Quasinormal modes and hidden conformal symmetry in the Reissner-Nordström black hole

Yong-Wan Kim $^{1,a}$, Yun Soo Myung$^{2,b}$, and Young-Jai Park$^{1,3,c}$

1 Center for Quantum Spacetime, Sogang University, Seoul 121-742, Korea
2 Institute of Basic Science and School of Computer Aided Science, Inje University, Gimhae 621-749, Korea
3 Department of Physics and Department of Global Service Management, Sogang University, Seoul 121-742, Korea

Abstract

It is shown that the scalar wave equation in the near-horizon limit respects a hidden SL(2,R) invariance in the Reissner-Nordström (RN) black hole space-times. We use the SL(2,R) symmetry to determine algebraically the purely imaginary quasinormal frequencies of the RN black hole. We confirm that these are exactly quasinormal modes of scalar perturbation around the near-horizon region of a near-extremal black hole.

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$^a$ywkim65@gmail.com
$^b$ysmyung@inje.ac.kr
$^c$yjpark@sogang.ac.kr
1 Introduction

It was known that the scalar wave equation in the near-region and low-energy limits enjoys a hidden conformal symmetry in the non-extremal Kerr black hole which is not an underlying symmetry of the spacetime itself [1, 2]. The existence of such a hidden symmetry originates from the observation that scattering amplitudes of scalar off a black hole are given in terms of hypergeometric functions [3, 4] which form representations of the conformal group SL(2,R). Importantly, this led to the conjecture that the non-extremal Kerr black hole with angular momentum $J$ is dual to a CFT$_2$ with the central charges $c_L = c_R = 12J$ [1], which provides exactly the Bekenstein-Hawking entropy of the Kerr black hole. It is also found that the low energy scalar-Kerr scattering amplitudes coincide with thermal correlators of a CFT$_2$.

Chen and Long [5] have shown that one can use the hidden conformal symmetry to algebraically determine quasinormal mode spectrums as descendants of a highest weight state in black hole spacetimes. On the other hand, the spin-2 and spin-3 quasinormal modes and frequencies around the BTZ black hole were constructed by the purely operator approach without any approximation [6, 7, 8].

Recently, the authors [9] have shown that the scalar wave equation in the near-region and low-energy limits enjoys a hidden SL(2,R) invariance in the Schwarzschild geometry. They have used the SL(2,R) symmetry to determine algebraically the quasinormal frequencies (QNFs) of the Schwarzschild black hole, and also shown that this yields the purely imaginary QNFs describing large damping. Explicitly, starting with the highest weight state $\Phi^{(0)}$ with $L_0 \Phi^{(0)} = h \Phi^{(0)}$ and $L_1 \Phi^{(0)} = 0$, all quasinormal modes could be constructed as descendants of $\Phi^{(n)} = (L_{-1})^n \Phi^{(0)}$ obtained by acting with $L_{-1}$ on the highest weight state $\Phi^{(0)}$. Then, one can read off the QNFs from the descendants.

We would like to mention that the method developed for the Kerr/CFT correspondence could not be directly applied to the Schwarzschild and RN black holes because there is no apparent AdS$_2$ structure in the near-horizon geometry of the non-extremal Schwarzschild and RN black holes. In this direction, a hidden conformal symmetry could be extracted by making a five-dimensional uplifted RN black hole [10]. However, this is not a genuine conformal symmetry developing in the four-dimensional RN black hole. Very recently, the authors [11] have found a hidden SL(2,R) symmetry in the near-region and low-energy limits of the RN black hole.

In this work, we will use the SL(2,R) symmetry to determine QNFs of the RN black hole algebraically. Employing the operator method, we derive quasinormal modes [12] and purely imaginary QNFs [11]. A key point in
deriving the quasinormal modes is to set the proper boundary conditions for the wave equation. As is well known, quasinormal modes are determined by solving a scalar wave equation around the RN black hole as well as imposing the boundary conditions: ingoing waves at the horizon and outgoing waves at infinity of asymptotically flat spacetime. It seems difficult to derive QNFs of a scalar propagating on the RN black hole by using a hidden conformal symmetry solely. The reason is that quasinormal modes do not satisfy the outgoing wave-boundary condition because these modes were developed using the near-region and low-energy limits [9] where the frequency $\omega$ should satisfy inequalities of $\omega \ll 1/r$ and $\omega \ll 1/r_+$. Importantly, the geometry modified in this way (the subtracted geometry) shows that it has the same near-horizon properties as the original RN black hole, but different asymptotes [12]. The latter is given by the asymptotically anti-de Sitter (AdS) spacetime. That is, developing the hidden conformal symmetry in the near-horizon is necessary to make change of the boundary condition at infinity from asymptotically flat to asymptotically AdS spacetime. It is suggested that quasinormal modes [12] satisfy the ingoing-boundary condition at the horizon and Dirichlet boundary condition at infinity.

