Lipschitz continuous dependence of piecewise constant Lamé coefficients from boundary data: the case of non-flat interfaces

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Abstract
We consider the inverse problem of determining the Lamé moduli for a piecewise constant elasticity tensor $C = \sum_j C_j \chi_{D_j}$, where $\{D_j\}$ is a known finite partition of the body $\Omega$, from the Dirichlet-to-Neumann map. We prove that Lipschitz stability estimates can be derived under $C^{1,\alpha}$ regularity assumptions on the interfaces.

Keywords: inverse boundary value problem, Lamé system, piecewise constant coefficients, Lipschitz stability

1. Introduction

An important inverse problem arising from engineering sciences consists in determining the elasticity coefficients of the material occupying a three-dimensional body from measurements of tractions and displacements taken on its accessible boundary.

The boundary value problem from which this inverse problem originates is as follows. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ made by a linearly elastic isotropic material, with Lamé moduli $\mu$ and $\lambda$ satisfying the strong convexity conditions $\mu(x) \geq \alpha_0 > 0$, $2\mu(x) + 3\lambda(x) \geq \beta_0$ in $\Omega$, for some positive constants $\alpha_0$ and $\beta_0$. For a given $\psi \in H^1(\partial \Omega)$, the direct problem consists in finding the displacement field $u \in H^1(\Omega)$ solution to the Dirichlet problem
\[ \begin{aligned}
\text{div} \left( \mathbf{C} \hat{\nabla} u \right) &= 0, \\
u &= \psi, & \text{in } \Omega, \\
u &= \psi, & \text{on } \partial \Omega,
\end{aligned} \tag{1} \]

where \( \mathbf{C} = \lambda(x) I_3 \otimes I_3 + 2\mu(x) I_{3\text{sym}} \) is the Lamé elasticity tensor.

We denote by \( \Lambda_\mathbf{C} : H^1(\partial \Omega) \to H^{-1}(\partial \Omega) \) the Dirichlet-to-Neumann map associated to the problem (1), that is the operator which maps the Dirichlet data \( u|_{\partial \Omega} = \psi \) onto the corresponding Neumann data \( (\mathbf{C} \hat{\nabla} u) \nu|_{\partial \Omega} \), where \( \nu \) is the outer unit normal to \( \Omega \).

An interesting inverse problem is the determination of the Lamé coefficients \( \mu \) and \( \lambda \) when \( \Lambda_\mathbf{C} \) is known. Most of the results available in the literature concern the uniqueness issue. A linearized version of this inverse problem was considered by Ikehata [Ik]. In [NU1], Nakamura and Uhlmann established that in two dimensions the Lamé moduli are uniquely determined by \( \Lambda_\mathbf{C} \), provided that they are smooth (e.g., \( C^\infty(\overline{\Omega}) \)) and sufficiently close to positive constants. The uniqueness in dimension three was proved in [NU2, ER, NU3], assuming that the Lamé moduli are \( C^\infty(\overline{\Omega}) \) and \( \mu \) close to a positive constant. Recent results concern the uniqueness in the case of partial Cauchy data, see [IUY] for details.

The stability issue for the above inverse problem is expected to be significantly more difficult than uniqueness and, to our knowledge, no general result is known. In the simpler context of an electric conductor, which involves the determination of a single smooth coefficient in a scalar elliptic equation from boundary measurements, it is well-known that the optimal rate of continuous dependence is of logarithmic type, see, for instance, [A] and [Ma]. It follows that logarithmic stability estimates, or even worse ones, are expected in our case. In addition, the situation is more complicated because, in several practical applications, the Lamé moduli are not smooth and, in some cases, may also be discontinuous.

In order to have better stability results, a possible way is based on the introduction of suitable a priori assumptions that are physically relevant and restore well-posedness. Following the approach suggested by Alessandrini and Vessella [AV] in the conductivity framework, in [BFV] the authors considered a class of piecewise constant elasticity tensors of the form

\[ \mathbf{C} = \sum_{j=1}^N \left( \lambda_j I_3 \otimes I_3 + 2\mu_j I_{3\text{sym}} \right) \chi_{D_j}(x), \tag{2} \]

where the collection of disjoint Lipschitz domains \( \{ D_j \}_{j=1}^N \) forms a known decomposition of the domain \( \Omega \), and \( \lambda_j, \mu_j, j = 1, \ldots, N \), are unknown constants to be determined from \( \Lambda_\mathbf{C} \).

Assuming that the boundaries of the domains \( D_j \) contain flat portions, the authors were able to prove a Lipschitz continuous dependence of the Lamé moduli from the local Dirichlet-to-Neumann map.

The structure (2) assumed for \( \mathbf{C} \) fits well in several problems arising in applications. Polyhedral partitions of \( \Omega \) appear frequently in finite element meshing used for effective reconstruction of the Lamé parameters [BJK]. In identification of material properties of masonry walls or concrete dams, for example, the actual elasticity coefficients are approximated by assuming that each finite element or group of finite elements is made by homogeneous Lamé material. The partition of the domain is often suggested by a priori information on different grades of the material or, in the case of dams, by the possible presence of natural joints inside the concrete [XJY]. Obviously, it is not always possible to ensure that the domains \( D_j \) have a flat portion of their boundary in common and, therefore, in order to address these more general inverse problems, it is necessary to remove this a priori assumption.
In this paper we prove a Lipschitz stability estimate assuming $C^{1,\alpha}$ regularity of some portions $\Sigma_j$ of the interfaces joining contiguous domains $D_{j-1}$, $D_j$ and on the portion $\Sigma$ of $\partial \Omega$ where the measurements are taken. The precise regularity conditions are given in section 2.2 (assumptions (A1)).

Our proof is inspired by the paper [BFV] and is mainly based on the use of unique continuation properties and on a refined local analysis, near the $C^{1,\alpha}$ interface $\Sigma_j$, of the behaviour of the corresponding biphase fundamental solution (see subsection 2.3.2 for the precise setting). To this aim, a new mathematical tool is the recent asymptotic approximation of this fundamental solution (see [AdCMR]) in terms of the biphase fundamental solution associated to a flat interface, which was determined in a close form by Rongved [R].

We follow a slightly different procedure to prove the stability estimate. In [BFV] the authors reformulate the direct problem in terms of the nonlinear forward map $F$ acting on a compact subset $K$ of $\mathbb{R}^{2N}$, and use an abstract lemma (see [BV]) which ensures that the inverse map $F^{-1}$ is Lipschitz continuous. Here, instead, we give a more direct proof following the lines in [AV] for the conductivity framework. As in [AV, BFV], also our proof proceeds by induction. However, in order to simplify the presentation and to emphasize the crucial points where new tools are needed, we focus on the first two steps of the induction process. Precisely, the key role of the asymptotic approximation of the biphase fundamental solution is emphasized in the first step, whereas the second step explains how to use the transmission conditions at the interface and the stability estimates for the Cauchy problem to propagate the smallness crossing an interface.

