Lyapunov stability analysis for the generalized Kapitza pendulum

O V Druzhinina, L A Sevastianov, S A Vasilyev, D G Vasilyeva

1 Federal Research Center Computer "Science and Control" of Russian Academy of Sciences (FRC CSC RAS), 44/2 Vavilov St, Moscow, 119333, Russian Federation and V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences (ICS RAS), 65 Profsoyuznaya St, Moscow, 117997, Russian Federation
2 Peoples Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation

E-mail: vasilyev.sa@rudn.university

Abstract. In this work generalization of Kapitza pendulum whose suspension point moves in the vertical and horizontal planes is made. Lyapunov stability analysis of the motion for this pendulum subjected to excitation of periodic driving forces and stochastic driving forces that act in the vertical and horizontal planes has been studied. The numerical study of the random motion for generalized Kapitza pendulum under stochastic driving forces has made. It is shown the existence of stable quasi-periodic motion for this pendulum.

1. Introduction
Kapitza pendulum is a rigid pendulum in which the pivot point vibrates in a vertical direction, up and down. The unique feature of the Kapitza pendulum is that the vibrating suspension can cause it to balance stably in an inverted position. A pendulum with vibrating point is a classical problem of perturbation theory. The phenomenon of stabilisation of the upper vertical position of the pendulum by fast vertical vibrations of the suspension point was discovered by A. Stephenson [13, 14]. P.L. Kapitsa developed a method of separation of slow and fast motions for the pendulum [5, 6]. Different aspects of this problem were discussed in many publications [1, 10, 11, 12]. In papers [3, 7, 9] the authors built various models of stochastic systems and considered their dynamics.

Without any doubt that time-varying systems, which are also referred to as non-autonomous systems and non-stationary systems, are more difficult to handle than time-invariant systems [15, 16]. For linear time-invariant system, it is known that there are only two kinds of stability concepts, namely, Lyapunov stability and asymptotic stability, which are totally determined by the eigenvalue set of the system matrix. However, for linear time-varying system, there are more stability concepts such as non-uniformly asymptotic stability, uniformly asymptotic stability, non-uniformly exponential stability and uniformly exponential stability. Moreover, differently from linear time-invariant systems, the stability of linear time-varying systems cannot be linked to the eigenvalue set of the system matrices directly [16]. Therefore, compared with time-invariant systems, study on the analysis and design of time-varying systems is very challenging and has been greatly retarded and only relatively less papers were available in the literature [2, 4, 8].
In this work generalization of Kapitza pendulum whose suspension point moves in the vertical and horizontal planes is made. Lyapunov stability analysis of the motion for this pendulum subjected to excitation of periodic driving forces and stochastic driving forces that act in the vertical and horizontal planes has been studied. The numerical study of the random motion for generalized Kapitza pendulum under stochastic driving forces has made. It is shown the existence of stable quasi-periodic motion of this pendulum.

2. Lyapunov stability analysis of the generalized Kapitza pendulum motion under periodic driving forces

2.1. Model of the generalized Kapitza pendulum under periodic driving forces

Let \( l \) and \( m \) be length of the massless rod and mass of the bob for this pendulum. Let \( x(t) \) and \( y(t) \) be horizontal and vertical Cartesian coordinates of the suspension point. Denote by \( \theta(t) \) the angle between the rod of the pendulum and the vertical. In this case we can written the coordinates of bob in the form:

\[
\begin{align*}
\dot{x}(t) &= l \sin \theta(t) + a \sin \omega_1 t, \\
\dot{y}(t) &= l \cos \theta(t) + b \sin \omega_2 t,
\end{align*}
\]

(1)

where \( \omega_1 \) and \( \omega_2 \) are the frequencies of the harmonic vertical and horizontal forced oscillations of the suspension, \( a \) and \( b \) are amplitudes of the forced oscillations along the axes \( x \) and \( y \). If in (1) we take the time derivative of \( x(t) \) and \( y(t) \) we can write

\[
\begin{align*}
\dot{v}_x(t) &= l \cos \theta(t) \dot{\theta}(t) + a \omega_1 \cos \omega_1 t, \\
\dot{v}_y(t) &= -l \sin \theta(t) \dot{\theta}(t) + b \omega_2 \cos \omega_2 t,
\end{align*}
\]

(2)

where \( v_x(t) = \dot{x}(t) \), \( v_y(t) = \dot{y}(t) \) are velocities along each of the axes.

