Erratum: Recovering parity-time symmetry in highly dispersive coupled optical waveguides (2016 New J. Phys. 18 125012)

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Due to an error in the production process, the term
‘Equation (8) predicts a ratio of $|\kappa_{\text{eff}}|/\Re\{\kappa_{\text{eff}}\}$’
should read:
‘Equation (8) predicts a ratio of $|\kappa_{\text{eff}}|/\Re\{\kappa_{\text{eff}}\}|$’
Recovering parity-time symmetry in highly dispersive coupled optical waveguides

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Abstract

Coupled photonic systems satisfying parity-time symmetry (PTS) provide flexibility to engineer the flow of light including non-reciprocal propagation, perfect laser-absorbers, and ultra-fast switching. Achieving the required index profile for an optical system with ideal PTS, i.e. \( n(x) = n(-x)^* \), has proven to be difficult due to the challenge of controlling gain, loss and material dispersion simultaneously. Consequently, most research has focused on dilute or low gain optical systems where material dispersion is minimal. In this paper, we study a model system of coupled inorganic semiconductor waveguides with potentially high gain (>1500 cm\(^{-1}\)) and dispersion. Our analysis makes use of coupled mode theory’s parameters to quantify smooth transitions between PTS phases under imperfect conditions. We find that the detrimental influence of gain-induced dispersion is counteracted and the key features of PTS optical systems are recovered by working with non-identical waveguides and bias pumping of the optical waveguides. Our coupled mode theory results show excellent agreement with numerical solutions, proving the robustness of coupled mode theory in describing various degrees of imperfection in systems with PTS.

1. Introduction

The field of non-Hermitian Hamiltonians is often linked to the concept of parity-time symmetry (PTS). In the early 1990s, several theoreticians noticed a class of non-Hermitian parity-time invariant Hamiltonians that possess real eigenspectra. Since PTS is a weaker constraint than Hermiticity, an abrupt transition between real and imaginary phases of eigenvalues can occur at the exceptional point (EP), which can be exploited for various applications [1, 2]. The resemblance between the Schrödinger equation and the wave equation has motivated many to apply PTS to optical systems. A parity-time symmetric optical system requires a particular complex refractive index distribution satisfying the condition, \( n(x) = n(-x)^* \), through the incorporation of gain and loss [2–4]. Various interesting consequences arising from this physical framework have been theoretically predicted and experimentally verified, including non-reciprocal propagation [5, 6], unidirectional reflection/transmission [7–9], perfect laser-absorbers [10–13], loss-induced lasers [14], single mode lasing [15, 16], dynamic memory [17] and fast switching [18–20]. Many of these demonstrations have achieved the required refractive index distribution by having equal amounts of gain and loss in otherwise identical coupled waveguides. In these works, the coupled mode theory (CMT) [21–23] of optical waveguides was used to explain the experimentally observed EP between eigenvalue phases in terms of a balance between loss, gain and inter-waveguide coupling.

The practicalities of controlling gain, loss and mode coupling experimentally have led to reports of imperfect PTS with arguably the absence of EPs [7]. In waveguide systems, imperfect PTS has been attributed to the imbalance of gain and loss [24] as well as modal asymmetry [25] while in resonant PTS systems, it has been attributed to the detuning between material and optical resonant frequencies [26]. Indeed, in most optical parity-time symmetric demonstrations the control over loss is somewhat inflexible as it is introduced in the form of additive metal layers [5–8, 14–16, 27, 28], while the gain is controlled by pumping a suitable active material.
such as transition metal or rare Earth doped solids \([5–7]\) and semiconductors \([15, 16, 28]\). In order to achieve close to ideal PTS, most of the above-mentioned experiments limited the gain to just a few hundreds \(\text{cm}^{-1}\) since gain-induced dispersion strongly affects any transition between the real and imaginary eigenvalue phases. To apply the concept of PTS to more highly functional and practical optical platforms based on optoelectronic or plasmonic components where very high loss/gain exist, strategies to manage gain-induced dispersion are required.

In this study, we theoretically investigate a test system of coupled semiconductor slab waveguides, which are capable of providing different levels of gain or loss via a pump-induced carrier inversion in each waveguide. Our proposed system consists of GaAs cores and AlGaAs claddings operating at a wavelength, \(\lambda = 850\ \text{nm}\) with \(n_{\text{AlGaAs}} = 3.1\) and \(n_{\text{GaAs}}(\lambda = 850\ \text{nm}) = 3.6468 – 0.044i\) (see figure 1). A realistic gain model for III–V semiconductor materials \([29–32]\) is used to model the variation in gain and loss with dispersion calculated based on the Kramers–Kronig relations. The advantage of modelling a simple 1D coupled waveguide system is that we are able to evaluate the parameters of CMT analytically, and then use them to understand the different degrees of imperfection in practical PTS devices. Thus, a direct comparison with rigorous numerical calculations can be made. At the chosen operating wavelength gain-induced dispersion in GaAs is extremely strong. We find that by working with non-identical waveguides and introducing either overall net gain or loss to the system, we can recover the key features of parity-time symmetric optical systems even in this high gain environment. The choice of materials and configuration is pertinent to practical applications of PTS for light manipulation functionality in semiconductor-based photonic systems.

