Research Article

The Cauchy Problem for Parabolic Equations with Degeneration

Ivan Pukal’skii and Bohdan Yashan

Department of Differential Equations, Yuriy Fedkovych Chernivtsi National University, 28 Universitetska St., Chernivtsi, Ukraine 58012

Correspondence should be addressed to Bohdan Yashan; b.yashan@chnu.edu.ua

Received 12 February 2020; Accepted 26 May 2020; Published 16 June 2020

Copyright © 2020 Ivan Pukal’skii and Bohdan Yashan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Annotation. For a second-order parabolic equation, the multipoint in time Cauchy problem is considered. The coefficients of the equation and the boundary condition have power singularities of arbitrary order in time and space variables on a certain set of points. Conditions for the existence and uniqueness of the solution of the problem in Hölder spaces with power weight are found.

1. Introduction

One of the important directions of development of the modern theory of partial differential equations is the study of nonlocal boundary value problems for different types of differential equations and partial differential equation systems and establishing the conditions for their correct solvability. Such problems arise when modeling different phenomena and processes in modern science, quantum mechanics, technology, economics, etc.

Multipoint problems for partial differential equations were studied in the papers by Ptashnyk and his disciples. In particular, paper [1] is dedicated to multipoint problems for partial differential equations, not resolved relative to the highest time derivative. The question of the existence and qualitative properties of solutions for equations with limited order of degeneration has been studied in papers [2–4]. The Dirichlet problem with impulse action for a parabolic equation with power singularities of arbitrary order on time and spatial variables is investigated in paper [5]. In paper [6], a multipoint one-sided boundary value problem is studied. In paper [7], presented is the result of research of optimal control of the system described by the problem with an oblique derivative and an integral condition on time variable for parabolic equations with power singularities of arbitrary order. For a second-order parabolic equation, a multipoint (in time) problem with oblique derivative is considered in paper [8]. Conditions for the existence and uniqueness of solution of the posed problem in Hölder spaces with power weight are established.

In this paper, we investigate a multipoint time-varying Cauchy problem for a parabolic second-order equation with power singularities and arbitrary order degenerations in the coefficients of spatial and time variables at some set of points. We also find conditions for the existence and uniqueness of the solution of formulated problem in Hölder spaces with power weight.

2. Statement of the Problem and Main Result

Let \(\eta, t_0, t_1, \ldots, t_N, t_{N+1}\) be fixed positive numbers, \(0 \leq t_0 < \cdots < t_{N+1}\), \(t_0 < \eta < t_{N+1}\), \(\lambda \in \{1, 2, \ldots, N\}\), and let \(\Omega\) be some bounded domain in \(R^{n-1}\), \(Q(0) = \{(t, x) \mid t \in [t_0 ; t_{N+1}], x \in \Omega\} \cup \{(t, x) \mid t = \eta, x \in \mathbb{R}^n \setminus \bar{\Omega}\}\).

Let us consider the problem of finding a function \(u(x, t)\) in the domain \(\Pi = [t_0, t_{N+1}] \times \mathbb{R}^d\), which, for \(t \neq t_3\), \((t, x) \in Q(0)\) satisfies the equation

\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{n} A_i(t, x) \frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} A_i(t, x) \frac{\partial}{\partial t} \frac{x_i}{\partial t} + A_0(t, x) u(t, x) = f(t, x),
\]

