A Generic Complementary Sequence Construction and Associated Encoder/Decoder Design

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Abstract—In this study, we propose a flexible construction of complementary sequences (CSs) that can contain zero-valued elements. To derive the construction, we use Boolean functions to represent a polynomial generated with a recursion. By applying this representation to recursive CS constructions, we show the impact of construction parameters such as sign, amplitude, phase rotation used in the recursion on the elements of the synthesized CS. As a result, we extend Davis and Jedwab's CS construction by obtaining independent functions for the amplitude and phase of each element of the CS, and the seed sequence positions in the CS. The proposed construction shows that a set of distinct CSs compatible with non-contiguous resource allocations for orthogonal frequency-division multiplexing (OFDM) and various constellations can be synthesized systematically. It also leads to a low peak-to-mean-envelope-power ratio (PMEPR) multiple accessing scheme in the uplink and a low-complexity recursive decoder. We demonstrate the performance of the proposed encoder and decoder through comprehensive simulations.

Index Terms—Complementary sequences, Gray code, peak-to-average-power ratio, pseudo-Boolean function, Reed-Muller code, non-contiguous resource allocation

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) enables a radio to transmit multiple modulation symbols on orthogonal channels separated in the frequency domain. One drawback of OFDM transmission is that it causes high peak-to-mean-envelope-power ratio (PMEPR) for arbitrary modulation symbols. In the literature, there are numerous approaches to reduce the PMEPR [1], [2]. One of the trends is to design codes such that the PMEPR of the resulting signal is bounded (see [3] and the references therein). In this direction, complementary sequences (CSs) from the cosets of Reed-Muller (RM) codes can limit the PMEPR of an OFDM symbol to be less than or equal to 3 dB while still providing coding gain [4]. However, these sequences have to be contiguous with phase shift keying (PSK) alphabet and limited choice of lengths. In this study, we address the issue of constructing CSs compatible with a set of flexible resource allocation in the frequency domain for OFDM while achieving expressions for not only phase but also amplitude of the elements of the CSs.

CSs were introduced by M. Golay in [5]. He established a framework to synthesize Golay complementary pairs (GCPs) with binary alphabet. His constructions were later extended by R. J. Turyn [6] and Sivasamy [7]. In [8], Budišin introduced the permutations of shifted phase-rotated sequences instead of simple concatenation seed CSs to obtain more CSs. In [9], he proposed another construction that explains how the scaling factors can be taken into account in a recursive method to generate multi-level GCPs. In [10] and [11], recursive constructions with phase factors and scaling factors were combined in a single recursion. Nevertheless, these earlier studies did not propose direct constructions for CSs or investigate CSs with certain alphabets.

Appealing applications of CSs on communication systems were first demonstrated in [4], [12]–[15]. By generalizing an earlier work of Boyd [12], Popovic showed that CSs were beneficial to control the peak instantaneous power of OFDM signals [13]. Davis and Jedwab proved that a large class of CSs over \( \mathbb{Z}_{2h} \) of length \( 2^m \) can be obtained through generalized Boolean functions related to RM codes [4], [14] for \( h \in \mathbb{Z}^+ \). Davis and Jedwab's result was also generalized to a larger class of CSs over \( \mathbb{Z}_H \) of length \( 2^m \) where \( H \) is even positive integer in [15]. In the literature, the CSs generated through Davis and Jedwab's construction is sometimes referred to as Golay-Davis-Jedwab (GDJ) or standard sequences.

The connection between CSs and RM codes has been a cornerstone of various research directions. The first major challenge has been the low code rate of the encoder proposed by Davis and Jedwab. Since the sequence length increases much faster than the number of distinct CSs for the encoder, the corresponding code rate decreases for a larger number of unencoded bits. To address this issue, one major direction has been exploring the CSs which cannot be generated through method in [4], i.e., nonstandard sequences. Until Li and Chu found 1024 quaternary nonstandard CSs of length 16 via a computer search in 2005 [16], there were no evidence on the existence of nonstandard CSs of length \( 2^m \). In [17], Li and Kao showed that these sequences contain third-order monomials and can be synthesized by concatenating or interleaving of quaternary GCP of length 8. In [18], Fiedler and Jedwab provided another explanation how these sequences arise by showing that some of the standard CSs in different cosets have identical aperiodic auto-correlation function (AACF). In [19], they also provided a construction leading to the nonstandard CSs in [16] by expressing CSs as a projection of a multi-dimensional array.

To increase the number of distinct CSs for a given sequence length, another direction has been the CS construction with larger alphabets such as finding quaternary amplitude modulation (QAM) constellations. In [20], Rößling and V. Tarokh showed that 16-QAM sequences which result in low PMEPR can be constructed through a weighted sum of two CSs with the alphabet of quadrature phase-shift keying (QPSK) constellation. In [21], [22], Chong et al. showed that a 16-QAM CS can...
be formed by introducing the "offset" sequences based on the Boolean functions to a "base" quaternary standard sequence and there exist at least \((14 + 12m)(m/2)4^{m+1}\) distinct CSs of length \(2^m\) with the alphabet of 16-QAM constellation. In [23], Li revised Chong et al. and Lee and Golomb’s theorems and introduced new offset pairs leading to 64-QAM CSs proposed in [23]. In [26], Li showed that there exist at least \(\left[\frac{m+1}{2}\right]4^{q(r-1)} - \left(m+1\right)4^{q(r-1)} + 2^{q-1}\left(m/2\right)4^{m+1}\) CSs for 4\(^{q}\)-QAM sequences. Although the offset method is helpful for synthesizing a large number of distinct QAM CSs, it typically does not give concise constructions as Davis and Jedwab’s methods for the standard sequences [22], [23]. This issue is one of our motivations to synthesize CSs by separating the functions for its amplitude and the phase terms. From this aspect, our study is in line with [27] and [28]. However, our work differs from [28] as we focus on a framework relying on linear operators, which enables us to generate non-contiguous CSs, i.e., CS with zero-valued elements, the CSs of length non-power-of-two, and the CS with various constellations. They also pave the way for developing an encoder and a decoder.

Another noteworthy direction for increasing the number of distinct sequences is to extend CSs to complementary sets or complete complementary codes at the expense of a larger PMEPR [29]–[31]. For these extensions, paraunitary matrices with larger sizes are often utilized. Since we focus on CSs in this study, we refer the reader to [29] and [30] and the references therein for further details.

A. Contributions and Organization

A framework based on linear operators: The first contribution is Lemma 1 that re-expresses the result of a recursion evolved with two linear operators at each step with Boolean functions. It describes how the operators applied to the input polynomials at the \(n\)th recursion step are distributed to the coefficients of the final polynomials.

A novel CS construction: By applying Lemma 1 to recursive CS constructions [8]–[11], [32], we show the impact of each parameter in the recursion on the synthesized CS. By removing the redundant parameters, we obtain a concise construction given in Theorem 2 as the main contribution of this study. It extends James and Davis’s result [4] by providing independent pseudo-Boolean functions for the amplitude, phase, seed sequences, and the support of the CS.

Distinct CSs with a wide range of constellations: We introduce a constellation family consisting of QAM, PSK, DVB-16APSK, DVB-32APSK, and a modified version IEEE 802.11ay 64-NUC, called equiangular sub-symmetric constellation (ESC). We show that distinct CSs can be enumerated systematically for a given ESC. We also discuss how to generate non-contiguous CSs where the non-zero-valued elements are in ESC.

Encoder/decoder for CSs with an ESC: We develop a generic encoder and maximum-likelihood (ML)-based decoder for CSs with ESC. We demonstrate its applicability for a practical wireless communication system through a low-PMEPR multiple accessing scheme in the uplink with non-contiguous resource allocation for OFDM.

The rest of the paper is organized as follows. In Section III preliminaries are provided. In Section IV we establish Lemma 1. In Section V Theorem 2 is discussed. In Section VI we enumerate CSs for an arbitrary ESC. In Section VII we discuss the encoder and the decoder. In Section VIII we provide numerical results. We conclude the paper in Section IX.

Notation: The sets of complex numbers, real numbers, non-negative real numbers, integers, non-negative integers, positive integers, and integers modulo \(H\) are denoted by \(\mathbb{C}, \mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}^+, \) and \(\mathbb{Z}_H\), respectively. The set of \(m\)-dimensional integers where each element is in \(\mathbb{Z}_H\) is denoted by \(\mathbb{Z}_H^m\). The transpose, Hermitian, the complex conjugation, the modulo 2, and the assignments are denoted by \((\cdot)^T, (\cdot)^H, (\cdot)^*, (\cdot)_2,\) and \(-\leftarrow\), respectively. The constant \(\sqrt{-1}\) is \(\mathbb{I}\).

II. Preliminaries and Definitions

A. Sequences, OFDM, and PMEPR

Let \(a = (a_i)_{i=0}^{N-1} \triangleq (a_0, a_1, \ldots, a_{N-1})\) be a sequence of length \(N\), where \(a_i \in \mathbb{C}\) and \(a_{N-1} \neq 0\). We associate the sequence \(a\) with the polynomial \(A(z) = a_{N-1}z^{N-1} + a_{N-2}z^{N-2} + \cdots + a_0\) in indeterminate \(z\). When it is clear from the context, \(A(z)\) is a polynomial function.

Let \(s_a(t) = \sum_{i=0}^{N-1} a_i e^{j2\pi i t}\) for \(t \in [0, T_s)\) be the continuous-time baseband OFDM symbol generated from the sequence \(a\) with the symbol duration \(T_s\). Since \(s_a(t) = A(e^{j2\pi t})\), the instantaneous envelope power of the OFDM symbol can be examined by evaluating the polynomial \(A(z)\) at \(|z| = 1\). When \(z\) is restricted to be on the unit circle in the complex plane, \(|A(z)|^2\) corresponds to \(\sum_{k=-N+1}^{N-k} \rho_k(k) z^k\), where \(\rho_k(k)\) is the AACF of the sequence \(a\) given by

\[
\rho_k(k) \triangleq \begin{cases} 
\sum_{i=0}^{k} a_i^* a_{i+k} & 0 \leq k \leq N - 1 \\
\sum_{i=0}^{N-k} a_i a_{i+k}^* & -N + 1 \leq k < 0 \\
0 & \text{otherwise}
\end{cases}
\]

Hence, an upper bound for the instantaneous power can be obtained as \(\max_{t \in [0, T_s)} |s_a(t)|^2 = \max_{i=1} \{|A(z)|^2 \leq \rho_0(0) + 2\sum_{k=1}^{N-1} |\rho_k(k)|, \) which indicates that a sequence with a smaller \(|\rho_0(k)|\) for \(k \neq 0\) also provides less power fluctuations when it is used for OFDM transmission.

Let \(C\) denote a code of length \(N\). Let \(c \in \mathbb{C}^N\) be an admissible sequence, i.e., a codeword, in \(C\). In this study, we define the PMEPR of the codeword \(c\) as \(\max_{t \in [0, T_s)} |s_c(t)|^2 / P_{av}\), where \(P_{av} = E_{c} [\rho_0(0)]\) is a constant that depends on the code.

