A DIRAC SEA AND THERMODYNAMIC EQUILIBRIUM
FOR THE QUANTIZED THREE WAVE INTERACTION

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Abstract. The classical version of the three wave interaction models the creation
and destruction of waves; the quantized version models the creation and destruction
of particles. The quantum three wave interaction is described and the Bethe Ansatz
for the eigenfunctions is given in closed form. The Bethe equations are derived in
a rigorous fashion and are shown to have a thermodynamic limit. The Dirac sea of
negative energy states is obtained as the infinite density limit. Finite particle/hole
excitations are determined and the asymptotic relation of energy and momentum is
obtained. The Yang-Yang functional for the relative free energy of finite density exci-
tations is constructed and is shown to be convex and bounded below. The equations
of thermal equilibrium are obtained.

1. INTRODUCTION

The creation and destruction of particles is a fundamental process in high energy
physics. Among other things, the presence of such processes implies that the en-
tries of the scattering matrix between states with different numbers of elementary
excitations (or physical particles) can be non-zero.

Integrable systems provide solvable paradigms for various physical phenomena: e.g.
for lattice statistical models, the algebraic decay of correlation functions at
a critical point may be observed. The creation and destruction of particle pairs
is modeled by the quantum three wave interaction (3WI), which is obtained by
quantizing a particular reduction of the $N = 3$ case of the AKNS hierarchy as
in Wadati and Ohkuma$^1$, or by taking the continuum limit of the $GL(3)$ lattice
magnet as in Kulish and Reshetikhin$^2$. The result is an interacting many body
system which does not conserve the number of particles: the interaction permits
two distinct types of particles to combine to form a third, and also the reverse
process.

The quantum 3WI has been considered in three forms. In one, all particles are
bosons, in the “Fermion I” model two fermions interact to form a boson, and in
the “Fermion II” model a boson and a fermion interact to form a fermion. In this
paper we consider the Fermion I model. We find the associated first quantization
problem, obtain the eigenfunctions of the Hamiltonian in closed form, and derive
the Bethe equations rigorously from the requirement that the wave functions be
periodic.
Because the model is first order in the derivatives, the energy is not bounded below. Nevertheless, as in Dirac’s model of the electron, a stable ground state may be constructed by filling all negative energy states. One can then consider all states with reference to the filled Dirac sea of negative energy states. This approach is used in Korepin\textsuperscript{3} and in Bergknoff and Thacker\textsuperscript{4} to calculate the excitation spectrum of the massive Thirring model from the Bethe ansatz states.

We prove the existence of a thermodynamic limit as the length of the interval tends to infinity with constant density. The Dirac sea is obtained in the infinite density limit. Finite particle/hole excitations from the Dirac sea correspond to the solutions of certain integral equations. Such excitations have positive relative energy, and we determine the asymptotic relation between energy and momentum. Following the treatment of a one-dimensional boson gas with delta-function interactions by Yang and Yang\textsuperscript{5}, we derive a functional for the relative free energy of finite (local) density perturbations from the Dirac sea. We prove that the functional is convex and bounded below and obtain the equation for thermal equilibrium.

Kulish and Reshetikhin\textsuperscript{2} used the quantum inverse scattering method (also called the algebraic Bethe Ansatz) to investigate $GL(N)$ invariant transfer matrices on a one dimensional lattice. They obtained quantized $N$ wave models as a formal continuum limit and found explicitly the Bethe equations for the bosonic 3WI. Wadati and Ohkuma\textsuperscript{1} also considered the bosonic case. Ohkuma and Wadati\textsuperscript{6} treated the Fermion I and Fermion II cases and gave an algorithm for constructing the Bethe wave functions.

Ohkuma\textsuperscript{7} investigated the Fermion I quantization of the three wave interaction, wrote down the Bethe equations, and calculated the free energy and excitations from a ground state. However a momentum cut-off is introduced in Ohkuka\textsuperscript{7}. The results depend on the cut-off and do not have a finite limit when the cut-off is taken to infinity. By constructing all physical states with reference to a Dirac sea we obtain that are finite and are independent of any cut-off. The methods provide a basis for calculating the free energy of other integrable models whose spectrum is unbounded below.

The quantized field of the 3WI may be rewritten as a first quantization for a many body problem. Such a reduction was carried out by Lieb and Liniger\textsuperscript{8} for the quantization of a Bose gas – essentially the quantization of the nonlinear Schrödinger equation. They obtained a Helmholtz equation in the interior of the fundamental domain, together with boundary conditions at the interfaces associated with the interactions; see also Korepin, Bogoliubov, and Izergin\textsuperscript{9}. A novel feature of the 3WI is that the system of equations links functions with different numbers of variables, corresponding to the fact that individual particle numbers are not conserved.

As in Lieb and Liniger\textsuperscript{8} we solve the differential equations in certain fundamental cones $\mathcal{C}^{m,n,p}$ and then extend the densities to all space, first by anti-symmetrizing across the boundaries of $\mathcal{C}^{m,n,p}$ and then by periodicity. The differential equations for the densities force discontinuities where fermions of different types interact. However like fermions do not interact, so to avoid discontinuities on the boundaries of the fundamental cones we impose a regularity requirement, (2.5). This requirement forces one to anti-symmetrize the basic solutions of the differential equations over momenta. This in turn is fundamental to obtaining the Bethe equations, as we show by example in §5.

In §§3,4 we sketch, in greater detail than in Wadati and Ohkuma\textsuperscript{1}, Wadati and
Ohkuma\textsuperscript{6}, a derivation of an algorithm for the eigenfunctions of the 3WI, and go on to obtain the explicit form of the eigenfunctions (cf. Theorem 5.1). This form generalizes the Bethe Ansatz (Bethe\textsuperscript{10}, Lieb and Liniger\textsuperscript{8}, Lieb\textsuperscript{11}, Yang and Yang\textsuperscript{12}) to systems in which the eigenvectors are superpositions of states having different numbers of particles. In §5 we derive the Bethe equations from the requirement of periodicity.

In §6 we reformulate the finite Bethe equations as a system of integral equations and prove existence of a certain thermodynamic limit at finite density. The Dirac sea is then obtained as an infinite density limit. In §7 integral equations for the particle/hole excited states are obtained, and energy/momentum asymptotics are found. Finite density perturbations of the Dirac sea are constructed in §8. In §9 we investigate the (relative) free energy functional. The formal Euler-Lagrange equations for its minimum are the same as those proposed by Ohkuma\textsuperscript{7} if one imposes a cut-off. Without a cut-off they have no strict solutions, but there is a natural interpretation which does yield meaningful equations for the thermal equilibrium state.

2. THE THREE WAVE INTERACTION

The Hamiltonian of the classical three-wave interaction is

\[ H = \int \left( \sum_{j=0}^{2} ic_j \frac{\partial \Psi_j^\dagger}{\partial x} \Psi_j + \gamma (\Psi_0^\dagger \Psi_1 \Psi_2 + \Psi_2^\dagger \Psi_1^\dagger \Psi_0) \right) dx \]

\[ = T + \gamma (V^\dagger + V), \]

where \( \Psi_j \) are complex valued functions and \( \dagger \) denotes complex conjugation. The constants \( c_j \) are wave speeds, and \( \gamma \) is a coupling constant. Zakharov and Manakov\textsuperscript{13} found a Lax pair for the classical system and Beals and Sattin ger\textsuperscript{14} gave a set of action-angle variables. However, integrability of the classical system will play no role in our analysis of the quantized model.

The system is formally quantized by taking the \( \Psi_j^\dagger, \Psi_j \) to be creation and annihilation operators on a Fock space, \( \dagger \) denoting Hermitian adjoints. We take \( \Psi_0(x,t) \) to be a boson field operator, and \( \Psi_1, \Psi_2 \) to be fermion fields. Thus the operators are required to satisfy the equal time commutation (\( [\ , \ ]_+ \)) or anti-commutation (\( [\ , \ ]_- \)) relations

\[ [\Psi_j(x,t), \Psi_j'(x',t)]_+ = 0, \quad [\Psi_j(x,t), \Psi_j^\dagger(x',t)]_+ = \delta_{jj'} \delta(x-x'), \]

\[ [\Psi_0(x,t), \Psi_0(x',t)]_- = 0, \quad [\Psi_0(x,t), \Psi_0^\dagger(x',t)]_- = \delta(x-x'), \]

\[ [\Psi_0(x,t), \Psi_j(x',t)]_- = 0, \quad [\Psi_0^\dagger(x,t), \Psi_j(x',t)]_- = 0, \quad j, j' = 1, 2. \] (2.2)

The Hermitian operator \( T \) represents the kinetic energy, with distinct wavespeeds \( c_j \) for each field. The interaction term \( V^\dagger \) formally annihilates the two particles with wavespeeds \( c_1 \) and \( c_2 \) (particles of type 1 and 2) and creates one with wavespeed \( c_0 \), while \( V \) destroys a type 0 particle and creates types 1 and 2.

We assume throughout that the speeds of the fermions bracket the boson speed:

\[ (c_1 - c_0)(c_2 - c_0) < 0. \]

\[ (2.3) \]
where \( \alpha \) is an arbitrary constant. Without loss of generality we assume throughout this paper that \( \alpha > 0 \).

The Hamiltonian is defined on a Hilbert space \( \mathcal{H} \) consisting of the tensor product of a boson and two fermion Fock spaces. There exists a non-zero vector \( |0\rangle \) in the Fock space, called the pseudovacuum, such that

\[
\Psi_j(x,t)|0\rangle = 0, \quad j = 0, 1, 2.
\]

Define particle number operators

\[
N_j = \int dx \Psi_j^\dagger \Psi_j, \quad j = 0, 1, 2.
\]

The coordinates of particles of types 0, 1, and 2 are denoted by \( x_j, y_j, z_j \) respectively. Let \( \Psi_0^\dagger(x) = \Psi_0^\dagger(x_1, \ldots, x_m) = \Psi_0^\dagger(x_1) \cdots \Psi_0^\dagger(x_m) \) and define \( \Psi_1^\dagger(y) \) and \( \Psi_2^\dagger(z) \) similarly. Then set

\[
\Psi^\dagger(x, y, z) = \Psi_0^\dagger(x) \Psi_1^\dagger(y) \Psi_2^\dagger(z), \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n, z \in \mathbb{R}^p.
\]

Fundamental states in \( \mathcal{H} \) are given formally by

\[
|\psi_{m,n,p}\rangle = \frac{1}{m!n!p!} \int_{\mathbb{R}^m,n,p} f_{mnp}(x, y, z) \Psi^\dagger(x, y, z)|0\rangle dx dy dz, \quad (2.4)
\]

where \( f_{mnp} \) belongs to \( L_2(\mathbb{R}^{m,n,p}) \) and \( dx = dx_1 dx_2 \ldots dx_m, etc. \). Since \( \Psi^\dagger(x, y, z) \) is symmetric in the \( x_j \) and anti-symmetric in the \( y_k \) and \( z_l \), the density function \( f_{mnp} \) may be assumed to possess these same symmetries.

If a density \( f_{mnp} \) is associated with an eigenstate, then, as shown in §4, it must be discontinuous across the hyperplanes \( y_k = z_l \), which we call the interaction manifolds. It is natural to assume that these are the only discontinuities. We therefore assume continuity across the hyperplanes \( y_k = y_{k+1} \), provided \( y_k \neq z_l \) for all \( z_l \), and similarly with respect to the \( z \) coordinates. Continuity plus anti-symmetry across these hyperplanes implies:

**Regularity condition**: \( f_{mnp} \) vanishes as \( y_k \rightarrow y_{k\pm 1} \) if \( y_{k\pm 1} \neq z_l \) for all \( l \) and similarly as \( z_l \rightarrow z_{l\pm 1} \neq y_k \), all \( k \). \( (2.5) \)

By symmetry, we may replace the integration in (2.4) by integration over the cone \( C^{m,n,p} \) given by

\[
x_1 < x_2 < \cdots < x_m; \quad y_1 < y_2 < \cdots < y_n; \quad z_1 < z_2 < \cdots < z_p,
\]

and write

\[
|\psi_{m,n,p}\rangle = \int_{C^{m,n,p}} f_{mnp}(x, y, z) \Psi^\dagger(x, y, z)|0\rangle dx dy dz. \quad (2.6)
\]
From now on, we write the state vectors in the form (2.6) and assume that the densities \( f_{mnp} \) are supported on the cone \( C^{m,n,p} \) and satisfy the regularity condition (2.5), which implies that the antisymmetric extensions of the \( f_{mnp} \) are continuous across the collision manifolds \( y_j = y_k \) and \( z_j = z_k \) of like fermions.

