Quantum massive conformal gravity

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Abstract

We first find the linear approximation of the second plus fourth order derivatives massive conformal gravity action. Then we reduce the linearized action to three separated second order derivatives terms, which allows us to quantize the massive conformal gravity fields by using the usual first order canonical quantization method. Finally, using the correct conformal normalization of the massive conformal gravity states, we show that the theory is unitary.

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1 Introduction

The massive conformal gravity \cite{1} is a recently developed theory of gravity in which the gravitational action is the sum of the fourth order derivatives Weyl action \cite{2} with the second order derivatives Einstein-Hilbert action conformally coupled to a scalar field \cite{3}. The gravitational potential of the theory, which is composed by an attractive Newtonian potential and a repulsive Yukawa potential, reproduces the rotation curves of the major number of galaxies. In addition, the momentum space propagators of massive conformal gravity have a good high energy behavior, which makes the theory power-counting renormalizable. However, one of these propagators has a negative sign between its terms, which is a common feature of fourth order derivatives theories of gravity. In such theories this negative sign imply the presence of a negative norm ghost state \cite{4}. This is because these theories are coordinate invariant and thus the normalization of its states is made by the usual Dirac conjugation.

In conformally invariant field theories there is another type of conjugation, namely, the Belavin-Polyakov-Zamolodchikov (BPZ) conjugation \cite{5}. In this article we use the BPZ conjugation in the normalization of the conformally invariant massive conformal gravity states. Additionally, we consider that the Dirac conjugation and the BPZ conjugation give the same state up to a sign. In section 2 we derive a second order derivatives linearized massive conformal gravity action. In section 3 we canonically quantize the massive conformal gravity fields and show that the theory is ghost-free. Finally, in section 4 we present our conclusions.

2 Linearized action

Let us consider the gravitational action of massive conformal gravity, which is given by \cite{1}

\[
S_g = -\frac{1}{2kc} \int d^4x \sqrt{-g} \left[ \alpha \lambda^2 C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} - \beta \left( \varphi^2 R + 6 \partial_\mu \varphi \partial^\mu \varphi \right) \right],
\]

where \( \alpha \) and \( \beta \) are dimensionless constants, \( \lambda = \hbar/mc \) (\( \hbar \) is the Planck constant and \( m \) is the graviton mass), \( k = 16\pi G/c^4 \) (\( G \) is the gravitational

\footnote{This action is equivalent to the action of Ref. \cite{1}. Both actions have the same dimensions Kg.m^2/s. The only difference is that the action of Ref. \cite{1} must have the mass measured in Kg/m^2.}
constant and $c$ is the speed of light in vacuum),

\begin{equation}
C_{\mu\beta\nu} = R_{\mu\beta\nu} + \frac{1}{2} \left( \delta^{\alpha}_{\nu} R_{\mu\beta} - \delta^{\alpha}_{\beta} R_{\mu\nu} + g_{\mu\beta} R_{\nu}^{\alpha} - g_{\mu\nu} R_{\beta}^{\alpha} \right) + \frac{1}{6} \left( \delta^{\alpha}_{\beta} g_{\mu\nu} - \delta^{\alpha}_{\nu} g_{\mu\beta} \right) R \tag{2}
\end{equation}

is the Weyl tensor, \( \phi \) is a scalar field, \( R^{\alpha}_{\mu\beta\nu} \) is the Riemann tensor, \( R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \) is the Ricci tensor and \( R = g_{\mu\nu} R_{\mu\nu} \) is the scalar curvature. It is worth noting that (1) is invariant under the conformal transformations

\begin{align*}
\tilde{g}_{\mu\nu} &= e^{2\theta(x)} g_{\mu\nu}, \\
\tilde{\phi} &= e^{-\theta(x)} \phi, 
\end{align*}

(3) (4)

where \( \theta(x) \) is an arbitrary function of the spacetime coordinates.

