The connection between conformal group and quantum states of photons

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This note is concerned with the connections between the conformal group and the quantum states of photons. It is shown that there exist analogies between the photonic quantum states and the conformal group, namely, the time-development operator (with a free Hamiltonian), displacement and squeezing operators of the vacuum state corresponds to the dilatation, translation, proper Lorentz transformations, respectively, and that the three quantum states of photons (i.e., Fock state, coherent state and squeezed state) in quantum optics thus bear some analogy to the above three transformations in the conformal group. Based on this comparison, we argue by analogy that if the phase transformation operators acting on a vacuum state (hence the Fock state, coherent state and squeezed state are generated) are truly exactly analogous to the conformal transformations, then a fourth quantum state of photons (referred to as the conformal state), which corresponds to the special conformal transformation (acceleration transformation) and will therefore be of special physical interest, can be suggested.

Keywords: photonic quantum state, conformal state

I. THE 15-PARAMETER CONFORMAL GROUP

The group under consideration is the 15-parameter Lie group often referred to as “conformal transformation” [1,2], which is defined as the set of those transformations that transforms flat space into flat space. The conformal group consists of the space-time translations ($x'^{\mu} = x^{\mu} + \alpha^{\mu}$, 4 parameters), the proper homogeneous Lorentz transformations (i.e., the Lorentz rotation, $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$, 6 parameters), the dilatation (or scale) transformation ($x'^{\mu} = s x^{\mu}$, 1 parameter) and the special conformal (acceleration) transformation ($x'^{\mu} = (1 + 2a^{\alpha} x_{\alpha} + x^{2} a^{2})^{-1}(x^{\mu} + a^{\mu} x^{2})$, 4 parameters). The corresponding algebraic generators can be realized in terms of the differential operators acting on the Minkowski space, which are as follows [1]:

\[
\begin{align*}
    P_\mu &= i \partial_\mu \quad \text{(translation)}, \\
    M_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad \text{(Lorentz transformation)}, \\
    D &= ix^\mu \partial_\mu \quad \text{(dilatation transformation)}, \\
    K_\mu &= i(2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu) \quad \text{(special conformal transformation)}. 
\end{align*}
\]

The algebraic commuting relations of the above 15 generators are given [1]

\[
\begin{align*}
    [P_\mu, P_\nu] &= 0, & [P_\lambda, M_{\mu\nu}] &= i(g_{\mu\lambda} P_\nu - g_{\nu\lambda} P_\mu), \\
    [P_\mu, D] &= i P_\mu, & [P_\mu, K_\nu] &= 2i(g_{\mu\nu} D - M_{\mu\nu}), \\
    [D, M_{\mu\nu}] &= 0, & [D, K_\mu] &= i K_\mu, \\
    [M_{\mu\nu}, M_{\sigma\rho}] &= i(g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} + g_{\mu\sigma} M_{\rho\nu} + g_{\nu\rho} M_{\sigma\mu}), \\
    [K_\mu, K_\nu] &= 0, & [K_\lambda, M_{\mu\nu}] &= i(g_{\mu\lambda} K_\nu - g_{\nu\lambda} K_\mu).
\end{align*}
\]

In what follows we will concern ourselves with the unitary phase transformation operators in the photonic quantum states, i.e., the Fock state, coherent state and squeezed state.

*This note is mainly devoted to a physically interesting comparison between the photonic quantum states and the conformal group. Because of the trivial and lengthy calculation involved, the detailed analysis of comparison made between photonic states and conformal transformations will be submitted nowhere else for publication, just uploaded at the e-print archives.

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II. PHOTONIC QUANTUM STATES

Historically, the fundamental concepts of the Fock state, coherent state and squeezed state were proposed by Dirac, Glauber and Stoler [3–5], respectively. In this section we will take into a comprehensive consideration these three photonic quantum states and discuss several topics such as the unitary phase transformation operators, the generators and the algebraic structures in quantum states of photons as well as the close relation to the conformal transformation [6].

