Rectification in Nonequilibrium Steady States of Open Many-Body Systems

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(Dated: September 3, 2020)

We study how translationally invariant couplings of many-body systems and nonequilibrium baths can be used to rectify particle currents. We propose novel setups to realize bath-induced currents in nonequilibrium steady states of one-dimensional open fermionic systems. We first analyze dissipative dynamics associated with a nonreciprocal Lindblad operator and identify a class of Lindblad operators that are sufficient to acquire a nonreciprocal current. Remarkably, we show that rectification can in general occur even when a Lindblad operator is reciprocal provided that the inversion symmetry and the time-reversal symmetry of the microscopic Hamiltonian are broken. We demonstrate this new mechanism on the basis of both analytical and numerical approaches including the Rashba spin-orbit coupling and the Zeeman magnetic field. Our findings will play fundamental roles for exploring rectification phenomena in homogeneous open many-body systems.

Introduction.— In recent years, open quantum systems are widely explored as exemplified by driven-dissipative many-body systems [15] and non-Hermitian phenomena [6]. They have revealed that dissipation can qualitatively change various aspects of many-body physics such as in quantum critical phenomena [7–10], phase transitions [11,14], magnetism [15,17], and quench dynamics [18,20]. In particular, experimental advances in controlling dissipation have allowed one to study nonequilibrium and non-Hermitian phenomena in trapped ions [21,22], photons [23,24], ultracold atoms [25,29], and exciton-polariton systems [30,35]. These remarkable developments have offered new opportunities for exploring intriguing phenomena unique to open quantum systems in homogeneous setups in contrast to, e.g., boundary-driven systems [56].

On another front, nonreciprocal phenomena, which have been a long-standing problem in condensed matter physics and nonequilibrium statistical mechanics, play a vital role in a variety of areas, including solid-state physics [37–41], photonics [42–46], acoustics [47–52], and active matter [53–56]. While p-n junctions are nonreciprocal devices of commercial success, there is significant interest in exploring alternative mechanisms. One common way to introduce rectification is to couple a system with two different baths at boundaries and use temperature gradients as exemplified by thermal diodes [57,62]. Meanwhile, recent discoveries have shed light on generating nonreciprocal flows without any temperature biases [56,63,67]. While it has long been recognized that dissipation is a key ingredient to control transport properties, Onsager’s reciprocal theorem [53] prohibits rectification by equilibrium baths and thus it is of central importance to introduce nonequilibrium baths. Despite its growing importance, previous studies solely focused on inhomogeneous setups such as current rectification by boundary driving [69,76]. In contrast, rectification induced by homogeneous dissipation of nonequilibrium baths has so far not been explored. Thus, this issue is currently a major challenge in open many-body systems.

In this Letter, we reveal new mechanisms to obtain a nonreciprocal current in nonequilibrium steady states (NESS) of one-dimensional open fermionic systems, where a nonequilibrium bath is uniformly coupled to the system in addition to an equilibrium heat bath (see Fig. 1). We first consider a nonreciprocal Lindblad operator, which is translationally invariant and conserves the particle number of the system, and elucidate a general condition to acquire a nonreciprocal current in NESS. We numerically calculate the current by considering a specific dissipator that can be realized in ultracold atoms [1,2]. Remarkably, we demonstrate that even a reciprocal Lindblad operator can rectify the current in NESS provided that the inversion symmetry and the time-reversal symmetry of the Hamiltonian are broken. We consider spin-dependent dephasing as a reciprocal Lindblad operator and evaluate the current in NESS by analytical and numerical methods in the presence of the Rashba spin-orbit coupling and the Zeeman magnetic field [77]. Our results demonstrate that open many-body systems that are uniformly coupled to nonequilibrium baths provide ideal platforms to explore novel mechanisms of realizing current rectification.

Model.— We consider a one-dimensional lattice model coupled to both an equilibrium heat bath and a nonequilibrium Markovian bath. Such a situation is described by the

![FIG. 1. Schematic illustration of our setup. Fermionic atoms are trapped in a one-dimensional lattice and uniformly coupled to a nonequilibrium bath. An equilibrium heat bath with the inverse temperature $\beta$ is also coupled to the system to ensure that the system goes to the Gibbs state in the absence of nonequilibrium driving. The coupling strength to each bath is given by $\gamma_{eq}$ and $\gamma_{neq} = 1 - \gamma_{eq}$, respectively. Nonreciprocal current $I$ can arise in NESS only when the system is driven out of equilibrium.]

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Model.— We consider a one-dimensional lattice model coupled to both an equilibrium heat bath and a nonequilibrium Markovian bath. Such a situation is described by the
Lindblad master equation
\[ \partial_t \rho = -i[H_0, \rho] + L_1 \rho, \]
\[ L_1 \rho = \epsilon(\gamma_{\text{eq}} D^{\text{eq}}(\rho) + \gamma_{\text{neq}} D^{\text{neq}}(\rho)), \]
with dissipators
\[ D^{(i)}(\rho) = \sum_m \left( L_m^{(i)} \rho L_m^{(i)\dagger} - \frac{1}{2} \left\{ L_m^{(i)\dagger} L_m^{(i)}, \rho \right\} \right), \]
where \( H_0 \) is a noninteracting Hamiltonian governing the internal dynamics, \( L_m \) is a so-called Lindblad operator, \( \gamma_{\text{eq}} \) and \( \gamma_{\text{neq}} = 1 - \gamma_{\text{eq}} \) denote the relative coupling strengths between two baths, \( \gamma_{\text{eq}} \in [0, 1] \). We assume that the baths are weakly coupled to the system with a small dimensionless parameter \( \epsilon \). Here and henceforth, we set \( \hbar = 1 \). We remark that the present model is an intrinsically interacting many-body problem because the dissipator cannot in general be expressed in terms of quadratic annihilation/creation operators as detailed below.

