|^{-1/(4p)} \| (1 - S) \tilde{v}_0\|_{L^2(\mathbb{D})} + \| S \tilde{v}_0\|_{L^2(\mathbb{D})} \right).
\]

### 4.2. Global existence

To make use of these dispersive properties in the nonlinear problem we make the following decomposition, which splits the baroclinic component into a linear part with initial data localized in Fourier space—in other words low frequency- and a nonlinear part with initial data whose support is bounded far away from the origin in Fourier space—that is high frequency. In particular, we set
\[
\tilde{v}(x, t) = \lambda (x, t) + r(x, t),
\]
where \( \lambda \) satisfies the linear Rossby equation
\[
\tilde{\partial}_t \lambda + \tilde{\Gamma} \tilde{P} \tilde{D} \tilde{\lambda} = \Delta \lambda, \quad \text{div} \lambda = 0,
\] (4.11)
with initial data \( \lambda_0 = \tilde{P}_K \tilde{v}_0 \) where \( \tilde{P}_k \) is the multiplier defined by
\[
\left( \tilde{P}_K \right)_n (k) = \chi \left( \frac{\sqrt{|k|^2 + (2\pi n)^2}}{R} \right) f_n (k),
\] (4.12)
and \( \chi \) is a smooth function with \( \chi(k) = 1 \) for all \( |k| < 1 \) and \( \chi(k) = 0 \) for \( |k| > 2 \). The remainder \( r \) must then solve

\[
\partial_t r + \Gamma^P L^p r + N_3 = \Delta r, \quad \text{div } r = 0,
\]

with \( N_3 = \mathbb{P}[(u \cdot \nabla)v + (\bar{u} \cdot \nabla)\bar{v} + (1 - Q)(\bar{u} \cdot \nabla)\bar{v}] \) and initial condition \( r_0 = (1 - P_{\Gamma})d_0 \).

The estimates of the previous section control the evolution of \( \lambda \), while we expect \( r \) will decay exponentially fast at the linear level (see remark 4.2). Such linear estimates are then sufficient to show that solution \( r(t) \) with \( r_0 \) small will remain so. Also note, we obtain the following local existence result using a standard fixed point argument.

**Proposition 4.5 (Local existence).** For any \( R > 0 \), there exists \( T_R > 0 \) such that for all \( \Omega, \Gamma \in \mathbb{R} \) and initial data \( v_0 \in X_d \) with \( \|v_0\|_{X_d} < R \) the equation (2.5) has a unique local solution \( v \in C^0([0, T_R], X_d) \) satisfying \( v(0) = v_0 \).

Note that this local existence result imposes no restriction on the size of the initial data. We now show that if the initial size of the quasi-geostrophic component of the solution is sufficiently small, the solution can be extended for all time, by deriving a bound on the solution in the \( X_d \)-norm which is uniform in time. We will prove

**Theorem 4.** For all \( v_0 \in X_d \) with \( S(1 - Q)v_0 \) sufficiently small, that is the projection onto the baroclinic quasi-geostrophic portion of \( u_0 \) is small, there exists \( \Omega_0 \) such that for all \( \Omega \in \mathbb{R} \) with \( |\Omega| > \Omega_0 \) the stably-stratified system (2.5) with stress-free boundary conditions has a unique global solution \( v \in C^0([0, \infty), X_d) \) with \( v(0) = v_0 \). Furthermore, there exists a constant \( C > 0 \) such that \( \|v(t)\|_{X_d} \leq C \) for all \( t > 0 \).

The proof of this theorem follows by showing that there exists a uniform bound of the functional

\[
\tilde{\Psi}(t) = \|\varpi_3(t)\|_{L^1} + \|\varpi_3(t)\|_{L^2}^2 + \|\tilde{\sigma}(t)\|_{H^1(\mathbb{D})}^2,
\]

which combines the enstrophy, energy, and the absolute integral of the vorticity. The inclusion of this last term could be seen as restricting to barotropic vorticity profiles with additional spatial localization, and is necessary in obtaining the barotropic asymptotics in section 4.3.

Such a bound implies that the \( X_d \)-norm of the solution is uniformly bounded and hence we can extend the local existence theorem indefinitely. Using the decomposition of \( \tilde{\sigma} \) defined above, a bound on \( \tilde{\Psi} \) is equivalent to a bound on

\[
\tilde{\Psi}(t) = \|\varpi_3(t)\|_{L^1} + \|\varpi_3(t)\|_{L^2}^2 + \|\tilde{\sigma}(t)\|_{H^1(\mathbb{D})}^2 + \|\lambda(t)\|_{H^1(\mathbb{D})}^2 .
\]

From the estimates in section 4.1, we conclude that \( \sup_{t>0} \|\lambda(t)\|_{H^1(\mathbb{D})}^2 \leq \|\lambda_0\|_{H^1(\mathbb{D})}^2 \). In addition, Poincaré’s inequality implies that \( \|\tilde{\sigma}(t)\|_{H^1(\mathbb{D})} \leq C \|\nabla \tilde{\sigma}(t)\|_{L^2(\mathbb{D})} \). Thus, our bound on the \( X_d \)-norm of the solution will follow from a bound on

\[
\tilde{\Psi}(t) = \|\varpi_3(t)\|_{L^1} + \|\varpi_3(t)\|_{L^2}^2 + \|\nabla \tilde{\sigma}(t)\|_{L^2}^2 .
\]

We control the evolution of the various terms in \( \tilde{\Psi}(t) \) using the following energy estimates modeled on [18]:

**Proposition 4.6.** There exists a constant \( C_1 \) such that if \( v \in C^0([0, T], X_d) \) is a solution of (2.5) for some \( \Gamma, \Omega \in \mathbb{R} \) and if \( v_0 \) as above for some \( R > 0 \), then the solutions of (4.11), (4.13) and (B.11) satisfy for any \( t \in (0, T] \) :
\[
\frac{d}{dt} \| \bar{\omega}_3(t) \|_{L^2(\mathbb{R}^3)}^2 \leq -\| \nabla \bar{\omega}_3(t) \|_{L^2(\mathbb{R}^3)}^2 + 8 \| \bar{u}(t) \| \| \nabla \bar{u}(t) \|_{L^2(\mathbb{D})},
\]
(4.17)

\[
\| \bar{\omega}_3(t) \|_{L^2(\mathbb{R}^3)} \leq \| \bar{\omega}_3(0) \|_{L^2(\mathbb{R}^3)} + 2 \int_0^t \| \bar{u}(s) \|_{L^2(\mathbb{D})} \| \Delta \bar{u}(s) \|_{L^2(\mathbb{D})} ds,
\]
(4.18)

\[
\frac{d}{dt} \| \nabla r(t) \|_{L^2(\mathbb{D})}^2 \leq -\| \Delta r(t) \|_{L^2(\mathbb{D})}^2 + C_1 \| \nabla r(t) \|_{L^2(\mathbb{D})}^2 \| \nabla \bar{u}(t) \|_{L^2(\mathbb{R}^3)} \| \Delta \bar{u}(t) \|_{L^2(\mathbb{R}^3)}
+ C_1 \left( \| \bar{u}(t) \|_{L^2(\mathbb{R}^3)}^2 \| \nabla \lambda(t) \|_{L^2(\mathbb{D})}^2 + \| \nabla \bar{u}(t) \|_{L^2(\mathbb{R}^3)}^2 \| \lambda(t) \|_{L^2(\mathbb{D})}^2 + \| \bar{u}(t) \| \| \nabla \bar{r}(t) \|_{L^2(\mathbb{D})}^2 \right).
\]
(4.19)

**Proof.** Note that for stress-free boundary conditions, the equations for \( \bar{u}_t \) and \( \bar{\omega}_3 \) are exactly the same as they are in the case of rotating fluids, so the proofs of (4.17) and (4.18) follow exactly as in proposition 2.5 of [18]. For (4.19) we must make a few small changes to account for the presence of the temperature term in our equation. If we compute \( \frac{d}{dt} \| \nabla r \|_{L^2}^2 \), the dissipative term in (4.13) gives rise to \( -\| \Delta r \|_{L^2}^2 \), while the term \( \Gamma P_{J,Q} r \) makes no contribution due to anti-symmetry. Thus, we need only estimate the contributions of the nonlinear term:
\[
\left| \int_\mathbb{D} (\Delta r) \cdot \tilde{N} \lambda dx \right| \leq \left| \int_\mathbb{D} (\Delta r) \cdot [(\bar{u} \cdot \nabla) \bar{v} + (\bar{u} \cdot \nabla) \bar{v} - (1 - Q)(\bar{u} \cdot \nabla) \bar{v}] dx \right|,
\]
(4.20)
and the Helmholtz projector has vanished due to the fact that \( \nabla \cdot r = 0 \). These terms are broken up and estimated in turn. For example:
\[
\left| \int_\mathbb{D} (\Delta r) \cdot (\bar{u} \cdot \nabla) \bar{v} \bar{\lambda} dx \right| = \left| \int_\mathbb{D} (\Delta r) \cdot (\bar{u} \cdot \nabla) \lambda dx + \int_\mathbb{D} (\Delta r) \cdot (\bar{u} \cdot \nabla) r dx \right|,
\]
(4.21)
and each of these terms is estimated in a fashion analogous to that used in [18], leading to the bound
\[
\left| \int_\mathbb{D} (\Delta r) \cdot (\bar{u} \cdot \nabla) \bar{v} \bar{\lambda} dx \right| \leq \frac{1}{16} \| \Delta r \|_{L^2}^2 + C \| \bar{u} \|_{L^2}^2 \| \nabla \lambda \|_{L^2}^2 + \frac{1}{16} \| \Delta r \|_{L^2}^2 + C \| \nabla r \|_{L^2}^2 \| \nabla \bar{u} \|_{L^2}^2 \| \Delta \bar{u} \|_{L^2}^2.
\]
(4.22)
The remaining terms are estimated in a similar way leading to (4.19). □

Note that (4.18) gives us control of the \( \| \bar{\omega}_3 \|_{L^2} \) term in \( \Psi(t) \) provided we can control the evolution of \( \bar{u} \) and \( \Delta \bar{u} \) and these are in turn controlled by the evolution of \( r \) and \( \lambda \). The evolution of \( \lambda \) is controlled by the estimates of the previous section, and thus, we turn our attention to the ‘reduced’ functional
\[
\Phi(t) = \| \bar{\omega}_3 \|_{L^2}^2 + \| \nabla r \|_{L^2}^2.
\]
(4.23)

**Remark 4.7.** Note that we expect the two terms in \( \Phi \) to have different properties—\( \bar{\omega}_3 \) may be large, but is not expected to grow much, while we can make \( \nabla r \) arbitrarily small (at least initially) by choosing \( R \) in (4.12) sufficiently large.

