Volterra-type Operators from analytic Morrey spaces to Bloch space

Zhengyuan Zhuo Shanli Ye†
(Department of Mathematics, Fujian Normal University, Fuzhou 350007, P. R. China)

Abstract
In this note, we study the boundedness and compactness of integral operators $I_g$ and $T_g$ from analytic Morrey spaces to Bloch space. Furthermore, the norm and essential norm of those operators are given.

Keywords Analytic Morrey space; Bloch space; Volterra type operator; essential norm
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1 Introduction

Let $\mathbb{D} = \{ z : |z| < 1 \}$ and $\partial \mathbb{D} = \{ z : |z| = 1 \}$ denote respectively the open unit disc and the unit circle in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$ and $dm(z) = \frac{1}{\pi} dx dy$ the normalized area Lebesgue measure.

The aim of this paper is to characterize the boundedness and compactness of two Volterra type operators $I_g$ and $T_g$ from the analytic Morrey spaces $L^{2,\lambda}$ to the classical Bloch space $B$, and from the little analytic Morrey spaces $L^{2,\lambda}_0$ to the little Bloch space $B_0$. Also, we estimate the essential norm of $I_g$ and $T_g$.

Morrey space was initially introduced in 1938 by Morrey [20] to show that certain systems of partial differential equations (PDEs) had Hölder continuous solutions. In the past, Morrey space has been studied heavily in different areas. For example, Adams and Xiao studied Morrey spaces which is defined on Euclidean spaces $\mathbb{R}^n$ by potential theory and Hausdorff capacity in [3, 4]. But here we will be mostly interested in the analytic Morrey spaces $L^{2,\lambda}$ in the unit disk. It was introduced and studied by Wu and Xie in [29].

For an arc $I \subset \partial \mathbb{D}$, let $|I| = \frac{1}{2\pi} \int_I |d\zeta|$ be the normalized arc length of $I$,

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}, \quad f \in H(\mathbb{D}),$$

and $S(I)$ be the Carleson box based on $I$ with

$$S(I) = \{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \}.$$

Clearly, if $I = \partial \mathbb{D}$, then $S(I) = \mathbb{D}$.

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†Corresponding author. E-mail address: shanliye@fjnu.edu.cn

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Denote $\mathcal{L}^{2,\lambda}(\mathbb{D})$ the analytic Morrey spaces of all analytic functions $f \in H^2$ on $\mathbb{D}$ such that
\[
\sup_{I \subset \partial \mathbb{D}} \left( \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \right)^{1/2} < \infty,
\]
where $0 < \lambda \leq 1$ and the Hardy space $H^2$ consists of analytic functions $f$ in $\mathbb{D}$ satisfying
\[
\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.
\]

Similarly to the relation between $BMOA$ space and $VMOA$ space, we have that $f \in \mathcal{L}^{2,\lambda}_0(\mathbb{D})$, the little Morrey spaces, if $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and
\[
\lim_{|I| \to 0} \left( \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \right)^{1/2} = 0.
\]

Xiao and Xu [33] studied the composition operators of $\mathcal{L}^{2,\lambda}(\mathbb{D})$ spaces. Cascante, Fàbrega and Ortega [12] studied the Corona theorem of $\mathcal{L}^{2,\lambda}$. It is a useful tools for the study of harmonic analysis and partial differential equations, We refer the readers to [20, 23, 37].

The following lemma gives some equivalent conditions of $\mathcal{L}^{2,\lambda}(\mathbb{D})$ (see Theorem 3.21 of [32] or Theorem 3.1 of [34]).

**Lemma 1.1** Suppose that $0 < \lambda < 1$ and $f \in H(\mathbb{D})$. Let $a \in \mathbb{D}$, $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$. Then the following statements are equivalent.

(i) $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$;

(ii) $\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} |f'(z)|^2 (1 - |z|^2) dm(z) < \infty$;

(iii) $\sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z) < \infty$.

From the lemma above, we can define the norm of function $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and equivalent formula as follows
\[
\|f\|_{\mathcal{L}^{2,\lambda}} = |f(0)| + \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dm(z))^{1/2}
\approx |f(0)| + \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z))^{1/2}.
\]

It is known that $\mathcal{L}^{2,1}(\mathbb{D}) = BMOA$ and if $0 < \lambda < 1$, $BMOA \subset \mathcal{L}^{2,\lambda}(\mathbb{D})$. For more information on $BMOA$ and $VMOA$, see [15].

A function $f$ analytic on the unit disk is said to belong to the Bloch space $B$ if
\[
\|f\|_B = \sup_{z \in \mathbb{D}} \{(1 - |z|^2) |f'(z)|\} < \infty,
\]
and to the little Bloch space $B_0$ if $f \in B$ and
\[
\lim_{|z| \to 1^-} (1 - |z|^2) |f'(z)| = 0.
\]

It is well known that $B$ is a Banach space under the norm $\|f\|_B = |f(0)| + \|f\|_B$ and $B_0$ is a closed subspace of $B$. See [5]. By [8, 31], together with Lemma 2.1 in [7], we have the following equivalent statements about the norm of $f \in B$. 

