A THEORY OF ELEMENTARY HIGHER TOPOSES

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ABSTRACT. We define an elementary ∞-topos that simuntaneously generalizes an elementary topos and Grothendieck ∞-topos. We then prove it satisfies the expected topos theoretic properties, such as descent, local Cartesian closure, locality and classification of univalent morphisms, generalizing results by Lurie [Lur09] and Gepner-Kock [GK17]. We also define ∞-logical functors and show the resulting ∞-category is closed under limits and filtered colimits, generalizing the analogous result for elementary toposes and Grothendieck ∞-toposes. Moreover, we give an alternative characterization of elementary ∞-toposes and their ∞-logical functors via their ind-completions. Finally we generalize these results by discussing the case of elementary (n,1)-toposes and give various examples and non-examples.

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INTRODUCTION

0.1 Set Theory as a Foundation for Mathematics. Since the work of Cantor in the late 19th century [Can74], set theory has established itself as a standard foundation for broad swaths of modern mathematics. As a result it is now fairly standard that in order to define a mathematical object, such as group, variety, manifold, or ..., one starts with a set and then adds the desired list of conditions. Having this common foundation allows mathematicians all over the world to effectively communicate ideas and introduce new objects.

The fact that set theory can now be comfortably used to do mathematics follows a century long development using various mathematical foundations. One such approach is axiomatic set theory first developed by Frege [Fre93], Zermelo [Zer08] and Fraenkel [Fra22], leading to the now famous Zermelo-Fraenkel (ZF) set theory. Another approach arose in the branch of categorical logic and originated with work of Lawvere [Law64] and Tierney [Tie73], leading to a foundation via elementary toposes. Another, third approach, comes from type theory, which has arose in the work of Russel [Rus08], as a way to avoid paradoxes in early versions of axiomatic set theory and is particularly suitable in constructive and computational mathematics [Chu40, ML75]. Moreover, there are ways to translate between these foundations: Every (higher-order) type theory gives us an elementary topos and vice versa [LS88]. Moreover, we can construct set theories both out of elementary topos theories [MLM94, Subsection VI.10] and type theories [Acz78].

The key aspect of set theory as a foundation for mathematics is not what a set is, but rather how it behaves in the sense that we want to know which axioms our chosen foundation satisfies. The development above allows mathematicians of all stripes to choose the axioms that fits their needs to develop their
mathematical theory, which might or might not include well-known axioms such as the axiom of choice, the continuum hypothesis, the law of excluded middle, and many other axioms, being assured of soundness of the theory without having to necessarily grapple with the underlying foundation and which one of the three is used to construct a specific model.

0.2 Homotopical Mathematics. Homotopical mathematics has its starting point in the work of Poincaré, who recognized that we can distinguish two topological spaces by analyzing homotopy classes of based loops, introducing the first homotopy invariant [Poi95]. This led to further invariants, such as higher homotopy groups [Čech32] and homology and cohomology theories [ES45].

The development of these invariants motivated the introduction of up-to-homotopy structures, such as $\infty$-spaces, which are monoids in spaces where the composition and associativity only hold up to a choice of coherent homotopies, a prominent example being the based loop space [Sta63]. This example and many other illustrate why set theoretical foundations are not suitable for the study of homotopical mathematics, as we would rather like to use a framework in which the loop space is directly a monoid (in the sense that the structure of a monoid is itself only defined up to coherent homotopies). This vision spurred a journey, which led to the development of various notions of homotopical or weak categories, such as quasi-categories [BV73], complete Segal spaces [Rez01], Kan enriched categories [Ber07a] and others, which are all now considered examples of the theory of $(\infty, 1)$-categories (or simply $\infty$-categories) [Ber10], the standard setting for homotopy coherent mathematics.

Similar to the previous subsection homotopical mathematical requires its own basic building block that can be used to characterize different objects of interest and we could analogously call a theory of spaces\(^1\). Again the key value is to determine how a space behaves rather than what a space is. Unlike the set-theoretical setting this foundation has not been fully established yet, however, we are currently witnessing its development. Ideally we would expect that the three foundations outlines above, namely axiomatic set theory, elementary topos theory and type theory, have a homotopy invariant analogues that also interact with each other in a similar manner.

A first step important step towards realizing this vision is a homotopy invariant analogue of type theory, known as homotopy type theory, which started with the work of Awodey, Warren [AW09] and Voevodsky [Voe14], and now covers many fundamental aspects of topology, such as fundamental groups, Blakers-Massey and truncations [Uni13].

0.3 Towards Elementary $\infty$-Toposes. The goal of this paper is to establish another pillar of homotopy invariant foundation, by generalizing elementary toposes to an appropriate notion of elementary $\infty$-topos. In order to understand how we can obtain a working definition, it is instructive to review how the definition of elementary toposes arose in the first place.

The story of topos theory starts with the work of Grothendieck and more generally Bourbaki, who needed a proper categorical framework to study algebro-geometric objects, introducing Grothendieck topologies and sheaves and hence defining a Grothendieck topos as a category of sheaves [sga72]. Lawvere and Tierney recognized that Grothendieck toposes can be an appropriate framework for categorical logic and an axiomatic study of sets if they could avoid the set-theoretical assumptions inherent in the definition of Grothendieck toposes (a technical condition known as local presentability). Their solution was to add a certain universal object, the subobject classifier, which allowed them to recover the desired results without assuming local presentability [Law64, Tie73], and defining elementary toposes as Cartesian closed categories with subobject classifier and now many important results can now be found in the famous work of Johnstone [Job02a, Job02b].

The $(\infty, 1)$-categorical story of topos theory starts again very similarly, namely with the desire by Lurie to study objects in derived algebraic geometry, motivating him to define Grothendieck $\infty$-toposes as appropriate generalizations of Grothendieck toposes [Lur09]. Around the same time, Rezk also developed the equivalent theory of model toposes, with the intention of better understanding the concept of descent, in particular introducing the (now standard) idea that Grothendieck $(\infty, 1)$-toposes are locally Cartesian closed presentable $(\infty, 1)$-categories that satisfy descent [Rez10]. The historical analogy would suggest that we can obtain our desired generalization by replacing the presentability condition with appropriately chosen universal objects.

\(^1\)To avoid ambiguity there have also been suggestions to call it animated set theory [CS19]
Before we explain how to choose the universal object, we will make a short detour into various characterizations of categories and \((\infty, 1)\)-categories. The data of a category can be characterized in two equivalent ways: Either as a set of objects along with a hom set for any two objects that has a composition, or as a set of objects and set of morphisms along with functions corresponding to source, target, composition, ... [Rie16, Subsection 1.1]. We can think of the first as a local characterization of categories and the second as a global characterization.

This philosophy carries over to the \((\infty, 1)\)-categorical setting. There we have models of \((\infty, 1)\)-categories that follow the local philosophy, such as relative categories [BK12], Kan enriched categories [Ber07a] or topologically enriched categories [Lur09], and models that follow the global philosophy, such as quasi-categories [BV73], complete Segal spaces [Rez01] and Segal categories [Ber07b]. There are prominent ways of translating between the local and global perspective, such as \((\mathcal{E}, N)\) between quasi-categories and Kan enriched categories [Lur09, Theorem 2.2.5.1], or \((F, R)\) between Segal categories and Kan enriched categories [Ber07b, Theorem 8.6], \((K_\mathcal{E}, N_\mathcal{E})\) between complete Segal spaces and relative categories [BK12, Theorem 6.1].

0.4 Elementary \(\infty\)-Topos: Local vs. Global. Having reviewed local and global approaches to \((\infty, 1)\)-categories, we claim that the appropriate notion of universal object is given by an internal \(\infty\)-category that classifies the morphisms over a given object. This universal object can be characterized using both the global perspective and the local perspective:

- **Local**: A universe, meaning an object \(U\), such that morphisms \(X \to U\) correspond (homotopically uniquely) to a space of morphisms over \(X\), along with local Cartesian closure. Here \(U\) should be thought of as the objects of our internal \(\infty\)-category and the local Cartesian closure gives us the morphism objects.

- **Global**: A complete Segal universe, meaning an internal complete Segal object \(\mathcal{U}_\bullet\), such that morphisms \(X \to \mathcal{U}_\bullet\) correspond (homotopically uniquely) to an \(\infty\)-category of objects over \(X\). Here, we are using the fact that the theory of complete Segal spaces can be internalized to give us a theory of internal \(\infty\)-categories via complete Segal objects [Ras17].

We claim that similar to before the local and global approach do in fact coincide. Concretely, we can combine (and slightly simplify) **Definition 4.1** and **Proposition 4.2**, fundamentally relying on **Theorem 3.27**, to give the following result.

**Theorem.** Let \(\mathcal{E}\) be a finitely (co)complete \(\infty\)-category. Then the following coincide:

- It is locally Cartesian closed and has sufficient universes\(^2\).
- It has sufficient complete Segal universes\(^2\).

An \(\infty\)-category that satisfies these equivalent conditions and has a subobject classifier is an elementary \(\infty\)-topos.

Using these characterizations we can deduce various valuable facts about elementary \(\infty\)-toposes. First of all we can study classify all univalent morphisms in an elementary \(\infty\)-topos, generalizing an analogous result by Gepner and Kock [GK17, Corolary 3.11]. In fact we can use univalent morphisms to give an alternative characterization of elementary \(\infty\)-toposes, which is closely related to the approach in [Ste20, Remark 3.27, Remark 4.21]. Moreover, using the local characterization, we realize that the definition (with some closure properties on the universe) coincides with the one suggested by Shulman [Shu17].

We can also witness that elementary \(\infty\)-toposes satisfy conditions known (or not known) from elementary topos theory. For example, similar to elementary toposes, we can prove the fundamental theorem of \(\infty\)-topos theory, which proves that the overcategory of an elementary \(\infty\)-topos is again an elementary \(\infty\)-topos (Theorem 4.9). Moreover, using [Ras21b], we can deduce that every elementary \(\infty\)-topos has a natural number object (Theorem 4.11). Finally, we can also use [Ras18], we can deduce that every elementary \(\infty\)-topos (with sufficiently closed universes), has a truncation functor (Theorem 4.12), generalizing the epi-mono factorization in every elementary topos. These results pave the way to tackle future problems, as we have discussed in greater detail in Subsection 0.6.

Having a definition of elementary \(\infty\)-toposes, we can also define an appropriate notion of functor, the \(\infty\)-logical functor. We can again use the local vs. global principle and combine **Definition 5.1** and **Proposition 5.2** to give the following result.

\(^2\) Here we are using several (complete Segal) universes in order to avoid size-related paradoxes.
Theorem. Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor of elementary \( \infty \)-toposes that preserves finite limits and colimits. Then the following coincide:

- \( F \) preserves the local Cartesian structure and universes.
- \( F \) preserves complete Segal universes.

A functor that satisfies these equivalent conditions and preserves the subobject classifier is an \( \infty \)-logical functor.

This generalizes logical functors of elementary toposes (Definition 2.2) and results in the \( \infty \)-category \( \mathcal{L} \mathrm{og}_{\infty} \). It is known that various notions of toposes and their functors are closed under limits and filtered colimits. Indeed for elementary toposes this can be found in [MLM94, Page 218], and for Grothendieck \( \infty \)-toposes in [Lur09, Proposition 6.3.2.3, Theorem 6.3.3.1]. We continue this tradition by proving the following result.

Theorem 5.9. The inclusion functor \( \mathcal{L} \mathrm{og}_{\infty}^P \to \mathcal{E} \mathrm{at}_{\infty} \) preserves small limits and filtered colimits.

There is a stronger result due to Dubuc and Kelly, that proves that the category of elementary toposes and logical functors is also cocomplete (and in fact a category of algebras of a finitary monad over \( \mathcal{C} \) [DK83]. This in particular implies the existence of an initial object, which is known as the free elementary topos and plays a key role in the translation between type theories and elementary toposes [Joh02b, Example D4.3.14] [LS80, Lam80, LS88]. Given our desire to better understand this connection, we can ask ourselves whether there is an free elementary \( \infty \)-topos. While it is expected to be true, we shall postpone it to later and work and leave it as Conjecture 5.10.

Even more generally, we can ask ourselves whether the inclusion functor \( \mathcal{L} \mathrm{og}_{\infty} \to \mathcal{E} \mathrm{at}_{\infty} \) has a left adjoint, that takes every category to its freely generated elementary \( \infty \)-topos, which would also give us the free elementary \( \infty \)-topos if we start with the initial \( \infty \)-category. Again, we shall postpone it to later and work and leave it as Conjecture 5.11.

While most of our work has focused on studying \( \infty \)-topos theory, there is also a relevant theory of \((n, 1)\)-toposes. The Grothendieck version was also developed by Lurie [Lur09, Section 6.4], and in Section 7 we introduce a theory of elementary \((n, 1)\)-toposes (Definition 7.3) and their \((n, 1)\)-logical functors (Definition 7.9), however only using the local approach (Remark 7.6).

Despite the fact that an elementary \((n, 1)\)-topos is not an elementary \((m, 1)\)-topos (Proposition 7.14), we can in fact obtain an elementary \((n + 1, 1)\)-topos as the subcategory of \( n \)-truncated objects (Theorem 7.15). While it is known that every Grothendieck \((n, 1)\)-topos can be obtained this way [Lur09, Theorem 6.4.1.5], it is not known whether this holds for elementary \((n, 1)\)-toposes and we leave the relevant conjecture (Conjecture 7.19) for future work.

0.5 External Universes and Models of Homotopy Type Theory. Having established a theory of elementary \( \infty \)-toposes one would similarly expect that it corresponds to homotopy type theories, the same way that elementary toposes correspond to higher-order type theories. The main challenge towards realizing this goal is to show that the universes in an elementary \( \infty \)-topos are as strict as required by type theory, which is complicated by the fact that there are many different universes that need to be strictified.

In the setting of Grothendieck \( \infty \)-toposes, this issue has been solved by Shulman [Shu19] by embedding the Grothendieck \( \infty \)-topos in a category of groupoid valued presheaves and constructing a single external universe, that can then be shown to satisfy the desired strictness results. This approach fundamentally hinges on the fact that Grothendieck \( \infty \)-toposes are all presentable and can be modeled by model categories [Dug01]. Arbitrary elementary \( \infty \)-toposes will not satisfy such strictness conditions and so we cannot hope to recover such strong results directly.

Hence, as a first step we would like to find a way to characterize elementary \( \infty \)-toposes in a way that requires only a single universe. One possible motivation could come from [ABSS14], where the authors use algebraic set theory [JM95] to associate to each elementary toposes a category of ideals using the directed system of inclusions. We will leave the \( \infty \)-categorical approach to algebraic set theory to future work and instead use the ind-completion of an \( \infty \)-category to obtain the desired characterization with a single universe. In fact, in Theorem 6.17 we prove a stronger result by giving a correspondence between various structured \( \infty \)-categories (such as universes, local Cartesian closure and subobject classifiers) and their ind-completions.
One immediate benefit of this correspondence is that, despite their complicated appearances, the characterizations of elementary $\infty$-toposes and their $\infty$-logical functors via sufficient universes and their preservation are in fact quite sensible, as they do correspond to the existence of a single universe and its preservation (Remark 6.19). Beyond that, the hope is that this alternative characterization can be employed in future work to prove that elementary $\infty$-toposes model homotopy type theories by strictifying the unique universe, generalizing the result by Shulman [Shu19].

0.6 Where to go from here. While we address some questions regarding elementary $\infty$-topos theory, many questions remain.

1. **Free Algebras and $\infty$-Operads:** One elegant result in elementary topos theory is the existence of free finitary algebras, and in particular free monoids, using the notion of $W$-types [Joh02b, Theorem D5.3.5]. The construction fundamentally relies on the existence of natural number objects, which suggests the possibility of similar free constructions in elementary $\infty$-topos theory, however, the $\infty$-categorical nature raises several new challenges that need to be addressed.

Unlike in the 1-categorical case, in an $\infty$-category $\mathcal{E}$, making an object $E$ into a monoid involves constructing an infinite tower of morphisms $\mu_n : E^n \to E$, which are appropriately compatible. This data is often managed via an appropriate choice of $\infty$-operad and the monoid is then an algebra for such $\infty$-operads [Lur17]. Given that an $\infty$-operad incorporates the data of operations $\mu_n$ for all $n \in \mathbb{N}$, the current definition relies on the natural numbers, which in an arbitrary elementary $\infty$-topos can differ, and, concretely, involve non-standard natural numbers [Ras21c]. As a result, before we can even talk about constructing free algebras, the first important step is to give a definition of $\infty$-operads internal to an elementary $\infty$-topos $\mathcal{E}$ based on the internal natural numbers of $\mathcal{E}$.

2. **Stabilizations:** One key development of algebraic topology is stable homotopy theory. It involves defining the $\infty$-category of spectra. Hence we would want a stabilization of an arbitrary elementary $\infty$-topos. There is an established notion of a stabilization of an $\infty$-category, however, in this case we face the same challenges as before, as the stabilization is indexed by the natural numbers [Lur17, Definition 1.4.2.8], which again can differ in an arbitrary elementary $\infty$-topos, and can in particular result in the existence of additional spheres. So what we would want is defining a stabilization operation that takes the additional natural numbers and spheres into account.

3. **Constructing Finite Colimits:** Another elegant result in elementary topos theory is the fact that we can construct finite colimits using Cartesian closure and the subobject classifier and so, unlike the $\infty$-categorical analogue, the definition of elementary toposes does not assume the existence of finite colimits [Mik72, Mik76, Par74]. One relevant question is whether a similar result holds in the $\infty$-categorical setting. As a first step, in joint work with Jonas Frey, we show that initial objects and finite coproducts can be constructed in every locally Cartesian closed $\infty$-category with subobject classifier, and so in particular in every elementary $\infty$-topos [FR21]. The remaining step towards recovering all finite colimits is to construct coequalizers using universes.

