RATES AND DISTRIBUTIONS IN TAU DECAY

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Abstract

Semileptonic decays of polarised τ leptons are investigated. Predictions for the rate, based on CVC and chiral Lagrangians, are contrasted with experiments. Predictions for the angular distributions of three meson final states are given. Emphasis is put on studies in electron-positron annihilation where the neutrino escapes detection and the τ restframe cannot be reconstructed. It is shown that the form factors can be measured in ongoing high statistics experiments. Of particular interest for the three meson case are the distribution of the normal to the Dalitz plane and the distribution around this normal. At LEP these distributions allow for an improved measurement of the τ polarisation. Implications are considered for an experiment where the τ restframe is reconstructed. It is shown that the measurement of impact parameters with the help of vertex detectors allows a full kinematic reconstruction, including the direction of the τ and the missing neutrino momentum.

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1 Introduction

Since the original discovery of the \( \tau \)-lepton our understanding of its production and decay properties has advanced continuously through improved experimental precision and through detailed theoretical investigations. All studies to date seem to be in excellent agreement with the predictions of the Standard Model (SM). Nevertheless there are a number of reasons why precision studies of the \( \tau \)-lepton ought to be pushed far beyond the present level. Pair production at \( e^+e^- \)-colliders seems to be the most promising experimental tool at present and in the foreseeable future. The problems and questions to be addressed in this connection can be broken down into three groups (Fig.1).

![Figure 1: Left: Production and decay of a \( \tau \) pair in electron-positron annihilation. Right: CVC relating electron positron annihilation and \( \tau \) decays.](image)

A The production mechanism through the virtual photon and the \( Z_0 \) boson is unambiguously fixed by the standard model. The dependence of the \( \tau \) polarisation as measured at LEP on the neutral vector coupling constant can be exploited for a determination of the weak mixing angle well comparable in its precision to that of the lepton forward backward asymmetry [1]. Of course one may and should, in addition, always test the SM, check lepton universality (which could be invalidated in certain variants of the two Higgs doublet model [2]) or search for an (CP violating) electric dipole moment [3]. These investigations are of course motivated by the large \( \tau \) mass which might enhance effects that are suppressed by the small masses in the case of electron or muon.

B Also the charged current coupling of the \( \tau \) to its neutrino and to the \( W \) should be scrutinized as far as possible. The determination of the neutrino helicity, the measurement of the Michel parameter or the search for a small admixture of \( V + A \) are obvious goals in this context [4].

C Semileptonic \( \tau \) decays are a unique tool for low energy hadron physics. Amplitudes and rates of exclusive decay modes can be predicted with chiral Lagrangians, supplemented by information about resonance parameters. Conversely, these decay modes may develop into a unique tool for the study of \( \rho, \rho' \) competing well with low energy \( e^+e^- \) colliders with energies tuned to the region below 1.7 GeV. \( \tau \) decays allow, furthermore, the direct production of resonances with \( J^{PC} = 0^- \) and 1++ with and without strangeness, a piece of information not directly accessible from any other experiment.

The topics A,B and C are of course closely connected in any actual analysis. Quite often theoretical or experimental information on the decay has to be included to pin down the production amplitude or vice versa. Theoretical models or low energy experiments (for example ARGUS or CLEO) may therefore provide important information for the \( \tau \) analysis e.g. at LEP.
or for the analysis of $\tau$’s from $W$ decay. Furthermore, with increasing event rates, one may study not only branching ratios or mass distributions but one may also analyse highly differential multidimensional angular distributions leading to rigid constraints on the theoretical input in the analysis.

These themes will be illustrated in the following sections by a few typical examples. In section 2 predictions for the rate and for the spectral functions will be discussed. The important role of angular distributions will be emphasized in section 3 and the important notion of structure functions will be introduced with emphasis on investigations based on momenta of the final hadrons only. The possibilities for full reconstruction of the kinematics, despite the missing neutrinos with the help of impact parameter measurements will be demonstrated in section 4.

