MULTI-OBJECTIVE HERGLOTZ’ VARIATIONAL PRINCIPLE AND
COOPERATIVE HAMILTON-JACOBI SYSTEMS

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ABSTRACT. We study a multi-objective variational problem of Herglotz’ type with cooperative linear coupling. We established the associated Euler-Lagrange equations and the characteristic system for cooperative weakly coupled systems of Hamilton-Jacobi equations. We also established the relation of the value functions of this variational problem with the viscosity solutions of cooperative weakly coupled systems of Hamilton-Jacobi equations. Comparing to the previous work in stochastic frame, this approach affords a pure deterministic explanation of this problem under more general conditions. We also showed this approach is valid for general linearly coupling matrix for short time.

1. INTRODUCTION

This paper is devoted to the study of the weakly coupled systems of Hamilton-Jacobi equations using a variational approach based on a vectorial form of the Herglotz’ principle. For technical reason, we will mainly discuss this problem under a general setting but a model with wide interests. We suppose our Lagrangians has the form

\[ L^i(x, v) - \sum_{j=1}^{n} a^i_{j} u_j, \quad i = 1, \ldots, n, \]

(1.1)

where \( A = (a^i_{j}) \) is an \( n \times n \) real matrix and each \( L^i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a Tonelli Lagrangian, i.e., each \( L^i \) is of class \( C^2 \) and each \( L^i(x, \cdot) \) is strictly convex and uniformly superlinear. We impose the following conditions on the coupling matrix \( A \):

(C1) \( a^i_{j} < 0 \) for all \( i \neq j \).

(C2) The matrix \( A \) is irreducible.

We remark that Condition (C1) means the matrix \(-A\) satisfies so called Kamke-Müller condition or cooperativeness condition in the literature (see [21, 25, 26, 9, 27]). We give the definition of irreducible matrix in Appendix. Please refer to Remark 2.1 and Section 5 for the discussion on more general conditions about \( A \).

For any \( x, y \in \mathbb{R}^n \) and \( \tau > 0 \), we set

\[ \Gamma^\tau_{x,y} = \{ \xi \in W^{1,1}([0, \tau], \mathbb{R}^n) : \xi(0) = x, \xi(\tau) = y \}. \]
From the theory of ordinary differential equation in the sense of Carathéodory ([8]), the following equation

\[
\begin{aligned}
\dot{u}_1(\xi, s) &= L^1(\xi(s), \dot{\xi}(s)) - \sum_{j=1}^n a_j^1 u_j(\xi, s), \\
\dot{u}_2(\xi, s) &= L^2(\xi(s), \dot{\xi}(s)) - \sum_{j=1}^n a_j^2 u_j(\xi, s), \\
&\vdots \\
\dot{u}_n(\xi, s) &= L^n(\xi(s), \dot{\xi}(s)) - \sum_{j=1}^n a_j^n u_j(\xi, s),
\end{aligned} \quad s \in [0, t]. \tag{1.2}
\]

with initial conditions \( u_i(0) = a_i, \ i = 1, \ldots, n \), admits a unique solution \( (u_1(\xi, \cdot), \ldots, u_n(\xi, \cdot)) \) for each \( \xi \in \Gamma'_{xy} \). Similar to the problem of the scalar case considered in [6, 5], we impose the variational problem

\[
\min \{ u_i(\xi, t) : \xi \in \Gamma'_{xy} \}.
\]

It is convenient to write the problem in a compact form. Let

\[
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, & \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, & \mathbf{v} &= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, & \mathbf{a} &= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, & \xi &= \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}
\end{aligned}
\]

and

\[
L(\mathbf{x}, \mathbf{v}) = \begin{pmatrix}
L^1(x_1, v_1) \\
L^2(x_2, v_2) \\
\vdots \\
L^n(x_n, v_n)
\end{pmatrix}.
\]

Therefore (1.2) becomes

\[
\dot{\mathbf{u}} = L(\xi, \dot{\xi}) - A \cdot \mathbf{u}, \quad \mathbf{u}(0) = \mathbf{a}.
\]

The uniqueness of the solution allows us to solve the equation above into

\[
\mathbf{u}(t) = e^{-At} \cdot \mathbf{a} + \int_0^t e^{A(t-s)} \cdot L(\xi(s), \dot{\xi}(s)) \, ds, \quad \mathbf{u}(0) = \mathbf{a}, \tag{1.3}
\]

or

\[
\begin{aligned}
u_i(\xi, t) &= \sum_{j=1}^n b_j^i(t) a_j + \int_0^t \sum_{k=1}^n b_k^i(t) e^k(s) L^j(\xi(s), \dot{\xi}(s)) \, ds
\end{aligned}
\]

where \( e^{-As} = (b^j(s)) \) and \( e^{As} = (c^j(s)) \). Our conditions (C1) and (C2) ensure that each integrand is a time-dependent Tonelli Lagrangian. Thus, we can deduce the existence and regularity of the minimizers for the functional \( u' \langle \xi, t \rangle \) by classical Tonelli’s theorem. Notice under our conditions there is no Lavrentiev phenomenon.

Now, we formulate main results of this paper. We always suppose conditions (C1) and (C2) except (d).
(a) For each \( i \), the functional \( u_{i}(\xi, t) \) admits a solution and each minimizer \( \xi_{i} \) is of class \( C^{2} \). Moreover, the system of minimizers \( (\xi_{1}, \ldots, \xi_{n}) \) satisfies the following Euler-Lagrange equations (Herglotz’ equations)

\[
\frac{d}{ds} L_{s}^{i}(\xi_{i}(s), \dot{\xi}_{i}(s)) = L_{x}^{i}(\xi_{i}(s), \dot{\xi}_{i}(s)) - \sum_{j=1}^{n} a_{j}^{i} L_{v_{j}}^{i}(\xi_{i}(s), \dot{\xi}_{i}(s)), \quad i = 1, \ldots, n.
\]

(b) Let \( \phi_{1}, \ldots, \phi_{n} \in BUC(\mathbb{R}^{n}, \mathbb{R}) \), the space of bounded uniformly continuous functions, then the value functions of the Bolza problem of Herglotz’ type, defined by

\[
u'(t, x) = \inf_{\xi \in \mathcal{A}_{t, x}} \left\{ \phi_{i}(\xi(0)) + \int_{0}^{t} \left( L_{s}^{i}(\xi(s), \dot{\xi}(s)) + \sum_{j=1}^{n} a_{j}^{i} u_{j}(\xi, s) \right) ds \right\}, \quad i = 1, \ldots, n,
\]

satisfies the weakly coupled system of Hamilton-Jacobi equations

\[
\begin{cases}
D_{t} u_{1}(t, x) + H_{1}(x, D_{x} u_{1}(t, x)) + \sum_{j=1}^{n} a_{j}^{1} u_{j}(t, x) = 0, \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
D_{t} u_{n}(t, x) + H_{n}(x, D_{x} u_{n}(t, x)) + \sum_{j=1}^{n} a_{j}^{n} u_{j}(t, x) = 0,
\end{cases}
\]

with initial conditions \( u_{i}(0, x) = \phi_{i}(x) \), in the sense of viscosity.

