SMOOTH HILBERT SCHEMES
THEIR CLASSIFICATION AND GEOMETRY
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ABSTRACT. Closed subschemes in projective space with a fixed Hilbert polynomial are parametrized by a Hilbert scheme. We classify the smooth ones. We identify numerical conditions on a polynomial that completely determine when the Hilbert scheme is smooth. We also reinterpret these smooth Hilbert schemes as generalized partial flag varieties and describe the subschemes being parametrized.

Overview

Hilbert schemes are crucial for compactifying families of subschemes and constructing moduli spaces. Among these parameter spaces, Hilbert schemes of points on a projective surface are exceptional. Being smooth, they have a wider range of applications including deep results in algebraic geometry, combinatorics, and representation theory; see [Bea83, Hai01, Gro96, Nak97]. In contrast, little is known about geometric properties of other Hilbert schemes. Even the geometry of $\text{Hilb}^p(\mathbb{P}^m)$, the Hilbert scheme parametrizing closed subschemes in projective $m$-space $\mathbb{P}^m$ with Hilbert polynomial $p$, is poorly understood when $m \geq 3$. Although Hartshorne [Har66] shows that each $\text{Hilb}^p(\mathbb{P}^m)$ is path-connected, celebrated insights into these Hilbert schemes typically highlight pathologies. For example, Mumford [Mum62] exhibits an irreducible component in $\text{Hilb}^{14-23}(\mathbb{P}^3)$ that is generically non-reduced, Ellia–Hirschowitz–Mezzetti [EHM92] show that the number of irreducible components in $\text{Hilb}^{d+c}(\mathbb{P}^3)$ is not bounded by a polynomial in $\mathbb{Q}[c,d]$, and Vakil [Vak06] proves that every singularity type appears in some $\text{Hilb}^p(\mathbb{P}^4)$. As a counterpoint, this article classifies the smooth Hilbert schemes $\text{Hilb}^p(\mathbb{P}^m)$ and describes their geometry.

Our primary theorem uses integer partitions to characterize smooth Hilbert schemes. An integer partition $\lambda$ is an $r$-tuple $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$.

Theorem A. For any positive integer $m$ and any polynomial $p$ in $\mathbb{Q}[t]$, the Hilbert scheme $\text{Hilb}^p(\mathbb{P}^m)$ is a smooth irreducible variety if and only if there exists an integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that

$$p(t) = \sum_{i=1}^{r} \left( \frac{t + \lambda_i - i}{\lambda_i - 1} \right)$$

and one of the following seven conditions holds:

1. $m = 2 \geq \lambda_1$,
2. $m \geq \lambda_1$ and $\lambda_r \geq 2$,
3. $\lambda = (1)$ or $\lambda = (m^{r-2}, \lambda_{r-1}, 1) = (m, m, \ldots, m, \lambda_{r-1}, 1)$ where $r \geq 2$ and $m \geq \lambda_{r-1} \geq 1$,
4. $\lambda = (m^{r-s-3}, \lambda_{r-s-2}^{s+2})$ where $r - 3 \geq s \geq 0$ and $m - 1 \geq \lambda_{r-s-2} \geq 3$,
5. $\lambda = (m^{r-s-5}, 2^{s+4}, 1)$ where $r - 5 \geq s \geq 0$,
6. $\lambda = (m^{r-3}, 1^3)$ where $r \geq 3$,
7. $\lambda = (m+1)$ or $r = 0$.

These combinatorial conditions encode the underlying geometry. For instance, the integer partition $\lambda = (\lambda_1)$ corresponds to the Grassmannian of $(\lambda_1 - 1)$-dimensional planes in $\mathbb{P}^m$ and $\lambda = (m^r) = (m, m, \ldots, m)$ corresponds to the Hilbert scheme parametrizing hypersurfaces of degree $m$. 

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tractable. What conditions on the integer partition $\lambda$ imply that Hilb$^P(\mathbb{P}^m)$ is irreducible? How does one extend this result to Quot schemes or nested Hilbert schemes? What is the right analogue if $\mathbb{P}^m$ is replaced with a smooth toric variety, a complete intersection, or a Grassmannian?
Strategy of proof. We analyze $\text{Hilb}^p(P^m)$ via the induced action of the general linear group. A point in this Hilbert scheme is Borel-fixed if the stabilizer of the corresponding closed subscheme contains all lower triangular matrices. Every nonempty $\text{Hilb}^p(P^m)$ has distinguished Borel-fixed point called the lexicographic point. This point shows that rewriting the polynomial $p$ in terms of the integer partition $\lambda$ is equivalent to $\text{Hilb}^p(P^m)$ being nonempty; see [Mac27] or [Har66, Corollary 5.7]. Reeves–Stillman [RS97, Theorems 1.4 and 4.1] establish that the lexicographic point is always smooth and determine the dimension of the unique irreducible component containing it. Hence, proving smoothness reduces, in principle, to computing the dimension of the tangent space at the other Borel-fixed points. Unfortunately, the rapid growth in the number of these points and the complexity of individual points overwhelm a brute-force attack.

To circumvent these complications, we identify a new family of subschemes in $P^m$ that correspond to singular points on $\text{Hilb}^p(P^m)$. Residual flags are used to describe these points and to prove that they are singular. By exploiting the geometry of residual flags, our analysis reduces to the Hilbert schemes with an integer partition $\lambda = ((m-1)^d, \ell^c, 1)$ where $m-2 \geq \ell \geq 1$; see Proposition 4.3. In this situation, we construct an explicit monomial ideal and exhibit a number of linearly independent deformations; see Lemma 4.2. Since this number exceeds the dimension of the lexicographic component, this ideal corresponds to a singular point on $\text{Hilb}^p(P^m)$.

In hindsight, the complete classification of smooth Hilbert schemes is obtain from just a few families of Borel-fixed points. Only three singular families, in addition to our new family, are required; see Examples 4.4–4.5. For smoothness, only one family other than the lexicographic points is needed; see Theorem 3.2 and Proposition 3.8. This reduction may be the most surprising development. The relevant Borel-fixed points also, fortuitously, avoid technicalities arising in positive characteristic, thereby producing uniform results over the integers.

Amusingly, all seven conditions in Theorem A correspond to Hilbert schemes that are already known to be smooth. Fogarty’s article [Fog68, Theorems 1.4 and 2.4] shows that Conditions 1 and 6 guarantee smoothness. Serving as our initial inspiration, Staal’s thesis [Sta20, Theorem 1.1] establishes that Conditions 2–3 correspond to smooth Hilbert schemes. Likewise, Ramkumar’s preprint [Ram19, Theorem A] proves that Conditions 4–5 are associated to smooth Hilbert schemes. Under Condition 7, the Hilbert scheme is just one point. Despite recognizing each condition, the consolidated list does not already appear in the literature and is not obviously complete.

Computational experience. Although independent of our proofs, calculations in Macaulay2 [M2] were indispensable in the discovery of our results. Recoding the Hilbert polynomial as an integer partition gives a novel method of sampling nonempty Hilbert schemes. Using Macaulay2, we made a systematic search of the Borel-fixed points, for all $3 \leq m \leq 7$, exposing Conditions 2–6. We learn, a posteriori, that Conditions 2–5 imply that Hilbert schemes have at most two Borel-fixed points. Our computational experiments suggest that the number of parts in the integer partition $\lambda$ equal to 1 governs the size of the intersection graph for the irreducible components in the Hilbert scheme.

1. Residual flags

In this section, we introduce the notion of a residual flag and show that the scheme parametrizing these objects is smooth and projective. Throughout, we work over a locally noetherian base scheme
S and E denotes a coherent $\mathcal{O}_S$-module. We write $\mathbb{P}(E) := \text{Proj}(\text{Sym}(E))$ for the projectivization of the graded symmetric algebra $\text{Sym}(E)$.

**Grassmannians.** For any $S$-scheme $T$, let $E_T$ denote the pull-back of $E$ to $T$. A surjection of coherent $T$-modules $E_T \to F$ gives a closed immersion

$$\mathbb{P}(F) \subseteq \mathbb{P}(E_T) = \mathbb{P}(E) \times_S T.$$ 

If $F$ is locally free of constant rank $n + 1$, then the subscheme $\mathbb{P}(F)$ is a $n$-plane in $\mathbb{P}(E_T)$. The set of $T$-valued points of the functor $\text{Gr}(n, \mathbb{P}(E))$ is the set of the $n$-planes in $\mathbb{P}(E_T)$. The $S$-scheme representing this functor is projective; see [EGA71, Proposition 9.8.4]. When $E$ is locally free of constant rank $m + 1$, the map $\text{Gr}(n, \mathbb{P}(E)) \to S$ is smooth of relative dimension $(n + 1)(m - n)$.

**Flag varieties.** Consider an $e$-tuple $n := (n_1, n_2, \ldots, n_e)$ of integers such that $n_1 > n_2 > \cdots > n_e \geq 0$. For any $S$-scheme $T$, a flag of type $n$ in $\mathbb{P}(E_T)$ is a chain of closed immersions

$$\mathbb{P}(F_e) \subseteq \mathbb{P}(F_{e-1}) \subseteq \cdots \subseteq \mathbb{P}(F_1)$$

where each $\mathbb{P}(F_i)$ is a $n_i$-plane in $\mathbb{P}(E_T)$ for all $1 \leq i \leq e$. The set of $T$-valued points of the functor $\text{Flag}(n, \mathbb{P}(E))$ is the set of flags of type $n$ in $\mathbb{P}(E_T)$. The $S$-scheme representing this functor is projective; see [EGA71, Proposition 9.9.3]. A flag is a succession of Grassmannians. Hence, when $E$ is locally free of constant rank $n_0 + 1$, it follows that the map $\text{Flag}(n, \mathbb{P}(E)) \to S$ is smooth of relative dimension $\sum_{i=1}^e (n_i + 1)(n_{i-1} - n_i)$.

**Relative divisors.** A closed subscheme $D \subseteq \mathbb{P}(E)$ is a relative effective Cartier divisor if it is flat over $S$ and its ideal sheaf is invertible. The divisor $D$ has degree $d$ if, for each geometric point $\text{Spec}(k) \to S$, the fibre $D \times_S \text{Spec}(k)$ is a hypersurface of degree $d$ in $\mathbb{P}^m \cong \mathbb{P}(E) \times_S \text{Spec}(k)$; compare with [KM85, Corollary 1.1.5.2]. Using the dual sheaf $E^* := \mathcal{H}om(E, \mathcal{O}_S)$, we may parametrize these divisors in $\mathbb{P}(E)$; see [Fog68, Proposition 1.2] and [Kol96, Exercise 1.4.1.4].

