Explicit lower bound of blow up time in a fully parabolic attraction-repulsion chemotaxis system with nonlinear terms

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Abstract

It is known that, for the parabolic-elliptic Keller-Segel type system in a smooth bounded domain in dimension $\mathbb{R}^n$, $n \geq 3$, the lower bound of a blow-up time of unbounded solution is given. This paper extends the previous works to deal with the fully parabolic Keller-Segel type system in any spacial dimension $n \geq 3$. Firstly, we prove that for any blow-up time of an energy function is also the classical blow-up time when its energy level is sufficiently large. Secondly, we give an explicit estimation for the lower bound of blow-up time for fully parabolic attraction-repulsion chemotaxis system with nonlinear term, under homogeneous Neumann boundary conditions, in a smooth bounded domain.

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1 Introduction

1.1 Problem and main results

We are concerned with the blow-up phenomenon of a Keller Segel type system with a logistic term

$$\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + g(u) \\
    v_t &= \Delta v - \alpha v + \beta u \\
    w_t &= \Delta w - \gamma w + \delta u
\end{align*}$$

$$x \in \Omega, t \in (0, T_{\text{max}}).$$

(1.1)

Throughout this paper, $n$ is a positive integer ($n \geq 3$), $\tau \in \{0, 1\}$, $\Omega$ is a bounded domain of $\mathbb{R}^n$ with smooth boundary, and $\alpha, \beta, \xi, \chi, \gamma, \delta$ are positive constants. System (1.1) is complemented with the initial nonnegative conditions in
We define the \( C^1(\Omega) \).

\[
\begin{align*}
    u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & w(x,0) &= w_0(x), & x \in \Omega, \\
\end{align*}
\]  

(1.2)

and the Neumann boundary conditions

\[
\begin{align*}
    \frac{\partial u}{\partial \nu}(x,t) &= \frac{\partial v}{\partial \nu}(x,t) = \frac{\partial w}{\partial \nu}(x,t) = 0, & x \in \partial \Omega, & t > 0,
\end{align*}
\]  

(1.3)

where \( \nu \) is the outward normal vector. The nonlinear logistic term

\[
g(u) = \mu_1 u - \mu_2 u^k, \tag{1.4}
\]

where \( \mu_1 \in \mathbb{R}, \mu_2 \geq 0 \). We restrict \( k \in (1, \frac{2 + 1}{n}) \) if \( n = 3 \) and \( k \in (1, 1 + \frac{1}{2(n-1)}) \) if \( n \geq 4 \) to guarantee the existence of an unbounded solution for a simpler version of (1.1) \((\tau = 0, w \equiv 0)\) as shown in [Winkler-2018].

From now on, we always assume \( \tau = 1 \), the aim of this paper is twofold

1. To ensure that the blow-up time for the unbounded classical solutions and energy functional defined as (1.5) is the same if the level energy is sufficiently high \((p > n/2, q \geq 1)\):

2. To estimate lower bounds for the blow-up time of unbounded solutions in these energy functionals.

The first result is to show the global boundedness of the solution to the system (1.1) under some suitable conditions.

**Theorem 1.1.** Let \( p > \frac{n}{2} \), and let \( (u,v,w) \) be a classical solution defined in (2.2) of (1.1)-(1.4) on \([0,T)\). If

\[
\sup_{0 < t < T} \| u(\cdot,t) \|_{L^p(\Omega)} < \infty,
\]

then

\[
\sup_{0 < t < T} \left\{ \| u(\cdot,t) \|_{L^\infty(\Omega)} + \| v(\cdot,t) \|_{W^{1,\infty}(\Omega)} + \| w(\cdot,t) \|_{W^{1,\infty}(\Omega)} \right\} < \infty.
\]

We define the \((p,q)\) energy function

\[
E_{p,q}(t) := \frac{1}{p} \int_{\Omega} u(x,t)^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v(x,t)|^q \, dx + \frac{1}{q} \int_{\Omega} |\nabla w(x,t)|^q \, dx. \tag{1.5}
\]

and the blow-up time

\[
T_{p,q} := \inf \{ T > 0 : \limsup_{t \to T} E_{p,q}(t) = \infty \}.
\]

As a consequence of Theorem 1.1, the following corollary briefly proven in Section 3 fulfills the first goal of this paper.

**Corollary 1.1.** If \( T_{\text{max}} < \infty \) is the classical blow-up time defined as in Definition 2.2 then

\[
\lim_{t \to T_{\text{max}}} E_{p,q}(t) = \infty, \tag{1.6}
\]

for all \( p > \frac{n}{2}, q \geq 1 \).

**Remark 1.1.** Notice that the condition \( p > \frac{n}{2} \) is sufficient to conclude that \( T_{\text{max}} = T_{p,q} \) defined as the blow up time of \( E_{p,q} \), and does not require any condition for \( q \). However, we will need to restrict \( q \) to satisfy the condition (1.7) stated below to obtain lower bounds of blow-up time.

Before stating the following theorem, let us introduce the following parameters

\[
\eta_0 := \frac{2s_2}{p}, \quad \eta_1 := \frac{s_1}{s_1 - 1}, \quad \eta_2 := \frac{s_2(q - 2)}{q(s_2 - 1)}, \quad \eta_3 := \frac{2s_1}{q},\tag{1.7}
\]

where \( p, q, s_1, s_2 \) satisfy the following conditions

\[
\begin{cases}
    n < q \\
    \max \left\{ 1 + \frac{2}{p} \cdot \frac{\eta}{s_2} \right\} < s_1 < (1 + \frac{2}{p}) \frac{\eta}{2} \\
    \max \left\{ \frac{q(n+2)}{2q+2n} \cdot \frac{\eta}{s_2} \right\} < s_2 < \min \left\{ (1 + \frac{2}{q}) \cdot \frac{\eta}{2}, \frac{\eta}{2} \right\} \\
    \frac{\eta}{n + q} < p.
\end{cases}
\]  

(C)
Explicit lower bounds for blow-up time of \((1.1)\) will be given in the following theorem shown in Section 4.

**Theorem 1.2.** Let \((u, v, w)\) be any solutions of \((1.1) - (1.4)\) such that blow-up time \(T_{\text{max}} < \infty\), and \(\eta_0 (i = 0, 1, 2, 3)\) are introduced in \((1.7)\). Assume that \(p, q, s_1, s_2\) satisfy the condition \((\text{C})\) then the following inequality holds

\[
T_{\text{max}} \geq \int_{E_{p,q}(0)} \left( m \sum_{i=0}^{3} s^{\eta_i} + \sum_{i=0}^{3} m_i s^{k(\eta_i)} + \mu_1 s + c \right)^{-1} ds
\]

where function

\[
k(\eta) := \frac{n - \eta(n - 2)}{n + 2 - n\eta},
\]

and \(c, m, m_i (i = 0, 1, 2, 3)\) are positive constants depending on \(\alpha, \beta, \chi, \xi, \Omega, n, p, q\). Furthermore, if \(\Omega\) is convex then the inequality \((1.3)\) holds with \(c = 0\).