2. CMT in coupled slab waveguides with gain/loss

CMT applies to weakly coupled systems where the field in one waveguide can be treated as a perturbation to the field in the other. Thus, the electric field of a mode in a waveguide takes the form of \(E_p(x, z) = A_p(z)E_0(x)e^{i\beta_p z}\), where the amplitude, \(A_p(z)\), varies in the propagation direction \([33–35]\). The field perturbation is then modelled by a linear polarisation term, \(PE\), which can be input into the Lorentz reciprocity theorem:

\[
\nabla \cdot (E_p \times H_p^* + E_0 \times H_0) = -\omega \varepsilon_0 \mu_0 \Delta P - E_p \cdot \Delta P_p.
\]

By integrating both sides of equation (1) over the cross-section of the waveguides, the familiar CMT equations and explicit functional forms of coupling coefficients are retrieved:

\[
\frac{dA_p}{dz} = \Sigma_{\mu \nu} \kappa_{\mu \nu} A_{\mu} e^{i(\beta_\mu - \beta_\nu)z},
\]

where the coupling coefficients, \(\kappa_{\mu \nu}\), can be expressed as:

\[
\kappa_{\mu \nu} = \frac{\hat{\kappa}_{\mu \nu} - \frac{e_m \kappa_{\mu \nu}}{\varepsilon_0}}{1 - \frac{e_m \varepsilon_{0\mu}}{\varepsilon_{0\nu}}} \approx \hat{\kappa}_{\mu \nu} \left(1 - \frac{\varepsilon_{0\nu} \hat{\kappa}_{\mu \nu}}{\varepsilon_{0\mu} \hat{\kappa}_{\nu \mu}}\right)^{-1},
\]

\[
\kappa_{\mu \nu} = \frac{\hat{\kappa}_{\mu \nu} - \frac{e_m \kappa_{\mu \nu}}{\varepsilon_0}}{1 - \frac{e_m \varepsilon_{0\mu}}{\varepsilon_{0\nu}}} \approx \hat{\kappa}_{\mu \nu} \left(1 - \frac{\varepsilon_{0\nu} \hat{\kappa}_{\mu \nu}}{\varepsilon_{0\mu} \hat{\kappa}_{\nu \mu}}\right)^{-1},
\]

with the analytic functions of the coefficients taking the form of:

\[
\hat{\kappa}_{\mu \nu} = \omega \int_{-\infty}^{\infty} E_\mu^p \Delta e_\nu(x)E_\nu dx \quad e_m = \int (E_\mu^p \times H_\nu^* + E_\nu \times H_\mu^*) dx.
\]
The approximations in equation (1) are valid when the two coupled modes are in the weak coupling limit. This implies that the overlap coefficients \( \gamma_{\mu \nu} \ll 1 \), the field normalisation \( \zeta_{\mu \nu} \approx 1 \), and self-coupling is ignored for the case where \( \kappa_{\mu \mu} \gg \kappa_{\mu \nu} \). We consider the simplest case of coupling between the two first order TE modes of semiconductor slab waveguides, shown in figure 1. Hence, the indices \( \mu \) and \( \nu \) take only the waveguide label values 1 and 2.

Explicit calculation of coupling coefficients requires the modal field distribution of each waveguide and the permittivity perturbation, \( \Delta \epsilon (x) \). For lossy waveguides of core width \( w = 2h \), the transcendental eigen-equation can be solved using an iterative method [36, 37] to determine the complex wavenumbers (see appendix A). For simplicity, the whole structure is clad with the same material of refractive index \( n_c \) and thus the perturbed permittivity \( \Delta \epsilon (x) = \epsilon_0 (\epsilon_{\mu} - n_c^2) \theta_{\mu} (x) \) where \( \theta_{\mu} (x) \) is the top-hat function. The coupling coefficients can be computed as:

\[
\tilde{\kappa}_{\mu \nu} = \omega \epsilon_0 (\epsilon_{\mu} - \epsilon_{\nu} ) \int_{-\infty}^{\infty} \text{E}_\mu^* \cdot \theta_{\mu} \left( \frac{x - x_{\mu}}{d_{\mu}} \right) \text{E}_\nu \, dx = \omega \epsilon_0 (\epsilon_{\mu} - \epsilon_{\nu} ) f_{\mu} (x).
\]

(5)

By expressing \( A(z) \) as a phasor with propagation phase, \( e^{i(\beta_{\mu} z + \kappa_{\mu \mu} z + \kappa_{\mu \nu} z) / 2} \), the coupled mode equations can be recast into the well-known matrix form:

\[
i \frac{\partial A}{\partial z} = \begin{bmatrix} \delta_{\text{mode}} & \kappa_{\text{eff}} \\ \kappa_{\text{eff}} & -\delta_{\text{mode}} \end{bmatrix} A = \mathcal{H} A,
\]

