LINEAR SYZYGIES OF CURVES WITH PRESCRIBED GONALITY

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Abstract. We prove two statements concerning the linear strand of the minimal free resolution of a $k$-gonal curve $C$ of genus $g$. Firstly, we show that a general curve $C$ of genus $g$ of non-maximal gonality $k \leq \frac{2g+1}{3}$ satisfies Schreyer’s Conjecture, that is, $b_{p-k,1}(C, K_C) = g - k$. This is a statement going beyond Green’s Conjecture and predicts that all highest order linear syzygies in the canonical embedding of $C$ are determined by the syzygies of the $(k-1)$-dimensional scroll containing $C$. Secondly, we formulate an optimal effective version of the Gonality Conjecture and prove it for general $k$-gonal curves. This generalizes the asymptotic Gonality Conjecture proved by Ein-Lazarsfeld and improves results of Rathmann in the case where $C$ is a general curve of fixed gonality.

0. Introduction

1. The effective gonality conjecture. Let $C$ be a smooth complex algebraic curve and $L$ a very ample line bundle on $C$ inducing an embedding $\psi_L : C \hookrightarrow \mathbb{P}H^0(C, L)$. In order to describe the equations of this embedding, after setting $r := r(L)$, we consider the finitely generated graded $S := \text{Sym} \ H^0(C, L) \cong \mathbb{C}[x_0, \ldots, x_r]$-module $\Gamma_C(L) := \bigoplus_n H^0(C, L^\otimes n)$. By the Hilbert Syzygy Theorem, one has a minimal free resolution

$$0 \to F_{r+1} \to F_r \to \cdots \to F_0 \to \Gamma_C(L) \to 0,$$

where

$$F_p = \bigoplus_{q > 0} K_{p,q}(C, L) \otimes S(-p-q),$$

with $K_{p,q}(C, L)$ being the Koszul cohomology group of $p$-th order syzygies of weight $q$. As usual, the graded Betti numbers of $(C, L)$ are defined by $b_{p,q} := \dim K_{p,q}(C, L)$. If $L$ is non-special, then $K_{p,q}(C, L) = 0$ for all $q \geq 3$. Accordingly, the graded Betti diagram of $(C, L)$ consists only of two non-trivial rows: the linear strand ($q = 1$) and the quadratic strand ($q = 2$).

The quadratic strand of the resolution is the subject of the Green-Lazarsfeld Secant Conjecture [GL] and has been studied extensively in [FK], [K2]. The linear row is the subject of the Gonality Conjecture formulated in the same paper [GL].

Assume $C$ is $k$-gonal and let $L$ be a line bundle on $C$ of degree $\deg(L) \geq 2g-1+k$. By the Green-Lazarsfeld Nonvanishing Theorem [G] Appendix, one has $K_{h^0(L)-k-1,1}(C, L) \neq 0$. In a major breakthrough, generalizing results in [AV] in the case of general $k$-gonal curves, Ein and Lazarsfeld [EL] proved that for an arbitrary smooth curve $C$ of gonality $k$, if $\deg(L) \gg 0$, then

$$K_{h^0(L)-k,1}(C, L) = 0.$$

This result has been significantly improved by Rathmann [R], who showed that the vanishing (1) holds for every smooth curve $C$ of genus $g$, when $\deg(L) \geq 4g-3$. As already indicated in the original paper [GL] Conjecture 3.7, one can ask for an effective version of the Gonality Conjecture. We put forward the following:

**Conjecture 0.1.** We say that a smooth curve $C$ of genus $g$ and gonality $k$ satisfies the Effective Gonality Conjecture if for each line bundle $L$ on $C$ of degree $\deg(L) \geq 2g-1+k$, one has

$$K_{h^0(L)-k,1}(C, L) = 0.$$
While the original Gonality Conjecture has been formulated as an asymptotic statement in $\deg(L)$, the Effective Gonality Conjecture is already raised as a possibility in [GL1, page 86]. Clearly Conjecture [0.1] implies $K_{p,1}(C,L) = 0$, for all $p \geq h^0(C,L) - k$. The bound on $\deg(L)$ in Conjecture [0.1] is optimal. Indeed, if $A \in W^1_k(C)$ is a pencil of minimal degree, then

$$K_{g-1,1}(C, \omega_C \otimes A) \neq 0,$$

by the Green–Lazarsfeld Nonvanishing Theorem, that is, on every curve there exist line bundles of degree $2g - 2 + k$ which do not verify the Gonality Conjecture. In light of this fact, the Effective Gonality Conjecture admits the following equivalent reformulation: If on a curve $C$ there exists a line bundle $L \in \Pic^{2g-1+k}(C)$ such that $K_{g,1}(C,L) \neq 0$, then $\gon(C) \leq k - 1$.

Our first result is then:

**Theorem 0.2.** The Effective Gonality Conjecture holds for a general $k$-gonal of genus $g \geq 4$.

The statement fails for $g = 3$. Indeed, in this case the general curve is trigonal and it is easy to see that $K_{3,1}(C, \omega_C^{\otimes 2}) \neq 0$, using the fact that the canonical linear system embeds $C$ in the plane. It remains an important question to determine precisely which curves satisfy the Effective Gonality Conjecture. For hyperelliptic curves, or when $g \geq 4$, our results are complete: for every smooth curve of genus $g \geq 4$, an arbitrary curve satisfies the Effective Gonality Conjecture by Green’s $K_{p,1}$-theorem, see [G, Theorem 3.c.1]. Similarly, Conjecture [0.1] holds for each 4-gonal curve of genus $g \geq 7$, see [Te, Proposition 3.8] or [AS]. It seems plausible that for each $k$, there is a constant $g(k)$ such that the Effective Gonality Conjecture is true for every curve of genus $g \geq g(k)$. Furthermore, extrapolating [G, Theorem 3.c.1], all exceptions to the Effective Gonality Conjecture should correspond to curves $C \hookrightarrow \mathbb{P}^r$ lying on a variety of low degree.

For curves of maximal gonality of odd genus $g \geq 5$, our results are complete:

**Theorem 0.3.** The Effective Gonality Conjecture is valid for every smooth curve of odd genus $g \geq 5$ and maximal gonality.

Theorem [0.3], which plays an essential role in the proof of Theorem [0.2], turns out to be intimately related to the divisorial case of the Green-Lazarsfeld Secant Conjecture proved in full generality [FK, Theorem 1.4]. We observe that using [FK], if $C$ is a smooth curve of genus $g = 2n + 1$ and gonality $n + 2$, the following equivalence holds for a line bundle $M \in \Pic^{2g}(C)$:

$$K_{n,1}(C,M) \neq 0 \iff M - K_C \in C_{n+1} - C_{n-1}. \tag{2}$$

The right hand side denotes the divisorial difference variety $C_{n+1} - C_{n-1} \subseteq \Pic^{2g}(C)$. An argument involving the geometry of secant varieties for line bundles on $C$ then shows that [2] implies the vanishing $K_{g,1}(C,L) = 0$, for every line bundle $L \in \Pic^{5n+3}(C)$, thus establishing Theorem [0.3]. In order to deduce Theorem [0.2], we fix a value for the gonality $k \leq \frac{2g+3}{2}$ and perform induction on the genus $g$; the initial step is Theorem [0.3]. By induction, assume that the general smooth curve $C$ of genus $g$ and gonality $k$ satisfies the Effective Gonality Conjecture. The stable curve $X$ of genus $g+1$ obtained by adding an elliptic curve $E$ at a point of ramification of degree $k$ pencil on $C$ lies in the limit in $\overline{\mathcal{M}}_{g+1}$ of the locus of smooth $k$-gonal curves of genus $g + 1$. An analysis of syzygies of line bundles of bidegree $(2g + k, 1)$ on $X$ allows us to deduce the Effective Gonality Conjecture for a smooth deformation of $X$ having gonality $k$.

2. Schreyer's Conjecture. Consider a general $k$-gonal curve canonically embedded curve $C \hookrightarrow \mathbb{P}^{g-1}$ of gonality $k$. Green’s Conjecture, known in this case, see [V1], [V2], [Ap2], and asserting that

$$K_{p,1}(C,K_C) = 0 \text{ if and only if } p \geq g - k + 1,$$

determines the length of the linear (as well as that of the quadratic) strand of the resolution of $C$. Schreyer’s Conjecture [Sch3, §6] and [SSW] addresses the more refined question of what actually
is the Betti diagram of $C$, that is, determine the values $b_{p,1}(C, K_C)$ for $k - 2 \leq p \leq g - k$. Note that in the case when $C$ has the same gonality as a general curve of genus $g$, that is, $\text{gon}(C) = \lceil \frac{2g+3}{2} \rceil$, and only in this case, Green’s Conjecture determines the entire resolution of $C$. Indeed, in this case Green’s Conjecture is equivalent to the statement that the resolution of $C \subseteq \mathbb{P}^{g-1}$ is natural, or equivalently
\[ b_{p,2}(C, K_C) \cdot b_{p+1,1}(C, K_C) = 0 \]
for all $p$. Since the differences $b_{p+1,1}(C, K_C) - b_{p,2}(C, K_C)$ are known and independent of $C$, knowing which Betti numbers vanish amounts to knowing the entire Betti diagram.

Assume now $\text{gon}(C) \leq \frac{2g+1}{2}$, that is, $C$ has non-maximal gonality. In this case, Green’s Conjecture predicts the following resolution, where we observe that $b_{p,1}(C, K_C) \cdot b_{p,2}(C, K_C) \neq 0$ for $k - 2 \leq p \leq g - k$.

| $p$ | $b_{1,1}$ | $b_{2,1}$ | $\ldots$ | $b_{k-3,1}$ | $b_{k-2,1}$ | $\ldots$ | $b_{g-k,1}$ | $0$ | $\ldots$ | $0$ |
|-----|------------|------------|------------|------------|------------|------------|------------|------|------------|------|
| 1   | 2          | $\ldots$   | $k - 3$    | $k - 2$    | $\ldots$   | $g - k$    | $g - k + 1$| $g - 2$|              |      |

Table 1. The Betti table of a general canonical $k$-gonal curve of genus $g$.

It is known [AC] that such a curve $C$ carries a unique pencil $A \in W^1_k(C)$ of minimal degree, inducing a $(k - 1)$-dimensional scroll $X \subseteq \mathbb{P}^{g-1}$ swept out by the fibres of $|A|$. The Betti numbers of $(X, \mathcal{O}_X(1))$ are determined by the Eagon-Northcott complex, see [Sch1]. Since $C \subseteq X \subseteq \mathbb{P}^{g-1}$, one has the following inequality (see also Section 4)
\[ b_{p,1}(C, K_C) \geq b_{p,1}(X, \mathcal{O}_X(1)) = p \cdot \left( \frac{g - k + 1}{p + 1} \right). \]

It was originally expected that the inequality (3) is always an equality for $p \geq \lceil \frac{2g+1}{2} \rceil$. This, however, is now known to fail. Indeed, Bopp [B] showed that the for a general $5$-gonal curve of sufficiently high genus, if $m := \lceil \frac{2g+1}{2} \rceil$, then $b_{m,1}(C, K_C) > b_{m,1}(X, \mathcal{O}_X(1))$. Schreyer’s Conjecture [SSW] concerns the value of the highest non-zero Betti number in the linear strand and predicts that in this case, under suitable generality assumptions, inequality (3) is an equality.