2. Main result

2.1. Notation and main definitions

For every $x \in \mathbb{R}^3$ we set $x = (x', x_3)$, where $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. For every $x \in \mathbb{R}^3$, $r$ and $L$ positive real numbers we will denote by $B_r(x)$, $B'_r(x')$ and $Q_{r,L}(x)$ the open ball in $\mathbb{R}^3$ centered at $x$ of radius $r$, the open ball in $\mathbb{R}^2$ centered at $x'$ of radius $r$ and the cylinder $B'_r(x') \times (x_3 - Lr, x_3 + Lr)$, respectively. In the sequel $B_r(0)$, $B'_r(0)$ and $Q_{r,L}(0)$ will be denoted by $B_r$, $B'_r$ and $Q_{r,L}$, respectively. We will also denote by $\mathbb{R}^3_+ = \{(x', x_3) \in \mathbb{R}^3 | x_3 > 0\}$, $\mathbb{R}^3_- = \{(x', x_3) \in \mathbb{R}^3 | x_3 < 0\}$, $B'_r = B_r \cap \mathbb{R}^3_+$, and $B''_r = B_r \cap \mathbb{R}^3_-$. For any subset $D$ of $\mathbb{R}^3$ and any $h > 0$, we denote by

$$(D)_h = \left\{ x \in D \mid \text{dist}(x, \mathbb{R}^3 \setminus D) > h \right\}.$$ 

Definition 2.1. ($C^{k,\alpha}$ regularity) Let $U$ be a bounded domain in $\mathbb{R}^3$. Given $k, \alpha$, with $k \in \mathbb{N}$ and $0 < \alpha \leq 1$, we say that $U$ is of class $C^{k,\alpha}$ with constants $r_0$, $L$ if, for any $P \in \partial U$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$U \cap B_r(0) = \left\{ x \in B_r(0) \mid x_3 > \phi(x') \right\},$$

where $\phi$ is a $C^{k,\alpha}$ function on $\mathbb{R}^2$ satisfying

$$\phi(0) = 0,$$

$$\nabla \phi(0) = 0,$$

when $k \geq 1$. 

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\[ \| \varphi \|_{C^k(\mathbb{R}^n)} \leq L r. \]

When \( k = 0, \alpha = 1 \), we also say that \( U \) is of \textit{Lipschitz class} with constants \( r_0, L \).

**Remark 2.2.** We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above is meant as follows

\[ \| \varphi \|_{C^k(\mathbb{R}^n)} = \sum_{i=0}^{k} r_0^i \| \nabla^i \varphi \|_{L^\infty(\mathbb{R}^n)} + r_0^k + \alpha \left| \nabla^k \varphi \right|_{\alpha, R^n}, \]

where

\[ \left| \nabla^k \varphi \right|_{\alpha, R^n} \equiv \sup_{x', y' \in \mathbb{R}^n, x' \neq y'} \frac{\left| \nabla^k \varphi(x') - \nabla^k \varphi(y') \right|}{|x' - y'|^\alpha}. \]

Similarly,

\[ \| u \|_{H^{2}(\Omega)} = \left( \sum_{i=0}^{m} r_0^{2^{i-3} \int \left| \nabla^i u \right|^2 \right)^{\frac{1}{2}}, \quad \| u \|_{C^k(\Omega)} = \sum_{i=0}^{k} r_0^i \| \nabla^i u \|_{L^\infty(\Omega)}, \]

\[ \| u \|_{L^2(\partial \Omega)} = \left( r_0^{-2} \int_{\partial \Omega} |u|^2 \right)^{\frac{1}{2}}, \]

where \( H^{0}(\Omega) = L^2(\Omega) \), and so on for trace norms such as \( \| \cdot \|_{H^{1/2}(\partial \Omega)} \| \cdot \|_{H^{-1/2}(\partial \Omega)} \), where \( \Omega \) is a bounded subset of \( \mathbb{R}^3 \) with regular boundary.

We will also make use of the following notation for matrices and tensors. Let \( \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3) \) be the linear space of \( 3 \times 3 \) matrices. For any \( A, B \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3) \), we set \( A: B = \sum_{i,j,k} A_{ij} B_{jk}, |A|^2 = A: A \) and \( \hat{A} = \frac{1}{2}(A + A^T) \). By \( I_3 \) we denote the \( 3 \times 3 \) identity matrix and by \( I_{\text{Sym}} \) we denote the fourth order tensor such that \( I_{\text{Sym}} A = \hat{A} \).

In the whole paper we are going to consider isotropic elastic materials, hence the fourth order elasticity tensor \( \mathcal{C} \) is given by

\[ \mathcal{C}(x) = \lambda(x) I_3 \otimes I_3 + 2 \mu(x) I_{\text{Sym}}, \quad \text{for a.e. } x \in \Omega, \quad (3) \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) of Lipschitz class and \( (I_3 \otimes I_3) A = (I_3: A) I_3 \) for every \( 3 \times 3 \) matrix \( A \). The real valued functions \( \lambda = \lambda(x) \) and \( \mu = \mu(x) \in L^\infty(\Omega) \) are the Lamé moduli, and satisfy the strong convexity condition

\[ a_0 \leq \mu(x) \leq a_0^{-1}, \quad \lambda(x) \leq a_0^{-1}, \quad 2 \mu(x) + 3 \lambda(x) \geq \beta_0, \quad \text{for a.e. } x \in \Omega, \quad (4) \]

where \( a_0 \in (0, 1], \beta_0 \in (0, 2] \) are given constants. Let us notice that the Poisson’s ratio \( \nu(x) = \frac{\lambda(x)}{2(\lambda(x) + \mu(x))} \) satisfies

\[ -1 + \frac{a_0 \beta_0}{4} \leq \nu(x) \leq \frac{1}{2} - \frac{a_0^2}{4}, \quad \text{for a.e. } x \in \Omega. \quad (5) \]

Under these assumptions, the elasticity tensor \( \mathcal{C} \) satisfies the minor and major symmetry conditions.
and the strong convexity condition
\[ CA : A \geq \xi_0 |A|^2, \]
where \( \xi_0 = \min \{2a_0, \beta_0\} \), for every \( A, B \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3) \).

In the sequel we will make use of the following norm in the linear space of bounded isotropic tensors:
\[ \| C \|_\infty = \max \left\{ \| \lambda \|_{L^\infty(\Omega)}, \| \mu \|_{L^\infty(\Omega)} \right\}. \]

Our boundary measurements are represented by the Dirichlet-to-Neumann map. As a matter of fact, since we will restrict our measurements to boundary data that have support on some subset of the boundary, we will make use of a local Dirichlet-to-Neumann map.

**Definition 2.3.** (The local Dirichlet-to-Neumann map) Let \( \Omega \) be a bounded domain of Lipschitz class and let \( \Sigma \) be an open portion of \( \partial \Omega \). We denote by \( H^2_0(\Sigma) \) the function space
\[ H^2_0(\Sigma) := \left\{ \phi \in H^2(\partial \Omega) \mid \text{supp } \phi \subset \Sigma \right\} \]
and by \( H^{-1}_0(\Sigma) \) the topological dual of \( H^2_0(\Sigma) \). We denote by \( < \cdot, \cdot > \) the dual pairing between \( H^2_0(\Sigma) \) and \( H^{-1}_0(\Sigma) \) based on the \( L^2(\Sigma) \) scalar product, that is \( < f, g > = r_0^{-2} \int_{\partial \Omega} f g \), for every \( f, g \in L^2(\partial \Omega) \). Then, given \( \psi \in H^1_0(\Sigma) \), there exists a unique vector valued function \( u \in H^1(\Omega) \) weak solution to the Dirichlet problem
\[ \begin{cases} \text{div} \left( C \nabla u \right) = 0, & \text{in } \Omega, \\ u = \psi, & \text{on } \partial \Omega. \end{cases} \]

We define the local Dirichlet-to-Neumann linear map \( \Lambda_{\Sigma}^\psi \) as follows:
\[ \Lambda_{\Sigma}^\psi : \psi \in H^1_0(\Sigma) \rightarrow \left( C \nabla u \right) n \big|_\Sigma \in H^{-1}_0(\Sigma), \]
where \( n \) is the exterior unit vector to \( \Omega \).

Note that for \( \Sigma = \partial \Omega \) we get the usual Dirichlet-to-Neumann map. For this reason we will set \( \Lambda_{\Sigma} := \Lambda_{\partial \Omega}^\psi \).

