Modulus of families of sets of finite perimeter and quasiconformal maps between metric spaces of globally $Q$-bounded geometry

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June 19, 2018

Abstract We generalize a result of Kelly [16] to the setting of Ahlfors $Q$-regular metric measure spaces supporting a 1-Poincaré inequality. It is shown that if $X$ and $Y$ are two Ahlfors $Q$-regular spaces supporting a 1-Poincaré inequality and $f : X \to Y$ is a quasiconformal mapping, then the $Q/(Q-1)$-modulus of the collection of measures $\mathcal{H}^{Q-1}\Sigma E$ corresponding to any collection of sets $E \subset X$ of finite perimeter is quasi-preserved by $f$. We also show that for $Q/(Q-1)$-modulus almost every $\Sigma E$, if the image surface $\Sigma f(E)$ does not see the singular set of $f$ as a large set, then $f(E)$ is also of finite perimeter. Even in the standard Euclidean setting our results are more general than that of Kelly, and hence are new even in there.

Key words and phrases: finite perimeter, quasiconformal mapping, modulus of families of surfaces, Ahlfors regular, Poincaré inequality.

Mathematics Subject Classification (2010): Primary: 30L10; Secondary: 26B30, 31E05.

1 Introduction

While classification of domains via conformal mappings gives a rich theory in the setting of planar domains, domains in higher dimensional Euclidean spaces support no non-Möbius conformal maps. The most suitable geometric classification in that setting is given by quasiconformal mappings. A homeomorphism $f : \Omega \to \Omega'$ between two domains $\Omega, \Omega' \subset \mathbb{R}^n$ is quasiconformal if $f \in W^{1,n}_{loc}(\Omega; \Omega')$ and there is a constant $K \geq 1$ such that whenever $x \in \Omega$, $\limsup_{r \to 0^+} \frac{\sup_{y \in B(x,r)} |f(y) - f(x)|}{\inf_{y \in \Omega \setminus B(x,r)} |f(y) - f(x)|} \leq K$.

The theory of quasiconformal mappings was extended by Heinonen and Koskela in [13] to the setting of metric measure spaces, and in this non-smooth setting properties of quasiconformal mappings have been studied extensively, see for example [12, 13, 14, 21, 6, 17]. In this paper

*The first and third authors’ research was partially supported by the grant DMS #1500440 of NSF (U.S.A.). Part of this research was conducted during the visit of the three authors to Linköping University in Spring 2017 and Spring 2018; the authors wish to thank that institution for its kind hospitality.
we continue this study by considering relationships between sets of finite perimeter and quasiconformal mappings in the spirit of [16].

The traditional perspective on quasiconformal mappings between Euclidean domains is that such a map is characterized by its ability to quasi-preserve the conformal modulus of families of rectifiable curves in the respective domains. Thus a homeomorphism \( f : \Omega \to \Omega' \) for two domains \( \Omega, \Omega' \subset \mathbb{R}^n \) is quasiconformal if there is a constant \( C \geq 1 \) such that whenever \( \Gamma \) is a family of non-constant compact rectifiable curves in \( \Omega \),

\[
\frac{1}{C} \text{Mod}_n(\Gamma) \leq \text{Mod}_n(f\Gamma) \leq C \text{Mod}_n(\Gamma).
\]

Here \( f\Gamma \) is the family of curves obtained as images of curves in \( \Gamma \) under \( f \). An excellent discussion about Euclidean quasiconformal mappings can be found in [20], see also [16, Theorem 2.6.1]. A less well-known fact is that quasiconformal mappings between two domains \( \Omega, \Omega' \subset \mathbb{R}^n \) quasi-preserve the \( n-1 \)-modulus of certain families of surfaces obtained as “essential boundaries” of sets of finite perimeter. This result is due to Kelly [16, Theorem 6.6]. In [16] the families considered were the classes of sets \( E \subset \Omega \subset \mathbb{R}^n \) of finite perimeter such that \( \mathcal{H}^{n-1}(\partial E) \) is finite and satisfies a double-sided cone condition at every point in \( \partial E \), see [16, Definition 6.1]. Kelly calls the boundary of such a set \( E \) a surface. Building upon the Federer theory of differential forms for sets of finite perimeter and the associated Gauss-Green theorem (see [8]), in [16] it is shown that if \( f \) is a quasiconformal mapping, then for \( \text{Mod}_{n/(n-1)} \)-almost every surface, there is a change of variables formula, see [16, Theorem 4.7.1]. Moreover, there is a family \( \Sigma_0 \) of sets of finite perimeter in \( \Omega \) with \( \text{Mod}_{n/(n-1)}(\Sigma_0) = 0 \) such that whenever \( \partial E \subset \Omega \) is a surface with \( E \notin \Sigma_0 \), then \( f(E) \) is of finite perimeter in \( \Omega' \) ([16, Theorem 6.3]). However, there is a gap in the proof of [16, Theorem 6.3], where the actual object studied is the reduced boundary of \( E \), denoted \( \beta(E) \) in [16, page 372], and this part of the boundary could be strictly smaller than the measure-theoretic boundary of \( E \). According to Federer’s characterization, a Euclidean set is of finite perimeter if and only if the \( (n-1) \)-dimensional Hausdorff measure of the measure-theoretic boundary of \( E \) is finite. In [16] it is shown that for almost every \( E \), \( \beta(f(E)) \) has finite \( (n-1) \)-dimensional Hausdorff measure, and this is not sufficient to conclude that \( f(E) \) is of finite perimeter.

The link between quasiconformal mappings and families of surfaces is natural also in light of the link between quasiconformal mappings and moduli of families of curves, for in the Euclidean setting it is known that there is a natural reciprocal connection between families of surfaces separating two compacta and families of curves connecting the two compacta, see [2]. This link was already portended in [1, Lemma 5] (in planar geometry, the separating “surfaces” are also curves). Motivated by the results in [16, 2], the goal of this paper is to prove a result similar to that of [16, Theorem 6.6] for quasiconformal mappings between two complete metric measure spaces equipped with an Ahlfors \( Q \)-regular (with \( Q > 1 \)) measure and supporting a 1-Poincaré inequality in the sense of Heinonen and Koskela [13], and indeed, we consider families of sets from the collection of all sets of finite perimeter without the additional geometric constraints considered in [16] (see Theorem 1.1). To do so, we use the tools developed in [13, 14] regarding first-order calculus on non-smooth spaces and the theory of BV functions first constructed in [19], together with the result from [17] that quasiconformal maps are characterized by quasi-preserving the measure density of measur-
able subsets of Ω. This latter result is itself a generalization of the work of Gehring and Kelly [10]. Unlike in the work of Kelly [16], the notions of the Gauss-Green theorem and differential forms are not available in the metric setting, and instead, we adapt the geometric measure theory tools developed in [6] in the metric setting to verify an analog of the change of variables formula for sets of finite perimeter in the non-smooth setting. The results of [6] are not directly applicable to our setting as neither the measure-theoretic boundary of variables formula for sets of finite perimeter in the non-smooth setting. The results of [6] measure theory tools developed in [6] in the metric setting to verify an analog of the change differential forms are not available in the metric setting, and instead, we adapt the geometric Kelly [10]. Unlike in the work of Kelly [16], the notions of the Gauss-Green theorem and able subsets of Ω. This latter result is itself a generalization of the work of Gehring and Kelly [10].

Let

\[ \text{Definition 1.1.} \]

\[ \text{Theorem 1.1.} \]

\[ \text{The following is the main result of this paper.} \]

\[ \text{Theorem 1.1. Let } X, Y \text{ be two complete Ahlfors } Q\text{-regular metric spaces, } Q > 1, \text{ that support a } 1\text{-Poincaré inequality, and let } f : X \rightarrow Y \text{ be a quasiconformal map. Then there exists } C > 0 \text{ such that for every collection } \mathcal{L} \text{ of bounded sets of finite perimeter measure in } X \text{ we have that} \]

\[ \text{Mod}_{Q/(Q-1)}(\mathcal{L}) \leq C \text{Mod}_{Q/(Q-1)}(f\mathcal{L}) \]

\[ \text{and} \]

\[ \text{Mod}_{Q/(Q-1)}(f\mathcal{L}') \leq C \text{Mod}_{Q/(Q-1)}(\mathcal{L}') \]

\[ \text{where } f\mathcal{L} \text{ denotes the collection of images under } f \text{ and } \mathcal{L}' \text{ consists of all } E \in \mathcal{L} \text{ for which } 0 < L_f(x) < \infty \text{ for } H^{Q-1}\text{-almost every } x \in \Sigma E. \]

\[ \text{The above theorem gives new results even in the Euclidean setting, addressing the wider class of all sets of finite perimeter rather than just those that satisfy a cone property at each point of the topological boundary, with the topological boundary of finite Hausdorff } (n-1)\text{-dimensional measure as considered in [16] Definition 6.1.} \]
In proving Theorem 1.1 we also show that for Mod\(_{Q/(Q-1)}\) almost every set \(E \subset X\) of finite perimeter the pull-back measure under \(f\) of \(\mathcal{H}^{Q-1}|_{\Sigma f(E) \cup \{\partial^* f(E) \setminus f(P)\}}\) is absolutely continuous with respect to \(\mathcal{H}^{Q-1}|_{\Sigma f(E)}\), with its Radon-Nikodym derivative estimated by \(J_f^{(Q-1)/Q}\), see Lemma 5.4 and Proposition 5.2. Here

\[
P = \{x \in X : L_f(x) = \infty \text{ or } L_f(x) = 0\}
\]

is the singular set of \(f\). We also address the question of whether images of sets of finite perimeter are of finite perimeter. There are examples of planar quasiconformal mappings that map the unit disk to the von Koch snowflake domain, and so map a set of finite perimeter to a set that is not of finite perimeter. Hence we cannot expect images of all sets of finite perimeter to be of finite perimeter, see also [6]. From the results in [13, 14] we know that \(\mathcal{H}^n(P) = 0\). Recall from there also that both \(f\) and \(f^{-1}\) satisfy Lusin’s condition \(N\), that is, for sets \(K \subset X\), \(\mathcal{H}^Q(f(K)) = 0\) if and only if \(\mathcal{H}^Q(K) = 0\), or equivalently, \(f_\# \mathcal{H}^Q_Y \ll \mathcal{H}^Q_X \ll f_\# \mathcal{H}^Q_Y\) (see [14, Theorem 8.12]). Following [14], the Radon-Nikodym derivative of \(f_\# \mathcal{H}^Q_Y\) with respect to \(\mathcal{H}^Q_X\) is denoted by \(J_f\). The quantities \(J_f\), \(L_f^Q\), and \(l_f^Q\) are comparable to each other almost everywhere in \(X\), see Lemma 4.6 below.

