Classification of 7-dimensional solvable Lie algebras having 5-dimensional nilradicals

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ABSTRACT
By combining basic techniques in Lie Theory and a computer algebra tool which is the so-called triangular decomposition, the class of 7-dimensional real and complex indecomposable solvable Lie algebras having 5-dimensional nilradicals is classified up to isomorphism. In association with Gong (1998), Parry (2007), Hindeleh and Thompson (2008), we achieve a full classification of 7-dimensional real and complex indecomposable solvable Lie algebras.

ARTICLE HISTORY
Received 09 May 2022
Communicated by P. Kolesnikov

KEYWORDS
Lie algebra; nilradical

2020 MATHEMATICS SUBJECT CLASSIFICATION
15A21; 16G60; 17B30; 20G05

1. Introduction

It is well-known that the Levi–Malcev theorem [16, 17] reduces the problem of classifying Lie algebras over a field of characteristic zero to the problem of classifying semi-simple Lie algebras and solvable ones. Semi-simple Lie algebras were fully classified by Cartan [4] (over the complex field \( \mathbb{C} \)) and Ganmatcher [9] (over the real field \( \mathbb{R} \)). However, classifying solvable Lie algebras is much harder, and in general, it still remains open.

In history, the classification of complex and real solvable Lie algebras was achieved up to dimension 6 by Lie [15], Bianchi [1], Dixmier [8], Morozov [18], Mubarakzyanov [19–21] and Turkowski [40] which were summarized by Šnobl and Winternitz [36]. For complex and real solvable Lie algebras of higher dimensions, several partial results are presented in [2, 3, 5, 11, 12, 14, 25, 28, 37–39]. Most of these results concern with nilpotent Lie algebras. So far, a full classification of solvable Lie algebras of dimension 7 has not been completed yet.

In this paper, we present a full classification of 7-dimensional real and complex indecomposable solvable Lie algebras. Our method is based on the fact that a given solvable Lie algebra \( L \) can be considered as an extension of its nilradical \( N(L) \), that is, the maximal nilpotent ideal of \( L \). Therefore, we start from a nilpotent Lie algebra and classify all 7-dimensional solvable Lie algebras which admit it as their nilradical. This method, perhaps, was initialized in 1963 by a series of articles of Mubarakzyanov [19–21] when he classified solvable Lie algebras of dimensions 4 and 5 over a field of characteristic zero. By using the same method, the results for the case of dimension 6 were also achieved by Mubarakzyanov [21] and Turkowski [40]. Furthermore, results in [23, 27, 29–35, 41] show that this method seems to be very effective.

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In case \( \dim L = 7 \), it follows from Mubarakzyanov [19, Theorem 5] that \( \dim N(L) \in \{ 4, 5, 6, 7 \} \). Therefore, the problem of classifying 7-dimensional solvable Lie algebras consists of four cases according to the possible values of \( \dim N(L) \). Moreover, three cases in which \( \dim N(L) \in \{ 4, 6, 7 \} \) were considered by Hindeleh and Thompson [12], Parry [25] and Gong [11], respectively. To our knowledge, the remaining case when \( \dim N(L) = 5 \) has just been solved partially, up to date.

According to Dixmier [8, Proposition 1], the class of 5-dimensional real and complex nilpotent Lie algebras consists of nine Lie algebras as follows:

| Algebras | Non-zero Lie brackets |
|----------|-----------------------|
| \((g_1)^5\) | The 5-dimensional abelian Lie algebra |
| \((g_1)^2 \oplus g_3\) | \([x_1, x_2] = x_3\) |
| \(g_1 \oplus g_4\) | \([x_1, x_2] = x_3, [x_1, x_3] = x_4\) |
| \(g_{5,1}\) | \([x_1, x_2] = x_5, [x_3, x_4] = x_5\) |
| \(g_{5,2}\) | \([x_1, x_2] = x_4, [x_1, x_3] = x_5\) |
| \(g_{5,3}\) | \([x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5\) |
| \(g_{5,4}\) | \([x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5\) |
| \(g_{5,5}\) | \([x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5\) |
| \(g_{5,6}\) | \([x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5\) |

It is also known that finite-dimensional real and complex indecomposable solvable extensions of \((g_1)^5, g_{5,1}, g_{5,3}, g_{5,5}\) and \(g_{5,6}\) were considered by Ndogmo and Winternitz [23], Rubin and Winternitz [27], Šnobl and Karásek [32], Šnobl and Winternitz [33, 34], respectively. Besides, finite-dimensional complex indecomposable solvable extensions of \(g_1 \oplus g_4\) were considered by Wang et al. [41].

The aim of this paper is to classify all 7-dimensional indecomposable solvable extensions of \((g_1)^2 \oplus g_3, g_1 \oplus g_4, g_{5,2}\) and \(g_{5,4}\) (see Theorem 1). Then, we combine our results with those of [23, 27, 32–34] to obtain a classification of 7-dimensional indecomposable solvable Lie algebras having 5-dimensional nilradicals (see Theorem 2). In association with Gong [11], Parry [25], Hindeleh and Thompson [12], we achieve a full classification of 7-dimensional indecomposable solvable Lie algebras.

Section 2 describes the classification procedure. In Section 3, we give explicit computations for nilradical \(g_{5,2}\). Afterwards, we formulate two main theorems of the paper in Section 4. Section 5 contains a full summary of the classification of 7-dimensional solvable Lie algebras. The final section presents the full lists of Lie algebras achieved in our classification with precisely isomorphic conditions.

2. The procedure of classification

The classification in this paper proceeds in two stages which will be described immediately as follows. From now on, \(\mathbb{F}\) will be \(\mathbb{R}\) or \(\mathbb{C}\).

2.1. Construction of Lie algebras

In the first stage, we construct four lists \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\) and \(\mathcal{L}_4\) which consist of 7-dimensional indecomposable solvable \(\mathbb{F}\)-Lie algebras \(L\) having nilradicals \((g_1)^2 \oplus g_3, g_1 \oplus g_4, g_{5,2}\) and \(g_{5,4}\), respectively. Note that solvable extensions of a given nilpotent Lie algebra is standard and can be found in many textbooks (see, e.g., [36]).

First of all, we fix a Lie algebra in \(\{(g_1)^2 \oplus g_3, g_1 \oplus g_4, g_{5,2}, g_{5,4}\}\) which plays a role as the input nilradical \(N(L)\). The basis of \(N(L)\) is always assume to be \([X_1, \ldots, X_5]\). By adding to the basis \([X_1, \ldots, X_5]\) two elements, say \(X\) and \(Y\), we obtain a basis \([X_1, \ldots, X_5, X, Y]\) of \(L\). Then, Lie brackets of \(L\) are absolutely determined by \([X, Y], [X, X_i]\) and \([Y, X_i]\) for \(i = 1, \ldots, 5\). Since the derived algebra of a solvable Lie algebra is contained in its nilradical (see [13, Chapter II, Section 7, Corollary 1]), these Lie brackets can be represented as follows:
[X, Y] = \sum_{j=1}^{5} \sigma_j X_j, \quad [X, X_i] = \sum_{j=1}^{5} a_{ij} X_j, \quad [Y, X_i] = \sum_{j=1}^{5} b_{ij} X_j, \quad 1 \leq i \leq 5.

Set A := (a_{ij}) and B := (b_{ij}). We call A, B \in \text{Mat}_5(\mathbb{F}) the \textit{structure matrices} of L. Then, all we have to do is to determine all possibilities of structure constants \sigma_i \in \mathbb{F} and the structure matrices A, B. To this end, the following techniques will be used.

1. First of all, twenty Jacobi identities involving \((X, X_i, X_j)\) and \((Y, X_i, X_j)\) initialize the original forms of A and B, respectively.
2. Next, five Jacobi identities involving \((X, Y, X_i)\) construct a relation between A and B. Moreover, \([A, B]\) is an inner derivation of \(N(L)\), i.e., we have

\[ [A, B] = \sum_{i=1}^{5} \sigma_i a_{X_i}, \quad a_{X_i} := (\text{ad}_{X_i})^T |_{N(L)}, \]

where \text{ad} is the adjoint operator and the superscript \(T\) indicates the transpose of a matrix.
3. Two supplemented elements X and Y must be \textit{linearly nil-independent} to ensure that the dimension of \(N(L)\) is not larger than 5. This is equivalent to the fact that A and B are linearly nil-independent, i.e., if \(\alpha A + \beta B\) is a nilpotent matrix then \(\alpha = \beta = 0\).
4. We use the three following types of transformations alternatively not only to eliminate \(\sigma_i\) or normalize \(\sigma_i \neq 0\) but also to simplify A and B:
   - Modifying A and B by inner derivations of \(N(L)\), i.e., \(A' = A + \sum_{i=1}^{5} \alpha_i a_{X_i}, B' = B + \sum_{i=1}^{5} \beta_i a_{X_i}\) with \(\alpha_i, \beta_i \in \mathbb{F}\).
   - The automorphisms of \(N(L)\): if \(G \in \text{Aut}(N(L))\) then it will transforms the structure matrices into \(A' = GAG^{-1}\) and \(B' = GBG^{-1}\).
   - The last one is the transformation concerning with X and Y.
5. Due to Mubarakzyanov [22, Corollary 2], the following relation holds for finite-dimensional solvable Lie algebras L over a field of characteristic zero

\[ 2 \dim N(L) \geq \dim L + \dim Z, \quad (2.1) \]

where \(Z\) is the center of \(L\). Inequation \(2.1\) gives an upper bound of \(\dim Z\) which is very useful to decide which cases can happen.

Afterwards, we repeat all techniques above, case by case, for other nilradicals in \(\{g_2, g_3, g_1 + g_4, g_5, g_5^A\}\). By this way, we can obtain four lists \(\mathcal{L}_i\) as desired.

### 2.2. Testing isomorphism for the obtained Lie algebras

To optimize four lists \(\mathcal{L}_i\), we need to test isomorphism between Lie algebras in the lists as well as refine parameters, if any. This stage is necessary since it makes \(\mathcal{L}_i\) more compact. Moreover, we can avoid redundancy, i.e., different Lie algebras in \(\mathcal{L}_i\) are non-isomorphic.