The classical version of the Hamiltonian (2.1) is integrable and possesses an infinite number of conservation laws: Zakharov and Manakov\(^{13} \). Three important conserved quantities for each version are

\[
N = \int 2\Psi_0^\dagger\Psi_0 + \Psi_1^\dagger\Psi_1 + \Psi_2^\dagger\Psi_2 \, dx
\]

\[
D = \int \Psi_1^\dagger\Psi_1 - \Psi_2^\dagger\Psi_2 \, dx
\]

\[
P = \int \sum_{j=0}^2 i\Psi_j^\dagger \frac{\partial \Psi_j}{\partial x} \, dx
\]

In the quantized version operators \( N \) and \( D \) give conserved particle numbers, and \( P \) is the momentum operator. Indeed it is not difficult to show from (2.2) that \( N, D, P \) commute with \( H \).

3. REDUCTION TO FIRST QUANTIZATION

The quantum field problem can be rewritten as a first quantization problem for a many body problem.

Denote the subspace of states \( |\psi_{m,n,p}\rangle \) by \( \mathcal{F}_{mnp} \). These are eigenstates of \( N \) and \( D \) with eigenvalues \( N = 2m + n + p \) and \( D = n - p \) respectively, but are not eigenstates of the Hamiltonian \( H \), since \( V^\dagger \) and \( V \) create and destroy particles. However, since \( N \) and \( D \) commute with \( H \), the quantum numbers \( N \) and \( D \) label a subspace that is invariant under \( H, N, \) and \( D \). We denote this space by \( \mathcal{F}_{N,D} \).

We now fix \( N \) and \( D \). Within \( \mathcal{F}_{N,D} \), densities can be labeled by \( m \) alone and we denote the corresponding subspace by \( \mathcal{F}_m \). Note that both of \( N \pm D \) must be non-negative even integers. When there are no bosons present, \( (m = 0) \), there are \( (N \pm D)/2 \) fermions of types 1 and 2 respectively. The maximum number of bosons occurs when the maximal number of fermions have combined, yielding \( M = \min(N \pm D)/2 = \frac{1}{2}(N - |D|) \).

The operators \( T, V^\dagger, \) and \( V \) act on the subspaces \( \mathcal{F}_m \) as follows:

\[
T : \mathcal{F}_m \to \mathcal{F}_m, \quad V^\dagger : \mathcal{F}_m \to \mathcal{F}_{m+1}, \quad V : \mathcal{F}_m \to \mathcal{F}_{m-1}.
\]

An eigenstate of the Hamiltonian in \( \mathcal{F}_{N,D} \) is a sum of vectors \( |\psi_m\rangle \in \mathcal{F}_m \), and the eigenvalue equation becomes the coupled system

\[
(E - T) |\psi_0\rangle = \gamma V |\psi_1\rangle
\]

\[
\vdots
\]

\[
(E - T) |\psi_m\rangle = \gamma V^\dagger |\psi_{m-1}\rangle + \gamma V |\psi_{m+1}\rangle
\]

\[
\vdots
\]

\[
(E - T) |\psi_{M}\rangle = \gamma V^\dagger |\psi_{M-1}\rangle
\]

\[
(E - T) |\psi_{M+1}\rangle = \gamma V |\psi_{M}\rangle
\]

\[
(E - T) |\psi_{M+2}\rangle = \gamma V^\dagger |\psi_{M+1}\rangle
\]

\[
\vdots
\]

\[
(E - T) |\psi_{|D|}\rangle = \gamma V^\dagger |\psi_{|D|-1}\rangle
\]

\[
(E - T) |\psi_{|D|+1}\rangle = \gamma V |\psi_{|D|}\rangle
\]
since $V|\psi_0\rangle = V^\dagger|\psi_M\rangle = 0$.

Let $f_m$ be the density corresponding to the vector $|\psi_m\rangle$. Equations (3.1) may be converted to a system of partial differential equations by calculating the action of $T$, $V^\dagger$ and $V$ on the densities $f_m$. The calculation of the action of $T$ is straightforward, while the calculation of $V$ and $V^\dagger$ is somewhat more involved; details are given in Beals, Sattinger, and Williams\textsuperscript{15}. The action of $T$ is:

\[
T|\psi_m\rangle = T \int_{C^{m,n,p}} f_m(x, y, z) \Psi^\dagger(x, y, z)|0\rangle \, dx \, dy \, dz = \int_{C^{m,n,p}} (X_m f_m(x, y, z)) \Psi^\dagger(x, y, z)|0\rangle \, dx \, dy \, dz
\]

where

\[
X_m f = ic_0 \sum_{j=0}^m \frac{\partial f}{\partial x_j} + ic_1 \sum_{k=0}^n \frac{\partial f}{\partial y_k} + ic_2 \sum_{l=0}^p \frac{\partial f}{\partial z_l}.
\] (3.4)

We use $V^\dagger$ to denote the action on densities as well as the action on vectors. Then

\[
V^\dagger f_m = \sum_{j,k,l=1}^{m+1,n,p} (-1)^{n(m) + k + l} f_m(\hat{x}_j; \langle y, x, j \rangle; \langle z, x, l \rangle),
\] (3.5)

where $(x, y, z) \in C^{m+1,n-1,p-1}$ and

\[
\hat{x}_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) \in C^m,
\]

\[
\langle y, x, j \rangle = (y_1, \ldots, y_{k-1}, x_j, y_k, \ldots, y_{n-1}) \in C^n,
\] etc.

Remarks. 1. Since $f_m$ is supported on $C^{m,n,p}$, $f_m(\hat{x}_j; \langle y, x, j \rangle; \langle z, x, l \rangle) = 0$ unless $x_j \in (y_{k-1}, y_k) \cap (z_{l-1}, z_l)$. Hence

\[
(V^\dagger f_m)(x; y; z) = \sum_{j=1}^{m+1} (-1)^{n(m) + k_j + l_j} f_m(\hat{x}_j; \langle y, x, j \rangle; \langle z, x, l_j \rangle),
\] (3.6)

where $k_j$ and $l_j$ are uniquely determined by the constraint

\[
x_j \in (y_{k_j-1}, y_{k_j}) \cap (z_{l_j-1}, z_{l_j}).
\] (3.7)

2. In (3.6) we interpret the value of $f_m$ on the interaction manifold $x_j = y_{k_j} = z_{l_j}$ as the average of its values as $y_{k_j} \to z_{l_j}^-$. 

3. The action of $V^\dagger$ can be thought of as gluing a type 1 fermion at $y_k$ and a type 2 fermion at $z_l$ to form a boson at $x_j$. The coordinates are relabeled to take into account the shift in numbers of variables, summed over all possible pairs. Similarly, the action of $V$ can be thought of as the decomposition of a boson into a pair of fermions, summed over the bosons. In fact the operation of $V$ on densities is given by

\[
(V f_m)(x, y, z) = \sum_{j,k,l=1}^{n+1,p+1} (-1)^{n(m) + k_j + l_j} f_m(\langle x, y_k, j_k \rangle; \langle \hat{y}_k, z_l \rangle \delta(y_k - z_l)).
\] (3.9)
where \( j_k \) is uniquely determined by the constraint \( y_k \in (x_{j_{k-1}}, x_{j_k}) \).

We illustrate in some low order cases. For \( 2m + n + p = N = 2, \) \( f_0 = f_0(y, z), \) \( f_1 = f_1(x) \):

\[
(V^\dagger f_0)(x) = -f_0(x, x), \quad Vf_0 = 0
\]
\[
V^\dagger f_1 = 0, \quad (Vf_1)(y, z) = f_1(y)\delta(y - z)
\]

For \( 2m + n + p = N = 4, \) \( f_0 = f_0(y_1, y_2, z_1, z_2), \) \( f_1 = f_1(x, y, z), \) \( f_2 = f_2(x_1, x_2) \):

\[
(V^\dagger f_0)(x, y, z) = f_0(x, y, x, z) - f_0(x, y, z, x) + f_0(y, x, z, x),
\]
\[
(V^\dagger f_1)(x_1, x_2) = -f_1(x_1, x_2, x_2) - f_1(x_2, x_1, x_1),
\]
\[
V^\dagger f_2 = 0;
\]
\[
Vf_0 = 0
\]
\[
(Vf_1)(y_1, y_2, z_1, z_2) = f_1(y_1, y_2, z_1)\delta(y_1 - z_2) + f_1(y_1, z_1, z_2)\delta(y_2 - z_1),
\]
\[
\quad - f_1(y_2, y_1, z_1)\delta(y_2 - z_2) + f_1(y_2, y_1, z_2)\delta(y_1 - z_2),
\]
\[
(Vf_2)(x, y, z) = [f_2(y, x) + f_2(x, y)]\delta(y - z).
\]

4. THE BETHE ANSATZ

The differential equations for the densities, supplemented with the regularity condition (2.5) at the boundaries of the cones \( C^{m,n,p} \), describe the spectral problem in first quantization.

From the results of \( \S 3 \) the differential equations for the densities are

\[
X_0 f_0 + \gamma V f_1 = E f_0,
\]
\[
\vdots
\]
\[
X_m f_m + \gamma V f_{m+1} + \gamma V^\dagger f_{m-1} = E f_m
\]
\[
\vdots
\]
\[
X_M f_M + \gamma V^\dagger f_{M-1} = E f_M
\]

Wadati and Ohkuma\(^1\) gave an algorithm for constructing the general eigenstates. In the next two sections we find these eigenstates in closed form, in two steps. In this section we construct “local solutions” of equations (4.1); on each of the cones \( C^{mnp} \) is a multiple of an exponential solution of the associated differential equation.

Because the operator \( V \) introduces delta function singularities on the interaction manifolds, solutions must have discontinuities across these manifolds. However the local solutions violate (2.5) and thus they have additional discontinuities if extended by anti-symmetry. We denote the densities of the local solutions by \( h_m \) to distinguish them from the regular densities, which will satisfy (2.5). We obtain these regular densities in \( \S 5 \) by taking linear combinations of the \( h_m \): anti-symmetrizing over the momenta.

Low Order Cases
Two low order cases illustrate the main features of these equations. For \( N = 2, D = 0 \), equations (4.1) are

\[
\begin{align*}
&ic_1 \frac{\partial h_0}{\partial y} + ic_2 \frac{\partial h_0}{\partial z} + \gamma h_1(y)\delta(y - z) = Eh_0(y, z) \quad (4.2a) \\
&ic_0 \frac{\partial h_1}{\partial x} - \gamma h_0(x, x) = Eh_1(x) \quad (4.2b)
\end{align*}
\]

where \( h_0 = h_0(y, z) \), \( h_1 = h_1(x) \). In the first equation the interaction term vanishes off the interaction manifold \( y = z \) so we take the solutions to be multiples of an exponential in each of the open sectors \( y < z \) and \( y > z \) of \( \mathbb{C}^{0,1,1} \):

\[
h_0(y, z) = \begin{cases} 
\phi_1^0 \exp i(\eta y - \zeta z), & y < z, \\
\phi_2^0 \exp i(\eta y - \zeta z), & y > z,
\end{cases}
\]

where \( E = -(c_1 \eta - c_2 \zeta) \), and \( \phi_1^0 \) and \( \phi_2^0 \) are constants to be determined.

In (4.2b) we take \( h_0(x, x) \) on the interaction manifold to be the average of the values of \( h_0(y, z) \) from the left and right. Hence the second equation is

\[
\left(ic_0 \frac{\partial}{\partial x} - E\right) h_1(x) = \gamma \frac{\phi_1^0 + \phi_2^0}{2} e^{i(\eta - \zeta)x}.
\]

We seek a solution of the form \( h_1(x) = Ae^{i(\eta - \zeta)x} \), so that

\[
h_1(x) = g(\eta, \zeta) \frac{\phi_1^0 + \phi_2^0}{2} \exp i(\eta - \zeta)x, \quad g(\eta, \zeta) = \frac{\gamma}{(c_1 - c_0)\eta - (c_2 - c_0)\zeta}.
\]

By (2.3),

\[
g(\eta, \zeta) = \frac{\gamma}{\alpha(\eta + \zeta)} := g(\eta + \zeta).
\]

We use this form of \( g \) from now on.