**Theorem 1.2.** Let \(X, Y\) be complete Ahlfors \(Q\)-regular metric spaces, \(Q > 1\), that support a 1-Poincaré inequality, and let \(f : X \to Y\) be a quasiconformal map. Let \(\mathcal{L}_P\) be the collection of all sets \(E \subset X\) of finite perimeter with \(\mathcal{H}^{Q-1}(\partial^* E \cap P) > 0\) or \(\mathcal{H}^{Q-1}(f((\partial^* E \setminus \Sigma E) \cap P)) > 0\). Then for \(\text{Mod}_{Q/(Q-1)}\)-almost every bounded set \(E \subset X\) of positive and finite perimeter in \(X\) that does not belong to \(\mathcal{L}_P\), the set \(f(E)\) is of finite perimeter in \(Y\) and the pull-back measure satisfies

\[
f_\# \mathcal{H}^{Q-1}_{\partial^* f(E)} \ll \mathcal{H}^{Q-1}_{\partial^* f(E)} \ll f_\# \mathcal{H}^{Q-1}_{\partial^* f(E)} \quad (1.4)
\]

with Radon-Nikodym derivative \(J_{f,E} \simeq J_f^{(Q-1)/Q}\) where the comparison constant \(C\) depends only on the Ahlfors regularity constants and the Poincaré inequality constants of \(X\) and \(Y\), and on the quasiconformality constant of \(f\). Furthermore,

\[
\frac{1}{C} \text{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_P) \leq \text{Mod}_{Q/(Q-1)}(f(\mathcal{L} \setminus \mathcal{L}_P)) \leq C \text{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_P). \quad (1.5)
\]

The above results are not as discordant with the theory of quasiconformal mappings as it might seem. The characterization of quasiconformal mappings by quasi-preservation of \(Q\)-modulus of families of curves is too strong; one only needs the quasi-preservation of \(Q\)-modulus of families of rectifiable curves that connect pairs of disjoint compact sets \(K, F\). Such classes of curves are associated with the relative capacity \(\text{cap}_Q(K, F)\), and the super-level sets of the potential associated with this capacity give sets of finite perimeter whose perimeter sets separate \(K\) from \(F\), and it is the families of such sets that connect quasiconformal maps to the BV theory.

We now consider the implications of the above Theorem 1.2 for the Euclidean setting \(\mathbb{R}^n\). An analog of the above theorem found in [16] is Theorem 6.3, but the proof of [16, Theorem 6.3] has a gap as explained above. However, in light of the restrictions placed on the sets \(E\) considered in [16], we know that those sets \(E\) satisfy \(\partial E = \Sigma E\), and for such \(E\) it follows from Proposition 5.1(3) that except for a \(\text{Mod}_{n/(n-1)}\)-null family, we know that
\[ H^{n-1}(f(P \cap \partial E)) = 0, \] and therefore [16, Theorem 6.3] follows from Theorem 1.2, that is, the gap in the proof found in [16] is filled by the above theorem.

The structure of the paper is as follows. In Section 2 we give the notations and definitions related to function spaces and measure-theoretic aspects of sets used in this paper. In Section 3 we give a brief background related to quasiconformal mappings between metric measure spaces, and in Section 4 we list the needed background results related to the concepts described in the previous two sections. Here we also give proofs and/or references to papers where the interested reader can find proofs of these results. We give a proof of Theorem 1.1 in Section 5, and the last section deals with the proof of Theorem 1.2. In the last section we also show that there are large families of sets of finite perimeter whose images under quasiconformal mappings are also of finite perimeter, and thus lie outside the collection \( \mathcal{L}_P \) of Theorem 1.2.

2 Notations and definitions

In this section we gather together the basic definitions we need in this paper. The definitions used here are extensions to the non-smooth setting of the natural notions in Euclidean setting discussed in the introduction. In this section, \((X, d, \mu)\) is a complete metric measure space with \(\mu\) a Radon measure.

Given \(x \in X\) and \(r > 0\), we denote an open ball by \(B(x, r) = \{ y \in X : d(y, x) < r \}\).

Given that in a metric space a ball, as a set, could have more than one radius and more than one center, we will consider a ball to be also equipped with a radius and center; thus two different balls might correspond to the same set. We then denote \(\text{rad}(B) = r\) as the pre-assigned radius of the ball \(B\), and \(aB := B(x, ar)\). If \(X\) is connected (as it must be in order to support a Poincaré inequality), and if \(X \setminus B\) is non-empty, then \(\text{rad}(B) \leq \text{diam}(B) \leq 2\text{rad}(B)\).

**Definition 2.1.** Let \(A \subset X\). Then for \(d \geq 0\), the \(d\)-dimensional Hausdorff measure of \(A\) is given by

\[
\mathcal{H}^d(A) = \lim_{r \to 0^+} \inf \left\{ \sum_{k \in I} \text{rad}(B_k)^d \ : \ A \subset \bigcup_{k \in I} B_k \text{ where } \text{rad}(B_k) \leq r \text{ and } I \subset \mathbb{N} \right\}.
\]

**Definition 2.2.** We say that \((X, d, \mu)\) is Ahlfors \(Q\)-regular if \(X\) has at least two points and there is a constant \(C_A \geq 1\) such that whenever \(x \in X\) and \(0 < r < 2\text{diam}(X)\), we have

\[
\frac{r^Q}{C_A} \leq \mu(B(x, r)) \leq C_A r^Q.
\]

As a consequence, we get

\[
\frac{\mu(A)}{C_A} \leq \mathcal{H}^Q(A) \leq C_A \mu(A).
\]

Given an open set \(U \subset X\), we write \(u \in L^1_{\text{loc}}(U)\) if \(u \in L^1(V)\) for every open \(V \subseteq U\); this expression means that \(\overline{V}\) is a compact subset of \(U\). Other local spaces are defined analogously.
A curve is a continuous mapping from an interval into $X$, and a rectifiable curve is a curve with finite length.

**Definition 2.3.** Let $Y$ be a metric space with metric $d_Y$. Given a function $u : X \to Y$, a Borel function $g_u : X \to [0, \infty]$ is said to be an upper gradient of $u$ if for every compact rectifiable curve $\gamma$
\[
d_Y(u(x), u(y)) \leq \int_{\gamma} g_u \, ds
\]
where $x$ and $y$ are the endpoints of $\gamma$. Let $1 \leq p < \infty$. A function $f : X \to Y$ is said to be in $N_{loc}^{1,p}(X; Y)$ if $f \in L^p_{loc}(X; Y)$ and there is an upper gradient $g$ of $f$ such that $g \in L^p_{loc}(X)$. If $Y = \mathbb{R}$ and $f, g \in L^p(X)$ then we say that $f \in N^{1,p}(X)$.

We refer the reader to [14, 15] for the details regarding mappings in $N_{loc}^{1,p}(X; Y)$.

**Definition 2.4.** Let $\mathcal{M}$ be a collection of measures on $X$. Then the admissible class of $\mathcal{M}$, denoted $\mathcal{A}(\mathcal{M})$, is the set of all positive Borel functions $\rho : X \to [0, \infty]$ such that
\[
\int_X \rho \, d\lambda \geq 1
\]
for all $\lambda \in \mathcal{M}$. Then the $p$-modulus of the family $\mathcal{M}$ is given by
\[
\text{Mod}_p(\mathcal{M}) = \inf_{\rho \in \mathcal{A}(\mathcal{M})} \int_X \rho^p \, d\mu.
\]

Mod$_p$ is an outer measure on the class of all measures, see [9]. There are two types of collections of measures associated with quasiconformal maps. Given a collection $\Gamma$ of curves in $X$, we set $\Gamma$ to also denote the arc length measures restricted to each curve in $\Gamma$; for this collection of measures, the above notion of Mod$_p(\Gamma)$ agrees with the standard notion of the $p$-modulus of the family $\Gamma$ of curves from [20, 13, 15]. For a collection $\mathcal{L}$ of sets of finite perimeter in $X$, we consider the measure $\mathcal{H}^{d-1}\chi_E$ for each $E \in \mathcal{L}$; it is known that this measure is comparable to the perimeter measure associated with $E$ as in Definition 2.10, see Theorem 4.1.

**Definition 2.5.** The relative $p$-capacity of two sets $E, F \subset X$ is given by
\[
\text{cap}_p(E, F) = \inf \int_X g_u^p \, d\mu
\]
where the infimum is over all upper gradients $g_u$ of all functions $u \in N_{loc}^{1,p}(X)$ such that $u|_E \leq 0$ and $u|_F \geq 1$.

**Definition 2.6.** We say that the space $X$ supports a $p$-Poincaré inequality if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for all open balls $B$ in $X$, all measurable functions $u$ on $\lambda B$ and all upper gradients $g_u$ of $u$,
\[
\int_B |u - u_B| \, d\mu \leq C_P \text{rad}(B) \left( \int_{\lambda B} g_u^p \, d\mu \right)^{1/p}.
\]
Here we denote the integral average of $u$ over $B$ by
\[
u_B := \int_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.
\]
One of the consequences of a space being complete and Ahlfors regular and supporting a Poincaré inequality is that such a metric space must necessarily be quasiconvex, that is, there is some constant $C_q \geq 1$ such that for every $x, y \in X$ there is a rectifiable curve $\gamma$ with end points $x, y$ and length $\ell(\gamma) \leq C_q d(x, y)$, see [11 Proposition 4.4] or [7 Theorem 4.32]. Thus a bi-Lipschitz change in the metric results in $X$ being a geodesic space, that is, a quasiconvex space with the quasiconvexity constant $C_q = 1$. Notions such as Poincaré inequality, quasiconformality, upper gradients and functions of bounded variation (see below), and Hausdorff measure are quasi-invariant under a bi-Lipschitz change in the metric, hence we do not lose generality by assuming that $X$ is a geodesic space. Geodesic spaces that support a Poincaré inequality do so even with $\lambda = 1$, see [11] or [7, Theorem 4.39].

**Definition 2.7.** For a measurable set $E \subset X$ and $x \in X$, we define the upper density of $E$ at $x$ by

$$D(E, x) = \limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}$$

and the lower density of $E$ at $x$ by

$$D(E, x) = \liminf_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.$$

**Definition 2.8.** For a set $E$, the measure-theoretic boundary is the set

$$\partial^* E = \{ x \in X : D(E, x) > 0 \text{ and } D(X \setminus E, x) > 0 \}.$$

**Definition 2.9.** For $u \in L^1_{\text{loc}}(X)$, the total variation of $u$ on an open set $U \subset X$ is given by

$$\|Du\|(U) = \inf \left\{ \liminf_{n \to \infty} \int_U g_{u_n} \, d\mu \mid (u_n)_{n \in \mathbb{N}} \subset \text{Lip}_{\text{loc}}(U), u_n \to u \text{ in } L^1_{\text{loc}}(U) \right\}.$$

In the above, $g_{u_n}$ stands for an upper gradient of $u_n$ in $U$ (here we consider $U$ to be the metric measure space with metric and measure inherited from $X$). We say $u$ is of bounded variation on $X$ (denoted $u \in BV(X)$) if $\|Du\|(X) < \infty$. We say that $u \in BV_{\text{loc}}(X)$ if $u \in BV(U)$ for each open set $U \subset X$.

It is shown in [19] that $\|Du\|$ is a Radon measure for any $u \in BV_{\text{loc}}(X)$. We call $\|Du\|$ the variation measure of $u$.

**Definition 2.10.** A measurable set $E \subset X$ has finite perimeter if $\chi_E$ is of bounded variation on $X$. We call $\|D\chi_E\|$ the perimeter measure of $E$ and we will denote it $P(E, \cdot)$.