(c) Each value function above also has the representation

\[
\nu'(t, x) = \inf_{\xi \in \mathcal{A}_{t, x}} \left\{ \sum_{j=1}^{n} b_{j}^{1}(t) \phi_{j}(\xi(0)) + \int_{0}^{t} \sum_{k,j=1}^{n} b_{k}^{j}(t) c_{k}^{j}(s) L_{s}^{j}(\xi(s), \dot{\xi}(s)) ds \right\}.
\]

(d) If \( A \) is an arbitrary real matrix with condition (C3) in Section 5, then all the statements in (a), (b) and (c) hold true for small time (see Theorem 5.1).

This paper is motivated by two types of work from different approaches. From PDE point of view, there have been a lot of works on this topic. The existence and uniqueness results for weakly coupled systems of Hamilton-Jacobi equations have been established by [12, 19]. We also note many works such as [3, 24] on long time behavior, [22, 10] on the random nature, [18, 17] on the vanishing discount problem and [11] on the Aubry set. We especially emphasize on the paper [23] by Matake and Tran. In their paper, from the previous work in the stochastic frame of this problem in [11], they deduced, for a concrete model, a new representation formula for the viscosity solutions of the weakly coupled system of Hamilton-Jacobi equations. This representation formula has quite interesting information on the deterministic nature of the problem. A recent work by Jin, Wang and Yan [20] also affords some new observation from the recent development of scalar Hamilton-Jacobi equations of contact type.

In the recent work on the weak KAM theory of contact type, there are mainly two ways to study the associated scalar Hamilton-Jacobi equation of contact type. One is the implicit variational principle developed in [28, 30, 29] and the other is the generalized variational principle of Gustav Herglotz ([14, 13, 2, 6, 5, 16]). In [20], the authors obtained a representation formula for the solution of the Cauchy problem with very general non-linear coupling, using the idea from implicit variational principle in the scalar case. But, one cannot find dynamical systems behind this problem there.

However, the dynamics behind may help us to understand the problem in a much deeper way, which is the main theme of this paper. We developed the characteristic system for the weakly...
coupled systems and gave a very clear connection between the Hamilton-Jacobi theory and the Lagrange-Hamilton theory in a complete deterministic way. We emphasize this approach indeed tells us some new observation from the viewpoint of dynamical system, geometry, mathematical physics and dynamic game theory under general conditions. The general theory of multi-contact system will be our future work.

The paper is organized as follows. In section 2, we raise a multi-objective variational problem of Herglotz’ type and obtain the regularity and some necessary conditions of optimality such as Euler-Lagrange equations. We also established the characteristic system of this problem. In Section 3, we consider the relevant Bolza problem and prove the value functions of the associated problem is the unique viscosity solutions of the associated weakly coupled systems of Hamilton-Jacobi equations. In Section 4, We introduce the Lax-Oleinik evolution of this cooperative Hamilton-Jacobi systems and discuss the relation between positive and negative type Lax-Oleinik evolutions. In Section 5, we obtained similar results on small time interval, removing condition (C1) and (C2) but adding condition (C3) on the Lagrangians. There is a short appendix on some basic knowledge on the cooperativeness condition of the coupling matrix.

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2. A HERGLOTZ-TYPE VARIATIONAL PROBLEM FOR SYSTEMS

For each \( i = 1, \ldots, n \), let \( L^i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be of class \( C^2 \) satisfying the following conditions:

(L1) \( L^i(x, v, u_1, \ldots, u_n) \) is a strictly convex in the \( v \)-variable for all \( x \in \mathbb{R}^n \) and \( (u_1, \ldots, u_n) \in \mathbb{R}^n \).

(L2) There exist two superlinear nondecreasing functions \( \overline{\theta}, \theta : [0, +\infty) \to [0, +\infty) \), \( \theta(0) = 0 \) and \( c_i > 0 \), such that

\[
\overline{\theta}(|v|) \geq L^i(x, v, 0, \ldots, 0) \geq \theta_0(|v|) - c_i, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

(L3) There exists \( K > 0 \) such that

\[
|L^i_{u_j}(x, v, u_1, \ldots, u_n)| \leq K, \quad (x, v, u_1, \ldots, u_n) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, j = 1, \ldots, n.
\]

2.1. A MULTI-OBJECTIVE VARIATIONAL PROBLEM OF HERGLOTZ TYPE. Consider an \( n \)-tuple \((\xi_1, \ldots, \xi_n) \) in \( \prod \Gamma_{x, y} \) and the following ordinary differential equation in the sense of Carathéodory

\[
\begin{aligned}
\dot{u}_1(s) &= L^1(\xi_1(s), \dot{\xi}_1(s), u_1(s), \ldots, u_n(s)), \\
\dot{u}_2(s) &= L^2(\xi_2(s), \dot{\xi}_2(s), u_1(s), \ldots, u_n(s)), \\
&\vdots \\
\dot{u}_n(s) &= L^n(\xi_n(s), \dot{\xi}_n(s), u_1(s), \ldots, u_n(s)),
\end{aligned} \quad s \in [0, t],
\tag{2.1}
\]

with initial conditions \( u_i(0) = a_i \in \mathbb{R}, i = 1, \ldots, n \). Equation (2.1) admits a unique solution \((u_i, \ldots, u_n)\) where

\[
u_i(s) = u_i(\xi_1, \ldots, \xi_n, s), \quad s \in [0, t], \ i = 1, \ldots, n.
\]
We denote \( L^i_0(x, v) = L^i(x, v, 0, \ldots, 0) \). It is clear that \( L^i_0(x, v) \) is a Tonelli Lagrangian. Each entry of Carathéodory equation (2.1) has the form
\[
\dot{u}_i(s) = L^i_0(\xi(s), \dot{\xi}(s), u_1(s), \ldots, u_n(s)) = \sum_{j=1}^{n} \hat{L}^i_{u_j}(s) \cdot u_j(s),
\]
where
\[
\hat{L}^i_{u_j}(s) = \int_0^1 L^i_{u_j}(\xi(s), \dot{\xi}(s), \lambda u_1(s), \ldots, \lambda u_n(s)) \, d\lambda.
\]
Let
\[
L_0(x, v) = \begin{pmatrix} L_0^1(x_1, v_1) \\ L_0^2(x_2, v_2) \\ \vdots \\ L_0^n(x_n, v_n) \end{pmatrix}, \quad \hat{L}_u = \begin{pmatrix} \hat{L}^1_{u_1} \\ \hat{L}^2_{u_2} \\ \vdots \\ \hat{L}^n_{u_n} \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}.
\]
Therefore (2.1) becomes
\[
\dot{u} = L_0(\xi, \dot{\xi}) + \hat{L}_u \cdot u, \quad u(0) = a.
\]
By solving this equation we have
\[
u(t) = e^{\int_0^t \hat{L}_u \, ds} \cdot a + \int_0^t e^{\int_0^s \hat{L}_u \, ds} \cdot L_0(\xi, \dot{\xi}) \, ds, \quad u(0) = a. \tag{2.2}
\]
We remark the matrix \( \hat{L}_u \) is implicitly determined by Carathéodory equation (2.1) in general. But the problem becomes much easier when \( L(x, v, u) \) is linear in \( u \).