4. **Localizations:** Another fascinating result about elementary toposes is our ability to construct all left-exact localizations of an elementary topos via certain endomorphisms of the subobject classifier, the so-called Lawvere-Tierney topologies [MLM94, Section V] and we would like to obtain a similar elegant characterization for elementary $\infty$-toposes. We do have some results in the case of Grothendieck $\infty$-toposes [Lur09, ABF22, Ste21], however, obtaining an appropriate generalization is greatly complicated by the fact that even in Grothendieck $\infty$-toposes not every left-exact accessible localization can be obtained via Grothendieck topologies, as they can have $\infty$-connected morphisms, and the ones that can be obtained are known as topological localizations [Lur09, Definition 6.2.1.4]. This suggests two reasonable next steps, that we should pursue:

- Proving that we can we can characterize topological localizations of elementary $\infty$-toposes via Lawvere-Tierney topologies on the subobject classifier.
- Proving that more general left-exact localizations can be classified by certain choice of topology on a specific universe.

5. **Models of Homotopy Type Theory:** Finally, as explained above, elementary toposes can be understood as models of various higher order type theories [Joh02b, Proposition D.4.3.15] and it is expected that there are similar connections between intensional type theories and $\infty$-categories, and particularly homotopy type theories with univalent universes and elementary $\infty$-toposes.
Making this connection precise is quite challenging as ∞-categories are not strict enough to model type theories directly and all existing results use the fact that we can strictify the ∞-category, either to model categories [Shu19] or at least fibration categories [KS19]. Hence one first step towards realizing this goal is to give a definition of elementary ∞-toposes in a strict model of ∞-categories, such as possibly fibration categories.

0.7 Background. We will assume general familiarity with the world of ∞-categories (any model) as presented in [RV22, Rez17, Lur09] and will only review some key concepts in Section 1. Some familiarity with ∞-topos theory as presented in [Lur09, Section 6] or [Rez19] would be helpful, however, we have reviewed most key results that we need in Section 2.

0.8 Acknowledgment. I want to thank my advisor Charles Rezk for suggesting this topic and for his helpful comments, in particular regarding complete Segal universes. I also want to thank Mike Shulman for many fruitful conversations that contributed to Theorem 3.27. Moreover, I want to thank Valery Isaev for helpful comments regarding closed universes now explained in Remark 4.3. I finally want to thank Nicola Gambino for pointing me to [ABSS14], which led to the material in Section 6.

FROM ∞-CATEGORIES TO REPRESENTABLE CARTESIAN FIBRATIONS

We will use the language of ∞-categories throughout. For that we will primarily rely on the theory of ∞-cosmoses introduced by Riehl and Verity [RV17, RV22] and to some extent on the theory of quasi-categories as studied by Joyal [Joy08a, Joy08b] and Lurie [Lur09, Lur17]. Finally, some results depend on the theory of representable Cartesian fibrations introduced in [Ras17] and further analyzed in [Ras21a, Ras21d] and the associated study of univalence [Ras21e]. We will review some of the key definitions and results here:

1. We fix three universes, which we call small, large and very large.
2. We denote the (very large) ∞-cosmos of (large) quasi-categories [Joy08a, Joy08b] by QC\text{at}. A theory of ∞-categories is an ∞-cosmos \( \mathcal{K} \) such that the underlying quasi-category functor \( (\_)_0 = \text{Hom}_{\mathcal{K}}(1, \_): \mathcal{K} \to QC\text{at} \) is a biequivalence of \( \infty \)-cosmosi with inverse the tensor \( 1 \otimes - : QC\text{at} \to \mathcal{K} \) (also called ∞-cosmos of \( (\infty, 1) \)-categories [RV22, Definition 1.3.10]). According to [RV22, Example 1.3.9], examples include the ∞-cosmos of quasi-categories itself, but also the ∞-cosmos of complete Segal spaces \( \text{CSS} \) [Rez01, JT07], Segal categories \( \text{SegCat} \) [Ber07b], and 1-complicial sets \( 1 - \text{Comp} \) [Lur09].
3. We define finitely complete ∞-categories as defined in [RV22]. In particular, by [RV22, Proposition 4.3.1], the limit of a diagram \( F: I \to \mathcal{C} \) is given by the terminal object in the pullback ∞-category
\[
\begin{array}{ccc}
1 & \xrightarrow{F} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C}/y & \to & \mathcal{C}/x
\end{array}
\]
Note that by [RV22, Appendix F] in the particular case of quasi-categories the definition coincides with alternative definitions given in [Lur09, GK17]. Moreover, we use the notation “finitely (co)complete ∞-category” to denote an ∞-category that has finite limits and colimits. Similarly, we use the notation “finitely (co)continuous” to denote a functor that preserves finite limits and colimits.
4. We say a finitely complete ∞-category \( \mathcal{C} \) is locally Cartesian closed if the pullback functor \( f^* : \mathcal{C}/y \to \mathcal{C}/x \) has a right adjoint [RV22, Definition 2.1.1] for every morphism \( f: x \to y \) in \( \mathcal{C} \). Again, by [RV22, Appendix F] in the particular case of quasi-categories the definition coincides with the other common definitions [Lur09, GK17].
5. If \( \mathcal{K} \) is a theory of ∞-categories, there is a biequivalence \( \text{CSS} = \text{nerve}(\text{Hom}_{\mathcal{K}}(1, \_)): \mathcal{K} \to \text{CSS} \) [RV22, Example 1.3.9], that takes an ∞-category in \( \mathcal{C} \) to its underlying complete Segal space.
6. We denote the underlying (very large) quasi-category of \( \mathcal{K} \) by \( \text{Cat}_{\infty} \) (which is not an object in the ∞-cosmos QC\text{at}) and up to equivalence does not depend on the choice of ∞-cosmos. In particular, for two ∞-categories \( \mathcal{C}, \mathcal{D} \) we denote the Kan complex of functors by \( \text{Map}_{\text{Cat}_{\infty}}(\mathcal{C}, \mathcal{D}) \). Moreover, we denote the full subcategory of (very large) ∞-groupoids (which we also call spaces) by \( \hat{\mathcal{C}} \).
7. The inclusion functor \( \hat{\mathcal{C}} \hookrightarrow \text{Cat}_{\infty} \) has a right adjoint, which takes an ∞-category to its underlying maximal ∞-groupoid, which we denote by \( (-)^{\ast} : \text{Cat}_{\infty} \to \hat{\mathcal{C}} \).
8. Recall a morphism \( f: c \to d \) in a finitely complete ∞-category \( \mathcal{C} \) is \((-2)\)-truncated if it is an equivalence and \( n\)-truncated (for \( n \geq -1 \)) if the diagonal \( f: c \to c \times_d c \) is \((n - 1)\)-truncated. An object is
$n$-truncated if the map to the final object is $n$-truncated. In particular, a morphism $c \to d$ is $(-1)$-truncated (also called mono) if $\Delta : c \to c \times_d c$ is an equivalence. We denote the full sub-category of $n$-truncated objects by $\tau_n \mathcal{C}$ and say $\mathcal{C}$ is an $(n + 1, 1)$-category if $\tau_n \mathcal{C} \simeq \mathcal{C}$.

(9) Let $F : I \to \mathcal{Cat}_\infty$ be a diagram, where $I$ is filtered. Then $I$ is weakly contractible [Lur09, Lemma 5.3.1.18] and so if $F$ is constant its colimit is constant. Moreover, by [Lur09, Proposition 5.3.3.3] and [Ras21c, Lemma 2.8], the colimit commutes with finite limits. As a result, if there is a functor $H : I \times [1] \to \mathcal{Cat}_\infty$, with $H(-,0) \simeq F$, $H(-,1) \simeq \mathcal{D}$ constant and $H(i,-)$ mono, then the induced map of colimits $\text{colim}_I F \to \mathcal{D}$ is also mono.

(10) For an $\infty$-category $\mathcal{C}$, a Cartesian fibration over $\mathcal{C}$ models a presheaf from $\mathcal{C}$ valued in $\infty$-categories and a right fibration over $\mathcal{C}$ (also called discrete Cartesian fibrations in [RV22, Section 5]) models a presheaf from $\mathcal{C}$ valued in spaces. They were first introduced in the context of quasicategories [Lur09, Section 2.4], but have been generalized to an arbitrary $\infty$-cosmos [RV22, Chapter 5], in a way that coincides with the original definition when we restrict to the $\infty$-cosmos of quasicategories [RV22, Appendix F]. For a fixed $\infty$-category $\mathcal{C}$, we denote the full sub-quasi-category of $(\mathcal{Cat}_\infty)_{/\mathcal{C}}$ consisting of Cartesian fibrations by $\mathcal{Cart}_{/\mathcal{C}}$ and the full subcategory of right fibrations by $\mathcal{RFib}_{/\mathcal{C}}$. Note $\mathcal{Cart}_{/\mathcal{C}}$ (up to categorical equivalence) is independent of the choice of $\infty$-cosmos, as biequivalences of $\infty$-cosmoi preserve and reflect (discrete) Cartesian fibrations [RV22, Proposition 10.3.6(x),(xi)].

(11) For every $\infty$-category $\mathcal{C}$ and object $c$, there exists a representable right fibration $\pi_c : \mathcal{C}_{/c} \to \mathcal{C}$ [RV22, Corollary 5.5.13]. Moreover, we have the Yoneda lemma: For a right fibration $\mathcal{R} \to \mathcal{C}$ there is an equivalence of spaces $\text{Map}_c(\mathcal{Cat}_\infty)_{/\mathcal{C}}(\mathcal{C}_{/c}, \mathcal{R}) \simeq \mathcal{Fib}_c(\mathcal{R})$, where $\mathcal{Fib}_c(\mathcal{R})$ is the fiber of $\mathcal{R}$ over $c$ [RV22, Theorem 5.7.1]. In particular, taking $\mathcal{R} = \mathcal{C}_{/d}$, we get an equivalence $\text{Map}_c(\mathcal{Cat}_\infty)_{/\mathcal{C}}(\mathcal{C}_{/c}, \mathcal{C}_{/d}) \simeq \text{Map}_c(c,d)$ [RV22, Corollary 5.7.16].

(12) Let $\mathcal{C}$ be a finitely complete $\infty$-category. A simplicial object $X_\bullet : \Delta^{op} \to \mathcal{C}$ is a complete Segal object if it satisfies the Segal condition: for all $n \geq 2$ the map

$$X_n \to X_1 \times_{X_0} \ldots \times_{X_0} X_1,$$

is an equivalence [Ras17, Definition 3.1], and the completeness condition: the square

$$\begin{array}{ccc}
X_0 & \xrightarrow{r} & X_3 \\
\downarrow & & \downarrow \\
X_1 & \rightarrow & X_1 \times_{X_0} X_1 \times_{X_0} X_1
\end{array}$$

is a pullback square [Ras17, Definition 3.3], where the bottom corner is defined as the limit of the diagram

$$X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_1} X_1.$$

Complete Segal objects are a model of internal $\infty$-categories and so have a notion of objects, morphisms, equivalences, ... as described in [Ras17, Section 3], which we will not require.

(13) For a given complete Segal object $X$, there is a Cartesian fibration $\mathcal{C}_{/X_\bullet}$ [Ras17, Proposition 2.4, Proposition 3.4], which models the functor $\text{Map}_c(-, X_\bullet) : \mathcal{C}^{op} \to \mathcal{Cat}_\infty$ [Ras17, Notation 2.5]. We have a Yoneda lemma for representable Cartesian fibrations: For two complete Segal objects $X_\bullet, Y_\bullet$, there is an equivalence $\text{Map}_{\mathcal{Cart}_{/\mathcal{C}}}(\mathcal{C}_{/X_\bullet}, \mathcal{C}_{/Y_\bullet}) \simeq \text{Map}_{\text{Fun}(\Delta^{op}, \mathcal{C})}(X_\bullet, Y_\bullet)$ [Ras17, Theorem 2.7].

(14) There is a fully faithful functor $\mathcal{RFib}_\bullet : \mathcal{Cart}_{/\mathcal{C}} \to \text{Fun}(\Delta^{op}, \mathcal{RFib}_{/\mathcal{C}})$ with essential image complete Segal objects in right fibrations, where $\mathcal{RFib}_0$ can be explicitly described as the right fibration associated to the functor

$$\mathcal{C}^{op} \to \mathcal{Cat}_\infty \xrightarrow{(-)|[0]} \mathcal{Cat}_\infty \xrightarrow{(-)_{\simeq}} \hat{S}.$$

This is the result of [Ras21a, Section 4] and [Ras21d, Section 3], and has also been reviewed in [Ras21e, Section 5]. The particular case of $\mathcal{RFib}_0$ is also known as the underlying right fibrations [Lur09, Corollary 2.4.2.5].
(15) A functor of Cartesian fibrations \( F : \mathcal{D} \to \mathcal{E} \) over \( \mathcal{C} \) is fully faithful if it is fiber-wise fully faithful \( \mathcal{Fib}_{\mathcal{D}} \hookrightarrow \mathcal{Fib}_{\mathcal{E}} \), which, by the argument in the proof of [Rez01, Proposition 7.6] is equivalent to the following being a pullback square of Cartesian fibrations

\[
\begin{array}{ccc}
\mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{D}} & \xrightarrow{\gamma} & \mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{F}} \\
\downarrow & & \downarrow \\
\mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{D}} \times_{\mathcal{E}} \mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{D}} & \xrightarrow{\mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{F}}, \mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{F}}} & \mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{E}} \times_{\mathcal{E}} \mathcal{R} \mathcal{F} \mathcal{b}_{\mathcal{E}}
\end{array}
\]

(16) Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category. Then the target fibration from the arrow \( \infty \)-category \( \mathcal{O}_{\mathcal{C}} \to \mathcal{C} \) is a Cartesian fibration [Lur09, Lemma 6.1.1.1], which represents the functor \( \mathcal{C}_{/-} : \mathcal{C}^{op} \to \mathcal{C} \mathcal{a} \mathcal{t}_{\infty} \), which takes an object \( c \) to the over-category \( \mathcal{C}_{/-} \). We denote the underlying right fibration \( \mathcal{O}_{\mathcal{C}} \) by \( \mathcal{O}_{\mathcal{C}}^{(all)} \), and \( \mathcal{O}_{\mathcal{C}} \) by \( \mathcal{O}_{\mathcal{C}}^{(all)} \) and, by Section 1(14), they have value \( \mathcal{O}_{\mathcal{C}}^{(all)} \) and \( \mathcal{O}_{\mathcal{C}}^{(all)} \), respectively. Moreover, if \( S \) is a class of morphisms in \( \mathcal{C} \) closed under pullbacks, then we denote the sub-fibration of \( \mathcal{O}_{\mathcal{C}} \) with objects morphisms in \( S \), by \( \mathcal{O}_{\mathcal{C}}^{S} \) and the analogous sub-fibration of \( \mathcal{O}_{\mathcal{C}}^{(all)} \) by \( \mathcal{O}_{\mathcal{C}}^{(all)} \). As a particular case, denote the full subcategory \( \mathcal{O}_{\mathcal{C}} \) (\( \mathcal{O}_{\mathcal{C}}^{(all)} \)) with objects \( n \)-truncated morphisms, by \( \mathcal{T}_{n} \mathcal{O}_{\mathcal{C}} \) (\( \mathcal{T}_{n} \mathcal{O}_{\mathcal{C}}^{(all)} \)).

(17) Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category. Then a morphism \( p : E \to B \) is univalent if the functor \( \mathcal{C}_{/-} \) given by Section 1(11) is fully faithful. Moreover, by [Ras21, Theorem 4.4], if \( \mathcal{C} \) is locally Cartesian closed and \( p : E \to B \) is univalent, then there exists a complete Segal object \( \mathcal{N}(p) : \Delta^{op} \to \mathcal{C} \), such that

- \( \mathcal{N}(p)_{0} \simeq B \)
- \( \mathcal{N}(p)_{1} \simeq [E \times B, B \times E]_{B \times B} \), the internal mapping object.
- There is a fully faithful inclusion of Cartesian fibrations \( \mathcal{C}_{\mathcal{N}(p)} \hookrightarrow \mathcal{O}_{\mathcal{C}} \), with essential image morphisms that can be obtained as a pullback of \( p \).

For more details regarding univalence in finitely complete \( \infty \)-categories see [Ras21, Section 2].

(18) [Ras21, Proposition 2.5] Let \( p : E \to B \) be a univalent morphism. Then in the pullback square

\[
\begin{array}{ccc}
D & \xrightarrow{r} & E \\
\downarrow q & & \downarrow p \\
A & \xrightarrow{i} & B
\end{array}
\]

\( q \) is univalent if and only if \( i \) is mono.

(19) Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category and \( p : E \to B \) be univalent. Then for any object \( c \in \mathcal{C} \), \( p \times \text{id}_{c} \) is univalent in \( \mathcal{C}_{/-} \). Indeed, for any morphism \( q : y \to x \to c \) we have an equivalence

\[
\text{Map}_{\mathcal{O}_{\mathcal{C}}}(q, p \times \text{id}_{c}) \simeq \text{Map}_{\mathcal{O}_{\mathcal{C}}}(q, p)
\]

and so the desired result follows from Section 1(17).

(20) Let \( \text{Idem} \) be the \( \infty \)-category described in [Lur09, Definition 4.4.5.2, Remark 4.4.5.3]. An \( \infty \)-category \( \mathcal{C} \) is idempotent complete if every functor \( \text{Idem} \to \mathcal{C} \) has a limit [Lur09, Corollary 4.4.5.14]. By [Lur09, Remark 4.4.5.3], \( \text{Idem} \) has a single non-degenerate morphism for each dimension. As a result, if \( \mathcal{C} \) is an \( n \)-truncated, a functor \( \text{Idem} \to \mathcal{C} \) corresponds to a functor \( \tau_{n} \text{Idem} \to \mathcal{C} \), which has a limit if and only if \( \mathcal{C} \) has finite limits. Hence all finitely complete \( n \)-truncated \( \infty \)-categories are idempotent complete.