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Proposition 1.1 For all $p \in (1, \infty)$,
\[
\|f\|_B \approx |f(0)| + \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} |f'(z)|^p (1 - |\varphi_a(z)|^2)^p \, dm(z) \right)^{1/2} \\
\approx |f(0)| + \sup_{I \subset \partial \mathbb{D}} \left( \frac{1}{|I|^p} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^p \, dm(z) \right)^{1/2}.
\]

Suppose that $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic map. The integral operator $T_g$, called Volterra-type operator, is defined as
\[
T_g f(z) = \int_0^z f(w) g'(w) \, dw, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).
\]

In [22] Pommelrenke introduced the operator $T_g$ and showed that $T_g$ is a bounded operator on the Hardy space $H^2$ if and only if $g \in BMOA$.

The companion integral operator $I_g$ is spontaneously defined as
\[
I_g f(z) = \int_0^z f'(w) g(w) \, dw, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).
\]

The boundedness, compactness or essential norm of $T_g$ and $I_g$ between spaces of analytic functions were investigated by many authors. Aleman and Siskakis in [11] studied the integral operator $T_g$ on the Bergman space, and then Aleman considered with Cima $T_g$ acting on the Hardy space in [2]. Siskakis and Zhao [24] also investigated $T_g$ on the space $BMOA$. $T_g$ on the $Q_p$ space was studied by Xiao in [30]. Li and Stević in [19] studied the boundedness and compactness of $T_g$ and $I_g$ on the Zygmund Spaces and the little Zygmund spaces. Constantin in [13] considered the boundedness and compactness of $T_g$ on $Fock$ spaces. Ye in [35] studied products of Volterra-type operators and composition operators on logarithmic Bloch space. Ye and Gao in [36] gave the boundedness and compactness of $T_g$ between different weighted Bloch spaces.

There are some articles about the integral operator acting on Morrey space. For example, Wu in [28] considered $T_g$ from Hardy to analytic Morrey spaces, Li, Liu and Lou [18] characterized the boundedness and essential norm of $T_g$ and $I_g$ on analytic Morrey spaces (see also the related references therein).

Now, we need two spaces. Let $\alpha > -1$. Recall that $f \in H(\mathbb{D})$ belongs to the weighted space $H^\infty_\alpha$ if it satisfies with
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.
\]

When $\alpha > -1$, $H^\infty_\alpha$ endowed with the norm $\|f\|_\alpha = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)|$ is a Banach space. This space is connected with the study of growth conditions of analytic functions and was also studied in detail, see [9, 10, 25, 26]. The space $H^\infty_\alpha$ is used in the characterizations of the boundedness and essential norm of $I_g$. Then we conclude the boundedness and essential norm of $T_g$ by introducing the following Bloch–Morrey type space.

**Definition 1.1** Let $0 < \lambda \leq 1$ and $p > 1$. The Bloch–Morrey type space $B\mathcal{L}^{p, \lambda}$ is the set of all $g \in H(\mathbb{D})$ such that
\[
M(g) = \sup_{I \subset \partial \mathbb{D}} \left( \frac{1}{|I|^{p - \lambda + 1}} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^p \, dm(z) \right)^{1/2} < \infty.
\]
The corresponding subspace $B_{0}^{p,\lambda}$, the little Bloch – Morrey type space, can be defined as

$$B_{0}^{p,\lambda} = \{ g \in BL^{p,\lambda}, \lim_{|I| \to 0} \left( \frac{1}{|I|^{p-\lambda+1}} \int_{S(I)} |g'(z)|^{2}(1-|z|^{2})^{p}dm(z) \right)^{1/2} = 0 \}. $$

It is easy to prove that $BL^{p,\lambda}$ is a Banach space under the norm

$$\|g\|_{BL^{p,\lambda}} = |g(0)| + M(g). $$

Clearly, $BL^{p,1} = L^{p,1}$. From [7], we know that $\|g\|_{BL^{p,\lambda}}$ is comparable with the norm

$$\|g(0)| + \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} \left( \frac{1-|a|^{2}}{1-|az|^{2}} \right)^{p+1-\lambda} |g'(z)|^{2}(1-|z|^{2})^{p}dm(z) \right)^{1/2}. $$

**Notations**: For two functions $F$ and $G$, if there is a constant $C > 0$ dependent only on indexes $p, \lambda...$ such that $F \leq CG$, then we say that $F \lesssim G$. Furthermore, denote that $F \asymp G$ ($F$ is comparable with $G$) whenever $F \lesssim G \lesssim F$.

## 2 $L^{2,\lambda}$ vs $B$

Evidently, when $0 < \lambda < 1$, $BMOA \subset L^{2,\lambda}(\mathbb{D})$. On the other hand $BMOA \subset B$. Does $L^{2,\lambda}$ and $B$ have the inclusion relation? We claim the answer is negative by the following two proposition. This makes our job more signally.

