Stabilized Doubly Robust Learning for Recommendation on Data Missing Not at Random

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Abstract
In recommender systems, users always choose the favorite items to rate, which leads to data missing not at random and poses a great challenge for unbiased evaluation and learning of prediction models. Currently, the doubly robust (DR) method and its variants have been widely studied and demonstrate superior performance. However, in this paper, we show that DR methods are unstable and have unbounded bias, variance, and generalization bounds to extremely small propensities. Moreover, the fact that DR relies more on extrapolation will lead to suboptimal performance. To address the above limitations while retaining double robustness, we propose a stabilized doubly robust (SDR) estimator with a weaker reliance on extrapolation. Theoretical analysis shows that SDR has bounded bias, variance, and generalization error bound simultaneously under inaccurate imputed errors and arbitrarily small propensities. In addition, we propose a novel learning approach for SDR that updates the imputation, propensity, and prediction models cyclically, achieving more stable and accurate predictions. Extensive experiments show that our approaches significantly outperform the existing methods.

1 Introduction
Modern recommender systems (RS) are rapidly evolving with the adoption of sophisticated deep learning models [39]. However, it is well documented that directly using advanced deep models usually achieves sub-optimal performance due to the existence of various biases in RS [25][38], and the biases would be amplified over time [35][4]. A large number of debiasing methods have emerged and gradually become a trend [38]. For many practical tasks in RS, such as rating prediction [25][33][31], post-view click-through rate prediction [6], post-click conversion rate prediction [40], and uplift modeling [21][23][24], a critical challenge is to combat the selection bias and confounding bias that leading to significantly difference between the trained sample and the targeted population [8][38]. Various methods were designed to address this problem and among them, the doubly robust (DR) method [31] plays a dominant role due to its better performance and theoretical properties.

The success of DR is attributed to its double robustness and joint-learning technique. However, the DR methods still have many limitations. Theoretical analysis shows that inverse probability scoring (IPS) and DR methods may have infinite bias, variance, and generalization error bounds, in the presence of extremely small propensity scores [25][31][6]. In addition, due to the fact that users are more inclined to evaluate the preferred items, the problem of data missing not at random (MNAR)
often occurs in RS. This would cause selection bias and results in inaccuracy for methods that more rely on extrapolation, such as error imputation based (EIB) [15, 26] and DR methods.

To overcome the above limitations while maintaining double robustness, we propose a stabilized doubly robust (SDR) estimator with a weaker reliance on extrapolation, which reduces the negative impact of extrapolation and MNAR effect on the imputation model. Through theoretical analysis, we demonstrate that the SDR has bounded bias and generalization error bound for arbitrarily small propensities, which further indicates that the SDR can achieve more stable predictions.

Furthermore, we propose a novel cycle learning approach for SDR. Figure 1 shows the differences between the proposed cycle learning of SDR and the existing unbiased learning approaches. Two-phase learning [15, 26, 25] first obtains an imputation/propensity model to estimate the ideal loss and then updates the prediction model by minimizing the estimated loss. DR-JL [31], MRDR-DL [6], and AutoDebias [5] alternatively update the model used to estimate the ideal loss and the prediction model. The proposed learning method cyclically uses different losses to update the three models with the aim of achieving more stable and accurate prediction results. We have conducted extensive experiments on two real-world datasets, and the results show that the proposed approach significantly improves debiasing and convergence performance compared to existing methods.

2 Preliminaries

2.1 Problem Setting

First, we formulate the counterfactual outcome prediction problem in RS using the potential outcome framework [15, 17, 9, 38]. Let \( \mathcal{U} = \{1, 2, ..., U\} \), \( \mathcal{I} = \{1, 2, ..., I\} \) and \( \mathcal{D} = \mathcal{U} \times \mathcal{I} \) be the index sets of users, items, all user-item pairs. For each \((u, i) \in \mathcal{D}\), we have a treatment (exposure) \( o_{u,i} \in \{0, 1\} \), a feature vector \( x_{u,i} \) and an outcome (feedback) \( r_{u,i} \), where \( o_{u,i} = 1 \) if item \( i \) is exposed to user \( u \), \( o_{u,i} = 0 \) otherwise. Let \( r_{u,i}(1) \) is defined as the be the outcome that would be observed if item \( i \) had been exposed to user \( u \), which is observable only for the exposed events \( \mathcal{O} = \{(u, i) \mid (u, i) \in \mathcal{D}, o_{u,i} = 1\} \); otherwise it is missing. Many tasks in RS can be formulated by predicting the potential outcome \( r_{u,i}(1) \) using feature \( x_{u,i} \) for each user-item pair \((u, i)\).

Let \( \hat{r}_{u,i}(1) = f(x_{u,i}; \phi) \) be a prediction model with parameters \( \phi \). If all the potential outcomes \( \{r_{u,i}(1) : (u, i) \in \mathcal{D}\} \) were observed, the ideal loss function for solving parameters \( \phi \) is given as

\[
L_{\text{ideal}}(\phi) = |\mathcal{D}|^{-1} \sum_{(u,i) \in \mathcal{D}} e_{u,i},
\]

where \( e_{u,i} \) is the prediction error, such as the squared loss \( e_{u,i} = (\hat{r}_{u,i}(1) - r_{u,i}(1))^2 \). \( L_{\text{ideal}}(\phi) \) can be regarded as a benchmark of unbiased loss function, even though it is infeasible due to the missingness of \( \{r_{u,i}(1) : o_{u,i} = 0\} \). As such, a variety of methods are developed through approximating \( L_{\text{ideal}}(\phi) \) to address the selection bias, in which the propensity-based estimators show the relatively superior performance [25, 31, 6, 5, 37], and the IPS and DR estimator is given as

\[
\mathcal{E}_{IPS} = |\mathcal{D}|^{-1} \sum_{(u,i) \in \mathcal{D}} \frac{o_{u,i} e_{u,i}}{\hat{p}_{u,i}} \quad \text{and} \quad \mathcal{E}_{DR} = |\mathcal{D}|^{-1} \sum_{(u,i) \in \mathcal{D}} \left[ \hat{e}_{u,i} + \frac{o_{u,i}(e_{u,i} - \hat{e}_{u,i})}{\hat{p}_{u,i}} \right].
\]
where \( \hat{p}_{u,i} \) is an estimate of propensity score 
\[
p_{u,i} := \mathbb{P}(o_{u,i} = 1 | x_{u,i})
\]
\( \hat{e}_{u,i} \) is an estimate of \( e_{u,i} \).

