GRAVITATIONAL INSTABILITY OF ROTATING, PRESSURE-CONFINED, POLYTROPIC GAS DISKS WITH VERTICAL STRATIFICATION

Jeong-Gyu Kim1,2, Woong-Tae Kim1,2, Young Min Seo2,3, and Seung Soo Hong2,4

1 Center for the Exploration of the Origin of the Universe (CEOU), Astronomy Program, Department of Physics and Astronomy, Seoul National University, Seoul 151-742, Republic of Korea; jgkim@astro.snu.ac.kr, wkim@astro.snu.ac.kr
2 FPRD, Department of Physics and Astronomy, Seoul National University, Seoul 151-742, Republic of Korea; seo3919@email.arizona.edu, sshong@astro.snu.ac.kr
3 Department of Astronomy and Steward Observatory, University of Arizona, Tucson, AZ 85721, USA
4 National Youth Space Center, Goheung, Jeolanamdo 548-951, Republic of Korea

Received 2012 August 8; accepted 2012 October 22; published 2012 December 4

Abstract

We investigate the gravitational instability (GI) of rotating, vertically stratified, pressure-confined, polytropic gas disks using a linear stability analysis as well as analytic approximations. The disks are initially in vertical hydrostatic equilibrium and bounded by a constant external pressure. We find that the GI of a pressure-confined disk is in general a mixed mode of the conventional Jeans and distortional instabilities, and is thus an unstable version of acoustic-surface-gravity waves. The Jeans mode dominates in weakly confined disks or disks with rigid boundaries. On the other hand, when the disk has free boundaries and is strongly pressure confined, the mixed GI is dominated by the distortional mode that is surface-gravity waves driven unstable under their own gravity and thus incompressible. We demonstrate that the Jeans mode is gravity-modified acoustic waves rather than inertial waves and that inertial waves are almost unaffected by self-gravity. We derive an analytic expression for the effective sound speed $c_{\text{eff}}$ of acoustic-surface-gravity waves. We also find expressions for the gravity reduction factors relative to a razor-thin counterpart that are appropriate for the Jeans and distortional modes. The usual razor-thin dispersion relation, after correcting for $c_{\text{eff}}$ and the reduction factors, closely matches the numerical results obtained by solving a full set of linearized equations. The effective sound speed generalizes the Toomre stability parameter of the Jeans mode to allow for the mixed GI of vertically stratified, pressure-confined disks.

Key words: hydrodynamics – instabilities – ISM: kinematics and dynamics – stars: formation – waves

Online-only material: color figures

1. INTRODUCTION

Gravitational instability (GI) plays a crucial role in the structure formation of various astronomical situations ranging from growth of primordial density fluctuations to star formation in disk galaxies (e.g., Zel’Dovich 1970; McKee & Ostriker 2007). GI in flattened systems is of particular importance in the formation of giant clouds in disk galaxies (e.g., Goldreich & Lynden-Bell 1965a; Elmegreen 1987; Kim et al. 2003), gas giant planets in protoplanetary disks (e.g., Boss 1997; Durisen et al. 2007), and bound clumps resulting from the fragmentation of shocked layers in star-forming regions (e.g., Elmegreen & Lada 1977; Elmegreen 1998).

While real astrophysical disks are vertically stratified, most analytic studies often neglect the vertical degree of freedom and adopt an isothermal equation of state (EOS) for simplicity. In a rotating, infinitesimally thin disk with surface density $\Sigma_0$ and sound speed $c_s$, the local dispersion relation for axisymmetric waves with frequency $\omega$ and wavenumber $k$ is given as

$$\omega^2 = c_s^2 k^2 - 2\pi G\Sigma_0 k + k_0^2, \quad (1)$$

where $k_0$ is the epicycle frequency and $G$ is the gravitational constant (e.g., Goldreich & Tremaine 1978; Binney & Tremaine 2008). If the disk is not rotating, sound waves with $k < k_1 = 2\pi G\Sigma_0/c_s^2$ would become gravitationally unstable. The Coriolis force arising from disk rotation stabilizes long-wavelength perturbations, making the disk Jeans stable if the Toomre stability parameter satisfies

$$Q_T \equiv \frac{c_s k_0}{\pi G\Sigma_0} > Q_{T,c} = 1 \quad (2)$$

for any $k$ (e.g., Toomre 1964). On the other hand, when $Q_T < 1$, thermal pressure and rotation are unable to stop the collapse of overdense regions.

The thin-disk approximation would be valid as long as vertical motions are unimportant and the scale of interest is much longer than the disk scale height. Nevertheless, it has been well known that finite disk thickness makes some quantitative changes to the characteristic wavelengths and critical $Q_T$ values. For instance, Ledoux (1951) showed that waves in self-gravitating, non-rotating, isothermal disks with scale height $H_0 = c_s^2/(\pi G\Sigma_0)$ become unstable if $k H_0 < 1$, which can be compared with the unstable condition $k H_0 < 2$ of the razor-thin counterpart (see also Simon 1965). For rotating disks, Goldreich & Lynden-Bell (1965a) considered the effect of the finite disk thickness and found that the stability condition changes to $Q_{T,c} = 0.676$. This decrease of the critical $Q_T$ value is due to the dilution of self-gravity at the disk midplane in a vertically stratified disk, which can approximately be treated by multiplying the gravity reduction factor $F = 1/(1 + k H_0)$ to the second term of the right-hand side of Equation (1) (e.g., Elmegreen 1987; Kim et al. 2002; Kim & Ostriker 2007).

While the results of GI in isothermal disks are informative, there are several issues that need clarification in more general situations. First of all, there are many classes of gaseous disks that do not have a constant temperature in the vertical direction. Examples include optically thick regions of planet-forming protoplanetary disks (e.g., Bell et al. 1997; D’Alessio et al. 1998; Boley et al. 2007) and accretion disks around compact objects (e.g., La Dous 1994; Lubow & Ogilvie 1998). These disks have often been modeled using a polytropic EOS (e.g., Lubow &
Ogilvie 1998; Nelson et al. 2000; Mamatsashvili & Rice 2010). While the behavior of various waves in polytropic disks has been studied extensively (e.g., Lin et al. 1990; Korycansky & Pringle 1995; Lubow & Ogilvie 1998; Ogilvie & Lubow 1999), the GI of such disks has not been explored in detail. Goldreich & Lynden-Bell (1965a) calculated a stability criterion of a uniformly rotating polytropic disk, but they were limited to a case with an adiabatic index of \( \gamma = 2 \). Larson (1985) studied the GI of polytropic disks with arbitrary \( \gamma \) by using approximate scaling relations instead of solving perturbation equations accurately. Recently, Mamatsashvili & Rice (2010) considered vertically stratified polytropic disks and argued that inertial modes rather than acoustic modes become unstable with self-gravitating perturbations. In this work, we shall use both analytic and numerical approaches to clarify that it is the acoustic modes rather than the inertial modes that become unstable.

Second, since the sound speed varies with height in polytropic disks, it is questionable what kind of average is most appropriate for \( c_s \) if one wants to use the stability condition (2). Several numerical studies used the vertically averaged sound speed (Mejia et al. 2005; Boley et al. 2006) or simply the midplane value (Rice et al. 2003, 2005) to evaluate the stability of protoplanetary disks, but the usage of these has yet to be justified. Mamatsashvili & Rice (2010) attempted to resolve this ambiguity by introducing \( Q_M = \Omega^2/(4\pi G \rho_{90}) \) as a three-dimensional stability parameter, where \( \Omega \) is the local orbital angular velocity and \( \rho_{90} \) is the midplane density, but the connection between \( Q_M \) and \( Q_T \) is uncertain since the former does not depend on \( c_s \) explicitly. In this work, we define the effective sound speed that represents the modal behavior of acoustic waves in a thermally stratified disk well and show that this naturally relates \( Q_T \) to \( Q_M \).

The third issue involves disk truncation by external pressure. Galactic disks are confined by the ram pressure of infalling gas (e.g., Dubois & Teyssier 2008) or hot halo gas (e.g., Goldreich & Lynden-Bell 1965a; Lee & Hong 2007). Thin shells produced by supernovae, stellar winds, or expanding H II regions are usually bounded by shocks (e.g., Deharveng et al. 2005; Churchwell et al. 2006, 2007). Elmegreen & Elmegreen (1978) showed that pressure-confined disks become unstable at scales smaller than the Jeans wavelength \( \lambda_J = c_s^2/(\pi G \Sigma_0) \) of unbounded disks. The mass of fragments produced by this GI is less than the critical Bonnor–Ebert mass, and thus may not necessarily experience gravitational runaway. When the confining pressure is very strong, Lubow & Pringle (1993) showed that the GI becomes essentially the same as that of an incompressible disk, which is in stark contrast to the conventional Jeans instability of compressional disks. Umekawa et al. (1999) and Wünsch et al. (2010) confirmed such predictions by using numerical simulations. Boyd & Whitworth (2005) suggested the GI of strongly confined disks as a potential candidate for forming “free-floating” planetary-mass objects (having a mass as low as \( \sim 0.003 M_\odot \)) detected in young star clusters (e.g., Zapatero Osorio et al. 2002). More elaborate theoretical and numerical models, including the effects of shell expansion and deceleration (e.g., Elmegreen 1989; Iwasaki et al. 2011; Dale et al. 2011), thermal and chemical processes (Hosokawa & Inutsuka 2006), boundary conditions (BCs, e.g., Elmegreen 1989; Usami et al. 1995; Dale et al. 2009), magnetic fields (Nagai et al. 1998), etc., examine the consequences of GI that occur under high external pressure.

