ATLAS AS SOLUTION OF SINCOV’S INEQUALITY

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Abstract: We find a general solution of Sincov’s inequality \( \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \subseteq \Phi_{\alpha\gamma} \) provided \( \Phi_{\alpha\beta}^{-1} \subseteq \Phi_{\beta\alpha} \), \( \Phi_{\alpha\alpha} \subseteq \text{id}_M \). Further, we prove that in the differentiable case we can interpret such solution as a differentiable manifold in the original sense of Lang. This allows to generalize the notion of atlas and transition map for non-differentiable and discontinuous case.

Keywords: Sincov’s inequality, differentiable manifold, binary relation, atlas, transition map, differential equation, difference equation.

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Sincov’s functional inequality

We are going to solve functional inequalities

\[(1) \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \subseteq \Phi_{\alpha\gamma}, \quad \Phi_{\alpha\beta}^{-1} \subseteq \Phi_{\beta\alpha}, \quad \Phi_{\alpha\alpha} \subseteq \text{id}_M \]

for an unknown map

\[(2) \quad \Phi : I \times I \ni (\alpha, \beta) \mapsto \Phi_{\alpha\beta} \in 2^{M \times M} \]

where \( I \) and \( M \) are sets, \( \circ \) denotes composition of binary relations and the exponent \(-1\) stands for inverse binary relation.

Motivation example: Dependence of the solution of an ordinary differential equation on initial condition. Consider an ordinary differential equation (ODE) \( \dot{x} = f(\tau, x) \) where \( x \) is a dependent variable, \( \tau \) an independent variable, \( f \) a differentiable map of class \( C^1 \) with domain \( \text{dom} f \subseteq \mathbb{R}^2 \) being an open set and codomain \( \text{cod} f = \mathbb{R} \). Since any Cauchy problem \( x(\alpha) = a \) where \( (\alpha, a) \in \text{dom} f \), has in this case a single maximal solution, it gives a map \( x = F(\tau, \alpha, a) \), where \( \text{dom} F \subseteq \mathbb{R}^3 \) is an open set and \( \text{cod} F = \mathbb{R} \) (see Theorem 14 in section 23 [1]). This map belongs to the differentiability class \( C^1 \) (see Corollary 4 in section 7 [2]). If variables \( \tau, \alpha, \beta \) and \( a \) satisfy conditions \( (\tau, \alpha, a) \in \text{dom} F \) and \( (\beta, \alpha, a) \in \text{dom} F \), then for the map \( F \) the following incidences hold:

1) Since it is a solution of Cauchy problem, we have

\[ F(\alpha, \alpha, a) = a. \]
2) Since Cauchy problem has a unique solution, we also have
\[ F(\tau, \alpha, a) = F(\tau, \beta, F(\beta, \alpha, a)). \]

Those incidences cannot be considered in general as functional equations of an unknown map \( F \) because they incorporate implications on the domain of \( F \), especially we have for 1) \((\tau, \alpha, a) \in \text{dom } F \Rightarrow (\alpha, \alpha, a) \in \text{dom } F \) and we have for 2) \((\tau, \alpha, a) \in \text{dom } F \land (\beta, \alpha, a) \in \text{dom } F \Rightarrow (\tau, \beta, F(\beta, \alpha, a)) \in \text{dom } F \). Thus, unless we know \( F \) we don’t know all possible values of variables \( \tau, \alpha, \beta \) and \( a \). Hence, the problem is about searching for \( F \) and \( \text{dom } F \) at the same time.

This problem can be rigorously treated by means of binary relations provided that by binary relation we mean the set of ordered pairs (see, e.g., Definition 1. v §3.1 [3] or 6.8 Definition in Chapter I [4]). In the sequel, we consider binary relations precisely in this sense. By composition \( \rho \circ \sigma \) of binary relations \( \rho \) and \( \sigma \) we mean a binary relation \( \rho \circ \sigma = \{(a, b) \mid \exists c: (c, b) \in \rho \land (a, c) \in \sigma\} \), by inverse \( \rho^{-1} \) of a binary relation \( \rho \) a binary relation \( \rho^{-1} = \{(a, b) | (b, a) \in \rho\} \), by domain \( \text{dom } \rho \) of a binary relation \( \rho \) the set \( \text{dom } \rho = \{a \mid \exists b: (a, b) \in \rho\} \) and by range \( \text{rng } \rho \) of a binary relation \( \rho \) the set \( \text{rng } \rho = \{b \mid \exists a: (a, b) \in \rho\} \).

