Gaussian Time-Dependent Variational Principle for Bosons

I - Uniform Case

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ABSTRACT

We investigate the Dirac time-dependent variational method for a system of non-ideal Bosons interacting through an arbitrary two body potential. The method produces a set of non-linear time dependent equations for the variational parameters. In particular we have considered small oscillations about equilibrium. We obtain generalized RPA equations that can be understood as interacting quasi-bosons, usually mentioned in the literature as having a gap. The result of this interaction provides us with scattering properties of these quasi-bosons including possible bound-states, which can include zero modes. In fact the zero mode bound state can be interpreted as a new quasi-boson with a gapless dispersion relation. Utilizing these results we discuss a straightforward scheme for introducing temperature.
1 Introduction

The objective of this paper is to exhibit the most general way of obtaining time dependent equations of motion in the Gaussian approximation [1]. This will lead to the so called generalized RPA, when one examines infinitesimal oscillations about equilibrium. The static solution in the uniform case can be obtained using several other methods [2] - [3] leading to a gap in the quasi-boson energy. We show here that the time dependent RPA equations lead to a gapless mode. In fact this must happen because particle number conservation symmetry was broken in the static solution, so the zero gap is exactly the associated Goldstone mode. This can be seen as an alternative to the functional derivative [4] method in the Girardeau-Arnowitt [5] approximation. We stress that to obtain this result the equations of motion must arise from a consistent variational scheme. An important element of this scheme is the fact that even in the uniform case the variational quantities have to depend on two momenta. It is actually this feature which leads to a gapless result. Since this solves the problem of the gap, truncations normally used to avoid it [6] are not necessary.

We would like to point out that due to the likelihood of increased densities in future experiments on Bose-Einstein condensation in atomic traps [7] - [9] our time dependent equations may be useful to study non-equilibrium evolution. The RPA equations also give us the possibility of investigating the scattering of two quasi-bosons in the medium. Information about scattering can be used in calculating the free energy at finite temperature [10] which takes into account the gapless dispersion relation in contrast to the so called Temperature Hartree Fock Bogoliubov method [13].

The structure of this paper is as follows. Section 2 reviews the time-dependent variational principle and the canonical nature of the equations of motion arising from it. In section 3 we discuss the static solution. Section 4 obtains the small oscillation approximation (RPA) for the equations of motion. The Goldstone mode is discussed in Section 5. In section 6
we show the connection with the Bogoliubov transformation and calculate the one and two quasi-boson wave functions. In section 7 we show how the two quasi-boson scattering affects the thermodynamics using well know methods os statistical mechanics.

2 General Formalism

In this section we shall review Time-dependent Variational Principle and show how it can be implemented in the non-relativistic many-body case. First we define an effective action functional for the time-dependent quantum system

\[ S = \int L(t)dt = \int dt\langle \Psi, t | (i\partial_t - \hat{H})|\Psi, t \rangle, \] (1)

where \( |\Psi, t \rangle \) is the quantum state of the system and \( \hat{H} \) is the Hamiltonian of the theory. For a system of non-relativistic interacting Bosons we have [we use the notation : \( \int_x f d^3x \)]

\[ \hat{H} = \int_{x,y} \hat{\psi}(x)^\dagger h(x,y)\hat{\psi}(y) + \frac{1}{2} \int_{x,y} \hat{\psi}(y)^\dagger \hat{\psi}(x)^\dagger V(x - y)\hat{\psi}(x)\hat{\psi}(y), \] (2)

where the one body Hamiltonian \( h(x,y) \) may include a one body external potential. The operators \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) can be written in the form

\[ \hat{\psi}(x) = \frac{1}{\sqrt{2}} \left[ \hat{\phi}(x) + i\hat{\pi}(x) \right] \] (3)

\[ \hat{\psi}(x)^\dagger = \frac{1}{\sqrt{2}} \left[ \hat{\phi}(x) - i\hat{\pi}(x) \right] \] (4)

where \( \hat{\phi}(x) \) is the field operator and \( \hat{\pi}(x) \) the canonical field momentum.

We can obtain the time dependent Schrödinger equation by requiring that \( S \) is stationary, supplemented by appropriate boundary conditions, under the most general variation of \( |\Psi, t \rangle \).

The variational scheme is implemented by chosing a trial wave functional describing the system. Working in the functional Shrödinger picture we replace the abstract state \( |\Psi, t \rangle \) by a wave functional of the field \( \phi'(x) \)

\[ |\Psi, t \rangle \rightarrow \Psi[\phi', t]. \] (5)

\footnote{This material is substantially the same as some sections of \cite{14} and it is included here for completeness}
The action of the operators $\hat{\phi}(x)$ and the canonical momentum $\hat{\pi}(x)$ are realized respectively by

$$\hat{\phi}(x)|\Psi, t\rangle \rightarrow \phi'(x)|\Psi, t\rangle,$$

$$\hat{\pi}(x)|\Psi, t\rangle \rightarrow -i \frac{\delta}{\delta \phi'(x)}|\Psi, t\rangle.$$

The mean value of any operator is calculated by the functional integral

$$\langle \Psi, t|O|\Psi, t\rangle = \int (D\phi') \Psi^*[\phi', t]O\Psi[\phi', t],$$

where $\Psi$ is normalized to unity. The Gaussian approximation consists in taking a Gaussian trial wave functional in its most general parametrization

$$\Psi[\phi', t] = N \exp \left\{- \int_{x,y} \delta \phi'(x, t) \left[ \frac{G^{-1}(x, y, t)}{4} - i \Sigma(x, y, t) \right] \delta \phi'(y, t) + i \int_{x} \pi(x, t) \delta \phi'(x, t) \right\},$$

with $\delta \phi'(x, t) = \phi'(x) - \phi(x, t)$. Due to the fact that the Hamiltonian commutes with the number of particles

$$[\hat{H}, \hat{N}] = 0$$

we can define a more general trial functional

$$|\Psi', t\rangle = e^{-i \hat{N} \theta(t)}|\Psi, t\rangle,$$

where $\theta(t)$ is another variational parameter introduced because of this continuous symmetry. Thus our variational parameters are $\phi(x, t)$, $\pi(x, t)$, $\theta(t)$, $G(x, y, t)$ and $\Sigma(x, y, t)$, with $G$ and $\Sigma$ being real symmetric matrices. These quantities are related to the following mean-values:

$$\langle \Psi', t|\hat{\phi}(x)|\Psi', t\rangle = \phi(x, t)$$

$$\langle \Psi', t|\hat{\pi}(x)|\Psi', t\rangle = \pi(x, t)$$

$$\langle \Psi', t|\hat{\phi}(x)\hat{\phi}(y)|\Psi', t\rangle = G(x, y, t) + \phi(x, t)\phi(y, t)$$
\[
\langle \Psi', t|\hat{\pi}(x, t)\hat{\pi}(y, t)|\Psi', t \rangle = \frac{G^{-1}(x, y, t)}{4} + 4 \int_{w, z} \Sigma(x, w, t)G(w, z, t)\Sigma(z, y, t) \\
+ \pi(x, t)\pi(y, t)
\]
\(15\)

\[
i\langle \Psi', t|\hat{\phi}(x, t)\hat{\pi}(y, t)|\Psi', t \rangle = -\frac{\delta(x - y)}{2} + i \int_z [\Sigma(x, z, t)G(z, y, t) + G(y, z, t)\Sigma(z, x, t)] \\
+ i\phi(x, t)\pi(y, t)
\]
\(16\)

\[
\langle \Psi', t|i\delta_t|\Psi', t \rangle = \int_{x, y} \Sigma(x, y, t)\dot{G}(y, x, t) + \int_x \pi(x, t)\dot{\phi}(x, t) \\
+ N\dot{\theta}(t) + \text{total time derivatives.}
\]
\(17\)

