ASYMPTOTICS OF A CUBIC SINE KERNEL DETERMINANT

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Abstract. The one-parameter family of Fredholm determinants det(I - γK_{csin}), γ ∈ R, is studied for an integrable Fredholm operator K_{csin} that acts on the interval (-s, s) and whose kernel is a cubic generalization of the sine kernel that appears in random matrix theory. This Fredholm determinant arises in the description of the Fermi distribution of semiclassical nonequilibrium Fermi states in condensed matter physics as well as in the random matrix theory. By using the Riemann–Hilbert method, the large s asymptotics of det(I - γK_{csin}) is calculated for all values of the real parameter γ.

§1. Introduction

Consider the real vector space \( \mathcal{M}(n) \simeq \mathbb{R}^{n^2} \) of \( n \times n \) Hermitian matrices \( M = (M_{ij}) = \bar{M}^t \) equipped with the probability distribution

\[
P^{(n,N)}(M) dM = ce^{-N \text{tr} V(M)} dM, \quad c \int_{\mathcal{M}(n)} e^{-N \text{tr} V(M)} dM = 1.
\]

Here \( dM \) denotes the Haar measure on \( \mathcal{M}(n) \), \( N \) is a fixed integer, and the potential \( V: \mathbb{R} \to \mathbb{R} \) is assumed to be real analytic and satisfying the growth condition

\[
\frac{V(x)}{\ln(x^2 + 1)} \to \infty \quad \text{as} \quad |x| \to \infty.
\]

The model \((\mathcal{M}(n), P^{(n,N)} dM)\) is commonly referred to as the unitary matrix model and it is well known (cf. [12, 32]) that basic statistical quantities such as

\[
E_{n,N}(s) = \text{Prob}(M \in \mathcal{M}(n) \text{ has no eigenvalues in the interval } (-s, s), s > 0)
\]

can be expressed as Fredholm determinants, indeed,

\[
E_{n,N}(s) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \int_{-s}^{s} \cdots \int_{-s}^{s} \det \left( K_{n,N}(x_k, x_l)_{k,l=1}^{j+1} \right) dx_1 \cdots dx_j \equiv \det(I - K_{n,N})
\]

where \( K_{n,N} \) is the finite rank operator with kernel

\[
K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)} e^{-\frac{N}{2}V(y)} \sum_{i=0}^{n-1} p_i(x)p_i(y), \quad \int_{\mathbb{R}} p_i(x)p_j(x)e^{-NV(x)} dx = \delta_{ij}
\]

acting on \( L^2((-s, s), d\lambda) \).

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The given assumption (1.2) enables one to study the scaling limits \( n, N \to \infty \) (compare \([12, 16]\)). First, the mean eigenvalue density \( \frac{1}{n} K_{n,N}(x,x) \) has a limit, that is, for all bounded continuous functions \( g: \mathbb{R} \to \mathbb{R} \),

\[
\lim_{n,N \to \infty} \frac{1}{n} \int g(x)K_{n,N}(x,x)\,dx = \int_{\Sigma_V} g(x)\rho_V(x)\,dx = \int_{\Sigma_V} g(x)\,d\mu_V(x)
\]

and the support \( \Sigma_V \) of the equilibrium measure \( \mu_V \) consists of a finite union of intervals. Here the density \( \rho_V(x) \) is determined by the potential \( V(x) \). At the same time, the local statistics of eigenvalues in the large \( n, N \) limit satisfies the so-called \textit{universality property}, i.e., it is determined only by the local characteristics of the eigenvalue density \( \rho_V \) (see \([37, 7, 11, 17]\)). For instance, choose a regular point \( x^* \in \Sigma_V \), i.e., \( \rho_V(x^*) > 0 \). Then the \textit{bulk universality} states that

\[
\lim_{n \to \infty} \frac{1}{np_V(x^*)} K_{n,n} \left( x^* + \frac{\lambda}{np_V(x^*)}, x^* + \frac{\mu}{np_V(x^*)} \right) = K_{\sin}(\lambda, \mu) = \frac{\sin \pi(\lambda - \mu)}{\pi(\lambda - \mu)}
\]

uniformly on compact subsets of \( \mathbb{R} \), or in other words for a regular point \( x^* \) we have

\[
\lim_{n \to \infty} \text{Prob} \left( M \in \mathcal{M}(n) \text{ has no eigenvalues in } (x^* - \frac{s}{np_V(x^*)}, x^* + \frac{s}{np_V(x^*)}) \right) = \det(I - K_{\sin}),
\]

where \( K_{\sin} \) is the trace class operator on \( L^2((-s, s); d\lambda) \) with the kernel \( K_{\sin}(\lambda, \mu) \) given in (1.5). The last equation shows that in double scaling limits the basic statistical properties of Hermitian random matrices are still expressible in terms of Fredholm determinants. On the other hand, the Fredholm determinant on the right-hand side of (1.6) admits the following asymptotic expansion \([19]\):

\[
\ln \det(I - K_{\sin}) = -\frac{(\pi s)^2}{2} - \frac{1}{4} \ln(\pi s) + \frac{1}{12} \ln 2 + 3\zeta'(-1) + O(s^{-1}), \quad s \to \infty,
\]

where \( \zeta(z) \) is the Riemann zeta-function. This formula yields one of the most important results in random matrix theory, i.e., an explicit evaluation of the \textit{large gap probability}, and its rigorous derivation was obtained in a series of papers \([14, 20, 21]\).

The focus of the current paper lies on the computation of similar expansions for a certain family of generalized sine kernels. Let \( K_{\sin} \) denote the trace class operator on \( L^2((-s, s); d\lambda) \) with kernel

\[
K_{\sin}(\lambda, \mu) = \frac{\sin \left( \frac{2}{\pi} (\lambda^3 - \mu^3) + x(\lambda - \mu) \right)}{\pi(\lambda - \mu)}, \quad x \in \mathbb{R},
\]

and \( \det(I - K_{\sin}) \) the corresponding Fredholm determinant. This Fredholm determinant appeared in \([9]\) in the asymptotic analysis of the Fredholm determinant

\[
\det(I - K_{\Pi})
\]

that corresponds to the first bulk critical universality class in the unitary matrix model. Assume that \( \rho_V(x) \) vanishes quadratically at an interior point \( x^* \in \Sigma_V \). Opposed to (1.5), the scaling limit is more complicated \([8, 11]\). Let \( \rho_V(x^*) = \rho'_V(x^*) = 0, \rho''_V(x^*) > 0 \), and let \( n, N \to \infty \) be such that the limit

\[
\lim_{n,N \to \infty} n^{2/3} \left( \frac{n}{N} - 1 \right) = C
\]

exists with \( C \in \mathbb{R} \). Then the \textit{critical bulk universality} guarantees the existence of positive constants \( c \) and \( c_1 \) such that

\[
\lim_{n,N \to \infty} \frac{1}{cn^{1/3}} K_{n,N} \left( x^* + \frac{\lambda}{cn^{1/3}}, x^* + \frac{\mu}{cn^{1/3}} \right) = K_{\Pi}(\lambda, \mu; x)
\]
uniformly on compact subsets of \( \mathbb{R} \), where the variable \( x \) is the scaling parameter defined by the relation
\[
\lim_{n,N \to \infty} n^{2/3} \left( \frac{n}{N} - 1 \right) = xc_1.
\]
Here the limiting kernel \( K_{\text{PII}}(\lambda, \mu; x) \) is constructed out of the \( \Psi \)-function associated with a special solution of the second Painlevé equation, the Hastings–McLeod solution \( u = u(x) \), see [9] for a precise definition of the Painlevé II kernel. The main result in [9] is the following analog of the Dyson formula (1.7) for the Painlevé II determinant. As \( s \to \infty \),
\[
\ln \det (I - K_{\text{PII}}) = -\frac{2}{3} s^6 - s^4 x - \frac{1}{2} (sx)^2 - \frac{3}{4} \ln s + \int_x^\infty (y-x) u^2(y) dy
\]
\[
- \frac{1}{6} \ln 2 + 3\zeta'(-1) + O(s^{-1}),
\]
(1.10)
and the error term in (1.10) is uniform on any compact subset of the set (1.11)
\[
\left\{ x \in \mathbb{R} : -\infty < x < \infty \right\}.
\]
The proof of the last expansion in [9] is based on a Riemann–Hilbert approach and an approximation argument, which allows one to derive the constant (with respect to \( s \)) term
\[
c_0 \equiv c_0(x) = -\ln F_{\text{TW}}(x) - \frac{1}{6} \ln 2 + 3\zeta'(-1),
\]
where \( F_{\text{TW}}(x) \) is the celebrated Tracy–Widom distribution function,
\[
F_{\text{TW}}(x) = e^{-\int_x^\infty (y-x) u^2(y) dy}.
\]
In more detail, the \( x \)-dependence of the constant term follows from the analysis of the relevant differential identities. Yet, these identities cannot produce the numerical part of the constant. This part follows from the fact that in the large positive \( x \)-limit we have
\[
K_{\text{PII}}(\lambda, \mu; x) = K_{\text{csin}}(\lambda, \mu; x) \left( 1 + O\left( x^{-1/4} e^{-\frac{3}{2} x^{3/2}} \right) \right), \quad x \to +\infty, \quad \lambda, \mu \in (-s, s).
\]
The last estimate, in conjunction with the Riemann–Hilbert analysis for the family \( K_{\text{csin}} \) with kernel
\[
\tilde{K}_{\text{csin}}(\lambda, \mu; s) = \frac{\sin \left( \frac{4}{3} t (\lambda^3 - \mu^3) + x(\lambda - \mu) \right)}{\pi (\lambda - \mu)}, \quad t \in [0, 1],
\]
leads to the constant term in (1.10). As a by-product of this analysis, [9] states, besides (1.10), the asymptotic relation
\[
\ln \det (I - K_{\text{csin}}) = -\frac{2}{3} s^6 - s^4 x - \frac{1}{2} (sx)^2 - \frac{3}{4} \ln s - \frac{1}{6} \ln 2 + 3\zeta'(-1) + O(s^{-1})
\]
(1.12)
valid as \( s \to \infty \) and the error term in (1.12) is uniform on any compact subset of the set (1.11).

In order to describe other spectral properties of large Hermitian matrices, one needs to study the Fredholm determinant
\[
\det (I - \gamma K_{\text{PII}})
\]
for the values of \( \gamma \) different from \( \gamma = 1 \). Such one-parameter families of determinants already appeared in connection with the sine-kernel determinant, for instance in the famous Montgomery–Odlyzko conjecture [35, 36] concerning the zeros of the Riemann zeta-function, in the description of the emptiness formation probability and other correlation functions in one-dimensional impenetrable Bose gas [24, 26, 27] as well as in a number of other important mathematical and theoretical physics applications.
The analytical challenge of the determinants (1.13) is once again the large \( s \) asymptotics. In the case of the sine-kernel determinants, the result is well known (see [30, 33, 34, 1, 6] and [14] for more on the history of the question).

1. As \( s \to \infty \),

\[
\ln \det (I - \gamma K_{\sin}) = -4\kappa \pi s + 2\kappa^2 \ln \pi s + \chi_{\sin} + O(s^{-1})
\]

uniformly on any compact subset of the set \( \{ \gamma \in \mathbb{R} : -\infty < \gamma < 1 \} \), where

\[
\kappa \equiv \kappa(\gamma) = -\frac{1}{2\pi} \ln(1 - \gamma).
\]

The constant \( \chi_{\sin} \equiv \chi_{\sin}(\gamma) \) is given by the equation

(1.14) \[
\chi_{\sin} = 2\kappa^2 + 4\kappa^2 \ln 2 - 4 \int_0^{\gamma} \kappa(t) \frac{d}{dt} \arg \Gamma(i\kappa(t)) dt,
\]

which was obtained by A. Budylin and V. Buslaev in [6] as a corollary to their main result in [6] — the asymptotics of the resolvent of the kernel \( \gamma K_{\sin}(\lambda, \mu) \).

Formula (1.14) also follows from the general theorem of E. Basor and H. Widom concerning the determinants of Toeplitz integral operators with piecewise continuous symbols [2].

2. For \( \gamma \) chosen in any compact subset of the set \( \{ \gamma \in \mathbb{R} : 1 < \gamma < \infty \} \), the Fredholm determinant \( \det (I - \gamma K_{\sin}) \) has infinitely many zeros \( \{ s_n \} \) that accumulate at infinity (see [40, 33, 34]).

The main results of the present paper are the following analogs for the cubic sine-kernel (1.8), which together with (1.12) state the large \( s \) behavior of \( \det (I - \gamma K_{\csin}) \) for all values of the parameter \( \gamma \).

**Theorem 1.** Let \( K_{\csin} \) denote the trace class operator on \( L^2((-s, s); d\lambda) \) with kernel as in (1.8). As \( s \to \infty \),

(1.15) \[
\ln \det (I - \gamma K_{\csin}) = -\kappa \left( \frac{16}{3} s^3 + 4xs \right) + 6\kappa^2 \ln s - \int_x^{\infty} (y-x)u^2(y) dy + \chi + O(s^{-1})
\]

uniformly on any compact subset of the set

(1.16) \[
\{(\gamma, x) \in \mathbb{R}^2 : -\infty < \gamma < 1, -\infty < x < \infty \},
\]

where

\[
\kappa \equiv -i\nu(\gamma) = -\frac{1}{2\pi} \ln(1 - \gamma),
\]

\[
\chi = 2\kappa^2 + 8\kappa^2 \ln 2 - 4 \int_0^{\gamma} \kappa(t) \frac{d}{dt} \arg \Gamma(i\kappa(t)) dt,
\]

with the Euler gamma-function \( \Gamma(z) \), and with \( u = u(x, \gamma) \) denoting the real-valued Ablowitz–Segur solution of the second Painlevé equation \( u_{xx} = xu + 2u^3 \) corresponding to the monodromy surface

\[
M = \{(s_1, \ldots, s_6) | s_1 = -i\gamma, s_2 = 0, s_3 = \bar{s}_1, s_{n+3} = -s_n \}.
\]

The Ablowitz–Segur solution \( u = u(x, \gamma) \) [38] used in the statement of Theorem 1 is given as a unique solution of the boundary-value problem

(1.17) \[
u_{xx} = xu + 2u^3, \quad u(x) \sim \gamma \text{Ai}(x), \quad x \to +\infty, \quad \gamma \neq 1.
\]

Such solutions are smooth if \( \gamma < 1 \), with exponentially fast decay as \( x \to +\infty \) and oscillatory behavior as \( x \to -\infty \), see, e.g., [22]. On the other hand, for \( \gamma > 1 \), the
solution of (1.17) has poles on the real axis, but is still pole-free for sufficiently large positive $x$, in fact (see [4]), for $(\gamma,x)$ chosen from any compact subset of the set

$$
\{(\gamma,x) \in \mathbb{R}^2 : 1 < \gamma < \infty, \ x > \left( \frac{3}{2} \ln \gamma \right)^{2/3} \}
$$

the solution $u = u(x,\gamma)$ to (1.17) is pole-free. In its turn, this leads to our second asymptotic result.

**Theorem 2.** For $(\gamma,x)$ chosen from any compact subset of the set (1.18), the Fredholm determinant $\det(I - \gamma K_{csin})$ has infinitely many zeros $\{s_n\}$ with asymptotic distribution

$$
\frac{8}{3}s_n^3 + 2x s_n + \frac{1}{\pi} \ln(\gamma - 1) \ln(16s_n^3 + 4xs_n) - \arg \frac{\Gamma(1 - i\kappa)}{\Gamma(i\kappa)} \sim \frac{\pi}{2} + n\pi, \quad n \to \infty.
$$

The proofs of Theorem 1 and 2 are based on the Riemann–Hilbert approach. This approach (compare [25, 15]) involves the integrable form of the Fredholm operator $K_{csin}$, allowing us to relate the resolvent kernel to the solution of a Riemann–Hilbert problem. The latter can be analyzed rigorously via the Deift–Zhou nonlinear steepest descent method.

It should be mentioned that a large class of generalized sine-kernel determinants was already considered in [30] (see equation (1.6) there). In the case where $\gamma < 1$ and after a proper rescaling, the determinant $\det(I - \gamma K_{csin})$ can be put in a form very close to that treated in [30]. However, an essential difference occurs: the fast phase function, the function $p(\lambda)$ in the notation of [30] (see (1.7)), which arises as a result of the rescaling, does not satisfy one of the key conditions of [30]; moreover, it becomes dependent on the large parameter. This means that the results of [30] are not directly applicable to our case. In fact, if one formally applies the main asymptotic formula of [30] to our case, then the first two terms of our asymptotic equation (1.15) are reproduced while the constant (in $s$) term is not. Most significantly, the integral term with the Painlevé function does not show up. The result of our Theorem 2, i.e., the asymptotic distribution (1.19) of the zeros of the cubic sine-kernel determinant cannot be compared with [30] because the techniques in [30] do not cover the situation where $\gamma \geq 1$.

We also want to point out that the analysis of the cubic sine-kernel determinant is also of interest without the random matrix theory background: the determinant $\det(I - K_{csin})$ appears in condensed matter physics [5], namely in the description of the Fermi distribution of semiclassical nonequilibrium Fermi states. In order to understand perturbations to a degenerate Fermi gas, one studies the one-parameter extension of determinants corresponding to the kernel (1.8), that is,

$$
\det(I - \gamma K_{csin}), \quad \gamma \in \mathbb{R}.
$$

Our formula (1.15) provides therefore an exact expression for the large perturbations.

Finally, we would like to emphasize the involvement of the general Ablowitz–Segur solution of the second Painlevé equation in our asymptotic results rather than merely of the usual Hastings–McLeod solution. This fact, in particular, poses the following intriguing question that Theorem 2 does not answer: What is the large $s$ asymptotic behavior of $\det(I - \gamma K_{csin})$ when $\gamma > 1$ and $x$ coincides with one of the poles of the corresponding Ablowitz–Segur second Painlevé transcendent?

We finish this Introduction with a brief outline for the rest of the paper. §2 starts with a short review of the Riemann–Hilbert approach to the asymptotics of integrable Fredholm operators. Then we apply the general framework to the Fredholm determinant $\det(I - \gamma K_{csin})$ and formulate the associated “master” Riemann–Hilbert problem (RHP). We also present logarithmic $s,x$, and $\gamma$ derivatives of the determinant $\det(I - \gamma K_{csin})$ and outline a derivation of an integrable system whose tau-function is represented by
det(I − γK_{\text{csin}}). In §15 following the Deift–Zhou roadmap, we construct an asymptotic solution of the master Riemann–Hilbert problem. At this point we will have to distinguish between γ < 1 and γ > 1, in fact for γ > 1 we are led to a solitonic Riemann–Hilbert problem. Using an additional “undressing” step, we overcome the singularities in the Riemann–Hilbert problem and derive the large zero distribution as stated in (1.19). The situation is similar to that dealt with in [10]. The calculations of §16 and §17 provide us with the asymptotics of ln det(I − γK_{\text{csin}}) including the constant term for γ < 1. The large zero distribution (1.19) will be derived in §18.

The authors dedicate this paper to the memory of Vladimir Savelievich Buslaev, whose pioneering works on the asymptotic analysis of integrable systems lie in the foundation of their modern asymptotic theory.

§2. RIEMANN–HILBERT APPROACH — SETUP AND REVIEW

The given integral kernel (1.8) belongs to an algebra of integrable operators first introduced in [25]. Let Γ be an oriented contour in the complex plane C such as a Jordan curve. We are interested in operators of the form λI + K on $L^2(\Gamma)$, where $K$ denotes an integral operator with kernel

$$K(\lambda, \mu) = \sum_{i=1}^{M} f_i(\lambda) h_i(\mu) \frac{1}{\lambda - \mu}, \quad \sum_{i=1}^{M} f_i(\lambda) h_i(\lambda) = 0, \quad M \in \mathbb{Z}_{\geq 1},$$

with functions $f_i, h_i$ that are smooth up to the boundary Γ. Given two operators $\lambda I + K, \lambda I + \tilde{K}$ of this type, the composition $(\lambda I + K)(\lambda I + \tilde{K})$ is again of the same form, hence we have a ring. Moreover, let $K^t$ denote the real adjoint of $K$, i.e.,

$$K^t(\lambda, \mu) = -\sum_{i=1}^{M} h_i(\lambda) f_i(\mu) \frac{1}{\lambda - \mu}.$$  

Our results are based on the following observations (see e.g. [25]). First, we state an algebraic lemma, showing that the resolvent of $I - K$ is again integrable.

**Lemma 1.** Consider an operator $I - K$ on $L^2(\Gamma)$ in the previous algebra with kernel (2.1). Suppose the inverse $(I - K)^{-1}$ exists; then $I + R = (I - K)^{-1}$ lies again in the same algebra with

$$R(\lambda, \mu) = \sum_{i=1}^{M} F_i(\lambda) H_i(\mu) \frac{1}{\lambda - \mu}, \quad \sum_{i=1}^{M} F_i(\lambda) H_i(\lambda) = 0,$$

and the functions $F_i, H_i$ are given by

$$F_i(\lambda) = ((I - K)^{-1} f_i) (\lambda), \quad H_i(\lambda) = ((I - K^t)^{-1} h_i) (\lambda).$$

Second, we present an analytic lemma, which relates integrable operators to a Riemann–Hilbert problem.

**Lemma 2.** Let $K$ be of integrable type such that $(I - K)^{-1}$ exists, and let $Y = Y(z)$ denote the unique solution of the following $M \times M$ Riemann–Hilbert problem (RHP):

- $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma$;
- on the contour $\Gamma$, the boundary values of the function $Y(z)$ satisfy the jump relation
  $$Y_+_z = Y_-(z)(I - 2\pi i f(z)h^t(z)), \quad z \in \Gamma,$$
  where $f(z) = (f_1(z), \ldots, f_M(z))^t$, and similarly
  $$h(z) = (h_1(z), \ldots, h_M(z))^t;$$
at an endpoint of the contour $\Gamma$, the function $Y(z)$ has at most a logarithmic singularity;

- as $z \to \infty$,

$$Y(z) = I + O(z^{-1}).$$

Then $Y(z)$ determines the resolvent kernel via

$$F(z) = Y(z)f(z), \quad H(z) = (Y'(z))^{-1}h(z),$$

and conversely, the solution of the above RHP is expressible in terms of the function $F(z)$ by using the Cauchy integral

$$Y(z) = I - \int_{\Gamma} F(w)h^I(w) \frac{dw}{w - z}.$$  

We use this general setup in the given situation (1.8). We have

$$\gamma K_{\text{csin}}(\lambda, \mu) = \frac{f'(\lambda)h(\mu)}{\lambda - \mu}, \quad f(\lambda) = \sqrt{\frac{\pi}{2i}} \left( e^{i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} - e^{-i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} \right), \quad h(\lambda) = \frac{\pi i}{2} \left( e^{-i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} - e^{i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} \right),$$

where $\sqrt{z}$ is defined on $\mathbb{C} \setminus (-\infty, 0]$ with its branch fixed by the condition $\sqrt{z} > 0$ if $z > 0$. Lemma 2 provides us with the following $Y$-RHP:

- $Y(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [-s, s]$;
- along the line segment $[-s, s]$, oriented from left to right, we have the following jump:

$$Y_+(\lambda) = Y_-(\lambda) \begin{pmatrix} 1 - \gamma & \gamma e^{2i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} \\ -\gamma e^{-2i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} & 1 + \gamma \end{pmatrix}, \quad \lambda \in [-s, s];$$

- at the endpoints $\pm s$, the function $Y(\lambda)$ has logarithmic singularities, i.e.,

$$Y(\lambda) = O(\ln(\lambda \mp s)), \quad \lambda \to \pm s;$$

- we have $Y(\lambda) \to I$ as $\lambda \to \infty$.

