Inverse diffusion theory of photoacoustics

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Abstract

This paper analyzes the reconstruction of diffusion and absorption parameters in an elliptic equation from knowledge of internal data. In the application of photoacoustics, the internal data are the amount of thermal energy deposited by high frequency radiation propagating inside a domain of interest. These data are obtained by solving an inverse wave equation, which is well studied in the literature. We show that knowledge of two internal data based on well-chosen boundary conditions uniquely determines two constitutive parameters in diffusion and Schrödinger equations. Stability of the reconstruction is guaranteed under additional geometric constraints of strict convexity. No geometric constraints are necessary when \(2n\) internal data for well-chosen boundary conditions are available, where \(n\) is spatial dimension. The set of well-chosen boundary conditions is characterized in terms of appropriate complex geometrical optics solutions.

1. Introduction

Photoacoustic tomography (PAT) is a recent hybrid medical imaging modality that combines the high resolution of acoustic waves with the large contrast of optical waves. When a body is exposed to short pulse radiation, typically emitted in the near-infrared region in PAT, it absorbs energy and expands thermo-elastically by a very small amount; this is the photoacoustic effect. Such an expansion is sufficient to emit acoustic pulses, which travel back to the boundary of the domain of interest where they are measured.

A first step in PAT is therefore to reconstruct the amount of deposited energy from the time-dependent boundary measurement of acoustic signals. Acoustic signals propagate in fairly homogeneous domains as the sound speed varies very little from one tissue to the next. The reconstruction of the amount of deposited energy is therefore quite accurate in many practical settings. For references on the practical and theoretical aspects of PAT, we refer the reader to e.g. [1, 5–7, 9–11, 13–16, 18, 20, 24, 25].

Once the amount of deposited energy has been reconstructed, a second step, sometimes called *quantitative photoacoustics*, consists of inferring the optical properties in the body.
Although this second step is less studied mathematically, it has received significant attention in the biomedical literature; see e.g. [6, 7, 19, 26] and their references. The reconstruction of the optical parameters is very useful in practice because the absorption properties of healthy and unhealthy tissues exhibit large contrast [7, 24]. The combination of high spatial resolution of the acoustic signals and large contrast of the optical parameters allows us e.g. to image blood vessels at high resolution, which is in turn important for assessing the status of cancerous tissues.

Near-infrared light is best modeled by radiative transfer equations. What we can reconstruct on the optical coefficients, seen as constitutive parameters in the radiative transfer equation, from measurements of heat deposition for all possible illuminations of the domain has recently been analyzed in [4]. The latter paper considers the continuous-illumination setting. In other words, heat deposition is measured for all possible incoming radiation condition, which can be controlled at the boundary of the domain.

In this paper, we consider the diffusion approximation to radiative transfer, which is accurate for radiation propagation in highly scattering media. Such an approximation is typically valid for propagations of radiation over 1 cm or more [2, 3]. In such a simplified setting, the unknown optical parameters are the spatially varying diffusion and absorption coefficients. Provided that the diffusion coefficient is known at the domain’s boundary, we show that these two coefficients are uniquely determined by two well-chosen illuminations at the boundary of the domain. An explicit reconstruction procedure is presented, which involves solving a first-order equation with a vector field explicitly constructed from the internal data (the amount of deposited energy).

The stability of the reconstruction of the optical coefficients from two internal data is established under geometric conditions of strict convexity on the domain of interest. In the presence of 2n well-chosen boundary conditions, where n is spatial dimension, such geometric constraints can be removed and constructions are possible on essentially arbitrary domains of interest with a well-defined boundary.

Mathematically, the inverse problem is an inverse diffusion problem with two internal measurements. By using the standard Liouville transformation, the diffusion equation is replaced by a Schrödinger equation with an unknown potential and with internal measurements involving a second unknown function. By adapting the theory of complex geometrical optics (CGO) solutions to this setting, we are able to obtain uniqueness and stability results for the inverse Schrödinger problem. By means of the inverse Liouville transformation, we are able to conclude on the uniqueness and stability of the reconstruction of the optical parameters for the PAT problem in the diffusive regime.

The rest of the paper is structured as follows. Section 2 presents the PAT problem and the main results we obtain in this paper. The inverse Schrödinger problem and the explicit reconstruction algorithms are addressed in section 3. How the latter results are used to solve the inverse diffusion problem is explained in section 4.

### 2. Photoacoustic tomography and main results

The propagation of radiation in scattering media is modeled by the following diffusion equation:

\[
\begin{align*}
\frac{1}{c} \frac{\partial u}{\partial t} - \nabla \cdot D(x) \nabla u + \sigma_a(x)u &= 0, & x \in X \subset \mathbb{R}^n, & t \geq 0 \\
\quad u &= g & x \in \partial X, & t \geq 0
\end{align*}
\]
where $X$ is an open, bounded, connected domain in $\mathbb{R}^n$ with a $C^1$ boundary $\partial X$ (embedded in $\mathbb{R}^n$); $n$ is the spatial dimension; $c$ is the light speed in tissues; $D(x)$ is a (scalar) diffusion coefficient and $\sigma_a(x)$ is an absorption coefficient. Throughout the paper, we assume that $D(x)$ and $\sigma_a(x)$ are bounded from above and below by (strictly) positive constants. The source of incoming radiation is prescribed by $g(t, x)$ on the boundary $\partial X$ and is assumed to be a very short pulse supported on an interval of time $(0, \eta)$ with $c\eta$ of order $O(1)$. The amount of energy deposited is proportional to absorption and is given by

$$H(t, x) = \sigma_a(x) u(t, x).$$

A thermal expansion (assumed to be proportional to $H$) results and acoustic waves are emitted. Such waves are modeled by

$$\frac{1}{c_s^2} \frac{\partial^2 p}{\partial t^2} - \Delta p = \beta \frac{\partial}{\partial t} H(t, x),$$

with $c_s$ the sound speed and $\beta$ a coupling coefficient assumed to be constant and known. The acoustic pressure $p(t, x)$ is then measured on $\partial X$ as a function of time.

Since light travels much faster than sound with $c_s \ll c$, we may assume for short light pulses that radiation propagation occurs over a very short time at the scale of the acoustic waves. This justifies the simplification (see [4] for additional details):

$$H(t, x) \sim H_0(x) \delta_0(t), \quad H_0(x) = \sigma_a(x) \int_{\mathbb{R}^+} u(t, x) \, dt.$$

We now have a well-posed inverse wave problem, where measurement of an $n$-dimensional information $p(t, x)$ for $t > 0$ and $x \in \partial X$ allows us to reconstruct the $n$-dimensional spatial map $H_0(x)$. We assume this step, which has been analyzed in great detail in the literature, completed; see e.g. [8, 10, 15–17, 20, 25] and references cited there.

Denoting by $g(x) = \int_{\mathbb{R}^+} g(t, x) \, dt$ and $u(x) = \int_{\mathbb{R}^+} u(t, x) \, dt$, we thus observe that the photoacoustic problem in the diffusive regime amounts to reconstructing $(D(x), \sigma_a(x))$ from knowledge of

$$H_j(x) = \sigma_a(x) u_j(x), \quad 1 \leq j \leq J,$$

for $J \in \mathbb{N}^*$ illumination maps $g_j(x)$, where $u_j$ is the solution to the steady-state equation

$$-\nabla \cdot D(x) \nabla u_j + \sigma_a(x) u_j = 0, \quad x \in X \subset \mathbb{R}^n, \quad u_j = g_j \quad x \in \partial X.\quad (4)$$

The set of internal data is given by

$$d = (d_j)_{1 \leq j \leq J}, \quad d_j(x) = \sigma_a(x) u_j(x).$$

Throughout the paper, we assume that $D$ is known at the domain’s boundary $\partial X$.

The main problem of interest in this paper is the uniqueness and the stability of the reconstruction for the inverse diffusion problem with internal data (IDID).

**IDID:** reconstruction of $(D(x), \sigma_a(x))$ from knowledge of $d = (d_j)_{1 \leq j \leq J}$ on $X$ for a fixed collection of illuminations $g = (g_j)_{1 \leq j \leq J}$ prescribed on $\partial X$.

We start with the case of two real-valued measurements $J = 2$. The reconstructions are based on the construction of vector fields that are well defined only when the optical coefficients are sufficiently smooth. More precisely, let $k \geq 1$ and assume that

$$\sqrt{D} \in Y = H^{2+k+2\varepsilon}(X) \subset C^{k+2}(\overline{X}), \quad \sigma_a \in C^{k+1}(\overline{X}), \quad \varepsilon > 0.$$

Let then $k \in \mathbb{R}^n$ be a constant vector with $|k|$ sufficiently large.
Let \( g = (g_1, g_2) \) be a given illumination and \( d = d_1 + id_2 = \sigma_d(x)(u_1 + iu_2) \) the corresponding internal data, where the real-valued solutions \( u_j \) solve (4) with boundary conditions \( g_j \) for \( j = 1, 2 \). We introduce the vector field and scalar quantity

\[
\beta = \frac{e^{-2kx}}{2|k|} \Im(d \nabla \bar{d} - \bar{d} \nabla d), \quad \gamma = \frac{e^{-2kx}}{|k|} \Im(d \Delta d - d \Delta \bar{d}).
\]

Here, \( \bar{d} = d_1 - id_2 \) is the complex conjugate of \( d \). Then we will see in section 3 that for \( \mu = D^{-\frac{1}{2}} \sigma_d \), we have

\[
\beta \cdot \nabla \mu + \gamma \mu = 0.
\]

Since \( \mu \) is known on \( \partial X \) as \( D \) and \( \sigma_d = d_j/g_j \) are known on \( \partial X \), the above equation is a well-posed equation for \( \mu \). When the integral curves of the vector field \( \beta \) map any point inside \( x \) to a point \( x_0(x) \) \( \in \partial X \), equation (7) uniquely characterizes \( \mu \) inside the domain.

