ON THE SPECTRAL RADIUS OF COMPACT OPERATOR COCYCLES

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Abstract. We extend the notions of joint and generalized spectral radii to cocycles acting on Banach spaces and obtain a version of Berger-Wang’s formula when restricted to the space of cocycles taking values in the space of compact operators. Moreover, we observe that the previous quantities depend continuously on the underlying cocycle.

1. Introduction

Let $M_d$ be the space of all real $d \times d$ matrices and consider a compact set $\mathcal{M} \subset M_d$. Then, we can define the joint spectral radius of $\mathcal{M}$ by

$$\hat{\rho}(\mathcal{M}) = \lim_{n \to \infty} \sup \{\|A_n \cdots A_1\|^{1/n} : A_i \in \mathcal{M}\}.$$ 

This notion was introduced by Rota and Strang in their seminal paper [RS60] and since then it has found its applications in several different fields like coding theory [MOS01] and stability theory [Dai12]. Furthermore, one can define a generalized spectral radius of $\mathcal{M}$ by

$$\bar{\rho}(\mathcal{M}) = \limsup_{n \to \infty} \sup \{\rho(A_n \cdots A_1)^{1/n} : A_i \in \mathcal{M}\},$$

where $\rho(A)$ denotes the usual spectral radius of $A \in M_d$.

The celebrated result of Berger and Wang [BW92] (usually called the Berger-Wang formula) asserts that those two quantities coincide, i.e. $\hat{\rho}(\mathcal{M}) = \bar{\rho}(\mathcal{M})$. Moreover, this equality holds even when $\mathcal{M}$ is just a bounded subset of $M_d$. In addition, several results concerned with the regularity of the map $\mathcal{M} \mapsto \hat{\rho}(\mathcal{M})$ (acting on the space of compact subsets of $M_d$) were obtained. Indeed, Wirth [W02] proved that this map is continuous and also established its local Lipschitz continuity on the space of irreducible compact sets $\mathcal{M} \subset M_d$ (explicit Lipschitz constant was given in [K10]).

It was natural to ask whether these results can be extended to the infinite-dimensional setting, where $\mathcal{M}$ is a compact subset of the space of all bounded operators acting on some Banach space $B$. It turns out that in this setting, the version of the Berger-Wang formula doesn’t hold in general. Indeed, Gurvits [Gu95] provided an explicit counterexample (with $\mathcal{M}$ consisting of only two operators). However, some partial extensions of the Berger-Wang formula were obtained in [ST00, ST02] with the best result to this date being that of Morris [IM12].

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It turns out that the previously mentioned results can be formulated in the context of ergodic theory. Indeed, it is possible to associate to \( M \), the so-called \textit{linear cocycle} (of matrices or operators) which acts over a full two-sided shift \( (M, f) \) and to give an alternative formulation of the Berger-Wang formula and related results. We refer to Remark 3.5 for a detailed discussion. This observation also opened possibilities of using tools from ergodic theory to study the notions of joint and generalized spectral radius from the point of view of dynamical systems. In this direction, Dai [Dai14] obtained the version of the Berger-Wang formula for linear cocycles of matrices acting over subshifts \( (M, f) \) of finite-type. More recently, Zou, Cao and Liao [ZCL18] extended the result of Dai by proving that the same conclusion holds whenever \( (M, f) \) is a topological dynamical system satisfying the so-called Anosov Closing property. Moreover, they showed that the joint spectral radius is a continuous function on the space of Hölder continuous cocycles.

The main objective of the present paper is to extend the results in [ZCL18] to the case of linear cocycles with values in the space of compact operators acting on arbitrary Banach spaces. This is achieved by carefully combining various results in the literature. In particular, our recent results dealing with the approximation of Lyapunov exponents in the infinite-dimensional setting [BD] play a central role.

2. Preliminaries

Throughout this paper \((M, d)\) will be a compact metric space and \( f : M \to M \) will be a homeomorphism that satisfies the \textit{Anosov Closing property}. We recall that the latter means that there exist \( C_1, \varepsilon_0, \theta > 0 \) such that if \( z \in M \) satisfies \( d(f^n(z), z) < \varepsilon_0 \) then there exists a periodic point \( p \in M \) such that \( f^n(p) = p \) and

\[
d(f^j(z), f^j(p)) \leq C_1 e^{-\theta \min(j, n-j)} d(f^n(z), z),
\]

for every \( j = 0, 1, \ldots, n \). We note that shifts of finite type, basic pieces of Axiom A diffeomorphisms and more generally, hyperbolic homeomorphisms are particular examples of maps satisfying the Anosov Closing property. We refer to [KH95, Corollary 6.4.17.] for details.

2.1. Semi-invertible operator cocycles. Let \((\mathcal{B}, \| \cdot \|)\) be a Banach space and let \( B(\mathcal{B}, \mathcal{B}) \) be the space of all bounded linear maps from \( \mathcal{B} \) to itself. Denote by \( B_0(\mathcal{B}, \mathcal{B}) \) the subset of \( B(\mathcal{B}, \mathcal{B}) \) formed by the compact operators of \( \mathcal{B} \). We recall that \( B(\mathcal{B}, \mathcal{B}) \) is a Banach space with respect to the norm

\[
\| T \| = \sup \{ \| Tv \| / \| v \| : \| v \| \neq 0 \}, \quad T \in B(\mathcal{B}, \mathcal{B})
\]

and \( B_0(\mathcal{B}, \mathcal{B}) \) is a closed subspace of \( B(\mathcal{B}, \mathcal{B}), \| \cdot \| \). Although we use the same notation for the norms on \( \mathcal{B} \) and \( B(\mathcal{B}, \mathcal{B}) \) this will not cause any confusion. Finally, consider a map \( A : M \to B(\mathcal{B}, \mathcal{B}) \).