We can factorize the jump matrix

$$\begin{pmatrix} 1 - \gamma & \gamma e^{2i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} \\ -\gamma e^{-2i\left(\frac{3}{4}\lambda^3 + x\lambda\right)} & 1 + \gamma \end{pmatrix} = e^{i\left(\frac{3}{4}\lambda^3 + x\lambda\right)\sigma_3} \begin{pmatrix} 1 - \gamma & \gamma \\ -\gamma & 1 + \gamma \end{pmatrix} e^{-i\left(\frac{3}{4}\lambda^3 + x\lambda\right)\sigma_3},$$

and employ the first transformation.

§3. First Transformation of the RHP

We make the following substitution in the original $Y$-RHP:

$$X(\lambda) = Y(\lambda)e^{i\left(\frac{3}{4}\lambda^3 + x\lambda\right)\sigma_3}, \quad \lambda \in \mathbb{C} \setminus [-s, s].$$

This leads to a RHP for the function $X(\lambda)$, our “master” RHP:

- $X(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [-s, s]$;
- the following jump holds:

$$X_+(\lambda) = X_-(\lambda) \begin{pmatrix} 1 - \gamma & \gamma \\ -\gamma & 1 + \gamma \end{pmatrix}, \quad \lambda \in [-s, s];$$

- from (2.6), we deduce the following refined endpoint behavior:

$$X(\lambda) = \tilde{X}(\lambda) \left[ I + \frac{\gamma}{2\pi i} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \ln \left( \frac{\lambda - s}{\lambda + s} \right) \right],$$

where $\tilde{X}(\lambda)$ is analytic at $\lambda = \pm s$ and the branch of the logarithm is fixed by the condition $-\pi < \arg \frac{\lambda - s}{\lambda + s} < \pi;
\[ X(\lambda) = (I + O(\lambda^{-1})) e^{i(\lambda^3 + x\lambda)} \gamma, \quad \lambda \to \infty. \tag{3.4} \]

The last master problem will be solved asymptotically as \( s \to \infty \) in the next sections by approximating its global solution with local model functions. Before we start this analysis in detail, first we relate the solution of the \( X \)-RHP to the Fredholm determinants \( \det(I - \gamma K_{\text{csin}}) \).

### §4. LOGARITHMIC DERIVATIVES — RELATIONSHIP WITH \( X \)-RHP

We shall express logarithmic derivatives of the determinant \( \det(I - \gamma K_{\text{csin}}) \) in terms of the solution of the \( X \)-RHP. To this end, recall the following classical identity, valid for any differentiable family of trace class operators \( [39] \):

\[ \frac{\partial}{\partial s} \ln \det(I - \gamma K_{\text{csin}}) = - \text{tr} \left( (I - \gamma K_{\text{csin}})^{-1} \frac{\partial}{\partial s} (\gamma K_{\text{csin}}) \right). \tag{4.1} \]

In our situation,

\[ \frac{\partial K_{\text{csin}}}{\partial s}(\lambda, \mu) = K_{\text{csin}}(\lambda, \mu)(\delta(\mu - s) + \delta(\mu + s)), \]

where, by definition,

\[ \int_{-s}^{s} \delta(w \mp s) f(w) \, dw = f(\pm s), \]

and therefore

\[ - \text{tr} \left( (I - \gamma K_{\text{csin}})^{-1} \frac{\partial}{\partial s} (\gamma K_{\text{csin}}) \right) = -R(s, s) - R(-s, -s) \]

with \( R(\lambda, \mu) \) denoting the kernel (see (2.2)) of the resolvent

\[ R = (I - \gamma K_{\text{csin}})^{-1} \gamma K_{\text{csin}}. \]

The latter derivative can be simplified by using the given definitions

\[ f_1(\lambda) = -h_2(\lambda), \quad f_2(\lambda) = h_1(\lambda) \]

as well as the identity \( \det Y(\lambda) \equiv 1 \), a direct consequence of the unimodularity of the jump matrix \( G(\lambda) \) and Liouville’s theorem:

\[ R(\lambda, \mu) = \frac{F_1(\lambda) H_1(\mu) + F_2(\lambda) H_2(\mu)}{\lambda - \mu} = \frac{F_1(\lambda) F_2(\mu) - F_2(\lambda) F_1(\mu)}{\lambda - \mu}. \]

Since \( R(\lambda, \mu) \) is continuous along the diagonal \( \lambda = \mu \) (see (2.2)), we obtain

\[ R(s, s) = F_1'(s) F_2(s) - F_2'(s) F_1(s), \]
\[ R(-s, -s) = F_1'(-s) F_2(-s) - F_2'(-s) F_1(-s), \tag{4.2} \]

provided \( F_i \) is analytic at \( \lambda = \pm s \). One way to see this is as follows. Use (3.1) and (2.4)

\[ F(\lambda) = X(\lambda)e^{-i(\frac{\gamma}{2} \lambda^3 + x\lambda)} \gamma f(\lambda) \]

to derive from (3.3) the local identity

\[ F(\lambda) = \tilde{X}(\lambda) \left[ I + \frac{\gamma}{2\pi i} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \ln \left( \frac{\lambda - s}{\lambda + s} \right) \right] e^{-i(\frac{\gamma}{2} \lambda^3 + x\lambda)} \gamma f(\lambda) = \tilde{X}(\lambda) \sqrt{\frac{\gamma}{2\pi i}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

valid in a neighborhood of \( \lambda = \pm s \). But this proves the analyticity of \( F(\lambda) \) at the endpoints and as we shall see later on, (4.3) will allow us to relate (4.1) via (4.2) to the solution of the \( X \)-RHP. We summarize.
Proposition 1. The logarithmic $s$-derivative of the given Fredholm determinant can be expressed as

\begin{equation}
\frac{\partial}{\partial s} \ln \det(I - \gamma K_{\text{csin}}) = -R(s, s) - R(-s, -s),
\end{equation}

where the relationship with the $X$-RHP is established through

\[ F(\lambda) = \bar{X}(\lambda) \sqrt{\frac{\gamma}{2\pi i}} (\frac{1}{1}) \quad \lambda \to \pm s. \]

Besides the logarithmic $s$-derivative, we also differentiate with respect to $x$:

\[ \frac{\partial}{\partial x} \ln \det(I - \gamma K_{\text{csin}}) = -\text{trace} \left( (I - \gamma K_{\text{csin}})^{-1} \frac{\partial}{\partial x} (\gamma K_{\text{csin}}) \right). \]

In our situation the kernel depends explicitly on $x$, indeed,

\[ \frac{\partial}{\partial x} (\gamma K_{\text{csin}}(\lambda, \mu)) = i(f_1(\lambda)h_1(\mu) - f_2(\lambda)h_2(\mu)) \]

and with (2.3)

\[ -\text{trace} \left( (I - \gamma K_{\text{csin}})^{-1} \frac{\partial}{\partial x} K_{\text{csin}} \right) = -\text{trace} \left( \frac{\partial}{\partial x} (f_1(\lambda)h_1(\lambda) - f_2(\lambda)h_2(\lambda)) \right) d\lambda. \]

On the other hand, the Cauchy integral (2.4) implies

\[ Y(\lambda) = I + \frac{m_1}{\lambda} + O(\lambda^{-2}), \quad \lambda \to \infty; \quad m_1 = \int_{-s}^{s} F(w)h^i(w)dw, \]

so

\[ \frac{\partial}{\partial x} \ln \det(I - \gamma K_{\text{csin}}) = i(m_{21} - m_1^{11}), \quad m_1 = (m_1^{ij}), \]

and the relationship with the $X$-RHP is established via (3.4). Again we summarize.

Proposition 2. The logarithmic $x$-derivative of the given Fredholm determinant can be expressed as

\begin{equation}
\frac{\partial}{\partial x} \ln \det(I - \gamma K_{\text{csin}}) = i\left( X_{12}^{\gamma} - X_{11}^{\gamma} \right)
\end{equation}

with

\[ X(\lambda) \sim \left( I + \frac{X_1}{\lambda} + \frac{X_2}{\lambda^2} + \frac{X_3}{\lambda^3} + O(\lambda^{-4}) \right) e^{i(\frac{3}{4}\lambda^3 + x\lambda)\sigma_3}, \quad \lambda \to \infty; \quad X_1 = (X_1^{ij}). \]

The logarithmic $s$ and $x$ derivatives are sufficient to determine the large $s$ asymptotics of $\det(I - \gamma K_{\text{csin}})$ up to the constant term. In order to evaluate the constant explicitly for $\gamma < 1$, we use the following approach (see [13] for a similar method used in the asymptotics of Toeplitz determinants).

Start with the logarithmic $\gamma$-derivative

\[ \frac{\partial}{\partial \gamma} \ln \det(I - \gamma K_{\text{csin}}) = -\text{trace} \left( (I - \gamma K_{\text{csin}})^{-1} K_{\text{csin}} \right) = -\frac{1}{\gamma} \int_{-s}^{s} R(\lambda, \lambda) d\lambda. \]

Our goal is to express the last integral over the resolvent kernel in terms of the solution of the underlying $X$-RHP. Recall (1.2), (2.4), the definition of the functions $f(\lambda), h(\lambda)$, and the unimodularity of $Y(\lambda)$:

\[ R(\lambda, \lambda) = F_1^1(\lambda)F_2(\lambda) - F_2^1(\lambda)F_1(\lambda) \]

\[ = \frac{\gamma}{\pi} \left( 4\lambda^2 + x \right) + \frac{\gamma}{2\pi i} \left( Y_{11}'(\lambda)Y_{22}(\lambda) - Y_{11}(\lambda)Y_{22}'(\lambda) + Y_{12}'(\lambda)Y_{21}(\lambda) - Y_{21}(\lambda)Y_{12}'(\lambda) \right) \]

\[ + (Y_{11}'(\lambda)Y_{21}(\lambda) - Y_{11}(\lambda)Y_{21}'(\lambda))f_1^2(\lambda) + (Y_{12}'(\lambda)Y_{22}(\lambda) - Y_{12}(\lambda)Y_{22}'(\lambda))f_2^2(\lambda), \]
where \( (') \) indicates differentiation with respect to \( \lambda \). Next, \((3.1)\) and unimodularity allow us to rewrite the previous expression for \( R(\lambda, \lambda) \) solely in terms of the X-RHP:

\[
R(\lambda, \lambda) = \frac{\gamma}{2\pi i} \left[ (X_{11}'(\lambda) + X'_{12}(\lambda))(X_{21}(\lambda) + X_{22}(\lambda)) - (X_{11}(\lambda) + X_{12}(\lambda))(X'_{21}(\lambda) + X'_{22}(\lambda)) \right].
\]

Our next move will replace all terms involving derivatives with respect to \( \lambda \) and this can be done by considering the differential equations associated with the X-RHP, see also \((4.5)\) below.

All jump matrices in the X-RHP are unimodular and constant with respect to \( \lambda \), thus the well-defined logarithmic derivatives \( X_{\lambda}X^{-1}(\lambda) \) are rational functions. Indeed, by \((3.4)\) and \((3.3)\),

\[
\frac{\partial X}{\partial \lambda} = \left[ 4i\lambda^2 \sigma_3 - 4i\lambda \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} + \begin{pmatrix} d & e \\ f & -d \end{pmatrix} + \frac{A}{\lambda - s} - \frac{B}{\lambda + s} \right] X \equiv A(\lambda, s, x)X,
\]

where

\[
A = \frac{\gamma}{2\pi i} \tilde{X}(s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(s))^{-1}; \quad B = \frac{\gamma}{2\pi i} \tilde{X}(-s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(-s))^{-1}
\]

and with parameters \( b, c, d, e, f \) that can be expressed in terms of the entries of \( X_1 \) and \( X_2 \),

\[
\begin{align*}
&b = 2X_1^{12}, \quad c = 2X_1^{21}, \quad d = ix + 8iX_1^{12}X_2^{21}, \\
e = 8i(X_1^{12}X_2^{22} - X_2^{12}), \quad f = -8i(X_1^{21}X_2^{11} - X_2^{21}).
\end{align*}
\]

Substituting \((4.7)\) in \((4.6)\) and recalling \((4.8)\) we obtain, with \( A = (A_{ij}), B = (B_{ij}) \):

\[
R(\lambda, \lambda) = \frac{\gamma}{2\pi i} \left[ 8i\lambda^2 + 2d + \frac{A_{11} - A_{22}}{\lambda - s} - \frac{B_{11} - B_{22}}{\lambda + s} \\
\times (X_{11}(\lambda) + X_{12}(\lambda))(X_{21}(\lambda) + X_{22}(\lambda)) \\
+ \left( -4i\lambda b + e + \frac{A_{12}}{\lambda - s} - \frac{B_{12}}{\lambda + s} \right)(X_{21}(\lambda) + X_{22}(\lambda))^2 \\
+ \left( -4i\lambda c - f + \frac{A_{21}}{\lambda - s} + \frac{B_{21}}{\lambda + s} \right)(X_{11}(\lambda) + X_{12}(\lambda))^2 \right].
\]

Next we \( \gamma \)-differentiate the X-RHP in \((3.2)\) to obtain the following additive RHP for the function \( H(\lambda) = \frac{\partial X}{\partial \gamma}(\lambda)(X(\lambda))^{-1} \):

- \( H(\lambda) \) is analytic for \( \lambda \in \mathbb{C} \setminus [-s, s] \);
- along the line segment \([-s, s]\), oriented from left to right, we have
  \[
  H_+(\lambda) = H_-(\lambda) + X_-(\lambda) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (X_-(\lambda))^{-1}, \quad \lambda \in [-s, s];
  \]
- \( H(\lambda) \) has at most logarithmic singularities at the endpoints \( \lambda = \pm s \),

\[
H(\lambda) = \frac{\partial \tilde{X}}{\partial \gamma}(\lambda)(\tilde{X}(\lambda))^{-1} + \tilde{X}(\lambda) \frac{1}{2\pi i} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(\lambda))^{-1} \ln \frac{\lambda - s}{\lambda + s}, \quad \lambda \rightarrow \pm s;
\]
- as \( \lambda \to \infty \), we have \( H(\lambda) \to 0 \):

If we let

\[
K(\lambda) = X(\lambda) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (X(\lambda))^{-1}, \quad \lambda \in \mathbb{C} \setminus [-s, s],
\]
then $K_+(\lambda) = K_-(\lambda)$, $\lambda \in [-s, s]$ and $K(\lambda)$ is bounded as $\lambda \to \pm s$. Hence, $K(\lambda)$ is entire and we have a solution of the H-RHP:

$$H(\lambda) = \frac{1}{2\pi i} \int_{-s}^{s} \frac{K_-(w)}{w - \lambda} \, dw = \frac{1}{2\pi i} \int_{-s}^{s} \frac{K(w)}{w - \lambda} \, dw$$

This solution enables us to rewrite $\int_{-s}^{s} R(\lambda, \lambda) \, d\lambda$ in (4.11), for instance,

$$\int_{-s}^{s} \lambda^n (X_{11}(\lambda) + X_{12}(\lambda)) (X_{21}(\lambda) + X_{22}(\lambda)) \, d\lambda = \int_{\Sigma} w^n H_{11}(w) \, dw, \quad n \in \mathbb{Z}_{\geq 0},$$

with $\Sigma$ denoting a closed Jordan curve around the interval $[-s, s]$ and where we have used the formula

$$\lambda^n = \frac{1}{2\pi i} \int_{\Sigma} \frac{w^n}{w - \lambda} \, dw, \quad \lambda \in [-s, s].$$

Similarly,

$$\int_{-s}^{s} \lambda^n (X_{21}(\lambda) + X_{22}(\lambda))^2 \, d\lambda = \int_{\Sigma} w^n H_{21}(w) \, dw,$$

$$\int_{-s}^{s} \lambda^n (X_{11}(\lambda) + X_{12}(\lambda))^2 \, d\lambda = -\int_{\Sigma} w^n H_{12}(w) \, dw$$

and we obtain

$$\frac{\partial}{\partial \gamma} \ln \det(I - \gamma K_{c\sin}) = -\frac{1}{\gamma} \int_{-s}^{s} R(\lambda, \lambda) \, d\lambda$$

$$= -\frac{1}{2\pi i} \left[ 8i \int_{\Sigma} w^2 H_{11}(w) \, dw + \int_{\Sigma} (2dH_{11}(w) + eH_{21}(w) + fH_{12}(w)) \, dw \right.$$

$$\left. - 4i \int_{\Sigma} w(bH_{21}(w) - cH_{12}(w)) \, dw \right.$$

$$\left. - \int_{-s}^{s} \left( (A_{11} - A_{22}) K_{11}(\lambda) + A_{12} K_{21}(\lambda) + A_{21} K_{12}(\lambda) \right) \frac{d\lambda}{\lambda - s} \right.$$

$$\left. + \int_{-s}^{s} \left( (B_{11} - B_{22}) K_{11}(\lambda) + B_{12} K_{21}(\lambda) + B_{21} K_{12}(\lambda) \right) \frac{d\lambda}{\lambda + s} \right],$$

Since

$$K(\lambda) = \frac{2\pi i}{\gamma} A + O(\lambda - s), \quad \lambda \to s, \quad K(\lambda) = \frac{2\pi i}{\gamma} B + O(\lambda + s), \quad \lambda \to -s,$$

and

$$(A_{11} - A_{22}) A_{11} + 2A_{12} A_{21} = 0 = (B_{11} - B_{22}) B_{11} + 2B_{12} B_{21},$$

we deduce that the last two integrals in (4.13) are indeed well defined. To evaluate them, let

$$\tilde{H}(\lambda) = H(\lambda) - \tilde{X}(s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(s))^{-1} \frac{1}{2\pi i} \ln \frac{\lambda - s}{\lambda + s}, \quad \lambda \in \mathbb{C} \setminus [-s, s],$$

$$\tilde{H}(\lambda) = H(\lambda) - \tilde{X}(-s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(-s))^{-1} \frac{1}{2\pi i} \ln \frac{\lambda - s}{\lambda + s}, \quad \lambda \in \mathbb{C} \setminus [-s, s].$$

From (4.12) we see that $\tilde{H}(\lambda)$ is bounded as $\lambda \to s$ and $\tilde{H}(\lambda)$ is bounded as $\lambda \to -s$, more precisely,

$$\tilde{H}(s) = \frac{\partial \tilde{X}(s)}{\partial \gamma}(\tilde{X}(s))^{-1}, \quad \tilde{H}(-s) = \frac{\partial \tilde{X}}{\partial \gamma}(-s)(\tilde{X}(-s))^{-1},$$

and

$$\tilde{H}(\lambda) = H(\lambda) - \tilde{X}(s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(s))^{-1} \frac{1}{2\pi i} \ln \frac{\lambda - s}{\lambda + s}, \quad \lambda \in \mathbb{C} \setminus [-s, s],$$

$$\tilde{H}(\lambda) = H(\lambda) - \tilde{X}(-s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(-s))^{-1} \frac{1}{2\pi i} \ln \frac{\lambda - s}{\lambda + s}, \quad \lambda \in \mathbb{C} \setminus [-s, s].$$
and also

$$\tilde{H}_+(\lambda) = \tilde{H}_-(\lambda) + K(\lambda) - K(s), \quad \tilde{H}_+(\lambda) = \tilde{H}_-(\lambda) + K(\lambda) - K(-s), \quad \lambda \in [-s, s].$$

Hence,

$$(4.14) \quad \hat{H}(\lambda) = \frac{1}{2\pi i} \int_{-s}^{s} \frac{K(w) - K(s)}{w - \lambda} \, dw, \quad \tilde{H}(\lambda) = \frac{1}{2\pi i} \int_{-s}^{s} \frac{K(w) - K(-s)}{w - \lambda} \, dw$$

and we conclude that

$$\int_{-s}^{s} \left( (A_{11} - A_{22}) K_{11}(\lambda) + A_{12} K_{21}(\lambda) + A_{21} K_{12}(\lambda) \right) \frac{d\lambda}{\lambda - s}$$

$$= (A_{11} - A_{22}) \int_{-s}^{s} \frac{K_{11}(\lambda) - K_{11}(s)}{\lambda - s} \, d\lambda + A_{12} \int_{-s}^{s} \frac{K_{21}(\lambda) - K_{21}(s)}{\lambda - s} \, d\lambda$$

$$+ A_{21} \int_{-s}^{s} \frac{K_{12}(\lambda) - K_{12}(s)}{\lambda - s} \, d\lambda$$

$$= 2\pi i \left[ (A_{11} - A_{22}) \tilde{H}_{11}(s) + A_{12} \tilde{H}_{21}(s) + A_{21} \tilde{H}_{12}(s) \right].$$

Similarly,

$$\int_{-s}^{s} \left( (B_{11} - B_{22}) K_{11}(\lambda) + B_{12} K_{21}(\lambda) + B_{21} K_{12}(\lambda) \right) \frac{d\lambda}{\lambda + s}$$

$$= 2\pi i \left[ (B_{11} - B_{22}) \tilde{H}_{11}(-s) + B_{12} \tilde{H}_{21}(-s) + B_{21} \tilde{H}_{12}(-s) \right].$$

To evaluate the remaining integrals in \( (4.13) \), we recall that

$$H(\lambda) = \frac{1}{\lambda} (X_1)_{\gamma} + \frac{1}{\lambda^2} ((X_2)_{\gamma} - (X_1)_{\gamma} X_1)$$

$$+ \frac{1}{\lambda^3} ((X_3)_{\gamma} + (X_1)_{\gamma} (X_1^2 - X_2) - (X_2)_{\gamma} X_1) + O(\lambda^{-4}), \quad \lambda \to \infty,$$

and apply the residue theorem:

$$\int_{\Sigma} H(w) \, dw = 2\pi i (X_1)_{\gamma}, \quad \int_{\Sigma} wH(w) \, dw = 2\pi i ((X_2)_{\gamma} - (X_1)_{\gamma} X_1),$$

$$\int_{\Sigma} w^2 H(w) \, dw = 2\pi i ((X_3)_{\gamma} + (X_1)_{\gamma} (X_1^2 - X_2) - (X_2)_{\gamma} X_1).$$

We summarize the previous computations.

