5-point CAT(0) spaces after Tetsu Toyoda

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Abstract
We give another proof of Toyoda’s theorem that describes 5-point sub-spaces in CAT(0) length spaces.

1 Introduction

The CAT(0) comparison is a certain inequality for 6 distances between 4 points in a metric space. The following descriptions, the so-called (2+2)-comparison, is the most standard, we refer to [2, 3] for other definitions and their equivalences.

Given a quadruple of points $p, q, x, y$ in a metric space $X$, consider two model triangles $\triangle(pxy)$ and $\triangle(qxy)$ with common side $[\hat{x}[\hat{y}]$.

If the inequality
\[ |p - q|_X \leq |\hat{p} - \hat{z}| + |\hat{z} - \hat{q}| \]
holds for any point $\hat{z} \in [\hat{x}[\hat{y}]$, then we say that the quadruple $p, q, x, y$ satisfies CAT(0) comparison; here $|p - q|_X$ denotes the distance from $p$ to $q$ in $X$.

If CAT(0) comparison holds for any quadruple (and any of its relabeling) in a metric space $X$, then we say that $X$ is CAT(0).

It is not hard to check that if a quadruple of points satisfies CAT(0) comparison for all relabeling, then it admits a distance-preserving inclusion into a length CAT(0) space. The following theorem generalizes this statement to 5-point metric spaces.

1.1. Toyoda’s theorem. Let $P$ be a 5-point metric space that satisfies CAT(0) comparison. Then $P$ admits a distance-preserving inclusion into a length CAT(0) space $X$.

Moreover, $X$ can be chosen to be a subcomplex of a 4-simplex such that (1) each simplex in $X$ has Euclidean metric and (2) the inclusion maps the 5 points on $P$ to the vertexes of the simplex.

A slightly weaker version of this theorem was proved by Tetsu Toyoda [8].

Our proof is shorter; it uses the fact that convex spacelike hypersurfaces in $\mathbb{R}^{3,1}$

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1that is, a plane triangle with the same sides.
equipped with the induced length metrics are CAT(0) spaces [5]. We construct a distance-preserving inclusion $\iota$ of $P$ into $\mathbb{R}^4$ or $\mathbb{R}^{3.1}$. In the case of $\mathbb{R}^4$ the convex hull $K$ of $\iota(P)$ can be taken as $X$; in the case of $\mathbb{R}^{3.1}$ we take as $X$ a spacelike part of the boundary of $K$.

It is expected that any 5 point metric space $P$ as in the theorem admits a distance-preserving inclusion in a product of trees.\(^2\)

An analog of Toyoda’s theorem does not hold for 6-point sets. It can be seen by using the so-called (4+2)-comparison introduced in [1]; this comparison holds for any length CAT(0) space, but may not hold for a space with CAT(0) comparison (if it is not a length space).

The (4+2)-comparison is not a sufficient condition for 6-point spaces. More precisely, there are 6-point metric spaces that satisfy (4+2) and (2+2)-comparisons but do not admit a distance-preserving embedding into a length CAT(0) space. An example was constructed by the first author; it is described in [1] right after 7.2. See the final section for related questions.

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\section{5-point arrays in 3-space}

Denote by $\mathcal{A}$ the space of all 5 point arrays in $\mathbb{R}^3$ that is nondegenerate in the following sense: (1) all 5 points do not lie on one plane and (2) no three points lie on one line. Note that $\mathcal{A}$ is connected.

A 5 point array $x_1, \ldots, x_5 \in \mathbb{R}^3$ defines an affine map from a 4-simplex to $\mathbb{R}^3$. Fix an orientation of the 4-simplex and consider the induced orientations on its 5 facets. Each facet may be mapped in an orientation-preserving, degenerate, or orientation-reversing way. For each array consider the triple of integers $(n_+, n_0, n_-)$, where $n_+$, $n_0$, and $n_-$ denote the number of orientation-preserving, degenerate, or orientation-reversing facets respectively.