(6)

where \( \delta_{\text{mode}} = (\beta_1 + \kappa_{11} + \beta_2 + \kappa_{22}) / 2 \), \( \delta_{\text{eff}} = (\beta_1 + \kappa_{11} - \beta_2 - \kappa_{22}) / 2 \) and \( \kappa_{\text{eff}} = \sqrt{\kappa_{12}^2 - \kappa_{11} \kappa_{22}} \) are the mutual and self-coupling coefficients respectively. \( \delta_{\text{mode}} \) accounts for a modal index difference between the two waveguides. The eigenvalues of \( \mathcal{H} \), corresponding to the formation of two supermodes, are \( \sigma_{\pm} = \beta_0 \pm \sqrt{\delta_{\text{mode}}^2 + \kappa_{\text{eff}}^2} = \beta_0 \pm \sigma_0 \).

In ideal PTS systems, identical waveguides are typically required. In the conventional case of lossless CMT, this leads to the condition \( \kappa_{12} = \kappa_{21}^* \) and in PTS systems with zero net gain/loss, \( \kappa_{\text{eff}} \) is always real. We note here that this condition holds subject to the approximation of weak coupling of CMT and that no gain-induced dispersion is present. Consequently, CMT predicts a perfect coalescing of the two supermodes, \( \Re \{ \sigma_0 \} = \Im \{ \sigma_0 \} = 0 \) at the EP of the system where \( |\kappa_{\text{eff}}| = |\delta_{\text{mode}}| \) (see figure 2), denotes a transition of the eigenvalues from being purely real to purely imaginary [2, 5]. Beyond the EP, one supermode is attenuated and the other is amplified while the two waveguides decouple. In an ideal parity-time symmetric situation, \( \kappa_{\text{eff}} \) must be purely real, \( \delta_{\text{mode}} \) purely imaginary. Any departure from these upsets the abrupt transition at the EP.

Explicit evaluation of eigenvalues using CMT with analytic modal field solutions for the ideal PTS scenario \((n_{f1} = n_{f2}^* \text{ and } f^*_j (x) = f_j^*(x))\) agrees very well with numerical simulations using COMSOL’s mode solver for weakly coupled systems (see appendix B). As shown in figure 2, there is no dramatic breakdown of CMT as the coupling strength increases. CMT predicts a smaller coupling compared to numerical solutions for cases of separation smaller than 100 nm. When plotting \( \sigma_0 \) as a function of \( n_{f1}^* \) for \( s < 100 \text{ nm} \) (not shown here), we notice that besides a horizontal discrepancy in the position of the EP there is also a vertical difference between CMT and numerical solutions.

3. Imbalance in gain and loss with non-dispersive materials

In practice, it is very difficult to achieve exactly equal gain and loss in a coupled system. The condition \( \kappa_{12} = n_{f1}^* \) relies on a delicate balance of CMT’s internal parameters. Explicitly, we find that:

\[
\kappa_{12} = n_{f1}^* \frac{\Delta \epsilon_1 f_2 (x)}{\Delta \epsilon_2 f_1^* (x)}.
\]

(7)

In the case of identical waveguides where \( \epsilon'' \ll \epsilon' \) and \( \Delta \epsilon = \epsilon'_1 - n_c^2 = \epsilon'_2 - n_c^2 \), we arrive at:

\[
\kappa_{12} \approx n_{f1}^* \left( 1 + i \frac{\epsilon''_1 + \epsilon''_2}{\Delta \epsilon} \right).
\]

(8)

The modal asymmetry, \( f_2 (x) / f_1^* (x) \), varies very weakly in this situation such that \( \left| \kappa_{\text{eff}} \right| / \left| \kappa_{12} \right| \approx (\epsilon''_1 + \epsilon''_2) / \Delta \epsilon \). While a number of theoretical reports have predicted that optical PTS systems with net gain/loss are mappable onto the PTS formalism [2, 19, 27], equations (7) and (8) suggest that an imbalance in gain and loss leads to the appearance of complex valued \( \kappa_{\text{eff}} \)—a departure from the ideal PTS coupled system.

We consider the case where \( n_{f1}'' = n_{f2}'' \) and \( |n_{f1}''| \neq |n_{f2}''| \). The difference in gain and loss between the two GaAs core layers can be conveniently expressed as: \( n'' = \delta_{0, \text{GaAs}} + \Delta_{\text{GaAs}} \), where \( \delta_{0, \text{GaAs}} \) can be either negative or positive depending on whether the overall system exhibits net loss or net gain. When \( \delta_{0, \text{GaAs}} = 0 \), we recover the ideal PTS system where the phase transition occurs at the location \( \sigma_{\text{EP}} \) (shown in figure 3) on the gain...
Figure 2. Simulation and analytic results of $\pm \sigma_n$ as a function of $n_{\text{GaAs}}$. The results are plotted for $d_1 = d_2 = d = 500$ nm (a, c) and $d_1 = d_2 = d = 900$ nm (b, d) under ideal PTS configuration for $\eta_1 = \eta_2$ at $\lambda = 850$ nm and $\eta_{1,2} = 3.6468 \pm i3$ ($n_{\text{GaAs}}$).