The \textit{semi-invertible operator cocycle} (or just \textit{cocycle} for short) generated by \( A \) over \( f \) is defined as the map \( A : \mathbb{N} \times M \to B(\mathcal{B}, \mathcal{B}) \) given by

\[
A^n(x) := A(n, x) = \begin{cases} A(f^{n-1}(x)) \ldots A(f(x))A(x) & \text{if } n > 0 \\
\text{Id} & \text{if } n = 0,
\end{cases}
\]

for all \( x \in M \). The term ‘semi-invertible’ refers to the fact that the action of the underlying dynamical system \( f \) is assumed to be an invertible transformation while the action on the fibers given by \( A \) may fail to be invertible.

We say that the cocycle generated by \( A \) over \( f \) is \textit{compact} if \( A \) take values in \( B_0(\mathcal{B}, \mathcal{B}) \), i.e. if \( A(x) \in B_0(\mathcal{B}, \mathcal{B}) \) for each \( x \in M \).
2.2. **Volume growth.** Let $T \in B(\mathcal{B}, \mathcal{B})$. For a subspace $V \subset \mathcal{B}$, set

$$m(T|_V) := \inf\{\|Tv\|; v \in V \text{ with } \|v\| = 1\}.$$ 

Furthermore, for each $k \in \mathbb{N}$ such that $k \leq d := \dim \mathcal{B}$, set

$$F_k(T) := \sup\{m(T|_V); V \subset \mathcal{B} \text{ is a } k\text{-dimensional subspace}\},$$

$$c_k(T) := \inf\{\|T|_V\|; V \subset \mathcal{B} \text{ is a } (k-1)\text{-codimensional subspace}\},$$

and

$$V_k(T) = \sup\left\{\frac{m_{TV}(T(B_V))}{m_V(B_V)}; V \subset \mathcal{B} \text{ is a } k\text{-dimensional subspace}\right\},$$

where $m_V$ denotes the Haar measure on the subspace $V$ normalized so that the unit ball $B_V$ in $V$ has measure given by the volume of the Euclidean unit ball in $\mathbb{R}^k$. We recall that quantities $F_k(T)$ are called Kolmogorov numbers of $T$, while quantities $c_k(T)$ are called Gelfand numbers of $T$.

We note that $V_k(T)$, $\prod_{j=1}^k F_j(T)$ and $\prod_{j=1}^k c_j(T)$ may be interpreted as the growth rates of $k$-dimensional volumes spanned by $\{Tv_i\}_{i=1}^k$, where $v_i \in \mathcal{B}$ are unit vectors. Below we present a result relating all the previous notions of volume growth. In fact, this result says that up to a multiplicative constant, all of them coincide.

**Lemma 2.1.** Given $k \in \mathbb{N}$ such that $k \leq \dim \mathcal{B}$, there exists $C > 0$ (depending only on $k$) such that for every $T \in B(\mathcal{B}, \mathcal{B})$,

$$\frac{1}{C} F_k(T) \leq c_k(T) \leq CF_k(T),$$

$$\frac{1}{C} V_k(T) \leq \prod_{j=1}^k F_j(T) \leq CV_k(T)$$

and

$$\frac{1}{C} V_k(T) \leq \prod_{j=1}^k c_j(T) \leq CV_k(T).$$

**Proof.** The first estimate is proved in [BM, Lemma 15.], while the second is established in the proof of [DFGT18, Lemma A.2]. Finally, the last assertion of the lemma is an easy consequence of the first two. \qed

We shall also need the following auxiliary result.

**Lemma 2.2.** For every $k \in \mathbb{N}$, the map $T \rightarrow V_k(T)$ is continuous on $B(\mathcal{B}, \mathcal{B})$. Moreover, it is also submultiplicative, i.e.

$$V_k(TS) \leq V_k(T)V_k(S) \text{ for every } T, S \in B(\mathcal{B}, \mathcal{B}).$$

**Proof.** The continuity of the map $T \rightarrow V_k(T)$ is established in [AB16, Lemma 2.20], while the submultiplicativity property was observed in the proof of [DFGT18, Lemma A.2]. \qed
3. Multiplicative ergodic theorem

We now recall the version of the multiplicative ergodic theorem established in [FLQ13] (see also [AB16, GTQ15]). We stress that we don’t state it in full generality. Indeed, we present a simplified version that will be sufficient for our purposes.

**Theorem 3.1.** Assume that $A$ is a continuous and compact cocycle over $f$. Furthermore, let $\mu$ be an $f$-invariant ergodic probability measure on $M$. Then, there exists a Borel set $R^\mu \subset M$ such that $\mu(R^\mu) = 1$ and either:

1. there is a finite sequence of numbers
   \[ \lambda_1(A, \mu) > \lambda_2(A, \mu) > \cdots > \lambda_k(A, \mu) > \lambda_\infty(A, \mu) = -\infty \]
   and a measurable decomposition
   \[ B = E_1(x) \oplus \cdots \oplus E_k(x) \oplus E_\infty(x) \]
   such that for $x \in R^\mu$,
   \[ A(x)E_i(x) = E_i(f(x)), \quad i = 1, \ldots, k, \]
   \[ A(x)E_\infty(x) \subset E_\infty(f(x)), \]
   and
   \[ \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(A, \mu), \]
   for $v \in E_i(x) \setminus \{0\}$, $i \in \{1, \ldots, k, \infty\}$. Moreover, each $E_i(x), i = 1, \ldots, k,$ is a finite-dimensional subspace of $B$;