**Conjecture 0.4** (Schreyer’s Conjecture). Let $C$ be a curve of genus $g$ and non-maximal gonality $3 \leq k \leq \frac{2g+1}{2}$. Assume $W^1_k(C) = \{A\}$ is a reduced single point and $A$ is the unique line bundle of degree at most $g - 1$ achieving the Clifford index. Then
\[ b_{g-k,1}(C, K_C) = g - k \text{ and } b_{p,1}(C, K_C) = 0, \text{ for } p > g - k. \]

The converse statement is straightforward. Indeed, if $W^1_k(C)$ does not consist of a reduced single point, then $b_{g-k,1}(C, K_C) > g - k$, see [SSW] Proposition 4.10]. As already pointed out, Green’s conjecture is known for general curves in each gonality stratum. Thus Schreyer’s Conjecture [14] in the case of generic $k$-gonal curves, purely concerns the condition
\[ b_{g-k,1}(C, K_C) = g - k. \]

In fact, Schreyer further conjectures that unless $C$ is isomorphic to a smooth plane quintic, the condition $b_{g-k,1}(C, K_C) = g - k$ automatically implies the vanishing statements $b_{p,1}(C, K_C) = 0$, for $p > g - k$; see [SSW] Conjecture 4.3. Conjecture 0.4 is known to hold for a general $k$-gonal curve provided $(k - 1)^2 < g$, see [Sch2]. An important piece of evidence for the conjecture is the case of general $k$-gonal curves of odd genus $2k - 1$. Such curves form a divisor $\mathfrak{Hur}$ in the moduli space $\mathcal{M}_{2k-1}$, much studied by Harris and Mumford in [H]. Combining results in [HR] and those in [V2], it follows that Conjecture 0.4 holds in this case. Outside this divisorial range, little has been known. The main result of this paper is the following:
Theorem 0.5. Schreyer’s Conjecture holds for a general $k$-gonal curve $C$ of genus $g \geq 2k - 1$:

$$b_{g-k,1}(C, K_C) = g - k.$$  

Part of Theorem 0.5 is that there is a canonical identification

$$K_{g-k,1}(C, K_C) \cong \bigwedge^{g-k+1} H^0(C, K_C \otimes A^V) \otimes \text{Sym}^{g-k-1} H^0(C, A) \otimes \bigwedge^2 H^0(C, A),$$

where $A$ is the unique degree $k$ pencil on $C$. All the $(g-k)$-th syzygies linear syzygies of the canonical curve $C \subseteq \mathbb{P}^{g-1}$ are of Eagon-Northcott type and can be written down explicitly. Precisely, if $(\tau_0, \ldots, \tau_{g-k})$ is a basis of $H^0(C, K_C \otimes A^V)$ and $\sigma \in H^0(C, A)$, then the syzygy corresponding to the power $\sigma^{g-k-1} \in \text{Sym}^{g-k-1} H^0(C, A)$ has the form

$$g-k \sum_{j=0}^n (-1)^j (\sigma_1 \wedge \ldots \wedge (\sigma_j) \wedge \ldots \wedge (\sigma_{g-k}) \wedge \{(\sigma_0) \otimes (\sigma')_{g-k} - (\sigma')_{g-k} \otimes (\sigma_j)\} \in \bigwedge^{g-k} H^0(C) \otimes H^0(C),$$

where $\sigma' \in H^0(C, A)$ is another section such that $(\sigma, \sigma')$ form a basis of $H^0(C, A)$.

The proof of Theorem 0.5 begins in Section 3 with the already mentioned observation that via $\mathbb{H}$ and $\mathbb{V}$, a smooth curve $C$ of genus $2k - 1$ and gonality $k$ satisfies $b_{k-1,1}(C, K_C) = k - 1$, provided $W^1_k(C)$ is integral of dimension zero. Consider the Hurwitz space $\mathcal{H}_{2k-1,k}$ of smooth curves of genus $g$ which are $k$-fold covers of $\mathbb{P}^1$. We define the Eagon-Northcott divisor $\mathcal{E}N$ on $\mathcal{H}_{2k-1,k}$ parametrizing moduli points $[f : C \to \mathbb{P}^1] \in \mathcal{H}_{2k-1,k}$ with $b_{k-1,1}(C, K_C) > k - 1$. In other words, points of $\mathcal{E}N$ correspond to canonical curves $C \subseteq \mathbb{P}^{g-1}$ having a $(g-k)$-th order linear syzygy which is not of Eagon-Northcott type. We also consider the Brill-Noether type divisor $\mathfrak{BN}$ on $\mathcal{H}_{2k-1,k}$ consisting of points $[f : C \to \mathbb{P}^1]$, such that $C$ has an extra pencil of degree $k$. By the above discussion these two divisors coincide set-theoretically, that is,

$$\mathcal{E}N = \mathfrak{BN}.$$

Now suppose we are no longer in the divisorial case and choose $k \leq \frac{g+1}{2}$. We follow a strategy reminiscent of $\mathbb{A}^2$. Starting with a general $k$-gonal curve $C$ of genus $g$, we form the irreducible nodal curve $[D] \in \mathcal{M}_{2g-2k+1}$ obtained by identifying $g - 2k + 1$ general pairs of points on $C$. Clearly $p_a(D) = 2g - 2k + 1$ and $\text{gon}(D) \leq g - k + 1$, that is, $[D]$ belongs to the closure $\mathfrak{B}N = \mathcal{M}_{2g-2k+1,g-k+1}$ of the Hurwitz divisor, already considered in $\mathbb{H}$, $\mathbb{HR}$ and $\mathbb{FK}$. Let

$$\pi : \overline{\mathcal{H}_{2g-2k+1,g-k+1}} \to \mathcal{M}_{2g-2k+1}$$

denote the forgetful map from the space of admissible covers of degree $g-k+1$ compactifying the Hurwitz space $\mathcal{H}_{2g-2k+1,g-k+1}$. Assuming the curve $C$ we started with is sufficiently general, one checks directly that set-theoretically $W^1_{g-k+1}(D)$ consists of one point (that is, $\pi^{-1}([D])$ consists of one admissible cover $[f]$). This point corresponds to the torsion free sheaf on $D$ given by pushing forward the unique degree $k$ pencil on $C$. By an argument inspired by limit linear series, we show that $[f] \notin \mathfrak{B}N$, therefore $[f] \notin \mathcal{E}N$. To conclude $b_{g-k,1}(C, K_C) = g - k$, we extend in Section 4 the determinantal structure of the Eagon-Northcott divisor $\mathcal{E}N$ over a partial compactification of $\mathcal{H}_{2g-2k+1,g-k+1}$ containing the moduli point of $[f]$. In the short Section 5 we then use $K3$ surfaces to show that this extended Eagon-Northcott divisor $\mathcal{E}N$ does not contain the unique boundary component of $\overline{\mathcal{H}_{2g-2k+1,g-k+1}}$ containing $[f]$. Since the injection $K_{g-k,1}(C, K_C) \hookrightarrow K_{g-k,1}(D, \omega_D)$ always holds, this completes the proof of Theorem 0.5.

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1It might be tempting to carry out this argument at the level of $\mathcal{M}_{2g-2k+1}$ rather than pass to the Hurwitz space. However, the scheme structure of $W^1_{g-k+1}(D)$ is difficult to analyse; in particular $[D]$ is a singular point of $\text{Im}(\pi) = \mathfrak{B}N$. Thus a degenerate version of results in $\mathbb{HR}$, does not quite lead to a proof of Conjecture 0.3.
The organisation of the paper is as follows: We first review some background on syzygies of curves in Section 1. In Section 2, we prove Theorem 0.2. We prove Theorem 0.5 in Sections 3 and 4.

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1. Background on Syzygies

We recall a few definitions and collect some basic results on syzygies that will be used throughout the paper. Let $X$ be a (possibly singular) projective variety and let $L, M \in \text{Pic}(X)$ be line bundles. Consider the graded $S := \text{Sym} H^0(X, L)$-module

$$\Gamma_X(M, L) := \bigoplus_{n \in \mathbb{Z}} H^0(X, L^\otimes n \otimes M).$$

One defines the Koszul cohomology groups $K_{p,q}(X, M; L)$ of $p$-th syzygies of weight $q$ by resolving the module $\Gamma_X(M, L)$ and computes them via the Koszul complex, see \cite{C}. When $M = \mathcal{O}_X$, we write $K_{p,q}(X, L) := K_{p,q}(X, \mathcal{O}_X; L)$. The following fact is surely well-known, we include a proof for the sake of completeness:

Lemma 1.1 (Semicontinuity). Let $\pi : \mathcal{X} \to S$ be a flat, projective morphism of schemes over an integral base. Let $L \in \text{Pic}(\mathcal{X})$ be a line bundle such that $h^0(X_s, \mathcal{L}_s) = c$, for each $s \in S$. Let $\mathcal{M} \in \text{Pic}(\mathcal{X})$ be a second line bundle, and assume

$$h^0(X_s, \mathcal{L}_s^\otimes (q-1) \otimes \mathcal{M}_s) = r_1, \quad h^0(X_s, \mathcal{L}_s^\otimes q \otimes \mathcal{M}_s) = r_2, \quad h^0(X_s, \mathcal{L}_s^\otimes (q+1) \otimes \mathcal{M}_s) = r_3$$

are also independent of $s \in S$. Then the function

$$\psi : s \mapsto \dim K_{p,q}(X_s, \mathcal{M}_s; \mathcal{L}_s)$$

is upper semicontinuous on $S$.

Proof. By Grauert’s Theorem, the sheaves $\mathcal{E} := \pi_* \mathcal{L}$, $\mathcal{F}_1 := \pi_*(\mathcal{L}^\otimes (q-1) \otimes \mathcal{M})$, $\mathcal{F}_2 := \pi_*(\mathcal{L}^\otimes q \otimes \mathcal{M})$ and $\mathcal{F}_3 := \pi_*(\mathcal{L}^\otimes (q+1) \otimes \mathcal{M})$ are all locally free. It is clearly enough to assume that $S = \text{Spec}(R)$ is affine. We have a Koszul complex

$$\bigwedge^{p+1} \mathcal{E} \otimes \mathcal{F}_1 \xrightarrow{\delta_1} \bigwedge^p \mathcal{E} \otimes \mathcal{F}_2 \xrightarrow{\delta_2} \bigwedge^{p-1} \mathcal{E} \otimes \mathcal{F}_3,$$

where both maps are $R$-module morphisms. For any $p \in \text{Spec}(R)$, the group $K_{p,q}(X_p, \mathcal{M}_p; \mathcal{L}_p)$ is given by the middle cohomology of the complex

$$\left(\bigwedge^{p+1} \mathcal{E} \otimes \mathcal{F}_1\right) \otimes k(p) \xrightarrow{\delta_1 \otimes k(p)} \left(\bigwedge^p \mathcal{E} \otimes \mathcal{F}_2\right) \otimes k(p) \xrightarrow{\delta_2 \otimes k(p)} \left(\bigwedge^{p-1} \mathcal{E} \otimes \mathcal{F}_3\right) \otimes k(p).$$

Hence

$$\dim K_{p,q}(X_p, \mathcal{M}_p; \mathcal{L}_p) = \dim \text{Ker}(\delta_2 \otimes k(p)) - \dim \text{Im}(\delta_1 \otimes k(p))$$

$$= r_2\left(c_p\right) - \dim \text{Im}(\delta_2 \otimes k(p)) - \dim \text{Im}(\delta_1 \otimes k(p)).$$

So it suffices to observe that for any morphism $\psi : A \to B$ of finitely generated, free $R$-modules, the function $p \mapsto \text{rank } \psi \otimes k(p)$ is lower semicontinuous, which is clear.
We collect some results on syzygies of curves which, taken together, reduce Conjecture \[0.1\] to the extremal case of line bundles of degree \(d = 2g - 1 + \text{gon}(C)\). We first quote from [Ap], Lemma 4.1:

**Lemma 1.2.** Let \(C\) be a smooth curve of genus \(g\) and \(L\) a line bundle of degree \(d \geq g\) with \(h^1(C, L) = 0\). Assume \(K_{p,1}(C, L) = 0\). Then \(K_{p+1,1}(C, L(x)) = 0\), for any point \(x \in C\) such that \(L(x)\) is base point free.