Note that [Yokota2] established a following lower bound of blow-up time \(T_{\text{max}}\) for the parabolic-elliptic attraction-repulsion chemotaxis system \((\tau = 0)\) with superlinear logistic degradation

\[
T_{\text{max}} \geq \int_{L(0)} \frac{ds}{B_1 s + B_2 s^{\frac{p+1}{p}} + B_3 s^{2\frac{\mu+1-n}{p-n}}},
\]

where \(B_1 \geq 0, B_2, B_3 > 0, \Omega = B_R(0)\) and

\[
L_1(0) = \int_{\Omega} u_0^p, \quad \text{where } p > \frac{n}{2}.
\]

Since the system is elliptic-parabolic Keller-Segel type \((\tau = 0)\), it was sufficient to analyze the energy function \(L_1(t) := \int_{\Omega} u^p(x, t) dx\). However this naive energy function may not work anymore in the fully parabolic case \((\tau = 1)\). Our analysis is much more involved as indicated in Section 4. Theorem 1.2 generalizes this result to the fully parabolic case, moreover, we derive a similar lower bound estimate for blow-up time to the parabolic-elliptic case by choosing \(q = 2p\) and \(p > \frac{n}{2}\). Precisely, we have the following result shown in Section 4.

**Corollary 1.2.1.** Let \((u, v, w)\) be any solutions of \((1.1) - (1.4)\) such that blow-up time \(T_{\text{max}} < \infty\) then the following inequality holds

\[
T_{\text{max}} \geq \int_{L(0)} \frac{ds}{A_0 + A_1 s + A_2 s^{\frac{p+1}{p}} + A_3 s^{2\frac{\mu+1-n}{p-n}}},
\]

where

\[
L_2(0) = \int_{\Omega} u_0^p + |\nabla u_0|^q + |\nabla v_0|^q, \quad \text{where } p > \frac{n}{2},
\]

and \(A_0, A_1, A_2, A_3\) are nonnegative constants depending on \(\alpha, \beta, \chi, \xi, \Omega, n\). Moreover, \(A_0 = 0\) if \(\Omega\) is a convex domain and \(A_1 = 0\) if \(q = 0\).

As shown in [Payne1; Payne2], the lower bounds estimate for blow-up time were also provided for parabolic-elliptic and fully parabolic chemotaxis model in 3-dimensional space as follows.

\[
T^* \geq \int_{\phi_1(0)} \frac{ds}{\sqrt{V s^2} + W s^b},
\]

where \(T^*\) is the blow-up time in \(\phi_1\)-measure, i.e., \(\limsup_{t \to T^*} \phi_1(t) = \infty\), where \(\phi_1(t)\) is defined as

\[
\phi_1(t) := \Gamma \int_{\Omega} u^2(\cdot, t) dx + \int_{\Omega} |\Delta v(\cdot, t)|^2 dx,
\]

with some \(\Gamma > 0\). These results were extended to the parabolic-parabolic chemotaxis system when \(n = 3\) with time dependency coefficients [Vigliarol1]. The following corollary improves the previous works to higher spacial dimension \((n \geq 4)\) and is also consistent with the previous results when \(n = 3\) by choosing \(p = n - 1\) and \(q = 2(n-1)\). The proof will briefly be given in Section 4.

**Corollary 1.2.2.** Let \((u, v, w)\) be any solutions of \((1.1) - (1.4)\) such that blow-up time \(T_{\text{max}} < \infty\) then the following inequality holds

\[
T_{\text{max}} \geq \int_{N(0)} \frac{ds}{B_1 s^{\frac{n}{n-1}} + B_2 s^{\frac{n}{n-2}}},
\]

where \(B_1, B_2\) are positive constants depending on \(\epsilon, \alpha, \beta, \chi, \xi, \Omega, n\) and

\[
N(0) = \int_{\Omega} u_0^{n-1} + |\nabla u_0|^{2(n-1)} + |\nabla v_0|^{2(n-1)}.
\]
1.2 Background on blow-up for parabolic attraction-repulsion chemotaxis system

Chemotaxis is the movement of an organism in response to a chemical stimulus. Somatic cells, bacteria, and other single-cell or multi-cellular organisms direct their movements according to certain chemicals in their environment. This is important for bacteria to find food by swimming toward the highest concentration of food molecules, or to flee from poisons. In multi-cellular organisms, chemotaxis is critical to early development (e.g., movement of sperm towards the egg during fertilization) and subsequent phases of development (e.g., migration of neurons or lymphocytes) as well as in normal function. In addition, it has been recognized that mechanisms that allow chemotaxis in animals can be subverted during cancer metastasis. One of the first mathematical models of chemotaxis known as Keller-Segel models, introduced in [Keller] can be simplified as follows

\[
\begin{align*}
    u_t &= \Delta u - a \nabla \cdot (u \nabla v), & x \in \Omega, t \in (0, T_{\text{max}}), \\
    \tau v_t &= \Delta v - v + u, & x \in \Omega, t \in (0, T_{\text{max}}).
\end{align*}
\]

(1.1)

Where \(u(x, t), v(x, t)\) represent the cell density and the concentration of chemical substances at position \(x\) at the time \(t\), respectively. In the following years, many variations of the system (KL) have been developed (see [Hillen; Gursamy]). Questions about local existence, global existence and especially blow-up solutions are intensively investigated in the last 40 years (see [Horstmann; Winkler-2015]). The Keller-Segel model (KL) has received so much attention in mathematics community over the last decades (cite their works) because of the critical mass phenomenon in 2 dimensional space. To be more precise, if the total mass \(M = \int_{\Omega} u < M_c\) then solution exists globally in time, whereas blow-up occurs in finite time if \(M > M_c\). However, this fact is no longer true in \(n\) dimensional space if \(n \geq 3\) (see ). There are several strategies to detect unbounded solutions to the system (KL), for instance, strategy 1: deriving an ordinary differential inequality for the \(n\)-th momentum as shown in [Nagai1; Nagai2], strategy 2: using large negative energy enforces the unboundedness as shown in [Winkler-2010], and strategy 3: deriving a superlinear ODI for the energy as shown in [Winkler-2013].

Now we shift our attention to the lower bound of blow-up time for unbounded solutions to chemotaxis models. There are several methods to detect lower bound of the blow-up time to the nonlinear parabolic equations, for instance, method 1: using gradient estimate in [Giga], method 2: using Neumann heat Kernel established in [Zhou] and method 3: deriving differential inequality for energy function established in [Payne2006, Payne2009]. In the following years, those methods have been adapted to serve the same purpose for Keller-Segel models, for more details see [Tao; Deng; Viglialoro1; Viglialoro2; Souplet-13].

Let us return to the system (1.1). In [Yokota2], the authors proved the existence of the finite time blow-up solution to the system (1.1) when \(\tau = 0\), and estimated the lower bound of blow-up for those solutions. When \(\tau = 1\) and \(g \equiv 0\) it was shown in [Lankeit] that the system (1.1) admits a finite time blow-up solution in 3 dimensional space if \(\xi \beta - \chi \delta > 0\), and extended to any space dimension \(n \geq 3\) in [Yokota3]. There are two questions motivated by those works:

- Does (1.1) admit an unbounded solution when \(\tau = 1\) and \(g\) is not identically 0?
- What is the lower bound of the blow-up time?

The answer for the second question when \(g \equiv 0\) will be provided in this paper, furthermore our method also works properly to detect life span of solutions or blow-up time of finite time blow-up solutions when \(g\) is not identically 0. The first question is still open, however, heuristically we believe that one can modifies the technique introduced in [Winkler-2013] to prove the first question.