**Proposition 3.** The logarithmic \( \gamma \)-derivative of the given Fredholm determinant can be expressed as

$$\frac{\partial}{\partial \gamma} \ln \det(I - \gamma K_{\text{csin}})$$

$$(4.15) \quad = -8i((X_3)_{\gamma} + (X_1)_{\gamma} (X_1^2 - X_2) - (X_2)_{\gamma} X_1)^{11} - 2d((X_1)_{\gamma})^{11}$$

$$+ 4ib((X_2)_{\gamma} - (X_1)_{\gamma} X_1)^{21} - 4ic((X_2)_{\gamma} - (X_1)_{\gamma} X_1)^{12} - e((X_1)_{\gamma})^{21} - f((X_1)_{\gamma})^{12}$$

$$+ ((A_{11} - A_{22}) \tilde{H}_{11}(s) + A_{12} \tilde{H}_{21}(s) + A_{21} \tilde{H}_{12}(s))$$

$$- ((B_{11} - B_{22}) \tilde{H}_{11}(-s) + B_{12} \tilde{H}_{21}(-s) + B_{21} \tilde{H}_{12}(-s))$$

with

$$X(\lambda) \sim \left( I + \frac{X_1}{\lambda} + \frac{X_2}{\lambda^2} + \frac{X_3}{\lambda^3} + O(\lambda^{-4}) \right) e^{i(\frac{\lambda}{4} + x\lambda) \sigma_3}, \quad \lambda \to \infty,$$

and with the functions \( b, c, d, e, f, A, B \) defined in \( (4.9), \ (4.10), \) and \( (4.8) \).
§5. Differential equations associated with the determinant $\det(I - \gamma K_{\sin})$

Our considerations rely only on the underlying Riemann–Hilbert problem. Nevertheless, before we continue the asymptotical analysis, we will take a short look into the differential equations associated with the $X$-RHP.

We already used the differential equation
\[
\frac{\partial X}{\partial \lambda} = A(\lambda, s, x) X,
\]
when we derived (4.15). Also, the logarithmic derivatives $X_s X^{-1}(\lambda)$ and $X_x X^{-1}(\lambda)$ are rational functions, indeed,
\[
\frac{\partial X}{\partial s} = \left[ -\frac{A}{\lambda - s} - \frac{B}{\lambda + s} \right] X \equiv \Theta(\lambda, s, x) X
\]
and furthermore
\[
\frac{\partial X}{\partial x} = \left[ i\lambda\sigma_3 - i \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \right] X \equiv \Pi(\lambda, s, x) X.
\]
Hence, we arrive at the Lax system for the function $X$:
\[
\begin{cases}
\frac{\partial X}{\partial \lambda} = A(\lambda, s, x) X, \\
\frac{\partial X}{\partial s} = \Theta(\lambda, s, x) X, \\
\frac{\partial X}{\partial x} = \Pi(\lambda, s, x) X.
\end{cases}
\]

Considering the compatibility conditions
\[
(5.1) \quad A_s - \Theta_{\lambda} = [\Theta, A], \quad A_x - \Pi_{\lambda} = [\Pi, A], \quad \Theta_x - \Pi_s = [\Pi, \Theta]
\]
we are led to a system of eighteen nonlinear ordinary differential equations. Since it is possible to express the previous derivatives of $\ln \det(I - \gamma K_{\sin})$ solely in terms of the unknowns $b, c, d, e, f, A$, and $B$, one can then try to derive a differential equation for the Fredholm determinant using (5.1). We plan to address this issue in a future publication.

§6. Second transformation of the RHP — rescaling and opening of lenses

Let us scale the variables in (3.1) as $\lambda = zs$, so that
\[
T(z) = X(zs)e^{-s^3 \vartheta(z)\sigma_3}, \quad z \in \mathbb{C} \setminus [-1, 1], \quad \vartheta(z) = i\left(\frac{4}{3}z^3 + \frac{xz}{s^2}\right),
\]
solves the following RHP, which up to the rescaling $\lambda = zs$, is identical to the initial $Y$-RHP:

- $T(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$;
- along the line segment $[-1, 1]$ oriented from left to right,

\[
T_+(z) = T_-(z)e^{s^3 \vartheta(z)\sigma_3} \begin{pmatrix} 1 - \gamma \gamma \gamma \\ -\gamma & 1 + \gamma \end{pmatrix} e^{-s^3 \vartheta(z)\sigma_3}, \quad z \in [-1, 1];
\]
- in a neighborhood of the endpoints $z = \pm 1$,

\[
T(z)e^{s^3 \vartheta(z)\sigma_3} = \tilde{X}(zs) \left[ I + \frac{\gamma}{2\pi i} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \ln \left( \frac{z - 1}{z + 1} \right) \right];
\]
- at infinity,
\[
T(z) = I + O(z^{-1}), \quad z \to \infty.
\]
Now we move to a RHP formulated in accordance with the sign-diagram of $\text{Re} \vartheta(z)$, which is depicted in Figure 1. In this figure we choose $x$ from a compact subset of the real line, $s > 0$ is sufficiently large, and

$$z_{\pm} = \pm i \sqrt{\frac{3x}{4s^2}}$$

denote the two vertices of the depicted algebraic curves.

![Figure 1. Sign-diagram for the function $\text{Re} \vartheta(z)$. In the left picture we indicate the location of $z_{\pm}$ as $x > 0$ and in the right picture for a particular choice of $x < 0$. Along the solid lines $\text{Re} \vartheta(z) = 0$, and the dashed lines resemble $\arg z = \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}$.]

With the matrix factorization

$$\begin{pmatrix} 1 - \gamma & \gamma \\ -\gamma & 1 + \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\gamma(1 - \gamma)^{-1} & 1 \end{pmatrix} (1 - \gamma)^{\sigma_3} \begin{pmatrix} 1 & \gamma(1 - \gamma)^{-1} \\ 0 & 1 \end{pmatrix} = SLSDSU,$$

valid as long as $\gamma \neq 1$, we perform opening of lenses as follows. Let $L^+_{jk}$ denote the upper (lower) lens, shown in Figure 2 that is bounded by the contours $\gamma_{jk}^\pm$ with

$$\gamma_{12}^+ = \left\{ z \in \mathbb{C} : \arg z = \frac{\pi}{6} \right\}, \quad \gamma_{21}^+ = \left\{ z \in \mathbb{C} : \arg z = \frac{5\pi}{6} \right\},$$

$$\gamma_{32}^- = \left\{ z \in \mathbb{C} : \arg z = -\frac{5\pi}{6} \right\}, \quad \gamma_{41}^- = \left\{ z \in \mathbb{C} : \arg z = \frac{\pi}{6} \right\}.$$

Define

$$S(z) = \begin{cases} T(z)S_U^{-1} & \text{if } z \in L^+_{1} \cup L^+_{2}, \\ T(z)S_L & \text{if } z \in L^-_{3} \cup L^-_{4}, \equiv S(z)L(z), \\ T(z) & \text{otherwise}, \end{cases}$$

then $S(z)$ solves the following RHP:

- $S(z)$ is analytic for $z \in \mathbb{C} \setminus (-1,1] \cup \mathcal{C}$, with $\mathcal{C} = \bigcup_{i,j} (\gamma_{ij}^+ \cup \gamma_{ij}^-)$;
- we have the following jumps, with orientation fixed as in Figure 2

$$S_+(z) = S_-(z)e^{s^3 \vartheta(z)\sigma_3} \tilde{G}_Se^{-s^3 \vartheta(z)\sigma_3}, \quad z \in \mathbb{C} \setminus (-1,1] \cup \mathcal{C},$$
Figure 2. The second transformation — opening of lenses.

where the piecewise constant matrix $\hat{G}_S$ is given by

$$\hat{G}_S = \begin{cases} 
S_U^{-1}, & z \in \gamma_{11}^+ \cup \gamma_{22}^+, \\
S_U, & z \in \gamma_{12}^+ \cup \gamma_{22}^+, \\
(1 - \gamma)^{\sigma_3}, & z \in [-1, 1], \\
S_L, & z \in \gamma_{31}^- \cup \gamma_{41}^+, \\
S_L^{-1}, & z \in \gamma_{32}^- \cup \gamma_{42}^+.
\end{cases}$$

- as $z \to \pm 1$, we have

$$S(z)L^{-1}(z)e^{s^3 \vartheta(z)\sigma_3} = \tilde{X}(zs) \left[ I + \frac{\gamma}{2\pi i} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \ln \left( \frac{z - 1}{z + 1} \right) \right];$$

- at infinity, $S(z) = I + O(z^{-1})$, $z \to \infty$.

Let us analyze the behavior of the jump matrix $G_S(z)$ in (6.5) along the infinite branches as $s \to \infty$. To this end, recall the sign-diagram of the function $\Re \vartheta(z)$, depicted in Figure 1. In the upper half-plane we have

$$G_S(z) = e^{s^3 \vartheta(z)\sigma_3} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} e^{-s^3 \vartheta(z)\sigma_3}, \quad z \in \gamma_{jk}^+, \ j, k = 1, 2,$n\right.$$n
with a constant $a \in \mathbb{C}$. Since we choose $x$ from a compact subset of the real line and $s > 0$ is sufficiently large, $\Re \vartheta(z)$ is always negative on $\gamma_{jk}^+, j, k = 1, 2$, outside small neighborhoods around the origin and the endpoints $z = \pm 1$. Hence, for such $z$,

$$G_S(z) \to I, \quad s \to \infty,$n\right.$$n
uniformly on any compact subset of the set (1.16) and the convergence is in fact exponentially fast. A similar statement holds true on the infinite branches in the lower half-plane. There we have

$$G_S(z) = e^{s^3 \vartheta(z)\sigma_3} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} e^{-s^3 \vartheta(z)\sigma_3}, \quad z \in \gamma_{jk}^-, \ j, k = 1, 2,$n\right.$$n
again with some constant $b \in \mathbb{C}$. In this situation, $\Re \vartheta(z) > 0$ outside a small neighborhood of the origin, and therefore,

$$G_S(z) \to I, \quad s \to \infty,$n\right.$$n
also uniformly on any compact subset of the set (1.16). From (6.7) and (6.8) we expect, and this will be justified rigorously, that as $s \to \infty$, $S(z)$ converges to a solution of the
model RHP, in which we only have to deal with the diagonal jump matrix $S_D$ on the line segment $[-1, 1]$. Let us now consider this model problem.

§7. The model RHP

Find the piecewise analytic $2 \times 2$ matrix-valued function $M(z)$ such that

- $M(z)$ is analytic for $\mathbb{C} \setminus [-1, 1]$;
- the following jump condition is fulfilled along $[-1, 1]$:
  $$M_+(z) = M_-(z)S_D, \quad z \in [-1, 1],$$
  where
  $$S_D = (1 - \gamma)^\sigma_3;$$
- $M(z)$ has at most square integrable singularities at the endpoints $z = \pm 1$;
- as $z \to \infty$, we have $M(z) = I + O(z^{-1})$.

Only assuming that $\gamma \neq 1$, we can always solve this diagonal and, thus, quasi-scalar RHP at once (see [22]):

$$M(z) = \exp \left[ \frac{1}{2\pi i} \int_{-1}^{1} \ln(1 - \gamma)^\sigma_3 \frac{dw}{w - z} \right] = \left( \frac{z + 1}{z - 1} \right)^{-\nu} \sigma_3,$$

where
$$\nu = \frac{1}{2\pi i} \ln(1 - \gamma), \quad \arg (1 - \gamma) \in (-\pi, \pi],$$
and $(\frac{z + 1}{z - 1})^\nu$ is defined on $\mathbb{C} \setminus [-1, 1]$ with its branch fixed by the condition $(\frac{z + 1}{z - 1})^\nu \to 1$ as $z \to \infty$.

Remark 1. We bring the reader’s attention to the important fact that in case $\gamma < 1$, we have $\arg(1 - \gamma) = 0$ and $\nu$ is therefore purely imaginary. However, if $\gamma > 1$, then $\arg(1 - \gamma) = \pi$ and $\nu$ equals
$$\nu = \frac{1}{2\pi i} \ln(\gamma - 1) + \frac{1}{2} \equiv \nu_0 + \frac{1}{2}, \quad \nu_0 \in i\mathbb{R}.$$
Later we shall see that this difference will have quite a substantial impact on the whole steepest descent analysis.

§8. Construction of a parametrix at the origin $z = 0$

In this section we construct a parametrix for the origin $z = 0$. This construction involves the Ablowitz–Segur solution $u = u(x, \gamma)$ of the boundary-value problem
$$u_{xx} = xu + 2u^3, \quad u(x) \sim \gamma \text{Ai}(x), \quad x \to +\infty,$$
where $\text{Ai}(x)$ is the Airy function. Viewing $x, u,$ and $u_x = \frac{du(x)}{dx}$ as real parameters, consider the $2 \times 2$ system of linear ordinary differential equations

$$\begin{align*}
\frac{\partial \Psi}{\partial \zeta} &= A(\zeta, x)\Psi, \\
A(\zeta, x) &= -4i\xi^2 \sigma_3 + 4i\xi \left( \begin{array}{cc} 0 & u \\ -u & 0 \end{array} \right) + \left( \begin{array}{cc} -ix - 2iu^2 & -2u_x \\ -2ux & ix + 2u^2 \end{array} \right).
\end{align*}$$

This system has precisely one irregular singular point of Poincaré rank 3 at infinity; thus, the classical theory of ordinary differential equations in the complex plane provides the existence of seven canonical solutions $\Psi_n(\zeta)$, which are fixed uniquely by their asymptotics (see [22] for more details),
$$\Psi_n(\zeta) \sim (I + O(\zeta^{-1})) e^{-i(\xi^3 + x\zeta)} \sigma_3, \quad \zeta \to \infty, \quad \zeta \in \Omega_n,$$
in the canonical sectors $\Omega_n$ shown in Figure 3

$$\Omega_n = \left\{ \zeta \in \mathbb{C} \mid \arg \zeta \in \left( \frac{\pi}{3}(n-2), \frac{\pi}{3}n \right), n = 1, \ldots, 7 \right\}.$$  

![Figure 3](image)

**Figure 3.** Canonical sectors of system (8.1) with the dashed lines indicating where $\text{Re} \zeta^3 = 0$.

Moreover, the presence of an irregular singularity gives us a nontrivial Stokes phenomenon described by the Stokes matrices $S_n$:

$$S_n = (\Psi_n(\zeta))^{-1} \Psi_{n+1}(\zeta).$$  

In the given situation (8.1) with the Ablowitz–Segur solution $u = u(x, \gamma)$, these multipliers are

$$S_1 = \begin{pmatrix} 1 & 0 \\ -i\gamma & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & i\gamma \\ 0 & 1 \end{pmatrix},$$  

$$S_3 = S_1, \quad S_5 = S_2, \quad S_6 = S_4.$$  

Now we choose $P_{II}(\zeta)$ to be the first canonical solution of (8.1) corresponding to the specific choice (8.3), i.e.,

$$P_{II}(\zeta) = \Psi_1(\zeta), \quad \arg \zeta \in \left( -\frac{\pi}{6}, \frac{\pi}{6} \right),$$

and assemble a piecewise analytic model function as follows:

$$\tilde{P}_{II}^{RH}(\zeta) = \begin{cases} P_{II}(\zeta), & \arg \zeta \in \left( -\frac{\pi}{6}, \frac{\pi}{6} \right) \cup \left( \frac{5\pi}{6}, \frac{7\pi}{6} \right), \\ P_{II}(\zeta)S_1, & \arg \zeta \in \left( \frac{\pi}{6}, \frac{5\pi}{6} \right), \\ P_{II}(\zeta)S_4, & \arg \zeta \in \left( \frac{2\pi}{6}, \frac{4\pi}{6} \right). \end{cases}$$

It can be checked directly that $\tilde{P}_{II}^{RH}(\zeta)$ solves the model RHP shown in Figure 4

- $\tilde{P}_{II}^{RH}(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \{\arg \zeta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$. 
\[(\begin{array}{cc} \frac{1}{i\gamma} & 0 \\ 0 & 1 \end{array}) \quad \quad \begin{array}{cc} 1 & 0 \\ -i\gamma & 1 \end{array}\]

**Figure 4.** The model RHP near \(z = 0\) that can be solved explicitly by using the real-valued Ablowitz–Segur solution of the second Painlevé equation.

- On the infinite rays, the following jumps occur:
  \[(\tilde{P}^{RH}_{II}(\zeta))_+ = (\tilde{P}^{RH}_{II}(\zeta))_- S_1, \quad \arg \zeta = \frac{\pi}{6},\]
  \[(\tilde{P}^{RH}_{II}(\zeta))_+ = (\tilde{P}^{RH}_{II}(\zeta))_- S_1^{-1}, \quad \arg \zeta = \frac{5\pi}{6},\]
  \[(\tilde{P}^{RH}_{II}(\zeta))_+ = (\tilde{P}^{RH}_{II}(\zeta))_- S_4, \quad \arg \zeta = \frac{7\pi}{6},\]
  \[(\tilde{P}^{RH}_{II}(\zeta))_+ = (\tilde{P}^{RH}_{II}(\zeta))_- S_4^{-1}, \quad \arg \zeta = \frac{11\pi}{6}.\]

- In terms of the previous choice (8.3), we have the following uniform asymptotics, valid in a full neighborhood of infinity:
  \[(8.5) \quad \tilde{P}^{RH}_{II}(\zeta) \sim \left(I + \frac{m_1}{\zeta} + \frac{m_2}{\zeta^2} + \frac{m_3}{\zeta^3} + O(\zeta^{-4})\right) e^{-i(\frac{1}{3}\zeta^3 + x\zeta)\sigma_3}, \quad \zeta \to \infty,\]
  with
  \[m_1 = \frac{1}{2} \begin{pmatrix} -iv & u \\ u & iv \end{pmatrix}, \quad m_2 = \frac{1}{8} \begin{pmatrix} u^2 - v^2 & 2i(u_x + uv) \\ -2i(u_x + uv) & u^2 - v^2 \end{pmatrix},\]
  and
  \[m_3 = \frac{1}{48} \begin{pmatrix} i(v^3 - 3vu^2 + 2(xv - uu_x)) & -3(u(u^2 + v^2) + 2(vu_x + xu)) \\ -3(u(u^2 + v^2) + 2(vu_x + xu)) & i(v^3 - 3vu^2 + 2(xv - uu_x)) \end{pmatrix},\]
  where we have introduced
  \[v = (u_x)^2 - xu^2 - u^4.\]

**Remark 2.** In view of our discussion in [1], the Ablowitz–Segur solution \(u = u(x, \gamma)\) might have poles on the real line. However, this can only happen in the case where \(\gamma > 1\), and in this situation we restrict ourselves to the values of \((\gamma, x)\) from [11,18]. Thus, in either of our cases, \(u = u(x, \gamma)\) is smooth in \(x\), and therefore the model function \(\tilde{P}^{RH}_{II}(\zeta) = \tilde{P}^{RH}_{II}(\zeta; x)\) is well defined.

Next, the model function (8.4) will be used to construct the parametrix to the solution of the original S-RHP in a neighborhood of \(z = 0\). First, set
\[P^{RH}_{II}(\zeta) = \begin{cases} e^{\pi i\sigma_3} \tilde{P}^{RH}_{II}(\zeta) e^{-\pi i\sigma_3}, & \text{Im} \zeta > 0; \\ e^{\pi i\sigma_3} \tilde{P}^{RH}_{II}(\zeta) e^{\pi i\sigma_3}, & \text{Im} \zeta < 0; \end{cases}\]
where \(\nu\) was introduced in (7.1). This leads to a model RHP with jumps on the positive oriented real line
\[(P^{RH}_{II}(\zeta))_+ = (P^{RH}_{II}(\zeta))_-(1 - \gamma)^{-\sigma_3}\]
as well as on the infinite rays \( \arg \zeta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \):

\[
\begin{align*}
(P_{II}^{RH}(\zeta))_+ &= (P_{II}^{RH}(\zeta))_+ \begin{pmatrix} 1 & 0 \\ -i\gamma(1-\gamma)^{-1} & 0 \end{pmatrix}, \quad \arg \zeta = \frac{\pi}{6}, \\
(P_{II}^{RH}(\zeta))_+ &= (P_{II}^{RH}(\zeta))_+ \begin{pmatrix} 1 & 0 \\ i\gamma(1-\gamma)^{-1} & 0 \end{pmatrix}, \quad \arg \zeta = \frac{5\pi}{6}, \\
(P_{II}^{RH}(\zeta))_+ &= (P_{II}^{RH}(\zeta))_- \begin{pmatrix} 1 & i\gamma(1-\gamma)^{-1} \\ 0 & 1 \end{pmatrix}, \quad \arg \zeta = \frac{7\pi}{6}, \\
(P_{II}^{RH}(\zeta))_+ &= (P_{II}^{RH}(\zeta))_- \begin{pmatrix} 1 & -i\gamma(1-\gamma)^{-1} \\ 0 & 1 \end{pmatrix}, \quad \arg \zeta = \frac{11\pi}{6},
\end{align*}
\]

and with the following behavior at infinity:

\[
P_{II}^{RH}(\zeta) = \left( I + \frac{\tilde{m}_1}{\zeta} + \frac{\tilde{m}_2}{\zeta^2} + \frac{\tilde{m}_3}{\zeta^3} + O(\zeta^{-4}) \right)e^{-i\left(\frac{4}{3}\zeta^3 + x\zeta\right)\sigma_3}\begin{pmatrix} I, & \text{Im } \zeta > 0, \\ e^{2\pi i\nu_3}, & \text{Im } \zeta < 0, \end{pmatrix},
\]

where

\[
\begin{align*}
\tilde{m}_1 &= \frac{1}{2}\begin{pmatrix} -iv & ue^{2\pi i\nu} \\ ue^{-2\pi i\nu} & iv \end{pmatrix}, \\
\tilde{m}_2 &= \frac{1}{8}\begin{pmatrix} u^2 - v^2 & 2i(u_x + uv)e^{2\pi i\nu} \\ -2i(u_x + uv)e^{2\pi i\nu} & u^2 - v^2 \end{pmatrix}, \\
\tilde{m}_3 &= \frac{1}{48}\begin{pmatrix} i(v^3 - 3vu^2 + 2(xv - xu)) & -3(u(u^2 + v^2) + 2(vu_x + xu)e^{2\pi i\nu}) \\ -3(u(u^2 + v^2) + 2(vu_x + xu)e^{2\pi i\nu}) & i(v^3 - 3vu^2 + 2(xv - xu)) \end{pmatrix}.
\end{align*}
\]

Second, we define

\[(8.6) \quad \zeta(z) = sz, \quad |z| < r,
\]

which yields a locally conformal change of variables and which allows us to define the parametrix \( U(z) \) near \( z = 0 \) by the formula

\[(8.7) \quad U(z) = \sigma_1 B_0(z)e^{-i\frac{\pi}{3}\sigma_3 P_{II}^{RH}(\zeta(z))}e^{i\left(\frac{\pi}{4}\zeta(z) + x\zeta(z)\right)\sigma_3}e^{i\frac{\pi}{6}\sigma_3} \sigma_1, \quad |z| < r, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

with \( \zeta(z) \) as in (8.6) and the matrix multiplier

\[(8.8) \quad B_0(z) = \left( \frac{z + 1}{z - 1} \right)^{\nu_3}e^{2\pi i\nu_3} \begin{pmatrix} I, & \text{Im } z > 0, \\ e^{-2\pi i\nu_3}, & \text{Im } z < 0, \end{pmatrix}, \quad B_0(0) = e^{-i\pi \nu_3}.
\]

By construction, in particular, since \( B_0(z) \) is analytic in a neighborhood of \( z = 0 \), the parametrix \( U(z) \) has jumps along the curves depicted in Figure 5, and we can always locally match these curves with the jump curves of the original RHP. Also, these jumps are described by the same Stokes matrices as in the original S-RHP, because the previously stated jumps of \( P_{II}^{RH}(\zeta) \) will be conjugated with \( e^{i\frac{\pi}{3}\sigma_3} \sigma_1 e^{-s^3\vartheta(z)\sigma_3} \), which precisely matches the model jumps with the original ones. Thus, the ratio of \( S(z) \) with \( U(z) \) is locally analytic, i.e.,

\[S(z) = N_0(z)U(z), \quad |z| < r < \frac{1}{2}.
\]

We explain the role of the left multiplier \( B_0(z) \) in the definition (8.7). Observe that

\[\sigma_1 B_0(z) \sigma_1 = M(z) \begin{pmatrix} I, & \text{Im } \zeta > 0, \\ e^{2\pi i\nu_3}, & \text{Im } \zeta < 0, \end{pmatrix}\]
These two observations suggest to use the confluent hypergeometric function

The confluent hypergeometric function is defined as the unique solution of Kummer’s equation



This idea can be justified rigorously as follows. Recall that the method, as we shall see later. This is the reason for choosing the left multiplier which is crucial for the successful implementation of the nonlinear steepest descent

This relation together with the asymptotic condition \( (8.5) \) implies that

\[
U(z) = \sigma_1 B_0(z)e^{-i\frac{\pi}{3}\sigma_3} \left( I + \frac{\tilde{m}_1}{\zeta} + \frac{\tilde{m}_2}{\zeta^2} + \frac{\tilde{m}_3}{\zeta^3} + O(\zeta^{-4}) \right) e^{i\frac{\pi}{3}\sigma_3} B_0^{-1}(z)\sigma_1 M(z)
\]

\[
= \left[ I + \frac{i}{2\zeta} B_0(z)^{-1} \left( \begin{array}{c} v + ue^{2\pi iv} - u e^{-2\pi iv} \\ u e^{2\pi iv} - u e^{-2\pi iv} \end{array} \right) B_0(z) \right. \\
\left. + \frac{1}{8\zeta^2} B_0(z)^{-1} \left( \begin{array}{c} 2(u_x + uv)ue^{-2\pi iv} \\ 2(u_x + uv)e^{2\pi iv} \end{array} \right) B_0(z) \right. \\
\left. + \frac{i}{48\zeta^3} B_0(z)^{-1} \left( \begin{array}{c} -(u^3 + 3vu^2 + 2(ux + uv)) e^{-2\pi iv} \\ 3(u^2 + v^2) + 2(vu_x + xu) e^{-2\pi iv} \end{array} \right) B_0(z) \right. \\
\left. + O(\zeta^{-3}) \right] M(z)
\]

as \( s \to \infty \) and \( 0 < r_1 \leq |z| \leq r_2 < 1 \) (so \( |\zeta| \to \infty \)). Since the function \( \zeta(z) \) is of order \( O(s^1) \) on this annulus and \( B_0(z) \) is bounded, equation \( (8.9) \) yields the matching relation between the model functions \( U(z) \) and \( M(z) \),

\[
U(z) = (I + o(1))M(z), \quad s \to \infty, \quad 0 < r_1 \leq |z| \leq r_2 < 1,
\]

which is crucial for the successful implementation of the nonlinear steepest descent method, as we shall see later. This is the reason for choosing the left multiplier \( B_0(z) \) in \( (8.7) \) in the form \( (8.8) \).