It is standard, see e.g. [AN], Corollary 2.13, that if \(L\) is base point free. Thus, for any \(L\) bundle on a smooth curve \(C\), one can read directly off the Betti table of an embedded curve such that \((C, L)\) fails property \(N\) classical properties \((M_n)\) defined in [GLI]. Precisely, \(\ell_2(C, L)\) is the smallest integer such that \((C, L)\) fails property \((N_{\ell_2(C,L)})\), whereas \(\ell_1(C, L)\) is the smallest integer such that \(L\) fails property \((M_{r(L)-\ell_1(C,L)})\).

Equally important invariants of the Betti table of \((C, L)\) are the extremal Betti numbers:

\[
b_1(C, L) := b_{\ell_1,1}(C, L) \quad \text{and} \quad b_2(C, L) := b_{\ell_2,2}(C, L).
\]

Using these invariants, the Effective Gonality Conjecture can be formulated as saying that \(\ell_1(C, L) \leq r(L) - \text{gon}(C)\), for every line bundle \(L\) of degree at least \(2g - 1 + \text{gon}(C)\). Similarly, Schreyer’s Conjecture is equivalent to \(b_1(C, K_C) = g - \text{gon}(C)\), for every curve of non-maximal gonality having a unique pencil of minimal degree.

## 2. The Effective Gonality Conjecture for generic curves

We start by proving Theorem \[0.3\]. It turns out that our proof of the generic Green-Lazarsfeld Secant Conjecture \([FK]\) takes us a long distance towards finding a complete solution.

**Proof of Theorem \[0.3\]** Let \(C\) be a curve of genus \(2n+1\) and gonality \(n+2\). Then using e.g. [HR, Remark 6.3], we observe that \(\text{Cliff}(C) = n\), that is, \(C\) has maximal Clifford index as well. We need to prove that for any line bundle \(L \in \text{Pic}(C)\) of degree at least \(5n+3\), we have \(K_{i,1}(C, L) = 0\) for \(i \geq h^0(C, L) - n - 2\). We may assume \(n \geq 2\) and as explained in the previous section, it is enough to prove that for any line bundle \(L \in \text{Pic}^{5n+3}(C)\), we have \(K_{2n+1,1}(C, L) = 0\).

Theorem 1.4 of [FK] establishes the following equivalence for any line bundle \(M \in \text{Pic}^{4n+2}(C)\):

\[
K_{n-1,2}(C, M) \neq 0 \iff M - K_C \in C_{n+1} - C_{n-1}.
\]

For any line bundle \(M \in \text{Pic}^{4n+2}(C)\) one has cf. [FK, formula (8)]

\[
\dim K_{n,1}(C, M) = \dim K_{n-1,2}(C, M).
\]

Thus, for any \(M \in \text{Pic}^{4n+2}(C)\), the equivalence

\[
K_{n,1}(C, M) \neq 0 \iff M - K_C \in C_{n+1} - C_{n-1}
\]

holds. Using Lemma \[1.2\] again, it thus suffices to show that for any line bundle \(L\) of degree \(5n + 3\), there exists an effective divisor \(D \in C_{n+1}\) such that

\[
L - D - K_C \notin C_{n+1} - C_{n-1}.
\]

Suppose this were not the case, that is,

\[
L - K_C - C_{n+1} \subseteq C_{n+1} - C_{n-1}.
\]
Then for every $D \in C_{n+1}$ there exists a divisor $E \in C_{n+1}$ such that $H^1(C, L(-D - E)) \neq 0$, that is, $D + E$ is an element of the (determinantal) secant variety $V_{2n+1}^{2n+2}(L)$ of effective divisors failing to impose independent conditions on $|L|$. In particular,
\[
\dim V_{2n+2}^{2n+1}(L) \geq n + 1,
\]
which is one higher than the expected dimension $n$. We observe that the morphism
\[
\psi : V_{2n+2}^{2n+1}(L) \rightarrow C_{n-1},
\]
is well-defined, since $h^0(C, K_C - L + A) = 1$, for $\text{gon}(C) > n - 1$. Let $I$ be any component of $V_{2n+2}^{2n+1}(L)$ of dimension $n + 1$ and set $r := n - 1 - \dim \psi(I)$. Then $\psi|_I$ must have fibres of dimension at least $2 + r$. As all divisors in the inverse image $\psi^{-1}(B)$ are clearly linearly equivalent, we have $h^0(C, A) \geq 3 + r$ for all $A \in V_{2n+1}^{2n+1}(L)$ such that $\psi(A) = B \in \psi(I)$. By Riemann–Roch, this implies $h^1(C, A) \geq 1 + r$, or $h^0(C, K_C - A) = h^0(2K_C - L - B) \geq 1 + r$. The latter inequality holds for any effective divisor $B \in \psi(I)$, so we must have
\[
\dim |2K_C - L| \geq r + \dim \psi(I) = n - 1.
\]
This implies $h^1(C, 2K_C - L) \geq 3$, or equivalently $L - K_C \in W_n^2(C)$. But then $\text{Cliff}(C) \leq n - 1$ (if $n = 2$, then compute the Clifford index of $2K_C - L$ rather than $L - K_C$). Since we have $\text{Cliff}(C) = n$, this is a contradiction. 

The proof of Theorem 0.3 gives a characterisation of those line bundles $L \in \text{Pic}^{2g-2+\text{gon}(C)}(C)$, such that $K_{h^0(L) - \text{gon}(C),1}(C, L) \neq 0$, in the case where $C$ has odd genus and maximal gonality.

**Proposition 2.1.** Let $C$ be a smooth curve of odd genus $2n + 1$ and gonality $n + 2$. Let $L \in \text{Pic}^{2n+2}(C)$ be such that $K_{2n+1}(C, L) \neq 0$. Then $L - K_C \in W_n^1(C)$. 

**Proof.** Following the proof of Theorem 0.3 we obtain $\dim V_{2n+1}^{2n+1}(L) \geq n$. By studying the morphism
\[
\psi : V_{2n+1}^{2n+1}(L) \rightarrow C_{n-1}, \quad A \mapsto K_C - L + A.
\]
and arguing as in Theorem 0.3, we are again led to the statement $h^0(C, 2K_C - L) \geq n$. The Riemann–Roch theorem gives $h^0(C, L - K_C) \geq 2$, as required. 

We will prove Theorem 0.2 by an induction on the genus, fixing the gonality. To perform the induction step, let $C$ be a smooth genus $g$ curve of gonality $k$ and denote by $f : C \rightarrow \mathbb{P}^1$ the induced degree $k$ cover. We assume that $C$ verifies the Effective Gonality Conjecture. Let $p \in C$ be a branch point of $f$, and consider the stable curve $X = C \cup_p E$ obtained by glueing a smooth, genus 1 curve at $p$. A standard argument with admissible covers or limit linear series shows that $X$ is a limit of smooth, genus $g + 1$ curves of gonality $k$, see [HM] §3.G.

**Proposition 2.2.** Let $X = C \cup_p E$ be the genus $g + 1$ stable curve as above. Let $L$ be a line bundle on $X$ such that $\deg(L_C) = 2g + k$ and $\deg(L_E) = 1$. Then, for a general point $q \in E \setminus \{p\}$, we have
\[
K_{g,1}(X, L(-q)) = 0.
\]
Further, for such a point, $h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0$. 

**Proof.** We have the Mayer-Vietoris sequence on $X$
\[
0 \rightarrow L_C(-p) \rightarrow L(-q) \rightarrow L_E(-q) \rightarrow 0.
\]
For a general point $q \in E \setminus \{p\}$, we have $h^0(E, L_E^{\otimes j}(-jq)) = h^1(E, L_E^{\otimes j}(-jq)) = 0$ for $j = 1, 2$, which implies $h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0$. Further, we have a natural isomorphism $H^0(C, L(-p)) \cong H^0(X, L(-q))$, and we know, by the assumptions on $C$, that
\[
K_{g,1}(C, L(-p)) = 0.
\]
We will use this to deduce $K_{g,1}(X, L(-q)) = 0$.

We have a natural commutative diagram

$$
\begin{array}{ccc}
\bigwedge^{g+1} H^0(C, L(-p)) & \xrightarrow{d} & \bigwedge^{g} H^0(C, L(-p)) \otimes H^0(C, L(-p)) \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\bigwedge^{g+1} H^0(X, L(-q)) & \xrightarrow{d} & \bigwedge^{g} H^0(X, L(-q)) \otimes H^0(X, L(-q)) \\
\end{array}
$$

where $\alpha, \beta$ are isomorphisms, and $\gamma$ is induced from the natural composition

$$
H^0(C, L^{\otimes 2}(-2p)) \hookrightarrow H^0(C, L^{\otimes 2}(-p)) \cong H^0(X, L^{\otimes 2}(-2q)).
$$

As $K_{g,1}(C, L(-p)) = 0$, the top row is exact and since $\beta$ is surjective and $\gamma$ is injective, the bottom row must also be exact, as required. \qed

From Proposition 2.2 we readily deduce Theorem 0.2.

**Proof.** Fix $k \geq 4$. Assume that the general curve $k$-gonal $C$ of genus $g$ has the property that, for any line bundle $L \in \text{Pic}^{2g-1+k}(C)$, one has $K_{g,1}(C, L) = 0$. We claim there exists a smooth curve $C'$ of genus $g+1$ and gonality $k$, such that, for each line bundle $L' \in \text{Pic}^{2g+1+k}(C')$, one has $K_{g+1,1}(C', L') = 0$. By performing induction on $g$ and noting that the initial step is Theorem 0.3 this suffices to prove the theorem. By Theorem 1.2 it further suffices to prove that there exists a smooth curve $C'$ of genus $g+1$ and gonality $k$ such that, for each line bundle $L' \in \text{Pic}^{2g+1+k}(C')$, there exists a point $q \in C'$ such that $K_{g,1}(C', L'(-q)) = 0$.

Let $X = C \cup_p E$ be the genus $g+1$ stable curve introduced in Proposition 2.2. Consider a flat family $\pi : C \to S$ of stable curves over a smooth, pointed, one dimensional base $(S, 0)$, such that the central fibre is $X$ and $\pi^{-1}(s)$ is a smooth curve of gonality $k$ for all $0 \neq s \in S$. As $X$ is a curve of compact type, after shrinking $S$ and performing a finite base change if necessary, we have a relative Picard scheme

$$
v : \text{Pic}^{2g+1+k}(C/S) \to S,
$$

with central fibre consisting of all line bundles of multidegree $(2g + k, 1)$ on $X = C \cup_p E$; this scheme is flat and proper over $S$, see [D, §4] and [H1], proof of Theorem 3.3.