**Proposition 2.1** $L^{2,\lambda} \nsubseteq B$.

**Proof** Considering $g(z) = (\log \frac{1}{1-z})^{2}$, which is obviously not a Bloch function. We claim that $g(z) \in L^{2,\lambda}$. Indeed,

$$\sup_{a \in \mathbb{D}} (1-|a|^{2})^{1-\lambda} \int_{\mathbb{D}} |g'(z)|^{2}(1-|\rho_{a}(z)|^{2})dm(z) \leq \sup_{a \in \mathbb{D}} (1-|a|^{2})^{1-\lambda} \int_{\mathbb{D}} \frac{1}{1-z}^{2} \log^{2} \frac{1}{1-|z|}(1-|\rho_{a}(z)|^{2})dm(z) \leq \int_{\mathbb{D}} \frac{1}{1-z}^{2} (1-|z|^{2})^{1-\lambda} \log^{2} \frac{1}{1-|z|}dm(z) = \int_{0}^{1} \int_{0}^{2\pi} \frac{1}{1-r} \log^{2} (1-r^{2})^{1-\lambda} \log \frac{1}{1-r} dr < \infty. $$

This finish the proof.

Conversely, the function $f(z) = \sum_{n=0}^{\infty} z^{2^{n}}$ is a Bloch function (see [6]) and it is well known that it has a radial limit almost nowhere. Consequently, $f$ does not belong to any of the Hardy spaces and so $f \notin L^{2,\lambda}$. So we have the following proposition.

**Proposition 2.2** $B \nsubseteq L^{2,\lambda}$. 
3 Boundedness of $I_g$ and $T_g$ from $L^{2,\lambda}$ to $B$

In this section, we prove the boundedness and estimate the norms of $I_g$ and $T_g$. The following lemmas will be used through this paper.

**Lemma 3.1** Let $0 < \lambda < 1$ and $b \in \mathbb{D}$. We set functions $f_b(z)$ and $F_b(z)$ as

$$f_b(z) = (1 - |b|^2)^{\frac{1 - \lambda}{2}} (\rho_b(z) - b), \quad F_b(z) = (1 - |b|^2)(1 - b\overline{z})^{\frac{1 - \lambda}{2}}.$$  

then $f_b(z) \in L^{2,\lambda}(\mathbb{D})$ and $F_b(z) \in L^{2,\lambda}(\mathbb{D})$. Particularly, we have $f_b(z) \in L^{2,\lambda}_0(\mathbb{D})$ and $F_b(z) \in L^{2,\lambda}_0(\mathbb{D})$. Moreover, $\|f_b\|_{L^{2,\lambda}} \lesssim 1$, $\|F_b\|_{L^{2,\lambda}} \lesssim 1$.

**Proof** See Lemma 4 in [18]. From its proof, we further deduce that $f_b(z) \in L^{2,\lambda}(\mathbb{D})$ and $F_b(z) \in L^{2,\lambda}(\mathbb{D})$.

we get a result about the growth rate of functions in $L^{2,\lambda}(\mathbb{D})$ from [18].

**Lemma 3.2** Let $0 < \lambda < 1$. If $f \in L^{2,\lambda}(\mathbb{D})$, then

$$|f(z)| \lesssim \frac{\|f\|_{L^{2,\lambda}}}{(1 - |z|^2)^{\frac{1 - \lambda}{4}}}, \quad z \in \mathbb{D}.$$

We first consider the boundedness of $I_g : L^{2,\lambda} \to B$.

**Theorem 3.1** Let $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. Then $I_g : L^{2,\lambda} \to B$ is bounded if and only if $g \in H_{\frac{\lambda - 1}{2}}$. Moreover the operator norm satisfies

$$\|I_g\| \approx \|g\|_{\frac{\lambda - 1}{2}}.$$

**Proof** For $0 < \lambda < 1$, $1 < 2 - \lambda$, we set $B = Q_{2-\lambda}$.

Sufficiency: let $g \in H_{\frac{\lambda - 1}{2}}$. For any $f \in L^{2,\lambda}(\mathbb{D})$, we have

$$\|I_g f\|_B \approx \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} |f'(z)|^2 |g(z)|^2 (1 - |\varphi_a(z)|^2)^{2 - \lambda} dm(z) \right)^{1/2}$$

$$= \sup_{a \in \mathbb{D}} \left( (1 - |a|^2)^{1 - \lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) g(z)^2 \left( \frac{1 - |z|^2}{|1 - a\overline{z}|^2} \right)^{1 - \lambda} dm(z) \right)^{1/2}$$

$$\lesssim \|g\|_{\frac{\lambda - 1}{2}} \cdot \|f\|_{L^{2,\lambda}}.$$  

These inequalities imply $I_g$ is bounded and $\|I_g\| \lesssim \|g\|_{\frac{\lambda - 1}{2}}$.

