Abstract

Let $K$ be the diagonal subgroup of $U(2)^\otimes n$. For the one-qubit state space $\mathcal{H}_1 = \mathbb{C}\{|0\} + \mathbb{C}\{|1\}$, we may view $\mathcal{H}_1$ as a standard representation of $U(2)$ and the $n$-qubit state space $\mathcal{H}_n = (\mathcal{H}_1)^\otimes n$ as the $n$-fold tensor product of standard representations. Representation theory then decomposes $\mathcal{H}_n$ into irreducible subrepresentations of $K$ parametrized by combinatorial objects known as Young diagrams. We argue that $n - 1$ classically controlled measurement circuits, each a Fredkin interferometer, may be used to form a projection operator onto a random Young diagram irrep within $\mathcal{H}_n$. For $\mathcal{H}_2$, the two irreps happen to be orthogonal and correspond to the symmetric and wedge product. The latter is spanned by $|\Psi^\pm\rangle$, and the standard two-qubit swap interferometer requiring a single Fredkin gate suffices in this case. In the $n$-qubit case, it is possible to extract many copies of $|\Psi^\pm\rangle$. Thus applying this process using nondestructive Fredkin interferometers allows for the creation of entangled bits (e-bits) using fully mixed states and von Neumann measurements.

1 Introduction

Theoretical quantum computing considers data to be stored within idealized quantum particles and then makes inferences on how such data may be manipulated in terms of the axioms of quantum mechanics. Since quantum measurement changes the state of the underlying particles, it is not merely a matter of input/output but rather a computational act. Consider for example how crucial the appropriate measurements are for quantum teleportation. More generally, a thread of recent research demonstrates that any quantum circuit may be emulated by a chain of carefully orchestrated measurements on a large highly entangled quantum register (e.g. [J05]).

This work is much more modest, in that it presents a sequence of classically controlled quantum circuits realizing a projector which seems to have been overlooked. The observation is related to but simpler than work of Bacon, Chuang, and Harrow on Clebsch-Gordon transforms [BCH04, BCH06]. Given $\mathcal{H}_1 = \mathbb{C}\{|0\} + \mathbb{C}\{|1\}$ and $\mathcal{H}_n = \mathcal{H}_1^\otimes n$ the $n$-qubit state space, both the Clebsch-Gordon transformation and this work make essential use of the (classical) representation theory of $\mathcal{H}_n$. For we may view $\mathcal{H}_1$ as the standard representation of $U(2)$, so that $\mathcal{H}_n$ is the $n$-fold tensor product of standard representations equipped with the left-multiplication by elements of $K$, the diagonal subgroup of $U(2)$:

$$K \overset{\text{def}}{=} \{ V^\otimes n ; V \in U(2) \}$$

(1)

Also, $\mathcal{H}_n$ inherits a standard representation by $S_n$, the group of permutations of the elements of the set $\{1, 2, \ldots, n\}$. Namely, if $\sigma \in S_n$, then we act by the permutation unitary $U_\sigma$ which satisfies $U_\sigma |b_1 b_2 \ldots b_n\rangle = |b_{\sigma(1)} \ldots b_{\sigma(n)}\rangle$ on all computational basis kets. Then $kU_\sigma |\psi\rangle = U_\sigma k |\psi\rangle$ for all $\sigma \in S_n, k \in K$. A subrepresentation w.r.t. $K$ is a linear subspace $L \subseteq \mathcal{H}_n$ preserved by all $k \in K$. The definition extends to considering subreprentations of subrepresentations, and a subrepresentation $L$ is irreducible if the only subrepresentations of $L$ are $\{0\}$ and $L$. Now suppose any decomposition of $\mathcal{H}_n$ into irreducible subrepresentations (henceforth irreps) of $K$ above. Each element $\sigma \in S_n$ will permute the factors, since $\sigma(L)$ is another irrep isomorphic to the original under $\sigma^{-1}$ and hence irreducible.

