Generic classification of homogeneous potentials of 
degree \(-1\) in the plane

Thierry COMBOT

IMCCE, 77 Avenue Denfert Rochereau 75014 PARIS

Abstract

We give a complete classification of integrable potentials in the plane \(V = r^{-1} U(\theta)\) with \(U\) meromorphic and real analytic. In the more general case \(V\) only meromorphic, we make a classification of all integrable potentials possessing a Darboux point \(c\) with \(V'(c) = -c\) and \(\text{Sp}(\nabla^2 V(c)) \subset \{-1, 0, 2\}\), and then prove the non existence of other meromorphically integrable potential such that \(\text{Sp}(\nabla^2 V(c)) \subset ] - \infty, 20]\). An algorithm is given to improve this bound arbitrary, and allows to find algorithmically all integrable potentials in any set \(E\) of potentials possessing a generic property, the eigenvalue bounded property, which corresponds to the fact that for any potential in \(E\), the lowest eigenvalue is always lower than some fixed explicit bound. We eventually present an analysis and some examples in the degenerate Darboux point case \(V'(c) = 0\).

Keywords: Morals-Ramis theory, homogeneous potential, D-finiteness, higher order variational equations

1. Introduction

We want to find all meromorphic integrable homogeneous potentials of degree \(-1\) in the plane. As we will not succeed to find all meromorphically integrable potential in this large class of potential, we will restrict ourselves to subsets of potentials satisfying a generic property, the “eigenvalue bounded” property.

Email address: combot@imcce.fr (Thierry COMBOT)
The set of meromorphic potentials is the most general reasonable one, excluding potentials involving algebraic functions. It contains in particular all the rational potentials with among them the co-linear 3 body problem, the co-linear 4 body problem with symmetric masses \((m_1, m_2, m_2, m_1)\), etc... and all problems in higher dimension which possess some particular restriction in the phase space corresponding to a 2 degrees of freedom potential. We will then be able to solve all these problems at once, providing they satisfy the condition of “eigenvalue bounded” and that we compute this bound. For the following, we will only consider complex potentials and complex functions, except if specified otherwise.

**Definition 1.** We will say that \(V\) is a homogeneous potential of degree \(-1\), meromorphic in \(q_1, q_2, r\) for \(r \neq 0\), if

\[
V : \{(q_1, q_2, r) \in \mathbb{C}^2 \times \mathbb{C}^* \text{ such that } r^2 = q_1^2 + q_2^2\} \longrightarrow \mathbb{C}
\]

and \(V\) is homogeneous of degree \(-1\) in \(q_1, q_2, r\).

We use this definition because we want also consider the potential \(V\) which is invariant by rotation, and which is not a meromorphic potential. This is why we want to add the dependence in \(r\), which will be needed by the way to consider polar coordinates.

**Theorem 1.** Let \(V\) be a homogeneous potential of degree \(-1\) meromorphic in \(q_1, q_2, r\) for \(r \neq 0\). Then it exists a \(2\pi\)-periodic meromorphic function \(U\) such that for all \((q_1, q_2, r) \in \mathbb{C}^2 \times \mathbb{C}^*, r^2 = q_1^2 + q_2^2\) we have

\[
V(q_1, q_2, r) = r^{-1}U(\theta) \quad \text{with} \quad r \cos \theta = q_1, \quad r \sin \theta = q_2
\]

Conversely, if \(V\) can be written as \((1)\), then \(V\) is a homogeneous potential of degree \(-1\) meromorphic in \(q_1, q_2, r\) for \(r \neq 0\).

Remark that \(r, \theta\) correspond to the polar coordinates, and that for a given \((q_1, q_2, r)\), \(\theta\) is only defined modulo \(2\pi\), which has no consequences because \(U\) is a \(2\pi\)-periodic meromorphic function.

**Proof.** Suppose that

\[
V(q_1, q_2, r) = r^{-1}U(\theta) \quad \text{with} \quad r \cos \theta = q_1, \quad r \sin \theta = q_2, \quad r^2 = q_1^2 + q_2^2
\]
with $U$ meromorphic $2\pi$-periodic. Then $U$ can be written $U(\theta) = f(\exp i\theta)$ with $f$ meromorphic. So we replace then these by their expressions in Cartesian coordinates. This gives

$$V(q_1, q_2, r) = r^{-1}f\left(\frac{q_1 + iq_2}{r}\right)$$

As $f$ is meromorphic, the function $V$ is meromorphic in its 3 variables for any $(q_1, q_2, r) \in \mathbb{C}^2 \times \mathbb{C}^*$. Conversely, we use homogeneity, and then we have

$$V(q_1, q_2, r) = r^{-1}V(\cos \theta, \sin \theta, 1) \quad \forall (q_1, q_2, r) \in \mathbb{C}^2 \times \mathbb{C}^*$$

using $r \cos \theta = q_1, r \sin \theta = q_2$. The function $V(\cos \theta, \sin \theta, 1)$ is meromorphic $2\pi$-periodic.

To a potential $V$ we will associate a Hamiltonian system and the following differential equation system

$$H(p_1, p_2, q_1, q_2, r) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2, r) \quad \text{with} \quad r^2 = q_1^2 + q_2^2$$

$$\dot{q}_1 = p_1 \quad \dot{q}_2 = p_2 \quad \dot{r} = r^{-1}(p_1 q_1 + p_2 q_2)$$

$$\dot{p}_1 = \frac{\partial}{\partial q_1}H + r^{-1}q_1 \frac{\partial}{\partial r}H \quad \dot{p}_1 = \frac{\partial}{\partial q_2}H + r^{-1}q_2 \frac{\partial}{\partial r}H$$

The function $H$ will be called the Hamiltonian, and this is a Hamiltonian system with 2 degrees of freedom. The function $H$ is meromorphic for $(p_1, p_2, q_1, q_2, r) \in \mathbb{C}^4 \times \mathbb{C}^*$ and $r^2 = q_1^2 + q_2^2$.

Thanks to Theorem 1, it is sufficient and even more general to study the potentials of the form $V = r^{-1}U(\theta)$ with $U$ a $2\pi$-periodic meromorphic function. We can also ask why one should do this generalization. This is because the potential invariant by rotation is not a meromorphic potential because of the square root (in the distance to the origin). It is still meromorphic in $q_1, q_2, r$, and as we will see, it is very important that this potential belongs to the class of potentials we are investigating to do our “generic classification”. We call it only “generic” because it is useful only if some generic property of “eigenvalue bounding” is satisfied. This is linked to the fact that this classification rely on a conjecture (conjecture 2) that depend on some integer $n$ and which can be proved algorithmically but only for a fixed $n$. 

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Definition 2. Let $V$ be a homogeneous potential of degree $-1$ in the plane such that $V = r^{-1}U(\theta)$ in polar coordinates with $U$ $2\pi$-periodic meromorphic function. We will say that $V$ is meromorphically integrable if there exists a first integral $I(q_1, q_2, p_1, p_2, r)$ where $I$ is meromorphic for $(p_1, p_2, q_1, q_2, r) \in \mathbb{C}^4 \times \mathbb{C}^*$ and $r^2 = q_1^2 + q_2^2$, and $I$ is independent almost everywhere with $H$.

In the following, we will often note $\|q\|^2 = q_1^2 + q_2^2$, even for complex values of $q$. In particular, this quantity is conserved by a rotation, even a complex one.

Definition 3. We say that $c = (c_1, c_2) \in \mathbb{C}^2$ is a Darboux point of $V$ if

$$\frac{\partial}{\partial q_1}V(c) = \alpha c_1 \quad \frac{\partial}{\partial q_2}V(c) = \alpha c_2$$

with $\alpha \in \mathbb{C}$ called multiplier. With homogeneity, we can always choose $\alpha = 0, -1$. We say that $c$ is non degenerated if $\alpha \neq 0$. A homothetic orbit associated to $c$ is given by

$$q_i(t) = c_i \phi(t) \quad p_i(t) = c_i \phi'(t) \quad i = 1, 2$$

with $\phi$ satisfying the following differential equation

$$\frac{1}{2} \phi'(t)^2 = -\frac{\alpha}{\phi(t)} + E \quad E \in \mathbb{C}$$

The first order variational equation of $H$ near a homothetic orbit is given by

$$\ddot{X}(t) = \frac{1}{\phi(t)^3} \nabla^2 V(c) X(t)$$

and after diagonalization (if possible) and variable change $\phi(t) \to t$, the equation simplifies to

$$2t^2(Et + 1)\dot{X}_i - t\ddot{X}_i = \lambda_i X_i, \quad \lambda_i \in \text{Sp} \left( \nabla^2 V(c) \right)$$

The main theorems of this article are the following:

Theorem 2. Let $V$ be a homogeneous potential of degree $-1$ in the plane such that $V = r^{-1}U(\theta)$ with $U$ a $2\pi$-periodic meromorphic function and $U(\mathbb{R}) \subset \mathbb{R}$. If $V$ is meromorphically integrable, then

$$V = \frac{a}{r} \quad a \in \mathbb{R}^*$$
Theorem 3. Let $V$ be a homogeneous potential of degree $-1$ in the plane such that

- $V = \frac{1}{r}U(\theta)$ with $U$ meromorphic $2\pi$-periodic
- $\exists c \in \mathbb{C}^2$ such that $c_1^2 + c_2^2 \neq 0$, $V'(c) = -c$ and $\text{Sp}(\nabla^2(V)(c)) \subset \{-1,0,2\}$

where Sp correspond the spectrum of a matrix. If $V$ is meromorphically integrable, then $V$ belongs after possibly rotation to one the following families

$$V = \frac{a}{q_1} + \frac{b}{q_2} \quad a,b \in \mathbb{C}, (a,b) \neq (0,0) \quad V = \frac{a}{r} \quad a \in \mathbb{C}^*$$

$$V = \frac{a(q_1^2 + q_2^2)}{(q_1 + ei q_2)^3} + \frac{a}{q_1 + ei q_2} \quad a \in \mathbb{C}^*, \epsilon = \pm 1 \quad \text{Hietarinta 1987 [4]}$$

To prove such theorems, we will mainly use Morales Ramis theory about meromorphic integrability

Theorem 4. (Morales, Ramis, Yoshida [2],[3],[4],[5]) If $V$ is meromorphically integrable, then the neutral component of the Galois group of the variational equation near a non degenerated homothetic orbit associated to a Darboux point $c$ with $E = 1$, $c_1^2 + c_2^2 \neq 0$ over the base field $\mathbb{C}(\phi(t))$ (where $\phi(t)$ is the time parametrization of the homothetic orbit) is abelian at all orders. Moreover, if we fix the multiplier of the associated Darboux point to $-1$, the Galois group of the first order variational equation has an abelian neutral component if and only if

$$\text{Sp} \left( \nabla^2 V(c) \right) \subset \left\{ \frac{1}{2}(k - 1)(k + 2), k \in \mathbb{N} \right\}$$

In fact, this is not exactly the same statement as the original theorem because we allow $r$ to appear in the potential and in the first integrals, and because we precise explicitly the base field we use for Galois group computations. There are two important arguments to use Morales Ramis proof

- The first integral needs to have an expansion in series (or the quotient of two series in the meromorphic case) on the curve
- The coefficients of this expansion will be functions of the time $t$, and the corresponding field will be the base field to be considered in Galois group computation.
The degenerate case will be studied in the last section.