B. Sequence Synthesis

A generalized Boolean function is a function \(f\) that maps from \(\mathbb{Z}_H^m = \{x_1, x_2, \ldots, x_m\} \forall x_j \in \mathbb{Z}_H\) to \(\mathbb{Z}_H\) as \(f : \mathbb{Z}_H^m \rightarrow \mathbb{Z}_H\), where \(H\) is an integer. It can be uniquely expressed as a linear combination of the monomials over \(\mathbb{Z}_H\), i.e.,

\[
f(x) = \sum_{k=0}^{2^m-1} c_k \prod_{j=1}^m x_j^{k_j} = c_0 1 + \cdots + c_{2^m-1} x_1 x_2 \cdots x_m ,
\]

where \(x \triangleq (x_1, x_2, \ldots, x_m)\) and \(c_k \in \mathbb{Z}_H\) for \(k = \sum_{j=1}^m k_j 2^{m-j}\) and \(k_j \in \mathbb{Z}_2\) (i.e., the coefficient of \((k + 1)\)th
monomial \( x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m} \) belongs to \( \mathbb{Z}_H \). The expression given in (2) is called algebraic normal form (ANF) of \( f(x) \) [23].

The co-domain of the function \( f \) can be extended to \( \mathbb{R} \) and \( \mathbb{Z}_0 \). In the former case, \( f : \mathbb{Z}_0^m \rightarrow \mathbb{R} \) is a pseudo-Boolean function and the sequences generated through the monomials construct a basis in \( \mathbb{R}^{2^m} \) and \( c_k \in \mathbb{R} \). In the latter case, i.e., \( f : \mathbb{Z}_0^m \rightarrow \mathbb{Z}_0 \), the monomial coefficients are in \( \mathbb{Z}_0 \). In this study, we use \( \hat{f}(i) \) as a basis for a function that maps the non-negative integers to the co-domain of \( f \) as \( \hat{f}(i) \triangleq f \circ e^{i}(i) \), where \( i = e(x) \triangleq \sum_{j=1}^{m} x_j2^{m-j} \), i.e., a decimal representation of the binary number constructed using all elements in the sequence \( x \), where the most significant bit is \( x_1 \).

Let \( f_r : \mathbb{Z}_2^m \rightarrow \mathbb{R} \), \( f_l : \mathbb{Z}_2^m \rightarrow \mathbb{R} \), and \( f_z : \mathbb{Z}_0^2 \rightarrow \mathbb{Z}_0^+ \). In this study, we generate a complex sequence \( a = \{a_i\}_{i=0}^{2^m-1} \) by listing \( a_i = \xi f_r(i)+f_l(i) \) for \( \xi \triangleq e^{i\pi} \). We utilize the function \( f_z(x) \) for generating complex sequences with zero-valued elements by modifying the associated polynomial of \( a = \{a_i\}_{i=0}^{2^m-1} \) as

\[
C(z) = \sum_{i=0}^{2^m-1} a_i z^{\hat{f}_r(i)+i}.
\]

Therefore, the sequence \( c \) can be characterized with \( f_r(x) \), \( f_l(x) \), and \( f_z(x) \).

Let \( f : \mathbb{Z}_0^m \rightarrow \mathbb{Z}_0 \) be a Boolean function and \( a \) and \( b \) be seed sequences of length \( N \). The polynomial \( S(x,z) = A(z)(1+f(x))z+B(z)f(x) \) is then equal to \( A(z) \) for \( f(x) = 0 \) and \( B(z) \) for \( f(x) = 1 \). For \( U \in \mathbb{Z}_0^+ \), consider the polynomial given by

\[
C(z) = \sum_{i=0}^{2^m-1} \tilde{S}(i,z)a_iz^{\hat{f}_r(i)+U_i}.
\]

For \( U = N \) and \( \tilde{f}_z(i) = 0 \) for all \( i \), the polynomial indicates the \( 2^m \) sequences (i.e., \( A(z)\xi^{\hat{f}_r(i)+\hat{f}_l(i)} \) or \( B(z)\xi^{\hat{f}_r(i)+\hat{f}_l(i)} \) based on \( f(i) \)) are concatenated. With the seed sequences generated by \( U - N \) zero-valued elements for \( U > N \), the \( \mathbb{N} - U \) elements of the two adjacent seed sequences are overlapped for \( U < N \).

### C. Constellations

Let \( p_u \in \mathbb{C} \) for \( u \in \{1, 2, \ldots, S\} \) and \( S \in \mathbb{Z}_+^+ \). Let \( r_u \) and \( \psi_u \) denote the amplitude and the phase of \( p_u \) subtracted by \( \pi/4 \) radian, respectively. We define \( \mathbb{S}_H \) as a set of distinct \( S \) complex numbers such that \(-jp_u^* \in \mathbb{S}_H \) if \( |\psi_u| < \pi/H \) for all \( p_u \in \mathbb{S}_H \) and \(-\pi/H < \psi_u \leq \pi/H \). In other words, the points in \( \mathbb{S}_H \) are symmetric with respect to the \( xy \)-diagonal of the complex plane (i.e., sub-symmetry), except the points on the line \( y = \tan(\pi/H + \pi/4)x \). By rotating all the points in \( \mathbb{S}_H \) by an integer multiple of \( 2\pi/H \) (i.e., equiangularity), we define ESC as follows:

**Definition 1.** Equiangular sub-symmetric constellation is \( \mathbb{S}_{esc} = \{p_u \times e^{ik2\pi/k} | p_u \in \mathbb{S}_H \text{ and } k \in \mathbb{Z}_H \} \).

QAM and PSK are ESCs. For example, let 4s\(^2\)-QAM constellation be a set of points given by \( \{(2u-1)+(2v-1)\mid u, v \in \mathbb{Z}, -s < u \leq s, -s < v \leq s \} \) for \( s \in \mathbb{Z}_+^+ \). A 4s\(^2\)-QAM constellation is an ESC for \( H = 4 \) and the corresponding \( \mathbb{S}_H \) is the set of \( S = s^2 \) points on the first quadrant as shown in Figure 1(a) for \( s = 4 \). HS-Star is another ESC defined as the points in \( \mathbb{S}_H \) satisfying \( r_u < r_{u+1} \) and \( \psi_u = \pi/H \times ((u-1) \mod 2) \) for \( u = 1, 2, \ldots, S \) as exemplified in Figure 1(b). DVB-16/32APSK defined in [34] are also ESCs for \( H = 4 \). DVB-32APSK is shown in Figure 1(c). IEEE 802.11ay 64-NUC constellation [35] is not an ESC. However, it can be modified to be an ESC for \( H = 4 \) by switching the real and imaginary components and calculating the mean of two closest points in 2-D complex plane. The modified IEEE 802.11ay 64-NUC is provided in Figure 1(d).

In this study, we synthesize complex sequences where their non-zero-valued elements in an ESC via \( \xi^{f_r(x)+f_l(x)} \) for \( f_r : \mathbb{Z}_2^m \rightarrow \mathbb{R} \) and \( f_l : \mathbb{Z}_2^m \rightarrow \mathbb{R} \). The fundamental strategy that we use is to offset the monomial coefficients of \( f_r(x) \) and \( f_l(x) \) such that the elements are in the targeted ESC. For instance, let \( f_r(x) = k' + k_1x_1 \) and \( f_l(x) = e_1x_1 \) for \( m = 2 \), \( k' = k' + \Delta_1 \), \( k_1 \in \mathbb{Z}_2 \), \( e_1 \in \mathbb{R} \), and \( H = 2 \). For \( \{\Delta_1 = 0, e_1 = 0\}, \{\Delta_1 = \pi/2, e_1 = 0\}, \{\Delta_1 = 0, e_1 = \ln(3/\pi)\}, \{\Delta_1 = \pi/2, e_1 = \ln(3/\pi)\} \), the elements of the complex sequence are in \( \{-1, 1\}, \{-1, 1, 3\} \), and \( \{-3, -1, 1, 3\} \), respectively, which forms an ESC of \( S = 8 \) and \( H = 2 \).

### D. Golay Complementary Pair and Complementary Sequence

The sequence pair \((a, b)\) of length \( N \) is a GCP if \( p_u(k) + p_b(k) = 0 \) for \( k \neq 0 \). The sequences \( a \) and \( b \) are referred to as CSs. By using the definition of GCP, a GCP \((a, b)\) satisfies \( |A(z)|^2 + |B(z)|^2 = p_a(0) + p_b(0) \) for
\[ |z| = 1. \text{ The instantaneous peak power of the corresponding OFDM signal generated through a CS } a \text{ is bounded since } \max_{t \in [0,T]} |s_a(t)|^2 \leq \rho_0(0) + \rho_1(0). \]

A general well-known recursive GCP construction can be given as follows:

**Theorem 1** \[8\] \(11\] \([32\]. Let \(a, b \) be a GCP of length \(N\). For \(\alpha_n, \beta_n \in \mathbb{R}^+ \), \(U \in \mathbb{Z}_0^+, \gamma_n, \eta_n \in \{ u : u \in \mathcal{C}, |u| = 1 \} \) for \(n = 1, 2, \ldots, m \), and the sequence \(\psi = \{\psi_n\}_{m=1}^{\infty}\) defined by a permutation of \(\{0, 1, \ldots, m-1\}\), assume that the following operations occur at the \(n\)th step of a recursion:

\[
\begin{align*}
A^{(n)}(z) &= \gamma_n \alpha_n A^{(n-1)}(z) + \eta_n \beta_n \omega_n B^{(n-1)}(z)z^{\tau_n+U2^{\psi_n}}, \\
B^{(n)}(z) &= \eta_n^* \alpha_n A^{(n-1)}(z) - \gamma_n^* \beta_n \omega_n B^{(n-1)}(z)z^{\tau_n+U2^{\psi_n}},
\end{align*}
\]

where \(A^{(0)}(z) = A(z), B^{(0)}(z) = B(z)\). The sequences \(a^{(m)}\) and \(b^{(m)}\) associated with the polynomials \(A^{(m)}(z)\) and \(B^{(m)}(z)\), respectively, construct a GCP for \(m \geq 1\).

**Theorem** \[1\] shows the Budišin’s recursive constructions in \[8\] Eq. (4) and \[9\] Eqs. (3-4), the Garcia’s constructions in \[10\] Eq. (2)] and \[11\] Eq. (5)], a special case of Golay’s concatenation in \[32\] Eq. (10)]. The expression in Theorem 1 can also be written in a matrix form as

\[
\begin{bmatrix}
A^{(n)}(z) \\
B^{(n)}(z)
\end{bmatrix} = 
\begin{bmatrix}
\gamma_n \alpha_n & \eta_n \beta_n \\
\eta_n^* \alpha_n & -\gamma_n^* \beta_n
\end{bmatrix}
\begin{bmatrix}
A^{(n-1)}(z) \\
B^{(n-1)}(z)
\end{bmatrix}z^{\tau_n+U2^{\psi_n}},
\]

which shows that CS construction can involve 2-by-2 paraunitary matrices. The special cases of the paraunitary matrices are investigated in the literature. For example, \[27\] and \[29\] consider the case \(a = b = 1, \tau_n = 0, U = 1\) for \(n = 1, 2, \ldots, m\). Both \[27\] and \[29\] show that Davis and Jedwab’s construction can also be derived as a special case of paraunitary matrices, which implies that Theorem \[1\] is general enough to construct standard CSs.

To utilize CSs in communications, their distinctness must be ensured. However, the recursion in Theorem \[1\] (and the corresponding paraunitary matrix representation) by itself does not guarantee distinct CSs for different parameters \[27\] Section IV] after \(m\) iterations.

