A NOTE ON NON-NOETHERIAN COHEN-MACaulAY RINGS

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1. Introduction

In this note, we study the Cohen-Macaulayness of non-Noetherian rings. We show that Hochster’s celebrated theorem that a finitely generated normal semigroup ring is Cohen-Macaulay does not extend to non-Noetherian rings. We also show that for any valuation domain \( V \) of finite Krull dimension, \( V[x] \) is Cohen-Macaulay in the sense of Hamilton-Marley. All rings are commutative with unity, and all \( R \)-modules are unitary.

In commutative algebra and algebraic geometry, Cohen-Macaulayness of a ring or module is a desirable property. If \( R \) is a Noetherian local ring, we say that \( R \) is Cohen-Macaulay if \( \text{depth} R = \text{dim} R \), where \( \text{depth} R \) denotes the maximal length of a regular sequence contained in the maximal ideal of \( R \). In the Noetherian case, there are several characterizations of Cohen-Macaulay rings, and the notion of Cohen-Macaulayness is rather well understood. In the general case, it is not clear what the “right” definition of a Cohen-Macaulay ring in the non-Noetherian case is. A direct extension of the definition to the non-Noetherian case seems to be a bit unnatural: For example, every valuation domain (which is not a field) has depth one, whereas the dimension can be arbitrarily large. Valuation domains belong to the class of coherent regular rings, which when Noetherian, are well-known to be Cohen-Macaulay. It is natural to search for a definition which generalizes the Noetherian case and includes meaningful non-Noetherian rings in the class of “Cohen-Macaulay” rings.

S. Glaz initiated the study of this extension question. In [Gla94], she surveyed the ascent and descent properties of the extension \( R^G \subset R \), where \( G \) is a group acting on a commutative ring \( R \), and \( R^G \) is the ring of invariants. In section 4 of [Gla94], she proposed a conjecture:

Let \( R \) be a coherent regular ring, and let \( G \) be a group of automorphisms of \( R \).
Assume that \( R^G \) is a module retract of \( R \) and that \( R \) is a finitely generated \( R^G \)-module, then \( R^G \) is a Cohen-Macaulay ring.

As \( R^G \) (and \( R \)) are not necessarily Noetherian (hence often not Cohen-Macaulay in the sense of \( \text{depth} R^G = \text{dim} R^G \)), she raised the question of finding a suitable extension of the definition of a Cohen-Macaulay ring which is not necessarily Noetherian.

Before explaining her definition and other generalizations, first we list characterizations of the Cohen-Macaulayness of a Noetherian local ring \( R \). The following conditions are equivalent.

[CM1] \( R \) is Cohen-Macaulay.
[CM2] For every proper ideal \( I \) of \( R \), \( \text{ht} I = \text{grade} I \).

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[CM3] For every proper ideal \( I \) of \( R \), if \( \text{ht} I = \mu(I) \), then \( I \) is unmixed.
[CM4] Every system of parameters of \( R \) is a regular sequence.

Here, \( \mu(I) \) denotes the minimal number of generators of \( I \), and the grade of an ideal \( I \) of \( R \), denoted \( \text{grade} I \), is the length of a maximal regular sequence in \( I \). These conditions will no longer be equivalent in the non-Noetherian case. Note that the notions of grade, unmixedness, and a system of parameters work well in the Noetherian case, but their useful properties do not hold, in general, in non-Noetherian rings. For instance, in the Noetherian case, \( \text{grade} I > 0 \) iff \( 0 :_R I = 0 \), but this equivalence can fail to hold in the non-Noetherian case, cf. [Nor76, Example on p. 145]. This issue can be fixed by considering the grade of \( I \) in the polynomial extension \( R[x] \).

**Definition 1 ([Hoc74]).** The polynomial grade of \( I \), denoted \( \text{p-grade} I \), is

\[
\text{p-grade} I = \sup \{ \text{grade} IR[x_1, \ldots, x_i] \mid i \in \mathbb{N} \}.
\]

(1)

Note that \( \text{grade} I \leq \text{p-grade} I \), and equality holds if \( R \) is Noetherian or \( I \) is a complete intersection ideal.

Later, in [Gla95, Sec. 6], Glaz defined an arbitrary ring \( R \) to be Cohen-Macaulay if each prime ideal \( p \) of \( R \), \( \text{ht} p = \text{p-grade} p \). Her goal was to have a definition of a Cohen-Macaulay ring such that

- [G1] the definition agrees in the Noetherian case,
- [G2] regular coherent rings are Cohen-Macaulay, and
- [G3] certain invariant rings of regular coherent rings are Cohen-Macaulay.

In [Ham04], T. Hamilton continued the study of non-Noetherian Cohen-Macaulayness by investigating condition [CM3]. We say \( I \) is unmixed if \( \text{ht} I = \text{ht} p \) for all associated primes \( p \) of \( I \). A prime \( p \) is associated to \( I \) if there exists \( x \in R \) such that \( p = I :_R x \). In the Noetherian case, every proper ideal has a non-empty set of associated primes. This does not hold for non-Noetherian rings. A natural generalization of the notion of associated primes are the weakly associated primes, which are the primes \( p \) that are minimal over \( I :_R x \) for some \( x \in R \). For any proper ideal \( I \) in an arbitrary ring \( R \), the set of weakly associated primes of \( I \) is non-empty, and when the ring is Noetherian, the weakly associated primes of \( I \) and the associated primes of \( I \) coincide. She defined \( R \) to be weakly-Bourbaki unmixed if for each finitely generated ideal \( I \) of \( R \) with \( \mu(I) \leq \text{ht} I \), the weakly associated primes of \( I \) are precisely the primes minimal over \( I \). In the same paper, she added two additional desirable conditions that should hold for \( R \) to be Cohen-Macaulay.