Then the kinetic \( K(t) \) and potential \( V(t) \) energies of the bob are

\[
\begin{align*}
K(t) &= 0.5 m \left( \dot{x}^2(t) + \dot{y}^2(t) \right), \\
V(t) &= mg y(t),
\end{align*}
\]

(3)

\[
\begin{align*}
K(t) &= 0.5 m \left[ \left( l \cos \theta(t) \dot{\theta}(t) + a \omega_1 \cos \omega_1 t \right)^2 + \left( -l \sin \theta(t) \dot{\theta}(t) + b \omega_2 \cos \omega_2 t \right)^2 \right], \\
V(t) &= mg \left( l \cos \theta(t) + b \sin \omega_2 t \right),
\end{align*}
\]

(4)

where \( m \) is mass of the bob and \( g \) is acceleration of free fall.

The total energy of the system is given by the sum of the kinetic and potential energies

\[
E(t) = K(t) + V(t) = 0.5 m \left( p_x^2(t) + p_y^2(t) \right) + mg y(t),
\]

(5)

where \( p_x = mv_x \) and \( p_y = mv_y \) are the components of momentum along the \( x \) and \( y \) axes.

Lagrangian of the system has the form:

\[
L(t) = K(t) - V(t) = \frac{m}{2} \left( \dot{x}^2(t) + \dot{y}^2(t) \right) - mgl \cos \theta(t).
\]

(6)

The Euler - Lagrange equation for the phase of the pendulum as follows

\[
\frac{d}{dt} \frac{\partial L(t)}{\partial \dot{\theta}} - \frac{\partial L(t)}{\partial \theta} = 0,
\]

(7)

and the equation of the pendulum motion has the form:

\[
\ddot{\theta}(t) - \frac{a \omega_1^2 \sin \omega_1 t}{l} \cos \theta + \frac{b \omega_2^2 \sin \omega_2 t}{l} \frac{g}{m} \sin \theta = 0.
\]

(8)
We can rewrite this equation in the form of the system of first order differential equations

\[
\begin{align*}
\dot{\theta}(t) &= \varphi(t), \\
\dot{\varphi}(t) &= l^{-1} a \omega_1^2 \sin \omega_1 t \cos \theta(t) - l^{-1} (b \omega_2^2 \sin \omega_2 t - g) \sin \theta(t),
\end{align*}
\]

where \(\varphi\) is an auxiliary variable.

### 2.2. Stability analysis of motion under periodic driving forces

In this section Lyapunov stability analysis of the motion for this pendulum subjected to excitation of periodic driving forces that act in the vertical and horizontal planes has been studied.

Let \(\theta_0\) and \(\varphi_0\) be parameters of the bob balance when equilibrium conditions can be obtained. Let \(\Delta \theta\) and \(\Delta \varphi\) be small deviations from the equilibrium state. In this case, the motion of the pendulum can be described as follows

\[
\begin{align*}
\theta(t) &= \theta_0 + \Delta \theta(t), \\
\varphi(t) &= \varphi_0 + \Delta \varphi(t).
\end{align*}
\]

Parameters \(\theta_0\) and \(\varphi_0\) can be found from

\[
\begin{align*}
\varphi_0 &= 0, \\
a \omega_1^2 \sin \omega_1 t \cos \theta_0 - (b \omega_2^2 \sin \omega_2 t - g) \sin \theta_0 &= 0.
\end{align*}
\]

We can get the coordinates of the equilibrium position for the pendulum in the form:

\[
\begin{align*}
\varphi_0 &= 0, \\
\tan \theta_0 &= a \omega_1^2 \sin \omega_1 t / (b \omega_2^2 \sin \omega_2 t - g).
\end{align*}
\]

