NEW EINSTEIN-MAXWELL FIELDS OF LEVI-CIVITA’S TYPE

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Abstract

The method based on the Horský-Mitskievitch conjecture is applied to the Levi-Civita vacuum metric. It is shown, that every Killing vector is connected with a particular class of Einstein-Maxwell fields and each of those classes is found explicitly. Some of obtained classes are quite new. Radial geodesic motion in constructed space-times is discussed and graphically illustrated in the Appendix.

1 Introduction

There exists a lot of methods how to generate Einstein-Maxwell (EM) fields from pure gravitational ones [1, 2, 3, 4, 5]. Some EM fields were obtained by means of the Horský-Mitskievitch (HM) conjecture [6] based on the connection between the four-potential of the electromagnetic field and the symmetries of the spacetimes described by Killing vectors. In this paper we apply the method outlined in [6] to the vacuum Levi-Civita (LC) metric which generally admits 3 Killing vectors and another Killing field for two special choices of the metric parameters.

The paper is divided into the following parts: we start by resuming basic characteristics of the LC solution, then we recapitulate the basic ideas of the HM conjecture adopting them to the LC seed metric. Gradually, we come to new five classes of the EM equations, each of which corresponds to one Killing vector of the seed LC vacuum metric. Finally, we add the sixth class which is interesting for another reason: for special values of its parameters it reduces to the Bonnor-Melvin (BM) universe filled not by magnetic but by electric background.

The Appendix deals in detail with radial geodesic motion in generated spacetimes. In each case we compare the radial motion with the situation in the seed LC solution and find out the way the radial geodesic motion indicates the presence of singularities.

2 The LC solution

The line element of the LC static vacuum spacetime can be written in the Weyl form [7, 8]

$$ds^2 = -r^{4\alpha} dt^2 + r^{4\alpha(2\sigma - 1)} (dr^2 + dz^2) + C^{-2} r^{2 - 4\sigma} d\varphi^2,$$

where \(\{t, r, \varphi, z\}\) are usual cylindrical coordinates: \(-\infty < t, z < \infty, r \geq 0, 0 \leq \varphi < 2\pi\), the hypersurfaces \(\varphi = 0, \varphi = 2\pi\) are identified. The expression (1) contains two arbitrary constants \(\alpha, C\), both of them are fixed by the internal composition of the physical source. The constant \(C\) refers to the deficit angle, and cannot be removed by scale transformations. The physical importance of the other parameter \(\alpha\) is mostly understood

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in accordance with the Newtonian analogy of the LC solution - the gravitational field of an infinite uniform line-mass (“infinite wire”) with the linear mass density $\sigma$. All orthonormal bases employed in the further calculations below are always chosen as a generalization of the set

$$\omega^{(0)} = r^{2\sigma} dt, \quad \omega^{(1)} = r^{2\sigma(2\sigma-1)} dr, \quad \omega^{(2)} = C^{-1} r^{1-2\sigma} d\phi, \quad \omega^{(3)} = r^{2\sigma(2\sigma-1)} dz,$$

probably the simplest tetrad one can use for the LC solution. The Kretschmann scalar

$$K = R^{(\alpha)(\beta)(\mu)(\nu)}R^{(\alpha)(\beta)(\mu)(\nu)} = 64\sigma^2(4\sigma^2 - 2\sigma + 1)(2\sigma - 1)^2 r^{-16\sigma^2 + 8\sigma - 4}, \quad (3)$$

where $R^{(\alpha)(\beta)(\mu)(\nu)}$ are the components of the Riemann tensor in a chosen orthonormal basis, is infinite at $r = 0$ for all $\sigma$, $C$ excluding $\sigma = 0$ and $\sigma = \frac{1}{2}$ when the spacetime is flat (see below). Thus metric [3] has a singularity along the $z$-axis $r = 0$ that is preferably interpreted as the infinite line source. There is evidently no horizon, the spacetime is asymptotically flat in the radial direction for $\sigma \neq 0, 1/2$.

The analytic form of non-zero Weyl scalars

$$\Psi_0 = \Psi_1 = (2\sigma - 1)(2\sigma + 1)\sigma r^{-8\sigma^2 + 4\sigma - 2}, \quad \Psi_2 = (2\sigma - 1)^2 \sigma r^{-8\sigma^2 + 4\sigma - 2} \quad (4.a, 4.b)$$

leads to the conclusion that the LC metric [3] belongs generally to the Petrov type $I$ with the exception of algebraically special cases belonging either to the Petrov type $0$ or to the Petrov type $D$:

| $\sigma$ | Petrov type | Weyl Scalars |
|----------|-------------|--------------|
| $0, \frac{1}{2}$ | 0 | all zero |
| $\frac{1}{2}$ | $D$ | $\Psi_2 = -\frac{2}{r^6}$ |
| 1 | $D$ | $\Psi_0 = \Psi_4 = \frac{3}{r^6}, \quad \Psi_2 = \frac{1}{r^6}$ |
| $\frac{1}{4}$ | $D$ | $\Psi_0 = \Psi_4 = -\frac{3}{16} r^{-3/2}, \quad \Psi_2 = \frac{1}{16} r^{-3/2}$ |

As any static cylindrically symmetric solution, the metric [3] admits three Killing vectors:

$$\xi_z = r^{4\sigma(2\sigma-1)} dz = r^{2\sigma(2\sigma-1)} \omega^{(3)} \leftrightarrow \partial_z, \quad (5.a)$$

$$\xi_\phi = C^{-2} r^{2-4\sigma} d\phi = C^{-1} r^{1-2\sigma} \omega^{(2)} \leftrightarrow \partial_\phi, \quad (5.b)$$

$$\xi_t = r^{4\sigma} dt = r^{2\sigma} \omega^{(0)} \leftrightarrow \partial_t; \quad (5.c)$$

these Killing vectors determine the integrals of motion for geodesic trajectories. In the case $\sigma = \frac{1}{2}$, the LC solution has another Killing vector $\xi_{1/4} = -\varphi r^{-4} dt + tr^2 d\phi$, resp. $\xi_{-1/2} = -z r^4 d\phi + \varphi r^4 dz$. The former corresponds to a Lorentz boost in $t - \varphi$ plane, the latter to a rotation in $\varphi - z$ plane. All mentioned Killing vectors can generate EM fields as is shown below.

Although the value of $\sigma$ is not bound by any mathematical relation, the physical interpretation sets strict limits. The LC solution is traditionally identified with an infinite line source only if $0 < \sigma < \frac{1}{4}$ (note: the value $\sigma = 1$ represents $10^{28}$ g⋅cm$^{-1}$ [3]). Negative values $\sigma < 0$ lead to the negative linear density of the Newtonian analogy and thus violate the energy conditions. Recently, it was shown (see [4]) that even for $0 \leq \sigma \leq 1$ the LC solution [3] can be produced by realistic cylindrical sources, namely cylindrical shells of an anisotropic fluid. However, in cases $\sigma = 0, \frac{1}{2}$ when the metric is flat, the parameter $\sigma$ cannot be interpreted as a linear density at all.