Let $C_0$ be the open set $C \setminus p$ of all points which are smooth in the fibres over $S$. By Proposition 2.2 together with semicontinuity for the dimension of Koszul groups, there is an open subset $U \subseteq \text{Pic}^{2g+1+k}(C/S) \times_S C_0$, such that, for each pair $(L', q') \in U$, one has $K_{g,1}(C', L'(-q)) = 0$, where $C' = \pi^{-1}(v(L'))$, and such that

$$
0 \notin v(\text{Pic}^{2g+1+k}(C/S) \setminus \text{pr}_1(U)),
$$

where $\text{pr}_1 : \text{Pic}^{2g+1+k}(C/S) \times_S C_0 \to \text{Pic}^{2g+1+k}(C/S)$ is the projection. As flat morphisms are open, $\text{pr}_1(U)$ is open, and since $v$ is proper, the image

$$
V := v(\text{Pic}^{2g+1+k}(C/S) \setminus \text{pr}_1(U))
$$

is closed. Thus if $0 \neq t \in S \setminus V$ and $C_t := \pi^{-1}(t)$, then, for each $L \in \text{Pic}^{2g+1+k}(C_t)$ there exists $q \in C_t$ with $K_{g,1}(C_t, L(-q)) = 0$, as required. \qed

3. Schreyer’s Conjecture for general curves of non-maximal gonality

In this section, we begin discussing Schreyer’s Conjecture for general $k$-gonal curves of genus $g \geq 2k - 1$. We start by explaining the relevance of [HR] for Conjecture 0.4.
For $g = 2k - 1$, we consider two divisors on $\mathcal{M}_g$, which already played a role in [Ap2] or [FK]:

$$\text{Sur} : = \{ [C] \in \mathcal{M}_g : W_k^1(C) \neq \emptyset \}.$$

Recall that $\text{Sur}$ has a structure of degeneracy locus whereas $\text{Sur}$ is the push-forward of the smooth Hurwitz space $\mathcal{H}_{2k-1,k}$ of degree $k$ covers of $\mathbb{P}^1$. We view both $\text{Sur}$ and $\text{Sur}$ as divisors on the moduli stack of smooth curves of genus $2k - 1$, rather than on the associated coarse moduli space. It is proved in [HR], that one has the following relation at stack level:

$$\text{Sur} = (k-1)\text{Sur} \in CH^1(\mathcal{M}_{2k-1}).$$

**Theorem 3.1.** (HR) Let $C$ be a curve of genus $2k - 1$ and gonality $k$ such that the point $W_k^1(C)$ consists of a reduced single point. Then $b_{k-1,1}(C, K_C) = k - 1$.

**Proof.** For a smooth curve $C$, we denote by $\phi : X \to (S,0)$ its versal deformation space, hence the associated moduli map $m(\phi) : S \to \mathcal{M}_g$ is an étale neighbourhood of the point $[C] \in \mathcal{M}_g$.

For $s \in S$, set $C_s := \phi^{-1}(s)$, thus $C_0 = C$. From [HR], there exist two vector bundles $V$ and $W$ of the same rank over $S$ together with a morphism $\chi : V \to W$, such that, for any $s \in S$, we may identify $K_{k-1,1}(C_s, \omega_{C_s}) = \text{Ker}(\chi_s)$. Then the divisor $\text{Sur}(\phi) \subseteq S$ is defined by $\text{det}(\chi)$. Suppose $b_{k-1,1}(C, K_C) \geq k$. Thus $\text{det}(\chi)$ vanishes to order at least $k$, cf. [HR] Lemma 6.1. By the equality of cycles $\text{Sur}(\phi) = (k-1)\text{Sur}(\phi)$ on $S$, the function defining $\text{Sur}(\phi)$ must vanish to order at least two. Thus $\text{Sur}(\phi)$ is not smooth at the point $0 \in S$. It is well-known, see [C], that this happens if and only if $W_k^1(C)$ consists of a single pencil $A$ and moreover $h^0(C,A^{\otimes 2}) = 3$. 

We now turn our attention to curves of genus $g$ and non-maximal gonality $k \leq \frac{2g-1}{2}$. Let $G_{d,\text{bpf}}^1(C) \subseteq G_d^1(C)$ be the subvariety of base point free pencils of degree $d$ on $C$. The following observation is a slight modification of the linear growth condition of [Ap2] Theorem 2):

**Lemma 3.2.** A general curve $C$ of genus $g$ and gonality $k \leq \frac{2g-1}{2}$ satisfies bpf-linear growth:

$$\dim G_{k+m}^1(C) \leq m, \quad \text{for} \quad 0 \leq m \leq g - 2k + 1$$

and, further, $$\dim G_{k+m}^1(C) < m, \quad \text{for} \quad 0 \leq m \leq g - 2k + 1.$$

**Proof.** From [Ap2], we have $\dim W_{k+m}^1(C) = m$, for $0 \leq m \leq g - 2k + 1$. It now suffices to observe that if $Z \subseteq W_d^r(C)$ is an irreducible component, then $Z \cap W_d^{r+1}(C)$ has codimension at least two in $Z$, provided $g - r + d \geq 0$. This follows from the fact that no component of $C_d$ is entirely contained in $C_{d+1}^r$, see [ACGH] §IV.1.

We claim $\dim G_{d+m}^1(C) \leq m$, for $0 \leq m \leq g - 2k + 1$. Take an irreducible component $J \subseteq G_{d+m}^1(C)$ and consider the restriction to $J$ of the surjection $e : G_{k+m}^1(C) \to W_{k+m}^1(C)$. Assume $c(J) \subseteq W_{d+j}^1(C)$ and choose $j \geq 0$ maximal with this property. Then by the above, $\dim \psi(J) \leq 2j$. The general fibre of $c_j$ is isomorphic to the Grassmannian $G(2,2+j)$, it follows $\dim(J) \leq 2j + \dim c(J) \leq m$. By an identical argument and using [AC] Theorem 2.6, we similarly obtain that $\dim G_{k+m}^1(C) < m$, in the range $0 < m \leq g - 2k + 1$. 

For an integral nodal curve $D$, we define $W_k^1(D) \subseteq \text{Pic}^1(D)$ to be the closed subset of the compactified Jacobian of rank one, torsion free sheaves $A$ of degree $k$ on $C$ with $h^0(D, A) \geq 2$.

**Proposition 3.3.** Let $C$ be a smooth curve of genus $g$ and gonality $k \leq \frac{2g-1}{2}$. Assume $C$ satisfies bpf-linear growth and $W_k^1(C)$ consists of a single point $A$. If $(x_i, y_i)$ are general pairs of points on $C$, where $1 \leq i \leq g - 2k + 1$, let $D$ be the nodal curve obtained by glueing $x_i$ to $y_i$ for all $i$. Then $W_k^1(D) = \{ \nu_* (A) \}$, where $\nu : C \to D$ is the normalisation morphism. Furthermore, $\text{gon}(D) = g - k + 1$. 


The morphism \( \square \) the total number of ramification points of the cover \( C \) that deg(\( f \)) positive genus. The restriction \( f \) be a component \( \tilde{P} \) respectively. As the stabilisation of \( B \) denote by \( R \) curve whose stable model is isomorphic to \( D \) curve. By admissibility, \( C_0 \) of \( B \) meeting \( C \) at \( x_i \) and \( y_i \) respectively. As the stabilisation of \( B \) is \( D \) and \( f(R_{x_i}) \cap f(R_{y_i}) = \emptyset \), for each such \( i \) there must be a component \( \tilde{R}_i \) of the subcurve \( \tilde{B} - C_0 \) of \( B \) such that \( f(\tilde{R}_i) = \mathbf{P}_1 \), or else \( T \) contains a loop. As deg(\( f \)) = \( g - k + 1 \), this implies that \( d := \deg(f_0) \leq k + j \).

Since the pairs \( (x_1, y_1), \ldots, (x_j, y_j) \) are general and \( f_0 \) gives rise to an element of \( C_d^{1, \text{bpf}}(C) \), it follows dim \( C_d^{1, \text{bpf}}(C) \geq j \). If \( d > k \), this contradicts the bpf-linear condition on \( C \), which implies that deg(\( f_0 \)) = \( k \) and \( f_0 \) is the map induced by the pencil of minimal degree \( A \in W^1_k(C) \). Each \( \tilde{R}_i \) maps isomorphically onto \( \mathbf{P}_1 \). Clearly deg(\( f_{|R_{x_i}} \)) \( \geq 2 \) and deg(\( f_{|R_{y_i}} \)) \( \geq 2 \), in particular \( f_{|R_{x_i}} \) and \( f_{|R_{y_i}} \) will both contain at least two ramification points of \( f \), for each \( i = 1, \ldots, g - 2k + 1 \) (Note that being general points, \( x_i, y_i \) are not among the ramification points of \( f_0 \)). Counting the total number of ramification points of the cover \( f \), it follows that deg(\( f_{|R_{x_i}} \)) = deg(\( f_{|R_{y_i}} \)) = 2.

The morphism \( f \) is now uniquely determined, for \( f^{-1}(\mathbf{P}_1) = C \cup \tilde{R}_1 \cup \ldots \cup \tilde{R}_{g - 2k + 1} \) and all the components of \( f^{-1}(f(R_{x_i})) \) and \( f^{-1}(f(R_{y_i})) \) other than \( R_{x_i} \) and \( R_{y_i} \) respectively are mapped isomorphically onto their images. \( \square \)

Let \( \mathfrak{W}' \subseteq \mathcal{H}_{2g - 2k + 1, g - k + 1} \times \mathcal{M}_{2g - 2k + 1} \mathcal{H}_{2g - 2k + 1, g - k + 1} \) be the closure of the locus of pairs

\[
([f : C \to \mathbf{P}^1], [g : C \to \mathbf{P}^1]),
\]

where \( C \) is a smooth curve of genus \( 2g - 2k + 1 \) and \( f \not\equiv g \). Applying [AC] Proposition 2.4], we know that dim \( \mathfrak{W}' \) = dim \( \mathcal{H}_{2g - 2k + 1, g - k + 1} -1 \). We introduce the Brill-Noether divisor on the
Hurwitz space of curves possessing an extra pencil:
\[ \BN := \text{pr}_1(\BN') \subseteq \mathcal{H}_{2g-2k+1,g-k+1}. \]

Since \( \BN' \) is birational to the Severi variety of nodal curves of type \((g-k+1, g-k+1)\) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) having geometric genus \(2g-2k+1\), using \([13]\), we conclude that \( \BN \) is an irreducible divisor. We also recall Coppens’ result \([9]\) saying that if a curve \([C] \in \mathcal{M}_{2g-2k+1}\) has a pencil \(A \in W_{g-k+1}^1(C)\) such that \(h^0(C, A^{\otimes 2}) \geq 4\), then \([C, A] \in \BN\). The locus of such pairs \([C, A] \in \mathcal{H}_{2g-2k+1,g-k+1}\) is of pure codimension one in \( \BN \).

Our next goal is to show that, in the notation of Proposition 3.4, the point \([f] \in \pi^{-1}(D)\) does not lie in the closure \(\BN' \subseteq \mathcal{H}_{2g-2k+1,g-k+1}\) provided the normalisation \(C\) is sufficiently general. This is achieved by degenerating to a \((k-2)\)-nodal curve such that its normalisation is hyperelliptic.

**Proposition 3.5.** Let \(C\) be a smooth hyperelliptic curve of genus \(g-k+2\) and set \([A] = W_2^1(C)\). Choose general points \(y_1\) and \(x_i^{g-k-1}\) on \(C\) and let \(y_i\) be the hyperelliptic conjugate of \(x_{i-1}\) for \(2 \leq i \leq g-k-1\). Consider the semistable curve \(D\) obtained by adjoining smooth rational curves \(R_i\) to \(C\) at \(x_i\) and \(y_i\) for \(1 \leq i \leq g-1-k\). If \(L \in \text{Pic}(D)\) is any line bundle such that

\[
\begin{align*}
&\bullet L_{|C} \cong A^{\otimes 2} \left( x_1 + y_1 + \cdots + x_{g-k-1} + y_{g-k-1} \right), \\
&\bullet L_{|R_i} \cong O_{R_i}, \text{ for all } 1 \leq i \leq g-1-k,
\end{align*}
\]

then \(h^0(D, L) = 3\).