However, it requires an appropriate picture described by the solution to a scalar wave equation around a specified RN black hole to derive QNFs. We should find the specified RN black hole which captures quasinormal modes [12] and QNFs [11]. It is found that the specified RN black hole is given by the near-horizon region of a near-extremal RN black hole. The wave equation (28) corresponds to (59) around the AdS segment of AdS$_2 \times S^2$, which is just the geometry of the near-extremal RN black hole [13]. We show in Appendix that the purely imaginary QNFs [11] could be found exactly by solving the scalar wave equation around the near-horizon region of near-extremal RN black hole. Also, quasinormal modes (73) as the solution to the scalar wave equation around the near-horizon region of a near-extremal RN black hole satisfy the ingoing-boundary condition at the horizon and Dirichlet boundary condition at infinity. Even the authors in [14] have used the different boundary condition, we obtain the same QNFs (66) when taking the massless and neutral limits. Finally, we will also discuss QNFs by adapting potential pictures.

2 Hidden conformal symmetry

First, let us introduce the RN black hole whose metric is given by

$$ds^2_{\text{RN}} = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$
with the metric function
\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \] (2)

Here, \( M \) and \( Q \) are the ADM mass and the electric charge of the RN black hole, respectively. Then, the inner \((r_-)\) and the outer \((r_+)\) horizons are obtained as
\[ r_{\pm} = M \pm \sqrt{M^2 - Q^2} \equiv M \pm r_0, \] (3)

which satisfy \( f(r_{\pm}) = 0 \). We note that \( r_0 \) is a non-extremal parameter, but a very small \( r_0 \ll M (\sim Q) \) corresponds to the near-extremal RN black hole. Also, we have an extremal RN black hole for \( r_0 = 0 \).

For the RN black hole, the relevant thermodynamic quantities are the Bekenstein-Hawking entropy and Hawking temperature
\[ S_{BH} = \frac{\pi r_+^2}{4}, \] (4)
\[ T_H = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{r_0}{2\pi r_+^2}, \] (5)

respectively. Note that the surface gravity is defined as
\[ \kappa = \frac{r_0}{r_+^2} = 2\pi T_H. \] (6)

Now, let us consider a minimally coupled massless scalar propagating in the spacetimes (1), which satisfies the Klein-Gordon equation
\[ \Box_{RN} \Phi = 0. \] (7)

Using the ansatz
\[ \Phi(t, r, \theta, \phi) = e^{-i\omega t} R(r) Y_m^l(\theta, \phi) \] (8)

together with the eigenvalue equation on \( S^2 \)
\[ \Delta_{S^2} Y_m^l(\theta, \phi) = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta Y_m^l(\theta, \phi)) + \frac{1}{\sin^2 \theta} \partial_\phi^2 Y_m^l(\theta, \phi) \]
\[ = -l(l + 1) Y_m^l(\theta, \phi), \] (9)

the Klein-Gordon equation (7) transforms into the Schrödinger equation
\[ \frac{d^2}{dr_+^2} R(r) + \left[ \omega^2 - V_{RN}(r) \right] R(r) = 0. \] (10)
Here the tortoise coordinate is defined by \( dr_\ast = dr/f(r) \) and the potential is given by

\[
V_{\text{RN}}(r) = f(r) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right].
\]  

(11)

Next, we consider a coordinate transformation of

\[
\rho \equiv -\frac{1}{2\kappa} \ln \left[ 1 - \frac{2r_0}{r - r_-} \right].
\]  

(12)

