HOMOGENEOUS TRANSFORMATION GROUPS OF THE SPHERE

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Abstract. In this paper, we study the structure of homogeneous subgroups of the homeomorphism group of the sphere, which are defined as closed groups of homeomorphisms of the sphere that contain the rotation group. We prove two structure theorems about the behaviour and properties of such groups and present a diagram of the structure of these groups partly on the basis of these results. In addition, we prove a number of explicit relations between the groups in the diagram.

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2010 Mathematics Subject Classification. 22F05 (primary), and 51H20 (secondary).
Key words and phrases. transformation groups, finite and infinite dimensional Lie groups, homeomorphisms and diffeomorphisms of surfaces.

Ferry Kwakkel was supported by FAPESP grant no. 2011/51671-7 and Fabio Tal was supported by CNPq grant no. 304474/2011-8 and FAPESP grant no. 2011/16265-8.
1. Introduction and statement of results

1.1. History. Let $\text{Homeo}(M)$ be the group of orientation preserving homeomorphisms of a closed orientable topological surface $M$, equipped with the topology of uniform convergence, with $M$ either the sphere $S^2$, the torus $\mathbb{T}^2$, or a higher genus surface $S_g$ of genus $g$, and let $\text{Diffeo}(M)$ be its subgroup of diffeomorphisms, equipped with the Whitney topology. A subgroup $G$ of $\text{Homeo}(M)$ of $\text{Diffeo}(M)$, is said to be \textit{closed} if it is a closed subset, in the uniform or Whitney topology, as a topological subspace and $G$ is said to be \textit{transitive} if for any two given $p,q \in M$, there exists a $h \in G$, such that $h(p) = q$. A closed transitive subgroup is said to be \textit{minimal}, respectively \textit{maximal}, if the group is minimal, respectively maximal, relative to inclusion of subgroups, with respect to the property of being a closed and transitive proper subgroup of $\text{Homeo}(M)$. It follows from Zorn’s Lemma that for every closed surface $M$, there exists at least one minimal closed and transitive subgroup other than the trivial group and at least one maximal closed and transitive subgroup other than the entire homeomorphism group of $M$. The general problem presented is to classify the closed and transitive subgroups, modulo conjugation, of the homeomorphism group $\text{Homeo}(M)$ of a closed surface and find its maximal subgroups. This problem can be seen as the global counterpart of the classification of Lie group actions on a closed surface, initiated by classical works of Lie [12] and further classified by Mostow [14]. Since $\text{Diffeo}^r(M) \subset \text{Homeo}(M)$ is dense for each $r \in [0,\omega]$ (see [15]), in many ways the distinction between considering homeomorphisms and diffeomorphisms in the context of classifying closed subgroups of $\text{Homeo}(M)$ can be blurred. Classifying closed and transitive subgroups of the homeomorphism group of a closed manifold is a particular problem in the study of general homeomorphism groups of manifolds, see the survey [5] by Fisher.

A closed subgroup $G \subseteq \text{Homeo}(S^2)$, with $S^2$ the two-sphere, that properly extends the rotation group is called a \textit{homogeneous transformation group}. A homogeneous transformation group $G \subset \text{Diffeo}(S^2)$ is said to be a smooth homogeneous transformation group. The purpose of this paper is to study homogeneous transformation groups, mainly topological but also smooth. In dimension one, subgroups of the homeomorphism group of the circle have been extensively studied, see [8] by Ghys and [16] by Navas, and in particular closed and transitive subgroups of the circle by Giblin-Markovic [7], answering a question posed in [8]. The results in [7] include that a closed and transitive subgroup of $\text{Homeo}(S^1)$, with $S^1$ the circle, that contains the rotation group $\text{Rot}(S^1)$, equals either $\text{Rot}(S^1)$, $\text{Möbiuss}(S^1)$, $\text{Homeo}(S^1)$, or one of its cyclic covers $\text{Möbiuss}_k(S^1)$ or $\text{Homeo}_k(S^1)$, with $k \geq 1$. In particular, the Möbius group $\text{Möbiuss}(S^1)$ acting on the circle is a maximal subgroup in $\text{Homeo}(S^1)$. Rebelo [20] using Lie algebra methods proved a similar result for diffeomorphisms acting transitively on four points on the circle. It
has been shown by Le Roux [21] that the group of area-preserving homeomorphisms of a closed triangulable manifold is a maximal subgroup of the homeomorphism group of that manifold.

1.2. Definitions and notation. The rotation group $\text{Rot}(S^2)$ is, modulo topological conjugation, the unique compact transitive and minimal subgroup of $\text{Homeo}(S^2)$ (see section 2.2) and is referred to as the kernel subgroup. The group acts transitively by isometries on the sphere. The projective plane $\mathbb{R}P^2$ is defined to be the quotient of the sphere $S^2$ with the antipodal action. Denote $\text{Lin}(\mathbb{R}P^2)$ the group of projective mappings, or collineations, which lifts to the degree two regular cover $S^2$ of $\mathbb{R}P^2$ as a group of orientation preserving homeomorphisms denoted $\text{Lin}(S^2)$, the linear group. The group $\text{Lin}(S^2)$ is a subgroup of $\text{Ant}(S^2)$, defined to be the centralizer of $\text{Homeo}(S^2)$ with respect to the antipodal action on the sphere. Note that $\text{Ant}(S^2) \cong \text{Homeo}(\mathbb{R}P^2)$. Further, there is the complex-projective action on the sphere, consisting of the group $\text{M"obius}(S^2)$ of Möbius transformations, which is precisely the group of all conformal homeomorphisms of $S^2$. To each of these closed and transitive groups of sphere corresponds its closed subgroup of area-preserving homeomorphisms. The area-preserving subgroups of the above groups that are transitive and not equal to $\text{Rot}(S^2)$ are $\text{Homeo}_\lambda(S^2) \subset \text{Homeo}(S^2)$, the subgroup of all area-preserving homeomorphisms of the sphere and, in the antipodal case, the group $\text{Ant}_\lambda(S^2) \subset \text{Ant}(S^2)$ of area-preserving and antipodal action preserving homeomorphisms of the sphere. We refer to section 2 for a detailed treatment of the basic groups. A group $G$ is said to be $n$-transitive, with $n \geq 1$, if the group $G$ can transport any $n$ given distinct points to any other $n$ distinct points by homeomorphisms in $G$, with $n$ being the maximal number of points for which this property holds. In the remainder of this paper, unless explicitly stated otherwise, $G$ denotes a homeomorphism group of the sphere containing $\text{Rot}(S^2)$, closed in either $\text{Homeo}(S^2)$ or $\text{Diffeo}(S^2)$. In particular, every such group $G$ is transitive.

1.3. Statement of results. It is known that a homogeneous group $G \subseteq \text{Homeo}(S^2)$ is homotopic to $\text{Rot}(S^2)$. The topological classification asks for the complete list and inclusions of the homogeneous subgroups of the sphere. Our first structure theorem is the following.

**Theorem A** (Structure of homogeneous transformation groups). Let $G \subseteq \text{Diffeo}(S^2)$ be a homogeneous transformation group, then either:

(I) $G$ is one-transitive on $S^2$ and $G \subseteq \text{Ant}(S^2)$, in which case $G$ is either:

(a) one-transitive on $\mathbb{R}P^2$ and $G = \text{Rot}(\mathbb{R}P^2)$,

(b) two-transitive on $\mathbb{R}P^2$ and $G = \text{Lin}(\mathbb{R}P^2)$,

(c) $G_2$ perturbs meridians and has essential support.

(II) $G$ is three-transitive on $S^2$ and $G = \text{M"obius}(S^2)$.

(III) $G_3$ has essential support.

The above types of groups are mutually exclusive.

In the above statement, we denote $G_k$ with $k \geq 1$, the stabilizer of $G$ on $k$ points respectively, where $k$ is counted in the sphere $S^2$ if $G$ is not contained in $\text{Ant}(S^2)$ and counted in $\mathbb{R}P^2$ if $G$ is contained in $\text{Ant}(S^2)$. Further, the condition of essential support for the stabilizer $G_3$ with $G$ of type III is defined as the support of the isotopies in $G_3$ that accumulate to $\{0, 1, \infty\}$ is contained in an essential continuum that has interior, where an essential continuum in $S^2 \setminus \{0, 1, \infty\}$ is a compact and connected set whose complement does not contain curves homotopic to one of the punctures. Similarly for type I. The known groups, taken together with Theorem A, suggest a diagram of homogeneous groups.
There is a more refined version of Theorem A in terms of sharp transitivity, where a group is said to be sharply \(k\)-transitive, for some \(k \geq 1\), if the stabilizer of \(k\) distinct points is the identity group.

**Theorem B** (Sharp transitivity of a homogeneous group). Let \(G \subseteq \text{Diffeo}(S^2)\) be a homogeneous transformation group. If \(G\) is sharply \(k\)-transitive, with \(k \geq 1\), and \(G\) is not contained in \(\text{Ant}(S^2)\), then \(G = \text{Möbius}(S^2)\) if \(k = 3\) and \(G\) is not sharply \(k\)-transitive for \(k \neq 3\). In addition, if \(G \subseteq \text{Ant}(S^2)\), then \(G\) is not sharply \(k\)-transitive, for any \(k \geq 1\).

Most arguments used in the proof of Theorem A and B are valid for topological homogeneous transformation groups, and only in certain details is some regularity required. The proof of Theorem B rests on the proof of Theorem A, together with dynamical properties of pseudo-Anosov homeomorphisms on surfaces [22], and the proofs are given in section 3. An arrow \(H \rightarrow K\) between two groups \(H, K\) is defined to be complete if \(H\) is maximal in \(K\), that is, if any closed group \(H \subset G \subseteq K\) has the property that \(G = K\). We prove the following set of implications in terms of the subgroups in the diagram.

**Theorem C** (Structure of the diagram). For topological homogeneous transformation groups of the sphere, each arrow in the diagram marked with \(*\) is complete and conversely each arrow not present between two groups in the diagram, other than \(\text{Rot}(S^2) \rightarrow \text{Homeo}(S^2)\), corresponds to a reduction to the rotation group.

These results form part of the problem that asks for a complete classification of the homogeneous groups of the sphere.

**Kernel subgroup problem.** Classify the topological homogeneous transformation groups of the sphere.

In particular, it would be interesting to determine whether the above diagram exhausts all possible homogeneous transformations groups in \(\text{Diffeo}(S^2)\), modulo taking the closure in \(\text{Homeo}(S^2)\). More specifically, as part of the above problem, we pose the following conjecture.

**Kernel subgroup conjecture.** The groups \(\text{Lin}(S^2)\) and \(\text{Möbius}(S^2)\) are maximal in \(\text{Ant}(S^2)\) and \(\text{Homeo}(S^2)\) respectively.
1.4. Remarks and applications. Most proofs we present are topological in nature. Several tools and applications that follow from our results are the following.

(*) Aleksandrov’s problem \[1\] asks whether a mapping \( f : X_1 \to X_2 \) between two metric spaces \((X_1, d_1)\) and \((X_2, d_2)\) such that \( f \) has one conservative distance \( \rho \), i.e. for all \( x, y \in X_1 \) with \( d_1(x, y) = \rho \) implies that \( d_2(f(x), f(y)) = \rho \), means that \( f \) has to be an isometry. One of the main topological tools used in the first part of the proof of Theorem A is a solution to this problem in the case of the sphere, which was in particular proved before in the case of Euclidean spaces in dimension at least two \[19\].

(*) The Möbius group \( G = \text{Möbius}(S^2) \) is sharply three-transitive, in that the stabilizer \( G_3 \) on three points is the identity group. As an application of Theorem A, this is the defining property of the Möbius group, in the sense that it is the unique group \( \text{Rot}(S^2) \subset G \subset \text{Diffeo}(S^2) \) that is sharply three-transitive.

(*) Similarly, the projective group \( G = \text{Lin}(S^2) \) descending to \( \mathbb{RP}^2 \) to a two-transitive action, sends great circles on the sphere to great circles. Conversely, by Theorem A, \( \text{Lin}(S^2) \) is the unique group \( \text{Rot}(S^2) \subset G \subset \text{Ant}(S^2) \), such that the stabilizer \( G_2 \) on two points on \( \mathbb{RP}^2 \) preserves meridians.

(*) Of the known explicit groups listed in the diagram, the groups contained in \( \text{Ant}(S^2) \) are precisely one-transitive and the other groups are at least three-transitive. As a further corollary to Theorem A, there are no two-transitive homogeneous groups in \( \text{Diffeo}(S^2) \), only one-transitive groups and at least three-transitive groups.

(*) Closed and transitive subgroups of the homeomorphism group of the sphere that do not arise as extensions of the rotation group are called inhomogenous groups. These groups are equivalently characterized as closed and transitive groups that do not contain a compact and transitive kernel group. It is a natural question whether or not there exist such inhomogeneous groups on the sphere. On the torus, the group of Hamiltonian homeomorphisms is an example of such a inhomogeneous group, see Appendix A, where we discuss several natural problems in the setting of higher genus surfaces. The homogeneous groups by definition, and the known inhomogeneous groups, on the sphere and higher genus surfaces, always contain isotopies.

(*) Call a group \( G \subset \text{Homeo}(S^2) \) acting on the sphere, or higher genus surface, an exotic group if the group \( G \) does not contain any arc. It would be interesting to prove the existence of an exotic closed and transitive group acting on the sphere or higher genus surface.

(*) Most proofs and results in this paper work for homeomorphism groups defined on higher dimensional closed manifolds as well.

2. Preliminary results

The purpose of this section is to collect basic properties of the groups under consideration that will be used at various stages. In particular, we consider groups preserving area and finite-dimensional Lie group actions. In what follows, fix the three points \( 0, 1, \infty \in S^2 \), where \( 0 \) is the southpole, \( \infty \) is the northpole and \( 1 \) is a point lying on the meridian \( \gamma_1 \) midway between \( 0 \) and \( \infty \), and \(-1\) is denoted to be antipodal point of \( 1 \), also lying on \( \gamma_1 \). Further, let \( 2, -2 \in \gamma_1 \) the

\[1\] The subgroup of Hamiltonian homeomorphisms, defined as the topological closure of the group of smooth Hamiltonian diffeomorphisms, themselves defined as the time-one maps of smooth Hamiltonian flows, is dense in \( \text{Homeo}_0(S^2) \), but this group of homeomorphisms is not dense in the topological sense on the torus or higher genus surfaces, due to the sphere being simply connected, see also Appendix A.
points midway between 1, −1. A circle on the sphere is the intersection of a Euclidean plane with the round sphere embedded in \(\mathbb{R}^3\), and this circle is a great circle if the plane passes through the origin of the sphere. Geodesics on the sphere are arcs of meridians, with meridian being synonymous to great circle. The connected components of the sphere minus a meridian are called hemispheres, and we denote \(H_0\) and \(H_\infty\) the connected components of \(S^2 \setminus \gamma_1\) containing 0 and \(\infty\) respectively. Given \(h_1, h_2 \in G\), we denote juxtaposition \(h_1 h_2\) as the composition \(h_1 \circ h_2 \in G\) of two homeomorphisms. Given a group \(G\), the isotopy subgroup denoted \(G^0 \subseteq G\) is defined as the subgroup of \(G\) of isotopies to the identity. Given \(G \subseteq \text{Homeo}(S^2)\), denote \(G_1 := \text{Stab}_G(\infty)\) the stabilizer of \(\infty \in S^2\) and similarly, \(G_2 := \text{Stab}_G(0, \infty)\) and \(G_3 := \text{Stab}_G(0, 1, \infty)\). Denote \(G^0_2 \subseteq G_2\) and \(G^0_3 \subseteq G_3\) the subgroups defined as the path connected component of the identity in \(G_2\) and \(G_3\) respectively. By definition, \(h \in G^0_2\) if there exists an isotopy \(H : [0, 1] \to G_2\) such that \(H(\cdot, 0) = \text{Id}\) and \(H(\cdot, 1) = h\), and similarly for \(G_3\). The orbit under \(G\) of a set \(V \subset S^2\) is denoted \(O_G(V)\). Given a group \(G\), denote \(\text{Stab}_G(\{p_1, \ldots, p_m\}) \subset G\) the stabilizer subgroup of \(G\) relative to the ordered tuple of points \(\{p_1, \ldots, p_m\} \subset S^2\) the subgroup of all homeomorphisms in \(G\) fixing the ordered tuple of points.

![Figure 2.1. The sphere with the canonical labeling of points, hemispheres, and meridians.](image-url)

### 2.1. Area-preserving homeomorphisms

An Oxtoby-Ulam measure is a finite Borel measure which gives zero measure to points and positive measure to sets containing interior. By von Neumann-Oxtoby-Ulam [17], given two Oxtoby-Ulam measures \(\mu_1, \mu_2\) on \(S^2\), there exists \(h \in \text{Homeo}(S)\) such that \(h_*(\mu_1) = \mu_2\) (see also [6]). A consequence in the setting of groups is the following, which justifies the use of the Lebesgue measure. A closed transitive group \(G\) preserves a positive finite Borel measure if and only if the measure is an Oxtoby-Ulam measure, in which case \(G\) is conjugate to a group preserving the Lebesgue measure. Furthermore, as \(\text{Rot}(S^2) \subset G\), the invariant measure necessarily has to be the Lebesgue measure. Indeed, if positive measure is given to a single point, by transitivity, the measure has to be infinite. Further, if zero measure is given to a set containing interior, taking a finite cover of the sphere by distributing this open set to cover the entire sphere, using compactness of the sphere and transitivity of the group, the entire sphere would have measure zero. Therefore, a closed transitive group \(G\) preserves a finite Borel measure if and only if it is an Oxtoby-Ulam measure and \(G\) is conjugate to a group \(G_\lambda\).
preserving the Lebesgue measure by the Oxtoby-Ulam theorem. Since $G$ contains $\text{Rot}(S^2)$ and the measure is invariant under $G$, it is in particular invariant under $\text{Rot}(S^2)$. Therefore, such a measure has to be constant over the entire sphere, and thus a positive scalar multiple of the Lebesgue measure, which we may assume is normalized by one. It is known that the group of area-preserving diffeomorphisms is a simple group (contains no proper normal subgroups), but this is still unknown for the group of area-preserving homeomorphisms, see [6].

2.2. **The rotation group.** It is a result by Belliart [3], that a Lie group acting continuously on the sphere with fixed points has to be either $\text{SO}(3, \mathbb{R})$, $\text{PGL}(3, \mathbb{R})$ or $\text{PGL}(2, \mathbb{C})$, see Appendix A. The sphere $S^2$ can be defined as the set of points in $\mathbb{R}^3$ with unit distance from the origin. Consider the general linear group $\text{GL}(3, \mathbb{R})$. The simplest transitive Lie subgroup is the subgroup $\text{SO}(3, \mathbb{R}) \subset \text{GL}(3, \mathbb{R})$, which acts by rotations, and hence by isometries, on the sphere $S^2$ and we have that $\text{SO}(3, \mathbb{R}) \cong \text{Rot}(S^2)$. The rotation group $\text{Rot}(S^2)$ is of special interest since it is the only compact transitive (Lie) subgroup of $\text{Homeo}(S^2)$, and it is minimal in that it does not contain proper compact transitive subgroups. To start with, we provide the basic structure of the rotation group used as a basis for all other closed and transitive subgroups we will be considering.

**Proposition 2.1** (The rotation group). The rotation group $G = \text{Rot}(S^2)$ acting on the sphere is minimal in $\text{Homeo}(S^2)$ and is the unique compact transitive subgroup of $\text{Homeo}(S^2)$

A rotation $r \in \text{Rot}(S^2)$ taking a point $p \in S^2$ to a point $q \in S^2$ can be decomposed into $r = r_a r_b r_c$, where $r_a \in \text{Stab}_{\text{Rot}}(p)$ is an element of the stabilizer of $p$, $r_b$ is a rotation moving $p$ to $q$ along a meridian passing through $p$ and $q$, and $r_c \in \text{Stab}_{\text{Rot}}(q)$ is an element of the stabilizer of $q$. Furthermore, this composition is unique up to pre-composition with an element of $\text{Stab}_{\text{Rot}}(p)$, the translation along the meridian is unique unless $p$ and $q$ are antipodal. In other words, every rotation can be decomposed as an action stabilizer-translation-stabilizer.

**Lemma 2.1.** The group $\text{Rot}(S^2)$ is a minimal closed and transitive subgroup of $\text{Homeo}(S^2)$.

**Proof.** To show that $\text{Rot}(S^2)$ is minimal, suppose that $G \subseteq \text{Rot}(S^2)$ is closed and transitive. Since $\text{Rot}(S^2)$ is compact and $G$ closed, $G$ is compact also. By transitivity of $G \subseteq \text{Rot}(S^2)$ and the stabilizer-translation-stabilizer decomposition of every rotation, it suffices to show that there exists $p \in S^2$ such that $\text{Stab}_G(p) \cong \text{SO}(2, \mathbb{R})$, since stabilizers are isomorphic at points along the same orbit, which is the entire sphere by transitivity. Suppose that there is a uniform upper bound on the order of the stabilizer of a $p \in S^2$. In that case we have $\text{Stab}_G(p) \cong \mathbb{Z}/n\mathbb{Z}$ for every $p \in S^2$. Take two meridians $\gamma_1$ and $\gamma_\alpha$ both passing through 0 and $\infty$ and whose angle is $0 < \alpha < \pi/2$, and let $\gamma_2$ be the horizontal meridian passing through 1 in $S^2$, and denote $p_\alpha$ the intersection point of $\gamma_\alpha$ with $\gamma_2$ nearest to 1. Denote $v_1 \in T_\infty S^2$ the vector tangent to $\gamma_1$. Given $\alpha > 0$, the parallel transport of the vector $v_1$ after transporting it along $\gamma_1$ to 1, then transporting it along $\gamma_2$ to $p_\alpha$ and then moving it up again along $\gamma_\alpha$ back to $\infty$ gives precisely an angle defect of $\alpha$. By transitivity of $G$, there exists a rotation $r_1 \in G$ taking $\infty$ to 1, and for each $\alpha$, there exists a rotation $r_\alpha \in G$ taking 1 to $p_\alpha$, and a rotation $r_\infty \in G$ taking $p_\alpha$ back to $\infty$. The composition $r_\infty r_\alpha r_1$ leaves $\infty$ fixed so is contained in the stabilizer.

Note that, since the groups acting on the sphere are groups of orientation preserving homeomorphisms by definition, the groups $\text{SO}(3, \mathbb{R})$, $\text{PGL}(3, \mathbb{R})$ and $\text{PGL}(2, \mathbb{C})$ are defined as the index two subgroup of matrices whose determinant is strictly positive.
of \( \infty \). Since the stabilizers at 0, 1, \( \infty \) are all periodic by assumption, by choosing \( \alpha \) irrational, we can produce an irrational element in the stabilizer of \( \infty \). Taking powers and subsequences, by compactness, we can produce any rotation we desire at \( \infty \), i.e. \( \text{Stab}_G(\infty) \cong SO(2, \mathbb{R}) \), and therefore \( \text{Stab}_G(p) \cong SO(2, \mathbb{R}) \) at every point \( p \in S^2 \). Subsequently, since \( S^2 = SO(3, \mathbb{R})/SO(2, \mathbb{R}) \), we have that \( G = SO(3, \mathbb{R}) \). \hfill \Box

Compact groups are conjugate to rotation groups, see e.g. [8] for circle groups.

Lemma 2.2. A compact group \( G \subset \text{Homeo}(S^2) \) is topologically conjugate to a subgroup of \( \text{Rot}(S^2) \).