**Proof.** Since \(h^0(C, A^{\otimes 2}) = 3\), by Riemann–Roch \(h^0(C, \omega_C \otimes A^{\otimes(-2)}) = g-k\). As \(y_1, x_1, \ldots, x_{g-1-k}\) are general, \(h^0(C, \omega_C \otimes A^{\otimes(-2)}(-y_1 - \sum_{i=1}^{g-1-k} x_i)) = 0\), thus \(h^0(C, A^{\otimes 2}(y_1 + \sum_{i=1}^{g-1-k} x_i)) = 3\), by Riemann-Roch. For each \(1 \leq i \leq g-1-k\), define the nodal subcurve of \(D\)

\[ D_i := C \cup R_1 \cup \ldots \cup R_i, \]

then set \(L_i := L_{|D_i} \left( -\sum_{j=i+1}^{g-1-k} (x_j + y_j) \right) \in \text{Pic}(D_i)\). We shall prove by induction on \(i\) that \(h^0(D_i, L_i) = 3\). The inequality \(h^0(D_i, L_i) \geq 3\) follows from the Mayer-Vietoris exact sequence

\[ 0 \rightarrow A^{\otimes 2} \rightarrow L_i \rightarrow \bigoplus_{j=1}^i O_{R_j} \rightarrow 0, \]

so we need to show that \(h^0(D_i, L_i) \leq 3\).

When \(i = 1\), we have the exact sequence \(0 \rightarrow O_{R_1}(-2) \rightarrow L_1 \rightarrow A^{\otimes 2}(x_1 + y_1) \rightarrow 0\), which gives \(h^0(D_1, L_1) \leq h^0(C, A^{\otimes 2}(x_1 + y_1)) \leq h^0(C, A^{\otimes 2}(y_1 + \sum_{i=1}^{g-1-k} x_i)) = 3\), so the claim holds.

We now prove the induction step. Assume the claim holds for \(i = j\). We claim that each section of \(L_j(x_{j+1})\) vanishes at \(x_{j+1}\), from which is follows that \(h^0(D_j, L_j(x_{j+1})) = h^0(D_j, L_j) = 3\), due to the induction hypothesis. We have the following short exact sequence on \(D_j\)

\[ 0 \rightarrow \bigoplus_{\ell=1}^j O_{R_\ell}(-2) \rightarrow L_j(x_{j+1}) \rightarrow A^{\otimes 2}(x_{j+1} + \sum_{\ell=1}^j (x_\ell + y_\ell)) \rightarrow 0. \]

Restriction to \(C\) gives an injection \(H^0(D_j, L_j(x_{j+1})) \hookrightarrow H^0(C, A^{\otimes 2}(x_{j+1} + \sum_{\ell=1}^j (x_\ell + y_\ell)))\) and it suffices to show the sections of the latter cohomology group vanish on \(x_{j+1}\). We have

\[ H^0(C, A^{\otimes 2}(x_{j+1} + \sum_{\ell=1}^j (x_\ell + y_\ell))) = H^0(C, A^{\otimes 2}(y_1 + (x_1 + y_2) + \cdots + (x_{j-1} + y_j) + x_{j+1})) = H^0(C, A^{\otimes(j+1)} + y_1 + x_{j+1}) \cong H^0(C, A^{\otimes(j+1)}), \]
for \(y_1, x_j\) and \(x_{j+1}\) are general points on \(C\). Therefore \(h^0(D_j, L_j(x_{j+1})) = 3\).

We now have two cases. If \(h^0(D_j, L_j(x_{j+1} + y_{j+1})) = 3\), then from the exact sequence
\[
0 \to \mathcal{O}_{R_{j+1}}(-2) \to L_{j+1} \to L_j(x_{j+1} + y_{j+1}) \to 0,
\]
we see \(h^0(D_{j+1}, L_{j+1}) \leq 3\) as required. In the second case, \(h^0(D_j, L_j(x_{j+1} + y_{j+1})) = 4\). In this case, there exists a section
\[
t \in H^0(D_j, L_j(x_{j+1} + y_{j+1}))
\]
which does not vanish at \(y_{j+1}\).

On the other hand, we claim that each section of \(L_j(x_{j+1} + y_{j+1})\) vanishes at \(x_{j+1}\). As above, it suffices to show that each global section of \(A^{\otimes 2}(\sum_{\ell=1}^{j+1}(x_\ell + y_\ell))\) vanishes at \(x_{j+1}\). We have
\[
A^{\otimes 2}(\sum_{\ell=1}^{j+1}(x_\ell + y_\ell)) = A^{\otimes (j+2)}(y_1 + x_{j+1}),
\]
so that
\[
H^0(C, A^{\otimes 2}(\sum_{\ell=1}^{j+1}(x_\ell + y_\ell))) \cong H^0(C, A^{\otimes (j+2)}(y_1 + x_{j+1})) \cong H^0(C, A^{\otimes (j+2)}(y_1 + x_{j+1} + 1))
\]
(note that \(j + 2 \leq g(C)\)). Consequently, each section of \(A^{\otimes 2}(\sum_{\ell=1}^{j+1}(x_\ell + y_\ell))\) vanishes at \(x_{j+1}\).

We now conclude the proof from the Mayer-Vietoris sequence
\[
0 \to L_{j+1} \to L_j(x_{j+1} + y_{j+1}) \oplus \mathcal{O}_{R_{j+1}} \xrightarrow{\alpha} \mathcal{O}_{x_{j+1}} \oplus \mathcal{O}_{y_{j+1}} \to 0.
\]
Indeed, it suffices to show \(H^0(\alpha)\) is surjective. By considering the image of \(t\), we see that \(\text{Im}(H^0(\alpha))\) contains the element \((0, 1)\). Next, by considering the image of constants elements in \(H^0(\mathcal{O}_{R_{j+1}})\), we note the image contains elements \((a, b)\) with \(a \neq 0\). Thus \(H^0(\alpha)\) is surjective. \(\square\)

**Remark 3.6.** Suppose we replace each curve \(R_i\) with a chain of smooth rational curves
\[
\Gamma_i := \Gamma_{i,1} \cup \ldots \cup \Gamma_{i,\ell_i},
\]
ordered in such way that \(\Gamma_{i,1}\) meets \(C\) at \(x_i\) (and nowhere else), \(\Gamma_{i,\ell_i}\) meets \(C\) at \(y_i\) (and nowhere else), no other components of \(\Gamma_i\) meets \(C\), whereas \(\Gamma_{i,j}\) intersects \(\Gamma_{i,j-1}\) in one point for \(j = 2, \ldots, \ell_i\), and there are no other intersections. The conclusion of Proposition 3.5 then holds with \(\Gamma_i\) replacing \(R_i\).

**Definition 3.7.** The **Eagon-Northcott divisor** \(\mathcal{E}N \subseteq \mathcal{H}_{2g-2k+1,g-k+1}\) is defined as the locus of covers \(\left[f : C \to \mathbb{P}^1\right]\) such that \(\dim K_{g-k+1}(C, \omega_C) > g - k\).

In the next section, we shall extend \(\mathcal{E}N\) as a determinantal locus over a partial compactification of \(\mathcal{H}_{2g-2k+1,g-k+1}\). From Theorem 3.4 and [SSW] Proposition 4.10, observe that we have the equality of subsets of \(\mathcal{H}_{2g-2k+1,g-k+1}\):
\[
\mathcal{B}\mathcal{M} = \mathcal{E}N.
\]

We now come to the main result of this section, showing that the admissible cover described in Proposition 3.4 does not lie in the closure of \(\mathcal{E}N\).

**Theorem 3.8.** Let \(C\) be a general curve of genus \(g\) and gonality \(k \leq \frac{2g+1}{3}\). Choose general pairs of points \((x_i, y_i)\) on \(C\) for \(1 \leq i \leq g - 2k + 1\) and let \(D\) be the nodal curve obtained by identifying \(x_i\) and \(y_i\) for all \(i\). Then \(\pi^{-1}([D]) \cap \mathcal{B}\mathcal{M} = \emptyset\).

**Proof.** In the course of proving Proposition 3.4, we have showed that the underlying set \(\pi^{-1}([D])\) consists of a single admissible cover \(f : B \to T\), so it suffices to establish that
\[
[f, f] \notin \mathcal{B}\mathcal{M} \subseteq \mathcal{H}_{2g-2k+1,g-k+1} \times \mathcal{H}_{2g-2k+1,g-k+1}.
\]
We specialise \(C\) to a nodal curve \(C_0\) with \(k - 2\) nodes, such that the normalization \(\tilde{C}_0\) of \(C_0\) is a hyperelliptic curve. Set \(\{A\} = W_2^1(C_0)\). Let \(\{((x_i, y_i))\}_{i=1}^{g-2k+1}\) be general pairs of points on
\( \tilde{C}_0 \) and let \((x_i, y_i)_{i=1}^{2g-2k} \) be the inverse images in \( \tilde{C}_0 \) of the nodes of \( C_0 \). If \( D_0 \) is the curve obtained by identifying \( x_i \) to \( y_i \) for all \( 1 \leq i \leq g - 1 - k \) and \( \nu : \tilde{C}_0 \to D_0 \) is the normalisation, then \( \nu_* (A) \in W_{g-1-k}^1 (D_0) \) and \( D_0 \) is a specialization of \( D \). Proposition \ref{prop:linear-syzygies} guarantees there is a unique cover \( \{ f_0 : B_0 \to T_0 \} \in \mathcal{F}_{2g-2k+1, g-k+1} \) such that the stable model of \( B_0 \) is \( D_0 \).

Further specialize by bringing the points \( \{(x_i, y_i)\}_{i=1}^{g-1-k} \) into the configuration from the hypothesis of Proposition \ref{prop:linear-syzygies}. This specializes \( f_0 \) to an admissible cover \( g_0 : B' \to T' \), where the stabilisation of \( B' \) is a nodal curve \( D'_0 \) with normalisation \( \tilde{C}_0 \). It suffices to show \( [g_0, g_0] \notin \mathfrak{BN}' \).

We record two properties of the admissible cover \( g_0 : B' \to T' \) which follow from the considerations in Proposition \ref{prop:linear-syzygies}. Firstly, \( B' \) has a unique non-rational component, which is isomorphic to \( \tilde{C}_0 \). The restriction \( g_0|_{\tilde{C}_0} \) is the degree 2 cover of \( \mathbb{P}^1 \) determined by \( A \in W_2^1 (\tilde{C}_0) \). Secondly, for each pair \((x_i, y_i)\), where \( 1 \leq i \leq g - 1 - k \), there is a chain of smooth rational curves in \( B' \) which meets \( \tilde{C}_0 \) at \( x_i, y_i \). This chain contains precisely one component \( \tilde{R}_i \) which is mapped isomorphically to \( \mathbb{P}^1 \) by \( g_0 \). The components \( \tilde{C}_0 \) and \( \tilde{R}_i \), where \( 1 \leq i \leq g - 1 - k \) are the only components mapped to \( \mathbb{P}^1 \).

Assuming for a contradiction that \([g_0, g_0]\) lies in the closure of \( \mathfrak{BN}' \), then there exists a smooth curve \( \Delta \) with chosen point \( 0 \in \Delta \), and a family of pairs of admissible covers

\[
(g_{1,t} : B_t \to T_t, g_{2,t} : B_t \to T_t)_{t \in \Delta}
\]

with \( g_{1,0} = g_{2,0} = g_0 \), and such that for \( t \neq 0 \), we have \( T_t \cong \mathbb{P}^1 \), the source curve \( B_t \) is smooth and \( g_{1,t} \neq g_{2,t} \). Each base curve \( T_t \) comes with the data of unlabelled branch points. Choose 3 marked points on the component \( \mathbb{P}^1 \subseteq T_0 \), ignore the other marked points and perform stabilisation on this family of 3-marked genus 0 curves. After possible replacing \( \Delta \) by an étale cover and contracting unstable components, we produce a family of pairs of stable maps

\[
(h_{1,t} : B'_t \to \mathbb{P}^1, h_{2,t} : B'_t \to \mathbb{P}^1)_{t \in \Delta},
\]

such that the general fibre is unchanged, that is, \( h_{i,t} = g_{i,t} \), for \( i = 1, 2 \) and \( t \in \Delta \setminus \{0\} \), but in addition, the special fibre is a morphism with smooth target.