In terms of \( \rho \), the event horizon \( r = r_+ \) is mapped into \( \rho \to \infty \), while the spatial infinity \( r \to \infty \) into \( \rho \to 0 \); \( r \in [r_+, \infty) \) is mapped to \( \rho \in [\infty, 0] \). In this work, we will only consider outside the event horizon because we wish to compute quasinormal modes. For the interior of the Cauchy horizon, see Ref. [11]. Using the new coordinate (12), the RN metric [11, 15] becomes

\[
ds_\rho^2 = -\tilde{f}(\rho)dt^2 + \tilde{f}^{-1}(\rho) \left( \frac{r_0}{\sinh(\kappa \rho)} \right)^2 \left[ \left( \frac{\kappa}{\sinh(\kappa \rho)} \right)^2 d\rho^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right]
\]  

(13)

with

\[
\tilde{f}(\rho) = \frac{1}{\left( e^{\kappa \rho} + \frac{r_0}{r_-} \sinh(\kappa \rho) \right)^2}.
\]  

(14)

Here we note a useful relation between \( r \) and \( \rho \)

\[
r^2 = \tilde{f}^{-1}(\rho) \left( \frac{r_0}{\sinh(\kappa \rho)} \right)^2.
\]  

(15)

Then, let us consider a minimally coupled massless scalar propagating in the spacetimes (13), which satisfies the Klein-Gordon equation

\[
\Box_\rho \Phi = 0.
\]  

(16)

Using the ansatz

\[
\Phi(t, \rho, \theta, \phi) = e^{-i\omega t} R(\rho) Y^l_m(\theta, \phi),
\]  

(17)

the Klein-Gordon equation (13) transforms into the second-order differential equation expressed in terms of \( \rho \)

\[
\left( \frac{\sinh(\kappa \rho)}{\kappa} \right)^2 \frac{d^2}{d\rho^2} R(\rho) + \left[ \frac{\omega^2}{\tilde{f}^2(\rho)} \left( \frac{r_0}{\sinh(\kappa \rho)} \right)^2 - l(l+1) \right] R(\rho) = 0,
\]  

(18)

which is again transformed into the Schrödinger-type equation [16]

\[
\frac{d^2}{d\rho^2} R(\rho) + \left[ \omega^2 - V_\omega(\rho) \right] R(\rho) = 0.
\]  

(19)
Here the $\omega$-dependent potential is given by

$$V_\omega(\rho) = \omega^2 \left[ 1 - \frac{(\kappa r_0)^2}{f^2(\rho) \sinh^4(\kappa \rho)} \right] + \frac{l(l + 1)\kappa^2}{\sinh^2(\kappa \rho)}. \quad (20)$$

In order to develop a hidden conformal symmetry, we introduce three vector fields

$$L_1 = \frac{1}{\kappa} e^{\kappa t} \left[ \cosh(\kappa \rho) \partial_t + \sinh(\kappa \rho) \partial_\rho \right],$$

$$L_0 = -\frac{1}{\kappa} \partial_t,$$

$$L_{-1} = \frac{1}{\kappa} e^{-\kappa t} \left[ \cosh(\kappa \rho) \partial_t - \sinh(\kappa \rho) \partial_\rho \right],$$

which are slightly different from the previous construction \[11\]. These satisfy the SL(2,R) commutation relations

$$[L_0, L_{\pm1}] = \mp L_{\pm1}, \quad [L_1, L_{-1}] = 2L_0. \quad (22)$$

Then, the SL(2,R) Casimir operator is constructed by

$$\mathcal{H}^2 = L_0^2 - \frac{1}{2} (L_1 L_{-1} + L_{-1} L_1)$$

$$= - \left( \frac{\sinh(\kappa \rho)}{\kappa} \right)^2 \partial_t^2 + \left( \frac{\sinh(\kappa \rho)}{\kappa} \right)^2 \partial_\rho^2. \quad (23)$$

We approximate the second term in Eq. (18) as

$$\frac{\omega^2}{f^2(\rho)} \left( \frac{r_0}{\sinh(\kappa \rho)} \right)^2 = \omega^2 r_+^4 \left( \frac{\sinh(\kappa \rho)}{r_0} \right)^2 \approx \omega^2 (r_+ + 2r_0 e^{-2\kappa \rho})^4 \left( \frac{\sinh(\kappa \rho)}{r_0} \right)^2 \approx \omega^2 r_+^4 \left( \frac{\sinh(\kappa \rho)}{r_0} \right)^2 = \omega^2 \left( \frac{\sinh(\kappa \rho)}{\kappa} \right)^2, \quad (24)$$

