Online Nonlinear Estimation via Iterative $L^2$-Space Projections: Reproducing Kernel of Subspace

Motoya Ohnishi, Student Member, IEEE, and Masahiro Yukawa, Member, IEEE

Abstract—We propose a novel online learning paradigm for nonlinear-function estimation tasks based on the iterative projections in the $L^2$ space with probability measure reflecting the stochastic property of input signals. The proposed learning algorithm exploits the reproducing kernel of the so-called dictionary subspace, based on the fact that any finite-dimensional space of functions has a reproducing kernel characterized by the Gram matrix. The $L^2$-space geometry provides the best decorrelation property in principle. The proposed learning paradigm is significantly different from the conventional kernel-based learning paradigm in two senses: (i) the whole space is not a reproducing kernel Hilbert space and (ii) the minimum mean squared error estimator gives the best approximation of the desired nonlinear function in the dictionary subspace. It preserves efficiency in computing the inner product as well as in updating the Gram matrix when the dictionary grows. Monotone approximation, asymptotic optimality, and convergence of the proposed algorithm are analyzed based on the variable-metric version of adaptive projected subgradient method. Numerical examples show the efficacy of the proposed algorithm for real data over a variety of methods including the extended Kalman filter and many batch machine-learning methods such as the multilayer perceptron.

Index Terms—online learning, metric projection, kernel adaptive filter, $L^2$ space, recursive least squares

I. INTRODUCTION

A. Background

Metric is a dominant factor in controlling convergence behaviors of online learning algorithms, as witnessed by the extensive studies on adaptive filtering [2]–[8] as well as the recent advances in stochastic optimization [9]–[12] (see also [13, Chapter 3] for a related idea called space dilation for accelerating the convergence of the subgradient method for minimization of nondifferentiable functions). Metric projection has been used extensively in adaptive/online learning algorithms [14]–[21] (see also the tutorial paper [22]). The main subject of the present study is the metric of online learning algorithms for nonlinear-function estimation tasks.

Kernel adaptive filtering is a powerful approach to the nonlinear-function estimation tasks [23]–[42], being an adaptive extension of the kernel ridge regression [43], [44] or Gaussian process [45]. Projection-based kernel adaptive filtering algorithms have been studied mainly by casting the nonlinear estimation as a minimization problem either (i) in the Euclidean space of coefficient vectors [30], or (ii) in the reproducing kernel Hilbert space (RKHS) [29], [31], [34], [35]. The two types of formulation induce two different geometries. The latter type is referred to as the functional approach, and its geometry in the dictionary subspace (i.e., the subspace spanned by the dictionary) can be expressed in the Euclidean space equivalently with a metric characterized by the kernel matrix [35]. The functional approach tends to exhibit better convergence behaviors (see, e.g., [34], [35], [46]) than the former approach. This has been supported theoretically in [47]. Specifically, provided that the dictionary can be considered as a set of realizations of the input vectors, the autocorrelation matrix can be approximated by a squared kernel matrix essentially, which indicates that its eigenvalue spread for the functional approach is reduced to a square root compared to the former approach in principle. The conventional kernel adaptive filtering methods employ a single kernel, thereby working efficiently only when all the three conditions are satisfied: (i) the target nonlinear function is sufficiently simple, (ii) its scale is known prior to adaptation so that one can design a Gaussian kernel with appropriate scale, and (iii) the scale is time-invariant.

Multikernel adaptive filtering [46], [48]–[50] is an efficient solution to the case in which some of the above conditions are violated, such as the case of multi-component/partially-linear functions (see [46]). A remarkable feature of multikernel adaptive filtering is that finding a well-fitting kernel and obtaining a compact representation (i.e., dictionary sparsification and parameter estimation) are simultaneously achieved within a convex analytic framework. The existing functional approach for multikernel adaptive filtering is called the Cartesian hyperplane projection along affine subspace (CHYPASS) algorithm [46], formulated in the Cartesian product of the RKHSs associated with the multiple kernels employed. Here, CHYPASS is a multikernel extension of the hyperplane projection along affine subspace (HYPASS) algorithm [34], [51], which is an efficient functional approach derived by formulating the normalized least mean square (NLMS) algorithm in the functional subspace. The decorrelation property of CHYPASS is however suboptimal since it counts no correlations among different kernels.

B. Motivation and Contributions

Suppose that the input (sample) is a real random vector. Our first primitive question is the following: what metric induces...
the best geometry having a perfect decorrelation property for online nonlinear-function estimation over a (possibly expanding) finite-dimensional subspace in general? An immediate answer to this question is the $L^2$ space (the set of square-integrable functions) under the probability measure determined by the probability density function of the input vector (see Sections II-A and III-A). Henceforth, we simply call it the $L^2$ space. In addition to its nice geometric property, the $L^2$ space is sufficiently large to accomodate the subspace even if it expands as time goes by (see Section II-A). The $L^2$ space, however, is not an RKHS because the function value at some specific point is not well-defined due to the presence of equivalence class. Now arises the central question penetrating this paper.

Should the learning space be an RKHS to achieve efficient online nonlinear estimation?

In this paper, we propose an efficient online nonlinear-function learning paradigm based on iterative projections in the $L^2$ space. In the proposed learning paradigm, the minimum mean squared error (MMSE) estimator gives the best approximation (in the $L^2$-metric sense) of the target nonlinear function in the dictionary subspace (Proposition 1 in Section III-A). We highlight the fact that the HYPASS algorithm implicitly exploits the reproducing kernel of the dictionary subspace for updating the estimates (see Section II-D). We then show the way of constructing the reproducing kernel of a finite-dimensional subspace in terms of the Gram matrix of its basis (Proposition 2 in Section III-A). We can thus extend the strategy of HYPASS to any space (which possibly has no reproducing kernel) in principle as long as the Gram matrix is computable at least approximately.

The key idea is the following: (i) we make the function values well-defined in the dictionary subspace by not considering the equivalence class, and (ii) we then define the reproducing kernel of the dictionary subspace of the $L^2$ space. For implementing the proposed method efficiently, we present three practical examples of computing the Gram matrix. 1) When the basis contains multiple Gaussian functions with different centers and scale parameters, the inner product can analytically be computed by assuming that the input vector obeys the normal distribution, or perhaps the improper constant distribution in analogy with a conjugate prior and its particular example, multikernel adaptive filtering. 2) The Gram matrix can be approximated with the atoms of the dictionary obeys the normal distribution, or perhaps the improper constant distribution in analogy with a conjugate prior and its particular example, multikernel adaptive filtering. 3) The Gram matrix can recursively be updated by using the matrix inversion lemma for rank-2 update. We show that the approximate linear dependency (ALD) condition [25] ensures a lower bound of the amount of the MMSE reduction due to the newly entering dictionary-element, keeping in mind the link between ALD and the coherence condition [30] (which we shall use for computational efficiency). See Lemma 1 and Proposition 3 in Section III-C. The computational complexity of the proposed algorithm has the same order as that of the Euclidean approach when the selective-update strategy is employed (see Section III-D). Monotone approximation, asymptotic optimization, and convergence of the proposed algorithm are proved for the full-updating case within the framework of the variable-metric adaptive projected subgradient method (APSM) [4], [52] (Theorem 2 in Section IV). Numerical examples show that (i) the proposed algorithm enjoys a better decorrelation property than CHYPASS [46] and the multikernel NLMS (MKNLMS) algorithm [48], and (ii) it outperforms, under the use of the selective-update strategy, the extended Kalman filter (EKF) for real data as well as 13 (out of 15) batch learning methods that have been compared in the literature [53], [54].

C. Relations to Bayesian and Stochastic Gradient Descent Approaches

The projection-based methods tend to show better tracking/convergence with low computational complexity compared to the Bayesian and stochastic gradient descent approaches. By using the well-known kernel trick, the rigorous framework of the projection-based linear adaptive filtering has been extended to kernel adaptive filtering [22], [34], [35], [48]. Monotone approximation is one of the most significant properties of the projection-based methods, ensuring stable tracking when the target function keeps changing. Convergence is also guaranteed when the target function is time-independent (see Theorem 2 in Section IV and its corresponding remark). Moreover, by virtue of the well-established algebraic properties of nonexpansive mappings [55, Chapter 17], the projection-based methods have high flexibility of the algorithm design, from the parallel-projection [34] and the multi-domain adaptive learning [21] to the sparsity-aware algorithms [35], [48].