Different colours correspond to the results of three separation distances, $s = 50–200$ nm. In this paper, $\sigma_n$, $\eta_{\text{eff}}$, $k_{\text{eff}}$ and $\kappa_{\text{mod}}$ have been normalised to $k_{\lambda/2} = \frac{2\pi}{\lambda}$.

Figure 3. (a) and (b) Real and imaginary parts of $\pm \sigma_n$ as a function of $\Delta_{\text{GaAs}}$ for three values of $g_{0,\text{GaAs}} = g_0$ (results from each $g_0$ case are shown as a different colour). Here, $d_1 = d_2 = d = 500$ nm, $s = 120$ nm at $\lambda = 850$ nm and $\eta_{1,2} = 3.6468 + i(g_{0,\text{GaAs}} \pm \Delta_{\text{GaAs}})$, $\kappa_{\text{GaAs}} = (\eta_1^2 - \eta_2^2)/2 = \Delta_{\text{GaAs}}$ (c) and (d) Extracted parameters from CMT. Comparison of the corresponding real and imaginary parts of the components of $\sigma_n$. In (c), $\Re{\{\omega_{\text{eff}}\}}$ and $\Im{\{\epsilon_{\text{mod}}\}}$ are insensitive to different values of $g_0$, leading to almost overlapping of the three coloured lines. (e) Assessing the quality of switching close to the EP of coupled system. Black vertical line indicates the position of $\eta_{\text{EP}}$. 

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spectrum. Figures 3(a) and (b) show the eigenspectra of the coupled system for various values of $g_{0,GaAs}$. We clearly observe a residual splitting of the modes near the position of $\alpha_{EP}$ which increases with the magnitude of $|g_{0,GaAs}|$. The two phases of the PTS system are no longer well-defined as $\sigma_0$ is always complex along the gain spectrum. A close look at the extracted parameters from CMT (figures 3(c) and (d)) reveals that this residual splitting is caused by the emergence of the imaginary part of $\kappa_{eff}$. Here, $\sigma_0$ can be approximated as:

$$\sigma_0 \approx \sqrt{[\kappa_{21}]^2 + \kappa_{\text{mode}}^2 + i[\kappa_{21}]^2 \left(\frac{\epsilon_1 + \epsilon_2}{\beta_\epsilon}\right)}.$$  \hfill (9)

Equation (8) predicts a ratio of $\Im \{\kappa_{eff}\}/\Re \{\kappa_{eff}\} \sim 0.03$, 0.06 for $g_{0,GaAs} = 0.013$, 0.031 respectively which agree well with values in figures 3(c) and (d). By allowing $\kappa_{eff}$ to take complex values, CMT agrees qualitatively with the numerical solutions. The larger the value of $g_{0,GaAs}$, the larger the mismatch between the two methods. This discrepancy arises as CMT requires an estimation of the average index of the whole system, which in this case has been taken to be the lossless cladding index ($n_0$). However, with the inclusion of $g_{0,GaAs}$, now the system has a net gain/loss, making this estimation less accurate. Non-zero values of $\Im \{\kappa_{eff}\}$ inhibit the occurrence of the EP. The abrupt transition of an ideal PTS system is replaced by a smooth bifurcation of supermode eigenvalues. This means that the effect of any applications relying on the sharp transitions between the two PTS phases is greatly reduced [7]. To characterise the quality of PTS phase transition, we define $\eta = \pi/(2\Re \{\sigma_0\})$. In the case of perfect PTS coupled systems, $\eta$ corresponds to the familiar coupling length of coupled waveguide systems. $\eta$ increases as gain/loss increases and reaches infinity at the EP. At this point, the two waveguides decouple while having exactly the same modal index. However, in an imperfect PTS system, the correspondence between coupling length and $\eta$ is less well-defined as the imaginary part of $\sigma_0$ is always non-zero. Only at points where $\Re \{\sigma_0\} \gg \Im \{\sigma_0\}$ can $\eta$ be used as an estimate for the coupling length. Equation (9) indicates that close to $\alpha_{EP}$ where $|\kappa_{21}|^2 + \kappa_{\text{mode}}^2 \approx 0$, the value of real and imaginary parts of $\sigma_0$ are equal. It is noteworthy that this residual $\sigma_0$ exists in all practical optical systems and quantifies the quality of PTS in those systems. Deviations of such imperfect PTS systems from ideal behaviour can be characterised by evaluating the term $|\kappa_{21}|^2 (\epsilon_1 + \epsilon_2)/\beta_\epsilon$. Beyond this point, $\eta$ represents an intrinsic residual modal mismatch between the two resultant decoupled modes. Unlike in a perfect PTS system, $\eta$ changes smoothly in the region around $\alpha_{EP}$ and slowly increases after. The value of $\eta$ increases by one order of magnitude close to $\alpha_{EP}$ (figure 3(e)).

