The role of singular values in single copy entanglement manipulations and unambiguous state discrimination

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Abstract
Unambiguous (non-orthogonal) state discrimination (USD) has a fundamental importance in quantum information and quantum cryptography. Various aspects of two-state and multiple-state USD are studied here using singular value decomposition of the evolution operator that describes a given state discriminating system. In particular, we relate the minimal angle between states to the ratio of the minimal and maximal singular values. This is supported by a simple geometrical picture in two-state USD. Furthermore, by studying the singular vectors population we find that the minimal angle between input vectors in multiple-state USD is always larger than the minimal angle in two-state USD in the same system. As an example we study what pure states can be probabilistically transformed into maximally entangled pure states in a given system.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Unambiguous state discrimination (USD [1–4]) refers to a process that is capable of detecting whether a quantum system was prepared in state \(|g_i\rangle\), or in state \(|g_j\rangle\), where the states are not orthogonal to each other (\langle g_i | g_j \rangle \neq 0), but they are priorly known. This process has prime importance in quantum information and quantum cryptography [1, 2, 5, 6]. In particular, in this work we focus on the application of USD to (probabilistic) single copy entanglement
distillation (SCED) [4]. That is, to a local process that with some success probability can transform a single copy of quantum pure state into a maximally entangled pure state with the same Schmidt rank. It is shown that the singular values of the ‘USD evolution operator’ that describes a given USD system, encapsulate all the USD properties of the system. We use the singular values analysis to derive general results on USD and SCED.

It is well known that standard projective measurements involve an intrinsic discrimination error that depends on the overlap of the states [7, 8]. Yet, this error can be avoided by using a different type of measurements called POVM (positive operator valued measure) [7, 8]. POVMs are considered to be a part of standard quantum theory since the Naimark dilation theorem [7] ensures that any POVM can be implemented as a standard projective measurement in a larger Hilbert space. While POVMs allow to have zero discrimination error they involve an intrinsic non-zero probability of obtaining an ‘inconclusive result’ from which the input state cannot be inferred.

Starting from the pioneering work of Ivanovich [9], Dieks [10], and Peres [11], USD has been extensively studied over the years ([1–4, 12] and references therein). Typically the question of interest in USD is how to minimize the inconclusive result probability. In this work, however, we are not interested in designing an optimal system for a given set of input states, but to study the USD properties of a given system. For example, in the context of SCED this translates to identifying the different states that can be probabilistically distilled into maximally entangled states by a given system. Another motivation for studying the singular values associated with a USD task, comes from a recent work where it is shown that the singular values of the aforementioned USD evolution operator determine the minimal time-energy cost needed to implement the USD by means of a unitary embedding [20].

The relation between POVM and USD to USD evolution operators, and lossy systems that do not conserve probability, was first presented and experimentally demonstrated in [13]. This relation has also been used in [14–17]. Recently, a complete equivalence and its details were studied in [18]. For this work it suffices to know that given a USD evolution operator $K$, one can find an equivalent POVM measurement [18]. Conversely, given USD POVM measurement operators, it is possible to find an equivalent $K$. We wish to emphasize that any system whose evolution can be described by a non-unitary USD evolution operator defined later on, is capable of USD and SCED.

After some background and preliminaries in section 2, in section 3 we show the key role of the singular values of $K$ in the analysis of the USD capabilities of $K$. In particular, we study the minimal angle between states in two-state USD. We relate it to the minimal and maximal singular values of $K$ and provide a geometrical picture of the process. Next we study multiple-state USD and find that the minimal angle between states in multiple state USD, must be larger than the minimal angle in two-state USD. While in section 3 we investigated the relation between a given input set of states and the discriminating operator $K$, in section 4 we discuss the relation between different sets of states that $K$ discriminates unambiguously. This has practical importance in USD-based SCED, as it relates all the states that can be distilled by a given physical system.