Those variants of the projection-based methods also lead to convergence speed comparable to the Bayesian approaches despite their low computational complexities. Compared to the stochastic gradient descent algorithms such as NORMA [24], (i) the projection-based methods offer tracking/convergence guarantees without elaborate step-size tuning, and (ii) can efficiently update the estimate even when the dictionary does not grow [51] (see Section II-D). In addition to the practical advantages, stable tracking capabilities and convergence guarantees for the variants can immediatly be analyzed, as witnessed by the present work itself. Comparisons of the projection-based methods to Bayesian approaches (online Gaussian processes (GPs) [23] and the kernel recursive least squares tracker (KRLS-T) [32]) and a stochastic gradient descent algorithm (NORMA) are summarized in Table I.

| Algorithm | Convergence speed | Tracking speed | Complexity | Variance information |
|-----------|------------------|----------------|-------------|-----------------------|
| Online GPs | very fast         | slow           | high        | yes                   |
| KRLS-T    | fast             | fast           | high        | yes                   |
| NORMA     | moderate         | moderate       | low         | uninvestigated        |
| Projection | fast             | very fast      | low         | uninvestigated        |

II. Preliminaries

We first present the nonlinear system model under study together with notation. We then present our nonlinear estimator and its particular example, multikernel adaptive filtering
model. We finally review the HYPASS algorithm from another angle based on a theorem on the reproducing kernel of a closed subspace.

A. Nonlinear System Model

Throughout, \( \mathbb{R}, \mathbb{N}, \) and \( \mathbb{N}^* \) are the sets of real numbers, nonnegative integers, and positive integers, respectively. We consider the following nonlinear system model:

\[
d_n := \psi(u_n) + \nu_n. \tag{1}
\]

Here, the input (sample) vector \( u_n \in \mathbb{R}^L \) is assumed to be a random vector with probability density function \( p(u), \nu_n \) is the additive noise at time \( n \in \mathbb{N} \), and the nonlinear function \( \psi \) is assumed to lie in the real Hilbert space \( \mathcal{H} := L^2(\mathbb{R}^L, d\mu) := \{ f \mid \| f \|_{\mathcal{H}} < \infty \} \) equipped with the inner product

\[
(f, g)_{\mathcal{H}} := \int_{\mathbb{R}^L} f(u)g(u) d\mu(u), \quad f, g \in \mathcal{H},
\]

and its induced norm \( \| f \|_{\mathcal{H}} := \sqrt{(f, f)_{\mathcal{H}}} \), where \( d\mu(u) := p(u)du \) is the probability measure. Assuming that there exists \( M \in (0, \infty) \) such that \( p(u) < M \) for all \( u \in \mathbb{R}^L \), we have

\[
\int |f(u)|^2 p(u)du \leq M \int |f(u)|^2 du,
\]

which implies that \( L^2(\mathbb{R}^L, d\mu) \subset L^2(\mathbb{R}^L, d\mu) = \mathcal{H} \). It is known that the space \( L^2(\mathbb{R}^L, d\mu) \) contains any Gaussian RKHS as its subset [56]. Hence, our assumption \( \psi \in \mathcal{H} \) is weaker than usually supposed in the literature of kernel adaptive filtering.

**Notation:** We denote by \( \theta \) the null vector of \( \mathcal{H} \). The metric projection of a point \( f \in \mathcal{H} \) onto a given closed convex set \( C \subset \mathcal{H} \) is defined by

\[
P_C(f) := \arg \min_{g \in C} \| f - g \|_{\mathcal{H}}. \tag{4}
\]

If, in particular, \( C \) is a linear variety (a translation of a linear subspace), \( P_C(f) \) is said to be the orthogonal projection. Given \( m \)-dimensional real vectors \( x, y \in \mathbb{R}^m \), define \( \langle x, y \rangle_{\mathbb{R}^m} := x^T y \) and \( \| x \|_{\mathbb{R}^m} := \sqrt{\langle x, x \rangle_{\mathbb{R}^m}} \), where \( (\cdot)^T \) stands for transposition. Given any pair of integers \( m, n \in \mathbb{N} \) such that \( m \leq n \), we denote by \( \overline{m, n} := \{ m, m+1, \cdots, n \} \). We denote the identity matrix by \( I \).

B. Nonlinear Estimator

Our nonlinear estimator takes the following form:

\[
\varphi_n := \sum_{i=1}^{r_n} h_{n,i} f_i^{(n)} \in \mathcal{M}_n := \text{span } \mathcal{D}_n, \quad n \in \mathbb{N},
\]

where \( \mathcal{D}_n := \{ f_1^{(n)}, f_2^{(n)}, \cdots, f_{r_n}^{(n)} \} \subset \mathcal{H} \) is the dictionary at time \( n \). We assume that the value \( f_i^{(n)}(x) \) of each \( f_i^{(n)}, i \in \overline{1, r_n} \), at an arbitrary point \( x \in \mathbb{R}^L \) is predefined, i.e., \( f_i^{(n)} \) is a representative of an equivalence class of functions in \( \mathcal{H} \). As will be seen in Section III, any set of functions in \( \mathcal{H} \) can be used as a dictionary in the proposed learning paradigm, as long as \( \langle f_i^{(n)}, f_j^{(n)} \rangle_{\mathcal{H}} \), \( i, j \in \overline{1, r_n} \), can be computed (or approximated) efficiently. The evaluation of \( \varphi_n \) at the current input \( u_n \) can be expressed as

\[
\varphi_n(u_n) = f_n(u_n)^T h_n, \tag{5}
\]

where \( h_n := [h_{1,n}, h_{2,n}, \cdots, h_{r_n,n}]^T \in \mathbb{R}^{r_n} \), and \( f_n(u) := [f_1^{(n)}(u), f_2^{(n)}(u), \cdots, f_{r_n}^{(n)}(u)]^T \in \mathbb{R}^{r_n} \) for any \( u \in \mathbb{R}^L \).

C. Multikernel Adaptive Filtering Model

We present a specific example of the dictionary \( D_n \). Let \( \mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_q \) be RKHSs equipped with the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}_q} \) and its induced norm \( \| \cdot \|_{\mathcal{H}_q} \), \( q \in \overline{1, Q} \). Let \( \kappa_q : \mathbb{R}^L \times \mathbb{R}^L \rightarrow \mathbb{R}, g \in \overline{1, Q} \), be the reproducing kernel of \( \mathcal{H}_q \). One of the celebrated examples is the Gaussian kernel

\[
\kappa_q(u, v) := \frac{1}{(2\pi\sigma^2)^{L/2}} \exp(-\frac{|u-v|^2}{2\sigma^2}), \quad u, v \in \mathbb{R}^L,
\]

where \( \sigma > 0 \) is the scale parameter with \( \sigma_1 > \sigma_2 > \cdots > \sigma_Q \). The existing kernel/multikernel adaptive filtering approaches exploit the properties of reproducing kernels: (i) \( \kappa_q(\cdot, u) \in \mathcal{H}_q \) and (ii) \( f(u) = (f, \kappa_q(\cdot, u))_{\mathcal{H}_q} f \in \mathcal{H}_q \), \( u \in \mathbb{R}^L \). We emphasize here that, given a space, different inner products give different reproducing kernels. This is important to follow the discussions presented in Section III. We assume that \( \mathcal{H}_q \subset \mathcal{H} \); e.g., this assumption holds in the case of Gaussian kernels (see Section II.A). For each \( q \in \overline{1, Q} \) and each time instant \( n \in \mathbb{N} \), let \( \mathcal{D}^{(q)} := \{ \kappa_q(\cdot, u_j) \}_{j=1}^{n_q} \),

\[
J^{(q)} := \{ (j_{q,n}, j_{2,n}, \cdots, j_{r_n,n}) \}_{n_q \in \overline{1, r_n}} \subset \mathcal{N}_n \]be the \( q \)-th dictionary of size \( r_n \) in \( \mathcal{N} \), \( q \in \overline{1, Q} \). The whole dictionary \( \mathcal{D} : = \bigcup_{q \in \overline{1, Q}} \mathcal{D}^{(q)} \) at time \( n \) is of size \( r_n := \sum_{q \in \overline{1, Q}} r_{n_q} \).

D. HYPASS Algorithm Revisited: A Fresh View

We start with the following theorem to find the reproducing kernel of a closed subspace of RKHS.

**Theorem 1** ([57, Theorem 11]). Let \( M \) be a closed subspace of an RKHS \( (X, \langle \cdot, \cdot \rangle_X) \) associated with the reproducing kernel \( \kappa \). Then, \( (M, \langle \cdot, \cdot \rangle_X) \) is an RKHS associated with the reproducing kernel \( \kappa_M \) given by

\[
\kappa_M(u, v) = P_M^X(\kappa(u, v))(u), \tag{6}
\]

where \( P_M^X \) denotes the projection operator defined with respect to the metric of \( X \). (Note that \( \kappa \) is not necessarily the reproducing kernel of \( (M, \langle \cdot, \cdot \rangle_X) \), because \( \kappa(u) \notin M \) in general.)

