Particle creation by strong fields and quantum anomalies

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Particle creation by strong and time-varying backgrounds is a robust prediction of quantum field theory. Another well-established feature of field theory is that classical symmetries do not always extend to the quantized theory. When this occurs, we speak of quantum anomalies. In this contribution we discuss the entwining relationship between both predictions, relating chiral anomalies with an underlying process of particle creation. Within this context, we will also argue that the symmetry under electric-magnetic duality rotations of the source-free Maxwell theory is anomalous in curved spacetime. This is a quantum effect, and it can be understood as the generalization of the fermion chiral anomaly to fields of spin one. This implies that the net polarization of photons propagating in a gravitational field could change in time.

Keywords: Chiral anomalies, gravitational particle creation, electromagnetic duality rotations, photon helicity

1. Introduction

Two fundamental predictions of quantum field theory in presence of strong field backgrounds were established in the sixties:

i) The spontaneous creation of particles out of the vacuum by time-varying gravitational fields. This was first discovered in the analysis of quantized fields in an expanding universe and some years later applied to black holes (for reviews see and for a historical perspective see). It also offered a better understanding of the pair creation phenomena induced by electric fields, as explored by Heisenberg and Euler and also Schwinger.

ii) The breaking of the axial symmetry of Dirac fermions by quantum effects. Classical symmetries of field theories may fail to survive quantization, leading to what is known in the literature as quantum anomalies (for a review see). The origin of the anomalies is rooted in the renormalization mechanism need to tame the ultraviolet divergences that affect most models of quantum field theory. Renormalization in presence of an electromagnetic background can spoil the classical conservation law of the axial Noether current $J_5^\mu$ of massless charged fermions. It is codified by the Adler-Bell-Jackiw anomaly \cite{ABJ} (\(\alpha = e^2/(4\pi\hbar)\) is the fine-structure constant; we take $c = 1$)

$$\partial_\mu \langle 0 | J_5^\mu | 0 \rangle = \frac{\alpha}{2\pi} F_{\mu\nu}^a F^{a\mu\nu}.$$ (1)
The axial anomaly can also be interpreted as a low-energy phenomena in terms of particle production. A time-varying electric field creates both left-handed and right-handed fermions. The net amount of created chirality $\Delta Q_5$ (i.e., the number of right-handed fermions $N_R$ minus the number of left-handed fermions $N_L$ created between $t_1$ and $t_2$) is evaluated according to

$$\Delta Q_5 = \hbar (N_R - N_L) = \frac{\alpha}{2\pi} \int_{t_1}^{t_2} \int_\Sigma d\sigma \langle 0 | J_5^\mu | 0 \rangle = \alpha \int_{t_1}^{t_2} \int_\Sigma d^4 x F_{\mu\nu} F^{\mu\nu}.$$  \hspace{1cm} (2)

It is important to stress that the minimum amount of created (mass less) fermions is obtained in the adiabatic limit\[^{17}\], and it is just encapsulated in the creation of the chirality accounted by the anomaly. This might be somewhat surprising since the particle number is naively expected to be an adiabatic invariant\[^{b}\]. The breaking of the adiabatic invariance of the particle number can then be understood as a signal of the chiral anomaly. Furthermore, the existence of the axial anomaly implies that there is necessarily a minimum amount of particle creation, even for an adiabatic process, to account for the creation of chirality. In more pedestrian words, the creation of chirality can be regarded as the “smoking gun” of the full particle creation process. We can summarize these features schematically:

**Quiral Anomaly $\rightarrow$ Particle creation**

**Adiabatic Particle creation $\rightarrow$ Quiral Anomaly**

The above results on the chiral anomaly can be extended to curved spacetime\[^{19}\]. Expression \(^1\) generalize to

$$\nabla_\mu \langle 0 | J_5^\mu | 0 \rangle = \frac{\alpha}{2\pi} F_{\mu\nu} F^{\mu\nu} + \frac{\hbar}{192\pi^2} R_{\mu\nu\alpha\beta}^* R^{\mu\nu\alpha\beta},$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor and $^* R^{\mu\nu\alpha\beta}$ its dual, and $\nabla_\mu$ is the covariant derivative. We note that the gravitational part of the chiral anomaly persists for all type of fermions, either charged or neutral. Therefore, the chirality of (neutral) fermions fails also to be conserved in curved spacetimes for which

$$\Delta Q_5 = \frac{\hbar}{192\pi^2} \int_{t_1}^{t_2} \int_\Sigma d^4 x \sqrt{-g} R_{\mu\nu\alpha\beta}^* R^{\mu\nu\alpha\beta}$$

is non-vanishing. This can be heuristically understood as a consequence of the universal character of gravity, as prescribed by Einstein’s equivalence principle. If \[^3\]

\[^{a}\]We assume that a consistent particle interpretation is available at early and late times.

\[^{b}\]It can be proved rigorously for a scalar field in an expanding universe\[^{18}\].

is valid for a type of massless spin-1/2 field it must also be valid for any other type. On physical grounds one can also argue that the universality of gravity suggests that these anomalies are not specific of spin 1/2 fermions. Therefore, one could also expect that a somewhat similar anomaly (also associated to an underlying particle production process) will arise for other fields admitting axial-type symmetries.

In this work we will further discuss the entwining relationship between particle creation and quantum anomalies. In particular, we are especially interested in exploring a new scenario in which (spontaneous and stimulated) particle creation can be relevant. The proposed scenario will be suggested by its link with quantum anomalies, as first explored in Ref. 20. Our heuristic and intuitive argument is based on the universality of gravity, which suggest that the chiral anomaly for spin 1/2 fields should also be extended to the electromagnetic field. In the language of particle creation, one would also expect that a chiral gravitational configuration will create photons with different helicities in unequal amounts, in the same way as it happens, according to Ref. 1, for spin 1/2 fermions.

2. Trace and Axial anomalies for massless fermions

Free massless Dirac spinors are highly symmetric. In addition to their Poincaré invariance in Minkowski spacetime, they exhibit two extra symmetries: conformal and axial invariance. The conformal (or Weyl symmetry) implies the tracelessness of the stress-energy tensor $T^{\mu\nu}$, while the axial symmetry $(\psi \rightarrow e^{i\theta \gamma^5} \psi)$ implies the conservation of the axial current $J_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$. Both symmetries cannot be extended to the quantum theory when the Dirac field is coupled to an electromagnetic background. The conservation of the axial current is broken according to (1). Furthermore, the trace of the stress-energy tensor also acquires a non-vanishing vacuum expectation value $\langle T^{\mu\mu} \rangle = \frac{\alpha}{6\pi} F_{\mu\nu} F^{\mu\nu}$, (for a more recent derivation, see Ref. 23),

$$\langle T^{\mu\mu} \rangle = \frac{\alpha}{6\pi} F_{\mu\nu} F^{\mu\nu},$$

which is usually interpreted in terms of the running of the coupling constant in quantum electrodynamics. In the same way, neither of the two symmetries is preserved when the Dirac field is coupled to a gravitational background. One also finds a trace anomaly

$$\langle T^\mu_\mu \rangle = \frac{\hbar}{2880\pi^2} [a C_{\mu
u\rho\sigma} C^{\mu\nu\rho\sigma} + b G + c \Box R],$$

where $C_{\mu
u\rho\sigma}$ is the Weyl tensor and $G$ is the integrand of the Gauss-Bonnet topological invariant. The numerical coefficients are given by $a = -9$, $b = 11/2$, $c = 6$, and they can be obtained by different methods. It is interesting to point out that the specific form of the trace anomaly implies that there are no massless fermions created in Friedman-Lemaître-Robertson-Walker (FLRW) universes (in this case
the trace anomaly is proportional to $G$, up to total derivatives, and no further term proportional to $R^2$ appears\(^\text{26}\)). This is fully consistent with earlier results\(^\text{1}\) showing the absence of particle creation for fields obeying conformally invariant equations.