Necessity: let $I_g$ is bounded. For any $b \in \mathbb{D}$, considering functions $f_b(z)$ in Lemma 3.1, we
have \( \|f_b\|_{\mathcal{L}^{2,\lambda}} \lesssim 1 \). Thus
\[
\|I_g\| \gtrsim \|I_gf_b\|_B \\
\approx \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} |f'_b(z)|^2 |g(z)|^2 (1 - |\varphi_a(z)|^2)^{2-\lambda}dm(z) \right)^{1/2} \\
= \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} \frac{(1 - |b|^2)^{1+\lambda}(1 - |\varphi_a(z)|^2)^{1+1-\lambda}dm(z)}{|1 - \overline{b}z|^4} \right)^{1/2} \\
\geq \left( \int_{\mathbb{D}} \frac{(1 - |b|^2)^2 |g(z)|^2 (1 - |z|)^{1-\lambda}(1 - |\rho_b(z)|^2)dm(z)}{|1 - \overline{b}z|^4} \right)^{1/2} \\
= \left( \int_{\mathbb{D}} |g(\rho_b(w))|^2 (1 - |\rho_b(w)|^2)^{1-\lambda}(1 - |w|^2)dm(w) \right)^{1/2} \\
\gtrsim \left| \frac{g(b)}{(1 - |b|^2)^{1/2}} \right|.
\]
where we used Lemma 4.12 of [38] in the last inequality. Since \( b \) is arbitrary, we have \( \|I_g\| \gtrsim \|g\|_{\mathcal{L}^{2,\lambda}} \). The proof is finished.

With the space \( B\mathcal{L}^{p,\lambda} \), we can establish the boundedness of \( T_g : \mathcal{L}^{2,\lambda} \to B \) as the following theorem.

**Theorem 3.2** Suppose that \( 0 < \lambda < 1 \) and \( g \in H(\mathbb{D}) \). Then the following conditions are equivalent:

(i) \( T_g : \mathcal{L}^{2,\lambda} \to B \) is bounded;

(ii) \( g \in B\mathcal{L}^{p,\lambda} \) for all \( p \in (1, \infty) \);

(iii) \( g \in B\mathcal{L}^{p,\lambda} \) for some \( p \in (1, \infty) \).

Moreover,
\[
\|T_g\| \approx M(g).
\]

**Proof** (i) \( \Rightarrow \) (ii). Suppose that \( T_g : \mathcal{L}^{2,\lambda} \to B \) is bounded. For any \( I \subset \partial\mathbb{D} \), let \( b = (1 - |I|)\zeta \in \mathbb{D} \), where \( \zeta \) is the centre of \( I \). Then
\[
(1 - |b|^2) \approx |1 - \overline{b}z| \approx |I|, \quad z \in S(I).
\]
Considering the functions \( F_b(z) \) in Lemma [3.1] \( \|F_b\|_{\mathcal{L}^{2,\lambda}} \lesssim 1 \). This together with Proposition [1.1] we obtain that for any \( p \in (1, \infty) \),
\[
\frac{1}{|I|^{p-\lambda+1}} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^p dm(z) \approx \frac{1}{|I|} \int_{S(I)} |F_b(z)|^2 |g'(z)|^2 (1 - |z|^2)^p dm(z) \lesssim \|T_gF_b\|_B^2 \\
\leq \|T_g\|^2 \|F_b\|^2_{\mathcal{L}^{2,\lambda}} \lesssim \|T_g\|^2.
\]
Since \( I \) is arbitrary, we have \( M(g) \lesssim \|T_g\| \).

(ii) \( \Rightarrow \) (iii). It is obvious.
As a result, \( \|T_g f\|_B \approx \sup_{I \subset \partial \mathbb{D}} \left( \frac{1}{|I|^p} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1 - |z|^2)^p \, dm(z) \right)^{1/2} \)

\[
\lesssim \|f\|_{L^2, \lambda} \cdot \sup_{I \subset \partial \mathbb{D}} \left( \frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^{p + \lambda - 1} \, dm(z) \right)^{1/2}.
\]

To the end, for a given subarc \( I \) of \( \partial \mathbb{D} \), let \( D_n(I) \) represent the set of \( 2^n \) subarcs of length \( 2^{-n}|I| \) obtained by \( n \) successive bipartition of \( I \). For each \( J \in D_n(I) \) write \( T(J) \) for the top half Carleson box of \( S(J) \), i.e.,

\[
T(J) = \{ z \in S(J) : \frac{z}{|z|} \in J, 1 - |J| < |z| < 1 - \frac{|J|}{2} \}.
\]

Then

\[
S(I) = \bigcup_{n=0}^{\infty} \bigcup_{J \in D_n(I)} T(J).
\]

Noting that \( z \in T(J) \), \( 1 - |z| \approx |J| \), one has

\[
\int_{S(I)} |g'(z)|^2 (1 - |z|^2)^{p + \lambda - 1} \, dm(z) = \sum_{n=0}^{\infty} \sum_{J \in D_n(I)} \int_{T(J)} |g'(z)|^2 (1 - |z|^2)^{p + \lambda - 1} \, dm(z)
\]

\[
\lesssim \sum_{n=0}^{\infty} \sum_{J \in D_n(I)} \int_{T(J)} |J|^\lambda - 1 |g'(z)|^2 (1 - |z|^2)^{p} \, dm(z)
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{J \in D_n(I)} \int_{S(J)} |J|^\lambda - 1 |g'(z)|^2 (1 - |z|^2)^{p} \, dm(z)
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{J \in D_n(I)} M(g)^2 |J|^{\lambda - 1} |J|^{p - \lambda + 1}
\]

\[
= \sum_{n=0}^{\infty} 2^n M(g)^2 |J|^p
\]

\[
= \sum_{n=0}^{\infty} (2^n)^{1-p} M(g)^2 |I|^p
\]

\[
\lesssim M(g)^2 |I|^p.
\]

Now invoking (3.2),

\[
\|T_g f\|_B \lesssim M(g) \cdot \|f\|_{L^2, \lambda}
\]

As a result, \( \|T_g\| \lesssim M(g) \).