**Proof.** We know that \( c_1^2 + c_2^2 \neq 0 \). The Morales Ramis theorem necessitate that the first integrals to be meromorphic along the curve. This is not the case but almost. In fact, a first integral \( I \) meromorphic in \( q_1, q_2, p_1, p_2, r \) will be meromorphic along the curve except maybe at the point 0. The parametrization of the curve is

\[
q_i(t) = c_i \phi(t) \quad p_i(t) = c_i \dot{\phi}(t) \quad i = 1, 2
\]

We remark in particular, that on this curve, we have \( r = \sqrt{c_1^2 + c_2^2} \phi(t) \) (this square root correspond to a choice on which branch of the potential the homothetic orbit is, because the potential is bi-valuated as a “function” of \( q_1, q_2 \)). Moreover, on the surface \( r^2 = q_1^2 + q_2^2 \) we have

\[
\partial_{q_1} r = \frac{q_1}{r} \quad \partial_{q_2} r = \frac{q_2}{r}
\]

Here we see that the formulas (2) are always well defined on the curve because \( \sqrt{c_1^2 + c_2^2} \neq 0 \). So, the initial form of \( I \) along this curve will still have coefficients meromorphic in \( (\phi, \dot{\phi}) \) (with possibly an essential singularity at \( \phi = 0, \infty \)), because such an initial form is computed by doing derivations and then evaluation on the curve. We now need to pass from meromorphic coefficients to rational coefficients. This is because all computations we are able to do are over a rational base field, and not a meromorphic one.

Let us note \( J \) the initial form of \( I \). After the variable change \( \phi(t) \rightarrow t \), the normal variational equation of order 1 becomes of the form

\[
2t^2(1 + t)y'' - ty' - \lambda y
\]

where \( \lambda \) is a parameter corresponding to an eigenvalue of the Hessian matrix \( \nabla^2 V(c) \). The function \( J \) is a homogeneous function in \( y, \dot{y} \), and its coefficients are in meromorphic in \( t, \sqrt{1 + t^{-1}} \) for \( t \neq 0, \infty \). We remark that this square root does not appear in the coefficients of the variational equation. So just by multiplicating \( J \) by itself changing the valuation of the square root, we can suppose that \( J \) is meromorphic in \( t \) for \( t \neq 0, \infty \). All singularities of equation (3) are regular. So the Galois group over \( \mathbb{C}(t) \) of equation (3) is exactly the Zariski closure of the monodromy group. If we remove small disks \( D_0, D_{-1}, D_{\infty} \) on the singularities, we will have that the first integral \( J \) will be meromorphic on

\[
\bar{\mathbb{C}} \setminus (D_0 \cup D_{-1} \cup D_{\infty})
\]
If we restrict ourselves to take paths in this set instead of taking path in \( \bar{C} \) to compute the monodromy group, this will not change anything (because we have chosen the disks small enough, and then all curves in \( \bar{C} \setminus \{0, -1, \infty\} \) are homotopic to curves in the set \( (4) \)). If the Galois group is \( SL_2(\mathbb{C}) \), then the monodromy group is non commutative, and then such a first integral \( J \) cannot exists.

In their theorem, Morales, Ramis use the Kimura table \( [6] \), which in fact correspond to Galois group computation over the base field \( \mathbb{C}(t) \) of equation \( (3) \). They found that the Galois group of equation \( (3) \) is \( SL_2(\mathbb{C}) \) except for the specific values of \( \lambda \) in Theorem \( [4] \) (for which the Galois group is abelian).

We will find in particular that a non degenerated Darboux point of \( V \) corresponds to some \( \theta_0 \in [0, 2\pi] \) such that

\[
U'(\theta_0) = 0 \quad \text{and} \quad U(\theta_0) \neq 0
\]

One of the main difficulties comes from the fact that function \( U \) can be arbitrary flat near \( \theta_0 \), and we then absolutely need higher variational equations at an arbitrary order. Indeed, previous classification of integrable homogeneous potentials in the plane for the degrees 3, 4 have been done for polynomial potentials \( [7], [8] \), and this corresponds to a finite number of parameters. Here the function \( U \) can be anything, and thus we need an infinity of integrability constraints to prove these theorems.

**Example**

\[
U(\theta) = (1 - \cos(\theta))^n - \frac{n2^n}{(2k - 1)(k + 1) + 1} - 2^n, \quad n, k \in \mathbb{N}^*
\]

This potential is integrable at order 2 near all Darboux points (using integrability table in \( [9] \)). Moreover, near \( \theta = 0 \) (which corresponds to a Darboux point), this potential is integrable at least at order \( 2n - 1 \). The analysis of the other Darboux point \( (\theta = \pi) \) is even more difficult because \( \text{Sp}(\nabla^2(V)(c)) \subset \{2, (2k - 1)(k + 1)\} \), integrable at order 2, and integrability at order 3 has an explicit dependence on \( k \). We will prove that it is in fact non integrable at order \( 4n - 3 \) near \( \theta = 0 \). For Theorem \( [2] \) we do not have as hypothesis the existence of a Darboux point. Because of this, we will need the following theorem.
Theorem 5. Let \( V \) be a meromorphic homogeneous potential in the plane of degree \(-1\) such that \( V = r^{-1}U(\theta) \) with \( U \) meromorphic \( 2\pi \)-periodic and real analytic. Then it exists \( \theta_0 \) such that

\[
U(\theta_0) \neq 0 \quad U'(\theta_0) = 0 \quad \frac{U''(\theta_0)}{U(\theta_0)} \leq 0
\]

Proof. We have that \( U(\mathbb{R}) \subset \mathbb{R} \) and \( U \) is \( C^\infty \). The function \( U \) is periodic, then it exists a minimum and a maximum for \( U \). Suppose first that \( U \) is not constant. Then \( \max U > \min U \). We have 3 cases

- \( \max U \geq \min U \geq 0 \). Then we pose \( \theta_0 \) such that \( U(\theta_0) = \max U \)
- \( \max U \geq 0 \geq \min U \). If \( \max U \neq 0 \), we pose \( \theta_0 \) such that \( U(\theta_0) = \max U \), otherwise we pose \( \theta_0 \) such that \( U(\theta_0) = \min U \)
- \( 0 \geq \max U \geq \min U \). We pose then \( \theta_0 \) such that \( U(\theta_0) = \min U \)

Knowing that \( \max U > \min U \), we get \( U(\theta_0) \neq 0 \). Then in all cases, we have

\[
\frac{U''(\theta_0)}{U(\theta_0)} \leq 0
\]

Knowing that \( \theta_0 \) is an extremum, we get

\[
U(\theta_0) \neq 0 \quad U'(\theta_0) = 0 \quad \frac{U''(\theta_0)}{U(\theta_0)} \leq 0
\]

which gives the theorem. \( \square \)

Remark 1. We have then that the first theorem is inside the second one. In particular, the computation in polar coordinates gives

\[
\text{Sp}(\nabla^2 V(c)) = \left\{ 2, \frac{U''(\theta_0)}{U(\theta_0)} - 1 \right\}
\]

If \( V \) is meromorphically integrable, then the eigenvalues at Darboux points should belong to

\[
\left\{ \frac{(p-1)(p+2)}{2}, \ p \in \mathbb{N} \right\} = \{-1, 0, 2, 5, 9, 14, \ldots \}
\]
Knowing that
\[
\frac{U''(\theta_0)}{U(\theta_0)} \leq 0
\]
we get the condition using Morales Ramis integrability condition [4]
\[
\text{Sp}(\nabla^2 V(c)) = \{2, -1\}
\]
This case is in the hypotheses of the second theorem.

2. Eigenvalue Bounding

We will denote
\[
\mathcal{M} = \{V(r, \theta) = r^{-1}U(\theta) \text{ with } U \text{ meromorphic } 2\pi \text{ periodic}\}
\]
Let \(V \in \mathcal{M}\). We note \(d(V)\) the set of Darboux points \(c\) of \(V\) with multiplier \(-1\) and such that \(\|c\|^2 \neq 0\). Let \(c \in d(V)\), then we have \(\text{Sp}(\nabla^2 V(c)) = \{2, \lambda\}\) and we note
\[
\Lambda(c) = \begin{cases} 
\lambda & \text{if } \lambda \in \mathbb{R} \\
-\infty & \text{otherwise}
\end{cases}
\]
We consider \(E \subset \mathcal{M}\) a subset of \(\mathcal{M}\) and we define the following
\[
\Lambda(E) = \sup_{V \in E, d(V) \neq \emptyset} \inf_{c \in d(V)} \Lambda(c)
\]
We say that the problem of finding all meromorphically integrable potentials in \(E\) is a bounded eigenvalue problem if
\[
\Lambda(E) < \infty
\]
We have \(\Lambda(\mathcal{M}) = \infty\) because of the following family
\[
V(r, \theta) = r^{-1} \left((1 + a) - 2ae^{i\theta} + ae^{2i\theta}\right), \quad a \in \mathbb{R},
\]
for which there exists only one Darboux point \(c = (1, 0)\) and the corresponding eigenvalue is \(\lambda = 2a - 1\).

Remark 2. Theorem [5] has then proved the following
\[
\lambda \left(\{V = r^{-1}U(\theta) \text{ } U \neq 0 \text{ meromorphic } 2\pi\text{-periodic } U(\mathbb{R}) \subset \mathbb{R}\}\right) = -1
\]
Here we want to present the following way to prove non integrability. First, you find a majoration of $\Lambda(E)$ of your problem. Then you run a program to check non existence of integrable potentials with eigenvalues up to this majoration (this program and associated conjecture are in the section “The other eigenvalues”). Finally you look if inside the family $E$, there is a potential given by Theorem 2. You do not even have to check Morales Ramis condition of order 1 given by Theorem 4. Thanks to this a lot of integrability problems can be proved. Moreover, this eigenvalue bounding property is more or less generic. This is because the Maciejewski-Pryzbylska relation on eigenvalues of Darboux points holds under generic assumption (even for non polynomial potentials), and this relation implies eigenvalue bounding. This is why we can call Theorem 3 and the partial proof of conjecture 2 used together as a “generic” classification, because it holds for all eigenvalue bounded potentials sets.