**Lemma 1.1.** Assume that $E$ is a locally free sheaf and let $E^*$ be its dual. For all nonnegative integers $d$, the $S$-scheme $\mathbb{P}(\text{Sym}^d(E^*))$ represents the functor of relative effective Cartier divisors in $\mathbb{P}(E)$ having degree $d$.

**Proof.** Let $T$ be an $S$-scheme and set $F_T := (\text{Sym}^d(E^*))_T = \text{Sym}^d((E^*)_T)$. Given a line bundle $L$ on $T$ and a surjection $F_T \to L$, we see that $L^*$ is an invertible subsheaf of $F_T^*$. The ideal sheaf generated by $L^*$ in $\text{Sym}(E^*)_T$ is invertible and determines a hypersurface of degree $d$ fiberwise in $\mathbb{P}(E_T)$. Thus, it defines a relative effective Cartier divisor in $\mathbb{P}(E_T)$ of degree $d$. \hfill $\Box$

**Residual scheme.** Consider the closed immersion $D \subseteq X$ in $\mathbb{P}(E)$ where $D$ is a relative effective Cartier divisor. Let $\mathcal{I}_D$ and $\mathcal{I}_X$ denote the ideal sheaves of the closed subschemes $D$ and $X$ in $\mathbb{P}(E)$. The residual scheme to $D$ in $X$ is the closed subscheme $Y$ in $\mathbb{P}(E)$ defined by the colon ideal sheaf $\mathcal{I}_Y := (\mathcal{I}_X : \mathcal{I}_D) = \mathcal{I}_X \cdot \mathcal{I}_D^{-1}$. It follows that $\mathcal{I}_X = \mathcal{I}_Y \cdot \mathcal{I}_D$ and $X$ is the union of the subschemes $D$ and $Y$; see [Ful98, Definition 9.2.1] and [Fog68, pp. 512–513].

**Definition 1.2.** For any positive integer $d$, a closed immersion $Y \subseteq X$ in $\mathbb{P}(E)$ is a $d$-residual inclusion if there exists a relative effective Cartier divisor $D$ in $\mathbb{P}(E)$ of degree $d$ such that the closed subscheme $Y$ is the residual scheme to $D$ in $X$ with respect to $\mathbb{P}(E)$. 
Lemma 1.3. Let \( d \) be a positive integer and let \( Y \subset X \) in \( \mathbb{P}(E) \) be a \( d \)-residual inclusion. The map \( Y \to S \) is flat if and only if the map \( X \to S \) is flat.

Proof. The existence of a relative effective Cartier divisor \( D \) in \( \mathbb{P}(E) \) such that \( Y \) is the residual scheme to \( D \) in \( X \) yields the short exact sequence

\[
0 \longrightarrow \mathcal{O}_Y(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0,
\]

where multiplication by a local equation for \( D \) defines the injective map. By hypothesis, the sheaf \( \mathcal{O}_D \) is flat over \( S \). We deduce that \( \mathcal{O}_Y \) and \( \mathcal{O}_Y(-D) \) are flat over \( S \) if and only if \( \mathcal{O}_X \) is flat over \( S \). \( \square \)

Definition 1.4. Let \( (n,d) := (n_1,d_1),(n_2,d_2),\ldots,(n_e,d_e) \) be a sequence of pairs of positive integers such that \( n_1 > n_2 > \cdots > n_e > 0 \). For any \( S \)-scheme \( T \), a residual flag of type \( (n,d) \) in \( \mathbb{P}(E_T) \) is a chain of closed immersions

\[
\emptyset = X_{e+1} \subset X_e \subset X_{e-1} \subset \cdots \subset X_1
\]

in \( \mathbb{P}(E_T) \) such that, for all \( 1 \leq i \leq e \), the following properties are satisfied:

(i) the scheme \( X_i \) is flat over \( T \),
(ii) the scheme \( X_i \) is contained in some \( n_i \)-plane \( \mathbb{P}(F_i) \subset \mathbb{P}(E_T) \), and
(iii) the closed immersion \( X_i+1 \subset X_i \) is a \( d_i \)-residual inclusion in \( \mathbb{P}(F_i) \).

Remark 1.5. For any residual flag, the third property for \( i = e \) asserts that closed immersion \( \emptyset = X_{e+1} \subset X_e \) is a \( d_e \)-residual inclusion. In other words, the closed subscheme \( X_e \) is a relative effective Cartier divisor of degree \( d_e \) in some \( n_e \)-plane \( \mathbb{P}(F_e) \subset \mathbb{P}(E_T) \).

Remark 1.6. Residual flags generalize flags of linear subspaces. To be more explicit, assume \( E \) is locally free of constant rank \( m + 1 \) and let \( T \) be an \( S \)-scheme. Given a flag \( \Lambda_e \subset \Lambda_{e-1} \subset \cdots \subset \Lambda_1 \) of type \( n \) in \( \mathbb{P}(E_T) \) where \( m > n_1 \), there exists a flag \( \mathbb{P}(F_e) \subset \mathbb{P}(F_{e-1}) \subset \cdots \subset \mathbb{P}(F_1) \) of type \( (n_1 + 1,n_2 + 1,\ldots,n_e + 1) \) in \( \mathbb{P}(E_T) \) such that the \( n_i \)-plane \( \Lambda_i \) is a hyperplane in \( \mathbb{P}(F_i) \) for all \( 1 \leq i \leq e \). Setting \( X_{e+1} := \emptyset \), we define \( X_i := \Lambda_i \cup X_{i+1} \) by a descending induction. It follows that \( X_{i+1} \subset X_i \) is a 1-residual inclusion for all \( 1 \leq i \leq e \). Thus, the chain of closed immersions \( \emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1 \) is a residual flag of type \( (n_1 + 1,1),(n_2 + 2,1),\ldots,(n_e + 1,1) \) in \( \mathbb{P}(E_T) \).

Example 1.7. We illustrate how to recursively construct the defining ideal for the closed subschemes in a residual flag. Let \( (n,d) := (3,2),(2,4) \) and set \( S = \text{Spec}(k) \) where \( k \) is a field. The residual flags of type \( (n,d) \) in \( \mathbb{P}^3 := \text{Proj}(k[x_0,x_1,x_2,x_3]) \) are nested pairs \( X_2 \subset X_1 \) in \( \mathbb{P}^3 \) such that \( X_2 \) is a planar curve of degree 4 and \( X_2 \) is a 2-residual scheme in \( X_1 \). The defining ideal of \( X_2 \) has the form

\[
I_{X_2} := \langle f_1, f_2 \rangle
\]

where \( f_1 \) is a homogeneous polynomial in \( k[x_0,x_1,x_2,x_3] \) of degree 1 and \( f_2 \) is a homogeneous polynomial in \( k[x_0,x_1,x_2,x_3] \) of degree \( d_2 = 4 \) that is not divisible by \( f_1 \). The 2-plane \( \mathbb{P}(F_2) \) containing \( X_2 \) is given by the vanishing of the linear form \( f_1 \). The defining ideal of the closed subscheme \( X_1 \) in \( \mathbb{P}^3 \) has the form

\[
I_{X_1} := g \cdot I_{X_2} = \langle g f_1, gf_2 \rangle
\]
where \( g \in k[x_0,x_1,x_2,x_3] \) is a homogeneous polynomial of degree \( d_1 = 2 \). Geometrically, the scheme \( X_1 \) is the union of the quadratic hypersurface defined by the vanishing of \( g \) and the planar quartic curve \( X_2 \). For the special configuration in which \( g = f_1^2 \), the defining ideal of the closed subscheme \( X_1 \) is \( \mathcal{I}_1 = \langle f_1^3, f_1^2 f_2 \rangle = \langle f_1^3 \rangle \cap \langle f_1^3, f_2 \rangle \).

**Functor of residual flags.** The pullback of a residual scheme is again a residual scheme; see [Fog68, Lemma 1.3]. It follows that residual flags define a contravariant functor from the category of \( S \)-schemes to the category of sets. Let \( (n,d) := (n_1,d_1), (n_2,d_2), \ldots, (n_e,d_e) \) be a sequence of pairs of positive integers such that \( n_1 > n_2 > \cdots > n_e \) and let \( E \) be coherent \( \mathcal{O}_S \)-module. For any \( S \)-scheme \( T \), the set of \( T \)-valued points of the functor \( \text{Flag}(n,d,\mathbb{P}(E)) \) is defined to be the set of residual flags of type \( (n,d) \) in \( \mathbb{P}(E_T) \).

**Lemma 1.8.** Assume that \( (n_1,d_1) \) is a pair of positive integers and \( E \) is a coherent sheaf on \( S \). Let \( F \) be the universal quotient sheaf on the Grassmannian \( \text{Gr}(n_1,\mathbb{P}(E)) \) and let \( F^* \) denote its dual.

(i) When \( d_1 > 1 \), the \( \mathbb{P}(\text{Sym}^{d_1}(F^*)) \) represents the functor of residual flags of type \( (n_1,d_1) \) in \( \mathbb{P}(E) \). The structure map of this \( S \)-scheme is the composition of the canonical maps \( \mathbb{P}(\text{Sym}^{d_1}(F^*)) \to \text{Gr}(n_1,\mathbb{P}(E)) \) and \( \text{Gr}(n_1,\mathbb{P}(E)) \to S \).

(ii) When \( d_1 = 1 \), the Grassmannian \( \text{Gr}(n_1-1,\mathbb{P}(E)) \) represents the functor of residual flags of type \( (n_1,1) \) in \( \mathbb{P}(E) \).

**Proof.** Let \( T \) be an \( S \)-scheme. Remark 1.5 shows that residual flags of type \( (n_1,1) \) in \( \mathbb{P}(E_T) \) are \( (n-1) \)-planes, so part (ii) follows immediately. Assume that \( d_1 > 1 \). A \( T \)-valued point of \( \mathbb{P}((\text{Sym}^{d_1}(F^*))) \) consists of a line bundle \( L_T \) on \( T \) and a surjection \( \text{Sym}^{d_1}((F_T)^*) \to L_T \) together with a \( T \)-valued point of \( \text{Gr}(n_1,\mathbb{P}(E)) \). Since \( (\text{Sym}^{d_1}(F^*))_T = \text{Sym}^{d_1}((F_T)^*) \), Lemma 1.1 demonstrates that \( L_T \) corresponds to a relative effective Cartier divisors of degree \( d_1 \) in the \( n_1 \)-plane \( \mathbb{P}(F_T) \). The \( T \)-valued point of \( \text{Gr}(n_1,\mathbb{P}(E)) \) corresponds to the \( n_1 \)-plane \( \mathbb{P}(F_T) \subseteq \mathbb{P}(E_T) \). Thus, the \( S \)-scheme \( \mathbb{P}((\text{Sym}^{d_1}(F^*))) \) represents the residual flags of type \( (n_1,d_1) \) in \( \mathbb{P}(E) \).