2. Background and preliminaries

2.1. USD evolution operator and POVM

Let $|\psi(t)\rangle \in \mathbb{C}^N$ be a state in an $N$-level system. The evolution of the state from $t = t_i$ to $t = t_f$ generated by a USD evolution operator, $K$, is given by:

$$|\psi(t_f)\rangle = K|\psi(t_i)\rangle,$$

(1)
where $K \in \mathbb{C}^{N \times N}$ is not unitary. The density matrix of the state evolves according to $K|\psi\rangle\langle\psi|K^\dagger$. The main difference in comparison to Kraus map evolution $\sum_i M_i |\psi\rangle\langle\psi| M_i^\dagger$ is that it does not conserve the trace of the density matrix (see (2)). Note that if $|\psi\rangle$ describes only a subsystem (e.g. there are other levels that are not accounted for in $|\psi\rangle$), then the trace of the density matrix of the subsystem need not be conserved (see [20] and references therein). Hence, this evolution may be referred to as ‘lossy’ evolution as in [13]. Such non-unitary operators frequently appear in optics when absorption cannot be ignored. Nevertheless, the loss mechanism does not have to be related to absorption. In quantum mechanics it can be due to tunneling or ionization that lead to probability loss in the subsystem of interest. There are two ways to generate such a $K$. The first is to use Schrödinger equations with some non-Hermitian Hamiltonian that takes loss effects into account. This formalism is very useful in the study of resonances and metastable states [19]. The other way to generate $K$ is by embedding it in a larger Hilbert space [13–17]. Typically this is done by coupling the system to an initially unpopulated ancillary system. For characteristics properties of such embeddings see [20, 21]. Although there is an intimate relation between USD and non-unitary evolution no prior knowledge of non-unitary evolution is needed in this paper. In the next subsection we give brief summary of the singular value decomposition (SVD) and introduce some notation.

2.2. Singular value decomposition and the spectral norm

Let $K \in \mathbb{C}^{N \times N}$ be a general linear operator in a Hilbert space of dimension $N$. According to the SVD ([22]) $K$ can always be written as:

$$K = \sum s_i |u_i\rangle \langle v_i|,$$

(3)

where $s_i \geq 0$ are called the singular values of $K$, and the vectors $|v_i\rangle$, $|u_i\rangle$ satisfy

$$K|v_i\rangle = s_i |u_i\rangle,$$

(4)

$$K^\dagger|u_i\rangle = s_i |v_i\rangle.$$

(5)

While $\langle v_i|v_j\rangle = \delta_{ij}$ and $\langle u_i|u_j\rangle = \delta_{ij}$, in general $\langle u_i|v_j\rangle \neq 0$. The singular values and singular vectors are calculated using:

$$K^\dagger K|v_i\rangle = s_i^2 |v_i\rangle,$$

(6)

$$KK^\dagger|u_i\rangle = s_i^2 |u_i\rangle.$$

(7)

Finally, the singular values induce a few important matrix norms. In this work, however, we shall always use the spectral norm [22]:

$$\|K\| = \max(s_i) = \sqrt{\text{max(eigenvalues}(K^\dagger K))}.$$  

(8)

Note that in systems with no gain (in contrast to amplifying medium in optics) $K$ can never increase the amplitude of a state. The passiveness of $K$ is given by the condition [23]:

$$\|K_{\text{passive}}\| \leq 1.$$

(9)

This can easily be seen from an equivalent definition of the spectral norm:

$$\|K\| = \max_{|\psi\rangle} \frac{\sqrt{\langle \psi|K^\dagger K|\psi\rangle}}{\sqrt{\langle \psi|\psi\rangle}} = \max_{|\psi\rangle} \frac{\sqrt{\langle \psi|K^\dagger K|\psi\rangle}}{\sqrt{\langle \psi|\psi\rangle}} = \max_{|\psi\rangle} \frac{\sqrt{\langle \psi|K^\dagger K|\psi\rangle}}{\sqrt{\langle \psi|\psi\rangle}}.$$  

(10)

This form shows that the spectral norm is equal to the amplification of the maximally amplified (or least attenuated) state. Hence, condition (9) ensures the system has no gain in it (the system is passive). Since we think of quantum applications (USD or SCED) where probability cannot be amplified, the USD evolution operator will always satisfy (9).
2.3. Structure of the USD evolution operator

