A reconstruction problem related to balance equations-II: the general case

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Discrete Mathematics 194, no. 1-3(1999) 281-284.
Mathematics Subject Classifications: 05C60

Abstract

A modified $k$-deck of a graph $G$ is obtained by removing $k$ edges of $G$ in all possible ways, and adding $k$ (not necessarily new) edges in all possible ways. Krasikov and Roditty asked if it was possible to construct the usual $k$-edge deck of a graph from its modified $k$-deck. Earlier I solved this problem for the case when $k = 1$. In this paper, the problem is completely solved for arbitrary $k$. The proof makes use of the $k$-edge version of Lovász’s result and the eigenvalues of certain matrix related to the Johnson graph.

This version differs from the published version. Lemma 2.3 in the published version had a typo in one equation. Also, a long manipulation of some combinatorial expressions was skipped in the original proof of Lemma 2.3, which made it difficult to follow the proof. Here a clearer proof is given.

1 Introduction

The graphs considered in this paper are simple and undirected, and are assumed to have $n$ vertices. The complement of $G$ is denoted by $G^c$. Let $N = \binom{n}{2}$. Let $U_m$ denote the collection of all unlabelled $n$-vertex, $m$-edge
graphs. We define three matrices $\Delta_i$, $D_i$ and $d_i$ as follows. The rows and columns of $\Delta_i$ and $D_i$ are indexed by the members of $U_m$. The $kl$-th entry of $\Delta_i$ is the number of graphs isomorphic to $G_k$ that can be obtained by removing $i$ edges from $G_l$ and then adding $i$ edges. Here the added edges need not be different from the removed edges. The entries of $D_i$ are similarly defined with an additional condition that the removed set of edges and the added set of edges be disjoint. The rows of $d_i$ are indexed by $F_k \in U_{m-i}$, and its columns are indexed by $G_l \in U_m$. The $kl$-th entry of $d_i$ is the number of $i$-edge deleted subgraphs of $G_l$ that are isomorphic to $F_k$. A set (or a multiset) $P$ of $m-i$-edge graphs is denoted by its characteristic vector $X_P$ of length equal to $|U_{m-i}|$. The characteristic vector of a singleton set $\{G\}$ is denoted by simply $X_G$. This has only one entry equal to 1 and other entries equal to 0. Thus, in our notation, the vector $d_kX_G$ represents the $k$-edge deck of $G$, (denoted by $k-ED(G)$), and the vector $\Delta_kX_G$ represents the modified $k$-deck of $G$, i.e., the collection of graphs obtained from $G$ by removing $k$ edges and then adding $k$ (not necessarily new) edges.

Krasikov and Roditty first introduced modified decks for the purpose of proving the reconstruction result of Müller. They asked if the $k$-edge deck of a graph could be constructed from its modified $k$-deck. In our notation, it is equivalent to asking if the vector $d_kX_G$ could be computed given the vector $\Delta_kX_G$. In [T], this problem was solved for the case when $k = 1$. Two proofs of this were offered there. In one proof, it was demonstrated that $\Delta_iX_G$ could be computed for $i > 1$ given $\Delta_1X_G$. The rest of the proof was based on the fact that Lovász’s edge reconstruction result in $k = 1$ case could be proved directly from modified decks, i.e., without knowing the 1-edge deck. In the second proof, which was based on the eigenvalues of Johnson graph, it was shown that Lovász’s result could be proved directly from $\Delta_1X_G$, thus avoiding the explicit construction of $\Delta_iX_G$, $i > 1$ in terms of $\Delta_1X_G$.

The proof for the general case presented here does involve construction of $\Delta_iX_G$ in terms of $\Delta_kX_G$, for $i \geq k$. But rest of the proof makes use of eigenvalues of Johnson graph.

## 2 Reconstructing $d_kX_G$ from $\Delta_kX_G$

In the following, we assume that for two graphs $G$ and $H$, we are given that $\Delta_kX_G = \Delta_kX_H$. We write $X = X_G - X_H$, therefore, $\Delta_kX = 0$. We first state two identities without proof. The first one - Lemma 2.1 - is equivalent
to Lemma 3.1 in [KR], and the second one - Lemma 2.2 - is Theorem 2.2 from [T].

