Numerical modelling of the bearing capacity of a beam cross-section

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Abstract. The paper presents a procedure for numerical modelling of the rod cross-section bearing capacity. Equilibrium between cross sectional forces and cross-sectional stresses is determined by iterative procedures. According to the described procedure, the load-bearing capacity of the cross-section is determined according to the isotropic linear and nonlinear behavior of the material, for homogeneous and inhomogeneous cross-sections. The nonlinear behavior of the material reduces the stiffness of the cross section of the rod $EA$ and $EI$, with a significant increase in the deformation values $\varepsilon$ and $\kappa$. The applicability of the calculation and analysis of obtained results is presented using numerical examples.

1. Introduction

During exploitation, building structures should be constructed to safely receive and transfer the projected loads to the base. It is necessary to ensure that the stresses and deformations of the structure remain below the limit state of bearing capacity, limit state of usability and limit state of durability. The aim of the calculation, according to the limit state method, is to achieve an appropriate probability that the structure will remain reliable for use during exploitation [8]. The limit state can be reached by: loss of balance of the system or one structural element, transformation of the structure into a mechanism, failure and excessive deformation of the critical cross-section and failure due to material fatigue. It is necessary to determine the load-bearing capacity of the cross section of the rod and required coefficient of safety against fracture for the purpose of ensuring the limit state of the cross section fracture [2]. The load-bearing capacity of the cross section of the rod is the basis for the analysis of the material non-linearity of the system. It depends on the material behavior, the cross-sectional geometry and the ratio of the cross-sectional forces that are acting at the centroid of the cross section. The load-bearing capacity of the cross section is ensured if all its parts are within the permissible stresses and strains. The load-bearing capacity of the cross section of the rod is determined by the elastic and plastic area of behavior of the material. Some parts of the cross-section under the
action of cross-sectional forces can be located in the elastic area, while others can be located in the plastic area. In this case, it is the elastic-plastic bearing capacity of the rod cross section. The stress and strain area of the rod cross section depends on the adopted material behavior diagram, or more of them. The behavior of the material under load is described by a stress-strain relation diagram. The dependence of normal stress and longitudinal deformation $\sigma=\sigma(\varepsilon)$, and shear stress and shear deformation $\tau=\tau(\gamma)$ is complex. The stress-strain ratio depends on the type of material and the stresses. Figure 1 presents the behavior diagrams of materials in structures [7].

Figure 1. a) Behavior diagram of steel, b) stress-strain ratio of concrete under pressure, c) stress-strain tension ratio of concrete.

Figure 1.a presents a behavior diagram characteristic of homogeneous isotropic materials such as steel. At lower stresses, the stress-strain relation is linear. The stress becomes constant and independent of deformation after reaching the yield strength $f_y$. An increase of load makes the material to harden therefore the stress-strain relation is nonlinear. In the case of in-homogeneous isotropic materials such as concrete (Figure 1.b), initially at low compressive stress the relation of stress-strain is linear. The function of the $\sigma_c-\varepsilon_c$ ratio with an increase of compressive stress on concrete is nonlinear. In the case of concrete, in addition to the concrete class, the duration of the load is significantly affected by the stress-strain ratio [6]. Stress-strain behavior diagrams are conveniently idealized [1]. During idealizing material behavior diagrams, we should strive to be as simple as possible and as close as possible to the actual material behavior under load. The manner of idealizing the diagram depends on its application in certain areas of construction.

2. Analysis of load capacity of homogeneous cross section rod
The behavior of the material in the form of a bilinear diagram describes the actual behavior of the material. The real behavior of the material is approximated by an ideal elastic-plastic diagram of the material behavior in the bilinear form, Figure 2.d. In the diagram, we distinguish between a linear stress-strain ratio in the range of $0-\varepsilon_y$, and a linear relation in the range of $\varepsilon_y-\varepsilon_u$. The secant modulus of elasticity of the material $E_s$ is valid up to the yield strength of the material $\varepsilon_y$. After the material softening, the tangent modulus of the material $E_t$ occurs [5].
The cross section is divided into equal layers of rib panels and flanges, thickness $h_j$. Each layer corresponds to a deformation $\varepsilon_j$ and a normal stress $\sigma_j$. The stress of individual layers in the elastic or plastic area depends on the deformation. Due to the different behavior of the material in the tension and compressed zone, the centroid of the cross section shifts, therefore the problem must be solved iteratively. In the initial iterative step, the secant modulus of elasticity of the material $E_s$ is adopted for the selected combination of section forces at the centroid of the cross section. On the basis of this, the deformation and curvature of the cross section are calculated in the form:

$$
\varepsilon^{(0)} = \frac{N}{E_s A_i}, \quad \kappa^{(0)} = \frac{M}{E_s I_i}
$$

