Abstract: The prime counting function inequality \( \pi(x + y) < \pi(x) + \pi(y) \), which is known as Hardy-Littlewood conjecture, has been established for a variety of cases such as \( \delta x \leq y \leq x \), where \( 0 < \delta \leq 1 \), and \( x \leq y \leq x \log x \log \log x \) as \( x \to \infty \). The goal in note is to extend the inequality to the new larger ranges \( \geq x \log^{-c} x \leq y \leq x \), where \( c \geq 0 \) is a constant, unconditionally; and for \( \geq x^{1/2} \log^3 x \leq y \leq x \), conditional on a standard conjecture.

Keyword: Distribution of prime; Prime in short interval; Hardy-Littlewood conjecture; Prime k-tuple.

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1 Introduction

Let \( x \geq 1 \) be a large number, and let \( \pi(x) = \# \{ p \leq x : p \text{ is prime} \} \). There are many partial results for the prime counting function inequality

\[
\pi(x + y) < \pi(x) + \pi(y).
\]

The range of parameter \( \delta x \leq y \leq x \), with a constant \( 0 < \delta \leq 1 \) as \( x \to \infty \), is proved in \[15\] Theorem 3]. The range of parameter \( x \leq y \leq x \log x \log \log x \) as \( x \to \infty \), is proved in \[6\]. Various other related inequalities are proved in \[15\], \[10\], \[19\], and by other authors. The Hardy-Littlewood conjecture states that inequality (1) is valid for any \( x \geq 2 \) and any \( y \geq 2 \). The goal in note is to extend the inequality to larger ranges than it is currently known. It is shown that it is valid for the range of parameter \( x \log^{-c} x \leq y \leq x \), where \( c \geq 0 \) is a constant, unconditionally. And for \( x^{1/2} \log^3 x \leq y \leq x \), conditional on the RH. These are new results in the mathematical literature.

Theorem 1.1. Let \( x \geq 2 \) be a large number, and let \( x \log^{-c} x \leq y \leq x \), with \( c \geq 0 \) an arbitrary constant. Then

\[
\pi(x + y) < \pi(x) + \pi(y).
\]

Theorem 1.2. Let \( x \geq 2 \) be a large number, and let \( x^{1/2} \log^3 x \leq y \leq x \). Assume the nontrivial zeros of the zeta function are on the line \( \Re(e) = 1/2 \). Then

\[
\pi(x + y) < \pi(x) + \pi(y).
\]

The proof of Theorem 1.1 is assembled in Section 2 and the proof of Theorem 1.2 is assembled in Section 3. The penultimate section inquires on the limitation of the prime counting inequality (1).

2 Unconditional Result

The unconditional result for the prime number theorem, see Theorem 5.2 together with the mean value theorem for integral, see Theorem ??, give a nice and simple proof for the prime counting function inequality
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(1) over the larger interval \([x, x+y]\) with \(x \log^{-c} x \leq y \leq x\), for any constant \(c \geq 0\).

Proof. (Theorem 1.1) By the prime number theorem, the prime counting function has the integral representation

\[
\pi(x) = \int_{2}^{x} \frac{1}{\log t} dt + O\left(xe^{-\sqrt{\log x}}\right),
\]

(4)

see Theorem 5.2. Accordingly, the reverse inequality \(\pi(x+y) \geq \pi(x) + \pi(y)\) has the integral representation

\[
\int_{x}^{x+y} \frac{1}{\log t} dt + O\left(xe^{-\sqrt{\log x}}\right) \geq \int_{t}^{y} \frac{1}{\log t} dt + O\left(xe^{-\sqrt{\log x}}\right).
\]

(5)

By the mean value theorem for integral, there is a value \(x_{0} \in (x, x+y)\), and a value \(x_{1} \in (2, y)\) such that

\[
y \log x_{0} + O\left(xe^{-\sqrt{\log x}}\right) = \int_{x}^{x+y} \frac{1}{\log t} dt + O\left(xe^{-\sqrt{\log x}}\right)
\]

\[
\geq \int_{x}^{y} \frac{1}{\log t} dt + O\left(xe^{-\sqrt{\log x}}\right)
\]