For any \( i = 1, \ldots, n \) we define
\[
J_i^*(\xi_1, \ldots, \xi_n) = \int_0^t L^i(\xi(s), \dot{\xi}(s), u_1(s), \ldots, u_n(s)) \, ds,
\]
where \( u_1(s), \ldots, u_n(s) \) are uniquely determined by (2.1). Our task is to find a way to understand the multi-objects optimization problem with respect to the functional \( \{ J_i^* \} \) for \( \xi_1, \ldots, \xi_n \in \Gamma_{x,y}^r \).

Keep in mind the dynamic programming principle for Bolza problem. So, it is natural to use a reduction of the functionals \( \{ J_i^* \} \) by setting
\[
J_i(\xi) = J_i^*(\xi, \ldots, \xi), \quad \xi \in \Gamma_{x,y}^r.
\]
Now, our problem becomes minimizing the functional
\[
J_i(\xi) \quad \text{for} \quad \xi \in \Gamma_{x,y}^r, \tag{2.3}
\]
for each \( i = 1, \ldots, n \), or equivalently,
\[
\text{minimize} \quad u_i(\xi, t)
\]
form we have that i.e.,

the general problem of contact type in the forthcoming paper. are absolutely nontrivial in general, not only because of technical difficulty. We will handle this with initial data \( u_{1D462} \). For arbitrary coupling matrix \( W \) we have to verify some points to ensure this variational problem can deduce the Remark 2.1.

– Under conditions (C1) and (C2) we also ensure that each \( L_i \) is a time-dependent Tonelli Lagrangian. In particular, \( \frac{d}{dt} L_i(t, x, v) \) is dominated by \( L_i(t, x, v) \). So, we can have full regularity of the minimizers.

– Condition (C2) also ensures that our Carathéodory system (2.4) cannot be split into subsystems. Indeed, this condition is not necessary and it just guarantees all the coefficients of the weighted Lagrangians do not vanish.

– For arbitrary coupling matrix \( A, e^{At} \sim I \) for \( t \ll 1 \). Then, we can deal with the problem in a small time interval for the family \( \{ L_i \} \) with certain uniform superlinear growth and uniform convexity similarly (see Section 5).

2.2. Linearly coupled systems. The existence and regularity of the minimizer of problem (2.3) are absolutely nontrivial in general, not only because of technical difficulty. We will handle this general problem of contact type in the forthcoming paper.

In the current paper, we will deal with a widely studied system with Lagrangians as in (1.1), i.e.,

\[
L(x, v) - Au \tag{2.5}
\]

where \( A = \{ a'_j \} \) is an \( n \times n \)-matrix with real entries satisfying conditions (C1) and (C2) in the introduction.

For the Lagrangians in the form (2.5), by solving Carathéodory equation (2.4) in a compact form we have that

\[
\begin{align*}
\dot{u}_1(s) &= L^1(\xi(s), \dot{\xi}(s), u_1(s), \ldots, u_n(s)), \\
\dot{u}_2(s) &= L^2(\xi(s), \dot{\xi}(s), u_1(s), \ldots, u_n(s)), \\
& \quad \vdots \\
\dot{u}_n(s) &= L^n(\xi(s), \dot{\xi}(s), u_1(s), \ldots, u_n(s)),
\end{align*}
\tag{2.4}
\]

with initial data \( u_i(0) = a_i, i = 1, \ldots, n. \)

We denote \( e^{-As} = \{ b'_j(s) \}, e^{As} = \{ c'_j(s) \} \) and \( e^{A(s-t)} = \{ d'_j(s) \} \). Thus, for each \( i \),

\[
J_i(\xi) = u_i(\xi, \ldots, \xi, t) = \sum_{j=1}^{n} b'_j(t) a_j + \int_0^t L'_i(s, \xi, \dot{\xi}) \, ds.
\]

where \( L'_i(s, x, v) \) is a time-dependent Lagrangian defined by

\[
L'_i(s, x, v) = \sum_{k,j=1}^{n} b'_k(t) c'_j(s) L'_j(x, v) = \sum_{j=1}^{n} d'_j(s)L'_j(x, v). \tag{2.7}
\]

Remark 2.1. We have to verify some points to ensure this variational problem can deduce the original one and we can solve it. Under conditions (C1) and (C2) we ensure that all the entries of \( e^{-At} \) are positive for all \( t > 0 \).

– Under conditions (C1) and (C2) we also ensure that each \( L_i \) is a time-dependent Tonelli Lagrangian. In particular, \( \frac{d}{dt} L_i(t, x, v) \) is dominated by \( L_i(t, x, v) \). So, we can have full regularity of the minimizers.

– Condition (C2) also ensures that our Carathéodory system (2.4) cannot be split into subsystems. Indeed, this condition is not necessary and it just guarantees all the coefficients of the weighted Lagrangians do not vanish.

– For arbitrary coupling matrix \( A, e^{At} \sim I \) for \( t \ll 1 \). Then, we can deal with the problem in a small time interval for the family \( \{ L_i \} \) with certain uniform superlinear growth and uniform convexity similarly (see Section 5).
2.2.1. **Herglotz equation.** In view of Remark 2.1, we always suppose conditions (C1) and (C2).

**Theorem 2.2.** Suppose conditions (C1) and (C2). Then, for each $i$, $u_i(\xi, t)$ admits a solution and each minimizer $\xi_i$ of $u_i(\xi, t)$ is of class $C^2$. Moreover, the system of extremals $(\xi_1, \ldots, \xi_n)$ satisfies the following Euler-Lagrange equations

$$
\frac{d}{ds} L_\nu^i(s, \xi_i, \dot{\xi}_i) = L_x^i(s, \xi_i, \dot{\xi}_i), \quad i = 1, \ldots, n.
$$

More precisely, the Euler-Lagrange equations have the form of Herglotz

$$
\frac{d}{ds} L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) = L_x^i(\xi_i(s), \dot{\xi}_i(s)) - \sum_{j=1}^n a_j^i L_\nu^i(\xi_i(s), \dot{\xi}_i(s)), \quad i = 1, \ldots, n. \tag{2.8}
$$

**Proof.** Indeed, the associated Euler-Lagrange equations have the form

$$
\sum_{j,k=1}^n b_{jk}^i \frac{d}{ds} c_j^i(s) L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) = \sum_{j,k=1}^n b_{jk}^i c_j^i(s) L_x^i(\xi_i(s), \dot{\xi}_i(s)), \quad i = 1, \ldots, n.
$$

Multiplying each $c_j^m(t)$ to the $m$th-line in the equality above and summing up, we obtain

$$
\sum_{j=1}^n \frac{d}{ds} c_j^i(s) L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) = \sum_{j=1}^n c_j^i(s) L_x^i(\xi_i(s), \dot{\xi}_i(s)), \quad i = 1, \ldots, n. \tag{2.9}
$$