With the help of the Lanczos identity, we can write (1) in the form

\begin{equation}
S_g = -\frac{1}{2kc} \int d^4x \sqrt{-g} \left[ 2\alpha \lambda^2 \left( R_{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) - \beta \left( \phi^2 R + 6 \partial_{\mu} \phi \partial^{\mu} \phi \right) \right]. 
\end{equation}

(5)

Then using the weak-field approximations

\begin{align*}
g_{\mu\nu} &= g_{\mu\nu}^{(0)} + h_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \\
\phi &= \phi^{(0)} (1 + \sigma) = \sqrt{\frac{2\alpha}{\beta}} (1 + \sigma),
\end{align*}

(6) (7)

and keeping only the terms of second order in \( h^{\mu\nu} \) and \( \sigma \), we find that (5) reduces to

\begin{equation}
S_g = -\frac{\alpha}{kc} \int d^4x \left[ \lambda^2 \left( \tilde{R}_{\mu\nu} \tilde{R}_{\mu\nu} - \frac{1}{3} \tilde{R}^2 \right) - \left( \tilde{R} + 2\sigma \tilde{R} + 6 \partial_{\mu} \sigma \partial^{\mu} \sigma \right) \right], 
\end{equation}

(8)

where

\begin{align*}
R_{\mu\nu} &= \frac{1}{2} (\partial_{\mu} \partial^{\rho} h_{\rho\nu} + \partial_{\nu} \partial^{\rho} h_{\rho\mu} - \partial_{\rho} \partial^{\rho} h_{\mu\nu} - \partial_{\rho} \partial_{\nu} h), \\
\tilde{R} &= \partial^{\mu} \partial_{\nu} h_{\mu\nu} - \partial_{\mu} \partial^{\mu} h, \\
\tilde{R} &= -\frac{1}{4} (\partial^{\rho} h^{\mu\nu} \partial_{\rho} h_{\mu\nu} - 2 \partial^{\rho} h^{\nu\rho} \partial_{\rho} h_{\mu\nu} + 2 \partial^{\rho} h_{\mu\nu} \partial^{\rho} h - \partial^{\rho} h \partial_{\rho} h),
\end{align*}

(9) (10) (11)

with \( h = h^{\mu}_{\rho} = \eta^{\mu\nu} h_{\mu\nu} \).
The linearized action (8) is invariant under the coordinate gauge transformation

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \]  

where \( \xi^\mu \) is an arbitrary spacetime dependent vector field, and under the conformal gauge transformations

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu} \Lambda, \]

\[ \sigma \rightarrow \sigma - \frac{1}{2} \Lambda, \]

where \( \Lambda \) is an arbitrary spacetime dependent scalar field. By imposing the coordinate gauge condition

\[ \partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h = 0 \]

and the conformal gauge condition

\[ \partial^\mu \partial^\nu h_{\mu\nu} - \partial_\rho \partial^\rho h - 6 \lambda^{-2} \sigma = 0 \]

to (8), and integrating by parts, we arrive at

\[ S_g = -\frac{\alpha}{2kc} \int d^4x \left[ \frac{1}{2} \Psi^{\mu\nu} (\lambda^2 \Box - 1) \Box \Psi_{\mu\nu} + 12 \sigma (\Box - \lambda^{-2}) \sigma \right], \]

where \( \Box = \partial_\rho \partial^\rho \) and

\[ \Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \]

In order to obtain a first order canonical form, we take the decomposition into oscillator variables method \[ \square \] and write the action (17) as

\[ S_g = \frac{\alpha}{2kc} \int d^4x \left[ \frac{1}{2} \Psi^{\mu\nu} \Box \Phi_{\mu\nu} + \frac{1}{8} \lambda^{-2} \Psi_{\mu\nu} \Psi^{\mu\nu} - \frac{1}{4} \lambda^{-2} \Psi_{\mu\nu} \Phi^{\mu\nu} \right. \\
+ \frac{1}{8} \lambda^{-2} \Phi_{\mu\nu} \Phi^{\mu\nu} - 12 \sigma (\Box - \lambda^{-2}) \sigma \right], \]