2.2 Related Work

Debiased learning in recommendation. The data collected in RS suffers from various biases \([4, 38]\), which are entangled with the true preferences of users and pose a great challenge to unbiased learning. There is increasing interest in coping with different biases in recent years \([41, 1, 13, 25]\) proposed using inverse propensity score (IPS) and self-normalized IPS (SNIPS) methods to address the selection bias on data missing not at random, and \([22, 19]\) extended them to implicit feedback data. \([15, 26]\) derived an error imputation-based (EIB) unbiased learning method. These three approaches adopt two-phase learning \([32]\), which first learns a propensity/imputation model and then applies it to construct an unbiased estimator of the ideal loss to train the recommendation model. A doubly robust joint learning (DR-JL) method \([31]\) was proposed by combining the IPS and EIB approaches. Subsequently, strands of enhanced joint learning methods were developed, including more robust doubly robust (MRDR) method \([6]\), multi-task learning \([40]\), collaborative targeted learning \([37]\), and uniform data-aware methods \([3, 12, 5, 32]\) that aimed to seek better recommendation strategies by leveraging a small uniform dataset. \([4]\) reviewed various biases in RS and discussed the recent progress on debiasing tasks. \([38]\) established the connections between the biases in causal inference and the biases in RS, thereby presenting the formal causal definitions of the biases in RS.

Stabilized causal effect estimation. The proposed method builds on the stabilized average causal effect estimation approaches in causal inference. \([16, 37]\) summarized the limitations of doubly robust methods, including unstable to small propensities \([36]\), unboundedness \([30]\), and large variance \([29]\). These issues inspired a series of stabilized causal effect estimation methods in statistics \([10, 2, 30, 16]\). Unlike previous work that focused only on achieving unbiased learning in RS, this paper proposes a doubly robust estimator with much more stable statistical properties.

3 Stabilized Doubly Robust Estimator

In this section, we elaborate the limitations of DR methods and propose a stabilized DR (SDR) estimator with a weaker reliance on extrapolation. Theoretical analysis shows that SDR has bounded bias and generalization error bound for arbitrarily small propensities, while IPS and DR doesn’t.

3.1 Motivation

Even though DR estimator \([31]\) has double robustness property, its performance could be significantly improved if the following three aspects can be enhanced.

1. **More stable to small propensities.** As shown in \([25, 31, 6]\), if there exist extremely small estimated propensity scores, the IPS/DR estimator and its tail bound are unbounded, deteriorating the prediction accuracy. However, the problem is difficult to be addressed, given the fact that there are many long-tailed users and items in RS, resulting in the presence of extreme propensities.

2. **More stable through weakening extrapolation.** DR relies more on extrapolation because the imputation model in DR is learned from the exposed events \( \mathcal{O} \) and extrapolated to the unexposed events. If the distributional disparity of \( e_{u,i} \) on \( o_{u,i} = 0 \) and \( o_{u,i} = 1 \) is large, the imputed errors are likely to be inaccurate on the unexposed events and incur bias of DR. Therefore, it is beneficial to reduce bias if we can develop an enhanced DR method with weaker reliance on extrapolation.

3. **More stable training process of updating a prediction model.** In general, alternating training between models results in better performance. From Figure \([1, 31]\) proposes joint learning for DR, alternatively updating the imputation and prediction models with given estimated propensities. Double learning \([6]\) further incorporates parameter sharing between the imputation and prediction models. Bi-level optimization \([32, 5]\) can be viewed as alternately updating the prediction model and the other parameters used to estimate the loss. To the best of our knowledge, this is the first paper that proposes a algorithm to update the three models separately using different optimizers, which may resulting in more stable and accurate rating predictions.
3.2 Stabilized Doubly Robust Estimator

We propose a stabilized doubly robust (SDR) estimator that has a weaker dependence on extrapolation and is robust to small propensities. The SDR estimator consists of the following three steps.

Step 1 (Initialize imputed errors). Pre-train an imputation model $\hat{e}_{u,i}$ and $\hat{\theta} \triangleq |D|^{-1} \sum_{(u,i) \in D} \hat{e}_{u,i}$.

Step 2 (Learn constrained propensities). Learn an propensity model $\hat{p}_{u,i}$ satisfying
\[
\frac{1}{|D|} \sum_{(u,i) \in D} \frac{o_{u,i}}{\hat{p}_{u,i}} (\hat{e}_{u,i} - \hat{\theta}) = 0.
\] (3)

Step 3 (SDR estimator). The SDR estimator is given as
\[
\mathcal{E}_{SDR} = \sum_{(u,i) \in D} \frac{o_{u,i} e_{u,i}}{\hat{p}_{u,i}} / \sum_{(u,i) \in D} \frac{o_{u,i}}{\hat{p}_{u,i}} \triangleq \sum_{(u,i) \in D} w_{u,i} e_{u,i},
\]
where $w_{u,i} = \frac{o_{u,i} e_{u,i}}{\sum_{(u,i) \in D} o_{u,i} / \hat{p}_{u,i}}$. It can be seen that SDR estimator has the same form as SNIPS estimator, but the propensities are learned differently. In SDR, the estimation of propensity model relies on the imputed errors, whereas not in SNIPS.