Difficulties in interpreting the LC solution were illustrated by Bonnor [5]. In case $\sigma = -1/2, C = 1$ the metric [3] can be transformed either into Taub’s plane symmetric solution, or into the Robinson-Trautman solution, or into the solution describing the gravitational field of a semi-infinite line-mass. Each of this possibilities suggests a different physical interpretation. More information about the LC metric, especially about the character of the source, can be found in [3, 4] and in the references cited therein.
The HM conjecture and its application

The HM conjecture proposed in [6] outlines an efficient and fruitful way, how to obtain solutions of EM equations as a generalization of some already known vacuum seed metrics. Its mathematical background is based on the striking analogy between equations satisfied by Killing vectors $\xi$ in vacuum spacetimes

$$\ast d\ast d\xi = 0,$$

and vacuum (sourceless) Maxwell equations for a testing electromagnetic four-potential

$$\ast d\ast dA = 0.$$

This connection is well known for a long time and is pointed out in many textbooks (see e.g. [9] p. 66, 326).

This suggestive coincidence inspired Horšký and Mıtskievitch in [6] to formulate the conjecture that can be expressed in the following way (quoted verbally according to [10]):

"The electromagnetic four-potential of a stationary self-consistent Einstein-Maxwell field is simultaneously proportional (up to constant factor) to the Killing covector of the corresponding vacuum spacetime when the parameter connected with the electromagnetic field of the self-consistent problem is set equal to zero, this parameter coinciding with the afore-mentioned constant factor."

Let $g$ denotes the metric tensor of a vacuum seed metric, $q$ parameter characterizing the strength of the electromagnetic field (mentioned in the quotation above) and $\bar{g} = \bar{g}(q)$ the metric tensor of an EM field, representing in fact one-parameter class of solutions. According to the conditions of the HM conjecture

$$\lim_{q \to 0} \bar{g} = g,$$

i.e. in the case of null electromagnetic field $q = 0$ one comes back to the original seed vacuum metric. The above quoted formulation of the conjecture was further generalized by Cataldo, Kunaradtya and Mıtskievitch [10] so that the four-potential $A$ need not be inevitably multiplied only by a constant factor, but also by a suitable scalar function $F$. The function is evidently not arbitrary; it must satisfy sourceless Maxwell equations

$$\ast d\ast d(F\xi) = 0$$

with respect to $\bar{g}$ (see [10]). Here the constant parameter $q$ must be involved in the analytical expression of the function $F$. From the point of this generalization it is not necessary to demand that $\xi$, a Killing vector with respect to $g$, must be also a Killing vector with respect to $\bar{g}$ as used to be argued (see e.g. [11]). This generalization leads to more vague connection between the Killing vectors and four-potentials. At the same time, however, it enables to find a wealth of situations in which the presumptions of this generalized HM conjecture are fulfilled.

One can find many particular examples in which the HM conjecture works, for example, EM fields listed in [6, 12, 13], the C-metric with electric field [14, 15], most electro-vacuum spacetimes of Petrov type $D$ studied in [1] (see p. 137, Eq. (11.61) (Datta), p. 158-159 (McVittie, Patnaik), p. 158 (Reissner-Nordstrom) p. 297 (Kowalczyński-Plebanski)) or even quite complicated metric studied by Kramer and Perjés [16]. It was also employed in generating new electromagnetic spacetimes from vacuum metrics [10, 11, 17]. Its applicability and possible limitations are still under systematic investigation. Although the original formulation quoted above restricts itself only to vacuum seed spacetimes $g$ and static or stationary EM fields $\bar{g}$, nowadays these conditions seem to become redundant. The possibility to construct new solutions also from non-vacuum seed metrics is supposed immediately in [6]. In section 8 a non-stationary solution will be obtained through the procedure of the generalized conjecture.

Probably the most important advantage of the HM conjecture is the opportunity to choose the character of the electromagnetic field we would like to obtain. It is obviously determined by the vector potential and thus by the geometrical substance of the Killing vector one uses for the generation of $g$. If the seed metric $g$ admits more than one Killing vector, there is usually possible to construct more EM fields, each of them corresponding to a different Killing vector. The situation is most lucid when the seed metric $g$ is static. In that case a straightforward calculation leads to the conclusion that rotational, as well as space-like translational Killing vectors, give magnetic EM fields, while timelike translational and the boost Killing vectors lead to the electric fields (in full analogy with the Minkowski spacetime). For the rotational Killing vector $\partial_\phi$ in common cylindrical coordinates the correspondence with longitudinal magnetic field was demonstrated by Wald (see e.g. [9], p. 66, 326). Let us remind that the electric or magnetic character of any obtained EM field determines the sign of the electromagnetic invariant

$$F_{\mu\nu}F^{\mu\nu} = 2B^2 - E^2,$$

(6)
where $F$ is an antisymmetric tensor of the electromagnetic field related to the components of electric and magnetic field strengths $E$ and $B$ in a standard way (see e.g. [8], p. 23, [18], p. 74). If $F_{(\mu)(\nu)} F^{(\mu)(\nu)} > 0$, the field is of magnetic type, if $F_{(\mu)(\nu)} F^{(\mu)(\nu)} < 0$ then it is of electric type.

Spacetimes treated below in this paper were found through the HM conjecture from the LC seed metric.

The condition of traceless Einstein tensor (8) determines the function

$$f(t, r, \varphi, z) = 1 + c_1 f_1(t, r, \varphi, z),$$

(7)

and the function $f_1(t, r, \varphi, z)$ must be a solution of the differential equation

$$G = G^{(\mu)}_{(\mu)} = -R = 0$$

(8)

arising from the well known fact that for a pure electromagnetic spacetime the Einstein tensor is traceless. Thus all obtained solutions have zero scalar curvature $R$. We shall see that the analytic form of $f_1$ coincides with the basis vector (3) collinear with the vector potential. The constant $c_1$ in (6) must naturally involve the parameter $q$ as the limit

$$\lim_{c_1 \to 0} f(t, r, \varphi, z) = 1$$

for any regular $f_1$ gives original seed metric.

c) Completing the steps a) and b) we can ensure the validity of sourceless Maxwell equations. Substituting $\tilde{g}$ into the Einstein equations we are able to fit the constant $c_1$ against $q$.

Although the simplicity of the above outlined algorithm is obviously caused by relatively high degree of symmetry characteristic for the LC metric, this scheme might contribute to the discussion about the application of the conjecture and the character of relation between Killing vectors and electromagnetic field. The following sections are devoted to the particular applications of the outlined scheme.