As remarked in the introduction one also finds an axial anomaly of the form

$$\nabla_\mu \langle J_\mu^5 \rangle = \frac{h}{192\pi^2} R_{\mu\nu\alpha\beta} \ast R^{\mu\nu\alpha\beta}.$$  \hspace{2cm} (7)

This anomaly can be interpreted in terms of particle creation induced by a chiral gravitational configuration (see\(^\text{27}\) for a cosmological setting). Here we want to remark that (7) is consistent with late-time black hole emission\(^\text{2}\). Stationary Kerr black holes emit fermions with a helicity-dependent angular distribution\(^\text{28-30}\), as one could heuristically expect from the local form of the anomaly (7). Fermions with positive helicity are emitted preferentially along the direction of rotation, while fermions with negative helicity in the opposite direction. [For instance, neutrinos are preferentially emitted in the direction opposite to hole’s rotation, while antineutrinos in the direction of rotation]. However, the net contribution when integrated over all angles is zero, in agreement with the vanishing of (4) for the Kerr metric. Nevertheless, in the transient process through the formation of a single Kerr black hole, as for instance the merger of two black holes (as the ones currently observed by LIGO-Virgo), the net contribution is not zero, as it has been evaluated using numerical relativity\(^\text{31}\). [For neutrinos/antineutrinos this means creation of matter-antimatter asymmetry]. Although the net creation of helicity is still small, due to the short duration of the process, it could be more significant in scenarios displaying an accumulative process over long periods of time. In any case, one should always take into account that the net creation of helicity represents only a lower bound of the full particle creation process. More particles could be produced without contributing to the creation of helicity.

The above discussion applies equally to the emission of photons. Right-handed photons are radiated more abundantly in the direction parallel to the axis of rotation of a Kerr black hole, and left-handed photons are emitted more abundantly in the opposite direction. This suggests the existence of an “axial anomaly” for spin-1 fields.

3. Electro-magnetic duality as an axial anomalous symmetry

Can we extend the above considerations to the electromagnetic field? Concerning the conformal symmetry, it is well-known that it is broken by quantum effects induced by the gravitational background. The trace of the renormalized stress-energy tensor can be expressed as

$$\langle T^\mu_\mu \rangle = -\frac{62h}{2880\pi^2} (R^\mu_\nu R^\nu_\mu - \frac{1}{3} R^2) + \frac{h}{16\pi^2} \Box R.$$  \hspace{2cm} (8)
where \(c\) is an ambiguous coefficient, which depend on the particular regularization method (\(c = -1/10\) for point-plitting and zeta-function regularization, and \(c = 1/15\) for dimensional regularization\(^{22}\)). We note again\(^{26}\) that, for a FLRW universe, the trace anomaly turns out to be proportional to the integrand of the Gauss-Bonnet invariant \(G\), up to the ambiguous and total derivative term \(\Box R\). The absence of an extra term proportional \(R^2\) is crucial to predict the absence of massless, spin-1 particle creation in an isotropically expanding universe.

