Roughness and avalanches in an extended Schelling model: an explanation of urban gentrification

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Residential segregation is analyzed via the Schelling model, in which two types of agents attempt to optimize their situation according to certain preferences and tolerance levels. Several variants of this work are focused on urban or social aspects. Whereas these models consider fixed values for wealth or tolerance, here we consider how sudden changes in the economic environment or the tolerance level affect the urban structure both in the closed city and open city frameworks, i.e. depending on whether migration processes are relevant or not. In the closed city framework, agents tend to group into clusters, whose boundary can be characterized using tools from kinetic roughening. On the other hand, in the open city approximation agents of a certain type may enter or leave the city in series of avalanches, whose statistical properties are discussed.

I. INTRODUCTION

People with similar features (culture, income, etc.) tend to group together in the same neighborhood, giving rise to segregation on a social scale. More than 40 years ago, Schelling put forward a seminal model that describes this reality, linking individual preferences to the macroscopic behaviour of the system [1]. Two different social groups, which we may call red and blue, are distributed over a square lattice with some vacancies on it. Agents are characterized by a tolerance $T$: the fraction of different agents in their neighborhood that he or she can tolerate. The model proceeds through the following dynamical rules: a random agent $i$ is selected and his/her fraction of diverse neighbors is evaluated. If this fraction value is lower or equal than $T$, the agent remains at his/her location. Otherwise, he/she relocates to the nearest vacancy that meets his/her demands. For intermediate values of $T$ we observe segregation, and clusters are formed with different types of agents.

This model has attracted a great deal of attention, due to its simplicity and insight, giving rise to a wealth of variants. System behaviour when one kind of agents are tolerant and the vacancies are differently priced was characterized in [2]. Differences between constrained models, where only unhappy individuals are allowed to move, and unconstrained ones, in which all agents can relocate to vacancies as long as they keep or increase their happiness, were also studied [3]. In [11], attempted relocations succeed with a probability which is modulated by a power-law linking their current happiness and the attractiveness of the offered place. The effect of the city shape, size and form is investigated in [5], finding that the properties of the system in equilibrium are weakly affected by these parameters. The authors of [8] proposed a thermodynamic approach to segregation based on their cluster geometry, and considered quantities analogous to the specific heat and susceptibility, along with a connection with spin-1 models. Moreover, an open city model in which agents can leave or enter the system was described in [9]. In addition to showing different kinds of interfaces between clusters, economic aspects of the system were introduced by means of a chemical potential. Recently, the use of different tolerance levels for the agents was proposed in [10], in a system with no vacancies, where agents could only exchange locations with agents of a different type. On the other hand, in [11] each cell of the system is considered a building containing many agents, and segregation was considered both at a microscopic and a macroscopic level, giving rise to a complex phase diagram. Some of the mentioned works take into account the importance of the initial conditions [2,10], and some others also consider migratory movements [9,10].

In this article we consider how a closed city, in which no agents can enter or leave, adapts to drops in the tolerance level. Under some circumstances, a vacancy interface is observed separating the main clusters. We establish its statistical properties, specially its roughness. On the other hand, we also consider an open city model in which the system becomes more hostile towards one type of agent (economic handicap) and more friendly towards the other (economic advantage). This phenomenon can give rise to a partial or total overtake of the favored type of agent, which may proceed through avalanches. These avalanches are shown to present a power-law behavior which is usual of similar processes [6,7]. Our general framework is established in similarity with the Blume-Emery-Griffiths (BEG) model in presence of an external magnetic field [8,9].

The paper is organized as follows. In Section [11] we define our BEG model and discuss the dynamics of both regimes (open and closed city) and the evolution process. In Section [12] we describe our results for the closed city model [11-17] and the open city model [12-18] linking them to well-known social mechanisms: confrontation between equal forces within the closed city or the overtaking of one kind of agents over the other due to economic superiority, which resembles a social process known as gentrification [12]. Our main conclusions and proposals for further work are discussed in section [14].
II. MODEL

The Blume-Emery-Griffith model [13] was introduced to study the behaviour of He$^3$-He$^4$ mixtures. In this model the spin values considered are $s_i = 0, \pm 1$. In the presence of a magnetic external field, the Hamiltonian can be written as:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} (J s_i s_j + K s_i^2 s_j^2) + \sum_i (D_B s_i^2 + H_B s_i),$$

(1)

where $\langle i, j \rangle$ stands for the eight nearest neighbors of a Moore neighborhood. This Hamiltonian represents a spin-1 Ising model with coupling constant $J$, biquadratic exchange constant $K$, crystal field $D_B$ and an external magnetic field of intensity $H_B$. The crystal field $D_B$ acts as a chemical potential that controls the entry of cells with non-zero spin value. Dissimilar entry fluxes for $s = +1$ and $s = -1$ are obtained by means of the magnetic field, $H_B$.

In our interpretation of the BEG model spin values will be associated with blue agents ($s_i = +1$), red agents ($s_i = -1$) and vacancies ($s_i = 0$). A positive value of $J$ yields a negative energy for each pair of neighboring agents of the same type, while a positive value of $K$ assigns a negative energy to every pair of neighboring agents, disregarding their type. If $D_B > 0$, the system reduces its energy by expelling agents, and if $H_B > 0$, the system reduces its energy either expelling blue agents or attracting red ones. Under certain conditions, the Hamiltonian provided by Eq. (1) can only decrease along the actual dynamical trajectories of the system, thus serving as a Lyapunov function [14].