Example 1: $L = \mathbb{C}\{|\psi_1\rangle = |0011\rangle - |1001\rangle - |0110\rangle + |1100\rangle\}$ is a subrepresentation of $\mathcal{H}_4$, since the ket...
lies within a decoherence-free-subspace (DFS) [ZR97, KLV00, Kea01] on which each \( k \in K \) satisfies \( k |\psi_1\rangle = \det(k)^2 |\psi_1\rangle \). Also, any one-dimensional subrepresentation is irreducible. If (23) denotes the flip permutation exchanging 2 and 3, then \( U_{(23)} = \mathbb{C}\{0101 \mid -1001 - |0110 + |1010\} \) spans a second dimension of the DFS.

Earlier works on quantum circuits for \( K \)-irreps [BCH04, BCH06] describe input-output register indexing schemes for the Clebsch-Gordon transform, which in particular carries each computational basis state into an irrep of \( \mathcal{H}_K \) indexed by a combinatorial object known as a Young diagram. The second work applies equally well to qubits and qudits. In contrast, this work attempts only gadgetry: a scheme exploiting classical control is outlined for projecting onto Young diagram irreps. Before outlining the general case in the next section, we argue that a Fredkin interferometer suffices in two qubits. The decomposition into irreps is as follows, where we have placed the appropriate Young diagram below each irrep.

\[
\mathcal{H}_2 = \text{Sym}^2(\mathcal{H}_1) \oplus \wedge^2(\mathcal{H}_1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \oplus \begin{bmatrix} 3 \end{bmatrix} \tag{2}
\]

Here, \( \text{Sym}^2(\mathcal{H}_1) = \text{span}_\mathbb{C}\{|00\rangle, |01\rangle + |10\rangle, |11\rangle\} = \text{span}_\mathbb{C}\{|\Phi^\pm\rangle, |\Psi^\mp\rangle\} \) is the symmetric irrep of \( \mathcal{H}_1 \) consisting of states invariant under \( \text{SWAP} \). The other irrep is anti-invariant under \( \text{SWAP} \). Setting notation, let \( |\Psi^-\rangle = 2^{-1/2}(|01\rangle - |10\rangle) \), so that \( \wedge^2(\mathcal{H}_1) = \mathbb{C}\{|\Psi^\mp\rangle\} \). For (12) the nontrivial element of \( S_2 \), we have \( U_{(12)} = \chi \) (the SWAP operator) and \( \chi \langle \Psi^\mp\rangle = -|\Psi^\mp\rangle \). Finally, consider the following quantum logic circuit, in this work described as a Fredkin gate interferometer.

Here, \( H = 2^{-1/2} \sum_{b_1, b_2 = 0}^{1} (-1)^{b_1 b_2} |b_1\rangle \langle b_1| \) is the one-qubit Hadamard unitary. By a direct computation or due to the interferometer-circuit literature [LLW03, B04, Bea05], when \( |\psi\rangle \) is placed on the input lines the measurement satisfies

\[
\text{Prob(ancilla = 0)} = \frac{1}{2} + \frac{1}{2} \text{Trace}(\chi |\psi\rangle \langle \psi|) \tag{3}
\]

Hence the two outputs of the Fredkin interferometer perform projections onto \( \text{Sym}^2(\mathcal{H}_1) \) and \( \wedge^2(\mathcal{H}_1) \). This work extends the above Young diagram projector into \( n \) qubits.

An application of a sort results. Recall that entangled-bits (or e-bits) are fully-entangled, pure two-qubit states. All such states are equivalent under the action of some \( U_1 \otimes U_2 \) for \( U_1, U_2 \) being two-by-two unitary matrices. For example, if the usual e-bit is \( |\Phi^+\rangle = 2^{-1/2}(|00\rangle + |11\rangle) \) and \( |\Psi^-\rangle = 2^{-1/2}(|01\rangle - |10\rangle) \), then \( \sigma_z \sigma_x \otimes I_2 |\Psi^\mp\rangle = |\Phi^\mp\rangle \). Such e-bits are required for quantum teleportation and are fundamental resources in theoretical works generalizing Shannon’s theory of classical channel capacities to quantum channels [BS98].

As is well-known, an e-bit may be produced by a unitary process. For \( |\Phi^+\rangle \) results from the following diagram.