4. Imbalance in modal index

We now investigate the scenario where this coupled system suffers from a geometric asymmetry of unequal corewidths. Material parameters are kept the same as in the ideal PTS case where $n_{21} = n_{12}^G$. With unequal corewidths, the modal indices of each waveguide are no longer complex conjugates of each other; a phase mismatch thus arises in the system. Figures 4(a) and (b) show the sensitivity of the coupled system to geometric mismatching. The residual splitting of the real eigenspectrum approaches $|\Re \{\kappa_{\text{mode}}\}|$ beyond $\alpha_{EP}$. Correspondingly, $\Im \{\sigma_0\}$ has non-zero values throughout the gain spectrum. Figures 4(c) and (d) reveal that $|\Re \{\kappa_{\text{mode}}\}|$ reaches 20% and 10% of the coupling strength for $\Delta d = \pm 10$ nm and $\pm 5$ nm respectively. For each coupling strength level, there will be a corresponding geometric mismatch that the system can tolerate before the detuning effect takes over. For example, $d_l = \pm 200$ nm, an equivalence of only 10% of the coupling strength mentioned above, the mismatch tolerance goes down to 2 nm, making this device impractical due to fabrication tolerance. The stronger the coupling, the greater the mismatch tolerance the system can take. Geometric asymmetry also causes a smooth variation of $\eta$ near the position of $\alpha_{EP}$ (figure 4(e)). While CMT still produces results closely resembling the numerical solutions, the slight disagreement can again be attributed to the estimation of average index of the system.

5. PTS in dispersive high gain material

The main problem with using semiconductors in optical waveguides for PTS applications is gain-induced dispersion. For our particular example, the dependence of the real part of $n_{GaAs}$ on its imaginary part has been calculated based on a realistic gain theory and the Kramers–Kronig relations (appendix C). Once we include this dynamic change of $\Re \{n_{GaAs}\}$ with $\Im \{n_{GaAs}\}$, the coupling mechanism is totally destroyed by dispersion effects (see figure C2). A PTS system ceases to have any significant effect on modal index manipulation. We now consider the use of asymmetry in gain/loss and waveguide geometry to compensate the gain-induced dispersion and recover the abrupt dynamics of PTS near $\alpha_{EP}$.

The dispersion curves of waveguides with different core widths are vertically shifted from each other (figure 5(a)). Along a single dispersion curve, there exist no two values of $\Im \{n_{GaAs}\}$ which give equal modal indices. Working with waveguides of dissimilar core widths, we find that equal values of modal indices can be
obtained at two distinct points on the gain spectrum, $g_{0,\text{GaAs}} \pm \Delta_{\text{GaAs}}$. The two sets of green and black double-headed arrows indicate the magnitude of $\Delta_{\text{GaAs}}$ when sweeping through the values of $3 \{n_{\text{GaAs}}\}$. Even though $k_{\text{GaAs}}$ varies for each $\Delta_{\text{GaAs}}$ value, the difference in material index between the two waveguides only depends on $\Delta_{\text{GaAs}}$. Figure 5(b) plots the variation of $|\delta_{\text{mode}}|$ with $g_{0,\text{GaAs}}$ for two cases of $\Delta d = \pm 50 \text{ nm}$. The points where $N_{\text{eff}}$ cuts $|\delta_{\text{mode}}|$ lines represent the occurrence of an abrupt change in the properties of these systems.

The recovery process has been carried out for $\Delta d = \pm 50 \text{ nm}$, $d_{1} = 500 \text{ nm}$ and $s = 150, 200 \text{ nm}$ with the results shown in figure 6. We note here that for any pair of waveguides with dissimilar core widths, by using...
suitable pumping conditions, perfect phase matching can always be achieved. The gain range at which splitting in the eigenspectrum of the coupled system occurs hence depends on the separation distance. This gives us great flexibility in choosing geometric parameters to tune the operating gain range of the system for different applications. When plotting $c_0$ as a function of $\Delta_{GaAs}$, an abrupt bifurcation of the two branches in both real and imaginary planes can be achieved (figures 6(a) and (c)). We have also shown in figures 6(b) and (d) the evolution of $n_{eff}$ of the supermodes as a function of $\Delta_{GaAs}$. While maintaining zero modal index difference between the two waveguides ($\Re \{ \delta_{mode} \} = 0$), their absolute values vary with $\Delta_{GaAs}$. Despite the net gain/loss, bifurcations in the eigenspectra can be observed. The excellent agreement between CMT and numerical solutions suggests that the recovery of the abrupt bifurcation is robust.