§9. CONSTRUCTION OF A PARAMETRIX AT THE EDGE POINT \( z = +1 \)

Fix a small neighborhood \( U \) of the point \( z = +1 \) and observe that

\[
\vartheta(z) = \vartheta(1) + i \left( 4 + \frac{x}{s^2} \right) (z - 1) + O((z - 1)^2)
\]

for \( z \in U \), and from \( (6.6) \) we have

\[
S(z) = O(\ln(z - 1)), \quad z \in U.
\]

These two observations suggest to use the confluent hypergeometric function \( U(a, b; \zeta) \) for our construction. This idea can be justified rigorously as follows. Recall that the confluent hypergeometric function is defined as the unique solution of Kummer’s equation

\[
zw'' + (b - z)w' - aw = 0
\]

satisfying the following asymptotic condition as \( \zeta \to \infty \) and \( -\frac{3\pi}{2} < \arg \zeta < \frac{3\pi}{2} \) (see \( 3 \)):

\[
U(a, b; \zeta) \sim \zeta^{-a} \left( 1 - \frac{a(1 + a - b)}{\zeta} + \frac{a(a + 1)(1 + a - b)(2 + a - b)}{2\zeta^2} + O(\zeta^{-3}) \right).
\]
Second, using the notation $U(a, \zeta) \equiv U(a, 1; \zeta)$, the following monodromy relation holds true on the entire universal covering of the punctured plane:

$$U(1 - a, e^{i\pi \zeta}) = e^{2\pi i a} U(1 - a, e^{-i\pi \zeta}) - e^{i\pi a} \frac{2\pi i}{a^2(1 - a)} U(a, \zeta)e^{-\zeta},$$

and finally we have an expansion at the origin (compare to (3.3)), namely,

$$U(a, \zeta) = c_0 + c_1 \ln \zeta + c_2 \zeta + c_3 \zeta \ln \zeta + O(\zeta^2 \ln \zeta), \quad \zeta \to 0,$$

with coefficients $c_i$ given by

$$c_0 = -\frac{1}{\Gamma(a)}(\psi(a) + 2\gamma_E), \quad c_1 = -\frac{1}{\Gamma(a)}, \quad c_2 = -\frac{a}{\Gamma(a)}(\psi(a + 1) + 2\gamma_E - 2), \quad c_3 = -\frac{a}{\Gamma(a)},$$

where $\gamma_E$ is Euler’s constant and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Retaining these properties, now we introduce the following matrix-valued function on the punctured plane $\zeta \in \mathbb{C} \setminus \{0\}$ (cf. [28]):

$$P_{CH}(\zeta) = \begin{pmatrix} U(\nu, e^{i\frac{\pi}{2}} \zeta) e^{2\pi i \nu} e^{-i\frac{\pi}{2}} & -U(1 - \nu, e^{-i\frac{\pi}{2}} \zeta) e^{\pi i \nu} e^{i\frac{\pi}{2}} \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} \end{pmatrix} e^{-i\frac{\pi}{2}(\frac{1}{2} - \nu)\sigma_3},$$

$$-\pi < \arg \zeta \leq \pi.$$

We collect the following asymptotic expansions. First, in the sector $-\frac{\pi}{2} < \arg \zeta < \frac{\pi}{2}$ we have

$$P_{CH}(\zeta) = \left[ I + \frac{i}{\zeta} \begin{pmatrix} \nu^2 & -\frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{\pi i \nu} \\ -\Gamma(1 + \nu) e^{-\pi i \nu} & -\nu^2 \end{pmatrix} \right] \begin{pmatrix} \frac{\Gamma(1 + \nu)}{\Gamma(\nu)} (1 + \nu)^2 e^{-\pi i \nu} & -\frac{\Gamma(1 - \nu)}{\Gamma(\nu)} (1 - \nu)^2 e^{\pi i \nu} \\ -\Gamma(1 + \nu) (1 + \nu)^2 e^{-\pi i \nu} & -\nu^2 (1 - \nu)^2 \end{pmatrix} + O(\zeta^{-3}) \right] \zeta^{-\nu \sigma_3} e^{-i\frac{\pi}{2}(\frac{1}{2} - \nu)\sigma_3} \begin{pmatrix} e^{i\frac{\pi}{2} \nu} & 0 \\ 0 & e^{-i\frac{\pi}{2} \nu} \end{pmatrix}, \quad \zeta \to \infty.$$

For another sector, say $\frac{\pi}{4} < \arg \zeta < \frac{3\pi}{4}$, we use (9.1) in the first column of (9.3) and obtain instead

$$P_{CH}(\zeta) = \left[ I + \frac{i}{\zeta} \begin{pmatrix} \nu^2 & -\frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{\pi i \nu} \\ -\Gamma(1 + \nu) e^{-\pi i \nu} & -\nu^2 \end{pmatrix} \right] \begin{pmatrix} \frac{\Gamma(1 + \nu)}{\Gamma(\nu)} (1 + \nu)^2 e^{-\pi i \nu} & -\frac{\Gamma(1 - \nu)}{\Gamma(\nu)} (1 - \nu)^2 e^{\pi i \nu} \\ -\Gamma(1 + \nu) (1 + \nu)^2 e^{-\pi i \nu} & -\nu^2 (1 - \nu)^2 \end{pmatrix} + O(\zeta^{-3}) \right] \zeta^{-\nu \sigma_3} e^{-i\frac{\pi}{2}(\frac{1}{2} - \nu)\sigma_3} \begin{pmatrix} e^{i\frac{\pi}{2} \nu} & 0 \\ 0 & e^{-i\frac{\pi}{2} \nu} \end{pmatrix}, \quad \zeta \to \infty,
as well as for \(-\frac{5\pi}{4} < \arg \zeta < -\frac{\pi}{4}\) with a similar argument in the second column of (9.3),

\[
P_{\text{CH}}(\zeta) = 1 + \frac{i}{\zeta} \left[ \frac{\nu^2}{\Gamma(1+\nu)} e^{-\pi i \nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} e^{\pi i \nu} \right] \left( \frac{-\nu^2}{\Gamma(1+\nu)} (1+\nu)^2 e^{-\pi i \nu} - \frac{\Gamma(1-\nu)}{\Gamma(\nu)} (1-\nu)^2 e^{\pi i \nu} \right) + O(\zeta^{-3})
\]

Also of interest is the following exact monodromy relation:

\[
P_{\text{CH}}(\zeta) = P_{\text{CH}}(e^{-2\pi i \nu}) \left( e^{2\pi i \nu} - \frac{2\pi e^{-i\pi \nu}}{\Gamma(1-\nu) \Gamma(\nu)} \right) - \frac{2\pi e^{-i\nu}}{\Gamma(1-\nu) \Gamma(\nu)}.
\]

Now we assemble the model function

\[
P_{\text{CH}}^{\text{RH}}(\zeta) = \begin{cases} 
P_{\text{CH}}(\zeta) \left( \frac{1}{\Gamma(1-\nu) \Gamma(\nu)} \right)^{1-\sigma_3} e^{-\frac{2\pi e^{-i\nu}}{\Gamma(1-\nu) \Gamma(\nu)}} & \text{for } \zeta \in \{ \pi, \pi \}, \\
P_{\text{CH}}(\zeta) \left( \frac{1}{\Gamma(1-\nu) \Gamma(\nu)} \right)^{1-\sigma_3} e^{\frac{2\pi e^{-i\nu}}{\Gamma(1-\nu) \Gamma(\nu)}} & \text{for } \zeta \in \left\{ -\frac{\pi}{3}, \frac{\pi}{3} \right\}, \\
P_{\text{CH}}(\zeta) \left( \frac{1}{\Gamma(1-\nu) \Gamma(\nu)} \right)^{1-\sigma_3} e^{\frac{2\pi e^{-i\nu}}{\Gamma(1-\nu) \Gamma(\nu)}} & \text{for } \zeta \in \left\{ -\frac{\pi}{3}, \frac{\pi}{3} \right\}, \\
\end{cases}
\]

that solves the RHP depicted in Figure 6.

Figure 6. The model RHP near \(z = +1\) that can be solved explicitly by using confluent hypergeometric functions.

In more detail, \(P_{\text{CH}}^{\text{RH}}(\zeta)\) possesses the following analytic properties:

- \(P_{\text{CH}}^{\text{RH}}(\zeta)\) is analytic for \(\zeta \in \mathbb{C} \setminus \{ \arg \zeta = -\pi, -\frac{\pi}{3}, \frac{\pi}{3} \}\);
- the following jumps occur, orienting the jump contours as shown in Figure 6:

\[
\begin{align*}
(P_{\text{CH}}^{\text{RH}}(\zeta))_+ &= (P_{\text{CH}}^{\text{RH}}(\zeta))_- (1-\gamma)^{-\sigma_3}, \quad \arg \zeta = -\pi, \\
(P_{\text{CH}}^{\text{RH}}(\zeta))_+ &= (P_{\text{CH}}^{\text{RH}}(\zeta))_- \left( \frac{1}{\Gamma(1-\nu) \Gamma(\nu)} \right)^{1-\sigma_3} e^{\frac{2\pi e^{-i\nu}}{\Gamma(1-\nu) \Gamma(\nu)}}, \quad \arg \zeta = -\frac{\pi}{3}; \\
(P_{\text{CH}}^{\text{RH}}(\zeta))_+ &= (P_{\text{CH}}^{\text{RH}}(\zeta))_- \left( \frac{1}{\Gamma(1-\nu) \Gamma(\nu)} \right)^{1-\sigma_3} e^{-\frac{2\pi e^{-i\nu}}{\Gamma(1-\nu) \Gamma(\nu)}}, \quad \arg \zeta = -\frac{\pi}{3},
\end{align*}
\]

and, by a classical identity

\[
\Gamma(1-\nu) \Gamma(\nu) = \frac{\pi}{\sin \pi \nu},
\]

the entry of the above triangular matrices equals

\[
\frac{2\pi e^{-i\pi \nu}}{\Gamma(1-\nu) \Gamma(\nu)} = i \gamma (1-\gamma)^{-1}.
\]
as $\zeta \to \infty$, the model function $P_{CH}^{RH}(\zeta)$ shows the following asymptotic behavior in the whole neighborhood of infinity:

$$
P_{CH}^{RH}(\zeta) = \left[ 1 + \frac{i}{\zeta} \left( \nu^2 \frac{(1+\nu)(1-\nu)e^{-\pi i \nu}}{1-\nu^2} - \frac{1}{\Gamma(1+\nu)} e^{-\pi i \nu} \right) \right] + O(\zeta^{-3})
$$

Next, the model function $P_{CH}^{RH}(\zeta)$ will be used in the construction of the parametrix near $z = +1$. Define

$$\zeta(z) = -2is^3(\vartheta(z) - \vartheta(1)), \quad |z - 1| < r,$$

and notice that

$$\zeta(z) = 2s^3\left(4 + \frac{x}{s^2}\right)(z - 1)(1 + O(z - 1)), \quad z \to 1,$$

yielding local conformality. This change of variables allows us to define the right parametrix $V(z)$ near $z = +1$ as follows:

$$V(z) = \sigma_1 e^{-i\frac{1}{2} s \sigma_3} B_r(z) e^{i\frac{1}{2} (\frac{1}{2} - \nu) s \sigma_3} e^{-s^3 \vartheta(1) s \sigma_3} P_{CH}^{RH}(\zeta(z)) e^{i\frac{1}{2} \vartheta(z) + s^3 \vartheta(1) s \sigma_3} e^{i\frac{1}{2} s \sigma_3} \sigma_1,$$

with $\vartheta(z)$ as in [9.6] and with the multiplier

$$B_r(z) = \left( \zeta(z) \frac{z + 1}{z - 1} \right)^{\nu \sigma_3}, \quad B_r(1) = \left(16s^3 + 4xs\right)^{\nu \sigma_3}.$$

Also here, from the analyticity of $B_r(z)$ and conjugation with $\sigma_1 e^{-i\frac{1}{2} s \sigma_3} e^{-s^3 \vartheta(z) s \sigma_3}$ in

$$e^{s^3 \vartheta(z) s \sigma_3}\left(\begin{array}{c} 1 - \gamma(1-\gamma)^{-1} \\ 1 - \gamma(1-\gamma)^{-1} \end{array}\right)e^{-s^3 \vartheta(z) s \sigma_3}
$$

**Figure 7.** Transformation of parametrix jumps to original jumps.

In [9.7], it follows that the parametrix jumps and jump curves (see Figure 7) match the original jumps and locally original jump curves. Furthermore, and we shall elaborate this in full detail very soon, the singular endpoint behavior of the parametrix $U(z)$ matches [6.6], i.e.,

$$V(z) = O\left(\ln(z - 1)\right), \quad z \to +1.$$

Hence, the ratio of $S(z)$ with $U(z)$ is locally analytic, i.e.,

$$S(z) = N_r(z)V(z), \quad |z - 1| < \frac{1}{2},$$
and again the role of the left multiplier $B_r(z)$ follows from the asymptotical match up

$$V(z) = \sigma_1 e^{-i\frac{\pi}{2} \sigma_3} B_r(z) e^{i\frac{\pi}{2} (\frac{1}{2} - \nu) \sigma_3} e^{-s^3 \theta(1) \sigma_3} \left[I + \frac{i}{\zeta} \left(\frac{-\nu^2}{10} (1 + \nu)^2 - \frac{\Gamma(1-\nu)}{10(\nu)} e^{-\pi i \nu} \right) \right]$$

$$+ \frac{1}{\zeta^2} \left(\frac{-\nu^2}{2} (1 + \nu)^2 e^{-\pi i \nu} \right) + O(\zeta^{-3})$$

(9.9)

This would mean that equation (9.9) yields a matching relation between the model and hence the matching case (9.10). We shall proceed with this analysis in (9.11) yields a very serious change in the further asymptotic analysis, compared to the matching case (9.10). We shall proceed with this analysis in

$$\beta_r(z) = \left(\zeta(z) \frac{z+1}{z-1}\right)^{\nu}.$$ 

If we are dealing with the case where $\gamma < 1$, then

$$\beta_r^{\pm 2}(z) \frac{1}{\zeta} = O(s^{-3+\Re \nu}) = o(1), \ s \to \infty.$$ 

This would mean that equation (9.9) yields a matching relation between the model functions $V(z)$ and $M(z)$,

(9.10) \hspace{1cm} V(z) = (I + o(1)) M(z), \ s \to \infty, \ 0 < r_1 \leq |z - 1| \leq r_2 < \frac{1}{2},$$

which is again crucial for the successful implementation of the nonlinear steepest descent method. However, if $\gamma > 1$, then

$$\nu = \frac{1}{2\pi i} \ln(\gamma - 1) + \frac{1}{2} \equiv \nu_0 + \frac{1}{2}$$

and hence

$$\beta_r^{\pm 2}(z) \frac{1}{\zeta} = O(1), \ s \to \infty; \ \hat{\beta}_r(z) = \left(\zeta(z) \frac{z+1}{z-1}\right)^{\nu_0}.$$ 

With this notation, we need to replace (9.10) in the case where $\gamma > 1$ by

(9.11) \hspace{1cm} V(z) = E_r(z)(I + o(1)) M(z), \ s \to \infty, \ 0 < r_1 \leq |z - 1| \leq r_2 < \frac{1}{2},$$

where

$$E_r(z) = \left(-i \frac{\Gamma(1-\nu)}{10(\nu)} e^{-2s^3 \theta(1) \beta_r^{\pm 2}(z) \frac{z+1}{z-1}} 0 \right).$$

The involvement of the nontrivial matrix term $E_r(z)$ instead of the unit matrix in estimate (9.11) yields a very serious change in the further asymptotic analysis, compared to the matching case (9.10). We shall proceed with this analysis in §§ 13 and 14.
§10. Construction of a parametrix at the edge point \( z = -1 \)

For now, we introduce the model RHP near the other endpoint \( z = -1 \). Opposed to \((9.3)\), consider

\[
\hat{P}_{CH}(\zeta) = \begin{pmatrix} U(\nu, e^{-i\frac{3\pi}{2}} \zeta) e^{-i\frac{\zeta}{2}} & U(1 + \nu, e^{-i\frac{3\pi}{2}} \zeta) e^{\pi i\nu} e^{i\frac{\zeta}{2} \Gamma(1+i\nu)} \\ U(1 - \nu, e^{-i\frac{3\pi}{2}} \zeta) e^{\pi i\nu} e^{-i\frac{\zeta}{2} \frac{\Gamma(1-i\nu)}{\Gamma(i
\nu)}} & U(\nu, e^{-i\frac{3\pi}{2}} \zeta) e^{2\pi i\nu} e^{i\frac{\zeta}{2}} \end{pmatrix} \times e^{i\frac{\pi}{4}(\frac{3}{2}-\nu)\sigma_3} = \sigma_2 P_{CH}(e^{-i\pi} \zeta) \sigma_2, \quad 0 < \arg \zeta \leq 2\pi,
\]

and

\[
\hat{P}^{RH}_{CH}(\zeta) = \begin{pmatrix} \hat{P}_{CH}(\zeta) \left( 1 - \frac{2\pi e^{-i\nu}}{\Gamma(1-i\nu)} \right) \left( e^{i\frac{\pi}{2}} e^{\frac{\nu}{2} \Gamma(1+i\nu)} \right) & \left( \frac{1}{2\pi e^{-i\nu}} \right) \left( e^{i\frac{\pi}{2}} e^{\frac{\nu}{2} \Gamma(1+i\nu)} \right) \\
\hat{P}_{CH}(\zeta) \left( 1 - \frac{2\pi e^{-i\nu}}{\Gamma(1-i\nu)} \right) \left( e^{-i\frac{\pi}{2}} e^{\frac{\nu}{2} \Gamma(1+i\nu)} \right) & \left( \frac{1}{2\pi e^{-i\nu}} \right) \left( e^{-i\frac{\pi}{2}} e^{\frac{\nu}{2} \Gamma(1+i\nu)} \right) \end{pmatrix}, \quad \arg \zeta \in (\frac{4\pi}{3}, 2\pi),
\]

(10.1)

The model function \( \hat{P}^{RH}_{CH}(\zeta) \) solves the RHP of Figure \[8\]

\[
\begin{array}{c}
\begin{pmatrix} -1 & 0 \\
0 & 1 \end{pmatrix}
\end{pmatrix}
\]

\[
(1-\gamma)^{-\sigma_3}
\]

\[
\begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}
\]

Figure 8. The model RHP near \( z = -1 \) that can be solved explicitly using confluent hypergeometric functions.