First of all, Lie algebras with different invariants (such as the dimensions of ideals in characteristic series or the dimensions of centers) are non-isomorphic. For Lie algebras sharing same invariants, we use computer algebra tools to verify their isomorphism. Gerdt and Lassner [10], perhaps, were the first authors considering the problem of testing isomorphism of real and complex Lie algebras from a computer algebra point of view. They reduce the problem of testing isomorphism of Lie algebras to the problem of testing the existence of solutions of a polynomial system. Gröbner basis technique is then used to solve the latter problem. However, since the complexity of computing Gröbner bases is very costly, this method is impractical when the dimension pass 6, especially, in case of parametric Lie algebras.
In this paper, we use another computer algebra tool which is the so-called triangular decomposition instead of Gröbner bases. Following the idea of Gerdt and Lassner [10], we also re-write isomorphic conditions between Lie algebras, even if parametric Lie algebras, in terms of polynomial systems and semi-algebraic systems. Afterwards, we use triangular decomposition to decide whether these systems admit roots or not; and if they do, we can find explicit roots to construct isomorphisms. This testing isomorphism procedure is specified by algorithms that are run by Maple software with supports of a hyper-computer. Details of these algorithms can be found in [24]. We also note that these algorithms in [24] are valid over fields of characteristic not only 0 but also prime. In this paper, we simply use them over \( \mathbb{F} \).

Remark 2.1. This stage has a further advantage as follows. Assume that there is a list \( \mathcal{L} \) which consists of real Lie algebras satisfying certain properties. Since each algebra in \( \mathcal{L} \) can be seen as a complex one, we first sweep out algebras that do not exist over \( \mathbb{C} \) by considering structure matrices. Afterwards, we check indecomposability since an indecomposable algebra over \( \mathbb{R} \) may be decomposable over \( \mathbb{C} \). Finally, we test their isomorphism over \( \mathbb{C} \). Consequently, a similar list over \( \mathbb{C} \) can be derived from \( \mathcal{L} \). Below is a demonstrative example.

Example 2.2. In 2007, Parry [25] classified 7-dimensional real indecomposable solvable Lie algebras with codimension one nilradicals. In case of nilradical \( \mathbb{R} \oplus \mathfrak{g}_{5,1} \), there is a family, namely \( [7, [6, 5], 1, 3] \), which is as follows (see [25, Appendix B.6]): \([e_3, e_5] = e_2, [e_4, e_6] = e_2, [e_1, e_7] = ae_1, [e_3, e_7] = -e_3, [e_4, e_7] = -ae_4, [e_5, e_7] = e_5, [e_6, e_7] = -e_1 + ae_6 \) with \( a \in \mathbb{R} \). We claim that the real parameter \( a \) can be reduced to \( a \geq 0 \) instead of \( a \in \mathbb{R} \). In fact, the testing isomorphism procedure above shows that all algebras depending on \( a \) are isomorphic to those on \(-a\) by the following isomorphism

\[
\begin{bmatrix}
-1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\] (we omit off-diagonal zeros).

Therefore, we can remove \([7, [6, 5], 1, 3]_{a<0}\) to avoid redundancy. We have not checked all other algebras in Parry’s list, however, similar situations may occur, especially, for algebras containing more than one parameter. Furthermore, if we consider \([7, [6, 5], 1, 3]\) over \( \mathbb{C} \), the similar thing also happens: \([7, [6, 5], 1, 3]\) is indecomposable, \( a \) and \(-a\) also determine isomorphic complex algebras, and we thus can reduce the complex parameter \( a \) to \( a = 0 \) or \( a \neq 0 \) with \( 0 < \arg(a) \leq \pi \).

3. A sample case: solvable extension of \( \mathfrak{g}_{5,2} \)

The goal of this section is to classify all 7-dimensional indecomposable solvable \( \mathbb{F}\)-Lie algebras having nilradical \( \mathfrak{g}_{5,2} \) with detailed computations by the procedure pointed out in Section 2. Recall that \( \mathfrak{g}_{5,2} \) is in Section 1.

First, twenty Jacobi identities involving \((X, X_i, X_j), (Y, X_i, X_j)\) and transformations \(X' := X - a_{24}X_1 + a_{14}X_2 + a_{15}X_3, Y' := Y - b_{24}X_1 + b_{14}X_2 + b_{15}X_3\) give (we omit off-diagonal zeros):

\[
A = \begin{bmatrix}
ad & e \\
b & f \\
h & c & k & l \\
a + b & f \\
\end{bmatrix}, \quad B = \begin{bmatrix}
u & p & q \\
p & r & t \\
 & x & w & y & z \\
 & u + v & r \\
 & x & u + w \\
\end{bmatrix}.
\]
Next, five Jacobi identities involving \((X, Y, X_i)\) give \(\sigma_1 = 0\) and
\[
\begin{align*}
(a - b)p + ex &= (u - v)d + hq \\
(a - c)q + dr &= (u - w)e + pf \\
fX &= hr \\
ax &= ul \\
(b - c)r &= (v - w)f \\
fY + gx &= ht + kr \\
(a - b + c)t + lr &= (u - v + w)g + fz \\
(b - c)x &= (v - w)h \\
(a + b - c)y + zh &= (u + v - w)k + lx
\end{align*}
\tag{3.1}
\]

Moreover, \([A, B] = -\sigma_2 a X_2 - \sigma_3 a X_3\), where \(a X_i := \left(\text{ad} X_i\right)^T\). Put \(C = \left[\begin{array}{cc} b & f \\ h & c \end{array}\right]\) and \(D = \left[\begin{array}{cc} v & r \\ x & w \end{array}\right]\). Then the third, fifth and eighth equations of (3.1) imply that \(C\) and \(D\) commute. Therefore, we can choose \(\alpha, \beta, \gamma, \delta\) in the following automorphism of \(g_{5,2}\)
\[
\begin{bmatrix}
X_1' \\
X_2' \\
X_3' \\
X_4' \\
X_5'
\end{bmatrix} = G
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{bmatrix}, \quad G = \begin{bmatrix}
1 & \alpha & \beta \\
\gamma & \delta & \alpha \\
\gamma & \delta & \beta
\end{bmatrix} \in \text{Aut}(g_{5,2}),
\]
such that the pair \((C, D)\) can obtain the three following types:
\[
\left(\begin{array}{cc}
\lambda_1 & 0 \\
0 & \lambda_2
\end{array}\right), \quad \left(\begin{array}{cc}
\lambda_1 & \mu_1 \\
0 & \mu_2
\end{array}\right), \quad \left(\begin{array}{cc}
\lambda_1 & \mu_1 \\
-\lambda_2 & \lambda_1
\end{array}\right), \quad \left(\begin{array}{cc}
\mu_1 & \mu_2 \\
-\mu_2 & \mu_1
\end{array}\right), \quad \lambda_2 \neq 0.
\]

We do not need to consider permutations of three types above since they will return to three original ones if we interchange \(X \leftrightarrow Y\). Now, these three types lead to three forms of structure matrices as follows:

(1) \(A = \begin{bmatrix}
a & d & e \\
\lambda_1 & k & g \\
\lambda_2 & a + \lambda_1 & l
\end{bmatrix}\) and \(B = \begin{bmatrix}
u & p & q \\
\mu_1 & t & \mu_1 \\
\mu_2 & y & z + u + \mu_1
\end{bmatrix};\)

(2) \(A = \begin{bmatrix}
a & d & e \\
\lambda_1 & 1 & g \\
\lambda & a + \lambda & l
\end{bmatrix}\) and \(B = \begin{bmatrix}
u & p & q \\
\mu_1 & t & \mu_1 \\
\mu & y & u + \mu_1 + \mu
\end{bmatrix};\)

(3) \(A = \begin{bmatrix}
a & d & e \\
-\lambda_2 & \lambda_1 & g \\
\lambda_1 & a + \lambda_1 & l
\end{bmatrix}\) and \(B = \begin{bmatrix}
u & p & q \\
\mu_1 & t & \mu_1 \\
-\mu_2 & y & u + \mu_1 + \mu
\end{bmatrix}.\)

The linearly nil-independent condition of $A$ and $B$ is as follows:

| Forms of $(A, B)$ | Linearly nil-independent conditions |
|-------------------|-------------------------------------|
| (1) and (2)       | rank $\begin{bmatrix} a & u \\ \lambda_1 & \mu_1 \end{bmatrix} = 2$ |
| (3)               | rank $\begin{bmatrix} a & u \\ \lambda & \mu \end{bmatrix} = 2$ |

To eliminate $\sigma_2, \sigma_4, \sigma_5$ we change $X' := X + \alpha X_4 + \beta X_5$ and $Y' := Y + \gamma X_4 + \delta X_5$. This transformation creates

$$[
X', Y'] = (\sigma_2 - t\beta + g\delta)X_2 + (\sigma_3 - y\alpha - z\beta + k\gamma + l\delta)X_3
+ (\sigma_4 - (u + v)\alpha - (a + b)\gamma + f\delta)X_4
+ [\sigma_5 - x\alpha - (u + w)\beta + h\gamma + (a + c)\delta]X_5.
$$

(3.3)

To destroy off-diagonal elements of $A$ and $B$, we use a basis changing which is an automorphism of $g_{5,2}$ as follows

$$
\begin{bmatrix}
X'_1 \\
X'_2 \\
X'_3 \\
X'_4 \\
X'_5
\end{bmatrix} = G_1
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{bmatrix},
\quad G_1 = \begin{bmatrix}
g_1 & g_2 & g_3 & 1 & 1 \\
1 & 1 & 1 & g_4 & g_5
\end{bmatrix} \in \text{Aut}(g_{5,2}).
\tag{G_1}
$$

Transformation $(G_1)$ will transform $A$ and $B$ into

$$G_1AG_1^{-1} = \begin{bmatrix}
a & d' & e' & m^A & n^A \\
b & f & g' & s^A & g' \\
h & c & k' & l' & a + b + c \\
- & f & + & g_1 & -(a - c)g_2
\end{bmatrix},
\quad G_1BG_1^{-1} = \begin{bmatrix}
u & p' & q' & m^B & n^B \\
v & r & s^B & t' \\
x & w & y' & z' \\
- & u + v & - & r & u + w
\end{bmatrix},
$$

in which

$$
d' = d - (a - b)g_1 + hg_2,
\quad e' = e + fg_1 - (a - c)g_2,
\quad g' = g + (a - b + c)g_3 - fg_5,
\quad k' = k + (a + b - c)g_4 + hg_5,
\quad l' = l - hg_3 + fg_4 + ag_5,
\quad p' = p - (u - v)g_1 + xg_2,
\quad q' = q + rg_1 - (u - w)g_2,
\quad t' = t + (u - v + w)g_3 - rg_5,
\quad y' = y + (u + v - w)g_4 + xg_5,
\quad z' = z - xg_3 + rg_4 + ug_5.
$$

(3.4)

Afterwards, we destroy $m^A, n^A, s^A, m^B, n^B, s^B$ by changing $X' := X - s^A X_1 + m^A X_2 + n^A X_3$ and $Y' := Y - s^B X_1 + m^B X_2 + n^B X_3$.