(1)

$E_s A_i$ - axial stiffness of the cross section of the rod; $E_s I_i$ - bending stiffness of the rod cross section. Deformations of layers in the initial iteration are determined by the expression:

$$
\varepsilon_j^{(0)} = \varepsilon + \kappa \cdot (y_i - y_j)
$$

(2)

The deformation of the $j$-th layer in the initial iteration corresponds to the stress in the first iteration, according to the stress diagram:

$$
\sigma_j^{(1)} = E_s \cdot \varepsilon_j^{(0)} \quad (0 \leq \varepsilon \leq \varepsilon_j)
$$

$$
\sigma_j^{(1)} = f_y + E_s \cdot (\varepsilon_j^{(0)} - \varepsilon_j) \quad (\varepsilon_j \leq \varepsilon \leq \varepsilon_u)
$$

(3)

An initial iterative step is sufficient if all the cross-sectional layers are in an elastic stress state. The iterative procedure continues in the case of plasticization of one or more layers. We determine the cross-sectional forces of the first iteration by integrating the stresses by layers:

$$
N^{(1)} = \sum_j \sigma_j^{(1)} \cdot A_j, \quad M_j^{(1)} = \sum_j \sigma_j^{(1)} \cdot A_i (y_i - y_j)
$$

(4)

In each iterative step, it is determined the stiffness of the cross section and the position of the new centroid of the cross-section. In the n-th iteration we get:
The increment of deformations and curvature of the cross-section of the shape corresponds to the new cross-sectional characteristics:

$$\Delta \varepsilon^{(n)} = \frac{\Delta N^{(n)}}{E^{(n)} A^{(n)}}, \quad \Delta \kappa^{(n)} = \frac{\Delta M^{(n)}}{E^{(n)} I^{(n)}}$$

(6)

the increment of normal force and bending moment is determined by expression (7):

$$\Delta N^{(n)} = N - N^{(n-1)}, \quad \Delta M_i^{(n)} = M_i - M_i^{(n-1)}$$

(7)

The deformation quantities in the n-th iterative step are:

$$\varepsilon^{(n)} = \varepsilon^{(n-1)} + \Delta \varepsilon^{(n)}, \quad \kappa^{(n)} = \kappa^{(n-1)} + \Delta \kappa^{(n)}$$

$$\varepsilon_j^{(n)} = \varepsilon_j^{(n)} + \kappa_j^{(n)} \cdot (y_j - y_j)$$

(8)

The stresses of the cross-sectional layers in the n+1 step are:

$$\sigma_j^{(n+1)} = E_y \cdot \varepsilon_j^{(n)} \quad (0 \leq \varepsilon \leq \varepsilon_y)$$

$$\sigma_j^{(n+1)} = f_y + E_y \cdot (\varepsilon_j^{(n)} - \varepsilon_y) \quad (\varepsilon_y \leq \varepsilon \leq \varepsilon_u)$$

(9)

The iterative procedure is repeated until the loading rate of forces and deformation becomes a small value.

3. Load capacity of inhomogeneous rod cross section

An analysis of nonlinear load-bearing capacity has been presented on the example of reinforced concrete section. In determining the relation of the M-k and N-ε cross-section, the following assumptions are adopted: a) the cross-sections remain straight and perpendicular to the axis of the girder before and after deformation; b) the concrete does not participate in the tensile of the load-bearing capacity; c) no slipping between concrete and reinforcement; the influence of transverse forces on bearing capacity is neglected; d) the participation of forks in bending capacity is neglected [3]. The cross section is loaded by the bending moment around the z axis and the normal force, which in interaction causes normal cross-sectional stresses. The concrete behavior diagram was adopted in the form of polynomials, while reinforcement is adopted as a bilinear stress-strain dependence, without taking into account the material safety coefficients (Figure 3.e, and 3.f). Normal stresses occur at each point of the section, depending on the deformation of the section when the combination of bending moment and normal force is combined. The solution of the cross-section load-bearing capacity problem is sought iteratively for the adopted load increment. The deformations of the section in the zero iterative step are described by the deformation $\varepsilon^{(0)}$ and the curve $\kappa^{(0)}$ which are determined at the centroid of the cross-section according to the expressions:

$$E^{(n)} A^{(n)} = \sum_j E_j^{(n)} \cdot A_j$$

$$E^{(n)} I_j^{(n)} = \sum_j E_j^{(n)} \cdot A_j (y_j - y_j)^2$$

$$\sum_j E_j^{(n)} \cdot A_j \cdot y_j$$

$$E^{(n)} A^{(n)}$$

(5)
\[ \varepsilon^{(0)} = \frac{N}{EA_y}, \quad \kappa^{(0)} = \frac{M_i}{EI_i} \]  \hspace{1cm} (10)

