A stationary heat conduction problem

Piotr Rybka* and Anna Zatorska-Goldstein†

Institute of Applied Mathematics and Mechanics
University of Warsaw
ul. Banacha 2, 02-097 Warsaw, Poland

October 12, 2018

Abstract

We study a basic linear elliptic equation on a lower dimensional rectifiable set $S$ in $\mathbb{R}^N$ with the Neumann boundary data. Set $S$ is a support of a finite Borel measure $\mu$. We will use the measure theoretic tools to interpret the equation and the Neumann boundary condition. For this purpose we recall the Sobolev-type space dependent on the measure $\mu$. We establish existence and uniqueness of weak solutions provided that an appropriate source term is given.

Key words and phrases: Neumann boundary problem, multijunction measures
Mathematics Subject Classification (2010): 35J20, 35J70, 28A33.

1 Introduction

We study here an old problem of determining the stationary heat distribution in a conductor $S$ in the ambient space $\Omega \subset \mathbb{R}^N$, when the conductivity tensor $A$ and the heat sources $Q$ are given. We assume that the conductor $S$ is insulated at the boundary of $\Omega$. The main objective of the paper is to investigate the case of "thin" $S$, i.e. the case when the set is a low dimensional structure. Very roughly, our problem may be written as,

$$\begin{align*}
\text{div} \ (A\nabla u) + Q &= 0 \quad \text{in } S, \\
[A\nabla u, \nu]|_{\partial \Omega} &= 0 \quad \text{on } S \cap \partial \Omega,
\end{align*}$$

(1.1)

where $\nu$ is the outward normal unit vector of $\partial \Omega$. Although the system of equations looks familiar the problem becomes non-standard, when $S$ is a low dimensional rectifiable set. To give proper meaning to these equations, we follow the approach proposed by Bouchitté, Buttazzo and

*P.Rybka@mimuw.edu.pl
†A.Zatorska-Goldstein@mimuw.edu.pl
Seppecher in [4] and we consider a measure \( \mu \), supported in \( S \), which is singular with respect to the Lebesgue measure in \( \mathbb{R}^N \). We assume further that \( Q \) and \( A \) are, respectively, a scalar and a tensor valued measure; both measures are absolutely continuous with respect to \( \mu \). Such measure-oriented point of view proved to be very fruitful when dealing with variational problems considered on low dimensional structures in \( \mathbb{R}^N \), for an example see the anisotropic shape optimization problem considered by Bouchitté and Buttazzo in [2]. We refer the reader to [6] for an introduction to the theory. The novelty of our paper is to consider the Neumann boundary condition (1.1); this will be explained later in more detail.

We employ Sobolev-type space \( H^{1,p}_\mu \) introduced by Bouchitté and Buttazzo, [4]. Its definition and basic properties are discussed in Section 2. In particular, we explain there the notion of the tangential gradient operator \( \nabla_\mu \) for functions from \( H^{1,p}_\mu \) and the notion of the tangent space of \( \mu \) at \( x \), denoted by \( T_\mu(x) \).

A natural way to handle the question of existence of solutions to (1.1) is to use a variational approach. One may expect (1.1) to be the Euler-Lagrange equation of the natural energy functional

\[
E(u) = \frac{1}{2} \int_\Omega (A(x) \nabla u, \nabla u) \, d\mu - \langle Q, u \rangle, \quad u \in C^\infty_c(\mathbb{R}^N).
\] (1.2)

Energy \( E \) is well-defined for smooth functions, but this space is not suitable from the point of view of the calculus of variations. As in [4], the relaxation of \( E \), denoted by \( E_\mu \), may be calculated explicitly, see (Section 3, Proposition 3.1). It is defined as

\[
E_\mu(u) = \frac{1}{2} \int_\Omega (A_\mu(x) \nabla_\mu u, \nabla_\mu u) \, d\mu - \langle Q, u \rangle, \quad u \in H^{1,2}_\mu,
\]

where \( A_\mu \) is given by (3.4).

As for the measure \( \mu \), we assume it to be a multijunction measure in the sense considered by Bouchitté and Fragalà in [7]. However, we point out that, contrary to the approach in the papers [5] and [7], we do not assume a Poincaré-type inequality to hold on \( S \), allowing in particular occurrence of multidimensional junctions. The issue of the Poincaré inequality is addressed further in Section 2 and examples are given in Section 5.

Our precise assumptions, denoted further by [S], are as follows:

- \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \);
- a compact set \( S \) contained in \( \overline{\Omega} \) is of the form
  \[
  \text{supp} \, \mu = S = \bigcup_{j=1}^J S_j,
  \]
  where each \( S_j \) is a compact manifold of class \( C^2 \) of dimension \( k_j < N \);
- the measure \( \mu \) is of the form
  \[
  \mu = \sum_{j=1}^J \mu_j, \quad \text{where} \quad \mu_j := \mathcal{H}^{k_j}_\mu S_j, \quad j = 1, \ldots J;
  \]
• these measures are mutually singular, i.e.
  \[ \mu_j(S_i) = 0 \quad \text{for all } j \neq i; \]

• for each \( i = 1, \ldots, J \), we assume that the boundary of the manifold \( S_i \) is contained in \( \partial \Omega \cup \bigcup_{j \neq i} \bar{S}_j \);

• the relative interior of \( S_i \) does not intersect the boundary \( \partial \Omega \).

The ambient space \( \Omega \) is, in fact, a design region restricting the position of set \( S \). Remark 4.1 at the end of Section 4 addresses the lack of influence of \( \Omega \) on the solutions to (1.1).

We employ the following definition:

Definition 1.1. Let us assume that the conditions [S] on measure \( \mu \), and (3.1) on measure \( Q \) hold. We say that a function \( u \in H^{1,2}_\mu \) is a distributional solution to the Neumann boundary problem (1.1) if

\[
\int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu \varphi) \, d\mu - \langle Q, \varphi \rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N). \tag{1.3}
\]

We stress that the meaning of the boundary condition (1.1) requires clarification. First of all, we should consider \( A_\mu \nabla_\mu u \), not \( A \nabla_\mu u \). Secondly, the way we defined the weak solutions implies that \( [A_\mu \nabla_\mu u, \nu] \) is the zero distribution over \( \partial \Omega \), i.e. \( [A_\mu \nabla_\mu u, \nu] \in \mathcal{D}(\partial \Omega) \). We will show that this a nicer object, but it cannot be considered in a pointwise manner.

To put it differently, \( A_\mu \nabla_\mu u \) is at best an element of \( L^2(\Omega, \mu) \). However, when \( u \) is a minimizer of \( E_\mu \), we can interpret \( \text{div} A_\mu \nabla_\mu u \) as a measure, which permits us to use the theory developed by Chen and Frid, [8]. In this way we are able to interpret \( [A_\mu \nabla_\mu u, \nu] \) as a normal trace of a measure with bounded divergence. The issue of boundary condition is discussed in Section 4.