Also, e-bits may be produced by cooling, since the Hamiltonian \( H_0 = -\sigma_z \otimes \sigma_z \) holds \( |\Phi^+\rangle \) within its groundstate and any small perturbation \( H'_\epsilon = -\sigma_z \otimes \sigma_z - \epsilon \sigma_x \otimes \sigma_x \) will split the degeneracy so that \( |\Phi^+\rangle \) is the unique groundstate. A third option is to prepare an e-bit using a von Neumann measurement within the Fredkin gate interferometer, as above. The yield would seem to be low. For say the input is a fully mixed state \( I_4/4 \). Then the probability of creating \( |\Psi^-\rangle \) is only \( 1/4 \), since \( \text{dim} \wedge^2(\mathcal{H}_1) = 1 \) and \( \text{dim} \text{Sym}^2(\mathcal{H}_1) = 3 \). Our generalization to \( n \)-qubit Young diagrams shows that much more efficient initialization of e-bits using von Neumann measurements is possible for \( n > 2 \).

This manuscript is organized as follows. In §2, we review the irreps of the two-qubit state space and their Young diagrams. In §3, we present an algorithm for choosing a sequence of (classically-controlled) von Neumann measurements. In §4, we present an application, namely harvesting copies of \( \wedge^2(\mathcal{H}_1) \) as e-bits.
2 Young diagram irreps

The representation theory of $\mathcal{H}_K$ under either $K$ or the symmetric group on $\{1,2,\ldots,n\}$ is both classical and well-known [GW98]. To begin, it suffices to produce a highest-weight $|\psi\rangle$ within each irrep. For given such a $|\psi\rangle$, it happens that $X^j |\psi\rangle$ span the irrep, where $X = \sum_{k=1}^n X_k \in \mathfrak{h}$ is a local Hamiltonian. Each (standard) Young diagram specifies such a highest-weight vector. For a two-level system, a diagram consists of two rows of $n$ boxes total. Each box holds an integer within $\{1,2,\ldots,n\}$, no integers are repeated, and the integers increase moving down each column and across both rows. We use the following notation, where $p + q = n$.

$$
\begin{array}{cccccccc}
  j_1 & j_2 & \cdots & j_q & |q+1\rangle & \cdots & |p-1\rangle & j_p \\
  k_1 & k_2 & \cdots & k_q
\end{array}
$$

(4)

For any Young diagram $p \geq q$, $j_1 = 1$, and $k_q = n$. Let $G$ denote the group of all permutations of $\{1,2,\ldots,n\}$ which preserve the subsets corresponding to the columns of the Young diagram. Let $b(\ell)$ denote the bit which is 0 if $\ell \in \{1,2,\ldots,n\}$ occurs in the top row and 1 if $\ell$ occurs in the bottom row. Then the highest-weight vector corresponding to the diagram $\Gamma$ is

$$
|\psi_\Gamma\rangle = (\#G)^{-1/2} \sum_{\sigma \in G} \text{sign}(\sigma) U_\sigma |b(1)b(2)\ldots b(n)\rangle.
$$

(5)

If $L_\Gamma$ is the corresponding irrep, then the basic result of classical representation theory states ([GW98, BCH06]) $\mathcal{H}_K = \oplus L_\Gamma$. Here, the nonorthogonal (vector-space) direct sum is taken over all Young diagrams.

**Example 1 (Cont.):** Consider the following four-qubit Young diagram.

Thus diagram irreps need not be orthogonal, given the earlier discussion. The $\Gamma$-shaped diagram in three-qubits provide another example.

**Example 2:** Consider the horizontal Young diagram for which $q = 0$. Then we iteratively apply $X$ to $|00\ldots0\rangle$. Now $X|00\ldots0\rangle$ is a singlet, $X^2|00\ldots0\rangle$ is a doublet, etc. The resulting irrep is $\text{Sym}^n(\mathcal{H}_1)$, i.e. the $K$-irrep consisting of those kets invariant under all $U_\sigma$ for $\sigma \in S_n$.