\( EA_y \) - axial stiffness of the rod cross section; \( EI_i \) - bending stiffness of the rod cross section. The described cross-sectional stiffness are determined on the basis of the initial modulus of elasticity of the material and the surface area of the entire cross-section of concrete and reinforcement. The stiffness of the cross section of the rod changes and the deformation size of the rod increases during use of nonlinear stress-strain ratios. In the \( n \)-th iterative step, the increment \( \Delta \varepsilon \) and \( \Delta \kappa \) is in the form of:

\[ \Delta \varepsilon^{(n)} = \frac{\Delta N^{(n)}}{E_y^{(n)} \cdot A_y^{(n)}}, \quad \Delta \kappa^{(n)} = \frac{\Delta M_i^{(n)}}{E_i^{(n)} \cdot I_i^{(n)}} \]  \hspace{1cm} (11)

\( E_y^{(n)} \cdot A_y^{(n)} \) - reduced axial stiffness of the rod cross section in the \( n \)-th iterative step; \( E_i^{(n)} \cdot I_i^{(n)} \) - reduced bending stiffness of the rod in the \( n \)-th iterative step. The increment of cross-sectional forces and bending moments is determined by the expression:

\[ \Delta N^{(n)} = N - N^{(n-1)}, \quad \Delta M_i^{(n)} = M_i - M_i^{(n-1)} \]  \hspace{1cm} (12)

The deformation sizes of the rod are in the form of:

\[ \varepsilon^{(n)} = \varepsilon^{(n-1)} + \Delta \varepsilon^{(n)}, \quad \kappa^{(n)} = \kappa^{(n-1)} + \Delta \kappa^{(n)} \]  \hspace{1cm} (13)

In order to calculate the ideal surface \( A_y \) and the ideal moment of inertia \( I_i \) of the cross section, the pressed part of the concrete and the longitudinal reinforcement are taken. The tangent modulus \( E_t \) of the material changes along the height of the cross section depending on \( \sigma \) and \( \varepsilon \) when the stress changes. In this case, the strain quantities \( \varepsilon \) and \( \kappa \) in expression (13) are defined through the functional relations of stresses and strains. The calculation of deformation quantities with complex functions can be simplified by dividing the cross section into layers. Each \( j \)-th cross-sectional layer at its centroid of the cross section corresponds to the stress, strain and tangent modulus of the material.

**Figure 3.** a) cross section; b) an elementary part of a rod length \( dx \); c) distribution of deformations; d) stresses and cross-sectional forces, e) relation diagram of \( \sigma_t \)-\( \varepsilon_t \) reinforcement; f) diagram of the relation of \( \sigma_c \)-\( \varepsilon_c \) concrete.
Static and geometric values of the cross section of the rod are determined in relation to the position of the ideal of the centroid cross section \( T_i \). The cross-sectional stiffness of the rod \( E_A \) and \( E_I \) is determined by the layers in relation to the ideal centroid of the cross-section according to the expressions:

\[
E_A = \sum_{j} E_{st,j} \cdot A_j + E_{st} \cdot A_1 + E_{nt} \cdot A_2 \\
E_I = \sum_{j} E_{st,j} \cdot A_j \left( y_j - y_i \right)^2 + E_{st} \cdot A_1 \left( y_i - y_{is} \right)^2 + E_{nt} \cdot A_2 \left( y_{i2} - y_i \right)^2
\]

(14)

On the basis of the deformation quantities \( \varepsilon \) and \( \kappa \), the distribution of deformations along the height of the cross section is in the form of:

\[
\varepsilon_{ij} = \varepsilon_{ci} + \kappa \cdot \left( y_i - y_j \right), \quad \varepsilon_s = \varepsilon_{ci} + \kappa \cdot \left( y_i - y_s \right)
\]

(15)

Sectional plasticization is based on one-axial material properties. The deformations of the cross-sectional layers from the stress-strain relation diagram determines the stress \( \sigma_{cj} \) of the concrete layers and \( \sigma_s \) of the reinforcement. On the basis of the obtained stress of concrete and reinforcement, the forces \( F_{cj} \) of concrete layers and \( F_s \) reinforcement are determined. The bending moments \( M_i \) and the force \( N \), which are not in equilibrium with the given cross-sectional forces, are obtained from the equilibrium of all forces in the direction of the x-axis and the sum of all moments around the z-axis in the centroid \( T_i \) of the cross section. Due to the increase in stress of the concrete layers and reinforcement, there is a change in the static and geometric characteristics of the cross section. In the next iterative step for the new position of the centroid cross section and static characteristics of the cross section, the increment of the deformation quantities \( \Delta \varepsilon \) and \( \Delta \kappa \) is determined, and the distribution of deformations along the height of the cross section. The deformation of loading rate is determined according to expression (11) due to the difference between the given cross-sectional forces and the forces in the \( (n-1) \) iterative step. The course of the iterative process of balancing forces and cross-sectional deformations is shown in Figure 4 [3].

