CORRECTION TERMS AND
THE NON-ORIENTABLE SLICE GENUS

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Abstract. By considering negative surgeries on a knot \( K \) in \( S^3 \), we derive a lower bound to the non-orientable slice genus \( \gamma_4(K) \) in terms of the signature \( \sigma(K) \) and the concordance invariants \( V_i(K) \), which strengthens a previous bound given by Batson, and which coincides with Ozsváth–Stipsicz–Szabó’s bound in terms of their \( \upsilon \) invariant for L-space knots and quasi-alternating knots. A curious feature of our bound is superadditivity, implying, for instance, that the bound on the stable non-orientable genus is sometimes better than the one on \( \gamma_4(K) \).

1. Introduction

Given a knot \( K \) in \( S^3 \), it is a very classical problem to determine the minimal genus of an orientable surface \( F \) in \( B^4 \) whose boundary is \( K \). More recently, some attention has been drawn to the case of non-orientable surfaces instead. Namely, one can define \( \gamma_4(K) \) as the minimal non-orientable genus among all such surfaces, where the non-orientable genus of \( F \) is defined as \( b_1(F) \).

Batson, and Ozsváth, Stipsicz, and Szabó, on the other hand, gave lower bounds in terms of Heegaard Floer data. More precisely, Batson proved that

\[
\gamma_4(K) \geq \sigma(K) - d(S^3_{-1}(K)),
\]

where \( d(S^3_{-1}(K)) \) is the Heegaard Floer correction term (or \( d \)-invariant) of the 3-manifold obtained as \((-1)\)-surgery along \( K \), in its unique spin\(^c\) structure (which is hence omitted from the notation) \([2]\).

Ozsváth, Szabó and Stipsicz proved that

\[
\gamma_4(K) \geq |\sigma(K) - \upsilon(K)|,
\]

where \( \upsilon \) is a concordance invariant defined in terms of the Floer homology package \([18\text{ Theorem 1.2}]\). Gilmer and Livingston gave lower bounds on \( \gamma_4 \) using Casson–Gordon invariants \([11]\).

The main goal of this manuscript is to provide a new lower bound that generalises Batson’s. It will be phrased in terms of the concordance invariants \( \{V_i(K)\}_i \), associated to the mirror \( \overline{K} \) of \( K \); these invariants were defined by Rasmussen \([21]\) and further studied by Ni and Wu \([16]\) (see also Section \([2]\) below). We will further package these invariants into a single integer-valued invariant that we call \( \varphi \),

\[
\varphi(K) = \min_{m \geq 0} \{m + 2V_m(K)\},
\]

Theorem 1.1. For every knot \( K \) in \( S^3 \),

\[
\gamma_4(K) \geq \frac{\sigma(K)}{2} - \varphi(K).
\]

The existence of such a bound was indicated, but not made explicit, by Batson in his PhD thesis \([3]\). Moreover, since \( d(S^3_{-1}(K)) = 2V_0(\overline{K}) \geq \varphi(K) \), this is a strengthening of \((1.1)\). Equation \((1.3)\) also implies the existence of a bound in terms of the invariant \( \nu^+ \) defined by Hom and Wu \([12]\). By definition, one has \( V_{\nu^+}(\overline{K}) = 0 \), so Theorem \((1.1)\) implies at once

\[
\gamma_4(K) \geq \frac{\sigma(K)}{2} - \nu^+(\overline{K}).
\]

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Note that this bound is formally identical to (1.1), (1.2) and (1.3); to the best of the authors’ knowledge, this bound never appeared in literature.

We will show below that the bound of Theorem 1.1 is sharp (see Remark 5.2), and agrees with the one of (1.2) in the case of alternating knots and L-space knots (see Proposition 6.2).

We note here that the bound (1.1) presents the following curious feature: it is superadditive in the knot $K$, in the sense that the bound for $K_1 \# K_2$ can be strictly larger than the sum of the two bounds for $K_1$ and $K_2$. As a special case, the bound for $nK$ can give more information on $\gamma_4(K)$ than the bound for $K$. In Proposition 7.1 we will exhibit an example where this phenomenon actually occurs.

Using superadditivity, we can optimise the bound above as follows:

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - \omega(K),$$

where $\omega(K)$ is defined as

$$\lim_{n \to \infty} \frac{1}{n} \varphi(nK) \leq \varphi(K).$$

**Organisation of the paper.** In Section 2 we recall some basic facts about spin$^c$ structures on 3- and 4-manifold and $d$-invariants, and we state all the results concerning them that we use in this paper. In Section 3 we fix the notation and we construct a cobordism $W_0$ from a particular 3-manifold $Q$ (defined in that section) to $S^3_n(K)$, which will be crucial to deduce the bound in Equation (1.3). In Section 4 we label spin$^c$ structures on $W_0$, compute their Chern classes, and understand their restrictions to $\partial W_0$. In Section 5 we apply a twisted version of Ozsváth–Szabó’s inequality (see Theorem 2.4) to $W_0$ to obtain the desired bound on $\gamma_4(K)$. In Section 6 we compare our bound to Batson’s and Ozsváth–Stipsicz–Szabó’s (see Equations (1.1) and (1.2)), and we refine it using superadditivity. Finally, in Section 7 we give an example of a knot $K$ where the bound for $nK$ is actually better the bound for $K$.

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## 2. ALL YOU NEED IS CORRECTION TERMS

Given an oriented manifold $M$ of dimension 3 or 4, recall that the set of spin$^c$ structures $\text{Spin}^c(M)$ is an affine space over $H^2(M; \mathbb{Z})$. Given an oriented 4-manifold $X$ with boundary $\partial X = Y$, the restriction map

$$\text{Spin}^c(X) \to \text{Spin}^c(Y),$$

is modelled over

$$H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z}).$$

To every spin$^c$ structure $s \in \text{Spin}^c(M)$ it is possible to associate an element in $H^2(M; \mathbb{Z})$, known as the (first) Chern class of $s$, and usually denoted by $c_1(s)$. The map

$$c_1: \text{Spin}^c(M) \to H^2(M; \mathbb{Z})$$

is injective if and only if $H^2(M; \mathbb{Z})$ has no 2-torsion. A spin$^c$ structure $s \in \text{Spin}^c(M)$ is called torsion if $c_1(s)$ is a torsion element in $H^2(M; \mathbb{Z})$.

Let $-M$ denote the manifold $M$ endowed with the opposite orientation. There is a canonical bijection

$$\iota: \text{Spin}^c(M) \to \text{Spin}^c(-M),$$

which is modelled over the canonical isomorphism $\iota: H^2(M; \mathbb{Z}) \to H^2(-M; \mathbb{Z})$ (see [9, Section 1.2.3]). If $s \in \text{Spin}^c(M)$, we will denote by the same letter $s$ the corresponding spin$^c$ structure on $-M$. It is worth noting that such a bijection commutes with the restriction map (see Equation (2.1)), and that

$$c_1(\iota(s)) = \iota(c_1(s)).$$
Remark 2.1. Let $X^4$ be the trace of the 2-handle cobordism from $S^3$ to $S^3_0(K)$, where $K$ is a knot in $S^3$ and $n > 0$ is a positive integer. Then we can label the spin$^c$ structures on $X$ as follows: we let $s_k$ denote the unique spin$^c$ structure on $X$ such that

$$\langle c_1(s_k), [\Sigma] \rangle = n + 2k,$$

where $\Sigma$ is a Seifert surface for $K$ in $S^3 \times I$, capped off with the core of the 2-handle. From the labelling above, we derive a labelling of spin$^c$ structures over $S^3_0(K)$ by $\mathbb{Z}/n\mathbb{Z}$, by setting

$$t_k := s_k|_{S^3_0(K)},$$

where we do not make the distinction between an integer and its class modulo $n$. Here and in the following, we refer the reader to [20] Section 2.4 for further details.

In what follows, we say that a pair $(Y, t)$ as above, where $t$ is a torsion spin$^c$ structure on the 3-manifold $Y$, is a torsion spin$^c$ 3-manifold.

