THE $k$-TUPLE DOMATIC NUMBER OF A GRAPH

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Abstract. For every positive integer $k$, a set $S$ of vertices in a graph $G = (V, E)$ is a $k$-tuple dominating set of $G$ if every vertex of $V - S$ is adjacent to at least $k$ vertices and every vertex of $S$ is adjacent to at least $k - 1$ vertices in $S$. The minimum cardinality of a $k$-tuple dominating set of $G$ is the $k$-tuple domination number of $G$. When $k = 1$, a $k$-tuple domination number is the well-studied domination number. We define the $k$-tuple domatic number of $G$ as the largest number of sets in a partition of $V$ into $k$-tuple dominating sets. Recall that when $k = 1$, a $k$-tuple domatic number is the well-studied domatic number.

In this work, we derive basic properties and bounds for the $k$-tuple domatic number.

Keywords: $k$-tuple dominating set, $k$-tuple domination number, $k$-tuple domatic number.

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1. Introduction

The notation we use is as follows. Let $G$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V$, the open neighborhood $N_G(v)$ is the set $\{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of $G$ has degree $k$, then $G$ is said to be $k$-regular. The complement of a graph $G$ is denoted by $\overline{G}$ which is a graph with $V(\overline{G}) = V(G)$ and for every two vertices $v$ and $w$, $vw \notin E(G)$ if and only if $vw \in E(G)$. The subgraph induced by $S$ in a graph $G$ is denoted by $G[S]$. We write $K_n$ for the complete graph of order $n$ and $K_{n,m}$ for the complete bipartite graph.

For every positive integer $k$, the $k$-join $G \circ_k H$ of a graph $G$ to a graph $H$, of order at least $k$, is the graph obtained from the disjoint union of $G$ and $H$ by joining each vertex of $G$ to at least $k$ vertices of $H$.

A dominating set of a graph $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex of $G$ is either in $S$ or has a neighbor in $S$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$ of $G$. It is well known that the complement of a dominating set of minimum cardinality of a graph $G$ without isolated vertices is also a dominating set. Hence one can partition the vertex set of $G$ into at least two disjoint dominating sets. The maximum number of dominating sets into which the vertex set of a graph $G$ can be partitioned is called the domatic number of $G$, and denoted by $d(G)$. This graph invariant was introduced by Cockayne and Hedetniemi [2]. They also showed that

$$\gamma(G) \cdot d(G) \leq n.$$  

(1)

To simplify matters of notation, a domatic partition of a graph $G$ into $\ell$ dominating sets is given by a colouring $f : V(G) \rightarrow \{1, 2, ..., \ell\}$ of the vertex set $V(G)$ with $\ell$ colors. The dominating sets are recovered from $f$ by taking the inverse, i.e. $D_i = f^{-1}(i)$, $i = 1, ..., \ell$. Clearly, a coloring $f$ defines a domatic partition of $G$ if and only if for every vertex $x \in V(G)$, $f(N(x)) = \{1, 2, ..., \ell\}$. Thus, any graph $G$ satisfies $d(G) \leq \delta(G) + 1$. The word domatic, an amalgamation of the words domination and chromatic, refers to an analogy between the chromatic number (partitioning of the vertex set

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into independent sets) and the domatic number (partitioning into dominating sets). For a survey of results on the domatic number of graphs we refer the reader to [12]. It was first observed by Cockayne and Hedetniemi [2] that for every graph without isolated vertices \(2 \leq \delta(G) \leq \delta(G) + 1\). The upper bound \(\delta(G) + 1\) is attained for interval graphs [8], for example.

Intuitively, it seems reasonable to expect that a graph with large minimum degree will have a large domatic number. Zelinka [13] showed that this is not necessarily the case. He gave examples for graphs of arbitrarily large minimum degree with domatic number 2. For more details about domatic number see the references [11, 22, 3] and [13].