In this subsection we write the general structure of a transformation that takes a non-orthogonal input set of states to a set of orthogonal output states. Let $G$ be a column matrix of the linearly-independent [24], non-orthogonal, non-normalized input vectors $|g_i\rangle$ we wish to discriminate, and let $G_\perp$ be a matrix whose columns are the bi-orthogonal vectors $|g_\perp^i\rangle$ so that:

$$G_\perp^\dagger G = I. \quad (11)$$

Clearly, $G_\perp^\dagger = G^{-1}$. $G$ is invertible due to the linear independence of its columns. Note that (11) can also be written as: $\langle g_\perp^i | g_j \rangle = \delta_{ij}$. A discriminating $K$ has the general form:

$$K = U_{\text{out}} \Lambda G^{-1}, \quad (12)$$

where $\Lambda$ is a diagonal matrix that affects the posterior probability to detect the vectors, and $U_{\text{out}}$ is a unitary matrix that determines the basis in which the results are expressed (the measurement basis). Without it, the results will appear in the computational basis (1,0,0,...), (0,1,0,...) and so on. The choice $\Lambda_{ii} = \text{const}$ is of particular importance as it used for SCED as we show in the next subsection. Equation (12) can also be written as:

$$K = \sum_{i=1}^{N} \Lambda_{ii} |\psi_i\rangle \langle g_\perp^i|, \quad (13)$$

where $|\psi_i\rangle$ are the columns of $U_{\text{out}}$. This transformation implements $|g_i\rangle K \rightarrow \Lambda_{ii} |\psi_i\rangle$. Note that $|g_\perp^i\rangle$ are not orthogonal to each other so (13) should not be confused with SVD (3).

2.4. USD-based single copy entanglement distillation

In quantum information theory the term ‘entanglement distillation’ refers to the study of efficiency of converting infinitely many copies of the same state into Bell states by means of local operations and classical communication (LOCC) [25]. However, the seminal work of Nielsen [26] and Vidal [27] (and references therein), clarified the limitation of such LOCC transformations for a single copy and not just in the asymptotic limit of many copies.

In this work we use the term SCED for a process where a pure entangled bipartite state is converted into a maximally entangled state of the same Schmidt rank by applying only local operations. According to [26] this cannot be done deterministically. There is some probability of success that depends on the needed entanglement increase. As described in detail by Chefles [4] SCED can be implemented by USD. Although to a large extent this subsection repeats the analysis of Chefles we find it worthwhile to repeat it in our notation and to use the USD evolution operator point of view.

Consider an entangled state constructed from non-orthogonal states $\langle x_j | x_k \neq j \rangle \neq 0$, $\langle y_j | y_k \neq j \rangle \neq 0$:

$$|\Psi\rangle = \sum_{i=1}^{N} c_k |x_k\rangle_A \otimes |y_k\rangle_B. \quad (14)$$

Our goal is to find a local transformation on side $A$ that will turn this state into a maximally entangled state of the form:

$$|\psi_{\text{max}}\rangle = b \sum_{k=1}^{N} |\phi_k\rangle \otimes |\psi_k\rangle \quad (15)$$
where \( \{|\phi_k\rangle\}_{k=1}^N \) and \( \{|\varphi_k\rangle\}_{k=1}^N \) are some orthonormal bases and \( b < 1/\sqrt{N} \). In a maximally entangled pure state in dimension \( N \), all the Schmidt coefficient are equal and different from zero. When normalized, this state maximizes the von Neumann entropy in dimension \( N \). The restriction \( b < 1/\sqrt{N} \) expresses the fact that the transformation is probabilistic, since \( \langle \psi_{\text{max}} | \psi_{\text{max}} \rangle < 1 \).

We start by spanning the \( |y_k\rangle \) in terms of some orthonormal basis \( \{|\phi_k\rangle\}_{k=1}^N \):

\[
|\Psi\rangle = \sum_{k=1}^N c_k |x_k\rangle_A \otimes \sum_i d_k \langle \phi_i | = \sum_{i=1}^N |g_i\rangle_A \otimes |\varphi_i\rangle_B,
\]

(16)

where the \( |g_i\rangle = \sum_{k=1}^N c_k d_k |x_k\rangle \) states are non-orthogonal and non-normalized. Now we operate locally on system \( A \) with the USD evolution operator:

\[
K_A = G^{-1} \frac{1}{\|G^{-1}\|},
\]

(17)

where, as before, \( G \) is a matrix whose columns are given by the \( |g_i\rangle \) states. The normalization factor \( 1/\|G^{-1}\| \) insures that \( K_A \) is a passive operator (see (9)). Operating with \( K_A \) on \( |\Psi\rangle \) we get:

\[
K_A |\Psi\rangle = \frac{1}{\|G^{-1}\|} \sum_{i=1}^N |e_i\rangle_A \otimes |\varphi_i\rangle_B,
\]