Theorem 3.2 has an interesting consequence.

**Corollary 3.1** Let \( 0 < \lambda < 1 \) and \( 1 < p < q < \infty \). Then \( B^{\lambda, \lambda} = B^{q, \lambda} \).
4 Essential norm of $I_g$ and $T_g$ from $\mathcal{L}^{2,\lambda}$ to $B$

Let $X$ and $Y$ be Banach spaces. The essential norm of a bounded operator $T : X \to Y$, $\|T\|_{e,X \to Y}$, is defined as the distance from $T$ to the space of compact operators,

$$\|T\|_{e,X \to Y} = \inf \{|T - K|_{X \to Y} : K \text{ is any compact operator}\},$$

where the norm of $T$ is denoted by $\|T\|_{X \to Y}$.

Since that $T$ is compact if and only if $\|T\|_{e,X \to Y} = 0$, then the estimation of $\|T\|_{e,X \to Y}$ indicates the condition for $T$ to be compact. For some recent results related to the essential norm, see [10] [17] [27] [18], and the references therein.

In this section, we estimate the essential norm of $I_g$ and $T_g$ from $\mathcal{L}^{2,\lambda}$ to $B$. We need some auxiliary results.

**Lemma 4.1** Let $0 < \lambda < 1$. For $0 < t < 1$, $z \in \mathbb{D}$, $f_t(z) = f(tz)$. If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$, then $f_t \in \mathcal{L}^{2,\lambda}_0(\mathbb{D})$ and $\|f_t\|_{\mathcal{L}^{2,\lambda}} \leq \|f\|_{\mathcal{L}^{2,\lambda}}$.

**Proof** If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and $0 < t < 1$, then, $f_t$ is analytic on the closed unit disk $\overline{\mathbb{D}}$. A simple computation shows that $f_t \in \mathcal{L}^{2,\lambda}_0(\mathbb{D})$. In addition, by Poisson formula, we have

$$f_t(z) = \int_0^{2\pi} f(ze^{i\theta}) \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.$$ 

So,

$$\sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f_t'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z)$$

$$\leq \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} \int_0^{2\pi} |f'(ze^{i\theta})|^2 \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi} (1 - |\varphi_a(z)|^2) dm(z)$$

$$= \int_0^{2\pi} \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(ze^{i\theta})|^2 (1 - |\varphi_a(z)|^2) dm(z) \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi}$$

$$\leq \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z) \int_0^{2\pi} \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi}$$

$$= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z).$$

Thus, $\|f_t\|_{\mathcal{L}^{2,\lambda}} \leq \|f\|_{\mathcal{L}^{2,\lambda}}$.

By Lemma 3.2 and standard arguments (see, e.g., [11], Proposition 3.11), the following lemma follows.

**Lemma 4.2** Assume that $g$ is an analytic function on $\mathbb{D}$. Then $T_g(\text{or} I_g) : \mathcal{L}^{2,\lambda} \to B$ is compact if and only if $T_g(\text{or} I_g) : \mathcal{L}^{2,\lambda} \to B$ is bounded, and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{L}^{2,\lambda}$ which converges to zero uniformly on $\mathbb{D}$ as $k \to \infty$, $\|T_g f_k\|_B \to 0$ (or $\|I_g f_k\|_B \to 0$) as $k \to \infty$.

**Lemma 4.3** Suppose that $0 < \lambda < 1$ and $p > 1$. For $g \in B\mathcal{L}^{p,\lambda}$, define the following operators $T_{g,r} : \mathcal{L}^{2,\lambda} \to B$:

$$T_{g,r} f(z) = \int_0^z f(rw)g'(w) dw.$$ 

where $r \in (0, 1)$. Then $T_{g,r}$ is compact.
Let \( \{f_n\} \) be such that \( \|f_n\|_{L^{2,\lambda}} \leq 1 \) and \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). We are required to show that \( \lim_{n \to \infty} \|T_g f_n\|_B = 0 \). In fact, since \( \|g\|_B \lesssim M(g) \), we have \( |g'(z)| \lesssim \frac{M(g)}{1-|z|^2} \). From \( \|f_n\|_{L^{2,\lambda}} \leq 1 \) and Lemma 3.2, it yields that \( (1 - |r|^2)(1-\lambda)/2 |f_n(rz)| \lesssim 1 \). Thus

\[
\|T_g f_n\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f_n(rz)||g'(z)| \\
\lesssim M(g) \sup_{z \in \mathbb{D}} \frac{1}{(1 - |z|^2)(1-\lambda)/2}.
\]

Accordingly, by Dominated Convergence Theorem one reaches \( \lim_{n \to \infty} \|T_g f_n\|_B = 0 \).

Now, we present the main result of this section.