In order to make an explicit connection between our physical model and social realities, let us now define a measure of the level of unhappiness of an agent in relation with the parameters of Eq. (1). The lack of happiness of agent $i$ is measured by the dissatisfaction index $I_{\text{dis}}(i)$

$$I_{\text{dis}}(i) = N_d(i) - T[N_s(i) + N_d(i)] + D + H(i),$$

(2)

where $N_s$ and $N_d$ are, respectively, the number of neighboring agents of the same (s) and different (d) type, $D$ is a measure of the global economic level and is the same for all agents. Meanwhile $H(i) = +H$ for blue agents and $-H$ for red ones. It can be understood as half the income gap between both types of agents. We will drop the dependency on $i$ when it is clear from context. The condition for satisfaction will be, therefore,

$$I_{\text{dis}}(i) \leq 0.$$  

(3)

Notice that when $D < 0$ the system is generally friendly towards agents, and unhappiness arising from their neighborhood can be endured. This is the common situation behind immigration processes; increased economic opportunities compensate the lack of homogeneity. On the other hand, when $D > 0$, the economic opportunities have disappeared and the system becomes hostile. Even when $N_s > N_d$, an agent might be forced to leave the system under such circumstances. Moreover, the possibility that the wealth levels of both communities are not similar is taken into account by the parameter $H$. When its absolute value is large, agents of one type might be forced to leave while the other type is attracted.

The number of similar and different neighbors can be easily obtained from the spin variables of sites neighboring $i$,

$$N_s(i) - N_d(i) = s_i \sum_{\langle i,j \rangle} s_j,$$

(4)

$$N_s(i) + N_d(i) = s_i^2 \sum_{\langle i,j \rangle} s_j^2,$$

(5)

where the sum over $\langle i, j \rangle$ should be understood as a sum over all $j$ which are neighbors of $i$. Substituting Eq. (4) and (5) into Eq. (1), we can rewrite our condition for the satisfaction of agent $i$, Eq. (3), as

$$-s_i \sum_{\langle j \rangle} s_j - (2T - 1) s_i^2 \sum_{\langle j \rangle} s_j^2 + 2D s_i^2 + 2H s_i \leq 0,$$

(6)

where $j$ runs over his/her eight closest neighbors in the Moore neighborhood. We will only allow moves that either preserve or reduce the dissatisfaction index for each agent, and their types are fixed. For constant values of $T$, $D$ and $H$, the energy of the BEG model becomes a Lyapunov function:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} (s_i s_j + (2T - 1)s_i^2 s_j^2) + 2 \sum_i (D s_i^2 + H s_i),$$

(7)

with an external magnetic field. This Hamiltonian represents a spin-1 Ising model with coupling constant $J = 1$, biquadratic exchange of strength $2T - 1$, crystal field of strength $2D$ and a magnetic field of intensity $2H$, as it can be seen by comparing with Eq. (1).

A. Closed city dynamics

Let us consider our agents to be living in an $N \times N$ square lattice with open boundaries and a fixed vacancy density $\rho$, in similarity to [3]. Agents will not be allowed to enter or leave the system and, therefore, the global economic level $D$ and income inequality $H$ need not be taken into account. In this case, Eq. (4) reduces to

$$I_{\text{dis}}(i) = N_d(i) - T[N_s(i) + N_d(i)].$$

(8)
At each iteration, a random occupied site $i$ and a random vacancy $j$ are selected. The proposed relocation is accepted if $I_{\text{dis}}(j) \leq 0$, where $I_{\text{dis}}(j)$ is calculated using Eq. (2). We must note that a relocation of an agent to another place where his/her happiness level is inferior is possible, if the destination environment verifies $T \geq N_d(j)/(N_s(j) + N_d(j))$. In this situation the energy,

$$E_S = \sum_{(i,j)} s_i s_j - (2T - 1) \sum_{(i,j)} s_i^2 s_j^2,$$

is not strictly a Lyapunov function, because relocations that increase the dissatisfaction level are allowed.

The parameters of the model in this situation are the system size $N$, the tolerance level $T$ and the vacancy density $\rho$. Typical values for $\rho$ in urban environments are under 0.1. All our simulations start with a random configuration.

**B. Open city dynamics**

Agents can enter or leave the $N \times N$ lattice depending on their dissatisfaction level, Eq. (2), and the economic environment $H$ and $D$ must be explicitly considered.

The system dynamics is similar to the one described in [9], and can be explained as follows: at each iteration we choose a random site $i$. If it corresponds to a vacancy we attempt to occupy it with an agent of a random type. The movement is accepted if the selected agent presents $I_{\text{dis}}(i) \leq 0$, see Eq. (2). On the other hand, if the selected site is occupied we attempt an internal or an external change with equal probabilities. For the internal change, a vacancy is randomly chosen at site $j$ (implying an infinite range interaction), and the dissatisfaction index for the agent in the offered place, $I_{\text{dis}}(j)$, is calculated. The internal change is accepted only if the dissatisfaction in the new place is preserved or reduced: $I_{\text{dis}}(j) \leq I_{\text{dis}}(i)$. If the external change is chosen, agent $i$ attempts to leave the system, which will take place if $I_{\text{dis}}(i) > 0$. Open city systems are controlled by a variety of parameters: the economic environment variables, $D$ and $H$, and the tolerance level $T$.

**III. RESULTS AND DISCUSSION**

**A. Closed city**

As we have discussed in Sec [11] two equal populations of agents inhabit an $N \times N$ square lattice. We will use $N = 50$ and a fixed vacancy ratio $\rho = 6\%$, unless otherwise specified. Agents can not enter or leave the system, i.e. no external changes are allowed. However, an agent $i$ is able to move into an empty cell $j$, randomly offered, if $I_{\text{dis}}(j) \leq 0$. Note that the relocation process may increase the dissatisfaction index of some of the old or new neighbors of the chosen agent.