(1)
and multipoint conditions for variable $t$

$$u(t_k + 0, x) = \varphi_k(x), \quad k \in \{0, 1, \ldots, N\}. \quad (2)$$

The power singularities of coefficients of the differential equation (1) at the point $P(t, x) \in \Pi \setminus Q_0$ characterize functions $s_1(\beta_1^{(1)}, t), s_2(\beta_2^{(2)}, x)$:

$$s_1(\beta_1^{(1)}, t) = \begin{cases} |t - \eta|^{p_1}, & |t - \eta| \leq 1, \\ 1, & |t - \eta| \geq 1, \end{cases} \tag{3}$$

$$s_2(\beta_2^{(2)}, x) = \begin{cases} \rho^{\beta_2^{(2)}}(x), & \rho(x) \leq 1, \\ 1, & \rho(x) \geq 1, \end{cases}$$

where $\rho(x) = \inf_{x \in \Omega} |x - z|$, $\beta_1^{(v)} \in (-\infty, \infty)$, $\beta_1^{(v)} = (\beta_1^{(v)}, \ldots, \beta_n^{(v)})$, $v \in \{1, 2\}$, $\beta = (\beta_1^{(1)}, \beta_2^{(2)})$.

Let us denote that $l, \alpha, q^{(1)}, q^{(2)}, y^{(1)}, y^{(2)}, \mu^{(1)}, \mu^{(2)}$ are real numbers, $q^{(v)} \geq 0$, $y^{(v)} \geq 0$, $l \geq 0$, $\mu^{(v)} \geq 0$, $j \in \{0, 1, \ldots, n\}$, $[l]$ is the integer part of $l$, $\{l\} = l - [l]$, $x^{(1)} = (x_1^{(1)}, \ldots, x_i^{(1)}, \ldots, x_n^{(1)})$, $x^{(2)} = (x^{(1)}_1, \ldots, x^{(1)}_{r_1}, x^{(2)}_r, x^{(1)}_{r+1}, \ldots, x_n^{(1)})$, $P(t, x)$, $P_1(t^{(1)}, x^{(1)})$, $P_2(t^{(2)}, x^{(2)})$, $R_j(t^{(1)}, x^{(1)})$, $R_0(t^{(2)}, x^{(2)})$ are arbitrary points of the domain $Q^{(k)}$, $i \in \{1, 2, \ldots, n\}$.

Let $D$ be an arbitrary closed domain in $R^n$, $Q^{(k)} = [t_k, t_{k+1}) \times D$. $Q^{(k)} \subset \Pi^{(k)} = [t_k, t_{k+1}) \times R^n$.

We define the functional space in which we study problem (1) and (2).

$H^{l}(\gamma; \beta; q; \Pi)$ is a set of functions $u(t, x)$ of space $L_1(\Pi)$, which are having continuous partial derivatives in $Q^{(k)} \setminus Q_0$ of the form $\partial_i \partial^r u$, $2i + |r| \leq [l]$ and a finite value of the norm

$$||u; y; \beta; q; \Pi|| = \sup_k \left\{ \sup_{Q^{(k)}} \left\| u \right\| \right\} = ||u, Q||_0,$$

$$||u; y; \beta; q; \Pi|| = \sup_k \left\{ \sum_{2i + |r| \leq [l]} \left\| u; y; \beta; q; Q^{(k)} \right\| \right\} + \left\{ u; y; \beta; q; Q^{(k)} \right\} / l$$

$$= \sup_k \left\{ \sum_{2i + |r| \leq [l]} \left\| u; y; \beta; q; Q^{(k)} \right\| \right\} \left( 1 + \sum_{(P_1, P_2) \in \Omega^{(k)}} \sum_{P_1} \sum_{P_2} \int_{\Pi^{(k)}} s_1 \left( \frac{q^{(1)} + (2i + |r|)y^{(1)}, t^{(1)}}{l} \right) \partial_i \partial^r u(P_1) \partial_i \partial^r u(P_2) - \partial_i \partial^r u(P_1) \right.$$ \n
$$\times \left( 1 + \frac{\partial_i \partial^r u(P_1)}{l} \right) s_2 \left( \frac{q^{(1)} + (2i + |r|)y^{(1)}, t^{(1)}}{l} \right) s_1 \left( \frac{q^{(1)} + (2i + |r|)y^{(1)}, t^{(1)}}{l} \right) \right\}.$$  