2 Decay Rates and hadron physics

Semileptonic rates and CVC

The production of $1^{--}$ states with strangeness zero in $\tau$ decay and in $e^+e^-$ annihilation are intimately connected (Fig.1). Semileptonic $\tau$ decays explore the mass region between $2m_\pi$ and $m_\tau$ with systematic errors different from those of $e^+e^-$ experiments. Already now they allow for example, a determination of the pion form factor with an accuracy comparable to a large number of low energy $e^+e^-$ experiments. To wit, the experimental results for the ratio $\text{Br}(\tau^- \rightarrow \pi^-\pi^0\nu)/\text{Br}(\tau^- \rightarrow e^-\nu\nu)$ of ARGUS [5] $(1.31 \pm 0.06)$ and ALEPH [6] $(1.38 \pm 0.04)$ are in nice agreement with the CVC prediction [7] of $1.32 \pm 0.05$. The impact of an improved pion form factor determination (and the corresponding information on $4\pi$, $\omega\pi$, ...) for $g-2$ predictions of the muon and for the running of $\alpha_{\text{QED}}$ are evident.

Based on the relation between the pion form factor, the $\tau$ decay rate and $\pi^+\pi^-$ production in $e^+e^-$ collisions, one may deduce a relation between the differential effective luminosity from $\tau$ decays and the true luminosity of a “$\tau$ factory” $L_\tau$.

$$\frac{1}{L_\tau} \frac{dL_{\text{eff}}}{d(Q^2/m_\tau^2)} = \frac{\Gamma_e}{\Gamma_\tau} \cos \theta \frac{Q^2}{m_\tau^2} (1 - \frac{Q^2}{m_\tau^2})^2 (1 + 2 \frac{Q^2}{m_\tau^2}) \frac{4m_\tau^2}{s_\tau} (1 + 2 \frac{m_\tau^2}{s}) \beta_\tau$$

(1)

On top of the $Z$ resonance this result has to be multiplied by $9 \times \text{Br}(Z \rightarrow \tau^+\tau^-)^2/\alpha^2$. The differential luminosity is shown in Fig.2 for $\sqrt{s} = 4.2\text{GeV}$. The shape of this curve is of course energy independent, the scale is modified by a factor 0.24 for $\sqrt{s} = 10\text{GeV}$ and by 0.56 on top of the $Z$ resonance, respectively.

Axial current

The theoretical prediction for three pion and quite generally for three meson states is on less safe grounds, and the rates cannot be related to other experimental observables. For small pion momenta one may invoke the prediction based on the chiral Lagrangian, which determines the form and the normalisation of the relevant matrix element [8]

$$\langle \pi^- (p_1)\pi^- (p_2)\pi^+ (p_3) | J_\alpha(0) | 0 \rangle \equiv J_\alpha = -i \frac{2\sqrt{2}}{3f_\pi} \left( g_{\alpha\beta} - \frac{Q_\alpha Q_\beta}{Q^2} \right) \left( p_1^\alpha - p_3^\alpha \right) \left( p_2^\beta - p_3^\beta \right)$$

(2)

This parameterfree prediction is reliable in the region of small $Q^2 = (p_1 + p_2 + p_3)^2$ where one is, however, confronted with low statistics. For increasing $Q^2$ and increasing $s_k = (p_i + p_j)^2$ these calculations have to be supplemented by form factors deduced from vector dominance models.
This introduces some dependence on factors which can be studied experimentally by investigating distributions in the Dalitz plot and angular distributions. Satisfactory agreement between theory and experiment is obtained for the three pion mode \cite{7}. Predictions for three meson states involving kaons and eta’s are also available \cite{9}. The rates and shapes of the distributions exhibit a more drastic dependence on the presently unknown form factors.

The decay rates involving three different mesons (for example $K^-\pi^-K^+$, $K^-\pi^-\pi^+$ or $\eta\pi^-\pi^0$) allow for axial vector and vector current induced amplitudes at the same time. The latter can be related to the anomaly \cite{10,9}, the former is again given by the chiral Lagrangian. As a consequence of the large kaon mass and the high threshold the predictions are fairly sensitive towards the (model dependent) assumptions on the interpolating form factors. These form factors have been implemented in the Monte Carlo program TAUOLA \cite{11} which allows to test and simulate the distributions in the Dalitz plot as well as the angular distributions discussed below.