A. Free-Hamiltonian time-evolution operator in Fock state

The operators $a^\dagger a$ and $aa^\dagger$ may be considered the generators of the Free-Hamiltonian time-evolution operator of photon state, which is of the form

$$P(\alpha) = \exp(aa^\dagger a - \alpha^* a a^\dagger), \quad P^\dagger(\alpha) = \exp[ -(a a^\dagger a - \alpha^* a a^\dagger)].$$  \hfill (3)

The unitary transformation operator $P(\alpha)$ leads to the following transformation

$$a \rightarrow a' = P^\dagger(\alpha)aP(\alpha) = \exp(\alpha^* - \alpha)a, \quad a^\dagger \rightarrow a'^\dagger = P^\dagger(\alpha)a^\dagger P(\alpha) = \exp[- (\alpha^* - \alpha)] a^\dagger,$$ \hfill (4)

and consequently the infinitesimal variations of $a$ and $a^\dagger$ under the transformation $P(\alpha)$ are

$$\delta a = (\alpha^* - \alpha)a, \quad \delta a^\dagger = -(\alpha^* - \alpha)a^\dagger,$$ \hfill (5)

which follows that $\delta a$, $a^\dagger$ and the infinitesimal variations of space-time coordinates

$$\delta x^\mu = \frac{1}{i}\alpha D x^\mu = \alpha x^\mu$$ \hfill (6)

under the dilatation transformation are alike in some way. The new states generated by the dilatation transformation $P(\alpha)$ are written

$$|\alpha, 0\rangle = P(\alpha)|0\rangle = \exp(\alpha^*)|0\rangle, \quad |\alpha, n\rangle = P(\alpha)|n\rangle = \exp[(\alpha^* - \alpha)a^\dagger a]|\exp(\alpha^*)|n\rangle.$$ \hfill (7)

It follows that $|\alpha, n\rangle = P(\alpha)|n\rangle = \exp[n(\alpha^* - \alpha)] \exp(\alpha^*)|n\rangle$. It is known that the quantum state $|\alpha, n\rangle$ can be realized by the time evolution with a free Hamiltonian $H = \frac{\omega}{2}(aa^\dagger + a^\dagger a)$ (governed by the time-dependent Schrödinger equation), namely, $|t, n\rangle = \exp[\frac{\omega}{2}(n + \frac{1}{2})\omega t]|n\rangle$. If the parameters in the unitary transformation operator $P(\alpha)$ are taken

$$\alpha = -\frac{1}{2i}\omega t, \quad \alpha^* = \frac{1}{2i}\omega t,$$ \hfill (8)

then $|t, n\rangle$ will be the photonic states characterized in (7). Thus it is concluded that the free-Hamiltonian time-evolution operator acting on the photon creation and annihilation operators closely resembles the dilatation (or scale) transformation in the conformal group, and that a stationary Fock state acted upon by the free-Hamiltonian time-evolution operator $P(\alpha)$ will be transformed into a time-dependent Fock state.

B. Displacement operator in coherent state

The coherent state of photons is defined to be $|\alpha\rangle = D(\alpha)|0\rangle$ with the displacement operator being

$$D(\alpha) = \exp(aa^\dagger - \alpha^* a).$$ \hfill (9)

The displacement operator acting on the vacuum state is equivalent to the following transformation

$$a \rightarrow a' = D^\dagger(\alpha)aD(\alpha) = a + \alpha, \quad a^\dagger \rightarrow a'^\dagger = D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*,$$ \hfill (10)

which will yield the infinitesimal variations

$$\delta a = \alpha, \quad \delta a^\dagger = \alpha^*.$$ \hfill (11)

It is physically interesting that the above $\delta a$ and $\delta a^\dagger$ are in analogy with the variations of the space-time coordinates $x^\mu$ under the infinitesimal translation transformation in the conformal group, i.e.,

$$\delta x^\mu = \frac{1}{i}\alpha^\mu P_\mu x^\mu = \alpha^\mu.$$ \hfill (12)
C. Squeezing operator in squeezed state