Despite these efforts, however, the physical mechanism behind the GI of strongly pressure-confined disks still remains controversial. For example, Lubow & Pringle (1993) argued that the instability is due to a neutral mode that exists because external pressure can hold the layer to any distorted shape, while Elmegreen (1989) and Umekawa et al. (1999) claimed that the distortion of the surfaces exerts a pinching force that causes the distorted disk to collapse. On the other hand, Wünsch et al. (2010) and Dale et al. (2011) considered the collapse of an oblate spheroid with uniform density pressurized by an ambient medium (see also Boyd & Whitworth 2005), and termed the enhanced GI “pressure-assisted GI,” although the physical processes responsible for it were not identified. Very recently, Iwasaki et al. (2011) studied the GI of pressure-confined shells around expanding H II regions taking into account boundary effects. They suggested from the vertical behaviors of eigenfunctions that the restoring force of the unstable mode comes from surface-gravity waves. We use both numerical and analytic approaches to confirm that it is indeed surface-gravity waves that become unstable in strongly confined disks.

In this paper, we investigate the GI of rotating, vertically stratified, pressure-confined gas disks. We adopt simple polytropic, rotating models of Goldreich & Lynden-Bell (1965a) and extend them to pressure-confined situations. This work also extends the case of non-rotating, isothermal models of Elmegreen & Elmegreen (1978) to rotating, polytropic disks. Our objectives are four-fold. First, we wish to distinguish conventional Jeans instability from the GI arising from the surface distortion of strongly pressure-confined disks, and give a clear physical explanation for the latter that has previously been confusing. Second, we provide analytic expressions for the effective sound speed \( c_{s,\text{eff}} \) that best represents the modal behavior of sound waves in polytropic disks. Third, we find the gravity reduction factors caused by finite disk thickness for the Jeans and distortional modes of GI. We show that the numerical dispersion relations found by solving a full set of perturbation equations are well matched by Equation (1) provided that \( c_s \) is changed to \( c_{s,\text{eff}} \) and that the appropriate reduction factor is considered in the gravity term. Finally, we generalize the Toomre stability parameter of razor-thin disks to vertically stratified, pressure-bounded disks. While our disk models are simple with a polytropic EOS and without considering external gravity, some qualitative properties of GI clarified in this paper can hold true in more general situations, such as disks with a barotropic EOS and/or with external gravity.

The remainder of this paper is organized as follows. In Section 2, we introduce basic equations, construct an equilibrium model, and present linearized perturbation equations. In Section 3, we derive expressions for the effective sound speed \( c_{s,\text{eff}} \) that describe the phase speeds of fundamental modes in non-self-gravitating, non-rotating disks. In Section 4, we present the gravity reduction factors \( F \) and show that an approximate dispersion relation using \( c_{s,\text{eff}} \) and \( F \) is in excellent agreement with the numerical results obtained by solving the perturbation equations. For rotating disks, we define a stability parameter using \( c_{s,\text{eff}} \) that is applicable also to the distortional mode. In Section 5, we summarize and discuss our results.

2. FORMULATIONS

We consider a rotating, self-gravitating gaseous disk in vertical hydrostatic equilibrium and analyze its stability to small-amplitude, axisymmetric perturbations. The disk is infinitely extended in the horizontal direction, but truncated at a finite height by a tenuous external medium with pressure \( p_{\text{ext}} \).
For local modes in the horizontal direction, it is advantageous to consider a local Cartesian reference frame whose center lies at a radius $R_0$ and orbits the disk center with a fixed angular velocity of $\Omega_0 = \Omega(R_0)$. In this local frame, radial, azimuthal, and vertical coordinates are represented by $x = R - R_0$, $\phi = \phi - \Omega_0 t$, and $z$, respectively, and the terms arising from the curvature effect are ignored (e.g., Goldreich & Lynden-Bell 1965b; Julian & Toomre 1966; Kim et al. 2002). Assuming that the shear rate $q \equiv -d \ln \Omega/d \ln R \mid_{R_0}$ is uniform, the equilibrium background velocity field is given by $u_0 = -q \Omega_0 x e_x$, and the local epicyclic frequency is $\kappa_0 = 4(1 - 2q) \Omega_0^2$. The equations of ideal hydrodynamics expanded in the local frame are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \psi + 2q \Omega_0^2 x e_x - 2 \Omega_0 \times \mathbf{u},$$

$$\nabla^2 \psi = 4\pi G \rho,$$

where $\rho$, $\mathbf{u}$, $p$, and $\psi$ denote mass density, velocity, thermal pressure, and the self-gravitational potential of the gas, respectively. To close the set of Equations (3)–(5), we assume that the gas follows a polytropic EOS of

$$p = K \rho^{\gamma},$$

where $K$ is an arbitrary constant and $\gamma$ is the polytropic exponent. If $\gamma$ is different from an adiabatic exponent of disturbances, the system under consideration exhibits complicated modal behaviors associated with convective motions. However, Mamatsashvili & Rice (2010) showed that the convective modes do not affect the properties of GI very much. In this work, we thus simply take $\gamma$ to be equal to the adiabatic exponent.

### 2.1. Initial Equilibria

We first explore the equilibrium profiles of density and temperature in purely self-gravitating, polytropic disks confined by an external medium. The pressure confinement model we adopt is a straightforward extension of isothermal disks studied by Elmegreen & Elmegreen (1978) to polytropic cases. For hydrostatic equilibrium in the vertical direction, Equations (4) and (5) require

$$\frac{1}{\rho_0} \frac{d \rho_0}{dz} = -\frac{d \psi_0}{dz},$$

$$\frac{d^2 \psi_0}{dz^2} = 4\pi G \rho_0,$$

where the subscript “0” denotes the unperturbed state.

Following Elmegreen & Elmegreen (1978), it is convenient to convert the vertical coordinate $z$ to the dimensionless variable $\mu$ defined by

$$\mu \equiv \frac{1}{4\pi G \rho_{00} H_0} \frac{d \psi_0}{dz},$$

where $\rho_{00} = \rho_0(0)$ and $H_0$ is the disk scale height

$$H_0^2 \equiv \frac{K}{2\pi G \rho_{00}^{2-\gamma}}.$$

In Appendix A, we show that Equations (7) and (8), subject to the symmetry condition $d \rho_0 / dz = 0$ at $z = 0$, yield the solution

$$\rho_0 = \rho_{00}(1 - \mu^2)^{1/\gamma},$$

(see also Harrison & Lake 1972). The corresponding local speed of sound is

$$c_s^2 = \gamma p_0 / \rho_0 = c_{00}^2(1 - \mu^2)^{1/\gamma},$$

where $c_{00} \equiv (\gamma p_{00}/\rho_{00})^{1/2} = (\gamma K \rho_{00}^{\gamma-1})^{1/2}$ is the midplane sound speed.

Figure 1 plots the equilibrium structures of self-gravitating polytropic disks with differing $\gamma$. For a softer EOS (smaller $\gamma$), the disk is bounded by a higher equilibrium density, and the column density of the bounded layer is given by

$$\Sigma_0 = \int_{-a}^{+a} \rho_0 dz = 2\rho_{00} AH_0 = 2\rho_{00} H,$$

where $H \equiv AH_0$ is the effective half-thickness of the layer. The relative density drop at the boundaries is $\rho_{0a}/\rho_{00} = (1 - A^2)^{1/\gamma}$ where $\rho_{0a} = \rho_0(z = a)$. Note that $A = 1$ when $a = |z_{cut}|$, corresponding to no pressure truncation. As $p_{ext}$ increases, the vertical sound crossing time ($\sim H/c_{00}$) becomes smaller than the internal free-fall time ($\sim 2\pi G \rho_{00}^{-1/2}$). In the limit of $A \rightarrow 0$, the disk takes a uniform density and $H \rightarrow a$ regardless of $\gamma$ in this case, self-gravity is so weak that the initial equilibrium is maintained by a balance between $p_{ext}$ and $p_{00}$.

### 2.2. Perturbation Equations

We apply small-amplitude perturbations to the initial equilibrium configurations described above. Equations (3)–(5) are linearized to

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho_0} \nabla p_0 - \frac{1}{\rho_0} \nabla \psi_1 + q \Omega_0 \times \mathbf{e}_y - 2 \Omega_0 \times \mathbf{u},$$

$$\nabla^2 \psi_1 = 4\pi G \rho_1,$$

where the subscript “1” indicates the perturbed quantities.
We further assume that perturbations are axisymmetric and of a plane-wave type

\[ q_1(x, z, t) = q_1(z) \exp(i \omega t + ik_x x), \]

where \( q_1 \) refers to any perturbed variable with frequency \( \omega \) and radial wavenumber \( k_x \). Then, Equations (16)–(18) can be reduced to four first-order ordinary differential equations

\[ \frac{d\xi_1}{dz} = \frac{k_x^2}{\omega^2 - \kappa_0^2} (h_1 + \psi_1) - \frac{1}{c_s^2} (h_1 - g_0 \xi_1), \]

\[ \frac{dh_1}{dz} = \omega^2 \xi_1 - \psi_1', \]

\[ \frac{d\psi_1}{dz} = \psi_1', \]

\[ \frac{d\psi_1'}{dz} = k_x^2 \psi_1 + \frac{4\pi G \rho_0}{c_s^2} h_1, \]

where \( \xi_1 \) is the vertical Lagrangian displacement defined through \( u_{1z} = \partial \xi_1 / \partial t, h_1 = p_1 / \rho_0 \) is the perturbed enthalpy, and \( \psi_1' = d\psi_1 / dz \). We take \( \xi_1, h_1, \psi_1, \) and \( \psi_1' \) as four independent variables.

The set of the perturbation equations (20)–(23) is to be integrated numerically along the vertical direction to yield a dispersion relation of \( \omega = \omega(k_x) \) that satisfies BCs as well as the constraints of even-symmetry modes. To separate the distortional modes of the GI from the conventional Jeans modes, we consider two distinct conditions at the boundaries: the rigid boundary that allows only the Jeans modes and the free boundary in which Jeans and distortional modes coexist. Appendix B describes the BCs that we adopt. Appendix C presents the numerical method we follow to obtain full dispersion relations, which will be compared with approximate dispersion relations in Sections 3 and 4.