3 Irrep projectors

When taking tensor products of irrep subspaces as specified by Young diagrams, one forms a direct sum of all possible concatenations of the diagrams which are themselves (standard) Young diagrams. The resulting direct summands are often orthogonal, since nonorthogonal Young diagram irreps must have the same shape. In symbols, suppose diagrams $T_1$ and $T_2$ with corresponding $p_1, q_1$ and $p_2, q_2$ respectively per Equation 4. Then Schur orthogonality demands $L_{T_1} \perp L_{T_2}$ whenever $p_1 \neq p_2$ (equivalently $q_1 \neq q_2$). In this way tensor products often produce orthogonal direct sums.

The simplest example of a tensor product involves rectangular diagrams. Suppose a diagram according to Equation 4, except that for purposes of induction say $p + q = n - 1$ rather than $n$, and let $K_{n-1}$ be the diagonal subgroup of $U(2)^{\otimes (n-1)}$ with $K_n$ similar. We tensor the Young diagram irrep with $\mathcal{H}_q$ on qubit $n$, whose corresponding Young diagram is a single box containing $n$. Suppose further that the original diagram is rectangular, i.e. $p = q$. Then the only valid concatenation is that diagram for which $j_{p+1} = n$, and the tensor product is itself this diagram irrep on $K_n$.

Consider next a qubit irrep of a Young diagram which is not rectangular. Arguing as above with a one-row diagram replacing $n$, we see that the diagram of Equation 4 is also the following tensor product:

$$
\begin{array}{cccccccc}
  j_1 & j_2 & \cdots & j_q & |q+1\rangle & \cdots & |p-1\rangle & j_p \\
  k_1 & k_2 & \cdots & k_q
\end{array}
$$

(7)

The latter factor is a copy of $\text{Sym}^{p-q}(\mathcal{H}_1)$, mapped into the $p - q$ qubits labelled by contents of the diagram boxes.
The rectangular factor is a one-dimensional representation. To see this, consider that the rectangular diagram is the only possible concatenation of the tensor product:

\[
j_1 \ j_2 \cdots \ j_q = \begin{array}{cccc}
k_1 & k_2 & \cdots & k_q \\
\end{array} \otimes \begin{array}{cccc}
j_1 & j_2 & \cdots & j_q \\
k_1 & k_2 & \cdots & k_q \\
\end{array} \otimes \cdots \otimes \begin{array}{cccc}
j_q \\
k_q \\
\end{array}
\]  \hspace{1cm} (8)

Each column describes a copy of \( |\Psi^-\rangle \) on the appropriate bits. Thus, \( k \in K \) acts as multiplication by \( \det(k)^q \) on the subspace spanned by qubits whose labels appear in the rectangular subrepresentation. We refer to the box containing \( j_{q+1} \) of Equation 4 as the hook-box of a diagram which is not rectangular.

What happens when we tensor the irrep of Young diagram which is not rectangular with \( H_1 \)? There are only two possible concatenations appear below, and they result by placing the box with contents \( n \) below or far to the right of the hook-box.

\[
j_1 \ j_2 \cdots \ j_q \ j_{q+1} \ j_{q-1} \ j_p \ n = \begin{array}{cccc}
j_1 & j_2 & \cdots & j_{q+1} \\
k_1 & k_2 & \cdots & k_q \\
\end{array} \oplus \begin{array}{cccc}
j_1 & j_2 & \cdots & j_{q+1} \\
k_1 & k_2 & \cdots & k_q \\
\end{array} \oplus \begin{array}{cccc}
j_p \\
k_p \\
|n\rangle \\
\end{array}
\]  \hspace{1cm} (9)

For simplicity, relabel \( j_{q+1} = j \). Due to comments of the last paragraph, any \( |\Psi\rangle \) within the irrep of the top direct summand will satisfy \( U_{(jn)} |\Psi\rangle = |\Psi\rangle \). On the other hand, \( U_{(jn)} \vec{X} = \vec{X} U_{(jn)} \), from which we may infer that any \( |\Psi\rangle \) within the irrep of the second diagram will satisfy \( U_{(jn)} |\Psi\rangle = -|\Psi\rangle \). For the same was true by construction of \( |\Psi_{\text{random}}\rangle \), the highest weight ket of the irrep of the lower Young diagram. Thus, the direct sum is as promised orthogonal, and each irrep lies in the +1 and -1 eigenspace of \( U_{(jn)} \) respectively.