Marked here, $|r| = r_1 + r_2 + \cdots + r_{10}$, $s_1(q, \tilde{t}) = \min \left( s_2(q, \tilde{t}), \frac{1}{2} \right)$, $s_2(q, \tilde{t}) = \min \left( s_2(q, \tilde{t}), s_2(q, \tilde{t}) \right)$. We assume that the initial problems (1) and (2) satisfy the following conditions:

(a) For the arbitrary vector $\xi = (\xi_1, \ldots, \xi_n)$, $\forall (t, x) \in \Pi \setminus Q_0$, the following inequality

$$c_1 |\xi|^2 \leq \sum_{i=1}^{n} s_1 \left( \beta_1^{(1)}, t \right) s_1 \left( \beta_1^{(1)}, t \right) s_2 \left( \beta_2^{(2)}, x \right) s_2$$

$$\cdot \left( \beta_2^{(2)}, x \right) A_{s_1(t, x)\xi, \xi} \leq c_2 |\xi|^2$$

is true, where $c_1, c_2$ are fixed positive constants and $s_1(\mu^{(1)}$, $t) s_2(\mu^{(2)}$, $x) A_{s} \in H^a(\gamma; \beta; 0; \Pi)$. $s_1(\mu^{(1)}, t) s_2(\mu^{(2)}, x) A_{s} \in H^a$
(γ; β; 0; Π), A_0 ≥ -a, a > 0, s_1(β_1(t), t) s_2(β_2(t), t) s_3(β_3(t), x) × s_4(β_4(t), x) A_{i j} ∈ H^6(γ; β; 0; Π), γ_0^v = max \{ max_i (1 + β_i^v), \max_i (1 + β_i^v), \mu_0^v, \mu_0^v, 2 \}, v ∈ \{1, 2\}.

(b) Functions f ∈ H^6(γ; β; μ_0; Π), Ψ_0 ∈ H^{2+n}(γ; β; 0; R^6), Ψ_1 ∈ H^{2+n}(γ; β; μ_0; Π \cap (t = t_k)), γ = (0; γ_0^2), \hat{β} = (0; β^2)

Let us formulate the main result of the paper.

**Theorem 1.** Let conditions (a) and (b) be satisfied for problem (1) and (2). Then, there exists a unique solution of problem (1) and (2) in the space H^{2+n}(γ; β; 0; Π), and the following estimate is correct:

\[ \| u_\gamma; β; 0; Π \|_{x \geq c} \leq \sup \{ \| Ψ_0 \|_{x \geq c}, \| f \|_{x \geq c} \} \]

To study problem (1) and (2), we construct a sequence of solutions of problems with a smooth coefficient limit value of which is the solution of problem (1) and (2).

**3. Evaluation of Solutions of Problems with Smooth Coefficients**

Let Π^{(k)}_m = Π \cap \{(t, x) ∈ Π | m^{-1}_1, s_2(1, x) ≥ m^{-1}_2 \}, m = (m_1, m_2), m_1 > 1, m_2 > 1 be a sequence of domains that for m_1 → ∞, m_2 → ∞ converges to Π^{(k)}.

In the domain Π_t, we consider the problem of finding the functions u_m(t, x) that satisfy the equations

\[ (L_1 u_m)(t, x) = \left( \partial_t - \sum_{i=1}^{n} a_{i j}(t, x) \partial_x \partial_x + \sum_{i=1}^{n} a_i(t, x) \partial_i \right) u_m(t, x) = f_m(t, x), \]

and the time variable t condition

\[ u_m(t_k + 0, x) = Ψ_0^{(k)}(x), \ x ∈ Q_k \cap (t = t_k). \]

Here, the coefficients a_{i j}, a_i, a_0, and functions f_m, Ψ_m^{(k)}, into the domains Π^{(k)}_m coincide with A_{i j}, A_i, A_0, f, Ψ_k, respectively, and in the domains Π \setminus Π^{(k)}_m are continuous prolongations of coefficients A_{i j}, A_i, A_0 and functions f, Ψ_k from domains Π^{(k)}_m into domains Π \setminus Π^{(k)}_m with preservation of their smoothness and norm ([9], p. 82).