We may ignore the total time derivatives because they do not contribute to the equations of motion. If now we write the action we will get

\[
S = \int dt \left( \int_x \pi(x, t)\dot{\phi}(x, t) + \int_{x, y} \Sigma(x, y, t)\dot{G}(y, x, t) + N\dot{\theta}(t) - \mathcal{H} \right),
\]

where

\[
\mathcal{H} = \langle \Psi', t|\hat{H}|\Psi', t \rangle
\]
\(19\)

and

\[
\mathcal{N} = \langle \Psi', t|\hat{N}|\Psi', t \rangle.
\]
\(20\)

From (18) we see that \((\mathcal{N}, \theta), (\pi, \phi)\) and \((\Sigma, \dot{G})\) are canonical pairs. Because of the symmetry \(\mathcal{H}\) has no dependence on \(\theta\) and it follows that \(\dot{N} = 0\) and \(\dot{\theta}(t) = \text{constant} \equiv \mu\). We can now write the remaining Hamilton equations,

\[
\dot{\phi}(x, t) = \delta(\mathcal{H} - \mu\mathcal{N}) \begin{pmatrix} \pi(x, t) \end{pmatrix},
\]
\(21\)

\[
\dot{\pi}(x, t) = -\delta(\mathcal{H} - \mu\mathcal{N}) \begin{pmatrix} \phi(x, t) \end{pmatrix},
\]
\(22\)

\[
\dot{G}(x, y, t) = \delta(\mathcal{H} - \mu\mathcal{N}) \begin{pmatrix} \Sigma(x, y, t) \end{pmatrix},
\]
\(23\)

\[
\dot{\Sigma}(x, y, t) = -\delta(\mathcal{H} - \mu\mathcal{N}) \begin{pmatrix} G(x, y, t) \end{pmatrix}.
\]
\(24\)

For convenience we introduce

\[
\psi(x, t) \equiv \langle \hat{\psi}(x) \rangle = \frac{\phi(x, t) + i\pi(x, t)}{\sqrt{2}},
\]
\(25\)
so that Eqs. (21)-(22) become

\[ \psi(x, t) = \frac{\delta(H - \mu N)}{\delta \psi^*(x, t)} \]  

(26)

It is well known that a Gaussian functional leads to the mean field factorization \[2\] i.e.

\[ H - \mu N = \int \rho(x, y) \left( \left[ h(x, y) - \mu \delta(x - y) \right] \rho(x, y, t) + \frac{1}{2} V(x - y) |\psi(x, t)|^2 |\psi(y, t)|^2 \right) + \int \frac{1}{2} \rho(x, y) \left[ \frac{1}{2} \psi^*(x, t) \psi(y, t) R(x, y, t) + \frac{1}{2} \psi^*(y, t) \psi(x, t) R(y, x, t) + |\psi(x, t)|^2 R(y, y, t) \right] - \frac{1}{2} \int \rho(x, y) \left[ \psi(x, t) \psi(y, t) D^*(x, y, t) + \psi^*(x, t) \psi^*(y, t) D(x, y, t) \right] \]  

(27)

and

\[ \rho(x, y, t) = \langle \psi^\dagger(x) \psi(y) \rangle = \psi^*(x, t) \psi(y, t) + R(x, y, t) \]  

(28)

\[ \Delta(x, y, t) = -\langle \psi(x) \psi(y) \rangle = -\psi(x, t) \psi(y, t) + D(x, y, t) \]  

(29)

with

\[ R(x, y, t) = \frac{1}{2} \left[ \frac{G^{-1}(x, y, t)}{4} + G(x, y, t) - \delta(x - y) \right] + 2 \int_{w, z} \Sigma(x, w, t) G(w, z, t) \Sigma(z, y, t) \]  

(30)

\[ D(x, y, t) = \frac{1}{2} \left[ \frac{G^{-1}(x, y, t)}{4} - G(x, y, t) \right] + 2 \int_{w, z} \Sigma(x, w, t) G(w, z, t) \Sigma(z, y, t) \]  

(31)

because of (12)-(14). We note that the density gets contributions from the condensate field \( \psi \) as well as from the fluctuations \( G, \Sigma \). The contribution from \( \psi^* \psi \) is the condensate density.

We introduce the generalized potentials

\[ U_d(x, y, t) = \delta(x - y) \int z \rho(z, z, t) V(x - z) \]  

(32)

\[ U_e(x, y, t) = \rho(x, y, t) V(x - y) \equiv U_e^d + i U_e^i \]  

(33)

\[ U_p(x, y, t) = \Delta(x, y, t) V(x - y) \equiv U_p^d + i U_p^i \]  

(34)
where the notation emphasizes real and imaginary parts of $U_p$. We also define the matrices

\begin{align*}
C(x, y, t) &= h(x, y) - \mu + U_e(x, y, t) + U_d(x, y, t) \quad (35) \\
A(x, y, t) &= C^e(x, y, t) + U'_p(x, y, t) \quad (36) \\
B(x, y, t) &= C^e(x, y, t) - U'_p(x, y, t) \quad (37)
\end{align*}

From Eq. (26)-(35) we obtain an abstract matrix form of the equations of motion

\begin{align*}
\dot{\Sigma} &= \frac{1}{8} G^{-1}AG^{-1} - 2\Sigma A\Sigma - \frac{B}{2} + \{U'_p, \Sigma\} - [[U'_e, \Sigma]] \quad (38) \\
\dot{G} &= \{A, \{G, \Sigma\}\} - \{U'_p, G\} - [[U'_e, G]] \quad (39) \\
\dot{\psi} &= C\psi - U_p\psi^* \quad (40)
\end{align*}

where we have used the fact that $(\Sigma, G)$ are symmetric matrices. These equations (38)-(40) are the nonlinear field equations for an arbitrary interaction $V$ between the particles and contain any external potential through $h$. As an example the matrix product $G^{-1}AG^{-1}$ can be written in coordinate representation as

\begin{equation}
\int_{z,w} G^{-1}(x, z, t) A(z, w) G^{-1}(w, y, t).
\end{equation}

The static equations can be obtained by setting the canonical momenta to zero, that is $\Sigma(x, t) = \pi(x, t) = 0, \dot{G}(x, y, t) = \phi(x, t) = 0$. From (35), (36) and (37) we then have

\begin{align*}
R(x, y, 0) &= R(x, y) = \frac{1}{2} \left[ \frac{G^{-1}(x, y)}{4} + G(x, y) - \delta(x - y) \right] \quad (42) \\
D(x, y, 0) &= D(x, y) = \frac{1}{2} \left[ \frac{G^{-1}(x, y)}{4} - G(x, y) \right] \quad (43) \\
\psi(x, 0) &= \psi(x) = \frac{\phi(x)}{\sqrt{2}} \quad (44)
\end{align*}

So that for the static case we have to self consistently solve

\begin{align*}
\frac{1}{4} \int_{z,w} G^{-1}(x, z) A(z, w) G^{-1}(w, y) - B(x, y) &= 0 \quad (45) \\
\int_{z} [B(x, z)\psi(z) - 2\psi(x)\psi^2(z)V(x - z)] &= 0. \quad (46)
\end{align*}
using (32)-(35). We note that the constraint \( G = 1/2 \) for the static solution leads to \( R = D = 0 \) and \( A = B \) so that equation (44) becomes the usual non linear equation for the single quantity \( \psi \) obtained from a many body permanent for bosons [12]. However the time dependent eqs. (38)-(40) are more general, because our trial Gaussian is actually a coherent state with an indefinite number of particles.