A phase diagram for different values of $T$ and $\rho$ was presented in [8], which we will take as our starting point. Once the density $\rho$ is fixed, $T$ remains as the only control parameter of the system. Thus, depending on its value, the system can be found in three different states, which we will call frozen, segregated and mixed, characterized by the stationary morphologies and the acceptance rate for relocations.

1. Low $T$, or frozen. Few changes are accepted and the system remains close to the random initial configuration: a random mixture of red agents, blue agents and vacancies.

2. Medium $T$ or segregated. Two big clusters are created and the accepted change rate is close to 50% in equilibrium.

3. High $T$ or mixed. Almost all changes are accepted, so no clusters are formed and the configuration remains close to random.

We will put special emphasis on the sizes of the different clusters, measured through the segregation coefficient $s.c.$, given by

$$s.c. = \frac{2}{N^4(1-\rho)^2} \sum_{\{c\}} n_c^2,$$

where $c$ indexes all the clusters in the system and $n_c$ is the number of agents in each cluster. This coefficient ranges from values close to 0, where clusterization has not taken place, to 1, where only two clusters remain and $n_c = N^2(1-\rho)/2$.

1. Continuous evolution of $T$

As it was discussed previously, we will focus on how the system adapts to changes in tolerance. So, first, we characterize the system behavior when the tolerance value $T$ decreases according to the following law

$$T(t) = 1 - \tanh \left( \frac{t}{t_0} \right),$$

where $t$ is the time measured in Monte Carlo steps and $t_0$ is a factor controlling the overall speed of the process. Eq. (11) describes the evolution of the tolerance in a city that changes from being extremely tolerant ($T \sim 1$) to totally intolerant ($T \sim 0$). Furthermore, it is also possible to consider the tolerance $T$ as an analogue of the system temperature [8], so the process can also be understood as a cooling process.
In order to characterize the interface we define the grouped vacancy ratio, which is defined as the proportion of vacancies that have either one or two more vacancies in their neighborhood.

We can provide a theoretical estimate for this magnitude as a function of the population density ratio. Notice that the grouped ratio should not depend on the tolerance before the cooling procedure, since vacancies follow a random distribution. We can assume a binomial distribution for the presence of one or two vacancies among the eight cells that comprise the Moore neighborhood,

\[ \binom{8}{1} \rho (1 - \rho)^7 + \binom{8}{2} \rho^2 (1 - \rho)^6. \]  

The grouped ratio is evaluated in four different situations in Fig. 2 (c). Measurements before the drop in \( T \) are shown as red circles, which can be compared with the theoretical estimates, shown as blue squares. Measures after the drop are shown for \( T = 1/2 \) using black circles and for \( T = 7/8 \) using purple circles. After the drop in \( T \), the grouped ratio presents major differences between the segregated society (black circles, \( T = 1/2 \)) and the mixed society (purple circles, \( T = 7/8 \)). As we can see in Fig. 2 (b), a straight line dividing the system into two big clusters requires at least \( N \) vacancies, i.e., the system length. The corresponding value of \( \rho \) is, therefore, \( \rho = 1/N \). In our case, \( N = 50 \) and the maximum grouped ratio value reached corresponds to \( \rho = 0.02 \), as we can see in Fig. 2 (c). If there are less vacancies, they will be randomly placed along the boundary between the clusters. However, when \( \rho > 1/N \), some vacancies may diffuse into the bulk of the clusters, while some other vacancies may allow the boundary to become rough. For \( \rho \gg 1/N \) most vacancies are located in the bulk, and the grouped ratio approach the predictions following the binomial distribution.

For \( T_1 = 7/8 \) (purple circles in Fig. 2 (c)) these arrangement effects on the boundary become negligible because clusters are not actually formed. Thus, the vacancies can be reordered during the process, but the grouped value only rises slightly along the procedure.

3. Characterization of the boundary

Even when the two clusters are well formed the interface between them is not static. It tends to be flat in average, but it always presents fluctuations. Thus, it is relevant to estimate its roughness, \( W \), defined as

\[ W = \sqrt{\frac{1}{N_b} \sum_i h_i^2}, \]  

where \( N_b \) is the total number of border vacancies, \( h \) its height, defined as its distance to the average flat line, and the summation index \( i \) runs over all the vacancies in the

Figure 1. Evolution of the segregation coefficient as a function of time (in Monte Carlo steps) for different \( t_0 \) values, shown in the figure key. The central lines correspond to the average over 50 runs, while the coloured zone represents the region for a confidence interval of 70%. Time ranges from 0 to 30\( t_0 \).

The evolution of the segregation coefficient for different values of \( t_0 \) ranging from 1 (long dash) to \( 10^4 \) (solid) is shown in Fig. 1. As \( t_0 \) increases, the system spends more time on intermediate values of \( T \), rising the clusterization effect. As a consequence, the segregation coefficient value becomes larger. For \( t \gg t_0 \), \( T \) becomes very low, the lattice freezes and the structure created in the previous stage remains. Time is measured in Monte Carlo (MC) steps, corresponding to \( N \times N \) iterations of the system dynamics.

2. Sudden change in \( T \)

Let us consider the possibility of a sudden drop in the tolerance towards a final value of \( T = 1/4 \). Two types of societies are considered, depending on the initial value of the tolerance denoted as \( T_i \): one is highly tolerant, with \( T_i = 7/8 \), and the other one is segregated, with \( T_i = 1/2 \). We have observed a phenomenon that also takes place under some conditions for the continuous variation: after the change in \( T \), the vacancies of the system are grouped at the interfaces between red and blue agents, see Fig. 2 (a) and (b).