(21) Let \( \mathcal{C} \mathcal{a} \mathcal{t}_{\text{idem}} \) denote the full subcategory of \( \mathcal{C} \mathcal{a} \mathcal{t} \) consisting of idempotent complete \( \infty \)-categories. By [Lur09, Proposition 5.1.4.2], the inclusion \( \mathcal{C} \mathcal{a} \mathcal{t}_{\text{idem}} \hookrightarrow \mathcal{C} \mathcal{a} \mathcal{t} \) has a left adjoint \( (-)_{\text{idem}} \) known as the idempotent completion [Lur09, Definition 5.1.4.1].

(22) A quasi-category \( \mathcal{C} \) is presentable (\( \omega \)-accessible) if it satisfies the equivalent conditions given in [Lur09, Theorem 5.5.1.1] ([Lur09, Proposition 5.4.2.2]). An \( \infty \)-category is presentable (\( \omega \)-accessible) if its underlying quasi-category (Section 1(2)) is presentable (\( \omega \)-accessible). We will simply call it accessible to simplify notation.
(23) Let $\kappa$ be a regular cardinal and $C$ an accessible $\infty$-category. Then an object $c$ in $C$ is $\kappa$-compact if $\text{Map}_C(c, -) : C \to S$ commutes with $\kappa$-filtered colimits. Let $\text{Acc}_\infty$ denote the very large $\infty$-category with objects accessible $\infty$-categories with large set of compact objects and morphism functors that preserve filtered colimits and compact objects. There is an equivalence $(-)^{\text{comp}} : \text{Acc} \to \widehat{\text{Cat}_{\text{idem}}^{\text{acc}}}$ that takes an accessible $\infty$-category to its full subcategory of compact objects and has inverse $\text{Ind}$, the ind completion [Lur09, Proposition 5.4.2.15], which takes $C$ to the full subcategory of $P(C) = \text{Fun}(C^{\text{op}}, S)$ consisting of presheaves that can be obtained via filtered colimits [Lur09, Definition 5.3.5.1].

(24) If $C$ has finite colimits, then $\text{Ind}(C)$ is presentable [Lur09, First paragraph of Subsection 5.5.7] and so has a $(-)^{\tau_1}$-truncation functor $C \to \tau_1 C$ [Lur09, Proposition 5.5.6.18].

(25) Let $C$ be an $\infty$-category and $P : C^{\text{op}} \to S$. Let $C_{/P} = C \times_P C$. Then the argument in [Lur09, Corollary 5.1.6.12] implies that the natural functor $P(C_{/P}) \to P(C)/P$ is an equivalence. Moreover, assuming $P$ is obtained from a filtered diagram, we can restrict both sides to presheaves obtained by filtered diagrams and get an equivalence $\text{Acc}(C_{/P}) \to \text{Acc}(C)/P$.

(26) Let $C$ be a presentable $\infty$-category. Then a functor $F : C^{\text{op}} \to S$ is representable if and only if it takes colimits to limits [Lur09, Proposition 5.5.2.2]. Moreover, an accessible functor of presentable $\infty$-categories $F : C \to D$ is a left adjoint if and only if it preserves colimits [Lur09, Corollary 5.5.2.9].

**Review of Topos Theory**

Before we move on to study the concepts necessary to study elementary $\infty$-toposes, we shortly review the three notions of topos that have already been established: elementary toposes, Grothendieck toposes and $\infty$-toposes. We will start with elementary toposes.

**Definition 2.1.** Let $C$ be a finitely complete 1-category. For a given object $x$, let $\text{Sub}_C(x)$ be the set of isomorphism classes of subobjects of $x$. Notice this gives us a functor $\text{Sub}_C : C^{\text{op}} \to \text{Set}$ with functoriality given by pullback. Now a subobject classifier is an object that represents $\text{Sub}_C$.

For more details regarding subobjects and subobject classifiers in categories see [MLM94, Section I.3].

**Definition 2.2.** An elementary topos is a locally Cartesian closed category with subobject classifier. Moreover, a logical functor is a functor of elementary toposes that preserves finite limits, local Cartesian structure and the subobject classifier. We denote the resulting category by $\text{Log}$.

For (far more) details regarding elementary toposes see [Joh02a, Joh02b, MLM94]. Elementary toposes have a special case, known as Grothendieck toposes. Recall that a category $\mathbb{S}$ is a Grothendieck topos if there exists a small category $C$ and adjunction

$$
\text{Fun}(C^{\text{op}}, \text{Set}) \xrightarrow{L} \mathbb{S},
$$

where $L$ is fully faithful and $L$ preserves finite limits [MLM94, Section III]. This notion has been generalized to the $\infty$-categorical setting by Lurie [Lur09, Definition 6.1.0.4].

**Definition 2.3.** An $\infty$-category $\mathbb{S}$ is presentable if there exists a small $\infty$-category $C$ and an adjunction

$$
\text{Fun}(C^{\text{op}}, \text{Set}) \xleftarrow{L} \mathbb{S},
$$

such that $L$ is accessible, and is a Grothendieck $\infty$-topos if $L$ is additionally left exact.

**Note on Terminology.** Grothendieck $\infty$-toposes are simply called $\infty$-toposes in [Lur09], however, we will always use the additional “Grothendieck” as we want to generalize it in later sections. On the other side, they are called model toposes in [Rez10], as Rezk is using model categorical techniques, and we will not use this terminology as well.

The definition of Grothendieck $\infty$-topos here is very dependent on the particular choice of $C$ and we want a more inherent definition, motivated by work of Rezk [Rez10], which is more amenable to eventual generalizations. This require us to review some concepts. Recall that colimits in an $\infty$-category $\mathbb{S}$ are universal if for every morphism $f : x \to y$ the pullback functor $f^* : \mathbb{S}/y \to \mathbb{S}/x$ preserves colimits [Lur09, Definition 6.1.1.2]. Now we have the following first result.
Section 1

Proposition 2.4. Colimits in a Grothendieck ∞-topos are universal \[\text{[Lur09, Theorem 6.1.0.6]},\] and so, by Section 1(26), \(S\) is locally Cartesian closed (Section 1(4)).

We want to generalize Definition 2.1 to Grothendieck ∞-toposes.

Definition 2.5. Let \(C\) be a finitely complete ∞-category. Let \(\text{Sub}_C : C^{op} \to \text{Set}\) be the composition of \(C\) (corresponding to the fibration \(\mathcal{O}_C\) from Section 1(16)) and the \((-1)\)-truncation functor \(\tau_{-1} : \text{Cat}_\infty \to \text{Set}\) (Section 1(8)). A subobject classifier is an object \(\Omega\) that represents \(\text{Sub}_C\).

We now have the following in a Grothendieck ∞-topos \[\text{[Lur09, Subsection 6.1.6]},\]

Proposition 2.6. \(S\) has a subobject classifier, meaning an object \(\Omega : S^{op} \to \text{Set}\) and a natural isomorphism \(\text{Sub}_C \cong \text{Hom}_C(-, \Omega)\).

We want to move on to the next key property of Grothendieck ∞-toposes, local morphisms. Let \(C\) be a finitely complete ∞-category and let \(S\) be a set of morphisms in \(C\) closed under pullbacks. Recall from Section 1(16) that \(\mathcal{O}_C^{S}\) the full subcategory of \(\mathcal{O}_C\) consisting of objects in \(S\) and similarly, \(\mathcal{O}_C^{(S)}\) the analogous subcategory of \(\mathcal{O}_C^{(all)}\). We now have the following definition.

Definition 2.7. Let \(C\) be a finitely complete ∞-category. Moreover, let \(S\) be a set of morphism in \(C\) closed under pullbacks. We say \(S\) is local if the inclusion functor \(\mathcal{O}_C^{S} \to \mathcal{O}_C^{S}\) preserves finite colimits.

Recall that the initial object in \(\mathcal{O}_C\) is just the identity map on the initial object, which is always included in \(S\) (as \(S\) is closed under pullbacks) hence, the locality condition reduces to \(\mathcal{O}_C^{S}\) being closed under pushouts, meaning for a given pushout square in \(\mathcal{O}_C^{S}\)

\[
\begin{array}{ccc}
\mathcal{O}_C^{S} & \xrightarrow{f} & \mathcal{O}_C^{S} \\
\downarrow{\beta} & \searrow{\gamma} & \\
\mathcal{O}_C & \downarrow{h} & \mathcal{O}_C^{S}
\end{array}
\]

if \(\alpha\) and \(\beta\) are in \(\mathcal{O}_C^{(S)}\) (meaning they are pullback squares), then \(\gamma\) and \(\delta\) are also pullback squares. We now have the following result \[\text{[Lur09, Theorem 6.1.0.6, Theorem 6.1.3.9]}\].

Proposition 2.8. Let \(S\) be a locally Cartesian closed presentable ∞-category. Then \(S\) is an ∞-topos if and only if the set of all morphisms in \(S\) is local.

We move on to another key property of ∞-toposes, descent.

Definition 2.9. Let \(C\) be a finitely complete ∞-category. Then \(\mathcal{C}\) satisfies descent if \((\mathcal{O}_{/})^\infty : \mathcal{C}^{op} \to \mathcal{S}\) takes colimits to limits.

We now have the following result \[\text{[Lur09, Lemma 6.1.3.7, Theorem 6.1.3.9]}\].

Proposition 2.10. Let \(S\) be a Grothendieck ∞-topos, then \(S\) satisfies descent. Moreover, if \(S\) is presentable with universal colimits and satisfies descent, then it is a Grothendieck ∞-topos.

Finally, a key property regarding ∞-toposes are universes. Let \(S\) be an ∞-topos and \(\kappa\) be a large enough cardinal. Denote by \((\mathcal{O}_{/})^{\kappa}\) the full subcategory of \(\kappa\)-compact objects in \(\mathcal{C}_{/}\) (Section 1(23)). Then, by Proposition 2.10, \((\mathcal{O}_{/})^{\kappa}\cong : \mathcal{C}^{op} \to \mathcal{S}\) takes colimits to limits and so, by Section 1(26), must be represented by an object \(\mathcal{U}^{\kappa}\). This motivates the following definition.

Definition 2.11. Let \(P\) be a presentable ∞-category and \(\kappa\) a cardinal. A universe or object classifier for \(\kappa\)-compact objects is an object \(\mathcal{U}^{\kappa}\) representing the functor \((\mathcal{O}_{/})^{\kappa}\cong : \mathcal{C}^{op} \to \mathcal{S}\).

The argument before the definition implies that ∞-toposes have universes. However, we do in fact, have the opposite result \[\text{[Lur09, Theorem 6.1.6.8]}\].

Proposition 2.12. A presentable ∞-category with universal colimits is a Grothendieck ∞-topos if and only if for every large enough cardinal \(\kappa\) there exists a universe \(\mathcal{U}^{\kappa}\).

Up until now, we have seen several different ways of characterizing a Grothendieck ∞-topos. We can combine them all into the following theorem.
Theorem 2.13. Let $\mathcal{G}$ be a presentable $\infty$-category. Then the following are equivalent:

1. $\mathcal{G}$ is an $\infty$-topos.
2. $\mathcal{G}$ is locally Cartesian closed and satisfies descent.
3. $\mathcal{G}$ is locally Cartesian closed and has object classifiers $\mathcal{U}_\kappa$ for large enough $\kappa$.
4. $\mathcal{G}$ is locally Cartesian closed and the set of morphisms in $\mathcal{G}$ is local.

We can also define an analogue of Grothendieck toposes for $(n,1)$-categories (Section 1(8)) and concretely we have the following result as given in [Lur09, Theorem 6.4.1.5].

Theorem 2.14. Let $\mathcal{G}$ be a presentable $(n,1)$-category. Then the following are equivalent:

1. $\mathcal{G}$ is an $(n,1)$-topos.
2. $\mathcal{G} \simeq \tau_{n-1}\mathcal{G}$ (Section 1(8)), where $\hat{\mathcal{G}}$ is a Grothendieck $\infty$-topos.
3. $\mathcal{G}$ is locally Cartesian closed and has $(n-2)$-truncated object classifiers $\mathcal{U}_{\leq n}$, meaning an object that represents the functor $\tau_{n-2}(\mathcal{G}/-)^{\leq n} : \mathcal{G}^{op} \to \mathcal{S}^{\leq n-2}$.
4. $\mathcal{G}$ is locally Cartesian closed and set of $(n-2)$-truncated morphisms in $\mathcal{G}$ is local.

Remark 2.15. Similar to Proposition 2.6, every Grothendieck $(n,1)$-topos $\mathcal{G}$ has a subobject classifier.

Our goal in the coming sections is to generalize these results from presentable $\infty$-categories to finitely complete $\infty$-categories.

Descend, Cartesian Closures and Universes

In the last section we reviewed a relation between universes, descent and Cartesian closure in the setting of presentable $\infty$-categories. In this section we want to generalize this relation by introducing a generalization of universes, complete Segal universes. This will culminate in Theorem 3.27, which tells us precisely how these notions interact in the presentable and non-presentable setting.

As a first step we need to generalize universes from the presentable setting in order to avoid the reliance on set-theoretical assumptions.

Definition 3.1. Let $\mathcal{C}$ be a finitely complete $\infty$-category. A universe is a tuple $(\mathcal{U}, i)$, where $\mathcal{U}$ is an object in $\mathcal{C}$ and $i : \mathcal{C}/\mathcal{U} \hookrightarrow \mathcal{O}_{\mathcal{C}}^{(\text{all})}$ is an inclusion of right fibrations.

By the Yoneda lemma (Section 1(11)) any functor $i : \mathcal{C}/\mathcal{U} \to \mathcal{O}_{\mathcal{C}}^{(\text{all})}$ is (homotopically) uniquely determined by a choice of morphism $p_{\mathcal{U}} : \mathcal{U}_* \to \mathcal{U}$, which we call the universal fibration. Now Section 1(17) directly implies the following result.

Lemma 3.2. $(\mathcal{U}, i)$ is a universe if and only if the corresponding universal fibration $p_{\mathcal{U}} : \mathcal{U}_* \to \mathcal{U}$ is univalent.

We can use this insight to understand the collection of universes.

Definition 3.3. Let $\mathcal{C}$ be a finitely complete $\infty$-category. Then $\mathcal{C}$ has sufficient universes if the induced map $\biguplus_{\mathcal{U} \in \mathcal{U}_{\text{univ}} \mathcal{C}} \mathcal{C}/\mathcal{U} \to \mathcal{O}_{\mathcal{C}}^{(\text{all})}$ is an essentially surjective functor of $\infty$-categories.
Concretely, this means that for every morphism \( f : Y \to X \) in \( \mathcal{C} \), there exists a universe \( \mathcal{U} \in \mathcal{U}_{\text{niv}} \) and a homotopically unique pullback square

\[
\begin{array}{ccc}
Y & \rightarrow & \mathcal{U} \\
f \downarrow & & \downarrow p_{\mathcal{U}} \\
X & \rightarrow & \mathcal{U}
\end{array}
\]

(3.5)

**Notation 3.6.** If such a pullback square exists then we say \( \mathcal{U} \) classifies \( f \).

We can combine the definition with Section 1(18) to obtain an elegant classification result for univalent morphisms that generalizes [GK17, Corollary 3.10] from \( \infty \)-toposes to finitely complete \( \infty \)-categories with sufficient universes.

**Lemma 3.7.** Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category with sufficient universes. Then a morphism \( f \) is univalent if and only if the classifying map \( \chi_f \) defined in 3.5 is mono.

**Proof.** Follows directly from combining the pullback square 3.5 with Section 1(18). \( \square \)

Notice the theorem does not depend on the choice of universe \( \mathcal{U} \) in \( \mathcal{U}_{\text{niv}} \) as the collection of universes form a poset. Next we observe that our definition of universe behaves well with respect to over-categories.

**Lemma 3.8.** Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category, let \( (\mathcal{U}, i) \) be a universe in \( \mathcal{C} \) and let \( c \) be an object in \( \mathcal{C} \). Then \( (\pi_2 : \mathcal{U} \times c \to c, i) \) is a universe in \( \mathcal{C}/c \). Moreover, if \( \mathcal{C} \) has sufficient universes then \( \mathcal{C}/c \) has sufficient universes.

**Proof.** First we observe that \( \pi_2 : \mathcal{U} \times c \to c \) is a universe in \( \mathcal{C}/c \). Indeed, this follows directly from Lemma 3.7 and the fact that the universal fibration \( p_{\mathcal{U}} \times \text{id}_c : \mathcal{U} \times c \to \mathcal{U} \times c \) is univalent in \( \mathcal{C}/c \) (Section 1(19)). Now, let us assume \( \mathcal{C} \) has sufficient universes and let \( f : d \to e \) be an arbitrary morphism over \( c \). By assumption, \( f \) is a pullback of \( p_{\mathcal{U}} : \mathcal{U}_* \to \mathcal{U} \), for some \( \mathcal{U} \) in \( \mathcal{U}_{\text{niv}} \) and so we have the pullback square

\[
\begin{array}{ccc}
d & \rightarrow & \mathcal{U}_* \times c \\
\downarrow & & \downarrow \mathcal{U}_* \times c \rightarrow c \\
e & \rightarrow & \mathcal{U} \times c
\end{array}
\]

over \( c \), finishing the proof. \( \square \)

In certain circumstances we can characterize the existence of sufficient universes in terms that closer resemble a representability condition.

**Lemma 3.9.** Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category with finite coproducts and sufficient universes. Then \( \mathcal{U}_{\text{niv}} \) is filtered.