4 The LC solution with azimuthal magnetic field

Let us start with the Killing vector $\xi$. The simplest possible choice of $f$ in Eq. (6) is $f = f(r)$. Therefore, in spirit of the above discussed scheme the modified tetrad takes the form

$$\omega^{(0)} = f(r)r^{2\sigma} dt, \quad \omega^{(1)} = f(r)r^{2\sigma(2\sigma - 1)} dr,$$

$$\omega^{(2)} = f(r)C^{-1} r^{1-2\sigma} d\varphi, \quad \omega^{(3)} = \frac{r^{2\sigma(2\sigma - 1)}}{f(r)} dz$$

(9)

and the corresponding line element reads as

$$ds^2 = -f(r)^2 r^{4\sigma} dt^2 + f(r)^2 r^{4\sigma(2\sigma - 1)} dr^2 + f(r)^2 C^{-2} r^{2-4\sigma} d\varphi^2 + \frac{r^{4\sigma(2\sigma - 1)}}{f(r)^2} dz^2.$$

(10)

For the four-potential we have

$$A = \frac{q r^{4\sigma(2\sigma - 1)}}{f(r)} dz = q r^{2\sigma(2\sigma - 1)} \omega^{(3)}.$$

(11)

The condition of traceless Einstein tensor (8) determines the function $f$

$$f(r) = 1 + c_1 r^{4\sigma(2\sigma - 1)}$$

(12)
and from the Einstein equations one obtains
\[ c_1 = q^2. \]
The electromagnetic field is of magnetic type, which can be demonstrated by the electromagnetic invariant
\[ F_{(\mu)(\nu)}F^{(\mu)(\nu)} = 32 \frac{q^2 \sigma^2 (2\sigma - 1)^2}{r^2 f(r)^2} \geq 0 \]
or directly by the electromagnetic field tensor
\[ F^{(1)(3)} = -B^{(2)} = \frac{4q\sigma(2\sigma - 1)}{rf(r)^2}. \]
The only non-zero component of the magnetic field strength \( B \) is the azimuthal one. In accordance with the accepted physical interpretation of the LC solution (at least for some values of \( \sigma \)) the line element (11) describes an EM field of an infinite line source with an azimuthal magnetic field, in other words, the gravitational field of an infinite line source with an electric current.

The metric (11) has a one-dimensional singularity along the \( z \)-axes \( r = 0 \) where the Kretschmann scalar
\[ R = \frac{64\sigma^2 (2\sigma - 1)^2}{f(r)^8 (4\sigma^2 - 2\sigma + 1)} \left[ g_1(r)^4 (4\sigma - 1)^2 (12\sigma^2 - 6\sigma + 1) - 12g_1(r)^3\sigma (2\sigma - 1)(4\sigma - 1)^2 + 2g_1(r)^2 (160\sigma^4 - 160\sigma^3 + 24\sigma^2 + 8\sigma - 1) + 12g_1(r)\sigma (2\sigma - 1) + 4\sigma^2 - 2\sigma + 1 \right]. \]
becomes infinite; \( g_1(r) = q^2 r^{4\sigma(2\sigma - 1)} \). Obviously, the only exceptions are the cases \( \sigma = 0, 1 = \frac{1}{2} \) for which the seed LC metric is flat. Then also the metric (11) becomes flat and does not include any electromagnetic field.

Eventually, the analytic expressions for the Weyl scalars have the form
\[ \Psi_0 = \Psi_4 = -\frac{2\sigma - 1}{r^2 f(r)^5 g_1(r)} \left( q^4 g_1(r)^2 - 1 \right) \left[ q^2 g_1(r) (24\sigma^2 - 10\sigma + 1) + (2\sigma + 1) \right], \]
\[ \Psi_2 = \frac{(2\sigma - 1)^2}{r^2 f(r)^7 g_1(r)} \left[ q^{10} g_1(r)^5 (4\sigma - 1) + q^8 g_1(r)^4 (8\sigma - 3) - 2q^6 g_1(r)^3 - 2q^4 g_1(r)^2 (4\sigma - 1) - q^2 g_1(r) (4\sigma - 3) + 1 \right]. \]

Thus the spacetime (11) is generally Petrov type \( I \), special cases being:

| \( \sigma \) | Petrov type | Weyl Scalars |
|---|---|---|
| 0, \( \frac{1}{2} \) | 0 | all zero |
| \( \frac{1}{4} \) | \( D \) | \( \Psi_0 = \Psi_4 = -3\Psi_2 = \frac{3}{16} \left( \frac{q^2 - \sqrt{r}}{q^2 + \sqrt{r}} \right) \) |

It should be noted that though (11) reminds of the general static cylindrically symmetric solution with azimuthal magnetic field (10), §20.2, Eq. 20.9a, it does not belong to this class of spacetimes. This inevitably means the metric 20.9a in (11) does not represent the most general cylindrical symmetric EM solution with azimuthal magnetic field. Putting \( q = 0 \), which means no electromagnetic field, one obtains the seed LC metric.

5 The LC solution with longitudinal magnetic field

Let us take the Killing vector \( \xi_\phi \) now. This requires the tetrad
\[ \omega^{(0)} = f(r) r^{2\sigma} dt, \quad \omega^{(1)} = f(r) r^{2\sigma(2\sigma - 1)} dr, \]
\[ \omega^{(2)} = \frac{r^{1 - 2\sigma}}{C f(r)} d\phi, \quad \omega^{(3)} = f(r) r^{2\sigma(2\sigma - 1)} dz \]

(17)
\[ \sigma = 0 \] into (18) one easily comes to described by Cataldo et. all \[10\] called “pencil of light in the Bonnor-Melvin Universe”. Moreover, substituting which is responsible for the background longitudinal magnetic field. The situation reminds us of the solution is flat and does not include any electromagnetic field. \[9\]\[A\]\[B\]\[C\] \[8\]\[D\]\[E\] Let us remind that for \[\sigma = 0\] the seed LC metric reduces to Minkowski spacetime. Thus, the Bonnor-Melvin solution of EM equations. Evidently, this metric with \(g = B_0/2\) and \(C = 1\) gives the well-known Bonnor-Melvin solution of EM equations. Let us remind that for \(\sigma = 0\), \(C = 1\) the seed LC metric reduces to Minkowski spacetime. Thus, the Bonnor-Melvin universe can be obtained through the HM conjecture straight from the Minkowski spacetime expressed in common cylindrical coordinates. This possibility was already mentioned by Cataldo at. all \[10\] called “pencil of light in the Bonnor-Melvin Universe”. Moreover, substituting \(\sigma = 0\) into (18) one easily comes to

\[ ds^2 = \left(1 + q^2 r^2/C^2\right)^2 \left[-dr^2 + dr^2 + dz^2\right] + \frac{r^2}{\left(1 + q^2 r^2/C^2\right) C^2} d\varphi^2. \]  