Orthogonality of the resulting irreps suggests that their kets might be distinguished by a projector. Indeed, consider the following Fredkin gate interferometer.

A reading of 0 on the classical wire output implies application of a projector onto the irrep of the top diagram of Equation 9, while a reading of 1 on the classical output implies a projector onto the irrep of the bottom diagram of Equation 9. Thus we obtain the following inductive algorithm for using Fredkin gate interferometers to project into the irrep of a random Young diagram on \( n \) qubits.
Algorithm for projecting into the irrep of a random Young diagram of \( n \)-qubits: If \( n = 1 \), do nothing. Else we suppose \( |\psi_{\ell_1-1}\rangle \) within the irrep of an \( \ell - 1 \) qubit Young diagram \( \mathcal{T}_{\ell-1} \) and continue as follows:

1. Add an 0th (pure) qubit in any state to the system.
2. If \( \mathcal{T}_{\ell-1} \) is rectangular, then form the only possible \( \mathcal{T}_\ell \) and return the system state \( |\psi_{\ell}\rangle \).
3. Else apply a Fredkin gate interferometer to project into an eigenspace of the SWAP unitary \( U_{(j\ell)} \), where \( j \) is the hook-qubit of \( \mathcal{T}_{\ell-1} \).

(a) If the eigenvalue is +1, then form \( \mathcal{T}_\ell \) by appending a box containing an \( n \) on the far right of the first row of \( \mathcal{T}_{\ell-1} \).

(b) If the eigenvalue is −1, then form \( \mathcal{T}_\ell \) by appending the new box below the hook-qubit box, i.e. below \( j_{q+1} \) per Equation 4.

The result is a valid \( \mathcal{T}_\ell \) and \( |\psi_{\ell}\rangle \). Continue until \( \ell = n \).

Note that the second paragraph of this section describes why there is no need to measure in Step 2. Indeed, a Young diagram which is rectangular except for one trailing box at the upper right describes \( [\Lambda^2(\mathbb{C}^2)]^\otimes(n-1)/2 \otimes \text{Sym}^1(\mathcal{H}_f) \). Yet the second factor is merely \( \mathcal{H}_f \). Thus we are tautologically in the irrep of this diagram upon adding any extra qubit, regardless of its state.

4 Creating e-bits

Recall the Algorithm describing a particular classically-controlled measurement process in the last section. Regardless of the initial state \( |\psi\rangle \), some random \( |\varphi\rangle \) known to be within the irrep \( L_T \) of some Young diagram \( T \) always results. The probability of a particular \( T \) depends on \( |\psi\rangle \). For example, in two qubits a single measurement is made which projects onto either the singlet \( |\Psi^-\rangle \) or its orthogonal complement, so that the probability of the resulting diagram \( T \) is \( |\langle \psi |\Psi^- \rangle|^2 \).

It is natural to consider averaging over all \( |\psi\rangle \), equivalently replacing \( |\psi\rangle \) by a fully decohered state, i.e. \( \eta = I_2/2^n = 2^{-n}\sum_{j=0}^{2^n-1} |j\rangle \langle j| \). In terms of the algorithm, since \( \eta = (I_2/2)^\otimes n \), we may equally well consider each added qubit in Step #1 to rather be the completely decoherent one-qubit state, \( (1/2)|0\rangle\langle 0| + |1\rangle\langle 1| \). For each Young diagram \( T \), we also choose an orthonormal basis \( \{ |\psi_{j,T}\rangle \} \) and define

\[
\rho_T = (\dim T)^{-1} \sum_{j=1}^{\dim T} |\psi_{j,T}\rangle \langle \psi_{j,T}| \tag{10}
\]

Then \( \rho_T \) does not depend on the choice of orthonormal basis, and the classically-controlled measurement sequence of the algorithm always maps \( \eta \mapsto \rho_T \) for some qubit Young diagram \( T \). In this way, the sequence of classically-controlled measurements gives rise to a random variable on Young diagrams, i.e. by assigning to each diagram the probability that the process carries the fully decoherent \( \eta \) to a decoherent mixture of the states of the associated irrep.