### 3 Uniform Static Solution

To solve equations (45) and (46) in the uniform case we go to the momentum basis where \( G \) is diagonal simultaneously with \( A, B \) and \( C \). There exists a straightforward generalization for the nonuniform case where \( A, B \) and \( C \) do not commute. In the uniform case \( \phi(k) = \phi\delta(k) \). The static equations (45) and (46) become

\[
\frac{1}{4} G^{-2}(k) A(k) - B(k) = 0 \tag{47}
\]

\[
\phi \left[ B(0) - \lambda \phi^2 \right] = 0 \tag{48}
\]

Where we have defined \( \lambda \equiv \tilde{V}(0) \), \( \tilde{V} \) indicates the Fourier transform of the potential and \( e(k) \equiv \frac{k^2}{2m} \). The diagonal matrices \( A \) and \( B \) (36)-(37) can be written as

\[
A(k) = e(k) - \mu + U_p(k) + U_e(k) + U_d(k) \tag{49}
\]

\[
B(k) = e(k) - \mu - U_p(k) + U_e(k) + U_d(k). \tag{50}
\]

In this representation the generalized potentials (33)-(34) become [we use the notation : \( \int_k = (2\pi)^{-3} \int d^3k \)]

\[
U_d(k) = \lambda \int_{k'} \rho(k') \tag{51}
\]

\[
U_e(k) = \int_{k'} \rho(k') \tilde{V}(k - k') \tag{52}
\]

\[
U_p(k) = \int_{k'} \Delta(k') \tilde{V}(k - k') \tag{53}
\]
The solutions for (47) and (48) are
\[ G(k) = \frac{1}{2} \sqrt{\frac{A(k)}{B(k)}} \] (54)
\[ \phi = 0 \quad \text{or} \quad B(0) = \lambda \phi^2, \] (55)
where the second solution is symmetry breaking because \( \langle \psi \rangle = \phi/\sqrt{2} \neq 0 \). For stability reasons we shall see (in section 4) that both \( A \) and \( B \) must be positive which means that using (55) there is no \( \phi \neq 0 \) solution for \( \lambda < 0 \). Using (54) with (42) and (43) we can express \( D \) and \( R \) as functions of \( A, B \)
\[ D(k) = \frac{1}{4} B(k) - A(k) \] (56)
\[ R(k) = \frac{1}{2} \left\{ \frac{B(k) + A(k)}{2 \sqrt{A(k)B(k)}} - 1 \right\}. \] (57)

From (28), (29), (49) and (50) we have
\[ A(k) - B(k) = 2 \int_{k'} D_{k'} \tilde{V}(k' - k) - \lambda \phi^2 \] (58)
\[ A(k) + B(k) = 2(e(k) - \mu) + \lambda \phi^2 + \int_{k'} \left[ \lambda + \tilde{V}(k' - k') \right] R(k'). \] (59)

For a given density \( \rho \) of the system we can write the constraint relation \( \mathcal{N}/V = \rho \) which gives us
\[ \rho = \phi^2 + \int_{k'} R(k'). \] (60)

To find a solution in the case \( \phi = 0 \) we have to solve eqs. (28), (59) and (60). Note that \( R \) and \( D \) are functions of \( A, B \) through (56) and (57). With these three equations one can obtain \( A, B \) and \( \mu \).

For the symmetry breaking solution \( \phi \neq 0 \), \( \mu \) can be determined using the equation \( B(0) = \lambda \phi^2 \) which gives us
\[ \mu = U_e(0) + U_d(0) - U_p(0) - \lambda \phi^2. \] (61)

Which means that solving (58)-(61) gives us \( A, B \) and \( \phi \).
4 Equations of Motion in the Small Oscillation Regime

Generalized RPA

To find the RPA equations we expand all the quantities around their equilibrium values, thus

\[ G(k, k', t) = G(k)\delta(k - k') + \delta G(k, k', t) \]  
\[ \Sigma(k, k', t) = \delta \Sigma(k, k', t) \]  
\[ \phi(k, t) = \phi\delta(k) + \delta \phi(k, t) \]  
\[ \pi(k, t) = \delta \pi(k, t) \].

For simplicity we write \( G \) and \( \Sigma \) in the basis where the equilibrium \( G \) is diagonal and keep terms up to first order in small quantities. For the uniform case the diagonal basis will be plane waves. It will be useful to introduce new momentum coordinates so that

\[ P = k - k' \]  
\[ q = \frac{k + k'}{2} \]

so that

\[ \delta G(k, k') \rightarrow \delta G(P, q) \]  
\[ \delta G^*(P, q) = \delta G(-P, q). \]

Note that from (18) the canonical variable to \( \delta G(P, q, t) \) is \( \delta \Sigma(-P, q) \). We will see that \( P \) and \( q \) can be interpreted as total and relative momenta respectively so that we can write the RPA equations in the form where \( P \) is diagonal and can be considered as a dummy variable

\[ \delta \dot{G}(q, P, t) = s_K(q, P)\delta \Sigma(q, P, t) + c_K(q, P)\delta \pi(P, t) + \int_{q'} S_K(q, q', P)\delta \Sigma(q', P, t) \]

In equations (62) and (64) the first delta is a Dirac delta and the second means an infinitesimal change.
\[
- \delta \dot{\Sigma}(q, P, t) = s_M(q, P) \delta G(q, P, t) + c_M(q, P) \delta \phi(P, t) + \int_{q'} S_M(q, q', P) \delta G(q', P, t) \tag{71}
\]

\[
\delta \dot{\phi}(P, t) = \delta \pi(P, t) A(P) + \int_{q'} c_K(q', P) \delta \Sigma(q', P, t) \tag{72}
\]

\[
- \delta \dot{\pi}(P, t) = \delta \phi(P, t) B(P) + \int_{q'} c_M(q', P) \delta G(q', P, t). \tag{73}
\]

Where we note that the \((\pi, \phi)\) degrees of freedom are coupled to the much more numerous degrees of freedom \((\Sigma, G)\) which are labeled by \(q\). Different \(q\) values among \((\Sigma, G)\) are also coupled. Introducing the notation \(f(q' + P/2) = f'_+\) and \(f(q - P/2) = f_-\), we find non-diagonal matrices in \((q, q')\)

\[
S_K(q, q', P) = 2 \tilde{V}(q - q') \left[ G_+ G'_+ + G_- G'_- \right] \tag{74}
\]

\[
S_M(q, q', P) = \frac{\tilde{V}(q - q')}{2} + \left[ 1 - \frac{G_+^{-1} G_-^{-1}}{4} \right] \frac{\tilde{V}(P)}{4} \left[ 1 - \frac{G'_+^{-1} G'_-^{-1}}{4} \right] + \left[ \frac{G_+^{-1} G_-^{-1}}{4} \right] \frac{V(q - q')}{2} \left[ \frac{G'_+^{-1} G'_-^{-1}}{4} \right] \tag{75}
\]

and diagonal elements

\[
s_K(q, P) = 2 [A_+ G_- + A_- G_+] \tag{76}
\]

\[
s_M(q, P) = \frac{G_+^{-2} G_-^{-1} A_+ + G_-^{-2} G_+^{-1} A_-}{8}. \tag{77}
\]

