1. Introduction

Although a systematic study of electromagnetic phenomena in media is not possible without methods of quantum mechanics, statistical physics and kinetics, in practice a standard mathematical model based on phenomenological Maxwell’s equations provides a good approximation to many important problems. As is well-known, one should be able to obtain the electromagnetic laws for continuous media from those for the interaction of fields and point particles [18], [34], [42], [51], [57], [66], [91]. As a result of the hard work of several generations of researchers and engineers, the classical electrodynamics, especially in its current complex covariant form, undoubtedly satisfies Dirac’s criteria of mathematical beauty, being a state of the art mathematical description of nature.

In macroscopic electrodynamics, the volume (mechanical or ponderomotive) forces, acting on a medium, and the corresponding energy density and energy flux are introduced with the help of the energy-momentum tensors and differential balance relations [24], [31], [51], [72], [86], [91]. These forces occur in the equations of motion for a medium or individual charges and, in principle, they can be experimentally tested [32], [69], [74], [92] (see also the references therein). But interpretation of the results should depend on the accepted model of the interaction between the matter and radiation.

In this methodological note, we discuss a complex version of Minkowski’s phenomenological electrodynamics (at rest or in a moving medium) without assuming any particular form of material equations as far as possible. Lorentz invariance of the corresponding differential balance equations is emphasized in view of long-standing uncertainties about the electromagnetic stresses and momentum density, the so-called “Abraham-Minkowski controversy” (see, for example, [5], [15], [19], [22], [24], [30], [31], [32], [34], [36], [51], [62], [63], [67], [68], [69], [72], [73], [74], [78], [80], [85], [89], [92], [93], [94], [95] and the references therein).
The paper is organized as follows. In sections 2 to 4, we describe the 3D-complex version of Maxwell’s equations and derive the corresponding differential balance density laws for the electromagnetic fields. Their covariant versions are given in sections 5 to 9. The case of a uniformly moving medium is discussed in section 10 and complex Lagrangians are introduced in section 11. Some useful tools are collected in appendices A to C for the reader’s benefit.

2. Maxwell’s Equations in 3D-Complex Form

Traditionally, the macroscopic Maxwell equations in a fixed frame of reference are given by

\[\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \quad \text{(Faraday)}, \quad \nabla \cdot B = 0 \quad \text{(no magnetic charge)} \quad (2.1)\]

\[\nabla \times H = \frac{1}{c} \frac{\partial D}{\partial t} + \frac{4\pi}{c} j_{\text{free}} \quad \text{(Biot&Savart)}, \quad \nabla \cdot D = 4\pi \rho_{\text{free}} \quad \text{(Coulomb)} \quad (2.2)\]

Here, \( E \) is the electric field, \( D \) is the displacement field; \( H \) is the magnetic field, \( B \) is the induction field. These equations, which are obtained by averaging of microscopic Maxwell’s equations in the vacuum, provide a good mathematical description of electromagnetic phenomena in various media, when complemented by the corresponding material equations. In the simplest case of an isotropic medium at rest, one usually has

\[D = \varepsilon E, \quad B = \mu H, \quad j = \sigma E, \quad (2.3)\]

where \( \varepsilon \) is the dielectric constant, \( \mu \) is the magnetic permeability, and \( \sigma \) describes the conductivity of the medium (see, for example, [1], [6], [7], [15], [16], [18], [21], [23], [28], [34], [37], [51], [57], [70], [72], [82], [86], [90], [91] for fundamentals of classical electrodynamics).

Introduction of two complex fields

\[F = E + iH, \quad G = D + iB \quad (2.4)\]

allows one to rewrite the phenomenological Maxwell equations in the following compact form

\[\frac{i}{c} \left( \frac{\partial G}{\partial t} + 4\pi j \right) = \nabla \times F, \quad j = j^*, \quad (2.5)\]

\[\nabla \cdot G = 4\pi \rho, \quad \rho = \rho^*, \quad (2.6)\]

where the asterisk stands for complex conjugation (see also [6], [47] and [79]). As we shall demonstrate, different complex forms of Maxwell’s equations are particularly convenient for study of the corresponding “energy-momentum” balance equations for the electromagnetic fields in the presence of the “free” charges and currents in a medium.

\footnote{From this point, we shall write \( \rho_{\text{free}} = \rho \) and \( j_{\text{free}} = j \). A detailed analysis of electromagnetic laws for continuous media from those for point particles is given in [34] (statistical description of material media).}
3. Hertz Symmetric Stress Tensor

We begin from a complex 3D-interpretation of the traditional symmetric energy-momentum tensor [72]. By definition,

\[ T_{pq} = \frac{1}{16\pi} \left[ F_p G_q^* + F_q G_p^* + F_q G_p^* + F_q G_p^* \right. \\
\left. - \delta_{pq} (F \cdot G + F^* \cdot G) \right] = T_{qp} \quad (p, q = 1, 2, 3) \tag{3.1} \]

and the corresponding “momentum” balance equation,

\[ \left( \rho E + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right)_p + \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c} (\mathbf{D} \times \mathbf{B})_p \right. \\
\left. + \frac{1}{16\pi} \left[ \mathbf{curl} (F \times G^* + F^* \times G) \right]_p \right. \\
\left. + \frac{1}{16\pi} \left[ F_q \frac{\partial G_q^*}{\partial x_p} - G_q \frac{\partial F_q^*}{\partial x_p} + F_q \frac{\partial G_q}{\partial x_p} - G_q \frac{\partial F_q}{\partial x_p} \right] \right) = \frac{\partial T_{pq}}{\partial x_q} \tag{3.2} \]

can be obtained from Maxwell’s equations (2.5)–(2.6) as a result of elementary but rather tedious vector calculus calculations usually omitted in textbooks. (We use Einstein summation convention over any two repeated indices unless otherwise stated. In this paper, Greek indices run from 0 to 3, while Latin indices may have values from 1 to 3 inclusive.)

**Proof.** Indeed, in a 3D-complex form,

\[ \frac{\partial}{\partial x_q} \left[ F_p G_q^* + F_q G_p^* - \delta_{pq} F \cdot G^* \right] = \frac{\partial F_p}{\partial x_q} G_q^* + \frac{\partial G_q}{\partial x_q} + \frac{\partial F_q}{\partial x_q} G_p^* + \frac{\partial G_p}{\partial x_q} - \frac{\partial}{\partial x_p} \left( F_q G_q^* \right) \tag{3.3} \]

\[ = F_q \left( \frac{\partial G_q}{\partial x_q} - \frac{\partial G_q^*}{\partial x_q} \right) + \left( \frac{\partial F_p}{\partial x_q} - \frac{\partial F_q}{\partial x_q} \right) G_q^* \\
+ F_p \text{ div } G^* + G_p^* \text{ div } F \]

\[ = F_p \text{ div } G^* - (F \times \text{ curl } G^*)_p + G_p^* \text{ div } F - (G^* \times \text{ curl } F)_p \]

due to an identity [86]:

\[ (\mathbf{A} \times \text{ curl } \mathbf{B})_p = A_q \left( \frac{\partial B_q}{\partial x_p} - \frac{\partial B_p}{\partial x_q} \right). \tag{3.4} \]

Taking into account the complex conjugate, we derive

\[ \frac{1}{2} \frac{\partial}{\partial x_q} \left[ F_p G_q^* + F_q G_p^* + F_q G_p^* + F_q G_p^* - \delta_{pq} (F \cdot G + F^* \cdot G) \right] = \frac{1}{2} \left( \text{ div } G^* - G^* \times \text{ curl } F + F^* \text{ div } G - G \times \text{ curl } F^* \right)_p \tag{3.5} \]

\[ + \frac{1}{2} \left( \text{ div } F^* - F^* \times \text{ curl } G + G^* \text{ div } F - F \times \text{ curl } G^* \right)_p \]

as our first important fact.

On the other hand, in view of Maxwell’s equations (2.5)–(2.6), one gets

\[ F \text{ div } G^* - G^* \times \text{ curl } F \tag{3.6} \]
\[ 4\pi \rho \mathbf{F} + \frac{i}{c} \left( \frac{\partial \mathbf{G}}{\partial t} \times \mathbf{G}^* + 4\pi \mathbf{j} \times \mathbf{G}^* \right) \]

and, with the help of its complex conjugate,
\[ \mathbf{F} \text{ div } \mathbf{G}^* - \mathbf{G}^* \times \text{curl } \mathbf{F} + \mathbf{F}^* \text{ div } \mathbf{G} - \mathbf{G} \times \text{curl } \mathbf{F}^* \]
\[ = 4\pi \rho (\mathbf{F} + \mathbf{F}^*) + \frac{i}{c} \frac{\partial}{\partial t} (\mathbf{G} \times \mathbf{G}^*) + \frac{4\pi i}{c} \mathbf{j} \times (\mathbf{G}^* - \mathbf{G}), \]

or
\[ \frac{1}{2} (\mathbf{F} \text{ div } \mathbf{G}^* - \mathbf{G}^* \times \text{curl } \mathbf{F} + \mathbf{F}^* \text{ div } \mathbf{G} - \mathbf{G} \times \text{curl } \mathbf{F}^*) \]
\[ = 4\pi \left( \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right) + \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}), \]

providing the second important fact. (Up to the constant, the first term in the right-hand side represents the density of Lorentz’s force acting on the “free” charges and currents in the medium under consideration [85], [86].)

In view of (3.8) and (3.5), we can write
\[ 4\pi \left( \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right) + \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) \]
\[ = \frac{1}{4} \partial_p \left[ \mathbf{F} \cdot \mathbf{G}^* - \mathbf{G}^* \times \text{curl } \mathbf{F} + \mathbf{F}^* \cdot \mathbf{G} \right] \]
\[ - \frac{1}{2} \left( \mathbf{G} \text{ div } \mathbf{F}^* - \mathbf{F}^* \times \text{curl } \mathbf{G} + \mathbf{G}^* \text{ div } \mathbf{F} - \mathbf{F} \times \text{curl } \mathbf{G}^* \right) \]
\[ - \frac{1}{4} \left( \mathbf{F} \text{ div } \mathbf{G}^* - \mathbf{G}^* \times \text{curl } \mathbf{F} + \mathbf{F}^* \text{ div } \mathbf{G} - \mathbf{G} \times \text{curl } \mathbf{F}^* \right) \]
\[ = 4\pi \partial_p \mathbf{T} + \frac{1}{4} \left( \mathbf{F} \text{ div } \mathbf{G}^* - \mathbf{G}^* \times \text{curl } \mathbf{F} + \mathbf{F}^* \text{ div } \mathbf{G} - \mathbf{G} \times \text{curl } \mathbf{F}^* \right) \]

Finally, in the last two lines, one can utilize the following differential vector calculus identity,
\[ [\mathbf{A} \text{ div } \mathbf{B} - \mathbf{B} \text{ div } \mathbf{A} + \mathbf{A} \times \text{curl } \mathbf{B} - \mathbf{B} \times \text{curl } \mathbf{A} - \text{curl} (\mathbf{A} \times \mathbf{B})]_p \]
\[ = \mathbf{A}_p \frac{\partial B_q}{\partial x_p} - \mathbf{B}_p \frac{\partial A_q}{\partial x_p}, \]

see (A.5), with \( \mathbf{A} = \mathbf{F}, \mathbf{B} = \mathbf{G}^* \) and its complex conjugates, in order to obtain (3.12) and/or (3.16), which completes the proof. (An independent proof will be given in section 7.)