where

\[ \Phi_{\mu\nu} = \Psi_{\mu\nu} - 2\lambda^2 \Box \Psi_{\mu\nu}. \]
Finally, with the change of variables
\[ \Psi_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu}, \quad (21) \]
\[ \Phi_{\mu\nu} = A_{\mu\nu} - B_{\mu\nu}, \quad (22) \]
we find
\[ S_g = \frac{\alpha}{2kc} \int d^4x \left[ \frac{1}{2} A^{\mu\nu} \Box A_{\mu\nu} - \frac{1}{2} B^{\mu\nu} (\Box - \lambda^{-2}) B_{\mu\nu} - 12\sigma (\Box - \lambda^{-2}) \sigma \right]. \quad (23) \]
This action is dynamically equivalent to action (8). The original coordinate gauge invariance of (8) has been replaced by the coordinate gauge invariance of \( A_{\mu\nu} \) alone in (23), with the coordinate gauge parameter restricted by \( \Box \xi_{\mu} = 0 \). The same occurs with the original conformal gauge invariance of (8), which has been replaced by the conformal gauge invariance of \( B_{\mu\nu} \) and \( \sigma \) in (23), with the conformal gauge parameter restricted by \( (\Box - \lambda^{-2}) \Lambda = 0 \). Note that the dynamical restrictions of the gauge parameters do not change the eight degrees of freedom of the theory.

### 3 Canonical quantization

Varying the action (23) with respect to \( A^{\mu\nu}, B^{\mu\nu} \) and \( \sigma \) in vacuum, we obtain the field equations
\[ \Box A_{\mu\nu} = 0, \quad (24) \]
\[ (\Box - \lambda^{-2}) B_{\mu\nu} = 0, \quad (25) \]
\[ (\Box - \lambda^{-2}) \sigma = 0. \quad (26) \]
Additionally, the fields \( A^{\mu\nu} \) and \( B^{\mu\nu} \) satisfy the constraints
\[ \partial_{\mu} A^{\mu\nu} = 0, \quad (27) \]
\[ \partial_{\mu} B^{\mu\nu} = 0, \quad B_\mu^\mu = 0. \quad (28) \]
We can write the most general real solutions of (24-28) as
\[ A_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p^4}} \sum_{r=1}^{2} \left[ a_r^p \epsilon_{\mu\nu}^r(p) e^{ip\cdot x} + c.c. \right], \quad (29) \]
\[ ^2 \text{In this section we use the Planck units of} \ c = \hbar = G = 1. \]
\[ B_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{5} [b_p^s \varepsilon_{\mu\nu}^s(p)e^{ip \cdot x} + \text{c.c.}], \quad (30) \]

\[ \sigma(x) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_q}} \left[ c_q e^{iq \cdot x} + \text{c.c.} \right], \quad (31) \]

where \( \omega_p^A = |p|, \ \omega_p^B = \sqrt{|p|^2 + m^2}, \ \omega_q = \sqrt{|q|^2 + m^2}, \) the creation and annihilation operators obey the commutation relations

\[ [a_p^r, a_{p'}^{r'}] = (2\pi)^3 \delta^3(p - p') \delta^{rr'}, \quad (32) \]

\[ [b_p^s, b_{p'}^{s'}] = (2\pi)^3 \delta^3(p - p') \delta^{ss'}, \quad (33) \]

\[ [c_q, c_{q'}^\dagger] = \frac{(2\pi)^3}{24} \delta^3(q - q'), \quad (34) \]

with all the others commutators equal zero, and the polarization tensors satisfy the orthonormality and completeness relations

\[ \varepsilon_{\mu\nu}^r \varepsilon_{\mu'\nu'}^{r'} = \delta^{rr'}, \quad (35) \]

\[ \varepsilon_{\mu\nu}^s \varepsilon_{\mu'\nu'}^{s'} = \delta^{ss'}, \quad (36) \]

\[ \sum_{r=1}^{2} \varepsilon_{\mu\nu}^r \varepsilon_{\alpha\beta}^{r} = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}), \quad (37) \]

\[ \sum_{s=1}^{5} \varepsilon_{\mu\nu}^s \varepsilon_{\alpha\beta}^{s} = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}). \quad (38) \]