Finally, we determine the ratio \( \phi_2^0 / \phi_1^0 \) by integrating (4.2a) across the interaction manifold \( y = z \). We change variables, letting \( d = y - z, \ s = y + z \), and obtain

\[
\begin{align*}
&i(c_1 - c_2) \frac{\partial h_0}{\partial d} + i(c_1 + c_2) \frac{\partial h_0}{\partial s} + \gamma h_1(d) = Eh_0, \\
&h_0 = h_0\left(\frac{s + d}{2}, \frac{s - d}{2}\right), \quad h_1 = h_1\left(\frac{s + d}{2}\right)
\end{align*}
\]

Integrating across the jump at \( d = 0 \) \( (y = z = x) \) we get

\[
i(c_1 - c_2)(\phi_2^0 - \phi_1^0) \exp ix(\eta - \zeta) + \gamma g(\eta + \zeta) \frac{\phi_1^0 + \phi_2^0}{2} \exp ix(\eta - \zeta) = 0;
\]

hence

\[
\frac{\phi_2^0}{\phi_1^0} = \frac{2i(c_1 - c_2) - \gamma g}{2i(c_1 - c_2) + \gamma g}.
\]

Using our assumption (2.3) we may write this in the simplified form

\[
\phi_2^0 = \theta(\eta + \zeta)\phi_1^0, \quad \theta(\lambda) = \frac{\lambda + i\omega}{\lambda - i\omega},
\]

where \( \lambda = c_1 \eta + c_2 \zeta \).
where
\[ \omega = \frac{\gamma^2}{4\alpha^2}. \]
Setting \( \phi_1^0 = 1 \), we have
\[ h_0(y, z) = \begin{cases} 
\exp i(\eta y - \zeta z), & y < z; \\
\theta(\eta - \zeta) \exp i(\eta y - \zeta z), & y > z; 
\end{cases} \]
and
\[ h_1(x) = \frac{1}{2} g(\eta + \zeta) (1 + \theta(\eta + \zeta)) \exp ix(\eta - \zeta). \]

For \( N = 3, D = 1 \) equations (4.1) are
\[ \begin{align*}
   &i c_1 \frac{\partial h_0}{\partial y_1} + i c_1 \frac{\partial h_0}{\partial y_2} + i c_2 \frac{\partial h_0}{\partial z} - E h_0(y_1, y_2; z) = \gamma(h_1(y_1; y_2)\delta(y_1 - z) - h_1(y_2; y_1)\delta(y_2 - z)) \\
   &i c_0 \frac{\partial h_1}{\partial x} + i c_1 \frac{\partial h_1}{\partial y} - Eh_1(x; y) = \gamma(h_0(y, x; x) - h_0(x, y; x)),
\end{align*} \tag{4.3a} \tag{4.3b} \]
with \( h_0 = h_0(y_1, y_2; z) \) and \( h_1 = h_1(x; y) \). Arguing as above, we obtain
\[ h_0(y; z) = \begin{cases} 
\exp i(\eta_1 y_1 + \eta_2 y_2 - \zeta z), & y_1 < y_2 < z; \\
\theta(\eta_2 + \zeta) \exp i(\eta_1 y_1 + \eta_2 y_2 - \zeta z), & y_1 < z < y_2; \\
\theta(\eta_1 + \zeta)\theta(\eta_2 + \zeta) \exp i(\eta_1 y_1 + \eta_2 y_2 - \zeta z), & z < y_1 < y_2;
\end{cases} \]
and
\[ h_1(x; y) = \frac{1}{2} \begin{cases} 
(1 + \theta(\eta_2 + \zeta)) g(\eta_2 + \zeta) \exp i((\eta_2 - \zeta)x + \eta_1 y) & y < x; \\
-\theta(\eta_2 + \zeta)(1 + \theta(\eta_1 + \zeta)) g(\eta_1 + \zeta) \exp i((\eta_1 - \zeta)x + \eta_2 y) & x < y.
\end{cases} \]

The General Case

The two preceding cases show that when \( y_k \) and \( z_l \) are transposed, (with \( y_k \) moving to the right of \( z_l \)), \( h_0 \) is multiplied by \( \theta(\eta_k + \zeta_l) \), where \( \eta_k, \zeta_l \) are the momenta of the corresponding fermions. Since the differential equations are local and involve only pairwise interactions, this situation extends to the general case.

We begin by introducing some notation and conventions. For fixed \( N \) and \( D \), \( C^{m,n,p} \) is uniquely determined by \( m \), so from now on we denote it by \( C^m \). The domain \( C^m \) is a union of
\[ \frac{(m + n + p)!}{m!n!p!} \]
open sectors determined by the interspacing of the particles of type 0,1,2. We label them by \( m + n + p \)-tuples \( \sigma = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n+p}) \) where \( \epsilon_j = 0, -1, \) or +1 if the particle in the \( j^{th} \) position along the real line is of type 0,1, or 2 respectively. For example
\[ \sigma = (1, 0, -1, 1) \leftrightarrow \{ z_1 < x_1 < y_1 < z_2 \}. \]
The boundary of \( C^m \) consists of the interaction manifolds \( y_j = z_k \) together with the sets \( y_j = y_{j+1} \) and \( z_k = z_{k+1} \), at which like fermions coalesce.
A transposition consists of interchanging a -1 with a +1 to its immediate right, e.g. \((1, 0, -1, 1) \mapsto (1, 0, 1, -1)\). Each transposition corresponds to interchanging an adjacent pair of coordinates \(y_k, z_l\); thus to every interaction manifold there is a corresponding transposition. The sectors of \(\mathcal{C}^m\) are partially ordered by the number of transpositions; thus, for example \((1, 0, -1, 1) < (1, 0, 1, -1)\). By definition, no transpositions occur between -1’s and 1’s separated by one or more zeroes.

Since \(Vh_1\) is supported on the interaction manifold, \(h_0 = h_0(y; z)\) satisfies

\[
X_0h_0 = ic_1 \sum_{j=1}^{n} \frac{\partial h_0}{\partial y_j} + ic_2 \sum_{k=1}^{p} \frac{\partial h_0}{\partial z_k} = Eh_0
\]

in each open sector \(\sigma \subset \mathcal{C}^0\).

Let

\[
\sigma_0 = \{y_1 < \cdots < y_n < z_1 < \cdots < z_p\} \sim (-1, \ldots, -1, 1, \ldots, 1).
\]

We take

\[
h_0 = \exp i \left( \sum_{j=1}^{n} \eta_j y_j - \sum_{k=1}^{p} \zeta_k z_k \right), \quad (y; z) \in \sigma_0
\]

with

\[
E = -c_1 \sum_{j=1}^{n} \eta_j + c_2 \sum_{k=1}^{p} \zeta_k.
\]  

(4.5)

The solution in a general sector \(\sigma \in \mathcal{C}^0\) is

\[
h_0 = \phi_{\sigma}^0 \exp i \left( \sum_{j=1}^{n} \eta_j y_j - \sum_{k=1}^{p} \zeta_k z_k \right), \quad y; z \in \sigma,
\]

(4.6)

where the constant \(\phi_{\sigma}^0\) is uniquely determined by \(\sigma\), and is given in (4.8) below.

**Lemma 4.1.** Let \(\sigma < \sigma'\) differ by a single transposition of fermions of types 1 and 2 at \(y_j, z_k\) with momenta \(\eta_j\) and \(-\zeta_k\) respectively. Then

\[
\phi_{\sigma'}^0 = \theta(\eta_j + \zeta_k)\phi_{\sigma}^0,
\]

(4.7)

**Remark.** For \(\sigma' < \sigma\) the phase factor is

\[
[\theta(\eta_j + \zeta_k)]^{-1} = \theta(-\eta_j - \zeta_k).
\]

Thus within a fixed particle number sector these conditions on the wave function are of the usual Bethe Ansatz type; the ratio of phases that differ by a permutation of neighboring particles is a function of the momenta.

**Proof.** All the interactions are pairwise and local, so the same computations for the case \(N = 2, D = 0\) extend to the general case. The ratio of the constants \(\phi_{\sigma'}^0, \phi_{\sigma}^0\) is determined entirely from the first and second equations in the hierarchy (4.1). \(\Box\)
Each $\sigma \in C^0$ is obtained from $\sigma_0 = (-1, \ldots, -1, 1, \ldots, 1)$ by a unique set of transpositions $(y_j, z_k)$, which we denote by $T_\sigma$. Thus
\[
\phi^0_\sigma = \prod_{(j,k) \in T_\sigma} \theta(\eta_j + \zeta_k). \tag{4.8}
\]

Now consider the higher order components of the density: $h_m$, $m \geq 1$. The sectors of $C^m$ are characterized by $(m + n + p)$-tuples $\sigma$ with $m$ zeroes. We assign momenta to $\sigma$ as follows. Regard each 0 in $\sigma$ as a pair $(-1, 1)$. Starting from the left, assign the -1’s momenta $\eta_1, \eta_2, \ldots$; the +1’s momenta $-\zeta_1, -\zeta_2, \ldots$; and assign the 0’s the sums of momenta $\eta_j - \zeta_k$, with the $\eta$’s and $\zeta$’s taken in their normal order. For example,
\[
(-1, 0, 1, 0) \leftrightarrow (\eta_1, \eta_2 - \zeta_1, -\zeta_2, \eta_3 - \zeta_3);
\]
\[
(-1, 1, 1, 0, 1, 0, -1) \leftrightarrow (\eta_1, -\zeta_1, -\zeta_2, \eta_2 - \zeta_3, -\zeta_4, \eta_3 - \zeta_5, \eta_4).
\]
For fixed locations of the zeroes in $\sigma$ the basic exponential solution is defined by
\[
e^m_\sigma = \exp i \left( \sum_{r=1}^{m} x_r(\eta_{i_r} - \zeta_{j_r}) + \sum_{s=1}^{n} y_s\eta_{a_s} - \sum_{t=1}^{p} z_t\zeta_{b_t} \right) \tag{4.9}
\]
where
\[
i_r (j_r) = \# \; x \;'s and y \;'s (z \;'s) to the left of $x_r$, inclusive of $x_r$; \tag{4.10}
\]
\[
a_s (b_t) = \# \; x \;'s and y \;'s (z \;'s) to the left of $y_s$, (z_t), inclusive of $y_s$, (z_t).
\]

**Remark.** From (3.7), $i_r = r + k_r - 1, \; j_r = r + l_r - 1$.

For example
\[
e^m_\sigma (x; y; z) = \exp i x_1 (x_1(\eta_2 - \zeta_1) + x_2(\eta_3 - \zeta_3) + y_1\eta_1 - z_1\zeta_2), \]
\[
\sigma = (-1, 0, 1, 0);
\]
\[
e^m_\sigma (x; y; z) = \exp i (x_1(\eta_2 - \zeta_3) + x_2(\eta_3 - \zeta_5) + y_1\eta_1 + y_2\eta_4 - z_1\zeta_1 - z_2\zeta_2 - z_3\zeta_4)
\]
\[
\sigma = (-1, 1, 1, 0, 1, 0, -1)
\]

We now describe the general case. For any $\sigma \in C^m$ define $g_\sigma$ by
\[
g_\sigma = \prod_{r=1}^{m} g_{i_r,j_r}, \quad g_{i_r,j_r} = g(\eta_{i_r} + \zeta_{j_r}), \tag{4.12}
\]
where $i_r, j_r$ are given in (4.10). For example, for $(-1, 0, 1, 0), i_1 = 2, j_1 = 1, i_2 = 3, j_2 = 2,$ and
\[
g_\sigma = g_{21}g_{32}.
\]
The components of $C^m$ are determined by the location of the zeroes and the numbers and orders of +1’s and -1’s between successive zeroes. The factor $g_\sigma$ is constant on each connected component of $C^m$. For $\sigma \in C^m$, we define
\[
\phi^m_\sigma = \phi^0_\Sigma \prod_{r=1}^{m} (-1)^{n(m) + k_r + l_r} \frac{1 + \theta(\eta_{i_r} + \zeta_{j_r})}{2}, \tag{4.13}
\]
where $\Sigma \in C^0$ is obtained from $\sigma$ by replacing each 0 in $\sigma$ by $(-1, 1)$, and $\phi^0_\Sigma$ is given by (4.8).
Theorem 4.2. The basic solutions of equations (4.1) are given by
\[ h_m(x, y, z) = \psi_m \psi_\sigma(x, y, z), \] (4.14)
where
\[ \psi_m = \phi_m \sigma = \phi_\Sigma \left( \frac{\gamma}{\alpha} \right)^m \prod_{r=1}^m \frac{(-1)^{n(m)+k_r+l_r}}{\eta_{\sigma_r} - i\omega} \]
and the \(\phi_m\) are given by (4.8), (4.13); the \(\sigma\) by (4.12); and the \(e_m\) by (4.9).