Note that \( \rho_T \) is pure if and only if the irrep associated to \( T \) is one-dimensional, i.e. \( \rho_T \) is pure iff the Young diagram \( T \) is rectangular. This is well-known in the two-qubit case. Namely, if we measure \( \rho_T \in \Lambda^2(\mathcal{H}_f) = \text{span}_\mathbb{C}\{ |\Psi^-\rangle \} \), then the measurement of the completely mixed state has produced a singlet (up to global phase.) This may be adjusted by local rotations to produce an e-bit or entangled bit, i.e. \( |\Psi^\otimes\rangle \). In the \( n \)-qubit case, each column of the rectangular Young diagram describes two
qubits in which the state must carry a copy of $|\Psi^-\rangle$, due to Equation 9. Hence a rectangular Young diagram shows that $\nu$ has been converted into a pure state of $n/2$ e-bits. Furthermore, suppose instead the generic $n$-qubit Young diagram of Equation 4. For convenience, we label one extra constant $r = n - 2q$ for the number of boxes on the top row of the diagram which do not have a lower neighbor. In particular, the irrep is a copy of $\lambda^\top(\mathcal{H}_L^\top)^{\otimes q} \otimes \text{Sym}^\top(\mathcal{H}_R^\top)$ and so has dimension $r + 1$. For the left hand factor is a copy of $\mathbb{C}$ while generically the symmetric representation is spanned by the $|00\ldots0\rangle$, the singlet, the doublet, $\ldots$, $|11\ldots1\rangle$. With this language set, the following Observation is an interpretation of Equation 7.

Observation: Let $|\psi\rangle \in \mathcal{L}_T$ be arbitrary, for $T$ a Young diagram per Equation 4. Then $q = (n - r)/2$ e-bits (in particular copies of $|\Psi^-\rangle$) may be harvested from $|\psi\rangle$ by pairwise selecting qubits $j_1, k_1$, then $j_2, k_2$, through $j_q, k_q$. Thus, a similar comment applies to any mixed $\rho$ supported within $\mathcal{L}_T$, e.g. the mixture $\rho_T$.

On average, how many e-bits might one harvest from the Algorithm? Label a random variable $R_n$ to the possible values of $r$ averaged over all Young diagrams according to their probability of arising from the Algorithm applied to $\nu$. Then the algorithm will produce $Q_n = (n - R_n)/2$ e-bits. Furthermore, $R_n$ is simpler to analyze than the random variable on Young diagrams, as follows. Suppose that after processing $n - 1$ qubits, the Algorithm has created $\rho_T$ for which the value of $R_{n-1} = d$. Let $\mathcal{T}_j, j = 1, 2$ be the two possible new Young diagrams which might result after measuring the eigenvalue of the swap gate $U_{(j\mu)}$ for $j$ the hook-qubit. Then the irrep decomposition is

$$\mathcal{L}_T \otimes \mathcal{H}_L^\top = \mathcal{L}_{T_1} \oplus \mathcal{L}_{T_2}$$

(11)

We may form an ensemble for $\rho_T \otimes (I_2/2)$ subordinate to the decomposition at right, i.e. consisting of a mixture of an orthonormal basis of the irrep of $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the probability of the two outcomes depends only on the dimension of the two vector spaces. Being specific, say $\mathcal{T}_1$ places the box containing $n$ below the hook qubit, while $\mathcal{T}_2$ places it to the right. Then $\dim \mathcal{T} = d + 1$, $\dim \mathcal{T}_1 = d$, and $\dim \mathcal{T}_2 = d + 2$. This provides a rule for incrementing the subscript of $R_n$ in terms of a Markov process.

$$\left\{ \begin{array}{ll}
\text{Prob}(R_n = d + 1 | R_{n-1} = d) &= (d + 2)/(2d + 2) \\
\text{Prob}(R_n = d - 1 | R_{n-1} = d) &= d/(2d + 2)
\end{array} \right.$$ 

(12)

This recursively defines $R_n$, after we observe that $\text{Prob}(R_1 = 1) = 1$.