Finally we see the coupling elements between \((\pi, \phi)\) and \((\Sigma, G)\)

\[
c_K(q, P) = \frac{\phi}{2} [G_+ + G_-] \left[ \tilde{V}_+ + \tilde{V}_- \right] \tag{78}
\]

\[
c_M(q, P) = \frac{\phi}{2} \left[ \tilde{V}_+ + \tilde{V}_- + \tilde{V}(P) \left( 1 - \frac{G_+^{-1} G_-^{-1}}{4} \right) \right], \tag{79}
\]

which vanish when the symmetry is conserved \((\phi = 0)\). As pointed out above the equations are diagonal in \(P\) so we can interpret it as the total momentum of a pair of quasi-bosons, where \(-\mathbf{k}'\) is thus interpreted as the momentum of a”hole”. Because \(\delta G, \delta \Sigma\) and \(\delta \phi, \delta \pi\) are canonical variables we may invert the definitions of momentum and coordinate, for reasons that will be clear in the next section, where the zero mode is discussed. We define column
vectors

$$\Theta(q, P, t) \equiv \begin{pmatrix} \Theta_{\Sigma} \\ \Theta_{\pi} \end{pmatrix} \equiv \begin{pmatrix} \delta \Sigma(q, P, t) \\ \delta \pi(P, t) \end{pmatrix} , \quad \Pi(q, P, t) \equiv \begin{pmatrix} \Pi_{G} \\ \Pi_{\phi} \end{pmatrix} \equiv - \begin{pmatrix} \delta G(q, P, t) \\ \delta \phi(P, t) \end{pmatrix} .$$

(80)

We can write a coupled oscillator Hamiltonian that corresponds to the RPA equations of motion in a matrix element form

$$H_{\text{RPA}} = \frac{1}{2} \langle \Theta | \mathcal{A} | \Theta \rangle + \frac{1}{2} \langle \Pi | \mathcal{B} | \Pi \rangle ,$$

(81)

where the matrixes $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathcal{A} = \begin{pmatrix} S_{K} + s_{K} & c_{K} \\ c_{K} & A \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} S_{M} + s_{M} & c_{M} \\ c_{M} & B \end{pmatrix} ,$$

(82)

and the equations of motion are

$$|\dot{\Theta} \rangle = \mathcal{B} |\Pi \rangle$$

(83)

$$|\dot{\Pi} \rangle = - \mathcal{A} |\Theta \rangle$$

(84)

which can be written as second order equations

$$|\ddot{\Theta} \rangle = - \mathcal{B} \mathcal{A} |\Theta \rangle$$

(85)

$$|\ddot{\Pi} \rangle = - \mathcal{A} \mathcal{B} |\Pi \rangle .$$

(86)

We may separate the diagonal part of $H_{\text{RPA}}$ so that

$$H_{\text{RPA}} = H_{0} + H_{\text{int}},$$

(87)

where

$$H_{0} = \frac{1}{2} \begin{pmatrix} \delta \Sigma^{*} & \delta \pi^{*} \end{pmatrix} \begin{pmatrix} s_{K} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \delta \Sigma \\ \delta \pi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta G^{*} & \delta \phi^{*} \end{pmatrix} \begin{pmatrix} s_{M} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \delta G \\ \delta \phi \end{pmatrix}$$

(88)

and $H_{\text{int}}$ is equal to

$$H_{\text{int}} = \frac{1}{2} \begin{pmatrix} \delta \Sigma^{*} & \delta \pi^{*} \end{pmatrix} \begin{pmatrix} S_{K} & c_{K} \\ c_{K} & 0 \end{pmatrix} \begin{pmatrix} \delta \Sigma \\ \delta \pi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta G^{*} & \delta \phi^{*} \end{pmatrix} \begin{pmatrix} S_{M} & c_{M} \\ c_{M} & 0 \end{pmatrix} \begin{pmatrix} \delta G \\ \delta \phi \end{pmatrix} .$$

(89)
Note that $A$ and $B$ must be positive for stability. If we introduce the trivial multiplicative canonical transformation

$$
\begin{align*}
\left(\frac{\delta \Sigma}{\delta \pi}\right) & \rightarrow \left(\frac{\delta \Sigma}{\delta \pi}\right)' = \left(\frac{\delta \Sigma \sqrt{s_M}}{\delta \pi \sqrt{B}}\right) \\
\left(\frac{\delta G}{\delta \phi}\right) & \rightarrow \left(\frac{\delta G}{\delta \phi}\right)' = \left(\frac{\delta G \sqrt{s_M}}{\delta \phi \sqrt{B}}\right)
\end{align*}
$$

we obtain

$$
H_0 = \frac{1}{2} \left(\delta \Sigma \delta \pi\right)' \left(\Omega_2 0 0\right) \left(\delta \Sigma \delta \pi\right)' + \frac{1}{2} \left(\delta G \delta \phi\right)' \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \left(\delta G \delta \phi\right)'.
$$

(92)

Where $\omega$ and $\Omega_2$ are

$$
\omega(P) = \sqrt{A(P)B(P)}
$$

(93)

$$
\Omega_2(q,P) = \sqrt{s_K(q,P)s_M(q,P)}.
$$

(94)

If we use the definitions of $s_K$ and $s_M$ from (76) and (77) we get after same algebra the remarkable result

$$
\Omega_2(q,P) = \sqrt{A_B + \sqrt{A_B} \sqrt{A_B}} = \omega(k) + \omega(k'),
$$

(95)

so that $\omega(P)$ and $\Omega_2(q,P)$ can be interpreted as the one and two free quasi-boson energies.

We note that $\Omega_2(0,P) = 2\omega(P/2)$, which means that at zero relative momentum $\Omega_2(P,0)$ corresponds to two quasi-bosons each with momentum $P/2$. For the $\phi \neq 0$ case the free quasi-boson energies $\omega(P)$ in general have a gap which means, that $\omega(0) \neq 0$. In fact if we use (49), (50) and (61) we get

$$
A(0) = 2U_p^c(0) + \lambda \phi^2
$$

(96)

$$
B(0) = \lambda \phi^2
$$

(97)

because of (29) we write

$$
U_p^c(0) = \int_{k'} \Delta(k')\tilde{V}(k') = -\frac{\lambda}{2} \phi^2 + \int_{k'} D(k')\tilde{V}(k').
$$

(98)
We have then
\[ \omega(0) = \sqrt{A(0)B(0)} = \sqrt{2\lambda\phi^2\int V(k')D(k')} \]
(99)