The total domatic number \(d_k(G)\) is similarly defined based on the concept of the total domination number \(\gamma_1(G)\). Sheikholeslami and Volkmann, in a similar manner, generalized in [11] the concept of total domatic number to the \(k\)-tuple total domatic number \(d_{x,k}(G)\) based on the concept of \(k\)-tuple total domination number \(\gamma_{x,k}(G)\), which is defined by Henning and Kazemi in [7]. We recall that for every positive integer \(k\), a \(k\)-tuple total dominating set, abbreviated kTDS, of a graph \(G\) is a subset \(S\) of the vertex set \(V(G)\) such that every vertex of \(G\) is adjacent to at least \(k\) vertices of \(S\). And the minimum cardinality of a kTDS of \(G\) is the \(k\)-tuple total domination number \(\gamma_{x,k}(G)\) of \(G\).

Here, we extend the concept of domatic number to \(k\)-tuple domatic number based on the concept of \(k\)-tuple domination number, which is defined by Harary and Haynes in [4]. For every positive integer \(k\), a \(k\)-tuple dominating set, abbreviated kDS, of a graph \(G\) is a subset \(S\) of the vertex set \(V(G)\) such that every vertex of \(G\) is either in \(S\) and is adjacent to at least \(k - 1\) vertices of \(S\) or is not in \(S\) and is adjacent to at least \(k\) vertices of \(S\). The minimum cardinality of a kDS of \(G\) is the \(k\)-tuple domination number \(\gamma_{x,k}(G)\) of \(G\). For a graph to have a \(k\)-tuple dominating set, its minimum degree is at least \(k - 1\). The \(k\)-tuple domatic number \(d_{x,k}(G)\) of \(G\) is the largest number of sets in a partition of \(V(G)\) into \(k\)-tuple dominating sets. If \(d = d_{x,k}(G)\) and \(V(G) = V_1 \cup V_2 \cup \ldots \cup V_d\) is a partition of \(V(G)\) into \(k\)-tuple dominating sets \(V_1, V_2, \ldots, V_d\), we say that \(\{V_1, V_2, \ldots, V_d\}\) is a \(k\)-tuple domatic partition, abbreviated kDP, of \(G\). The \(k\)-tuple domatic number is well-defined and

\[
d_{x,k}(G) \geq 1,\]

for all graphs \(G\) with \(\delta(G) \geq k - 1\), since the set consisting of \(V(G)\) forms a \(k\)-tuple domatic partition of \(G\).

To simplify matters of notation, a \(k\)-tuple domatic partition of a graph \(G\) into \(\ell\) \(k\)-tuple dominating sets is given by a coloring \(f : V(G) \rightarrow \{1, 2, \ldots, \ell\}\) of the vertex set \(V(G)\) with \(\ell\) colors. The \(k\)-tuple dominating sets are recovered from \(f\) by taking the inverse, i.e. \(D_i = f^{-1}(i), i = 1, \ldots, \ell\). Clearly, a coloring \(f\) defines a \(k\)-tuple domatic partition of \(G\) if and only if for every vertex \(x \in V(G)\), \(f(N(x)) = \{f(y) \mid y \in N(x)\}\) contains the multiset \(\{t_1, t_2, \ldots, t_k\}\) such that for every \(i, t_i \in \{k - 1, k\}\) and for an index \(i\), if \(t_i = k - 1\), then \(f(x) = i\). Clearly, each graph \(G\) satisfies

\[
d_{x,k}(G) \leq \frac{\delta(G) + 1}{k}.\]

Graphs for which \(d_{x,k}(G)\) achieves this upper bound \(\frac{\delta(G) + 1}{k}\) we call \(k\)-tuple domatically full.

In this work, we derive basic properties and bounds for the \(k\)-tuple domatic number.

The following observations are useful.