The map \( \Lambda_{\Sigma}^\psi \) can be identified with the bilinear form on \( H^2_0(\Sigma) \times H^{-1}_0(\Sigma) \) by
\[ \Lambda_{\Sigma}^\psi (\psi, \phi) := < \Lambda_{\Sigma}^\psi \psi, \phi > = r_0^{-2} \int_{\partial \Omega} C \nabla u : \nabla v, \]
for all \( \psi, \phi \in H^2_0(\Sigma) \) and where \( u \) solves (8) and \( v \) is any \( H^1(\Omega) \) function such that \( v = \psi \) on \( \partial \Omega \).

We shall denote by \( \| \cdot \|_\ast \) the usual norm in the linear space \( \mathcal{L} \left( H^2_0(\Sigma), H^{-1}_0(\Sigma) \right) \). Let us observe that, from our convention on the homogeneity of the norms, we have in particular that
\[ \| \Lambda_{\Sigma}^\psi \|_\ast = \sup \left\{ r_0^{-2} \int_{\partial \Omega} C \nabla u : \nabla v \right\}, \]
where the sup is taken for \( \phi, \psi \in H^2_0(\Sigma) \), \( \| \phi \|_{H^2_0(\Sigma)} = \| \psi \|_{H^2_0(\Sigma)} = 1 \), being \( u \) the solution to (1) and \( v \in H^1(\Omega) \) any extension of \( \phi \).
2.2. *A priori* assumptions and statement of the main result

Our main assumptions are:

(A1) \( \Omega \subset \mathbb{R}^3 \) is an open bounded domain such that \( \Omega \) is of class \( C^{0,1} \), with constants \( r_0, L \), and we assume that

\[
\Omega = \bigcup_{j=1}^{N} \mathcal{D}_j,
\]

where \( \mathcal{D}_j, j = 1, \ldots, N \), are connected and pairwise disjoint domains of class \( C^{0,1} \) with constants \( r_0, L \), such that there exists a constant \( A > 0 \) such that

\[
|\mathcal{D}_j| \leq Ar_0^3, \quad j = 1, \ldots, N.
\]

We also assume that there exists one region, say \( \mathcal{D}_1 \), such that \( \Omega \cap \partial \mathcal{D}_1 \) contains the open portion \( \Sigma \) where the measurements are taken. Moreover, for every \( j \in \{2, \ldots, N\} \) there exist \( \mathcal{D}_j, j = 1, \ldots, M \) such that

\[
\mathcal{D}_j = \mathcal{D}_1, \quad \mathcal{D}_{j_0} = \mathcal{D}_j,
\]

and, for every \( k = 2, \ldots, M \), the set

\[
\partial \mathcal{D}_{j_{k-1}} \cap \partial \mathcal{D}_j
\]

contains a portion \( \Sigma_k \subset \Omega \).

Furthermore, for \( k = 1, \ldots, M \), we assume there exists \( R_k \in \Sigma_k \) and a rigid transformation of coordinates such that \( P_k = 0 \) and for all \( k = 1, \ldots, M \)

\[
\Sigma_k \cap Q_{r_0,L} = \left\{ x \in Q_{r_0,L} \middle| x_3 = \varphi_k(x') \right\},
\]

\[
\mathcal{D}_j \cap Q_{r_0,L} = \left\{ x \in Q_{r_0,L} \middle| x_3 < \varphi_k(x') \right\},
\]

\[
\mathcal{D}_{j_{k-1}} \cap Q_{r_0,L} = \left\{ x \in Q_{r_0,L} \middle| x_3 > \varphi_k(x') \right\},
\]

where \( \Sigma_1 \subset \Sigma \), with \( \varphi_k \in C^{1,\alpha}(\mathbb{R}^2) \) such that

\[
\varphi_k(0) = \left| \nabla \varphi_k(0) \right| = 0, \quad \| \varphi_k \|_{C^{1,\alpha}(\mathbb{R}^2)} \leq L_0,
\]

and where we set \( \mathcal{D}_0 := \mathbb{R}^3 \setminus \overline{\Omega} \). Finally, let

\[
\mathcal{D}_0 := \left\{ x \in Q_{r_0,L} \middle| \varphi_k(x') < x_3 < \frac{2}{3}r_0L \right\}
\]

and let \( \Omega_0 = \text{Int}(\overline{\Omega} \cup \mathcal{D}_0) \). Observe that \( \mathcal{D}_0 \) and \( \Omega_0 \) are of class \( C^{0,1} \), with constants \( r_1, L_1 \) such that \( L_1 = \tan(\frac{\pi}{4} + \frac{1}{3}) \) and \( r_1 = \frac{2}{3} \sqrt{1 + \frac{2}{3}L_0^2} - r_0 \).

For simplicity we will call \( \mathcal{D}_1, \ldots, \mathcal{D}_{j_0} \) a *chain of domains* connecting \( \mathcal{D}_1 \) to \( \mathcal{D}_j \). For any \( k \in \{1, \ldots, M\} \) we will denote by \( n_k \) the exterior unit vector to \( \partial \mathcal{D}_k \) in \( P_k \).

(A2) We assume that the tensor \( \mathcal{C} \) is piecewise constant

\[
\mathcal{C} = \sum_{j=1}^{N} \mathcal{C}_j \chi_{\mathcal{D}_j}(x), \tag{10}
\]
where
\[ C_j = \lambda_j I_3 \otimes I_3 + 2 \mu_j I_{3\text{sym}}, \]
with constant Lamé coefficients \( \lambda_j \) and \( \mu_j \) satisfying (4). In what follows we shall refer to the constants \( L, \alpha, A, N, \alpha_0, \beta_0 \) as to the a priori data.

In the sequel we will introduce a number of constants that we will always denote by \( C \). The values of these constants might differ from one line to the other.

The main result of this paper is the following stability result.

**Theorem 2.4.** Let \( \Omega \) and \( \Sigma \) satisfy (A1) and let the tensors \( C \) and \( C \) satisfy (A2). Then there exists a positive constant \( C \) depending only on the a priori data such that
\[ \|C - \Pi\|_\infty \leq C r_0 \left\| A \right\|_*, \]

(11)

2.3. Some basic properties of the Lamé system

2.3.1. Alessandrini’s identity. Alessandrini’s identity is a key relation connecting the Dirichlet-to-Neumann maps and a volume integral. It was originally derived in [A] within the conductivity framework. Its extension to our context is as follows. Given \( u_1 \) and \( u_2 \) solutions to
\[ \mathcal{C}_k \nabla u_k = 0, \quad \text{in} \quad \Omega, \quad k = 1, 2, \]
we have
\[ \int_\Omega (\mathcal{C}_1 - \mathcal{C}_2) \nabla u_1 \cdot \nabla u_2 = n_1 \cdot (A_{\mathcal{C}_1} - A_{\mathcal{C}_2}) u_1, \]
where \( A_{\mathcal{C}_1}, A_{\mathcal{C}_2} \) denotes the Dirichlet-to-Neumann map corresponding to \( \mathcal{C}_1, \mathcal{C}_2 \) respectively.

2.3.2. Singular solutions. In a suitable coordinate system, let us consider the set
\[ D = \{ (x', x_3) \mid x_3 < \phi(x') \}, \]
where \( \phi \in C^{1,\alpha}(\mathbb{R}^2) \) is such that
\[ \phi(0) = |\nabla\phi(0)| = 0, \quad \|\phi\|_{C^{1,\alpha}(\mathbb{R}^2)} \leq L_0. \]
Let
\[ C_b = C + (\mathcal{C}_D - C)_{\mathcal{X}_D}, \]
where \( C \) and \( \mathcal{C}_D \) are constant isotropic elasticity tensors satisfying (4). Given \( y \in \mathbb{R}^3 \), let us consider the normalized fundamental solution \( \Gamma^D(\cdot, y) \) defined by
\[ \begin{cases} \text{div} \left( C_b \nabla \Gamma^D(\cdot, y) \right) = -\delta_y I_3, \\ \lim_{|x| \to \infty} \Gamma^D(x, y) = 0. \end{cases} \]
(13)

The following result, derived in [AdCMR], holds.