When the integrability of the translationally invariant many-particle system is weakly broken, the time evolution of the system can be described by a time-dependent generalized Gibbs ensemble (tGGE) \[ [73][80] \], which is justified for times \( t \) of the order of \( 1/\epsilon \) and larger,
\[ \rho_{\text{GGE}}(t) = \frac{e^{-\sum_p \lambda_q(t) I_q}}{\text{Tr} \left[ e^{-\sum_p \lambda_q(t) I_q} \right]}, \]
where \( I_q \) is an approximately conserved quantity as a consequence of weak driving. Previous studies \[ [78][80] \] have shown that, by applying a perturbation theory to the Lindblad equation, one can obtain a simple differential equation that determines the dynamics of Lagrange parameters up to the order of \( \epsilon \)
\[ \dot{\lambda}_q = -\sum_p (\chi(t)^{-1})_{qp} \text{tr} \left[ I_p L_1 \rho_{\text{GGE}}(t) \right], \]
\[ \chi_{qp}(t) = \langle I_p I_q \rangle_{\text{GGE}} - \langle I_q \rangle_{\text{GGE}} \langle I_p \rangle_{\text{GGE}}, \]
where \( \langle \cdots \rangle_{\text{GGE}} = \text{tr} \left[ \cdots \rho_{\text{GGE}}(t) \right] \). In the following, we evaluate the current in NESS by using tGGE approach with Eqs. (4)–(6).

Rectification by nonreciprocal dissipator.— We first consider the one-dimensional tight-binding model
\[ H_0 = -J \sum_{j=0}^{L-1} (c_j^{\dagger} c_{j+1} + \text{H.c.}) = -\sum_{-\pi \leq k < \pi} \epsilon_k c_k^{\dagger} c_k, \]
where \( J \) is the hopping amplitude and \( \epsilon_k = -2J \cos(k) \) is the eigenspectrum. The system is subject to periodic boundary conditions and periodic dissipation of length \( L \). In this case, \( I_p \) in Eq. (4) is given by the local number operator in the momentum space \( I_p = c_p^{\dagger} c_p \).

The Lindblad operators corresponding to the equilibrium heat bath satisfy \[ \left[ L_m, H_0 \right] = \zeta_m L_m \] with \( \zeta_m = \epsilon_k - \epsilon_l \), \( m = (k, l) \in \{-\pi, -\pi + 2\pi/N, \ldots, \pi - 2\pi/N\} \) to ensure the detailed balance condition \( L_{ik}^d = L_{ik} e^{-\beta(\epsilon_k - \epsilon_i)/2} \) \[ [5][81] \] in such a way that, without nonequilibrium driving, the system goes to the Gibbs state \( \rho_{\text{can}} = e^{-\beta H_0}/\text{tr} \left( e^{-\beta H_0} \right) \) irrespective of the initial state [see Fig. 2(a) and Supplemental Materials]. For the sake of simplicity, we here employ the following Lindblad operator corresponding to the equilibrium heat bath
\[ L_{ik}^{\text{eq}} = \sqrt{J/L} c_k e^{\beta(\epsilon_k - \epsilon_i)/4}. \]

To realize current rectification in NESS, we consider a nonreciprocal Lindblad operator corresponding to the nonequilibrium bath and assume that it is translationally invariant and conserves the particle number of the system. In this case, the Lindblad operator can in general be labeled by a wave number with coefficients \( \Delta_{kq} \) as
\[ L_{kq}^{\text{neq}} = \sqrt{J/L} \sum_{-\pi \leq k < \pi} \Delta_{kq} c_{k-q}^{\dagger} c_k. \]
Using Eq. (5) and Lindblad operators (8) and (9), we obtain the rate equation that governs the dynamics of the system \[ [82] \]
\[ \dot{\lambda}_q = -\epsilon J \frac{1 + e^{-\lambda_q}}{L} (\gamma_{\text{eq}} F_q^{\text{eq}} + \gamma_{\text{neq}} F_q^{\text{neq}}), \]
where
\[ F_q^{\text{eq}} = \sum_{-\pi \leq k < \pi} \frac{\epsilon^\beta(\epsilon_k - \epsilon_q)/2 - \lambda_k - \epsilon^\beta(\epsilon_k - \epsilon_q)/2 - \lambda_q}{1 + e^{-\lambda_k}}, \]
\[ F_q^{\text{neq}} = \sum_{-\pi \leq k < \pi} \frac{|\Delta_{k,q-k}^2 e^{-\lambda_q} - |\Delta_{q,k}^2 e^{-\lambda_k}|}{1 + e^{-\lambda_k}}. \]
We numerically solve the rate equation (10) to obtain the dynamics of Lagrange parameters \[ [82] \] and their steady state values [see Fig. 2(a)]. We see that Lagrange parameters depart from the Gibbs state when the system is driven out of equilibrium as the nonequilibrium dissipation rate \( \gamma_{\text{neq}} \) is increased.