**Observation 1.** Let \(K_n\) be the complete graph of order \(n \geq 1\). Then

\[
d_{x,k}(K_n) = \left\lceil \frac{n}{k} \right\rceil.
\]

**Observation 2.** Let \(G\) be a bipartite graph with \(\delta(G) \geq k - 1 \geq 1\). If \(X\) and \(Y\) are the bipartite sets of \(G\), then \(\gamma_{x,k}(G) \geq 2k - 2\) with equality if and only if \(G = K_{k-1,k-1}\).

**Proof.** Let \(D\) be a \(\gamma_{x,k}(G)\)-set, and let \(w \in X\) and \(z \in Y\) be two arbitrary vertices. The definition implies that \(|D \cap N(w)| \geq k - 1\) and \(|D \cap N(z)| \geq k - 1\). Since \(N(w) \cap N(z) = \emptyset\), we deduce that \(|D| \geq 2k - 2\) and thus \(\gamma_{x,k}(G) \geq 2k - 2\). Obviously, we can see that \(\gamma_{x,k}(G) = 2k - 2\) if and only if \(G = K_{k-1,k-1}\). \(\square\)
2. Properties of the k-Tuple Domatic Number

Here, we present basic properties of \( d_{xk}(G) \) and bounds on the k-tuple domatic number of a graph. We start our work with a theorem that characterizes graphs \( G \) with \( \gamma_{xk}(G) = m \), for some \( m \geq k - 1 \).

**Theorem 3.** Let \( G \) be a graph with \( \delta(G) \geq k - 1 \). Then for any integer \( m \geq k - 1 \), \( \gamma_{xk}(G) = m \) if and only if \( G = K_m \) or \( G = F \circ_k K_m' \), for some graph \( F \) and some spanning subgraph \( K_m' \) of \( K_m \) with \( \delta(K_m') \geq k - 1 \) such that \( m \) is minimum in the set

\[
\{ t \mid G = F' \circ_k K_t', \text{ for some graph } F' \text{ and some spanning subgraph } K_t' \text{ of } K_t \text{ with } \delta(K_t') \geq k - 1 \}.
\]

**Proof.** Let \( S \) be a \( \gamma_{xk}(G) \)-set and \( \gamma_{xk}(G) = m \), for some \( m \geq k - 1 \). Then, \( |S| = m \) and every vertex in \( V - S \) has at least \( k \) neighbors in \( S \) and otherwise \( k - 1 \) neighbors. Then \( G[S] = K_m' \), for some spanning subgraph \( K_m' \) of \( K_m \) with \( \delta(K_m') \geq k - 1 \). If \( |V| = m \), then \( G = K_m' \). If \( |V| > m \), then let \( F \) be the induced subgraph \( G[V - S] \). Then \( G = F \circ_k K_m' \). Also by the definition of k-tuple domination number, \( m \) is minimum in the set given in (4).

Conversely, let \( G = K_m' \) or \( G = F \circ_k K_m' \), for some graph \( F \) and some spanning subgraph \( K_m' \) of \( K_m \) with \( \delta(K_m') \geq k - 1 \) such that \( m \) is minimum in the set given in (4). Then, since \( V(K_m') \) is a kDS of \( G \) with cardinal \( m \), \( \gamma_{xk}(G) \leq m \). If \( \gamma_{xk}(G) = m' < m \), then the previous paragraph concludes that for some graph \( F' \) and some spanning subgraph \( K_m' \) of \( K_m' \) with \( \delta(K_m') \geq k - 1 \), \( G = F' \circ_k K_m' \), that is contradiction with the minimality of \( m \). Therefore \( \gamma_{xk}(G) = m \). \( \square \)

**Corollary 4.** Let \( G \) be a graph with \( \delta(G) \geq k - 1 \). Then \( \gamma_{xk}(G) = k - 1 \) if and only if \( G = K_{k-1} \) or \( G = F \circ_k K_{k-1} \), for some graph \( F \).