We can write the action (23) as

\[ S_g = \frac{\alpha}{32\pi} \int d^4x \mathcal{L}, \quad (39) \]

where

\[ \mathcal{L} = \frac{1}{2} A_{\mu\nu} \Box A_{\mu\nu} - \frac{1}{2} B_{\mu\nu} \left( \Box - m^2 \right) B_{\mu\nu} - 12\sigma \left( \Box - m^2 \right) \sigma \quad (40) \]

is the Lagrangian density. Using this Lagrangian density, we find the canonical momenta

\[ \Pi_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu\nu}} = -\dot{A}_{\mu\nu}, \quad (41) \]
\[ \Theta_{\mu\nu} = \frac{\partial L}{\partial \dot{B}^{\mu\nu}} = \dot{B}_{\mu\nu}, \]  
(42)

\[ \pi = \frac{\partial L}{\partial \dot{\sigma}} = 24 \dot{\sigma}, \]  
(43)

where the dot denotes time derivative.

It follows from (29-31) and (41-43) that

\[ \Pi_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_A}{2}} \sum_{r=1}^{2} [a_r^e \varepsilon_{\mu\nu}^e(p)e^{ip \cdot x} + c.c.], \]  
(44)

\[ \Theta_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} i\sqrt{\frac{\omega_B}{2}} \sum_{s=1}^{5} [b_s^e \varepsilon_{\mu\nu}^e(p)e^{ip \cdot x} + c.c.], \]  
(45)

\[ \pi(x) = \int \frac{d^3q}{(2\pi)^3} 24i \sqrt{\frac{\omega_q}{2}} [c_q e^{iq \cdot x} + c.c.]. \]  
(46)

Thus, by imposing the commutation rules (32-34), we arrive at

\[ [A_{\mu\nu}(x), \Pi_{\alpha\beta}(x')] = \frac{i}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) \delta^3(x - x'), \]  
(47)

\[ [B_{\mu\nu}(x), \Theta_{\alpha\beta}(x')] = -\frac{i}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) \delta^3(x - x'), \]  
(48)

\[ [\sigma(x), \pi(x')] = -i\delta^3(x - x'), \]  
(49)

with all the others commutators equal zero.

The Hamiltonian of massive conformal gravity reads

\[ H = \int d^3x \mathcal{H}, \]  
(50)

where

\[ \mathcal{H} = \Pi_{\mu\nu}\dot{A}^{\mu\nu} + \Theta_{\mu\nu}\dot{B}^{\mu\nu} + \pi \dot{\sigma} - \eta_{\mu\nu}L \]  
(51)

is the Hamiltonian density. Substituting the Lagrangian density (40) and the canonical momenta (41-43) into (51), we obtain

\[ \mathcal{H} = \frac{1}{2} (-\Pi_{\mu\nu} \Pi^{\mu\nu} + \partial_i A_{\mu\nu} \partial^i A^{\mu\nu}) + \frac{1}{2} (\Theta_{\mu\nu} \Theta^{\mu\nu} - \partial_i B_{\mu\nu} \partial^i B^{\mu\nu}) \]

\[ -m^2 B_{\mu\nu} B^{\mu\nu} \]  
\[ + \frac{1}{2} \left( \frac{1}{24} \pi^2 - 24 \partial_i \sigma \partial^i \sigma - 24 m^2 \sigma^2 \right), \]  
(52)
where we consider $\eta_{\mu \nu} = (+, -, -, -)$ and $\partial_i$ denotes spacial derivatives, with $i = 1, 2, 3$. Finally, after some calculation, we find

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} (\omega_p^A a_{p}^{\dagger} a_{p}^{\alpha}) - \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{5} (\omega_p^B b_{p}^{\dagger} b_{p}^{s}) - \int \frac{d^3q}{(2\pi)^3} (\omega_q c_{q}^{\dagger} c_{q}), \tag{53}$$

where we have dropped the infinite constants terms that comes from $[a_p^r, a_p^{\dagger r}], [b_p^s, b_p^{\dagger s}]$ and $[c_q, c_q^{\dagger}]$.

