DOUBLE TOTAL RAMIFICATIONS FOR CURVES OF GENUS 2

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Abstract. Inside the moduli space of curves of genus 2 with 2 marked points we consider the loci of curves admitting a map to \( \mathbb{P}^1 \) of degree \( d \) totally ramified over the two marked points, for \( d \geq 2 \). Such loci have codimension two. We compute the class of their compactifications in the moduli space of stable curves. Several results will be deduced from this computation.

A classical way of producing subvarieties of moduli spaces of curves is by means of Hurwitz theory. Let \( \mathcal{M}_g \) be the moduli space of smooth curves of genus \( g \). If \( 1 < d < (g + 2)/2 \), the locus of curves of genus \( g \) admitting a map to \( \mathbb{P}^1 \) of degree \( d \) has codimension \( g - 2d + 2 \) inside \( \mathcal{M}_g \) ([AC81]).

Similarly one can produce subvarieties of the moduli spaces \( \mathcal{M}_{g,n} \) of smooth \( n \)-pointed curves of genus \( g \) by imposing exceptional ramifications at the marked points. An important example is the double ramification locus, that is, the locus of curves admitting a map to \( \mathbb{P}^1 \) with prescribed ramification profile over two points. More precisely, fix a multi-index \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) of degree 0, that is, \( \sum_i d_i = 0 \), and denote by \( d_+ \) and \( d_- \) the multi-indices composed respectively of the positive \( d_i \) and the negative \( d_i \) in \( d \). For \( g > 0 \) the double ramification locus \( DR_g(d) \) in \( \mathcal{M}_{g,n} \) is the locus of curves admitting a map to \( \mathbb{P}^1 \) with marked ramification profiles \( d_+ \) and \( -d_- \) over two points \( 0, \infty \in \mathbb{P}^1 \) and simple ramifications elsewhere. Such a locus has codimension \( g \) in \( \mathcal{M}_{g,n} \).

Another possible description is the following. Let \( \mathcal{J}_{g,n} \to \mathcal{M}_{g,n} \) be the universal Jacobian variety over \( \mathcal{M}_{g,n} \) and consider the section \( \varphi_d : \mathcal{M}_{g,n} \to \mathcal{J}_{g,n} \) defined by \( \varphi_d([C, p_1, \ldots, p_n]) = O_C(\sum_i d_i p_i) \). The double ramification locus \( DR_g(d) \) in \( \mathcal{M}_{g,n} \) can be viewed as the pull-back of the zero section \( Z_g \) in \( \mathcal{J}_{g,n} \) via \( \varphi_d \).

Double ramification loci have played a crucial role in the study of topological recursive relations in [Ion02] and in the proof of the \( r \)-spin Witten conjecture in [FSZ10]. The problem of computing the classes of closures of double ramification loci is generally known as the Eliashberg’s problem and its interest is heightened by applications in symplectic field theory.

The universal Jacobian variety over \( \mathcal{M}_{g,n} \) can be extended over the moduli space \( \mathcal{M}_{g,n}^{ct} \) of stable curves of compact type. Given a multi-index \( d \in \mathbb{Z}^n \) of degree 0, Hain computed the class in \( H^{2g}(\mathcal{M}_{g,n}^{ct}, \mathbb{Q}) \) of the pull-back of the zero section \( Z_g \) via the map \( \varphi_d : \mathcal{M}_{g,n}^{ct} \to \mathcal{J}_{g,n} \) using normal functions ([Hai13]).

Moreover, Hain’s formula holds in the Chow group \( A^9(\mathcal{M}_{g,n}^{ct}) \) (see for instance [GZ12]) and Grushevsky and Zakharov have extended it over the open subset of \( \mathcal{M}_{g,n}^{ct} \) parametrizing curves with at most one non-disconnecting node ([GZ13]).

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Note however that the pull-back of the zero section of the universal Jacobian variety \( J_{g,n} \to M_{g,n}^{ct} \) is not irreducible and contains other components besides the compactification of the double ramification locus.

This phenomenon has been investigated in genus 1 in [CC13]. For \( d \in \mathbb{Z}^n \) such that all \( d_i \) are non-zero, one has
\[
[\varphi^*_d \mathcal{Z}_2] = [\overline{\text{DR}_1}(d)] + \delta_{0,\{1,\ldots,n\}} \in \text{Pic}_Q(\overline{M}_{1,n})
\]
where \( \delta_{0,\{1,\ldots,n\}} \) is the class of the divisor of curves with a rational tail containing all marked points.

The study of the compactification of Hurwitz loci in the moduli space of stable pointed curves \( \overline{M}_{g,n} \) can be carried out using the theory of admissible covers ([HM82]). For instance, in \( \overline{M}_g \) explicit expressions are known for the classes of the closures of Hurwitz divisors ([HM82]), and Hurwitz loci of codimension two ([Tar13]).

In this paper we study the compactification of double ramification loci by means of admissible covers. After the study of the case \( g = 1 \) in [CC13], we carry out the next non-trivial case, that is, the case \( g = n = 2 \).

Let \( \text{DR}_2(d) \) be the locus of pointed curves \( [C, p_1, p_2] \) in \( M_{2,2} \) admitting a map to \( \mathbb{P}^1 \) of degree \( d \) totally ramified at \( p_1 \) and at \( p_2 \). In other words,
\[
\text{DR}_2(d) = \{ [C, p_1, p_2] \in M_{2,2} : dp_1 \equiv dp_2 \in \text{Pic}^d(C) \}.
\]
We obtain an explicit formula for the class of its compactification in \( M_{2,2} \).

**Theorem 1.** For \( d \geq 2 \) the class of \( \overline{\text{DR}}_2(d) \) in \( A^2(\overline{M}_{2,2}) \) is
\[
[\overline{\text{DR}}_2(d)] = (d^2 - 1) \left[ \frac{d^2}{2} \psi_1 \psi_2 + \frac{2 - d^2}{4} (\psi_1^2 + \psi_2^2) \right. \\
+ (\psi_1 + \psi_2) \left( \frac{3d^2 + 2}{20} \delta_{1,1} + \frac{d^2 - 6}{10} \left( \delta_{1,2} + \frac{1}{12} \delta_0 \right) \right) \right].
\]
We use the same notation for divisor classes and codimension-two classes of \( \overline{M}_{2,2} \) as in [Get98]. We recall the definitions in §2. As a first consequence of the explicit expression of the class of \( \overline{\text{DR}}_2(d) \), we have the following result.

**Corollary 1.** The locus \( \overline{\text{DR}}_2(d) \) is not a complete intersection in \( \overline{M}_{2,2} \).

This last result highlights the difference with the pull-back of the zero section. Indeed, one has
\[
[\varphi^*_{(d,-d)} \mathcal{Z}_2] = \frac{1}{2} \left[ \varphi^*_{(d,-d)} \mathcal{T} \right]^2 \in A^2(\mathcal{M}_{2,2}^{ct})
\]
where \( \mathcal{T} \) is the universal symmetric theta divisor trivialized along the zero section (see for instance [GZ12]), while from Corollary 1 the class of \( \overline{\text{DR}}_2(d) \) cannot be expressed as a product of two effective divisor classes.