(18)

where \( |e_i\rangle \) is the computational basis. The distillation success probability is given by the square of the state’s norm:

\[
P_{\text{SCED}} = \frac{N}{\|G^{-1}\|^2}.
\]

(19)

Like in USD, for SCED we also use a non-unitary evolution operator that turns a non-orthogonal set of states into an orthogonal set. The difference is that in the USD setup the operator works on a single input state in each experiment, while in SCED \( K \) operates on all the input states simultaneously in each experiment. Another difference is that for SCED we insist that the output vectors will have the same weights (i.e. that \( \Lambda_{ii} = \text{const} \) in (13)).

Before moving on, another relation between USD and SCED should be mentioned.

A system which is prepared in one of the non-orthogonal normalized states \( \{|h_i\rangle\}_{i=1}^N \) with probability \( p_i \) is described by the density matrix: \( \rho_{\text{USD}} = \sum_{i=1}^N p_i |h_i\rangle \langle h_i| \). This positive matrix has an eigenvalue decomposition:

\[
\rho_{\text{USD}} = \sum_{i=1}^N \sigma_i |\phi_i\rangle \langle \phi_i|,
\]

(20)

where \( 0 \leq \sigma_i \leq 1 \). Next we point out that any pure bipartite state can be written in the Schmidt form [8]:

\[
|\Psi\rangle = \sum_{i=1}^N \lambda_i |\xi_i\rangle_A \otimes |\chi_i\rangle_B,
\]

(21)

where \( \{|\xi_i\rangle\}_{i=1}^N \) and \( \{|\chi_i\rangle\}_{i=1}^N \) are orthogonal bases and the coefficients \( 0 \leq \lambda_i \leq 1 \) are the Schmidt coefficients. The reduced density matrix of system \( A \) (B) is obtained by taking the partial trace on \( B \) (A) so that:

\[
\rho_A = \sum_{i=1}^N \lambda_i |\xi_i\rangle_A \langle \xi_i|_A,
\]

(22)
which has the same form as (20). Therefore the eigenvalues of the density matrix in USD play the same role as the Schmidt coefficients in pure entangled states. This analogue will become useful in section 4.

Although we selected SCED as our leading physical example our findings are relevant to any USD application.

3. Singular values and unambiguous state discrimination

Before explicitly studying the role of singular values in USD we wish to present a general argument why singular values capture the essence of USD. We shall do so by comparing two systems $A$ and $B$ (not to be confused with $A$ and $B$ of section 2.4) and checking when they are ‘USD equivalent’.

3.1. Equivalence of USD systems

Consider a USD evolution operator $K_A$ that transform a non-orthogonal set of states in system $A$ \{+$|g_i\rangle_A\}_{i=1}^N$ to orthogonal states \{+$|\psi_i\rangle_A\}_{i=1}^N$ (13). Let the USD evolution operator in system $B$, be $K_B = K_A U_R$ where $U_R$ is unitary matrix. Clearly the operator $K_B$ will discriminate the set \{+$|g_i\rangle_B\}_{i=1}^N = \{U_R^\dagger |g_i\rangle_A\}_{i=1}^N$. This set has exactly the same properties as \{+$|g_i\rangle_A\}_{i=1}^N$. The angles and inner products between all the states are exactly the same. Hence $K_A$ and $K_B$ have the same discrimination properties. Multiplying of $K_B$ from the left by $U_L$ simply change the output measurement basis to \{+$|\psi_i\rangle_B\}_{i=1}^N = \{U_L |\psi_i\rangle_A\}_{i=1}^N$ but it does not affect the discrimination properties. In conclusion, since $K_A$ and $K_B = U_L K_A U_R$ have exactly the same discrimination properties we call them USD equivalent. What quantities are invariant under such double unitary transformations? From (3) or (6) it is easy to see that the singular values are invariant under any such transformation. Consequently, two systems are USD equivalent if they have the same singular values. Furthermore, we expect that any discrimination property that depends only on $K$ and not on the input states will be a function of the singular values. For example, in the next section we study the minimal discrimination angle between input states and express it in term of the singular values. It should be noted, though, that not all quantities of interested in USD are input independent. The success or detection probability does depend on the input\(^1\) so one cannot expect an exact expression for it using only the singular values of $K$.