Derivation of the corresponding differential “energy” balance equation is much simpler. By (2.5),
\[ \mathbf{F} \cdot \frac{\partial \mathbf{G}^*}{\partial t} + \mathbf{F}^* \cdot \frac{\partial \mathbf{G}}{\partial t} + 4\pi \mathbf{j} \cdot (\mathbf{F} + \mathbf{F}^*) = \frac{c}{\ell} \text{ div } \mathbf{F} \mathbf{F}^* \]
due to a familiar vector calculus identity (A.1):

$$\text{div} \left( A \times B \right) = B \cdot \text{curl} \, A - A \cdot \text{curl} \, B.$$  \hfill (3.12)

In a traditional form,

$$\frac{1}{4\pi} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \mathbf{j} \cdot \mathbf{E} + \text{div} \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \right) = 0$$  \hfill (3.13)

(see, for example, [18], [86]), where one can substitute

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left( \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} \right) + \frac{1}{2} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t} \right).$$  \hfill (3.14)

As a result, 3D-differential “energy-momentum” balance equations are given by

$$\frac{\partial}{\partial t} \left( \frac{1}{8\pi} \left( \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} \right) \right) + \text{div} \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \right) + \mathbf{j} \cdot \mathbf{E}$$

$$+ \frac{1}{8\pi} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t} \right) = 0$$  \hfill (3.15)

and

$$- \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c} \left( \mathbf{D} \times \mathbf{B} \right) \right]_p + \frac{\partial T_{pq}}{\partial x_q} - \left( \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right)_p$$

$$+ \frac{1}{8\pi} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial x_p} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial x_p} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial x_p} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial x_p} \right) = 0,$$

respectively (see also [32], [62]). The real form of the symmetric stress tensor (3.1), namely,

$$T_{pq} = \frac{1}{8\pi} \left[ E_p D_q + E_q D_p + H_p B_q + H_q B_p - \delta_{pq} \left( \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} \right) \right] \quad (p, q = 1, 2, 3),$$  \hfill (3.17)

is due to Hertz [72].

Equations (3.15)–(3.16) are related to a fundamental concept of conservation of mechanical and electromagnetic energy and momentum. Here, these balance conditions are presented in differential forms in terms of the corresponding local field densities. They can be integrated over a given volume in $\mathbb{R}^3$ in order to obtain, in a traditional way, the corresponding conservation laws of the electromagnetic fields (see, for example, [50], [51], [88], [90], [91]). These laws made it necessary to ascribe a definite linear momentum and energy to the field of an electromagnetic wave, which can be observed, for example, as light pressure.

Note. At this point, the Lorentz invariance of these differential balance equations is not obvious in our 3D-analysis. But one can introduce the four-vector $x^\mu = (ct, \mathbf{r})$ and try to match (3.15)–(3.16) with the expression,

$$\frac{\partial}{\partial x^\nu} T^\nu_\mu = \frac{\partial T^0_\mu}{\partial x_0} + \frac{\partial T^q_\mu}{\partial x_q} \quad (\mu, \nu = 0, 1, 2, 3; \quad p, q = 1, 2, 3),$$  \hfill (3.18)
as an initial step, in order to guess the corresponding four-tensor form. An independent covariant derivation will be given in section 7.

**Note.** In an isotropic nonhomogeneous variable medium (without dispersion and/or compression), when \( \mathbf{D} = \varepsilon(\mathbf{r}, t) \mathbf{E} \) and \( \mathbf{B} = \mu(\mathbf{r}, t) \mathbf{H} \), the “ponderomotive forces” in (3.15) and (3.16) take the form [86]:

\[
\mathbf{E} \cdot \partial \mathbf{D} / \partial x^\nu - \mathbf{D} \cdot \partial \mathbf{E} / \partial x^\nu + \mathbf{H} \cdot \partial \mathbf{B} / \partial x^\nu - \mathbf{B} \cdot \partial \mathbf{H} / \partial x^\nu
\]

\[
= \partial \varepsilon / \partial x^\nu \mathbf{E}^2 + \partial \mu / \partial x^\nu \mathbf{H}^2 = \left( \frac{1}{c} \left( \frac{\partial \varepsilon}{\partial t} \mathbf{E}^2 + \frac{\partial \mu}{\partial t} \mathbf{H}^2 \right) \right)
\]

\[
\mathbf{E}^2 \nabla \varepsilon + \mathbf{H}^2 \nabla \mu
\]

which may be interpreted as a four-vector “energy-force” acting from an inhomogeneous and time-variable medium. Its covariance is analyzed in section 7.

4. “Angular Momentum” Balance

The 3D-“linear momentum” differential balance equation (3.16), can be rewritten in a more compact form,

\[
\frac{\partial \mathbf{T}_{pq}}{\partial x_q} = \mathbf{F}_p + \frac{\partial \mathbf{G}_p}{\partial t}, \quad \mathbf{\vec{G}} = \frac{1}{4\pi c}(\mathbf{D} \times \mathbf{B}),
\]

with the help of the Hertz symmetric stress tensor \( \mathbf{T}_{pq} = T_{qp} \) defined by (3.17). A “net force” is given by

\[
\mathbf{F}_p = \left( \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right)_p - \frac{1}{8\pi} \left[ \text{curl} \left( \mathbf{E} \times \mathbf{D} + \mathbf{H} \times \mathbf{B} \right) \right]_p
\]

\[
- \frac{1}{8\pi} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial x_p} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial x_p} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial x_p} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial x_p} \right).
\]

In this notation, we state the 3D-“angular momentum” differential balance equation as follows

\[
\frac{\partial \mathbf{M}_{pq}}{\partial x_q} = \mathbf{T}_p + \frac{\partial \mathbf{L}_p}{\partial t}, \quad \mathbf{\vec{L}} = \mathbf{r} \times \mathbf{\vec{G}}, \quad \mathbf{\vec{T}} = \mathbf{r} \times \mathbf{\vec{F}},
\]

where the “field angular momentum density” is defined by

\[
\mathbf{\vec{L}} = \frac{1}{4\pi c} \mathbf{r} \times (\mathbf{D} \times \mathbf{B})
\]

and the “flux of angular momentum” is described by the following tensor [37]:

\[
\mathbf{M}_{pq} = e_{prs} x_r \mathbf{T}_{sq}.
\]

(Here, \( e_{pqr} \) is the totally anti-symmetric Levi-Civita symbol with \( e_{123} = +1 \)). An elementary example of conservation of the total angular momentum is discussed in [86].

**Proof.** Indeed, in view of (4.1), one can write

\[
\frac{\partial \mathbf{M}_{pq}}{\partial x_q} = e_{prs} T_{sr} + e_{prs} x_r \frac{\partial T_{sq}}{\partial x_q}
\]

\[
= e_{pqr} x_q \mathbf{F}_r + \frac{\partial}{\partial t} (e_{pqr} x_q \mathbf{G}_r),
\]
which completes the proof.

**Note.** Once again, in 3D-form, the Lorentz invariance of this differential balance equation for the local densities is not obvious. An independent covariant derivation will be given in section 8.

5. **Complex Covariant Form of Macroscopic Maxwell’s Equations**

With the help of complex fields \( \mathbf{F} = \mathbf{E} + i \mathbf{H} \) and \( \mathbf{G} = \mathbf{D} + i \mathbf{B} \), we introduce the following anti-symmetric four-tensor,

\[
Q^\mu\nu = -Q^\nu\mu = 
\begin{pmatrix}
0 & -G_1 & -G_2 & -G_3 \\
G_1 & 0 & iF_3 & -iF_2 \\
G_2 & -iF_3 & 0 & iF_1 \\
G_3 & iF_2 & -iF_1 & 0
\end{pmatrix} \tag{5.1}
\]

and use the standard four-vectors, \( x^\mu = (ct, \mathbf{r}) \) and \( j^\mu = (\rho, \mathbf{j}) \) for contravariant coordinates and current, respectively.

Maxwell’s equations then take the covariant form [47], [54]:

\[
\partial_\nu Q^{\mu\nu} = -\frac{4\pi}{c} j^\mu \tag{5.2}
\]

with summation over two repeated indices. Indeed, in block form, we have

\[
\partial Q^{\mu\nu} = \partial_x \left( G_\rho^{\ 0} - i e_{\rho\tau\rho\sigma} F_{\mu\nu} \right) = \begin{pmatrix} -\text{div} \mathbf{G} = -4\pi \rho \\ \frac{1}{c} \frac{\partial \mathbf{G}}{\partial t} + i \text{curl} \mathbf{F} = -\frac{4\pi}{c} \mathbf{j} \end{pmatrix}, \tag{5.3}
\]

which verifies this fact. The continuity equation,

\[
0 \equiv \frac{\partial^2 Q^{\mu\nu}}{\partial x^\mu \partial x^\nu} = -\frac{4\pi}{c} \frac{\partial j^\mu}{\partial x^\mu}, \tag{5.4}
\]

or in the 3D-form,

\[
\frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0, \tag{5.5}
\]

describes conservation of the electrical charge. The latter equation can also be derived in the complex 3D-form from (2.5)–(2.6).

**Note.** In vacuum, when \( \mathbf{G} = \mathbf{F} \) and \( \rho = 0, \mathbf{j} = 0 \), one can write due to (B.5)–(B.6):

\[
Q^{\mu\nu} = F^{\mu\nu} - \frac{i}{2} e^{\mu\nu\sigma\tau} F_{\sigma\tau}, \quad F^{\mu\nu} = g^{\mu\sigma} g^{\nu\tau} F_{\sigma\tau}, \quad g_{\mu\sigma} g_{\nu\tau} Q^{\sigma\tau} = Q^{\mu\nu}. \tag{5.6}
\]

As a result, the following self-duality property holds

\[
e_{\mu\nu\sigma\tau} Q^{\sigma\tau} = 2iQ_{\mu\nu}, \quad 2iQ^{\mu\nu} = e^{\mu\nu\sigma\tau} Q_{\sigma\tau} \tag{5.7}
\]

(see, for example, [8], [48] and appendix B). Two covariant forms of Maxwell’s equations are given by

\[
\partial_\nu Q^{\mu\nu} = 0, \quad \partial^\mu Q_{\mu\nu} = 0, \tag{5.8}
\]

where \( \partial^\mu = g^{\mu\nu} \partial_\nu \), \( \partial_\mu = \partial / \partial x^\mu \) and \( g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \). The last equation can be derived from a more general equation, involving a rank three tensor,

\[
g^{\alpha\epsilon} e_{\alpha\mu\tau} \partial^\mu Q^{\tau\beta} - g^{\alpha\epsilon} e_{\beta\mu\tau} \partial^\mu Q^{\alpha\tau} = -i \partial_\mu Q^{\alpha\beta} \tag{5.9}
\]
\[ \alpha, \beta = 0, 1, 2, 3 \text{ are fixed; no summation is assumed over these two indices}, \] which is related to the Pauli-Lubanski vector from the representation theory of the Poincaré group \[47\]. Different spinor forms of Maxwell’s equations are analyzed in \[48\] (see also the references therein).

