Effective action method for computing next to leading corrections of $O(N)$ models

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ABSTRACT

We compute the corrections of next to leading order in the $\frac{1}{N}$ expansion to the effective potential of a system described by a Ginzburg-Landau model with $N$ components and quartic interaction, in the case of spontaneous symmetry breaking. The method we apply allows to generalize in a simple way the so-called Self-Consistent Screened Approximation (SCSA).
1 The CJT formalism

Variational principles provide a very useful tool for studying field theories and a valid alternative to perturbative methods. The simplest of these principles is perhaps the Hartree method, which consists in finding the best quadratic approximation to a given Hamiltonian, by rendering the variational estimate of the effective potential minimal with respect to some adjustable parameters. In order to go beyond the quadratic approximation several methods have been devised, such as two or three loop approximations post-gaussian methods etc. [1]. In the present paper, we derive and generalize, to the case of a spontaneously broken symmetry, the so-called Self Consistent Screened Approximation (SCSA), originally introduced by Bray [2], in the case of an $O(N)$ invariant model by means of a general method introduced by Cornwall, Jackiw and Tomboulis (CJT) [3] in Quantum Field Theory (see also [4] for previous work). Our analysis makes also use of the $1/N$ expansion method which has been already employed in the $O(N)$ theory [3][5][7]. We believe that the compactness of the present derivation provides a simpler route to obtain results for generic $O(N)$ models.

The basic idea, behind this effective action method for composite operators, is to introduce, in the generating functional, sources which couple to the composite operators, to be studied, and then perform a double Legendre transform.

Let us consider for the sake of generality the generating functional for the Green functions

$$Z[J, K] = \int D\phi \exp \left[ -S(\phi) + J\phi + \frac{1}{2}\phi K\phi \right]$$

where $J$ and $K$ are respectively a local and a bilocal source:

$$J\phi = \int d^Dx J(x)\phi(x)$$

and

$$\frac{1}{2}\phi K\phi = \frac{1}{2} \int d^Dx \int d^Dy \phi(x)K(x,y)\phi(y)$$

By considering $W = -\ln Z$ and defining the variables

$$\phi_c = \frac{\delta W}{\delta J}, \quad G = \frac{\delta^2 W}{\delta J \delta J} = 2\frac{\delta W}{\delta K} - \phi_c \phi_c$$

one can eliminate the sources $J$ and $K$ in favor of the fields $\phi_c$ and $G$ and obtain the generating functional for 2PI (Two Particle Irreducible) Green functions

$$\Gamma(\phi_c, G) = W - J\phi_c - \frac{1}{2}\phi_c K\phi_c - \frac{1}{2}GK$$

2PI meaning that the graphs cannot be disconnected by cutting only two propagator lines.

The stationarity conditions for the functional $\Gamma$ read:

$$\frac{\delta \Gamma}{\delta \phi_c} = -J - K\phi_c, \quad \frac{\delta \Gamma}{\delta G} = -\frac{1}{2}K$$

and physical processes correspond to vanishing sources $J = 0$ and $K = 0$. 
As shown in [3] it is possible to get a formal series for $\Gamma$

$$\Gamma(\phi_c, \mathcal{G}) = -S(\phi_c) - \frac{1}{2} \text{Tr} \ln \mathcal{G}^{-1} - \frac{1}{2} \text{Tr} \mathcal{D}^{-1} \mathcal{G} + \Gamma_2(\phi_c, \mathcal{G})$$  \hspace{1cm} (1.7)

where

$$\mathcal{D}^{-1}(\phi_c; x, y) = \frac{\delta^2 S(\phi)}{\delta \phi(x) \delta \phi(y)}|_{\phi_c} = D^{-1}(x, y) + \frac{\delta^2 S_{\text{int}}(\phi)}{\delta \phi(x) \delta \phi(y)}|_{\phi_c}$$  \hspace{1cm} (1.8)