We now derive a general condition to realize a nonzero nonreciprocal current in NESS. The current \( I \) generally consists of two terms including Hamiltonian current and dissipative current of order \( \epsilon \). For such a small \( \epsilon \) that justifies

![Figure 2](image-url)

FIG. 2. (a) Lagrange parameters in NESS that are driven out of equilibrium as the nonequilibrium dissipation rate \( \gamma_{\text{neq}} \) is increased. The grey dashed line denotes the Gibbs state. (b) Current \( I \) in NESS as a function of \( \gamma_{\text{neq}} \) with Lindblad operators (8) and (15). The parameters are set to \( \beta = 2/J, \delta = 1 + i, \) and \( \delta' = 1 + 0.5i. \)
the tGGE approach, the dissipative current can be ignored, which is consistent with a general description of the current in open quantum systems [33]. We obtain the current from the continuity equation for the density matrix as $I = 2/L \sum_j \text{Im}(c_j^\dagger c_{j+1} - H.c.)_{\text{GGE}}$, where $(H_0)_{j,j+1}$ denotes the coefficient of $c_j^\dagger c_{j+1}$ in $H_0$. In the present model, the current is then given by

$$I = \frac{iJ}{L} \sum_{j=0}^{L-1} \langle c_j^\dagger c_j - H.c. \rangle_{\text{GGE}} = \frac{2J}{L} \sum_{-\pi \leq q < \pi} \sin(q) e^{-\lambda_q} \frac{1}{1 + e^{-\lambda_q}}. \quad (13)$$

Thus, to obtain a nonreciprocal current, the Lagrange parameter $\lambda_q$ must not be an even function of $q$. More specifically, as inferred from Eq. (12), this condition requires a set of $(k, q) \in [-\pi, \pi]$ to satisfy (at least) one of the following conditions:

$$|\Delta_{q,k+q}| \neq |\Delta_{-q,k-q}|, \quad |\Delta_{k,k-q}| \neq |\Delta_{k,k+q}|. \quad (14)$$

We note that an even function $\lambda_q$ prohibits the rectification of the current even when the dissipative current of order $\epsilon$ is included because it leads to a parity-even distribution of particles in real space and thus the current (that is parity-odd) cannot exist.

Let us apply the condition (14) for obtaining the nonreciprocal current to a specific example, which is proposed in ultracold atoms [1, 2].

$$L_j^{\text{neq}} = \sqrt{J} (c_{j+1}^\dagger + \delta c_{j+1}^\dagger + \delta' c_{j+1}), \quad (15)$$

where the subscript $j$ denotes the lattice site. After the Fourier transformation, it is rewritten as

$$L_q^{\text{neq}} = \sqrt{J} \sum_k (1 + \delta e^{-i(k-q)} - \delta' e^{i(k-q)}) c_{k-q}^\dagger c_k, \quad (16)$$

where we set the lattice constant $a = 1$. From Eq. (14), the Lindblad operator (16) should give rise to a nonreciprocal current when either $\delta$ or $\delta'$ has the imaginary part. This is demonstrated in Fig. 2(b), where the current in NESS is plotted as a function of $\gamma_{\text{neq}}$ for $\delta = 1 + i$, $\delta' = 1 + 0.5i$. We see that the current rectifies as it is driven out of equilibrium [32] though it exactly vanishes in equilibrium ($\gamma_{\text{neq}} = 0$).

**Rectification by reciprocal dissipator.**—We now consider a more intriguing mechanism to realize a nonzero nonreciprocal current in NESS, namely, rectification by a *reciprocal* Lindblad operator at the expense of the broken inversion and time-reversal symmetries of the internal Hamiltonian. To be concrete, we include the Rashba spin-orbit coupling and the Zeeman magnetic field into the one-dimensional tight-binding model [17].

$$H_0 = -J \sum_{j} (c_j^\dagger c_{j+1} + H.c.) + \hbar \sum_{j=0}^{L-1} (n_{j\uparrow} - n_{j\downarrow})$$

$$- \alpha_z \sum_{j} (c_j^\dagger (i\sigma_y)_\sigma c_j^\dagger + H.c.)$$

$$+ \alpha_y \sum_{j} (c_j^\dagger i\sigma_z c_j^\dagger + H.c.)$$

$$= \sum_{-\pi \leq k < \pi} \sum_{\nu = \pm} \epsilon_{k\nu} \eta_{k\nu}^\dagger \eta_{k\nu}, \quad (17)$$