6. Dual Electromagnetic Field Tensors

Two dual anti-symmetric field tensors of complex fields, \( F = E + iH \) and \( G = D + iB \), are given by

\[ Q_{\mu\nu} = \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_1 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix} = R_{\mu\nu} + iS_{\mu\nu} \tag{6.1} \]

and

\[ P_{\mu\nu} = \begin{pmatrix} 0 & F_1 & F_2 & F_3 \\ -F_1 & 0 & iG_3 & -iG_2 \\ -F_2 & -iG_3 & 0 & iG_1 \\ -F_3 & iG_2 & -iG_1 & 0 \end{pmatrix} = F_{\mu\nu} + iG_{\mu\nu} \tag{6.2} \]

The real part of the latter represents the standard electromagnetic field tensor in a medium \[6\], \[72\], \[91\]. As for the imaginary part of (6.1), which, ironically, Pauli called an “artificiality” in view of its non-standard behavior under spatial inversion \[72\], the use of complex conjugation restores this important symmetry for our complex field tensors.

The dual tensor identities are given by

\[ e_{\mu\nu\sigma\tau}Q^{\sigma\tau} = 2iP_{\mu\nu}, \quad 2iQ^{\mu\nu} = e_{\mu\nu\sigma\tau}P_{\sigma\tau}. \tag{6.3} \]

Here \( e^{\mu\nu\sigma\tau} = -e_{\mu\nu\sigma\tau} \) and \( e_{0123} = +1 \) is the Levi-Civita four-symbol \[27\]. Then

\[ 6i \frac{\partial Q^{\mu\nu}}{\partial x^{\nu}} = e^{\mu\nu\lambda\sigma} \left( \frac{\partial P_{\lambda\sigma}}{\partial x^{\nu}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\sigma}} + \frac{\partial P_{\sigma\nu}}{\partial x^{\lambda}} \right) \tag{6.4} \]

and both pairs of Maxwell’s equations can also be presented in the form \[47\]

\[ \frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial P_{\lambda\mu}}{\partial x^{\nu}} = -\frac{4\pi i}{c} e_{\mu\nu\lambda\sigma} j^{\sigma} \tag{6.5} \]

in addition to the one given above

\[ \frac{\partial Q^{\mu\nu}}{\partial x^{\nu}} = -\frac{4\pi i}{c} j^{\mu}. \tag{6.6} \]
The real part of the first equation traditionally represents the first (homogeneous) pair of Maxwell’s equation and the real part of the second one gives the remaining pair. In our approach both pairs of Maxwell’s equations appear together (see also [6], [8], [9], [54], and [87] for the case in vacuum). Moreover, a generalization to complex-valued four-current may naturally represent magnetic charge and magnetic current not yet observed in nature [79].

An important cofactor matrix identity,

\[ P^{\mu \nu} Q_{\nu \lambda} = (F \cdot G) \delta_{\lambda}^{\mu} = \frac{1}{4} \left( P^{\sigma \tau} Q_{\tau \sigma}^* \right) \delta_{\lambda}^{\mu}, \]

was originally established, in a general form, by Minkowski [65]. Once again, the dual tensors are given by

\[ P^{\mu \nu} = \begin{pmatrix} 0 & F_q \\ -F_p & i e_{pqr} G_r \end{pmatrix}, \quad Q^{\mu \nu} = \begin{pmatrix} 0 & -G_q \\ G_p & i e_{pqr} F_r \end{pmatrix}, \]

in block form. A complete list of relevant tensor and matrix identities is given in appendix B.

7. Covariant Derivation of Energy-Momentum Balance Equations

7.1. Preliminaries. As has been announced in [17] (see also [18]), the covariant form of the differential balance equations can be presented as follows

\[ \frac{\partial}{\partial x^\nu} \left[ \frac{1}{16\pi} \left( P_{\mu \lambda} Q^{\lambda \nu} + P_{\mu \lambda} Q^*_{\lambda \nu} \right) \right] \]

\[ + \frac{1}{32\pi} \left( P^*_{\sigma \tau} \frac{\partial Q^{\tau \sigma}}{\partial x^\nu} + P_{\sigma \tau} \frac{\partial Q^*_{\tau \sigma}}{\partial x^\nu} \right) \]

\[ = -\frac{1}{c} F_{\mu \lambda} j^\lambda = \begin{pmatrix} -j \cdot E/c \\ \rho E + j \times B/c \end{pmatrix}. \]

In our complex form, when \( F = E + iH \) and \( G = D + iB \), the energy-momentum tensor is given by

\[ 16\pi T^{\mu \nu}_{\text{E-M}} = P^{*}_{\mu \lambda} Q^{\lambda \nu} + P_{\mu \lambda} Q^*_{\lambda \nu} \]

\[ = \begin{pmatrix} F \cdot G^* + F^* \cdot G & 2i (F \times F^*)_q \\ -2i (G \times G^*)_p & 2 \left( F_p G^*_q + F^*_p G_q \right) - \delta_{pq} (F \cdot G^* + F^* \cdot G) \end{pmatrix}. \]

Here, we point out for the reader’s convenience that

\[ i (F \times F^*) = 2 (E \times H), \quad i (G \times G^*) = 2 (D \times B), \]

and, in real form,

\[ 4\pi T^{\mu \nu}_{\text{E-M}} = \begin{pmatrix} (E \cdot D + H \cdot B)/2 & (E \times H)_q \\ - (D \times B)_p & E_p D_q + H_p B_q - \delta_{pq} (E \cdot D + H \cdot B)/2 \end{pmatrix}. \]
The covariant form of the differential balance equation allows one to clarify the meanings of different energy-momentum tensors. For instance, it is worth noting that the non-symmetric Maxwell and Heaviside form of the 3D-stress tensor [72],

$$\tilde{T}_{pq} = \frac{1}{4\pi} (E_pD_q + H_pB_q) - \frac{1}{8\pi} \delta_{pq} (E \cdot D + H \cdot B),$$

(7.5)

appears here in the corresponding “momentum” balance equation [86]:

$$- \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c} (D \times B) \right]_p + \frac{\partial \tilde{T}_{pq}}{\partial x^q} - \left( \rho E + \frac{1}{c} j \times B \right)_p \quad (7.6)$$

$$+ \frac{1}{8\pi} \left( E \cdot \frac{\partial D}{\partial x_p} - D \cdot \frac{\partial E}{\partial x_p} + H \cdot \frac{\partial B}{\partial x_p} - B \cdot \frac{\partial H}{\partial x_p} \right) = 0.$$

At the same time, in view of (3.16), use of the form (7.5) differs from Hertz’s symmetric tensors in (3.1) and (3.17) only in the case of anisotropic media (crystals) [72], [85]. Indeed,

$$8\pi \frac{\partial}{\partial x^q} (\tilde{T}_{pq} - T_{pq}) = [\text{curl} (E \times D + H \times B)]_p.$$

(7.7)

Moreover, with the help of elementary identities,

$$[\text{curl} (A \times B)]_p = \frac{\partial}{\partial x_q} (A_pB_q - A_qB_p)$$

(7.8)

and

$$2 \frac{\partial}{\partial x_q} (A_pB_q) = \frac{\partial}{\partial x_q} (A_pB_q + A_qB_p) + [\text{curl} (A \times B)]_p,$$

(7.9)

one can transform the latter balance equation into its “symmetric” form, which provides an independent proof of (3.16).

7.2. Proof. The fact that Maxwell’s equations can be united with the help of a complex second rank (anti-symmetric) tensor allows us to utilize the standard Sturm-Liouville type argument in order to establish the energy-momentum differential balance equations in covariant form. Indeed, by adding matrix equation

$$P^\ast_{\mu\lambda} \left( \frac{\partial Q^{\lambda\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\lambda \right)$$

(7.10)

and its complex conjugate

$$P_{\mu\lambda} \left( \frac{\partial Q^{*\lambda\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\lambda \right)$$

(7.11)

one gets

$$P^\ast_{\mu\lambda} \frac{\partial Q^{\lambda\nu}}{\partial x^\nu} + P_{\mu\lambda} \frac{\partial Q^{*\lambda\nu}}{\partial x^\nu} = -\frac{8\pi}{c} F_{\mu\lambda} j^\lambda.$$

(7.12)

A simple decomposition,

$$f \frac{\partial g}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} (fg) + \frac{1}{2} \left( f \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} g \right)$$

(7.13)
with \( f = P^*_\mu \lambda \) and \( g = Q^\lambda \nu \) (and their complex conjugates), results in
\[
\frac{\partial}{\partial x^\nu} \left[ \frac{1}{16\pi} \left( P^*_\mu \lambda Q^\lambda \nu + P^*_\mu \lambda Q^\lambda \nu \right) \right] + \frac{1}{16\pi} \left( \left( P^*_\mu \lambda \frac{\partial Q^\lambda \nu}{\partial x^\nu} - \frac{\partial P^*_\mu \lambda}{\partial x^\nu} Q^\lambda \nu \right) + \text{(c.c.)} \right) = -\frac{1}{c} F^\mu \lambda j^\lambda .
\] (7.14)

By a direct substitution, one can verify that
\[
Z^\mu = P^*_\mu \lambda \frac{\partial Q^\lambda \nu}{\partial x^\nu} - \frac{\partial P^*_\mu \lambda}{\partial x^\nu} Q^\lambda \nu = \frac{1}{2} P^*_\sigma \lambda \frac{\partial Q^\sigma \tau}{\partial x^\mu} (7.15)
\]
\[
= -\frac{1}{2} Q^\sigma \tau \frac{\partial P^*_\sigma \tau}{\partial x^\mu} = \mathbf{F}^* \cdot \frac{\partial \mathbf{G}}{\partial x^\mu} - \mathbf{G}^* \cdot \frac{\partial \mathbf{F}}{\partial x^\mu}.
\]
(An independent covariant proof of these identities is given in appendix C.) Finally, introducing
\[
X^\mu = Z^\mu + Z^\star \mu \]
we obtain (7.1) with the explicitly covariant expression for the ponderomotive force (3.19), which completes the proof.

As a result, the covariant energy-momentum balance equation is given by
\[
\frac{\partial}{\partial x^\nu} T^\mu \nu + X^\mu = -\frac{1}{c} F^\mu \lambda j^\lambda \]
(7.17)
in a compact form. If these differential balance equations are written for a stationary medium, then the corresponding equations for moving bodies are uniquely determined, since the components of a tensor in any inertial coordinate system can be derived by a proper Lorentz transformation [72].

### 8. Covariant Derivation of Angular Momentum Balance

By definition, \( x^\mu = g^\mu \nu x^\nu = (ct, -\mathbf{r}) \) and \( T^\mu \lambda = T^\nu \mu g^\nu \lambda \), where \( g^\mu \nu = \text{diag}(1, -1, -1, -1) = \partial x^\mu / \partial x^\nu \). In view of (7.17), we derive
\[
\frac{\partial}{\partial x^\nu} (x^\lambda T^\mu \nu - x^\mu T^\lambda \nu) = (T^\mu \lambda - T^\lambda \mu)
\]
\[
- (x^\lambda X^\mu - x^\mu X^\lambda) - \frac{1}{c} (x^\lambda F^\mu \nu - x^\mu F^\lambda \nu) j^\nu
\]
as a required differential balance equation.