**Theorem 5.** If \( G \) is a graph of order \( n \) and \( \delta(G) \geq k - 1 \), then

\[
\gamma_{xk}(G) \cdot d_{xk}(G) \leq n.
\]

Moreover, if \( \gamma_{xk}(G) \cdot d_{xk}(G) = n \), then for each kDP \( \{V_1, V_2, ..., V_d\} \) of \( G \) with \( d = d_{xk}(G) \), each set \( V_i \) is a \( \gamma_{xk}(G) \)-set.

**Proof.** Let \( \{V_1, V_2, ..., V_d\} \) be a kDP of \( G \) such that \( d = d_{xk}(G) \). Then

\[
d \cdot \gamma_{xk}(G) = \sum_{i=1}^{d} \gamma_{xk}(G) \leq \sum_{i=1}^{d} |V_i| = n.
\]

If \( \gamma_{xk}(G) \cdot d_{xk}(G) = n \), then the inequality occurring in the proof becomes equality. Hence for the kDP \( \{V_1, V_2, ..., V_d\} \) of \( G \) and for each \( i, |V_i| = \gamma_{xk}(G) \). Thus each set \( V_i \) is a \( \gamma_{xk}(G) \)-set. \( \square \)

The case \( k = 1 \) in Theorem 5 leads to the well-known inequality (1), given by Cockayne and Hedetniemi in 1977.

An immediate consequence of Corollary 4 and Theorem 5 now follows.

**Corollary 6.** If \( G \) is a graph of order \( n \) with \( \delta(G) \geq k - 1 \), then

\[
d_{xk}(G) \leq \frac{n}{k - 1},
\]

with equality if and only if \( G = K_{k-1} \) or \( G = F \circ_k K_{k-1} \), for some graph \( F \).

For bipartite graphs, we can improve the bound given in Corollary 6 by Observation 2.

**Corollary 7.** Let \( G \) be a bipartite graph of order \( n \) with vertex partition \( V(G) = X \cup Y \) and \( \delta(G) \geq k - 1 \). Then

\[
d_{xk}(G) \leq \frac{n}{2k - 2},
\]

with equality if and only if \( G = K_{k-1,k-1} \).
Theorem 8. If $G$ is a graph of order $n$ and $\delta(G) \geq k - 1 \geq 2$, then
\[ \gamma_{xk}(G) + d_{xk}(G) \leq n + 1. \]

Proof. Applying Theorem 5 we obtain
\[ \gamma_{xk}(G) + d_{xk}(G) \leq \frac{n}{d_{xk}(G)} + d_{xk}(G). \]
Since $d_{xk}(G) \geq 1$, by inequality (2), and $k \geq 3$, Corollary 6 implies that $d_{xk}(G) \leq \frac{n}{2}$. Using these inequalities, and the fact that the function $g(x) = x + \frac{n}{2}$ is decreasing for $1 \leq x \leq n^{1/2}$ and increasing for $n^{1/2} \leq x \leq \frac{n}{2}$, we obtain
\[ \gamma_{xk}(G) + d_{xk}(G) \leq \max\{n + 1, \frac{n}{2} + 2\} = n + 1, \]
and this is the desired bound. \hfill \Box

If $G = tK_k$ for integers $\ell \geq 1$ and $k \geq 3$, then $\gamma_{xk}(G) = n(G) = \ell k$ and $d_{xk}(G) = 1$. Therefore $\gamma_{xk}(G) + d_{xk}(G) = n + 1$, and so the upper bound $n + 1$ in Theorem 8 is sharp.

By closer look at the proof of Theorem 8 we have:

Theorem 9. Let $G$ be a graph of order $n$ with $\delta(G) \geq k - 1 \geq 2$. If $d_{xk}(G) \geq 2$, then
\[ \gamma_{xk}(G) + d_{xk}(G) \leq \frac{n}{2} + 2. \]
If $G = K_{2k}$, then $\gamma_{xk}(G) = k$ and $d_{xk}(G) = 2$. Therefore $\gamma_{xk}(G) + d_{xk}(G) = n/2 + 2$, and so the upper bound $n/2 + 2$ in Theorem 8 is sharp.

