Nonlinear diffusion equations with Robin boundary conditions as asymptotic limits of Cahn–Hilliard systems

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Abstract

Condition imposed on the nonlinear terms of a nonlinear diffusion equation with Robin boundary condition is the main focus of this paper. The degenerate parabolic equations, such as the Stefan problem, the Hele–Shaw problem, the porous medium equation and the fast diffusion equation, are included in this class. By characterizing this class of equations as an asymptotic limit of the Cahn–Hilliard systems, the growth condition of the nonlinear term can be improved. In this paper, the existence and uniqueness of the solution are proved. From the physical view point, it is natural that, the Cahn–Hilliard system is treated under the homogeneous Neumann boundary condition. Therefore, the Cahn–Hilliard system subject to the Robin boundary condition looks like pointless. However, at some level of approximation, it makes sense to characterize the nonlinear diffusion equations.

Key words: Cahn–Hilliard system, degenerate parabolic equation, Robin boundary condition, growth condition.

AMS (MOS) subject classification: 35K61, 35K65, 35K25, 35D30, 80A22.
1 Introduction

We consider the initial boundary value problem of a nonlinear diffusion system (P), comprising a parabolic partial differential equation with a Robin boundary condition:

(P) \[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta \xi = g, & \xi \in \beta(u) \quad \text{in } Q := \Omega \times (0,T), \\
\partial_\nu \xi + \kappa \xi = h & \text{on } \Sigma := \Gamma \times (0,T), \\
u(0) = u_0 & \text{in } \Omega,
\end{cases}
\]

where \(\kappa\) is a positive constant. In an asymptotic form, it is characterized as the limit of the Cahn–Hilliard system with a Robin boundary condition,

\[(P)_\varepsilon \begin{cases}
\frac{\partial u_\varepsilon}{\partial t} - \Delta \mu_\varepsilon = 0 & \text{in } Q, \\
\mu_\varepsilon = -\varepsilon \Delta u_\varepsilon + \xi_\varepsilon + \pi_\varepsilon(u_\varepsilon) - f, & \xi_\varepsilon \in \beta(u_\varepsilon) \quad \text{in } Q, \\
\partial_\nu u_\varepsilon + \kappa u_\varepsilon = 0, & \partial_\nu \mu_\varepsilon + \kappa \mu_\varepsilon = 0 \quad \text{on } \Sigma, \\
u_\varepsilon(0) = u_{0\varepsilon} & \text{in } \Omega
\end{cases}\]

as \(\varepsilon \downarrow 0\) with \(\xi := \mu + f\), where \(0 < T < +\infty\), \(\Omega\) is a bounded domain of \(\mathbb{R}^d\) \((d = 2, 3)\) with smooth boundary \(\Gamma := \partial \Omega\), \(\partial_\nu\) denotes the outward normal derivative on \(\Gamma\), \(\Delta\) is the Laplacian. Functions \(g : Q \to \mathbb{R}\), \(h : \Sigma \to \mathbb{R}\), \(u_0 : \Omega \to \mathbb{R}\) and \(u_{0\varepsilon} : \Omega \to \mathbb{R}\) are given as the boundary and initial data. \(f : Q \to \mathbb{R}\) is constructed from \(g\) and \(h\) later. Moreover, in the nonlinear diffusion term, \(\beta\) is a maximal monotone graph and \(\pi_\varepsilon\) is an anti-monotone function that tends to 0 as \(\varepsilon \downarrow 0\). It is well known that the Cahn–Hilliard system is characterized by the nonlinear term \(\beta + \pi_\varepsilon\), a simple example being \(\beta(r) = r^3\) and \(\pi_\varepsilon(r) = -\varepsilon r\) for all \(r \in \mathbb{R}\). In this way, we choose a suitable \(\pi_\varepsilon\) that depends on the definition of \(\beta\) yielding the structure of the Cahn–Hilliard system for \((P)_\varepsilon\). Alternatively, by choosing a suitable \(\beta\), the degenerate parabolic equation \((P)\) characterizes various types of nonlinear problems, such as the Stefan problem, Hele-Shaw problem, porous medium equation, and fast diffusion equation (see, e.g., [7, pp.6935–6937, Examples]).

In analyzing the well-posedness of system \((P)\), there are two standard approaches, the “\(L^1\)-approach” and the ‘Hilbert space approach’ (see, e.g., [4, Chapter 5]). With respect to the “Hilbert space approach”, the pioneering result [9] concerns the enthalpy formulation for the Stefan problem with a Dirichelet–Robin boundary condition, essentially of Robin-type. Afterwards, the Neumann boundary condition was treated in [15,18] and the dynamic boundary condition in [1,3,14]. See also [13,19,20] for a more general space setting. For all of these results, the growth condition for \(\beta\) is a very important assumption, such as

\[\hat{\beta}(r) \geq c_1 r^2 - c_2 \quad \text{for all } r \in \mathbb{R},\]

where \(c_1\) and \(c_2\) are positive constants, and \(\hat{\beta}\) is a proper lower semicontinuous convex function satisfying the subdifferential form \(\partial R \hat{\beta} = \beta\). However, it is too restricted in regard to application; indeed, for fast diffusion or nonlinear diffusion of Penrose–Fife type are excluded. A drawback of the “Hilbert space approach” compared with the “\(L^1\)-approach” is detailed in [5, Chapter 5]. With that as motivation, the improvement of the growth condition subject to the Robin boundary condition, was studied in [10].
using a certain technique called the “lower semicontinuous convex extension”. For porous medium, that is, \( \beta(r) = |r|^{q-1}r, \ (q > 1) \), a different approach to the doubly nonlinear evolution equation was studied in [4]. Regarding a recent result, the characterization of the nonlinear diffusion equation as an asymptotic limit of the Cahn–Hilliard system with dynamic boundary conditions was introduced in [11,12], and the same problem subject to the Neumann boundary condition was given in [7]. In these instances, we do not need any growth condition; see [7, Chapter 6] or [12]. This is one of the big advantages of this approach; indeed, we do not need techniques such as the lower semicontinuous convex extension by [10].

The main objective of this paper is to improve on the pioneering result give in [9,10], more precisely, an improvement of the growth condition subject to the Robin boundary condition without using the lower semicontinuous convex extension. Up to a certain level of approximation, we consider the Cahn–Hilliard system subject to the Robin boundary condition (see, e.g., [21]). From a physical perspective, the Cahn–Hilliard system is more naturally treated under the homogeneous Neumann boundary condition. Therefore the Cahn–Hilliard system with the Robin boundary condition imposed looks to be without points. However, up to a given level of approximation, characterizing the nonlinear diffusion equations with the Robin boundary condition imposed does make sense. Moreover, we obtain the order of convergence for the solutions of (P) with the Robin boundary condition as that for (P) with the Neumann boundary condition, that is, letting \( \kappa \) tend to 0, where \( \kappa \) is the constant in the boundary condition.

The outline of the paper is as follows. In Section 2, the main theorems are stated. For this purpose, we present the notation used in this paper and define a suitable duality map and the \( H^1 \)-norm equivalent to the standard norm. Next, we introduce the definition of a weak solution of (P) and \( (P)_\varepsilon \); the principal theorems are then given. In Section 3, to prove the convergence theorem, we deduce the uniform estimates of the approximate solution of \( (P)_\varepsilon \). We use Moreau–Yosida regularization of \( \tilde{\beta} \) employing the second-order approximate of parameter \( \lambda \). In Section 4, to obtain the weak solution of \( (P)_\varepsilon \), we first pass to the limit \( \lambda \downarrow 0 \). Second, we prove the existence of weak solutions by passing to the limit \( \varepsilon \downarrow 0 \). We also discuss the uniqueness of solutions. In Section 5, we improve the assumption for \( \beta \) subject to a strong assumption for the heat source \( f \). From this results, we can avoid the growth condition for \( \beta \). In Section 6, we obtain the order of convergence related to the Neumann problem from the Robin problem as \( \kappa \downarrow 0 \).

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Appendix

2 Main results

In this section, we state the main theorem. Hereafter, $\kappa$ is a positive constant.

2.1 Notation

We employ spaces $H := L^2(\Omega)$ and $H^\Gamma := L^2(\Gamma)$, with standard norms $|\cdot|_H$ and $|\cdot|_{H^\Gamma}$, along with inner products $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_{H^\Gamma}$, respectively. We also use the space $V := H^1(\Omega)$ with norm $|\cdot|_V$ and inner product $(\cdot, \cdot)_V$.