As we have remarked, the restriction of \( \overline{\text{DR}}_2(d) \) to the locus of curves of compact type is one of the components of the pull-back of the zero section \( \mathcal{Z}_2 \) via \( \varphi_{(d,-d)} \). Using the computation of the class \( \varphi^*_{(d,-d)} \mathcal{T} \) from [Hai13] and [GZ12], we deduce the following.
Corollary 2. We have the following equality in $A^2(M_{2,2}^t)$

$$\left[\omega_{(d, -d)}^*\mathbb{Z}_2\right] = [\overline{DR}_2(d)] + \delta_{22} + (2d^2 - 1)\delta_{11} + \left(d^2 - \frac{6}{5}\right)\delta_{11|12}. \quad (1)$$

We recall the definitions of the classes $\delta_{22}, \delta_{11|1}, \delta_{11|12}$ in §3. We remark that, as in the $g = 1$ case, the difference of the pull-back of the zero section and the compactification of the double ramification locus in $\overline{M}_{2,2}$ is supported on loci of curves with a rational component containing the two marked points.

Note that the class appearing on the left-hand side of (1) has one more interpretation: by the results of [CMW12] and [MW13], the class of the pull-back of the zero section coincides with the push-forward in $A^g(M_{g,n})$ of the virtual fundamental class of the space of relative stable maps to an unparameterized rational curve. See also [Pan13] for a discussion on different geometric ways to extend such classes to $M_{g,n}$.

By varying $d$, Theorem 1 gives us infinitely many rays in the pseudo-effective cone of codimension-two classes in $\overline{M}_{2,2}$. A natural problem is to study the cone spanned by the classes of $[\overline{DR}_2(d)]$ for $d \geq 2$. Let $[\overline{DR}_2(\infty)]$ be the class defined as follows:

$$[\overline{DR}_2(\infty)] := \lim_{d \to \infty} \frac{[\overline{DR}_2(d)]}{d^4}$$

$$= \frac{1}{2} \psi_1\psi_2 - \frac{1}{4} (\psi_1^2 + \psi_2^2) + (\psi_1 + \psi_2) \left(-\frac{3}{20} \delta_{1,1} + \frac{1}{10} \delta_{1,2} + \frac{1}{120} \delta_0\right).$$

Corollary 3. The classes of the loci $[\overline{DR}_2(d)]$ for $d \geq 2$ lie in a two-dimensional cone spanned by the classes $[\overline{DR}_2(2)]$ and $[\overline{DR}_2(\infty)]$.

In [CC13] Chen and Coskun show that infinitely many double ramification classes are extremal and rigid in the pseudo-effective cone of divisors of $\overline{M}_{1,n}$, for $n \geq 3$, hence proving that $\overline{M}_{1,n}$ is not a Mori dream space for $n \geq 3$. In §4 we show that the classes of $[\overline{DR}_2(d)]$ are not extremal in the pseudo-effective cone of codimension-two classes of $\overline{M}_{2,2}$. In §1 we obtain stronger results for the push-forward of the classes of $[\overline{DR}_2(d)]$ via the map $\pi_i: \overline{M}_{2,2} \to \overline{M}_{2,1}$ forgetting the point $i$: the divisor class $(\pi_i)_* [\overline{DR}_2(d)]$ is extremal and rigid for $d = 2$, and big and moving for $d \geq 3$.

In this paper all classes are stack fundamental classes and all cohomology and Chow groups are taken with rational coefficients.

**Contents**

1. Forgetting a point \hspace{1cm} 4
2. Counting admissible covers \hspace{1cm} 6
3. Comparing with the pull-back of the zero section \hspace{1cm} 12
4. In the cone of effective classes in codimension two \hspace{1cm} 13
Appendix A. Non-polynomiality for $n \geq 3$ \hspace{1cm} 15
Acknowledgments \hspace{1cm} 15
References \hspace{1cm} 15
1. Forgetting a point

Let \( \pi_i : \overline{M}_{2,2} \to \overline{M}_{2,1} \) be the map forgetting the point \( i \), for \( i = 1, 2 \). In this section we study the closures in \( \overline{M}_{2,1} \) of the divisors

\[
(\pi_i)_* (\mathcal{DR}_2(d)) = \{ [C, p] \in \mathcal{M}_{2,1} : \exists x \in C \setminus \{ p \} \text{ such that } dp \equiv dx \}
\]

for \( d \geq 2 \). Note that we have

\[
(\pi_1)_* (\mathcal{DR}_2(d)) = (\pi_2)_* (\mathcal{DR}_2(d))
\]

since the loci \( \mathcal{DR}_2(d) \) are symmetric in the two marked points. The classes of these divisors will be one of the ingredients in the proof of Theorem 1 in the next section.

We will need the following result about the enumerative geometry of pencils on the general curve, in the spirit of [HM82, Theorem A,B], [Har84, §2], [Far09, §2.1].

**Proposition 1.** Let \( (C, p) \) be a general pointed curve of genus \( g \geq 1 \). The number of pairs \( (L, x) \in W_{g+1}^1(C) \times C \) satisfying the conditions

\[
h^0(L \otimes \mathcal{O}(-(g+1)x)) \geq 1 \quad \text{and} \quad h^0(L \otimes \mathcal{O}(-2p)) \geq 1
\]

is

\[
m(g) := (g + 2)g^2.
\]

**Proof.** By [HM82, Theorem B], a smooth Brill-Noether general curve \( \tilde{C} \) of genus \( g + 1 \) has

\[
g(g+1)(g+2)
\]

pairs \( (L, x) \in W_{g+1}^1(\tilde{C}) \times \tilde{C} \) such that \( h^0(L \otimes \mathcal{O}(-(g+1)x)) \geq 1 \).

Let \( (C, p) \) be a general pointed curve of genus \( g \geq 1 \) and let us consider the curve obtained from \( C \) by attaching an elliptic tail \((E, p)\) at the point \( p \in C \). Since \( p \) is general in \( C \), the curve \( C \cup_p E \) is a Brill-Noether general curve. It follows that the curve \( C \cup_p E \) admits \( g(g+1)(g+2) \) admissible covers of degree \( g+1 \) totally ramified at a certain point \( x \in C \cup_p E \).

We distinguish two cases. If \( x \in E \), then the admissible cover is totally ramified at \( x \) and the restriction to \( x \) is uniquely determined by \( |\mathcal{O}((g+1)p)| \), and \( p-x \) is a non-trivial \((g+1)\)-torsion point in \( \text{Pic}^0(E) \). There are \((g+1)^2-1\) choices for \( x \) and each determines a unique admissible cover. If \( x \in C \), then the restriction of the admissible cover to \( C \) is totally ramified at \( x \) and has a simple ramification at \( p \). That is, it corresponds to a pair \( (L, x) \) as in the statement. The restriction to \( E \) is uniquely determined by \( |\mathcal{O}(2p)| \). It follows that

\[
g(g+1)(g+2) = ((g+1)^2 - 1) + m(g)
\]

hence the statement. \( \square \)

We are now ready to study the classes of the divisors \((\pi_i)_* (\mathcal{DR}_2(d))\). We recall that the classes \( \psi, \delta_0, \delta_1 \) form a basis for \( \text{Pic}(\overline{M}_{2,1}) \), and we have the following equality \( \lambda = \frac{1}{10}\delta_0 + \frac{6}{5}\delta_1 \). Let \( \mathcal{W} \) be the Weierstrass divisor in \( \mathcal{M}_{2,1} \)

\[
\mathcal{W} = \{ [C, p] \in \mathcal{M}_{2,1} : p \in C \text{ is a Weierstrass point} \}.
\]

The class of the closure of \( \mathcal{W} \) in \( \overline{M}_{2,1} \) is

\[
[\overline{\mathcal{W}}] = 3\psi - \frac{1}{10}\delta_0 - \frac{6}{5}\delta_1
\]

(see [EH87, Theorem 2.2]). If \( d = 2 \), it is easy to see that the class of the locus \((\pi_i)_* (\mathcal{DR}_2(2))\) is \( 5 \cdot [\overline{\mathcal{W}}] \). Indeed for every smooth pointed curve \( (C, p) \) of genus 2
with \( p \in C \) a Weierstrass point, there are 5 points in \( \mathcal{D} \mathcal{R}_2(2) \) lying over \( (C, p) \), and they correspond to the other 5 Weierstrass points of \( C \). Using admissible covers, also stable pointed curves have 5 points lying over them.