2. there exists an infinite sequence of numbers
   \[ \lambda_1(A, \mu) > \lambda_2(A, \mu) > \cdots > \lambda_k(A, \mu) > \cdots > \lambda_\infty(A, \mu) = -\infty, \]
   \[ \lim_{k \to \infty} \lambda_k(A, \mu) = \lambda_\infty(A, \mu), \]
   and for each $k \in \mathbb{N}$ a measurable decomposition
   \[ B = E_1(x) \oplus \cdots \oplus E_k(x) \oplus \cdots \oplus E_\infty(x) \]
   such that for $x \in R^\mu$,
   \[ A(x)E_i(q) = E_i(f(x)), \quad i \in \mathbb{N}, \]
   \[ A(q)E_\infty(x) \subset E_\infty(f(x)), \]
   and
   \[ \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(A, \mu), \]
   for $v \in E_i(x) \setminus \{0\}$, $i \in \mathbb{N} \cup \{\infty\}$. Moreover, each $E_i(x), i \in \mathbb{N}$ is a finite-dimensional subspace of $B$.

We recall that numbers $\lambda_i(A, \mu)$ are called *Lyapunov exponents* of $A$ with respect to $\mu$. Moreover, $d_i(A, \mu) := \dim E_i(x)$ is said to be a multiplicity of $\lambda_i(A, \mu)$. 
3.1. The Joint and Generalized spectral radii. Finally, we recall some notions that will be of central importance to our paper. Given \( s > 0 \) and \( T \in \mathcal{B}(\mathcal{B}, \mathcal{B}) \), let \( d = \dim(\mathcal{B}) \) and consider

\[
\varphi^s_c(T) = \begin{cases} 
  c_1(T)c_2(T) \cdots c_{[s]}(T)c_{[s]+1}(T)^{s-[s]} & \text{for } s < d; \\
  |\det T|^{s/d} & \text{for } s \geq d \text{ and } d < \infty;
\end{cases}
\]

and

\[
\varphi^s_f(T) = \begin{cases} 
  F_1(T)F_2(T) \cdots F_{[s]}(T)F_{[s]+1}(T)^{s-[s]} & \text{for } s < d; \\
  |\det T|^{s/d} & \text{for } s \geq d \text{ and } d < \infty.
\end{cases}
\]

Remark 3.2. Assume that \( d < \infty \). Then, \( c_i(T) \) is precisely the \( i \)-th singular value of \( T \) for \( i = 1, \ldots, d \) (see [BM] for example). Hence, \( \varphi^s_c(T) \) coincides with \( \varphi^s(T) \), where \( \varphi^s \) is a singular value function (see [ZCL18, p.2]).

Then, we define the \( s \)-joint spectral radius of \((A, f)\) by

\[
\hat{\rho}_s(A) := \lim_{n \to +\infty} \sup_{x \in M} \varphi^s_f(A^n(x))^{1/s}
\]

\[
= \lim_{n \to +\infty} \sup_{x \in M} \varphi^s_f(A^n(x))^{1/s},
\]

where the second equality follows from Lemma 2.1. Finally, the \( s \)-generalized spectral radius of \((A, f)\) is defined by

\[
\overline{\rho}_s(A) := \limsup_{n \to +\infty} \left( \sup_{x \in M} \rho_s(A^n(x)) \right)^{1/s},
\]

where

\[
\rho_s(T) := \lim_{n \to +\infty} \varphi^s_f(T^n)^{1/s} \quad \text{for any } T \in \mathcal{B}(\mathcal{B}, \mathcal{B}).
\]

Remark 3.3. It follows from Remark 3.2 that in the case when \( d < \infty \), \( \hat{\rho}_s(A) \) and \( \overline{\rho}_s(A) \) coincide respectively with the values of the joint spectral radius and the generalized spectral radius of \( A \) which were studied in [ZCL18].

3.2. Main results. We are now in position to formulate the main results of our paper. For \( \alpha > 0 \), we say that \( A : M \to \mathcal{B}(\mathcal{B}, \mathcal{B}) \) is an \( \alpha \)-Hölder continuous map if there exists a constant \( C_2 > 0 \) such that

\[
\| A(x) - A(y) \| \leq C_2 d(x, y)^{\alpha},
\]

for all \( x, y \in M \).

The following is our first result. It can be described as an extension of [ZCL18, Theorem C.] to the case of compact cocycles acting on arbitrary Banach spaces.

Theorem 3.4. Let \( f : M \to M \) be a homeomorphism satisfying the Anosov Closing property and \( A : M \to \mathcal{B}_0(\mathcal{B}, \mathcal{B}) \) an \( \alpha \)-Hölder continuous map. Then,

\[
\hat{\rho}_s(A) = \overline{\rho}_s(A)
\]

for every \( s > 0 \).

Remark 3.5. The first result in the spirit of Theorem 3.4 was obtained by Berger and Wang [BW92] and, in fact, their result is a particular case of Theorem 3.4. More precisely, let \( \mathcal{M} \) be a compact subset of \( M_d \), where \( M_d \) denotes the space of all real matrices of order \( d \). Set \( M = M^2 \) and equip \( M \) with the product topology so that it becomes a compact metric space. Furthermore, let \( f : M \to M \) be a
two-sided shift given by \( f((M_i)_{i \in \mathbb{Z}}) = (M_{i+1})_{i \in \mathbb{Z}} \) for \((M_i)_{i \in \mathbb{Z}} \in M\). Finally, let 
\( A: M \to M_d \) be given by \( A((M_i)_{i \in \mathbb{Z}}) = M_0 \), \((M_i)_{i \in \mathbb{Z}} \in M\). It turns out that the 
main result from [BW92] can be recovered from Theorem 3.4 for this particular 
choice of \( M, f \) and \( A \) and \( s = 1 \). We note that strictly speaking, the main result 
from [BW92] requires only that \( \mathcal{M} \subset M_d \) is bounded. However, it turns out that 
this general version can be deduced from the one previously stated by replacing \( \mathcal{M} \) 
with its closure (see [IM12, p.8]). Similar results for the case when \( A \) is as above 
but when \((M, f)\) is a subshift of finite type were obtained by Dai [Dai14]. Finally, 
as we already mentioned, the general case that corresponds to our Theorem 3.4 
when \( \mathcal{B} \) is finite-dimensional was treated in [ZCL18].