With the help of familiar dual relations (B.4), one can get another covariant form of the angular momentum balance equation:
\[
\frac{\partial}{\partial x^\nu} \left( e^{\mu \lambda \sigma \tau} x^\sigma T^\tau \nu \right) + e^{\mu \lambda \sigma \tau} T^\sigma \tau
\]
\[
+ e^{\mu \lambda \sigma \tau} x^\sigma X^\tau + \frac{1}{c} e^{\mu \lambda \sigma \tau} x^\sigma F^\tau \nu j^\nu = 0^\mu \lambda.
\] (8.2)

In 3D-form, the latter relation can be reduced to (4.3)–(4.5).

Indeed, when \( \mu = 0 \) and \( \lambda = p = 1, 2, 3 \), one gets
\[
-\frac{1}{4\pi c} \frac{\partial}{\partial t} [e_{pqr} x^q (\mathbf{D} \times \mathbf{B})_r] + \frac{\partial}{\partial x^s} \left( e_{pqr} x^q \tilde{T}^\tau \nu \right)
\]
(8.3)
$+\epsilon_{pqr} \tilde{T}_{qr} + \epsilon_{pqr} x_q (X_r + Y_r) = 0,$

where $-Y = \rho E + j \times B/c$ is the familiar Lorentz force. Substitution, $\tilde{T}_{rs} = T_{rs} + (\tilde{T}_{rs} - T_{rs})$, results in (4.3) in view of identity (7.7). The remaining cases, when $\mu, \nu = p, q = 1, 2, 3$, can be analyzed in a similar fashion. In 3D-form, the corresponding equations can be reduced to (5.15) and (7.6). Details are left to the reader.

Thus the angular momentum law has the form of a local balance equation, not a conservation law, since in general the energy-momentum tensor will not be symmetric [34]. A torque, for instance, may occur, which cannot be compensated for by a change in the electromagnetic angular momentum, though not in contradiction with experiment [72].

9. Transformation Laws of Complex Electromagnetic Fields

Let $\mathbf{v}$ be a constant real-valued velocity vector representing uniform motion of one frame of reference with respect to another one. Let us consider the following orthogonal decompositions,

$$F = F_\parallel + F_\perp, \quad G = G_\parallel + G_\perp,$$

such that our complex vectors $\{F_\parallel, G_\parallel\}$ are collinear with the velocity vector $\mathbf{v}$ and $\{F_\perp, G_\perp\}$ are perpendicular to it (Figure 1). The Lorentz transformation of electric and magnetic fields $\{E, D, H, B\}$ take the following complex form

$$F' = F, \quad G' = G$$

and

$$F'_\perp = \frac{F_\perp - i}{c} (\mathbf{v} \times G), \quad G'_\perp = \frac{G_\perp - i}{c} (\mathbf{v} \times F).$$

Although this transformation was found by Lorentz, it was Minkowski who realized that this is the law of transformation of the second rank anti-symmetric four-tensors [58], [65]; a brief historical overview is given in [72].) This complex 3D-form of the Lorentz transformation of electric and magnetic fields was known to Minkowski (1908), but apparently only in vacuum, when $G = F$ (see also [88]). Moreover,

$$r'_\parallel = \frac{r_\parallel - \mathbf{v} t}{\sqrt{1 - v^2/c^2}}, \quad r'_\perp = r_\perp, \quad t' = \frac{t - (\mathbf{v} \cdot \mathbf{r})/c^2}{\sqrt{1 - v^2/c^2}}.$$

in the same notation [72].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Complex electromagnetic fields decomposition.}
\end{figure}
The latter equations can be rewritten as follows
\[ r' = r + \left[ (\gamma - 1) \frac{v \cdot r}{v^2} - \gamma t \right] v, \quad t' = \gamma \left( t - \frac{v \cdot r}{c^2} \right), \quad (9.5) \]
where \( \gamma = (1 - v^2/c^2)^{-1/2} \). In a similar fashion, one gets
\[ F' = \gamma \left( F - \frac{i}{c} v \times G \right) - (\gamma - 1) \frac{v \cdot F}{v^2} v, \quad (9.6) \]
\[ G' = \gamma \left( G - \frac{i}{c} v \times F \right) - (\gamma - 1) \frac{v \cdot G}{v^2} v, \quad (9.7) \]
as a compact 3D-version of the Lorentz transformation for the complex electromagnetic fields.

In complex four-tensor form,
\[ Q'_{\mu \nu}(x') = \Lambda^\mu_{\sigma} \Lambda^\nu_{\tau} Q^{\sigma \tau}(x), \quad x' = \Lambda x. \quad (9.8) \]

Although Minkowski considered the transformation of electric and magnetic fields in a complex 3D-vector form, see Eqs. (8)–(9) and (15) in [65] (or Eqs. (25.5)–(25.6) in [50]), he seems never to have combined the corresponding four-tensors into the complex forms (6.1)–(6.2). In the second article [66], Max Born, who used Minkowski’s notes, didn’t mention the complex fields. As a result, the complex field tensor seems only to have appeared, for the first time, in [51] (see also [87]). The complex identity, \( F \cdot G = \) invariant under a similarity transformation, follows from Minkowski’s determinant relations (B.23)–(B.25).

**Figure 2.** Example of moving frame velocity.

**Example.** Let \( \{e_k\}_{k=1}^3 \) be an orthonormal basis in \( \mathbb{R}^3 \). We choose \( v = v e_1 \) and write \( x'^\mu = \Lambda^\mu_{\nu} x^\nu \) with
\[ \Lambda^\mu_{\nu} = \begin{pmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (9.9) \]
for the corresponding Lorentz boost (Figure 2). In view of (9.8), by matrix multiplication one gets
\[
\begin{pmatrix}
\gamma & -\beta\gamma & 0 & 0 \\
-\beta\gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -G_1 & -G_2 & -G_3 \\
G_1 & 0 & iF_3 & -iF_2 \\
G_2 & -iF_3 & 0 & iF_1 \\
G_3 & iF_2 & -iF_1 & 0
\end{pmatrix}
\begin{pmatrix}
\gamma & -\beta\gamma & 0 & 0 \\
-\beta\gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]}
\begin{align}
&= \begin{pmatrix}
\gamma G_2 + i\beta\gamma F_3 & -\beta\gamma G_2 - i\beta\gamma F_3 & -\gamma G_2 - i\beta\gamma F_3 & -\gamma G_2 + i\beta\gamma F_2 \\
\gamma G_3 - i\beta\gamma F_2 & -\beta\gamma G_3 + i\gamma F_2 & -\beta\gamma G_3 + i\beta\gamma F_2 & 0 \\
G_1 & 0 & \beta\gamma G_2 + i\gamma F_3 & 0 \\
G_1 & 0 & \beta\gamma G_2 + i\gamma F_3 & 0 \\
\end{pmatrix}.
\end{align}
(9.10)
Thus \(G'_1 = G_1\) and
\[
G'_2 = \gamma G_2 + i\beta\gamma F_3 = \frac{G_2 + i(v/c) F_3}{\sqrt{1 - v^2/c^2}} = \frac{G_2 - \frac{i}{c} (v \times F)_2}{\sqrt{1 - v^2/c^2}},
(9.11)
\]
\[
G'_3 = \gamma G_3 - i\beta\gamma F_2 = \frac{G_3 - i(v/c) F_2}{\sqrt{1 - v^2/c^2}} = \frac{G_3 - \frac{i}{c} (v \times F)_3}{\sqrt{1 - v^2/c^2}}.
\]
In a similar fashion, \(F'_1 = F_1\) and
\[
F'_2 = \gamma F_2 + i\beta\gamma G_3 = \frac{F_2 - \frac{i}{c} (v \times G)_2}{\sqrt{1 - v^2/c^2}},
(9.12)
\]
\[
F'_3 = \gamma F_3 - i\beta\gamma G_2 = \frac{F_3 - \frac{i}{c} (v \times G)_3}{\sqrt{1 - v^2/c^2}}.
\]
The latter relations are in agreement with the field transformations (9.2)–(9.3).

In block form, one gets
\[
\begin{pmatrix}
F'_1 \\
F'_2 \\
G'_3 \\
G'_2 \\
F'_3 \\
G'_1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos (i\theta) & \sin (i\theta) & 0 & 0 & 0 \\
0 & -\sin (i\theta) & \cos (i\theta) & 0 & 0 & 0 \\
0 & 0 & 0 & \cos (i\theta) & \sin (i\theta) & 0 \\
0 & 0 & 0 & -\sin (i\theta) & \cos (i\theta) & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
F_1 \\
F_2 \\
G_3 \\
G_2 \\
F_3 \\
G_1
\end{pmatrix},
(9.13)
\]
where, by definition,
\[
\cos (i\theta) = \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \sin (i\theta) = i\beta\gamma = \frac{i\beta}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}.
(9.14)
\]
As a result, the transformation law of the complex electromagnetic fields \(\{F, G\}\) under the Lorentz boost can be thought of as a complex rotation in \(\mathbb{C}^6\), corresponding to a reducible representation of the one-parameter subgroup of \(SO(3, \mathbb{C})\). (Cyclic permutation of the spatial indices cover the two remaining cases; see also [88].)
10. Material Equations, Potentials, and Energy-Momentum Tensor for Moving Isotropic Media

Electromagnetic phenomena in moving media are important in relativistic astrophysics, the study of accelerated plasma clusters and high-energy electron beams [15], [16], [26], [91].

10.1. Material equations. Minkowski’s field- and connecting-equations [65], [66] were derived from the corresponding laws for the bodies at rest by means of a Lorentz transformation (see [15], [18], [34], [51], [67], [72], [91]). Explicitly covariant forms, which are applicable both in the rest frame and for moving media, are analyzed in [15], [16], [34], [39], [40], [67], [71], [72], [75], [77], [88], [91] (see also the references therein). In standard notation, \( \beta = v/c, \ \gamma = (1 - \beta^2)^{-1/2}, \ v = |v|, \ \kappa = \varepsilon\mu - 1, \) (10.1)
on can write [15], [16], [18], [91]:

\[
D = \varepsilon E + \frac{\kappa\gamma^2}{\mu} \left[ \beta^2 E - \frac{v}{c^2} (v \cdot E) + \frac{1}{c} (v \times B) \right],
\]

\[
H = \frac{1}{\mu} B + \frac{\kappa\gamma^2}{\mu} \left[ -\beta^2 B + \frac{v}{c^2} (v \cdot B) + \frac{1}{c} (v \times E) \right].
\]

In covariant form, these relations are given by

\[
R^{\lambda\nu} = \epsilon^{\lambda\nu\sigma\tau} F_{\sigma\tau} = \frac{1}{2} \left( \epsilon^{\lambda\nu\sigma\tau} - \epsilon^{\lambda\nu\tau\sigma} \right) F_{\sigma\tau}
\]

\[
= \frac{1}{4} \left( \epsilon^{\lambda\nu\sigma\tau} - \epsilon^{\lambda\nu\tau\sigma} + \epsilon^{\mu\lambda\sigma\tau} - \epsilon^{\mu\nu\lambda\sigma} \right) F_{\sigma\tau}
\]

(see [14], [15], [16], [39], [40], [75], [77], [91] and the references therein). Here,

\[
\epsilon^{\lambda\nu\sigma\tau} = \frac{1}{\mu} \left( g^{\lambda\sigma} + \kappa u^{\lambda} u^{\sigma} \right) \left( g^{\nu\tau} + \kappa u^{\nu} u^{\tau} \right) = \epsilon^{\nu\lambda\sigma\tau}
\]

(10.4)
is the four-tensor of electric and magnetic permeabilities\(^3\) and

\[
u^{\lambda} = (\gamma, \gamma v/c), \quad u^{\lambda} u_{\lambda} = 1
\]

(10.5)
is the four-velocity of the medium ([75], [77], a computer algebra verification of these relations is given in [52]). In a complex covariant form,

\[
\left( Q^{\mu\nu} + Q^{\mu\nu\ast} \right) = \epsilon^{\mu\nu\sigma\tau} \left( P_{\sigma\tau} + P_{\sigma\tau\ast} \right).
\]

(10.6)

In view of (10.3) and (B.5)–(B.6), we get

\[
Q^{\mu\nu} = \left( \epsilon^{\mu\nu\sigma\tau} - \frac{i}{2} \epsilon^{\mu\nu\sigma\tau} \right) F_{\sigma\tau}, \quad P_{\mu\nu} = \left( \delta^{\lambda\rho}_{\mu} \delta^{\nu}_{\tau} - \frac{i}{2} \epsilon_{\mu\nu\sigma\tau} \epsilon^{\sigma\tau\lambda\rho} \right) F_{\lambda\rho},
\]

(10.7)
in terms of the real-valued electromagnetic field tensor.