The limiting stable map \( h := h_{1,0} = h_{2,0} \) is easy to describe. The curve \( B''_0 \) consists of the smooth, hyperelliptic curve \( \tilde{C}_0 \) together with rational components \( R_i \) meeting \( \tilde{C}_0 \) precisely at \( x_i, y_i \) for \( i = 1, \ldots, g - 1 - k \). Let \( B'' \to \Delta \) be the total family with fibre over \( t \) given by \( B''_t \). Then \( B'' \) may have isolated singularities over the nodes of the central fibre \( B''_0 \). We have two line bundles, \( L_1 \) and \( L_2 \), on \( B'' \) with \( L_{i,t} \cong h_{i,t}^*(\mathcal{O}_{\mathbb{P}^1}(1)) \) for \( i = 1, 2 \). Consider \( N' := L_1 \otimes L_2 \). The morphisms \( h_{1,t} : B_t = B''_t \to \mathbb{P}^1 \) and \( h_{2,t} : B_t = B''_t \to \mathbb{P}^1 \) are distinct, so \( h^0(B''_t, N'_t) \geq 4 \).

Assume firstly \( B'' \) is smooth. The rational components \( R_i \) of the central fibre define Cartier divisors on \( B'' \) for \( 1 \leq i \leq g - 1 - k \). Consider the line bundle on \( B'' \)

\[
N' := N' \left( \sum_{i=1}^{g-1-k} R_i \right).
\]

The line bundle \( N'_0 \) on the central fibre \( B''_0 \) satisfies the hypothesis of Proposition \ref{prop:linear-syzygies} so that \( h^0(B''_0, N'_0) = 3 \). On the other hand, \( N'_t \cong N'_t \) for \( t \neq 0 \), which contradicts semicontinuity.

In the general case, blow \( B'' \) up over the nodes on the central fibre to obtain a smooth surface \( \overline{B} \to \Delta \). This introduces chains

\[
\Gamma_{x_i} = \Gamma_{x_i,1} \cup \ldots \cup \Gamma_{x_i,\ell_i}, \quad \Gamma_{y_i} = \Gamma_{y_i,1} \cup \ldots \cup \Gamma_{y_i,m_i}
\]

of rational curves into the central fibre for all \( 1 \leq i \leq g - 1 - k \). Here \( \Gamma_{x_i,1} \) respectively \( \Gamma_{y_i,1} \) are the components of \( \Gamma_{x_i} \) respectively \( \Gamma_{y_i} \), meeting \( \tilde{C}_0 \) precisely at \( x_i \) and \( y_i \) respectively. Furthermore, \( \Gamma_{x_i,\ell_i} \) and \( \Gamma_{y_i,m_i} \) are the components meeting \( R_i \). Finally, the components are
ordered in such a way that $\Gamma_{x_1,j} \cap \Gamma_{x_1,j-1} \neq \emptyset$ for $j = 2, \ldots, \ell_i$ and $\Gamma_{y_1,j} \cap \Gamma_{y_1,j-1} \neq \emptyset$ for $j = 2, \ldots, m_j$.

If the two chains have different lengths, say $\ell_i < m_i$, then we increase the length of $\Gamma_{x_1}$ as follows. First, blow up $B''$ at $x_i$. The central fibre is no longer reduced, so follow stable reduction by performing a degree two base change and then normalizing. This has the effect of increasing the length of the chain $\Gamma_{x_1}$ by one (the total family remains smooth). By repeating this procedure, we may assume $\ell_i = m_i$.

Let $\mathcal{N}'$ denote the pull-back of $\mathcal{N}$ to $B''$. For $i = 1, \ldots, g - k - 1$, set

$$Z_i := (\Gamma_{x_1,1} + \Gamma_{y_1,1}) + 2(\Gamma_{x_1,2} + \Gamma_{y_1,2}) + \cdots + \ell_i(\Gamma_{x_1,\ell_i} + \Gamma_{y_1,\ell_i}) + (\ell_i + 1)R_i,$$

then consider the line bundle $\mathcal{N}'' := \mathcal{N}' \left( \sum_{i=1}^{g-1-k} Z_i \right)$. Then one checks that

$$\mathcal{N}''|_{\Gamma_{x_1}} \cong \mathcal{O}_{\Gamma_{x_1}}, \quad \mathcal{N}''|_{\Gamma_{y_1}} \cong \mathcal{O}_{\Gamma_{y_1}}, \quad \mathcal{N}''|_{R_i} \cong \mathcal{O}_{R_i},$$

whereas $\mathcal{N}''|_{\mathcal{C}_0} \cong A^{\otimes 2} \left( \sum_{i=1}^{g-1-k} (x_i + y_i) \right)$. We now reach a contradiction from Remark 3.6. \qed

4. Extending the Eagon-Northcott Divisor

In this section we construct an extension of the Eagon-Northcott divisor $\mathcal{E}N$ on a partial compactification of the Hurwitz space $\mathcal{H}_{2g-2k+1,g-k+1}$. We keep the notation of the previous section, set $a := g - k + 1$, and further assume $a \geq 3$. In order to describe the partial compactification of the Hurwitz space $\mathcal{H}_{2a-1,a-1}$ it is convenient to work with stable maps.

Let $\overline{\mathcal{M}}_{2a-1}^{ns}(\mathbb{P}^1, a)$ denote the moduli space of finite stable maps $f : C \to \mathbb{P}^1$ of degree $a$ such that $C$ has genus $2a - 1$ and only non-separating nodes and with $h^0(C, f^*\mathcal{O}_{\mathbb{P}^1}(1)) = 2$. Set

$$\overline{\mathcal{G}}_{2a-1,a}^{ns} := \overline{\mathcal{M}}_{2a-1}^{ns}(\mathbb{P}^1, a)/\text{PGL}(2),$$

where PGL(2) acts on the target. Then $\overline{\mathcal{G}}_{2a-1,a}^{ns}$ is an open subset of the projective moduli space

$$\overline{\mathcal{G}}_{2a-1,a} := \overline{\mathcal{M}}_{2a-1}(\mathbb{P}^1, a)/\text{PGL}(2)$$

of stable maps $f : C \to \mathbb{P}^1$ with $f_*[C] = a[\mathbb{P}^1]$, modulo isomorphisms of the target. Note that the Hurwitz space $\mathcal{H}_{2a-1,a}$ can be regarded as an open set of $\overline{\mathcal{G}}_{2a-1,a}^{ns}$.

We shall construct the extended Eagon-Northcott divisor

$$\overline{\mathcal{E}}N \subseteq \overline{\mathcal{G}}_{2a-1,a}^{ns}$$

by studying the minimal free resolutions of the scrolls attached to a cover $[f : C \to \mathbb{P}^1] \in \overline{\mathcal{G}}_{2a-1,a}^{ns}$. Set $A := f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \in \mathcal{W}_a(C)$. Since $f$ is finite and flat, $f_*\mathcal{O}_C$ is locally free and we write $f_*\mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{E}_f$, where $\mathcal{E}_f$ is the so-called Tschirnhausen bundle of $f$, admitting a splitting

$$\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_{a-1}),$$

where $e_1 \leq \cdots \leq e_{a-1}$ are the scroller invariants of $f$ and satisfy $e_1 + \cdots + e_{a-1} = 3a - 2$. Dualising the morphism $f_*\mathcal{O}_C$ leads to an exact sequence

$$0 \to \mathcal{E}_f \to f_*\omega_f \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

We tensor the morphism $f^*(\mathcal{E}_f) \to \omega_f$ by $f^*\omega_{\mathbb{P}^1}$ and produce a morphism $f^*(\mathcal{E}_f(-2)) \to \omega_C$, inducing a closed immersion, see \cite{Sch1}, or \cite{CE}

$$j : C \to \mathbb{P}(\mathcal{E}_f(-2)).$$
Note that $\mathcal{E}_f(-2)$ is a globally generated vector bundle on $\mathbb{P}^1$ with $\deg(\mathcal{E}_f(-2)) = a$. Denoting by $\varphi : X := \mathbb{P}(\mathcal{E}_f(-2)) \to \mathbb{P}^1$ the associated $(a - 1)$-dimensional scroll, we have a morphism

$$\iota : X \to \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{E}_f(-2))) \cong \mathbb{P}^{2a - 2},$$

such that $\iota \circ j : C \to \mathbb{P}^{2a - 2}$ is the canonical morphism of $C$, cf. [Sch1]. Observe that since $C$ has no disconnected nodes, $\omega_C$ is globally generated. Also observe that if $h^0(C, A^{\otimes 2}) = 3$, then $e_1 \geq 3$ and $\iota$ is a closed immersion.

The Picard group of the scroll $X$ is generated by the class of a ruling $R := \varphi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ together with $H := \mathcal{O}_X(1)$. Note that $H^0(X, H) \cong H^0(C, \omega_C)$, whereas $H^0(X, R) \cong H^0(C, A)$ and $H^0(X, \mathcal{O}_X(H - R)) \cong H^0(C, \omega_C \otimes A^\vee)$. As already mentioned in the Introduction, the Eagon-Northcott complex, explicitly describes the minimal free resolution of

$$\Gamma_X(H) := \bigoplus_{q \in \mathbb{Z}} H^0(X, H^{\otimes q}),$$

as a $\text{Sym} H^0(X, H)$-module, see [Sch1]. This gives that

$$K_{p,0}(X, H) = 0 \text{ for } p > 0, \quad \text{whereas } K_{p,q}(X, H) = 0, \text{ for } q \geq 2 \text{ and any } p,$$

as well as the canonical identifications

$$K_{p,1}(X, H) \cong \bigwedge^{p+1} H^0(X, H - R) \otimes \text{Sym}^{p-1} H^0(X, R) \otimes \bigwedge^2 H^0(X, R)$$

$$\cong \bigwedge^{p+1} H^0(C, \omega_C \otimes A^\vee) \otimes \text{Sym}^{p-1} H^0(C, A) \otimes \bigwedge^2 H^0(C, A).$$

In particular, $b_{p,1}(X, H) = p \binom{a}{p+1}$.

We record the following lemma, while skipping the proof:

**Lemma 4.1.** We have the vanishing $H^i(X, H^{\otimes q}) = 0$, for $i \geq 1$ and $q \geq 0$. Furthermore, $H^i(X, \mathcal{O}_X(-H)) = 0$, for $i \geq 2$.

Define the kernel bundles $M_H$ and $M_{\omega_C}$ on $X$ and $C$ respectively by the exact sequences

$$0 \to M_H \to H^0(X, H) \otimes \mathcal{O}_X \to H \to 0$$

$$0 \to M_{\omega_C} \to H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C \to 0.$$

As $C \subseteq X$ is linearly normal, $j^*M_H \cong M_{\omega_C}$. Note that $H^0(X, \bigwedge^p M_H) = H^0(C, \bigwedge^p M_{\omega_C}) = 0$, for $p \geq 1$. Further, we record the following short exact sequences:

$$(4) \quad 0 \to \bigwedge^{p+1} M_H \otimes \mathcal{O}_X((q - 1)H) \to \bigwedge^{p+1} H^0(X, H) \otimes \mathcal{O}_X((q - 1)) \to \bigwedge^p M_H \otimes \mathcal{O}_X(qH) \to 0,$$

$$(5) \quad 0 \to \bigwedge^{p+1} M_{\omega_C} \otimes \omega_C^{\otimes (q-1)} \to \bigwedge^{p+1} H^0(C, \omega_C) \otimes \omega_C^{\otimes (q-1)} \to \bigwedge^p M_{\omega_C} \otimes \omega_C^{\otimes q} \to 0.$$

We shall make use of the following vanishing statement.