3. Non degeneracy of higher variational equations

We will first recall some properties of the solutions of the first order variational equations. After diagonalization and in the integrable case, the equation is the following (after fixing the energy $E = 1$)

$$2t^2(1 + t)y'' - ty' - \frac{1}{2}(n - 1)(n + 2)y \quad n \in \mathbb{N}$$  \hspace{1cm} (5)

We make then the change of variables

$$\sqrt{1 + t^{-1}} \longrightarrow t$$

and the differential equation becomes

$$\frac{1}{2}(t^2 - 1)\ddot{y} + 2t\dot{y} - \frac{1}{2}(n - 1)(n + 2)y = 0$$

A basis of solutions is given by $(P_n, Q_n)$ where $P_n$ are polynomials in $t$ and the functions $Q_n$ are

$$Q_n(t) = P_n(t) \int \frac{1}{(t^2 - 1)^2 P_n(t)^2} dt$$

The functions $Q_n$ are multivalued except for $n = 0$ which will be a special case. Indeed, the Galois group of $Q_n$ in this case is $Id$ instead of $C$. 10
The polynomials $P_n$ can be computed using the “Rodrigues” type formula

$$P_n(t) = \frac{1}{t^2 - 1} \frac{\partial^{n-1}}{\partial t^{n-1}}(t^2 - 1)^n$$

which gives a normalization for the leading term of $P_n$ that we will choose now and the functions $Q_n$ can be written as

$$Q_n(t) = \epsilon_n P_n(t) \arctanh \left( \frac{1}{t} \right) + \frac{W_n(t)}{t^2 - 1}$$

with $W_n$ being polynomials, and $\epsilon_n$ a real sequence given by

$$\epsilon_n = \frac{4^{-n}n(n+1)}{n!^2}$$

**Lemma 6.** (already proved in [9]) Let $F \in \mathbb{C}(z_1)[z_2]$ and

$$f(t) = F \left( t, \arctanh \left( \frac{1}{t} \right) \right)$$

We consider the following field extension and monodromy group

$$K = \mathbb{C} \left( t, \arctanh \left( \frac{1}{t} \right), \int f dt \right), \quad G = \sigma(K, \mathbb{C}(t))$$

If $G$ is abelian, then

$$\frac{\partial}{\partial \alpha} \text{Res}_{t=\infty} F \left( t, \arctanh \left( \frac{1}{t} \right) + \alpha \right) = 0 \quad \forall \alpha \in \mathbb{C}$$

where Res correspond to the residue.

**Proof.** We will consider two paths, the “eight” path $\sigma_1$ around the singularities $-1$ and $1$, and the path $\sigma_2$ around infinity. At infinity, the function $F \left( t, \arctanh \left( \frac{1}{t} \right) + \alpha \right)$ will have a series expansion of the kind

$$\int F \left( t, \arctanh \left( \frac{1}{t} \right) + \alpha \right) dt = \sum_{n=n_0}^{\infty} a_n(\alpha)t^n + r(\alpha)\ln t$$

because the function $\arctanh \left( \frac{1}{t} \right)$ has a regular point at infinity. Let us now consider the monodromy commutator

$$\sigma = \sigma_2^{-1} \sigma_1^{-\frac{\pi}{2\pi}} \sigma_2 \sigma_1^{\frac{\pi}{2\pi}}$$
We have that $\sigma_1^\beta (f) = F(t, \text{arctanh} \left( \frac{1}{t} \right) + \beta)$ and $\sigma_2(\ln t) = \ln t + 2i\pi$. We deduce that

$$\sigma(f) = f + r(\beta) - r(0)$$

This $r(\alpha)$ corresponds to the residue of $F \left( t, \text{arctanh} \left( \frac{1}{t} \right) + \alpha \right)$ at infinity. If the monodromy is commutative, then the commutator $\sigma$ should act trivially on $f$. This is the case only if $r(\beta) - r(0) \forall \beta \in \mathbb{Z}$. The function $r$ is a polynomial in $\beta$, so $r(\beta) - r(0) \forall \beta \in \mathbb{C}$. This gives us the formula of the lemma

$$\frac{\partial}{\partial \alpha} \text{Res}_{t=\infty} F \left( t, \text{arctanh} \left( \frac{1}{t} \right) + \alpha \right) = 0 \forall \alpha \in \mathbb{C}$$

\[\square\]

**Definition 4.** Let $V = r^{-1}U(\theta)$ be a meromorphic homogeneous potential of degree $-1$ such that $c = (1, 0)$ is a Darboux point with multiplier $-1$, and we note $\text{Sp} \left( \nabla^2 V(c) \right) = \{2, 1/2(i-1)(i+2)\}$. Let $k \in \mathbb{N}^*$ be fixed and $(V_E_k)$ the $k$-th order variational equation near $c$. We suppose $(V_{E_{k-1}})$ integrable. Then we will say that the integrability constraint of $(V_E_k)$ is non degenerated if

$$\frac{\partial}{\partial \alpha} \text{Res}_{t=\infty} (t^2 - 1)^k(Q_i + \epsilon_i \alpha P_i)^{k+1} \neq 0$$

Now we can ask why such definitions. This is because non degeneracy implies a rigidity property. If we take two integrable potentials “close” enough (meaning that enough derivative on some Darboux point are equals), then they should be equal. Remark that our non degeneracy constraint could be generalized in other cases.

Behind this, there is the following philosophy. Let $V : \mathbb{C}^n \rightarrow \mathbb{C}$ be a homogeneous potential such that $c = (1, 0, \ldots, 0)$ be a non degenerated Darboux point. Let $k \in \mathbb{N}^*$ be fixed and $(V_E_k)$ the $k$-th order variational equation near $c$. We suppose $(V_{E_{k-1}})$ integrable. Then we will say that the integrability constraint of $(V_E_k)$ is non degenerated if the abelianity of the identity component of the Galois group of $(V_E_k)$ depend explicitly on the $k+1$-th derivatives of $V$ on $c$. This notion of “explicit dependence” depend on the degree and dimension you consider, and also the rigidity result you want. Such a procedure in fact suppose that we want to use integrability conditions to express the $k+1$-th derivatives with lower order derivatives, and then apply a recurrence property. Once first derivatives are determined,
the higher ones will be determined every time this non degeneracy condition is satisfied (and in practice, it is generically satisfied if the first order Galois group is not too simple).

**Theorem 7.** Let \( V : \mathbb{C}^2 \to \mathbb{C} \) be a integrable meromorphic homogeneous potential of degree \(-1\) such that \( c = (1, 0) \) be a Darboux point with multiplier \(-1\). Suppose it exists \( k_0 \) such that integrability constraint of \((VE_k)\) is non degenerated \( \forall k \geq k_0 \). Let \( \tilde{V} \) be another integrable meromorphic homogeneous potential of degree \(-1\) such that

\[
\frac{\partial^{i+j}}{\partial q_1^i \partial q_2^j} V(c) = \frac{\partial^{i+j}}{\partial q_1^i \partial q_2^j} \tilde{V}(c) \quad \forall i + j \leq k_0
\]

Then \( V = \tilde{V} \)

**Proof.** Let us prove it by recurrence. Suppose that all \((VE_i)\) are integrable up to \( i = k - 1 \geq k_0 - 1 \) and non degeneracy of the integrability condition at all orders \( \geq k_0 \). The \( k \)-th variational equation is written (one of them)

\[
\ddot{X}(t) = \frac{1}{\phi(t)^3} \begin{pmatrix} 2 & 0 \\ 0 & \lambda \end{pmatrix} X(t) + \left( \sum_{i=2}^{k} \frac{1}{\phi(t)^{i+2}} \sum_{j=0}^{i} a^{(j)}_i X^j_1 X^{i-j}_1 \right) \left( \sum_{i=2}^{k} \frac{1}{\phi(t)^{i+2}} \sum_{j=0}^{i} b^{(j)}_i X^j_2 X^{i-j}_1 \right)
\]

(6)

Here we use the following abusive notation. The functions \( X^j_1 X^{i-j}_1 \) are not unknowns of the equation. They corresponds to solutions of linear differentials equations systems, with possibly non homogeneous terms. A more complete description of the construction of higher variational equations can be found in [1]. We also suppose by recurrence that the coefficients \( a^{(j)}_i, b^{(j)}_i \) are already determined. These coefficients corresponds to the derivatives \( \partial_1^i \partial_2^j V(c), \ i + j \leq k \). Let us now prove that the coefficients \( a^{(j)}_k, b^{(j)}_k \) are uniquely determined.

We have that the constants \( a^{(j)}_k, b^{(j)}_k \) are only in equation (6). They do not appear in the linear differentials equations systems whose solutions are the \( X^j_2 X^{i-j}_1 \). This is because they correspond to the term of highest order in the series expansion. Then the monodromy of \((VE_k)\) is an affine function in \( a^{(j)}_k, b^{(j)}_k \). We now use Euler relation

\[
q_1 \partial_{q_1} V + q_2 \partial_{q_2} V = -V
\]

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and we derive it in $q_1$ or $q_2$ enough, and then evaluate it on $(q_1, q_2) = (1, 0)$. This gives us all derivatives of order $\leq k + 1$ except

$$\frac{\partial^{k+1}}{\partial q_2^{k+1}} V(c)$$

Then all coefficients are known except $b_k^{(0)}$. We need to determine it using the fact that $V$ is integrable. We write the solution of the second equation of (6)

$$X(t) = X_{\text{hom}}(t) + X_{\text{part}_1}(t) + b_k^{(0)} X_{\text{part}_2}(t)$$

and we apply to it a monodromy commutator $\sigma = \sigma_1^{-\alpha} \sigma_2^{-1} \sigma_1^\alpha \sigma_2$ (with the same notation as in the proof of Lemma 6). This gives

$$\sigma(X) = \sigma(X_{\text{hom}}) + \sigma(X_{\text{part}_1}) + b_k^{(0)} \sigma(X_{\text{part}_2})$$

Now let us look at $X_{\text{part}_2}$. It can be computed and one solution is

$$X_{\text{part}_2}(t) = \int (t^2 - 1)^k Q_i^{k+1} dt$$

We now apply lemma 6 which says that the monodromy element $\sigma$ add to such function the constant

$$G(\alpha) = \text{Res}_{t=\infty} (t^2 - 1)^k Q_i^{k+1} - \text{Res}_{t=\infty} (t^2 - 1)^k Q_i^{k+1}$$

This is not zero by hypothesis and so we get this equation

$$0 = \sigma(X) - X = \sigma(X_{\text{hom}}) - X_{\text{hom}} + \sigma(X_{\text{part}_1}) - X_{\text{part}_1} + b_k^{(0)} G(\alpha)$$

that we can solve in $b_k^{(0)}$. So $b_k^{(0)}$ is uniquely determined. The meromorphic property of $V$ allows us to conclude. All derivatives on $c$ are uniquely determined, then $V$ is uniquely determined.

This rigidity allows to produce uniqueness theorems, but never existence theorems because we never know if all integrability conditions are compatible or not. For classification results as Theorem 3, there is then a “praying” phase, where it is needed to find explicit integrable potentials in the literature, whose first derivatives can be arbitrary. Direct search help a lot like [1], but if it is not successful, all previous work is almost useless (because
only unicity is proved, not existence). This suggests to carefully read the literature before beginning to prove similar theorems as by using the non degeneracy condition. This phase cannot be algorithmically made because computing higher variational equation will only give a series expansion of the corresponding potential (and because computer limitations, with often less than 10 terms).