Proof. Let \( \lambda_G \) be the Haar measure on \( G \), which exists as \( G \) is compact and assume it to be normalized so that \( \int_G d\lambda_G = 1 \). Each homeomorphism \( g \in G \) induces a measure \( g_*(\lambda) \), defined by \( g_*(\lambda)(U) := \lambda(g^{-1}(U)) \), for measurable \( U \subset S^2 \), where \( \lambda \) is the Lebesgue measure on \( S^2 \). Since \( G \) is compact, we can define the average \( \mu := \int_G g_*(\lambda) d\lambda_G \), which is a Borel probability measure on \( S^2 \), invariant under \( G \), that has no atoms and gives positive finite measure to sets that have interior. Indeed, suppose that \( \mu \) gives zero measure to a set containing interior. Since \( \mu \) is defined as an average over induced measures \( g_*(\lambda) \), there must exist a sequence \( g_n \in G \) of homeomorphisms such that their induced measure on a set that has interior converging to zero. Since \( G \) is compact, there exists a limiting homeomorphism \( g_n \to g_0 \) whose induced measure over a set containing interior has zero measure, which is impossible as this set is pulled back by \( g_0 \) to a set containing interior, which must have positive Lebesgue measure. Similarly, since each homeomorphism in the average gives zero measure to points, the average \( \mu \) gives zero measure to points as well. Therefore, by the Oxtoby-Ulam theorem, there exist a homeomorphism \( h: S^2 \to S^2 \) such that \( h_*(\mu) = \lambda \) and this homeomorphism \( h \) conjugates \( G \) to a group \( G_0 \) of homeomorphisms preserving the Lebesgue measure, and therefore \( G_0 \subseteq \text{Rot}(S^2) \) is a subgroup of the rotation group, as required. \hfill \Box

Proof of Proposition 2.1 Since \( G \) is compact, by Lemma 2.2 \( G \) is conjugate to a group \( G_0 \subseteq \text{Rot}(S^2) \). Since \( G \) is transitive, \( G_0 \) is transitive and thus \( G_0 = \text{Rot}(S^2) \) by Lemma 2.1. Therefore, \( G \) is conjugate to \( \text{Rot}(S^2) \) as required. \hfill \Box

### 2.3. The real-linear action.

If we take an element \( T \in \text{GL}(3, \mathbb{R}) \), then \( T \) does not leave the unit sphere invariant unless \( T \in SO(3, \mathbb{R}) \). The action of \( \text{PSL}(3, \mathbb{R}) \), which is defined to be the centralizer in \( \text{GL}(3, \mathbb{R}) \) of the homothetic action \( p \sim q \) if and only if \( p = q \) with \( p, q \in \mathbb{R}^3 \) and \( \lambda \in \mathbb{R}_* \), does leave the sphere \( S^2 \) invariant and induces an action by homeomorphisms on the sphere. The real-projective plane \( \mathbb{RP}^2 \) is defined as the sphere modulo the antipodal action \( p \sim q \) if and only if \( p = -q \). The action \( \text{PSL}(3, \mathbb{R}) \) acts on \( S^2 \) and passes to the quotient \( \mathbb{RP}^2 \). A fundamental domain \( U \) of \( \mathbb{RP}^2 \) in the sphere \( S^2 \) is a domain \( U \subset S^2 \) not containing antipodal points. The action of an element \( T \in \text{GL}(3, \mathbb{R}) \)

\[
T = \begin{pmatrix} c_{11} & c_{12} & \tau_1 \\ c_{21} & c_{22} & \tau_2 \\ d_1 & d_2 & 1 \end{pmatrix},
\]

where \( \det(c_{ij}) > 0 \), induces an action \( h_T \) on an affine chart \( \mathbb{R}^2 \subset \mathbb{RP}^2 \) given by

\[
h_T(x, y) \mapsto \left( \frac{\tau_1 + c_{11}x + c_{12}y}{1 + d_1x + d_2y}, \frac{\tau_2 + c_{21}x + c_{22}y}{1 + d_1x + d_2y} \right),
\]
The mapping $h_T$ defined by (2.1) takes the line
\[ \ell = \{(x, y) \in \mathbb{R}^2 \mid 1 + d_1 x + d_2 y = 0\} \subset \mathbb{R}^2 \]
to the line at infinity. Thus the projective plane is obtained as the affine plane $\mathbb{R}^2$ augmented with a line at infinity, i.e. $\mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1$. We have $\text{Lin}(\mathbb{R}P^2) \cong \text{PGL}(3, \mathbb{R})$ and $\text{Lin}(\mathbb{R}P^2) \subset \text{Homeo}(\mathbb{R}P^2)$ the group of orientation-preserving collineations, or projective transformations of $\mathbb{R}P^2$. As the sphere $S^2$ is the two-sheeted regular cover of the projective plane $\mathbb{R}P^2$, the group $\text{Lin}(\mathbb{R}P^2)$ lifts to a group of homeomorphisms $\text{Lin}(S^2)$ of the sphere $S^2$. A hemisphere in $S^2$ corresponds precisely to an affine chart $\mathbb{R}^2 \subset \mathbb{R}P^2$. After a rotation of the sphere, we may assume that $0 \in \mathbb{R}^2$ corresponds to the south-pole of the sphere, that is, to $0 \in S^2$. An action $h \in \text{Lin}(S^2)$, since it preserves the antipodal action and meridians, can modulo rotations always be assumed to fix $0$ and one vertical meridian passing through $0$ and $\infty$ in $S^2$. There is a natural group isomorphism $\text{SL}(3, \mathbb{R}) \cong \text{PGL}(3, \mathbb{R})$, with $\text{SO}(3, \mathbb{R})$ descending to an action $\text{PSO}(3, \mathbb{R})$.

**Lemma 2.3.** Area-preserving actions in $\text{Lin}(S^2)$ correspond precisely to rotations.

**Proof.** Without loss of generality, we may assume that $h \in \text{Stab}_{\text{Lin}}(0, \infty)$. In order for the action to be area-preserving, since $0$ and $\infty$ are preserved, the vector $(0, 0, -1)$ corresponds to $0 \in S^2$, which is an eigenvector with eigenvalue $1$ of $T$, at $0$ we must have $(c_{ij}) \in \text{SL}(2, \mathbb{R})$. That is,
\[
(2.3) \quad T = \begin{pmatrix}
c_{11} & c_{12} & 0 \\
c_{21} & c_{22} & 0 \\
d_1 & d_2 & 1
\end{pmatrix} \in \text{SL}(3, \mathbb{R}),
\]
since $\det(T) = \det(c_{ij}) = 1$. After pre- and postcomposition by rotations, we may assume that $T$ is of diagonal form with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$, with $\lambda_1 < 1$ and $\lambda_3 > 1$, and corresponding to mutually perpendicular eigenvectors $v_1, v_2, v_3$. $T$ maps the unit sphere in $\mathbb{R}^3$ to an ellipsoid in $\mathbb{R}^3$. Take the axis corresponding to the smallest eigenvalue $\lambda_1$ and take the plane $P_{12}$, $P_{13}$ passing through the sphere through the axes corresponding to vectors $(v_1, v_2)$ and $(v_1, v_3)$ respectively. In each plane, the mapping $T$ acts as a linear mapping with eigenvalues in the corresponding eigenvector direction. Since the direction $v_1$ corresponds to the smallest eigenvalue, the ellipse in the plane $P_{12}$ is squashed most in the direction $v_1$. Similarly in the plane $P_{13}$, and in this case the squashing is strictly larger. It is seen that in the two perpendicular directions the length of a small arc is increased in the projective action, and this is strict in the $(v_1, v_3)$ direction. Since this product equals (infinitesimally) the Jacobian of the mapping, the Jacobian at this point is strictly positive. Therefore, the mapping does not preserve area, unless $\lambda_1 = \lambda_2 = \lambda_3$. Since $T \in \text{SL}(3, \mathbb{R})$, we must have $\lambda_1 = \lambda_2 = \lambda_3 = 1$, which corresponds precisely to the mapping being a rotation. \qed

Record the transitivity properties of the linear group.

**Lemma 2.4** (Transitivity of $\text{Lin}(S^2)$). The group $\text{Lin}(S^2) \subset \text{Ant}(S^2)$ is one-transitive on $S^2$ and $\text{Lin}(\mathbb{R}P^2)$ is two-transitive on $\mathbb{R}P^2$. Further, $\text{Lin}(\mathbb{R}P^2)$ is almost three-transitive on $\mathbb{R}P^2$, meaning that every three non-collinear points can be sent to any three other non-collinear points.

**Proof.** A hemisphere $H \subset S^2$ can be represented as a copy of $\mathbb{R}^2$ in affine coordinates. Two points in the hemisphere then correspond to two points in $\mathbb{R}^2$ lying on a unique straight line in $\mathbb{R}^2$, which is verified can be sent to any other two points in $\mathbb{R}^2$. Sharpness of 2-transitivity follows since three points on a straight line can not be sent to points not lying on a straight
line, since in affine coordinates the action of Lin($S^2$) sends straight lines to straight lines. To
prove the last claim, note that every conic in $\mathbb{R}^2$ can be sent by an element of Lin($S^2$) back to
the unit circle centered at $0 \in \mathbb{R}^2$ corresponding to $0 \in S^2$. Therefore, three points lying on a
conic can be sent to three points on the unit circle centered at $0 \in \mathbb{R}^2$. Taking a pair of such
conics, since the group of collineations leaving invariant the unit circle is the Klein group which
is 3-transitive on the circle, there is an induced action between the two conics sending one triple
of points onto the other triple of points.

Lastly, linear mappings are defined in terms of their action on meridians.

**Lemma 2.5** (Linear mappings and meridians). $G$ preserves meridians if and only if $G \subseteq Lin(S^2)$.

**Proof.** First, we claim that if $G$ preserves meridians, then $G \subseteq Ant(S^2)$. Indeed, suppose that
$G$ preserves meridians and take $h \in G$ with the property that $h(\infty) = \infty$. Then any meridian
passing through $\infty$ also has to pass through 0, since a meridian contains it antipodal points as
well. Therefore, if we take another meridian also through $\infty$, then the intersection of the two
meridians has to equal $\{0, \infty\}$. Since $h$ is a homeomorphism preserving meridians, the images of
the two original meridians are again two meridians and intersection points of the two meridians
are sent to intersection points. Since $\infty$ is preserved by $h$, and the two image meridians pass
through $\infty$, these intersect in the unique antipodal points $0 \in S^2$. Since intersection points
have to be mapped to intersection points, the claim follows. This proves the first claim that
$G \subseteq Ant(S^2)$.

Since Ant($S^2$) $\cong$ Homeo($\mathbb{R}P^2$), identify $h \in G \subseteq Ant(S^2)$ with $h \in G \subseteq Homeo(\mathbb{R}P^2)$. Consider
the action of $h$ in an affine chart $\mathbb{R}^2$, where the chart is chosen so that $0 \in S^2$
corresponds to $0 \in \mathbb{R}^2$. The meridians passing through 0 and $\infty$ in $S^2$ correspond precisely
to the pencil of full straight lines passing through $0 \in \mathbb{R}^2$. Since Rot($S^2$) induces a transitive
action on $\mathbb{R}P^2$ (and preserves meridians), we may assume that $h$ in $S^2$ preserves the horizontal
meridian and is thus a homeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$ in the domain and image chart. If
$h$ sends straight line segments to straight line segments in $\mathbb{R}^2$, then $h$ is the restriction to $\mathbb{R}^2$
of a projective mapping $[18]$. Therefore, either $h \in Lin(S^2)$, or else there exist a line segment
$\ell \subset \mathbb{R}^2$ such that the image $h(\ell)$ is not a straight line segment. The action induced by Rot($S^2$)
by projective mappings on $\mathbb{R}P^2$ sends straight lines to straight lines, and thus we can pre- and
postcompose the action of $h$ by projective mappings to assume that $\ell$ and $h(\ell)$ pass through the
origin. Since $\ell$ can be completed to a full straight line lifting to a geodesic in $S^2$ and since the
image segment $h(\ell)$ is not a straight line segment, the action of $h$ on the sphere $S^2$ is such that
it sends a meridian into a non-meridian, contrary to our assumption. Therefore, we must have
that $h \in Lin(S^2)$, as required.

2.4. The complex-linear action. Next, consider the complex actions on the sphere induced
by GL(2, $\mathbb{C}$). Identify the sphere $S^2$ with the complex plane $\mathbb{C}$ compactified with a point $\{\infty\}$
at infinity. Taking the centralizer of SL(2, $\mathbb{C}$) by the homothetic action, one obtains PSL(2, $\mathbb{C}$),
which leaves the sphere $S^2$ invariant and induces an action by homeomorphisms on the sphere.
Specifically, this action is precisely isomorphic to the full group of Möbius transformations,
which in turn is the full group of conformal homeomorphisms of the sphere $S^2$. The action of
an element $A \in \text{SL}(2, \mathbb{C})$ is given by
\begin{equation}
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad z \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}},
\end{equation}
with $z \in \mathbb{C} \cup \{\infty\} \cong S^2$. Topologically, the behaviour of a Möbius transformation is either elliptic, parabolic or hyperbolic, according to whether the action is conjugate to a rotation of the sphere around two fixed points, a horocycle action of circles around a unique fixed point, or a northpole-southpole action relative to two fixed points, respectively. The group $\text{Möbius}(S^2)$ is sharply 3-transitive on $S^2$ and consists precisely of the conformal homeomorphisms of $S^2$.

**Lemma 2.6 (Antipodal Möbius transformations).** Antipodal Möbius transformations are rotations.

**Proof.** Take $h \in \text{Ant}(S^2)$. Composing with a rotation $r \in \text{Rot}(S^2)$, we may assume that $h(0) = 0$, and thus also $h(\infty) = (\infty)$. Consider the image $\gamma = h(\gamma_1)$ where $\gamma_1$ is the horizontal meridian passing through $1 \in S^2$. As $h \in \text{Möbius}(S^2) \cap \text{Ant}(S^2)$, $\gamma$ is again a meridian as it can not be contained in a single hemisphere. In particular, $\gamma \cap \gamma_1$ has at least two intersection points. Taking a rotation $r \in \text{Stab}_{\text{Rot}}(0, \infty)$ rotating one of the points $\gamma \cap \gamma_1$ back to $1 \in S^2$, the homeomorphism thus obtained fixes $0, 1, \infty$. Since the only Möbius transformation fixing $0, 1, \infty$ is the identity by sharp 3-transitivity of the Möbius group, the claim follows. \hfill \Box

The Möbius group is defined by their action on circles.

**Lemma 2.7 (Groups preserving circles).** The group $\text{Möbius}(S^2)$ is the unique three-transitive group preserving circles.

**Proof.** Consider $G$ which is at least three-transitive and preserves circles. It is sufficient to prove that the stabilizer $G_3 \subset G$, with $G$ being 3-transitive and preserving circles, is the identity group. Indeed, suppose that there exist $G_0 \neq \text{Möbius}(S^2)$ with the properties as claimed. Then the group $G$ generated by both $G_0$ and $\text{Möbius}(S^2)$ has these properties also. Take $g \in G \setminus \text{Möbius}(S^2)$. Since $\text{Möbius}(S^2)$ is sharply three-transitive, by postcomposing $g$ with a
$h \in \text{M"obius}(S^2)$ so that $g$ fixes three points, the stabilizer $G_3$ of $G$ is non-trivial as $hg \neq \text{Id}$, contrary to the initial claim. To prove the claim, suppose that $G_3$ stabilizes the three points 0, 1, $\infty$. If this is the case, since 0 and $\infty$ are fixed and circles are sent to circles, every element of $G_3$ preserves the collection, parametrized by the circle $S^1$, of meridians passing through 0 and $\infty$. Denote $\gamma_\infty$ and $\gamma_0$ the circle of minimal radius passing through 1, $\infty$ and 1, 0 respectively and denote $\gamma_{0,\infty}$ the meridian passing through 0, $\infty$ tangent to the other two circles defined before. As $G_3$ stabilizes 0, 1, $\infty$, the circles $\gamma_{0,\infty}, \gamma_\infty, \gamma_0$ are preserved, since this is the unique configuration of circles passing through the given points that are pairwise tangent at each point, this configuration has to be preserved by the homeomorphism. It can be shown that from this configuration can be constructed infinitely many circles within this configuration whose topology, i.e. intersection pattern, entirely determines the position and geometry of the circle, see the sketch. Since this collection of circles, and their intersection points, fills the sphere densely and has to be preserved by a circle-preserving homeomorphism, such a homeomorphism has to be identity. □

3. The structure of homogeneous subgroups

The purpose of this section is to give the proof of Theorem $A$, which is proved by combining two theorems in this main section, namely Theorem $[1]$ and Theorem $[2]$ below. The first part of Theorem $A$ is the following. The second part concerning groups acting on the real projective plane is treated below.

**Theorem 1** (Transitivity of groups acting on $S^2$). Let $G$ denote a smooth homogeneous transformation group. Then precisely one of the following three hold:

(I) $G$ is one-transitive and $G \subseteq \text{Ant}(S^2)$.

(II) $G$ is three-transitive and $G = \text{M"obius}(S^2)$.

(III) $G$ is three-transitive and $G_3$ has essential support.

3.1. Aleksandrov’s problem for the sphere. We first prove the following result, which forms the main part of the proof of Proposition $[3.2]$ below. A conservative distance $\rho \in (0, \pi)$ for a homeomorphism $h \in \text{Homeo}(S^2)$ is a distance for which the property $d(h(p), h(q)) = \rho$ holds for every $p, q \in S^2$ for which $d(p, q) = \rho$.

**Proposition 3.1** (Aleksandrov’s problem for the sphere). Let $h \in \text{Homeo}(S^2)$ with a conservative distance $\rho \in (0, \pi)$, then $h \in \text{Rot}(S^2)$.

Let $\gamma_1$ be the meridian through 1 \in $S^2$. Given $p \in S^2$ and $r \in (0, \pi)$, denote $D_p(r)$ the closed topological disk centered at $p \in S^2$ with radius $r$ and $C_p(r)$ the boundary of the disk $D_p(r)$.

**Lemma 3.1.** If $h \in \text{Homeo}(S^2)$ preserves a distance $\rho$, then $h^{-1}$ also preserves the distance $\rho$.

**Proof.** Take a point $q \in S^2$ and take the disk $D_q(\rho)$ of radius $\rho$ centered at $q$. We need to show that $h^{-1}(D_q(\rho)) = D_p(\rho)$, with $p = h^{-1}(q)$. Indeed, since $h$ preserves the distance $\rho$, $h$ takes the disk $D_p(\rho)$ to the disk $D_q(\rho)$, and it takes the boundary of one disk to the boundary of the image disk. Since $h$ is a homeomorphism, we have that $D_p(\rho) = h^{-1}(h(D_p(\rho))) = h^{-1}(D_q(\rho))$. Since this holds for each such disk, $h^{-1}$ preserves the distance $\rho$ as well. □

**Lemma 3.2.** If $h \in \text{Homeo}(S^2)$ preserves a sequence $(\rho_n)_{n \in \mathbb{N}}$ of distances with either $\rho_n \to 0$ or $\rho_n \to \pi$, as $n \to \infty$, then $h \in \text{Rot}(S^2)$. 

Proof. Given $h \in \text{Homeo}(S^2)$, it suffices to show that $h$ is not distance decreasing, that is, for any $p, q \in S^2$ we have $d(h(p), h(q)) \leq d(p, q)$. By Lemma 3.1 the same inequality holds in the opposite direction and thus $h$ preserves all distances yielding that $h \in \text{Rot}(S^2)$. Choose small $\epsilon > 0$ and let $\delta > 0$, depending on $h$, such that $d(h(p), h(q)) < \epsilon$ whenever $d(p, q) < \delta$. Fix $p, q \in S^2$, and assume these to be non-antipodal, and let $\gamma$ the unique meridian passing through $p$ and $q$. Since $h$ has conservative distances $\rho_n \to 0$ as $n \to \infty$, choose $N$ large enough so that $\rho_N < \delta$ and let $M := \lfloor d(p, q)/\rho_N \rfloor$. Define $p_0 = p$ and $(p_i)_{i=1}^M$ such that $p_i \in \gamma$, with $d(p_{i-1}, p_i) = \rho_N$ and $d(p_M, y) < \delta$. We have that

$$d(h(p), h(q)) \leq \sum_{i=1}^M d(h(p_{i-1}), h(p_i)) + d(h(p_M), q) \leq \sum_{i=1}^M d(p_{i-1}, p_i) + \epsilon \leq d(p, q) + \epsilon.$$ 

Since for smaller choices of $\epsilon > 0$ exist smaller constants $\delta > 0$, again depending on $h$, but as small as desired, the claim follows for $\rho_n \to 0$. Similarly, if $\rho_n \to \pi$, then $h$ must preserve antipodal points. As $d(p, q) + d(\gamma, q) = \pi$ for fixed $p$ and all $q$, and as $h$ preserves the distance $\pi$ and distances $\rho_n \to \pi$, it also preserves distances $\rho_n = \pi - \rho_n$, with $\rho_n \to 0$ as $n \to \infty$, in which case the previous argument proves the case.

Lemma 3.3. If $h \in \text{Homeo}(S^2)$ preserves a distance $\rho \in (0, \pi)$, then it also preserves the distance $2\rho$ if $2\rho \leq \pi$ or $2\pi - 2\rho$ if $2\rho > \pi$.

Proof. First, assume that $2\rho < \pi$, and let $p_1, p_2 \in S^2$ be such that $d(p_1, p_2) = 2\rho < \pi$ and let $p_0 \in \gamma$ with $\gamma$ the unique geodesic between $p_1$ and $p_2$ and $p_0$ equidistant from $p_1$ and $p_2$. Denote $q_0, q_1, q_2 \in S^2$ the image points under $h$ of $p_0, p_1, p_2$. Since $h$ preserves the distance $\rho$, by the triangle inequality, the distance $d(q_1, q_2) \leq d(q_1, q_0) + d(q_0, q_2) = 2\rho$. Suppose that $d(q_1, q_2) < 2\rho$, then the three points $q_0, q_1, q_2$ form a triangle in the sphere. Furthermore, the circles $C_{p_1}(\rho)$ and $C_{p_2}(\rho)$ bound disks $D_{p_1}(\rho)$ and $D_{p_2}$ that have a unique common point at $p_0$, and are each sent to disks $D_{q_1}(\rho)$ and $D_{q_2}(\rho)$ of the same radius. The disks $D_{p_1}(\rho)$ and $D_{p_2}(\rho)$ are disjoint except at $p_0$ in the domain, but since $q_0 = h(p_0)$ does not lie on the geodesic between $q_1$ and $q_2$, the interiors of image disks $D_{q_1}(\rho)$ and $D_{q_2}(\rho)$ intersect around $q_0$, a contradiction. Therefore, we must have that $d(q_1, q_2) = 2\rho$.

Second, in case $2\rho = \pi$, the two disks $D_{p_1}(\pi/2)$ and $D_{p_2}(\pi/2)$ whose centers at a distance $\pi$ apart form precisely two hemispheres whose common boundary is a meridian, the configuration of which is preserved by $h$ and thus $h$ sends antipodal points to antipodal points and thus preserves the distance $\pi$. Third, assume that $2\rho > \pi$ and denote $\bar{\rho} := 2\pi - 2\rho$. We claim that $h$ preserves $\bar{\rho}$. Indeed, take any $p_1 \in S^2$ and consider $p_2 \in S^2$ with $d(p_1, p_2) = \bar{\rho}$. Denote $\gamma \subset S^2$ the unique meridian through $p_1$ and $p_2$ and denote $p_0 \in \gamma$ the unique point midway between $p_1$ and $p_2$ and having distance $\rho$ to both $p_1, p_2$ and $p_0$. Considering the antipodal points $-p_1$ and $-p_2$, the disks $D_{-p_1}(\bar{\rho}/2)$ and $D_{-p_2}(\bar{\rho}/2)$ are such that their interiors are disjoint and their boundary curves $C_{-p_1}(\bar{\rho}/2)$ and $C_{-p_2}(\bar{\rho}/2)$ intersect in the unique point $p_0$. After composing $h$ with a rotation, we may assume that $p_0 = \infty = h(p_0)$ and that $p_1 = h(p_1)$ lies on the vertical meridian through $\infty$ and $0$. Since $p_1$ is fixed under $h$, we must have that $C_{-p_1}(\bar{\rho}/2)$ is invariant under $h$. The image of $p_2$ under $h$ has to lie on the circle $C_0(\bar{\rho}/2)$. Denoting $h(p_2) = q_2$, we must have that $q_2$ lies at a distance $\bar{\rho}$ from $p_1$ on the opposite side of the circle $C_0(\bar{\rho}/2)$, since otherwise the interior of the disk $D_{-q_2}(\bar{\rho}/2)$ intersects $D_{q_2}(\bar{\rho}/2)$, but their preimages under $h$ do not, a contradiction. 

\qed
Lemma 3.4. If \( h \in \text{Homeo}(S^2) \) preserves a distance \( \rho \leq \pi/2 \), then \( h \in \text{Rot}(S^2) \).

Proof. If \( h \) preserves the distance \( \rho = \pi/2 \), then the claim follows from [19]. The case where \( \rho \neq \pi/2 \) is proved by showing that, under the given hypothesis, a distance strictly smaller than \( \rho \) is preserved. Since the set of distances preserved by \( h \) is closed, this implies there exists a sequence of distances \( \rho_n \) with \( \rho_n \to 0 \) as \( n \to \infty \). The argument is a modification of that given in [19], which fails in the case \( \rho = 2\pi/3 \). Divide the proof into two subcases.

The first case is to assume that \( h \) preserves the distance \( \rho = \pi/3 \), and therefore also preserves \( \rho = 2\pi/3 \) by Lemma 3.3. Placing the sphere \( S^2 \subset \mathbb{R}^3 \) with the center at the origin \((0,0,0) \in \mathbb{R}^3 \), each point \( p \) on the sphere \( S^2 \) can be represented as \( p = (x,y,z) \), with \( \|p\| = 1 \). By composing \( h \) with a rotation, we can assume that \( h \) fixes the points \( 0 = (0,0,1) \in S^2 \) and \( p_1 = (\sqrt{3}/2,0,-1/2) \in S^2 \). The circle \( C := C_0(\pi/3) \), which equals the intersection of \( S^2 \) with the horizontal plane of height \( z = -1/2 \), is kept invariant by \( h \). As \( p_2 = (-\sqrt{3}/2,0,-1/2) \in C \) is the only point in \( C \) at distance \( 2\pi/3 \) of \( p_1 \), the point \( p_2 \) must also be fixed by \( h \). Furthermore, if \( p_3 \in C \) is a point with \( d(p_1,p_3) = \pi/3 \), then \( p_3 \) is fixed by \( h \). Similarly, the point \( p_4 = (-\sqrt{3}/6,\sqrt{6}/3,-1/2) \in S^2 \) must be fixed by \( h \) since \( d(p_2,p_4) = \pi/3 \). Since \( d(p_3,p_4) < \pi/3 \), we must have that \( \rho < \pi/3 \) is preserved under \( h \) as well.

The second case is to assume that \( h \) preserves \( \rho \neq \pi/3 \). Since, if \( h \) preserves the distance \( \rho \) it also preserves the distance \( 2\rho \) by Lemma 3.3. First, compose \( h \) with an element of \( \text{Rot}(S^2) \) so that \( h \) fixes \( \infty \). Since \( \rho \neq \pi/3, \pi/2 \), the diameter of the circumferences \( C_\infty(\rho) \) and \( C_\infty(2\rho) \) must be different. Assume that the diameter of \( C_\infty(\rho) \) is less than the diameter of \( C_\infty(2\rho) \). Using the triangle-inequality, the points \( p_1, p_2 \in C_\infty(2\rho) \) and \( q_1, q_2 \in C_\infty(\rho) \) are fixed under \( h \), if these are subject to the condition that \( \infty, p_1, q_1 \) contained in the meridian through \( 0, \infty \) and similarly \( \infty, p_2, q_2 \) contained in the meridian through \( 0, \infty \) and \( d(p_1,p_2) = \rho \). It follows that \( d(q_1,q_2) < \rho \) and this distance is also preserved by \( h \). In case the diameter of \( C_\rho \) is larger than that of \( C_{2\rho} \), let again \( h \) fix \( \infty \) and follow the above argument, where the points \( p_1, p_2 \) and \( q_1, q_2 \) are interchanged to yield the same conclusion.