Differentiating with respect to \( t \) and using the estimates of proposition 4.6, we obtain
\[
\frac{d}{dt} \Phi(t) \leq -\left( \| \nabla \bar{\omega}_3(t) \|_{L^2}^2 + \| \Delta r(t) \|_{L^2}^2 \right) + C \| \bar{u}(t) \| \| \nabla \bar{u}(t) \|_{L^2}^2
+ C_4 \left( \| \nabla r(t) \|_{L^2}^2 \| \nabla \bar{u}(t) \|_{L^2}^2 + \| \Delta \bar{u}(t) \|_{L^2}^2 \| \lambda(t) \|_{L^2}^2 + \| \nabla \bar{u}(t) \|_{L^2}^2 \| \lambda(t) \|_{L^2}^2 \right).
\]
(4.24)
We bound the term
\[ ||\bar{u}||\nabla \bar{v}||_{L^2} \leq C (||\nabla v||_{L^2} ||\Delta r||_{L^2} + ||\nabla r||_{L^2} (||\nabla \lambda||_{L^\infty} + ||\lambda||_{L^\infty}^2) ) + C||\nabla \lambda||_{L^2} ||\lambda||_{L^\infty}, \] (4.25)
while we bound \( \bar{u} \) with the aid of the Biot–Savart law (see [19, appendix B])
\[ ||\bar{u}||_{L^2}^2 \leq C||\bar{w}_3||_{L^2}^2 \leq C||\bar{w}_3||_{L^2}||\bar{w}_3||_{L^2}. \] (4.26)
Recalling remark 4.7 about the expected relative sizes of \( \bar{w}_3 \) and \( r \), we see that (4.25) implies

**Lemma 4.8.** There exist constants \( C_2, C_3, C_4 \) such that the following holds. Let \( v \in C^0([0, T], X_d) \) be a solution of (2.5) which is decomposed \( v = \bar{v} + \lambda + r \) as above for some \( R > 0 \). Assume as well that there exist \( K \geq 1 \) and \( \epsilon \in (0, 1] \) such that the corresponding components satisfy
\[ ||\bar{w}_3(t)||_{L^2(\mathbb{R}^2)} \leq K, \quad ||\nabla r(t)||_{L^2(\mathbb{R}^2)} < \epsilon, \] (4.27)
for all \( t \in [0, T] \). Then
\[ \frac{d}{dt} \Phi(t) \leq - \left( ||\nabla \bar{w}_3(t)||_{L^2(\mathbb{R}^2)}^2 + ||\Delta r(t)||_{L^2(\mathbb{R}^2)}^2 \right) \]
\[ + C_2 \epsilon^2 K^2 ||\Delta \bar{u}(t)||_{L^2(\mathbb{R}^2)}^2 + C_3 \epsilon^2 ||\Delta r(t)||_{L^2(\mathbb{R}^2)}^2, \]
\[ + \Phi(t)G(t) + F(t), \]
\[ \leq - \frac{1}{2} \left( ||\nabla \bar{w}_3(t)||_{L^2(\mathbb{R}^2)}^2 + ||\Delta r(t)||_{L^2(\mathbb{R}^2)}^2 \right) + \Phi(t)G(t) + F(t), \] (4.28)
for all \( t \in (0, T] \). Here
\[ F(t) = C_4 (||\lambda(t)||_{L^\infty(\mathbb{R}^2)} ||\Delta \lambda(t)||_{L^2(\mathbb{R}^2)} + ||\bar{w}_3||_{L^2} ||\nabla \lambda||_{L^2}), \]
\[ G(t) = C_4 (||\nabla \lambda(t)||_{L^\infty(\mathbb{R}^2)} + ||\lambda(t)||_{L^\infty(\mathbb{R}^2)}^2 + ||\nabla \lambda(t)||_{L^2(\mathbb{R}^2)}). \] (4.29)

We now prove that with the aid of Gronwall’s inequality, the \( X_d \)-norm of the solution of (2.5) remains uniformly bounded for all time, and hence, the local existence theorem can be extended without limit, completing the proof of theorem 4. More precisely, we show

**Proposition 4.9.** For any initial conditions, \( \bar{v}_0 \in X_d \) of (2.5), there exists \( K, \Omega_0 > 0 \) and \( \epsilon \in (0, 1) \), such that if \( |\Omega| > \Omega_0 \), and if the projection onto the quasi-geostrophic mode, \( \bar{S_0} \), is sufficiently small, then for any \( T > 0 \),
\[ \sup_{0 \leq t \leq T} \{ ||\bar{w}_3(t)||_{L^2} + ||\bar{w}_3(t)||_{L^2} \} \leq K \] (4.30)
\[ \sup_{0 \leq t \leq T} ||\nabla r(t)||_{L^2} \leq \epsilon. \] (4.31)

**Remark 4.10.** Note that theorem 4 follows immediately from this proposition and the estimates on \( \lambda \) in section 4.1.

**Proof.** Choose \( K \) and \( R \) large enough that
\[ ||\bar{w}_3||_{L^2} + ||\bar{w}_3||_{L^2} \leq \frac{1}{16} K, \] (4.32)
and
\[ \| \nabla r \|_{L^2} \leq \frac{1}{16} \epsilon. \]  
(4.33)

By the local existence theorem, there exists \( T^* > 0 \) such that (4.30) and (4.31) hold for \( 0 \leq t \leq T^* \). Let \( T^* \) be the supremum over the set of values of \( T \) for which these estimates hold. We claim that \( T^* = \infty \).

Suppose instead that \( T^* < \infty \). Note that from the estimates of section 4.1, if we choose \( \Omega_0 \) sufficiently large, and quasi-geostrophic projection sufficiently small, we can insure that
\[ e^{\int_0^T G(s)ds} \leq 2, \quad \int_0^T F(s)ds < \frac{1}{16} K, \]
for any \( T > 0 \). Furthermore, using the estimates on \( \lambda \) from the previous section, plus (4.31) and (4.18), we can insure that both
\[ \| \bar{\omega}_3(t) \|_{L^1} \leq 2 \| \bar{\omega}_3 \|_{L^1} \leq \frac{1}{8} K \]
for all \( 0 \leq t \leq T^* \). If one then applies Gronwall’s inequality to (4.28), one finds that
\[ \Phi(t) \leq \frac{1}{2} K, \]  
(4.34)
for all \( 0 \leq t \leq T^* \). If we then apply a similar argument with Gronwall’s inequality to \( \| \nabla r(t) \|_{L^2} \), we also find that for \( \Omega_0 \) sufficiently large, and the quasi-geostrophic projection sufficiently small, we have
\[ \| \nabla r(t) \|_{L^2} \leq \frac{1}{2} \epsilon, \]  
(4.35)
for all \( 0 \leq t \leq T^* \). However, these two estimates imply that we could extend the time for which (4.30) and (4.31) hold beyond \( T^* \), contradicting its definition. Hence, \( T^* = \infty \) as desired.

4.3. Asymptotics

Having proved solutions exist, we now consider their asymptotics. For the baroclinic component, we find exponential convergence as in the local analysis of section 3. For the barotropic component, we still find that solutions converge to an Oseen vortex in the vertical vorticity at leading order. Moreover, because of the global techniques required to treat large initial data, our second result does not give precise estimates for how solutions approach this solution in the barotropic subspace.

4.3.1. Exponential decay of \( \bar{V} \). To study decay of the baroclinic component, we use the energy estimates from the existence analysis above, to derive a differential inequality for the quantities \( \| \nabla \bar{V}(t) \|_{L^2} \), \( \| \Delta \bar{V}(t) \|_{L^2} \), from which we can readily conclude the following exponential decay.
Proposition 4.11. For all $0 < \mu < 2\pi^2$, we have
\begin{align}
\sup_{t \geq 0} e^{\mu t} \| \nabla \tilde{v}(t) \|_{L^2} &< \infty, \quad (4.36) \\
\sup_{t \geq 1} e^{\mu t} \| \Delta \tilde{v}(t) \|_{L^2} &< \infty. \quad (4.37)
\end{align}

Proof. This once again follows in a similar way as in [18]. We already have by Poincare’s inequality that
\[ \| \lambda(t) \|_{H^s} \leq C e^{-4\pi^2 t}. \]
Then using the global bound derived in theorem 4, along with dispersive estimates for $\lambda$, we find that (4.2) and (4.19) implies
\[ \frac{d}{dt} \| \nabla r(t) \|_{L^2(\mathbb{D})} + \frac{1}{2} \| \Delta r(t) \|_{L^2(\mathbb{D})} \leq C \| \nabla r(t) \|_{L^2(\mathbb{D})} \| \Delta \bar{u}(t) \|_{L^2(\mathbb{R}^2)} + C_2 e^{-8\pi^2 t}. \]
(4.38)

We follow [18] and set $f(t) = e^{\mu t} \| \nabla r(t) \|_{L^2(\mathbb{D})}^2$ obtaining
\[ f'(t) \leq C f(t) \| \Delta \bar{u}(t) \|_{L^2(\mathbb{R}^2)}^2 + C_2 e^{-(8\pi^2 - \mu) t}. \]
(4.39)

Then pairing this with the fact that $\int_0^\infty \| \Delta \bar{u}(t) \|_{L^2(\mathbb{R}^2)} dt < \infty$ (from the Gronwall estimate on $\Psi(t)$), we obtain that $f(t) \leq C_3$ and thus
\[ \| \nabla r(t) \|_{L^2(\mathbb{D})} \leq C_3 e^{-\mu t/2}. \]
(4.40)

From this we conclude the exponential decay of $\| \tilde{v}(t) \|_{H^s}$ as $t \to \infty$ with any rate $0 < \mu < 2\pi^2$ as
\[ \| \tilde{v}(t) \|_{H^s(\mathbb{D})} \sim \| \nabla \tilde{v}(t) \|_{L^2(\mathbb{R}^2)} \leq \| \nabla r(t) \|_{L^2(\mathbb{D})} + \| \nabla \lambda \|_{H^s(\mathbb{D})}. \]

We also remark that optimal decay rates can be found in a similar way by multiplying (B.12) with $\Delta \bar{v}$ to find
\[ \sup_{t \geq 0} e^{\mu t} \| \nabla \tilde{v}(t) \|_{L^2} < \infty, \quad \text{for all } \mu < 4\pi^2, \quad (4.41) \]
and furthermore, by differentiating in space, that
\[ \sup_{t \geq 1} e^{\mu t} \| \Delta \tilde{v}(t) \|_{L^2} < \infty, \quad \text{for all } \mu < 4\pi^2. \]
\[ \square \]

4.3.2. Diffusive decay of $\bar{\omega}_3$. In this section we show that the solution of the barotropic vertical vorticity, $\bar{\omega}_3(t)$, of (B.11) converges to Oseen’s vortex, $\bar{\omega}_0$, as $t \to +\infty$. This can be obtained using the approach in [18, section 3.4] and thus we only outline the argument. One first introduces scaling variables by defining
\[ \dot{\omega}_3(x_h, t) = \frac{1}{1 + t} \dot{\omega}_3 \left( \frac{x_h}{\sqrt{1 + t}}, \log(1 + t) \right), \]
\[ \ddot{u}_h(x_h, t) = \frac{1}{\sqrt{1 + t}} \ddot{u}_h \left( \frac{x_h}{\sqrt{1 + t}}, \log(1 + t) \right), \]
\[ \xi = \frac{x_h}{\sqrt{1 + t}}, \quad \tau = \log(1 + t). \]

It then follows that \( \dot{\omega}_3 \) satisfies the equation
\[ \ddot{w}_3,_\tau = \mathcal{L} \dot{w}_3 - (\Pi_h \cdot \nabla \xi) \dot{w}_3 - \bar{N}, \tag{4.42} \]
where \( \mathcal{L} := \Delta_\xi + \frac{1}{2} (\xi \cdot \nabla \xi) + 1, \quad \bar{N} = e^{2\tau} \bar{N}(v(\xi e^{\tau}, x_h, e^{\tau} - 1)), \) and \( \bar{N} \) is defined in (B.11). The exponential decay of \( \bar{\varnothing} \) gives that
\[ \int_0^\infty \frac{\| \bar{N}(\cdot, \tau) \|_2^2}{|\xi|} d\tau < \infty. \]

Solutions \( \ddot{w}_3(\tau) \) of (4.42) with initial data \( \ddot{w}_3,0 \) in \( L^1(\mathbb{R}^2) \) are thus globally defined for \( \tau \geq 0 \) satisfying
\[ \ddot{w}_3 \in C^0([0, \infty), L^1(\mathbb{R}^2)), \quad \| \ddot{w}_3(\tau) \|_{L^1(\mathbb{R}^2)} \leq C, \]
for all \( \tau \geq 0 \). Asymptotics for the solution \( \ddot{w}_3(\tau) \) are then obtained by explicitly characterizing its omega-limit set \( \Omega_\infty := \{ w_\infty \in L^1(\mathbb{R}^2) \mid \exists \tau_n \rightarrow \infty, \ddot{w}_3(\tau_n) \rightarrow w_\infty \} \). As in [18, lemma 3.1], one uses Duhamel’s formula to express the solution \( \ddot{w}_3(\tau) \) of (4.42) in terms of the explicit semi-flow, \( \Phi(\tau) \), associated with the purely barotropic equation
\[ \ddot{w}_3,\tau = \mathcal{L} \ddot{w}_3 - \Pi_h \cdot \nabla \ddot{w}_3 \]
which one can use to prove that the trajectory \( \{ \ddot{w}_3(\tau) \}_{\tau \geq 0} \) is relatively compact in \( L^1(\mathbb{R}^2) \).