For future purposes, we note that the total perturbed density \( \rho_1 \) is a superposition of the perturbed density \( \rho_{1,i} \) inside the disk due to wave motions and the perturbed density \( \rho_{1,s} \) at \( z = \pm a \) due to the surface distortion:

\[ \rho_1 = \rho_{1,i} + \rho_{1,s}, \]

where

\[ \rho_{1,i} = \rho_{1,i} + \rho_{1,s}, \]

\[ \rho_{1,s} = \rho_{0s} \delta(|z| - a) \]

for even-symmetry modes. Here, \( \delta \) is the Dirac delta function. The corresponding perturbations in the surface density can be written as \( \Sigma_1 = \Sigma_{1,i} + \Sigma_{1,s} \). We define

\[ w \equiv \frac{\Sigma_{1,i}}{\Sigma_{1,i} + \Sigma_{1,s}} \]

as the fraction of the perturbed surface density inside the disk relative to the total perturbed surface density. We will show that \( w \), defined in Equation (26), is a key parameter that controls the relative importance of the Jeans modes to the distortional modes.
2.3. Classification of Local Modes

Ignoring waves arising from surface distortion, the perturbation equations (20)–(23) give rise to two principal types of waves: inertial waves and acoustic waves, both modified by self-gravity. The former is characterized by epicycle motions, while the latter is based on gas compressibility. Mamatsashvili & Rice (2010) argued that inertial modes, rather than acoustic modes, are strongly influenced by self-gravity to become unstable. In Appendix D, we classify two types of waves using local dispersion relations and compare them with the numerical results. We directly demonstrate that while the inertial modes are not strongly affected by self-gravity and thus remain stable, it is the fundamental acoustic modes that can become unstable in the presence of self-gravity. This makes sense since both acoustic waves and self-gravity rely on density perturbations, while inertial waves are incompressible in nature (e.g., Latter & Balbus 2009).

Before exploring numerical dispersion relations, some physical insight into the fundamental acoustic waves in disks can be gleaned by integrating Equation (16) and the horizontal component of Equation (17) multiplied by $\rho_0$ over $z$. With the help of Equation (19), we obtain

$$\omega^2 = \frac{k_y^2}{\Sigma_1} \hat{h}_1 + \frac{k_z^2}{\Sigma_1} + \frac{\hat{h}_1}{\Sigma_1} + \frac{\hat{\psi}_1}{\Sigma_1},$$

(27)

where the hat indicates the density-weighted vertical averages (i.e., $\hat{h}_1 = \int \rho_0 h_1 dz/\Sigma_0$). Because $\hat{h}_1$, $\hat{\psi}_1$, and $\Sigma_1$ are interrelated with each other and generally depend on $k_x$ and $\omega$, one needs to solve Equations (20)–(23) numerically to obtain full dispersion relations. Nevertheless, Equation (27) implies that $\omega^2$ consists of three terms, each responsible for the effect of pressure (including thermal pressure and surface distortion), self-gravity, and the Coriolis force, as in Equation (1). This opens a possibility that by finding the solutions of $\hat{h}_1$ and $\hat{\psi}_1$ independent of $\omega$ in some suitable limits, one can obtain an approximate dispersion relation that matches the numerical results closely. We will focus on this in the remainder of this paper.

3. NON-Self-Gravitating Fundamental Modes

As mentioned before, the sound speed of polytropic disks varies with vertical height, so it is interesting to find an effective sound speed that best represents modal behavior of waves propagating throughout the disk. Also of interest is the effect of surface distortion on sound waves in the case of free boundaries. To address these issues, in this section we limit ourselves to the fundamental acoustic modes in non-self-gravitating disks. For simplicity, we ignore the effect of disk rotation, which merely adds $\kappa_0^2$ to $\omega^2$ (see Equation (27)).

3.1. Rigid Boundary

In non-self-gravitating, non-rotating disks, Equation (27) is reduced to

$$\omega^2 = \frac{\Sigma_1 \hat{h}_1}{\Sigma_1} + \frac{k_y^2}{\Sigma_1} = \int \rho_0 h_1 dz/\Sigma_0 = \frac{c_{\text{eff}}^2 k_x^2}{\Sigma_1},$$

(28)

where $c_{\text{eff}}$ is the effective sound speed. When the boundaries are rigid or $A = 1$, $\Sigma_{1,0} = 0$ and

$$c_{\text{eff}}^2 = \frac{\int \rho_1 h_1^2 dz}{\int \rho_1 dz} = \frac{\int_{-a}^{a} \rho_0 h_1 dz}{\int_{-a}^{a} \rho_0 dz/c_{\text{eff}}^2 dz}.$$  

(29)

Figure 2. Dispersion relations of the fundamental acoustic modes in non-self-gravitating, non-rotating, vertically stratified, polytropic disks with $A = 1$ and $\gamma = 0.8, 1.5, 2$. The frequency $\omega$ in the ordinate is normalized by $c_{\text{eff}} k_x$. Horizontal dotted lines mark $c_{\text{eff}}^2/\rho_0$ (see $a$) (Equation (31)) in the long-wavelength limit.

(A color version of this figure is available in the online journal.)

Equation (29) shows that excited waves are pure acoustic modes ($p$ modes) whose restoring force is thermal pressure. Obviously, $c_{\text{eff}} = c_0$ in an isothermal disk. Note that we did not make any approximation in deriving Equations (28) and (29) except for ignoring the effects of self-gravity and rotation.

Now consider the fundamental mode in the long-wavelength limit. Appendix E shows that the fundamental acoustic waves have a simple solution of $h_1(z) = \text{constant}$ in the limit of $k_x \rightarrow 0$. Equation (29) is then simplified to

$$c_{\text{eff}}^2 \equiv c_x^2 = \frac{\Sigma_1}{\rho_0 c_{\text{eff}}^2} \int_{-a}^{a} \rho_0 dz = \frac{1}{\alpha},$$

(30)

indicating that the effective sound speed corresponds to the density-weighted, harmonic mean of the local sound speeds. For polytropic disks, the ratio of the effective sound speed to the midplane sound speed is given analytically by

$$\alpha \equiv \frac{c_{\text{eff}}^2}{c_x^2} = 2F_1 \left(\frac{1}{2}, 1 - \frac{1}{\gamma}; 3; A^2 \right)^{-1},$$

(31)

where $2F_1$ denotes the Gaussian hypergeometric function. Disks with $\gamma > 1$ have $\alpha < 1$ since the temperature decreases monotonically with height. When there is no external confining medium ($A = 1$), $\alpha$ varies smoothly from 3/2 to $2/\pi$ for $0.5 < \gamma < 2$. Larger external pressure makes $\alpha$ closer to unity.

To check the applicability of Equation (30) for waves with finite wavelengths, we obtain numerical dispersion relations of pure acoustic waves in non-self-gravitating, non-rotating disks for $A = 1$ and $\gamma = 0.8, 1, 1.2, 2$, by following the procedures given in Appendix C. Figure 2 plots the resulting numerical dispersion relations using solid lines, which can be compared with the approximate dispersion relation (28) together with Equation (30) shown as dotted lines. When $\gamma = 1$, the dispersion relation is simply $\omega^2 = c_x^2 k_x^2$. When $\gamma \neq 1$, the...
numerical dispersion relations progressively deviate from the analytic results with increasing $k_x H$. When $k_x H \lesssim 1$, however, the two agree within $\sim 4\%$ for $0.8 < \gamma < 2$. The agreement becomes better for disks with smaller $A$. Since GI occurs for waves with $k_x H \lesssim 1$ when self-gravity is included, this suggests that $c_{\text{eff}}$ given in Equation (30) is a good representative of the average sound speed.

### 3.2. Free Boundary

In the case of free BCs, $\Sigma_{i,s}/\Sigma_{i} = \rho_{00} k_x \int_{0}^{\infty} \rho_{0s} d z$. Then, the effective sound speed in Equation (28) becomes

$$c_{\text{eff}}^2 = \left( \frac{\int_{0}^{\infty} \rho_{0} h_{1} d z}{\int_{0}^{\infty} \rho_{0} h_{1} d z} + \frac{\rho_{0} h_{1a}}{\rho_{0a} h_{1a}} \right)^{-1},$$

(32)

where Equation (B3) is used. Equation (32) indicates that waves in a pressure-confined disk with free boundaries make use of two restoring forces: thermal pressure and vertical gravity. The former drives acoustic waves, while the latter is responsible for surface-gravity waves. We term these mixed waves acoustic-surface-gravity waves. The fact that the effective sound speed has two terms, each arising from sound waves and surface-gravity waves, is completely analogous to longitudinal sound waves propagating in a distensible tube where the effective compressibility of a gas is the sum of the true gas compressibility and the distensibility of the tube (e.g., Lighthill 1978).

Since $\rho_{0a} = \rho_{00}(1 - A^2)\gamma$ and $\rho_{0a} = 4\pi G \rho_{00} A H_0$, it is apparent that the acoustic term in Equation (32) dominates for $A$ close to unity. On the other hand, the surface-gravity term becomes important in disks with $A \ll 1$. Appendix E shows that in highly confined disks, even fundamental modes have a particular solution of $h_1 \propto \cosh(k_x z)$. Equation (32) is then reduced to

$$c_{\text{eff}}^2 = \left( \frac{1}{c_s^2} + \frac{\rho_{0} / \rho_{00}}{g_{0a} H \tanh(k_x H)} \right)^{-1} = w \tilde{c}_s^2$$

(33)

with $w$ given by Equation (26). The sonic contribution $\tilde{c}_s$ is valid for $k_x H \lesssim 1$, as in the rigid BC case. The contribution from surface-gravity waves accounts for the reduced density at the surfaces. In the limit of $A \ll 1$, Equations (28) and (33) recover the usual dispersion relation of surface-gravity waves with

$$\omega^2 = g_{0a} k_x \tanh(k_x H)$$

(34)

in an incompressible medium (e.g., Goldreich & Lynden-Bell 1965a).