In the context of SCED, USD equivalence means that the set of input states that can be distilled by $A$, is related to the distillable input states of $B$ by a local unitary transformation. Furthermore, the maximally entangled output states of $A$ are related to those of $B$ by another local unitary transformation.

After discussing the equivalence of two systems we now turn to study the set of states that can be discriminated by a given system $K$.

3.2. The basic relation between singular values and USD

In this section we show that in the most basic USD example in Hilbert space of dimension 2 (two levels or one qubit), the angle between the non-orthogonal states that can be discriminated by $K$ is a function of the singular values ratio of $K$. In the next section we generalize this

\(^1\) For example it depends on the prior probability of each input state. This information is contained in the input density matrix not in $K$. 

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Figure 1. Singular values geometrical interpretation of two state USD. By a unitary rotation the two-state discrimination problem in an arbitrary size Hilbert space can be reduced to 2D real vectors transformation as depicted above. The $x$ and $y$ component of the non-orthogonal blue vectors are squeezed by different factors so that after the squeezing the two vectors become orthogonal. The smallest discrimination angle is determined by the smallest and largest singular values of the transformation.

and show that this example is also important in multiple-state USD. For two-level systems the SVD of the USD evolution operator (3) is:

$$K = K_{2\times2} = s_{\text{min}}|u_{\text{min}}\rangle\langle v_{\text{min}}| + s_{\text{max}}|u_{\text{max}}\rangle\langle v_{\text{max}}|.$$  \hspace{1cm} (23)

Consider now the special input vectors:

$$|g_{\pm}\rangle = s_{\text{min}}|v_{\text{max}}\rangle \pm s_{\text{max}}|v_{\text{min}}\rangle.$$  \hspace{1cm} (24)

The corresponding output vectors are:

$$K|g_{\pm}\rangle = s_{\text{min}}s_{\text{max}}(|u_{\text{max}}\rangle \pm |u_{\text{min}}\rangle).$$  \hspace{1cm} (25)

Since the $|u_i\rangle$'s are orthonormal, we get that the output vectors are orthogonal while the input vectors are not. The input angle is:

$$\cos \theta_0 = \frac{|\langle g_-|g_+\rangle|}{\sqrt{|\langle g_+|g_+\rangle||\langle g_-|g_-\rangle|}} = \frac{1 - \left(\frac{s_{\text{min}}}{s_{\text{max}}}\right)^2}{1 + \left(\frac{s_{\text{min}}}{s_{\text{max}}}\right)^2}. \hspace{1cm} (26)$$

Figure 1 offers a geometrical interpretation that differs from the standard geometrical interpretation of USD. Usually USD is explained by a unitary rotation of the vectors in a higher dimensional Hilbert space (e.g. [28]). Here, the vectors remain in the same space but different axes are squeezed by different factors. In figure 1 the non-orthogonal thin-blue vectors undergo a non-uniform stretching. The $x$ axis is squeezed by a factor $s_{\text{min}} < 1$, and the $y$ axis is stretched by a factor $s_{\text{max}} > 1$ (for the illustration we used $s_{\text{max}} > 1$ but for passive systems $s_{\text{max}} \leq 1$). Due to the anisotropic stretching, the resulting thick-red vectors are orthogonal to each other.

An easier way to obtain (26) follows immediately from figure 1. According to the figure:

$$\cos \frac{\theta_0}{2} = \frac{s_{\text{min}}}{s_{\text{max}}}.$$  \hspace{1cm} (26)

Squaring it and using half angle formula leads directly to (26). The fact
that the vectors may be complex does not matter, as will be explained in the next section. Alternatively, \((26)\) can be written in a simpler way using half the angle:

\[
\tan \frac{\theta_0}{2} = \frac{s_{\text{min}}}{s_{\text{max}}} = \frac{1}{\| K \| \| K^{-1} \|}. \tag{27}
\]

We comment that using the singular values approach and expressions \((24), (25)\) it is straightforward to obtain the known result of the detection probability for optimal two-state discrimination ([10] or p 154 of [2]): \(P_D = 1 - |\langle g_i | g_j \rangle|\). In the next section we study two-state discrimination in a scenario of multiple USD.