In [19], Ozsváth and Szabó introduce a Heegaard Floer theoretical invariant $d(Y, t)$, called the correction term or $d$-invariant, associated to a pair $(Y, t)$, where $Y$ is a rational homology 3-sphere equipped with a spin$^c$ structure $t$. In [19] Section 9, they explain how it is possible to generalise it to invariants $d_b$ and $d_t$ (bottom and top) associated to a torsion spin$^c$ 3-manifold $(Y, t)$, where $Y$ is now a 3-manifold with standard $HF^\infty$ (which is equivalent to having trivial triple cup product [15]). See also [14] Section 3 for an introduction to $d$-invariants of arbitrary 3-manifolds with standard $HF^\infty$.

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In the case of rational homology 3-spheres we have

$$d(Y, t) = d_b(Y, t) = d_t(Y, t) = d(Y, t).$$

More generally, one has the following.

**Theorem 2.2** ([19] Proposition 4.2, [14] Proposition 3.7, and [4] Proposition 3.8]. Let $(Y, t)$ be a torsion spin$^c$ 3-manifold, and suppose that $Y$ has standard $HF^\infty$. Then, under the canonical identification $\text{Spin}^c(Y) \cong \text{Spin}^c(-Y)$,

$$d_b(Y, t) = -d_t(-Y, t) = d(Y, t).$$

In the rest of this section, we state the results that we need about $d$-invariants.

The following result by Ni and Wu allows us to compute $d$-invariants for surgeries on a knot $K \subseteq S^3$ in terms of some knot invariants $V_i$, which were first introduced in [21] with the name of $h_i$. We refer to [16] Section 2.2 for the definition of $V_i$.

**Theorem 2.3** ([16] Proposition 1.6 and Remark 2.10]. Given positive integers $0 \leq k < n$, then

$$d(S^3_n(K), t_k) = -\frac{n - (2k - n)^2}{4n} - 2 \max \{V_k(K), V_{n-k}(K)\}.$$

Correction terms can be used to give restrictions to intersection forms of 4-manifolds bounding a given 3-manifold (compare also with [19] Theorem 9.15]).

**Theorem 2.4** ([4] Theorem 4.1]. Let $(W, s)$ be a negative semi-definite spin$^c$ cobordism from $(Y, t)$ to $(Y', t')$, two torsion spin$^c$ 3-manifolds, such that the map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ induced by the inclusion is injective. Then

$$c_1(s)^2 + b_2^-(X) \leq 4d(Y', t') + 2b_1(Y') - 4d(Y, t) - 2b_1(Y).$$

3. Notation and Construction

Let $K$ be a knot in $S^3$. If we consider $S^3$ as the boundary of the 4-ball $B^4$, the (orientable) slice genus $g_s$ is defined as the minimum genus of a smooth orientable surface is $B^3$ whose boundary is $K$, and it is a well-studied invariant of $K$. More recently, the non-orientable slice genus $\gamma_4$ has been studied. We have the following definition.
Figure 1. The figure shows the 4-manifold $W$ obtained by attaching a $(-n)$-framed 2-handle (whose trace we denote by $X$) to $B^4$ along a knot $K \subseteq S^3$. $N = N_W(\hat{F})$ denotes a neighbourhood of $\hat{F}$ in $W$, and $Q = \partial N$.

**Definition 3.1.** Given a knot $K$ in $S^3$, we define its *non-orientable slice genus* as

$$\gamma_4(K) = \min \left\{ b_1(F) \mid F \hookrightarrow B^4 \text{ smooth, non-orientable, } \partial F = K \right\},$$

where $b_1(F)$ denotes the first Betti number of $F$.

**Remark 3.2.** With this definition of $\gamma_4$, one always has $\gamma_4(K) \geq 1$. One could also consider the 4-dimensional crosscap number instead; this is the minimal number $h$ such that $K$ bounds a punctured $\#^h \mathbb{R}P^2$ in $B^4$. The two definitions are indeed equivalent except when $K$ is slice, in which case our definition yields $\gamma_4(K) = 1$, while the 4-dimensional crosscap number is 0. We note here that, when $K$ is slice, the bound in (1.3) is in any case $\gamma_4(K) \geq 0$, so this is in fact a bound for the crosscap number as well; this is true since, when $K$ is slice, both $\sigma(K)$ and $\varphi(L)$ vanish (see Proposition 6.1(2) below). Our proof, however, actually uses the definition of $\gamma_4$ given above, to which therefore we stick.

In [2], Batson proved that the non-orientable slice genus can be arbitrarily large. More specifically, for a non-orientable surface $F$ as in Definition 3.1, Batson gives the following inequality (see [2, Theorem 4]):

$$b_1(F) + 2d(S^3_{-1}(K)) \geq \frac{\varepsilon(F)}{2}.$$  

Here $d(S^3_{-1}(K))$ denotes the $d$-invariant of $S^3_{-1}(K)$ in the unique spin$^c$ structure, whereas $\varepsilon(F)$ is the *normal Euler number* of $F$: given a non-vanishing section $s$ of the normal bundle $\nu_F$ (which always exists since $F$ deformation retracts on a 1-complex), we let

$$\varepsilon(F) = -\text{lk}(K, s(K)).$$

In [2], Batson combines Equation (3.1) and the 'signature' inequality

$$b_1(F) \geq \sigma(K) - \frac{\varepsilon(F)}{2}$$  

to derive the bound for the non-orientable slice genus in Equation (1.1). The main result of this paper is a generalisation of Equation (3.1), where instead of the $(-1)$-surgery along $K$ we consider $(-n)$-surgeries for arbitrary integers $n \geq 1$. Inspired by [2] and [14], we construct a negative semi-definite cobordism from a 3-manifold $Q$ to $S^3_{-n}(K)$, and use Theorem 2.4 to give a lower bound to $b_1(F)$.

We now give the details of the construction, illustrated in Figure 1. Let $K$ be a knot in $S^3 = \partial B^4$, and let $F$ denote a smooth non-orientable surface properly embedded in $B^4$ such that $\partial F = K$. Fix
an integer \( n > 0 \). Let \( W \) denote the 4-manifold obtained by attaching a \((-n)\)-framed 2-handle to \( B^4 \), along \( K \subset \partial B^4 \). We denote with \( Y \) the boundary of \( W \), i.e. \( Y = S^3_{-n}(K) \). Then the surface \( F \) can be capped off with the core of the 2-handle to obtain a closed surface \( \hat{F} \subseteq W \). Notice that

\[
b_1(\hat{F}) + 1 = b_1(F) =: h.
\]

If \( e = e(F) \) denotes the normal Euler number of \( F \), and \( e(\hat{F}) \) denotes the Euler number of the closed surface \( \hat{F} \), then we have

\[
e(\hat{F}) = e - n.
\]

As already noticed in [2], \( e \) is even, because the self-intersection of \( F \) in \( B^4 \) can be computed algebraically over \( \mathbb{Z}/2\mathbb{Z} \).

Let \( N = N_W(\hat{F}) \) denote a regular neighbourhood of \( \hat{F} \) in \( W \). We define \( Q = \partial N \). Notice that \( Q \) (resp. \( N \)) is a circle (resp. disc) bundle over the closed surface \( \hat{F} \cong (\mathbb{RP}^2)^{\#h} \) of Euler number \( e - n \). According to the notation in [14, Section 2], we have \( N \cong P_{h,e-n} \) and \( Q \cong Q_{h,e-n} \), and moreover \( Q \) has standard HF\(^\infty\).

The manifold \( W^\circ := W \setminus N \) is a cobordism between \( Q \) and \( S^3_{-n}(K) \). Since the labelling of spin\(^c\) structures is better understood for positive surgeries, we consider also the manifold \(-W\), obtained from \( W \) by reversing the orientation; \(-W\) is the 4-manifold obtained by attaching an \( n \)-framed 2-handle to \( B^4 \) along \( \overline{K} \). This allows us to label the spin\(^c\) structures on \( W \) and on \( Y \); by a slight abuse of notation, we write \( \mathcal{G}_k \) and \( \mathcal{T}_k \), dropping the identifications \( \text{Spin}^c(W) \cong \text{Spin}^c(-W) \) and \( \text{Spin}^c(Y) = \text{Spin}^c(-Y) \).