As for the source term \( Q \), in order to ensure solvability of (1.1), it is clear that \( Q \) must be perpendicular to the kernel of \( \text{div} A_\mu \nabla_\mu \). The source term \( Q \) must be also somehow subordinate to \( \mu \). The first guess might be that \( Q \in (H^{1,2}_\mu)^* \). However, as explained in Section 4, the interpretation of the boundary condition requires \( Q \) that be a measure of a finite total variation.

Our main result is then stated in the following theorem:

**Theorem 1.1.** Let us assume that the conditions [S] on measure \( \mu \), and (3.1) on measure \( Q \) hold. In addition, the function \( A : S \to M(N \times N) \) takes values in symmetric, non-negative matrices satisfying (3.3). Moreover, \( \langle Q, h \rangle = 0 \) for all \( h \) in the kernel of \( \nabla_\mu \). Then there exists a unique distributional solution to (1.1) which is perpendicular (in the \( L^2 \) scalar product) to \( \ker \nabla_\mu \). Moreover, the solution satisfies the boundary condition in the following sense: \( [A_\mu \nabla_\mu u, \nu] = 0 \) as a continuous linear functional on \( C^2(\partial \Omega) \).

This Theorem follows from Theorem 3.1 in Section 3 and Theorem 4.1 in Section 4. Uniqueness is addressed in Corollary 3.1. Examples of explicit boundary problems are discussed in Section 5.
2 Sobolev spaces $H^{1,p}_\mu$

2.1 Definitions

In the problem we study, $\Omega$ is the ambient space, containing $S$, which is the support of a measure $\mu$. Set $S$ interpreted as the heat conductor. We study an elliptic problem in $S$, however, we are interested in the Neumann-type boundary conditions on $S \cap \partial \Omega$.

We need properly defined Sobolev spaces to study weak solvability of differential equations like (1.1). We require a definition, which is general, yet easy to use. It seems that the Sobolev space $H^{1,p}_\mu$ introduced for any $p \in [1, \infty)$ by Bouchitte-Buttazzo-Seppecher in [4] is the right tool. We will briefly recall this definition. A recent discussion on other possible definitions of Sobolev space on $S$ is in [9].

We begin with the notion of a tangent space to a measure, there is a number of ways to introduce it. We use the approach presented in [4] and further exposed in [6], [10] in a more reader friendly way.

We assume that $\mu$ is a positive, Radon measure on $\mathbb{R}^N$. If we set $N_\mu = \{ v \in C^\infty_c(\mathbb{R}^N) : v = 0 \text{ in supp } \mu \}$, then we introduce $N_\mu : \mathbb{R}^N \to \mathbb{R}^N$ by

$$N_\mu(x) := \{ w(x) \in \mathbb{R}^N : \exists v \in N_\mu, w = \nabla v \text{ in supp } \mu \}.$$ 

We define the multifunction $N_\mu : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ by

$$N_\mu(x) := \{ w(x) \in \mathbb{R}^N : w \in N_\mu \}.$$ 

The vector subspace of $\mathbb{R}^N$ is called the normal space to $\mu$ at $x$. The vector space $T_\mu(x) \subset \mathbb{R}^N$ is defined as the orthogonal complement of $N_\mu(x)$ and it is called the tangent space to $\mu$ at $x$.

We introduce $\Pi_\mu(x, \cdot)$ as the orthogonal projection of $\mathbb{R}^N$ onto $T_\mu(x)$. Due to measurability of $T_\mu(\cdot)$, see [4], the projection is $\mu$-measurable.

For any $u \in D(\mathbb{R}^n)$, we define, for $\mu$-a.e. $x \in \mathbb{R}^n$, the tangential gradient

$$\nabla_\mu u(x) = \Pi_\mu(x, \nabla u(x)).$$

The space $H^{1,p}_\mu$ is defined as a completion of $D(\mathbb{R}^n)$ in the following norm

$$\|u\|_{1,p,\mu} = \left( \|u\|_{L^p(\Omega)}^2 + \|\nabla_\mu u\|_{L^p(\Omega)}^2 \right)^{1/2}.$$ 

It turns out to be a reflexive Banach space for $p \in (1, \infty)$. We should stress that we could define the Sobolev space by weak derivatives, this is done in [4]. This space is denoted by $W^{1,p}_\mu$, in general $H^{1,p}_\mu$ is closed subspace of $W^{1,p}_\mu$, see [3] for details.

The weak convergence on space $H^{1,p}_\mu$ is introduced in a natural way,

$$u_k \rightharpoonup u \text{ weakly in } H^{1,p}_\mu \iff \begin{cases} u_k \rightharpoonup u \text{ weakly in } L^p(\Omega), \\ \nabla_\mu u_k \rightharpoonup \nabla_\mu u \text{ weakly in } L^p(\Omega). \end{cases}$$
In a typical situation we deal with, described in the Introduction, i.e. $\mu$ being supported on a finite union of smooth compact manifolds and agreeing locally with the natural Hausdorff measure, the definitions of the tangent space and the Sobolev space $H^{1,p}_\mu$ are intuitive.

**Proposition 2.1.** (see [4, Corollary 5.4]) Let $S$ be a $C^2$ compact manifold in $\mathbb{R}^N$ of dimension $k \leq N$, let $T_S(x)$ be the tangent space at every $x \in S$, let $\mu = H^k \llcorner S$. Then, for every $p \in [1, +\infty)$ we have

$$T_{\mu}(x) = T_S(x) \quad \mu - a.e.$$

**Proposition 2.2.** (see [7, Lemma 2.2]) Assume that $S$ is a finite union of $C^2$ compact manifolds, $S = \bigcup_{j=1}^J S_j$, $\dim S_j = k_j \leq N$ and the measures $\mu_j = H^{k_j} \llcorner S_j$ are mutually singular. We set $\mu = \sum_{j=1}^J \mu_j$ and we denote by $T_{S_j}(x)$ the tangent space to $S_j$ at $x$. Then, for every $p \in [1, +\infty)$ we have

$$T_{\mu}(x) = T_{S_j}(x) \quad \mu_j - a.e.$$

If $S = \text{supp} \mu \subset \overline{\Omega}$ and $\Omega$ is an open domain in $\mathbb{R}^N$ and $\mu(S \cap \partial \Omega) = 0$, then

$$H^{1,p}_\mu = \{u \in L^p(S, \mu) : u|_{S_j} \in H^{1,p}(S_j), j = 1, \ldots, J\}.$$

### 2.2 On validity of the Poincaré inequality

The important tool in the analysis of the well posedness of the problem (1.1) is the Poincaré inequality. In case $S$ is a smooth, compact manifold of dimension $k < N$, the global Poincaré inequality holds, i.e. there is $C_p > 0$ such that

$$\|u - u_S\|_{L^2(\Omega, \mu)}^2 \leq C_p \|\nabla \mu u\|_{L^2(\Omega, \mu)}^2 \quad u \in H^{1,2}_\mu \quad u_S = \frac{1}{\mu(S)} \int_{\Omega} u \, d\mu. \quad (2.1)$$

However, we are particularly interested in junctions between several thin structures of possibly different dimension, i.e. when $S$ is a finite union of compact manifolds of possibly different Hausdorff dimensions. In such a case (2.1) may not be true.