**Remark 1.9.** When the base scheme \( S \) is the spectrum of a field \( k \) and \( d_1 > 1 \), the parameter space for the residual flags of type \( (n_1,d_1) \) in \( \mathbb{P}^{m} := \text{Proj}(k[x_0,x_1,\ldots,x_m]) \) is the variety of degree \( d_1 \) hypersurfaces in \( n_1 \)-planes in \( \mathbb{P}^{m} \); see [Ful98, Example 14.7.12].

**Latent planes.** The definition of a residual flag \( \varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1 \) of type \( (n,d) \) in \( \mathbb{P}(E_T) \) includes the existence of flag of linear subspaces. For all \( 1 \leq i \leq e \), the scheme \( X_i \) lies in some \( n_i \)-plane \( \mathbb{P}(F_i) \subseteq \mathbb{P}(E_T) \). When \( d_e > 1 \), we refer to the set \( \{ \mathbb{P}(F_i) \mid 1 \leq i \leq e \} \) as the latent planes of the residual flag. In the special case \( d_e = 1 \), the scheme \( X_e \) is itself a \( (n_e-1) \)-plane and the latent planes are \( \{ X_e \} \cup \{ \mathbb{P}(F_i) \mid 1 \leq i \leq e-1 \} \).

**Lemma 1.10.** Let \( (n,d) \) be the type of a residual flag.

(i) When \( d_e > 1 \), there exists a morphism \( \text{Flag}(n,d,\mathbb{P}(E)) \to \text{Flag}(n,\mathbb{P}(E)) \) sending a residual flag to its flag of latent planes.

(ii) When \( d_e = 1 \), there exists a morphism \( \text{Flag}(n,d,\mathbb{P}(E)) \to \text{Flag}(n^e,\mathbb{P}(E)) \) sending a residual flag to its flag of latent planes where \( n^e := (n_1,n_2,\ldots,n_{e-1},n_e-1) \).

**Proof.** We need to show that the latent planes are unique and form a flag. Let \( T \) be an \( S \)-scheme and let \( \varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1 \) be a residual flag of type \( (n,d) \) in \( \mathbb{P}(E_T) \). Suppose that, for
some $1 \leq i \leq e$, the scheme $X_i$ is contained in two distinct $n_i$-planes $\mathbb{P}(F_i)$ and $\mathbb{P}(F'_i)$. The closed immersion $X_{i+1} \subset X_i$ is a $d_i$-residual inclusion in $\mathbb{P}(F_i)$, so there exists a relative effective Cartier divisor $D_i$ in $\mathbb{P}(F_i)$ such that $X_i$ is the union of $D_i$ and $X_{i+1}$. Since the codimension of $X_i$ in $\mathbb{P}(F_i)$ equals 1, we deduce that $X_i = D_i = \mathbb{P}(F_i) \cap \mathbb{P}(F'_i)$. It follows that either the $n_i$-plane containing $X_i$ is unique or $i = 1$, $D_1 = X_1$, and $d_1 = 1$. Thus, each scheme $X_i$ is contained in a unique plane having the dimension of its corresponding latent plane, so both assertions follow.

**Representability.** The pivotal result in this section shows that the functor of residual flags is representable. Moreover, it realizes this parameter space as a generalization of a partial flag variety.

**Proposition 1.11.** Assume that $(n, d) := (n_1, d_1), (n_2, d_2), \ldots, (n_e, d_e)$ is the type of a residual flag and $E$ is a coherent sheaf on $S$. For all $1 \leq i \leq e$, let $F_i^*$ denote the dual of the universal quotient sheaf on the Grassmannian $\text{Gr}(n_i, \mathbb{P}(E))$.

(i) When $d_e > 1$, we have the Cartesian square

$$
\text{Flag}(n, d, \mathbb{P}(E)) \longrightarrow \mathbb{P}(\text{Sym}^{d_1}(F_1^*)) \times_S \mathbb{P}(\text{Sym}^{d_2}(F_2^*)) \times_S \cdots \times_S \mathbb{P}(\text{Sym}^{d_e}(F_e^*))
$$

$$
\downarrow
$$

$$
\text{Flag}(n, \mathbb{P}(E)) \longrightarrow \text{Gr}(n_1, \mathbb{P}(E)) \times_S \text{Gr}(n_2, \mathbb{P}(E)) \times_S \cdots \times_S \text{Gr}(n_e, \mathbb{P}(E)).
$$

(ii) When $d_e = 1$, setting $n^* := (n_1, n_2, \ldots, n_{e-1}, n_e - 1)$ gives the Cartesian square

$$
\text{Flag}(n, d, \mathbb{P}(E)) \rightarrow \mathbb{P}(\text{Sym}^{d_1}(F_1^*)) \times_S \mathbb{P}(\text{Sym}^{d_2}(F_2^*)) \times_S \cdots \times_S \mathbb{P}(\text{Sym}^{d_{e-1}}(F_{e-1}^*)) \times_S \text{Gr}(n_e - 1, \mathbb{P}(E))
$$

$$
\downarrow
$$

$$
\text{Flag}(n^*, \mathbb{P}(E)) \longrightarrow \text{Gr}(n_1, \mathbb{P}(E)) \times_S \text{Gr}(n_2, \mathbb{P}(E)) \times_S \cdots \times_S \text{Gr}(n_{e-1}, \mathbb{P}(E)) \times_S \text{Gr}(n_e - 1, \mathbb{P}(E)).
$$

In both cases, the functor $\text{Flag}(n, d, \mathbb{P}(E))$ is represented by a projective $S$-scheme.

**Proof.** The bottom arrows in the diagrams are closed immersions; see [EGA71, Proposition 9.9.3]. To prove the projectivity of $\text{Flag}(n, d, \mathbb{P}(E))$, it is enough to show that these squares are Cartesian.

Assume that $d_e > 1$. Let $T$ be an $S$-scheme and let $\emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ be a residual flag of type $(n, d)$ in $\mathbb{P}(E_T)$. For all $1 \leq i \leq e$, there exists a relative effective Cartier divisor $D_i$ in $\mathbb{P}(F_i)$ such that $X_i = D_i \cup X_{i+1}$. Lemma 1.8 implies that the product of projective bundles

$$
P := \mathbb{P}(\text{Sym}^{d_1}(F_1^*)) \times_S \mathbb{P}(\text{Sym}^{d_2}(F_2^*)) \times_S \cdots \times_S \mathbb{P}(\text{Sym}^{d_e}(F_e^*))
$$

over $G := \text{Gr}(n_1, \mathbb{P}(E)) \times_S \text{Gr}(n_2, \mathbb{P}(E)) \times_S \cdots \times_S \text{Gr}(n_e, \mathbb{P}(E))$ parametrizes $e$-tuples of relative effective Cartier divisors of degree $d_i$ contained in some $n_i$-plane in $\mathbb{P}(E)$ for all $1 \leq i \leq e$. Hence, we have a morphism from $\text{Flag}(n, d, \mathbb{P}(E))$ to $P$ sending the residual flag $\emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ to the $e$-tuple $(D_1 \subset \mathbb{P}(F_1), D_2 \subset \mathbb{P}(F_2), \ldots, D_e \subset \mathbb{P}(F_e))$. Combined with the morphism in Lemma 1.10, we obtain a morphism from $\text{Flag}(n, d, \mathbb{P}(E))$ to the fibre product $\text{Flag}(n, \mathbb{P}(E)) \times_G P$.

We want to exhibit the inverse of this morphism. A $T$-valued point of this fibre product is a flag $\mathbb{P}(F_e) \subset \mathbb{P}(F_{e-1}) \subset \cdots \subset \mathbb{P}(F_1)$ of type $n$ in $\mathbb{P}(E_T)$ together with an $e$-tuple of relative effective Cartier divisors $D_i$ in $\mathbb{P}(F_i)$ for all $1 \leq i \leq e$. Setting $X_{e+1} := \emptyset$, we define $X_i := D_i \cup X_{i+1}$ for all $1 \leq i \leq e$ by descending induction. By construction, we have a chain $\emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ of closed immersions in $\mathbb{P}(E_T)$. Since Lemma 1.3 shows that, for all $1 \leq i \leq e$, the scheme $X_i$
is flat over $T$, this chain is a residual flag of type $(n,d)$ in $\mathbb{P}(E_T)$. As the pullback of a residual scheme is again a residual scheme, this construction is also functorial. We conclude that the scheme $\text{Flag}(n,d,\mathbb{P}(E))$ and the fibre product $\text{Flag}(n,\mathbb{P}(E)) \times_G P$ are isomorphic.

The proof for the case $d_e = 1$ is very similar. Since the scheme $X_e$ is a $(n_e - 1)$-plane in $\mathbb{P}(E_T)$, we have a natural morphism from $\text{Flag}(n,d,\mathbb{P}(E))$ to the product

$$\mathbb{P}(\text{Sym}^{d_1}(F^*_1)) \times_S \mathbb{P}(\text{Sym}^{d_2}(F^*_2)) \times_S \cdots \times_S \mathbb{P}(\text{Sym}^{d_{e-1}}(F^*_{e-1})) \times_S \text{Gr}(n_e, \mathbb{P}(E))$$

sending the residual flag to the tuple $(D_1 \subset \mathbb{P}(F_1), D_2 \subset \mathbb{P}(F_2), \ldots, D_{e-1} \subset \mathbb{P}(F_{e-1}), X_e \subset \mathbb{P}(E_T))$. Using Lemma 1.10, we obtain a morphism from $\text{Flag}(n,d,\mathbb{P}(E))$ to the appropriate fibre product. As above, one exhibits the inverse morphism by defining $X_i := D_i \cup X_{i+1}$ for all $1 \leq i \leq e - 1$. □

**Corollary 1.12.** Let $(n,d) := (n_1,d_1),(n_2,d_2),\ldots,(n_e,d_e)$ be the type of a residual flag and let $E$ be a locally free sheaf on $S$ of constant rank $n_0 + 1$ where $n_0 \geq n_1$.