- [H1] \( R \) is Cohen-Macaulay iff \( R[x] \) is Cohen-Macaulay;
- [H2] \( R \) is Cohen-Macaulay iff \( R_p \) is Cohen-Macaulay for all prime ideals \( p \).

With this notion of Cohen-Macaulayness, she was able to show that weak Bourbaki rings satisfy [G1] and that the if parts of both [H1],[H2] hold, see [Ham04, Theorems 1-3].

Most recently, Hamilton and Marley in [HM07] defined the notion of a strong parameter sequence which generalizes the notion of a system of parameters (see section 2 for the definition). They define an arbitrary ring \( R \) to be Cohen-Macaulay if every strong parameter sequence of \( R \) is a regular sequence of \( R \). With this definition, the authors were able to show the following:

- Condition [G1] holds ([HM07, Proposition 3.6, Proposition 4.2]).
• Condition [G2] holds locally ([HM07, Theorem 4.8]). That is a local coherent regular ring is Cohen-Macaulay. (Thus, any valuation domain of finite Krull dimension is Cohen-Macaulay.)
• Condition [G3] holds if \( \dim R \leq 2 \) (see [HM07, Corollary 4.15] for the precise statement).
• The if parts of Hamilton’s conditions [H1],[H2] hold ([HM07 Corollary 4.6, Proposition 4.7]).

Furthermore, the two notions of Cohen-Macaulayness introduced by Glaz and Hamilton are both stronger condition than that of Hamilton-Marley cf. [HM07, Proposition 2.6, Proposition 4.10], and there are in fact examples showing that these notions of Cohen-Macaulayness need not be equivalent cf. [AT09, Section 4]. The primary goal of this paper is to study this definition of Cohen-Macaulay introduced by Hamilton and Marley. In the sequel, unless otherwise stated, when we say a ring is Cohen-Macaulay, we always mean that it is Cohen-Macaulay in the sense of Hamilton and Marley.

Our main result is on the Cohen-Macaulayness of semigroup rings. A semigroup is called affine if it is finitely generated. Hence an affine semigroup ring is Noetherian. In [HM07, Example 4.9], the authors show that the ring \( K + xK[[x,y]] \) is non-Noetherian and Cohen-Macaulay, where \( K \) is a field. This was extended to the semigroup rings \( K + xK[x,y] \) in [ADT14, Theorem 4.10]. Note that this semigroup ring is normal, and a normal affine semigroup ring is Cohen-Macaulay [Hoc72]. To this end, in [ADT14], the authors proposed the following question.

**Question 1** ([ADT14, Question 5.1]). Let \( R \) be a semigroup ring of finite Krull dimension which is not necessarily Noetherian. Then is \( R \) Cohen-Macaulay if it is normal?

To answer this question, we study a family of rings \( R = k + Q \), where \( Q \) is an ideal of a finitely generated \( k \)-algebra \( S \), where \( k \) is a field. We note that this is a special case of amalgamated algebras in [DFF10]. One of the challenging parts in studying the Cohen-Macaulayness in the sense of Hamilton and Marley is the verification of the condition of being a parameter sequence. In this setup, we give a method that can quickly verify this condition (Theorem 14 and Lemma 16). With this construction, we are able to answer Question 1 negatively.

**Theorem 2** (Theorem 22). *The semigroup ring \( K + xK[x,y,z] \) is normal. The sequence \( xy, xz \) is a strong parameter sequence, but it is not a regular sequence. In particular, this ring is non-Noetherian and normal but not Cohen-Macaulay.*

Note that our theorem also addresses [ADT14, Remark 5.3] negatively. Our result shows that Hochster’s theorem fails to hold in the non-Noetherian case if one uses any of the notions of Cohen-Macaulayness mentioned thus far. Considering this, it is natural to ask the following question.

**Question 2.** Is there a notion of Cohen-Macaulayness satisfying [G1]-[G3] such that Hochster’s theorem holds?

Our second result concerns the only if part of condition [H1]; that is, the question of \( R[x] \) is Cohen-Macaulay if \( R \) is Cohen-Macaulay. We have already mentioned that Hamilton and Marley [HM07] showed the converse. We answer the only if part of [H1] for the class of valuation domains (or Prüfer domains) of finite Krull dimension. Note that even though a valuation domain is always quasi-local, its Krull dimension can be infinite (in the Noetherian case, the Krull dimension of a local ring is finite). We believe that this might be known to the experts, but to the best of our knowledge, the statement and proof is not available.
Theorem 3 (Theorem [24]). Let \( R \) be a Prufer domain of finite Krull dimension, then \( R[x_1, \ldots, x_n] \) is (locally) Cohen-Macaulay. In particular, \( V[x_1, \ldots, x_n] \) is Cohen-Macaulay if \( V \) is a valuation domain.

2. COHEN-MACAULAYNESS IN THE SENSE OF HAMILTON AND MARLEY

In this section, we review the definition of a Cohen-Macaulay ring as in [HM07] and some preliminary results.