**Proof.** Let \( \mathcal{U}, \mathcal{U}' \) be two universes. As \( \mathcal{C} \) has sufficient universes, there exists a universe \( \mathcal{V} \) that classifies \( p_{\mathcal{U}} \coprod p_{\mathcal{U}'} \), meaning we have pullback diagrams

\[
\begin{array}{ccc}
\mathcal{U}_* & \rightarrow & \mathcal{U}_* \coprod \mathcal{U}'_* \\
\downarrow p_{\mathcal{U}} & & \downarrow p_{\mathcal{U}} \coprod p_{\mathcal{U}'} \\
\mathcal{U} & \rightarrow & \mathcal{U} \coprod \mathcal{U}' \rightarrow \mathcal{V} \cdot
\end{array}
\]

As \( p_{\mathcal{U}}, p_{\mathcal{V}} \) are univalent the map \( \mathcal{U} \to \mathcal{V} \) is mono (Section 1(18)), meaning \( \mathcal{U} \leq \mathcal{V} \) in \( \mathcal{U}_{\text{niv}} \) and similarly \( \mathcal{U}' \leq \mathcal{V} \), hence we are done. \( \square \)

**Lemma 3.10.** Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category with finite coproducts. Then \( \mathcal{C} \) has sufficient universes if and only if \( \colim_{\mathcal{U} \in \mathcal{U}_{\text{niv}}} \mathcal{C}/\mathcal{U} \to \mathcal{O}^{(\text{all})}_\mathcal{C} \) is an equivalence of \( \infty \)-categories.


Proof. Assume $\mathbb{C}$ has sufficient universes then, by Lemma 3.9, $\mathcal{U}_{niv}^c$ is filtered. As the diagram is filtered, the colimit is still a full sub-category of $\mathcal{O}_c^{(all)}$ (Section 1(9)). Moreover, the existence of sufficient universes means it is also essentially surjective, proving it is an equivalence.

On the other hand, if \[ \text{colim}_{\mathcal{U}\in \mathcal{U}_{niv}^c} \mathcal{E}/\mathcal{U} \to \mathcal{O}_c^{(all)} \] is an equivalence, then it is in particular essentially surjective and so for every object $f$ in $\mathcal{O}_c^{(all)}$, there exists a universe $\mathcal{U}$ that classifies $f$, proving $\mathbb{C}$ has sufficient universes. \[ \square \]

General universes help us recover certain objects in the category, however, we often also want to those objects to be closed under certain categorical constructions.

**Definition 3.11.** Let $\mathbb{C}$ be an $\infty$-category. A universal property $\mathcal{P}$ is a collection of functors $P : \mathbb{C}^{op} \to \mathbb{S}$. We say $\mathbb{C}$ is closed under the universal property $\mathcal{P}$ if all functors $P$ are representable in $\mathbb{C}$.

Examples of such universal properties includes (finite) (co)limits and Cartesian closure.

**Definition 3.12.** Let $\mathbb{C}$ be a finitely complete $\infty$-category closed under the universal property $\mathcal{P}$ and let $\mathcal{U}$ be a universe. We say $\mathcal{U}$ is closed under $\mathcal{P}$, if for all objects $c$, the full subcategory of $\mathbb{C}/c$ classified by $\mathcal{U}$ is closed under $\mathcal{P}$. We denote the sub-poset of $\mathcal{P}$-closed universes of $\mathbb{C}$ by $\mathcal{U}_{niv}^c$.

Let us present one example explicitly.

**Example 3.13.** Let $\mathbb{C}$ be a locally Cartesian closed $\infty$-category and a $\mathcal{U}$ a universe. We say $\mathcal{U}$ is Cartesian closed if it closed under the universal property of Cartesian closure.

The existence of universes has valuable implications about the $\infty$-category, however, as proven in Lemma 3.10, they at best can represent the right fibration $\mathcal{O}_c^{(all)}$ rather than the actual Cartesian fibration $\mathcal{O}_c$. Hence, in order to obtain a better representability result we need to generalize universes appropriately.

**Definition 3.14.** Let $\mathbb{C}$ be a finitely complete $\infty$-category. A complete Segal universe is a pair $(\mathcal{U}_* : \Delta^{op} \to \mathbb{C}, i)$, where $\mathcal{U}_*$ is a simplicial object in $\mathbb{C}$ and $i : \mathbb{C}/\mathcal{U}_* \to \mathcal{O}_c$ is an inclusion of Cartesian fibrations. Here $\mathbb{C}/\mathcal{U}_*$ is the representable Cartesian fibration defined in Section 1(13).

**Warning 3.15.** Notice our definition of complete Segal universe does not coincide with complete Segal objects internal to the category $\mathcal{U}_{niv}^c$ (as defined in Section 1(12)). Indeed $\mathcal{U}_{niv}^c$ is a poset and so any complete Segal object would be constant.

Despite the possible confusion we can justify our naming convention via the following two results that relate complete Segal universes both to universes and complete Segal objects.

**Lemma 3.16.** Let $\mathbb{C}$ be a finitely complete $\infty$-category. There is a functor $\mathcal{U}_{niv}^c : \mathbb{C}^{sI} \to \mathcal{U}_{niv}^c$ that takes a complete Segal universe $(\mathcal{U}_*, i)$ to the universe $(\mathcal{U}_0, i)$, which we call the underlying universe.

**Proof.** It suffices to show that $\mathcal{U}_{niv}^c$ is functorial. If $\mathcal{U}_0 \leq \mathcal{U}_1$, then there is an inclusion of Cartesian fibrations $\mathbb{C}/\mathcal{U}_0 \to \mathbb{C}/\mathcal{U}_1$, which induces an inclusion of right fibrations $\mathcal{O}_c/\mathcal{U}_0 \to \mathcal{O}_c/\mathcal{U}_1$, proving that $\mathcal{U}_0 \leq \mathcal{U}_1$. \[ \square \]

**Lemma 3.17.** Let $(\mathcal{U}_*, i)$ be a complete Segal universe. Then $\mathcal{U}_*$ is a complete Segal object.

**Proof.** By definition $\mathbb{C}/\mathcal{U}_*$ is a Cartesian fibration and so the result follows from Section 1(13). \[ \square \]

**Definition 3.18.** Let $\mathbb{C}$ be a finitely complete $\infty$-category. Denote by $\mathbb{C}^{sI} \mathcal{U}_{niv}^c$ the poset with objects complete Segal universes and $\mathcal{U}_0 \leq \mathcal{U}_1$ if there is a diagram of Cartesian fibrations

\[
\begin{array}{ccc}
\mathbb{C}/\mathcal{U}_0 & \xrightarrow{\Delta^{op}} & \mathbb{C}/\mathcal{U}_1 \\
\downarrow{i} & & \downarrow{i'} \\
\mathcal{O}_c & & \mathcal{O}_c
\end{array}
\]

**Definition 3.19.** Let $\mathbb{C}$ be a finitely complete $\infty$-category. Then $\mathbb{C}$ has **sufficient complete Segal universes** if the map

\[
\prod_{\mathcal{U}_* \in \mathbb{C}^{sI} \mathcal{U}_{niv}^c} \mathbb{C}/\mathcal{U}_* \to \mathcal{O}_c
\]

is an essentially surjective map of $\infty$-categories.

We now have the result analogous to Lemma 3.8 for over-categories.
Lemma 3.20. Let $\mathcal{E}$ be a finitely complete $\infty$-category, let $(\mathcal{U}_c, i)$ be a complete Segal universe in $\mathcal{E}$ and let $c$ be an object in $\mathcal{E}$. Then $(\pi_2 : \mathcal{U}_c \times c \to c, i)$ is a complete Segal universe in $\mathcal{E}/c$. Moreover, if $\mathcal{E}$ has sufficient complete Segal universes then $\mathcal{E}/c$ has sufficient complete Segal universes.

We also have results analogous to Lemma 3.9 and Lemma 3.10 for complete Segal universes.

Lemma 3.21. Let $\mathcal{E}$ be a finitely complete $\infty$-category with finite coproducts and sufficient complete Segal universes. Then $\mathcal{E}\text{Univ}_c$ is filtered.

Lemma 3.22. Let $\mathcal{E}$ be a finitely complete $\infty$-category with finite coproducts. Then $\mathcal{E}$ has sufficient complete Segal universes if and only if \( \text{colim}_{\mathcal{U} \in \mathcal{E}\text{Univ}_c} \mathcal{E}/\mathcal{U} \to \mathcal{O}_c \) is an equivalence of $\infty$-categories.

We can also additionally assume that complete Segal universes in $\mathcal{E}$ are closed, generalizing Definition 3.12.

Definition 3.23. Let $\mathcal{E}$ be a finitely complete $\infty$-category closed under the universal property $\mathcal{P}$ and let $\mathcal{U}_c$ be a complete Segal universe. We say $\mathcal{U}_c$ is closed under $\mathcal{P}$, if for all objects $c$, the full subcategory of $\mathcal{E}/c$ classified by $\mathcal{U}_c$ is closed under $\mathcal{P}$. We denote the sub-poset of $\mathcal{P}$-closed complete Segal universes of $\mathcal{E}$ by $\mathcal{E}\text{Univ}_c^{\mathcal{P}}$.

By Definition 3.12, we immediately have the following lemma.

Lemma 3.24. Let $\mathcal{E}$ be a finitely complete $\infty$-category closed under the universal property $\mathcal{P}$. Then a complete Segal universe $\mathcal{U}_c$ is closed under $\mathcal{P}$ if and only if the underlying universe is closed under $\mathcal{P}$.

Moreover, complete Segal universes can have an additional property that we need later on.

Definition 3.25. Let $\mathcal{E}$ be a finitely complete $\infty$-category and let $\mathcal{U}_c$ be a complete Segal universe. We say $\mathcal{U}_c$ is a locally small if $\mathcal{U}_0$ classifies the map $(d_1, d_0) : \mathcal{U}_1 \to \mathcal{U}_0 \times \mathcal{U}_0$.

The terminology is motivated by locally small categories, which might not have small sets of objects $\mathcal{O}$ and morphisms $\mathcal{M}$, but the source target projection $(s, t) : \mathcal{M} \to \mathcal{O} \times \mathcal{O}$ is small [Rie16, Definition 1.1.7]. Before we can prove the main result, we have the following key proposition.

Proposition 3.26. Let $\mathcal{E}$ be a locally Cartesian closed $\infty$-category. Then the functor $\text{lUnd} : \mathcal{E}\text{Univ}_c \to \mathcal{U}\text{Univ}_c$ has an inverse $\text{Lift} : \mathcal{U}\text{Univ}_c \to \mathcal{E}\text{Univ}_c$. Moreover, for a given universal property $\mathcal{P}$, it restricts to an inverse $\text{Lift} : \mathcal{U}\text{Univ}_c^{\mathcal{P}} \to \mathcal{E}\text{Univ}_c^{\mathcal{P}}$. In particular, there are sufficient universes (closed under $\mathcal{P}$) if and only if there are sufficient complete Segal universes (closed under $\mathcal{P}$).

Proof. Let $\mathcal{U}$ be a universe in $\mathcal{E}$ and $p_{\mathcal{U}} : \mathcal{U}_c \to \mathcal{U}$ the universal fibration. Let $N(p_{\mathcal{U}})$ be the complete Segal object constructed in Section 1(17), which satisfies $N(p_{\mathcal{U}})_0 \simeq \mathcal{U}$. This implies that $\text{lUnd} \circ \text{Lift}$ is the identity. On the other side, for a given complete Segal universe $\mathcal{U}_c$, $\text{Lift}\text{lUnd}(\mathcal{U}_c)$ is a complete Segal universe with an equivalence of Cartesian fibrations over $\mathcal{E}$

$$\mathcal{E}/\text{Lift}\text{lUnd}(\mathcal{U}_c) \simeq \mathcal{O}_{\mathcal{U}} \simeq \mathcal{E}/\mathcal{U}_c,$$

where $\mathcal{O}_{\mathcal{U}}$ is the full subcategory of $\mathcal{O}_c$ consisting of morphisms classified by $\mathcal{U}$. Hence, by the Yoneda lemma for complete Segal objects (Section 1(13)) $\text{Lift}\text{lUnd}(\mathcal{U}_c) \simeq \mathcal{U}_c$, which proves that $\text{Lift}\text{lUnd}$ is also the identity. This proves that $\text{lUnd}$ and $\text{Lift}$ are inverses.

Now, let $\mathcal{P}$ be a universal property. Then, by Lemma 3.24, a complete Segal universe $\mathcal{U}_c$ is $\mathcal{P}$-closed if and only if $\text{lUnd}(\mathcal{U})$ is $\mathcal{P}$-closed. Hence, the bijection $(\text{lUnd}, \text{Lift})$ between $\mathcal{U}\text{Univ}_c$ and $\mathcal{E}\text{Univ}_c^{\mathcal{P}}$ restricts to a bijection of posets $\mathcal{U}\text{Univ}_c^{\mathcal{P}}$ and $\mathcal{E}\text{Univ}_c^{\mathcal{P}}$.

We are now ready to state and prove the main theorem.

Theorem 3.27. Let $\mathcal{E}$ be a finitely complete $\infty$-category with finite coproducts closed under the (possibly empty) universal property $\mathcal{P}$. Consider the following statements:

1. $\mathcal{E}$ is locally Cartesian closed with sufficient universes closed under $\mathcal{P}$.
2. $\mathcal{E}$ has sufficient Cartesian closed complete Segal universes closed under $\mathcal{P}$.
3. Colimits in $\mathcal{E}$ are universal and the morphisms in $\mathcal{E}$ are local.
4. Colimits in $\mathcal{E}$ are universal and $\mathcal{E}$ satisfies descent.

Then we have the following implications:
(1) and (2) are equivalent and, assuming they hold, the complete Segal universes are Cartesian closed if and only if they are locally small.

(3) and (4) are equivalent.

(1), (2) imply (3), (4).

If \( E \) is presentable and \( \kappa \)-compact morphisms are closed under \( P \) for \( \kappa \) large enough, then (3), (4) imply (1), (2).

Proof. (1) \( \Leftrightarrow \) (2) If (1) holds, then (2) follows directly from Proposition 3.26. On the other side, the existence of sufficient universes closed under \( P \) follows directly from Lemma 3.24. Hence we only need to prove that \( E \) is locally Cartesian closed. Before we can do so we need to construct the appropriate right fibration.

Let \( \mathcal{A}_E^{(\text{all})} \rightarrow \mathcal{E} \) be the right fibration introduced in Section 1(16). Notice, the source and target projection map gives us a functor of right fibrations \( \mathcal{A}_E^{(\text{all})} \rightarrow \mathcal{O}_E^{(\text{all})} \times \mathcal{E} \). Finally, by the Yoneda lemma, a functor of right fibrations \( \mathcal{E}_{/x} \rightarrow \mathcal{O}_E^{(\text{all})} \times \mathcal{E} \) is given by a choice of two morphisms \( f : y \rightarrow x, g : z \rightarrow x \) in \( \mathcal{E} \). The resulting pullback \( \mathcal{A}_E^{(\text{all})} \times \mathcal{O}_E^{(\text{all})} \times \mathcal{E} \) is the right fibration which takes an object \( c \) to \( \text{Map} \_c(y \times x, z \times x) \) and by [GK17, Proposition 2.1] is representable if and only if the internal mapping object \( [y, z]_x \) exists.

Let \( f : y \rightarrow x, g : z \rightarrow x \) be two arbitrary objects in \( \mathcal{E}_{/x} \) and fix a complete Segal universe \( \mathcal{U}_* \) that classifies \( f \mid g \) and hence both \( f, g \). Denote by \( [y, z]_x \) the pullback of \( (f, g) : x \rightarrow \mathcal{U}_0 \times \mathcal{U}_0 \) along \( (d_1, d_0) : \mathcal{U}_1 \rightarrow \mathcal{U}_0 \times \mathcal{U}_0 \) and notice we now have the following pullback diagram of right fibrations over \( \mathcal{E} \):

\[
\begin{array}{ccc}
\mathcal{E}_{/[y,z]_x} & \xrightarrow{r} & \mathcal{E}_{/\mathcal{U}_1} \\
\downarrow & & \downarrow \\
\mathcal{E}_{/x} & \xrightarrow{(f,g)} & \mathcal{E}_{/\mathcal{U}_0} \times \mathcal{E}_{/\mathcal{U}_0} \xrightarrow{} \mathcal{O}_E^{(\text{all})} \times \mathcal{E} \mathcal{O}_E^{(\text{all})}
\end{array}
\]

where the left hand side is a pullback square by definition of \( [y, z]_x \) and the right hand side is a pullback square as the functor \( \mathcal{E}_{/\mathcal{U}_*} \rightarrow \mathcal{O}_E \) is fully faithful (Section 1(15)). This means the rectangle is a pullback, which proves that \( [y, z]_x \) is the internal mapping object of \( y, z \) over \( x \). This proves that (2) implies (1).

Finally, if (1), (2) hold, then, by Proposition 3.26, a complete Segal object \( \mathcal{U}_* \) is of the form \( \text{Lift} \mathcal{U}_* \), where \( \mathcal{U} \) is a universe and in particular \( \mathcal{U}_1 \simeq [\mathcal{U}_0 \times \mathcal{U}_0, \mathcal{U} \times \mathcal{U}]_{\mathcal{U} \times \mathcal{U}} \). Hence, if the complete Segal object is Cartesian closed (Definition 3.23), then \( [\mathcal{U}_0 \times \mathcal{U}_0, \mathcal{U} \times \mathcal{U}]_{\mathcal{U} \times \mathcal{U}} \rightarrow \mathcal{U} \times \mathcal{U} \) is classified by \( \mathcal{U}_* \), proving \( \mathcal{U}_* \) is locally small (Definition 3.25). On the other side, if \( \mathcal{U}_1 \rightarrow \mathcal{U}_0 \times \mathcal{U}_0 \) is classified by \( \mathcal{U}_0 \), then for all \( f : y \rightarrow x, g : z \rightarrow x \) classified by \( \mathcal{U}_0 \), \( [y, z]_x \) is the pullback of \( \mathcal{U}_1 \rightarrow \mathcal{U}_0 \times \mathcal{U}_0 \) and so also classified by \( \mathcal{U}_0 \), meaning \( \mathcal{U}_* \) is also Cartesian closed.