- \( \hat{P}^{RH}_{CH}(\zeta) \) is analytic for \( \zeta \in \mathbb{C} \setminus \{ \arg \zeta = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \} \).
- Along the contour in Figure \[8\] the following jumps occur (recall \((9.1)\) and the symmetry relation \( \hat{P}^{RH}_{CH}(\zeta) = \sigma_2 P_{CH}(e^{-i\pi} \zeta) \sigma_2 \)):

\[
(\hat{P}^{RH}_{CH}(\zeta))_+ = (\hat{P}^{RH}_{CH}(\zeta))_- e^{-2\pi i \nu \sigma_3}, \quad \arg \zeta = 0,
\]

\[
(\hat{P}^{RH}_{CH}(\zeta))_+ = (\hat{P}^{RH}_{CH}(\zeta))_- \left( \frac{1}{2\pi e^{-i\nu}} \right) \left( e^{\frac{\pi}{2}} \right), \quad \arg \zeta = \frac{2\pi}{3},
\]

\[
(\hat{P}^{RH}_{CH}(\zeta))_+ = (\hat{P}^{RH}_{CH}(\zeta))_- \left( \frac{1}{2\pi e^{-i\nu}} \right) \left( e^{-\frac{\pi}{2}} \right), \quad \arg \zeta = \frac{4\pi}{3}.
\]

- From the symmetry \( \hat{P}^{RH}_{CH}(\zeta) = \sigma_2 P_{CH}(e^{-i\pi} \zeta) \sigma_2 \) and the asymptotic information derived earlier for \( P_{CH}(\zeta) \) in different sectors, we deduce the following behavior, valid in a full neighborhood of infinity:

\[
\hat{P}^{RH}_{CH}(\zeta) = \left[ I + i \frac{\nu^2}{\zeta^2} \left( \frac{\Gamma(1+i\nu)}{\Gamma(i\nu)} e^{\pi i\nu} - \nu^2 \right) \Gamma(1-i\nu) \right] \times e^{i\pi \zeta} \sigma_3 e^{-i\frac{\pi}{4}(\frac{3}{2}-\nu)\sigma_3}, \quad \zeta \to \infty.
\]
Now, much as we did in the construction of $V(z)$, define
\begin{equation}
\zeta(z) = -2is^3\bigl(\vartheta(z) - \vartheta(-1)\bigr), \quad |z + 1| < r,
\end{equation}
with
\begin{equation}
\zeta(z) = 2s^3\left(4 + \frac{x}{s^2}\right)(z + 1)(1 + O(z + 1)), \quad z \to -1,
\end{equation}
hence a locally conformal change of variables. With this change, the parametrix $W(z)$ near the left endpoint $z = -1$ will be defined as
\begin{equation}
W(z) = \sigma_1 e^{-i\frac{\pi}{2}\sigma_3} B_l(z) e^{-i\frac{\pi}{2}(1 - \nu)\sigma_3} e^{-s^3\vartheta(-1)\sigma_3} \tilde{P}_{\text{RH}} \left(\zeta(z)\right) e^{i\frac{\pi}{2}\varsigma(z) + s^3\vartheta(-1)\sigma_3} e^{i\frac{\pi}{2}\sigma_3} \sigma_1,
\end{equation}
with $\zeta(z)$ as in (10.2) and
\begin{equation*}
B_l(z) = \left(\frac{e^{-i\pi \zeta(z)} z - 1}{z + 1}\right)^{-\nu\sigma_3},
\end{equation*}
\begin{equation*}
B_l(-1) = (16s^3 + 4sx)^{-\nu\sigma_3}.
\end{equation*}
Since
\begin{equation*}
\frac{2\pi e^{-i\pi \nu}}{\Gamma(1 - \nu)\Gamma(\nu)} = i\gamma(1 - \gamma)^{-1},
\end{equation*}
the stated conjugation with $\sigma_1 e^{-i\frac{\pi}{2}\sigma_3} e^{-s^3\vartheta(z)\sigma_3}$ will again match parametrix jumps with original jumps locally on the original jump contour (see Figure 9).
\begin{equation*}
e^{s^3\vartheta(z)\sigma_3} \begin{pmatrix} 1 & \gamma(1 - \gamma)^{-1} \\ 0 & 1 \end{pmatrix} e^{-s^3\vartheta(z)\sigma_3} \begin{pmatrix} 1 \\ -\gamma(1 - \gamma)^{-1} \end{pmatrix} e^{s^3\vartheta(z)\sigma_3}
\end{equation*}
\begin{equation*}
(1 - \gamma)^{\sigma_3}
\end{equation*}
\begin{equation*}
e^{s^3\vartheta(z)\sigma_3} \begin{pmatrix} 1 \\ -\gamma(1 - \gamma)^{-1} \end{pmatrix} e^{s^3\vartheta(z)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{-s^3\vartheta(z)\sigma_3}
\end{equation*}
\begin{equation*}
\text{Figure 9. Transformation of parametrix jumps to original jumps.}
\end{equation*}

Moreover, the singular endpoint behavior
\begin{equation}
W(z) = O\left(\ln(|z + 1|)\right), \quad z \to -1,
\end{equation}
matches (3.3), leading to a locally analytic ratio of $S(z)$ to $W(z)$,
\begin{equation*}
S(z) = N_l(z)W(z), \quad |z + 1| < \frac{1}{2}.
\end{equation*}
Also, here the left multiplier $B_l(z)$ provides us with an asymptotical matchup
\begin{equation}
W(z) = \left[I + \frac{i}{\zeta} \begin{pmatrix} -\nu^2 & \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} e^{2s^3\vartheta(-1)\beta_l^2} \\ \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{-2s^3\vartheta(-1)\beta_l^2} & \nu^2 \end{pmatrix} \right] M(z)
\end{equation}
\begin{equation}
+ \frac{1}{\zeta^2} \begin{pmatrix} -\nu^2(1 + \nu)^2 & -\frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{2s^3\vartheta(-1)\beta_l^2} \\ \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} e^{-2s^3\vartheta(-1)\beta_l^2} & -\nu^2(1 - \nu)^2 \end{pmatrix} M(z)
\end{equation}
\begin{equation}
+ O(\zeta^{-3})
\end{equation}
as $s \to \infty$, valid on the annulus $0 < r_1 \leq |z + 1| \leq r_2 < 1$ (thus $|\zeta| \to \infty$), and we introduce the abbreviation

$$\beta_l(z) = \left( e^{-i\pi} \frac{\zeta(z) z - 1}{z + 1} \right)^\nu .$$

As in the previous situation, this implies that, on the annulus for $\gamma < 1$,

$$W(z) = (I + o(1)) M(z), \quad s \to \infty,$$

whereas for $\gamma > 1$ we have

(10.6) \quad \begin{align*}
W(z) &= E_l(z)(I + o(1)) M(z), \quad s \to \infty, \\
E_l(z) &= \left( \begin{array}{cc} 1 & -i \Gamma(1-\nu) \frac{z^2 \vartheta(z)}{\vartheta(z) - 1} \beta_l(z) \frac{z - 1}{z + 1} \\ 0 & 1 \end{array} \right), \quad \hat{\beta}_l(z) = \left( e^{-i\pi} \frac{\zeta(z) z - 1}{z + 1} \right)^{\nu_0}.
\end{align*}

At this point we can use the model functions $M(z), U(z), V(z),$ and $W(z)$ to employ a further transformation.

§11. Third Transformation of the RHP

In this transformation we put

(11.1) \quad \begin{align*}
R(z) &= S(z) \begin{cases} 
(V(z))^{-1}, & |z - 1| < r_1, \\
(U(z))^{-1}, & |z| < r_2, \\
(W(z))^{-1}, & |z + 1| < r_1, \\
(M(z))^{-1}, & |z - 1| > r_1, |z + 1| > r_1, |z| > r_2,
\end{cases}
\end{align*}

where $0 < r_1, r_2 < \frac{1}{2}$ are fixed. With $C_{0,r,l}$ denoting the clockwise oriented circles shown in Figure 10, the ratio-function $R(z)$ solves the following RHP:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{The jump graph for the ratio-function $R(z)$.}
\end{figure}

- $R(z)$ is analytic for $z \in \mathbb{C} \setminus \{C_{0,r,l} \cup \bigcup_{i=1}^{8} \gamma_i\}$;
- for the jumps, along the infinite branches $\gamma_i$, we have

$$R_+(z) = R_-(z) M(z) e^{s^3 \vartheta(z) \sigma_3} \hat{G} e^{-s^3 \vartheta(z) \sigma_3} (M(z))^{-1},$$
with \( \hat{G}_S \) denoting the corresponding jump matrices from (6.5); on the clockwise oriented circles \( C_0 \) and \( C_{r,t} \), the jumps are described by the formulas
\[
R_+(z) = R_-(z)U(z)(M(z))^{-1}, \quad z \in C_0,
\]
\[
R_+(z) = R_-(z)V(z)(M(z))^{-1}, \quad z \in C_r,
\]
\[
R_+(z) = R_-(z)W(z)(M(z))^{-1}, \quad z \in C_l;
\]
- \( R(z) \) is analytic at \( z = \pm 1 \) (this observation will follow directly from (9.8) and (10.4), which will be proved in 11.6);
- in a neighborhood of infinity, we have \( R(z) \to I \).

We emphasize that, by construction, \( R(z) \) has no jumps inside of the circles \( C_{0,r,l} \) and across the line segment in-between. In order that the Deift–Zhou nonlinear steepest descent be applicable to the ratio-RHP, all its jump matrices must be close to the unit matrix, as \( s \to \infty \), compare [18]. Hence, it is now important to recall the previously stated behavior of the jump matrices as \( s \to \infty \). As mentioned before, due to the triangularity of \( S_U, S_L \) and the sign-diagram of \( \Re, \vartheta(z) \), the jump matrices corresponding to the infinite parts \( \bigcup_{i=1}^{9} \gamma_i \) of the \( R \)-jump contour are in fact exponentially close to the unit matrix:
\[
\left\| Me^{s^3\vartheta(\sigma_3)}\hat{G}_Se^{-s^3\vartheta(\sigma_3)}(M)^{-1} - I \right\|_{L^2 \cap L^\infty(\gamma_i)} \leq c_1 \left\{ \begin{array}{ll}
e^{-c_2s^3|z|} & \text{from } C_0, \\
e^{-c_3s^3|z+1|} & \text{from } C_{r,t}, \end{array} \right.
\]
as \( s \to \infty \), with constants \( c_i > 0 \) whose values are not important. Also by (8.9), \( U(z)(M(z))^{-1} \) approaches the unit matrix as \( s \to \infty \),
\[
\left\| U(M)^{-1} - I \right\|_{L^2 \cap L^\infty(C_0)} \leq c_4 s^{-1}
\]
with a constant \( c_4 > 0 \). The jumps on \( C_{r,l} \) however have to be treated more carefully. As we already mentioned in 8.9 and 10.4 estimates (9.9) and (10.5) in the case where \( \gamma < 1 \) yield
\[
\left\| V(M)^{-1} - I \right\|_{L^2 \cap L^\infty(C_r)} \leq c_5 s^{-3}, \quad \left\| W(M)^{-1} - I \right\|_{L^2 \cap L^\infty(C_l)} \leq c_6 s^{-3}
\]
as \( s \to \infty \). Estimates (11.2), (11.3), and (11.4) in the case where \( \gamma < 1 \) yield
\[
\left\{ (\gamma, x) \in \mathbb{R}^2 : -\infty < \gamma < 1, \ -\infty < x < \infty \right\},
\]
enable us to solve the ratio-RHP iteratively in that particular situation.

\section{Solution of the RHP for \( R(z) \) via Iteration, \( \gamma < 1 \)}

Let \( G_R \) denote the jump matrix in the ratio-RHP and \( \Sigma_R \) the underlying contour. The stated RHP for the function \( R(z) \), i.e.,
- \( R(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma_R \);
- along the contour depicted in Figure 10 we have
  \[
  R_+(z) = R_-(z)G_R(z), \quad z \in \Sigma_R;
  \]
- as \( z \to \infty \), we have \( R(z) = I + O(z^{-1}) \),

is equivalent to the singular integral equation
\[
R_-(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} R_-(w)(G_R(w) - I) \frac{dw}{w - z},
\]
and by the previous estimates (11.2), (11.3), and (11.4), we have
\[
\left\| G_R - I \right\|_{L^2 \cap L^\infty(\Sigma_R)} \leq c_6 s^{-1}
\]
uniformly on any compact subset of the set \([1.16]\). By standard arguments (see \([18]\)), we know that for sufficiently large \(s\), the relevant integral operator is a contraction and equation \((12.1)\) can be solved iteratively in \(L^2(\Sigma_R)\). Moreover, its unique solution satisfies

\[
\|R_\gamma - I\|_{L^2(\Sigma_R)} \leq cs^{-1}, \quad s \to \infty.
\]

This information is all we need to compute the asymptotic expansion for the Fredholm determinant \(\det(I - \gamma K_{\text{sin}})\) in case \(\gamma < 1\). Before we derive the relevant asymptotics, first we discuss the situation where \(\gamma > 1\). In this case

\[
\|V(M)^{-1} - I\|_{L^2(\Lambda_{const})} \to 0, \quad \|W(M)^{-1}\|_{L^2(\Lambda_{const})} \to 0
\]

and we need to employ further transformations.

§13. Fourth transformation of the RHP — undressing

The presence of the multipliers \(E_r(z)\) and \(E_l(z)\) in \((9.11)\) and \((10.6)\) requires further transformations leading to a singular or solitonic type of Riemann–Hilbert problem. Following \([21, 10]\), we shall show how to deal with the singular structure.

A key observation for our next move is that the jump matrices \(G_r(z) = V(z)(M(z))^{-1}\) and \(G_l(z) = W(z)(M(z))^{-1}\) admit the following algebraic factorizations:

\[
G_r(z) = E_r(z)\hat{G}_r(z) = \begin{pmatrix} 1 & 0 \\ -i\Gamma(1-\nu)e^{-2s^3\arg(1)}\beta_\gamma^2(z)\frac{z+1}{z-1} & 1 \end{pmatrix}
\times \left[ I + \frac{i}{\zeta} \begin{pmatrix} -\nu^2 & 0 \\ -i\Gamma(1-\nu)e^{-2s^3\arg(1)}\frac{z+1}{z-1}(2\nu - 1) \end{pmatrix} + O(\zeta^{-2}) \right],
\]

\[
G_l(z) = E_l(z)\hat{G}_l(z) = \begin{pmatrix} 1 & 0 \\ -i\Gamma(1-\nu)e^{-2s^3\arg(-1)}\beta_\gamma^2(z)\frac{z+1}{z-1} & 1 \end{pmatrix}
\times \left[ I + \frac{i}{\zeta} \begin{pmatrix} -\nu^2 & 0 \\ -i\Gamma(1-\nu)e^{-2s^3\arg(-1)}\frac{z+1}{z-1}(1 - 2\nu) \end{pmatrix} + O(\zeta^{-2}) \right]
\]

as \(s \to \infty\) and \(0 < r_1 \leq |z| = 1 \leq r_2 < 1\). We observe that \(\|\hat{G}_{r,l} - I\| \to 0\) as \(s \to \infty\); in fact, since \(|\zeta(z)| \geq cs^3\) on \(C_r \cup C_l\), we have

\[
\|\hat{G}_{r,l} - I\|_{L^2(\Lambda_{const})} \leq c_7s^{-3}, \quad s \to \infty.
\]

Hence, the natural idea is to pass from the function \(R(z)\) to the function \(P(z)\) defined by the equations

\[
P(z) = \begin{cases} \begin{array}{ll} R(z)E_r(z), & |z - 1| < r_1, \\ R(z)E_l(z), & |z + 1| < r_1, \\ R(z), & |z| = 1 > r_1, \end{array} \end{cases}
\]

with \(0 < r_1 < \frac{1}{2}\) chosen as in \((11.3)\). By definition, the function \(P(z)\) solves the following RHP.

- \(P(z)\) is analytic for \(z \in \mathbb{C} \setminus (C_{0,r,l} \cup \{\pm 1\} \cup \bigcup_{i=1}^{8} \gamma_i)\).
- \(P_+(z) = P_-(z)G_P(z)\), where

\[
G_P(z) = \begin{cases} \begin{array}{ll} \hat{G}_{r,l}(z), & z \in C_{r,l}, \\ U(z)(M(z))^{-1}, & z \in C_0, \\ M(z)e^{s^3\arg(z)}\hat{G}_S e^{-s^3\arg(z)}\sigma_3(M(z))^{-1}, & z \in \gamma_i, \quad i = 1, \ldots, 8. \end{array} \end{cases}
\]
Proof. The residue relations (13.4) and (13.5) imply unique solution.

Proposition 4. The singular Riemann–Hilbert problem for $P(z)$ stated above has a unique solution.

Proof. The residue relations (13.4) and (13.5) imply

$$\text{res}_{z=-1} P^{(2)}(z) = P^{(1)}(-1) \left( 2 i \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{2 s^3 \vartheta(-1) \beta^2_{\nu}(-1)} \right)$$

$$\text{res}_{z=+1} P^{(1)}(z) = P^{(2)}(1) \left( -2 i \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{-2 s^3 \vartheta(1) \beta^2_{\nu}(1)} \right)$$

where the $\hat{P}^{(\pm)}(z)$ are analytic at $z = \pm 1$. Hence, $\det P(z) \equiv 1$ by the Liouville theorem, using the normalization at infinity and unimodularity of the jump matrices. From this and the representation (13.6), the ratio of any two solutions $P_1(z)$ and $P_2(z)$ of the given $P$-RHP, i.e.,

$$P_1(z) (P_2(z))^{-1},$$

is an entire function approaching identity at infinity; hence $P_1 = P_2$, showing uniqueness.

As $z \to \infty$, we have $P(z) \to I$.

First, four properties determine $P(z)$ uniquely.

Second, all jump matrices in the $P$-RHP approach the identity matrix as $s \to \infty$; however $P(z)$ has singularities at $z = \pm 1$ whose structure is described by the residue relations (13.4) and (13.5). This type of Riemann–Hilbert problem is a known one in the theory of integrable systems. The way to deal with such RHPs is to use a certain “dressing procedure”, which reduces the problem to the one without the pole singularities.

§14. Fifth and final transformation of the RHP — dressing

We put

$$P(z) = (zI + B)Q(z) \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix},$$

where $B \in \mathbb{C}^{2 \times 2}$ is constant and see immediately that $Q(z)$ solves the following RHP.

- $Q(z)$ is analytic for $z \in \mathbb{C} \setminus (C_{0, r, l} \cup \bigcup_{i=1}^{8} \gamma_i)$.
- $Q_{\pm}(z) = Q_{\pm}(z)G_Q(z)$, where

$$G_Q(z) = \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix} \hat{G}_{r,l}(z) \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}, \quad z \in C_{r,l},$$

and

$$G_Q(z) = \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix} U(z) (M(z))^{-1} \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}, \quad z \in C_0,$$
as well as
\[ G_Q(z) = \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix} M(z) e^{s^3 \sigma_3 (M(z))^{-1}} \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}, \quad z \in \gamma_i. \]

- \( Q(z) \to I \) as \( z \to \infty \).

The \( Q \)-jump matrix \( G_Q(z) \) is uniformly close to the unit matrix; therefore, the \( Q \)-RHP admits direct asymptotic analysis, which will be performed after finding of the unknown matrix \( B \). Using conditions (13.4) and (13.5), we obtain
\[
\text{res } z = 1 P(1) = (I + B) Q(1),
\]
\[
\text{res } z = -1 P(2) = (-I + B) Q(2),
\]
so
\[
(14.2) \quad B = \left( \begin{pmatrix} Q(1) & Q(-1) \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} Q(1) & Q(-1) \end{pmatrix} \right)^{-1},
\]
where \( p \) was introduced in (13.6). We want to see for which values of \( s \) the last matrix inverse is well defined.

\section*{§15. Solution of the RHP for \( Q(z) \) via iteration, \( \gamma > 1 \)}

Since
\[
\| G_Q - I \|_{L^2(\Sigma_R)} \leq c_s s^{-1}, \quad s \to \infty,
\]
we can solve the \( Q \)-RHP via iteration. This problem is equivalent to the singular integral equation
\[
(15.2) \quad Q_-(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} Q_-(w) (G_Q(w) - I) \frac{dw}{w - z},
\]
which can be solved via iteration in \( L^2(\Sigma_R) \), and its unique solution satisfies
\[
(15.3) \quad \| Q_- - I \|_{L^2(\Sigma_R)} \leq c_s s^{-1}, \quad s \to \infty.
\]
Combining the integral representation
\[
Q(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} Q_-(w) (G_Q(w) - I) \frac{dw}{w - z}, \quad z \notin \Sigma_R,
\]
with (15.1) and (15.3), we conclude that
\[
Q(\pm1) = I + O(s^{-1}), \quad s \to \infty.
\]
Hence, the matrix inverse on the right-hand side of (14.2) exists for all sufficiently large \( s \) lying outside of the zero set of the function
\[
1 - p^2,
\]
which consists of the points \( \{s_n\} \) determined by the equation
\[
\frac{8}{3} s_n^3 + 2x s_n + \frac{1}{\pi} \ln(\gamma - 1) \ln (16s^3 + 4xs) - \arg \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} = \frac{\pi}{2} + n\pi, \quad n = 1, 2, \ldots,
\]
and which will eventually form the zeros of the Fredholm determinant as written in Theorem 1.19. From now on, when dealing with the situation where \( \gamma > 1 \), we shall assume that \( s \) stays away from the small neighborhood of the points \( s_n \). We have gathered enough information to prove Theorem 1 and 2.
§16. Asymptotics of \( \ln \det (I - \gamma K_{\text{sin}}) \) — Preliminary Steps

In order to prove the stated theorems, we shall use Propositions 1, 2, and 3 which, in particular, requires us to relate \( \tilde{X}(\pm s) \) and \( \tilde{X}'(\pm s) \) to the solution of either the R-RHP or the Q-RHP, see (16.1) and (16.15). For this, we recall (9.6), (11.1), and Figure 2 which implies for \(|z - 1| < r_1\) that

\[
(16.1) \quad R(z)V(z)L^{-1}(z)e^{s\theta(z)}\sigma_3 = \tilde{X}(zs) \left[ I + \frac{\gamma}{2\pi i} \left( \begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right) \ln \left( \frac{z-1}{z+1} \right) \right].
\]

On the other hand, for \(|z + 1| < r_1\) we have

\[
(16.2) \quad R(z)W(z)L^{-1}(z)e^{s\theta(z)}\sigma_3 = \tilde{X}(zs) \left[ I + \frac{\gamma}{2\pi i} \left( \begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right) \ln \left( \frac{z-1}{z+1} \right) \right].
\]

This shows that the required values of \( \tilde{X}(\pm s) \) and \( \tilde{X}'(\pm s) \) can be determined via comparison in (16.1) and (16.2) once we know the local expansions for \( V(z) \) (respectively, \( W(z) \)) at \( z = \pm 1 \). Our starting point is (9.2),

\[
(16.3) \quad P_{CH}(\zeta) = \left[ \begin{array}{cc} d_1(\zeta, \nu)e^{2\pi i\nu} & -d_2(\zeta, 1-\nu)e^{\pi i\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ -d_1(\zeta, 1+\nu)e^{\pi i\nu} \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & d_2(\zeta, -\nu) \end{array} \right] + \zeta \left( \begin{array}{cc} d_3(\zeta, \nu)e^{2\pi i\nu} & -d_4(\zeta, 1-\nu)e^{\pi i\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ -d_3(\zeta, 1+\nu)e^{\pi i\nu} \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & d_4(\zeta, -\nu) \end{array} \right) + O(\zeta^2 \ln \zeta) e^{-i\zeta^2 (\frac{1}{2}-\nu)}\sigma_3, 
\]

\( \zeta \to 0 \),

with (recall (9.2))

\[
d_1(\zeta, \nu) = c_0(\nu) + c_1(\nu) \left( \ln \zeta + i\frac{\pi}{2} \right), \quad d_2(\zeta, \nu) = c_0(\nu) + c_1(\nu) \left( \ln \zeta - i\frac{\pi}{2} \right)
\]

and

\[
d_3(\zeta, \nu) = -\frac{i}{2} d_1(\zeta, \nu) + i \left( c_2(\nu) + c_3(\nu) \left( \ln \zeta + i\frac{\pi}{2} \right) \right)
\]

as well as

\[
d_4(\zeta, \nu) = \frac{i}{2} d_2(\zeta, \nu) - i \left( c_2(\nu) + c_3(\nu) \left( \ln \zeta - i\frac{\pi}{2} \right) \right).
\]