(23)

In accordance with the accepted physical interpretation of LC solution (at least for some values of \(\sigma\)) the line element (18) describes an EM field of an infinite line source with a longitudinal magnetic field, in other words, the gravitational field of an infinite line source in a Bonnor-Melvin-like universe (see e.g. \[4\], \(\sigma = 1\), Eq. 20.10) which is responsible for the background longitudinal magnetic field. The situation reminds us of the solution described by Cataldo et. all \[10\] called “pencil of light in the Bonnor-Melvin Universe”. Moreover, substituting \(\sigma = 0\) into (18) one easily comes to

\[ ds^2 = -f(r)^2 r^{4\sigma} dt^2 + f(r)^2 r^{4\sigma(2\sigma - 1)} \left[dr^2 + dz^2\right] + \frac{r^{2-4\sigma}}{f(r)^2 C^2} d\varphi^2. \]  

(18)

Killing vector \(\xi_\varphi\) induces the four-potential \[22\]

\[ A = \frac{qr^{2(1-2\sigma)}}{C^2 f(r)} d\varphi = qC^{-1} r^{1-2\sigma} \omega^{(2)}. \]  

(19)

Following the scheme described in the section \[8\] one gets

\[ f(r) = 1 + c_1 r^{2(1-2\sigma)}, \quad c_1 = \frac{q^2}{C^2}. \]  

(20)

The electromagnetic field is again of magnetic type since

\[ F_{(\mu)(\nu)} F^{(\mu)(\nu)} = \frac{8q^2 (2\sigma - 1)^2}{C^2 f(r)^4 r^{4\sigma}} \geq 0, \]  

(21)

while the only non-zero component of magnetic field strength being the longitudinal one in direction along the \(z\)-axis

\[ F^{(1)(2)} = B^{(3)} = -\frac{2q(2\sigma - 1)}{C f(r)^2 r^{4\sigma}}. \]  

(22)

The expressions for the Weyl scalars

\[ \Psi_0 = \Psi_4 = \frac{(2\sigma - 1)}{C^4 f(r)^4 r^{4\sigma - 12\sigma + 2}} \left[r^{4\sigma} C^2 \sigma (2\sigma + 1) + r^{2-4\sigma} (2\sigma - 5\sigma + 3)\right], \] \[25\] \[a\] \[b\] \[c\] \[d\] \[e\] \[f\] \[g\] again leads to the conclusion that the metric is generally Petrov type I with the exception of algebraically special cases
\[ \Psi_0 = \Psi_4 = 3 \Psi_2 = \frac{3q^2 (q^2r^2 - C^2)}{C^4 (C^2 + q^2r^2)^4} \]

The metric (18) reminds of the general static cylindrically symmetric solution with longitudinal magnetic field [4], §20.2, Eq. 20.9b, but it does not belong to this class of spacetimes (the only exception being the Bonnor-Melvin universe [23]). This definitely means that the metric 20.9b in [4] does not represent the most general cylindrical symmetric EM field with longitudinal magnetic field.

6 The LC solution with radial electric field

The last from the Killing vectors (5) is \( \xi_t \). The tetrads
\[
\begin{align*}
\omega^{(0)} &= \frac{r^{2\sigma}}{f(r)} dt, \\
\omega^{(1)} &= f(r)r^{2\sigma(2\sigma - 1)} dr, \\
\omega^{(2)} &= f(r)C^{-1}r^{1 - 2\sigma} d\varphi, \\
\omega^{(3)} &= f(r)r^{2\sigma(2\sigma - 1)} dz
\end{align*}
\]
determines the metric
\[
ds^2 = -\frac{r^{4\sigma}}{f(r)^2} dt^2 + f(r)^2 r^{4\sigma(2\sigma - 1)} \left[ dr^2 + dz^2 \right] + f(r)^2 C^{-2} r^{2 - 4\sigma} d\varphi^2,
\]
where
\[
f(r) = 1 + c_1 r^{4\sigma},
\]
and
\[ c_1 = -q^2. \]

The vector potential
\[
A = -\frac{qr^{4\sigma}}{f(r)} dt = -qr^{2\sigma} \omega^{(0)}
\]
sets the field of electric type
\[
F_{(\mu)(\nu)} F^{(\mu)(\nu)} = -32 \frac{q^2 r^{2 - 8\sigma + 8\sigma - 2}}{f(r)^4} \leq 0
\]
with non-zero radial component of electric field strength
\[
F^{(0)(1)} = E^{(1)} = -\frac{4q r^{4\sigma - 4\sigma^2 + 4\sigma - 1}}{f(r)^2}.
\]

One can conclude the metric (27) describes EM field of a charged infinite line source.

Unlike the seed metric (1) and the solutions (10), (18), the spacetime (27) contains not only one dimensional singularity along the \( z \)-axes but also a singularity at a radial distance \( r_s \) for which
\[
f(r_s) = 1 - q^2 r_s^{4\sigma} = 0.
\]

The Kretschmann scalar
\[
K = \frac{64\sigma^2}{f(r)^8 r^{16\sigma^2 - 8\sigma + 4}} \left[ g_3(r)^4 \left( 4\sigma^2 + 2\sigma + 1 \right) \left( 2\sigma + 1 \right)^2 + 12g_3(r)^3 \sigma \left( 2\sigma + 1 \right)^2 - 2g_3(r)^2 \left( 16\sigma^4 - 48\sigma^2 + 1 \right) - 12g_3(r)\sigma \left( 2\sigma + 1 \right)^2 \left( 4\sigma^2 - 2\sigma + 1 \right) \right],
\]
where \( g_3(r) = q^2 r^{4\sigma} \), proves that this surface represents physical singularity that cannot be removed by any coordinate transformation.
The solution \(27\) is generally Petrov type \(I\) with Weyl scalars
\[
\Psi_0 = \Psi_4 = (2\sigma - 1)(2\sigma + 1)\sigma r^{-8\sigma^2 + 4\sigma - 2}\frac{1 + q^2 r^{4\sigma}}{f(r)^3},
\]
\[
\Psi_2 = -\frac{\sigma r^{-8\sigma^2 + 4\sigma - 2}}{f(r)^4}[q^2 r^{4\sigma}(2\sigma + 1)^2 - (2\sigma - 1)^2 - 8q^2\sigma r^{4\sigma}];
\]
the only exceptions belonging to other Petrov classes are

| \(\sigma\) | Petrov type | Weyl Scalars |
|-----|-------------|--------------|
| 0   | 0           | all zero     |
| \(-\frac{1}{2}\) | \(D\) | \(\Psi_2 = -2 \frac{q^2 (1 + q^2 r^2)}{(1 - q^2 r^2)^4}\) |
| \(-\frac{1}{2}\) | \(D\) | \(\Psi_2 = -2 \frac{(q^2 + r^2)}{(q^2 - r^2)^4}\) |

The metric \(27\) represents a special case of the general cylindrically symmetric solutions with radial electric field \([4], \S 20.2, \text{Eq. 20.9c}\). Nevertheless, the approach within the framework of the HM conjecture provides a promising possibility for its physical interpretation and for understanding the nature of sources (at least for some values of \(\sigma\)).