(3) \( \Leftrightarrow \) (4) This follows directly from [Lur09, Lemma 6.1.3.7]. Notice the lemma assumes presentability, however, the proof relies on [Lur09, Lemma 6.1.3.5], which never uses the presentability assumption.

(3) \( \Rightarrow \) (1) Assuming \( \mathcal{E} \) is presentable, this follows directly from Theorem 2.13.

Remark 3.28. In the proof of Theorem 3.27 if \( \mathcal{E} \) has sufficient complete Segal universes, by Lemma 3.20, the over-category \( \mathcal{E}_{/x} \) also has sufficient complete Segal universes for all objects \( x \). Hence it would have sufficed to prove that \( \mathcal{E} \) is Cartesian closed and the case for \( \mathcal{E}_{/x} \) would have followed directly. However, our approach above has the additional benefit of giving us an explicit construction of the internal mapping objects as a pullback.

We will use these various characterizations in the next section to define elementary versions of \( \infty \)-toposes.

**Elementary \( \infty \)-Topos Theory**

In this section we give a definition of an elementary \( \infty \)-topos, using the motivation from Section 2 and results from Section 3.

**Definition 4.1.** Let \( P \) be a universal property. A \( \mathcal{P} \)-closed elementary \( \infty \)-topos \( \mathcal{E} \) is a finitely (co)complete \( \infty \)-category that satisfies \( P \) with subobject classifier and sufficient \( \mathcal{P} \)-closed complete Segal universes in \( \mathcal{E} \).
Theorem 3.27 permits the following alternative characterization as well as valuable implications.

**Proposition 4.2.** Let \( \mathcal{P} \) be a universal property. A finitely complete \( \infty \)-category that satisfies \( \mathcal{P} \) with subobject classifier is a \( \mathcal{P} \)-closed elementary \( \infty \)-topos if and only if it is locally Cartesian closed and has sufficient \( \mathcal{P} \)-closed universes. In particular all colimits are universal.

**Remark 4.3.** Proposition 4.2 implies that our definition of elementary \( \infty \)-topos closed under Cartesian closure and finite (co)limits precisely coincides with the definition suggested by Shulman [Shu17].

**Note on Terminology.** We will often suppress the notation of the universal property \( \mathcal{P} \) in order to simplify notation.

Theorem 3.27 also has the following valuable implication.

**Corollary 4.4.** Let \( \mathcal{E} \) be an elementary \( \infty \)-topos. Then \( \mathcal{E} \) satisfies descent (Definition 2.9) and the morphisms in \( \mathcal{E} \) are local (Definition 2.7).

Besides universes elementary \( \infty \)-toposes can also be characterized via univalence, giving us a connection to homotopy type theory [Uni13, Subsection 2.10].

**Corollary 4.5.** Let \( \mathcal{P} \) be a universal property. A finitely cocomplete locally Cartesian closed \( \infty \)-category that satisfies \( \mathcal{P} \) with subobject classifier \( \mathcal{E} \) is an elementary \( \infty \)-topos if and only if it has sufficient univalent morphisms.

We can use this result to characterize all univalent morphisms, generalizing [GK17, Corollary 3.10] from Grothendieck \( \infty \)-toposes, as discussed in Lemma 3.7.

**Corollary 4.6.** Let \( \mathcal{E} \) be an elementary \( \infty \)-topos. A map \( f : x \to y \) is univalent if and only if any classifying map \( \chi : y \to \mathbf{1} \) given in 3.5 is mono.

**Remark 4.7.** In [Ras21e, Section 6] we showed that we can already classify all mono univalent maps in any \( \infty \)-category with subobject classifier and so in particular elementary toposes. However, we also showed that there are non mono univalent maps in elementary toposes that cannot be classified this way [Ras21e, Example 6.9]. The result above shows that an elementary \( \infty \)-topos is a suitable generalization as it allows us to classify all univalent morphisms.

We will discuss several examples (and non-examples) of elementary \( \infty \)-toposes in Section 8, however, in order to help understanding we will give one very explicit example right now.

**Example 4.8.** Let \( \mathcal{S} \) be the \( \infty \)-category of spaces. We want to show that it satisfies the conditions given in Definition 4.1. The existence of finite limits and colimits is evident and the subobject classifier is given by the two element discrete space \( \{0, 1\} \) and so we focus on showing there are sufficient complete Segal universes.

Let \( \kappa \) be a large enough cardinal. Then, \( \mathcal{S}^\kappa \) is a small \( \infty \)-category and by Section 1(5) corresponds to a small complete Segal space, which we simply denote by \( \mathcal{CSS}(\mathcal{S}^\kappa) : \Delta^{op} \to \mathcal{S} \). Notice, the two functors

\[
\text{Map}(\mathcal{S}^\kappa, \mathcal{CSS}(\mathcal{S}^\kappa)) : \Delta^{op} \to \mathcal{S} \quad \text{and} \quad \text{Map}(\mathcal{S}^\kappa, \mathcal{CSS}(\mathcal{S}^\kappa)) : \Delta^{op} \to \mathcal{S}
\]

are colimit preserving and take the point to \( \mathcal{CSS}(\mathcal{S}^\kappa) \). As any two colimit preserving functors out of \( \mathcal{S} \) which agree on the point are equivalent [Lur09, Theorem 5.1.5.6], we get the desired equivalence of \( \infty \)-categories \( \mathcal{CSS}(\mathcal{S}^\kappa) \cong (\mathcal{S}^\kappa)^\kappa \). This proves that \( \mathcal{CSS}(\mathcal{S}^\kappa) \) is the desired complete Segal universe. Hence, assuming there are sufficiently large cardinals, there are also sufficient complete Segal universes, proving \( \mathcal{S} \) is an elementary \( \infty \)-topos.

Let us move on to the fundamental theorem of topos theory [McL92] in the \( \infty \)-categorical setting, also generalizing the analogous result for Grothendieck \( \infty \)-toposes [Lur09, Proposition 6.3.5.1].

**Theorem 4.9.** Let \( \mathcal{P} \) be a universal property and let \( \mathcal{E} \) be a \( \mathcal{P} \)-closed elementary \( \infty \)-topos and \( x \) an object. Then \( \mathcal{E}/_x \) is also a \( \mathcal{P} \)-closed elementary \( \infty \)-topos.

**Proof.** We need to check that \( \mathcal{E}/_x \) satisfies the conditions in Definition 4.1. The case for finite limits and colimits is evident. The case for subobject classifier follows analogous to [MLM94, Theorem VI.7.1]. The case for complete Segal universes follows from Lemma 3.20. \( \square \)
We end this section with various implications regarding elementary ∞-toposes. First of all we have natural number objects.

**Definition 4.10.** Let \( \mathcal{C} \) be a finitely complete \( \infty \)-category. A triple \((\mathbb{N}, s : \mathbb{N} \rightarrow \mathbb{N}, o : 1 \rightarrow \mathbb{N})\) is a natural number object if for every triple \((x, u : x \rightarrow x, b : 1 \rightarrow x)\) in \( \mathcal{C} \), the space of of morphisms \( f : \mathbb{N} \rightarrow X \) that fit into the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{N} \\
\downarrow & & \downarrow f \\
1 & \rightarrow & X
\end{array}
\]

is contractible.

For a more detailed explanation of the definition see [Ras21b, Subsection 2.1]

**Theorem 4.11 ([Ras21b, Theorem 4.1.2]).** Every elementary \( \infty \)-topos has a natural number object.

Notice this result is trivial if the elementary \( \infty \)-topos has countable colimits, as \( \coprod_{\mathbb{N}} 1 \) is always a natural number object [Ras21b, Proposition 5.3.2]. However, there are non-trivial examples, as we shall see in Example 8.13. One key implication of the existence of natural number objects is the existence of truncations. To each natural number \( n : 1 \rightarrow \mathbb{N} \) we can associate a sphere \( S^n \) in \( \mathcal{E} \) and a subcategory of \( n \)-truncated objects \( \tau_n \mathcal{E} \) (as the \( S^n \)-local objects) [Ras18, Definition 3.5]. We now have the following result.

**Theorem 4.12 ([Ras18, Corollary 4.26]).** Let \( \mathcal{E} \) be an elementary \( \infty \)-topos with sufficient Cartesian closed universes closed under finite (co)limits. Let \( n : 1 \rightarrow \mathbb{N} \) be a natural number. Then the inclusion functor \( \tau_n \mathcal{E} \rightarrow \mathcal{E} \) has a left adjoint \( \tau_n : \mathcal{E} 
\rightarrow \tau_n \mathcal{E} \).

We will use this result extensively when studying elementary \( (n,1) \)-toposes in Section 7.

**The \( \infty \)-Category of Elementary \( \infty \)-Toposes**

In the last section we defined elementary \( \infty \)-toposes. We now want to move on and construct a (very large) \( \infty \)-category of elementary \( \infty \)-toposes, which requires introducing an appropriate notion of functor. In Definition 2.2 we observed that the appropriate functors of elementary toposes, the logical functors, preserved finite limits, Cartesian closure and subobject classifier. Taking this as a motivation we want to define \( \infty \)-logical functors as functors that preserve the appropriate structure.

Notice that a functor \( F : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) induces a functor of arrow categories \( \mathcal{O}_F : \mathcal{O}_{\mathcal{C}_1} \rightarrow \mathcal{O}_{\mathcal{C}_2} \). With this in mind we introduce the following terminology.

**Definition 5.1.** Let \( \mathcal{P} \) be a universal property. A finitely (co)continuous functor \( F : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) of \( \mathcal{P} \)-closed elementary \( \infty \)-toposes is an \( \infty \)-logical functor if \( \mathcal{O}_F : \mathcal{O}_{\mathcal{C}_1} \rightarrow \mathcal{O}_{\mathcal{C}_2} \) restricts to a map of posets \( \mathcal{CSU}_F : \mathcal{CSU}_{\mathcal{C}_1} \rightarrow \mathcal{CSU}_{\mathcal{C}_2} \), which preserves the subobject classifier.

The second condition concretely means that for a given complete Segal universe \( \mathcal{U} \) in \( \mathcal{CSU}_{\mathcal{C}_1} \), the simplicial object \( F(\mathcal{U}) \) is a complete Segal universe in \( \mathcal{CSU}_{\mathcal{C}_2} \).

**Note on Terminology.** Similar to the previous section we will often suppress the notation of the universal property \( \mathcal{P} \) in order to simplify notation.

Recall that logical functors of elementary toposes involves preservation of power objects. As before we can deduce this condition rather than assuming it. Recall that a functor \( F : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) is locally Cartesian closed if there is a natural equivalence \( F[Y, Z]_{\mathcal{E}_1} \cong [FY, FZ]_{\mathcal{E}_2} \).

**Proposition 5.2.** Let \( F : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) be an \( \infty \)-logical functor of elementary \( \infty \)-toposes. Then \( F \) is locally Cartesian closed.

**Proof.** First of all, by Lemma 3.9, \( \lniv_{\mathcal{E}_1} \) is filtered. Hence, the equivalence \( \text{colim}_{\mathcal{U} \in \lniv_{\mathcal{E}_1}} (\mathcal{E}_1)_{/\mathcal{U}} \simeq \mathcal{O}_{\mathcal{E}_1}^{(\text{all})} \) (from Lemma 3.10) induces an equivalence

\[
\text{colim}_{\mathcal{U} \in \lniv_{\mathcal{E}_1}} (\mathcal{E}_1)_{/\mathcal{U} \times \mathcal{U}} \simeq \text{colim}_{\mathcal{U} \in \lniv_{\mathcal{E}_2}} (\mathcal{E}_2)_{/\mathcal{U}} \times \mathcal{E}_{/\mathcal{U}} \simeq \mathcal{O}_{\mathcal{E}_1}^{(\text{all})} \times \mathcal{E}_{2} \simeq \mathcal{O}_{\mathcal{E}_1}^{(\text{all})} \times \mathcal{O}_{\mathcal{E}_2}^{(\text{all})}.
\]
Now, by the proof in Theorem 3.27, the functor \([-,-]_\ast : O_{E_1}^{(all)} \times_{E_1} O_{E_2}^{(all)} \to E_1\) is given by pulling back along \((d_1,d_0) : \mathcal{U}_1 \to \mathcal{U}_0 \times \mathcal{U}_0\). Now the fact that \(F\) commutes with pullbacks implies that the following diagram commutes

\[
\begin{array}{ccc}
O_{E_1}^{(all)} \times_{E_1} O_{E_2}^{(all)} & \xrightarrow{(d_1,d_0)_\ast} & E_1 \\
F_{E_1} \times F_{E_2} \downarrow & & \downarrow F \\
O_{E_2}^{(all)} \times_{E_2} O_{E_2}^{(all)} & \xrightarrow{(d_1,d_0)_\ast} & E_2
\end{array}
\]

which proves that \(F\) is locally Cartesian closed. \(\square\)

Using this result we can give an alternative characterization of \(\infty\)-logical functors, similar to Proposition 4.2.

**Proposition 5.3.** Let \(\mathcal{P}\) be a universal property. A finitely (co)continuous functor \(F : \mathcal{E}_1 \to \mathcal{E}_2\) of elementary \(\infty\)-toposes is an \(\infty\)-logical functor if and only if \(O_{\mathcal{E}_1}^{(all)} : \mathcal{E}_1 \to \mathcal{E}_2\) restricts to a map of posets \(\mathfrak{Univ}_{\mathcal{E}_1}^\mathcal{P} : \mathfrak{Univ}_{\mathcal{E}_1} \to \mathfrak{Univ}_{\mathcal{E}_2}\), which preserves the subobject classifier, and \(F\) is a locally Cartesian closed functor.

**Proof.** If \(F : \mathcal{E}_1 \to \mathcal{E}_2\) is an \(\infty\)-logical functor, then by Proposition 5.2, it is a locally Cartesian closed functor. On the other side, we need to prove for every complete Segal universe \(\mathcal{U}_\bullet\) in \(\mathcal{E}_1\), \(F(\mathcal{U}_\bullet)\) is a complete Segal universe by assumption \(F(\mathcal{U}_0)\) is a universe in \(\mathcal{E}_2\). Moreover, by Section 1(17), \(\mathcal{U}_1 \simeq [(\mathcal{U}_0)_\ast \times \mathcal{U}_0, \mathcal{U}_0 \times (\mathcal{U}_0)_\ast]_{\mathcal{U}_0 \times \mathcal{U}_0}\), and so by assumption \(F(\mathcal{U}_1) \simeq [F(\mathcal{U}_0)_\ast \times F\mathcal{U}_0, F\mathcal{U}_0 \times F(\mathcal{U}_0)_\ast]_{F\mathcal{U}_0 \times F\mathcal{U}_0}\) which, again by Section 1(17), proves that \(F\mathcal{U}_\bullet\) is a complete Segal universe. This proves that \(F\) is an \(\infty\)-logical functor. \(\square\)

Similarly, we have a characterization of \(\infty\)-logial functors via univalent morphisms.

**Proposition 5.4.** A finitely (co)continuous functor \(F : \mathcal{E}_1 \to \mathcal{E}_2\) of elementary \(\infty\)-toposes is an \(\infty\)-logical functor if and only if \(F\) preserves univalent morphisms and the subobject classifier and is a locally Cartesian closed functor.

**Proof.** By Proposition 5.3 it suffices to prove that if \(F\) is (co)continuous, locally Cartesian closed and preserves the subobject classifier, then \(F\) preserves universes if and only if it preserves univalent morphisms. However, this follows directly from Lemma 3.7. \(\square\)

Finally, an \(\infty\)-logical functor also interacts well with the natural number object.

**Proposition 5.5.** Let \(F : \mathcal{E}_1 \to \mathcal{E}_2\) be \(\infty\)-logical and \((\mathbb{N}, s, o)\) a natural number object in \(\mathcal{E}_1\). Then \((FN, Fs, Fo)\) is a natural number object in \(\mathcal{E}_2\).

**Proof.** By [Ras21b, Definition 2.1.4, Theorem 4.1.2] a natural number object can be characterized as a triple such that \((s, o) : 1] \mathbb{N} \to \mathbb{N}\) is an equivalence and the coequalizer of \(s\) and \(id_\mathbb{N}\) is the terminal object. These two properties are preserved by \(F\), as \(F\) preserves finite limits and colimits. Hence the result follows. \(\square\)

Let us give a class of examples of \(\infty\)-logical functors.

**Proposition 5.6.** Let \(\mathcal{E}\) be an elementary \(\infty\)-topos and \(f : x \to y\) a morphism. Then \(f^\ast : \mathcal{E}/y \to \mathcal{E}/x\) is logical.

**Proof.** \(\mathcal{E}\) preserves finite limits by definition. Moreover, it preserves finite colimits because \(\mathcal{E}\) is locally Cartesian closed and hence has a right adjoint. Finally, \(f\) preserves complete Segal universes, as \(f^\ast(\mathcal{U}_\bullet \times y \to y) = \mathcal{U}_\bullet \times x \to x\) and Lemma 3.20. \(\square\)

Notice the identity functor is \(\infty\)-logical and it is closed under composition giving us the following definition.

**Definition 5.7.** Let \(\mathcal{P}\) be a universal property. Let \(\mathcal{Log}_{\mathcal{P}}\) be the (non-full) subcategory of \(\mathcal{Cat}_{\infty}\) with objects \(\mathcal{P}\)-closed elementary \(\infty\)-toposes and morphisms \(\infty\)-logical functors.