When \( d = 3 \), the class of the locus \( (\pi_i)_* (\mathcal{D} \mathcal{R}_2(3)) \) has been computed by Farkas in [Far09, Proposition 4.1]. The computation in the case \( d \geq 3 \) is a straightforward generalization of the case \( d = 3 \). In the following we work out the details.

In [Dia86] Diaz studied the closure in \( \mathcal{M}_g \) of the following divisor of exceptional Weierstrass points
\[
\mathcal{D}_{g-1} = \{ \{C\} \in \mathcal{M}_g : \exists x \in C \text{ such that } h^0(O_C(g-1)x) \geq 2 \}
\]
for \( g \geq 3 \), and computed its class in \( \text{Pic} (\mathcal{M}_g) \):
\[
\overline{\mathcal{D}}_{g-1} = \frac{g^2(g-1)(3g-1)}{2} \lambda - \frac{(g-1)^2g(g+1)}{6} \delta_0 - \sum_{i \geq 1} \frac{i(g-i)g(g^2 + g - 4)}{2} \delta_i.
\]

For \( d \geq 2 \) let \( \chi_d : \mathcal{M}_{2,1} \to \mathcal{M}_{d+1} \) be the map obtained by attaching a fixed general pointed curve of genus \( d-1 \) at the marked point.

**Proposition 2.** For \( d \geq 2 \) we have
\[
\chi_d^* (\overline{\mathcal{D}}_{d-1}) \equiv (\pi_i)_* (\mathcal{D} \mathcal{R}_2(d)) + (d+1)(d-1)^2 \cdot \overline{W} \in \text{Pic} (\mathcal{M}_{2,1}).
\]

**Proof.** The proof is identical to the \( d = 3 \) case shown in [Far09]. Let \( [C_2, p] \) be a point in \( \mathcal{M}_{2,1} \) and let \([C_{d-1}, p]\) be a general pointed curve of genus \( d-1 \). Suppose that \([C_2 \cup_p C_{d-1}]\) is contained in \( \chi_d^* (\overline{\mathcal{D}}_{d-1}) \). Then there exists an admissible cover of \( C_2 \cup_p C_{d-1} \) of degree \( d \) totally ramified at a smooth point \( x \in C_2 \cup_p C_{d-1} \). There are two cases. If \( x \in C_{d-1} \), then the admissible cover has a simple ramification at the point \( p \), the restriction on \( C_2 \) is uniquely determined by \([O(2p)]\), \( p \in C_2 \) is a Weierstrass point, and by Proposition 1 there are \( m(d-1) \) choices for the restriction of the cover on \( C_{d-1} \). If \( x \in C_2 \), then the admissible cover is totally ramified at \( x \) and \( p \), hence the restriction on \( C_2 \) is uniquely determined, and the restriction on \( C_{d-1} \) is uniquely determined by \([O(dp)]\). \qed

**Corollary 4.** For \( d \geq 2 \) we have
\[
(\pi_i)_* (\mathcal{D} \mathcal{R}_2(d)) = (d^2 - 1) \left( (d^2 + 1) \psi - \frac{d^2 + 6}{5} \left( \frac{1}{12} \delta_0 + \delta_1 \right) \right).
\]

**Proof.** The statement follows from Proposition 2 using the following well known formulae: \( \chi^*(\lambda) = \lambda \), \( \chi^*(\delta_0) = \delta_0 \), \( \chi^*(\delta_1) = \delta_1 \), and \( \chi^*(\delta_2) = -\psi \). \qed

In [Rul06] Rulla shows that the pseudo-effective cone of divisors classes in \( \mathcal{M}_{2,1} \) is generated by the classes \([\overline{W}]\), \( \delta_0 \), \( \delta_1 \); the nef cone is generated by the classes \( \psi \), \( \lambda = \frac{1}{10} \delta_0 + \frac{1}{5} \delta_1 \); and finally the moving cone is generated by the nef cone together with the classes \( D = 30 ( [\overline{W}] + \psi ) \) and \( E = 20 [\overline{W}] + 6 \delta_0 + 6 \delta_1 \).

**Corollary 5.** The classes \( (\pi_i)_* (\mathcal{D} \mathcal{R}_2(d)) \) lie in the two-dimensional cone spanned by the classes \( (\pi_i)_* (\mathcal{D} \mathcal{R}_2(2)) = 5 [\overline{W}] \) and \( (\pi_i)_* (\mathcal{D} \mathcal{R}_2(\infty)) = \frac{1}{6} [\overline{W}] + \frac{1}{3} \psi \) inside the pseudo-effective cone of divisors classes in \( \mathcal{M}_{2,1} \). In particular, the class \( (\pi_i)_* (\mathcal{D} \mathcal{R}_2(d)) \) is extremal and rigid for \( d = 2 \), and big and moving for \( d > 2 \).
Proof. It is easy to verify the following two equalities

\[(\pi_i)_* [\mathcal{DR}_2(d)] = (d^2 - 1) \left( \frac{d^2 + 6}{6} [W] + \frac{d^2 - 4}{2} \psi \right) \]

\[= (d^2 - 1) \left( \frac{1}{3}(\pi_i)_* [\mathcal{DR}_2(2)] + (d^2 - 4)(\pi_i)_* [\mathcal{DR}_2(\infty)] \right).\]

It follows that the classes \((\pi_i)_* [\mathcal{DR}_2(d)]\) are in the cone spanned by the classes \((\pi_i)_* [\mathcal{DR}_2(2)]\) and \((\pi_i)_* [\mathcal{DR}_2(\infty)]\). Since the class \(\psi\) is big, from the first equality above we have that the classes \((\pi_i)_* [\mathcal{DR}_2(d)]\) are big for \(d \geq 3\). Note that, for \(d = 3\), the class \((\pi_i)_* [\mathcal{DR}_2(3)]\) is a multiple of the class \(D\), and for \(d \geq 3\) the classes \((\pi_i)_* [\mathcal{DR}_2(d)]\) lie in the cone spanned by \(D\) and \(\psi\), hence by Rulla’s result, they are in the boundary of the moving cone. \(\square\)

2. Counting admissible covers

By [FP05] the classes of the loci \(\mathcal{DR}_g(d)\) lie in the tautological group \(R^g(\overline{\mathcal{M}}_{g,n}) \subset A^g(\overline{\mathcal{M}}_{g,n})\). By the results of [Get08] we have that the rational cohomology ring of \(\overline{\mathcal{M}}_{2,2}\) is generated by \(\text{Pic}_0(\overline{\mathcal{M}}_{2,2})\), hence in particular \(RH^2(\overline{\mathcal{M}}_{2,2}) = H^4(\overline{\mathcal{M}}_{2,2}, \mathbb{Q})\).