### 3.3. The best discrimination between two states

In a larger Hilbert space, are there two input vectors \(| g_1, 2 \rangle\) that lead to a smaller discrimination angle than \(\theta_0\) obtained in the previous section? To answer this we define the sum-difference basis:

\[
| \pm \rangle = \frac{| g_1 \rangle \pm | g_2 \rangle}{\sqrt{1 + 2 \langle g_2 | g_1 \rangle}}, \tag{28}
\]

\(< g_2 | g_1 \rangle > 0. \tag{29}\]

Without loss of generality we assumed the states are normalized \((< g_1 | g_1 \rangle = < g_2 | g_2 \rangle = 1)\) to simplify the writing. Condition \((29)\) ensures the orthogonality of this basis \(| - | + \rangle = 0.\) One can always change the relative global phase of the states so that \((29)\) holds\(^2\). In this new basis, the original input complex \(N\)-component vectors \(| g_1, 2 \rangle\) have the form:

\[
| \tilde{g}_1, 2 \rangle = (x_{\text{in}}, \pm y_{\text{in}}), \tag{30}\]

where \(x_{\text{in}}, y_{\text{in}}\) are real numbers and the tilde denote representation in the \(| \pm \rangle\) basis. These are the coefficients in the new basis, but one can apply a unitary transformation \(U_R\) and rotate \(| \pm \rangle\) to the computational basis \((1, 0)\) and \((0, 1)\). By applying the same type of rotation \(U_L\) to the output vectors we have:

\[
\tilde{K} | \tilde{g}_1, 2 \rangle = (x_{\text{out}}, \pm y_{\text{out}}), \tag{31}\]

where \(\tilde{K} = U_L K U_R^\dagger\). As explained before, such unitary rotations have no impact on the discrimination properties of \(K\). That is, \(\tilde{K}\) is USD equivalent to \(K\). If so, the problem of two-state discrimination in \(N\) dimensional Hilbert space was reduced to two-dimensional real vector space. In USD we want the output vectors to be orthogonal so:

\[
\tilde{K} | \tilde{g}_1, 2 \rangle = (x_{\text{out}}, \pm x_{\text{out}}). \tag{32}\]

The input angle satisfies:

\[
\tan \frac{\phi}{2} = \frac{y_{\text{in}}/x_{\text{in}}}{x_{\text{out}}/x_{\text{out}}} = \frac{y_{\text{in}}/x_{\text{in}}}{y_{\text{out}}/x_{\text{out}}} = \frac{x_{\text{out}}}{x_{\text{in}}} \frac{y_{\text{in}}}{y_{\text{out}}}. \tag{33}\]

Since \(\frac{y_{\text{in}}}{x_{\text{in}}} \geq s_{\text{min}}\) and \(\frac{y_{\text{out}}}{x_{\text{out}}} \geq 1/s_{\text{max}}\) we get:

\[
\tan \frac{\phi}{2} \geq \frac{s_{\text{min}}}{s_{\text{max}}}. \tag{34}\]

Comparing to \((27)\) we see that an equality takes place when \(\phi = \theta_0\). Therefore, the best discrimination angle is the one obtained in the previous section. The input states that achieve

\(^2\) This condition can be relaxed, but then several phase factors must be included in the definition of the \(| \pm \rangle\) basis \((28)\).
this angle are given by (24). We conclude that the best discrimination of $K$ can generate is given by:

$$\theta_{\text{best}} = 2 \arctan \frac{s_{\min}}{s_{\max}} = 2 \arctan \frac{1}{\|K\|_{sp} \|K^{-1}\|_{sp}}.$$  

(35)

After some algebra and using $b = 1$ and $a = 2$ in formula (1.7) of [29] we obtain:

$$\frac{3}{2} \frac{1}{\|K\|_{sp} \|K^{-1}\|_{sp}} \leq \theta_{\text{best}} \leq \frac{2}{\|K\|_{sp} \|K^{-1}\|_{sp}}.$$  

(36)

Significantly tighter inequalities can be written by making other choices of $a$ and $b$, but here we prefer to present the simplest expressions.

Interestingly, this result carries over directly to the inverse problem of ‘non-Hermitian cooling’ [23]. There, the goal is to cool a statistical mixture of orthogonal states by making them more parallel to each other. The best discrimination angle of $K$ is the best cooling angle of $K^{-1}$. However formula (35) is invariant to $K \leftrightarrow K^{-1}$ transformation, so the best discrimination angle of $K$ is also the best cooling angle of $K$.