**Lemma 2.1**
\[ \Delta_s = \sum_{i=0}^{s} \binom{m - i}{s - i} D_i \]

**Lemma 2.2**
\[ D_1 D_i = (m - i + 1)(N - m - i + 1)D_{i-1} + i(N - 2i)D_i + (i + 1)^2 D_{i+1}. \]

**Lemma 2.3**
\[ \Delta_{i+1} = \frac{1}{(i + 1)^2} \{ i(2m - N - i - 1)\Delta_0 + \Delta_1 \} \Delta_i \]

**Proof** From Lemma 2.2 we write
\[ (i + 1)^2 D_{i+1} = D_1 D_i - (m - i + 1)(N - m - i + 1)D_{i-1} - i(N - 2i)D_i \]

Substituting for \( D_{i+1} \) and \( D_i \) from Lemma 2.1 we have
\[
(i + 1)^2 \left( \Delta_{i+1} - \sum_{j=0}^{i} \binom{m - j}{i + 1 - j} D_j \right)
\]
\[ = D_1 \left( \Delta_i - \sum_{j=0}^{i-1} \binom{m - j}{i - j} D_j \right) - (m - i + 1)(N - m - i + 1)D_{i-1} \]
\[ - i(N - 2i) \left( \Delta_i - \sum_{j=0}^{i-1} \binom{m - j}{i - j} D_j \right) \]

In the first term on the RHS, we substitute \( D_1 \Delta_i = (\Delta_1 - m\Delta_0)\Delta_i, D_1 D_j; j > 0 \) from Lemma 2.2 and \( D_1 D_0 = D_1 \). Therefore,
\[(i + 1)^2 \Delta_{i+1} \]
\[= (\Delta_1 - m \Delta_0) \Delta_i - \binom{m}{i} D_1 \]
\[- \sum_{j=1}^{i-1} \binom{m-j}{i-j} (m-j+1)(N-m-j+1)D_{j-1} \]
\[- \sum_{j=1}^{i-1} \binom{m-j}{i-j} j(N-2j)D_j \]
\[- \sum_{j=1}^{i-1} \binom{m-j}{i-j} (j+1)^2 D_{j+1} \]
\[- (m-i+1)(N-m-i+1)D_{i-1} \]
\[- i(N-2i) \left( \Delta_i - \sum_{j=0}^{i-1} \binom{m-j}{i-j} D_j \right) + (i+1)^2 \sum_{j=0}^{i} \binom{m-j}{i+1-j} D_j \]

Two terms on the RHS contribute to \(D_i\) - the summation in the fourth line on the RHS, for \(j = i-1\), and the last summation in the last line on the RHS, for \(j = i\). Both these \(D_i\) terms are replaced by \(\Delta_i - \sum_{j=0}^{i-1} \binom{m-j}{i-j} D_j\). This leaves only terms containing \(D_j; j \leq i-1\). One can then verify that, after simplification of the RHS, all terms containing \(D_j; j \leq i-1\) cancel out, and we get

\[(i + 1)^2 \Delta_{i+1} = (i(2m-N-i-1)\Delta_0 + \Delta_1) \Delta_i \]

This completes the proof.

**Corollary 2.4** If \(\Delta_k X = 0\) then \(\Delta_i X = 0\) for all \(i \geq k\).

The following lemma is the \(k\)-edge version of Lovász’s result. This may be found in [GKR], but we only note here that the bound in the following result doesn’t depend upon the number of graphs in the collection \(P\).
Lemma 2.5 Let \(2p - k + 1 > N\), and let \(P\) and \(Q\) be collections of \(p\)-edge graphs such that \(d_kX_P = d_kX_Q\), then \(X_P = X_Q\).

Now we prove the main result of this section.