Then, as in [18, lemma 3.2], one compares the solution \( \ddot{w}_3(\tau) \) to that of the limiting equation, \( \Phi(\tau) \bar{w}_3,0 \), to obtain that \( \Phi(\tau) \Omega_\infty = \Omega_\infty \) for all \( \tau \geq 0 \). Continuing as in the proof of [20, proposition 3.4] then gives the following result.

**Proposition 4.12.** Let \( A = \int_{\mathbb{R}^2} \ddot{w}_3,0,0 d\xi = \int_{\mathbb{R}^2} \ddot{\omega}(x_h, 0) dx_h \), then the omega-limit set of a solution, \( \ddot{w}_3(\xi, \tau) \), of (4.42) with initial condition \( \ddot{w}_3(\xi, 0) = \ddot{w}_3,0(\xi) \) in \( L^1(\mathbb{R}^2) \) satisfies \( \Omega_\infty = \{ A \varphi_0 \} \).

It follows from this proposition that \( \| \ddot{w}_3(\tau) - A \varphi_0 \|_{L^1(\mathbb{R}^2)} \rightarrow 0 \) as \( \tau \rightarrow \infty \). Translating this result into unscaled variables one can obtain leading order asymptotics for the solution \( \ddot{\omega}(x_h, t) \) (and thus \( \ddot{u}_h(x_h, t) \)) to complete the proof of theorem 2.

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Appendix A. Estimates on the linear evolution

In this appendix we derive estimates on the semigroup generated by linearizing the Boussinesq equations around an Oseen vortex. The estimates are nontrivial because the perturbation created by this linearization is large in the operator norm, if the vortex around which we linearize has large amplitude. However, because of the explicit form of the velocity and vorticity fields for the Oseen vortex, we are able to compute explicit information about the spectral properties of this linearized operator and then use them to derive estimates on the associated semigroup.

A.1. Barotropic evolution

We derive estimates on the solution operators for the linear equations $W_t = (\bar{L}(\tau) + \bar{\Lambda})W$, $W(\sigma) = W_0$, defined for $\tau \geq \sigma$ in (3.28), (3.30) and (3.32) in a step-by-step fashion. We begin by focussing on the barotropic part of the solution. In this section we build, in a step-by-step fashion, the solution operators for the linear equations $W_t = (\bar{L}(\tau) + \bar{\Lambda})W$, $W(\sigma) = W_0$, defined for $\tau \geq \sigma$ in (3.28), (3.30) and (3.32) above. The barotropic linear evolution can be characterized and estimated more easily in the rotating frame as discussed in section 3.1. That is, converting the linear part of (3.1) into the rotating frame with coordinates

$$\begin{pmatrix} \bar{\omega} \\ \bar{\Theta} \end{pmatrix} := e^{\bar{\Gamma}t}J_1 \begin{pmatrix} \bar{\omega}_0 \\ \bar{\Theta}_0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

(A.1)

one obtains

$$\begin{align*}
(\bar{\omega}_0)_t &= \Delta \bar{\omega}_0 \\
(\bar{\Theta}_0)_t &= \Delta \bar{\Theta}_0 \\
\bar{\Theta}_0 &= \Delta \bar{\Theta}.
\end{align*}$$

(A.2)

Then moving into scaling variables

$$\bar{\omega}(x_0, t) = (1 + t)^{-1} \bar{\omega}(x(1 + t)^{-1/2}, \log(1 + t)), \quad \bar{\Theta}(\bar{x}_0, t) = (1 + t)^{-1} \bar{\Theta}(x(1 + t)^{-1/2}, \log(1 + t)),$$

we obtain a linear system for $\bar{W} := (\bar{w}, \bar{\Phi})^T$,

$$\bar{W}_t = \bar{L} \bar{W},$$

which generates the strongly continuous semigroup, $\bar{S}_d(\tau) := e^{\bar{L}t}$ on $L^2_{\mathbb{D}}(m)^\mathbb{R}$. Since the linear operator is diagonal, we can study spectral properties component-wise. First we have that any $\lambda$ in the point spectrum satisfies $\text{Re} \lambda \leq 0$. Due to the incompressibility conditions, $\nabla_h \cdot \bar{\Phi} = \nabla_h \cdot \bar{w}_0$, we have that the 0-eigenspace is spanned by the vector $(0, 0, \bar{\varphi}_0, 0, 0)^T$ and the $(-1/2)$-eigenspace is spanned by the vectors $(\partial_2 \varphi_0, -\partial_1 \varphi_0, 0, 0, 0)^T$, $(0, 0, \partial_2 \varphi_0, -\partial_1 \varphi_0)^T$, and $(0, 0, \partial_2 \varphi_0, 0, 0)^T$ for $i = 1, 2$.

Since we must study different components of the evolution we define the following projection operators

$$e_i^+ = 1 - e_i e_i^T, \quad i = 1, \ldots, s, \quad e_i^+ = e_i e_i^T + e_i e_i^T, \quad e_i^+ = 1 - e_i e_i^T, \quad e_i^+ = e_i e_i^T + e_i e_i^T, \quad e_i^+ = 1 - e_i^T,$$

where $e_i$ are the standard basis vectors in $\mathbb{R}^5$. Also recall that $b = (1 + |\xi|^2)^{1/2}$. Next, recall that

$$Y = Y_{h,1} \times Y_{3,0} \times Y_{p,1}, \quad Y_{h,n} = L^2_{2D,n}(m)^2 \cap \{ \nabla_h \cdot \bar{w}_0 = 0 \}, \quad Y_{3,n} = L^2_{2D,n}(m), \quad Y_{p,n} = L^2_{2D,n}(m)^2 \cap \{ \nabla_h \cdot \bar{\Phi} = 0 \}.$$
with the standard $L^2_{2D}(m)$ norm. We first state asymptotics for the strongly continuous semi-group $e^{\tau L}$ generated by the operator $L$ on $L^2_{2D}(m)$.

**Proposition A.1.** Let $f \in L^2_{2D}(m)$, $m > 1$, $q \in [1, 2]$, $\alpha \in \mathbb{N}^2$, then there exists a constant $C > 0$ such that

\[
\|b^m \partial^n e^{\tau L} Q_0 f\|_{L^2(\mathbb{R}^2)} \leq C \frac{e^{-\gamma \tau}}{a(\tau)^{1/q-1/2+|\alpha|/2}} \|b^m f\|_{L^q(\mathbb{R}^2)}
\]  

(A.4)

where $a(\tau) = 1 - e^{-\gamma \tau}$ and

\[
\gamma = \begin{cases} 
\frac{m-1}{2}, & \text{if } n + 1 < m \leq n + 2 \\
\frac{n+1}{2}, & \text{if } m > n + 2.
\end{cases}
\]

**Proof.** See proposition 4.2.2 of [45] or propositions A.2 and A.5 of [19]. \(\square\)

One can then readily conclude the following temporal estimates on $S_0$.

**Proposition A.2.** Let $f \in Y$, $m > 3$, $q \in [1, 2]$, $\alpha \in \mathbb{N}^2$. If in addition $b^m f \in L^q$ then there exists a constant $C > 0$ such that

\[
\|e^T \partial^n S_0(\tau) f\|_{m} \leq C \frac{e^{-\gamma \tau}}{a(\tau)^{1/q-1/2+|\alpha|/2}} \|b^m e^T f\|_{L^q},
\]

(A.5)

\[
\|e^T \partial^n S_0(\tau) f\|_{m} \leq C \frac{e^{-\gamma \tau/2}}{a(\tau)^{1/q-1/2+|\alpha|/2}} \|b^m e^T f\|_{L^q},
\]

(A.6)

\[
\|e^T \partial^n S_0(\tau) f\|_{m} \leq C \frac{e^{-\gamma \tau}}{a(\tau)^{1/q-1/2+|\alpha|/2}} \|b^m e^T f\|_{L^q}.
\]

(A.7)

Transforming the nonlinear equation (3.7) on the barotropic subspace \{\(\bar{\nu} = 0\)\} into scaling variables, we obtain an equation in the rotating frame for \((w, \Phi)\).

\[
w_\tau = Lw - u_h \cdot \nabla w + w_h \cdot \nabla \left( \begin{array}{c} u_h \\ 0 \end{array} \right), \quad \Phi_\tau = L\phi - u_h \cdot \nabla \Phi + \Phi \cdot \nabla u_h.
\]

(A.8)

We use the ansatz

\[
w = w_{\text{app}} + w_R, \quad \Phi = \Phi_{\text{app}} + \Phi_R, \quad u = u_{\text{app}} + u_R,
\]

with the approximate solution

\[
w_{\text{app}} = (0, 0, A \varphi_0) + B_1 e^{-\gamma/2} (\partial_2 \varphi_0, -\partial_1 \varphi_0, 0), \quad \Phi_{\text{app}} = B_2 e^{-\gamma/2} (\partial_2 \varphi_0, -\partial_1 \varphi_0), \quad u_{\text{app}} = (A u_{\text{app}}^0, B_1 e^{-\gamma/2} \varphi_0, B_2 e^{-\gamma/2} \varphi_0)^T,
\]

where $\varphi_0$ is the Gaussian, $u_{\text{app}}^0$ is its corresponding velocity profile, and $w_{R,1}, \Phi_R \in L^3_{2D,1}(m)^2$, $w_{R,3} \in L^2_{2D,0}(m)$. Note that $w_{\text{app}}$ and $\Phi_{\text{app}}$ correspond to the explicit vortex solution discussed in section 3.2. Alternatively, they can be obtained from the stationary frame approximate solution by re-writing (3.26) in the rotating frame. Using the fact that the approximate solution solves (A.8) we find that the perturbation of the linear evolution satisfies

\[
(W_R)_\tau = L W_R + \Lambda W_R
\]

(A.9)
where
\[ \Lambda W_R := -u_{app,h} \cdot \nabla W_R - u_{R,h} \cdot \nabla W_{app} + \left( \begin{array}{c} \Phi_{app,h} \cdot \nabla u_{R,h} \\ 0 \\ \Phi_{app,h} \cdot \nabla u_{R,h} \end{array} \right) \cdot \left( \begin{array}{c} \Phi_{app,h} \\ 0 \\ \Phi_{app,h} \end{array} \right) \cdot W_{app} \]
Estimating the integral terms as we did the corresponding expressions in section 3, one readily shows that $T_1(\tau)$ defines a strongly continuous semigroup such that for any $T > 0$, there exists $C_T > 0$ such that for any $0 \leq \tau \leq T$, and multi-index $\alpha$ with $|\alpha| = 1$,

$$
\|T_1(\tau)W_0\|_Y \leq C_T \|W_0\|_Y, \quad \|\partial^\alpha (T_1(\tau)W_0)\|_Y \leq \frac{C_T}{\alpha(\tau)^{1/2}} \|W_0\|_Y.
$$

(A.15)

The more refined estimates of the evolution given in proposition A.3 will follow from a more detailed analysis of the semigroup.

We begin the estimates of $T_1(\tau)$ by noting that, because of the form of $\Lambda_0$ and $\Lambda_1$, it splits into three independent parts which govern the evolution of $W_h = (W_1, W_2)^T, W_3$ and $W_p = (W_4, W_5)^T$. We refer to these three parts of $T_1$ as $T_{1,h}, T_{1,3}$ and $T_{1,p}$ and will estimate each of them separately. For later use, we will refer to the pieces of $\mathcal{L} + (\Lambda_0 + \Lambda_1)$ which generate each of these semigroups as $(\mathcal{L} + (\Lambda_0 + \Lambda_1))_h$, and so forth.

As a first step, note that $T_{1,3}(\tau)$ is exactly the semigroup studied in section 4 of [20], as it is the linearization of the 2D Navier–Stokes equation. Thus, estimates (A.11) follow from the results of proposition 4.13 of that work.

To prove the estimates in (A.10) and (A.12) we use methods similar to those in section 4 of [20]. We will give the details for (A.10), as the case of (A.12) follows in the exact same way. The proof of (A.10) begins by noting that these estimates are exactly those we would obtain for the horizontal components of the diagonal semigroup $S_0(\tau)$ (which we will denote $S_{0,h}$). Thus, we proceed in two steps:

(i) We first show that the difference between $S_{0,h}(\tau)$ and $T_{1,h}(\tau)$ is a compact operator for any $\tau > 0$. As a consequence, the essential spectral radius of $T_{1,h}(\tau)$ is the same as that of $S_{0,h}(\tau)$.