The oscillations of the \( \delta\phi, \delta\pi \) pair can be interpreted as a quasi-boson mode with a gap while the oscillations of \( \delta G, \delta\Sigma \) can be interpreted as an interacting pair of these same quasi-bosons. When \( \phi = 0 \), we get \( S_3 = S_6 = 0 \) and the one and two quasi-bosons systems are calculated independently. When \( \phi \neq 0 \) we must rediagonalize so that our final modes will be mixtures of one and two quasi-bosons. The variable \( q \) represents the internal motion of the quasi-boson pair with interaction given by the quantities \( S \). In general this is a scattering problem and we must search for the scattering amplitude at a given energy and \( P \), where the boundary conditions are determined by (94). We can write the equations for \( \ddot{\delta G} \) and \( \ddot{\delta\phi} \) using (70)-(73)

\[ \ddot{\delta G}(q, P, t) + \int_{q'}^{q''} S_K(q, q', P)S_M(q', q'', P)\delta G(q'', P, t) + c_K(q, P) \int_q^{q'} c_M(q', P)\delta G(q', P, t) \]
\[ = - \left[ B(P)c_K(q, P) + \int_{q'}^{q''} S_K(q, q', P)c_M(q', P) \right]\delta\phi(P, t) \]
(100)

\[ \ddot{\delta\phi}(P, t) + \left[ A(P)B(P) + \int_{q'}^{q''} c_K(q', P)c_M(q', P) \right]\delta\phi(P, t) = -A(P) \int_q^{q'} c_M(q', P)\delta G(q', P, t) - \int_{q', q''} c_K(q', P)S_M(q', q'', P)\delta G(q'', P, t) \]
(101)

where \( S \) contain both \( S \) and \( s \) defined in (74)-(77) i.e.

\[ S_K(q, q', P) = S_K(q, (q', P) + s_K(q, q', P)\delta(q' - q) \]
(102)

\[ S_M(q, q', P) = S_M(q, q', P) + s_M(q, q', P)\delta(q' - q). \]
(103)

If we look for oscillatory solutions for the equations (100) and (101)

\[ \delta G(q, P, t) = \delta G(q, P, t = 0)e^{i\Omega t} \]
(104)

\[ \delta\Sigma(q, P, t) = \delta\Sigma(q, P, t = 0)e^{i\Omega t} \]
(105)

\[ \delta\pi(P, t) = \pi(P, t = 0)e^{i\Omega t} \]
(106)

\[ \delta\phi(P, t) = \phi(P, t = 0)e^{i\Omega t} \]
(107)
we may rewrite (100) and (101) with the compact mode notation of (86)

\[
\begin{align*}
(\Omega^2 - h_2)\Pi_G &= \left( cKB + \int S_K \right) \Pi_\phi \\
(\Omega^2 - h_1)\Pi_\phi &= \left( A \int c_M + \int c_K S_M \right) \Pi_G.
\end{align*}
\]

We have defined \( h_1 \) and \( h_2 \) as

\[
\begin{align*}
h_1 &= AB + \int_q c_K c_M \\
h_2 &= S_K S_M + c_K c_M
\end{align*}
\]

where \( h_1 \) depends only on \( P \) and \( h_2 \) is understood as a \( P \) dependent operator in \( q \) and \( q' \) with \( c_K c_M \) being of separable form. We may write \( h_1 \) and \( h_2 \) in terms of the free quasi-boson frequencies defined in (93) and (94).

\[
\begin{align*}
h_1 &= \omega^2 + \int_q c_K c_M \\
h_2 &= \Omega_2^2 + S_K S_M + s_K S_M + S_K S_M \equiv \Omega_2^2 + V_0.
\end{align*}
\]

We note that, in this form, \( h_2, (108) \) and \( (109) \) are not hermitian. However it can be shown that they correspond to a fully hermitian form which leads to a unitary \( S \) matrix. We do not enter into this discussion here.

Eliminating \( \Pi_\phi \) from (108) and (109) we get

\[
\left[ (\Omega^2 - h_2) - (cKB + S_K c_M) \frac{1}{(\Omega^2 - h_1 + i\epsilon)} \left( A \int c_M + \int c_K S_M \right) \right] \delta G = 0
\]

Defining the separable potential

\[
V_1 \equiv (cKB + S_K c_M) \frac{1}{(\Omega^2 - h_1 + i\epsilon)} (A c_M + c_K S_M)
\]

we can write

\[
(\Omega^2 - \Omega_2^2) = (V_0 + V_1) \delta G
\]

\[\text{If this were a problem with a discrete spectrum the non hermitian structure has the form } AB \text{ with } A \text{ and } B \text{ both hermitian but non commuting. It is well known that the secular determinant } \text{Det}|\Omega^2 - AB| \text{ is equivalent to that of the hermitian form } \text{Det}|\Omega^2 - \sqrt{AB} \sqrt{A}|, \text{ corresponding to a transformation of the variables } \Pi.\]

15
which lead us to the standard form for the scattering for a given value for \( \Omega = \Omega^2(P, q) \) and \( \Omega' = \Omega^2(P, q') \).

\[
\Pi_G(q, q') = \delta(q - q') + \frac{1}{\Omega^2 - \Omega'^2 - i\epsilon} V\Pi_G \equiv \delta(q - q') + \mathcal{G}_0^{(+)} V\Pi_G. \tag{117}
\]

In the usual fashion we can define a \( T \) matrix as

\[
T = V\Pi_G \tag{118}
\]

so that the equation that determines the matrix \( T \) is

\[
T = V + V\mathcal{G}_0^{(+)} T. \tag{119}
\]

In principle for a given interaction we can solve for the \( T \) matrix obtaining all the scattering properties like phase shifts and bound states. The existence of bound states will be determined by the poles of the \( T \) matrix. Because of the coupling between one and two quasi-bosons there can be one "bound" state which can be interpreted as the dispersion relation of a new RPA-boson \( \Omega_b \). Again using (108) and (109) we can eliminate \( \delta G \) in favor of \( \delta \phi \) and obtain a formal form for the dispersion relation for \( \Omega_b(P) \)

\[
\left( \Omega_b^2 - h_1 \right) - (A_{CM} + c_K S_M\frac{1}{\Omega^2 - h_2} (B_{CM} + c_M S_K) = 0. \tag{120}
\]

This involves the Greens function for \( h_2 \) below the threshold so that the \( i\epsilon \) in not necessary. Because of the broken symmetry (see next section) this dispersion relation will start at zero, in contrast to \( \omega(P) \) given in (13). This can be seen schematically in Fig. 1. If the interaction is attractive enough there can of course be additional bound states. If any bound state actually leads to a negative value for \( \Omega^2 \) the system is unstable, as is usual for the RPA.
5 The Goldstone mode