To solve problem (7) and (8), we have a correct theorem.

**Theorem 2.** Let u_m(t, x) be the classical solutions of problem (7) and (8) in the domain Π and let conditions (a) and (b) be satisfied. Then, for u_m(t, x), the following estimate

\[ \| u_m; Π^{(k)} \|_0 ≤ \| Ψ_m^{(k)}; Π^{(k)} \cap (t = t_k) \|_0 + \| f_m a_0^{-1}; Π^{(k)} \|_0 \]

is true.

**Proof.** Let max_{Q_{x}} u_m(t, x) = u_m(M_1). If M_1 ∈ Q^{(k)}, then at the point M_1, the following correlations

\[ \partial_t u_m(M_1) ≥ 0, \]

\[ \partial_\gamma u_m(M_1) = 0, \]

\[ \sum_{i,j=1}^{n} a_{i j}(M_1) \partial_x \partial_y u_m(M_1) ≤ 0 \]

are true and equation (7) is satisfied. Considering (10) and equation (7) at the point M_1, the inequality

\[ u_m(M_1) ≤ \sup (f a_0^{-1}) \]

is correct.

Let min_{Q_{x}} u_m(t, x) = u_m(M_2). If M_2 ∈ Q^{(k)}, then at the point M_2, the following correlations

\[ \partial_t u_m(M_2) ≤ 0, \]

\[ \partial_\gamma u_m(M_2) = 0, \]

\[ \sum_{i,j=1}^{n} a_{i j}(M_2) \partial_x \partial_y u_m(M_2) ≥ 0 \]

are true and equation (7) is satisfied. Considering correlation (12) and equation (7) at the point M_2, we have

\[ u_m(M_2) ≥ \inf (f a_0^{-1}). \]

In the case of M_1 ∈ Q^{(k)} \cap (t = t_k) or M_2 ∈ Q^{(k)} \cap (t = t_k) from condition (8), we obtain

\[ \| u_m \|_0 ≤ \| Ψ_m^{(k)}; Q^{(k)} \cap (t = t_k) \|_0. \]

Considering inequality (11), (13), and (14), we obtain

\[ \| u_m; Π^{(k)} \|_0 ≤ \| f_m a_0^{-1}; Π^{(k)} \|_0 + \| Ψ_m^{(k)}; Π^{(k)} \cap (t = t_k) \|_0 \]

The theorem is proved.

Now, we find estimates of the derivatives of solutions u_m(t, x). In the space C^1(Π), we introduce a norm \[ \| u_m; γ; β; q; Π \|_l \] which is equivalent, at fixed m_1, m_2, to the Hölder norm, which is defined by the same way as the
norm \( \| u; y; \beta; q; \Pi \|_p \); only, instead of functions \( s_1(q^{(1)}, t) \) and \( s_2(q^{(2)}, x) \), we take \( d_1(q^{(1)}, t) \) and \( d_2(q^{(2)}, x) \), respectively,

\[
\begin{align*}
d_1(q^{(1)}, t) &= \begin{cases} 
\max (s_1(q^{(1)}, t), m_1^{-q^{(1)}}), \quad q^{(1)} \geq 0, \\
\min (s_1(q^{(1)}, t), m_1^{-q^{(1)}}), \quad q^{(1)} < 0,
\end{cases} \\
d_2(q^{(2)}, x) &= \begin{cases} 
\max (s_2(q^{(2)}, x), m_2^{-q^{(2)}}), \quad q^{(2)} \geq 0, \\
\min (s_2(q^{(2)}, x), m_2^{-q^{(2)}}), \quad q^{(2)} < 0.
\end{cases}
\end{align*}
\]

(16)

**Theorem 3.** Let conditions (a) and (b) be satisfied. Then, for the solution of problem (7) and (8), the estimate

\[
\begin{align*}
\frac{1}{2} \| u_m; y; \beta; 0; \Pi^{(k)} \|_{L^2} &\leq \epsilon \left( \| f; y; \beta; u_0; \Pi^{(k)} \|_a \right. \\
+ &\left. \| \Phi k; y; \beta; 0; \Pi^{(k)} \cap (t = t_{k-1}) \|_{L^2} \right)
\end{align*}
\]

(17)

is true.