On close scrutiny, we notice a small residual splitting in the real and imaginary planes of $c_0$ beyond $\alpha_{EP}$. This is because an imbalance in gain and loss is required to achieve equal modal indices, giving rise to the familiar limitation of $\eta$ value close to $\alpha_{EP}$. The effectiveness of this recovery mechanism is determined by the magnitude of the term $|\kappa_2|^2 \left( f_1^*(x) \left( \epsilon_{\rho}^f + \epsilon_{\rho}^s \right) / f_1^2(x) \delta_{\epsilon} \right)$ in equation (10):

$$
\sigma_0 \approx \sqrt{ |\kappa_2|^2 \frac{f_1^*(x)}{f_1^2(x)} + \delta_{\kappa_2}^2 + i |\kappa_2|^2 \frac{f_1^*(x)}{f_1^2(x)} \left( \epsilon_{\rho}^f + \epsilon_{\rho}^s \right) / \delta_{\epsilon} }.
$$

The increase in $\eta$ value reaches three orders of magnitude. As both $\Delta_{GaAs}$ and $\epsilon_{\kappa_2, GaAs}$ are varied, the value of $\Re \{ \kappa_{eff} \}$ also changes. The point where $\Im \{ \kappa_{eff} \} = 0$ signifies a crossing of the two branches; whether it occurs in real or imaginary planes of the eigenspectrum depends precisely on the interplay between coupling strength and $\delta_{mode}$ at that point. For the case of $s = 200$ nm, a spike in the evolution of $\eta$ (figure 7(a)) indicates a perfect crossing point in the real plane of the eigenspectrum while this crossing point appears in the imaginary plane for $s = 150$ nm.

A parameter often used in the literature to characterise the distinguishability between these two supermodes is the phase rigidity, defined as $r = \left( A_L^f A_S^f \right) / \left( A_L^s A_S^s \right)$, where $A_L^f$ is the corresponding normalised left and right eigenvectors established on the basis of the bilinear product of the non-Hermitian system [25, 40, 41]. $|r|$ varies between 0 and 1 with $|r_L| = 1$ corresponding to orthogonal modes and $|r_L| = 0$ for the perfect coalescence of the two modes. The dependence of the phase rigidity on the gain spectrum is plotted in figure 7(b), where $|r_L| = |r_L| = |r|$. We notice that for imperfect PTS systems, the phase rigidity is greatly suppressed, meaning $|r|$ does not reach 0 at $\alpha_{EP}$ and a smoother minimum occurs instead of a sharp one observed for perfect PTS (see appendix D for details). With our recovery procedure, the sharp minimum behaviour of the phase rigidity near $\alpha_{EP}$ is recovered. However, due to an existing small imbalance in gain/loss, $|r|$ does not reach zero at $\alpha_{EP}$.

Further investigation of the system’s parameters could potentially bring $|r|$ closer to zero at $\alpha_{EP}$. 

Figure 6. (a) and (c) $\Re \{ \eta_0 \}$ and $\pm |\eta_0|$, (b) and (d) $\Re \{ \eta_{d\epsilon} \}$ and $\Im \{ \eta_{d\epsilon} \}$ as a function of $\Delta_{GaAs}$ for the two cases of $\Delta d = 50$ nm (right panel) and $\Delta d = -50$ nm (left panel) with $d_4 = 500$ nm. In each panel, the results of two separation distances are plotted, $s = 150$ nm (black lines), and $s = 200$ nm (green lines) at $\lambda = 850$ nm. Insets in (c) show the corresponding enlarged locations where bifurcation of the system occurs. $g_{s, \kappa}$ lie in the range of $-1000$ to $2000$ cm$^{-1}$. As seen in these plots, the numerical and CMT results agree very well, leading to almost overlapping dash and dotted lines.
6. Conclusion

CMT is robust to various degrees of asymmetry introduced to an ideal PTS system and produces excellent agreement with COMSOL simulations. CMT also provides an intuitive description of imperfect PTS systems in which CMT's parameters can be used to understand the detrimental effects of complex coupling and detuning to the achievement of a perfect EP behaviour. The arising of a complex coupling coefficient is associated with asymmetric coupling produced by an imbalance in gain and loss. Here, we find that the effects of a gain/loss imbalance can be conveniently described by \((\epsilon_1^p + \epsilon_2^p)/\beta_c\). For a system to retain parity-time symmetric behaviour, this ratio must be significantly less than 1. Gain-induced dispersion completely overpowers any coupling mechanism existing in a PTS system. Nonetheless, we found that by introducing a net gain/loss into the system together with a geometric asymmetry, we could shift the dispersion curve of a single waveguide and thus retrieve the condition of phase matching even when the core medium is highly dispersive. Inevitably, unequal gain and loss are introduced into the system, leading to a small residual splitting. The effectiveness of this recovery mechanism relies on our ability to engineer suitable modal asymmetry with bias pumping of the two waveguides such that

\[
\alpha_{\text{EP}} = \frac{(f_1^+(x) \epsilon_1^p + f_1^-(x) \epsilon_2^p)}{(f_1^+(x) \beta_c)} \ll 1.
\]

In this case, the characteristic sharp phase transition of parity-time symmetric systems is recovered with \(\eta\) increasing by three orders of magnitude near the position of \(\alpha_{\text{EP}}\). The effectiveness of the recovery procedure is also confirmed by a sharp minimum in the phase rigidity behaviour of the system at \(\alpha_{\text{EP}}\). The proposed system can potentially be used as a fast switching mechanism in a high gain regime \(1500 \sim \text{cm}^{-1}\) in the presence of strong dispersion with great flexibility in geometry and materials selection.