(ii) The first point implies that the only way that the decay rates of the two semigroups can differ is if $T_{1,h}(\tau)$ has an isolated eigenvalue lying outside the disc which gives the essential spectral radius. Thus, the second step in our analysis is to determine the location of isolated eigenvalues of $(\mathcal{L} + (\Lambda_0 + \Lambda_1))_h$.

Thus we shall prove the following two results

**Proposition A.4.** Assume the hypotheses of proposition A.3. Then for any $\tau > 0$, the linear operator $K_h(\tau) = T_{1,h}(\tau) - S_{0,h}(\tau)$ is compact.

**Proposition A.5.** Assume that $m > 3$. Then any eigenvalue, $\lambda$, of $(\mathcal{L} + \Lambda_0 + \Lambda_1)_h$ on $Y_{h,1}$ satisfies

$$
\text{Re}(\lambda) \leq -1.
$$

(A.16)

Combining these two propositions, the estimates (A.10) follow exactly as the proof of proposition 4.13 in [20]. The proof of proposition A.4 follows immediately from Rellich’s criterion and the following lemma.

**Lemma A.6.** Assume the hypotheses of proposition A.3. Then for any $T > 0$, there exists a constant $C_T > 0$ such that for $0 \leq \tau \leq T$, for any $\tau > 0$, the linear operator $K_h(\tau) = T_{1,h}(\tau) - S_{0,h}(\tau)$ satisfies
\[ \|K_h(\tau)W_{\alpha}\|_{m+1} \leq C_T \|W_{\alpha}\|_m, \quad \|\partial_j(K_h(\tau)W_{\alpha})\|_m \leq \frac{C_T}{a(\tau)^{1/2}} \|W_{\alpha}\|_m, \text{ for } j = 1, 2. \] \tag{A.17}

**Proof.** From Duhamel’s formula we can write
\[ K_h(\tau)W_{\alpha,0} = -\int_0^\tau S_{0,h}(\tau - \sigma)(W_{\alpha}(\sigma) \cdot \nabla)W_{\alpha,0} d\sigma - \int_0^\tau S_{0,h}(\tau - \sigma)(u_{h,0}^\alpha \cdot \nabla)W_{\alpha}(\sigma) d\sigma. \tag{A.18} \]

The estimates of the lemma now follow by estimating each of the integral terms. We first have
\[
\left\| \int_0^\tau S_{0,h}(\tau - \sigma)(u_{h,0}^\alpha \cdot \nabla)W_{\alpha}(\sigma) \right\|_{m+1} \leq C \int_0^\tau e^{-(\tau - \sigma)} \left( \|u_{h,0}^\alpha\| \|W_{\alpha}(\sigma)\| \right)_{m+1} d\sigma \\
\leq C \int_0^\tau e^{-(\tau - \sigma)} \|\nabla W_{\alpha}(\sigma)\|_m d\sigma \\
\leq C C_T \int_0^\tau \frac{e^{-(\tau - \sigma)}}{a(\sigma)^{1/2}} d\sigma \|W_{\alpha,0}\|_m \\
\leq C \|W_{\alpha,0}\|_m. \tag{A.19}
\]

Here, the first inequality used our estimates on the semigroup \(S_{0}(\tau)\), the second inequality used the fact that \(\|u_{h,0}^\alpha\|_{m+1} \leq C \|f\|_m\) due to the decay of \(u_{h,0}^\alpha(\xi)\) as \(|\xi| \to \infty\), the third inequality used (A.15), and the last one the fact that the singularity in \(a(\tau)^{-1/2}\) is integrable.

Estimating the \(\| \cdot \|_{m+1}\)-norm of the first integral term in (A.18) is similar, but even easier since derivatives of \(u_{h,0}^\alpha(\xi)\) decay more rapidly as \(|\xi| \to \infty\) than \(u_{h,0}^\alpha(\xi)\) itself, and we leave this estimate to the reader. Now we turn to the second estimate in the lemma, namely the estimate of the derivatives of \(K_h\). Differentiating the expression in (A.18) we must estimate
\[
\partial_j(K_h(\tau)W_{\alpha,0}) = -\int_0^\tau \partial_j(S_{0,h}(\tau - \sigma)(W_{\alpha}(\sigma) \cdot \nabla)W_{\alpha,0}) d\sigma - \int_0^\tau \partial_j(S_{0,h}(\tau - \sigma)(u_{h,0}^\alpha \cdot \nabla)W_{\alpha}(\sigma)) d\sigma. \tag{A.20}
\]

Once again, because of the explicit formulas for the derivatives of \(u_{h,0}^\alpha(\xi)\), the estimates of the first term are easier than the second, so we focus on estimating the second. Using the estimates on the semi-group \(S_{0,h}\), plus the estimates in (A.15), we have
\[
\left\| \int_0^\tau \partial_j(S_{0,h}(\tau - \sigma)(u_{h,0}^\alpha \cdot \nabla)W_{\alpha}(\sigma)) \right\|_m \leq C C_T \left( \int_0^\tau \frac{e^{-(\tau - \sigma)}}{a(\tau - \sigma)^{1/2} a(\sigma)^{1/2}} d\sigma \right) \|W_{\alpha,0}\|_m. \tag{A.21}
\]

Now, using the fact that
\[
\int_0^\tau \frac{e^{-(\tau - \sigma)}}{a(\tau - \sigma)^{1/2} a(\sigma)^{1/2}} d\sigma \leq \frac{C}{a(\tau)^{1/2}}, \tag{A.22}
\]
the estimate, and the lemma follows. \qed

We now turn to the proof of proposition A.5. We begin with:

**Lemma A.7.** \(\lambda\) is an eigenvalue of the operator
\[ L_h w_h := L w_h + w_h \cdot \nabla u_{h,0}^\alpha - u_{h,0}^\alpha \cdot \nabla w_h, \]
posed on \(Y_{h,t}\), if and only if \(\lambda\) is an eigenvalue of the operator

\[ \text{R39} \]
\[ L_3 u_3 := (\mathcal{L} - 1/2) u_3 - A u_3^0 \cdot \nabla u_3 \]
on $L^2_{2D,0}(m')$, where $\widetilde{w}_{R,h} = (\partial_2 u_3, -\partial_1 u_3)^T$ and $0 < m' \leq m - 1$.

**Remark A.8.** Note that our estimates of the Biot–Savart kernel imply that the velocity field $u_3 \in L^2_{2D,0}(m')$.

**Proof.** The proof of the lemma follows by noting that if $W$ and $\lambda$ satisfy

\[ L_3 W = \lambda W, \]

then we set $U = u_3,\text{BS}(W)$ where $u_3,\text{BS}$ is the Biot–Savart mapping (3.4) defined above so that $(\partial_2 U, -\partial_1 U) = W$, and $U \in L^2_{2D,0}(m')$. Indeed our choice of $m, m', \text{ proposition B.1 of [19]},$ and the generalized Hölder inequality gives

\[ \|u_3,\text{BS}(\widetilde{w}_h)\|_{m'} \leq \|b^{m'-m''} L_p\|_{L^q} \|u_3,\text{BS}(\widetilde{w}_h)\|_{L^q}, \]

(A.23)

where $1/2 = 1/p + 1/q'$ with $q' \in (2, \infty)$, chosen so that $m' - m'' > 2/p$ (and hence $b^{m'-m''} \in L^p$) and $m'' = m - 2/q'$, (so [19, proposition B.1] holds). Now, we compute the derivatives of $h = L_3 U$ to find

\[ (\partial_2 h, -\partial_1 h)^T = L_3 W = \lambda W. \]

Applying the Biot–Savart law we then find by linearity that $h = \lambda U$ and thus

\[ L_3 U = \lambda U. \]

\[ \square \]

Proposition A.5 now follows from

**Lemma A.9.** Let $m'$ be as in lemma A.7. Then any eigenvalue, $\lambda$, of the operator $L_3$ whose eigenfunction lies in the space $L^2_{2D,0}(m')$, satisfies

\[ \text{Re}(\lambda) \leq -1. \]

(A.24)

**Proof.** To prove this lemma, we first note that any eigenfunction of $L_3$ decays as a Gaussian as $|\xi| \to \infty$. This is proven in a fashion almost identical to the analogous result in [20]. More precisely, if we write the eigenfunction $\psi$ in polar coordinates we find that $\psi = \Psi(r) e^{i\theta}$ where $\Psi$ satisfies the ODE

\[ \Psi''(r) + \left( \frac{r}{2} + \frac{1}{r} \right) \Psi'(r) + (1 - \lambda - \frac{\ell^2}{r^2} - i\ell \eta(r)) \Psi(r) = 0, \]

(A.25)

where $\eta(r) = \frac{1}{2\pi \gamma}(1 - e^{-r^{2}/\gamma})$. This is almost identical to equation (70) of [20] (in fact, significantly simpler than that case, because the nonlocal term denoted $\Omega(r)$ in that reference is absent) and using the same sort of ODE estimates applied there one finds that there exists $\gamma > 0$ such that

\[ |\Psi(r)| \leq C (1 + r^2)^{\gamma} e^{-r^{2}/\gamma}. \]

(A.26)
To localize the eigenvalues of $L_\sigma$ we conjugate the two pieces of the operator with the Gaussian $\varphi_0^{-1/2}$. Alternatively, we study the operator on the weighted Hilbert space

$$X_0 = \{ f \in L^2(\mathbb{R}^2) : \varphi_0^{-1/2}f \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} f(\xi) d\xi = 0 \}. \quad (A.27)$$

From lemma 4.7 of [20], we know that $L - 1/2$ is a self-adjoint operator on this space with pure point spectrum $-1, -3/2, -2, \ldots$. A direct calculation shows that the operator $f \mapsto u_0^\sigma \cdot \nabla f$ is anti-symmetric in this space. Explicitly, we have

$$\langle \tilde{f}, u_0^\sigma \cdot \nabla f \rangle_X := \int_{\mathbb{R}^2} \varphi_0^{-1} \tilde{f} \cdot (u_0^\sigma \cdot \nabla f) d\xi = -\int_{\mathbb{R}^2} \varphi_0^{-1} f \cdot (u_0^\sigma \cdot \nabla \tilde{f}) d\xi \quad (A.28)$$

where we have used the fact that $\text{div} (\varphi_0^{-1} u_0^\sigma) = 0$. But now we use the fact that if an anti-symmetric operator is added to a self-adjoint operator with pure point spectrum, the real part of the spectrum of the resulting operator is less than the largest eigenvalue of the self-adjoint part—in this case $-1$ (see the proof of proposition 4.1 in [20]).