We consider the monokernel case of \( Q = 1 \). Taking a fresh look at the update equation of the HYPASS algorithm [34], [51] under the light of Theorem 1 we obtain

\[
\varphi_{n+1} = \varphi_n + \lambda_n (P_{\mathcal{H}_n}(\varphi_n) - \varphi_n) \tag{7}
\]

\[
= \varphi_n + \lambda_n \frac{d_n - \langle \varphi_n, \kappa_{1,M_n}(\cdot, u_n) \rangle_{\mathcal{H}_1}}{\| \kappa_{1,M_n}(\cdot, u_n) \|_{\mathcal{H}_1}^2} \kappa_{1,M_n}(\cdot, u_n), \tag{8}
\]

\[
\text{where } d_n := \langle \varphi_n, \kappa_{1,M_n}(\cdot, u_n) \rangle_{\mathcal{H}_1} \tag{9}
\]
where $\lambda_n \in (0, 2)$ is the step size, $\kappa_{1,M_n}$ is the reproducing kernel of $(M_n, \langle \cdot, \cdot \rangle_{H_n})$, and

$$\Pi_n := \left\{ f \in M_n \mid f(u_n) = \langle f, \kappa_{1,M_n}(\cdot, u_n) \rangle_{H_n} = d_n \right\}. \quad (10)$$

Here, the orthogonal decomposition [58] indicates that $f(u) = \langle f, \kappa_{1,\cdot}(\cdot, u_n) \rangle_{H_n} = \langle f, \kappa_{1,M_n}(\cdot, u_n) \rangle_{H_n}$ for any $f \in \mathcal{M}_n$, and HYPASS can be regarded as projecting the current estimate $\varphi_n$ onto the hyperplane $\Pi_n$ in the RKHS $(\mathcal{M}_n, \langle \cdot, \cdot \rangle_{H_n})$ of which the reproducing kernel is $\kappa_{1,M_n}$. Note here that, if $\kappa_1, u_n) \in \mathcal{M}_n$, then it holds that $\kappa_1(\cdot, u_n) = \kappa_{1,M_n}(\cdot, u_n)$.

### III. PROPOSED ONLINE LEARNING METHOD

We first clarify why to use the $L^2$ metric for online learning, and how to implement it. We then present how to compute/approximate the autocorrelation matrix efficiently, learning, and how to implement it. We then present how to formulate a projection-based online nonlinear estimation in the sense of decorrelation under the possibly expanding dictionary subspace. This is the core motivation of the present study. Note that it is well known that a better-conditioned correlation matrix leads to faster convergence for linear adaptive filter (see [16], for example).

The question now is how to formulate a projection-based online learning algorithm working in $\mathcal{H}$. We have seen in Section II-D that the normal vector $\kappa_{M_n}(\cdot, u_n)$ of the hyperplane $\Pi_n$ gives the direction of update in $[2]$, and it is readily available if the reproducing kernel $\kappa_{M_n}$ is known. As widely known, the $L^2$ space $\mathcal{H}$ has no reproducing kernel because the value $f(u)$ of $f \in \mathcal{H}$ at a given point $u \in \mathbb{R}^L$ is not well defined due to the presence of equivalence classes (i.e., those functions which coincide except for a measure-zero set are regarded to be the same point). Fortunately, however, what we need is the reproducing kernel of the dictionary subspace $M_n$, as already mentioned. In fact, if one regards $\varphi_n$ as an element of $\mathcal{H}$, its value $\varphi_n(u)$ at some specific point $u \in \mathbb{R}^L$ is not well defined. Nevertheless, we define it by $\varphi_n(u) := \sum_{i=1}^{n} h_{i,n} f_i^{(u)}(u)$ as the value $f_i^{(u)}(u)$ is assumed to be predefined. By doing so, $(\mathcal{M}_n, \langle \cdot, \cdot \rangle_{H_n})$ becomes a finite-dimensional real Hilbert space in which the value of each function at each point is well defined. In this case, there is a systematic way to construct the reproducing kernel of the space, as shown below.

**Proposition 2.** Let $D := \{f_1, f_2, \ldots, f_r\} \subset \mathcal{H}$, $r \in \mathbb{N}^*$, be an independent set, and $G$ the Gram matrix with its $(k,l)$ entry $g_{k,l} := \langle f_k, f_l \rangle_{H_n}$. Define $f(u) := \langle f, (\cdot, u) \rangle_{H_n}$ as the (inner product) in the sense of the inner product. Then

\[ f(u) = \sum_{k=1}^{r} g_{k,l} f_k(u) \quad \text{for all} \quad u \in \mathbb{R}^L. \]
where set can be found in [57, page 7 - Example 1].

We mention for clarity that the bounded-instantaneous-error hyperslab $C_n := \{ f \in \mathcal{M}_n | |f(u_n) - d_n| \leq \rho \},$ where $\rho \geq 0$. For the initial estimate $\varphi_0 := \theta$, generate the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of nonlinear estimators by

$$\varphi_{n+1} := \varphi_n + \lambda_n (P_{C_n}(\varphi_n) - \varphi_n) = \varphi_n + \lambda_n \text{sgn}(e_n(u_n)) \max\{|e_n(u_n)| - \rho, 0\} \frac{1}{\|\kappa_{M_n}(\cdot, u_n)\|_{\mathcal{H}}} \kappa_{M_n}(\cdot, u_n),$$

(17)

where $\text{sgn}(\cdot)$ is the sign function, $e_n(u_n) := d_n - \varphi_n(u_n)$, and $\lambda_n \in (0, 2)$ is the step size.

We have shown how to update our nonlinear estimator $\varphi_n$, given a dictionary $D_n$. The remaining issues to be discussed are how to compute $R$ in (4) efficiently (Section III-B) and how to construct the dictionary $D_n$ (Section III-C). We shall also present the selective-update strategy to reduce the computational complexity in Section III-D.

### B. Practical examples of computing $R$ efficiently

We present three options to estimate/approximate $R$ efficiently. The first option assumes the use of multiple Gaussian functions with different scales (see Section II-C), while the other two options can be applied to the general case.

1) **Analytical approach:** We present two examples in which analytical expressions of inner product can be obtained by using the analogy to a conjugate prior and a noninformative prior [60].

**Proposition 3.** Let $\kappa_p(\cdot, u)$, $p \in \mathbb{L}$, $u \in \mathbb{R}^L$, and $\kappa_q(\cdot, v)$, $q \in \mathbb{L}$, $v \in \mathbb{R}^L$, be two Gaussian functions with scale parameters $\sigma_p, \sigma_q > 0$, respectively.

(a) **Case of Gaussian input:** Assume that the input vector $u \in \mathbb{R}^L$ follows the normal distribution with variance $\sigma^2$, i.e., the probability density function for the input vector is given by

$$p(u) := \frac{1}{(2\pi \sigma^2)^{L/2}} \exp \left( -\frac{|u|^2}{2\sigma^2} \right), \quad \sigma > 0.$$  

(18)

Then, the inner product can be given analytically by

$$\langle \kappa_p(\cdot, u), \kappa_q(\cdot, v) \rangle_{\mathcal{H}} = \frac{1}{(2\pi \sigma^2)^{L/2}} \exp \left( -\frac{|u-v|^2}{2\sigma^2} + \frac{\sigma^2}{4\sigma^2} |u|^2 + \frac{\sigma^2}{4\sigma^2} |v|^2 \right),$$

(19)

where $v := \sigma^2 \sigma_p^2 + \sigma^2 \sigma_q^2 + \sigma^2 \sigma_r^2$.

(b) **Case of unknown input distribution:** Suppose that there is no available information about the input distribution. In this case, by using the analogy to a noninformative prior which is improper, let $d\mu(u) := du$, i.e., the input is assumed to distribute uniformly over the infinite interval. The inner product is then given by

$$\langle \kappa_p(\cdot, u), \kappa_q(\cdot, v) \rangle_{\mathcal{H}} = \frac{1}{(2\pi \sigma^2)^{L/2}} \exp \left( -\frac{|u-v|^2}{2(\sigma_p^2 + \sigma_q^2)} \right).$$

(20)

**Proof.** See Appendix C.

2) **Finite-sample approach — use of sample average:** It is also possible to approximate $R$ by a sample average. Let $\{u_j\}_{j \in \mathcal{J}_n}$, where $\mathcal{J}_n := \{1, 2, \ldots, j_n\} \subset \overline{\mathbb{N}}$, be a fixed set of realizations of the input vectors $u_n$. Then, at time $n$, the matrix $R$ is approximated by

$$R \approx \frac{1}{l_n} F_n F_n^T,$$

(21)

where $l_n \in \mathbb{N}^*$ is the size of $\mathcal{J}_n$, and $F_n := \left[ f_n(u_{j_1}), f_n(u_{j_2}) \cdots f_n(u_{j_{l_n}}) \right]$. When the dictionary elements are associated with input vectors, the set $\{u_j\}_{j \in \mathcal{J}_n}$ might be given as the set of dictionary data (e.g., $\mathcal{J}_n = \cup_{q \in \mathbb{Q}} \mathcal{J}_n(q)$ for the case of multikernel adaptive filtering).