**Proposition 2.5.** There exists a unique normalized fundamental solution \( \Gamma^D(\cdot, y) \in C^0(\mathbb{R}^3 \setminus \{y\}) \). Moreover, for every \( x \in \mathbb{R}^3, x \neq y \), we have
\[ \Gamma^D(x, y) = \left( \Gamma^D(y, x) \right)^T, \quad (14) \]
\[ \left| \Gamma^D(x, y) \right| \leq C \left| x - y \right|^{-1}, \quad (15) \]
\[ \left| \nabla_x \Gamma^D(x, y) \right| \leq C \left| x - y \right|^{-2}, \quad (16) \]

where the constant \( C > 0 \) only depends on \( L, \alpha, \alpha_0 \) and \( \beta_0 \).

In particular, for \( \varphi = 0 \), we have \( D = \mathbb{R}^3 \) and we will denote the fundamental solution by \( \Gamma \). An explicit expression of \( \Gamma \) has been obtained by Rongved in \([R]\).

A crucial result in our analysis is the following asymptotic estimate of \( \Gamma^D \) in terms of \( \Gamma \) derived in \([AdCMR, \text{theorem 8.1}]\).

**Proposition 2.6.** Let \( y = (0, 0, h) \), where \( h \) such that \( \frac{L_0}{h^3} < 1 \). Then
\[ \left( \Gamma^D - \Gamma \right)(x, y) \leq C \left( \frac{\left| x - y \right|}{r_0} \right)^{-1+\alpha}, \quad \forall x \in Q \bigcap D, \quad (17) \]
\[ \left( \nabla_x \Gamma^D - \nabla_x \Gamma \right)(x, y) \leq C \left( \frac{\left| x - y \right|}{r_0} \right)^{-2+\frac{\alpha}{1+\alpha}}, \quad \forall x \in Q \bigcap D, \quad (18) \]

where \( C \) only depends on \( L, \alpha, \alpha_0 \) and \( \beta_0 \).

Let \( C \) be an isotropic elasticity tensor satisfying (A2). We still denote by \( C \) its extension to \( \Omega_0 \) such that \( C_{\mathbb{R}^3} = C_0 \) is the isotropic tensor with Lamé parameters \( \lambda_0 = 0 \) and \( \mu_0 = 1 \). This extended tensor is still an isotropic elasticity tensor of the form
\[ C = \sum_{j=0}^N C^j \delta_D_j(x), \quad (19) \]
where each \( C^j, j = 0, \ldots, N, \) has Lamé parameters satisfying (4).

For all possible interfaces \( \Sigma_k \) introduced in (A1) let
\[ \Sigma^k = \Sigma_k \bigcap Q_{x, y}^r, \]
and denote by \( F := \bigcup_k \Sigma^k \). Let
\[ y \in \bigcup_{j=0}^N D_j \bigcup F, \quad r^* = r^*(y) = \min \left\{ \frac{r_0}{3}, \text{dist}(y, \bigcup_{j=0}^N \partial D_j \setminus F) \right\}. \]

Let us consider the sphere \( B_r^*(y) \). Then, either \( B_r^*(y) \cap F = \emptyset \), so that \( B_r^*(y) \subset D_j \) for some \( j \in \{1, \ldots, N\} \) and we define \( C_y = C_j \), or \( B_r^*(y) \cap F \neq \emptyset \) so that, under our regularity assumptions there exist exactly two domains, say, \( D_{j-1} \) and \( D_j \), intersecting \( B_r^*(y) \) and, in this case, we define \( C_y = C_{j-1} + (C_j - C_{j-1})\chi_{\{x \prec \varphi_j(y)\}} \). In the latter expression, \( \varphi_j \) is the function whose graph contains \( \Sigma_y \), according to (A1).

Let \( \Gamma^* (\cdot, y) \) denote the normalized fundamental solution to
\[ \text{div} \left( C_y \nabla \Gamma^*(\cdot, y) \right) = -\delta_y I_3, \quad \text{in } \mathbb{R}^3. \quad (20) \]

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Proposition 2.7. Let \( \Omega_0 \) and \( \mathcal{C} \) satisfy (A1) and (A2). Then, for any \( y \in \bigcup_{j=0}^{N} D_j \cup \mathcal{F} \), there exists a unique matrix-valued function \( G(\cdot, y) \in C^0(\Omega_0 \setminus \{y\}, L(\mathbb{R}^3, \mathbb{R}^3)) \) such that

\[
\int_{\Omega_0} \nabla \cdot \mathbf{G}(\cdot, y) : \nabla \mathbf{v} = \phi(y), \quad \forall \mathbf{v} \in C^0_{\Omega_0}(\Omega_0, L(\mathbb{R}^3, \mathbb{R}^3)),
\]

for all \( \phi \in C^0_{\Omega_0}(\Omega_0, L(\mathbb{R}^3, \mathbb{R}^3)) \) and

\[
G(\cdot, y) = 0, \quad \text{on } \partial \Omega_0,
\]

and

\[
\|G(\cdot, y)\|_{H^1(\Omega_0 \setminus B_R(y))} \leq \frac{C}{\sqrt{r}} , \quad \forall r \leq r^* .
\]

where \( C > 0 \) depends only on \( \alpha_0, \beta_0, A, N, \alpha \) and \( L \).

Furthermore, for every \( c_1 > 1 \), if \( \text{dist}(y, \bigcup_{j=0}^{N} \partial D_j \setminus \mathcal{F}) \geq \frac{\alpha_0}{c_1} \), then

\[
\|G(\cdot, y) - G(\cdot, y)\|_{H^1(\Omega_0)} \leq \frac{C}{r_0},
\]

where \( C > 0 \) depends only on \( \alpha_0, \beta_0, A, N, \alpha, L \) and \( c_1 \).

Finally

\[
G(x, y) = G(y, x)^T, \quad \text{for every } x, y \in \left( \bigcup_{j=0}^{N} D_j \right) \cup \mathcal{F}, \quad x \neq y.
\]

The proof follows the lines of the proof of proposition 3.1 in [BFV].

2.3.3. Three spheres inequality. A mathematical tool which plays an important role in the proof of theorem 2.4 is the following three spheres inequality for solutions to the Lamé system.

Proposition 2.8. Let \( u \in H^1(B_R) \) be a solution to the Lamé system

\[
\text{div} \left( \mathbf{C} \nabla u \right) = 0, \quad \text{in } B_R,
\]

where \( \mathbf{C} \) is a constant isotropic elasticity tensor satisfying (4). For every \( \rho_1, \rho_2, \rho_3 \) with \( 0 < \rho_1 < \rho_2 < \rho_3 \leq R \), we have

\[
\|u\|_{L^\infty(B_{\rho_2})} \leq C \|u\|_{L^\delta(B_{\rho_1})} \|u\|_{L^\infty(B_{\rho_3})},
\]

where \( C > 0 \) and \( \delta \in (0, 1) \) only depend on \( \alpha_0, \beta_0, \frac{\alpha_0}{\rho_1}, \frac{\alpha_0}{\rho_3} \).

For a proof, see [AM].