(6)

\[
= \frac{y-2}{\log x_{1}} + O\left(xe^{-\sqrt{\log x}}\right).
\]

Dividing the left and right sides of (6) by \(y \geq x \log^{-c} x\), and multiplying it by \(\log x_{1}\) give

\[
\frac{\log x_{1}}{\log x_{0}} + O\left(\frac{(\log x_{1}) \log^{c} x}{e^{\sqrt{\log x}}}\right) \geq 1 - \frac{2}{y}.
\]

(7)

Based on the data

\[
x \leq x_{0} \leq x+y = x + x \log^{-c} x \quad \text{and} \quad 2 \leq x_{1} \leq y = x \log^{-c} x,
\]

(8)

the upper bound of the left side of (7) is

\[
\frac{\log x_{1}}{\log x_{0}} + O\left(\frac{(\log x_{1}) \log^{c} x}{e^{\sqrt{\log x}}}\right) \leq \frac{\log (x \log^{-c} x)}{\log x} + O\left(\frac{\log^{c+1} x}{e^{\sqrt{\log x}}}\right)
\]

\[
\leq 1 - \frac{c \log \log x}{\log x} + O\left(\frac{\log^{c+1} x}{e^{\sqrt{\log x}}}\right).
\]

(9)

And the upper bound of the right side of (7) is

\[
1 - \frac{2}{y} \leq 1 - \frac{2 \log^{c} x}{x}.
\]

(10)

Replacing (9) and (10) into (7) yield

\[
1 - \frac{2 \log^{c} x}{x} \leq 1 - \frac{c \log \log x}{\log x} + O\left(\frac{\log^{c+1} x}{e^{\sqrt{\log x}}}\right).
\]

(11)

Since the left side increases at a faster rate than the right side, this is a contradiction as \(x \to \infty\). Ergo, \(\pi(x+y) < \pi(x) + \pi(y)\).

3 Conditional Result

The conditional results for the zeta function and the prime number theorem, see Theorem 5.2, together with the mean value theorem for integral, are sufficient to extend the prime counting function inequality (1) to the larger interval \([x, x+y]\) with \(x^{1/2} \log^{3} x \leq y \leq x\). This is a new result in the mathematical literature.
Proof. (Theorem 1.2) The prime counting function has the integral representation

\[ \pi(x) = \int_{2}^{x} \frac{1}{\log t} dt + O\left(\frac{1}{\log x}\right), \]

(12) see Theorem 5.2. Accordingly, the reverse inequality \( \pi(x + y) \geq \pi(x) + \pi(y) \) has the integral representation

\[ \int_{x}^{x+y} \frac{1}{\log t} dt + O\left(\frac{1}{\log x}\right) \geq \int_{2}^{y} \frac{1}{\log t} dt + O\left(\frac{1}{\log x}\right). \]

(13)

By the mean value theorem for integral, there is a value \( x_0 \in (x, x+y), \) and a value \( x_1 \in (2, y) \) such that

\[ \frac{y}{\log x_0} + O\left(\frac{1}{\log x}\right) = \int_{x}^{x+y} \frac{1}{\log t} dt + O\left(\frac{1}{\log x}\right) \]

\[ \geq \int_{2}^{y} \frac{1}{\log t} dt + O\left(\frac{1}{\log x}\right) \]

\[ = \frac{y - 2}{\log x_1} + O\left(\frac{1}{\log x}\right). \]

(14)

Dividing the left and right sides of (14) by \( y \geq x^{1/2} \log^3 x, \) and multiplying it by \( \log x_1 \) give

\[ \frac{\log x_1}{\log x_0} + O\left(\frac{\log x_1}{\log^2 x}\right) \geq 1 - \frac{2}{y}. \]

(15)

Based on the data

\[ x \leq x_0 \leq x + y = x + x^{1/2} \log^3 x \quad \text{and} \quad 2 \leq x_1 \leq y = x^{1/2} \log^3 x, \]