A.1.2. Evolution generated by $L + \Lambda$. We now can find and estimate solutions of the full linear equation (A.9), which being non-autonomous, generates an evolutionary family of operators $\mathcal{S}(\tau, \sigma)$ which satisfy the following temporal estimates for $q \in [1, 2]$, $|\alpha| \leq 1$, $m > 3$, $\mu \in (0, 1/2)$, $f \in Y$, and $b^m f \in L^3(\mathbb{R}^2)$,

$$\|e_t^\lambda (\partial^\alpha \mathcal{S}(\tau, \sigma) f)\|_m \leq \frac{e^{-(1/2+\mu)(\tau - \sigma)}}{a(\tau - \sigma)^{1/2q} + |\alpha|/2} \|b^m f\|_L,$$

$$\|e_t^\lambda (\partial^\alpha \mathcal{S}(\tau, \sigma) f)\|_m \leq \frac{e^{-(\tau - \sigma)/2}}{a(\tau - \sigma)^{1/2q} + |\alpha|/2} \|b^m e_t^{\lambda^3} f\|_L^q.$$ 

Proof. First, the existence and uniqueness of $\mathcal{S}(\tau, \sigma)$ can be obtained using a fixed point argument with the mapping

$$\mathcal{W}(\tau) \mapsto \mathcal{S}_0(\tau - \sigma) f + \int_{\tau}^{\sigma} \mathcal{S}_0(\tau - s) \Lambda \mathcal{W}(s) ds,$$

where we have set $\mathcal{W}(\sigma) = f$. To obtain the temporal estimates we use a Gronwall type argument which uses the fact that the coefficients of $\Lambda$, decay with rate $e^{-\tau/2}$. Due to the uniqueness of the above fixed point, $\mathcal{S}(\tau, \sigma)$ can also be represented by the following fixed point formula

$$\mathcal{W}(\tau) := \mathcal{S}(\tau, \sigma) f = \mathcal{T}_1(\tau - \sigma) f + \int_{\tau}^{\sigma} \mathcal{T}_1(\tau - s) \Lambda \mathcal{W}(s) ds. \quad (A.29)$$
We then wish to show that
\[
\Psi(\tau, \sigma) = e^{(r-\sigma)/2} \left[ e^{\mu(r-\sigma)} \left( \left\|e_3^T W(\tau)\right\|_m + a(\tau - \sigma)^{1/2} \left\|\nabla(e_3^T W(\tau))\right\|_m \right) \right. \\
\left. + \left\|W_\tau(\sigma)\right\|_m + a(\tau - \sigma)^{1/2} \left\|\nabla(W_\tau(\sigma))\right\|_m \right] \\
= e^{(r-\sigma)/2} \left[ e^{\mu(r-\sigma)} \left( \left\|W_\tau(\sigma)\right\|_m + \left\|W_\sigma(\tau)\right\|_m + a(\tau - \sigma)^{1/2} \left( \left\|\nabla W_\tau(\sigma)\right\|_m + \left\|\nabla W_\sigma(\tau)\right\|_m \right) \right) \right. \\
\left. + \left\|W_\tau(\sigma)\right\|_m + a(\tau - \sigma)^{1/2} \left\|\nabla(W_\tau(\sigma))\right\|_m \right]
\tag{A.30}
\]
is bounded for all $\tau \geq \sigma$. Here we weaken the temporal decay rate on the $W_\tau$ and $W_\sigma$ components in order to be able to close the Gronwall argument for $\Psi$. We use proposition A.3 to obtain
\[
\left\|W_\tau(\sigma)\right\|_m \leq C e^{-(1/2+\mu)(\tau-\sigma)} e^{\tau f_m} \left[ e^{\mu\tau f_m} \left( \left\|W_{\text{app},h}(\sigma)\right\|_m + \left\|b^\mu \Phi_{\text{app},h}(\sigma)\right\|_m \right) \left\|\nabla W_{\text{app},h}(\sigma)\right\|_{L^2} \right. \\
\left. + \left\|W_{\text{app},h}\right\|_{L^2} \left( \left\|\nabla W_{\text{app},h}(\sigma)\right\|_m + \left\|\Phi_{\text{app},h}(\sigma)\right\|_m \right) \right] \int_\sigma^\tau e^{-(\tau-s)} e^{-\tau/2} \left\|\Psi_s(\sigma)\right\|_{L^{1/2}} ds \leq C e^{-(1/2+\mu)(\tau-\sigma)} e^{\tau f_m} + C_B \int_\tau^\tau e^{-(\tau-s)} e^{-\tau/2} \left\|\Psi_s(\sigma)\right\|_{L^{1/2}} ds \leq C e^{-(1/2+\mu)(\tau-\sigma)} e^{\tau f_m} + C_B \int_\tau^\tau e^{-(\tau-s)} e^{-\tau/2} \left\|\Psi_s(\sigma)\right\|_{L^{1/2}}, \tag{A.31}
\]
Note in the second line we used the estimate $\left\|\nabla W_{\text{app},h}\right\|_{L^2} \leq C \left\|W_{\text{app},h}\right\|_{L^2} \Psi_1$ which holds for all $q \in [1, \infty]$ (see [45, proposition 4.1.2]) as well as the Biot–Savart estimate (3.22) to obtain $\left\|W_{\text{app},h}\right\|_{L^2} \leq C \left\|W_{\text{app},h}\right\|_{L^2} \Psi_1$. Also, $C_B > 0$ is a constant dependent on the amplitudes $B_i$ for $i = 1, 2$. We can then conclude
\[
e^{(1/2+\mu)(\tau-\sigma)} \left\|W_\tau(\sigma)\right\|_m \leq C e^{\tau f_m} + C_B \int_\tau^\tau e^{(\mu-1/2)\Psi_1} \Psi(s, \sigma) ds \tag{A.32}
\]
Using the fact that the third component $W_3$ is unaffected by $\Delta$, we similarly obtain
\[
e^{(r-\sigma)/2} \left\|W_\tau(\sigma)\right\|_m \leq C e^{\tau f_m},
\]
e as well as the gradient estimates
\[
a(\tau - \sigma)^{1/2} e^{(1/2+\mu)(\tau-\sigma)} \left\|\nabla(W_\tau(\sigma))\right\|_m \leq C \left\|e_3^T W(\tau)\right\|_m + C_B \int_\tau^\tau e^{(\mu-1/2)\Psi(s, \sigma)}\frac{a(\tau - \sigma)^{1/2}}{a(\tau - s)^{1/2}} ds,
\]
\[
a(\tau - \sigma)^{1/2} e^{(r-\sigma)/2} \left\|\nabla(W_\tau(\sigma))\right\|_m \leq C \left\|e_3^T W(\tau)\right\|_m + C_B \int_\tau^\tau e^{(\mu-1/2)\Psi(s, \sigma)}\frac{a(\tau - \sigma)^{1/2}}{a(\tau - s)^{1/2}} ds,
\]
\[
a(\tau - \sigma)^{1/2} e^{(r-\sigma)/2} \left\|\nabla(W_\tau(\sigma))\right\|_m \leq C \left\|e_3^T W(\tau)\right\|_m + C_B \int_\tau^\tau e^{(\mu-1/2)\Psi(s, \sigma)}\frac{a(\tau - \sigma)^{1/2}}{a(\tau - s)^{1/2}} ds.
\]

Combining these together we obtain the inequality
\[ \Psi(\tau, \sigma) \leq C \|f\|_m + C_{A,B} \int_{\sigma}^{\tau} \frac{e^{(\mu-1/2)s}a(\tau - s)^{1/2}a(\tau - \sigma)^{1/2}}{a(\tau - s)^{1/2}} \Psi(s, \sigma) \, ds. \] (A.33)

Then, applying Gronwall’s inequality, we obtain that
\[ \Psi(\tau, \sigma) \leq C_{A,B} \|f\|_m \exp \left( \int_{\sigma}^{\tau} \frac{e^{(\mu-1/2)s}a(\tau - s)^{1/2}}{a(\tau - s)^{1/2}} \, ds \right), \]
\[ \leq C_{A,B} e^{(1/2-\mu)\|f\|_m}, \] (A.34)
which since \( \mu < 1/2 \) implies that \( \Psi(\tau, \sigma) \) is bounded for all \( \tau \geq \sigma \). We thus obtain the temporal estimates for \( q = 2 \) on each component of \( W(\tau) \) by unraveling the definition of \( \Psi \) and taking note of the invariance properties of \( L + \Lambda \). A similar argument then gives the desired \( L^q \) estimates.

**A.1.3. Evolution generated by \( L + \bar{\Lambda} \).** We now translate the estimates of proposition A.10 back into the stationary frame in order to conclude estimates on the linear flow for \( L(\tau) + \bar{\Lambda} \).

**Proposition A.11.** Let \( m > 3, \alpha \in \mathbb{N}^2 \) with \( |\alpha| \leq 1 \), and \( W_0 \in Y \), then the operator \( L(\tau) + \bar{\Lambda} \) generates a evolutionary family of operators, \( \bar{S}(\tau, \sigma) \) for \( \tau \geq \sigma \geq 0 \) which satisfies the following decay estimates
\[ \|e^\xi \left( \partial^\alpha \bar{S}(\tau, \sigma)W_0 \right)\|_m \leq \frac{e^{-(1/2+\mu)(\tau-\sigma)}}{a(\tau - \sigma)^{1/2+|\alpha|/2}} \|b^\alpha W_0\|_{L^t}, \]
\[ \|e^\xi \left( \partial^\alpha \bar{S}(\tau, \sigma)W_0 \right)\|_m \leq \frac{e^{-(\tau-\sigma)\|b^\alpha W_0\|_{L^t}}}{a(\tau - \sigma)^{1/2+|\alpha|/2}} \|b^\alpha W_0\|_{L^t}. \]

**Proof.** We set \( W(\xi, \sigma) = W_0(\xi) \) as the initial condition of the linear evolution \( \bar{W}(\xi, \tau) = \bar{S}(\tau, \sigma)W(\xi, \sigma) \) which solves \( \bar{W}_\tau = (L(\tau) + \bar{\Lambda}) \bar{W} \). In order to use proposition A.10, we translate to the corresponding evolution in the rotating frame. We first set \( \bar{w}_0(\xi, \sigma) = \bar{W}_0(\xi, \sigma) \). Since it is untouched by the rotating frame transformation. For the other components, by translating back to unscaled variables, we set
\[ \left( \bar{w}_0, \bar{\Phi}(\xi, \sigma) \right) = (1 + s)(\bar{w}_0, \bar{\Theta})(x,s) \]
\[ = (1 + s)e^{-\Gamma J_2}(\bar{w}_0, \bar{\Theta})(x,s) \]
\[ = e^{-(\tau-\sigma)\|b^\alpha W_0\|_{L^t}}(\bar{w}_0, \bar{\Phi})(\xi, \sigma). \]
with \( J_2 \) as in (A.1). We then let \( \bar{W}(\tau) := \bar{S}(\tau, \sigma)W(\sigma) \), be the solution of (A.9) with initial data \( \bar{W}(\sigma) \) as defined above. Note \( \bar{W}_3(\tau) := \bar{w}_3(\tau) \) as the third vorticity component is not affected by the rotating frame. Next, defining \( R(\tau) = e^{(\tau-\sigma)\|b^\alpha W_0\|_{L^t}} \), we have
$$\mathbf{\bar{w}}(\tau) = \begin{bmatrix} R(\tau) \\ \Phi(\tau) \end{bmatrix}_{1,2}$$
$$= c(\tau)\mathbf{w}_0(\tau) - s(\tau)\Phi(\tau)$$
$$\mathbf{\bar{\Phi}}(\tau) = \begin{bmatrix} R(\tau) \\ \Phi(\tau) \end{bmatrix}_{3,4}$$
$$= s(\tau)\mathbf{w}_0(\tau) + c(\tau)\Phi(\tau)$$

where the subscripts at the end of the first and third lines denote the component of the vector and we recall that $c(\tau) := \cos(\Gamma(e^\tau - 1))$, $s(\tau) := \sin(\Gamma(e^\tau - 1))$. Next we use the estimates in the rotating frame to estimate different components of $\mathbf{\bar{w}}$. Using the estimates of proposition A.10 we find

$$\|\mathbf{\bar{w}}(\tau)\|_m \leq C\|\mathbf{w}_0(\tau)\|_m + |s(\tau)|\|\Phi(\tau)\|_m,$$
$$\leq Ce^{-(1/2+\mu)(\tau-\sigma)}\|\mathbf{W}(\sigma)\|_m$$
$$\leq Ce^{-(1/2+\mu)(\tau-\sigma)}\|\mathbf{W}_0\|_m \quad \text{(A.35)}$$

$$\|\Phi(\tau)\|_m \leq C\left(\|\mathbf{w}_0(\tau)\|_m + \|\Phi(\tau)\|_m\right),$$
$$\leq Ce^{-(\mu+1/2)(\tau-\sigma)}\|\mathbf{W}_0\|_m \quad \text{(A.36)}$$

since the rotational transformation does not affect the norm of $\mathbf{W}_0$. The estimates for the gradients, as well as $L^q$ data, follow in a similar way.