To normalize non-zero off-diagonal elements of $A$ and $B$, we also use an automorphism of $g_{5,2}$ which is as follows

$$
\begin{bmatrix}
X'_1 \\
X'_2 \\
X'_3 \\
X'_4 \\
X'_5
\end{bmatrix} = G_2
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{bmatrix},
\quad G_2 = \text{diag}(h_1, h_2, h_3, h_1 h_2, h_1 h_3) \in \text{Aut}(g_{5,2}).
\tag{G_2}
$$
Transformation \((G_2)\) will transform \(A\) and \(B\) into

\[
G_2A G_2^{-1} = \begin{bmatrix}
  a & b & c \\
  h_{12} & h_{22} & h_{32} \\
  h_{13} & h_{23} & h_{33}
\end{bmatrix}, \quad G_2B G_2^{-1} = \begin{bmatrix}
  u & v & w \\
  h_{14} & h_{24} & h_{34} \\
  h_{15} & h_{25} & h_{35}
\end{bmatrix}.
\]

### 3.1. The structure matrices are of form (1)

In this case, we have

\[
A = \begin{bmatrix}
  a & d & e \\
  \lambda_1 & k & g \\
  a + \lambda_1 & \lambda_2 & l
\end{bmatrix}, \quad B = \begin{bmatrix}
  u & p & q & t \\
  \mu_1 & \mu_2 & y & z \\
  u + \mu_1 & u + \mu_2
\end{bmatrix}.
\]

Transformation \((G_1)\) transforms \(A\) and \(B\) by (3.4) which becomes

\[
d' = d - (a - \lambda_1)g_1 \\
e' = e - (a - \lambda_2)g_2 \\
g' = g + (a - \lambda_1 + \lambda_2)g_3 \\
k' = k + (a + \lambda_1 - \lambda_2)g_4 \\
l' = l + ag_5 \\
p' = p - (u - \mu_1)g_1 \\
q' = q - (u - \mu_2)g_2 \\
t' = t + (u - \mu_1 + \mu_2)g_3 \\
y' = y + (u + \mu_1 - \mu_2)g_4 \\
z' = z + u g_5
\]

and (3.1) becomes

\[
\begin{cases}
  (a - \lambda_1)p = (u - \mu_1)d \\
  (a - \lambda_2)q = (u - \mu_2)e \\
  (a - \lambda_1 + \lambda_2)t = (u - \mu_1 + \mu_2)g \\
  (a + \lambda_1 - \lambda_2)y = (u + \mu_1 - \mu_2)k \\
  az = ul
\end{cases}
\]

By (3.1a) and (3.5a) we can see that:

| If | we choose | then |
|----|-----------|------|
| \((a - \lambda_1)^2 + (u - \mu_1)^2 \neq 0\) | \(g_1 \in \{ \frac{d}{a - \lambda_1}, \frac{p}{u - \mu_1}\} \) | \(d' = p' = 0\) |
| \((a - \lambda_2)^2 + (u - \mu_2)^2 \neq 0\) | \(g_2 \in \{ \frac{e}{a - \lambda_2}, \frac{q}{u - \mu_2}\} \) | \(e' = q' = 0\) |
| \((a - \lambda_1 + \lambda_2)^2 + (u - \mu_1 + \mu_2)^2 \neq 0\) | \(g_3 \in \{ \frac{g}{a - \lambda_1 + \lambda_2}, \frac{t}{u - \mu_1 + \mu_2}\} \) | \(g' = t' = 0\) |
| \((a + \lambda_1 - \lambda_2)^2 + (u + \mu_1 - \mu_2)^2 \neq 0\) | \(g_4 \in \{ \frac{k}{a + \lambda_1 - \lambda_2}, \frac{y}{u + \mu_1 - \mu_2}\} \) | \(k' = y' = 0\) |
| \(a^2 + u^2 \neq 0\) | \(g_5 \in \{ \frac{1}{a}, \frac{1}{u} \} \) | \(l' = z' = 0\) |

According to (2.1), the center \(Z\) of \(L\) satisfies \(\dim Z \leq 3\). However, Lie brackets of \(g_{5,2}\) show that there are only \(X_4, X_5\) can belong to \(Z\). Moreover, if \(X_4, X_5 \in Z\) then \(a = -\lambda_1 = -\lambda_2\) and \(u = -\mu_1 = -\mu_2\) which conflict (3.2). Therefore, we only have \(\dim Z = 1\) or \(\dim Z = 0\).
3.1.1. \( \dim \mathcal{Z} = 1 \)
In this subcase, we have \( \mathcal{Z} = \text{span}(X_4) \) or \( \mathcal{Z} = \text{span}(X_5) \). However, if we interchange \( X_2 \leftrightarrow X_3 \) and \( X_4 \leftrightarrow X_5 \) then they will coincide. Therefore, without loss of generality, we can assume \( \mathcal{Z} = \text{span}(X_4) \), i.e.,
\[
a + \lambda_1 = 0 = u + \mu_1, \quad (a + \lambda_2)^2 + (u + \mu_2)^2 \neq 0.
\]
Since (3.2) guarantees the left-hand side of \((*)\), its right-hand side is always valid. In other words, we can always transform the structure matrices into the following diagonal forms:
\[
A = \text{diag}(a, -a, \lambda_2, 0, a + \lambda_2), \quad B = \text{diag}(u, -u, \mu_2, 0, u + \mu_2).
\]

First, we have \( \sigma_2 = \sigma_3 = 0 \) as \([A, B] = 0\). Next, we can choose appropriately \( \beta, \delta \) in (3.3) to destroy \( \sigma_5 \), i.e., \([X, Y] = \sigma_4 X_4 \). Since \( a^2 + u^2 \neq 0 \), we can assume \( a \neq 0 \), otherwise, we interchange \( X \leftrightarrow Y \). Thus, we normalize \( a = 1 \) by scaling \( X \rightarrow \frac{1}{a} X \) and then destroy \( u \) by changing \( Y' := Y - uX \). Since \( \mu_2 \neq 0 \), we normalize \( \mu_2 = 1 \) by scaling \( Y \rightarrow \frac{1}{\mu_2} Y \) and then destroy \( \lambda_2 \) by changing \( X' := X - \lambda_2 Y \). It creates the following Lie algebras:
\[
L^0_1: \quad A = \text{diag}(1, -1, 0, 0, 1), \quad B = \text{diag}(0, 0, 1, 0, 1), \quad [X, Y] = \sigma X_4.
\]

Remark 3.1. Lie brackets of \( L^0_1 \) can be easily read off due to their structure matrices. Beyond the original ones of \( g_{5,2} \) and \([X, Y]\), we have additionally
\[
[X, X_1] = X_1, \quad [X, X_2] = -X_2, \quad [X, X_5] = X_5, \quad [Y, X_3] = X_3, \quad [Y, X_5] = X_5.
\]

In our view, using structure matrices has an advantage that is a global view of the obtained Lie algebras’ structures, such as decomposability or grouping Lie algebras for testing isomorphism (see Subsection 3.4 below), becomes more easier. Therefore, from now on, we use the structure matrices instead of Lie brackets.

3.1.2. \( \dim \mathcal{Z} = 0 \)
In this subcase, \( (a + \lambda_1)^2 + (u + \mu_1)^2 \neq 0 \) and \( (a + \lambda_2)^2 + (u + \mu_2)^2 \neq 0 \). Due to \((*)\), we can divide this subcase into two mutually-exclusive subcases as follows. Note that in two subcases below, we always have \([A, B] = 0\) which implies \( \sigma_2 = \sigma_3 = 0 \). Moreover, \( \sigma_4, \sigma_5 \) can always be eliminated by (3.3), i.e., \([X, Y] = 0\) in all two subcases.

A. All of \( d, e, g, k, l, p, q, t, y, z \) are zero. It happens when all of \( d, e, g, k, l, p, q, t, y, z \) are automatically zero or five inequalities on the left-hand side of \((*)\) hold. This means that
\[
A = \text{diag}(a, \lambda_1, \lambda_2, a + \lambda_1, a + \lambda_2), \quad B = \text{diag}(u, \mu_1, \mu_2, u + \mu_1, u + \mu_2).
\]

1. If \( \lambda_1 = \mu_1 = 0 \) then the linearly nil-independent condition of \( A \) and \( B \) becomes rank \[
\begin{bmatrix} a & u \\
\lambda_2 & \mu_2 
\end{bmatrix}
\]
2 which implies \( a^2 + u^2 \neq 0 \) and \( \lambda_2^2 + \mu_2^2 \neq 0 \). Without loss of generality, we can assume \( a \neq 0 \), otherwise, we interchange \( X \leftrightarrow Y \). Thus, we normalize \( a = 1 \) by scaling \( X \rightarrow \frac{1}{a} X \) and then destroy \( u \) by changing \( Y' := Y - uX \). Then, \( \mu_2 \neq 0 \), we normalize \( \mu_2 = 1 \) by scaling \( Y \rightarrow \frac{1}{\mu_2} Y \) and then destroy \( \lambda_2 \) by changing \( X' := X - \lambda_2 Y \).