The squeezed state is defined to be \( |\zeta\rangle = S(\zeta)|0\rangle \) with the squeezing operator being

\[
S(\zeta) = \exp \left[ \frac{1}{2} \zeta^* a^2 - \frac{1}{2} \zeta (a^\dagger)^2 \right], \quad \zeta = s \exp(i\theta).
\]

The corresponding variations of \( a \) and \( a^\dagger \) are

\[
a \rightarrow a' = S(\alpha) a S(\alpha) = a \cosh s - a \exp(i\theta) \sinh s, \quad a^\dagger \rightarrow a'^\dagger = S(\alpha) a^\dagger S(\alpha) = a^\dagger \cosh s - a \exp(-i\theta) \sinh s.
\]

The infinitesimal variations

\[
\delta a = \left[ -\left( \frac{1}{2} \zeta^* a^2 - \frac{1}{2} \zeta (a^\dagger)^2 \right), a \right] = -\zeta a^\dagger, \quad \delta a^\dagger = \left[ -\left( \frac{1}{2} \zeta^* a^2 - \frac{1}{2} \zeta (a^\dagger)^2 \right), a^\dagger \right] = -\zeta^* a,
\]

which shows some analogy with the two-dimensional infinitesimal Lorentz rotation, \( \delta x^\mu = \frac{1}{2} \epsilon^{\nu\mu} M_{\nu\mu} x^\mu = \epsilon^{0\mu} x_\mu \), i.e.,

\[
\delta x^0 = -e^{01} x^1, \quad \delta x^1 = -e^{10} x^0.
\]

III. DEFINING A KIND OF OPERATOR INTEGRAL TO OBTAIN THE GENERATORS OF QUANTUM STATES OF PHOTONS

In this section, we will define a kind of operator integral technique to get the algebraic generators of the above three quantum states of photons.

The generators of displacement operator \( D(\alpha) = \exp(aa^\dagger - \alpha^* a) \) can be obtained via the following two integrals (with c-number \( \alpha \) and \( \alpha^\dagger \) being the integrands)

\[
\alpha^* a = \int \alpha^* da, \quad \alpha a^\dagger = \int \alpha da^\dagger,
\]

where the integral constant (unit matrix) is omitted due to its triviality. In (17) we obtain the linear-form operators (generators) \( a \) and \( a^\dagger \).

In the similar fashion, we calculate the following operator integrals (note that the definition of the operator integral is implied in the following calculation)

\[
\mathcal{F} = \int da \left( \zeta a^\dagger + \zeta^* a \right) = \frac{1}{2} \zeta (aa^\dagger + a^\dagger a) + \frac{1}{2} \zeta^* a^2,
\]

\[
\mathcal{F}^\dagger = \int da^\dagger \left( \zeta a^\dagger + \zeta^* a \right) = \frac{1}{2} \zeta^* (aa^\dagger + a^\dagger a) + \frac{1}{2} \zeta (a^\dagger)^2.
\]

So, one can arrive at the generators of squeezed state (and Fock state), which are involved in

\[
\mathcal{F} - \mathcal{F}^\dagger = \frac{1}{2} \left[ \zeta^* a^2 - \zeta (a^\dagger)^2 \right] + \frac{1}{2} \left( \zeta - \zeta^* \right) (aa^\dagger + a^\dagger a).
\]

Note that the generators in (19) are quadratic-form operators. In Eq.(19) \( aa^\dagger \) and \( a^\dagger a \) can be viewed as the generators of the dilatation transformation \( F(\alpha) \) in (3). If, for example, \( \frac{1}{2} (\zeta - \zeta^*) = \Im \alpha \) and \( \Re \alpha = 0 \), then we have \( \frac{1}{2} (\zeta - \zeta^*) (aa^\dagger + a^\dagger a) = aa^\dagger a - \alpha^* aa^\dagger \).

The calculations in Eq.(17) and (19) shows that one can obtain the generators of photonic quantum states by using such operator integrals just defined above.