Recall $(c_j^i(s)) = e^{A^i t}$ and $((c_j^i)')(s)) = e^{A^i t}$. That is

$$(c_j^i)'(s) = \sum_{k=1}^n c_j^i(s)a_k^i. \tag{2.10}$$

It follows

$$
\sum_{j=1}^n \frac{d}{ds} c_j^i(s) L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) = \sum_{j=1}^n c_j^i(s) \frac{d}{ds} L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) + \sum_{j=1}^n (c_j^i(s))' L_x^i(\xi_i(s), \dot{\xi}_i(s))
$$

$$
= \sum_{j=1}^n c_j^i(s) \frac{d}{ds} L_x^i(\xi_i(s), \dot{\xi}_i(s)) + \sum_{j,k=1}^n c_j^i(s)a_k^i L_x^i(\xi_i(s), \dot{\xi}_i(s)).
$$

Thus (2.9) becomes

$$
\sum_{j=1}^n c_j^i(s) \frac{d}{ds} L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) + \sum_{j,k=1}^n c_j^i(s)a_k^i L_\nu^i(\xi_i(s), \dot{\xi}_i(s)) = \sum_{j=1}^n c_j^i(s) L_x^i(\xi_i(s), \dot{\xi}_i(s)). \tag{2.10}
$$

This leads to (2.8) by multiplying each $b_{jk}^m(s)$ to $m$th-line in (2.10) and summing up. This completes the proof. \qed

2.2.2. **Inf-convolution of convex functions.** To understand the Hamiltonians $\{H^i\}$’s associated to $\{L^i\}$’s, we need recall more facts from convex analysis (see, for instance, [15]).

**Proposition 2.3.** Suppose $f_1, \ldots, f_k, g$ are $C^2$, strictly convex functions with superlinear growth. Let $\lambda_1, \ldots, \lambda_k, \lambda > 0$ and $f = \sum_{j=1}^k \lambda_j f_j$.

(a) $f$ is also a $C^2$ strictly convex function with superlinear growth as well as $f^*$, the convex dual of $f$. 


(b) \((f^*)_* = f\).
(c) \((\lambda g*)_*(y) = \lambda g_*(y/\lambda)\).
(d) \((f_1 + \cdots + f_k)_*(y) = (f_1^* \oplus \cdots \oplus f_k^*)(y)\) where \(\oplus\) stands for the inf-convolution of convex functions defined by
\[
(f_1 \oplus \cdots \oplus f_k)(x) = \inf \{f_1(x_1) + \cdots + f_k(x_k) : x_1 + \cdots + x_k = x\}. \tag{2.11}
\]
(e) \(\nabla (f_1 \oplus \cdots \oplus f_k)(x) = \nabla f_1(x_1^*) = \cdots = \nabla f_k(x_k^*)\) where \((x_1^*, \ldots, x_k^*)\) is the unique minimizer in (2.11) such that \(x_1^* + \cdots + x_k^* = x\).

We introduce the Hamiltonians with respect to the Lagrangians \(\{L^i\}\) in (2.5). Let for each \(i\)
\[
H^i(x, p) = \sup_{v \in \mathbb{R}^n} \{p \cdot v - L^i(x, v)\}.
\]
Due to Proposition 2.3, for each \(i\), the Hamiltonian \(\mathbb{H}^i\) associated to \(\mathbb{L}^i\) in (2.7) is
\[
\mathbb{H}^i(s, x, p) = \inf \left\{ \sum_{j=1}^{n} d_j^i(s) H^i(x, q_j/d_j^i(s)) : q_1, \ldots, q_n \in \mathbb{R}^n, \sum_{j=1}^{n} q_j = p \right\}. \tag{2.12}
\]
In the definition of \(\mathbb{H}^i\), we conclude the infimum can be achieved since \(\sum_{j=1}^{n} c_j^i(s) H^i(x, q_j/c_j^i(s))\) as a function of \((q_1, \ldots, q_n)\) is superlinear, and the minimizer is unique because \(\mathbb{H}^i\) is smooth. Given \((s, x, p)\), there exists a unique \((q_1^i(s), \ldots, q_n^i(s))\) such that
\[
p = \sum_{j=1}^{n} q_j^i(s), \quad \mathbb{H}^i(s, x, p) = \sum_{j=1}^{n} d_j^i(s) H^i(x, q_j^i(s)/d_j^i(s)),
\]
\[
\mathbb{H}_p^i(s, x, p) = H^i_p(x, q_1^i(s)/d_1^i(s)) = \cdots = H^i_n(x, q_n^i(s)/d_n^i(s)) \quad \tag{2.13}
\]
\[
\mathbb{H}_x^i(s, x, p) = \sum_{j=1}^{n} d_j^i(s) H^i_x(x, q_j^i(s)/d_j^i(s)).
\]

2.2.3. Lie equation and characteristic systems. We also write down the Hamiltonian equations with respect to \(\mathbb{H}^i\) \((i = 1, \ldots, n)\) as
\[
\begin{cases}
\dot{\xi}_i(s) = \mathbb{H}^i_p(s, \xi_i(s), p_i(s)), \\
\dot{p}_i(s) = -\mathbb{H}^i_x(s, \xi_i(s), p_i(s)).
\end{cases}
\]

Lemma 2.4. Let \(p_i(s) := \mathbb{L}_p^i(s, \xi_i(s), \dot{\xi}_i(s))\) be the dual arc. Then
\[
p_i(s) = \sum_{j=1}^{n} d_j^i(s)p_j^i(s), \quad i = 1, \ldots, n,
\]
where \(p_j^i(s) = L_v^i(\xi_i(s), \dot{\xi}_i(s))\) for all \(i, j = 1, \ldots, n\). Moreover, we have
\[
\mathbb{H}^i(s, \xi_i(s), p_i(s)) = \sum_{j=1}^{n} d_j^i(s) H^i(s, \xi_i(s), p_j^i(s)), \quad i = 1, \ldots, n.
\]

Remark 2.5. The curves \(\{d_j^i(s)p_j^i(s)\}_{j=1}^{n}\) are exactly the minimizers for \(\mathbb{H}^i(s, \xi(s), p(s))\) in the inf-convolution representation (2.12) by (2.13), for each \(i\).
Proof. From the definition of \( L^i \), we obtain

\[
p_i(s) = L^i_{\nu}(s, \xi_i(s), \dot{\xi}_i(s)) = \sum_{j=1}^{n} d_j^i(s)L^j_{\nu}(\xi_j(s), \dot{\xi}_j(s)) = \sum_{j=1}^{n} d_j^i(s)p_j^i(s),
\]

Therefore

\[
\mathcal{H}(s, \xi_i(s), \mathcal{L}_\nu^i(s, \xi_j(s), \dot{\xi}_j(s))) = \mathcal{L}_\nu^i(s, \xi_j(s), \dot{\xi}_j(s)) \cdot \dot{\xi}_j(s) - \mathcal{L}_\nu^i(s, \xi_j(s), \dot{\xi}_j(s)) = \sum_{j=1}^{n} d_j^i(s)\{L_j^i(\xi_j(s), \dot{\xi}_j(s)) \cdot \dot{\xi}_j(s) - L_j^i(\xi_j(s), \dot{\xi}_j(s))\} = \sum_{j=1}^{n} d_j^i(s)H^j(\xi_j(s), p_j^i(s)).
\]

This completes our proof. \( \square \)

By the observation above the associated Hamiltonian systems of \((2.8)\) are more complicated. Similar to Theorem \(2.2\) we have the following characteristic system in Hamiltonian form.