It is also worth noticing that, as pointed out by Gurvits [Gu95, Theorem A.1], 
the previously described result by Berger and Wang (and consequently our Theorem 
3.4) doesn’t hold, in general, in the infinite-dimensional case. In fact, Gurvits 
presented an example of two operators \( T_1, T_2 \in B(\mathcal{B}, \mathcal{B}) \) for which the generalized 
spectral radius is strictly smaller than the joint spectral radius. However, some 
partial extensions of Berger-Wang’s formula to the infinite dimensional setting were 
obtained in [ST00, ST02], with the most general result being that of Morris [IM12, 
Theorem 1.4]. In the particular case when dealing with compact operators, the 
result of Morris is covered by Theorem 3.4 and it corresponds to the case when 
\( M = \mathcal{M}^\mathbb{Z} \), where \( \mathcal{M} \) is a (pre)compact subset of \( B_0(\mathcal{B}, \mathcal{B}) \) and with \( f \) and \( A \) as 
in the previous paragraph and \( s = 1 \). In Theorem 3.4, we deal with a general 

For \( \alpha > 0 \), set

\[
C^\alpha(M, B_0(\mathcal{B}, \mathcal{B})) := \left\{ A: M \to B_0(\mathcal{B}, \mathcal{B}) : A \text{ is an } \alpha \text{-Hölder continuous map} \right\}.
\]

We note that \( C^\alpha(M, B_0(\mathcal{B}, \mathcal{B})) \) is a Banach space with respect to the norm

\[
\|A\|_\alpha := \sup_{x, y \in M} \|A(x) - A(y)\| + \sup_{x \neq y} \left( \frac{\|A(x) - A(y)\|}{d(x, y)^\alpha} \right).
\]

The following is our second result. It represent an extension of [ZCL18, Theorem 
A.] to the case of compact cocycles acting on arbitrary Banach spaces.

**Theorem 3.6.** Let \( f: M \to M \) be a homeomorphism satisfying the Anosov Closing 
property. Then, the map

\[
A \to \hat{\rho}_s(A) = \overline{\rho}_s(A)
\]

is continuous on \( C^\alpha(M, B_0(\mathcal{B}, \mathcal{B})) \).

**Remark 3.7.** We stress that the conclusion of Theorem 3.6 can fail if \( f \) doesn’t 
satisfy the Anosov Closing property. Indeed, explicit counterexamples were 
constructed in [DHH17, WY13] (see [ZCL18, p.2] for a detailed discussion).

**Remark 3.8.** Finally, we would like to explain why we restricted our attention to 
the case of compact cocycles. It turns out (see [Deg08, Theorem 2.1.]) that the 
spectral radius mapping is not a continuous function on \( B(\mathcal{B}, \mathcal{B}) \). Hence, 
Theorem 3.6 doesn’t hold if one replaces \( C^\alpha(M, B_0(\mathcal{B}, \mathcal{B})) \) by \( C^\alpha(M, B(\mathcal{B}, \mathcal{B})) \) even in 
the case when \( A \) is a constant map.
4. Proofs

In this section we present proofs of our main results.

4.1. Proof of Theorem 3.6. We can assume that $\mathcal{B}$ is infinite-dimensional since the case when $d < \infty$ is covered by [ZCL18, Theorem A]. We first present several auxiliary results.

Lemma 4.1. For any $s \in \mathbb{N}$,
\[
\log \hat{\rho}_s(A) = \max_{\mu \in \mathcal{M}_f} \left\{ \inf_n \frac{1}{n} \int \log V_s(A^n(x)) d\mu \right\}
\]
\[
= \max_{\mu \in \mathcal{M}_f} \left\{ \inf_n \frac{1}{n} \int \log \varphi_s(A^n(x)) d\mu \right\}
\]
where $\mathcal{M}_f$ denotes the set of all $f$-invariant probability measures.

Proof. The last two equalities follow directly from Lemma 2.1. Let us now proof that the first equality holds. By Lemma 2.1, we have that
\[
\log \hat{\rho}_s(A) = \log \left( \lim_{n \to +\infty} \sup_{x \in \mathcal{M}} V_s(A^n(x))^{1/n} \right)
\]
\[
= \lim_{n \to +\infty} \sup_{x \in \mathcal{M}} \left( \frac{1}{n} \log V_s(A^n(x)) \right).
\]
It follows from Lemma 2.2 that we can apply [IM13, Lemma A.3] for $f_n(x) = \log V_s(A^n(x))$ and we obtain that
\[
\lim_{n \to +\infty} \sup_{x \in \mathcal{M}} \left( \frac{1}{n} \log V_s(A^n(x)) \right) = \inf \sup_n \left( \frac{1}{n} \log \mu(V_s(A^n(x))) \right)
\]
\[
= \max_{\mu \in \mathcal{M}_f} \left\{ \inf_n \frac{1}{n} \int \log V_s(A^n(x)) d\mu \right\}.
\]
The proof of the lemma is completed. $\square$