**Definition 2.11.** We say that $X$ supports a relative isoperimetric inequality if there exist constants $C_I > 0$ and $\lambda \geq 1$ such that for all balls $B(x, r)$ and for all measurable sets $E$, we have

$$\min\{\mu(B(x, r) \cap E), \mu(B(x, r) \setminus E)\} \leq C_I r P(E, B(x, \lambda r)).$$

Again, with $X$ a geodesic space, we can choose $\lambda = 1$. We know that if $X$ is Ahlfors regular and supports a 1-Poincaré inequality, then it supports a relative isoperimetric inequality, see for example [3 Theorem 4.3].
Definition 2.12. For $\beta > 0$, let

$$\Sigma_\beta E = \{x \in \partial^* E \mid D(E, x) \geq \beta \text{ and } D(X \setminus E, x) \geq \beta\},$$

and set

$$\Sigma E = \bigcup_{\beta \in (0,1)} \Sigma_\beta E.$$

See [3] or Theorem 4.1 below for connections between $\Sigma_\beta E$, $\partial^* E$, and the perimeter measure $P(E, \cdot)$.

Standing assumptions on the metric spaces: Throughout this paper we will assume that both $(X, d_X, \mu_X)$ and $(Y, d_Y, \mu_Y)$ are complete metric spaces that are Ahlfors $Q$-regular for some $Q > 1$ and support a 1-Poincaré inequality. We will also, without loss of generality, assume that $X$ and $Y$ are geodesic spaces. We will use the letter $C$ to denote various constants that depend, unless otherwise specified, only on the Ahlfors regularity constants and the Poincaré inequality constants of $X$, and the value of $C$ could differ at each occurrence.

3 Quasiconformal mappings

In this section we gather together definitions related to the notion of quasiconformal mappings between two metric spaces. Here, $(X, d_X, \mu_X)$ and $(Y, d_Y, \mu_Y)$ are complete metric measure spaces with $\mu_X$, $\mu_Y$ Radon measures and $Q > 1$ such that $\mu_X \approx \mathcal{H}^Q_X = H_Q^X$ and $\mu_Y \approx \mathcal{H}^Q_Y = H_Q^Y$ as in the standing assumptions from Section 2. Recall also the definitions of $L_f$ and $l_f$ from (1.1):

Definition 3.1. Define $L_f : X \to \mathbb{R}$ by

$$L_f(x) = \limsup_{r \to 0} \frac{L_f(x, r)}{r} \quad \text{where} \quad L_f(x, r) = \sup_{y \in B(x, r)} d_Y(f(x), f(y)).$$

Similarly, define $l_f : X \to \mathbb{R}$ by

$$l_f(x) = \liminf_{r \to 0} \frac{l_f(x, r)}{r} \quad \text{where} \quad l_f(x, r) = \inf_{y \notin B(x, r)} d_Y(f(x), f(y)).$$

When $f$ is a homeomorphism, we always have $l_f(x, r) \leq L_f(x, r)$. When $f$ is a quasiconformal homeomorphism, there is a constant $K_D$ such that $L_f(x, r) \leq K_D l_f(x, r)$, see for example [13] or (4.2) below.

There are different geometric notions of quasiconformal maps on metric spaces.

Definition 3.2. The homeomorphism $f$ is metric quasiconformal if there is a constant $K_D \geq 1$ such that for all $x \in X$ we have

$$\limsup_{r \to 0^+} \frac{L_f(x, r)}{l_f(x, r)} \leq K_D.$$

When we need to emphasize the constant $K_D$ we say that $f$ is $K_D$-quasiconformal.

The map $f$ is geometric quasiconformal if there is a constant $K \geq 1$ such that whenever $\Gamma$ is a family of non-constant compact rectifiable curves in $X$, we have

$$\frac{1}{K} \text{Mod}_Q(f\Gamma) \leq \text{Mod}_Q(\Gamma) \leq K \text{Mod}_Q(f\Gamma).$$
Definition 3.3. A homeomorphism $f : X \to Y$ is quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for every distinct triple of points $x, y, z \in X$ and $t > 0$,

$$\frac{d_X(x, y)}{d_X(x, z)} \leq t \implies \frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta(t).$$

The notions of quasisymmetry, metric quasiconformality, and geometric quasiconformality are connected, see [13, 21] and Theorem 4.3 below.

Definition 3.4. A homeomorphism $f : X \to Y$ between Ahlfors $Q$-regular metric spaces satisfies Lusin’s condition (N) if whenever $A \subset X$ is such that $\mathcal{H}^Q(A) = 0$ then $\mathcal{H}^Q(f(A)) = 0$. We say that $f$ satisfies condition $(N^{-1})$ if its inverse satisfies condition (N).

Quasiconformal maps satisfy both Lusin’s condition (N) and $(N^{-1})$, see Theorem 4.3 below.

Definition 3.5. If $\nu_X$ is a Radon measure on $X$ and $\nu_Y$ is a Radon measure on $Y$ and $f : X \to Y$ is a homeomorphism, then the pull-back of the measure $\nu_Y$ is the measure on $X$ given by

$$f_\# \nu_Y(D) := \nu_Y(f(D))$$

whenever $D$ is a Borel subset of $X$. Note that since $f$ is a homeomorphism, $f_\# \nu_Y$ defines a Borel measure.

Definition 3.6. We define the (generalized) Jacobian of $f$ at the point $x \in X$ as follows:

$$J_f(x) = \limsup_{r \to 0^+} \frac{\mathcal{H}^Q(f(B(x, r)))}{\mathcal{H}^Q(B(x, r))}.$$

Note that $J_f$ is the Radon-Nikodym derivative of the pull-back measure $f_\# \mathcal{H}^Q$ with respect to $\mathcal{H}^Q_X$ and $\mathcal{H}^Q$ is a doubling measure, and so the limit supremum in the definition of $J_f$ is actually a limit at $\mathcal{H}^Q$-almost every $x$.

Definition 3.7. For a set $E \subset X$ of finite perimeter, should $f_\# \mathcal{H}^{Q-1}_{\Sigma f(E)} \ll \mathcal{H}^{Q-1}_{\Sigma E}$, we define the $(Q-1)$-Jacobian of $f$ with respect to $\Sigma E$ by

$$J_{f,E}(x) = \lim_{r \to 0^+} \frac{\mathcal{H}^{Q-1}_{\Sigma f(E)}(f(B(x, r)))}{\mathcal{H}^{Q-1}_{\Sigma E}(B(x, r))}.$$

Given that $\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma E) = 0$ (see Theorem 4.1), we can equivalently consider $J_{f,E}$ to be the Radon-Nikodym derivative on $\partial^* E$.

Further standing assumptions: In this paper, in addition to the standing assumptions listed at the end of Section 2 we will also assume that $f : X \to Y$ is a quasiconformal mapping.
4 Background results

In this section we will gather together some of the background results needed in the paper.

**Theorem 4.1** ([3, Theorem 5.3, Theorem 5.4], [5, Theorem 4.6]). There exists $\gamma > 0$ (depending only on the Ahlfors regularity constant of $\mu$ and the Poincaré inequality constants) such that for any set of finite perimeter $E$, the perimeter measure $P(E, \cdot)$ is concentrated on $\Sigma_\gamma E$. Furthermore, $\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma_\gamma E) = 0$ and there exist constants $\tilde{\alpha} > 0$ and $C > 0$ (again depending only on $C_A, C_P$, and $\lambda$) and a Borel function $\Theta_E : X \to [\tilde{\alpha}, C]$ such that

$$P(E, B(x, r)) = \int_{B(x, r) \cap \partial^* E} \Theta_E \, d\mathcal{H}^{Q-1} = \int_{B(x, r) \cap \Sigma_\gamma E} \Theta_E \, d\mathcal{H}^{Q-1},$$

for any $x \in X$ and $r > 0$. Consequently we have that

$$\tilde{\alpha} \mathcal{H}^{Q-1}(B(x, r) \cap \partial^* E) \leq P(E, B(x, r)) \leq C \mathcal{H}^{Q-1}(B(x, r) \cap \partial^* E) \quad (4.1)$$

and

$$\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma E) = 0.$$

The results of [3] did not need the measure to be Ahlfors regular, only that the measure be doubling, but as Ahlfors regularity is a stronger condition, the results of [3] hold here as well.

**Theorem 4.2** ([18, Theorem 1.1]). If $E \subset X$ is measurable and $\mathcal{H}^{Q-1}(\partial^* E) < \infty$ then $E$ is of finite perimeter.

Now we turn to preliminary results related to quasiconformal mappings needed in the paper.

**Theorem 4.3** ([14, Theorem 9.8]). Let $f : X \to Y$ be a homeomorphism between metric spaces of locally $Q$-bounded geometry. Then the following conditions are quantitatively equivalent:

1. $f$ is $H$-quasiconformal for some $H \geq 1$,
2. There is a homeomorphism $\eta$ such that $f$ is locally $\eta$-quasisymmetric,
3. $f \in N^{1,Q}_\text{loc}(X : Y)$ and $L_f(x)^Q \leq KJ_f(x)$ for almost every $x \in X$ and a constant $K > 0$,
4. There is some $L > 0$ such that for all curve families $\Gamma$ in $X$,

$$L^{-1}\text{Mod}_Q(\Gamma) \leq \text{Mod}_Q(f\Gamma) \leq L\text{Mod}_Q(\Gamma).$$

Furthermore, if any one of these conditions holds, then both $f$ and $f^{-1}$ are quasiconformal and satisfy Lusin’s condition (N).

**Remark 4.4.** A metric space is said to be of locally $Q$-bounded geometry if $X$ is separable, path connected, locally compact, locally uniformly Ahlfors $Q$-regular and satisfies the Loewner condition locally uniformly. Under our assumptions, $X$ is locally (even globally) of $Q$-bounded geometry.
In fact, under our assumptions the quasiconformal mapping \( f \) is necessarily quasisymmetric.

**Proposition 4.5.** The quasiconformal mapping \( f \) is quasisymmetric.

**Proof.** Whenever \( A \subset X \) is bounded, \( \overline{A} \) is compact, and then since \( f \) is continuous, \( f(\overline{A}) \) is compact and thus bounded, that is, \( f \) maps bounded sets to bounded sets. Since \( f : X \to Y \) is a homeomorphism, \( f^{-1} \) also maps bounded sets to bounded sets, and so \( X \) is bounded if and only if \( Y \) is bounded. Now if \( X \) is unbounded, by [13, Corollary 4.8, Theorem 5.7] we know that \( f \) is quasisymmetric; if \( X \) is bounded, we know this by [13, Theorem 4.9]. \( \Box \)

The next lemma follows also from Theorem 4.3(3) together with the chain rule applied to \( f \) and to \( f^{-1} \), but as the proof is simple we provide it here for the convenience of the reader.