The proof is given in Beals, Sattinger, and Williams. In addition to Lemma 4.1, it relies on the identity
\[ (-1)^{n(m)+k_r+l_r} \phi^m_{\sigma_r} + \phi^m_{\sigma'_r} = \phi^{m+1}_\sigma, \quad 1 \leq r \leq m + 1 \]
where \(\sigma_r\) and \(\sigma'_r\) are obtained from \(\sigma\) by replacing the \(r\)-th zero by \((-1,1)\) and by \((1,-1)\) respectively. This condition insures compatibility between the phases in different particle number sectors; it has no analogue in the usual Bethe Ansatz.

5. THE BETHE EQUATIONS

In §4 we constructed basic solutions \(h_m\) of equations (4.1). As noted in §2 we want to impose the regularity condition (2.5) that eigendensities have no unnecessary discontinuities, i.e. that their anti-symmetric extensions be continuous across the collision manifold of like fermions. To obtain this property, we antisymmetrize the basic solutions over the momenta.

Permuting the momenta of the basic solution of (4.4), (4.5) gives another solution. The general solution of (4.4, 4.5) is a linear combination
\[ e_0(J, K) = \exp i \left( \sum_{j=1}^n y_j \eta_{\sigma_j} - \sum_{k=1}^p z_k \zeta_{\sigma_k} \right), \]
where \(J, K\) are permutations of \(1, \ldots, n\) and \(1, \ldots, p\) respectively. In particular,
\[ f_0(y; z) = \sum_{J,K} \text{sgn}(J) \text{sgn}(K) e_0(J, K), \]
satisfies \(X_0 f_0 = E f_0\) in
\[ \sigma_0 = \{y_1 < \cdots < y_n < z_1 < \cdots < z_p\}, \]
and satisfies the regularity condition (2.5).

The density in the other sectors is given by
\[ f_0(y; z) = \sum_{J,K} \text{sgn}(J) \text{sgn}(K) \phi^0_{\sigma}(J, K) e_0(J, K), \] (5.1)
where \((y; z) \in \sigma\), and
\[ \phi^0_{\sigma}(J, K) = \prod_{i=1}^n \theta(\eta_{\sigma_i} + \zeta_{K_i}). \]
The corresponding densities for higher \( m \) are consequently

\[
f_m(x; y; z) = \sum_{J,K} \text{sgn}(J) \text{sgn}(K) \phi^m_\sigma(J, K) g_\sigma(J, K) e^m_\sigma(J, K),
\]

(5.2)

where \( e^m_\sigma(J, K) \), \( g_\sigma(J, K) \) and \( \phi^m_\sigma(J, K) \) are obtained from (4.9), (4.12), and (4.13) by making the substitutions \( j, k \mapsto J, K \). For example, to obtain densities which vanish when \( y_1 = y_2 \neq z \) in the case \( N = 3, D = 1 \), we antisymmetrize the base function over the momenta: setting

\[
e_0(j, k) = \exp i(\eta_j y_1 + \eta_k y_2 - \zeta z),
\]

we take

\[
f_0(y_1, y_2, z) = \begin{cases} 
e_0(1, 2) - \ne_0(2, 1), & y_1 < y_2 < z; \\
\theta(\eta_2 + \zeta)e_0(1, 2) - \theta(\eta_1 + \zeta)e_0(2, 1), & y_1 < z < y_2; \\
\theta(\eta_1 + \zeta)\theta(\eta_2 + \zeta)[e_0(1, 2) - e_0(2, 1)], & z < y_1 < y_2. 
\end{cases}
\]

Now \( f_0(y_1, y_2, z) \) vanishes when \( y_1 = y_2 \neq z \); but we can not assert anything for the case \( y_1 < z < y_2 \), since \( y_1 \) and \( y_2 \) cannot coalesce without equaling \( z \). The corresponding density \( f_1 \) is also obtained as a superposition: setting

\[
w(j, k) = g(\eta_k + \zeta)(1 + \theta(\eta_k + \zeta)) \exp i(\eta_k - \zeta)x + \eta_j y
\]
we find

\[
f_1(x, y) = \begin{cases} w(1, 2) - w(2, 1), & y < x; \\
-\theta(\eta_2 + \zeta)w(2, 1) + \theta(\eta_1 + \zeta)w(1, 2), & x < y. 
\end{cases}
\]

**Theorem 5.1.** The antisymmetrized wave functions (5.1) and (5.2) satisfy the regularity condition (2.5) and thus determine a basis of eigenstates for the Hamiltonian (2.1).

**Proof.** Consider a pair \((y_k, y_{k+1})\) (the proof is the same for contiguous \( z \)'s) and assume there are no intervening \( x \)'s or \( z \)'s. Group the terms in the sum (5.1) into pairs corresponding to \( J = (\ldots, j, j+1 \ldots) \) and \( J' = (\ldots, j+1, j \ldots) \). At \( y_j = y_{j+1} \) we have \( e_0(J, K) = e_0(J', K) \), and the coefficient of this common value is

\[
\phi^0_\sigma(J, K) - \phi^0_\sigma(J', K).
\]

For any given pair \((j, k)\) either \( y_j < y_{j+1} < z_k \) or \( z_k < y_j < y_{j+1} \). In the first case, neither \((j, k)\) nor \((j+1, k)\) belong to \( T_\sigma \), so neither of the factors \( \theta(\eta_j - \zeta K_k) \) belong to the products \( \phi^0_\sigma(J, K) \), \( \phi^0_\sigma(J', K) \). In the second case, both \((j, k)\) and \((j+1, k)\) belong to \( T_\sigma \), so both factors belong to both the products, and \( \phi^0_\sigma(J, K) = \phi^0_\sigma(J', K) \). Therefore the coefficient of \( e_0(J, K) = e_0(J', K) \) vanishes in either case, and the sum is zero. This establishes the result for \( f_0 \).

We proceed in the same way to verify that the higher order densities \( f_m \), \( m \geq 1 \) vanish when \( y_j \) and \( y_{j+1} \) coincide. In view of (4.16), we must show that

\[
\phi^m_\sigma(J, K) e_0(J, K) = \phi^m_\sigma(J', K) e_0(J', K).
\]
From (4.12) the constants $g_\sigma(J,K)$ depend only on the momenta associated with the bosons (the zeroes of $\sigma$), and so are unchanged under any permutation of the fermion momenta. The same applies to the product in (4.13): it depends only on the boson, not the fermion momenta. Therefore the issue for the higher order densities reduces to the proposition $\phi^0_\Sigma(J,K) = \phi^0_\Sigma(J',K)$, which was established in the first step of this proof. □

Given the $f_m$ defined on the cells

$$\{ 0 \leq x_1 < \cdots < x_m \leq L, \ 0 \leq y_1 < \cdots < y_n \leq L, \ 0 \leq z_1 < \cdots < z_p \leq L \},$$

we extend them to the hypercube $0 \leq x_i, y_j, z_k \leq L$ by

$$h_m(x; y; z) = \text{sgn}(J)\text{sgn}(K)f_m(Mx,Jy,Kz) \quad (5.3)$$

where $Mx, Jy, Kz$ are the unique permutations of the variables $x, y, z$ for which

$$x_{M_1} < \cdots < x_{M_m}, \quad y_{J_1} < \cdots < y_{J_n}, \quad z_{K_1} < \cdots < z_{K_p}.$$  

We now impose periodic boundary conditions on the wave functions $h_m$. These periodicity conditions lead to constraints on the momenta, known as the Bethe equations.

For example, in the simplest case $N = 2, D = 0$ (cf. §4) we obtained

$$f_0(y,z) = \begin{cases} \exp i(\eta y - \zeta z), & y < z; \\ \theta(\eta + \zeta) \exp i(\eta y - \zeta z), & y > z. \end{cases}$$

The periodicity conditions

$$f_0(0,z) = f_0(L,z), \quad f_0(y,0) = f_0(y,L)$$

lead immediately to the Bethe equations

$$1 = \theta(\eta + \zeta)e^{i\eta L} \quad e^{i\zeta L} = \theta(\eta + \zeta).$$

For the case $n = 2, p = 1$, the conditions (5.3) lead to

$$f_0(y_1,y_2,0) = f_0(y_1,y_2,L) \quad f_0(0,y_2,z) = -f_0(y_2,L,z).$$

From the definition of $f_0$ in this case we get, respectively,

$$\theta(\eta_1 + \zeta) \theta(\eta_2 + \zeta) (\exp i(\eta_1 y_1 + \eta_2 y_2) - \exp i(\eta_1 y_2 + \eta_2 y_1))$$

$$= \exp i(\eta_1 y_1 + \eta_2 y_2 - L\zeta) - \exp i(\eta_1 y_2 + \eta_2 y_1 - L\zeta)$$

and (say for $y_2 < z$)

$$\exp i(\eta_2 y_2 - \zeta z) - \exp i(\eta_1 y_2 - \zeta z) = -\theta(\eta_2 + \zeta) \exp i(\eta_1 y_2 + \eta_2 y_2 - \zeta)$$

$$+ \theta(\eta_1 + \zeta) \exp i(\eta_1 y_1 + \eta_2 y_2 - \zeta).$$
From these two equations we get the Bethe equations

\[ \exp iL\zeta = \theta(\eta_1 + \zeta) \theta(\eta_2 + \zeta); \]
\[ 1 = \theta(\eta_1 + \zeta) \exp i\eta_1 L, \quad 1 = \theta(\eta_2 + \zeta) \exp i\eta_2 L. \]

All other periodicity conditions follow from these.

**Remark.** The second example shows that one must anti-symmetrize in momenta to achieve periodicity. For example, suppose \( N = 3, D = 1, \) and \( 0 < y_2 < z \). The condition \( h_0(0, y_2, z) = -h_0(y_2, L, z) \) for the basic wave function obtained in §4 becomes

\[ h_0(0, y_2, z) = \exp i(\eta_2 y_2 - \zeta z) = -h_0(y_2, L, z) = \theta(\eta_2 + \zeta) \exp i(\eta_1 y_2 + \eta_2 L - \zeta z), \]

leading to the contradiction

\[ \exp i\eta_2 y_2 = -\theta(\eta_2 + \zeta) \exp i(\eta_1 y_2 + \eta_2 L). \]

**Theorem 5.2.** Let

\[ e^{i\eta_k L} = \prod_{l=1}^{p} \theta(\zeta_l + \eta_k) \quad k = 1, \ldots, n, \quad (5.4) \]
\[ e^{i\zeta_l L} = \prod_{k=1}^{n} \theta(\eta_k + \zeta_l) \quad l = 1, \ldots, p \quad (5.5) \]

Then the densities \( h_m \) defined by (5.1), (5.2), and (5.3) are \( L \)-periodic in each variable.

The equations (5.4), (5.5) are known as the Bethe equations. The proof is given in Beals, Sattinger, and Williams\(^{15}\).

6. CONSTRUCTION OF A DIRAC GROUND STATE

A solution of the Bethe equations gives an eigenvector of the Hamiltonian with energy

\[ E = -c_1 \sum_{k=1}^{n} \eta_k + c_2 \sum_{l=1}^{p} \zeta_l, \quad (6.1) \]

where \( n \) and \( p \) denote the maximum number of particles of types 1 and 2 for states in the space \( \mathcal{F}_{N,D} \), i.e. \( n = \frac{1}{2}(N + D), \) \( p = \frac{1}{2}(N - D); \) cf. (4.5). The energy is not bounded below. We construct a stable ground state by filling all negative energy states.

Taking logarithms in the Bethe equations (5.4), (5.5), we get

\[ \eta_k - \frac{2\pi}{L} I_k = \sum_{l=1}^{p} \frac{1}{iL} \log \theta(\eta_k + \zeta_l), \quad k = 1, \ldots, n \quad (6.2) \]
\[ \zeta_l - \frac{2\pi}{L} J_l = \sum_{k=1}^{n} \frac{1}{iL} \log \theta(\eta_k + \zeta_l) \quad l = 1, \ldots, p \]
where
\[ \theta(\lambda) = \frac{\lambda + i\omega}{\lambda - i\omega}, \quad \omega = \frac{\gamma^2}{4\alpha^2} \tag{6.3} \]

and \( I_k \) are distinct integers, as are \( J_l \).