### 7 The magnetovacuum solution for \(\sigma = -\frac{1}{2}\)

Before proceeding on the Killing vector \(\xi_{-\frac{1}{2}}\) several explanatory notes should be added. The vector \(\xi_{-\frac{1}{2}}\) obviously generates rotation in the \(\varphi - z\) plane, so it is geometrically equivalent to \(\xi_\varphi\) and should lead to magnetic EM field. The cylindrical coordinates used so far are very convenient for the expression of \(\xi_\varphi\). To follow the scheme outlined in section \([4]\) one should preferably start with the coordinate transformation
\[
\varphi = CX \cos Y, \quad z = X \sin Y,
\]
which turn the LC metric \([4]\) for \(\sigma = -\frac{1}{2}\) into the form
\[
ds^2 = -\frac{dt^2}{r^2} + r^4 dr^2 + r^4 dX^2 + r^4 X^2 dY^2
\]
and the Killing vector
\[
\xi_{-\frac{1}{2}} = \partial_Y = r^4 X^2 dY.
\]

Now in analogy with all preceding cases let us take the basis
\[
\omega^{(0)} = \frac{F(X, r)}{r} dt, \quad \omega^{(1)} = F(X, r) r^2 dr, \quad \omega^{(2)} = F(X, r) r^2 dX, \quad \omega^{(3)} = \frac{r^2 X}{F(X, r)} dY
\]
inducing a metric
\[
ds^2 = -\frac{F(X, r)^2}{r^2} dt^2 + F(X, r)^2 r^4 dr^2 + F(X, r)^2 r^4 dX^2 + \frac{r^4 X^2}{F(X, r)^2} dY^2
\]
and set the four-potential
\[
A = q \frac{r^4 X^2}{F(X, r)} dY.
\]

The solution of Einstein and sourceless Maxwell equations then provides
\[
F(X, r) = 1 + q^2 X^2 r^4.
\]

Although the coordinates \((t, r, X, Y)\) are extremely suitable for calculation of tensor components and solving EM equations, the cylindrical coordinates \((t, r, \varphi, z)\) fit better the aim of physical interpretation, namely because
of the evident relation to the LC seed metric. Therefore, let us consequently transform all above computed objects back into the cylindrical coordinates. We obtain

\[ A = q r^4 \frac{r f(r, \varphi, z)}{C} (-z d\varphi + \varphi dz) = q r^2 \sqrt{\varphi^2 / C^2 + z^2} \omega^{(3)}, \]

\[ \omega^{(0)} = \frac{f(r, \varphi, z)}{r^2 f(r, \varphi, z)} \frac{dt}{dt}, \quad \omega^{(1)} = f(r, \varphi, z) r^2 d\varphi, \]

\[ \omega^{(2)} = \frac{r^2 f(r, \varphi, z)}{\sqrt{\varphi^2 / C^2 + z^2}} \left( \frac{\varphi}{C^2} d\varphi + z dz \right), \]

\[ \omega^{(3)} = \frac{1}{\sqrt{\varphi^2 / C^2 + z^2}} f(r, \varphi, z) \left( -\frac{z}{C} d\varphi + \frac{\varphi}{C} dz \right); \]

and

\[ ds^2 = -\frac{f(r, \varphi, z)^2}{r^2} dt^2 + f(r, \varphi, z)^2 r^2 dr^2 + \frac{r^2}{\varphi^2 / C^2 + z^2} \left[ f(r, \varphi, z)^2 \left( \frac{\varphi}{C^2} d\varphi + z dz \right)^2 + \frac{1}{f(r, \varphi, z)^2} \left( -\frac{z}{C} d\varphi + \frac{\varphi}{C^2} dz \right)^2 \right], \]

where

\[ f(r, \varphi, z) = 1 + q^2 \left( \varphi^2 / C^2 + z^2 \right) r^4. \]

The electromagnetic field is of magnetic type, since

\[ F_{(\mu)(\nu)} F^{(\mu)(\nu)} = \frac{8 q^2 \left[ 4 \left( \varphi^2 / C^2 + z^2 \right) + r^2 \right]}{r^2 f(r, \varphi, z)^4} \geq 0, \]

the tetrad components of the magnetic field strength read as

\[ F^{(1)(3)} = -B^{(2)} = \frac{4 q \sqrt{\varphi^2 / C^2 + z^2}}{r f(r, \varphi, z)^2}, \quad F^{(2)(3)} = B^{(1)} = \frac{2 q}{f(r, \varphi, z)^2}. \]

The physical interpretation of (38) is rather ambiguous because of the negative value of \( \sigma \). As we have already mentioned in section 2, in case of negative \( \sigma \) the problem of physical cylindrical symmetric sources was not solved in a satisfactory way even for the seed LC metric. Bonnor (see references in [7]) has proved that the seed LC metric for \( \sigma = -1/2, C = 1 \) is locally isometric to Taub’s plane solution. Therefore, a more suitable alternative seems to be a point of view preferred by Wang at all [8], that both the LC solution in case of negative \( \sigma \) and the metric (38) are plane symmetric.

The Kretschmann scalar

\[
\mathcal{R} = \frac{64}{f(r, \varphi, z)^8} \left[ 3q^6 r^{16} \left( \varphi^2 / C^2 + z^2 \right)^2 \times \right. \\
\left. \times \left( 21 \varphi^4 / C^4 + 42 \varphi^2 z^2 / C^2 + 6 \varphi^2 r^2 / C^2 + 6 \varphi^2 z^2 + 21 \varphi^4 + r^4 \right) - \\
- 6q^6 r^{12} \left( \varphi^4 / C^4 + 7 \varphi^2 r^2 / C^2 + 36 \varphi^2 z^2 / C^2 + \\
+ 7r^2 \varphi^2 + 18z^4 + r^4 \right) + q^4 r^8 \left( 62 \varphi^4 + 124 \varphi^2 z^2 / C^2 + 6 \varphi^4 / C^4 + \\
+ 46 \varphi^2 r^2 / C^2 + 46 \varphi^2 z^2 + 5r^4 \right) + 6q^2 r^4 \left( 2z^2 - r^2 + 2 \varphi^2 / C^2 \right) + 3 \right].
\]