One of the consequences of the global Poincaré inequality (2.1) is the following property:

$$u \in H^{1,2}_\mu(\Omega), \quad \nabla \mu u = 0 \quad \mu - a.e. \Rightarrow u = \text{const.} \quad \mu - a.e. \quad (2.2)$$

In other words

$$\dim \ker \nabla \mu = \dim \{u \in H^{1,2}_\mu(\Omega) : \nabla \mu u = 0\} = 1,$$

so $\ker \nabla \mu$ contains only constant functions. In a general situation of a multijunction measure this is not necessarily true, i.e. it may happen that

$$\dim \ker \nabla \mu > 1.$$

The important feature is "how big" the junction area is. Every space that supports a global Poincaré inequality is connected. We expect that removing a set of zero capacity from the space should not affect the Poincaré inequality. In particular, removing a set of zero capacity should
not disconnect the space. Examples show that our expectation need not be true. We consider the situation presented on Figure 1. Since the capacity of a point on a plane is equal to zero, the global Poincaré inequality cannot be valid for this set. A similar situation occurs for the set presented on Figure 2. We do not want to recall the definition of a (Sobolev) capacity here (we refer the interested reader for instance to the book [1], chapter 4). However, in the situation presented on the Figure 1 the argument is simple and straightforward.

Consider the unit disk on a plane $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and the function

$$f(x, y) = -\log \log (1 + (x^2 + y^2)^{-1/2}).$$

This is an unbounded function which belongs to the Sobolev space $H^{1,2}(D)$. Set

$$\zeta(t) = \begin{cases} 
0 & t < 0, \\
t & t \in [0, 1], \\
1 & t > 1.
\end{cases}$$

Then $\zeta \circ f \in H^{1,2}(D)$ as well. Define a sequence of functions $f_n$ as

$$f_n(x, y) = \zeta(f(x, y) - n + 1).$$

Since the $H^{1,2}(D)$ energy of $f$ is finite, the $H^{1,2}(D)$-energy of $f_n$ tends to zero.

Now, we consider a metric space $(S, \mu)$, such that $S = X \cup_p Y$, where

$$X = \{(x, y, z) \in \mathbb{R}^3 : z = 0; x^2 + y^2 \leq 1\}$$
and $\mu|_X$ is a 2-dimensional Hausdorff measure;

$$Y = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0; z \in [0, 1]\}$$

and $\mu|_Y$ is a 1-dimensional Hausdorff measure and $P = (0, 0, 0)$.

**Proposition 2.3.** Let us suppose that $S$ and $\mu$ are defined above. Then, $\dim \ker \nabla_{\mu} > 1$.

**Proof.** We define a sequence of functions $u_n : S \to \mathbb{R}$ as

$$u_n(s) = \begin{cases} \zeta(f(x, y) - n + 1) & s = (x, y, 0) \in X; \\ 1 & s = (0, 0, z) \in Y \end{cases}$$

and a function

$$u_o(s) = \begin{cases} 0 & s = (x, y, 0) \in X; \\ 1 & s = (0, 0, z) \in Y. \end{cases}$$

Since

$$\int_S |u_n(s)| \, d\mu = \int_X |u_n(s)| \, dH^2 + \int_Y |u_n(s)| \, dH^1$$

and

$$\int_S |\nabla_{\mu} u_n(s)| \, d\mu = \int_X |\nabla_{\mu} u_n(s)| \, dH^2 + \int_Y |\nabla_{\mu} u_n(s)| \, dH^1 = 0,$$

then it follows that $u_n \in H^{1,2}_\mu$. By the construction, we also have

$$\|u_n - u_o\|_{H^{1,2}_\mu} \to 0,$$

which implies $u_o \in H^{1,2}_\mu$ which in turn falsifies the Poincaré inequality (2.1) on $S$ and it shows that (2.2) in this situation is not true. As a result, our claim holds.

Let us denote

$$V = \ker \nabla_{\mu} \subset H^{1,2}_\mu \subset L^2(\Omega, \mu).$$

This is a finite dimensional subspace of $L^2(\Omega, \mu)$. We define

$$P : H^{1,2}_\mu \to V \quad (2.3)$$

as be the orthogonal projection onto $V$ (with respect to $L^2$ scalar product). Set $d = \dim V$. We can split the set $\{1, \ldots, J\}$ from the condition [S] into a final family of disjoint sets of indices

$$\{1, \ldots, J\} = J_1 \cup \ldots \cup J_d, \quad d \leq J, \quad J_l \cap J_m = \emptyset, \quad l \neq m.$$ 

The sets $J_l, l = 1, \ldots, d$ are defined in the following way. The set

$$\tilde{S}_l = \bigcup_{i \in J_l} S_i$$
is such that
\[ \chi_{\tilde{S}_l} \in \ker \nabla \mu, \]
and \( J_l \) are "maximal" in the sense that for any proper subset \( K \subset J_l \),
the characteristic function of \( \bigcup_{i \in K} S_i \) does not belong to \( V \).

Then the projection \( P \) in (2.3) can be expressed as
\[ P = \sum_{l=1}^{d} P_l, \quad \text{where} \quad P_l u = \chi_{\tilde{S}_l} \int_{\tilde{S}_l} u \, d\mu. \] (2.4)

For a multijunction measure satisfying \([S]\) the following weaker version of the Poincaré inequality holds.

**Theorem 2.1.** Let us suppose that a multijunction measure satisfies \([S]\). Then there is \( C_l > 0 \) such that for \( u \in H_{\mu}^{1,2} \)
\[ \sum_{i \in J_l} \int_{\Omega} |u - P_l u|^2 \, d\mu_i \leq C_l \sum_{i \in J_l} \int_{\Omega} |\nabla \mu u|^2 \, d\mu_i, \quad \text{for } l = 1, \ldots, d, \] (2.5)
where \( P_l \) is defined in (2.4).

Let us mention in advance that, the regularity of solutions depends in an essential way on whether the set \( S \) supports the inequality (2.1) or (2.5). It is known that if a metric measure space (the set \( S \) may be treated as such) supports the Poincaré inequality (2.1), one can expect solutions to be continuous. The weaker version is not enough to ensure this, as shown on examples presented in Section 5.

### 3 The elliptic problem

We would like to use the calculus of variations to establish existence of solutions to (1.1) in the space \( H_{\mu}^{1,2} \). For this purpose we recall the definition of \( E(u) \) for \( H_{\mu}^{1,2} \). We have to specify the assumptions on \( A \) and \( Q \).

In principle, \( u \) is an element of \( L^2(\Omega, \mu) \), so we must specify properties of \( Q \) appropriately. This is why we make the following assumption,
\[ Q \text{ is a measure of a finite total variation supported in } \Omega, \]
absolutely continuous with respect to \( \mu, \quad \text{(3.1)} \)
and there is \( f \in L^2(\mu) \) such that \( Q = f \mu \).