Lemma 3.5. If \( h \in \text{Homeo}(S^2) \) preserves a distance \( \rho \not\leq 2\pi/3 \) and \( \rho < \pi \), then \( h \in \text{Rot}(S^2) \).
Figure 3.2. Proof of Lemma 3.4: preserving a distance different from $\pi/3$.

Proof. If $\rho \leq \pi/2$, then we are done by Lemma 3.4. Thus assume that $\pi/2 < \rho < \pi$ and denote $\rho_0 := \rho$. Define the map

$$T(\rho) = (2\pi - 2\rho) \mod 2\pi.$$  

(3.2)

The unique fixed point of the angle map (3.2) is $\rho = 2\pi/3$. By Lemma 3.3, the distance $T(\rho_0) = 2(\pi - \rho_0)$ is also preserved. By induction, if $h$ preserves $\rho_0$ and

$$\rho_k := T^k(\rho_0) = (2\pi + (-2)^k \rho_0) \mod 2\pi > \pi/2,$$  

(3.3)

then $h$ preserves $\rho_{k+1}$ as well. But if $\rho_0 \neq 2\pi/3$, then for some $k > 0$, $\rho_k \leq \pi/2$, after which Lemma 3.4 proves the case. \qed

Lemma 3.6. If $h \in \text{Homeo}(S^2)$ preserves a distance $\rho = 2\pi/3$, then it also preserves the distance $\pi/3$.

Proof. In order to prove the lemma, we show that, assuming that $h(0) = 0$, we have $h(\infty) = \infty$. Since $h(0) = 0$, the restriction $h|_C$ of $h$ to the circle $C_{2\pi/3} \subset S^2$ is also a homeomorphism. Also, for any $p = (x, y, 1/2) \in C$, there exists a single point in $C$ which is at distance $2\pi/3$, namely $q = (-x, -y, 1/2)$. Writing $r_\pi$ the rotation fixing 0 and $\infty$ by angle $\pi$, we thus have $q = r_\pi(p)$. Thus we must have

$$d(h(p), hr_\pi(p)) = d(p, r_\pi(p)) = \frac{2\pi}{3},$$  

(3.4)

so that $h(r_\pi(p)) = r_\pi(h(p))$ for every $p \in C$. Now, given any $z \in S^2$ such that $0 < d(z, 0) < 2\pi/3$, then there exist exactly two points $I_1(z), I_2(z) \in C$ with

$$d(z, I_1(z)) = d(z, I_2(z)) = \frac{2\pi}{3}.$$  

(3.5)

Since $2\pi/3$ is a preserved distance by $h$, we have that

$$d(h(z), h(I_1(z))) = d(h(z), h(I_2(z))) = \frac{2\pi}{3},$$  

(3.6)
and thus \( I_1(h(z)) = h(I_1(z)) \) and \( I_2(h(z)) = h(I_2(z)) \). Further, note that \( r_\pi(I_1(z)) = I_1(r_\pi(z)) \) and similarly for \( I_2 \) and that, given \( w_1, w_2 \in C \) with \( d(w_1, w_2) \neq 2\pi/3 \), there exists a unique point \( y \neq 0 \) such that \( d(y, w_1) = d(y, w_2) = 2\pi/3 \). Therefore, we have
\[
I_1(hr_\pi(z)) = h(I_1(r_\pi(z))) = r_\pi(I_1(h(z))) = I_1(r_\pi(h(z))),
\]
and likewise \( I_2(hr_\pi(z)) = I_2(r_\pi(h(z))) \), so that
\[
r_\pi h(z) = hr_\pi(z),
\]
for arbitrary \( z \in D_0(2\pi/3) \). That is, the actions of \( r_\pi \) and \( h \) commute in \( D_0(2\pi/3) \). To finish the proof, assume for contradiction, that \( h(\infty) \neq \infty \). Since \( h(D_0(2\pi/3)) = D_0(2\pi/3) \) and \( h(D_\infty(2\pi/3)) = D_h(\infty)(2\pi/3) \), there exists a unique \( y \in D_0(2\pi/3) \) with the property that \( d(y, h(\infty)) = 2\pi/3 \) and for every other \( z \neq y \) with \( d(z, h(\infty)) = 2\pi/3 \), we have that \( d(z, 0) < d(y, 0) \). In other words, \( y \) is the unique point of maximal height in \( C_h(\infty)(2\pi/3) \). Let \( z \in S^2 \) be a point with the property that \( d(z, \infty) = 2\pi/3 \) and \( h(z) = y \). Then \( d(z, \infty) = d(r_\pi(z), \infty) = 2\pi/3 \). However, denoting \( y_0 = h(r_\pi(z)) \), we have that
\[
d(y_0, 0) = d(h(r_\pi(z)), 0) = d(r_\pi(h(z)), 0) = d(h(z), 0) = d(y, 0),
\]
whereas \( y \) was supposed to be the unique point of maximal distance from 0, a contradiction. This concludes the proof.

**Proof of Proposition 3.1.** Let \( \rho \) be a distance preserved by \( h \in \text{Homeo}(S^2) \). Lemma 3.4 proves the case where \( \rho \leq \pi/2 \), Lemma 3.5 proves the case where \( \rho \neq 2\pi/3 \) and Lemma 3.6 in turn proves the case where \( \rho = 2\pi/3 \), exhausting all cases. This completes the proof.

**3.2. One-transitive groups.** We now turn to the first result specific to groups that pertains to the special role of groups preserving the antipodal action.

**Proposition 3.2** (Groups one-transitive on the sphere). The group \( G \subset \text{Homeo}(S^2) \) is transitive on one point if and only if \( G \subset \text{Ant}(S^2) \) and \( G \) is transitive on \( \mathbb{RP}^2 \).

The main use of Proposition 3.1 is the following.
Lemma 3.7 ($\rho$-lemma). Suppose $G \subset \text{Homeo}(S^2)$ has the property that $G_1$ preserves a distance $\rho$, then $G = \text{Rot}(S^2)$.

Proof. Suppose that $G_1 \subset G$ preserves a distance $\rho > 0$, then we claim that each $h \in G$ preserves the distance $\rho$. Indeed, since $\text{Rot}(S^2) \subseteq G$, if there exists $h \in G$ and $p \in S^2$ such that relative to the point $p$ the homeomorphism does not preserve the distance $\rho$, then by pre- and postcomposing by elements of $\text{Rot}(S^2)$, we can construct homeomorphisms in $G_1$ that do not preserve $\rho$ with respect to $0 \in S^2$ either, contrary to our assumption. Therefore, each $h \in G$ has to preserve the distance $\rho$. Therefore, by Proposition 3.1, if $h \in G \subset \text{Homeo}(S^2)$ preserves a distance $\rho$, then $h \in \text{Rot}(S^2)$ and the claim follows.

Next, using the $\rho$-lemma, we prove Proposition 3.2. Let $O_{G_1}(p)$ be the orbit of $p \in S^2$ under $G_1$ and denote $G^0_1 := \text{Stab}_G(0) \cap G^0$, the group of isotopies with the identity stabilizing $0 \in S^2$.

Lemma 3.8. If $G \subset \text{Homeo}(S^2)$ is not contained in $\text{Ant}(S^2)$, then $O_{G_1}(\infty)$ is open, non-empty, and connected. Furthermore, if $p \in O_{G_1}(\infty)$, then there exists an isotopy $F : [0, 1] \to G^0_1$ with the identity, such that $F(\infty, 1) = p$.

Proof. First note that, since $G$ is not contained in $\text{Ant}(S^2)$, there exists $g \in G$ and $p \in S^2$ such that $g(-p) \neq -g(p)$. Using the rotation action $\text{Rot}(S^2)$, we may assume that $g(0) = 0$ and $g(\infty) \neq \infty$. In particular, we may assume that $g \in G_1$, so that $O_{G_1}(\infty) \neq \{\infty\}$. Moreover, since all rotations that fix 0 and $\infty$ are contained in $G_1$, for every $q \in O_{G_1}(\infty)$, we have that $\ell_q \subset O_{G_1}(\infty)$, where $\ell_q \subset S^2$ is the horizontal latitude passing through $q \in S^2$, which is an equidistant set for the point $0 \in S^2$. Specifically, if $p \in O := O_{G_1}(\infty)$, then there exists $g \in G_1$ such that $g(p) = \infty$ by definition of $G_1$. Define $F(\cdot, t) = g^{-1} r_t g$, with $r_t \in G^0_1, t \in [0, 1]$ an isotopy of rotations around 0, $\infty$ with $r_0 = r_1 = \text{Id}$, moving $p$ around in the circle $\ell_p$. Note that $F(\cdot, 0) = \text{Id}$ so that $F \in G^0_1$. This shows that $O_{G^0_1}(\infty)$ is rotationally symmetric and non-empty. Write $h_t(\cdot) := F(\cdot, t) \in G_1$, for brevity.

![Figure 3.4. Proof of Lemma 3.8](image_url)

To prove the first claim $O_{G^0_1}$ is open, it suffices to show that, for some $t_1, t_2 \in (0, 1)$, we have that $d(h_{t_1}(\infty), 0) < d(p, 0)$ and $d(h_{t_2}(\infty), 0) > d(p, 0)$. Indeed, as above, denote $p = g^{-1}(\infty)$.
and denote \( q := g(\infty) \), and denote \( \gamma := g(\ell_p) \subset S^2 \), which is a simple closed curve dividing the sphere \( S^2 \) into two topological disks \( D_1, D_2 \subset S^2 \). Then \( q \) is contained in one such disk, which we denote \( D_2 \), and we denote \( D_1 \) the component containing \( 0 \in S^2 \). Denoting \( q_t := r_t(q) \), since \( \infty \in \gamma \), there exists a time \( t_1 \in (0, 1) \) and \( t_2 \in (0, 1) \), such that \( q_{t_1} \in D_1 \) and \( q_{t_2} \in D_2 \). Pulling back these points using \( g^{-1} \), these points will have the properties as stated. To prove the claim that for each \( p \in O_{G_1}(\infty) \), there exists an isotopy \( F: [0, 1] \to G^\theta_1 \) with the identity such that \( F(\infty, 1) = p \), construct an isotopy as follows. Let \( q_0 \in \ell_q \cap \gamma \). There exists an isotopy of rotations \( R^1: [0, 1] \to G^0_1 \), with \( r^1_0 = \text{Id} \) and \( r^1_1 = r^0_1 \) such that \( g(r^0_1(q)) = q_0 \in \ell_q \). Denote \( p_0 := g^{-1}(q_0) \in \ell_p \). There exists a second isotopy of rotations \( R^2: [0, 1] \to G^0_1 \) such that \( r^2_0 = \text{Id} \) and \( r^2_1 = r^0_1 \). Then the isotopy \( H: [0, 1] \to G^0_1 \), defined by \( H(\cdot, t) := r^2_1g^{-1}r^1_1g \) has the property that \( H(\cdot, 0) = \text{Id} \) and \( H(\infty, 1) = p \), as required. Connectedness of \( \mathcal{O} \) follows readily. \( \square \)

**Lemma 3.9.** If \( G \subset \text{Homeo}(S^2) \) is not contained in \( \text{Anti}(S^2) \) and \( G \neq \text{Rot}(S^2) \), then \( \mathcal{O} := \mathcal{O}_{G_1}(\infty) = S^2 \setminus \{0\} \). In particular, every such \( G \) is at least two-transitive.

**Proof.** Suppose, to derive a contradiction, that \( \mathcal{O} \neq S^2 \setminus \{0\} \) and let \( d_1 := \inf_{p \notin \mathcal{O}} d(p, \infty) \). Since \( \mathcal{O} \) is open and connected by Lemma 3.8 and since \( \mathcal{O} \) is also invariant by rotations fixing \( 0 \) and \( \infty \), it follows that for all \( p \in S^2 \), \( d(p, \infty) = d_1 \) if and only if \( p \in \partial \mathcal{O} \) and that \( 0 < d_1 < \pi \). Since \( \mathcal{O} \) is invariant by \( G_1 \) by Lemma 3.8, we have that \( \partial \mathcal{O} \) is invariant by \( G_1 \) as well. Let \( h \in G_1 \setminus \text{Rot}(S^2) \). Then, by Lemma 3.7, there exist \( p_1, p_2 \in S^2 \) with the property that \( d(p_1, p_2) = \pi - d_1 \), but \( d(h(p_1), h(p_2)) \neq \pi - d_1 \). Also, there are rotations \( r^0_1, r^0_2 \in \text{Rot}(S^2) \), such that \( r^0_1(0) = p \) and \( r^0_2(h(p)) = 0 \). By construction, we have \( g := r^0_2h^0_1 \in G_1 \) and if we define \( p := r^0_2(p_2) \), then

\[
(3.10) \quad d(p(0), 0) = d(g(p), g(0)) = d(h(r^0_1(p)), h(r^0_1(0))) = d(h(p_1), h(p_2)) \neq \pi - d_1,
\]

but \( d(p, 0) = \pi - d_1 \), a contradiction. This proves that \( \mathcal{O} := \mathcal{O}_{G_1}(\infty) = S^2 \setminus \{0\} \). To prove that \( G \) is at least two-transitive, given points \( p_1, p_2, q_1, q_2 \in S^2 \), with \( p_1 \neq p_2 \) and \( q_1 \neq q_2 \), first take an isotopy \( R^1: [0, 1] \to \text{Rot}(S^2) \) with the identity that takes \( p_1 \) to \( 0 \) and \( p_2 \) to \( y_1 \in \gamma_0 \) respectively, where \( d(p_1, p_2) = d(0, y_1) \) on the vertical meridian passing through \( 0, 1, \infty \). Similarly, let \( R^2: [0, 1] \to \text{Rot}(S^2) \) be the isotopy with the identity that takes \( 0 \) to \( q_1 \) and \( y_2 \in \gamma_0 \) to \( q_2 \) where \( d(0, y_2) = d(q_1, q_2) \). Finally, let \( F: [0, 1] \to G^\theta_1 \) be the above isotopy with the identity, with the property that \( F(y_1, 1) = y_2 \). Composing the three isotopies \( R_1 \) and \( F \) and \( R_2 \), one obtains the desired isotopy. \( \square \)

**Proof of Proposition 3.4.** If \( G \subset \text{Anti}(S^2) \), then \( G \) can not be transitive on more than one point. Conversely, if \( G \not\subset \text{Anti}(S^2) \), then \( G \) is at least two-transitive. Therefore, \( G \subset \text{Anti}(S^2) \) if and only if \( G \) is one-transitive. Since \( G \) contains \( \text{Rot}(S^2) \) which descends to the transitive action \( \text{Rot}(\mathbb{R}P^2) \) on \( \mathbb{R}P^2 \), the induced action of \( G \) is transitive on \( \mathbb{R}P^2 \). \( \square \)

The following construction is a crucial tool in what follows.

**Proposition 3.3 (Monotone isotopy in \( G_1 \)).** Let \( G \subset \text{Diffeo}(S^2) \) and \( G \) not contained in \( \text{Anti}(S^2) \). Then there exists an isotopy \( F: \mathbb{R}^+ \to G^\theta_1 \), which is continuous in \( \text{Diffeo}(S^2) \), and such that \( F(\cdot, 0) = \text{Id} \), \( \lim_{t \to \infty} F(\infty, t) = 0 \) and \( d(F(\infty, t), 0) \) is a strictly decreasing function.

Proposition 3.3 requires a number of additional technical lemmas. Assume, until the end of the proof of Proposition 3.3 that \( G \subset \text{Diffeo}(S^2) \) is not contained in \( \text{Anti}(S^2) \).
Lemma 3.10. There exists $\delta_\pi > 0$, $\delta > 0$ and an isotopy $F^\pi: [0, \delta] \to G^0_1$, such that $F(\cdot, t) = \text{Id}$ and $d(t) = d(F^\pi(\infty, t), 0)$ is strictly decreasing and $(\pi - \delta, \pi]$ is contained in the image of $d$.

Proof. Let $\gamma_0$ be the equator, let $h \in G_1$ be a diffeomorphism such that $h(1) = \infty$, and let $R \in \text{Rot}(S^2) \cap G^0_2$ be the isotopy of rotations $r_t$ of angle $t$, fixing $0$ and $\infty$. Note that, since $h$ is a diffeomorphism, the curve $\alpha: [0, 1] \to S^2, \alpha(t) = h(r_t(1))$ is differentiable in $t$, and $\alpha(0) = \infty$.

Furthermore, since $\alpha'(0)$ is non-vanishing, and since $\alpha(0)$ is the antipodal point of $0$, the $C^1$ function $d: [0, 1] \to \mathbb{R}$, defined by $d(t) = d(\alpha(t), 0)$ is such that $d'(0) = -||\alpha'(0)||$. In particular, there exists $\delta > 0$ such that the restriction of $d$ to $[0, \delta]$ is strictly decreasing, and let $\delta_\pi = d(\delta)$.

Defining $F^\pi(\cdot, t) := hr_t h^{-1}$, and noting that,

$$(3.11) \quad F^\pi(\infty, t) = hr_t h^{-1}(\infty) = hr_t(1) = \alpha(t),$$

and $F^\pi(0) = 0$ since $r_t(0) = 0$, proving the claim.

Lemma 3.11. Let $0 < \ell < \pi$ and $p \in S^2 \setminus \{0, \infty\}$. There exists $\delta_\ell > 0$, $\delta > 0$ and an isotopy $F^\ell: (-\delta, \delta) \to G^0_1$, continuous in $\text{Diffeo}(S^2)$, such that $F^\ell(\cdot, 0) = \text{Id}$ and such that $d_t(t) = d(F^\ell_t(p), 0)$ is strictly decreasing, with $(t - \delta_\ell, t + \delta_\ell)$ contained in the image of $d_t$.

Proof. Let $h_1 \in G_1$ be such that $h_1(p) = \infty$, with $p \in S^2 \setminus \{0, \infty\}$, let $r_t$ be the rotation of angle $t$ fixing $0$ and $\infty$, and let $h_2 \in G_1$, $h_2 = r_{\pi/2} h_1$. Define

$$(3.12) \quad \alpha(t), \beta(t) : (-\pi, \pi) \to S^2, \quad \alpha(t) = h_1(r_t(p)), \quad \beta(t) = h_2(r_t(p)).$$

Note that, at time $t = 0$, both $\alpha(0) = \beta(0) = \infty$ and that both curves are perpendicular. In particular, since $h_1$ is a diffeomorphism, the curves $h_1^{-1}(\alpha)$ and $h_1^{-1}(\beta)$ are transversal at time $t = 0$. Since the image by $h_1^{-1}$ of $\alpha$ is contained in the latitude of $p$, the curve $h_1^{-1}(\beta)$ is transversal to this latitude at $t = 0$ and so, if we define

$$(3.13) \quad d(t) = d(h_1^{-1}(\beta(t)), 0),$$

then $d$ is $C^1$ and $d(0) = \ell$ and $d'(0) \neq 0$. We assume, with no loss of generality that $d'(0) < 0$, otherwise below we take $H^\ell_t = F^\ell_t$. Note that there exists $\delta > 0$ such that, if $-\delta < t < \delta$, $d'(t) < 0$. Let $\delta_\ell$ be such that $(t - \delta_\ell, t + \delta_\ell) \subset d((-\delta, \delta))$. Consider now the isotopy

$$(3.14) \quad F^\ell: (\pi, \pi) \to G^0_1, \quad F^\ell_t = h_1^{-1} h_2 r_t h_2^{-1} h_1.$$  

We note that $F^\ell(\cdot, 0) = \text{Id}$ and that

$$(3.15) \quad F^\ell_t(p) = h_1^{-1} h_2 r_t h_2^{-1} h_1(p) = h_1^{-1} h_2 r_t h_2^{-1}(\infty) = h_1^{-1} h_2 r_t(p) = h_1^{-1}(\beta(t)),$$

so that $d(F^\ell_t(p), 0) = d(t)$.

Lemma 3.12. If $G \subset \text{Diffeo}(S^2)$ and $G$ is not contained in $\text{Ant}(S^2)$, then for every $0 < \epsilon < \pi$, there exists an isotopy $F: [0, 1] \to G^0_1$ such that, if $d(t) = d(F(\infty, t), 0)$ then $d(t)$ is strictly decreasing as a function of $t$ and $d(1) = \epsilon$.

Proof. Since the interval $[\epsilon, \pi]$ is compact, by Lemma 3.10 and Lemma 3.11 there exist a finite sequence

$$(3.16) \quad \pi = r_0 = \ell_0 > r_1 > \ell_1 > r_2 > ... > r_{n-1} > \ell_{n-1} > r_n = \epsilon,$$

and sequences $(s_i), 0 \leq i \leq n-1, (t_i), 0 \leq i \leq n-1$, points $(p_i), 1 \leq i \leq n-1$ and isotopies $(F^\ell_i)_{0 \leq i \leq n-1}$, such that

(i) $d(F^\ell_i(p_i), 0)$ is strictly decreasing for $t \in [s_i, t_i]$
\( (\text{ii}) \ d(F^{\ell_i}_{s_i}(p_i),0) = r_i, \ d(F^{\ell_i}_{t_i}(p_i),0) = r_{i+1} \)

Let \( a_i = \sum_{j=0}^{i} (t_j - s_j) \). Define, inductively, the isotopies \( H^i_t : [0, a_i] \rightarrow G^0_1, 0 \leq i \leq n - 1 \) as:

\[
H^{i+1}_t = \begin{cases} 
H^i_t & \text{if } t \leq a_i, \\
H^{i+1}_t = r^{-1}_0 F^{\ell_i+1}_{t-a_i+s_i} (F^{\ell_i+1}_{s_i})^{-1} r_0 H^i_{a_i}, & \text{if } t \in [a_i, a_{i+1}] 
\end{cases}
\]

where \( r_0 \) is the rotation fixing 0 such that \( r_0 H^i_{a_i}(\infty) = F^{\ell_i+1}_{s_i+1}(p_{t+1}) \). With this definition, it can be shown that \( H(\cdot, t) = H^{n-1}(\cdot, t) \in G^0_1 \), with \( t \in [0, a_n] \), is continuous with respect to \( t \) in \( \text{Diff}(S^2) \) and such that, if \( a_{i-1} \leq t \leq a_i \), then \( d(t) := d(H(\infty, t), 0) = d(F^{\ell_i}_{t-a_i}(p_i), 0) \) and so \( d(t) \) is a strictly decreasing function with \( d(a_{n-1}) = \epsilon \). Then the isotopy \( F : [0, 1] \rightarrow G^0_1 \), defined by \( F(\cdot, t) := H(\cdot, a_n-t) \), satisfies the requirements.

**Proof of Proposition 3.3.** Let \( F^n : [0, 1] \rightarrow G^0_1 \) be isotopies such that \( d_n(t) = (d(F^n(\infty, t), 0) \) is strictly decreasing, and such that \( d_n(1) = 1/n \), given by Lemma 3.12. Let \( t_1 = 0 \) and, for \( n > 1 \), let \( t_n \) be the exact time such that \( d_n(t_n) = 1/(n-1) \). Let \( a_1 = 1, a_{n+1} = a_n + (1 - t_{n+1}) \). Define, inductively, the isotopy \( H^i_t : [0, a_n] \rightarrow G^0_1, n \in \mathbb{N} \) as: \( H^1_t = F^1_t \) and

\[
H^{n+1}_t = \begin{cases} 
H^n_t & \text{if } t \leq a_n, \\
H^{n+1}_t = r^{-1}_n F^{n+1}_{t-a_n+t_{n+1}} (F^{n+1}_{t_{n+1}})^{-1} r_n H^n_{a_n}, & \text{if } t \in [a_n, a_{n+1}] 
\end{cases}
\]

where \( r_n \) is the rotation fixing 0 such that \( F^{n+1}_{t_{n+1}}(\infty) = r_n H^n_{a_n}(\infty) \), and note that, by construction, \( d(H^n_{a_n}(\infty), 0) = 1 \) and

\[
d(H^{n+1}_{a_{n+1}}(\infty), 0) = d(r^{-1}_n F^{n+1}_{t_{n+1}-a_n+t_{n+1}} (F^{n+1}_{t_{n+1}})^{-1} (F^{n+1}_{t_{n+1}}(\infty)), 0) = d(r_n F^{n+1}_{t_{n+1}}(\infty), 0) = \frac{1}{n+1}.
\]

The desired isotopy is \( F : \mathbb{R}^+ \rightarrow G^0_1 \), where \( F(\cdot, t) = H^n(\cdot, t) \) if \( t \leq a_n \). \( \square \)

### 3.3. Two-transitive groups

Using the above results, we next show that there are no two-transitive groups containing \( \text{Rot}(S^2) \).

**Proposition 3.4 (Two-transitive groups).** Let \( G \subset \text{Diff}(S^2) \) and \( G \) not contained in \( \text{Ant}(S^2) \). Then given any \( p, q \in S^2 \setminus \{0, \infty\} \), there exists an isotopy \( H : [0, 1] \rightarrow G^0_1 \), such that \( H(0, t) = 0 \), \( H(\infty, t) = \infty \) for all \( t \in [0, 1] \), and \( H(\cdot, 0) = \text{Id} \) and \( H(p, 1) = q \). In particular, every such group \( G \) which is transitive on two points, is transitive on three points.