where we use [15] to obtain (24), and use the near-horizon approximation of $r \approx r_+ + 2r_0 e^{-2\kappa \rho}$ to derive (25). Finally, we obtain (26) by employing the low-energy approximation [9]. In the near-region and low-energy approximations which are necessary to develop the hidden conformal symmetry, the first two terms of the $\omega$-dependent potential (20) disappear, leading to the last term only.
As a result, comparing (18) with (23), the Klein-Gordon equation in the near-region and low-energy approximations can be rewritten in terms of the SL(2,R) Casimir operator $\mathcal{H}^2$ as

$$\Box_\rho \Phi = 0 \rightarrow \mathcal{H}^2 \Phi = l(l + 1)\Phi.$$  \hfill (27)

The latter can be rewritten as the Schrödinger equation

$$\frac{d^2}{d\rho^2} R(\rho) + \left[ E - V_{\text{HCS}}(\rho) \right] R(\rho) = 0,$$  \hfill (28)

where the energy $E$ is

$$E = \omega^2,$$  \hfill (29)

and the HCS-potential takes the form

$$V_{\text{HCS}}(\rho) = \frac{l(l + 1)\kappa^2}{\sinh^2(\kappa\rho)}.$$  \hfill (30)

Therefore, the massless scalar wave equation carries the hidden conformal symmetry which is not a spacetime symmetry. We note that the hidden conformal symmetry is realized only when approximating the $\omega$-dependent potential $V_{\omega}(\rho)$ \hspace{1em} (20) by the HCS-potential $V_{\text{HCS}}(\rho)$ \hspace{1em} (30) in the Schrödinger equation. See Fig. 1-(b) for their difference.

At this stage we would like to emphasize that in the near-horizon limit ($\rho \rightarrow \infty$) the HCS-potential takes a form of $V_{\text{HCS}} \sim e^{-2\kappa\rho}$, while at infinity ($\rho \rightarrow 0$) it goes to a form of $V_{\text{HCS}} \sim \frac{1}{\rho^2}$. It shows that $V_{\text{HCS}}(\rho)$ is similar to the potential of a scalar field around the AdS-black hole. This may imply that its asymptote is changed from a flat spacetime implied by the RN black hole to an AdS spacetime.

### 3 Quasinormal modes constructed by operator method

We are now in a position to use the hidden conformal symmetry to derive QNFs of the RN black hole. First, we define the primary state by $\Phi^{(0)}$ which satisfies

$$L_0 \Phi^{(0)} = \hbar \Phi^{(0)},$$  \hfill (31)

and the highest weight condition

$$L_1 \Phi^{(0)} = 0.$$  \hfill (32)
Since \( \Phi^{(0)} \) takes the form
\[
\Phi^{(0)} = e^{-i\omega_0 t} R^{(0)}(\rho) Y^l_m(\theta, \phi),
\] (33)
on one has a conformal weight
\[
h = i \frac{\omega_0}{\kappa} = i \frac{\omega_0}{2\pi T_H}.
\] (34)
On the other hand, for \( \Phi^{(0)} \), the SL(2,R) Casimir operator satisfies
\[
\mathcal{H}^2 \Phi^{(0)} = h(h + 1) \Phi^{(0)}.
\] (35)
Comparing Eq. (35) with Eq. (27), one has
\[
h = \frac{1}{2} [1 \pm (2l + 1)].
\] (36)
Together with Eq. (34), one can find
\[
\omega_0 = -i \frac{\kappa}{2} [1 \pm (2l + 1)].
\] (37)
Proposing that the QNFs are purely imaginary \( \omega_I < 0 \) (\( \omega = \omega_R + \omega_I \)) with \( \omega_R = 0 \), we choose the upper sign as
\[
\omega_0 = -i \kappa (l + 1).
\] (38)
Then, all the descendants are constructed by
\[
\Phi^{(n)} = (L_-)^n \Phi^{(0)}
\] (39)
so that we have
\[
\Phi^{(n)} = e^{-i\omega_n t} R^{(n)}(\rho) Y^l_m(\theta, \phi),
\] (40)
where the QNFs are read off as
\[
\omega_n = \omega_0 - i\kappa n = -i\kappa \left[ n + l + 1 \right].
\] (41)
which is our main result.
Moreover, the \( n \)-th radial eigenfunction \( R^{(n)}(\rho) \) takes the form
\[
R^{(n)}(\rho) = \left( \kappa \right)^{-n} \left( -i \omega_{n-1} \cosh (\kappa \rho) - \sinh (\kappa \rho) \frac{d}{d\rho} \right) \times \left( -i \omega_{n-2} \cosh (\kappa \rho) - \sinh (\kappa \rho) \frac{d}{d\rho} \right) \times \cdots \times \left( -i \omega_0 \cosh (\kappa \rho) - \sinh (\kappa \rho) \frac{d}{d\rho} \right) R^{(0)}(\rho).
\] (42)
We also have

\[ L_0 \Phi^{(n)} = (h + n) \Phi^{(n)}, \]  