Each step in the construction of SDR estimator plays a different role. Specifically, the Step 2 is designed to enable double robustness property as shown in Theorem 1.

Theorem 1 (Double Robustness). $\mathcal{E}_{SDR}$ is a consistent estimator of $\mathcal{E}_{ideal}$, when either the learned propensities $\hat{p}_{u,i}$ or the imputed errors $e_{u,i}$ is accurate for all user-item pairs.

To demonstrate the double robustness of the SDR, first note that $P (\lim_{|D| \to \infty} \mathcal{E}_{SDR} = \mathcal{E}_{ideal}) = 1$ if propensity model is correctly specified [25], since $|D|^{-1} \sum_{(u,i) \in D} o_{u,i} / \hat{p}_{u,i}$ converges to 1 almost surely as $|D|$ goes to infinity and IPS is unbiased. Besides, the constraint (3) is constructed to ensure the unbiasedness of $\mathcal{E}_{SDR}$ if the error imputation model is correctly specified. In fact, $\mathcal{E}_{SDR}$ satisfies
\[
\frac{1}{|D|} \sum_{(u,i) \in D} \frac{o_{u,i} (e_{u,i} - \mathcal{E}_{SDR})}{\hat{p}_{u,i}} = 0.
\] (4)

Combining the constraint (3) and equation (4) gives
\[
\frac{1}{|D|} \sum_{(u,i) \in D} \left[ o_{u,i} (e_{u,i} - \hat{e}_{u,i}) / \hat{p}_{u,i} + o_{u,i} (\hat{\theta} - \mathcal{E}_{SDR}) / \hat{p}_{u,i} \right] = 0,
\] (5)
where the first term equals to 0 when the imputation model is correctly specified, it implies that $\mathcal{E}_{SDR} = \hat{\theta}$, then the unbiasedness of $\mathcal{E}_{SDR}$ follows immediately from the unbiasedness of $\hat{\theta}$. The double robustness of the SDR estimator is presented in Theorem 1.

In addition, Step 3 is designed for two main reasons to achieve stability. First, it is more robust to extrapolation compared with DR. This is because the propensities are learned from the entire data and thus have less requirement on extrapolation. Second, it is more stable to small propensities, since the self-normalization imposes the weight $w_{u,i}$ to fall on the interval [0,1].

3.3 Theoretical Analysis of Stableness

Through theoretical analysis, we note that previous debiasing estimators such as IPS [25] and DR-based methods [31, 6] tend to have infinite biases, variances, tail bound, and corresponding generalization error bounds, in the presence of extremely small estimated propensities. Remarkably, the proposed SDR estimator doesn’t suffer from such problems and is stable to arbitrarily small propensities, as shown in the following Theorems (see Appendix for proofs).

Theorem 2 (Bias of SDR). Given imputed errors $\hat{e}_{u,i}$ and learned propensities $\hat{p}_{u,i}$ satisfying the stabilization constraint (3), with $\hat{p}_{u,i} > 0$ for all user-item pairs, the bias of $\mathcal{E}_{SDR}$ is
\[
\text{Bias}(\mathcal{E}_{SDR}) = \frac{1}{|D|} \sum_{(u,i) \in D} \left( \delta_{u,i} - \frac{\sum_{(u,i) \in D} \delta_{u,i} \hat{p}_{u,i} / \hat{p}_{u,i}}{\sum_{(u,i) \in D} \hat{p}_{u,i} / \hat{p}_{u,i}} \right) + O(|D|^{-1}),
\]
where $\delta_{u,i} = e_{u,i} - \hat{e}_{u,i}$ is the error deviation.
Theorem \[\text{2}\] shows the bias of the SDR estimator consisting of a dominant term given by the difference between \(\delta_{u,i}\) and its weighted average, and a negligible term of order \(O(|D|^{-1})\). The fact that the \(\delta_{u,i}\) and its convex combinations are bounded, shows that the bias is bounded for arbitrarily small \(\hat{p}_{u,i}\). Compared to the Bias (\(\hat{E}_{\text{IPS}}\)) = \(|D|^{-1}|\sum_{u,i\in D}(\hat{p}_{u,i} - p_{u,i})e_{u,i}/\hat{p}_{u,i}|\) and Bias (\(\hat{E}_{\text{DR}}\)) = \(|D|^{-1}|\sum_{u,i\in D}(\hat{p}_{u,i} - p_{u,i})\delta_{u,i}/\hat{p}_{u,i}|\), it indicates that IPS and DR will have extremely large bias when there exist an extremely small \(\hat{p}_{u,i}\).

**Theorem 3 (Variance of SDR).** Under the conditions of Theorem \[\text{2}\] the variance of \(\hat{E}_{\text{SDR}}\) is

\[
\text{Var}(\hat{E}_{\text{SDR}}) = \sum_{(u,i)} p_{u,i}(1 - p_{u,i})\frac{h_{u,i}}{\hat{p}_{u,i}^2} + O(|D|^{-2}),
\]

where \(h_{u,i} = (e_{u,i} + \hat{E} - \hat{e}_{u,i}) - \frac{\sum_{(u,i)} p_{u,i}(e_{u,i} - \hat{e}_{u,i})}{\hat{p}_{u,i}}\) is a bounded difference between \(e_{u,i} - \hat{e}_{u,i}\) and its weighted average.

Theorem \[\text{3}\] shows the variance of the SDR estimator consisting of a dominant term and a negligible term of order \(O(|D|^{-2})\). The boundedness of the variance for arbitrarily small \(\hat{p}_{u,i}\) is given directly from the fact that SDR has a bounded range given by the self-normalized form. Compared to the Var (\(\hat{E}_{\text{IPS}}\)) = \(|D|^{-2}\sum_{u,i\in D} p_{u,i}(1 - p_{u,i})e_{u,i}/\hat{p}_{u,i}^2\) and Var (\(\hat{E}_{\text{DR}}\)) = \(|D|^{-2}\sum_{u,i\in D} p_{u,i}(1 - p_{u,i})(\hat{e}_{u,i} - \hat{e}_{u,i})^2/\hat{p}_{u,i}^2\), it indicates that IPS and DR will have extremely large variance (tend to infinity) when there exist an extremely small \(\hat{p}_{u,i}\) (tends to 0).