where $h$ denotes the Zeeman splitting, $\sigma_{y,z}$ are the Pauli matrices, $\alpha_y, \alpha_z$ denote the Rashba hopping with spin-flip, $\gamma = \uparrow \downarrow$ and $\nu = \pm$ label spin and band indices, respectively, and the system is subject to periodic boundary conditions and periodic dissipation of length $L$. The Rashba spin-orbit coupling and the Zeeman magnetic field break the inversion symmetry and the time-reversal symmetry of the Hamiltonian, respectively (see Fig. 3). The Hamiltonian is diagonalized with eigenvalues $\epsilon_{k\pm} = -2J \cos(k) \pm \sqrt{(2\alpha_y \sin(k) + h)^2 + 4\alpha_z^2 \sin^2(k)}$ and quasiparticle operators $\eta_{k\nu}$, which are given by a unitary transformation as $c_{k\sigma} = \sum_{\nu} \eta_{k\nu}(k) \eta_{k\nu}^\dagger$ and obey the anticommutation relation $\{\eta_{k\mu}, \eta_{k'\nu}\} = \delta_{kk'} \delta_{\mu\nu}$. In this case, local conservation laws of few-body observables are given by the number operators of quasiparticles $I_{\nu \nu} = \eta_{\nu \nu}^\dagger \eta_{\nu \nu}$ [cf. Eq. (3)]

To identify the Lindblad operators $L_{\nu \nu}^{\text{neq}}$ that satisfy the detailed balance condition, we consider $\mu(\nu)$ dependence for the energy bands of quasiparticles in addition to Eq. (3).

$$L_{\nu \nu}^{\text{eq}} = \sqrt{J} \sum_k \eta_{k\nu}^\dagger \eta_{k\nu} e^{\beta(\epsilon_{k\nu} - \epsilon_{\mu})}/4. \quad (18)$$

As the reciprocal Lindblad operator of the nonequilibrium bath, we consider the spin-dependent dephasing given by

$$L_{j\sigma}^{\text{neq}} = \sqrt{J} \gamma_{\sigma} c_{j\sigma}^\dagger c_{j\sigma}, \quad (19)$$

where $j$ labels the lattice site and the dissipation rates of up and down spins satisfy $\gamma_{\uparrow} + \gamma_{\downarrow} = 1$. We calculate the rate
up spins leads to the nonreciprocal current \( I \). As shown in Fig. 4(a), the dephasing applied to up spins (right) causes population changes to be enhanced near the Fermi surface due to dephasing (marked by grey dotted circles). When dephasing is applied to up spins [see the right panel in Fig. 3], they heat up and those near the Fermi surface are most likely to move to the other eigenstates of the Hamiltonian. As a result, the number of particles near the Fermi surface where up spins exist decreases, thereby contributing to the current in the positive direction [see also Eqs. (S12) and (S13) in Supplemental Materials].

**Discussions.**— We demonstrate a new type of rectification mechanism unique to homogeneous open quantum systems, which arises from the interplay between nonequilibrium dissipation and the system parameters, e.g., the Zeeman magnetic field \( \gamma \) and the Rashba spin-orbit coupling \( \alpha \).

We have confirmed that, by numerical calculations using Eq. (23), the current \( I \) is nonzero only when both the Zeeman magnetic field and the Rashba spin-orbit coupling exist. This can be understood as follows. Since dissipation by an equilibrium bath does not rectify the current, one must resort to a nonequilibrium bath for obtaining a nonzero nonreciprocal current. From Eq. (22), we see that nonreciprocity of the distribution of Lagrange parameters is determined from the property of the unitary transformation of quasiparticles, namely, the symmetry of the internal Hamiltonian \( H_0 \). In fact, due to the structure of the matrix component \( \langle \sigma \rangle \) (see Supplemental materials), dephasing by the nonequilibrium bath in Eq. (22) contributes to the Lagrange parameters as an even function with respect to \( q \) if either one of the Zeeman magnetic field or the Rashba spin-orbit coupling is broken. As inversion-symmetric Lagrange parameters give the parity-even distribution of particles in real space, the current does not rectify even if dissipative correction of the order of \( \epsilon \) is included.

Figures 4(a) and (b) show the currents in NESS in the presence of the Rashba spin-orbit coupling and the Zeeman magnetic field. As shown in Fig. 4(a), the dephasing applied to up spins leads to the nonreciprocal current \( I \) in NESS and it becomes larger as the system is driven out of equilibrium. We recall that the system goes to the Gibbs state for \( \gamma_{\text{neq}} = 0 \) and the infinite temperature state for \( \gamma_{\text{neq}} = 1 \), both of which do not rectify the total current \( I \). When the dephasing is applied to both up and down spins with equal rates [see Fig. 4(b)], the total current \( I \) vanishes irrespective of the dissipation rate \( \gamma_{\text{neq}} \). As up spins and down spins contribute to the currents \( I \) in the opposite directions and cancel out \[ \gamma_{\text{neq}} = 1 \]. Here, we note that the sharp peak of the current in Fig. 4(a) comes from the sudden heating up to the infinite temperature due to the nonequilibrium bath and the peak position can be controlled by the system parameters, e.g., the Zeeman magnetic field \( h \).

Physically, rectification of the current in NESS can be understood from the change of spin distribution near the Fermi surface. As shown in the left panel of Fig. 4(c), the spin distribution forms an effective Fermi surface in the steady state [see also the right panel in Fig. 3], reflecting the half-filled initial state. When dephasing is applied to up spins [see the right panel in Fig. 4(c)], they heat up and those near the Fermi surface are most likely to move to the other eigenstates of the Hamiltonian. As a result, the number of particles near the Fermi surface where up spins exist decreases, thereby contributing to the current in the positive direction [see also Eqs. (S12) and (S13) in Supplemental Materials].