Although Equation (33) is derived for a disk with either $A \ll 1$ or $A \sim 1$, we find that it is applicable for arbitrary $A$ as long as $k_x H \lesssim 1$. Figure 3 compares Equation (33) (dotted lines) with the direct numerical solutions (solid lines) found by solving the full perturbation equations (with self-gravity and rotation neglected) for moderately confined disks with $A = 0.8$ and $\gamma = 0.8, 1, 1.5, 2$. The discrepancies between the two are less than 0.5% for $k_x H \lesssim 1$. This proves that $c_{\text{eff}}$ given in Equation (33) is excellent in describing the phase velocity of the acoustic-surface-gravity waves in the long-wavelength regime.

From Equation (33), it is now clear that the variable $w$ defined by Equation (26) is a weight factor representing the relative importance of acoustic to surface-gravity terms in the dispersion relation of acoustic-surface-gravity waves. In the long-wavelength limit, Equation (32) leads to

$$w_0 \equiv w|_{k_x \rightarrow 0} = \left[ 1 + \gamma \alpha (1 - A^2)^{1/\gamma} \right]^{-1} / 2 A^2.$$  

(35)

When $\gamma = 1$, $w_0 = 2 A^2 / (1 + A^2) = (1 + p_{\text{ext}} / \pi G \Sigma_0^2)^{-1}$. Figure 4 plots $w_0$ as functions of $A$ for disks with $\gamma = 0.8, 1, 1.5, 2$. As expected, surface-gravity modes dominate for smaller $A$ and...
larger $\gamma$. From Equations (26) and (35), one finds that the mean amplitude of density perturbation inside the disk is given by

$$\Sigma_{1,i} = \frac{2A^2 \xi_{1,i}|_{z=a}}{\gamma \alpha H},$$  

while $\Sigma_{1,i}/\Sigma_t = (1 - A^2)^{1/\gamma} \xi_{1,i}|_{z=a}/H$ at the disk surfaces. Therefore, $|\Sigma_{1,i}/\Sigma_t| \ll 1$ for small $A$. This clearly demonstrates that the fundamental mode and its unstable version (upon inclusion of self-gravity) are incompressible in highly pressure-confined disks (e.g., Elmegreen & Elmegreen 1978; Lubow & Pringle 1993; Nagai et al. 1998; Umekawa et al. 1999).

4. GRAVITATIONAL INSTABILITY

In the preceding section, we derive expressions for the effective sound speed for fundamental acoustic or acoustic-surface-gravity waves in non-self-gravitating and non-rotating disks. In this section, we derive the gravity reduction factors caused by finite disk thickness that explain the numerical dispersion relations of GI fairly well. We begin by considering non-rotating, zero-temperature disks, and then include the effects of thermal pressure and rotation.

4.1. Pressureless Disks

Finite disk thickness is known to stabilize the conventional Jeans instability by diluting self-gravity at the disk midplane. As mentioned in the introduction, $F = (1 + k_x H_0)^{-1}$ is often used as the gravity reduction factor for an unbounded, exponential disk with scale height $H_0$ (e.g., Elmegreen 1982, 1987; Kim et al. 2002). In this section, we derive reduction factors for conventional Jeans modes and the distortional modes in truncated polytropic disks.

We first consider a pressureless, non-rotating disk (i.e., $\tilde{h}_1 = k_0^2 = 0$) for simplicity. For the given $\rho_1$, Equation (18) has a formal solution

$$\psi_1(z) = -\frac{2\pi G}{k_x} \int \rho_1(z') e^{-k_x |z - z'|} dz'$$  

(e.g., Kim & Ostriker 2007). Substituting Equation (37) into Equation (27), one obtains

$$\omega^2 = -\frac{2\pi G \Sigma_0 H}{k_x} F(k_x),$$  

where

$$F(k_x) = \frac{1}{\Sigma_0 \Sigma} \int \rho_0(z) \rho_1(z') e^{-k_x |z - z'|} dz dz'$$

is the generalized reduction factor of self-gravity at wavenumber $k_x$. It is apparent from Equation (39) that $F = 1 + O(k_x H)$ in the long-wavelength limit, regardless of density distributions.

Using Equations (25) and (26), we decompose Equation (39) into two parts as

$$F(k_x) = w F_1(k_x) + (1 - w) F_D(k_x),$$

where

$$F_1(k_x) = \frac{1}{\Sigma_0 \Sigma_{1,i}} \int \rho_0(z) \rho_{1,i}(z') e^{-k_x |z - z'|} dz dz'$$

and

$$F_D(k_x) = \frac{e^{-k_x a}}{\Sigma_0} \int \rho_0(z) \cosh(k_x z) dz$$

represent the reduction factors of self-gravity for the conventional Jeans modes and distortional modes, respectively.

We integrate Equation (41) for an unbounded ($A = 1$), isothermal disk with $\rho_0 \propto \sech^2(|z|/H_0)$. The resulting $F_1$ with $\rho_{1,i} \propto \rho_0$ is plotted in Figure 5 as a dot-dashed line, while the case with $\rho_{1,i}$, numerically obtained from the full solution of Equations (20)–(23), is shown as a solid line. Note that both are closely approximated by $(1 + k_x H)^{-1}$, as shown as a dotted line. For pressure-confined disks, we empirically found by varying $A$ as well as the BCs that

$$F_1 = \frac{1}{1 + k_x H A^{1/2}} \quad \text{for rigid BC}$$

and

$$F_1 = \frac{1}{1 + k_x H} \quad \text{for free BC}$$

match the numerical dispersion relations of GI quite well (see below). The difference between Equations (43) and (44) is due to the fact that $\rho_{1,i}$ in Equation (41) is affected by the BCs.

For small $A$, $F$ is mostly from $F_D$ (see Equation (40)). In this case, the distortional modes resulting from $\rho_{1,i}$ dominate, and it is reasonable to take $\rho_0 \approx \text{constant}$ and $a \approx H$ in the evaluation of $F_D$. Equation (42) then becomes

$$F_D = \frac{e^{-k_x a} \sinh k_x H}{k_x H},$$

representing a correction factor averaged along the $z$-direction. Nevertheless, Equation (43) matches the reduction factor for $\sech^2(|z|/H)$ disks better than $F_1^{\exp}$.

Figure 5. Various gravity reduction factors as functions of $k_x H$. The solid line plots $F_1$ by taking $\rho_0 \propto \sech^2(|z|/H_0)$ and $\rho_1$ from the eigenfunctions of Equations (20)–(23), while the dot-dashed line gives $F_1$ by assuming $\rho_{1,i} \propto \rho_0 \propto \sech^2(|z|/H_0)$. The reduction factor $F_D$ (Equation (45)) of the distortional modes is plotted as a dashed line. For comparison, the commonly used $F = (1 + k_x H)^{-1}$ is plotted as a dotted line.
which is plotted as a dashed line in Figure 5. Incidentally, $\mathcal{F}_D$ is not much different from $(1 + k_x H)^{-1}$, either.

4.2. Non-rotating Disks

We now explore the GI of non-rotating, pressure-confined disks with the effect of thermal pressure included. Combining the results of Sections 3 and 4.1, we write an approximate dispersion relation of

$$\omega^2 = c_{\text{eff}}^2 k_x^2 - 2\pi G \Sigma_0 k_x (w \mathcal{F}_1 + (1 - w) \mathcal{F}_D).$$

(46)

where $c_{\text{eff}}$ is given by either Equation (30) or Equation (33) depending on the BCs.

4.2.1. Rigid Boundary

First, we impose the rigid BCs with $w = 1$. Figure 6(a) plots the dispersion relations of waves with (solid lines) and without (dashed lines) self-gravity in disks with $\gamma = 1$ and $A = 0.8$ for the fundamental acoustic modes (thick) as well as the first harmonics (thin) obtained numerically by solving the perturbation equations (20)–(25). The approximate dispersion relation (46) for the fundamental mode is plotted as dotted lines, which is in good agreement with the numerical results for $k_x H \lesssim 2$. Clearly, self-gravity lowers the frequency, and it is the fundamental mode that becomes unstable upon inclusion of self-gravity. GI under the rigid BCs is the conventional Jeans instability since surface distortion is absent.

Figure 7 plots as solid lines the numerical dispersion relations of pure Jeans modes for some selected values of $A$ and $\gamma$, which agree quite well with Equation (46) drawn as dotted lines. The relative differences between the true and approximate values of the maximum growth rates and critical wavenumbers are less than 5% for $0.8 \leq \gamma \leq 2$ and $0 \leq A \leq 1$. For fixed $\Sigma_0$ and $c_{\text{eff}}$, the Jeans modes are more unstable for smaller $A$ because the disk becomes geometrically thinner. In the limit of $A \to 0$, the dispersion relations of the GI in pressure-confined disks become identical to those of an infinitesimally thin disk with the same $\Sigma_0$ and $c_{\text{eff}}$, shown as dashed lines in Figures 7(a)–(c). For fixed $A$, disks with larger $\gamma$ are more unstable owing to a steeper temperature gradient as well as a smaller vertical scale height ($H_0 = A c_{\text{eff}}^2/(\pi G \Sigma_0 \gamma) \propto \gamma^{-1}$). Figure 7(d) replots Figure 7(a) by scaling $\omega$ and $k_x$ in terms of $c_{\text{eff}}$ rather than the midplane sound speed $c_{\text{eff}}$. Close agreement among the curves with different $\gamma$ shows that the effective sound speed defined in Equation (30) is really a good representative of the mean averaged sound speed of polytropic disks.

