Towards the phase diagram of dense two-color matter

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We study two-color QCD with two flavors of Wilson fermion as a function of quark chemical potential $\mu$ and temperature $T$. We find evidence of a superfluid phase at intermediate $\mu$ and low $T$ where the quark number density and diquark condensate are both very well described by a Fermi sphere of nearly-free quarks disrupted by a BCS condensate. Our results suggest that the quark contribution to the energy density is negative (and balanced by a positive gauge contribution), although this result is highly sensitive to details of the energy renormalisation. We also find evidence that the chiral condensate in this region vanishes in the massless limit. This region gives way to a region of deconfined quark matter at higher $T$ and $\mu$, with the deconfinement temperature, determined from the renormalised Polyakov loop, decreasing only very slowly with increasing chemical potential. The quark number susceptibility $\chi_q$ does not exhibit any qualitative change at the deconfinement transition. We argue that this is because $\chi_q$ is not an appropriate measure of deconfinement for 2-color QCD at high density.

I. INTRODUCTION

Despite over a decade of intensive efforts to unveil the phase structure of strongly interacting matter at high density (beyond a few times the nuclear saturation density) and low temperature, even the question of which phases exist remains unanswered. A quantitative knowledge of this region would allow us to answer many questions regarding the structure and properties of compact stars, including the question of whether deconfined quark matter can exist inside such stars. The reason for the lack of definite progress on this issue is that standard weak-coupling methods are inapplicable except at asymptotically high densities, while the various model approaches that have been employed have not been sufficiently constrained by input from experiment or first-principles theoretical calculations to yield reliable information in the region of interest. Thus, while a wealth of information exists regarding possible phases and their properties in various models, no reliable, quantitative results are available as yet. For a recent review of high-density QCD, see Ref. [1].

Many of the outstanding questions could in principle be answered by lattice QCD simulations, but these have been hindered by the notorious sign problem. While no method has as yet been shown to solve the sign problem for QCD, lattice simulations may still constrain model calculations by providing first-principles, nonperturbative results for QCD-like theories without a sign problem. This is the main aim of the present study.

Among these theories, QCD with gauge group SU(2) (two-color QCD or QC2D) is of particular interest in that it shares most of the salient features of real QCD (eg, confinement, dynamical chiral symmetry breaking and long-range interactions). It differs from QCD in that the baryons of the theory are bosons, and the lightest baryon is a pseudo-Goldstone boson, degenerate with the pion (note though, that SU(2) models with adjoint matter [2] and G2 with fundamental matter [3], both of which are free from a sign problem, are expected to contain fermionic baryons in the physical spectrum). Therefore, instead of a normal nuclear matter phase this theory has a superfluid state characterised by condensation of these baryons, which at this point become true Goldstone bosons. This has been observed in a number of lattice simulations; in particular, the excitation spectrum including the Goldstone bosons has been studied in Refs. [4, 5]. A transition to a state of deconfined quark matter is expected at high chemical potential $\mu$ (see however [6]), and evidence of this was found in [7, 8]. The precise nature of this transition remained unclear, however, and in this paper we will attempt to answer some of the outstanding questions about this.

An intriguing possibility is that in an intermediate régime, strongly interacting matter may enter a chirally symmetric and confined phase, dubbed quarkyonic [9]. In [9], it was suggested that the scaling of thermodynamic quantities with $\mu$ in the intermediate régime could be a sign of such a phase. It was not possible to draw any further conclusions, not least because the presence of a non-zero diquark source $\jmath \neq 0$, introduced to stabilise the simulations, distorted the $\mu$-dependence of the relevant quantities. This will be remedied in the present paper.

This paper is organised as follows. In Section II we present results from simulations at zero chemical potential. These results allow us to map out lines of constant physics, including the line of zero quark mass, which will in the future allow us to perform controlled extrapolations to the continuum and chiral limits, and also by varying $N_f$ at fixed cutoff to estimate the critical temperature $T_d$ for deconfinement at $\mu = 0$. In addition, these results form a large part of the input into the renormalisation of energy densities, which is described and carried out in Section III. Section IV contains the bulk of our results for the $(\mu, T)$ phase diagram. After addressing some general technical issues in Section IVA, we present in Section IVB results for the order parameters for superfluidity and deconfinement, giving us an
Table I: Simulation parameters, pi and rho masses and lattice spacing at $\mu = j = 0$.

| $\beta$ | $\kappa$ | $N_\tau$ | $N_x$ | $N_{\tau a j}$ | $m_{\pi}$ | $m_{\rho}$ | $a$ (fm) |
|--------|--------|--------|--------|-------------|--------|--------|--------|
| 1.7    | 0.1780 | 12     | 24     | 500         | 0.779(7)| 0.804(10)| 0.229(3)|
| 1.7    | 0.1790 | 12     | 24     | 1050        | 0.683(5)| 0.783(12)| 0.213(8)|
| 1.7    | 0.1810 | 12     | 24     | 500         | 0.438(15)| 0.61(5) | 0.189(4)|
| 1.8    | 0.1740 | 12     | 24     | 2000        | 0.640(4)| 0.778(7) | 0.178(8)|
| 1.8    | 0.1750 | 12     | 24     | 880         | 0.490(9)| 0.67(2) | 0.174(8)|
| 1.9    | 0.1680 | 12     | 24     | 1570        | 0.645(8)| 0.805(9) | 0.178(6)|
| 1.9    | 0.1685 | 12     | 24     | 2000        | 0.589(4)| 0.789(9) | 0.153(18)|
| 1.9    | 0.1690 | 12     | 24     | 1000        | 0.517(11)| 0.71(2) | 0.144(8)|
| 2.0    | 0.1620 | 12     | 24     | 1000        | 0.638(7)| 0.839(9) | 0.164(5)|
| 2.0    | 0.1625 | 16     | 32     | 2000        | 0.586(3)| 0.820(8)|
| 2.0    | 0.1627 | 16     | 32     | 2000        | 0.562(4)| 0.809(8)|
| 2.0    | 0.1630 | 12     | 24     | 1000        | 0.524(10)| 0.758(16)| 0.145(3)|
| 2.1    | 0.1570 | 16     | 32     | 1600        | 0.536(3)| 0.836(8)|
| 2.1    | 0.1580 | 16     | 32     | 2100        | 0.405(5)| 0.770(12)|

Table II: Critical hopping parameter $\kappa$, given by $m_\pi / (k_c) = 0$, for different values of $\beta$.

| $\beta$ | $\kappa_\tau$ |
|---------|-------------|
| 1.7     | 0.19226 $^{+8}_{-5}$ |
| 1.8     | 0.17644 $^{+15}_{-15}$ |
| 1.9     | 0.17089 $^{+20}_{-19}$ |
| 2.0     | 0.16456 $^{+13}_{-15}$ |
| 2.1     | 0.15935 $^{+15}_{-8}$ |

II. SIMULATION DETAILS AND VACUUM PHASE STRUCTURE

We study QC2D with a conventional Wilson action for the gauge fields and two flavours of Wilson fermion. The fermion action is augmented by a gauge- and iso-singlet diquark source term which serves the dual purpose of lifting the low-lying eigenvalues of the Dirac operator and allowing a controlled study of diquark condensation. The quark action is

$$S_Q + S_f = \sum_{i=1,2} \bar{\psi}_i M \psi_i + \kappa \sum_{j} [\bar{\psi}_j^{l/r} (C \gamma_5) \tau_j \psi_1 - h.c.] ,$$

(1)

with

$$M_{xy} = \delta_{xy} - \kappa \sum_{\nu} \left[ (1 - \gamma_\nu) e^{i \delta_{\nu a} U_{\nu} (x) \delta_{y,x+\nu}} + (1 + \gamma_\nu) e^{-i \delta_{\nu a} U_{\nu} (y) \delta_{y,x-\nu}} \right] .$$

(2)

Further details about the action and the Hybrid Monte Carlo algorithm used can be found in [7].

We have performed an extensive exploration of the parameter space in the vacuum ($T = \mu = j = 0$) in the range $\beta = 1.7 - 2.1$. The parameters used are shown in Table I together with the values obtained for the pion (pseudoscalar meson) mass $m_\pi$, ratio of pion to rho (vector meson) mass $m_\pi / m_\rho$ and lattice spacing $a$. The lattice spacing was determined by fitting the static quark potential to the Cornell form $V(r) = C + \alpha/r + \sigma r$ and taking the string tension to be $\sqrt{\sigma} = 440$ MeV.

We can determine the value $\kappa_c(\beta)$ where the quark mass vanishes by performing a linear extrapolation of $m_\pi$ in $1/\kappa$ for each value of $\beta$. The results of this are shown in Table II.