The internal data satisfy that \( d_1 = \sigma_d u_1 = \mu v \) for \( v = \sqrt{D} u_1 \), which is now known since \( \mu \) is known. Define then \( q = -v^{-1} \Delta v \). Then it turns out (see section 4) that

\[
-\Delta \sqrt{D} - q \sqrt{D} = \mu, \quad x \in X.
\]

This provides an elliptic equation for \( \sqrt{D} \), which admits a unique solution since \( \sqrt{D} \) is known on \( \partial X \). This provides a constructive algorithm to reconstruct \( (D, \sigma_d) \) from knowledge of \( d_j \), \( j = 1, 2 \).

The principal difficulty is to ensure that \( \beta \) is constructed in such a way that (7) allows for a unique solution \( \mu \). We will show that there exists an open set of boundary conditions \( g \) on \( \partial X \) that allows us to do so. For each fixed \( M > 0 \), we define the set of coefficients \( (D, \sigma_d) \in M \) as

\[
M = \{(D, \sigma_d) \in Y \times C^{k+1}(\bar{X}), \|\sqrt{D}\|_Y + \|\sigma_d\|_{C^{k+1}(\bar{X})} \leq M\}.
\]

The main results for the IDID are then as follows. We start with a uniqueness result.

**Theorem 2.1.** Let \( X \) be an open, bounded, domain with a \( C^2 \) boundary \( \partial X \). Assume that \( (D, \sigma_d) \) and \( (D, \tilde{\sigma}_d) \) are in \( M \) with \( D_{\partial X} = \tilde{D}_{\partial X} \). Let \( d \) and \( \tilde{d} \) be the internal data in (5) for the coefficients \( (D, \sigma_d) \) and \( (D, \tilde{\sigma}_d) \), respectively, and with boundary conditions \( (g_j)_{j=1,2} \). Then there is an open set of illuminations \( g \in (C^{k,\alpha}(\partial X))^2 \) for some \( \alpha > \frac{1}{2} \) such that if \( d = \tilde{d} \), then \( (D, \sigma_d) = (D, \tilde{\sigma}_d) \).

The above result shows uniqueness of the reconstruction of the optical coefficients but does not imply stability. Without additional geometric information about \( \partial X \), the above procedure may yield unstable reconstructions. However, provided that \( 2n \) well-chosen measurements \( d_j \) for \( 1 \leq j \leq J = 2n \) are available, \( n \) vector fields similar to \( \beta \) above may be constructed in order to form locally a basis of vectors in \( \mathbb{R}^n \). In such a setting, the following stability result holds.

**Theorem 2.2.** Let \( k \geq 2 \) and let \( X \) be an arbitrary bounded domain with a boundary \( \partial X \) of class \( C^{k+1} \). Assume that \( (D, \sigma_d) \) and \( (D, \tilde{\sigma}_d) \) are in \( M \) with \( D_{\partial X} = \tilde{D}_{\partial X} \). Let \( d = (d_1, \ldots, d_{2n}) \) and \( \tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_{2n}) \) be the internal data constructed in (5) for the coefficients \( (D, \sigma_d) \) and \( (D, \tilde{\sigma}_d) \), respectively, and with boundary conditions \( g = (g_j)_{1 \leq j \leq 2n} \). Then there is an open set of illuminations \( g \in (C^{k,\alpha}(\partial X))^2 \) and a constant \( C \) such that

\[
\|D - \tilde{D}\|_{C^k(X)} + \|\sigma_d - \tilde{\sigma}_d\|_{C^k(X)} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\partial X))^{2n}}.
\]

Stability is therefore ensured without geometric constraints provided that enough measurements are available. When only two measurements are available and are of the
form described in theorem 2.1, we can still get a stability result under the following geometric hypothesis.

**Hypothesis 2.3.** There exists \( R < \infty \) such that for each \( x_0 \in \partial X \), which we assume is of class \( C^2 \), we have \( X \subset B_{x_0}(R) \) where \( B_{x_0}(R) \) is a ball of radius \( R \) that is tangent to \( \partial X \) at \( x_0 \in \partial X \).

Then we can show the following result.

**Theorem 2.4.** Let \( k \geq 3 \). Let \( X \) satisfy hypothesis 2.3 with a boundary \( \partial X \) of class \( C^{k+1} \). Assume that \((D, \sigma_a)\) and \((\tilde{D}, \tilde{\sigma}_a)\) are in \( \mathcal{M} \) with \( D|_{\partial X} = \tilde{D}|_{\partial X} \). Let \( d \) and \( \tilde{d} \) be internal data as above for the coefficients \((D, \sigma_a)\) and \((\tilde{D}, \tilde{\sigma}_a)\), respectively, and with boundary conditions \( g = (g_j)_{j=1,2} \). Then there is an open set of illuminations \( g \in (C^{k,\alpha}(\partial X))^2 \) and a constant \( C \) such that

\[
\|D - \tilde{D}\|_{C^{k-1}(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(X)} \leq C\|d - \tilde{d}\|_{(C^k(X))^2}.
\]

The above three theorems are proved in the following two sections. They show that the internal data \( d \) for illuminations \( g \) with \( J \geq 2 \) provide stable reconstructions of the optical coefficients under the assumption that the illuminations are well chosen. How these illuminations are chosen will become more explicit in the next two sections. The characterization of the open set of illuminations is however not very explicit. The main features of the result are as follows. For coefficients in \( \mathcal{M} \), there is a minimal value of \( |k| \) that ensures that we can construct a vector field \( \beta \) with property \( P \): the integral curves of \( \beta \) map any point in \( X \) to a point in \( \partial X \). Such a vector field is constructed by means of CGO solutions, which depend on the unknown optical parameters. The illuminations must then be chosen sufficiently close to the trace on \( \partial X \) of the above CGO solutions to ensure that they generate another vector field with property \( P \). Closedness is therefore not characterized in a very explicit manner. It remains an interesting question to obtain a priori constraints on the illumination that will ensure that the resulting vector field satisfies property \( P \) globally on the whole domain \( X \) or more locally on part of the domain.

The theory developed in this paper focuses on Dirichlet conditions for the diffusion equation, which is an accurate model to describe photon propagation in highly scattering media. More general boundary conditions such as Neumann and Robin boundary conditions could be handled by the theory with minor modifications. Our results require sufficient smoothness on the optical coefficients. How such regularity hypotheses may be weakened is currently investigated.

In the next section, we consider a similar inverse problem for the Schrödinger equation, which is derived from the diffusion equation by a Liouville transformation. The results presented above in this section are then the consequences of the results obtained for the Schrödinger equation using the inverse Liouville transformation. Their derivation is described in section 4.

### 3. Inverse Schrödinger with internal data

Let \( X \) be an open, bounded, connected, domain in \( \mathbb{R}^n \), where \( n \) is the spatial dimension, with a smooth boundary \( \partial X \). We consider the Schrödinger equations

\[
\begin{align*}
\Delta u_j + qu_j &= 0 & X
u_j &= g_j & \partial X,
\end{align*}
\]

(11)
for $1 \leq j \leq J$. Here, $J \in \mathbb{N}^*$ is the number of illuminations and $q$ is an unknown potential. We assume that the homogeneous problem with $g_j = 0$ admits the unique solution $u \equiv 0$ so that $\lambda = 0$ is not in the spectrum of $\Delta + q$. We assume that $q$ on $X$ is the restriction to $X$ of a function $\tilde{q}$ compactly supported on $\mathbb{R}^n$ and such that $\tilde{q} \in H^2_{\text{loc}}(\mathbb{R}^n)$ with $\varepsilon > 0$ for $k \geq 1$. Moreover we assume that the extension is chosen so that

$$
\|q\|_{\mathcal{H}^2_{\text{loc}}(X)} \leq C(X, k, n)\|\tilde{q}\|_{H^2_{\text{loc}}(\mathbb{R}^n)},
$$

(12)

for some constant $C(X, k, n)$ independent of $q$. That such a constant exists may be found e.g. in [21, chapter VI, theorem 5].

We assume that $g_j \in C^k,\alpha(X)$ with $\alpha > \frac{1}{2}$ and $\partial X$ is of class $C^{k+1}$ so that (11) admits a unique solution $u_j \in C^{k+1}(X)$ [12, theorem 6.19]. The internal data are of the form

$$
d_j(x) = \mu(x)u_j(x), \quad X, \quad 1 \leq j \leq J.
$$

(13)

Here $\mu \in C^{k+1}(\tilde{X})$ verifies $0 < \mu_0 \leq \mu(x) \leq \mu_0^{-1}$ for a.e. $x \in X$.

The inverse Schrödinger problem with internal data (ISID) consists of reconstructing $(q, \mu)$ in $X$ from knowledge of $d = (d_1, \ldots, d_J) \in (C^{k+1}(X))^J$ for a given illumination $g = (g_j)_{1 \leq j \leq J}$. We will mostly be concerned with the case $J = 2$ and $J = 2n$ with $g_j$, and hence $d_j$ real-valued measurements.