Suppose that $\mathcal{H}$ is an RKHS with $f_j^{(n)} := \kappa(\cdot, u_j)$, $j \in \mathcal{J}_n (l_n = r_n)$, where $\kappa$ is supposed to be the reproducing kernel of $\mathcal{H}$. Then, $F_n$ is the Gram matrix of the dictionary $D_n$. Hence, the approximation in (21) is a natural extension of the $G^2$-metric studied in [61].

3) **Recursive approach:** The inverse autocorrelation matrix $R^{-1}$ appearing in (14) can be approximated, recursively, by using a similar trick to the kernel recursive least squares (KRLS) algorithm [25]. We assume that the dictionary may only change in an incremental way; i.e., $D_{n-1} \subseteq D_n$ and $r_n \in \{r_{n-1}, r_{n-1} + 1\}$. (It is straightforward to scale down the size of the autocorrelation matrix when some elements are excluded from the dictionary.) When the dictionary is unchanged, the estimate of the autocorrelation matrix can be updated as $R_n := R_{n-1} + f_n f_n^T$, and its inverse $R_n^{-1}$ can be updated recursively as

$$R_n^{-1} = R_{n-1}^{-1} - \frac{R_{n-1}^{-1} f_n f_n^T R_{n-1}^{-1}}{1 + f_n^T R_{n-1}^{-1} f_n}.$$  

(22)
When a new basis function is added and the dictionary is changed, we define the estimate of the augmented autocorrelation matrix as

\[
R_n := \begin{bmatrix} R_{n-1} & 0 \\ 0 & 0 \end{bmatrix} + f_n(u_n) f_n(u_n)^T = A + BC,
\]

where \( A := \begin{bmatrix} R_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \), \( B := [f_n e_n] \), \( C := [f_n - e_n]^T \), and \( e_n := [0, 0, \ldots , 0, 1]^T \in \mathbb{R}^{n+1} \). Assuming that \( f_n^{(n)}(u_n) \neq 0 \) to ensure the nonsingularity of \( I + CA^{-1}B \), one can apply the matrix inversion lemma to compute the inverse of the rank-2 update \((23)\), obtaining the following recursion:

\[
R_{n+1}^{-1} = A^{-1} - A^{-1} B (I + CA^{-1}B)^{-1} CA^{-1}.
\]

The idea of this “Recursive approach” comes certainly from the recursive least squares (RLS) algorithm, which iteratively minimizes the sum of the squared errors. In fact, RLS can be viewed as a variable-metric projection algorithm with nearly-unity step size \([62]\) (see Appendix [E]).

C. Dictionary Construction with Novelty Criterion

The dictionary is constructed based on some novelty criterion as follows: a function \( f_{u_n} \in \mathcal{H} \) depending on the new measurement \( u_n \) is added into the dictionary if it satisfies some prespecified novelty criterion. In the particular case of multiple Gaussian functions (see Section [II-C]), a possible option is the following:

1) The coarsest Gaussian function \( \kappa_1(\cdot, u_n) \) is added into the dictionary when it satisfies the novelty criterion.

2) A finer Gaussian \( \kappa_i(\cdot, u_n), i \geq 2 \), is added into the dictionary if it satisfies the novelty criterion but all the coarser Gaussians \( \kappa_1(\cdot, u_n), \kappa_2(\cdot, u_n), \ldots , \kappa_{i-1}(\cdot, u_n) \), do not.

In analogy with Platt’s criterion \([63]\), we consider two novelty conditions both of which need to be satisfied: (i) the coherence condition (elaborated below) and (ii) the large-normalized-error (LNE) condition

\[
|d_n - \varphi_n(u_n)|^2 = |e_n(u_n)|^2 > \epsilon |\varphi_n(u_n)|^2, \epsilon \geq 0.
\]

Given a threshold \( \delta \in [0, 1] \), the coherence condition is given as follows:

\[
\max_{\mathcal{D}_n} c(f, f_{u_n}) \leq \delta,
\]

where \( c(f, g) := \frac{|\langle f, g \rangle|_{\mathcal{H}}}{\|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}} \), \( f, g \in \mathcal{H} \). Note here that, by definition, those factors \( \|f\|_{\mathcal{H}} \), \( \|f\|_{\mathcal{H}} \), and \( \|f_{u_n}\|_{\mathcal{H}} \) involve expectation, which brings the same issue as for the computation of \( R \) discussed in Section [III-B]. When Analytical approach is employed for the computation of \( R \), Proposition [3] can be applied. When Finite-sample/Recursive approach is employed, one may use sample averages with the \( s_n \in \mathbb{N}^* \) most-recent measurements, for instance, as

\[
\langle f, f_{u_n} \rangle_{\mathcal{H}} \approx \frac{1}{s_n} \sum_{i=n-s_n+1}^{n} f(u_i) f_{u_n}(u_i).
\]

The following lemma links the coherence condition to the ALD condition.

**Lemma 1.** Assume that \( (r_n - 1)\delta < 1 \). Then, the coherence condition \((26)\) ensures the following ALD condition:

\[
\frac{\|f_{u_n} - P_{\mathcal{M}_n}(f_{u_n})\|_{\mathcal{H}}^2}{\|f_{u_n}\|_{\mathcal{H}}^2} \geq 1 - \frac{(r_n - 1)\delta^2}{1 - (r_n - 2)\delta}. \tag{28}
\]

**Proof.** The assertion can be verified by \([30, \text{Equation (16)}]\) with the simple observation that the left-hand side of \((28)\) equals to

\[
\frac{\|f_{u_n} - P_{\mathcal{M}_n}(f_{u_n})\|_{\mathcal{H}}^2}{\|f_{u_n}\|_{\mathcal{H}}^2} \geq 1 - \frac{(r_n - 1)\delta^2}{1 - (r_n - 2)\delta}. \tag{29}
\]

Due to the property \( \psi^*_{\mathcal{M}_n} = P_{\mathcal{M}_n}(\psi) \) presented in Proposition [1], the proposed online learning algorithm with the \( L^2 \) metric takes a particular benefit from ALD, as indicated by the following proposition.

**Proposition 4.** Suppose that \( f_{u_n} \in \mathcal{D}_{n+1} \), i.e., \( f_{u_n}^{(n+1)} := f_{u_n} \), and \( \mathcal{M}_{n+1} = r_{n+1} \). Assume that \( E[f_{u_n}^2] = 0 \), and \( E\psi^*_{\mathcal{M}_{n+1}} = 0 \). Assume also that the ALD condition

\[
\frac{\|f_{u_n} - P_{\mathcal{M}_n}(f_{u_n})\|_{\mathcal{H}}^2}{\|f_{u_n}\|_{\mathcal{H}}^2} \geq \eta
\]

is satisfied for a given threshold \( \eta \in [0, 1] \). Then, for the MMSE estimators \( \psi^*_{\mathcal{M}_n} := \arg\min_{f \in \mathcal{M}_n} E[d_n - f(u_n)^2] \) and \( \psi^*_{\mathcal{M}_{n+1}} \), it holds that

\[
E[d_n - \psi^*_{\mathcal{M}_n}(u_n)]^2 - E[d_n - \psi^*_{\mathcal{M}_{n+1}}(u_n)]^2 \geq (h_n^*)^2 \|f_{u_n}\|_{\mathcal{H}}^2 \eta, \tag{30}
\]

where \( h_n^* \in \mathbb{R} \) is the coefficient of \( f_{u_n} \) in the expansion of \( \psi^*_{\mathcal{M}_{n+1}} \).

**Proof.** See Appendix [D].

Proposition [4] states that the amount of MMSE reduction is at least \((h_n^*)^2 \|f_{u_n}\|_{\mathcal{H}}^2 \eta \) under the ALD condition in the space \( \mathcal{H} \). The coherence condition actually ensures the ALD condition for \( \eta = 1 - \frac{(r_n - 1)\delta^2}{1 - (r_n - 2)\delta} \) as long as \((r_n - 1)\delta < 1 \) (see Lemma [1]), thereby yielding efficient MMSE reduction. When the condition \((r_n - 1)\delta < 1 \) is violated, an alternative option that could have a better performance-complexity tradeoff is to select \( s_n \in \mathbb{N}^* \) elements from \( \mathcal{D}_n \) that are maximally coherent to the \( f_{u_n} \) \([34]\) and check the ALD condition with respect to the selected elements.