**Example 1.** Assume that \(a = b = 1, U = 1, m = 3, \psi = (1, 2, 3), \tau_n = 0, \gamma_n = 0, \) and \(\eta_n = 0\) for \(n = 1, 2, 3\). Consider two different configurations: \(\{ (\alpha_1, \alpha_2, \alpha_3) = (1, 2, 3), (\beta_1, \beta_2, \beta_3) = (1/2, 2, 3) \} \) and \(\{ (\alpha_1, \alpha_2, \alpha_3) = (3, 1, 2), (\beta_1, \beta_2, \beta_3) = (3/2, 1, 2) \} \). Both configurations result in the same CS \(a^{(3)} = (6, 3, 3, -6, 6, 3, -3, 6).\)

To resolve this issue, in \[27\], the distinct CSs are generated by using the properties of Gaussian integers for QAM alphabet. In this study, we eliminate the redundancy in Theorem 1 by using pseudo-Boolean functions, which leads to a generalization of Davis and Jedwab’s construction.

**E. Operators**

An operator \(O\) transforms one polynomial to another polynomial. If \(O\{A(z) + bB(z)\} = aO\{A(z)\} + bO\{B(z)\}\) for all \(A(z)\) and \(B(z)\) and \(a, b \in \mathbb{C}\), the operator \(O\) is linear. Several examples of linear operators are \(O\{R(z)\} = z^n R(z)\), \(O\{R(z)\} = -R(z)\), \(O\{R(z)\} = \xi R(z)\), and \(O\{R(z)\} = \xi^b R(z)\) for \(d \in \mathbb{Z}_0^+\) and \(a, b \in \mathbb{R}\), where \(R(z)\) is an arbitrary polynomial.

Let \(O_0\) and \(O_1\) be two linear operators and \(f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2^n\) be a Boolean function. Consider the polynomial given by \(P^{(n)}(z) = \sum_{i=0}^{2^n-1} f_i(\psi) R^{(i)}(z) z^i\). The Boolean function \(f(i)\) determines which one of the two operators affects the coefficient of \(z^i\). Thus, one can simply define the polynomial \(P^{(n)}(z)\) with the operators \(O_0\) and \(O_1\) and the Boolean function \(f\). We use this notation to define the polynomials as shown in Table \[I\].

**III. REPRESENTATION OF A RECURSION**

Let \(P^{(n)}(z)\) and \(Q^{(n)}(z)\) for \(n = 1, 2, \ldots, m\) be the polynomials generated by

\[
P^{(n)}(z) = O^{(1)}_{11} \{ P^{(n-1)}(z) \} + O^{(1)}_{12} \{ Q^{(n-1)}(z) \} z^{2^{\psi_n}},
\]

\[
Q^{(n)}(z) = O^{(2)}_{11} \{ P^{(n-1)}(z) \} + O^{(2)}_{22} \{ Q^{(n-1)}(z) \} z^{2^{\psi_n}},\tag{3}
\]

where \(\psi_n\) is the \(n\)th element of the sequence \(\psi = \{\psi_n\}_{m=1}^{\infty}\) defined by a permutation of \(\{0, 1, \ldots, m-1\}\).

| \(O(\cdot)\) | \(\mathbb{Z}_2^+\) |
|----------------|----------------|
| \(\mathbb{Z}_2\) | \(\mathbb{Z}_2\) |
| \(\mathbb{Z}_2^+\) | \(\mathbb{Z}_2^+\) |

where the Boolean functions \(f_n : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2\) and \(g_n : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2\) indicate which of the two operators, \(O^{(0)}_0\) and \(O^{(0)}_1\), are linear operators, \(P^{(m)}(z)\) and \(Q^{(m)}(z)\) can be obtained as

\[
P^{(m)}(z) = \sum_{i=0}^{2^m-1} O^{(m)}_{f_n(i)} ... O^{(2)}_{f_2(i)} O^{(1)}_{f_1(i)} \{ R(z) \} z^i, \tag{4}
\]

\[
Q^{(m)}(z) = \sum_{i=0}^{2^m-1} O^{(m)}_{g_n(i)} ... O^{(2)}_{g_2(i)} O^{(1)}_{g_1(i)} \{ R(z) \} z^i, \tag{5}
\]

where the Boolean functions \(f_n : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2\) and \(g_n : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2\) indicate which of the two operators, \(O^{(0)}_0\) and \(O^{(0)}_1\), are involved in the constructions of the composite operators \(F_n \{ R(z) \}\) and \(G_n \{ R(z) \}\) by setting the subscripts of \(O^{(n)}_0\) and \(O^{(n)}_1\), respectively. The Boolean functions can be obtained in closed-form as follows:

**Lemma 1.** Let \(b_n\) be a configuration vector defined by \(b_n^T = [b_1(b_n)] b_2(b_n) b_2(b_n) \in \mathbb{Z}_2^4\), where \(b_1(i) = 0\) if \(O^{(n)}_{ij} = O^{(n)}_{11}\) and \(b_1(i) = 1\) if \(O^{(n)}_{ij} = O^{(n)}_{11}\). Then, \(f_n(x)\) and \(g_n(x)\) are the Boolean functions given by

\[
f_n(x) = \begin{cases}
\sum_{i=1}^{n} f_1(i) x_{i+1} + b_{11} x_{i+1} & \text{if } n = m \\
\sum_{i=1}^{n} f_1(i) x_{i+1} & \text{if } n < m
\end{cases}, \tag{6}
\]

\[
g_n(x) = \begin{cases}
\sum_{i=1}^{n} g_1(i) x_{i+1} + b_{11} x_{i+1} & \text{if } n = m \\
\sum_{i=1}^{n} g_1(i) x_{i+1} & \text{if } n < m
\end{cases}. \tag{7}
\]
for \( n = 1, 2, \ldots, m \), where \( \pi_n = m - \psi_n \) is the \( n \)th element of the sequence \( \pi = (\pi_n)_{n=1}^m \).

The proof for Lemma 1 is given in Appendix A. Since there exist 16 possible options for the \( n \)th configuration vector \( b_n \), there also exist 16 different Boolean functions for each step in \( \{3\} \).

**Example 2.** Assume the operator configuration given by \( O^{(n)}_1 = O^{(n)}_2 = O^{(n)}_3 = O^{(n)}_4 = O^{(n)}_5 = O^{(n)}_6 \) for all iterations, \( m = 2 \), and \( \psi = (0, 1) \). When \( n = 1 \), \( P^{(1)}(z) = O^{(1)}_1(\{R(z)\}) + O^{(1)}_2(\{R(z)\}) \psi \) and \( Q^{(1)}(z) = O^{(1)}_3(\{R(z)\}) + O^{(1)}_4(\{R(z)\}) \psi \). When \( n = 2 \), \( P^{(2)}(z) = O^{(2)}_1(\{R(z)\}) + O^{(2)}_2(\{R(z)\}) \psi \) and \( Q^{(2)}(z) = O^{(2)}_3(\{R(z)\}) + O^{(2)}_4(\{R(z)\}) \psi \) are obtained as \( (3) \) to identify underlying code. For example, the underlying \( H \)-bit Gray code can be obtained as follows:

\[
0 \to 0_0, 1, 1 \to 0_0 + 1, 2, \ldots, m \to m_0, m_1, m_2, \ldots, m_{m-1},
\]

\[
= m - \psi_n \\text{ and } \psi = (0, 1), \ \text{the Boolean functions that lead to the sequences } f_1, f_2, g_1, \ \text{and } f_2 \text{ are obtained as } f_1(x) = x_1, x_2 = x_2x_1, f_2(x) = 0, \ \text{and } g_2(x) = x_1x_2x_3 = x_2x_1, \text{ and } f_2(x) = x_2x_1 = x_1 = x_1, \text{ respectively, which result in the same sequences obtained in Example 2. Now assume that } O^{(m)}_1(\{R(z)\}) = R(z) \text{ and } O^{(m)}_2(\{R(z)\}) = -R(z). \text{ Since } \xi^H/2 = -1, \text{ we can then express the composite operator as } F_{\xi_1} = \xi_1+f(i) \times R(z), \text{ where the underlying function determining the sign is } f(x) = \frac{\sum n=1}{n} x_0 x_{n+1}.
\]

Lemma 1 can be used for any recursion in the form given in \( \{3\} \) to identify underlying code. For example, the underlying code for \( m \)-bit Gray code can be obtained as follows:

**Example 4 (Gray Code).** Assume the operator configuration given by \( O^{(n)}_1 = O^{(n)}_2 = O^{(n)}_3 = O^{(n)}_4 = O^{(n)}_5 = O^{(n)}_6 \) for all iterations, \( \psi = (0, 1, \ldots, m - 1) \). In this case, the configuration vector is \( b^n = [0 1 1 0] \) for \( n = 1, 2, \ldots, m \). Hence, by using \( \{3\} \), \( f_n(x) \) is obtained as

\[
f_n(x) = \begin{cases} x \pi_n, & \text{if } n = m \\ (x \pi_n + x_{n+1})_2, & \text{if } n < m, \end{cases}
\]

where \( \pi = (m, m - 1, \ldots, 1) \). As the recursion combines two polynomials after applying \( O^{(n)}_1 \) to the second polynomial and the positions of \((n-1)\)th operators are reflected in \( F^{(n-1)}(z) \) and \( Q^{(n-1)}(z) \), \( f_n(x) \) for \( n = 1, 2, \ldots, m \) form the basis vectors of \( m \)-bit Gray code, e.g., \( f_1 = (0, 1, 1, 0, 0, 1, 1, 0) \), \( f_2 = (0, 0, 1, 1, 1, 0, 0, 1) \), and \( f_3 = (0, 0, 0, 0, 1, 1, 1, 1) \) for \( m = 3 \).

**IV. A GENERIC CS CONSTRUCTION**

In this section, we first utilize Lemma 1 to develop a direct GCP construction by re-expressing the recursion given in Theorem 1. We then discuss how to generate distinct CS without an auxiliary method and compare it with other constructions in the literature through examples.

**Theorem 2 (Main Theorem).** Let \((a, b)\) be a GCP of length \( N \geq 1 \) and \( \pi = (\pi_n)_{n=1}^m \) be a sequence defined by a permutation of \([1, 2, \ldots, m]\). For any \( H \in \mathbb{Z}^+ \), \( e_n, e'_n \in \mathbb{R} \), and \( k_n, k'_n \in [0, H] \) for \( n = 1, 2, \ldots, m \) and \( \xi = e\pi \), let

\[
f_i(x) = e_m x_\pi + \sum_{n=1}^{m-1} e_n (x_\pi + x_{\pi+n+1})_2 + e',
\]

\( g_i(x) = e_m (1 + x_\pi) + \sum_{n=1}^{m-1} e_n (x_\pi + x_{\pi+n+1})_2 + e', \)

\( f_i(x) = \frac{H}{2} \sum_{n=1}^{m-1} x_\pi x_{\pi+n+1} + \sum_{n=1}^{m-1} k_n x_\pi + k', \)

\( g_i(x) = \frac{H}{2} \left( x_\pi + \sum_{n=1}^{m-1} x_\pi x_{\pi+n+1} \right)_2 + \sum_{n=1}^{m-1} k_n x_\pi + k'', \)

\( f_s(x) = \sum_{n=1}^{m} d_n x_\pi, \)

\( S(x, z) = A(z)(1 + x_\pi) + B(z) x_\pi. \)

Then, the sequences \( c \) and \( d \) associated with the polynomials given by

\[
C(z) = \sum_{i=0}^{2^m-1} \tilde{S}(i, z) \xi^{f(i)} + f(i) + f(i) + U_i,
\]

\[
D(z) = \sum_{i=0}^{2^{m-1}-1} \tilde{S}(i, z) \xi^{g(i)} + g(i) + f(i) + U_i,
\]

construct a GCP.

The proof is given in Appendix B in detail.