### 3.1. Complex geometrical optics solutions

The analysis of ISID carried out in this paper is based on the construction of CGO solutions [22, 23]. When $q = 0$, CGOs are harmonic solutions of the form $e^{\rho \cdot x}$ for $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$. For $\rho = \rho_0 + i\rho_0$ with $\rho_0$ and $\rho_0$ vectors in $\mathbb{R}^n$, this means that $|\rho_0|^2 = |\rho_0|^2$ and $\rho_0 \cdot \rho_0 = 0$. When $q \neq 0$, CGOs are solutions of the following problem:

$$
\Delta u_\rho + qu_\rho = 0, \quad u_\rho \sim e^{\rho \cdot x} \text{ as } |x| \to \infty.
$$

(14)

More precisely, we say that $u_\rho$ is a solution of the above equation with $\rho \cdot \rho = 0$ and the proper behavior at infinity when it is written as

$$
u_\rho(x) = e^{\rho \cdot x}(1 + \psi_\rho(x)),
$$

(15)

for $\psi_\rho \in L^2_\delta$ a weak solution of

$$
\Delta \psi_\rho + 2\rho \cdot \nabla \psi_\rho = -q(1 + \psi_\rho).
$$

(16)

The space $L^2_{\delta}$ for $\delta \in \mathbb{R}$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{L^2_{\delta}}$ defined as

$$
\|u\|_{L^2_{\delta}} = \left(\int_{\mathbb{R}^n} (\langle x \rangle^2 |u|^2 \, dx) \right)^{\frac{1}{2}}, \quad \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.
$$

(17)

Let $-1 < \delta < 0$ and $q \in L^2_{\delta+1}$ and $\langle x \rangle q \in L^\infty$. One of the main results in [22] is that there exists $\eta = \eta(\delta)$ such that the above problem admits a unique solution with $\psi \in L^2_\delta$ provided that

$$
\|\langle x \rangle q\|_{L^\infty} + 1 \leq \eta|\rho|.
$$

Moreover, $\|\psi\|_{L^2_{\delta}} \leq C|\rho|^{-1}\|q\|_{L^2_{\delta+1}}$ for some $C = C(\delta)$. In the analysis of ISID, we need smoother CGOs than what was recalled above. We introduce the spaces $H^s_\delta$ for $s \geq 0$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{H^s_\delta}$ defined as

$$
\|u\|_{H^s_\delta} = \left(\int_{\mathbb{R}^n} (\langle x \rangle^2 \langle I - \Delta \rangle^s |u|^2 \, dx) \right)^{\frac{1}{2}}.
$$

(18)
Here \( (I - \Delta)^{\frac{1}{2}} u \) is defined as the inverse Fourier transform of \( (\xi)^{\frac{1}{2}} \hat{u}(\xi) \), where \( \hat{u}(\xi) \) is the Fourier transform of \( u(x) \). Then we have the following.

**Proposition 3.1.** Let \(-1 < \delta < 0 \) and \( k \in \mathbb{N}^* \). Let \( q \in H_{1+\delta}^{\frac{1}{2}+k+\epsilon} \) and hence in \( H_{s+1}^{\frac{1}{2}+k+\epsilon} \) and \( \rho \) be such that

\[
\|q\|_{H_{s+1}^{\frac{1}{2}+k+\epsilon}} + 1 \leq \eta|\rho|.
\]

Then \( \psi_{\rho} \) the unique solution to (16) belongs to \( H_{s+1}^{\frac{1}{2}+k+\epsilon} \) and

\[
|\rho|\|\psi_{\rho}\|_{H_{s+1}^{\frac{1}{2}+k+\epsilon}} \leq C\|q\|_{H_{s+1}^{\frac{1}{2}+k+\epsilon}},
\]

for a constant \( C \) that depends on \( \delta \) and \( \eta \).

**Proof.** We recall [22] that for \( |\rho| \geq c > 0 \) and \( f \in L_{s+1}^{2} \) with \(-1 < \delta < 0 \), the equation

\[
(\Delta + 2\rho \cdot \nabla)\psi = f
\]

admits a unique weak solution \( \psi \in L_{s+1}^{2} \) with

\[
\|\psi\|_{L_{s+1}^{2}} \leq C(\delta, c)|\rho|^{-1}\|f\|_{L_{s+1}^{2}}.
\]

Now since \((\Delta + 2\rho \cdot \nabla)\) and \((I - \Delta)^{\epsilon}\) are the constant coefficient operators and hence commute, we deduce that when \( f \in H_{s+1}^{\delta} \) for any \( s > 0 \),

\[
\|\psi\|_{H_{s+1}^{\delta}} \leq C(\delta, c)|\rho|^{-1}\|f\|_{H_{s+1}^{\delta}}.
\]

The solution to (16) is known to admit the decomposition

\[
\psi_{\rho} = \sum_{j=0}^{\infty} \psi_{j}, \quad \Delta \psi_{j} + 2\rho \cdot \nabla \psi_{j} = -q\psi_{j-1},
\]

with \( \psi_{-1} = 1 \). Let \( s = \frac{\delta}{2} + k + \epsilon \). Assume \( q\psi_{j-1} \in H_{s+1}^{\delta} \) which is true for \( j = 0 \) by assumption on \( q \). Then \( \psi_{j} \in H_{s+1}^{\delta} \). Since \( H_{s+1}^{\delta} \) is an algebra, we want to prove that

\[
\|q\psi_{j}\|_{H_{s+1}^{\delta}} \leq \|q\|_{H_{s+1}^{\delta}}\|\psi_{j}\|_{H_{s+1}^{\delta}}.
\]

Indeed, decompose \( \mathbb{R}^{n} \) into cubes. On each cube \( B \), \( (x)^{2\epsilon} \) is more or less constant up to a \( C^{\pm 2\epsilon} \). Now \( H_{s+1}^{\delta} \) is an algebra so that \( \|qu\|_{H_{s+1}^{\delta}(B)} \leq \|q\|_{H_{s+1}^{\delta}(B)}\|u\|_{H_{s+1}^{\delta}(B)} \). Since \( (x) \) is more or less constant and equal to \( (xB) \),

\[
(xB)^{2\epsilon + 2}\|qu\|_{H_{s+1}^{\delta}(B)}^{2} \leq C\|(x)(I - \Delta)^{\frac{1}{2}}\|_{H_{s+1}^{\delta}(B)}\|\psi_{j}\|_{H_{s+1}^{\delta}(B)}\|(x)^{\epsilon}(I - \Delta)^{\frac{1}{2}}u\|_{H_{s+1}^{\delta}(B)}^{2}.
\]

It remains to sum over all the cubes \( B \) to get the result. When the size of the cubes tends to 0, the constant \( C \) tends to 1, which yields (23).

This and (22) show that

\[
\|\psi_{j}\|_{H_{s+1}^{\delta}} \leq C|\rho|^{-1}\|q\|_{H_{s+1}^{\delta}}\|\psi_{j-1}\|_{H_{s+1}^{\delta}}.
\]

By selecting \( \eta \) such that \( C|\rho|^{-1}\|q\|_{H_{s+1}^{\delta}} < \frac{1}{2} \), we obtain

\[
\|\psi_{j}\|_{H_{s+1}^{\delta}} \leq \frac{1}{2}\|q\|_{H_{s+1}^{\delta}}.
\]

It remains to sum the geometric series to obtain the result. \( \square \)

We now want to obtain estimates for \( \psi_{\rho} \) and \( u_{\rho} \) restricted to \( X \). We have the following result.
Corollary 3.2. Let us assume the regularity hypotheses of the previous proposition. Then we find that
\[ |\rho| \|\psi_\rho\|_{H^{\frac{\alpha}{2}}(\bar{X})} + \|\psi_\rho\|_{H^{\frac{\alpha}{2}+\kappa}} \leq C \|q\|_{H^{\frac{\alpha}{2}+\kappa}}. \] (24)

Proof. On the bounded domain \( X \), we have thanks to \( (12) \) that
\[ |\rho| \|\psi_\rho\|_{H^{\frac{\alpha}{2}}(\bar{X})} \leq C \|q\|_{H^{\frac{\alpha}{2}}(\bar{X})} \leq C(X) \|q\|_{H^{\frac{\alpha}{2}}(\bar{X})}. \]

Now we have
\[ \Delta_1 \psi_\rho = -2\rho \cdot \nabla \psi_\rho - q(1 + \psi_\rho). \]

By elliptic regularity, with \( X' \) a smooth domain in \( \mathbb{R}^n \) such that \( \bar{X} \subset X' \), we find for all \( s = \frac{n}{2} + k + \varepsilon > 0 \) that
\[ \|\psi_\rho\|_{H^{s+1}(\bar{X})} \leq C \|q\|_{H^{s+1}(\bar{X})} \|\psi_\rho\|_{H^{s}(\bar{X})}. \]

The latter is bounded by \( |\rho| \|\psi_\rho\|_{H^{s}(\bar{X})} + \|q\|_{H^{s+1}(\bar{X})} \|\psi_\rho\|_{H^{s+1}(\bar{X})} \) since \( H^{s+1}(\bar{X}) \) is a Banach algebra. By using the above bound on \( \|\psi_\rho\|_{H^{s}(\bar{X})} \), with \( C(X) \) replaced by the larger \( C(X') \), we get the result. \( \square \)

By Sobolev embedding, we have just proved the following.

Proposition 3.3. Under the hypotheses of corollary 3.2, the restriction to \( X \) of the CGO solution verifies that
\[ |\rho| \|\psi_\rho\|_{C^k(\bar{X})} + \|\psi_\rho\|_{C^{k+1}(\bar{X})} \leq C \|q\|_{H^{s+1}(\bar{X})}. \] (25)

We recall that \( q \) satisfies constraint \( (19) \).

We are now in a position to prove the main result of this section.

Theorem 3.4. Let \( u_{\rho_j} \) for \( j = 1, 2 \) be CGO solutions with \( q \) such that \( (19) \) holds for both \( \rho_j \) and \( k \geq 1 \) and with \( c_0 \leq |\rho_1| \leq |\rho_2| \leq c_0 |\rho_1| \) for some \( c_0 > 0 \). Then we have
\[ \hat{\beta} := \frac{1}{2|\rho_1|} e^{-((\rho_1+\rho_2)+s)} \left( u_{\rho_1} \nabla u_{\rho_2} - u_{\rho_2} \nabla u_{\rho_1} \right) = \frac{\rho_1 - \rho_2}{2|\rho_1|} + \hat{h}, \] (26)

where the vector field \( \hat{h} \) satisfies the constraint
\[ \|\hat{h}\|_{C^s(\bar{X})} \leq \frac{C_0}{|\rho_1|} \] for some constant \( C_0 \) independent of \( \rho_j \), \( j = 1, 2 \).