Using the substitution variables (10) and linearization we can write the system

\[
\begin{align*}
\Delta \dot{\theta} &= \Delta \varphi(t), \\
\Delta \dot{\varphi} &= -l^{-1} \left( \sqrt{ (a \omega_1^2 \sin \omega_1 t)^2 + (b \omega_2^2 \sin \omega_2 t - g)^2 } \right) \Delta \theta(t).
\end{align*}
\]

The characteristic polynomial of the linear system (13) has the form

\[
D(p) = p^2 + l^{-1} \sqrt{ (a \omega_1^2 \sin \omega_1 t)^2 + (b \omega_2^2 \sin \omega_2 t - g)^2 }.
\]

The necessary and sufficient condition for stability of generalized Kapitza pendulum motion is given by \(D(p) > 0\). This condition of stability takes place for any \(\omega_1, \omega_2\) and \(t \geq 0\).

### 3. Lyapunov stability analysis of the generalized Kapitza pendulum motion under stochastic driving forces

#### 3.1. Model of the generalized Kapitza pendulum under stochastic driving forces

Let \(l\) and \(m\) be length of the massless rod and mass of the bob for this pendulum. Let \(x(t)\) and \(y(t)\) be horizontal and vertical Cartesian coordinates of the suspension point. Denote by \(\theta(t)\) the angle between the rod of the pendulum and the vertical. In this case we can written the coordinates of bob in the form:

\[
\begin{align*}
x(t) &= l \sin \theta(t) + a(t; \omega), \\
y(t) &= l \cos \theta(t) + b(t; \omega).
\end{align*}
\]
where $a(t; \omega)$ and $b(t; \omega)$ are random functions that associated with acting upon the pendulum of stochastic forces that drive in the vertical and horizontal planes oscillations along $x$, $y$ axes and $(t; \omega) \in \mathcal{T} \times \Omega$ ($\Omega; \mathcal{F}; \mathbb{P}$) is an abstract probability space.

Then the kinetic and potential energies of the bob are

\[
\begin{align*}
K(t) &= \frac{m}{2} \left[ (l \cos \theta(t) \dot{\theta}(t) + \dot{a}(t; \omega))^2 + \left( -l \sin \theta(t) \dot{\theta}(t) + \dot{b}(t; \omega) \right)^2 \right], \\
V(t) &= mg \left( l \cos \theta(t) + b(t; \omega) \right).
\end{align*}
\]

The total energy of the system is given by the sum of the kinetic and potential energies

\[
E(t) = K(t) + V(t) = \frac{m}{2} \left( \dot{x}^2(t) + \dot{y}^2(t) \right) + mg \left( l \cos \theta(t) + b(t; \omega) \right),
\]

and then Lagrangian of the system has the form:

\[
L(t) = K(t) - V(t) = \frac{m}{2} \left( \dot{x}^2(t) + \dot{y}^2(t) \right) - mg \left( l \cos \theta(t) + b(t; \omega) \right).
\]

The equation of the pendulum motion has the form:

\[
\ddot{\theta}(t) + \frac{\dot{a}(t; \omega)}{l} \cos \theta(t) - \frac{\dot{b}(t; \omega) + g}{l} \sin \theta(t) = 0.
\]

We can rewrite this equation in the form of the system differential equations first order

\[
\begin{align*}
\dot{\theta}(t) &= \varphi(t), \\
\dot{\varphi}(t) &= -l^{-1} \dot{a}(t; \omega) \cos \theta(t) + l^{-1} \left( \dot{b}(t; \omega) + g \right) \sin \theta(t),
\end{align*}
\]

where $\varphi$ is an auxiliary variable.

\section{Stability analysis of motion under stochastic driving forces}

In this section using Lyapunov stability analysis of the motion for this pendulum subjected to excitation of stochastic driving forces that random act in the vertical and horizontal planes has been studied.