3.1. Application to the eigenvalue 0

**Theorem 8.** Let \( V \) be a meromorphic homogeneous potential of degree \(-1\) in the plane. Suppose that \( c = (1,0) \) is a Darboux point with multiplier \(-1\), \( \text{Sp}(\nabla^2 V(c)) = \{2, 0\} \) and that \( V \) is meromorphically integrable. Then \( V = 1/q_1 \).

**Proof.** We just need to prove the non degeneracy theorem. The functions \( P_1, Q_1 \) for the eigenvalue 0 are the following

\[
P_1 = 1 \quad Q_1 = \text{arctanh} \left( \frac{1}{t} \right) - \frac{t}{t^2 - 1}
\]

We need to look at the following residue

\[
\text{Res}_{t=\infty} \left( (t^2 - 1)^{k+1} \left( \text{arctanh} \left( \frac{1}{t} \right) + \alpha - \frac{t}{t^2 - 1} \right) \right)^{k+2}
\]

and it should be independent of \( \alpha \). The easiest coefficient to study (and non trivial) seems to be the coefficient in \( \alpha^{k+1} \). We find after simplification

\[
S_k = \text{Res}_{t=\infty} \left( (t^2 - 1)^{k+1} \left( (k+2) \text{arctanh} \left( \frac{1}{t} \right) - \frac{kt + 2t}{t^2 - 1} \right) \right)
\]

By expanding, we remark that the second term always give a zero residue. Indeed, in the expansion, the faction simplifies and we get a polynomial. Then, we will compute

\[
S_k = \text{Res}_{t=\infty} \left( (k+2) (t^2 - 1)^{k+1} \text{arctanh} \left( \frac{1}{t} \right) \right) = \frac{k+2}{2} \int_{-1}^{1} (t^2 - 1)^{k+1} dt > 0
\]

The last equality is produced with the expansion of \( \text{arctanh} \left( \frac{1}{t} \right) \) at infinity. We deduce that

\[
S_k \neq 0 \quad \forall k \geq 1
\]
So we now know that there is a unique potential with \((1, 0)\) as Darboux point with multiplier \(-1\) and eigenvalue 0. The potential \(1/q_1\) satisfy these conditions, and is integrable because it is invariant by translation. \(\square\)

**Conclusion.** So, after rotation, an integrable potential \(V\) with a zero eigenvalue near a non degenerate Darboux point correspond to the potential

\[
V = \frac{a}{q_1}, \quad a \in \mathbb{C}^*
\]

3.2. Application to the eigenvalue 2

In the case of the eigenvalue 2, the Hessian matrix should be diagonalizable. This is a condition for integrability of first order variational equation. First we will prove the non degeneracy condition.

**Theorem 9.** For eigenvalue 2, the integrability condition is non degenerate at order \(k \geq 3\), and degenerated at order \(k = 2\).

**Proof.** We need to look at the following residue

\[
\text{Res}_{t=\infty} (t^2 - 1)^{k+1} \left( -\frac{6t^2 - 4}{t^2 - 1} + 6t \arctanh \left( \frac{1}{t} \right) \right)^{k+2}
\]

and this residue should be independent of \(\alpha\) to prove non degenerescence. We will look at

\[
S_k^{(1)} = \text{Res}_{t=\infty} (t^2 - 1)^{k+1} t^{k+1} \left( -\frac{6t^2 - 4}{t^2 - 1} + 6t \arctanh \left( \frac{1}{t} \right) \right)
\]

which corresponds to the coefficient in \(\alpha^{k+1}\) (after simplifying a non zero factor). But this sequence is not always non zero. We will also need to look at another one

\[
S_k^{(2)} = \text{Res}_{t=\infty} (t^2 - 1)^{k+1} t^{k+1} \left( -\frac{6t^2 - 4}{t^2 - 1} + 6t \arctanh \left( \frac{1}{t} \right) \right)^2
\]

Then, we want to prove

\[
S_k^{(1)} \neq 0 \text{ or } S_k^{(2)} \neq 0 \quad \forall k \geq 2
\]

More precisely, we will prove that

\[
S_{2k}^{(1)} \neq 0 \text{ and } S_{2k+1}^{(2)} \neq 0 \quad \forall k \geq 1
\]
We use Mgfun to build a recurrence and then an explicit expression of these sequences. We find the following recurrence for $S^{(1)}_{2n}$

$$
64(2n + 3)(2n + 1)(6n + 11)(n + 1)^2 f(n) - 
(20736n^5 + 152064n^4 + 439200n^3 + 622752n^2 - 431784n - 116328)
$$

$$
f(n + 1) + 36(6n + 3)(3n + 5)(3n + 4)(6n + 13)(6n + 17) f(n + 2)
$$

This recurrence can be solved explicitly and gives the formula

$$
S^{(1)}_{2n} = -\frac{\pi \Gamma(2n + 2) 27^{-n}}{72\Gamma(n + \frac{7}{6}) \Gamma(n + \frac{11}{6})}
$$

This expression never vanish. We do the same for $S^{(2)}_{2n+1}$. We find a 3-th order recurrence and solve it

$$
S^{(2)}_{2n+1} = -\frac{\pi 27^{-n} \Gamma(2n + 3)}{3456\Gamma(n + \frac{7}{6}) \Gamma(n + \frac{3}{6})} \sum_{k=0}^{n-1} \left( \frac{(3k + 4) \Gamma(k + 5/3) \Gamma(k + 7/3)}{(k + 1)(k + 2)(2k + 3) \Gamma(k + \frac{13}{6}) \Gamma(k + \frac{11}{6})} \right)
$$

Using this expression, we find that $S^{(2)}_{2n+1}$ never vanish for $n \geq 1$. This prove the non degeneracy condition for order $\geq 3$. At order 2, the two formulas vanish. Since in this case the residue is a polynomial in $\alpha$ of degree at most 2, this implies that the residue is constant. So the $\alpha$ derivative is zero and integrability condition is degenerated. 

Now we know that after fixing the 3-th order derivatives of the potential $V$, the integrability condition completely determine the potential. Still we need to find such potentials. We already know the potential

$$
\frac{a}{q_1} + \frac{b}{q_2}, \quad a, b \in \mathbb{C}^*
$$

which is integrable. Computation gives that Darboux points have the eigenvalue 2. So we need to prove that after rotation, all possible 3-th order derivatives can be produced. As we will see, this is only “almost” true. There is an unexpected set of parameters impossible to produce. These parameters correspond in fact to the Hietarinta potential.

**Theorem 10.** Let $V$ be a meromorphic homogeneous potential of degree $-1$ in the plane with $c = (1,0)$ a Darboux point of $V$ with multiplier $-1$ and $\nabla^2 V(c) = 2I_2$. Then it corresponds to a potential of the form

$$
\frac{a}{q_1} + \frac{b}{q_2}, \quad a, b \in \mathbb{C}^* \quad (7)
$$
after rotation except if $V$ possess the following expansion

$$V(c + q) = V(c) + aq_1 + (q_1^2 + q_2^2) + dq_1q_2^2 + 3dq_1q_2^2 \pm 2idq_2^3 + o(q^3)$$

**Proof.** We expand $V$ on $c = (1, 0)$ which gives

$$V(c + q) = V(c) - q_1 + (q_1^2 + q_2^2) + aq_1^3 + bq_1q_2 + cq_1q_2^2 + dq_2^3 + o(q^3)$$

We use Euler equation

$$q_1\partial_{q_1}V + q_2\partial_{q_2}V = -V$$

and after derivation, we have

$$2\partial_{1,1}V + q_1\partial_{1,1,1}V + q_2\partial_{1,1,2}V = -\partial_{1,1}V$$

$$2\partial_{1,2}V + q_1\partial_{1,1,2}V + q_2\partial_{1,2,2}V = -\partial_{1,2}V$$

$$2\partial_{2,2}V + q_1\partial_{1,2,2}V + q_2\partial_{2,2,2}V = -\partial_{2,2}V$$

Thanks to this, we have the following values

$$\partial_{1,1,1}V(c) = -6 \quad \partial_{1,1,2}V(c) = 0 \quad \partial_{1,2,2}V(c) = -6$$

Then the series expansion of $V$ on $c$ is always of the form

$$V(c + q) = 1 - q_1 + (q_1^2 + q_2^2) - (q_1^3 + 3q_1q_2^2 + dq_2^3) + o(q^3)$$

where $d \in \mathbb{C}$. We want now prove that such an expansion can correspond to the expansion of the potential (7) after rotation. So we will make a rotation of the coordinates $q_1, q_2$. After rotation, the potentials (7) can be written

$$\frac{a}{zq_1 - wq_2} + \frac{b}{wq_1 + zq_2}, \quad w^2 + z^2 = 1, a, b \in \mathbb{C}^*$$

The condition of possessing a Darboux point at $c = (1, 0)$ with multiplier $-1$ implies that this family of potentials can be written

$$\frac{z^3}{(z^2 + w^2)(zx - wy)} + \frac{w^3}{(z^2 + w^2)(wx + wy)}, \quad w^2 + z^2 = 1$$

We make series expansion near $c = (1, 0)$ and by identification, we get

$$-\frac{z^2 + w^2}{wz} = d \quad w^2 + z^2 = 1$$

This produce the solution

$$w = \frac{1}{\sqrt{2}}\sqrt{\frac{4 + d^2 + \sqrt{4d^2 + d^4}}{4 + d^2}}$$

which is valid for $d \neq \pm 2i$. For $d = \pm 2i$, there are no solutions. \qed
3.3. The irreducible case

There is still one open possibility. It corresponds in fact to the Hietarinta potential

\[ V = \frac{a(q_1^2 + q_2^2)}{(q_1 + \epsilon iq_2)^3} + \frac{a}{q_1 + \epsilon iq_2}, \quad a \in \mathbb{C}^*, \quad \epsilon = \pm 1 \]

Contrary to the other ones, this one can not be decoupled after rotation. It still possess an additional first integral quadratic in momenta, but the quadratic term is \((p_1 + ip_2)^2\), and effectively, this quadratic form can never be decoupled after rotation (any rotation, even a complex one). This is why it is a special case.