Lemma 4.2. For any $s \in \mathbb{N}$,
\[
\inf \frac{1}{n} \int \log V_s(A^n(x)) d\mu = \gamma_1(A, \mu) + \gamma_2(A, \mu) + \cdots + \gamma_s(A, \mu),
\]
where $\gamma_j(A, \mu)$ stands for the $j$-th Lyapunov exponent of $(A, f)$ with respect to $\mu$ counted with multiplicities. In particular,
\[
\log \hat{\rho}_s(A) = \max_{\mu \in \mathcal{M}_f} \left\{ \gamma_1(A, \mu) + \gamma_2(A, \mu) + \cdots + \gamma_s(A, \mu) \right\}.
\]

Proof. The first claim was established in the proof of [DFGTV18, Lemma A.3], while the second is a direct consequence of the first one together with Lemma 4.1. $\square$

Lemma 4.3. For any $s \in \mathbb{N}$ and $T \in \mathcal{B}(\mathcal{B}, \mathcal{B})$, set
\[
r_s(T) := \lim_{n \to +\infty} \frac{1}{n} \log V_s(T^n).
\]
Then, $T \to r_s(T)$ is a continuous map on $B_0(\mathcal{B}, \mathcal{B})$. 

Proof. We first observe that it follows from Lemma 2.2 that \( r_s(T) \) is well-defined for each \( T \in B(\mathcal{B}, \mathcal{B}) \). By Lemma 2.1, we have that
\[
r_s(T) = \lim_{n \to +\infty} \frac{1}{n} \log V_s(T^n) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{s} \log c_j(T^n).
\]
In particular, \( r_1(T) \) is just the logarithm of the spectral radius of \( T \). Thus, it follows from [CM79] (see also [Deg08, Theorem 2.1]) that \( T \to r_1(T) \) is a continuous map on \( B_0(\mathcal{B}, \mathcal{B}) \).

In order to treat the general case, we start by recalling some classical facts about compact operators on Banach spaces. Let \( T \) be a compact operator acting on \( \mathcal{B} \). Since we assumed that \( \mathcal{B} \) is infinite-dimensional, it follows from the Fredholm’s Alternative (see [RR00, Theorem 6.2.8]) that its spectrum \( \sigma(T) \) can be written as
\[
\sigma(T) = \{0\} \cup \{\lambda_i : i \in \mathbb{N}\}
\]
with
\[
|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots \quad \text{and} \quad \lim_{n \to \infty} \lambda_n = 0
\]
where each \( \lambda_i \) is an eigenvalue of \( T \). Moreover, from the Riesz Decomposition Theorem for compact operators (see [RR00, Theorem 6.4.11] and [RR00, Corollary 6.4.12]), we conclude that for every \( i \in \mathbb{N} \),
\[
\mathcal{B} = \mathcal{N}_{\lambda_i} \bigoplus \mathcal{R}_{\lambda_i},
\]
where \( \mathcal{N}_{\lambda_i} = \ker(T - \lambda_i)^N \) and \( \mathcal{R}_{\lambda_i} = \Im(T - \lambda_i)^N \) for some \( N \in \mathbb{N} \). Furthermore, \( \mathcal{N}_{\lambda_i} \) and \( \mathcal{R}_{\lambda_i} \) are invariant under \( T \), \( \mathcal{N}_{\lambda_i} \) is finite-dimensional, \( \sigma(T|_{\mathcal{N}_{\lambda_i}}) = \{\lambda_i\} \) and \( \sigma(T|_{\mathcal{R}_{\lambda_i}}) = \sigma(T) \setminus \{\lambda_i\} \). Set
\[
\xi_j(T) := \lim_{n \to +\infty} \frac{1}{n} \log c_j(T^n) = \lim_{n \to +\infty} \frac{1}{n} \log V_j(T^n) - \lim_{n \to +\infty} \frac{1}{n} \log V_{j-1}(T^n),
\]
for \( j \in \mathbb{N} \). Observe that
\[
r_s(T) = \sum_{j=1}^{s} \xi_j(T).
\]
We claim that
\[
\xi_j(T) = \log |\lambda_k|,
\]
for \( k \in \mathbb{N} \) and \( j \in \{\dim(N_{\lambda_0}) + \ldots + \dim(N_{\lambda_{k-1}}) + 1, \ldots, \dim(N_{\lambda_0}) + \ldots + \dim(N_{\lambda_k})\} \), where \( \dim(N_{\lambda_0}) := 0 \). Indeed, the fact that (2) holds for \( j = 1 \) was already observed. Take now \( j = 2 \). If \( \dim(\mathcal{N}_{\lambda_1}) = 1 \), then one can conclude that \( \xi_2(T) = \log |\lambda_2| \) by applying (2) for \( j = 1 \) and \( T|_{\mathcal{R}_{\lambda_1}} \) (instead of \( T \)). Assume now that \( \dim(\mathcal{N}_{\lambda_1}) \geq 2 \). Then, if \( V \subset \mathcal{B} \) is a 1-codimensional subspace we have that \( V \cap \mathcal{N}_{\lambda_1} \neq \{0\} \). Let us fix \( v_V \in \mathcal{N}_{\lambda_1} \cap V \) such that \( \|v_V\| = 1 \). We have that
\[
c_2(T^n) = \inf_{v_V} \|(T^n)v_V\| \geq \inf_{v_V} \|T^n v_V\| \geq \|(T|_{\mathcal{N}_{\lambda_1}})^{-1} n\|^{-1}.
\]
Hence,
\[
\xi_2(T) = \lim_{n \to +\infty} \frac{1}{n} \log c_2(T^n) \geq - \lim_{n \to +\infty} \frac{1}{n} \log \|(T|_{\mathcal{N}_{\lambda_1}})^{-1} n\| = \log |\lambda_1|,
\]
since \( r_1((T_{N\lambda})^{-1}) = |\lambda_1|^{-1} \). This easily implies that that \( \xi_2(T) = \log |\lambda_1| \). Hence, (2) holds for \( j = 2 \). By iterating the above argument one can conclude that (2) holds for each \( j \in \mathbb{N} \).