(i) When $d_e > 1$, the structure map $\text{Flag}(n,d,\mathbb{P}(E)) \rightarrow S$ is smooth of relative dimension

$$\sum_{i=1}^{e} \left( \frac{n_i + d_i}{d_i} - 1 + (n_i + 1)(n_{i-1} - n_i) \right).$$

(ii) When $d_e = 1$, the structure map $\text{Flag}(n,d,\mathbb{P}(E)) \rightarrow S$ is smooth of relative dimension

$$-(n_{e-1} - n_e) + \sum_{i=1}^{e} \left( \frac{n_i + d_i}{d_i} - 1 + (n_i + 1)(n_{i-1} - n_i) \right).$$

**Proof.** For all $1 \leq i \leq e$, the sheaf $\text{Sym}^{d_i}(F^*_i)$ on the Grassmannian $\text{Gr}(n_i, \mathbb{P}(E))$ is locally free of constant rank $(n_i + d_i)$. Using the relative dimensions of flag varieties and Grassmannians, Proposition 1.11 shows that the map $\text{Flag}(n,d,\mathbb{P}(E)) \rightarrow S$ is smooth of the claimed relative dimensions. □

**Question 1.14.** When $E$ is locally free, the fibre product interpretation leads to a presentation for the Chow ring of $\text{Flag}(n,d,\mathbb{P}(E))$. Specifically, one combines the formula for the Chow ring of a projective bundle with the description of the Chow ring of a partial flag variety; see [Ful98, Examples 8.3.4 and 14.7.16]. Moreover, the cycle map on $\text{Flag}(n,d,\mathbb{P}(E))$ from its Chow ring to its integral cohomology ring is an isomorphism; see [Ful98, Example 19.1.11]. Which aspects of Schubert calculus on partial flag varieties extend to the parameter space of residual flags?

**2. HILBERT POLYNOMIALS AND RESIDUAL FLAGS**

Using the geometry of residual flags, we explain the combinatorial formula for the Hilbert polynomials. We show that the type of a residual flag encodes the Hilbert polynomial of its largest subscheme. We also prove that every lexicographic ideal (other than the zero ideal and unit ideal) determines a residual flag. Let $R := k[x_0,x_1,\ldots,x_m]$ denote the standard graded polynomial ring over a field $k$ and set $\mathbb{P}^m := \text{Proj}(R)$.
**Integer partitions.** We repackage the type of a residual flag as a single integer partition. Given a sequence \((n_1, d_1), (n_2, d_2), \ldots, (n_e, d_e)\) of pairs of positive integers such that \(n_1 > n_2 > \cdots > n_e > 0\), the integer partition \(\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)\) satisfies \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1\) and

\[
\lambda = (n_1, n_1, \ldots, n_1, n_2, n_2, \ldots, n_2, \ldots, n_e, n_e, \ldots, n_e).
\]

The length of \(\lambda\) is \(r := d_1 + d_2 + \cdots + d_e\). It can be convenient to use a notation for integer partitions that indicates the number of times each integer occurs as a part; see [Mac15, Subsection 1.1]. The expression \(\lambda = (\ldots, i^{a_i}, \ldots, 2^{a_2}, 1^{a_1})\) means that, for all positive integers \(i\), exactly \(a_i\) of the parts in \(\lambda\) are equal to \(i\). For instance, we have \(\lambda = (n_1^{d_1}, n_2^{d_2}, \ldots, n_e^{d_e})\).

To describe Hilbert polynomials, we treat a binomial coefficient with a variable in its numerator as a polynomial. Specifically, for all integers \(c\), we set

\[
\binom{t}{c} := \begin{cases} 
\frac{1}{c!} (t)(t-1) \cdots (t-c+1) & \text{if } c \geq 0 \\
0 & \text{if } c < 0.
\end{cases}
\]

The polynomial \(\binom{t}{c}\) has rational coefficients and degree \(c\).

**Lemma 2.1.** Let \((n, d)\) be the type of a residual flag and let \(\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)\) be its associated integer partition. For any residual flag \(\emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1\) of type \((n, d)\) in \(\mathbb{P}^m\), the Hilbert polynomial of the closed scheme \(X_1\) in \(\mathbb{P}^m\) is

\[
p(t) = \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1}.
\]

**Proof.** We proceed by induction on \(e\). The closed immersion \(X_2 \subset X_1\) is a \(d_1\)-residual inclusion in some \(n_1\)-plane contained in \(\mathbb{P}^m\). Hence, there exists a relative effective Cartier divisor \(D_1\) in this \(n_1\)-plane and a short exact sequence of sheaves

\[
0 \longrightarrow \mathcal{O}_{X_2}(-D_1) \longrightarrow \mathcal{O}_{X_1} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0.
\]

When \(e = 1\), we have \(X_2 = \emptyset\). The closed scheme \(X_1\) is the divisor \(D_1\), so its Hilbert polynomial is

\[
p(t) = \binom{t + n_1}{n_1} - \binom{t + n_1 - d_1}{n_1} = \sum_{i=1}^{d_1} \binom{t + n_1 - i + 1}{n_1} - \binom{t + n_1 - i}{n_1} = \sum_{i=1}^{d_1} \binom{t + n_1 - i}{n_1 - 1}.
\]

The integer partition associated to the residual flag \(\emptyset \subset X_1\) is \(\lambda = (n_1^{d_1})\), so the base case holds.

Suppose that \(e > 1\). The residual flag \(\emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_2\) has \(e - 1\) subschemes and its integer partition is \((n_2^{d_2}, n_3^{d_3}, \ldots, n_e^{d_e})\). The induction hypothesis implies that the Hilbert polynomial of the closed scheme \(X_2\) is \(\sum_{i=d_1+1}^{r} \binom{t + \lambda_i - i + d_1}{\lambda_i - 1}\). From the short exact sequence (2.1.1), we deduce that the Hilbert polynomial of the closed scheme \(X_1\) is

\[
p(t) = \sum_{i=d_1+1}^{r} \binom{t + \lambda_i - i + d_1}{\lambda_i - 1} + \sum_{i=1}^{d_1} \binom{t + n_j - i}{n_j - 1} = \sum_{i=1}^{r} \binom{t + \lambda_i - i}{\lambda_i - 1},
\]

\(\Box\)
Remark 2.2. Let $X$ be a closed subscheme in $\mathbb{P}^m$ having Hilbert polynomial $p(t) = \sum_{i=1}^r \binom{t+i}{\lambda_i-1}$. The dimension of $X$ is $\lambda_1 - 1$ and the degree of $X$ is the number $d_1$ of parts in $\lambda$ equal to $\lambda_1$. One verifies that the arithmetic genus of $X$ is

$$(-1)^{\lambda_1 - 1} \sum_{i=2}^r \binom{\lambda_i - i}{\lambda_i - 2} = \sum_{i=2}^r (-1)^{\lambda_1 - \lambda_i} \binom{i - 2}{\lambda_i - 1}.$$ 

Remarkably, the Gotzmann Regularity Theorem shows that the length $r$ of $\lambda$ is an upper bound on the Castelnuovo-Mumford regularity of the saturated ideal defining the closed subscheme $X$ in $\mathbb{P}^m$; see [Got78, Lemma 2.9] or [BH93, Theorem 4.3.2].

Lexicographic ideals. We identify a special residual flag by recognizing the geometric properties of a distinguished monomial ideal. The lexicographic order on the monomials in $R = k[x_0, x_1, \ldots, x_m]$ is defined by declaring $x_0^b_0 x_1^{b_1} \cdots x_m^{b_m} > x_0^{c_0} x_1^{c_1} \cdots x_m^{c_m}$ whenever the first nonzero entry in the integer sequence $(b_0 - c_0, b_1 - c_1, \ldots, b_m - c_m)$ is positive. A lexicographic ideal $I$ is a monomial ideal in $R$ such that, for all integers $j$, the homogeneous component $I_j$ is the $k$-vector space spanned by the largest $\dim_k I_j$ monomials in lexicographic order.

As the cornerstone of our approach, we recount a variant of the Macaulay characterization [Mac27] of the Hilbert functions for homogeneous ideals in polynomial ring $R$. Recall that the Hilbert function $h_{R/I}: \mathbb{Z} \to \mathbb{N}$ of a homogeneous ideal $I$ in $R$ is defined, for all integers $j$, by $h_{R/I}(j) := \dim_k (R/I)_j$.

Lemma 2.3. Let $p$ be a numerical polynomial having degree less than $m$. The following statements are equivalent.

(a) There exists a closed subscheme $X$ in $\mathbb{P}^m$ whose Hilbert polynomial is $p$.

(b) There exists an integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $p(t) = \sum_{i=1}^r \binom{t+i}{\lambda_i-1}$.

(c) There exists an integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $p(t) = \sum_{i=1}^r \binom{t+i}{\lambda_i-1}$ and a lexicographic ideal $L(\lambda)$ in $R$ such that $h_{R/L(\lambda)}(j) = \sum_{i=1}^r \binom{j+i}{\lambda_i-1}$ for all integers $j$.

Outline of Proof. We sketch the details because a proof may be derived from other accounts of Macaulay’s work via elementary identities for binomial coefficients; see [GKP94, Table 174].

$(a \Rightarrow b)$ Let $\ell$ be a fixed sufficiently large integer. One uses the $\ell$-th Macaulay representation for the integer $p(\ell)$ to obtain an integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $p(t) = \sum_{i=1}^r \binom{t+i}{\lambda_i-1}$; see [BH93, Lemma 4.2.6 and Corollary 4.2.14] or [Sta20, Proposition 2.3].

$(b \Rightarrow c)$ One verifies that the function $h: \mathbb{Z} \to \mathbb{N}$ defined, for all integers $j$, by $h(j) := \sum_{i=1}^r \binom{j+i}{\lambda_i-1}$ satisfies the Macaulay inequality $(h(j))^{(j)} \geq h(j)$; see [BH93, Theorem 4.2.10].

$(c \Rightarrow a)$ One takes $X$ to be the closed subscheme of $\mathbb{P}^m$ defined by the lexicographic ideal $L(\lambda)$. □

Remark 2.4. By using the conjugate integer partition, Corollary 5.7 in [Har66] gives an alternative condition equivalent to Lemma 2.3 (b); see [Sta20, Lemma 2.4].

We specify monomial generators and a primary decomposition of the lexicographic ideal $L(\lambda)$; these generators are also listed in [RS97].

Proposition 2.5. Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition and let $a_j$ be the number of parts in $\lambda$ equal to $j$, for all positive integers $j$. When $n \geq \lambda_1$, the corresponding lexicographic ideal is

$$L(\lambda) = \langle x_0^{a_0 m+1} x_1^{a_0 m-1+1}, \ldots, x_0^{a_m m-1}, \ldots, x_0^{a_m m-1}, \ldots, x_0^{a_0 m-1} x_1^{a_0 m-1} \cdots x_m^{a_m m-1} \rangle.$$
Moreover, the unique irredundant irreducible decomposition of this monomial ideal is
\[ L(\lambda) = \bigcap_{1 \leq i \leq m, a_i \neq 0} \langle x_0^{a_{m-1}+1}, x_1^{a_{m-2}+1}, \ldots, x_{m-i}^{a_i+1}, x_{m-i-1} \rangle. \]

Proof. We first establish the decomposition. As each intersectand is generated by powers of the variables and no two have the same dimension, this intersection is the irredundant irreducible decomposition of some monomial ideal. It remains to show that this ideal is \( L(\lambda) \). Since \( a_j = 0 \) for all \( j > \lambda_1 \), each irreducible ideal contains \( \langle x_0, x_1, \ldots, x_{m-\lambda_1-1} \rangle \) and we may assume that \( \lambda_1 = m \).