For later sections it is helpful to have some intermediary notions between general \(\infty\)-categories and elementary \(\infty\)-toposes.
Notation 5.8. We denote the subcategory of $\mathcal{C} \mathcal{a} t_\infty$ with objects finitely complete $\infty$-categories with sufficient universes and functors that preserve finite limits and universes by $(\mathcal{C} \mathcal{a} t_\infty)_U$. Moreover, denote the subcategory of $(\mathcal{C} \mathcal{a} t_\infty)_U$ where objects additionally have a subobject classifier and functors that preserve it by $(\mathcal{C} \mathcal{a} t_\infty)_U, \text{Prop}$. Finally, denote the subcategory of $(\mathcal{C} \mathcal{a} t_\infty)_U$ where the objects are additionally locally Cartesian closed and functors that preserve it by $(\mathcal{C} \mathcal{a} t_\infty)_U, \mathcal{E} \mathcal{S} \mathcal{U}$.

We know the inclusion of Grothendieck $\infty$-toposes and geometric morphisms in the $\infty$-category of $\infty$-categories preserves limits and filtered colimits [Lur09, Proposition 6.3.2.3, Theorem 6.3.3.1]. Moreover, we have a similar result for the inclusion of elementary toposes in the category of categories (as stated in [MLM94, Page 218]). We want to observe that $\mathcal{L} \text{og}_{\infty}$ behaves similarly.

**Theorem 5.9.** The inclusion functor $\mathcal{L} \text{og}_{\infty}^P \to \mathcal{C} \mathcal{a} t_\infty$ preserves small limits and filtered colimits.

**Proof.** Let $F : I \to \mathcal{L} \text{og}_{\infty}^P$ be a diagram. Then the limit in $\mathcal{C} \mathcal{a} t_\infty$, $\mathcal{L} = \lim_i F$, has a terminal object given by the tuple of terminal objects. Moreover, for a finite diagram $d : K \to \mathcal{L}$, by Section 1(3) the limit is given as the terminal object in the pullback $\infty$-category $1 \xrightarrow{d} \mathcal{L}^K \xrightarrow{\Delta} \mathcal{L}$, which evidently commutes with limits. The case for finite colimits is analogous.

Now, for $i \in I$, let $\Omega_i$ be a subobject classifier in $\mathcal{E}_i$ and notice for every morphism $i \to j$ in $I$, the corresponding functor $\mathcal{E}_i \to \mathcal{E}_j$ maps $\Omega_i$ to $\Omega_j$ as the functor is $\infty$-logical. Hence, we have an object $(\Omega_i)_{i \in I}$ in $\mathcal{L}$. Now, for an arbitrary object $(x_i)_{i \in I}$ we have

$$\text{Map}_\mathcal{L}(\lim_i (x_i), (\Omega_i)_{i \in I}) \simeq \text{Map}_{\mathcal{E}_i}(x_i, \Omega_i)_{i \in I} \cong \text{Sub}_{\mathcal{E}_i}(x_i)_{i \in I} \cong \text{Sub}_\mathcal{L}((x_i)_{i \in I})$$

which proves that $(\Omega_i)_{i \in I}$ is the subobject classifier. The case for complete Segal universes follows analogously. This proves that $\mathcal{L} \text{og}_{\infty}^P$ is closed under small limits in $\mathcal{C} \mathcal{a} t_\infty$.

Next, we want to observe that the filtered colimit of elementary $\infty$-toposes along $\infty$-logical functors is an elementary $\infty$-topos. Here we use [Ras21c, Theorem 2.26]. The theorem is stated for filter quotients, however, only uses the fact that the diagram is filtered and that the functors are $\infty$-logical.

Notice the inclusion $\mathcal{L} \text{og}_{\infty}^P \to \mathcal{C} \mathcal{a} t_\infty$ does not preserve the initial object as the initial $\infty$-category is the empty $\infty$-category, which is evidently not an elementary $\infty$-topos. However, we do expect $\mathcal{L} \text{og}_{\infty}^P$ to have an initial object.

**Conjecture 5.10.** $\mathcal{L} \text{og}_{\infty}^P$ has an initial object.

This initial object would be called the free elementary $\infty$-topos, in analogy to the free elementary topos [Joh02b, Example D4.3.14]. In fact there is a stronger version of the conjecture, which assigns to every category $\mathcal{E}$ the free elementary $\infty$-topos $L(\mathcal{E})$.

**Conjecture 5.11.** For every $\infty$-category $\mathcal{E}$, $(\mathcal{L} \text{og}_{\infty}^P)_{/\mathcal{E}}$ has an initial object. Equivalently the inclusion $\mathcal{L} \text{og}_{\infty}^P \to \mathcal{C} \mathcal{a} t_\infty$ has a left adjoint.

We hope to return to these questions in future work. Finally, note on the other side that the inclusion functor does not preserve the initial object and so cannot have a right adjoint, meaning there cannot be an underlying elementary $\infty$-topos.

**SUFFICIENT VS. EXTERNAL UNIVERSES**

One challenging aspect of the definition of an elementary $\infty$-topos is the existence of sufficient universes as it involves the existence of various classifying objects. It would have been preferable to have a single universe that classifies all objects, however, this would result in size related paradoxes and hence is impossible.

On the other hand having a better understanding of universes can be of great benefit in particular as we want to eventually strictly the universe and relate it to homotopy type theory (as has been done for Grothendieck $\infty$-toposes by Shulman [Shu19]). Hence, we here want to introduce a way to obtain such a result using the ind-completion. This requires some aspects of accessible $\infty$-categories. By Section 1(23), there is an equivalence $(-)^\text{emp} : \mathcal{A} \mathcal{C} \mathcal{C} \mathcal{O} \mathcal{O}_\infty \to \mathcal{C} \mathcal{a} t_\infty^{\text{idempotent}}$, from accessible $\infty$-categories (Section 1(22)) to idempotent complete $\infty$-categories (Section 1(20)). We want to generalize this equivalence to all $\infty$-categories, which requires us to generalize accessible $\infty$-categories.
**Definition 6.1.** An accessible ∞-category with small objects is a pair \((A, C)\), where \(A\) is an accessible category and \(C^{idem} \simeq A^{cmp}\) where \(C^{idem}\) is the idempotent completion (Section 1(21)). A functor of accessible ∞-categories with small objects \(F : (A, C) \to (A', C')\) is an accessible functor \(F : A \to A'\) that restricts to a functor of small objects \(F : C \to C'\). We will denote the ∞-category of accessible ∞-categories with small objects by \(Acc_{sma}^{\infty}\).

Notice, there is a fully faithful functor \(Acc_{sma}^{\infty} \to Acc_{idem}^{\infty}\) that takes an accessible ∞-category \(A\) to the pair \((A, A^{sma})\). Similarly \(\hat{Cat}_{sma}\) is a full subcategory of \(\hat{Cat}_{idem}\). Moreover, let \((-)^{sma} : Acc_{idem}^{\infty} \to \hat{Cat}_{idem}\) be the functor that takes \((A, C)\) to the category of small objects \(C\) and a functor \(F : (A, C) \to (A', C')\) to its restriction \(F : C \to C'\). Finally, let \(\text{Ind} : \hat{Cat}_{idem} \to Acc_{sma}^{\infty}\) be the functor that takes \(C\) to \((\text{Ind}C, C)\). We now have the following result.

**Lemma 6.2.** The functors \((-)^{sma} : Acc_{idem}^{\infty} \to \hat{Cat}_{idem}\) and \(\text{Ind} : \hat{Cat}_{idem} \to Acc_{sma}^{\infty}\) are inverses.

**Proof.** On the one side \((-)^{sma} \circ \text{Ind}\) is the identity. On the other side, \(\text{Ind} \circ (-)^{sma}(A, C) \simeq (\text{Ind}C, C)\), where the last step follows from Section 1(23).

This new equivalence does in fact generalize the original equivalence \((\text{Ind}, (-)^{cmp})\) in the sense that we have the following commutative diagram

\[
\begin{array}{ccc}
\hat{Cat}_{idem} & \xrightarrow{\text{Ind}} & Acc_{idem}^{\infty} \\
\downarrow & & \downarrow \\
\hat{Cat}_{sma} & \xrightarrow{(-)^{sma}} & Acc_{sma}^{\infty}
\end{array}
\]

We now use this equivalence to give an alternative characterization of elementary ∞-toposes. For a given accessible ∞-category with small objects \((A, C)\) and object \(x\) in \(A\), we use the notation \(C/_{/x} \to C\) for the right fibration obtained via the pullback

\[
\begin{array}{ccc}
C/_{/x} & \xrightarrow{\pi} & A/_{/x} \\
\downarrow & & \downarrow \\
C & \xrightarrow{\pi} & A
\end{array}
\]

We want to impose various conditions on the objects and morphisms in \(Acc_{sma}^{\infty}\) in order to obtain a subcategory equivalent to \(\text{Log}_{\infty}\).

**Definition 6.3.** Let \((A, C)\) be an accessible ∞-category with small objects. Then it is an accessible ∞-category with small objects and (complete Segal) universe, if \(C\) is closed under finite limits and colimits and there is a (simplicial) object \(\mathcal{U} (\mathcal{U}_{\bullet})\) in \(A\) that represents \(O^{(all)}_{C} (O_{\mathcal{U}})\), meaning there is an equivalence of right (Cartesian) fibrations \(C/_{/U} \simeq O^{(all)}_{C} (C/_{/U}_{\bullet} \simeq O_{\mathcal{U}})\) over \(C\).

We have the following lemma regarding universes.

**Lemma 6.4.** Let \((A, C)\) be an accessible ∞-category with small objects and universe \(\mathcal{U}\). Then the corresponding universal fibration \(p_{\mathcal{U}} : U_{\bullet} \to \mathcal{U}\) is univalent in \(A\). Hence \(\mathcal{U}\) is a universe in \(A\) as defined in Definition 3.1.

**Proof.** By Section 1(25), we have an equivalence \(A/_{/U} \simeq \text{Ind}(C/_{/U})\). Moreover, by assumption \(C/_{/U} \simeq O^{(all)}_{C}\), hence \(A/_{/U} \simeq \text{Ind}(O^{(all)}_{C})\). Now \(\text{Ind}(O^{(all)}_{C})\) is the full subcategory of \(O^{(all)}_{A}\) with objects morphisms \(f\) of the form \(\text{colim}_{i \in I} f_{i}\), where \(I \to O^{(all)}_{C}\). Hence, \(p_{\mathcal{U}}\) is univalent by Section 1(17).

To better comprehend the role of the universe it is instructive to understand the essential image of \(A/_{/U} \to O^{(all)}_{A}\).

**Lemma 6.5.** Let \(C\) be closed under finite limits. An object \(f : Y \to X\) in \(O^{(all)}_{A}\) is in the essential image of \(A/_{/U}\) if and only if for every \(Z\) in \(C\) and morphism \(Z \to X\), the pullback \(Z \times_{X} Y\) is in \(C\). In particular, a morphism \(Y \to 1\) is in the essential image if and only if \(Y\) is in \(C\).
Proof. If \( f : Y \to X \) is in \( \mathcal{A}/\mathcal{U} \), then \( f \simeq \colim_I f_i : Y_i \to X_i \). Let \( Z \) be in \( \mathcal{C} \), then \( Z \) is compact and hence every map \( Z \to X \) is of the form \( Z \to X_i \to X \) for some \( i \in I \). Hence \( Z \times X \simeq Z \times_{X_i} Y_i \) which is in \( \mathcal{C} \), as \( \mathcal{C} \) is closed under finite limits.

On the other side, if \( f : Y \to X \) satisfies the condition of the lemma, then
\[
Y \simeq X \times X Y \simeq \colim_I (X_i \times X Y),
\]
where the last step follows from the fact that filtered colimits commute with finite limits (Section 1(9)). Hence, \( f \simeq \colim_I \pi_1 : X_i \times X Y \to X_i \). \( \square \)

Remark 6.6. We should think of the morphisms classified by \( \ul{u} \) in \( \mathcal{A} \) as “small morphisms”. In [ABSS14] those are chosen axiomatically as part of the definition of the category of classes.

We can use Lemma 6.4 to define functors.

Definition 6.7. A functor \( F : (\mathcal{A}, \mathcal{C}) \to (\mathcal{A}', \mathcal{C}') \) of accessible \( \infty \)-categories with small objects and universes is a functor of accessible \( \infty \)-categories such that \( FU \) is a universe, \( \mathcal{U} \leq \mathcal{U}' \) in \( \mathcal{U} \text{Univ}_{\mathcal{A}'} \) and \( F \), if restricted to \( \mathcal{C} \), preserves finite limits and colimits. We denote the subcategory of \( \text{Acc}_{\mathcal{C}'}, \mathcal{C} \) with objects accessible \( \infty \)-categories with small objects and universe and morphism their functors by \( (\text{Acc}_{\mathcal{C}'}, \mathcal{C})) \).

Notice the functors in \( (\text{Acc}_{\mathcal{C}'}, \mathcal{C})) \) permit following alternative characterization, which follows immediately from Lemma 3.2.

Lemma 6.8. Let \( F : (\mathcal{A}, \mathcal{C}) \to (\mathcal{A}', \mathcal{C}') \) be a functor of accessible \( \infty \)-category with small objects such that \( F \) restricted to \( \mathcal{C} \) preserves finite limits and colimits. Then \( F \) is a functor of accessible \( \infty \)-categories with small objects and universes if and only if \( FU \to FU \) is univalent and \( \mathcal{U} \leq \mathcal{U}' \).

We want to restrict further. For that we need the following lemma.

Lemma 6.9. Let \( (\mathcal{A}, \mathcal{C}) \) be an accessible \( \infty \)-category with small objects, such that \( \mathcal{C} \) has finite coproducts. Then the following are equivalent:

1. \( (\mathcal{A}, \mathcal{C}) \) has a complete Segal universe.
2. \( (\mathcal{A}, \mathcal{C}) \) has a universe and \( \mathcal{C} \) is locally Cartesian closed.
3. \( \mathcal{C} \) has sufficient complete Segal universes.
4. \( \mathcal{C} \) has sufficient universes and is locally Cartesian closed.

Proof. (1) \( \Rightarrow \) (3) If \( \mathcal{C} \) has sufficient complete Segal universes, and finite coproducts, then \( \mathcal{C}_{\text{SlUniv}} \) is a directed poset (Lemma 3.9) and so it has a colimit \( \mathcal{C}_{/\mathcal{U}'} = \colim (\mathcal{U}_{\mathcal{U}})_{/\mathcal{U}'} \) and we have
\[
\mathcal{C}_{/\mathcal{U}'} \simeq \colim (\mathcal{U}_{\mathcal{U}})_{/\mathcal{U}'} \simeq \mathcal{C}_{\mathcal{U}}.
\]
where the last step follows from Lemma 3.22, proving that \( \mathcal{U}_{\mathcal{U}} \) is a complete Segal universe in \( (\text{Ind} \mathcal{C}, \mathcal{C}) \).

(3) \( \Rightarrow \) (1) If \( (\text{Ind} \mathcal{C}, \mathcal{C}) \) has a complete Segal universe \( \mathcal{U}_{\mathcal{U}} \), then by definition there exists an equivalence \( \colim_I (\mathcal{U}_{\mathcal{U}})_{/\mathcal{U}} \simeq \mathcal{U}_{\mathcal{U}} \), where \( I \) is filtered. By construction of the filtered colimit the map \( (\mathcal{U}_{\mathcal{U}})_{/\mathcal{U}} \to \mathcal{U}_{\mathcal{U}} \) is mono and so, by Section 1(18) and Lemma 3.2, \( (\mathcal{U}_{\mathcal{U}})_{/\mathcal{U}} \) is a complete Segal universe. Now, let \( f : y \to x \) be an arbitrary morphism in \( \mathcal{C} \), then there exists \( x \to \mathcal{U}_{\mathcal{U}} \) that classifies \( f \). The fact that \( x \) is compact implies that there exists a factorization \( x \to (\mathcal{U}_{\mathcal{U}})_{0} \to \mathcal{U}_{\mathcal{U}} \) (Section 1(23)). Hence, the complete Segal universe \( (\mathcal{U}_{\mathcal{U}})_{/\mathcal{U}} \) classifies \( f \), proving \( \mathcal{C} \) has sufficient complete Segal universes.

(2) \( \Leftrightarrow \) (4) Follows from the same argument as in the previous two steps.

(3) \( \Leftrightarrow \) (4) Follows from Theorem 3.27. \( \square \)

This now motivates the following definition.

Definition 6.10. An accessible elementary model \( (\mathcal{A}, \mathcal{E}) \) is an accessible \( \infty \)-category with small objects that satisfies the equivalent conditions of Lemma 6.9.

We now move on to the appropriate notion of functors.

Lemma 6.11. Let \( (\mathcal{A}, \mathcal{E}) \) and \( (\mathcal{A}', \mathcal{E}') \) be two accessible elementary models. Let \( F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}') \) be a functor of accessible \( \infty \)-categories with small objects such that \( F : \mathcal{E} \to \mathcal{E}' \) preserves finite limits and colimits. Then the following are equivalent:
(1) $F$ takes the complete Segal universe $\mathcal{U}_\bullet$ to the complete Segal universe $F(\mathcal{U}_\bullet) \leq \mathcal{U}'_\bullet$. 
(2) $F$ takes the universe $\mathcal{U}$ to the universe $\text{Fl}(\mathcal{U}) \leq \mathcal{U}'$ and $F$ is locally Cartesian closed. 
(3) $F : \mathcal{E} \to \mathcal{E}'$ induced a map of complete Segal universes $\mathcal{E} \text{niv}_F : \mathcal{E} \text{niv}_F \to \mathcal{E}' \text{niv}_F$. 
(4) $F : \mathcal{E} \to \mathcal{E}'$ induced a map of universes $\mathcal{U} \text{niv}_F : \mathcal{U} \text{niv}_F \to \mathcal{U}' \text{niv}_F$ and $F$ is locally Cartesian closed.