Let \( R \) be a ring and \( \underline{x} = x_1, \ldots, x_\ell \) a sequence of elements of \( R \). For any \( n \in \mathbb{N} \), let \( \underline{x}^n \) denote the sequence \( x_1^n, \ldots, x_\ell^n \). By \( \mathbb{K}_* (\underline{x}) \), we mean the Koszul complex with respect to \( \underline{x} \). For any \( n \geq m \), there is a map of chain complexes

\[
\varphi_m^n : \mathbb{K}_* (\underline{x}^n) \to \mathbb{K}_* (\underline{x}^m)
\]

induced by multiplication of \( (x_1 x_2 \cdots x_\ell)^{n-m} \) on \( R \). A sequence of elements \( \underline{x} = x_1, \ldots, x_\ell \) of a ring \( R \) is said to be weakly proregular, if for each \( m \geq 0 \), there is an \( n \geq m \) so that the maps

\[
H_i(\varphi_m^n) : H_i(\mathbb{K}_* (\underline{x}^n)) \to H_i(\mathbb{K}_* (\underline{x}^m))
\]

are 0 for all \( i \geq 1 \). For \( M \) an \( R \)-module, define the \( \check{\text{Cech}} \) complex to be the complex

\[
\check{C}_*(M) : 0 \to M \to \bigoplus_i M_{x_i} \to \bigoplus_{i < j} M_{x_ix_j} \to \cdots \to M_{x_1 \cdots x_\ell} \to 0,
\]

where the maps are natural up to a sign convention. For each \( i \in \mathbb{Z} \), the \( i \)th \( \check{\text{Cech}} \) cohomology module of \( M \) with respect to the sequence \( \underline{x} \) is \( H_i(\check{C}_*(M)) := H_i(\check{C}_*(\underline{x}^1)) \). Schenzel [Sch03, Theorem 1.1] showed that weakly proregular sequences \( \underline{x} \) in a ring \( R \) are precisely the sequences, where for \( I = (\underline{x})R \), \( \check{\text{Cech}} \) cohomology \( H_i^*(-) \) and local cohomology \( H_i^*(-) := \lim_{\to n} \text{Ext}_R^n(R/I^n, -) \) are canonically isomorphic functors. These two functors are isomorphic in the Noetherian case, but they are not isomorphic in general, cf. [HM07, paragraph after Theorem 2.3]. For any sequence of elements \( \underline{x} = x_1, \ldots, x_\ell \), we define \( \ell(\underline{x}) \) to be the length of the sequence \( \underline{x} \).

Definition 4 ([HM07] Definition 3.1). Let \( R \) be a ring. A sequence of elements \( \underline{x} = x_1, \ldots, x_\ell \) is said to be a parameter sequence on \( R \) if the following conditions hold:

1. \( (\underline{x})R \neq R \).
2. \( \underline{x} \) is weakly proregular.
3. \( H_i^*(R)_p \neq 0 \) for each \( p \in V_R(\underline{x}R) \).

Here, \( V_R(I) \) denotes the set of prime ideals of \( R \) containing the ideal \( I \), and for a sequence of elements \( f_1, \ldots, f_s \in R \), we define \( V_R(f_1, \ldots, f_s) := V_R((f_1, \ldots, f_s)R) \). Moreover, we say \( \underline{x} \) is a strong parameter sequence on \( R \) if \( x_1, \ldots, x_i \) is a parameter sequence on \( R \) for all \( 1 \leq i \leq \ell \).

Note that there is an example [HM07, Proposition 2.9] of a parameter sequence that is not a strong parameter sequence. The next proposition states that a strong parameter sequence is part of a system of parameters if the ring is local and Noetherian.

Proposition 5 ([HM07] Proposition 3.6]). Let \( R \) be a ring, and let \( \underline{x} \) be a parameter sequence of \( R \). Then \( \text{ht}(\underline{x})R \geq \ell(\underline{x}) \).

Definition 6 ([HM07]). We say that a ring \( R \) is Cohen-Macaulay if every strong parameter sequence on \( R \) is a regular sequence on \( R \).
Hamilton-Marley showed the following characterizations of their notion of Cohen-Macaulayness.

**Theorem 7** ([HM07, Proposition 4.2]). Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is Cohen-Macaulay.
2. $\text{grade}(x)R = \ell(x)$ for every strong parameter sequence $x$ of $R$.
3. $\text{p-grade}(x)R = \ell(x)$ for every strong parameter sequence $x$ of $R$.
4. $H^i_x(R) = 0$ for all $i < \ell(x)$ and for every strong parameter sequence $x$ of $R$.
5. $H_i(x; R) = 0$ for all $i \geq 1$ and for every strong parameter sequence $x$ of $R$. Here, $H_i(x; R)$ denotes the Koszul homology module with respect to the sequence $x$.

**Remark 8** ([HM07, Proposition 3.3(f)]). In any ring, every regular sequence on $R$ is a parameter sequence on $R$.

Lastly, we record some results on local cohomology modules which we will use in the following sections. For the proofs of the following results and basic facts on local cohomology, we refer the reader to [BS13, ILL+07].

**Proposition 9.** Let $R$ be a Noetherian ring, and let $I, J$ be $R$-ideals.

1. [Mayer-Vietoris sequence] One has the following long exact sequence of local cohomology modules

   $$
   0 \rightarrow H^0_{I+J}(R) \rightarrow H^0_I(R) \oplus H^0_J(R) \rightarrow H^0_{I\cap J}(R) \\
   \quad \rightarrow H^1_{I+J}(R) \rightarrow H^1_I(R) \oplus H^1_J(R) \rightarrow H^1_{I\cap J}(R) \\
   \quad \rightarrow H^2_{I+J}(R) \rightarrow \cdots.
   $$

2. $\text{grade}(I, R) = \inf\{i \mid H^i_I(R) \neq 0\}$. In particular, $H^i_I(R) = 0$ for all $i < \text{grade}(I, R)$.
3. If $R$ is local and $I$ is an ideal generated by a system of parameters of $R$, then $H^i_I(R) \neq 0$.