This assumption in particular makes elements of \( H_{\mu}^{1,2} \) measurable with respect to \( Q \).
Now, we specify our assumptions on $A$. We consider

$$A \in L^\infty(\Omega; M(N \times N), \mu) \text{ and for a.e. } x \text{ the matrix } A(x) \text{ is symmetric.} \quad (3.2)$$

We have to present the positivity of $A$ in a way suitable for dealing with $\nabla_\mu u$ in the tangent space to $\mu$. We require that at $\mu$-almost every $x \in \Omega$, we have

$$(A(x)\xi, \xi) \geq \lambda |\xi|^2, \quad \text{for all } \xi \in T_\mu(x). \quad (3.3)$$

After these preparations, we state our first result.

**Proposition 3.1.** Let us suppose that $A$ satisfies $(3.2)$ and for each $x \in \Omega$ matrix $A(x)$ is non-negative. Suppose further that the measure $\mu$ satisfies the assumptions $[S]$. Then, the relaxation of $E$, defined by $(1.2)$, is given by

$$E_\mu(u) = \frac{1}{2} \int_\Omega (A_\mu(x) \nabla_\mu u, \nabla_\mu u) \, d\mu - \int_\Omega f u \, d\mu.$$  

Here, matrix $A_\mu(x)$ is defined by

$$A_\mu(x) = (I - \Pi(x, \cdot)) A(I - \Pi(x, \cdot)), \quad (3.4)$$

where $(I - \Pi(x, \cdot))$ is the orthogonal projection of $\mathbb{R}^N$ onto $T_\mu(x)$, hence $A_\mu(\cdot)$ is $\mu$-measurable.

**Remark 3.1.** In other words, $A_\mu(x)$ is the projection of $A(x)$ onto the tangent space $T_\mu(x)$.

**Proof. Step 1.** Let us set $F^B(x, p) := (B(x)p, p)$, where $B \in L^\infty(\Omega, M(N \times N), \mu)$ and for $\mu$-a.e. $x \in \Omega$ matrix $B(x)$ is symmetric and positive definite, then $E^B(u) = \frac{1}{2} \int_\Omega F^B(x, \nabla u) \, d\mu$. By [4], $E^B_\mu$, the relaxation of $E^B$, is given by

$$E^B_\mu(u) = \frac{1}{2} \int_\Omega F^B_\mu(x, \nabla_\mu u) \, d\mu,$$

where

$$F^B_\mu(x, p) = \inf \{ F^B(x, p + \xi) : \xi \in (T_\mu(x))^\perp \}.$$  

Let us suppose that $e_i$, $i = 1, \ldots, N - k$ span $(T_\mu(x))^\perp$. We may choose them so that

$$(Be_i, e_j) = 0, \quad i \neq j, \quad (Be_i, e_i) > 0, \quad i = 1, \ldots, N - k.$$  

Then, the minimization problem in the definition of $F_\mu$ may be written as

$$\inf \{ F^B(x, p + \sum_{i=1}^{N-k} t_i e_i) : t_i \in \mathbb{R}, i = 1, \ldots, N - k \}.$$  

A simple differentiation yields optimality conditions

$$t^o_i = -(Bp, e_i)/(Be_i, e_i), \quad i = 1, \ldots, l.$$
We easily see that
\[ F^B_\mu(x,p) = (B_\mu(x)p, p), \]
where \( B_\mu(x) \) is given by
\[ B_\mu(x) = B(x) - \sum_{i=1}^{N-k} \frac{B(x)e_i(x) \otimes B(x)e_i(x)}{(B(x)e_i(x), e_i(x))}, \tag{3.5} \]
where \( e_i, i = 1, \ldots, N - k \) are linearly independent and span \( (T_\mu(x))^\perp \). It is easy to check that
\[ B_\mu(x) = (I - \Pi(x, \cdot))B(I - \Pi(x, \cdot)). \]
Hence, due to measurability of projection \( \Pi \) the mapping \( x \mapsto B_\mu(x) \) is \( \mu \)-measurable.

**Step 2.** We assumed that \( A \) satisfies (3.3), so if we set \( F(x,p) = (A(x)p, p) \), then this intergrand does not satisfy the lower estimate
\[ F(x,p) \geq c_0|p|^2 \]
for any \( c_0 > 0 \), which is in the assumptions of [4, Theorem 3.1]. For this reason we take any symmetric matrix \( W \in L^\infty(\Omega, M(N \times N), \mu) \), whose kernel at \( x \) is the image of \( A(x) \). Moreover, we require that there is \( c_0 > 0 \) such that
\[ (W(x)\xi, \xi) \geq c_0|\xi|^2, \]
for all \( \xi \in \ker A(x) \). Then, \( B(x) = A(x) + W(x) \) is symmetric and for \( \xi \) in the image of \( A(x) \) and \( \zeta \in \ker A(x) \) we have
\[ (B(x), \xi + \zeta, \xi + \zeta) = (A(x)\xi, \xi) + (W(x)\zeta, \zeta), \]
because the image of a symmetric matrix is perpendicular to its kernel. Hence,
\[ (B(x), \xi + \zeta, \xi + \zeta) \geq \lambda|\xi|^2 + c_0|\zeta|^2 \geq \min\{\lambda, c_0\}|\xi + \zeta|^2. \]
and formula (3.5) applies to \( B = A + W \).

**Step 3.** We claim that if \( B = A + W \), then \( B_\mu = A_\mu \), where \( A_\mu \) is given by (3.4). Indeed, the summation in (3.5) may be split, the first \( l \leq N - k \) vectors belong to \( \text{Im} A \cap (T_\mu)^\perp \), while vectors \( e_i, i = l + 1, \ldots, N - k \) span the kernel of \( A \). Then, \( B_\mu \) takes the following form,
\[ B_\mu = (A + W) - \sum_{i=1}^{l} \frac{Ae_i \otimes Ae_i}{(Ae_i, e_i)} - \sum_{i=l+1}^{N-k} \frac{We_i \otimes We_i}{(We_i, e_i)} = A_\mu + W_\mu. \]
It is easy to see that for any \( \xi \in \mathbb{R}^N \) we have \( B_\mu \xi \in T_\mu \). Moreover, one can check that if \( \zeta \in \ker A \), then \( W_\mu \zeta = 0 \) and if \( \xi \in \text{Im} A \), then \( W_\mu \xi = 0 \). Thus, our claim follows.