4. Labelling Spin\(^c\) structures

4.1. (Co-)homological computations. The aim of this subsection is to compute \( H^2(W^\circ; \mathbb{Z}) \), in order to understand spin\(^c\) structures on \( W^\circ \). Consider the Mayer–Vietoris long exact sequence in cohomology associated to \( W = W^\circ \cup Q N \). When we do not specify it, we assume that we are using \( \mathbb{Z} \) coefficients.

| \( W \) | \( W^\circ \) | \( \cup \) | \( N \) | \( Q \) |
|---|---|---|---|---|
| \( H^0 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \oplus \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |
| \( H^1 \) | 0 | 0 | \( \mathbb{Z}^{h-1} \) | \( \mathbb{Z}^{h-1} \) |
| \( H^2 \) | \( \mathbb{Z} \) | ? | \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}^{h-1} \oplus T \) |
| \( H^3 \) | 0 | \( \mathbb{Z} \oplus 0 \) | \( \mathbb{Z} \) |

The cohomology of \( W \) can be easily obtained by recalling that \( W \) is constructed by attaching a 2-handle on a \( B^4 \). The cohomology of \( N \) is also straightforward, since \( N \) deformation retracts on \( \overline{F} = (\mathbb{RP}^2)^{\#h} \). As for \( Q \), its cohomology can be deduced from [14, Lemma 2.1], and it is written in the table above. \( T \) is the torsion subgroup of \( H_1(Q) \), which is, according to [14, Lemma 2.1],

\[
T = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } e(\hat{F}) \text{ is even,} \\
\mathbb{Z}/4\mathbb{Z} & \text{if } e(\hat{F}) \text{ is odd.}
\end{cases}
\]

In both cases, the map \( H^2(N) \cong \mathbb{Z}/2\mathbb{Z} \rightarrow T \) is non-trivial. From the cohomology groups that we already know (and the fact that the map \( H^1(N) \rightarrow H^1(Q) \) is an isomorphism) we can deduce almost all the cohomology groups of \( W^\circ \). \( H^2(W^\circ) \) will depend on the parity of \( e(\hat{F}) \), according to the following lemma.

Lemma 4.1. We have that

\[
H^2(W^\circ) = \begin{cases} 
\mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } e(\hat{F}) \text{ is even,} \\
\mathbb{Z}^h & \text{if } e(\hat{F}) \text{ is odd.}
\end{cases}
\]

Proof. From the long exact sequence above we have an exact sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow H^2(W^\circ) \rightarrow \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} \rightarrow 0,
\]

regardless of the parity of \( e(\hat{F}) \). The two possible extensions are \( \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}^h \). If we consider the reduction modulo 2, which maps \( H^2(W^\circ) \) to \( H^2(W^\circ; \mathbb{F}_2) \), the two possible extensions are mapped
Lemma 4.2. The intersection form \( Q \) on \( H_2(W^\circ) \) is given by \( Q_W = (-n) \). The intersection form \( Q^W \) on \( H^2(W) \) is given by \( Q^W = (-\frac{1}{n}) \).

Proof. The intersection form on \( H_2(W) \) is \((-n)\) because the 4-manifold \( W \) is obtained by attaching a \((-n)\)-framed 2-handle to \( B^4 \).

The intersection form on \( H^2(W) \) can be worked out by considering the following portion of the long exact sequence in homology associated to the couple \((W,Y)\), where \( Y = S^3_{-n}(K) \):

\[
0 \to H_2(W) \to H^2(W) \to H_1(Y) \to 0.
\]

Such a short exact sequence is isomorphic to

\[
0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.
\]

The generator of \( H_2(W) \) is mapped to \( n \) times the generator of \( H^2(W) \), so the intersection form on \( H^2(W) \) is represented by the matrix \((-\frac{1}{n})\).

It is also worth noting that for each \( c \in H^2(W) \), \( c|_{W^\circ} \) restricts to a torsion spin\(^c\) structure on both boundary components, and therefore it makes sense to consider its square. We claim that:

\[
Q^W(c|_{W^\circ}) = Q^W(c).
\]

Indeed, the class \( nc \in H^2(W) \cong H_2(W,Y) \) is in the image of the map \( H_2(W) \to H_2(W,Y) \). Now, the map \( \iota : H_2(W^\circ) \to H_2(W) \) is surjective: this comes from the Mayer–Vietoris sequence for \( W = W^\circ \cup N \), since the connecting morphism \( \partial : H_2(W) \to H_1(Y) \) vanishes (see the proof of Lemma 1.1) and \( H_2(N) = 0 \).

Therefore, there is an element \( d \in H_2(W^\circ) \) such that \( \iota(d) = nc \). The elements \( d \) and \( nc \) can be represented by some copies of a surface \( S \subseteq W^\circ \). The squares \( Q^W(c) \) and \( Q^W(nc) \) can be computed as the algebraic self-intersection \( S \cdot S \) of \( S \), which in turn can be computed in an arbitrarily small neighbourhood of \( S \).
4.3. Spin$^c$ structures. Recall (see Remark 2.1) that spin$^c$ structures on $-W$ are labelled by integers as follows:

$$\langle c_1(s_k), [\Sigma] \rangle = 2k + n.$$  

By symmetry we also get a labelling for Spin$^c(W)$, and we still denote the spin$^c$ structures on $W$ by $s_k$. $s_k$ and $s_{k'}$ restrict to the same spin$^c$ structure on $Y$ if and only if $n \mid (k - k')$. In such a case we denote the restriction to $Y$ by $t_k = t_{k'}$.

It is worth noting that we have isomorphisms $H^2(W) \cong \mathbb{Z}$ and $H^2(Y) \cong \mathbb{Z}/n\mathbb{Z}$ such that, under these identifications, the restriction map is the usual projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, and $c_1(t_k) \equiv 2k \pmod{n}$.

In order to apply Theorem 2.4 we need a spin$^c$ structure on the cobordism $W^\circ$ that restricts to a torsion spin$^c$ structure on $Q$. Therefore, we introduce the following notation:

**Definition 4.3.** Given a 4-manifold $X$, we define Spin$^c_{\text{tor}}(X)$ to be the subset of Spin$^c(X)$ of elements that restrict to torsion spin$^c$ structures on $\partial X$.

Notice that in our case Spin$^c_{\text{tor}}(W^\circ)$ is given by all spin$^c$ structures that restrict to torsion spin$^c$ structures on $Q$, because all spin$^c$ structures on $Y$ are already torsion. We will now give a classification of Spin$^c_{\text{tor}}(W^\circ)$ in the case of $e(\widehat{F})$ odd (or, equivalently, $n$ odd).

4.4. The case $e(\widehat{F})$ odd. By Lemma 4.1 we have that $H^2(W^\circ) \cong \mathbb{Z}^h$. From the Mayer–Vietoris exact sequence associated to $W = W^\circ \cup Q$, $N$ we find:

$$0 \to H^2(W) \xrightarrow{\alpha} H^2(W^\circ) \oplus H^2(N) \xrightarrow{\beta} H^2(Q) \to 0$$

We have that $\alpha(1) = (c, 1)$ for some nonzero $c \in \mathbb{Z}^h$, otherwise the quotient would contain a $\mathbb{Z}/2\mathbb{Z}$ summand. Then we have that

$$\mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z} \cong \frac{\mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z}}{\langle (c, 1) \rangle} \cong \frac{\mathbb{Z}^h}{\langle 2c \rangle}.$$  

This implies that $c = 2d$, where $d \in \mathbb{Z}^h$ is a primitive element. We denote by $x \in H^2(W^\circ)$ the element that corresponds to $d$, and we let $A = \langle x \rangle \subseteq H^2(W^\circ)$. Therefore, Spin$^c_{\text{tor}}(W^\circ)$ is an affine space over $A$.