4.2.2. Free Boundary

Under free BCs, perturbations experience two types of restoring forces: compressibility and surface gravity. The former and latter dominate when $A \sim 1$ and $A \ll 1$, respectively. For arbitrary $A$ these are mixed together to become acoustic-surface-gravity waves. Figure 6(b) plots the numerical (solid lines) and approximate (dotted lines; Equation (46)) dispersion relations for the fundamental modes (thick) and the first harmonics (thin) in disks with $A = 0.8$ and $\gamma = 1$. Non-self-gravitating waves are plotted as dashed lines, while solid lines draw self-gravitating waves. The close agreement between the numerical and analytic results shows that the fundamental modes under the free BCs are really acoustic-surface-gravity waves that become unstable when self-gravity is included. In what follows, we call the unstable acoustic-surface-gravity modes the mixed GI.

Figure 8 plots as solid lines the numerical dispersion relations of the acoustic-surface-gravity modes in non-rotating, pressure-confined disks with various values of $A$ and $\gamma$. The wavenumber and growth rate are normalized by the effective half-thickness $H$ and the free-fall time at the midplane, $(2\pi G \rho_0)^{1/2}$, respectively. The cases with $\gamma = 1$ recover the results of earlier studies (e.g., Simon 1965; Elmegreen & Elmegreen 1978; Usami et al. 1995; Iwasaki et al. 2011). Shown also as dotted lines are the approximate dispersion relation in good agreement with the numerical results. Regardless of $A$, the maximum growth rate, $|\omega_{\text{max}}|$, of unstable modes occurs at $k_x H \sim 0.3$–0.4. However, the physical nature of GI is markedly different depending on the degree of pressure confinement. When the external pressure is weak ($A \sim 1$), unstable modes are dominated by acoustic-type perturbations ($w \sim 1$), making the mixed GI under free BCs similar to the pure Jeans mode that occurs under rigid BCs. When perturbations from surface distortion dominate ($A \ll 1$),
The Astrophysical Journal, 761:131 (17pp), 2012 December 20
Kim et al.

Figure 7. (a)–(c) Dispersion relations of the pure Jeans modes in non-rotating, pressure-confined, polytropic disks under rigid BCs. The wavenumber and growth rate are normalized by using the midplane sound speed. The solid lines are the numerical results, while the approximate dispersion relations (Equation (46)) are plotted as dotted lines. The razor-thin dispersion relation ($A \rightarrow 0$) is shown as a long dashed line in each panel. (d) The $A = 1$ case plotted in panel (a) is redrawn using $c_{\text{eff}}$ in the normalization of $\omega$ and $k_x$. (A color version of this figure is available in the online journal.)

however, Equation (46) is reduced to

$$\omega^2 = g_0 k_x \tanh(k_x H) - 2\pi G \Sigma_0 k_x \frac{1 - e^{-2k_x H}}{2k_x H}$$

(47)

for the pure distortional instability (e.g., Goldreich & Lynden-Bell 1965a).

Figure 9 compares the marginal wavenumber $k_{x,\text{c}}$, the most unstable wavenumber $k_{x,\text{max}}$, and the maximum growth rate $\omega_{\text{max}}$ of the pure Jeans instability with those of the mixed GI of isothermal disks as functions of $A$. The mixed GI is identical to the pure Jeans mode when $A = 1$, but the former in general has a higher growth rate and occurs at a shorter wavelength than the latter for arbitrary $A$. This is because the effective sound speed varies with $A$ differently depending on the BCs. For $A \ll 1$, $c_{\text{eff}} \sim c_0$ for the pure Jeans modes (Equation (31)), while $c_{\text{eff}} \sim \sqrt{2} \alpha \gamma \pi G \rho_0 H_0 \propto A c_0$ for the mixed GI (Equation (33)). Since both types of GIs have the characteristic wavenumber $k_{x,\text{c}} \sim \pi G \Sigma_0 / c_{\text{eff}}^2$ and a growth rate of $|\omega| \sim k_{x,\text{c}} / c_{\text{eff}}$, the critical wavenumber and maximum growth rate of the Jeans modes are smaller by a factor of $A^2$ and $A$, respectively, compared to those of the mixed GI.

4.3. Rotating Disks and Stability Criteria

Finally, we include the effect of disk rotation and study the stability condition of axisymmetric GI. In view of Equation (27), one may assume that the gas motions arising from thermal pressure and self-gravity are separable from the epicycle motions, which is shown as valid for $k_x H \lesssim 1$ in the preceding sections. The approximate dispersion relation of GI then becomes

$$\omega^2 = c_{\text{eff}}^2 k_x^2 - 2\pi G \Sigma_0 k_x \mathcal{F}(k_x) + \kappa_0^2,$$

(48)

which is a generalization of Equation (1) to pressure-confined, polytropic disks with vertical stratification. Figure 10 compares Equation (48), shown as dotted lines, with the full numerical results plotted as solid lines for disks with $A = 0.7$, $\gamma = 0.8, 1, 1.5, 2$, and $\kappa_0^2 = 0.2(\pi G \Sigma_0 / c_0)^2$, which are in good agreement with each other, demonstrating again that Equation (48) is accurate in describing the GI of pressure-confined polytropic disks provided $c_{\text{eff}}$ and $\mathcal{F}$ are chosen appropriately.

We generalize Toomre’s stability parameter, originally defined for a razor-thin disk to vertically stratified, pressure-confined disks, as

$$Q_{\text{eff}} = \frac{c_{\text{eff}} k_0}{\pi G \Sigma_0}.$$  

(49)

Using $c_{\text{eff}} / H_0 = (2\alpha \gamma \pi G \rho_0)^{1/2}$ for $A = 1$, one can show that the stability parameter $Q_M = \Omega_0^2 / (4\pi G \rho_0)$ used by Mamatsashvili & Rice (2010) is equal to $\sqrt{Q_{\text{eff}} / (8\alpha \gamma)}$. Since $\mathcal{F} \lesssim 1$, the critical value of $Q_{\text{eff}}$ is less than unity. Note that in
Figure 8. Dispersion relations of the mixed GI in non-rotating, pressure-confined polytropic disks with $\gamma = 0.8, 1.5, 2$ and $A = 0.4, 0.6, 0.8$ under the free BCs. Solid lines draw the numerical results, while dotted lines plot the approximate dispersion relation (Equation (46)). In each panel, the pure distortional modes in the limit of $A \to 0$ are shown as a long dashed line.

Figure 9. (a) Maximum ($k_{x,\text{max}}$; dashed lines) and critical ($k_{x,c}$; solid lines) wavenumbers, and (b) the squared maximum growth rate as functions of $A$ for $\gamma = 1$. The thick lines correspond to the mixed GI under the free BCs, while the thin lines are for the pure Jeans modes under the rigid BCs.

In the limit of strong pressure confinement, $Q_{\text{eff}}$ is reduced to

$$Q_{\text{eff}} \to \sqrt{\frac{\bar{\rho}_0}{\pi G \Sigma_0}} = \frac{k_0}{\sqrt{\pi G \rho_0}} \pi G \Sigma_0 = \kappa_0 \sigma_0,$$

for $A \to 0$, (50)

which is applicable to the pure distortional modes.

Figure 11 plots the critical values of $Q_{\text{eff},c}$ for the axisymmetric GI in the $(\gamma, A)$ plane. Figure 11(a) is for pure Jeans instability under rigid BCs, while the case of the mixed GI with the free BCs is shown in Figure 11(b). As $A \to 0$, $Q_{\text{eff},c}$ always converges to 1, for pure Jeans modes, corresponding to a razor-thin disk, and to 0.756 for the mixed GI, corresponding to an incompressible disk (Goldreich & Lynden-Bell 1965a). In the pure Jeans instability, finite disk thickness weakens self-gravity, decreasing $Q_{\text{eff},c}$ with increasing $A$. For example, an
unbounded isothermal disk has \( Q_{\text{eff,c}} \) \( \approx 0.676 \) (Goldreich & Lynden-Bell 1965a). For the mixed GI, \( Q_{\text{eff,c}} \sim 0.68-0.75 \) is largely insensitive to \( \gamma \).

5. SUMMARY AND DISCUSSION

We investigate the GI of rotating, pressure-confined, vertically stratified gas disks to axisymmetric perturbations. As an initial equilibrium, we consider a self-gravitating, polytropic disk in hydrostatic equilibrium. The disk is truncated by a constant external pressure characterized by a dimensionless parameter \( A \) (Equation (14)). To distinguish distortional modes of GI occurring in highly confined disks from the conventional Jeans modes, we adopt the rigid-surface BCs that allow only Jeans modes as well as the free-surface BCs under which the two modes coexist. Our model disks do not include convective motions.

By deriving approximate dispersion relations and comparing them with the numerical results from a set of the perturbation equations (20)–(23), we find that GI-unstable modes are in general even-symmetry fundamental modes that have no vertical node. Under the rigid BCs, the fundamental modes are simply acoustic waves that propagate via thermal pressure only. Under the free BCs, the acoustic waves arising from surface distortion provide an additional restoring force, forming acoustic-surface-gravity waves. In the presence of self-gravity, the acoustic waves become unstable to pure Jeans modes, while the surface-gravity waves become unstable to incompressible, distortional modes. Therefore, disks under the free BCs are unstable to the mixed GI in which the Jeans and distortional modes coexist. When pressure confinement is weak (\( A \sim 1 \)), the Jeans modes dominate, while the distortional modes become prevalent in strongly confined disks. The relative importance of the Jeans modes can be measured by the dimensionless parameter \( w_0 \) defined by Equation (35).

While polytropic disks are vertically stratified both in density and temperature, we find that the effective sound speed \( c_{\text{eff}} \), defined either by Equation (30) for the rigid BCs or Equation (32) for the free BCs, represents the mean propagation speed of acoustic-surface-gravity waves quite well. Under rigid BCs, \( c_{\text{eff}} \) is given by the density-weighted, harmonic mean of the local sound speeds, while it has a contribution from the surface-gravity waves under the free BCs.