**Theorem 11.** Let \( V \) be a meromorphic homogeneous potential of degree \(-1\) in the plane with the following expansion at \( c = (1, 0) \)

\[ V(c + q) = V(c) + bq_1 + (q_1^2 + q_2^2) + dq_1^3 + 3dq_1q_2^2 \pm 2idq_2^3 + o(q^3) \]

If \( V \) is meromorphically integrable, then it corresponds after rotation to a potential of the form

\[ V = \frac{a(q_1^2 + q_2^2)}{(q_1 + \epsilon iq_2)^3} + \frac{a}{q_1 + \epsilon iq_2}, \quad a \in \mathbb{C}^*, \quad \epsilon = \pm 1 \]

**Proof.** We already know that if such potential satisfy these conditions, it is unique (after fixing the sign of \( \epsilon \)). Let us check that the Hietarinta potential satisfied them.

\[ V = a \frac{(q_1^2 + q_2^2)}{(q_1 + \epsilon iq_2)^3} + \frac{1}{q_1 + \epsilon iq_2}, \quad \epsilon = \pm 1 \]

We will study the case \( \epsilon = +1 \). After rotation, we get

\[ V = a(\frac{q_1^2 + q_2^2}{(q_1 + iq_2)^3}) + ab \frac{1}{q_1 + iq_2}, \quad a, b \in \mathbb{C}^* \]

The condition of having a Darboux point at \( c = (1, 0) \) with multiplier \(-1\) gives

\[ V = -\frac{1}{2} \frac{q_1^2 + q_2^2}{(q_1 + iq_2)^3} + \frac{3}{2(q_1 + iq_2)} \]

We compute the series expansion at \( c = (1, 0) \) and this gives exactly the good expansion. \( \blacksquare \)
4. Case of the eigenvalue $-1$

The case of eigenvalue $-1$ is much more difficult, because higher variational equations do not seem to have particular properties, and the non degenerescence theorem does not apply. In fact, we will even prove the reverse, that the integrability condition is always degenerated. We need to use a completely different method. We already guess that this case will correspond to the potential invariant by rotation. Then, to see something special, it seems to be a good idea to compute all these higher variational equations in polar coordinates, because it corresponds to the action-angles coordinates for this potential. This will be a priori much more difficult, but in fact, the eigenvalue $-1$ will give us lots of simplifications.

**Theorem 12.** Let $V = r^{-1}U(\theta)$ be a meromorphic potential homogeneous of degree $-1$ in the plane with $c = (1,0)$ a Darboux point of $V$ with multiplier $-1$. Suppose that $\text{Sp}(\nabla^2 V(c)) = \{2,-1\}$. If $V$ is integrable, then

$$V = r^{-1}$$

**Proof.** The potential $V = r^{-1}U(\theta)$ gives the following differential equations in polar coordinates

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2}U(\theta), \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^2}U'(\theta).$$

Let us linearize this equation near a homothetic orbit corresponding to the critical point $0$ of $U$. We suppose moreover that $U''(0) = 0$, which corresponds to $\text{Sp}(\nabla^2 V(c)) = \{2,-1\}$. We get at first order

$$\ddot{r} = \frac{2U(0)}{\phi(t)^3}r, \quad \phi(t)\ddot{\theta} + 2\dot{\phi}(t)\dot{\theta} = \frac{U''(0)}{\phi(t)^2}\theta = 0.$$ 

We choose the normalization of $c$ such that the multiplier is $-1$ and we make the variable change $\phi(t) \rightarrow t$ which gives

$$2 \left(\frac{1}{t} + 1\right)\ddot{r} - \frac{1}{t^2}\ddot{r} = \frac{2}{t^3}r, \quad 2(t + 1)\ddot{\theta} + \left(\frac{3}{t} + 4\right)\dot{\theta} = 0.$$ 

Then we make the change of variables $\sqrt{1 + t^{-1}} \rightarrow t$ which gives

$$\frac{1}{2} \left(t^2 - 1\right)\ddot{r} + 2\dot{r}t - 2r = 0, \quad \frac{1}{2} \left(t^2 - 1\right)\ddot{\theta} = 0.$$
Of course these equations are integrable (because they correspond to an integrable case of Theorem 4 and the solutions are
\[ r(t) = C_1 P_2 + C_2 Q_2, \quad \theta(t) = C_3 t + C_4 \]

Now we take a look at second-order variational equations. We find
\[ \frac{1}{2} (t^2 - 1) \ddot{r} + 2t \dot{r} - \frac{1}{2} \dot{\theta}^2 = 2r - 3(t^2 - 1)r^2 \\
\frac{1}{2} \ddot{\theta} + \frac{1}{2} (t^2 - 1) r \ddot{\theta} + 2tr \dot{\theta} + (t^2 - 1) r \dot{\theta} = \frac{1}{2} t^2 - 1 \frac{U^{(3)}(0) \theta^2}{t^2 - 1} \]

by using the notation abuse consisting to consider that the terms of degree 2 in \( r \) and \( \theta \) are replaced by the solutions of the first-order variational equations. The first equation integrates because
\[ -\frac{1}{2} \dot{\theta}^2 = -\frac{1}{2} C_3^2 \]
corresponds to add to the solution the constant \(-\frac{1}{4} C_3^2\) and that the monodromy of
\[ \int (t^2 - 1) Q_2^3 dt \]
is commutative. For the second equation, we find
\[ \frac{1}{2} \ddot{\theta} + 2(t(C_1 P_2 + C_2 Q_2) C_3 + (t^2 - 1)(C_1 \dot{P}_2 + C_2 \dot{Q}_2) C_3 = \frac{1}{2} \frac{1}{t^2 - 1} U^{(3)}(0)(C_3 t + C_4)^2 \]

The solution can be written as
\[ 2 \int \left[ -2(t(C_1 P_2 + C_2 Q_2) C_3 -(t^2 - 1)(C_1 \dot{P}_2 + C_2 \dot{Q}_2) C_3 + \frac{1}{2} U^{(3)}(0) \frac{(C_3 t + C_4)^2}{t^2 - 1} \right] dt^2 \]

We have that
\[ P_2 = 4t \quad Q_2 = \frac{3}{8} t \arctanh \left( \frac{1}{t} \right) + \frac{1}{4} - \frac{3t^2}{8(t^2 - 1)} \]

The terms in \( P_2 \) are polynomials and integrate well. For the terms in \( Q_2 \), we find the following expression
\[ \int \int 2t Q_2 + (t^2 - 1) \dot{Q}_2 dt dt = \int \int \frac{3}{8} (3t^2 - 1) \arctanh \left( \frac{1}{t} \right) - \frac{9}{8} t dt dt = \frac{3}{32} (t^2 - 1)^2 \arctanh \left( \frac{1}{t} \right) - \frac{3}{32} t^3 \]
Then the only term left is

\[ \int \int 1^2 U^{(3)}(0) \frac{(C_3 t + C_4)^2}{t^2 - 1} dt \, dt \]

For we integrate and we find that

\[ \int \int 1^2 U^{(3)}(0) \frac{(C_3 t + C_4)^2}{t^2 - 1} dt \, dt \in \mathbb{C} \left[ t, \arctanh \left( \frac{1}{t} \right), \ln \left( t^2 - 1 \right) \right] \quad (9) \]

Then the second-order variational equation is always integrable, and moreover we have that

- If \( U^{(3)}(0) \neq 0 \) then the Galois group of \( \mathbf{S} \) is \( \mathbb{C}^2 \)
- If \( U^{(3)}(0) = 0 \) then the Galois group of \( \mathbf{S} \) is \( \mathbb{C} \)

We have in fact already more or less proved the following

**Theorem 13.** Let \( V \) be a meromorphic potential homogeneous of degree \(-1\) in the plane with \( c = (1, 0) \) a Darboux point of \( V \) with multiplier \(-1\). Suppose that \( \text{Sp}(\nabla^2 V(c)) = \{2, -1\} \). Suppose that \( U^{(i)}(0) = 0 \) \( \forall i = 1..k \) then the fact that the identity component of the Galois group is abelian or not does not depend on the value of \( U^{(k+1)}(0) \).

**Remark 3.** This imply the degenerescence of the integrability condition at all orders.

**Proof.** The case of order 2 correspond to the previous proof. Let us look now at order \( k \). We pick in the equations the non homogeneous terms where \( U^{(k+1)}(0) \) appear. The only equation where such term appear is the following

\[ \frac{1}{2} \ddot{\theta} = \frac{1}{2} \frac{U^{(k+1)}(0)}{t^2 - 1} \theta^k \]

(we have suppressed all other non homogeneous terms for which \( U^{(k+1)}(0) \) do not appear). We conclude with the following property

\[ \int \int \frac{(at + b)^k}{t^2 - 1} dt \, dt \in \mathbb{C}[t, \arctanh \left( \frac{1}{t} \right), \ln \left( t^2 - 1 \right)] \]

which can be checked using recursive integration by parts. This term has then a commutative monodromy, and then the fact that the identity component of the Galois group is abelian or not does not depend on the value of \( U^{(k+1)}(0) \). \( \square \)
Remark 4. Still we remark that in fact this integral will grow a little the Picard Vessiot extension, and the Galois group become at least $\mathbb{C}^2$ for a generic value of $U^{(k+1)}(0)$, but this does not give us the non integrability. This is precisely why this case is particularly difficult. We cannot use non degenerescence properties, and so we need to keep these unknown coefficients and go higher in the order of variational equations.

Let us look now at order 3. We find

$$\frac{1}{2} \ddot{\theta} + \frac{1}{2}(t^2-1)r \ddot{\phi} + 2tr \dot{\phi} + (t^2-1) \dot{r} \dot{\phi} = \frac{1}{2} \frac{U^{(3)}(0)}{t^2 - 1} y_1(t) + \frac{1}{6} \frac{U^{(4)}(0) \theta^3}{(t^2 - 1)^2} r \theta^2$$

and $y_1$ satisfies the following system of differential equations

$$\dot{y}_1 = 2y_2 \quad \dot{y}_2 = y_3 + \frac{z\theta(t)^3}{t^2 - 1} \quad \dot{y}_3 = 2\frac{z\theta(t)^2 \theta'(t)}{t^2 - 1}$$

with $z = U^{(3)}(0)$ and by using the notation abuse consisting to consider that the terms of degree 3 in $r$ and $\theta$ are replaced by the solutions of the first order variational equations. We will only consider a “normal perturbation”, which consists to put $r = 0$ in all terms of degree 3. Moreover, to simplify, we will choose $\theta(t) = 1$ as solution for the first order variational equation. We also have that the terms

$$\frac{1}{2} \ddot{\theta} = \frac{1}{2}(t^2 - 1)r \ddot{\phi} + 2tr \dot{\phi} + (t^2 - 1) \dot{r} \dot{\phi}$$

will have no impact on the Galois group because these terms are solutions of a linear differential system of equations with non-homogeneous terms, but these non-homogeneous terms vanish for $r = 0, \dot{\phi} = 0$. Then the solutions of these systems will be solutions of the second order variational equation, and we have already determined that the corresponding Galois group is abelian. Because of that, we come down to the study of the equation

$$\frac{1}{2} \ddot{\theta} = \frac{1}{2} \frac{U^{(3)}(0)}{t^2 - 1} y_1(t) + \frac{1}{6} \frac{U^{(4)}(0) \theta^3}{t^2 - 1}.$$