D. Complexity Reduction by Selective Update

Although matrix inversion requires cubic complexity in general, the complexity is \( O(r_n^2) \) when Analytical approach (Section [III-B.1]) or Recursive approach (Section [III-B.3]) are adopted. However, it is still computationally expensive when the dictionary size becomes large. Therefore, in practice, one may use the selective-update strategy, i.e., select a subset \( \bar{\mathcal{D}}_n \subseteq \mathcal{D}_n \) of cardinality \( |\bar{\mathcal{D}}_n| = s_n \in \Gamma, \) such that

\[
c(f, \kappa_{\mathcal{M}_n}(\cdot, u_n)) \geq c(g, \kappa_{\mathcal{M}_n}(\cdot, u_n)) \text{ for any } f \in \bar{\mathcal{D}}_n \text{ and for any } g \in \mathcal{D}_n \setminus \bar{\mathcal{D}}_n, \text{ where}
\]

\[
c(f, \kappa_{\mathcal{M}_n}(\cdot, u_n)) = \frac{\|f(u_n)\|_{\mathcal{H}}^2}{\|f\|_{\mathcal{H}} \sqrt{\kappa_{\mathcal{M}_n}(u_n, u_n)}}. \tag{31}
\]
Algorithm 1 Online Nonlinear Estimation via Iterative $L^2$-Space Projections

**Requirement:** $(\lambda_n)_{n \in \mathbb{N}} \subset [\epsilon_1, 2 - \epsilon_2] \subset (0, 2)$, $\exists \epsilon_1, \epsilon_2 > 0$, $\rho \geq 0$ (hyperslab), $\gamma \in (0, 1)$ (regularization for $R_n$), $\gamma_{update} \geq 0$ (regularization for coefficient updates), $\delta \in [0, 1]$ (coherence), $\epsilon \geq 0$ (LNE), and $s_n \in \mathbb{N}$, $r_n$ (efficiency factor)

**Initialization:** $\varphi_0 := \theta, \mathcal{D}_0 = \emptyset$

**Output:** $\varphi_n(u_n) := \sum_{i=1}^{r_n} h_i, \tilde{f}^{(n)}(u_n)$

for $n \in \mathbb{N}$ do
- Receive $u_n \in \mathbb{R}^L$ and $d_n \in \mathbb{R}$
- Check if the novelty criterion is satisfied for a candidate function $f(u_n)$
  - Coherence computation \( \triangleright \) Proposition [3], or [26]
  - If Novelty criterion is satisfied then
    - Dictionary increment: $\mathcal{D}_n = \mathcal{D}_{n-1} \cup \{f_n\}$, $h_{r_n, n} = 0$
    - Select $s_n$ coefficients to update \( \triangleright \) Proposition [3], or [33]
    - Compute $R_n$ \( \triangleright \) Proposition [3], or [34]
    - Update $h_n$ \( \triangleright \) Proposition [3]
end if
end for

Let $\tilde{D}_n := \{f_1^{(n)}, f_2^{(n)}, \ldots, f_{s_n}^{(n)}\}$ without loss of generality. The update equation is then given in a parametric form as

$$
\tilde{h}_{n+1} = \tilde{h}_n + \lambda_n \operatorname{sgn}(e_n(u_n)) \max \left\{ \frac{|e_n(u_n)|}{\rho}, 0 \right\} \tilde{R}_n^{-1} \tilde{f}_n,
$$

(32)

where $\gamma_{update} \geq 0$ is the regularization parameter, $\tilde{h}_n := [h_1, h_2, \ldots, h_{s_n}]^T \in \mathbb{R}^{s_n}$ is the coefficient vector corresponding to the selected basis functions, $\tilde{f}_n := \tilde{f}_n(u_n) := [f_1^{(n)}(u_n), f_2^{(n)}(u_n), \ldots, f_{s_n}^{(n)}(u_n)]^T \in \mathbb{R}^{s_n}$, and $\tilde{R}_n$ is the submatrix of $R_n$ corresponding to the selected dictionary $\tilde{D}_n$.

It is straightforward to obtain $R_n$ by applying Proposition [3] when Analytical approach is employed. Otherwise, only the submatrix $\tilde{R}_n$ of $R_n$ is updated at time $n$ as

$$
\tilde{R}_n := \tilde{R}_{n-1} + \tilde{f}_n \tilde{f}_n^T,
$$

(33)
or, it is approximated by using Finite-sample approach as

$$
\tilde{R}_n \approx \frac{1}{l_n} \tilde{F}_n \tilde{F}_n^T,
$$

(34)

where $\tilde{F}_n := [\tilde{f}_n(u_{j_1}), \tilde{f}_n(u_{j_2}), \ldots, \tilde{f}_n(u_{j_{l_n}})]$. The proposed online learning algorithm, including the selective-update strategy and dictionary constructions, is summarized in Algorithm [1].

**Complexity:** We discuss the computational complexity in terms of the number of multiplications required at each iteration when the normalized Gaussian functions are used. Suppose that the coherence condition is employed. The coherence condition only requires $O(r_n)$ complexity whereas the ALD condition requires $O(r_n^2)$ complexity. Suppose also that we employ the selective-update strategy with the efficiency factor $s_n$, and that $\gamma_{update}$ and $\gamma_{coherence}$ are used for $l_n := s_n$. Table II summarizes the overall per-iteration complexity of the proposed algorithm, NLMS [14], KNLMS [30], KRLS-T [32], HYPASS [34], MKNLMS, and CHYPASS. Here, Analytical, Finite-sample, and Recursive in the table correspond respectively to Analytical, Finite-sample, and Recursive approaches presented in Section III-B. The complexity required for the inverse of an $s_n \times s_n$ matrix is denoted by $\text{inv}(s_n)$ in Table II. Figure 2 shows the evolutions of computational complexities of the algorithms for $L = 2$ and $s_n = 7$. The proposed algorithm is of linear order to the dictionary size as implied in Table II.

| Algorithm       | Computational complexities of the algorithms |
|-----------------|---------------------------------------------|
| NLMS            | $3L + 2 + (L + 6)r_n + 1$                  |
| KNLMS           | $(L + 4)r_n + 5^2$                         |
| KRLS-T          | $(L + 3)r_n + 2$                           |
| HYPASS          | $(L + 6)r_n + 2$                           |
| MKNLMS          | $(L + 5)r_n + 2$                           |
| Analytical      | $(L + 6)r_n + 2$                           |
| Finite-sample   | $(L + 10 + (L + 5)s_n + (L + 4)^2 s_n) r_n$ | $+ 2 + \text{inv}(s_n)$ |
| Recursive       | $(L + 11 + (L + 5)s_n + (L + 4)^2 s_n) r_n$ | $+ 2 + \text{inv}(s_n)$ |

IV. CONVERGENCE ANALYSIS

In this section, convergence analysis (together with monotone approximation) of the proposed algorithm is presented for the full-updating case; i.e., the case of $s_n = r_n$. (Note here that the analysis for $s_n < r_n$ is intractable [34].) Before presenting the analysis, we show how the proposed algorithm can be derived from APSM [52]. Let $\Theta_n : H \to [0, \infty), n \in \mathbb{N}$, be continuous convex functions and $K \subset H$ a nonempty closed convex subset. For an arbitrary $\phi_n \in H$, APSM [52] generates the sequence $(\phi_n)_{n \in \mathbb{N}} \subset K$ as

$$
\phi_{n+1} := \left\{ \begin{array}{ll}
P_K \left( \phi_n - \lambda_n \Theta_n'(\phi_n) \Theta_n(\phi_n) \right), & \text{if } \Theta_n'(\phi_n) \neq 0, \\
\phi_n, & \text{if } \Theta_n'(\phi_n) = 0.
\end{array} \right.
$$

(35)

where $\lambda_n \in [0, 2], n \in \mathbb{N}$, and $\Theta_n'(\phi_n)$ is a subgradient of $\Theta_n$ at $\phi_n$. Letting

$$
\Theta_n(\phi) := \| \phi - P_C(\phi) \|_H
$$

(36)
and $K := \mathcal{H}$ in APSM reproduces the proposed algorithm. More precisely, the metric of $\mathcal{H}$ is characterized by the auto-correlation matrix $R$ in the dictionary subspace (cf. Fact 1), and the proposed algorithm exploits the efficiently computable $R_n$ in lieu of $R$ (which is unavailable in practice). This means that the metric used is fairly close to that of $\mathcal{H}$ but it involves time variations. We therefore present our analysis based on the variable-metric version of APSM [4]. We first present a set of assumptions (see [34, Assumption 1] and [4, Assumption 2]).