Clearly $n_+ + n_0 + n_- = 5$ and since all 5 points cannot lie in one plane, we have that $n_+ \geq 1$, $n_- \geq 1$, and $n_0 \leq 1$. Therefore, the value $m = n_- - n_+$ can take an integer value between $-3$ and $3$; in this case, we say that an array belongs to $\mathcal{A}_m$.

It defines a subdivision of $\mathcal{A}$ into 7 subsets $\mathcal{A}_{-3}, \ldots, \mathcal{A}_3$ with combinatorial configuration as on the diagram; quadruples in one plane are marked in gray and the triple $(n_+, n_0, n_-)$ is written below.

Every two quadrilaterals in the array have 3 common points that define a plane. If the remaining two points lie on opposite sides from the plane, then the corresponding facets have the same orientation; if they lie on one side, then the orientations are opposite. Therefore, the 7 subsets $\mathcal{A}_{-3}, \ldots, \mathcal{A}_3$ can be described in the following way:

\(^2\)Later we found a counterexample [7].
A−3 — a tetrahedron with preserved orientation and one point inside.
A−2 — a tetrahedron with preserved orientation and one point on a facet.
A−1 — a double triangular pyramid formed by two tetrahedrons with preserved orientation.
A0 — a pyramid over a convex quadrilateral
A1 — a double triangular pyramid formed by two tetrahedrons with reversed orientation.
A2 — a tetrahedron with reversed orientation and one point on a facet.
A3 — a tetrahedron with reversed orientation and one point inside.

Note that the complement $A\setminus A_0$ has two connected components formed by $A_{−3} \cup A_{−2} \cup A_{−1}$ and $A_{+} = A_3 \cup A_2 \cup A_1$. Observe that each array in $A_{−}$ has at least 3 positively oriented facets and each array in $A_{+}$ has at least 3 negatively oriented facets.

2.1. Observation. Let $Q$ be a connected subset of $A$ that does not intersect $A_0$. Then either $Q \subset A_{+}$ or $Q \subset A_{−}$.

3 Associated form

In this section we recall some facts about the so-called associated form introduced in [6]; it is a quadratic form $W_x$ on $\mathbb{R}^{n-1}$ associated to a given $n$-point array $x = (x_1, \ldots , x_n)$ in a metric space $X$.

Construction. Let $\triangle$ be the standard simplex $\triangle$ in $\mathbb{R}^{n-1}$; that is, the first $(n-1)$ of its vertices $v_1, \ldots , v_n$ form the standard basis on $\mathbb{R}^{n-1}$, and $v_n = 0$.

Recall that $|a-b|_X$ denotes the distance between points $a$ and $b$ in the metric space $X$. Set

$$W_x(v_i - v_j) = |x_i - x_j|_X^2$$

for all $i$ and $j$. Note that this identity defines $W_x$ uniquely.

The constructed quadratic form $W_x$ will be called the form associated to the point array $x$.

Note that an array $x = (x_1, \ldots , x_n)$ in a metric space $X$ is isometric to an array in Euclidean space if and only if $W_x(v) \geq 0$ for any $v \in \mathbb{R}^{n-1}$.

In particular, the condition $W_x \geq 0$ for a triple $x = (x_1, x_2, x_3)$ means that all three triangle inequalities for the distances between $x_1$, $x_2$, and $x_3$ hold. For an $n$-point array, it implies that $W_x(v) \geq 0$ for any vector $v$ in a plane spanned by a triple $v_i, v_j, v_k$. In particular, we get the following:

3.1. Observation. Let $W_x$ be a form on $\mathbb{R}^{n-1}$ associated with a point array $x = (x_1, \ldots , x_n)$. Suppose that $L$ is a subspace of $\mathbb{R}^{n-1}$ such that $W_x(v) < 0$
for any nonzero vector $v \in L$. Then the projections of any 3 vertices of $\triangle$ to the quotient space $\mathbb{R}^{n-1}/L$ are not collinear.