The term in $\theta^3$ integrates like (9) and so this term will not give a non-abelian Galois group. The terms left are

$$\frac{1}{2} \ddot{\theta} = \frac{1}{2} \frac{z}{t^2 - 1} y_1 \quad \dot{y}_1 = 2y_2 \quad \dot{y}_2 = y_3 + \frac{z\theta^3}{t^2 - 1} \quad \dot{y}_3 = 2\frac{z\theta^2 \dot{\theta}}{t^2 - 1}$$
We solve and we find the following solution (a well chosen one)

\[ y_1(t) = 2z \int \frac{t \arctanh \left( \frac{1}{t} \right)}{t^2 - 1} dt + \frac{\theta(t)^3}{t^2 - 1} dt \quad \theta(t) = \int \frac{z}{t^2 - 1} y_1(t) dt \]

The \( \theta(t) \) in terms of degree 3 is replaced by the solution of the first order variational equation, and we had chosen \( \theta(t) = 1 \). We have then

\[ y_1(t) = 2z \left( -t \arctanh \left( \frac{1}{t} \right) - \frac{1}{2} \ln \left( t^2 - 1 \right) \right) \]

We know that \( \int \frac{z}{t^2 - 1} y_1(t) dt \) is in the Picard-Vessiot field and we have

\[ \int \frac{2}{t^2 - 1} \left( t \arctanh \left( \frac{1}{t} \right) + \frac{1}{2} \ln \left( t^2 - 1 \right) \right) dt = \]

\[ \frac{1}{4} \ln^2(t - 1) - 2 \ln(t - 1) - \text{dilog}(t + 1) - \]

\[ \frac{1}{4} \ln^2(t + 1) - \frac{1}{2} \ln(t - 1) \ln(t + 1) \]

All the terms are in \( \mathbb{C}[t, \arctanh \left( \frac{1}{t} \right), \ln \left( t^2 - 1 \right)] \) except one, the dilogarithmic term

\[ \text{dilog} (t + 1) = \int \frac{\ln(t + 1)}{t} dt \]

With the same idea as in Lemma 6, we get that this term has a noncommutative monodromy because of the following residue in 0

\[ \text{Res}_{t=0} \frac{\ln(t + 1) + \alpha}{t} = \alpha \]

which depends explicitly in \( \alpha \).

Here we have then proved that the 3-th order integrability condition is \( U^{(3)}(0) = 0 \). We remark this is a very special case because the 4-th order derivatives do not appear in the constraint. We will now prove that a very similar scheme appear at higher order.

Let us prove now Theorem 12 by recurrence. Suppose that \( U^{(i)}(0) = 0 \) \( \forall i = 1..k \) and let us look at the \( 2k - 1 \)-th order variational equation. We want to prove that \( U^{(k+1)}(0) = 0 \). We have already initialized this recurrence
with \( k = 2 \). In this equation, there are the derivatives \( U^{(i)}(0) \) for \( i = k + 1 \ldots 2k \). We have already proved that the coefficient \( U^{(2k)}(0) \) will have no impact on integrability. We get then (ignoring terms with no influence on integrability, and considering only the normal perturbation)

\[
\frac{1}{2} \dddot{\theta} + \frac{1}{2} (t^2 - 1) r \dot{\theta} + 2tr \dot{\theta} + (t^2 - 1) \ddot{r} \dot{\theta} = \sum_{i=0}^{k-1} \frac{1}{(k + i)!} \frac{U^{(k+1+i)}(0)}{t^2 - 1} \theta^{k+i}
\]

We use the abusing notation as before and so the \( \theta^{k+i} \) are solutions of differential equations system with non homogeneous terms coming from lower orders. As before, we will take \( \theta(t) = 1 \) as solution for the first order variational equation, and thus we will have \( \dot{\theta} = 0 \). This produce a huge simplification. We can choose for example the solution 0 for the terms

\[
\frac{1}{2} (t^2 - 1) r \ddot{\theta} + 2tr \dot{\theta} + (t^2 - 1) \ddot{r} \dot{\theta}
\]

which are solutions of homogeneous linear differential systems. We then only have the following

\[
\frac{1}{2} \dddot{\theta} = \sum_{i=0}^{k} \frac{1}{k!} \frac{U^{(k+1+i)}(0)}{t^2 - 1} \theta^{k+i}
\]

Still, these \( \theta^{k+i} \) are not easy to compute a priori. We build the differential system for these \( \theta^{k+i} \) by posing

\[
y^{(i)}_j = \dot{\theta}^i \theta^{k+i-j} \quad j = 0 \ldots k + i
\]

and we have then

\[
y^{(i)}_j = (k + i - j)y^{(i)}_{j+1} + j \sum_{n=0}^{k} \frac{2}{k!} \frac{U^{(k+1+n)}(0)}{t^2 - 1} \theta^{2k+n+i-j} \dot{\theta}^{i-1}
\]

The degree in \( \theta \) of the second term is \( 2k + n + i - 1 \). But we study the variational equation at order \( 2k - 1 \), then we can suppress all terms of order higher than \( 2k - 1 \). Then there is only one term in the sum

\[
y^{(0)}_j = (k - j)y^{(0)}_{j+1} + j \frac{2}{k!} \frac{U^{(k+1)}(0)}{t^2 - 1} \theta^{2k-j} \dot{\theta}^{i-1} \quad y^{(i)}_j = (k - j)y^{(i)}_{j+1} \quad i \geq 1
\]
The only non homogeneous terms are of maximal degree $2k - 1$, and we had chosen $\theta(t) = 1$ as solution for first order variational equation. Then all term with $\dot{\theta}$ vanish. We have then only

$$y_1^{(i)} = (k - 1)y_{1+1}^{(i)} + \frac{2}{k!} \frac{U^{(k+1)}(0)}{t^2 - 1} \quad y_j^{(i)} = (k - j)y_{j+1}^{(i)} \quad (i, j) \neq (0, 1)$$

The system of equation number $i, i \neq 0$ for the $y_j^{(i)}$, is a linear homogeneous differential equation system. We can then take the following solution

$$y_j^{(i)} = 0 \quad \forall i \neq 1$$

We look now for $i = 0$. We have

$$y_0^{(i)} = k y_1^{(0)}$$
$$y_1^{(0)} = (k - 1)y_2^{(0)} + \frac{2}{k!} \frac{U^{(k+1)}(0)}{t^2 - 1}$$
$$y_j^{(0)} = (k - j)y_{j+1}^{(0)} \quad j = 2 \ldots k$$

We find the following solution of this system (very well chosen)

$$y_0^{(0)} = k \int \int \frac{2}{k!} \frac{U^{(k+1)}(0)}{t^2 - 1} \, dt \, dt = -\frac{2U^{(k+1)}(0)}{(k - 1)!} \left( t \, \text{arctanh} \left( \frac{1}{t} \right) + \frac{1}{2} \ln \left( t^2 - 1 \right) \right)$$

and we remplace it in the initial equation

$$\ddot{\theta} = -\frac{4U^{(k+1)}(0)}{(k - 1)!} \frac{1}{t^2 - 1} \left( t \, \text{arctanh} \left( \frac{1}{t} \right) + \frac{1}{2} \ln \left( t^2 - 1 \right) \right)$$

After solving it, we get that

$$\frac{U^{(k+1)}(0)}{(k - 1)!} \int \frac{1}{t^2 - 1} \left( t \, \text{arctanh} \left( \frac{1}{t} \right) + \frac{1}{2} \ln \left( t^2 - 1 \right) \right) \, dt$$

should belong to the Picard-Vessiot field. This is the same integral as before, and its monodromy is not commutative. Then a necessary integrability condition is that $U^{(k+1)}(0) = 0$, which completes the recurrence.

The function $U$ is meromorphic, we have that $U^{(k)}(0) = 0 \forall k \in \mathbb{N}^*$ and so this implies that $U$ is constant. Then

$$V = r^{-1}$$
5. The other eigenvalues

To find all integrable potentials, we would need to study all the other possible eigenvalues. But in the other cases, there are no special simplifications, but moreover, we do not know integrable potentials with these eigenvalues. Then we can suppose that such potentials do not exist, and we can also make a stronger supposition that at some order, there is a condition impossible to satisfy. First of all, we will give some application theorems for the non degenerescence property. This property cannot be checked for all possibilities for the moment, but with an axi-symmetric assumption, such property can be proved.

**Theorem 14.** Let $V = \frac{1}{r}U(\theta)$ be a homogeneous potential of degree $-1$ in the plane in polar coordinates with $U$ meromorphic $2\pi$-periodic. We suppose that

- $V$ is symmetric with respect to the straight line generated by $\theta = 0$
- $U(0) \neq 0$

If $V$ is meromorphically integrable, then $V$ belong to a discrete family

$$\left\{ V_0 = \frac{a_0}{r}, V_1 = \frac{a_1}{r \cos(\theta)}, V_2 = \frac{a_2}{r \cos(2\theta)}, V_3, V_4, V_5, \ldots \right\}$$

More precisely, $V = aV_i, \ a \in \mathbb{C}^*$ with

$$i = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 8 \frac{U''(0)}{U(0)}}$$

**Remark 5.** We say nothing about existence of the potentials $V_i$ (for $i \geq 3$). We only prove that for each eigenvalue $\frac{1}{2}(k-1)(k+2), \ k \in \mathbb{N}$, it exists at most one axi-symmetric integrable potential.

**Proof.** We just need to prove the non degenerescence property for odd orders. Indeed, for even orders, we use Euler relation which gives us all derivatives except one, the derivative in the normal direction to the straight line $\theta = 0$. But for the variational equation of even order $k$, this maximal order derivative is then of odd order $k+1$. This derivative is then automatically 0 because we suppose the symmetry. The non degenerescence is written

$$\frac{\partial}{\partial \alpha} \text{Res}_{t=\infty} (t^2 - 1)^k (Q_n + \alpha \epsilon_n P_n)^{k+1} \neq 0$$
We look at coefficient $\alpha^k$ of the residue.

$$\epsilon_n^k(k + 1) \text{Res}_{t=\infty} (t^2 - 1)^k Q_n P_n = \frac{1}{2} \epsilon_n^k(k + 1) \int_{-1}^{1} P_{n+1}^{k+1} dt$$

by doing the Taylor expansion of $\text{arctanh} \left( \frac{1}{t} \right)$ at infinity and recognizing that this sum can be written as this integral. The integer $k$ is odd, the polynomials $P_n$ are never identically 0, then this coefficient never vanish. This prove non degenerescence.

**Conjecture 1.** Let $V = \frac{1}{r} U(\theta)$ be a homogeneous potential of degree $-1$ in the plane in polar coordinates with $U$ meromorphic $2\pi$-periodic. We suppose it exists $\theta_0$ such that $U'(\theta_0) = 0$ and

$$\frac{U''(\theta_0)}{U(\theta_0)} - 1 \in \left\{ \frac{1}{2}(n - 1)(n + 2), \; k \in \mathbb{N}, \; n \geq 3 \right\}$$

Then it exists a $k \in \mathbb{N}^*$ such that the variational equation at order $k$ is not integrable.

