A Sharpened Nuclearity Condition and the Uniqueness of the Vacuum in QFT

Wojciech Dybalski

Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany. E-mail: dybalski@theorie.physik.uni-goettingen.de

Received: 30 August 2007 / Accepted: 10 December 2007
Published online: 29 May 2008 – © The Author(s) 2008

Abstract: It is shown that only one vacuum state can be prepared with a finite amount of energy and it appears, in particular, as a limit of physical states under large timelike translations in any theory which satisfies a phase space condition proposed in this work. This new criterion, related to the concept of additivity of energy over isolated subsystems, is verified in massive free field theory. The analysis entails very detailed results about the momentum transfer of local operators in this model.

1. Introduction

Since the seminal work of Haag and Swieca [1] restrictions on the phase space structure of a theory formulated in terms of compactness and nuclearity conditions have proved very useful in the structural analysis of quantum field theories [2–6] and in the construction of interacting models [7,8]. However, the initial goal of Haag and Swieca, namely to characterize theories which have a reasonable particle interpretation, has not been accomplished to date. While substantial progress was made in our understanding of the timelike asymptotic behavior of physical states [9–15], several important convergence and existence questions remained unanswered. As a matter of fact, it turned out that the original compactness condition introduced in [1] is not sufficient to settle these issues.

Therefore, in the present article we propose a sharpened phase space condition, stated below, which seems to be more appropriate. We show that it is related to additivity of energy over isolated subregions and implies that there is only one vacuum state within the energy-connected component of the state space, as one expects in physical spacetime [16]. We stress that there may exist other vacua in a theory complying with our condition, but, loosely speaking, they are separated by an infinite energy barrier and thus not accessible to experiments. The convergence of physical states to the vacuum state under large timelike translations is a corollary of this discussion. A substantial part of this work is devoted to the proof that the new condition holds in massive scalar free field theory. As a matter of fact, it holds also in the massless case which will be treated
elsewhere. These last results demonstrate that the new criterion is consistent with the basic postulates of local relativistic quantum field theory [17] which we now briefly recall.

The theory is based on a local net \( \mathcal{O} \rightarrow \mathfrak{A}(\mathcal{O}) \) of von Neumann algebras, which are attached to open, bounded regions of spacetime \( \mathcal{O} \subset \mathbb{R}^{s+1} \) and act on a Hilbert space \( \mathcal{H} \). The global algebra of this net, denoted by \( \mathfrak{A} \), is irreducibly represented on this space. Moreover, \( \mathcal{H} \) carries a strongly continuous unitary representation of the Poincaré group \( \mathbb{R}^{s+1} \times L^1_x \ni (x, \Lambda) \rightarrow U(x, \Lambda) \) which acts geometrically on the net

\[
\alpha_{(x, \Lambda)} \mathfrak{A}(\mathcal{O}) = U(x, \Lambda) \mathfrak{A}(\mathcal{O}) U(x, \Lambda)^{-1} = \mathfrak{A}(\Lambda \mathcal{O} + x). \tag{1.1}
\]

We adopt the usual notation for translated operators \( \alpha_x A = A(x) \) and functionals \( \alpha_x^* \varphi(A) = \varphi(\alpha_x(A)) \), where \( A \in \mathfrak{A} \), \( \varphi \in \mathfrak{A}^* \), and demand that the joint spectrum of the generators of translations \( H, P_1, \ldots, P_s \) is contained in the closed forward lightcone \( \overline{\mathbb{V}}_+ \).

We denote by \( P_E \) the spectral projection of \( H \) (the Hamiltonian) on the subspace spanned by vectors of energy lower than \( E \). Finally, we identify the predual of \( B(\mathcal{H}) \) with the space \( T \) of trace-class operators on \( \mathcal{H} \) and denote by \( \dot{T}_E = P_E TP_E \) the space of normal functionals of energy bounded by \( E \). We assume that there exists a vacuum state \( \omega_0 \in \dot{T}_E \) and introduce the subspace \( \dot{T}_E = \{ \varphi - \varphi(I)\omega_0 \mid \varphi \in \dot{T}_E \} \) of functionals with the asymptotically dominant vacuum contribution subtracted.

The main object of our investigations is the family of maps \( \Pi_E : \dot{T}_E \rightarrow \mathfrak{A}(\mathcal{O})^* \) given by

\[
\Pi_E(\varphi) = \varphi|_{\mathfrak{A}(\mathcal{O})}, \quad \varphi \in \dot{T}_E. \tag{1.2}
\]

Fredenhagen and Hertel argued in some unpublished work that in physically meaningful theories these maps should be subject to the following restriction:

**Condition \( C_E \):** The maps \( \Pi_E \) are compact for any \( E \geq 0 \) and double cone \( \mathcal{O} \subset \mathbb{R}^{s+1} \).

This condition is expected to hold in theories exhibiting mild infrared behavior [19]. In order to restrict the number of local degrees of freedom also in the ultraviolet part of the energy scale, Buchholz and Porrmann proposed a stronger condition which makes use of the concept of nuclearity\(^1\) [19]:

**Condition \( N_E \):** The maps \( \Pi_E \) are \( p \)-nuclear for any \( 0 < p \leq 1 \), \( E \geq 0 \) and double cone \( \mathcal{O} \subset \mathbb{R}^{s+1} \).

This condition is still somewhat conservative since it does not take into account the fact that for any \( \varphi \in \dot{T}_E \) the restricted functionals \( \alpha_{x_k}^* \varphi|_{\mathfrak{A}(\mathcal{O})} \) should be arbitrarily close to zero apart from translations varying in some compact subset of \( \mathbb{R}^{s+1} \), depending on \( \varphi \). It seems therefore desirable to introduce a family of norms on \( \mathcal{L}(\dot{T}_E, X) \), where \( X \) is some Banach space, given for any \( N \in \mathbb{N} \) and \( x_1, \ldots, x_N \in \mathbb{R}^{s+1} \) by

\[
\| \Pi \|_{x_1, \ldots, x_N} = \sup_{\varphi \in \dot{T}_{E,1}} \left( \sum_{k=1}^{N} \| \Pi(\alpha_{x_k}^* \varphi) \| \right)^{1/2}, \quad \Pi \in \mathcal{L}(\dot{T}_E, X), \tag{1.3}
\]

\(^1\) We recall that a map \( \Pi : X \rightarrow Y \) is \( p \)-nuclear if there exists a decomposition \( \Pi = \sum_n \Pi_n \) into rank-one maps s.t. \( \nu^p := \sum_n \| \Pi_n \|^p < \infty \). The \( p \)-norm \( \| \Pi \|_p \) of this map is the smallest such \( \nu \) and it is equal to zero for \( p > 1 \) [18]. Note that for any norm on \( \mathcal{L}(X, Y) \) one can introduce the corresponding class of \( p \)-nuclear maps. Similarly, we say that a map is compact w.r.t. a given norm on \( \mathcal{L}(X, Y) \) if it can be approximated by finite rank mappings in this norm.