(43)

which implies that \( \Phi^{(n)} \) forms a principal discrete highest weight representation of the SL(2,R). Now we wish to solve the highest weight condition \( (32) \) to determine the highest weight state \( R^{(0)}(\rho) \)

\[ \left[ -i \omega_0 \cosh (\kappa \rho) + \sinh (\kappa \rho) \frac{d}{d\rho} \right] R^{(0)}(\rho) = 0. \]  

(44)

The solution is given by

\[ R^{(0)}(\rho) = C \left[ \sinh (\kappa \rho) \right]^{\frac{i \omega_0}{\kappa}}. \]  

(45)

Here we note that the tortoise coordinate \( r_* \) given by

\[ r_* = r - \frac{r^2}{2r_0} \ln(r - r_-) + \frac{1}{2\kappa} \ln(r - r_+) \]  

(46)

approaches

\[ r - r_+ \sim e^{2\kappa r_*}, \quad \text{as} \quad r \to r_+. \]  

(47)

On the other hand, \( \rho \)-coordinate \( (12) \) goes to

\[ r - r_+ \sim e^{-2\kappa \rho} \]  

(48)

in the near-horizon region so that \( \rho \) behaves as

\[ \rho \sim -r_* \]  

(49)

in this region. This gives us the solution \( (45) \) which behaves as

\[ R^{(0)} \sim e^{-i \omega_0 r_*}, \]  

(50)

for \( r \to r_+ \). This is the ingoing mode propagating into the horizon. For the \( n \)-th radial eigenfunction, one can easily show by induction

\[ R^{(n)} \sim e^{-i \omega_0 r_*}, \quad \text{as} \quad r_* \to -\infty. \]  

(51)

Finally, we observe that \( R^{(0)}(0) = 0 \) at infinity \( \rho \to 0 \) \( (r_* \to \infty) \), which shows that it is not the outgoing wave at infinity but satisfies the Dirichlet boundary condition as like at the infinity of AdS spacetime. Moreover, the first radial eigenfunction \( R^{(1)}(\rho) \) can be explicitly constructed as

\[ R^{(1)}(\rho) = -2i C \omega_0 \cosh (\kappa \rho) \left[ \sinh (\kappa \rho) \right]^{\frac{i \omega_0}{\kappa}}, \]  

(52)

which also satisfies the Dirichlet boundary condition at infinity. One can easily show that the \( n \)-th radial eigenfunction \( R^{(n)}(\rho) \) behaves as the same way as \( R^{(1)}(\rho) \) likewise.
4  QNFs around near-extremal RN black hole

We know that the literature of quasinormal modes for the RN black hole is vast. It is unclear that the purely imaginary QNFs (41) capture what kind of RN black hole, even though they have obtained from the near-region and low-energy approximations of a scalar wave equation around the RN black hole. This requires to know an appropriate picture of a scalar perturbation around a specified RN black hole. We wish to show that the specified RN black hole is exactly the near-horizon region of a near-extremal RN black hole.

For this purpose, we briefly mention a known way to obtain the QNFs of the near-extremal RN black hole [14]. Then we compare the previous results with the ones obtained from the near-extremal RN black hole. By taking the near-horizon and the near-extremal limits,

\[ r \to Q + \tilde{\rho}, \quad M \to Q + r_0, \]

the RN metric (1) can be rewritten by two parameters \( r_0 = \sqrt{2Q(M - Q)} \) and \( Q \) as

\[ ds_{NHNE}^2 = -\frac{\tilde{\rho}^2 - r_0^2}{Q^2} dt^2 + \frac{Q^2}{\tilde{\rho}^2 - r_0^2} d\tilde{\rho}^2 + Q^2 d\Omega_2^2, \]