3. Proof of the main result

Let \( j \in \{1, \ldots, N\} \) be such that

\[
\|\mathbf{C} - \mathbf{C}\|_{L^\infty(D_j)} = \|\mathbf{C} - \mathbf{C}\|_{L^\infty(\Omega_0)}
\]

and let \( D_{j_1}, \ldots, D_{j_M} \) be a chain of domains, defined according to (A1), connecting \( D_1 \) to \( D_j \). For the sake of brevity, set \( D_k = D_{j_k}, k = 1, \ldots, M \). Let
\[ W_k = \text{Int}\left( \bigcup_{j=0}^k D_j \right), \quad U_k = \Omega_0 \setminus \overline{W_k}, \quad k = 0, \ldots, M - 1. \]  

(28)

Note that here the tensors \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) are extended as in (19) in all of \( \Omega_0 \). Finally, for \( y, z \in (\bigcup_{j=0}^k D_j) \cup (\bigcup_{j=1}^k \Sigma_j) \), let us define the matrix-valued function

\[ S_k(y, z) := \int_{U_k} \left( \mathcal{C} - \overline{\mathcal{C}} \right)(x) \nabla G(x, y) : \nabla \overline{G}(x, z) \, dx, \]

whose entries are given by

\[ S_k^{p, q}(y, z) := \int_{U_k} \left( \mathcal{C} - \overline{\mathcal{C}} \right)(x) \nabla G(x, y) e_p : \nabla \overline{G}(x, z) e_q \, dx, \quad p, q = 1, 2, 3, \]

and where \( G(\cdot, y) \) and \( \overline{G}(\cdot, z) \) denotes respectively the singular solution of proposition 2.7 corresponding to the tensors \( \mathcal{C} \) and \( \overline{\mathcal{C}} \), respectively. Let us denote \( S_k^{i, q} = \sum_{i=1}^3 S_k^{i, q, e_i} \) and \( S_k^{p, \cdot} = \sum_{i=1}^3 S_k^{p, i, e_i} \). Here \( e_i, i = 1, 2, 3, \) are the fundamental unit vectors of \( \mathbb{R}^3 \).

Proceeding similarly to in [BFV, proposition 4.4], one can see that the functions \( S_k^{i, q}(\cdot, \cdot) \), \( S_k^{p, \cdot}(\cdot, \cdot) \) are solutions to the Lamé system with elasticity tensor \( \mathcal{C} \) defined in (19) in the weak sense clarified below.

**Proposition 3.1.** Let \( R_k := (\bigcup_{j=0}^k \partial D_j) \setminus (\bigcup_{j=1}^k \Sigma_j) \) and let \( y, z \in (\bigcup_{j=0}^k D_j) \cup (\bigcup_{j=1}^k \Sigma_j) \). Then, \( S_k^{i, q}(\cdot, z) \) and \( S_k^{p, \cdot}(y, \cdot) \) belong to \( H^1_0(W_k \setminus R_k, \mathbb{R}^3) \) and

\[ \int_{W_k} \nabla S_k^{i, q}(y, z) : \nabla \phi(y) \, dy = 0, \quad \forall \phi \in C_0^\infty \left( W_k \setminus R_k, \mathbb{R}^3 \right), \quad \forall q \in \{1, 2, 3\}, \]  

(29)

\[ \int_{W_k} \nabla S_k^{p, \cdot}(y, z) : \nabla \phi(z) \, dz = 0, \quad \forall \phi \in C_0^\infty \left( W_k \setminus R_k, \mathbb{R}^3 \right), \quad \forall p \in \{1, 2, 3\}. \]  

(30)

**Proof.** Proof of theorem 2.4

Let

\[ E := \| \mathcal{C} - \overline{\mathcal{C}} \|_{L^\infty(\Omega_0)} \]

\[ \epsilon := r_0 \left\| \Lambda \mathcal{C} - \Lambda \overline{\mathcal{C}} \right\|, \]

and let

\[ K_0 := \left\{ x \in D_0 \mid \text{dist}(x, \partial \Omega) \geq \frac{r_0}{24} \right\}. \]

By Hölder inequality and (23) we have

\[ \left| S_k^{p, q}(y, z) \right| \leq \frac{CE}{\sqrt{d(y)d(z)}}, \quad \forall p, q = 1, 2, 3, \]

(31)

where \( d(y) = d(y, U_k) \), \( d(z) = d(z, U_k) \) and \( C > 0 \) only depends on \( a_0, \beta_0, A, N, \alpha \) and \( L \).

By Alessandrini’s identity (12) applied to \( u_1(\cdot, \cdot) = G(\cdot, y)l \) and \( u_2(\cdot, \cdot) = \overline{G}(\cdot, z)m \), for \( y, z \in K_0 \) and for \( l, m \in \mathbb{R}^3 \) with \( ll = lm = l \), we get
where $C > 0$ depends on the \textit{a priori} data only.

We now proceed iteratively with respect to the index $k$.

**First step:** $k = 0$.

For $y, z \in K_0$, let us consider

$$S_0^{(p, q)}(y, z) := \int_{L_0} (C - \mathcal{T})(x) \mathcal{V}G(x, y) e_p : \mathcal{V}G(x, z) e_q \, dx, \quad p, q \in \{1, 2, 3\}.$$  

From (32) we get

$$\left| S_0^{(p, q)}(y, z) \right| \leq \frac{C}{r_0} \varepsilon, \quad \forall y, z \in K_0,$$

where $C > 0$ depends only on the \textit{a priori} data. Let us fix $z \in K_0$ and $q \in \{1, 2, 3\}$.

Recalling that, for fixed $q \in \{1, 2, 3\}$, $S_0^{(p, q)}(\cdot, z)$ is solution to (29), we shall propagate the smallness with respect to the first variable from the point $Q_1 = P_1 + \frac{n_0}{24} L n_1$ to $y_1 = P_1 + m_1$, for every $r \in \left(0, \frac{n_0}{24 \sqrt{1 + L^2}}\right)$, by iterating the three spheres inequality (27) over a chain of balls of decreasing radius and contained in a suitable cone with vertex at $P_1$ and axis in the direction $n_1$, obtaining

$$\left| S_0^{(p, q)}(y, z) \right| \leq \frac{CE}{\sqrt{r_0}} \left( \frac{\varepsilon}{E} \right)^{y_1},$$

where

$$y_1 = P_1 + m_1, \quad \eta = b \left| \ln \left( \frac{r}{r_0} \right) \right| + 1, \quad r \in \left(0, \frac{n_0}{24 \sqrt{1 + L^2}}\right).$$

$\delta \in (0, 1)$ only depends on $a_0$, $b_0$, and $a > 0$, $b > 0$ only depend on $L$.

Now, for fixed $p \in \{1, 2, 3\}$, let us consider a solution $S_0^{(p, \cdot)}(y, \cdot)$ to (30). Then, a similar procedure leads to

$$\left| S_0^{(p, q)}(y, z_r) \right| \leq \frac{CE}{\sqrt{r_1}} \left( \frac{\varepsilon}{E} \right)^{z_1},$$

where

$$z_r = P_1 + mn_1, \quad \sigma = b \left| \ln \left( \frac{r}{r_0} \right) \right| + 1, \quad r \in \left(0, \frac{n_0}{24 \sqrt{1 + L^2}}\right).$$

Hence, for every $r, \sigma \in \left(0, \frac{n_0}{24 \sqrt{1 + L^2}}\right)$

$$\left| S_0^{(p, q)}(y, z_r) \right| \leq \frac{CE}{\sqrt{r}} \left( \frac{\varepsilon}{E} \right)^{y_1}, \quad \forall p, q = 1, 2, 3.$$  