We proceed by induction on the number \( e \) of positive entries in \( (a_1, a_2, \ldots, a_m) \). When \( e = 1 \), the integer partition is \( (m^{a_m}) \) and the ideal is \( \langle x_0^{a_m} \rangle \). The principal ideal \( \langle x_0^{a_m} \rangle \) is lexicographic. Since the monomials \( \{ 1, x_0, \ldots, x_0^{a_m-1} \} \) form a basis as free \( k[x_1, x_2, \ldots, x_m] \)-module for the quotient \( R/\langle x_0^{a_m} \rangle \), it follows that \( h_{R/\langle x_0^{a_m} \rangle} (j) = \sum_{i=1}^{a_m} \binom{j+m-1}{j-i+1} \). By Lemma 2.3 (c), we deduce that \( L(m^{a_m}) = \langle x_0^{a_m} \rangle \) and the base case holds.

Now, assume that \( e > 1 \). Set \( I := L(m^{a_m}) = \langle x_0^{a_m} \rangle \). The induction hypothesis implies that
\[ J := \bigcap_{1 \leq i \leq m-1, a_i \neq 0} \langle x_0^{a_{m-1}+1}, x_1^{a_{m-2}+1}, \ldots, x_{m-i}^{a_i+1}, x_{m-i-1} \rangle \]
is the lexicographic ideal associated to the integer partition \( \nu := (\lambda_{a_m+1}, \lambda_{a_m+2}, \ldots, \lambda_r) \). Since the intersection of lexicographic ideals is again lexicographic, it is enough to prove that the ideals \( I \cap J \) and \( L(\lambda) \) have the same Hilbert function. From the short exact sequences of graded \( R \)-modules
\[
\begin{align*}
0 & \longrightarrow \frac{R}{I \cap J} \longrightarrow \frac{R}{I} \oplus \frac{R}{J} \longrightarrow \frac{R}{I+J} \longrightarrow 0 \quad \text{and} \\
0 & \longrightarrow \frac{R(-a_n)}{J} \longrightarrow \frac{R}{J} \longrightarrow \frac{R}{I+J} \longrightarrow 0,
\end{align*}
\]
we see that \( h_{R/\langle I \cap J \rangle} (j) = h_{R/I} (j) + h_{R/J} (j) - h_{R/(I+J)} (j) = h_{R/I} (j) + h_{R/J} (j-a_n) \) for all integer \( j \).

The equality \( J = L(\nu) \) implies that \( h_{R/J} (j) = \sum_{i=a_m+1}^{r} \binom{j+i-a_m-1}{j-i} \), so we deduce that
\[
h_{R/I \cap J} (j) = \sum_{i=1}^{a_m} \binom{j+\lambda_i - i}{j-i+1} + \sum_{i=a_m+1}^{r} \binom{j+\lambda_i - i}{j-i+1} = \sum_{i=1}^{r} \binom{j+\lambda_i - i}{j-i+1},
\]
which by Lemma 2.3 (c) completes the induction.

Lastly, we establish that the given set of monomials generate \( L(\lambda) \). The intersection of monomial ideals is generated by the least common multiples of their monomial generators, so we observe
\[
\begin{align*}
x_0^{a_{m-1}} x_1^{a_{m-2}} x_{m-i}^{a_{i+1}} & = \text{lcm}(x_0^{a_{m-1}}, x_1^{a_{m-2}}, \ldots, x_{m-i}^{a_{i+1}}, x_{m-i}^{a_{i+1}-1}, \ldots, x_0^{a_m}) \\
x_0^{a_{m-1}} x_1^{a_{m-2}} x_{m-2}^{a_{i+1}} & = \text{lcm}(x_0^{a_{m-1}}, x_1^{a_{m-2}}, \ldots, x_{m-2}^{a_{i+1}}, x_{m-2}^{a_{i+1}-1}, \ldots, x_0^{a_m}),
\end{align*}
\]
for all \( 2 \leq i \leq m \). Hence, each of the given monomials is a least common multiple of a generator from the irreducible components. Since each variable \( x_i \) appears as a minimal generator in an irreducible component with exponent either \( a_{m-i} \) or \( a_{m-1} + 1 \), we see that the least common multiple of any subset of generators for the irreducible components is divisible by at least one of the given monomials, so the opposite inclusion also holds.
We complete our converse to Lemma 2.1 by relating lexicographic ideals to residual flags. This geometric interpretation for the lexicographic ideal \( L(\lambda) \) appears to be new.

**Corollary 2.6.** Let \((n,d)\) be the type of a residual flag in \( \mathbb{P}^m \). For all \( 1 \leq i \leq e \), let \( X_i \) be the closed subscheme in \( \mathbb{P}^m \) defined by the lexicographic ideal \( L(n_i^{d_i}, n_{i+1}^{d_{i+1}}, \ldots, n_e^{d_e}) \). The chain of closed immersions \( \emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1 \) forms a residual flag of type \((n,d)\) in \( \mathbb{P}^m \).

**Proof.** For all \( 1 \leq i \leq e \), the irredundant irreducible decomposition in Proposition 2.5 implies that \( X_{i+1} \subset X_i \) and the monomial generators in Proposition 2.5 establish that the closed subscheme \( X_i \) is contained in the \( n_i \)-plane in \( \mathbb{P}^m \) defined by the monomial ideal \( \langle x_0, x_1, \ldots, x_{m-n_i} \rangle \). For all positive integers \( j \), let \( a_j \) denote the number of parts in the integer partition \((n_j^{d_j}, n_{j+1}^{d_{j+1}}, \ldots, n_e^{d_e})\) equal to \( j \). Restricting to the linear subspace \( \mathbb{P}^m := \text{Proj}(k[x_{m-n_i}, x_{m-n_i+1}, \ldots, x_m]) \) in \( \mathbb{P}^m \), Proposition 2.5 also shows that the closed subscheme \( X_i \) is defined by the monomial ideal

\[
I_i := \langle x_{m-n_i}^{a_{n_i}+1}, x_{m-n_i}^{a_{n_i}+1} x_{m-n_i-1}^{a_{n_i}+1}, \ldots, x_{m-n_i}^{a_{n_i}+1} x_{m-n_i-1}^{a_{n_i}+1} x_{m-3}^{a_3+1}, x_{m-n_i}^{a_{n_i}+1} x_{m-n_i-1}^{a_{n_i}+1} x_{m-2}^{a_2+1}, x_{m-n_i}^{a_{n_i}+1} x_{m-n_i-1}^{a_{n_i}+1} x_{m-1}^{a_1+1} \rangle.
\]

It follows that \( I_i = x_{m-n_i}^{a_{n_i}} J \) where

\[
J := \langle x_{m-n_i}^{a_{n_i}+1}, x_{m-n_i-1}^{a_{n_i}+1}, \ldots, x_{m-n_i-1}^{a_{n_i}+1} x_{m-3}^{a_3+1}, x_{m-n_i-1}^{a_{n_i}+1} x_{m-2}^{a_2+1}, x_{m-n_i-1}^{a_{n_i}+1} x_{m-1}^{a_1+1} \rangle
= \langle x_{n-n_i}, x_{n-n_i}, \ldots, x_{n-n_i} \rangle + I_{i+1}.
\]

Since \( a_{n_i} = d_i \), the closed immersion \( X_{i+1} \subset X_i \) is a \( d_i \)-residual inclusion in \( \mathbb{P}^m \). Therefore, the chain of closed immersions \( \emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1 \) is a residual flag of type \((n,d)\) in \( \mathbb{P}^m \).

**Remark 2.7.** By definition, the lexicographic point in a Hilbert scheme corresponds to the closed subscheme in \( \mathbb{P}^m \) defined by the lexicographic ideal. Theorem 1.4 in [RS97] proves that the lexicographic point is smooth, so this point lies on a unique irreducible component called the lexicographic component. Theorem 4.1 in [RS97] computes the dimension of this component. When \((n,d) := (n_1, d_1), (n_2, d_2), \ldots, (n_e, d_e)\) is the type of a residual flag, \( n_0 := m \), and \( \lambda \) is its associated integer partition, the lexicographic component determined by \( L(\lambda) \) in \( R := k[x_0, x_1, \ldots, x_m] \) is

\[
\begin{aligned}
\sum_{i=1}^{e} \left[ \left( \begin{array}{c} n_i + d_i \\ d_i \end{array} \right) - 1 + (n_i + 1)(n_{i-1} - n_i) \right] & \quad \text{if } n_e > 1 \text{ and } d_e > 1, \\
-(n_{e-1} - n_e) + \sum_{i=1}^{e} \left[ \left( \begin{array}{c} n_i + d_i \\ d_i \end{array} \right) - 1 + (n_i + 1)(n_{i-1} - n_i) \right] & \quad \text{if } n_e > 1 \text{ and } d_e = 1, \\
n_0 d_e + \sum_{i=1}^{e-1} \left[ \left( \begin{array}{c} n_i + d_i \\ d_i \end{array} \right) - 1 + (n_i + 1)(n_{i-1} - n_i) \right] & \quad \text{if } n_e = 1 \text{ and } d_{e-1} > 1, \\
n_0 d_e - (n_{e-2} - n_{e-1}) + \sum_{i=1}^{e-1} \left[ \left( \begin{array}{c} n_i + d_i \\ d_i \end{array} \right) - 1 + (n_i + 1)(n_{i-1} - n_i) \right] & \quad \text{if } n_e = 1 \text{ and } d_{e-1} = 1, \\
n_0 d_e & \quad \text{if } n_e = 1 \text{ and } e = 1.
\end{aligned}
\]

**Question 2.8.** The saturated monomial ideals defining residual flags of type \((n,d)\) in \( \mathbb{P}^m \) determine the torus-fixed points in the parameter space \( \text{Flag}(n,d,\mathbb{P}(E)) \). Following Proposition 2.5, the irredudant irreducible decomposition for each such monomial ideal has a combinatorial description. Can one use this perspective to count these monomial ideals and, thereby, compute the Euler characteristic of the projective scheme \( \text{Flag}(n,d,\mathbb{P}(E)) \)?
3. Geometry of smooth Hilbert schemes

In this section, we identify the Hilbert schemes isomorphic to a parameter space of residual flags. Exploiting this identification, we describe the closed subscheme corresponding to a general point on any smooth Hilbert scheme. Thus, we obtain a birational description of all smooth Hilbert schemes.

**Geometry.** Let $E$ be a locally free sheaf on a locally noetherian scheme $S$. The projective bundle $P(E)$ carries a tautological invertible sheaf that is relatively ample over $S$. We compute the Hilbert polynomials for closed subschemes in $P(E)$ relative to this tautological bundle. For any numerical polynomial $p$, the set of $T$-valued points of the functor $\text{Hilb}^p(P(E))$ is the set of closed subschemes $X \subseteq P(E_T)$ that are flat over the $S$-scheme $T$ and have Hilbert polynomial $p$. The $S$-scheme representing this functor is projective; see [Kol96, Theorem 1.4].