2 Preliminaries

In this section, we recall some basic inequalities and known results supporting for the proof of our main result. Let us begin with definition of a classical solution.

**Definition 2.1.** A triplet \((u, v, w)\) is called a classical solution of (1.1) if

\[
\begin{align*}
    u &\in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times [0, T_{\text{max}})), \\
    v, w &\in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times [0, T_{\text{max}})) \cap L^\infty([0, T_{\text{max}}); W^{1,\infty}(\Omega))
\end{align*}
\]

and \(u, v, w\) satisfy (1.1) in the classical sense.

We clearly define the classical blow-up time as the following

**Definition 2.2.** Let \(T_{\text{max}}\) be a maximal time for which a solution of (1.1) exists for \(0 \leq t < T_{\text{max}}\). Then \(T_{\text{max}}\) is called a blow-up time in the classical sense if \(T_{\text{max}} < \infty\) and

\[
\lim_{t \to T_{\text{max}}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]
Now we ensure existence of solutions with the help of the following lemma.

**Lemma 2.1.** Let \( u_0 \in C(\bar{\Omega}) \) and \( v_0, w_0 \in C^1(\bar{\Omega}) \). Then there exist \( T_{\text{max}} \in (0, \infty] \) and a uniquely determined triplet \((u, v, w)\) of nonnegative functions in \( C(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\text{max}}]) \) solving (1.1) classically in \( \Omega \times (0, T_{\text{max}}) \). Additionally we either have
\[
T_{\text{max}} = \infty, \quad \text{or} \quad \limsup_{t \to T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]
Moreover, \((u, v, w)\) is radically symmetric for every \( t \in (0, T_{\text{max}}) \) if \( u_0, v_0, w_0 \) are radically symmetric. Finally, \( w \geq 0 \) and \( v \geq 0 \) in \( \Omega \times (0, T_{\text{max}}) \).

**Proof.** The local existence result can be proven by the fixed point argument well-established as in [Winkler-2015] or in [Tao-2013]; nonnegativity for the first and third component follow from the comparison principle. \( \square \)

Let us introduce the following lemma obtained by a minor modification of the Gigliari-Nirenberg inequality (see [Yokota1]).

**Lemma 2.2.** Let \( \Omega \) be a bounded and smooth domain of \( \mathbb{R}^n \) with \( n \geq 1 \). Let \( r \geq 1, 1 \leq q < p < \infty, s \geq 1 \). Then there exists a constant \( C_{GN} > 0 \) such that
\[
\|f\|_{L^p(\Omega)} \leq C_{GN} \left( \|\nabla f\|_{L^r(\Omega)}^p \|f\|_{L^s(\Omega)}^{(1-a)} \right)
\]
for all \( f \in L^q(\Omega) \) with \( \nabla f \in (L^r(\Omega))^n \), and \( a = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{n}{q} + \frac{n}{r} - m} \in [0, 1] \).

The next lemma using Young’s inequality is one of the main ingredients for the proof of Theorem 1.2.

**Lemma 2.3.** For any \( \epsilon > 0, \) and \( 1 < \eta < 1 + 2/n \)
\[
\int_{\Omega} |f|^{2\eta} \leq \epsilon C_1(\eta) \int_{\Omega} |\nabla f|^2 + C_2 \left( \int_{\Omega} f^2 \right)^{-\eta} + C_3(\eta) \epsilon^{-h(\eta)} \left( \int_{\Omega} f^2 \right)^{k(\eta)},
\]
where \( k(\eta) \) is defined as in (1.9) and
\[
\begin{align*}
   h(\eta) & := \frac{2(\eta-1)n}{n+2-n\eta}, \\
   C_1(\eta) & := n(\eta-1)/2, \\
   C_2 & := C_{GN}, \\
   C_3(\eta) & := \frac{n+2-n\eta}{2} C_{GN}^{\frac{2-2\eta}{2-n\eta}}.
\end{align*}
\]

**Proof.** Apply Lemma 2.2 with parameters \( p = 2\eta, q = 2, r = 2 \) and \( s = 2 \), we have
\[
\int_{\Omega} |f|^{2\eta} \leq C_{GN} \left( \int_{\Omega} |
\end{align*}
\]
Since \( \frac{2}{n(\eta-1)} > 1 \) when \( \eta \in (1, 1 + 2/n) \), by Young’s inequality (2.3)
\[
\left[ \epsilon^{\eta-1/2} \left( \int_{\Omega} \nabla f|^2 \right)^{\frac{\eta-1}{2}} \right] \left[ \epsilon^{-\eta-1/2} C_{GN} \left( \int_{\Omega} |f|^2 \right)^{-\frac{\eta-1}{2}} \right] \leq \frac{n(\eta-1)}{2} \epsilon \int_{\Omega} \nabla f|^2 + \frac{2}{2 - n(\eta-1)} \epsilon^{-\frac{2(\eta-1)n}{n+2-n\eta}} \left( \int_{\Omega} |f|^2 \right)^{\frac{n-2(n-2)}{n+2-n\eta}}.
\]
We finally complete the proof by making use of (2.3) and (2.4). \( \square \)

**Remark 2.1.** Our result slightly extends Lemma 2.2 in [Deng] in the sense that it provides us a larger room of \( \eta \) to choose when needed. For convenience, let us recall the previous work shortly as follows
\[
\int_{\Omega} |f|^{2\eta} \leq \epsilon C_1 \int_{\Omega} |\nabla f|^2 + C_2 \epsilon^{-\eta/(2-\eta)} \left( \int_{\Omega} f^2 \right)^{\frac{2}{2-\eta}} + C_3 \left( \int_{\Omega} f^2 \right)^{\eta},
\]
where \( \eta = \frac{n}{n-1} \). To be more precise, if \( \eta \in (1, \frac{n}{n-1}) \), we have

\[
\frac{n - \eta(n - 2)}{n + 2 - n\eta} < \frac{\eta}{2 - \eta} < \frac{n}{n - 2},
\]

and the equality happens when \( \eta = \frac{n}{n-1} \)

\[
\frac{n - \eta(n - 2)}{n + 2 - n\eta} = \frac{\eta}{2 - \eta} = \frac{n}{n - 2}.
\]

Compare to our result, the inequality (2.3) is the same as (2.5) when \( \eta = \frac{n}{n-1} \in (1, 1 + \frac{2}{n}) \).

The next lemma supports the proof of Theorem 1.2

**Lemma 2.4.** The condition (C) holds if and only if \( \eta_i \in (1, 1 + \frac{2}{n}) \) for all \( i = 0, 1, 2, 3 \).