**Theorem 4.1** Suppose \( 0 < \lambda < 1 \) and \( g \in H(\mathbb{D}) \). If \( I_g : L^{2,\lambda} \to B \) is bounded, then

\[
\|I_g\|_{e,L^{2,\lambda} \to B} \approx \|g\|_{\frac{\lambda-1}{2}}.
\]

**Proof** Choose the zero operator \( O : L^{2,\lambda} \to B : f \mapsto 0 \). Since \( O \) is compact and \( \|O\| = 0 \), we get

\[
\|I_g\|_{e,L^{2,\lambda} \to B} = \inf_{K} \|I_g - K\| \leq \|I_g\| \lesssim \|g\|_{\frac{\lambda-1}{2}}.
\]

Conversely, choose the sequence \( \{b_n\} \subset \mathbb{D} \) such that \( |b_n| \to 1 \) as \( n \to \infty \). Considering the sequence of functions \( f_n(z) = (1 - |b_n|^2) \frac{1}{1 - \overline{b_n} z} (p_{b_n}(z) - b_n) \), we obtain \( \|f_n\|_{L^{2,\lambda}} \lesssim 1 \) by Lemma 3.1. By easy calculation, \( f_n(z) = -\frac{(1 - |b_n|^2)}{1 - \overline{b_n} z} \int_0^z \frac{b_n^z}{(1 - b_n^z)^2} \), and thus \( f_n \) converges to zero uniformly on compact subsets of \( \mathbb{D} \). Then \( \|K f_n\|_B \to 0 \) as \( n \to \infty \) for any compact operator \( K \). So

\[
\|I_g - K\| \gtrsim \limsup_{n \to \infty} \|(I_g - K) f_n\|_B \geq \limsup_{n \to \infty} (\|I_g f_n\|_B - \|K f_n\|_B) \geq \limsup_{n \to \infty} \|I_g f_n\|_B.
\]

By (3.1), we have

\[
\|I_g - K\| \gtrsim \limsup_{n \to \infty} \left| \frac{g(b_n)}{(1 - |b_n|^2)^{\frac{\lambda-1}{2}}} \right|.
\]

The arbitrary choice of the sequence \( \{b_n\} \) implies

\[
\|I_g\|_{e,L^{2,\lambda} \to B} \gtrsim \|g\|_{\frac{\lambda-1}{2}}.
\]

**Theorem 4.2** Suppose \( 0 < \lambda < 1 \) and \( g \in H(\mathbb{D}) \). If \( T_g : L^{2,\lambda} \to B \) is bounded, then

\[
\|T_g\|_{e,L^{2,\lambda} \to B} \approx \limsup_{|a| \to 1} \left( \int_{\mathbb{D}} \frac{1 - |a|^2}{1 - |az|^2} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2}.
\]

**Proof** For any \( r_n \in (0, 1) \) such that \( r_n \to 1 \) as \( n \to \infty \), we introduce \( T_{g,r_n} : L^{2,\lambda} \to B \) which is
Since the sequence \( \{ T_g \} \) is compact. Let \( s \in (0, 1) \), we have

\[
\| T_g \|_{e, L^{2, \lambda} \rightarrow B} \\
\leq \| T_g - T_{g, r_n} \| \\
\approx \sup_{\| f \|_{L^{2, \lambda}} = 1} \| T_g - T_{g, r_n} \|_B \\
= \sup_{\| f \|_{L^{2, \lambda}} = 1} \sup_{a \in D} \left( \int_D |f(z) - f(r_n z)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^{p + 1 - \lambda} dm(z) \right)^{1/2} \\
\leq \sup_{\| f \|_{L^{2, \lambda}} = 1} \sup_{a \in D} \left( \int_D |f(z) - f(r_n z)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^{p + 1 - \lambda} dm(z) \right)^{1/2} \\
+ \sup_{\| f \|_{L^{2, \lambda}} = 1} \sup_{a \in D} \left( \int_D |f(z) - f(r_n z)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^{p + 1 - \lambda} dm(z) \right)^{1/2} \\
\leq \sup_{\| f \|_{L^{2, \lambda}} = 1} \sup_{a \in D} \left( \int_D |f(z) - f(r_n z)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^{p + 1 - \lambda} dm(z) \right)^{1/2} \\
+ 2 \sup_{\| f \|_{L^{2, \lambda}} = 1} \sup_{a \in D} \left( \int_D \left( \frac{1 - |a|^2}{|1 - az|^2} \right)^{p + 1 - \lambda} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2} \\
\triangleq K_1 + K_2.
\]

Since \( |a| \leq s \) is a closed set of \( D \) and \( g \in BL^{p, \lambda} \), the Dominated Convergence Theorem yields \( K_1 \to 0 \) as \( n \to \infty \). Now, letting \( n \to \infty \) and then letting \( s \to 1 \), we get

\[
\| T_g \|_{e, L^{2, \lambda} \rightarrow B} \lesssim \lim_{s \to 1} \sup_{|a| \leq s} \left( \int_D \left( \frac{1 - |a|^2}{|1 - az|^2} \right)^{p + 1 - \lambda} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2}.
\]

Conversely, Let \( I_n \subset \partial D \) such that \( |I_n| \to 0 \) as \( n \to \infty \). \( \zeta_n \) is the center of arc \( I \) and \( b_n = (1 - |I_n|) \zeta_n \), so

\[
(1 - |b_n|^2) \approx |1 - \overline{b_n} z| \approx |I_n|, \quad z \in S(I_n).
\]