The Picard group \(\text{Pic}(\overline{\mathcal{M}}_{2,2})\) is generated by the following classes: \(\psi_1, \psi_2\) are the cotangent line classes at the two marked points; \(\delta_0\) is the class of the closure of the divisor \(\Delta_0\) of nodal irreducible curves; \(\delta_2\) is the class of the closure of the divisor \(\Delta_2\) whose general element has a component of genus 2 meeting transversally a component of genus 0 with the two marked points; \(\delta_{1,1}\) is the class of the closure of the divisor \(\Delta_{1,1}\) consisting of curves having two elliptic components meeting transversally, each with a marked point; finally \(\delta_{1,2}\) is the class of the closures of the divisor \(\Delta_{1,2}\) of curves consisting of two elliptic components meeting transversally, with the two markings on just one component.

Getzler shows that the following relations among product of divisor classes

\[\delta_{1,2}(12\delta_{1,1} + 12\delta_{1,2} + \delta_0) = \delta_{1,1}(12\delta_{1,1} + 12\delta_{1,2} + \delta_0) = 0\]

\[\delta_{1,1}(\psi_1 + \psi_2 + \delta_{1,1}) = \psi_1\delta_2 = \psi_2\delta_2 = \delta_{1,1}\delta_2 = 0\]

\[(\psi_1 - \psi_2)(10\psi_1 + 10\psi_2 - 2\delta_{1,1} - 12\delta_{1,2} - \delta_0) = 0\]

hold in \(RH^2(\overline{\mathcal{M}}_{2,2}, \mathbb{Q})\). Moreover, \(R^2(\overline{\mathcal{M}}_{2,2})\) is isomorphic to \(RH^2(\overline{\mathcal{M}}_{2,2})\) via the cycle map (see for instance [AL07]). It follows that we can write

\[\mathcal{DR}_2(d) = A_{\psi_1,\psi_2}\psi_1\psi_2 + A_{\psi^2}\psi_1^2 + A_{\psi_2}\psi_2^2 + A_{\psi_1,\delta_1,1}\psi_1\delta_{1,1} + A_{\psi_2,\delta_1,1}\psi_2\delta_{1,1} + A_{\psi_1,\delta_2,\psi_1}\delta_1,1,2 + A_{\psi_2,\delta_2,\psi_2}\delta_1,1,2 + A_{\psi_1,\delta_0,\psi_1}\delta_0 + A_{\psi_2,\delta_0,\psi_2}\delta_0 \]

\[+ A_{\delta_1,2}\delta_0\delta_{1,2} + A_{\delta_2}\delta_0^2 + A_{\delta_1,2}\delta_0\delta_{1,2} + A_{\delta_0,\delta_2}\delta_0\delta_2 + A_{\delta_0,\delta_1,1}\delta_0\delta_{1,1} + A_{\delta_0,\delta_1,2}\delta_0\delta_{1,2} + A_{\delta_0,\delta_1,2}\delta_0\delta_{1,2}\]

in \(R^2(\overline{\mathcal{M}}_{2,2})\).

2.1. Test surfaces. In the following we are going to intersect both sides of the formula in (3) with 10 test surfaces in \(\overline{\mathcal{M}}_{2,2}\). Each intersection will produce a linear equation in the coefficients \(A\).

(1) Let \(C\) be a general curve of genus 2 and consider the family of curves in \(\overline{\mathcal{M}}_{2,2}\) whose fibers are obtained by varying two points in \(C\).

The base of this family is \(C \times C\). To construct the family, start with \(C \times C \times C\) and blow up the diagonal \(\{(p,p,p) \mid p \in C\}\).
Let \( \pi_i : C \times C \rightarrow C \) be the projection on the \( i \)-th factor, for \( i = 1, 2 \). On the surface \( C \times C \) we have

\[
\begin{align*}
\psi_1 &= \pi_1^*(K_C) + \Delta_{C \times C} \\
\psi_2 &= \pi_2^*(K_C) + \Delta_{C \times C} \\
\delta_2 &= \Delta_{C \times C}
\end{align*}
\]

and all other divisor classes restrict to zero.

The fibers of this family are either smooth or consist of the curve \( C \) attached at a rational tail containing the two marked points. An admissible cover for one of the singular fibers has to have a ramification of order at least 2 at the singular point. By the Riemann-Hurwitz formula, the singular fibers do not admit an admissible cover totally ramified at the marked points. A smooth fiber \((C, p, q)\) has an admissible cover of degree \( d \) totally ramified at \( p \) and \( q \) if and only if \( p - q \) is a non-trivial \( d \)-torsion point in \( \text{Pic}^0(C) \). Let \( \varphi : C \times C \rightarrow \text{Pic}^0(C) \) be the difference map defined as \( \varphi(p, q) = \mathcal{O}_C(p - q) \). The map \( \varphi \) is surjective of degree 2 (see [ACGH85, Chapter VI]). It follows that \( 2(d^2 - 1) \) fibers of this family are in \( \overline{\text{DR}_2(d)} \). Since the admissible cover is uniquely determined, each fiber contributes with multiplicity one. We deduce the following relation

\[
4A_{\psi_1^2 + \psi_2^2} + 6A_{\psi_1\psi_2} - 2A_{\delta_2^2} = 2(d^2 - 1).
\]

(2) Let \( (E_1, p_1, x_1) \) and \( (E_2, p_2, x_2) \) be two elliptic curves. Identify the points \( p_1 \) and \( p_2 \). Consider the family of curves whose fibers are obtained by varying \( x_1 \in E_1 \) and \( x_2 \in E_2 \).

The basis of this family is \( E_1 \times E_2 \). To construct this family, let \( \overline{E_i \times E_i} \) be the blow-up of \( E_i \times E_i \) at \((p_i, p_i)\), let \( \Gamma_i \) be the proper transform of \( p_i \times E_i \), and let \( \sigma_{\Delta_i} \) be the section corresponding to the proper transform of the diagonal \( \Delta_i \subset E_i \times E_i \), for \( i = 1, 2 \). Consider \( \overline{E_1 \times E_1} \times E_2 \) and \( E_1 \times \overline{E_2 \times E_2} \) and finally identify \( \Gamma_1 \times E_2 \times E_1 \times E_2 \). The sections \( \sigma_{\Delta_1} \) and \( \sigma_{\Delta_2} \) give rise to two sections of this family over \( E_1 \times E_2 \).