Let us consider a vacancy at a flat segment of the border between red and blue agents. We can calculate the critical tolerance value for its location as \( T^* = 4/(4+4) = 1/2 \). Before the drop in tolerance, \( T = 1/2 \), so agents can accept to move into these vacancies. After the drop, when \( T = 1/4 \), no relocation can take place anymore towards this place, because \( T < T^* \), so the vacancy must remain empty. After a short time, most vacancies inside clusters have been transferred into the interface and it becomes flat. From a social perspective, nobody wants to be placed between two intolerant social groups: red agents find this place too close to the blue ones and vice versa.

\[ h = \sqrt{\frac{1}{N_b} \sum_i h_i^2}, \]  

where \( N_b \) is the total number of border vacancies, \( h \) its height, defined as its distance to the average flat line, and the summation index \( i \) runs over all the vacancies in the
interface. This roughness typically scales with system size, \( W \sim N^\alpha \), where \( \alpha \) is usually called the roughness exponent \[16\], as we can see in Fig. 3.

Figure 3. Roughness of the cluster interface as a function of the system size \( N \) for flat boundary configurations, in log-log scale. Each value represents the average of 50 runs over 150000 MC steps.

The interface is subject both to a smoothing effect and a random noise. In other words, the system will tend to flatten the boundary in order to minimize the dissatisfaction of the agents, yet agent relocations are random events that will disturb the shape of the interface. The balance between both effects is reminiscent of the Edward-Wilkinson (EW) and Kardar-Parisi-Zhang (KPZ) universality classes \[16\]. Both of them present a roughness exponent \( \alpha = 1/2 \). Our estimate is close to this value: \( \alpha = 0.562 \pm 0.006 \), as shown in Fig. 3. From a social point of view, this roughness may describe the tension between two intolerant groups in an enclosed location.

4. Avalanche processes

The arrival of an agent into a new neighborhood can decrease the satisfaction of one or more agents who may wish to be relocated. When these agents occupy new places, other agents close to them may feel frustrated and desire a new location in the system, giving rise to a chain reaction. Such avalanche phenomena in this setup were already anticipated by Schelling \[1\].

In our closed city model, agents are relocated when their new level of satisfaction is not acceptable, independently of the effects on the neighborhood, as discussed in Sec. III A. Under these circumstances, the energy is not a Lyapunov function. This fact is illustrated in Fig. 4 where we can see the time evolution of the energy of a single run with \( T = 1/2 \) in the closed city dynamics scheme (black line). Indeed, the energy curve is not strictly monotonic, as expected. On the other hand, in the open city dynamics the only allowed changes are those which lower the dissatisfaction level for agents, thus making it a Lyapunov function (red line).
Figure 4. Normalized energy evolution in MC steps for closed and open city dynamics. Two single runs are considered for a system with $N = 50$, $\rho = 6\%$ and constant $T = 1/2$. Energy is normalized per agent and link by the factor $4N^2(1 - \rho)$.

B. Open city

Let us discuss the case in which the number of agents of either type is not conserved. Again, as discussed in Sec. II, we will consider agents of two types in a $100 \times 100$ square lattice. The initial configuration is random, with red agents, blue agents and vacancies constituting each $1/3$ of the total system. Let us stress that the happiness level of the agents depends on the tolerance value $T$, the economical level of the system $D$, and also the gap between the financial incomes for each type of agent, $H$.

The system undergoes the following process: in the first stage, with fixed values for $T$ and $D$, with $H = 0$, the system evolves allowing both internal and external relocations, until a stationary state is reached with both agent populations settled inside clusters, as we can see in Fig. 5. Now, we proceed to increase $H$ gradually, in order to provide an economical advantage to one type of agents over the other. Thus, a gentrification process will start. As a consequence one of the agent types will tend to leave the system, while the other will fill these vacancies, giving rise to avalanches. From this moment on, we will not allow internal relocations, because our aim is to characterize migratory movements inside and outside the city.

1. Types of borders

Starting from a random configuration, we set a finite value for $T$ and $D$, with $H = 0$. Given enough time, the system reaches an equilibrium state where the borders have both straight and curved segments. In this scenario, $N_s$ can be either 4 or 5 for agents at the boundary, depending on whether the agent stands at a straight or a curved point of the boundary, as we can readily see in Fig. 6. The dissatisfaction index of each agent diminishes when the number of similar neighbors increases (see Eq. 2). Thus, any agent on a curved spot will leave the lattice with higher probability than one at a straight spot. This difference is the key to the creation of different types of borders. Notice that, for a given value of $T$, an agent for which $D > T(N_s + N_d) - N_d$ becomes unsatisfied and is transferred out of the lattice. Thus, different types of borders can arise, associated with their value for $N_d$.

- $N_d = 4$. If $D < 0$ the system is economically very advantageous for all agents. Thus, there are almost no vacancies in the system and they are located in the corners between clusters. We will call this a $C$-type border, see Fig. 5 (a).
- $N_d = 3$. Vacancies appear at straight segments separating both types of agents, in addition to corner sites (Fig. 5 (b)). This kind of interface location will be termed a $\theta$-type border.
- $N_d = 2$. Straight segments of vacancies are completed, but contacts along diagonals between distinct clusters are allowed (I-type border [11]), as in Fig. 5 (c).
- $N_d = 1$. Diagonal contacts between clusters are not allowed any more (II-type border [11]), see Fig. 5 (d).
- $N_d = 0$. This is the usual situation in economically handicapped systems, where $D > 0$. When $D$ is above this threshold the system expels all agents.

In previous work [9] borders of types I and II were described and characterized by means of geometric con-
structions. In this paper we have chosen to determine them via threshold values, including borders of type C and 0, which have not been reported previously.