We have also investigated the thermal deconfinement transition at $\mu = 0$ using the fixed-scale approach. We have generated configurations with $N_\tau = 4 - 10$ at $\beta = 1.9, \kappa = 0.168$, corresponding to a temperature range of 113–281 MeV. At each temperature we have computed the Polyakov loop $\langle L \rangle$, which is an order parameter for deconfinement of static color charges in the pure gauge theory, and exhibits a rapid crossover in a theory with dynamical fermions. It is related to the free energy $F_q$ of a static quark by

$$L = e^{-F_q (T)/T} ,$$

(3)

The free energy $F_q$ is only defined up to an additive renormalisation constant $\Delta F$, which depends on the bare couplings $\beta, \kappa$. Different prescriptions for determining this constant correspond to different renormalisation schemes. We have imposed the condition that the renormalised Polyakov loop on our $N_\tau = 4$ lattice ($T = 263$ MeV) is equal to 1, or in other words, the free energy is zero at this temperature. We can then compute the renormalised Polyakov loop $L_R(T)$ at any other temperature $T$ from the bare Polyakov loop $L_0$ via

$$L_R(T) = e^{-F_R (T)/T} = e^{-(F_0 + \Delta F)/T}$$

$$= L_0(T) e^{-\Delta F/T} = Z_L^{N_\tau} L_0(T=1/aN_\tau) ,$$

(4)

where $Z_L = \exp(-\alpha \Delta F) = L_0(N_\tau = 4)^{-1/4}$ (this procedure was first outlined in Ref. [11]). The results are shown in Fig. 4 as a function of $aT = 1/N_\tau$. The red (solid) curve in Fig. 4 is a cubic spline interpolation between the data points. Taking the derivative of this (denoted by the green, dashed curve), we find the maximum at $T a = 0.193$. If we instead use an Akima spline to interpolate, the maximum of the derivative appears at $Ta = 0.183$. Taking the cubic spline as our best estimate and conservatively estimating the uncertainty to be twice the difference between the Akima and cubic spline estimates, our result for the deconfinement temperature is $T_d(\mu = 0) = 0.193(20)$ or $T_d = 217(23)$ MeV.
The red (solid) band is a cubic spline interpolation where we have used \( V \) and \( L \) is kept fixed. In our case, this means that the physical quark mass, and therefore the ratio of the partial derivatives must be taken with all other physical parameters kept fixed. In our case, this means that the physical quark mass, and therefore the ratio \( m_\pi/m_q \), is kept fixed.

The anisotropic action \( S = S_G + S_Q + S_J \) describing \( N_f = 2 \) Wilson quark flavors is given by

\[
S_G = -\frac{\beta}{N_c} \left[ \frac{1}{\gamma_2} \sum_{x,i<j} \text{Re Tr} \, U_{ij}(x) + \gamma_q \sum_x \text{Re Tr} \, U_{i0}(x) \right],
\]

\[
S_Q = \sum_{x,\alpha} \left[ \bar{\psi}^\alpha(x)\psi^\alpha(x) + \gamma_q \kappa \bar{\psi}^\alpha(x)(D_0\psi)^\alpha(x) \right],
\]

\[
S_J = \kappa \sum_x [\bar{\psi}^{2tr}(x)C\gamma_5\tau_2\psi^3(x) - \bar{\psi}^1(x)C\gamma_5\tau_2\bar{\psi}^{2tr}(x)],
\]

with

\[
(D_4\psi)^\alpha(x) = (\gamma_4 - 1)U_i(x)\psi^\alpha(x + i)
- (\gamma_4 + 1)U^\dag_i(x - i)\psi^\alpha(x - i),
(10)
\]

\[
(D_0\psi)^\alpha(x) = (\gamma_0 - 1)U_0(x)\psi^\alpha(x + 0)
- (\gamma_0 + 1)U_0^\dag(x - 0)\psi^\alpha(x - 0).
(11)
\]

We also define

\[
\beta_s = \frac{\beta}{\gamma_g}; \quad \beta_t = \gamma_g\beta; \quad \kappa_t = \gamma_q\kappa_s = \gamma_q\kappa.
(12)
\]

The parameters \( \gamma_g \) and \( \gamma_q \) are the bare gluon and quark anisotropies, which in our formalism will be taken to be independent.

Substituting these expressions into (10) (and dropping the \( |a_\tau \) from all partial derivatives as it will be understood), we then readily derive

\[
\frac{\varepsilon_g}{T^4} = -\xi \left( \frac{N_f a_\tau}{N_c a_s} \right)^3 \left\{ \frac{\partial S_G}{\partial \xi} \right\},
(5)
\]

\[
= \frac{3N_f^2}{\xi^2 N_c} \left[ \langle \text{Re Tr} \, U_{ij} \rangle \left( \kappa_{g^{-1}} \frac{\partial \beta}{\partial \xi} + \frac{\gamma_g^{-1}}{\partial \xi} \right) \right.
+ \langle \text{Re Tr} \, U_{i0} \rangle \left( \gamma_g \frac{\beta}{\partial \xi} + \frac{\partial \gamma_g}{\partial \xi} \right).
(13)
\]

This coincides with the first part of Eq. (17) of Ref. [13]. The terms in angled brackets are the average spatial and temporal plaquettes respectively, and the terms multiplying them are what are usually known as the Karsch coefficients. In the weak coupling isotropic limit \( \beta \rightarrow \infty \), \( \gamma_g = 1 \) we have

\[
\frac{\partial \gamma_g}{\partial \xi} = -\frac{\partial \gamma_g^{-1}}{\partial \xi} = 1; \quad \frac{\partial \beta}{\partial \xi} = -a \frac{\partial \beta}{\partial a} = 0,
(14)
\]

and we recover the expression used in [13]:

\[
\frac{\varepsilon_g}{T^4} = \frac{3N_f^2 \beta}{N_c} \left[ \langle \text{Re Tr} \, U_{i0} \rangle - \langle \text{Re Tr} \, U_{ij} \rangle \right].
(15)
\]

The quark contribution to the energy density is given by

\[
\frac{\varepsilon_q}{T^4} = -\xi \left( \frac{N_f a_\tau}{N_c a_s} \right)^3 \left\{ \frac{\partial S_Q}{\partial \xi} \right\},
(16)
\]

\[
= -\frac{N_f^2}{\xi^2} \left[ \langle \sum_i \bar{\psi} D_i \psi \rangle \frac{\partial \kappa}{\partial \xi} + \langle \psi D_0 \psi \rangle \left( \gamma_g \frac{\partial \kappa}{\partial \xi} + \kappa \frac{\partial \gamma_g}{\partial \xi} \right) \right].
\]

The terms in angled brackets are calculated using a stochastic estimator. Note a potentially useful identity

\[
\gamma_q \kappa \langle \bar{\psi} D_0 \psi \rangle + \kappa \sum_i \langle \bar{i} \psi D_i \psi \rangle + \langle \bar{i} \psi \rangle = -\text{Tr} \, 1 = -4N_c N_f.
(17)
\]

Note that we have taken explicit account of the minus sign associated with closed fermion loops in the definition of the bilinear expectation values, i.e. \( \langle \bar{i} \psi \rangle \equiv
\[-\text{Tr}(\Gamma M^{-1})\]. It is therefore sufficient to evaluate the first and third terms on the LHS, enabling the second term, which enters into Eq. \((14)\), to be estimated. In the isotropic limit \(\gamma_q = \xi = 1\) this reduces to

\[
\frac{\varepsilon_q}{T^4} = N_f^4 \left[ (4N_f N_c + \langle \bar{\psi}\psi \rangle) \kappa^{-1} \frac{\partial \kappa}{\partial \xi} - \kappa \frac{\partial \gamma_q}{\partial \xi} \langle \bar{\psi}D_0 \psi \rangle \right],
\]

(18)

In the weak coupling isotropic limit \(\partial \kappa/\partial \xi = 0, \partial \gamma_q/\partial \xi = 1\) and we recover

\[
\frac{\varepsilon_q^0}{T^4} = -N_f^4 \kappa \langle \bar{\psi}D_0 \psi \rangle,
\]

(19)

which coincides up to an overall sign with the expression in \([7]\), where the fermion’s Grassmann nature was ignored.

Finally, the diquark contribution is given by

\[
\frac{\varepsilon_J}{T^4} = \frac{N_f^4}{\xi^2} \left( \frac{\partial \langle \bar{\epsilon}_j \rangle}{\partial \xi} \right) (-\langle \bar{\psi}^1 C \gamma_5 T_2 \bar{\psi}^2 \rangle + \langle \bar{\psi}^2 \rangle C \gamma_5 T_2 \psi^1 \rangle
\]

= \frac{2N_f^4}{\xi^2} \left( \frac{\partial \langle j \rangle}{\partial \xi} + \frac{j}{\kappa} \frac{\partial \kappa}{\partial \xi} \right) \langle \bar{\psi} \rangle \langle \psi \rangle
\]

(20)

in the notation of \([7]\). However, in the U(1)\(B\)-symmetric limit \(j \to 0\) the second term inside the brackets vanishes, and since this limit is always found at \(j = 0\) for any anisotropy \(\xi\), the first term also vanishes here.

Similarly, the trace anomaly is given by

\[
T_{\mu\mu} = \varepsilon - 3p = \frac{T}{V} \left[ a_s \frac{\partial S}{\partial a_s} \right].
\]

(21)

With our anisotropic action the quark and gluon contributions are given by

\[
(T_{\mu\mu})_g = \frac{3}{\xi^2 N_c} \left[ \langle \text{Re} \text{ Tr} U_{ij} \rangle \left( \gamma_g^{-1} a \frac{\partial \gamma_g^{-1}}{\partial a} + \beta a \frac{\partial \gamma_g}{\partial a} \right) \right.
\]

\[+ \langle \text{Re} \text{ Tr} U_{io} \rangle \left( \gamma_g a \frac{\partial \gamma_g}{\partial a} + \beta a \frac{\partial \gamma_g}{\partial a} \right) \left. \right] \frac{\partial \kappa}{\partial \xi},
\]

(22)

\[
(T_{\mu\mu})_q = \frac{1}{\xi^2} \left[ \langle \sum_i \bar{\psi} D_i \psi \rangle a \frac{\partial \kappa}{\partial a} \right.
\]

\[+ \langle \bar{\psi} \rangle \langle \psi \rangle \left( \gamma_q a \frac{\partial \kappa}{\partial a} + \kappa \frac{\partial \gamma_q}{\partial a} \right) \left. \right].
\]

(23)

However, in the isotropic limit, the bare anisotropies are always 1, and hence the derivatives \(\partial \gamma_q/\partial a\) vanish. We are then left with the standard expressions for the trace anomaly,

\[
(T_{\mu\mu})_g = -a \frac{\partial \beta}{\partial a} \frac{3}{N_c} \langle \text{Re} \text{ Tr} U_{ij} + \text{Re} \text{ Tr} U_{io} \rangle,
\]

(24)

\[
(T_{\mu\mu})_q = -a \frac{\partial \kappa}{\partial a} \kappa^{-1} \left( 4N_f N_c + \langle \bar{\psi}\psi \rangle \right).
\]

(25)

Eqs. \((23)\)\((25)\) differ from the expressions used in \([3, 8]\) by an overall factor \(\beta\) and an overall sign respectively; the resulting error is corrected in this paper.