**Theorem 2.6.** Let \((\xi_1, \cdots, \xi_n)\) be the minimizers of \((2.3)\) for \( L \) as in \((2.5)\). For each \(i, j = 1, \cdots, n\), let \( p_j^i(s) = L_j^i(\xi_j(s), \dot{\xi}_j(s)) \). Then, the curves \(\{\xi_i\}\) and \(\{p_j^i\}\) satisfy

\[
\dot{\xi}_j(s) = H^j_p(\xi_j(s), p_j^i(s)), \quad i, j = 1, \ldots, n,
\]

\[
\dot{p}_j^i(s) = -H_j^i(\xi_j(s), p_j^i(s)) + \sum_{j=1}^{n} d_j^i p_j^i(s), \quad i = 1, \ldots, n.
\]

To obtain a clear picture of the characteristic system, we also need consider the \(u\)-curves. Comparing to the scalar case, the \(u\)-curves are much complicated as well as \(p\)-curves explained in the last subsection. Suppose \((\xi_1, \cdots, \xi_n)\) are minimizers of \((2.3)\) for \( L \) as in \((2.5)\). Then, Carathéodory equation \((2.4)\) defines for each pair \((i, j)\) a curve \(u_j^i\) and such \(n \times n\) number of curves \(\{u_j^i\}\) satisfy

\[
\dot{u}_j^i = L_j^i(\xi_i, \dot{\xi}_i) + \sum_{j=1}^{n} a_j^i u_j^k, \quad i, j = 1, \ldots, n.
\]

In the Hamiltonian formalism, they have the form

\[
\dot{u}_j^i = p_j^i \cdot H_j^i(\xi_i, p_j^i) - H(\xi_i, p_j^i) + \sum_{j=1}^{n} a_j^i u_j^k,
\]

where \(\{p_j^i\}\) are determined by Theorem \((2.6)\). Because of linear coupling, the \(u\)-factor of the characteristic systems is determined by the \((\xi, p)\)-factor. But it is not the case in general if we consider general nonlinear coupling.
3. Weakly coupled system of Hamilton-Jacobi equations

Let \( \phi_i \in BUC(\mathbb{R}^n, \mathbb{R}) \), \( i = 1, \ldots, n \). For any \( (t, x) \in (0, \infty) \times \mathbb{R}^n \) we define the set of accessible arcs by

\[
\mathcal{A}_{t,x} = \{ \xi \in W^{1,1}([0, t], \mathbb{R}^n) : \xi(t) = x \}.
\]

Consider the following Bolza problem for systems: for \( t > 0 \) and \( x \in \mathbb{R}^n \),

\[
u^i(t, x) = \inf_{\xi \in \mathcal{A}_{t,x}} \left\{ \sum_{j=1}^n b_j^i(t)\phi_j(\xi(x)) + \int_0^t \sum_{j=1}^n d_j^i(s)L^j(\xi, \dot{\xi}) \, ds \right\} \tag{3.1}
\]

**Theorem 3.1.** Each \( \nu^i \) defined in (3.1) is locally semiconcave, \( i = 1, \ldots, n \), and \( (\nu^1, \ldots, \nu^n) \) satisfies the following weakly coupled systems of Hamilton-Jacobi equations

\[
\begin{aligned}
&D_t \nu^i(t, x) + H^i(x, D_x \nu^i(t, x)) + \sum_{j=1}^n a_j^i \nu^j(t, x) = 0,
&D_t \nu^n(t, x) + H^n(x, D_x \nu^n(t, x)) + \sum_{j=1}^n a_j^n \nu^j(t, x) = 0
\end{aligned}
\]

on \( (0, +\infty) \times \mathbb{R}^n \) in the sense of viscosity. Moreover, if \( (t, x) \in (0, \infty) \times \mathbb{R}^n \) is a point of differentiability for all \( \nu^i \)'s, then for each \( i \) we have

\[
D_x \nu^i(t, x) = L^i_{\nu^i}(\xi^i(t), \dot{\xi}^i(t))
\]

\[
D_t \nu^i(t, x) = -H^i(\xi^i(t), L^i_{\nu^i}(\xi^i(t), \dot{\xi}^i(t))) - \sum_{k=1}^n a_k^i \nu^k(t, x),
\]

where the \( C^2 \) curve \( \xi^i \in \mathcal{A}_{t,x} \) is the unique minimizer for \( \nu^i(t, x) \), \( i = 1, \ldots, n \).

**Proof.** One can rewrite (3.1) as

\[
u^i(t, x) = \inf_{\xi \in \mathcal{A}_{t,x}} \left\{ \sum_{j=1}^n b_j^i(t)\phi_j(z) + A_{0,i}(z, x) \right\}, \tag{3.2}
\]

where

\[
A_{0,i}(x, y) = \inf_{\xi \in \mathcal{A}_{t,x}} \int_0^t L^i(s, \xi(s), \dot{\xi}(s)) \, ds, \quad x, y \in \mathbb{R}^n, t > 0.
\]

Invoking Lemma 3.1 in [4], the infimum in (3.2) can be achieved. Observe we represent each \( \nu^i \) as a family of functions \( \sum_{j=1}^n b_j^i(t)\phi_j(z) + A_{0,i}(z, x) \) (\( z \) as a parameter). We conclude \( \nu^i \) is locally semiconcave since for each \( (t, x) \in (0, \infty) \times \mathbb{R}^n \), the function \( (z, x) \mapsto \sum_{j=1}^n b_j^i(t)\phi_j(z) + A_{0,i}(z, x) \) is uniformly semiconcave on any compact subset of \( (0, \infty) \times \mathbb{R}^n \) by Theorem 3.4.4 in [7].

Now, suppose \( (t, x) \in (0, \infty) \times \mathbb{R}^n \) is a point of differentiability for all \( \nu^i \)'s. Applying Theorem 3.4.4 in [7] again we know there exists a unique \( z \) such that

\[
u^i(t, x) = \sum_{j=1}^n b_j^i(t)\phi_j(z) + A_{0,i}(z, x),
\]
and

\[ D_t u'(t, x) = \sum_{j=1}^{n} (b_j'(t))' \phi_j(z) - \mathbb{H}(t, \xi_i(t), L_i^t(t, \xi_i(t), \dot{\xi}_i(t))) + \sum_{j=1}^{n} (b_j'(t))' \int_{0}^{t} \sum_{k=1}^{n} c_k^j(s)L^k(\xi_i, \dot{\xi}_i) \, ds \]

\[ D_x u'(t, x) = \sum_{j,k=1}^{n} b_j'(t)c_k^j(t)L^k_i(\xi_i(t), \dot{\xi}_i(t)), \]

where \( \xi_{x,x}^i \) is the unique minimal curve for \( A_{0,i}(z, x) \). Observe \( \frac{d}{dt} e^{-A t} = -A \cdot e^{-A t} \), i.e.