**Theorem 4 (Tail Bound of SDR).** Under the conditions of Theorem \[\text{2}\] for any prediction model, with probability \(1 - \eta\), the deviation of the SDR estimator from its expectation has the following tail bound

\[
|\hat{E}_{\text{SDR}} - E_{\eta}(\hat{E}_{\text{SDR}})| \leq \sqrt{\frac{2}{\log \left(\frac{4}{\eta}\right)} \sum_{(u,i)\in D} \left(\frac{\delta_{\max} - \delta_{u,i}}{1 + \hat{p}_{u,i}}\right)^2 + \left(\frac{\delta_{u,i} - \delta_{\min}}{\hat{p}_{u,i} - \epsilon'}\right)^2/\sum_{(u,i)\in D\setminus(u,i)} 1/\hat{p}_{u,i}^2,}
\]

where \(\delta_{\min} = \min_{(u,i)\in D} \delta_{u,i}\), \(\delta_{\max} = \max_{(u,i)\in D} \delta_{u,i}\), \(\epsilon' = \sqrt{\log(4/\eta)/2 \cdot \sum_{(u,i)\in D\setminus(u,i)} 1/\hat{p}_{u,i}^2}\), and \(D\setminus(u,i)\) is the set of \(D\) excluding the element \((u, i)\).

Note that \(\sum_{D\setminus(u,i)} p_{u,i}/\hat{p}_{u,i} = O(|D|)\) and \(\epsilon' = O(|D|^{1/2})\) in Theorem \[\text{4}\] it follows that the tail bound of the SDR estimator converges to 0 for large samples. In addition, the tail bound is bounded for arbitrarily small \(\hat{p}_{u,i}\). Compared to the tail bound of IPS and DR, with probability \(1 - \eta\), we have

\[
|\hat{E}_{\text{IPS}} - E_{\eta}(\hat{E}_{\text{IPS}})| \leq \sqrt{\frac{2}{\log \left(\frac{4}{\eta}\right)} \sum_{u,i\in D} \left(\frac{e_{u,i}}{\hat{p}_{u,i}}\right)^2,}
\]

\[
|\hat{E}_{\text{DR}} - E_{\eta}(\hat{E}_{\text{DR}})| \leq \sqrt{\frac{2}{\log \left(\frac{4}{\eta}\right)} \sum_{u,i\in D} \left(\frac{\delta_{u,i}}{\hat{p}_{u,i}}\right)^2}
\]

which are both unbounded when \(\hat{p}_{u,i} \to 0\). For SDR in the prediction model training phase, the boundedness of the generalization error bound follows immediately from the boundedness of the bias and tail bound. The above analysis demonstrates that SDR can comprehensively mitigate the negative effects caused by extreme propensities and results in more stable predictions.

## 4 Cycle Learning with Stabilization

In this section, we propose a novel SDR-based cycle learning approach, that not only exploits the stable statistical properties of the SDR estimator itself, but also carefully designs the updating process among various models to achieve higher stability. In general, inspired by the idea of value iteration in reinforcement learning \[\text{22}\], alternatively updating the model tends to achieve better predictive performance, as existing de-biasing training approaches suggested \[\text{31, 6, 5}\]. As shown in Figure \[\text{1}\] the proposed approach dynamically interacts with three models, utilizing the propensity model and imputation model simultaneously in a differentiated way, which can be regarded as an extension of these methods. In cycle learning, given pre-trained propensities, the inverse propensity weighted imputation error loss is used to first obtain an imputation model, and then take the equation \[\text{3}\] as the regularization term to train a stabilized propensity model and ensure the double robustness of SDR.
Finally, the prediction model is updated by minimizing the SDR loss and used to readjust the imputed errors. By repeating the above update processes cyclically, the cycle learning approach can fully utilize and combine the advantages of the three models to achieve more accurate rating predictions.

Specifically, the data MNAR leads to the presence of missing \( r_{u,i}(1) \), so that all \( e_{u,i} \) cannot be used directly. Therefore, we obtain imputed errors by learning a pseudo-labeling model \( \hat{r}_{u,i}(1) \) parameterized by \( \beta \), and the imputed errors \( \hat{e}_{u,i} = CE(\hat{r}_{u,i}(1), \hat{r}_{u,i}(1)) \) are updated by minimizing

\[
\mathcal{L}_e(\phi, \alpha, \beta) = |\mathcal{D}|^{-1} \sum_{(u,i) \in \mathcal{D}} \frac{o_{u,i}(\hat{e}_{u,i} - e_{u,i})^2}{\pi(x_{u,i};\alpha)} + \lambda_e \| \beta \|^2_F,
\]

where \( e_{u,i} = CE(r_{u,i}(1), \hat{r}_{u,i}(1)) \), \( \lambda_e \geq 0 \), \( \hat{p}_{u,i} = \pi(x_{u,i};\alpha) \) is the propensity model, \( \| \cdot \|^2_F \) is the Frobenius norm. For each observed ratings, the inverse of the estimated propensities are used for weighting to account for MNAR effects. Next, we consider two methods for estimating propensity scores, which are Naive Bayes with Laplace smoothing and logistic regression. The former provides a wide range of opportunities for achieving stability constraints through the selection of smoothing coefficients. The latter requires user and item embeddings, which are obtained by employing MF before performing cycle learning. The learned propensities need to both satisfy the accuracy, which is evaluated with cross entropy, and meet the constraint (3) for stabilization and double robustness. The propensity model \( \pi(x_{u,i};\alpha) \) is updated by using the loss \( \mathcal{L}_{cc}(\phi, \alpha, \beta) + \eta \cdot \mathcal{L}_{stable}(\phi, \alpha, \beta) \), where \( \mathcal{L}_{cc}(\phi, \alpha, \beta) \) is cross entropy loss of propensity model and