2. If \( \lambda_1^2 + \mu_1^2 \neq 0 \) then we can assume \( \lambda_2^2 + \mu_2^2 \neq 0 \) since on the contrary, we interchange \( X_2 \leftrightarrow X_3 \) and \( X_4 \leftrightarrow X_5 \) and return to 3.1.2. If \( a = u = 0 \) then we can assume \( \lambda_1 \neq 0 \). Thus, we normalize \( \lambda_1 = 1 \) by scaling \( X \rightarrow \frac{1}{\lambda_1} X \) and then destroy \( \mu_1 \) by changing \( Y' := Y - \mu_1 X \). Then, \( \mu_2 \neq 0 \), we normalize \( \mu_2 = 1 \) by scaling \( Y \rightarrow \frac{1}{\mu_2} Y \) and destroy \( \lambda_2 \) by changing \( X' := X - \lambda_2 Y \). If \( a^2 + u^2 \neq 0 \) then we can assume \( a \neq 0 \), otherwise, we interchange \( X \leftrightarrow Y \). Thus, we normalize \( a = 1 \) by scaling \( X \rightarrow \frac{1}{a} X \) and then destroy \( u \) by changing \( Y' := Y - uX \). Then, \( \mu_1^2 + \mu_2^2 \neq 0 \), we can assume \( \mu_1 \neq 0 \), otherwise, we interchange \( X_2 \leftrightarrow X_3 \) and \( X_4 \leftrightarrow X_5 \). Thus, we normalize \( \mu_1 = 1 \) by scaling \( Y \rightarrow \frac{1}{\mu_1} Y \) and then destroy \( \lambda_1 \) by changing \( X' := X - \mu_1 Y \).
To sum up, we obtain the following Lie algebras:

\[
\begin{align*}
L_2: \quad & A = \text{diag}(1, 0, 0, 1), & B = \text{diag}(0, 0, 1, 0), \\
L_3: \quad & A = \text{diag}(1, 0, 1, 0), & B = \text{diag}(0, 0, 1, 0), \\
L_{4b}^{ab}: \quad & A = \text{diag}(1, 0, a, 1 + a), & B = \text{diag}(0, 1, b, 1); \quad (a, b) \neq (0, -1).
\end{align*}
\]

**B. There exists at least one of \(d, e, g, k, l, p, q, t, y, z\) which is non-zero.** It happens when \(A\) or \(B\) consists of non-zero off-diagonal elements \(d, e, g, k, l, p, q, t, y, z\) and five inequalities on the left-hand side of (*) do not hold. However, (3.2) guarantees that there is at most one of them cannot hold. This means that \((A, B)\) can only contain at most one pair of non-zero off-diagonal elements which is \((d, p)\) or \((e, q)\) or \((g, t)\) or \((k, y)\) or \((l, z)\). Furthermore, if we interchange \(X_2 \leftrightarrow X_3\) and \(X_4 \leftrightarrow X_5\) then the pairs \((e, q)\) and \((k, y)\) will return to \((d, p)\) and \((g, t)\), respectively. Therefore, we have three situations as follows.

1. **\((A, B)\) contains the pair \((d, p)\).** We have \(a = \lambda_1\) and \(u = \mu_1\). First, we can assume \(p \neq 0\), otherwise, we interchange \(X \leftrightarrow Y\). Then we destroy \(d\) by changing \(X' := X - \frac{d}{p} Y\). Note that (3.2) implies \(a^2 + \lambda_2^2 \neq 0\). If \(a = 0\), we normalize \(\lambda_2 = 1\) by scaling \(X \rightarrow \frac{1}{\mu_2} X\) and then destroy \(\mu_2\) by changing \(Y' := Y - \mu_2 X\); afterwards, we normalize \(u = p = 1\) by \(Y \rightarrow \frac{1}{u} Y\) and \(G_2 = \text{diag} \left( \frac{u}{p}, 1, 1, \frac{u}{p}, \frac{u}{p} \right)\). If \(a \neq 0\), we normalize \(a = 1\) by scaling \(X \rightarrow \frac{1}{a} X\) and then destroy \(u\) by changing \(Y' := Y - u X\); afterwards, we normalize \(\mu_2 = p = 1\) by scaling \(Y \rightarrow \frac{1}{\mu_2} Y\) and \(G_2 = \text{diag} \left( \frac{\lambda_2}{p}, 1, 1, \frac{\lambda_2}{p}, \frac{\lambda_2}{p} \right)\). To sum up, we obtain the following Lie algebras:

\[
L_5: \quad A = \text{diag}(0, 0, 1, 0), \quad B = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 2 \\
2 & 1
\end{bmatrix},
\]

\[
L_6^a: \quad A = \text{diag}(1, 1, a, 2 + a), \quad B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

2. **\((A, B)\) contains the pair \((g, t)\).** We have \(\lambda_1 = a + \lambda_2\) and \(\mu_1 = u + \mu_2\). First, we can assume \(t \neq 0\) and then destroy \(g\) by changing \(X' := X - \frac{t}{\mu_2} Y\). By a similar way as above, we obtain the following Lie algebras:

\[
L_7: \quad A = \text{diag}(0, 1, 1, 1), \quad B = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 2 \\
2 & 1
\end{bmatrix},
\]

\[
L_8^b: \quad A = \text{diag}(1, 1 + a, a, 2 + a, 1 + a), \quad B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1
\end{bmatrix}.
\]
3. (A, B) contains the pair (l, z). We have a = u = 0. First, we can assume z ≠ 0 and then destroy l by changing $X' := X - \frac{l}{z}Y$. By a similar way as above, we obtain the following Lie algebras:

$L_9: \quad A = \text{diag}(0, 0, 1, 0, 1), \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & 1 \\ \lambda & k & l \\ a + \lambda & 0 \\ a + \lambda \end{bmatrix}$,

$L_{10}: \quad A = \text{diag}(0, 1, a, 1, a), \quad B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 \\ u + \mu & u + \mu \\ t & y & z \\ u + \mu & u + \mu \end{bmatrix}$.

### 3.2. The structure matrices are of form (2)

In this case, we first destroy $\mu_1$ by changing $Y' := Y - \mu_1X$ to get

$$A = \begin{bmatrix} a & d & e \\ \lambda & 1 \\ \lambda & k & l \\ a + \lambda & 1 \\ a + \lambda \end{bmatrix}, \quad B = \begin{bmatrix} u & p & q \\ \mu & y & t \\ \mu & y & z \\ u + \mu & u + \mu \end{bmatrix}.$$}

Transformation $(G_1)$ transforms $A$ and $B$ by (3.4) which becomes

\[ \begin{align*}
    d' &= d - (a - \lambda)g_1 \\
    e' &= e + g_1 - (a - \lambda)g_2 \\
    g' &= g + ag_3 - g_5 \\
    k' &= k + ag_4 + g_5 \\
    l' &= l + g_4 \\
    p' &= p - (u - \mu)g_1 \\
    q' &= q - (u - \mu)g_2 \\
    t' &= t + ug_3 \\
    y' &= y + ug_4 \\
    z' &= z + ug_5
\end{align*} \tag{3.5b} \]

and (3.1) becomes

\[ \begin{align*}
    (a - \lambda)p &= (u - \mu)d \\
    (a - \lambda)q + d\mu_1 &= (u - \mu)e + p \\
    at + \mu_1l &= ug + z \\
    y &= k\mu_1 \\
    ay &= uk \\
    az &= ul
\end{align*} \tag{3.1b} \]

Now, (3.2) implies $(a - \lambda)^2 + (u - \mu)^2 \neq 0$ and $a^2 + u^2 \neq 0$. Taking account of (3.1b) and (3.5b), we can choose $g_1 \in \left\{ \frac{d}{a - \lambda}, \frac{p}{a - \mu} \right\}$, $g_2 \in \left\{ \frac{(a - \lambda)e + d}{(a - \lambda)^2}, \frac{q}{a - \mu} \right\}$, $g_3 \in \left\{ \frac{k - al - a^2g}{a^3}, -\frac{t}{u} \right\}$, $g_4 \in \left\{ \frac{k}{a}, -\frac{y}{u} \right\}$, $g_5 \in$
\[
\left\{ \frac{k-al}{a^2}, -\frac{z}{u} \right\}
\]
to destroy all \(d, e, p, q, g, k, l, t, y, z\). Therefore, the structure matrices are transformed into

\[
A = \begin{bmatrix}
a & \lambda & 1 \\
\lambda & a + \lambda & 1 \\
a + \lambda & 1 & a + \lambda \\
\end{bmatrix}, \quad B = \text{diag}(u, \mu, u + \mu, u + \mu).
\]

We have \(\sigma_2 = \sigma_3 = 0\) as \([A, B] = 0\). Besides, we can choose \(\alpha, \beta, \gamma, \delta\) in (3.3) to destroy \(\sigma_4, \sigma_5\), i.e. \([X, Y] = 0\). If \(u = 0\), we normalize \(\mu = 1\) by scaling \(Y \to \frac{1}{\mu} Y\) and then destroy \(\lambda\) by changing \(X' := X - \lambda Y\) and normalize \(a = 1\) by scaling \(X \to \frac{1}{a} X\) and \(G_2 = (1, a, 1, a, 1)\); otherwise, we normalize \(u = 1\) by scaling \(Y \to \frac{1}{u} Y\) and then destroy \(a\) by changing \(X' := X - aY\), and normalize \(\lambda = 1\) by scaling \(X \to \frac{1}{\lambda} X\) and \(G_2 = \text{diag}(1, \lambda, 1, \lambda, 1)\). We interchange \(X \leftrightarrow Y\) to get a good look and obtain the following Lie algebras:

\[
L_{11}: \quad A = \text{diag}(0, 1, 1, 1, 1), \quad B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix},
\]

\[
L_{12}^a: \quad A = \text{diag}(1, a, a, 1 + a, 1 + a), \quad A = \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}.
\]