Consider the function \( F_n(z) = (1 - |b_n|^2)(1 - \overline{b_n} z)^{\frac{p+1-\lambda}{2}} \). Then \( \| F_n \|_{L^{2, \lambda}} \lesssim 1 \) and \( F_n \to 0 \) uniformly on the compact subsets of \( D \) as \( n \to \infty \) by Lemma [3.3]. Thus \( \| K F_n \|_B \to 0 \) for any compact operator \( K \). Therefore

\[
\| T_g - K \| \gtrsim \lim_{n \to \infty} \| (T_g - K) F_n \|_B \geq \lim_{n \to \infty} \sup_{n \to \infty} (\| T_g F_n \|_B - \| K F_n \|_B) \geq \lim_{n \to \infty} \sup_{n \to \infty} \| T_g f_n \|_B \\
\gtrsim \lim_{n \to \infty} \sup_{n \to \infty} \left( \frac{1}{|I_n|^p} \int_{S(I_n)} |F_n(z)|^2 |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2} \\
\approx \lim_{n \to \infty} \sup_{n \to \infty} \left( \frac{1}{|I_n|^p} \int_{S(I_n)} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2}.
\]

Since the sequence \( \{ I_n \} \) is arbitrary, we conclude

\[
\| T_g \|_{e, L^{2, \lambda} \rightarrow B} \gtrsim \lim_{|a| \to 0} \left( \frac{1}{|I|^{p+1-\lambda}} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2} \\
\approx \lim_{|a| \to 1} \left( \int_D \left( \frac{1 - |a|^2}{|1 - az|^2} \right)^{p + 1 - \lambda} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2}.
\]

This completes the proof.
We have the following corollary about their compactness.

**Corollary 4.1** Suppose that $0 < \lambda < 1$ and $p > 1$. Then

1. $I_g : L^{2,\lambda} \to B$ is compact if and only if $g = 0$.
2. $T_g : L^{2,\lambda} \to B$ is compact if and only if $g \in BL^{p,\lambda}_0$.

## 5 Boundedness and essential norm of $I_g$ and $T_g$ from $L^{2,\lambda}_0$ to $B_0$

**Theorem 5.1** Let $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. Then $I_g : L^{2,\lambda}_0 \to B_0$ is bounded if and only if $g \in H^\infty_{\lambda - 1}$. Moreover,

$$\|I_g\| \approx \|g\|_{\lambda - 1}.$$

**Proof** Necessity: assume that $I_g : L^{2,\lambda}_0 \to B_0$ is bounded. Then it is clear that $I_g : L^{2,\lambda}_0 \to B$ is bounded. The necessity of Theorem 4.1 together with $f_\lambda(z) \in L^{2,\lambda}_0(\mathbb{D})$, proves $g \in H^\infty_{\lambda - 1}$.

Sufficiency: let $g \in H^\infty_{\lambda - 1}$. Then from Theorem 4.1 $I_g : L^{2,\lambda} \to B$ is bounded and hence $I_g : L^{2,\lambda}_0 \to B$ is bounded. It suffices to prove that for any $f \in L^{2,\lambda}_0$, $I_g f \in B_0$. In fact, for any $f \in L^{2,\lambda}_0$, we have

$$\lim_{|a| \to 1} \int_\mathbb{D} |f'(z)|^2 |g(z)|^2 (1 - |\varphi_a(z)|^2)^{2-\lambda} dm(z)$$

$$= \lim_{|a| \to 1} (1 - |a|^2)^{1-\lambda} \int_\mathbb{D} |f'(z)|^2 (1 - |\varphi_a(z)|^2) |g(z)|^2 (1 - |z|^2) (1 - |\varphi_a(z)|^2)^{1-\lambda} dm(z)$$

$$\leq \|g\|_{\lambda - 1} \cdot \lim_{|a| \to 1} (1 - |a|^2)^{1-\lambda} \int_\mathbb{D} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z) = 0.$$ 

Consequently, $I_g : L^{2,\lambda}_0 \to B_0$ is bounded.

**Theorem 5.2** Suppose that $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. If $I_g : L^{2,\lambda}_0 \to B_0$ is bounded, then

$$\|I_g\|_{L^{2,\lambda}_0 \to B_0} \approx \|g\|_{\lambda - 1}.$$

**Proof** As a matter of fact, if $g \in H^\infty_{\lambda - 1}$, then for any $f \in L^{2,\lambda}_0$, $I_g f \in B_0$. Since $f_\lambda(z) \in L^{2,\lambda}_0$ (see Theorem 4.1), we complete the proof as the same as in the proof of Theorem 4.1.

**Theorem 5.3** Suppose that $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. Then the following conditions are equivalent.

1. $T_g : L^{2,\lambda}_0 \to B_0$ is bounded;
2. $g \in BL^{p,\lambda}_0$ for all $p \in (1, \infty)$;
3. $g \in BL^{p,\lambda}_0$ for some $p \in (1, \infty)$.

