Computational Complexity of the Monoid Frobenius Problem with Compressed Input

Jeffrey Shallit, Zhi Xu

Abstract

The following problem is NP-hard: given a regular expression $E$, decide if $E^*$ is not co-finite. This problem is also in PSPACE.

1 Introduction

Given $k$ positive integers $x_1, x_2, \ldots, x_k$ with $\gcd(x_1, x_2, \ldots, x_k) = 1$, the Frobenius Problem is to find the largest integer that cannot be represented as a non-negative integer linear combination of the given integers. This largest integer is called the Frobenius number of the given integers, and is denoted as $g(x_1, x_2, \ldots, x_k)$. One can refer to [1] for a good survey on the Frobenius Problem.

There have been different generalizations of the Frobenius Problem in the literature. One generalization of this problem to a free monoid as follows: [3] given a finite set $S$ of words on a given alphabet, find the (length of) the longest word(s) not in $S^*$ if $S^*$ is co-finite. One natural variation of this Monoid Frobenius Problem is to give the input set of words $S$ in a compressed way.

**Problem 1.** Given a regular expression $E$, find the length of the longest word(s) not in $E^*$ if $E^*$ is co-finite.

2 NP-hardness Proof

The NP-hardness of the following decision problem will be proved by giving a polynomial reduction from 3SAT, using similar techniques in the NP-completeness proof of star-free regular expression inequivalence.

**Problem 2.** Given a star-free regular expression $E$, decide if $E^*$ is not co-finite.

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of variables and $C = \{c_1, c_2, \ldots, c_m\}$ be a set of clauses making up an arbitrary instance of 3SAT. Without loss of generality, suppose each variable appears in at least one clause. Then $n \leq 3m$. For each clause $c_i$, a star-free regular expression $e_i = u_1^i u_2^i \cdots u_n^i$ can be constructed, where $u_j^i = F$ if $u_j$ appears in $c_i$, and
\[ u_j^i = T \] if \( u_j^i \) appears in \( c_i \), and \( u_j^i = (T + F) \) otherwise. Let \( E' = e_1 + e_2 + \cdots + e_m \), \( E'' = (T + F)(T + F) \cdots (T + F) = (T + F)^n \), \( E = E' + E''(T + F) \). It is easy to check this construction can be performed in polynomial time. It remains to see that, in the above construction, the clauses \( C \) are satisfiable if and only if \( E^* \) is not co-finite over the alphabet \( \Sigma = \{T, F\} \). The following lemma is required.

**Lemma 3.** \([3]\) Suppose \( S \subseteq \Sigma^m \cup \Sigma^n \), \( 0 < m < n \), and \( S^* \) is co-finite. Then \( \Sigma^m \subseteq S \).

Let \( S \) be the set of words in \( E \). Then \( \Sigma^{n+1} \subseteq S \subseteq \Sigma^n \cup \Sigma^{n+1} \), which follows by the construction of \( E \). If \( C \) is satisfiable, then one can check that \( E' \neq E'' \), so \( \Sigma^n \nsubseteq S \). Therefore, \( S^* \) cannot be co-finite. If \( S^* \) is not co-finite, since \( \gcd(n, n+1) = 1 \), then \( \Sigma^n \nsubseteq S \), which leads to \( E' \neq E'' \). So, \( C \) is satisfiable. This finishes the NP-hardness proof of Problem 2.

As seen in the given polynomial reduction, restricting Problem 2 to the binary alphabet and/or demanding that the language of \( E \) consist of words of only two different length results in a problem that is also NP-hard. The following problems can also easily be seen to be NP-hard, as consequences of the NP-hardness of Problem 2.

**Problem 4.** Given a regular expression \( E \), decide if \( E^* \) is not co-finite.

**Problem 5.** Given a NFA \( M \), decide if \( S^* \) is not co-finite, where \( S = L(M) \).

### 3 PSPACE Proof

Both Problem 4 and Problem 5 are in PSPACE. A NPSAPCE algorithm for Problem 4 will be presented, and the NPSAPCE proof for Problem 5 follows in a straightforward manner. Since NPSAPCE = PSAPCE by Savitch’s theorem, the result will follow.

Let \( E \) be an arbitrary regular expression, and \( t \) be the length of \( E \). An NFA \( M \) can be constructed to accept \( E^* \) with at most \( t + 1 \) states[5]. Let \( n = 2^{t+1} \). Then, nondeterministically guess a word \( w \) of length \( i \) for \( n \leq i < 2n \), and verify that it is rejected by \( M \). \( E^* \) is not co-finite if and only if there exists a \( w \) of length \( i \) for \( n \leq i < 2n \) such that \( M \) rejects \( w \). To verify the word \( w \) is rejected by \( M \), a boolean matrix of dimension at most \( t + 1 \) is stored to keep track of reachability. The entry \((p, q)\) is true if and only if \( q \) is reachable from \( p \) on a given word. At the beginning, this matrix is initialized as the identity matrix, for the word \( \epsilon \). In each step, the matrix is updated to process one guessed letter. This requires only polynomial space.

To prove the correctness of this algorithm, notice that an NFA can be constructed to accept the language \( E^* \) with at most \( t + 1 \) states. Hence a DFA \( M' \) accepting \( E^* \) has at most \( n = 2^{t+1} \) states[4]. Interchanging final and nonfinal states, the DFA \( M'' \) for \( E^* \) has at most \( n \) states. By a classical result [2], \( M'' \) accepts an infinite language if and only if it accepts a word of length \( i \), \( n \leq i < 2n \). Hence \( M' \)’s language is co-infinite if and only if it rejects some word of length \( i \), \( n \leq i < 2n \).

**appendix**

This report is to be a part of a future paper.
References

[1] J. L. Ramírez Alfonsín: The Diophantine Frobenius Problem. Oxford University Press, 2005.

[2] J. Hopcroft and J. Ullman, Introduction to Automata Theory, Languages, and Computation, Addison-Wesley, 1979.

[3] Jui-Yi Kao, Jeffrey Shallit, Zhi Xu: The Frobenius Problem in a Free Monoid. STACS2008: pp.421-432.

[4] Sheng Yu, Qingyu Zhuang, Kai Salomaa: The State Complexities of Some Basis Operations on Regular Languages. Theoretical Computer Science, Volume 125, 1994: pp.315-328.

[5] Markus Holzer, Martin Kutrib: Nondeterministic Descriptive Complexity of Regular Languages. International Journal of Foundations of Computer Science, Volume 14, 2003: pp.1087-1102.