RECOLLEMENTS OF COHEN-MACAULAY AUSLANDER ALGEBRAS AND GORENSTEIN DERIVED CATEGORIES

JAVAD ASADOLLAHI, RASOOL HAFEZI AND RAZIEH VAHED

Abstract. Let $A$, $B$ and $C$ be associative rings with identity. Using a result of Koenig we show that if we have a $\mathbb{D}^b$(mod-) level recollement, writing $A$ in terms of $B$ and $C$, then we get a $\mathbb{D}^{-}$(Mod-) level recollement of certain functor categories, induces from the module categories of $A$, $B$ and $C$. As an application, we generalise the main theorem of Pan [Sh. Pan, Derived equivalences for Cohen-Macaulay Auslander algebras, J. Pure Appl. Algebra, 216 (2012), 355-363] in terms of recollements of Gorenstein artin algebras. Moreover, we show that being Gorenstein as well as being of finite Cohen-Macaulay type, are invariants with respect to $\mathbb{D}_G^b$(mod-) level recollements of virtually Gorenstein algebras, where $\mathbb{D}_G^b$ denotes the Gorenstein derived category.

1. Introduction

Rcollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [BBD] in geometric contexts for derived categories of sheaves. Recollements with all three triangulated categories being derived categories of rings are in fact short exact sequences of derived categories. Using these tools, recently Angeleri Hügel, Koenig and Liu proved a Jordan-Hölder theorem for derived module categories of certain algebras and also study stratifications of derived module categories, see [AKL2] and [AKL3]. Recollements also have applications in tilting theory [AKL1].

On the other hand, study of invariants of recollements have been the subject of several researches. For example, Wiedemann [W, Lemma 2.1] proved that if we have a $\mathbb{D}^b$(mod-) level recollement of finite dimensional algebras over a field, writing $A$ in terms of $B$ and $C$, i.e.

$\mathbb{D}^b(\text{mod-}B) \rightarrow \mathbb{D}^b(\text{mod-A}) \rightarrow \mathbb{D}^b(\text{mod-C})$

then gldim($A$) is finite if and only if gldim($B$) and gldim($C$) are finite. Similar result for the finitistic dimension is proved by Happel [H, 3.3]. In these cases, we say that the finiteness of global dimension and also finitistic dimension are invariants of recollements. There is an example showing that being of finite representation type is not an invariant of recollements, see Section 4 below.

In the first part of this paper, using functor category techniques, we get a recollement of functor categories and use it to generalise the main result of Pan [P] in terms of recollements of Gorenstein artin algebras. More precisely, Pan proved that if $A$ and $B$ are Gorenstein algebras of finite Cohen-Macaulay type that are derived equivalent, then their Cohen-Macaulay Auslander algebras are also derived equivalent. We show that if we have a recollement
of algebras such that $B$ and $C$ are Gorenstein of finite Cohen-Macaulay type, then we get a recollement

$$
\mathbb{D}^b(\text{mod-}\mathrm{End}_B(G)) \longrightarrow \mathbb{D}^b(\text{mod-}\mathrm{End}_A(H)) \longrightarrow \mathbb{D}^b(\text{mod-}\mathrm{End}_C(G'))
$$

in which $\mathrm{End}_B(G)$, resp. $\mathrm{End}_C(G')$, is the Cohen-Macaulay Auslander algebra of $B$, resp. $C$. This easily implies that, if $A$ is also of finite Cohen-Macaulay type and $A$ and $B$ are derived equivalent, then so are their Cohen-Macaulay Auslander algebras. Moreover, we prove that Gorensteiness is an invariant of a $\mathbb{D}^b(\text{mod-})$ level recollement of artin algebras as above in which $B$ and $C$ are of finite Cohen-Macaulay type.

Let $\mathcal{A}$ be an abelian category. The concept of Gorenstein derived category of $\mathcal{A}$, $\mathbb{D}_{\mathcal{GP}}(\mathcal{A})$ is introduced and studied by Gao and Zhang in [GZ] based on the idea to have a category in which Gorenstein quasi-isomorphisms are isomorphisms. It also can be interpreted as the Verdier quotient of the homotopy category $\mathbb{K}^*(\mathcal{A})$ with respect to the thick triangulated subcategory $\mathbb{K}^*_\mathcal{GP}_{\text{ac}}(\mathcal{A})$ of $\mathcal{GP}$-acyclic complexes.

In Section 4, we show that if we have a $\mathbb{D}^b(\text{mod-})$ level recollement of virtually Gorenstein algebras, then being Gorenstein as well as being of finite Cohen-Macaulay type, are invariants of recollements.

Let us end this introduction by mentioning that in a private communication with Nan Gao, she point out that she also has proofs for Lemma 3.7 and Theorem 4.3 of this paper. Her proofs, that are obtained independently, are quite different from the proofs appeared in this paper. We would like to thank her for letting us know her proofs.

2. Preliminaries

Let $A$ be a ring, associative with identity and Mod-$A$, resp. mod-$A$, denote the category of all, resp. all finitely presented, $A$-modules.

For an $A$-module $M$, add-$M$ is the additive subcategory of Mod-$A$ consisting of all direct summands of finite direct sums of copies of $M$. For a class $\mathcal{M}$ of $A$-modules, Sum-$\mathcal{M}$, resp. sum-$\mathcal{M}$, denotes the subcategory of Mod-$A$ consisting of all, resp. all finite, direct sums of copies of objects in $\mathcal{M}$.

Let $\mathcal{A}$ be an additive category. $\mathbb{C}(\mathcal{A})$ denotes the category of complexes over $\mathcal{A}$. We write complexes cohomologically, so every object of $\mathbb{C}(\mathcal{A})$ is of the form

$$
\cdots \rightarrow C^{n-1} \rightarrow C^n \rightarrow C^{n+1} \rightarrow \cdots
$$

$\mathbb{K}(\mathcal{A})$ denotes the homotopy category of $\mathcal{A}$. Moreover, $\mathbb{K}^- (\mathcal{A})$, resp. $\mathbb{K}^b (\mathcal{A})$, denotes the full subcategory of $\mathbb{K}(\mathcal{A})$ formed by all bounded above, resp. bounded, complexes. As well, we let $\mathbb{K}^- (\mathcal{A})$ denote the full subcategory of $\mathbb{K}^- (\mathcal{A})$ consisting of all complexes $X$ in which there is an integer $n = n_X$ such that $H^i(X) = 0$, for all $i \leq n$.

The derived category of $\mathcal{A}$ will be denoted by $\mathbb{D}(\mathcal{A})$. We write $\mathbb{D}^- (\mathcal{A})$, resp $\mathbb{D}^b (\mathcal{A})$, for the full subcategory of $\mathbb{D}(\mathcal{A})$ consisting of all homologically bounded above, resp. homologically bounded, complexes.