2. Avalanche processes

At this point, we proceed to increase $H$ gradually by a fixed amount $\Delta H \geq T/2$ every 50 time-steps. This value is approximately half of the one needed to start any avalanche process for a given neighborhood configuration. In the economic interpretation, the system offers less and less financial resources to blue agents, whose economic level is given by $\mu_b = D + H$. Meanwhile, red agents become more and more prosperous, $\mu_r = D - H$, thus enlarging the economic gap. Both expressions account for the dissatisfaction or unhappiness of an agent due to their economic conditions. Moreover, if we define the happiness associated to being in a specific neighborhood as $\lambda = T(N_s + N_d) - N_d$, the satisfaction condition, Eq. (2), can be expressed now as

$$\mu_{r,b} \leq \lambda. \quad (14)$$

The interpretation of Eq. (14) is straightforward: when the satisfaction of being into a neighborhood, $\lambda$ compensates the unhappiness arising from the economic situation, $\mu_{r,b}$, the agent remains in the system. As we keep increasing $H$, the gap between $\mu_r$ and $\mu_b$ opens up. The system becomes more hostile towards blue agents, which increase $\mu_b$, and some of them are forced to leave when Eq. (14) ceases to hold. Therefore, there appear some new vacancies in the lattice. At the same time, the effect is opposite for red agents: $\mu_r$ decreases and the lattice is more satisfactory for them, because they are getting more and more economic advantages. Thus, red agents come from outside and occupy the vacancies previously created. Now, some of the blue agents close to these occupied locations cease to verify Eq. (14) because $\lambda$ has decreased for them due to the arrival of red agents. Thus, they may be transferred out of the system on subsequent time-steps. This gives rise to further new vacancies that are filled again with red agents and the process goes on in a self-sustained way. This is what we called a blue avalanche, because it originates with blue agents leaving the lattice.

Yet, there is another way to generate an avalanche. We depart from an equilibrium situation with some vacancies, which requires a mid-ranged economic environment, check Fig. 5 (b), (c) and (d). Before $H$ is strong enough to force blue agents out, red agents may be able to fill these vacancies up. Blue agents next to these locations may cease to verify Eq. (14), as in the previous situation, and are forced out of the system. These new vacancies are occupied by red agents from outside, making further blue agents leave. The process goes on, giving rise to what we will call a red avalanche.

The value of $H$ remains constant while these avalanches take place. After the avalanches are finished, if there are still some blue agents in the system the value of $H$ is increased further until a new avalanche takes place.

The similarity with a gentrification process is clear: one of the agent types gets richer while the other is becoming economically handicapped. The latter is forced out of the system, leaving vacancies on the borders between the two clusters. Meanwhile, red agents with more economic power enter the system and occupy these vacancies. The process becomes self-sustained because other agents from the less favoured group are forced to leave the system due to their proximity to members of the wealthy class. Of course, not all processes in the system give rise to an avalanche.

Avalanches will be characterized by their size $s$, defined as the total number of blue agents that have left the lattice as a consequence of the departure of the first one. We have obtained the avalanche size histograms, and fitted them to a probability density function (PDF) of the form

$$p(s) = C x^{\alpha} \exp(-x/x_0), \quad (15)$$

where $C$ is the normalization constant, $\alpha$ accounts for the scaling exponent for the avalanche sizes and $x_0$ acts as a maximal cutoff value. For reasons of numerical stability we focus on the complementary cumulative distribution function (CCDF) $17$, which is fitted by the expression $C_x x^{\alpha+1} \exp(-(x/x_0))$. Data from 100 complete extinctions of the blue agents in the system have been measured for each $D$ value. For more numerical details, see Appendix A.

The next sections are devoted to the analysis of avalanche distributions for three values of $T$: low ($T = 1/4$), medium ($T = 1/2$) and high ($T = 3/4$). For each value of $T$ we have characterized the behavior of the system for a wide range of fixed values of $D$, ranging from the situation where no vacancies are present to the extinction point in which, due to the high hostility of the medium all agents leave. We will vary $H$ in order to observe the avalanche processes for each $D$ value considered.

Let us introduce a convenient notation to describe neighborhoods, using $s$ and $d$ to denote similar and different neighbors, respectively, e.g. $3s + 2d$ means that a certain agent has 3 similar and 2 dissimilar neighbors.

3. Avalanche distributions for $T = 1/4$

We begin our study in a city with a high economic interest, $D = -2.125$ and low tolerance value $T = 1/4$.

Under these circumstances the system exhibits a stationary state with big clusters of red and blue agents and no vacancies, due to the low value of $D$. Now, we increase $H$ gradually, $\mu_b$ becomes higher, and some blue agents
may abandon the lattice. In fact, these blues agents are the ones which do not verify Eq. 14 and have $4s + 4d$ neighbors. It must be noted that knowing the value of $T$ and the number of different and similar neighbors, $\lambda$ is fixed. Meanwhile, the situation for red agents keeps improving: $\mu_r$ becomes lower, so the vacancies created by the blue agents are filled by red agents coming from outside. This is what we defined previously as a blue avalanche (Sec. III B 2).

The system is so interesting economically for blue agents that they will adopt different strategies in order to remain, such as the formation of special cluster configurations: triangles and rectangles in contact with the system border. Thus, up to three avalanche processes can be needed to deplete the system of blue agents, as can be seen in Fig. 6 (a), where $D = -2.125$. To clarify the order in which each avalanche occurs and its kind (blue or red), we use the notation $\lambda_{n,k}$, where $n$ denotes the order and $k$ takes value $r$ for red and $b$ for blue ones. The fitted values for $\alpha$ and $x_0$ in all these cases are shown in Table I.