It follows from the exact sequence above that the image of the map

$$\text{Spin}^c(W) \to \text{Spin}^c(W^\circ)$$

is contained inside Spin$^c_{\text{tor}}(W^\circ)$. Moreover, the map is modelled on the map

$$H^2(W) \cong \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \cong A.$$  

It follows from the naturality of the first Chern class that $c_1(s_k|W^\circ) = (2n + 4k)x$:

$$\begin{align*}
\text{Spin}^c(W) & \xrightarrow{c_1} \text{Spin}^c_{\text{tor}}(W^\circ) \\
\mathbb{Z}/2\mathbb{Z} & \xrightarrow{2} 2n + 4\mathbb{Z} \subseteq A
\end{align*}$$

The Chern classes of all spin$^c$ structures in Spin$^c_{\text{tor}}(W^\circ)$ form the subset $2n + 2\mathbb{Z} = 2\mathbb{Z} \subseteq \mathbb{Z} \cong A$. This motivates the following definition.

**Definition 4.4.** We define $s_k^\circ \in \text{Spin}^c_{\text{tor}}(W^\circ)$ to be the spin$^c$ structure on $W^\circ$ that restricts to a torsion spin$^c$ structure on $Q$ and that satisfies

$$c_1(s_k^\circ) = (2n + 2k)x.$$
**Remark 4.5.** It follows from the computations above that 
\[ \text{Spin}^c(W) \to \text{Spin}_{\text{tor}}^c(W^\circ) \]
\[ s_k \mapsto s_{2k}^0 \]
and that \( s_{2k}^0 \in \text{Spin}_{\text{tor}}^c(W^\circ) \) extends to a spin\(^c\) structure on \( W \) if and only if \( k \) is even.

We now want to understand the restriction of the spin\(^c\) structure \( s_k^0 \) to \( Y \). This is done in the following lemma. Instead of \( W \), we use \( W_n = -W \) and \( S^3_n(\mathcal{K}) = -Y \) to label the spin\(^c\) structure, so we can stick to the usual positive surgery conventions.

**Lemma 4.6.** For all \( k \in \mathbb{Z} \), we have that 
\[ s_{2k}^0 \big|_{S^3_n(\mathcal{K})} = t_k \quad \text{and} \quad s_{n+2k}^0 \big|_{S^3_n(\mathcal{K})} = t_k. \]

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^2(W_n) & \xrightarrow{\pi} & A \\
\downarrow{\pi} & & \downarrow{r} \\
H^2(S^3_n(\mathcal{K})) & \xrightarrow{\pi} & \mathbb{Z}^2 \\
\end{array}
\]

Recall that we chose isomorphisms \( H^2(W_n) \cong \mathbb{Z} \) and \( H^2(S^3_n(\mathcal{K})) \cong \mathbb{Z}/n\mathbb{Z} \) such that \( \pi(1) = 1 \in \mathbb{Z}/n\mathbb{Z} \). Then
\[ c_1(t_k) = \pi(c_1(s_k)) = n + 2k = 2k. \]
Since \( n \) is odd, \( 2 \) is invertible modulo \( n \), so every spin\(^c\) structure on \( S^3_n(\mathcal{K}) \) is determined by its first Chern class.

By the naturality of the Chern class we have that for every \( k \in \mathbb{Z} \), the following diagram commutes:

\[
\begin{array}{ccc}
c_1(s_k) & \xrightarrow{\pi} & c_1(s_{2k}^0) \\
\downarrow{\pi} & & \downarrow{r} \\
c_1(t_k) & \xrightarrow{\pi} & \mathbb{Z}/n\mathbb{Z}^2 \\
\end{array}
\]

From this, we obtain that \( s_{2k}^0 \big|_{S^3_n(\mathcal{K})} = t_k \).

For the case of \( s_{n+2k}^0 \), recall that \( c_1(s_{n+2k}) = 4n + 4k \). From the commutativity of the diagram below we deduce that \( s_{n+2k}^0 \big|_{S^3_n(\mathcal{K})} = t_k \).

\[
\begin{array}{ccc}
2n + 2k & \xrightarrow{\pi} & 4n + 4k \\
\downarrow{\pi} & & \downarrow{2k} \\
2k & \xrightarrow{r} & \\
\end{array}
\]

\[ \square \]

4.5. **The case \( e(\mathcal{F}) \) even.** When \( e(\mathcal{F}) \) is even, \( H^2(W^\circ) \cong \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z} \) by Lemma 4.1. One can check that \( \text{Spin}_{\text{tor}}^c(W^\circ) \) is an affine space over a submodule 
\[ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z}, \]
where the \( \mathbb{Z} \) summand is generated by a primitive element \( x \). One can then define 
\[ s_k^0 := s_k|_{W^\circ} \in \text{Spin}_{\text{tor}}^c(W^\circ), \]
and, if \( \gamma \) denotes the generator of the \( \mathbb{Z}/2\mathbb{Z} \) summand,
\[ \tilde{s}_k^0 := s_k^0 + \gamma \in \text{Spin}_{\text{tor}}^c(W^\circ). \]
One can check that \( \tilde{s}_k \) restricts to \( Q \) to a non-extendible spin\(^c\) structure \( \tilde{t} \), and to \( Y \) to the spin\(^c\) structure \( t_{k+\frac{1}{2}} \). Moreover, we have that 
\[ c_1(\tilde{s}_k)^2 = -\frac{(n+2k)^2}{n}. \]
Note that $n$ is even because so is $e(\tilde{F})$, so $k + \frac{n}{2}$ is an integer.

5. A BOUND FOR THE NON-ORIENTABLE SLICE GENUS

We now prove Theorem 1.1 that we restate here. Recall that we have defined $\varphi(K)$ to be the quantity $\min_{m \geq 0} \{ m + 2V_m(K) \}$.

**Theorem 1.1.** For every knot $K$ in $S^3$,

\[(5.3) \quad \gamma_4(K) \geq \frac{\sigma(K)}{2} - \varphi(K).\]

*Proof.* Choose an odd integer $n > 0$, and let $k$ be any integer. We denote by $[k]$ the representative for the residue class of $k$ modulo $n$ such that $0 \leq [k] < n$. By Remark 1.5 and Lemma 1.6 the spin$^c$ structure $\mathfrak{s}_{n+2k}$ restricts to a non-extendible spin$^c$ structure on $Q$, that we denote by $t$, and to $t_k$ on $\mathcal{Y}$.

We apply Theorem 2.4 to the cobordism $(W^e, \mathfrak{s}_{n+2k}^e)$ turned upside down, i.e. seen as a cobordism from $(-Y, t_k)$ to $(-Q, \tilde{t})$: the assumption that the map $H_1(Y; \mathbb{Q}) \to H_1(W^e; \mathbb{Q})$ be injective is automatically satisfied, since $Y$ is a rational homology sphere. The inequality of Theorem 2.4 then reads as follows:

\[(5.1) \quad c_1(\mathfrak{s}_{n+2k}^e)^2 + b^2_2(W^e) \leq 4d(-Q, \tilde{t}) - 4d(-S_{-n}^3(K), t_k).\]

We now compute each term of Equation (5.1). We have that $b^2_2(W^e) = 1$ and $b_1(Q) = h - 1$. Moreover,

\[c_1(\mathfrak{s}_{n+2k}^e)^2 = ((4n + 4k)x)^2 = \frac{1}{4n} \cdot (4n + 4k)^2 = \frac{4}{n} (n + k)^2,\]

where we used the fact that $Q^{W^e}(2x, 2x) = -\frac{1}{n}$.

As for the $d$-invariant of $S_{-n}^3(K)$, by Theorems 2.2 and 2.3 we have

\[d(-S_{-n}^3(K), t_k) = d(S_{-n}^3(K), t_k) = -\frac{n - (2[k] - n)^2}{4n} - 2 \max \{ \overline{V}_{[k]}, \overline{V}_{n-[k]} \},\]

where we set $\overline{V}_i := V_i(K)$.