**Proof.** Using the definition of the norm and interpolation inequalities from [9], we have

\[
\begin{align*}
\| u_m; y; \beta; 0; \Pi^{(k)} \|_{L^2} &\leq (1 + \epsilon^n) \left( \| u_m; y; \beta; 0; \Pi^{(k)} \|_a \right.
\end{align*}
\]

(18)

where \( \epsilon \) is an arbitrary real number \( \epsilon \in (0, 1) \). Hence, it is sufficient to estimate the seminorm \( \langle u_m; y; \beta; 0; \Pi^{(k)} \rangle_{L^2} \). As follows from the definition of seminorm, there exist in \( \Pi^{(k)} \) points \( P_1, P_2, R_0 \), for which one of the inequalities

\[
\begin{align*}
\frac{1}{2} \left\| u_m; y; \beta; 0; \Pi^{(k)} \right\|_{L^2} &\leq E_{\mu}, \quad \mu \in \{1, 2\},
\end{align*}
\]

(19)

is true, where

\[
E_2 \equiv \sum_{2\sigma|t| = 2} d_1 \left( (2 + \alpha) y^{(1)}; \tilde{t} \right) d_2 \left( r(\gamma^{(2)} - \beta_v^{(2)}); x^{(1)} \right) \prod_{i=1}^n d_i
\]

\[
\cdot \left( -r(\beta_v^{(1)}; t^{(2)}) \right) \times d_2 \left( r(\gamma^{(2)} - \beta_v^{(2)}); \check{x} \right) \left| \partial \tilde{t} \partial x u_m(P_2) \right|
\]

\[
\cdot \left( \alpha(\gamma^{(2)} - \beta_v^{(2)}); \check{x} \right),
\]

(20)

If \( |x_v^{(1)} - x_v^{(2)}| \geq (en^{-1}/4) d_1(y^{(1)}, \tilde{t}) d_2(y^{(2)} - \beta_v^{(2)}, \check{x}) \equiv T_1 \), and \( e_1 \) is an arbitrary real number from (0,1), then

\[
E_2 \leq 2e_1 \| u_m; y; \beta; 0; \Pi^{(k)} \|_{L^2}.
\]

(21)

If \( |t^{(1)} - t^{(2)}| \geq (c_1^2/16) d_1(2y^{(1)}, \tilde{t}) d_2(y^{(2)} - \beta_v^{(2)}, \check{x}) \equiv T_2 \), then

\[
E_2 \leq 2e_1 \| u_m; y; \beta; 0; \Pi^{(k)} \|_{L^2}.
\]

(22)

Applying the interpolation inequalities to (21) and (22), we find

\[
E_\mu \leq \epsilon \| u_m; y; \beta; 0; \Pi^{(k)} \|_{L^2} + c(\epsilon) \| u_m; \Pi^{(k)} \|_{L^2}.
\]

(23)

Let \( |x_v^{(1)} - x_v^{(2)}| \leq T_1 \), and let \( |t^{(1)} - t^{(2)}| \leq T_2 \). We assume that \( |x_v^{(1)} - \xi_v| \geq 4T_1, \xi \in \partial D \) and

\[
\begin{align*}
d_2(y^{(2)}, \check{x}) &= \min \left( d_1(y^{(2)}, x^{(1)}), d_2(y^{(2)}, x^{(3)}) \right) \\
d_1(y^{(2)}, \tilde{t}) &= \min \left( d_1(y^{(1)}, t^{(1)}), d_1(y^{(1)}, t^{(2)}) \right)
\end{align*}
\]