**Lemma 4.6.** If \( \eta \) is the homeomorphism of quasisymmetry for \( f \), then there exists \( C > 0 \), depending only on the Ahlfors regularity constant of the measures on \( X \) and \( Y \) and on \( \eta(1) \), such that at every \( x \in X \),

\[
\frac{L_f(x)^Q}{C} \leq J_f(x) \leq C L_f(x)^Q.
\]

**Proof.** Fix \( x \in X \), and choose \( r > 0 \) small enough so that \( X \setminus \overline{B}(x, r) \) is non-empty. Let \( w \in B(x, r) \) and \( z \in X \setminus B(x, r) \). Then

\[
\frac{d_Y(f(x), f(w))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, w)}{d_X(x, z)}\right) \leq \eta(1)
\]

because \( d(x, w) \leq d(x, z) \). This holds for all \( w \in B(x, r) \) and \( z \in X \setminus B(x, r) \), and so

\[
\frac{L_f(x, r)}{l_f(x, r)} \leq \eta(1).\tag{4.2}
\]

Now for \( y \in Y \setminus f(B(x, r)) \) we have \( d_X(f^{-1}(y), x) \geq r \), and so

\[
d_Y(y, f(x)) \geq l_f(x, r) \geq \frac{L_f(x, r)}{\eta(1)}.
\]

It follows that \( y \notin B(f(x), L_f(x, r)/\eta(1)) \). Hence, \( B(f(x), L_f(x, r)/\eta(1)) \subset f(B(x, r)) \). Now at every \( x \in X \), \( J_f(x) \) is given by

\[
J_f(x) = \limsup_{r \to 0^+} \frac{\mathcal{H}^Q(f(B(x, r)))}{\mathcal{H}^Q(B(x, r))} \geq \limsup_{r \to 0^+} \frac{\mathcal{H}^Q\left( f(x), \frac{L_f(x, r)}{\eta(1)}\right)}{\mathcal{H}^Q(B(x, r))}
\]

\[
\geq \limsup_{r \to 0^+} \frac{1}{C_A \eta(1)^Q} \left( \frac{L_f(x, r)}{\eta(1)} \right)^Q
\]

\[
= \limsup_{r \to 0^+} \frac{1}{C_A^2 \eta(1)^Q} \left( \frac{L_f(x, r)}{r} \right)^Q
\]

\[
= \frac{1}{C_A^2 \eta(1)^Q} L_f(x)^Q,
\]
and
\[ J_f(x) \leq \limsup_{r \to 0^+} \frac{\mathcal{H}^Q(B(f(x), L_f(x, r)))}{\mathcal{H}^Q(B(x, r))} \leq C_A^2 L_f(x)^Q. \]

Letting \( C = C_A^2 \max\{\eta(1)^Q, 1\} \), the conclusion follows. \( \square \)

**Lemma 4.7** ([14, Lemma 2.7]). For every ball \( B(f(x), s) \subset Y \) there exists \( r > 0 \) such that
\[ B(f(x), s) \subset f(B(x, r)) \subset f(B(x, 10r)) \subset B(f(x), \eta(10)s). \]

Furthermore, if \( f^{-1} \) is uniformly continuous with modulus of continuity \( \omega(\cdot) \), then we can choose \( r \leq \omega(s) \).

**Proof.** Since \( Y \) is proper and \( f^{-1} \) is continuous, there exists \( r > 0 \) such that the first inclusion holds. Let \( r = \inf\{r' \mid f(B(x, r')) \supset B(f(x), s)\} \); since \( f \) is a homeomorphism, \( B(f(x), s) \subset f(B(x, r)) \). Then for any \( 0 < c < 1 \), there exists a point \( z_c \in f^{-1}(B(f(x), s)) \setminus B(x, cr). \) Then \( f(z_c) \in B(f(x), s) \setminus f(B(x, cr)) \) which implies that \( d_Y(f(x), f(z_c)) \leq s \) and \( d_X(x, z_c) \geq cr. \) Let \( w \in B(x, 10r) \). Now \( f \) is quasisymmetric by Proposition 4.5 with an associated homeomorphism \( \eta \), and so
\[ d_Y(f(x), f(w)) \leq \eta \left( \frac{d_X(x, w)}{d_X(x, z_c)} \right) d_Y(f(x), f(z_c)) \leq \eta \left( \frac{10}{c} \right) s. \]

Letting \( c \) tend to 1, we get that \( d_Y(f(x), f(w)) \leq \eta(10)s \). Thus the last inclusion holds. Now if \( \omega(t) \) is a modulus of continuity of \( f^{-1} \), then \( r \leq \omega(s) \) since we chose \( r \) minimally. \( \square \)

Recall that \( \Sigma E = \bigcup_{\beta \in (0, 1)} \Sigma_\beta E \), see Definition 2.12

**Lemma 4.8.** Let \( E \subset X \) be measurable. For each \( \beta \in (0, 1) \) there exists \( \beta_0 \in (0, 1) \) such that \( \Sigma_\beta f(E) \subset f(\Sigma_{\beta_0} E) \). Consequently, \( \Sigma f(E) = f(\Sigma E) \). Also, \( \partial^* f(E) = f(\partial^* E) \).

**Proof.** Let \( x \in X \). By [17, Theorem 6.2] we know that for all sufficiently small balls \( B_1 \) centered at \( x \), we have for some \( a, b > 0 \)
\[ \frac{\mu_Y(f(E) \cap B_2)}{\mu_Y(B_2)} \leq b \left( \frac{\mu_X(E \cap B_1)}{\mu_X(B_1)} \right)^a, \tag{4.3} \]
where \( B_2 \) denotes the largest open ball in \( f(B_1) \) with center \( f(x) \).

Suppose \( \beta \in (0, 1) \) and \( f(x) \in \Sigma_\beta f(E) \), that is, both \( D_f(E, f(x)) \) and \( D(Y \setminus f(E), f(x)) \) are at least as large as \( \beta \). As the radius of \( B_1 \) converges to 0, so does the radius of \( B_2 \), and so it follows from (4.3) that
\[ \beta_0 := \left( \frac{\beta}{b} \right)^{1/a} \leq D(E, x). \]

By using the fact that \( D(Y \setminus f(E), f(x)) \geq \beta \), and (4.3) with \( E \) replaced by \( X \setminus E \), we get also \( \beta_0 \leq D(X \setminus E, x) \). Thus \( x \in \Sigma_{\beta_0} E \), and so we have proved that \( \Sigma_\beta f(E) \subset f(\Sigma_{\beta_0} E) \). It follows that \( \Sigma f(E) \subset f(\Sigma E) \). Since \( f^{-1} \) is also quasiconformal, we also get
\[ \Sigma f^{-1}(f(E)) \subset f^{-1}(\Sigma f(E)) \]
and so \( f(\Sigma E) \subset \Sigma f(E) \). We therefore have \( f(\Sigma E) = \Sigma f(E) \).

Next suppose that \( f(x) \in \partial^* f(E) \). It follows from (4.3) that

\[
0 < \left( \frac{D(f(E), x)}{b} \right)^{1/a} \leq D(E, x).
\]

From the fact that \( D(Y \setminus f(E), f(x)) > 0 \), and (4.3) with \( E \) replaced by \( X \setminus E \), we get also \( 0 < D(E \setminus E, x) \). Thus \( x \in \partial^* E \), and so we have proved that \( \partial^* f(E) \subset f(\partial^* E) \). Since \( f^{-1} \) is also quasiconformal, we also get

\[
\partial^* f^{-1}(f(E)) \subset f^{-1}(\partial^* f(E))
\]

and so \( f(\partial^* E) \subset \partial^* f(E) \), whence we conclude that \( f(\partial^* E) = \partial^* f(E) \).

\[\square\]

**Lemma 4.9.** There exists \( \alpha > 0 \) such that for every \( E \subset X \) of finite perimeter and for \( \mathcal{H}^{Q-1} \)-almost every \( x \in \partial^* E \),

\[
\liminf_{r \to 0} \frac{P(E, B(x, r))}{r^{Q-1}} > \alpha.
\]

Moreover, if \( 0 < \beta < 1 \), then there is some \( \alpha(\beta) > 0 \) such that whenever \( E \subset X \) is a measurable set and \( x \in \Sigma_\beta E \), we have

\[
\liminf_{r \to 0} \frac{P(E, B(x, r))}{r^{Q-1}} > \alpha(\beta).
\]

**Proof.** By Ahlfors \( Q \)-regularity we know that \( r^{-1} \mu(B(x, r)) \) is comparable to \( r^{Q-1} \). Recall that \( C_I \) is the constant from the isoperimetric inequality and \( C_A \) is the Ahlfors regularity constant. By Theorem [4.1] there exists \( \gamma > 0 \) such that for \( \mathcal{H}^{Q-1} \)-almost every \( x \in \partial^* E \),

\[
\gamma \leq \liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \quad \text{and} \quad \gamma \leq \liminf_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))}.
\]

Then by the relative isoperimetric inequality,

\[
\gamma \leq \liminf_{r \to 0^+} \frac{\min\{\mu(B(x, r) \cap E), \mu(B(x, r) \setminus E)\}}{\mu(B(x, r))} \leq \liminf_{r \to 0^+} \frac{rc_I P(E, B(x, r))}{\mu(B(x, r))} \leq \liminf_{r \to 0^+} \frac{C_I C_A P(E, B(x, r))}{r^{Q-1}}.
\]

Letting \( \alpha = \frac{\gamma}{C_I C_A} \) concludes the proof of the first part of the lemma. The second part of the lemma is proved in the same way as the first part, with

\[
\alpha(\beta) := \frac{\beta}{C_I C_A}.
\]

\[\square\]
Definition 4.10. For $\alpha > 0$, define
\[
\partial^* E := \left\{ x \in \partial^* E \mid \liminf_{r \to 0} \frac{P(E, B(x, r))}{r^{Q-1}} > \alpha \right\}.
\]

Remark 4.11. By Lemma 4.9 above, we have that for each $0 < \beta < 1$,
\[
\Sigma_\beta E \subset \partial^*(\beta) E.
\]
We need the following “continuity from below” for families of measures, with the families not necessarily measurable with respect to the outer measure $\Mod_p$.

Lemma 4.12 (Ziemer’s lemma [6] Lemma 3.1(3)). Let $\{\mathcal{L}_i\}_{i \in \mathbb{N}}$ be a sequence of families of measures in $X$ such that for each $i$, $\mathcal{L}_i \subset \mathcal{L}_{i+1}$. Then for $1 < p < \infty$,
\[
\Mod_p \left( \bigcup_{i \in \mathbb{N}} \mathcal{L}_i \right) = \lim_{i \to \infty} \Mod_p (\mathcal{L}_i).
\]

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1. To do so, we adapt the tools given in [6] to study boundaries of sets of finite perimeter, which are not Ahlfors regular in general. The adaptation of the tools to this setting is given in Proposition 5.1.

Recall that $P = \{ x \in X : L f(x) = \infty \text{ or } L f(x) = 0 \}$. Note from Lemma 4.6 that $J_f(x) = 0$ for $\mathcal{H}^Q$-almost every $x$ for which $L f(x) = 0$. So by the fact that $f$ and $f^{-1}$ satisfy condition (N) (from Theorem 4.3), we must have that the zero set of $J_f$ and hence the zero set of $L f$ is a null set. As $L f \in L^1_{loc}(X)$, we know that $\mathcal{H}^Q(P) = 0$. We remind the reader of the standard assumptions set forth at the end of Sections 2 and 3. Recall also the definition of $\partial^* E$ from Definition 4.10. As noted in the introduction, with a slight abuse in notation, we will use $\mathcal{L}$ to denote both the family of sets $E$ of finite perimeter and the family of measures $\mathcal{H}^{Q-1}|_{\Sigma E}$.