$\Gamma_2$ is given by the 2PI vacuum diagrams of a theory with interactions determined by $S_{\text{int}}$ and propagators $\mathcal{G}$, where $S_{\text{int}}$ is defined by the shifted action

$$S(\phi_c + \psi) - S(\phi_c) - \psi \cdot \frac{\delta S}{\delta \phi_c} = \frac{1}{2} \psi \cdot \mathcal{D}^{-1} \psi + S_{\text{int}}(\psi, \phi_c)$$  \hspace{1cm} (1.9)

Notice that this procedure represents a dressed loop expansion, all the propagators being fully radiative corrected and can thus exhibit non-perturbative effects even for a small number of dressed loops. The second of eqs. (1.6) together with eq. (1.7) represents the Schwinger-Dyson equation for the propagator $\mathcal{G}$

$$\mathcal{G}^{-1} = \mathcal{D}^{-1} + \Sigma(\phi_c, \mathcal{G})$$  \hspace{1cm} (1.10)

where the last term is the self-energy contribution

$$\Sigma(\phi_c, \mathcal{G}) = -2 \frac{\delta \Gamma_2(\phi_c, \mathcal{G})}{\delta \mathcal{G}}$$  \hspace{1cm} (1.11)

As a final remark, frequently one is interested in translationally invariant solutions $\phi(x) = \phi$ and $\mathcal{G}(x, y) = \mathcal{G}(x - y)$. One therefore defines the so-called effective potential

$$V(\phi, \mathcal{G}) = -\Omega_D \Gamma(\phi, \mathcal{G})|_{\text{trans. inv.}}$$  \hspace{1cm} (1.12)

where $\Omega_D$ is a volume factor.

## 2 The effective potential for the $O(N)$ model

We apply, now, the above construction to the Ginzburg-Landau $O(N)$ model with quartic interaction, which is defined by the classical action

$$S(\phi) = \int d^D x \left[ \frac{1}{2} \sum_\alpha (\partial_\mu \phi_\alpha)^2 + \frac{1}{2} \mu^2 \sum_\alpha \phi_\alpha^2 + \frac{\lambda}{4!N} (\sum_\alpha \phi_\alpha^2)^2 \right]$$  \hspace{1cm} (2.1)

One constructs first the shifted action and then obtains $\Gamma_2$ in the following way. One starts by drawing all graphs at a given order in $\lambda$, using the vertices determined by the shifted action (eq.1.9) and propagator $\mathcal{G}$, and neglecting all diagrams which are two-particle reducible.

To illustrate the previous rules we consider the effective action up to $\lambda^2$ order. The list of graphs contributing to $\Gamma_2$ is given in Fig. 1. As it is well known in the limit $N \to \infty$ the Hartree approximation, which corresponds to considering only the first diagram of Fig. 1, becomes exact. However, if one considers the $\frac{1}{N}$ corrections, one sees that an infinite set of 2PI diagrams must be kept.
For future applications we are mainly interested in the effective potential, therefore we consider translationally invariant solutions $\phi(x) = \phi$ and $\mathcal{G}(x,y) = \mathcal{G}(x-y)$. Under such hypothesis, the following decomposition of the tensor $\mathcal{G}$ is convenient:

$$\mathcal{G}_{\alpha\beta}(x-y) = \frac{\phi_{\alpha}\phi_{\beta}}{\phi^2} \mathcal{G}_L(x-y) + \left[ \delta_{\alpha\beta} - \frac{\phi_{\alpha}\phi_{\beta}}{\phi^2} \right] \mathcal{G}_T(x-y)$$

(2.2)

where $\phi^2 = \sum_{\gamma} \phi_{\gamma}^2$.