Proof. Some algebra shows that
\[ \hat{h} = \frac{(\rho_1 - \rho_2)}{2|\rho_1|} (\psi_{\rho_1} + \psi_{\rho_2} + \psi_{\rho_1} + \psi_{\rho_2}) + \frac{\nabla \psi_{\rho_1} (1 + \psi_{\rho_1}) - \nabla \psi_{\rho_2} (1 + \psi_{\rho_2})}{2|\rho_1|}. \] (28)

We know from proposition 3.3 that \( |\rho_j| \|\psi_{\rho_j}\| \) and \( |\nabla \psi_{\rho_j}| \) are bounded in \( C^s(\bar{X}) \) for \( j = 1, 2 \). This concludes the proof of the theorem. \( \square \)
3.2. Construction of vector fields and uniqueness result

Let us consider two internal complex-valued data \( d_j(x), \ j = 1, 2 \) obtained as follows. We assume that we can impose the complex-valued boundary conditions \( g_j \in C^{k,\alpha}(\partial X; \mathbb{C}) \) and define the solution \( u_j \) of

\[
\Delta u_j + qu_j = 0, \quad X, \quad u_j = g_j, \quad \partial X, \quad j = 1, 2.
\]

Note that the real and imaginary parts of \( u_j \) may be solved independently since (29) is a linear equation. We then assume that we have access to the complex-valued internal data \( d_j = \mu u_j \) on \( X \) for \( j = 1, 2 \), where \( u_j \) is the solution of (29) with a boundary condition \( g_j \). We recall that \( \mu \in C^{k+1}(\bar{X}) \) and is bounded above and below by positive constants. We verify that

\[
u_1 \Delta u_2 - u_2 \Delta u_1 = 0.
\]

Introducing \( v = \frac{1}{\mu} \), which is well defined since \( \mu \) is bounded away from 0, and using \( u_j = vd_j \), we obtain that

\[
2(d_1 \nabla d_2 - d_2 \nabla d_1) \cdot \nabla v + (d_1 \Delta d_2 - d_2 \Delta d_1)v = 0.
\]

This is equivalent to

\[
\beta_d \cdot \nabla \mu + \gamma_d \mu = 0,
\]

where

\[
\beta_d := \chi(x)(d_1 \nabla d_2 - d_2 \nabla d_1), \quad \gamma_d := \frac{1}{2}\chi(x)(d_2 \Delta d_1 - d_1 \Delta d_2) = -\frac{\beta_d \cdot \nabla \mu}{\mu}.
\]

Here, \( \chi(x) \) is a smooth known complex-valued function with \(|\chi(x)| \) uniformly bounded from below by a positive constant on \( \bar{X} \). Note that by assumption on \( \mu \), we have that \( \beta_d \in (C^k(\bar{X}; \mathbb{C}))^n \) and \( \gamma_d \in C^k(\bar{X}; \mathbb{C}) \).

A methodology for the reconstruction of \((\mu, q)\) is therefore as follows: we first reconstruct \( \mu \) using the real part or the imaginary part of (30); then \( \Re \beta_d \) and \( \Im \beta_d \) are the real-valued vector fields since \( \mu = d/\bar{g} \) is known on \( \partial X \). When \( \mu \) is reconstructed, this gives us explicit reconstructions for \( u_j = d_j/\mu \) for \( j = 1, 2 \) and we may then reconstruct \( q \) from the Schrödinger equation. Such a method provides a unique reconstruction provided that the integral curves of the real part of the imaginary part of \( \beta_d \) join any point in \( x \) to a point \( x_0(x) \in \partial X \), where \( \mu \) is known. We thus need the vector field \( \beta \) to satisfy such properties. CGO solutions will allow us to construct families of vector fields \( \beta_d \) with the required properties.

Let us consider two CGOs \( u_{\rho_j} \) with parameters \( \rho_j, j = 1, 2 \). Let \( d_j \) be the complex-valued corresponding internal data. Let us decompose as before

\[
u_{\rho_j}(x) = e^{\rho_jx}(1 + \psi_{\rho_j}(x)), \quad \nabla u_{\rho_j}(x) = e^{\rho_jx}((1 + \psi_{\rho_j})\rho_j + \nabla \psi_{\rho_j}).
\]

Let us choose \( \chi(x) = e^{-(\rho_1+\rho_2)x} \) in (31). Then we find after some algebra that \( \beta_d \) in (31) is given by

\[
\beta_d = \mu^2((\rho_1 - \rho_2)(1 + \psi_{\rho_1})(1 + \psi_{\rho_2}) + \nabla \psi_{\rho_1}(1 + \psi_{\rho_2}) - \nabla \psi_{\rho_2}(1 + \psi_{\rho_1})).
\]

We may then define

\[
\beta := \frac{1}{2|\rho_1|}\beta_d = \mu^2\frac{\rho_1 - \rho_2}{2|\rho_1|} + \mu^2\hat{h}, \quad \gamma := \frac{1}{2|\rho_1|}\gamma_d,
\]

where \( \hat{h} \) is defined as in (28). Then, we deduce from theorem 3.4 that \(|\rho_1|\mu^2\hat{h} \) is bounded uniformly in \( C^k(\bar{X}; \mathbb{C}) \). When \(|\rho_1| \) is sufficiently large, \( \beta \) is close to \( \mu^2\frac{\rho_1 - \rho_2}{2|\rho_1|} \), which is a non-vanishing vector when \( \rho_1 \neq \rho_2 \). Provided that the real part or the imaginary part of \( \beta \)
does not vanish, (30) gives an equation for \( \mu \) that can be uniquely solved since \( \mu \) is known on \( \partial X \).

Note that the data \( d_2 \) are complex valued. The only possibility of constructing two different complex-valued data with two real-valued data is to assume that \( d_2 = \tilde{d}_1 \), the complex conjugate of \( d_1 \). For the construction of CGOs, this implies that we choose \( \rho_2 = \overline{\rho_1} \).

Indeed, we verify that \( u_{\rho_2} = u_\rho \) since for \( \rho = k + ik \) with \( |k| = |k^\perp| \) and \( k \cdot k^\perp = 0 \), we have \( e^{ikx} = e^{i|x|} \) and \( \overline{\psi_\rho} = \psi_\rho \) by uniqueness of the solution to the equation satisfied by \( \psi_\rho \). This implies then that \( \tilde{\beta} \) defined in (33) with \( \rho_2 = \overline{\rho_1} \) is given by

\[
\tilde{\beta}_\rho = i\mu^2k^\perp + \mu^2 \bar{h}.
\]

As soon as \( |\rho| > C_0 \) so that \( \|h\|_{C^1(\bar{X})} < 1 \), we obtain that any point in \( X \) is connected to a point in \( \partial X \) by an integral curve of \( \tilde{\beta}_\rho := \bar{\beta}_\rho \).

Note that \( u_\rho \) solves (29) with the unknown boundary condition \( u_{\rho|\partial X} \in C^{k,\alpha}(\partial X; \mathbb{C}) \) for some \( \alpha > \frac{1}{2} \) since \( u_\rho \) is known to be a little more regular than being of class \( C^{k+1}(\bar{X}; \mathbb{C}) \) by construction (since \( \varepsilon > 0 \)).

Let us now define the boundary conditions \( g \in C^{k,\alpha}(\partial X; \mathbb{C}) \) such that

\[
\|g - u_{\rho|\partial X}\|_{C^{k,\alpha}(\partial X; \mathbb{C})} \leq \varepsilon,
\]

for some \( \varepsilon > 0 \) sufficiently small. Let \( u \) be the solution of (29) with \( g \) as in (35). By elliptic regularity, we thus have

\[
\|u - u_{\rho}\|_{C^{k,\alpha}(\bar{X}; \mathbb{C})} \leq C\varepsilon,
\]

for some positive constant \( C \). Define the complex-valued internal data \( d = \mu u \). Since \( \mu \in C^{k+1}(\bar{X}) \), we deduce that

\[
\|d - d_{\rho}\|_{C^{k,\alpha}(\partial X; \mathbb{C})} \leq C_0\varepsilon,
\]

for \( C_0 > 0 \). Once \( d \) is constructed, define \( d_1 = d \) and \( d_2 = \bar{d} \) and define \( \tilde{\beta}_d \) and \( \tilde{\gamma}_d \) as in (31) with \( \chi(x) = e^{-2k\cdot x} \) and the normalized quantities \( \beta \) and \( \gamma \) as in (33). Note that \( \chi(x) \) is positive and bounded on \( \bar{X} \).

Let us define

\[
\beta := \bar{\gamma} = \frac{1}{2|k|}\gamma d,
\]

and define \( \gamma \) as in (33).

Thanks to (37) and (34), we obtain the error estimate

\[
\|\beta - \mu^2k\|_{C^1(\bar{X})} \leq C \frac{1 + \varepsilon}{|k|}.
\]

As a consequence, as soon as \( |k| \) is sufficiently large and \( \varepsilon \) sufficiently small, we obtain that \( \beta \cdot k \geq \xi > 0 \) so that any point \( x \in X \) is mapped to a point in \( \partial X \) in a time less than \( |\xi|^{-1} \text{diam}(X) \) by an integral curve of \( \beta \).

Moreover, we have the equation with real-valued coefficients:

\[
\beta \cdot \nabla u + \gamma \mu = 0.
\]

Since \( \mu = d/g \) is known on \( \partial X \), this equation provides a unique reconstruction for \( \mu \).

Let us define the set of parameters

\[
P = \{(\mu, q) \in C^{k+1}(\bar{X}) \times H^{2, k+\varepsilon}(X); 0 \not\text{ an eigenvalue of } \Delta + q, \|\mu\|_{C^{k+1}(\bar{X})} + \|q\|_{H^{2, k+\varepsilon}(X)} \leq P < \infty\}.
\]

The above construction of the vector field allows us to obtain the following uniqueness result.