diverges at \( r = 0 \); this one-dimensional singularity along the z-axis might indicate the location of the infinite line source. The Petrov type is I with Weyl scalars

\[ \Psi_0 = \Psi_4 = -12 q^2 \left( \varphi^2 / C^2 + z^2 \right) \left[ q r^4 \left( \varphi^2 / C^2 + z^2 \right) - 1 \right] / r^2 f(r, \varphi, z)^4, \]

\[ \Psi_1 = -\Psi_3 = -6 q^2 \sqrt{\varphi^2 / C^2 + z^2} \left[ q r^4 \left( \varphi^2 / C^2 + z^2 \right) - 1 \right] / r f(r, \varphi, z)^4, \]

\[ \Psi_2 = \frac{2q^2 r^4}{r^6 f(r, \varphi, z)^4} \left( \left[ q^2 r^4 \left( \varphi^2 / C^2 + z^2 \right) - 1 \right] \times \right. \\
\left. \left[ 3q^2 r^4 \left( \varphi^2 / C^2 + z^2 \right) - q^2 r^{10} + 1 \right] \right). \]
8 The electrovacuum solution for $\sigma = 1/4$

The last Killing vector listed in section 3 is $\xi_{1/4}$ characterizing a boost in $t - \phi$ plane. As in the previous section, one should preferably perform coordinate transformation

$$t = X \cosh Y, \quad \phi = CX \sinh Y$$

to simplify the calculus. This transformation does not map the whole spacetime but only the interior of the light cone $t^2 - \phi^2 > 0$. The rest of the spacetime can be mapped analogously when we interchange the hyperbolic functions and then one should formulate suitable boundary conditions to couple those maps smoothly together. Here we shall restrict ourselves only to the interior of the light cone. After substituting $\sigma = 1/4$ Eq. (1) turns into

$$ds^2 = -rdX^2 + \frac{dr^2}{\sqrt{r}} + rX^2 dY^2 + \frac{dz^2}{\sqrt{r}}.$$ 

It is worth mentioning that for $\sigma = 1/4$ the LC metric represents a transformation of one of the Kinnersley’s type $D$ metric (his Case IVB with his $C = 1$). For the Killing vector one gets $\xi_{1/4} = \partial_Y = rX^2 dY$. The choice of the tetrad

$$\omega^{(0)} = \sqrt{r}F(X,r) dX, \quad \omega^{(1)} = \frac{F(X,r)}{r^{1/4}} dr,$$
$$\omega^{(2)} = \frac{\sqrt{r}X}{F(X,r)} dY, \quad \omega^{(3)} = \frac{F(X,r)}{r^{1/4}} dz$$

(44)

gives the metric

$$ds^2 = -F(X,r)^2 rdX^2 + \frac{F(X,r)^2}{\sqrt{r}} dr^2 + \frac{rX^2}{F(X,r)^2} dY^2 + \frac{F(X,r)^2}{\sqrt{r}} dz^2,$$

$$F(X,r) = 1 + q^2 X^2 r;$$

and set the four-potential

$$A = q \frac{r X^2}{F(X,r)} dY.$$ (45)

Returning back to the cylindrical coordinates one obtains the vector potential

$$A = q \frac{r}{Cf(t,r,\phi)} (\varphi dt + t d\phi) = q \sqrt{r} \sqrt{t^2 - \varphi^2/C^2} \omega^{(2)},$$ (46)

the basis tetrad

$$\omega^{(0)} = \frac{\sqrt{r}f(t,r,\phi)}{\sqrt{t^2 - \varphi^2/C^2}} \left(t dt - \frac{\varphi}{C^2} d\phi\right), \quad \omega^{(1)} = \frac{f(t,r,\phi)}{r^{1/4}} dr,$$
$$\omega^{(2)} = \frac{\sqrt{r}}{\sqrt{t^2 - \varphi^2/C^2}} f(t,r,\phi) \left(\varphi \frac{dt}{C} + t \frac{d\phi}{C}\right), \quad \omega^{(3)} = \frac{f(t,r,\phi)}{r^{1/4}} dz,$$ (47)

and the line element

$$ds^2 = \frac{r}{t^2 - \varphi^2/C^2} \left[-f(t,r,\phi)^2 \left(t dt - \frac{\varphi}{C^2} d\phi\right)^2 + \frac{1}{f(t,r,\phi)^2} \left(\frac{\varphi}{C} dt + t \frac{d\phi}{C}\right)^2 + \frac{f(t,r,\phi)^2}{\sqrt{r}} (dr^2 + dz^2), \right.$$ (48)

where

$$f(t,r,\phi) = 1 + q^2 \left(t^2 - \varphi^2/C^2\right) r. \quad \quad \quad \quad \quad \quad (49)$$

This time the electromagnetic field is neither purely electric, nor purely magnetic but its type is different at various places and at various time, which can be demonstrated by the invariant

$$F_{(\mu)(\nu)} F^{(\mu)(\nu)} = -\frac{2q^2 \left[4\sqrt{r} - (t^2 - \varphi^2/C^2)\right]}{\sqrt{r} f(t,r,\phi)^4}$$ (50)

and components of the electromagnetic field tensor

$$F^{(0)(2)} = E^{(2)} = \frac{2q}{f(t,r,\phi)^2}, \quad F^{(1)(2)} = B^{(3)} = \frac{q}{r^{1/4} f(t,r,\phi)^2}.$$ (51)
This rather strange behaviour originates in the fact that the tetrad \([17]\) is carried by an observer moving round the infinite line source in azimuthal direction, that means, rotating round \(z\)-axis. In this sense the indefinite character of electromagnetic field can be understood as a special-relativistic effect. The EM field \([18]\) can be interpreted as an infinite line source with linear density \(2\) in the external electromagnetic field, part of which is analogous to the Bonnor-Melvin longitudinal magnetic background. The solution \([18]\) is non-stationary and in contrast to the seed LC metric it is neither cylindrically, nor axially symmetric.