(24)

\[
P_i(t^{(1)}, x^{(1)}) \in \Pi^{(k)}, k \in \{0, 1, \ldots, N\}. \] In the domain \( Q^{(k)} \), we write the problem (7) and (8) in the form

\[
\begin{align*}
\partial_t - \sum_{i=1}^n a_{ij}(P_i) \partial_{\xi_i} u_m = \sum_{i=1}^n (a_{ij}(P_i) - a_{ij}(P_0)) \partial_{\xi_i} u_{m0} \\
- \sum_{i=2}^n a_{ij}(P_i) \partial_{\xi_i} u_m - a_{ij}(P_0) u_{m0} + f_m(t, x)
\end{align*}
\]

(25)

\[
u_m(t_k + 0, x) = Q_m(n, t_k, x).
\]

(26)

Let \( \Pi_t^{(k)} \) be a domain from \( Q^{(k)}, \Pi_t^{(k)} = \{ (t, x) \in Q^{(k)} \mid x_v - x_v^{(1)} \leq \tau T_1, v \in \{1, 2\}, |t - t^{(1)}| \leq \tau^2 T_2 \}. \) Performing the substitution,

\[
\begin{align*}
u_m(t, x) &= v_m(t, y), \\
x_v &= d_1(\beta_v^{(1)}, t^{(1)}) d_2(\beta_v^{(2)}, x^{(1)}) y_v
\end{align*}
\]

(27)
in problem (25) and (26), we obtain
\[
(L_2 v_m)(t, y) = \left[ \partial_t - \sum_{i,j=1}^n a_{ij}(P_1) d_1 (\beta_i^{(1)}, t^{(1)}) d_1 (\beta_j^{(2)}, t^{(1)}) \right] \nu_m
d_2 \left( \beta_i^{(2)}, x^{(1)} \right) d_2 \left( \beta_j^{(2)}, x^{(1)} \right) \partial_y \nu_m
= F_m(t, \tilde{y}), \quad \nu_m(t_k + 0, y) = \Phi_m^{(k)}(t_k, \tilde{y}), \quad \nu_m(t, y) = \Phi_m^{(k)}(t_k, \tilde{y}),
\]
where \( \tilde{y} = (d_1(-\beta_1^{(1)}, t^{(1)}) d_2(-\beta_2^{(2)}, x^{(1)}), \ldots, d_1(-\beta_n^{(1)}, t^{(1)}) d_2(-\beta_n^{(2)}, x^{(1)}) x_n) \).

We denote \( \psi(t, y) = \{ (t, y) \in V_{1/2}^{(k)} : 0 \leq \psi(t, y) \leq 1 \} \)
\[
\psi(t, y) = \begin{cases} 
1, & (t, y) \in V_{1/2}^{(k)} : 0 \leq \psi(t, y) \leq 1, \\
0, & (t, y) \in V_{1/4}^{(k)} : |\partial_t \partial_y \psi| \leq c_n d_1 \left( -2i + |r| \right) y^{(1)} ; t^{(1)} \times d_2 \left( -2i + |r| \right) y^{(2)} ; x^{(1)} \right).
\]

We denote the function \( Z_m(t, y) = v_m(t, y) \psi(t, y) \) which
is a solution of the Cauchy problem
\[
(L_2 Z_m)(t, y) = \sum_{i,j=1}^n a_{ij}(P_1) d_1 (\beta_i^{(1)}, t^{(1)}) d_1 (\beta_j^{(2)}, t^{(1)}) d_2 \left( \beta_i^{(2)}, x^{(1)} \right) d_2 \left( \beta_j^{(2)}, x^{(1)} \right) \partial_y \psi
+ \left[ \partial_y \nu_m \partial_y \psi + \nu_m \partial_y \nu_m \partial_y \psi + v_m \partial_y \nu_m \partial_y \psi \right] - v_m \partial_y \psi + \psi F_m(t, \tilde{y})
= F_m^{(k)}(t, \tilde{y}, \psi),
\]
\[
Z_m(t_k + 0, x) = \Phi_m^{(k)}(t_k, \tilde{y}) \psi(t_k, y).
\]