So, in order to evaluate the full energy density (ignoring \(j \neq 0\)) from Eqs. \((13)\)\((16)\) we need the following, which go into the definition of the “Karsch coefficients”:

\[
\frac{\partial \beta}{\partial \xi} \bigg|_{\xi=1}; \frac{\partial \gamma_q}{\partial \xi} \bigg|_{\xi=1}; \frac{\partial \kappa}{\partial \xi} \bigg|_{\xi=1}; \frac{\partial \gamma_q}{\partial \xi}. \]

(26)

These are computed using the method presented in \([13, 14]\). In addition to the bare anisotropies \(\xi_q = a_g/a_s\) as determined from gluonic observables such as the “sideways potential” \([13]\), and \(\xi_q = a_g/a_s\) as determined from a meson dispersion relation. For a parameter set corresponding to a physical system \(\xi_q\) and \(\xi_q\) should be equal, since otherwise a massless meson would not propagate at the correct speed of light; choosing the bare parameters to bring this about is a non-trivial tuning problem \([13, 14]\). In attempting to calculate the Karsch coefficients for the parameter set \(\beta = 1.9, \kappa = 0.168\), we do not attempt this tuning, but rather simulate unphysical ensembles with either \(\gamma_q\) or \(\gamma_q\) set to unity; the parameters are given in Table \(\text{III}\). In addition we use the isotropic ensembles given in Table \(\text{II}\).

For each ensemble we compute the ratio \(M = (m_r/m_\rho)^2\), the lattice spacing \(a = a_s\), the gluon anisotropy \(\xi_q\) (from the “sideways potential”), and the quark anisotropy \(\xi_q\) (from the pion dispersion relation). The quark and gluon anisotropies are combined to form the average anisotropy \(\xi_+ = \frac{1}{2}(\xi_q + \xi_q)\) and the anisotropy mismatch \(\xi_- = \xi_q - \xi_q\). Each of these quantities is fitted to a linear function in the bare parameters,

\[
\xi_+ = 1; a_1 \Delta \gamma_+ + b_1 \Delta \gamma_q + c_1 \Delta \beta + d_1 \Delta \kappa,
\]

(27)

\[
a_2 \frac{a-a_0}{a_0} = a_2 \Delta \gamma_q + b_2 \Delta \gamma_q + c_2 \Delta \beta + d_2 \Delta \kappa,
\]

(28)

\[
M - M_0 = a_3 \Delta \gamma_q + b_3 \Delta \gamma_q + c_3 \Delta \beta + d_3 \Delta \kappa,
\]

(29)

\[
\xi_- = a_4 \Delta \gamma_q + b_4 \Delta \gamma_q + c_4 \Delta \beta + d_4 \Delta \kappa,
\]

(30)

where \(a_0\) and \(M_0\) are the values of \(a\) and \(M\) at the reference point \(\beta = 1.9, \kappa = 0.168\), \(\gamma_q = 1\), and \(\Delta x\) is the deviation of the bare parameter \(x\) from its value at the same reference point. Inverting the 4 \(\times\) 4 matrix of coefficients \((a_1, b_1, c_1, d_1)\) gives us the “generalised Karsch coefficients”, which are the derivatives of the bare parameters with respect to the “physical” parameters \((\xi_+, \xi_-, a_0, M)\). The first column gives us the Karsch coefficients \((29)\), while the second column gives us the beta functions \(\partial \beta/\partial a, \partial \kappa/\partial a\).

Since we do not need to renormalise the pressure, knowledge of the beta-functions is not required here. However, we can use information about them to perform consistency checks. In the isotropic limit, two of the Karsch coefficients can be expressed in terms of beta-functions, since

\[
\frac{\partial \beta}{\partial \xi} \bigg|_{\xi=1} = -a_2 \frac{\partial \beta}{\partial a}; \frac{\partial \kappa}{\partial \xi} \bigg|_{\xi=1} = -a_2 \frac{\partial \kappa}{\partial a}.
\]

(31)

We can also independently estimate the beta-functions from the isotropic results in Sec. \(\text{III}\) by taking derivatives wrt \(a\) along lines of constant physics.
illustrate the determination of the quark and gluon anisotropies respectively.

Fig. 2: The pion dispersion relation from the anisotropic $12^3 \times 24$ lattices in Table III

Results for the observables on our anisotropic lattices as well as the isotropic lattices used in this study, are given in Table III Figs 2 and 3 illustrate the determination of the quark and gluon anisotropies respectively.

The gluon anisotropy in Fig. 3 was computed using

$$\xi_g = \frac{V_{xt}(R_2) - V_{xt}(R_1)}{V_{xy}(R_2) - V_{xy}(R_1)}, \quad \beta = 0.157, \beta = 0.150, \gamma = 0.150,$$

where $V_{xt}(x), V_{xy}(x)$ are the potentials obtained from Wilson loops in the $(x,t)$ and $(x,y)$ plane respectively,

$$W_{xt}(x, y) \sim Z_{xy} e^{-V_{xy}(x)}, W_{xt}(x, t) \sim Z_{xt} e^{-V_{xt}(x)}.$$

which is valid for large $x$ and $t, y$. The fermion anisotropy is determined from the pion dispersion relation,

$$a_t^2 E_t^2 = a_t^2 m_t^2 + \frac{a_t^2 \kappa^2}{\xi_g^2}.$$

Hence, a straight-line fit of $a_t^2 E_t^2$ vs $a_t^2 \kappa^2$, as shown in Fig. 2 will give the anisotropy $\xi_g$.

The results of the fits to Eqs. (27)–(30) are shown in Table IV We see that the $\chi^2$ per degree of freedom is very high, especially for the average anisotropy and the mass ratio fits. This indicates that our linear approximation breaks down in this region, something which in the case of the anisotropy may be seen directly from the numbers in Table III where a nonlinear response of the physical anisotropies (and, indeed the lattice spacing) to the bare anisotropies is evident. To account for this, we would need to either include nonlinear terms in our Ansatz or employ smaller anisotropies (which would again require much higher statistics to determine the coefficients with sufficient precision). That is beyond the scope of this study.

The generalised Karsch coefficients are presented in Table V We see that although the anisotropy deriva-

| $\beta_s$ | $\beta_t$ | $\kappa_s$ | $\kappa_t$ | $\gamma_s$ | $\gamma_t$ | $\xi_s$ | $\xi_t$ | $m_s/m_g$ | $a_s$ (fm) |
|----------|----------|----------|----------|----------|----------|----------|----------|-----------|-----------|
| 1.90     | 1.90     | 0.1680   | 0.1680   | 1.0      | 1.0      | 0.9687   | 1.0753   | 0.8075    | 0.1785    |
| 2.37     | 1.52     | 0.168    | 0.168    | 0.8      | 1.0      | 0.720-12 | 0.853-10 | 0.809-5   | 0.177-3   |
| 1.27     | 2.83     | 0.168    | 0.168    | 1.5      | 1.0      | 1.321^5  | 1.32^5   | 0.648^-12 | 0.125^-6  |
| 1.90     | 1.90     | 0.180    | 0.157    | 1.0      | 0.87     | 0.747^-44 | 0.75^4   | 0.746^-21 | 0.107^-3  |
| 1.90     | 1.90     | 0.147    | 0.192    | 1.0      | 1.3      | 1.146^-44 | 1.53^-2   | 0.94^-1   | 0.229^-13 |
| 1.80     | 1.80     | 0.1740   | 0.1740   | 1.0      | 1.0      | 0.989^-3  | 1.03^-3   | 0.777^-8   | 0.177^-8  |
| 1.90     | 1.90     | 0.1685   | 0.1685   | 1.0      | 1.0      | 0.947^-6  | 0.98^-3   | 0.760^-18 | 0.153^-18 |
| 2.00     | 2.00     | 0.1620   | 0.1620   | 1.0      | 1.0      | 0.921^-5  | 0.99^4   | 0.829^-5  | 0.166^-13 |
| 2.00     | 2.00     | 0.1630   | 0.1630   | 1.0      | 1.0      | 0.881^4   | 1.04^-1   | 0.773^11  | 0.148^-2  |

TABLE III: Anisotropic lattice parameters and anisotropy results. The uncertainties are purely statistical.

| $\chi_{xy}$ | $\chi_{xt}$ | $\chi_{yxt}$ | $\chi_{xyt}$ |
|------------|-----------|-------------|-------------|
| 1.27       | 2.83      | 0.168       | 0.168       |
| 1.90       | 1.90      | 0.180       | 0.157       |
| 2.37       | 1.52      | 0.168       | 0.168       |
| 1.90       | 1.90      | 0.147       | 0.192       |

FIG. 3: The gauge anisotropy from the anisotropic $12^3 \times 24$ lattices in Table III computed according to Eq. (32).