**Lemma 3.13.** Let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of homeomorphisms \( g_n \in \text{Homeo}(S^2) \) such that

\[
\lim_{n \to \infty} (g_n(0), g_n(\infty)) = 0.
\]

Then for every \( 0 < \ell < \pi \), there exists \( \epsilon = \epsilon(\ell) > 0 \) and sequences \( p_n, q_n \in S^2 \) with the properties

\[
\lim_{n \to \infty} d(g_n(p_n), g_n(-p_n)) = 0 \quad \text{and} \quad \limsup_{n \in \mathbb{N}} d(g_n(p_n), g_n(q_n)) \geq \epsilon,
\]

and \( d(p_n, q_n) = \ell \).
Proof. Suppose, for a contradiction, that the statement is false. We claim that there exist $\ell > 0$ such that for every $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon)$ and $m_\epsilon = m_\epsilon(\delta_1)$ such that whenever $n > m_\epsilon$ and $d(p, q) = \ell$, then

$$
(3.21) \quad \text{if } d(g_n(p), g_n(-p)) < \delta_1, \text{ then } d(g_n(p), g_n(q)) < \epsilon.
$$

Assume this is false. Then, for every $0 \leq \ell < \pi$, there exists $\epsilon > 0$ and a sequence $(p_{n_k}, q_{n_k})$ with $n_k \to \infty$ such that $d(g_{n_k}(p_{n_k}), g_{n_k}(-p_{n_k})) < 1/k$ and $d(g_{n_k}(p_{n_k}), g_{n_k}(q_{n_k})) > \epsilon$. Let $w$ be such that $d(w, 0) = \ell$ and let $p_i, q_i$ be defined as $p_{n_k}, q_{n_k}$ if $i = n_k$ for some integer $k$, and $p_i = 0, q_i = w$ otherwise. Then the sequences $(p_i, q_i)$ satisfy (3.20), which we assumed was not the case. Therefore, condition (3.21) holds. To prove the lemma, we use an induction argument.

![Figure 3.5. Proof of Lemma 3.13; constructing $\delta_2$ from $\delta_1$.](image)

First, we claim that, for the $\ell$ in condition (3.21), there exists $\delta_2 = \delta_2(\epsilon)$ and $m_2 = m_2(\delta_2)$ such that, if $n > m_2$ and $d(p, q) \leq 2\ell$,

$$
(3.22) \quad \text{if } d(g_n(p), g_n(-p)) < \delta_2, \text{ then } d(g_n(p), g_n(q)) < \epsilon.
$$

First, by the triangle inequality, we have that

$$
(3.23) \quad d(g_n(p), g_n(q)) \leq d(g_n(p), g_n(w)) + d(g_n(w), g_n(q)).
$$

To prove claim (3.22), we will show that the right hand side of (3.23) is bounded by $\epsilon$. First note that, if $p, q \in S^2$ such that $d(p, q) \leq 2\ell$, there exists $w \in S^2$ such that $d(p, w) = d(w, q) = \ell$. By condition (3.21), given $\epsilon$, we can choose $\delta_2(\epsilon) > 0$ small enough so that

$$
(3.24) \quad \text{if } d(g_n(p), g_n(-p)) < \delta_2, \text{ then } d(g_n(p), g_n(q)) < \frac{\delta_1(\epsilon/3)}{3},
$$

and further

$$
(3.25) \quad \delta_2 < \frac{\delta_1(\epsilon/3)}{3} \quad \text{and} \quad \delta_1(\epsilon/3) \leq 2\epsilon.
$$

By this choice of $\delta_2$, if $d(g_n(p), g_n(-p)) < \delta_2$, we have

$$
(3.26) \quad d(g_n(p), g_n(w)) < \frac{\delta_1(\epsilon/3)}{3} \quad \text{and} \quad d(g_n(-p), g_n(-w)) < \frac{\delta_1(\epsilon/3)}{3},
$$

and

$$
(3.27) \quad d(g_n(p), g_n(q)) < \frac{\delta_1(\epsilon/3)}{3}.
$$

Hence, $d(g_n(p), g_n(q)) < \epsilon$.

Thus, we have shown that the sequences $(p_i, q_i)$ satisfy (3.22) for all $\epsilon$ and $\ell$. Therefore, condition (3.21) holds for all $\epsilon$. To prove the lemma, we use an induction argument.
since \( d(p,w) = d(-p,-w) = \ell \). Using again the triangle inequality, we have
\[
d(g_n(w), g_n(-w)) \leq d(g_n(w), g_n(p)) + d(g_n(p), g_n(-p)) + d(g_n(-p), g_n(-w)) \leq \frac{\delta_1(\epsilon/3)}{3} + \delta_2 + \frac{\delta_1(\epsilon/3)}{3} < \delta_1(\epsilon/3),
\]
using (3.24). Therefore, the condition that \( d(g_n(w), g_n(-w)) < \delta_1(\epsilon/3) \) applied to (3.21), we obtain that \( d(g_n(w), g_n(q)) \leq \epsilon/3 \), since \( d(w,q) = \ell \). Combining (3.22) with (3.24) and (3.25), we obtain
\[
d(g_n(p), g_n(q)) \leq d(g_n(p), g_n(w)) + d(g_n(w), g_n(q)) < \frac{\delta_1(\epsilon/3)}{3} + \frac{\epsilon}{3} = \epsilon.
\]
proving claim (3.22). To finish the proof, by an induction argument, given \( k \in \mathbb{N} \), there exists \( \delta_k(\epsilon) > 0 \), with \( \delta_k(\epsilon) \) a decreasing sequence of positive real numbers, and \( m_k \) such that, if \( n > m_k \) and \( d(p,q) < 2^k \ell \) and \( d(g_n(p), g_n(-p)) < \delta_k \) then \( d(g_n(p), g_n(q)) < \epsilon \). Let \( \epsilon < \pi/2 \) and choose a finite \( k \) large enough so that \( 2^k \ell > \pi \geq 2^{-k-1} \ell \) holds. There exists \( n \in \mathbb{N} \) such that \( d(g_n(0), g_n(\infty)) < \delta_k(\epsilon) \) but this implies that, since for every \( q \in S^2 \) with the property that \( d(q,\infty) \leq \pi \), for every \( q \in S^2 \), the condition \( d(g_n(q), g_n(\infty)) < \pi/2 \) holds. However,
\[(3.27) \quad d(g_n(q), g_n(\infty)) = \pi \quad \text{if} \quad q = g_n^{-1}(g_n(\infty)),\]
for every \( n \in \mathbb{N} \), a contradiction. This concludes the proof. \(\square\)

*Proof of Proposition 3.4* Given two points \( z_1, z_2 \in S^2 \), assume that \( d(z_1, \infty) > d(z_2, \infty) \), the other case is analogous. Define \( \ell_1 := d(z_1, \infty) \) and \( \ell_2 := d(z_2, \infty) \). Since Rot\( (S^2) \subset G \), it suffices to show the existence of an isotopy \( H : [0,1] \rightarrow G^0 \), i.e. with the property that \( H(0,t) = 0 \), \( H(\infty,t) = \infty \), and that for some \( t_0 > 0 \), we have \( d(H(z_1,t_0), \infty) < \ell_2 \). First, by Proposition 3.2 there exists \( h \in \text{Homeo}(S^2) \) such that \( h(\infty) = \infty \) and \( h(z_1) = 0 \). Let \( w = h(0) \) and \( \ell_0 = d(w, \infty) \).

Let, by Proposition 3.3 \( F_t : [0,\infty) \rightarrow G^0 \) be a monotone isotopy to the identity such that \( F(0,t) = 0 \), and \( d(0,F_t(\infty)) \) is strictly decreasing, and \( \lim_{t \rightarrow \infty} F_t(\infty) = 0 \), for \( t \in \mathbb{R}^+ \). By Lemma 3.13 there exists \( \epsilon > 0 \) and a sequence \( p_k,q_k \in S^2 \) with \( d(p_k,q_k) = \ell_0 \),
\[(3.28) \quad \lim_{k \rightarrow \infty} \inf_{k} d(F_k(p_k), F_k(-p_k)) = 0 \quad \text{and} \quad \inf_{k} d(F_k(p_k), F_k(q_k)) > \epsilon.\]

Let now \( \epsilon_1 > 0 \) be such that, if \( d(w,y) > \epsilon \) then \( d(0,h^{-1}(y)) > \epsilon_1 \). Let \( \lambda_0 > 0 \) be such that \( d(F_{\lambda_0},0) = \epsilon_1 \). By continuity of \( F_t \) on the interval \( t \in [0,\lambda_0] \), there exists \( \delta_1 > 0 \) such that for all \( t \in [0,\lambda_0] \), if \( d(v_1,v_2) > \delta_1 \) then \( d(F_t^{-1}(v_1), F_t^{-1}(v_2)) < \ell_2 \), and let \( \delta_2 > 0 \) be such that, whenever \( d(v_1,v_2) < \delta_2 \), then \( d(h^{-1}(v_1), h^{-1}(v_2)) < \delta_1 \). Let \( k_0 \) be such that
\[(3.29) \quad d(F_{k_0}(p_{k_0}), F_{k_0}(-p_{k_0})) < \delta_2.\]

Define the continuous choice of points \( p,q : \mathbb{R}^+ \rightarrow S^2 \), such that \( d(p(t),q(t)) = \ell_0 \) and \( p(k) = p_k \) and \( q(k) = q_k \) with \( k \in \mathbb{N} \). Define \( r_{\theta_1} : \mathbb{R}^+ \rightarrow \text{Rot}(S^2) \) be the continuous family of rotations with the property that
\[(3.30) \quad r_{\theta_1(t)}(w) = q(t) \quad \text{and} \quad r_{\theta_1(t)}(\infty) = p(t),\]
and let \( r_{\theta_2} : \mathbb{R}^+ \rightarrow \text{Rot}(S^2) \) be a continuous family of rotations such that
\[(3.31) \quad r_{\theta_2(0)} = r_{\theta_1(0)}^{-1} \quad \text{and} \quad r_{\theta_2(t)}(F_t(q(t))) = w.\]
Define $Q_t := h^{-1} r_{\theta_2(t)} F_t r_{\theta_1(t)} h$, and note that $Q: [0, 1] \to G'_1$ using (3.31) and since

\begin{equation}
Q(0, t) = h^{-1} r_{\theta_2(t)} F_t r_{\theta_1(t)} (w) = h^{-1} r_{\theta_2(t)} F_t (q(t)) = h^{-1} (w) = 0.
\end{equation}

Since $F_t r_{\theta_1(t)} h(\infty) = F_t (p(t))$ and $F_t r_{\theta_1(t)} h(\infty) = F_t (q(t))$, and since

\begin{equation}
d(r_{\theta_2(k_0)} (F_{k_0}(p_{k_0})), r_{\theta_2(k_0)} (F_{k_0}(q_{k_0}))) = d(F_{k_0}(p_{k_0})), F_{k_0}(q_{k_0})) > \epsilon,
\end{equation}

we have that $d(Q_{k_0}(0), d(Q_{k_0}(+\infty)) > \epsilon_1$. Note that, since $h(z_1) = 0$, $r_{\theta_1(t)}(0) = -p(t)$ and

\begin{equation}
d(F_{k_0}(p_{k_0})), F_{k_0}(-p_{k_0})) < \delta_2,
\end{equation}

it follows that $d(Q_{k_0}(z_1), Q_{k_0}(\infty)) < \delta_1$. Finally, since $d(F_t(\infty), 0)$ is strictly decreasing, for any $t \in [0, k_0]$ there exists a unique $\lambda(t)$ such that

\begin{equation}
d(F_{\lambda(t)}(\infty), 0) = d(Q_t(\infty), 0)
\end{equation}

Let $r_{\theta_3} : [0, k_0] \to \text{Rot}(S^2)$ be an isotopy of rotations and define $\lambda : [0, k_0] \to \mathbb{R}^+$ the continuous function with the property that $r_{\theta_3(t)} (Q_t(\infty)) = F_{\lambda(t)}(\infty)$ and such that $r_{\theta_3(t)}(0) = 0$. Note that, as $d(0, Q_{k_0}(\infty)) > \epsilon_1$, $\lambda(k_0) < \lambda_0$. Define $H_t := F_{\lambda(t)} r_{\theta_3(t)} Q_t$. By construction, for $t \subseteq [0, k_0]$, we have that (i) $H_t \in G$, (ii) $H_t(0) = 0$ and (iii) $H_t(\infty) = \infty$ and (iv), since $d(Q_{k_0}(z_1), Q_{k_0}(\infty)) < \delta_1$ and $\lambda(k_0) < \lambda_0$, we have

\begin{equation}
d(H_{k_0}(z_1), H_{k_0}(\infty)) = d(H_{k_0}(z_1), \infty) < \ell_2,
\end{equation}

as required. \hfill \Box

3.4. Virtual Möbius groups. To proceed to the proof of Theorem A, we consider the groups of type II and III. We divide the groups not of type I into two types: the first type is called virtual Möbius, which satisfies the condition that $G_2 := \text{Stab}_G(0, \infty)$ preserves latitudes, and the second complementary type is to consider groups for which $G_2$ does not preserve latitudes. Note that the Möbius group is virtual Möbius in this definition. We first characterize the virtual Möbius groups and then proceed to study the non-virtual Möbius case.

**Proposition 3.5** (Virtual Möbius groups). If $G \subseteq \text{Diffeo}(S^2)$ is virtual Möbius, then it is Möbius.

Denote $\mathcal{F}$ the foliation of the punctured sphere $S^2 \setminus \{0, \infty\}$ by the latitudes $\gamma_s$, where $s \in (0, \pi)$ is defined to be the distance of $\gamma_s$ to $\infty$. In this paragraph, define the height function $\phi_h : (0, \pi) \to (0, \pi)$, with $h \in G_2$, to be the homeomorphism that associates to the height $s$ of the leaf $\gamma_s$ the height of image leaf, i.e. the homeomorphism defined by

$$
\gamma_{\phi_h(s)} = h(\gamma_s).
$$

First, we observe that the stabilizer $G_2$ contains an isometric copy of every homeomorphism $g \in G$, in the following sense.

**Lemma 3.14** (Stabilizer on two points in $S^2$). The stabilizer $G_2$ of any group $G \subseteq \text{Homeo}(S^2)$ has the property that, for each $g \in G$, there exist $r_1, r_2 \in \text{Rot}(S^2)$ and $h \in G_2$, such that $g = r_1 h r_2$. 

Proof. To prove the result, we claim that for every $g \in \text{Homeo}(S^2)$, there exist a pair of antipodal points $\{p, -p\}$ in $S^2$ with the property that $g(-p) = -g(p)$. The pairs $\{p, -p\}$ and $\{g(p), -g(p)\}$ can be brought by rotations $r_1, r_2 \in \text{Rot}(S^2)$ back to $\{0, \infty\}$ respectively, to prove the desired result. To prove the claim, suppose that $g(-p) \neq -g(p)$ for all $p \in S^2$. Define a line field $X : S^2 \to T^1 S^2$ on the sphere $S^2$ as follows. For each $q \in S^2$, denote $p := g^{-1}(q)$ and let $\gamma_q$ be the geodesic passing through $q = g(p)$ and $w = g(-p)$. The geodesic passing through $q$ is unique and assigns a line element at $q$ that depends continuously on the basepoint $q$ since antipodal points $\{p, -p\}$ are mapped by $g$ to points whose distance is bounded away from 0 and $\pi$, for all $p \in S^2$ by uniform continuity of $g$ combined with the assumption that $g$ does not send antipodal points to antipodal points. The line field $X$ is a globally defined line field on the sphere and is everywhere continuous, which is impossible by the hairy ball theorem. □

Lemma 3.15. Suppose that $G \subset \text{Homeo}(S^2)$ is virtual Möbius, then $G_3 = \text{Id}$.

We use the following two technical results to prove this.

Lemma 3.16. Let $G_0 \subseteq \text{Homeo}(S^1)$ be generated by $\text{Rot}(S^1)$ and $\psi \in \text{Homeo}(S^1) \setminus \text{Rot}(S^1)$. Then $G$ is locally at least three-transitive, meaning that there exists an interval $J \subset S^1$ such that any ordered triple of points in $J$ can be sent to another ordered triple of points in $J$ by a homeomorphism in $G_0$.

Proof. This is a corollary of the classification of transitive circle groups, see [7]. □

Lemma 3.17. The action of $G_3$ on $\gamma_{\pi/2}$ is the identity.

Proof. Denote $G_{2,0} \subset G_2$ the subgroup of homeomorphisms that leave invariant $\gamma_{\pi/2} \cong S^1$ in $G_2$ and note that this action includes the rotation action $\text{Rot}(S^2) \cap G_2$. Suppose, to derive a contradiction, that the action is not the rotation action. Then by Lemma 3.16 the group $G_{2,0}$ acting on the circle $\gamma_{\pi/2}$ is locally three-transitive and there exists a small interval $J \subset \gamma_{\pi/2}$ on which the action is three-transitive. Associate to each $h \in G_{2,0}$ leaving invariant $\gamma_{\pi/2}$ the induced action $\psi$ on the circle. By the above remarks, for every ordered triple of points $(p_1, 1, p_2)$ and $(p_1, 1, q_2)$ in $J$, there exists here exists $h \in G_{2,0}$ inducing $\psi$ with the property that $\psi(1) = 1$ and $\psi(p_1) = p_1$ and $\psi(p_2) = q_2$. Note that the corresponding $h \in G_2$ is contained in $G_3$ since 1 is preserved by $h$ as well. By three-transitivity of $G$, there exist $g_0 \in G$ with the property that fixes $0 \in S^2$ and interchanges 1 and $\infty$, i.e. $g_0(1) = \infty$ and $g_0(\infty) = 1$. Given the above $h \in G_3$, $h$ preserves the latitudinal foliation which we denote by $\mathcal{F}$ as well as the foliation $\mathcal{F}_1$ defined as the inverse image under $g_0$ of the foliation $\mathcal{F}$. The foliation $\mathcal{F}_1$ consists of simple closed curves winding around 1 in $S^2$.

Take a round open disk $U \subset S^2$ centered at 1 whose diameter is at most the length of the interval $J$ on which the action is locally three-transitive. Take a leaf $\gamma \in \mathcal{F}_1$ small enough so that it is contained in a round open disk $U_0 \subset U$ centered at 1 of half the diameter of $U$. Let $p_1, p_2 \in \gamma_{\pi/2}$ be points in $\gamma$. Take $h \in G_3$ such that the induced action $\psi$ is such that $\psi(1) = 1$, $\psi(p_1) = p_1$ and $\psi(p_2) = q_2 \in U \setminus U_0$. However, this contradicts that the foliation $\mathcal{F}_1$ is left invariant since points of one leaf are sent to two different leaves. Therefore, $G_{2,0}$ can only consist of rigid rotations and thus $G_3$ is the identity on $\gamma_{\pi/2}$. □

Proof of Lemma 3.17 To finish the proof, we need to show that $G_3 = \text{Id}$ on $S^2$. First we claim that each leaf $\gamma_s$ with $s \in (0, \pi/2)$ is left invariant by $G_3$. Indeed, by Lemma 3.17 the action
of $G_3$ is the identity on $\gamma_{\pi/2}$. Given $h \in G_3$, $h$ leaves invariant the foliations $\mathcal{F}$ and $\mathcal{F}_1$, in the notation of Lemma 3.17. Given a leaf $\gamma_s \in \mathcal{F}$ with $s \in (0, \pi/2)$, the case where $s \in (\pi/2, \pi)$ is identical, the leaves of $\mathcal{F}_1$ close enough to 1 will be disjoint from $\gamma_s$. Furthermore, there exist leaves in $\mathcal{F}_1$ that intersect the leaf $\gamma_s$. Therefore, there exists a unique leaf of $\mathcal{F}_1$ which is closest, relative to the transverse distance in the foliation, to 1 that intersects $\gamma_s$. Since the action $G_3$ on $\gamma_{\pi/2}$ is the identity, this leaf is invariant by $h$ and thus the leaf $\gamma_s$ has to be fixed by $h$ as well. Since this holds for every leaf $\gamma_s$, with $s \in (0, \pi/2)$, the claim follows. Next, we claim that the action of $G_3$ on each leaf has to be a rigid rotation. Indeed, suppose that there exist $h \in G_3$ such that the action on $\gamma_s$ for some $s \in (0, \pi/2)$ is not a rigid rotation. Since $\text{Rot}(S^2) \cap G_2$ also leaves invariant $\gamma_s$, then there exists a subgroup in $G_2$ leaving invariant $\gamma_s$ that by Lemma 3.16 is locally three-transitive. Conjugating, using that $G$ is three-transitive, the action on the leaf $\gamma_s$ to the leaf $\gamma_{\pi/2}$, the action on the leaf $\gamma_{\pi/2}$ is also locally three-transitive, which by the argument of Lemma 3.17 was excluded. Therefore, the action of $G_3$ on each latitude is by rigid rotations modulo rescaling.

To complete the proof, given $h \in G_3$, denote $\rho_h(s)$ the rotation number of the rigid rotation action of $h$ on the leaf $\gamma_s$ with $s \in (0, \pi/2]$ and note that $\rho_h(\pi/2) = 0$ by Lemma 3.17. Since the rotation number $\rho_h(s)$ is continuous with respect to $s$ and $\rho_h(\pi/2) = 0$, if for some $s_0 \in (0, \pi/2)$, we have that $\rho_h(s_0) \neq 0$, then there exists $s_1 \in (s_0, \pi/2)$ such that $\rho_h(s_1) \notin \mathbb{Q}$. Therefore, taking iterates of $h$, each point $p \in \gamma_{s_1}$ is mapped densely around the leaf $\gamma_{s_1}$, which is readily seen to contradict that the action $G_3$ has to preserve the foliation $\mathcal{F}_1$, whose leaves wind around 1 and not $\infty$.

□

Lemma 3.18. Suppose that $G \subset \text{Homeo}(S^2)$ is virtual Möbius, then the action of each $h \in G_2$ is isometric, modulo constant rescaling, along each latitude.

Proof. By the previous lemma, if $G_2$ preserves latitudes, then $G_3 = \text{Id}$. Take any $h \in G_2$ and take an interval $J \subset \gamma_{\pi/2}$ of length $0 < \epsilon < 1$. Rotate this interval by angle $\theta \in [0, 2\pi)$ along $\gamma_{\pi/2}$ to another interval $J_\theta = r_\theta(I) \subset \gamma_{\pi/2}$. Take $h \in G_2$ and suppose that the lengths of the image intervals $I := h(J) \subset \gamma_s$ and $I_\theta := h(J_\theta) \subset \gamma_s$ for some $s \in (0, \pi/2]$, are not equal. From this it follows that, upon pre- and postcomposing $h$ with suitable rotations, there exist $h_1, h_2 \in G_2$ such that $h_1$ and $h_2$ send $J$ to intervals $I_1, I_2 \subset \gamma_s$ with the property that the left endpoints of $I_1$ and $I_2$ are equal, and such that $I_1$ and $I_2$ have different lengths. Taking $g := h_1 h_2^{-1}$, we obtain
that $g \in G_2$ and $g(1) = 1$ so that $g \in G_3$, but since the action of $g$ on $\gamma_{\pi/2}$ is strictly non-trivial, this contradicts Lemma 3.17 stating that $G_3 = \text{Id}$. Therefore, the lengths of $I$ and $I_0$ had to be equal. Furthermore, since this arguments works for all $s \in (s, \pi/2]$ and for all $\theta \in [0, 2\pi)$, taking $\theta = 2\pi/n$, for some integer $n \in \mathbb{N}$, since $n$ such consecutive intervals fill the meridian, we conclude that the length of the image interval has to be multiplied by the quotient of the length of the $\gamma_s$ and the length of $\gamma_{\pi/2}$.

For a topological $G \subset \text{Homeo}(S^2)$ being virtual Möbius, the above arguments in essence show that $G_2$ is an action along latitudes, isometric modulo rescaling on each leaf, and the action is embedded in a one-dimensional flow. If $G \subset \text{Diffeo}(S^2)$, we can conclude more.

Lemma 3.19. Suppose that $G \subset \text{Diffeo}(S^2)$ is virtual Möbius, then $g$ is conformal at $0, \infty$ for each $g \in G_2$.

Proof. This follows from the action on the latitudes being isometric, modulo constant rescaling on each latitude, using Lemma 3.18.

Lemma 3.20. Let $G \subset \text{Diffeo}(S^2)$ be not contained in $\text{Ant}(S^2)$. For each point $p \in S^2 \setminus \{0, \infty\}$, whose distance to $\infty$ is small, there exist $q_1 \in G_1$ fixing $0$ and such that $q_1(p) = \infty$, where the $C^1$-norm of $q_1$ decreases monotonically to zero as $p$ approaches $\infty$ in distance.

Proof. Using the construction of Lemma 3.10 which constructs the isotopy acting on the points as required, observing that the isotopy is $C^1$ in the time variable, the claim follows.

Lemma 3.21. Suppose that $G \subset \text{Diffeo}(S^2)$ is virtual Möbius, then $G_2$ is conformal.

Proof. Suppose, to derive a contradiction, that there exist $h \in G_2$ and $p \in S^2 \setminus \{0, \infty\}$ such that $h$ is not conformal at $p$. Define rotations $r_1, r_2 \in \text{Rot}(S^2)$ such that $r_1(0) = p$ and $r_2(h(p)) = 0$ so that $h_1 := r_2hr_1$ has the property that $h_1(0) = 0$ and thus $h_1 \in G_1$. Define $q_1 = h_1(\infty)$. Since $G$ is three-transitive, there exists $g \in G_2$ such that $q_2 := g(q_1)$ is close enough to $\infty$ so that the quasiconformal dilatation of the mapping $g_1$, which exists by Lemma 3.20 taking $q_2$ back to $\infty$, is strictly smaller than that of $h$. Now, the composition

\[
(3.37) h_2 := g_1^{-1}gh_1 = g_1^{-1}gr_2hr_1,
\]

has the property that $h_2(0) = 0$ and $h_2(\infty) = \infty$, so that $h_2 \in G_2$. The dilatation of $h_2$ at 0 should vanish by Lemma 3.19, however since (i) $r_1, r_2$ are conformal, (ii) the dilatation of $g_1$ is strictly smaller than that of $h$, and (iii) the dilatation of $g$ vanishes at 0, by (3.37) the dilatation of $h_2$ at 0 cannot vanish, a contradiction.