Theorem 10. If $G$ is a graph with $\delta(G) \geq k - 1$, then
\[ d_{xk}(G) \leq \frac{\delta(G) + 1}{k}. \]
This bound is sharp and moreover, if $d_{xk}(G) = (\delta(G) + 1)/k$, then for each kDP $\{V_1, V_2, ..., V_d\}$ of $G$ with $d = d_{xk}(G)$ and for all vertices $v$ of degree $\delta(G)$, $|V_i \cap N_G[v]| = k$ for each $1 \leq i \leq d$.

Proof. Let $\{V_1, V_2, ..., V_d\}$ be a kDP of $G$ such that $d = d_{xk}(G)$, and let $v$ be a vertex of degree $\delta(G)$. Since $|V_i \cap N_G[v]| \geq k$ for each $1 \leq i \leq d$, then
\[
k \cdot d_{xk}(G) \leq \sum_{i=1}^{d} |V_i \cap N_G[v]| = |N_G[v]| = \delta(G) + 1,\]
as desired. This bound is sharp for the complete graphs which their orders are multiple of $k$. Since $d_{xk}(G) = (\delta(G) + 1)/k$ follows that the inequality occurring in the above becomes equality, which leads to the property given in the statement. \hfill \Box

Corollary 11. Let $k \geq 1$ be an integer, and let $G$ be a graph. If $k - 1 \leq \delta(G) \leq 2k - 2$, then $d_{xk}(G) = 1$.

As a further application of Theorem 10 we will prove the following result.

Theorem 12. For every graph $G$ of order $n$ in which $\min\{\delta(G), \delta(G)\} \geq k - 1,$
\[ d_{xk}(G) + d_{xk}(\overline{G}) \leq \frac{n + 1}{k}, \]
and this bound is sharp.

Proof. Theorem 10 follows that
\[ d_{xk}(G) + d_{xk}(\overline{G}) \leq \frac{\delta(G) + \delta(\overline{G}) + 2}{(\delta(G) + 1) + (n - \Delta(G))} \leq \frac{n + 1}{k}, \]
as desired.

If $G$ is the complete bipartite graph $K_{k,k}$, where $k \geq 2$, then $d_{xk}(G) + d_{xk}(\overline{G}) = 1 + 1 = \left\lfloor \frac{2k + 1}{k} \right\rfloor$, and so the upper bound $\frac{n + 1}{k}$ is sharp. \hfill \Box
Now we derive some structural properties on graphs with equality in the inequality of Theorem 12.

**Theorem 13.** Let $G$ be a graph of order $n$ with $\min\{\delta(G), \delta(G)\} \geq k - 1$ which
\[ d_{xk}(G) + d_{xk}(\overline{G}) = \frac{n + 1}{k}, \]
and $d_{xk}(G) \geq d_{xk}(\overline{G})$. Then $G$ is regular and
\[ \frac{n}{r + 1} + \frac{1}{k} \leq d_{xk}(G) \leq \frac{n}{r} \]
for an integer $k - 1 \leq r \leq 2k - 1$.

**Proof.** According to Theorem 10, we have
\[ d_{xk}(G) + d_{xk}(\overline{G}) \leq \delta(G) + \delta(G) + 2. \]
If $G$ is not regular, then $\delta(G) + \delta(G) \leq n - 2$, and we obtain the upper bound $d_{xk}(G) + d_{xk}(\overline{G}) \leq \frac{n}{k} < \frac{n + 1}{k}$, a contradiction. Thus $G$ is regular.