Let $\mathcal{T}$ be a triangulated category and $T$ be an object of $\mathcal{T}$. The smallest triangulated full subcategory of $\mathcal{T}$ containing $T$ and closed under taking direct summands is denoted by $\langle T \rangle$. It is known fact that the subcategory $\langle A \rangle$ of $\mathbb{D}^b(\text{mod-}A)$ is just $\mathbb{K}^b(\text{prj-}A)$.
Let $\mathcal{A}$ be an abelian category and $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory of $\mathcal{A}$ which is closed under direct summands. For every object $C$ of $\mathcal{A}$, a morphism $f : X \to C$ is called a right $\mathcal{X}$-approximation of $C$, if $X \in \mathcal{X}$ and for every $X′ \in \mathcal{X}$ the induced map $\text{Hom}_{\mathcal{A}}(X′, X) \to \text{Hom}_{\mathcal{A}}(X′, C)$ is surjective. The subcategory $\mathcal{X}$ of $\mathcal{A}$ is called contravariantly finite if every object in $\mathcal{A}$ admits a right $\mathcal{X}$-approximation.

2.1. Let $\mathcal{X}$ be an additive category. A complex $X \in \mathcal{C}(\mathcal{X})$ is called $\mathcal{X}$-totally acyclic if for every $X \in \mathcal{X}$, the induced complexes $\text{Hom}(X, X)$ and $\text{Hom}(X, X)$ are acyclic.

Let $\mathcal{A}$ be an abelian category having enough projective, resp. injective, objects and $\mathcal{X} = \text{Prj-}\mathcal{A}$, resp. $\mathcal{X} = \text{Inj-}\mathcal{A}$, be the class of projectives, resp. injectives. In this case, an $\mathcal{X}$-totally acyclic complex is called totally acyclic complex of projectives, resp. totally acyclic complex of injectives. An object $G$ in $\mathcal{A}$ is called Gorenstein projective, resp. Gorenstein injective, if $G$ is a syzygy of a totally acyclic complex of projectives, resp. totally acyclic complex of injectives. We denote the class of all Gorenstein projective, resp. Gorenstein injective, objects in $\mathcal{A}$ by $\mathcal{GP-}\mathcal{A}$, resp. $\mathcal{GI-}\mathcal{A}$. In case $\mathcal{A} = \text{Mod-}\mathcal{A}$, we abbreviate the notations to $\mathcal{GP-}\mathcal{A}$ and $\mathcal{GI-}\mathcal{A}$. We set $\mathcal{GP-}\mathcal{A} = \mathcal{GP-}\mathcal{A} \cap \text{mod-}\mathcal{A}$ and $\mathcal{GI-}\mathcal{A} = \mathcal{GI-}\mathcal{A} \cap \text{mod-}\mathcal{A}$.

2.2. Virtually Gorenstein algebras. Let $A$ be an artin algebra over a commutative artinian ring $k$. $A$ is called Gorenstein if $\text{id}_A A < \infty$ and $\text{id}_A A < \infty$. For a class $\mathcal{X}$ of $A$-modules, the left and right orthogonals of $\mathcal{X}$ are defined as follows

\[ \mathcal{X}^{\perp} := \{ M \in \text{Mod-}A \mid \text{Ext}^1_A (M, Y) = 0, \text{ for all } Y \in \mathcal{Y} \} \]

and

\[ \mathcal{X}^{\perp} := \{ M \in \text{Mod-}A \mid \text{Ext}^1_A (X, M) = 0, \text{ for all } X \in \mathcal{X} \}. \]

$A$ is called virtually Gorenstein if $(\mathcal{GP-}A)^{\perp} = (\mathcal{GI-}A)$. These algebras were introduced in [BR] as a common extension of Gorenstein algebras and algebras of finite representation type. Beligiannis proved that if $A$ is virtually Gorenstein, then $\mathcal{GP-}A$ is a contravariantly finite subcategory of $\text{mod-}A$, provided $A$ is virtually Gorenstein [B, Proposition 4.7].

2.3. Cohen-Macaulay Auslander algebras. Let $A$ be an artin algebra. $A$ is called of finite Cohen-Macaulay type, CM-finite for short, if there exist only finitely many indecomposable finitely generated Gorenstein projective $A$-modules, up to isomorphism. In other words, an artin algebra $A$ is CM-finite if there is an $A$-module $M$ such that $\text{add-}M = \mathcal{GP-}A$. In this case, $\Gamma = \text{End}_A(M)$ is called the Cohen-Macaulay Auslander algebra of $A$.

2.4. Functor category. Let $\mathcal{C}$ be an additive skeletally small category. We denote by $\text{Mod}(\mathcal{C})$ the category of additive contravariant functors from $\mathcal{C}$ to the category of abelian groups, which is called the category of modules on $\mathcal{C}$. It known that $\text{Mod}(\mathcal{C})$ is an abelian category with arbitrary direct sums.

For an object $C \in \mathcal{C}$, one may apply Yoneda lemma to show that the representable functor $\text{Hom}_\mathcal{C}(\cdot, C)$ is a projective object in $\text{Mod}(\mathcal{C})$. Moreover, for any functor $F \in \text{Mod}(\mathcal{C})$, there is an epimorphism $\coprod \text{Hom}_\mathcal{C}(\cdot, C_i) \to F \to 0$, where $C_i \in \mathcal{C}$ for all $i$. So the abelian category $\text{Mod}(\mathcal{C})$ has enough projective objects. More precisely, the set $\{ \text{Hom}_\mathcal{C}(\cdot, C) \mid C \in \mathcal{C} \}$ is a set of projective generators for the category $\text{Mod}(\mathcal{C})$.

2.5. Recollements of triangulated categories. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories. $\mathcal{T}$ is called a recollement of $\mathcal{T}'$ and $\mathcal{T}''$ if there exits a diagram consisting of six
triangulated functors as follows

\[
\begin{array}{ccc}
T' & \xleftarrow{i^*} & T \\
\downarrow{i_*=i} & & \downarrow{j= j_*} \\
T'' & \xrightarrow{j^*} & T
\end{array}
\]

satisfying the following conditions:

(i) \((i^*, i_*)\), \((i^!, i^!)\), \((j^!, j_*!)\) and \((j^*, j_*)\) are adjoint pairs.
(ii) \(i^! j_* \equiv 0\), and hence \(j^! i_* \equiv 0\) and \(i^* j_! \equiv 0\).
(iii) \(i_* j_*\) and \(j_* j_!\) are full embeddings.
(iv) for any object \(T \in T\), there exist the following triangles

\[i^! i^*(T) \rightarrow T \rightarrow j_* j^!(T) \rightarrow \text{ and } j^! j^*(T) \rightarrow i_* i^*(T) \rightarrow \text{ in } T.\]

3. Recollements of Cohen-Macaulay Auslander algebras

Let \(\mathcal{A}\) be an abelian category and \(\text{Prj}-\mathcal{A}\) be the class of projective objects in \(\mathcal{A}\). A complex \(\mathcal{T} \in \mathbb{K}^b(\text{Prj}-\mathcal{A})\) is called a partial tilting complex if it satisfies the following properties