Finally, by [14] Theorem 5.1 and Theorem 2.2 above, we have that

\[d(-Q, \tilde{t}) = d_4(Q, \tilde{t}) = -\left( \frac{e(\tilde{F}) - 2}{4} + a \right) \leq -\frac{e - n - 2}{4}.\]

Therefore, Equation (5.1) becomes

\[-\frac{4}{n} \cdot (n + k)^2 + 1 \leq \frac{n - (2[k] - n)^2}{n} + 8 \max \{ \overline{V}_{[k]}, \overline{V}_{n-[k]} \} - (e - n - 2) + 2h - 2,

which can be re-written as follows:

\[(5.2) \quad 2h + 8 \max \{ \overline{V}_{[k]}, \overline{V}_{n-[k]} \} \geq e - n - \frac{4(n + k)^2 - (2[k] - n)^2}{n}\]

By combining it with Equation (5.2) as in [2], we obtain:

\[(5.3) \quad 4h + 8 \max \{ \overline{V}_{[k]}, \overline{V}_{n-[k]} \} \geq 2\sigma(K) - n - \frac{4(n + k)^2 - (2[k] - n)^2}{n}\]

Given a fixed integer $m \geq 0$, it is not difficult to check that the best bound for $h$ coming from Equation (5.3) and involving $\overline{V}_m$ is obtained by setting $n = 2m + 2j + 1$ and $k = -n \pm m$ (where $j$ is an arbitrary non-negative integer). The bound for $\gamma_4(K)$ that we obtain in this case is then:

\[(5.4) \quad \gamma_4(K) \geq \frac{\sigma}{2} - m - 2\overline{V}_m.\]

By taking the maximum over $m \geq 0$ we conclude the proof of the theorem. \(\square\)

*Remark 5.1.* By setting $m = 0$ in Equation (5.3), we obtain exactly Batson’s inequality [11].
Remark 5.2. For every $m \geq 0$, the bound in Equation (5.4) is sharp, in the sense that for each $m$ there exists a knot $K_m$ such that $\gamma_4(K_m) = \frac{\sigma(K)}{2} - m - V_m(K)$. The knot $K_0 = T_{3, -4}$ exhibits that the inequality is sharp for $m = 0$, as already shown by Batson [2].

For $m \geq 1$, consider the torus knot $K = T_{3, -5}$, whose signature is 8. Since $K = T_{3, 5}$ is a positive torus knot, hence an L-space knot, the invariants $V_i(T_{3, 5})$ coincide with the torsion coefficients [19, Corollary 7.5]:

$$V_i(K) = \sum_{j>0} j a_{j+i},$$

where

$$\Delta_{K}(t) = a_0 + \sum_{j>0} a_j (t^j + t^{-j})$$

is the Alexander polynomial of $K$. One can explicitly compute that, for $K = T_{3, 5}$,

$$\Delta_{T_{3, 5}}(t) = t^4 - t^3 + t - 1 + t^{-1} - t^{-3} + t^{-4}.$$  

It follows that $V_1(K) = 1$ and that Equation (5.4) for $m = 1$ gives

$$\gamma_4(K) \geq \frac{8}{2} - (1 + 2) = 1.$$  

Since $K$ bounds a Moebius band in $B^4$, as shown in Figure 2 (see also [2, Section 4]), it follows that (5.4) is sharp for $m = 1$.

For all $m > 1$, consider the knot $mK$, the connected sum of $m$ copies of $K$. Recall from [5, Proposition 6.1] that the sequence $\{V_i(K)\}$ satisfies the following subadditivity property: $V_{k+l}(K\#L) \leq V_k(K)\#V_l(L)$ for each pair $(k, l)$ of non-negative integers and each pair $(K, L)$ of knots. By subadditivity of $\gamma_4$, subadditivity of the $V_i$, and additivity of the signature, we obtain

$$m = m\gamma_4(K) \geq \gamma_4(mK) \geq \frac{\sigma(mK)}{2} - (m + 2V_m(mK)) \geq m\left(\frac{\sigma(K)}{2} - (1 + 2V_1(K))\right) = m.$$  

It follows that all the inequalities above are actually equalities, and that therefore (5.4) is sharp for every $m \geq 1$.

Remark 5.3. In the proof of Theorem 1.1 we only considered surgery with some odd framing $n > 0$. If we considered the case of even $n$, and applied Theorem 2.4 to the torsion spin$^c$ structure $s_k^0$ (defined in Section 4.3), we would have obtained exactly the same bound as Equation (5.4) for all $m \geq 0$.

6. Comparison to other bounds

In this section we study some properties of the functions $\varphi$ and $\omega$ defined in the introduction, and discuss the relationship between the bounds given by (1.1), (1.2), and (1.3).

Proposition 6.1. The invariant $\varphi$ is a concordance invariant, with values in the non-negative integers. It has the following properties:
Proposition 6.2. When we need to compare \( \nu \), we may use \( \varphi \) instead.

Proof. The sequence \( \{V_i(K)\}_i \) is a concordance invariant, hence so is \( \varphi \); moreover, the quantity \( m + 2V_m(K) \) is a non-negative integer for each \( m \), and hence so is \( \varphi(K) \).

1. When \( m = 0 \), \( m + 2V_m(K) = 2V_0(K) \), while for \( m = \nu^+(K) \), \( m + 2V_m(K) = \nu^+(K) \). By definition, \( \varphi(K) \leq 2V_0(K) \) and \( \varphi(K) \leq \nu^+(K) \).

2. Observe that \( m + 2V_m(K) \) is always strictly positive if \( m > 0 \); hence, if \( \varphi(K) = \min_m \{m + 2V_m(K)\} = 0 \), the minimum can only be attained at \( m = 0 \), and in that case \( V_0(K) = 0 \), which implies \( \nu^+(K) = 0 \). The converse is obvious.

3. By [3] Lemma 5.1 we have that, under the given assumptions, \( V_{n+g}(K_1) \leq V_m(K_2) \) for each non-negative integer \( m \). It follows that \( m + g + 2V_{m+g}(K_1) \leq m + 2V_m(K_2) + g \), hence, minimising over \( m \),

\[
\varphi(K_1) \leq \min_{m \geq g} \{m' + 2V_{m'}(K_2)\} \leq \varphi(K_2) + g.
\]

Exchanging the roles of \( K_1 \) and \( K_2 \), we obtain the symmetric inequality.

4. Observe that there is a genus-1 cobordism from \( K_+ \) to \( K_- \), obtained by smoothing the double point in the trace of the crossing change homotopy. Thus, point (3) above shows that \( \varphi(K_-) - 1 \leq \varphi(K_+) \). Using [7] Theorem 6.1 we also obtain:

\[
V_m(K_+) \leq V_m(K_-),
\]

from which, for each \( m \geq 0 \),

\[
m + 2V_m(K_+) \leq m + 2V_m(K_-),
\]

and minimising over all values of \( m \) yields the desired inequality.

5. For each \( k, l \) non-negative integers, \( V_{k+l}(K_1 \# K_2) \leq V_k(K_1) + V_l(K_2) \) by [5] Proposition 6.1, hence

\[
\varphi(K_1 \# K_2) = \min_n \{n + 2V_n(K_1 \# K_2)\}
\]

\[
\leq \min_{k+l=n} \{k + l + 2V_k(K_1) + 2V_l(K_2)\}
\]

\[
= \min_k \{k + 2V_k(K_1)\} + \min_l \{l + 2V_l(K_2)\}
\]

\[
= \varphi(K_1) + \varphi(K_2).
\]

We will compare our bound with [1,2] obtained by Ozsváth–Stipsicz–Szabó, and in order to do so we need to compare \( \nu(K) \) with \( \varphi(K) \). We say that a knot is Floer-thin if its knot Floer homology is supported on the diagonal \( i-j = -\tau(K) \).

Proposition 6.2. When \( K \) is a Floer-thin knot with \( \tau(K) \geq 0 \) or an L-space knot, then \( \varphi(K) = -\nu(K) \) and \( \varphi(K) = 0 \).