(16)

the upper bound of the left side of (15) is

\[ \frac{\log x_1}{\log x_0} + O\left(\frac{\log x_1}{\log^2 x}\right) \leq \frac{\log \left(\frac{x^{1/2} \log^3 x}{\log x}\right)}{\log x} + O\left(\frac{1}{\log x}\right) \]

\[ \leq \frac{1}{2} + \frac{\log \log^3 x}{\log x} + O\left(\frac{1}{\log x}\right). \]

(17)

And the upper bound of the right side of (15) is

\[ 1 - \frac{2}{y} \leq 1 - \frac{2}{x^{1/2} \log^3 x}. \]

(18)

Replacing (17) and (18) into (15), yield

\[ 1 - \frac{2}{x^{1/2} \log^3 x} \leq \frac{1}{2} + \frac{\log \log^3 x}{\log x} + O\left(\frac{1}{\log x}\right). \]

(19)

Trivially, this is a contradiction for all large numbers \( x \geq 2. \) Ergo, \( \pi(x + y) < \pi(x) + \pi(y). \)

\[ \square \]

4 Limits Of the Primes Counting Function Inequality

There are several results for the oscillations of the primes counting function over small intervals, confer Theorem 5.2 and Theorem 5.4. The oscillations of the values of the prime counting function seems to force some limits on the conjectured Hardy-Littlewood inequality

\[ \pi(x + y) < \pi(x) + \pi(y). \]

(20)
Conjecture 4.1. Let \( x \geq 2 \) be a large number. Let \( y = \log^r x \), where \( r > 0 \) a real number. Then
\[
\pi(x + \log^r x) < \pi(x) + \pi(\log^r x) \tag{21}
\]
and
\[
\pi(x + \log^r x) > \pi(x) + \pi(\log^r x) \tag{22}
\]
independently often as \( x \to \infty \).

It is not clear if it can be proved or disproved by elementary methods, for example, using Theorem 5.4 or Theorem 5.2. In synopsis, this inequality is likely to fail on very short intervals. This has some relevance to the prime \( k \)-tuples conjecture, see [1]. Some detailed information on the hierarchy of prime \( k \)-tuples conjectures are explicated in a new survey, [9, p. 10].

5 Prime Numbers Theorems

The omega notation \( f(x) = g(x) + \Omega(\pm h(x)) \) means that both \( f(x) > g(x) + c_0 h(x) \) and \( f(x) < g(x) - c_1 h(x) \) occur infinitely often as \( x \to \infty \), where \( c_0 > 0 \) and \( c_1 > 0 \) are constants, see [16, p. 5], and similar references.

The set of prime numbers is denoted by \( \mathbb{P} = \{2, 3, 5, \ldots\} \), and for a real number \( x \geq 1 \), the standard prime counting function is denoted by
\[
\pi(x) = \#\{p \leq x : p \text{ prime}\} = \sum_{\substack{p \leq x}} 1. \tag{23}
\]

In addition, the logarithm integral and multiple logarithm integral are defined by \( \text{li}(x) = \int_2^x \frac{1}{\log t} \, dt \) and \( \text{li}_k(x) = \int_2^x \frac{1}{\log^k t} \, dt \), \( k \geq 1 \), respectively. The weighted primes counting functions, psi \( \psi(x) \) and theta \( \theta(x) \), are defined by
\[
\theta(x) = \sum_{\substack{p \leq x}} \log p \quad \text{and} \quad \psi(x) = \sum_{\substack{p^k \leq x}} \log p^k \tag{24}
\]
respectively.

Theorem 5.1. Uniformly for \( x \geq 2 \) the psi and theta functions have the following asymptotic formulae.

(i) Unconditionally,
\[
\theta(x) = x + O\left(x e^{-c_0 \sqrt{\log x}}\right).
\]

(ii) Unconditional oscillation,
\[
\theta(x) = x + \Omega\left(x^{1/2} \log \log x\right).
\]

(iii) Conditional on the RH,
\[
\theta(x) = x + O\left(x^{1/2} \log^2 x\right).
\]

Proof. (ii) The oscillations form of the theta function is proved in [16, p. 479], ■

The same asymptotics hold for the function \( \psi(x) \). Explicit estimates for both of these functions are given in [3, 20], [1, Theorem 5.2], and related literature.