(i) \(\text{Hom}_{\mathbb{K}^b(\text{Prj}-\mathcal{A})}(\mathcal{T}, \mathcal{T}[i]) = 0\), for all \(i \neq 0\).
(ii) If \(\{\mathcal{T}_i\}_{i \in I}\) is an index family of copies of \(\mathcal{T}\), then

\[\text{Hom}_{\mathbb{K}^b(\text{Prj}-\mathcal{A})}(\mathcal{T}, \bigoplus_{i \in I} \mathcal{T}_i) \cong \bigoplus_{i \in I} \text{Hom}_{\mathbb{K}^b(\text{Prj}-\mathcal{A})}(\mathcal{T}, \mathcal{T}_i).\]

For any \(\mathcal{X} \in \mathbb{D}^-(\mathcal{A})\), \(\mathcal{X}^+\) is a full triangulated subcategory of \(\mathbb{D}^-(\mathcal{A})\) generated by all complexes \(\mathcal{Y} \in \mathbb{D}^-(\mathcal{A})\) in which \(\text{Hom}_{\mathbb{D}^-(\mathcal{A})}(\mathcal{X}, \mathcal{Y}[i]) = 0\), for all \(i \in \mathbb{Z}\). Let \(\mathcal{T}\) be a subcategory of \(\mathbb{D}^-(\mathcal{A})\). We set \(\mathcal{T}^+ = \bigcap_{\mathcal{X} \in \mathcal{T}} \mathcal{X}^+\).

Let \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{C}\) denote rings that are associative with identity. Koenig proved that derived category \(\mathbb{D}^-(\text{Mod}-\mathcal{A})\) of \(\mathcal{A}\) admits a recollement

\[\mathbb{D}^-(\text{Mod-B}) \xrightarrow{i^*} \mathbb{D}^-(\text{Mod-A}) \xrightarrow{j^*} \mathbb{D}^-(\text{Mod-C})\]

relative to \(\mathbb{D}^-(\text{Mod-B})\) and \(\mathbb{D}^-(\text{Mod-C})\) if and only if there exist partial tilting complexes \(\mathcal{B} \in \mathbb{K}^b(\text{Prj}-\mathcal{A})\) and \(\mathcal{C} \in \mathbb{K}^b(\text{prj}-\mathcal{A})\), where \(\text{prj}-\mathcal{A}\) denotes the category of finitely generated projective \(\mathcal{A}\)-modules, satisfying the following properties

(i) \(\text{End}_\mathcal{A}(\mathcal{B}) \cong \mathcal{B}\),
(ii) \(\text{End}_\mathcal{A}(\mathcal{C}) \cong \mathcal{C}\),
(iii) \(\text{Hom}_{\mathbb{D}^-(\text{Mod-A})}(\mathcal{C}, \mathcal{B}[i]) = 0\) for all \(i \in \mathbb{Z}\),
(iv) \(\mathcal{B}^+ \cap \mathcal{C}^+ = 0\).

We shall need a generalised version of this theorem in terms of functor categories for the proof of our main theorem. The proof of this version is just a simple modification of the Koenig’s argument in the proof of the above theorem.

**Theorem 3.1.** Let \(\mathcal{S}_A, \mathcal{S}_B\) and \(\mathcal{S}_C\) be three full skeletally small subcategories of \(\text{Mod}-\mathcal{A}\), \(\text{Mod}-\mathcal{B}\) and \(\text{Mod}-\mathcal{C}\). If there exists a recollement

\[\mathbb{D}^-(\text{Mod}(\mathcal{S}_B)) \xrightarrow{i^*} \mathbb{D}^-(\text{Mod}(\mathcal{S}_A)) \xrightarrow{j^*} \mathbb{D}^-(\text{Mod}(\mathcal{S}_C)),\]

then...
of derived categories, then for every $X \in \mathcal{S}_B$, resp. for every $Y \in \mathcal{S}_C$, there is a partial tilting complex $\mathfrak{B}_X \in K^b(\text{Prj-Mod}(S_A))$, resp. $\mathfrak{C}_Y \in K^b(\text{prj-Mod}(S_A))$, satisfying the following conditions

(i) If we set $\mathfrak{B} = \{ \mathfrak{B}_X \mid X \in \mathcal{S}_B \}$, then $\text{Mod}(\mathfrak{B})$ is equivalent to $\text{Mod}(\mathcal{S}_B)$.
(ii) If we set $\mathcal{C} = \{ \mathfrak{C}_Y \mid Y \in \mathcal{S}_C \}$, then $\text{Mod}(\mathcal{C})$ is equivalent to $\text{Mod}(\mathcal{S}_C)$.
(iii) $\text{Hom}(\mathfrak{C}_Y, \mathfrak{B}_X[i]) = 0$, for any $X \in \mathcal{S}_B$, $Y \in \mathcal{S}_C$ and $i \in \mathbb{Z}$.
(iv) $\mathfrak{B}^\perp = \mathfrak{C}^\perp = \{0\}$.

The converse holds if, furthermore, we have that there exists fixed integer $n$, resp. $m$, such that $\mathfrak{B}^i_X = 0$ for $|i| > n$ and for every $X \in \mathcal{S}_B$, resp. $\mathfrak{C}^i_Y = 0$ for $|i| > m$ and for every $Y \in \mathcal{S}_C$.

Proof. The result follows by modifying the proof of Theorem 1 of [K] to this context. Just note that, for every $X \in \mathcal{S}_B$, resp. $Y \in \mathcal{S}_C$, one should take $B_X = i_*(\text{Hom}_B(-, X))$, resp. $\mathfrak{C}_Y = j_!(\text{Hom}_C(-, Y))$.

For the converse, consider complexes $\bigoplus_{X \in \mathcal{S}_B} \mathfrak{B}_X$ and $\bigoplus_{Y \in \mathcal{S}_C} \mathfrak{C}_Y$. The same techniques used in the Rickard’s proof [Ric, Theorem 6.4] can be applied to define the fully faithful functor $i_* : \mathbb{D}^- (\text{Mod}(\mathcal{S}_B)) \rightarrow \mathbb{D}^- (\text{Mod}(S_A))$, resp. $j_* : \mathbb{D}^- (\text{Mod}(\mathcal{S}_C)) \rightarrow \mathbb{D}^- (\text{Mod}(S_A))$, which has a right adjoint $i^*: \mathbb{D}^- (\text{Mod}(S_A)) \rightarrow \mathbb{D}^- (\text{Mod}(\mathcal{S}_B))$, resp. $j^*: \mathbb{D}^- (\text{Mod}(S_A)) \rightarrow \mathbb{D}^- (\text{Mod}(\mathcal{S}_C))$. Now, the same argument as in [K, Theorem 1] works to construct the other functors and prove that these functors form a recollement. □

We point out that similar techniques as what is used in the ‘converse’ part of the above proof, also has been used in the proof of Proposition 4.7 of [AHV].