3 Structure functions and angular distributions

The information that can be deduced from a full analysis is encoded in the hadronic tensor $H_{\mu\nu} = J_\mu J^*_\nu$. For a final state consisting of (pseudo-)scalar mesons only, $H_{\mu\nu}$ corresponds to 16 real functions of $Q^2$, $s_1$ and $s_2$. For a three pion state in the spin one configuration, the most general current is given by

$$J_\alpha = (g_{\alpha\beta} - \frac{Q\alpha Q\beta}{Q^2})(p_1^\alpha - p_3^\alpha)F(s_1, s_2, Q^2) + (1 \leftrightarrow 2)$$

In the three pion rest frame $J_0 = 0$ and $\vec{J}$ is confined to the plane spanned by $\vec{q}_1$ and $\vec{q}_2$ and therefore has only two independent components. The tensor $H_{\mu\nu}$ is therefore determined by four real functions, which in turn allow to reconstruct the form factors $F$ introduced above. In the hadronic rest frame and with the coordinates 1 and 2 in the $\vec{q}_1, \vec{q}_2$ plane a convenient choice

Figure 2: Left: Differential effective luminosity at 4.2 GeV as defined in (1).
Right: Normalised structure functions $w_C/w_A$, $w_D/w_A$ and $w_E/w_A$ as functions of $Q^2$. 
for the structure functions $W_i$ which build up the hadronic tensor reads as follows

\[
W_A = H^{11} + H^{22}
\]

\[
W_C = H^{11} - H^{22}
\]

\[
W_D = 2\text{Re}H^{12}
\]

\[
W_E = 2\text{Im}H^{12}
\]

$W_A$ governs the rate and the distribution in the Dalitz plot. The angular distributions are most easily characterized in the hadron rest frame. $W_C$ and $W_E$ multiply the odd and even parts of the distribution of the normal on the three pion plane and $W_D$ describes the rotations within this plane. (In passing it should be mentioned that also $\tau \to \nu\pi\omega(\rightarrow 3\pi)$ can be studied with similar techniques.) For an unpolarized $\tau$ one predicts

\[
\frac{dN}{d\cos\beta d\gamma} \propto [(1 - \frac{m^2_{\tau}}{Q^2})(1 + \cos^2\beta) + 2\frac{m^2_{\tau}}{Q^2}]W_A - (1 - \frac{m^2_{\tau}}{Q^2})\sin^2\beta(\cos 2\gamma W_C - \sin 2\gamma W_D) + 2\cos\beta W_E
\]

with the angles $\beta$ and $\gamma$ as defined in Fig. 3. The structure functions, averaged over $s_1$ and $s_2$, are shown in Fig. 2 as functions of $Q^2$.

In practice the $\tau$ rest frame and hence $\vec{n}_{\tau}$, that is the direction of the $\tau$ as seen from the hadron rest frame, are unknown as a consequence of the missing neutrino momentum. However, one may in this case replace $\vec{n}_{\tau}$ by $\vec{n}_L$, the direction of the lab as seen from the hadron rest frame. The analysis then involves the angle $\theta$ between $\vec{n}_{\tau}(\text{lab})$, (the direction of flight of the $\tau$ as seen from the lab), and $\vec{n}_H(\tau)$ (the direction of the hadronic system as seen from the $\tau$) and, furthermore, the angle $\psi$ between the lab and the $\tau$ directions, as seen from the hadronic
system. Both angles can be expressed by $x$, the energy of the hadronic system in the lab

$$
\cos \theta = \tilde{n}_{\tau}(\text{lab}) \tilde{n}_H(\tau) = \frac{2xm_{\tau}^2 - m_{\tau}^2 - Q^2}{(m_{\tau}^2 - Q^2)\sqrt{1 - 4m_{\tau}^2/s}}
$$

$$
\cos \psi = \tilde{n}_L(\text{had}) \tilde{n}_\tau(\text{had}) = \frac{x(m_{\tau}^2 + Q^2) - 2Q^2}{(m_{\tau}^2 - Q^2)\sqrt{1 - 4m_{\tau}^2/s}}
$$