**Proof.** The restriction of \( \eta_1 = \frac{s_1}{s_1-1} < 1 + \frac{2}{n} \) implies that \( s_1 > 1 + \frac{2}{n} \). Moreover, because \( \eta_3 = \frac{2s_3}{q} \in (1, 1 + \frac{2}{n}) \), we have

\[
\frac{q}{2} < s_1 < \left(1 + \frac{2}{n}\right) \frac{q}{2}.
\]

Therefore, we obtain the first inequality of (C).

\[
\max\left\{1 + \frac{n}{2}, \frac{q}{2}\right\} < s_1 < \left(1 + \frac{2}{n}\right) \frac{q}{2}. \tag{2.6}
\]

The inequality \( 1 + \frac{n}{2} < \left(1 + \frac{2}{n}\right) \frac{q}{2} \) entails that

\[
q > n. \tag{2.7}
\]

Because \( \eta_0 = \frac{2s_0}{p} \in (1, 1 + \frac{2}{n}) \), we obtain \( \frac{q}{2} < s_2 < (1 + \frac{2}{n}) \frac{q}{2} \). In addition, we also have \( \frac{(n+2)n}{2(n+2)} < s_2 < \frac{q}{2} \) due to the condition of \( \eta_2 = \frac{s_2(q-2)}{q(q-2)} \in (1, 1 + \frac{2}{n}) \). Hence,

\[
\max\left\{\frac{q(n+2)}{2q+2n}, \frac{p}{2}\right\} < s_2 < \min\left\{\left(1 + \frac{2}{n}\right) \frac{p}{2}, \frac{q}{2}\right\} \tag{2.8}
\]

The inequality \( (1 + \frac{2}{n}) \frac{p}{2} > \frac{q(n+2)}{2q+2n} \) entails that

\[
\frac{nq}{n + q} < p. \tag{2.9}
\]

From (2.6) to (2.9), we prove (C).

To be more specific, let us introduce the following figure representing for all selections of \((p, q)\).

![Figure 1](image-url)
Remark 2.2. We have some remarks as follows

1. The inequalities (2.7) and (2.9) are equivalent to the following
\[
\begin{cases}
  n < q < \frac{np}{n-p}, & \text{if } p \in \left(\frac{n}{2}, n\right) \\
  n < q < \infty, & \text{if } p \in [n, \infty).
\end{cases}
\]

2. With the selection \( q = 2p \) was used in [Yokota1] to obtain lower bounds for blow-up time.

Next let us derive an estimate for a particular boundary integral that enables us to cover possibly non-convex domains [Li].

Lemma 2.5. Let \( \Omega \) be a bounded domain with smooth boundary, let \( q \in [1, \infty) \). Then for any \( \theta > 0 \) there is \( C_\theta > 0 \) such that for any \( f \in C^2(\bar{\Omega}) \) satisfying \( \frac{\partial f}{\partial \nu} = 0 \) on \( \partial \Omega \), the inequality
\[
\int_{\partial \Omega} |\nabla f|^{2(q-1)} \nabla(|\nabla f|^2) \cdot \nu \leq \theta \int_{\Omega} |\nabla(|\nabla f|^q)|^2 + C_\theta.
\]

holds.

If \( \Omega \) is a convex bounded domain, then the following holds (see [Winkler-2011]):

Lemma 2.6. Assume that \( \Omega \) is a convex bounded domain, and that \( f \in C^2(\Omega) \) satisfies \( \frac{\partial f}{\partial n} = 0 \) on \( \partial \Omega \). Then
\[
\frac{\partial |\nabla f|^2}{\partial n} \leq 0 \quad \text{on } \partial \Omega.
\]

The next result helps us to handle the difficulties of the parabolic equations. For more details, see [Freitag-2018]

Lemma 2.7. Let \( p \geq 1 \) and \( q \geq 1 \) satisfy
\[
\begin{cases}
  q < \frac{np}{n-p}, & \text{when } p < n, \\
  q < \infty, & \text{when } p = n, \\
  q = \infty, & \text{when } p > n.
\end{cases}
\]

Assuming \( g_0 \in W^{1,q}(\Omega) \) and \( g \) is a weak solution to the following system
\[
\begin{aligned}
  g_t &= \Delta g - \zeta_1 g + \zeta_2 f & & \text{in } \Omega \times (0, T), \\
  \frac{\partial g}{\partial \nu} &= 0 & & \text{on } \partial \Omega \times (0, T), \\
  g(\cdot, 0) &= g_0 & & \text{in } \Omega
\end{aligned}
\]

for some \( T \in (0, \infty] \). If \( f \in L^\infty ((0, T); L^p(\Omega)) \), then \( g \in L^\infty ((0, T); W^{1,q}(\Omega)) \).

3 From blow-up time of high level energy to blow-up of the classical sense.

In this section, we show that the classical blow-up time for solutions to the system (1.1) is also the blow-up time for \( E_{p,q} \) for all sufficiently large \( p,q \). A method due to [Alikakos1; Alikakos2] based on Moser-type iterations is implemented to prove Theorem (1.1). We will rely on the following proposition.

Proposition 3.1. Let \( (u, v, w) \) be a classical solution of (1.1) on \([0, T]\), \( r > \frac{n}{2} \) and
\[
U_r := \max \left\{ \|u_0\|_{L^\infty(\Omega)}, \sup_{t < T} \|u(\cdot, t)\|_{L^\infty(\Omega)} \right\} < \infty.
\]

Then there exists a constant \( C_1, C_2 > 0 \) independent of \( r \) such that
\[
U_{2r} < (C_1 r^{C_2})^{\frac{n}{r}} U_r
\]

Proof of Proposition 3.1. Multiplying the first equation in the system (1.1) by \( u^{2r-1} \) we obtain
\[
\frac{1}{2r} \frac{d}{dt} \int_{\Omega} u^{2r} = \int_{\Omega} u^{2r-1} u_t
\]
where

\[ J := \int_{\Omega} u^{2r-1} \left( -\chi \nabla (u \nabla v) + \xi \nabla (u \nabla w) + \mu_1 u - \mu_2 u^k \right) \]

Applying Holder's inequality to the first two terms of the right hand side of (3.2),

\[ J \leq \epsilon \int_{\Omega} |\nabla u^r|^2 + \left( \frac{(2r-1)^2}{4r^2 \epsilon} (\xi^2 + \chi^2) \int_{\Omega} u^{2r} (|\nabla v|^2 + |\nabla w|^2) + \mu_1 \int_{\Omega} u^{2r} \right) . \]

for all \( \epsilon > 0 \). Here we divide the proof in two cases.

**Case 1:** \( r > n \), Lemma 2.7 implies that \( v, w \) are in \( L^\infty \left( (0, T); W^{1,\infty}(\Omega) \right) \). Thus,

\[ \sup_{0 < t < \tau} \left( \|\nabla v\|_{L^\infty}^2 + \|\nabla w\|_{L^\infty}^2 \right) = c_1 < \infty, \]

It follows from (3.1) and (3.3) that

\[ \frac{1}{2r} \int_{\Omega} u^{2r} \leq \left( -\frac{2r-1}{r^2} + \epsilon \right) \int_{\Omega} |\nabla u^r|^2 + \left( \frac{(2r-1)^2}{4r^2 \epsilon} (\xi^2 + \chi^2) c_1 + \mu_1 \right) \int_{\Omega} u^{2r}. \]

Substituting \( \epsilon = \frac{r-1}{r^2} \) in (3.4), and notice that

\[ \frac{(2r-1)^2}{4r^2 \epsilon} = \frac{(2r-1)^2}{4r-4} < 2r, \quad \forall r > \frac{3}{2}. \]

Therefore,

\[ \frac{1}{2r} \frac{d}{dt} \int_{\Omega} u^{2r} \leq -\frac{1}{r} \int_{\Omega} |\nabla u^r|^2 + \left( 2r(\xi^2 + \chi^2)c_1 + \mu_1 \right) \int_{\Omega} u^{2r}. \]