**FIG. 4.** (a,b) NESS current and its spin dependence as a function of \( \gamma_{\text{neq}} \) in the presence of the Zeeman magnetic field and the Rashba spin-orbit coupling. Dephasing is applied to up spins in (a), and to both up and down spins with equal rates in (b). (c) Distribution of the upper band (blue) and the lower band (red) in NESS for the equilibrium Gibbs state (left) and the nonequilibrium state where dephasing is applied to up spins (right). Population changes are enhanced near the Fermi surface due to dephasing (marked by grey dotted circles). The parameters are set to \( \beta = 2J \), \( \alpha_y = 1.1J \), \( \alpha_z = 0.9J \), and \( h = J \). The initial state is at infinite temperature.
sipator and internal Hamiltonian dynamics. Our finding is distinct from most of the previous studies that focused on inhomogeneous setups, where a system is coupled to different baths at its boundaries, thus relying on temperature biases or boundary driving. In particular, our open-system formulation provides a new versatile platform for studying current rectification, which gives a completely different framework from the conventional approach to nonreciprocal phenomena, such as magnetoechiral effect, i.e., unidirectional electrical resistance due to DC electric fields [33] [86] [87], or transmissions of an electron current in the presence of a potential barrier [77] [88]. From an experimental perspective, our results can be tested in ultracold atoms; the use of Raman-type spin-orbit coupling can be another possible candidate to break the inversion symmetry. One can also consider semiconductor quantum dots in GaAs as possible experimental candidates, where the spin-relaxation time is very long [89] [92]. spin-resolved dephasing should be realized by using the Zeeman shift.

To summarize, we have proposed a novel mechanism to realize a nonreciprocal current in open many-body systems. In contrast to conventional approaches, our finding provides a unique avenue for rectification, namely, the current is neither generated by temperature gradients nor boundary driving, but via the translationally invariant couplings to nonequilibrium baths. We have demonstrated that a nonreciprocal Lindblad operator in general rectifies the current in NESS. Importantly, we have revealed that even a reciprocal Lindblad operator can be used to rectify the current when the inversion symmetry and the time-reversal symmetry of the internal Hamiltonian are broken. The present analysis opens up various avenues of possible future research such as current rectification in higher dimensions or changes on transport properties by strong integrability breaking.

We are grateful to Takahiro Morimoto and Sota Kitamura for fruitful discussions through the TMS junior researcher visiting program. This work was supported by KAKENHI (Grants No. JP18H01140 and No. JP19H01838) and a Grant-in-Aid for Scientific Research on Innovative Areas (KAKENHI Grant No. JP15H05855) from the Japan Society for the Promotion of Science. K.Y. was supported by WISE Program, MEXT and JSPS KAKENHI Grant-in-Aid for JSPS fellows Grant No. JP20J21318. Y.A. acknowledges support from the Japan Society for the Promotion of Science through Grant No. JP19K23424.

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Supplemental Material for
"Rectification in Nonequilibrium Steady States in Open Many-Body Systems"

Detailed calculations of rate equations for Lagrange parameters

We here explain the detailed calculations to obtain the rate equations for Lagrange parameters. For the first model discussed in the main text, the local conservation laws of few-body observables are given by \( I_q = c_q^\dagger c_q \). Thus, \( \chi_{pq} \) in Eq. (6) in the main text is nonzero only for the diagonal components, given by

\[
\chi_{qq}(t) = \langle c_q^\dagger c_q c_q^\dagger c_q \rangle - \langle c_q^\dagger c_q \rangle^2 = \langle c_q^\dagger c_q \rangle \langle c_q c_q^\dagger \rangle = \frac{e^{-\lambda_q}}{(1 + e^{-\lambda_q})^2}, \tag{S1}
\]

where we have omitted the subscript \( \langle \cdots \rangle_{GGE} \) and the same applies hereafter. Then, we calculate \( \langle \dot{I}_q \rangle = \text{tr} \left[ I_q \mathcal{L}_{1} \rho_{GGE} \right] \) on the right hand side of the rate equation (5) for the Lindblad operator (8) as

\[
\langle \dot{I}_q \rangle^{\text{eq}} = \frac{\epsilon_{\text{eq}}}{L} \text{tr} \left[ I_q \sum_{kl} \left( L_{kl}^{\text{eq}} \rho_{GGE} L_{kl}^{\text{eq}} - \frac{1}{2} \left( L_{kl}^{\text{eq}} L_{kl}^{\dagger} \rho_{GGE} \right) \right) \right]
\]

\[
= \frac{\epsilon_{\text{eq}}}{L} \text{tr} \left[ \sum_{kl} e^{\beta(\epsilon_k - \epsilon_l)/2} c_l^\dagger c_q \left( c_k^\dagger c_l \rho_{GGE} c_l^\dagger c_k - \frac{1}{2} \left( c_k^\dagger c_k c_l^\dagger c_l \rho_{GGE} \right) \right) \right]
\]