**Theorem 3.5.** Let \( X \) be a bounded, open subset of \( \mathbb{R}^n \) with a boundary \( \partial X \) of class \( C^2 \). Let \( (\mu, q) \) and \( (\tilde{\mu}, \tilde{q}) \) be two elements in \( P \). Let \( k \in \mathbb{R}^n \) with \( |k| \geq |k_0| \) and \( |k_0| \) sufficiently large.
and define \( \rho = k + ik^\perp \) so that \( \rho \cdot \rho = 0 \). Let \( u_\rho \) be the corresponding CGO for \( q \) and \( u \) constructed as above with \( d = \mu d \) and with \( \epsilon \) sufficiently small. Let \( \bar{d} \) be constructed similarly with the parameters \((\bar{\mu}, \bar{q})\).

Then \( d = \bar{d} \) implies that \((\mu, q) = (\bar{\mu}, \bar{q})\).

**Proof.** Since the two measurements \( d = \bar{d} \), we have that \( \mu \) and \( \bar{\mu} \) solve the same equation \((40)\). Since \( \mu = \bar{\mu} = d/g \) on \( \partial X \), we deduce that \( \mu = \bar{\mu} \) since the integral curves of \( \beta \) map any point \( x \in X \) to the boundary \( \partial X \). More precisely, consider the flow \( \phi_x(t) \) associated with \( \beta \), i.e. the solution to

\[
\phi_x(t) = \beta(\phi_x(s)), \quad \phi_x(0) = x \in \bar{X}.
\]

By the Picard–Lindelöf theorem, the above equations admit unique solutions since \( C^1 \). And by hypothesis on \( \beta \) since \( \beta \cdot k \geq \xi > 0 \) for \( |k| \) sufficiently large, any point \( x \) is mapped to two points (for positive and negative values of \( t \)) on \( \partial X \) by the flow \( \phi_x(t) \) in a time less than \( \xi^{-1}\text{diam}(X) \). For \( x \in X \), let us define \( x_{\pm}(x) \in \partial X \) and \( \pm t_{\pm}(x) > 0 \) such that

\[
\phi_x(t_{\pm}(x)) = x_{\pm}(x) \in \partial X.
\]

Then by the method of characteristics, the \( \mu(x) \) solution of \((40)\) is given by

\[
\mu(x) = \mu_0(x_{\pm}(x))e^{-\int_{t_{\pm}(x)}^{t_{\pm}(\bar{x})} \gamma(\phi_x(t))dt},
\]

(44)

The solution \( \bar{\mu}(x) \) is given by the same formula since \( \phi_x(t) = \bar{\phi}_x(t) \) so that \( \bar{\mu} = \mu \). This implies that \( u = \bar{u} \since d = \bar{d} \). It remains to use the equation for \( u \) to deduce that \( q = \bar{q} \) on the domain where \( u \neq 0 \). By unique continuation, \( u \) cannot vanish on an open set in \( X \) different from the empty set for otherwise \( u \) vanishes everywhere and this is impossible to satisfy the boundary conditions. This shows that the set \( F \subset X \) where \( |u| > 0 \) is open and \( \bar{F} = \bar{X} \) since the complement of \( \bar{F} \) has to be empty. By continuity, this shows that \( q \) is known on \( \bar{X} \).

The above result shows that there exists an open set of boundary conditions \( g \) close to \( u|_{\partial \bar{X}} \) so that the data \( d_1 = d \) and \( d_2 = \bar{d} \) obtained from one complex-valued solution \( u \) or equivalently from two real-valued solutions \( \Re u \) and \( \Im u \) uniquely determine the parameters \((\mu, q)\). A more explicit characterization of the open set of illuminations is lacking. However, we observe that larger values of \( q \) require larger values of \( |k| \) in order to straighten the vector field \( \beta \). Although \((35)\) seems to be independent of \( k \) and \( \rho \), in fact \( u_\rho \) itself grows exponentially with \( |k| = |\rho| \) so that \( g \) has to be in the \( \epsilon \) vicinity of an exponentially growing quantity. This means that \( |k| \) has to be sufficiently large that the field \( \beta \) is sufficiently flat while at the same time not so large that the imposed illuminations become physically infeasible.

The above uniqueness result does not guaranty stability in the reconstruction. We easily verify that the construction provides stability of the reconstruction of \( \mu \) in most of the domain \( X \). However, small changes in the data may generate small changes in the field \( \beta \). This may in turn modify the value of the reconstructed function \( \mu \) at points where \( \beta \) is ‘almost’ tangent to the boundary \( \partial X \). We will see below that under some geometric constraints of sufficient convexity of \( X \), the above procedure provides a stable reconstruction of the parameters \((\mu, q)\). When such conditions are not met, we can still obtain stability by acquiring more measurements. Indeed, if a sufficient number of vector fields \( \beta \) can be constructed at every point so that the span of these vector fields is exactly \( \mathbb{R}^n \), then we face a significantly more favorable situation. We now consider such a case where \( 2n \) real-valued measurements are available. Later, we will derive stability results in the two-measurement setting under additional geometric constraints.
3.3. ISID with 2n real-valued internal data

Let us consider first the setting in which we can access 2n real-valued internal data viewed as n complex-valued internal data (since the measurements are linear in \(u\), we can measure the real and imaginary parts separately).

Let us define \(k_j = |k|e_j\) where \((e_1, \ldots, e_n)\) is an orthonormal basis. We define the complex vectors

\[
\rho_j = k_1 + ik_j, \quad 2 \leq j \neq n, \quad \rho_1 = -k_1 - ik_2 = -\rho_2.
\]

Let \(u_{\rho_j}\) be the corresponding CGOs. We choose the boundary conditions \(g_j\), such that

\[
\|g_j - u_{\rho_j|ax}\|_{C^0(\partial X, \mathbb{C})} \leq \epsilon,
\]

for \(\epsilon\) sufficiently small. We define \(u_j\) as the solutions to (29) with boundary conditions \(g_j\). These are n complex-valued solutions whose real and imaginary parts consist of 2 real-valued solutions. For \(1 \leq j \leq n\), we define \(d_j = \mu u_j\). We now construct the n vector field \(\beta_j\). For \(2 \leq j \leq n\), the real-valued vector fields and scalar terms are constructed as in the preceding section; for \(j = 1\), the vector field is constructed by using \(\rho_2 = -\rho_1\):

\[
\begin{align*}
\beta_1 &= \frac{1}{2|k|} \text{Im}(d_2 \nabla d_1 - d_1 \nabla d_2), \\
\gamma_1 &= \frac{1}{4|k|} \text{Im}(d_1 \Delta d_2 - d_2 \Delta d_1), \\
\beta_j &= \frac{e^{-2k_j \cdot x}}{2|k|} \text{Im}(d_j \nabla \tilde{d}_j - \tilde{d}_j \nabla d_j), \\
\gamma_j &= \frac{e^{-2k_j \cdot x}}{4|k|} \text{Im}(\tilde{d}_j \Delta d_j - d_j \Delta \tilde{d}_j),
\end{align*}
\]

for \(2 \leq j \leq n\). As in the preceding section, we verify that

\[
\|\beta_j - \mu^2 \beta_j\|_{C^1(\mathcal{X})} \leq C \frac{1 + \epsilon}{|k|}.
\]

For \(|k|\) sufficiently large, and thanks to the bound \(\mu_0^{-1} \leq \mu \leq \mu_0\), we obtain that at each point \(x \in \mathcal{X}\), the vectors \(\beta_j(x)\) form a basis. Moreover, the matrix \(a_{jk}\) such that \(\beta_j = \sum a_{jk} e_k\) is an invertible matrix with an inverse of class \(C^k(\mathcal{X})\). In other words, we have constructed a vector-valued function \(\Gamma(x) \in (C^k(\mathcal{X}))^n\) such that (40) may be recast as

\[
\nabla \mu + \Gamma(x)\mu = 0.
\]

Finally, the construction of \(\Gamma\) is stable under small perturbations in the data \(d_j\). Indeed, invertibility of \(a_{jk}\) is ensured for vector fields close to \(\beta_j\). Let \(\Gamma\) and \(\bar{\Gamma}\) be two vector fields constructed from knowledge of two sets of internal data \(d = \{d_j, 1 \leq j \leq n\}\) and \(\bar{d} = \{ar{d}_j, 1 \leq j \leq n\}\). Then we find that

\[
\|\Gamma - \bar{\Gamma}\|_{C^0(\mathcal{X})} \leq C\|d - \bar{d}\|_{C^1(\mathcal{X})},
\]

provided the right-hand side is sufficiently small.

Let us now assume that \(X\) is connected (otherwise, the method applies to each connected component) and \(\mu\) is known and equal to \(\mu_0 = d/g\) for some point \(x_0 \in \partial X\). In other words, we want to solve the over-determined problem

\[
\nabla \mu + \Gamma(x)\mu = 0, \quad X, \quad \mu(x_0) = \mu_0(x_0), \quad x_0 \in \partial X.
\]

Let \(x \in X\) be an arbitrary point and assume that \(X\) is bounded and connected and \(\partial X\) is smooth. Then we find a smooth curve that links \(x\) to the point \(x_0 \in \partial X\). Restricted to this curve, (51) becomes a stable ordinary differential equation. The solution of the ordinary differential equation is then stable with respect to modifications in \(\Gamma\) (the curve between \(x\) and \(x_0\) is kept constant). The solution \(\mu\) then clearly inherits the smoothness of \(\Gamma(x)\) directly from (51). Moreover, since \(\mu_0(x_0) - \bar{\mu}_0(x_0)\) (with \(\bar{\mu}_0 = \bar{d}/g\) on \(\partial X\)) is small and equation (50) is
stable with respect to changes in the value of \( \mu_0(x_0) \), we deduce that the reconstruction of \( \mu \) is stable with respect to perturbations in \( d \).

We may thus state the main result of this section.