$|z|_V := (|\nabla z|_H^2 + \kappa |z|_{H^\Gamma}^2)^{\frac{1}{2}}$ for all $z \in V$.

By virtue of the Poincaré inequality and the trace theorem, there exist positive constants $c_P, c'_P$ such that

$c_P |z|_V^2 \leq z|_V^2 \leq c'_P |z|_V^2$ for all $z \in V$,

where $\|\cdot\|_V$ stands for the standard $H^1$-norm. Moreover, we set

$W := \{ z \in H^2(\Omega) : \partial_\nu z + \kappa z = 0 \text{ a.e. on } \Gamma \}$.

The symbol $V^*$ denotes the dual space of $V$; the duality pairing between $V^*$ and $V$ is denoted by $(\cdot, \cdot)_{V^*,V}$. Also, for all $z \in V$, let $F : V \to V^*$ be the duality mapping defined by

$(Fz, \tilde{z})_{V^*,V} := \int_\Omega \nabla z \cdot \nabla \tilde{z} dx + \kappa \int_{\Gamma} z \tilde{z} d\Gamma$ for all $z, \tilde{z} \in V$.

Moreover, we define an inner product of $V^*$ by

$(z^*, \tilde{z}^*)_{V^*} := (z^*, F^{-1}\tilde{z}^*)_{V^*,V}$ for all $z^*, \tilde{z}^* \in V^*$.

Then, the dense and compact embeddings $V \hookrightarrow \hookrightarrow H \hookrightarrow \hookrightarrow V^*$ hold; that is, $(V, H, V^*)$ is a standard Hilbert triplet. As a remark, for all of these settings, $\kappa > 0$ is essential. For a Neumann boundary condition, $\kappa = 0$, see [7].

2.2 Definition of the solution and main theorem

We next define our solution for (P) and then state the main theorem.
Definition 2.1. The pair $(u, \xi)$ is called the weak solution of (P) if
\[ u \in H^1(0, T; V^*) \cap L^\infty(0, T; H), \quad \xi \in L^2(0, T; V), \]
and they satisfy
\[ u(0) = u_0 \quad a.e. \ in \ \Omega. \tag{2.2} \]

In this definition, the Robin boundary condition for $\xi$ is hidden in the weak formulation (2.1). The strategy behind the proof of the main theorem is the characterization of our nonlinear diffusion equation (P) as an asymptotic limit of the Cahn–Hilliard system. Therefore, for each $\varepsilon \in (0, 1]$, we define the approximate problem of Cahn–Hilliard type with Robin boundary condition as follows:

Definition 2.2. The triplet $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ is called the weak solution of $(P)_\varepsilon$ if
\[ u_\varepsilon \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \]
\[ \mu_\varepsilon \in L^2(0, T; V), \quad \xi_\varepsilon \in L^2(0, T; H), \]
and they satisfy
\[ \langle u_\varepsilon'(t), z \rangle_{V^*,V} + \int_{\Omega} \nabla \mu_\varepsilon(t) \cdot \nabla z \, dx + \kappa \int_{\Gamma} \mu_\varepsilon(t) z \, d\Gamma = 0 \quad \text{for all } z \in V, \tag{2.4} \]
\[ \mu_\varepsilon(t) = -\varepsilon \Delta u_\varepsilon(t) + \xi_\varepsilon(t) + \pi_\varepsilon(u_\varepsilon(t)) - f(t) \quad \text{in } H, \tag{2.5} \]
for a.a. $t \in (0, T)$, with
\[ u_\varepsilon(0) = u_{0\varepsilon} \quad a.e. \ in \ \Omega. \]

The Robin boundary condition for $u_\varepsilon$ is stated with regard to the class of $W$, that is, the regularity (2.3); that for $\mu_\varepsilon$ is hidden in the weak formulation (2.4). The Cahn–Hilliard structure is characterized by the nonlinear term $\beta + \pi_\varepsilon$. The conditions for these terms are given as assumptions:

(A1) $\beta : \mathbb{R} \to 2^\mathbb{R}$ is a maximal monotone graph, which is the subdifferential $\beta = \partial_R \hat{\beta}$ of some proper lower semicontinuous convex function $\hat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\hat{\beta}(0) = 0$ with the effective domain $D(\beta) := \{ r \in \mathbb{R} : \beta(r) \neq \emptyset \}$;

(A2) there exist positive constants $c_1, c_2$ such that $\hat{\beta}(r) \geq c_1 r^2 - c_2$ for all $r \in \mathbb{R}$;

(A3) $\pi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function for all $\varepsilon \in (0, 1]$. Moreover, there exist a positive constant $c_3$ and strictly increasing continuous function $\sigma : [0, 1] \to [0, 1]$ such that $\sigma(0) = 0$, $\sigma(1) = 1$, and
\[ |\pi_\varepsilon(0)| + |\pi_\varepsilon'|_{L^\infty(\mathbb{R})} \leq c_3 \sigma(\varepsilon) \quad \text{for all } \varepsilon \in (0, 1]. \tag{2.6} \]
In particular, (A1) yields $0 \in \beta(0)$. Assumption (A2) is improved in Section 6. The assumptions pertaining to the given data are as follows:

(A4) $g \in L^2(0, T; H)$, $h \in L^2(0, T; H_T)$;

(A5) $u_0 \in H$ with $\widehat{\beta}(u_0) \in L^1(\Omega)$. Moreover, let $u_{0\varepsilon} \in V$; then there exists a positive constant $c_4$ such that, for all $\varepsilon \in (0, 1]$,

$$|u_{0\varepsilon}|_H^2 \leq c_4, \quad \int_{\Omega} \widehat{\beta}(u_{0\varepsilon}) dx \leq c_4, \quad \varepsilon|\nabla u_{0\varepsilon}|_H^2 \leq c_4, \quad \varepsilon|u_{0\varepsilon}|_{H_T}^2 \leq c_4. \quad (2.7)$$

In addition, $u_{0\varepsilon} \to u$ strongly in $H$ as $\varepsilon \searrow 0$ (cf. [7, Lemma A.1]).

From assumption (A4), we see from the Lax–Milgram theorem that there exists a unique function $f \in L^2(0, T; V)$ such that

$$\int_{\Omega} \nabla f(t) \cdot \nabla z dx + \kappa \int_{\Gamma} f(t) z d\Gamma = \int_{\Omega} g(t) z dx + \int_{\Gamma} h(t) z d\Gamma$$

for all $z \in V$ and for a.a. $t \in (0, T)$. Therefore, introducing new variable $\mu := \xi - f \in L^2(0, T; V)$, we rewrite the weak formulation (2.1) as follows:

$$\langle u'(t), z \rangle_{V', V} + \int_{\Omega} \nabla \mu(t) \cdot \nabla z dx + \kappa \int_{\Gamma} \mu(t) z d\Gamma = 0 \quad \text{for all } z \in V, \quad (2.8)$$

for a.a. $t \in (0, T)$.

The proof of main theorem follows that in [7,11] for the Robin boundary condition. The characterization of the nonlinear diffusion equation from the asymptotic limits of Cahn–Hilliard system [7, 11–13] furnishes a big advantage in regard to the growth condition for $\beta$. Because of this, we can improve the result [9] and widen the setting for $\beta$ using the different approach described in [10] starting from the lower semicontinuous convex extension. To do so, we replace assumption (A4) by the following (A6):

(A6) $g \in L^2(0, T; H)$, $h = 0$ a.e. on $\Sigma$.

Then, we see that there exists a unique function $f \in L^2(0, T; V)$ such that,

$$\int_{\Omega} \nabla f(t) \cdot \nabla z dx + \kappa \int_{\Gamma} f(t) z d\Gamma = \int_{\Omega} g(t) z dx \quad \text{for all } z \in V,$$

for a.a. $t \in (0, T)$. Now, taking a test function $z \in D(\Omega)$, we have $-\Delta f(t) = g(t)$ in $D'(\Omega)$. Specifically, by comparison, $-\Delta f(t) = g(t)$ in $H$. This yields

$$\partial_{\nu} f(t) + \kappa f(t) = 0 \quad \text{a.e. on } \Gamma,$$

for a.a. $t \in (0, T)$. Therefore, under assumption (A6), we have $-\Delta f \in L^2(0, T; H)$ and $\partial_{\nu} f \in L^2(0, T; H_T)$. These higher regularities are essential to improve the growth condition (A2).