In the interior of the light cone \(t^2 - \varphi^2/C^2 > 0\) the metric \([18]\) has again one-dimensional singularity at \(r = 0\) where the Kretschmann scalar

\[
\mathcal{R} = \frac{1}{4f(t, r, \varphi)} \left[ 3q^2 \left( t^2 - \varphi^2/C^2 \right) \left( 21t^4 r^{9/2}/C^4 - 42t^2 r^{9/2}/C^2 + 56t^4 r^{9/2}/C^2 + 256t^4 - 9t^4 r^{9/2} - 96t^2 r^5 \right) - 12q^2 \left( t^2 - \varphi^2/C^2 \right) \left( 9t^4 r^{7/2}/C^4 + 56t^2 r^{7/2} - 368t^2 r^5/C^2 - 62t^2 r^5/2/C^2 \right) + 12q^2 \left( t^2 r^{3/2} - \varphi^2 r^{3/2}/C^2 + 8r^2 \right) + 3\sqrt{r} \right].
\]

becomes infinite. The Petrov type is \(I\) with the Weyl scalars

\[
\Psi_0 = \Psi_1 = \frac{3}{4} \frac{q^2 \left( t^2 - \varphi^2/C^2 \right) t^2 - \varphi^2/C^2 - 1}{\sqrt{f(t, r, \varphi)}},
\]

\[
\Psi_1 = -\Psi_3 = -\frac{3q^2 t^2 - \varphi^2/C^2 - t^2 - \varphi^2/C^2}{2 r^{1/4} \sqrt{t^2 - \varphi^2/C^2 f(t, r, \varphi)}},
\]

\[
\Psi_2 = \sqrt{\frac{1}{8r^3 f(t, r, \varphi)^2}} \left[ 3q^2 t^2 - \varphi^2/C^2 \right] + 16q^2 r^5/2 \left( t^2 - \varphi^2/C^2 \right) - 16q^2 r^3/2 - 2q^2 t^2 - \varphi^2/C^2 - 1.
\]

9 The LC solution with longitudinal electric field

In preceding sections all Killing vectors of the LC metric were exhausted to generate EM field. As a by-product of those calculations one more EM field was found, or surprisingly, new possible interpretation was assigned to the metric \([18]\) derived in section \(3\). It should be pointed out that though the next steps and considerations follow the scheme of the HM conjecture described above, there is a key difference: we do not employ any Killing vector of the seed LC metric as a vector four-potential. On the other hand, one should emphasize that this fact does not contradict the conjecture, which has never been considered as the only possibility of generating EM fields.

Let us take the boost vector potential

\[
A = q (z dt - t dz) = \frac{q}{f(r)} \left( z r^{-2\sigma} \omega^{(0)} - t r^{2\sigma} \omega^{(3)} \right)
\]

with tetrad \([17]\) inducing metric \([18]\) and with the function \(f(r)\) in the form \([21]\). Solving EM equations one derives the constant \(c_1\)

\[
c_1 = \frac{q^2}{(2\sigma - 1)^2}.
\]

The electromagnetic invariant

\[
F_{(\mu)(\nu)} F^{(\mu)(\nu)} = -\frac{8q^2 C^2 (2\sigma - 1)^2}{r^{4\sigma^2} f(r)^4} < 0,
\]

so the electromagnetic field represents an electric type with longitudinal electric field strength oriented along the \(z\)-axis

\[
F^{(0)(3)} = E^{(3)} = -\frac{2q}{f(r)^2 r^4\sigma^2}.
\]
The Kretschmann scalar

\[
\mathcal{R} = \frac{64 (2\sigma - 1)^2}{f(r)^8 r^8 \sigma (2\sigma - 1)^2} \left[ g_4(r)^4 (\sigma - 1)^2 (4\sigma^2 - 6\sigma + 3) + 
+ 6g_4(r)^3 (2\sigma - 1)(\sigma - 1)^2 - 
- g_4(r)^2 (8\sigma^4 - 16\sigma^3 - 12\sigma^2 + 20\sigma - 5) - 
- 6g_4(r)(\sigma (2\sigma - 1) + \sigma^2 (4\sigma^2 - 2\sigma + 1)) \right],
\]

where \(g_4(r) = \frac{q^2 r^2 - 4\sigma}{(2\sigma - 1)^2}\) is again singular at \(r = 0\) with the exception \(\sigma = 1/2\), in which case the spacetime is flat and does not include any electric field.

Analogously to (58) the metric is Petrov type I with Weyl scalars

\[
\Psi_0 = \Psi_4 = -\frac{q^2 r^2 - 4\sigma (2\sigma - 1)^2}{r^8 \sigma^4 + 16\sigma^3 - 10(2\sigma - 1)^4 f(r)^4} \left[ r^4 \sigma (8\sigma^4 - 4\sigma^3 - 2\sigma^2 + \sigma) + 
+ q^2 r^2 (2\sigma^2 - 5\sigma + 3) \right],
\]

\[
\Psi_2 = -\frac{q^2 r^2 - 4\sigma (2\sigma - 1)^2}{r^8 \sigma^4 + 16\sigma^3 - 10(2\sigma - 1)^2 f(r)^4} \left[ r^4 \sigma (2\sigma - 1)^2 + q^2 r^2 (\sigma - 1) \right],
\]

algebraic special cases are summarized in the following table:

| \(\sigma\) | Petrov type | Weyl Scalars |
|-----------|-------------|--------------|
| 0         | D           | \(\Psi_0 = 3\Psi_2 = \frac{3q^2 (qr + 1)(qr - 1)}{(q^2 r^2 + 1)^4}\) |
| 1/2       | 0           | all zero     |
| 1         | D           | \(\Psi_0 = 3\Psi_2 = \frac{3(r^2 - q^2)}{(r^2 + q^2)^4}\) |

10 Conclusion

The application of the HM conjecture to the LC seed metric revealed several interesting features:

(i) All the Killing vectors of the seed vacuum solution were employed to obtain electromagnetic fields of both electric and magnetic type using HM conjecture. The process of generation is marked by common algorithmic steps (though not applicable generally) allowing to devise an instructive scheme (section 3).

(ii) The cylindrical symmetric EM fields 20.9a and 20.9b in [4] do not represent the most general cases since they do not include solutions 20.9c, 20.9d.

(iii) The Bonnor-Melvin universe (23) need not necessarily contain a longitudinal magnetic EM field as is usually supposed. According to the results of section 3 the background field might be both electric and magnetic. Thus, using the HM conjecture we have obtained a qualitatively new interpretation of the Bonnor-Melvin universe.

(iv) Setting \(\sigma = -1/2\) and following the transformation found by Bonnor [5] one can reduce each of the generated spacetimes (with the exception (48)) to some plane symmetric solution of the Einstein-Maxwell equations. Thus we have obtained also Einstein-Maxwell fields of Taub’s type (either electric or magnetic ones) as special cases of found solutions.
The analysis of radial geodesic motion in the appendix supports the above interpretation of one-dimensional singularity located along the $z$-axis. The singularity has an attractive character that can be explained naturally by the presence of an infinite line source, the character of which is described in 8. In comparison with the LC seed metric the presence of electromagnetic field generally results in a stronger singularity’s attraction.