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Appendix A. Calculation of modal indices in lossy waveguides

For first order TE modes, the field function of each single waveguide of core width \(d = 2h\) takes the form of:

\[
E_j(x, z) = E_{TE} e^{ikz} \begin{cases} 
C e^{-\gamma (x-h)} & (x > h) \\
\cos(\kappa h) & (|x| \leq h) \\
C e^{(x+h)} & (x < -h) 
\end{cases},
\]

(A.1)

Boundary conditions at \(x = \pm h\) and conservation of momentum dictate that \(\cos(k_h) = C\) and

\[
\gamma = \sqrt{k_f^2 - k_c^2 - k_d^2}\text{ where } k_{f,c} = n_{f,c} k_0.\]

In a lossy environment, \(\gamma, k, \beta, C\) are allowed to be complex. To solve this transcendental eigenvalue, we employed an iterative steepest descent method with linear line search [36, 37]. Field normalisation gives \(E_{TE} = \sqrt{\frac{n_0}{2\pi(\gamma)}}\text{ where } d_{TE} = d + 2/\Re\{\gamma\}.\)
Appendix B. COMSOL simulation

We used mode analysis under the RF module in COMSOL v4.3a to solve for the effective refractive indices of the supermodes of this coupled gain-lossy system. PEC and PMC boundary conditions were used to help define predominantly TE modes in the system. A maximum element mesh size of 0.1 μm and a minimum element mesh size of 2 × 10⁻⁴ μm were implemented to achieve convergence. Gain and loss were modelled as negative and positive values of the imaginary parts of the material index. The refractive index of GaAs as a function of gain used in the simulation were the results of the calculation of carrier-induced refractive index change based on free carrier theory (see appendix C).

Appendix C. Dispersion property of GaAs in high gain environment

We introduce a gain model based on free carrier theory and use Kramers–Kronig relations to estimate the carrier induced refractive index change, χ as a function of gain in a bulk semiconductor, in this case, GaAs [32]. The amount of gain in a material can be directly linked to the imaginary part of its refractive index, χ = n'k₀ where k₀ is the propagation constant. Our gain model approximates the lineshape function as a hyperbolic secant distribution in order to remove absorption below bandgap. Taking into account spin–orbit interaction, we arrive at the following expression for G:  

\[ G = C \int_{0}^{\infty} \sqrt{E^2} \left( f_c(E') - f_c(E') \right) S(E') dE', \]  

(C.1)

where \( C = \frac{2\pi}{4\mu^2} \left( \frac{2m_e}{\epsilon_0} \right)^2 \) and \( S(E') = \left( \frac{E' - E_0 - E}{\gamma} \right) \). Amongst III–V semiconductors, GaAs is one of the most promising gain media, possessing high resistivity and carrier mobility. The following parameters have been used to calculate gain in bulk GaAs: \( E_0 = 1.424 \text{ ev}, \mu = 0.473 \text{ nm}, m_e^* = 0.067m_0, m_h^* = 0.52m_0, \gamma = 10^{-3} \text{ s}^{-1}. \) Our main findings are summarised in figure C1. Most importantly, our results correctly predict the trend in changing χ with gain, in agreement with values reported in the literature which consider many more complex effects such as bandgap shrinkage and intraband free-carrier absorption [38, 39]. For emission of GaAs at 850 nm, figures C1 (b) and (c) clearly show the drawback of having high gain in a PTS device. With more and more carriers made available, the increase in gain slows down, reaching a saturating region while χ continues to change rapidly.

Figure C2 shows χ for a PTS coupled system where both semiconductor waveguides have the same corewidth of 500 nm and a separation of 150 nm. The changing refractive index with increasing gain completely destroys any eigenvalue phase change in comparison with figures 6(a) and (c) where dispersion is compensated.
Appendix D. Characterisation of modes coalescence using the phase rigidity

The phases of the eigenfunctions are not fixed when gain/loss is introduced into the system [25, 40, 41]. In the case of a lossless coupled system, \( \sum_{n_{GaAs}} = 0 \), the two supermodes are distinct and orthogonal as the system is Hermitian. Once gain/loss is introduced, the value of \( r \) decreases as the modes become highly mixed and \( r \) sharply approaches zero at the EP where the two eigenfunctions are perfectly coalesced. This is clearly seen in figure D1(a) where a perfect PTS system is considered. We observe that for the case of imperfect PTS, these characteristics of \( r \) are significantly suppressed (figures D1(b) and (c)). In particular, when material dispersion is introduced into this identical coupled waveguides system with balanced gain/loss, \( |r| \) stays very close to 1 (figure D1(d)). This means that the huge effect of modal detuning generated by material dispersion overpowers any coupling effects and thus the two modes remain isolated throughout the gain spectrum.
References