Assumption 1. 1) Step-size condition: there exist $\epsilon_1, \epsilon_2 > 0$ such that $(\lambda_n)_{n\in\mathbb{N}} \subseteq [\epsilon_1, 2 - \epsilon_2] \subseteq (0, 2)$.
2) Boundedness of dictionary size: there exists some $N_0 \in \mathbb{N}$ such that $\mathcal{M}_n = \mathcal{M}_{N_0}$ for all $n \geq N_0$.
3) Data consistency: there exists some $M \in \mathcal{M}$ such that the metric used is fairly close to that of $\mathcal{H}$ in APSM reproduces the proposed algorithm.

Remark 1 (On Assumption 1). Assumption 1 is reasonable, because it is almost impossible to guarantee convergence in case the dictionary subspace keeps changing indefinitely. When the input space is compact and the coherence condition is used to construct the dictionary, for instance, the dictionary size remains finite as the time index goes to infinity [30].

Remark 2 (On Assumption 2). The assumption requires that there exists a small open ball in $\bigcap_{n \geq N_1} C_n$ with respect to $\mathcal{M}_{N_0}$. In an ideal case where the noise $\nu$ is zero and $\psi \in \mathcal{M}_{N_0}$, it is clear that $\nu \in C_n$ for all $n \geq N_0$, because $|\psi(u_n) - d_n| = |\psi(u_n) - \psi(u_n)| = 0 \leq \rho$ in [79] for any $\rho \geq 0$. Since the evaluation functional over an RKHS is linear; continuous, and hence bounded [57, page 9 - Theorem 1], there exists a constant $M_1 > 0$ such that $|\psi(u_n)| \leq M_1$. Therefore, it follows that $|\psi(u_n)| \leq \mathcal{M}_n \mathcal{M}_{N_0}$. Therefore, it follows that $|\psi(u_n)| \leq M_1 \mathcal{M}_{N_0}$. Then, for any $\rho > 0$, $B_{\rho} := \{f \in \mathcal{M}_{N_0} \mid \|f \psi\| \leq \rho\} \subseteq \mathcal{M}_{N_0}$ for all $n \geq N_0$. $\epsilon_3 := \frac{\rho}{M_1}$, because $\|f \psi\| \leq \epsilon_3$ implies $|\psi(u_n)| \leq \epsilon_3$.

In the general case where $\nu \neq 0$ and/or $\psi \notin \mathcal{M}_{N_0}$, it is necessary that $|\nu_n| \leq M_2$, $\psi_{\mathcal{M}_{N_0}}(u_n) \leq M_3$ for some constants $M_2, M_3 \in (0, \infty)$, and $\psi_{\mathcal{M}_{N_0}}(u_n)$ is in the dictionary subspace because $\mathcal{M}_{N_0} = P_{\mathcal{M}_{N_0}}(\psi)$. Then, for any $\rho > M_2 + M_3$, $B_{\rho} := \{f \in \mathcal{M}_{N_0} \mid \|f \psi\| \leq \rho\} \subseteq \mathcal{M}_{N_0}$ for $\epsilon_3 := \frac{\rho}{M_2 + M_3}$, because $\|f \psi\| \leq \epsilon_3$ implies $|\psi(u_n)| \leq \epsilon_3$.

Remark 3 (On Assumption 2). For Analytical approach, the metric is fixed (i.e., $E_n = O$) after the time instant $n = N_0$ due to Assumption 1. Then, the approximation becomes tight as $n$ increases under Assumption 1.2, and thus the assumption is reasonable.

Now we are ready to prove the following theorem.

Theorem 2. The sequence $(\phi_n)_{n\in\mathbb{N}}$, or $(h_n)_{n\in\mathbb{N}}$, generated by Algorithm 1 satisfies the following properties.
(a) Monotone approximation:
For any $h_n^* \in \{h \in \mathbb{R}^n \mid \sum_{i=1}^n h_i \phi_i \in C_n\}$, it holds that $\|h_n - h_n^*\|^2_{R_n} = \|h_n + \epsilon^2\|^2_{R_n}$. (38)

(b) Convergence and asymptotic optimality:
The sequence $(\phi_n)_{n\in\mathbb{N}}$ converges to some point $\phi \in H$, and $\lim_{n \to \infty} \Theta_n(\phi_n) = \lim_{n \to \infty} \Theta_n(\phi) = 0$ if Analytical approach is employed under Assumptions 1.1 - 1.3, or if Finite-sample/Recursive approach is employed under Assumptions 4.1 - 4.5.

Proof. (a) The claim is verified by [52, Theorem 2(a)]. Note that the analysis of APSM is directly applied to the dictionary subspace because Algorithm 1 updates the current estimate within the dictionary subspace at each time instant.

(b) For Analytical approach, the argument in [34, Theorem 2(a)] can be applied by considering the $L^2$ space $H$ instead of an RKHS. For Finite-sample/Recursive approach, the same argument of the variable-metric APSM [4, Theorem 1(c)] can be applied by considering the fixed dictionary subspace after the dictionary has been well constructed. Specifically, [4, Assumption 1] is validated by Assumptions 4.1, 4.2, and 4.3, and [4, Assumption 2] is validated by Assumptions 4.4 and 4.5 to apply [4, Theorem 1(c)].
target functions. Since the primary focus of the present study is an online learning for possibly time-varying target functions, analyzing the convergence rate is out of the scope. The interested readers are referred to the detailed analysis of APSM [52], which gives the bound of how close the estimate will get to an optimal point at each iteration.

V. NUMERICAL EXAMPLES

We first show the decorrelation property of the proposed algorithm. We then show the efficacy of the proposed algorithm in applications to online predictions of two real datasets. The kernel adaptive filtering toolbox [64] is used in the experiment. Throughout the experiments, the set of dictionary data is used to compute the sample average for Finite-sample approach.

A. Decorrelation Property

We compare the eigenvalue spreads of the modified autocorrelation matrices of the proposed algorithm and the existing multikernel adaptive filtering algorithms, namely MKNLMS and CHYPASS. Input vectors are drawn from the i.i.d uniform distribution within $[-1,1] \subset \mathbb{R}$, and Gaussian functions with scale parameters $\sigma_1 := 1.0$, $\sigma_2 := 0.5$, and $\sigma_3 := 0.05$ are employed (see Section II-C). Dictionary is constructed by the sole use of the coherence condition (i.e., $\delta = 0.8$). For meaningful comparison, all the algorithms share the same dictionary which is constructed based on the coherence condition defined in the Cartesian product of Gaussian RKHSs (see [46]).

To avoid numerical errors in computing matrix inverses, the metric matrix $G_n$ is approximated as $G_n := \gamma G_n + (1 - \gamma) I$, $\gamma := 0.99$. The modified autocorrelation matrix $R$ is then computed as $\hat{R}_n := \hat{G}_n^{-\frac{1}{2}} R \hat{G}_n^{-\frac{1}{2}}$, where $R$ is approximated as $R \approx \frac{1}{\gamma} \sum_{n=1}^{N} f_n f_n^T$, $N := 10000$ at each iteration (see the arguments below Fact 1 in Section III-A). Figure 3 plots the evolutions of the eigenvalue spreads of $\hat{R}$ for each algorithm. One can see that the proposed algorithm attains a smaller eigenvalue spread of $\hat{R}$, having a better decorrelation property.

For Analytical approach, it works relatively well despite the use of (possibly inappropriate) noninformative distribution for the input vector. Although Recursive approach shows degradations during the initial phase when the dictionary size increases rapidly, the eigenvalue spread tends to decrease successfully as the iteration number increases. Finite-sample approach shows stable performance at the expense of high computational complexity of $O(\tau_n^2)$. In practice, one may use the selective-update strategy to reduce the complexity (see Section III-D). For further clarification, $R_n$s for MKNLMS, CHYPASS, and the proposed algorithm (Analytical approach) are illustrated in Figure 4. Here, “jet colormap array” in MATLAB_R2017b is used for the illustrations. In particular, we can observe that the off-diagonal elements of $\hat{R}_n$ for the proposed algorithm are suppressed better than the other algorithms, as supported quantitatively by Figure 3.