The well-posedness of Cahn–Hilliard system has been treated in many studies (see, e.g., [21]). In regard to the abstract theorem of the evolution equation, we refer the reader to [16,17]. Based on these results, we obtain the following proposition:
Proposition 2.1. Given assumptions (A1)–(A5) or (A1), (A3) with \( \sigma(\varepsilon) = \varepsilon^{1/2} \), (A5), (A6), then for each \( \varepsilon \in (0, 1] \), there exists a unique weak solution of \((P)_\varepsilon\).

This proposition implies that, for the well-posedness of the Cahn–Hilliard system, the growth condition (A2) is not essential. It can be recovered by the strong assumption (A6). The proof of this proposition is given in Section 4. Indeed, we can prove this proposition by considering the approximate problem given in Proposition 3.1.

Our main theorem is now given.

Theorem 2.1. With assumptions (A1)–(A5), for each \( \varepsilon \in (0, 1] \), let \( (u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon) \) be the weak solution of \((P)_{\varepsilon}\) obtained in Proposition 2.1. Then, there exists a weak solution \((u, \xi)\) of \((P)\) and \((u, \xi)\) characterized by \((u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)\) in the following sense:

\[
\begin{align*}
    u_\varepsilon &\to u \quad \text{strongly in } C([0, T]; V^*) \text{ and weakly in } H^1(0, T; V^*) \cap L^\infty(0, T; H), \\
    \xi_\varepsilon &\to \xi \quad \text{weakly in } L^2(0, T; H), \\
    \mu_\varepsilon &\to \mu := \xi - f \quad \text{weakly in } L^2(0, T; V)
\end{align*}
\]

as \( \varepsilon \searrow 0 \). Moreover, the component \( u \) of the solution of \((P)\) is uniquely determined. Also, if \( \beta \) is single-valued, then the component \( \xi \) of the solution of \((P)\) is also unique.

The second theorem relates to improving the well-posedness result of [9,10].

Theorem 2.2. Given assumptions (A1), (A3) with \( \sigma(\varepsilon) = \varepsilon^{1/2} \), (A5), (A6), the same statement as in Theorem 2.1 holds.

3 Approximate problem and uniform estimates

The proof of the main theorems exploits the characterization of the nonlinear diffusion equation through the asymptotic limits of the Cahn–Hilliard system [7,11]. To apply it, we consider the second-order approximation of the nonlinear term \( \beta \). At this level of approximation, we obtain a uniform estimate independent of the approximation parameters.

3.1 Approximate problem for \((P)_\varepsilon\)

We consider an approximate problem to show the well-posedness of \((P)_\varepsilon\). For each \( \lambda \in (0, 1] \), we define \( \beta_\lambda : \mathbb{R} \to \mathbb{R} \) by

\[
\beta_\lambda(r) := \frac{1}{\lambda} \left( r - J_\lambda(r) \right) \quad \text{for all } r \in \mathbb{R},
\]

where the resolvent operator \( J_\lambda : \mathbb{R} \to \mathbb{R} \) is given by

\[
J_\lambda(r) := (I + \lambda \beta)^{-1}(r) \quad \text{for all } r \in \mathbb{R}.
\]

Also, we define the Moreau–Yosida regularization \( \tilde{\beta}_\lambda \) of \( \tilde{\beta} : \mathbb{R} \to \mathbb{R} \) by

\[
\tilde{\beta}_\lambda(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\lambda} |r - s|^2 + \tilde{\beta}(s) \right\} = \frac{1}{2\lambda} |r - J_\lambda(r)|^2 + \tilde{\beta}(J_\lambda(r)) \quad \text{for all } r \in \mathbb{R}.
\]
Now, we consider the problem \((P)_{\varepsilon,\lambda}\) for the viscous Cahn–Hilliard like system:

\[
\begin{aligned}
\frac{\partial u_{\varepsilon,\lambda}}{\partial t} - \Delta \mu_{\varepsilon,\lambda} &= 0 \quad \text{a.e. in } Q, \\
\mu_{\varepsilon,\lambda} &= \lambda \frac{\partial u_{\varepsilon,\lambda}}{\partial t} - \varepsilon \Delta u_{\varepsilon,\lambda} + \beta_{\lambda}(u_{\varepsilon,\lambda}) + \pi_{\varepsilon}(u_{\varepsilon,\lambda}) - f \quad \text{a.e. in } Q, \\
\partial_{\nu} u_{\varepsilon,\lambda} + \kappa u_{\varepsilon,\lambda} &= 0, \quad \partial_{\nu} \mu_{\varepsilon,\lambda} + \kappa \mu_{\varepsilon,\lambda} = 0 \quad \text{a.e. on } \Sigma, \\
u_{\varepsilon,\lambda}(0) &= u_{0\varepsilon} \quad \text{a.e. in } \Omega.
\end{aligned}
\]

Define \(A : D(A) \to H\) by \(Au = -\Delta u\) in \(H\) with \(D(A) = W\); the treatment of \(A\) is given in Appendix. From the well-known abstract theory of the doubly nonlinear evolution equation [8], we obtain the following well-posedness result (see, also [6,16,17]):

**Proposition 3.1.** For each \(\lambda \in (0,1]\), there exists a unique

\[
u_{\varepsilon,\lambda} \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)
\]

such that \(\nu_{\varepsilon,\lambda}\) satisfies the following Cauchy problem

\[
\begin{aligned}
(\lambda I + F^{-1})u_{\varepsilon,\lambda}'(t) + \varepsilon Au_{\varepsilon,\lambda}(t) &

= -\beta_{\lambda}(u_{\varepsilon,\lambda}(t)) - \pi_{\varepsilon}(u_{\varepsilon,\lambda}(t)) + f(t) \quad \text{in } H, \quad \text{for a.a. } t \in (0,T), \\
u_{\varepsilon,\lambda}(0) &= u_{0\varepsilon} \quad \text{in } H.
\end{aligned}
\]

With this level of abstractness, the Cahn–Hilliard system with Robin boundary condition is essentially the same as in previous studies. Therefore, we omit the proof of this proposition.

Now, for each \(\lambda \in (0,1]\), we put

\[
\mu_{\varepsilon,\lambda}(t) := \lambda u_{\varepsilon,\lambda}'(s) + \varepsilon Au_{\varepsilon,\lambda}(t) + \beta_{\lambda}(u_{\varepsilon,\lambda}(t)) + \pi_{\varepsilon}(u_{\varepsilon,\lambda}(t)) - f(t) \quad \text{in } H,
\]

for a.a. \(t \in (0,T)\). Then, we can rewrite the evolution equation (3.1) as

\[
F^{-1}u_{\varepsilon,\lambda}'(t) + \mu_{\varepsilon,\lambda}(t) = 0 \quad \text{in } V,
\]

for a.a. \(t \in (0,T)\). We remark here that we do not need the projection to \(\mu_{\varepsilon,\lambda}\) because of the boundary condition (cf. [6,16,17]). This is different from that for the Neumann boundary condition.

### 3.2 Uniform estimates

To prove the convergence theorem, we now obtain the uniform estimates independent of \(\varepsilon,\lambda\).
Lemma 3.1. There exists a positive constant $M_1$ and two values $\bar{\lambda}, \varepsilon \in (0,1)$, depending only on the data, such that

$$\int_0^t \left| u'_{\varepsilon,\lambda}(s) \right|^2_{V^*} ds + 2\lambda \int_0^t \left| u'_{\varepsilon,\lambda}(s) \right|^2_H ds + \varepsilon \left| u_{\varepsilon,\lambda}(t) \right|^2_V + \left| \bar{\beta}_\lambda(u_{\varepsilon,\lambda}(t)) \right|_{L^1(\Omega)} + \frac{c_1}{4} \left| u_{\varepsilon,\lambda}(t) \right|^2_H \leq M_1,$$

for all $t \in [0,T]$, $\lambda \in (0, \bar{\lambda})$ and $\varepsilon \in (0, \bar{\varepsilon}]$.