In particular, the bound given by [5,4] for both \( K \) and \( \overline{K} \) is at most as strong as the one given by \( \nu \), when \( K \) is an L-space knot or an alternating knot.
Definition 6.6. We call \( \text{lemma } \) \[10 \]. By property (5) of Proposition 6.1, the function \( v(K) = v(T_{2,\pm(2n+1)}) \). It follows that it is enough to prove the statement for L-space knots. When \( K \) is an L-space knot, then a direct computation from the knot Floer complex shows that \( V_i(K) = 0 \) for every \( i \); hence \( \varphi(K) = 0 \). On the other hand, Borodzik and Hedden have shown in [7, Proposition 4.6] that

\[
v(K) = \Upsilon_K(1) = -\min_n \{ n + 2V_n(K) \} = -\varphi(K),
\]
as desired. \( \square \)

In the case of Floer-thin knots we can actually say more about \( \varphi \).

Proposition 6.3. If \( K \) is a Floer-thin knot with \( \tau(K) \geq 0 \), then we have

\[
\varphi(K) = \nu^+(K) = \tau(K) - \nu(K).
\]

If, additionally, \( K \) is quasi-alternating, then \( \varphi(K) = -\sigma(K)/2 \), and in this case the bounds (1.2) and (1.3) – applied to \( K \) and \( \overline{K} \) – yield

\[
\gamma_4(K) \geq 0.
\]

Proof. By [1, Equation (8)], we know that the minimum of \{ \( m + 2V_m(K) \) \} is attained at \( m = \tau(K) = \nu^+(K) \). This implies at once that \( \varphi(K) = \tau(K) \). The equality with \( \nu(K) \) follows from Proposition 6.2.

When \( K \) is quasi-alternating, \( \tau(K) = -\sigma(K)/2 \), and the second part of the statement readily follows. \( \square \)

In many instances, the bound given by \( \nu \) is better than the one given by \( \varphi \); this is true, for example, for many knots of the form \( K_1 \# K_2 \), where \( K_1 \) and \( K_2 \) are L-space knots.

Example 6.4. Consider the two knots \( K_1 = T_{2,3} \), \( K_2 = T_{5,6} \), and let \( K = K_1 \# K_2 \). One computes

\[
\sigma(K_1) = -2, \quad \sigma(K_2) = -16, \quad \nu(K_1) = -1 \quad \text{and} \quad \nu(K_2) = -6.
\]

Using the techniques from [13] as in [5], we can also compute \( \varphi(K) = 6 \) and \( \varphi(\overline{K}) = 0 \).

It follows that the bound given by (1.3), applied to both \( K \) and \( \overline{K} \), gives \( \gamma_4(K) \geq 1 \), while the bound given by (1.2) is \( \gamma_4(K) \geq 2 \).

As a consequence of Proposition 6.1, we deduce the following interesting feature of \( \varphi \).

Corollary 6.5. The invariant \( \varphi(K) \) is subadditive. In particular, the following identity holds:

\[
\lim_{n \to \infty} \frac{1}{n} \varphi(nK) = \inf \frac{1}{n} \varphi(nK).
\]

Proof. By property (5) of Proposition 6.1, the function \( n \mapsto \varphi(nK) \) is subadditive, in the sense that \( \varphi(aK + bK) \leq \varphi(aK) + \varphi(bK) \) for every \( a, b \geq 0 \). The existence of the limit follows from Fekete's lemma [10]. \( \square \)

Definition 6.6. We call \( \omega(K) = \lim_n \frac{1}{n} \varphi(nK) \).

We now introduce the stable non-orientable 4-genus \( \gamma_4^s(K) \) of \( K \), i.e. the limit \( \lim_{n \to \infty} \frac{1}{n} \gamma_4(nK) \). Notice that the limit exists since the sequence \( (\gamma_4(nK))_n \) is subadditive, and that \( \gamma_4^s(K) \leq \gamma_4(K) \).

Theorem 6.7. The invariant \( \omega(K) \) is a concordance invariant of \( K \), and it descends to a subadditive, homogeneous function \( \omega: \mathcal{C} \to \mathbb{R}_{\geq 0} \). Additionally:

1. \( \gamma_4^s(K) \geq \frac{\sigma(K)}{2} - \omega(K) \); 2. if there is an orientable genus-\( g \) cobordism between \( K_1 \) and \( K_2 \), then \( |\omega(K_1) - \omega(K_2)| \leq g \); 3. if there is a crossing change (from negative to positive) from \( K_- \) to \( K_+ \), then \( \omega(K_-) - 1 \leq \omega(K_+) \leq \omega(K_-) \).

As an immediate corollary to the theorem, we get the following:

Corollary 6.8. If the inequality in Theorem 6.7 is sharp, then \( \gamma_4(nK) = n\gamma_4(K) \) for each \( n \); in particular \( \gamma_4^s(K) = \gamma_4(K) \).
As remarked for \( \varphi \) above, \( \omega \) is not a homomorphism, since it takes only non-negative values. Note also that \( \omega \) is not identically 0, since, by Proposition 6.3 applied to \( nK \) for all \( n \geq 0 \), \( \omega(K) \) coincides with \( \sigma(K)/2 \) for Floer-thin knots with positive signature.

Also, by definition, \( \omega(K) \leq \varphi(K) \), and in particular the bound for \( \gamma^\text{st}_4(K) \) given by \( \varphi \) can be better than the bound given by \( \omega \) on \( \gamma_4(K) \) (see Proposition 7.1 for an example). This is by contrast with the bound given, for example, by \( \tau \), \( s \), or \( \nu^+ \) on the stable orientable slice genus: the first two are linear, while the third is sublinear in \( K \) [3, Theorem 1.4].

**Proof of Theorem 6.7.** The invariant \( \omega \) is a concordance invariant, since \( \varphi \) is, and it takes non-negative values, since \( \varphi \) does. Moreover, it is subadditive by construction:

\[
\omega(K#L) = \lim_n \left\{ \frac{1}{n} \varphi(n(K#L)) \right\} \leq \lim_n \left\{ \frac{1}{n} (\varphi(nK) + \varphi(nL)) \right\} = \\
= \lim_n \left\{ \frac{1}{n} \varphi(nK) \right\} + \lim_n \left\{ \frac{1}{n} \varphi(nL) \right\} = \omega(K) + \omega(L),
\]

where the inequality follows from the subadditivity of \( \varphi \) (Property (5) of Proposition 6.1).

It is also homogeneous, in the sense that \( \omega(nK) = n \omega(K) \):

\[
\omega(nK) = \lim_m \frac{1}{m} \varphi(mnK) = n \lim_m \frac{1}{mn} \varphi(mnK) = n \lim_m \frac{1}{m'} \varphi(m'K) = n \omega(K).
\]

(1) Applying 5.4 for \( nK \) we obtain, for each \( n \geq 1 \):

\[
\gamma_4(nK) \geq \frac{\sigma(nK)}{2} - \varphi(nK) = n \frac{\sigma(K)}{2} - \varphi(nK),
\]

from which

\[
\gamma^\text{st}_4(K) = \lim_n \frac{\gamma_4(nK)}{n} \geq \frac{\sigma(K)}{2} - \varphi(nK) = \frac{\sigma(K)}{2} - \omega(K).
\]

Properties (2) and (3) follow immediately from the corresponding properties of \( \varphi \), stated in Proposition 6.1 above. \( \square \)

### 7. An example

An interesting feature of \( \omega \) is that — by contrast with \( \varphi \) — it can attain non-integer values, as we shall see presently.

To this end, we study an example in detail: we show that \( \omega(T_{2,3} - T_{5,6}) = \frac{26}{3} \). Before doing so, we recall some facts about Krcatovich’s reduced knot Floer complex.

In [13], Krcatovich associates to each knot \( J \subset S^3 \) a reduced version of the knot Floer complex, denoted by \( \text{CFK}^- (J) \). The reduced knot Floer complex for L-space knots is of a particularly simple form, in that it only consists of a single tower, i.e. it is isomorphic to \( F[U] \) as an \( F[U] \)-module, but *not* as a graded module (see [13, Corollary 4.2]).