It is well known that the normalization of a complete basis of states $|\psi\rangle$ in a coordinate invariant field theory is given by

$$\langle \psi | \psi \rangle = 1, \tag{54}$$

where $\langle \psi | = (|\psi\rangle)^{\dagger}$ are the Dirac conjugate states. However, in a conformally invariant field theory the normalization of $|\psi\rangle$ must be replaced by \(^7\)

$$\langle \psi^{bpz} | \psi \rangle = 1, \tag{55}$$

where $\langle \psi^{bpz} |$ are the BPZ conjugate states. The normalization (55) ensures the unitarity of conformal theories. In addition, the reality condition

$$\langle \psi | = -\langle \psi^{bpz} | \tag{56}$$

must be imposed on conformal fields.

Using (54-56) in (53), we see that the energy expectation value of any physical state $|\psi\rangle$ in massive conformal gravity is given by

$$\langle \psi | H | \psi \rangle = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \omega_p^A \langle \psi | a_{p}^{\dagger} a_{p}^{\alpha} | \psi \rangle + \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{5} \omega_p^B \langle \psi^{bpz} | b_{p}^{\dagger} b_{p}^{s} | \psi \rangle + \int \frac{d^3q}{(2\pi)^3} \omega_q \langle \psi^{bpz} | c_{q}^{\dagger} c_{q} | \psi \rangle. \tag{57}$$

There are thus no states with negative norm or negative energy in massive conformal gravity, and thus the theory is fully consistent and unitary.

The Feynman propagators of the theory reads

$$D_{A}^{\mu \nu, \alpha \beta} (x - x') = \langle 0 | T(A^{\mu \nu}(x) A^{\alpha \beta}(x')) | 0 \rangle = -\frac{i}{2} (\eta^{\mu \alpha} \eta^{\nu \beta} + \eta^{\mu \beta} \eta^{\nu \alpha} - \eta^{\mu \nu} \eta^{\alpha \beta}) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i p \cdot (x - x')}}{p^2 - i\epsilon}, \tag{58}$$
\[ D_{B}^{\mu\nu,\alpha\beta}(x-x') = \langle 0|T(B^{\mu\nu}(x)B^{\alpha\beta}(x'))|0\rangle = -\langle 0^{bpz}|T(B^{\mu\nu}(x)B^{\alpha\beta}(x'))|0\rangle = \frac{i}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta}) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-x')}}{p^2 + m^2 - i\chi}, \]

(59)

\[ D_{\sigma}(x-x') = \langle 0|T(\sigma(x)\sigma(y))|0\rangle = -\langle 0^{bpz}|T(\sigma(x)\sigma(y))|0\rangle = i \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq \cdot (x-x')}}{q^2 + m^2 - i\chi}, \]

(60)

where \( T \) denotes time-ordered product and \( \chi \) is an infinitesimal parameter. It follows from (21) and (58-59) that the Feynman propagator of the field \( \Psi^{\mu\nu} \) is given by

\[ D_{\Psi}^{\mu\nu,\alpha\beta} = \frac{i}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta}) \times \int \frac{d^4p}{(2\pi)^4} \frac{m^2}{(p^2 - i\chi)(p^2 + m^2 - i\chi)} e^{-ip \cdot (x-x')} \]

(61)

This propagator have a good \( p^{-4} \) behavior at high momenta, making massive conformal gravity power-counting renormalizable.

### 4 Final remarks

We have shown in this article that massive conformal gravity is a viable candidate for a quantum theory of gravity. The fact that the action of massive conformal gravity have a fourth order derivatives term makes the theory renormalizable. At the same time, the conformal symmetry of the action ensures that massive conformal gravity is unitary. Furthermore, the theory presents so far an important classical result. The gravitational potential of massive conformal gravity describes well the galaxies rotation curves [1]. Of course, the theory must still solve many gravity problems such as the dark energy problem. However, we are investigating some of these problems right now and we hope that this investigation help to show that massive conformal gravity is a complete theory of gravity.
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