Proof We test (3.1) at time $s \in (0,T)$ for $u'_{\varepsilon,\lambda}(s) \in H$. Then, we see that

$$\lambda \left| u'_{\varepsilon,\lambda}(s) \right|^2_H + \left| u'_{\varepsilon,\lambda}(s) \right|^2_{V^*} - \varepsilon (\Delta u_{\varepsilon,\lambda}(s), u'_{\varepsilon,\lambda}(s))_H$$
$$+ \frac{d}{ds} \int_{\Omega} \bar{\beta}_\lambda(u_{\varepsilon,\lambda}(s)) dx = -\left( \pi_\varepsilon(u_{\varepsilon,\lambda}(s)), u'_{\varepsilon,\lambda}(s) \right)_H + (f(s), u'_{\varepsilon,\lambda}(s))_H$$

for a.a. $s \in (0,T)$. We have now from the boundary condition of $u_{\varepsilon,\lambda}(s)$

$$-(\Delta u_{\varepsilon,\lambda}(s), u'_{\varepsilon,\lambda}(s))_H = -\int_{\Gamma} \partial_n u_{\varepsilon,\lambda}(s) u'_{\varepsilon,\lambda}(s) d\Gamma + \int_{\Omega} \nabla u_{\varepsilon,\lambda}(s) \cdot \nabla u'_{\varepsilon,\lambda}(s) dx$$
$$= \kappa \int_{\Gamma} u_{\varepsilon,\lambda}(s) u'_{\varepsilon,\lambda}(s) d\Gamma + \int_{\Omega} \nabla u_{\varepsilon,\lambda}(s) \cdot \nabla u'_{\varepsilon,\lambda}(s) dx$$
$$= \frac{\kappa}{2} \frac{d}{ds} \int_{\Gamma} \left| u_{\varepsilon,\lambda}(s) \right|^2 d\Gamma + \frac{1}{2} \frac{d}{ds} \int_{\Omega} \left| \nabla u_{\varepsilon,\lambda}(s) \right|^2 dx$$

for a.a. $s \in (0,T)$. Integrating (3.6) with respect to $s$ over interval $[0,t]$, and using the above, we infer that

$$\int_0^t \left| u'_{\varepsilon,\lambda}(s) \right|^2_{V^*} ds + \lambda \int_0^t \left| u'_{\varepsilon,\lambda}(s) \right|^2_H ds + \frac{\varepsilon}{2} \left| u_{\varepsilon,\lambda}(t) \right|^2_V + \int_{\Omega} \bar{\beta}_\lambda(u_{\varepsilon,\lambda}(t)) dx$$
$$= \frac{\varepsilon}{2} \left| u_{\varepsilon,\lambda}(0) \right|^2_V + \int_{\Omega} \bar{\beta}_\lambda(u_{\varepsilon,\lambda}(0)) dx + \int_{\Omega} \widehat{\pi}_\varepsilon(u_{\varepsilon,\lambda}) dx - \int_{\Omega} \widehat{\pi}_\varepsilon(u_{\varepsilon,\lambda}(t)) dx$$
$$+ \int_0^t \left( u'_{\varepsilon,\lambda}(s), f(s) \right)_{V^*} ds$$

for all $t \in [0,T]$, where $\widehat{\pi}_\varepsilon$ is the primitive of $\pi_\varepsilon$ given by $\widehat{\pi}_\varepsilon(r) := \int_0^r \pi_\varepsilon(\tau) d\tau$ for all $r \in \mathbb{R}$. Now, aided by assumption (A2), we have

$$\bar{\beta}_\lambda(r) = \frac{1}{2\lambda} \left| r - J_\lambda(r) \right|^2 + \beta_\lambda(J_\lambda(r))$$
$$\geq \frac{1}{2\lambda} \left| r - J_\lambda(r) \right|^2 + c_1 |J_\lambda(r)|^2 - c_2$$

for all $r \in \mathbb{R}$. 

Hence, putting \( \bar{\lambda} := \min\{1, 1/(2c_1)\} \), we see that \( c_1/2 \leq 1/(4\bar{\lambda}) \). It follows that

\[
\int_\Omega \tilde{\beta}_\lambda(u_{\varepsilon, \lambda}(s)) \geq \frac{1}{2} \int_\Omega \tilde{\beta}_\lambda(u_{\varepsilon, \lambda}(s)) \, dx + \frac{1}{4\lambda}\left| u_{\varepsilon, \lambda}(s) - J_\lambda(u_{\varepsilon, \lambda}(s)) \right|^2_H \\
+ \frac{c_1}{2} \left| J_\lambda(u_{\varepsilon, \lambda}(s)) \right|^2_H - \frac{c_2}{2} |\Omega| \geq \frac{1}{2} \int_\Omega \tilde{\beta}_\lambda(u_{\varepsilon, \lambda}(s)) \, dx + \frac{c_1}{4} \left| u_{\varepsilon, \lambda}(s) \right|^2_H - \frac{c_2}{2} |\Omega|
\]

for a.a. \( s \in (0, T) \). Next, we use the Maclaurin expansion and (A3) to obtain

\[
|\tilde{\pi}_\varepsilon(r)| \leq |\pi_\varepsilon(0)| |r| + \frac{1}{2} |\pi_\varepsilon'| L^\infty(\mathbb{R}) r^2 \leq c_3 \sigma(\varepsilon)(1 + r^2)
\]

for all \( r \in \mathbb{R} \). Also, with assumption (A3), there exists \( \bar{\varepsilon} \in (0, 1] \) such that

\[
\sigma(\varepsilon) \leq \frac{c_1}{8c_3(1 + |\Omega|)}
\]

for all \( \varepsilon \in (0, \bar{\varepsilon}] \); that is, we deduce that

\[
- \int_\Omega \tilde{\pi}_\varepsilon(u_{\varepsilon, \lambda}(t)) \, dx \leq c_3 \sigma(\varepsilon) \int_\Omega \left( 1 + \left| u_{\varepsilon, \lambda}(t) \right|^2 \right) \, dx \\
\leq \frac{c_1}{8} \left( 1 + \left| u_{\varepsilon, \lambda}(t) \right|^2 \right)
\]

for all \( t \in [0, T] \). Hence, applying the Young inequality to (3.7) and collecting all of the above, we see that there exists a positive constant \( M_1 \) depending only on \( c_1, c_2, c_4, |\Omega|, \) and \( |f|_{L^2(0, T; V)} \), independent of \( \varepsilon \in (0, \bar{\varepsilon}], \lambda \in (0, \bar{\lambda}] \) such that (3.4) holds. Finally, we have from (3.3)

\[
\int_0^t \left| \mu_{\varepsilon, \lambda}(s) \right|^2_V \, ds = \int_0^t \left| u'_{\varepsilon, \lambda}(s) \right|^2_{V^*} \leq M_1
\]

for all \( t \in [0, T] \). \( \square \)

**Lemma 3.2.** There exist positive constants \( M_2 \) and \( M_3 \), independent of \( \varepsilon \in (0, \bar{\varepsilon}] \) and \( \lambda \in (0, \bar{\lambda}] \), such that

\[
\int_0^t \left| \beta_\lambda(u_{\varepsilon, \lambda}(s)) \right|^2_H \, ds \leq M_2,
\]

\[
\int_0^t \left| \varepsilon \Delta u_{\varepsilon, \lambda}(s) \right|^2_H \, ds \leq M_3
\]

for all \( t \in [0, T] \).

**Proof** We test (3.2) for times \( s \in (0, T) \) by \( \beta_\lambda(u_{\varepsilon, \lambda}(s)) \in H \). Then, we see that

\[
(\mu_{\varepsilon, \lambda}(s), \beta_\lambda(u_{\varepsilon, \lambda}(s)))_H = -\varepsilon \left( \Delta u_{\varepsilon, \lambda}(s), \beta_\lambda(u_{\varepsilon, \lambda}(s)) \right)_H + \left| \beta_\lambda(u_{\varepsilon, \lambda}(s)) \right|^2_H \\
+ \left( \lambda u'_{\varepsilon, \lambda}(s) + \pi_\varepsilon(u_{\varepsilon, \lambda}(s)) - f(s), \beta_\lambda(u_{\varepsilon, \lambda}(s)) \right)_H
\]
for a.a. $s \in (0, T)$. Calculating the first term on the right-hand side of the above equation, we infer from the boundary condition of $u_{\varepsilon, \lambda}(s)$ that

$$-\varepsilon (\Delta u_{\varepsilon, \lambda}(s), \beta_{\lambda}(u_{\varepsilon, \lambda}(s))) = \varepsilon \int_{\Omega} \beta'_{\lambda}(u_{\varepsilon, \lambda}(s)) |\nabla u_{\varepsilon, \lambda}(s)|^2 dx + \kappa \varepsilon \int_{\Gamma} u_{\varepsilon, \lambda}(s) \beta_{\lambda}(u_{\varepsilon, \lambda}(s)) d\Gamma \geq 0$$

for a.a. $s \in (0, T)$ because $\beta$ is monotonic. Hence, applying the Young inequality, we obtain

$$|\beta_{\lambda}(u_{\varepsilon, \lambda}(s))|^2_H \leq \left( \mu_{\varepsilon, \lambda}(s) - \lambda u'_{\varepsilon, \lambda}(s) - \pi_{\varepsilon}(u_{\varepsilon, \lambda}(s)) + f(s), \beta_{\lambda}(u_{\varepsilon, \lambda}(s)) \right)_H$$

$$\leq \frac{1}{2} |\beta_{\lambda}(u_{\varepsilon, \lambda}(s))|^2_H + 2 \left( |\mu_{\varepsilon, \lambda}(s)|^2_H + \lambda^2 |u'_{\varepsilon, \lambda}(s)|^2_H + |\pi_{\varepsilon}(u_{\varepsilon, \lambda}(s))|^2_H + |f(s)|^2_H \right)$$

for a.a. $s \in (0, T)$. Integrating with respect to $s$ over $[0, t]$ and applying Lemma 3.2, we see that there exists a positive constant $M_2$ depending only on $M_1$, $c_3$, and $|f|_{L^2(0,T;H)}$ such that estimate (3.10) holds.