At this point we use the changes of variables

\[
\zeta = \zeta(z) = -2is^3(\vartheta(z) - \vartheta(1)), \quad |z - 1| < r_1, \quad \lambda = zs,
\]

and recall (9.6) to deduce from (16.3) that, for \(-\frac{\pi}{3} < \arg(\lambda - s) < \frac{\pi}{3}\),

\[
P_{CH}^{RH}(\zeta(z)) = \left[ P_1(\ln(\lambda - s)) + (\lambda - s)P_2(\ln(\lambda - s)) + O((\lambda - s)^2 \ln(\lambda - s)) \right]
\times \left( \begin{array}{cc} e^{-i\frac{\pi}{2}\nu} & 0 \\ 0 & e^{i\frac{\pi}{2}\nu} \end{array} \right), \quad \lambda \to s,
\]

where the matrix functions \( P_1(\lambda) = (P_1^{ij}(\lambda)) \) and \( P_2(\lambda) = (P_2^{ij}(\lambda)) \) can be determined from (16.3). For the remaining sectors \(-\pi < \arg(\lambda - s) < -\frac{\pi}{3} \) and \( \frac{\pi}{3} < \arg(\lambda - s) < \pi \) we can derive similar expansions, they differ from the last one only by multiplication by a triangular matrix. Now we combine the last expansion with (9.7) and (11.1), so that,
as \( \lambda \to s \) in the sector \(-\pi/3 < \arg(\lambda - s) < \pi/3\), the left-hand side of (16.1) reads as

\[
R(z)V(z)L^{-1}(z)e^{s^3\vartheta(s)\sigma_3}\bigg|_{z=\lambda} = R(\lambda)\sigma_1 e^{-iz\pi/3} B_r(z) e^{i\pi/4} (1-\nu)\sigma_3 e^{-s^3\vartheta(1)\sigma_3} P_{C_H}^R(\zeta(z)) e^{i\pi/4} \sigma_3 \sigma_1
\]

\[
= (1)\sigma_1 e^{-i\pi/3} B_r(1) e^{i\pi/4} (1-\nu)\sigma_3 e^{-s^3\vartheta(1)\sigma_3} P_1(\ln(\lambda - s))(e^{-i\pi/4} 0 e^{i\pi/4}) e^{i\pi/4} \sigma_3 \sigma_1
\]

\[
+ (\lambda - s) \left( R'(1)\sigma_1 e^{-i\pi/3} B_r(1) + R(1)\sigma_1 e^{-i\pi/3} B_r'(1) \right)
\]

\[
\times e^{i\pi/4} (1-\nu)\sigma_3 e^{-s^3\vartheta(1)\sigma_3} \frac{1}{s} P_1(\ln(\lambda - s))(e^{-i\pi/4} 0 e^{i\pi/4}) e^{i\pi/4} \sigma_3 \sigma_1
\]

\[
+ R(1)\sigma_1 e^{-i\pi/3} B_r(1) e^{i\pi/4} (1-\nu)\sigma_3 e^{-s^3\vartheta(1)\sigma_3} P_2(\ln(\lambda - s))(e^{-i\pi/4} 0 e^{i\pi/4}) e^{i\pi/4} \sigma_3 \sigma_1
\]

\[+ O((\lambda - s)^2 \ln(\lambda - s)).\]

In the other sectors \(-\pi < \arg(\lambda - s) < -\pi/3\) and \(\pi < \arg(\lambda - s) < \pi\), we can derive similar identities, they differ from the last one only by multiplication by a triangular matrix, see [9.5]. The right-hand side in (16.1) implies that, in the sector \(-\pi/3 < \arg(\lambda - s) < \pi/3\), we have

\[
R(z)V(z)L^{-1}(z)e^{s^3\vartheta(s)\sigma_3}\bigg|_{z=\lambda} = \tilde{X}(s)\left[I + \frac{\gamma}{2\pi i} \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right) \ln\left(\frac{\lambda - s}{\lambda + s}\right)\right]
\]

\[
= \left(\tilde{X}(s) + (\lambda - s)\tilde{X}^t(s) + O((\lambda - s)^2)\right)\left[I + \frac{\gamma}{2\pi i} \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right) \ln\left(\frac{\lambda - s}{\lambda + s}\right)\right]
\]

with constants \(\alpha_{ij}\) depending on \(\tilde{X}_{ij}(s)\), and \(\beta_{ij}\) involving both \(\tilde{X}_{ij}(s)\) and \(\tilde{X}^t_{ij}(s)\). Comparing now the left-hand side and right-hand side in (16.1), we deduce that

\[
\tilde{X}_{11}(s) = [R(1)(B_r(1))^{-1}]_{11} e^{i\pi/4} e^{s^3\vartheta(1)} \left(c_0(-\nu) + c_1(-\nu) \left(\ln(16s^3 + 4xs) - i\pi/2\right)\right)
\]

\[
- [R(1)(B_r(1))^{-1}]_{12} e^{i\pi/4} e^{s^3\vartheta(1)} \left(c_0(1-\nu) \frac{\Gamma(1-\nu)}{\Gamma(\nu)} + c_1(\nu) \left(\ln(16s^3 + 4xs) - i\pi/2\right)\right),
\]

\[
\tilde{X}_{12}(s) = -[R(1)(B_r(1))^{-1}]_{11} e^{i\pi/4} e^{s^3\vartheta(1)} \left(c_0(1+\nu) \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} + c_1(\nu) \left(\ln(16s^3 + 4xs) + i\pi/2\right)\right)
\]

\[
+ [R(1)(B_r(1))^{-1}]_{12} e^{i\pi/4} e^{-s^3\vartheta(1)} \left(c_0(\nu) + c_1(\nu) \left(\ln(16s^3 + 4xs) + i\pi/2\right)\right).
\]

Hence, in particular,

\[
\tilde{X}_{11}(s) + \tilde{X}_{12}(s) = \frac{2\pi i}{\gamma} e^{i\pi/4} \left([R(1)(B_r(1))^{-1}]_{11} e^{s^3\vartheta(1)} - [R(1)(B_r(1))^{-1}]_{12} e^{-s^3\vartheta(1)}\right).
\]
Furthermore,
\[ \tilde{X}_{21}(s) = [R(1)(B_r(1))^{-1}]_{21} e^{i \frac{\pi}{2} \nu} e^{s \vartheta(1)} (c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i \frac{\pi}{2} \right)) \]
\[ - [R(1)(B_r(1))^{-1}]_{22} e^{i \frac{\pi}{2} \nu} e^{-s \vartheta(1)} (c_0(1 - \nu) \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} + c_1(\nu) \left( \ln(16s^3 + 4xs) - i \frac{\pi}{2} \right)) \]
and
\[ \tilde{X}_{22}(s) = - [R(1)(B_r(1))^{-1}]_{21} e^{i \frac{\pi}{2} \nu} e^{s \vartheta(1)} (c_0(1 + \nu) \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i \frac{\pi}{2} \right)) \]
\[ + [R(1)(B_r(1))^{-1}]_{22} e^{i \frac{\pi}{2} \nu} e^{-s \vartheta(1)} (c_0(\nu) + c_1(\nu) \left( \ln(16s^3 + 4xs) + i \frac{\pi}{2} \right)) \]
which implies
\[ \tilde{X}_{21}(s) + \tilde{X}_{22}(s) = \frac{2\pi i}{\gamma} e^{i \frac{\pi}{2} \nu} \left( [R(1)(B_r(1))^{-1}]_{21} e^{s \vartheta(1)} \right) \left[ R(1)(B_r(1))^{-1} \right]_{21} \Gamma(-\nu) - \left[ R(1)(B_r(1))^{-1} \right]_{22} \Gamma(\nu) \right) \]
Comparing after that terms of order \( O(\lambda - s) \ln(\lambda - s) \), we obtain
\[ \tilde{X}_{11}(s) + \tilde{X}_{12}(s) = \frac{2\pi i}{\gamma} e^{i \frac{\pi}{2} \nu} \left( \left[ R'(1) - R(1) \frac{\nu \sigma_3}{2} \frac{1}{1 + \frac{x}{4s^2}} \right] (B_r(1))^{-1} \right) \left[ R(1)(B_r(1))^{-1} \right]_{11} \Gamma(-\nu) - \left[ R(1)(B_r(1))^{-1} \right]_{12} \Gamma(\nu) \right) \]
and
\[ \tilde{X}_{21}(s) + \tilde{X}_{22}(s) = \frac{2\pi i}{\gamma} e^{i \frac{\pi}{2} \nu} \left( \left[ R'(1) - R(1) \frac{\nu \sigma_3}{2} \frac{1}{1 + \frac{x}{4s^2}} \right] (B_r(1))^{-1} \right) \left[ R(1)(B_r(1))^{-1} \right]_{21} \Gamma(-\nu) - \left[ R(1)(B_r(1))^{-1} \right]_{22} \Gamma(\nu) \right) \]
\[ + (8s^2 + 2x) \left[ R(1)(B_r(1))^{-1} \right]_{11} \left( \frac{i}{2} + i\nu \right) \Gamma(\nu) \right) \]
Although we derived the previous identities from a comparison in the sector \(-\frac{\pi}{3} < \arg(\lambda - s) < \frac{\pi}{3}\), the same identities are also valid in the other two sectors. There, the correct triangular matrices in (9.33) should be used on the left-hand side. Thus, all identities derived previously follow in fact from comparison in a full neighborhood of \( \lambda = +s \). When we move on to a neighborhood of \( \lambda = -s \), a completely similar analysis provides us with
\[ \tilde{X}_{11}(-s) + \tilde{X}_{12}(-s) \]
\[ = \frac{2\pi i}{\gamma} e^{i \frac{\pi}{2} \nu} \left[ R(-1)(B_l(-1))^{-1} \right]_{12} \frac{e^{-s \vartheta(-1)}}{\Gamma(-\nu)} - \left[ R(-1)(B_l(-1))^{-1} \right]_{11} \frac{e^{s \vartheta(-1)}}{\Gamma(\nu)} \]
and
\[ \tilde{X}_{21}(-s) + \tilde{X}_{22}(-s) = \frac{2\pi i}{\gamma} e^{i\pi \nu} \left( [R(-1)(B_{1}(-1))^{-1}]^{22} \frac{e^{-s^2 \vartheta(-1)}}{\Gamma(-\nu)} - [R(-1)(B_{1}(-1))^{-1}]^{21} \frac{e^{s^2 \vartheta(-1)}}{\Gamma(\nu)} \right). \]

Also, comparing terms of order \( O((\lambda + s) \ln(\lambda + s)) \), we obtain
\[ \tilde{X}'_{11}(-s) + \tilde{X}'_{12}(-s) = \frac{2\pi i}{\gamma} e^{i\pi \nu} \left( [R'(-1) - R(-1) \frac{\nu \sigma_3 + \frac{x}{2} + \frac{x^2}{4s^2}}{1 + \frac{x^2}{4s^2}}]^{22} \frac{e^{-s^2 \vartheta(-1)}}{\Gamma(-\nu)} \right) \]
\[ - \left[ [R'(-1) - R(-1) \frac{\nu \sigma_3 + \frac{x}{2} + \frac{x^2}{4s^2}}{1 + \frac{x^2}{4s^2}}]^{21} \frac{e^{s^2 \vartheta(-1)}}{\Gamma(\nu)} \right]_{12} \left( \frac{3i}{2} + i\nu \right) e^{-s^2 \vartheta(-1)} \Gamma(\nu) \]
\[ + (8s^2 + 2x) \left[ [R(-1)(B_{1}(-1))^{-1}]^{11} \frac{e^{s^2 \vartheta(-1)}}{\Gamma(\nu)} \right] \]
\[ - \left[ R(-1)(B_{1}(-1))^{-1} \right]_{12} \left( \frac{3i}{2} + i\nu \right) e^{-s^2 \vartheta(-1)} \Gamma(-\nu) \} \]
and
\[ \tilde{X}'_{21}(-s) + \tilde{X}'_{22}(-s) = \frac{2\pi i}{\gamma} e^{i\pi \nu} \left( [R'(-1) - R(-1) \frac{\nu \sigma_3 + \frac{x}{2} + \frac{x^2}{4s^2}}{1 + \frac{x^2}{4s^2}}]^{22} \frac{e^{-s^2 \vartheta(-1)}}{\Gamma(-\nu)} \right) \]
\[ - \left[ [R'(-1) - R(-1) \frac{\nu \sigma_3 + \frac{x}{2} + \frac{x^2}{4s^2}}{1 + \frac{x^2}{4s^2}}]^{21} \frac{e^{s^2 \vartheta(-1)}}{\Gamma(\nu)} \right]_{22} \left( \frac{3i}{2} + i\nu \right) e^{-s^2 \vartheta(-1)} \Gamma(-\nu) \} \]
We finish this section with evaluating the resolvent kernel at \( \lambda = \pm s \).

Recall (4.3) and deduce
\[ F_{1}(\pm s) = \sqrt{\frac{\gamma}{2\pi i}} (\tilde{X}_{11}(\pm s) + \tilde{X}_{12}(\pm s)), \quad F_{2}(\pm s) = \sqrt{\frac{\gamma}{2\pi i}} (\tilde{X}_{21}(\pm s) + \tilde{X}_{22}(\pm s)), \]
\[ F'_{1}(\pm s) = \sqrt{\frac{\gamma}{2\pi i}} (\tilde{X}'_{11}(\pm s) + \tilde{X}'_{12}(\pm s)), \quad F'_{2}(\pm s) = \sqrt{\frac{\gamma}{2\pi i}} (\tilde{X}'_{21}(\pm s) + \tilde{X}'_{22}(\pm s)). \]
Since
\[ R(s, s) = F'_{1}(s)F_{2}(s) - F'_{2}(s)F_{1}(s), \]
we use the identities derived previously and obtain
\[ R(s, s) = \frac{2\pi i}{\gamma} e^{i\pi \nu} \left( [R'_{11}(1)R_{21}(1) - R'_{12}(1)R_{11}(1)] (16s^3 + 4xs) \right)^{-2\nu} \frac{e^{2s^2 \vartheta(1)}}{\Gamma^2(-\nu)} \]
\[ + [R'_{12}(1)R_{22}(1) - R'_{22}(1)R_{12}(1)] (16s^3 + 4xs)^{2\nu} \frac{e^{-2s^2 \vartheta(1)}}{\Gamma^2(\nu)} \]
\[ - [R'_{11}(1)R_{12}(1)R_{22}(1) + R_{12}(1)R_{21}(1) - R'_{21}(1)R_{12}(1)] \frac{3 + \frac{x}{4s^2}}{1 + \frac{x^2}{4s^2}} \frac{1}{s\Gamma(\nu)\Gamma(-\nu)} \]
\[ - [R_{11}(1)R_{22}(1) - R_{21}(1)R_{12}(1)]i \frac{8s^2 + 2x}{\Gamma(\nu)\Gamma(-\nu)}. \]
Proposition 5. \( R(z) \) is unimodular for any \( x, \gamma \in \mathbb{R} \), i.e., \( \det R(z) \equiv 1 \).

Proof. Since \( \det P^R_{CH} (\zeta) \equiv 1 \), we have \( \det U(z) = 1 \). Similarly, \( \det P^R_{CH} (\zeta) = 1 = \det P^R_{CH} (\zeta) \), leading to \( \det V(z) = 1 = \det W(z) \). Thus, the ratio RHP has a unimodular jump matrix \( G_R(z) \), which means that the function \( \det R(z) \) is entire. By normalization at infinity, we end up with

\[
\det R(z) = R_{11}(z)R_{22}(z) - R_{12}(z)R_{21}(z) \equiv 1, \quad z \in \mathbb{C}.
\]

In the light of the last proposition, we obtain

\[
R(s, \bar{s}) = -i e^{i\pi \nu} \frac{2\pi i}{\gamma} \frac{8s^2 + 2x}{\Gamma(\nu)\Gamma(-\nu)} + \frac{2\pi i}{\gamma} \frac{\nu e^{i\pi \nu}}{s\Gamma(\nu)\Gamma(-\nu)} \frac{3 + \frac{x}{4\pi}}{1 + \frac{x}{4\pi}} - \frac{2\pi i}{\gamma} \frac{e^{i\pi \nu}}{\Gamma(\nu)\Gamma(-\nu)}
\]

\[
\times [R'_{11}(1)R_{22}(1) - R'_{22}(1)R_{11}(1) + R'_{12}(1)R_{21}(1) - R'_{21}(1)R_{12}(1)]
\]

\[
+ \frac{2\pi i}{\gamma} e^{i\pi \nu} \left( [R'_{11}(1)R_{22}(1) - R'_{22}(1)R_{11}(1)] (16s^3 + 4xs)^{2\nu} e^{2s^3\vartheta(1)} s^{-2\gamma/2} \right)
\]

\[
+ [R'_{12}(1)R_{22}(1) - R'_{22}(1)R_{12}(1)] (16s^3 + 4xs)^{-2\nu} e^{-2s^3\vartheta(1)} s^{1/2} \right).
\]

Similarly, using Proposition 5 once again, we get

\[
R(-s, -s) = -i e^{i\pi \nu} \frac{2\pi i}{\gamma} \frac{8s^2 + 2x}{\Gamma(\nu)\Gamma(-\nu)} + \frac{2\pi i}{\gamma} \frac{\nu e^{i\pi \nu}}{s\Gamma(\nu)\Gamma(-\nu)} \frac{3 + \frac{x}{4\pi}}{1 + \frac{x}{4\pi}} - \frac{2\pi i}{\gamma} \frac{e^{i\pi \nu}}{\Gamma(\nu)\Gamma(-\nu)}
\]

\[
\times [R'_{11}(-1)R_{22}(-1) - R'_{22}(-1)R_{11}(-1) + R'_{12}(-1)R_{21}(-1) - R'_{21}(-1)R_{12}(-1)]
\]

\[
+ \frac{2\pi i}{\gamma} e^{i\pi \nu} \left( [R'_{11}(-1)R_{22}(-1) - R'_{22}(-1)R_{11}(-1)] (16s^3 + 4xs)^{2\nu} e^{2s^3\vartheta(1)} s^{-2\gamma/2} \right)
\]

\[
+ [R'_{12}(-1)R_{22}(-1) - R'_{22}(-1)R_{12}(-1)] (16s^3 + 4xs)^{-2\nu} e^{-2s^3\vartheta(1)} s^{1/2} \right).
\]

§17. ASYMPTOTICS OF \( \ln \det (I - \gamma K_{\text{csin}}) \) — PROOF OF THEOREM 4

The asymptotics (1.15) is a direct consequence of Proposition 3. Since the relevant estimates (12.2) and (12.3) were uniform on any compact subset of the set (1.16), the asymptotic series for

\[
\frac{\partial}{\partial \gamma} \ln \det (I - \gamma K_{\text{csin}})
\]

can be integrated with respect to \( \gamma \), which leads to (1.15) including the constant term \( \chi \).

We trace back the transformations

\[
X(\lambda) \mapsto T(z) \mapsto S(z) \mapsto R(z)
\]

and deduce the first asymptotic expansions for the coefficients \( X_1, X_2, \) and \( X_3 \) in (1.15). First,

\[
X_1 = \lim_{\lambda \to \infty} \left( \lambda \left( X(\lambda) e^{-i(\frac{x}{2} + y) \lambda^3 - I} \right) \right)
\]

\[
= -2\nu \sigma_3 s + \frac{is}{2\pi} \int_{\Sigma_R} R_-(w)(G_R(w) - I) \, dw \]

\[
= -2\nu \sigma_3 s + \frac{is}{2\pi} \int_{\Sigma_R} (R_-(w) - I)(G_R(w) - I) \, dw + \frac{is}{2\pi} \int_{\Sigma_R} (G_R(w) - I) \, dw
\]
and for \( z \in \Sigma_R \) from (12.1), (12.3), and (8.9) we obtain
\[
R_-(z) - I = \frac{1}{2\pi i} \int_{C_0} (G_R(w) - I) \frac{dw}{w-z} + O(s^{-2})
\]
\[
= \frac{1}{2\pi i} \int_{C_0} (B_0(w))^{-1} \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} \begin{pmatrix} -2\pi iv \\ -v \end{pmatrix} B_0(w) \frac{idw}{2sw(w-z)} + O(s^{-2})
\]
\[
= \frac{i}{2sz} \left[ \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} - (B_0(z))^{-1} \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} B_0(z) \right] + O(s^{-2}).
\]

The last expansion will be improved via iteration,
\[
R_-(z) - I = \frac{1}{2\pi i} \int_{C_0} (R_-(w) - I) (G_R(w) - I) \frac{dw}{w-z} 
+ \frac{1}{2\pi i} \int_{C_0} (G_R(w) - I) \frac{dw}{w-z} + O(s^{-3}), \quad s \to \infty.
\]

If \([\cdot, \cdot]\) denotes the usual matrix commutator on \( \mathbb{C}^{2 \times 2} \), (8.8) implies that, for any constant matrix \( \Lambda \in \mathbb{C}^{2 \times 2} \),
\[
(B_0(0))^{-1} \Lambda B_0(0) = (B_0(0))^{-1} \Lambda B_0(0) + 2\nu w \left[ (B_0(0))^{-1} \Lambda B_0(0), \sigma \right]
+ 4\nu^2 w^2 \left[ (B_0(0))^{-1} \Lambda B_0(0), \sigma \right] \sigma + O(w^3), \quad w \to 0.
\]

In its turn, this leads to
\[
R_-(z) - I = \frac{i}{2sz} \left[ \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} - (B_0(z))^{-1} \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} B_0(z) \right]
- \frac{1}{4s^2z^2} \left[ \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} - (B_0(z))^{-1} \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} B_0(z) \right]
+ \frac{1}{8s^2z^2} \left[ \begin{pmatrix} u^2 - v^2 & 2(u_x + uv) \\ 2(u_x + uv) & u^2 - v^2 \end{pmatrix} - (B_0(z))^{-1} \begin{pmatrix} u^2 - v^2 & 2(u_x + uv) \\ 2(u_x + uv) & u^2 - v^2 \end{pmatrix} B_0(z) \right]
+ \frac{\nu}{s^2z} \begin{pmatrix} u^2 & -u_x \\ u_x & -u^2 \end{pmatrix} + O(s^{-3}), \quad s \to \infty,
\]
for any \( z \in \Sigma_R \). Back to (17.1), one starts with
\[
I_1 = \int_{\Sigma_R} (R_-(w) - I) (G_R(w) - I) \, dw = \frac{2\pi i\nu}{s^2} \begin{pmatrix} u^2 & -uv \\ u & u^2 \end{pmatrix} + \frac{i(-2\pi i)\nu^2}{s^3} \begin{pmatrix} 2u & -2u^2 + v(u_x + uv) + \frac{u}{2}(u^2 - v^2) \\ 2u^3 - v(u_x + uv) + \frac{u}{2}(u^2 - v^2) - 2u^2 & -2uv \end{pmatrix} + O(s^{-4}),
\]
\[
\quad s \to \infty,
\]
and moves on to
\[
I_2 = \int_{\Sigma_R} (G_R(w) - I) \, dw = \frac{(-2\pi i)i}{2s} \begin{pmatrix} v & \sigma \\ -u & -v \end{pmatrix} + \frac{(-2\pi i)\nu}{s^2} \begin{pmatrix} 0 & -(u_x + uv) \\ u_x + uv & 0 \end{pmatrix}
+ \frac{i(-2\pi i)\nu^2}{s^3} \begin{pmatrix} \frac{u}{2} & 0 \\ 0 & \frac{u}{2} \end{pmatrix} \begin{pmatrix} (u^2 + v^2) + vuvx + xu \\ 0 \end{pmatrix}
+ \frac{(-2\pi i)i}{4s^3 + xs} \begin{pmatrix} -\nu^2 & \sqrt{-\nu^2 \cos \sigma} \\ \sqrt{-\nu^2 \cos \sigma} & \nu^2 \end{pmatrix} + O(s^{-4})
\]
with
\[
\sigma = \sigma(s, x, \gamma) = \frac{8}{3} s^3 + 2xs + \frac{\ln|1 - \gamma|}{\pi} \ln(16s^3 + 4xs) - \arg\left(\frac{\Gamma(1 - \nu)}{\Gamma(\nu)}\right).
\]
Together,
\[
I_1 + I_2 = \frac{(-2\pi i)i}{2s} \begin{pmatrix} v & u \\ -u & -v \end{pmatrix} + \frac{(-2\pi i)\nu}{s^2} \begin{pmatrix} u^2 & -u_x \\ u_x & -u^2 \end{pmatrix} + \frac{(-2\pi i)i\nu^2}{s^3} \begin{pmatrix} 2wu_x & -u_{xx} \\ u_{xx} & -2uu_x \end{pmatrix}
\]
\[+ \frac{(-2\pi i)i}{4s^3 + xs} \begin{pmatrix} -\nu^2 & \sqrt{-\nu^2 \cos \sigma} \\ -\sqrt{-\nu^2 \cos \sigma} & \nu^2 \end{pmatrix} + O(s^{-4}),
\]
where we have used the fact that \( u = u(x, \gamma) \) solves the second Painlevé equation. The previous line leads us to the following expansion for \( X_1 \):
\[
X_1 = -2\nu \sigma_3 s + i \frac{1}{2} \left( \frac{v}{-u} - \nu \right) + \frac{\nu}{s} \left( \begin{array}{c}
u \left( u^2 - v^2 \right) \\ 2u^2v + uu_x \end{array} \right) + \frac{i\nu^2}{s^2} \left( \begin{array}{c} 2wu_x \\ 2uu_x \end{array} \right) + O(s^{-4}).
\]
(17.3)