\(^3\)Originally introduced by Tamm for a general case of the moving anisotropic medium [83], [84].
10.2. **Potentials.** In practice, one can choose

\[ F_{\sigma\tau} = \frac{\partial A_\tau}{\partial x^\sigma} - \frac{\partial A_\sigma}{\partial x^\tau}, \]  

(10.8)

for the real-valued four-vector potential \( A_\lambda (x) \). Then

\[ \partial_\nu Q^{\lambda\nu} = \epsilon^{\lambda\nu\sigma\tau} (\partial_\sigma A_\tau - \partial_\tau A_\sigma) - \frac{i}{2} \epsilon^{\lambda\nu\sigma\tau} (\partial_\nu A_\tau - \partial_\tau A_\sigma) \]

\[ = \frac{1}{\mu} (g^{\lambda\sigma} + \kappa u^{\lambda} u^\sigma) (g^{\nu\tau} + \kappa u^{\nu} u^\tau) \partial_\nu (\partial_\sigma A_\tau - \partial_\tau A_\sigma) \]

by (10.4). Substitution into Maxwell’s equations (6.6) or (6.5) results in

\[ \left( g^{\lambda\sigma} + \kappa u^{\lambda} u^\sigma \right) \{ - [\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] A_\sigma + \partial_\sigma (\partial^\tau A_\tau + \kappa u^{\nu} u^\tau \partial_\nu A_\tau) \} = - \frac{4\pi\mu}{c} j^\lambda, \]

where \(-\partial^\tau \partial_\tau = -g^{\sigma\tau} \partial_\sigma \partial_\tau = \Delta - (\partial/c\partial t)^2 \) is the D’Alembert operator. In view of an inverse matrix identity,

\[ (g^{\lambda\rho} - \kappa (u^{\lambda} u^\rho)) \frac{\kappa}{1 + \kappa} u_{\lambda} u_\rho \]

\[ (g^{\lambda\sigma} + \kappa u^{\lambda} u^\sigma) = \delta^\rho_\sigma, \]

(10.10)

the latter equations take the form

\[ \left[ \partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2 \right] A_\sigma - \partial_\sigma (\partial^\tau A_\tau + \kappa u^{\nu} u^\tau \partial_\nu A_\tau) \]

\[ = \frac{4\pi\mu}{c} \left( g_{\sigma\lambda} - \kappa \frac{1}{1 + \kappa} u_{\lambda} u_\sigma \right) j^\lambda. \]

(10.11)

Subject to the subsidiary condition,

\[ \partial^\tau A_\tau + \kappa u^{\nu} u^\tau \partial_\nu A_\tau = (g^{\nu\tau} + \kappa u^{\nu} u^\tau) \partial_\nu A_\tau = 0, \]

(10.12)

these equations were studied in detail for the sake of development of the phenomenological classical and quantum electrodynamics in a moving medium (see [13], [14], [15], [16], [71], [75], [76], [77], [91] and the references therein). In particular, Green’s function of the photon in a moving medium was studied in [39], [75], [76] (with applications to quantum electrodynamics).

10.3. **Hertz’s tensor and vectors.** We follow [15], [16], [91] with somewhat different details. The substitution,

\[ A^\mu (x) = \left( \frac{\kappa}{1 + \kappa} u^{\mu} u_\lambda - \delta^\mu_\lambda \right) \partial_\sigma Z^{\lambda\sigma} (x) \]

(10.13)

(a generalization of Hertz’s potentials for a moving medium [15], [91]), into the gauge condition (10.12) results in \( Z^{\lambda\sigma} = -Z^{\sigma\lambda} \), in view of

\[ (g^{\nu\mu} + \kappa u^{\nu} u_\mu) \partial_\nu A^\mu \]

\[ = (g^{\nu\mu} + \kappa u^{\nu} u_\mu) \left( \frac{\kappa}{1 + \kappa} u^{\mu} u_\lambda - \delta^\mu_\lambda \right) \partial_\nu \partial_\sigma Z^{\lambda\sigma} \]

\[ = -g_{\nu\lambda} \partial_\nu Z^{\lambda\sigma} = -\partial_\nu \partial_\sigma Z^{\lambda\sigma} \equiv 0. \]

Equations (10.8) and (10.11), together with the gauge condition (10.12), may be considered as the fundamentals of the theory [39]. Our complex fields are given by (10.7).
Then, equations (10.11) take the form

\[
[\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] \partial_\sigma Z^{\lambda \sigma} = -\frac{4\pi \mu}{c} j^\lambda. \tag{10.14}
\]

Indeed, the left-hand side of (10.11) is given by

\[
[\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] A_\sigma = [\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] g_{\sigma \mu} A^\mu
\]

\[
= [\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] g_{\sigma \mu} \left( \frac{\kappa}{1 + \kappa} u_\mu u_\lambda - \delta^\mu_\lambda \right) \partial_\rho Z^{\lambda \rho}
\]

\[
= [\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] \left( \frac{\kappa}{1 + \kappa} u_\sigma u_\lambda - g_{\sigma \lambda} \right) \partial_\rho Z^{\lambda \rho}
\]

\[
= \frac{4\pi \mu}{c} \left( g_{\sigma \lambda} - \frac{\kappa}{1 + \kappa} u_\sigma u_\lambda \right) j^\lambda,
\]

from which the result follows due to (10.10).

Finally, with the help of the standard substitution,

\[
j^\lambda = c \partial_\sigma p^{\lambda \sigma}, \quad p^{\lambda \sigma} = -p^{\sigma \lambda} \tag{10.15}
\]

(in view of \( \partial_\lambda j^\lambda = c \partial_\lambda \partial_\sigma p^{\lambda \sigma} \equiv 0 \)), we arrive at

\[
\partial_\sigma \left\{ [\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] Z^{\lambda \sigma} + 4\pi \mu p^{\lambda \sigma} \right\} = 0. \tag{10.16}
\]

Therefore, one can choose

\[
[\partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2] Z^{\lambda \nu} = -4\pi \mu p^{\lambda \nu}. \tag{10.17}
\]

Here, by definition,

\[
p^{\lambda \nu} = \begin{pmatrix}
0 & -p_1 & -p_2 & -p_3 \\
p_1 & 0 & m_3 & -m_2 \\
p_2 & -m_3 & 0 & m_1 \\
p_3 & m_2 & -m_1 & 0
\end{pmatrix} \tag{10.18}
\]

is an anti-symmetric four-tensor [15, 16, 91]. The “electric” and “magnetic” Hertz vectors, \( Z^{(e)} \) and \( Z^{(m)} \), are also introduced in terms of a single four-tensor,

\[
Z^{\lambda \nu} = \begin{pmatrix}
0 & Z_1^{(e)} & Z_2^{(e)} & Z_3^{(e)} \\
-Z_1^{(e)} & 0 & -Z_3^{(m)} & Z_2^{(m)} \\
-Z_2^{(e)} & Z_3^{(m)} & 0 & -Z_1^{(m)} \\
-Z_3^{(e)} & -Z_2^{(m)} & Z_1^{(m)} & 0
\end{pmatrix}. \tag{10.19}
\]

In view of (10.13), for the four-vector potential, \( A^\lambda = (\varphi, \mathbf{A}) \), we obtain

\[
\varphi = - \left( 1 - \frac{\kappa \gamma^2}{1 + \kappa} \right) \text{div} Z^{(e)} + \frac{\kappa \gamma^2}{(1 + \kappa) c} \mathbf{v} \cdot \left( \frac{\partial Z^{(e)}}{c \partial t} + \text{curl} Z^{(m)} \right) \tag{10.20}
\]

and

\[
\mathbf{A} = \frac{\partial Z^{(e)}}{c \partial t} + \text{curl} Z^{(m)} \tag{10.21}
\]

\[
+ \frac{\kappa \gamma^2 \mathbf{v}}{(1 + \kappa) c^2} \left[ c \text{div} Z^{(e)} + \frac{\partial}{c \partial t} (\mathbf{v} \cdot Z^{(e)}) + \mathbf{v} \cdot \text{curl} Z^{(m)} \right].
\]
Then, equations (10.17) take the form
\[\left[ \partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2 \right] Z^{(e)} = 4\pi \mu p, \quad \left[ \partial^\tau \partial_\tau + \kappa (u^\tau \partial_\tau)^2 \right] Z^{(m)} = 4\pi \mu m\] (10.22)
and, for the four-current, \(j^\lambda = (cp, j)\), one gets
\[\rho = - \text{div } p, \quad j = \frac{\partial p}{\partial t} + c \text{ curl } m\] (10.23)
(see [15], [16], [91] for more details).

The Hertz vector and tensor potentials, for a moving medium and at rest, were utilized in [15], [16], [28], [41], [86], [91], [96] (see also the references therein). Many classical problems of radiation and propagation can be consistently solved by using these potentials.

10.4. Energy-momentum tensor. In the case of the covariant version of the energy-momentum tensor given by (7.2), the differential balance equations under consideration are independent of the particular choice of the frame of reference. Therefore, our relations (10.7) are useful for derivation of the expressions for the energy-momentum tensor and the ponderomotive force for moving bodies from those for bodies at rest which were extensively studied in the literature. For example, one gets
\[4\pi T_{\mu}^{\nu} = F_{\mu\lambda}\epsilon^{\lambda\nu\sigma\tau} F_{\sigma\tau} + \frac{1}{4} \delta_{\nu}^{\lambda} F_{\sigma\tau} \epsilon^{\sigma\tau\lambda\rho} F_{\lambda\rho}\] (10.24)
with the help of (10.3)–(10.4) and (B.14) (see also [84]).

11. Real versus Complex Lagrangians

In modern presentations of the classical and quantum field theories, the Lagrangian approach is usually utilized.