Next, by comparing with (3.2), we obtain (3.11) for some positive constant $M_3$. $\square$

## 4 Proof of convergence theorem

We next show the existence of solution of $(P)_\varepsilon$ by passing to the limit for the approximate problem $(P)_{\varepsilon, \lambda}$.

### 4.1 Passage to the limit $\lambda \searrow 0$

**Proof of Proposition 2.1.** From previous estimates established in Lemmas 3.1 and 3.2, we see that there exists a subsequence $\{\lambda_k\}_{k \in \mathbb{N}}$ with $\lambda_k \searrow 0$ as $k \nearrow +\infty$ and some limit functions $u_\varepsilon \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W)$, $\mu_\varepsilon \in L^2(0, T; V)$, and $\xi_\varepsilon \in L^2(0, T; H)$ such that

$$u_{\varepsilon, \lambda_k} \to u_\varepsilon \quad \text{weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; V),$$

$$\lambda_k u_{\varepsilon, \lambda_k} \to 0 \quad \text{strongly in } H^1(0, T; H),$$

$$\mu_{\varepsilon, \lambda_k} \to \mu_\varepsilon \quad \text{weakly in } L^2(0, T; V),$$

$$\beta_{\lambda_k}(u_{\varepsilon, \lambda_k}) \to \xi_\varepsilon \quad \text{weakly in } L^2(0, T; H),$$

$$\varepsilon \Delta u_{\varepsilon, \lambda_k} \to \varepsilon \Delta u_\varepsilon \quad \text{weakly in } L^2(0, T; H)$$

as $k \nearrow +\infty$. From (4.12) and well-known compactness results (see, e.g., [22]), we obtain

$$u_{\varepsilon, \lambda_k} \to u_\varepsilon \quad \text{strongly in } C([0, T]; H)$$

as $k \nearrow +\infty$. From (4.17), we deduce that $u_\varepsilon(0) = u_{0\varepsilon}$ a.e. in $\Omega$. Moreover, from (4.17) and the Lipschitz continuity of $\pi_{\varepsilon}$, we deduce that

$$\pi_{\varepsilon}(u_{\varepsilon, \lambda_k}) \to \pi_{\varepsilon}(u_\varepsilon) \quad \text{strongly in } C([0, T]; H)$$

(4.18)
as \( k \downarrow +\infty \). Also, applying (4.15), (4.17), and the monotonicity of \( \beta \), we obtain \( \xi \in \beta(u) \) a.e. in \( Q \). Finally, given the level of approximation associated with the weak formulations of (3.2), (3.3), taking the limit as \( k \uparrow +\infty \), and using (4.12)–(4.18), we find that \( (u,\mu,\xi) \) satisfies (2.4)–(2.5).

### 4.2 Passage to the limit \( \varepsilon \searrow 0 \)

**Proof of Theorem 2.1.** From weakly and strongly convergence, (4.12)–(4.16), the same kind of uniform estimates in Lemmas 3.1 and 3.2 holds for \( (u,\mu,\xi) \), that is,

\[
\int_0^t \left| u'(s) \right|^2_V ds + \varepsilon \left| u(t) \right|^2_V + \frac{c_1}{4} \left| u(t) \right|^2_H \leq M_1,
\]

\[
\int_0^t \left| \mu(s) \right|^2_V ds \leq M_1,
\]

\[
\int_0^t \left| \xi(s) \right|^2_H ds \leq M_2,
\]

\[
\int_0^t \varepsilon \Delta u(s) \left| H \right|^2 ds \leq M_3
\]

for all \( t \in [0,T] \). Hence, there exists a subsequence \( \{\varepsilon_k\}_{k \in \mathbb{N}} \) with \( \varepsilon_k \searrow 0 \) as \( k \uparrow +\infty \) and some limits functions \( u \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \), \( \mu \in L^2(0,T;V) \), \( \xi \in L^2(0,T;H) \) such that

\[
u_{\varepsilon_k} \to u \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;H),
\]

\[
\varepsilon_k \Delta u_{\varepsilon_k} \to 0 \quad \text{strongly in } L^\infty(0,T;V),
\]

\[
\mu_{\varepsilon_k} \to \mu \quad \text{weakly in } L^2(0,T;V),
\]

\[
\xi_{\varepsilon_k} \to \xi \quad \text{weakly in } L^2(0,T;H)
\]

as \( k \uparrow +\infty \). From (4.19) and the well-known Ascoli–Arzelà theorem (see, e.g., [22]), we obtain

\[
u_{\varepsilon_k} \to u \quad \text{strongly in } C([0,T];V^*).
\]

Moreover, by virtue of (4.20),

\[
\varepsilon_k \Delta u_{\varepsilon_k} \to 0 \quad \text{weakly in } L^2(0,T;H)
\]

as \( k \uparrow +\infty \). Aided by assumption (A3), we see that

\[
|\pi_{\varepsilon_k}(u_{\varepsilon_k})| \leq |\pi'_{\varepsilon_k}|_{L^\infty(\mathbb{R})} |u_{\varepsilon_k}| + |u_{\varepsilon_k}(0)| \leq c_3 \sigma(\varepsilon_k)(1 + |u_{\varepsilon_k}|) \quad \text{a.e. in } Q.
\]

Therefore, we obtain

\[
\pi_{\varepsilon_k}(u_{\varepsilon_k}) \to 0 \quad \text{strongly in } L^\infty(0,T;H)
\]

as \( k \uparrow +\infty \). Now, from (2.4), we have

\[
\left\langle u'(t),z \right\rangle_{V^*,V} + \left\langle Fz,\mu(t) \right\rangle_{V^*,V} = 0 \quad \text{for all } z \in V,
\]
for a.a. \( t \in (0, T) \). Therefore, using (4.19), (4.21) and (4.23), we obtain (2.8) by passing to the limit in the above, that is, (2.1). On the other hand, in (2.5), using (4.21), (4.22), (4.24), (4.25) and performing a comparison, we obtain \( \mu = \xi - f \in L^2(0, T; H) \). Moreover, by comparison, this gives us the additional regularity \( \xi = \mu + f \in L^2(0, T; V) \) because \( \mu, f \in L^2(0, T; V) \). We now have \( u \in C([0, T]; V^*) \cap L^\infty(0, T; H) \); the function \( u \) is thus weakly continuous from \([0, T] \) to \( H \), that is, (2.2) holds. Finally, it remains to show that \( \xi \in \beta(u) \) a.e. in \( Q \). For this purpose, we show that

\[
\limsup_{k \to +\infty} \int_0^T \langle \xi_{\varepsilon_k}(t), u_{\varepsilon_k}(t) \rangle_H \, dt \leq \int_0^T \langle \xi(t), u(t) \rangle_H \, dt. \tag{4.26}
\]

To this aim, testing (2.5) for \( u_{\varepsilon_k}(t) \) and integrating over \([0, T] \), we deduce from the boundary condition that

\[
\int_0^T \langle \xi_{\varepsilon_k}(t), u_{\varepsilon_k}(t) \rangle_H \, dt = \int_0^T \langle \mu_{\varepsilon_k}(t) + f(t), u_{\varepsilon_k}(t) \rangle_H \, dt + \varepsilon_k \int_0^T \langle \Delta u_{\varepsilon_k}(t), u_{\varepsilon_k}(t) \rangle_H \, dt
\]

\[
- \int_0^T \langle \pi_{\varepsilon_k}(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t) \rangle_H \, dt
\]

\[
= \int_0^T \langle u_{\varepsilon_k}(t), \mu_{\varepsilon_k}(t) + f(t) \rangle_{V^*, V} \, dt - \varepsilon_k \int_0^T |\nabla u_{\varepsilon_k}(t)|_{H^1}^2 \, dt
\]

\[
- \varepsilon_k \kappa \int_0^T |u_{\varepsilon_k}(t)|_{H^1}^2 \, dt - \int_0^T \langle \pi_{\varepsilon_k}(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t) \rangle_H \, dt
\]

\[
\leq \int_0^T \langle u_{\varepsilon_k}(t), \mu_{\varepsilon_k}(t) + f(t) \rangle_{V^*, V} \, dt - \int_0^T \langle \pi_{\varepsilon_k}(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t) \rangle_H \, dt.
\]