We derive the reduction factors \( F(k_x) \) responsible for the reduced gravity due to finite disk thickness for the pure Jeans modes (Equations (43) and (44)) and for the distortional modes (Equation (45)). The approximate dispersion relation (48) matches the numerical results very closely as long as the effective sound speed and the gravity reduction factors are properly chosen. Disk rotation introduces inertial waves associated with epicyclic motion, but it is again the fundamental mode of acoustic-surface-gravity waves that are subject to GI. Using the effective sound speed, we define the stability parameter \( Q_{\text{eff}} \) as in Equation (49), which generalizes the Toomre’s parameter into pressure-confined, polytropic disks. The pure Jeans...
modes have critical values of $Q_{\text{eff,c}} = 0.67\pm1$, with the larger value corresponding to a razor-thin disk, while the mixed GI has $Q_{\text{eff,c}} = 0.67\pm0.76$ insensitive to $\gamma$.

As mentioned in the introduction, the origin of the mixed GI of pressure-confined disks has been uncertain. Various mechanisms, such as neutral modes (Lubow & Pringle 1993), pinching force (Elmegreen 1989; Umekawa et al. 1999; Wünsch et al. 2010), and surface-gravity waves (Iwasaki et al. 2011), have been proposed as the cause of GI. In this paper, we clearly show that the mixed GI that occurred under the free BCs is a combination of pure Jeans and distortional modes, the latter of which is generated by deformation of free surfaces. The existence of an additional restoring force due to the vertical gravity excites low-frequency fundamental modes, as opposed to high-frequency acoustic modes. With a reduced effective sound speed, the mixed GI has a larger growth rate and a smaller length scale than the pure Jeans modes.

Recently, Iwasaki et al. (2011) have performed a linear stability analysis of expanding isothermal shells around H II regions. They considered combinations of the rigid and free BCs as well as an asymmetric density profile caused by the bulk deceleration of a shell. They found that the numerically calculated dispersion relations are well described by an approximate relation of $\omega^2 = c_{\text{eff}}^2 k_x^2 - 2\pi G\Sigma k_z$, where $c_{\text{eff},\text{I}}^2 = [B \times 4\pi G\Sigma_0, H + (c_s/2)^2]$. Although the dimensionless parameter $B = 0.39$ in their effective sound speed $c_{\text{eff, I}}$ gives the best fit to their numerical growth rates, its physical nature is somewhat ambiguous. Noting that $H = A^2 c_0^2 / (\pi G \rho_0)$, the above relation can be arranged as $|\omega|^2 = c_{\text{eff}}^2 k_z^2 - 2\pi G\Sigma_0 k_z$, similar to Equation (46), with the effective sound speed of acoustic-surface-gravity waves $c_{\text{eff}} = 2A^2/(1 + A^2)^2$ in the long-wavelength limit (Equation (33)), and the gravity reduction factor modified to $F_1(k_x) = 1 - k_x H \times [2B + (1 - 7A^2)/(8A^2(1 + A^2))]$. For $A \gtrsim 0.4$ (or, $\rho_{\text{gas}}/(\pi G \Sigma_0) \lesssim 21/8$) considered by Iwasaki et al. (2011), $F_1 = 1 - (0.4 - 0.6)k_x H$ in reasonable agreement with the reduction factors that we found (see Figure 5). This suggests that $c_{\text{eff, I}}$ in the approximate dispersion relation of Iwasaki et al. (2011), contains a contribution from the reduced gravity due to finite shell thickness.

While the mixed GI of pressure-confined disks has received little attention compared to pure Jeans modes, it may be responsible for the structure formation via fragmentation of thin shells in the interstellar medium. For example, Boyd & Whitworth (2005) claimed that the GI of shock-compressed layers can explain the formation of Jupiter-mass (significantly smaller than the Jeans mass) objects detected in young star clusters (Zapatero Osorio et al. 2002). Recently, Wünsch et al. (2012) have reported that the observed slopes of the mass spectrum of molecular clouds in the Carina Flare are indeed consistent with those expected from the GI of pressure-confined shells. Distortional modes in flattened systems can be thought of as arising from the tendency of lowering the gravitational potential energy by the transformation into a spherical configuration (Usami et al. 1995). Thus, the mixed GI would occur not only in a planar geometry but also in a cylindrical gaseous column. In the presence of magnetic fields, distortional instability is suppressed in the direction parallel to the magnetic fields (Nagai et al. 1998), naturally leading to cloud formation in filamentary shapes.

We are grateful to the referee for helpful comments. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MEST), No. 2010-0000712.

### APPENDIX A

#### HYDROSTATIC EQUILIBRIUM

In this appendix, we derive an analytic solution of Equations (7) and (8) for the density distributions of self-gravitating polytropic disks. We begin by defining the dimensionless variables

$$\xi = z/H_0,$$  \hfill (A1)

$$\Theta(\xi) = (\rho_0/\rho_{\text{eq}})^{1/n},$$  \hfill (A2)

where $n \equiv 1/(\gamma - 1)$ is the polytropic index. Then, Equations (7) and (8) result in

$$\frac{d^2 \Theta}{d\xi^2} = \left(\frac{2}{1+n}\right)\Theta^n.$$  \hfill (A3)

Equation (A3) is the usual Lane–Emden equation in the planar geometry (e.g., Viala & Horedt 1974), which can be solved with the proper BCs $\Theta(0) = 1$ and $d\Theta(0)/d\xi = 0$.

Now, we combine Equations (8) and (9) together with Equations (A1) and (A2) to obtain

$$\frac{d}{d\xi} = \Theta^n \frac{d}{d\mu}.$$  \hfill (A4)

On the other hand, by integrating Equation (8) over $z$ and using Equation (A3), we express $\mu$ in Equation (9) as

$$\mu = -\left(\frac{1+n}{2}\right)d\Theta = -\left(\frac{n+1}{2}\right)\Theta^n \frac{d\Theta}{d\mu},$$  \hfill (A5)

where the second equality utilizes Equation (A4). The solution of Equation (A5) that satisfies the BCs is simply

$$\Theta = (1 - \mu^2)^{1/(n+1)}$$  \hfill (A6)

or

$$\rho_0(\mu) = \rho_{\text{eq}}(1 - \mu^2)^{1/\gamma}$$  \hfill (A7)

(see also Harrison & Lake 1972).

To find $\rho_0$ in terms of $z$, we substitute Equations (A1) and (A6) into Equation (A4) to obtain

$$\frac{z}{H_0} = \int_0^\mu \frac{d\mu}{(1 - \mu^2)^{1/\gamma}}.$$  \hfill (A8)

A formal solution of Equation (A8) is given by

$$\frac{z}{H_0} = _2F_1\left(\frac{1}{2}, \frac{1}{\gamma}; \frac{3}{2}; \mu^2\right)\mu,$$  \hfill (A9)

where $_2F_1$ denotes the Gaussian hypergeometric function. Equation (A8) is integrable for a few particular values of $\gamma$:

1. When $\gamma = 2/3$,

$$\frac{z}{H_0} = \mu/\sqrt{1 - \mu^2} \quad \text{and} \quad \rho_0 = \rho_{\text{eq}}\left(1 + (z/H_0)^2\right)^{-3/2},$$  \hfill (A10)

where $H_0^3 = K/(2\pi G\rho_{\text{eq}}^{4/3})$ (e.g., Larson 1985).
2. When $\gamma = 1$,
\[ \frac{z}{H_0} = \tanh^{-1} \mu \quad \text{and} \quad \rho_0 = \rho_{00}\sech^2(z/H_0), \quad (A11) \]
where $H_0^2 = K/(2\pi G\rho_{00})$ (e.g., Ledoux 1951; Goldreich & Lynden-Bell 1965a).

3. When $\gamma = 2$,
\[ \frac{z}{H_0} = \sin^{-1} \mu \quad \text{and} \quad \rho_0 = \rho_{00}\cos(z/H_0), \quad (A12) \]
where $H_0^2 = K/(2\pi G)$ (e.g., Goldreich & Lynden-Bell 1965a).

4. When $\gamma = \infty$,
\[ \frac{z}{H_0} = \mu \quad \text{and} \quad \rho_0 = \rho_{00} \quad (A13) \]
(e.g., Harrison & Lake 1972).

**APPENDIX B**

**BOUNDARY CONDITIONS**

Equations (20)–(23) possess a reflection symmetry with respect to the $z = 0$ plane, that is, they are invariant under the transformations $z \rightarrow -z$, $\xi_{1z} \rightarrow -\xi_{1z}$, $\psi' \rightarrow -\psi'$, $h_1 \rightarrow h_1$, and $\gamma \rightarrow \gamma$. Thus, perturbations are in general a superposition of even-symmetry modes and odd-symmetry modes. Of these, odd-symmetric modes satisfying $\xi_{1z}(z) = \xi_{1z}(-z)$, $h_1(z) = -h_1(-z)$, $\psi_1(z) = -\psi_1(-z)$, and $\gamma(z) = \gamma(-z)$ are shown to be stable against GI (e.g., Simon 1965; Elmegreen & Elmegreen 1978; Mamatsashvili & Rice 2010). Therefore, in this work we consider only even-symmetry modes for which it is suffice to write only two conditions
\[ \xi_{1z}(0) = \psi'_1(0) = 0. \quad (B1) \]
The other two conditions on $h_1(0)$ and $\psi_1(0)$ are automatically satisfied from Equations (20)–(23).

At the boundaries truncated by the external pressure, we consider two different types of BCs: the rigid and free conditions. In the rigid BC, the boundaries are assumed to be fixed with a vanishing Lagrangian displacement
\[ \xi_{1z}\big|_{z=a} = 0 \quad \text{for rigid BC}, \quad (B2) \]
(e.g., Voit 1988). This condition may be appropriate to describe boundaries of a thin layer, as produced by colliding clouds, since the shocked interfaces are stable against distortion (Usami et al. 1995; Iwasaki et al. 2011).