TABLE IV: Results for the fits to Eqs. (27)–(30). $\chi^2/N_{df}$ is the $\chi^2$ per degree of freedom for each fit.

| $\xi_s$ | $\xi_t$ | $m_s/m_g$ | $a_s$ (fm) |
|----------|----------|-----------|-----------|
| 0.157    | 0.150    | 0.150    | 0.150    |
| 0.157    | 0.150    | 0.150    | 0.150    |
shows the computational effort a fairly zero within errors, the central values in Table IV C are reasonably well determined, other quantities, including the beta functions, have quite large uncertainties. The same has been found previously in real QCD with anisotropic lattices [14]. It is likely that the extraction of the lattice spacing from the static quark potential is the main limiting factor here, and that a high-precision lattice spacing determination from for example the Wilson flow [18] (which may also be used to determine the gauge anisotropy [19]) would help in this respect.

A surprising result is the small value for the coefficient \( \partial \gamma / \partial \xi \), which comes out between 0.1 and 0.2, in contrast to \( \partial \gamma / \partial \xi \), which has a value close to 1 as expected. It is possible that this is related to the breakdown of the linear approximation, and that including non-linear terms might bring this coefficient closer to 1. As we shall see in Sec. IV C, this has a significant impact on the resulting energy density.

The coefficients \( a \partial \gamma / a \partial \mu \) should be zero in the isotropic limit. While consistent or nearly consistent with zero within errors, the central values in Table IV are fairly large. If we could constrain these to be exactly zero, our overall uncertainties might be reduced. We also see that Eq. (23) is satisfied within the admittedly large uncertainties. Again, it might improve the accuracy of our determination if these equations could be constrained to hold exactly.

We may also use \( m_\pi / m_\rho \) instead of \( M = (m_\pi / m_\rho)^2 \) as our mass observables in the fits. We find that repeating the analysis above with this choice does not change the results for the Karsch coefficients and beta functions by much.

We have also computed the beta functions separately from a 2-dimensional fit to the isotropic ensembles only. The results are shown in Table IV. As we can see, the two approaches give consistent results, suggesting that the systematic uncertainties of the method are under reasonable control. The numbers are also roughly consistent with (but somewhat larger than) the crude estimates used in Ref. [8], where a simple backward derivative approximation was used.

### IV. RESULTS AT \( \mu \neq 0 \)

We now focus on the \((\beta = 1.9, \kappa = 0.1680)\) parameter set, and explore the interior of the \((T, \mu)\) plane for these bare couplings. Results for \( j = 0.04 \) on the \( 12^3 \times 24 \) lattices were already presented in [8]. Now, with the addition of data for \( j = 0.02 \) and, for some selected \( \mu \)-values, \( j = 0.03 \), we can extrapolate all our results to the \( j = 0 \) limit. The details of this extrapolation will be discussed in Section IV A as will our treatment of finite lattice spacing and finite volume lattice artefacts.

We have also explored higher temperatures with data at \( N_\tau = 16, 12, 8, \) and studied finite volume effects with the addition of a \( 16^3 \) spatial volume. The temperatures are \( T = 47, 70, 94 \) and 141 MeV for \( N_\tau = 24, 16, 12 \) and 8 respectively. Details of our data sets are given in Tables IV, V, and VI. Figure 3 shows the computational effort for the \( N_\tau = 24 \) lattices in terms of the number of conjugate gradient iterations per inversion and the molecular dynamics stepsize. It is evident from this figure that simulations in the dense region at the lowest \( j \)-value are 1–2 orders of magnitude more costly than those of the vacuum.

#### IV A. Diquark source extrapolation and lattice artefacts

In Fig. 4 we show the diquark condensate \((qq)\) as a function of the diquark source \( j \) for \( \mu a = 0.3, 0.5, 0.6, 0.9 \) on the \( 12^3 \times 24 \) lattice. We have attempted to fit the behaviour with three different functional forms: linear

| \( c_1 \) | \( \partial c_1 / \partial \gamma \) | \( \partial c_1 / \partial \beta \) | \( \partial c_1 / \partial \kappa \) | \( M \) | \( \partial c_1 / \partial M \) | \( \partial c_1 / \partial \mu \) |
|---|---|---|---|---|---|---|
| \( \gamma \) | \( 0.90^{+4.4}_{-1.4} \) | \( -0.51^{+19}_{-10} \) | \( 0.13^{+32}_{-18} \) | \( 1.4^{+1.6}_{-1.2} \) | \( 0.90^{+4.4}_{-1.4} \) | \( -0.51^{+19}_{-10} \) |
| \( \beta \) | \( 0.13^{+5}_{-3} \) | \( 0.22^{+12}_{-5} \) | \( -0.55^{+29}_{-20} \) | \( -2.9^{+0.6}_{-0.6} \) | \( 0.13^{+5}_{-3} \) | \( 0.22^{+12}_{-5} \) |
| \( \kappa \) | \( 0.05^{+15}_{-5} \) | \( 0.07^{+15}_{-5} \) | \( -0.22^{+35}_{-20} \) | \( -0.39^{+8.8}_{-2.8} \) | \( 0.05^{+15}_{-5} \) | \( 0.07^{+15}_{-5} \) |

TABLE V: Results for the generalised Karsch coefficients \( \partial c_1 / \partial \mu \). The numbers in the first column are the actual Karsch coefficients, while the second column gives the beta functions.

| \( c_1 \) | \( \partial c_1 / \partial \gamma \) | \( \partial c_1 / \partial \beta \) | \( \partial c_1 / \partial \kappa \) | \( M \) | \( \partial c_1 / \partial M \) | \( \partial c_1 / \partial \mu \) |
|---|---|---|---|---|---|---|
| \( \beta \) | \( -1.02^{+29}_{-32} \) | \( 0.73^{+15}_{-13} \) | \( -0.47^{+8.9}_{-10} \) | \( 0.73^{+15}_{-13} \) | \( -0.47^{+8.9}_{-10} \) |

TABLE VI: Results for the beta functions \( a \partial c_1 / a \partial \mu \) and mass derivatives \( M \partial c_1 / \partial M \), computed from fits to the isotropic data sets.
TABLE VII: Number of trajectories for $\mu \neq 0$, $\beta = 1.9, \kappa = 0.168, N_r = 24 \ (T = 47$ MeV). The $ja = 0.02, 0.03$ configurations all have $N_s = 12$. All trajectories have average length 0.5.

| $a\mu$ | $ja = 0.02$ | $ja = 0.03$ | $ja = 0.04$ |
|---|---|---|---|
| $N_s = 12$ | 250 | 250 | 560 |
| $N_s = 16$ | 250 | 315 | 632 |

TABLE VIII: Chemical potential values and number of trajectories for the $12^3 \times 16$ lattices ($T = 70$ MeV). The diquark source is $ja = 0.04$ in all cases. All trajectories have average length 0.5.

| $a\mu$ | $ja = 0.04$ | $ja = 0.03$ | $ja = 0.02$ |
|---|---|---|---|
| $N_{traj} = 500$ | 1000 | 2520 | 2000 |

TABLE IX: Chemical potential values and number of trajectories for the $16^3 \times 12$ lattices ($T = 94$ MeV). $N(0.04)$ and $N(0.02)$ are the number of trajectories for $ja = 0.04$ and 0.02 respectively. All trajectories have average length 0.5.

| $a\mu$ | $ja = 0.04$ | $ja = 0.03$ | $ja = 0.02$ |
|---|---|---|---|
| $N_{traj} = 500$ | 1000 | 2520 | 2550 |

TABLE X: As Table IX for the $16^3 \times 8$ lattices ($T = 141$ MeV).

| $a\mu$ | $ja = 0.04$ | $ja = 0.03$ | $ja = 0.02$ |
|---|---|---|---|
| $N(0.04)$ | 1000 | 1000 | 1000 |

FIG. 5: The diquark condensate $\langle qq \rangle$ as a function of diquark source $j$, for the $12^3 \times 24$ lattice, together with extrapolations to $j = 0$. The dotted lines denote the 68% confidence interval for each fit. At $\mu = 0.50$ the central value lies outside the 68% confidence interval.
The results for other observables are similar. As an illustration of this, the corresponding fits for the quark number density \( n_q \) summarised in Table XII. Based on these findings, we use a linear function as our default extrapolation model for all observables, keeping in mind that this will distort the results somewhat in the régime \( \mu a \lesssim 0.5 \).

Next, we discuss our treatment of lattice artefacts in the context of the quark number density \( n_q \). As in previous works, it will prove convenient to express results in terms of dimensionless ratios, eg. \( n_q/n_{SB} \), where \( n_{SB} \) is the result for non-interacting quarks. However, even for free quarks artifacts due to non-zero lattice spacing and finite spatial volume are non-negligible, resulting in significant departures from the result in continuum and thermodynamic limits, and very careful discussion is required. Insight into both UV and IR artefacts can be gleaned by considering the ratio \( n_{lat}^{SB}/n_{cont}^{SB} \), calculated for two different volumes using the formula given in [7], and shown in Fig. 7. The correction is numerically large across extensive portions of the \( \mu a \)-axis. The oscillatory behaviour seen for \( \mu a < 0.8 \) is an IR artefact known to arise from the non-sphericity of the Fermi surface resulting from the discretisation of momentum space [21].