We introduce a system \( \{\Phi_{\alpha\beta}\}_{(\alpha, \beta) \in \mathbb{R}^2} \) of binary relations \( \Phi_{\alpha\beta} \) defined by
\[ (b, a) \in \Phi_{\alpha\beta} \iff a = F(\alpha, \beta, b). \]
Then the inequalities (1) are for any choice of dependent variables \( \alpha, \beta, \gamma \in \mathbb{R} \) equivalent to the incidences for \( F \) and implications for \( \text{dom } F \) that are considered in this example. Hence, we can treat them as functional inequalities for the map \( \Phi \) defined by (2) where \( I = M = \mathbb{R} \).

Once we find the map \( \Phi \), we can return to the map \( F \). Then the domain of \( F \) is given by
\[ b \in \text{dom } \Phi_{\alpha\beta} \iff (\alpha, \beta, b) \in \text{dom } F. \]
Moreover, we can return also to the original differential equation. To do that, we differentiate the incidence 2) with respect to \( \tau \) and we set \( \beta = \tau \), then \( \dot{x} = f(\tau, x) \), where
\[ f(\tau, x) = \frac{\partial F(\tau, \beta, x)}{\partial \tau} \bigg|_{\beta=\tau}. \]

**General solution**

If we have equalities in (1), then we deal with a modified Sincov’s equation \( \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} = \Phi_{\alpha\gamma} \) with known solution (see, e.g., Theorem 2 v 8.1.3 [3]). In our work we consider another modification of the modified Sincov’s equation that we call modified Sincov’s inequality.
Before we formulate the result, we need to define an *injective binary relation* or shortly *injection*. By that we understand a binary relation $\rho$ with the property $((b_1, a) \in \rho \land (b_2, a) \in \rho) \Rightarrow b_1 = b_2$. This is connected with the notion of co-injection. A *co-injective binary relation* or shortly *co-injection* is such a binary relation that $\rho^{-1}$ is an injection. We can easily see that the injectivity criterion of the relation $\rho$ is $\rho^{-1} \circ \rho = \text{id}_{\text{dom} \rho}$ and its co-injection criterion $\rho \circ \rho^{-1} = \text{id}_{\text{rng} \rho}$.

**Theorem 1:** The map (2) is a solution of functional inequalities (1) if and only if there exist such a system $\{\varphi_\alpha\}_{\alpha \in I}$ of injective co-injections $\varphi_\alpha$ such that

$$(3) \quad \Phi: I \times I \ni (\alpha, \beta) \mapsto \varphi_\alpha \circ \varphi_\beta^{-1} \in 2^{M \times M}.$$

**Proof:** We assume that the map (2) is a solution of functional inequalities (1) and we define a binary relation $\psi$ by

$$((\beta, b), (\alpha, a)) \in \psi \iff (b, a) \in \Phi_{\alpha \beta}.$$

Since $\Phi_{\alpha \beta} \circ \Phi_{\beta \gamma} \subseteq \Phi_{\alpha \gamma}$, the relation is transitive. Since further $\Phi_{\alpha \beta}^{-1} \subseteq \Phi_{\beta \alpha}$, this relation is also symmetric. Any binary relation that is transitive and symmetric, is an equivalence on its range, hence we can decompose $\psi$ by means of the corresponding quotient projection $\pi: \text{rng} \psi \to \text{rng} \psi/\psi$ into a composition $\psi = \pi^{-1} \circ \pi$. Now, we define for any $\alpha \in I$ a binary relation $\varphi_\alpha$ by

$$(z, a) \in \varphi_\alpha \iff ((\alpha, a), z) \in \pi.$$

Hence, we have $(b, a) \in \varphi_\alpha \circ \varphi_\beta^{-1} \iff ((\beta, b), (\alpha, a)) \in \psi$, and therefore (3) is proved. Since the projection $\pi$ is a co-injection, then any binary relation $\varphi_\alpha$ is injective. And since $\varphi_\alpha \circ \varphi_\alpha^{-1} = \Phi_{\alpha \alpha} \subseteq \text{id}_M$, any relation $\varphi_\alpha$ is a co-injection. This proves one implication.