Moreover,

$$\|T_g\| \approx M(g).$$

**Proof** (i) $\Rightarrow$ (ii). Suppose that $T_g$ is bounded. For any $I \subset \partial \mathbb{D}$, let $b = (1 - |I|)\zeta$, where $\zeta$ is the centre of $I$. Then

$$(1 - |b|^2) \approx |1 - b z| \approx |I|,$$ 

$z \in S(I)$. 

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Concerning the functions $F_b(z)$ in Lemma 5.1 we have $F_b(z) \in \mathcal{L}_{0}^{2,\lambda}(\mathbb{D})$, thus $T_g(F_b(z)) \in B_0$. Furthermore, from Proposition 1.1 it yields $g \in B\mathcal{L}_{0}^{2,\lambda}$.

(ii) $\implies$ (iii). It is obvious.

(iii) $\implies$ (i). Let fix $p \in (1, \infty)$ and $g \in B\mathcal{L}_{0}^{2,\lambda}$. Then from Theorem 5.2 $T_g : \mathcal{L}_{0}^{2,\lambda} \to B$ is bounded and hence $T_g : \mathcal{L}_{0}^{2,\lambda} \to B$ is bounded. It suffices to prove that for any $f \in \mathcal{L}_{0}^{2,\lambda}$, $T_g f \in B_0$. Indeed, $g \in B\mathcal{L}_{0}^{2,\lambda}$, for every $\varepsilon > 0$ there is an integer $\delta > 0$ such that as $|J| < \delta$,

$$\frac{1}{|J|^{p-\lambda+1}} \int_{S(J)} |g'(z)|^2 (1 - |z|^2)^p dm(z) < \varepsilon.$$ 

With the above $\delta$, for any $|I|$ satisfying $|I| < \delta$, we break up $S(I)$ in the same way in Theorem 3.2 then by (3.2)

$$\int_{S(I)} |g'(z)|^2 (1 - |z|^2)^{p+\lambda-1} dm(z) \lesssim \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} |J|^{\lambda-1} |g'(z)|^2 (1 - |z|^2)^p dm(z)$$

$$\leq \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} \varepsilon |J|^{\lambda-1} |J|^{p-\lambda+1}$$

$$\lesssim \varepsilon |I|^p.$$ 

namely,

$$\lim_{|I| \to 0} \frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^{p+\lambda-1} dm(z) = 0.$$ 

Now, it is easy to see that

$$\lim_{|I| \to 0} \left( \frac{1}{|I|^p} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2}$$

$$\lesssim \|f\|_{\mathcal{L}_{0}^{2,\lambda}} \cdot \lim_{|I| \to 0} \left( \frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^{p+\lambda-1} dm(z) \right)^{1/2} = 0.$$ 

In conclusion, $T_g : \mathcal{L}_{0}^{2,\lambda} \to B_0$ is bounded.

Lemma 5.1 Suppose that $0 < \lambda < 1$, $1 < p$ and $g \in B\mathcal{L}_{0}^{p,\lambda}$, the operator $T_{g,r} : \mathcal{L}_{0}^{2,\lambda} \to B_0$ satisfies

$$T_{g,r} f(z) = \int_{0}^{z} f(w) g'(w) dw.$$ 

where $r \in (0, 1)$. Then $T_{g,r} : \mathcal{L}_{0}^{2,\lambda} \to B_0$ is compact.

Proof Since $g \in B\mathcal{L}_{0}^{p,\lambda}$, it follows from Lemma 4.3 that $T_{g,r} : \mathcal{L}_{0}^{2,\lambda} \to B$ is compact and hence $T_{g,r} : \mathcal{L}_{0}^{2,\lambda} \to B$ is compact. As a matter of fact in Theorem 5.3 if $g \in B\mathcal{L}_{0}^{p,\lambda}$, then for any $f \in \mathcal{L}_{0}^{2,\lambda}$, $T_g f \in B_0$, together with Lemma 4.1, we conclude $f \in \mathcal{L}_{0}^{2,\lambda}$, $T_{g,r} f \in B_0$, so that $T_{g,r} : \mathcal{L}_{0}^{2,\lambda} \to B_0$ is compact.

Theorem 5.4 Let $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. If $T_g : \mathcal{L}_{0}^{2,\lambda} \to B_0$ is bounded, then

$$\|T_g\|_{\mathcal{L}_{0}^{2,\lambda} \to B_0} \approx \lim_{|a| \to 1} \left( \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{1 - |az|^2} \right)^{p+1-\lambda} |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2}.$$ 

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**Proof** Based on the fact that if \( g \in BL_0^{p,\lambda} \), then for any \( f \in L_0^{2,\lambda} \), \( T_g f \in B_0 \), together with \( F_n(z) \in L_0^{2,\lambda} \) and the compact operator \( T_g,r \) in Lemma 5.1 similar to the proof of Theorem 1.2 we obtain the desired result.

The following corollary is an immediate consequence of the above theorem.

**Corollary 5.1** Let \( 0 < \lambda < 1 \) and \( p > 1 \). Then

(i) \( I_g : L_0^{2,\lambda} \to B_0 \) is compact if and only if \( g = 0 \).

(ii) \( T_g : L_0^{2,\lambda} \to B_0 \) is compact if and only if \( T_g : L_0^{2,\lambda} \to B_0 \) is bounded if and only if \( g \in BL_0^{p,\lambda} \).

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