From (4.21) and (4.23), we see that

\[
\lim_{k \to \infty} \int_0^T \langle u_{\varepsilon_k}(t), \mu_{\varepsilon_k}(t) + f(t) \rangle_{V^*, V} \, dt = \int_0^T \langle u(t), \mu(t) + f(t) \rangle_{V^*, V} \, dt
\]

\[
= \int_0^T \langle u(t), \mu(t) + f(t) \rangle_H \, dt
\]

\[
= \int_0^T \langle \xi(t), u(t) \rangle_H \, dt.
\]

Moreover, from (4.19) and (4.25), we have

\[
\lim_{k \to \infty} \int_0^T \langle \pi_{\varepsilon_k}(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t) \rangle_H \, dt = 0.
\]

Therefore, (4.26) holds. Applying the closedness theorem with respect to weakly-weakly convergence (see, e.g., [3 Lemma 2.3]), we obtain the fact that \( \xi \in \beta(u) \) a.e. in \( Q \). Thus, the proof of existence stated in Theorem 2.1 is complete.

Next, we show that the component \( u \) of the weak solution of (P) is unique. Now, for \( i = 1, 2 \), let \( (u^{(i)}, \xi^{(i)}) \) be weak solutions of (P), respectively. Then, from (2.8), we have

\[
\langle u'_1(t) - u'_2(t), z \rangle_{V^*, V} + \int_\Omega \nabla (\mu_1(t) - \mu_2(t)) \cdot \nabla z \, dx
\]

\[
+ \kappa \int_\Gamma (\mu_1(t) - \mu_2(t)) z \, d\Gamma = 0 \quad \text{for all } z \in V.
\]
Nonlinear diffusion equations with Robin boundary conditions

Here, putting $z := F^{-1}(u_1(t) - u_2(t))$ in $V$ at time $t \in (0, T)$, using the monotonicity of $\beta$, and $\mu_1 - \mu_2 = \xi_1 - \xi_2$ a.e. in $Q$, we infer that

$$
\int_{\Omega} \nabla (\mu_1(t) - \mu_2(t)) \cdot \nabla F^{-1}(u_1(t) - u_2(t)) \, dx + \kappa \int_{\Gamma} (\mu_1(t) - \mu_2(t)) F^{-1}(u_1(t) - u_2(t)) \, d\Gamma \\
= \langle FF^{-1}(u_1(t) - u_2(t)), \xi_1(t) - \xi_2(t) \rangle_{V^*, V} \\
= \langle u_1(t) - u_2(t), \xi_1(t) - \xi_2(t) \rangle_{V^*, V} \\
= (\xi_1(t) - \xi_2(t), u_1(t) - u_2(t))_H \\
\geq 0.
$$

Therefore, for all $t \in [0, T]$, we deduce that

$$
\frac{1}{2} \|u_1(t) - u_2(t)\|_{V^*}^2 + \int_0^t (\xi_1(t) - \xi_2(t), u_1(t) - u_2(t))_H \, dt \leq 0,
$$

which implies that the component $u$ is unique. Thus, we obtain our result.

\[\square\]

**Remark 4.1.** (i) Allowing the maximal monotone graph $\beta$ to be multi-valued, the component $\xi$ is therefore not uniquely determined. However, if $\beta$ is single-valued, then $\xi = \beta(u)$ is unique.

(ii) The argument for the proof of Theorem 2.1 is essentially the same as in [7]. Also, we can obtain the error estimates by some reinforcement. Indeed, if we add the assumption in (2.6) of (A3) by $\sigma(\varepsilon) := \varepsilon^{1/2}$. Moreover, if we add the following assumption in (A5), then there exists $c_4 > 0$ such that

$$
|u_{0\varepsilon} - u_0|_{V^*} \leq c_4 \varepsilon^{\frac{1}{4}} \text{ for all } \varepsilon \in (0, 1].
$$

Then we obtain error estimate

$$
|u_{\varepsilon} - u|_{C([0, T]; V^*)}^2 + \int_0^T (\xi_\varepsilon(t) - \xi(t), u_\varepsilon(t) - u(t))_H \, dt \leq C^* \varepsilon^{\frac{1}{4}}
$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$, where $C^*$ is a positive constant depending only on the data (see, [7, Theorem 5.1]).

## 5 Improvement of the results

As mentioned in Introduction, for all of the results from the “Hilbert space approach” to nonlinear diffusion equation, the growth condition (A2) for $\beta$ is a very important assumption. Nevertheless, it is too restricted from the perspective of applications. For this reason, improving the growth condition was studied in [10] using the “lower semicontinuous convex extension”. See also a different approach given in [4]. In this section, we consider the improved growth condition (A2) for $\beta$. Hereafter, we assume (A1), and (A3) with $\sigma(\varepsilon) = \varepsilon^{1/2}$, (A5), and (A6); that is, assumption (A2) is avoided.
Proof of Theorem 2.2. Recall (3.3) in the following form
\[ u_{\varepsilon,\lambda}'(s) + F_{\mu_{\varepsilon,\lambda}}(s) = 0 \quad \text{in } V^*, \]
for a.a. \( s \in (0, T) \). Multiplying \( u_{\varepsilon,\lambda}(s) \) by the above, we see that
\[ \langle u_{\varepsilon,\lambda}'(s), u_{\varepsilon,\lambda}(s) \rangle_{V^*, V} + \int_{\Omega} \nabla \mu_{\varepsilon,\lambda}(s) \cdot \nabla u_{\varepsilon,\lambda}(s) \, dx + \kappa \int_{\Gamma} \mu_{\varepsilon,\lambda}(s) u_{\varepsilon,\lambda}(s) \, d\Gamma = 0 \quad (5.29) \]
for a.a. \( s \in (0, T) \). Moreover, testing (3.2) with \( -\Delta u_{\varepsilon,\lambda}(s) \) and using boundary conditions and assumption (A6), we deduce that
\[
\begin{align*}
\int_{\Omega} \nabla \mu_{\varepsilon,\lambda}(s) \cdot \nabla u_{\varepsilon,\lambda}(s) \, ds + \kappa \int_{\Gamma} \mu_{\varepsilon,\lambda}(s) u_{\varepsilon,\lambda}(s) \, d\Gamma \\
= \frac{\lambda}{2} d_s |\nabla u_{\varepsilon,\lambda}(s)|^2_{H^d} + \frac{\kappa \lambda}{2} d_s |u_{\varepsilon,\lambda}(s)|^2_{H^r} + \varepsilon |\Delta u_{\varepsilon,\lambda}(s)|^2_{H} + \int_{\Omega} \beta_{\Lambda}(u_{\varepsilon,\lambda}(s)) |\nabla u_{\varepsilon,\lambda}(s)|^2 \, dx \\
+ \kappa \int_{\Gamma} u_{\varepsilon,\lambda}(s) \beta_{\Lambda}(u_{\varepsilon,\lambda}(s)) \, d\Gamma - (\pi_{\varepsilon}(u_{\varepsilon,\lambda}(s)), \Delta u_{\varepsilon,\lambda}(s))_H + (\Delta f(s), u_{\varepsilon,\lambda}(s))_H \\
\end{align*}
\]
(5.30)
for a.a. \( s \in (0, T) \). Here, we used
\[
(f(s), \Delta u_{\varepsilon,\lambda}(s))_H = - \int_{\Omega} \nabla f(s) \cdot \nabla u_{\varepsilon,\lambda}(s) \, dx + \int_{\Gamma} f(s) \partial_{\nu} u_{\varepsilon,\lambda}(s) \, d\Gamma \\
= - \int_{\Omega} \nabla f(s) \cdot \nabla u_{\varepsilon,\lambda}(s) \, dx - \kappa \int_{\Gamma} f(s) u_{\varepsilon,\lambda}(s) \, d\Gamma \\
= - \int_{\Omega} \nabla f(s) \cdot \nabla u_{\varepsilon,\lambda}(s) \, dx + \int_{\Gamma} \partial_{\nu} f(s) u_{\varepsilon,\lambda}(s) \, d\Gamma \\
= (\Delta f(s), u_{\varepsilon,\lambda}(s))_H 
\]
for a.a. \( s \in (0, T) \). Here, using (A3) and the Young inequality, we infer that
\[
(\pi_{\varepsilon}(u_{\varepsilon,\lambda}(s)), \Delta u_{\varepsilon,\lambda}(s))_H \leq \int_{\Omega} \left( |\pi_{\varepsilon}'|_{L^\infty(\mathbb{R})} |u_{\varepsilon,\lambda}(s)| + |\pi_{\varepsilon}(0)| \right) |\Delta u_{\varepsilon,\lambda}(s)| \, dx \\
\leq \int_{\Omega} c_3 \varepsilon \frac{1}{2} \left( 1 + |u_{\varepsilon,\lambda}(s)| \right) |\Delta u_{\varepsilon,\lambda}(s)| \, dx \\
\leq \varepsilon |\Delta u_{\varepsilon,\lambda}(s)|^2_{H} + c_3^2 \left( |\Omega| + |u_{\varepsilon,\lambda}(s)|^2_{H} \right) 
\]
(5.31)
for a.a. \( s \in (0, T) \). Then, combining (5.29) - (5.31) and integrating the resultant with respect to \( s \) over interval \([0, t]\), we obtain
\[
\begin{align*}
\frac{1}{2} |u_{\varepsilon,\lambda}(t)|^2_{H} + \frac{\lambda}{2} |\nabla u_{\varepsilon,\lambda}(t)|^2_{H^d} + \frac{\kappa \lambda}{2} |u_{\varepsilon,\lambda}(t)|^2_{H^r} + \varepsilon \int_{0}^{t} |\Delta u_{\varepsilon,\lambda}(s)|^2_{H} \, ds \\
\leq \frac{1}{2} |u_{0e}|^2_{H} + \frac{\lambda}{2} |\nabla u_{0e}|^2_{H^d} + \frac{\kappa \lambda}{2} |u_{0e}|^2_{H^r} \\
+ c_3^2 \int_{0}^{t} \left( |\Omega| + |u_{\varepsilon,\lambda}(s)|^2_{H} \right) \, ds + \frac{1}{2} \int_{0}^{t} |\Delta f(s)|^2_{H} \, ds + \frac{1}{2} \int_{0}^{t} |u_{\varepsilon,\lambda}(s)|^2_{H} \, ds 
\end{align*}
\]
(5.32)
for all \( t \in [0, T] \). Now, note that we can take \( \lambda \leq \epsilon \), because \( \lambda \) tends to 0 with fixed \( \epsilon \).