### 3.3. The structure matrices are of form (3)

In this case, we first normalize \(\lambda_2 = 1\) by scaling \(X \to \frac{1}{\lambda_2} X\) and then destroy \(\mu_2\) by changing \(Y' := Y - \mu_2 X\) to get

\[
A = \begin{bmatrix}
a & d & e \\
\lambda_1 & 1 & 0 \\
-1 & \lambda_1 & 0 \\
a + \lambda_1 & 1 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
u & p & q & t \\
\mu_1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

Transformation \((G_1)\) transforms \(A\) and \(B\) by (3.4) which becomes

\[
\begin{align*}
d' &= d - (a - \lambda_1)g_1 & g_2 \\
e' &= e + g_1 & -(a - \lambda_1)g_2 \\
g' &= g + a g_3 & -g_5 \\
k' &= k + a g_4 & -g_5 \\
l' &= l + a g_3 & +a g_5 \\
p' &= p - (u - \mu_1)g_1 & -g_2 \\
q' &= q & -(u - \mu_1)g_2 \\
t' &= t + ug_3 & +ug_5 \\
y' &= y & +ug_4 \\
z' &= z & +ug_5
\end{align*}
\]

(3.5c)
and (3.1) becomes
\[
\begin{align*}
(a - \lambda_1)p &= (u - \mu_1)d - q \\
(a - \lambda_1)q &= (u - \mu_1)e + p \\
y &= -t \\
at &= ug + 2z \\
ay - az &= uk \\
az &= ul.
\end{align*}
\tag{3.1c}
\]

By (3.1c) and (3.5c), we take \( g_1 \in \left\{ \frac{d+e(a-\lambda_1)}{1+(a-\lambda_1)^2}, \frac{p}{u-\mu_1} \right\} \), \( g_2 \in \left\{ \frac{d+a(e-a-\lambda_1)}{1+(a-\lambda_1)^2}, \frac{q}{u-\mu_1} \right\} \) to destroy \( d', e', p', q' \). Moreover, if \( a^2 + u^2 \neq 0 \), we take \( g_3 \in \left\{ \frac{k-al-g(1+a^2)}{a(a^2+2)}, -\frac{t}{u} \right\} \), \( g_4 \in \left\{ \frac{g-al-k(1+a^2)}{a(a^2+2)}, -\frac{y}{u} \right\} \), \( g_5 \in \left\{ \frac{g+k-al}{a^2+2}, -\frac{z}{u} \right\} \) to further destroy \( g', k', l', t', y', z' \). Therefore, we divide this case into two mutually-exclusive subcases as follows. Note that in two subcases below, we always have \([A, B] = 0\) and \( \sigma_4, \sigma_5 \) can always be eliminated by (3.3), i.e., \([X, Y] = 0\) in all two subcases.

**A. All of \( g, k, l, t, y, z \) are zero.** It happens when \( g, k, l, t, y, z \) are automatically zero or \( a^2 + \mu^2 \neq 0 \). This means that

\[
A = \begin{bmatrix}
a & 1 & 0 & 0 \\
\lambda_1 & -1 & \lambda_1 & 1 \\
a + \lambda_1 & 0 & a + \lambda_1 & 0 \\
-1 & 0 & a + \lambda_1 & 0
\end{bmatrix}, \quad B = \text{diag}(u, \mu_1, \mu_1, u + \mu_1, u + \mu_1).
\]

Note that (3.2) implies \( u^2 + \mu_1^2 \neq 0 \). If \( u = 0 \), we normalize \( \mu_1 = 1 \) by scaling \( Y \rightarrow \frac{1}{\mu_1}Y \) and then destroy \( \lambda_1 \) by changing \( X' := X - \lambda_1Y \); otherwise, we normalize \( u = 1 \) by scaling \( Y \rightarrow \frac{1}{u}Y \) and then destroy \( a \) by changing \( X' := X - aY \). We interchange \( X \leftrightarrow Y \) to get a good look. It creates the following Lie algebras:

\[
L_{13}^a: \quad A = \text{diag}(0, 1, 1, 1, 1), \quad B = \begin{bmatrix} 0 & 1 \\
-1 & 0 \\
a & 1 \\
-1 & a \\
\end{bmatrix},
\]

\[
L_{14}^{ab}: \quad A = \text{diag}(1, a, a, 1 + a, 1 + a), \quad B = \begin{bmatrix} 0 & b & 1 \\
-1 & b & 1 \\
\end{bmatrix}.
\]

**B. There exists at least one of \( g, k, l, t, y, z \) which is non-zero.** It happens when \( A \) and \( B \) consists of non-zero elements \( g, k, l, t, y, z \) and \( a = u = 0 \). In this subcases, (3.1c) gives \( z = 0 \) and \( y = -t \) and we have

\[
A = \begin{bmatrix}
0 & 1 & g & 0 \\
\lambda_1 & -1 & \lambda_1 & k \\
-1 & \lambda_1 & \lambda_1 & 1 \\
\lambda_1 & 1 & 0 & \lambda_1
\end{bmatrix}, \quad B = \begin{bmatrix} \mu_1 & 0 \\
0 & \mu_1 \\
t & 0 \\
\mu_1 & t \\
\end{bmatrix}.
\]

Now, we take \( g_3 = g_4 = -\frac{1}{2} \) in (3.5c) to destroy \( l \). Furthermore, we can destroy \( g \) or \( k \) by taking \( g_5 = g \) or \( g_5 = k \) in (3.5c), respectively. However, if we change \( X_2 \leftrightarrow X_3 \) and \( X_4 \leftrightarrow X_5 \) then they will
coincide. So, we take \( g_5 = k \) to destroy \( k \) and get

\[
A = \begin{bmatrix}
0 & 0 & g \\
-1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -t \\
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

Since \( g^2 + t^2 \neq 0 \) to avoid subcase 3.3, we normalize \( t = 1 \) by \( G_2 = \text{diag}(1, 1, t, t) \) if \( g = 0 \); otherwise, we normalize \( g = 1 \) by \( G_2 = \text{diag}(g, 1, 1, g) \). It creates the following Lie algebras:

\[
L_{15}: \quad A = \begin{bmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & 1 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

\[
L_{16}^a: \quad A = \begin{bmatrix}
0 & 0 & 1 \\
-1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & a \\
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

3.4. Testing isomorphism

So far, we have done stage 1 in Subsection 2.1 for \( g_{5,2} \). Results in Subsections 3.1, 3.2 and 3.3 show that we have constructed the list \( \mathcal{L}_3 \) which consists of sixteen families of 7-dimensional indecomposable solvable \( \mathbb{F} \)-Lie algebras having nilradical \( g_{5,2} \). However, \( \mathcal{L}_3 \) is not optimal since some families may be redundant. The goal of this subsection is to proceed stage 2, i.e., to test isomorphism between the obtained Lie algebras by the technique pointed out in Subsection 2.2. There are two steps as follows.

A. The first step is to reduce \( L_1^a, L_4^ab, L_6^a, L_8^a, L_{10}^a, L_{12}^a, L_{13}^a, L_{14}^a \) and \( L_{16}^a \).

(a) For \( L_1^a \), since \( \text{diag}(1, 1, 1, 1, 1) \) is an isomorphism \( L_1^a \neq 0 \cong L_1^1 \), we can reduce \( \sigma \) to \( \sigma \in \{0, 1\} \).

(b) For \( L_4^ab \), the transformation

\[
\begin{bmatrix}
-1 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & a & b
\end{bmatrix} \quad (b \neq 0)
\]

gives rise to an isomorphism \( L_4^ab \cong L_4^{(-\frac{a}{b})} \). This means that two pairs \((a, b)\) and \((-\frac{a}{b}, \frac{1}{b})\) determine isomorphic algebras.

(c) \( L_6^a \) and \( L_8^a \) cannot be reduced since each parameter determines a unique Lie algebra, i.e., \( L_6^a \cong L_6^b \) and \( L_8^a \cong L_8^b \) if and only if \( a = b \).
(d) $L_{10}^a \cong L_{10}^\frac{1}{a}$ by
\[
\begin{bmatrix}
\frac{1}{a} & 0 & -\frac{1}{a} \\
0 & a & 0 \\
a & 0 & 1 \\
1 & 0 & \frac{1}{a}
\end{bmatrix}
\] (a \neq 0).

(e) $L_{12}^a$ cannot be reduced: $L_{12}^a \cong L_{12}^b$ if and only if $a = b$.

(f) We have $L_{13}^a \cong L_{13}^{-a}$ and $L_{14}^{a-b} \cong \text{diag}(1,-1,1,-1,1,1)$.

(g) For $L_{16}^a$, we have $L_{16}^a \cong L_{16}^{-a}$ by diag$(1,-1,1,-1,1,1,1)$.  

B. In the second step, we test isomorphism between Lie algebras in different families. To this end, we first group families into three groups with respect to their forms of structure matrices, i.e., three groups corresponding to Subsections 3.1, 3.2 and 3.3. Afterwards, these groups can be split into subgroups by using the dimensions of centers and further forms of structure matrices. By this way, we need to test isomorphism of families in the inside of eight groups as follows:

$A_{1.1} := \{L_1\}$, $A_{1.2} := \{L_2, L_3, L_4^{ab}\}$, $A_{1.3} := \{L_5, L_6^a\}$,

$A_{1.4} := \{L_7, L_8^a\}$, $A_{1.5} := \{L_9, L_{10}^a\}$, $A_2 := \{L_{11}, L_{12}^a\}$,

$A_{3.1} := \{L_{13}, L_{14}^{ab}\}$, $A_{3.2} := \{L_{15}, L_{16}^a\}$.

For $A_{1.1}$, the test does not arise. To save more testing times, we further use the characteristic series of Lie algebras, i.e., the derived series and the lower central series. After checking these series, we do not need to test isomorphism in $A_{1.2}$ because of different dimensions of ideals in the derived series. Consequently, we just need to test isomorphism in six remaining groups. The result is that we cannot reduce $L_3$, i.e., $L_3$ consists of 16 families of $F$-Lie algebras.