\[ (b_j'(t))' = -\sum_{k=1}^{n} a_k^j b_j'(t), \quad i, j = 1, \ldots, n. \]  \hspace{1cm} (3.3)

Thus,

\[ \sum_{j=1}^{n} (b_j'(t))' \phi_j(z) + \sum_{j=1}^{n} (b_j'(t))' \int_{0}^{t} \sum_{k=1}^{n} c_k^j(s)L^k(\xi_i, \dot{\xi}_i) \, ds \]

\[ = -\sum_{j=1}^{n} a_j^i \left\{ \sum_{k=1}^{n} b_k'(t)\phi_k(z) + \sum_{j=1}^{n} b_j'(t) \int_{0}^{t} \sum_{k=1}^{n} c_k^j(s)L^m(\xi_i, \dot{\xi}_i) \, ds \right\} \]

\[ = -\sum_{j=1}^{n} a_j^i u'(t, x) \]

Thus, by Lemma 2.4 we conclude

\[ D_x u'(t, x) = \sum_{j,k=1}^{n} b_j'(t)c_k^j(t)L^k_i(\xi_i(t), \dot{\xi}_i(t)) = L^j_i(\xi_i(t), \dot{\xi}_i(t)) \]  \hspace{1cm} (3.4)

and

\[ D_t u'(t, x) = -\sum_{j=1}^{n} a_j^i u'(t, x) - \sum_{k,j=1}^{n} b_k'(t)c_j^k(t)H^k_i(\xi_k(t), L^j_i(\xi_i(t), \dot{\xi}_i(t))) \]

\[ = -\sum_{j=1}^{n} a_j^i u'(t, x) - H^i(\xi_i(t), L^j_i(\xi_i(t), \dot{\xi}_i(t))), \]  \hspace{1cm} (3.5)

since \( \sum_{k=1}^{n} b_k'(t)c_j^k(t) = \delta_j^i \), the Kronecker symbol, \( i, j = 1, \ldots, n \). Recall that \( u' \)'s are all locally semiconcave. Then, the combination of (3.4) and (3.5) together with Proposition 5.3.1 in [7] completes the proof. \( \square \)
4. LAX-OLENIK EVOLUTION OF WEAKLY COUPLED SYSTEMS

Fix \( x, y \in \mathbb{R}^n, t > 0 \) and \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \). For any \( \xi \in \Gamma_{x,y}^t \) we will consider two kinds of problem of the following Carathéodory system

\[
\begin{aligned}
\dot{u}_1(\xi, s) &= L^1(\xi(s), \dot{\xi}(s)) - \sum_{j=1}^n a_j^1 u_j(\xi, s), \\
\dot{u}_2(\xi, s) &= L^2(\xi(s), \dot{\xi}(s)) - \sum_{j=1}^n a_j^2 u_j(\xi, s), \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
**Proof.** Fix \( x, y \in \mathbb{R}^n, t > 0 \). Let \( \xi \in \Gamma^t_{y,x} \) and let \( \{ u_i(\xi, \cdot) \} \) be determined by (4.1) with respect to \( \bar{L} + Au \), with initial conditions \( u_i(\xi, 0) = -a_i, i = 1, \ldots, n \), i.e.,

\[
h_i(\bar{L} + Au, t, x, y, -a) = \inf_{\eta} \int_0^t \left\{ L^i(\eta(s), -\dot{\xi}(s)) + \sum_{j=1}^n a_j u_j(\eta(s), s) \right\} \, ds
\]

Notice that \( \dot{\eta}(s) = -\dot{\xi}(t-s), u_i(\eta, s) = u_i(\xi, t-s) \) with \( \{ u_i(\eta, \cdot) \} \) determined by (4.1) with initial conditions \( u_i(\eta, t) = a_i, i = 1, \ldots, n \). Therefore

\[
\bar{h}_i(L - Au, t, x, y, a) \leq \int_0^t \left\{ L^i(\eta(s), \dot{\eta}(s)) - \sum_{j=1}^n a_j u_j(\eta(s), s) \right\} \, ds
\]

\[
= \int_0^t \left\{ L^i(\eta(t - \tau), \dot{\eta}(t - \tau)) - \sum_{j=1}^n a_j u_j(\eta(t - \tau), \tau) \right\} \, d\tau
\]

\[
= \int_0^t \left\{ L^i(\dot{\xi}(\tau), -\dot{\xi}(\tau)) + \sum_{j=1}^n a_j u_j(\dot{\xi}(\tau), \tau) \right\} \, d\tau
\]

\[
= h_i(\bar{L} + Au, t, y, x, -a).
\]

The opposite inequality can be obtained similarly. \( \square \)

Now we can define the Lax-Oleinik evolution.

**Definition 4.4.** For any \( \phi_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, n \), and \( t > 0 \), we define

\[
T^i_\tau \phi_i(x) = \inf_{y \in \mathbb{R}^n} \{ \phi_i(y) + h_i(L - Au, t, x, y, \phi_1(y), \ldots, \phi_n(y)) \},
\]

\[
\bar{T}^i_\tau \phi_i(x) = \sup_{y \in \mathbb{R}^n} \{ \phi_i(y) - \bar{h}_i(L - Au, t, y, x, \phi_1(y), \ldots, \phi_n(y)) \}.
\]

In a compact form, we set \( \phi = (\phi_1, \ldots, \phi_n) \) and write

\[
\mathbb{T}_i \phi(x) = \begin{pmatrix} T^1_\tau \phi_1(x) \\ \vdots \\ T^n_\tau \phi_n(x) \end{pmatrix}, \quad \bar{T}_i \phi(x) = \begin{pmatrix} \bar{T}^1_\tau \phi_1(x) \\ \vdots \\ \bar{T}^n_\tau \phi_n(x) \end{pmatrix}.
\]

We call \( \mathbb{T}_i \) and \( \bar{T}_i \) the negative and positive type Lax-Oleinik evolution respectively.

As an easy consequence of Lemma 4.3 and the definition of \( \mathbb{T}_i \) and \( \bar{T}_i \), we have the following consequence.

**Corollary 4.5.** For any \( \phi_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, n \), and \( t > 0 \), we have

\[
-\bar{T}^i_\tau \phi_i = T^i_\tau (-\phi_i).
\]

In Section 3, we have already verified that, if all \( \phi_i \)’s are bounded and uniformly continuous, then \( u^i = T^i_\tau \phi_i, i = 1, \ldots, n \), are indeed the viscosity solutions of weakly coupled systems of Hamilton-Jacobi equations (HJ, S). This implies the following result with wide interests in the literature.
Theorem 4.6. Suppose \( \phi_1, \ldots, \phi_n \in \text{BUC}(\mathbb{R}^n, \mathbb{R}) \). Then \( u_i(t, x) = -\tilde{T}_i \phi_i(x) \), \( i = 1, \ldots, n \), are solutions of the following weakly coupled systems of Hamilton-Jacobi equations

\[
\begin{aligned}
D_t u_1(t, x) + H^1(x, -D_x u_1(t, x)) - \sum_{j=1}^n a_{ij}^1 u_j(t, x) &= 0, \\
&\vdots \\
D_t u_n(t, x) + H^n(x, -D_x u_n(t, x)) - \sum_{j=1}^n a_{ij}^n u_j(t, x) &= 0,
\end{aligned}
\]

\((\text{HJS}_+)\)

on \((0, \infty) \times \mathbb{R}^n\) with initial conditions \( u_i(0, x) = -\phi_i(x) \), in the sense of viscosity.