**Lemma 3.1.** Let $E$ be locally free sheaf on $S$ of constant rank $m + 1$ and fix a polynomial $p$ in $\mathbb{Q}[t]$. The Hilbert scheme $\text{Hilb}^p(P(E))$ is smooth over $S$ if and only if the fibre $\text{Hilb}^p(P^m)$ is nonsingular over every geometric point of $S$.

**Proof.** Let $X \subseteq P^m$ be a closed subscheme in the fibre of $P(E)$ over a geometric point in $S$. When $X$ corresponds to a smooth point on $\text{Hilb}^p(P^m)$, Theorem 2.10 in [Kol96] proves that the structure map $\text{Hilb}^p(P(E)) \to S$ is flat at this geometric point. Therefore, this structure map is smooth if and only if its the fibre is nonsingular over every geometric point.

**Theorem 3.2.** Let $(n,d)$ be the type of a residual flag and let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be its associated integer partition. Set $p(t) = \sum_{i=1}^r \binom{t + \lambda_i - 1}{\lambda_i - 1}$. Assume that $E$ is a locally free sheaf on $X$ of constant rank $m + 1$. The natural morphism

$$\pi: \text{Flag}(n,d,P(E)) \to \text{Hilb}^p(P(E))$$

sending a residual flag $\emptyset \subseteq X_e \subseteq X_{e-1} \subseteq \cdots \subseteq X_1$ to the closed subscheme $X_1 \subseteq P(E)$, is an isomorphism if and only if one of the two conditions holds:

1. $m \geq \lambda_1$ and $\lambda_r \geq 2$,
2. $\lambda = (1)$ or $\lambda = (m^{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $m \geq \lambda_{r-1} \geq 1$.

In both cases, the Hilbert scheme $\text{Hilb}^p(P(E))$ is smooth over $S$.

**Proof.** Suppose that $S = \text{Spec}(k)$ for some algebraically closed field $k$. Theorem 1.1 in [Sta20] demonstrates that Conditions 2 and 3 characterize when a nontrivial Hilbert scheme $\text{Hilb}^p(P^m)$ has a unique Borel-fixed point. Moreover, Lemma 5.6 in [Sta20] proves that the target $\text{Hilb}^p(P^m)$ is nonsingular and irreducible in this situation. Proposition 1.11 and Corollary 1.12 show that the source $\text{Flag}(\kappa,d,P(E))$ is a smooth projective variety. Since $\pi$ is injective, it is enough to certify that the dimensions of the source and target agree. Using Corollary 1.12 and equation (2.7.2), one verifies that the dimension of the lexicographic component in $\text{Hilb}^p(P^m)$ equals the dimension of the parameter space of residual flags if and only if Conditions 2 or 3 holds.

Suppose that $S$ is any locally noetherian scheme. Lemma 3.1 implies that the Hilbert scheme is smooth. Hence, the source and target of the morphism $\pi$ are smooth. Since the induced morphism on fibres over any geometric point is an isomorphism, the result follows. $\square$
Example 3.3. The conditions in Theorem 3.2 cover the well-known cases of hypersurfaces and Grassmannians. Consider an integer partition $\lambda = (\lambda'_1)$ and set $p(t) := \sum_{i=1}^{s} (t^{i+\lambda'_1-1})$. When $\lambda_1 = m$, each point in $\text{Hilb}^p(\mathbb{P}(E))$ corresponds to a hypersurface of degree $r$ in $\mathbb{P}(E)$; see Lemma 1.8. More generally, each point in $\text{Hilb}^p(\mathbb{P}(E))$ corresponds to a hypersurface of degree $r$ lying some $\lambda_1$-dimensional linear subspace of $\mathbb{P}(E)$. In the special case $r = 1$, the Hilbert scheme $\text{Hilb}^p(\mathbb{P}(E))$ is the Grassmannian parametrizing $(\lambda_1 - 1)$-dimensional linear subspaces in $\mathbb{P}(E)$. \hfill \Box

Example 3.4. For the integer partition $\lambda = (1^3)$, Theorem 3.2 shows that each point in the Hilbert scheme $\text{Hilb}^2(\mathbb{P}(E))$ correspond to a hypersurface of degree 2 lying on some line in $\mathbb{P}(E)$. Additionally, the Hilbert scheme of two points in $\mathbb{P}(E)$ is also known to be the blow-up of the diagonal of the quotient scheme $\mathbb{P}(E) \times_5 \mathbb{P}(E)/\mathfrak{S}_2$, where the symmetric group $\mathfrak{S}_2$ on two elements acts by permuting the factors in the product $\mathbb{P}(E) \times_5 \mathbb{P}(E)$.

Remark 3.5. The geometry of residual flags also explains why the morphism $\pi$ cannot be surjective when Conditions 2 and 3 fail to hold. To avoid these conditions, we may assume that $\lambda_r = 1$, $r \geq 3$, and $m > \lambda_{r-2}$. The smallest scheme $X_e$ in the residual flag is a degree $d_e$ hypersurface in a line. Hence, the map $\pi$ cannot be surjective when $d_e \geq 3$. When $d_e \leq 2$, there exists a plane $\Lambda_e$ containing $X_e$. The defining properties of a residual flag require the line $\Lambda_e$ to be contained in the latent plane $\Lambda_e - 1$ which by assumption has dimension less than $m$. Since a general line is not contained such a plane, the map $\pi$ also not surjective in this case.

Remark 3.6. The strategy outlined in Question 1.13 also leads to a description of the Chow ring (and integral cohomology ring) of the Hilbert schemes classified in Theorem 3.2.

Example 3.7. Two trivial Hilbert schemes, not covered by Theorem 3.2, are nevertheless particular Grassmannians. When $\lambda = (m+1)$ and $p(t) = t^{1+m}$, the Hilbert scheme $\text{Hilb}^p(\mathbb{P}(E))$ is a one point corresponding to closed subscheme $\mathbb{P}(E)$ itself. When $r = 0$, the Hilbert scheme $\text{Hilb}^0(\mathbb{P}(E))$ is a one point corresponding to empty scheme in $\mathbb{P}(E)$.

Birationality. Before examining the birational geometry of the other smooth Hilbert schemes, we remember that some Hilbert schemes split into a product. Let $\tilde{\lambda} = (m, \lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_r)$ be an integer partition with $m > \lambda_{s+1}$ and set $p(t) := \sum_{i=1}^{s} (t^{i+\lambda_{s+1}-1})$. We see that $p(t) = q_1(t) + q_2(t-s)$ where $q_1(t) := \sum_{i=1}^{s} (t^{i+\lambda_{s+1}-1})$ is the Hilbert polynomial for a hypersurface of degree $s$ in $\mathbb{P}(E)$ and $q_2(t) := \sum_{i=1}^{r-s} (t^{i+\lambda_{s+1}-1})$. Lemma 1.1 shows that the Hilbert scheme parametrizing of these hypersurfaces is $\mathbb{P}(\text{Sym}^s(E^*))$ and Remark 2 in [Fog68, p. 514] yields there is a natural splitting

\begin{equation}
\text{Hilb}^p(\mathbb{P}(E)) \cong \mathbb{P}(\text{Sym}^s(E^*)) \times \mathbb{P}(\text{Hilb}^q(\mathbb{P}(E))).
\end{equation}

Given an integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$, the new integer partition $\lambda \cup (1)$ is defined to be $(\lambda_1, \lambda_2, \ldots, \lambda_r, 1)$ and has length $r + 1$; see [Mac15, Subsection 1.1].

Proposition 3.8. Let $(n, d)$ be a residual type and $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be its integer partition. Set $p(t) := \sum_{i=1}^{s} (t^{i+\lambda_{i+1}-1})$. Assume that one of the following conditions holds:

1. $\lambda \cup (1) = (m^{r-s-3}, \lambda_{s+2}^{s+2}, 1)$ where $r - 3 \geq s \geq 0$, and $m - 1 \geq \lambda_{r-3-2} \geq 3$, or
2. $\lambda \cup (1) = (m^{r-s-5}, 2^{s+4}, 1)$ where $r - 5 \geq s \geq 0$. 


The Hilbert scheme \( \text{Hilb}^{n+1}(\mathbb{P}(E)) \) is smooth over \( S \). Moreover, a general point on this Hilbert scheme corresponds to the disjoint union of a residual flag of type \((n,d)\) and a point.

**Proof.** From the splitting in (3.7.3), it is enough to consider an integer partition \( \lambda = (\lambda_i) \) where \( r \geq 1 \) and \( \lambda_1 \geq 2 \), excluding the two integer partitions \((2^2)\) and \((2^3)\). By Lemma 3.1, we may assume that the base scheme is the spectrum of an algebraically closed field. Over a field of characteristic zero, the Theorem A in [Ram19] classifies all Hilbert schemes with precisely two Borel-fixed points; Theorem B in [Ram19] and Theorem 1.1 in [Sta19] also describe this classification over any algebraically closed field. Conditions (4) and (5) guarantee that the Hilbert scheme \( \text{Hilb}^{n+1}(\mathbb{P}(E)) \) has two Borel-fixed points. By computing the dimension of the tangent space at the non-lexicographic Borel-fixed point, Theorem A in [Ram19] demonstrates that \( \text{Hilb}^{n+1}(\mathbb{P}(E)) \) is nonsingular.

To understand a general point, consider the universal flag \( X_1 \) of type \((\lambda_1,s)\) in \( \mathbb{P}(E) \) and the universal closed subscheme \( Z \) having length one on \( \mathbb{P}(E) = \text{Hilb}^1(\mathbb{P}(E)) \). As the structure map is proper, their intersection determines a closed subset in the product \( \text{Flag}(\lambda_1,s,\mathbb{P}(E)) \times_S \mathbb{P}(E) \). Let \( U \) denote the open complement. There is a morphism \( \psi: U \rightarrow \text{Hilb}^{n+1}(\mathbb{P}(E)) \) induced by sending the pair \((X_1,Z)\) to their disjoint union. The source and target of \( \psi \) are smooth \( S \)-schemes and, using Corollary 1.12 and equation (2.7.2), one verifies that they have the same relative dimension. Over each geometric point in \( S \), the induced morphism on the fibres is an open immersion. We conclude that \( \psi: \text{Flag}(\lambda_1,s,\mathbb{P}(E)) \times_S \mathbb{P}(E) \rightarrow \text{Hilb}^{n+1}(\mathbb{P}(E)) \) is birational map. \( \square \)

**Remark 3.9.** Under the hypothesis of Proposition 3.8, the Hilbert scheme \( \text{Hilb}^{n+1}(\mathbb{P}(E)) \) and the product \( \text{Flag}(m,d,\mathbb{P}(E)) \times_S \mathbb{P}(E) \) are birational. However, these schemes are not isomorphic. For instance, the existence of two different Borel-fixed points on the Hilbert scheme implies that there is more than one way to embedded a point of multiplicity 1 into the lexicographic ideal; see Proposition 2.5 and Proposition 4.3.