Finally, since \( T \to \sigma(T) \) is continuous on \( B_0(\mathcal{B}, \mathcal{B}) \) (see [CM79]), the conclusion of the lemma follows from (1) and (2).

We are now in position to complete the proof of Theorem 3.6. Suppose initially that \( s \in \mathbb{N} \). By Lemma 2.2 we know that \((A, \mu) \to \frac{1}{n} \int V_s(A^n(x)) d\mu \) is a continuous map for every \( n \in \mathbb{N} \). In particular, \( A \to \inf_n \frac{1}{n} \int V_s(A^n(x)) d\mu \) is upper-semicontinuous. Thus, the compactness of \( \mathcal{M}_f \) combined with Lemma 4.1 implies that \( A \to \hat{\rho}_s(A) \) is upper-semicontinuous.

On the other hand, by Lemma 4.2 combined with [BD, Theorem 2.5] we obtain that

\[
\log \hat{\rho}_s(A) = \max_{\mu \in \mathcal{M}_f} \{ \gamma_1(A, \mu) + \gamma_2(A, \mu) + \cdots + \gamma_s(A, \mu) \} = \max_{\mu \in \mathcal{M}_f(Per)} \{ \gamma_1(A, \mu) + \gamma_2(A, \mu) + \cdots + \gamma_s(A, \mu) \},
\]

where \( \mathcal{M}_f(Per) \) denotes the set of all \( f \)-invariant probability measures supported on periodic orbits. Now, if \( p \in \mathcal{M} \) satisfies \( f^k(p) = p \) and \( \mu_p \) is the \( f \)-invariant measure supported on the orbit of \( p \), then

\[
\gamma_1(A, \mu_p) + \gamma_2(A, \mu_p) + \cdots + \gamma_s(A, \mu_p) = \frac{1}{k} r_s(A^k(p)).
\]

Indeed, it follows from Lemmas 2.2 and 4.2 together with Kingman’s subadditive ergodic theorem that

\[
\gamma_1(A, \mu_p) + \gamma_2(A, \mu_p) + \cdots + \gamma_s(A, \mu_p) = \inf_n \frac{1}{n} \int V_s(A^n(x)) d\mu_p
\]

\[
= \lim_{n \to +\infty} \frac{1}{n} \log V_s(A^n(p))
\]

\[
= \lim_{n \to +\infty} \frac{1}{nk} \log V_s(A^{nk}(p))
\]

\[
= \frac{1}{k} \lim_{n \to +\infty} \frac{1}{n} \log V_s(A^k(p)^n)
\]

\[
= \frac{1}{k} r_s(A^k(p)).
\]

We observe that Lemma 4.3 implies that the map \( A \to \frac{1}{k} r_s(A^k(p)) \) is continuous and consequently, the map \( A \to \log \hat{\rho}_s(A) \) is lower-semicontinuous which yields the conclusion of the theorem in the case when \( s \in \mathbb{N} \).

Take now an arbitrary \( s > 0 \). Observe that

\[
\varphi_c^s(T) = (\varphi_c^{[s]} + 1(T))^{s-[s]}(\varphi_c^{[s]}(T))^{1-s+[s]},
\]

for any \( T \in B(\mathcal{B}, \mathcal{B}) \). By setting \( V_s(T) := (V_{[s]} + 1(T))^{s-[s]}(V_{[s]}(T))^{1-s+[s]} \), one can repeat the arguments in the proof of Lemma 4.1 to show that

\[
\log \hat{\rho}_s(A) = \max_{\mu \in \mathcal{M}_f} \left\{ \inf_n \frac{1}{n} \int V_s(A^n(x)) d\mu \right\} = \max_{\mu \in \mathcal{M}_f} \left\{ \inf_n \frac{1}{n} \int \log \varphi_c^s(A^n(x)) d\mu \right\}.
\]

Arguing as in the case when \( s \in \mathbb{N} \), we obtain that \( A \to \log \hat{\rho}_s(A) \) is upper-semicontinuous.
On the other hand, it follows from (5) that
\[
\log \hat{\rho}_s(A) = \max_{\mu \in \mathcal{M}} \left\{ \left( s - |s| \right) \inf \frac{1}{n} \int \log \varphi_c^{[s]}(A^n(x)) \, d\mu \right. \\
+ \left( 1 - s + |s| \right) \inf \frac{1}{n} \int \log \varphi_c^{[s]}(A^n(x)) \, d\mu \\
\geq \max_{f^k(p) = p, k \in \mathbb{N}} \left\{ \left( s - |s| \right) \inf \frac{1}{n} \int \log \varphi_c^{[s]}(A^n(x)) \, d\mu_p \\
+ \left( 1 - s + |s| \right) \inf \frac{1}{n} \int \log \varphi_c^{[s]}(A^n(x)) \, d\mu_p \right\}.
\]

Thus,
\[
\log \hat{\rho}_s(A) \geq \max_{f^k(p) = p, k \in \mathbb{N}} \left\{ \left( s - |s| \right) r_{[s]+1}(A^k(p)) + \frac{1 - s + |s|}{k} r_{[s]}(A^k(p)) \right\}.
\]

By applying [BD, Theorem 2.5] one can easily conclude that
\[
\log \hat{\rho}_s(A) \leq \max_{f^k(p) = p, k \in \mathbb{N}} \left\{ \left( s - |s| \right) r_{[s]+1}(A^k(p)) + \frac{1 - s + |s|}{k} r_{[s]}(A^k(p)) \right\},
\]
and therefore
\[
\log \hat{\rho}_s(A) = \max_{f^k(p) = p, k \in \mathbb{N}} \left\{ \left( s - |s| \right) r_{[s]+1}(A^k(p)) + \frac{1 - s + |s|}{k} r_{[s]}(A^k(p)) \right\}.
\]

Arguing as in the case when \( s \in \mathbb{N} \), we obtain that the map \( A \mapsto \hat{\rho}_s(A) \) is lower-\( \sigma \)-semicontinuous. Hence, the conclusion of the theorem holds for arbitrary \( s > 0 \).