\[
= \frac{\epsilon_{\text{eq}}}{L} \sum_{kl} \sum_{k \neq q} e^{\beta(\epsilon_k - \epsilon_l)/2} \left( c_k^\dagger c_k^\dagger c_q c_k c_l^\dagger c_l - \frac{1}{2} \left( c_k^\dagger c_k^\dagger c_q c_k c_l + c_k^\dagger c_k^\dagger c_k c_l^\dagger c_l \right) \right)
\]

\[
= \frac{\epsilon_{\text{eq}}}{L} \sum_{k} \frac{1}{(1 + e^{-\lambda_k})(1 + e^{-\lambda_q})} \left( e^{\beta(\epsilon_k - \epsilon_q)/2 - \lambda_k} - e^{\beta(\epsilon_q - \epsilon_k)/2 - \lambda_q} \right). \tag{S2}
\]

where we used Wick’s theorem. Here, we note that the terms that do not include \( q \) in \( \sum_{kl} \) becomes zero because such terms correspond to flows \( k \rightarrow l \) or \( l \rightarrow k \) \( (k, l \neq q) \) and do not contribute to the dynamics of \( I_q \). In the same way, we calculate \( \langle \dot{I}_q \rangle \) on the right hand side of the rate equation (5) for the Lindblad operator (9) as

\[
\langle \dot{I}_q \rangle^{\text{neq}} = \frac{\epsilon_{\text{neq}}}{L} \sum_{q,k} \sum_{k'\neq k} \Delta_{q,k'} \Delta_{k'q} \left( c_{k'}^\dagger c_k c_{q'}^\dagger c_k^\dagger c_{k'} c_k - \frac{1}{2} \left( c_{q'}^\dagger c_k c_k^\dagger c_{q'} c_k + c_k^\dagger c_k^\dagger c_k c_{q'} c_k c_k \right) \right)
\]

\[
= \frac{\epsilon_{\text{neq}}}{L} \sum_{k} \left( -|\Delta_{q,k}|^2 \langle c_k^\dagger c_k \rangle \langle c_k c_k^\dagger \rangle + |\Delta_{q+k,k}|^2 \langle c_{q+k}^\dagger c_{q+k} \rangle \langle c_{q+k} c_{q+k}^\dagger \rangle \right)
\]

\[
= \frac{\epsilon_{\text{neq}}}{L} \sum_{k} \left( -|\Delta_{q,k-k'}|^2 \frac{e^{-\lambda_k}}{(1 + e^{-\lambda_k})(1 + e^{-\lambda_q})} + |\Delta_{k,k-\Delta_{q,k}}|^2 \frac{e^{-\lambda_k}}{(1 + e^{-\lambda_k})(1 + e^{-\lambda_q})} \right). \tag{S3}
\]

By using Eqs. (S1) - (S3), we obtain the rate equation (10) for the first model in the main text.

For the second model (with the reciprocal dissipator), we can calculate the rate equation almost in the same way as discussed above. As the local conservation law is given by \( I_{q\nu} = \eta_{q\nu}^\dagger \eta_{q\nu} \), \( \chi_{q\nu,p\mu} \) in Eq. (6) in the main text is zero for the off-diagonal components and the diagonal component is calculated as

\[
\chi_{q\nu,q\nu}(t) = \langle \eta_{q\nu}^\dagger \eta_{q\nu} \eta_{q\nu}^\dagger \eta_{q\nu} \rangle - \langle \eta_{q\nu}^\dagger \eta_{q\nu} \rangle \langle \eta_{q\nu}^\dagger \eta_{q\nu} \rangle = \frac{e^{-\lambda_{q\nu}}}{(1 + e^{-\lambda_{q\nu}})^2}. \tag{S4}
\]

We see from Eq. (S4) that the degrees of freedom in momentum space are doubled by upper and lower energy bands compared to Eq. (S1). Then, \( \langle I_{q\nu} \rangle \) on the right hand side of the rate equation (5) for the Lindblad operator (18) is calculated by doubling
the momentum space as [see also Eq. (S2)]

\[
\langle \dot{I}_{q \nu} \rangle_{eq} = \frac{\epsilon \gamma_{eq} J}{L} \sum_{kl} \sum_{\mu,\kappa=\pm} e^{\beta(\epsilon_{k\mu} - \epsilon_{k\nu})/2} \left( \eta_{\kappa\nu}^\dagger \eta_{\kappa\mu} \eta_{k\nu}^\dagger \eta_{k\mu}^\dagger - \frac{1}{2} (\eta_{q\nu}^\dagger \eta_{q\nu}^\dagger \eta_{k\mu}^\dagger \eta_{k\mu}^\dagger + \eta_{\kappa\nu}^\dagger \eta_{\kappa\mu} \eta_{k\nu}^\dagger \eta_{k\mu}^\dagger) \right) (e^{\beta(\epsilon_{k\mu} - \epsilon_{k\nu})/2} - e^{\beta(\epsilon_{q\nu} - \epsilon_{k\nu})/2}) .
\]

(S5)

