Characterization of uniaxial crystals through the study of fringe patterns

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Abstract. The fringe pattern obtained when a divergent (or convergent) beam goes through a sample of birefringent crystal between two polarizers contains information which is inherent of the crystalline sample under study. On the other hand, by considering the design details of the experience and the parameters that are characteristic of the uniaxial plate, it is possible to theoretically obtain the luminous intensity corresponding to each point on a screen or CCD. Thus, this theoretical model allows us to obtain, from images experimentally obtained and theoretical expressions, the parameters that are characteristic of the plates. The results obtained proved the concordance between theory and experience while endorsing this technique for the characterization of birrefringent crystals and devices based on this type of materials.

1. Introduction
The accurate characterization of birefringent crystals, either for crystallographic study or for the design and testing of devices, is of great importance. However, manufacturers of optical components rarely provide a complete description of birefringent crystals. Usually, the specifications (with mean and error) of the spatial direction of the optical axis, the thickness or the indices are either missing or not accurate enough. Several methods of characterization have been proposed by different authors. Some of them use white light and make the most of the different wavelengths involved for their measurements [1]. It is also possible to make measurements at different angles of incidence by means of rotation stage and determine the direction of the optical axis and the indices [2].

In this paper we examine a method that simultaneously combines different angles of incidence for a single wavelength by means of a converging or diverging beam. Similar procedures have been widely used for many purposes in several fields under different names such as conoscopic measurements, polarization microscope, polarization interferometers. This technique can be used for birefringence mapping in large area wafers [3, 4] as well as for optical mineralogy and designing and testing of polarization control devices (e.g. plate retarders). It provides a fringe interference pattern that identifies the crystal sample under test the same way a fingerprint identifies a person. These fringe patterns can also be calculated by a theoretical model that we have developed. This way, by means of experimental fringe patterns and a theoretical formula, we find the parameters that best fit our expressions.

Firstly, we will explain the experimental setup and the model behind it. Secondly, we study a crystal sample and its fringe pattern in order to estimate the direction of the optic axis and...
finally we show the validity of both the method and the model.

2. Birefringent polarization interferometers
The birefringent polarization interferometers have been extensively studied. Birefringent crystals are anisotropic media where the incident beam is generally split in two linearly polarized beams with different directions. We consider the particular case where the birefringent element only consists of a plane parallel plate of uniaxial crystal with arbitrary orientation of the optical axis.

2.1. Experimental setup
The operation of these devices is based on the interference between ordinary waves and extraordinary waves produced by the crystal plate. The setup consists of a source of monochromatic linearly polarized light (HeNe Laser $\lambda = 632.8\text{nm}$). Then a diffuser is placed following the beam axis in order to expand the beam divergence. After that, we place the crystal sample of thickness $H$ followed by a crossed polarizer and then a projecting screen or photographic film at a distance $D$ from the crystal (Fig. 1).

![Figure 1](image)

Figure 1. Scheme of the experience. Light propagates in the direction of the x axis, while the screen is placed perpendicular to it.

2.2. Theoretical model
The model of the experience is based on the calculation of the position of the geometrical images by ray tracing [5], [6]. This way, the interference fringes are obtained by superposing the wavefronts emerging from the ordinary image and the extraordinary image. The ordinary image can be found as in isotropic plane parallel plates giving a point image (under paraxial considerations), whereas the extraordinary image gives rise to a first-order astigmatic image. This method allows us to work with the wavefronts emerging from the plate without having to calculate the optical path difference within the birefringent medium each time. The ordinary wavefronts are spherically shaped, whereas the extraordinary wavefronts may be written to first order as the superposition of two crossed cylindrical wavefronts (Fig. 2).

In [6], it is shown that the phase shift $\phi$ between the ordinary and the extraordinary waves over the screen located in the $(y, z)$ plane is given by an hyperbolic formula (Eqs. (42–44) in ref [6])

$$\phi = ay^2 + bz^2 + cz + d$$

(1)

where $a$, $b$, $c$ and $d$ depend on the characteristics of both the uniaxial crystal sample (refractive indices, orientation of the optical axis and thickness) and the design of the experience.
Figure 2. Images and wavefronts. For the ordinary case there is a point image that originates spherical wavefronts and for the extraordinary case there is an astigmatic image that originates cylindrical wavefronts. Once the images are located, it is possible to construct the fringe patterns by superposing the wavefronts that emerge from them.

1) The intensity of the light over the screen (plane \((y, z)\)) can be calculated as the resulting interference of waves by replacing \(\phi\) (Eq. 1) in

\[
I = I_o + I_e + 2\sqrt{I_oI_e}\cos \phi
\]

where the intensities \(I_o\) and \(I_e\) correspond to the ordinary and the extraordinary beams respectively. We do not consider differences neither between the intensities of both beams nor in the intensity due to the location of the point over the screen. These effects would require a complete study of the divergent beam and the reflection and transmission coefficients along the different media involved. However, our simple model allows us to generate a computer image of a fringe pattern due to a uniaxial crystal piece with arbitrary direction of the optical axis and refractive indices (Fig. 3). It also allows us to solve the inverse problem of finding the parameters of the crystal sample that best fits an experimental image as we will show in the following section.