Let $r = cr$, with $c \in \left[\frac{3}{2}, \frac{5}{4}\right], p = q = 3$ and let $n = \frac{n_0}{12 \sqrt{1 + L^2}}$. Let us write

$$S_0(y, z_r) e_3 : e_3 = I_1 + I_2,$$
where
\[ I_1 = \int_{B_{r_1} \cap D_1} (C - \overline{C})(x) \nabla G(x, y_j) e_3 : \nabla \overline{G} (x, z_r) e_3 \, dx \] (38)
\[ I_2 = \int_{D \setminus (B_{r_1} \cap D_1)} (C - \overline{C})(x) \nabla G(x, y_j) e_3 : \nabla \overline{G} (x, z_r) e_3 \, dx . \] (39)

Here and in the sequel, $B_{r_1}$ denotes $B_{r_1}(R)$. Then, from (23) we have
\[ |I_2| \leq \frac{CE}{n_0} . \] (40)

From (36), (37) and (40) we derive
\[ |I_1| \leq CE \left( \frac{1}{r} \left( \frac{e}{E} \right)^{\delta^{\#\#}} + \frac{1}{n_0} \right) . \] (41)

We rewrite $I_1$ as follows
\[ I_1 = \int_{B_{r_1} \cap D_1} (C_1 - \overline{C}_1) \nabla \Gamma'(x, y_j) e_3 : \nabla \overline{\Gamma}'(x, z_r) e_3 \, dx + A_1 + A_2 + A_3 , \] (42)
where $\Gamma'$, $\overline{\Gamma}'$ is the normalized fundamental solution to (13) corresponding to the pair of elasticity tensors $(C = C_0, C^D = C_1)$, $(C = \overline{C}_0, C^D = \overline{C}_1)$ respectively, $D = D_1$ and
\[ A_1 = \int_{B_{r_1} \cap D_1} (C_1 - \overline{C}_1)(\nabla G - \nabla \Gamma')(x, y_j) e_3 : (\nabla \overline{G} - \nabla \overline{\Gamma}')(x, z_r) e_3 \, dx , \]
\[ A_2 = \int_{B_{r_1} \cap D_1} (C_1 - \overline{C}_1)(\nabla G - \nabla \Gamma')(x, y_j) e_3 : \nabla \overline{\Gamma}'(x, z_r) e_3 \, dx , \]
\[ A_3 = \int_{B_{r_1} \cap D_1} (C_1 - \overline{C}_1)\nabla \Gamma'(x, y_j) e_3 : (\nabla \overline{G} - \nabla \overline{\Gamma}')(x, z_r) e_3 \, dx . \]

By (24) and (16) we obtain
\[ |A_1| \leq \frac{CE}{n_0} , \] (43)
\[ |A_2|, |A_3| \leq \frac{CE}{\sqrt{n_0}} , \] (44)
where $C > 0$ only depends on the a priori data. From (42)–(44) we get
\[ |I_1| \geq \left| \int_{B_{r_1} \cap D_1} (C_1 - \overline{C}_1) \nabla \Gamma'(x, y_j) e_3 : \nabla \overline{\Gamma}'(x, z_r) e_3 \, dx \right| - \frac{CE}{\sqrt{n_0}} , \] (45)
where $C > 0$ only depends on the a priori data. From (41) and (45) we obtain
\[ \left| \int_{B_{r_1} \cap D_1} (C_1 - \overline{C}_1) \nabla \Gamma'(x, y_j) e_3 : \nabla \overline{\Gamma}'(x, z_r) e_3 \, dx \right| \leq \frac{CE}{r} \left( \frac{e}{E} \right)^{\delta^{\#\#}} + \left( \frac{r}{n_0} \right)^{1/2} , \] (46)
where $C > 0$ only depends on the a priori data.

Let us denote by $\Gamma$ and $\overline{\Gamma}$ the Rongved fundamental solutions corresponding to the tensors $C_0\chi_{R^1} + C_1\chi_{R^1}$ and $\overline{C}_0\chi_{R^1} + \overline{C}_1\chi_{R^1}$, respectively. Let
\[
\int_{B_1 \cap \partial B_1} (C_1 - C_1) \nabla \cdot \Gamma (x, y) \, e_3 \approx \nabla \cdot \Gamma (x, z) \, e_3 \, dx = B_1 + B_2 + B_3, \quad (47)
\]

where
\[
B_1 = \int_{B_1 \cap \partial B_1} (C_1 - C_1) \nabla \cdot \Gamma (x, y) \, e_3 \approx \nabla \cdot \Gamma (x, z) \, e_3 \, dx,
\]
\[
B_2 = \int_{B_1 \cap \partial B_1} (C_1 - C_1) \nabla \cdot (\Gamma' - \Gamma)(x, y) \, e_3 \approx \nabla \cdot \Gamma (x, z) \, e_3 \, dx,
\]
\[
B_3 = \int_{B_1 \cap \partial B_1} (C_1 - C_1) \nabla \cdot (\Gamma' - \Gamma')(x, y) \, e_3 \approx \nabla \cdot (\nabla \Gamma' - \nabla \Gamma') (x, z) \, e_3 \, dx.
\]

To estimate \( B_2 \) and \( B_3 \), we observe that
\[
B_n \cap D_l \subset \left( D_1 \cap Q_{-a}^{+a} \right) \cup \left\{ (x', x_3) : 0 \leq x_3 \leq \frac{L}{r_0} |x'|^{\alpha} \right\}.
\]

In \( D_1 \cap Q_{-a}^{+a} \), we can apply the asymptotic estimate (18) so that, recalling also (16), we have
\[
|B_2| \leq \frac{C E}{r_0^2} \int_{\mathbb{R}^3} |x - y|^{-2} |x - z|^{-2} dx + C E \int_{0 \leq l \leq \frac{L}{r_0} |x'|^{\alpha}} |x - y|^{-2} |x - z|^{-2} dx,
\]
where \( \gamma = \frac{n^2}{2(n+2)} < \frac{1}{2} \) and \( C > 0 \) depend only on \( a_0, \beta_0, L \) and \( \alpha \). The first integral can be easily estimated by passing to cylindrical coordinates and by applying Hölder inequality, obtaining
\[
\frac{C E}{r_0^2} \int_{\mathbb{R}^3} |x - y|^{-2} |x - z|^{-2} dx \leq \frac{C E}{r_0^2} \left( \frac{r}{r_0} \right)^{-1}.
\]

The estimate of the second integral is not straightforward. First, by performing the change of variables \( y = \frac{x}{r} \), we have
\[
C E \int_{0 \leq l \leq \frac{L}{r_0} |x'|^{\alpha}} |x - y|^{-2} |x - z|^{-2} dx
\]
\[
\leq \frac{C E}{r} \int_{\mathbb{R}^2} \left( \frac{1}{|y'|^2 + (\gamma_1 - 1)^2} \right) dy_1 dy_2.
\]

By splitting \( \mathbb{R}^2 \) as the union of \( A = \{ y' \in \mathbb{R}^2 | l y' l \geq \left( \frac{2l}{r_0} \right)^{\frac{1}{\gamma-1}} r^{-\frac{1}{\gamma}} \} \) and \( B = \mathbb{R}^2 \setminus A \), we have
\[
C E \int_{0 \leq l \leq \frac{L}{r_0} |x'|^{\alpha}} |x - y|^{-2} |x - z|^{-2} dx \leq \frac{C E}{r_0^2} \left( \frac{r}{r_0} \right)^{\frac{\gamma-1}{\gamma}} + \left( \frac{r}{r_0} \right)^{\frac{\gamma-1}{\gamma}}
\]
with \( C > 0 \) only depending on \( a_0, \beta_0, L \) and \( \alpha \), where we have used the fact that \( |y_3| < \frac{1}{2} \) in \( B \), so that \( |y_3| < 1 \) and \( |y_3 - c| \geq c - \frac{1}{2} \geq \frac{1}{6} \).
The term \( B_3 \) in (47) can be estimated similarly, obtaining

\[
\left| B_2 \right|, \left| B_3 \right| \leq \frac{CE}{\eta_0} \left( \frac{r}{r_0} \right)^{\gamma-1} + \left( \frac{r}{r_0} \right)^{\alpha-1}, \tag{48}
\]

where \( C > 0 \) only depends on \( \alpha_0, \beta_0, L, \alpha \), and \( \gamma = \frac{\alpha^2}{3\alpha + 2} < \frac{1}{2} \).