### 3. Cohen-Macaulayness of $k + Q$ rings

In this section, we will study the Cohen-Macaulayness of rings of type $k + Q$, where $k$ is a field and $Q$ is an ideal in a Noetherian ring. We first state a lemma which will be useful in our setup.

**Lemma 10.** Let $R \subset S$ be a ring extension, and suppose $f = f_1, \ldots, f_d \in (R :_S S)$. Then the following hold:

1. We have an exact sequence of Čech cohomology modules

   $$
   0 \rightarrow H^0_U(R) \rightarrow H^0_U(S) \rightarrow H^0_U(S/R) \rightarrow H^1_U(R) \rightarrow H^1_U(S) \rightarrow 0, \tag{2}
   $$

   and $H^i_U(R) = H^i_U(S)$ for each $i \geq 2$.

2. $f$ is a weakly proregular sequence in $R$ if and only if $f$ is a weakly proregular sequence in $S$.

**Proof.** The statement (a) follows from the long exact sequence of Čech cohomology modules induced from the short exact sequence of $R$-modules $0 \rightarrow R \rightarrow S \rightarrow S/R \rightarrow 0$ and from the fact that $(f)(S/R) = 0$ implies $H^i_U(S/R) = 0$ for $i \geq 1$.

For (b), the short exact sequence $0 \rightarrow R \rightarrow S \rightarrow S/R \rightarrow 0$ yields a short exact sequence of inverse systems of Koszul homology modules for any $i \geq 0$:

$$
0 \rightarrow \{H_i(f^n; R)\} \rightarrow \{H_i(f^n; S)\} \rightarrow \{H_i(f^n; S/R)\} \rightarrow 0.
$$
For $n \geq m$, the multiplication map $K_\bullet(f^n; S/R) \xrightarrow{(f_1 \cdots f_d)^{n-m}} K_\bullet(f^n; S/R)$ is identically zero, so that \( H_i(f^n; S/R) \) is pro-zero, i.e., \( H_i(f^n; S/R) = 0 \) for \( i > 0 \). Thus, by [Har67, Remark 2, p. 24], for each \( T \geq 0 \), the inverse system \( \{ H_i(f^n; S) \} \) is pro-zero if and only if the inverse system \( \{ H_i(f^n; S) \} \) is pro-zero.

**Setup:** We will use the following setup for the rest of the section. Let \( S \) be a domain which is essentially of finite type over a field \( k \). That is, \( S \) is a localization of a finitely generated \( k \)-algebra. In particular, \( S \) is a Noetherian ring. An \( S \)-ideal \( Q \) gives rise to a sub \( k \)-algebra \( R = k + Q \subset S \). We call \( R \) the amalgamated \( k \)-algebra of the ring \( S \) along the ideal \( Q \). We will consider the case where \( Q \) is at most two-generated.

**Remark 11.** We present examples of Noetherian and non-Noetherian amalgamated \( k \)-algebras. Let \( S = k[x, y] \). For \( Q = (x, y^2)S \), \( R = k + Q = k[x, xy, y^2, y^3] \) is Noetherian. For \( Q = (x)S \), \( R = k + Q = k[x, xy, xy^2, xy^3, \ldots] \) is not Noetherian.

**Remark 12.** For any ring extension \( A \subset B \) of domains having the same field of fractions \( K \), we call \( c_{B/A} := A :_K B \) the conductor of the extension. It is well-known that \( c_{B/A} = A :_B B = A :_A B \), and \( c \) is the largest \( B \)-ideal contained in \( A \). Furthermore, any prime ideal \( p \) of \( A \), \( (c_{B/A})_p \subset c_{B_p/A_p} \).

With the setup above, if \( R \neq S \), then \( Q \) is the conductor of the extension \( R \subset S \), i.e., \( R :_R S = Q \). As a consequence, for any prime ideal \( p \neq Q \), \( Q_p = R_p \subset c_{S_p/R_p} \), so \( R_p = c_{S_p/R_p} \), i.e., \( R_p = S_p \).

**Proposition 13.** Let \( R, S \), and \( Q \) be as above. If \( Q \neq S \), then we have the following.

(a) \( Q \) is a maximal ideal of \( R \).

(b) For any prime ideal \( p \in \text{Spec } R \setminus \{ Q \} \), \( R_p \) is Noetherian.

(c) Assume that \( S \) is Cohen-Macaulay. Then \( R \) is Cohen-Macaulay in the sense of Hamilton-Marley if \( R_Q \) is Cohen-Macaulay in the sense of Hamilton-Marley.

**Proof.** (a): Since \( R/Q \cong k \) is a field, \( Q \) is a maximal ideal.

(b): If \( p \neq Q \), then \( R_p :_{R_p} S_p = (R :_R S)_p = Q_p = R_p \). So, \( R_p = S_p \) is Noetherian since \( S \) is Noetherian.

(c): This follows from [HM07, Proposition 4.7].

We note that we do not know if \( R_Q \) is Cohen-Macaulay under the condition of \( R \) and \( S \) being Cohen-Macaulay. This is a special case of an implication of condition [H2].

**Question 3.** Is \( R_Q \) Cohen-Macaulay if \( R \) and \( S \) are Cohen-Macaulay?

We first treat the case where \( Q \) is generated by two elements in \( S \).