Taking
\[ \frac{1}{L} \log \theta(\lambda) = \pi - 2 \tan^{-1} \frac{\lambda}{\omega}, \tag{6.4} \]
we write the Bethe equations in the equivalent form
\[ \eta_k = \frac{2\pi}{L} I_k + \frac{2}{L} \sum_{l=1}^{p} \tan^{-1} \frac{\eta_l + \zeta_l}{\omega}, \tag{6.5} \]
\[ \zeta_l = \frac{2\pi}{L} J_l + \frac{2}{L} \sum_{k=1}^{n} \tan^{-1} \frac{\eta_k + \zeta_l}{\omega} \]

where the \( I_k \) are distinct half integers if \( p \) is odd, distinct integers if \( p \) is even, and similarly for the \( J_l \) in relation to \( n \). We assume for convenience that both are even and, without loss of generality, that
\[ I_1 < I_2 < \cdots < I_n; \quad J_1 < J_2 < \cdots < J_p. \tag{6.6} \]

These conditions imply that there is a solution to (6.5) for which the \( \eta_k \) and the \( \zeta_l \) are themselves strictly increasing. Indeed equations (6.7) are the variational equations for the functional
\[ \Sigma = \frac{1}{2} \left( \sum_{k=1}^{n} \eta_k^2 + \sum_{l=1}^{p} \zeta_l^2 \right) + \frac{\pi}{L} \left( \sum_{k=1}^{n} I_k \eta_k + \sum_{l=1}^{p} J_l \zeta_l \right) - \sum_{k,l=1}^{n,p} M(\eta_k + \zeta_l) \]

where
\[ M(\lambda) = \int_{0}^{\lambda} 2 \frac{2}{L} \tan^{-1} \frac{s}{\omega} \, ds. \]

For convenience in this section we take \( \omega = 1 \). (This amounts to fixing a length scale, since \( \omega \) has the dimension 1/length.)

**Theorem 6.1.** The Bethe equations have at least one solution for each set of distinct integers \( I_k \) and distinct integers \( J_l \).

**Proof.** The term \( M(\lambda) \) grows only linearly in its argument, so the quadratic terms dominate \( \Sigma \) as \( \eta, \zeta \to \infty \). It follows that \( \Sigma \) is bounded below and tends to infinity for large values of its argument. It therefore always has a minimum. □

Uniqueness of the solutions is a subtler question. For large values of the densities, \( \Sigma \) is not convex, so in principle it could have multiple critical points. In the limiting cases of interest, however, we will obtain uniqueness results. In fact we use the finite case to motivate the limiting cases. Therefore we save space and smooth the exposition by stating most of the finite case results without proof and leaving mathematical details for the limiting cases.
We now turn to the construction of a stable ground state, or vacuum state. For the ground state we assume that $D = 0$ (hence $n = p = N/2$) and we consider the case when the velocities of the fermions have opposite signs:

$$c_2 < 0 < c_1. \quad (6.7)$$

Thus $\eta > 0$ and $\zeta > 0$ mean negative energy for the corresponding particles. Our procedure will be to fill the available negative energy states from the top down by choosing consecutive integers $I_k, J_k$ such that the solution $\{\eta_k, \zeta_k\}$ is positive but minimally so; we then take the infinite length and infinite density limits. The equations (6.5) are symmetric, so we take the data $I_k = J_k$ and look for a solution with $\eta_k = \zeta_k$. Thus we want

$$I_k = J_k = k - \kappa_n \quad (6.8)$$

where $\kappa_n$ is an integer to be chosen.

**Theorem 6.2.** There is a largest integer $\kappa_n = O(n)$ such that (6.5) with data (6.8) has a positive solution. This solution is unique and satisfies $\eta_k = \zeta_k$, and $\eta_1 < \cdots < \eta_n$. Moreover, $\eta_1 \to 0$ as $n \to \infty$, $L \to \infty$ with $n/L$ bounded above.

We refer to the solution in Theorem 6.2 as the *cut-off ground state* (CGS). Our interest is the infinite length, finite density limit, and the infinite density limit. For this purpose we recast (6.5) as a pair of integral equations. Assuming still that $n = p$, we let $\Delta = 2\pi n/L$ and define functions on the interval $[0, \Delta)$ by

$$I(\xi) = \frac{2\pi}{L} I_k, \quad J(\xi) = \frac{2\pi}{L} J_k, \quad \frac{2\pi(k-1)}{L} \leq \xi < \frac{2\pi k}{L}; \quad \eta(\xi) = \eta_k, \quad \zeta L(\xi) = \zeta_k \quad \frac{2\pi(k-1)}{L} \leq \xi < \frac{2\pi k}{L}. \quad (6.9)$$

(We hope that context will prevent confusion between the use of $\eta$ and $\zeta$ to denote functions and the use of $\eta$ as $\zeta$ as pure variables.)

Equations (6.2) with $n = p$ are equivalent to integral equations

$$\eta(\xi) = I(\xi) + \frac{1}{\pi} \int_0^{2\pi n/L} \tan^{-1}(\eta(\xi) + \zeta(\xi')) d\xi', \quad (6.10)$$

$$\zeta(\xi) = J(\xi) + \frac{1}{\pi} \int_0^{2\pi n/L} \tan^{-1}(\eta(\xi') + \zeta(\xi)) d\xi'.$$

It can be shown that for every set of data $I, J$ corresponding to sets of integers indexed as in (6.6), there is a solution $(\eta, \zeta)$ with positive jumps at each of the points of discontinuity $2\pi k/L$. We define approximate inverse functions accordingly:

$$\xi(\eta) = \inf\{\xi : \eta(\xi) \geq \eta\}; \quad \xi(\zeta) = \inf\{\xi : \zeta(\xi) \geq \zeta\}. \quad (6.11)$$

**Remark.** These functions have jumps of size $2\pi/L$, so their difference quotients with respect to $\eta$ and to $\zeta$ respectively count the number of particles associated to a given momentum interval.
With the Bethe equations in the form (6.10), it is clear how to pass to the infinite length, finite density limit. To pass also to the infinite density limit we observe that (6.10) is equivalent to

\[\eta(\xi) - \eta(0) = I(\xi) - I(0)\]

\[+ \frac{1}{\pi} \int_0^{2\pi n/L} [\tan^{-1}(\eta(\xi) + \zeta(\xi')) - \tan^{-1}(\eta(0) + \zeta(0))] \, d\xi';\]

\[\zeta(\xi) - \zeta(0) = J(\xi) - J(0)\]

\[+ \frac{1}{\pi} \int_0^{2\pi n/L} [\tan^{-1}(\eta(\xi') + \zeta(\xi')) - \tan^{-1}(\eta(\xi') + \zeta(0))] \, d\xi'.\]

The data (6.8) give rise to the functions

\[I(\xi) = J(\xi) = \xi - c \Delta\text{ in the } L \to \infty \text{ limit, } \Delta \text{ finite, and in this limit } \eta(0) = z(0) = 0.\]

Thus in the form (6.12) we pass to the \(\Delta = \infty\) limit and find the equations for the ground state \((\eta_0, \zeta_0 = \eta_0):\)

\[\eta_0(\xi) = \xi + \frac{1}{\pi} \int_0^{\infty} [\tan^{-1}(\eta_0(\xi) + \eta_0(\xi')) - \tan^{-1}(\eta_0(\xi'))] \, d\xi'.\]  

(6.13)

We obtain existence and uniqueness of the solution to (6.13) by converting it to a linear integral equation for the derivative of the inverse function:

\[\rho_0(\eta(\xi)) = \frac{1}{(d\eta_0/d\xi)(\xi)}.\]  

(6.14)

Differentiate (6.9), multiply by \(\rho_0\), and change the variable of integration to obtain:

\[\rho_0(\eta) = 1 - \frac{1}{\pi} \int_0^{\infty} \frac{\rho_0(\eta') \, d\eta'}{1 + (\eta + \eta')^2} = 1 - K(\rho_0).\]  

(6.15)

**Remarks.**

1. In view of the remark after (6.9), \(\rho_0\) may be thought of as the *particle density function* for the ground state.

2. Note that the kernel of the integral operator \(K\) satisfies

\[\sup_{\eta \geq 0} \frac{1}{\pi} \int_0^{\infty} \frac{d\eta'}{1 + (\eta + \eta')^2} = \frac{1}{2}.\]  

(6.16)

Therefore \(T\) has norm \(1/2\) as an operator in \(L^\infty(\mathbb{R}_+)\) and (6.16) has a unique solution, obtainable by successive approximations. The solution is positive and bounded.

In the next section we need both the analytic continuation \(\rho_{0, t}\) of \(\rho\) to the negative axis and a dual function \(\rho_{0, t}^*\) that is the analytic continuation of the solution to the same equation but with a change of sign:

\[\rho_0^*(\eta) = 1 + \frac{1}{\pi} \int_0^{\infty} \frac{\rho_0^*(\eta') \, d\eta'}{1 + (\eta + \eta')^2} = 1 + K\rho_{0, t}(\eta), \eta \geq 0.\]  

(6.17)

In terms of these functions we define functions \(\xi_0\) and \(\xi_0^*\)

\[\xi_0(\eta) = \int_0^\eta \rho_{0, t}(s) \, ds; \quad \xi_0^*(\eta) = \int_0^\eta \rho_{0, t}^*(s) \, ds, \quad \eta \in \mathbb{R}.\]  

(6.18)
Theorem 6.3. The extension $\rho_{0,t}$ of the particle density function $\rho_0$ and the dual function $\rho^*_{0,t}$ satisfy the estimates

$$0 < \rho_{0,t}(\eta) < 1 < \rho^*_{0,t} < 2, \quad \eta \in \mathbb{R};$$

$$\rho_{0,t}(\eta) = 1 + O(1/\eta), \quad \rho^*_{0,t}(\eta) = 2 + O(1/\eta) \quad \text{as} \quad \eta \to \infty;$$

$$\pi \eta \rho_{0,t}(\eta) = -1 + O(\log |\eta|/|\eta|), \quad \rho^*_{0,t} = 1 + O(1/\eta) \quad \text{as} \quad \eta \to -\infty.$$ 

The functions $\xi_0$ and $\xi^*_0$ have asymptotics

$$\xi_0(\eta) = \eta + O(1), \quad \xi^*_0(\eta) = 2\eta + O(1) \quad \text{as} \quad \eta \to +\infty;$$

$$\pi \xi_0(\eta) = -1 + O(\log |\eta|/|\eta|), \quad \xi^*_0(\eta) = \eta + O(1) \quad \text{as} \quad \eta \to -\infty.$$ 

In particular the function $\xi_0$ is a diffeomorphism of the line. Its inverse function $\eta_0$ satisfies the integral equation (6.13) on $\mathbb{R}^+$. 

Proof. It follows from (6.16) that the solution extends analytically. The kernel is positive, so $0 < \rho_0 < 1$. The asymptotics of $\rho_0$ and of $\xi_0$ are easily verified from (6.17). The argument for $\rho^*_{0,t}$ and $\xi^*_0$ are similar. Finally, (6.13) results from substituting (6.16) in the definition of $\xi_0$ and using $\rho_0 d\eta' = d\xi'$ to change the variable of integration. 

We conclude this section with the integral expressions of various quantities associated with a state described by a pair of functions $\eta, \zeta$. For a finite number of particles the momentum $P$ and energy $E$ are

$$P = \sum_{k=1}^{n} \eta_k - \sum_{l=1}^{n} \zeta_l = \frac{L}{2\pi} \int_{0}^{2\pi n/L} \eta(\xi) d\xi - \frac{L}{2\pi} \int_{0}^{2\pi n/L} \zeta(\xi) d\xi;$$

$$E = -c_1 \sum_{k=1}^{n} \eta + c_2 \sum_{l=1}^{n} \zeta_l = -c_1 \frac{L}{2\pi} \int_{0}^{2\pi n/L} \eta(\xi) d\xi + c_2 \frac{L}{2\pi} \int_{0}^{2\pi n/L} \zeta(\xi) d\xi.$$ 

The particle number $n = p$ itself can be expressed in terms of the inverse functions:

$$n = \frac{L}{2\pi} \sup \xi(\eta) = \frac{L}{2\pi} \sup \xi(\zeta).$$ 

Therefore in the infinite length limit the particle densities per unit length are

$$\frac{1}{2\pi} \sup \xi(\eta) = \frac{1}{2\pi} \int \rho(\eta) d\eta; \quad \frac{1}{2\pi} \sup \xi(\zeta) = \frac{1}{2\pi} \int \sigma(\zeta) d\zeta.$$ 

7. FINITE PARTICLE EXCITED STATES

In this section we construct particle-hole excitations of the ground state. These have finite positive energy with respect to the ground state in the infinite density limit, confirming the stability of the ground state.