We set \( \psi = u^r \) and substitute it in (3.5),

\[ \frac{d}{dt} \int_{\Omega} \psi^2 \leq -2 \int_{\Omega} |\nabla \psi|^2 + c_2 r^2 \int_{\Omega} \psi^2 \]

where \( c_2 := 4(\xi^2 + \chi^2) + \max \{ \mu_1, 0 \} \). By Holder’s inequality, we have

\[ \int_{\Omega} \psi^2 \leq \left( \int_{\Omega} \psi^{\frac{2}{n-2}} \right)^{\frac{n-2}{2}} \left( \int_{\Omega} \psi^{\frac{2}{n+2}} \right)^{\frac{n+2}{2}} . \]

The Sobolev embedding \( W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega) \) implies that there exists a constant \( C_S > 0 \) such that

\[ \left( \int_{\Omega} \psi^{\frac{2}{n-2}} \right)^{\frac{n-2}{2}} \leq C_S \|\psi\|_{W^{1,2}(\Omega)} . \]

Hence,

\[ \int_{\Omega} \psi^2 \leq C_3 \left( \int_{\Omega} \psi^{\frac{2}{n-2}} \right)^{\frac{n-2}{2}} \|\psi\|_{W^{1,2}(\Omega)^{\frac{2n}{2n-2}}} . \]

where \( C_3 = C_{\frac{2n}{2n-2}} \). We make use of Young’s inequality to (3.8) and obtain that

\[ \int_{\Omega} \psi^2 \leq \frac{n(c_4 \epsilon)^{\frac{1}{2}}}{n+2} \|\psi\|_{W^{1,2}(\Omega)^{\frac{2n}{2n-2}}} + \frac{2}{n+2} \epsilon^{\frac{1}{2}} \left( \int_{\Omega} \psi \right)^2 . \]
Let $\epsilon$ be sufficiently small ($\epsilon \leq c_3^{1-n/2}$) such that $\frac{n^{1+\frac{n}{1-n/2}}}{n+2} \leq \frac{1}{4}$. We plug in (3.9) to have

$$\int_{\Omega} \psi^2 \leq \epsilon \int_{\Omega} |\nabla \psi|^2 + \frac{4}{n+2} \epsilon^{1-n/2} \left( \int_{\Omega} \psi \right)^2. \tag{3.10}$$

From (3.6) and (3.10), it follows that that

$$\frac{d}{dt} \int_{\Omega} \psi^2 \leq (-2 + c_2 r^2 \epsilon) \int_{\Omega} |\nabla \psi|^2 + \frac{4c_2 r^2}{n+2} \epsilon^{1-n/2} \left( \int_{\Omega} \psi \right)^2. \tag{3.11}$$

Let fix $\epsilon = \frac{n^2}{4r^2 c_3^{1-n/2}}$, we have

$$\frac{d}{dt} \int_{\Omega} \psi^2 \leq c_4 r^{n+4} \left( \int_{\Omega} \psi \right)^2, \tag{3.11}$$

where

$$c_4 = \frac{4c_2}{n+2} \left( \frac{n^2}{4c_3^{1-n/2}} \right)^{-1-n/2}. \tag{3.11}$$

Integration both sides of (3.11), the following inequality holds

$$\int_{\Omega} u^{2r}(x, t) \, dx \leq c_4 T r^{n+4} U^2 r.$$ 

That leads to 

$$U_{2r} \leq \left( c_4 T r^{n+4} \right)^{\frac{2}{r-n}} U_r, \quad \forall r > n. \tag{3.12}$$

**Case 2:** \( \frac{n}{2} < r \leq n \), Lemma 2.7 implies that \( v, w \) are in \( L^\infty \left( (0, T); W^{1,q}(\Omega) \right) \) for \( q < \frac{n}{n-r} \) if \( r < n \) and any \( q < \infty \) if \( r = n \). We set \( \lambda := \frac{2n}{n-r} \) and apply Holder’s inequality to obtain

$$\int_{\Omega} u^{2r} |\nabla v|^2 \leq \left( \int_{\Omega} u^{2r+\lambda} \right)^{\frac{2r}{2r+\lambda}} \left( \int_{\Omega} |\nabla v|^{\frac{2r\lambda}{r+\lambda}} \right)^{\frac{r}{2r+\lambda}}. \tag{3.13}$$

For \( r < n \), we find that

$$\frac{2(2r+\lambda)}{\lambda} \leq \frac{rn}{n-r},$$

therefore \( v, w \) belong to \( L^\infty \left( (0, T); W^{1,\frac{2r(2r+\lambda)}{r+\lambda}}(\Omega) \right) \). It is obvious that \( v, w \) belong to \( L^\infty \left( (0, T); W^{1,\frac{2r(2r+\lambda)}{r+\lambda}}(\Omega) \right) \) when \( r = n \). Thus,

$$\sup_{0 < t < T} \left( ||v||_{L^\frac{2r(2r+\lambda)}{r+\lambda}(\Omega)}^2 + ||w||_{L^\frac{2r+\lambda}{r+\lambda}(\Omega)}^2 \right) = c_5 < \infty,$$

notice that

$$\int_{\Omega} u^{2r} \leq |\Omega|^{\frac{2r}{2r+\lambda}} \left( \int_{\Omega} u^{2r+\lambda} \right)^{\frac{2r}{2r+\lambda}} \leq \max \{1, |\Omega|\} \left( \int_{\Omega} u^{2r+\lambda} \right)^{\frac{2r}{2r+\lambda}}. \tag{3.14}$$

By the similar argument as in (3.4) and combine with (3.14), it follows that

$$\frac{1}{2r} \frac{d}{dt} \int_{\Omega} u^{2r} \leq -\frac{1}{r} \int_{\Omega} |\nabla u|^2 + (2r(\xi^2 + \chi^2)c_5 + \mu_1 \max \{1, |\Omega|\}) \left( \int_{\Omega} u^{2r+\lambda} \right)^{\frac{2r}{2r+\lambda}}. \tag{3.15}$$

Hence,

$$\frac{d}{dt} \int_{\Omega} \psi^2 \leq -2 \int_{\Omega} |\nabla \psi|^2 + c_6 r^2 \left( \int_{\Omega} \psi^{2+\frac{\lambda}{r}} \right)^{\frac{2r}{2r+\lambda}} \tag{3.15},$$

where \( c_6 := 2 \left( (\xi^2 + \chi^2)c_5 + \mu_1 \max \{1, |\Omega|\} \right) \) is independent of \( r \). We find that