**Step 4.** We will check that the lower semicontinuous envelope of \( E \) is \( E_\mu \).
By the definition we have $E(u) \leq E^B(u)$. If we denote by bar the lower semicontinuous envelope, then we see,

$$\bar{E}(u) \leq \bar{E}^B(u) = E^B_\mu(u).$$

If we look at the definition of $E_\mu$, then we see that $E_\mu(u) \leq E(u)$. Moreover, $E_\mu$ is lower semicontinuous, hence

$$E_\mu(u) \leq \bar{E}(u) \leq \bar{E}^B(u) = E^B_\mu(u),$$

where the last equality follows from step 3. Finally,

$$E_\mu(u) = \bar{E}(u).$$

We have to describe $\ker \text{div } A_\mu \nabla \nu$, what is necessary, before we can specify $Q$. Actually, we prove:

**Proposition 3.2.** $\ker \text{div } A_\mu \nabla \nu = \ker \nabla \nu$.

**Proof.** The inclusion $\supset$ is obvious. Let us suppose that $u \in H^{1,2}_\mu$ satisfies

$$\text{div } (A_\mu \nabla \mu u \mu) = 0 \quad \text{in } D'(\mathbb{R}^N).$$

In other words,

$$\int_\Omega A_\mu \nabla \mu u \nabla \varphi d\mu = 0 \quad \text{for all } \varphi \in D(\mathbb{R}^N).$$

We notice that $\nabla \varphi(x) = \nabla \mu \varphi(x) + (\text{Id} - \Pi_\mu(x, \nabla \varphi))$ for $\mu$-a.e. $x \in S$. Since $A_\mu \nabla \mu u(x) \in T_\mu(x)$ for $\mu$-a.e. $x \in S$, we have

$$0 = \int_\Omega A_\mu \nabla \mu u \nabla \varphi d\mu = \int_\Omega A_\mu \nabla \mu u \nabla \mu \varphi d\mu.$$

We may take a sequence $\varphi_n \in D(\mathbb{R}^N)$ such that $\nabla \mu \varphi_n$ converges to $\nabla \mu u$. This yields,

$$0 = \int_\Omega A_\mu \nabla \mu u \nabla \mu u d\mu.$$

Combining this with ellipticity of $A_\mu$, see (3.3), yields the claim. \qed

After these preparations we can state one of our main results.

**Theorem 3.1.** Let us assume that the conditions: [S] on $\mu$ and (3.1) on $Q$ as well as (3.3) on $A$ hold. In addition, we assume that $(Q, h) = 0$ for all $h$ in the kernel of $\nabla \mu$. Then, there exists a minimizer $u$ of the functional $E_\mu$ defined on the linear subspace

$$H = \{u \in H^{1,2}_\mu : P u = 0\},$$

where $P$ is the orthogonal projection onto $\ker \nabla \mu u$ described by (2.4).
Proof. We have already computed $E_\mu$, the relaxation of $E$. Actually, we study minimizers of $E_\mu$. Let us suppose that $\{u_k\} \subset H$ is a sequence minimizing $E_\mu$, i.e.,
\[
\lim_{k \to \infty} E_\mu(u_k) = \inf \{ E_\mu(v) : v \in H \} =: R.
\]
We may assume that for all $k \in \mathbb{N}$ we have,
\[
R \leq \frac{1}{2} \int_\Omega (A_\mu(x) \nabla_\mu u_k, \nabla_\mu u_k) \, d\mu - \int_\Omega f u_k \, d\mu \leq R + 1.
\]
Due to (3.3), we end up with,
\[
\lambda \frac{1}{2} \int_\Omega |\nabla_\mu u_k|^2 \, d\mu - \int_\Omega f u_k \, d\mu \leq R + 1.
\]
Now, the Theorem 2.1 and $P u_k = 0$ yield that there exists $C > 0$ such that
\[
\int (u_k^2 + |\nabla_\mu u_k|^2) \, d\mu \leq C \int |\nabla_\mu u_k|^2 \, d\mu.
\]
Combining this with (3.6) and Young’s inequality yields,
\[
(1 - \epsilon \frac{C}{2\lambda}) \|u_k\|_{H_\mu^{1,2}}^2 \leq \frac{C}{\lambda 2\epsilon} \|f\|^2_{L^2(\Omega_\mu)} + R + 1.
\]
In other words the minimizing sequence is bounded in $H_\mu^{1,2}$. Due to the results of [4] we deduce that there is a weakly convergent subsequence (not relabeled), with limit $u$. Now, we invoke the lower semicontinuity results so that we deduce,
\[
\lim_{k \to \infty} E_\mu(u_k) \geq \frac{1}{2} \int_\Omega (A_\mu(x) \nabla_\mu u, \nabla_\mu u) \, d\mu - \int_\Omega f u \, d\mu.
\]
Uniqueness of minimizers will be treated separately. \qed

Establishing the Euler-Lagrange equations requires further assumptions on $A_{\mu}$ and $\mu$.

**Proposition 3.3.** Let us suppose that $u$ is a minimizer of $E_{\mu}$ on $H^{1,2}_{\mu}$, then the following weak form of Euler-Lagrange equation holds,
\[
\int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu \varphi) \, d\mu - \int_\Omega f \varphi \, d\mu = 0 \quad \text{for all } \varphi \in C^1_0(\mathbb{R}^N). \tag{3.7}
\]

**Proof.** Since $u$ is a minimizer, then for any test function $\varphi \in H^{1,2}_{\mu}$, (in particular we may take $\varphi \in C^1_0(\mathbb{R}^N)$), we have
\[
E_\mu(u) \leq E_\mu(u + \varphi).
\]
Thus,
\[
0 \leq \int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu \varphi) \, d\mu + \frac{1}{2} \int_\Omega (A_\mu \nabla_\mu \varphi, \nabla_\mu \varphi) \, d\mu - \int_\Omega f \varphi \, d\mu.
\]
After replacing $\varphi$ with $t\varphi$, where $t$ is real, we will deduce that (3.7) holds. \qed
Corollary 3.1. Let us suppose that the assumptions of the previous theorem hold. If \( u_1 \) and \( u_2 \) are two minimizers of \( E_\mu \), which are perpendicular to \( \ker \nabla_\mu \), then \( u_1 = u_2 \).

Proof. Let us take \( u = u_2 - u_1 \) and take the difference of (3.7) corresponding to \( u_1 \) and \( u_2 \),

\[
\int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu \varphi) \, d\mu = 0.
\]

Since \( u = u_2 - u_1 \) can be approximated in the \( H^{1,2}_\mu \)-norm by a sequence of \( C^1 \) functions, we may take \( \varphi = u \) in the identity above,

\[
0 = \int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu u) \, d\mu \geq \lambda \| \nabla_\mu u \|_{L^2_\mu},
\]

i.e. \( u \in \ker \nabla_\mu u \). Since we assumed that \( u \in (\ker \nabla_\mu)^\perp \), we deduce that \( u = 0 \), i.e. \( u_2 = u_1 \). \( \square \)

We do not specify any boundary conditions so we expect that the solution satisfies the so-called natural boundary condition, however, in a weak form suitable for \( H^{1,2}_\mu \).

Remark 3.2. Let us remark that in the case \( S \) is a smooth compact manifold of codimension 1, \( \mu = H^{N-1}_\mu \mathcal{L} S \), and \( A \) is the identity matrix, the equation (1.1) takes the familiar form of

\[
\Delta u + f = 0 \quad \text{on } S,
\]

where \( \Delta \) is the Laplace-Beltrami operator on \( S \).

4 On the boundary condition

After [8], we introduce the notion of a vector valued measure with bounded divergence.