Second,
\[
X_2 = 2\nu^2 s^2 I - \frac{i\nu s^2}{\pi} (I_1 + I_2) \sigma_3 + \frac{is^2}{2\pi} \int_{\Sigma_R} R_-(w)(G_R(w) - I) w \, dw.
\]
(17.4)

We need to compute
\[
I_3 = \int_{\Sigma_R} (R_-(w) - I)(G_R(w) - I) w \, dw = \frac{(-2\pi i)i\nu}{2s^3} \begin{pmatrix} u^2v + uu_x \\ \frac{u}{2}(u^2 - v^2) \\ u^2v + uu_x \end{pmatrix} + O(s^{-4})
\]
as well as
\[
I_4 = \int_{\Sigma_R} (G_R(w) - I) w \, dw = \frac{(-2\pi i)i\nu}{8s^2} \begin{pmatrix} u^2 - v^2 & 2(u_x + uv) \\ 2(u_x + uv) & u^2 - v^2 \end{pmatrix} + \frac{(-2\pi i)i\nu}{4s^3} \begin{pmatrix} 0 \\ u(u^2 + v^2) + 2(vu_x + xu) \\ u^2(u^2 + v^2) + 2(vu_x + xu) \end{pmatrix} + \frac{2\pi i}{4s^3 + xs} \begin{pmatrix} 0 \\ -\sqrt{-\nu^2 \sin \sigma} \\ 0 \end{pmatrix} + O(s^{-4}),
\]
to obtain
\[
I_3 + I_4 = \frac{(-2\pi i)i\nu}{8s^2} \begin{pmatrix} 2(u_x + uv) & u^2 - v^2 \\ u^2 - v^2 & 2(u_x + uv) \end{pmatrix} + \frac{2\pi i}{4s^3 + xs} \begin{pmatrix} 0 \\ -\sqrt{-\nu^2 \sin \sigma} \\ 0 \end{pmatrix} + O(s^{-4}), \quad s \to \infty.
\]

Adding together in (17.4), we arrive at
\[
X_2 = 2\nu^2 s^2 I - \frac{i\nu s^2}{\pi} (I_1 + I_2) \sigma_3 + \frac{is^2}{2\pi} \int_{\Sigma_R} R_-(w)(G_R(w) - I) w \, dw
\]
(17.5)
\[
\begin{aligned}
&+ \frac{1}{8} \begin{pmatrix} u^2 - v^2 & 2(u_x + uv) \\ 2(u_x + uv) & u^2 - v^2 \end{pmatrix} - \frac{2i\nu^3}{s} \begin{pmatrix} 2uu_x & xu + 2u^3 \\ xu + 2u^3 & 2uu_x \end{pmatrix} \\
&+ \frac{i\nu}{2s} \begin{pmatrix} u^2v + uu_x \\ u^3 + uv_x + xu \end{pmatrix} + \frac{2i\nu s}{4s^2 + x} \begin{pmatrix} \nu^2 & \sqrt{-\nu^2 \cos \sigma} \\ \sqrt{-\nu^2 \cos \sigma} & \nu^2 \end{pmatrix} \\
&- \frac{s}{4s^2 + x} \begin{pmatrix} 0 \\ \sqrt{-\nu^2 \sin \sigma} \\ 0 \end{pmatrix} + O(s^{-2}), \quad s \to \infty.
\end{aligned}
\]
Finally, the computation of $X_3$ yields
\begin{equation}
X_3 = -\frac{2\nu}{3}s^3(1 + 2\nu^2)\sigma_3 + i\nu^2 s^3 \frac{1}{\pi} (I_1 + I_2) - \frac{i\nu s^3}{\pi} (I_3 + I_4) \sigma_3
\end{equation}
(17.6)
\[ + \frac{is^3}{2\pi} \int_{\Sigma_R} R_-(w) (G_R(w) - I) w^2 dw. \]

Since
\[ I_5 = \int_{\Sigma_R} (R_-(w) - I) (G_R(w) - I) w^2 dw = O(s^{-4}) \]
and
\[ I_6 = \int_{\Sigma_R} R_-(w) (G_R(w) - I) w^2 dw = \frac{(-2\pi i)i}{4s^3} \left( -\frac{\nu^2}{s} \frac{\sqrt{-\nu^2 \cos \sigma}}{\nu^2} \right) \]
\[ + \frac{(-2\pi i)i}{48s^3} \left( (v^3 - 3\nu v^2 + 2(xv - uu_x)) -3(\nu^2 + 2(vu_x + xu)) \right) + O(s^{-4}), \]
we add together and obtain in (17.7), using the previous results,
\[ X_3 = -\frac{2\nu}{3}s^3(1 + 2\nu^2)\sigma_3 + i\nu^2 s^2 \left( \frac{v}{u} \frac{u}{-u} - \frac{u}{v} \right) + 2\nu^2 s \left( \frac{u}{u_x} - \frac{u}{u_x} \right) \]
\[ - \nu s \left( \left( \frac{u^2 - v^2}{4(u_x + uv)} -2(u_x + uv) \right) + 2iv^2 \left( \frac{2uu_x}{u_x} - \frac{u}{uu_x} \right) \right) \]
\[ + \frac{2iu\nu^2}{4s^2 + x} \left( \left( -\frac{\nu^2}{s} \frac{\sqrt{-\nu^2 \cos \sigma}}{\nu^2} \right) \right) \]
\[ + \frac{is^2}{4s^2 + x} \left( -\frac{\nu^2}{s} \frac{\sqrt{-\nu^2 \cos \sigma}}{\nu^2} \right) + 2\nu^2 s \left( \frac{0}{4s^2 + x} + \frac{-\nu^2}{s} \frac{\sqrt{-\nu^2 \sin \sigma}}{\nu^2} \right) \]
\[ - i\nu^2 \left( \frac{u^2 v + uu_x}{u^3} u + uu_x + xu \right) - \nu \left( \frac{u^3 + uu_x + xu}{u^2 v + uu_x} \right) \]
\[ + \frac{i}{48} \left( \left( \frac{3(v^3 - 3\nu v^2 + 2(xv - uu_x))}{3(u^2 + v^2) + 2(vu_x + xu)} \right) -3(\nu^2 + 2(vu_x + xu)) \right) + O(s^{-1}). \]

With the given information at hand, (4.9) and (4.10) lead to
\[ b = 2X_1^{12} = iu - \frac{2\nu}{s} \frac{2iu^2 uu_x}{s^2} + \frac{2i\sqrt{-\nu^2}}{4s^2 + x} \cos \sigma + O(s^{-3}) \]
(17.7)
as well as
\[ c = 2X_1^{21} = -b + O(s^{-3}), \quad s \to \infty, \]
(17.8)
and
\[ d = ix + 8iX_1^{12}X_1^{21} \]
\[ = ix + 2iu^2 - \frac{8\nu}{s} uu_x - \frac{8i\nu^2}{s^2} \left( \left( \nu^2 \right)^2 + xu^2 + 2u^4 \right) + \frac{8iu\sqrt{-\nu^2}}{4s^2 + x} \cos \sigma + O(s^{-3}). \]
(17.9)
Furthermore,
\[ e = 8i(X_1^{12}X_1^{22} - X_2^{12}) = -2iu_x + \frac{4\nu}{s} (xu + u^3) + \frac{8is\sqrt{-\nu^2}}{4s^2 + x} \sin \sigma + O(s^{-2}) \]
and
\[ f = -e + O(s^{-2}), \]
(17.10)
where we have used the following identities, see (17.3) and (17.5):
\[ X_1^{21} = -X_1^{12} + O(s^{-3}), \quad X_1^{11} = -X_1^{12} + O(s^{-3}), \quad X_2^{21} = X_2^{12} + O(s^{-2}), \quad s \to \infty. \]
(17.11)
We have now derived enough information to evaluate the first terms listed in Proposition 3. Put

\[ \mathcal{P}_1(s, x, \gamma) = -8i((X_3)_\gamma + (X_1)_\gamma (X_1^2 - X_2) - (X_2)_\gamma X_1)^{11} - 2d((X_1)_\gamma)^{11} \]
\[ + 4ib((X_2)_\gamma - (X_1)_\gamma X_1)^{21} - 4ic((X_2)_\gamma - (X_1)_\gamma X_1)^{12} - e((X_1)_\gamma)^{21} - f((X_1)_\gamma)^{12} \]

and notice that

\[ \left( \frac{\partial}{\partial \gamma} X_k \right)^{ij} = \frac{\partial}{\partial \gamma} (X_k^{ij}), \quad i, j, k = 1, 2. \]

Therefore, from (17.10) and (17.11) we obtain

\[-e((X_1)_\gamma)^{21} = -f((X_1)_\gamma)^{12} + O(s^{-2}), \]

and via (17.7) and (17.11),

\[ 4ib((X_2)_\gamma)^{21} = -4ic((X_2)_\gamma)^{12} + O\left( \frac{\ln s}{s^2} \right). \]

Since also

\[-4ib((X_1)_\gamma X_1)^{21} = -4ib((X_1^{21})_\gamma X_1^{11} + (X_1^{22})_\gamma X_1^{21}) = 4ic((X_1^{12})_\gamma X_1^{22} + (X_1^{11})_\gamma X_1^{12}) + O\left( \frac{\ln s}{s^2} \right) \]
\[ = 4ic((X_1)_\gamma X_1)^{12} + O\left( \frac{\ln s}{s^2} \right) \]

as \( s \to \infty \) uniformly on any compact subset of the set \( \{1, 16\} \), we can simplify the expression for \( \mathcal{P}_1(s, x, \gamma) \) asymptotically:

\[ \mathcal{P}_1(s, x, \gamma) = -8i((X_3)_\gamma + (X_1)_\gamma (X_1^2 - X_2) - (X_2)_\gamma X_1)^{11} - 2d((X_1)_\gamma)^{11} \]
\[ + 8ib((X_2)_\gamma - (X_1)_\gamma X_1)^{21} - 2e((X_1)_\gamma)^{21} + O\left( \frac{\ln s}{s^2} \right). \]

Next, from (17.3) we get

\[ X_1^2 = \left[ 4\nu^2 s^2 - 2i\nu s - 4\nu^2 u^2 + \frac{u^2 - v^2}{4} - \frac{8iu^3}{s} uu_x + \frac{4iv^3 s}{4s^2 + x} + \frac{i\nu}{s} (vu^2 + uu_x) \right] I + O(s^{-2}) \]

with \( I \) denoting the \( 2 \times 2 \) identity matrix. Thus,

\[ ((X_1)_\gamma X_1^{21})^{11} = \left[ -2\nu \gamma s + \frac{i}{2} \nu \gamma + \frac{(\nu u^2)}{s} \right] + \frac{i}{4s^2} \left( \nu^2 uu_x \right) - \frac{i}{4s^2 + x} \left( \nu^2 \gamma \right) + O\left( \frac{\ln s}{s^3} \right) \]
\[ \times \left[ 4\nu^2 s^2 - 2i\nu s - 4\nu^2 u^2 + \frac{u^2 - v^2}{4} - \frac{8iu^3}{s} uu_x + \frac{4iv^3 s}{4s^2 + x} + \frac{i\nu}{s} (vu^2 + uu_x) + O(s^{-2}) \right] \]

\[ = -8\nu^2 \nu s^3 + 4iv\nu s^2 + 2i\nu^2 s^2 s v + 8\nu^2 \nu s^2 s - \nu s^\gamma \left( \frac{u^2 - v^2}{2} + v s v \gamma \right) \]
\[ + 4\nu^2 (\nu u^2) + 16i\nu s^3 u u x - \frac{8iu^3 s^3}{4s^2 + x} - 2i\nu s^2 (vu^2 + uu_x) + 2i\nu^2 v \gamma \]
\[ + \frac{i}{2} \nu \gamma \left( 4v u^2 - 2i\nu (\nu v \nu)^2 \right) + 4i\nu^2 (2\nu u^2 u_x) - \frac{4i\nu^2 s^3}{4s^2 + x} \left( \nu^2 \gamma \right) + O\left( \frac{\ln s}{s} \right). \]

Moving on, we use

\[ ((X_1)_\gamma X_2 + (X_2)_\gamma X_1)^{11} = (X_1^{11} X_2^{11})_\gamma + (X_1^{12})_\gamma X_2^{21} + (X_2^{12})_\gamma X_1^{21} \]
and obtain
\[
((X_1)_\gamma X_2 + (X_2)_\gamma X_1)^{11} = \left[-4i^2 s^3 + 3i^2 s^2 v + 6i^2 u^2 s - 3i^2 u^2 v + 12i^4 u u_x \right]
- iv^2 u u_x - \frac{6i^4 s^2}{4 s^2 + x} - i s \frac{u^2 - v^2}{4} + \frac{i v u^2 - v^2}{2} + O(s^{-1})_{\gamma}
- \frac{1}{4} \left[ \left( \nu v - \nu u_x \right) - \frac{i v}{4} \right] - \frac{1}{4} \left( \nu v - \nu u_x \right) - \nu \left( \nu u_x \right) + O(s^{-1}),
\]

Furthermore,
\[
(X_1)_\gamma X_1 = 4\nu s^2 I - i \nu s \left( \begin{array}{ll} v & u \\ u & v \end{array} \right) - i \nu s \left( \begin{array}{ll} v \gamma & -u \gamma \\ -u \gamma & v \gamma \end{array} \right) - 2i \nu \left( \begin{array}{ll} u^2 & -u_x \\ -u_x & u^2 \end{array} \right)
- \frac{1}{4} \left( \nu v - \nu u_x \right) - 2i \nu \left( \nu u_x \right) + O(s^{-1}),
\]

and, thus,
\[
((X_2)_\gamma - (X_1)_\gamma X_1)^{21} = i s (\nu u)_\gamma + i \nu s u - i \nu s u_\gamma - 2i (\nu u^2)_\gamma + \frac{(u_x + u v)_\gamma}{4}
- 2i \nu u_x + \frac{1}{4} (\nu v - \nu u_x) + 2i (\nu u_x)_\gamma + O(s^{-1}),
\]

which implies
\[
b((X_2)_\gamma - (X_1)_\gamma X_1)^{21} = - s (\nu u)_\gamma - s^2 u_\gamma + s (\nu u)_\gamma - 2i (\nu u^2)_\gamma
+ \frac{i u (u_x + u v)}{4} - 2i \nu u_x + \frac{u v - v u_x}{4} + 2i \nu u (\nu u_x)_\gamma
- 2i \nu (\nu u)_\gamma - 2i \nu u_x + 2i \nu u_x u_\gamma + O(s^{-1}).
\]

Finally,
\[
d((X_1)_\gamma)^{11} = -2i \nu s x - 4i \nu u^2 s - \frac{sv_x}{2} - u^2 v_\gamma + 16i \nu u u_x + O(s^{-1})
\]

and
\[
e((X_1)_\gamma)^{21} = -u_x u_\gamma + O(s^{-1}).
\]

We write
\[
\mathcal{P}_1(s, x, \gamma) = s^3 \mathcal{P}_1^{(3)}(x, \gamma) + s^2 \mathcal{P}_1^{(2)}(x, \gamma) + s \mathcal{P}_1^{(1)}(x, \gamma) + \mathcal{P}_1^{(0)}(x, \gamma) + O\left(\frac{\ln s}{s}\right),
\]

where the \(\mathcal{P}_1^{(i)}(x, \gamma)\) are independent of \(s\). Since
\[
\nu|_{\gamma=0} = 0,
\]

from \textit{[17.12]} and the previous computations we get
\[
\int_0^\gamma \mathcal{P}_1^{(3)}(x, t) \, dt = \int_0^\gamma \left[-8i \left( -\frac{2\nu}{3} (1 + 2\nu^2) \right) \right] \, dt = \frac{16}{3} i^{\nu}.
\]

Next,
\[
\int_0^\gamma \mathcal{P}_1^{(2)}(x, t) \, dt = \int_0^\gamma \left[-8i (i^{\nu^2} v) - 8i (4i^{\nu} v + 2i^{\nu^2} v) + 8i (3i^{\nu^2} v) \right] \, dt = 0,
\]

\]
and
\[ \int_{\gamma}^{\gamma} P_1^{(1)}(x,t) \, dt = \int_{0}^{\gamma} \left[ -8i \left( 2t^3 u^2 - \frac{t}{4} (u^2 - v^2) \right) \right] \, dt \]
\[ - 8i \left( 8\nu^2 v_t u^2 - \nu t u^2 - v^2 + \nu \nu_v t + 4\nu^2 (\nu u^2)_t \right) \]
\[ + 8i \left( 6\nu^3 u^2 - \nu u^2 - 3v^2 \right)_t \]
\[ + 8i \left( -\frac{\nu}{2} u_{tt} + \frac{u}{2} (\nu u)_t \right) \]
\[ + 8i \left( iu (i(\nu u)_t + iv_t u - iu_v t) \right) - 2 \left( -2i\nu_t x - 4i\nu_t u^2 \right) \]
\[ dt = 4i\nu x. \]

We conclude the computations for \( P_1(s, x, \gamma) \) by evaluating
\[ \int_{\gamma}^{\gamma} P_1^{(0)}(x,t) \, dt = \int_{0}^{\gamma} \left[ -8i \left( 4i\nu^4 u_{xx} - \frac{2i\nu^4 s^2}{4s^2 + x} - \frac{is^2 \nu^2}{4s^2 + x} - i\nu^2 (u^2 v + uu_x) \right) \right] \, dt \]
\[ - \frac{i}{48} \left( v^3 - 3v^2 + 2(xv - uu_x) \right)_t \]
\[ - 8i \left( 16i\nu^3 v_t uu_x - \frac{8i\nu^3 v_s s^2}{4s^2 + x} - 2i\nu \nu_v (\nu u^2 + uu_x) \right) \]
\[ - 2i\nu^2 u_t v_t + iv_t \frac{u^2 - v^2}{8} - 2i\nu \nu_v (\nu u^2)_t + 8i \nu^2 (\nu u^2 u_x)_t - \frac{4i\nu^2 s^2}{4s^2 + x} (\nu^2)_t \]
\[ + 8i \left( -3i\nu^2 u^2 v + 12i\nu^4 uu_x - i\nu^2 uu_x - \frac{6i\nu^4 s^2}{4s^2 + x} + i\nu \frac{u^2 - v^2}{16} \right)_t \]
\[ + 8i \left( -i\nu^2 u_x u_t + iu \frac{u_x + uu}{8} - i\nu \nu_v (uu)_t + i\nu u_x (\nu u)_t + iu (\nu^2 u)_t - iu \frac{(u_x + uu)_t}{8} \right) \]
\[ - 2 \left( -\frac{xv}{2} - u^2 v_t + 16i\nu \nu_t uu_x \right) + 8i \left( -2i\nu (\nu^2 u)_t + iu \frac{(u_x + uu)_t}{4} - 2i\nu \nu_t uu_x \right) \]
\[ + iu \frac{uu_t - uu}{4} + 2i\nu \nu_v (uu)_t - 2i\nu u_x (\nu u)_t - 2i\nu \nu_t uu_x + 2i\nu^2 u_x u_t - 2 \left( -u_x u_t \right) \]
\[ dt = -2\nu^2 + \frac{2}{3} (xv - uu_x) + 2 \int_{0}^{\gamma} u_x u_t \, dt. \]

The following observation is useful.