If $\mathcal{L}$ contains a set $E$ with $P(E, X) = 0$, i.e. $\mathcal{H}^{Q-1}(\partial^* E) = 0$, then $\Mod_{Q/(Q-1)}(\mathcal{L}) = \infty$ as there can be no admissible test function $\rho$ for the class $\mathcal{L}$. In addition, by the fact that $X$ supports a 1-Poincaré inequality we will also have that $\mathcal{H}^Q(X \setminus E) = 0$. It then follows that $\mathcal{H}^Q(f(E)) = 0$ or $\mathcal{H}^Q(f(X \setminus E)) = \mathcal{H}^Q(Y \setminus f(E)) = 0$, whence it follows that $f(E)$ is of finite perimeter with $P(f(E), Y) = 0$. We conclude that $\Mod_{Q/(Q-1)}(f\mathcal{L}) = \infty$.

In this case Theorem 1.1 trivially holds true. So in the proof of the next proposition (and in the next section) we will assume that every $E \in \mathcal{L}$ satisfies $0 < P(E, X) < \infty$.

Proposition 5.1. Let $\mathcal{L}$ denote the given collection of bounded sets $E \subset X$ of positive and finite perimeter measure. Then the following hold true:

1. for each $\alpha > 0$ and $\Mod_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$ we have $\mathcal{H}^{Q-1}(f(\partial^* E)) < \infty$,
2. if $U_0 \subset X$ with $\mathcal{H}^Q(f(U_0)) = 0$ and $\alpha > 0$, then we have that $\mathcal{H}^{Q-1}(f(\partial^* E \cap U_0)) = 0$ for $\Mod_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$,
3. if $U_0 \subset X$ with $\mathcal{H}^Q(f(U_0)) = 0$, then with $\mathcal{L}_{\text{bad}}$ the collection of all $E \in \mathcal{L}$ with $\mathcal{H}^{Q-1}(\Sigma f(E) \cap f(U_0)) > 0$, we have $\Mod_{Q/(Q-1)}(\mathcal{L}_{\text{bad}}) = 0 = \Mod_{Q/(Q-1)}(f\mathcal{L}_{\text{bad}})$. 

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Thus we have that $\int \partial^* E = \partial^* f(E)$ and $\int \Sigma E = \Sigma f(E)$.

Let $U \subseteq X$ be a bounded measurable set. Then since $f$ is a homeomorphism and closed and bounded subsets of $X$ are compact, $f(U)$ is also bounded and $f$ and $f^{-1}$ are uniformly continuous on the sets $U$ and $f(U)$, respectively. Let $\omega(\cdot)$ be a modulus of continuity for $f^{-1}$ on $f(U)$. Let $\varepsilon > 0$. By the definition of Hausdorff measure, there exist $y_i \in f(U)$ and $0 < s_i < \varepsilon$ such that $\{B(y_i, s_i)\}_{i \in \mathbb{N}}$ covers $f(U)$ and $\sum_{i \in \mathbb{N}} s_i^Q \leq (H^Q(f(U)) + \varepsilon)$. By Lemma 4.7, for every $x_i := f^{-1}(y_i)$, there exists $0 < r_i \leq \omega(\varepsilon)$ such that

$$B(y_i, s_i) \subset f(B(x_i, r_i)) \subset f(B(x_i, 10r_i)) \subset B(y_i, \eta(10)s_i).$$

Set $B_i = B(x_i, r_i)$ and $B_i' = B(y_i, s_i)$ and define $g : X \to \mathbb{R}$ by

$$g(x) = \sup_{i \in \mathbb{N}} \left( \frac{s_i}{r_i} \right)^{Q-1} \chi_{2B_i}(x).$$

Fix $\alpha > 0$ and define

$$\partial^*_\alpha E = \left\{ x \in \partial^* E \left| \frac{P(E, B(x, r))}{r^{Q-1}} > \alpha \text{ for all } 0 < r \leq \delta \right. \right\}$$

and note that $\partial^*_\alpha E = \bigcup_{k \in \mathbb{N}} \partial^*_{1/k} E$. For each $M > 0$, define

$$\mathcal{L}^M_{\alpha \varepsilon} = \left\{ E \in \mathcal{L} \left| \frac{H^Q(f(\partial^*_{\omega(\varepsilon)} E \cap U)) > M} \right. \right\}.$$

We want to show that $C\eta(10)^{Q-1}[\alpha M]^{-1} g$ is admissible for calculating $\text{Mod}_{\eta(10)}(\mathcal{L}^M_{\alpha \varepsilon})$. Let $E \in \mathcal{L}^M_{\alpha \varepsilon}$ and set $I_E = \{i \in \mathbb{N} \mid B_i \cap (\partial^*_{\omega(\varepsilon)} E \cap U) \neq \emptyset\}$. Note that $\partial^*_{\omega(\varepsilon)} E \cap U \subset \bigcup_{i \in I_E} 2B_i$. Then by the 5-Covering Lemma, there exists $J_E \subseteq I_E$ such that $\{2B_j\}_{j \in J_E}$ is pairwise disjoint and $\partial^*_{\omega(\varepsilon)} E \cap U \subset \bigcup_{j \in J_E} 10B_j$. Since $B_j \cap \partial^*_{\omega(\varepsilon)} E \cap U \neq \emptyset$ for each $j \in J_E$, there exists $z_j \in B_j \cap \partial^*_{\omega(\varepsilon)} E \cap U$. So $B(z_j, r_j) \subset 2B_j$. Moreover, $z_j \in \partial^*_{\omega(\varepsilon)} E$ and $r_j \leq \omega(\varepsilon)$ imply that $r_j^{-(Q-1)} P(E, B(z_j, r_j)) > \alpha$, and hence $P(E, 2B_j) \geq P(E, B(z_j, r_j)) > \alpha r_j^{Q-1}$. Then by the pairwise disjointness property of $\{2B_j\}_{j \in J_E}$,

$$\int_{\Sigma E} g \ dH^{Q-1} \geq \int_{\Sigma E} \left( \sup_{j \in J_E} \left( \frac{s_j}{r_j} \right)^{Q-1} \chi_{2B_j} \right) \ dH^{Q-1} = \int_{\Sigma E} \left( \sum_{j \in J_E} \left( \frac{s_j}{r_j} \right)^{Q-1} \chi_{2B_j} \right) \ dH^{Q-1} \geq \frac{1}{C} \sum_{j \in J_E} \left( \frac{s_j}{r_j} \right)^{Q-1} P(E, 2B_j) \text{ by (4.1)}$$

$$\geq \frac{1}{C} \sum_{j \in J_E} \left( \frac{s_j}{r_j} \right)^{Q-1} \alpha r_j^{Q-1}$$

$$= \frac{\alpha}{C} \sum_{j \in J_E} s_j^{Q-1}.$$

Thus we have that

$$\int_{\Sigma E} g \ dH^{Q-1} \geq \frac{\alpha}{C} \sum_{j \in J_E} s_j^{Q-1}. \quad (5.1)$$
Because $U \cap \partial \alpha^\varepsilon E \subset \bigcup_{j \in J_E} 10B_j$,

$$f(\partial \alpha^\varepsilon E \cap U) \subset f\left( \bigcup_{j \in J_E} 10B_j \right) \subset \bigcup_{j \in J_E} B(y_j, \eta(10)s_j).$$

Since $E \in L^M_{U,\varepsilon}$, we have that $H_Q^{-1} \left( \frac{M}{\eta(10)Q-1} \right) > M$. As $s_j < \varepsilon$, $\bigcup_{j \in J_E} \eta(10)B'_j$ is an admissible cover for computing $H_Q^{-1} \left( f(\partial \alpha^\varepsilon E \cap U) \right)$, and hence

$$M < H_Q^{-1} \left( f(\partial \alpha^\varepsilon E \cap U) \right) \leq \sum_{j \in J_E} (\eta(10)s_j)^{Q-1} = \eta(10)^{Q-1} \sum_{j \in J_E} s_j^{Q-1}.$$

So

$$\frac{M}{\eta(10)Q-1} < \sum_{j \in J_E} s_j^{Q-1}. \quad \text{(5.2)}$$

Combining (5.1) and (5.2), we get

$$\int_{\Sigma_E} g \ dH^Q \geq \frac{\alpha}{C} \left( \frac{M}{\eta(10)Q-1} \right).$$

Therefore

$$\frac{C \eta(10)^{Q-1}}{\alpha M} g$$

is admissible for computing $\text{Mod}_{Q/(Q-1)}(L^M_{U,\varepsilon})$. Setting

$$C_M = \frac{C \eta(10)^{Q-1}}{\alpha M},$$

we obtain

$$\text{Mod}_{Q/(Q-1)}(L^M_{U,\varepsilon}) \leq C_A \int_X (C_M g)^{Q/(Q-1)} \ dH^Q$$

$$= C_M^{Q/(Q-1)} C_A \int_X \sup_{i \in \mathbb{N}} \left( \frac{s_i}{r_i} \right)^{Q-1} \chi_{2B_i} \ dH^Q$$

$$\leq C_M^{Q/(Q-1)} C_A \int_X \left( \sum_{i \in \mathbb{N}} \left( \frac{s_i}{r_i} \right)^Q \chi_{2B_i} \right) \ dH^Q$$

$$\leq C_M^{Q/(Q-1)} C_A \sum_{i \in \mathbb{N}} \left( \frac{s_i}{r_i} \right)^Q \mathcal{H}^Q(2B_i)$$

$$\leq C_M^{Q/(Q-1)} C_A \sum_{i \in \mathbb{N}} \left( \frac{s_i}{r_i} \right)^Q C_A (2r_i)^Q$$

$$= C_M^{Q/(Q-1)} C_A^2 2^Q \sum_{i \in \mathbb{N}} s_i^Q$$

$$\leq C_M^{Q/(Q-1)} C_A^2 2^Q (\mathcal{H}^Q \left( f(U) + \varepsilon \right)).$$

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Let \( \mathcal{L}_U^M = \bigcup_{k \in \mathbb{N}} \mathcal{L}_{U,1/k}^M \). Then
\[
\mathcal{L}_U^M = \{ E \in \mathcal{L} \mid \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U)) > M \}.
\] (5.3)

Note that if \( m < k \) then \( \mathcal{L}_{U,1/m}^M \subset \mathcal{L}_{U,1/k}^M \). Applying Lemma 4.12 we get that
\[
\Mod_{Q/(Q-1)}(\mathcal{L}_U^M) = \lim_{k \to \infty} \Mod_{Q/(Q-1)}(\mathcal{L}_{U,1/k}^M) \leq \lim_{k \to \infty} C_M^{Q/(Q-1)} C_A^2 2^Q \left( \mathcal{H}^Q(f(U)) + \frac{1}{k} \right) = C_M^{Q/(Q-1)} C_A^2 2^Q \mathcal{H}^Q(f(U)).
\] (5.4)

**Proof of Claim 1:** Set \( \mathcal{L}_U^\infty = \{ E \in \mathcal{L} \mid \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U)) = \infty \} \). Then for each \( M > 0 \) we have \( \mathcal{L}_U^M \supset \mathcal{L}_U^\infty \), and so
\[
\Mod_{Q/(Q-1)}(\mathcal{L}_U^\infty) \leq \lim_{M \to \infty} \Mod_{Q/(Q-1)}(\mathcal{L}_U^M) \leq \lim_{M \to \infty} C_M^{Q/(Q-1)} C_A^2 2^Q \mathcal{H}^Q(f(U)) = \lim_{M \to \infty} \frac{1}{M^{Q/(Q-1)}} \left( \frac{C \eta(10)^Q}{\alpha Q/(Q-1)} \right) C_A^2 2^Q \mathcal{H}^Q(f(U)) = 0.
\]

Fix \( x_0 \in X \) and considering \( U_i = B(x_0, i) \) for each \( i \in \mathbb{N} \), we set \( \mathcal{L}_i^\infty = \bigcup_{\mathcal{L}_U^\infty} \). Then as the sets in \( \mathcal{L} \) are bounded, \( \mathcal{L}_U^\infty = \bigcup_{\mathcal{L}_U^\infty} \) and
\[
\Mod_{Q/(Q-1)}(\mathcal{L}_U^\infty) = \Mod_{Q/(Q-1)}(\bigcup_{i \in \mathbb{N}} \mathcal{L}_U^i) \leq \sum_{i \in \mathbb{N}} \Mod_{Q/(Q-1)}(\mathcal{L}_U^i) = 0.
\]

Therefore the set for which \( \mathcal{H}^{Q-1}(f(\partial^\alpha E)) \) is not finite has \( Q/(Q-1) \)-modulus zero. This proves Claim 1.