![Figure 4](image-url)

**Figure 4.** a) The course of the iterative procedure of the M-\( \kappa \) relation; b) the course of the iterative procedure of N-\( \varepsilon \) relation.

### 4. Numerical examples

#### 4.1. Example 1

In the first numerical example, the bearing capacity of a homogeneous cross-section of a HEA profile made of S 235 JR steel \( (f_y=235 \text{ MPa}, F_{eu}=360 \text{ MPa}) \) is analysed. Load capacity is determined by the
interaction of bending moments and normal forces for different N/N\textsubscript{pl} ratios. Reference values of bending moments at the yield strength M\textsubscript{y}, plasticity moments M\textsubscript{pl} with and without the action of normal force according to expressions (16) are used [2, 4] for the purpose of comparing the results:

\[ M_y = f_y \cdot W_{el}, \quad M_{pl} = f_y \cdot W_{pl} \]
\[ M_y = \left( f_y - \frac{N}{A} \right) W_{el}, \quad M_{pl,N} = 1.2 \cdot M_{pl} \left( 1 - \frac{N}{N_{pl}} \right), \quad N_{pl} = f_y \cdot A \]

\[ N = 0 \]
\[ N \neq 0 \]
\[ 0.15N_{pl} \leq N \leq N_{pl} \]  

(16)

![Figure 5. Bending-curvature ratio of HEA 300 profile.](image)

The calculation results are given in Table 1.

**Table 1. Results of the calculation of the cross-sectional forces of the HEA profile.**

| Bar     | N/N\textsubscript{pl} | A [cm\textsuperscript{2}] | W\textsubscript{el} [cm\textsuperscript{3}] | W\textsubscript{pl} [cm\textsuperscript{3}] | N\textsubscript{pl} [kN] | M\textsubscript{y} [kNm] | M\textsubscript{el} [kNm] | M\textsubscript{pl} [kNm] | M\textsubscript{pl} [kNm] |
|---------|------------------------|-----------------------------|---------------------------------------------|---------------------------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| HEA 200 | 0                      | 53.8                        | 389                                         | 430                                         | 1264.30                  | 91.42                    | 101.05                   | 89.03                    | 142.80                   |
|         | 0.2                    | 73.13                       | 97.01                                       | 70.20                                       | 129.05                   | 73.13                    | 97.01                    | 70.20                    | 129.05                   |
|         | 0.4                    | 54.85                       | 72.76                                       | 51.50                                       | 91.30                    | 54.85                    | 72.76                    | 51.50                    | 91.30                    |
|         | 0.6                    | 36.57                       | 48.50                                       | 32.70                                       | 78.81                    | 36.57                    | 48.50                    | 32.70                    | 78.81                    |
|         | 0.8                    | 18.28                       | 24.25                                       | 13.96                                       | 17.35                    | 18.28                    | 24.25                    | 13.96                    | 17.35                    |
|         | 1                      | 0.00                        | 0.00                                        | 0.00                                        | 4.70                     | 0.00                     | 0.00                     | 0.00                     | 4.70                     |
| HEA 300 | 0                      | 296.10                      | 325.24                                      | 286.70                                      | 458.00                   | 296.10                   | 325.24                   | 286.70                   | 458.00                   |
|         | 0.2                    | 236.88                      | 312.23                                      | 225.70                                      | 413.60                   | 236.88                   | 312.23                   | 225.70                   | 413.60                   |
|         | 0.4                    | 177.66                      | 234.17                                      | 164.80                                      | 296.00                   | 177.66                   | 234.17                   | 164.80                   | 296.00                   |
|         | 0.6                    | 118.44                      | 156.12                                      | 103.70                                      | 250.80                   | 118.44                   | 156.12                   | 103.70                   | 250.80                   |
|         | 0.8                    | 59.22                       | 78.06                                       | 42.80                                       | 52.80                    | 59.22                    | 78.06                    | 42.80                    | 52.80                    |
|         | 1                      | 0.00                        | 0.00                                        | 0.00                                        | 18.00                    | 0.00                     | 0.00                     | 0.00                     | 18.00                    |
| HEA 400 | 0                      | 542.85                      | 601.60                                      | 533.45                                      | 859.60                   | 542.85                   | 601.60                   | 533.45                   | 859.60                   |
|         | 0.2                    | 434.28                      | 577.54                                      | 422.35                                      | 780.50                   | 434.28                   | 577.54                   | 422.35                   | 780.50                   |
|         | 0.4                    | 325.71                      | 433.15                                      | 311.30                                      | 601.00                   | 325.71                   | 433.15                   | 311.30                   | 601.00                   |
|         | 0.6                    | 217.14                      | 288.77                                      | 200.20                                      | 477.50                   | 217.14                   | 288.77                   | 200.20                   | 477.50                   |
|         | 0.8                    | 108.57                      | 144.38                                      | 89.10                                       | 95.00                    | 108.57                   | 144.38                   | 89.10                    | 95.00                    |
|         | 1                      | 0.00                        | 0.00                                        | 0.00                                        | 25.00                    | 0.00                     | 0.00                     | 0.00                     | 25.00                    |
4.2. Example 2
In the second example, the load-bearing capacity of the reinforced concrete section is determined. The cross section is made of concrete class C25/30 (E\textsubscript{c} = 3 x 10\textsuperscript{7} kN/m\textsuperscript{2}), reinforced with bars 4 Φ16 B 500 B (E\textsubscript{s} = 2 x 10\textsuperscript{8} kN/m\textsuperscript{2}). The stress-strain relation for reinforcement was adopted in the form of a bilinear diagram (f\textsubscript{y} = 500 MPa, F\textsubscript{u} = 550 MPa), and concrete in the form of a third degree polynomial (f\textsubscript{ck} = 25 MPa). Cross-section failure occurs when a stress limit occurs in a single concrete cross-sectional layer or in reinforcement. The stiffness of the cross section of the bending rod is minimal at the fracture boundary of the cross section, Figure 6.b. The load-bearing capacity of the cross-section is defined by the bending-curvature ratio, Figure 6.a. The change in static and geometric cross-sectional sizes of the rod is shown in Tables 2 and 3.

