Quantum Hall effects of exciton condensate in topological flat bands

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Tunable exciton condensates in two-dimensional electron gas systems under strong magnetic field exhibits anomalous Hall transport owing to mutual Coulomb coupling, and have attracted a lot of research activity. Here, we explore another framework using topological flat band models in the absence of Landau levels, for realizing the many-body exciton phases of two-component fermions under strong intercomponent interactions. By developing new diagnosis based on the state-of-the-art density-matrix renormalization group and exact diagonalization, we show the theoretical discovery of the emergence of Halperin (111) quantum Hall effect at a total filling factor \( \nu = 1 \) in the lowest Chern band under strong Hubbard repulsion, which is classified by the unique ground state with bulk charge insulation and spin superfluidity. The topological nature is further characterized by one edge branch of chiral propagating Luttinger modes with level counting 1, 1, 2, 3, 5, 7 in consistent with the conformal field theory description. Moreover, with nearest-neighbor repulsions, we propose the Halperin (331) fractional quantum Hall effect at a total filling factor \( \nu = 1/3 \) in the lowest Chern band.

I. INTRODUCTION

As extensions of Laughlin wavefunction, the intercomponent correlation of two-component quantum Hall effects was proposed as Halperin’s two-component \((mnm)\) wavefunction in Ref. [1], described by the \( K = \begin{pmatrix} m & n \\ n & m \end{pmatrix} \) matrix within the framework of the field theory [2]. Two-component quantum Hall effects, where the phrase “two-component” represents a generic label for spin or pseudo-spin (bilayer, double well, etc.) quantum number, could exhibit tremendously richer physics than single-component systems, attributing to intercomponent interaction in two-component systems. Experimentally, such exotic intercomponent correlation for two-component integer and fractional quantum Hall states at \( \nu = 1 \) and \( \nu = 1/2 \) in the lowest Landau level had been confirmed [3, 4], in consistency with several theoretical proposals of Halperin’s (111) exciton superfluid [5–7] and (331) fractional quantum Hall state [12–15].

To gain a better understanding of the internal structure of multicomponent quantum Hall states, it is highly desirable to investigate the diagonal and off-diagonal elements of the \( K \) matrix which describe intracomponent and intercomponent Chern-Simons gauge-field couplings respectively, and can be derived from the inverse of the Chern number matrix for gapped quantum Hall states [16–18]. However a peculiar property of Halperin’s \((mnm)\) quantum Hall states is that they host intercomponent tunneling and counterflow transport anomalies in Hall resistance measurements [20], due to the superfluidity of exciton condensate in which particles in one component are coupled to holes in the other component. Such coherent exciton transport, which serves as a hallmark signature of Halperin (111) quantum Hall effect, has inspired a series of motivating quantum Hall drag measurements in different setups [21]. The intercomponent Coulomb drag transport in most recent experiments also serves as a primary proof in searching new types of correlated many-body topological states in two parallel graphene layers [22, 23].

Recently the rise of topological flat band models has fostered an excitingly new platform for studying the quantum Hall effect without the conventional Landau level [24–26] (See the recent reviews in Refs. [36, 37] for extended literatures). On the experimental side, realization of topological Haldane-honeycomb band provides a highly tunable system to explore intercomponent correlated states of two-component quantum fermionic \( ^{40}\text{K} \) gases with strong Hubbard repulsion [27], and the two-component correlated charge pumping can be implemented using spin- and density-resolved microscopy [28]. Also, various types of topological bands are proposed in multi-layer heterostructure, such as moiré flat bands in twisted multilayer graphene [29–31] with tunable correlated ferromagnetism observed [32], and some of fractionalised interacting phases in such topological bands dubbed “fractional Chern insulators” have been experimentally observed [33]. These related experimental advances, thus enabling the control and study of multi-layered systems, would open up new relevant prospects across a broader class of two-component Halperin \((mnm)\) quantum Hall effects for interacting fermions in topological lattice models, where a compelling theoretical evidence of them is still lacking and demanding which is the focus of our work.

In this work, we theoretically propose two-component Halperin \((mnm)\) quantum Hall effects emerging in topological flat bands through the state-of-the-art density-matrix renormalization group (DMRG) and exact diagonalization (ED) simulations with strong interactions in the thermodynamic limit, and elucidate the interaction controlling of the topological exciton condensate of two-component systems. We have studied the characteris-
tic topological degeneracy, the excitation gap, the topologically invariant Chern number, charge pumping, off-diagonal long range order and entanglement spectrum of the ground states, which depict the topological information from $K$ matrix.