Let $A$ admits a recollement of the form

$$
\mathbb{D}^b(\text{mod-}B) \xrightarrow{i_*} \mathbb{D}^b(\text{mod-}A) \xleftarrow{j^!} \mathbb{D}^b(\text{mod-}C).
$$

The argument of the proof of Theorem 1 of [K] shows that $i_*(B)$ and $j!(C)$ both belong to $K^b(\text{prj-}A)$. Without loss of generality, we can assume that $i_*(B)$ and $j!(C)$ are complexes of the form

$$
0 \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots \rightarrow B^n \rightarrow 0 \quad \text{and}
$$

$$
0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^n \rightarrow 0,
$$

respectively, where $B^i, C^i \in \text{prj-}A$, for all $0 \leq i \leq n$.

Throughout, for a ring $A$, we set

$$
\perp A := \{ M \in \text{mod-}A \mid \text{Ext}^i(M, A) = 0, \quad \forall \ i > 0 \}.
$$

Lemma 3.2. Let $i_* : \mathbb{D}^b(\text{mod-}B) \rightarrow \mathbb{D}^b(\text{mod-}A)$ and $j! : \mathbb{D}^b(\text{mod-}C) \rightarrow \mathbb{D}^b(\text{mod-}A)$ be functors appearing in the above recollement.

(i) For every finitely generated $B$-module $X$ in $\perp B$, the complex $\mathfrak{B}_X = i_*(X)$ in $\mathbb{D}^b(\text{mod-}A)$ is isomorphic to a complex of the form

$$
0 \rightarrow B^0_X \rightarrow B^1_X \rightarrow \cdots \rightarrow B^n_X \rightarrow 0
$$

with $B^n_X \in \perp A$ and $B^i_X$ projective $A$-module, for all $i = 1, 2, \cdots, n$. 

For every finitely generated C-module $Y$ in $\mathcal{C}$, the complex $C_Y = J(Y)$ in $D^b(\text{mod}-A)$ is isomorphic to a complex of the form

$$0 \rightarrow C^0_Y \rightarrow C^1_Y \rightarrow \cdots \rightarrow C^n_Y \rightarrow 0$$

with $C^0_Y \in \perp A$ and $C^i_Y$ finitely generated projective $A$-module, for all $i = 1, 2, \ldots, n$.

**Proof.** We just prove the first part, the proof of the second part is similar. The shape of $i_*(B)$ together with the same argument as in the proof of [HX1, Lemma 3.1] imply that $i_*(X)$ is of the form

$$0 \rightarrow B^0_X \rightarrow B^1_X \rightarrow \cdots \rightarrow B^n_X \rightarrow 0,$$

in which $B^i_X \in \text{prj-}A$ for all $1 \leq i \leq n$. So, to complete the proof, we should show that $B^i_X \in \perp A$, i.e. $\text{Ext}^i_A(B^0_X, A) = 0$, for $i > 0$.

Let $\mathcal{B}^{\geq 1}_X$ denote the complex

$$0 \rightarrow 0 \rightarrow B^1_X \rightarrow \cdots \rightarrow B^n_X \rightarrow 0.$$

Consider the exact triangle

$$\mathcal{B}^{\geq 1}_X \rightarrow \mathcal{B}_X \rightarrow B^0_X \rightarrow \mathcal{B}^{\geq 1}_X[1]$$

in $K^b(\text{mod-}A)$ and apply the cohomological functor $\text{Hom}_{K^b(\text{mod-}A)}(\cdot, A[i])$ to it, to get the exact sequence

$$\cdots \rightarrow \text{Hom}_{K^b(\text{mod-}A)}(\mathcal{B}^{\geq 1}_X[1], A[i]) \rightarrow \text{Hom}_{K^b(\text{mod-}A)}(B^0_X, A[i]) \rightarrow \text{Hom}_{K^b(\text{mod-}A)}(\mathcal{B}_X, A[i]) \rightarrow \cdots$$

of abelian groups. Since $\text{Ext}_B^i(X, B) = 0$ for all $i > 0$, by [Ka, Lemma 1.6],

$$\text{Hom}_{K^b(\text{mod-}A)}(\mathcal{B}_X, A[i]) \cong \text{Hom}_{K^b(\text{mod-}B)}(X, i_!(A)[i]) = 0,$$

for all $i > 0$. Moreover,

$$\text{Hom}_{K^b(\text{mod-}A)}(\mathcal{B}^{\geq 1}_X[1], A[i]) \cong \text{Hom}_{K^b(\text{mod-}A)}(\mathcal{B}^{\geq 1}_X[1], A[i]) = 0,$$

for all $i > 0$. Thus,

$$\text{Ext}_B^i(B^0_X, A) \cong \text{Hom}_{K^b(\text{mod-}A)}(B^0_X, A[i]) = 0,$$

for all $i > 0$. This means that $B^i_X \in \perp A$ and the proof is hence complete. \qed

The proof of the following lemma is exactly the same as [P, Lemma 3.7], so we skip the proof.

**Lemma 3.3.** Let $X$ and $Y$ be two complexes in $K^b(\text{Mod-}A)$ of the following forms

$$X : \quad 0 \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^n \rightarrow 0,$$

$$Y : \quad 0 \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \rightarrow Y^n \rightarrow 0,$$

with $X^0, Y^0 \in \perp A$ and $X^i, Y^i \in \text{Prj-}A$ for all $1 \leq i \leq n$. Then $\text{Hom}_{K^b(\text{Mod-}A)}(X, Y[i]) = 0$ for all $i \neq 0$.

We also need the following lemma, that is quoted from [HX1, Lemma 2.2].

**Lemma 3.4.** Let $X$ be a bounded above and $Y$ be a bounded below complex of $A$-modules. If there is an integer $m$, such that $X^i$ is projective for all $i > m$ and $Y^j = 0$ for all $j < m$, then $\text{Hom}_{D(\text{Mod-}A)}(X, Y) \cong \text{Hom}_{K(\text{Mod-}A)}(X, Y)$. 

For any finitely generated $B$-module $X$ in $\perp B$, Lemma 3.2 implies that $i_*(X)$ is isomorphic to a complex of the form

$$0 \to B^0_X \to B^1_X \to \cdots \to B^n_X \to 0$$

with $B^i_X \in \perp A$ and $B^i_X$ finitely generated projective $A$-module, for $1 \leq i \leq n$. Set $i_!(X) = B^0_X$. Similarly, for any finitely generated $C$-module $Y$ in $\perp C$, we let $j_!(Y)$ to be the zeroth term of the complex $j_!(Y)$ that belongs to $\perp C$. So we fix the following notations.

**Notation 3.5.** Consider two subcategories $S_B \subseteq \perp B$ and $S_C \subseteq \perp C$ that contain $B$ and $C$, respectively. We let $\mathcal{H}$ denote the subcategory of mod-$A$ consisting of all finitely generated projective $A$-modules and all finitely generated $A$-modules $G$ that satisfy one of the following conditions

(i) There is a $B$-module $X$ in $S_B$ such that $G = i_*(X)$.