For $D = -2.125$ and increasing $H$ gradually, we find that the first three avalanches start at threshold values $\lambda_{1,b} = -2.0$, $\lambda_{2,b} = -1.75$ and $\lambda_{3,b} = -1.0$. Extending our notation, we may say that each avalanche set has a dominant process, which is characterized by the typical neighborhood of the expelled blue agents and their $\lambda$ value. The values previously calculated correspond with certain neighborhood types: $4s + 4d$, $2s + 3d$ and $5s + 3d$, respectively. Of course, no avalanche consists of a single type of process in practice. We must note that when blue agents with a higher $\lambda$ start their expulsion (such as $2s + 3d$) those with an inferior threshold (for example $4s + 4d$) can be still be in process of leaving. Thus, it is possible for various processes to take place simultaneously in the same avalanche. This is the situation for the first and second avalanche sets, $\lambda_{1,b}$ and $\lambda_{2,b}$ in Fig. 5 (a). The process $4s + 4d$ dominates both avalanches so their curves are close to each other. As the avalanche for $\lambda_{3,b}$ is the last one for this $D$ value, it has a smaller size due to the small number of blue agents remaining on the lattice.

The social meaning is clear: the less favoured agents will try to stand on an economic advantageous environment despite their income gap with the other group. To achieve this goal diverse neighborhood structures will be created (ghettos). This is the case scenario for neighborhoods as Harlem or Clinton Hills [18].

Now, let us focus our attention on higher values of $D$: $-1.875$, $-1.625$ and $-1.375$, as shown in Fig. 6 (b). Although the environment is still economically advantageous, the economic conditions are not as good as before. Some vacancies appear in the lattice and the system presents a C type of border (Fig. 5 (a)). As we explained before, different kinds of avalanches are still possible. The first one takes place for $D = -1.875$. Red agents coming from outside fill the existing vacancies on the system when $\mu_r \lesssim -2.0$. That points at a $4s + 4d$ red avalanche with $\lambda_{1,r} = -2.00$ (Sec. III B 2). The next one, $D = -1.625$, is what we may call a purple avalanche: a simultaneous red avalanche with $\lambda_{1,r} = -2.00$ as in the former case, and a blue avalanche, $4s + 3d$ with $\lambda_{1,b} = -1.25$, due to the overcoming of both thresholds at the same time $(D \pm H)$. We must note that the $4s + 3d$ is the ice avalanche associated with a $0$-type border (Fig. 5(b)). So, while the red agents coming from outside are filling the vacancies with $4s + 4d$ which correspond to a $C$-type border, blue agents with $4s + 3d$ are leaving the system creating a 0-type border. Finally, we have blue avalanches for the first and second avalanche sets with $D = -1.375$, being their associated neighborhoods $4s + 3d$ and $3s + 1d$, respectively. The behavior of blue avalanches is similar to the ones explained for $D = -2.125$. Interestingly, red avalanches are more frequent than blue ones, although they present smaller sizes, as we can readily see in Fig. 6 (b) and Table I. Red avalanches interfere among them, thus reducing the maximum cut-off, $x_0$.

The last value that we analyze for $T = 1/4$ is $D = 0.875$. The behaviour of the blue avalanche suffers an important change because the condition for a predominantly vacancy state is verified, $D/T \geq 3$ [9]. Combining eq. 2, with $H = 0$, and Eq. 3 the condition for an agent to remain on the system can be written as $N_0 \leq (N_0 - D/T)/(1/T - 1)$. As $T$ is lower than unity, the denominator stays positive, so the numerator of the former equation dictates the agent behaviour. Any agent must fulfill the condition $N_0 \geq D/T$ to remain on the lattice, even if no different agent is in the vecindary. As the system initial configuration is random and both kinds of agents and vacancies are equally likely, the probability for an agent to have more than three similar neighbors is small. Therefore, when $D/T > 3$ the system becomes very hostile, agents leave massively, and only a small number of them remain inside clusters, shortening $x_0$ (Fig 7(a)). The border between clusters is not relevant anymore: red clusters grow with a vecindary as $(3s + 0d)$ and blue ones become smaller when agents have a neighborhood $(4s + 0d)$ (Fig 7 (b) and (c)).

Socially, the situation for blue agents might be compared to the Chicago suburbs where the population could increase their personal ties via a community network [9]. These ties may prevent a massive exodus despite the lack of attractiveness of the environment. However, in our model, the evolution of the system departs from this situation and develops in two opposite directions: blue agents are removed from the system, in contrast to red agents, which increase their population. This could be understood as the unrelated behavior of two communities. One of them chooses to cooperate and the ensuing growth overcomes economical hardships. The other one does not strengthen their links and is forced to leave the city.

Finally, we show the values of the fitted parameters in Table I being $x_c$ the chosen size for the fitting (Appendix A). We can find $\alpha$ values in the range
$D=-2.125 \lambda_2, b=-1.75$