A.2. Baroclinic evolution

A.2.1. Unperturbed evolution of $\mathbf{L}(\tau)$. First we estimate the unscaled velocity $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{\theta})^T$ under the linear evolution, $\tilde{\mathbf{v}} := \mathbf{M}\tilde{\mathbf{v}} := \Delta\tilde{\mathbf{v}} + \mathbf{P}[\Omega, T_\tau]\tilde{\mathbf{v}}$, in algebraically weighted spaces. One readily finds that this equation generates a $C^0$-semigroup which we denote by $e^{\mathbf{M}t}$. The approach used in [45, proposition 4.4.2] can be used to readily find

**Proposition A.12.** Let $m \geq 0$, $\alpha \in \mathbb{N}^3$, $q \in [1,2]$, and $f \in (1-Q)L^2(m)^4$. If also $b^m f \in L^q(\mathbb{R})$ then

$$\|\partial^\alpha e^{\mathbf{M}(t-s)}f\|_m \leq Ce^{-4\pi^2(t-s)}\left(\frac{1}{t-s}\right)^{1+|\alpha|/2} \|b^m f\|_{L^q(\mathbb{R})} \quad \text{(A.37)}$$

To transfer these results to the vorticity formulation, we need to show that the unscaled Biot–Savart law, which maps $\tilde{\omega}$ to $\partial_j\tilde{\mu}$ for $j = 1, 2, 3$ is a bounded operator on the $b^m$-weighted $L^q$ spaces. We view this operator as a matrix Fourier multiplier and aim to apply the theory of weighted $L^p$ multipliers developed over the past several decades; see [21] for a review. In particular, a weight function $\rho \geq 0$ with $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be Muckenaupt class $A_4(\mathbb{R}^n)$ for some $q \in (1,\infty)$ if for all bounded cubes $Q$ with faces parallel to coordinate axes,

$$\sup_{Q} \left(\frac{1}{|Q|} \int_Q \rho(x)^q \, dx\right) \left(\frac{1}{|Q|} \int_Q \rho(x)^{-1/(q-1)} \, dx\right)^{q-1} < \infty \quad \text{(A.38)}$$

where $|Q|$ denotes the Lebesgue measure of $Q$. For our weight $b^m$ considered in $\mathbb{R}^2$ we first have
Lemma A.13. The weight \( b^m := (1 + |x_k|^2)^{m/2} \) is of class \( A_q(\mathbb{R}^2) \) for all \( q > 1 \) and \( 0 \leq m \leq 2(q - 1) \).

Proof. See [16, lemma 2.3 (v)].

Following [47] we can then define a Muckenhaus class of weights, \( A_q(\mathbb{D}) \), for our spatial domain \( \mathbb{D} = \mathbb{R}^2 \times T \) in the same way as above; see [47, p 338]. From this we then can conclude the following lemma:

Lemma A.14. For all \( q > 1 \) and \( 0 \leq m < 2(q - 1) \), the weight \( b^m := (1 + |x_k|^2)^{m/2} \) is of class \( A_q(\mathbb{D}) \).

Proof. Let \( Q = I_1 \times I_2 \times I_3 \), for some bounded intervals \( I_1, I_2, I_3 \subset \mathbb{R}, I_1 \subset T \). Then we use the fact that \( b^m \) does not depend on \( x_3 \) and lemma A.13 to obtain

\[
\text{sup}_{Q} \left( \frac{1}{|Q|} \int_Q b^m \, dx \right) \left( \frac{1}{|Q|} \int_Q b^{-m/(q-1)} \, dx \right)^{q-1} = \text{sup}_{Q} \left( \frac{1}{|I_1 \times I_2|} \int_{I_1 \times I_2} b^m \, dx \right) \left( \frac{1}{|I_1 \times I_2|} \int_{I_1 \times I_2} b^{-m/(q-1)} \, dx \right)^{q-1} \leq C
\]

(A.39)

for some positive constant \( C > 0 \).

For this class of weights, the results of [47] can be used to show that \( \partial \tilde{\mu} \), for \( j = 1, 2, 3 \), are bounded operators in the \( b^m \) weighted \( L^q \) space. Namely we obtain the following:

Lemma A.15. Let \( Q \in (1 - Q)|L^2(m) \) satisfy \( b^m \tilde{w} \in L^q(\mathbb{D}) \) for some \( m \geq 0 \) and \( q \in [3/2, \infty) \). Also let \( \tilde{u} \) be determined by the Biot–Savart law given in (2.9) above. Then there exists a \( C > 0 \) such that for \( j = 1, 2, 3 \),

\[
\|b^m \partial \tilde{u}\|_{L^q(\mathbb{D})} \leq C\|b^m \tilde{w}\|_{L^q(\mathbb{D})}.
\]

(A.40)

Proof. First we find observe that \( \partial \tilde{\mu} \), when defined on \( \mathbb{R}^3 \), is obtained from \( \tilde{\omega} \) via a Fourier matrix multiplier \( B_j(k) \) with components of the form \( \frac{k_j}{k} \), \( k \in \mathbb{R}^3 \). This implies that \( B_j(k) \) is bounded for \( k \in \mathbb{R}^3 \), is smooth away from the origin, and satisfies the Mikhlin condition \( |k|^{3/2} |\partial_k^3 B_j(k)| \leq C \) for some \( C > 0 \). Thus \( B_j(k) \) defines a bounded operator on \( L^q(\mathbb{R}^3) \).

Then, following [47, proposition 6], we take a smooth bump function \( \chi(k) \), with \( \chi(0) = 1 \) and \( \text{supp} \chi \subset (-1/2, 1/2) \), and define a smoothed operator \( \bar{B}_j(k) := (1 - \chi(k)) B_j(k) \). Observe that \( \bar{B}_j(k) \equiv B_j(k) \) for \( k \in \mathbb{R}^2 \times \{0\} \). Then, since \( \mathbb{D} \) is a locally compact abelian group (so that the Fourier transform can be defined on it), lemma A.14 and the results of [47, proposition 4, remark 5] (see also [1]) can be used to show that \( B_j(k) \), defined on \( k \in \mathbb{R}^2 \times \{0\} \), defines a bounded operator on the weighted space \( L^q(\mathbb{D}) := \{ f \in L^q(\mathbb{D}) \mid \|bf\|_{L^q(\mathbb{D})} < \infty \} \) for all \( 0 < \ell < 2(q - 1) \). We note that the derivatives \( \partial_k^m \bar{B}_j(k) \), being smooth away from the origin, also define bounded operators on \( L^q(\mathbb{D}) \).

Next, letting \( \beta \in \mathbb{N}^2 \) denote a multi-index and \( m_1 \in \mathbb{N} \), it can readily be found that there exists constants \( C, C' > 0 \) such that
Denoting $\mathcal{F}$ as the Fourier transform on $\mathbb{D}$, $m_1 = \lfloor m \rfloor$ the greatest integer below $m$ we then have
\[
\|b^m \partial_\beta \tilde{u}\|_{L^q(\mathbb{D})} = \|b^m \partial_\beta \tilde{u}\|_{L^q(m - m_1)} \\
\leq C \sum_{|\beta| \leq m_1} \|x_\beta^m \partial_\beta \tilde{u}\|_{L^q(m - m_1)} \\
\leq C \sum_{|\beta| \leq m_1} \left\|\mathcal{F}^{-1} \left[ \partial_\beta^a (B_j \tilde{\omega}(k)) \right] \right\|_{L^q(m - m_1)} \\
\leq C \sum_{|\beta| \leq m_1} \left\|\mathcal{F}^{-1} \left[ \tilde{\omega}(k) \right] \right\|_{L^q(m - m_1)} \\
= C \sum_{|\beta| \leq m_1} \left\|x_\beta^m \tilde{\omega}\right\|_{L^q(m - m_1)} \\
\leq C \|b^m \partial_\beta \tilde{\omega}\|_{L^q(m - m_1)} = C \|b^m \tilde{\omega}\|_{L^q(\mathbb{D})}.
\]

Here we have used equivalencies (A.41) in the second and last lines and the $L^q(m - m_1)$ boundedness of the multipliers $\partial_\beta^a B_j$, for all $m - m_1 < 1 \leq 2(q - 1)$, in the fourth line.

Next let $\tilde{W}(\tau)$ be the solution of the linear equation (3.31) in vorticity formulation. It can readily be found that this formulation also defines an evolutionary family of operators $\tilde{S}_0(\tau, \sigma)$ on $L^2(m)^3 \times H^1(m)$. Also let $\tilde{U} := (\tilde{u}, \tilde{\phi})^T$ be its corresponding velocity profile. We can then obtain

**Proposition A.16.** Let $m \geq 0$, $\alpha \in \mathbb{N}^3$, $q \in [\frac{4}{3}, 2]$, and $f \in (1 - Q)L^2(m)^3 \times H^1(m)$. If also $b^m f \in L^q(\mathbb{D})$ then, for any $\delta \in (0, 4\pi^2)$,
\[
\|\partial^\delta \tilde{S}_0(\tau, \sigma) f\|_{*, m} \leq C e^{-(4\pi^2 - \delta)(e^\tau - e^\sigma)} \frac{\alpha^{1/2} \pi^{1/2}}{\alpha^{1/2} \pi^{1/2}} \left( \|b^m e^f\|_{L^q(\mathbb{D})} + \|b^m \nabla e^f\|_{L^q(\mathbb{D})} \right).
\]

**Proof.** Denote $\tilde{W}(\tau) \equiv \tilde{S}_0(\tau, \sigma) f$ and set $\tilde{W}(\sigma) = f$. Recall the norm has the form $\|\tilde{W}(\tau)\|^{2}_{*, m} = \|\tilde{u}\|^{2}_{m} + \|\tilde{\phi}\|^{2}_{H^1(m)}$. Furthermore, it is readily seen that
\[
\|\tilde{W}(\tau)\|_{*, m} \leq C \left( \|\nabla \tilde{u}\|_{m} + \|\tilde{\phi}\|_{H^1(m)} \right) \\
\leq C \left( \|\nabla \tilde{u}\|_{m} + \|\partial_\beta \tilde{u}\|_{m} + \|\nabla \tilde{\phi}\|_{L^q(m)} + \|\partial_\beta \tilde{\phi}\|_{L^q(m)} + \|\tilde{\phi}\|_{L^q(m)} \right).
\]

We shall then estimate each of these pieces by relating them to their unscaled components which we can estimate using proposition A.12. We find, setting $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{\phi})^T$ and using the fact that $\nabla$ and $M$ commute, the estimate
\[ ||\nabla_x \tilde{U}(\tau)||_m^2 = \int_D \left( 1 + \left| \frac{x_0}{\sqrt{1 + t}} \right|^2 \right)^{m/2} \left( 1 + t \right)^{1/2} \nabla \tilde{U} \left( \frac{x_0}{\sqrt{1 + t}}, x_3, \log(1 + t) \right) \left| \frac{x_0}{\sqrt{1 + t}} \right|^2 \frac{dx}{1 + t} \]
\[ = \int_D \left( 1 + \left| \frac{x_0}{\sqrt{1 + t}} \right|^2 \right)^{m/2} \left( 1 + t \right) \nabla \tilde{v}(x, t) \left| \frac{x_0}{\sqrt{1 + t}} \right|^2 \frac{dx}{1 + t} \]
\[ = \left( 1 + t \right) \int_D \left( 1 + \left| \frac{x_0}{\sqrt{1 + t}} \right|^2 \right)^{m/2} \left( 1 + t \right) e^{(t-s)M} \nabla \tilde{v}(x, s) \frac{dx}{1 + t} \]
\[ \leq \frac{C e^{-8\sigma^2(t-s)\delta \rho/2}}{(t-s)^{2(1/q-1/2)}a(t-s)^{(3/4-1/2)}} \left( \int_D \left( 1 + \left| \frac{x}{\sqrt{1 + s}} \right|^2 \right)^{m/2} \nabla \tilde{v}(x, s) \frac{dx}{1 + s} \right)^{2/q} \]
\[ \leq \frac{C e^{-8\sigma^2(t-s)\delta \rho/2}}{(t-s)^{2(1/q-1/2)}a(t-s)^{(3/4-1/2)}} \left( \int_D \left( 1 + \left| \frac{x}{\sqrt{1 + s}} \right|^2 \right)^{m/2} \tilde{\omega}(x, s) \frac{dx}{1 + s} \right)^{2/q} \]
\[ + \left( \int_D \left( 1 + \left| \frac{x}{\sqrt{1 + s}} \right|^2 \right)^{m/2} \nabla \tilde{\theta}(x, s) \frac{dx}{1 + s} \right)^{2/q} \]

where \( \nabla_\xi = (\partial_{\xi_1}, \partial_{\xi_2})^T, \nabla \tilde{\nu} = (\partial_{\tilde{\nu}_1}, \partial_{\tilde{\nu}_2})^T, \) and the last inequality was obtained using lemma A.15. Changing back to scaled variables we then obtain for some small \( \delta > 0, \)
\[ ||\nabla_x \tilde{U}(\tau)||_m^2 \leq C e^{-8\sigma^2(t-s)\delta \rho/2} e^{-\delta \rho/2} e^{2(1/2-1/q)(t-s)} \left( ||b^\alpha \tilde{w}||_{L^2(D)}^2 + ||b^\alpha \nabla \tilde{\nu}||_{L^2(D)}^2 \right) \]
\[ \leq C e^{-8\sigma^2(t-s)\delta \rho/2} e^{-\delta \rho/2} e^{2(1/2-1/q)(t-s)} \left( ||b^\alpha \tilde{\omega}||_{L^2(D)}^2 + ||b^\alpha \nabla \tilde{\omega}||_{L^2(D)}^2 \right). \]  

(A.44)

The estimates for \( ||\partial_j \tilde{U}(\tau)||_m, ||\partial^\alpha \tilde{\theta}(\tau)||_m \) and \( ||\partial^\alpha \tilde{\theta}(\tau)||_m \) follow in the same way.