**Proof of Claim 2:** Let \( U_0 \subset X \) such that \( \mathcal{H}^Q(f(U_0)) = 0 \). Then for any bounded \( U \subset U_0 \), \( \mathcal{H}^Q(f(U)) = 0 \). Recall \( \mathcal{L}_U^M \) from (5.3) for \( M > 0 \), and let \( \mathcal{L}_U^+ = \bigcup_{m \in \mathbb{N}} \mathcal{L}_U^{1/m} \). Then
\[
\mathcal{L}_U^+ = \{ E \in \mathcal{L} \mid \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U)) > 0 \},
\]
and by (5.4) and by the outer measure property of modulus,
\[
\Mod_{Q/(Q-1)}(\mathcal{L}_U^+) = \Mod_{Q/(Q-1)}(\bigcup_{m \in \mathbb{N}} \mathcal{L}_U^{1/m}) \leq \sum_{m \in \mathbb{N}} \Mod_{Q/(Q-1)}(\mathcal{L}_U^{1/m}) \leq \sum_{m \in \mathbb{N}} C_M^{Q/(Q-1)} C_A^2 2^Q \mathcal{H}^Q(f(U)) = 0.
\]

Define \( V_i = B(x_0, i) \cap U_0 \) and set \( \mathcal{L}_i^+ = \bigcup_{\mathcal{L}_U^+} \). Then
\[
\mathcal{L}_i^+ = \{ E \in \mathcal{L} \mid \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U_0)) > 0 \},
\]
and
\[
\Mod_{Q/(Q-1)}(\mathcal{L}_i^+) \leq \sum_{i \in \mathbb{N}} \Mod_{Q/(Q-1)}(\mathcal{L}_i^+) = 0.
\]

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Proof of Claim 3: Again, suppose \( H^Q(f(U_0)) = 0 \). By Lemma \[4.10\] for any \( 0 < \beta < 1 \) there exists \( \alpha(\beta) > 0 \) such that \( \Sigma E \subset \partial \alpha(\beta) E \). Then by Claim 2, for \( \text{Mod}_{Q/(Q-1)} \)-almost every \( E \)

\[
H^{Q-1}(f(S E \cap U_0)) = H^{Q-1}\left( \bigcup_{\beta \in (0,1)} f(\Sigma E \cap U_0) \right) \leq H^{Q-1}\left( \bigcup_{\beta \in (0,1) \cap Q} f(\partial \alpha(\beta) E \cap U_0) \right) \leq \sum_{\beta \in (0,1) \cap Q} H^{Q-1}(f(\partial \alpha(\beta) E \cap U_0)) = 0.
\]

Finally, by considering the function \( \infty_{X(f(U_0))} \), we see that \( \text{Mod}_{Q/(Q-1)}(f^{\text{bad}}) = 0 \).

Proposition 5.2. Let \( \mathcal{L} \) denote the given collection of bounded sets of finite perimeter in \( X \). For \( \text{Mod}_{Q/(Q-1)} \)-almost every \( E \in \mathcal{L} \) we have

\[
f_{\#} H^{Q-1}|_{\Sigma f(E)} = f_{\#} H^{Q-1}|_{f((\partial E \cap \Sigma E) \cup \Sigma E)} \ll H^{Q-1}|_{\partial E} = H^{Q-1}|_{\Sigma E}
\]

Proof. The last equality follows from Theorem \[4.1\]. To prove the absolute continuity, first recall that \( P = \{ x \in X \mid L_f(x) \in \{0, \infty\} \} \). For \( n, m \in \mathbb{N} \), set

\[
A_{n,m} = \left\{ x \in X \mid \frac{1}{m} \leq \frac{L_f(x, r)}{r} \leq m \quad \text{for} \quad 0 < r < \frac{1}{n} \right\},
\]

Then \( \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A_{n,m} = X \setminus P \). If \( x, y \in A_{n,m} \) such that \( d_X(x, y) < 1/n \), then

\[
d_Y(f(x), f(y)) \leq m d_X(x, y),
\]

that is, \( f \) is locally \( m \)-Lipschitz on \( A_{n,m} \). Furthermore, from \[4.2\] we have that \( l_f(x, r) \geq L_f(x, r)/\eta(1) \), and so

\[
d_Y(f(x), f(y)) \geq \frac{1}{\eta(1)m} d_X(x, y),
\]

that is, \( f \) is locally \( \max\{1, \eta(1)\} \)-biLipschitz on \( A_{n,m} \). (In this proof, we only need \( f \) to be Lipschitz, not biLipschitz.) Note that \( H^Q(P) = 0 \) and thus \( H^Q(f(P)) = 0 \), and so by Proposition \[5.13\], for \( \text{Mod}_{Q/(Q-1)} \)-almost every \( E \in \mathcal{L} \) we have \( H^{Q-1}(f(P \cap \Sigma E)) = 0 \). For such \( E \in \mathcal{L} \), fix \( N \subset \partial E \) such that \( H^{Q-1}(N) = 0 \). Then for each \( m, n \in \mathbb{N} \), by the local \( m \)-Lipschitz property of \( f \) on \( A_{n,m} \), we have

\[
H^{Q-1}(f(N \cap A_{n,m})) \leq m^{Q-1} H^{Q-1}(N \cap A_{n,m}) = 0.
\]

Thus

\[
H^{Q-1}(f(N \setminus P)) = H^{Q-1}\left( f(N \setminus \bigcup_{n,m \in \mathbb{N}} A_{n,m} \right) \leq \sum_{n,m \in \mathbb{N}} m^{Q-1} H^{Q-1}(N \cap A_{n,m}) = 0,
\]

and therefore

\[
H^{Q-1}(f((\partial E \setminus P) \cup \Sigma E) \cap N)) \leq H^{Q-1}(f(N \setminus P)) + H^{Q-1}(f(N \cap \Sigma E \cap P)) = 0.
\]

Finally, note that \( H^{Q-1}(\partial E \setminus (P \cup \Sigma E)) \leq H^{Q-1}(\partial E \setminus \Sigma E) = 0 \), and so the first identity in the proposition also holds by the absolute continuity of the pull-back measure together with the fact that \( \Sigma f(E) = f(\Sigma E) \) (see Lemma \[4.8\]).
Recall the definition of $J_{f,E}$ from Definition 3.7, and note that it is a function on $\Sigma E$.

**Lemma 5.3.** For $\text{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$ we have $\mathcal{H}^{Q-1}(\Sigma f(E)) < \infty$.

**Proof.** By Lemma 4.9 there exists $\alpha_0 > 0$ such that $\mathcal{H}^{Q-1}(\Sigma E \setminus \partial^{\alpha_0}E) = 0$ for every $E \in \mathcal{L}$. Then by Proposition 5.2 we know that $\mathcal{H}^{Q-1}(\Sigma f(E) \setminus f(\partial^{\alpha_0}E)) = 0$ for $\text{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$. Finally, by Proposition 3.7(1) we have $\mathcal{H}^{Q-1}(f(\partial^{\alpha_0}E)) < \infty$ after eliminating a further family of $\text{Mod}_{Q/(Q-1)}$-zero from $\mathcal{L}$. \hfill \Box

**Lemma 5.4.** There exists $C > 0$ such that for $\text{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$,

$$J_{f,E}(x) \leq C J_f(x)^{(Q-1)/Q}$$

for $\mathcal{H}^{Q-1}|_{\Sigma E}$-almost every $x$.

**Proof.** From Proposition 5.2 we know that for $\text{Mod}_{Q/(Q-1)}$-almost every $E$, $f_\# \mathcal{H}^{Q-1}|_{\Sigma f(E)}$ is absolutely continuous with respect to $\mathcal{H}^{Q-1}|_{\partial E} = \mathcal{H}^{Q-1}|_{\Sigma E}$. Furthermore, from Lemma 5.3 we know that $\mathcal{H}^{Q-1}(\Sigma f(E)) < \infty$ for $\text{Mod}_{Q/(Q-1)}$-almost every $E$. We focus only on such $E \in \mathcal{L}$ now.

Note that $L_f$ is a locally integrable function on $X$, so by Lusin’s Theorem, for each $k \in \mathbb{N}$ there is some open set $U_k \subset X$ such that $\mu(U_k) \leq 2^{-k}$ and $L_f|_{X \setminus U_k}$ is continuous. By enlarging $U_k$ if necessary, we can also assume that for each $k \in \mathbb{N}$, $P \subset U_k$. Let $U = \bigcap_{k \in \mathbb{N}} U_k$. Then $\mu(U) = 0$. Let $\mathcal{L}_0 = \{E \in \mathcal{L} : \mathcal{H}^{Q-1}(U \cap \Sigma E) > 0\}$. Then $\infty \chi_U$ is admissible for computing $\text{Mod}_{Q/(Q-1)}(\mathcal{L}_0)$, and so $\text{Mod}_{Q/(Q-1)}(\mathcal{L}_0) \leq \int_X \infty \chi_U \, d\mu = 0$. We ignore such $E$ as well.