**Lemma 4.2.** We have $H^i(X, \bigwedge^p M_H \otimes H^{\otimes q}) = 0$ for $i \geq 2$ and arbitrary $p, q \geq 0$.

*Proof.* By the sequence (4) and Lemma 4.1 it suffices to show $H^{i-1}(\bigwedge^{p-1} M_H \otimes H^{\otimes (q+1)}) = 0$. Continuing in this fashion, it suffices to show $H^1(X, \bigwedge^{p-i+1} M_H \otimes H^{\otimes (q+i-1)}) = 0$. Since $H^1(X, H^{\otimes (q+i-1)}) = 0$, this amounts to $K_{p,q+i}(X, H) = 0$, which holds as $q + i \geq 2$. □
Lemma 4.3. There is an injective restriction map of linear syzygies

\[ \alpha_f : K_{a-1,1}(X, H) \to K_{a-1,1}(C, \omega_C). \]

The map \( \alpha_f \) is surjective if and only if the restriction map

\[ \beta_f : H^0\left( X, \bigwedge^a M_H \otimes H \otimes 2 \right) \to H^0\left( C, \bigwedge^a M_{\omega_C} \otimes \omega_C \otimes 2 \right) \]

is injective.

Proof. The map \( \alpha_f \) fits into a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \bigwedge^a H^0(X, H) & \longrightarrow & H^0(X, \bigwedge^{a-1} M_H \otimes H) & \longrightarrow & K_{a-1,1}(X, H) & \longrightarrow & 0.
\end{array}
\]

\[ \cong \]

\[
\begin{array}{cccccc}
0 & \longrightarrow & \bigwedge^a H^0(C, \omega_C) & \longrightarrow & H^0(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C) & \longrightarrow & K_{a-1,1}(C, \omega_C) & \longrightarrow & 0.
\end{array}
\]

Since \( C \subseteq X \) is linearly normal, it follows that \( \text{res}_C \) is injective, therefore \( \alpha_f \) is injective as well. On the other hand, by the snake lemma the surjectivity of \( \alpha_f \) is equivalent to the surjectivity of \( \text{res}_C \). From the kernel bundle description of Koszul cohomology, we write

\[
K_{a-2,2}(X, H) = \text{Ker}\left\{ H^1(X, \bigwedge^{a-1} M_H \otimes H) \to \bigwedge^a H^0(X, H) \otimes H^1(X, H) \right\}.
\]

Since \( H^1(X, H) = 0 \) and \( K_{a-2,2}(X, H) = 0 \), it follows \( H^1(X, \bigwedge^{a-1} M_H \otimes H) = 0 \). We write the following diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(X, \bigwedge^{a-1} M_H \otimes H) & \longrightarrow & \bigwedge^a H^0(X, H) \otimes H^0(X, H) & \longrightarrow & H^0(X, \bigwedge^{a-2} M_H \otimes H \otimes 2) & \longrightarrow & 0,
\end{array}
\]

\[ \cong \]

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C) & \longrightarrow & \bigwedge^a H^0(C, \omega_C) \otimes H^0(C, \omega_C) & \longrightarrow & H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C \otimes 2) & \longrightarrow & 0.
\end{array}
\]

By the snake lemma, the surjectivity of \( \text{res}_C \) is equivalent to the injectivity of \( \beta_f \). \( \square \)

Koszul duality gives an isomorphism \( K_{a-2,2}(C, \omega_C) \cong K_{a-1,1}(C, \omega_C)^\vee \), therefore we have a surjection

\[
H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C) \to H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C) / \bigwedge^a H^0(C, \omega_C) \cong K_{a-1,1}(C, \omega_C)^\vee.
\]

The composition of this map with \( \alpha_f^\vee \) gives rise to a surjection

\[
\psi_f : H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C) \to K_{a-1,1}(X, H)^\vee.
\]

Because \( K_{a-2,2}(X, H) = 0 \), from the second diagram in the proof of Lemma 4.3, it follows \( \psi_f \circ \beta_f = 0 \).

Lemma 4.4. We have a natural isomorphism \( \text{Ker}(\psi_f) \cong H^2(X, \bigwedge^a M_H \otimes I_{C/X})^\vee \).

Proof. Since \( H^1(X, \mathcal{O}_X) = 0 \), the description of Koszul cohomology via kernel bundles yields the identification \( K_{a-1,1}(X, H)^\vee \cong H^1(X, \bigwedge^a M_H)^\vee \). Using that \( \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C \cong \bigwedge^a M_{\omega_C}^\vee \), Serre-Duality gives the isomorphism

\[
H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C) \cong H^1(C, \bigwedge^a M_{\omega_C}).
\]
which enables us to identify the dual map $\psi_f^\dagger$ with the restriction
\[
H^1(X, \bigwedge^a M_H) \to H^1(C, \bigwedge^a M_{\omega_C}).
\]
Then $\text{Ker}(\psi_f) \cong \text{Coker}(\psi_f)^\dagger \cong H^2(X, \bigwedge^a M_H \otimes I_{C/X})^\dagger$, using also Lemma 4.2. \qed

Putting the above pieces together, we have constructed a natural map
\[
\beta_f : H^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) \to H^2(X, \bigwedge^a M_H \otimes I_{C/X})^\dagger
\]
such that $b_{a-1,1}(C, \omega_C) > a - 1$ if and only if $\beta_f$ fails to be injective. We shall see that both sides of this map have the same dimension. This allows us to construct $\tilde{E}_N$ as the degeneracy locus of a morphism between vector bundles of the same rank on the space of stable maps.

**Lemma 4.5.** We have:
\[
h^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) = h^2(X, \bigwedge^a M_H \otimes I_{C/X}) = (2a - 2)\binom{2a - 1}{a} - a + 1.
\]

**Proof.** As already pointed out $H^1(X, \bigwedge^{a-1} M_H \otimes H) = 0$. Therefore
\[
h^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) = (2a - 1)\binom{2a - 1}{a} - h^0(X, \bigwedge^{a-1} M_H \otimes H),
\]
by the short exact sequence \[\text{[4]}\]. We further have a short exact sequence
\[
0 \longrightarrow \bigwedge^a H^0(X, H) \longrightarrow H^0(X, \bigwedge^{a-1} M_H \otimes H) \longrightarrow K_{a-1,1}(X, H) \longrightarrow 0,
\]
thus using that $b_{a-1,1}(X, H) = a - 1$, we find $h^0(X, \bigwedge^{a-1} M_H \otimes H) = a - 1 + \binom{2a-1}{a}$, which leads to the claimed formula for $h^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2})$.

Using Lemma 4.3 we compute:
\[
h^2(X, \bigwedge^a M_H \otimes I_{C/X}) = \dim(\text{Ker} \ \psi_f) = h^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_{C}^{\otimes 2}) - b_{a-1,1}(X, H).
\]
Recall that $b_{a-1,1}(X, H) = a - 1$. The Riemann-Roch theorem (still valid for a nodal curve $C$ with no disconnecting nodes) gives
\[
h^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_{C}^{\otimes 2}) = \chi(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_{C}^{\otimes 2}) = (4a - 2)\binom{2a - 2}{a},
\]
which finishes the proof. \qed

We now explain how the above considerations can be carried out in a relative setting. Let
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{P} \\
\nu & \downarrow & \mu \\
\tilde{G}_{2a-1,a}^{\text{ns}} & \to & \mathcal{P}\end{array}
\]
be the universal degree $a$ cover, where $\mathcal{P} \to \tilde{G}_{2a-1,a}^{\text{ns}}$ is a $\mathbb{P}^1$-bundle. The universal Tschirnhausen bundle $\mathcal{E}_f$ on $\mathcal{P}$ fits into an exact sequence:
\[
0 \longrightarrow \mathcal{E}_f \longrightarrow f_* \omega_f \longrightarrow \mathcal{O}_\mathcal{P} \longrightarrow 0.
\]
We further have the projective bundle \( \varphi : X := \mathbb{P}(\mathcal{E}_f \otimes \omega_{\mu}) \to \mathcal{P} \) and a closed immersion \( j : C \hookrightarrow X \). Set \( h := \mu \circ \varphi : \mathbb{P}(\mathcal{E}_f \otimes \omega_{\mu}) \to \mathcal{G}^{\text{ns}}_{2a-1,a} \). By Grauert’s Theorem, \( h_*(\mathcal{O}_X(1)) \) is a vector bundle of rank \( 2a - 1 \). Define the determinant \( \xi := \det(f_\ast) \). The evaluation map \( h^*h_*\mathcal{O}_X(1) \to \mathcal{O}_X(1) \) is furthermore surjective, thus we can define the kernel bundle \( \mathcal{M} \) by

\[
0 \to \mathcal{M} \to h^*h_*\mathcal{O}_X(1) \to \mathcal{O}_X(1) \to 0.
\]

Then \( \mathcal{M} \) restricts to the kernel bundle \( M_H \) for each scroll induced by an element \([C \to \mathbb{P}^1]\). Note that \( j \) is defined by the surjection \( f^*\mathcal{E}_f \to \mathcal{O}_f \otimes \omega_{\mu} \cong \omega_f \), hence \( \mathcal{O}_C(1) \cong \omega_f \). Set

\[
\mathcal{F}_1 := h_*(\bigwedge^a_\mathcal{M} \otimes \mathcal{O}_X(2)) \otimes \xi^\vee
\]

which is a vector bundle of rank \( (2a - 2)(2a - 1) - a + 1 \), by Lemma 4.5. Set

\[
\mathcal{F}_2 := h_*(\bigwedge^a_\mathcal{M} \otimes \mathcal{O}_C(2)) \otimes \xi^\vee,
\]

which is a vector bundle of rank \( (2a - 2)(2a - 1) \). Restriction to \( C \) induces a morphism \( \beta : \mathcal{F}_1 \to \mathcal{F}_2 \).

Relative duality gives the isomorphism

\[
R^1\nu_*\bigwedge^a_\mathcal{M}|_C \cong \left(\nu_*\bigwedge^a_\mathcal{M}|_C \otimes \omega_f\right)^\vee \cong \mathcal{F}_2^\vee,
\]

using \( \det(\mathcal{M}) \cong h^*\xi \otimes \mathcal{O}_X(-1) \). Define the rank \( a - 1 \) vector bundle by \( \mathcal{F}_3 := R^1\nu_*\left(\bigwedge^a_\mathcal{M}\right)^\vee \).

The dual of the restriction morphism \( \psi^\vee : R^1h_*\left(\bigwedge^a_\mathcal{M}\right) \to R^1\nu_*\left(\bigwedge^a_\mathcal{M}|_C\right) \) gives a morphism

\[
\psi : \mathcal{F}_2 \to \mathcal{F}_3
\]

with fibre over a moduli point \([f : C \to \mathbb{P}^1]\) equal to \( \psi_f \). As already explained, \( \psi \circ \beta = 0 \).

We get a short exact sequence of vector bundles over \( \mathcal{G}^{\text{ns}}_{2a-1,a} \):

\[
0 \to R^1h_*\left(\bigwedge^a_\mathcal{M}\right) \to R^1\nu_*\left(\bigwedge^a_\mathcal{M} \otimes \mathcal{O}_C\right) \to R^2h_*\left(\bigwedge^a_\mathcal{M} \otimes I_{C/X}\right) \to 0,
\]

where \( \mathcal{F}_4 := R^2h_*\left(\bigwedge^a_\mathcal{M} \otimes I_{C/X}\right) \) is a vector bundle of rank \( (2a - 2)(2a - 1) - a + 1 \) by Lemma 4.5. Thus we may canonically identify

\[
\text{Ker}(\psi) \cong \mathcal{F}_4^\vee
\]

and we have an induced morphism between vector bundles \( \beta : \mathcal{F}_1 \to \mathcal{F}_4^\vee \) globalizing the morphisms \( \beta_f \) as the moduli point \([f] \in \mathcal{G}^{\text{ns}}_{2a-1,a}\) varies. Since \( \text{rk}(\mathcal{F}_1) = \text{rk}(\mathcal{F}_4) \), we define the extended Eagon-Northcott divisor

\[
\mathcal{E}\mathcal{N} \subseteq \mathcal{G}^{\text{ns}}_{2a-1,a}
\]

as the degeneracy locus of \( \beta \). By the results of the previous chapter, this is a genuine divisor.