$$2 + \frac{\lambda}{r} < \frac{2n}{n-2} \quad \forall r > \frac{n}{2},$$

which enables us to apply Holder’s inequality as follows

$$\left( \int_{\Omega} \psi^{2+\frac{\lambda}{r}} \right)^{\frac{2r}{2r+\lambda}} \leq \|\psi\|^2_{L^1(\Omega)} \|\psi\|_{L^{\frac{2n}{n-2}}(\Omega)}.$$
where $\theta := \frac{(n-2)r+2n}{r(n+2)} \in (0, 1)$. From (3.7) we find that
\[
\left( \int_{\Omega} \psi^{2^*} \right)^{\frac{2^*}{2^* - \lambda}} \leq c_7 \| \psi \|_{L^1(\Omega)}^{2^* \lambda} \| \psi \|_{W^{1,2}(\Omega)}^{2^* r - \lambda},
\]
(3.16)
where $c_7 = \max\{1, c_S^2 \} \geq C_S$ is independent of $r$, and $\theta^*$ is defined as
\[
\frac{1}{\theta^*} := \frac{2rn\theta}{(n-2)(2r+\lambda)}.
\]
Since $\theta^* \in (0, 1)$ if $r > \frac{\lambda}{\theta}$, applying Young’s inequality to the right hand side of (3.16) entails
\[
\| \psi \|_{L^1(\Omega)}^{2^* r - \lambda} \| \psi \|_{W^{1,2}(\Omega)}^{2^* \lambda} \leq \left( \theta^* \right)^{-1} \left( c_7 \epsilon \| \psi \|_{W^{1,2}(\Omega)}^{2/\theta^*} \right)^{\theta^*} + \frac{\theta^* - 1}{\theta^*} \left( \epsilon^{-1} \| \psi \|_{L^1(\Omega)}^{2^* r - \lambda} \right)^{\frac{\theta^*}{\theta^* - 1}}
\]
\[
\leq \epsilon^{\theta^*} \| \psi \|_{W^{1,2}(\Omega)}^{2} + \epsilon^{-\frac{\theta^*}{\theta^* - 1}} \left( \int_{\Omega} \psi \right)^{2},
\]
(3.17)
where we have used the following identity
\[
\frac{r(1 - \theta)\theta^*}{(2r+\lambda)(\theta^* - 1)} = 1.
\]
From (3.16) and (3.17), we have
\[
\left( \int_{\Omega} \psi^{2^*} \right)^{\frac{2^*}{2^* - \lambda}} \leq \epsilon^{\theta^*} \| \psi \|_{W^{1,2}(\Omega)}^{2} + \epsilon^{-\frac{\theta^*}{\theta^* - 1}} \left( \int_{\Omega} \psi \right)^{2}. \tag{3.18}
\]
By similar procedure as in case 1, we can choose $\epsilon$ sufficiently small and plug into (3.15) and (3.18), to obtain
\[
U_{2r} \leq (c_8 T r c_9)^{\frac{1}{p}} U_r, \quad \forall r \in \left( \frac{n}{2}, n \right), \tag{3.19}
\]
where $c_8, c_9 > 0$ independent of $r$. Finally, we complete the proof by combining (3.12) and (3.19). \hfill \square

Now we apply the iteration scheme to prove Theorem 1.1

**Proof of Theorem 1.1.** Proposition 3.1 ensures that the following inequality holds for all $r > \frac{n}{2}$
\[
U_{2r} \leq (C_1 r C_2)^{\frac{1}{p}} U_r. \tag{3.20}
\]
Reiterate Proposition 3.1 with $r = 2^k p$ for all positive integers $k$, one can easily verify that
\[
U_{2^{k+1} p} \leq (C_1^k)^{A_k} (2 C_2^k)^{B_k} U_p, \quad \forall k \geq 1. \tag{3.21}
\]
where
\[
A_k := \sum_{j=1}^{k} \frac{1}{2^j} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,
\]
\[
B_k := \sum_{j=1}^{k} \frac{j}{2^j} \leq \sum_{j=1}^{\infty} \frac{j}{2^j} = 2.
\]
Let $k \to \infty$, we have
\[
U_{\infty} \leq C_0^\frac{1}{p} C_1 C_2 U_p, \quad \forall p > \frac{n}{2}.
\]
Which implies that $u \in L^{\infty}((0, T); L^{\infty}(\Omega))$. Thank to Lemma 2.7 we assert that $v, w$ are in $L^{\infty}(0, T); W^{1,\infty}(\Omega))$. Hence
\[
\|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C
\]
for all $t \in (0, T)$ with some $C > 0$. \hfill \square
We can finally claim that the energy function at level \((p, q)\) blows up at \(T_{\text{max}}\) if \(p > \frac{n}{2}\), \(q \geq 1\) by proving Corollary 1.1.1.

**Proof of Corollary 1.1.1.** We first claim that \(u\) blows up at \(T_{\text{max}}\), otherwise by Lemma 2.1

\[
\sup_{0 < t < T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.
\]

By Lemma 2.7

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C
\]

for all \(t \in (0, T_{\text{max}})\) with some \(C > 0\), which contradicts the definition of the classical blow-up time. Next, we suppose that there exists some \(p > \frac{n}{2}\) and \(q \geq 1\) such that

\[
\limsup_{t \to T_{\text{max}}} E_{p,q}(t) < \infty.
\]

implying that \(\|u(\cdot, t)\|_{L^p(\Omega)} < K_1\) for all \(t \in (0, T_{\text{max}})\) where \(K_1 > 0\). By Theorem 1.1 we obtain that

\[
\sup_{0 < t < T_{\text{max}}} \|u\|_{L^\infty(\Omega)} < \infty,
\]

which is a contradiction. Therefore

\[
\limsup_{t \to T_{\text{max}}} E_{p,q}(t) = \infty,
\]

which completes the proof. \(\square\)

### 4 Lower bound of blow-up time

We are now in a position to complete the proof of Theorem 1.2.

**Proof.** Let us recall the energy function

\[
E_{p,q}(t) = \frac{1}{p} \int_\Omega u(x,t)^p \, dx + \frac{1}{q} \int_\Omega |\nabla v(x,t)|^q \, dx + \frac{1}{q} \int_\Omega |\nabla w(x,t)|^q \, dx
\]

(4.1) \(= I_1(t) + I_2(t) + I_3(t).\)

We are going to estimate each \(I_i\) \((i = 1, 2, 3)\), and notice that estimates for \(I_2\) and \(I_3\) are similar under some modifications of parameters. Let us begin with \(I_1\),

\[
I_1'(t) = \int_\Omega u^{p-1} (\Delta u - \chi \nabla (u \nabla v) + \xi \nabla (u \nabla w) + g(u))
\]

\[
= -\frac{4(p-1)}{p^2} \int_\Omega |\nabla u|^2 \frac{\partial \psi}{\partial u} + \chi(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v - \xi(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla w + \int_\Omega u^{p-1} g(u).
\]

(4.2)

By Holder inequality,

\[
\chi(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v \leq \chi^2(p-1) \frac{2(p-1)}{p} \int_\Omega u^{\frac{p}{2}} |\nabla v|^2 + \frac{p-1}{p^2} \int_\Omega |\nabla u|^2.
\]

(4.3)

Similarly,

\[
-\xi(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla w \leq \xi^2(p-1) \int_\Omega u^{p} |\nabla w|^2 + \frac{p-1}{p^2} \int_\Omega |\nabla u|^2.
\]

(4.4)

We also have

\[
\int_\Omega u^{p-1} g(u) \leq \mu_1 \int_\Omega u^p.
\]

(4.5)

From (4.1) to (4.5), we obtain

\[
I_1'(t) \leq -\frac{2(p-1)}{p^2} \int_\Omega |\nabla u|^2 + \chi^2(p-1) \int_\Omega u^p |\nabla v|^2 + \xi^2(p-1) \int_\Omega u^p |\nabla w|^2 + \mu_1 \int_\Omega u^p.
\]

(4.6)
Now, we give an estimate for $I_2$,

$$I_2'(t) = \int_{\Omega} |\nabla v|^q \nabla v \cdot \nabla v_t$$

(4.7)

Note that

$$\nabla v \cdot \nabla v_t = \nabla v \cdot \nabla \Delta v - \alpha |\nabla v|^2 + \beta \nabla v \cdot \nabla u,$$