(ii) There is a $C$-module $Y$ in $S_C$ such that $G = j_!(Y)$.

**Theorem 3.6.** Let $S_B$, $S_C$ and $\mathcal{H}$ be as above. If there is a recollement as follows

$$\mathbb{D}^b(\text{mod-}B) \xrightarrow{i_*} \mathbb{D}^b(\text{mod-}A) \xleftarrow{j_*} \mathbb{D}^b(\text{mod-}C),$$

then $\mathbb{D}^-(\text{Mod}(\mathcal{H}))$ admits the following recollement

$$\mathbb{D}^-(\text{Mod}(S_B)) \xrightarrow{i_*} \mathbb{D}^-(\text{Mod}(\mathcal{H})) \xleftarrow{j_*} \mathbb{D}^-(\text{Mod}(S_C)).$$

**Proof.** For any $X \in S_B$, resp. $Y \in S_C$, let $\mathcal{B}_X$, resp. $\mathcal{C}_Y$, be the complex $\text{Hom}_A(-, i_*(X))$, resp. $\text{Hom}_A(-, j_!(Y))$.

Since $\mathcal{B}_X \in \mathbb{K}^b(\text{Prj-Mod}(\mathcal{H}))$, $\text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(\mathcal{B}_X, \mathcal{B}_X[i]) \cong \text{Hom}_{\mathbb{K}^b(\text{Mod}(\mathcal{H}))}(\mathcal{B}_X, \mathcal{B}_X[i])$. By using Yoneda lemma, $\text{Hom}_{\mathbb{K}^b(\text{Mod}(\mathcal{H}))}(\mathcal{B}_X, \mathcal{B}_X[i]) \cong \text{Hom}_{\mathbb{K}^b(\mathcal{H})}(i_*(B), i_*(B)[i])$ and hence vanishes for all $i \neq 0$ and all $X \in S_B$. Moreover, since $S_B \subseteq \text{mod-}B$, $\text{Hom}_{\mathbb{K}^b(\mathcal{H})}(\mathcal{B}_X, \oplus_i \mathcal{B}_X) \cong \oplus_i \text{Hom}_{\mathbb{K}^b(\mathcal{H})}(\mathcal{B}_X, \mathcal{B}_X)$ thanks to the Yoneda lemma. Then $\text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(\mathcal{B}_X, \oplus_i \mathcal{B}_X) \cong \oplus_i \text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(\mathcal{B}_X, \mathcal{B}_X)$, because $\mathcal{B}_X \in \mathbb{K}^b(\text{Prj-Mod}(\mathcal{H}))$. So for every $X \in S_B$, $\mathcal{B}_X$ is a partial tilting complex. Similarly, for every $Y \in S_C$, $\mathcal{C}_Y$ is a partial tilting complex.

Set $\mathcal{B} = \{ \mathcal{B}_X \mid X \in S_B \}$ and $\mathcal{C} = \{ \mathcal{C}_Y \mid Y \in S_C \}$. One may use Yoneda lemma to show that there is an equivalence between $S_B$ and $\mathcal{B}$ which assigns any $X$ in $S_B$ to $\mathcal{B}_X$. Therfore, $\text{Mod}(S_B) \simeq \text{Mod}(\mathcal{B})$ and $\text{Mod}(S_C) \simeq \text{Mod}(\mathcal{C})$.

Now, it follows from Lemma 3.3 that $\text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(\mathcal{B}_Y, \mathcal{B}_X[i]) = 0$ for every $Y \in S_C$ and $X \in S_B$ and for all $i \neq 0$. In addition,

$$\text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(\mathcal{C}_Y, \mathcal{B}_X) = \text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(\text{Hom}_A(-, j_!(Y)), \text{Hom}_A(-, i_*(X)))$$

$$\cong \text{Hom}_{\mathbb{K}^b(\text{Mod}(\mathcal{H}))}(\text{Hom}_A(-, j_!(Y)), \text{Hom}_A(-, i_*(X)))$$

$$\cong \text{Hom}_{\mathbb{K}^b(\text{Mod}(\mathcal{H}))}(j_!(Y), i_*(X)).$$

Lemmas 3.2 and 3.4 together imply that

$$\text{Hom}_{\mathbb{K}^b(\text{Mod}(\mathcal{H}))}(j_!(Y), i_*(X)) \cong \text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(j_!(Y), i_*(X)).$$

The adjoint pair $(j_!, j^*)$ implies that

$$\text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(j_!(Y), i_*(X)) \cong \text{Hom}_{\mathbb{D}^-(\text{Mod}(\mathcal{H}))}(Y, j^*(i_*(X))) = 0.$$

So in view of Theorem 3.1, to complete the proof, we need to show that $\mathcal{B}^\perp \cap \mathcal{C}^\perp = 0$. First note that an standard argument shows that $\mathbb{D}^-(\text{Mod}(\mathcal{H})) \simeq \mathbb{K}^-(\text{Sum-}\mathcal{H})$. Now, if
\( F \in \mathfrak{B}^\perp \cap \mathfrak{C}^\perp \), then for every \( X \in \mathcal{S}_B \), \( Y \in \mathcal{S}_C \) and each \( i \in \mathbb{Z} \), \( \text{Hom}_{\mathcal{B}^-(\text{Mod}(\mathcal{H}))}(\mathfrak{B}_X, F[i]) = 0 \) and \( \text{Hom}_{\mathcal{B}^-(\text{Mod}(\mathcal{H}))}(\mathfrak{C}_Y, F[i]) = 0 \).

Let \( \tilde{F} \in \mathbb{K}^-(\text{Sum-H}) \) be the image of \( F \) under the above equivalence. So for every \( X \in \mathcal{S}_B \), \( Y \in \mathcal{S}_C \) and \( i \in \mathbb{Z} \), we have

\[
\text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(i_*(X), \tilde{F}[i]) = 0 = \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(j_!(Y), \tilde{F}[i]).
\]

Since \( B \in \mathcal{S}_B \) and \( C \in \mathcal{S}_C \), \( \tilde{F} \in i_*(B)^\perp \cap j!(C)^\perp \) and hence is exact. So it is enough to show that, for every \( X \in \mathcal{S}_B \) and \( Y \in \mathcal{S}_C \), the induced complexes \( \text{Hom}_A(i_*(X), \tilde{F}) \) and \( \text{Hom}_A(j!(Y), \tilde{F}) \) are exact.

Observe that by a criterion of Rickard [Ric, Proof of Proposition 8.1], we get that \( i_*(A) \in \mathbb{K}^b(\text{prj-B}) \) and \( j^*(A) \in \mathbb{K}^b(\text{prj-C}) \). Moreover, it is known that \( \mathbb{K}^b(\text{prj-B}) = \langle B \rangle \) and \( \mathbb{K}^b(\text{prj-C}) = \langle C \rangle \). So the standard triangle

\[
j_1j^1_!(A) \to A \to i_!i^*(A) \to 1
\]

which is induced from the original recollement, implies that \( i_*(B) \oplus j!(C) \) generates \( \mathbb{K}^b(\text{prj-A}) \).