Proof of Proposition 3.5. Since each $g \in G_2$ is conformal if $G \subset \text{Diffeo}(S^2)$ is virtual Möbius, and since each $h \in G$ can be decomposed as $r_1gr_2$ with $g \in G_2$, by Lemma 3.14 we have that $h$ is conformal since $g \in G_2$ and $r_1, r_2 \in \text{Rot}(S^2)$ are conformal. Since each $h \in G$ is conformal, we thus have that the group $G$ is conformal and by classical principles we have that $G$ is a subgroup of Möbius($S^2$) but $G \neq \text{Rot}(S^2)$ and thus $G = \text{Möbius}(S^2)$, by Proposition 4.2 below.

3.5. Non-virtual Möbius groups. Consider next the non-virtual Möbius groups with the aim of showing that the stabilizer on three points of these groups have a rich structure, in contrast to the (virtual) Möbius case. By definition of non-virtual Möbius, after the previous paragraph, $G_2$ does not preserve latitudes. In fact, the following much stronger conclusion holds.
Proposition 3.6 (Non-virtual M"obius groups). Let \( G \subset \text{Diffeo}(S^2) \) be non-virtual M"obius. Then \( G_3^0 \) has essential support.

Proposition 3.6 is proved with the aid of a number of further lemmas. If \( h \in G_3^0 \) and \( r_\pi \in \text{Rot}(S^2) \) is a rotation of angle \( \pi \) fixing 1 \( \in \) \( S^2 \), then \( r_\pi hr_\pi^{-1} \) is also in \( G_3^0 \) and since \( r_\pi(\gamma_1) = \gamma_1 \), we have

\[
d_{\infty} := \inf_{h \in G_3^0} d(\infty, h(\gamma_1)) = \inf_{h \in G_3^0} d(0, h(\gamma_1)),
\]

the minimal distance a point in the equator can approach the poles by isotopies in \( G_3^0 \). To prove Proposition 3.6, we show that either \( d_{\infty} = 0 \) or else \( G_3^0 = \text{Id} \). However, we construct below explicit non-trivial elements of \( G_3^0 \), so that we must have \( d_{\infty} = 0 \). Define

\[
\psi_\infty: (0, \pi/2) \to \mathbb{R}^+, \quad \psi_\infty(s) := \inf_{h \in G_3^0} d(h(\gamma_s), \infty),
\]

where \( \gamma_s \) is the latitude of height \( s > 0 \), with the height \( s \) measured as the distance from \( \infty \). Note that \( \psi_\infty(s) \) is a non-decreasing and continuous function of \( s \) and that, by the symmetry by \( r_\pi \) mentioned above, we have that \( \psi_\infty(s) = \psi_0(\pi - s) \).

Lemma 3.22. The meridian \( \gamma_1 \) is taken by \( G_3^0 \) to any height, that is, we have that \( d_{\infty} = 0 \).

The proof of Lemma 3.22 requires a number of more technical steps to complete.

Lemma 3.23. For every \( s \in (0, \pi/2] \), there exists \( h_s \in G_2 \) such that \( h(\gamma_s) \) is contained in both connected components of \( S^2 \setminus \gamma_s \) and \( h_s \) with this property can be chosen arbitrarily close to the identity.

Proof. First, there exists \( s_0 \in (0, \pi/2] \) and \( h \in G_2 \) such that \( h(\gamma_{s_0}) \neq \gamma_{s_0} \) is not a latitude and \( h \) with this property exist arbitrarily close to the identity. By definition, since \( G \) is non-virtual M"obius, there exists a latitude \( \gamma_{s_0} \subset S^2 \) of height \( s_0 \in (0, \pi) \) and \( h \in G_2 \) such that \( h(\gamma_{s_0}) \neq \gamma_{s_0} \) is not a latitude. By conjugating with \( r_\pi \in \text{Rot}(S^2) \) fixing 1 and interchanging 0 and \( \infty \), we may assume that \( s \in (0, \pi/2] \). Denote \( \gamma := h(\gamma_{s_0}) \). Since \( \gamma \) is not a latitude, it is not invariant by rotations around 0 and \( \infty \). In fact, \( \gamma \) can not be invariant under isotopy \( R: [0, \epsilon] \to G_2 \), with \( R(\cdot, \theta) = r_\theta \in G_2 \cap \text{Rot}(S^2) \) for any \( \epsilon > 0 \). Indeed, if there would exist such \( \epsilon > 0 \), then taking a finite iterate, we generate the entire circle group of rotations around 0 and \( \infty \) and invariance of the simple closed curve \( \gamma \) under the entire circle action means \( \gamma \) has to be a latitude, contrary to our assumptions. Therefore, for given \( \epsilon > 0 \), for a dense set of angles in this interval, we have that \( r_\theta(\gamma) \neq \gamma \). Define the mapping \( g_\theta := h^{-1}r_\theta h \). Since \( r_\theta \) is an isometry taking the topological disks, defined as the connected components of \( S^2 \setminus \gamma \), to new topological disks, \( \gamma \) has to properly enter both connected components of \( S^2 \setminus r_\theta(\gamma) \), as otherwise one such component has to be mapped strictly into itself, contradicting that the rotation is an ambient isometry. Thus for arbitrarily small \( \theta > 0 \), we have that \( \gamma_\theta := g_\theta(\gamma_{s_0}) \neq \gamma_{s_0} \) is not a latitude, since it is contained in both connected components of \( S^2 \setminus \gamma_{s_0} \), and by choosing \( \theta \) small enough, \( g_\theta \) is arbitrarily close to the identity in the \( C^1 \)-sense. To prove the claim holds for every \( s \in (0, \pi/2] \), since \( G \) is three-transitive, for any \( s \in (0, \pi/2] \), there exists \( g \in G_2 \) such that \( g(p_0) = p_s \), where \( p_0 \in \gamma_{s_0} \) and \( p_s \in \gamma_s \). Then either \( g(\gamma_s) \) is not a latitude, in which case the argument of Lemma 3.23 proves the case using \( g \in G_2 \), or \( g(\gamma_s) = \gamma_{s_0} \), in which case Lemma 3.23 applies with respect to the mapping \( h_s := g^{-1}h_{s_0}g \in G_2 \). In both cases, we have the desired conclusion. \( \square \)
Let $\ell \subset S^2$ be the geodesic arc connecting 0 and $\infty$ contained in the meridian passing through 0, 1, $\infty$.

**Lemma 3.24 (Monotone isotopy in $G_2$).** Let $G$ be non-virtual Möbius. Then there exists an isotopy $F: \mathbb{R} \to G_0^2$, continuous in $\text{Diffeo}(S^2)$, such that $F$ moves points strictly monotonously downwards on $\ell$ with increasing $t \in \mathbb{R}$. Furthermore, given $p_0 \in \ell$, we have that $F(p_0, t) \to 0, \infty$ for $t \to \pm \infty$.

The proof is an adaptation of that of the monotone isotopy in Proposition 3.3. First, we observe the following.

**Lemma 3.25.** Let $0 < s < \pi$ and $p \in S^2 \setminus \{0, \infty\}$. There exists $\delta > 0, \delta > 0$ and an isotopy $F_s^* : (-\delta, \delta) \to G_0^2$, continuous in $\text{Diffeo}(S^2)$, such that $F_s^*(\cdot, 0) = \text{Id}$ and such that $d_s'(t) = d(F_t^*(p), 0)$ is strictly decreasing, with $(s - \delta, s + \delta)$ contained in the image of $d_s$.

**Proof.** By Lemma 3.23 there exists a $g_s \in G_2$ such that $g_s(\gamma_s)$ is not a latitude. Let $r_t$ be the rotation of angle $t$ fixing $[0, \infty)$. Given $q \in \gamma_s$, let $\alpha(t) = g_s(r_t(q))$ which is a $C^1$ curve and since $g_s(\gamma_s) = \alpha(0, 2\pi)$ is not a latitude, there exists some $t_0$ such that, if $d(t) = d(\alpha(t), 0)$, then $d'(t_0) < 0$. Let $p \in \gamma_s$ and consider the curve $\beta : (-\pi, \pi) \to S^2$, defined by $\beta(s) = r_t g_s r_{t_1}(p)$, where $r_t(p) = r_{t_0}(q)$. Note that $\beta(0) = \alpha(t_0)$ and since $d(\beta(s), 0)$ is constant, $\beta$ intersects $\alpha$ transversely at $s = 0$. Therefore, the curves $r_{-t_1} g_s^{-1}(\alpha)$, whose image is contained in $\gamma_s$, and $r_{(-t_1)}(\beta)$ are transversal at $p = r_{-t_1}(\beta(0))$. The remainder of the argument is precisely that of Lemma 3.11 by taking

$$F_t^s := r_{-t_1} g^{-1}_s r_t g_s r_{t_1},$$

and sufficiently small $\delta > 0$ and $\delta_s > 0$.

**Proof of Lemma 3.24.** Apply the argument of Proposition 3.3 using Lemma 3.25 instead of Lemma 3.11. \qed

This in particular implies the following technical lemma.

**Lemma 3.26 (Tracing a curve by isotopies).** Given a continuous $\alpha : [0, 1] \to S^2 \setminus \{0, \infty\}$ with $\alpha(0) = 1$, there exists an isotopy $H : [0, 1] \to G^0_\infty$ such that $H(\alpha(t), t) = 1$.

**Proof.** Write the curve $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, where the first component $\alpha_1$ describes the height of the curve and the second component $\alpha_2$ the angle relative to the circle defined by 0, 1, $\infty$. Define the isotopy $H : [0, 1] \to G_2^0$ by using $F : [0, 1] \to G_0^2$ by Lemma 3.24 to trace the vertical movement of $\alpha$ and the rotation isotopy $R : [0, 1] \to G_0^2$ to trace the angle direction, and a suitable composition gives the desired isotopy. \qed

**Lemma 3.27.** If $d_\infty > 0$, then $\psi_\infty(s) > 0$ for all $s > 0$.

**Proof.** Suppose that $d_\infty > 0$ and suppose, for a contradiction, that $\inf\{s \mid \psi_\infty(s) > 0\} = s_0 > 0$. By definition of $s_0$, there exists a point $p_0 \in \gamma_{s_0}$, a small round open disk $U$ centered at $p_0$, such that arbitrarily close to $p_0$ in the upper component of $U \setminus \eta$, where $\eta := U \cap \gamma_{s_0}$, there exist points $p \in U$ such that $\Omega U^1(p) \to \infty$ and such that each point in the lower component of $U \setminus \eta$ stays in a uniformly bounded annulus $V_{\varepsilon_1} \subset S^2$ containing 1 in $S^2$ whose boundary curves are at distance $\varepsilon_1 > 0$ away from 0 and $\infty$, for some small $\varepsilon_1 > 0$. Denote $U^-$ and $U^+$ the upper and lower half disk defined above, and let $p \in U^+$ the point whose orbit accumulates to $\infty$ introduced above, and a point $q \in U^-$ whose orbit stays in the annulus $V_{\varepsilon_1}$ close to $p$. We
include the degenerate case where \( \varepsilon_1 = \pi/2 \) so that \( V_{\pi/2} = \gamma_{\pi/2} \), in which case the orbit stays in \( \gamma_{\pi/2} \) and we choose \( q \) to lie in \( \gamma_{\pi/2} \). By three-transitivity, there exists \( g \in G_2 \) such that \( g(q) = 1 \) and \( g(p) = p_0 \), which lies close to 1. Given an isotopy \( F(\cdot,t) \in G_0^2 \) taking the point \( p \) close to \( \infty \), define the conjugate isotopy \( L := g^{-1}FG \) and \( Q(\cdot,t) := H(L(\cdot,t),t) \) where \( H(\cdot,t) \in G_2^2 \) is an isotopy that traces the continuous curve \( \alpha(t) := L(1,t) \) back to 1, i.e. the isotopy with the property that \( H(\alpha(t),t) = 1 \in S^2 \) for all \( t \in [0,1] \). By Lemma 3.26 the isotopy \( H \) exists.

By construction, we have that \( Q(\cdot,t) \in G_3^0 \). Furthermore, since the orbit under \( F(\cdot,t) \) of \( q \) stays within \( V_{\varepsilon} \), and conjugating with and isotopy in \( G_2^0 \) that takes a point \( p \in V_{\varepsilon-\delta} \) for a small \( \delta << \varepsilon \) back to 1 is uniformly continuous by a compactness argument, the orbit of 1 under \( L(\cdot,t) \) stays within a uniform annulus \( V_{\varepsilon_2} \) where \( \varepsilon_2 > 0 \). Since the isotopy \( H \) traces the curve \( \alpha(t) \) which stays within \( V_{\varepsilon_2} \) back to 1 can be chosen to be a compact family of isotopies, and since by assumption, for each \( \varepsilon > 0 \) there exists an isotopy \( F(\cdot,t) \) such that \( F(p,1) = p_1 \) with \( d(p_1,\infty) < \varepsilon \), there exists \( 0 < \varepsilon_3 < \min\{\varepsilon_1,\varepsilon_2\} \), such that if \( F(\cdot,t) \) has the property that \( F(p,1) = p_1 \) with \( d(p_1,\infty) < \varepsilon_3 \), then the endpoint \( q_0 := Q(p_0,1) \) is such that \( d(q_0,\infty) << d_\infty \). Notice that the starting point \( p_0 \) we constructed lies as close to 1 as we desire. To complete the proof, we need to make sure we can carry out this construction while making sure that the orbit of \( p_0 \) indeed passes through \( \gamma_{\pi/2} \). For this, since the orbit of \( p_0 \) under \( Q(\cdot,t) \) gets very close to \( \infty \), it is sufficient by a continuity argument that the initial point \( p_0 \) lies slightly below \( \gamma_{\pi/2} \) for \( t = 0 \).

To ensure this condition, either the conjugation \( g \in G_2 \) defined above sends the horizontal tangent of \( \gamma_s \) in the image to a curve passing through 1 with a non-trivial slope in a small neighborhood of 1, in which case the points \( p \) and \( q \) in \( U \) can be positioned such that \( q \) is sent to 1 and \( p \) is sent to \( p_0 \) below \( \gamma_{\pi/2} \). If \( g \) sends a small portion of \( \gamma_s \) to an small portion of \( \gamma_{\pi/2} \), then by Lemma 3.24 since for every \( s \in (0,\pi) \), there exists a \( C^1 \)-small \( h \in G_2 \) such that \( \gamma := h(\gamma_s) \) is a curve intersecting both connected components of \( S^2 \setminus \gamma_s \), there also exists \( h \in G_2 \) such that \( h \) fixes any given point of \( \gamma_s \) (using the rotation action around 0 and \( \infty \)) and sends a curve of positive slope passing through the fixed point to the a curve of negative slope. Conjugating the original \( g \) suitably with \( h \), we obtain we new conjugation \( g \) that has the property that taking

![Figure 3.7. Proof of Lemma 3.27](image-url)
a point $p \in S^2$ slightly above $\gamma_s$ and $q$ slightly below $\gamma_s$, with $p$ and $q$ suitably positioned, the mapping $g$ sends $q$ to 1 and $p$ to a point slightly below $\gamma_{\pi/2}$, as required.

Lemma 3.28. If $\psi_\infty(s) > 0$ for all $s > 0$, then the group $G^0_3$ is equicontinuous.

Proof. First, we claim that there exists a function $\psi_1 : (0, \pi/2) \to \mathbb{R}^+$ such that, for every $h \in G^0_3$ and $p \in S^2$ with the property that $d(p, 1) > s$ implies that $d(h(p), 1) > \psi_1(s)$. Indeed, by Lemma 3.27 there exists a function $\psi_\infty : (0, \pi/2) \to \mathbb{R}^+$ with the properties as stated. As $G$ is three-transitive, we can choose $g \in G_s$ such that $g(1) = \infty$ and $g(\infty) = 1$. Since this switch mapping is uniformly continuous, and since $ghg^{-1} \in G^0_3$ for every $h \in G^0_3$, the uniform function $\psi_\infty(s)$ relative to $s \in S^2$ of Lemma 3.27 is transformed into a new such function $\psi_1$ relative to $1 \in S^2$ under conjugation with $g$, as required.

In order to prove equicontinuity of $G^0_3$, we have to show that, given a sequence $(h_k)_{k \in \mathbb{N}}$ in $G^0_3$, and points $p_1 \neq p_2$ and $q_1, q_2$ such that $(h_k(p_1))_{k \in \mathbb{N}}$ and $(h_k(p_2))_{k \in \mathbb{N}}$ are convergent sequences of points in the sphere $S^2$, with $\lim_{k \to \infty} h_k(p_1) = q_1 \in S^2$ and $\lim_{k \to \infty} h_k(p_2) = q_2 \in S^2$, then $q_1 \neq q_2$. Assume, for a contradiction, this is false, and denote $H^k(\cdot, t)$, with $t \in [0, 1]$, the isotopy associated to $h_k = H^k(\cdot, 1)$. Since $\psi_\infty(s)$ and $\psi_0(s)$ are both positive for every $s \in (0, \pi)$, the orbits $H^k(p_1, t)$ and $H^k(p_2, t)$ of $p_1$ and $p_2$ stay uniformly bounded away from $\infty$ and 0, for all $t \in [0, 1]$ and $k \in \mathbb{N}$. By conjugating $H^k(\cdot, t) \in G^0_3$ with $g \in G_2$, with the property that $g(p_1) = 1$ and $g(p_2) := p$, we obtain a new isotopy which we denote again $H^k(\cdot, t) \in G^0_3$ that now has the property that the points 1 and $p$ are sent under the sequence $(h_k)_{k \in \mathbb{N}}$ to points that converge in distance to zero as $k \to \infty$. For each $k$, let $\alpha_k : [0, 1] \to S^2$ the arc traced by the image under $H^k$ of 1 in $S^2$, that is $\alpha_k(t) := H^k(1, t)$. Applying Lemma 3.24 and Lemma 3.26 to this setup, for each $k \in \mathbb{N}$, there exists an isotopy $F^k(\cdot, t) \in G^0_2$, such that $F^k(\alpha(t), t) = 1$, for all $t \in [0, 1]$. Furthermore, since the orbits $\alpha_k$ are uniformly bounded away from $\infty$ and 0, by a compactness argument, the isotopies $F^k$ are uniformly continuous for all $t \in [0, 1]$ and $k \in \mathbb{N}$. Therefore, if we take the composition of $H^k$ and $F^k$, we obtain an isotopy $Q^k(\cdot, t) \in G^0_3$, for each $k \in \mathbb{N}$, such that $p \in S^2 \setminus \{0, 1, \infty\}$ with $d(1, p) = s > 0$ is sent to a sequence of points $q_k := Q^k(p, 1)$ with $q_k \to 1$, by the aforementioned uniform continuity and since the original sequence of points $h_k(p_1)$ and $h_k(p_2)$ had the property of converging to the same point as $k \to \infty$. However, this contradicts the conclusion of the first paragraph in which we constructed the function $\psi_1(s)$ that has to be positive for all $s > 0$.

Lemma 3.29. The group $G^0_3$ contains non-trivial elements.

Proof. Denote $\gamma_0 := h(\gamma_{\pi/2})$ and $\gamma(\theta) := r_\theta(\gamma_0)$, where $r_\theta \in \text{Rot}(S^2) \cap G_2$. To construct the desired isotopy in $G^0_3$, for all $\theta \in [0, \epsilon]$ for sufficiently small $\epsilon > 0$, we need to find a continuous arc of points $p : [0, \epsilon) \to S^2$, where $p(0) \in \gamma = \gamma(0)$ and $p(\theta) \in \gamma \cap \gamma(\theta)$. Indeed, in this case, since $p(\theta)$ moves continuously on both $\gamma$ and $\gamma(\theta)$, the point $q(\theta) := h^{-1}(p(\theta))$ moves continuously on $\gamma_0$, for $\theta \in [0, \epsilon]$. Furthermore, as $q(\theta)$ moves continuously on $\gamma_0$, there exists a continuous arc of rotations $r_{\phi(\theta)} \in \text{Rot}(S^2) \cap G_2$, such that $r_{\phi(\theta)}(q(\theta)) = 1 \in S^2$ and $r_{\phi(0)} = \text{Id}$. Therefore, given the above continuous section of points, assuming that $p_0 = 1 \in \gamma_0$, which can be achieved by precomposing by a single rotation in $G_2$, the isotopy is defined by

$$(3.40) \quad H : [0, 1] \to G^0_3, \quad H(\cdot, t) := r_{\phi(\epsilon)}h^{-1}r_{\epsilon h}.$$

Choosing $\epsilon > 0$ slightly smaller if necessary, we may assume that $h(\gamma_1) := H^s(\gamma_1, 1) \neq \gamma_1$ is not a latitude and (thus) intersects both connected components of $S^2 \setminus \gamma_1$. To show this
continuous section exists, we need the following setup. Choosing \( h \) sufficiently close to the identity using Lemma 3.23, the image curve \( \gamma_0 \) is a simple closed curve \( C^1 \) close to \( \gamma_{\pi/2} \), such that the derivative \( \gamma'_0(t) \) is everywhere in the range \([1 - \delta, 1 + \delta]\), with \( 0 \leq \delta \ll 1 \). Denote \( s_1 \in (0, \pi/2) \) the unique smallest value of \( s \) for which \( \gamma_s \cap \gamma_0 \neq \emptyset \). Note that such \( s_1 \) always exists. Define \( \omega := \gamma_{s_1} \cap \gamma_0 \subset S^2 \), which is a closed subset of \( \gamma_{s_1} \). If \( \omega \) contains an interval, then the section is readily seen to exist. Therefore, suppose that \( \omega \) is nowhere dense. In this case it is either a Cantor set, or it has isolated points. Either way, there exists a point \( x_0 \in \gamma_{s_1} \) such that to the left of \( x_0 \) in \( \gamma_{s_1} \) there are no points of \( \omega \) for some small open interval. In addition, since at points of \( \omega \) the curve \( \gamma_0 \) has derivative 1, if we denote \( x_0 = \gamma_0(t_0) \), there exists a small time interval \((t_0 - \varepsilon_1, t_0)\) of positive length, on which the condition \( \gamma'_0(t) > 1 \) holds. Since \( \gamma'_0(t) > 0 \) the interval \((t_0 - \varepsilon_1, t_0)\), \( \gamma_0(t) \) is monotonically increasing in height as \( t \to t_0 \) on this interval and moving monotonically to the right. Define \( 0 < \varepsilon_r < \varepsilon_l \), such that \( \gamma_0(t) \) has height bounded below from \( \gamma_0(t_0 - \varepsilon_1, t_0) \) for all \( t \in (t_0, t_0 + \varepsilon_r) \). Further, since \( \gamma'_0(t) \approx 1 \) for all \( t \), the piece \( \gamma_0(t) \) with \( t \in (t_0, t_0 + \varepsilon_r) \) moves strictly monotonically to the right. Cutting locally the piece of the curve \( \gamma_0(t) \) into the two pieces \( \gamma_l \) and \( \gamma_r \) defined by the time intervals \((t_0 - \varepsilon_1, t_0)\) and \((t_0, t_0 + \varepsilon_r)\) and rotating the defined left piece of the curve by a small rotation of angle \( \theta \leq \varepsilon_r \) to the right, the intersection point of \( r_\theta(\gamma_l) \) with the above defined right piece of the curve \( \gamma_r \subset \gamma_0 \), is unique and moves continuously for \( \theta \in [0, \varepsilon_r] \) on both pieces simultaneously, as required. \( \square \)

**Proof of Lemma 3.22.** Either \( O_{G^0_3}(\gamma_1) \) reaches any height, or else the group \( G^0_3 \) is equicontinuous by Lemma 3.28. In the latter case, every cyclic subgroup generated by an element \( h \in G^0_3 \) is uniformly equicontinuous over all iterates and thus topologically conjugate to a rotation [10]. But since every such homeomorphism \( h \) in \( G^0_3 \) fixes at least three distinct points, this implies that the conjugate rotation fixes three distinct points as well. Since the only rotation of the sphere fixing three distinct points has to be the identity, and mappings conjugate to the identity are the identity, \( h \) itself has to be the identity. Since this holds for each \( h \in G^0_3 \), we must have that \( G^0_3 = \text{Id} \). However, by Lemma 3.29 we have that \( G^0_3 \neq \text{Id} \), yielding the desired contradiction that proves the claim. \( \square \)

The condition that certain points in \( \gamma_1 \) are taken by \( G^0_3 \) to any given height is now improved to the essential support condition.

**Lemma 3.30** (Essential support of the stabilizer). If \( G \) is non-virtual Möbius, then \( G_3 \) has essential support.

This is proved with the aid of the following lemmas. Define \( \Omega := \Omega_{0,1} \cup \Omega_{0,\infty} \cup \Omega_{1,\infty} \), where \( \Omega_{p,q} \subset S^2 \) is defined as the closure of the points in the sphere \( S^2 \setminus \{0, 1, \infty\} \) for which there exist isotopies in \( G^0_3 \) whose orbits accumulate to \( \{p, q\} \), with \( p, q \in \{0, 1, \infty\} \).