Theorem 2 shows how each element of the generated CSs is formed explicitly as compared to Theorem 1. As described in Section II, while \( f_i(x) \) alters the amplitude of \( \xi^{f(i)} + f(i) \), \( f_i(x) \) changes its phase. The function \( f_s(x) \) determines the position of the seed sequences in the constructed CS by offsetting the order of the polynomial \( S(x, z) \). The seed sequences are also scaled by different complex values, i.e., \( \xi^{f(i)} + f(i) \) and \( \xi^{g(i)} + g(i) \).

The proof of Theorem 2 results in the following conclusions:

**Corollary 1.** Let \( \alpha_n = \xi^\alpha_n \), \( \beta_n = \xi^\beta_n \), \( \gamma_n = \xi^\gamma_n \), \( \eta_n = \xi^\eta_n \), and \( \omega_n = \xi^\omega_n \). Theorem 4 and Theorem 2 lead to identical GCP if \( \pi_n = m - \psi_n \), \( d_n \pi_n = \tau_n \), \( e_n \pi_n = h_n - a_n \) for \( n = \ldots, 1 \).
1, 2, ..., m, \epsilon' = \sum_{n=1}^{m} a_n, \ k_1 = c_1 + \epsilon_1 - \epsilon_1, \ k_{n|n=2,3,...,m} = c_n + \epsilon_n - \epsilon_n - 1 - \epsilon_n, \ k' = \sum_{n=1}^{m} \epsilon_m, \ and \ k'' = -\epsilon_m + \sum_{n=1}^{m-1} \epsilon_n.

Corollary 2. The length of the constructed CSs in (15) and (16) are \( U(2^m - 1) + N + \sum_{n=1}^{m} d_n \).

Proof. The length is equal to \( \max(U_i + N + f_s(i)) \), which gives \( U(2^m - 1) + N + \sum_{n=1}^{m} d_n \) since the function \( f_s(i) \) gets its maximum and minimum values when \( i \) is maximum, i.e., \( 2^m - 1 \), and minimum, i.e., 0, respectively.

In the rest of the study, we provide our discussions based on \( C(z) \) as \( C(z) \) and \( D(z) \) share the same properties.

A. Distinct CSs with a Fixed Support

1) Avoiding Overlapping: For a given \( i \), the order of the polynomial \( \hat{S}(i, z) = U(i) + f_s(i) \) is \( U + f_s(i) + N - 1 \). Hence, the seed sequences are placed side-by-side for \( U = N \) and \( f_s(i) = 0 \) for all \( i \). On the other hand, for different values of \( i \), the resulting polynomials can consist of the same monomials for \( f_s(i) \neq 0 \) or \( U \neq N \). Thus, \( f_s(x) \) can change the phase and amplitude of the elements through the superposition of the seed sequences. This issue can be avoided if

\[
d_k|\pi_k=a \geq \sum_{\ell \in (|\pi|)>a} d_\ell, \ U \geq N, \quad (17)
\]

for \( 1 \leq a \leq m - 1 \). For example, for \( \pi = (m, m - 1, ..., 1) \), several conditions can be listed as \( d_\pi = d_m - 1 + d_m - 2 + \cdots + d_1 = 0 \) for \( a = 1 \), \( d_m - 1 + d_m - 2 + \cdots + d_1 = 1 \) for \( a = 2 \), and \( d_1 \geq d_1 \) for \( a = m - 1 \). The conditions in (17) can be derived by evaluating the Boolean functions in (15). For instance, \( x_1 \) in (15) shifts the last \( 2^{m-1} \) seed sequences by \( d_{k|\pi_k=a} = 1 \) as the last \( 2^{m-1} \) elements of the corresponding sequence for \( x_1 \) is 1. Therefore, it gives a room of size \( d_{k|\pi_k=a} \) for other shift parameters for \( U = N \) as expressed in (17) for \( a = 1 \).

2) Fixed Support: Let \( f_s(x) \) be a fixed function for all \( \pi \) by using a fixed set of \( d_n \) for \( n = 1, 2, ..., m \). In this case, the supports of the seed sequences in the CS do not change for different \( \pi \) values, which is instrumental to generate CSs with a certain support.

3) Distinct Sequences: Let \( f_s(x) \) be a fixed function where its coefficients satisfy (17). Then, the superpositions of the seed sequences are avoided and the supports of the seed sequences are fixed. Therefore, the functionality of \( f_s(x) \) in (15) is completely separated the ones for \( f_1(x) \), \( f_2(x) \), and \( S(x, z) \). Hence, for distinct \( f_1(x) \), \( f_2(x) \), or \( S(x, z) \), the CSs are distinct.

The functions \( f_1(x), f_1(x), \) or \( S(x, z) \) lead to sets of distinct sequences for a given \( \pi \). On the other hand, for different \( \pi \), their combined impact result in distinct CSs under certain conditions:

- For \( N = 1 \), when the elements of \( \pi \) are reversed, \( \sum_{n=1}^{m} x_{n}x_{n+1} \) does not change. Therefore, \( \pi \) and the reversed \( \pi \) do not change the set of sequences generated by \( f_1(x) \) if \( k_n, k' \in \mathbb{Z}_H \). However, if one of \( \{k_n\} \) is offset by a non-integer value for even \( m \), \( \pi \) and the reversed \( \pi \) lead to different set of sequences. For odd \( m \), \( \pi \) and the reversed \( \pi \) can be used if one of \( \{k_n|n=2\} \) is offset by an non-integer value. Also, all permutations generate a set of distinct CSs when \( e_m \neq e_{m,n} \) since \( f_s(x) \) yields different functions for \( \pi \) and reversed \( \pi \) even \( f_1(x) \) remains unchanged.

- For \( N > 1 \) (a \( \neq kb \) for \( k \in \mathbb{C} \)), each permutation lead to a distinct set of CSs. This is because the polynomial \( S(x, z) \) is a function of \( x_{n} \). When \( \pi \) is reversed, the placements of the seed sequences in the CS changes.

B. Comparisons with Other Constructions and Examples

Theorem\( \Box \) under the condition (17) provides further insight into CSs as follows:

1) Independent functions for amplitude and phase: Theorem\( \Box \) extends the Davis and Jedwab’s construction\( \Box \) by providing independent functions that manipulate the amplitude and phase of the elements of the generated CSs, where the one for phase coincides with the result in \( \Box \). The parameters \( e_n \) for \( n = 1, 2, ..., m \) and \( \epsilon' \) change the real part of the exponent of \( \xi \), which is beneficial to synthesize CSs with various amplitude levels without affecting the phase of the elements. The offset method, e.g., [20–26, 36–39], and para-unitary matrix-based construction, e.g., [27], do not yield concise expressions related to the phase, amplitude, or seed sequences for synthesizing CSs. As we shown in Section \( \Box \) and Section \( \Box \), the independent pseudo-Boolean functions for amplitude and phase can be exploited to synthesize CSs with various constellation and develop algorithms for receiver.

Example 5. Let \( m = 3, U = 1, \pi = (3, 2, 1), e_1 = \frac{1}{\sqrt{\pi}} \frac{\ln(3)}{\pi}, e_{n=2,3} = 0, e' = 0, H = 4, k_{n|n=1,2,3} = 0, k' = 0, d_{n|n=1,2,3} = 0, a = b = 1 \). Based on Theorem\( \Box \) the basis vectors for \( f_1(x) \) can be listed as \( x_{n_3} = (0, 0, 0, 1, 1, 1, 1), x_{n_2} + x_{n_3} = (0, 0, 1, 1, 1, 0, 1), x_{n_2} + x_{n_3} = (0, 1, 1, 1, 1, 1, 0), x_{n_1} = (0, 0, 0, 0, 0, 0, 1) \). The function \( f_1(x) \) leads to the sequence \( 0, 0, 0, 0, 0, 0, 0, 0 \) while the function \( f_2(x) \) gives the sequence \( 0, 0, 0, 0, 0, 0, 0, 0 \) independently. The function \( S(x, z) = A(z)(1 - x_{n_2}) + B(z)x_{n_3} = A(z)(1 - x_{n_3}) + B(z)x_{n_3} = 1 \) results in \( 1, 1, 1, 1, 1, 1, 1, 1 \). As \( d_{n|n=1,2,3} = 0 \), \( f_s(x) = 0 \). Finally, combining all functions with (15), the sequence \( e \) is obtained as \( 0, 1, 3, 3, -1, 1, 3, -3, 1 \).

Note that for \( \pi = (m, m - 1, ..., 1) \), the amplitude of the elements of the CS are determined by the linear combinations of the columns of the \( m \)-bit Gray code (see\( \Box \) and\( \Box \)) over \( \mathbb{R} \) since the variables that change the amplitudes in Theorem\( \Box \) alternate as in Example\( \Box \).

2) Enumeration under the presence of a seed GCP: Theorem\( \Box \) gives contiguous CSs of length \( N \cdot 2^m \) when a seed GCP\( (a, b) \) of length \( N > 1 \) is utilized for \( U = N \) and \( f_s(x) = 0 \). It reduces to the method in \( \Box \) for \( f_1(x) = 0, k_n, k' \in \mathbb{Z}_H \), and \( a = b = 1 \). However, under the same configuration for \( N > 1 \), it determines \( N! \cdot H^{2m-1} \) CSs, i.e., doubles the number in \( \Box \), as all permutations are valid for generating distinct CSs in the presence of the seed sequences.

Example 6. Consider the parameters in Example\( \Box \) and replace \( a \) and \( b \) as \( a = (1, j, 1, 1, 1, -1) \) and \( b = (1, j, 1, 1, -1, -1, 1) \).
For $\pi = (3, 2, 1)$ and $U = 6$, the function $S(x, z) = A(z)(1-x_1)z_2 + B(z)x_1$ results in $(A(z), B(z), A(z), B(z), A(z), B(z))$. Therefore, the resulting sequence $c$ is length of 48 and can be expressed as $(a, 3 \cdot b, 3 \cdot a, -b, a, 3 \cdot b, -3 \cdot a, b)$ from (16). Now assume the order of the elements in $\pi$ are reversed, $e_{n=1,3} = 0$, and $e_2 = \frac{2}{\pi} \ln(3)$. The function $S(x, z) = A(z)(1-x_1)z_2 + B(z)x_1$ then gives $(A(z), A(z), A(z), A(z), B(z), B(z), B(z))$, while $f_1(x)$ and $f_2(x)$ remain the same. The final sequence is obtained as $(a, 3 \cdot a, 3 \cdot a, -a, b, 3 \cdot b, -3 \cdot b, b)$, which is different from the original one as the positions of the seed sequences changes.

In [19], the multi-dimensional arrays were exploited for generating CSs by projecting the arrays to a lower dimension, where the projection operation re-orders the elements of the multi-dimensional arrays. As compared to [19], we assume that the seed CSs are given in this study. Our construction can be inferred as a projection from a two-dimensional array to a single dimension in [19] for $U = N$. However, for $U \neq N$, our construction extends the definition of projection by allowing zero-valued elements or summation of the elements of the seed CSs. We also identify the seed sequence positions for a given $\pi$, which is useful for multiplexing as done in Section [VII].

3) Flexible support: In the literature, the constructions mainly focus on contiguous CSs, i.e., CSs with no zero-valued elements [20-27, 36-49]. In contrast, Theorem 2 explains how to generate distinct non-contiguous CSs under the condition given in (17). In [30], various resource allocations for complementary sets were considered for preamble design. However, the maximum PMEPR of the resulting waveform reaches 4 dB. In [41], several correlation bounds under non-contiguous resource allocation and several constructions were discussed. However, systematic construction of the zero-valued elements for CSs with [13] was not investigated in these studies.