\textbf{Remark 4.4.} We note that it was not necessary to refer to [ZCL18] for the case when \( d < \infty \). Indeed, our arguments can be easily modified to cover the finite-dimensional case also. In fact, one only needs to modify slightly (actually simplify) the proof of Lemma 4.3.

4.2. \textbf{Proof of Theorem 3.4.} Let us again assume that \( \mathcal{B} \) is infinite-dimensional (Remark 4.4 applies for the proof of this theorem also). We start with two auxiliary lemmas.

\textbf{Lemma 4.5.} For any \( T \in B_0(\mathcal{B}, \mathcal{B}) \) and \( j, n \in \mathbb{N} \), we have
\[
\xi_j(T^n) = n \xi_j(T).
\]

\textit{Proof.} Using the same notation as in the proof of Lemma 4.3 one has that
\[
\sigma(T^n) = \{ \lambda^n : i \in \mathbb{N} \}.
\]

Furthermore, \( \dim N_{\lambda_i} = \dim N_{\lambda^n} \). Hence, the conclusion of the lemma follows directly from (2) (applied both for \( T \) and \( T^n \)). \hfill \Box

\textbf{Lemma 4.6.} For any \( s \in \mathbb{N} \) and \( T \in B_0(\mathcal{B}, \mathcal{B}) \),
\[
\lim_{n \to +\infty} \varphi_c^{[s]}(T^n) = \lim_{n \to +\infty} \rho_s(T^n) = \lim_{n \to +\infty} \frac{1}{n} \log V_s(T^n).
\]

\textit{Proof.} Observe that
\[
\lim_{n \to +\infty} \frac{1}{n} \log \varphi_c^{[s]}(T^n) = \lim_{n \to +\infty} \frac{1}{n} \log V_s(T^n) = r_s(T).
\]
On the other hand,
\[
\frac{1}{n} \log \rho_s(T^n) = \frac{1}{n} \log \lim_{m \to +\infty} V_s((T^n)^m)^{1/m} = \frac{1}{n} \lim_{m \to +\infty} \frac{1}{m} \log V_s((T^n)^m) = \frac{1}{n} r_s(T^n).
\]

Thus, since
\[
r_s(T^n) = \sum_{j=1}^{s} \xi_j(T^n),
\]
it follows from the previous lemma that for each \(n \in \mathbb{N}\),
\[
r_s(T^n) = n \sum_{j=1}^{s} \xi_j(T) = nr_s(T).
\]
This completes the proof of the lemma. \(\square\)

We start observing that from Lemma 2.1 and the submultiplicativity of \(V_s\) (see Lemma 2.2) we get that for any \(T \in B(B, B)\) and \(s \in \mathbb{N}\),
\[
\rho_s(T) = \lim_{n \to +\infty} \left( c_1(T^n)c_2(T^n) \ldots c_s(T^n) \right)^{1/n} = \lim_{n \to +\infty} V_s(T^n)^{1/n} = \inf_n V_s(T^n)^{1/n} \leq V_s(T).
\]
Take now any \(s > 0\). It follows easily from (5) that
\[
\rho_s(T) = \rho_{\lfloor s \rfloor + 1}(T)^{s-\lfloor s \rfloor} \rho_{\lfloor s \rfloor}(T)^{1-s+\lfloor s \rfloor}, \quad \text{for every } T \in B(B, B).
\]
Using Lemma 2.1 and the previous observation, we have that
\[
\overline{\rho}_s(A) = \limsup_{n \to +\infty} \left( \sup_{x \in M} \rho_s(A^n(x)) \right)^{1/n} = \limsup_{n \to +\infty} \left( \sup_{x \in M} \rho_{\lfloor s \rfloor + 1}(A^n(x))^{s-\lfloor s \rfloor} \rho_{\lfloor s \rfloor}(A^n(x))^{1-s+\lfloor s \rfloor} \right)^{1/n} \leq \limsup_{n \to +\infty} \left( \sup_{x \in M} V_{\lfloor s \rfloor + 1}(A^n(x))^{s-\lfloor s \rfloor} V_{\lfloor s \rfloor}(A^n(x))^{1-s+\lfloor s \rfloor} \right)^{1/n} = \limsup_{n \to +\infty} \left( \sup_{x \in M} \varphi^s_{c}(A^n(x)) \right)^{1/n} = \limsup_{n \to +\infty} \sup_{x \in M} \varphi^s_{c}(A^n(x))^{1/n},
\]
and therefore
\[
\overline{\rho}_s(A) \leq \hat{\rho}_s(A). \tag{7}
\]
Let us now establish the converse inequality. Take an arbitrary $p \in \text{Fix}(f^k)$, where $k \in \mathbb{N}$. By Lemma 4.6, we have that

$$\log \rho_s(A) = \lim_{n \to +\infty} \sup_{x \in M} \log \rho_s(A^n(x))^{\frac{1}{n}}$$