In order to understand the structure of the Goldstone mode we recognize from (85) that for \( \Omega = 0 \) we must have
\[
A|\theta^{(0)}(0) = 0,
\]
(121)
going back to the more explicit notation we have
\[
\int q' S_K(q, q', P)\theta^{(0)}_\Sigma(q', P) + s_K(q, P)\theta^{(0)}_\Sigma(q, P) + c_K(q, P)\theta^{(0)}_\pi(P) = 0
\]
(122)
\[
\int q' c_K(q', P)\theta^{(0)}_\Sigma(q', P) + \theta^{(0)}_\pi(P)A(P) = 0.
\]
(123)
Eliminating \( \theta^{(0)}_\pi(P) \) from (123) we get an equation for \( \theta^{(0)}_\Sigma \)
\[
\int q' S_K(q, q', P)\theta^{(0)}_\Sigma(q', P) + s_K(q, P)\theta^{(0)}_\Sigma(q, P) - \frac{1}{A(P)}c_K(q, P)\int q' c_K(q', P)\theta^{(0)}_\Sigma(q', P) = 0.
\]
(124)
If \( P = 0 \) we can find a solution for (124)
\[
\theta^{(0)}_\Sigma(q, 0) = 1 - \frac{G^{-2}(q, 0)}{4},
\]
(125)
which arises from the properties of the static solutions. To do so we substitute (125) in (124) and using the static properties (58)-(59) we compute each one of the integrals separately. For convenience we omit the index \( (P = 0) \)
\[
\int q' S_K(q, q') \left[ 1 - \frac{G^{-2}(q)}{4} \right] = 4G(q) \int q' \bar{V}(q - q') \left[ G(q') - \frac{G^{-1}(q')}{4} \right]
\]
(126)
\[
= 4G(q) \left[ B(q) - A(q) - \phi^2\bar{V}(q) \right]
\]
and
\[
\int q' c_K(q') \left[ 1 - \frac{G^{-2}(q)}{4} \right] = 2\phi \int q' \bar{V}(q') \left[ G(q') - \frac{G^{-1}(q')}{4} \right]
\]
(127)
\[
= 2\phi \left[ B(0) - A(0) - \lambda\phi^2 \right] = -2\phi A(0).
\]
It is easy to see that the substitution of (126) and (127) leads to satisfying (124). Once we know $\theta_{\Sigma}^{(0)}$ we can find $\theta_{\pi}^{(0)}$ using (123) so that
\[
\theta_{\pi}^{(0)} = -\frac{1}{A(0)} \int_{q'} c_{\pi}(q') \left[ 1 - \frac{G^{-2}(q)}{4} \right] = 2\phi.
\] (128)
This gives us a particular normalized column vector $\theta^{(0)}$ for the zero mode
\[
\theta^{(0)} = C \left( \left[ \frac{1}{2} \left[ 1 - \frac{G^{-2}(q,0)}{\phi} \right] \right], \phi \right).
\] (129)
with
\[
C = \frac{1}{\sqrt{\int_{q'} \frac{1}{2} \left[ 1 - \frac{G^{-2}(q,0)}{4} \right]^2 + \phi^2}}.
\] (130)
This corresponds to a new coordinate and momentum which are linear combinations of the previous $(\delta G, \delta \phi)$ and $(\delta \Sigma, \delta \pi)$ and can be written as inner products.
\[
Q \equiv (\theta^{(0)}|\Theta) = C \left( \left[ \frac{1}{2} \left[ 1 - \frac{G^{-2}(q,0)}{4} \right] \right], \phi \right) \left( \delta \Sigma \delta \pi \right),
\] (131)
\[
P \equiv (\theta^{(0)}|\Pi) = C \left( \left[ \frac{1}{2} \left[ 1 - \frac{G^{-2}(q,0)}{4} \right] \right], \phi \right) \left( \delta G \delta \phi \right).
\] (132)
To better understand these coordinates we expand $\mathcal{N}$, the mean number of particles, up to first order
\[
\mathcal{N} = \mathcal{N}_0 + \delta \mathcal{N},
\] (133)
where $\mathcal{N}_0$ is the static result given by
\[
\mathcal{N}_0 = \frac{1}{2} \int_{q'} \left[ \frac{G^{-1}(q',0)}{4} + G(q',0) - 1 \right] + \frac{\phi^2}{2}.
\] (134)
The first order term $\delta \mathcal{N}$ is
\[
\delta \mathcal{N} = \frac{1}{2} \int_{q'} \left( 1 - \frac{G^{-2}(q',0)}{4} \right) \delta G(q',0) + \phi \delta \phi.
\] (135)
Which we see is proportional to our new canonical momentum
\[
P = C \delta \mathcal{N}.
\] (136)
From (84) and (121) we have

$$\langle \dot{\theta}^{(0)} | \Pi \rangle = \dot{P} = -(\dot{\theta}^{(0)} | A | \Theta) = 0.$$  \hfill (137)

Indeed this is the general result ($\dot{\delta N} = 0$) proved in section 1 and, as expected, the zero mode corresponds to a "translational" motion in $N$ with no oscillation.

In fact because we have the RPA Hamiltonian (81) it is possible to see the explicit form for the mass coefficient $M$ in the quadratic expression for the $\delta N$ dependence. To see this we first rewrite the RPA Hamiltonian (81) in a basis where $A$ is diagonal, because it contains the zero mode.

$$H_{\text{RPA}} = \frac{1}{2}\theta^* (i | A | i) \theta + \frac{1}{2} P^* (i | B | j) P^j.$$  \hfill (138)

with

$$(0|A|0) = 0.$$  \hfill (139)

This shows that there is no dependence on $\theta^{(0)}$ which corresponds to our original parameter $\theta$ canonical to $N$. The summation on $(i, j)$ is implicit. We now separate the contributions $P^{(0)}$ from the kinetic piece as

$$K \equiv \frac{1}{2} P^* (i | B | j) P_j = \frac{1}{2} P^{m*} (m | B | m') P^{m'} + \frac{1}{2} P^{0*} (0 | B | m) P^m + \frac{1}{2} P^{m*} (m | B | 0) P^0 + \frac{1}{2} P^{0*} (0 | B | 0) P^0.$$  \hfill (140)

where $m$ and $m'$ are different from zero. We can diagonalize $B$ in the subspace getting

$$K = \frac{1}{2} P^{\lambda*} (\lambda | B | \lambda) P^\lambda + \frac{1}{2} P^{0*} (0 | B | \lambda) P^\lambda + \frac{1}{2} P^{\lambda*} (\lambda | B | 0) P^0 + \frac{1}{2} P^{0*} (0 | B | 0) P^0.$$  \hfill (141)

Using the fact that $P^0$ is constant we complete squares

$$K = \frac{1}{2} \left[ P^\lambda + P^0 X^\lambda \right]^* (\lambda | B | \lambda) \left[ P^\lambda + X^\lambda P^0 \right] + \frac{1}{2} P^{0*} (0 | B | 0) P^0 - \frac{1}{2} P^{0*} X^\lambda (\lambda | B | \lambda) X^{\lambda*} P^0.$$  \hfill (142)
where the vector $X$ has to be equal to

$$X^\lambda = (0|\mathcal{B}|\lambda) \frac{1}{(\lambda|\mathcal{B}|\lambda)}.$$  \hspace{1cm} (143)

So that the coefficient of $(P^0)^2/2$ will be

$$\frac{1}{\mathcal{M}} = (0|\mathcal{B}|0) - (0|\mathcal{B}|\lambda) \frac{1}{(\lambda|\mathcal{B}|\lambda)}(\lambda|\mathcal{B}|0)$$

which gives us

$$\mathcal{M} = \frac{1}{(0|\mathcal{B}|0) - (0|\mathcal{B}|\lambda) \frac{1}{(\lambda|\mathcal{B}|\lambda)}(\lambda|\mathcal{B}|0)} \equiv (0|\mathcal{B}|0) \neq \frac{1}{(0|\mathcal{B}|0)}.$$ \hspace{1cm} (145)