The point now is: what is the analog of axial symmetry for the free electromagnetic field? A natural candidate are the well-known electric-magnetic duality transformations, defined by
\[
F^{\mu\nu} \rightarrow F'^{\mu\nu} = \sin \theta \ast F^{\mu\nu} + \cos \theta \ast F^{\mu\nu}.
\]
(9)
The above transformations leave the action
\[
S_{\text{Maxwell}} = -\frac{1}{4} \int d^4x \sqrt{-g} \left( \nabla_\mu A_\nu - \nabla_\nu A_\mu \right) \left( \nabla^\mu A^\nu - \nabla^\nu A^\mu \right)
\]
(10)
invariant, as first proved in Refs.\(^{32,33}\) in Minkowski space and in Refs.\(^{34,35}\) in curved space. The associated Noether current can be expressed as
\[
j^\mu_D = \frac{1}{2} [ A_\nu \ast F^{\mu\nu} - Z_\nu F^{\mu\nu} ],
\]
(11)
where the auxiliary field \(Z_\mu\) is defined as \(\nabla_\mu Z_\nu - \nabla_\nu Z_\mu = *F_{\mu\nu}\). \(j^\mu_D\) is gauge invariant, but it cannot be expressed locally in terms of the field strength, in sharp contrast with the stress-energy tensor. The conserved charge
\[
Q_D = \int_{\Sigma_t} d\Sigma_t j^\mu_D
\]
(12)
evaluates the electromagnetic helicity of a classical electromagnetic configuration\(^{33}\).

We note that \(Q_D\) can be decomposed in two contributions\(^{37}\)
\[
Q_D = Q_m + Q_e,
\]
(13)
where
\[
Q_m = \frac{1}{2} \int_{\Sigma_t} d\Sigma_t A_\nu \ast F^{\mu\nu},
\]
\[
Q_e = -\frac{1}{2} \int_{\Sigma_t} d\Sigma_t Z_\nu F^{\mu\nu}.
\]
(14)
\(Q_m\) is the magnetic helicity and \(Q_e\) the electric helicity. Neither \(Q_m\) or \(Q_e\) are conserved quantities, in general.

\(^{cQ_D\ can\ be\ written\ in\ terms\ of\ a\ non-local\ integral\ involving\ only\ electric\ and\ magnetic\ fields.\)
At the quantum level $Q_D$ is proportional to the difference between the number of right-handed and left-handed photons:

$$Q_D = \hbar (N_R - N_L) . \quad (15)$$

Therefore, this quantity is no longer conserved if the symmetry is afflicted by an anomaly. This issue was worked out in Ref.\textsuperscript{20,39–41} from different methods and viewpoints, arguing that the symmetry under electric-magnetic duality rotations becomes anomalous in curved spacetime. The result is somewhat parallel to that found for spin-1/2 fields

$$\nabla_\mu \langle j^\mu_D \rangle \propto \hbar R_{\mu\nu\alpha\beta}^* R^{\mu\nu\alpha\beta} . \quad (16)$$

We want to remark here that previous works in the eighties\textsuperscript{42–44} computed the divergence of the Pauli-Lubansky vector $K_\mu \propto A_\nu^* F_{\mu\nu}$ in curved space

$$\nabla_\mu \langle A_\nu^* F_{\mu\nu} \rangle = \frac{1}{2} \langle F_{\mu\nu}^* F^{\mu\nu} \rangle = \frac{\hbar}{96\pi^2} R_{\mu\nu\alpha\beta}^* R^{\mu\nu\alpha\beta} . \quad (17)$$

This result is indeed related to the (classically non-conserved) magnetic helicity, instead of the electromagnetic helicity. We can easily realize this from the fact that the Pauli-Lubansky vector is proportional to the current

$$j^\mu_m \equiv \frac{1}{2} A_\nu^* F_{\mu\nu} . \quad (18)$$

This current gives the magnetic helicity as the integral

$$Q_m(t) = \int_{\Sigma_t} d\Sigma_{\mu} j^\mu_m . \quad (19)$$

$j^\mu_m$ is not a Noether current, hence $Q_m$ is not time-independent. The proper Noether current associated to the electro-magnetic duality symmetry involves an extra contribution $j^\mu_e \equiv -\frac{1}{2} Z_\nu F_{\mu\nu}$. This additional term is crucial to produce a classical conserved current

$$j^\mu_D = j^\mu_m + j^\mu_e , \quad (20)$$

which can be physically interpreted in terms of the spin-1 axial anomaly if $\nabla_\mu (j^\mu_D)$ is different from zero. Since $\nabla_\mu (j^\mu_D) = -(F_{\mu\nu}^* F^{\mu\nu}) - \frac{1}{2} (Z_\nu \nabla_\mu F_{\mu\nu})$, one gets

$$\nabla_\mu (j^\mu_D) = -\frac{1}{2} (Z_\nu \nabla_\mu F_{\mu\nu}) . \quad (21)$$