**Theorem 3.6.** Let \( k \geq 1 \). We assume that we have access to \( n \) well-chosen complex-valued measurements and that \((\mu, q)\) and \((\tilde{\mu}, \tilde{q})\) are the elements in \( \mathcal{P} \). Under the hypotheses outlined above, and provided that \( \|u_{\rho_j}|x| = g_j\|_{C^1(\bar{X}; C)} \) is sufficiently small, we have the following stability result:

\[
\|\mu - \tilde{\mu}\|_{C^1(\bar{X})} + \|q - \tilde{q}\|_{C^{1,1}(\bar{X})} \leq C\|d - \tilde{d}\|_{(C^{1,1}(\bar{X}))^n}. \tag{52}
\]

**Proof.** The inequality for \( \mu - \tilde{\mu} \) is a direct consequence of the results proved above. This provides a stability result for \( v = \mu^{-1} \) and for \( u_j = \nu d_j \) from the data \( d_j \). We thus have a stability result for \( \Delta u_j = -u_j q \) and hence the above stability result for \( u_j(q - \tilde{q}) \) since \( (u_j - \tilde{u}_j)\tilde{q} \) is small.

Now, \( u_{\rho} = e^{\rho_x}(1 + \psi_{\rho}) \) does not vanish on \( X \) when \( |\rho| \) is sufficiently large since \( |\rho|\psi_{\rho} \) is bounded. When the boundary condition \( g_j - u_{\rho_j}|_{\partial X} \) is small, by the maximum principle, \( u_j \) does not vanish on \( X \) either. This means that either its real part or its imaginary part does not vanish everywhere in \( X \). This provides control of \( q - \tilde{q} \) in \( X \) as given in (52).

3.4. Vector fields and stability of solutions

The above construction allows one to stably reconstruct the two functions \( \mu \) and \( q \) provided that we have constructed \( J = 2n \) well-chosen real-valued boundary conditions and collected \( 2n \) corresponding internal data. We now return to the reconstruction of \( \mu \) and \( q \) in the presence of \( J = 2 \) well-chosen real-valued internal data. Such internal data are obtained as theorem 3.5.

We recall that \( k \) is fixed and \( \rho = k + ik^\perp \). We define \( u \) as the solution to (29) with \( g \) close to \( u_{\rho_j}|x| \). The complex-valued internal data are then \( d = \mu u \). The vector field \( \beta \) and the scalar \( \gamma \) are then given by

\[
\beta = \frac{e^{-2k x}}{2|k|} \Im(d\nabla d - \bar{d}\nabla \bar{d}), \quad \gamma = \frac{e^{-2k x}}{4|k|} \Im(d\Delta d - d\Delta \bar{d}), \tag{53}
\]

and we verify that

\[
\beta \cdot \nabla \mu + \gamma \mu = 0 \quad X. \tag{54}
\]

As earlier, we verify that

\[
\|\beta - \mu^2 k^\perp\|_{C^1(X)} \leq C\frac{1 + \epsilon}{|k|}. \tag{55}
\]

As a consequence, the integral curves of \( \beta \) given by \( \varphi_\beta(t) \) map any point \( x \in X \) to two points on \( \partial X \) when \( |k| \) is sufficiently large as was mentioned earlier.

However, the stability of equation (54) with respect to changes in \( \beta \) and \( \gamma \) is not as good as in the presence of \( n \) complex internal data. The stability of the reconstruction degrades for points \( x \) close to \( x_0 \in \partial X \) where \( \beta_{\mu}(x_0) \cdot n(x_0) \) is close to 0. The stability of the reconstruction of \( \mu \) will however be good when \( X \) is a convex domain with ‘sufficient’ convexity as established in hypothesis 2.3. We prove the following result.

**Proposition 3.7.** Let \( k \geq 1 \). Let \( \mu \) and \( \tilde{\mu} \) be solutions of (54) corresponding to the coefficients \( (\beta, \gamma) \) and \( (\tilde{\beta}, \tilde{\gamma}) \), respectively, where

\[
\beta = \mu^2 k^\perp + \frac{1}{|k|} h, \quad \tilde{\beta} = \mu^2 k^\perp + \frac{1}{|k|} \tilde{h},
\]

for \( h, \gamma, \tilde{h}, \) and \( \tilde{\gamma} \) bounded in \( C^4(\bar{X}) \).

13
Let us assume that $\mu|_{\partial X} = \mu_0$ and $\tilde{\mu}|_{\partial X} = \tilde{\mu}_0$ on $\partial X$ for some functions $\mu_0, \tilde{\mu}_0 \in C^k(\partial X)$. Let us assume that $X$ is sufficiently convex so that hypothesis 2.3 holds for some $R < \infty$. We also assume that $|k| \geq k_0$ is sufficiently large. Then there is a constant $C$ such that
\[ \|\mu - \tilde{\mu}\|_{C^{k-1}(\bar{X})} \leq C\|\mu_0\|_{C^k(\partial X)}(\|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})} + \|\gamma - \tilde{\gamma}\|_{C^{k-1}(\bar{X})}) + C\|\mu_0 - \tilde{\mu}_0\|_{C^k(\partial X)}. \] (56)

**Proof of Proposition 3.7.** We recall that $\psi_\alpha(t)$ is the flow defined in (42) and that $x_\pm(x)$ and $t_\pm(x)$ are defined in (43). By the method of characteristics, the $\mu(x)$ solution of (54) is given by
\[ \mu(x) = \mu_0(x_\pm(x)) e^{-\int_0^t \beta(\psi_\alpha(s)) ds}. \] (57)

The solution $\tilde{\mu}(x)$ is given similarly. We first assume that $\tilde{\mu}_0 = \mu_0$. From the equality
\[ |\psi_\alpha(t) - \tilde{\psi}_\alpha(t)| = \int_0^t [\beta(\psi_\alpha(s)) - \beta(\tilde{\psi}_\alpha(s))] ds, \]
and using the Lipschitz continuity of $\beta$ and Gronwall’s lemma, we thus deduce the existence of a constant $C$ such that
\[ |\psi_\alpha(t) - \tilde{\psi}_\alpha(t)| \leq Ct\|\beta - \tilde{\beta}\|_{C^0(X)} \]
uniformly in $t$ knowing that all characteristics exit $X$ in finite time and provided that $\psi_\alpha(t)$ and $\tilde{\psi}_\alpha(t)$ are in $\bar{X}$.

Such estimates are stable with respect to modifications in the initial conditions. Let us define $W(t) = D_\alpha \psi_\alpha(t)$. Then classically, $W$ solves the equation $W = D_\alpha \beta(\psi_\alpha)W$ with $W(0) = I$ and by using Gronwall’s lemma once more, we deduce that
\[ |W - \tilde{W}(t)| \leq Ct\|D_\alpha \beta - D_\alpha \tilde{\beta}\|_{C^0(X)}, \]
for all times provided that $\psi_\alpha(t)$ and $\tilde{\psi}_\alpha(t)$ are in $\bar{X}$. As a consequence, since $\beta$ and $\tilde{\beta}$ are of class $C^k(\bar{X})$, we obtain similarly that
\[ |D_{\psi_\alpha}^{-1} \psi_\alpha(t) - D_{\tilde{\psi}_\alpha}^{-1} \tilde{\psi}_\alpha(t)| \leq Ct\|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})}, \]
and this again for all times provided that $\psi_\alpha(t)$ and $\tilde{\psi}_\alpha(t)$ are in $\bar{X}$.

However, this does not imply that $x_\alpha(x)$ is close to $\tilde{x}_\alpha(x)$. When $\partial X$ is flat for instance, we may very well have that $x_\alpha(x)$ is such that $n(x_\alpha(x)) : \psi_\alpha(t_\alpha(x))$ is very small and that $x_\alpha(x) - \tilde{x}_\alpha(x)$ is arbitrarily large if $\tilde{\beta}$ is parallel to the surface $\partial X$ for instance. This behavior, however, cannot occur when both $\beta$ and $\tilde{\beta}$ are sufficiently flat, which is the case when $|k|$ is sufficiently large, and when $\partial X$ is sufficiently curved, which is obtained from the existence of $R < \infty$ in hypothesis 2.3. In such a setting, we can obtain the following result. \hfill \Box

**Lemma 3.8.** Let $k \geq 1$ and assume that $\beta$ and $\tilde{\beta}$ are the $C^k(\bar{X})$ vector fields that are sufficiently flat, i.e. that $|k|$ is sufficiently large. Let us assume that $\partial X$ is sufficiently convex so that hypothesis 2.3 holds for some $R < \infty$. Then we have that
\[ \|x_\alpha - \tilde{x}_\alpha\|_{C^{k-1}(\bar{X})} + \|t_\alpha - \tilde{t}_\alpha\|_{C^{k-1}(\bar{X})} \leq C\|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})}, \] (58)
where $C$ is a constant that depends on $|k|$ and $R$.

The above lemma is mostly a consequence of the following result:

**Lemma 3.9.** Let $C_0$ be the constant defined such that
\[ |\psi_\alpha(t)| = |\nabla \beta(\psi_\alpha(t))\beta(\psi_\alpha(t))| \leq \frac{C_0}{|k|}. \]
Let $t_M$ be the maximal time spent by any trajectory in $X$, which we know is bounded. Assume that $|k|$ is sufficiently large that for all $x \in X$,

$$\left( \frac{C_0 R}{|k|} + \frac{C_2 t_M^2}{4|k|^2} \right) \frac{1}{|\beta(x_+(x))|^2} = \rho < 1.$$ 

Then we have that for all $x \in X$,

$$t_+(x) \leq \frac{2R}{|\beta(x_+(x))|^2(1 - \rho)} n(x_+(x)) \cdot \beta(x_+(x)).$$

(59)

In other words, the vector field $\beta(x_+(x))$ is close to being tangent to $\partial X$ only when the time spent in $X$ is small.