A.2.2. Perturbed evolution of \( \tilde{L}(\tau) + \tilde{\Lambda} \). We now wish to prove proposition 3.6 which estimates the decay of the linear baroclinic evolution perturbed by the barotropic approximate solution. To begin we construct the evolutionary family of operators, which we shall denote as \( \tilde{S}(\tau, \sigma) \), generated by \( \tilde{L}(\tau) + \tilde{\Lambda} \). This can be done by studying the integral formulation,
\[ \tilde{W}(\tau) = \tilde{S}_{0}(\tau, \sigma) \tilde{W}_0 + \int_0^\tau \tilde{S}_0(\tau, \sigma) (1 - Q) \left[ \tilde{\pi}_{app, 3} \cdot \nabla \tilde{W} + \tilde{u}_h \cdot \nabla \tilde{W}_{app} - \tilde{w}_h \cdot \nabla \tilde{K} \left( \tilde{\pi}_{app, 3} \cdot \tilde{u} \right) \right] d\tau \]
\[ + \int_0^\tau e^{\sqrt{2}S_0(\tau, \sigma) (1 - Q)} \left[ \tilde{\pi}_{app, 3} \cdot \partial_3 \tilde{W} - \tilde{w}_{app, 3} \cdot \partial_3 \tilde{u} \right] d\sigma. \]  

(A.45)

Existence and uniqueness can then be obtained via a fixed-point argument similar to ones in previous sections. To obtain the temporal estimates we use another Gronwall type argument. That is we show that the following quantity is bounded uniformly for all \( \tau \geq \sigma \geq 0 \). For ease of notation we set \( c = 4\pi^2 - \delta, \tilde{c} = c - \delta' \) with \( \delta' > 0 \) to be chosen so that \( \tilde{c} > 0, \)
\[ \tilde{\Psi}(\tau, \sigma) = e^{(e^\tau - e^\sigma)} \left( \| \tilde{W}(\tau) \|_{s,m} + a(\tau - \sigma)^{1/2} \| \nabla_b \tilde{W}(\tau) \|_{s,m} + a(e^\tau - e^\sigma)^{1/2} \| \partial_3 \tilde{W}(\tau) \|_{s,m} \right). \]  

(A.46)

Using estimates similar to those used in the proof of theorem 3, as well as the fact that \( \| \nabla \tilde{\phi} \|_{m} \leq C \| \tilde{\Phi} \|_{m} \), we then estimate with \( q = 3/2 \),

\[
e^{(e^\tau - e^\sigma)} \| \tilde{W}(\tau) \|_{s,m} \leq C e^{-\delta (e^\tau - e^\sigma)} \| \tilde{W}_0 \|_{s,m} + C_{A,B} \int_0^\tau e^{(e^\tau - e^\sigma)} a(\tau - s)^{1/2} d(e^\tau - e^\sigma)^{1/2} \left( \| \nabla_b \tilde{W}(\tau) \|_{s,m} + \| \tilde{W}(\tau) \|_{s,m} \right) ds
\]

(A.47)

where \( C_{A,B} \) is a constant dependent on the approximate solution amplitudes \( A, B_1, B_2 \) and

\[
g(\tau, s, \sigma) = \frac{1 + e^{\tau/2} + a(s - \sigma)^{-1/2} + a(e^\tau - e^\sigma)^{-1/2}}{a(\tau - s)^{1/2}m(e^\tau - e^\sigma)^{1/2}}.
\]

In a similar manner we find

\[
a(\tau - \sigma)^{1/2} e^{(e^\tau - e^\sigma)} \| \nabla_b \tilde{W}(\tau) \|_{s,m} \leq C e^{-\delta (e^\tau - e^\sigma)} \| \tilde{W}_0 \|_{s,m}
\]

\[
+ C_{A,B} \int_0^\tau e^{(e^\tau - e^\sigma)} a(\tau - s)^{1/2} \tilde{\Psi}(\tau, s, \sigma) \tilde{W}(s, \sigma) ds
\]

\[
a(e^\tau - e^\sigma)^{1/2} e^{(e^\tau - e^\sigma)} \| \partial_3 \tilde{W}(\tau) \|_{s,m} \leq C e^{-\delta (e^\tau - e^\sigma)} \| \tilde{W}_0 \|_{s,m}
\]

\[
+ C_{A,B} \int_0^\tau e^{(e^\tau - e^\sigma)} a(e^\tau - e^\sigma)^{1/2} d(e^\tau - e^\sigma)^{1/2} g(\tau, s, \sigma) \tilde{\Psi}(\tau, s, \sigma) ds.
\]  

(A.48)

Combining these all together we obtain

\[
\tilde{\Psi}(\tau, \sigma) \leq C e^{-\delta (e^\tau - e^\sigma)} \| \tilde{W}_0 \|_{s,m} + C_{A,B} \int_0^\tau e^{-\delta (e^\tau - e^\sigma)} \tilde{g}(\tau, s, \sigma) \tilde{\Psi}(\tau, s, \sigma) ds,
\]

(A.49)

with

\[
\tilde{g}(\tau, s, \sigma) = g(\tau, s, \sigma) \left( 1 + \frac{a(\tau - s)^{1/2} + a(e^\tau - e^\sigma)^{1/2}}{a(e^\tau - e^\sigma)^{1/2}} \right).
\]

Then applying Gronwall’s inequality we obtain

\[
\tilde{\Psi}(\tau, \sigma) \leq C_{A,B} \| \tilde{W}_0 \|_{s,m} \exp \left[ \int_0^\tau e^{-\delta (e^\tau - e^\sigma)} \tilde{g}(\tau, s, \sigma) ds \right].
\]

(A.50)

where \( \tilde{C} > 0 \) is a constant dependent on \( \delta' \) and \( q \). The integral inside the exponential can be bounded, uniformly in \( \tau \geq \sigma \), using the argument of [45, proposition 4.5.2] and using the fact that the singularities at \( s = \tau \) have the form \( a(\tau - s)^{1/2} \) or \( a(e^\tau - e^\sigma)^{1/2} \) and those at \( s = \sigma \) take the form \( a(s - \sigma)^{1/2} \) or \( a(e^\tau - e^\sigma)^{1/2} \).
Appendix B. The form of the equations with stress-free boundary conditions

In this appendix we give the details of the form of the equations in the case of stress-free boundary conditions, analogous to those for periodic boundary conditions derived in section 2.1. This formulation to prove theorem 2 in sections 4.1–4.3.

In the stress-free case, the boundary conditions imply that \( \hat{u}_3 = \hat{\theta} = 0 \), so
\[
\mathbf{v} = (\hat{u}_1, \hat{u}_2, 0, 0)^T + (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{\theta})^T, \quad \hat{u}_h = (u_1, u_2)^T,
\]
and
\[
\tilde{u}_h(x_h) = Qu_h = \int_0^1 u_h(x_h, x_3) \, dx_3, \quad \tilde{u}_h = (1 - Q)u_h.
\]
Hence \( \tilde{u}_h \) is \( x_3 \)-independent, \( \tilde{u}_h \) has zero vertical mean, and \( \tilde{u}_3 \) and \( \tilde{\theta} \) satisfy Dirichlet boundary conditions. Also note that the incompressibility condition implies that \( \text{div} \, \tilde{u} = 0 \) and \( \text{div} \, \tilde{v} \tilde{u}_h = 0 \). Furthermore, the corresponding vorticity \( \omega = \text{curl} \, u \), has the decomposition \( \omega = (0, 0, \omega_3)^T + (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)^T \) with \( \tilde{\omega}_3 \) mean-zero in \( z \), \( \tilde{\omega}_i = \text{curl} \, \tilde{u}_h \), and \( \tilde{\omega}_i \) satisfying Dirichlet boundary conditions in \( x_3 \) for \( i = 1, 2 \).

With this decomposition, the \( x_3 \)-dependent terms have the following Fourier decomposition,

\[
\tilde{u}_i(x) = \int_{\mathbb{R}^2} \sum_{n=1}^{\infty} \hat{\tilde{u}}_i(k)e^{i k \cdot x_3} \cos(n\pi x_3) \, dk, \quad i = 1, 2
\]

(B.1)

\[
\tilde{u}_3(x) = \int_{\mathbb{R}^2} \sum_{n=1}^{\infty} \hat{\tilde{u}}_3(k)e^{i k \cdot x_3} \sin(n\pi x_3) \, dk,
\]

(B.2)

\[
\tilde{\theta}(x) = \int_{\mathbb{R}^2} \sum_{n=1}^{\infty} \hat{\tilde{\theta}}(k)e^{i k \cdot x_3} \sin(n\pi x_3) \, dk,
\]

(B.3)

\[
\tilde{\omega}_i(x) = \int_{\mathbb{R}^2} \sum_{n=1}^{\infty} \hat{\tilde{\omega}}_i(k)e^{i k \cdot x_3} \sin(n\pi x_3) \, dk, \quad i = 1, 2
\]

(B.4)

\[
\tilde{\omega}_3(x) = \int_{\mathbb{R}^2} \sum_{n=1}^{\infty} \hat{\tilde{\omega}}_3(k)e^{i k \cdot x_3} \cos(n\pi x_3) \, dk,
\]

(B.5)

where the \( \hat{u}_i \) and \( \hat{\omega}_i \) terms can be obtained from the inverse Fourier transform. The Biot–Savart law for these boundary conditions can be found to be

\[
\hat{u}(k) = \frac{1}{|k_h|^2 + 4\pi^2 n^2} \begin{pmatrix} 0 & n\pi & ik_2 \\ -n\pi & 0 & -ik_1 \\ -ik_2 & ik_1 & 0 \end{pmatrix} \hat{\omega}(k),
\]

(B.6)

while the skew-Hermitian term \( \mathbb{P}J_\eta \mathbb{P} \) takes the form

\[
\mathbb{P}J_\eta \mathbb{P} = \frac{1}{|k|^2} \begin{pmatrix} 0 & (n\pi)^2 \eta & ik_2 n \pi \eta & ik_1 n \pi \\ -(n\pi)^2 \eta & 0 & -ik_1 n \pi \eta & ik_2 n \pi \\ ik_2 n \pi \eta & -ik_1 n \pi \eta & 0 & |k_h|^2 \\ ik_1 n \pi & ik_2 n \pi & -|k_h|^2 & 0 \end{pmatrix}.
\]

(B.7)
where \(|k|^2 := |k_h|^2 + \pi n\)^2 and \(|k_\eta|^2 := |k_h|^2 + (\pi n)^2\). The matrix defined above once again has a double eigenvalue at 0 and a pair of eigenvalues \(\pm i|k_\eta|/|k|\) with corresponding eigenvectors

\[
a_\pm(k) = \pm \frac{1}{|k_\eta|} \begin{pmatrix} \frac{i k_2}{|k_\eta|} & 0 \\ -\frac{i k_1}{|k_\eta|} & \pm \pi \end{pmatrix}, \quad a_0(k) = \frac{1}{|k|} \begin{pmatrix} i k_1 & i k_2 \\ -\pi & 0 \end{pmatrix},
\]

(B.8)

We can then decompose (2.5) into the following system

\[
\partial_t \tilde{u}_h = \Delta \tilde{u}_h - \nabla \cdot \{ \tilde{u}_h \cdot \nabla \tilde{u}_h + Q \tilde{u} \cdot \nabla \tilde{u}_h \}
\]

\[
\partial_t \tilde{\omega}_3 = \Delta \tilde{\omega}_3 - \tilde{u}_h \cdot \nabla \tilde{\omega}_3 - \tilde{N}(v),
\]

(B.9)

(B.10)

(B.11)

(B.12)

where

\[
\tilde{N}(v) = Q \left[ (\tilde{u} \cdot \nabla) \tilde{\omega}_3 - (\tilde{\omega} \cdot \nabla) \tilde{u}_h \right], \quad \tilde{N}(v) = P(1 - Q) \left[ (\tilde{u} \cdot \nabla) \tilde{v} + (\tilde{u}_h \cdot \nabla) \tilde{v} + (\tilde{u} \cdot \nabla) \tilde{v} \right]
\]

and \(\tilde{u}\) and \(\tilde{\omega}\) can be related via the Biot–Savart law (B.6).