Now we continue calculating the following operator integrals

\[
\mathcal{G} = \int da \left[ \varrho a^2 + \varrho (a^\dagger)^2 \right] = \frac{\varrho}{3} \left[ a (a^\dagger)^2 + a^\dagger aa^\dagger + (a^\dagger)^2 a \right] + \frac{\varrho^*}{3} a^3 = \varrho aa^\dagger a + \varrho^* a^3,
\]

\[
\mathcal{G}^\dagger = \int da^\dagger \left[ \varrho a^2 + \varrho (a^\dagger)^2 \right] = \frac{\varrho}{3} \left[ (a^\dagger)^2 a^2 + aa^\dagger a + a^\dagger a \right] + \frac{\varrho^*}{3} (a^\dagger)^3 = \varrho^* aa^\dagger a + \varrho a^3.
\]
In the same manner as (19), we have
\[ G - G^\dagger = \varrho Q - \varrho^* Q^\dagger, \]  
where
\[ Q = a^\dagger a a^\dagger - \frac{1}{3} (a^\dagger)^3, \quad Q^\dagger = a a^\dagger a - \frac{1}{3} a^3, \]
which are nonlinear (cubic-form) generators.

IV. CUBIC-NONLINEARITY PHASE TRANSFORMATION OPERATOR

It is of physical interest to consider the so-called cubic-nonlinearity phase transformation operator \( C(\varrho) \), which is defined to be
\[ C(\varrho) = \exp \left( \varrho Q - \varrho^* Q^\dagger \right). \]  
In the meanwhile, we define a new photonic quantum state as follows
\[ |\varrho\rangle = C(\varrho)|0\rangle. \]  
The variations of \( a \) and \( a^\dagger \) under the nonlinear unitary transformation \( C(\varrho) \) are
\[ \delta a = \left[ - (\varrho Q - \varrho^* Q^\dagger), a \right] = \varrho \left( aa^\dagger + a^\dagger a \right) - \left[ \varrho^* a^2 + \varrho (a^\dagger)^2 \right], \]
\[ \delta a^\dagger = \left[ - (\varrho Q - \varrho^* Q^\dagger), a^\dagger \right] = \varrho^* \left( aa^\dagger + a^\dagger a \right) - \left[ \varrho (a^\dagger)^2 + \varrho^* a^2 \right]. \]  
Accordingly, here we may take into account the special conformal transformation in the conformal group, i.e.,
\[ \delta x^\mu = \frac{1}{i} \varrho^\nu K_\nu x^\mu = 2 \varrho^\nu x_\nu x^\mu - x^2 \varrho^\mu, \]  
that is, the two-dimensional infinitesimal special conformal transformation is of the form
\[ \delta x^0 = 2 \varrho^\nu x_\nu x^0 - x^2 \varrho^0 = - \left\{ 2 \varrho^1 x^1 x^0 - \varrho^0 \left[ (x^0)^2 + (x^1)^2 \right] \right\}, \]
\[ x^1 = 2 \varrho^\nu x_\nu x^1 - x^2 \varrho^1 = 2 \varrho^0 x_0 x^1 - \varrho^1 \left[ (x^0)^2 + (x^1)^2 \right]. \]  
It follows that the quantum state \( C(\varrho)|0\rangle \) and the special conformal transformation (characterized by the generators \( K_\nu \)'s) are alike in some way. If, for example, in the particular case the two-dimensional infinitesimal parameters agree with \( \varrho^0 = -\varrho^1 \), and \( \varrho = \varrho^* \), then the two variations (25) and (27) are of the same mathematical form. Both (25) and (27) are their respective extensions of this same special case. Thus we may think of \( |\varrho\rangle = C(\varrho)|0\rangle \) as the fourth quantum state of photons and refer to it as the optical conformal state. Differing from the three photonic quantum states studied previously, the conformal state is a nonlinear one. It is reasonably believed that such quantum state may be of special physical interest and deserves further investigation.

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