Example 7. Consider the parameters given in Example 5. Let $a = (1, j, 1)$, $b = (1, 1, -1)$ and $U = 3$. The final sequence $c$ is $(a, 3 \cdot b, 3 \cdot a, -b, a, 3 \cdot b, -3 \cdot a, b)$. Now assume that $d_1 = 60$ and $d_{n=1,2} = 0$. The function $f_5(x)$ leads to $(0, 0, 0, 0, 60, 60, 60, 60)$. Hence, from (15), the support of the last four parts of $e$ are shifted by 60. The resulting sequence can be expressed as $c = (a, 3 \cdot b, 3 \cdot a, -b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. In other words, a non-contiguous CS with two clusters of length of 12 separated by $d_2 = 60$ zero-valued elements is obtained where the alphabet on the clusters remains the same. Under the same condition, assume that $\pi = (1, 2, 3)$. Let $d_1 = 60$ and $d_{n=2,3} = 0$. Since $f_5(x)$ still gives $(0, 0, 0, 0, 60, 60, 60, 60)$ the resulting sequence keeps the original support as $c = (a, 3 \cdot a, 3 \cdot a, -a, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. While the position of the seed sequence changes.

4) Continuous parameters: Theorem 2 provides a framework for synthesizing CSs with various alphabets through the continuous parameters $e_n$ and $e_1$, and $k_n$ and $k_1'$, as opposed to existing work on CSs [13, 29, 27]. For example, since $k_n$ and $k_1'$ are in $[0, H)$, i.e., a range rather than $\mathbb{Z}_H$, it offers a flexibility in phase. Hence, it is possible to generate CSs with various alphabets as discussed in Section [V] in detail.

V. SEQUENTIAL CONSTRUCTIONS WITH EQUIANGULAR SUB-SYMMETRIC CONSTELLATIONS

Consider two points $p_u$ and $p_v$ in $\mathbb{S}_H$ indexed by $u, v \in \{1, 2, \ldots, S\}$, where $H$ is an even number. By choosing $e' = \frac{H}{2\pi} \ln r_u$ and $e_1 = \frac{H}{2\pi} \ln \frac{r_v}{r_u}$ while $e_{n=1,2} = 0$ for $n = \{1, 2, \ldots, m\}$ in (9) for $l \in \{1, \ldots, m\}$, we equate the amplitudes of the elements of the CSs to either $r_u$ or $r_v$. Since our construction also admits real values for (11), we set $k' = \kappa' + \beta' \Delta'$ and $k_n = \kappa_n + \frac{H}{2\pi} \Delta_n$ for $\Delta' \in [0, \pi]$ and $k', k_n \in \mathbb{Z}_H$ for $n = \{1, 2, \ldots, m\}$. While $l$ determines the parameters to be offset, $u$ and $v$ identify where to offset. Consider the case $u = v$. Due to the equiangular property, an ESC consists of the points in $\mathbb{S}_H$ rotating with an integer multiple of $2\pi/H$ radian. Hence, by offsetting $\kappa'$ by $\psi_u$, the rotated and scaled H-PSK GDJ sequences on an ESC alphabet can be obtained. Since there are $S$ elements in $\mathbb{S}_H$, only $S$ valid permutations exist for $(u, v)$, where each permutation generates $G = \frac{H}{2\pi} \mathbb{Z}_H$ sequences for $N = 1$ and $2G$ sequences for $N > 1$, based on Section [IV-A3].

Now, consider the case $u \neq v$. Assume that $k_1 = q_1$, $k_n = q_n \pm q_{n-1}$ for $n = 2, \ldots, m$, and $k' = q'$. We then write $\sum_{n=1}^{m} q_n x_{n+1} + k' = q_n x_n + \sum_{n=1}^{m-1} q_n \left( (x_{n+1})^2 \pm (x_{n+1})^2 \right) + q'$, which allows us to express the phase and amplitude functions in similar forms. For example, if $k_n = q_n - q_{n-1},$ 

\[ (x_{n+1}) - (x_{n+1})^2 \in \begin{cases} \{0\}, & \text{if } (x_n + x_{n+1})^2 = 0 \\ \{-1, 1\}, & \text{if } (x_n + x_{n+1})^2 = 1 \end{cases} \]

(18)

Therefore, $q_1$ and $e_1$ alter the same elements of the sequence. Since the non-zero values of $(x_{n+1})^2 - (x_{n+1})^2$ are either $-1$ or $1$, the phase function rotates the elements scaled with $e_1$ in the counterclockwise and the clockwise directions equally on the unit circle. For $q_1 = \psi_u$, the amount of the phase rotation is $\psi_u$. If $k_n = q_n + q_{n-1},$

\[ 1 - (x_{n+1})^2 + (x_{n+1})^2 \in \begin{cases} \{0\}, & \text{if } (x_n + x_{n+1})^2 = 1 \\ \{-1, 1\}, & \text{if } (x_n + x_{n+1})^2 = 0 \end{cases} \]

(19)

which implies that the amount of phase rotation in the counterclockwise and the clockwise directions for the elements that are not scaled by $e_1$ is $\psi_u$ for $q' = \psi_u$ and $q_1 = -\psi_u$.

The rotation in the counterclockwise and the clockwise directions equally is exactly aligned with the sub-symmetry property of an ESC. Equations (18) and (19) allow us to alter the different elements of the sequence, for given $l$, $u$, and $v$.

As the coset term in (11) can flip the sign of the elements of CSs, we ensure that $p_u$ and $-p_u$ are in $\mathbb{S}_H$ with an even $H$ as in (15).
Their combined impacts on the phase offset parameters can be calculated as \( \Delta' = \psi_u, \Delta_l = \psi_v - \psi_u, \Delta_{l+1} = -\psi_v - \psi_u \) if \( l < m \), and \( \Delta_{n|n \neq l|l+1} = 0 \) for \( l \in \{1, 2, \ldots, m\} \).

For \( N > 1 \), the total number of \((u, v)\) permutations where \( u \neq v \) is \( S^2 - S \) and \( l \in \{1, 2, \ldots, m\} \). This implies that the total number of CSs with an ESC generated through the sub-symmetry is \( 2((S^2 - S) m + A + B) \cdot G \). For \( N = 1 \), \( \pi \) and the reversed \( \pi \) give different sequences when \( r_u \neq r_v \) for \( l = m \) based on the discussions in Section [2, A]. In addition, \( \pi \) and the reversed \( \pi \) yield different sequences when \( l = m \) if \( \Delta_m = \psi_u - \psi_v \neq 0 \) and \( \psi_u + \psi_v \neq 0 \), which implies that \( |\psi_u| \neq |\psi_v| \) should hold.

Since these special cases generate 2G different CSs for \( l = m \) while others lead to \( G \) different CSs, the total number of CSs can be calculated as \(( (S^2 - S)m + A + B) \cdot G \), where \( A \) and \( B \) are the number of \((u, v)\) permutations for \( r_u \neq r_v \) and the number of \((u, v)\) permutations for \( r_u = r_v \) and \( |\psi_u| \neq |\psi_v| \), respectively.

We summarize how the phase and amplitude parameters are modified for given \( u, v, l \) and for an arbitrary ESC in Algorithm 1. The number of distinct CSs are calculated as follows:

**Corollary 3.** The number of distinct CSs with an ESC of \( H \cdot S \) points are at least \(( (S^2 - S)m + A + B + S) \cdot G \) and \( 2((S^2 - S)m + S) \cdot G \) for \( N = 1 \) and \( N > 1 \), respectively.

Note that non-contiguous CSs with a certain support can be generated by fixing \( f(x) \) as in Example 2. Hence, for a given support, the number of distinct non-contiguous CSs is equal to the number of distinct contiguous CSs.

### A. Spectral Efficiency and Minimum Distance

Based on Corollary 3 the spectral efficiency (SE) \( \rho \) with CSs with an arbitrary ESC can be as high as \( \log_2((S^2 - S)m + A + B + S) \cdot G/2^m \) bit/s/Hz and \( \log_2(2((S^2 - S)m + S) \cdot G)/(2^m \cdot N) \) bit/s/Hz for \( N = 1 \) and \( N > 1 \), respectively. To infer the minimum Euclidean distance, denoted by \( d_{\text{min}} \), let \( r_{\text{min}} = \min_{p, q \in S_u} |p_u - q_u| \) and \( \delta_{\text{min}} = \min_{p, q \in S_u, p \neq q} |p_u - q_u| \). In [4, Theorem 10], it was shown that the minimum Lee distances for the phase function \( f(x) \) are \( 2^{m-1} \) and \( 2^{m-2} \) for \( H = 2 \) and \( H = 2, \) respectively. Therefore, it can be shown that the minimum Euclidean distance for all sequences generated with \( r_{\text{min}} = \min_{p, q \in S_u} |p_u - q_u| \) and \( \delta_{\text{min}} = \min_{p, q \in S_u, p \neq q} |p_u - q_u| \). In [4, Theorem 10], it was shown that the minimum Lee distances for the phase function \( f(x) \) are \( 2^{m-1} \) and \( 2^{m-2} \) for \( H = 2 \) and \( H = 2, \) respectively. Therefore, it can be shown that the minimum Euclidean distance for all sequences generated with \( r_{\text{min}} = \min_{p, q \in S_u} |p_u - q_u| \) and \( \delta_{\text{min}} = \min_{p, q \in S_u, p \neq q} |p_u - q_u| \).

The seed sequences exist, \( d_{\text{min}} \) is a function of the seed sequences in general.

Our numerical analysis on \( d_{\text{min}} \) shows that \( d_{\text{min}} \) is exact in certain scenarios, e.g., \( 4s^2\) QAM for \((s, m) = \{(2, 2), (2, 3), (3, 2), (3, 3)\} \), \( H\) s-STAR for \((s, m) = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\} \) for

![Algorithm 1: Offsetting amplitude and phase parameters for an arbitrary ESC](image)

**B. Examples**

1) \( 4s^2\)-QAM: The number of \((u, v)\) permutations for \( r_u = r_v \) and \( |\psi_u| = |\psi_v| \) can be calculated as \( s + 4s(s - 1)/2 = 2s^2 - s \). Hence, \( A + B \) must be equal to \( s^4 - (2s^2 - s) = s^4 - 2s^2 + s \). Thus, the total number \( 4s^2\)-QAM CSs can be calculated as \(( (s^2 - s^2)(m + 1) + s) \cdot G \) and \( 2((s^4 - 2s^2 + s + s^2 - s)m + s) \cdot G \) for \( N = 1 \) and \( N > 1 \), respectively. For \( 4s^2\)-QAM, \( r_{\text{min}} = \sqrt{3}/(4s^2 - 1) \) and \( \delta_{\text{min}} = \sqrt{3}/(4s^2 - 1) \). Therefore, \( d_{\text{min}} \leq 2m^2/2 \sqrt{3}/(4s^2 - 1) \) for \( N = 1 \).

Our result extends the enumerations in [26] via offset method and Case I-II results in [27] via para-unitary matrices. In [26], for \( N = 1 \), Li showed that there exist at least \((m + 1)4^{2(q-1)} - (m + 1)4^{(q-1)} + 2^{q-1} \) G CSs for \( 4s^2\)-QAM alphabet. For \( s = 2^{q-1} \), we obtain the same result when it is plugged into \( (s^4 - s^2)(m + 1) + s \) without using an auxiliary method such as Gaussian integers. As a new result, for \( N > 1 \), we show that the total number of sequences is \( 2(m4^{2(q-1)} - m4^{(q-1)} + 2^{q-2}) \cdot G \) for \( 4s^2\)-QAM alphabet. Note that our framework also enumerates the CSs for non-square constellations, e.g., \( 4s^2 = 36 \) points in \( S_{\text{sec}} \) for \( s = 3 \).