(4.8)

and

$$\Delta(|\nabla v|^2) = 2|D^2v|^2 + 2\nabla v \cdot \nabla \Delta v.$$ 

(4.9)

From (4.7), (4.8) and (4.9),

$$I_2'(t) = \int_{\Omega} |\nabla v|^q \left( \nabla v \cdot \nabla \Delta v - \alpha |\nabla v|^2 + \beta \nabla v \cdot \nabla u \right)$$

$$= \int_{\Omega} |\nabla v|^q \left( \frac{1}{2} \Delta(|\nabla v|^2) - |D^2v|^2 - \alpha |\nabla v|^2 + \beta \nabla v \cdot \nabla u \right)$$

$$= -\frac{1}{2} \int_{\Omega} \nabla(|\nabla v|^q) \cdot \nabla(|\nabla v|^2) + \frac{1}{2} \int_{\partial \Omega} |\nabla v|^q |\nabla v| \cdot \nu$$

$$- \int_{\Omega} |D^2v|^2 |\nabla v|^q - \alpha \int_{\Omega} |\nabla v|^2 + \beta \int_{\Omega} |\nabla v|^q \nabla v \cdot \nabla u.$$ 

(4.10)

From Lemma (2.5), there exists a positive constant $C$ such that

$$\frac{1}{2} \int_{\partial \Omega} |\nabla v|^q |\nabla v| \cdot \nu \leq \frac{2(q - 2)}{q^2} \int_{\Omega} |\nabla(|\nabla v|^\frac{2}{q})|^2 + C. 

(4.11)$$

Since $|D^2v|^2 \geq \frac{|\Delta u|^2}{n}$, from (4.10) and (4.11) we have

$$I_2'(t) \leq -\frac{2(q - 2)}{q^2} \int_{\Omega} |\nabla(|\nabla v|^\frac{2}{q})|^2 - \frac{n}{q^2} \int_{\Omega} (\Delta v)^2 |\nabla v|^q - \alpha \int_{\Omega} |\nabla v|^2 + \beta \int_{\Omega} |\nabla v|^q \nabla v \cdot \nabla u + C. 

(4.12)$$

We make use of Holder’s inequality to the first and second terms of the right hand side of (4.11) to obtain

$$\beta \int_{\Omega} |\nabla v|^q \nabla v \cdot \nabla u = -\beta \int_{\Omega} |\nabla v|^q \nabla v \cdot \nabla u - \frac{\beta(q - 2)}{2} \int_{\Omega} u |\nabla v|^q \nabla v \cdot \nabla v$$

$$\leq \frac{1}{n} \int_{\Omega} (\Delta v)^2 |\nabla v|^q - \frac{\beta q^2}{4} \int_{\Omega} |\nabla v|^q - \frac{1}{n} \int_{\Omega} u^2 |\nabla v|^q + \frac{\beta(q - 2)}{16} \int_{\Omega} |\nabla v|^q |\nabla v|^2.$$ 

(4.13)

Note that

$$\nabla(|\nabla v|^\frac{2}{q}) = \frac{q - 2}{4} |\nabla v|^\frac{q - 2}{2} \nabla(|\nabla v|^2),$$

which leads to

$$\frac{q - 2}{16} \int_{\Omega} |\nabla v|^q |\nabla v|^2 = \frac{q - 2}{q^2} \int_{\Omega} |\nabla(|\nabla v|^\frac{2}{q})|^2.$$ 

(4.14)

Combining (4.12), (4.13) and (4.14) we have,

$$I_2'(t) \leq -\frac{q - 2}{q^2} \int_{\Omega} |\nabla(|\nabla v|^\frac{2}{q})|^2 + \beta^2 (n/4 + q - 2) \int_{\Omega} u^2 |\nabla v|^q + C. 

(4.15)$$

Similarly,

$$I_3'(t) \leq -\frac{q - 2}{q^2} \int_{\Omega} |\nabla(|\nabla w|^\frac{2}{q})|^2 + \alpha^2 (n/4 + q - 2) \int_{\Omega} u^2 |\nabla w|^q + C. 

(4.16)$$
Hence, from (4.6), (4.15), and (4.16)

\[
E_p(t) \leq -\frac{2(p-1)}{p} \int_\Omega |\nabla u|^2 + \chi^2 p \int_\Omega u^p |\nabla v|^2 + \xi^2 p \int_\Omega u^p |\nabla w|^2 - \frac{q-2}{q} \int_\Omega |\nabla (|\nabla v|^{1/2})|^2 + \beta^2 (n/4 + q - 2) \int_\Omega u^2 |\nabla v|^{q-2} - \frac{q-2}{q} \int_\Omega |\nabla (|\nabla w|^{1/2})|^2 + \alpha^2 (n/4 + q - 2) \int_\Omega u^2 |\nabla w|^{q-2} + \mu_1 \int_\Omega u^p + 2C.
\]

In light of Young’s inequality, we obtain

\[
\int_\Omega u^p |\nabla v|^2 \leq \frac{1}{s_1} \int_\Omega |\nabla v|^{2s_1} + \frac{s_1 - 1}{s_1} \int_\Omega u^{\frac{2s_1}{s_1 - 1}},
\]

for any arbitrary \(s_1 > 1\), and

\[
\int_\Omega u^2 (|\nabla v|^{q-2} + |\nabla w|^{q-2}) \leq \frac{1}{s_2} \int_\Omega u^{2s_2} + \frac{s_2 - 1}{s_2} \int_\Omega |\nabla v|^{\frac{q_2(q-2)}{q_2 - 1}} + |\nabla w|^{\frac{q_2(q-2)}{q_2 - 1}}
\]

for any arbitrary \(s_2 > 1\).

We will make use of Lemma 2.3 in order to bound quantities: \(\int_\Omega u^{2s_2}, \int_\Omega u^{\frac{2s_1}{s_1 - 1}}, \int_\Omega |\nabla v|^{\frac{q_2(q-2)}{q_2 - 1}}, \int_\Omega |\nabla w|^{\frac{q_2(q-2)}{q_2 - 1}}\), \(\int_\Omega |\nabla v|^{2s_1}\), and \(\int_\Omega |\nabla w|^{2s_1}\) by the terms appearing in \(E_{p,q}\). \(\int_\Omega u^p, \int_\Omega |\nabla v|^q, \int_\Omega |\nabla w|^q, \int_\Omega |\nabla u|^\frac{q}{2}, \int_\Omega |\nabla v|^\frac{q}{2}, \int_\Omega |\nabla v|^\frac{q}{2}\), and \(\int_\Omega |\nabla w|^\frac{q}{2}\). Recall \(\eta_0 := \frac{2q}{p}, \eta_1 := \frac{2q}{q_2 - 1}, \eta_2 := \frac{2q}{q_3 - 1}\), and \(\eta_3 := \frac{2q}{q_3}\), it is verified by Lemma (2.4) that \(\eta_i \in (1, 1 + \frac{2}{n})\) for all integers \(0 \leq i \leq 3\) when \(p, q, s_1, s_2\) satisfy (C). Substitute \(f = u^\frac{q}{2}, \eta = \eta_0\) into inequality (2.1), we obtain

\[
\int_\Omega u^{2s_2} \leq c_1 (\eta_0) \int_\Omega |\nabla u|^2 + c_2 (\int_\Omega u^p)^{\eta_0} + C_3 (\eta_0)^{-h(\eta_0)} \left( \int_\Omega u^p \right)^{k(\eta_0)}.
\]