The inverse implication follows from substituting (2), (3) into (1) and using equalities

$$\varphi_\alpha^{-1} \circ \varphi_\alpha = \text{id}_{\text{dom} \varphi_\alpha}, \quad \varphi_\alpha \circ \varphi_\alpha^{-1} = \text{id}_{\text{rng} \varphi_\alpha}, \quad \text{rng} \varphi_\alpha \subseteq M,$$

which express that $\{\varphi_\alpha\}_{\alpha \in I}$ is a system of injective co-injections satisfying (3).

**Properties of injective co-injections**

Injective co-injections play an important role in finding solutions of functional inequalities (1). Therefore, we will first recapitulate some of
the most important corresponding properties that are easy to verify from definitions.

First of all, injective co-injections form a subalgebra of the algebra of binary relations together with composition and inverse operation. In the context of Theorem 1, this means that not only the relation \( \varphi_\alpha \) but also all relations \( \Phi_{\alpha\beta} \) solving the inequalities (1) are injective co-injections.

To link it with known results it is also important to realize connections with maps. In particular, co-injections are closely related to surjective maps and injective co-injections to bijective maps. We will describe it in detail in the following.

For any binary relation \( \rho \) holds \( \rho \subseteq \text{dom} \rho \times \text{rng} \rho \). By definition of the domain, for any \( b \in \text{dom} \rho \) there exists \( a \) such that \((b, a) \in \rho\). If the relation \( \rho \) is co-injective, there is exactly one such \( a \).

This allows to define a map \( \tilde{\rho} : \text{dom} \rho \rightarrow \text{rng} \rho \) that assigns to any \( b \in \text{dom} \rho \) a unique \( a \in \text{rng} \rho \) such that
\[
(4) \quad (b, a) \in \rho \iff \tilde{\rho}(b) = a.
\]
By definition of the range, this map is surjective. On the other hand, any surjective map \( \tilde{\rho} : \text{dom} \tilde{\rho} \rightarrow \text{cod} \tilde{\rho} \) determines a co-injective binary relation \( \rho \subseteq \text{dom} \tilde{\rho} \times \text{cod} \tilde{\rho} \) satisfying (4).

Since the surjection \( \tilde{\rho} \) is fully described by the co-injection \( \rho \), without loss of accuracy in the following, we can identify the co-injection \( \rho \) with \( \text{rng} \rho \) and use for co-injections the notation \( \rho(b) = a \) as alternative to \((b, a) \in \rho\).

Moreover, if the co-injection \( \rho \) is injective, then it is bijective as a map in the previously described sense.

The next theorem treats the relation between collections of injective co-injections giving by means of (3) the same solution of functional inequalities (1).

**Theorem 2:** Let \( \{ \varphi_\alpha \}_{\alpha \in I} \) and \( \{ \tilde{\varphi}_\alpha \}_{\alpha \in I} \) be two sets of injective co-injections such that for any \((\alpha, \beta) \in I \times I \) holds
\[
(5) \quad \varphi_\alpha \circ \varphi_\beta^{-1} = \tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1}.
\]
Then there exists an injective co-injection \( \omega \) such that for any \( \alpha \in I \)
\[
(6) \quad \varphi_\alpha = \tilde{\varphi}_\alpha \circ \omega.
\]

**Proof:** Let \( \chi \) be a binary relation \( \chi \) defined by
\[
((\alpha, a), z) \in \chi \iff (z, a) \in \varphi_\alpha.
\]
Since the relation $\varphi_\alpha$ is injective, the relation $\chi$ is a co-injection. Therefore, the symmetric relation $\chi^{-1} \circ \chi$ is transitive and we have an equivalence on $\text{rng}(\chi^{-1} \circ \chi) = \text{dom} \chi$. The canonical decomposition of co-injection $\chi$ treated as a surjective map $\chi \colon \text{dom} \chi \to \text{rng} \chi$ is of the form $\chi = \nu \circ \vartheta \circ \pi$. The surjectivity implies the identity of the embedding $\nu$, hence we can compute the projection $\pi = \vartheta^{-1} \circ \chi$, where $\vartheta$ is a bijection of the canonical decomposition of surjection $\chi$.