We begin with a single particle/hole excited state for an $\eta$ particle, with momentum $p_\eta < 0$ for the particle and $p_\eta > 0$ for the hole. Consider first a cut-off ground
state with a high particle density, so that there are large momentum particles. Denote the functions for this CGS by $\eta, \zeta = \eta$. We change the CGS data by choosing $k$ such that $\eta_k$ is close to $\eta_h$ and changing $I_k$ to $I_k - s$, where the integer $s$ is chosen so that $I_k - s$ is close to $\xi_0(\eta)$. In the integral equation picture (6.10) the new data is given by functions $I, J = I$. Here $I = I$ except on an interval of length $2\pi/L$ located near $\xi_0(\eta)$, on which it differs from $I$ by approximately $\xi_0 - \xi_0$, $\xi = \xi_0(\eta)$. This change in data changes the $\eta$ solution by a bounded amount on that interval and otherwise changes both $\eta(\xi)$ and $\zeta(\xi)$ by $O(1/L)$. Therefore we write the solution $(\eta, \zeta)$ of the perturbed equation as

$$\tilde{\eta}(\xi) = \eta(\xi) + e_1(\xi)/L + f(\xi), \quad \tilde{\zeta}(\xi) = \zeta(\xi) + e_2(\xi)/L$$  \quad (7.1)$$

where $e_1$ vanishes on the critical interval and $f$ is supported on the critical interval. With a similar notation for energy and momentum, the excitation momentum and energy are

$$\delta P = \bar{P} - P = \frac{1}{2\pi} \int_0^\Delta [e_1(\xi) - e_2(\xi)] d\xi + f;$$  \quad (7.2)$$

$$\delta E = \bar{E} - E = \frac{1}{2\pi} \int_0^\Delta [e_1(\xi) - e_2(\xi)] d\xi + c_1 f.$$  

Here we abuse notation and write $f$ also for the value of $\tilde{\eta} - \eta$ on the critical interval.

With large $L$ the integral equations (6.10) give

$$e_1(\xi) \approx \frac{1}{\pi} \int_0^\Delta \frac{[e_1(\xi) + e_2(\xi')]}{1 + (\eta(\xi) + \eta(\xi'))^2} d\xi'$$ \quad (7.3a)$$

$$f \approx \xi - \xi + \frac{1}{\pi} \int_0^\Delta [\tan^{-1}(\eta + f + \eta(\xi')) - \tan^{-1}(\eta + \eta(\xi'))] d\xi';$$ \quad (7.3b)$$

$$e_2(\xi) \approx 2[\tan^{-1}(\eta + f + \eta(\xi)) - \tan^{-1}(\eta + \eta(\xi))]
+ \frac{1}{\pi} \int_0^\Delta \frac{[e_1(\xi') + e_2(\xi')] d\xi'}{1 + (\eta(\xi) + \eta(\xi'))^2}.$$ \quad (7.3c)$$

Here (7.3a) is valid outside the critical interval.

In the infinite length, infinite density limit (6.13) and (7.3b) show that $f = \eta_h - \eta$, so $\tilde{\eta} \approx \eta_h$ on the critical interval, as desired. In this limit the remaining equations (7.3) become

$$e_1(\eta) = \frac{1}{\pi} \int_0^\infty \frac{[e_1(\eta) + e_2(\eta')]}{1 + (\eta + \eta')^2} d\eta';$$ \quad (7.4)$$

$$e_2(\eta) = \frac{1}{\pi} \int_0^\infty \frac{[e_1(\eta') + e_2(\eta)] \rho_0(\eta') d\eta'}{1 + (\eta + \eta')^2}
+ 2 \tan^{-1}(\eta + \eta) - 2 \tan^{-1}(\eta_h + \eta).$$

We write $\tilde{\eta}_j(\eta)$ for $e_j(\xi_0(\eta)) \rho_0(\eta)$: The integral equation (6.18) for $\rho_0$ allows (7.4) to be rewritten:

$$\tilde{e}_1(\eta) = K(\tilde{e}_2)(\eta);$$ \quad (7.5)$$

$$\tilde{e}_2(\eta) = 2 \tan^{-1}(\eta + \eta) - 2 \tan^{-1}(\eta_h + \eta) + T(\tilde{e}_2)(\eta).$$
In the limit the excitation momentum and energy are

\[ \delta P = \frac{1}{2\pi} \int_0^\infty [\tilde{e}_1(\eta) - \tilde{e}_2(\eta)] \, d\eta + (\eta_p - \eta_h); \tag{7.6} \]
\[ \delta E = \frac{1}{2\pi} \int_0^\infty [-c_1\tilde{e}_1(\eta) + c_2\tilde{e}_2(\eta)] \, d\eta - c_1(\eta_p - \eta_h) \]
\[ = -\alpha \left( \frac{1}{2\pi} \int_0^\infty [\tilde{e}_1(\eta) + \tilde{e}_2(\eta)] \, d\eta + \eta_p - \eta_h \right) \]
\[ + c_0 \left( \frac{1}{2\pi} \int_0^\infty [\tilde{e}_1(\eta) - \tilde{e}_2(\eta)] \, d\eta - \eta_p + \eta_h \right). \]

(The last part of (7.6) follows from the second part, since \(2\alpha = c_1 - c_2\) and \(2c_0 = c_1 + c_2\).)

The system (7.5) and the equations (7.6) thus characterize this single particle/hole excitation of the ground state. Again the fact that \(K\) has norm 1/2 implies that (7.6) has, for each choice of \(\eta_h\) and \(\eta_p\), a unique solution. Positivity of the kernel implies that the solution \((e_1, e_2)\) is negative when \(\eta_p - \eta_h\) is negative, so the excitation energy is positive. We can be more precise.

**Theorem 7.1.** The excited state momentum and the excitation energy can be expressed in terms of the (extended) functions \(\xi_0, \xi_0^\ast\) of Theorem 6.3. They have the form

\[ \delta P = P_p - P_h = P(\eta_p) - P(\eta_h) = \xi_0(\eta_p) - \xi_0(\eta_h); \tag{7.7} \]
\[ \delta E = E_p - E_h = E(\eta_p) - E(\eta_h) \]
\[ = [-\alpha \xi_0^\ast(\eta_p) - c_0 \xi_0(\eta_p)] - [-\alpha \xi_0^\ast(\eta_h) - c_0 \xi_0(\eta_h)]. \]

These functions have the asymptotics

\[ E_p \sim 2\alpha e^{\pi |P_p|} \quad \text{as} \quad P_p \to -\infty, \quad (7.8) \]
\[ E_h \sim -c_1 P_h \quad \text{as} \quad P_h \to +\infty. \]

**Proof.** The inhomogeneous term for the pair \((\eta_p, \eta_h)\) is the sum of the inhomogeneous term for the pair \((\eta_p, 0)\) and the inhomogeneous term for the pair \((0, \eta_h)\) and is an odd function of its two arguments. Therefore it suffices to consider the case \(\eta_p = \tilde{\eta}, \eta_h = 0\). We make the \(\tilde{\eta}\) dependence explicit by writing \(\tilde{e}_1(\eta, \tilde{\eta}), \tilde{e}_2(\eta, \tilde{\eta})\). Set

\[ f_1 = \frac{1}{2\pi} \frac{\partial \tilde{\eta}}{\partial \eta} [\tilde{e}_1 + \tilde{e}_2]; \quad f_2 = \frac{1}{2\pi} \frac{\partial \tilde{\eta}}{\partial \eta} [\tilde{e}_2 - \tilde{e}_1]. \tag{7.9} \]

(Note that the \(\tilde{e}_j\) vanish at \(\tilde{\eta} = 0\), so the sum and difference can be recovered by integrating the \(f_j\) from \(\tilde{\eta} = 0\).) Adding and subtracting first derivatives with respect to \(\tilde{\eta}\) in (7.5), we find that as functions of \(\eta\) the \(f_j\) satisfy

\[ f_1 = K f_1 + k, \quad f_2 = -K f_2 + k, \quad k(\eta, \tilde{\eta}) = \frac{1}{\pi} \frac{1}{1 + (\eta + \tilde{\eta})^2}. \tag{7.10} \]

Note that \(k\) is the kernel of the operator \(K\). Let \(F_j\) denote the integral operator that has kernel \(f_j\) and \(I_d\) the identity operator. Then (7.10) is equivalent to the operator equations

\[ (I_d - K)(I_d + F_1) = I_d, \quad (I_d + K)(I_d - F_2) = I_d. \tag{7.11} \]
The integral \( \int_0^\infty f_j(\eta, \tilde{\eta}) \, d\eta \) is the same as the function \( F_j(1)(\tilde{\eta}) \), where 1 denotes the function that is identically 1. It follows from (7.9) and (7.10) that
\[
\frac{1}{2\pi} \int_0^\infty \tilde{e}_1(\eta, \tilde{\eta}) \, d\eta + \tilde{\eta} = \frac{1}{2} \int_0^{\tilde{\eta}} \left[ \int_0^\infty (f_1 - f_2) + 2 \right] \, d\eta
\]
\[= \frac{1}{2} \int_0^{\tilde{\eta}} [(F_1(1) + 1) + (1 - F_2(1))];
\]
\[
\frac{1}{2\pi} \int_0^\infty \tilde{e}_2(\eta, \tilde{\eta}) \, d\eta = \frac{1}{2} \int_0^{\tilde{\eta}} \left[ \int_0^\infty (f_1 + f_2) \right] \, d\eta
\]
\[= \frac{1}{2} \int_0^{\tilde{\eta}} [F_1(1) + F_2(1)] \, d\eta = \frac{1}{2} \int_0^{\tilde{\eta}} [(F_1(1) + 1) - (1 - F_2(1))] \, d\eta
\]
In view of (7.11), \( F_1(1) + 1 \) is the solution of \( u = Ku + 1 \), while \( 1 - F_2(1) \) is the solution of \( v = -Kv + 1 \). By (6.17) and (6.16) these functions are \( \rho_{0,i}^* \) and \( \rho_0 \), respectively. It follows that
\[
\frac{1}{2\pi} \int_0^\infty \tilde{e}_1(\eta, \tilde{\eta}) \, d\eta + \tilde{\eta} = \frac{1}{2}[\xi^*_0(\tilde{\eta}) + \xi_0(\tilde{\eta})],
\]
\[
\frac{1}{2\pi} \int_0^\infty \tilde{e}_2(\eta, \tilde{\eta}) \, d\eta = \frac{1}{2}[\xi^*_0(\tilde{\eta}) - \xi_0(\tilde{\eta})].
\]
The momentum term \( P(\tilde{\eta}) \) in (7.6) is the difference of the two expressions (7.13) and (7.14). Similarly, the excitation energy term \( E(\tilde{\eta}) \) in (7.6) is is the linear combination of (7.13) and (7.14) with coefficients \(-c_1\) and \( c_2 \) respectively. This proves (7.7). The asymptotics (7.8) follow from (7.7) and the asymptotics (6.20) of \( \xi_0 \) and \( \xi^*_0 \). □

Exactly similar considerations apply to a single particle/hole \( \zeta \) excitation, with the roles of \( e_1 \) and \( e_2 \) interchanged. In particular, the indices 1 and 2 in (7.9) should be interchanged throughout.