Note that this a non-trivial result, and it is the effect of the linear terms in $P^0$ in (140). If we use (136) we can actually see that

$$\frac{(P^0)^2}{2\mathcal{M}} = \frac{C^2 \delta N^2}{2\mathcal{M}}$$

which means we have calculated the coefficient of $\delta N^2$. To see the physical meaning of this coefficient we expand our original $\mathcal{H} - \mu \mathcal{N}$ around the equilibrium value $N_0$

$$\mathcal{H}(\mathcal{N}) - \mu(N_0)\mathcal{N} = \mathcal{H}(N_0) + \delta \mathcal{N} \frac{d\mathcal{H}}{dN_0} + \frac{1}{2} \delta \mathcal{N}^2 \frac{d^2\mathcal{H}}{dN_0^2} - \mu(N_0)\mathcal{N} + \cdots.$$ \hspace{1cm} (147)

Using

$$\mu = \frac{d\mathcal{H}}{dN_0}$$ \hspace{1cm} (148)

we have

$$\mathcal{H}(\mathcal{N}) - \mu(N_0)\mathcal{N} = \mathcal{H}(N_0) - \mu(N_0)N_0 + \frac{1}{2} \delta \mathcal{N}^2 \frac{d^2\mathcal{H}}{dN_0^2}$$ \hspace{1cm} (149)

comparing the $\delta N^2$ coefficients in (146) and (149) we have

$$\frac{d^2\mathcal{H}}{dN_0^2} = \frac{d\mu}{dN_0} = \frac{C^2}{\mathcal{M}}.$$ \hspace{1cm} (150)

Which means that once we have computed $\mathcal{M}$ using (145) we also get the value of the second derivative of the energy with respect to the number of particles.
For the remaining modes we have to work with a Hamiltonian in the sub-space that excludes the zero mode i.e.

$$H_{RPA}^s = \frac{1}{2} P^*_s (i | B_s | j) P^s_j + \frac{1}{2} Q^*_s (i | A_s | j) Q^s_j. \quad (151)$$

If we introduce the canonical transformation

$$P_s \rightarrow \frac{1}{\sqrt{B_s}} P_s \quad (152)$$
$$Q_s \rightarrow \sqrt{B_s} Q_s \quad (153)$$

we get

$$H_{RPA}^s = \frac{1}{2} P^*_s P^s_s + \frac{1}{2} Q^*_s \left( i \sqrt{B_s} A_s \sqrt{B_s} | j \right) Q^s_j. \quad (154)$$

If we now diagonalize the matrix $\sqrt{B_s} A_s \sqrt{B_s}$ we have the final form to be used in subsection 6.2.

$$H_{RPA}^s = \frac{1}{2} \mathcal{P}^\Omega \mathcal{P}^\Omega + \frac{1}{2} \mathcal{Q}^\Omega \Omega^2 \mathcal{Q}^\Omega, \quad (155)$$

where $\Omega^2$ is real for stability and the $\mathcal{P}^\Omega$ and $\mathcal{Q}^\Omega$ are in general complex.

6 General Remarks

6.1 Connection with the Bogoliubov Transformation

One can construct the operator that annihilates our trial wave functional (9) so that

$$\hat{\xi}(t) \Psi[\phi', t] = 0 \quad (156)$$

where

$$\hat{\xi}(t) = \frac{1}{2} \left[ \frac{1}{G^{1/2}} - 4iG^{1/2} \Sigma \right] \delta \hat{\phi} + i \sqrt{2} G^{1/2} \delta \hat{\pi} \quad (157)$$

and

$$\delta \hat{\phi} = \hat{\phi} - \phi, \quad \delta \hat{\pi} = \hat{\pi} - \pi. \quad (158)$$
Note that this result is valid for any particular time \( t \). For the present purpose we will use the static annihilation operator

\[
\hat{\xi}(0) \equiv \hat{\xi} = \frac{1}{2} \frac{1}{G^{1/2}} \delta \hat{\phi} + i G^{1/2} \hat{\pi}
\]

where now both \( G \) and \( \phi \) are the time independent equilibrium results. This will be our quasi-boson annihilation operator. The corresponding creation operator is then

\[
\hat{\xi}^\dagger = \frac{1}{2} \frac{1}{G^{1/2}} \delta \hat{\phi} - i G^{1/2} \hat{\pi}
\]

and as usual

\[
[\hat{\xi}_a, \hat{\xi}_b] = \delta_{a,b} \quad \text{(161)}
\]

\[
[\hat{\xi}_a, \hat{\xi}_b] = [\hat{\xi}_a^\dagger, \hat{\xi}_b^\dagger] = 0 \quad \text{(162)}
\]

then

\[
\delta \hat{\phi} = \sqrt{G} \left[ \hat{\xi} + \hat{\xi}^\dagger \right] \quad \text{(163)}
\]

and

\[
\hat{\pi} = -\frac{i}{2\sqrt{G}} \left[ \hat{\xi} - \hat{\xi}^\dagger \right]. \quad \text{(164)}
\]

We can now make the connection with the Bogoliubov transformation writing the quasi-boson operator as a function of the original \( \hat{\psi} \) as

\[
\hat{\xi} = X \left[ \hat{\psi} - \psi \right] + Y \left[ \hat{\psi}^\dagger - \psi^* \right] \quad \text{(165)}
\]

with

\[
X = \frac{1}{2\sqrt{2}} \left[ \frac{1}{G^{1/2}} + 2G^{1/2} \right] \quad \text{(166)}
\]

\[
Y = \frac{1}{2\sqrt{2}} \left[ \frac{1}{G^{1/2}} - 2G^{1/2} \right] \quad \text{(167)}
\]

Equation (165) is a generalized version of the Bogoliubov transformation where the commutation rules (161) imply that the matrix relation

\[
X^2 - Y^2 = 1 \quad \text{(168)}
\]
\[ D = XY \]  
\[ R = Y^2 \]  

We may write the original Hamiltonian in terms of these quasi-boson operators \( \hat{\xi} \) and \( \hat{\xi}^\dagger \). It is easy to see that there will be terms like \( \phi \hat{\xi} \hat{\xi} \hat{\xi} \hat{\xi}^\dagger \) which turn two quasi-bosons into one justifying our interpretation of the generalized RPA. In the next subsection we will show how the small oscillations correspond to modes that are mixtures of one and two quasi-bosons.

### 6.2 One and two quasi-boson coupling

We proceed by expanding our wave functional around the equilibrium values for \( G \), \( \Sigma \), \( \phi \) and \( \pi \)

\[
\langle \phi' | \Psi(t) \rangle \equiv \Psi[\phi', t] \approx \left[ 1 - \int \delta \phi' \left( \frac{1}{4G} \delta G - i \delta \Sigma \right) \delta \phi' - \int \delta \phi' \left( \frac{1}{2G} \delta \phi - i \delta \pi \right) \right] \Psi_0[\phi']
\]

(171)

where \( \Psi_0[\phi'] \) is the ground state wave functional that is annihilated by \( \hat{\xi} \). If we use (163) and (164) we can write

\[
\Psi[\phi', t] \approx \left\{ 1 - \int \left( \frac{1}{4\sqrt{G}} \delta G \frac{1}{\sqrt{G}} - i \sqrt{G} \delta \Sigma \right) \right. \\
\left. \left( \hat{\xi}^\dagger \hat{\xi}^\dagger + \hat{\xi} \hat{\xi} \right) + \int \left( \frac{1}{2\sqrt{G}} \delta \phi - i \sqrt{G} \delta \pi \right) \right\} \Psi_0[\phi']
\]

(172)