We split \( B_1 \) as follows

\[
B_1 = C_1 + C_2 + C_3,
\]

where

\[
C_1 = \int_{B_1^+} (C_1 - \bar{C}_1) \tilde{v} \Gamma(x, y) e_3; \tilde{v} \mathcal{T}(x, z_r)e_3 \, dx,
\]

\[
C_2 = \int_{B_1^+ \cap D_1} (C_1 - \bar{C}_1) \tilde{v} \Gamma(x, y) e_3; \tilde{v} \mathcal{T}(x, z_r)e_3 \, dx,
\]

\[
C_3 = \int_{B_1^+ \setminus D_1} (C_1 - \bar{C}_1) \tilde{v} \Gamma(x, y) e_3; \tilde{v} \mathcal{T}(x, z_r)e_3 \, dx.
\]

Since

\[
B_1^+ \cap D_1 \subseteq \left\{ (x', x_3) \, | \, 0 \leq x_3 \leq \frac{L}{r_0} |x'|^{\alpha+a} \right\}
\]

and

\[
B_1^+ \setminus D_1 \subseteq \left\{ (x', x_3) \, | \, -\frac{L}{r_0} |x'|^{\alpha+a} \leq x_3 \leq 0 \right\},
\]

we can estimate the terms \( C_2 \) and \( C_3 \) similarly to the second addend of \( B_2 \), getting

\[
\left| C_2 \right|, \left| C_3 \right| \leq \frac{CE}{\eta_0} \left( \frac{r}{r_0} \right)^{\alpha-1}, \tag{49}
\]

where \( C > 0 \) only depends on \( \alpha_0, \beta_0, L, \alpha \).

Finally, to estimate \( C_1 \), we use the following property of the Rongved fundamental solution

\[
\Gamma\left( \xi, y_0 \right) = h\Gamma\left( h\xi, hy_0 \right), \quad \mathcal{T}\left( \xi, y_0 \right) = h\mathcal{T}\left( h\xi, hy_0 \right), \quad \forall \xi \neq y_0, \quad \forall h > 0.
\]

Then

\[
C_1 = \frac{1}{r} \int_{B_1^2} (C_1 - \bar{C}_1)(x) \tilde{v} \Gamma(x, e_3) e_3; \tilde{v} \mathcal{T}(x, ce_3)e_3 \, dx \tag{50}
\]

and by (46)–(49) we obtain

\[
\left| \int_{B_1^2} (C_1 - \bar{C}_1)(x) \tilde{v} \Gamma(x, e_3) e_3; \tilde{v} \mathcal{T}(x, ce_3)e_3 \, dx \right| \leq CE \left( \frac{r}{r_0} \right)^{\gamma} + \left( \frac{c}{E} \right)^{\delta_{1+\epsilon}}. \tag{51}
\]

where \( C > 0 \) only depends on the \textit{a priori} data. From (16) and since \( c \in \left[ \frac{2}{3}, \frac{4}{3} \right] \) and \( n_1 = \frac{n_1}{12\xi_1 + L^2} \), we derive
where $C > 0$ only depends on $\alpha_0$, $\beta_0$, $L$, $\alpha$. From (51), (52), since $\gamma < \alpha$, $\gamma < \frac{1}{2}$ and $c \geq \frac{2}{3}$, by (33) we have

$$\left| \int_{\mathbb{R}^2 \setminus \mathbb{B}_r} (C_1 - \mathcal{C}_1)(x) \nabla \Gamma(x, e_3) e_3 \cdot \nabla \mathcal{T}(x, ce_3) e_3 \, dx \right| \leq CE \frac{r}{n_0},$$

(52)

where $C > 0$ only depends on $\alpha_0$, $\beta_0$, $L$, $\alpha$. From (51), (52), since $\gamma < \alpha$, $\gamma < \frac{1}{2}$ and $c \geq \frac{2}{3}$, by (33) we have

$$\left| \int_{\mathbb{R}^2} (C_1 - \mathcal{C}_1)(x) \nabla \Gamma(x, e_3) e_3 \cdot \nabla \mathcal{T}(x, ce_3) e_3 \, dx \right| \leq CE \left( \frac{r}{n_0} \right),$$

(53)

where

$$f(\rho) = \rho^r + \left( \frac{e}{E} \right)^{\frac{1}{2} + \frac{1}{\alpha_1} + \rho_1},$$

where $0 < \rho \leq \frac{1}{24\sqrt{1 + L^2}}$ and $A, B > 0$ only depend on $L$.

By an appropriate choice of $\rho = \rho(\epsilon)$, we get

$$\left| \int_{\mathbb{R}^2} (C_1 - \mathcal{C}_1)(x) \nabla \Gamma(x, e_3) e_3 \cdot \nabla \mathcal{T}(x, ce_3) e_3 \, dx \right| \leq CE \left| \ln \frac{\epsilon}{E} \right|^{\frac{1}{\alpha_1}},$$

(54)

where $C > 0$ only depends on $\alpha_0$, $\beta_0$, $L$, $\alpha$. Applying proposition 3.2 of [BFV] we have

$$\left| \int_{\mathbb{R}^2} (C_1 - \mathcal{C}_1)(x) \nabla \Gamma(x, e_3) e_3 \cdot \nabla \mathcal{T}(x, ce_3) e_3 \, dx \right| = \left| \Gamma(e_3, ce_3) - \mathcal{T}(e_3, ce_3) \right| e_3 \cdot e_3$$

and then

$$\left| \Gamma(e_3, ce_3) - \mathcal{T}(e_3, ce_3) \right| \leq CE \left| \ln \frac{\epsilon}{E} \right|^{\frac{1}{\alpha_1}},$$

(55)

where $C > 0$ only depends on the a priori data. Now, by using the explicit form of the Rongved fundamental solution and proceeding as in [BFV] (Section 4.2), it can be shown that (55) implies

$$\|C_1 - \mathcal{C}_1\|_\infty \leq CE \omega_1 \left( \frac{\epsilon}{E} \right),$$

(56)

where

$$\omega_1(t) = \left| \ln \left| \frac{\epsilon}{E} \right| \right|^{\frac{1}{\alpha_1}},$$

and $C > 0$ only depends on the a priori data. If $\|C - \mathcal{C}\|_\infty = \|C_1 - \mathcal{C}_1\|_\infty$, then we get

$$\|C - \mathcal{C}\|_\infty = E \leq \frac{\epsilon}{\omega_1^{-1} \left( \frac{1}{C} \right)},$$

and the claim follows. Otherwise, we proceed with the next step.