Observe that for every $x \in X \setminus U$ we have $L_f(x) > 0$. For $n \in \mathbb{N}$ we set

$$A_n = \{x \in \Sigma E : L_f(x, r)/r \leq 2L_f(x) \text{ for } 0 < r < 1/n\}.$$

It is not difficult to show that $A_n$ is a Borel set by writing it as an intersection of Borel sets, one for each rational $r \in (0, 1/n)$. Now by Proposition 5.2 and the Radon-Nikodym Theorem,

$$\int_{\Sigma E} J_{f,E} \, d\mathcal{H}^{Q-1} = \mathcal{H}^{Q-1}(f(\Sigma E)).$$

Since $\mathcal{H}^{Q-1}(f(\Sigma E)) = \mathcal{H}^{Q-1}(\Sigma f(E))$ is finite, $J_{f,E} \in L^1(\Sigma E, \mathcal{H}^{Q-1})$. For $k, n \in \mathbb{N}$ let $E_n^k$ denote the collection of all $x \in \Sigma E$ that are not Lebesgue points of $\chi_{A_n \setminus U_k} J_{f,E}$. Then for each $k \in \mathbb{N}$ we have $\mathcal{H}^{Q-1}(\bigcup_{n \in \mathbb{N}} E_n^k) = 0$. For $x \in \Sigma E \setminus \bigcap_{k \in \mathbb{N}} \left(U_k \cup \bigcup_{n \in \mathbb{N}} E_n^k\right)$, there is some $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $x \in A_n$ but $x \notin E_n^k \cup U_k$. Therefore

$$J_{f,E}(x) = \lim_{r \to 0^+} \int_{B(x, r) \cap \Sigma E} \chi_{A_n \setminus U_k} J_{f,E} \, d\mathcal{H}^{Q-1}$$

$$= \lim_{r \to 0^+} \frac{\int_{B(x, r) \cap \Sigma E} J_{f,E} \, d\mathcal{H}^{Q-1}}{\mathcal{H}^{Q-1}(B(x, r) \cap \Sigma E)}$$

$$= \lim_{r \to 0^+} \frac{f_\# \mathcal{H}^{Q-1}(B(x, r) \cap \Sigma E)}{\mathcal{H}^{Q-1}(B(x, r) \cap \Sigma E)}. \quad (5.8)$$
From an argument similar to (5.6) we know that $f$ is $2\sup_{B(x,r)\setminus U_k} L_f$-Lipschitz continuous on $B(x, r) \cap A_n \setminus U_k$ when $0 < r < 1/2n$, and so
\[
 f_\# \mathcal{H}^{Q-1}(B(x, r) \cap A_n \setminus U_k) = \mathcal{H}^{Q-1}(f(B(x, r) \cap A_n \setminus U_k))
\]
\[
 \leq \left[ 2 \sup_{B(x,r)\setminus U_k} L_f \right]^{Q-1} \mathcal{H}^{Q-1}(B(x, r) \cap A_n \setminus U_k)
\]
\[
 \leq \left[ 2 \sup_{B(x,r)\setminus U_k} L_f \right]^{Q-1} \mathcal{H}^{Q-1}(B(x, r) \cap \Sigma E),
\]
and so by (5.8) and the continuity of $L_f$ in $X \setminus U_k$,
\[
 J_{f,E}(x) \leq \limsup_{r \to 0^+} \left[ 2 \sup_{B(x,r)\setminus U_k} L_f \right]^{Q-1} = 2^{Q-1} L_f(x)^{Q-1}.
\]
Now by applying Lemma 4.6 we obtain
\[
 J_{f,E}(x) \leq C J_f(x)^{(Q-1)/Q}.
\]
This holds for all $x \in \Sigma E \setminus (\bigcap_{k \in \mathbb{N}} (U_k \cup \bigcup_{n \in \mathbb{N}} E_n^k))$. As $E \notin \mathcal{L}_0$ and $\mathcal{H}^{Q-1}(\bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} E_n^k) = 0$, we have
\[
 \mathcal{H}^{Q-1}\left( \Sigma E \setminus \bigcap_{k \in \mathbb{N}} \left( U_k \cup \bigcup_{n \in \mathbb{N}} E_n^k \right) \right) = 0,
\]
and so the claim holds.

\textit{Proof of Theorem 1.1.} Let $\mathcal{L}_0$ be the set of $E \in \mathcal{L}$ for which the conclusion of either Proposition 5.2 or Lemma 5.4 fails. Then by those results, we know that $\text{Mod}_{Q/(Q-1)}(\mathcal{L}_0) = 0$. Let $E \in \mathcal{L} \setminus \mathcal{L}_0$.

From Proposition 5.2 and Lemma 4.8 we know that the pull-back measure $f_\# \mathcal{H}^{Q-1}|_{\Sigma E} = f_\# \mathcal{H}^{Q-1}|_{f(\Sigma E)}$ is absolutely continuous with respect to $\mathcal{H}^{Q-1}|_{\Sigma E}$. Therefore, whenever $\varphi$ is a nonnegative Borel function on $Y$, we have
\[
 \int_{f(\Sigma E)} \varphi \, d\mathcal{H}^{Q-1} = \int_{\Sigma E} \varphi \circ f \, J_{f,E} \, d\mathcal{H}^{Q-1}.
\]
From Lemma 5.4 we know that the corresponding Radon-Nikodym derivative is dominated by $C_0 J_{f}^{(Q-1)/Q}$. Suppose $\rho : Y \to [0, \infty]$ is admissible for computing $\text{Mod}_{Q/(Q-1)}(f \mathcal{L})$. Define $\tilde{\rho} : X \to [0, \infty]$ by
\[
 \tilde{\rho} = C_0 (\rho \circ f) J_{f}^{(Q-1)/Q}.
\]
Then $\tilde{\rho}$ is admissible for calculating $\text{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_0)$ since by Proposition 5.2 and the change of variables formula,
\[
 \int_{\Sigma E} \tilde{\rho} \, d\mathcal{H}^{Q-1} = \int_{\Sigma E} C_0 (\rho \circ f) J_{f}^{(Q-1)/Q} \, d\mathcal{H}^{Q-1} \geq \int_{\Sigma E} (\rho \circ f) J_{f,E} \, d\mathcal{H}^{Q-1}
\]
\[
 = \int_{f(\Sigma E)} \rho \, d\mathcal{H}^{Q-1}
\]
\[
 = \int_{\Sigma f(E)} \rho \, d\mathcal{H}^{Q-1} \geq 1.
\]
It follows that
\[
\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}) = \operatorname{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_0) \leq C_A \int_X \tilde{\rho}^{Q/(Q-1)} \, d\mathcal{H}^Q
\]
\[
= C_A \int_X \left( C_0 (\rho \circ f) J_f^{(Q-1)/Q} \right)^{Q/(Q-1)} \, d\mathcal{H}^Q
\]
\[
\leq C \int_X (\rho \circ f)^{Q/(Q-1)} J_f \, d\mathcal{H}^Q
\]
\[
= C \int_Y \rho^{Q/(Q-1)} \, d\mathcal{H}^Q.
\]
Taking the infimum over admissible \(\rho\), we obtain
\[
\frac{1}{C} \operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}) \leq \operatorname{Mod}_{Q/(Q-1)}(f\mathcal{L}).
\]

We cannot directly apply the argument to \(f^{-1}\) with respect to \(f\mathcal{L}'\) as we do not know that that family consists solely of sets of finite perimeter. For \(E \in \mathcal{L}\) we know that \(f(E) \in \mathcal{L}\) is of finite perimeter if and only if \(\mathcal{H}^{Q-1}(\partial^* f(E)) < \infty\), see Theorem \([12]\). A priori we do not know whether \(\mathcal{H}^{Q-1}(f(\Sigma) E)) = \infty\). So instead we provide a direct proof. Recall that \(P = \{x \in X : L_f(x) = 0\} \cup \{x \in X : L_f(x) = \infty\}\), and as in \([5.5]\), we set
\[
A_{n,m} = \left\{ x \in X \mid \frac{1}{m} \leq \frac{L_f(x,r)}{r} \leq m \text{ for } 0 < r < \frac{1}{n} \right\}.
\]

Then, as noted in \([5.7]\), \(f\) is \(\max\{1, \eta(1)\} m\)-biLipschitz on \(A_{n,m}\) and
\[
X \setminus P = \bigcup_{n,m \in \mathbb{N}} A_{n,m}.
\]

By the definition of \(\mathcal{L}'\) we have that \(\mathcal{H}^{Q-1}(P \cap \Sigma E) = 0\). Therefore
\[
\mathcal{H}^{Q-1} \left( \Sigma E \setminus \bigcup_{n,m \in \mathbb{N}} A_{n,m} \right) = 0.
\]

As \(f\) is locally (that is, at scales smaller than \(1/n\)) biLipschitz on each \(A_{n,m}\), we now have that
\[
\mathcal{H}^{Q-1} \ll f^{-1}_# \mathcal{H}^{Q-1} \ll \mathcal{H}^{Q-1} \ll \mathcal{H}_{\Sigma f(E)}.
\]

Equivalently, \(f^{-1}_# \mathcal{H}^{Q-1} \ll \mathcal{H}^{Q-1} \ll \mathcal{H}_{\Sigma f(E)}\). Now a direct adaptation of the proof of Lemma \([5.3]\) tells us that the Radon-Nikodym derivative of \(f^{-1}_# \mathcal{H}^{Q-1} \ll \mathcal{H}^{Q-1} \ll \mathcal{H}_{\Sigma f(E)}\), denoted \(J_{f^{-1},E}\), satisfies for \(\operatorname{Mod}_{Q/(Q-1)}\)-almost every \(f(E)\), with \(E \in \mathcal{L}'\), that
\[
J_{f^{-1},E}(y) \leq C \, J_{f^{-1},E}(y)^{(Q-1)/Q},
\]
for \(\mathcal{H}^{Q-1} \ll \mathcal{H}_{\Sigma f(E)}\)-almost every \(y\). Now we can apply the proof of the first part of this theorem to obtain that if \(\rho : X \to [0, \infty]\) is admissible for computing \(\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}')\), then
\[
\tilde{\rho} = C_1 (\rho \circ f^{-1}) \, J_{f^{-1},E}^{(Q-1)/Q}
\]
is admissible for computing \(\operatorname{Mod}_{Q/(Q-1)}(f\mathcal{L}')\). Thus we get the desired comparison stated in the last part of the theorem. \(\square\)
6 Proof of Theorem 1.2

In this section we focus on the proof of Theorem 1.2. The proof uses the tools developed in the previous section. Recall that

$$\mathcal{L}_P = \{ E \in \mathcal{L} : \mathcal{H}^{Q-1}(f((\partial^* E \setminus \Sigma E) \cap P)) > 0 \text{ or } \mathcal{H}^{Q-1}(P \cap \Sigma E) > 0 \}.$$ 

**Proposition 6.1.** For Mod$_{Q/(Q-1)}$-almost every $E \in \mathcal{L} \setminus \mathcal{L}_P$, $\mathcal{H}^{Q-1}(\partial^* f(E))$ is finite and hence $f(E)$ is of finite perimeter.

**Proof.** By Lemma 4.9 there exists $\alpha > 0$ such that $\mathcal{H}^{Q-1}(\partial^* E \setminus \partial^0 E) = 0$. By Proposition 5.1, for Mod$_{Q/(Q-1)}$-almost every $E$ we have that $\mathcal{H}^{Q-1}(f(\partial^0 E))$ is finite. Then by Proposition 5.2 we have $\mathcal{H}^{Q-1}(f(\partial^* E \setminus (P \cup \partial^0 E))) = 0$. For $E \in \mathcal{L} \setminus \mathcal{L}_P$ that satisfy these conditions (and Mod$_{Q/(Q-1)}$-almost all of them do),

$$\mathcal{H}^{Q-1}(f(\partial^* E)) \leq \mathcal{H}^{Q-1}(f(\partial^0 E)) + \mathcal{H}^{Q-1}(f(\partial^* E \setminus (P \cup \partial^0 E)))$$

$$= \mathcal{H}^{Q-1}(f(\partial^0 E)) < \infty.$$ 

An application of Theorem 1.2 and Lemma 4.8 completes the proof. \qed

From the above chain of inequalities we see that for Mod$_{Q/(Q-1)}$-almost all $E \in \mathcal{L} \setminus \mathcal{L}_P$ we have $\mathcal{H}^{Q-1}(f(\partial^* E)) = \mathcal{H}^{Q-1}(f(\partial^0 E))$ and thus

$$\mathcal{H}^{Q-1}(f(\partial^* E)) = \mathcal{H}^{Q-1}(f(\Sigma E)). \tag{6.1}$$

**Proof of Theorem 1.2.** The above proposition implies the validity of the first claim in the theorem.

By Proposition 5.2 and (6.1), the first absolute continuity of (1.1) holds for Mod$_{Q/(Q-1)}$-almost every $E \in \mathcal{L} \setminus \mathcal{L}_P$. By (5.9) and the fact that $\mathcal{H}^{Q-1}|_{\partial^* E} = \mathcal{H}^{Q-1}|_{\Sigma E}$, the second holds for Mod$_{Q/(Q-1)}$-almost every $E \in \mathcal{L}' \supset \mathcal{L} \setminus \mathcal{L}_P$.