References

[1] Andersen K and Mohanty P 2009 Restriction and extension of Fourier multipliers between weighted \(L^p\) spaces on \(\mathbb{R}^n\) and \(\mathbb{T}^n\) Proc. Am. Math. Soc. 137 1689–97
[2] Babin A, Mahalov A and Nicolaenko B 1998 On nonlinear baroclinic waves and adjustment of pancake dynamics Theor. Comput. Fluid Dyn. 11 215–35
[3] Babin A, Mahalov A and Nicolaenko B 1999 On the regularity of three-dimensional rotating Euler–Boussinesq equations Math. Models Methods Appl. Sci. 9 1089–121
[4] Babin A, Mahalov A and Nicolaenko B 2002 Fast singular oscillating limits of stably-stratified 3D Euler and Navier–Stokes equations and ageostrophic wave fronts Large-Scale Atmos.–Ocean Dyn. 1 126–201
[5] Balci N, Isenberg A M and Jolly M S 2018 Turbulence in vertically averaged convection Physica D: Nonlinear Phenomena 376–377 216–27
[6] Boffetta G and Ecke R E 2012 Two-dimensional turbulence Annu. Rev. Fluid Mech. 44 427–51
[7] Cao C and Titi E S 2007 Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics Ann. Math. 245–67
[8] Carlen E A and Loss M 1996 Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2d Navier–Stokes equation Duke Math. J. 81 135–57
Charney J 1948 On the scale of atmospheric motions Geofysiske Publikasjoner (Oslo: Grundahl & Sons Boktr., I Kommission Hos Cammermeyer)

Charve F 2005 Global well-posedness and asymptotics for the geophysical fluid system Fac. Sci. Toulouse Math. 6 221–38

Charve F 2006 Asymptotics and vortex patches for the quasigeostrophic approximation J. Math. Pures Appl. 85 493–539

Charve F and Roussier-Michon V 2009 Global existence and long-time asymptotics for rotating fluids in a 3D layer J. Math. Anal. Appl. 360 14–34

Charve F and Wayne C 2002 Invariant manifolds and the long-time asymptotics of the Navier–Stokes and vorticity equations on $\mathbb{R}^2$ Arch. Ration. Mech. Anal. 163 209–58

Charve F and Wayne C 2005 Global stability of vortex solutions of the two-dimensional Navier–Stokes equation Commun. Math. Phys. 255 97–129

Charney J 1948 On the scale of atmospheric motions Geofysiske Publikasjoner (Oslo: Grundahl & Sons Boktr., I Kommission Hos Cammermeyer)

Charve F 2005 Global well-posedness and asymptotics for the primitive system of the quasigeostrophic equations Asymptotic Anal. 42 173–209

Charve F 2006 Asymptotics and vortex patches for the quasigeostrophic approximation J. Math. Pures Appl. 85 493–539

Charve F and Wayne C 2008 Global well-posedness for the primitive equations with less regular initial data Ann. Fac. Sci. Toulouse Math. 6 221–38

Charve F et al 2011 Global existence for the primitive equations with small anisotropic viscosity Rev. Mat. Iberoamericana 27 1–38

Chemin J Y, Desjardins B, Gallagher I and Grenier E 2006 New formulations of the primitive equations of atmosphere–ocean dynamics: dynamic bifurcation and periodic solutions Discrete Continuous Dyn. Syst. 5 97–132

Chemin J Y, Desjardins B, Gallagher I and Grenier E 2006 On the equations of the large scale ocean Rep. Prog. Phys. 69 209–58

Chemin J Y, Desjardins B, Gallagher I and Grenier E 2006 On the equations of the large scale ocean Nonlinearity 5 1007

Chemin J Y, Desjardins B, Gallagher I and Grenier E 2006 On the equations of the large scale ocean Nonlinearity 5 1007

Charney J 1948 On the scale of atmospheric motions Geofysiske Publikasjoner (Oslo: Grundahl & Sons Boktr., I Kommission Hos Cammermeyer)

Charve F 2005 Global well-posedness and asymptotics for the primitive system of the quasigeostrophic equations Asymptotic Anal. 42 173–209

Charve F 2005 Global well-posedness and asymptotics for a geophysical fluid system Commun. PDE 29 1919–40

Charve F 2006 Asymptotics and vortex patches for the quasigeostrophic approximation J. Math. Pures Appl. 85 493–539

Charve F 2008 Global well-posedness for the primitive equations with less regular initial data Ann. Fac. Sci. Toulouse Math. 6 221–38

Charve F et al 2011 Global existence for the primitive equations with small anisotropic viscosity Rev. Mat. Iberoamericana 27 1–38

Chemin J Y, Desjardins B, Gallagher I and Grenier E 2006 Mathematical Geophysics: an Introduction to Rotating Fluids and the Navier–Stokes Equations vol 32 (Oxford: Oxford University Press)

Farwig R and Sohr H 1997 Weighted $L^q$-theory for the Stokes resolvent in exterior domains J. Math. Soc. Japan 49 251–88

Favier B, Silvers L and Proctor M 2014 Inverse cascade and symmetry breaking in rapidly rotating Boussinesq convection Phys. Fluids 26 096605

Gallay T and Roussier-Michon V 2009 Global existence and long-time asymptotics for rotating fluids in a 3D layer J. Math. Anal. Appl. 360 14–34

Gallay T and Wayne C E 2002 Invariant manifolds and the long-time asymptotics of the Navier–Stokes and vorticity equations on $\mathbb{R}^2$ Arch. Ration. Mech. Anal. 163 209–58

Gallay T and Wayne C E 2005 Global stability of vortex solutions of the two-dimensional Navier–Stokes equation Commun. Math. Phys. 255 97–129

García-Cuerva J and de Francia J L R 1985 Weighted Norm Inequalities and Related Topics vol 104 (Amsterdam: Elsevier)

Gill A E 1982 Atmosphere–Ocean Dynamics vol 30 (New York: Academic)

Greenleaf A 1981 Principal curvature and harmonic-analysis Indiana Univ. Math. J. 30 519–37

Guillén-González F et al 2001 Anisotropic estimates and strong solutions of the primitive equations Differ. Integral Equ.-Athens 14 1381–408

Herring J R and M'étaius O 1989 Numerical experiments in forced stably stratified turbulence J. Fluid Mech. 202 97–115

Holm D D and Long B 1989 Lyapunov stability of ideal stratified fluid equilibria in hydrostatic balance Nonlinearity 2 23

Hsia C H, Ma T and Wang S 2007 Stratified rotating Boussinesq equations in geophysical fluid dynamics: dynamic bifurcation and periodic solutions J. Math. Phys. 48 065602

Hu C, Temam R and Ziane M 2003 The primitive equations on the large scale ocean under the small depth hypothesis Discrete Continuous Dyn. Syst. 9 97–132

Iwabuchi T, Mahalov A and Takada R 2017 Global solutions for the incompressible rotating stably stratified fluids Math. Nachr. 290 613–31

Julien K, Rubio A, Grooms I and Knobloch E 2012 Statistical and physical balances in low Rossby number Rayleigh–Bénard convection Geophys. Astrophys. Fluid Dyn. 106 392–428

Knobloch E 2015 Spatial localization in dissipative systems Annu. Rev. Condens. Matter Phys. 6 325–59

Koba A, Mahalov H and Yoneda T 2012 Global well-posedness of the rotating Navier–Stokes–Boussinesq equations with stratification effects Adv. Math. Sci. Appl. 22 61–90

Kraichnan R H and Montgomery D 1980 Two-dimensional turbulence Rep. Prog. Phys. 43 547

Li J and Wang S 2008 Some mathematical and numerical issues in geophysical fluid dynamics and climate dynamics Commun. Comput. Phys. 3 759–93

Lions J L, Temam R and Wang S 1992 New formulations of the primitive equations of atmosphere and applications Nonlinearity 5 237

Lions J L, Temam R and Wang S 1992 On the equations of the large-scale ocean Nonlinearity 5 1007

Lorenz E N 1986 On the existence of a slow manifold J. Atmos. Sci. 43 1547–58

Lorenz E N and Krishnamurthy V 1987 On the nonexistence of a slow manifold J. Atmos. Sci. 44 2940–50
[39] Majda A 2003 *Introduction to PDEs and Waves for the Atmosphere and Ocean* vol 9 (Providence, RI: American Mathematical Society)

[40] Franzke C L E, Oliver M, Rademacher J D M and Badin G 2019 Multi-scale methods for geophysical flows *Energy Transfers in Atmosphere and Ocean* ed C Eden and A Iske (Berlin: Springer) accepted (https://doi.org/10.1007/978-3-030-05704-6)

[41] Pazy A 2012 *Semigroups of Linear Operators and Applications to Partial Differential Equations* vol 44 (New York: Springer)

[42] Pedlosky J 1979 *Geophysical Fluid Dynamics* (New York: Springer)

[43] Petcu M, Temam R and Wirosoetisno D 2005 Renormalization group method applied to the primitive equations *J. Differ. Equ.* **208** 215–57

[44] Riley J J and Lelong M P 2000 Fluid motions in the presence of strong stable stratification *Annu. Rev. Fluid Mech.* **32** 613–57

[45] Roussier-Michon V 2003 Sur la stabilité des ondes sphériques et le mouvement d’un fluide entre deux plaques infinies *PhD Thesis* Paris

[46] Rubio A M, Julien K, Knobloch E and Weiss J B 2014 Upscale energy transfer in three-dimensional rapidly rotating turbulent convection *Phys. Rev. Lett.* **112** 144501

[47] Sauer J 2015 Weighted resolvent estimates for the spatially periodic Stokes equations *Ann. Univ. Ferrara* **61** 333–54

[48] Sprague M, Julien K, Knobloch E and Werne J 2006 Numerical simulation of an asymptotically reduced system for rotationally constrained convection *J. Fluid Mech.* **551** 141–74

[49] Temam R and Wirosoetisno D 2010 Stability of the slow manifold in the primitive equations *SIAM J. Math. Anal.* **42** 427–58

[50] Temam R and Wirosoetisno D 2011 Slow manifolds and invariant sets of the primitive equations *J. Atmos. Sci.* **68** 675–82

[51] Temam R and Ziane M 2005 Some mathematical problems in geophysical fluid dynamics *Handbook of Mathematical Fluid Dynamics* vol 3 (Amsterdam: Elsevier) pp 535–658

[52] Whitehead J P and Wingate B A 2014 The influence of fast waves and fluctuations on the evolution of the dynamics on the slow manifold *J. Fluid Mech.* **757** 155–78

[53] Wingate B A, Embid P, Holmes-Cerfon M and Taylor M A 2011 Low Rossby limiting dynamics for stably stratified flow with finite Froude number *J. Fluid Mech.* **676** 546–71