whose geometry is given by AdS\(_2\) \( \times S^2 \). It is pointed out that (54) with \( r_0 = 0 \) describes the near-horizon region of the extremal RN black hole which was known to be the Bertotti-Robinson geometry. We note that the geometry (54) describes the near-horizon region of a near-extremal RN black hole only, but not the whole geometry of a near-extremal black hole because the geometry at infinity is an asymptotically flat spacetime. This point is important to understand why QNFs (41) are purely imaginary. Here \( \tilde{\rho} \) describes outside the black hole \((\rho \in [r_0, \infty])\), which is an extended geometry of the near-horizon region. For this near-extremal RN black hole, the surface gravity is given by

\[ \tilde{\kappa} = \frac{r_0}{Q^2}. \]

Here we note that the surface gravity (55) is obtained directly from the near-extremal solution (54), while one can approximate the surface gravity (6) to give

\[ \kappa \approx \tilde{\kappa}\left[1 - O\left(\frac{2r_0}{M}\right)\right]. \]

We observe that for the near-extremal RN black hole, its surface gravity is given by \( \tilde{\kappa} \). Then, the Klein-Gordon equation (16) with the ansatz (17) can be written by

\[ \frac{d}{d\tilde{\rho}} \left( (\tilde{\rho}^2 - r_0^2) \frac{d}{d\tilde{\rho}} \right) R(\tilde{\rho}) + \left( \frac{\omega^2 Q^4}{\tilde{\rho}^2 - r_0^2} - l(l + 1) \right) R(\tilde{\rho}) = 0. \]

9
Introducing the tortoise coordinate defined by
\[
\rho_* = \frac{1}{2\tilde{\kappa}} \ln \left( \frac{\tilde{\rho} + r_0}{\tilde{\rho} - r_0} \right), \quad \rho_* \in [\infty, 0],
\] (58)
the Klein-Gordon equation becomes the Schrödinger-type equation
\[
\frac{d^2}{d\rho_*^2} R(\rho_*) + \left[ \omega^2 - V_{\text{NE}}(\tilde{\rho}) \right] R(\rho_*) = 0,
\] (59)
where the near-extremal RN potential is given by
\[
V_{\text{NE}}(\tilde{\rho}) = \frac{l(l+1)(\tilde{\rho}^2 - r_0^2)}{Q^4}.
\] (60)
Moreover, solving the tortoise coordinate in terms of \(\tilde{\rho}\) as
\[
\tilde{\rho} = r_0 \coth(\tilde{\kappa} \rho_*),
\] (61)
one can easily show that the near-horizon and near-extremal RN potential leads to
\[
V_{\text{NHNE}}(\rho_*) = \frac{l(l+1)\tilde{\kappa}^2}{\sinh^2(\tilde{\kappa} \rho_*)},
\] (62)
which is the exactly same form of \(V_{\text{HCS}}(\rho)\) in Eq. (30) when replacing \(\tilde{\kappa}\) and \(\rho_*\) by \(\kappa\) and \(\rho\). See Fig. 1-(b) for \(V_{\text{NE}}(\rho_*)\). Here \(\rho_*\) mimics exactly \(\rho\) for describing the region outside the event horizon. This means that our previous results is valid for the near-extremal RN black hole only where the surface gravity of the RN black hole \(\kappa\) is replaced by the surface gravity of the near-extremal RN black hole \((\kappa \to \tilde{\kappa})\).

As obtained in [14], the QNFs of a massive charged scalar with mass \(m\) and charge \(q\) around the near-horizon region of a near-extremal RN black hole is given by
\[
\tilde{\omega}_n = -\tilde{\kappa} b - i\tilde{\kappa} \left( n + \frac{1}{2} \right),
\] (63)
where the parameter \(b\) is given by
\[
b = \sqrt{(q^2 - m^2)Q^2 - \left( l + \frac{1}{2} \right)^2}.
\] (64)
In this work, since we are considering the uncharged massless scalar field, the parameter \(b\) takes the purely imaginary value
\[
b = i \left( l + \frac{1}{2} \right).
\] (65)
Plugging $b$ into (63), one obtains the QNFs
\[ \tilde{\omega}_n = -i\tilde{\kappa} [n + l + 1], \] (66)
for the scalar propagating on the near-extremal RN black hole. Also, there are other works which support our results. Kim and Oh \[17\] have computed QNFs ($\omega = -i\tilde{\kappa}(n + 1)$) from the scalar equation around the AdS$_2$ black hole obtained by making the dimensional reduction of the Einstein-Maxwell theory on $S^2$. Because of the dimensional reduction on $S^2$, the angular quantum number $l$ was missed in $\omega$. Hod \[18\] has shown that QNFs of a massive charged scalar propagating around the near-extremal RN black hole are given by (66) in the massless and uncharged cases.