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A Appendix: radial geodesic motion

This part concentrates only on the static cylindrically symmetric cases, namely the metrics (1), (10), (13) and (22) treated in sections 4, 5, 6 and 7 respectively and on the solution derived in the section 9. In all these cases the metric coefficients depend only on the radial distance from the $z$-axis $r$, so one has three integrals of motion at his disposal: the covariant coordinate components of particles four-velocity $u_t, u_\varphi, u_z$ defined in the common way (here we use the fact that all the metrics are diagonal)

$$u_t = g_{t t} \frac{d t}{d \tau}, \quad u_\varphi = g_{\varphi \varphi} \frac{d \varphi}{d \tau}, \quad u_z = g_{z z} \frac{d z}{d \tau},$$

where $\tau$ is the particle’s proper time. The normalization condition

$$u_\mu u^\mu = g_{t t} u_t^2 + g_{\varphi \varphi} (u_\varphi)^2 + g_{z z} u_z^2 = -1$$

enables to express the square of the contravariant radial four-velocity component

$$(u_\tau^2) = \left( \frac{d r}{d \tau} \right)^2 = \frac{1}{g_{\tau \tau}} \left( -1 - g_{t t} u_t^2 - g_{\varphi \varphi} u_\varphi^2 - g_{z z} u_z^2 \right).$$

The purely radial motion (there is, evidently, no dragging effect) is set by putting $u_\varphi = u_z = 0$. Here, unfortunately, it is not possible to introduce an effective potential independent of the particle energy per unit mass $-u_0$. Therefore, we use the absolute value of radial velocity instead. The scale of the radial velocity is not important for qualitative discussion, and thus it is not explicitly introduced in the drawings. The boundary between the region with zero and non-zero radial velocity physically determines the turning points for radial motion. To detect the position of the turning points more exactly, the contour lines are drawn in the base plane of each figure. The top flat part of the plots corresponds to regions with high radial velocities.

Each figure is related to a different spacetime and includes four subplots. These subplots correspond with two different values of particle energy (first and second row of subplots), and with weaker or stronger electromagnetic field (left and right column respectively). In this way we can illustrate the influence of the electromagnetic field on radial geodesics and compare the situation with motion in the seed LC metric. The first value of energy $(u_0 = 1)$ characterizes a particle at rest in Minkowski spacetime.

The plots in Fig. 1 represent the dependence of the $|u_\tau|$ on $r$ for various values of $\sigma$, that means, for different spacetimes from the class of solutions (13). When the spacetime is flat ($\sigma = 0$) then the radial motion with constant energy must result in constant radial velocity which equals zero in cases (a), (b) and is non-zero in cases (c),(d) (the top of the “ridge”). While for negative $\sigma$ (which is probably not relevant to any real situation) the singularity is not attractive and particle is kept at some distance from the $z$-axis, for $\sigma > 0$ the radial velocity rapidly increases towards the singularity with an evident attractive effect. The Figs. 1(a) and 1(c) illustrating the situation in presence of weak magnetic field are qualitatively identical to appropriate plots for the seed LC solution (13).

Quite an analogous situation can be found in Fig. 2 belonging to the solution (18) with longitudinal magnetic field which degenerates to flat spacetime for $\sigma = 1/2$. Apparently, one should expect that in this case we again recognize the motion with a constant radial velocity in the plots. There is, however, an important difference originating in the form of the seed LC metric. Evidently, the metric (13) for $\sigma = 1/2$ turns into

$$ds^2 = -r^2 dt^2 + dr^2 + d\varphi^2 + dz^2$$

with interchanged components $g_{tt}$ and $g_{\varphi \varphi}$ compared to the Minkowski spacetime; it rather corresponds to the frame of an accelerated observer. Such an observer certainly will not measure a constant radial velocity for a considered radial motion. On the other hand, comparing Figs. 1 and 2 subplots (a), (c), we can see there is no crucial difference between the motion in a weak azimuthal and longitudinal magnetic field, the positions of
turning points nearly coincide. Some differences can be detected in stronger fields (Figs. 1 and 2, subplots (b), (d)).

The plots become slightly more complicated in presence of electric field in Fig. 3. The presence of another singularity in (27) results in the fact that the radial geodesic motion of particles with given energy is restricted to two separated regions perceptible in all subplots (a)-(d). In case of negative $\sigma$ the singularity gets an attractive character (the parts with increasing radial velocity at the back of subplots). Moreover, in case of positive $\sigma$ and stronger electric field (subplots (b),(d)) the radial motion is possible also at larger radial distances from the $z$-axis (compare to corresponding subplots in Figs. 1, 2 which are different). In subplots 3(a),(c) we can again recognize motion with a constant radial velocity for $\sigma = 0$, when (27) becomes flat.

The solution including the longitudinal electric field (section 9) has many features in common with the corresponding magnetic one (18). Comparing Figs. 2 and 4 one finds out that the electric case in Fig. 4 differs in the particular detail that for the case of flat spacetime $\sigma = 1/2$ the radial velocity equals zero and the plane of constant $\sigma = 1/2$ strictly divides the surfaces on all subplots into two parts. The subplots 4(a), (c) for a weak longitudinal electric field are analogous to those ones corresponding to a weak longitudinal magnetic field in Fig. 3(a), (c).

There should be stressed one more interesting point in connection with Fig. 4(b). For stronger electromagnetic field, even for positive $\sigma$, the singularity qualitatively changes its behaviour: there is a turning point close to the $z$-axis so that the particle cannot reach the singularity (see the right part of the subplot (b)). This might be another argument supporting the problematic interpretation of the LC solution for $\sigma > 1$ (and consequently, all the solutions generated from the LC metric in this paper).

Our discussion of the radial geodesic motion is, of course, far from being exhaustive. Its aim is to summarize the physical interpretation of the generated spacetimes and to emphasize the most essential points. The physical qualities of those spacetimes are determined most of all by the character of the seed metric. This conclusion is in full accordance with the principles of the HM conjecture: all the generated EM fields represent a generalization of the seed metric that must be their limiting case for a zero electromagnetic field.

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Figure 1: Absolute value of radial velocity for the solution with azimuthal magnetic field. (a) $u_0 = 1$, $q = 0.1$. (b) $u_0 = 1$, $q = 1$. (c) $u_0 = 2$, $q = 0.1$. (d) $u_0 = 2$, $q = 1$. 
Figure 2: Absolute value of radial velocity for the solution with longitudinal magnetic field. (a) $u_0 = 1$, $q = 0.1$. (b) $u_0 = 1$, $q = 1$. (c) $u_0 = 2$, $q = 0.1$. (d) $u_0 = 2$, $q = 1$. 
Figure 3: Absolute value of radial velocity for the solution with radial electric field. (a) $u_0 = 1, \ q = 0.1$. (b) $u_0 = 1, \ q = 1$. (c) $u_0 = 2, \ q = 0.1$. (d) $u_0 = 2, \ q = 1$. 
Figure 4: Absolute value of radial velocity for the solution with longitudinal electric field. (a) $u_0 = 1, \ q = 0.1$. (b) $u_0 = 1, \ q = 1$. (c) $u_0 = 2, \ q = 0.1$. (d) $u_0 = 2, \ q = 1$. 