[1] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[2] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H 2011 Int. J. Theor. Phys. 50 1019–41
[3] El-Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 Opt. Lett. 32 2632–4
[4] Bender C M 2003 Contemp. Phys. 46 277–92
[5] Rüter C E, Makris K G, El-ganainy R, Christodoulides D N, Segev M and Kip D 2010 Nat. Phys. 6 192–5
[6] Chang L, Jiang X, Hua S, Yang C, Wen J, Jiang L, Li G, Wang G and Xiao M 2014 Nat. Photon. 8 524–9
[7] Peng B, Özdemir K A, Lei F, Monifi F, Gianfreda M, Long G L, Fan S, Nori F, Bender C M and Yang L 2014 Nat. Phys. 10 394–8
[8] Peng L, Xu Y L, Fegadolli W S, Lu M H, Oliveira J E B, Almeida V R, Chen Y F and Scherer A 2013 Nat. Mater. 12 108–13
[9] Lin Z, Ramezani H, Eichelkraut T, Kottos T, Cao H and Christodoulides D N 2011 Phys. Rev. Lett. 106 213901
[10] Sun Y, Tan W, Li H Q, Li and Chen H 2014 Phys. Rev. Lett. 111 1–5
[11] Baum B, Aalaeian H and Dionne J 2015 J. Appl. Phys. 117 063106
[12] Longhi S 2010 Phys. Rev. A 82 031801
[13] Chong Y D, Ge L and Stone D A 2011 Phys. Rev. Lett. 106 035902
[14] Peng B, Özdemir K, Rottet S, Yilmaz H, Liertzer M, Monifi F, Bender C M, Nori F and Yang L 2014 Science 346 328–32
[15] Feng L, Wong Z J, Ma R M, Wang Y and Zhang X 2014 Science 346 972–5
[16] Hodaei H, Miri M A, Heinrich M, Christodoulides D N and Khajavikhan M 2014 Science 346 975–8
[17] Lupu A, Benisty H and Degiron A 2013 7th Int. Congress on Advanced Electromagnetic Materials in Microwaves and Optics (METAMATERIALS) ((16–21 September 2013)) pp 6–6
[18] Lupu A, Benisty H and Degiron A 2014 Photonics and Nanostructures - Fundamentals and Applications 12 305–11
[19] Lupu A, Benisty H and Degiron A 2015 Opt. Express 21 192–5
[20] Benisty H et al 2011 Opt. Express 19 18094
[21] Liu J M 2005 Photonic Devices (Cambridge: Cambridge University Press) (doi:10.1017/cbo9780511614255)
[22] Kong A 1986 Electromagnetic Wave Theory (New York: Wiley)
[23] Marcus D 1974 Theory of Dielectric Optical Waveguides (New York: Academic)
[24] Benisty H, Yan C, Degiron A and Lupu A 2012 J. Lightwave Technol. 30 2675–83
[25] Liu Z, Zhang Q, Liu X, Yao Y and Xiao J 2016 Sci. Rep. 6 22711
[26] Phang S, Vukovic A, Creagh S, Benson T M, Sewell P and Gradoni G 2015 Opt. Express 23 11493–307
[27] Guo a, Salerno G J, Duchesne D, Morandotti R, Volatier–Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 Phys. Rev. Lett. 103 093902
[28] Brandstetter M, Liertzer M, Deutsch C, Klang P, Schöberl J, Türeci H E, Strasser G, Unterrainer K and Rottet S 2014 Nat. Commun. 5 4034
[29] Lafone L, Sidiropoulos T P, Hamm J M and Oulton R F 2014 IET Optoelectron. 8 122–8
[30] Sigh J 1995 Semiconductor Optoelectronics Physics and Technology (New York: McGraw-Hill International Editions)
[31] Suhara T 2004 Semiconductor Laser Fundamentals (New York: Marcel Dekker INC)
[32] Chow W W and Koch S W 1999 Semiconductor–Laser Fundamentals: Physics of the Gain Materials 1st edn (Berlin: Springer)
[33] Haus H A and Huang W 1991 Proc. IEEE 79 1505–18
[34] Huang W P 1994 J. Opt. Soc. Am. A 11 963
[35] Little R E and Huang W P 1995 Prog. Electromagn. Res. 10 217–70
[36] Nagel J R, Blair S and Scarcellas A M 2011 Opt. Express 19 20159–71
[37] Snyman J A 2005 Practical Mathematical Optimization: An Introduction to Basic Optimization Theory and Classical and New Gradient-Based Algorithms (Applied Optimization) (Berlin: Springer)
[38] Bennett B R, Soref R A and Del Alamo J A 1990 IEEE J. Quantum Electron. 26 113–22
[39] Zoroochichi J and Butler J K 1973 J. Appl. Phys. 44 3697–9
[40] Rottet I 2007 J. Phys. A: Math. Theor. 40 41451
[41] Ding K, Ma G, Xiao M, Zhang Z and Chan C 2016 Phys. Rev. X 6 021007