**Second step:** $k = 1$
In this case, let us consider the function

\[
S^{(p, q)}(y, z) = \int_{\Omega} \left( C - \nabla \right)(x) \nabla G(x, y)e_p; \nabla G(x, z)e_q \, dx
- \int_{D_i} \left( C - \nabla \right)(x) \nabla G(x, y)e_p; \nabla G(x, z)e_q \, dx, \quad p, q = 1, 2, 3.
\]

From (23), (32) and (56) we get

\[
\left| S^{(p, q)}(y, z) \right| \leq \frac{CE}{r_0} \omega_1 \left( \frac{c}{E} \right), \quad \forall y, z \in K_0,
\]

(57)

where \( C > 0 \) only depends on the \textit{a priori} data. Proceeding similarly to what previously described for \( S_0^{(p, q)} \), and by regularity estimates for elliptic systems, we derive that for every \( P \in \Sigma_i \) and every \( r, 0 < r \leq \frac{r_0}{24 + L^2} \),

\[
\left| S^{(p, q)}(y', z) \right| + r_0 \left| \nabla S^{(p, q)}(y', z) \right| \leq \frac{CE}{r_0} \omega_1 \left( \frac{c}{E} \right), \quad \forall z \in K_0,
\]

(58)

where \( y' = P + m_1 \) and \( \eta \) is the same constant defined in (34). Let us recall the following regularity estimates due to Li and Nirenberg [LN]

\[
\| \nabla S^{(p, q)}(\cdot, z) \|_{L^2(P_\delta \cap Q_{0, r}^+)} + r_0 \left| \nabla S^{(p, q)}(\cdot, z) \right|_{L^2(P_\delta \cap Q_{0, r}^+)} \leq \frac{CE}{r_0^2},
\]

(59)

where \( \beta = \frac{a}{2(1 + a)} \) and \( C > 0 \) only depends on the \textit{a priori} data.

From (58) and (59) we have that, for every \( P \in \Sigma_i \), \( z \in K_0 \) and \( r \in \left( 0, \frac{r_0}{24 + L^2} \right) \),

\[
\left| S^{(p, q)}(P, z) \right| + r_0 \left| \nabla S^{(p, q)}(P, z) \right| \leq \frac{CE}{r_0} \left( \frac{r}{r_0} \right)^\beta + \left( \frac{c}{E} \right),
\]

(60)

where \( C \) only depends on the \textit{a priori} data. By a suitable choice of \( r = r(c) \), we get

\[
\left| S^{(p, q)}(P, z) \right| + r_0 \left| \nabla S^{(p, q)}(P, z) \right| \leq \frac{CE}{r_0} \tilde{a}_2 \left( \frac{c}{E} \right),
\]

(61)

where

\[
\tilde{a}_2(t) = \ln \ln t \| \ln \ln t \|^{\frac{1}{2(1 + a)}}.
\]

By the transmission conditions

\[
S^{(p, -)}(\cdot, z)|_{\partial \Omega} = S^{(p, -)}(\cdot, z)|_{\partial \Omega},
\]

\[
C^0 \nabla S^{(p, -)}(\cdot, z)n_1|_{\partial \Omega} = C^1 \nabla S^{(p, -)}(\cdot, z)n_1|_{\partial \Omega}, \quad \text{on} \ \Sigma_i,
\]

and by (61) we have, for every \( p \in \{1, 2, 3\} \),

\[
\| S^{(p, -)}(\cdot, z) \|_{H^{\frac{1}{2}}(\Sigma_i)} + r_0 \| \nabla S^{(p, -)}(\cdot, z) \|_{H^{\frac{1}{2}}(\Sigma_i)} \leq \frac{CE}{r_0} \tilde{a}_2 \left( \frac{c}{E} \right),
\]

(62)

where \( C \) only depends on the \textit{a priori} data.

We can adapt the arguments in [ARRV] (see in particular lemma 6.1 and theorem 6.2) to obtain the following stability estimate for the Cauchy problem for \( S^{(p, -)}(\cdot, z) \) in \( Q_{0, r}^+ \cap D_i \).
where \( R_1 = R - d_{n_1}, \quad d = \frac{n_1}{\sqrt{1 + L^2}}, \quad \overline{\mathcal{P}} = \frac{L}{12(1 + \sqrt{1 + L^2})}, \) and the constants \( \xi \in (0, 1) \) and \( C > 0 \) only depend on the \textit{a priori} data. Observe that by proposition 5.5 in [ARRV] there exists \( h_0 \), such that \( (D_h)_0 \) is connected, \( \forall h \leq h_0 \).

Let \( \overline{h} = \min \{ h_0, \frac{n_1}{\sqrt{1 + L^2}} \} \). Then, \( \frac{h}{n_0} \) depends only on \( L \), \( (D_h)_0 \) is connected and contains the points \( R_1 \) and \( Q_2 = P_2 + \frac{\overline{h} \sqrt{1 + L^2}}{2} n_2 \), with \( P_2 \in \Sigma_2 \) as in (A.1). Let \( \gamma \) be an arc contained in \((D_h)_0 \) connecting \( R_1 \) with \( Q_2 \). By iterating the three spheres inequality (27) first over a chain of balls with centers on \( \gamma \) and then over a chain of balls of decreasing radius and contained in a suitable cone with vertex at \( P_2 \) and axis in the direction \( n_2 \), we obtain

\[
\| S^{(p, q)}(\cdot, z) \|_{L^\infty(\rho(R_1))} \leq C \left( \frac{\rho_2(\frac{e}{E})}{\rho_0} \right)^{\frac{\xi}{\sqrt{3} + 3}},
\]

where \( \delta = \frac{p}{\rho_0} = \frac{n_1}{\sqrt{1 + L^2}}, \) and \( \tau_2 \) depends only on \( L \), \( (D_h)_0 \) is connected and contains the points \( R_1 \) and \( Q_2 = P_2 + \frac{\overline{h} \sqrt{1 + L^2}}{2} n_2 \), with \( P_2 \in \Sigma_2 \) as in (A.1). Let \( \gamma \) be an arc contained in \((D_h)_0 \) connecting \( R_1 \) with \( Q_2 \). By iterating the three spheres inequality (27) first over a chain of balls with centers on \( \gamma \) and then over a chain of balls of decreasing radius and contained in a suitable cone with vertex at \( P_2 \) and axis in the direction \( n_2 \), we obtain

\[
\| S^{(p, q)}(\cdot, z) \|_{L^\infty(\rho(R_1))} \leq C \left( \frac{\rho_2(\frac{e}{E})}{\rho_0} \right)^{\frac{\xi}{\sqrt{3} + 3}},
\]

where \( C \) only depend on the \textit{a priori} data and

\[
\omega_2^*(t) = \ln \left| \ln \left| \ln \left( \frac{e}{E} \right) \right| \right|^{1/t}.
\]

Now, choosing a coordinate system centered at \( P_2 \), with \( e_3 = n_2 \), and denoting by \( \Gamma \) and \( \mathbf{T} \) the Rongved solutions corresponding to the tensors \( \mathbf{C}_{\mathcal{U}R_2} + \mathbf{C}_{\mathcal{Z}R_2} \) and \( \mathbf{C}_{\mathcal{U}R_2} + \mathbf{C}_{\mathcal{Z}R_2} \), we get that

\[
\left( \Gamma(e_3, ce_3) - \mathbf{T}(e_3, ce_3) \right) e_3 \cdot e_3 \leq CE\omega_2(\frac{e}{E}),
\]

where \( C \) only depend on the \textit{a priori} data and

\[
\omega_2(t) = \ln \left| \ln \left| \ln \left( \frac{e}{E} \right) \right| \right|^{1/t},
\]

so that, proceeding as in [BFV], we have

\[
\| \mathbf{C}_2 - \mathbf{C}_2 \|_\infty \leq CE\omega_2(\frac{e}{E}).
\]

If \( E = \| \mathbf{C}_2 - \mathbf{C}_2 \|_\infty \), then

\[
\| \mathbf{C} - \mathbf{C}_2 \|_\infty = E \leq \frac{e}{\omega_2^2(\frac{1}{E})}
\]

and the claim follows. Otherwise, we proceed similarly iterating the procedure up to \( k = j \) obtaining the desired result. \( \square \)
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