11.1. Complex forms. We introduce two quadratic “Lagrangian” densities
\[\mathcal{L}_0 = \mathcal{L}_0^* = \frac{1}{2} \left( P_{\sigma\tau} Q^{\tau\sigma} + P^*_{\sigma\tau} Q^{\tau\sigma} \right)\] (11.1)
\[= \frac{i}{4} e^{\sigma\tau\kappa\rho} \left( P_{\sigma\tau} P_{\kappa\rho} - P^*_{\sigma\tau} P^*_{\kappa\rho} \right)\]
\[= F_{\sigma\tau} R^{\tau\sigma} - G_{\sigma\tau} S^{\tau\sigma} = 2 F_{\sigma\tau} R^{\tau\sigma} = 4 (E \cdot D - H \cdot B)\]
and
\[\mathcal{L}_1 = -\mathcal{L}_1^* = P^*_{\sigma\tau} Q^{\tau\sigma} = \frac{1}{2} \left( P^*_{\sigma\tau} Q^{\tau\sigma} - P_{\sigma\tau} Q^{\tau\sigma} \right)\] (11.2)
\[= \frac{i}{2} e^{\sigma\tau\kappa\rho} P_{\sigma\tau} P^*_{\kappa\rho} = 4i (E \cdot B - H \cdot D).\]

Then, by formal differentiation,
\[\frac{\partial \mathcal{L}_0}{\partial P^\alpha_{\beta}} = Q^\beta_\alpha, \quad \frac{\partial \mathcal{L}_0^*}{\partial P^*_{\alpha\beta}} = Q^{*\beta}_\alpha\] (11.3)
and
\[ \frac{\partial L_1}{\partial P_{\alpha \beta}^*} = Q^{*\alpha}, \quad \frac{\partial L_1^*}{\partial P_{\alpha \beta}} = Q^{\alpha} \] (11.4)
in view of (13.7).

The complex covariant Maxwell equations (11.2) take the forms
\[ \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_0}{\partial P_{\nu \mu}} \right) = -\frac{4\pi}{c} j^{\mu}, \quad \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_1}{\partial P_{\nu \mu}} \right) = \frac{4\pi}{c} j^{\mu} \] (11.5)
and the covariant energy-momentum balance relations (7.1) are given by
\[ \frac{\partial}{\partial x^\nu} \left[ \frac{1}{16\pi} \left( P_{\mu \lambda}^* \frac{\partial L_0}{\partial P_{\nu \lambda}^*} + P_{\mu \lambda} \frac{\partial L_0^*}{\partial P_{\nu \lambda}} \right) \right] \]
\[ + \frac{1}{32\pi} \left[ P_{\sigma \tau}^* \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_0}{\partial P_{\sigma \tau}} \right) + P_{\sigma \tau} \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_0^*}{\partial P_{\sigma \tau}} \right) \right] = -\frac{1}{c} F_{\mu \lambda} j^{\lambda} \] (11.6)
and
\[ \frac{\partial}{\partial x^\nu} \left[ \frac{1}{16\pi} \left( P_{\mu \lambda} \frac{\partial L_1}{\partial P_{\nu \lambda}} + P_{\mu \lambda}^* \frac{\partial L_1^*}{\partial P_{\nu \lambda}^*} \right) \right] \]
\[ + \frac{1}{32\pi} \left[ P_{\sigma \tau} \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_1}{\partial P_{\sigma \tau}} \right) + P_{\sigma \tau}^* \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_1^*}{\partial P_{\sigma \tau}} \right) \right] = \frac{1}{c} F_{\mu \lambda} j^{\lambda} \] (11.7)
in terms of the complex Lagrangians under consideration, respectively.

Finally, with the help of the following densities,
\[ L_0 = \mathcal{L}_0 - \frac{4\pi}{c} j^\mu A_\nu, \quad L_1 = \mathcal{L}_1 + \frac{4\pi}{c} j^\mu A_\nu, \] (11.8)
one can derive analogs of the Euler-Lagrange equations for electromagentic fields in media:
\[ \frac{\partial}{\partial x^\nu} \left( \frac{\partial L_{0,1}}{\partial P_{\nu \mu}} \right) - \frac{\partial L_{0,1}}{\partial A_\mu} = 0. \] (11.9)
In the case of a moving isotropic medium, a relation between \( P_{\nu \mu} \) and \( A_\mu \) is given by our equations (10.7) - (10.8).

11.2. **Real form.** Taking the real and imaginary parts, Maxwell’s equations (6.6) can be written as follows
\[ \partial_\nu R^{\mu \nu} = -\frac{4\pi}{c} j^\mu, \quad \partial_\nu S^{\mu \nu} = 0. \] (11.10)
Here,
\[ -6 \partial_\nu S^{\mu \nu} = \epsilon^{\mu \nu \lambda \sigma} (\partial_\nu F_{\lambda \sigma} + \partial_\sigma F_{\nu \lambda} + \partial_\lambda F_{\nu \sigma}) \equiv 0, \]
with the help of (6.4) and (10.8). Thus the second set of equations is automatically satisfied when we introduce the four-vector potential. For the inhomogeneous pair of Maxwell’s equations, the Lagrangian density is given by
\[ L = \frac{1}{4} F_{\sigma \tau} R^{\tau \sigma} - \frac{4\pi}{c} j^\sigma A_\sigma \]
\[ = \frac{1}{4} F_{\sigma \tau} \epsilon^{\tau \sigma \lambda \rho} F_{\lambda \rho} - \frac{4\pi}{c} j^\sigma A_\sigma, \] (11.11)
in view of (10.3). Then, for “conjugate momenta” to the four-potential field \( A_\mu \), one gets
\[
\frac{\partial L}{\partial (\partial_\nu A_\mu)} = \frac{\partial L}{\partial F_{\sigma\tau}} \frac{\partial F_{\sigma\tau}}{\partial (\partial_\nu A_\mu)} = R^{\mu\nu} \tag{11.12}
\]
and the corresponding Euler-Lagrange equations take a familiar form
\[
\partial_\nu \left( \frac{\partial L}{\partial (\partial_\nu A_\mu)} \right) - \frac{\partial L}{\partial A_\mu} = 0. \tag{11.13}
\]
The latter equation can also be derived with the help of the least action principle [72], [88], [90]. The corresponding Hamiltonian and quantization are discussed in [35], [39], [75] among other classical accounts.

In conclusion, it is worth noting the role of complex fields in quantum electrodynamics, quadratic invariants and quantization (see, for instance, [2], [8], [9], [20], [39], [40], [44], [46], [53], [55], [56], [75], [76], [77], [90], [97]). The classical and quantum theory of Cherenkov radiation is reviewed in [3], [11], [13], [29], [31], [81], [85]. For paraxial approximation in optics, see [28], [43], [45], [60], [61], and the references therein. Maxwell’s equations in the gravitational field are discussed in [17], [27]. One may hope that our detailed mathematical consideration of several aspects of macroscopic electrodynamics will be useful for future investigations and pedagogy.

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**Appendix A. Formulas from Vector Calculus**

Among useful differential relations are
\[
\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B). \tag{A.1}
\]
\[
\nabla \cdot (fA) = (\nabla f) \cdot A + f(\nabla \cdot A). \tag{A.2}
\]
\[
\nabla \times (fA) = (\nabla f) \times A + f(\nabla \times A). \tag{A.3}
\]
\[
A \cdot (\nabla \times (f \nabla \times B)) - B \cdot (\nabla \times (f \nabla \times A)) \tag{A.4}
\]
\[
= \nabla \cdot (f(B \times (\nabla \times A) - A \times (\nabla \times B))).
\]
\[
A(A \cdot \nabla) - B(\nabla \cdot A) + A \times (\nabla \times B) - B \times (\nabla \times A) \tag{A.5}
\]
\[
- \nabla \times (A \times B) = \sum_{\alpha=1}^{3} A_{\alpha}^2 \nabla \left( \frac{B_{\alpha}}{A_{\alpha}} \right) = - \sum_{\alpha=1}^{3} B_{\alpha}^2 \nabla \left( \frac{A_{\alpha}}{B_{\alpha}} \right).
\]
(See also [2], [79] and [90].) Here, \( \text{div} \ A = \nabla \cdot A \) and \( \text{curl} \ A = \nabla \times A \).
In this article, \( \varepsilon^{\mu\nu\sigma\tau} = -\varepsilon_{\mu\nu\sigma\tau} \) and \( \varepsilon_{0123} = +1 \) is the Levi-Civita four-symbol \([27]\) with familiar contractions:

\[
\varepsilon^{\mu\nu\sigma\tau} \varepsilon_{\mu\kappa\lambda\rho} = -\begin{vmatrix} \delta^\mu_\kappa & \delta^\mu_\lambda & \delta^\mu_\rho \\ \delta^\nu_\kappa & \delta^\nu_\lambda & \delta^\nu_\rho \\ \delta^\sigma_\kappa & \delta^\sigma_\lambda & \delta^\sigma_\rho \\ \delta^\tau_\kappa & \delta^\tau_\lambda & \delta^\tau_\rho \end{vmatrix},
\]

(B.1)

\[
\varepsilon^{\mu\nu\sigma\tau} \varepsilon_{\mu\kappa\lambda\rho} = -2 \begin{vmatrix} \delta^\nu_\kappa & \delta^\nu_\lambda & \delta^\nu_\rho \\ \delta^\tau_\kappa & \delta^\tau_\lambda & \delta^\tau_\rho \end{vmatrix} = -2 \left( \delta^\tau_\kappa \delta^\nu_\lambda - \delta^\nu_\kappa \delta^\tau_\lambda \right),
\]

(B.2)

\[
\varepsilon^{\mu\nu\sigma\tau} \varepsilon_{\mu\kappa\sigma\rho} = -6 \delta^\nu_\rho, \quad \varepsilon^{\mu\nu\sigma\tau} \varepsilon_{\mu\nu\sigma\rho} = -24.
\]

(B.3)

Dual second rank four-tensor identities are given by \([27]\):

\[
\varepsilon^{\mu\nu\sigma\tau} A_{\sigma\tau} = 2 B^{\mu\nu}, \quad \varepsilon^{\mu\nu\sigma\tau} B_{\sigma\tau} = A_{\nu\mu} - A_{\mu\nu}.
\]

(B.4)

In particular,

\[
Q^{\mu\nu} = R^{\mu\nu} + i S^{\mu\nu} = R^{\mu\nu} - \frac{i}{2} \varepsilon^{\mu\nu\sigma\tau} F_{\sigma\tau},
\]

(B.5)

\[
P^{\mu\nu} = F^{\mu\nu} + i G^{\mu\nu} = F^{\mu\nu} - \frac{i}{2} \varepsilon^{\mu\nu\sigma\tau} R_{\sigma\tau}.
\]

(B.6)

\[
2 R^{\mu\nu} = \varepsilon^{\mu\nu\sigma\tau} G_{\sigma\tau}, \quad 2 S^{\mu\nu} = \varepsilon^{\mu\nu\sigma\tau} F_{\sigma\tau}.
\]

(B.7)

\[
2 G^{\mu\nu} = -\varepsilon^{\mu\nu\sigma\tau} R_{\sigma\tau}, \quad 2 F^{\mu\nu} = \varepsilon^{\mu\nu\sigma\tau} S_{\sigma\tau}.
\]

(B.8)

\[
P^{\mu\nu} Q^{\mu\nu} = 2 F^{\mu\nu} R^{\mu\nu} - \frac{i}{2} (\varepsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau} + \varepsilon^{\mu\nu\sigma\tau} R^{\mu\nu} R_{\sigma\tau}).
\]