This paper is organized as follows. In Sec. II we give a description of the model Hamiltonian of interacting two-component fermions in different topological lattice models, such as $\pi$-flux checkerboard and Haldane-honeycomb lattices. In Sec. III we study the many-body ground states of these two-component fermions in the strong interaction regime, present detailed numerical results of Halperin (111) state at fillings $\nu = 1$ in Sec. IIIA and further discuss the topological signatures of Halperin (333) state under nearest-neighboring repulsion at fillings $\nu = 1/3$ in Sec. IIIB. Finally, in Sec. IV we summarize our results and discuss the prospect of investigating topological exciton condensate in two-component systems.

II. MODEL AND METHOD

Here, we start from the following Hamiltonian of interacting spinful fermions in two typical topological lattice models, such as the $\pi$-flux checkerboard (CB) lattice,

$$H_{CB} = \sum_{<r,r'>} \epsilon_{\sigma} c_{r',\sigma}^\dagger c_{r,\sigma} - \sum_{<r,r'>} t'_{r',r} c_{r',\sigma}^\dagger c_{r,\sigma} - \sum_{<r,r'>} t''_{r',r} c_{r',\sigma}^\dagger c_{r,\sigma} + V_{\text{int}},$$

and Haldane-honeycomb (HC) lattice

$$H_{HC} = \sum_{<r,r'>} \epsilon_{\sigma} c_{r',\sigma}^\dagger c_{r,\sigma} - \sum_{<r,r'>} t'_{r',r} c_{r',\sigma}^\dagger c_{r,\sigma} + V_{\text{int}},$$

where $c_{r,\sigma}$ is the particle creation operator of spin $\sigma = \uparrow, \downarrow$ at site $r$, $(\ldots, \langle \ldots \rangle)$ and $(\langle \ldots \rangle, \langle \ldots \rangle)$ denote the nearest-neighbor, the next-nearest-neighbor, and the next-next-nearest-neighbor pairs of sites, respectively. The flat band limit is taken with the tunnel couplings $t' = 0.3t$, $t'' = -0.2t$, $\phi = \pi/4$ for checkerboard lattice [33], while $t' = 0.6t$, $t'' = -0.58t$, $\phi = 2\pi/5$ for honeycomb lattice [34]. We take the on-site and nearest-neighbor interactions with SU(2) symmetry,

$$V_{\text{int}} = U \sum_r n_{r,\uparrow} n_{r,\downarrow} + V \sum_{\sigma,\sigma'} \sum_{<r,r'>} n_{r',\sigma} n_{r,\sigma},$$

where $n_{r,\sigma}$ is the particle number operator of spin $\sigma$ at site $r$. Here, $U$ is the strength of the onsite interaction while $V$ is the strength of nearest-neighbor interaction.

We perform ED calculations on the many-body ground state of the model Hamiltonian Eqs. (1) and (2) in a finite system of $N_x \times N_y$ unit cells (the total number of sites is $N_s = 2 \times N_x \times N_y$), up to $N_s = 20$. The total filling of the lowest Chern band is $\nu = \nu_1 + \nu_2 = 2(N_1 + N_1)/N_s$, where $N_1, N_2$ are the particle numbers with $U(1) \times U(1)$-symmetry. With the translational symmetry, the energy states are labeled by the total momentum $K = (K_x, K_y)$ in units of $(2\pi/N_x, 2\pi/N_y)$ in the Brillouin zone. For larger systems, we exploit both finite and infinite DMRG on the cylindrical geometry. We keep the maximal bond dimension up to $M = 8000$ in infinite DMRG, which leads to excellent convergence for the results we report here. In infinite DMRG, the geometry of cylinders is open boundary condition in the $x$ direction and periodic boundary condition in the $y$ direction. Our comprehensive DMRG and ED studies can access large system sizes to establish the emergence of quantum Hall states at filling factors $\nu = 1/m$ for two-component fermions (odd $m = 1, 3$).

III. THE MANY-BODY GROUND STATES

In this section, we first examine systematically the topological properties of the many-body ground states at $\nu = 1, U \gg t, V = 0$, which becomes an easy-plane ferromagnet. Having identified the properties of Halperin (111) state which is most favored, we next describe numerical studies of Halperin (333) state at $\nu = 1/3, U, V \gg t$.