Let \( \pi_i : E_1 \times E_2 \rightarrow E_i \) be the projection on the \( i \)-th factor for \( i = 1, 2 \). The non-zero restrictions of the divisor classes are

\[
\begin{align*}
\psi_1 &= \pi_1^*[p_1] \\
\psi_2 &= \pi_2^*[p_2] \\
\delta_{1,1} &= -\pi_1^*[p_1] - \pi_2^*[p_2] \\
\delta_{1,2} &= \pi_1^*[p_1] + \pi_2^*[p_2].
\end{align*}
\]

A fiber of this family has an admissible cover of degree \( d \) totally ramified at the two marked points if and only if \( p_i - x_i \) is a non-trivial \( d \)-torsion point in \( \text{Pic}^0(E_i) \) for \( i = 1, 2 \). For each such fiber \((C, x_1, x_2)\), the admissible cover \( C \rightarrow D \) is unique. By the argument of [HM82, Theorem 6], the map from a neighborhood of the point \([C \rightarrow D]\) in the moduli space of admissible covers to the universal deformation space of \((C, x_1, x_2)\) is transverse to \( \delta_{1,1} \). It follows that each fiber in the intersection of this family with \( \overline{\text{DR}_2(d)} \) counts with multiplicity one. We deduce

\[
A_{\psi_1\psi_2} - A_{\psi_1\delta_{1,1}} - A_{\psi_2\delta_{1,1}} + A_{\psi_1\delta_{1,2}} + A_{\psi_2\delta_{1,2}} = (d^2 - 1)^2.
\]

(3) Consider a general four-pointed rational curve and attach at two of the marked points elliptic tails varying in pencils of degree 12.
To construct an elliptic pencil of degree 12, blow up \( \mathbb{P}^2 \) in the nine points of intersection of two general cubics. Let \( Y_1 \to \mathbb{P}^1 \) and \( Y_2 \to \mathbb{P}^1 \) be two such elliptic pencils with zero sections \( \sigma_1 \) and \( \sigma_2 \). Let \( R \) be a rational curve and choose four general sections of \( R \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \). Consider \( Y_1 \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times Y_2 \) and identify \( \sigma_1 \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times \sigma_2 \) with two of the sections of \( R \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \).

The base of the family is \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( \pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the projection on the \( i \)-th factor for \( i = 1, 2 \), and let \( x \) be the class of a point in \( \mathbb{P}^1 \). The divisors classes restrict as follows

\[
\delta_{1,2} = -\pi_1^*(x) - \pi_2^*(x) \\
\delta_0 = 12\pi_1^*(x) + 12\pi_2^*(x).
\]

For every fiber the two marked points are in the same rational component. An admissible cover for a fiber of this family admits an admissible cover totally ramified at the two marked points. We deduce the following relation

\[ 288A_{\delta_0} - 24A_{\delta_0\delta_{1,2}} = 0. \]

(4) We consider a chain of three curves: a central rational curve with two elliptic tails. We fix a first marked point on the rational component, we vary one elliptic tail in a pencil of degree 12 and we consider a moving second marked point on the other elliptic tail.

Let \((E, p)\) be a pointed elliptic curve. Let \( E \times E, \Gamma \), and \( \sigma_\Delta \) as in (2). Let \( Y \to \mathbb{P}^1 \) be an elliptic pencil of degree 12 as in (3) with zero section \( \sigma \). The base of the family is \( E \times \mathbb{P}^1 \). Consider \( E \times E \times \mathbb{P}^1 \), \( E \times Y \), and the trivial family \( R \times E \times \mathbb{P}^1 \) with three general sections \( \tau_1, \tau_2, \tau_3 \) for a rational curve \( R \). Finally identify the section \( \tau_2 \) with \( \Gamma \times \mathbb{P}^1 \) and the section \( \tau_3 \) with \( E \times \sigma \). The sections \( \tau_1 \) and \( \sigma_\Delta \) give rise to two sections of the family over \( E \times \mathbb{P}^1 \). We let \( \tau_1 \) be the section corresponding to the first marked point, and \( \sigma_\Delta \) the one corresponding to the second marked point.

Let \( x \) be the class of a point in \( \mathbb{P}^1 \). The divisors classes restrict as follows

\[
\psi_2 = \pi_1^*[p] \\
\delta_{1,1} = -\pi_1^*[p] \\
\delta_{1,2} = \pi_1^*[p] - \pi_2^*(x) \\
\delta_0 = 12\pi_2^*(x).
\]

Suppose there exists an admissible cover for a fiber of this family totally ramified at the two marked points. Then the restriction of the cover to the central rational component is totally ramified at the marked point and at one of the singular points, plus it has at least a simple ramification at the other singular point, a contradiction. Hence this family has empty intersection with \( \mathcal{R}_2(d) \) and we obtain

\[-A_{\psi_2\delta_{1,2}} + 12A_{\psi_2\delta_0} - 12A_{\delta_0\delta_{1,1}} + 12A_{\delta_0\delta_{1,2}} = 0.\]

(5) Pick a general element in the divisor \( \Delta_{1,1} \) and consider the surface obtained by varying the first marked point on the corresponding elliptic tail and the \( j \)-invariant of that elliptic tail.

The base of the family is the blow-up of \( \mathbb{P}^2 \) in the nine points of intersection of two general cubics. Let \( H \) be the pullback of the hyperplane class in \( \mathbb{P}^2 \), let \( \Sigma \) denote the sum of the classes of the nine exceptional divisors, and let \( E_0 \) be the class of one of the exceptional divisors.
The divisors classes of $\overline{M}_{2,2}$ restrict as follows

\begin{align*}
\psi_1 &= 3H - \Sigma + E_0 \\
\delta_{1,1} &= -3H + \Sigma - E_0 \\
\delta_{1,2} &= E_0 \\
\delta_0 &= 36H - 12\Sigma.
\end{align*}

Since the second marked point is general, the surface is disjoint from the locus $\overline{DR}_2(d)$, hence we deduce the following relation

$$A_{\psi_2^2} + A_{\psi_1\delta_{1,1}} + 12A_{\psi_1\delta_0} - 12A_{\delta_0\delta_{1,1}} + 12A_{\delta_0\delta_{1,2}} = 0.$$  

(6) Choose a general element in the locus of curves with a rational and an elliptic component meeting at two non-disconnecting nodes, with both marked points on the rational components, and consider the surface obtained by varying the two moduli of the two-pointed elliptic component.

The base of this family is the same surface as for the family (5). The restriction of the divisor classes are

\begin{align*}
\delta_{1,2} &= E_0 \\
\delta_0 &= 30H - 10\Sigma - 2E_0.
\end{align*}

Since we can take the two singular points to be general in the rational component, this family is disjoint from $\overline{DR}_2(d)$. We deduce the following relation

$$12A_{\delta_0\delta_{1,2}} - 44A_{\delta_2^2} = 0.$$  

(7) Pick a general element in the intersection of the two divisors $\delta_{1,2}$ and $\delta_2$ and vary the two moduli of the central two-pointed elliptic component. Once again the base of this family is the same surface as for the family (5). Suppose there exists an admissible cover for a fiber of this family, totally ramified at the two marked points. Then the restriction of the cover to the rational component is totally ramified at the two marked points and has at least a simple ramification at the singular point, a contradiction. It follows that the family is disjoint from $\overline{DR}_2(d)$. The divisors classes restrict as follows

\begin{align*}
\delta_2 &= -3H + \Sigma - E_0 \\
\delta_{1,2} &= -3H + \Sigma \\
\delta_0 &= 36H - 12\Sigma
\end{align*}

hence we obtain

$$A_{\delta_2^2} + A_{\delta_{1,2}\delta_2} - 12A_{\delta_0\delta_2} = 0.$$  

(8) Again pick a general element in the intersection of the two divisors $\delta_{1,2}$ and $\delta_2$. This time vary the $j$-invariant of the external elliptic component and vary one of the two nodes in the central elliptic component.

Denote by $E$ the central fixed elliptic component. The base of this family is $\mathbb{P}^1 \times E$. Let $x$ be class of a point in $\mathbb{P}^1$ and let $y$ be the class of the point in $E$ corresponding to the fixed node. The divisors classes are

\begin{align*}
\delta_2 &= -\pi_2^*(y) \\
\delta_{1,2} &= -\pi_1^*(x) \\
\delta_0 &= 12\pi_1^*(x).\end{align*}
As for the family (7), this family has zero intersection with $\overline{DR}_2(d)$, hence we obtain

$$A_{\delta_1,2\delta_2} - 12A_{\delta_0\delta_2} = 0.$$  

(9) Consider a 4-pointed rational curve $(R, p_1, p_2, s, t)$. Identify the point $s$ with the base point of an elliptic pencil and identify the point $t$ with a moving point in $R$.