In this study, \( f_j(x) \) is utilized to generate CSs with two distinct amplitude levels. By exploiting the grid nature of QAM alphabet, it is possible to obtain the phase and amplitude parameters with a computer search for CSs with more than two amplitude levels, as done in [27] with para-unitary matrix construction. However, we do not have a systematic rule to enumerate QAM-CSs with more than two amplitude levels. Since these sequences also require a large \( q \), it can decrease \( d_{\text{min}} \) drastically without a particular CS selection.

2) \( H\) s-STAR: For this case, \( (A, B) = (S(S - 1), 0) \). Thus, there are at least \( ((S^2 - S)(m + 1) + S) \cdot G \) and \( 2((S^2 - S)m + S) \cdot G \) CSs with an \( H\) s-STAR alphabet for \( N = 1 \) and \( N > 1 \), respectively.

3) \( DVB-16/32APS\): For DVB-16PSK and DVB-32PSK, \( (A, B) = (6, 4) \) and \( (A, B) = (38, 14) \), respectively.

Thus, the numbers of CSs with DVB-16PSK constellation...
are at least \((240m + 26) \cdot G\) and \((480m + 32) \cdot G\) for \(N = 1\) and \(N > 1\). For DVB-32APSK constellation, there are at least \((992m + 84) \cdot G\) and \((1984m + 64) \cdot G\) CSs for \(N = 1\) and \(N > 1\), respectively.

4) Modified IEEE 802.11ay 64-NUC: For modified IEEE 802.11ay 64-NUC constellation, \((A, B) = (228, 0)\) and the numbers of CSs are at least \((4032m + 292) \cdot G\) and \((8064m + 128) \cdot G\) for \(N = 1\) and \(N > 1\), respectively.

C. Constant-Modulus CSs with an ESC

Using a constellation different from PSK for CSs can increase the mean power of the OFDM symbol. This issue denotes the main motivation of using CSs with a larger constellation for a practical system since the PMEPR can be higher than 3 dB \[26\]. On the other hand, our framework provides a way to control the mean power of OFDM symbol through \(\{ (u, v), l \} \) for a constellation different from PSK. The choice of \(e = \frac{2}{m} \ln r, c_l = \frac{2}{m} \ln r, and e_n \neq 0 \) in Theorem 2 scales halves of the elements to \(r_u\) and \(r_v\). Hence, the OFDM symbol power can be fixed through constant-modulus CSs if \((r_u^2 + r_v^2)/2 = 1\) holds. To retain the PMEPR benefit of CSs, we consider only \(\{ (u, v), l \} \) combinations that satisfies \((r_u^2 + r_v^2)/2 = 1\). Under this constraint, the number of distinct CSs reduces, but it can still be high if the constellation is designed well. For example, if \(S = 4\) for \(H\)-Star constellation, \((r_u^2, r_v^2, r_0^2, r_0^2)\) can be chosen as \((2 - b_2^2, 2 - a_2^2, a_2^2, b_2^2)\) for any \(1 < a < b < \sqrt{2}\). For QAM, there exist \((u, v)\) permutations that satisfy \((r_u^2 + r_v^2)/2 = 1\) and the valid permutations can be easily identified through a computer search. It is also worth noting that ensuring a larger \(d_{\text{min}}\) with a constellation optimization or a constant-modulus CS selection algorithm is not considered in this work.

VI. ENCODER AND DECODER

A. Encoder

The proposed encoder encodes the information bits with \(\{ (u, v), l \} \) combinations, \(\kappa', \kappa_n \in \mathbb{Z}_H\) for \(n \in \{ 1, 2, \ldots, m \}\), and the permutations of \(\pi\), where \(k' = \kappa' + \frac{2}{m} \Delta', k_n = \kappa_n + \frac{2}{m} \Delta_n\) for \(n \in \{ 1, 2, \ldots, m \}\) and \(\Delta', \Delta_n, \epsilon\), and \(e_n\) are functions of \(\{ (u, v), l \} \) based on Algorithm 1 and 2. For \(H\)-Star constellation, it is assumed to be power-of-two. We utilize orthogonal seed GCPs to support orthogonal multiple access in the uplink by using the properties of modulated unimodular sequences \[42\] and use a fixed \(\pi_1\) to keep the locations of the seed sequences at a cost of maximum \(\log_2 m\) bits. Therefore, the same subcarriers can be shared by \(N\) users for a seed GCP of length \(N\).

Based on Corollary 3 we transmit \(\log_2 (N_{u,v,l}(m-1)!)) + (m+1)\log_2 H\) information bits over \(N_{u,v,l}\) subcarriers, where \(N_{u,v,l}\) is the number of \(\{ (u, v), l \} \) combinations and equal to \((S^2 - S) + S\). To utilize all available constellation points, \(K_{u,v,l} = \log_2 (N_{u,v,l}(m-1)!))\) information bits is first converted to an integer \(i_{u,v,l,\pi} \in \{ 0, 1, \ldots, 2K_{u,v,l} - 1 \}\). It is then decomposed as \(i_{u,v,l,\pi} = i_{\pi} N_{u,v,l} + i_{u,v,l}\), where \(i_{u,v,l} \in \{ 0, 1, \ldots, N_{u,v,l} - 1 \}\). The indices \(i_{u,v,l}\) and \(i_{\pi}\) are utilized to identify a combination of \(\{ (u, v), l \} \) and a permutation of \(\pi\), respectively. We use factoradic via Lehmer code to construct a bijective mapping from integers to permutations.

B. Decoder

At the receiver side, we first separate the users by using the orthogonality of the seed sequences and exploiting the fixed \(S\langle x, z \rangle\). We then concatenate the matched filter outputs, i.e., the sequence length reduces to \(2^m\) for each user. The 4th element of the sequence of a user can be expressed as \(r_i = c_i\xi_{i,j}^i + n_i\), where \(c_i\) and \(n_i\) are the complex channel and the noise coefficients, respectively. Assuming that \(c_i\) is available at the receiver, an ML decoder is equivalent to a minimum distance decoder for additive white Gaussian noise (AWGN), i.e.,

\[
\{ \hat{\theta}_u, \hat{\theta}_l \} = \arg \min_{\theta_u, \theta_l} \sum_{i=0}^{2^m-1} |c_i\xi_{i}^j + \hat{f}_i^j| - r_i^2,
\]

\[
= \arg \max_{\theta_u, \theta_l} \Re \left( \sum_{i=0}^{2^m-1} \xi_{i}^j - f_i^j s_i^\ell 2^{\ell-1} \right) h_i^j,
\]

(21)

where \(\theta_u, \theta_l = \{ (u, v), l, \pi, k', k_1, \ldots, k_m \}\), \(s_i = e_i^r r_i\), and \(h_i = |e_i|/2\). However, solving (21) is not easy task and the complexity with a brute-force search can be very high.

1) Principle: To achieve a low-complexity ML detector, we decompose \(f_i^j(x; \theta_1)\) and \(f_i^j(x; \theta_1)\) as \(f_i^j(x; \theta_1) = g_i^j(x^1; \psi) + x_i^j f_i^j(x^1; \phi)\) and \(f_i^j(x; \theta_1) = g_i^j(x^1; \psi) + x_i^j f_i^j(x^1; \phi)\) for \(\ell = \{ 1, 2, \ldots, m \}\), respectively, where \(x^\ell = (x_1, x_2, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_m)\). While the order of \(g_i^j(x^1)\) is \(r, f_i^j(x^1)\) is a polynomial with the order of \(r - 1\) and can be calculated as \(f_i^j(x^1) = \partial f_i^j(x)/\partial x\). Both \(f_i^j(x^1)\) and \(g_i^j(x^1)\) lead to the sequences of length \(2^{m-1}\). By using these decompositions, the maximization in (21) can be re-expressed as

\[
\max_{\psi_1, \psi_2, \psi_3, \psi_4} \Re \left( \sum_{i=0}^{2^m-1} \xi_{i}^j - f_i^j s_i^\ell 2^{\ell-1} \right) h_i^j,
\]

(22)

where

\[
s_i^\ell = s_{I_1(i)} + s_{I_2(i)} + \sum_{j=m}^{2^m-1} \xi_{i}^j - f_i^j s_i^\ell 2^{\ell-1},
\]

(23)

\[
h_i^j = h_{I_1(i)} + h_{I_2(i)} + \sum_{j=m}^{2^m-1} \xi_{i}^j - f_i^j s_i^\ell 2^{\ell-1},
\]

(24)

and \(I_1(i)\) maps the integers from 0 to \(2^{m-1} - 1\) to the values of \(\sum_{j=1}^{m} x_j 2^{m-j}\) in ascending order for \(x_i = 0\). The variables \(s_i^\ell\) and \(h_i^j\) can be considered as the element of a new received sequence and the half of absolute square of channel gain as in the original expression in (21), where the original sequence length is halved.

Theorem 2 provides \(f_i^j(x)\) and \(f_i^j(x)\). By using (27), \(f_i^j(x)\) can be re-written as

\[
f_i^j(x) = e^c + e_1 x_{\pi_n} + \sum_{n=2}^{m} (e_{n+1} + e_{n-1}) x_{\pi_n} - 2 \sum_{n=1}^{m-1} e_{n+1} x_{\pi_n} x_{\pi_{n+1}}.
\]
Therefore, \( f^i_\ell(x^\ell) \) can be calculated as

\[
f^i_\ell(x^\ell) = \begin{cases} 
 e_{m-1} - 2e_{m-1}x_{\pi_m-1} + e_m, & \pi_m = \ell \\
 e_{n-1} - 2e_{n-1}x_{\pi_n-1} + e_n - 2e_n x_{\pi_{n+1}}, & \pi_n = \ell \\
 e_1 - 2e_1 x_{\pi_1}, & \pi_1 = \ell 
\end{cases}
\]

Similarly, we can express \( f_1(x) = g^i_1(x^\ell) + x_1 f^i_\ell(x) \), where

\[
f^i_1(x) = \begin{cases} 
 k_1 + \frac{H_2}{2} x_{\pi_1-1}, & \pi_1 = \ell \\
 k_n + \frac{H_2}{2} x_{\pi_n-1} + H_2 x_{\pi_{n+1}}, & \pi_n = \ell \\
 k_1 + \frac{H_2}{2} x_{\pi_2}, & \pi_1 = \ell 
\end{cases}
\]

As \( \pi_1 \) is fixed and available at the receiver, \( f^i_1(x) = e_1 - 2e_1 x_{\pi_1} \) and \( f^i_\ell(x) = k_1 + \frac{H_2}{2} x_{\pi_2} \) for \( \pi = \ell \). Hence, \( \phi_i = \{k_1, \pi_2\} \) and \( \phi_{i_1} = \{e_1, \pi_1, \ell\} \), where \( k_1 = \kappa_1 + \Delta_1 \) and \( e_1 \) depend on \( i_u,v,l \). Hence, for a hypothesized \( i_u,v,l \), we can calculate (23) and (24) for different \( \kappa_1 \) and \( \pi_2 \). This procedure can done continuously, where the sequence length is halved for each step. By backtracking the branch that maximizes (22), a recursive ML detector can be achieved. We adopt this principle and address the exponential growth by terminating unpromising branches early.