(B.10)

By direct calculation,

\[
F^{\mu\nu} R^{\mu\nu} = 2 (H \cdot B - E \cdot D),
\]

(B.11)

\[
\varepsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau} = 8 E \cdot B, \quad \varepsilon^{\mu\nu\sigma\tau} R^{\mu\nu} R_{\sigma\tau} = 8 H \cdot D.
\]

(B.12)

As a result,

\[
\frac{1}{4} P^{\mu\nu} Q^{\mu\nu} = H \cdot B - E \cdot D - i (E \cdot B + H \cdot D).
\]

(B.13)

An important decomposition,

\[
P^{*}_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda} Q^{*}_{\lambda\nu} = 2 \left( F^{\mu\lambda} R^{\lambda\nu} + G_{\mu\lambda} S^{\lambda\nu} \right)
\]

(B.14)

\[
= 4 F^{\mu\lambda} R^{\lambda\nu} + \delta^\nu_\mu F_{\sigma\tau} R^{\sigma\tau}
\]

\[
= 4 F^{\mu\lambda} R^{\lambda\nu} - 2 \delta^\nu_\mu (E \cdot D - H \cdot B),
\]

is complemented by an identity,

\[
P^{*}_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda} Q^{*}_{\lambda\nu} = \frac{1}{4} \left( P_{\sigma\tau} Q^{\sigma\tau} + P^{*}_{\sigma\tau} Q^{*\sigma}_{\tau}\right) \delta^\nu_\mu
\]

(B.15)

\[
= \frac{1}{2} (E \cdot D - H \cdot B) \delta^\nu_\mu.
\]

In matrix form,

\[
P Q = (F + i G) (R + i S) = (F R - G S) + i (F S + G R),
\]

(B.16)
\[ P^*Q = (F - iG) (R + iS) = (FR + GS) + i (FS - GR). \] (B.17)

Here,

\[ FS = \frac{1}{4} \text{Tr} (FS) I = (E \cdot B) I, \] (B.18)

\[ GR = \frac{1}{4} \text{Tr} (GR) I = (H \cdot D) I. \] (B.19)

\[ FR - GS = \frac{1}{2} \text{Tr} (FR) I = (E \cdot D - H \cdot B) I, \] (B.20)

\[ FR + GS = \frac{1}{2} \text{Tr} (FR) I \]
\[ = 2FR - (E \cdot D - H \cdot B) I. \]

\[ \text{Tr} (FR + GS) = 0, \] (B.22)

where \( I = \text{diag}(1, 1, 1, 1) \) is the identity matrix.

Also,

\[ PQ = QP = (F \cdot G) I, \] (B.23)

\[ \det P = \det Q = -(F \cdot G)^2 \] (B.24)

and

\[ F \cdot G = (E + iH) \cdot (D + iB) \]
\[ = (E \cdot D - H \cdot B) + i (E \cdot B + H \cdot D). \] (B.25)

Other useful dual four-tensor identities are given by [27]:

\[ \epsilon^{\mu\nu\lambda\sigma} A_{\nu\sigma\tau} = 6B^\mu, \quad A_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\sigma} B^\sigma. \] (B.26)

In particular,

\[ 6i \frac{\partial Q^{\mu\nu}}{\partial x^\nu} = \epsilon_{\mu\nu\lambda\sigma} \left( \frac{\partial P_{\lambda\sigma}}{\partial x^\nu} + \frac{\partial P_{\nu\lambda}}{\partial x^\sigma} + \frac{\partial P_{\nu\mu}}{\partial x^\lambda} \right), \] (B.27)

and

\[ \frac{\partial P_{\mu\nu}}{\partial x^\lambda} + \frac{\partial P_{\nu\lambda}}{\partial x^\mu} + \frac{\partial P_{\nu\mu}}{\partial x^\lambda} = i \epsilon_{\mu\nu\lambda\sigma} \frac{\partial Q^{\sigma\tau}}{\partial x^\tau}. \] (B.28)

(see also [47]).

**Appendix C. Proof of Identities**

In view of (6.3), or (B.7), and (B.28), we can write

\[ \left( \frac{\partial P_{\mu\nu}}{\partial x^\lambda} + \frac{\partial P_{\nu\lambda}}{\partial x^\mu} + \frac{\partial P_{\lambda\mu}}{\partial x^\nu} = i \epsilon_{\mu\nu\lambda\sigma} \frac{\partial Q^{\sigma\tau}}{\partial x^\tau} \right) Q^{\lambda\nu}, \] (C.1)

or

\[ 2Q^{\lambda\nu} \frac{\partial P_{\mu\nu}}{\partial x^\lambda} + Q^{\lambda\nu} \frac{\partial P_{\nu\lambda}}{\partial x^\mu} \]
\[ = i \left( \epsilon_{\mu\nu\lambda\sigma} Q^{\lambda\nu} \right) \frac{\partial Q^{\sigma\tau}}{\partial x^\tau} = -2P_{\mu\nu} \frac{\partial Q^{\sigma\tau}}{\partial x^\tau}. \]
by (B.7). Therefore,

\[ P_{\mu \lambda} \partial Q^{\lambda \nu} - \partial P_{\mu \lambda} Q_{\nu}^* = -\frac{1}{2} Q_{\sigma \tau}^* \partial P_{\sigma \tau}. \]  

(C.2)

In addition, with the help of (B.7) one gets

\[ 2i \left( P_{\sigma \tau}^* \partial Q^{\sigma \tau} \right) = P_{\sigma \tau}^* e^{\sigma \tau \nu \lambda} \partial P_{\nu \lambda} \partial x^{\mu} = e^{\sigma \tau \nu \lambda} P_{\sigma \tau}^* \partial P_{\nu \lambda} \partial x^{\mu} = -2i \left( Q_{\sigma \tau}^* \partial P_{\sigma \tau} \right), \]

which completes the proof.

REFERENCES

[1] M. Abraham, The Classical Theory of Electricity and Magnetism, Translation of the Eighth German Edition, Blackie & Son, Glasgow, 1948.
[2] P. B. Acosta-Humánez, S. I. Kryuchkov, E. Suazo, and S. K. Suslov, Degenerate parametric amplification of squeezed photons: explicit solutions, statistics, means and variances, J. Nonlinear Opt. Phys. Mater. 24 (2015) # 2, 1550021 (27pp).
[3] G. N. Afanasiev, Vavilov-Cherenkov and Synchrotron Radiation: Foundations and Applications, Kluwer, Dordrecht, 2004.
[4] K. Alladi, J. R. Klauder, and C. R. Rao (Eds.), The Legacy of Alladi Ramakrishnan in the Mathematical Sciences, Springer, New York, 2010.
[5] S. M. Barnett, Resolution of the Abraham-Minkowski dilemma, Phys. Rev. Lett. 104 (2010) # 7, 070401 (4pp).
[6] A. O. Barut, Electrodynamics and Classical Theory of Fields and Particles, Dover, New York, 1980.
[7] R. Becker, Electromagnetic Fields & Interactions, Blaisdell Pub. Co., New York, 1964.
[8] I. Białynicki-Birula and Z. Białynicki-Birula Z Quantum Electrodynamics, Pergamon and PWN–Polish Scientific Publishers, Oxford, New York, Toronto, Sydney, Warszawa,1975.
[9] Białynicki-Birula I and Białynicki-Birula Z The role of Riemann-Silberstein vector in classical and quantum theories of electromagnetism, J. Phys. B: At. Mol. Opt. Phys. 46 (2013), 053001 (32 pp).
[10] N. N. Bogolubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, General Principles of Quantum Field Theory, Seventh Edition, Kluwer, Dordrecht, 1990.
[11] B. M. Bolotovskii, Theory of Vavilov-Cherenkov effect (I–II), Usp. Fiz. Nauk 62 (1957) # 3, 201–246 [in Russian].
[12] B. M. Bolotovskii, Theory of Cherenkov radiation (III), Sov. Phys.–Usp. 4 (1962) # 5, 781–811.
[13] B. M. Bolotovskii, Vavilov-Cherenkov radiation: its discovery and application, Phys.–Usp. 52 (2009) # 11, 1099–1110.
[14] B. M. Bolotovskii and A. A. Rukhadze, Field of a charged particle in a moving medium, Sov. Phys. JETP 10 (1960) # 5, 958–961.
[15] B. M. Bolotovskii and S. N. Stolyarov, Current status of the electrodynamics of moving media (infinite media), Sov. Phys.–Usp. 17 (1975) # 6, 875–895; see also a revised version with the annotated bibliography in: Einstein’s Collection 1974, pp. 179–275, Nauka, Moscow, 1976 [in Russian].
[16] B. M. Bolotovskii and S. N. Stolyarov, The Fields of Emitting Sources in Moving Media, in: Einstein’s Collection 1978–1979, pp. 173–277, Nauka, Moscow, 1983 [in Russian].
[17] M. Carmeli, Classical Fields: General Relativity and Gauge Theory, World Scientific, New Jersey, 2001.
[18] V. I. Denisov, Introduction to Electrodynamics of Material Media, Izd. MGU, Moscow State University, Moscow, 1989 [in Russian].
[19] I. Y. Dodin and N. J. Fisch, Axiomatic geometrical optics, Abraham-Minkowski controversy, and photon properties derived classically, Phys. Rev. A 86 (2012) # 5, 053834 (16pp).
[20] V. V. Dodonov and V. I. Man’ko, Invariants and Correlated States of Nonstationary Quantum Systems, in: Invariants and the Evolution of Nonstationary Quantum Systems, Proceedings of Lebedev Physics Institute, vol. 183, pp. 71–181, Nauka, Moscow, 1987 [in Russian]; English translation published by Nova Science, Commack, New York, 1989, pp. 103–261.
[21] A. Einstein, Zur Electrodynamik der bewegter Körper, Ann. Phys. 17 (1905), 891–921.
[22] A. Einstein, *Eine neue formale Deutung der Maxwellschen Feldgleichungen der Electrodynamik*, Sitzungsber. preuss Akad. Wiss. 1 (1916), 184–188.

[23] A. Einstein and J. Laub, *Über die elektromagnetischen Grundgleichungen für bewegte Körper*, Ann. Phys. 26 (1908), 532–540; *Bemerkungen zu unserer Arbeit <<Über die elektromagnetischen Grundgleichungen für bewegte Körper>>*, Ann. Phys. 28 (1908), 445–447.

[24] A. Einstein and J. Laub, *Über die im elektromagnetischen Felde auf ruhende Körper ausgeübtenpondenmotorischen Kräfte*, Ann. Phys. Wiss. 26 (1908), 541–550.

[25] G. Farmelo, *The Strangest Man: The Hidden Life of Paul Dirac*, Mystic of the Atom, Faber & Faber, London, 2009.

[26] G. D. Fleishman and I. N. Toptygin, *Cosmic Electrodynamics: Electrodynamics and Magnetic Hydodynamics of Cosmic Plasmas*, Springer-Verlag, New York, Heidelberg, Dordrecht, London, 2013.

[27] V. A. Fock, *The Theory of Space, Time and Gravitation*, Pergamon Press, New York, 1964.

[28] V. A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, London, 1965.

[29] V. L. Ginzburg, *Quantum theory of electromagnetic radiation of an electron uniformly moving in a medium*, Sov. Phys. JETP 10 (1940) # 6, 589–600 [in Russian].