**Theorem 14.** Let \( S \) be a domain which is essentially of finite type over a field \( k \). \( Q \) an \( S \)-ideal, and \( R = k + Q \). If \( Q = (f, g) \subset S \) for some nonzero elements \( f, g \) of \( S \), then \( f, g \) is a strong parameter sequence in \( R \) if and only if \( H^2_{f, g}(S) \neq 0 \). Furthermore, if \( R \) is Cohen-Macaulay, then \( H^2_{f, g}(S) = 0 \).

**Proof.** As \( f \) is a regular element of \( R \), it is a parameter sequence. Observe that since \( Q = (R :_S S) \) is the conductor ideal and \( S \) is Noetherian, by Lemma [10(b)], \( f, g \) is a weakly proregular sequence.

\footnote{The names comes from [DFF10]. According to their terminologies, \( R \) is the amalgamated \( k \)-algebra of \( S \) along \( Q \) under the embedding \( k \to S \).}
Remark 15. Let \((f, g)\) be a parameter sequence on \(R\). Thus, to prove the first statement, we show that for \(f, g\) to be a parameter sequence, it is equivalent to say \(H^2_{f,g}(R)_p \neq 0\) for each \(p \in V_R(f, g)\). Note that \(V_R(f, g) = \{Q\}\) by [DFF10, Prop. 2.6(2)]. Thus, \(H^2_{f,g}(R) \neq 0\) if and only if \((H^2_{f,g}(R))_Q \neq 0\). Then the equivalence is clear since \(H^2_{f,g}(R) = H^2_{f,g}(S)\) by Lemma [10](a).

Now we show the second statement. Suppose by way of contradiction that \(f, g\) is a strong parameter sequence on \(R\). Since \(R\) is Cohen-Macaulay, \(f, g\) is a regular sequence on \(R\). Thus, \(H^1_{f,g}(R) = 0\) by Proposition [9](b). By Lemma [10](a), we have the exact sequence

\[ H^0_{f,g}(S) \to H^0_{f,g}(S/R) \to H^1_{f,g}(R). \]

Since \(\operatorname{grade} (f, g)S > 0\), \(H^0_{f,g}(S) = 0\) by Proposition [9](b), and since \(R \neq S\) and \(Q = R :_S S_1\), one has \(H^0_{f,g}(S/R) \neq 0\). Therefore, \(H^1_{f,g}(R) \neq 0\), but this is a contradiction.

We do not know if the converse of the last statement is true. It is natural to ask if it is true when \(S\) is a polynomial ring.

Question 4. Let \(S = k[x_1, \ldots, x_n]\) be a polynomial ring over a field \(k\), \(Q\) an \(S\)-ideal, and \(R = k + Q\). Let \(Q = (f, g) \subset S\) for some nonzero elements \(f, g\) of \(S\). Is the ring \(R\) Cohen-Macaulay if \(H^2_{f,g}(S) = 0\)?

Remark 15. It is worth noting the contrapositive of Theorem [14]. That is, if \(H^2_{f,g}(S) \neq 0\), then \(R\) is not Cohen-Macaulay. We will use this fact to give examples of non-Cohen-Macaulay rings.

In Lemma [16] below, we give an ideal-theoretic characterization of the condition \(H^2_{f,g}(S) \neq 0\), and this provides a quick way to verify the condition of \(H^2_{f,g}(S) \neq 0\). The essential idea of this criterion appeared in [HKM09], and the equivalence of (a) and (c) in Lemma [16] is due to them, [HKM09, Prop. 2.6]. For an \(S\)-ideal \(I\), let \(\text{ara} I := \min\{\ell \in \mathbb{Z}_{\geq 0} \mid \text{there exist } z_1, \ldots, z_{\ell} \in S, \sqrt{z_1, \ldots, z_{\ell}} = \sqrt{I}\}\) denote the arithmetic rank of the ideal \(I\).

Lemma 16. Let \(S\) be a Noetherian UFD and \(f, g\) non-zero elements in \(S\). Let \(d = \gcd(f, g)\). Then we have

\[ H^2_{f,g}(S) = H^2_{f,g}(S_d) = H^2_{\sqrt{f,g}}(S_d). \]

Furthermore, the following statements are equivalent:

(a) \(H^2_{f,g}(S) = 0\).

(b) \(\text{ara}(f/d, g/d)S = 1\) or \(d\) is in the radical of \((f/d, g/d)S\).

(c) \((f/g)^n \in (f^{n+1}, g^{n+1})\) for some positive integer \(n\).

In particular, \(H^2_{f,g}(S) \neq 0\) iff \(\text{ara}(f/d, g/d)S = 2\) and \((f/d, g/d)S_d \neq S_d\).

Proof. Note that \(H^2_{f,g}(S)\) is the cokernel of the map

\[ \pi : S_f \oplus S_g \to S_{fg}. \]