5. Concluding Remark

The method developed in this paper can also be applied for arbitrary real coupling matrix \( A \). For arbitrary coupling matrix \( A, e^{A t} \sim I \) for \( t \ll 1 \). Then, we can deal with the problem in a small time interval. We impose a condition as follows.

(C3) There exist two superlinear functions \( \theta_0, \theta_1 : [0, \infty) \to [0, \infty) \) and \( c_0 > 0 \) such that

(i) \( \theta_1(|v|) \geq L^i(x, v) \geq \theta_0(|v|) - c_0 \) for all \( (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \) and \( i = 1, \ldots, n \), and there exists \( C_1 > 0 \) such that \( \theta_1(v) \leq C_1 (1 + \theta_0(v)) \) for all \( v \in [0, \infty) \).

(ii) There exists \( C_2 > 0 \) such that \( \sum_{j=1}^n D^2 L^i(x, \cdot) \leq C_2 D^2 L^i(x, \cdot) \) in the sense of distribution for all \( x \in \mathbb{R}^n \) and \( i = 1, \ldots, n \).

Recall that \( \mathbb{L}^i(x, v) = \sum_{j=1}^n d_j^i(s) L^j(x, v) \).

Theorem 5.1. If \( A \) is arbitrary and condition (C3) is satisfied, then there exists \( C, \tilde{t} > 0 \) such that each \( \mathbb{L}^i(x, v) \), \( i = 1, \ldots, n \), is a time-dependent Lagrangian with \( \frac{d}{dt} \mathbb{L}^i(s, x, v) \leq C(1 + \mathbb{L}^i(s, x, v)) \) on \((0, \tilde{t}) \times \mathbb{R}^n \times \mathbb{R}^n\). In addition the following statements hold true.

(1) For any \( i, x, y \in \mathbb{R}^n \) and \( t \in (0, \tilde{t}) \), \( u_i(\xi, t) \) admits a minimizer \( \xi_i \in \Gamma^i_{x, y} \). Any such a minimizer \( \xi_i \) is of class \( C^2 \) and satisfies Herlotz equation (2.8) on \([0, t]\).

(2) Let \( x, y \in \mathbb{R}^n \), \( t \in (0, \tilde{t}) \) and \( \xi_i \) be a minimizer of \( u_i(\cdot, t) \), \( i = 1, \ldots, n \). Set \( p_i^j(s) = L_v^i(\xi_i(s), \dot{\xi}_i(s)) \), \( i, j = 1, \ldots, n \). Then \( \{\xi_i\} \) and \( \{p_i^j\} \) satisfy Lie equation (2.14) on \([0, t]\).

(3) For \( \phi_1, \ldots, \phi_n \in \text{BUC}(\mathbb{R}^n, \mathbb{R}) \) and let \( u^1, \ldots, u^n \) be the value functions of the Bolza problem (3.1). Then, each \( u^i \) is locally semiconcave on \((0, \tilde{t}) \times \mathbb{R}^n\) and \( (u^1, \ldots, u^n) \) is the unique set of viscosity solutions of \((\text{HJS}_-)\) on \([0, \tilde{t}] \times \mathbb{R}^n\).

Proof. Observe that for \( s \ll 1 \), we have

\[
\sum_{j=1}^n d_j^i(s) L^j(x, v) = d_i^i(s) L^i(x, v) + \sum_{j \neq i} d_j^i(s) L^j(x, v)
\geq d_i^i(s) L^i(x, v) - \sum_{j \neq i} |d_j^i(s)| \cdot \theta_1(|v|)
\geq d_i^i(s) (\theta_0(|v|) - c_0) - \sum_{j \neq i} |d_j^i(s)| (C_1 + C_1 \theta_0(|v|))
= \left\{ d_i^i(s) - C_1 \sum_{j \neq i} |d_j^i(s)| \right\} \theta_0(|v|) - \left\{ C_1 \sum_{j \neq i} |d_j^i(s)| + c_0 |d_i^i(s)| \right\}.
\]
Now, take $t_0 > 0$ and $c_1, \kappa_1 > 0$ such that

$$d_i'(s) - C_1 \sum_{j \neq i}^n |d_j'(s)| \geq \kappa_1, \quad C_1 \sum_{j \neq i}^n |d_j'(s)| + c_0 |d_i'(s)| \leq c_1, \quad \forall s \in [0, t_0].$$

In follows for each $i$

$$\sum_{j=1}^n d_j'(s)L^i(x, \nu) \geq \kappa_1 \theta_0(\|\nu\|) - c_1, \quad (x, \nu) \in \mathbb{R}^n \times \mathbb{R}^n, \quad s \in [0, t_0]. \quad (5.1)$$

Similarly, in the sense of distribution we have

$$\sum_{j=1}^n d_j'(s)D_v^2 L^i(x, \nu) = d_i'(s)D_v^2 L^i(x, \nu) + \sum_{j \neq i}^n d_j'(s)D_v^2 L^i(x, \nu) \geq d_i'(s)D_v^2 L^i(x, \nu) - C_2 \sum_{j \neq i}^n |d_j'(s)|D_v^2 L^i(x, \nu) \geq (d_i'(s) - C_2 \sum_{j \neq i}^n |d_j'(s)|)D_v^2 L^i(x, \nu).$$

Similarly, take $t_1 > 0$ and $\kappa_2 > 0$ such that

$$d_i'(s) - C_2 \sum_{j \neq i}^n |d_j'(s)| \geq \kappa_2, \quad \forall s \in [0, t_1].$$

In follows for each $i$

$$D_v^2 \left( \sum_{j=1}^n d_j'(s)L^i(x, \nu) \right) \geq \kappa_2 D_v^2 L^i(x, \nu) > 0, \quad (x, \nu) \in \mathbb{R}^n \times \mathbb{R}^n, \quad s \in [0, t_1]. \quad (5.2)$$

Thus, each $L^i$ is a time-dependent Tonelli Lagrangian by (5.1) and (5.2). The fact $\frac{d}{dt}L^i(s, x, \nu) \leq C(1 + L^i(s, x, \nu))$ on $(0, \bar{t}) \times \mathbb{R}^n \times \mathbb{R}^n$ shows there is no Lavrentiev phenomenon and we have full regularity for the minimizers.

The rest part is directly from the proofs of Theorem 2.2, Theorem 2.6 and Theorem 3.1 respectively. This completes the proof. \hfill \square

**Remark 5.2.** To extend Theorem 5.1 to a large time, we need use the technique of broken geodesic by the conjunction of a sequence of fundamental solutions, which is based on the local analysis of the fundamental solutions such as semiconcavity and convexity estimate. We will touch this part in the future.