The importance of the angle $\theta$ was originally observed in [17], the angle $\psi$ was introduced in [18]. In the nonrelativistic approximation the relation between $\theta, \psi$ and $\theta_L$ is indicated in Fig. 3. Including longitudinal $\tau$ polarisation $P_\tau$ the experimentally observable angular distribution can be cast into the following form [12, 13]:

$$
\frac{dN}{d\cos \theta d\cos \beta d\gamma} \propto [K_1(1 + \cos^2 \beta) + 2K_2 - (K_1 \sin^2 \psi + K_4 \sin 2\psi)(3 \cos^2 \beta - 1)/2]W_A
$$

$$
- [K_1(3 \cos^2 \psi - 1)/2 - 3/2K_4 \sin 2\psi] \sin^2 \beta (\cos^2 \gamma W_C - \sin^2 \gamma W_D)
$$

$$
+ 2[K_3 \cos \psi - K_5 \sin \psi] \cos \beta W_E
$$

with the coefficients $K_i$ defined as functions of $\theta, \psi$ and of $P_\tau$. The most general case, including a spin zero contribution of the hadronic matrix element, and the corresponding predictions relevant for the general three mesons final state can be found in [13]. These distributions can be exploited to determine the $\tau$ polarisation in $Z$ decays. Experimental studies of the sensitivity of the method demonstrate that the sensitivity is increased from about 0.23 to 0.45 (corresponding to a statistical error of $1/0.45\sqrt{N_{\text{evt}}}$) if the full angular distribution is incorporated [14, 13, 16]. The dependence on the hadronic matrix element is weak and could furthermore be reduced by the corresponding measurements at ARGUS and CLEO.

4 Tau kinematics from impact parameters

Figure 4: Kinematic configuration indicating the relative orientation of the hadronic tracks, the $\tau$ directions and the vector $\vec{d}_{\text{min}}$. 
A further increase in sensitivity could be achieved if the \( \tau \) direction could be reconstructed experimentally, such that \( \vec{n}_L \) could be replaced by \( \vec{n}_\tau \) and the simpler eq. 4 would be applicable. Theoretical predictions of the full hadron distribution from the decay of an arbitrarily polarised \( \tau \) are given in appendix B of [13].

As shown in [19], the \( \tau \) direction can be reconstructed if the hadron tracks are measured with the help of microvertex detectors, even if the production vertex is unknown as a consequence of the large beam spot.

Let us assume that both \( \tau \) decay into one charged hadron each and that both charged tracks can be measured with high precision. The direction \( \vec{d}_{\min} \) of the minimal distance between the two nonintersecting charged tracks (Fig.4) resolves the ambiguity and introduces two additional constraints that can be used to reduce the measurement errors. The \( \tau^+ \) and \( \tau^- \) decay points and their original direction of flight can then be determined as follows.

The angles \( \theta_L^\pm \) between the \( \tau^\pm \) and the hadron \( h^\pm \) directions respectively as defined in the lab frame are given by the energies of \( h^+ \) and \( h^- \) [18]:

\[
\cos \theta_L^- = \frac{\gamma x_- - (1 + r_-^2)/2\gamma}{\beta \sqrt{\gamma^2 x_-^2 - r_-^2}} \\
\sin \theta_L^- = \sqrt{\frac{(1 - r_-^2)^2/4 - (x_- - (1 + r_-^2)/2)^2/\beta^2}{\gamma^2 x_-^2 - r_-^2}} \\
x_- = E_{h^-}/E_\tau \quad r_- = m_{h^-}/m_\tau
\]

(5)

(6)

(7)

and similarly for \( \cos \theta_L^+ \) and \( \sin \theta_L^+ \).

The velocity \( \beta \), and the boost factor \( \gamma \) refer to the \( \tau \) in the lab frame.