The free BC allows for a deformed Lagrangian surface on which the pressure is maintained constant to its initial value $p_{ext}$ (Goldreich & Lynden-Bell 1965a; Elmegreen & Elmegreen 1978; Lubow & Pringle 1993; Iwasaki et al. 2011). Since $p_{ext} = p(a + \xi_{1z}\big|_{z=a}) = p_0\big|_{z=a} + p_1\big|_{z=a} + (\xi_{1z}d\rho_0/dz)\big|_{z=a}$ to the first order, one obtains
\[ h_{1z}\big|_{z=a} = g_{0a}\xi_{1z}\big|_{z=a} \quad \text{for free BC}, \quad (B3) \]
where $g_{0a} \equiv d\rho_0/dz\big|_{z=a}$ is the vertical gravity at the disk boundary (e.g., Goldreich & Lynden-Bell 1965a).

In self-gravitating disks, another condition comes from applying the divergence theorem to the perturbed Poisson equation at the interface. Integrating Equation (18) over the volume of a thin shell placed at the interface gives
\[ -k_\xi\psi_1\big|_{z=a} - \frac{d\psi_1}{dz}\big|_{z=a} = 4\pi G\rho_0\xi_{1z}\big|_{z=a}, \quad (B4) \]
where the relation $d\psi_1/dz\big|_{z=a} = -k_\xi\psi_1\big|_{z=a}$ is used since the gravitational potential outside the disk satisfies Laplace’s equation.

**APPENDIX C**

**METHOD OF NUMERICAL INTEGRATION**

For a given set of disk parameters ($\gamma, A, k_\xi^2$), we wish to find a dispersion relation $\omega = \omega(k_\xi)$ that satisfies the perturbation equations (20)–(23) subject to the two BCs and two even-symmetry constraints. Since the perturbation equations are linear in terms of the independent variables ($\xi_{1z}, h_1, \psi_1, \psi'_1$), we are free to choose one of the variables arbitrarily. We thus fix $h_{1z}\big|_{z=a} = 1$ (or $\xi_{1z}\big|_{z=a} = 1$) and take trial values for $\psi_1\big|_{z=a}$ and $\omega^2$, while obtaining $\xi_{1z}\big|_{z=a}$ (or $h_{1z}\big|_{z=a}$) and $\psi'_1\big|_{z=a}$ from Equations (B2)–(B4) depending on the adopted BCs. We then integrate Equations (20)–(23) from $z = a$ to $z = 0$ using the fourth-order Runge–Kutta–Gill method (Abramowitz & Stegun 1972). At the $z = 0$ plane, we check the even-symmetry conditions (B1); if these are not fulfilled, we go back to the disk boundary, change $\psi_1\big|_{z=a}$ and $\omega^2$ slightly based on the Newton–Raphson technique, and repeat the integrations. Solutions with an accuracy of $10^{-4}$, that is, with $|\xi_{1z}(0)/\xi_{1z}(a/2)|, |\psi_1(0)/\psi_1(a)| < 10^{-4}$, are obtained typically within less than three iterations.

**APPENDIX D**

**MODE CLASSIFICATION AND THE EFFECTS OF SELF-GRAVITY**

Here, we first derive the local dispersion relations of waves in the WKB limit and compare them with the numerical results. This will help us verify that acoustic modes become unstable in the presence of self-gravity. We then discuss our results in comparison with the results of Mamatsashvili & Rice (2010), who claimed that gravity makes inertial modes unstable.

**D.1. Local Analysis**

We seek the solutions of Equations (20)–(23) for local waves with $k_x, k_z \gg H_0^{-1}$. Defining $k_\xi^2 \equiv -k_x^{-1}d\chi_1/dz^2$ with $\chi_1 = h_1 + \psi_1$, it is straightforward to derive
\[ \omega^4 - \omega^2(\beta c_s^2 k_\xi^2 + k_\xi^2) + k_\xi^2 \beta c_s^2 k_\xi^2 = 0, \quad (D1) \]
where $k^2 = k_x^2 + k_z^2$ and $\beta \equiv 1 + \psi'/h_1 = 1 - \pi G\rho_0/(c_s^2 k^2)$ is the correction factor for self-gravity (e.g., Chandrasekhar 1961). The corresponding eigenfunctions are
\[ \left( \begin{array}{c} p_1/p_0 \\ u_{1x}/c_s \\ u_{1z}/c_s \end{array} \right) = A \left( \begin{array}{c} 1 \\ -\beta c_s^2 k_\xi^2(\omega^2 - k_\xi^2) \\ -\beta c_s^2 k_\xi^2/\omega \end{array} \right) e^{i(\omega t + k_x x + k_z z)}, \quad (D2) \]
where $A$ is an arbitrary constant.

For waves with $|\beta|c_s^2 k_\xi^2 \gg k_\xi^2$ or $k \gg k_\xi$, Equation (D1) yields two approximate solutions
\[ \omega_\rho^2 = \beta c_s^2 k_\xi^2 + k_\xi^2, \quad (D3) \]
\[ \omega_r^2 = k_\xi^2 \frac{\beta c_s^2 k_\xi^2}{\beta c_s^2 k_\xi^2 + k_\xi^2}. \quad (D4) \]
It can be shown that the group and phase velocities of the waves associated with $\omega_\rho^2$ are parallel to each other, while those with $\omega_r^2$ are perpendicular. This demonstrates that the former is acoustic waves ($\rho$ modes) boosted by epicycle motions, while the latter
is inertial waves (r modes). Note that inertial waves require non-vanishing $k_r$ for propagation.

It is apparent from Equation (D3) that the local acoustic waves become gravitationally unstable provided self-gravity is sufficiently strong with $\beta < -\kappa_0^2/(c_s^2 k^2)$. On the other hand, the inertial waves usually have $k_x H_0 \gg 1$ so that $\omega_2^2 \approx \kappa_0^2 k_x^2 / k^2$ regardless of $\beta$, indicating that the character and frequencies of inertial waves are almost unchanged by self-gravity. Figure 12 demonstrates this by directly comparing Equations (D3) and (D4), shown as dashed lines with the numerical dispersion relations plotted as dots obtained by integrating Equations (20)–(23) subject to the rigid BCs. An isothermal disk with $A = 0.8$ and $k_x^2/(2\pi G \rho_0 H_0) = 0.2$ is considered for the numerical calculations using the method described in Appendix C. Only the results for the fundamental acoustic modes (i.e., no node in the $z$-direction) and the lowest order inertial modes with a single node are shown. The inset in Figure 12(b) enlarges the regions with $0.7 \leq k_x H \leq 0.8$ and $0.15 \leq \omega_2^2/(2\pi G \rho_0 H) \leq 0.21$, where the acoustic and inertial modes cross each other. In plotting Equations (D3) and (D4), we utilize the eigenfunctions and their derivatives at $z = 0$ to calculate $\beta$ and $k_z$. Note that agreement between the local and numerical dispersion relations is excellent for the non-self-gravitating case. Self-gravity makes $\beta \lesssim -0.4$ for the acoustic modes, making them unstable for $0.12 < k_x H < 0.63$. On the other hand, the inertial modes have $\beta c_s k_x^2 / \kappa_0^2 \gtrsim 38$ and thus remain stable even in the presence of self-gravity.

As in our present work, Mamatsashvili & Rice (2010) also calculated the dispersion relations of waves in vertically stratified, polytropic, rotating disks. Except for the internal gravity waves arising from convective motions that are missing in our models, their numerical results are qualitatively similar to ours. In interpreting their results, however, they argued that the influence of self-gravity is strongest on the inertial mode and the GI results from the first harmonics of the inertial mode.

This finding seemingly results from the observation that the long-wavelength unstable branch of the dispersion curves is smoothly connected to the short-wavelength part of the inertial modes, which occurs at $k_x H \simeq 0.77$ in Figure 12(b). They claimed that the frequency ordering $\omega_2^2 < \kappa_0^2 < \omega_1^2$, which is valid in the non-self-gravitating case, holds true also for the self-gravitating counterpart, so that the lowest-frequency modes are always the inertial modes regardless of $k_x$. However, our local dispersion relations (Equations (D3) and (D4)) show that this is correct only when $\beta > 0$, i.e., only for waves with a relatively short wavelength. For gravitationally unstable modes with $\beta < -\kappa_0^2/(c_s^2 k^2)$, $\omega_2^2 < 0 < \omega_1^2 \lesssim \kappa_0^2$. In Figure 12(b), the lower-frequency mode at $k_x H < 0.77$ is connected to the higher-frequency acoustic mode at $k_x H > 0.77$, while the higher-frequency mode at $k_x H < 0.77$ is connected to the lower-frequency inertial mode at $k_x H > 0.77$. When the inertial and acoustic modes are mixed together by having similar frequencies and wavenumbers, it is quite ambiguous to tell which one is near $k_x H = 0.77$.

### D.2. Comparison of Eigenfunctions

A clear way to distinguish the modes is to compare the corresponding eigenfunctions since they contain information on mode characteristics. Inertial modes, being incompressible in nature, should involve very weak density perturbations. This can be readily seen from Equation (D2)

$$\frac{|\rho_1/\rho_0|}{|u_{1z}/c_s|} = \frac{|\omega_2 - \kappa_0^2|}{|\omega_1^2|} \approx \frac{\kappa_0 k_x}{\beta c_s k_z^2} \ll 1$$

for inertial modes with $k_x/k_z \ll 1$. On the other hand, acoustic modes with $k_x/k_z \gg 1$ have $|\rho_1/\rho_0|/|u_{1z}/c_s| \approx c_s k_x/|\omega_1| \sim O(1)$, implying that they rely on relatively large density perturbations.

---

6 Note that Mamatsashvili & Rice (2010) considered disks with $\gamma > 1$ and $\rho_{ext} = 0$ that automatically truncated at $z = z_{cut}$ (see Equation (15)). With the imposed free BCs, they identified surface-gravity waves (f mode), although the destabilizing effect of surface distortion is absent in their models.

7 This is usually the case in stellar interiors since $v^2 \sim \pi G \rho R^2$ from the condition of hydrostatic equilibrium, with $R$ denoting the stellar radius. Then, $\beta = 1 - 0.1(\lambda/R)^2 > 0$ for waves with wavelength $\lambda < R$. 