Also, from (2.7) of assumption (A5), we have

\[
\frac{1}{2} |u_\epsilon|_H^2 + \frac{\lambda}{2} |\nabla u_\epsilon|_{H^d}^2 + \frac{\kappa \lambda}{2} |u_\epsilon|_{H^d}^2 \leq \frac{3}{2} (1 + \kappa)c_4.
\]  

(5.33)

Then, using (5.32), (5.33), and the Gronwall inequality, it follows that

\[
\left| u_{\epsilon, \lambda}(t) \right|_H^2 + \lambda |\nabla u_{\epsilon, \lambda}(t)|_{H^d}^2 + \kappa \lambda |u_{\epsilon, \lambda}(t)|_{H^d}^2 \\
\leq \left\{ 3(1 + \kappa)c_4 + \int_0^T |\Delta f(s)|_H^2 ds + 2c_3^2 |\Omega| T \right\} \exp \left\{ (2c_3^2 + 1)T \right\} =: M_4
\]

for all \( t \in [0, T] \). Moreover,

\[
\epsilon \int_0^t |\Delta u_{\epsilon, \lambda}(s)|_H^2 ds \leq M_4 (1 + 2c_3^2 T + T)
\]

for all \( t \in [0, T] \). Hence, according to Lemma 3.1, using (3.7) and (3.9) without (3.8), we obtain

\[
\frac{1}{2} \int_0^t |u_{\epsilon, \lambda}'(s)|_V^2 ds + \lambda \int_0^t |u_{\epsilon, \lambda}'(s)|_H^2 ds + \frac{\epsilon}{2} |u_{\epsilon, \lambda}(t)|_V^2 + |\tilde{\beta}_\lambda(u_{\epsilon, \lambda}(t))|_{L^1(\Omega)} \\
\leq \frac{3}{2} c_4 + \frac{c_1}{8} (1 + c_4) + \frac{c_1}{8} (1 + M_4) + \frac{1}{2} |f|_{L^2(0, T; V)}^2
\]

for all \( t \in [0, T] \). Therefore, we obtain the same kind of uniform estimates (3.4) and (3.5) in Lemma 3.1 without the growth condition (A2). The rest of the proof of Theorem 2.2 is the same as that of Theorem 2.1.

\[ \square \]

**Remark 5.1.** As stated in Remark 4.1, we also can improve the error estimate, by assuming (A1), (A3) with \( \sigma(\epsilon) := \epsilon^{1/2} \), (A5) and (4.27). Then, (4.28) is improved by stating it in the form

\[
|u_\epsilon - u|^2_{C([0, T]; V \ast)} + \int_0^T \left( \xi_\epsilon(t) - \xi(t), u_\epsilon(t) - u(t) \right)_H dt \leq C^* \epsilon^{\frac{1}{4}}
\]

for all \( \epsilon \in (0, \bar{\epsilon}] \) (see, [7, Theorem 6.1]).

### 6 Asymptotic limits to solutions of Neumann problem

In this section, we establish the order of convergence between the Robin problem and the Neumann problem related to (P).

We rewrite our problem for (P), hereafter denoted (P)\(_R\), as follows:

\[
(P) \quad \begin{cases}
\frac{\partial u_\kappa}{\partial t} - \Delta_\kappa = g_\kappa, & \kappa \in \beta(u_\kappa) \quad \text{in } Q, \\
\partial_\nu \xi_\kappa + \kappa \xi_\kappa = h_\kappa \quad \text{on } \Sigma, \\
u_\kappa(0) = u_{0\kappa} \quad \text{in } \Omega.
\end{cases}
\]
for \( \kappa > 0 \), where \( g_\kappa : Q \to \mathbb{R}, \ h_\kappa : \Sigma \to \mathbb{R}, \) and \( u_{0\kappa} : \Omega \to \mathbb{R} \) are given data, and \( \beta \) is same maximal monotone graph as in (A1). Moreover, we recall the previous result \([7]\) for problem (P) subject to the Neumann boundary condition,

\[
(P)_N \begin{cases} \\
\frac{\partial u}{\partial t} - \Delta \xi = g, & \xi \in \beta(u) \quad \text{in } Q, \\
\partial_\nu \xi = h & \text{on } \Sigma, \\
u(0) = u_0 & \text{in } \Omega
\end{cases}
\]

which is denoted \((P)_N\), where \( g : Q \to \mathbb{R}, \ h : \Sigma \to \mathbb{R}, \) and \( u_0 : \Omega \to \mathbb{R} \) are given data. The well-posedness for \((P)_N\) has already discussed in \([7, \text{Theorem } 2.3]\), specifically, there exists a pair \((u, \xi)\) such that

\[
\langle u'(t), z \rangle_{V^*,V} + \int_\Omega \nabla \xi(t) \cdot \nabla z dx = \int_\Omega g(t) z dx + \int_{\Gamma} h(t) z d\Gamma
\]

for all \( z \in V \), for a.a. \( t \in (0, T) \),

\[
u(0) = u_0 \quad \text{a.e. in } \Omega.
\]

For each \( \kappa > 0 \), we assume in addition that,

(A7) \( g, g_\kappa \in L^2(0, T; H) \), \( h, h_\kappa \in L^2(0, T; H_\Gamma) \) and \( u_0, u_{0\kappa} \in H \) with \( \hat{\beta}(u_0), \hat{\beta}(u_{0\kappa}) \in L^1(\Omega) \). Moreover, there exists a positive constant \( c_5 \) such that

\[
|g_\kappa - g|_{L^2(0,T;H)} \leq \kappa c_5, \quad |h_\kappa - h|_{L^2(0,T;H_\Gamma)} \leq \kappa c_5, \quad |u_{0\kappa} - u_0|_{V^*} \leq \kappa c_5. \quad (6.35)
\]