The hypothesis $d_{xk}(G) \geq d_{xk}(\overline{G})$ and the hypothesis $d_{xk}(G) + d_{xk}(\overline{G}) = \frac{n + 1}{k}$ lead to
\[ d_{xk}(G) \geq \frac{n + 1}{2k}. \]
Let $\{V_1, V_2, ..., V_d\}$ be a kDP of $G$ such that $d = d_{xk}(G)$ and $r = |V_1| \leq |V_2| \leq ... \leq |V_d|$. Clearly, $r \geq k - 1$ and
\[ r \cdot d_{xk}(G) \leq n. \]
If $r \geq 2k$, then
\[ n \geq 2k \cdot \frac{n + 1}{2k} > n. \]
Therefore we have shown that $k - 1 \leq r \leq 2k - 1$. Since $V_1$ is a $k$-tuple dominating set and $G$ is regular, we deduce that
\[ r \cdot \Delta(G) = \sum_{v \in V_1} \deg(v) \geq k(n - r) + (k - 1)r = kn - r \]
and thus $\Delta(G) = \delta(G) \geq \frac{kn}{r + 1}$ and so
\[ \delta(G) + 1 = \frac{n - \delta(G)}{} \leq \frac{n - \frac{kn}{r + 1}}{\frac{n(r + 1) - kn}{r + 1}}. \]
Applying Theorem 10, we thus obtain
\[ d_{xk}(G) \leq \frac{\delta(G) + 1}{k} \leq \frac{\delta(G) + 1}{n(r + 1) - kn}. \]
Now $d_{xk}(G) + d_{xk}(\overline{G}) = \frac{n + 1}{k}$ leads to
\[ d_{xk}(G) = \frac{n + 1}{k} - d_{xk}(\overline{G}) \geq \frac{n}{r + 1} + \frac{1}{k}. \]

**Corollary 14.** Let $G$ be a graph of order $n$ with $\min\{\delta(G), \delta(G)\} \geq k - 1 \geq 1$ which
\[ d_{xk}(G) + d_{xk}(\overline{G}) = \frac{n + 1}{k}, \]
and $d_{xk}(G) \geq d_{xk}(\overline{G})$. Then
\[ \frac{n}{2k} + \frac{1}{k} \leq d_{xk}(G) \leq \frac{n}{k - 1}. \]
We now present a sharp lower bound on the $k$-tuple domatic number, which generalizes the bound due to Zelinka [13] in 1983.

**Theorem 15.** For every graph $G$ of order $n$ with $\delta(G) \geq k - 1$, 
\[ d_{x,k}(G) \geq \left\lfloor \frac{n}{k(n - \delta(G))} \right\rfloor, \]
and this bound is sharp.

**Proof.** If $k(n - \delta(G)) > n$, then there is nothing to prove. Thus we assume in the following that $n \geq k(n - \delta(G))$. Now let $S \subseteq V(G)$ be any subset with $|S| \geq k(n - \delta(G))$. It follows that 
\[ |S| \geq k(n - \delta(G)) \geq n - \delta(G) + k - 1 \]
and therefore $|V(G) - S| \leq \delta(G) - k + 1$. This inequality implies that 
\[ N_G(u) \cap S \geq \delta(G) - (\delta(G) - k) = k \]
for $u \in V(G) - S$ and 
\[ |N_G(u) \cap S| \geq \delta(G) - (\delta(G) - k + 1) = k - 1 \]
for $u \in S$. Hence $S$ is a $k$-tuple dominating set of $G$. Let $n = \ell k(n - \delta(G)) + r$ with integers $\ell \geq 1$ and $0 \leq r \leq k(n - \delta(G)) - 1$, then one can take any $\ell$ disjoint subsets, $\ell - 1$ of cardinality $k(n - \delta(G))$ and one of cardinality $k(n - \delta(G)) + r$, and all these subsets are $k$-tuple dominating sets of $G$. This yields a $k$-tuple domatic partition of cardinality $\ell = \left\lceil \frac{n}{k(n - \delta(G))} \right\rceil$, and thus our Theorem is proved.