---

Figure 12. Comparison of the local dispersion relations (dashed lines) with the numerical dispersion relations (dots) for acoustic and inertial modes in (a) non-self-gravitating and (b) self-gravitating cases. For the numerical results, an isothermal disk with $A = 0.8$ and $k_x^2/(2\pi G \rho_0 H_0) = 0.2$ is taken and rigid BCs are adopted. In panel (a), the local dispersion relations are almost identical to the numerical results. In panel (b), the acoustic and inertial modes cross each other near $k_x H = 0.77$, and the inset zooms in on the crossing regions. While self-gravity changes the acoustic mode dramatically, the inertial modes are almost intact.
Figure 13. Comparisons of the eigenfunctions for the acoustic modes (left panels) and the lowest order inertial modes (right panels) with $k_x H = 0.37$ shown in Figure 12. The solid and dashed lines correspond to the self-gravitating and non-self-gravitating modes, respectively. All eigenfunctions are normalized such that $u_{1x}/c_s = 1$ at $z = 0$. Note that the inertial modes involve very little density perturbations and are almost unaffected by self-gravity, while the acoustic modes have $|u_z/u_{1x}| \ll 1$ since they propagate primarily along the horizontal direction.

Figure 13 plots the vertical profiles of the numerically calculated eigenfunctions $u_{1x}, u_{1z},$ and $\rho_1/\rho_0$ of the fundamental acoustic modes (left panels) and the lowest order inertial modes (right panels). The background state is the same as in Figure 12. The most unstable wavenumber $k_x H = 0.37$, well away from $k_x H = 0.77$ where the two modes are mixed, is chosen for all modes. The solid and dashed lines correspond to the cases with and without self-gravity, respectively. The normalization is such that $u_{1x}/c_s = 1$ at $z = 0$. The non-self-gravitating acoustic mode has $\omega^2/(2\pi G \rho_0) = 0.41$, $k_z = 0$, $u_{1x}/c_s = 1$, $u_{1z} = 0$, and $\rho_1/\rho_0 = 0.57$, independent of $z$, which are in fact exact solutions (see Appendix E). When self-gravity is included, the acoustic mode becomes unstable with $\omega^2/(2\pi G \rho_0) = -0.16$; the corresponding eigenfunctions vary with $z$ only slightly with $k_x H = 0.11$, but they do not have any nodes. For acoustic modes, $|u_{1z}/u_{1x}| \ll 1$ for $k_z/k_x \ll 1$, consistent with the prediction of the local analysis. The amplitude of the perturbed density for the acoustic modes
is of an order of unity relative to the velocity perturbations. On the other hand, the inertial modes have a node at around $z/H \sim 0.57 - 0.61$, regardless of the presence of self-gravity. Gravity makes little change in $u_{1x}$ and small changes in $u_{1z}$ and $\rho_t$. The corresponding eigenfrequencies and vertical wavenumbers are $\omega^2/(2\pi G\rho_0) = 0.1954, 0.1949$ and $k_H = 2.424, 2.421$ in non-self-gravitating and self-gravitating cases, respectively. Note that $|\rho_t/\rho_0|/|u_{1x}/c_s| \ll 1$ for the inertial modes, just as expected.

All of the above results suggest that the acoustic modes are strongly affected by self-gravity to be unstable when self-gravity is sufficiently strong, and that the inertial modes are not influenced much by self-gravity and remain stable.

APPENDIX E

SOLUTIONS FOR WAVES IN NON-SELF-GRAVITATING DISKS

When $\psi_1 = \kappa_0 = 0$, Equations (20) and (21) become

$$\frac{d\xi_1}{dz} = \frac{k^2}{\omega^2} h_1 - \frac{1}{c_s^2} (h_1 - g_0 \xi_1), \quad (E1)$$

$$\frac{dh_1}{dz} = \omega^2 \xi_1. \quad (E2)$$

In Equation (E1), the vertical gravity $g_0$ should be considered as being external. Substituting Equation (E2) into Equation (E1), we obtain

$$\frac{d}{dz} \left( \rho_0 \frac{d}{dz} h_1 \right) + \left( \frac{\omega^2}{c_s^2} - k_z^2 \right) \rho_0 h_1 = 0, \quad (E3)$$

where the equilibrium condition $d\rho_0/dz = -g_0 \rho_0/c_s^2$ is used. Note that Equation (E3), together with the rigid BCs and the symmetry constraint at the midplane, is in the Sturm–Liouville form with an eigenvalue of

$$k_z^2 \equiv \frac{\omega^2}{c_s^2} - k_1^2, \quad (E4)$$

indicating that the fundamental modes have the lowest frequency (thus most susceptible to GI).

For the time being, we limit ourselves to an isothermal disk with $\rho_0 \propto \text{sech}^2(z/H_0)$ for which analytic solutions of Equation (E3) can be derived. In terms of the dimensionless variable $\mu$ (Equation (A11)), Equation (E3) becomes

$$(1 - \mu^2) \frac{d^2 v}{d\mu^2} - 2 \mu \frac{dv}{d\mu} + \left( \frac{2}{1 - \mu^2} - \frac{\lambda^2}{1 - \mu^2} \right) v = 0, \quad (E5)$$

where $v \equiv (1 - \mu^2)^{1/2} h_1$ and $\lambda^2 \equiv 1 - (k_H H_0)^2$. The above equation is of the associated Legendre type whose solution is given by

$$v = C_1 P_{1}^{\lambda} + C_2 P_{1}^{-\lambda}, \quad (E6)$$

where $C_1$ and $C_2$ are constants to be determined, and $P_{1}^{\pm\lambda}$ is the Ferrers’ associated Legendre function defined as

$$P_{1}^{\pm\lambda} = (\lambda \mp \mu) \left( \frac{1 + \mu}{1 - \mu} \right)^{\pm\lambda/2}. \quad (E7)$$

(Whittaker & Watson 1963).

The even-symmetry condition, $dv/d\mu|_{\mu=0}$, at the midplane requires $C_1 = C_2$. After some manipulations, we find $h_1$ in terms of $z$ as

$$h_1 = \frac{\sinh(z/H_0)}{\sinh(k_z z)} - \bar{k}_z H_0 \cosh(z/H_0) \cos(k_z z), \quad (E8)$$

where $\bar{k}_z \equiv \sqrt{k^2_H - \frac{\omega^2}{c_s^2}}$ and the proportionality constant $C_1$ or $C_2$ is omitted. The vertical wavenumber that satisfies the rigid BC, $dh_1/dz|_{z=0} = 0$, is then given by

$$k_{z,n}^2 = \frac{n^2 \pi^2}{a^2} + 1/H_0^2, \quad \text{for harmonics with order } n = 1, 2, 3, \ldots. \quad (E9)$$

Now we consider general polytropic disks and seek fundamental-mode solutions for (E3). While Equation (E3) cannot be solved in a closed form for arbitrary $\gamma$, it has the simplest but still important solution,

$$h_1 \rightarrow \text{constant}, \quad (E10)$$

in the long-wavelength limit regardless of the BCs. This corresponds to Equation (E8) with $k_z = 0$ for isothermal disks. For waves with $k_z H \ll 1$, gas motions are restricted mostly to the horizontal direction. In this case, the acceleration of the gas along the vertical direction ($-\omega^2 \xi_1$, in Equation (E2)) becomes relatively unimportant, resulting in $dh_1/dz \approx 0$. As Figure 2 shows, $h_1 \approx \text{constant}$ is reasonably good for $k_z H \lesssim 1$.

Another limiting case is a strongly confined disk with $A \ll 1$, for which $\rho_0(z) = \text{constant}$ and $\omega^2/c_s^2 \ll k_z^2$, the latter of which shall be verified a posteriori. Then, the even-symmetry solution of Equation (E3) is

$$h_1 \propto \cosh(k_z z), \quad (E11)$$

which corresponds to Equation (E8) in the limit of $\bar{k}_z \rightarrow k_z$, and $z/H_0 \ll 1$. In this case, $\nabla \cdot \xi_1 = 0$ and the system is essentially incompressible. Note that Equation (34) gives $\omega^2/c_s^2 < (g_0 H/c_s^2) k_z^2 \ll k_z^2$, as expected.

Up to now we have ignored the effect of disk rotation, but it is a simple matter to show that the above results are valid also for rotating disks provided $k_z$ in Equation (E4) is changed to

$$k_z^2 = \frac{\omega^2}{c_s^2} - \frac{\omega^2}{\omega^2 - \kappa_0^2 k_z^2}, \quad (E12)$$

which is identical to Equation (D1). This implies that the local solutions presented in Appendix D are exact for non-self-gravitating, isothermal disks.

REFERENCES

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions (New York: Dover)
Bell, K. R., Cassen, P. M., Klahr, H. H., & Henning, T. 1997, ApJ, 486, 372
Binney, J., & Tremaine, S. 2008, Galactic Dynamics (2nd ed.; Princeton, NJ: Princeton Univ. Press)
Boley, A. C., Durisen, R. H., Nordlund, Å., & Lord, J. 2007, ApJ, 665, 1254
Boley, A. C., Mejía, A. C., Durisen, R. H., et al. 2006, ApJ, 651, 517
Boss, A. P. 1997, Science, 276, 1836
Boyd, D. F. A., & Whitworth, A. P. 2005, A&A, 430, 1059
Chandrasekhar, S. 1961, Hydrodynamic and Hydromagnetic Stability (Oxford: Clarendon)
Churchwell, E., Povich, M. S., Allen, D., et al. 2006, ApJ, 649, 759
Churchwell, E., Watson, D. F., Povich, M. S., et al. 2007, ApJ, 670, 428
Dale, J. E., Wünsch, R., Smith, R. J., Whitworth, A., & Padoan, P. 2011, MNRAS, 411, 2230
