A remark on Kähler forms on symmetric products of Riemann surfaces

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Abstract

Users of Heegaard Floer homology may be reassured to know that it can be made to conform exactly to the standard analytic pattern of Lagrangian Floer homology. This follows from the following remark, which we prove using an argument of J. Varouchas: the natural singular Kähler form $\text{Sym}^n(\omega)$ on the $n$th symmetric product of a Kähler curve $(\Sigma, \omega)$ admits a cohomologous smoothing to a Kähler form which equals $\text{Sym}^n(\omega)$ away from a chosen neighbourhood of the diagonal.

In this note we consider branched coverings $\pi: X \to X'$ of complex manifolds—that is, holomorphic maps which are proper, surjective, and finite. The branch locus $B_\pi \subset X'$ of such a map is

$$B_\pi = \{ \pi(x) : x \in X, \ ker \, D_x(\pi) \neq 0 \}.$$ 

A $(C^\infty)$ smooth Kähler form $\omega$ on $X$ can be pushed forward—in the sense of currents, that is, of 2-forms with $L^1$ coefficients—to a closed current $\pi^* \omega$ on $X'$ which is smooth on $X' \setminus B_\pi$.

The following theorem is essentially due to Varouchas [2].

**Theorem 1.** Let $\pi: X \to X'$ be a branched covering of complex manifolds, and $\omega$ a Kähler form on $X$. Let $N$ be a neighbourhood of the branch locus in $X'$. Then there exists a Kähler form $\omega'$ on $X'$ such that

1. $(\pi_* \omega - \omega')|_{X' \setminus N} = 0$, and
2. $[\omega'] = \pi_* [\omega] \in H^2(X'; \mathbb{R})$.

The stated conclusion in [2] is simply that $X'$ admits a Kähler form. The purpose of this note is to explain this minor modification of Varouchas' argument, and to draw attention to the following example:

**Corollary 2.** Let $\Sigma$ be a Riemann surface with Kähler form $\omega$. Let $\pi: \Sigma^r \to \text{Sym}^r(\Sigma) = \Sigma^r / S_r$ be the projection map. Suppose that $N \subset \text{Sym}^r(\Sigma)$ is an open subset containing the (large) diagonal. Then there exists a Kähler form $\eta$ on $\text{Sym}^r(\Sigma)$ such that

1.
1. outside $\overline{\mathcal{N}}$, $\eta$ is the smooth push-forward $\pi_* (\omega \times r)$ of the product form;

2. $[\eta] = \pi_* [\omega \times r] \in H^2 (\text{Sym}^r (\Sigma); \mathbb{R})$.

As advertised in the abstract, this has a direct application to the Heegaard Floer theory of Ozsváth-Szabó [1]. There one has tori $T_\alpha$ and $T_\beta$ in $\text{Sym}^g (\Sigma)$, disjoint from the diagonal and Lagrangian for product forms $\omega \times r$; they remain Lagrangian for a globally-defined Kähler form $\eta$. The construction of Ozsváth-Szabó differs from the standard analytic framework of Lagrangian Floer homology only in the handling of energy bounds for the holomorphic disks; if one uses the form $\eta$, the energy has its usual cohomological interpretation, and so is constant on each homotopy class.

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**Definition 3.** Let $X$ be a complex manifold. A *Kähler cocycle* on $X$ is a collection $(U_i, \varphi_i)_{i \in I}$, where $(U_i)_{i \in I}$ is an open cover of $X$, and $\varphi_i : U_i \to \mathbb{R}$ is a function, such that for all $i, j \in I$,

1. $\varphi_i$ is strictly plurisubharmonic on $U_i$; and

2. $\varphi_i - \varphi_j$ is pluriharmonic on $U_i \cap U_j$.

One ascribes to the cocycle a property (continuity, smoothness, etc.) possessed by all the $\varphi_i$. Kähler cocycles are, by definition, upper semicontinuous.

Condition (1) means that the 2-current $dd^c \varphi_i$ is strictly positive on $U_i$; (2) means that these currents agree on overlaps, and are therefore restrictions of a 2-current $\omega$ on $X$ (closed and strictly positive). If the cocycle is $C^\infty$ then $\omega$ will be a Kähler form.

Varouchas’ *lemme principal* is the following. The proof uses the “regularised maximum” technique of Richberg and Demailly.

**Lemma 4.** Let $U, V, W, \Omega$ be open subsets of $\mathbb{C}^n$ with

$$U \subseteq V \subseteq W, \quad \Omega \subseteq W.$$ 

Let $\phi : W \to \mathbb{R}$ be continuous, strictly plurisubharmonic, and smooth on $\Omega$. Then there exists a function $\psi : W \to \mathbb{R}$, again continuous and strictly plurisubharmonic, equal to $\phi$ on $W \setminus \overline{\Omega}$ and smooth on $U \cup \Omega$.

One then passes from local to global by the following argument, which I give in detail since Varouchas’ stated conclusion is weaker here.
**Lemma 5.** Let \((U_i, \varphi_i)_{i \in I}\) be a continuous Kähler cocycle on the complex manifold \(X\). Suppose that \(X = X_1 \cup X_2\), with \(X_1\) and \(X_2\) open, and that the functions \(\varphi_i|_{U_i \cap X_1}\) are smooth. Then there exists a continuous function
\[
\chi: X \to \mathbb{R}, \quad \text{Supp}(\chi) \subset X_2,
\]
and a locally finite refinement
\[
V_j \subset U_{i(j)} \quad (j \in J)
\]
so that the family
\[
(V_j, \varphi_{i(j)}|_{V_j} + \chi|_{V_j})_{j \in J}
\]
is a smooth Kähler cocycle.