\[
\mathcal{L}_{stable}(\phi, \alpha, \beta) = |\mathcal{D}|^{-1} \left\{ \sum_{(u,i) \in \hat{\mathcal{D}}} \frac{o_{u,i}}{\pi(x_{u,i};\alpha)} \left( \hat{e}_{u,i} - \hat{\pi} \right)^2 \right\} + \lambda_{stable} \| \alpha \|^2_F,
\]

where \( \lambda_{stable} \geq 0 \), and \( \eta \) is a hyper-parameter for trade-off. Finally, the prediction model \( f(x_{u,i};\phi) \) is updated by minimizing the SDR loss

\[
\mathcal{L}_{sdr}(\phi, \alpha, \beta) = \left[ \sum_{(u,i) \in \mathcal{D}} \frac{o_{u,i}(\hat{e}_{u,i} - e_{u,i} + \hat{e})}{\pi(x_{u,i};\alpha)} \right] / \left[ \sum_{(u,i) \in \mathcal{D}} \frac{o_{u,i}}{\pi(x_{u,i};\alpha)} \right] + \lambda_{sdr} \| \phi \|^2_F,
\]

where the first term is equivalent to (5), and \( \lambda_{sdr} \geq 0 \). In cycle learning, the updated prediction model will be used for re-update the imputation model using the next sample batch. We summarized the cycle learning approach in Alg. 1.
5 Real-world Experiments

In this section, several experiments are conducted to evaluate the proposed methods on two real-world benchmark datasets. We conduct experiments to answer the following questions:

RQ1. Do the proposed Stable-DR and Stable-MRDR improve in debiasing performance compared to the existing studies?

RQ2. Do our methods stably perform well for the various propensity models? How does the performance of our method change under different strengths of the stabilization constraint?

RQ3. Do our methods have stabler convergence rates and values during the training phase?

RQ4. How does the embedding size affect the performance of the proposed methods in practice?

5.1 Experimental Setup

Dataset and preprocessing. To answer the above RQs, we need to use the datasets that contain both MNAR ratings and missing-at-random (MAR) ratings. Following the previous studies [25, 31, 6, 5], we conduct our experiments on the two commonly used datasets: Coat Shopping contains ratings from 290 users to 300 items. Each user evaluates 24 items, containing 6,960 MNAR ratings in total. Meanwhile, each user evaluates 16 items randomly, which generates 4,640 MAR ratings. Yahoo! R3 contains totally 54,000 MAR and 311,704 MNAR ratings from 15,400 users to 1,000 items.

Baselines. In our experiments, we take Matrix Factorization (MF) [11] and Neural Collaborative Filtering (NCF) [7] as the base model respectively, and compare against the proposed methods with the following baselines: Base Model [11, 7], IPS [22, 25], SNIPS [23], CVIB [34], DR [20], DR-JL [31], and MRDR-JL [6]. In addition, Naive Bayes with Laplace smoothing and logistic regression are used to establish the propensity model respectively.

Experimental protocols and details. The following four metrics are used simultaneously in the evaluation of debiasing performance: MSE, AUC, NDCG@5, and NDCG@10. The default values of both user and item embedding size are set to 4. All the experiments are implemented on PyTorch with Adam as the optimizer. We tune the learning rate in \{0.005, 0.01, 0.05, 0.1\}, weight decay in \{1e-6, 5e-6, ..., 5e-3, 1e-2\}, constrain parameter eta in \{50, 100, 150, 200\} for Coat and \{500, 1000, 1500, 2000\} for Yahoo! R3, and batch size in \{128, 256, 512, 1024, 2048\} for Coat and \{1024, 2048, 4096, 8192, 16384\} for Yahoo! R3. In addition, for the Laplacian smooth parameter in Naive Bayes model, the initial value is set to 0 and learning rate is tuned in \{5, 10, 15, 20\} for Coat and in \{50, 100, 150, 200\} for Yahoo! R3.

5.2 Performance Comparison (RQ1)

Table I summarizes the performance of the proposed Stable-DR and Stable-MRDR methods compared with existing debiasing methods. First, the causally inspired methods perform better than the base model, verifying the necessity of handling the selection bias in rating prediction. Second, the proposed Stable-DR and Stable-MRDR have the best performance in all four metrics. On the one hand, our methods outperform SNIPS, attributed to the inclusion of the propensity model in the training process, as well as the boundedness and double robustness of SDR. On the other hand, our methods outperform DR-JL and MRDR-JL, attributed to the stabilization constraint introduced in the training of the propensity model. This further demonstrates the benefit of cycle learning, in which the propensity model is acted as the mediation between the imputation and prediction model during the training process, rather than updating the prediction model from the imputation model directly.