Therefore, for every complex \( P \in \mathbb{K}^b(\text{prj-A}) \) and every \( i \in \mathbb{Z} \), \( \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(P, \tilde{F}[i]) = 0 \).

Now, for every \( X \in \mathcal{S}_B \), there is the following triangle

\[
\mathfrak{B}_X^1 \to \mathfrak{B}_X \to i_*(X) \to \mathfrak{B}_X^1[1].
\]

Applying the cohomological functor \( \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(\mathcal{C}[1], \tilde{F}[i]) \) to it, we get the following long exact sequence

\[
\text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(\mathfrak{B}_X^1[1], \tilde{F}[i]) \to \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(i_*(X), \tilde{F}[i]) \to \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(\mathfrak{B}_X, \tilde{F}[i]) \to \cdots.
\]

By Lemma 3.2, \( \mathfrak{B}_X^1 \in \mathbb{K}^b(\text{prj-A}) \) and so \( \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(\mathfrak{B}_X^1[1], \tilde{F}) = 0 \), for all \( i \in \mathbb{Z} \). Therefore, \( \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(i_*(X), \tilde{F}) = 0 \) for all \( i \in \mathbb{Z} \). In a similar way, one can deduce that for every \( Y \in \mathcal{S}_C \) and all \( i \in \mathbb{Z} \), \( \text{Hom}_{\mathbb{K}^-(\text{Sum-H})}(j!(Y), \tilde{F}) = 0 \).

Consequently, \( \mathfrak{B} \) and \( \mathfrak{C} \) satisfy all the conditions of Theorem 3.6 and hence we have the desired recollement.

If we consider \( \mathcal{S}_B \) and \( \mathcal{S}_C \), in the theorem above, to be finite sets, then we obtain a \( \mathbb{D}^-(\text{Mod}) \) level recollement of rings from a \( \mathbb{D}^b(\text{mod}) \) level recollement.

Following lemma will be used in the proof of our next corollary.

**Lemma 3.7.** Let \( A \) be a CM-finite Gorenstein algebra. Then \( \text{Mod}(Gp-A) \) has finite global dimension.

**Proof.** Let \( F \in \text{Mod}(Gp-A) \). There is the following projective presentation

\[
\oplus_i \text{Hom}_A(\mathcal{C}, L_i) \longrightarrow \oplus_i \text{Hom}_A(\mathcal{C}, M_i) \longrightarrow F \longrightarrow 0,
\]

of \( F \), where \( L_i, M_i \in \mathcal{GpA} \) and \( i \) runs through all isomorphism classes of objects of \( \mathcal{GpA} \). So we have the exact sequence

\[
\text{Hom}_A(\mathcal{C}, \oplus_i L_i) \xrightarrow{\text{Hom}_A(\mathcal{C}, \varphi)} \text{Hom}_A(\mathcal{C}, \oplus_i M_i) \longrightarrow F \longrightarrow 0
\]

in \( \text{Mod}(Gp-A) \). Set \( K := \text{Ker} \varphi \). Since \( A \) is Gorenstein, there is the following finite \( Gp \)-acyclic resolution of \( K \)

\[
0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow K \longrightarrow 0,
\]
with \( Q_i \in \mathcal{GP} - A \) and \( n = \text{id}_A A \). In view of [B, Theorem 4.10], \( \text{Add} \mathcal{GP} - A = \mathcal{GP} - A \). Hence the functor \( \text{Hom}_A(-, Q_i) \) in \( \text{Mod}(\mathcal{GP} - A) \) is projective, for \( 0 \leq i \leq n \). Therefore, we obtain a finite projective resolution

\[
0 \to \text{Hom}_A(-, Q_n) \to \cdots \to \text{Hom}_A(-, Q_0) \to \text{Hom}_A(-, \oplus_i L_i) \to \text{Hom}_A(-, \oplus_i M_i) \to F \to 0
\]

of \( F \) in \( \text{Mod}(\mathcal{GP} - A) \). The proof is now complete. \( \square \)

Let \( \mathcal{X} \) be a subcategory of \( \text{mod-} A \) and \( n \) be a positive integer. For an \( A \)-module \( M \), we say that the \( \mathcal{X} \)-dimension of \( M \) is less than or equal to \( n \), \( \mathcal{X} \)-dim \( M \leq n \), if there exists an exact sequence \( 0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0 \), with all \( X_i \in \mathcal{X} \). If such sequence does not exist, we set \( \mathcal{X} \)-dim \( M = \infty \). In case \( \mathcal{X} = \perp A \), the \( \perp A \)-dimension of \( M \) is called the left orthogonal dimension of \( M \).

**Remark 3.8.** Let \( A \) be a ring. It is proved in [HH, Theorem 1.4] that \( A \) is Gorenstein with \( \text{id}_A A = \text{id}_A A_{\text{op}} = n < \infty \) if and only if every module in \( \text{mod-} A \) and every module in \( \text{mod-} A_{\text{op}} \) have left orthogonal dimension at most \( n \). We use this fact in the proof of the following corollary.

**Corollary 3.9.** Suppose that \( A \) is an artin algebra admitting a \( \mathbb{D}^b(\text{mod-}) \) level recollement

\[
\mathbb{D}^b(\text{mod-} B) \xrightarrow{i_*} \mathbb{D}^b(\text{mod-} A) \xrightarrow{j_*} \mathbb{D}^b(\text{mod-} C),
\]

with \( B \) and \( C \) CM-finite artin algebras. If \( B \) and \( C \) are Gorenstein, then so is \( A \).

**Proof.** In view of Theorem 3.6, we have the following recollement of derived categories

\[
\mathbb{D}^-(\text{Mod}(\mathcal{GP} - B)) \xrightarrow{i_*} \mathbb{D}^-(\text{Mod}(\mathcal{H})) \xrightarrow{j_*} \mathbb{D}^-(\text{Mod}(\mathcal{GP} - C)),
\]

where \( \mathcal{H} \) is a subcategory of \( \perp A \) which is defined with respect to \( \mathcal{GP} - B \) and \( \mathcal{GP} - C \), as in the Notation 3.5.

By Lemma 3.7, \( \text{Mod}(\mathcal{GP} - B) \) and \( \text{Mod}(\mathcal{GP} - C) \) both have finite global dimension. So \( \text{Mod}(\mathcal{H}) \) has finite global dimension, due to [K, Corollary 5]. Moreover, since \( \mathcal{GP} - B \) and \( \mathcal{GP} - C \) are of finite representation type, so is \( \mathcal{H} \). Thus \( \mathcal{H} \) is contravariantly finite subcategory of \( \text{mod-} A \). Now, standard techniques show that every finitely generated \( A \)-module has finite \( \mathcal{H} \)-dimension. Therefore, \( \text{mod-} A \) has finite left orthogonal dimension, because \( \mathcal{H} \subseteq \perp A \).