**Proof.** Refine the cover \((U_i)_{i \in I}\) to a countable, locally finite cover \((V_i)_{i \in I_{1 \cup I_2}}\) with the property that
\[
i \in I_\alpha \Rightarrow V_i \subset X_\alpha, \quad \alpha = 1, 2.
\]
For definiteness let us suppose both \(I_1\) and \(I_2\) are infinite; say \(I_\alpha = \mathbb{N} \times \{\alpha\}\). Find open subsets
\[
V''_i \Subset V'_i \Subset V_i
\]
such that \((V''_i)\) still covers \(X\), and set
\[
A_1 = \emptyset, \quad A_n = V''_{(1,2)} \cup \cdots \cup V''_{(n-1,2)}.
\]
So the sets \(A_n\) exhaust \(X_2 \setminus X_1\). Let \((V_i, \psi^n_i)_{i \in I_{1 \cup I_2}}\) be the Kähler cocycle induced from \((U_i, \varphi_i)_{i \in I}\) by the refinement.

**Claim:** there are Kähler cocycles \((V_i, \psi^n_i)\), where \(n = 1, 2, \ldots\) indexes the elements of \(I_2\), such that the following hold for all \(i \in I_1 \cup I_2\) and all \(n > 1\):

1. \(\psi^n_i\) is smooth on the set \(V_i \cap (X_1 \cup A_n)\).
2. There is a continuous function \(\chi_n: X \to \mathbb{R}\), with \(\text{Supp}(\chi_n) \subset V''_{(n-1,2)}\), such that
\[
\psi^n_i = \psi^{n-1}_i + \chi_n.
\]

We prove the claim by induction on \(n\). Apply the previous lemma to
\[
(U, V, W, \Omega) = (V''_{(n-1,2)}, V'_{(n-1,2)}, V_{(n-1,2)}, V_{(n-1,2)} \cap (X_1 \cup A_{n-1}))
\]
and to the function \(\psi^{n-1}_{(n-1,2)}\), obtaining a new function \(\psi^n_{(n-1,2)}\); let \(\chi_n = \psi^n_{(n-1,2)} - \psi^{n-1}_{(n-1,2)}\), extended by zero to all of \(X\), and use (2) to define the new cocycle. We have to verify (1), i.e. to prove smoothness of \(\psi^n_i\) at each \(x \in V_i \cap (X_1 \cup A_n)\). If \(x \notin V'_{(n-1,2)}\) then \(\chi_n(x) = 0\), but \(\psi^{n-1}_i\) was already smooth. If \(x \in V'_{(n-1,2)}\) then, near \(x\), \(\psi^n_i = (\psi^n_i - \psi^n_{(n-1,2)}) + \psi^n_{(n-1,2)} = (\psi^n_i - \psi^n_{(n-1,2)}) + \psi^n_{(n-1,2)}\), which is the sum of a pluriharmonic function and a smooth plurisubharmonic one. But a pluriharmonic function is smooth. By a similar argument, \(\psi^n_i\) is strictly plurisubharmonic.
Now define a function $\chi: X \to \mathbb{R}$ by the locally finite sum

$$\chi(x) = \sum_{n \geq 1} \chi_n(x).$$

Then $\psi_\infty^i(x) := \psi_1^i(x) + \chi(x)$ defines a Kähler cocycle. It is smooth, since on $X \setminus \bigcup V_{(n,2)} \subset X_1$, the original cocycle was smooth and has not been modified, while

$$V_{(n,2)} \subset X_1 \cup \bigcup A_k,$$

so smoothness on $V_{(n,2)}$ is guaranteed by (1). Hence $\chi$ has the required properties.

**Proof of Theorem 1.** Each fibre $\pi^{-1}(x')$, being finite, has a neighbourhood which is a disjoint union of open balls. Hence, using the $dd^c$-lemma, one can find a smooth Kähler cocycle $(U_i, \varphi_i)$ on $X$ such that each $U_i$ contains a fibre of $\pi$, with $\omega|_{U_i} = dd^c \varphi_i$. One can then find a locally finite cover $(U'_i)$ of $X'$ such that $U_i \supset \pi^{-1}(U'_i)$.

A general property of branched covers is that the push-forward $\pi_* f$ of a continuous function $f: X \to \mathbb{R}$ is again continuous (it is given by $\pi_* f(x') = \sum_{x \in \pi^{-1}(x')} f(x)$, where the points $x$ are taken with multiplicities). The family $(U'_i, \pi_* \varphi_i)$ on $X'$ is thus a continuous Kähler cocycle: plurisubharmonicity is clear away from $B_\pi$, hence everywhere by density; similarly for pluriharmonicity on overlaps.

Now let $N' \subset N$ be a smaller open neighbourhood of $B_\pi$. Apply the global smoothing lemma to $(U'_i, \pi_* \varphi_i)$ on $X'$, taking $X_1 = X' \setminus \overline{N}$ and $X_2 = N$. The output is a function $\chi: X' \to \mathbb{R}$ (as well as a refinement of $(U'_i)$, which we omit from the notation) such that

$$\omega_{X'} := dd^c (\pi_* \varphi_i + \chi) = \pi_* dd^c \varphi_i + dd^c \chi$$

is a well-defined 2-form with the right properties. Notice that, since $\chi$ is continuous, $dd^c \chi$ represents the zero cohomology class.

**References**

[1] Ozsváth, P., and Szabó, Z., *Holomorphic disks and topological invariants for closed three-manifolds*, math.SG/0101206, Ann. Math., to appear.

[2] Varouchas, J., *Stabilité de la classe des variétés Kähleriennes par certains morphismes propres*, Invent. Math. 77 (1984), 117–127.

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