As an illustration of these effects, in Fig. 7 we show the normalised quark number density \( n_q/n_{SB} \) at fixed diquark source \( j a = 0.04 \), with two different choices for \( n_{SB} \). In the upper panel we have normalised by \( n_{SB} \) for the corresponding lattice volumes, while in the lower panel we have used the same normalisation for all lattices. We have chosen to use \( n_{SB} \) for a \( 16^3 \times 24 \) lattice for this normalisation; note that this choice is purely a matter of convenience, the purpose being to easily compare the raw numbers for \( n_q \) from different lattices. We see that there is no difference between our raw numbers for \( n_q \) on the \( 12^3 \times 24 \) and \( 16^3 \times 24 \) lattices at \( j = 0.04 \); however \( n_{SB} \) for the \( 12^3 \) lattice has a dip around \( \mu a \approx 0.4 \), while on the \( 16^3 \) lattice this feature has moved to smaller \( \mu a \). By contrast, the correction factor coincides on the two volumes for \( \mu a > \mathcal{O}(1) \), suggesting that the considerable departure from unity at large \( \mu \) is due to UV effects. As we can see in the inset of Fig. 6 the diquark source has a negligible effect on the noninteracting quark density, and hence any significant \( j \)-dependence in our results must arise from interactions.

Based on these findings, we will in the following present our results for \( n_q \) and the pressure \( p \), as well as the quark number susceptibility \( \chi_q \), using both the noninteracting lattice and continuum expressions to normalise our data. This will allow us to assess the magnitude of IR and UV lattice artefacts. For the energy density and trace anomaly, where gluonic contributions are significant, we will instead normalise by \( \mu^4 \).

**TABLE XI:** Parameters for \( j = 0 \) extrapolations of the diquark condensate \( \langle qq \rangle \). Note that the power + constant fit is a 3-parameter fit to 3 data points, and hence there is no \( \chi^2 \) for this fit.

| \( \mu a \) | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| Linear fit \( A + Bj \) | A | 0.0068(1) | 0.0206(3) | 0.0557(5) | 0.1418(8) |
| | \( \chi^2 \) | 7.5 | 3.3 | 0.06 | 1.05 |
| Power law fit \( B_j^\alpha \) | \( \alpha \) | 0.700(6) | 0.376(7) | 0.261(6) | 0.104(5) |
| | \( \chi^2 \) | 0.23 | 2.1 | 15.1 | 1.03 |
| Power + constant fit \( A + B_j^\alpha \) | A | 0.0274(21) | 0.0254(21) | 0.5484(5) | 0.1294(6) |
| | \( \alpha \) | 0.367(6) | 0.504(6) | 1.00(4) | 0.217(4) |

**TABLE XII:** Parameters for \( j = 0 \) extrapolations of the quark number density \( n_q \), for the \( 12^3 \times 24 \) lattice.

| \( \mu a \) | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| Linear fit \( A + Bj \) | A | 0.0000(5) | 0.0128(9) | 0.0407(17) | 0.190(3) |
| | \( \chi^2 \) | 6.8 | 3.7 | 0.04 | 0.04 |
| Power law fit \( B_j^\alpha \) | \( \alpha \) | 0.95(17) | 0.18(5) | 0.20(3) | 0.076(15) |
| | \( \chi^2 \) | 6.7 | 4.8 | 0.07 | 0.11 |
| Power + constant fit \( A + B_j^\alpha \) | A | 0.0000(24) | 0.0162(24) | 0.0254(26) | 0.1854(6) |
| | \( \alpha \) | 0.214(6) | -0.0171(6) | 0.111(6) | 0.291(1) |

**FIG. 6:** Ratio \( n_{lat}^{SB}/n_{cont}^{SB} \) evaluated for free massless quarks on both \( 12^3 \times 24 \) and \( 16^3 \times 24 \) lattices. The inset shows the same ratio for the \( 12^3 \times 24 \) lattice, for four different values of the diquark source \( j \).
densate at the Fermi surface, the diquark condensate, which is the number density of Cooper pairs, should be proportional to the area of the Fermi surface, i.e. \( \langle qq \rangle \sim \mu^2 \). This is to be contrasted with chiral perturbation theory (\( \chi PT \)), which for \( \mu \gg \mu_o \) at leading order predicts \( \langle qq \rangle \) to be \( \mu \)-independent.

For the lowest temperature \( T = 47 \text{ MeV} (N_\tau = 24) \) we see an almost perfect proportionality in the region \( 0.35 \lesssim \mu a \lesssim 0.6 \). The lower limit of this region roughly coincides with the onset chemical potential \( \mu_o \approx m_\pi/2 \approx 0.33 a^{-1} \), below which both the quark number density and diquark condensate is expected to be zero. The reason we see a gradual rise from \( \mu a \approx 0.25 \) is our use of a linear Ansatz for the \( j \to 0 \) extrapolation, which is not valid in this régime, as discussed in Section IV A. For \( \mu a \gtrsim 0.6 \), \( \langle qq \rangle/\mu^2 \) rises again before possibly reaching a new plateau at \( \mu a \approx 1.0 \). This is possible evidence of a transition to a new state of matter at high density, but at these large densities the impact of lattice artifacts cannot be excluded.

At \( T = 70 \text{ MeV} (N_\tau = 16) \) we are not in a position to perform a \( j \to 0 \) extrapolation, but from the \( ja = 0.04 \) data we see only a mild suppression in \( \langle qq \rangle \), and only for \( \mu a \gtrsim 0.8 \). Since the results are almost indistinguishable from those at \( T = 47 \text{ MeV} \) we do not show them here.

At \( T = 94 \text{ MeV} (N_\tau = 12) \) we see that \( \langle qq \rangle \) is significantly smaller for all values of \( \mu \) and drops dramatically above \( \mu a \gtrsim 0.7 \). This gives us the first indications of the transition between the diquark-condensed and the normal phase. At \( T = 141 \text{ MeV} (N_\tau = 8) \) we find that the diquark condensate is zero at all \( \mu \), confirming that the system is in the normal phase at this temperature. A systematic investigation including more temperatures and an extrapolation to \( j = 0 \) at all temperatures will be required to establish the exact location and nature of this transition.

Finally, comparing the numbers from the \( 12^3 \times 24 \) and \( 16^3 \times 24 \) lattices, no evidence of any significant finite volume effects are found, except at \( \mu a = 0.9 \) where the condensate on the smaller volume is slightly suppressed.

Figure 8 shows the order parameter for deconfinement, the Polyakov loop \( \langle L \rangle \), for our four different temperatures. It has been renormalised using (4), using the \( \mu \)-independent renormalisation constant \( Z_L \) already computed in Sec. 11. We see that for each temperature \( T, \langle L \rangle \) increases rapidly from zero above a chemical potential \( \mu_d(T) \) which we may identify with the chemical potential for deconfinement. However, since \( L \) is a convex function of \( \mu \) at all \( T \), it is not possible to use the variation of \( L \) with \( \mu \) to define \( \mu_d(T) \). In the absence of a more rigorous criterion, we have taken the point where \( L \) crosses the value it takes at \( T_d(\mu = 0) \), \( L_d = 0.6 \), to define \( \mu_d(T) \). The results are shown in Fig. 11, with error bars denoting the range \( L_d = 0.5 \)–0.7. To more accurately locate the deconfinement line, we will need to perform a temperature scan for fixed \( \mu \)-values, as was done for \( \mu = 0 \).

For our lowest temperature \( (N_\tau = 24) \), the renormalised Polyakov loop is too noisy for any quantitative
we will present re-
shows the quark number
we also show our estimate of the transition
alisation factor
would not cross.
the diquark condensed
from the renormalised Polyakov loop, and the transition to
micros, and the transition to
PT, the deconfinement transition in the (µ, T)
curves for the renormalised Polyakov loop at the differ-
ent temperatures cross, so that at higher µ, L is smaller
for higher temperatures. This, however, depends on the
renormalisation scheme: if we had instead imposed the
condition that L_R = 0.5 at N_r = 4, µ = 0, the curves
would not cross.
The estimates of critical chemical potentials for both
deconfinement and superfluidity can be translated into a
tentative phase diagram, shown in Fig. [10]. It is worth
reiterating that the points on the phase boundaries are
rough estimates only, since we do not have a precise crite-
riion for the transition. In Section [IV D] we will present re-
results for a different measure of deconfinement, the quark
number susceptibility. We also show the estimate from
the coarser lattice in Ref. [7]. Clearly, a combination of
temperature effects and lattice artefacts is responsible for
the discrepancy between the µ_d-values quoted in [7, 8].
In Fig. [10] we also show our estimate of the transition
between the superfluid and the normal phase. Again,
since we do not yet have j → 0 extrapolated data at
all temperatures, and because our temperature grid is
fairly coarse, these transition points are also only rough
estimates.
In summary, from the order parameters we find sig-
atures of three different regions (or phases): a nor-
mal (hadronic) phase with ⟨qq⟩ = 0, ⟨L⟩ ≈ 0; a BCS
(quarkyonic) region with ⟨qq⟩ ∼ µ^2 at low T and inter-
mediate to large µ; and a deconfined, normal phase with
⟨qq⟩ = 0, ⟨L⟩ ≠ 0 at large T and/or µ. We cannot exclude a
deconfined superfluid phase with ⟨L⟩ > 0, ⟨qq⟩ ≠ 0 at
large µ and intermediate T.
After extrapolating our results to zero diquark source,
we see no evidence of a BEC region described by χPT,
with ⟨qq⟩ ∼ 1 − µ^2/µ_T^2 [21], in contrast with earlier
work with staggered lattice fermions [2]. This may be be-
cause we do not have a clear separation of scales between
the Goldstone diquark scale and more massive states,
and hence the region of tightly bound diquarks is very
narrow. A more pessimistic scenario is that the BEC
region is masked by the poor chiral properties of Wil-
son fermions. Simulations with lighter quarks may help
clarify this.