The contribution from the nonequilibrium bath, denoted as \( \langle \dot{I}_{q \nu} \rangle_{neq} \), can also be simplified by using the expression of the Lindblad operator (19):

\[
\langle \dot{I}_{q \nu} \rangle_{neq} = \frac{\epsilon \gamma_{neq} J}{L} \sum_{kk'} \sum_{\mu,\kappa=\pm} \sum_{\sigma=\uparrow,\downarrow} \gamma_\sigma \left( \eta_{k\sigma}^\dagger \eta_{k'\sigma} \eta_{q\nu}^\dagger \eta_{q\nu}^\dagger c_{k\sigma}^\dagger c_{k'\sigma} c_{k-q',\sigma} c_{k-q',\sigma}^\dagger - \frac{1}{2} \eta_{q\nu}^\dagger \eta_{q\nu}^\dagger c_{k\sigma}^\dagger c_{k\sigma} c_{k-q',\sigma}^\dagger c_{k-q',\sigma}^\dagger \right) (1 + e^{-\lambda_{k\mu}}) (1 + e^{-\lambda_{q\nu}}).
\]

(S6)

To use Wick’s theorem, we substitute the Bogoliubov transformation \( c_{k\sigma} = u_{\sigma \nu}(k) \eta_{k\nu} \) in Eq. (S6) (for the detailed form of \( u_{\sigma \nu}(k) \), see the section “Detailed derivation of the quasiparticle operators” in Suppplemental Materials below). We note that, though we have to calculate \( 2^4 \) times as many terms as Eq. (S6) as a result of the substitution, many of which become zero since tGGE ensemble is defined by local conservation quantities. Then, we obtain

\[
\langle \dot{I}_{q \nu} \rangle_{neq} = \frac{\epsilon \gamma_{neq} J}{L} \sum_{k} \sum_{\mu=\pm} \sum_{\sigma=\uparrow,\downarrow} \gamma_\sigma |u_{\sigma \nu}(q)|^2 |u_{\sigma \mu}(k)|^2 e^{-\lambda_{k\mu}} e^{-\lambda_{q\nu}} (1 + e^{-\lambda_{k\mu}}) (1 + e^{-\lambda_{q\nu}}).
\]

(S7)

Finally, Eq. (20) in the main text follows from Eqs. (S4)--(S7).

**Dynamics of Lagrange parameters without nonequilibrium driving**

We numerically verify that Lagrange parameters go to the Gibbs state in NESS if there is no nonequilibrium driving. Figure S1 shows such relaxation dynamics of Lagrange parameters for the first model in the main text, which obeys Eq. (10) with \( \gamma_{eq} = 1 \) satisfying the detailed balance condition Eq. (8). We see that, though we have to calculate \( 2^4 \) times as many terms as Eq. (S6) as a result of the substitution, many of which become zero since tGGE ensemble is defined by local conservation quantities. Then, we obtain

\[
\langle \dot{I}_{q \nu} \rangle_{neq} = \frac{\epsilon \gamma_{neq} J}{L} \sum_{k} \sum_{\mu=\pm} \sum_{\sigma=\uparrow,\downarrow} \gamma_\sigma |u_{\sigma \nu}(q)|^2 |u_{\sigma \mu}(k)|^2 e^{-\lambda_{k\mu}} e^{-\lambda_{q\nu}} (1 + e^{-\lambda_{k\mu}}) (1 + e^{-\lambda_{q\nu}}).
\]

(Fig. S1) Dynamics of Lagrange parameters \( \lambda_q \) without nonequilibrium driving obtained from Eq. (10) in the main text, which satisfies the detailed balance condition Eq. (8). The system goes to the Gibbs state (dashed lines) after sufficiently long-time evolution. The initial state is set to infinite temperature.
Detailed derivation of the quasiparticle operators

Here, we explain the detailed derivation of the quasiparticle operators for the second model in the main text. The tight-binding Hamiltonian with the Rashba spin-orbit coupling and the Zeeman magnetic field (Eq. (17) in the main text) is diagonalized as

\[
H_0 = -J \sum_{j\sigma}(c_{j+1\sigma}^{\dagger}c_{j\sigma} + \text{H.c.}) + \hbar \sum_j (n_{j\uparrow} - n_{j\downarrow}) - \alpha_z \sum_{j\sigma\sigma'}(c_{j+1\sigma}(i\sigma_y)_{\sigma\sigma'}c_{j\sigma'} + \text{H.c.}) + \alpha_y \sum_{j\sigma\sigma'}(c_{j+1\sigma}(i\sigma_z)_{\sigma\sigma'}c_{j\sigma'} + \text{H.c.})
\]

\[
= \sum_k \begin{pmatrix} c_{k\uparrow}^{\dagger} & c_{k\downarrow}^{\dagger} \end{pmatrix} \begin{pmatrix} 2J \cos k + 2\alpha_y \sin k + \hbar & 2i\alpha_z \sin k \\ -2i\alpha_z \sin k & -2J \cos k - 2\alpha_y \sin k - \hbar \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}
\]

\[
= \sum_{k,\nu=\pm} \epsilon_{k\nu} \eta_{k\nu}^{\dagger} \eta_{k\nu},
\]

with eigenvalues

\[
\epsilon_{k\pm} = -2J \cos(k) \pm \sqrt{(2\alpha_y \sin(k) + \hbar)^2 + 4\alpha_z^2 \sin^2(k)},
\]

and quasiparticles, which are given by the unitary transformation,

\[
\begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} = U(k) \begin{pmatrix} \eta_{k+} \\ \eta_{k-} \end{pmatrix},
\]

\[
U(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} u_{k}^{+}(k) & u_{k}^{-}(k) \\ u_{k}^{+}(k) & u_{k}^{-}(k) \end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} -i \sqrt{\frac{2\alpha_y \sin k + \hbar}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k}} + 1 & -i \sqrt{\frac{-2\alpha_y \sin k - \hbar}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k}} + 1 \\ -\frac{2\alpha_y \sin k - \hbar}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k} + 1 & -\frac{-2\alpha_y \sin k + \hbar}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k} + 1 \end{pmatrix}.
\]