Krcatovich also observed that, if one is only concerned with correction terms, the connected sum of two L-space knots behaves as an L-space knot [13, Example 2]; more specifically, he showed that if \( K \) and \( K' \) are L-space knots, then \( \text{CFK}^- (K#K') \) fits in a short exact sequence of complexes:

\[ 0 \to T \to \text{CFK}^- (K#K') \to A \to 0, \]

where \( T \) is a tower and \( A \) is acyclic. In this case, we will write \( \text{CFK}^- (K#K') \approx T \); moreover, if \( C \) is another chain complex such that \( C \approx T \), we will also write \( \text{CFK}^- (K#K') \approx C \). In Krcatovich’s terminology, \( \text{CFK}^- (K#K') \) has a representative staircase, which is determined by \( T \); conversely, the staircase determines \( T \) and the collection \( \{V_i(K#K')\} \). Moreover, for any other knot \( L \), we can use \( T \) as a substitute for \( \text{CFK}^- (K#K') \) to compute \( \text{CFK}^- (K#K'\#L) \), in the sense that there is a filtered quasi-isomorphism

\[ T \otimes \text{CFK}^- (L) \cong \text{CFK}^- (K#K') \otimes \text{CFK}^- (L). \]

**Proposition 7.1.** Let \( K = T_{2,3} - T_{5,6} \). Then \( \omega(K) = \frac{26}{3} < \varphi(K) = 6 \). Moreover, \( \omega(K) < \frac{\varphi(nK)}{n} \) for all \( n \in \mathbb{Z}_{>0} \), so the limit in Definition 6.6 is not attained at any \( n \).
Before proving the proposition, recall that it is proven in [3] that, in the case of torus knots $T_{p,q}$, the representative staircase for $nT$ here in a special case.

That is, the representative staircase for $nT_5$ is the staircase of $T_{5,n+1}$.

We will also need a lemma about $nT_2$. This is true in wider generality (see [3]), but we prove it here in a special case.

**Lemma 7.2.** For each positive integer $n$, the complex $\CFK^\infty(±nT_2)\equiv A_{±n}$ is filtered chain homotopy equivalent to $\CFK^\infty(±T_2)\otimes A_{±n}$, where $A_{±n}$ is an acyclic complex over $\mathbb{F}[U]$. 

**Proof.** It suffices to prove the statement for $\CFK^\infty(nT_2)$, since the corresponding statement for $\CFK^\infty(−nT_2)$ follows by taking duals: in fact, $\CFK^\infty(K)$ is isomorphic to the dual of $\CFK^\infty(K)$, and taking duals preserves direct sums and acyclicity.

We will now prove the statement for $\CFK^\infty(nT_2)$ by induction on $n$: recall that $\CFK^\infty((n+1)T_2)$ is filtered quasi-isomorphic to $\CFK^\infty(nT_2)\otimes\CFK^\infty(T_2)$, and that $\CFK^\infty(T_2)$ is filtered quasi-isomorphic to $(\mathbb{F}[U, U^{-1}]a \oplus \mathbb{F}[U, U^{-1}]b) \oplus \mathbb{F}[U, U^{-1}]c, \partial_i)$, where $\partial_i b = U a + c$ and $a$ and $c$ are cycles; moreover, the Alexander gradings of the generators are $A(a) = 1, A(b) = 0, A(c) = −1$.

By induction, we can assume that $\CFK^\infty(nT_2) = \CFK^\infty(T_{2n+2})\oplus A_n$, where $\CFK^\infty(T_{2n+2})$ is generated over $\mathbb{F}[U, U^{-1}]$ by $x_1, \ldots, x_{2n+1}$, is equipped with the differential $\partial_n$ defined by

$$\partial_n x_{2i} = U x_{2i-1} + x_{2i+1}, \quad \partial_n x_{2i+1} = 0,$$

and the Alexander grading is $A(x_i) = n + 1 - i$.

We observe that, whenever $A$ is acyclic, $A \otimes C$ is acyclic for every other complex $C$. Therefore, in order to prove the theorem, it suffices to show that $\CFK^\infty(T_{2n+2})\otimes\CFK^\infty(T_2) \cong \CFK^\infty(T_{2n+3})\oplus A$, where $A$ is acyclic.

To this end, consider the subspace $V$ of $\CFK^\infty(T_{2n+2})\otimes\CFK^\infty(T_2)$ spanned by:

$$V = \text{Span}_{\mathbb{F}[U, U^{-1}]} \{x_1a, x_1b, x_1c\},$$

where we drop the $\otimes$ between generators to ease readability, so that $x_1 a$ really means $x_1 \otimes a$. It is easy to check that $V$ is in fact a subcomplex of $\CFK^\infty(T_{2n+2})\otimes\CFK^\infty(T_2)$, and that $V$ is indeed isomorphic to $\CFK^\infty(T_{2n+3})$. In fact, an explicit isomorphism is given by $x_1 a \mapsto x_1, x_1 b \mapsto x_2, x_1 c \mapsto x_{i+2}$.

We claim that $V$ has a complement, which is the direct sum of copies of rank-4 subspaces $W_{2i}$, for $i = 1, \ldots, n$.

$$W_{2i} = \text{Span}_{\mathbb{F}[U, U^{-1}]} \{x_{2i}b, x_{2i}a, x_{2i}b + x_{2i+1}b + x_{2i}c, x_{2i+1}c\}.$$ 

It is easy to prove that $W_{2i}$ is in fact an acyclic subcomplex for each $i$, and that the $W_{2i}$ together with $V$ span all of $\CFK^\infty(T_{2n+2})\otimes\CFK^\infty(T_2)$.

Moreover, since the ranks of $V$ and $W_{2i}$ add up to the rank of $\CFK^\infty(T_{2n+2})\otimes\CFK^\infty(T_2)$, this is actually a direct sum decomposition of complexes. Since the $W_{2i}$ are acyclic, we have exhibited the desired decomposition.

We can now turn to the proof of Proposition 7.1.

**Proof of Proposition 7.1.** Let $K_1 = T_{2,3}$ and $K_2 = T_{5,6}, K = K_1 - K_2$. The fact that $\varphi(K) = 6$ was already observed in Example 6.4. Let now $L_n = nK = nK_1 - nK_2$, and $n = 5\ell$. We will prove that for $\ell \in \mathbb{Z}_{>0}$ we have

$$\varphi(L_\ell) = 26\ell + 1.$$ 

This implies at once that $\omega(K) = \lim_{n \to \infty} \frac{\varphi(L_n)}{n} = \frac{26}{3}$, and that $\varphi(L_\ell) > \omega(K) \cdot 5\ell$ for each $\ell$. Moreover, by definition, for each $n$ we have

$$\varphi(L_n) \geq \frac{26}{5}n.$$ 


for all $n \in \mathbb{Z}_{>0}$; since right-hand side is an integer only if $n$ is a multiple of 5, the inequality is strict also for all $n$ not divisible by 5, hence the limit is never attained.

We now set out to prove that $\varphi(L_{5\ell}) = 26\ell + 1$.

Since $\overline{\text{CFK}}^-(nK_2) \simeq \overline{\text{CFK}}^-(T_{5,5n+1})$, we can use Lemma 7.2 and results from [5] to compute the invariants $V_i(nK_2 - nK_1)$, treating $nK_2$ as $T_{5,5n+1}$ and $-nK_1$ as $-T_{2,2n+1}$. Indeed, let $J_i = 5\ell K_i$ for $i = 1, 2$.