Therefore we get

$$\frac{1}{2} \text{Tr} \ln \mathcal{G}^{-1} = \frac{1}{2} (N-1) \text{Tr} \ln \mathcal{G}_T^{-1} + \frac{1}{2} \text{Tr} \ln \mathcal{G}_L^{-1}$$

(2.3)

$$\frac{1}{2} \text{Tr} \mathcal{D}^{-1} \mathcal{G} = \frac{1}{2} \int d^Dx \int d^Dy [(N-1)(-\Delta + \mu^2 + \frac{\lambda}{6N} \sum_{\alpha} \phi_{\alpha}^2)\delta(x-y)\mathcal{G}_T(x-y)$$

$$+(-\Delta + \mu^2 + \frac{\lambda}{2N} \sum_{\alpha} \phi_{\alpha}^2)\delta(x-y)\mathcal{G}_L(x-y)]$$

(2.4)

where we have used the inverse propagator

$$\mathcal{D}^{-1}_{\alpha\beta}(x,y) = \left[ (-\Delta + \mu^2 + \frac{\lambda}{6N} \sum_{\gamma} \phi_{\gamma}^2)\delta_{\alpha\beta} + \frac{\lambda}{3N} \phi_{\alpha}\phi_{\beta} \right] \delta(x-y)$$

(2.5)

and

$$-\Gamma_2(\phi, \mathcal{G}) = \frac{\lambda N}{24} \int d^Dx \mathcal{G}_{T}^2(0)$$

$$+\frac{\lambda}{12} \int d^Dx \mathcal{G}_T(0)\mathcal{G}_L(0)$$

$$-\frac{\lambda^2 \phi^2}{36N} \int \int d^Dx d^Dy \mathcal{G}_{T}^2(x-y)\mathcal{G}_L(x-y)$$

$$-\frac{\lambda^2}{144} \int \int d^Dx d^Dy \mathcal{G}_{T}^4(x-y)$$

(2.6)

Eq. (2.6) is exact to order $\lambda^2$ and to all orders in $N$.

A more systematic way of improving upon the Hartree approximation is to observe an interesting aspect of the effective potential, i.e. that the leading corrections in the $\frac{1}{N}$ expansion of the effective action $\Gamma$ form two simple classes of diagrams with respect to all possible graphs determined by $S_{int}$.

In Fig. 2 we draw the series for $\Gamma_2$ up to $O(N^0)$ to all orders in $\lambda$. We have neglected graphs like the one in Fig. 3, which gives a contribution $O(1/N)$.