Since \(d | f\), we have a natural map \(S_d \to S_f = (S_d)_f\), and similarly for \(S_d \to S_g\). Hence \(\text{coker} \pi = \text{coker} \pi_d\), where \(\pi_d : (S_d)_f \oplus (S_d)_g \to (S_d)_{fg}\). This shows the first equality. The second equality follows from [HM07, Prop. 2.1(e)] since \(\sqrt{(f/g)}S_d = \sqrt{(f/d, g/d)}S_d\).
Next, we show the equivalence. The equivalence of (a) and (c) is \[\text{[HKM09, Prop 2.6]}\]. We will show that \(H^2_{J, S}(d) = 0\) iff \(\text{ara}(f, g) = 1\) or \(d\) is in the radical of \((f/d, g/d)\) in \(S\). If \(\text{ara}(f, g) = 1\), then \(H^2_{J, S}(d) = 0\) by definition. If \(d \in \sqrt{(f/d, g/d)}\), then \((f/d, g/d)S_d = S_d\). Thus, \(H^2_{J, S}(d) = H^2_{J, S}(d) = 0\). Conversely, suppose \(H^2_{J, S}(d) = 0\). It suffices to show that if \(H^2_{J, S}(d) = 0\) and \(\text{ara}(f, g) \neq 1\), then \(\text{ara}(f, g) = 2\) and \(d\) is in the radical of \((f/d, g/d)\) in \(S\). Since \((f, g)S\) is two generated, \(\text{ara}(f, g) \leq 2\), and since \(f, g\) are not zero in an integral domain, \(\text{ara}(f, g)S \geq 1\). Thus, we have \(\text{ara}(f, g)S = 2\). Let \(J' = (f/d, g/d)S\). Assume to the contrary that \(d\) is not in the radical of \(J'S\). Then \(J'S_d\) is a proper ideal of \(S_d\). We claim that \(htJ'S_d = 2\). Since \(S_d\) is Noetherian, \(htJ'S_d \leq 2\) and since \(J'S_d \neq 0\), \(htJ'S_d \geq 1\). Hence, the claim follows immediately once we have shown that \(J'S_d\) is not contained in any prime ideal of height 1. Since \(S\) is a UFD, \(S_d\) is a UFD, and under \(S \to S_d\), \(f/d\), \(g/d\) remain as relatively prime elements. Hence \(\gcd(f/d, g/d) = 1\) in \(S_d\), and there is no principal prime ideal containing the ideal \(J'S_d\). Therefore, \(htJ'S_d = 2\).

Let \(p\) be a minimal prime of \(htJ'S_d\) of height 2. Hence, \(H^2_{J, S}(d)p = H^2_{J, S}(d)p \neq 0\) by Proposition\[\text{[DFF10, Prop 2.6]}\], and this is a contradiction. This completes the proof. \(\square\)

**Example 17.** Let \(S = k[x, y]\), where \(k\) is a field, and let \(Q = (x, y^2)S\). Then \(\text{ara}(x, y^2) = 2\) and \(\gcd(x, y^2) = 1\) is not in \((x, y) = \sqrt{(x, y^2)}\). By Lemma\[\text{[16]}\] \(H^2_Q(S) \neq 0\). Therefore, by Theorem\[\text{[14]}\], the ring \(k + Q = k[x, xy, y^2, y^3]\) is not Cohen-Macaulay.

**Example 18.** Let \(S = k[x, y, z]\). Consider ideals \(Q_1 = (x, y)S, Q_2 = (x, yz)S, Q_3 = (xy, xz)S\). Define \(R_i = k + Q_i\) for \(i = 1, 2, 3\). Then \(R_i\) are not Cohen-Macaulay.

By Theorem\[\text{[14]}\] it suffices to show that \(H^2_{Q_i}(S) \neq 0\). We apply Lemma\[\text{[16]}\] to \(Q_1, Q_2, Q_3\). That is \(Q_iS_{\gcd} \neq S_{\gcd}\). For \(Q_1, Q_2\), notice that \(\gcd(x, y) = \gcd(x, yz) = 1\), and for \(Q_3\), \(\gcd(xy, xz) = x\) and \((xy, xz)S_x = (y, z)S_x \neq S_x\). Therefore, the rings \(R_1, R_2, R_3\) are not Cohen-Macaulay.

Next, we treat the case where \(Q\) is principal.

**Lemma 19.** Let \(S\) be a domain which is essentially of finite type over a field \(k\) and \(Q = fS\) for some nonzero element \(f\) in \(S\). If \(f, g, h \in S\) is a regular sequence on \(S\), then \(fg, fh\) is a strong parameter sequence of \(R\) that is not a regular sequence of \(R\). In particular, the ring \(R := k + QS\) is not Cohen-Macaulay.

**Proof.** First, we claim that \(fg, fh\) is a strong parameter sequence on \(S\) that is not regular. By Lemma\[\text{[10]}\], it is clear that \(fg, fh\) is a weakly proregular sequence since \(S\) is Noetherian. We show that for each \(p \in V_R(fg, fh)\), we have that \(H^2_{fg, fh}(p) = 0\). First, suppose that \(p \neq Q\). Then \(f \notin p\) \((\sqrt{JR} = Q)\), so that \(H^2_{fg, fh}(R)p = H^2_{g,h}(R)p = H^2_{g,h}(S)p \neq 0\), since the regular sequence \(f, g\) of \(S\) extends to a regular sequence in \(S_p = R_p\). It remains to show \(H^2_{fg, fh}(R)Q \neq 0\). Choose a prime \(q \in V_S(f, g, h)\). Thus \(q \cap R = Q\) by \[\text{[DFF10, Prop 2.6(2)]}\]. Since \(H^2_{fg, fh}(R)Q = H^2_{fg, fh}(S)Q\) by Lemma\[\text{[10]}\], it is enough to show that \(H^2_{fg, fh}(S)_q \neq 0\), since \(R - Q \subseteq S - q\). We note that since \((f, g, h)\) is a regular sequence of length 3, one has \((f) \cap (g, h) = (fg, fh)\).

From the Mayer-Vietoris sequence, we have the following exact sequence
\[H^2_{fg, fh}(S_q) \to H^3_{fg, fh}(S_q) \to H^3_{g,h}(S_q) \oplus H^3_f(S_q) = 0.\]
Here, the right most cohomology modules are zero because 3 > ara\((g, h), ara(f)\). Since \(S_\mathfrak{g}\) is a Noetherian ring of dimension 3, by Proposition\([9]\)c, \(H^3_{f, g, h}(S_\mathfrak{g}) \neq 0\). Thus \(H^2_{f, g, h}(S_\mathfrak{g}) \neq 0\), and so \(fg, fh\) is a strong parameter sequence in \(R\).