We also note that this lower bound is sharp for the complete graph $K_{\ell k}$. \hfill $\square$

**Corollary 16.** [13] For every graph $G$ of order $n$, 
\[ d(G) \geq \left\lfloor \frac{n}{n - \delta(G)} \right\rfloor. \]

Finally, we compare the $k$-tuple domatic number of a graph with its $k$-tuple total domatic number.

**Theorem 17.** Let $G$ be a graph with $\delta(G) \geq k \geq 1$. Then 
\[ d_{x,k,t}(G) \leq d_{x,k}(G) \leq 2d_{x,k,t}(G), \]
and this bounds are sharp.

**Proof.** Since every $k$-tuple total dominating set of $G$ is a $k$-tuple dominating set and the union of at least two disjoint $k$-tuple dominating sets is a $k$-tuple total dominating set, then $d_{x,k,t}(G) \leq d_{x,k}(G) \leq 2d_{x,k,t}(G)$.

The lower bound is sharp for the complete bipartite graph $K_{mk,mk}$, where $k \geq 2$ and $m \geq 1$. Because $d_{x,k,t}(G) = d_{x,k}(G) = m$. Also for the cycle $C_4$, we have $d(C_4) = d_{x}(C_4) = 2$.

The upper bound is sharp for the graphs $G$ which is obtained as follow: let $H_1$, $H_2$, $H_3$ and $H_4$ be four disjoint copies of the complete graph $K_k$, where $k \geq 1$. Let $G$ be the union of the four graphs $H_1$, $H_2$, $H_3$ and $H_4$ such that for each $1 \leq i \leq 3$ every vertex of $H_i$ is adjacent to all vertices of $H_{i+1}$. Obviously $V(H_2) \cup V(H_3)$ is the unique $\gamma_{x,k,t}(G)$-set, and so $d_{x,k,t}(G) = 1$. This follows that $d_{x,k}(G) \leq 2d_{x,k,t}(G) = 2$. Since the sets $V(H_2) \cup V(H_3)$ and $V(H_1) \cup V(H_4)$ are two disjoint $\gamma_{x,k}(G)$-sets, then $d_{x,k}(G) = 2 = 2d_{x,k,t}(G)$. \hfill $\square$

**Corollary 18.** [14] Let $G$ be a graph with no isolated vertices. Then 
\[ d_t(G) \leq d(G) \leq 2d_t(G). \]

**Theorem 19.** Let $k \geq 1$ be integer. If one of the numbers $d_{x,k}(G)$ and $d_{x,k,t}(G)$ for a graph $G$ is infinite, then 
\[ d_{x,k}(G) = d_{x,k,t}(G). \]

**Proof.** Let $d_{x,k}(G) = \alpha$, where $\alpha$ is an infinite cardinal number. Then there exists a $k$-tuple domatic partition $\mathcal{R}$ having $\alpha$ classes. The family $\mathcal{R}$ can be partitioned into two subfamilies $\mathcal{R}_1$ and $\mathcal{R}_2$ which both have the cardinality $\alpha$. There exists a bijection $f : \mathcal{R}_1 \to \mathcal{R}_2$. Let $\mathcal{R}_0 = \{ D \cup f(D) \mid D \in \mathcal{R}_1 \}$. This is evidently a $k$-tuple total domatic partition of $G$ having $\alpha$ classes and thus $d_{x,k,t}(G) \geq \alpha = d_{x,k}(G)$. Since $d_{x,k,t}(G) \leq d_{x,k}(G)$, we have $d_{x,k,t}(G) = d_{x,k}(G) = \alpha$. If $d_{x,k,t}(G)$ is infinite, then so is $d_{x,k}(G)$ and also $d_{x,k,t}(G) = d_{x,k}(G)$. \hfill $\square$
Corollary 20. If one of the numbers \(d(G)\) and \(d_t(G)\) for a graph \(G\) is infinite, then
\[
d(G) = d_t(G).
\]

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