In Appendix, we show explicitly that purely imaginary QNFs (66) could be obtained as (77) by solving the scalar wave equation (59) around the near-horizon region of a near-extremal RN black hole directly. Furthermore, it is proven that quasinormal modes as the solution to the scalar wave equation around the near-horizon region of a near-extremal RN black hole satisfy the ingoing-boundary condition at the horizon and Dirichlet boundary condition at infinity.

The above states clearly that the QNFs of the scalar perturbation around the RN black hole with the hidden conformal symmetry are those obtained from the scalar perturbation around the near-horizon region of a near-extremal RN black hole. In this context, our QNFs (11) are the same as (66) obtained from the scalar perturbation around the near-horizon region of a near-extremal RN black hole (54) whose geometry is AdS$_2 \times S^2$. This productive geometry describes effectively the two-dimensional black hole whose quasinormal frequencies are usually given by purely imaginary QNFs [19].

## 5 Potential Picture

Now we wish to support the validity of the QNFs $\omega_n$ in Eq. (11) by comparing the relevant potentials. In Fig.1, we depict these potentials with $l = 10, r_0 = 1$ ($r_+ = 2.1, r_- = 0.1, M = 1.1, Q = 0.46, \kappa = 0.23$) for $V_{RN}(r)$, $V_{\omega}(\rho)$ with $\omega = 1$, $V_{HCS}(\rho)$, and $V_{NE}(\rho_*)$.

First, as is shown in Fig.1-(a), $V_{RN}(r)$ is the potential barrier existing outside the event horizon ($r = r_+$) of the RN black hole. It is hard to find the analytic expression for the QNFs because of asymptotic behavior of $V_{RN} \sim 1/r^2$ as $r \to \infty$. Accordingly, the literature of quasinormal modes QNFs for the RN black hole is vast.

Second, as depicted by a solid curve in Fig.1-(b), the $\omega$-dependent potential $V_{\omega}(\rho)$ shows a negative potential around $\rho = 0$ ($r \to \infty$). At this stage,
Figure 1: Potential pictures for $l = 10$: (a) the RN potential $V_{\text{RN}}(r)$ with $M = 1.10$, $Q = 0.46$, (b) the $\omega$-dependent potential $V_{\omega}(\rho)$ in solid curve with $\omega = 1$, $r_0 = 1$, $r_- = 0.10$, and the HCS potential $V_{\text{HCS}}(\rho)$ in dashed curve with $r_0 = 1$. We note that the potential $V_{\text{NHNE}}(\rho_*)$ is the same graph as in $V_{\text{HCS}}(\rho)$ when replacing $\rho$ by $\rho_*$. It seems that we do not know the appearance of the negative potential which may induce the instability of the RN black hole. In order to understand the negativeness of the potential, we take its limit of $\rho \to 0$ as

$$V_{\omega}^{\rho \to 0}(\rho) = \omega^2 - \omega^2 r_+^4 \left( \frac{1}{\rho} + \frac{r_+ + r_-}{2r_+^2} \right)^4 + \frac{l(l + 1)}{\rho^2},$$

(67)

where the second term is responsible for the negativeness in the solid curve in Fig.1-(b), while the last term makes the potential positively infinite as drawn by the dashed curve in Fig.1-(b). Hence, imposing the near-region and low-energy limits, one may neglect the second term in favor of the last term, effectively leading to the HCS-potential depicted as the dashed curve in Fig.1-(b).

Third, concerning the $V_{\text{HCS}}(\rho)$ potential, we have to say that this potential is not a genuine potential which is valid for the whole RN black hole but a form of the potential obtained by the approximation (26) to develop the hidden conformal symmetry near the event horizon. As was shown in section 4, $V_{\text{HCS}}(\rho)$ coincides with the potential $V_{\text{NHNE}}(\rho_*)$ in (62) of the scalar perturbation around the near-horizon region of a near-extremal RN black hole.