Now, we show that \(fg, fh\) is not a regular sequence. From the Mayer-Vietoris sequence, we have the following exact sequence

\[ H^1_{f, g, h}(S) \rightarrow H^1_{g, h}(S) \oplus H^1_f(S) \rightarrow H^1_{f, g, fh}(S). \]

Since grade\((f, g, h)S\) \(\geq 2\), \(H^1_{f, g, h}(S) = 0\) by Proposition\([9]\)c, then \(H^1_f(S) = 0 \Rightarrow H^1_{f, g, fh}(S) = 0\). By Lemma\([10]\)a, we have the surjective map \(H^1_{f, g, fh}(R) \rightarrow H^1_{f, g, fh}(S)\). Thus, \(H^1_{f, g, fh}(R) \neq 0\) and p-grade\((fg, fh)R, R\) = 1 by [HM07, Prop. 2.7]. By Theorem\([7]\), \(R\) is not Cohen-Macaulay. □

**Example 20.** Let \(S = k[x, y, z]\), and consider the regular sequence \(x, y, z\) on \(S\). Then by Lemma\([19]\), the ring \(R = k + xk[x, y, z]\) is not Cohen-Macaulay, as \(xy, xz\) is a strong parameter sequence that is not a regular sequence. It is worth mentioning that the ring \(T = k + xk[x, y]\) is Cohen-Macaulay.

**Theorem 21.** Let \(S\) be a domain which is essentially of finite type over a field \(k\), and let \(f\) be a nonzero element in \(S\). If \(S\) is Cohen-Macaulay and \(\dim S \geq 3\), then the ring \(k + fS\) is not Cohen-Macaulay.

**Proof.** By Lemma\([19]\), it suffices to show there exists \(g, h \in S\) such that \((g, h)S\) is a regular sequence of length 3. Notice that \(S/fS\) is Cohen-Macaulay. Hence \(\text{Min}(S/fS) = \text{Ass}(S/fS)\). Since these sets are finite, by the prime avoidance lemma, one can find an element \(g\) which is not in the union of the associated primes of \(S/fS\). Then \(S/(f, g)S\) is Cohen-Macaulay. Thus one can find \(h\), similarly. □

4. **AN EXAMPLE OF A NON COHEN-MACAUŁAY NORMAL SEMIGROUP RING**

Let \(H\) be a semigroup in \(\mathbb{Z}^n\). We say that \(H\) is *normal* if for any \(m \in \mathbb{N}\) and \(s \in \mathbb{Z}H\), we have \(ms \in H \Rightarrow s \in H\). A celebrated theorem of Hochster [Hoc72] states that if \(H\) is a finitely generated normal semigroup in \(\mathbb{Z}^n\), then \(k[H]\) is Cohen-Macaulay, where \(k\) is any field. In this section, we will present an example of a semigroup ring \(k[H]\), where the ring \(K[H]\) is normal (so, the semigroup \(H\) is normal\(^2\)), yet \(k[H]\) is not Cohen-Macaulay. It is necessary that \(H\) is not finitely generated by the result of Hochster. For more details on (affine) semigroup rings, we refer the reader to [Hoc72] or [BH93, Section II.6].

Recall the question we mentioned in the introduction.

**Question 5 ([ADT14, Question 5.1]).** Let \(R\) be a semigroup ring of finite Krull dimension which is not necessarily Noetherian. Then is \(R\) Cohen-Macaulay if it is normal?

We answer this question negatively.

**Theorem 22.** Let \(S := k[x, y, z] = k[N_0^3]\) and \(R := k + xS = k[H]\), where \(k\) is a field and \(H = \mathbb{N}_0 \times \mathbb{N}^2 \cup \{(0, 0, 0)\}\). Then we have the following.

(a) The semigroup ring \(k[H]\) is normal.
(b) \(R\) is a non-Noetherian subring of \(S\) of dimension 3.

\(^2\)We do not know if \(H\) is normal is sufficient to conclude that \(K[H]\) is normal.
(c) $xy, xz$ is a strong parameter sequence of $R$ that is not a regular sequence of $R$.

In particular, $k[H]$ is a non-Noetherian normal domain which is not Cohen-Macaulay in the sense of Hamilton-Marley.

Proof. (a): Notice that since $S$ is a UFD, $S$ is integrally closed. Let $\overline{R}$ denote the integral closure of a domain in its field of fractions. Suppose that $f \in \overline{R}$. Then since $R \subseteq S$, we have that $\overline{R} \subseteq S = S$, so that $f \in S$. Now, for some $v \in \mathbb{N}$ and $g_0, \ldots, g_v \in R$, we have an equation of the form

$$f^v + g_1f^{v-1} + \cdots + g_{v-1}f + g_v = 0. \tag{4}$$

For each $i \in \{1, \ldots, v\}$, write $g_i = a_i + xh_i$, where $a_i \in k$, and $h_i \in S$. Going modulo $xS$, Equation (4) induces an equation of integrality of the image of $f$ in $S/xS$ over $R/(xS \cap R) = k$. Since $k$ is a field, it is integrally closed. Therefore $f \in k + xS = R$. Hence $R$ is integrally closed.