Remark 3.2. All 16 families above are also valid over $\mathbb{C}$, except for $L_{13}^{ab}, L_{14}^a, L_{15}^a$ and $L_{16}^a$. They disappear since the Jordan block $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ in the structure matrices does not exist over $\mathbb{C}$.

4. Two main theorems

In this section, we formulate the two main results of this paper. The first one is four lists $L_1, L_2, L_3$ and $L_4$ which consist of 7-dimensional real and complex indecomposable solvable Lie algebras with nilradicals $(g_1)^2 \oplus g_3, g_1 \oplus g_4, g_5,2$ and $g_5,4$, respectively. Detailed computations of $L_3$ are given in Section 3. For $L_1, L_2$ and $L_4$, the computations are absolutely similar to that of Section 3, however, they are quite long. For convenience, we sum up these lists in Theorem 1.

Theorem 1. Four lists $L_1, L_2, L_3$ and $L_4$ are as follows.

1. $L_1$ contains 99 (resp., 57) families of real (resp., complex) Lie algebras which are in Table A.1.
2. $L_2$ contains 12 families of real and complex Lie algebras which are in Table A.2.
3. $L_3$ contains 16 (resp., 12) families of real (resp., complex) Lie algebras which are in Table A.3.
4. $L_4$ contains precisely one real and complex Lie algebra which is in Table A.4.

Tables A.1, A.2, A.3 and A.4 are given in the Appendix.
extensions of \((g_1)^5, g_{5,1}, g_{5,3}, g_{5,5}\) and \(g_{5,6}\) were investigated. For the sake of completeness, we sum up these results here.

1. **Nilradical** \((g_1)^5\). Ndogmo and Winternitz [23] presented a procedure to classify all finite-dimensional solvable Lie algebras with abelian nilradical. By this procedure, we obtain 31 (resp., 23) families of 7-dimensional real (resp., complex) indecomposable Lie algebras with 5-dimensional abelian nilradicals.

2. **Nilradical** \(g_{5,1}\). The nilradical \(g_{5,1}\) is the 5-dimensional Heisenberg Lie algebra \(h_5\). Rubin and Winternitz [27, Table A2] presented a table which consists of 27 (resp., 8) families of 7-dimensional real (resp., complex) indecomposable solvable Lie algebras with nilradical \(h_5\).

3. **Nilradical** \(g_{5,3}\). Šnobl and Karásek [32] classified solvable extension of nilradical \(n_{5,3}\) in which \(g_{5,3} \cong n_{5,3}\). Due to [32, Theorem 2], there is precisely one 7-dimensional real and complex indecomposable solvable Lie algebra with nilradical \(n_{5,3}\) as follows: \(A = \text{diag}(1, 0, 1, 0, 1), B = \text{diag}(2, 2, 1, 1, 0), [X, Y] = 0\). By the procedure in Section 2, we obtain the same algebra.

4. **Nilradical** \(g_{5,5}\). Šnobl and Winternitz [33] classified solvable extension of nilradical \(n_{5,1}\) in which \(g_{5,5} = n_{5,1}\). Due to [33, Theorem 3], there is precisely one 7-dimensional real and complex indecomposable solvable Lie algebra with nilradical \(n_{5,1}\) as follows: \(A = \text{diag}(3, 2, 1, 0, 1), B = \text{diag}(1, 1, 1, 0), [X, Y] = 0\). By the procedure in Section 2, we also obtain the same algebra.

5. **Nilradical** \(g_{5,6}\). Šnobl and Winternitz [34] classified solvable extension of nilradical \(n_{5,2}\) in which \(g_{5,6} \cong n_{5,2}\). Due to [34, Theorem 1], all real and complex solvable extensions of \(n_{5,2}\) must be \((n + 1)\)-dimensional. In other words, there is no 7-dimensional real and complex indecomposable solvable Lie algebra with nilradical \(n_{5,2}\). By the procedure in Section 2, we also obtain the same result.

Combining all above results with Theorem 1, our second main result is:

**Theorem 2.** There are 188 (resp., 115) families of 7-dimensional real (resp., complex) indecomposable solvable Lie algebras with 5-dimensional nilradicals. These amounts are distributed as follows:

| Nilradicals | \((g_1)^5\) | \((g_1)^2 \oplus g_3\) | \(g_1 \oplus g_4\) | \(g_{5,1}\) | \(g_{5,2}\) | \(g_{5,3}\) | \(g_{5,4}\) | \(g_{5,5}\) | \(g_{5,6}\) |
|-------------|-------------|-------------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Over \(\mathbb{R}\) | 31          | 99               | 12               | 27          | 16          | 1           | 1           | 1           | 0           |
| Over \(\mathbb{C}\) | 23          | 57               | 12               | 8           | 12          | 1           | 1           | 1           | 0           |

5. **Complete classification of 7-dimensional solvable Lie algebras**

As mentioned in Section 1, for a 7-dimensional Lie algebras \(L\), its nilradical \(N(L)\) satisfies \(\dim N(L) \in \{4, 5, 6, 7\}\). Hindeleh and Thompson [12], Parry [25] and Gong [11] classified 7-dimensional Lie algebras \(L\) such that \(\dim N(L) \in \{4, 6, 7\}\). Here, we summarize these results.

- Gong [11] classified 7-dimensional indecomposable nilpotent Lie algebras over \(\mathbb{R}\) and algebraically closed fields in which there are 149 (resp., 125) families of real (resp., complex) Lie algebras. Precisely isomorphic conditions for families containing parameters are also included.
- Parry [25] classified 7-dimensional real indecomposable solvable Lie algebras with 1-codimensional nilradicals in which there are 594 families of real Lie algebras. By performing a procedure as in Remark 2.1, we obtained 525 families of complex Lie algebras. As mentioned in Example 2.2, Parry’s list should be refined more to avoid redundancy.
- Hindeleh and Thompson [12] classified 7-dimensional real and complex indecomposable solvable Lie algebras with 4-dimensional nilradicals. There are 8 (resp., 2) families of real (resp., complex) Lie algebras. This results also should be refined more, in particular, parameters’ conditions to avoid decomposability and redundancy. For instance, the condition of parameters of algebra 7.2(ab) in [12, Section 6] should be \(ab \neq 0\) instead of \(a^2 + b^2 \neq 0\) since both 7.2(0b) and 7.2(a0) are decomposable. Moreover, by using our testing isomorphism procedure in Subsection 2.2, we can see that two pairs
(a, b) and (b, a) determine isomorphic Lie algebras. Hence, we can reduce parameters to \( a \geq b \) and \( ab \neq 0 \) (over \( \mathbb{R} \)) or \( |a| \geq |b| > 0 \) (over \( \mathbb{C} \)). The other families can also be refined appropriately.

To sum up, we have the following theorem:

**Theorem 3.** The class of 7-dimensional solvable Lie algebras consists of 939 and 767 families of real and complex Lie algebras, respectively.

**Funding**

This research is supported by the project SPD2019.01.37.

**References**

[1] Bianchi, L. (1903). *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*. Pisa: E Spoerri Libraio-Editore.

[2] Boza, L., Echarte, F. J., Núez, J. (1994). Classification of complex filiform Lie algebras of dimension 10. *Algebras Groups Geom.* 11(2):53–276.

[3] Boza, L., Fedriani, E. M., Núez, J. (2003). Complex filiform Lie algebras of dimension 11. *Appl. Math. Comput.* 141(2–3):611–630.

[4] Cartan, E. (1984). Sur la structure des groupes de transformations finis et continus [Thèse]. Paris: Faculté des sciences de Paris, Académie de Paris.

[5] Chen, L. (2012). A class of solvable Lie algebras with triangular decompositions. *Commun. Algebra.* 40(7):2285–2300.

[6] Cox, D. A., Little, J., O’Shea, D. (2015). *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Cham: Springer.

[7] de Graaf, W. A. (2005). Classification of solvable Lie algebras. *Exp. Math.* 14(1):15–25.

[8] Dixmier, J. (1958). Sur les représentations unitaires des groupes de Lie nilpotents III. *Canad. J. Math.* 10:321–348.

[9] Gantmacher, F. R. (1939). On the classification of real simple Lie groups. *Sb Math.* 5(2):217–250.

[10] Gerdt, V. P., Lassner, W. (2003). Isomorphism verification for complex and real Lie algebras by Gröbner basis technique. In: Ibragimov, N. H., Torrisi, M., Valenti, A., eds. *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*. Proceedings of the International Workshop; 1992 October 27–31; Acireale, Catania, Italy. The Netherlands: Springer, pp. 245–254.

[11] Gong, M. P. (1998). Classification of nilpotent Lie algebras of dimension 7 (Over algebraically closed fields and \( \mathbb{R} \)) [PhD dissertation]. Waterloo, Ontario, CA: University of Waterloo. [http://hdl.handle.net/10012/1148](http://hdl.handle.net/10012/1148).

[12] Hindeleh, F., Thompson, G. (2008). Seven dimensional Lie algebras with a four-dimensional nilradical. *Algebras Groups Geom.* 25(3):243–265.

[13] Jacobson, N. (1962). *Lie Algebras*. New York: Dover.

[14] Kobotis, A., Tsagas, G. (1990). Invariants of nilpotent Lie algebras of dimension eight. *J. Inst. Math. Comput. Sci. Math. Ser.* 3(3):345–356.

[15] Lie, M. S., Engel, F. (1893). *Theorie der Transformationsgruppen III*. Leipzig: B. G. Teubner.

[16] Levi, E. E. (1905). Sulla struttura dei gruppi finiti e continui. *Atti Accad Sci Torino Cl Sci Fis Mat Natur.* 40:551–565.

[17] Malcev, A. I. (1945). On solvable Lie algebras. *Izv Ross. Akad. Nauk. Ser. Mat.* 9(5):329–356.

[18] Morozov, V. V. (1958). Classification of nilpotent Lie algebras of dimension 6. *Izv Vyssh Uchebn Zaved Mat.* 4:161–171.