The original \( \tau^- \) direction must therefore lie on the cone of opening angle \( \theta_L^- \) around the direction of \( h^- \) and on the cone of opening angle \( \theta_L^+ \) around the reflected direction of \( h^+ \). The extremal configuration where \( \theta_L^+ \) or \( \theta_L^- \) assume the values 0 or \( \pi \), or where the two cones touch in one line, leads to a unique solution for the \( \tau \) direction. In general a twofold ambiguity arises, as is obvious from this geometric argument. The cosine of the relative azimuthal angle \( \varphi \) between the directions of \( h^+ \) and \( h^- \) denoted by \( \vec{n}_- \) and \( \vec{n}_+ \) can be calculated from the momenta and energies of \( h^+ \) and \( h^- \) as follows: In the coordinate frame (see Fig.4) with the \( z \) axis pointing along the direction of \( \tau^- \) and with \( \vec{n}_- \) in the \( xz \) plane and positive \( x \) component

\[
\vec{p}_- / |\vec{p}_-| \equiv \vec{n}_- = \begin{pmatrix} \sin \theta_L^- \\ 0 \\ \cos \theta_L^- \end{pmatrix} \\
\vec{p}_+ / |\vec{p}_+| \equiv \vec{n}_+ = \begin{pmatrix} \sin \theta_L^+ \cos \varphi \\ \sin \theta_L^+ \sin \varphi \\ -\cos \theta_L^+ \end{pmatrix}
\]

(8)

and \( \cos \varphi \) can be determined from

\[
\vec{n}_- \vec{n}_+ = -\cos \theta_L^- \cos \theta_L^+ + \sin \theta_L^- \sin \theta_L^+ \cos \varphi
\]

(9)

The well-known twofold ambiguity in \( \varphi \) is evident from this formula.

Additional information can be drawn from the precise determination of tracks close to the production point. Three-prong decays allow to reconstruct the decay vertex and the ambiguity can be trivially resolved.

However, single-prong events may also serve this purpose. Let us first consider decays into one charged hadron on each side. Their tracks and in particular the vector \( \vec{d}_{\min} \) of closest
approach (Fig.4) can be measured with the help of microvertex detectors. The vector pointing from the \( \tau^- \) to the \( \tau^+ \) decay vertex

\[
\vec{d} \equiv \vec{\tau}^+ - \vec{\tau}^- = - l \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]  

(10)
is oriented by definition into the negative \( z \) direction (\( l > 0 \)). The vector \( \vec{d}_{\text{min}} \) can on the one hand be measured, on the other hand calculated from \( \vec{d}, \vec{n}_+, \text{and} \vec{n}_-: \)

\[
\vec{d}_{\text{min}} = \vec{d} + [(\vec{d}\vec{n}_+ \vec{n}_+ - \vec{d}\vec{n}_-)\vec{n}_- + (\vec{d}\vec{n}_- \vec{n}_+ - \vec{d}\vec{n}_+)\vec{n}_+)/[(1 - (\vec{n}_-\vec{n}_+))^2] \]

(11)
The sign of the projection of \( \vec{d}_{\text{min}} \) on \( \vec{n}_+ \times \vec{n}_- \) then determines the sign of \( \varphi \) and hence resolves the ambiguity.

\[
\vec{d}_{\text{min}}(\vec{n}_+ \times \vec{n}_-) = l \sin \theta^L_+ \sin \theta^L_- \sin \varphi
\]

(12)
The length of the projection determines \( l \) and hence provides a measurement of the lifetimes of \( \tau^+ \) plus \( \tau^- \). Exploiting the fact that \( \vec{d}\vec{n}_- = -l \cos \theta^L_- \) and \( \vec{d}\vec{n}_+ = l \cos \theta^L_+ \) the direction of \( \vec{d} \) can be geometrically constructed by inverting (11):

\[
\vec{d}/l = \vec{d}_{\text{min}}/l - [(\cos \theta^L_+ \vec{n}_+ \vec{n}_- + \cos \theta^L_-)\vec{n}_- + (- \cos \theta^L_- \vec{n}_+ \vec{n}_- - \cos \theta^L_+)\vec{n}_+]/(1 - (\vec{n}_-\vec{n}_+)^2)
\]

(13)
The generalization of this method to decays into multihadron states with one or several neutrals and a more detailed discussion of the constraints resulting from this method can be found in [19].

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