Note that the equilibrium \( G \) can be taken diagonal in a particular representation while in this same representation \( \delta G \) and \( \delta \Sigma \) are non-diagonal. If we expand \( \delta G \) , \( \delta \phi \), \( \delta \Sigma \) and \( \delta \pi \) in the \( \Omega \) modes found by solving the RPA equations we have the general structure. Thus

\[
\begin{align*}
\begin{pmatrix} \delta \Sigma \\ \delta \pi \end{pmatrix} &= \sum_\Omega \rho_\Omega \begin{pmatrix} \rho_\Omega^{\Sigma} \\ \rho_\Omega^{\pi} \end{pmatrix} e^{i \Omega t} \\
\begin{pmatrix} \delta G \\ \delta \phi \end{pmatrix} &= \sum_\Omega Q_\Omega \begin{pmatrix} Q_\Omega^{G} \\ Q_\Omega^{\phi} \end{pmatrix} e^{i \Omega t}
\end{align*}
\]

(173)  
(174)
with the hamiltonian (155) for $P^\Omega$ and $Q^\Omega$. The state vector looks like

$$|\Psi(t)\rangle \approx \left\{ |0\rangle + \sum_\Omega [|1\rangle + |2\rangle] e^{i\Omega t} \right\}$$ (175)

where we have taken into account that $\hat{\xi}$ annihilates the ground state and have defined

$$|1\rangle = \int \Phi^\Omega_1 \xi^\dagger |0\rangle$$ (176)
$$|2\rangle = \int \Phi^\Omega_2 \xi^\dagger \xi^\dagger |0\rangle$$ (177)

where $\Phi^\Omega_1$ and $\Phi^\Omega_2$ are the one and two particle wave functions given by

$$\Phi^\Omega_1 = P^\Omega \frac{1}{2\sqrt{G}} \theta^\Omega_{\pi} - \frac{Q^\Omega}{\sqrt{G}} \theta^\Omega_{\pi}$$ (178)
$$\Phi^\Omega_2 = P^\Omega \frac{1}{2\sqrt{G}} \theta^\Omega_{\Sigma} \frac{1}{2\sqrt{G}} - \frac{Q^\Omega}{\sqrt{G}} \sqrt{G} \theta^\Omega_{\Sigma} \sqrt{G}.$$ (179)

Note that for each value of $\Omega$ (either the bound state $\omega$ or the the continuum $\Omega_2$ ) we have, as expected, a mixture of one and two quasi-bosons. We take into account that for each given mode $\Omega$ have a relationship between $P^\Omega$ and $Q^\Omega$

$$Q^\Omega = \frac{P^\Omega}{i\Omega}.$$ (180)

Then the unnormalized wave functions can be written as

$$\Phi^\Omega_1 = \frac{1}{2\sqrt{G}} \theta^\Omega_{\pi} - \frac{\sqrt{G}}{\Omega} \theta^\Omega_{\pi}$$ (181)
$$\Phi^\Omega_2 = \frac{1}{2\sqrt{G}} \theta^\Omega_{\Sigma} \frac{1}{2\sqrt{G}} - \frac{1}{\Omega} \sqrt{G} \theta^\Omega_{\Sigma} \sqrt{G}.$$ (182)

In order to understand the significance of the bound state for $P = 0$ we can examine its time independent two body component, $\Phi^\omega(P = 0, q)$. In this particular case

$$\Phi^\omega_2 = \frac{1}{\sqrt{G}} \theta^\omega_{\Sigma} \frac{1}{\sqrt{G}}.$$ (183)

\[^4\text{In fact as we note before in section 5 this relation does not hold for the zero mode where } P^0 = \text{constant and } Q^0 = (P/M)t\]
From (125) we get

$$\Phi_0^2 = \frac{1}{\sqrt{G(q)}} \left( 1 - \frac{1}{4G^2(q)} \right) \frac{1}{\sqrt{G(q)}}.$$  \hspace{1cm} (184)

If it is to represent the bound state of a pair of quasi-bosons the integral

$$\int q (\Phi_0^2)^2 = \int \frac{B(q)}{A(q)} \left[ \frac{A(q) - B(q)}{A(q)^2} \right]$$  \hspace{1cm} (185)

must be finite. To show this, we examine the behaviour of $(\Phi_0^2)^2$. For $q \to 0$, it goes to a finite value because of the gap as discussed in (99). For $q \to \infty$ all the generalized potentials go to zero and both $A$ and $B$ go to infinity as $q^2$ plus a constant. This leads to $(\Phi_0^2)^2 \to \alpha/k^4$ where $\alpha$ is the difference of these constants. This behavior assures the convergence of (185).

For the case when $P \neq 0$ we expect this normalizable bound state to evolve with $P$ and to exist for at least a finite range of $P$. We note that it is not separable in $P$ and $q$ so that the relative wave function changes with $P$ and could become unbound for some critical $P$.

7 Temperature Dependent Calculation

Temperature is often introduced [13] in the so called temperature dependent Hartree Fock Bogoliubov approximation by generalizing the static quantities $R$ and $D$ defined in (42) and (43) as

$$D(P) = X(P)Y(P) \to X(P)Y(P)(1 + 2\nu(P))$$  \hspace{1cm} (186)

$$R(P) = Y^2(P) \to X^2(P)\nu(P) + Y^2(P)(1 + \nu(P))$$  \hspace{1cm} (187)

with the occupation $\nu(P)$ given by

$$\nu(P) = \frac{1}{e^{\omega(P)/kT} - 1}$$  \hspace{1cm} (188)

This corresponds to the usual grand canonical ensemble where $\omega$ is the one quasi-boson energy defined in (93), with its gap. This obviously describes the physics as having free
quasi-bosons where neither the existence of the gapless bound state nor the interaction between the quasi-bosons is taken into account.

Our approach to the problem is different in that we use the canonical ensemble. We have seen that the RPA Hamiltonian consists of a free quasi-boson part $H_0$ and $H_{\text{int}}$ that treats the interaction between the quasi-bosons. So our first approximation in the canonical ensemble for the free energy will be

$$F^{(1)}(\mathcal{N},T) = \mathcal{H} + \frac{1}{\beta} \int_{\mathbf{P}} \log(1 - e^{-\beta \omega(\mathbf{P})}).$$

where $\omega$ is the free quasi-boson energy given in (93). We can also extract more information due to the interaction of the quasi-bosons and this can be done by computing the second virial coefficient

$$F^{(2)}(\mathcal{N},T) = \frac{1}{\beta} \int_{\mathbf{P}} e^{-\beta \Omega(\mathbf{P})} + \frac{1}{4\pi i} \int_{\mathbf{P},\mathbf{q}} e^{-\beta \Omega_2} T \partial S \left. \frac{\partial}{\partial \Omega_2} S \right|_{\Omega(\mathbf{P})}. $$

Where $\Omega(\mathbf{P})$ is the zero mode bound state dispersion relation and $S$ can be determined, as mention before, through the scattering of the quasi-bosons using (113). Note that for the cases where we have two variational solutions $\phi = 0$ and $\phi \neq 0$ we can calculate using (189) and (190) for a given $\mathcal{N}$ obtaining two free energies and this allow us to determine the critical temperature by setting their derivatives with respect to $\mathcal{N}$ equal i.e looking for the same value of the chemical potential.
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Figure Caption

Fig. 1: Schematic plot of the one $\omega$ and two $\Omega^2$ free quasi-boson as a function of the total momentum. When the interaction between them is taken into account (RPA) we have a scattering region and a gapless bound state $\Omega$