Conjecture 5.1. Assuming the RH and the LI conjecture, the suprema are
\[
\liminf_{x \to \infty} \frac{\psi(x) - x}{\sqrt{x (\log \log x)^2}} = -\frac{1}{\pi} \quad \text{and} \quad \limsup_{x \to \infty} \frac{\psi(x) - x}{\sqrt{x (\log \log x)^2}} = \frac{1}{\pi}. \tag{25}
\]
More details on the Linear Independence conjecture appear in [12, 7, Theorem 6.4], and recent literature. The LI conjecture asserts that the imaginary parts of the nontrivial zeros \( \rho_n = 1/2 + i\gamma_n \) of the zeta function \( \zeta(s) \) are linearly independent over the set \( \{-1, 0, 1\} \). In short, the equations

\[
\sum_{1 \leq n \leq M} r_n \gamma_n = 0,
\]

where \( r_n \in \{-1, 0, 1\} \), have no nontrivial solutions.

**Theorem 5.2.** (Prime number theorem) Let \( x \geq 1 \) be a large number. Then

(i) Unconditionally,

\[
\pi(x) = \text{li}(x) + O\left(xe^{-c_0 \sqrt{\log x}}\right).
\]

(ii) Unconditional oscillation,

\[
\pi(x) = \text{li}(x) + \Omega_{\pm} \left(\frac{x^{1/2} \log \log \log x}{\log x}\right).
\]

(iii) Conditional on the RH,

\[
\pi(x) = \text{li}(x) + O\left(x^{1/2} \log x\right).
\]

**Proof.** (i) The unconditional part of the prime counting formula arises from the delaVallee Poussin form \( \pi(x) = \text{li}(x) + O\left(xe^{-c_0 \sqrt{\log x}}\right) \) of the prime number theorem, see [16, p. 179]. Recent information on the constant \( c_0 > 0 \) and the sharper estimate \( \pi(x) = \text{li}(x) + O\left(xe^{-c_0 \log x^{3/5} (\log \log x)^{-2/5}}\right) \) appears in [8] and [13, p. 307]. The constant \( c = .2018 \) is computed in [8].

(ii) The unconditional oscillations part arises from the Littlewood form \( \pi(x) = \text{li}(x) + \Omega_{\pm} \left(\frac{x^{1/2} \log \log \log x}{\log x}\right) \) of the prime number theorem, consult [11, p. 51], [16, p. 479], et cetera.

(iii) The conditional part arises from the Riemann form \( \pi(x) = \text{li}(x) + O\left(x^{1/2} \log x\right) \) of the prime number theorem. In [20, Corollary 1] there is an explicit version.

New explicit estimates for the number of primes in arithmetic progressions are computed in [2].

**Theorem 5.3.** ([15]) For all real numbers \( x > 1 \), and any monotonically increasing function \( \theta(x) \geq 2 \),

\[
\pi(x + \theta(x)) - \pi(x) \leq \frac{2\theta(x)}{\log \theta(x)}.
\]

**Theorem 5.4.** ([14]) Let \( \theta(x) = (\log x)^r \), where \( r > 1 \). Then

\[
\liminf_{x \to \infty} \frac{\pi(x + \theta(x)) - \pi(x)}{\theta(x)/\log x} < 1 \quad \text{and} \quad \limsup_{x \to \infty} \frac{\pi(x + \theta(x)) - \pi(x)}{\theta(x)/\log x} > 1.
\]

For the range \( 1 < r < e^\gamma \), the limit supremum is

\[
\limsup_{x \to \infty} \frac{\pi(x + \theta(x)) - \pi(x)}{\theta(x)/\log x} \geq \frac{e^\gamma}{r},
\]

where \( \gamma \) denotes Euler constant.
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