Conical Kähler–Einstein Metrics Revisited

Chi Li\textsuperscript{1,2}, Song Sun\textsuperscript{2,3}

\textsuperscript{1} Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
\textsuperscript{2} Current address: Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651, USA. E-mail: chi.li@stonybrook.edu; song.sun@stonybrook.edu
\textsuperscript{3} Department of Mathematics, Imperial College, London SW7 2AZ, UK

Received: 7 October 2012 / Accepted: 27 May 2014
Published online: 29 July 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: In this paper we introduce the “interpolation–degeneration” strategy to study Kähler–Einstein metrics on a smooth Fano manifold with cone singularities along a smooth divisor that is proportional to the anti-canonical divisor. By “interpolation” we show the angles in \((0, 2\pi]\) that admit a conical Kähler–Einstein metric form a connected interval, and by “degeneration” we determine the boundary of the interval in some important cases. As a first application, we show that there exists a Kähler–Einstein metric on \(\mathbb{P}^2\) with cone singularity along a smooth conic (degree 2) curve if and only if the angle is in \((\pi/2, 2\pi]\). When the angle is \(2\pi/3\) this proves the existence of a Sasaki–Einstein metric on the link of a three dimensional \(A_2\) singularity, and thus answers a question posed by Gauntlett–Martelli–Sparks–Yau. As a second application we prove a version of Donaldson’s conjecture about conical Kähler–Einstein metrics in the toric case using Song–Wang’s recent existence result of toric invariant conical Kähler–Einstein metrics.

Contents

1. Introduction ........................................ 928
2. Existence Theory on Conical Kähler–Einstein Metrics ............................... 932
   2.1 Space of admissible potentials .......................... 932
   2.2 Energy functionals and analytic criterions .................. 934
   2.3 Alpha-invariant and small cone angles .................... 939
   2.4 Proof of Proposition 1, Theorem 1.1 and Corollary 1.7 ........... 940
3. Obstruction to Existence: Log-K-Stability .................................. 941
   3.1 Log-Futaki invariant and log-K-(semi)stability .............. 941
   3.2 Log-Mabuchi-energy and log-Futaki-invariant ............... 943
   3.3 Log-slope stability and log-Fano manifold ................... 946
4. Special Degeneration to Kähler–Einstein Svarieties ............................. 950
   4.1 Kähler metrics on singular varieties ....................... 950
1. Introduction

The existence of Kähler–Einstein metrics on a smooth Kähler manifold $X$ is a main problem in Kähler geometry. For the case when the first Chern class of $X$ is negative, this problem was solved by Aubin [3] and Yau [74]. For the case when the first Chern class is zero, this problem was settled by Yau [74]. The main interest at present lies in the case of Fano manifolds, when the first Chern class is positive. There is the famous Yau–Tian–Donaldson program which relates the existence of Kähler–Einstein metrics to algebro-geometric stability.

More generally one could look at a pair $(X, D)$ where $D$ is a smooth divisor in a Kähler manifold $X$, and study the existence of Kähler–Einstein metrics on $X$ with cone singularities along $D$ and smooth away from $D$. This problem was classically studied on the Riemann surfaces [45,48,72] (see also a recent paper [20]) and was first considered in higher dimensions by Tian in [65]. Recently, there is much new interest on this generalized problem, mainly due to Donaldson’s program (see [25]) on constructing smooth Kähler–Einstein metrics on $X$ by varying the cone angle along an anti-canonical divisor. There are many subsequent works, see, for example, [6,34].

From now on in this paper, we assume $X$ is a smooth Fano manifold, and $D$ is a smooth divisor which is $\mathbb{Q}$-linearly equivalent to $-\lambda K_X$ with $0 < \lambda \in \mathbb{Q}$. $\beta$ will always be a number in $(0, 1]$. We say $(X, D)$ is log canonical (resp. log Calabi–Yau, resp. log $\mathbb{Q}$-Fano) polarized if $\lambda > 1$ (resp. $\lambda = 1$, resp. $\lambda < 1$). We will study Kähler–Einstein metrics in $2\pi c_1(X)$ with cone singularities along $D$. The equation is given by

$$Ric(\omega) = r(\beta)\omega + 2\pi(1 - \beta)\{D\},$$

where $2\pi\beta$ is the angle along $D$. For brevity we say $\omega$ is a conical Kähler–Einstein metric on $(X, (1 - \beta)D)$. Note that when $\beta = 1$, conical Kähler–Einstein metrics become smooth Kähler–Einstein metrics.

Recall that the Ricci curvature form of a Kähler metric $\omega$ can be calculated as

$$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \omega^n.$$

In other words, the volume form $\omega^n$ determines a Hermitian metric on $K_X^{-1}$ whose Chern curvature is the Ricci curvature. So in particular, it represents the cohomology class $2\pi c_1(X)$. By taking cohomological class on both sides of the equation $(\ast)$, we obtain

$$r(\beta) = 1 - (1 - \beta)\lambda.$$  

We will use the above notation throughout this paper. Given a pair $(X, D)$, we define the set

$$E(X, D) = \{\beta \in (0, 1] \mid \text{There is a conical Kähler–Einstein metric on } (X, (1 - \beta)D)\}.$$