**CAT(0) condition.** Consider a point array $x$ with 4 points. From 3.1, it follows that $W_x$ is nonnegative on every plane parallel to a face of the tetrahedron $\triangle$. In particular, $W_x$ can have at most one negative eigenvalue.

Assume $W_x(w) < 0$ for some $w \in \mathbb{R}^3$. From 3.1, the line $L_w$ spanned by $w$ is transversal to each of 4 planes parallel to a face of $\triangle$.

Consider the projection of $\triangle$ along $L_w$ to a transversal plane. The projection of the 4 vertices of $\triangle$ lie in general position; that is, no three of them lie on one line. Therefore, we can see one of two combinatorial pictures shown on the diagram. Since the set of lines $L_w$ with $W_x(w) < 0$ is connected, the combinatorics of the picture does not depend on the choice of $w$.

3.2. **Claim.** If CAT(0) comparison holds in $X$, then the diagram on the right cannot appear.

(The converse holds as well, but we will not need it.)

**Proof.** Suppose we see the picture on the right.

Let $[v_1, v_3]$ and $[v_2, v_4]$ be the line segments of $\triangle$ that correspond to the diagonals on the picture. Denote by $m$ the point of $[v_1, v_3]$ that corresponds to the point of intersection.

In the plane spanned by $[v_2, v_4]$ and $w$, the vector $w$ is timelike. Therefore, we have the following reversed triangle inequality:

$$|v_2 - m| + |v_4 - m| < |v_2 - v_4|;$$

here we use shortcut $|a - b| = \sqrt{W(a - b)}$.

Note that the triangles $[v_1v_2v_3]$ and $[v_1v_3v_4]$ with metric induced by $W$ are isometric to model triangles of $[x_1x_2x_3]$ and $[x_1x_3x_4]$. Whence (2+2)-point comparison does not hold. \hfill $\square$

The claim implies the following:

3.3. **Observation.** Suppose a metric on $x = (x_1, \ldots, x_n)$ satisfies CAT(0) comparison and $W_x$ is its associated form on $\mathbb{R}^{n-1}$. Assume that $L$ is a subspace of $\mathbb{R}^{n-1}$ such that $W_x(v) < 0$ for any nonzero vector $v \in L$. Then if the projections of 4 vertices of $\triangle$ to the quotient space $\mathbb{R}^{n-1}/L$ lies in one plane, then its projection looks like the picture on the left; that is, one of the points lies in the triangle formed by the remaining three points.

3.4. **Corollary.** Suppose a metric on $x = (x_1, \ldots, x_5)$ satisfies CAT(0) comparison and $W_x$ is its associated form on $\mathbb{R}^4$. Assume that $L$ is a subspace of $\mathbb{R}^4$ such that $W_x(v) < 0$ for any nonzero vector $v \in L$. Then $\dim L \leq 1$.

Moreover, if $\dim L = 1$, then the projections of the vertices of $\triangle$ to the quotient space $\mathbb{R}^3 = \mathbb{R}^4/L$ belong to $A \setminus A_0$ (defined in the previous section).
Proof. If \( \dim L \geq 2 \), then \( \dim(\mathbb{R}^4/L) \leq 2 \). By 3.1, these 5 projections lie in a general position; that is, no three of these projections lie on one line. Therefore, \( \mathbb{R}^4/L = 2 \) is the plane.

Any 5 points in a general position on the plane include 4 vertices of a convex quadrangle. The latter contradicts 3.3.

4 Convex spacelike surfaces

Let \( W \) be a quadratic form on \( \mathbb{R}^4 \). Suppose that \( W \) has exactly one negative eigenvalue. Choose future and past cones \( C^+ \) and \( C^- \) for \( W \); that is, \( C^+ \) and \( C^- \) are connected components of the set \( \{ v \in \mathbb{R}^4 \mid W(v) < 0 \} \). A subset \( S \) in \( \mathbb{R}^4 \) will be called spacelike if \( W(x - y) \geq 0 \) for any \( x, y \in S \).