Finally, we mention that the graphs in Fig.1 are designed for a non-extremal RN black hole of $r_+ = 2.1$, $r_- = 0.1$, $M = 1.1$, $Q = 0.46$, $\kappa = 0.23$. Even for the near-extremal RN black hole, the behaviors of $V_{\omega}(\rho)$ and $V_{\text{HCS}}(\rho)$ are similar to those in Fig.1-(b) because $\rho$-coordinate was used to draw the pictures whose range is from $\rho = \infty$ (event horizon) to $\rho = 0$ (infinity).
6 Conclusion

In conclusion, we have derived the purely imaginary QNFs of the RN black hole by making use of the hidden conformal symmetry developed in the near-region and low-energy approximations of the massless Klein-Gordon equation. We see that the operator approach used to derive QNFs has a limitation because the $\omega$-dependent potential is just replaced by the HCS potential, as is shown in Fig.1-(b). This means that developing the hidden conformal symmetry in the near-region and low-energy approximations implies losing the large $r$ behavior of the potential in the whole RN black hole. In deriving quasinormal modes, thus, the outer-boundary condition at infinity was changed from outgoing modes to Dirichlet boundary condition.

Consequently, the imaginary QNFs based on the hidden conformal symmetry are nothing but the quasinormal frequencies of the scalar perturbation around the near-horizon region of a near-extremal RN black hole whose geometry is described by AdS$_2 \times$ S$^2$. Since this geometry is not the whole geometry of near-extremal RN black hole whose geometry at infinity is asymptotically flat, QNFs are purely imaginary, in comparison with the complex value [22].

Appendix: A computation of QNFs around the near-extremal RN black hole

In appendix, we show that a scalar propagating around the near-horizon region of a near-extremal RN black hole has purely imaginary QNFs. Let us start with the Schrödinger equation with the near-horizon and near-extremal RN potential. Introducing a new variable $x = \frac{1}{\cosh^2(\tilde{\kappa} r)}$, $x \in [0, 1]$, Eq. (59) can be written as

$$x(1-x) \frac{d^2}{dx^2} R + \left(1 - \frac{3}{2} x \right) \frac{d}{dx} R + \left(\frac{\omega^2}{4\tilde{\kappa}^2 x} - \frac{l(l+1)}{4(1-x)} \right) R = 0. \quad (69)$$

By changing to a new wave function $y$ through

$$R = x^{-\frac{1}{4}} (1-x)^{-\frac{1}{4}} y, \quad (70)$$

Eq. (69) can be transformed into a standard hypergeometric equation

$$x(1-x)y'' + [c - (a + b + 1)x]y' - aby = 0 \quad (71)$$
with
\[ a = \frac{-i\omega}{2\kappa} - \frac{l}{2} + \frac{1}{2}, \quad b = \frac{-i\omega}{2\kappa} - \frac{l}{2}, \quad c = 1 - \frac{i\omega}{\kappa}. \tag{72} \]

In order to obtain the quasinormal modes, we have to check whether or not solutions of Eq. (69) satisfy the boundary conditions: ingoing waves near the horizon \((x = 0)\) and zero (Dirichlet condition) at infinity of \(x = 1\). Taking into account \(e^{-i\omega t}\), such a solution takes the form of
\[ R = x^{-\frac{i\omega}{2\kappa}}(1 - x)^{-\frac{l}{2}} \binom{a, b, c}{x}, \tag{73} \]
where \(\binom{a, b, c}{x}\) is a standard hypergeometric function. Imposing the boundary condition at infinity \((x \rightarrow 1, \rho_\ast \rightarrow 0, \tilde{\rho} \rightarrow \infty)\), we recall a property of the hypergeometric function
\[ \lim_{x \rightarrow 1} \binom{a, b, c}{x} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{74} \]
The Dirichlet boundary condition can be achieved when choosing
\[ c - a = -n \text{ or } c - b = -n \tag{75} \]
with \(n = 0, 1, 2, \cdots\). Therefore, using (75) together with (72), the QNFs are given by
\[ \omega_n = -i\kappa(2n + l + 1), \quad \omega_n = -i\kappa(2n + l + 2), \tag{76} \]
which are combined to give a single expression of purely imaginary QNFs
\[ \tilde{\omega}_n = -i\kappa(n + l + 1). \tag{77} \]
This is the exactly same form as found in the QNFs (66).

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