Furthermore, if we introduce a binary relation $\bar{\chi}$ by

$$(\alpha, a, \bar{z}) \in \bar{\chi} \iff (\bar{z}, a) \in \varphi_\alpha$$

we analogously obtain the projection $\bar{\pi} = \tilde{\vartheta}^{-1} \circ \bar{\chi}$, where $\tilde{\vartheta}$ is a bijection of the canonical decomposition of surjection $\bar{\chi}$.

Since by (5) we have $\chi^{-1} \circ \chi = \bar{\chi}^{-1} \circ \bar{\chi}$ then $\pi = \bar{\pi}$ and therefore $\chi = \vartheta \circ \tilde{\vartheta}^{-1} \circ \bar{\chi}$. Definitions of relation $\chi$ and relation $\bar{\chi}$ give (6), where

$$\omega = \tilde{\vartheta} \circ \tilde{\vartheta}^{-1} : \cup_{\beta \in I} \text{dom} \varphi_\beta \to \cup_{\beta \in I} \text{dom} \varphi_\beta$$

is a bijection that is an injective co-injection if considered as a binary relation. □

Connections to solution of Sincov’s equation on a group

Given the composition operation of binary relations, the codomain $\text{cod} \Phi = 2^{M \times M}$ has a structure of a monoid with identity element $\text{id}_M$. Replacing in (1) all inequalities by equalities, the image $\text{img} \Phi$ with regard to the same operation has a group structure because for any $\Phi_{\alpha\beta} \in \text{img} \Phi$ there exists an inverse element which is coincidentally the inverse relation $\Phi^{-1}_{\alpha\beta} = \Phi_{\beta\alpha} \in \text{img} \Phi$. If we fix in (1) the index value $\gamma \in I$ and we denote $\varphi_\alpha = \Phi_{\alpha\gamma}$, we obtain $\Phi_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha$. Hence, applying group properties it follows that

$$\Phi_{\alpha\beta} = \varphi_\alpha \circ \varphi^{-1}_\beta$$

and this is the general solution of Sincov’s equation $\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} = \Phi_{\alpha\gamma}$ in case $\circ$ is a group operation. This solution has been already published and described in detail, e.g., in Theorem 2 in 8.1.3 [5] and has used only the fact that associativity and invertibility hold for any group and not only for Abelian groups as originally considered by Sincov [6].

Equality (11) gives $\Phi_{\alpha\beta} \circ \Phi^{-1}_{\alpha\beta} = \Phi^{-1}_{\alpha\beta} \circ \Phi_{\alpha\beta} = \text{id}_M$, which means that $\Phi_{\alpha\beta}$ and then consequently also $\varphi_\alpha$ are bijections $M \to M$. As such they are also injective co-injections, hence the general solution of the modified Sincov’s equation 8.1.3(9) [5] is a particular case of our general solution (3) of inequality (1).
Differentiable manifolds

Let \( p \geq 0 \) be an integer, let \( M \) be a Banach space. Since we can consider the values \( \Phi_{\alpha\beta} \) of solution \( \Phi \) of functional inequalities (1) as maps, we can impose differentiability requirements on them. If we insist all the values \( \Phi_{\alpha\beta} \) of general solution (3) are \( C^p \)-isomorphisms of open sets, then such solution has a structure of a differentiable manifold of class \( C^p \). In particular, the system of pairs \( \{ (\text{dom } \varphi_\alpha, \varphi_\alpha) \}_{\alpha \in I} \) is therefore by (2) and (3) a \( C^p \)-atlas on the set \( X = \bigcup_{\alpha \in I} \text{dom } \varphi_\alpha \) in the sense of Lang’s axioms [7], Chapter II, §1:

**AT1.** Each \( U_\alpha \) is a subset of \( X \) and \( \{ U_\alpha \}_{\alpha \in I} \) cover \( X \).

**AT2.** Each \( \varphi_\alpha \) is a bijection of \( U_\alpha \) onto an open subset \( \varphi_\alpha(U_\alpha) \) of the Banach space \( M \) and for any \( \alpha, \beta \), \( \varphi_\alpha(U_\alpha \cap U_\beta) \) is open in \( M \).