When evaluating the Markov process numerically, we abbreviate $\text{Prob}(R_n = j) = P(t, j)$. By reversing a step, the recursion rule may be restated as

$$P(t, j) = \frac{j + 1}{2j} P(t - 1, j - 1) + \frac{j + 1}{2j + 4} P(t - 1, j + 1)$$

(13)

Table 1 then shows some sample means and deviations of $R_n$, rounded to three decimal places. Note that the expected value of $R_n$ grows slowly with $n$. Indeed, for large $d$ Equation 12 is approximately a coin toss, corresponding to a random walk in the integers with expected value equal to its starting point. Hence, it is unsurprising that $R_n$ grows slowly, since any increase is due to the bias between the two probabilities of Equation 12. This justifies the assertion of the introduction in that it is cheaper to prepare e-bits by von Neumann measurement in bulk. For $R_n$ is the number of $n$-qubits processed which remain outside e-bits, i.e. within the copy of $\text{Sym}^\top(\mathcal{H}_R^\top)$ contained within the irrep of $\mathcal{T}_n$.

A graph of the probability density function of $R_n$ for $n = 500$ qubits appears in Figure 1. In particular, suppose that one has a quantum processor capped at 100 qubits, and one attempts to fuse 500 fully mixed qubits into as many e-bits as possible using the algorithm. Any anti-symmetric measurement causes the resulting singlet to be removed. Then the graph suggests that one is unlikely to overrun the processor with this procedure, and only rarely would fewer than 210 singlet pairs result. By Table 1, we estimate that of 500 qubits an average of about 465 would be fused into e-bits, for a yield of 93%. In contrast, if $n = 2$, then the probability that the SWAP measurement produces an e-bit is 1/4, for a 25% yield.

5 Conclusions and Open Questions

We have presented a classically controlled algorithm requiring $n - 1$ Fredkin gate interferometers, whose placement depends on the outcomes of earlier classical measurements. It accomplishes the projection onto the subspace of $\mathcal{H}_n = (\mathcal{H}_R^\top)^{\otimes n}$ described by a random Young diagram, which may be inferred from the classical measurements. As an application, we show that applying this process to a complete mixture $\rho = I_{2^n}/2^n$ produces a number
of entangled bits equal to the width of the two-row subdiagram. A large percentage of the qubits tend to lie in this subdiagram for $n > 50$.

A generalization to qutrits or qudits is not obvious. In this context, Young diagrams have three (or $d$) rows, and there are multiple hook qudits. Distinguishing the location of the new box is no longer possible with a SWAP, although the irreps are orthogonal due to their shape. Thus, the problem would be choosing unitaries which map the computational basis into the space of each irrep and demonstrating polynomial size circuits for these unitaries.

In a broader context, this sequence of Fredkin-gate interferometers appear to be a new application of measurement and classical control, distinct from stabilizer code checks, quantum teleportation, and computation with cluster states. Ongoing work seeks to find other ways in which measurement gates might be exploited to perform computations in quantum circuit diagrams, e.g. to observe spin-flip symmetries.

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A Analytic consideration of the Markov process $R_n$

Recall $P(t, j) = \operatorname{Prob}(R_t = j)$ satisfies the recursion

$$P(t, j) = \frac{j+1}{2j} P(t-1, j-1) + \frac{j+1}{2j+4} P(t-1, j+1)$$

Label $S(t, j) = 2^{j+1} P(t, j)$. Then

$$S(t, j) = S(t-1, j-1) + S(t-1, j+1)$$

This recursion is much simpler to consider analytically. The base case $P(1, 1) = 1$ and $P(1, *) = 0$ else becomes $S(1, 1) = 2(1/2) = 1$ and $S(1, *) = 0$ else. The simplified recursion quickly produces the Table 2, starting the first row with $t = 1$. The table is most easily analyzed by considering the (nonzero) diagonals. By protracted inspection, the diagonal terms are differences of binomial coefficients, where each term of the difference lies on the same row and they are two apart. The appropriate formula

\begin{align*}
S(t, j) &= S(t-1, j-1) + S(t-1, j+1) \\
&= \binom{t}{j} - \binom{t-1}{j-1} - \binom{t-1}{j+1}
\end{align*}
In this equation only, we use the summation. In order to evaluate the last expression, recall the following:

Table 2: This table lists values of $S(t, j)$, with $t = 1$ the top row and $j = 0$ the left column.

is as follows.

$$S(t, j) = \begin{cases} 
0, & t - j \text{ odd} \\
\frac{t - 1}{(t - j)/2}, & t - j \text{ even}
\end{cases}$$

This formula may be verified by a double induction on $j$ within $t$. Hence we have the following for $P(t, j)$.

$$P(t, j) = 2^{-t}(j+1)\times \left[\frac{t - 1}{(t - j)/2} - \frac{t - 1}{(t - j)/2 - 2}\right]$$

if $t - j$ is even.

We can now derive an analytic formula for the expectation of $R_t$. For convenience, we consider only the case in which $t$ is even. Also, we prefer $E(R_t)$ to $\langle R_t \rangle$.

$$E(R_t) = 2^{-t} \sum_{j=0}^{(t-1)/2} j(j+1)S(t, j) = 2^{-t}(j+1)\times \left[\frac{t - 1}{(t - j)/2} - \frac{t - 1}{(t - j)/2 - 2}\right]$$

In order to evaluate the last expression, recall the following summation. In this equation only, we use $n$ and $j$ out of their context in this manuscript.

$$\sum_{j=0}^{(n-1)/2} j \binom{n}{j} = n\left[2^n - 2 - \frac{1}{2}\binom{n-1}{(n-1)/2}\right]$$

To verify this, consider the summand. For the product of $j$ by the appropriate choose is equivalently $n$ times $n-1$ choose $j-1$, and the rewritten sum now (almost) calculates the sum of half a row of Pascal’s triangle.

Returning to the expression for $E(R_t)$, note the difference of binomial coefficients on the RHS produces a telescoping difference up to adjustments. Collecting and emphasizing the cancelling $j^2$ terms,

$$2^t E(R_t) = 6 \left(\frac{t - 1}{t/2 - 1}\right) + 20 \left(\frac{t - 1}{t/2 - 2}\right) + \sum_{j=3}^{t/2} (4j^2 - 4j^2 + 16j - 12) \left(\frac{t - 1}{t/2 - j}\right)$$

Now one might substitute $j \mapsto t/2 - j$ into the index of the final summation and collect terms appropriately. Certain cancellations occur, with the final result being the following:

$$E(R_t) = \frac{(2t + 1)}{2^{t-1}} \left(\frac{t - 1}{t/2 - 1}\right) - 1, \text{ for } t \text{ even}$$

The result of the similar analysis for $t$ odd is

$$E(R_t) = \frac{4t}{2^t} \left(\frac{t - 1}{t/2 - 1/2}\right) - 1, \text{ for } t \text{ odd}$$

We briefly remark that the above formula produces asymptotics for $E(R_t)$ as well. This is due to a corollary of Stirling’s approximation of the gamma function:

$$\sqrt{2\pi} t^{t + 1/2} e^{-t} < t! < \sqrt{2\pi} t^{t + 1/2} e^{-t} [1 + (4n)^{-1}]$$

Using this to estimate the combination above in terms of factorials with $t$ even yields

$$E(R_t) \approx \frac{(2t + 1)\sqrt{2}}{\sqrt{\pi t}} - 1, \text{ for } t \text{ large}$$

Due to Equation 23, the error in the approximation of the binomial coefficient is roughly $O(t^{-1})$, so that the error in the above is $O(t^{-1/2})$. Hence we might simplify further to $E(R_t) \approx -1 + 2\sqrt{2\pi t}$. This has technically only been shown for $t$ even; the odd case is similar. The above estimates also argue that $E(R_t) \in O(\sqrt{t})$ and indeed $E(R_t) \in \Theta(\sqrt{t})$. 