Then, we obtain the order of convergence between the Robin problem \((P)_R\) and the Neumann problem \((P)_N\) as \( \kappa \searrow 0 \):

**Theorem 6.1.** Under assumptions (A1), (A2), (A7) or (A1), (A7) with \( h = h_\kappa \equiv 0 \). Let \((u, \xi)\) be the weak solution \((P)_N\) and \((u_\kappa, \xi_\kappa)\) be the weak solution of \((P)_R\). Then, there exists a positive constant \( M^* \), depending only the data, such that

\[
|u_\kappa - u|^2_{C([0,T];V^*)} + 2 \int_0^T \langle \xi_\kappa(s) - \xi(s), u_\kappa(s) - u(s) \rangle_H ds \leq M^* \kappa^2 \quad (6.36)
\]

for all \( \kappa > 0 \). Moreover, if \( \beta \) is Lipschitz continuous, then it follows that

\[
\int_0^T |\xi_\kappa(s) - \xi(s)|_H^2 ds \leq C_\beta M^* \kappa^2
\]

for all \( \kappa > 0 \), where \( C_\beta \) is the Lipschitz constant for \( \beta \).
Proof. We take the difference between (2.1) for $u_\kappa$ and (6.34) for $u$ at $t = s$ and test it setting $z := F^{-1}(u_\kappa(s) - u(s))$ at time $s \in (0, T)$. Then, we have
\[
\frac{1}{2} \frac{d}{ds} |u_\kappa(s) - u(s)|^2_{V^*, V} + \int_\Omega \nabla (\xi_\kappa(s) - \xi(s)) \cdot \nabla F^{-1}(u_\kappa(s) - u(s)) \, dx \\
+ \kappa \int_\Gamma \xi_\kappa(s) F^{-1}(u_\kappa(s) - u(s)) \, d\Gamma \\
= \int_\Omega (g_\kappa(s) - g(s)) F^{-1}(u_\kappa(s) - u(s)) \, dx + \int_\Gamma (h_\kappa(s) - h(s)) F^{-1}(u_\kappa(s) - u(s)) \, d\Gamma.
\]
Now, note that
\[
\int_\Omega \nabla (\xi_\kappa(s) - \xi(s)) \cdot \nabla F^{-1}(u_\kappa(s) - u(s)) \, dx + \kappa \int_\Gamma \xi_\kappa(s) F^{-1}(u_\kappa(s) - u(s)) \, d\Gamma \\
= \langle FF^{-1}(u_\kappa(s) - u(s)), \xi_\kappa(s) - \xi(s) \rangle_{V^*, V} + \kappa \langle \xi(s), F^{-1}(u_\kappa(s) - u(s)) \rangle_{H'}, \\
= \langle u_\kappa(s) - u(s), \xi_\kappa(s) - \xi(s) \rangle_H + \kappa \langle \xi(s), F^{-1}(u_\kappa(s) - u(s)) \rangle_{H'}.
\]
Therefore, using the Young inequality and the trace theorem $| \cdot |_{H'}^2 \leq c'_p | \cdot |_V^2$, we deduce that
\[
\frac{1}{2} \frac{d}{ds} |u_\kappa(s) - u(s)|^2_{V^*, V} + (\xi_\kappa(s) - \xi(s), u_\kappa(s) - u(s))_H \\
= -\kappa \langle \xi(s), F^{-1}(u_\kappa(s) - u(s)) \rangle_{H'} + (g_\kappa(s) - g(s), F^{-1}(u_\kappa(s) - u(s)))_H \\
+ (h_\kappa(s) - h(s), F^{-1}(u_\kappa(s) - u(s)))_{H'} \\
\leq \frac{3}{2} \kappa^2 c'_p |\xi(s)|_V^2 + \frac{1}{6c'_p} |F^{-1}(u_\kappa(s) - u(s))|_{H'}^2 + \frac{3}{2} |g_\kappa(s) - g(s)|_H^2 \\
+ \frac{1}{6} |F^{-1}(u_\kappa(s) - u(s))|_{H'}^2 + \frac{3}{2} c'_p |h_\kappa(s) - h(s)|_H^2 + \frac{1}{6c'_p} |F^{-1}(u_\kappa(s) - u(s))|_{H'}^2 \\
\leq \frac{3}{2} \kappa^2 c'_p |\xi(s)|_V^2 + \frac{3}{2} |g_\kappa(s) - g(s)|_H^2 + \frac{3}{2} c'_p |h_\kappa(s) - h(s)|_H^2 + \frac{1}{2} |u_\kappa(s) - u(s)|^2_{V^*.} \quad (6.37)
\]
for a.a. $s \in (0, T)$. Here, using (6.35) of (A7) and the Gronwall inequality, we infer that
\[
|u_\kappa(t) - u(t)|^2_{V^*.} \\
\leq \left\{ |u_0 - u_0|_{V^*.} + 3c'_p \kappa^2 |\xi|^2_{L^2(0,T;V)} + 3|g_\kappa - g|_{L^2(0,T;H)} + 3c'_p |h_\kappa - h|_{L^2(0,T;H')} \right\} \exp(T) \\
\leq \kappa^2 \left\{ c'_p^2 + 3c'_p^2 |\xi|^2_{L^2(0,T;V)} + 3c'_p^2 + 3c'_p c'_p^2 \right\} \exp(T)
\]
for all $t \in [0, T]$. Moreover, by integrating (6.37) over $[0, T]$ with respect $s$, there exists a positive constant $M^*$, such that the estimates (6.36) holds. If $\beta$ is Lipschitz continuous, we infer from its monotonicity that
\[
\int_0^T |\xi_\kappa(s) - \xi(s)|_H^2 \, ds \leq \int_0^T \int_\Omega C_\beta |u_\kappa(s) - u(s)| |\xi_\kappa(s) - \xi(s)| \, dx \, ds \\
= C_\beta \int_0^T \langle \xi_\kappa(s) - \xi(s), u_\kappa(s) - u(s) \rangle_H \, ds \\
\leq C_\beta M^* \kappa^2.
\]
This completes the proof. □
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Appendix

We use the same notation as in the previous sections. To characterize the operator $A$ as in Section 3, we employ the maximal monotone theory related to the subdifferential. To do so, we define a proper lower semicontinuous convex functional $\varphi : H \to [0, +\infty]$, 

$$
\varphi(z) := \begin{cases}
\frac{1}{2} \int_\Omega |\nabla z|^2 \, dx + \frac{\kappa}{2} \int_\Gamma |z|^2 \, d\Gamma & \text{if } z \in V, \\
+\infty & \text{otherwise}.
\end{cases}
$$

We now present the representation of subdifferential operator $\partial \varphi$ (see e.g., [5, pp.62–65, Proposition 2.9]).

Lemma A. The subdifferential $\partial \varphi$ on $H$ is characterized by

$$
\partial \varphi(z) = -\Delta z \quad \text{in } H \quad \text{for all } z \in D(\partial \varphi) = W.
$$

Proof Let $z \in D(\varphi) = V$. For all $z^* \in \partial \varphi(z)$ in $H$. From the definition of the subdifferential $\partial \varphi$, we have

$$(z^*, \tilde{z})_H = (\nabla z, \nabla \tilde{z})_{H^1} + \kappa(z, \tilde{z})_{H^1} \quad \text{for all } \tilde{z} \in V.
$$

Now, taking $\tilde{z} \in D(\Omega)$ as the test function, we deduce $z^* = -\Delta z$ in $D'(\Omega)$, that is, $z^* = -\Delta z$ in $H$ by comparison. Hence, for all $\tilde{z} \in V$, we have

$$(z^*, \tilde{z})_H = (-\Delta z, \tilde{z})_H + \langle \partial \nu z, \tilde{z} \rangle + \kappa \int_\Gamma z \tilde{z} \, d\Gamma,
$$

specifically, by comparison, $\partial \nu z = -\kappa z$ in $H^{1/2}(\Gamma)$. Therefore, from the theory of elliptic regularity for Neumann problem, it follows that $z \in H^2(\Omega)$. Thus, we infer that $\partial \varphi(z)$ is singleton and is equal to $-\Delta z$ with $D(\partial \varphi) = \{ z \in H^2(\Omega) : \partial \nu z + \kappa z = 0 \text{ a.e. on } \Gamma \}$.

Finally, we can define $A$ by $\partial \varphi$ and then find the same abstract structure for the evolution equation (3.1), which has also been treated in previous works [6,16,17].

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