**Conjecture 2.** Let $V = \frac{1}{r} U(\theta)$ be a homogeneous potential of degree $-1$ in the plane in polar coordinates with $U$ meromorphic $2\pi$-periodic. We suppose it exists $\theta_0$ such that $U'(\theta_0) = 0$ and

$$\frac{U''(\theta_0)}{U(\theta_0)} - 1 = \frac{1}{2}(n - 1)(n + 2) \; \; \text{n odd } n \geq 3$$

then the 5-th variational equation is not integrable. If

$$\frac{U''(\theta_0)}{U(\theta_0)} - 1 = \frac{1}{2}(n - 1)(n + 2) \; \; \text{n even } n \geq 4$$

then the 7-th variational equation is not integrable.

The second conjecture seems to be extremely difficult. Solving it would classify completely integrable homogeneous potentials of degree $-1$ in the plane, and would probably allow with some generalization for other degrees to close completely the search of integrable homogeneous potentials (with at least some assumption on Darboux points). A partial proof up to the 5-th variational equation would lead to classification of axi-symmetric integrable
potentials. This would also lead to numerous theorems in higher dimension for potentials possessing discrete symmetry groups. We will prove these conjectures for \( n = 3, 4, 5, 6 \), and give all corresponding expansions of \( V \) integrable at order \( k \) (with \( k \leq 4 \) or \( \leq 6 \) depending on parity). This proves by the way the nonexistence of the potentials \( V_3, V_4, V_5, V_6 \) of Theorem 14. Moreover, we present an algorithm to prove this conjecture for an arbitrary fixed number of eigenvalues.

**Proof.** (Partial proof of conjectures 1 and 2) We will suppose that \( c = (1, 0) \) is a Darboux point and that \( V \) has been normalized such that the multiplier of \( c \) is \(-1\). We note \( \text{Sp}(\nabla^2 V(c)) = \{2, \lambda\} \). The corresponding angle in polar coordinates is \( \theta = 0 \). We begin by odd index eigenvalues, because we will just have to look at order 5. This correspond to \( \lambda = 5, 14 \). We know that the 3-th derivative in \( q_2 \) should be zero for integrability at order 2. The Euler relation for \( V \) gives the following expansion near \( c \)

\[
V = q_1 + \frac{1}{2}(2q_1^2 + \lambda q_2^2) - q_1^3 - \frac{3}{2}\lambda q_1 q_2^2 + q_1^4 + 3\lambda q_1^2 q_2^2 + \lambda q_1 q_2^4 + o(q^4)
\]

where \( z \) is an unknown coefficient. This is this coefficient that our program will determine. To make more readable the results, we will write the solution with an expansion in polar coordinates (even if the variational equations are computed in Cartesian coordinates). We have for the moment (let us recall that the \( k \)-th order variational equation gives constraints on the \( k + 1 \)-th derivatives)

\[
V_3 = \frac{1}{r} \left( 1 + 3\theta^2 + o(\theta^3) \right) \quad V_5 = \frac{1}{r} \left( 1 + \frac{15}{2}\theta^2 + o(\theta^3) \right)
\]

We want now to compute variational equations to very high order to see if we can always be integrable at high order by choosing good values for the derivatives. Let us build a general algorithm for computation of higher variational equation. We already know that we will need to compute the differential systems associated to

\[
y_{i,j,k,l} = \dot{X}_1^i \dot{X}_2^j X_1^k X_2^l
\]

This produce linear differential equations in \( y_{i,j,k,l} \) with \( i + j + k + l \leq n \)
where \( n \) is the order of the expansion. We use the following formula
\[
\dot{y}_{i,j,k,l} = ky_{i+1,j,k+1,l+1} + 4iy_{i+1,j,k,l} - \frac{4ity_{i,j,k,l}}{t^2 - 1} - \frac{4jty_{i,j,k+1,l}}{t^2 - 1} + i\text{ add}(\text{coeff}(\text{coeff}(W_1^n, q^n_1) y_{i-1,j,k+l+p}, n = 0..\infty), p = 0..\infty) +
\]
\[
\text{add}(\text{coeff}(\text{coeff}(W_2^n, q^n_2) y_{i,j-1,k+l+p}, n = 0..\infty), p = 0..\infty)
\]

where
\[
W_1 = \text{subs}\left(q_1 = (t^2 - 1)q_1, q_2 = (t^2 - 1)q_2, \frac{2}{(t^2 - 1)^2} \partial_{q_1} V\right)
\]
\[
W_2 = \text{subs}\left(q_1 = (t^2 - 1)q_1, q_2 = (t^2 - 1)q_2, \frac{2}{(t^2 - 1)^2} \partial_{q_2} V\right)
\]

To simplify, we look at the system with \( i + k \) and \( j + l \) fixed. The \( y_{i,j,k,l} \) in the equations have all \( i + k \) and \( j + l \) equal or higher. This produces a block triangularization of our system.

**Practical computation**

We already know the solutions for \( i + j + k + l = n \). We can then take
\[
y_{i,j,k,l} = \hat{Q}^j_p Q^l_p \quad \text{if } i = k = 0
\]
\[
y_{i,j,k,l} = 0 \quad \text{otherwise}
\]

This choice corresponds to a normal perturbation, with as much as possible functions \( Q \) (this is because they are more complicated, so we can expect more integrability conditions from them). By linearity, we also can make the even better choice
\[
y_{i,j,k,l} = \text{coeff}((\hat{Q}^j_p + \epsilon_p \alpha \hat{P}^j_p) (Q^l_p + \epsilon_p \alpha P^l_p), \alpha^m) \quad \text{if } i = k = 0
\]
\[
y_{i,j,k,l} = 0 \quad \text{otherwise}
\]

which select in advance the coefficient in \( \alpha \) in the final residue to compute. This allows to simplify the computation by doing it separately. Another difficulty is that even if the system should always have a solvable Galois group, it is very difficult to solve. We will then just search solutions of the form
\[
y_{i,j,k,l} \in \mathbb{C}(t) \left[ \text{arctanh} \left( \frac{1}{t} \right) \right]
\]
with the degree in arctanh less than $m$, except for the equation in $i = k = 0$
and $j + l = 1$. This is in fact a very reasonable hypothesis. Behind this, there
is the supposition that the Galois group will not grow before becoming non
abelian. This hypothesis is reasonable, because if the Galois group grows,
then some logarithm should appear at infinity. We have already seen that this
phenomenon is exceptional in [9]. We also suppose that the degree in arctanh
does not grow. The question of integrability at order $n$ arrives only when
solving the last equation. This last equation is solved with standard method
of variation of the constant, and the integrability condition is established
using Lemma 6. Let us precise that all these conditions are sufficient to
assure the termination of the algorithm, but are in no way necessary. In
all cases, if the algorithm gives a solution, the solution is always valid. One
important precision is that we need to use **only one solution** (although well
chosen).

We find the following expansions

$$V_3 = \frac{1}{r} \left(1 + 3\theta^2 + \frac{125}{12}\theta^4 + o(\theta^5)\right)$$

$$V_5 = \frac{1}{r} \left(1 + \frac{15}{2}\theta^2 + \frac{374495}{5352}\theta^4 + o(\theta^5)\right)$$

and that $V_3, V_5$ are never integrable at order 5. To prove this, the easiest way
is to consider the case $m = 1$, and $m = 3$, and this give two incompatible
constraints for the 6-th derivative.

In the case of the even index, we have the following expansions at first
order

$$V_4 = \frac{1}{r} \left(1 + 5\theta^2 + b\theta^3 + o(\theta^4)\right) \quad V_6 = \frac{1}{r} \left(1 + \frac{21}{2}\theta^2 + b\theta^3 + o(\theta^4)\right)$$

We have here a free parameter $b$. This is the main reason of the greater
difficulty. When we arrive at order 5, we also get two incompatible conditions
for the 6-th derivative, except for some well chosen $b$. We find then the
following expansions (for $r = 1$)

$$\frac{1}{5!}\partial_\theta^5 V_4 = \frac{363467}{4824000}b^3 + \frac{112035}{8576}b$$
\[
\frac{1}{6!} \partial^6_V V_4 = \frac{216926052083}{10224685080000} b^4 + \frac{279352141289}{54531653760} b^2 + \frac{4715685295}{24563808}
\]

with
\[
R_4(b) = \frac{158469311}{97702546320000} b^4 + \frac{372429603}{868467078400} b^2 + \frac{45927}{2729312} = 0
\]

\[
\frac{1}{5!} \partial^5_V V_6 = \frac{68250852673}{42577251500000} b^3 + \frac{98831601}{3475694} b
\]

\[
\frac{1}{6!} \partial^6_V V_6 = \frac{10915637473609903}{5190230823727250000} b^4 + \frac{6605928379884787}{1271076936423000} b^2 + \frac{19638863047783}{10039110960}
\]

with
\[
R_6(b) = \frac{1987151111646995383}{27724359406485352625000000} b^4 + \frac{1448561702310687}{270431334600} b^2 + \frac{3128145145507}{7921245544710100750} = 0
\]

So for each expansions for eigenvalues 9, 20 there are exactly 4 possible expansion integrable at order 5. We remark that this set of possible expansions is invariant by the transformation \( \theta \rightarrow -\theta \) which is expected because this transformation is a symmetry and does not change integrability or non integrability of the potential. For higher orders, we need then to suppose that \( R_i(b) = 0 \). The explicit expression of \( b \) is too complicated to do efficiently the computations at higher order. So, we first remark that \( b \) appear only in the non homogeneous part of the linear differentials systems we have to solve. This means that all our computations can be done in
\[
\mathbb{Q}(b) \left[ t, \frac{1}{t^2 - 1}, \text{arctanh} \left( \frac{1}{t} \right) \right] \simeq \mathbb{Q} \left[ X, t, \frac{1}{t^2 - 1}, \text{arctanh} \left( \frac{1}{t} \right) \right] / (R_i(X))
\]

if \( R_i \) is irreducible (which in our case has always been true but it can of course be tested for each eigenvalue). We then just have to write for each differential system the non homogeneous part as a linear combination of 1, \( b \), \( b^2 \), \( b^3 \), then solve it for each term and then sum. Indeed, the solutions of each term will be in (at least we assume it with our assumption on Galois groups)
\[
\mathbb{Q} \left[ t, \frac{1}{t^2 - 1}, \text{arctanh} \left( \frac{1}{t} \right) \right]
\]
When we reinject the solution in the next differential system, there could be multiplications by $b$, so we reduce the non homogeneous part modulo $R_i$. We find at the end

$$\frac{1}{7!} \theta^7 V_4 = \frac{57826741017348283}{893377392990720} b + \frac{25932696791821703}{100504956711456000} b^3$$

$$\frac{1}{7!} \theta^7 V_6 = \frac{118828154548524498748866853503827777}{431797756299715943933989778480280} b + \frac{8633140425176867273801758981735627411}{528952251467152031319137477863834300000} b^3$$

There is only one condition at order 6 and it can be satisfied. At order 7, we have again two constraints, and this time we do not have a free parameter to fix anymore. These two conditions cannot be simultaneously satisfied.