**Lemma 3.31.** If \( G \) is non-virtual Möbius, then \( 1 \in \Omega_{0,\infty} \cap \gamma_{\pi/2} \).

**Proof.** By Lemma 3.22 there exist isotopies \( F^n \in G^0_3 \) and \( x_n \in \gamma_{\pi/2} \), such that if we define \( h_n := F^n(\cdot, 1) \), then \( h_n(x_n) \to \infty \). Since \( \gamma_{\pi/2} \) is compact, there exists at least one limit point \( x \in \gamma_{\pi/2} \) of the sequence \( (x_n)_{n \in \mathbb{N}} \). and the claim follows. To show that \( 1 \in \gamma_{\pi/2} \), the closure of points limiting on \( \infty \), suppose this is not the case, and take \( x_0 \in \gamma_{\pi/2} \) the nearest point to \( 1 \in \gamma_{\pi/2} \) still contained in \( \Omega_{0,\infty} \), which we may assume lies to the right of \( 1 \) with respect to the ordering on the circle, and denote \( J \subset \gamma_{\pi/2} \) the interval \( J := (1, x_0) \). Choose a small \( 0 < \varepsilon < x_0 \)
Lemma 3.32. If $G$ is non-virtual M"obius, then $\Omega$ is an essential continuum.

Proof. To prove that $\Omega$ is an essential continuum, we show that each of the three constituent subsets of $\Omega$ is a continuum (i.e. compact and connected set) and that these pairwise intersect, so that the union is compact and connected as well. Then we show that $\Omega$ is essential.

To prove these claims, first, by Lemma 3.31, $1 \in \Omega_{0,\infty} \cap \gamma_{\pi/2}$. Further, by conjugating by a switch $g \in G_1$ that switches 1 and $\infty$, and using that $g(\Omega_{0,\infty}) = \Omega_{0,1}$, we have that $\infty \in \Omega_{0,1}$. We claim that $\Omega_{0,\infty}$ is connected. To prove this, we show that each point in $\Omega_{0,\infty}$ is connected to $\infty \in \Omega_{0,\infty}$. Take a point $p \in \Omega_{0,\infty}$ and take a connected component $C$ of $\Omega_{0,\infty}$. Given a latitude $\gamma_s$ of height $s \in (0,\pi/2)$, by definition of $\Omega_{0,\infty}$, there exist points $p_n$ and arcs $\alpha_n$ emanating from $p_n$ such that the Hausdorff distance of $\alpha_n \to 0$ is smaller than $s$ and such that $p_n \to p$ as $n \to \infty$. Since each $\alpha_n$ is a continuum and the Hausdorff limit (which exists by compactness of the sphere $S^2$) is a continuum, the point $p$ is connected to a point $q \in \gamma_s$, where $q$ is a limit point of $\alpha_n \cap \gamma_s$, which again by compactness exists, after possibly passing to a subsequence. Since this argument holds for each $s \in (0,\pi/2)$, the claim follows.

To show that $\Omega_{0,\infty}$ is essential, suppose that there exists a simple closed curve $\gamma$ in the complement of $\Omega_{0,\infty}$. Applying the above to $\Omega_{0,\infty}$ and using that $r_\pi(\Omega_{0,\infty}) = \Omega_{0,\infty}$, with $r_\pi \in \text{Rot}(S^2)$ the rotation of angle $\pi$ around 1, and $1 \in \Omega_{0,\infty}$, thus $\Omega_{0,\infty}$ does not contain curves homotopic to $\infty$ and 0. By taking a switch as above that switches 1 and $\infty$, conjugating the action relative to 0, $\infty$ to the action accumulating at 1, it is seen that there can be no curves homotopic to 1 in the complement of $\Omega$ either. This concludes the proof.

Lemma 3.33. If $G$ is non-virtual M"obius, then $\Omega$ contains interior.

Proof. In order to prove that $\Omega_{\infty}$ has interior, we show that if $g \in G_2^0$ such that $g(p) = 1 \in \Omega^c$, then $g(1) \in \text{Int}(\Omega)$. From this we readily below conclude the desired claim. Assume that $p$ has bounded orbit and $q := g(1) \in \Omega^c$ has bounded orbit as well. Then we claim that $\Omega^c = \emptyset$. Suppose to the contrary, that there exists $p \in S^2 \setminus \{0, \infty\}$ such that there exists a neighborhood $U \supset p$ with the property that $U \cap \Omega = \emptyset$. Relative to such $p$, denote $G_{3,p}^0 := \text{Stab}_{\gamma_p}(0,p,\infty)$ the group of isotopies stabilizing pointwise $\{0,p,\infty\}$, where $G_3^0 := G_{3,1}^0$. Take $g \in G_2$ such that $g(p) = 1$ and define the map

$$g^* : G_3^0 \to G_{3,p}^0, \quad g^*(F) = g^{-1}Fg,$$
which is an isomorphism between the two groups. Since \( d(\{0, \infty\}, F(p, t)) > \varepsilon_1 \) for all \( F \in \mathcal{G}_3^0 \), \( d(\{0, \infty\}, H(1, t)) > \varepsilon_2 \), for every \( H \in \mathcal{G}_3^{0, p'} \).

Indeed, using the isomorphism (3.41), denote \( h_1 \in G_3^0 \) and \( h_p \in G_3^{0, p} \) and let \( g(p) = 1 \), where \( h_p = g^{-1}h_1g \). Then \( gh_p(1) = h_1(g(1)) \). Since \( g(1) = q \in \Omega^c \) and thus \( h_1(q) \) stays in the annulus \( V_{\varepsilon_1} \) and thus has strictly positive distance from \( \{0, \infty\} \), we have that \( h_p(1) \) has strictly positive distance from \( \{0, \infty\} \) as well, and this distance is uniform over all \( h_p \in G_3^{0, p'} \). By Lemma 5.22 there exist \( F^n \in G_3^0 \) and \( x_n \to 1 \) such that \( F^n(x_n, \infty) \to 0 \). Define \( H^n := g^*(F^n) \in G_3^{0, p} \) using (3.41), and define \( p_n := g(x_n) \to p \), for each \( n \in \mathbb{N} \). Further, define \( a^n : [0, 1] \to S^2 \) the continuous arc by \( a^n(t) := H^n(1, t) \). By (3.42), the arc \( a^n \) stays in a bounded annulus \( V_{\varepsilon_2} \), for all \( t \in [0, 1] \) and \( n \in \mathbb{N} \). By Lemma 3.24 and Lemma 3.26, there exists \( L^n \in G_2^0 \) such that \( L^n(a^n(t), t) = 1 \) for all \( t \in [0, 1] \) and \( n \in \mathbb{N} \). Furthermore, since the arcs \( a^n \) stay in a bounded annulus, we have that \( h_{n, t} := L^n(\cdot, t) \) are uniformly continuous over all \( t \) and \( n \). Define the isometry \( Q^n(\cdot, t) := L^n(F^n(\cdot, t), t) \in G_2^0 \). Then \( Q^n(p_n, 1) \to \infty \) as \( n \to \infty \), contrary to our assumption. To finish the proof, either \( \Omega = S^2 \), or else \( \Omega^c \) is open and thus contains an open disk. In the latter case, if \( g(p) = 1 \in \Omega^c \), then \( g(1) \in \text{Int}(\Omega) \). Indeed, if \( q \in \Omega^c \) then the above argument shows this is impossible. And if \( q \in \partial \Omega \), then by using small perturbations in \( G_2^0 \) using Lemma 5.24 we can move \( g \) slightly away into \( \Omega^c \) without moving the perturbed initial point \( p \) out of \( \Omega^c \), so that the initial argument applies again to show that \( \Omega^c = \emptyset \). \hfill \Box

Proof of Lemma 3.30. Combine Lemma 3.32 with Lemma 3.33. \hfill \Box

This concludes the proof of Proposition 3.6.

3.6. Virtual linear groups. The previous paragraph has shown that the type I groups corresponding to the one-transitive actions on \( S^2 \) correspond precisely to the groups contained in \( \text{Ant}(S^2) \equiv \text{Homeo}(\mathbb{R}P^2) \). We need to show the following.

Theorem 2 (Transitivity of groups acting on \( \mathbb{R}P^2 \)). Let \( G \subseteq \text{Ant}(S^2) \), then either

(a) \( G \) is one-transitive on \( \mathbb{R}P^2 \) and \( G = \text{Rot}(\mathbb{R}P^2) \).
(b) \( G \) is two-transitive on \( \mathbb{R}P^2 \) and \( G = \text{Lin}(\mathbb{R}P^2) \).
(c) \( G \) perturbs meridians and has essential support.

We first show that every \( G \subseteq \text{Homeo}(\mathbb{R}P^2) \) with \( G \neq \text{Rot}(\mathbb{R}P^2) \) is at least two-transitive.

Proposition 3.7. Let \( \text{Rot}(S^2) \subset G \subseteq \text{Ant}(S^2) \), then \( G \) is at least two-transitive on \( \mathbb{R}P^2 \) by isotopies.

Lemma 3.34. Given a meridian \( \gamma \subset S^2 \) and \( g \in \text{Ant}(S^2) \), we have that \( g(\gamma) \cap \gamma \neq \emptyset \).

Proof. Indeed, the meridian \( \gamma_1 \) divides the sphere \( S^2 \) into two hemispheres. If \( g(\gamma_1) \) is disjoint from \( \gamma_1 \), then \( g(\gamma_1) \) has to lie entirely in either the northern or southern hemisphere. Taking a point of \( g(\gamma_1) \) in one such hemisphere, the antipodal point has to lie in the other hemisphere. Since \( g(\gamma_1) \) is a continuous curve, it has to cross \( \gamma_1 \) at some point. \hfill \Box

Lemma 3.35. Let \( \text{Rot}(S^2) \subset G \subseteq \text{Ant}(S^2) \) and \( \mathcal{O}(\gamma_1) := \mathcal{O}_{G_1}^G(\gamma_1) \). Then for every \( p \in \mathcal{O} \), there exists an isotopy to the identity \( H \in G_3^0 \) and \( x \in \gamma_1 \) such that \( H(x, 0) = x \) and \( H(x, 1) = p \). Moreover, \( \mathcal{O} \) is connected.
Proof. Since rotations fixing 0, ∞ are elements of $G_1$, it suffices to prove the existence of an isotopy $H(\cdot, t)$ such that $\max d(H(\gamma_1, 1), 0) \geq d(p, 0) \geq \min d(H(\gamma_1, 1), 0)$. Since $p \in \mathcal{O}$, there exists $h \in G_1$ such that $p \in h(\gamma_1)$. Let $q = h^{-1}(p)$, and let $x \in \gamma_1$ such that $h^{-1}(x) \in \gamma_1$, which by Lemma 3.34 exists. Consider $H(\cdot, t) := hr_{\theta(t)}^{-1}(\cdot)$, where $r_{\theta(t)}$ is a continuous family of rotations parametrized by $t \in [0, 1]$ with $r_{\theta(0)} = \text{Id}$ and $r_{\theta(1)}(h^{-1}(x)) = h^{-1}(p)$. Since $H \in G_1$ is an isotopy with $H(\cdot, 0) = \text{Id}$ and $H(x, 1) = p$ by construction, the first claim follows. Since $G_1^0$ is the group of isotopies to the identity containing the stabilizer of the rotation group around a point, the orbit $\mathcal{O}(\gamma_1)$ is readily seen to be connected. □

Proof of Proposition 3.7. We claim that if $G \subseteq \text{Ant}(S^2)$, with $G \neq \text{Rot}(S^2)$, then $\mathcal{O}(\gamma_1) = S^2 \setminus \{0, \infty\}$. Indeed, since $\mathcal{O}(\gamma_1)$ is connected and invariant by rotations, there exists $d_1 \geq 0$ and $d_2 \leq \pi$ such that

\begin{equation}
\partial \mathcal{O} = \{ p \mid d(p, 0) = d_1, d(p, 0) = d_2 \}.
\end{equation}

Since the boundaries of this annulus are invariant by the stabilizer by Lemma 3.35, we must have that $d_1 = 0$ and $d_2 = \pi$ by Lemma 3.7. Since $\gamma_1$ can have any height under the action of $G_1$, by using the rotation action around 0, $\infty$, we can move any point $p \neq 0, \infty$ to any other point $q \neq 0, \infty$ and thus $G$ is two-transitive. □

If $G \subseteq \text{Lin}(\mathbb{RP}^2)$, then either $G = \text{Rot}(\mathbb{RP}^2)$ or $G = \text{Lin}(\mathbb{RP}^2)$, which are one and two transitive respectively. The group $G_2 := \text{Stab}_G(0, 1)$ is defined if $G \subseteq \text{Ant}(S^2)$ and $G \neq \text{Rot}(S^2)$, by Proposition 3.7. The property virtual linear of $G \subseteq \text{Ant}(S^2)$ is defined as $G_2 \subseteq G$ leaving invariant the meridian $\gamma_0$ passing through $0, \infty, 1, -1$. Note that the groups $\text{Rot}(S^2)$ and $\text{Lin}(S^2)$ are virtual linear in this definition. Conversely, we have the following.

Proposition 3.8 (Virtual linear groups). If $G \subseteq \text{Ant}(S^2)$ is virtual linear, then either $G = \text{Rot}(S^2)$ or $G = \text{Lin}(S^2)$.

Lemma 3.36. Let $G \subseteq \text{Ant}(S^2)$. For any $g \in G$ and any given meridian $\gamma \subset S^2$, there exist $p_1, p_2 \in \gamma$ with the property that $d(p_1, p_2) = \pi/2$ and $d(q_1, q_2) = \pi/2$ with $q_1 = g(p_1)$ and $q_2 = g(p_2)$.

Proof. Since $g \in \text{Ant}(S^2)$, the curve $g(\gamma_0)$ splits the sphere into two symmetric parts. Take two points $p_1, p_2$ with the given distance between them and consider the distance $d(q_1, q_2)$ in the image. If this distance is less than $\pi/2$, moving the points $p_1, p_2$ along $\gamma_0$, at some point the distance between the image points has to become at least $\pi/2$. Indeed, if this was not the case, then dividing the meridian $\gamma$ into four quarters, and mapping these to the image curve $g(\gamma)$, the distance between each two consecutive image points is strictly less that $\pi/2$ each. Taking the piecewise geodesic curve connecting the four vertices, this piecewise geodesic curve must be symmetric with respect to the antipodal action. However, since each of the constituent geodesic arcs of this curve is strictly less in length than $\pi/2$, its total length is strictly less than $2\pi$. However, each such piecewise geodesic curve that is symmetric with respect to the antipodal involution would have to have length at least $2\pi$, with equality in this inequality if and only if the piecewise geodesic curve is in fact geodesic. In addition, there always exists such $p_1$ and $p_2$ for which the images lie a distance at most $\pi/2$ apart. Indeed, since $g \in \text{Ant}(S^2)$ and the points $p_1$ and $-p_1$ lie a distance $\pi$ apart, the distance between $g(p_1)$ and $g(-p_1) = -g(p_1)$ is also $\pi$. Therefore, the midpoint on $\gamma$ between $p_1$ and $p_2$ denoted by $p_0$ has distance $\pi/2$ from both $p_1$
and $p_2$. In the image the distance between the point $q_0 = g(p_0)$ and either $g(p_1)$ or $-g(p_1)$ is at most $\pi/2$, as required.

\begin{lemma}
\textbf{(Stabilizer on two points in $\mathbb{RP}^2$).} The stabilizer $G_2$ in $\mathbb{RP}^2$ has the following relation with $G$. For every $g \in G \subseteq \text{Ant}(S^2)$, there exist $r_1, r_2 \in \text{Rot}(S^2)$ and $h \in G_2$ such that $g = r_2hr_1$.
\end{lemma}

\begin{proof}
Take a meridian $\gamma$, the points $p_1, p_2 \in \gamma$ constructed above and corresponding image points $q_1, q_2 \in g(\gamma)$, there exist rotations taking these points to 0,1 and the induced mapping $h \in G_2$ has the property as claimed.
\end{proof}

\begin{proof}[Proof of Proposition 3.8]
By Lemma 2.5 if $G$ preserves meridians, then $G \subseteq \text{Lin}(S^2)$. Furthermore, by Proposition 4.1 below, if $G \subseteq \text{Lin}(S^2)$, then either $G = \text{Rot}(S^2)$ or else $G = \text{Lin}(S^2)$. Therefore, it suffices to show that $G \subseteq \text{Ant}(S^2)$ not preserving meridians implies that there exist $g \in G_2$ that send $\gamma_0$ to a non-meridian. To prove this, take any $g \in G$ with the property that $\gamma \subset S^2$ is a meridian, but $g(\gamma)$ is not a meridian. By Lemma 3.30, there exist points $p_1, p_2 \in \gamma$ with $d(p_1, p_2) = \pi/2$ such that the image points $d(q_1, q_2) = \pi/2$ for the image points $q_1$ and $q_2$ of $p_1$ and $p_2$ under $g$. Precompose $g$ with a rotation $r_1 \in \text{Rot}(S^2)$ so that $\gamma = \gamma_0$ the vertical meridian and such that $p_1 = 0$ and $p_2 = 1$, so that $-p_1 = \infty$ and $-p_2 = -1$. Postcompose with a rotation $r_2 \in \text{Rot}(S^2)$ so that $q_1 = 0$ and $q_2 = 1$, so that again $-q_1 = \infty$ and $-q_2 = -1$. Then $h := r_2g r_1 \in G_2$ and $h(\gamma_0)$ is not a meridian by construction, as required.
\end{proof}

### 3.7. Non-virtual linear groups.

The third part of the proof of Theorem 2 is to show the following. Consider the subgroup of isotopies to the identity $G_2^0 \subseteq G_2$ fixing two points in $\mathbb{RP}^2$.

\begin{proposition}[Non-virtual linear groups]
Let $G \subseteq \text{Diff}(\mathbb{RP}^2)$ be non-virtual linear. Then $G_2$ acting on $\mathbb{RP}^2$ perturbs meridians and $G_2$ has essential support.
\end{proposition}

\begin{proof}
Since the proof is similar to the proof of the non-virtual Möbius case, we only remark on the differences needed to adapt the proof. First, by Proposition 3.8 as $G$ is non-virtual linear, the group $G_2 \subset G$ has to perturb meridians into non-meridians, which proves the first claim.

Therefore, we need to prove that $G_2$ has essential support. If $G \neq \text{Rot}(\mathbb{RP}^2)$, using that $\mathcal{O}_{G_2}(\gamma_{\pi/2}) = S^2 \setminus \{0, \infty\}$ by Proposition 3.7, we obtain that $G_2^0$ is transitive on $S^2 \setminus \{0, \infty\}$, so that in particular we can send a point $p \in \gamma_{\pi/2}$ into any height $\gamma_s$, where $s \in (0, \pi/2)$ by isotopies. Then using a wiggle argument in $G_1^0$ that wiggles $\gamma_{\pi/2}$ to obtain by pullback a curve passing through $\gamma_s$, for any given $s$, that passes through $\gamma_s$ transversely, we obtain in $\text{Ant}(S^2)$ the analogue of Lemma 3.26 as the main step in showing the existence in $\text{Ant}(S^2)$ of a monotone isotopy analogous to Lemma 3.24 that moves by isotopies in $G_1^0$ monotonously in height a point up and down. Further, by the same reasoning as before, we can trace a continuous arc $\alpha : [0,1] \rightarrow S^2 \setminus \{0, \infty\}$ with $\alpha(0) = 1$ by an isotopy in $G_1^0$. Since fixing two points in $\mathbb{RP}^2$ amounts to fixing two pairs of antipodal points in the sphere, $\{-1,1\}$ and $\{0, \infty\}$, and since every rotation leaving fixed all four points has to be the identity, the analogue in $\text{Ant}(S^2)$ of Lemma 3.28 holds in this setting, showing that either $\gamma_{\pi/2}$ reaches any height under $G_2^0$ or else we have that $G_2^0 = \text{Id}$. However, by Proposition 3.7 the group $G_1$ sends the meridian $\gamma_{\pi/2}$ to any given height relative to $\infty$ and 0, and since $G \subset \text{Ant}(S^2)$ and using $\text{Rot}(S^2) \cap G_1$, we can use the argument of Lemma 3.29 to construct non-trivial isotopies in $G_2^0$ to the identity that move $\gamma_{\pi/2}$. Therefore, we have that $G_2^0 \neq \text{Id}$, and thus $\gamma_{\pi/2}$ reaches any height under $G_2^0$. Finally,
that the support of $G_2$ is essential follows the same argument, as does the final step in showing that the support $\Omega$ has interior proved in Lemma 3.33. This concludes the proof for the case of $\text{Ant}(S^2)$.

\[\square\]

3.8. Proof of Theorem A. Theorem A follows by combining Theorem 1 and Theorem 2.

3.9. Proof of Theorem B. To prove Theorem B, consider a homogeneous transformation group $G \subset \text{Diffeo}(S^2)$ and assume that $G$ is not contained in $\text{Ant}(S^2)$ and is $k$-transitive, with $k \geq 1$.

Lemma 3.38. Let $G \subset \text{Homeo}(S^2)$ not be contained in $\text{Ant}(S^2)$ and suppose that $k \geq 4$. Then there exist $h \in G$ such that $h$ contains an action homotopic to a pseudo-Anosov mapping relative to the four-punctured sphere $S^2 \setminus \{0, 1, -1, \infty\}$.

Proof. First, consider four points $\{1, 2, 3, \infty\}$, where $\infty$ is the north-pole as before and place the other three points $\{1, 2, 3\}$ on the equator $\gamma_{\pi/2}$ equidistant one from the other. Then the points 1, 2, 3 can be permuted cyclically by the order three rotation $r \in \text{Rot}(S^2)$ of angle $2\pi/3$ and fixing 0 and $\infty$. The subgroup of the mapping class group of four-punctured sphere $S^2 \setminus \{1, 2, 3, \infty\}$ of isotopy classes of homeomorphisms keeping $\infty$ fixed is isomorphic to the mapping class group of the three times punctured disk $D \setminus \{1, 2, 3\}$, with $D$ a closed disk and three distinct points 1, 2, 3 $\in D$, and its mapping class group is generated by two Dehn-twists about the pairs $\{1, 2\}$ and $\{2, 3\}$. By four-transitivity, there exist $h \in G$ such that $h$ sends the ordered tuple $\{1, 2, 3\}$ to $\{2, 1, 3\}$, i.e. $h$ switches the points 1 and 2. We can write $h \in G$ as a finite composition of Dehn-twists $h \cong e_1 \cdots e_m$, where for each $1 \leq i \leq m$, $e_i$ is a Dehn-twist about either $\{1, 2\}$ or $\{2, 3\}$. Similarly, using the rotation action and the symmetry of the placement of the marked points, we can produce a conjugate mapping $g := r^{-1}hr \in G$ that sends the ordered tuple $\{1, 2, 3\}$ to $\{1, 3, 2\}$ and again we can write $g \cong f_1 \cdots f_n$, where $1 \leq j \leq n$, over the same basis of Dehn-twists about $\{1, 2\}$ and $\{2, 3\}$. Since the composition of $h$ and $g$ can now be written as a word, over the two generating of Dehn-twists, where the word contains at least one Dehn-twist interchanging $\{1, 2\}$ and at least one other Dehn-twist interchanging $\{2, 3\}$, using an analogue of Sharkovski’s theorem for train-tracks, it can now be shown [11], that the composition $hg \in G$ contains a pseudo-Anosov action, as required.

\[\square\]

Lemma 3.39. Let $G \subset \text{Ant}(S^2) \subset \text{Homeo}(S^2)$ and suppose that $k \geq 3$, then there exist $h \in G$ such that $h$ contains an action homotopic to a pseudo-Anosov relative to the three times punctured projective plane.

Proof. Indeed, take $G$ that acts three-transitively on $\mathbb{R}P^2$. Identifying $G$ to the group acting the cover $S^2$ of $\mathbb{R}P^2$ and three-transitively on antipodal points on $S^2$. Place two pairs of antipodal points $\{1, -1\}$ and $\{2, -2\}$ symmetrically on the meridian $\gamma_1 \subset S^2$ and take $h$ that fixes $\{0, \infty\}$ and $\{1, -1\}$ and interchanges $\{2, -2\}$, using three-transitivity of $G$. This setup can be identified with mappings of the disk $D$ punctured at four symmetrically placed points $\{1, -1, 2, -2\} \subset D$ around the origin (which is kept fixed). Since the action is antipodal, sending 2 to $-2$ automatically means sending $-2$ back to 2. By conjugating $h$ with a rotation around $0, \infty$ by angle $\pi$, we obtain a conjugate homeomorphism $g$ fixing $\{2, -2\}$ and interchanging $\{1, -1\}$. Due to their action neither $h$ nor $g$ can be homotopic to an isometry. The homeomorphism $h$
has a mapping class consisting of disjointly supported Dehn-twists, or else $h$ contains a pseudo-Anosov action \cite{22}, and similarly for $g$, in which case we have the desired conclusion. In the former case, it is verified that $hg^{-1}$ contains a pseudo-Anosov action. □

**Proof of Theorem B.** Suppose that $G$ is $k$-transitive, for some $k \geq 1$. Since $G$ contains $\text{Rot}(S^2)$ and $\text{Rot}(S^2)$ is one-transitive, but not sharply one-transitive, there are no sharply one-transitive groups $G$. By Theorem A, there exist no two-transitive groups $G$, and therefore no such sharply two-transitive groups either. The case where $k = 3$, by Theorem A, we can only have $G = \text{M"obius}(S^2)$ and this group is sharply three-transitive. Therefore, to conclude the proof, let us consider the case where $k \geq 4$. In this case, by Lemma 3.38, there exist $h \in G$ such that $h$ contains an action homotopic to a pseudo-Anosov on the four-punctured sphere. In particular, since any such homeomorphism has periodic points of arbitrarily high period by classical theory \cite{22}, given $N \in \mathbb{N}$, there exists a sufficiently high iterate of $h$, such that $h$ fixes at least $N$ distinct points, but $h$ is not the identity on the complement of these points. As $N$ can be taken as large as desired by taking powers of $h$, this proves the claim.