Let \( K \) be a convex body in \( \mathbb{R}^4 \); denote by \( \Sigma \) the surface of \( K \). A point \( p \) lies on the upper side of \( \Sigma \) (briefly \( p \in \Sigma^+ \)) if there is a spacelike hyperplane in \( \mathbb{R}^4 \) that supports \( \Sigma \) at \( p \) from above; more precisely if the Minkowski sum \( \{ p \} + C^+ \) does not intersect \( K \).

Similarly, we define the lower side of \( \Sigma \) denoted by \( \Sigma^- \). Note that \( \Sigma^+ \) and \( \Sigma^- \) might have common points. The subsets \( \Sigma^+ \) and \( \Sigma^- \) are spacelike; in particular, the length of any Lipschitz curve in these subsets can be defined and it leads to induced intrinsic pseudometrics on \( \Sigma^+ \) and \( \Sigma^- \). Abusing notation, we will not distinguish a pseudometric space and the corresponding metric space.

4.1. Lemma. Let \( \Sigma \) be the surface of a convex set \( K \) in \( \mathbb{R}^4 \) and \( C^\pm \) be the future and past cones for a quadratic form \( W \). Then the upper and lower sides \( \Sigma^+ \) and \( \Sigma^- \) of \( \Sigma \) equipped with the induced intrinsic metric are \( \text{CAT}(0) \) length spaces.

Moreover, if a line segment \( [pq] \) in \( \mathbb{R}^4 \) lies on \( \Sigma^\pm \), then \( [pq] \) is a minimizing geodesic in \( \Sigma^\pm \); that is,
\[
|p - q|^2_{\Sigma^\pm} = W(p - q).
\]

This lemma is essentially stated by Anatolii Milka [5, Theorem 4]; we give a sketch of alternative proof based on smooth approximation.

Sketch. We can assume that \( W \) is nondegenerate; that is, after a linear change of coordinates it is the standard form on \( \mathbb{R}^{3,1} \). If not, then there is a \( W \)-preserving projection of \( \mathbb{R}^4 \) to a \( W \)-nondegenerate subspace; apply this projection and note that this subspace is isometric a subspace of \( \mathbb{R}^{3,1} \).

Assume \( S \) is a smooth strictly spacelike hypersurface in \( \mathbb{R}^{3,1} \) with convex epigraph. By Gauss formula, \( S \) has nonpositive sectional curvature.

Suppose a strictly spacelike hyperplane \( \Pi \) cuts from \( S \) a disc \( D \). Recall that Liberman’s lemma [5, Theorem 3] implies that time coordinate is convex on any geodesic in \( S \). We may assume that time is vanishing on \( \Pi \); therefore, by the
lemma, $D$ has a convex set in $S$. Therefore the Cartan–Hadamard theorem [3] implies that that $D$ is CAT(0).

Now suppose $D_n$ is a sequence of smooth discs of the described type that converges to a (possibly nonsmooth) disc $D$. Note that the metric on $D_n$ converges to the induced pseudometric on $D$. It follows that the metric space $D'$ that corresponds to $D$ is CAT(0).

The disc $D$ might contain lightlike segments which have zero length. Note that every maximal lightlike segment in $D$ starts at its interior point and goes to the boundary. Consider the map $\iota: D \to D$ that sends each maximal lightlike segment to its starting point. Note that the sublemma below implies that $\iota$ is length-nonincreasing. Since $|x - \iota(x)|_D = 0$, we get that the $D'$ is isometric to the image of $\iota$ with the induced metric.