5.3 In-depth Analysis

Ablation Study (RQ2, RQ3). The debiasing performance under different stabilization constraint strength and propensity models is shown in Figure 2. First, the proposed Stable-DR and Stable-MRDR outperform DR-JL and MRDR-JL, when both Naive Bayes with Laplace smoothing and

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2https://www.cs.cornell.edu/~schnabts/mnar/

3http://webscope.sandbox.yahoo.com/

4For all experiments, we use NVIDIA GeForce RTX 2060 as the computing resource.
Table 1: Evaluation metrics of MSE, AUC, NDCG@5, and NDCG@10 on the MAR test set of Coat and Yahoo!R3 using Naive Bayes with Laplace smoothing as the propensity model and taking MF (Top) and NCF (Bottom) as the base model respectively. The best two results are bolded.

| Datasets | Coat |       |       |       |       |       |       |
|----------|------|-------|-------|-------|-------|-------|-------|
|          | MSE  | AUC   | NDCG@5| NDCG@10| MSE  | AUC   | NDCG@5| NDCG@10|
| MF       | 0.2428| 0.7063| 0.6025| 0.6774| 0.2500| 0.6722| 0.6374| 0.7634 |
| + IPS    | 0.2316| 0.7166| 0.6184| 0.6937| 0.2194| 0.6742| 0.6304| 0.7556 |
| + SNIPS  | 0.2333| 0.7070| 0.6222| 0.6851| 0.1931| 0.6831| 0.6348| 0.7608 |
| + CVIB   | 0.2195| 0.7239| 0.6285| 0.6947| 0.2625| 0.6853| 0.6513| 0.7729 |
| + DR     | 0.2298| 0.7132| 0.6243| 0.6918| 0.2093| 0.6873| 0.6574| 0.7741 |
| + DR-JL  | 0.2254| 0.7209| 0.6252| 0.6961| 0.2194| 0.6863| 0.6525| 0.7701 |
| + Stable-DR (Ours) | 0.2159| 0.7508| 0.6511| 0.7073| 0.2090| 0.6946| 0.6620| 0.7786 |
| + MRDR-JL| 0.2252| 0.7318| 0.6375| 0.6989| 0.2173| 0.6830| 0.6437| 0.7652 |
| + Stable-MRDR (Ours) | 0.2076| 0.7548| 0.6532| 0.7105| 0.2087| 0.6915| 0.6585| 0.7757 |
| NCF      | 0.2050| 0.7670| 0.6228| 0.6954| 0.3215| 0.6782| 0.6501| 0.7672 |
| + IPS    | 0.2042| 0.7646| 0.6327| 0.7054| 0.1777| 0.6719| 0.6548| 0.7703 |
| + SNIPS  | 0.1904| 0.7707| 0.6271| 0.7062| 0.1694| 0.6903| 0.6693| 0.7802 |
| + CVIB   | 0.2042| 0.7655| 0.6176| 0.6946| 0.3088| 0.6715| 0.6669| 0.7793 |
| + DR     | 0.2081| 0.7578| 0.6119| 0.6900| 0.1705| 0.6886| 0.6628| 0.7768 |
| + DR-JL  | 0.2115| 0.7600| 0.6272| 0.6967| 0.2452| 0.6818| 0.6516| 0.7678 |
| + Stable-DR (Ours) | 0.1896| 0.7712| 0.6337| 0.7095| 0.1664| 0.6907| 0.6756| 0.7861 |
| + MRDR-JL| 0.2046| 0.7609| 0.6182| 0.6992| 0.2367| 0.6778| 0.6465| 0.7664 |
| + Stable-MRDR (Ours) | 0.1899| 0.7710| 0.6380| 0.7082| 0.1671| 0.6910| 0.6734| 0.7846 |

logistic regression are used as propensity models. It indicates that our methods have better debiasing ability in the feature containing and collaborative filtering scenarios simultaneously. Second, when the strength of the stabilization constraint is zero, our method performs similarly to SNIPS and slightly worse than the DR-JL and MRDR-JL, which indicates that simply using cross-entropy loss to update the propensity model is not effective in improving the model performance. However, as the strength of the stabilization constraint increases, Stable-DR and Stable-MRDR using cycle learning have a stable and significant improvement compared to DR-JL and MRDR-JL. Our methods achieve the best performance at the appropriate constraint strength, which can be interpreted as simultaneous consideration of accuracy and stability to ensure boundedness and double robustness of SDR.

To further illustrate the stability of the proposed methods in the convergence process, Figure 3 shows the average MSE and AUC against training epochs on Coat. When MF is used as the base model, our methods significantly outperform SNIPS and DR-JL in terms of convergence rate and value, resulting in smaller MSE and larger AUC in the training phase. When NCF is used as the base model, same as the results of the previous study [34], SNIPS performed competitively in terms of convergence values. However, compared to the SNIPS method, our method can converge faster and slightly outperforms SNIPS in terms of convergence values. Compared to the DR-JL method, our methods have smaller volatility after convergence, attributed to SDR being more robust to small propensities while maintaining double robustness. In conclusion, the proposed cycle learning can not only improve debiasing ability but also stabilize the model performance during the training phase.

Parameter Sensitivity Study (RQ4). We conducted repeated experiments on Yahoo! R3 to investigate the effect of embedding size on prediction performance. Specifically, on the one hand, the user and item embeddings used in Stable-DR and Stable-MRDR are obtained through MF; on the other hand, cycle learning has a more complex training process and it is necessary to discuss the influence of model complexity on the performance as distinct from alternative learning. As shown in Figure 4, AUC, NDCG@5 and NDCG@10 reach the optimal performance at embedding size $k = 4$.

6 Conclusion

In this paper, we propose an SDR estimator for data MNAR that maintains double robustness and improves the stability of DR in the following three aspects: first, we show that SDR has a weaker
extrapolation dependence than DR and can result in more stable and accurate predictions in the presence of MNAR effects. Next, through theoretical analysis, we show that the proposed SDR has bounded bias, variance, and generalization error bounds under inaccurate imputed errors and arbitrarily small estimated propensities, while DR does not. Finally, we propose a novel learning approach for SDR that updates the imputation, propensity, and prediction models cyclically, achieving more stable and accurate predictions. Extensive experiments show that our approach significantly outperforms the existing methods in terms of both convergence and prediction accuracy.

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Figure 4: The performance of AUC (Left), NDCG@5 (Middle) and NDCG@10 (Right) with varying user and item embedding size of MF-Stable-DR and MF-Stable-MRDR. Shaded regions show the 90% confidence intervals of the test AUC.