[19] Mubarakzyanov, G. M. (1963). On solvable Lie algebras. *Izv. Vyssh. Uchebn. Zaved. Mat.* 1:114–123.

[20] Mubarakzyanov, G. M. (1963). Classification of real structures of Lie algebras of fifth order. *Izv. Vyssh. Uchebn. Zaved. Mat.* 3:99–106.

[21] Mubarakzyanov, G. M. (1963). Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element. *Izv. Vyssh. Uchebn. Zaved. Mat.* 4:104–116.

[22] Mubarakzyanov, G. M. (1966). Some theorems on solvable Lie algebras. *Izv. Vyssh. Uchebn. Zaved. Mat.* 6:95–98.

[23] Ndogmo, J. C., Winternitz, P. (1994). Solvable Lie algebras with Abelian nilradicals. *J. Phys. A Math. Gen.* 27:405–423.

[24] Nguyen, A. T., Le, A. V., Vo, N. T. (2021). Testing isomorphism of complex and real Lie algebras. [https://arxiv.org/abs/2102.10770](https://arxiv.org/abs/2102.10770), arXiv:2102.10770, 14pp.

[25] Parry, A. R. (2007). A classification of real indecomposable solvable Lie algebras of small dimension with codimension one nilradicals [Master thesis]. Logan, UT: Utah State University. [https://digitalcommons.usu.edu/etd/7145](https://digitalcommons.usu.edu/etd/7145).

[26] Romdhani, M. (1989). Classification of real and complex nilpotent Lie algebras of dimension 7. *Linear Multilinear Algebra.* 24(3):167–189.

[27] Rubin, J. R., Winternitz, P. (1993). Solvable Lie algebras with Heisenberg ideals. *J. Phys. A Math. Gen.* 26:1123–1138.
Appendix

All of the four following tables will consist of three columns as follows.

- Algebras in the first column denoted by $L$ exist both over $\mathbb{C}$ and $\mathbb{R}$, while those denoted by $R$ indicate that they only exist over $\mathbb{R}$.
- The second column contains triples $(A, B, [X, Y])$ in which the disappearance of $[X, Y]$ means that $[X, Y] = 0$. For convenience, we denote respectively by $(a_1, \ldots, a_5)$, $E_{ij}$ and $S_{ab}$ the diagonal matrix \( \text{diag}(a_1, \ldots, a_5) \), the 5-square matrix whose only non-zero entry is 1 in row $i$ column $j$, and $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.
- The final column contains additional conditions of parameters in which “$\equiv$” means that these parameters yield isomorphic algebras. If there is no condition then parameters are arbitrary, and if “$\equiv$” disappears then parameters are optimal in the sense that different parameters give rise to non-isomorphic algebras.
- Conventions: $\sigma, \sigma' \in \{0, 1\}$, $\epsilon \in \{0, \pm 1\}$ and $\delta = \pm 1$.

For instance, $L_{1,1}^\sigma$ having structure $(1, -1, 0, 0, 0), (0, 0, 0, 0, 1), \sigma X_3 + X_4$ is a family of complex and real Lie algebras such that

$$A = \text{diag}(1, -1, 0, 0, 0), \quad B = \text{diag}(0, 0, 0, 0, 1), \quad [X, Y] = \sigma X_3 + X_4,$$

in which $\sigma \in \{0, 1\}$. Similarly, $R_{2,1}^{\sigma\epsilon\delta}$ having structure $(a, -a, 0, S_{01}, (0, 0, 0, 1, 1), \sigma X_3$ with $a \geq 0, (a, \sigma) \neq (0, 0)$ indicates a family of real Lie algebras such that

$$A = \begin{bmatrix} a & -a \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \text{diag}(0, 0, 0, 1, 1), \quad [X, Y] = \sigma X_3,$$

in which $a \geq 0, \sigma \in \{0, 1\}$ and $(a, \sigma) \neq (0, 0)$. 

\[28\] Seeley, C. (1993). 7-dimensional nilpotent Lie algebras. Trans. Amer. Math. Soc. 335(2):479–496.

\[29\] Shabanskaya, A., Thompson, G. (2013). Solvable extensions of a special class of nilpotent Lie algebras. Arch. Math. (Brno). 49(3):141–159.

\[30\] Shabanskaya, A. (2016). Solvable indecomposable extensions of two nilpotent Lie algebras. Commun. Algebra. 44(3):3626–3667.

\[31\] Šnobl, L. (2011). Maximal solvable extensions of filiform algebras. Arch. Math. (Brno). 47(5):405–414.

\[32\] Šnobl, L., Karásek, D. (2010). Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras. Linear Algebra Appl. 432:1836–1850.

\[33\] Šnobl, L., Winternitz, P. (2009). All solvable extensions of a class of nilpotent Lie algebras of dimension $n$ and degree of nilpotency $n - 1$. J. Phys. A Math. Gen. 2009:42:1–16.

\[34\] Šnobl, L., Winternitz, P. (2012). Solvable Lie algebras with Borel nilradicals. J Phys A Math Theor. 45(095202):18.

\[35\] Šnobl, L., Winternitz, P. (2014). Classification and Identification of Lie Algebras, Vol. 33 of CRM Monograph Series. Providence, RI: American Mathematical Society.

\[36\] Tsagas, G. (1999). Classification of nilpotent Lie algebras of dimension eight. J. Inst. Math. Comput. Sci. Math. Ser. 12(3):179–183.

\[37\] Tsagas, G., Kobotis, A. (2000). Classification of nilpotent Lie algebras of dimension nine whose maximum abelian ideal is of dimension seven. Int. J. Comput. Math. 74(1):5–28.

\[38\] Wang, Y., Lin, J., Deng, S. (2008). Solvable Lie algebras with quasifiliform nilradicals. Commun. Algebra. 36(11):4052–4067.
### Table A.1. Solvable Lie algebras with nilradical \((g_1)^2 \oplus g_3\)

| No. | \((A, B, [X, Y])\) | Notes |
|-----|---------------------|-------|
| \(L_{1,1}^{\sigma}\) | \((1, -1, 0, 0, 0), (0, 0, 0, 0, 1), \sigma X_3 + X_4\) | \((a, b), (a, \sigma), (b, \sigma) \neq (0, 0); \ (a, b, \sigma) \equiv \left( \pm \frac{\sigma}{B}, \frac{1}{B}, \sigma \right)\) |
| \(L_{1,2}^{\sigma}\) | \((0, 0, 0, 0, 1), (0, 0, 0, 0, 1), X_3\) | |
| \(L_{1,3}^{\sigma}\) | \((1, -1, 0, 0, a), (0, 0, 0, 1, b), \sigma X_3\) | |
| \(L_{1,4}\) | \((0, 0, 0, 0, 1), (1, -1, 0, 0, 0) + E_{14}, X_3\) | |
| \(L_{1,5}^{\rho}\) | \((1, -1, 0, 0, a), (0, 0, 0, 0, 1) + E_{14}, \sigma X_3\) | |
| \(L_{1,6}\) | \((0, 0, 0, 0, 1), (1, -1, 0, 0, 0) + E_{43}, X_4\) | |
| \(L_{1,7}^{\rho}\) | \((1, -1, 0, 0, a), (0, 0, 0, 0, 1) + E_{43}, \sigma X_4\) | |
| \(L_{1,8}^{\rho}\) | \((a, b) \neq (0, -1); (a, b) \equiv \left(-\frac{a}{B}, \frac{1}{B}\right)\) | |
| \(L_{1,9}\) | \((0, 0, 0, 1, 0), (1, 1, 0, 0, 0) + E_{14}, X_4\) | |
| \(L_{1,10}\) | \((0, 1, 1, 0, a), (0, 0, 0, 0, 1) + E_{14}, \sigma X_4\) | |
| \(L_{1,11}\) | \((0, 1, 1, 0, 0), (1, 1, 0, 1) + E_{15}, X_4\) | |
| \(L_{1,12}\) | \((1, a, 1 + a, 0, 1), (0, 1, 1, 0, 0) + E_{15}, X_4\) | |
| \(L_{1,13}\) | \((a, 1 + a, 0, 1, a), (0, 1, 1, 1, 1) + E_{33}, X_4\) | |
| \(L_{1,14}^{abc}\) | \((0, a, b, 1), (1, c, 1 + c, 0, 0)\) | |
| \(L_{1,15}^{abcd}\) | \((a, b, a + b, 0, 1), (c, d, c + d, 1, 0)\) | |
| \(L_{1,16}\) | \((0, 1, 1, 0, a), (1, 0, 1, 1, b) + E_{14}\) | |
| \(L_{1,17}\) | \((1, a, 1 + a, b), (0, 0, 0, 0, 1) + E_{14}\) | |
| \(L_{1,18}\) | \((b, c) \neq (0, 0)\) | |
| \(L_{1,19}\) | \((a, b, c) \neq (0, 0, 0)\) | |
| \(L_{1,20}\) | \((a, b, a + b, 0, 1), (1, c, 1 + c, 1 + c, 0) + E_{43}\) | |
| \(L_{1,21}\) | \((1, 0, 1, 1, 1) + E_{14}, (0, 1, 1, 0, 0) + E_{15}\) | |
| \(L_{1,22}\) | \((0, 0, 0, 0, 1) + E_{14}, (1, 1, 2, 1, 0) + E_{24}\) | |
| \(L_{1,23}\) | \((0, 1, 1, 0, 1) + E_{14} + E_{25}, (1, 0, 1, 0, 1) + aE_{14} + bE_{25}\) | |
| \(L_{1,24}\) | \((0, 1, 1, 0, 1) + E_{25}, (1, 0, 1, 1, 0) + E_{14}\) | |
| \(L_{1,25}\) | \((0, 1, 1, 0, 1) + E_{14}, (1, 0, 1, 1, 0) + E_{25}\) | |
| \(L_{1,26}\) | \((0, 0, 0, 0, 1) + E_{43}, (1, 0, 1, 1, 0) + E_{14} + aE_{43}, \sigma X_2\) | |
| \(L_{1,27}\) | \((0, 0, 0, 0, 1) + E_{14}, (1, 0, 1, 1, 0) + aE_{14} + E_{43}, X_2\) | |
| \(L_{1,28}\) | \((0, 0, 0, 0, 1) + E_{14} + E_{43}, (1, 0, 1, 0, 1) + aE_{14} + bE_{43}, (b - a)X_2\) | |
| \(L_{1,29}\) | \((1, 0, 1, 1, 1), (0, 1, 1, 0, 1) + E_{14} + E_{53}\) | |
| \(L_{1,30}\) | \((0, 1, 1, 1, 1) + E_{53}, (0, 1, 1, 0, 1) + E_{14} + aE_{53}\) | |
| \(L_{1,31}\) | \((1, 0, 1, 1, 1) + E_{14}, (0, 1, 1, 0, 1) + aE_{14} + E_{53}\) | |
| \(L_{1,32}\) | \((1, 0, 1, 1, 1) + E_{14} + E_{53}, (0, 1, 1, 0, 1) + aE_{14} + bE_{53}\) | |
| \(L_{1,33}\) | \((1, 1, 0, 1, 1) + E_{43}, (0, 1, 1, 1, 1) + E_{53}\) | |
| \(L_{1,34}\) | \((1, 0, 1, 1, 1) + E_{43}, (0, 1, 1, 1, 1) + E_{53}\) | |
| \(L_{1,35}\) | \((1, -1, 0, 0, 0) + E_{55}, (0, 0, 0, 0, 1), \sigma X_3\) | |
Table A.1. (Continued)