**Appendix A. Kamke-Müller Condition and Irreducible Matrix**

It is already known that Kamke-Müller condition is closely related to the theory of non-negative matrix which plays an essential role in the analysis of evolutionary differential or integral equations of monotone type especially on the long time behavior ([21, 25, 26, 27]). A good reference on non-negative matrix is [1].

**Definition A.1.**
(1) An \( n \times n \) matrix \( A \) is said to be non-negative if all the entries are non-negative real numbers.

(2) An \( n \times n \) matrix \( A \) is cogredient to a matrix \( E \) if for some permutation matrix \( P \), \( PAP^T = E \). \( A \) is reducible if it is cogredient to
\[
E = \begin{bmatrix}
B & 0 \\
C & D
\end{bmatrix},
\]
where \( B \) and \( D \) are square matrices, or if \( n = 1 \) and \( A = 0 \).

(3) An \( n \times n \) matrix \( A \) is said to be irreducible if \( A \) is not reducible.

(4) An \( n \times n \) real matrix \( A = (a_{ij}) \) is essentially nonnegative if \( a_{ij} \geq 0 \) for all \( i \neq j \).

**Proposition A.2.** Let \( A \) be an \( n \times n \) real matrix. Then \( e^{At} \geq 0 \) for all \( t \geq 0 \) if and only if \( A \) is essentially nonnegative. In addition, if \( A \) is also irreducible then all the entries of \( e^{At} \) are positive for all \( t > 0 \).

**REFERENCES**

[1] Abraham Berman and Robert J. Plemmons. *Nonnegative matrices in the mathematical sciences*, volume 9 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994. Revised reprint of the 1979 original.

[2] Alessandro Bravetti. Contact Hamiltonian dynamics: the concept and its use. *Entropy*, 19(10):Paper No. 535, 12, 2017.

[3] Fabio Camilli, Olivier Ley, Paola Loreti, and Vinh Duc Nguyen. Large time behavior of weakly coupled systems of first-order Hamilton-Jacobi equations. *NoDEA Nonlinear Differential Equations Appl.*, 19(6):719–749, 2012.

[4] Piermarco Cannarsa and Wei Cheng. Generalized characteristics and Lax-Oleinik operators: global theory. *Calc. Var. Partial Differential Equations*, 56(5):Art. 125, 31, 2017.

[5] Piermarco Cannarsa, Wei Cheng, Liang Jin, Kaizhi Wang, and Jun Yan. Herglotz’ variational principle and Lax-Oleinik evolution. *J. Math. Pures Appl.* (9), 141:99–136, 2020.

[6] Piermarco Cannarsa, Wei Cheng, Kaizhi Wang, and Jun Yan. Herglotz’ generalized variational principle and contact type Hamilton-Jacobi equations. In *Trends in control theory and partial differential equations*, volume 32 of *Springer INdAM Ser.*, pages 39–67. Springer, Cham, 2019.

[7] Piermarco Cannarsa and Carlo Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2004.

[8] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[9] W. A. Coppel. *Stability and asymptotic behavior of differential equations*. D. C. Heath and Co., Boston, Mass., 1965.

[10] Andrea Davini, Antonio Siconolfi, and Maxime Zavidovique. Random Lax-Oleinik semigroups for Hamilton-Jacobi systems. *J. Math. Pures Appl.* (9), 120:294–333, 2018.

[11] Andrea Davini and Maxime Zavidovique. Aubry sets for weakly coupled systems of Hamilton-Jacobi equations. *SIAM J. Math. Anal.*, 46(5):3361–3389, 2014.

[12] Hans Engler and Suzanne M. Lenhart. Viscosity solutions for weakly coupled systems of Hamilton-Jacobi equations. *Proc. London Math. Soc.* (3), 63(1):212–240, 1991.

[13] Mariano Giaquinta and Stefan Hildebrandt. *Calculus of variations. II: The Hamiltonian formalism*, volume 311 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1996.

[14] R. B. Guenther, C. M Guenther, and J. A. Gottsch. *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems*. Juliusz Schauder Center for Nonlinear Studies. Nicholas Copernicus University, 1995.

[15] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2001. Abridged version of it Convex analysis and minimization algorithms. I [Springer, Berlin, 1993; MR1261420 (95m:90001)] and II [ibid.; MR1295240 (95m:90002)].
[16] Jiahui Hong, Wei Cheng, Shengqing Hu, and Kai Zhao. Representation formulas for contact type Hamilton-Jacobi equations. Journal of Dynamics and Differential Equations, online, 2021.
[17] Hitoshi Ishii. The vanishing discount problem for monotone systems of Hamilton-Jacobi equations. Part 1: linear coupling. *Math. Eng.*, 3(4):Paper No. 032, 21, 2021.
[18] Hitoshi Ishii and Liang Jin. The vanishing discount problem for monotone systems of Hamilton-Jacobi equations: part 2—nonlinear coupling. *Calc. Var. Partial Differential Equations*, 59(4):Paper No. 140, 28, 2020.
[19] Hitoshi Ishii and Shigeaki Koike. Viscosity solutions for monotone systems of second-order elliptic PDEs. *Comm. Partial Differential Equations*, 16(6-7):1095–1128, 1991.
[20] Liang Jin, Lin Wang, and Jun Yan. A representation formula of viscosity solutions to weakly coupled systems of Hamilton-Jacobi equations with applications to regularizing effect. *J. Differential Equations*, 268(5):2012–2039, 2020.
[21] E. Kamke. Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II. *Acta Math.*, 58(1):57–85, 1932.
[22] H. Mitake, A. Siconolfi, H. V. Tran, and N. Yamada. A Lagrangian approach to weakly coupled Hamilton-Jacobi systems. *SIAM J. Math. Anal.*, 48(2):821–846, 2016.
[23] H. Mitake and H. V. Tran. A dynamical approach to the large-time behavior of solutions to weakly coupled systems of Hamilton-Jacobi equations. *J. Math. Pures Appl. (9)*, 101(1):76–93, 2014.
[24] Hiroyoshi Mitake and Hung V. Tran. Remarks on the large time behavior of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton-Jacobi equations. *Asymptot. Anal.*, 77(1-2):43–70, 2012.
[25] Max Müller. Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen. *Math. Z.*, 26(1):619–645, 1927.
[26] Hal L. Smith. *Monotone dynamical systems*, volume 41 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1995. An introduction to the theory of competitive and cooperative systems.
[27] Wolfgang Walter. *Differential and integral inequalities*. Translated from the German by Lisa Rosenblatt and Lawrence Shampine. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 55. Springer-Verlag, New York-Berlin, 1970.
[28] Kaizhi Wang, Lin Wang, and Jun Yan. Implicit variational principle for contact Hamiltonian systems. *Nonlinearity*, 30(2):492–515, 2017.
[29] Kaizhi Wang, Lin Wang, and Jun Yan. Aubry–Mather Theory for Contact Hamiltonian Systems. *Comm. Math. Phys.*, 366(3):981–1023, 2019.
[30] Kaizhi Wang, Lin Wang, and Jun Yan. Variational principle for contact Hamiltonian systems and its applications. *J. Math. Pures Appl. (9)*, 123:167–200, 2019.

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