The base of the family is $R \times \mathbb{P}^1$. To construct the surface, let $R \times R$ be the blow-up of $R \times R$ at the points $(p_1, p_1), (p_2, p_2), (s, s), (t, t)$. Let $E_{p_1}, E_{p_2}, E_s, E_t,$ and $E_\Delta$ be the proper transform respectively of $\{p_1\} \times R, \{p_2\} \times R, \{s\} \times R, \{t\} \times R,$ and $\Delta \subset R \times R$. Let $Y \to \mathbb{P}^1$ be as in the family (2) and $\sigma$ be the zero section. Consider $R \times R \times \mathbb{P}^1$ and $R \times Y$ and identify $E_s \times \mathbb{P}^1$ with $R \times \sigma$ and $E_\Delta \times \mathbb{P}^1$ with $E_t \times \mathbb{P}^1$.

By an argument similar to the one used for the family (7), this family has zero intersection with the locus $\overline{DR}_2(d)$. The divisor classes restrict as follows

$$\begin{align*}
\psi_1 &= \pi_1^*[p_1] \\
\psi_2 &= \pi_1^*[p_2] \\
\delta_{1,2} &= -\pi_2^*(x) \\
\delta_0 &= 12\pi_2^*(x).
\end{align*}$$

We deduce

$$-A_{\psi_1\delta_{1,2}} + 12A_{\psi_1\delta_0} - A_{\psi_2\delta_{1,2}} + 12A_{\psi_2\delta_0} = 0.$$  

(10) Let $(R, q_1, q_2, q_3, q_4, q_5)$ be a 5-pointed rational curve. Attach at $q_1$ a rational curve with two marked points. Finally identify $q_2$ with $q_3$, and $q_4$ with $q_5$. Consider the surface obtained by varying the two moduli of the 5-pointed rational curve.

The base of this surface is $\overline{M}_{0,5}$. For $i, j \in \{1, 2, 3, 4, 5\}$ let $D_{i,j}$ be the class of the divisor in $\overline{M}_{0,5}$ corresponding to singular rational curves with two components, the markings $i, j$ in one component and the other three markings in the other component. Note that $D_{i,j}^2 = -1$, and $D_{i,j} \cdot D_{k,l} = 1$ if $\{i, j\} \cap \{k, l\} = \emptyset$, otherwise $D_{i,j} \cdot D_{k,l} = 0$.

Let $\bar{\psi}_i$ be the cotangent line class in $\overline{M}_{0,5}$ corresponding to the marking $i$. If we fix two markings $j, k$ different from $i$ and if $l, m$ are such that $\{1, 2, 3, 4, 5\} \neq \{i, j, k, l, m\}$, then note that we can write

$$\bar{\psi}_i = D_{j,k} + D_{i,l} + D_{i,m}.$$  

The restrictions of the divisor classes in $\overline{M}_{2,2}$ to this family are

$$\begin{align*}
\delta_2 &= -\bar{\psi}_1 \\
&= -D_{2,3} - D_{1,5} - D_{1,4} \\
\delta_{1,2} &= D_{4,5} + D_{2,3} \\
\delta_0 &= -\bar{\psi}_2 - \bar{\psi}_3 - \bar{\psi}_4 - \bar{\psi}_5 \\
&+ D_{1,2} + D_{1,3} + D_{1,4} + D_{1,5} \\
&+ D_{2,4} + D_{2,5} + D_{3,4} + D_{3,5} \\
&= -2D_{4,5} - D_{2,3} - D_{3,4} - D_{1,2} + D_{1,4}.
\end{align*}$$

As for the family (7), this family is disjoint from the locus $\overline{DR}_2(d)$ and we obtain

$$A_{\delta_2^2} - 2A_{\delta_{1,2}\delta_2} + 4A_{\delta_0\delta_2} = 0.$$
2.2. **Symmetry.** The definition of the locus $\mathcal{DR}_2(d)$ is symmetric in the two marked points. We deduce that, for a choice of basis of $R^2(\overline{M}_{2,2})$ symmetric with respect to the classes $\psi_1$, $\psi_2$, the expression of the class of $\mathcal{DR}_2(d)$ is symmetric in $\psi_1$, $\psi_2$. Hence we have the following three relations

\[
A_{\psi_1\delta_1} = A_{\psi_2\delta_1},
\]

\[
A_{\psi_1\delta_2} = A_{\psi_2\delta_2},
\]

\[
A_{\psi_1\delta_0} = A_{\psi_2\delta_0}.
\]

2.3. **The push-forward to $\overline{M}_{2,1}$.** It is straightforward to compute the pushforward of products of divisor classes via the maps $\pi_i$: when $i = 1$ we have

\[
(\pi_1)_*(\psi_1^2) = \kappa_1 = \psi + \frac{1}{5}\delta_0 + \frac{7}{5}\delta_1
\]

\[
(\pi_1)_*(\psi_1\delta_1) = \delta_1
\]

\[
(\pi_1)_*(\psi_1\delta_0) = 3\delta_0
\]

\[
(\pi_1)_*(\delta_1\delta_2) = \delta_1
\]

and all other products have zero pushforward. It follows that the pushforward of the expression in (3) is

\[
(\pi_1)_*[\mathcal{DR}_2(d)] = \left(3A_{\psi_1\psi_2} + 2A_{\psi_1^2 + \psi_2^2} - A_{\delta_0^2}\right)\psi
\]

\[
+ \left(\frac{1}{5}A_{\psi_1^2 + \psi_2^2} + 3A_{\psi_1\delta_0} + A_{\psi_2\delta_0} + A_{\delta_0\delta_2}\right)\delta_0
\]

\[
+ \left(\frac{7}{5}A_{\psi_1^2 + \psi_2^2} + A_{\psi_1\delta_1} + 2A_{\psi_1\delta_1} + A_{\psi_2\delta_1} + A_{\delta_1\delta_2}\right)\delta_1.
\]

Comparing with Corollary 4, we deduce the following three relations

\[
3A_{\psi_1\psi_2} + 2A_{\psi_1^2 + \psi_2^2} - A_{\delta_0^2} = d^4 - 1
\]

\[
\frac{1}{5}A_{\psi_1^2 + \psi_2^2} + 3A_{\psi_1\delta_0} + A_{\psi_2\delta_0} + A_{\delta_0\delta_2} = \frac{-(d^2 - 1)(d^2 + 6)}{60}
\]

\[
\frac{7}{5}A_{\psi_1^2 + \psi_2^2} + A_{\psi_1\delta_1} + 2A_{\psi_1\delta_1} + A_{\psi_2\delta_1} + A_{\delta_1\delta_2} = \frac{-(d^2 - 1)(d^2 + 6)}{5}.
\]

2.4. **Proof of Theorem 1.** In order to determine the class of $\mathcal{DR}_2(d)$ we need to determine the coefficients $A$ in formula (3). In §2.2 we have found 10 relations using test surfaces. In §2.2 we have 3 relations coming from the symmetry of $\mathcal{DR}_2(d)$ in the two marked points. Finally in §2.3 we have deduced 3 more relations from the study of the push-forward of $\mathcal{DR}_2(d)$ to $\overline{M}_{2,1}$. In total we have a system of 16 linear relations in the coefficients $A$. The associated matrix has rank 14 and solving the system we prove the statement. The linear system is consistent, hence there are two redundant relations which verify our computation. \[\square\]

2.5. **Test.** Although classes of closures of double ramification cycles are not known in general, in [BSSZ12] all intersections in $\overline{M}_{g,n}$ of double ramification cycles with monomials in $\psi$-classes have been computed, when the intersection is zero-dimensional. It is interesting to note that the authors show that for $d = (d_1, \ldots, d_n)$ with all $d_i$ non-zero, the closures of double ramification loci $\mathcal{DR}_g(d)$ by means of admissible covers and the push-forward of the virtual fundamental class of the
space of relative stable maps to an unparametrized \( \mathbb{P}^1 \) have same intersections with monomials in \( \psi \)-classes.