C. Equation of state

We now turn to the bulk thermodynamics of the sys-
tem: the quark number density n_q, the pressure p and
the energy density ε. Figure [11] shows the quark number
density n_q for N_r = 24, 12 and 8, extrapolated to zero
diquark source. In the top panel we have normalised by
the density n_{SB}^\text{int} for noninteracting fermions on the same
lattice volumes (12^3 × 24, 16^3 × 12, 16^3), as was done in
[7, 8]. In the bottom panel, we have instead divided by the
continuum, infinite-volume expression for noninter-
acting fermions at the same temperature and chemical
potential. The difference between the two gives an in-
dication of the lattice artefacts. We see that the
density rises from zero at µ ≈ µ_0 = 0.32a^{-1}, and for the
two lower temperatures is roughly constant and approxi-
mately equal to the noninteracting fermion density in the
region 0.4 ≤ µa ≤ 0.7. The peak at µa ≈ 0.4 in the
N_r = 24 data in the upper panel is an artefact of the
normalisation with n_{SB} for a finite lattice volume, as discussed in Sec. [IV A] it would be absent if we instead
We cannot say anything at this point about chiral symmetry restoration, another characteristic of the quarkyonic phase conjectured in Ref. [9]. We will come back to this issue in Sec. IV E.

Next we discuss pressure, which as the negative of the free energy density, may be calculated via the integral of any thermodynamic observable along an appropriate contour. It is particularly convenient to integrate along the $\mu$-axis via $p = \int_{\mu_0}^{\mu} n_q(\mu')d\mu'$, since the cutoff does not change. Here $\mu_0$ is chosen so that $p(\mu_0) = 0$ to good approximation; in the limit $T \to 0$ $\mu_0$ should coincide with the onset $\mu_o$.

In our analysis the integral is readily approximated by a trapezoidal rule; as always, we present data normalised by the free field value $p_{SB}$, a procedure not uniquely defined away from the continuum limit. We have examined three schemes:

$$
\left( \frac{p}{p_{SB}} \right)_0 = (p_{SB}(\mu'))^{-1} \int_{\mu_0}^{\mu} n_q(\mu')d\mu',
$$

$$
\left( \frac{p}{p_{SB}} \right)_I = (p_{SB}(\mu))^{-1} \int_{\mu_0}^{\mu} n_q(\mu')d\mu',
$$

$$
\left( \frac{p}{p_{SB}} \right)_{II} = (p_{SB}(\mu))^{-1} \int_{\mu_0}^{\mu} n_{qSB}(\mu)n_q(\mu')d\mu',
$$

where

$$
p_{SB} = \frac{N_f N_c}{12\pi^2} \left( \mu^4 + 2\pi^2 \mu^2 T^2 + \frac{7\pi^4 T^4}{15} \right)
$$

is the continuum pressure for a free gas of quarks, and $p_{SB}$ the corresponding value obtained by summing over free quark modes on the finite lattice. Versions (37) and (38) were both studied in [7], whereas only $(p/p_{SB})_{II}$ was used in [8, 22].

Fig. 11 shows the results for data taken with $\mu a = 0.04$, as well as the $j \to 0$ extrapolated data for $N_f = 24$. In scheme II lattice data are “corrected” for artefacts before integrating. The results clearly inherit the bump at $\mu a \approx 0.45$ also manifest in Fig. 7 which we now believe to be an IR artefact. However, this bump is absent (or strongly suppressed) in the $j \to 0$ limit, mirroring the absence of a significant bump in the upper panel of Fig. 11. This extrapolation reduces the ratio $p/p_{SB}$ from approximately 1.5 to approximately one in the quarkyonic regime. By contrast the scheme 0 data have the ratio $p/p_{SB}$ substantially exceeding unity in the large-$\mu$ regime above deconfinement, which probably reflects the fact that UV artefacts are not being fully corrected here. For this reason we now prefer scheme I, where for the coldest lattice $p/p_{SB}$ has a plateau with value $\approx 1$ (after $j \to 0$) in the suspected quarkyonic region, only rising to $\approx 2$ at large $\mu$. Again, therefore, we conclude that for low $T$ there is a range of $\mu$ where thermodynamic quantities scale approximately the same as free quarks; that the evidence for a peak above onset matching the expectations of $\chi$PT has substantially diminished; and

FIG. 11: The quark number density at $j = 0$, divided by the density for a noninteracting gas of lattice quarks (top) and continuum quarks (bottom).
that \(p/p_{SB}\) rises above unity in the deconfined regime. By \(T = 141\) MeV \((N_\tau = 8)\), however, the ratio rises monotonically and the distinction between these different regimes is largely washed out. It is clear, however, that the full story will only emerge once the continuum and thermodynamic limits are both taken with care.

The quark and gluon contributions to the energy density, for \(ja = 0.04\), are shown in the upper panel of Fig. 13. We see that the quark energy density is almost independent of temperature for all temperatures, while the gluon energy density shows a clearly different behaviour only for the highest temperature. We find that the gluon energy density is independent of the diquark source within errors, so these results are representative for the \(j \to 0\) extrapolated data. The quark contribution is sensitive to the diquark source in the low-\(\mu\) region, as can be seen from the \(j \to 0\) extrapolated data also shown in Fig. 13.

Comparing these results with the unrenormalised (and unextrapolated) results in Figs 1 and 3 of Ref. [8], we see a dramatic difference. Clearly, the proper renormalisation is crucial to any reliable determination of the energy density, and in particular it is clear that the terms proportional to \(\partial \beta / \partial \xi\) and \(\partial \kappa / \partial \xi\) in (13) and (16) respectively cannot be ignored. To illustrate this more clearly, we show in Fig. 14 the quark contribution to the energy density on the \(12^3 \times 24\) lattice at \(ja = 0.04\), computed using different values for the Karsch coefficients. The open circles correspond to the unrenormalised energy density which was presented in Ref. [8] (note that the normalisation is different). The other data sets correspond to different values of \(\partial \gamma_q / \partial \xi\), with \(\partial \kappa / \partial \xi\) set to the value of \(-0.052\) that was determined in Sec. III. We have chosen to use the tree-level value of 1, the value 0.131 determined in Sec. III and a value of 0.8, which is similar to the value found for \(\partial \gamma_g / \partial \xi\), and at the margins of our 95% confidence interval. We see that using the correct (non-zero) value for \(\partial \kappa / \partial \xi\) is most important at low \(\mu\), where this alone changes the sign of \(\varepsilon_q\). At large \(\mu\), the \(\partial \gamma_q / \partial \xi\) term will dominate, as it does at tree level.

It should be noted that the uncertainties in the Karsch coefficients are not included in the total uncertainties in the plots shown here. On the basis of Fig. 14 one may conclude that these uncertainties will have an effect of \(\mathcal{O}(100\%)\) in the energy density.

Although \(\varepsilon_q\) appears to be negative at least for low \(\mu\), and possibly for all \(\mu\)-values considered here, the total energy density \(\varepsilon = \varepsilon_g + \varepsilon_q\), shown in the bottom panel of Fig. 13 remains positive or consistent with zero everywhere in the \(j \to 0\) limit. Although on the face of it a
negative value for $\varepsilon_q$ is surprising, it is notable that the renormalised quark energy density shown in Fig. 13 has a qualitative resemblance to the unrenormalised energy density measured for QC$_2$D with $N_f = 4$ Wilson quark flavors in Fig. 5 of Ref. [22]. The parameters used in that study correspond to a much finer lattice, with $a/\sqrt{\sigma}$ having a value approximately one-third that used here. It is conceivable, therefore, that the Karsch coefficients for $N_f = 4$ fall far closer to their free-field values, and hence their neglect in [22] is much better justified, reinforcing the conclusion that $\varepsilon_q(\mu) < 0$.

Finally, we consider the trace anomaly, computed according to Eqs. (24, 25), which is shown in Fig. 14. With the correct expression for $\epsilon_j$ it had erroneously been presented as positive. Since the beta-functions only enter into the expressions as overall constants, and our updated values are not dramatically different from those used in Ref. [8], the qualitative behaviour of the $N_f = 24, ja = 0.04$ data is the same as previously reported in Ref. [8], apart from the sign of the quark contribution.

For small and intermediate $\mu$, the gluon and quark contributions have opposite signs and similar magnitudes, leading to a nearly vanishing total trace anomaly in the region $\text{O} (= a, \mu) \lesssim 0.7$. The gluon contribution decreases for $\mu \gtrsim 0.5$ and becomes negative for $\mu a \gtrsim 0.75$, while the quark contribution has a plateau for $0.5 \lesssim \mu a \lesssim 0.75$ and increases rapidly in magnitude thereafter. This leads to a negative total trace anomaly at large $\mu$, which corresponds to the positive and increasing pressure $p = (\varepsilon - T \mu \eta)/3$ observed in Fig. 12.

We see no difference in the trace anomaly between the two lowest temperatures, $T = 47$ and 70 MeV. At $T = 94$ MeV and 141 MeV ($N_f = 12$ and 8) the gluon contribution becomes larger (or less negative) and the quark contribution becomes more negative at large $\mu$. The net effect of this, however, is to leave the total trace anomaly nearly unchanged.

We find that the trace anomaly depends only weakly on the diquark source for nearly all $T$ and $\mu$. The main effect is to increase the gluon contribution at large $\mu$ and $T$, and to decrease the magnitude of the quark contribution at low $T$, for large and small $\mu$. It is quite striking that there appears to be little or no dependence on either temperature or diquark source in the region $\mu \gtrsim 0.55/a$.