Proof. (1) $\Rightarrow$ (3) Let $(\mathcal{U}_\bullet)_i$ be a complete Segal universe in $\mathcal{E}$. By definition of the complete Segal universe $\mathcal{U}_\bullet$, there is a map $(\mathcal{U}_\bullet)_i \to \mathcal{U}_\bullet$, which by Section 1(18), it mono. As $F$ preserves finite limits it preserves monos and so we have 

$$F((\mathcal{U}_\bullet)_i) \hookrightarrow F(\mathcal{U}_\bullet) \hookrightarrow \mathcal{U}'_\bullet,$$

which, again by Section 1(18), proves that $F$ induces a functor $\mathcal{E} \text{niv}_F : \mathcal{E} \text{niv}_F \to \mathcal{E}' \text{niv}_F$.

(3) $\Rightarrow$ (1) Let $\mathcal{U}_\bullet$ be the complete Segal universe in $\mathcal{A}$ and notice $\mathcal{U}_\bullet \simeq \text{colim}_{i} (\mathcal{U}_\bullet)_i$, where $I$ is filtered. As $F$ is accessible it preserves filtered colimits and so $F(\mathcal{U}_\bullet) \simeq \text{colim}_i F((\mathcal{U}_\bullet)_i)$. As we have $F((\mathcal{U}_\bullet)_i) \leq \mathcal{U}'_\bullet$, we have $F(\mathcal{U}_\bullet) \leq \mathcal{U}'_\bullet$ by Section 1(9), finishing the proof.

(2) $\Leftrightarrow$ (4) This follow from an analogous argument to the previous two steps.

(3) $\Leftrightarrow$ (4) Follows from Proposition 5.3. \qed

We can now give the desired definition.

**Definition 6.12.** A functor $F : (\mathcal{A}, \mathcal{C}) \to (\mathcal{A}', \mathcal{C}')$ of accessible elementary models is a functor of accessible $\infty$-categories that satisfies the equivalent conditions of Lemma 6.11. We denote the subcategory of $\text{Acc}^{\text{sma}}_{\infty}$ with objects accessible elementary models and their functors by $\text{Acc}^{\text{lem}}_\infty$.

**Definition 6.13.** An accessible $\infty$-category with small objects $(\mathcal{A}, \mathcal{C})$ has a subobject classifier if there exists an object $\Omega$ that represents the functor $\text{Sub}_{\mathcal{C}}$, as defined in Definition 2.5, meaning there is a natural isomorphism $\text{Hom}_{\mathcal{A}}(c, \Omega) \cong \text{Sub}_{\mathcal{C}}(c)$ for all objects $c$ in $\mathcal{C}$.

**Lemma 6.14.** Let $(\mathcal{A}, \mathcal{E})$ be an accessible $\infty$-category with small locally Cartesian closed small objects that has a universe $\mathcal{U}$. Then the inclusion functor $\tau_{-1} : \mathcal{E} \to \mathcal{E}$ has a left adjoint $\tau_{-1} : \mathcal{E} \to \tau_{-1} \mathcal{E}$.

Proof. $\mathcal{E}$ has finite limits and colimits, is locally Cartesian closed, and satisfies descent, where the last condition follows from Theorem 3.27. Hence, by [Ras21b, Theorem 4.1.2], $\mathcal{E}$ has a natural number object and so in particular a $(-1)$-truncation [Ras18, Theorem 2.29]. \qed

**Lemma 6.15.** Let $(\mathcal{A}, \mathcal{E})$ be an accessible $\infty$-category with small objects that has a universe $\mathcal{U}$. Then $(\mathcal{A}, \mathcal{E})$ has a subobject classifier.

Proof. By Lemma 6.14 there is a functor $\tau_{-1} : \mathcal{O}_{\mathcal{E}}^{(\text{all})} \to \mathcal{O}_{\mathcal{E}}^{(\text{all})}$ over $\mathcal{E}$. We can extend it to a functor $\text{Ind}\tau_{-1} : \text{Ind}\mathcal{O}_{\mathcal{E}}^{(\text{all})} \to \text{Ind}\mathcal{O}_{\mathcal{E}}^{(\text{all})}$. Combining Definition 6.3 and Section 1(25), we have $\text{Ind}\mathcal{O}_{\mathcal{E}}^{(\text{all})} \simeq \mathcal{E}_{/\mathcal{U}} \simeq A_{/\mathcal{U}}$ and so by the Yoneda lemma for right fibrations (Section 1(11)) we get a morphism $\tau_{-1} : \mathcal{U} \to \mathcal{U}$ in $\mathcal{A}$. Now, $\mathcal{E}$ has finite limits and colimits and so, by Section 1(24), $\mathcal{E} \simeq \mathcal{A}$ is presentable and has a $(-1)$-truncation functor $\tau_{-1} : \mathcal{A} \to \tau_{-1} \mathcal{A}$. Applying it to $\tau_{-1} : \mathcal{U} \to \mathcal{U}$ gives us a factorization $\mathcal{U} \hookrightarrow \Omega \to \mathcal{U}$, such that $\text{Map}(X, \Omega) \simeq \tau_{-1}(\mathcal{E}_{/X}) \cong \text{Sub}_{\mathcal{E}}(X)$, proving $\Omega$ is a subobject classifier. \qed

**Definition 6.16.** An accessible $\infty$-category with small objects $(\mathcal{A}, \mathcal{C})$ with universe $\mathcal{U}$ satisfies propositional resizing if the object $\Omega$ is small. Let $(\text{Acc}^{\text{sma}}_{\infty})_{\mathcal{U} \text{Prop}}$ be the full subcategory of $(\text{Acc}^{\text{sma}}_{\infty})_{\mathcal{U}}$ consisting of objects that satisfy propositional resizing and functors in $(\text{Acc}^{\text{sma}}_{\infty})_{\mathcal{U}}$ that preserve the small subobject classifier. Define $(\text{Acc}^{\text{lem}}_{\infty})_{\mathcal{U} \text{Prop}}$ similarly.

We are now ready to state and prove the main theorem.

**Theorem 6.17.** There is a diagram of $\infty$-categories
where the vertical functors are faithful and the horizontal functors are equivalences. Here the $\infty$-categories on the left hand side were defined in Notation 5.8.

**Proof.** By Lemma 6.2 we have an equivalence $\text{Ind} : \hat{\text{Cat}}_\infty \to \text{Acc}_{\infty}^\sm$. We need to prove the equivalence restricts appropriately and for that it suffices to prove that $\text{Ind}$ and its inverse $(-)^\sm$ preserve the additional structure.

1. **(Co)limits and Cartesian Closure:** First of all if $\mathcal{C}$ has finite (co)limits or is locally Cartesian closed, then the accessible $\infty$-category with small objects $(\text{Ind}\mathcal{C}, \mathcal{C})$ also has the same property for small objects and the same applies to functors.

2. **Sufficient Universes vs. Universe:** By Lemma 6.9, $(\mathcal{A}, \mathcal{E})$ has a universe if and only if $\mathcal{C}$ has sufficient universes. Moreover, by the analogous argument to the one in Lemma 6.11 (where we ignore the additional locally Cartesian closed assumption), a functor $F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}')$ preserves the universe if and only if $F : \mathcal{E} \to \mathcal{E}'$ restricts to a map $\lniv_F : \lniv_{\mathcal{E}} \to \lniv_{\mathcal{E}'}$.

3. **Subobject classifier vs. Propositional Resizing:** By Definition 6.16, $\mathcal{C}$ has a subobject classifier if and only if $(\mathcal{A}, \mathcal{E})$ has a subobject classifier that satisfies propositional resizing. Moreover, it is evident that $F : \mathcal{E} \to \mathcal{E}'$ preserves the subobject classifier if and only if $F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}')$ preserves the small subobject classifier, as it is the same object.

As we have confirmed that all properties are preserved appropriately, our proof is finished.

**Remark 6.18.** The equivalence relates elementary $\infty$-toposes with universes and elementary models. One additional question is whether we can give a similar result for elementary $\infty$-toposes with $\mathcal{P}$-closed universes. This would require using a different approach as the one given in the proof of Theorem 6.17. Indeed, the proof uses the fact that the universe $\mathcal{U}$ in $(\mathcal{A}, \mathcal{E})$ can be recovered as a filtered colimit of universes $\mathcal{U}_i$ in $\mathcal{E}$, however, we cannot choose those universes to be closed under $\mathcal{P}$.

**Remark 6.19.** The definition of $\infty$-logical functor given in Definition 5.1 involves an intricate condition because of the existence of sufficient universes. On the other hand, the definition of functors of accessible elementary models is straightforward and very much modeled on the definition of logical functors of elementary toposes. The fact that these two notions coincide (as proven in Theorem 6.17) gives a strong indication that the definition of $\infty$-logical functor is in fact appropriately chosen.

**Elementary $(n,1)$-Topos Theory**

Up until now our work has focused on generalizing elementary 1-toposes (Definition 2.2) to elementary $\infty$-toposes in a way that appropriately generalizes Grothendieck $\infty$-toposes. However, there similarly are Grothendieck $(n,1)$-toposes for all $0 \leq n \leq \infty$, which sit in between the two boundary cases and have been studied extensively in [Lur09, Section 6.4]. In this section we generalize the results from Section 4, Section 5 and Section 6, from $\infty$-categories to $(n,1)$-categories (Section 1(8)), in particular introducing elementary $(n,1)$-topos theory.

**Note on Terminology.** It is an unfortunate fact of higher categorical terminology that an $\infty$-category by now refers to an $(\infty, 1)$-category, whereas an $n$-category usually refers to an $(n, n)$-category creating possible confusion. In order to avoid such confusion, we will consistently use the notation $(n,1)$-category, when $n < \infty$. 

\[ (\hat{\text{Cat}}_\infty)_{\mathcal{U}} \cong \text{Ind} \rightarrow (\text{Acc}_{\infty}^{\sm})_{\mathcal{U}} \]

\[ (\hat{\text{Cat}}_\infty)_{\mathcal{C} \mathcal{U}} \cong \text{Ind} \rightarrow \text{Acc}\text{Elem}_\infty \]

\[ (\hat{\text{Cat}}_\infty)_{\mathcal{U}, \text{Prop}} \cong \text{Ind} \rightarrow (\text{Acc}_{\infty}^{\sm})_{\mathcal{U}, \text{Prop}} \]

\[ \text{Log}_\infty \cong \text{Ind} \rightarrow (\text{Acc}\text{Elem}_\infty)_{\text{Prop}} \]
We want to start with an appropriate notion of universe for \((n,1)\)-categories. As every object in an \((n,1)\)-category is \((n-1)\)-truncated, for a given morphism \(p : E \to B\), the space \(\text{Map}_c(-,B)\) is \((n-1)\)-truncated and so the induced map (Section 1(11)) \(E/B \to \Omega^\text{all}_c\) can only be fully faithful if it takes value in the full subcategory \(\tau_{n-2}\Omega^\text{all}_c\) (Section 1(16)). This suggests the following notion of universe (Definition 3.1) in the setting of \((n,1)\)-categories.

**Definition 7.1.** Let \(\mathcal{E}\) be a finitely complete \((n,1)\)-category and \(\mathcal{P}\) a universal property. A \(\mathcal{P}\)-closed \((n-2)\)-truncated universe is a pair \((\mathcal{U}, i)\), where \(\mathcal{U}\) is an object in \(\mathcal{E}\) and \(i : E/\mathcal{U} \to \tau_{n-2}\Omega^\text{all}_c\) is a fully faithful functor, such that the full subcategory of \(\tau_{n-2}\Omega^\text{all}_c\) with objects in the essential image of \(i\) satisfy \(\mathcal{P}\) (Definition 3.11).

Similar to Definition 3.3 we denote the poset of \(\mathcal{P}\)-closed \((n-2)\)-truncated universes by \(\text{Univ}_{\mathcal{E}}^{\mathcal{P}, n-2}\) and, following Definition 3.4, a finitely complete \((n,1)\)-category has sufficient universes if every \((n-2)\)-truncated morphism in \(\mathcal{E}\) is classified by a universe. We now have the following appropriate result corresponding to Theorem 3.27.

**Corollary 7.2.** Let \(\mathcal{E}\) be a finitely complete locally Cartesian closed \((n,1)\)-category with finite coproducts closed under the (possibly empty) universal property \(\mathcal{P}\). If \(\mathcal{E}\) has sufficient \((n-2)\)-truncated universes closed under \(\mathcal{P}\) then colimits in \(\mathcal{E}\) are universal and \((n-2)\)-truncated morphisms in \(\mathcal{E}\) are local. The opposite holds if \(\mathcal{E}\) is presentable and \(\kappa\)-compact morphisms are closed under \(\mathcal{P}\) for \(\kappa\) large enough.

**Proof.** If \(\mathcal{E}\) has sufficient universes closed under \(\mathcal{P}\), then the proof follows using the same argument as in Theorem 3.27. On the other side, assuming \(\mathcal{E}\) is presentable the other direction follows from [Lur09, Theorem 6.4.1.5], as reviewed in Theorem 2.14.

We now have all the necessary ingredients to define elementary \((n,1)\)-toposes.

**Definition 7.3.** Let \(\mathcal{P}\) be a universal property, and \(n > 0\). A finitely (co)complete locally Cartesian closed \((n,1)\)-category is an elementary \((n,1)\)-topos if it has a subobject classifier and sufficient \(\mathcal{P}\)-closed \((n-2)\)-truncated universes.

This definition is in fact an appropriate generalization of the previous cases.

**Example 7.4.** If \(n = 1\), then the subobject classifier itself is precisely a universe for all \((-1)\)-truncated morphisms. Hence, the existence of sufficient universes becomes vacuous and elementary \((1,1)\)-toposes in the sense of Definition 7.3 directly correspond to elementary toposes, as defined in Definition 2.2.

**Example 7.5.** If \(n = \infty\), then it is a finitely (co)complete locally Cartesian closed \(\infty\)-category with subobject classifier and sufficient universes for \((n-2)\)-truncated morphisms, which corresponds to all morphisms as \(n = \infty\) and hence recovers the definition of an elementary \(\infty\)-topos, by Proposition 4.2.

**Remark 7.6.** Unlike in the case of \(\infty\)-toposes, we cannot recover the Cartesian closure from the universes, as morphisms that are not \((n-2)\)-truncated are not classified.

We also have an analogous result to Corollary 4.6.

**Corollary 7.7.** Let \(\mathcal{E}\) be an elementary \((n,1)\)-topos. An \((n-2)\)-truncated map \(f : x \to y\) is univalent if and only if any classifying map \(\chi_f : y \to \mathcal{U}\) is mono.

Notice for \(n = 1\) Corollary 7.7 recovers the already well-known fact that univalent monos are classified by the subobject classifier, as discussed in Remark 4.7. There is also a fundamental theorem of elementary \((n,1)\)-toposes, with the same proof as in Theorem 4.9.

**Corollary 7.8.** Let \(\mathcal{P}\) be a universal property and let \(\mathcal{E}\) be a \(\mathcal{P}\)-closed elementary \((n,1)\)-topos and \(x\) an object. Then \(\mathcal{E}/x\) is also a \(\mathcal{P}\)-closed elementary \((n,1)\)-topos.

We move on to the \((n,1)\)-categorical analogue of Section 5 and study \((n,1)\)-logical functors.

**Definition 7.9.** A finitely (co)continuous locally Cartesian closed functor \(F : \mathcal{E}_1 \to \mathcal{E}_2\) of elementary \((n,1)\)-toposes is an \((n,1)\)-logical functor if \(0_F : O_{\mathcal{E}_1} \to O_{\mathcal{E}_2}\) restricts to a map of posets \(\text{Univ}_F^{\mathcal{P}, n-2} : \text{Univ}_{\mathcal{E}_1}^{\mathcal{P}, n-2} \to \text{Univ}_{\mathcal{E}_2}^{\mathcal{P}, n-2}\), which preserves the subobject classifier.
**Example 7.10.** In the case \( n = 1 \), this precisely reduces to the case of logical functors of elementary toposes \( \text{Definition 2.2} \), whereas in the case of \( n = \infty \) it coincides with \( \infty \)-logical functors, by \( \text{Proposition 5.3} \).

Similarly we denote the \( \infty \)-category with objects \( \mathcal{P} \)-closed elementary \((n,1)\)-toposes and morphisms \((n,1)\)-logical functors by \( \text{Log}^\mathcal{P}_{(n,1)} \). We now have the analogous result to \( \text{Theorem 5.9} \) with the same proof.

**Corollary 7.11.** The inclusion functor \( \text{Log}^\mathcal{P}_{(n,1)} \to \hat{\text{Cat}}_{(n,1)} \) preserves small limits and filtered colimits.

Finally, we want to adapt the results in \( \text{Section 6} \). First of all, let \( \hat{\text{Cat}}_{(n,1)} \to \hat{\text{Cat}}_{(n,1), \text{Prop}} \) be defined analogous to \( \hat{\text{Cat}}_{\infty} \to \hat{\text{Cat}}_{\infty, \text{Prop}} \) in \( \text{Notation 5.8} \), respectively.

On the other hand, following \( \text{Section 1} \), if \( \mathcal{E} \) is a finitely complete \((n,1)\)-category, where \( n < \infty \), then \( \mathcal{E} \) is idempotent complete. Hence, we do not need to use the notion of accessible \( \infty \)-category with small objects \( \text{Definition 6.1} \). Rather we have the following.

**Definition 7.12.** Let \( \text{Acc}^\text{cmp}_{(n,1)} \) be the full subcategory of \( \text{Acc}_{\infty} \), consisting of accessible \( \infty \)-categories such that \( \text{Acc}^\text{cmp}_{(n,1)} \) is an \((n,1)\)-category and notice the equivalence in \( \text{Section 1(23)} \) restricts to an equivalence \( \text{Ind} : \hat{\text{Cat}}_{(n,1)} \to \text{Acc}^\text{cmp}_{(n,1)} \).