Given a semigroup $\Gamma \subseteq \mathbb{N} = \{0, 1, \ldots, \}$, we denote by $\Gamma(\cdot)$ its \textit{enumerating function}, i.e. the unique strictly increasing function

$$\Gamma : \mathbb{N} \to \mathbb{N}$$

which is surjective on $\Gamma$. Note that $\Gamma(0) = 0$. Given an integer $x$, we denote $(x)_+ = \max\{0, x\}$. Since $\overline{\text{CFK}}^\infty(-nT_{2,3})$ is, up to an acyclic summand, $\overline{\text{CFK}}^\infty(-T_{2,2n+1})$, we can apply [5] Theorem 3.1 and Remark 3.3 and obtain:

$$\nu_v^+(5\ell K) := \min \{i \mid V_i(5\ell K) \leq v\} = \left(\max_{k \geq 0} \{g(J_2) - g(J_1) + \Gamma_{J_1}(k) - \Gamma_{J_2}(k + v)\}\right)_+,$$

where $\Gamma_{J_1}(\cdot)$ and $\Gamma_{J_2}(\cdot)$ are the enumerating functions associated to the semigroups

$$\Gamma_{J_1} = (2, 10\ell + 1); \quad \Gamma_{J_2} = (5, 25\ell + 1).$$

The genera of the knots $J_1$ and $J_2$ are respectively $5\ell$ and $50\ell$, so the formula for $\nu_v^+$ becomes

$$(7.1) \quad \nu_v^+(L_{5\ell}) = \left(45\ell - \min_{k \geq 0} \{\Gamma_{J_2}(k + v) - \Gamma_{J_1}(k)\}\right)_+.$$ 

Note that, with this notation, we have that

$$(7.2) \quad \varphi(L_{5\ell}) = \min_{v \geq 0} \{\nu_v^+(L_{5\ell}) + 2v\},$$

which we are now going to compute.

The enumerating functions above can be expressed in the following equations:

$$\begin{align*}
\Gamma_{J_1}(k) &= \begin{cases} 2k & 0 \leq k \leq 5\ell \\ 5\ell + k & k \geq 5\ell \end{cases} \\
\Gamma_{J_2}(k) &= \begin{cases} 5k & 0 \leq k \leq 5\ell \\ 25\ell + 5\left\lfloor \frac{k - 5\ell}{2} \right\rfloor + \left[k - 5\ell\right]_2 & 5\ell \leq k \leq 15\ell \\ 50\ell + 5\left\lfloor \frac{k - 15\ell}{4} \right\rfloor + \left[k - 15\ell\right]_3 & 15\ell \leq k \leq 30\ell \\ 75\ell + 5\left\lfloor \frac{k - 30\ell}{6} \right\rfloor + \left[k - 30\ell\right]_4 & 30\ell \leq k \leq 50\ell \\ 50\ell + k & k \geq 50\ell \end{cases}
\end{align*}$$

Note that in Equation (7.1) we can in fact take the minimum over $0 \leq k \leq 5\ell$, because for $k \geq 5\ell$ the function $\Gamma_{J_1}(k)$ increases at a lesser or equal rate than any translate of $\Gamma_{J_2}$: specifically, $\Gamma_{J_1}(k + j) - \Gamma_{J_2}(k) \geq j \leq \Gamma_{J_2}(k + v + j) - \Gamma_{J_2}(k + v)$. Therefore

$$\nu_v^+(L_{5\ell}) = \left(45\ell - \min_{0 \leq k \leq 5\ell} \{\Gamma_{J_2}(k + v) - \Gamma_{J_1}(k)\}\right)_+.$$ 

Now we return to the proof of Proposition 7.1. Recall that we want to prove that $\varphi(5\ell K) = 26\ell + 1$. By (7.2) we have

$$\varphi(L_{5\ell}) = \min_{v \geq 0} \{\nu_v^+(L_{5\ell}) + 2v\}.$$ 

As shown in Lemma 7.3 below, the choice $v = 13\ell$ gives $\nu_v^+(L_{5\ell}) + 2v = 26\ell + 1$. Moreover, it also follows from Lemma 7.3 that $V_0(L_{5\ell}) = 13\ell + 1$, hence choosing $v \geq 13\ell + 1$ yields $2v \geq 26\ell + 2 > 26\ell + 1$.

We now distinguish between $v \leq 5\ell - 1$ and $v \geq 5\ell$. By Lemma 7.4 below, for $v \in [0, 5\ell - 1]$ we have

$$\nu_v^+(L_{5\ell}) + 2v = 45\ell - 3v \geq 45\ell - 15\ell + 3 > 26\ell + 1;$$
by Lemma 7.3 on the other hand, for $v \in [5\ell, 13\ell - 1]$ we have

$$\nu_v^+(L_{5\ell}) + 2v \geq 2(13\ell - v) + 1 + 2v = 26\ell + 1.$$ 

This shows that $\varphi(L_{5\ell}) = 26\ell + 1$, as desired. \hfill \square

**Lemma 7.3.** $\nu_{13\ell}(L_{5\ell}) = 1$.

**Proof.** Note that, since $k \leq 5\ell$, $k + 13\ell \in [13\ell, 18\ell]$. Therefore, the difference of the enumerating functions is

$$f(k) := \Gamma_{J_2}(k + 13\ell) - \Gamma_{J_1}(k) = \begin{cases} 45\ell + 5\lceil \frac{k}{2} \rceil + [k]_2 - 2k & 0 \leq k \leq 2\ell \\ 50\ell + 5\lceil \frac{k - 2\ell}{3} \rceil + [k - 2\ell]_3 - 2k & 2\ell \leq k \leq 5\ell \end{cases}$$

In the first interval $f(k + 2) \geq f(k)$, while in the second interval $f(k + 3) \leq f(k)$. It follows that the minimum is attained for some $k \in \{0, 1, 5\ell - 2, 5\ell - 1, 5\ell\}$. A direct computation for these five values shows that the minimum is $45\ell - 1$, attained both at $k = 1$ and at $k = 5\ell - 1$. It follows that $\nu_{13\ell}(L_{5\ell}) = 45\ell - (45\ell - 1) = 1$. \hfill \square

**Lemma 7.4.** For each $v = 0, \ldots, 5\ell - 1$, $\nu_v^+(L_{5\ell}) = 45\ell - 5v$.

**Proof.** Note that, since we only need to test $k \leq 5\ell$ when computing the minimum in (7.1), we can assume that for each value of $v$ in the statement $k + v \leq 10\ell - 1$. Therefore, the difference of the enumerating functions is

$$f(k) := \Gamma_{J_2}(k + v) - \Gamma_{J_1}(k) = \begin{cases} 5v + 3k & 0 \leq k \leq 5\ell - v \\ 25v + 5\lceil \frac{k + v - 5\ell}{2} \rceil + [k + v - 5\ell]_2 - 2k & 5\ell - v \leq k \leq 5\ell \end{cases}$$

Such a function is increasing on the interval $0 \leq k \leq 5\ell - v$, and on the second interval it satisfies the condition $f(k + 2) - f(k) \geq 1$. It follows that the minimum is attained for some $k = 0, 5\ell - v$ or $5\ell - v + 1$. A direct computation for these values shows that the minimum is $5v$, attained at $k = 0$. Therefore, $\nu_v^+(L_{5\ell}) = 45\ell - 5v$. \hfill \square

**Lemma 7.5.** Let $v = 13\ell - s$ for some $0 < s \leq 8\ell$. Then $\nu_v^+(L_{5\ell}) \geq 2s + 1$.

**Proof.** Choosing $k = 0$ in Equation (7.1), we obtain:

$$\nu_v^+(L_{5\ell}) \geq 45\ell - \Gamma_{J_2}(13\ell - s).$$

Since $13\ell - s \in [5\ell, 13\ell] \subseteq [5\ell, 15\ell]$, we have

$$\Gamma_{J_2}(13\ell - s) = 45\ell + 5\lceil -\frac{s}{2} \rceil + [s]_2.$$ 

If $s \geq 2$ is even, then $\Gamma_{J_2}(13\ell - s) = 45\ell - \frac{s}{2}s \leq 45\ell - 2s - 1$. If $s$ is odd, then $\Gamma_{J_2}(13\ell - s) = 45\ell - \frac{s}{2}(s + 1) + 1 \leq 45\ell - 2s - 1$. In both cases we have $\Gamma_{J_2}(13\ell - s) \leq 45\ell - 2s - 1$, so we obtain

$$\nu_v^+(L_{5\ell}) \geq 45\ell - \Gamma_{J_2}(13\ell - s) \geq 2s + 1.$$ \hfill \square

With techniques similar to the ones used in Proposition 7.1 one can to show that $\omega$ attains many other positive non-integer values. We conclude with two questions concerning the image of $\omega$.

**Question 7.6.** Is $\mathbb{Q}_{\geq 0} \subseteq \text{im}(\omega)$? Can $\omega$ take irrational values?

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