Definition 4.1. (see [8, Definition 1.1]) Let us suppose that \( F \) is a vector-valued Radon measure. We set

\[
|\text{div } F|(\Omega) := \sup \{ \langle F, \nabla \psi \rangle : \psi \in C^1_c(\Omega), |\psi(x)| \leq 1 \}.
\]

If \( |\text{div } F|(\Omega) \) is finite, we say that the divergence of \( F \) has a finite total variation.

The advantage of measures whose divergence has a finite total variation is that one can define the trace of their normal component of \( \partial \Omega \). We recall, see [8, Theorem 2.2],

Proposition 4.1. We assume that the boundary of \( \Omega \) is smooth. If both \( F \) and its divergence have a bounded total variation, there exists a continuous linear functional \( [F, \nu]|_{\partial \Omega} \) over \( C^2(\partial \Omega) \) such that

\[
\langle [F, \nu]|_{\partial \Omega}, \varphi \rangle = \langle \text{div } F, \varphi \rangle + \langle F, \nabla \varphi \rangle \quad \text{for any } \varphi \in C^2(\overline{\Omega}).
\]
Actually, the original statement of the Proposition is more general, in particular less smoothness of the boundary of $\Omega$ is sufficient. Namely, it suffices if $\partial \Omega$ is Lipschitz deformable, see [8, Definition 2.1]. In addition, $[F, \nu]|_{\partial \Omega}$ could be defined as a functional on the space Lip$(\gamma, \Omega)$, $\gamma > 1$, which is larger than $C^2(\partial \Omega)$, see [8]. However, such generality is not necessary here, so we choose the simplicity of exposition.

We want to show that if $u$ is a minimizer of $E$, then $F := A_\mu \nabla_\mu u$ has divergence with a finite total variation.

**Theorem 4.1.** Let us suppose that the structural assumption [S] as well as the condition (3.1) hold. If $u$ is the minimizer of $E_\mu$, then $F := A_\mu \nabla_\mu u \in M(\Omega, \mathbb{R}^N)$ has divergence with finite total variation. As a result, $[A_\mu \nabla_\mu u, \nu]|_{\partial \Omega} = 0$ in the sense of Proposition 4.1.

**Proof.** We have to reconcile Definition 4.1 with the Euler-Lagrange equation (3.3): in (3.3) the inner product of $A_\mu \nabla_\mu u$ with $\nabla_\mu u$ is taken, while in the definition $A_\mu \nabla_\mu u$ is multiplied with the full gradient of the test function. The relaxation result, Proposition 3.1, makes this difference apparent. In fact, no matter what the matrix $A(x)$ is, Proposition 3.1 implies that

$$A_\mu : T_\mu \rightarrow T_\mu. \quad (4.1)$$

Observation (4.1) makes (3.7) coincide with

$$\langle F, \nabla \psi \rangle = \langle Q, \psi \rangle \quad \forall \psi \in C^1_c(\Omega).$$

Indeed, if $\Pi_\mu(x)$ is the projection onto $T_\mu(x)$ at each $x \in \Omega$, then for $\psi \in C^1_c(\Omega)$ we have $\nabla \psi(x) = \Pi_\mu(x) \nabla \psi(x) + (Id - \Pi_\mu(x)) \nabla \psi(x)$. As a result we notice,

$$\langle F, \nabla \varphi \rangle = \int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu \psi + (Id - \Pi_\mu) \nabla \psi) \, d\mu$$

$$= \int_\Omega (A_\mu \nabla_\mu u, \nabla_\mu \psi) \, d\mu + 0$$

$$= \langle Q, \psi \rangle,$$

where we used (4.1). Now, we can see that $|\text{div } F|(\Omega)$ is finite, because

$$\sup\{\langle Q, \psi \rangle : \psi \in C^1_c(\Omega), |\psi(x)| \leq 1\} = \int_\Omega |f| \, d\mu < \infty.$$  

Thus, we immediately deduce that $[A_\mu \nabla_\mu u, \nu]|_{\partial \Omega}$ exists in the sense explained in Theorem [8, Theorem 2.2]. Now, we have to show that it vanishes, i.e.

$$\langle \text{div } A_\mu \nabla_\mu u, \varphi \rangle + \langle A_\mu \nabla_\mu u, \nabla \varphi \rangle = 0 \quad \text{for any } \varphi \in C^2(\overline{\Omega}).$$

Let us suppose that $\eta_k \in C^\infty_c(\Omega)$ are the cut-off functions such that $\eta_k(x) = 0$ for $x \in \Omega$ at a distance smaller than $1/(k + 1)$ from $\partial \Omega$ and equal to 1 for $x$ further than $1/k$ from $\partial \Omega$. Then,
since $\eta_k \varphi \in C^1_c(\Omega)$ and by using of the Euler-Lagrange equation (3.7) with $(1 - \eta_k) \varphi$ as a test function, we obtain

$$\langle \text{div} A_{\mu} \nabla_{\mu} u, \varphi \rangle + \langle A_{\mu} \nabla_{\mu} u, \nabla \varphi \rangle = \langle \text{div} A_{\mu} \nabla_{\mu} u, (1 - \eta_k) \varphi \rangle + \langle A_{\mu} \nabla_{\mu} u, \nabla((1 - \eta_k) \varphi) \rangle$$

If $\omega$ is a measure with a finite variation, then

$$\lim_{k \to \infty} \int_{\Omega} (1 - \eta_k) \varphi \, d\omega = 0,$$

because of the Lebesgue dominated convergence Theorem. If we apply this remark to $\omega = Q$ and $\omega = \text{div} A_{\mu} \nabla_{\mu} u$, then we deduce that

$$\langle [A_{\mu} \nabla_{\mu} u, \nu]_{\partial \Omega}, \varphi \rangle = 0$$

for all $\varphi \in C^2(\bar{\Omega})$. Our claim follows.

**Remark 4.1.** Theorem 4.1 tells us about the meaning of $[A_{\mu} \nabla_{\mu} u, \nu]$ only at the boundary of $\Omega$. So, if a part of $\partial S$ does not touch $\partial \Omega$, then we have no information about the boundary behavior of $u$. This is why we impose the third condition of $[S]$. On the other hand, the fourth condition in $[S]$ rules out spurious conditions.

## 5 Examples of Neumann boundary problems

We present below several examples of Neumann boundary problems on various sets $S$. The first two examples deal with situations when the Poincaré inequality (2.1) holds. We further present an example (see Proposition 5.3) showing that discontinuous solutions actually occur, even for regular data, when the set $S$ supports only the weaker inequality (2.5).

Let us suppose that $\Omega$ is a ball centered at 0 with radius 1, $T$ is an inscribed isosceles triangle and $S_1$, $S_2$, $S_3$ is the set of radii connecting the center of $\Omega$ with the vertices of $T$ (see Fig. 3). We set $S = S_1 \cup S_2 \cup S_3$. Moreover, $\mu = \mathcal{H}^{1}_{\partial \Omega}$. We set $Q = \sum_{i=1}^{3} a_i \chi_{S_i}$.