It has been argued in\textsuperscript{33,38} that the above vacuum expectation value is nonvanishing and that it is proportional, as expected, to $R_{\mu\nu\alpha\beta}^* R^{\mu\nu\alpha\beta}$, in parallel with the fermionic case. As stressed above, and in the language of particles, this would imply that the difference in the number of photons with positive and negative helicities, $N_R - N_L$, is not necessarily conserved in curved spacetimes.
Nevertheless, we want to remark that the result (17) contains an indirect signal of the electromagnetic axial anomaly found in\textsuperscript{39–41}. If the invariance of the electromagnetic field equations under the duality transformation $F_{\mu\nu} \rightarrow F'_{\mu\nu} = \sin \theta \ast F_{\mu\nu} + \cos \theta F_{\mu\nu}$ is strictly translated to composite quantum operators one would get
\begin{equation}
\langle F_{\mu\nu}^* F_{\mu\nu} \rangle = (\cos^2 \theta - \sin^2 \theta) \langle F_{\mu\nu}^* F_{\mu\nu} \rangle - \sin \theta \cos \theta (\langle F_{\mu\nu} F_{\mu\nu} \rangle - \langle \ast F_{\mu\nu} \ast F_{\mu\nu} \rangle).
\end{equation}
(22)
This would force $\langle F_{\mu\nu}^* F_{\mu\nu} \rangle = 0 = \langle F_{\mu\nu} F_{\mu\nu} \rangle - \langle \ast F_{\mu\nu} \ast F_{\mu\nu} \rangle$. Since neither $\langle F_{\mu\nu}^* F_{\mu\nu} \rangle$ or $(\langle F_{\mu\nu} F_{\mu\nu} \rangle - \langle \ast F_{\mu\nu} \ast F_{\mu\nu} \rangle)$ are zero (see\textsuperscript{45} for a detailed discussion on this), we should conclude that electro-magnetic duality fails for non-linear vacuum expectation values. This is the underlying reason permitting the result
\begin{equation}
\nabla_{\mu} \langle j_{\mu}^e \rangle + \nabla_{\mu} \langle j_{\mu}^\mu \rangle \neq 0.
\end{equation}
(23)

### 3.1. Chiral anomalies and gravitational radiation

We end this section by outlining an very interesting connection between chiral anomalies and gravitation radiation\textsuperscript{31,46}. It is well-know that the right-hand side of (16), as for anomalies in gauge theories, is a total divergence. This simple fact suggests to reinterpret the result (16), or the analogous one for massless fermions, in a physically appealing manner. To evaluate the produced quirality induced by particle creation one should evaluate four-dimensional integrals of the form (4). In doing this one gets crucial contributions from the boundary of the spacetime (i.e., null infinity) involving the outgoing flux of gravitational waves. The contribution of the chiral anomaly can then be exactly related to the amount of circular polarization of the outoging gravitation radiation. Following\textsuperscript{31,46} one gets the intriguing relation (see\textsuperscript{46} for details)
\begin{equation}
\int d^4x R_{\mu\nu\lambda\sigma} \ast R^{\mu\nu\lambda\sigma} \propto \int_0^\infty \frac{d\omega \omega^3}{24\pi^4} \sum_{\ell m} \left[ |h_+^{\ell m}(\omega) + ih_\times^{\ell m}(\omega)|^2 - |h_+^{\ell m}(\omega) - ih_\times^{\ell m}(\omega)|^2 \right],
\end{equation}
(24)
where $h_+$, $h_\times$ are the standard gravitational waves polarization modes. The right-hand-side is related to the difference in the intensity between right and left circularly-polarized gravitational waves reaching future null infinity. This shows that a flux of circularly polarized gravitational waves triggers the spontaneous creation of quanta with net helicity. Note the similarity with the Hawking emission by rotating black holes. The angular momentum of the Kerr black hole triggers the spontaneous creation of quanta with net angular momentum.