Now let $x_0 \in X$ and $x \in \partial X$ and define $v_0 = \beta(x_0)$. Assume moreover that

$$|x - x_0| \leq C_1 \delta^2, \quad |n(x) \cdot v_0| \leq C_2 \delta.$$ 

(60)

Then we have

$$t_+(x_0) \leq C_3 \delta,$$

(61)

for some constant $C_3 > 0$ independent of $x_0$. In other words, a trajectory close to $\partial X$ and almost tangent to $\partial X$ exits $X$ in a short time.

We postpone the proof of the above two lemmas to the end of the section. Let us conclude the proof of the proposition. We recall that $\mu(x) = \mu_0(x_+(x)) e^{-\int_0^{t(x)} \gamma(\psi_s(x)) \, ds}$, with a similar expression for $\tilde{\mu}$. Since $x \to e^{-s}$ is smooth, by the Leibniz rule it is sufficient to prove the stability result for $\mu_0(x_+(x))$ and for $\int_0^{t(x)} \gamma(\psi_s(x)) \, ds$. It is clear from the above lemmas that

$$\|\mu_0(x_+(x)) - \mu_0(\tilde{x}_+(x))\|_{C^s(\bar{X})} \leq \|\mu_0\|_{C^s(\partial X)} \|x_+ - \tilde{x}_+\|_{C^{s-1}(\bar{X})} \leq C \|\mu_0\|_{C^s(\partial X)} \|\beta - \tilde{\beta}\|_{C^{s-1}(\bar{X})}.$$ 

Let us now assume without loss of generality that $\tilde{t}_+(x) \geq t_+(x)$. Then we have

$$\int_0^{\tilde{t}_+(x)} (\gamma(\psi_s(x)) - \tilde{\gamma}(\tilde{\psi}_s(x))) \, ds = \int_0^{t_+(x)} (\gamma(\psi_s(x)) - \gamma(\tilde{\psi}_s(x)) + (\gamma - \tilde{\gamma})(\psi_s(x))) \, ds.$$ 

We verify that the above expression has $k-1$ derivatives uniformly bounded since (i) $x \to t_+(x)$ is $C^{k-1}(\bar{X})$; (ii) $\gamma$ has $C^k$ derivatives bounded on $\bar{X}$; (iii) $(\psi_s - \tilde{\psi}_s)(x)$ has $k-1$ derivatives bounded by $\|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})}$.

It thus remains to handle the term

$$v(x) = \int_{\tilde{t}_+(x)}^{t_+(x)} \tilde{\gamma}(\tilde{\psi}_s(x)) \, ds.$$ 

The function $x \to \tilde{\gamma}(\tilde{\psi}_s(x))$ is of class $C^{k-1}(\bar{X})$ by regularity of the flow and because $\tilde{\gamma}$ is of class $C^k(\bar{X})$. Derivatives of order $k-1$ of $v(x)$ thus involve terms of size $\tilde{t}_+(x) - t_+(x)$ and terms of the form

$$D^m_s \{ \tilde{\gamma}(\tilde{\psi}_s(\tilde{t}_+)) - t_+(x)D^{k-1-m}_s \tilde{\gamma}(\tilde{\psi}_s(\tilde{t}_+)) \}, \quad 0 \leq m \leq k - 1.$$ 

Because $\tilde{\beta}$ is of class $C^k(\bar{X})$, then so is $x \to \tilde{\gamma}(\tilde{\psi}_s(x))$. Since the latter function has $k-1$ derivatives that are Lipschitz continuous, we thus find that

$$|D^k_s v(x)| \leq C \|\tilde{t}_+ - t_+\|_{C^{k-1}(\bar{X})}.$$
This concludes the proof of the proposition when \( \tilde{\mu}_0 = \mu_0 \). Applying lemma 3.8 as before, we verify that

\[
|D_t^{k-1}[(\tilde{\mu}_0(x_\pm(x)) - \mu_0(x_\pm(x))) e^{-\int_0^t \nu(\rho_s(x)) ds}]| \leq C \| \mu_0 - \tilde{\mu}_0 \|_{C^k(\bar{X})},
\]

By the triangle inequality, we deduce the error estimate on \( \mu - \tilde{\mu} \) described in the proposition.

**Proof of Lemma 3.8.** Let us assume without loss of generality that \( t_\ast(x) \leq \bar{t}_\ast(x) \). We have seen that

\[
|\phi_x(t_\ast(x)) - \tilde{\phi}_x(t_\ast(x))| \leq C t_\ast(x) \delta, \quad \delta = \| \beta - \tilde{\beta} \|_{C^0(\bar{X})},
\]

We also have that \( |\beta(x_\ast(x)) - \tilde{\beta}(x_\ast(x))| \leq C_2 \delta \). From lemma 3.9, we know that \( \beta(x_\ast(x)) \cdot n(x_\ast(x)) \geq 2C_1 t_\ast(x) \) for some constant \( C_1 > 0 \).

Let us assume first that \( C_1 t_\ast \geq C_2 \delta \) so that \( v_0 \cdot e_1 \geq C_1 t_\ast(x) \) where \( x_0 = \tilde{\phi}_x(t_\ast) \), \( v_0 = \tilde{\beta}(x_0) \) and \( e_1 = n(x_\ast(x)) \). We want to show that the integral curve of \( \tilde{\beta} \) starting at \( x_0 \) at time 0 with velocity \( v_0 \) exists in \( X \) in a time of order \( \delta \) so that \( x_\ast(x) - \tilde{x}_\ast(x) \) is of order \( \delta \).

In an appropriate system of coordinates, we have \( x_0 = (-y, 0) \) and \( x_\ast = (0, c) \) where \( 0 \leq y \leq C \delta t_\ast \). We verify that

\[
\tilde{\phi}_x(x_0) - x_0 - tv_0 = \int_0^t \int_0^s \delta^2 \tilde{\phi}_x(u) \, du \, ds.
\]

\( \tilde{\phi}_x(x_0) \) will be outside of the convex domain \( X \) as soon as its first component becomes non-negative, which implies that

\[
-y + tv \cdot e_1 + e_1 \cdot \int_0^t \int_0^s \delta^2 \tilde{\phi}_x(u) \, du \, ds \leq 0,
\]

or equivalently

\[
t_\ast t \leq C_3 C \delta + C_4 t^2.
\]

Now take \( t = 2C_3 \delta \) so that the above constraint becomes

\[
t_\ast \leq 4C_4 C \delta.
\]

It remains to choose \( t_\ast > 4C_4 C \delta \) to obtain the existence of a time \( t \) so that \( \tilde{\phi}_x(x_\ast(2C_3 \delta)) \) is outside of \( X \). This shows that the distance traveled by \( \phi_x \) is of size \( \delta \) so that \( |x_\ast(x) - \tilde{x}_\ast(x)| \leq C \delta \).

We have treated the case \( t_\ast(x) > 0 \delta \) for some constant \( \alpha \) sufficiently large. It remains to address the case \( t_\ast < 0 \delta \). For this, a sufficiently positive curvature of \( \partial X \) is necessary and we use lemma 3.9. Indeed, we know that

\[
|\phi_x(t_\ast(x)) - \tilde{\phi}_x(t_\ast(x))| \leq C \alpha \delta^2.
\]

We also know that \( \beta(x_\ast(x)) \cdot n(x_\ast(x)) \geq 2\alpha C_1 \delta \). We can then invoke the second result of lemma 3.9 and obtain that \( |x_\ast(x) - \tilde{x}_\ast(x)| \leq C \delta \).

At this stage, we have thus proved that independent of \( t_\ast(x) \), \( |x_\ast(x) - \tilde{x}_\ast(x)| \leq C \delta \) with \( C \) a constant independent of \( \delta \). The proof of the above result shows that \( |t_\ast(x) - \bar{t}_\ast(x)| \leq C \delta \) as well. Higher order derivatives are now treated in a similar fashion. We have seen that

\[
|W(t_\ast) - \tilde{W}(t_\ast)| \leq C \| \beta \|_{C^1(\bar{X})}.
\]

Since \( \tilde{W}(t) \) is of class \( C^1 \) and \( |t_\ast(x) - \bar{t}_\ast(x)| \leq C \delta \), we deduce that

\[
|W(t_\ast) - \tilde{W}(\bar{t}_\ast)| \leq C \| \beta \|_{C^1(\bar{X})}
\]

which is equivalent to

\[
\|D_\beta x_\ast - D_\beta \bar{x}_\ast\|_{C^0(\bar{X})} \leq C \| \beta \|_{C^1(\bar{X})}.
\]
Higher order derivatives are treated in exactly the same manner providing a bound for \( k - 1 \) derivatives of \( x_+ - \tilde{x}_+ \) in the uniform norm.

The error on \( t_+ \) is obtained as follows. We note that
\[
\varphi_+(t_+) - \tilde{\varphi}_+(t_+) = (x_+ - \tilde{x}_+)(x).
\]
After differentiation in space, we obtain
\[
W(t_+)(D_t x_+ - D_t \tilde{x}_+) = D_x(x_+ - \tilde{x}_+)(x).
\]
Since \( W \) is Lipschitz, and \( t_+ - \tilde{t}_+ \) is small, this implies that
\[
W(t_+)(D_t x_+ - D_t \tilde{x}_+) = O(\delta).
\]
We have that \( \dot{W} \) is of order \(|k|^{-1}\) and that \( W(0) = I \) so that for \(|k| \) sufficiently large, \( W(t_+) \) is invertible.

This implies
\[
(D_t x_+ - D_t \tilde{x}_+) = O(\delta).
\]
Higher order derivatives are treated in the same manner by using the Leibniz product rule and the invertibility of \( W(t_+) \). This concludes the proof of the lemma.

□

Proof of Lemma 3.9. Instead of running the characteristics forward from \( x \) to \( x_+(x) \), we run the characteristics backwards from \( x_+(x) \) to \( x-(x) \) and show that the time spent in \( X \) is controlled by the angle the trajectories make with the normal to \( X \) at \( x_+(x) \in \partial X \). More precisely, we set \( y = x_+(x) \) and \( v = \dot{\varphi}_+(x_+(x)) = \beta(x_+(x)) \) and run characteristics backwards.