The first inequality of (1.3) is verified by applying the first part of Theorem 1.1 to $\mathcal{L} \setminus \mathcal{L}_P$ and $f$. Next note that if $E \in \mathcal{L} \setminus \mathcal{L}_P$, then $E \in \mathcal{L}'$. Thus the second inequality of (1.3) follows from the second part of Theorem 1.1. To complete the proof, we note from Lemma 5.4 and (1.4), (1.5) that $J_{f,E}(x) \leq C J_f^{(Q-1)/Q}(x)$ and $J_{f^{-1},f(E)}(f(x)) \leq C J_f^{(Q-1)/Q}(f(x))$ for $\mathcal{H}^{Q-1}|_{\Sigma E}$-almost every $x$, for Mod$_{Q/(Q-1)}$-almost every $E \in \mathcal{L} \setminus \mathcal{L}_P$. By the chain rule for Radon-Nikodym derivatives, we know that $J_{f^{-1}}(f(x)) = 1/J_f(x)$ for $\mathcal{H}^Q$-almost every $x \in X$ and thus for $\mathcal{H}^{Q-1}|_{\Sigma E}$-almost every $x$, for Mod$_{Q/(Q-1)}$-almost every $E \in \mathcal{L} \setminus \mathcal{L}_P$. Similarly, $J_{f^{-1},f(E)}(f(x)) = 1/J_{f,E}(x)$. Thus we have that

$$J_{f,E} \leq C J_f^{(Q-1)/Q} = \frac{C}{(J_{f^{-1}} \circ f)^{(Q-1)/Q}} \leq \frac{C^2}{J_{f^{-1},f(E)} \circ f} = C^2 J_{f,E}.$$ 

This completes the proof of the theorem. \qed

**Remark 6.2.** We now complete the discussion in this paper by considering the reasonable-ness of excluding $\mathcal{L}_P$. The natural question to ask here is: how much are we losing by neglecting $\mathcal{L}_P$? Suppose that for all $E \in \mathcal{L}$ we have $f(E)$ is of finite perimeter in $Y$. Then
\( \mathcal{H}^{Q-1}(\partial^* f(E) \setminus \Sigma f(E) \cap f(P)) = 0 \). Since \( \mathcal{H}^Q(P) = 0 \), we have \( \mathcal{H}^{Q-1}(P \cap \Sigma E) = 0 \) for \( \text{Mod}_{Q/(Q-1)} \)-almost every \( E \in \mathcal{L} \). Note also that as \( \mathcal{H}^Q(P) = \mathcal{H}^Q(f^{-1}(f(P))) = 0 \), for \( \text{Mod}_{Q/(Q-1)} \)-almost every \( f(E) \) we have that \( \mathcal{H}^{Q-1}(f^{-1}(f(P)) \cap \Sigma E) = 0 \) by Proposition 5.1(3) applied to the family \( f(E), E \in \mathcal{L} \), and so \( E \notin \mathcal{L}_P \).

There is a large family of sets of finite perimeter whose quasiconformal images are also sets of finite perimeter. Indeed, thanks to the BV co-area formula (see [19, Proposition 4.2]) and the fact that \( N^{1,Q}(X) \subset BV_{loc}(X) \), we know that if \( u \in N^{1,Q}(X) \) is compactly supported, then for \( \mathcal{H}^1 \)-almost every \( t \in \mathbb{R} \) we have that the superlevel set \( E_t := \{ u > t \} \) is of finite perimeter in \( X \). Here, by \( N^{1,Q}(X) \) we mean the function class \( N^{1,Q}(X; \mathbb{R}) \) from Definition 2.3. By [14, Theorem 9.10] we know that \( u \circ f^{-1} \in N^{1,Q}(Y) \) since \( f \) is quasiconformal. Therefore for \( \mathcal{H}^1 \)-almost every \( t \in \mathbb{R} \) we have that \( f(E_t) = \{ u \circ f^{-1} > t \} \) is also of finite perimeter, and so the collection of all \( t \in \mathbb{R} \) for which either \( E_t \) is not of finite perimeter or \( f(E_t) \) is not of finite perimeter is of \( \mathcal{H}^1 \)-measure zero. Hence there are plenty of sets of finite perimeter in \( X \) whose image under \( f \) is of finite perimeter in \( Y \). The remaining part of this section is devoted to make concrete the notion of “plenty”, see Remark 6.5.

**Proposition 6.3.** Let \( u \in N^{1,Q}(X) \) be compactly supported such that \( \int_X g_u^Q \, d\mu > 0 \), where \( g_u \) is the minimal \( Q \)-weak upper gradient of \( u \) (see e.g. [14, Section 6]), and let \( \mathcal{L} \) be the collection of all sets \( E_t = \{ x \in X : u(x) > t \}, t \in \mathbb{R} \), for which \( 0 < P(E_t, X) < \infty \). Then \( \text{Mod}_{Q/(Q-1)}(\mathcal{L}) > 0 \).

**Proof.** By employing truncation of \( u \) and by adding a constant to \( u \) if necessary, we may assume without loss of generality that \( 0 \leq u \leq 1 \) on \( X \) and by the monotonicity of \( \text{Mod}_{Q/(Q-1)} \), we may replace \( \mathcal{L} \) with the collection of all \( E_t \) with \( 0 < P(E_t, X) < \infty \), \( 0 < t < 1 \). If \( P(E_t, X) = 0 \) for almost every \( t \in [0, 1] \), then by the 1-Poincaré inequality we know that \( E_t \) is either almost all of \( X \) or is of measure zero, whence we would have \( u \) is constant, violating that \( \int_X g_u^Q \, d\mu > 0 \). Therefore \( \mathcal{L} \) has many sets, one for each \( t \) in a positive \( \mathcal{H}^1 \)-measure subset of \( [0, 1] \).

Let \( \rho \) be admissible for computing \( \text{Mod}_{Q/(Q-1)}(\mathcal{L}) \). Then for almost every \( t \in [0, 1] \),

\[
\int_{\Sigma E_t} \rho \, dP(E_t, \cdot) \geq 1.
\]

Integrating over \( t \in [0, 1] \) and applying the co-area formula and Hölder’s inequality, we obtain

\[
\mathcal{H}^1(\{ t \in [0, 1] : E_t \in \mathcal{L} \}) \leq \int_0^1 \int_{\Sigma E_t} \rho \, dP(E_t, \cdot) \, dt
\]

\[
= \int_X \rho \, \| Du \|
\]

\[
\leq C \int_X \rho \, g_u \, d\mathcal{H}^Q
\]

\[
\leq C \left( \int_X \rho^{Q/(Q-1)} \, d\mathcal{H}^Q \right)^{1-\frac{1}{Q}} \left( \int_X g_u^Q \, d\mathcal{H}^Q \right)^{\frac{1}{Q}}.
\]

Taking the infimum over all such \( \rho \) gives

\[
0 < \sqrt[Q/(Q-1)]{\mathcal{H}^1(\{ t \in [0, 1] : E_t \in \mathcal{L} \})^{Q/(Q-1)}} \leq \text{Mod}_{Q/(Q-1)}(\mathcal{L}).
\]  

(6.2)
Suppose $\int_X g_u^Q d\mu > 0$ and that $0 \leq u \leq 1$ on $X$. If there is some $0 < t_0 < 1$ for which $P(E_{t_0}, X) = 0$, then either $u > t_0$ almost everywhere on $X$, or else $u \leq t_0$ almost everywhere on $X$. In the former case we have $P(E_t, X) = 0$ for all $0 < t \leq t_0$, and in the latter case we have $P(E_t, X) = 0$ for all $t_0 \leq t < 1$. Thus the set of all $t \in (0, 1)$ for which $0 < P(E_t, X) < \infty$ is a full-measure subset of a subinterval of $[0, 1]$.

Remark 6.4. If $t_1 < t_2$ and $\int_{\{t_1 < u < t_2\}} g_u^Q d\mathcal{H}^Q > 0$, then for $\mathcal{L}(t_1, t_2)$, which consists of all $E_t \in \mathcal{L}$ as in the above proposition for $t_1 \leq t \leq t_2$, we have

$$0 < \frac{\mathcal{H}^1(\{t_1 < t < t_2 : E_t \in \mathcal{L}\})}{C} \leq \text{Mod}_{\mathcal{L}}(\mathcal{L}(t_1, t_2)) \frac{\mathcal{H}^Q}{\mathcal{H}^Q(t_1, t_2)} \left(\int_{\{t_1 < u < t_2\}} g_u^Q d\mathcal{H}^Q\right)^{\frac{1}{Q}}.$$

Remark 6.5. As above, we consider the family of sets $E_t$ with $E_t$ superlevel sets for a given compactly supported function $u \in N^{1,Q}(X)$. Then, with $f^{-1} = F : Y \to X$ quasiconformal, we must have $u \circ F \in N^{1,Q}(Y)$, and so for almost every $t \in \mathbb{R}$ we have both $E_t$ and $F(E_t)$ are of finite perimeter. Moreover, as noted above, $P(E_t, X) = 0$ if and only if $P(f(E_t), Y) = 0$. Furthermore, note that $f(E_t) = \{y \in Y : u \circ F(y) > t\}$, and so the discussion in Remark 6.2 shows that for almost every $t > 0$ we have $P(f(E_t), Y)$ is finite.

Now the proof of the inequality (6.2) tells us that the $Q/(Q - 1)$-modulus of the family of all $E_t$ for which $P(E_t, X) < \infty$ and $P(f(E_t), Y) < \infty$ is positive. Indeed, if $\mathcal{L}_0$ is the collection of all $E_t$ for which $0 < P(E_t, X) < \infty$ but $P(f(E_t), Y) = \infty$, then whenever $\rho_0$ is admissible for computing $\text{Mod}_{Q/(Q - 1)}(\mathcal{L} \setminus \mathcal{L}_0)$, we then have $1 \leq \int_{E_t, \rho} d\mu E_t$ for almost every $t \in [0, 1]$ with $E_t \in \mathcal{L}$; thus the computation that derives (6.2) also gives the validity of (6.2) with the role of $\rho$ played by $\rho_0$. That is, $\text{Mod}_{Q/(Q - 1)}(\mathcal{L} \setminus \mathcal{L}_0) > 0$. Therefore there are plenty of sets of positive and finite perimeter in $X$ whose image under $f$ is also of positive and finite perimeter; that is, with $\mathcal{L}_u$ the collection of all $E_t$ for which $0 < P(E_t, X) < \infty$ and $P(f(E_t), Y) < \infty$ (and hence $0 < P(f(E_t), Y) < \infty$) satisfies $\text{Mod}_{Q/(Q - 1)}(\mathcal{L}_u) > 0$ and $\text{Mod}_{Q/(Q - 1)}(f(\mathcal{L}_u)) > 0$ provided $\int_X g_u^Q d\mathcal{H}^Q > 0$.

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