**AT3.** The map

\[
\Phi_{\alpha\beta} : \varphi_\beta(U_\beta \cap U_\alpha) \ni a \mapsto \varphi_\alpha(\varphi_\beta^{-1}(a)) \in \varphi_\alpha(U_\alpha \cap U_\beta)
\]

is a \( C^p \)-isomorphism for each pair of indices \( \alpha, \beta \),

where \( U_\alpha \) is another notation for \( \text{dom } \varphi_\alpha \). The other way around, any \( C^p \)-atlas satisfying these axioms generates by (3) solution of functional inequalities (1).

The set \( X = \bigcup_{\alpha \in I} \text{dom } \varphi_\alpha \) equipped with a \( C^p \)-atlas \( \{ (\text{dom } \varphi_\alpha, \varphi_\alpha) \}_{\alpha \in I} \) is called a differentiable manifold of class \( C^p \) or shortly a \( C^p \)-manifold, and the maps \( \Phi_{\alpha\beta} \) appearing in the axiom **AT3** are called transition maps (see, e.g., Chapter 1 in [8]). The co-injection \( \omega : X \to \bar{X} \) in Theorem 2 is therefore an isomorphism of \( C^p \)-manifolds.

An example of a \( C^1 \)-manifold is the manifold of solutions of ordinary differentiable equation described in our motivation example above.

**Remark on Hausdorff property.** From the above cited Lang’s axioms as they appear in the first edition of its book [7], the Hausdorff condition of a suitable differentiable manifold \( X \) does not follow. But thanks to axiom **AT3** we get

\[
\Phi_{\alpha\beta} \subseteq \text{rng } \Phi_{\beta\beta} \times \text{rng } \Phi_{\alpha\alpha}
\]

and it is easy to see, that \( X \) is a Hausdorff space if and only if for any pair \( (\alpha, \beta) \in I \times I \) the set \( \Phi_{\alpha\beta} \) is closed in \( \text{rng } \Phi_{\beta\beta} \times \text{rng } \Phi_{\alpha\alpha} \). The Hausdorff property of the manifold can be achieved in a similar way as its differentiability by means of demands placed only on transition maps.
ATLAS AS SOLUTION OF SINCOV’S INEQUALITY

SINCOV’S INEQUALITY FOR TRANSITION RELATIONS

From the above it follows that if the solution values $\Phi_{\alpha\beta}$ (2) of inequalities (1) are $C^p$–isomorphisms of open sets, then we can interpret them as transition maps and the system of injective co-injections $\{\varphi_\alpha\}_{\alpha \in I}$ from Theorem 1 as their corresponding atlas on the set $X = \bigcup_{\alpha \in I} \text{dom} \varphi_\alpha$.

The above interpretation makes sense even though $\Phi_{\alpha\beta}$ are not $C^p$-isomorphisms of open sets. Hence, we are able to generalize the notion of system of transition maps to the term system of transition relations, that we define as a system of binary relations $\{\Phi_{\alpha\beta}\}_{(\alpha,\beta) \in I \times I}$ solving inequalities (1) representing therefore Sincov’s inequality for transition relations. By Theorem 1, for any system of transition relations there exists an atlas $\{\varphi_\alpha\}_{\alpha \in I}$ which is by Theorem 2 unique up to some bijection $\omega$.

Consequently, Sincov’s inequality for transition relations represents a generalization of the theory on differentiable manifolds to non-differentiable and discontinuous case. This allows to pose many important questions for future work. A direct application is expected for generalization in the analysis of our motivation example from differential to difference equations.

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