Furthermore, for any artin algebra \( \Lambda \), the duality \( D = \text{Hom}_\Lambda(-, E) \), where \( E \) is an injective envelope of \( \Lambda/\text{rad} \Lambda \), induces the duality \( RD : \mathbb{D}^b(\text{mod-} \Lambda) \to \mathbb{D}^b(\text{mod-} \Lambda_{\text{op}}) \) of bounded derived categories. Besides, the functor \( \text{Hom}_B(-, B) \), resp. \( \text{Hom}_C(-, C) \), induces an equivalence \( \text{Hom}_B(-, B) : \mathcal{GP} - B \to \mathcal{GP} - B_{\text{op}} \), resp. \( \text{Hom}_C(-, C) : \mathcal{GP} - C \to \mathcal{GP} - C_{\text{op}} \) and hence \( B_{\text{op}} \), resp. \( C_{\text{op}} \), is a CM-finite Gorenstein algebra. Therefore, there exists the following recollement

\[
\mathbb{D}^b(\text{mod-} B_{\text{op}}) \xrightarrow{i_*} \mathbb{D}^b(\text{mod-} A_{\text{op}}) \xrightarrow{j_*} \mathbb{D}^b(\text{mod-} C_{\text{op}}).
\]

So, in the same way as above, one can prove that every module in \( \text{mod-} A_{\text{op}} \) also has finite left orthogonal dimension. Now, Remark 3.8 yields the assertion. \( \square \)

As a consequence of Theorem 3.6, we have the following result that can be considered as a generalization of results of Pan and Xi [P, Theorem 3.11] and [HX2, Corollary 3.13].
Corollary 3.10. Let $A$ be an artin algebra admitting a recollement of the following form

$$\text{D}^b(\text{mod-}B) \xrightarrow{i^*} \text{D}^b(\text{mod-}A) \xrightarrow{j_*} \text{D}^b(\text{mod-}C).$$

If $B$ and $C$ are CM-finite Gorenstein algebras, then there is a recollement

$$\text{D}^b(\text{mod-End}_B(G)) \xrightarrow{i^*} \text{D}^b(\text{mod-End}_A(H)) \xrightarrow{j_*} \text{D}^b(\text{mod-End}_C(G')),$$

where $\text{End}_B(G)$, resp. $\text{End}_C(G')$, is the Cohen-Macaulay Auslander algebra of $B$, resp. $C$ and $H = i_*(G) \oplus j_!(G')$.

Proof. By Theorem 3.6, there is the following recollement

$$\text{D}^-(\text{mod-End}_B(G)) \xrightarrow{i^*} \text{D}^-(\text{mod-End}_A(H)) \xrightarrow{j_*} \text{D}^-(\text{mod-End}_C(G')).$$

By Theorem 3.15 of [GZ], $\text{End}_B(G)$ and $\text{End}_C(G')$ are artin algebras of finite global dimension. So Corollary 5 of [K] implies that $\text{End}_A(H)$ has finite global dimension. Now, [AKL3, Lemma 4.3] and [AKL2, Corollary 2.7] imply the desired recollement. \qed

Corollary 3.11. Let $A$ and $B$ be CM-finite Gorenstein artin algebras. If $A$ and $B$ are derived equivalent, then their Cohen-Macaulay Auslander algebras are also derived equivalent.

Proof. Since $A$ and $B$ are derived equivalent, the right hand side of the recollement in Corollary 3.10, vanishes. So, by Corollary 3.10, we get an equivalence

$$\text{D}^b(\text{mod-End}_B(G)) \xrightarrow{i^*} \text{D}^b(\text{mod-End}_A(H)),$$

where $\text{End}_B(G)$ is the Cohen-Macaulay Auslander algebra of $B$ and $H = i_*(G)$. Now, [Ka, Proposition 5.4] implies that $\text{add-}H = \mathcal{G}_p$ and hence the proof is complete. \qed

4. Recollements of Gorenstein derived categories

Let $k$ be an algebraically closed filed and $R = k[x]/(x^n)$ with $n > 5$. It is proved in [L, Proposition 3.6] that for any ring $R$ there is the following recollement

$$\text{D}^b(\text{mod-}R) \xrightarrow{i^*} \text{D}^b(\text{mod-T}_2(R)) \xrightarrow{j_*} \text{D}^b(\text{mod-}R),$$

where $T_2(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ is the $T_2$-extension of $R$. By [B, Example 4.17], we know that $T_2(R)$ if of infinite CM-type, while it is known that $R$ is of finite representation type. That is, being of finite representation type is not an invariant of recollements.

In this section, we show that being CM-finite and also being Gorenstein are invariants of recollements of Gorenstein derived categories of virtually Gorenstein algebras.

Let us recall briefly the definition of Gorenstein derived categories. Let $A$ be an artin algebra. A complex $X$ of finitely generated $A$-modules is called $\mathcal{G}_p$-proper if for every $G \in \mathcal{G}_p$, the induced complex $\text{Hom}_A(G, X)$ is exact. We denote by $\mathcal{K}^b_{\mathcal{G}_p}(\text{mod-}A)$ the class of all $\mathcal{G}_p$-proper complexes in $\mathcal{K}^b(\text{mod-}A)$. The bounded Gorenstein derived category of $\mathcal{G}_p$, denoted by $\text{D}^b_{\mathcal{G}_p}(\text{mod-}A)$, is the quotient category $\mathcal{K}^b(\text{mod-}A)/\mathcal{K}^b_{\mathcal{G}_p}(\text{mod-}A)$. The Gorenstein derived category was first introduced and studied by Gao and Zhang [GZ]. Recently this category has been studied more in [AHV].
The following lemma will be used in the proof of the next theorem. Its proof is exactly similar to the proof of Lemma 2.1 of [W]. Hence we skip the proof.

**Lemma 4.1.** Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ be abelian categories with enough projectives. Assume that there exist a recollement as follows

$$
\text{D}^b(\mathcal{A}') \xrightarrow{i^*} \text{D}^b(\mathcal{A}) \xleftarrow{j_!} \text{D}^b(\mathcal{A}'').
$$

Then $\mathcal{A}$ has finite global dimension if and only if $\mathcal{A}'$ and $\mathcal{A}''$ have finite global dimension.