Similarly, we substitute \(f = u^\frac{q}{2}\) and \(\eta = \eta_1\) into inequality (2.1) to imply

\[
\int_\Omega u^{\frac{2s_1}{s_1 - 1}} \leq c_1 (\eta_1) \int_\Omega |\nabla u|^2 + c_2 (\int_\Omega u^p)^{\eta_1} + C_3 (\eta_1)^{-h(\eta_1)} \left( \int_\Omega u^p \right)^{k(\eta_1)}.
\]

Plugging \(f = |\nabla v|^\frac{q}{2}, f = |\nabla w|^\frac{q}{2}\) and \(\eta = \eta_2\) into (2.1) and adding them together yields

\[
\int_\Omega \left( |\nabla v|^\frac{q_2(q-2)}{q_2 - 1} + |\nabla w|^\frac{q_2(q-2)}{q_2 - 1} \right) \leq c_1 (\eta_2) \int_\Omega |\nabla v|^{\frac{q_2(q-2)}{q_2 - 1}} + |\nabla w|^{\frac{q_2(q-2)}{q_2 - 1}}^2
\]

\[
+ C_2 \left[ \left( \int_\Omega |\nabla v|^q \right)^{\eta_2} + \left( \int_\Omega |\nabla w|^q \right)^{\eta_2} \right]
\]

\[
+ C_3 (\eta_2)^{-h(\eta_2)} \left[ \left( \int_\Omega |\nabla v|^q \right)^{k(\eta_2)} + \left( \int_\Omega |\nabla w|^q \right)^{k(\eta_2)} \right].
\]

Similarly, we plug \(f = |\nabla v|^\frac{q}{2}, f = |\nabla w|^\frac{q}{2}\) and \(\eta = \eta_3\) into (2.1) and add them together to obtain

\[
\int_\Omega \left( |\nabla v|^\frac{q_3(q-2)}{q_3 - 1} + |\nabla w|^\frac{q_3(q-2)}{q_3 - 1} \right) \leq c_1 (\eta_3) \int_\Omega |\nabla v|^{\frac{q_3(q-2)}{q_3 - 1}} + |\nabla w|^{\frac{q_3(q-2)}{q_3 - 1}}^2
\]

\[
+ C_2 \left[ \left( \int_\Omega |\nabla v|^q \right)^{\eta_3} + \left( \int_\Omega |\nabla w|^q \right)^{\eta_3} \right]
\]

\[
+ C_3 (\eta_3)^{-h(\eta_3)} \left[ \left( \int_\Omega |\nabla v|^q \right)^{k(\eta_3)} + \left( \int_\Omega |\nabla w|^q \right)^{k(\eta_3)} \right].
\]
Where \( h(\eta) \), \( k(\eta) \), \( C_1(\eta) \), \( C_2 \), \( C_3(\eta) \) are given in (1.9) and (2.2). Now we are in a position to derive an differential inequality for \( E_{p,q} \). We first make use of inequalities from (4.20) to (4.23) into the right hand side of (4.18), (4.19), thereafter plug into the right hand side of (4.17) to obtain

\[
E'_{p,q}(t) \leq \zeta_1 \int_\Omega |\nabla u|_w^2 + \zeta_2 \left( \int_\Omega |\nabla(|\nabla v|_w^\frac{\eta}{2})|^2 + \int_\Omega |\nabla(|\nabla u|_w^\frac{\eta}{2})|^2 \right) + C_2 M \sum_{i=0}^{3} E^i_{p,q} + M \sum_{i=0}^{3} C_3(\eta_i) e^{-h(\eta_i)} E_{p,q}^{k(\eta_i)}
\]

\[
(4.24)
\]

Where

\[
\zeta_1 := (\alpha^2 + \beta^2)(n/4 + q - 2)eC_1(\eta_0) + p(\chi^2 + \xi^2)^{\frac{s_1-1}{s_1}} eC_1(\eta_1) - \frac{2(p-1)}{p^2}
\]

\[
\zeta_2 := eC_1(\eta_2)(\alpha^2 + \beta^2)(n/4 + q - 2) + eC_1(\eta_3) \frac{1}{s_1}(\chi^2 + \xi^2)p - \frac{q - 2}{q^2}
\]

\[
M := \max \left\{ p(\xi^2 + \chi^2), (n/4 + q - 2)(\alpha^2 + \beta^2) \right\}.
\]

Choosing a sufficiently small \( \epsilon > 0 \) such that \( \zeta_1, \zeta_2 < 0 \) yields

\[
E'_{p,q}(t) \leq m \sum_{i=0}^{3} E^i_{p,q} + \sum_{i=0}^{3} m_i E_{p,q}^{k(\eta_i)} + m_1 E_{p,q} + c
\]

\[
(4.25)
\]

where \( m := C_2 M, c = 2C \) and \( m_i := MC_3(\eta_i) e^{-h(\eta_i)} \) for all \( i = 0, 1, 2, 3 \). In addition, Theorem [1.1] implies that

\[
\lim_{t \to T_{\text{max}}} E_{p,q} = \infty,
\]

for all \( p > \frac{\theta}{q} \) and \( q > 1 \). By integrating from 0 to \( T_{\text{max}} \) of (4.25), we arrive at (1.8). In conclusion, the proof of Theorem 1.2 is completed.

**Proof of Corollary 1.2.1.** We just need to verify that there exist \( p, q, s_1, s_2 \) satisfying the following equation

\[
\eta_0 = \eta_1 = \eta_2 = \eta_3 \in (1, 1 + \frac{2}{n}).
\]

Equivalently,

\[
\frac{2s_1}{q} = \frac{2s_2}{p} = \frac{s_1}{s_1-1} = \frac{(q-2)s_2}{(s_2-1)q}
\]

Solving this equation we obtain

\[
\begin{align*}
q &= 2p > n \\
\frac{s_1}{s_1-1} &= \frac{(q-2)s_2}{(s_2-1)q} \\
\frac{s_1}{s_1-1} &= \frac{p+1}{s_2-1} \\
p &= \frac{4p^2}{2p^2-4p-4}.
\end{align*}
\]

Therefore, the following holds

\[
\left\{ \begin{array}{l}
\eta_i = \frac{p+1}{n+2-n\eta_i} = 2p-n+2, \\
n = n(n-2) \\
s = 2s_2 = n \quad \text{for all } i \in \{0, 1, 2, 3\}. \end{array} \right.
\]

Substituting (4.26) into (1.8) completes the proof.

**Proof of Corollary 1.2.2.** Similar to the proof of Corollary 1.2.1, we solve the following equation

\[
\eta_0 = \eta_1 = \eta_2 = \eta_3 = \frac{n}{n-1}.
\]

Which is equivalent to

\[
\frac{2s_1}{q} = \frac{2s_2}{p} = \frac{s_1}{s_1-1} = \frac{(q-2)s_2}{(s_2-1)q} = \frac{n}{n-1}.
\]

Hence, we obtain

\[
\left\{ \begin{array}{l}
q = 2p = 2(n-1) \\
s_1 = 2s_2 = n \\
\eta_i = \frac{n}{n+2-n\eta_i} = \frac{n}{n-2}.
\end{array} \right.
\]

We finish the proof by substituting (4.27) into (1.8).
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