The proof in the case $G \subseteq \text{Ant}(S^2)$ is analogous to the above argument, where now we use Lemma 3.39 and we observe that neither $\text{Rot}(\mathbb{R}P^2)$ nor $\text{Lin}(\mathbb{R}P^2)$ is sharply transitive for $k = 1$ and $k = 2$ respectively. □

4. The structure of the diagram

In Theorem A, we approach the classification of homogeneous groups by providing a general mechanism underlying all homogeneous groups in terms of their behaviour in various shapes and forms. The second, and complementary, way is to find explicit relations between groups so as to reduce the possibilities further. Specifically, the purpose of this section is to prove completeness of the arrows marked with $\star$ in the diagram as expressed by Theorem C. This result complements Theorem A, and shows how in a number of concrete cases perturbing the symmetries of one group leads in a constructive way to generating a much larger group. In the proofs below, an $\epsilon$-grid $\Gamma(\epsilon) \subset S^2$ on the sphere is defined to be the union of vertical meridians passing through 0 and $\infty$ and horizontal latitudes perpendicular to the meridians, such that the complement $S^2 \setminus \Gamma(\epsilon)$ consists of finitely many open disks of diameter at most $\epsilon$.

4.1. Area-preserving groups. The purpose of this section is to consider the following area-preserving actions.

- $\text{Homeo}_\lambda(S^2)$
- $\text{Rot}(S^2)$
- $\text{Ant}_\lambda(S^2)$
- $\text{Homeo}(S^2)$

We start by showing the following arrow $\text{Homeo}_\lambda(S^2) \rightarrow \text{Homeo}(S^2)$ is complete. The result was proved independently by Le Roux \cite{21}, for the group of area-preserving homeomorphisms of any closed triangulable manifold, in every dimension $n \geq 2$.

**Theorem 3.** The area-preserving group $\text{Homeo}_\lambda(S^2)$ is maximal in $\text{Homeo}(S^2)$.

Let $h \in \text{Homeo}(S^2)$ be any homeomorphism and let $\epsilon > 0$ be the error allowed in approximating $h$ with homeomorphisms in $G$. Take an $\epsilon$-grid $\Gamma_0 := \Gamma(\epsilon) \subset S^2$ on the sphere, and let
$\Gamma = h(\Gamma_0)$ be the distorted image grid. The argument now consists in being able to move this perturbed grid back into a straight grid using only homeomorphisms from the group $G$ containing $\text{Homeo}(S^2)$ and at least one homeomorphism that does not preserve area. Since the $\epsilon > 0$ is taken arbitrarily small, this means every homeomorphism $h$ can be approximated arbitrarily well by homeomorphisms in the group $G$. Since the $\epsilon > 0$ is taken arbitrarily small, this means every homeomorphism $h$ can be approximated arbitrarily well by homeomorphisms in the group $G$. A good arc or curve $\eta, \gamma \subset S^2$ is an arc or curve that has zero area and a good disk is a closed topological disk whose boundary curve is a Jordan curve of zero area. Since every homeomorphism can be approximated uniformly by piecewise linear homeomorphisms, by a diagonal argument, we may assume that $\Gamma = h(\Gamma_0)$ has no area.

Given a closed topological disk $D \subset S^2$, denote $\text{Homeo}_\lambda(D) \subset \text{Homeo}(D) \subset \text{Homeo}(S^2)$ the group of area-preserving homeomorphisms of the sphere that leave invariant the good disk $D \subset S^2$ and have support contained in the interior of $D$. In particular, these homeomorphisms are the identity on the boundary $\partial D$ of the disk $D$.

**Lemma 4.1** (Elementary move). There exists a $\delta > 0$, such that given a good disk $D \subset S^2$ of area at most $\delta$, and two good arcs $\eta_1, \eta_2 \subset D$ having the same two endpoints in the boundary of $D$, dividing $D$ into two good disks, there exist $g \in G$ such that $g$ is the identity on $S^2 \setminus D$ and $g(\eta_1) = \eta_2$.

![Figure 4.1. The elementary move of Lemma 4.1.](image)

This is proved in several steps.

**Lemma 4.2.** Let $D_1, D_2 \subset S^2$ be two good disks of equal area. Then there exists $h \in \text{Homeo}_\lambda(S^2)$ such that $h(D_1) = D_2$.

**Proof.** This follows from the Jordan curve theorem and the Oxtoby-Ulam extension property of area-preserving homeomorphisms [17].

**Lemma 4.3.** Given two good simple arcs $\eta_1, \eta_2 \subset D$ having the same endpoints $p \neq q$ in $\partial D$ and each partitioning $D$ into two topological disks $D_j^2$, the two connected components of $D \setminus \eta_i$. Suppose that the disks $D_1^1, D_2^1$ both have area $m_1$ and that the disks $D_1^2, D_2^2$ both have area $m_2$, with $m_1 + m_2 = \lambda(D)$. Then there exists $g \in G$ such that $G$ is the identity off $D$, and restricted to $D$ it leaves invariant $\partial D$, takes $D_1^1$ to $D_1^2$ and $D_2^1$ to $D_2^2$ and it maps $\eta_1$ onto $\eta_2$. 
This is a consequence of Lemma 4.2.\hfill \Box

Let $h \in \text{Homeo}(S^2)$ and let $\eta \subset S^2$ be a good arc and suppose that $h(\eta)$ has positive measure. There exists a homeomorphism $h_0 \in \text{Homeo}(S^2)$ that sends $\eta$ to a geodesic arc $\ell \subset S^2$. Taking a compactly supported isotopy moving this arc to a family of parallel arcs, this isotopy can be constructed to be everywhere small in the uniform topology. Pre-composing $h$ with this isotopy, one constructs a family of arcs $\eta_t$ with $t \in I$ and $I = [0, 1]$ a compact interval, embedded through a family of homeomorphisms of the ambient sphere. Since $[0, 1]$ has uncountably many points, and for only countably many $t \in I$, the arc $\bar{\eta}_t := h(\eta_t)$ can have positive measure since the arcs $\eta_{t_1}$ is disjoint from $\eta_{t_2}$ if $t_1 \neq t_2$, arbitrarily close to the initial arc $\eta$, we can find an arc for which the image under $h$ has zero area, and is thus a good arc again. Similarly for a simple closed curve. More precisely, we have the following.

**Lemma 4.4.** Let $D \subset S^2$ be a Jordan disk. If $\partial D$ has positive area, then arbitrarily close to $D$, in the Hausdorff distance, there are good disks with the same area.

**Proof.** Given a Jordan disk $D$ for which $\partial D$ has positive area, construct a nearby good curve $\gamma$ with zero area enclosing a region whose area equals the area of $D$ as follows. To construct $\gamma$, suppose that the disk $D \subset S^2$ with $\gamma = \partial D$ has area $\lambda(D)$, and take a disk $D_1 \supset D$ such that $D_1 \setminus D$ has area small compared to $D$ and $\partial D_1$ is a good curve. Choose a disk $D_2 \subset D_1$ whose area equals that of $D$, but cutting a small half-disk out of $D_1$ whose diameter is much smaller than that of $D_1$. If the boundary of this half-disk is a good curve, then we are done. If it is not, then repeat the process. Since the modifications tend to zero in diameter, the limiting boundary curve will be a good simple closed curve enclosing an area equal to that of $D$. Since this construction can be carried out so as to produce a curve $\gamma$ as close to $\partial D$ as we desire, the claim follows.\hfill \Box

**Lemma 4.5.** For given $0 < \delta < \epsilon$, there exist $g \in G$ and a good disk $D \subset S^2$ with area at most $\delta$ such that the support of $g$ is contained in $D$, and $g$ does not preserve area in $D$.

**Proof.** For a given $0 < \delta < \epsilon$, it suffices to construct a homeomorphism $g_0 \in G$ such that $g_0(D_1) = D_2$ with good topological disks $D_1, D_2$ and $\lambda(D_1) = \lambda(D_2) \leq \delta$ that does not preserve area from $D_1$ onto $D_2$. Using Lemma 4.2, pre- and postcompose with $h_1, h_2 \in \text{Homeo}_\lambda(S^2)$, so that $h_1 h_0 h_2 \in \text{Homeo}_\lambda(S^2)$ fixes $D$ and does not preserve area when it maps $D$ onto itself. There exist good disks $W_1, W_2 \subset D_1$ and $\delta_1 \leq \delta$ with

$$W_1 \cap W_2 = \emptyset \quad \text{and} \quad \lambda(W_1) = \lambda(W_2) = \delta_1/3,$$

such that

$$\lambda(V_1) = \lambda(W_1)(1 + \epsilon_1) \quad \text{and} \quad \lambda(V_2) = \lambda(W_2)(1 - \epsilon_2),$$

with $\epsilon_1 > 0$ and $0 < \epsilon_2 < 1$, where $V_i = g_0(W_i)$ are good disks with $i = 1, 2$ as well, by small perturbations of the original choice of disks if necessary, see the remark above. Since proving the result for sufficiently small $\delta$ counts, we may assume that $\delta_1 = \delta$. Connect the two disks $W_1$ and $W_2$ by a thin tube $T_1$, homeomorphic to a disk, bounded by two good disjoint arcs $\eta_1, \eta_2 \subset D_1$ whose interiors are disjoint from $W_1$ and $W_2$, and have precisely one endpoint in the boundary of $W_1$ and $W_2$ each. Joined by arcs in the boundary of the disks $W_1$ and $W_2$, the tube $T_1$ itself can be constructed, again by small perturbations, to be a good disk, such that the
image $T_2 := g_0(T_1)$ is a good disk as well. By construction, both
\begin{equation}
D_1 := W_1 \cup W_2 \cup T_1 \text{ and } D_2 := V_1 \cup V_2 \cup T_2 = g_0(D_1)
\end{equation}
are good disks. In addition to $T_1$ and $T_2$ being constructed to be good disks, the maximal area of $T_1$ and $T_2$ can be chosen to be as small as desired. In particular, we can choose these areas to be much smaller than $\lambda(W_1 \cup W_2)$ and $\lambda(V_1 \cup V_2)$. In case that $\lambda(D_1) = \lambda(D_2)$, the claim is proved. If not, then suppose that $\lambda(D_1) > \lambda(D_2)$. Start with $T_1 \cup W_2$, where the area of $V_1 = g_0(W_1)$ is larger than the area of $W_1$ and the area of $T_1$ and $T_2$ small compared to either $W_1$ and $V_1$. Exhaust $W_1$ by a continuous family of good disks $W_{1,t} \subset W_1$ with $t \in [0,1]$ and define
\begin{equation}
D_{1,t} := W_{1,t_1} \cup W_2 \cup T_1 \text{ and } D_{2,t_1} := V_{1,t} \cup V_2 \cup T_2,
\end{equation}
where $W_{1,t}$ has the same boundary segment with $T_1$ in common as $W_1$ does. For some $t_1 \in (0,1)$, the domain and image areas of $D_{1,t_1}$ and $D_{2,t_1}$ are the same. Though it may happen that the image under $g_0$ of $\partial W_{1,t_1}$ has positive area, construct a simple closed curve close to $\partial W_{1,t_1}$ and still contained in $W_1$ using Lemma 4.3 for which equality of areas hold. In all, $D_{1,t_1}$ has area equal to its image, both of which are good disks. Furthermore, since the restriction of $g_0$ to $W_2$ is area-decreasing, $g_0$ does not preserve area on the union. In case that $\lambda(D_1) < \lambda(D_2)$, start with $T_1$ and $W_2$ and drain $W_1$ until an area balance is reached, and proceed to same conclusion. Denoting the desired good disks with equal area $D_1$ and $D_2$, uniformizing the disks $D_1$ and $D_2$ to the desired disk $D$, and denoting again $g_0 \in \text{Homeo}(S^2)$ the mapping that leaves invariant the disk $D$, the mapping $g_0$ is constructed to change areas in the interior of $D$. Using the action $\text{Homeo}_\lambda(D) \subset \text{Homeo}_\lambda(S^2)$ of area-preserving mappings in $D$ that have support contained in the interior of $D$, we obtain the family of mappings $g := g_0 h g_0^{-1} \in \text{Homeo}(D) \setminus \text{Homeo}_\lambda(D)$ with $h \in \text{Homeo}_\lambda(D)$, as required. \hfill \square

Proof of Lemma 4.4 It suffices to prove the following. Given a good disk $D \subset S^2$ and given $m_1, m_2 > 0$ and $n_1, n_2 > 0$ such that $m_1 + m_2 = n_1 + n_2 = \lambda(D)$, suppose that a good arc $\eta \subset D$ separates $D$ into two disks $D_1, D_2$ with area $m_1$ and $m_2$, there exists a $g \in G \cap \text{Homeo}(D)$ with support in $\text{Int}(D)$, and the images $g(D_1)$ and $g(D_2)$ have area $n_1$ and $n_2$ respectively. If the areas of $D_1$ and $D_2$ are correct, then Lemma 4.3 can move the arcs to the desired position. To prove the claim, by Lemma 4.5 there exist $h \in G$ such that the support of $h$ is contained in $\text{Int}(D)$ and such that $h$ does not preserve area by twisting the action $h$ by area-preserving homeomorphisms in $\text{Homeo}_\lambda(D) \subset \text{Homeo}_\lambda(S) \subset G$. Twisting by elements $\text{Homeo}_\lambda(D)$ that have support contained in a small subdisk $D_0 \subset D$ of small area, to create homeomorphisms with small support that do not preserve area, we can find a good arc $\eta_0 \subset D_0$ separating $D_0$ into two disks, one is increased in area and the other is decreased in area. Conjugating by $\text{Homeo}_\lambda(D)$ this action such that the disk that is increased in area sits in $D_1$ and the disk that is decreased in area sits in $D_2$ (or conversely, depending on which disk needs an area increase), the area of one disk can be increased by any small amount at the expense of the area of the other, as required. \hfill \square

Proof of Theorem 3 We have to move the grid $\Gamma$ back to the straight grid $\Gamma_0$ using the two moves. First, since $\text{Homeo}_\lambda(S^2) \subset G$ and $\text{Homeo}_\lambda(S^2)$ is $n$-transitive for any $n \in \mathbb{N}$, there exists a homeomorphism $g_0 \in \text{Homeo}_\lambda(S^2) \subset G$ that sends all vertices of $\Gamma$ back to the corresponding vertices of $\Gamma_0$. As $g \in \text{Homeo}_\lambda(S^2)$, the grid $\Gamma$ has zero area and thus every arc and closed
curve in $\Gamma$ is good and thus the closure of every complementary disk, defined as a connected component of $S^2 \setminus \Gamma$, is a good disk. After $g$, the grid $\Gamma$ has the vertices moved back to the correct vertices, but the arcs between the vertices are still distorted. Applying finitely many elementary moves on the grid $\Gamma$, it is verified that $\Gamma$ can be moved back to $\Gamma_0$. \hfill \square

4.2. Antipodal action groups. Consider the following actions and implications.

The first arrow under consideration is the rotation group as a subgroup of the linear group.

**Proposition 4.1.** The group $\text{Rot}(S^2)$ is maximal in $\text{Lin}(S^2)$.

By the singular value decomposition of matrices, every element $T \in \text{GL}(3, \mathbb{R})$ can be written as $R_1D R_2$, with $R_1, R_2 \in \text{SO}(3, \mathbb{R})$ and $D \in \text{GL}(3, \mathbb{R})$ a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+$ on the diagonal.

**Lemma 4.6.** Given $T \in \text{PGL}(3, \mathbb{R})$ and $R_1, R_2 \in \text{SO}(3, \mathbb{R})$, where $R_1, R_2, T$ correspond to $r_1, r_2, h \in \text{Lin}(S^2)$, with $r_1, r_2 \in \text{Rot}(S^2)$. Then $R_1TR_2$ corresponds to $r_1hr_2 \in \text{Lin}(S^2)$.

**Proof.** The inclusion $\text{Rot}(S^2) \subset \text{Lin}(S^2)$ is induced by the inclusion $\text{SO}(3, \mathbb{R}) \subset \text{GL}(3, \mathbb{R})$, which in turn induces the inclusion $\text{PSO}(3, \mathbb{R}) \subset \text{PGL}(3, \mathbb{R})$. Since the group homomorphism $\text{GL}(3, \mathbb{R}) \to \text{Lin}(S^2)$, becomes an isomorphism after projectivizing $\text{GL}(3, \mathbb{R}) \to \text{PGL}(3, \mathbb{R})$, the map $T \mapsto h_T$ in the quotient passes to a group isomorphism of $\text{PSO}(3, \mathbb{R}) \subset \text{PGL}(3, \mathbb{R})$ correspond to rotations on the sphere, the claim follows. \hfill \square

**Lemma 4.7.** The group $\langle \text{SO}(3, \mathbb{R}), D \rangle$ generated by $\text{SO}(3, \mathbb{R})$ and a non-orthogonal $D \in \text{SL}(3, \mathbb{R})$ equals $\text{SL}(3, \mathbb{R})$.

**Proof.** First, assume that $\lambda_2 = 1$ for $T$, so that $\lambda_1 < 1 < \lambda_3$, where $\lambda := \lambda_3$ and $\lambda^{-1} := \lambda_1$. Iterating $T$, one gets $T_n := T^n$ for which $\lambda_1^n \to 0$ and $\lambda_3^n \to \infty$, and $\lambda_2^n = 1$ for all $n \in \mathbb{N}$. For fixed $n \in \mathbb{N}$, rotating $T_n$ along the axis defined by $v_2$ and taking compositions, one obtains $T$ with largest eigenvalue $t \in [1, \lambda^n]$ and smallest eigenvalue $t^{-1}$. Letting $n \to \infty$, we thus obtain all possible combinations for $\lambda_1$ and $\lambda_3 = \lambda_1^{-1}$. To produce a non-orthogonal element $T$ for which $\lambda_2 = 1$, take $T_0$ with general eigenvalues, where $\lambda_1 < 1 < \lambda_3$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3$, rotate by a small amount along the axis defined by $v_2$, and take the inverse of $T_0$ with the rotated version, so that the eigenvalue corresponding to the direction $v_2$ is one, but the new eigenvalues lying in the plane spanned by $v_1$ and $v_3$ are not both one. This produces the desired non-orthogonal element $T$ with the properties as mentioned.

To produce any $T \in \text{SL}(3, \mathbb{R})$, take an element $T_1$ with eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, 1, \lambda_1^{-1})$ and compose with $T_2$ with eigenvalues $(1, \lambda_2, \lambda_2^{-1})$, so that $T_1T_2$ has eigenvalues $(\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1})$. Pre- and postcomposing $T_1T_2$ with rotations, we obtain any $T \in \text{SL}(3, \mathbb{R})$ as desired. \hfill \square
Proof of Proposition 4.1. Since the groups $\text{SL}(3, \mathbb{R})$ and $\text{PGL}(3, \mathbb{R})$ are group isomorphic, combining Lemma 4.6 with Lemma 4.7, the claim follows. □

To prove the other implications, we first prove the following.

**Theorem 4.** The arrow $\text{Ant}_\lambda(S^2) \rightarrow \text{Ant}(S^2)$ is complete.

**Proof.** The group $\text{Ant}_\lambda(S^2)$ is maximal in $\text{Ant}(S^2)$ since these closed groups of homeomorphisms pass to the quotient to closed groups $\text{Homeo}_\lambda(\mathbb{RP}^2)$ and $\text{Homeo}(\mathbb{RP}^2)$ of $\mathbb{RP}^2$ which is a topological two-manifold without boundary. Therefore, $\text{Homeo}_\lambda(\mathbb{RP}^2)$ is maximal in $\text{Homeo}(\mathbb{RP}^2)$ by a straightforward adaptation of Theorem 3, or alternatively by Le Roux [21]. □

The next step is to prove the following.

**Theorem 5.** The arrow $\text{Ant}(S^2) \rightarrow \text{Homeo}(S^2)$ is complete, that is, the group $\text{Ant}(S^2)$ is maximal in $\text{Homeo}(S^2)$.

In what follows, let $G$ be a group with the property that $\text{Ant}(S^2) \subset G \subset \text{Homeo}(S^2)$. Denote $\gamma_1 = \partial \mathcal{H}_\infty$ the meridian passing through $1 \in S^2$. The idea of the proof is to first pull a hemisphere $\mathcal{H}_\infty$ over the equator $\gamma_1 = \partial \mathcal{H}_\infty$ by iterating a perturbation away from being antipodal. Using the antipodal action, we can pull the hemisphere over almost the entire sphere, and choose the homeomorphism in this region in any way we desire. By closedness, this is sufficient to prove.

**Lemma 4.8.** There exists $h_0 \in G$ such that $h_0(\mathcal{H}_\infty) \subset \mathcal{H}_\infty$ and if $p \in \gamma_\infty$ and $h_0(p) \neq p$, then $h_0(p) \in \mathcal{H}_\infty$.

**Proof.** Let $g \in G \setminus \text{Ant}(S^2)$ and let $p \in \gamma_1 \subset S^2$ such that $g(-p) \neq -g(p)$. Let $q_1 = g(p)$ and $q_2 = g(-p)$ and we may assume that $q_1 \in \gamma_1$ and $q_2 \in \mathcal{H}_0$, after a suitable rotation. Let $\epsilon > 0$ be sufficiently small so that

\begin{equation}
(4.5) \quad g(D_{-p}(\epsilon)) \subset \mathcal{H}_0 \quad \text{and} \quad g(D_p(\epsilon)) \cap g(D_{-p}(\epsilon)) = \emptyset,
\end{equation}

and denote $D_1 = g(D_p(\epsilon))$ and $D_2 = g(D_{-p}(\epsilon))$. Since $q_1 \in \gamma_1$, the disk $D_1$ intersects both $\mathcal{H}_\infty$ and $\mathcal{H}_0$, so there exists $h \in \text{Ant}(S^2)$ with support contained in $D_p(\epsilon) \cup D_{-p}(\epsilon)$, such that

\begin{equation}
\mathcal{H}_\infty
\end{equation}
Lemma 4.9. For every $\delta > 0$, there exist $h_1 \in G$ that maps the hemisphere $H_0 \subset S^2$ onto $S^2 \setminus D_\infty(\delta)$.

Proof. First, we construct $h_1 \in G$ with the property that it sends the lower hemisphere $H_0$ to a domain $U \subset S^2$ containing 0 and the meridian $\gamma_1$. Indeed, applying finitely many perturbations of the type constructed in Lemma 4.8 going around $\gamma_1$, by compactness of $\gamma_1$, a finite composition of such perturbations yields the desired $h_1$ with the properties as mentioned. Since the action of $\text{Ant}(S^2)$ includes the action of the antipodal double of homeomorphisms with compact support in the disk $H_\infty$, to construct $h_1 \in G$ with the properties mentioned in the statement, first take $h_1 \in G$ that maps $H_0$ to $U$ containing 0 and $\gamma_1$ and postcompose with $g_1 \in \text{Ant}(S^2)$ such that $\gamma := \partial U$ is sent by $g_1$ into $D_\infty(\delta)$, for a given $\delta > 0$, to obtain the desired mapping, which we denote again $h_1$. □

Proof of Theorem 6. Given $g \in \text{Homeo}(S^2)$, it suffices to show that for each $\epsilon > 0$, there exists $h \in G$ such that the distance between $h$ and $g$ is at most $\epsilon$. To this end, we may assume that $g(\infty) = \infty$ by composing with a rotation $r \in \text{Rot}(S^2)$. There exists small $\delta > 0$ for which $g(D_\infty(\delta))$ is contained in the interior of $D_\infty(\epsilon/2)$. There exists $g_0 \in \text{Ant}(S^2)$ whose support in $H_\infty$ is a compact disk $D \subset H_0$ whose boundary $\gamma = \partial D$ is close to $\gamma_1$ and $h_1 \in G$ by Lemma 4.9 with the property that $h_1(\gamma) \subset D_\infty(\delta/2)$, such that $g_0$ has the property that $h := h_1 g_0 h_1^{-1}$ for each $p \in S^2 \setminus D_\infty(\delta)$. By construction, we have that $h(p) = g(p)$ for every $p \in S^2 \setminus D_\infty(\delta)$ and $d(h(p), g(p)) < \epsilon$ for $p \in D_\infty(\delta)$, and thus the $C^0$ distance between $h$ and $g$ on the entire sphere $S^2$ is at most $\epsilon$, as required. □

Next, we prove the following.

Theorem 6. The arrow $\text{Ant}_\lambda(S^2) \longrightarrow \text{Homeo}(S^2)$ is complete, that is, a closed group $G$ extending $\text{Ant}_\lambda(S^2)$ containing homeomorphisms not preserving the antipodal action and not preserving area, equals $\text{Homeo}(S^2)$.

The proof is an improved version of the proof of Theorem 3. Let $\text{Ant}_\lambda(S^2) \subset G \subset \text{Homeo}(S^2)$. Take a homeomorphism $h \in \text{Homeo}(S^2)$, which by a diagonal argument we may approximate by a piecewise-linear homeomorphism, and choose $\epsilon > 0$. Take an $\epsilon$-grid $\Gamma_0 := \Gamma(\epsilon) \subset S^2$ and consider $\Gamma := h(\Gamma_0)$. By a finite composition of elementary moves in $G$, we move the grid $\Gamma$ back to $\Gamma_0$, which is sufficient to prove density, using the following moves.

Lemma 4.10 (Elementary move I). Given $\epsilon$, there exists $\delta$ with $0 < \delta < \epsilon$, such that given a good disk $D \subset S^2$ of area $\delta$ not containing antipodal points, and given points $p_1, \ldots, p_n \in \text{Int}(D)$ and $q_1, \ldots, q_n \in \text{Int}(D)$, with $n \in \mathbb{N}$, there exists $g \in G$ with support contained in $D \cup D_\infty(\delta)$, such that $g(p_k) = q_k$ for every $1 \leq k \leq n$.