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Throughout, following existing work [25, 31, 6], we assume that the indicator matrix $O$ contains independent random variables and each $o_{u,i}$ follows a Bernoulli distribution with probability $p_{u,i}$.

### A Proof of Theorem 2

**Proof of Theorem 2.** Equation (5) implies that $E_{SDR}$ can be expressed as

$$E_{SDR} = \frac{1}{|D|} \sum_{(u,i) \in D} \frac{o_{u,i}(c_{u,i} - \tilde{c}_{u,i} + \tilde{\epsilon})}{\tilde{p}_{u,i}} \left[ \frac{1}{|D|} \sum_{(u,i) \in D} o_{u,i} \right] \tag{6}$$

For notational simplicity, let

$$w_{u,i} \triangleq o_{u,i}/\tilde{p}_{u,i} \quad \text{and} \quad v_{u,i} \triangleq o_{u,i}(c_{u,i} - \tilde{c}_{u,i} + \tilde{\epsilon})/\tilde{p}_{u,i},$$

then $E_{SDR}$ can be written as a ratio statistic

$$E_{SDR} = \frac{1}{|D|} \sum_{(u,i) \in D} v_{u,i} \div \frac{1}{|D|} \sum_{(u,i) \in D} w_{u,i} \triangleq f(\bar{v}, \bar{w}),$$

where $f(v, w) = v/w, \bar{v} = |D|^{-1} \sum_{(u,i) \in D} v_{u,i}$, and $\bar{w} = |D|^{-1} \sum_{(u,i) \in D} w_{u,i}$.

Applying the Taylor expansion around $(\mu_v, \mu_w) \triangleq (E[\bar{v}], E[\bar{w}])$ yields that

$$f(\bar{v}, \bar{w}) = f(\mu_v, \mu_w) + f_v'(\mu_v, \mu_w) (\bar{v} - \mu_v) + f_w'(\mu_v, \mu_w) (\bar{w} - \mu_w)$$

$$+ \frac{1}{2} \left\{ f_{vv}''(\mu_v, \mu_w) (\bar{v} - \mu_v)^2 + 2f_{vw}''(\mu_v, \mu_w) (\bar{v} - \mu_v) (\bar{w} - \mu_w) + f_{ww}''(\bar{w} - \mu_w)^2 \right\}$$

$$+ R(\bar{v}, \bar{w}),$$

where $R(\bar{v}, \bar{w})$ is the remainder term. Note that $f_{vv}''(\mu_v, \mu_w) = 0, f_{vw}''(\mu_v, \mu_w) = -1/\mu_w^3$, and $f_{ww}''(\bar{w} - \mu_w)^2 = 2\mu_v/\mu_w^3$, then taking an expectation on both sides of the Taylor expansion leads to

$$E(\bar{v}/\bar{w}) = \frac{\mu_v}{\mu_w} - \frac{\text{Cov}(\bar{v}, \bar{w})}{(\mu_w)^2} + \frac{\text{Var}(\bar{w})}{(\mu_w)^3} + E[R(\bar{v}, \bar{w})].$$

By some calculations, we have $\text{Cov}(\bar{v}, \bar{w}) = O(|D|^{-1}), \text{Var}(\bar{w}) = O(|D|^{-1}), E[R(\bar{v}, \bar{w})] = o(|D|^{-1})$. Thus, the bias of $E_{SDR}$ is given as

$$\text{Bias}(E_{SDR}) = \left| \frac{1}{|D|} \sum_{(u,i) \in D} \left( \delta_{u,i} - \frac{\sum_{(u,i) \in D} \delta_{u,i} p_{u,i} / \tilde{p}_{u,i}}{\sum_{(u,i) \in D} p_{u,i} / \tilde{p}_{u,i}} \right) \right| + O(|D|^{-1}).$$

\[\square\]

### B Proof of Theorem 3

**Proof of Theorem 3.** According to the proof of Theorem 2, we have

$$E[\bar{v}/\bar{w}] - \mu_v/\mu_w = O(|D|^{-1}). \tag{7}$$

Then the variance of $E_{SDR}$ can be decomposed into as

$$\text{Var}(E_{SDR}) = \text{Var}(\bar{v}/\bar{w}) = E \left[ (\bar{v}/\bar{w} - E(\bar{v}/\bar{w}))^2 \right]$$

$$= E \left[ (\bar{v}/\bar{w} - \mu_v/\mu_w)^2 - 2O(|D|^{-1}) \cdot \{\bar{v}/\bar{w} - \mu_v/\mu_w\} + O(|D|^{-2}) \right],$$

$$= \mathcal{V}_1 + \mathcal{V}_2 + O(|D|^{-2}),$$

where $\mathcal{V}_1 \triangleq E[(\bar{v}/\bar{w} - \mu_v/\mu_w)^2], \mathcal{V}_2 \triangleq -2O(|D|^{-1}) \cdot [E(\bar{v}/\bar{w}) - \mu_v/\mu_w]$. Equation (7) implies that $\mathcal{V}_2 = O(|D|^{-2})$. 