A. Halperin (111) state

For Halperin quantum Hall state characterized by the $K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ matrix, the charge channel remain insulating, and the system displays the quantized Hall effect, similar to usual $\nu = 1$ integer quantum Hall effect. However the spin channel condenses and the system becomes a gapless superfluid [47]. First, we demonstrate

![FIG. 1. (Color online) Numerical ED results for the low energy spectrum of two-component fermionic systems $\nu = 1$ with $U(1) \times U(1)$-symmetry in different topological lattices at $U \gg t, V = 0$ for (a) $\pi$-flux checkerboard lattice and (b) Haldane-honeycomb lattice.](image-url)
the unique ground state degeneracy on periodic lattice under strong onsite Hubbard repulsion $U \gg t, V = 0$. In Figs. 1(a) and 1(b), for different topological systems in strong interacting regime, we find that, there exists a well-defined single ground state separated from higher energy levels by a robust gap. In the spin notation, a variational ansatz describing the particle-hole pairs of Halperin (111) state can be written as

$$|\psi\rangle = \prod_{\mathbf{k}} \frac{1}{\sqrt{2}} (\chi_{\mathbf{k},\uparrow}^\dagger + e^{i\varphi} \chi_{\mathbf{k},\downarrow}^\dagger) |0\rangle,$$

where $\chi_{\mathbf{k},\sigma}$ creates a Bloch fermion of spin $\sigma$ and momentum $\mathbf{k}$ in the lowest Chern band and $\varphi$ is the phase for particle-hole pairs, which can take certain value without any energy cost. For strong Hubbard repulsion $U \gg t$, each of the single-particle orbitals in the lowest Chern band is occupied with only one particle at $\nu = 1$ and the total momentum $K = (\sum_{i=1}^{N_\uparrow} k_{i,\uparrow}, \sum_{i=1}^{N_\downarrow} k_{i,\downarrow})$ of the ground state can be easily determined, in consistency with the ED results in Figs. 1(a) and 1(b). Meanwhile, we calculate the charge-hole gap in the charge channel $\Delta_q = (E_0(N_\uparrow+1, N_\downarrow) + E_0(N_\uparrow-1, N_\downarrow) - 2E_0(N_\uparrow, N_\downarrow))/2$, and the magnon spin gap in the spin channel $\Delta_s = E_0(N_\uparrow+1, N_\downarrow-1) - E_0(N_\uparrow, N_\downarrow)$ for different system sizes. As shown in Figs. 2(a) and 2(b) for different system sizes, the finite size scaling of $\Delta_q$ remains a large finite value, which serves as a primary signature of an incompressible charge Hall phase, while the finite size scaling of $\Delta_s$ goes to a vanishing small value in the thermodynamic limit, signaling a compressible spin superfluid.

Next we extract the Chern number matrix $C = \begin{pmatrix} C_{\uparrow\uparrow} & C_{\uparrow\downarrow} \\ C_{\downarrow\uparrow} & C_{\downarrow\downarrow} \end{pmatrix}$ for a two-component system, related to the Hall conductance $48 \ 50$. With twisted boundary conditions $\psi(r_x + N_\alpha) = \psi(r_x) \exp(i\theta_\alpha^\sigma)$ where $\theta_\alpha^\sigma$ is the twisted angle for spin $\sigma$ particles in the $\alpha$ direction, the Chern number of the many-body ground state wavefunc-

$FIG. 2. (Color online) Numerical results for the charge-hole gap $\Delta_q$ and spin excitation gap $\Delta_s$ of two-component fermionic systems $\nu = 1$ at $U \gg t, V = 0$ for (a) $\pi$-flux checkerboard lattice and (b) Haldane-honeycomb lattice. Results are obtained using ED for $N_x = 16, 20$, and results are obtained using DMRG for $N_x = 24, 32, 40$.\n
$FIG. 3. (Color online) Numerical ED results for Berry curvatures $F^{xy} \Delta_{\nu}^x \Delta_{\nu'}^y / 2\pi$ of the $K = (\pi, 0)$ ground state of two-component fermionic systems $N_\uparrow = N_\downarrow = 5, N_s = 2 \times 2 \times N_\uparrow$ at $U \gg t, V = 0$ on the checkerboard lattice in the parameter plane: (a) $(\theta_\uparrow^\sigma = \theta_\downarrow^\sigma = \theta^x, \theta_\uparrow^y = \theta^y)$ and (b) $(\theta_\uparrow^\sigma = -\theta_\downarrow^\sigma = \theta^x, \theta_\uparrow^y = -\theta_\downarrow^y = \theta^y)$.\n
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$0.005 \ 0.015 \ 0.01 \ 0.005 \ 0.015$
hosts only one chiral branch of edge mode [47, 54]. Here
itive while the other is zero, that is, Halperin (111) state
icharacteristic chiral edge mode which can be revealed
is discussed in Chern bands with high Chern number [52].

Another “fingerprint” of Halperin (111) state is the
characteristic chiral edge mode which can be revealed
through the low-lying entanglement spectrum in the
bulk [52]. Different from usual two-component quantum
Hall states, one of the eigenvalues of \( K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) is positive while the other is zero, that is, Halperin (111) state
hosts only one chiral branch of edge mode [47, 54]. Here
we examine the structure of momentum-resolved entan-
glement spectrum for different cylinder widths \( N_y = 4, 6 \).
On the \( N_y = 4 \) cylinder, we observe only one forward-
moving branch of low-lying bulk entanglement spectrum
with the level counting 1, 1, 2, 3 for different charge and
spin sectors. Similarly, as shown in Figs. [5]a) and [5]b),
only one forward-moving branch of low-lying bulk entan-
glement spectrum with the level counting 1, 1, 2, 3, 5, 7 is
obtained for different charge sectors on the \( N_y = 6 \) cylin-
der. Nevertheless, this level counting is consistent with
\( SU(2)_1 \) Wess-Zumino-Witten conformal field theory with

the central charge \( c = 1 \), implying the gapless nature of
ciral edge modes.