| No. | \((A, B, [X, Y])\) | Notes |
|-----|-----------------|-------|
| \(L_{4.1}\) | \((0, 0, 0, 0, 1) + E_{12}, (0, 0, 0, 1, a), \epsilon X_3\) | \((a, \epsilon) \neq (0, 0); \overline{\mathbb{R}}: (a > 0, \epsilon) \equiv \left(\frac{1}{2}, \epsilon\right), \overline{\mathbb{C}}: \epsilon \in (0, 1), (a, \epsilon) \equiv \left(\frac{1}{2}, \epsilon\right)\) |
| \(L_{4.2}\) | \((1, 1, 2, 0, 0) + E_{12}, (0, 0, 0, 1, 0), X_5\) | \(a, b \neq 0; (a, b) \equiv \left(-\frac{a}{b}, 1\right)\) |
| \(L_{4.3}\) | \((0, 0, 0, 1, 0) + E_{12}, (1, 1, 2, 0), X_5\) | \((a, c) \neq (0, 0); (a, b, c) \equiv \left(\frac{1}{2}, c, b\right)\) |
| \(L_{4.4}\) | \((1, 1, 2, 0, a) + E_{12}, (0, 0, 0, 1, b)\) | |
| \(L_{4.5}\) | \((0, 0, 0, 1, a) + E_{12}, (1, 1, 2, b, c)\) | |
| \(L_{4.6}\) | \((1, 1, 2, 1, 0) + E_{12} + aE_{24}, (0, 0, 0, 1, 1) + E_{14}\) | |
| \(L_{4.7}\) | \((0, 0, 0, 0, 1) + E_{12}, (1, 1, 2, 1, a) + E_{14}\) | |
| \(L_{4.8}\) | \((0, 0, 0, 1, 0) + E_{12} + E_{24}, (1, 1, 2, 1, a) + bE_{14}\) | |
| \(L_{4.9}\) | \((1, 1, 2, 2, 0) + E_{12} + aE_{43}, (0, 0, 0, 1, 1) + E_{43}\) | |
| \(L_{4.10}\) | \((0, 0, 0, 0, 1) + E_{12}, (1, 1, 2, 2, a) + E_{43}\) | |
| \(L_{4.11}\) | \((0, 0, 0, 1, 0) + E_{12} + E_{43}, (1, 1, 2, 2, a) + bE_{43}\) | |
| \(L_{4.12}\) | \((1, 1, 2, 2, 0) + E_{12} + E_{45}, (0, 0, 0, 1, 1) + aE_{45}\) | |
| \(L_{4.13}\) | \((0, 0, 0, 1, 1) + E_{12} + E_{45}, (1, 1, 2, a, b) + bE_{45}\) | |

\(\sigma, \alpha, \gamma, \epsilon, \delta, \sigma, \sigma', \alpha'\) refer to specific conditions or values.

Continued...
### Table A.1. (Continued)

| No. | $(A,B,[X,Y])$ | Notes |
|-----|---------------|-------|
| $\alpha_{8.8}^{ab}$ | $(S_{20}, 2a, 0, 0) + E_{45}, (0, 0, 0, 1, 1) + bE_{45}$ | $a > 0, b \geq 0$ |
| $\beta_{8.8}^{ab}$ | $(S_{20}, 0, a, a), (1, 1, 2, b) + \alpha E_{45}$ | $a \geq 0, (a,b) \neq (0,0)$ |
| $\gamma_{8.7}^{ab}$ | $(S_{21}, 0, a, a) + E_{45}, (1, 1, 2, b) + cE_{45}$ | $a,c \geq 0, (a,b) \neq (0,0)$ |
| $\delta_{8.8}^{ab}$ | $(S_{21}, 2a, 2a, 2a) + E_{45}, (1, 1, 2, 2, 2) + E_{43} + bE_{45}$ | $(a,0) \equiv (c,0); (0,b) \equiv (0,-b); (a, \frac{1}{2}) \equiv (b, \frac{1}{2})$; $(a,b) \equiv (a, \pm \frac{b}{1-ab}); ab \neq 1; (a,b) \equiv (c,d)$, $\frac{b}{a} = \pm \frac{1}{1-ab} \neq 0, ab, cd \neq 1$ |

### Table A.2. Solvable Lie algebras with nilradical $g_1 \oplus g_4$

| No. | $(A,B,[X,Y])$ | Notes |
|-----|---------------|-------|
| $L_1$ | $(0,0,0,0,1), (1, -2, -1, 0, 0), X_4$ | |
| $L_2$ | $(0,1,1,1,0), (1,0,1,2,0), X_5$ | |
| $L_3$ | $(0,1,1,0,0), (a,0,2a,1)$ | $a \neq 0$ |
| $L_4^{ab}$ | $(1,a,1+a,2+a,0), (0,b,b,b,1)$ | $b \neq 0; L_4^{ab \neq 0}$ should be added to Wang et al. [41, Theorem 3] |
| $L_5^{ab}$ | $(1,1,2,3,a), (0,0,0,0,1) + E_{12}$ | |
| $L_6^{ab}$ | $(0,1,1,1,0), (1,0,1,2,1) + E_{15}$ | |
| $L_7^{ab}$ | $(1,a,1+a,2+a,1), (0,1,1,1,0) + E_{15}$ | |
| $L_8^{ab}$ | $(0,1,1,0,a), (0,0,0,0,1) + E_{24}$ | Over $C^*$; $\delta = 1$ |
| $L_9^{ab}$ | $(0,1,1,1), (1,0,1,2,0) + E_{25}$ | Should be added to Wang et al. [41, Theorem 3] |
| $L_{10}^{ab}$ | $(1,a,1+a,2+a,a), (0,1,1,1,1) + E_{25}$ | Should be added to Wang et al. [41, Theorem 3] |
| $L_{11}^{ab}$ | $(0,1,1,1,1), (1,0,1,2,2) + E_{54}$ | |
| $L_{12}^{ab}$ | $(1,a,1+a,2+a,2+a), (0,1,1,1,1) + E_{54}$ | |

### Table A.3. Solvable Lie algebras with nilradical $g_{5.2}$

| No. | $(A,B,[X,Y])$ | Notes |
|-----|---------------|-------|
| $L_1^{a1}$ | $(1,-1,0,0,1), (0,0,1,0,1), \sigma X_4$ | |
| $L_2^{a2}$ | $(1,0,1,1), (0,0,1,1)$ | |
| $L_3^{a3}$ | $(0,1,0,1,0), (0,0,1,0,1)$ | |
| $L_4^{ab}$ | $(1,0,a,1,1+a), (0,1,b,1,b)$ | $(a,b) \neq (-1,0); (a,b) \equiv \left( -\frac{a}{b}, \frac{1}{b} \right)$ |
| $L_5^{a5}$ | $(0,0,1,0,1), (1,0,2,1) + E_{12}$ | |
| $L_6^{a6}$ | $(1,1,a,2,1+a), (0,0,1,0,1) + E_{12}$ | |
| $L_7^{a7}$ | $(0,1,1,1,1), (1,1,0,2,1) + E_{25}$ | |
| $L_8^{a8}$ | $(1,1+a,a,2+a,1+a), (0,1,1,1,1) + E_{25}$ | |
| $L_9^{a9}$ | $(0,0,1,0,1), (0,1,0,1,0) + E_{35}$ | $a \equiv \frac{1}{a}$ |
| $L_{10}^{a10}$ | $(0,a,1,a,0), (0,0,1,0,1) + E_{35}$ | |
| $L_{11}^{a11}$ | $(0,1,1,1,1), (1,0,0,1,1) + E_{23} + E_{45}$ | |
| $L_{12}^{a12}$ | $(1,a,a,1+a,1+a), (0,1,1,1,1) + E_{23} + E_{45}$ | |
| $L_{13}^{a13}$ | $(0,1,1,1,1), (a, S_{50}, S_{51})$ | $a \geq 0$ |
| $L_{14}^{a14}$ | $(a, a, 1+a, 1+a), (0, S_{50}, S_{51})$ | $b \geq 0$ |
| $L_{15}^{a15}$ | $(0, S_{50}, S_{51}), (0,1,1,1,1) + E_{25} - E_{34}$ | |
| $L_{16}^{a16}$ | $(0, S_{50}, S_{51}), (0,1,1,1,1) + a(E_{25} - E_{34})$ | $a \geq 0$ |

### Table A.4. Solvable Lie algebras with nilradical $g_{5.4}$

| No. | $(A,B,[X,Y])$ | Notes |
|-----|---------------|-------|
| $L_1$ | $(1,0,1,2,1), (0,1,1,1,2)$ | |