We can state the following fact.

**Proposition 5.1.** Let us assume that $S$, $\mu$ and $Q$ are defined above and $a_1 + a_2 + a_3 = 0$. If $A = I_d$, then the relaxed form of equation (1.1) takes the form,

$$\frac{d^2 u}{ds^2} = -a_i \quad \text{in the interior of } S_i, \quad i = 1, 2, 3 \quad (5.1)$$

and

$$\frac{du}{ds} = 0 \quad \text{on } S_i \cap \partial \Omega, \quad i = 1, 2, 3. \quad (5.2)$$
Here, $s$ is the arc-length parameter counted from the center of $\Omega$.

Finally, the solution of (5.1-5.2) with \( \int_S u \, d\mathcal{H}^1 = 0 \) has the following form,

\[
    u = -\sum_{i=1}^{3} a_i \left( \frac{s^2}{2} - s \right) \chi_{S_i}.
\]

Proof. By Proposition 2.1 \( T_\mu(x) = T_{S_i}(x) \), except for $x$ being the center of the ball. Hence, Proposition 3.1 yields \( \text{div} A_\mu \nabla_\mu u = \frac{d^2 u}{ds^2} \) in the interior of $S_i$, $i = 1, 2, 3$. Also the form, (5.2), of boundary data follows.

The assumption $a_1 + a_2 + a_3 = 0$ means that $Q$ is perpendicular to the kernel of $\nabla_\mu$ so that we may apply Theorem 1.1.

The form of any solution to (5.1-5.2) follows from solving the ODE’s (5.1). While doing so, we keep in mind the requirement that the solution must be continuous at $s = 0$, i.e. at the center of the ball $\Omega$. The solution is determined up to an additive constant, which can be computed due to the zero average condition, $\int_S u \, d\mathcal{H}^1 = 0$.

We can also study (1.1) on sums of manifolds of dimension $k \geq 1$. We describe the set $S$ depicted on Fig. 4.
Proposition 5.2. We assume that $A = Id$, $\Omega$ is the unit ball $B(0, 1) \subset \mathbb{R}^3$ and $S = D_1 \cup D_2$ are two great disks of $B(0, 1)$.

$$D_1 = \{(x_1, x_2, x_3) \in B(0, 1) : x_3 = 0\}, \quad D_2 = \{(x_1, x_2, x_3) \in B(0, 1) : x_1 = 0\}.$$ Moreover, $Q = Q_1 \chi_{D_1} + Q_2 \chi_{D_2}$, where $\int_{D_i} dQ_i = 0$, $i = 1, 2$. Then, the relaxed equations (1.3) take the form

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + Q_1 = 0 \quad \text{in } D_1,$$

and

$$\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + Q_2 = 0 \quad \text{in } D_2,$$

where $\nu_i$ are normal to $D_i$ in the plane containing $D_i$, $i = 1, 2$. The condition of orthogonality to the kernel of $\nabla u$ is

$$\int_{D_i} u d\mathcal{H}^2 = 0, \quad i = 1, 2.$$ 

Proof. We easily see that, due to Proposition 3.1, we will have $A \mu = (e_1 \otimes e_1 + e_2 \otimes e_2) \chi_{D_1} + (e_3 \otimes e_3 + e_2 \otimes e_2) \chi_{D_2}$. Hence, the form of relaxed equations follows. The boundary conditions are addressed in Corollary 5.1.

We notice that $A \mu$ depends upon $x \in \Omega$ despite $A$ being constant. Moreover, the kernel of $A \mu$ is two-dimensional.

Due to the generalized Poincaré inequality (2.5), we can study (1.1) on sums of manifolds of different dimensions. We analyze an example of an equation on the domain depicted on Figure 5. We define the design region $\Omega$ to be $B(0, \sqrt{2}) \subset \mathbb{R}^3$. We set,

$$S_i = \Omega \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = (-1)^i\}, \quad i = 1, 2, \quad S_3 = \{(x_1, 0, 0) \in \mathbb{R}^3 : |x_1| \leq 1\}.$$ 

In the case of this domain, we do not expect the global Poincaré inequality (2.1) to hold.

Proposition 5.3. We assume that $A = Id$ and

$$\mu = \mathcal{H}^2 \ll S_1 \cup S_2 + \mathcal{H}^1 \ll S_3.$$ 

We assume that $Q = Q_1 \chi_{S_1} + Q_2 \chi_{S_2}$, i.e. $Q|_{S_i} \equiv 0$. We will require that

$$\int_{S_i} Q_i d\mu = 0, \quad i = 1, 2.$$ 

We set

$$Q_1(x_1, x_2, x_3) = q(\sqrt{x_2^2 + x_3^2}) \quad \text{and} \quad Q_2(x_1, x_2, x_3) = -q(\sqrt{x_2^2 + x_3^2}).$$

Then, $u$, the solution to (1.1) in $S$ with $\int_{S_i} u d\mathcal{H}^2 = 0$, $i = 1, 2$ and $\int_{S_3} u d\mathcal{H}^2 = 0$ is discontinuous.
Proof. We notice that the relaxation of the functional (Proposition 3.1) yields,

$$A_\mu = (e_2 \otimes e_2 + e_3 \otimes e_3)\chi_{S_1 \cup S_2} + (e_1 \otimes e_1)\chi_{S_3}.$$ 

Thus, eq. \( \text{div} A_\mu \nabla u + Q = 0 \) becomes

$$\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + q = 0 \quad \text{on } S_1, \tag{5.3}$$

$$\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} - q = 0 \quad \text{on } S_2$$

and

$$\frac{\partial^2 u}{\partial x_1^2} = 0 \quad \text{on } S_3.$$

The radial symmetry of \( q \) implies that

$$0 = \int_{\{x_2^2 + x_3^2 \leq 1\}} Q_1 \, d\mu = 2\pi \int_0^R rq(r) \, dr. \tag{5.4}$$

The above equation \( (5.3) \), considered on \( S_1 \), due to the radial symmetry takes the following form

$$u_{rr} + \frac{1}{r} u_r + q = 0.$$  

After multiplying by \( r \) and integration, we obtain,

$$ru_r(r) = a - \int_0^r sq(s) \, ds.$$ 

In order to solve it, we have to pay a bit of attention to the boundary conditions. Since \( \nu = 2^{-1/2}(-1, x_2, x_3) \) for \((-1, x_2, x_3) \in \partial S_1\), then the Neumann boundary condition

$$[A_\mu \nabla u, \nu]_{\partial \Omega} = 0$$
takes the form
\[(0, u_{x_2}, u_{x_3}) \cdot \nu = \nabla_\mu u \cdot \mathbf{n} = 0,\]
where \(\mathbf{n}\) is the outer normal to \(S_1\) in the plane \(x_1 = -1\). We notice that due to (5.4), the above boundary conditions are automatically satisfied, provided that \(a = 0\). Moreover, \(a = 0\) is necessary for \(u\) to be an element of \(H^{1,2}_\mu\). The formula for the solution is as follows,
\[u(r) = b - \int_0^r \frac{1}{\rho} \int_0^\rho sq(s) \, ds \, d\rho.\]
The zero mean condition imposed on the solution on \(S_1\) implies that
\[b = \frac{2}{R^2} \int_0^R r \int_0^r \frac{1}{\rho} \int_0^\rho sq(s) \, ds \, d\rho \, dr.\]
We may choose \(q\) so that \(b > 0\). Due to the symmetry of \(Q\), we deduce that \(u(-1, x_2, x_3) = -u(1, x_2, x_3)\). Thus, in particular \(u(-1, 0, 0) = b = -u(1, 0, 0)\).