$$\geq \lim_{n \to +\infty} \sup_{x \in M} \log \rho_s(A^n(p))^{\frac{1}{n}}$$

$$\geq \lim_{n \to +\infty} \sup_{x \in M} \log \rho_s(A^k(p)^n)^{\frac{1}{n}}$$

$$= \frac{s - |s|}{k} \lim_{n \to +\infty} \sup_{x \in M} \log \rho_{|s|+1}(A^k(p)^n)^{\frac{1}{n}}$$

$$+ \frac{1 - s + |s|}{k} \lim_{n \to +\infty} \sup_{x \in M} \log \rho_{|s|}(A^k(p)^n)^{\frac{1}{n}}$$

$$= \frac{s - |s|}{k} \lim_{n \to +\infty} \sup_{x \in M} \log \varphi_{c[s]}^{|s|+1}(A^k(p)^n)^{\frac{1}{n}}$$

$$+ \frac{1 - s + |s|}{k} \lim_{n \to +\infty} \sup_{x \in M} \log \varphi_{c[s]}^{|s|}(A^k(p)^n)^{\frac{1}{n}}$$

$$= \frac{s - |s|}{k} r_{[s]+1}(A^k(p)) + \frac{1 - s + |s|}{k} r_{[s]}(A^k(p)).$$

Hence, (6) implies that $\log \overline{\rho}_s(A) \geq \log \hat{\rho}_s(A)$. Therefore,

$$\overline{\rho}_s(A) \geq \hat{\rho}_s(A),$$

which together with (7) yields the conclusion of the theorem.

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**References**

[BD] L. Backes and D. Dragičević, *Periodic approximation of Lyapunov exponents for semi-invertible cocycles*, To appear in Annales Academiae Scientiarum Fennicae.

[BW92] M. A. Berger and Y. Wang, *Bounded semigroups of matrices*, Linear Algebra Appl., 166 (1992), 21–27.

[AB16] A. Blumenthal, *A volume-based approach to the multiplicative ergodic theorem on Banach spaces*, Discrete Contin. Dyn. Syst. 36 (2016), 2377–2403.

[BM] A. Blumenthal and I. Morris, *Characterization of dominated splittings for operator cocycles acting on Banach spaces*, Preprint ArXiv 2015.

[CM79] J. Conway and B. Morrel, *Operators that are points of spectral continuity*, Integral Equations and Operator Theory, 2 (1979), 174–198.

[Dai12] X. Dai, *A Gel’fand-type spectral-radius formula and stability of linear constrained switching systems*, Linear Algebra Appl. 436 (2012), 1099–1113.

[Dai14] X. Dai, *Robust periodic stability implies uniform exponential stability of Markovian jump linear systems and random linear ordinary differential equations*, J. Franklin Inst. 351 (2014), 2910–2937.

[DHH17] X. Dai, T. Huang and Y. Huang, *Exponential stability of matrix-valued Markov chains via nonignorable periodic data*, Trans. Amer. Math. Soc. 369 (2017), 5271–5292.

[Deg08] G. Degla, *An overview of semi-continuity results on the spectral radius and positivity*, J. Math. Anal. Appl. 338 (2008), 101–110.

[DFGT18] D. Dragičević, G. Froyland, Cecilia Gonzalez-Tokman and S. Vaienti, *A spectral approach for quenched limit theorems for random expanding dynamical systems*, Comm. Math. Phys. 360 (2018), 1121–1187.
[FLQ13] G. Froyland, S. LLoyd, and A. Quas, A semi-invertible Oseledets Theorem with applications to transfer operator cocycles, Discrete and Continuous Dynamical Systems, 33 (2013), 3835–3860.

[GTQ15] C. González-Tokman and A. Quas, A concise proof of the multiplicative ergodic theorem on Banach spaces, Journal of Modern Dynamics, 9 (2015), 237–255.

[Gur95] L. Gurvits, Stability of discrete linear inclusion, Linear Algebra Appl. 231 (1995), 47–85.

[KH95] A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press, London-New York, 1995.

[K10] V. Kozyakin, An explicit Lipshitz constant for the joint spectral radius, Linear Algebra Appl. 433 (2010), 12–18.

[MOS01] B. E. Moision, A. Orlitsky and P. H. Siegel, On codes that avoid specified differences, IEEE Trans. Inform. Theory 47 (2001), 433–442.

[IM12] I. D. Morris, The generalized Berger-Wang formula and the spectral radius of linear cocycles, J. Funct. Anal. 262 (2012), 811–824.

[IM13] I. D. Morris, Mather sets for sequences of matrices and applications to the study of joint spectral radii, Proc. London Math. Soc. (3) 107 (2013), 121–150.

[ST00] V. S. Shulman and Y. V. Turovski˘ı, Joint spectral radius, operator semigroups, and a problem of W. Wojty´ nski, J. Funct. Anal. 177 (2000), 383–441.

[ST02] V. S. Shulman and Y. V. Turovski˘ı, Formulae for joint spectral radius of sets of operators, Studia Math. 149 (2002), 23–37.

[RR00] H. Radjavi and P. Rosenthal, Simultaneous Triangularization, Springer New York, New York, 2000.

[RS60] G.-C. Rota and G. Strang, A note on the joint spectral radius, Indag. Math. 22 (1960), 379–381.

[WY13] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasiperiodic cocycles, Duke Math. J. 162 (2013), 2363–2412.

[W02] F. Wirth, The generalized spectral radius and extremal norms, Linear Algebra Appl. 342 (2002), 17–40.

[ZCL18] R. Zou, Y. Cao and G. Liao, Continuity of spectral radius over hyperbolic systems, Discrete Contin. Dyn. Syst. 38 (2018), 3977–3991.