From the equality
\[
\dot{\varphi}_+(\cdot - t) = v - \int_0^t \ddot{\varphi}_+(\cdot - s) \, ds,
\]
we deduce that
\[
|\dot{\varphi}_+(\cdot - t) - v| \leq C_0 |t|,
\]
and hence
\[
|\dot{\varphi}_+(\cdot - t) - (y - tv)| \leq C_0 |t|^2.
\]
Let \( t_m \) be the time it takes from \( x_+(x) \) to \( x-(x) \). We obviously have that \( t_+(x) \leq t_m \). Let \( B_y(R) \) be the (unique) ball of radius \( R \) tangent to \( \partial X \) at \( y \in \partial X \) and such that \( X \subset B_y(R) \).

In a system of coordinates with \( B_y(R) \) centered at \( 0 \) and \( v = e_1 \), we find that
\[
|y - tv|^2 = R^2 + |v|^2 t^2 - 2rtv \cdot n(y).
\]
We want \( \dot{\varphi}_+(\cdot - t) \in X \subset B_y(R) \). This imposes that
\[
|y - tv| \leq R + \frac{C_0}{2|k|} |t|^2.
\]
Let us define \( t_M \) as the maximal time a trajectory spends in \( X \), which is a bounded quantity. Then the above imposes that
\[
|v|^2 t^2 - 2Rtv \cdot n(y) \leq \frac{RC_0}{|k|} t^2 + \frac{C_0^2 |t|^2}{4|k|^2} \leq |v|^2 \rho t^2,
\]
and in other words that \( t_M \) is bounded by \( \frac{2R}{\rho|v| \cdot n(y)} \).

For the second result, we define \( B_2(R) \) as the ball of radius \( R \) tangent to \( \partial X \) at \( x \) and such that \( X \subset B_2(R) \). We again have that \( |\varphi_0(t) - (x_0 + \tilde{x}_0t)| \) is bounded by \( \rho \|x_0\|^2 t^2 \). In a system
of coordinates where \( n(x) = e_2, x = Re_2 \), and \( v_0 = (v_0 \cdot e_1 + (v_0 \cdot e_2)e_2 \), we obtain that \( \varphi_n(t) \in X \subset B_2(R) \) implies that

\[
|x + tv_0| \leq |x - x_0| + R + \frac{C_0}{2|k|^2}t^2.
\]

This is equivalent to

\[
2t R v_0 \cdot e_2 + t^2|v_0|^2 \leq |x - x_0|^2 + \frac{C_0^2 t^2}{2|k|^2} + 2|x - x_0|R + 2|x - x_0| \frac{C_0}{|k|} + \frac{R C_0}{|k|} t^2.
\]

This implies that

\[
(1 - \rho)|v|^2 \leq C|x - x_0| + 2t R v \cdot e_2 \leq C(\delta^2 + t \delta),
\]

for some constants \( C \) that can be made explicit. Solving this quadratic inequality yields that \( t_m \) is bounded as prescribed. \( \square \)

### 3.5. ISID with two real-valued measurements

We are now in a position to state our main stability result in the presence of one well-chosen complex-valued internal data. We fix \( k \) and let \( \rho = k + i k^\perp \). We define \( u \) as the solution to (29) with the complex-valued illumination \( g \) close to \( u_{\rho|x} \). The complex-valued internal data are then \( d = \mu u \). As before, we assume that \( k \geq 1 \) and \( \mu \in C^{k+1}(\bar{X}) \) so that \( d \in C^{k+1}(\bar{X}; \mathbb{C}) \), which implies that \( \beta \) and \( \gamma \) defined in (53) are of class \( C^k(\bar{X}) \).

The results of proposition 3.7 yield the following result.

### Theorem 3.10.

Let us assume that \( (\mu, q) \) and \( (\tilde{\mu}, \tilde{q}) \) are the elements in \( \mathcal{P} \) and that \( \| g - u_{\rho|x} \|_{C^1(\bar{X})} \) is sufficiently small so that \( u \) does not vanish on \( X \). Under the hypotheses of proposition 3.7 and assuming that \( |k| \geq |k_0| \) sufficiently large, we have that

\[
\| \mu - \tilde{\mu} \|_{C^{k-1}(\bar{X})} \leq C\|d - \tilde{d}\|_{C^k(\bar{X}; \mathbb{C})}.
\]

Moreover, we have the following stability result provided that \( k \geq 3 \):

\[
\| q - \tilde{q} \|_{C^{k-1}(\bar{X})} \leq C\|d - \tilde{d}\|_{C^k(\bar{X}; \mathbb{C})}.
\]

**Proof.** Let us define \( \mu_0 = d\partial_X / g \) and \( \tilde{\mu} = \tilde{d}\partial_X / g \). By assumption, \( g \) does not vanish on \( \partial X \).

We thus deduce that \( \| \mu_0 - \tilde{\mu}_0 \|_{C^1(\bar{X})} \) is controlled by \( \|d - \tilde{d}\|_{C^1(\bar{X}; \mathbb{C})} \). The rest of the proof of the theorem is a direct consequence of the results obtained in the preceding section and of the proof of theorem 3.6. \( \square \)

We now make a few comments on the differences between the \( 2n \)- and \( 2 \)-internal data settings. In the presence of \( n \) fields, the geometry of \( X \) is allowed to be rather general. In contrast, the \( 2 \)-data setting requires much stronger convexity assumptions on \( X \) to avoid that integral curves of the vector field be too close to the boundary \( \partial X \) for too long, which would result in a severe lack of stability. Since the integral curves of the vector fields \( \beta \) and \( \bar{\beta} \) are not known a priori, more (Lipschitz) regularity is required on the vector fields to ensure that information propagates along near-by trajectories. This is the reason for the replacement of \( k \) in theorem 3.6 by \( 'k - 1' \) in theorem 3.10.
4. Inverse diffusion with internal data

We now return to the diffusion equation with unknown diffusion coefficient $D$ and unknown absorption coefficient $\sigma_a$:

$$-\nabla \cdot D \nabla u + \sigma_a u = 0, \quad X, \quad u = g, \quad \partial X.$$  

The theorems stated in section 2 are straightforward consequences of the results presented in this section in a slightly more general setting.

We easily verify (this is standard Liouville transformation) that

$$v = \sqrt{D} u$$ solves

$$\Delta v + q v = 0,$$

with

$$q = -\frac{\Delta \sqrt{D}}{\sqrt{D}} - \frac{\sigma_a}{D}.$$  

The internal data in photoacoustics are given by

$$d = \sigma_a u = \sqrt{D} v = \mu v, \quad \mu := \frac{\sigma_a}{\sqrt{D}}.$$  

We assume that we know $\sqrt{D}$ on $\partial X$. This allows us to prescribe $v$ on $\partial X$ and thus to reconstruct $\mu$ and $q$ as in the preceding section. Then we find that

$$-\Delta \sqrt{D} - q \sqrt{D} = \mu,$$  

so we can solve for $\sqrt{D}$ and then get $\sigma_a = \mu \sqrt{D}$. Let us recall that $\sqrt{D} \in Y = H^{\frac{2k+1}{2}}(X) \subset C^{k+1}(\bar{X})$. We assume that $(D, \sigma_a) \in \mathcal{M}$ for some $M$. This implies that $(q, \mu) \in \mathcal{P}$ for some $P$. Indeed, that $\sqrt{D}$ solves (64) implies that $\Delta + q$, whose resolvent is compact, does not have 0 as an eigenvalue. Here, as always, $k \geq 1$.

The above calculations show the unique reconstruction of $(D, \sigma_a)$ from internal data for well-chosen boundary distributions as stated in theorem 2.1. From theorem 3.6 in the 2n-internal data setting, we get the following result.

**Theorem 4.1.** Let $k \geq 2$ and assume that $(D, \sigma_a)$ and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D|_{\partial X} = \tilde{D}|_{\partial X}$ on $\partial X$. Then there is an open set of $2n$ real-valued boundary values $g$ in $C^{k,\alpha}(\partial X)$ for $\alpha > \frac{1}{2}$ such that we have the stability estimate

$$\|D - \tilde{D}\|_{C^{k}(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k/2}(X)} \leq C(M, k)\|d - \tilde{d}\|_{C^{\alpha,1}(X)}.$$  

**Proof.** The main result consists of getting the stability on $D$ mentioned above. Since $k \geq 2$, we have stability of the reconstruction of $q$ in $C^{k-2}(\bar{X})$ and of $\mu$ in $C^{k}(\bar{X})$ provided that the boundary conditions are well chosen. Thus we have

$$-(\Delta + q)(\sqrt{D} - \sqrt{\tilde{D}}) = \mu - \tilde{\mu} + (q - \tilde{q})\sqrt{\tilde{D}}.$$  

By elliptic regularity, we deduce that $(\sqrt{D} - \sqrt{\tilde{D}})$ is bounded in $C^{k}(\bar{X})$, and hence the result.

Finally, we deduce from theorem 3.10 the following result.

**Theorem 4.2.** Under the hypotheses of theorem 3.10 and those of theorem 4.1, we obtain in the 2n-internal data setting the following result. Let $k \geq 3$ and $(D, \sigma_a)$ and $(\tilde{D}, \tilde{\sigma}_a)$ be in $\mathcal{M}$. Let us assume that $D|_{\partial X} = \tilde{D}|_{\partial X}$.

Then there is an open set of two real-valued boundary conditions $g$ in $C^{k,\alpha}(\partial X)$ for $\alpha > \frac{1}{2}$ such that we have the stability estimate

$$\|D - \tilde{D}\|_{C^{k-1}(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k/2}(X)} \leq C(M, k)\|d - \tilde{d}\|_{C^{\alpha,1}(X)}.$$  

The proof of the above theorem is the same as that of theorem 4.1.
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