On the basis of Theorem 5.3 from [1], p. 364, for the
solution of problem (31) and (32) and arbitrary points \((M_1, M_2) \subset V_{1/2}^{(k)} \) inequalities
\[
|d^\alpha (M_1, M_2) [\partial_t \partial_y \nu_m(M_1) - \partial_t \partial_y \nu_m(M_2)]|
\leq c \left( \| F_m^{(k)} \|_{C^{2+\alpha}(V_{1/2}^{(k)})} + \| \psi \Phi_m^{(k)} \|_{C^{2+\alpha}(V_{1/4}^{(k)})} \right)
\]
are true, where \( 2i + |r| = 2, d(M_1, M_2) \) is the parabolic distance between the points \( M_1 \) and \( M_2 \). With regard to the
properties of function \( \psi(t, y) \), we find estimates of the norms

The definition of the space \( H^l(y; \beta; q; Q) \) implies the satisfaction of inequalities
\[
c_1 \left\| v_m ; y : 0 ; 0 ; V_{3/4}^{(k)} \right\|_{l} \leq u_m ; y ; \beta : 0 ; \Pi_{3/4}^{(k)} \left\|_{l} \leq c_2 \left\| v_m ; y : 0 ; 0 ; V_{3/4}^{(k)} \right\|_{l}.
\]
Substituting (34) and (35) into (33) and returning to the variables \((t, x)\), we obtain inequalities
\[
E_m \leq c_1 \left( \left\| F_m ; \gamma ; \beta ; 2 \gamma ; \Pi^{(k)} \right\|_a + \left\| \Phi^{(k)} ; \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right\|_{2\alpha} + \left\| u_m ; \Pi^{(k)} \right\|_0 + \left\| u_m ; \gamma ; \beta ; 0 ; \Pi^{(k)} \right\|_2 \right) + \phi_k^{(k)} \left( \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right) \right) \right) \] (37)

Given the interpolation inequalities and estimates of the norm of each additive of the expressions \(F_m, \Phi^{(k)}\), we obtain the inequalities
\[
E_m \leq \left( \varepsilon^k (n + 2) + \varepsilon_1 C_1 \right) \left\| u_m ; \gamma ; \beta ; 0 ; \Pi^{(k)} \right\|_a + \phi_k^{(k)} \left( \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right) \right) \right) \] (38)

Using inequalities (23) and (37) and choosing \(\varepsilon_1\) and \(\varepsilon\) sufficiently small, we obtain the estimate
\[
\left\| u_m ; \gamma ; \beta ; 0 ; \Pi^{(k)} \right\|_a \leq \left( \left\| F_m ; \gamma ; \beta ; 0 ; \Pi^{(k)} \right\|_a + \phi_k^{(k)} \left( \gamma ; \beta ; 0 ; \Pi^{(k)} \cap (t = t_k) \right) \right) \right) \] (39)

Given the values of the expression \(F_m(t, x)\) and \(\Phi^{(k)}(t_k, x)\), for \(k = 1, 2, \cdots, N\), we have
\[
\left\| F_m ; \gamma ; \beta ; 2 \gamma ; \Pi^{(k)} \right\|_a \leq c \left( \left\| f_m ; \gamma ; \beta ; \mu_0 ; \Pi^{(k)} \right\|_a \right) \right) \] (40)

Because
\[
\left\| f_m ; \gamma ; \beta ; \mu_0 ; \Pi^{(k)} \right\|_a \leq c \left( \left\| f ; \gamma ; \beta ; \mu_0 ; \Pi^{(k)} \right\|_a \right) \right) \] (41)

then, given the estimate (9) and inequalities (39) and (40) for \(k = 0, 1, \cdots, N\), the estimate (17) is true. The theorem is proved.