(b): The fact that $\dim R = 3$ follows from a result of Gilmer, [And06, Proposition 3.3]. Suppose $R$ is Noetherian. Then by Krull’s principal ideal theorem, $ht\, m = 1$. Let $n = (x, y, z)S$. Since $n \cap R = m$, the dimension formula [Mat89, Theorem 15.5] implies that $ht\, n \leq ht\, m$. But this is a contradiction.

(c): Notice that $S$ is Cohen-Macaulay and $x, y, z$ forms a regular sequence in $S$. Thus by Lemma 19, $xy, xz$ is a strong parameter sequence of $R$ which is not a regular sequence of $R$ (cf. Example 20).

Let $k$ be a field. We showed that $R = k + xk[x, y, z]$ is a non-Noetherian normal domain, but $R$ is not Cohen-Macaulay as $xy, xz$ is a strong parameter sequence which is not a regular sequence. On the other hand, the ring $A = k + xk[x, y]$ is Cohen-Macaulay. One might be tempted to say that $A[z]$ is not Cohen-Macaulay since one has the sequence $xy, xz$ in $A[z]$. However, $xy, xz$ is not a strong parameter sequence in $A[z]$, and we do not know whether or not $A[z]$ is Cohen-Macaulay. In the following section, we discuss the Cohen-Macaulayness of $R[x]$, where $x$ is an indeterminate over $R$.

5. A REMARK ON CONDITION [H1]

Recall Condition [H1]: $R$ is Cohen-Macaulay iff $R[x]$ is Cohen-Macaulay, where $x$ is an indeterminate over $R$. The if part was shown in [HM07, Corollary 4.6], but it is not known that $R[x]$ is Cohen-Macaulay if $R$ is Cohen-Macaulay. Unlike in the Noetherian case, several properties of the ring $R$ do not pass to its polynomial extension $R[x]$. In particular, $R[x]$ need not be coherent even if $R$ is coherent. In this section, we collect a few results to prove that $R[x]$ is Cohen-Macaulay if $R$ is a Prüfer domain of finite Krull dimension. We believe that Theorem 24 might be known to experts, but it was not written anywhere. We note that there are examples of coherent regular quasi local rings such that $R[x]$ is not even coherent, see Example 25. Here, we list some definitions, theorems, and a remark we will use in the proof of Theorem 24.

Remark 23. Let $R$ be a commutative ring.

(a) [Sab74, Prop. 3] If $R$ is a Prüfer domain, then $R[x_1, \ldots, x_n]$ is coherent for any $n$.

(b) [CE56, p.122] The weak dimension of $R$, denoted $\text{w.d.} R$, is

$$\sup\{i \in \mathbb{Z}_{\geq 0} \mid \text{Tor}_i^R(M, N) \neq 0, M, N \text{ R-modules}\}.$$
Furthermore, \( w. \dim R \leq \text{gl.} \dim R := \sup \{ i \in \mathbb{Z}_{\geq 0} \mid \text{Ext}^i_R(M, N) \neq 0, M, N \text{ } R\text{-modules} \} \).

(c) \[ \text{Gla89b, Lemma 3} \] If \((R, \mathfrak{m})\) is a quasi-local coherent regular ring, then \( \text{grade}(\mathfrak{m}, R) = w. \dim R \). In particular, \( w. \dim R \) is finite.

(d) \[ \text{Vas76, Theorem 0.14} \] If \( R[x_1, \ldots, x_d] \) is coherent, then \( w. \dim R[x_1, \ldots, x_d] = w. \dim R + d \).

(e) If \( R \) is a Prüfer domain of finite Krull dimension, then \( w. \dim R < \infty \).

(f) \[ \text{Gla87} \] A coherent ring of finite weak dimension is a regular ring, although not every coherent regular ring has finite weak dimension (e.g., \( k[[x_1, x_2, \ldots]] \), where \( k \) is a field).

**Theorem 24.** Let \( R \) be a Prüfer domain of finite Krull dimension, then \( R[x_1, \ldots, x_n] \) is (locally) Cohen-Macaulay. In particular, \( V[x_1, \ldots, x_n] \) is Cohen-Macaulay if \( V \) is a valuation domain.

**Proof.** Since \( R \) is a Prüfer domain of finite Krull dimension, \( w. \dim R < \infty \) (Remark 23(e)). By Remark 23(a), \( R[x_1, \ldots, x_n] \) is coherent. Therefore, Remark 23(d) implies that \( w. \dim R[x_1, \ldots, x_n] = w. \dim R + n < \infty \). Thus, \( R[x_1, \ldots, x_n] \) is a coherent ring with finite weak dimension. So, it is coherent regular by Remark 23(f). Then we are done by [HM07, Theorems 4.7, 4.8]. \( \square \)

We end the paper with an example of Soublin. He constructed an example of a 2 dimensional coherent regular ring \( R \) such that \( R[x] \) is not coherent. Alfonsi used this example to construct a 2 dimensional coherent quasi-local regular domain such that \( R[x] \) is not coherent. It is an interesting question whether \( R[x] \) is Cohen-Macaulay. We do not have an answer to this question.

**Example 25** ([Gla89a, §7.1.13], [Sou70], [Alf81]). Let \( \mathbb{N} \) and \( \mathbb{Q} \) denote the set of natural numbers and the set of rational numbers, \( S = \mathbb{Q}[[t, u]] \) a power series ring in two variables over \( \mathbb{Q} \), and \( R = \prod_{\alpha \in \mathbb{N}} S_\alpha \), where \( S_\alpha \cong S \). Then \( R \) is a coherent ring of weak dimension 2, but \( R[x] \) is not coherent.

**Question 6.** Is \( R[x] \) Cohen-Macaulay if \( R \) is the ring in the example of Soublin and Alfonsi?

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