**Remark 4.2.** Let $\mathbb{K}^-(\mathcal{Gp}-\mathcal{A})$ denote the full subcategory of $\mathbb{K}^-(\mathcal{Gp}-\mathcal{A})$ consisting of all complexes $X$ such that there is an integer $n = nX$ such that $H^i(\text{Hom}_B(G, X)) = 0$, for all $i \leq n$ and every $G \in \mathcal{Gp}-\mathcal{A}$. Let $X$ be a complex in $\mathbb{K}^-(\mathcal{Gp}-\mathcal{A})$. An analogous argument as in [Ric, Proposition 6.2] shows that $X$ lies in $\mathbb{K}^b(\mathcal{Gp}-\mathcal{A})$ if and only if for every $Y \in \mathbb{K}^-(\mathcal{Gp}-\mathcal{A})$, $\text{Hom}_{\mathbb{K}^-(\mathcal{Gp}-\mathcal{A})}(X, Y[n]) = 0$ for large $n$.

**Theorem 4.3.** Let $A$, $B$ and $C$ be virtually Gorenstein algebras. Let the bounded Gorenstein derived category $\mathbb{D}_{\mathcal{Gp}}(\text{mod}-A)$ admits the following recollement

$$
\mathbb{D}_{\mathcal{Gp}}(\text{mod-}B) \xrightarrow{i^*} \mathbb{D}_{\mathcal{Gp}}(\text{mod-}A) \xleftarrow{j_!} \mathbb{D}_{\mathcal{Gp}}(\text{mod-}C).
$$

Then

(i) $A$ is CM-finite if and only if $B$ and $C$ are so.

(ii) $A$ is Gorenstein if and only if $B$ and $C$ are so.

**Proof.** (i) Since $A$, $B$ and $C$ are virtually Gorenstein algebras, $\mathcal{Gp}-\mathcal{A}$, $\mathcal{Gp}-\mathcal{B}$ and $\mathcal{Gp}-\mathcal{C}$ are contravariantly finite subcategories of $\text{mod}-\mathcal{A}$, $\text{mod}-\mathcal{B}$ and $\text{mod}-\mathcal{C}$, respectively. So the given recollement can be stated in the following form

$$
\mathbb{K}^-(\mathcal{Gp}-\mathcal{B}) \xrightarrow{i^*} \mathbb{K}^-(\mathcal{Gp}-\mathcal{A}) \xleftarrow{j_!} \mathbb{K}^-(\mathcal{Gp}-\mathcal{C}).
$$

Let $A$ be CM-finite. So there is an $A$-module $G$ such that $\text{add-}G = \mathcal{Gp}-\mathcal{A}$. Observe that the functor $i_*$, resp. $j_!$, sends any finitely generated Gorenstein projective $B$-module $G'$, resp. $C$-module $G''$, to a bounded complex over $\mathcal{Gp}-\mathcal{A}$. Indeed, for any $Y \in \mathbb{K}^-(\mathcal{Gp}-\mathcal{A})$, $\text{Hom}_{\mathbb{K}^-(\mathcal{Gp}-\mathcal{A})}(i_*(G'), Y[n])$, resp. $\text{Hom}_{\mathbb{K}^-(\mathcal{Gp}-\mathcal{A})}(j_!(G''), Y[n])$, is isomorphic to $\text{Hom}_{\mathcal{Gp}-\mathcal{B}}(G', i_!Y[n])$, resp. $\text{Hom}_{\mathcal{Gp}-\mathcal{C}}(G'', j_!Y[n])$, and hence vanishes for large $n$. So Remark 4.2 implies that $i_*(G')$ and $j_!(G'')$ belong to $\mathbb{K}^b(\mathcal{Gp}-\mathcal{A})$.

Since $\text{add-}G$ generates $\mathbb{K}^b(\mathcal{Gp}-\mathcal{A})$, the natural isomorphism $i^*i_* \cong \text{id}_{\mathbb{K}^-(\mathcal{Gp}-\mathcal{B})}$ implies that $\text{add-}\mathcal{Gp}-\mathcal{B} = \text{add}-\bigoplus_{n \in \mathbb{Z}} i_*(G)^n$. Note that $i^*$ has a right adjoint and so it can be easily checked, by using the criterion above, that $i^*(G)$ is in $\mathbb{K}^b(\mathcal{Gp}-\mathcal{B})$. So the direct sum $\bigoplus_{n \in \mathbb{Z}} i_*(G)^n$ has only finitely many non-zero terms and hence $B$ is CM-finite. Similarly, one can prove that $C$ is CM-finite.

For the converse, let $G'$, resp. $G''$, be a $B$-module, resp. $C$-module, in which $\text{add-}G' = \mathcal{Gp}-\mathcal{B}$, resp. $\text{add-}G'' = \mathcal{Gp}-\mathcal{C}$. As we saw above $i_*(G')$ and $j_!(G'')$ lie in $\mathbb{K}^b(\mathcal{Gp}-\mathcal{A})$. Moreover, the standard triangles, that are induced by the recollement, together with the technique which is used in the proof of Theorem 3.6 implies that $i_*(G') \oplus j_!(G'')$ generates the bounded homotopy category $\mathbb{K}^b(\mathcal{Gp}-\mathcal{A})$. Therefore, every complex $X \in \mathbb{K}^b(\mathcal{Gp}-\mathcal{A})$ is homotopy isomorphic to a bounded complex $Y$ with all terms in $\text{add-}(G' \oplus G'')$, where $\bar{G'} = \bigoplus_{n \in \mathbb{Z}} i_*(G')^n$ and...
\(\vec{G}'' = \oplus_{n \in \mathbb{Z}} j_t(G'')^n\). Now, if we pick a finitely generated Gorenstein projective \(A\)-module \(G\) and consider it as a stalk complex with \(G\) in degree zero, then \(G\) is homotopy equivalent to a bounded complex \(Y\) with all terms in \(\text{add}(-G' \oplus \vec{G}'')\). This, in turn, implies that \(G\) should belong to \(\text{add}(-G' \oplus \vec{G}'')\) and hence \(A\) is CM-finite.

(ii) In view of [ABHV, Proposition 5.2], we have the following recollement

\[
\mathbb{D}^b(\text{mod}(Gp-B)) \longrightarrow \mathbb{D}^b(\text{mod}(Gp-A)) \longrightarrow \mathbb{D}^b(\text{mod}(Gp-C)).
\]

By Lemma 4.1, \(\text{mod}(Gp-A)\) has finite global dimension if and only if \(\text{mod}(Gp-C)\) and \(\text{mod}(Gp-B)\) have so. This, in turn, implies that \(A\) is Gorenstein if and only if \(B\) and \(C\) are so, see [ABHV, Lemma 5.3].

\[\square\]

Acknowledgments

The authors would like to thank the Center of Excellence for Mathematics (University of Isfahan). Part of this work is carried out in the IPM-Isfahan Branch, Isfahan, Iran.

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Department of Mathematics, University of Isfahan, P.O.Box: 81746-73441, Isfahan, Iran and School of Mathematics, Institute for Research in Fundamental Science (IPM), P.O.Box: 19395-5746, Tehran, Iran
E-mail address: asadollahi@ipm.ir
E-mail address: vahed@ipm.ir

School of Mathematics, Institute for Research in Fundamental Science (IPM), P.O.Box: 19395-5746, Tehran, Iran
E-mail address: hafezi@ipm.ir