Compound excitations may be treated in exactly the same way. Consider the case of particle/hole excitations in \( \eta \) described by \( \eta_{h,i} \geq 0, \eta_{p,i} \leq 0, \, i = 1, \ldots, r \) and in \( \zeta \) described by \( \zeta_{h,j} \geq 0, \zeta_{p,j} \leq 0, \, j = 1, \ldots, s \). Then in the infinite density limit the Bethe equations become
\[
\tilde{e}_1(\eta) = K(\tilde{e}_2)(\eta) + 2 \sum_{j=1}^s [\tan^{-1}(\eta + \zeta_{p,j}) - \tan^{-1}(\eta + \zeta_{h,j})],
\]
\[
\tilde{e}_2(\eta) = K(\tilde{e}_1)(\eta) + 2 \sum_{i=1}^r [\tan^{-1}(\eta + \eta_{p,i}) - \tan^{-1}(\eta + \eta_{h,i})].
\]
This state is a superposition of corresponding single particle/hole excited states. Therefore the associated momentum and excitation energy are given by
\[
\delta P = \sum_{i=1}^r [\xi_0(\eta_{p,i}) - \xi_0(\eta_{h,i})] + \sum_{j=1}^s [\xi_0(\zeta_{h,j}) - \xi_0(\zeta_{p,j})];
\]
\[
\delta E = -\alpha \left( \right) \sum_{i=1}^r [\xi^*_0(\eta_{p,i}) - \xi^*_0(\eta_{h,i})] + \sum_{j=1}^s [\xi^*_0(\zeta_{p,j}) - \xi^*_0(\zeta_{h,j})] \right)
\]
\[+ c_0 \left( \sum_{i=1}^r [\xi_0(\eta_{p,i}) - \xi_0(\eta_{h,i})] - \sum_{j=1}^s [\xi_0(\zeta_{p,j}) - \xi_0(\zeta_{h,j})] \right) \]
8. FINITE DENSITY PERTURBATIONS OF THE GROUND STATE

In this section we study large deviations from the ground state. Again we moti- 
vate the procedure by looking briefly at the finite length, finite particle case in 
the form of the integral equations (6.10). Denote by $I_0 = J_0$ the data that corre-

sponds to the CGS data (6.8). The CGS data uses all the integers from an interval 
with endpoints $\pm O(n/2)$ and we are temporarily applying a low energy cut-off that 
corresponds to the upper end of this interval, so the data that corresponds to an 
admissible perturbation of the CGS satisfies $I, J \leq I_0$. The solution can be found 
by iteration from the integral equations (6.10). The monoto-

nicity properties of the 
mapping imply that the solution is dominated by the CGS $(\eta_0, \zeta_0)$:

$$
\eta(\xi) \leq \eta_0(\xi), \quad \zeta(\xi) \leq \zeta_0(\xi). \quad (8.1)
$$

Moreover if the perturbation is to have finite relative energy density in the limit,
we expect that $I, J \approx I_0$ for large $\xi$. Note also that the jumps in the data must be 
at least as great as the jumps $2\pi/L$ in $I_0$. We pass to the infinite length, infinite 
(total) density limit and look for a solution of an analogue of (6.12):

$$
\eta(\xi) - I(\xi) + \frac{1}{\pi} \int_0^\infty [\tan^{-1}(\eta(\xi) + \zeta(\xi')) - \tan^{-1}(\eta_0(0) + \zeta_0(\xi'))] d\xi';
$$

$$
\zeta(\xi) - J(\xi) + \frac{1}{\pi} \int_0^\infty [\tan^{-1}(\eta(\xi') + \zeta(\xi)) - \tan^{-1}(\eta_0(\xi') + \zeta_0(\xi))] d\xi'.
$$

According to the preceding discussion the natural assumptions on the data are

$$
\frac{dI}{d\xi} \geq 1, \quad \frac{dJ}{d\xi} \geq 1; \quad I(\xi) \approx J(\xi) \approx \xi, \quad \xi >> 0. \quad (8.3)
$$

It can be shown that finite particle solutions whose data is such that the formal 
limits are (8.2) converge to solutions of (8.2). However it is more convenient for 
what follows to write the equations for the particle density functions $\rho = d\xi/d\eta$, 
$\sigma = d\xi/d\zeta$:

$$
\rho(\eta) \cdot I'(\eta) = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma(\zeta) d\zeta}{1 + (\eta + \zeta)^2} = 1 - (K\sigma)(\eta); 
$$

$$
\sigma(\zeta) \cdot J'(\zeta) = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\rho(\eta) d\eta}{1 + (\eta + \zeta)^2} = 1 - (K\rho)(\zeta),
$$

where

$$
I'(\eta) = \frac{dI}{d\xi}(\eta(\xi)); \quad J'(\zeta) = \frac{dJ}{d\xi}(\zeta(\xi)).
$$

We have removed the upper bound on particle energies, so the integrations in (8.4) 
need to be taken over the whole line. Now $K$ has norm 1 as an operator in $L^\infty(\mathbb{R})$ 
so solvability of (8.4) becomes an issue. We rewrite (8.4) in the form

$$
\rho = \Theta(1 - K\sigma), \quad \sigma = \Psi(1 - K\rho) \quad (8.4')
$$

and adopt the point of view that the functions $\Theta$ and $\Psi$ are the data. For the 
ground state the data are $\Psi_0 = \Theta_0$ the Heaviside function: $\Theta_0(s) = 0, s \leq 0; = 1,$
\[ s \geq 0. \] More generally we can express the conditions (8.3) and the assumption that our states are perturbations of the ground state by requiring

\[ 0 \leq \Theta, \; \Psi \leq 1; \quad \lim_{|\eta| \to \infty} (\Theta - \Theta_0) = \lim_{|\zeta| \to \infty} (\Psi - \Psi_0) = 0, \tag{8.5} \]

while (8.1) gives an implicit constraint on the data \((\Theta, \Psi)\) in terms of the solution. In fact (8.1) implies the reverse relation between the inverse functions, leading to

\[ \int_{-\infty}^{\eta} (\rho - \rho_0) \geq 0, \quad \int_{-\infty}^{\zeta} (\sigma - \sigma_0) \geq 0, \quad \text{all } \eta, \sigma. \tag{8.6} \]

Moreover the cut-off solution \((\eta, \zeta)\) and the corresponding CGS \((\eta_0, \zeta_0)\) at any given step were defined over the same \(\xi\) interval, so the relative particle density should be zero:

\[ \int_{-\infty}^{\infty} (\rho - \rho_0) = 0 = \int_{-\infty}^{\infty} (\sigma - \sigma_0). \tag{8.7} \]

We show below that (8.7) is necessary for the relative energy density to be finite.

**Theorem 8.1.** Under assumption (8.5) the equations (8.4) have a unique solution. For the solution we have

\[ 0 \leq \rho \leq 1, \quad 0 \leq \sigma \leq 1; \tag{8.8} \]

\[ \lim_{\eta \to \infty} \rho(\eta) = \lim_{\zeta \to \infty} \sigma(\zeta) = 1; \quad \lim_{\eta \to -\infty} \rho(\eta) = \lim_{\zeta \to -\infty} \sigma(\zeta) = 0. \]

**Proof.** We use \(\Theta\) and \(\Psi\) also to denote the operations of multiplication by the corresponding functions. Then the equations are

\[ \rho + \Theta K \sigma = \Theta, \quad \sigma + \Psi K \rho = \Psi. \]

We apply the operator \(Id - \Psi K\) to the first equation and \(Id - \Theta K\) to the second and reorganize to obtain

\[ (Id - \Theta K \Psi K) \rho = \Theta - \Theta K \Psi, \quad (Id - \Psi K \Theta K) \sigma = \Psi - \Psi K \Theta. \tag{8.9} \]

It is enough to show that \(\Theta K \Psi\) and \(\Psi K \Theta\) have norm < 1. The assumptions on \(\Theta\) and \(\Psi\) imply easily that the integral of the kernel of either operator against either variable is bounded away from 1, so the operators have norm < 1 in any \(L^p\), \(1 \leq p \leq \infty\). The asymptotics follow easily from the assumptions about \(\Theta\) and \(\Psi\), together with the trivial estimates

\[ 0 \leq \rho \leq \Theta; \quad 0 \leq \sigma \leq \Psi. \quad \square \]

One must make stronger assumptions to ensure finiteness of the integrals (8.6) and (8.7) and the relative energy density

\[ E = -\frac{c_1}{2\pi} \int_{-\infty}^{\infty} (\rho - \rho_0) \eta \, d\eta + \frac{c_2}{2\pi} \int_{-\infty}^{\infty} (\sigma - \sigma_0) \zeta \, d\zeta. \tag{8.10} \]
Theorem 8.2. The integrals (8.6), (8.7) are finite if \((\Theta - \Theta_0)\) and \((\Psi - \Psi_0)\) are integrable.

If \(\eta(\Theta_0 - \Theta)\) and \(\zeta(\Psi_0 - \Psi)\) are integrable, then \(E\) is finite if and only if (8.7) holds. If so, then \(E \geq 0\).

Proof. Note that

\[\rho - \rho_0 = (\Theta - \Theta_0)(1 - K\sigma_0) - \Theta K(\sigma - \sigma_0),\]  

(8.11)

with a similar equation for \(\sigma - \sigma_0\). Analyzing this system as above, we see that integrability of \(\rho - \rho_0\) and \(\sigma - \sigma_0\) follows from integrability of the data in the analogue of (8.9), which follows in turn from integrability of \(\Theta - \Theta_0\) and \(\Psi - \Psi_0\).

Note that integrability of \(\eta(\rho_0 - \rho)\) at \(-\infty\) follows immediately from (8.11) and integrability of \(\eta(\Theta_0 - \Theta)\). At \(+\infty\) we observe from (8.11) that the necessary and sufficient condition is the integrability of \(\eta K(\sigma - \sigma_0)\). Assume that \(\zeta^2(\Psi - \Psi_0)\) is integrable at \(-\infty\). Then \(\zeta^2(\sigma - \sigma_0)\) is also integrable at \(-\infty\) and Lebesgue’s Dominated Convergence Theorem implies that

\[\lim_{\eta \to +\infty} K(\sigma - \sigma_0) = \int_{-\infty}^{\infty} (\sigma - \sigma_0) d\zeta.\]  

(8.13)

This proves the necessity of the conditions (8.7).

Conversely one can deduce the integrability of \(\eta K(\sigma - \sigma_0)\) and \(\zeta K(\rho - \rho_0)\) from conditions (8.7) together with integrability of \(\eta(\rho_0 - \rho)\) and \(\zeta(\sigma_0 - \sigma)\). To see this, note that the mean-value zero condition allows us to replace the kernel of \(K\) with the kernel

\[\tilde{k}(\eta, \zeta) = \frac{1}{\pi} \left[ \frac{1}{1 + (\eta + \zeta)^2} - \frac{1}{1 + \eta^2} \right].\]  

(8.14)

The corresponding operator takes \(L^1(\mathbb{R}, (1 + |\eta|) d\eta)\) to \(L^1(\mathbb{R}, (1 + |\eta|) d\eta)\), since we can multiply \(\tilde{k}\) by \((1 + |\eta|)/(1 + |\eta|)\), take the absolute value, and estimate the integral with respect to \(\eta\).

Finally, to show positivity of the relative energy density we let \(g_1\) and \(g_2\) denote the integrals in (8.6), so that

\[\frac{dg_1}{d\eta} = \rho, \quad \frac{dg_2}{d\zeta} = \sigma; \quad g_j \geq 0, \quad g_j(\pm \infty) = 0.\]  

(8.15)

Integrating by parts in (8.10) we find that

\[E = \frac{c_1}{2\pi} \int_{-\infty}^{\infty} g_1(\eta) d\eta - \frac{c_2}{2\pi} \int_{-\infty}^{\infty} g_2(\zeta) d\zeta \geq 0.\]  

\[\square\]

9. THERMODYNAMICS

We consider now the thermodynamics of the finite density perturbations of the ground state that were constructed in the preceding section. We follow, with appropriate modifications, Yang and Yang\textsuperscript{5} and Okhuma\textsuperscript{7}. Consider first the case of...
finite \( L \), a state with a very large number of particles, and data \( I, J \) as in (8.1). Define
\[
h(s) = s - \frac{1}{\pi} \int_0^\Delta [\tan^{-1}(s + \zeta(\xi')) - \tan^{-1}(\eta(0) + \zeta(\xi'))] d\xi', \quad s \geq \eta(0);
\]
\[
k(t) = t - \frac{1}{\pi} \int_0^\Delta [\tan^{-1}(\eta(\xi') + t - \tan^{-1}(\eta(0) + \zeta(\xi'))] d\xi', \quad t \geq \zeta(0).
\]

These are strictly increasing functions with the properties that at the (discrete) values taken by the data the values are
\[
h(\eta(\xi)) = I(\xi), \quad k(\zeta(\xi)) = J(\xi).
\]

The number \( m \) of \( \eta \) particles from our state with momenta in a small interval \((\eta, \eta + d\eta)\) is proportional to the length of the corresponding \( \xi \) interval, i.e. if \( \eta = \eta(\xi), \eta + d\eta = \eta(\xi + d\xi) \) then
\[
m \approx \frac{L}{2\pi} d\xi.
\]

The function \( I \) has jumps that are positive integer multiples of \( L/2\pi \), so
\[
m \leq n = \frac{L}{2\pi} (I(\xi + d\xi) - I(\xi)).
\]

If data \( I \) at the \( m \) jump sites in the interval \((\xi, \xi + d\xi)\) is changed in such a way that \( I \) remains strictly increasing and the values of \( LI/2\pi \) are integers, the effect on the total energy of the corresponding state will be negligible: \( O((d\eta)^2) \). The possible sites are indexed by the \( n \) integers between \( LI(\xi)/2\pi \) and \( L(I\xi)/2\pi \). Thus the contribution to entropy from the interval is \( \log \binom{n}{m} \).