**Proof of Theorem 4.** The right-hand side of inequality (17) is independent of \(m\), and the sequences
\[
\left\{ u_m^{(0)} \right\} = \{ u_m \}, \quad \left\{ u_m^{(1)} \right\} = \{ d_1 \left( y^{(1)} - \beta_i^{(1)}, t \right) d_2 \left( y^{(2)} - \beta_j^{(2)}, x \right) \right\}_{i, j, \alpha}, \quad \left\{ u_m^{(2)} \right\} = \{ d_1 \left( 2y^{(1)}, t \right) d_2 \left( 2y^{(2)}, x \right) \right\}_{\alpha}, \quad \left\{ u_m^{(3)} \right\} = \{ d_1 \left( y^{(1)} - \beta_i^{(1)}, t \right) d_2 \left( y^{(2)} - \beta_j^{(2)}, x \right) d_1 \left( y^{(1)} - \beta_i^{(1)}, t \right) \}
\] (42)

are uniformly bounded and equicontinuous in \(Q^{(k)}\). According to Arcel’s theorem, there exist subsequences \(\{ u_m^{(i)} \}\) which are uniformly convergent to \(\{ u_m^{(j)} \}\) in \(Q^{(k)}\) for \(m(j) \to \infty\), \(\mu \in \{ 0, 1, 2, 3 \}\). Passing to the limit as \(m(j) \to \infty\) in problem (7) and (8), we obtain that \(u(t, x) = u_0^{(0)}\) is the unique solution of problem (1) and (2) and \(u \in H^{2\alpha}(\gamma ; \beta ; 0 ; \Pi)\). The theorem is proved.

**4. Conclusions**

The necessary and sufficient conditions for the existence of the unique solution of a multipoint problem for parabolic equations with degeneration are established. Estimates of derivatives of the solution of the problem in the Hölder spaces with power weight are found. The order of the degree weight depends on the power of the degree features of the coefficients of the equation.

**Data Availability**

The article contains theoretical material. This paper has links to articles and textbooks from other researchers. The article does not use any data.

**Conflicts of Interest**

The authors declare that there is no conflict of interests regarding the publication of this article.

**References**

[1] I. S. Klyus and B. I. Ptashnyk, “A multipoint problem for partial differential equations unresolved with respect to the higher time derivative,” *Ukrainian Mathematical Journal*, vol. 51, no. 12, pp. 1813–1823, 1999.

[2] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural’tceva, *Linear and Quasilinear Equations of Parabolic Type*, Science, Moscow, 1967.

[3] M. I. Mattyuchuk, *Parabolic and Elliptic Problems with Singularities*, Ukrainian, Prut, Chernivtsi, 2003.

[4] I. D. Pukalskyi, *The Boundary Value Problems for Unevenly Parabolic and Elliptic Equations with Degeneration and Singularities*, Ukrainian, Chernivtsi, 2008.

[5] B. Yashan, “The boundary value dirichlet problem for parabolic equation with a pulse effect,” *Bukovinian Mathematical Journal*, vol. 6, no. 1-2, 2018.

[6] B. Yashan and I. Pukalskyi, “Multi-point boundary value problem within equalities for parabolic equations with degeneration,” *Bukovinian Mathematical Journal*, vol. 6, no. 3-4, pp. 118–126, 2018.

[7] I. Pukal’skii and B. Yashan, “Optimal control in a nonlocal boundary value problem with integral conditions for parabolic
equation with degeneration,” *Bukovinian Mathematical Journal*, vol. 7, no. 1, 2019.

[8] I. D. Pukal’s’kyi and B. O. Yashan, “Nonlocal multipoint (in time) problem with oblique derivative for a parabolic equation with degeneration,” *Journal of Mathematical Sciences*, vol. 247, no. 1, pp. 43–57, 2020.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, NJ, USA, 1964.