Consider the Minkowski sum

$$K^- = K + C^+;$$

it has a convex spacelike boundary $\partial K^-$. Choose a strictly spacelike hyperplane $\Pi$ that lies above $K$. Denote by $D$ the subset of $\partial K^-$ below $\Pi$. Let us equip $D$ with induced intrinsic pseudometric. By construction $\Sigma^-$ is isometric to $\iota(D)$. It follows that $\Sigma^-$ is CAT(0).

Now suppose a line segment $[pq]$ in $\mathbb{R}^4$ lies on $\Sigma^-$. Choose a supporting hyperplane $\Pi$ at the midpoint of $[pq]$. Choose time coordinate that vanish on $\Pi$; by Liberman's lemma, every shortest path in $\Sigma^-$ between $p$ and $q$ has to lie on $\Pi$; that is, the intersection $\Sigma^- \cap \Pi$ is a convex subset of $\Sigma^-$. Therefore $[pq]$ is convex in $\Sigma^-$ which implies the second statement.

4.2. Sublemma. Let $u$ and $v$ be two lightlike vectors in $\mathbb{R}^{3,1}$. Suppose that the union of two half-lines $s \mapsto p + s \cdot u$ and $t \mapsto q + t \cdot v$ for $s, t \geq 0$ is a spacelike set. Then the function $(s, t) \mapsto |(p + s \cdot u) - (q + t \cdot v)|$ is nondecreasing in both arguments, where $|w| := \sqrt{(w, w)}$ for a spacelike vector $w$.

Proof. Since $u$ and $v$ are lightlike, $(u, u) = (v, v) = 0$. Since the union of two half-lines is spacelike, $(p + s \cdot u) - (q + t \cdot v)$ is spacelike for any $s, t \geq 0$. It follows that

$$0 \leq |(p + s \cdot u) - (q + t \cdot v)|^2 =$$

$$= |p - q|^2 - 2 \cdot s \cdot (u, q - p) - 2 \cdot t \cdot (v, p - q) - 2 \cdot s \cdot t \cdot (u, v)$$

for any $s, t \geq 0$. Therefore

$$(u, q - p) \leq 0, \quad (v, p - q) \leq 0, \quad (u, v) \leq 0.$$

Whence the result.\[\square\]
Assume $v$ is a nonzero vector in $\mathbb{R}^4$ and $p \in \Sigma$. We say that $p$ lies on the upper side of $\Sigma$ with respect to $v$ (briefly $p \in \Sigma^+(v)$) if $p + t \cdot v \notin K$ for any $t > 0$. Correspondingly, $p$ lies on the lower side of $\Sigma$ with respect to $v$ (briefly $p \in \Sigma^-(v)$) if $p + t \cdot v \notin K$ for any $t < 0$.

4.3. Observation. Let $K$ be a compact convex set in $\mathbb{R}^4$ and $C^\pm$ be the future and past cones for a quadratic form $W$. Then the upper (lower) side of the boundary surface $\Sigma$ of $K$ can be described as the intersection of the upper (respectively lower) sides of $\Sigma$ with respect to all vectors $v \in C^+$; that is,

$$\Sigma^\pm = \bigcap_{v \in C^+} \Sigma^\pm(v).$$

5 Proof assembling

Proof of Toyoda’s theorem. Let $\{x_1, \ldots, x_5\}$ be the points in $P$. Choose a 5-simplex $\Delta$ in $\mathbb{R}^4$; denote by $W$ the form associated with the point array $(x_1, \ldots, x_5)$.

If $W \geq 0$, then $P$ admits a distance preserving embedding into Euclidean 4-space, so one can take the convex hull of its image as $X$.

Suppose $W(v) < 0$ for some $v \in \mathbb{R}^4$. Since $P$ is CAT(0), 3.4 implies that $W$ has exactly one negative eigenvalue. Moreover, if a line $L$ is spanned by a vector $v$ such that $W(v) < 0$, then the projection of the vertices of the simplex to $\mathbb{R}^3 = \mathbb{R}^4/L$ belongs to $A \backslash A_0$.