2) Algorithm: The proposed algorithm has four different phases given as follows:

a) Preparation: We calculate \( N_{u,v,l} \) offset parameters, i.e., \( \Delta_1, \Delta_2, \Delta_m, \ell, e_1, \ldots, e_m \) based on the ESC for all \( \{(u,v),l\} \) combinations. Since \( e' \) and \( \Delta' \) alter the amplitude and phase of all elements in the sequence, we combine them with the channel coefficients as \( c_{i_1} \leftarrow c_i e^{\ell} + \Delta' \). Then we calculate \( s_1 \leftarrow c_1 r_1 + h_1 \leftarrow c_1 |\ell|/2 \) for each offset and store \( N_{u,v,l} \) resulting sequences in \( S_m \) and \( H_m \), respectively. The parameter \( \ell \) (i.e., the fixed \( \pi_1 \) for the preparation) and \( i_{u,v,l} \) are stored in \( L_m \) and \( O_m \), respectively, for identifying the offsets in the following steps.

b) Enumeration: There are \( m - 1 \) and \( H \) options for \( \pi_2 \) and \( \kappa_1 \) for each sequence in \( S_m \), respectively. By taking \( e_1 \) and \( \Delta_1 \) into account, we calculate (23) and (24) for all combinations of \( \pi_2 \) and \( \kappa_1 \). We populate \((m-1)HL\) resulting sequences in \( S_{m-1} \) and \( H_{m-1} \), respectively, where \( L \) is the number of sequences given to the algorithm. We populate the corresponding \( i_{u,v,l} \) and the information related to \( \ell \) for the next recursion (i.e., the enumerated values for \( \pi_2 \)) in \( O_{m-1} \) and \( L_{m-1} \), respectively.

c) Pruning: To address the exponential growth of the enumerated sequences, we keep only \( N_{best} \) enumerated sequences for the next step. By using the high signal-to-noise ratio (SNR) approximations of \( \xi_{k_1}(x^{\hat{\pi}_1}) + j \xi_{k_1}(x^{\hat{\pi}_1}) \cong s_i/\hat{h}_i \) and \( \xi_{k_2}(x^{\hat{\pi}_2}) \cong |s_i|^2/H_i^2 \) in (21), we calculate an estimate of (21) and choose \( N_{best} \) sequences based on the resulting metric, i.e., \( \sum_{i=1}^{\hat{\pi}_1} |s_i|^2/\hat{h}_i \). The indices of the best sequences are populated in \( N_m \) for backtracking.

d) Backtracking: We recursively run the enumeration and pruning phases till \( m = 1 \). For \( m = 1 \), we perform an ML detection based on (22). The detected parameters, i.e., \( (\hat{\kappa}_1, \hat{\pi}_1) \), \( \pi = (1, \ldots, 1) \), and the value of (22) for all sequences are stored in \( \Pi_m \), \( \Theta_m \), and \( S_m \), where \( \hat{\kappa}_1 \) and \( \hat{\pi}_1 \) correspond to the detected \( \kappa_1 \) and \( \pi_1 \), respectively. The algorithm then returns \( \Pi_m \), \( \Theta_m \), and \( S_m \) for the step above. For \( m \geq 2 \), the detected parameters from the previous finalized step are utilized to calculate (22) for each enumerated sequence. The best \( \hat{\kappa}_1 \) and \( \hat{\pi}_1 \) for the sequences indicated in \( N_m \) are obtained by using \( O_m \) and \( L_m \), and the results are combined with the parameters from the previous step in \( \Pi_m \) and \( \Theta_m \). The scores with (22) for different sequences are also populated in \( S_m \).

After the recursions are finalized, the sequence index that returns the maximum (22) among \( N_{u,v,l} \) sequences is assigned to \( i_{u,v,l} \). The parameters \( \hat{i} \) and \( (\hat{\kappa}_1, \ldots, \hat{\kappa}_m) \) are detected from \( \Pi_m \) and \( \Theta_m \). After mapping \( \pi_1 \) to an integer \( i_{u,v,l,\pi} = \pi \sum_{u,v,l} + i_{u,v,l} \) is calculated. The binary representation of \( i_{u,v,l,\pi} \) corresponds to the received bits over \( \pi \) and \( \{(u,v),l\} \). The pseudocode and an example for the phases are given in Algorithm 2 and Figure VI-B26, respectively.

3) Complexity Reduction with Pruning: With pruning, the remaining complexity is due to the generation of \( max(m-1,1)H \min(L,N_{best}) \) sequences of length \( 2^{m-1} \) at the 6th step, the process of choosing \( N_{best} \) sequences, and the calculation of (22) for \( N_{best} \) sequences. For instance, for \( N_{best} = 400 \), maximum 400H(i − 1) sequences of length \( 2^{m-1} \) are enumerated and 400 of them survive for the \((i+1)\)th step, which corresponds to a substantially reduced search space.

4) Comparisons with Other Decoders for CSs: Although there are many studies that focus on the enumeration of CSs, the literature is not rich with decoding algorithms for the codes related to CSs. Particularly, there are only few studies that consider the coset term changing based on \( \pi \). In
a supercode decoder that subtracts each possible coset representative and decodes the remaining sequences with a fast Hadamard transformation (FHT) were proposed. In [43], FHT was extended for decoding the first-order generalized RM codes. In [44], for a given coset term, a recursive decoding which can reduce the complexity of generalized FHT further is proposed. In [45], \( f'(x^t) \) is defined as \( f(x|x_1) = f(x|x_2 = 0) \) and utilized for developing efficient decoding algorithms. Although Algorithm 8 in [45] considers the coset term, it consists of a step (i.e., Step 2) that multiplies two noisy terms, which would cause noise enhancement.

The proposed decoder can be considered as an extension of the decoders in [44] and [45]. The recursion steps in our decoder follow the same rationale discussed in [44] and exploit the order reduction proposed in [45]. However, it differs from [44] as we consider the coset term. As opposed to the decoder in [45], we utilize \( f'(x^t) \) such that the decoder does not cause noise enhancement. We also use the derivatives of both phase and amplitude functions in Theorem 2 which allow us to develop the decoder for different constellations and selective fading channels.

VII. NUMERICAL RESULTS

In this section, we evaluate the proposed encoder and decoder for an uplink scenario numerically. For comparison, we also generate results with a polar code and binary phase-shift keying (BPSK) under similar SE conditions. We assume that multiple users access the communication medium with a single OFDM symbol synchronously over a non-contiguous resource allocation in the frequency domain to exploit the frequency diversity. Considering the compatibility with 3GPP Fifth Generation (5G) New Radio (NR) waveform parameters, we assume that each user utilizes 384 subcarriers distributed over 8 clusters in the frequency domain. Each cluster consists of 48 subcarriers. We assume that 132 subcarriers at the center of the band are allocated for random access channel and left unoccupied as shown in Figure 3(c). The spacing between the clusters on the left and right portions of the band is set to 96 subcarriers. Hence, three interlaced groups are constructed. We support multiple access on each group by exploiting orthogonal seed sequences of length \( N = 3 \). Therefore, 9 users can be multiplexed in the uplink within a single OFDM symbol. To align with the resource allocation, we set \( m = 7 \) for the proposed encoder and polar code. We choose \( \pi_1 = 7 \). For ESCs, we consider \( HS\text{-Star} \) for \( H = \{4,8\} \) and \( S = \{1,2,4\} \) and \( 4s^2\text{-QAM} \) for \( s = \{2,4\} \). For \( HS\text{-Star} \), \( (r_1, r_2, r_3, r_4, r_5, r_6, r_7) \) are chosen as \( (2 - a^2, a^2, 0, 0, 0, 0, 0) \) where \( a = 1.256 \) for \( S = 2 \) and \( (2 - b^2, 2 - a^2, a^2, b^2) \) where \( a = 1.1 \) and \( b = 1.33 \) for \( S = 4 \), respectively. For the proposed decoder, we set \( N_{\text{best}} = 400 \). For polar decoder, we consider soft decoding with successive interference cancellation (SIC) and the equalizer is minimum mean square error (MMSE) frequency-domain equalization (FDE). The design SNR for the polar code is set to 3 dB. The demodulation and equalization are combined in the proposed decoder for the CSs (see Algorithm 2). The subcarrier spacing is set to 30 kHz.

In Figure 3(d), we compare PMEPR distributions of different schemes. While PMEPR of OFDM symbols with the polar code reaches to 12 dB at the rate of 1e−2, the maximum PMEPR is 6 dB for all cases of CSs. When the OFDM symbol power is fixed by limiting the \( (u,v) \) combinations, the maximum PMEPRs of the schemes based on CSs reduce to 3 dB, which leaves a 9 dB extra room for transmit power as compared to OFDM with the polar code. It is worth noting that the PMEPR of the OFDM symbols generated with the CSs is not affected by the non-contiguous resource allocation as the support can be achieved via \( f_s(x) \) given in [13], i.e., \( d_{n=1} = 288 \), \( d_{n=2} = 96 \), \( d_{n=3} = 48 \), and \( d_{n\neq1,2,3} = 0 \). The SEs of the schemes are provided in Figure 3(d) for the fixed OFDM symbol power. While the number of bits sacrificed for fixed OFDM symbol power is only 1 bit for 16-QAM and \( HS\text{-Star} \) for \( S = 4 \) and \( H = \{4,8\} \), it is 3 bits for 64-QAM. No bit is sacrificed with \( HS\text{-Star} \) for \( S = 2 \) and \( H = \{4,8\} \).

In Figure 3(c) and Figure 3(d), we provide bit-error rate (BER) and block-error rate (BLER) curves in AWGN and fading channel, respectively. We consider ITU Vehicular A for the multi-path channel model without mobility. We perform the evaluations with constant-modulus CSs by restricting \( (u,v) \) permutations. For \( S = 1 \), \( HS\text{-Star} \) is equivalent to \( H\text{-PSK} \) constellation and our encoder reduces to the one in [4]. The BERs are at \( 1e−3 \) when \( E_b/N_0 = 2 \) dB and \( E_b/N_0 = 4 \) dB for \( H = 4 \) and \( H = 8 \) in AWGN channel as shown in Figure 3(c). In the fading channel, \( 1e−2 \) BLER is achieved when SNRs are \( −7 \) dB and \( −3 \) dB for \( H = 4 \) and \( H = 8 \), respectively, as in Figure 3(d). For \( S = \{2,4\} \), the SE increases gradually as more distinct CSs are synthesized without sacrificing 3 dB PMEPR. For \( H = 4 \), \( 1e−2 \) BLER is achieved at \( −5.6 \) dB and \( −4.5 \) dB when \( S = 2 \) and \( S = 4 \), respectively. For \( H = 8 \), the number of information bits reach up to 37 and 38 bits per user for \( S = 2 \) and \( S = 4 \), respectively, where \( 1e−2 \) BLER is achieved at \( −2 \) dB and 0 dB. The results indicate that \( HS\text{-Star} \) performs better than QAM in average BLER. While both \( HS\text{-Star} \) for \( H = 4 \) and \( S = 4 \) and 16-QAM result in the same SE under PMEPR constraint, \( HS\text{-Star} \) provides 1 dB improvement at \( 1e−2 \) BLER. The SNR gap between \( HS\text{-Star} \) for \( H = 1 \) and \( S = 8 \) and 64-QAM reaches to 3 dB. The OFDM with polar code provides similar BER versus \( E_b/N_0 \) curves for both \( \rho = 25/384 \) bit/s/Hz and \( \rho = 38/384 \) bit/s/Hz (i.e., the code rates are 25/128 and 38/128, respectively) as shown in Figure 3(c). It performs 2 dB worse than the QPSK CSs as the decoder for polar code relies on SIC. The polar code is more reliable than the CSs with high-order constellations in AWGN. It outperforms CSs with \( HS\text{-Star} \) for \( S = \{2,4\} \) for \( H = 8 \) with 2 dB and 4 dB, respectively. Nevertheless, the proposed encoder under a similar SE provides a 9 dB room for transmit power and can offset the degradation due to the SNR loss in practice.