Let $\theta_0$ and $\varphi_0$ be parameters of the bob balance when equilibrium conditions can be obtained. Let $\Delta \theta$ and $\Delta \varphi$ be small deviations from the equilibrium state. In this case, the motion of the pendulum can be described as follows

\[
\begin{align*}
\dot{\theta} &= \theta_0 + \Delta \theta(t), \\
\dot{\varphi} &= \varphi_0 + \Delta \varphi(t).
\end{align*}
\]

Parameters $\theta_0$ and $\varphi_0$ can be found from

\[
\begin{align*}
\varphi_0 &= 0, \\
\dot{a}(t; \omega) \cos \theta_0 - (\dot{b}(t; \omega) + g) \sin \theta_0 &= 0.
\end{align*}
\]

If the mathematical expectation $M_a = M[\dot{a}(t)]$ and $M_b = M[\dot{b}(t)]$ exist we can get the coordinates of the equilibrium position for the pendulum

\[
\begin{align*}
\varphi_0 &= 0, \\
\tan \theta_0 &= M_a / (M_b + g).
\end{align*}
\]
Using the substitution variables (21) and linearization we can write the system

\[
\begin{align*}
\Delta \dot{\theta} &= \Delta \phi(t), \\
\Delta \dot{\phi} &= -l^{-1} \left( \sqrt{M_a^2 + (M_b + g)^2} \right) \Delta \theta(t).
\end{align*}
\] (24)

The characteristic polynomial of the linear system (24) has the form

\[
D(p) = p^2 + l^{-1} \sqrt{M_a^2 + (M_b + g)^2}.
\] (25)

The necessary and sufficient condition for average stability of generalized Kapitza pendulum motion is given by \( D(p) > 0 \). This condition of stability takes place for any \( M_a, M_b \) and \( t \geq 0 \).

5. Numerical study of the generalized Kapitza pendulum motion

In this section the numerical study of the random motion for generalized Kapitza pendulum under stochastic driving forces is presented. Suppose that random functions \( a(t) \) and \( b(t) \) can be represented in the form \( a(t) = A(t; \omega) \sin \nu_1 t, \) \( b(t) = B(t; \omega) \sin \nu_2 t \) where \( A(t; \omega) \) and \( B(t; \omega) \) are random functions of amplitudes, \( \nu_1 \) and \( \nu_2 \) are the frequencies of the harmonic vertical and horizontal forced oscillations of the pendulum suspension. We can rewrite the system differential equations (20) in the form
\[
\begin{align*}
\dot{\theta} &= \varphi(t), \\
\dot{\varphi} &= -l^{-1} \left( \ddot{A}(t) \sin \nu_1 t + 2 \dot{\dot{A}}(t) \nu_1 \cos \nu_1 t - A(t) \nu_1^2 \sin \nu_1 t \right) \cos \theta(t) + \\
&\quad + l^{-1} \left( \ddot{B}(t) \sin \nu_2 t + 2 \dot{\dot{B}}(t) \nu_2 \cos \nu_2 t - B(t) \nu_2^2 \sin \nu_2 t + g \right) \sin \theta(t),
\end{align*}
\]

where \( \dot{A}, \dot{B} \) are random velocities and \( \ddot{A}, \ddot{B} \) are random accelerations of the pendulum suspension long each of the axes.

The numerical example is presented in the figure (see Fig. 1) where \( l = 0.5 \text{ m.} \) is length of the massless rod, \( \nu_1 = 40 \text{ sec.}^{-1} \), \( \nu_2 = 10 \text{ sec.}^{-1} \) are the frequencies of the harmonic vertical and horizontal forced oscillations of the pendulum suspension, \( g = 10 \text{ m./sec.}^2 \) is acceleration of free fall. The random functions of amplitudes \( A(t) \) and \( B(t) \) were taken in the form

\[
\begin{align*}
A(t) &= a_0 + \epsilon_t, \\
B(t) &= b_0 + \eta_t,
\end{align*}
\]

where parameters \( a_0 = 0.1, b_0 = 0.05, \epsilon_t N(0, \sigma_x^2) \) and \( \eta_t N(0, \sigma_y^2) \) are independent normally distributed random variables (in the numerical example \( \sigma_x = 0.01 \) and \( \sigma_y = 0.001 \)). In this numerical example it is shown the existence of stable quasi-periodic motion of this pendulum.