In the infinite length and infinite density limits the density functions for the numbers \( n \) and \( m \) above have densities with respect to the momentum \( \eta \) given by \( \rho = d\xi/d\eta \) and \( \rho_t = dh/d\eta \) respectively. As before \( \rho \) is the particle density. We take \( \rho_h = \rho_t - \rho \) to be the hole density and \( \rho_t \) to be the (formal) total density. Similar considerations apply to \( \zeta \) particles and give particle density \( \sigma \) and hole density \( \sigma_h = \sigma_t - \sigma \). In the notation of §8 the relations among these functions are
\[
\rho = \Theta \rho_t, \quad \rho_h = (1 - \Theta) \rho_t; \quad \sigma = \Psi \sigma_t, \quad \sigma_h = (1 - \Psi) \sigma_t;
\]
\[
\rho_t = 1 - K \sigma = 1 - K \Psi \sigma_t; \quad \sigma_t = 1 - K \rho = 1 - K \Theta \rho_t.
\]

Returning to the interval \((\xi, \xi + d\xi)\), let \((\eta, \eta + d\eta)\) denote its image under \( h^{-1} \). According to the preceding argument and Stirling’s formula, the entropy density over \((\eta + d\eta)\) is approximately
\[
\frac{L}{2\pi} [\rho_t \log \rho_t - \rho \log \rho - \rho_h \log \rho_h] \]
\[
= \frac{L}{2\pi} [\rho \log(\rho_t/\rho) + \rho_h \log(\rho_t/\rho_h)]
\]
\[
= -\rho_t [\Theta \log \Theta + (1 - \Theta) \log(1 - \Theta)].
\]
A similar formula holds for particles of the second type. Thus in the infinite length, infinite density limit the entropy per unit length is

\[
S = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \rho_t \log \rho_t - \rho \log \rho - \rho_h \log \rho_h \right] d\eta \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \sigma_t \log \sigma_t - \sigma \log \sigma - \sigma_h \log \sigma_h \right] d\zeta.
\] (9.4)

The entropy density for the ground state is zero.

At a given finite temperature \( T \) the free energy density of a perturbation \((\rho, \sigma)\) of the ground state relative to that of the ground state is

\[
F_T(\rho, \sigma) = E - T S
\] (9.5)

where the relative energy density \( E \) is given by (8.9). In order for \( E \) to be finite, we showed that the relative particle densities must vanish:

\[
\int_{-\infty}^{\infty} (\rho - \rho_0) d\eta = 0 = \int_{-\infty}^{\infty} (\sigma - \sigma_0) d\zeta.
\] (9.6)

Thus the equilibrium solution should minimize \( F \) subject to the constraints (9.6).

We consider briefly the formal argument for an equilibrium. Equations (9.2) relate variations in \( \rho_t, \rho_h, \sigma_t, \) and \( \sigma_h \) to variations in the particle densities \( \rho, \Theta \):

\[
\dot{\rho}_t = -K \dot{\sigma}, \quad \dot{\rho}_h = -\dot{\rho} - K \dot{\sigma}; \\
\dot{\sigma}_t = -K \dot{\rho}, \quad \dot{\sigma}_h = -\dot{\sigma} - K \dot{\rho}.
\] (9.7)

The constraints (9.6) require

\[
\int_{-\infty}^{\infty} \dot{\rho} d\eta = 0 = \int_{-\infty}^{\infty} \dot{\sigma} d\zeta.
\] (9.8)

Taking into account (9.7) and the symmetry of the operator \( K \), we can express the variation in the free energy as

\[
\dot{F} = \int_{-\infty}^{\infty} \dot{\rho} \left[ -c_1 \eta + T \log(\rho/\rho_h) + TK \log(\sigma_t/\sigma_h) \right] d\eta \\
+ \int_{-\infty}^{\infty} \dot{\sigma} \left[ c_2 \zeta + T \log(\sigma/\sigma_h) + TK \log(\rho_t/\rho_h) \right] d\zeta.
\] (9.9)

Following Yang and Yang\(^5\), one can introduce functions \( \epsilon_j \):

\[
e^{-\epsilon_1/T} = \rho/\rho_h, \quad e^{-\epsilon_2/T} = \sigma/\sigma_h.
\] (9.10)

and derive Euler-Lagrange equations in the form

\[
\epsilon_1 = A_1 T - c_1 \eta + TK \log(1 + e^{-\epsilon_2/T}), \\
\epsilon_2 = A_2 T + c_2 \zeta + TK \log(1 + e^{-\epsilon_1/T}),
\] (9.11)
where the $A_j$ are the Lagrange multipliers that correspond to \((9.8)\).

As noted in the introduction, there are two difficulties with this argument. First, equations \((9.11)\) have no solution: taking derivatives one sees that the $\epsilon_j$ should be decreasing functions, so the functions \(\log(1 + e^{-\epsilon_j/T})\) are increasing and bounded above, but they are taken by $K$ to decreasing functions and thus one needs $\epsilon_1 \leq -c_1 \eta + O(1)$ at $+\infty$. Therefore $\log(1 + e^{-\epsilon_1/T})$ is nonnegative and grows like $\eta$ at $+\infty$, so $K \log(1 + e^{-\epsilon_1/T})$ cannot be defined. Second, it is not clear that the free energy is bounded below.

The first difficulty is easily dealt with. We may assume that $\dot{r}$ and $\dot{s}$ are test functions, and \((9.9)\) simply means that they are derivatives of test functions. Therefore the condition $\dot{F} = 0$ simply means that the derivatives of the two bracketed terms vanish. Now $K$ anti-commutes with derivation, so the correct form of \((9.11)\) is

$$
\frac{d\epsilon_1}{d\eta} = -c_1 + K\left(\Psi \frac{d\epsilon_2}{d\zeta}\right);
\frac{d\epsilon_2}{d\zeta} = c_2 + K\left(\Theta \frac{d\epsilon_1}{d\eta}\right).
$$

Combining this with

$$
\Theta = \frac{e^{-\epsilon_1/T}}{1 + e^{-\epsilon_1/T}}, \quad \Psi = \frac{e^{-\epsilon_2/T}}{1 + e^{-\epsilon_2/T}}
$$

\((9.13)\) yields a well-behaved system of equations for the thermal equilibrium particle distribution $(\rho, \sigma) = (\rho^{(T)}, \sigma^{(T)})$.

To deal with the second difficulty we prove the following.

**Theorem 9.1.** The free energy functional $F_T$ is convex and bounded below by $-CT^2$, where $C$ is a constant that depends only on the velocities $c_j$.

**Proof.** From \((9.9)\), the second derivative of the free energy for the variation $(\dot{\rho}, \dot{\sigma})$ is

$$
T \int_{-\infty}^\infty \left[ \dot{\rho} - \frac{\dot{\rho}}{\rho} \right] d\eta + T \int_{-\infty}^\infty \left[ \frac{\dot{\sigma}}{\sigma} - \frac{\dot{\sigma}}{\sigma_h} \right] d\zeta = T \int_{-\infty}^\infty \left[ \dot{\sigma} \left( \frac{\dot{\rho}}{\rho} - \frac{\dot{\rho}}{\rho_h} \right) + (K \dot{\sigma}) \left( \frac{\dot{\rho}}{\rho_t} - \frac{\dot{\rho}}{\rho_h} \right) \right] d\eta
$$

$$
+ T \int_{-\infty}^\infty \left[ \dot{\rho} \left( \frac{\dot{\sigma}}{\sigma} - \frac{\dot{\sigma}}{\sigma_h} \right) + (K \dot{\rho}) \left( \frac{\dot{\sigma}}{\sigma_t} \right) \frac{\dot{\sigma}_h}{\sigma_h} \right] dz.
$$

\((9.14)\)

We use \((9.7)\) and reorganize to find that the second derivative is

$$
T \int_{-\infty}^\infty \left( \frac{\dot{\rho}_t}{\sqrt{\rho_t \rho_h}} \dot{\rho} + \frac{\dot{\rho}}{\sqrt{\rho_t \rho_h}} K \dot{\sigma} \right)^2 d\eta + T \int_{-\infty}^\infty \left( \frac{\dot{\sigma}_t}{\sqrt{\sigma_t \sigma_h}} \dot{\sigma} + \frac{\dot{\sigma}}{\sqrt{\sigma_t \sigma_h}} K \dot{\rho} \right)^2 d\zeta.
$$

\((9.15)\)

To prove boundedness we introduce the function

$$
C(s) = C(1-s) = -\log s - (1-s) \log(1-s), \quad 0 \leq s \leq 1.
$$
\[ G(\Theta) = G(|\Theta_0 - \Theta|) \] and the entropy density can be written
\[ 2\pi S(\rho, \sigma) = \int_{-\infty}^{\infty} \rho_t G(\Theta) \, d\eta + \int_{-\infty}^{\infty} \sigma_t G(\Psi) \, d\zeta. \quad (9.16) \]

For the first integral we partition the line into three sets
\[ I_1 = \{ \eta : G(\Theta) \leq |\eta(\Theta_0 - \Theta)|/\delta \}, \quad (9.17) \]
\[ I_2 = \{ \eta : |\Theta_0 - \Theta| \geq 1/2, \, G(\Theta) > |\eta(\Theta_0 - \Theta)|/\delta \}, \quad \]
\[ I_3 = \{ \eta : |\Theta_0 - \Theta| < 1/2, \, G(\Theta) > |\eta(\Theta_0 - \Theta)|/\delta \}, \]

where the constant \( \delta \) is to be chosen. Note that \( 0 \leq G(t) \leq \log 2 \) so \( \eta \in I_2 \) implies \(|\eta| < 2\delta \log 2\). Also
\[ -\log t \geq -2t \log t \geq G(t), \quad 0 \leq t \leq 1/2 \]
so \( \eta \in I_3 \) implies \(|\Theta_0 - t| < \exp(-|\eta|/\delta)\). Now \(-t \log t \) is increasing on the interval \([0, 1/e]\), so we conclude that \( \eta \in I_3 \) implies that either \(|\eta|/\delta < 1\) or
\[ G(\Theta) = G(|\Theta_0 - \Theta|) \leq -2|\Theta_0 - \Theta| \log |\Theta_0 - \Theta| \leq \frac{2|\eta|}{\delta} e^{-|\eta|/\delta}. \]

Since \( 1 < 2 \log 2 \) we can combine these estimates and obtain
\[ \int_{-\infty}^{\infty} \rho_t G(\Theta) \, d\eta \leq \frac{1}{\delta} \int_{-\infty}^{\infty} \rho_t (\Theta_0 - \Theta) \eta \, d\eta + 4\delta \log 2 + 2 \int_{-\infty}^{\infty} \frac{|\eta|}{\delta} e^{-|\eta|/\delta} \, d\eta \]
\[ = \int_{-\infty}^{\infty} \rho_t (\Theta_0 - \Theta) \eta \, d\eta + 4\delta \log 2 + 4\delta. \quad (9.18) \]

We claim that
\[ \int_{-\infty}^{\infty} \rho_t (\Theta_0 - \Theta) \eta \, d\eta \leq \int_{-\infty}^{\infty} (\rho_0 - \rho) \eta \, d\eta + \int_{-\infty}^{\infty} (\sigma_0 - \sigma) \zeta \, d\zeta. \quad (9.19) \]

In fact (8.11) can be rewritten
\[ \rho_t (\Theta_0 - \Theta) = \rho - \rho_0 + \Theta_0 K(\rho - \rho_0). \quad (9.20) \]

Using \( \langle , \rangle \) to denote the standard inner product, we want to prove that
\[ \int_{-\infty}^{\infty} \Theta_0 K(\sigma - \sigma_0) \eta \, d\eta = \langle K(\sigma - \sigma_0), \Theta_0 \eta \rangle \]
\[ \leq \langle \sigma_0 - \sigma, \zeta \rangle = -\langle Dg_2, z \rangle = \langle g_2, 1 \rangle, \quad (9.21) \]
where \( D \) denotes differentiation and \( g_2 \) is the function in (8.15). We use the fact that derivation anti-commutes with \( K \) and integrate by parts to obtain
\[ \langle K(\sigma - \sigma_0), \Theta_0 \eta \rangle = \langle KDg_2, \Theta_0 \eta \rangle = -\langle DKg_2, \Theta_0 \eta \rangle \]
\[ = \langle K(\Theta), \eta \rangle = \langle g_2, 1 \rangle, \]
Now $0 \leq K \theta_0 \leq 1$ and $g_2 \geq 0$, so this proves (9.19). Using (9.18), (9.19), and the analogous results for $\sigma$, we obtain

$$F_T(\rho, \sigma) \leq \left(c_1 - \frac{2T}{\delta}\right) \int_{-\infty}^{\infty} (\rho_0 - \rho) d\eta + 4\delta(1 + \log 2)$$

$$+ \left(-c_2 - \frac{2T}{\delta}\right) \int_{-\infty}^{\infty} (\sigma_0 - \sigma) d\zeta + 4\delta(1 + \log 2).$$

We take $\delta = 2T/(c_1 - c_2)$ and obtain $C = 8(1 + \log 2)/(c_1 - c_2)$. □

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