**Example 3.10.** The integer partition \( \lambda = (1^r) \) is associated the constant Hilbert polynomial \( r \). The Hilbert scheme \( \text{Hilb}^r(\mathbb{P}(E)) \) is known to be smooth in two cases: Theorem 2.4 in [Fog68] applies when \( m = 2 \) and Equation (0.2.1) in [Che98] applies when \( r \leq 3 \). In either case, this Hilbert scheme is birational to the \( r \)-fold symmetric product \( \mathbb{P}(E) \times_S \mathbb{P}(E) \times_S \cdots \times_S \mathbb{P}(E) / \mathcal{G}_r \), where the symmetric group \( \mathcal{G}_r \) on \( r \) elements acts by permuting the factors in the product \( \mathbb{P}(E) \times_S \mathbb{P}(E) \times_S \cdots \times_S \mathbb{P}(E) \).

Using the splitting (3.7.3), this analysis extends to the integer partition \( \lambda = (m^{r-s},1^s) \) where \( r \geq s \geq 0 \). Set \( p(t) := \sum_{i=1}^{r} \binom{t+\lambda_i-1}{\lambda_i-1} \). Assuming \( m = 2 \) or \( r \leq 3 \), a general point on \( \text{Hilb}^d(\mathbb{P}(E)) \) corresponds to the disjoint union of a hypersurface of degree \( r-s \) and \( s \) isolated points. \( \diamond \)

**Remark 3.11.** The seven conditions in Theorem A imply that the Hilbert scheme \( \text{Hilb}^\eta(\mathbb{P}(E)) \) is smooth over \( S \): Example 3.10 handles Conditions 1 and 6, Theorem 3.2 handles conditions 2 and 3, Proposition 3.8 handles Conditions 4 and 5, and Example 3.7 handles Condition 7. In particular, we have a birational description for all of these smooth Hilbert schemes.

### 4. General classification

The final section completes our classification of smooth Hilbert schemes. By identifying enough singular points on Hilbert schemes, we prove that our list of smooth Hilbert schemes is exhaustive.
Nearly lexicographic points. We specify a novel point on a Hilbert scheme by perturbing a lexicographic ideal. Geometrically, this nearly lexicographic point corresponds to a residual flag with an embedded point whose nilpotent elements do not lie in the smallest linear subspace containing the residual flag.

Lemma 4.1. Let \( \lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r) \) be an integer partition and set \( p(t) := \sum_{i=1}^{r} (t^{i+\lambda_i-1}) \). Fix \( m > \lambda_1 \) and consider both the lexicographic ideal \( L(\lambda) \) and the monomial ideal

\[
J := \langle x_0, x_1, \ldots, x_{n-\lambda_1-2}, x_{m-\lambda_1-1}^2, x_{m-\lambda_1}, x_{n-\lambda_1+1}, \ldots, x_{n-1} \rangle
\]

in the polynomial ring \( R = k[x_0, x_1, \ldots, x_m] \). The closed subscheme in \( \mathbb{P}^m \) defined by the homogeneous ideal \( K := L(\lambda) \cap J \) has Hilbert polynomial \( p + 1 \) and corresponds to a point on the lexicographic component of the Hilbert scheme \( \text{Hilb}^{p+1}(\mathbb{P}^m) \).

Proof. By Lemma 2.3, the Hilbert polynomial of the closed subscheme defined by the ideal \( L(\lambda) \) is \( p(t) = \sum_{i=1}^{r} (t^{i+\lambda_i-1}) \). Proposition 2.5 implies that \( L(\lambda) + J = \langle x_0, x_1, \ldots, x_{m-1} \rangle \). For all integers \( j \) greater than 1, the sets \( \{ x_{m-\lambda_1-1}x_m^{j-1}, x_m^j \} \) and \( \{ x_m^j \} \) form bases for the \( j \)-th homogeneous components of \( R/J \) and \( R/(L(\lambda) + J) \) respectively. It follows that their Hilbert polynomials are the constants 2 and 1. From the short exact sequence of graded \( R \)-modules

\[
0 \longrightarrow \frac{R}{L(\lambda) \cap J} \longrightarrow \frac{R}{L(\lambda)} \oplus \frac{R}{J} \longrightarrow \frac{R}{L(\lambda) + J} \longrightarrow 0,
\]

we deduce that \( p + 1 \) is the Hilbert polynomial of the closed subscheme in \( \mathbb{P}^m \) corresponding to the monomial ideal \( K = L(\lambda) \cap J \).

Using Proposition 2.5, we also deduce that the saturation \( (L(\lambda) : x_m^{\infty-1}) \) is equal to the saturation \( (K : x_m^{\infty-1}) \). Since both \( L(\lambda) \) and \( K \) are Borel-fixed ideals, Theorem 6 in [Ree95] establishes that the point corresponding to the homogeneous ideal \( K \) lies on the lexicographic component. \( \square \)

Tangent spaces. We show that these nearly lexicographic points are singular for a special class of integer partitions. The Zariski tangent space at the point in the Hilbert scheme \( \text{Hilb}^{p}(\mathbb{P}^m) \) corresponding to the closed subscheme \( \mathcal{X} \) in \( \mathbb{P}^m \) with ideal sheaf \( \mathcal{I}_X \) is naturally isomorphic to \( \text{Hom}_{\mathbb{P}^m}(\mathcal{I}_X, \mathcal{O}_X) = \text{Hom}_X(\mathcal{I}_X, \mathcal{I}_X^2, \mathcal{O}_X) \); see [Kol96, Theorem 2.8].

Lemma 4.2. Let \( \lambda := ((m-1)^{r-s-1}, (m-n)^{s+1}) \) be an integer partition where \( r - 2 \geq s \geq 0 \) and \( m - 1 \geq n \geq 2 \). Set \( p(t) := \sum_{i=1}^{r} (t^{i+\lambda_i-1}) \). The Hilbert scheme \( \text{Hilb}^{p+1}(\mathbb{P}^m) \) is singular at the point corresponding to the saturated monomial ideal

\[
K := x_0 \cdot \langle x_0, x_1, \ldots, x_{m-1} \rangle + x_1^{r-s-1} \cdot \langle x_1, x_2, \ldots, x_{n-1}, x_n^{s+1} \rangle = L(\lambda) \cap \langle x_0^2, x_1, x_2, \ldots, x_{m-1} \rangle
\]

in the polynomial ring \( R = k[x_0, x_1, \ldots, x_m] \).

Proof. From the monomial generators for the lexicographic ideal \( L(\lambda) \) appearing in Proposition 2.5, we see that the given monomials generate the ideal \( K \). It remains to show that the dimension of the Zariski tangent space at the nearly lexicographic point is larger than the dimension of the Zariski
tangent space at the lexicographic point. Equation (2.7.2) establishes that the dimension of later is
less than or equal to (with equality holding when \( s > 0 \))

\[
(4.2.4) \quad \left( \frac{m + r - s - 2}{r - s - 1} \right) + \left( \frac{m - n + s + 1}{s + 1} \right) + (m - n + 1)(n - 1) + 2m - 2.
\]

To estimate the dimension of the Zariski tangent space at the nearly lexicographic point, we
examine the sheaf on \( \mathbb{P}^m \) corresponding to the graded \( R \)-module \( \text{Hom}_R(K, R/K) \). Since the variable \( x_m \) does not does any of the generators of the ideal \( K \), the dimension of this tangent space is
greater than or equal to \( \text{dim}_k \text{Hom}_R(K, R/K) \); see [RS97, Lemma 3.1]. Because \( K \) is a stable
monomial ideal, the Eliahou–Kervaire resolution [PS08, Theorem 2.3] yields a homogeneous free
presentation. The minimal syzygies among the generators of the ideal \( K \) are given by the block
matrix \( \Theta := [A_1 \ A_2 \ \cdots \ A_{m-1} \ B_1 \ B_2 \ \cdots \ B_{n-1} \ C] \) where

\[
A_i^T := \begin{bmatrix}
x_0^s x_0 x_1 & \cdots & x_0 x_n & x_0 x_{n+1} & \cdots & x_0 x_{m-1} & x_1^s x_1 x_2 & x_1 x_2 x_3 & \cdots & x_1 x_{m-1} x_{m+1} & \cdots & x_1 x_{m-1} x_{m+n} & x_1 x_{m-1} x_{m+n+1} & \cdots & x_1 x_{m-1} x_{m+n+s} \\
0 & \cdots & 0 & -x_j & x_j & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & -x_{j+1} & 0 & x_{j+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -x_{m-1} & 0 & 0 & \cdots & x_{m-1} & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

\[
B_j^T := \begin{bmatrix}
x_0^s x_0 x_1 & \cdots & x_0 x_n & x_0 x_{n+1} & \cdots & x_0 x_{m-1} & x_1^s x_1 x_2 & x_1 x_2 x_3 & \cdots & x_1 x_{m-1} x_{m+1} & \cdots & x_1 x_{m-1} x_{m+n} & x_1 x_{m-1} x_{m+n+1} & \cdots & x_1 x_{m-1} x_{m+n+s} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -x_j & x_j & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -x_{j+1} & 0 & x_{j+1} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -x_{n-1} & 0 & 0 & \cdots & x_{j-1} & 0 \\
\end{bmatrix}
\]

\[
C^T := \begin{bmatrix}
x_0^s x_0 x_1 & \cdots & x_0 x_n & x_0 x_{n+1} & \cdots & x_0 x_{m-1} & x_1^s x_1 x_2 & x_1 x_2 x_3 & \cdots & x_1 x_{m-1} x_{m+1} & \cdots & x_1 x_{m-1} x_{m+n} & x_1 x_{m-1} x_{m+n+1} & \cdots & x_1 x_{m-1} x_{m+n+s} \\
0 & \cdots & 0 & -x_j^s x_j x_n & 0 & 0 & 0 & 0 & \cdots & 0 & x_0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -x_{j+1} & 0 & 0 & \cdots & 0 & x_1 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -x_{j+1} & 0 & \cdots & 0 & x_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & x_{n-1} & 0 & \cdots & -x_{n+1} & x_{n-1} & 0 \\
\end{bmatrix}
\]

for all \( 1 \leq i \leq m - 1 \) and all \( 1 \leq j \leq n - 1 \). It follows that \( \text{Hom}_R(K, R/K) = \text{Ker}(\text{Hom}_R(\Theta, R/K)) \).