In view of the current Haldane-honeycomb experiment
where \( t'' \) is negligible and the topological band becomes
significantly dispersive [53], it is natural and important
to take into account the effect of dispersive band struc-
tures controlled by the weak next-nearest-neighbour
tunnel coupling \( t'' \), on the stability of this exciton con-
densate under strong Hubbard repulsion. As shown in Figs.
[5]a) and [5]b), when \( t'' \) is tuned down away from
the flat band limit, we observe that the unique ground
state evolves smoothly and persists with a moderately
large protecting energy gap of the order 0.1\( t \) even in the
weakly tunneling regime \( |t''| \ll t, t' \) at \( \nu = 1, U \gg t \).
The robustness of Halperin (111) state, regardless of the
longer range hopping or band dispersion in different topo-
logal lattices, makes it very promising and straigh-

FIG. 4. (Color online) (a) The charge and spin transfer
for two-component fermions on the \( N_y = 4 \) cylinder at
\( \nu = 1, U \gg t, V = 0 \) with inserting flux \( \theta^\nu = \theta^S = \theta \)
for both topological checkerboard and honeycomb lattices.
(b) The off-diagonal long range order of the particle-hole
pair \( \langle c^\dagger_{\mathbf{r}, \uparrow} \mathbf{c}_{\mathbf{r'}, \downarrow} \rangle \) versus the lattice distance \( \mathbf{r} \) along the
x direction. As the maximal bond dimension is increased,
\( \langle c^\dagger_{\mathbf{r}, \uparrow} \mathbf{c}_{\mathbf{r'}, \downarrow} \rangle \) tends to a finite large value for \( \mathbf{r} \gg 1 \) when
the DMRG results are more and more converged.

FIG. 5. (Color online) Chiral edge mode identified from
the momentum-resolved entanglement spectrum for two-
component fermions on the \( N_y = 6 \) cylinder at \( \nu = 1, U \gg t, V = 0 \). The horizontal axis shows the relative momentum
\( \Delta \mathbf{K} = \mathbf{K}_y - \mathbf{K}_0 \) (in units of \( 2\pi/N_y \)). The numbers below the
red dashed line label the nearly degenerating pattern with
different momenta: 1, 1, 2, 3, 5, 7, · · · for different charge and
spin sectors (a) \( \Delta q = 0, \Delta S = 0 \) and (b) \( \Delta q = 0, \Delta S = 1 \).

FIG. 6. (Color online) Numerical ED results for the low
energy evolution of two-component fermionic systems as a
function of \( t'' \) in different topological lattices at \( \nu = 1, U \gg t, V = 0 \) for (a) \( \pi \)-flux checkerboard lattice and (b) Haldane-
honeycomb lattice.
a well-quantized value $\Delta Q \approx 1/3 = C_q$, consistent with the analysis of ED study.

IV. CONCLUSION

In summary, we have proved numerically that two-component fermions in topological lattice models could realize Halperin (mmm) quantum Hall states at commensurate partial fillings $\nu = 1/m$ (odd $m = 1, 3$) in the lowest Chern band, with topological properties characterized by the $K = \left( \begin{array}{ccc} m & m & m \\ m & m & m \end{array} \right)$ matrix. For Halperin (111) state, we demonstrate that it is an intercomponent exciton condensate of particle and hole pairs bound by the effective attractive interaction between particles and holes, when the onsite Hubbard interaction between intercomponent particles is repulsive, along with integer quantized Hall conductance and one chiral edge mode. For Halperin (333) state, we qualitatively identify its fractionally quantized topological nature from the degenerate ground state manifold and one-third quantized Hall conductance, similar to Laughlin $\nu = 1/3$ fractional quantum Hall effect. At experimental side, our two-component flat band models would be paradigmatic examples of a Hamiltonian featuring topological exciton condensation purely driven by local interaction, which is of sufficient feasibility to be realized for current optical Haldane-honeycomb lattice experiments in cold atoms. We believe that this work would offer an alternative route for the study of exotic topological excitonic insulator on designed band structures [57, 59], and excite a more extensive investigation of the fate of topological exciton condensate in many other topological band systems, such as moiré exciton [55, 60] in twisted multilayer graphene when strong electronic correlation is introduced.

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