**Classification of Galois groups**

| Order: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------|---|---|---|---|---|---|---|
| $\lambda$ odd $\neq 0$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | n.a. |
| $\lambda$ even $\neq -1,2$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | n.a. |
| $\lambda = 0$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ |
| $\lambda = 2$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ |

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The graph is built using the following rules: At each order $k$, if there is branching, the branch choice depend on some condition on the $k + 1$-th derivatives or lower. The branch noted n.a. correspond to non abelian Galois groups. The tree is infinite for eigenvalues $-1, 0, 2$, but at the end, only the lower branch correspond to integrable potentials at all orders (and these are truly integrable).

Remark. - These expansions of the potentials are unique et allow for each given potential to precisely compute the order at which it is integrable (here in the case of the eigenvalues $5, 9, 14, 20$). This classification is then only proved for the eigenvalues $-1, 0, 2, 5, 9, 14, 20$, and for bigger eigenvalues, this corresponds the conjecture 2. But the algorithm we used can compute the same for an arbitrary fixed eigenvalue. The proof of conjecture 2 is then possible for many eigenvalues, even if the computation cost is great. This leads to a practical way to study in all generality eigenvalue bounded problems, at least in the case where the bounding is effective (known explicitly).

6. Other type of potentials

We have here only written about non degenerated Darboux points. Now we will take a look at the degenerate case.

**Theorem 15.** Let $V = \frac{1}{r} U(\theta)$ be a meromorphic homogeneous potential of degree $-1$ in the plane in polar coordinates with $U$ meromorphic $2\pi$-periodic. We suppose it exists $\theta_0$ such that

$$U(\theta_0) = U'(\theta_0) = 0$$

If $V$ possess a first integral $I$, meromorphic for $(p_1, p_2, q_1, q_2, r) \in \mathbb{C}^5$, $r^2 = q_1^2 + q_2^2$ and independent almost everywhere with $H$, then $U''(\theta_0) = 0$. Conversely, if $U''(\theta_0) = 0$, then the identity component of the Galois group of variational equation near $\theta_0$ is abelian at any order.
The fact that the first integral should be meromorphic for \( r = 0 \) is in fact a serious limitation. It is possible that the potential itself is not meromorphic for \( r = 0 \), for example when \( U \) is not a rational function in \( \exp i\theta \). So in this sense we cannot call this a “reasonnable” definition of integrability, because the class of possible new first integral does not contain the Hamiltonian. So it seems that such a definition is reasonnable only when \( U \) is a rational function in \( \exp i\theta \), which corresponds to \( V \) rational in \( q_1, q_2, r \). Remark that the proof cannot be so easy as Theorem [4] because the variational equation is not Fuchsian, and then we cannot use the trick with the monodromy group.

**Proof.** Suppose first that \( U''(\theta_0) \neq 0 \). The first order variational equation is the following

\[
\ddot{X}_1 = 0 \quad \ddot{X}_2 = \frac{U''(\theta_0)}{t^3} X_2
\]

The first equation is clearly integrable. For the second one, we make a linear variable change and this gives

\[
\ddot{y} = \frac{1}{t^3} y
\]

This equation is not Fuchsian because 0 is not a regular singularity. Using the Kovacic algorithm is in this case a priori not enough. Indeed, the Galois group over the field \( \mathbb{C}(t) \) will detect only rational first integral. Let us prove now that if (10) admit a first integral of the form \( I(y, \dot{y}) \) with \( I \) an homogeneous rational fraction with meromorphic coefficients in \( t \), then \( I \) has rational coefficients in \( t \).

Infinity is a regular singularity of (10) and the Frobenius expansion near infinity gives (after variable change \( t \rightarrow \frac{1}{t} \))

\[
y_{c_1, c_2}(t) = c_2 \ln t \left( \frac{1}{4} + \frac{1}{32} t + \frac{1}{768} t^2 \right) + c_2 \left( \frac{1}{t} - \frac{3}{64} t + \frac{7}{2304} t^2 \right) + c_1 \left( \frac{1}{4} + \frac{1}{32} t + \frac{1}{768} t^2 \right) + o(t^2)
\]

There is a logarithmic singularity. Suppose it exists a first integral meromorphic in \( t \) and homogeneous in \( y, \dot{y} \) of (10). After variable change \( t \rightarrow \frac{1}{t} \), there could be an essential singularity in 0 (but not multivaluation) in the
coefficients of $I$. We pose

$$I_0 = t^2 \left( \frac{1}{8} + \frac{1}{96} t + \frac{1}{3072} t^2 + o(t^2) \right) y - t^2 \left( 1 + \frac{1}{8} t + \frac{1}{192} t^2 + o(t^2) \right) \dot{y} = t^2 y_{1,0} y - t^2 y_{1,0} \dot{y}$$

The function $I_0$ is the only first integral of degree 1 whose coefficients are univaluated in 0. Indeed, if there was another one (independent), then $y, \dot{y}$ would be univaluated for all choices of $c_1, c_2$, which is not true because $\ln t$ is infinitely multivaluated. Let us look now at higher degrees. We consider $I(y, \dot{y})$ a first integral and $y_0$ a solution such that $I(y_0, \dot{y}_0) = 0$. Then $\dot{y}_0/y_0$ is an algebraic combination of the coefficients, so $\dot{y}_0/y_0$ is at most finitely multivaluated, so this implies that

$$y_0 = y_{a,0} \quad a \neq 0$$

Then we can factorize in $I$

$$I(y, \dot{y}) = (t^2 y_{1,0} y - t^2 y_{1,0} \dot{y}) \hat{I}(y, \dot{y})$$

This strictly decrease the degree of the numerator. We can also do the same procedure with the denominator by considering $I^{-1}$ instead of $I$. We conclude by recurrence that

$$I = AI_0^k \quad A \in \mathbb{C}^*, \quad k \in \mathbb{Z}$$

Then the coefficients of $I$ are meromorphic at infinity (so they are rational). The Galois group of $[10]$ is $SL_2(\mathbb{C})$, and then it do not exists such first integrals (non constant). Then $V$ is not meromorphically integrable.

Now suppose the reverse, that $U''(\theta_0) = 0$. The first order variational equation is written

$$\ddot{X}_1 = 0 \quad \ddot{X}_2 = 0$$

We now use the lemma

**Lemma 16.** The algebra $\mathbb{C}[t, \frac{1}{t}, \ln t]$ is stable by integration.

**Proof.** We consider $f \in \mathbb{C}[t, \frac{1}{t}, \ln t]$ and we write it as a linear combination of terms of the type

$$t^n \ln(t)^m \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}$$
If $n \geq 0$, then we use integration by parts to decrease $m$ until 0. If $n < 0$, we use integration by part to increase $n$ up to $n = -1$. We then have the formula
\[
\int \frac{1}{t} \ln(t)^m dt = \frac{1}{m+1} \ln(t)^{m+1}
\]
Then all functions in $\mathbb{C}[t, \frac{1}{t}, \ln t]$ have a primitive in $\mathbb{C}[t, \frac{1}{t}, \ln t]$.  

To conclude, we remark the following phenomenon: The solutions of higher variational equation are in fact solutions of non homogeneous linear differential equations and the non homogeneous terms are produced only using products of lower order solutions and functions $t^{-k}$. So the solutions always live in some algebra in which we take recursively integrations. So we apply the method of variation of constants to find the solutions. Moreover, the wronskian of $X_2 = 0$ equal to 1 (and also for the higher variational equations matrices), then we never have to divide. So all solutions belong to $\mathbb{C}[t, \frac{1}{t}, \ln t]$ which is stable by integration. Then the Picard Vessiot field is
\[
K_i = \mathbb{C}(t) \text{ or } \mathbb{C}(t, \ln t) \quad G_i = id \text{ or } \mathbb{C}
\]
The Galois group is virtually abelian at all order.

**Remark 6.** This type of proof appears to be very general. Indeed, if some potential appears to be integrable at all order near a particular solution, without known first integral, the Picard Vessiot field is often not growing. If the coefficients of the potential are not well adjusted to avoid creating further monodromy, it is probably because it is not possible. The solutions of higher variational equations are in fact solutions of non homogeneous linear differential equations and the non homogeneous terms are produced only using products. So the solutions always live in some algebra in which we take recursively integrations. For homogeneous potentials and non degenerate Darboux point, the algebra is given by the following process
\[
\mathcal{A}_0 = \mathbb{C} \left[ t, \frac{1}{t^2 - 1} \right], \quad \mathcal{A}_{i+1} = \int \mathcal{A}_i dt
\]
where $\int \mathcal{A}_i dt \supset \mathcal{A}_i$ is the algebra generated by all primitivations of functions in $\mathcal{A}_i$. We have then in particular that the Picard Vessiot of $(VE_i)$ (in the case of all the eigenvalues belong to Morales Ramis table) is always contained in the fraction field of $\mathcal{A}_i$. These algebras contains in particular all the polylogarithms functions that give integrability constraints (however, it is not clear that $\mathcal{A}_i$ is only generated by polylogarithms).
7. Conclusion

Eventually using this analysis, all eigenvalue bounded problems can be algorithmically studied. The two conjectures represent the last open questions about homogeneous potentials in the plane of degree $-1$. They could in principle be proved using the $D$-finiteness property of the functions $P_n, Q_n$ (the fact that they satisfy linear polynomial recurrence and differential equations), but in practice direct computation seems to be way out of reach for the moment. No counter example have been found to conjectures 1, 2, and numerical computations strongly suggest that the integrability conditions at order 5 and 7 are never compatible. With these theorems we see that very often, the difficulty is not the number of parameters, but specific family of potentials (with even as low as 1 parameters) which exhibit the non eigenvalue bounding phenomenon.

Some examples of potentials integrable at all order near all Darboux points
In the case of non degenerated Darboux points, if we admit conjecture 2, it will not be possible to find non integrable potentials which are integrable all orders. We then need to find homogeneous potentials either having no Darboux points at all, either only multiple degenerated Darboux points (the second derivative should vanish). The functions $U(\theta) = F (e^{i\theta})$ with

$$F(z) = h(z^n) \quad h \text{ fractional linear, } n \in \mathbb{N}^*$$

have no critical points. The functions $U(\theta) = F (e^{i\theta})$ with

$$F(z) = f(z^n) \text{ with } f(z) = \int \frac{az^i}{(z - \alpha)^j} dz \quad 0 \leq i \leq j - 2, \quad n \in \mathbb{N}^* \quad \alpha \in \mathbb{C}^*$$

have only degenerated critical points satisfying the integrability constraint.

These examples show that there are still open questions about integrability, but the difficulties do not rely on Morales Ramis theory but on the search of Darboux points. Potentials without non degenerate Darboux points are not common, but they still exist and a complete classification of them seems to be difficult.
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