### Table 2. Results of the change in cross-sectional stiffness depending on M and N = 0 kN.

| Beam (N =0 kN) | M [kNm] | 0   | 50  | 100 | 150 | 175 | 190 |
|----------------|---------|-----|-----|-----|-----|-----|-----|
| EI[kNm\textsuperscript{2}] | 84813.17 | 23146.31 | 22932.17 | 22678.11 | 2324.28 | 1376.95 |
| κ [1/m] | 0       | 0.002186 | 0.004434 | 0.006771 | 0.016738 | 0.062313 |

### Table 3. Results of the change in cross-sectional stiffness as a function of M and N=100 kN.

| Beam (N =100 kN) | M [kNm] | 0   | 50  | 100 | 150 | 160 | 170 |
|----------------|---------|-----|-----|-----|-----|-----|-----|
| EI[kNm\textsuperscript{2}] | 84813.17 | 23284.50 | 23086.35 | 22730.36 | 2359.62 | 1664.49 |
| κ [1/m] | 0       | 0.001516 | 0.003739 | 0.006077 | 0.012655 | 0.051915 |

**Figure 6.** a) Bending moment-curvature ratio of cross-section b/h = 25/50 cm, b) change of stiffness of EI concrete cross-section.

5. Conclusion
It is shown in the paper that the bearing capacity of the cross section can be determined numerically, with linear and nonlinear behavior of the material. The bending stiffness up to the yield strength of the material f\textsubscript{y} is constant in the rod made of a homogeneous material. The cross-sectional stiffness decreases after reaching the yield point. During a further increase in bending moments and cross-sectional forces, the curvature of the cross-section has a multi-stage increment until the section break. The action of normal force on the cross section reduces the bending moment at the yield point of the material. Analysis of the inhomogeneous reinforced concrete cross-section concluded that after the
appearance of the first cracks in the tension zone, the stiffness of the EI cross-section decreases significantly. Thereafter, the cross-sectional stiffness does not change significantly, until the stress in the reinforcement reaches the yield strength $f_y$. There is a decrease in stiffness and an increase in the curvature of the cross section after the yield strength. Analysis of the load-bearing capacity of the cross-section under the nonlinear behavior of the material showed that the cross-sections have a significant plastic reserve in the load-bearing capacity. The consequence of the cross-sectional stress in the nonlinear range of material behavior are deformations of the cross-sections of the rods and the bearing system.

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