**Theorem 2.6** For collections \(P\) and \(Q\) of graphs, if \(\Delta_XP = \Delta_XQ\) then \(d_kX_P = d_kX_Q\).

**Proof** This is done by induction on \(k\). The result was proved in [T] for \(k = 1\). Let the result be true for \(k \leq r - 1\). Let \(\mathcal{P}' = \{F^c; F \in r - ED(P)\}\) and \(\mathcal{Q}' = \{F^c; F \in r - ED(Q)\}\). Here \(r - ED(P)\) denotes the multiunion of \(r\)-edge decks of graphs in \(P\). Note that \(\Delta_rX_P = \Delta_rX_Q\) is equivalent to \(d_rX_P' = d_rX_Q'\). This follows from the fact that for any \(F, A \in E(F)\) and \(B\) disjoint with \(E(F) - A\), \((F - A + B)^c = (F - A)^c - B\). Now, if \(2(N - m + r) - r + 1 > N\), then \(X_P' = X_Q'\), and \(d_rX_P = d_rX_Q\). Therefore, we assume the contrary that \(2m - r - 1 \geq N\), i.e., \(2m - r + 1 \geq N + 2\).

Now we demonstrate that either \(\Delta_{r-1}X_G = \Delta_{r-1}X_H\) or \(2m-r+1 \leq N+1\). We write,

\[\Delta_r = \frac{1}{r^2} \{(r - 1)(2m - N - r)\Delta_0 + \Delta_1\} \Delta_{r-1}\]

We are interested in the invertibility of \((r - 1)(2m - N - r)\Delta_0 + \Delta_1\).

**Definition 2.7** Johnson graph is a simple graph whose vertex set is the family of \(m\)-sets of an \(N\)-set. Two vertices \(U\) and \(V\) are adjacent if and only if \(|U \cap V| = m - 1\).

Let \(J\) be the adjacency matrix of the Johnson graph with parameters \(N = \binom{n}{2}\) and \(m\). Let the square matrix \(B\) be defined as follows. The rows and columns of \(B\) are indexed by all the labelled \(m\)-edge graphs on a fixed set of \(n\) vertices, and \(ij\)-th entry is the number of ways of removing an edge from \(G_j\) and adding an edge to get \(G_i\). Note that the diagonal entry is \(m\), since we can add the same edge that is removed. Other entries of \(B\) are either 0 or 1. The matrix \(A\) is defined similarly for unlabelled graphs with \(m\) edges and \(n\) vertices. Thus matrix \(A\) is the matrix \(\Delta_1\). Matrix \(P\) is defined by indexing the rows by unlabelled graphs and columns by labelled graphs, and the \(ij\)-th entry is 1 if the labelled graph \(G_j\) is isomorphic to the unlabelled graph \(G_i\). Other entries are 0. As in [ER], one can verify
that $AP = PB$, and every eigenvalue of $A$ is also an eigenvalue of $B$. But $B = mI + J$, therefore, its eigenvalues are $m + (m - j)(N - m - j) - j$, where $j \leq \min(m, N - m)$. Thus, eigenvalues of $(r - 1)(2m - r - N)\Delta_0 + \Delta_1$ are $(m - j)(N - m - j + 1) + (r - 1)(2m - r - N)$. If 0 is not an eigenvalue, then $\Delta_{r-1}(X_P - X_Q) = 0$, therefore, by induction hypothesis, $d_{r-1}(X_P - X_Q) = 0$, and $d_r(X_P - X_Q) = 0$ by Kelly’s lemma, (see [BH]). For one of the eigenvalues to be 0, $(r - 1)(2m - r - N) \leq 0$. Therefore, $r = 1$ (for which the problem is solved independently in [T]) or $2m \leq N + r$, i.e., $2m - r + 1 \leq N + 1$. This contradicts the inequality assumed earlier.

The theorem implies that the $k$-edge deck of a graph can be reconstructed from its modified $k$-deck.

Acknowledgements

This work was done while I was at Indian Institute of Technology, Guwahati, India. I would like to thank Philip Maynard for pointing out an error in one of the equations in the published version of the paper.

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