### 4. Final remarks

It is well-known that spontaneous emission induces stimulated emission in presence of bosons. This is also true for gravitational particle creation\textsuperscript{1}. One can write the simple and basic result
\begin{equation}
\langle N_i(t) \rangle \equiv \langle \Psi[a_i^\dagger(t)a_i(t)|\Psi \rangle = \langle N_i^0 \rangle + (1 + 2\langle N_i^0 \rangle)\beta_i(t)\rangle\rangle.
\end{equation}
(25)
where $N_0^i$ is the initial number of quanta in mode $i$ contained in the quantum state $|\Psi\rangle$ (the effect is reversed for fermions). This was also considered for black hole radiation in $^{47,48}$, and more recently in $^{49,50}$ for non-gaussianities during inflation.

The stimulated counterpart effect is the main difference in the consequences of the axial anomaly for spin-1/2 fermions and photons. In the latter case, the presence of photons in a given mode will trigger the creation of photons of the same mode. It is not easy to evaluate quantitatively the consequences of this effect on a macroscopic pulse of radiation. But it not difficult to guess that it will change the circular polarization of light-rays, with trajectory $x^\mu = x^\mu(\tau)$, propagating through a gravitational field with non-trivial $R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}(x(\tau))$. This is a quantum effect, probably very tiny, to be added to the classical gravitational redshift and the deflection of light rays by massive bodies.$^{51}$