$D=-2.125 \lambda_1, b=-2.00$

$D=-2.125 \lambda_3, b=-1.00$

$1.375 \lambda_1, b$

$1.375 \lambda$

$D=-1.875$

$D=1.625$

$D=1.25$

$\lambda_1, r=-2.00$

$\lambda_1, p=\pm 0.375$

$2, b=0.00$

$\alpha$

$1.40$

$-16$

$23$

$79$

$1-150$

$-b$

$-p$

$30$

$1-15$

$1-15$

$100$

$240$

$40$

$B$

$A$

$D$

$\alpha$

$x_0$

$x_e$

$C$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| N | b | -2.125 | $-1.781 \pm 0.002$ | 16.2 $\pm 0.3$ | 40 | 1.20 $\pm 0.03$ |
| C | r | -1.875 | $-1.77 \pm 0.02$ | 15.0 $\pm 0.3$ | 30 | 1.15 $\pm 0.02$ |
| C | p | -1.625 | $-1.50 \pm 0.01$ | 24.7 $\pm 0.6$ | 50 | 1.20 $\pm 0.03$ |
| C | b | -1.375 | $-1.38 \pm 0.02$ | 23.7 $\pm 0.5$ | 50 | 1.21 $\pm 0.03$ |
| 0 | r | -1.125 | $-1.37 \pm 0.03$ | 3.30 $\pm 0.04$ | 20 | 1.42 $\pm 0.04$ |
| 0 | p | -0.875 | $-1.716 \pm 0.008$ | 42.8 $\pm 0.9$ | 50 | 1.13 $\pm 0.02$ |
| 0 | b | -0.625 | $-1.618 \pm 0.004$ | 140 $\pm 1$ | 240 | 1.23 $\pm 0.02$ |
| I | r | -0.375 | $-1.35 \pm 0.02$ | 18.0 $\pm 0.3$ | 40 | 1.14 $\pm 0.02$ |
| I | p | -0.125 | $-1.463 \pm 0.004$ | 79.2 $\pm 0.6$ | 150 | 1.15 $\pm 0.01$ |
| I | b | 0.125 | $-1.495 \pm 0.005$ | 181 $\pm 4$ | 200 | 1.26 $\pm 0.02$ |
| II | r | 0.375 | $-1.136 \pm 0.007$ | 20.9 $\pm 0.2$ | 50 | 1.12 $\pm 0.01$ |
| II | p | 0.625 | $-1.41 \pm 0.03$ | 69.9 $\pm 0.4$ | 150 | 1.081 $\pm 0.008$ |
| V | r | 0.875 | $-1.33 \pm 0.02$ | 9.6 $\pm 0.3$ | 20 | 1.15 $\pm 0.02$ |

Table I. Avalanche distribution parameters $(T = 1/4)$ for $D$ values from (Section IIIB). $B$ is the type of border, with $V$ meaning vacancy dominated regime. $A$ is the avalanche type, (‘b’ is blue, ‘p’ is purple and ‘r’ is red), $\alpha$ is the power law exponent, $x_0$ accounts for cutoff value and $x_e$ is the chosen size for the fitting (see Appendix A).

Figure 6. CCDF of the avalanche distribution sizes for $T = 1/4$. For each curve the $D$ value is specified. Threshold values are given as $\lambda_n, k$, where $n$ is the avalanche set index and $k$ its kind: $r$ for reds, $b$ for blues and $p$ for purple ones. For purple avalanches upper and lower thresholds are expressed as $D \pm \lambda_1, p$. The fitted power-law cutoff functions are depicted with lines.

Figure 7. From left to right: snapshot of the system evolution for $T=1/4$ and $D=0.875$. Equilibrium (a), 6 MC steps (b) and 13 MC steps (c).

$[-1.78, -0.98]$ previously reported in references [19–23], concerning self-organized criticality in different systems.

Figure 8. CCDF of the avalanche distributions for $T = 1/2$ in the predominant vacancy state, represented by symbols. The fitted power-law cutoff functions are depicted with lines.

4. Avalanche distributions for $T = 1/2$ and $T = 3/4$

Most of the processes that arise for these values of $T$ are avalanches with similar parameter values and behaviours to those explained in the previous section (Sec. III B 3).

Nevertheless, as this $T$ fixed value is larger than before, the dissatisfaction index diminishes and Eq. 14 is verified for higher values of $D$. Thus, despite being in a less economically interesting system, agents populate the lattice. As a consequence the range of values of $D$ for which the system is vacancy dominated increases.
The two last avalanches for $T = 1/2$, $D = 1.750$ (short dash) and $D = 1.875$ (long dash) are in the predominant vacancy state (Fig. 8). They are also really close to each other, suggesting that the same mechanism takes place and it is related to $4s + 0d$ neighborhoods with $\lambda_{1,b} = 2.00$. This process implies that the blue agents will be expelled when they are surrounded by vacancies.

The system evolution for $D = 1.750$ is apparently similar to the one in Fig. 6. Nonetheless, the environment has become so hard that red agents are not able to expand as the blue agents progressively leave the city, converting a purple avalanche into a blue one. From a social perspective it can be explained as people leaving the city looking for a better financial situation somewhere else, abandoning the neighborhood into obliteration.

One of the most interesting processes in our work takes place for $D = 1.625$ (continuous), and is also found in the predominant vacancy state (Fig. 5). It is convenient to depict its evolution in Fig. 3. As it is shown, the first procedure that takes place is the red cluster expansion (Fig. 3(b)) for the neighborhood $3s + 0d$ which takes place when $\mu_r \lesssim 1.5$. This mechanism proceeds until a blue cluster is found. As the red cluster grows, the population increases. When both kinds of agents are close, the $4s + 1d$ red avalanche removes all blue agents. For this $D$ value, $\alpha = -0.97 \pm 0.01$, $x_0 = 17.4 \pm 0.3$.

This two stage process may be interpreted as the expansion of some kind of neighborhoods that improve their economical status and are able to increase their size, wrapping around other devaluated zones. After that, a gentrification process starts. This process is reminiscent of the transformation of small towns or villages close to growing commuting zones [24].

Finally, for $T = 3/4$, multiple $D$ values are found in the vacancy dominated regime, but all the situations are related to those previously explained. Type of borders 0, I and II are found to be in this situation.

When the system is in the predominant vacancy state, there are only two processes that guarantee the complete removal of blue agents. The first one is associated with $3s + 0d$ neighborhoods and consist of the red cluster expansion, which changes the environment. After that, other complementary processes take place (as in Fig. 9). The other one is related to $4s + 0d$ neighborhoods, which implies the total expulsion of blue agents surrounded by vacancies (Fig. 7). As a consequence, fitted parameters are close to the ones previously calculated for other values of $T$ in the same situations.