For instance, the authors in [BSSZ12] show the following

\[
[DR_2(d)] \cdot \psi^3 = [DR_2(d)] \cdot \psi_2^3 = \frac{(d^2 - 1)(3d^2 - 7)}{5760}.
\]

Using the explicit expression of the class of \( [DR_2(d)] \) from Theorem 1, we verify the intersections in (4) as a test for our computation:

\[
[DR_2(d)] \cdot \psi^3 = [DR_2(d)] \cdot (\pi_2^* \psi + \delta_2)^3 = [DR_2(d)] \cdot \pi_2^* \psi^3 = (\pi_2)_* [DR_2(d)] \cdot \psi^3 = (d^2 - 1) \left( (d^2 + 1)\psi - \frac{d^2 + 6}{5} \left( \frac{1}{12} \delta_0 + \delta_1 \right) \right) \cdot \psi^3 = \frac{(d^2 - 1)(3d^2 - 7)}{5760}.
\]

Note that in (5) we have used that \( \psi_1 \cdot \delta_2 = 0 \) in \( \overline{M}_{2,2} \), and in (6) we have used the following intersections in \( \overline{M}_{2,1} \):

\[
\psi^4 = \frac{1}{1152}, \quad \psi^3 \delta_0 = \frac{1}{48}, \quad \psi^3 \delta_1 = 0
\]

computed in [Fab88, Chapter 3].

3. Comparing with the pull-back of the zero section

The class in \( A^q(M_{g,n}^{ct}) \) of the pull-back of the zero section \( \overline{Z}_g \) of the universal Jacobian variety \( J_{g,n} \rightarrow M_{g,n}^{ct} \) has been computed in [Hai13] and [GZ12]. The space \( R^2(M_{2,2}^{ct}) \) has dimension 5. We can consider the basis formed by degree-two monomials in the divisor classes \( \psi_1, \psi_2, \delta_{1,1}, \delta_{1,2}, \delta_2 \) modulo the restrictions to \( M_{2,2}^{ct} \) of the relations (2) in \( \overline{M}_{2,2} \) and the following relations which additionally hold in \( M_{2,2}^{ct} \):

\[
\psi_1 \psi_2 = \frac{3}{2} (d_1^2 + d_2^2) - \frac{9}{10} (\psi_1 + \psi_2) \delta_{1,1} - \frac{2}{5} (\psi_1 + \psi_2) \delta_{1,2},
\]

\[
\psi_1^2 = \frac{7}{10} (\psi_1 (\delta_{1,1} + \delta_{1,2}) - \delta_{1,2} \delta_2) - \delta_2^2.
\]

To prove the above relations it is enough to express the product of divisor classes in terms of decorated boundary strata classes as in [Get98]. To have a basis symmetric with respect to the two marked points, we choose the basis given by the following classes:

\( (\psi_1 + \psi_2) \delta_{1,1}, \psi_1 \delta_{1,2}, \psi_2 \delta_{1,2}, \delta_2, \delta_{1,2} \delta_2 \).

Proof of Corollary 2. From [Hai13] and [GZ12] the class of \( \varphi^*_{(d,-d)} \overline{Z}_2 \) in \( A^2(M_{2,2}^{ct}) \) can be expressed as follows

\[
\left[ \varphi^*_{(d,-d)} \overline{Z}_2 \right] = \frac{1}{2} \left( \frac{d^2}{2} \left( (\psi_1 - \delta_2) + (\psi_2 - \delta_2) \right) + d^2 \delta_2 - \frac{d^2}{2} \delta_{1,1} \right)^2.
\]
Using the relations in (2) and (7), we have

\[
[\varphi^*_C(d,-d)\mathcal{Z}_2] = d^4 \left( \frac{1}{4} (\psi_1 + \psi_2) (\delta_{22} - 1) \right) - \frac{7}{10} \delta_{22} \delta_2
\]

and the restriction of the class of \( \mathcal{D} \mathcal{R}_2(d) \) from \( \mathcal{M}_{2,2} \) is

\[
[\mathcal{D} \mathcal{R}_2(d)] = (d^2 - 1) \left( \frac{d^2 - 1}{4} (\psi_1 + \psi_2) (\delta_{12} - \delta_{1,1}) \right) - (d^2 + 1) \delta_2 \left( \delta_2 + \frac{7}{10} \delta_{1,2} \right).
\]

It follows that

\[
[\varphi^*_C(d,-d)\mathcal{Z}_2] - [\mathcal{D} \mathcal{R}_2(d)] = \frac{2d^2 - 1}{4} (\psi_1 + \psi_2) (\delta_{12} - \delta_{1,1}) - \delta_2 \left( \delta_2 + \frac{7}{10} \delta_{1,2} \right).
\]

In [Get98] Getzler describes also a basis for \( R^2(\mathcal{M}_{2,2}) \) made of decorated boundary strata classes and explains the change of basis. In terms of decorated boundary strata classes, using the notation in [Get98] (see below), we have

\[
[\varphi^*_C(d,-d)\mathcal{Z}_2] = d^4 \left( \delta_{22} + \delta_{1,1} - \frac{1}{5} \delta_{1,12} \right)
\]

\[
[\mathcal{D} \mathcal{R}_2(d)] = (d^2 - 1) \left( (d^2 + 1) \delta_{22} + (d^2 - 1) \delta_{1,1} - \frac{d^2 + 6}{5} \delta_{1,12} \right)
\]

in \( R^2(\mathcal{M}_{2,2}) \), hence the statement. \( \square \)

We recall that the class \( \delta_{22} \) is the pushforward of the \( \psi \)-class in \( \mathcal{M}_{2,1} \) via the gluing map \( \mathcal{M}_{2,1} \times \mathcal{M}_{0,1} \rightarrow \Delta_2 \subset \mathcal{M}_{2,2} \); the class \( \delta_{1,1} \) is the class of the closure of the locus of curves with two elliptic tails attached at a rational component containing the two marked points; finally we have \( \delta_{1,12} = \delta_{1,2} \cdot \delta_2 \).

The set \( \{ \delta_{22}, \delta_{1,1}, \delta_{1,12} \} \) extends to a basis of the 5-dimensional space \( R^2(\mathcal{M}_{2,2}^d) \) by means of the classes \( \delta_{1,11}, \delta_{1,12} \) of the two components of the intersection \( \delta_{1,1} \cdot \delta_{1,2} \); for \( i \in \{ 1, 2 \} \), \( \delta_{1,1} \) is the class of the closure of the locus of curves with a central rational component and two elliptic tails, with the point \( i \) on one of the elliptic components and the other point on the central rational component.