Once again, it is instructive to compare with the $N_f = 4$ study of Ref. [22]. In that case (see figs. 6 and 8 of [22]), after taking into account the incorrect sign for the quark contribution, the gluon unrenormalised contribution to $T_{\mu \eta}$ is negative for all $\mu \lesssim \mu_d$, while the quark contribution is positive, which is the opposite of what we observe here. However, this still leaves open the possibility of the two contributions nearly cancelling, giving rise to nearly-conformal matter in the quarkyonic region.
D. Quark number susceptibility

In a mathematical sense the Polyakov loop is a well-defined signal for deconfinement, at least in pure gauge theories; physically it reveals something about the behaviour of static color sources, which are well approximated by heavy quarks, in a baryonic medium. Recent studies of a non-relativistic formulation of QC2D \[23\] took the first step beyond the static approximation, and revealed a non-trivial \( T \)- and \( \mu \)-dependence for \( s \)-wave states formed from heavy quarks. Another observable related to confinement is the quark number susceptibility \( \chi_q \equiv \frac{\partial n_q}{\partial \mu} \). This observable is usually thought of as encoding the fluctuations in the baryon (or quark) number, and is of particular interest as a measure of confinement or deconfinement of light quark degrees of freedom \[23,24\]. If quarks are confined inside hadrons, the fluctuations of the quark number and hence the susceptibility will be suppressed, since increasing the quark number entails exciting a baryon, which requires a large amount of energy. If quarks are not confined, it is possible to excite a single quark, which requires much less energy, giving a larger quark number susceptibility.

This link between \( \chi_q \) and deconfinement is clear in the case of QCD, where all baryons are heavy. In the case of QC2D the situation is less clear, since the lightest baryons are the pseudo-Goldstone diquarks, and large fluctuations are possible even in the confined phase. Nonetheless, it is of great interest to study fluctuations in quark number at large density and low temperature. The only previous such study is Ref. \[28\], where the Dyson–Schwinger equation in the rainbow approximation was employed. Hence QC2D offers an opportunity for a first systematic non-perturbative study of \( \chi_q \) in this régime.

For an ideal gas of massless (continuum) quarks and gluons, at temperature \( T \) and chemical potential \( \mu \), we have:

\[
n_{SB}^{\text{cont}} = N_f N_c \left( \frac{\mu T^2}{3} + \frac{\mu^3}{3\pi^2} \right),
\]

\[
\chi_{SB}^{\text{cont}} = N_f N_c \left( \frac{T^2}{3} + \frac{\mu^2}{\pi^2} \right).
\]

Now consider the quark action \( \Box \) rewritten in the form \( \bar{\Psi} \mathcal{M} \Psi \), where we have introduced the bispinors \( \Psi \equiv (\psi_1, C^{-1} \tau_2 \psi_2^r)^r \), \( \bar{\Psi} \equiv (\bar{\psi}_1 - \bar{\psi}_2^r C \tau_2) \); see Ref. \[10\] for details. From the definition of \( \chi_q \) we have:

\[
\chi_q = \frac{\partial n_q}{\partial \mu} = \frac{T}{V_s} \left\{ -\left\langle \left[ -\bar{\Psi} \frac{\partial \mathcal{M}}{\partial \mu} \Psi \right] \right\rangle^2 + \left\langle \left[ -\bar{\Psi} \frac{\partial \mathcal{M}}{\partial \mu} \Psi \right] \right\rangle^2 \right\}.
\]
From this equation we can identify four different terms:

\[
T_1 = -\left\langle \left[ -\bar{\Psi} \frac{\partial M}{\partial \mu} \Psi \right]^2 \right\rangle = -\left\langle \text{Tr} \left[ M^{-1} \frac{\partial M}{\partial \mu} \right]^2 \right\rangle, \tag{43}
\]

\[
T_2 = +\left\langle \left[ -\bar{\Psi} \frac{\partial M}{\partial \mu} \Psi \right]^2 \right\rangle_{\text{disc}} = \left\langle \text{Tr} \left[ M^{-1} \frac{\partial M}{\partial \mu} \right] \cdot \text{Tr} \left[ M^{-1} \frac{\partial M}{\partial \mu} \right] \right\rangle, \tag{44}
\]

\[
C_1 = +\left\langle \left[ -\bar{\Psi} \frac{\partial M}{\partial \mu} \Psi \right]^2 \right\rangle_{\text{conn}} = -\left\langle \text{Tr} \left[ M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} \right] \right\rangle, \tag{45}
\]

\[
T_3 = +\left\langle \left[ -\bar{\Psi} \frac{\partial^2 M}{\partial \mu^2} \Psi \right] \right\rangle = \left\langle \text{Tr} \left[ M^{-1} \frac{\partial^2 M}{\partial \mu^2} \right] \right\rangle. \tag{46}
\]

The second term of Eq. (42) yields two terms, \(T_2\) and \(C_1\), because there are two ways to contract the spinors.

The calculation of the traces is done using unbiased estimators, introducing \(N_q\) complex noise vectors \(\eta\) with the properties: \(\langle \eta \rangle = 0\) and \(\langle \eta_\alpha \eta_\beta \rangle = \delta_{\alpha \beta}\). For example, the determination of the trace, used for \(T_1\) and \(T_2\), is based on the following relation:

\[
\text{Tr} \left[ M^{-1} \frac{\partial M}{\partial \mu} \right] = \frac{1}{N_q} \sum_{\alpha \xi} \eta_{\alpha \xi} \left( \frac{\partial M}{\partial \mu} \right)_{\alpha \xi ; \xi \beta} M^{-1}_{\beta \gamma ; \gamma k} \eta_{\gamma k}. \tag{47}
\]

Because two independent source vectors are required to compute \(T_2\), we refer to this term as “disconnected”; the other three “connected” terms need only one source vector.

It turns out that the connected term gives an important contribution to \(\chi_q\) at low and high values of the chemical potential and therefore cannot be considered negligible. Moreover, it changes sign around \(\mu a \approx 0.66\). On the other hand, the terms \(T_1\) and \(T_2\) are equal within errors but with opposite sign, i.e., their net contribution is consistent with zero everywhere, except possibly around the onset transition.

All the systematic issues discussed in Sec. [VIII] regarding the normalisation of data with the same quantity calculated for free quarks, are also relevant for \(\chi_q\). In Fig. 16 we plot the ratio \(\chi_q/\chi_{SB}^{\text{cont}}\) for four different temperatures, versus the chemical potential. For an ideal gas of quarks and gluons this ratio would be a constant, \(\chi_q = \chi_{SB}\), whereas the Polyakov loop is easily obtained. Fig. 17 plots the ratio \(\chi_q/\chi_{SB}^{\text{cont}}\) for two values of the quark mass used in the determination of \(\chi_{SB}^{\text{cont}}\), the subtracted bare quark mass \(m_q = 1/2k - 1/2k_c\) and the ‘constituent’ quark mass \(m = m_p/2\). In this case we observe a different behaviour for \(m a \lesssim 0.45\), but now in the quarkyonic regime there is a discernable plateau with a ratio compatible with one, i.e., the system is behaving as free fermions, with an increase for higher values of \(\mu\).

FIG. 16: Ratio \(\chi_q/\chi_{SB}^{\text{cont}}\) versus \(\mu\), for \(m = 0.04\). The vertical dashed line marks the position of \(\mu_o\).

In account the finite volume and the lattice discretisation. In Eq.(26) of Ref. [7], the expression for the quark number density \(n_{SB}^{\text{lat}}\) for free Wilson fermions on the lattice is presented, from which \(\chi_{SB}^{\text{lat}}\) is easily obtained. Fig. 17 plots the ratio \(\chi_q/\chi_{SB}^{\text{lat}}\) for two values of the quark mass used in the determination of \(\chi_{SB}^{\text{lat}}\), the subtracted bare quark mass \(m_q = 1/2k - 1/2k_c\) and the ‘constituent’ quark mass \(m = m_p/2\). In this case we observe a different behaviour for \(m a \lesssim 0.45\), but now in the quarkyonic regime there is a discernable plateau with a ratio compatible with one, i.e., the system is behaving as free fermions, with an increase for higher values of \(\mu\).

Fig. 17 demonstrates that the value of the mass used for the free fermions has a quantitative effect for this observable, in that the value of the plateau is shifted when the mass is increased, but this does not change the qualitative considerations. The results using \(m_q\) are almost identical those obtained setting \(m = 0\). These plots again confirm the above scenario: we do not see any abrupt change for \(\chi_q\) as a function of \(T\), whereas the Polyakov loop becomes different from zero at a \(T\)-dependent \(\mu_d\). Again, something different seems to emerge for the highest temperature, \(T = 141\) MeV (\(N_c = 8\)).

The effect of the diquark source is illustrated in Fig. 18.
is set by fermions for two values of the fermion mass: \( m = 0.05 \) (top) and \( m = 0.42 \) (bottom). The vertical dashed lines mark the position of \( \mu_o \).

where we show \( \chi_q/\chi_{SB}^{\text{lat}}(m_c) \) for the 12\(^3\) × 24 lattice and \( ja = 0.04, 0.02, \) and 0. We find that the diquark source only has a significant effect for low \( \mu \), where it increases the value of \( \chi_q \) slightly.