The $O(N^0)$ truncated series has a simple structure and it is therefore possible to resum it completely. In fact, one recognizes $\Gamma_2$ to be the sum of two types of contributions: the first containing only two trilinear vertices and an arbitrary number of quadrilinear vertices, the second only quadrilinear vertices. By simple combinatorics one can write the
symmetry factors of each diagram contained in Fig. 4. The symmetry factor associated to the diagrams in Fig. 4a and 4b are respectively given by
\[
2^{n-3}(n-2)! \quad 2^{n-1}(n-1)!
\]
Thus we get the following contribution from the amplitude corresponding to Fig. 4a to \( \Gamma_2 \) (we assume a constant field \( \phi_0(x) = \phi_n \))
\[
\Gamma_2^{(n)} = (-1)^n \phi_0^2 N \left( \frac{\lambda}{6} \right)^n \int dx_1 \ldots \int dx_{n-1} \int dx_n G_L(x_1 - x_n) G_T^2(x_1 - x_2) \ldots G_T^2(x_{n-1} - x_n) \tag{2.8}
\]
and
\[
\Gamma_2^{(n)} = \frac{(-1)^n}{2n} \left( \frac{\lambda}{6} \right)^n \int dx_1 \ldots \int dx_{n-1} \int dx_n G_T^2(x_1 - x_n) G_T^2(x_1 - x_2) \ldots G_T^2(x_{n-1} - x_n) \tag{2.9}
\]
where \( n \geq 2 \) from the amplitude corresponding to Fig. 4b. Under the assumption of translation invariance the series corresponding to eq. (2.8) turns out to be a geometrical series and thus we can recast the contribution in the following simple form
\[
\frac{\Gamma_2}{\Omega_D} = \frac{\phi^2 \lambda}{6N} \int \frac{d^Dq}{(2\pi)^D} \hat{G}_L(q) - \frac{\phi^2 \lambda}{6N} \int \frac{d^Dq}{(2\pi)^D} \hat{G}_T(q) 1 + \frac{\lambda}{6} \Pi(q) \tag{2.10}
\]
where one has introduced the so called vacuum polarization
\[
\Pi(q) = \int \frac{d^Dk}{(2\pi)^D} \hat{G}_L(k) \hat{G}_T(-k - q) \tag{2.11}
\]
and \( \Omega_D \) is a volume factor. In an analogous way one gets for the contribution from eq.(2.9)
\[
\frac{\Gamma_2}{\Omega_D} = \frac{\lambda}{12} \int \frac{d^Dq}{(2\pi)^D} \Pi(q) - \frac{1}{2} \int \frac{d^Dq}{(2\pi)^D} \ln \left[ 1 + \frac{\lambda}{6} \Pi(q) \right] \tag{2.12}
\]
This last result was already obtained by Bray [2]. Notice that the factor \( \left[ 1 + \frac{\lambda}{6} \Pi(q) \right]^{-1} \) is reminiscent of the improved propagator for the auxiliary field [7]. In fact, an alternative way to study \( \phi^4 \) model is by the trick of the auxiliary field. The new field \( \alpha(x) \) is coupled to \( \phi \) via trilinear vertices and has a tree-level propagator equal to 1 (in given units). Finally taking into account all the contribution we get the final expression for the effective potential
\[
V = -\frac{\Gamma}{\Omega_D} = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{24N} \phi^4 + \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \hat{G}_L(k) \left[ k^2 + \mu^2 + \frac{\lambda}{2N} \phi^2 \right] + \frac{N-1}{2} \int \frac{d^Dk}{(2\pi)^D} \hat{G}_T(k) \left[ k^2 + \mu^2 + \frac{\lambda}{6N} \phi^2 \right] - \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \ln \hat{G}_L(k) - \frac{N-1}{2} \int \frac{d^Dk}{(2\pi)^D} \ln \hat{G}_T(k) + \frac{\lambda N}{24} \int \frac{d^Dq}{(2\pi)^D} \frac{d^Dk}{(2\pi)^D} \hat{G}_T(k) \hat{G}_T(k - q)
\]
We notice that the previous expression must be taken up to $O(N^0)$ in the large $N$ limit.

To conclude in order to determine the equilibrium value of $\Gamma$ one has to solve explicitly the stationarity conditions of CJT (see eqs. (1.6) with vanishing external sources). By differentiating with respect to the fields $\phi, G_T$ and $G_L$ one gets (when the vacuum expectation value of $\phi_\alpha$ is different from zero)

$$
\mu^2 + \frac{\lambda}{6N} \phi^2 + \frac{\lambda}{6N} [G_L(0) + (N-1)G_T(0)] + \frac{\lambda}{3N} \int \frac{d^D q}{(2\pi)^D} \frac{G_L(q)}{1 + \frac{\lambda}{6} \Pi(q)} = 0 \quad (2.14)
$$

$$
\hat{G}_L^{-1}(p) - \hat{D}_T^{-1}(p) - \frac{\lambda}{6} G_T(0) - \frac{\lambda \phi^2}{3N} \frac{1}{1 + \frac{\lambda}{6} \Pi(p)} = 0 \quad (2.15)
$$

$$
\hat{G}_T^{-1}(p) - \hat{D}_T^{-1}(p) - \frac{\lambda}{6} G_T(0) - \frac{\lambda}{6(N-1)} [G_L(0) - G_T(0)] + \frac{\lambda}{3(N-1)} \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_T(p-q)}{1 + \frac{\lambda}{6} \Pi(q)}
$$

$$
+ \frac{\lambda^2 \phi^2}{9N(N-1)} \int \frac{d^D q}{(2\pi)^D} \left[ \frac{\hat{G}_L(q) \hat{G}_T(p-q)}{1 + \frac{\lambda}{6} \Pi(q)} \right]^2 = 0 \quad (2.16)
$$