4. IN THE CONE OF EFFECTIVE CLASSES IN CODIMENSION TWO

In this section we study the position of the classes of the loci \( \mathcal{D} \mathcal{R}_2(d) \) inside the cone in \( A^2(\mathcal{M}_{2,2}) \) spanned by classes of effective codimension-two loci in \( \mathcal{M}_{2,2} \).

The first result is Corollary 1, that is, the classes of the loci \( \mathcal{D} \mathcal{R}_2(d) \) are outside the cone of classes of complete intersections.

**Proof of Corollary 1.** Suppose that the class of \( \mathcal{D} \mathcal{R}_2(d) \) is \( [D_1] \cdot [D_2] \), where \( D_1, D_2 \) are two effective divisors in \( \mathcal{M}_{2,2} \). For \( i = 1, 2 \), the class \( [D_i] \) can be expressed as

\[
[D_i] = c^{(i)}_{\psi_1} \psi_1 + c^{(i)}_{\psi_2} \psi_2 - c^{(i)}_{\delta_0} \delta_0 - c^{(i)}_{\delta_2} \delta_2 - c^{(i)}_{\delta_{1,1}} \delta_{1,1} - c^{(i)}_{\delta_{1,2}} \delta_{1,2} \in \text{Pic}(\mathcal{M}_{2,2})
\]

where the coefficients \( c^{(i)} \) are non-negative. Using the relations (2), the product \( [D_1] \cdot [D_2] \) can be expressed in terms of the basis chosen in (3). The coefficient of the class \( \psi_1^2 + \psi_2^2 \) is

\[
\frac{1}{2} \left( c^{(1)}_{\psi_1} c^{(2)}_{\psi_1} + c^{(1)}_{\psi_2} c^{(2)}_{\psi_2} \right)
\]

hence non-negative, a contradiction. \( \square \)
An immediate consequence of Theorem 1 is Corollary 3, that is, the classes of the loci $\overline{DR}_2(d)$ lie in the two-dimensional cone spanned by the classes $[\overline{DR}_2(2)]$ and $[\overline{DR}_2(\infty)]$.

**Proof of Corollary 3.** It is easy to verify that

$$[\overline{DR}_2(d)] = (d^2 - 1) \left( \frac{1}{3} \cdot [\overline{DR}_2(2)] + (d^2 - 4) \cdot [\overline{DR}_2(\infty)] \right)$$

hence the statement. $\square$

In §1 we have studied the cone of the classes $(\pi_i)_* (\overline{DR}_2(d))$ in $\overline{M}_{2,1}$. In particular we have seen that the class of the push-forward of $\overline{DR}_2(2)$ lies in an extremal ray of the cone of effective divisor classes of $\overline{M}_{2,1}$. It is natural to ask whether the class $\overline{DR}_2(2)$ lie in an extremal ray of the cone of effective codimension-two classes of $\overline{M}_{2,2}$. In the following we prove that this is not the case.

Let $HC$ be the divisor in $\overline{M}_{2,2}$ of hyperelliptic pairs, that is,

$$HC = \{ [C, p_1, p_2] \in \overline{M}_{2,2} : h^0(p_1 + p_2) \geq 2 \}.$$

The class of the closure of $HC$ in $\overline{M}_{2,2}$ is a special case of the formula in [Log03, Theorem 4.5]

$$[\overline{HC}] = \psi_1 + \psi_2 - 3\delta_2 - \frac{1}{10}\delta_0 - \frac{1}{5}\delta_{1,1} - \frac{6}{5}\delta_{1,2} \in \text{Pic}_Q(\overline{M}_{2,2}).$$

Let $\delta_{01}$ be the class of the closure of the locus of curves obtained by attaching an elliptic tail at a rational curve with two marked points and an irreducible node. For $1 \in \{1, 2\}$ let $\delta_{01i}$ be the class of the locus of curves with a rational and an elliptic component meeting at two non-disconnecting nodes, with the point $i$ on the elliptic component, and the other marked point on the rational component. Finally consider the class $\delta_{11}$ defined at the end of the previous section. We have verified that the following equality holds for every $d \geq 2$:

$$\frac{[\overline{DR}_2(d)]}{d^2 - 1} = \frac{5}{4}\delta_2^2 + 3\frac{d^2 - 4}{40}\delta_{1,2}\delta_2 + \frac{d^2 - 4}{100}\delta_0\delta_2 + \frac{31d^2 - 24}{4800}\delta_0^2 + \frac{7d^2 + 22}{100}\delta_{1,2}$$

$$+ \frac{3d^2 - 2}{40}\delta_{11} + \frac{16d^2 + 11}{50}\delta_{1,1}\delta_{1,2} + \frac{3d^2 - 2}{24}\delta_{01}$$

$$+ \frac{9d^2 - 11}{600}\delta_0\delta_{1,1} + \frac{d^2}{48}\left(\delta_{01} + \delta_{0|2}\right)$$

$$+ \frac{7d^2 + 12}{16}\pi_* \left( [W] \right) \cdot \delta_2 + \frac{5d^2}{16}[\overline{HC}]^2.$$

In particular, for every $d \geq 2$ the class of $\overline{DR}_2(d)$ is expressed as a linear combination with non-negative coefficients of the effective codimension-two classes appearing in the right-hand side of the above formula. This proves the following statement.

**Proposition 3.** The classes $[\overline{DR}_2(2)]$ and $[\overline{DR}_2(\infty)]$ are not extremal in the cone of effective codimension-two classes of $\overline{M}_{2,2}$. 

Appendix A. Non-polynomiality for $n \geq 3$

By varying the multi-index $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, the class in $A^0(M_{g,n})$ of the pull-back of the zero section $\mathbb{Z}_g$ of the universal Jacobian $J_g \to M_{g,n}$ can be viewed as a polynomial in the indices $d_i$ ([Hai13]). The push-forward of the virtual fundamental class of the space of relative stable maps to an unparametrized $\mathbb{P}^1$ coincides with the class of the pull-back of $\mathbb{Z}_g$ in $A^0(M_{g,n})$ ([CMW12]) and similarly its extension to $A^0(\overline{M}_{g,n})$ is conjectured to be a polynomial in $d_i$.

On the other side, the closure of the double ramification loci computed by means of admissible covers are not expected to be polynomials in the indices $d_i$ ([BSSZ12]). In this section we remark this fact in the case $g = 2, n \geq 3$ by the following easy computation.

Let $\underline{d} = (d_1, \ldots, d_n)$ be a multi-index of degree 0 for $n \geq 3$. Consider the following surface in $\overline{M}_{2,n}$. Attach a rational curve containing the marked points $1, \ldots, n-1$ to a moving point on a general curve $C$ of genus 2, and choose the point $n$ to be a moving point in $C$. The family is constructed as family (1) in §2.

Let us study the intersection of this surface with the locus $\overline{DR}_2(\underline{d})$ in $\overline{M}_{2,n}$. If $d_n \neq 0$, the intersection is analogous to the one studied for the family (1) in §2, that is, $2(d_n^2-1)$. On the other hand, the intersection is empty if $d_n = 0$. Since the result of this intersection is not a polynomial in $d_n$, we deduce the non-polynomiality of the class of $\overline{DR}_2(\underline{d})$ in the indices $d_i$.

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