Since we set \(Q|_{S_3} = 0\), then we study the problem of minimizing
\[\int_{-1}^1 u_{x_1}^2 \, dx,\]
on \(S_3\), where \(u \in H^{1,2}(-1, 1)\) and \(\int_{-1}^1 u \, dx_1 = 0\). As a result \(u|_{S_3} = 0\).

We conclude that we constructed a discontinuous solution when the forcing term is continuous. \(\square\)

Here is another example.

**Proposition 5.4.** We take
\[\Omega = B(0, \sqrt{2}), \quad S_1 = \{x \in \Omega; x_3 = -1\}, \quad S_2 = \{x \in \Omega; x_1 = 1\},\]
\[\mu = \mathcal{H}^2_\perp(S_1 \cup S_2)\) and
\[A = e_2 \otimes e_2 + \frac{1}{2}(e_1 + e_3) \otimes (e_1 + e_3).\]
Then,
\[A_\mu = e_2 \otimes e_2 + \frac{1}{2}e_1 \otimes e_1 \chi_{S_1} + \frac{1}{2}e_3 \otimes e_3 \chi_{S_2}\]
and the boundary conditions take the following form,
\[0 = [A_\nu \nabla_\mu u, \nu_1]|_{\partial \Omega} = 2^{-1/2}(\frac{1}{2}x_1 u_{x_1} + x_2 u_{x_2}) \quad \text{on } S_1 \quad (5.5)\]
\[0 = [A_\nu \nabla_\mu u, \nu_2]|_{\partial \Omega} = 2^{-1/2}(\frac{1}{2}x_3 u_{x_3} + x_2 u_{x_2}) \quad \text{on } S_2. \quad (5.6)\]
The condition of orthogonality to the kernel of \(\nabla_\mu\) is \(\int_{S_1} u \, d\mathcal{H}^2 = 0\).
Proof. Obviously, $\bar{S}_1 \cap \bar{S}_2 = \{(1, 0, -1)\}$. When we take $\mu = \mathcal{H}^2_\text{L}(S_1 \cup S_2)$, then we see that an application of the Relaxation Theorem yields

We shall compute $[A_\mu \nabla u, \nu]|_{\partial \Omega}$, where $\nu$ is normal to $\Omega$. We notice that

$$\nabla u = (u_{x_1}, u_{x_2}, 0) \chi_{S_1} + (0, u_{x_2}, u_{x_3}) \chi_{S_2}.$$  

Similarly, we see that

$$\nu_1 = 2^{-1/2}(x_1, x_2, -1), \quad x_1^2 + x_2^2 = 1$$

on $S_1$ and

$$\nu_2 = 2^{-1/2}(1, x_2, x_3), \quad x_3^2 + x_2^2 = 1$$

on $S_2$.

Thus, $[A_\mu \nabla u, \nu]|_{\partial \Omega}$ leads to the conclusion that (5.5) and (5.6) hold.  

We may interpret the last computations as the lack of dependence of the boundary conditions on $\Omega$.

**Corollary 5.1.** Let us suppose $S$ is a $k$-dimensional manifold contained in $\Omega \subset \mathbb{R}^N$, $k < N$, and the boundary of $\Omega$ is smooth with outer normal $\nu$. Moreover, $A_\mu$ is a relaxation of matrix $A$. Then, $[A_\mu \nabla u, \nu]|_{\partial \Omega} = 0$ on $\partial \Omega$ is equivalent to $[A_\mu \nabla u, n]|_{\partial S} = 0$ on $\partial \Omega \cap \partial S$, where $n$ is the normalized projection of $\nu$ to the normal bundle of $S$.

**Proof.** Since $S$ intersects $\partial \Omega$ transversally, then $\nu$ is not tangential to $S$, i.e. $\nu = \alpha n + \beta n^\perp$, $\alpha \neq 0$. Since $A_\mu$ maps $\mathbb{R}^N$ into the tangent space of $S$, then we can see that,

$$[A_\mu \nabla u, \nu]|_{\partial \Omega} = [A_\mu \nabla u, \alpha n + \beta n^\perp]|_{\partial \Omega} = [\alpha A_\mu \nabla u, n]|_{\partial S}.$$  

The last equality holds because $n^\perp$ is perpendicular to $T_x S$ at $x \in \partial S$.  

**Acknowledgement**

The work of the first author was in part supported by the National Science Centre, Poland, through the grant number 2017/26/M/ST1/00700.

**References**

[1] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[2] Guy Bouchitté and Giuseppe Buttazzo. Characterization of optimal shapes and masses through Monge-Kantorovich equation. *J. Eur. Math. Soc. (JEMS)*, 3(2):139–168, 2001.

[3] Guy Bouchitté, Giuseppe Buttazzo, and Ilaria Fragalà. Convergence of Sobolev spaces on varying manifolds. *J. Geom. Anal.*, 11(3):399–422, 2001.
[4] Guy Bouchitte, Giuseppe Buttazzo, and Pierre Seppecher. Energies with respect to a measure and applications to low-dimensional structures. *Calc. Var. Partial Differential Equations*, 5(1):37–54, 1997.

[5] Guy Bouchitté and Ilaria Fragalà. Homogenization of thin structures by two-scale method with respect to measures. *SIAM J. Math. Anal.*, 32(6):1198–1226, 2001.

[6] Guy Bouchitté and Ilaria Fragalà. Variational theory of weak geometric structures: the measure method and its applications. In *Variational methods for discontinuous structures*, volume 51 of *Progr. Nonlinear Differential Equations Appl.*, pages 19–40. Birkhäuser, Basel, 2002.

[7] Guy Bouchitté and Ilaria Fragalà. Second-order energies on thin structures: variational theory and non-local effects. *J. Funct. Anal.*, 204(1):228–267, 2003.

[8] Gui-Qiang Chen and Hermano Frid. On the theory of divergence-measure fields and its applications. *Bol. Soc. Brasil. Mat. (N.S.*), 32(3):401–433, 2001. Dedicated to Constantine Dafermos on his 60th birthday.

[9] J. Louet. Some results on Sobolev spaces with respect to a measure and applications to a new transport problem. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 411(Teoriya Predstavleni˘ı, Dinamicheskie Sistemy, Kombinatornye Metody. XXII):63–84, 241, 2013.

[10] Jean-Philippe Mandallena. Quasiconvexification of geometric integrals. *Ann. Mat. Pura Appl. (4)*, 184(4):473–493, 2005.