E. A first look at chiral symmetry in the dense phase

An important issue which we have been hitherto unable to address is the chiral properties of the ground state once \( \mu \geq \mu_o \). This issue is of course of general theoretical interest when the phase diagram of any non-abelian gauge theory is discussed; in the current context it is of particular interest since the original description of the quarkyonic phase in SU(\( N_c \)) gauge theory was in terms of a chirally symmetric but confined medium, i.e. one in which the chiral condensate \( \langle \bar{\psi}\psi \rangle \rightarrow 0 \) as the bare quark mass \( m \rightarrow 0 \) [48]. Later this picture was modified: chiral symmetry breaking via a translationally non-invariant “chiral spiral” was postulated in [29]. For theories of the class exemplified by QC\(_2\)D where the relevant mass scale is set by \( m_\pi \), chiral symmetry is necessarily always broken explicitly by a bare quark mass \( m \); in this case the question is how the condensate \( \langle gg \rangle \) scales with \( m \) as \( m \rightarrow 0 \).

It is clearly desirable to determine the fate of chiral symmetry breaking for the case of QC\(_2\)D by a lattice calculation. Indeed, \( \langle \bar{\psi}\psi \rangle \) was examined in early studies such as [2] using staggered lattice fermions, and reasonable quantitative agreement found over a decade of quark mass with the prediction of leading order \( \chi \)PT for \( T \rightarrow 0 \), namely that for \( \mu < \mu_o \) the chiral condensate is \( \mu \)-independent, and for \( \mu \geq \mu_o \)

\[
\langle \bar{\psi}\psi \rangle \propto \frac{m}{\mu^2}.
\]

Unfortunately, since the global symmetries of staggered fermions do not coincide with those of continuum QC\(_2\)D [2], these results are not directly applicable. In any case, no attempt was made to explore beyond the regime of applicability of \( \chi \)PT.

However, our use of Wilson fermions precludes any direct study in the current simulation, since this formulation violates chiral symmetry explicitly. Our strategy therefore is to calculate a chiral order parameter using a fermion formulation with manifest chiral and baryon number symmetries using the gauge backgrounds ensembles generated with Wilson quarks. The disparity between valence and sea quarks violates unitarity; we mitigate this uncontrolled approximation by tuning the mass of the valence quarks so that the pion mass coincides with that used in the simulation; once \( \mu \neq 0 \) the onset transition of the valence quarks should then at least coincide with the true value.

Rather than the obvious choice of staggered fermions for the valence quarks, we found it expedient to use the existing code for \( N_f = 2 \) Wilson fermions with the parameter \( r \) (which has the conventional value of unity in [2]) set to zero. For \( j = 0 \) this is equivalent to eight identical staggered fermions with mass.
m = (2\kappa)^{-1}. For non-zero lattice spacing and \mu \neq 0 the action has a U(8) \otimes U(8) global symmetry which is broken by \mu \neq 0 (explicitly) or \langle \bar{\psi} \psi \rangle \neq 0 (spontaneously) to U(8)_V (the subscript denotes vectorlike), which incorporates the U(1)_B of baryon number. A diquark source \mu \neq 0 breaks U(8)_V to a SU(2) \otimes SU(2) which preserves isospin but no longer includes U(1)_B.

By studying effective mass plots as the valence \kappa_V was varied we found that \kappa_V = 8.0 gave the closest match to the value \kappa_{V,B} = 0.66(2) found for \beta = 1.9, \kappa = 0.168. Fig. 19 then shows the resulting chiral condensate as a function of \mu for the various lattices studied. Note that \mu_a = 0.04 throughout, since this was found to yield a less noisy and more stable signal – hence these results are not reproducible using pure staggered fermions. Two things are apparent; first the shape of the curve is in qualitative agreement with the old staggered results of [48] over the whole range of \mu studied, and thus consistent with [48] assuming an onset \mu_{o,a} \approx 0.3. Secondly, the results are independent of temperature even up to \tau = 141 MeV (N_\tau = 8). It is also apparent that volume effects are negligible.

It appears that the chiral symmetry properties of the dense phase are well-described by \chi PT. In a sense the issue of “chiral symmetry restoration” in QC\(_2\)D is academic, since the onset scale is set on the assumption that chiral symmetry is explicitly broken. Nonetheless, we can characterise the dense phase by whether \lim_{\tau \to 0} \langle \bar{\psi} \psi(m_V) \rangle vanishes or not. We determine this by using three different values \kappa_V = 8, 16 and 40, and observing that with the field normalisations implicit in [49],

\[
\frac{\kappa_V^2 \langle \bar{\psi} \psi \rangle_1}{\kappa_V^2 \langle \bar{\psi} \psi \rangle_2} = \frac{m_2 \langle \bar{q} q \rangle_1}{m_1 \langle \bar{q} q \rangle_2} \begin{cases} 1 & \langle \bar{\psi} \psi \rangle_0 = 0; \\ < 1 & \langle \bar{\psi} \psi \rangle_0 \neq 0, m_2 < m_1. \end{cases}
\]

(49)

Here \langle \bar{q} q \rangle denotes the scalar quark bilinear with conventional normalisation and \langle \bar{\psi} \psi \rangle_0 is the chiral condensate in the massless limit. Fig. 20 shows this ratio plotted for both (8,16) and (8,40) valence mass pairs on the 12\(^3\) \times 24 and 16\(^3\) \times 24 lattices as a function of \mu, and clearly indicates symmetry restoration for \mu \gtrsim \mu_o. Very similar plots are found for the other temperatures explored. We therefore conclude that the gauge field backgrounds at high baryon density in QC\(_2\)D are consistent with chiral symmetry being unbroken by a scalar condensate, although the exotic translationally-non-invariant scenario of [29] is not ruled out.

We find no significant difference between our results for the 12\(^3\) and 16\(^3\) lattices. This suggests that the chiral order parameter responds smoothly as \mu increases, with no indication at this stage of a phase transition (indeed the results are compatible with the predictions of chiral perturbation theory). However, in the absence of any systematic finite volume scaling study, and in light of the uncontrolled systematic uncertainties involved in our use of different actions and quark masses for sea and valence quarks, this should, like all the other results in this section, be taken as merely indicative.

**V. CONCLUSIONS AND OUTLOOK**

We have carried out the first extensive exploration of the phase diagram of two-color QCD (QC\(_2\)D) in the (T, \mu) plane using first-principles lattice simulations. Our main findings are summarised in the tentative phase diagram of Fig. 10. We find evidence of three distinct regions:

1. A vacuum/hadronic phase, with \langle q q \rangle = 0, \langle L \rangle = 0, \langle \bar{\psi} \psi \rangle = 0, \mu_a \approx 0, \text{ at low } T \text{ and } \mu \lesssim \mu_0 = m_\pi/2;

2. A quarkyonic phase at low T and intermediate to large \mu, which is confined (\langle \bar{L} \rangle \approx 0) and characterised by a chiral condensate which vanishes in
the chiral limit, Stefan–Boltzmann scaling of bulk thermodynamic quantities (including a nearly vanishing trace anomaly) and BCS scaling of the diquark condensate;

3. A deconfined quark–gluon plasma phase at high $T$ (and/or large $\mu$).

The main difference from our previous studies is that the BEC region has disappeared as a consequence of the $j \to 0$ extrapolation and a better understanding of the volume dependence and appropriate normalisation of our results. The BEC window would be expected to reappear for smaller $m_\pi/m_\rho$.

While we have clearly defined the finite-temperature deconfinement transition at $\mu = 0$, and find clear evidence of a deconfinement temperature that decreases as $\mu$ increases, the exact nature and location of this transition at large $\mu$ remain elusive. In order to pin down this transition, and also to precisely locate the superfluid-to-normal transition, we need to perform fine temperature scans by varying $N_c$ at fixed chemical potential. This is currently underway. We are also studying the static quark potential, which should give further insight into the nature of this transition.

For the first time in this paper we have attempted to calculate renormalised energy densities via an estimate of Karsch coefficients obtained from simulations on anisotropic lattices. We find the resulting corrections to our earlier results are substantial, and indeed strongly anisotropic lattices. We find the resulting corrections to mate of Karsch coefficients obtained from simulations on the nature of this transition.

For smaller $\mu$, we need to perform fine temperature scans by varying $N_c$ at fixed chemical potential. This is currently underway. We are also studying the static quark potential, which should give further insight into the nature of this transition.

The main shortcoming of this study is that it has been performed with a single, relatively coarse lattice spacing. Although, as observed in Ref. [8], the main results are in qualitative agreement with the earlier results [7] obtained on a coarser lattice with $\alpha = 0.23$fm, we also observe significant quantitative discrepancies, and substantial lattice artefacts for $\mu \rho \gtrsim 0.75$. To get this under control it will be necessary to repeat our simulations on a finer lattice. Thanks to the extensive investigation of parameter space reported in Sec. [II] we are in a good position to carry this out, and these simulation are underway.

The large quark mass is another source of systematic uncertainty; moreover our discussion of chiral symmetry in Sec. [IV E] is at best exploratory, and must in due course be supplemented by a calculation respecting unitarity. It is clear that QC$_2$D must be treated as a separate theory and cannot be viewed as an approximation to QCD – indeed, the differences between the two theories become most stark in the chiral limit – and there is hence no need to attempt to match quark masses to those in the real world. Still, many analytical results have been obtained in or near the chiral limit. Also, as already mentioned, we would expect a BEC region to open up near $\mu_o$ for smaller values of $m_\pi/m_\rho$, and a fuller understanding of the BEC–BCS crossover would be valuable. For all these reasons, simulations with smaller quark masses would be of great interest, and such simulations are underway.

In addition to the quantities considered here, we are in the process of computing the Landau-gauge gluon and quark propagators. This will allow us to check the assumptions involved in model solutions of the superfluid or superconducting gap equation, and may form a direct link with functional methods such as the functional renormalisation group and Dyson–Schwinger equations. These do not suffer from the sign problem, but rely on assumptions regarding the form of propagators and higher order vertices. This will be addressed in a forthcoming publication.
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