We see from Eq. (22) in the main text and Eq. (S11) that the contribution to Lagrange parameters from the nonequilibrium bath is inversion symmetric with respect to \( q \) if either one of the Zeeman magnetic field or the Rashba spin-orbit coupling is absent, which does not rectify the current. As a result, we need to break both the inversion symmetry and the time-reversal symmetry of the Hamiltonian to obtain the nonreciprocal current in NESS. We can also calculate the current (24) in the main text by using these quasiparticle operators as

\[
I_\tau = \frac{2J}{L} \sum_q \cos(q) \langle c_{q\uparrow}^{\dagger}c_{q\uparrow} \rangle + \frac{2\alpha_y}{L} \sum_q \cos(q) \langle c_{q\uparrow}^{\dagger}c_{q\uparrow} \rangle + \frac{2i\alpha_z}{L} \sum_q \cos(q) \langle c_{q\uparrow}^{\dagger}c_{q\downarrow} \rangle
\]

\[
= \frac{1}{L} \sum_q (J \sin(q) + \alpha_y \cos(q)) \left\langle \sqrt{\frac{2\alpha_y \sin(q) + \hbar}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k}} + 1 \right\rangle \eta_{k+}^{\dagger} \eta_{k+}
\]

\[
+ \left\langle \sqrt{\frac{-2\alpha_y \sin(q) - \hbar}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k}} + 1 \right\rangle \eta_{k-}^{\dagger} \eta_{k-}
\]

\[
+ \frac{2\alpha_z}{L} \sum_q \cos(q) \sin(q) \left\langle \sqrt{\frac{\eta_{k+}^{\dagger} \eta_{k+} - \eta_{k-}^{\dagger} \eta_{k-}}{(2\alpha_y \sin k + \hbar)^2 + 4\alpha_z^2 \sin^2 k}} \right\rangle.
\]

(S12)
\( I_\downarrow = \frac{2J}{L} \sum_q \sin(q) \langle c_{q\uparrow}^\dagger c_{q\uparrow} \rangle - \frac{2\alpha_y}{L} \sum_q \cos(q) \langle c_{q\uparrow}^\dagger c_{q\downarrow} \rangle - \frac{2i\alpha_z}{L} \sum_q \cos(q) \langle c_{q\uparrow}^\dagger c_{q\uparrow} \rangle \)

\[ = \frac{1}{L} \sum_q (J \sin(q) - \alpha_y \cos(q)) \left( \frac{-2\alpha_y \sin(q) - h}{\sqrt{(2\alpha_y \sin k + h)^2 + 4\alpha_y^2 \sin^2 k}} + 1 \right) \eta_k^+ \eta_{k^-}^+ \\
+ \left( \frac{2\alpha_y \sin(q) + h}{\sqrt{(2\alpha_y \sin k + h)^2 + 4\alpha_y^2 \sin^2 k}} + 1 \right) \eta_k^- \eta_{k^-}^- \right) \\
+ \frac{2\alpha_y^2}{L} \sum_q \cos(q) \sin(q) \left( \frac{\eta_k^+ \eta_{k^+}^- \eta_{k^-}^- - \eta_k^- \eta_{k^+}^+ \eta_{k^-}^+}{\sqrt{(2\alpha_y \sin k + h)^2 + 4\alpha_y^2 \sin^2 k}} \right). \]  

(S13)

**Results of the nonreciprocal current in NESS with down-spin dephasing**

We here give the numerical results of the current in NESS when dephasing is applied to down spins in the second model discussed in the main text. From Fig. S2(a), we see that the current rectifies in the opposite direction and the total current \( I \) has the reversed value of that in Fig. 4(a) in the main text. As shown in Fig. S2(b), the change of population near the Fermi surface where dephasing is applied becomes large (grey dotted circles) compared to the Gibbs state, which contributes to the current in the negative direction [see Eqs. (S12) and (S13)].

![Fig. S2](image_url)

**FIG. S2.** (a) Current in NESS and its spin dependence as a function of \( \gamma_{\text{neq}} \) in the presence of the Zeeman magnetic field and the Rashba spin-orbit coupling (model 2 in the main text), where dephasing is applied to down spins. (b) Distribution of the upper band (blue) and the lower band (red) in NESS for the equilibrium Gibbs state (left) and the nonequilibrium state (right) corresponding to (a). Change of population near the Fermi surface where dephasing is applied becomes large (marked by grey dotted circles). The parameters are set to \( \beta = 2/J \), \( \alpha_y = 1.1J \), \( \alpha_z = 0.9J \), and \( h = J \) for the initial state at infinite temperature.