In the next section we show why typically in USD of more than two non-orthogonal states this angle cannot be achieved.

3.4. Minimal angle in multiple-state USD and singular vector population

Let $\{|\alpha_i\rangle\}_{i=1}^N$ be a ‘completely non-orthogonal’ set of states ( $\langle \alpha_i|\alpha_j \rangle \neq 0$ for all $i, j$) and let $K$ be the USD evolution operator that discriminates them. Furthermore, let us make an ansatz that a subset $\{|\alpha_k\rangle\}_{k=1}^L$ is spanned by $L$ singular vectors:

$$|\alpha_k\rangle = \sum_{i=1}^L a_{ki} |v_i\rangle,$$  

(37)

where the coefficients $a_{ki}$ form an invertible matrix. An important observation on the singular values population will follow by showing that this last ansatz contradicts the complete non-orthogonality ansatz. Since the output states $|\beta_k\rangle = K|\alpha_k\rangle = \sum_{i=1}^L s_i a_{ki} |u_i\rangle$ span the image subspace $\{u_i\}_{i=1}^L$ (the $|\beta_k\rangle$ are linearly independent) and $\langle \beta_{k' > L}|\beta_{k \leq L} \rangle = 0$ we get that:

$$0 = \langle \beta_{k' > L}|u_{k \leq L} \rangle = s_k \langle \alpha_{k' > L}|v_{k \leq L} \rangle,$$  

(38)

where we used (5). Using this in our second ansatz (37) one obtains:

$$\langle \alpha_{k' > L}|\alpha_{k \leq L} \rangle = 0.$$  

(39)

That is, if some $L$ input states are spanned by $L$ singular vectors, all other vectors that can be discriminated must be initially orthogonal to the aforementioned $L$ vectors. This contradicts the first complete non-orthogonality ansatz. The implication is that if all the input states are non-orthogonal to each other, each input vector must populate all the singular vectors $|v_i\rangle$: $\langle \alpha_k|v_i \rangle \neq 0 \forall k, i$. Another implication is that in $N > 2$ USD for which $\langle \alpha_i|\alpha_{j \neq i} \rangle \neq 0$, the smallest angle between vectors satisfies:

$$\theta_{\text{min}} > \theta_{\text{best}} = 2 \arctan \frac{s_{\min}}{s_{\max}}.$$  

(40)

The strong inequality follows from the fact that the minimal and maximal singular vectors cannot be exclusively populated in a completely non-orthogonal multiple-state USD setup. Alternatively stated, in multiple-state USD the minimal discrimination angle can never be as small as the optimal two-state discrimination in the same system. The optimal two-state discrimination states are given by (24).
4. Families of discriminable states and singular value degeneracy

In this section we ask what states can be discriminated by a given USD evolution $K$. In SCED this translates to the question which states can be distilled by a given USD system. Our approach is based on finding transformations that are applied to the input states and leave the output states orthogonal to each other. These transformations do not have to be carried out in practice, so there is no reason they should be unitary. In fact, we shall explore both unitary and non-unitary transformations. The unitary transformations will be used to find a family of discriminable states that have the same USD density matrix eigenvalues (or the same Schmidt coefficients in SCED). The non-unitary transformation, on the other hand, will be useful to move from one family (described by density matrix eigenvalues/Schmidt coefficients) to a different family.

4.1. Special unitary transformations

Let us write the states of an input set $\{|g_k\rangle\}_{k=1}^N$ in terms of the singular vectors of $K$ (3):

$$|g_k\rangle = \sum_{i} a_{ki}|v_i\rangle.$$  \hfill (41)

The overlap of the output states $|h_j\rangle = K|g_j\rangle$ is:

$$\langle h_k|h_j \rangle = \langle g_k|K^\dagger K|g_i \rangle = \sum_{i} s_i^2 a_{ki}^* a_{ji}.$$  \hfill (42)

This overlap will not change if the following transformation is applied to the input states:

$$|\tilde{g}_k\rangle = W|g_k\rangle = \left(\sum_{i} e^{i\phi_i}|v_i\rangle\langle v_i|\right)|g_k\rangle.$$  \hfill (43)

Although the output vectors are modified, they remain orthogonal to each other. Notice that $W$ is a unitary matrix, but a very specific one. Its eigenvectors are the $|v_i\rangle$ singular vectors of $K$. If the set $\{|g_i\rangle\}$ can be discriminated, the set $\{|W|g_i\rangle\}$ can be discriminated as well by the same $K$. This should not be confused with the earlier discussion about USD equivalence of two systems. Here $K$ is set, and we ask what are the families of states that can be discriminated by this $K$.