Moving points is one elementary move, in addition to the following move, which is the analogue in the antipodal action groups case of the elementary move used before.
Lemma 4.11 (Elementary move II). Given $\epsilon$, there exists $\delta$ with $0 < \delta < \epsilon$, such that given a good disk $D \subset S^2$ of area at most $\delta$ and not containing antipodal points, and given two good arcs $\eta_1, \eta_2 \subset D$ having the same two endpoints in the boundary of $D$, dividing $D$ into two disks, there exist $g \in G$ with support in $D$ and $D_{\infty}(\delta)$, such that $g(\eta_1) = \eta_2$.

![Figure 4.3. Elementary move II of Lemma 4.11, where $q := g(\infty)$.](image)

Define $D_{0,\infty}(\epsilon) := D_0(\epsilon) \cup D_{\infty}(\epsilon) \subset S^2$ the union of the two round closed disjoint disks centered at 0 and $\infty$ in $S^2$ both of radius $\epsilon$. The difference with the case $\text{Homeo}_\lambda(S^2) \to \text{Homeo}(S^2)$ is that now the small perturbation domains have to be uniformized with the group $\text{Ant}_\lambda(S^2)$, rather than $\text{Homeo}_\lambda(S^2)$.

Proof of Lemma 4.10. Take $h \in G$ for which $h(\infty) = \infty$ and $h(0) = q \neq 0$. Denote $D_1 = h(D_{\infty}(\delta))$ and $D_2 = h(D_0(\delta))$. Choose $\delta > 0$ small enough so that $-D_{\infty}(\delta) \cap D_2 = \emptyset$. Given any good disk $D \subset S^2$ not containing antipodal points and disjoint from $D_{0,\infty}(\epsilon)$, since $D_2$ and $D_{\infty}(\delta)$ are contained in a fundamental domain of $\mathbb{R}P^2$, by Lemma 4.12 using the action $\text{Ant}_\lambda(S^2)$, we can conjugate an action on the disks $D_2$ and $D_{\infty}(\delta)$ onto an action on the disks $D$ and $D_{\infty}(\delta)$. Given tuples of points $p_1, \ldots, p_n \in D$ and $q_1, \ldots, q_n \in D$, pull these back to $n$-tuples of points in $D_0(\delta)$. There exist $g_0 \in \text{Ant}_\lambda(S^2)$ with local support contained $D_{0,\infty}(\delta)$ that send any $n$-tuple of points onto any other $n$-tuple of points, and this action pushes forward to an $n$-transitive action on the designated disk $D$, as required. □

To construct elementary move II, we first need the following. In what follows, non-trivial area distortion on a domain is defined as the mapping not inducing a positive multiple of the Lebesgue measure.

Lemma 4.12. For every $\delta > 0$, and for each pair of neighborhoods $U_0, U_\infty \subset S^2$ of 0, $\infty$ of area at less than $\delta$, there exist $h \in G$, such that $h(\infty) = \infty$, $h(0) = q \neq 0$ and $V_0 := h(U_0)$ and $V_\infty := h(U_\infty)$ do not contain antipodal points and the area distortion on either $U_0$ or $U_\infty$ is non-trivial.

Proof. Suppose not, then for each $g \in G$ and for each pair $\{p, -p\} \subset S^2$ mapped to $q_1$ and $q_2 \neq -q_1$, there exist neighborhoods $U_p$ and $U_{-p}$ such that the areas of disks in these neighborhoods
are multiplied by a constant depending only on the basepoint $p$ and $-p$. Passing to smaller $U_p$ and $U_{-p}$ if necessary, we may assume that $V_p \cap V_{-p} = \emptyset$, with $V_p := h(U_p)$ and $V_{-p} := h(U_{-p})$. Then the same conclusion must hold if $q_1$ and $q_2 = -q_1$ are antipodal. Indeed, if there would exist $g \in G$ and points $p, -p$ sent to $q, -q$ for which for arbitrarily small neighborhoods $U_p$ and $U_{-p}$ of $p$ and $-p$, small disks of equal area are sent to disks of differing area in the image $V_p$ and $V_{-p}$, then the composition $hg$ has the property of distorting areas in arbitrarily small neighborhoods nested around antipodal points whose images under $hg$ are not antipodal.

Take any $h \in G$ and consider the function that to each $p \in S^2$ assigns the above constructed multiplier of the areas. Since this function is constant on each small enough open neighborhood that exists for each point, and since finitely many of these neighborhoods cover the sphere by compactness, observing that the function has to be constant on overlaps, it has to be constant on the entire sphere, and thus equal to 1. Thus $h \in \text{Homeo}_\lambda(S^2)$ and by generality of $h$ it follows that $G \subseteq \text{Homeo}_\lambda(S^2)$, contrary to our assumption. Since the area-distortion property holds for arbitrarily small neighborhoods around antipodal points, we may assume that the image neighborhoods $V_p$ and $V_{-p}$ of the disjoint neighborhoods $U_p$ and $U_{-p}$ around the antipodal points $\{p, -p\}$ not containing antipodal points, do not contain antipodal points. □

Proof of Lemma 4.11 By Lemma 4.12 there exist $h \in G$ with the properties that $h(\infty) = \infty$ and $h(0) = q \neq 0$ and arbitrarily small neighborhoods $U_0$ and $U_\infty$ containing 0 and $\infty$ respectively, so that both the images $V_0$ and $V_\infty$ under $h$ are disjoint, contained in one fundamental domain and $h$ distorts the area of $V_0$ non-trivially. Since the domains $V_0$ and $V_\infty$ are contained in the same fundamental domain of $\mathbb{RP}^2$, we can conjugate the action by an element of $\text{Ant}_\lambda(S^2)$ to send $V_\infty$ to a round disk centered at $\infty$ and having radius less than $\epsilon$ and applying locally supported in $\text{Ant}_\lambda(S^2)$, we can uniformize $D_1 = D_2 = D$ any good disk.

Since the area-distortion is non-trivial, perturbing by an element $g_0 \in \text{Ant}_\lambda(S^2)$ which has local support on a good disk $D_0 \subset U_0$, and its antipodal image, mappings $g := h g_0 h^{-1}$ can be constructed that have support in $V_0$ and $V_\infty$, both of which lie in one fundamental domain of $\mathbb{RP}^2$, and that distort areas in $V_0$. Since the domain $V_0$ is left invariant by $g$, there must exist, for $\delta > 0$ small enough, disjoint good disks $W_1, W_2 \subset V_0$ of area $\delta$ and $V_1 = g(W_1), V_2 = g(W_2)$, of area greater than, and less than, $\delta$ respectively. Applying the argument in Lemma 4.5 there exist good disks $D_1, D_2 \subset V_0$ of equal area, and such that $g(D_1) = D_2$ and $g$ does not preserve area from $D_1$ to $D_2$. Applying the argument in Lemma 4.1 any area balance can be created and uniformized to send any good arc $\eta \subset D$ separating $D$ into good disks $D_1$ and $D_2$ into any two other good disks in the image and the argument proceeds as before. □

Proof of Theorem 4 Given the distorted grid $\Gamma = h(\Gamma_0)$, every move of type I and II have support contained in two small disks, one of which can be thrown into either of the two $\epsilon$-disks containing $0, \infty$ and coincide with $D_{0,\infty}(\epsilon)$. To good disk that perturbs in the desired way can be placed anywhere away from $D_{0,\infty}(\epsilon)$. First apply finitely many moves of type I to move the finite set of vertices of the distorted grid $\Gamma$ back to the original position. Then use finitely many moves of type II to move the edges of the distorted grid $\Gamma$ back into their position, to obtain $\Gamma_0$ as required. □

4.3. Conformal groups. The conformal groups fit in the following sequence of groups

\begin{equation}
\text{Rot}(S^2) \rightarrow \text{Möbius}(S^2) \rightarrow \text{Homeo}(S^2).
\end{equation}
Adding any Möbius transformation which is not a rotation of the sphere generates the full Möbius group. In what follows, let $G$ have the property that $\text{Rot}(S^2) \subset G \subseteq \text{Möbius}(S^2)$.

**Proposition 4.2.** The rotation group $\text{Rot}(S^2)$ is maximal in $\text{Möbius}(S^2)$.

**Proof of Proposition 4.2 using Theorem A.** Any proper extension $\text{Rot}(S^2) \subset G \subseteq \text{Möbius}(S^2)$ is by Lemma 4.6 not contained in $\text{Ant}(S^2)$. Therefore, by Theorem A, $G$ has to be at least three-transitive. Since $\text{Möbius}(S^2)$ is sharply three-transitive, it follows that $G = \text{Möbius}(S^2)$. □

The second proof of Proposition 4.2 is a direct construction in line with the proof for the case of the linear group. Let $G = \langle \text{Rot}(S^2), g \rangle$ denote a group generated by the rotation group $\text{Rot}(S^2)$ and any $g \in \text{Möbius}(S^2)$ which is not an isometry.

**Lemma 4.13.** The group $G \subseteq \text{Möbius}(S^2)$ contains a subgroup $G_1 \subseteq G$ of transformations leaving invariant the northern hemisphere $\mathcal{H}_\infty \subset S^2$ bounded by the horizontal meridian $\gamma_1 \subset S^2$, where $G_1$ contains the subgroup $\langle \text{Rot}(\mathcal{H}_\infty), h \rangle$, where $h \in \text{Möbius}(\mathcal{H}_\infty) \setminus \text{Rot}(\mathcal{H}_\infty)$.

**Proof.** Since $\text{Ant}(S^2) \cap \text{Möbius}(S^2) = \text{Rot}(S^2)$, adding $g \in \text{Möbius}(S^2)$ which is not an isometry, $g$ has the property that there exist points $p \in S^2$ such that $h(p) \neq h(p)$ by Lemma 4.6. Take a rotation $r \in \text{Rot}(S^2)$ around 0 and $\infty$ and conjugate $r$ with $g$ to obtain an elliptic transformation $g_0 \in \text{Möbius}(S^2)$ which is not an isometry. The elliptic transformation $g_0$ will have a unique invariant meridian, which upon a conjugation with an element of $\text{Rot}(S^2)$ we can assume to be the horizontal meridian $\gamma_1$. In that case, $g_0$ leaves invariant the upper hemisphere $\mathcal{H}_\infty$ containing $\infty$ and the induced action on $\gamma_1$ by $g_0$ is not isometric. The subgroup $\text{Rot}(S^2)$ of rotations leaving invariant $\gamma_1$, which is isomorphic to $\text{SO}(2, \mathbb{R})$, acts isometrically on $\gamma_1$. Therefore, $G$ contains a subgroup $G_1 \subset G$ acting on $\gamma_1$ with the required properties. □

**Lemma 4.14.** We have that $G_1 := \langle \text{Rot}(\mathcal{H}_\infty), h \rangle = \text{Möbius}(\mathcal{H}_\infty)$, for any $h \in \text{Möbius}(\mathcal{H}_\infty)$ which is not an isometry.

**Proof.** Indeed, project the hemisphere $\mathcal{H}_\infty$ conformally on the unit disk $\mathbb{D}$ and denote $\gamma = \partial \mathbb{D}$. The full Möbius group $\text{Möbius}(\mathbb{D})$ is isomorphic to $\text{PGL}(2, \mathbb{R})$. In turn, the group $\text{PGL}(2, \mathbb{R})$ is isomorphic to $\text{SL}(2, \mathbb{R})$. The action $\text{PSO}(2, \mathbb{R})$ identifies in this isomorphism with $\text{SO}(2, \mathbb{R})$. Therefore, the statement reduces to the claim that $\text{SL}(2, \mathbb{R}) = \langle \text{SO}(2, \mathbb{R}), A \rangle$ for a non-orthogonal $A \in \text{SL}(2, \mathbb{R})$, which follows as in the case of $\text{SL}(3, \mathbb{R})$ readily from the singular value decomposition theorem. □

To complete the proof, we need to reconstruct the full group $\text{Möbius}(S^2)$ given $\text{Möbius}(\mathbb{D})$.

**Proof of Proposition 4.2.** To produce the full group $\text{Möbius}(S^2)$, we use Lemma 4.13 and the group $\text{Rot}(S^2)$. First, an elliptic element is determined by an axis of rotation $\alpha$ and a rotation angle $\theta \in S^1$. The axis $\alpha$ is the unique geodesic in the ball enclosed by $S^2$ passing through two given points $p, q \in S^2$. Unless $p$ and $q$ are antipodal, in which case the corresponding elliptic element is contained in $\text{Rot}(S^2)$, define $\gamma$ be the unique meridian passing through $p$ and $q$. By Lemma 4.13 there exists $g \in \text{Möbius}(S^2)$ leaving invariant $\gamma$ and sending $p$ and $q$ respectively. Conjugating the rotation $r_\theta \in \text{Rot}(S^2) \subset \text{Möbius}(S^2)$ by $g \in \text{Möbius}(S^2)$ produces the desired elliptic element. A hyperbolic element in $\text{Möbius}(S^2)$ is determined uniquely by the translation axis passing through two different points $p, q \in S^2$, the translation length and the rotation angle around the axis. Given a translation length $T$, by Lemma 4.14 there exists a hyperbolic transformation $\text{Möbius}(\mathbb{D})$ with translation length $T$ and whose translation axis is a
geodesic passing through the origin of \( \mathbb{D} \). Taking the double of this transformation, we obtain a hyperbolic element in \( \text{M"{o}bius}(S^2) \) with an axis whose endpoints are antipodal, which we may assume upon a rotation, to be \( \infty \) and \( 0 \). Applying again Lemma A.13 we can find \( g \in \text{M"{o}bius}(S^2) \) sending \( 0, \infty \) to \( p, q \) respectively. Conjugating the original hyperbolic element with antipodal fixed points with \( g \), one obtains a hyperbolic transformation in \( \text{M"{o}bius}(S^2) \) with the required axis and translation length. Precomposing this transformation with a rotation of angle \( \theta \in \mathbb{S}^1 \) around \( \infty \) and \( 0 \), one obtains the required general hyperbolic (or loxodromic) element. Parabolic elements are constructed similarly by passing to the double and conjugating. 

The second half of the sequence is the arrow \( \text{M"{o}bius}(S^2) \rightarrow \text{Homeo}(S^2) \). As a special case of the kernel subgroup conjecture, we conjecture that the group \( \text{M"{o}bius}(S^2) \) is maximal in \( \text{Homeo}(S^2) \). The group \( \text{M"{o}bius}(S^2) \) is special in that all elements have full support, in the sense that the support of each homeomorphism other than the identity mapping is the entire sphere \( S^2 \). Therefore, conjecturally a group \( G \) containing \( \text{M"{o}bius}(S^2) \) that contains homeomorphisms that have local support, equals \( \text{Homeo}(S^2) \). In dimension one, the group \( \text{M"{o}bius}(S^1) \) is maximal in \( \text{Homeo}(\mathbb{S}^1) \) [7].

4.4. **Proof of Theorem C.** Completeness of \( \text{Rot}(S^2) \rightarrow \text{M"{o}bius}(S^2) \) and \( \text{Rot}(S^2) \rightarrow \text{Lin}(S^2) \) is proved in Proposition 4.2 and Proposition 4.1. Maximality of \( \text{Ant}(S^2) \) in \( \text{Homeo}(S^2) \) is proved in Theorem 5. Completeness of the arrows \( \text{Homeo}_\lambda(S^2) \rightarrow \text{Homeo}(S^2) \), \( \text{Ant}_\lambda(S^2) \rightarrow \text{Ant}(S^2) \) and \( \text{Ant}_\lambda(S^2) \rightarrow \text{Homeo}(S^2) \) is proved in Theorem 3, Theorem 4, and Theorem 6 respectively. The arrows not present in the diagram, other than \( \text{Rot}(S^2) \rightarrow \text{Homeo}(S^2) \), are

(i) \( \text{M"{o}bius}(S^2) \rightarrow \text{Lin}(S^2) \), \( \text{M"{o}bius}(S^2) \rightarrow \text{Ant}_\lambda(S^2) \), \( \text{M"{o}bius}(S^2) \rightarrow \text{Ant}(S^2) \),

(ii) \( \text{Lin}(S^2) \rightarrow \text{Ant}_\lambda(S^2) \), \( \text{Lin}(S^2) \rightarrow \text{Homeo}_\lambda(S^2) \),

as the intersection of the two groups reduces to the rotation group. Indeed, it follows from combining Lemma 2.6 stating that \( \text{Ant}(S^2) \cap \text{M"{o}bius}(S^2) = \text{Rot}(S^2) \), with the inclusions \( \text{Lin}(S^2) \subset \text{Ant}(S^2) \) and \( \text{Ant}_\lambda(S^2) \subset \text{Ant}(S^2) \) that the first set of arrows do not exist and for the second set of arrows this follows from combining Lemma 2.3 stating that \( \text{Lin}(S^2) \cap \text{Homeo}_\lambda(S^2) = \text{Rot}(S^2) \), with the inclusion \( \text{Ant}_\lambda(S^2) \subset \text{Homeo}_\lambda(S^2) \). This completes the proof of Theorem C.

**Appendix A. Homeomorphism Groups on Higher Genus Surfaces**

Since Lie groups, finite or infinite-dimensional, are particularly well understood, a correspondence between homeomorphism groups and Lie group actions is therefore of interest. In the case of the sphere, which has trivial fundamental group, this correspondence is quite explicit in terms of the classification of finite-dimensional Lie groups. In case of higher genus surfaces, with non-trivial fundamental group, the global topology of the surface obstructs certain global geometrical actions, substituting it with a subgroup of the mapping class group.

**A.1. Transitive Lie group actions on closed surfaces.** A subgroup of the homeomorphism group of the sphere containing \( \text{Rot}(S^2) \) is homotopic to \( \text{Rot}(S^2) \), a subgroup of the homeomorphism group of the torus containing the translation group is homotopic to \( \mathbb{S}^1 \times \mathbb{S}^1 \) and for higher genus surfaces subgroups of the homeomorphism group are always discrete groups (in the form of subgroups of the mapping class group). The classification of continuous (local) Lie group actions on a surface, initiated by Lie, and completed by Mostow [14], gives a complete list of all finite-dimensional continuous Lie groups acting locally on \( \mathbb{R}^2 \). The list produced by Mostow gives all infinitesimal actions, however only a strict subset of these infinitesimal actions
can actually be integrated on a closed surface to homeomorphisms due to global topological restrictions. The actions that do integrate have been classified more recently by Belliart [3], resulting in the following. The finite-dimensional Lie groups acting by homeomorphisms on a closed surface without fixed points consists precisely of the following:

(S) The sphere: the action induced by \( SO(3, \mathbb{R}) \), \( PGL(2, \mathbb{C}) \) and \( PGL(3, \mathbb{R}) \).

(T) The torus: a circle \( S^1 \)-action by \( (x, y) \mapsto (x + \alpha, y) \mod \mathbb{Z}^2 \).

(H) Hyperbolic surfaces: none.

The corresponding dimensions of the groups are \( \dim_{\mathbb{R}}(SO(3, \mathbb{R})) = 3 \), \( \dim_{\mathbb{R}}(PGL(2, \mathbb{C})) = 6 \) and \( \dim_{\mathbb{R}}(PGL(3, \mathbb{R})) = 8 \). Note that a composition of circle actions can give any desired translation and thus generate, in particular, the full translation group of the torus. The possible local infinitesimal geometry of a homeomorphism is described by \( GL(2, \mathbb{R}) \), and gives rise to infinite-dimensional Lie group actions by homeomorphisms on a closed surface. The possible \( GL(2, \mathbb{R}) \) actions and their global homeomorphisms are (i) \( O(2, \mathbb{R}) \subset GL(2, \mathbb{R}) \) corresponding to holomorphic homeomorphisms, (ii) \( SL(2, \mathbb{R}) \subset GL(2, \mathbb{R}) \) corresponding to area-preserving homeomorphisms, and (iii) \( U(2, \mathbb{R}) \subset GL(2, \mathbb{R}) \) the upper-triangular matrices, corresponding to homeomorphisms fibering over the circle \( S^1 \); these are skew-product homeomorphisms in the coordinates induced by the foliation by circles of the form \( g(x, y) = (\varphi(x), \psi(x, y)) \).

On higher genus surfaces, finite-dimensional Lie group actions do not have a place. The loss of finite-dimensional Lie groups acting by homeomorphisms on a closed surface without fixed points consists precisely of the following. The finite-dimensional Lie groups acting by homeomorphisms on a closed surface of genus at least two is endowed with the Goldman Lie-bracket [9], putting a Lie algebra on the (subgroups) of the fundamental group.

A.2. Homeomorphism groups of the torus. In terms of the above described Lie groups, let us now consider the subgroups of the homeomorphism group of the torus \( T^2 \).

(T.1) **Holomorphic mappings.** The group of holomorphic mappings equals the translation group, analogous to the rotation group of the sphere.

(T.2) **Invariant foliations.** Groups leaving invariant \( n = 1, 2, \ldots \) pairwise transversal foliations, corresponding to skew-product and product groups for \( n = 1, 2 \).

(T.3) **Area-preserving mappings.** The group \( \text{Homeo}_d(T^2) \) of area-preserving homeomorphisms, maximal in \( \text{Homeo}(T^2) \) as in the case of sphere.

(T.4) **Mapping class group.** The mapping class group \( \text{MCG}(T^2) \) isomorphic to \( \text{SL}(2, \mathbb{Z}) \) that represents \( \text{MCG}(T^2) \) by area-preserving homeomorphisms.

(T.5) **Finite degree covers.** Subgroups \( \text{Homeo}_{n,m}(T^2) \subset \text{Homeo}(T^2) \) commuting with a finite degree \( (n, m) \) regular covering action of the torus.

(T.6) **Special rotation sets.** The rotation set of a closed group \( G \subset \text{Homeo}(T^2) \) is defined as the union \( \bigcup_{g \in G} \rho(g) \) over the rotation sets \( \rho(g) \) of all elements \( g \in G \), each of which is a convex and compact subset of \( \mathbb{R}^2 \). Taking iterates of a single homeomorphism means adding the rotation set, and thus \( G \) for which the rotation set is a proper subset of \( \mathbb{R}^2 \), and closed under addition, is a proper subgroup of \( \text{Homeo}(T^2) \). Examples include the sets \( \{0\} \) or a line \( \ell \subset \mathbb{R}^2 \) passing through the origin.

\[\text{If } g, h \in G \text{ and } \rho(g) = \rho(h) = \{0\}, \text{ then } \rho(hg) \text{ may or may not equal } \{0\}. \text{ However if the group } G \text{ preserves an Oxtoby-Ulam measure } \mu, \text{ and } g, h \in G \text{ with } \rho_\mu(h) = \rho_\mu(g) = \{0\}, \text{ where } \rho_\mu(h) \text{ denotes the integral over a fundamental domain of } \pi^2 \text{ of the displacement function } \delta(p) := h(p)−p \text{ in the lift, then it is true that } \rho_\mu(hg) = \{0\}.\]
Taking combinations of the above actions gives closed and transitive subgroups of Homeo($\mathbb{T}^2$), such as the group of area-preserving skew-product mappings. The combination of Homeo$_\lambda$($\mathbb{T}^2$) and zero mean rotation homeomorphisms homotopic to the identity gives the group containing the Hamiltonian homeomorphisms which are the topological closure of Hamiltonian diffeomorphisms. This is a closed and transitive subgroup of Homeo($\mathbb{T}^2$), but it does not contain Trans($\mathbb{T}^2$), therefore, it is an example of an inhomogeneous group, and it has a rich structure. In case the group $G$ is homogeneous, i.e. $\text{Trans}(\mathbb{T}^2) \subseteq G$, then the rotation set $\rho(G) = \mathbb{R}^2$.

A.3. Homeomorphism groups of higher genus surfaces. In the case of the homeomorphism group Homeo($S$) of a genus $m \geq 2$ surface $S$, several interesting closed transitive subgroups arise, in particular those arising as subgroups of the mapping class group, see [4].

(H.1) Holomorphic mappings. A closed surface $S$ admits only a finite group of isometries, and consequently this group is never transitive for any genus $m \geq 2$.

(H.2) Area-preserving mappings. The group Homeo$_\lambda$(S) of area-preserving homeomorphisms, where every homotopy class of a homeomorphism admits an area-preserving representative.

(H.3) Mapping class groups. Denote $H \subset \text{MCG}(S)$ a subgroup of the mapping class group. Write $\text{Homeo}_H(S) \subset \text{Homeo}(S)$ the subgroup of homeomorphisms whose homotopy classes lie in the subgroup $H \subset \text{MCG}(S)$ (see e.g. [4]). Homeo$_H(S) \subset \text{Homeo}(S)$ is never maximal in Homeo(S) due to global topological obstructions.

(H.4) Finite degree covers. The group of homeomorphism groups commuting with the full homeomorphism group, or one of its subgroups, of a closed surface with lower genus of which it is a regular finite degree cover.

(H.5) Special rotation sets. The rotation set of a closed group $G \subset \text{Homeo}_0(S^2)$ is defined as the union over all $g \in G$ of the rotation sets $\rho(g)$, each of which is a convex and compact subset of $\mathbb{R}^{2m}$. Similar to the torus, if $G$ has a rotation set which is a proper subset of $\mathbb{R}^{2m}$ that is closed under addition, the group must be proper closed and transitive subgroup of the group of homeomorphisms isotopic to the identity. The Grassmannian of $\mathbb{R}^{2m}$ describes all possible affine rotation sets a group $G$ can have.

Taking the area-preserving homeomorphisms isotopic to the identity whose rotation set is a singleton $\{0\}$, this group includes the group of Hamiltonian homeomorphisms, defined as the closure of the group of Hamiltonian diffeomorphisms, and is as in the case of the torus a proper closed subgroup of Homeo$_\lambda(S)$.

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