B. Online prediction of electrical power output

We consider the online prediction of electrical power output analyzed in [53], [54]. The target variable, namely the full load electrical power output, depends highly on ambient temperature (AT). Because AT is strongly correlated with the target variable and can individually predict the target variable [53], [54], AT is employed as a sole input variable in the present experiment for the comparison purpose. In [53], [54], different machine learning regression methods are compared to each other in terms of the root mean squared error (RMSE). The same dataset and problem settings are used to compare the RMSE performance of the proposed algorithm with linear NLMS, KNLMS, HYPASS, CHYPASS, and the machine learning regression methods analyzed in [53], [54].

Note here that the proposed algorithm is designed for online learning, while those analyzed in [53], [54] are batch methods. It is observed in [53], [54], that AT affects the target variable more than the other variables, and that the model trees rules (M5R) achieves the lowest RMSE 5.085 among 15 machine learning regression methods.

Following [53], [54], $5 \times 2$ cross-validation is employed, i.e., datasets are equally partitioned into two sets with the same size and each of the sets is trained to validate the other (2-fold cross-validation), and it is repeated five times by shuffling the datasets. For the comparison purposes, the same five shuffled datasets as those in [53], [54] are used. The RMSEs over the test set for $5 \times 2 = 10$ runs are then averaged to obtain the final results. Note that the estimator is trained in an online fashion with the first half of the dataset, and the trained estimator is applied to the other half. To choose the best parameters for each algorithm, we first use a coarse search to find rough regions of good parameters, and then exploit a fine random...
search [65] of 100 combinations to find the best parameters achieving the best RMSE averaged over the 10 runs. For the nonlinear estimators, Gaussian functions are employed with fixed scale parameters because an advantage of using multiple Gaussian functions is that no elaborative parameter tuning is needed. For the monokernel methods, the best scale parameter \( \sigma \) is selected.

Table III summarizes the parameter settings and the means and standard deviations of RMSEs over the 10 runs including those of the batch methods. It is observed that Finite-sample approach achieves lower RMSE than the batch methods excluding the top-two methods (M5R and the model trees regression). Moreover, it can be observed that the use of multiple Gaussian functions leads to significantly better results than their monokernel counterparts. The normalized MSE (NMSE) learning curves averaged over the 5 \( \times \) 2 runs are smoothed and plotted in Figure 5(a), and the evolutions of dictionary size are plotted in Figure 5(b). Figure 5(c) shows an instance of the estimate of each algorithm over the test set of the final run and the target values for 235 input data (AT) selected from the test set of the final run.

C. Online Prediction of GPS Measurements

We use the real trajectory data of GPS positions, the dynamics of a true vehicle, and the simulated pseudo range...
functions are employed with fixed scale parameters. For the monokernel methods, the best scale parameter $\sigma$ is selected. The coherence threshold $\delta$ is tuned so that the final dictionary sizes become the same among HYPASS, CHYPASS and the proposed algorithm. The regularization parameter $\gamma_{\text{update}}$ is tuned carefully only for KRLS-T because of sensitivity.

Table IV summarizes the parameter settings. Here, $M$, $\xi$, $\gamma_{\text{update}}$ are the budget, the forgetting factor, and the regularization parameter for KRLS-T, respectively. The MSE learning curves are plotted in Figure 6(a), and the evolutions of dictionary size are plotted in Figure 6(b). It can be observed that HYPASS, CHYPASS, and the proposed algorithm outperform EKF despite the use of less information. Finite-sample approach performs worse than Analytical/Recursive approach because of the small dictionary size.

VI. CONCLUSION

The online learning paradigm presented in this paper is a significant extension of the conventional kernel adaptive filtering framework from RKHS to the space $L^2(\mathbb{R}^L, d\mu)$ which has no reproducing kernel and which induces the best geometry in the sense of decorrelation. The proposed algorithm was built upon the fact that the reproducing kernel of the dictionary subspace can be obtained in terms of the Gram matrix. Three approaches to computing the Gram matrix were presented. A remarkable difference from kernel adaptive filtering is that the whole space $L^2(\mathbb{R}^L, d\mu)$ has no reproducing kernel. In $L^2(\mathbb{R}^L, d\mu)$, the MSE estimator gives the best approximation of the target nonlinear function in the dictionary subspace in contrast to the case of kernel adaptive filtering. Also, the ALD condition in $L^2(\mathbb{R}^L, d\mu)$ ensures a lower bound of the amount of the MMSE reduction due to the newly entering atom. The selective-update strategy was presented to reduce the computational complexity. The analysis was presented to show the monotone approximation, asymptotic optimality, and convergence of the proposed algorithm for the full-updating case. The numerical examples demonstrated the efficacy of the proposed algorithm using the selective-update strategy.
for two real datasets, showing its superior performance to the extended Kalman filter and comparable performance with the best batch machine-learning method that was tested. We finally emphasize that the proposed paradigm can be extended straightforwardly to any other functional spaces as long as the Gram matrix can be computed efficiently.

**APPENDIX A**

**PROOF OF PROPOSITION [1]**

Let \( P_{M_n}(\psi) := \sum_{i=1}^{r_n} h_i f_i(\alpha_n) \), \( h_i \in \mathbb{R} \), then the coefficient vector \( \mathbf{h} := [h_1, h_2, \cdots, h_{r_n}]^T \in \mathbb{R}^{r_n} \) is characterized by the following normal equation [58]:

\[
R\mathbf{h} = \mathbf{b}, \tag{A.1}
\]

where \( R \) is the Gram matrix of the dictionary (see Fact [1]), and \( \mathbf{b} := [(f_1, \psi)_H, (f_2, \psi)_H, \cdots, (f_{r_n}, \psi)_H]^T \in \mathbb{R}^{r_n} \). Here, it holds that \( \mathbf{b} = p(= E [f_n(u_n)d_n]) \) because

\[
\begin{align*}
E[f_i(u_n)d_n] &= E[f_i(u_n)(\psi(u_n) + \nu_n)] \\
&= E[f_i(u_n)\psi(u_n)] + E[f_i(u_n)\nu_n] \\
&= E[f_i(u_n)\psi(u_n)] + 0 \\
&= \int_{\mathbb{R}^L} f_i(\mathbf{u})\psi(\mathbf{u}) p(\mathbf{u}) d\mathbf{u} = (f_i, \psi)_H. \tag{A.2}
\end{align*}
\]

Hence, (A.1) is equivalent to \( R\mathbf{h} = \mathbf{p} \), which is nothing but the Wiener-Hopf equation derived directly from (11) to obtain the MMSE estimator.

**APPENDIX B**

**PROOF OF PROPOSITION [2]**

It is clear that \( \kappa(\cdot, \mathbf{u}) \in \text{span} \mathcal{D} \) for any \( \mathbf{u} \in \mathbb{R}^L \). By definition of \( (\cdot, \cdot)_H \), it can be readily verified that

\[
\begin{align*}
\langle \kappa(\cdot, \mathbf{u}), \kappa(\cdot, \mathbf{v}) \rangle_H &= \int_{\mathbb{R}^L} f^T(\mathbf{u})G^{-1} f(\mathbf{w}) G^{-1} f(\mathbf{v}) d\mu(\mathbf{w}) \\
&= f^T(\mathbf{u})G^{-1} \int_{\mathbb{R}^L} f(\mathbf{w}) G^{-1} f(\mathbf{v}) d\mu(\mathbf{w}) \\
&= \kappa(\mathbf{u}, \mathbf{v}). \tag{B.1}
\end{align*}
\]

For any \( \mathbf{u} \in \mathbb{R}^L \) and \( \phi := \sum_{i=1}^{r} \alpha_i f_i, \alpha_i \in \mathbb{R} \), the reproducing property holds:

\[
\begin{align*}
\langle \phi, \kappa(\cdot, \mathbf{u}) \rangle_H &= \int_{\mathbb{R}^L} \phi(\mathbf{w}) f^T(\mathbf{w}) G^{-1} f(\mathbf{v}) d\mu(\mathbf{w}) \\
&= \sum_{i=1}^{r} \alpha_i \int_{\mathbb{R}^L} f_i(\mathbf{w}) f^T(\mathbf{v}) d\mu(\mathbf{w}) G^{-1} f(\mathbf{v}) \\
&= \sum_{i=1}^{r} \alpha_i \mathbf{e}_i^T f(\mathbf{u}) = \phi(\mathbf{u}), \tag{B.2}
\end{align*}
\]

where \( \{\mathbf{e}_i\}_{i=1}^{r} \) is the standard basis of \( \mathbb{R}^r \).