Define \( \mathcal{E}\mathcal{N}^{\text{sm}} \) as the union of all components of \( \mathcal{E}\mathcal{N} \) containing an element \([f : C \to \mathbb{P}^1]\), with \( C \) being a smooth curve. The following lemma is a direct consequence of Theorem 3.8.

**Lemma 4.6.** Let \( C \) be a general curve of genus \( g \) and gonality \( k \leq \frac{g+1}{2} \). For \( 1 \leq i \leq g-2k+1 \), choose \((x_i,y_i)\) to be general pairs of points on \( C \) and let \( B \) be the semi-stable curve given as the union of \( C \) with \( g-2k+1 \) smooth rational curves \( R_i \) such that each \( R_i \) meets the rest of \( B \) precisely at \( x_i, y_i \) for \( 1 \leq i \leq g-2k+1 \). Let

\[
[f : B \to \mathbb{P}^1] \in \mathcal{G}^{\text{ns}}_{2g-2k+1,g-k+1}
\]

be a morphism with \( \deg(f|_C) = k \) and \( f|_{R_i} \) an isomorphism. Then \([f] \notin \mathcal{E}\mathcal{N}^{\text{sm}}\).
Proof. Consider the closure \( \mathcal{EN}^{sm} \subseteq \mathcal{G}_{2g-2k+1,g-k+1} \) in the moduli space of stable maps. We have the projections \( \pi' : \mathcal{G}_{2g-2k+1,g-k+1} \to \mathcal{M}_{2g-2k+1} \), as well as the projection \( \pi \) from the space of admissible covers. There is an equality of closed sets \( \pi' (\mathcal{EN}^{sm}) = \pi (\mathcal{EN}) \), since \( \mathcal{G}_{2g-2k+1,g-k+1} \) and \( \mathcal{F}_{2g-2k+1,g-k+1} \) are isomorphic over the open set of morphisms with smooth source. By Theorem 3.8 the point \([D] \in \mathcal{M}_{2g-2k+1}\) defined by the stabilization of \( B \) does not lie in \( \pi' (\mathcal{EN}^{sm}) \), therefore, \([f] \notin \mathcal{EN}^{sm}\).

To complete the proof of Theorem 0.3 we need to show that, in the situation of Lemma 4.6 the point \([f] \) does not lie in the extended Eagon-Northcott divisor \( \mathcal{EN} \). Note that \([f] \) lies in precisely one of the three boundary divisors, namely the divisor \( \Delta \) parametrising covers with singular base. Hence we need to show that the divisor \( \mathcal{EN} \) does not contain the boundary divisor \( \Delta \). We carry this out in the next section, using \( K3 \) surfaces.

5. \( K3 \) Surfaces and Schreyer’s Conjecture

We start by considering a \( K3 \) surface \( X = X_d \) with Picard group generated by two classes \( L \) and \( E \) with self intersections given by \( (L)^2 = 4d - 4 \), \( (E)^2 = 0 \) and \( (L \cdot E) = d \), for \( d \geq 3 \). By performing Picard-Lefschetz transformations and a reflection if necessary, we may assume that \( L \) is big and nef.

Lemma 5.1. For \( X \) as above, the class \( L \) is base point free and \( E \) is the class of a smooth elliptic curve.

Proof. We firstly show that \( L \) is base point free. As \( L \) is big and nef, it suffices to show there is no smooth elliptic curve \( F \) with \( (L \cdot F) = 1 \). Assume such \( F \) exists, and write \( F = aL + bE \) for \( a, b \in \mathbb{Z} \). As \( F \) is smooth and elliptic, \( (F)^2 = 0 \), implying \( 0 = (aL + bE) \cdot F = a + b(E \cdot F) = a(1 + db) \). If \( a = 0 \), then \( (L \cdot F) = bd \neq 1 \), since \( d \geq 2 \), so \( db = -1 \), which is again impossible. Thus \( L \) is base point free.

We next show that \( E \) is the class of a smooth elliptic curve. As \( (E)^2 = 0 \) and \( E \) is primitive, it suffices to show that \( E \) is nef. Since \( (E \cdot L) > 0 \), and \( L \) is big and nef, \( E \) is effective. Suppose \( E \) is not nef. Then there exists a smooth, rational curve \( R \) with \( (R \cdot E) < 0 \). Write \( R = aL + bE \) for \( a, b \in \mathbb{Z} \). Then \( (R \cdot E) < 0 \) implies \( a < 0 \). As \( (R)^2 = -2 \) and \( R \) is effective, we must have \( b > 0 \). We have \(-2 = (R)^2 = R \cdot (aL + bE) = a(R \cdot L) + b(R \cdot E) = a((R \cdot L) + bd) \), which is impossible for \( d \geq 3 \).

We now discuss the Brill-Noether theory of a smooth curve \( C \in |L| \). To that end, we follow [K1, §2] which works in the situation of a higher rank Picard lattice containing the lattice \( \text{Pic}(X_d) \).

Lemma 5.2. Let \( D \in \text{Pic}(X_d) \) be effective with \( (D)^2 \geq 0 \). Assume in addition \( L - D \) is effective and \( (L - D)^2 > 0 \). Then \( D = cE \), for some integer \( c \).

Proof. This is a slight modification of [K1, Lemma 2.5]. Write \( D = aL + bE \). As \( L - D \) is effective and \( E \) nef, \( (L - D) \cdot E = (1 - a)(L \cdot E) \geq 0 \), so \( a \leq 1 \). From \( (D \cdot E) \geq 0 \), we obtain \( a \geq 0 \). If \( a = 1 \), then \( (L - D)^2 = b^2(E)^2 = 0 \), so we must have \( a = 0 \) as required.

The next lemma describes the Brill-Noether behaviour of curves in the linear system \(|L|\).

Lemma 5.3. Let \( C \in |L| \) be a smooth curve. Then \( \text{Cliff}(C) = d - 2 \) and \( W^1_d(C) \) is reduced and consists of the single point \( \mathcal{O}_C(E) \).

Proof. The proof that \( \text{Cliff}(C) = d - 2 \) is essentially the same as [K1, Lemma 2.6]. Arguing as in [K1, Lemmas 2.7, 2.8], we see that \( W^1_d(C) \) is set-theoretically a single point, namely \( \mathcal{O}_C(E) \).
It remains to establish that $W_d^i(C)$ is reduced, which amounts to showing that $h^0(\mathcal{O}_C(2E)) = 3$. From the exact sequence

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_X(2E) \to \mathcal{O}_E(2E) \cong \mathcal{O}_E \to 0,$$

we deduce $h^1(X, 2E) = 1$ and then $h^0(X, 2E) = 3$ by Riemann–Roch. By the exact sequence

$$0 \to \mathcal{O}_X(2E - C) \to \mathcal{O}_X(2E) \to \mathcal{O}_C(2E) \to 0,$$

it suffices to show $h^0(X, 2E - C) = h^1(X, 2E - C) = 0$. As $(C - 2E)^2 = -4$, by Riemann–Roch, it suffices to show that neither $2E - C$ nor $C - 2E$ are effective. As $(E \cdot 2E - C) < 0$ and $E$ is nef, $2E - C$ is not effective. Now suppose $C - 2E$ is effective with integral components $R_1, \ldots, R_\ell$, for $\ell \geq 1$. We write $R_i = a_iL + b_iE$, for integers $a_i, b_i$, with $\sum_{i=1}^\ell a_i = 1$ and $\sum_{i=1}^\ell b_i = -2$. As $(E \cdot R_i) \geq 0$, we find $a_i \geq 0$ for all $i$. Without loss of generality, we may assume $a_1 = 1$ and $a_i = 0$ for $2 \leq i \leq \ell$. As $R_i$ is integral, we must then have $b_i = 1$ for $i > 1$. Thus $R_1 = L - (\ell + 1)E$, which implies $(R_1)^2 = 4d - 4 - 2d(\ell + 1) \leq -4$, contradicting that $R_1$ is integral.

We can now prove Theorem 0.5.

**Proof of Theorem 0.5.** Let $[f : B \to \mathbb{P}^1]$ be as in the statement of Lemma 4.6. By an argument along the lines of [V1 Corollary 1], we have an injection $K_{g-k,1}(C, \omega_C) \hookrightarrow K_{g-k,1}(B, \omega_B)$. For the sake of completeness we recall the proof.

The Mayer-Vietoris sequence induces an injection $H^0(C, \omega_C) \hookrightarrow H^0(B, \omega_B)$, as well as the composition of injections $H^0(C, \omega_C^2) \hookrightarrow H^0(C, \omega_C^2(\sum_{i=1}^\ell (x_i + y_i))) \hookrightarrow H^0(B, \omega_B^2)$. We then get a commutative diagram

$$
\begin{array}{ccc}
\bigwedge^{g-k+1} H^0(\omega_C) & \delta_0 & \bigwedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C) \\
\downarrow & & \downarrow \\
\bigwedge^{g-k+1} H^0(\omega_B) & \delta_0 & \bigwedge^{g-k} H^0(\omega_B) \otimes H^0(\omega_B)
\end{array}
$$

The conclusion now follows from the existence of maps, see also [AN Lemma 7.1] $
\wedge : \bigwedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C) \to \bigwedge^{g-k+1} H^0(\omega_C)$ and $\wedge' : \bigwedge^{g-k} H^0(\omega_B) \otimes H^0(\omega_B) \to \bigwedge^{g-k+1} H^0(\omega_B)$, with $\wedge \circ \delta_0 = \pm (g-k)\text{Id}$ and $\wedge' \circ \delta_0 = \pm (g-k)\text{Id}$.

We secondly claim that $[f]$ does not lie in the extended Koszul divisor $\tilde{\mathcal{N}}$. In light of the injective map above, this will complete the proof. As $[f]$ lies in exactly one boundary divisor, namely $\Delta$, all that remains is to show that the divisor $\tilde{\mathcal{N}}$ does not contain $\Delta$. By Lemma 5.3, we know that any smooth curve $C \in |L|$ on the K3 surface $X = X_{g-k+1}$ satisfies $K_{g-k+1}(C, \omega_C) = g - k$. By the Lefschetz Theorem [C], the same holds for any integral nodal curve $C_0 \in |L|$. As any integral, nodal curve $C_0$ (with at least one node) defines a point in $\Delta$, it suffices to show that such curves exist for the general $X_{g-k+1}$.

To do this, it suffices to take $2g - 2k + 1 \geq 8$, as the conclusion of the Theorem is well-known for $g \leq 8$ by [Sch1]. Then the class $L - E$ is very ample for a general K3 surface $X_d$ general with the given Picard lattice, by degenerating to the K3 surface $Y_{g-n}$ from [K1 Lemma 2.3]. Choose a curve $C_1 \in |L - E|$ meeting a smooth elliptic curve $E_0 \in |E|$ transversally, and consider the nodal curve $C_1 \cup E_0$. Pick any node $p_1 \in C_1 \cup E_0$. Then, by [La3 Theorem 3.8], the moduli space $\tilde{\mathcal{N}}_1(X_d)$ parametrising deformation of $C_1 \cup E_0$ preserving the assigned node $p_1$ is smooth near $(C_1 \cup E_0, p_1)$ of dimension $2g - 2k$. As $\dim |L - E| + \dim |E| = g - k + 1 < g - 1$ for $k \geq 3$, there exist integral, nodal curves $C_0 \in |L|$ with exactly one node, completing the proof.

$\square$
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