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References

1. L. Parker, *The creation of particles in an expanding universe*, Ph.D. thesis, Harvard University (1966). Dissexpress.umi.com, Publication Number 7331244; *Phys. Rev. Lett.* 21, 562 (1968); *Phys. Rev. D* 183, 1057 (1969); *Phys. Rev. D* 3, 346 (1971).
2. S. W. Hawking, *Commun. Math. Phys.* 43, 199 (1975).
3. N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge, UK (1982).
4. S. A. Fulling, *Aspects of quantum field theory in curved space-time*, Cambridge University Press, Cambridge, UK (1989).
5. R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago University Press, Chicago, USA (1994).
6. L. Parker and D. J. Toms, *Quantum field theory in curved spacetime: quantized fields and gravity*, Cambridge University Press, Cambridge, UK (2009).
7. V. P. Frolov and I. D. Novikov, *Black hole physics*, Kluwer Academic Publishers, Dordrecht, The Netherlands (1998).
8. A. Fabbri and J. Navarro-Salas, *Modeling black hole evaporation*, ICP-World Scientific, London, UK (2005).
9. L. Parker, *J.Phys.Conf.Ser.* 600, 1, 012001 (2015); arXiv: 1503.00359.
10. L. Parker and J. Navarro-Salas, *Fifty years of cosmological particle creation*, arXiv:1702.07132.
11. W. Heisenberg and H. Euler, *Z. Phys.* 98, 714 (1936). English translation: arXiv:physics/0605038.
12. G. V. Dunne, *Int. J. Mod. Phys. A* 27, 1260004 (2012); F. Karbstein, *Particles* 3, 39 (2020).
13. J. Schwinger, *Phys. Rev.* 82, 664 (1951).
14. S. L. Adler, *Phys. Rev.* 82, 2426 (1969).
15. J. S. Bell and R. Jackiw, *Nuovo Cimento A* 51, 47 (1969).
16. R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Oxford University Press, Oxford.
17. P. Beltran-Palau, A. Ferreiro, J. Navarro-Salas and S. Pla, Phys.Rev.D **100**, 085014 (2019).
18. L. Parker, J.Phys.A **45**, 374023 (2012).
19. T. Kimura, Progress. Theor. Phys. **42**, 1191 (1969).
20. I. Agullo, A. del Rio and J. Navarro-Salas, Int. J. Mod. Phys. D **26**, 1742001 (2017).
21. R.J. Crewther, Phys. Rev. Lett. **28**, 1421 (1972).
22. M. J. Duff, Class. Quantum Grav. **11** 1397 (1994).
23. P. Beltran-Palau, A. Ferreiro, J. Navarro-Salas and S. Pla, Phys.Rev.D **101**, 105014 (2020).
24. S.L. Adler, J.C. Collins and A. Duncan, Phys. Rev. D **15**, 1712 (1977).
25. A. Landete, J. Navarro-Salas and F. Torrenti, Phys.Rev.D **88** (2013) 061501; Phys. Rev. D **89** 044030 (2014). J. F. Barbero G., A. Ferreiro, J. Navarro-Salas and E. J. S. Villaseñor, Phys. Rev. D **98**, 025016 (2018).
26. L. Parker, *Aspects of quantum field theory in curved spacetime: effective action and energy-momentum tensor*, in Recent Developments in Gravitation, Cargese 1978, ed. M. Lévy and S. Deser (Plenum Press, New York), 219-273.
27. G.W. Gibbons, Phys. Lett. **B84**, 431 (1979).
28. Don N. Page, Phys. Rev. D **14**, 3260 (1976).
29. A. Vilenkin, Phys. Rev.D **20**, 1807 (1979).
30. D.A. Leahy and W.G. Unruh, Phys.Rev.D **19**, 3509 (1979).
31. A. del Rio et al.,Phys. Rev. Lett. **124**, 211301 (2020).
32. D. M. Lipkin, J. of Math. Phys. **5**, 696 (1964).
33. M. G. Calkin, Am. J. Phys. **33**, 958 (1965).
34. S. Deser, and C. Teitelboim, Phys. Rev. D **13**, 1592 (1976).
35. S. Deser, J. Phys. A **15**, 1053 (1982).
36. J. Bernabeu and J. Navarro-Salas, Symmetry **11** (10), 1191 (2019); arXiv:1910.05041
37. M. Galaverni and G.S.J. Gabriele, Gen. Relativ. Gravit. **53**, 46 (2021).
38. J.L. Trueba and A. F. Rañada, European Journal of Physics **17**, 141 (1996).
39. I. Agullo, A. del Rio and J. Navarro-Salas, Phys. Rev. Lett. **118**, 111301 (2017).
40. I. Agullo, A. del Rio and J. Navarro-Salas, Phys. Rev. D **98**, 125001 (2018).
41. I. Agullo, A. del Rio and J. Navarro-Salas, Symmetry **10**, 763 (2018).
42. A.D. Dolgov, I. B. Khriplovich and V. I. Zakharov, JETP Lett. **45**, 651 (1987).
43. A.D. Dolgov, I. B. Khriplovich, A.I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **315**, 138 (1989).
44. M. Reuter, Phys. Rev. D**37**, 1456 (1988).
45. I. Agullo, A. Landete and J. Navarro-Salas, Phys. Rev. D **90**, 124067 (2014).
46. A. del Rio, Phys. Rev. D **104**, 065012 (2021).
47. R. M. Wald, Phys. Rev. D **13**, 3176 (1976).
48. J. D. Bekenstein and A. Meisels, Phys. Rev. D **15**, 2775 (1977).
49. I. Agullo and L. Parker, Phys. Rev. D**83**, 063526 (2011); Gen. Rel. Grav. **43**, 2541 (2011).
50. I. Agullo, J. Navarro-Salas and L. Parker, J. Cosmol. Astropart. Phys. **05**, 019 (2012).
51. C. W. Misner, K.S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco, USA (1973).