Let \( \text{Acc}^\text{prop}_{(n,1)} \) be defined as the full subcategories of \( \text{Acc}^\text{sm}_{\infty} \) \( \text{Def. 6.7} \), \( \text{Acc}^\text{sm}_{\infty} \) \( \text{Def. 6.12} \), \( \text{Acc}^\text{prop}_{\infty} \) \( \text{Def. 6.16} \), respectively, with objects in \( \text{Acc}^\text{prop}_{(n,1)} \). Now, we have the following result.

**Corollary 7.13.** \( \text{Let } n < \infty. \) There is a diagram of \( \infty \)-categories

\[
\begin{array}{ccc}
\hat{\text{Cat}}_{(n,1)} & \xrightarrow{\cong} & \text{Acc}_{(n,1)} \\
\downarrow \cong & \text{Ind} & \downarrow \text{Ind} \\
\text{Acc}^\text{prop}_{(n,1)} & \cong & \text{Acc}^\text{prop}_{(n,1)} \\
\downarrow \cong & \text{Ind} & \downarrow \text{Ind} \\
\text{Log}^\mathcal{P}_{(n,1)} & \cong & \text{Log}^\mathcal{P}_{(n,1)}
\end{array}
\]

where the vertical functors are faithful and the horizontal functors are equivalences.

We end this section with a discussion how the various truncation levels compare to each other. It is well established that for all \( m \leq n \leq \infty \), every \((m,1)\)-category is an \((n,1)\)-category. Now that we have established a notion of elementary \((m,1)\)-topos, we want to understand whether the various dimensions are related similarly.

**Proposition 7.14.** \( \text{Let } -1 \leq n < m \leq \infty \) and \( \mathcal{E} \) an \((n,1)\)-category. Then \( \mathcal{E} \) is not an elementary \((m,1)\)-topos.

**Proof.** By \( \text{Definition 7.1} \), a universe for \((m-2)\)-truncated objects is necessarily \((m-1)\)-truncated, but not \((m-2)\)-truncated. Hence, if \( n < m \), then every object in \( \mathcal{E} \) is \((n-1)\)-truncated and so also \((m-2)\)-truncated and so cannot be a universe, meaning \( \mathcal{E} \) cannot be an elementary \((m,1)\)-topos.

Hence, elementary \((n,1)\)-toposes are sensitive to dimension. However, these notions are still related.

**Theorem 7.15.** \( \text{Let } \mathcal{P} \text{ be the universal property consisting of finite limits, finite colimits and local Cartesian closure. Then for all } n \geq 0 \) we have a functor \( \tau_n : \text{Log}^\mathcal{P}_{(n+1,1)} \to \text{Log}^\mathcal{P}_{(n,1)} \).

**Proof.** We need to prove that for every \( \mathcal{P} \)-closed elementary \( \infty \)-topos \( \mathcal{E} \), \( \tau_n \mathcal{E} \) is a \( \mathcal{P} \)-closed elementary \((n+1,1)\)-topos and for every \( \infty \)-logical functor \( F : E_1 \to E_2 \), \( \tau_n F : \tau_n E_1 \to \tau_n E_2 \) is an \((n+1,1)\)-logical functor. By \( \text{Theorem 4.12} \), there is an \( n \)-truncation functor \( \tau_n : \mathcal{E} \to \tau_n \mathcal{E} \), which is left adjoint to the inclusion \( \tau_n \mathcal{E} \to \mathcal{E} \). This implies that \( \tau_n(\mathcal{E}) \) is closed under finite limits and colimits. Moreover, by [Ras21e, Proposition 2.11], \( \tau_n \) is a locally Cartesian closed localization, which implies that \( \tau_n \mathcal{E} \) is locally Cartesian closed as well.
Next, the subobject classifier $\Omega$ is 0-truncated and so in particular an object in $\tau_0(\mathcal{E})$. Finally, by Section 1(11), the truncation functor $\tau_0^{-1} : \mathcal{O}_0^{(\text{all})} \to \mathcal{O}_0^{(\text{all})}$ induces a map of universes $\tau_0^{-1} : \mathcal{U} \to \mathcal{U}$ and applying epimono factorization (which exists by Theorem 4.12 for $n = -1$) gives us a factorization $\mathcal{U} \to \mathcal{U}_{\leq n-1} \to \mathcal{U}$. By definition, $\operatorname{Map}_\mathcal{E}(x, \mathcal{U}_{\leq n-1}) \simeq \tau_{n-1}(\mathcal{E}_x)$, which proves that $\mathcal{U}_{\leq n-1}$ is a universe for $(n-1)$-truncated objects in $\tau_n\mathcal{E}$. As $\mathcal{E}$ has sufficient universes and this process constructs a $(n-1)$-truncated universe out of every universe, this proves that $\tau_n\mathcal{E}$ also has sufficient $(n-1)$-truncated universes. Hence $\tau_n\mathcal{E}$ is an elementary $(n+1,1)$-topos.

Now, let $F : \mathcal{E}_1 \to \mathcal{E}_2$ be an $\mathcal{O}$-logical functor. Then $\tau_n F : \tau_n\mathcal{E}_1 \to \tau_n\mathcal{E}_2$ preserves finite limits and colimits and, again by [Ras21e, Proposition 2.11], the local Cartesian closure. Finally, as $F(\mathcal{U})$ is a universe and $F$ respects truncation levels, $F(\mathcal{U}_{\leq n-1})$ is again a universe for $(n-1)$-truncated objects, proving $\tau_nF$ is $(n+1,1)$-logical.

Remark 7.16. The result from Theorem 7.15 would also hold if $\mathcal{P}$ is a universal property that includes finite limits, finite colimits and local Cartesian closure and such that for all $n \geq 0$, if $\mathcal{E}$ satisfies $\mathcal{P}$, then $\tau_n\mathcal{E}$ also satisfies $\mathcal{P}$.

This result has some fascinating implications and possibilities for generalizations. By Theorem 4.12, for a given elementary $\infty$-topos $\mathcal{E}$, with natural number $n : 1 \to \mathbb{N}$ we have an $n$-truncation $\tau_n : \mathcal{E} \to \tau_n\mathcal{E}$. We would like to consider $\tau_n\mathcal{E}$ an example of an elementary $(n+1,1)$-topos. However, this is not possible if the natural number is non-standard. This suggest the following interesting conjecture.

Conjecture 7.17. Let $\mathcal{E}$ be an elementary $\infty$-topos. Let $\mathcal{Cat}_{\mathcal{E}}$ denote the $\infty$-category of $\mathcal{E}$-enriched $\infty$-categories (as defined by Gepner and Haugseng in [GH15]) and notice that $\mathcal{E}$ is itself $\mathcal{E}$ enriched [GH15, Corollary 7.4.10]. Fix an internal natural number $n$ in the natural number object $\mathbb{N}$ of $\mathcal{E}$. There is a notion of elementary $(n+1,1)$-topos and applying the $n$-truncation functor $\tau_n : \mathcal{Cat}_{\mathcal{E}} \to \mathcal{Cat}_{\mathcal{E}_{\leq n}}$ gives us an example thereof.

Notice Theorem 7.15 has an inverse in the context of Grothendieck $\infty$-toposes, meaning every Grothendieck $(n+1,1)$-topos is equivalent to the full subcategory of $n$-truncated objects in a Grothendieck $\infty$-topos, as explained in Theorem 2.14. At this level of generality, this result does not generalize to elementary $(n,1)$-toposes.

Example 7.18. The category of finite sets is an elementary topos, but is not equivalent to the category of 0-truncated objects of an elementary $\infty$-topos. Indeed, by Theorem 4.11, every elementary $\infty$-topos has a natural number object, which is by definition 0-truncated and so an element in the sub-category of 0-truncated objects.

What prevents the existence of a lift is the lack of universes, which by definition exist if $n > 1$, suggesting the following conjecture.

Conjecture 7.19. Let $n > 1$. For every elementary $(n+1,1)$-topos $\mathcal{E}$, there exists an elementary $\infty$-topos $\hat{\mathcal{E}}$, such that $\tau_n\hat{\mathcal{E}} \simeq \mathcal{E}$.

Examples and Non-Examples

In this section we cover various examples and non-examples of elementary $(n,1)$-toposes.

Grothendieck $(n,1)$-Toposes: Let us start with one key example, for which we have the following result.

Proposition 8.1. Let $n \leq \infty$, let $\mathcal{P}$ be a universal property and $\mathcal{P}$ a presentable $(n,1)$-category that satisfies $\mathcal{P}$. If there exists sufficient cardinals $\kappa$, such that the full subcategory of $\kappa$-compact objects is closed under $\mathcal{P}$, then $\mathcal{P}$ is an elementary $(n,1)$-topos with $\mathcal{P}$-closed universes if and only if it is a Grothendieck $(n,1)$-topos.

Proof. If $\mathcal{P}$ is a Grothendieck $(n,1)$-topos, then, by Theorem 2.14, it is locally Cartesian closed and has sufficient universes $\mathcal{U}^\kappa$ for $\kappa$-small $(n-2)$-truncated objects and a subobject classifier (Remark 2.15), and hence an elementary $(n,1)$-topos, by Corollary 7.2. On the other side, if $\mathcal{P}$ is an elementary $(n,1)$-topos, then it is locally Cartesian closed and has sufficient universes for $(n-2)$-truncated objects and so a Grothendieck $(n,1)$-topos, by Theorem 2.14. 

Example 8.2. Assuming sufficiently large cardinals, $\mathcal{S}_{\leq n-1}$ is an elementary $(n,1)$-topos.
Finite Spaces: As the category of finite sets is an elementary topos, we might expect that finite spaces are an elementary $\infty$-topos. However, that is not the case.

Example 8.3. The category of $\omega$-compact spaces is not an elementary $\infty$-topos. In fact it is not even closed under finite limits and colimits. Indeed, if it has finite limits it has the terminal object $\ast$. Moreover, if it has finite colimits it has the suspension of $\ast \coprod \ast$, which is $S^1$, then finite limits imply it has $\Omega S^1 \cong \mathbb{Z}$, which is certainly not $\omega$-compact.

1-Inaccessible Spaces: There is a way to correct the previous example that is due to Lo Monaco [LM21]. Recall that a cardinal $\kappa_{\text{acc}}$ is 1-inaccessible if it is weakly inaccessible and for every $\kappa < \kappa_{\text{acc}}$, there exists a weakly inaccessible cardinal $\kappa < \kappa_1 < \kappa_{\text{acc}}$. We now have the following result as stated in [LM21, Proposition 7.3].

Example 8.4. Let $\kappa_{\text{acc}}$ be a 1-inaccessible cardinal. The $\infty$-category of $\kappa_{\text{acc}}$-small spaces $S^{\kappa_{\text{acc}}}$ is an elementary $\infty$-topos, using the same argument as in Example 4.8. Indeed, for every weakly inaccessible cardinal $\kappa < \kappa_{\text{acc}}$, we have a complete Segal universe $\mathbb{E}S/S^{\kappa}$ (Section 1(5)).

We can in fact generalize the result by Lo Monaco in the expected manner.

Example 8.5. Let $\kappa_{\text{acc}}$ be a 1-inaccessible cardinal. The $(n+1,1)$-category of $\kappa_{\text{acc}}$-small $n$-truncated spaces $(S_{\leq n})^{\kappa_{\text{acc}}}$ is an elementary $(n+1,1)$-topos. Indeed, as $\kappa_{\text{acc}}$ is weakly inaccessible, it is closed under finite limits, finite colimits and locally Cartesian closed. Moreover, $\{0,1\}$ is evidently $\kappa_{\text{acc}}$-small and so $(S_{\leq n})^{\kappa_{\text{acc}}}$ has a subobject classifier. Finally, any weakly inaccessible cardinal $\kappa < \kappa_1$, let $((S_{\leq n-1})^{\kappa_{\text{acc}}})$, be the maximal subgroupoid (Section 1(7)) of the $\infty$-category of $\kappa$-small $n-2$-truncated spaces. Then by the same argument as the one given in Example 4.8 we have an equivalence

$$\text{Map}(X, ((S_{\leq n-1})^{\kappa_{\text{acc}}}) \simeq (\pi_{\leq n-1}/X)^{\kappa_{\text{acc}}}$$

which proves that $((S_{\leq n-1})^{\kappa_{\text{acc}}})^{\kappa_{\text{acc}}}$ is a universe for $(n-1)$-truncated spaces.

Notice, the inclusion functor $(S_{\leq n})^{\kappa_{\text{acc}}} \to S_{\leq n}$ is the identity functor on all objects, immediately giving us the following result.

Corollary 8.6. The inclusion functor $(S_{\leq n})^{\kappa_{\text{acc}}} \to S_{\leq n}$ is $(n+1,1)$-logical.

We can think of this result as a generalization of the fact that the inclusion from finite sets to sets is logical.

$\pi$-Finite Spaces: There is an alternative notion of smallness of spaces, known as $\pi$-finite spaces [Lur18, Definition E.0.7.8] or truncated coherent spaces [Ane21], which are spaces for which the disjoint union of all homotopy groups is finite. We have the following result similar to Example 8.3.

Example 8.7. The $\infty$-category of bounded coherent spaces does not form an elementary $\infty$-topos, as it again not closed under finite limits and colimits, using the fact that $\mathbb{Z}$ is not bounded coherent.

Remark 8.8. The title of [Ane21] might create the impression that $\pi$-finite spaces are an elementary $\infty$-topos, however, as the abstract explains, the goal is not to prove that it satisfies all conditions in Definition 4.1, but rather that it satisfies many of its conditions (such as finite limits and local Cartesian closure).

Kan Objects: In [GHSS21], the authors construct an $\infty$-category of Kan objects out of a wide range of categories with countable coproducts and finite limits. While the resulting $\infty$-category always has finite limits and colimits and satisfies descent [GHSS21, Proposition 10.1], we do in fact have an example that does not have a subobject classifier [GHSS21, Example 11.8]. However, it is not known whether they have sufficient complete Segal universes.

Filter Quotient $(n,1)$-Toposes: One interesting example of elementary toposes that are not Grothendieck are filter quotients [AJ82]. They have been generalized to non-trivial examples of elementary $\infty$-toposes in [Ras21c]. We can now use our results to generalize to elementary $(n,1)$-toposes that are not Grothendieck $(n,1)$-toposes, proving that the opposite to Proposition 8.1 does not hold.

First we need to review several concepts.

Definition 8.9. A filter is a partially ordered set $(\Phi, \leq)$ that satisfies the following three conditions:

1. $\Phi \neq \emptyset$. 

(2) $\Phi$ is downward directed, meaning that for any two objects $x, y \in \Phi$ there exists $z \in \Phi$ such that $z \leq x$ and $z \leq y$.

(3) $\Phi$ is upward closed, meaning that if $x \leq y$ and $x \in \Phi$, then $y \in \Phi$.

Notice $\Phi^{op}$ is a filtered category and if $\Phi$ has a maximal object 1, then (1) and (3) implies that $1 \in \Phi$.

**Definition 8.10.** A finitely complete $\infty$-category with product-closed filter of subobjects is a finitely complete $\infty$-category $\mathcal{C}$ along with a filter $\Phi$ on the poset $\text{Sub}_C(1)$ that is closed under products.

For a given filter, let $\mathcal{E}_{/\_} : \Phi^{op} \to \text{Cat}_{\infty}$ be the restriction of $\mathcal{E}_{/\_}$ (Section 1(16)) with the inclusion $\Phi^{op} \to \mathcal{E}^{op}$. Concretely, it is the functor that takes $U \in \Phi$ to $\mathcal{E}_{/U}$ and $U \leq V$ to $- \times U : \mathcal{E}_{/V} \to \mathcal{E}_{/U}$. We now have the following definition.

**Definition 8.11.** Let $(\mathcal{E}, \Phi)$ be a finitely complete $\infty$-category with product-closed filter of subobjects. Define the *filter quotient* $\mathcal{E}_\Phi$ as the colimit $\text{colim}_{U \in \Phi^{op}} \mathcal{E}_{/U}$ and denote $P_\Phi : \mathcal{E} \simeq \mathcal{E}_1 \to \mathcal{E}_\Phi$ as the *quotient functor*.

We now have the following result generalizing [Ras21c].

**Proposition 8.12.** Let $(\mathcal{E}, \Phi)$ be a finitely complete $(n, 1)$-category with product-closed filter of subobjects. Then $P_\Phi : \mathcal{E} \to \mathcal{E}_\Phi$ preserves

1. finite limits and colimits
2. subobject classifiers
3. locally Cartesian closure
4. $(n - 2)$-truncated universes

In particular if $\mathcal{E}$ is an elementary $(n, 1)$-topos, then $\mathcal{E}_\Phi$ is one as well and $P_\Phi$ is $(n, 1)$-logical.

**Proof.** The diagram $\Phi^{op}$ is filtered (Definition 8.9) and so $\mathcal{E}_\Phi$ is a filtered colimit and so the result follows from the same argument given in Corollary 7.11. \hfill $\Box$

Let us give one explicit example motivated by [Ras21c, Subsection 3.2].

**Example 8.13.** Let $\mathbb{N}$ be the set of natural numbers and $\Phi$ the Fréchet filter of cofinite subsets [Ras21c, Example 1.47]. Then this induces a filter of subobjects on the $(n + 1, 1)$-category $(S_{<n})^\mathbb{N}$ and we call the resulting filter quotient the *filter product* elementary $(n + 1, 1)$-topos and denote it by $\prod_{\Phi} S_{<n}$. Following the explanation in [Ras21c, Subsection 3.2], it does not have countable products and coproducts and the natural number object is non-standard, giving us a very explicit example of an elementary $(n + 1, 1)$-topos that is not Grothendieck.

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