The space of such lines $L$ is connected. By 2.1, we can assume that all the projections belong to $A_-$. That is, we can choose timelike orientation such that for any $v \in C^+$ the lower part $\Sigma^-(v)$ of $\Sigma = \partial \Delta$ has at least 3 facets of $\Delta$.

In particular, $\Sigma^-(v)$ contains all edges of $\Delta$ for any $v \in C^+$. By 4.3, $\Sigma^-$ contains all edges of $\Delta$. By 4.1, $\Sigma^-$ with induced (pseudo)metric is a length CAT(0) space.

Since all edges of $\Delta$ lie in $\Sigma^-$, the inclusion $P \hookrightarrow \Sigma^-$ is distance preserving. Whence we can take $X = \Sigma^-$.

Finally, observe that in each case $X$ is a subcomplex of $\Delta$ that includes all edges and has a model metric on each simplex. \hfill $\square$

6 Remarks

Let us recall the definition of graph comparison given by Vladimir Zolotov and the authors [4] and use it to formulate a few related questions.

Let $\Gamma$ be a graph with vertices $v_1, \ldots, v_n$. A metric space $X$ is said to meet the $\Gamma$-comparison if for any set of points in $X$ labeled by vertices of $\Gamma$ there is a model configuration $\tilde{v}_1, \ldots, \tilde{v}_n$ in the Hilbert space $\mathbb{H}$ such that if $v_j$ is adjacent to $v_j$, then

$$|\tilde{v}_i - \tilde{v}_j|\mathbb{H} \leq |v_i - v_j|_X$$
and if \( v_j \) is nonadjacent to \( v_j \), then

\[
|\tilde{v_i} - \tilde{v_j}| \geq |v_i - v_j|.
\]

The \( C_4 \)-comparison (for the 4-cycle \( C_4 \) on the diagram) defines CAT(0) comparison. Tetsu Toyoda have shown that \( C_4 \)-comparison implies graph comparisons for all cycles \( C_n \) [9]; remarkably, the metric space is not assumed to be intrinsic. The \( O_3 \)-comparison (for the octahedron graph \( O_3 \) on the diagram)

\[
C_4 \quad O_3
\]

defines another comparison. Since \( O_3 \) contains \( C_4 \) as an induced subgraph, we get that \( O_3 \)-comparison is stronger than \( C_4 \)-comparison.

6.1. Open question. Is it true that octahedron-comparison holds in any 6 points in a length CAT(0) space?

And, assuming the answer is affirmative, what about the converse: is it true that any 6-point metric space that satisfies octahedron-comparison admits a distance preserving embedding in a length CAT(0) space?

The analogous questions for spaces with nonnegative curvature in the sense of Alexandrov (briefly CBB(0)) are open as well. The CBB(0) comparison is equivalent to the 3-tree comparison (for the tripod-tree shown first on the following diagram). It turns out that any length CBB(0) space satisfies the

\[
\begin{align*}
&\text{3-tree} & \text{4-tree} & \text{5-tree} & \text{6-tree} & \cdots \\
&\text{2(2)-tree} & \text{3(1)-tree}
\end{align*}
\]

comparison for the other trees on the diagram; it is formed by an infinite family of star-shaped trees and two trees with 6 vertices [1, 4]. (The 4-tree comparison (the second tree on the diagram) is equivalent to the so-called \((4+1)\)-point comparison in the terminology of [1].)

We expect that this comparison provides a necessary and sufficient condition for 5-point sets. Namely, we expect an affirmative answer to the following stronger question.

6.2. Question. Suppose a 5-point metric space \( P \) satisfies the 4-tree comparison. Is it true that \( P \) admits a distance preserving embedding into a length CBB(0) space?
Finally, let us mention a related question about a 6-point condition.

6.3. Question. Suppose a 6-point metric space $P$ satisfies the 5-tree, 2(2)-tree, and 3(1)-tree comparisons. Is it true that $P$ admits a distance preserving embedding into a length CBB(0) space?

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