**APPENDIX C**

**PROOF OF PROPOSITION [3]**

(a) The inner product can be computed as follows:

\[
\langle \kappa_p(\cdot, \mathbf{u}), \kappa_q(\cdot, \mathbf{v}) \rangle_H = \frac{1}{(2\pi\sigma_p^2)^{L/2}} \frac{1}{(2\pi\sigma_q^2)^{L/2}} \int \exp \left[ -\left( \frac{\|\mathbf{u} - \mathbf{w}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{v} - \mathbf{w}\|_2^2}{2\sigma_q^2} + \frac{\|\mathbf{w}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{w}\|_2^2}{2\sigma_q^2} \right) \right] d\mathbf{w}. \tag{C.1}
\]

Here, \( A(w) = \frac{\|\mathbf{u}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{v}\|_2^2}{2\sigma_q^2} + \frac{\|\mathbf{w}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{w}\|_2^2}{2\sigma_q^2} \) and \( \alpha := \frac{1}{\sigma_p^2} + \frac{1}{\sigma_q^2} > 0 \), and \( z := \frac{\mathbf{u}}{\sigma_p^2} + \frac{\mathbf{v}}{\sigma_q^2} \). From it follows that

\[
\int \exp(-A(w)) d\mathbf{w} = \frac{1}{(2\pi\alpha)^{L/2}} \int \exp \left( -\left( \frac{\|\mathbf{z}\|_2^2}{2\alpha} + \frac{\|\mathbf{u}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{v}\|_2^2}{2\sigma_q^2} \right) \right) d\mathbf{w}. \tag{C.2}
\]

(b) Since \( d\mu(\mathbf{u}) = d\mathbf{u} \), it follows that

\[
\langle \kappa_p(\cdot, \mathbf{u}), \kappa_q(\cdot, \mathbf{v}) \rangle_H = \frac{1}{(2\pi\sigma_p^2)^{L/2}} \frac{1}{(2\pi\sigma_q^2)^{L/2}} \int \exp \left[ -\left( \frac{\|\mathbf{u} - \mathbf{v}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{v} - \mathbf{w}\|_2^2}{2\sigma_q^2} \right) \right] d\mathbf{w}. \tag{C.4}
\]

Here, \( C(w) = \frac{\|\mathbf{u}\|_2^2}{2\sigma_p^2} + \frac{\|\mathbf{v}\|_2^2}{2\sigma_q^2} + \frac{2(\mathbf{u}, \mathbf{v})_H}{\sigma_p^2 \sigma_q^2} \).
\[ \beta \left( \left\| w \right\|_{L^2}^2 - 2 \left( \frac{\beta}{\beta - 1}, w \right)_{L^2} \right), \]
where \( \beta := \frac{1}{\sigma^2} + \frac{1}{\sigma^2} > 0, \) and \( z := \frac{u_n}{\sigma^2} + \frac{v_n}{\sigma^2}. \) Therefore, (C.4) becomes
\[ \langle \kappa_p(\cdot, u), \kappa_q(\cdot, v) \rangle_H \]
\[ = \frac{1}{(2\pi \sigma^2)^L/2} \frac{1}{(2\pi \sigma^2)^L/2} \int \exp \left( - \frac{1}{2\beta} \left\| w - \beta \right\|_{L^2}^2 \right) \, dw \exp \left\{ - \frac{1}{2\beta} \left[ \left\| z \right\|_{L^2}^2 + \left\| u \right\|_{L^2}^2 + \left\| v \right\|_{L^2}^2 \right] \right\} \]
\[ = \frac{1}{(2\sigma^2 \sigma^2 + \sigma^2 + \sigma^2)^L/2} \exp \left( - \frac{1}{2\beta} \left\| u - v \right\|_{L^2}^2 \right). \] (C.5)

We mention that the result in (20) is also obtained in the Gaussian RKHS by taking the limit of its scale parameter towards zero in [69, Equation (25)] because Gaussian RKHSs have a nested structure [70], [71].

**APPENDIX D**

**PROOF OF PROPOSITION 1**

By the independence assumptions and the definition of \( \left\| \cdot \right\|_H \), we have
\[ E \left[ d_n - \psi^*_{M_n}(u_n) \right]^2 = E \left[ \psi(u_n) - \psi^*_{M_n}(u_n) \right]^2 + E(v_n^2) \]
\[ = \left\| \psi - \psi^*_{M_n} \right\|_H^2 + E(v_n^2). \] (D.1)
\[ E \left[ d_n - \psi^*_{M_{n+1}}(u_n) \right]^2 = \left\| \psi - \psi^*_{M_{n+1}} \right\|_H^2 + E(v_n^2). \] (D.2)

By Pythagorean theorem and the assumed ALD condition, it follows that
\[ \Delta \text{MMSE} = \left\| \psi - \psi^*_{M_n} - \psi^*_{M_{n+1}} \right\|_H^2 \]
\[ = \left\| \psi^*_{M_{n+1}} - \psi^*_{M_{n}} \right\|_H^2 + \left\| \psi^*_{M_{n+1}} - \psi_{M_{n+1}} \right\|_H^2 \]
\[ = \left\| \psi^*_{M_{n+1}} - \psi_{M_{n+1}} \right\|_H^2 + \left\| \psi_{M_{n+1}} - \psi_{M_{n+1}} \right\|_H^2 \]
\[ = \left\| h_{u_n}^* f_{n} - P_{M_{n+1}}(h_{u_n}^* f_{n}) \right\|_H^2 \]
\[ = \left\| h_{u_n}^* f_{n} - P_{M_{n+1}}(f_{n}) \right\|_H^2 \geq \frac{\left( h_{u_n}^* \right)^2 \left\| f_{n} - P_{M_{n+1}}(f_{n}) \right\|_H^2}{\eta} \] (D.3)

**APPENDIX E**

**RLS AS ITERATIVE VARIABLE-METRIC PROJECTION METHOD**

We first write down a variant of RLS for reference:
\[ x_{n+1} = x_n + \lambda_n \frac{d_n - u_n^T x_n}{u_n^T R_n^{-1} u_n} R_n^{-1} u_n, \] (E.1)
where \( x_n \in \mathbb{R}^L \) is the coefficient vector, \( R_n = R_{n-1} + u_n u_n^T \) and \( \lambda_n = \frac{u_n^T R_n^{-1} u_n}{u_n^T R_n^{-1} u_n + 1} \). Although RLS in (E.1) iteratively minimizes
\[ J(h) = \sum_{i=1}^{n} \left( d_i - u_i^T h \right)^2, \] (E.2)

it can also be viewed as a variable-metric projection with the time-varying step size \( \lambda_n \) under the framework of [4] as pointed out in [62].

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Motoya Ohnishi (S’15) received the B.S. degree in Electronics and Electrical Engineering from Keio University, Tokyo, Japan, in 2016. He is currently working toward the M.S. degrees both in Electronics and Electrical Engineering from Keio University, Tokyo, Japan, and Electrical Engineering from KTH Royal Institute of Technology, Stockholm, Sweden. He was a research assistant at the Department of Automatic Control at KTH Royal Institute of Technology, and was a visiting researcher at GRITSlab at Georgia Institute of Technology, Atlanta, USA, in 2017, and is currently a research assistant at RIKEN AIP center, Tokyo, Japan. His research interests include mathematical signal processing, machine learning, and robotics.
Masahiro Yukawa (S’05–M’06) received the B.E., M.E., and Ph.D. degrees from Tokyo Institute of Technology in 2002, 2004, and 2006, respectively. He studied as Visiting/Guest Researcher with the University of York, U.K., for half a year, and with the Technical University of Munich, Germany, for four months. He worked with RIKEN, Japan, as Special Postdoctoral Researcher for three years, and with Niigata University, Japan, as Associate Professor for another three years. In 2016, he studied with Machine Learning Group of the Technical University of Berlin as Visiting Professor. He is currently an Associate Professor with the Department of Electronics and Electrical Engineering, Keio University, Japan. He has been Associate Editor for the IEEE TRANSACTIONS ON SIGNAL PROCESSING (since 2015), Multidimensional Systems and Signal Processing (2012–2016), and the IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences (2009–2013). His research interests include mathematical adaptive signal processing, convex/sparse optimization, and machine learning. Dr. Yukawa was a recipient of the Research Fellowship of the Japan Society for the Promotion of Science (JSPS) from April 2005 to March 2007. He received the Excellent Paper Award and the Young Researcher Award from the IEICE in 2006 and in 2010, respectively, the Yasujiro Niwa Outstanding Paper Award in 2007, the Ericsson Young Scientist Award in 2009, the TELECOM System Technology Award in 2014, the Young Scientists’ Prize, the Commendation for Science and Technology by the Minister of Education, Culture, Sports, Science and Technology in 2014, the KDDI Foundation Research Award in 2015, and the FFIT Academic Award in 2016. He is a member of the IEICE.