The orthogonal output vectors will appear in a different basis than $|h_j\rangle$. Using (13) one can easily see that the new output basis is related to the old one by:

$$|\tilde{h}_k\rangle = \left(\sum_{i} e^{i\phi_i}|u_i\rangle\langle u_i|\right)|h_k\rangle.$$  \hfill (44)

Strictly speaking when the output basis is changed the POVM operators are modified as well [18]. However, in embedding realizations ([20] and also [14, 15, 17, 28]), the change of basis corresponds to a unitary rotation on the ‘system’ levels (i.e. without the ancilla levels) after the embedding part has been completed. So physically the $\{|\tilde{g}_k\rangle\}$ states describe family of discriminable states related to the same device even though the POVM operator are modified by $W$.

Positive $K$ operators have a special property. In [20] it was shown that the positivity is a necessary condition for minimizing the time-energy resources needed for unitary embedding of the desired USD. Note that if $K$ is not positive, it can be made positive by applying a unitary from the left. Since positive $K$ and $W$ have the same eigenvectors they commute. This means that the $W$ rotation can be performed before or after the distillation/USD and the result will be the same.
4.2. Singular value degeneracy

It may happen that two or more singular vectors have the same singular values. Such degeneracies appear automatically in unitary embedding realizations of USD evolution (or lossy evolution) when the ancilla dimension is smaller than the system dimension by more than one level [20]. In the presence of singular value degeneracy, on top of $W$, it is possible to apply any unitary $V$ that mixes the degenerate singular vectors.

When the degenerate singular values are equal to one (as in the unitary embedding scheme [20]) there is an interesting physical consequence. In the case of an inconclusive result, the degeneracy determines the rank of the density matrix after the measurement. The inconclusive density matrix is $\rho_{\text{inc}} = M_{\text{inc}} \rho M_{\text{inc}}^\dagger$ where $M_{\text{inc}} = \sqrt{I - K^* K}$ (see [18]). In the extreme case where $N - 1$ singular values are equal to 1, the rank of $\rho_{\text{inc}}$ is 1. This means that the remaining density matrix contains no information at all. If an inconclusive result is obtained, the state of the system is always the same regardless of the input. This case naturally appears in the embedding scheme when the ancilla has only one level [20]. This is consistent with the fact that no information can be encoded in a single quantum level.

The local unitary transformation $W$ and $V$ do not change the entanglement of the original state (Schmidt coefficients are invariant to local unitary transformations). Alternatively in USD, $W$ and $V$ do not change the eigenvalues of the density matrix. In what follows we describe a non-unitary transformation that change the Schmidt coefficients and extend the family of states that can be distilled/discriminated by a given system.

4.3. The entanglement distillation transformation

Given $K$, it is clear that the columns of $G = K^{-1}$ describe a possible set of vectors that $K$ can discriminate since $I = KK^{-1}$. This corresponds to the case $\Lambda_{ii} = \text{const}$ (see (13)) that is used for distillation. Multiplying both sides from the right by a unitary matrix $U_0$ we get:

$$U_0 = KK^{-1}U_0 = K \tilde{G},$$

where:

$$\tilde{G} = K^{-1}U_0.$$  \hspace{1cm} (46)

$\tilde{G}$ describes a new set of input states (column vectors) that can be discriminated by $K$. Notice that unitary transformation acting from the right is not a unitary rotation for the columns of $G$. Therefore, in contrast to the previous unitary transformations, here the relative angles between the new input states $\tilde{G}$ are different for different choices of $U_0$. Since this transformation modifies the Schmidt coefficients it extends the family of states that can be distilled with respect the family formed by $W$ and $V$.

5. Concluding remarks

This work shows that various insights to USD can be gained from singular value analysis of the USD evolution operator. It is likely that deriving the same findings directly from the POVM operators and without singular values would prove to be rather difficult. It is interesting to see what other USD and SCED features can be unraveled from the singular value analysis of the USD evolution operator.

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