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C Proof of Theorem 4

Proof of Theorem 4. The McDiarmid’s inequality states that for independent bounded random variables $X_1, X_2, \ldots, X_n$, where $X_i \in \mathcal{X}_i$ for all $i$ and a mapping $f : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$. Assume there exist constant $c_1, c_2, \ldots, c_n$ such that for all $i$,

\[
\sup_{x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n} |f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)| \leq c_i.
\]

Then, for any $\epsilon > 0$,

\[
\mathbb{P}(|f(X_1, X_2, \cdots, X_n) - \mathbb{E}[f(X_1, X_2, \cdots, X_n)]| \geq \epsilon) \leq 2 \exp \left( - \frac{2\epsilon^2}{\sum_{i=1}^n c_i^2} \right).
\]

In fact, equation (6) implies that the SDR estimator can be written as

\[
\mathcal{E}_{SDR} = \frac{\sum_{(u,i) \in \mathcal{D}} o_{u,i}(e_{u,i} - \hat{e}_{u,i}) / \hat{p}_{u,i}}{\sum_{(u,i) \in \mathcal{D}} \hat{p}_{u,i}} + \hat{\mathcal{E}},
\]

denoted as $f(o_{1,1}, \ldots, o_{u,i}, \ldots, o_{U,1})$. Note that

\[
\sup_{o_{u,i}, o'_{u,i}} |f(o_{1,1}, \ldots, o_{u,i}, \ldots, o_{U,1}) - f(o_{1,1}, \ldots, o'_{u,i}, \ldots, o_{U,1})| \leq \begin{cases} 
\delta_{\max} - \frac{\delta_{u,i} / \hat{p}_{u,i} + \sum_{D \setminus (u,i)} o_{u,i} / \hat{p}_{u,i} \delta_{\max}}{1 / \hat{p}_{u,i} + \sum_{D \setminus (u,i)} o_{u,i} / \hat{p}_{u,i}}, & \text{if } \delta_{u,i} \leq (\delta_{\min} + \delta_{\max})/2, \\
\sum_{D \setminus (u,i)} o_{u,i} / \hat{p}_{u,i} + \delta_{\min} + \delta_{u,i} / \hat{p}_{u,i} - \delta_{\min}, & \text{if } \delta_{u,i} > (\delta_{\min} + \delta_{\max})/2,
\end{cases}
\]

\[(8)\]
where $D \setminus (u, i)$ is the set of $D$ excluding the element $(u, i)$.

Next, we focus on analyzing the $\sum_{D \setminus (u, i)} o_{u,i}/\hat{p}_{u,i}$. The Hoeffding’s inequality states that for independent bounded random variables $X_1, \ldots, X_n$ that take values in intervals of sizes $\rho_1, \ldots, \rho_n$ with probability 1 and for any $\epsilon > 0$, 

$$P\left( \left| \sum_k X_k - E(\sum_k X_k) \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{2\epsilon^2}{\sum_k \rho_k^2} \right).$$

For $\sum_{D \setminus (u, i)} o_{u,i}/\hat{p}_{u,i}$, we have 

$$P\left( \left| \sum_{D \setminus (u, i)} o_{u,i}/\hat{p}_{u,i} - \sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{2\epsilon^2}{\sum_{D \setminus (u, i)} \frac{1}{\hat{p}_{u,i}^2}} \right).$$

Setting the last term equals to $\eta/2$, and solving for $\epsilon$ gives that with probability at least $1 - \eta/2$, the following inequality holds 

$$\left| \sum_{D \setminus (u, i)} o_{u,i}/\hat{p}_{u,i} - \sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} \right| \leq \sqrt{\frac{1}{2} \log \frac{4}{\eta} \sum_{D \setminus (u, i)} \frac{1}{\hat{p}_{u,i}^2} \delta \leq \epsilon^2. (9)$$

Therefore, combining (8) and (9) yields that with probability at least $1 - \eta/2$, 

$$\sup_{o_{1, \ldots, o_{u,i}, o_{U,1}, \ldots, o_{U,T}}} |f(o_{1,1}, \ldots, o_{u,i}, \ldots, o_{U,T}) - f(o_{1,1}, \ldots, o_{u,i}^*, \ldots, o_{U,T})|$$

$$\leq \begin{cases} \delta_{\max} - \delta_{u,i}/\hat{p}_{u,i} + \left( \sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon \right)\delta_{\max}, & \text{if } \delta_{u,i} \leq (\delta_{\min} + \delta_{\max})/2, \\ \left( \frac{1}{\hat{p}_{u,i}} + \sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon \right)\delta_{\min} + \delta_{u,i}/\hat{p}_{u,i}, & \text{if } \delta_{u,i} > (\delta_{\min} + \delta_{\max})/2, \\ \left( \delta_{u,i} - \delta_{\min}/\hat{p}_{u,i} + \sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon \right), & \text{if } \delta_{u,i} \leq (\delta_{\min} + \delta_{\max})/2, \\ \left( \delta_{u,i} - \delta_{\min}/\hat{p}_{u,i} + \sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon \right), & \text{if } \delta_{u,i} > (\delta_{\min} + \delta_{\max})/2, \\ \end{cases}$$

where $\delta_{u,i} = \epsilon_{u,i} - \hat{\epsilon}_{u,i}$ is the error deviation, $\delta_{\min} = \min_{(u,i) \in D} \delta_{u,i}$, and $\delta_{\max} = \max_{(u,i) \in D} \delta_{u,i}$.

Invoking McDiarmid’s inequality leads to that 

$$P\left( (\mathcal{E}_{\text{SR}} - \mathbb{E}(\mathcal{E}_{\text{SR}})) \geq \epsilon \right) \leq 2 \exp \left\{ -\frac{2\epsilon^2}{\sum_{(u,i), \delta_{u,i} \leq \delta_{\min} + \delta_{\max}} (1 + \hat{p}_{u,i}/\sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon)^2} \right\}$$

$$+ \left\{ \sum_{(u,i), \delta_{u,i} > \delta_{\min} + \delta_{\max}} (1 + \hat{p}_{u,i}/\sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon)^2 \right\}$$

$$\leq 2 \exp \left( -\frac{2\epsilon^2}{\sum_{(u,i)} \{(\delta_{\max} - \delta_{u,i})^2 + (\delta_{u,i} - \delta_{\min})^2\}/(1 + \hat{p}_{u,i}/\sum_{D \setminus (u, i)} p_{u,i}/\hat{p}_{u,i} - \epsilon)^2} \right).$$

Setting the last term equals to $\eta/2$, and solving for $\epsilon$ complete the proof. 

\[ \square \]