Figure 3. Computer generated image. \(\lambda = 632.8nm\), \(n_o = 1.5426\), \(n_e = 1.5516\), \(x_j = 0\), \(H = 3mm\), \(D = 10cm\). (a) \(\theta = 45^\circ\). (b) \(\theta = 0^\circ\)

3. Experiment and results
We show some tests that we carried out on a \(3mm\) thick sample of quartz crystal. The combination of both birefringence and plate thickness allows us to work with several fringes, as can be seen in Fig. 4. The recorded pattern size is \(0.0493m \times 0.0490m\) and the resolution of the image is \(146 \times 145\) pixels (i.e. pixel size = \(338\mu m \times 338\mu m\)).
3.1. Preprocessing

Equation 1 shows that the $z$ axis is an axis of symmetry. The $z$ axis corresponds the projection of the optical axis on the interface. We rotate the figure until the horizontal axis becomes an axis of symmetry. We can estimate the location of the axes $y$ and $z$ by calculating the intensity center of mass in our picture. Thus, we slightly modify the model function by adding some extra parameters for the location of the axes: $y_0$ and $z_0$

$$I = I_o + I_e + 2\sqrt{I_o I_e} \cos[a(y - y_0)^2 + b(z - z_0)^2 + c(z - z_0) + d]$$

where $y_0$ and $z_0$ are the location of the horizontal axis and the location of the vertical axis respectively. The image has some saturated pixels, some speckle and background illumination that are not represented by our simple model of interference. These noisy features are removed by first filtrating the image by means of contrast adjustment, equalization and frequency filtering. Next, we proceed to binaryze it (Fig. 5(a)). After that, we apply a low pass filter in order to recover the low frequency information (Fig. 5(b)).

![Figure 4. Experimental data. Our initial guess: $\theta = 45^\circ$, $\lambda = 632.8nm$, $n_o = 1.5426$, $n_e = 1.5516$, $x_j = 0$, $H = 3mm$, $D = 10cm$](image)

![Figure 5. Image processing](image)

3.2. Image fitting

We make use of different routines implemented in Python that belong to the library SciPy.optimize. We only estimate three parameters: the location of the horizontal axis $y_0$, the location of the vertical axis $z_0$ and the angle between the optical axis and the interfaces of the plate $\theta$ (Fig. 1). The first two parameters allow the algorithms to vary the location
of the center of the image, while the third one corresponds to a feature of the crystal sample under study. Some of the algorithms admit the use of bounds so we define a range for each parameter according to our confidence on the initial guess. We also restrict the useful pixels by their location (radius) or intensity (grey level) in order to fit the image around a specific region as well as reduce the computer processing time. We run the algorithms several times varying the initial conditions and the selection of pixels (radius and intensity). Some of the results of the estimation of $\theta$ are shown in Table 1.

### Table 1. Estimation of $\theta$ [°].

| script exec. | radius [m] | intensity (0, 1) | Lev-Marq | TNC | BFGS | NM | BF | $\hat{\theta}$ | $\sigma_{\theta}$ |
|--------------|------------|-----------------|----------|-----|------|----|----|----------------|----------------|
| A. 0.015     | (0, 1)     |                 | 52.6     | 46.6| 52.6 | 52.6| 46.6| 50.2           | 3.3            |
| B. 0.015     | (0, 0.1)   |                 | 52.5     | 55.0| 52.5 | 52.5| 46.3| 51.8           | 3.2            |
| C. 0.030     | (0, 1)     |                 | 41.1     | 41.1| 40.4 | 41.0| 39.8| 40.7           | 0.6            |
| D. 0.030     | (0, 0.1)   |                 | 41.8     | 41.8| 41.8 | 41.8| 41.8| 41.8           | 0              |

Each execution of the script tried five different algorithms. The first one is based on the Levenberg-Marquardt algorithm. The second one is the Truncated Newton Code (TNC) algorithm. The third one is a variation of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. The fourth is the Nelder-Mead simplex algorithm and the fifth is a global optimization routine called brute force. Figure 6 shows the a priori location of the coordinate axes (intensity center of mass) and the different algorithms estimations obtained in the script execution A. This time, three of the algorithms converged to $\theta = 52.6^\circ$, $y_0 = 0.42mm$ and $z_0 = -0.45mm$.

![Figure 6](image1.png)  
**Figure 6.** Estimation of Center. ◇ A priori. △Levenberg-Marquardt, L-BFGS-B, Nealder-Mead, ♦TNC, Brute-Force.

![Figure 7](image2.png)  
**Figure 7.** Computer generated image with estimated parameters $\theta_{est} = 46.6^\circ$ (TNC, BF)

As shown in Table 1, there are no significant differences between the mean values obtained in script execution A and B. This feature can also be observed if we compare the values of $\theta$ obtained in C and D. Instead, we can see that there are significant differences between the values obtained for each radius (i.e. by comparing A with C and C with D). Moreover, it seems that the enlargement of the radius reduces the dispersion of the estimated parameter. As a particular case, it is shown that in the script execution D all the algorithms converge to the same value of $\theta$. 
\( \theta \) (this case corresponds to the larger radius and the smaller range of intensities). From Table 1 it also follows that the mean values obtained do not depend strongly on the selection of the pixels because of their intensity but because of their location (radius).

4. Conclusions

The results show certain variations regarding the estimated parameter. The values of \( \theta \) given by the algorithms range from 39.8° to 52.6°. The dependence on the initial guess, the range of bounds and the selected pixels was fairly strong. If we compare the fringe patterns generated with the estimated parameters and the experimental image (for example Figs. 6 and 7), we will find that the approximation of the curvature of the fringes in the center of the figure is better than in the periphery (even for values of \( \theta \) above or below 45°). The paraxial approximation predicts the behavior of the crystals allowing us to obtain the well known hyperbola shaped fringe pattern for \( \theta = 0° \) (Fig. 3(b)) or the ring shaped fringe pattern for \( \theta = 90° \). However, since the selection of the central pixels leads to more dispersion of the estimated parameter, we believe that including the periferic pixels and using a non-paraxial theory would improve the performance of the algorithms. It would also be desirable to estimate more parameters such as its thickness, \( H \) and the distances \( D \) and \( x_j \). Due to the apparent complexity of the fitting problem, the utilization of genetic algorithms will also be considered in future analysis.

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