IV. CONCLUSION

Summing up, interesting results which can be correlated to socio-economical situations emerge when a variation of parameters such as the tolerance in the closed city framework or $H$ in the open city model are considered.

The generalization of the open city model provides a new framework for the study and understanding of a broad class of segregation processes. Besides analyzing its results from a social and economical perspective, the model is also linked with the physics of the BEG model under the influence of an external magnetic field. We have considered this model under two different and complementary angles: the closed city (Sec. III A) and the open city approximation (Sec. III B).

In the closed city approximation the only relevant social variable is the tolerance. Since agents can not enter or leave the city, economic environment variables do not play a role in this case. We characterize the behaviour of the system when the tolerance falls in a continuous or a sudden way, after the system has reached equilibrium for an intermediate or high tolerance value. For the continuous decay we have found that the final clusterization degree of the system, measured via the segregation coefficient, depends on the drop rate: for slow rates the system has enough time to create clusters, so the segregation coefficient is close to unity (Fig. 1). The main feature is that once the system has reached equilibrium for $T = 1/2$ a sudden drop in tolerance ($T = 1/4$) creates a vacancy border between the two big remaining clusters (Fig. 3). The interface rugosity, $W$, scales with the system length $N$ as $W \sim N^\alpha$ with $\alpha = 0.562 \pm 0.006$. The equilibrium state is not static and from a social point of view it might describe the tension between two intolerant groups in an enclosed location.

Nonetheless, this approach presents some handicaps. The decision to accept or reject a change does not take into account the agent current happiness level, which does not seem realistic. Usually, happy people do not want to move to another location. Besides, the characterized system corresponds to the situation with a flat interface. Yet, a different disposition of the clusters is possible: one of the agent types may concentrate around a system corner developing a circular border. The analysis of this situation goes beyond the aim of the present paper.

For the open city model, in which agents can leave or enter the city, not only tolerance but two economic terms are taken into consideration: $D$ is associated with the mean economic city level, and $H$ stands for the economic gap between both types of agents. The dynamical rules are the following: once the system has evolved into an equilibrium state, fixing $T$ and $D$, $H$ progressively decreases. In the economical sense, there are less financial resources available for blue agents, meanwhile, red agents are becoming prosperous, so an economic separation is created. Some blue agents, generally the ones that are closer to the red agents, begin to leave the city. These new vacancies are occupied by red agents coming from outside, so the process goes on in a self-sustained way, and resembles gentrification.

A power law with exponential cutoff expression has been used to describe the avalanche size histograms. While the cutoff length depends on the system size, the power law exponents are in the range $[-1.781, -0.97]$.
values that can be found in the literature for diverse avalanche processes.

In our modified Schelling model gentrification processes could also help understand the formation of ghettos, as special configuration of the less favoured class in order to remain in the cities. Moreover, our results highlight the importance of tolerance and ties for people to be satisfied despite harsh conditions. On one hand, collaboration inside of a neighborhood implies an improving in economic and trading exchanges, making growth possible. On the other hand the lack of ties in a disfavorable environment results in the neighborhood progressive degradation.

Further studies should focus on variants in which agents consider their future happiness perspectives \[25\], or the influence of altruistic behaviour \[26\] and the balance between cooperative and individual dynamics \[27\]. The transfer rules from these works combined with the extended open city model presented in this paper could lead to a framework where evolution of a city could be analyzed during several stages.

Acknowledgments

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Appendix A: Fitting the avalanche histograms

Some numerical difficulties associated with direct estimation of $\alpha$ and $x_0$ from the probability density function (PDF) of the histograms are known to arise \[17\]. In order to address them, we have resorted to the use of the complementary cumulative distribution function (CCDF). Our analytical form is given by Eq. (A1).

\[
P(s) = Pr(X > s) = \int_s^\infty C x^{-\alpha} \exp(-x/x_0) dx = F(\alpha, x_0) + G(\alpha, x_0)s^{\alpha+1} \exp(-x/x_0) \left[ 1 + \frac{1}{2 + \alpha} \frac{s}{x_0} + \ldots \right]. \tag{A1}
\]

The terms inside the brackets are an expansion of $M(1, 2 + \alpha, s/x_0)$, where $M$ is the confluent hypergeometric function. Once $\alpha$ and $x_0$ have been estimated, both $F(\alpha, x_0)$ and $G(\alpha, x_0)$ take fixed values. In fact, $F(\alpha, x_0)$ does not introduce significative changes in the fit, and it can be neglected. Under these circumstances, and retaining only the leading term of the series, we have

\[
P(s) \approx G(\alpha, x_0)x^{\alpha+1} \exp(-x/x_0). \tag{A2}
\]

Consequently, we can infer the exponent $\alpha$ and the length $x_0$ of an avalanche from the CCDF of the avalanche histogram.

The experimental CCDF data series in our avalanches have been calculated from 100 complete extinctions of the blue agents. After that, data are plotted with a constant bin size by means of the \texttt{poweRlaw} R package \[28\]. Then an expression of the type $C^* x^{\alpha+1} \exp(-x/x_0)$, where $C^* \approx G(\alpha, x_0)$ is fitted by the nonlinear least square method from the \texttt{stats} subroutines \[29\]. Although the curves have been depicted for a wide avalanche size range, they have been fitted inside the interval $[1, x_c)$, being $x_c$ a chosen value to uphold precision. Deviations between data and fit can appear for $s \gg x_0$, due to the series expansion approximation and the own nature of the tail distribution, but they are not relevant. As a practical rule, if $x_0 < 60$ then $x_c \leq 2x_0$ (see Table I). Choosing $x_c$ in this way, the precision of the fit improves for a wide range of avalanche size values.