6. Conclusions

In this work generalization of Kapitza pendulum whose suspension point moving in the vertical and horizontal planes is made. Lyapunov stability analysis of the motion for this pendulum subjected to excitation of periodic driving forces and stochastic driving forces that random act in the vertical and horizontal planes has been studied. The numerical study of the random motion for generalized Kapitza pendulum under stochastic driving forces has made. It is shown the existence of stable quasi-periodic motion of this pendulum. The vertical and horizontal motion simulation of the pendulum can be used for analysis of buildings and structure stability during earthquake.

7. Acknowledgments

The publication was prepared with the support of the RUDN University Program 5-100 and partially funded by RFBR grants No 15-07-08795, No 16-07-00556.

References

[1] Bardin B S and Markeyev A P 1995 On the stability of equilibrium of a pendulum with vertical oscillations of its suspension point (J. Appl. Math. Mech. vol 59) pp 879–886
[2] Du H, Qian C, Yang S and Li S 2013 Recursive design of finite-time convergent observers for a class of time-varying nonlinear systems (Automatica vol 49, No 2) pp 601-609
[3] Gaidamaka Y, Sopin E and Talanova M 2016 Approach to the analysis of probability measures of cloud computing systems with dynamic scaling (Communications in Computer and Information Science vol 601) pp 121–131
[4] Jiang C, Teo K L and Duan G 2012 A suboptimal feedback control for nonlinear time-varying systems with continuous inequality constraints (Automatica vol 48, No 4) pp 660–665
[5] Kapitsa P L 1951 Dynamic stability of a pendulum with oscillating point of suspension (Sov. Phys. JETP (in Russian) vol 21) pp 588–597
[6] Kapitsa P L 1951 A pendulum with vibrating point of suspension Usp. Phys. Nauk (in Russian) vol 44) pp 7–20
[7] Korolkova A V, Eferina E G, Laneev E B, Gudkova I A, Sevastianov L A and Kulyabov D S 2016 Stochasticization of one-step processes in the occupations number representation (Proceedings - 30th European Conference on Modelling and Simulation, ECMS) pp 698 — 704
[8] Mazenc F, Andrieu V and Malisoff M 2015 Design of continuous/discrete observers for time-varying non-linear systems (Automatica vol 57) pp 135–144
[9] Samouylov K, Naumov V, Sopin E, Gudkova I and Shorgin S 2016 *Sojourn time analysis for processor sharing loss system with unreliable server* (Lecture Notes in Computer Science, including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics vol 9845) pp 284—297

[10] Levi M 1988 *Geometry of Kapitsa’s potential* (Nonlinearity vol 11) pp 1365–1368

[11] Neishtadt A I and Shenga K 2016 *Bifurcations of phase portraits of pendulum with vibrating suspension point* [https://arxiv.org/abs/1605.09448](https://arxiv.org/abs/1605.09448)

[12] Ovseyevich A I 2006 *The stability of an inverted pendulum when there are rapid random oscillations of the suspension point* (J. Appl. Math. Mech. vol 70) pp 761–768

[13] Stephenson A 1908 *On induced stability* (Philosophical Magazine Series vol 6 No 15) pp 233–236

[14] Stephenson A 1908 *On a new type of dynamical stability* (Mem. Proc. Manch. Lit. Phil. Soc. vol 52 No 8) pp 1–10

[15] Zhang X, Liu L and Feng G 2015 *Leaderfollower consensus of time-varying nonlinear multi-agent systems* (Automatica vol 52) pp 8–14

[16] Zhou B, Cai G, and Duan G 2013 *Stabilization of time-varying linear systems via Lyapunov differential equations* (International Journal of Control vol 86, No 2) pp 332–347

[17] Zhou B 2015 *Global stabilization of periodic linear systems by bounded controls with applications to space-craft magnetic attitude control* (Automatica vol 60) pp 145–154