**Proposition 6.** Let \( u = u(x, \gamma), \gamma < 1 \), denote the Ablowitz–Segur solution of the second Painlevé equation \( u_{xx} = xu + 2u^3 \) as described in Theorem \( \square \). Put
\[ F(x, \gamma) = \frac{2}{3} (xv(x, \gamma) - u(x, \gamma) u_x(x, \gamma)) + 2 \int_{0}^{\gamma} u_x(x, t) u_t(x, t) \, dt, \]
then
\[ F(x, \gamma) = -\int_{x}^{\infty} (y - x) u^2(y, \gamma) \, dy. \]

**Proof.** Using the differential equation for \( u \) and integration by parts, we obtain
\[ \frac{\partial}{\partial x} F(x, \gamma) = v(x, \gamma) = (u_x(x, \gamma))^2 - xu^2(x, \gamma) - u^4(x, \gamma) \]
and hence, after integration,
\[ F(x, \gamma) = -\int_{x}^{\infty} (y - x) u^2(y, \gamma) \, dy + C(\gamma) \]
with a constant \( C \) only depending on \( \gamma \). Since \( u \) decays exponentially fast as \( x \to +\infty \), the same limit on both sides of (17.13) gives us the claimed identity. \( \square \)

At this point we summarize the previous computations. As \( s \to \infty \), uniformly on any compact subset of \( \mathbb{R} \), we have
\[ \int_{0}^{\gamma} P_1(s, x, t) \, dt = iv \left( \frac{16}{3} s^3 + 4sx \right) - \int_{x}^{\infty} (y - x) u^2(y, \gamma) \, dy + 2(iv)^2 + O \left( \frac{\ln s}{s} \right). \]
To move further ahead, we define
\[ \mathcal{P}_2(s, x, \gamma) = (A_{11} - A_{22}) \hat{H}_{11}(s) + A_{12} \hat{H}_{21}(s) + A_{21} \hat{H}_{12}(s), \]
\[ \hat{H}(s) = \frac{\partial \tilde{X}}{\partial \gamma}(s) (\tilde{X}(s))^{-1} \]
with (compare (1.8))
\[ A = \frac{\gamma}{2\pi i} \tilde{X}(s) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (\tilde{X}(s))^{-1}. \]
Since
\[ A_{11} = -\frac{\gamma}{2\pi i} (\tilde{X}_{11}(s) + \tilde{X}_{12}(s))(\tilde{X}_{21}(s) + \tilde{X}_{22}(s)), \]
now we can use the identities derived in §16 for \( \tilde{X}_{ij}(s) \). With
\[ R(\pm 1) = I + \frac{1}{2\pi i} \int_{\Sigma_R} R_-(w)(G_R(w) - I) \frac{dw}{w+1} \]
\[ = I + \frac{1}{2\pi i} \int_{\Sigma_R} (G_R(w) - I) \frac{dw}{w+1} + O(s^{-2}) \]
(17.15)
and the classical identity
\[ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \in \mathbb{C} \setminus \mathbb{Z}, \]
we see that
\[ A_{11} = \nu + O(s^{-1}), \quad A_{22} = -\nu + O(s^{-1}) \]
uniformly on any compact subset of the set (1.16). Also,
(17.16)
\[ A_{12} = \nu(16s^3 + 4xs)^{-2\nu} e^{2s^3 \vartheta(1)} \frac{\Gamma(\nu)}{\Gamma(-\nu)} + O(s^{-1}) \]
and
(17.17)
\[ A_{21} = -\nu(16s^3 + 4xs)^{2\nu} e^{-2s^3 \vartheta(1)} \frac{\Gamma(-\nu)}{\Gamma(\nu)} + O(s^{-1}). \]
Next, we use (17.15) to simplify the identities for \( \tilde{X}_{ij}(s) \) obtained in §16
\[ \tilde{X}_{11}(s) = (16s^3 + 4xs)^{-\nu} e^{i \frac{\pi}{3} \nu} e^{s^3 \vartheta(1)} \left( c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i \frac{\pi}{2} \right) \right) + O\left( \frac{\ln s}{s} \right), \]
\[ \tilde{X}_{12}(s) = -(16s^3 + 4xs)^{-\nu} e^{i \frac{\pi}{3} \nu} e^{s^3 \vartheta(1)} \times \left( c_0(1+\nu) \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i \frac{\pi}{2} \right) \right) + O\left( \frac{\ln s}{s} \right), \]
\[ \tilde{X}_{21}(s) = -(16s^3 + 4xs)^{\nu} e^{i \frac{\pi}{3} \nu} e^{-s^3 \vartheta(1)} \times \left( c_0(1-\nu) \frac{\Gamma(1-\nu)}{\Gamma(\nu)} + c_1(\nu) \left( \ln(16s^3 + 4xs) - i \frac{\pi}{2} \right) \right) + O\left( \frac{\ln s}{s} \right), \]
\[ \tilde{X}_{22}(s) = (16s^3 + 4xs)^{\nu} e^{i \frac{\pi}{3} \nu} e^{-s^3 \vartheta(1)} \left( c_0(\nu) + c_1(\nu) \left( \ln(16s^3 + 4xs) + i \frac{\pi}{2} \right) \right) + O\left( \frac{\ln s}{s} \right). \]
Combining with (17.16) and (17.17), we deduce the following asymptotics for \( \mathcal{P}_2(s, x, \gamma) \):
\[ \mathcal{P}_2(s, x, \gamma) = 2\nu e^{i\nu} \left( c_0(\nu) + c_1(\nu) \left( \ln(16s^3 + 4xs) + i \frac{\pi}{2} \right) \right) \times \left[ \nu \left( i \frac{\pi}{2} - \ln(16s^3 + 4xs) \right) \left( c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i \frac{\pi}{2} \right) \right) \right]. \]
\[
(\c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \bigg] \\
- 2\nu e^{i\pi\nu} \left( c_0(1 - \nu) \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} + c_1(\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \\
\times \left[ \nu_s \left( \frac{i\pi}{2} - \ln(16s^3 + 4xs) \right) \left( c_0(1 + \nu) \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \\
+ \left( c_0(1 + \nu) \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \bigg] \\
- \nu e^{i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} \left( c_0(\nu) + c_1(\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \\
\times \left( c_0(1 - \nu) \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} + c_1(\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \bigg] \\
+ \nu e^{i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} \left( c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \\
\times \left( c_0(1 + \nu) \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \\
- \nu e^{i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} \left( c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \\
\times \left( c_0(1 + \nu) \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) + O \left( \frac{(\ln s)^3}{s} \right), \quad s \to \infty.
\]

What is left in the identity stated in Proposition 3 is the term
\[
P_3(s, x, \gamma) = (B_{11} - B_{22}) \tilde{H}_{11}(-s) + B_{12} \tilde{H}_{21}(-s) + B_{21} \tilde{H}_{12}(-s),
\]
with
\[
B_{11} = \nu + O(s^{-1}), \quad B_{12} = \nu(16s^3 + 4xs)^{2\nu} e^{-2s^3 \partial(1)} \frac{\Gamma(-\nu)}{\Gamma(\nu)} + O(s^{-1})
\]
and
\[
B_{21} = -\nu(16s^3 + 4xs)^{-2\nu} e^{2s^3 \partial(1)} \frac{\Gamma(\nu)}{\Gamma(-\nu)} + O(s^{-1}), \quad s \to \infty,
\]
which, also here, is valid uniformly on any compact subset of (1.16). Again, (17.15) allows us to simplify the identities for \( \tilde{X}_{ij}(-s) \) obtained in §16 leading to the following asymptotics for \( P_3(s, x, \gamma) \):
\[
P_3(s, x, \gamma) = 2\nu e^{i\pi\nu} \left( c_0(-\nu) + c_1(-\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \\
\times \left[ \nu_s \left( \frac{i\pi}{2} + \ln(16s^3 + 4xs) \right) \left( c_0(\nu) + c_1(\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \\
+ \left( c_0(\nu) + c_1(\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \bigg] \\
- 2\nu e^{i\pi\nu} \left( c_0(1 + \nu) \frac{\Gamma(1 + \nu)}{\Gamma(-\nu)} + c_1(-\nu) \left( \ln(16s^3 + 4xs) + i\frac{\pi}{2} \right) \right) \\
\times \left[ \nu_s \left( \frac{i\pi}{2} + \ln(16s^3 + 4xs) \right) \left( c_0(1 - \nu) \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} + c_1(\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \\
+ \left( c_0(1 - \nu) \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} + c_1(\nu) \left( \ln(16s^3 + 4xs) - i\frac{\pi}{2} \right) \right) \bigg] \]
For this evaluation it is important to recall the definitions of combined with (17.14) implies (1.15) with an error term of as well as the functional equation of the Digamma function (see, e.g., [3])

\[ (17.18) \]

\[ \int_0^\gamma (P_2(s, x, t) - P_3(s, x, t)) \, dt. \]

For this evaluation it is important to recall the definitions of \( c_0(\nu) \) and \( c_1(\nu) \),

\[ c_0(\nu) = -\frac{1}{\Gamma(\nu)}(\psi(\nu) + 2\gamma_E), \quad c_1(\nu) = -\frac{1}{\Gamma(\nu)} \]

as well as the functional equation of the Digamma function (see, e.g., [3])

\[ \psi(z) = \psi(z + 1) - \frac{1}{z} = \psi(1 - z) - \pi \cot \pi z, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots \}. \]

It implies

\[ c_0(\nu)c_1(-\nu) + c_1(\nu)c_0(-\nu) - c_0(1 + \nu)\frac{\Gamma(1 + \nu)}{\Gamma(-\nu)}c_1(\nu) - c_1(-\nu)c_0(1 - \nu)\frac{\Gamma(1 - \nu)}{\Gamma(\nu)} = 0 \]

and shows therefore that all terms of order \( O((\ln s)^2) \) in (17.18) vanish. The remaining terms of order \( O(\ln s) \) and \( O(1) \) can be computed in a similar way. Thus, we obtain

\[ (17.19) \]

\[ \int_0^\gamma (P_2(s, x, t) - P_3(s, x, t)) \, dt = 6(i\nu)^2 \ln s + 8(i\nu)^2 \ln 2 \]

\[ + 2 \int_0^\gamma \nu(t)(\ln \Gamma(\nu(t)) - \ln \Gamma(-\nu(t))) \, dt + O\left(\frac{(\ln s)^3}{s}\right) \]

as \( s \to \infty \) uniformly on any compact subset of the set (1.16). The last statement combined with (17.14) implies (1.15) with an error term of

\[ O\left(\frac{(\ln s)^3}{s}\right). \]
In order to improve this error term, we use Proposition \[1\] and, in particular, the identities for \(R(\pm s, \pm s)\) derived in §16. With \(17.15\) after simplification, we have
\[
R(s, s) = -i\nu(8s^2 + 2x) - \frac{3(i\nu)^2}{s} + O(s^{-2})
\]
and also
\[
R(-s, -s) = -i\nu(8s^2 + 2x) - \frac{3(i\nu)^2}{s} + O(s^{-2}),
\]
which implies via \((4.4)\)
\[
\frac{\partial}{\partial s} \ln \det(I - \gamma K_{\text{csin}}) = i\nu(16s^3 + 4x) + \frac{6(i\nu)^2}{s} + O(s^{-2}), \quad s \to \infty,
\]
uniformly on any compact subset of the set \((1.16)\). Integrating the last equation with respect to \(s\) and comparing with \((17.19)\), we complete the proof of Theorem \[1\].

§18. Asymptotics of \(\ln \det(I - \gamma K_{\text{csin}})\) — Proof of Theorem \[2\]

The statement on the asymptotic distribution of the zeros of \(\det(I - \gamma K_{\text{csin}})\), \(\gamma > 1\), will follow from an asymptotic expansion for this quantity, which we derive from Proposition \[1\] and \[2\]. To this end, we trace back the transformations
\[
X(\lambda) \mapsto T(z) \mapsto S(z) \mapsto R(z) \mapsto P(z) \mapsto Q(z)
\]
and use the identities \((16.4)\) and \((16.5)\), which were derived independently of the choice of \(\gamma \in \mathbb{R}\). First, the residue conditions \((13.4)\) and \((13.5)\) show explicitly that \(R(z)\) is analytic at \(z = 1:\)
\[
(18.1) \quad R(1) = \left( Q(1) + (I + B)Q'(1) \right) \left( \begin{array}{cc} 1 & 0 \\ p & 0 \end{array} \right) + (I + B)Q(1) \left( \begin{array}{cc} 0 & 0 \\ \nu_0 p^{3 + \frac{x}{4s^2}} & 0 \end{array} \right),
\]
where \(\nu_0\) and \(p\) were introduced earlier as
\[
\nu_0 = \frac{1}{2\pi i} \ln |1 - \gamma| = \frac{1}{2\pi i} \ln (\gamma - 1), \quad p = \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{-2s^3 \vartheta(1)} \tilde{\beta}^2(1).
\]
Furthermore,
\[
(18.2) \quad R'(1) = \left( Q(1) + (I + B)Q'(1) \right) \left( \begin{array}{cc} 0 & 0 \\ \nu_0 p^{3 + \frac{x}{4s^2}} & 1/2 \end{array} \right)
\]
\[
+ \left( Q'(1) + (I + B)Q''(1) \right) \left( \begin{array}{cc} 1 & 0 \\ p & 0 \end{array} \right) + (I + B)Q(1) \left( \begin{array}{cc} 0 & 0 \\ \nu_0 p \kappa(s, x) & 0 \end{array} \right),
\]
where
\[
\kappa(s, x) = \frac{1}{2} \left( \frac{10}{3} \left( 1 + \frac{x}{4s^2} \right) + \left( \nu_0 - \frac{1}{2} \right) \left( 3 + \frac{x}{4s^2} \right)^2 \right) \left( 1 + \frac{x}{4s^2} \right)^{-2}.
\]
Similarly
\[
(18.3) \quad R(-1) = \left( Q(-1) - (I - B)Q'(1) \right) \left( \begin{array}{cc} 0 & p \\ 0 & 1 \end{array} \right) - (I - B)Q(-1) \left( \begin{array}{cc} -1/2 & -\nu_0 p^{3 + \frac{x}{4s^2}} \\ 0 & 0 \end{array} \right)
\]
and
\[
R'(-1) = \left( Q(-1) - (I - B)Q'(1) \right) \left( \begin{array}{cc} -1/2 & -\nu_0 p^{3 + \frac{x}{4s^2}} \\ 0 & 0 \end{array} \right)
\]
\[
+ \left( Q'(-1) - (I - B)Q''(-1) \right) \left( \begin{array}{cc} 0 & p \\ 0 & 1 \end{array} \right) - (I - B)Q(-1) \left( \begin{array}{cc} -1/4 & \nu_0 p \kappa(s, x) \\ 0 & 0 \end{array} \right).
\]
To evaluate $Q(\pm 1)$, we iterate. First, for any $z \in \Sigma_R$, from (15.1), (15.3), and the residue theorem we get

$$Q_1(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{Q_1(w)(G_Q(w) - I)\, dw}{w - z}$$

$$= I + \frac{i}{2sz} \left( \begin{pmatrix} v & -u \\ u & -v \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{z+1} \right) \left( B_0(z) \right)^{-1} \frac{v \cdot e^{-2\pi i v}}{-u e^{-2\pi i v}} B_0(z) \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix} + O(s^{-2}) \right),$$

where we have used the relation

$$\left( \begin{pmatrix} w-1 \\ 0 \end{pmatrix} 0 \begin{pmatrix} \frac{1}{w+1} \end{pmatrix} \right) \left( B_0(w) \right)^{-1} = \left( \begin{pmatrix} v & -u \\ u & -v \end{pmatrix} \right) + 4w \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \nu_0 + O(w^2), \quad w \to 0.$$

Thus,

$$Q(\pm 1) = I + \frac{1}{2\pi i} \int_{\Sigma_R} (Q_1(w) - I)(G_Q(w) - I)\, dw$$

$$+ \frac{1}{2\pi i} \int_{\Sigma_R} (G_Q(w) - I)\, dw = I \pm \frac{i}{2s} \begin{pmatrix} v & -u \\ u & -v \end{pmatrix} \nu_0 \begin{pmatrix} -u^2 & -u_x \\ u_x & u^2 \end{pmatrix}$$

$$+ \frac{1}{8s^2} \begin{pmatrix} w^2 - u^2 & -2(u_x + uv) \\ -2(u_x + uv) & u^2 - v^2 \end{pmatrix} + O(s^{-3}), \quad s \to \infty,$$

uniformly on any compact subset of the set (11.18), and similarly,

$$Q'(\pm 1) = -\frac{i}{2s} \begin{pmatrix} v & -u \\ u & -v \end{pmatrix} + \nu_0 \begin{pmatrix} -u^2 & -u_x \\ u_x & u^2 \end{pmatrix}$$

$$+ \frac{1}{4s^2} \begin{pmatrix} w^2 - u^2 & -2(u_x + uv) \\ -2(u_x + uv) & u^2 - v^2 \end{pmatrix} + O(s^{-3}).$$

Since

$$p = i \frac{\Gamma(1 - \nu)}{\Gamma(\nu)} e^{-2s^{\alpha(1)} \beta_2(1)} = ie^{-i\sigma}, \quad \bar{p} = p^{-1},$$

with $\sigma = \sigma(s, x, \gamma)$ as in (17.2), for the matrix

$$\Theta = \begin{pmatrix} Q(1) & (p) \\ Q(-1) & (p) \end{pmatrix},$$

which appears in (14.2), we obtain

$$\det \Theta = -2ip \begin{pmatrix} \cos \sigma - \frac{v}{s} \sin \sigma + \frac{u}{s} \\ \frac{2i\nu_0}{s^2} \left( u_x + u^2 \sin \sigma \right) + \frac{u^2 - v^2}{2s^2} \cos \sigma + O(s^{-3}) \end{pmatrix}.$$

Since we agreed that $s$ stays away from the small neighborhood of the points $\{s_n\}$ defined by

$$\cos \sigma(s_n, x, \gamma) = 0,$$
we see that for all sufficiently large $s$ lying outside of the zero set of this transcendental equation, the stated determinant is nonzero. Back to (14.2), this implies

$$B = - \frac{2p}{\det \Theta} \left[ \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} + \frac{\cos \sigma}{s} \begin{pmatrix} v & -u \\ u & -v \end{pmatrix} \\ + \frac{1}{s^2} \left\{ 2i\nu_0 \cos \sigma \begin{pmatrix} -u^2 & -u_x \\ u_x & u^2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -v^2 \sin \sigma & -v^2 \\ v^2 & v^2 \sin \sigma \end{pmatrix} \\ + \frac{u_x}{2} \begin{pmatrix} -1 & \sin \sigma \\ -\sin \sigma & 1 \end{pmatrix} + uv \sin \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} + O(s^{-3}) \right], \quad s \to \infty,$$

and with

$$- \frac{2p}{\det \Theta} = - \frac{i}{\cos \sigma} \left[ 1 + \frac{1}{s} (v \tan \sigma - u) + \frac{1}{s^2} \left( v^2 \tan^2 \sigma - \frac{2uv \sin \sigma}{\cos^2 \sigma} + \frac{u^2}{\cos^2 \sigma} \\ - \frac{u^2 - v^2}{2} - 2i\nu_0 u^2 \tan \sigma - \frac{2\nu_0 u^2}{\cos \sigma} \right) + O(s^{-3}) \right]$$

we obtain

$$B = -i \left[ \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} + \frac{1}{s} \left\{ \frac{v \sin \sigma - u}{\cos^2 \sigma} \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} + \begin{pmatrix} v & -u \\ u & -v \end{pmatrix} \right\} \\ + \frac{1}{s^2} \left\{ \frac{1}{2 \cos \sigma} \begin{pmatrix} 2v^2 \sin \sigma - u^2 \sin \sigma - u_x \\ u^2 - u_x \sin \sigma \end{pmatrix} - \frac{1}{2} v^2 + u_x \sin \sigma \\ - 2v^2 \sin \sigma + u^2 \sin \sigma + u_x \right\} \\ - \frac{2\nu_0 (u^2 \sin \sigma + u) \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} + (v \sin \sigma - u)^2 \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} \\ + \left( -\frac{uv \cos \sigma - 2i\nu_0 u^2}{u^2 \cos \sigma + 2i\nu_0 u_x} \sin \sigma \right) + O(s^{-3}) \right], \quad s \to \infty,$$

and all expansions are uniform on any compact subset of the set (1.18). Let us go back to (16.4) and (16.5). Since $\nu = \nu_0 + \frac{1}{2}$, we notice that

$$R(s, s) = -i\nu_0 (8s^2 + 2x) - i(4s^2 + x) + ip(16s^2 + 4x) [R'_{12}(1)R_{22}(1) - R'_{22}(1)R_{12}(1)] + O(s^{-1})$$

and similarly

$$R(-s, -s) = -i\nu_0 (8s^2 + 2x) - i(4s^2 + x) + ip(16s^2 + 4x) \times [R'_{11}(-1)R_{21}(-1) - R'_{21}(-1)R_{11}(-1)] + O(s^{-1}).$$

Next,

$$R'_{12}(1) = \frac{1}{2} (Q(1) + (I + B)Q'(1))_{12} - \frac{1}{4} ((I + B)Q(1))_{12},$$

$$R_{22}(1) = \frac{1}{2} ((I + B)Q(1))_{22},$$

$$R'_{22}(1) = \frac{1}{2} (Q(1) + (I + B)Q'(1))_{22} - \frac{1}{4} ((I + B)Q(1))_{22},$$

$$R_{12}(1) = \frac{1}{2} ((I + B)Q(1))_{12}$$

and, therefore,

$$R'_{12}(1)R_{22}(1) - R'_{22}(1)R_{12}(1) = \frac{1}{4} [(Q(1) + (I + B)Q'(1))_{12}((I + B)Q(1))_{22} - (Q(1) + (I + B)Q'(1))_{22}((I + B)Q(1))_{12}].$$
Now, we combine the information on $Q(1), Q'(1),$ and $B$ derived previously to deduce that
\[
i p(16s^2 + 4x) [R_{12}(1)R_{22}(1) - R_{22}'(1)R_{12}(1)] = (4s^2 + x)(i + \tan \sigma) + \alpha_+ + O(s^{-1})
\]
with a function $\alpha_+ = \alpha_+(s, x, \gamma)$ such that $\int \alpha_+(s, x, \gamma) \, ds = O(\ln s), \ s \to \infty$. Following the same computations for $R(-s, -s)$, we get
\[
R(-s, -s) = -i v_0(8s^2 + 2x) + (4s^2 + x) \tan \sigma + \alpha_- + O(s^{-1}),
\]
where $\int \alpha_-(s, x, \gamma) \, ds = O(\ln s)$, and together with Proposition 1.

(18.7) $\frac{\partial}{\partial s} \ln \det(I - \gamma K_{\text{csin}}) = iv_0(16s^2 + 4x) - (8s^2 + 2x) \tan \sigma - \alpha_+ - \alpha_- + O(s^{-1})$.

Opposed to the last equation, now we recall Proposition 2 and evaluate the logarithmic $x$-derivative. For $\gamma > 1$,
\[
X_1 = \lim_{\lambda \to \infty} (\lambda(X(\lambda)e^{-i\frac{\lambda^3 + x\lambda}{\lambda^2}}) - I)
\]
\[
= -2v_0s\sigma_3 + s(\sigma_3 + B) + \frac{is}{2\pi} \int_{\Sigma_R} Q_-(w)(G_Q(w) - I) \, dw,
\]
where the expansion for $B$ has already been computed. From this and the residue theorem, we obtain
\[
X_1 = -2v_0s\sigma_3 + s\sigma_3 - \frac{is}{\cos \sigma} \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} - \frac{i}{2} \begin{pmatrix} v & -u \\ u & -v \end{pmatrix}
\]
\[
- \frac{i(v \sin \sigma - u)}{\cos^2 \sigma} \begin{pmatrix} \sin \sigma & -1 \\ 1 & -\sin \sigma \end{pmatrix} + O(s^{-1}),
\]
whence
\[
(18.9) \frac{\partial}{\partial x} \ln \det(I - \gamma K_{\text{csin}}) = 4iv_0s + v - 2s \tan \sigma - \frac{2v}{\cos^2 \sigma} + \frac{2u \sin \sigma}{\cos^2 \sigma} + O(s^{-1}).
\]

Integrating both identities (18.8), (18.9) and comparing the result, we conclude that, for $s \to \infty$ outside the zero set (18.6), we have
\[
\ln \det(I - \gamma K_{\text{csin}}) = iv_0 \left(\frac{16}{3}s^3 + 4sx\right) + \ln |\cos \sigma(s, x, \gamma)| + \chi_1 \ln s
\]
\[
- \int_x^\infty (y - x)u^2(y, \gamma) \, dy + \chi_2 + O(s^{-1}),
\]
with real-valued constants $\chi_i$ solely depending on $\gamma$, and where the error term is uniform on any compact subset of the set (1.18). The expansion (18.10) verifies the claim on the asymptotic distribution of the zeros of the Fredholm determinant as given in Theorem 1.

Remark 3. We want to emphasize that our proof of Theorem 2 produces in fact an entire asymptotic series for $\det(I - \gamma K_{\text{csin}})$ in the case where $\gamma > 1$, like what we obtained in Theorem 1. However, due to the increased amount of computations within the Riemann–Hilbert analysis, we chose not to compute the constants $\chi_1(\gamma)$ and $\chi_2(\gamma)$.

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