where

$$
G_{T,L}(0) = \int \frac{d^D q}{(2\pi)^D} \hat{G}_{T,L}(q) \quad (2.17)
$$

In order to solve the coupled set of eqs. (2.14) (2.16) one proceeds by expanding about the zeroth order, $O(N^0)$, Hartree solution. Since the effective action contains $G_L$ in terms which are $O(N^0)$ we write

$$
\phi^2 = N(\chi_0^2 + \frac{1}{N} \chi_1^2)
$$

$$
G_T = G_T^{(0)} + \frac{1}{N} G_T^{(1)}
$$

$$
G_L = G_L^{(0)} \quad (2.18)
$$

Substituting in eqs.(2.14-16) we find

$$
\mu^2 + \frac{\lambda}{6} \chi_0^2 + \frac{\lambda}{6} G_T^{(0)}(0) = 0
$$

$$
\chi_1 + G_L(0) + G_T^{(1)}(0) - G_T^{(0)}(0) + 2 \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_L(q)}{1 + \frac{\lambda}{6} \Pi(q)} = 0 \quad (2.19)
$$
and

\[(\hat{G}_L)^{-1}(p) = p^2 + \mu^2 + \frac{\lambda}{6} \chi_0^2 + \frac{\lambda}{6} \hat{G}_T^{(0)}(0) + \frac{\lambda}{3} \frac{\chi_0^2}{1 + \frac{\lambda}{6} \Pi(p)}\]

\[(\hat{G}_T^{(0)})^{-1}(p) = p^2 + \mu^2 + \frac{\lambda}{6} \chi_0^2 + \frac{\lambda}{6} \hat{G}_T^{(0)}(0)\]

\[\hat{G}_T^{(1)}(p)/[\hat{G}_T^{(0)}(p)]^2 = -\frac{\lambda}{6} \left[ \chi_1^2 + \hat{G}_L(0) + \hat{G}_T^{(1)}(0) - \hat{G}_T^{(0)}(0) \right] - \frac{\lambda}{3} \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_T^{(0)}(p-q)}{1 + \frac{\lambda}{6} \Pi(q)} + \frac{\lambda^2 \chi_0^2}{9} \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_L(q) \hat{G}_T^{(0)}(p-q)}{\left[1 + \frac{\lambda}{6} \Pi(q)\right]^2}\]

In the absence of magnetic field, i.e. at two phase coexistence in the thermodynamic language, the above equations reduce to:

\[(\hat{G}_L)^{-1} = p^2 + \frac{\lambda}{3} \frac{\chi_0^2}{1 + \frac{\lambda}{6} \Pi(p)}\]

\[(\hat{G}_T^{(0)})^{-1} = p^2\]

a result already known \[8\] and

\[\hat{G}_T^{(1)}(p)/[\hat{G}_T^{(0)}(p)]^2 = \frac{\lambda}{3} \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_L(q)}{1 + \frac{\lambda}{6} \Pi(q)} - \frac{\lambda}{3} \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_T^{(0)}(p-q)}{1 + \frac{\lambda}{6} \Pi(q)} + \frac{\lambda^2 \chi_0^2}{9} \int \frac{d^D q}{(2\pi)^D} \frac{\hat{G}_L(q) \hat{G}_T^{(0)}(p-q)}{\left[1 + \frac{\lambda}{6} \Pi(q)\right]^2}\]

To summarize we have derived by means of the CJT formalism the next to leading order corrections to the effective potential for the \(\phi^4\) theory. The method is easily generalizable to more complex interactions of the type \((\phi^2)^n\) with \(O(N)\) symmetry. The case \((\phi^2)^3\), relevant for tricritical points, is currently under study.

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Figure Captions

Fig. 1 Graphs contributing to $\Gamma_2$ up to $\lambda^2$ order.

Fig. 2 Series of graphs $\Gamma_2$ up to $O(N^0)$ to all orders in $\lambda$.

Fig. 3 Example of neglected graph, which gives a contribution $O(1/N)$.

Fig. 4 General graphs of $O(N^0)$ contributing to $\Gamma_2$