[30] V. L. Ginzburg, *The laws of conservation of energy and momentum in emission of electromagnetic waves (photons) in a medium and the energy-momentum tensor in macroscopic electrodynamics*, Sov. Phys.–Usp. 16 (1973) # 3, 434–439.

[31] V. L. Ginzburg, *Applications of Electrodynamics in Theoretical Physics and Astrophysics*, 2nd ed., Gordon and Breach, New York, 1989.

[32] V. L. Ginsburg and V. A. Ugarov, *Remarks on forces and the energy-momentum tensor in macroscopic electrodynamics*, Sov. Phys.–Usp. 19 (1976) # 1, 94–101.

[33] M. V. Gorkunov and A. V. Kondratov, *Microscopic view of light pressure on a continuous medium*, Phys. Rev. A 88 (2013) # 1, 011804(R) (5pp).

[34] S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics*, North-Holland, Amsterdam, 1972.

[35] J. D. Jackson, *Classical Electrodynamics*, Second Edition, John Wiley & Sons, New York, London, Sydney, 1962.

[36] M. Jacob, *Antimatter*, https://cds.cern.ch/record/294366/files/open-96-005.pdf

[37] J. M. Jauch and K. M. Watson, *Phenomenological quantum-electrodynamics*, Phys. Rev. 74 (1948) # 8, 950–957.

[38] J. M. Jauch and K. M. Watson, *Phenomenological quantum electrodynamics. Part II. Interaction of the field with charges*, Phys. Rev. 74 (1948) # 10, 1485–1493.

[39] L. Kannenberg, *A note on the Hertz potentials in electromagnetism*, Am. J. Phys. 18 (1987) # 4, 370–372.

[40] A. N. Kaufman, *Maxwell equations in nonuniformly moving media*, Ann. Phys. 18 (1962) # 2, 264–273.

[41] A. P. Kiselev and A. B. Plachenov, *Laplace-Gauss and Helmholtz-Gauss paraxial modes in media with quadratic refraction index*, J. Opt. Soc. Am. A 33 (2016) # 4, 663–666.

[42] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics*, W. A. Benjamin, Inc., New York, Amsterdam, 1968.

[43] C. Krattenthaler, S. I. Kryuchkov, A. Mahalov, and S. K. Suslov, *On the problem of electromagnetic-field quantization*, Int. J. Theor. Phys. 52 (2013) # 12, 4445–4460.

[44] J. Kryuchkov, N. A. Lanfear, and S. K. Suslov, *The Pauli-Lubański vector, complex electrodynamics, and photon helicity*, Phys. Scripta 90 (2015), 074006 (8pp).

[45] J. Kryuchkov, N. A. Lanfear, and S. K. Suslov, *The role of the Pauli-Lubański vector for the Dirac, Weyl, Proca, Maxwell and Fierz-Pauli equations*, Phys. Scripta 91 (2016), 035301 (15pp).

[46] S. Kryuchkov, S. K. Suslov, and J. M. Vega-Guzmán, *The minimum-uncertainty squeezed states for atoms and photons in a cavity*, J. Phys. B: At. Mol. Opt. Phys. 46 (2013), 104007 (15pp); IOP Select and Highlight of 2013.
[50] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn, Butterworth-Heinemann, Oxford, 1975.
[51] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd edn, Pergamon, Oxford, New York, Beijing, Frankfurt, 1989.
[52] N. Lanfear, Mathematica notebook: ComplexElectrodynamics.nb.
[53] N. Lanfear, R. M. López and S. K. Suslov, *Exact wave functions for generalized harmonic oscillators*, Journal of Russian Laser Research 32 (2011) # 4, 352–361.
[54] O. Laporte and G. E. Uhlenbeck, *Applications of spinor analysis to the Maxwell and Dirac equations*, Phys. Rev. 37 (1931), 1380–1397.
[55] R. M. López, S. K. Suslov, and J. M. Vega-Guzmán, *Reconstructing the Schrödinger groups*, Physica Scripta 87 (2013) # 3, 038118 (6pp).
[56] R. M. López, S. K. Suslov, and J. M. Vega-Guzmán, *On a hidden symmetry of quantum harmonic oscillators*, Journal of Difference Equations and Applications 19 (2013) # 4, 543–554.
[57] H. A. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat*, 2nd edn, Teubner, Leipzig, 1916.
[58] H. A. Lorentz, A. Einstein, and H. Minkowski, *Das Relativitätsprinzip, eine Sammlung von Abhandlungen*, Teubner, Leipzig, Berlin, 1913; English translation in: A. Einstein et al, *The Principle of Relativity, A Collection of Original Papers on the Special and General Theory of Relativity*, 4th edn, Dover, New York, 1932.
[59] A. Mahalov, E. Suazo, and S. K. Suslov, *Spiral laser beams in inhomogeneous media*, Opt. Lett. 38 (2013) # 15, 2763–2766.
[60] A. Mahalov and S. K. Suslov, An “Airy gun”: *Self-accelerating solutions of the time-dependent Schrödinger equation in vacuum*, Phys. Lett. A 377 (2012), 33–38.
[61] A. Mahalov and S. K. Suslov, *Solution of paraxial wave equation for inhomogeneous media in linear and quadratic approximation*, Proc. Amer. Math. Soc. 143 (2015) # 2, 595–610.
[62] V. P. Makarov and A. A. Rukhadze, *Force acting on a substance in an electromagnetic field*, Phys.–Usp. 52 (2009) # 9, 937–943.
[63] V. P. Makarov and A. A. Rukhadze, *Negative group velocity electromagnetic waves and the energy-momentum tensor*, Phys.–Usp. 54 (2011) # 12, 1285–1296.
[64] H. Minkowski, *Raum und Zeit*, in: Jahresbericht der Deutschen Mathematiker-Vereinigung, (1909), 75–88.
[65] H. Minkowski, *Die Grundlagen für die elektromagnetischen Vorgänge in bewegten Körpern*, Nachr. König. Ges. Wiss. Göttingen, math.-phys. Kl. (1908), 53–111; *The fundamental equations for electromagnetic processes in moving bodies*, English translation in: The Principle of Relativity (M. Saha, ed.), University Press, Calcutta, 1–69, 1920.
[66] H. Minkowski, *Eine Ableitung der Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern vom Standpunkt der Elektronentheorie*, Math. Ann. 68 (1908), 526–556.
[67] V. V. Nesterenko and A. V. Nesterenko, *Symmetric energy-momentum tensor: The Abraham form and the explicit covariant formula*, J. Math. Phys. 57 (2016) # 3, 032901 (14pp).
[68] Y. N. Obukhov, *Electromagnetic energy and momentum in moving media*, Ann. Phys. (Berlin) 17 (2008) # 9–10, 830–851.
[69] Y. N. Obukhov and F. W. Hehl, *Electromagnetic energy-momentum and forces in matter*, Phys. Lett. A 311 (2003), 277–284.
[70] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Second Edition, Addison-Wesley, Reading, London, 1962.
[71] C. H. Papas, *Theory of Electromagnetic Wave Propagation*, Dover, New York, 2011.
[72] W. Pauli, *Theory of Relativity*, Pergamon Press, Oxford, 1958.
[73] V. I. Pavlov, *On discussions concerning the problem of ponderomotive forces*, Sov. Phys.–Usp. 21 (1978) # 2, 171–173.
[74] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, *Momentum of an electromagnetic wave in dielectric media*, Rev. Mod. Phys. 79 (2007), 1197–1216.
[75] M. I. Riazanov, *Phenomenological study of the effect of nonconducting medium in quantum electrodynamics*, Sov. Phys. JETP 5 (1957) # 5, 1013–1015.
[76] M. I. Riazanov, *Radiative corrections to Compton scattering taking into account polarization of the surrounding medium*, Sov. Phys. JETP 7 (1958) # 5, 869–875.
[77] M. I. Ryazanov, Covariant Formulation of Phenomenological Quantum Electrodynamics, in: Topics in Theoretical Physics, pp. 75–86, Atomizdat, Moscow, 1958 [in Russian].

[78] G. L. J. A. Rikken and B. A. van Tiggelen, Observation of the intrinsic Abraham force in time-varying magnetic and electric fields, Phys. Rev. Lett. 108 (2012) # 23, 230402 (4pp).

[79] J. Schwinger, L. L. DeRoad, Jr., K. A. Milton, and Wu-y. Tsai, Classical Electrodynamics, Perseus Books, Reading, Massachusetts, 1998.

[80] D. V. Skobel’tsyn, The momentum-energy tensor of the electromagnetic field, Sov. Phys.–Usp. 16 (1973) # 3, 381–401.

[81] A. Sokolov, Quantum theory of radiation of elementary particles, Doklady Phys. 28 (1940) # 5, 415–417 [in Russian].

[82] J. A. Stratton, Electromagnetic Theory, McGraw Hill, New York and London, 1941.

[83] I. E. Tamm, Electrodynamics of anisotropic medium in special theory of relativity, J. Russ. Phys. Chem. Soc. 56 (1924) # 2–3, 248–262 [in Russian].

[84] I. E. Tamm, Crystal optics of the relativity theory in connection with geometry of a biquadratic form, J. Russ. Phys. Chem. Soc. 57 (1925) # 3–4, 209–214 [in Russian].

[85] I. E. Tamm, Radiation induced by uniformly moving electrons, J. Phys. USSR 1 (1939) # 5–6, 439–454.

[86] I. E. Tamm, Fundamentals of the Theory of Electricity, Mir, Moscow, 1979.

[87] N. W. Taylor, A simplified form of the relativistic electromagnetic equations, Australian Journal of Scientific Research, Series A: Physical Sciences 5 (1952), 423–429.

[88] Ya. P. Terletskii and Yu. P. Rybakov, Electrodynamics, 2nd edn, Vysshaya Skola, Moscow, 1990 [in Russian].

[89] M. Testa, The momentum of an electromagnetic wave inside a dielectric, Ann. Phys. 336 (2013) # 1, 1–11.

[90] I. N. Toptygin, Foundations of Classical and Quantum Electrodynamics, Wiley-VCH, Weinheim, 2014.

[91] I. N. Toptygin, Electromagnetic Phenomena in Matter: Statistical and Quantum Approaches, Wiley-VCH, Weinheim, 2015.

[92] I. N. Toptygin and K. Levina, Energy-momentum tensor of the electromagnetic field in dispersive media, Phys.–Usp. 186 (2016) # 2, 146–158 [in Russian].

[93] V. G. Veselago, Energy, linear momentum, and mass transfer by an electromagnetic wave in a negative-refraction medium, Phys.–Usp. 52 (2009) # 6, 649–654.

[94] V. G. Veselago, Waves in metamaterials: their role in physics, Phys.–Usp. 54 (2011) # 11, 1161–1165.

[95] V. G. Veselago and V. V. Shchavlev, Force acting on a substance in an electromagnetic field, Phys.–Usp. 53 (2010) # 3, 317–318.

[96] M. B. Vinogradova, O. V. Rudenko, and A. P. Sukhorukov, Theory of Waves, Second Edition, Nauka, Moscow, 1990 [in Russian].

[97] K. M. Watson and J. M. Jauch, Phenomenological quantum electrodynamics. Part III. Dispersion, Phys. Rev. 75 (1949) # 8, 1249–1261.

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