The two \( (m + n) \times 1 \)-matrices defined by

\[
D_0^T := \begin{bmatrix}
x_0^s x_0 x_1 & x_0 x_2 & \cdots & x_0 x_n & 0 & 0 & \cdots & 0 & x_0 \\
x_1 & x_2 & \cdots & x_n & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

\[
D_1^T := \begin{bmatrix}
x_0^s x_0 x_1 & x_0 x_2 & \cdots & x_0 x_n & 0 & 0 & \cdots & 0 & x_0 \\
0 & 0 & 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_{n-1} & x_{n+1} \\
\end{bmatrix}
\]

satisfy \( \Theta^T D_0 = 0 \) and \( \Theta^T D_1 = 0 \). Thus, for all \( 2 \leq i \leq m \), the column in the product \( x_i D_0 \) represents
a nonzero element in \( \text{Hom}_{\mathbb{P}^m}(K, R/K) \). The column in the product of the matrix \( D_1 \) with any
monomial of degree \( r - s - 1 \) in the variables \( x_1, x_2, \ldots, x_m \) (excluding \( x_1^{r-s-1} \)) also represents a
nonzero element in Hom$_R(K, R/K)_0$. There are $(m - 1) + \binom{m+n-s-2}{r-s-1} - 1$ columns of these products. Since all entries in the product $x_0 \Theta$ lie in the ideal $K$, the $m+n$ columns of the square matrix

\[
\begin{pmatrix}
  1 & 2 & \cdots & m & 1 & 2 & \cdots & n-1 & n \\
  x_0x_m & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
  0 & x_0x_m & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & x_0x_m & 0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 0 & x_0x_n^{r-s-1} & 0 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 0 & 0 & x_0x_n^{r-s-1} & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & x_0x_n^{r-s-1} & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & x_0x_n^{r-s-1} \\
\end{pmatrix}
\]

represent nonzero elements in Hom$_R(K, R/K)_0$. Similarly, all of entries in the bottom $n$ rows of the matrix $\Theta$ when multiplied by the monomial $x_1^{r-s-1}$ lie in the ideal $K$. Hence, for all $n \leq j \leq m$, the columns of the matrices

\[
\begin{pmatrix}
  1 & 2 & \cdots & n-1 \\
  0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

represent nonzero elements in Hom$_R(K, R/K)_0$. Each nonzero entry in the bottom row of the second matrix is the product of $x_1^{r-s-1}$ and a monomial of degree $s+1$ in the variables $x_n, x_{n+1}, \ldots, x_m$ (excluding $x_n^{s+1}$). Hence, there are $(m-n+s+1) - 1$ columns in this matrix.

Our total number of distinct columns representing nonzero elements in Hom$_R(K, R/K)_0$ is

\[
N := (m-1) + \binom{m+r-s-2}{r-s-1} - 1 + (m+n) + (m-n+1)(n-1) + \binom{m-n+s+1}{s+1} - 1.
\]

By comparing their nonzero monomial entries, we see that these $N$ columns are linearly independent. The difference between the number $N$ and (4.2.4) is $n - 1$. As $n \geq 2$, we conclude that the Hilbert scheme is singular at the point corresponding to the monomial ideal $K$. 

**Proposition 4.3.** Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be an integer partition such that $\lambda \cup (1)$ has at least three distinct parts or $\lambda := (\lambda_1^{r-s-1}, 1^{s+1})$ where $r - 2 \geq s \geq 0$ and $\lambda_1 > 1$. Fix $m > \lambda_1$, set $p(t) := \sum_{i=1}^{\lambda_1} \binom{t+i-1}{\lambda_1-1}$, and consider the monomial ideal

\[
J := \langle x_0, x_1, \ldots, x_{m-\lambda_1-2}, x_{m-\lambda_1-1}, x_{m-\lambda_1}, x_{m-\lambda_1+1}; \ldots, x_{m-1} \rangle.
\]
We may deform the scheme \( \mathbb{P}^m \). The Hilbert scheme \( \text{Hilb}^{p+1}(\mathbb{P}^m) \) is singular at the point corresponding to the saturated monomial ideal \( K := L(\lambda) \cap J \).

**Proof.** Lemma 4.1 shows that the nearly lexicographic point lies on the lexicographic component of the Hilbert scheme \( \text{Hilb}^{p+1}(\mathbb{P}^m) \). We reduce the analysis to a special case.

The inclusion \( (x_0, x_1, \ldots, x_{m-\lambda_1-2}) \subseteq K \) implies that the nearly lexicographic point is contained in a \((\lambda_1 + 1)\)-plane. Hence, we may assume that \( m = \lambda_1 + 1 \).

Assume that \( \lambda \cup (1) \) has at least three distinct parts. The lexicographic ideal \( L(\lambda \cup (1)) \) is the flat limit of a one-parameter family whose general member is the sum of the lexicographic ideal \( L(\lambda) \) and the ideal of a disjoint point. Since the dimension of the Zariski tangent space at a point in family is an upper-semicontinuous function, we may also assume that \( \lambda_r > 1 \). Applying Corollary 2.6, the lexicographic ideal \( L(\lambda \cup (1)) \) determines a residual flag \( \emptyset \subset X_e \subset X_{e-1} \subset \cdots \subset X_1 \) in \( \mathbb{P}^m \).

The hypotheses ensure that \( e \geq 3 \). The closed subscheme \( X_{e-2} \) lies in some linear space \( \Lambda \subseteq \mathbb{P}^m \). We may deform the scheme \( X_{e-2} \) in the linear space \( \Lambda \) and leave the rest of the residual flag \( X_{e-3} \subset X_{e-4} \subset \cdots \subset X_1 \) unchanged. If the closed scheme \( X_{e-2} \) corresponds to a singular point on the Hilbert scheme in \( \Lambda \), then it follows that the closed scheme \( X_1 \) corresponds to a singular point on the Hilbert scheme. Thus, we may assume that \( e = 3 \).

With these reductions, it suffices to consider the integer partition \( \lambda = ((m-1)^{r-s-1}, (m-n)^{s+1}) \) where \( r-2 \geq s \geq 0 \) and \( m-1 \geq n \geq 2 \). In this special case, Lemma 4.2 proves that the dimension of the Zariski tangent space at the nearly lexicographic point exceeds the dimension of the lexicographic component. Therefore, the Hilbert scheme \( \text{Hilb}^{p+1}(\mathbb{P}^m) \) is singular at the point corresponding to the saturated monomial ideal \( K := L(\lambda) \cap J \).

**Other singular examples.** In addition to our family of singular Hilbert schemes, the classification of smooth Hilbert schemes relies on three other singular families.

**Example 4.4.** Two familiar Hilbert schemes explain the curious gap between Conditions 4 and 5 in Proposition 3.8. By appealing to the splitting in (3.7.3), it is enough to understand integer partitions \((2,1),(2^2,1)\) and \((2^3,1)\). The first of these is already covered by both Theorem 3.2 and Example 3.10. In contrast, the Hilbert schemes in the other two cases are singular.

The integer partition \((2^2,1)\) is associated to the Hilbert polynomial \( 2r + 2 \). The Hilbert scheme \( \text{Hilb}^{2r+2}(\mathbb{P}^m) \) is singular; it has two irreducible components. A general point on one component corresponds to a pair of skew lines and a general point on the other corresponds to the union of a plane conic and an isolated point; compare with [CCN11, Theorem 1.1].

The integer partition \((2^3,1)\) is associated to the Hilbert polynomial \( 3r + 1 \). The Hilbert scheme \( \text{Hilb}^{3r+1}(\mathbb{P}^m) \) is again singular because it has two irreducible components. A general point in first component corresponds to a twisted cubic curve and a general point in the other corresponds to the union of a plane cubic and an isolated points; compare with [PS85, Theorem].

**Example 4.5.** For completeness, we also add an explicit description of another well-known singular Hilbert scheme. For any nonnegative integer \( s \), consider the integer partition \( \lambda = (s+4) \) whose associated Hilbert polynomial is the constant \( s + 4 \). For all \( m \geq 3 \), the Hilbert scheme \( \text{Hilb}^{s+4}(\mathbb{P}^m) \) is singular at the point corresponding to the saturated homogeneous ideal

\[
B(s) := \langle x_0, x_1, \ldots, x_{m-4}, x_{m-3}^2, x_{m-3}x_{m-2}, x_{m-3}x_{m-1}, x_{m-2}^2, x_{m-2}x_{m-1}, x_{m-1}^2 \rangle
\]
in the polynomial ring $R = k[x_0, x_1, \ldots, x_m]$; compare with [Che98, Lemma 1.4].

\textbf{Theorem 4.6.} Let $E$ be a locally free sheaf on a locally noetherian scheme $S$ of constant rank $m + 1$ and let $p$ be a numerical polynomial. The Hilbert scheme $\text{Hilb}^p(\mathbb{P}(E))$ is smooth and irreducible over $S$ if and only if there exists an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $p(t) = \sum_{i=1}^{r} \left( \frac{t^{i+1} - t}{\lambda_i} \right)$ and one of the following seven conditions holds:

1. $m > \lambda_1$
2. $m \geq \lambda_1$ and $\lambda_r \geq 2$,
3. $\lambda = (1)$ or $\lambda = (m^{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $m \geq \lambda_{r-1} \geq 1$,
4. $\lambda = (m^{r-s-3}, \lambda_{r-s-2}, 1)$ where $r - 3 \geq s \geq 0$ and $m - 1 \geq \lambda_{r-s-2} \geq 3$,
5. $\lambda = (m^{r-s-5}, 2s+4, 1)$ where $r - 5 \geq s \geq 0$,
6. $\lambda = (m^{r-3}, 1^3)$ where $r \geq 3$,
7. $\lambda = (m+1)$ or $r = 0$.

\textbf{Proof.} Remark 3.11 already shows that each condition implies that the Hilbert scheme is smooth. Hence, it suffices to prove that the Hilbert scheme has a singular point when the integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)$ does not satisfy Conditions 1–7. To bypass Conditions 1 and 2, we must have $m \geq 3$ and $\lambda_r = 1$. By Lemma 3.1, we may assume that $S$ is the spectrum of an algebraically closed field. Using the splitting in (3.7.3), we may also assume that $m > \lambda_1$. For the remaining integer partitions with one distinct part, Example 4.5 describes a singularity. When the integer partition has two distinct parts, there are three outstanding cases, namely $\lambda = (2^2, 1)$, $\lambda = (2^3, 1)$, or $\lambda = (\lambda_r^{r-3-2}, 1^{s+2})$ where $r - 1 \geq s \geq 0$ and $\lambda_1 > 1$. Example 4.4 and Proposition 4.3 exhibit their singularities. Finally, Proposition 4.3 also identifies a singular point whenever the integer partition has at least three distinct parts. \hfill \Box

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