VIII. CONCLUSION

In this study, we introduce a method that allows one to re-express a polynomial generated with a recursion consisting of linear operators by indicating how the operators are distributed to coefficients of the polynomial via Boolean functions. By
applying this method to a recursive GCP constructions, we show how the seed sequences, phase rotations, signs, real scalars, and the shifting factors applied at each step of the recursion alter the elements of the sequences in the GCP. As a result, we obtain a generic CS construction that extends David and Jedwab’s construction in [4] by deriving separate pseudo-Boolean functions for the amplitude, the phase, the seed sequences, and the support of the CS. We show that distinct CSs with a wide range of constellations can be enumerated systematically. We also present an encoder and an ML decoder. We show that the proposed encoder can achieve low-PMEPR OFDM symbols without utilizing any optimization technique while supporting multiple users, different constellations, and non-contiguous resource allocation. The proposed decoder also decodes the information bits on the second-order coset and Jedwab’s construction in [4] by deriving separate pseudo-Boolean functions for the amplitude, the phase, the seed sequences, and the support of the CS. We show that distinct CSs when the OFDM symbol power is fixed through \((u,v)\) can be decomposed as

\[
1(z) = \prod_{n=1}^{m} (1 + z^{2^{v_n}})
\]

Also, the proposed encoder offers a 9 dB room for increasing the transmit power without any clipping, which can offset the performance difference in practice.

**APPENDIX A**

**PROOF OF LEMMA**

**Proof.** The sequence \(1 \equiv (1)^{2^{m-1}} \) can be decomposed as

\[
1(z) = \prod_{n=1}^{m} (1 + z^{2^{v_n}})
\]

\[
= \prod_{n=1}^{m} (1 + 2^{v_{n+1}}) + z^{2^{v_{n+1}}} \prod_{n=1}^{m} (1 + 2^{v_{n+1}})
\]

\[
= \prod_{n=1}^{m} (1 + z^{2^{v_{n+1}}}) + z^{2^{v_{n+1}}} \prod_{n=1}^{m} (1 + z^{2^{v_{n+1}}})
\]

\[
+ z^{2^{v_{n+1}}} \prod_{n=1}^{m} (1 + z^{2^{v_{n+1}}}) + z^{2^{v_{n+1}}+2^{v_{n+1}}} \prod_{n=1}^{m} (1 + z^{2^{v_{n+1}}})
\]
By the definitions, $H_1(z)$ and $H_2(z)$ have no common monomials, i.e., $h_1 + h_2 = 1$. The polynomial $H_2(z)$ corresponds to a polynomial where the degree of each monomial in $H_1(z)$ is increased by $2^v$ by their definitions. Hence, the sequence $h_2$ should be the shifted version of the sequence $h_1$ by $2^v$. Since $h_1 + h_2 = 1$ and $h_2$ is the shifted version of $h_1$ by $2^v$, the $v$th element of the sequence $h_1$ should be 1 for $n = s^2+1, 2, \ldots, 2^v$ for even $s$ and 0 for odd $s$ where $0 \leq s < 2^{m-v}$. Thus, the Boolean functions generating $h_1$ and $h_2$ can be obtained as $x_n$, and $1 - x_n$, respectively. The polynomials $H_1(z)$ and $H_2(z)$ can also be decomposed as $H_1(z) + H_2(z)$ and $H_1(z) + H_2(z)$, respectively, i.e., $h_1 + h_2 = 1$ and $h_1 + h_2 = h_2$. The sequences $h_1$ and $h_2$ are the shifted version of the sequence $h_1$ and $h_1$, respectively. Thus, the Boolean functions generating $h_1$, $h_2$, $h_1$, and $h_2$ can then be expressed as $(1 - \pi_{1}) (1 - \pi_{1})$, $(1 - \pi_{1}) (1 - \pi_{1})$, and $(1 - \pi_{1}) (1 - \pi_{1})$, respectively.

Now, consider a new recursion given by $P(n)$, $Q(n)$, and $R(n)$ for $n = 1, \ldots, m, \ldots, m$. Otherwise, $P(n) = Q(n) = R(n)$ for $n = 1, \ldots, m, \ldots, m$. The definition, $P(n)$, and $Q(n)$ correspond to the associated polynomials for the sequences $f_i$ and $g_i$. By the definition, $P(n)$ and $Q(n)$ can be expressed as

$$P(n) = \begin{cases} \sum_{i=0}^{2^n-1} S(i, z) \xi F(i) + \eta F(i), & 0 \leq i < 2^m, \\ \sum_{i=0}^{2^n-1} S(i, z) \xi F(i) + \eta F(i), & 2^m \leq i < 2^{2m} \end{cases}$$

$$Q(n) = \begin{cases} \sum_{i=0}^{2^n-1} S(i, z) \xi G(i) + \eta G(i), & 0 \leq i < 2^m, \\ \sum_{i=0}^{2^n-1} S(i, z) \xi G(i) + \eta G(i), & 2^m \leq i < 2^{2m} \end{cases}$$

where $i = \sum_{j=1}^{m} 2^{j-1} m$, $j(x)$, $\xi(x)$, and $\eta(x)$ can be calculated as $\xi(x)$ and $\eta(x)$ for $n = 1, \ldots, m, \ldots, m$. Hence, $f_i(x)$ and $g_i(x)$ can be written as $\xi(x)$ and $\eta(x)$ for $n = 1, \ldots, m, \ldots, m$.

**APPENDIX B**

**Proof of Theorem**

We re-write $A(n, z)$ and $B(n, z)$ in Theorem 1 as

$$P(n) = \begin{cases} \sum_{i=0}^{2^n-1} S(i, z) \xi F(i) + \eta F(i), & 0 \leq i < 2^m, \\ \sum_{i=0}^{2^n-1} S(i, z) \xi F(i) + \eta F(i), & 2^m \leq i < 2^{2m} \end{cases}$$

$$Q(n) = \begin{cases} \sum_{i=0}^{2^n-1} S(i, z) \xi G(i) + \eta G(i), & 0 \leq i < 2^m, \\ \sum_{i=0}^{2^n-1} S(i, z) \xi G(i) + \eta G(i), & 2^m \leq i < 2^{2m} \end{cases}$$

using the operators defined in Table 1. The operators related to the scalars, i.e., $\alpha_n$, $\beta_n$, and the phase rotations, i.e., $\gamma_n$, $\delta_n$, $\eta_n$, $\theta_n$, and $\omega_n$, are denoted by $A(n)$, $B(n)$, $\gamma(n)$, $\delta(n)$, $\eta(n)$, $\theta(n)$, and $\omega(n)$, respectively. The signs for $A(n, z)$ and $B(n, z)$ in Theorem 1 are generated with the operators $S(n)$. While the operators $D(n)$, $E(n)$, and $F(n)$ apply a shift to their arguments, the operators $O(n)$, $P(n)$, and $Q(n)$ generate the polynomials $A(z)$ and $B(z)$ for $n = 1, \ldots, m$. For the sake of unifying the operator formats, we express the operators by using $\xi$ as $\alpha_n = \xi \alpha_n$, $\beta_n = \xi \beta_n$, $\gamma_n = \xi \gamma_n$, $\eta_n = \xi \eta_n$, $\omega_n = \xi \omega_n$, $\xi^{\prime} = \xi^{\prime}$, and $\xi^{\prime} = \xi^{\prime}$, and exploit the identity $\xi^{\prime} = 1$ in Table 1. The configuration vectors $b_i$ for the sub-recursions are also given in Table 1. By using (5) and (7), $f_i(x)$ and $g_i(x)$ are obtained for each sub-recursion as in Table 1. Finally, by composing all operators based on $f_i(x)$ and $g_i(x)$, the polynomials $P(n, z)$ and $Q(n, z)$ can be calculated as

$$P(n, z) = \begin{cases} \sum_{i=0}^{2^n-1} S(i, z) \xi F(i) + \eta F(i), & 0 \leq i < 2^m, \\ \sum_{i=0}^{2^n-1} S(i, z) \xi F(i) + \eta F(i), & 2^m \leq i < 2^{2m} \end{cases}$$

$$Q(n, z) = \begin{cases} \sum_{i=0}^{2^n-1} S(i, z) \xi G(i) + \eta G(i), & 0 \leq i < 2^m, \\ \sum_{i=0}^{2^n-1} S(i, z) \xi G(i) + \eta G(i), & 2^m \leq i < 2^{2m} \end{cases}$$

Based on (27) in Lemma 2, $a_n(1 - x_n - x_{n+1}) = a_n - a_n(1 - x_n - x_{n+1})$, $a_m(1 - x_n - x_{n+1}) = a_m - a_m x_{n+1}$, and $b_m(1 - x_n - x_{n+1}) = b_m - b_m x_{n+1}$. Hence, $f_i(x)$ and $g_i(x)$ can be re-written as (9) and (10), respectively, where $\xi_n = (b_n - a_n)$ for $n = 1, \ldots, m, \ldots, m$. By using the identities in (27) and (28), $\xi_n(1 - x_n) + \xi_n x_{n+1} = \xi_n + (\xi_n - \xi_n) x_{n+1}$, $\xi_n x_n + \xi_n x_{n+1} = \xi_n + (\xi_n - \xi_n) x_{n+1}$, $\xi_n(1 - x_n) + \xi_n x_n x_{n+1} = \xi_n + (\xi_n - \xi_n) x_{n+1}$, and $\xi_n x_n + \xi_n x_n x_{n+1} = \xi_n + (\xi_n - \xi_n) x_{n+1}$. Hence, $f_i(x)$ and $g_i(x)$ can be re-expressed as (11) and (12), respectively, where $k_1 = c_1 + \xi_n - \xi_n$, $k_m(n) = c_m + \xi_n - \xi_n - \xi_n - \xi_n$, and $k_{n+1} = \xi_n - \xi_n - \xi_n - \xi_n$, respectively.
### Table I

| \(\alpha_n\) | \(O^{(\alpha)}(R(z)) \doteq \xi^{\alpha}R(z)\) | \(b_{\gamma}^0\) | \(f_n(x)\) | \(g_n(x)\) |
|---|---|---|---|---|
| Scalar \(\alpha_n\) | \(A^{(\alpha)}(R(z)) \doteq \xi^{\alpha}R(z)\) | \[1 \ 0 \ 0 \ 1\] | \(1 - x_{m\!\!m}, \ n = m\) | \(x_{m\!\!m}, \ n = m\) |
| | | | \(1 - x_{m\!\!m} - x_{m\!\!m+1} , \ n < m\) | \(1 - x_{m\!\!m} - x_{m\!\!m+1} , \ n < m\) |
| Phase \(\gamma_n\) | \(O^{(\gamma)}(R(z)) \doteq \xi^{\gamma}R(z)\) | \[0 \ 1 \ 1 \ 0\] | \(x_{m\!\!m}, \ n = m\) | \(0, \ n < m\) |
| | | | \(x_{m\!\!m} + x_{m\!\!m+1}, \ n < m\) | \(x_{m\!\!m} + x_{m\!\!m+1}, \ n < m\) |

**Proof.** The right hand side of (27) is 0 for \(f_1(x) = f_2(x)\) and \(k_1\) for \(f_1(x) \neq f_2(x)\). For (28), after the arguments of \(\gamma_2\) are arranged as the summations of two Boolean functions, (27) is applied.

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