Precompact abelian groups and topological annihilators

Gábor Lukács

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Abstract
For a compact Hausdorff abelian group $K$ and its subgroup $H \leq K$, one defines the $g$-closure $g_K(H)$ of $H$ in $K$ as the subgroup consisting of $\chi \in K$ such that $\chi(a_n) \to 0$ in $T = \mathbb{R}/\mathbb{Z}$ for every sequence $\{a_n\}$ in $\hat{K}$ (the Pontryagin dual of $K$) that converges to 0 in the topology that $H$ induces on $\hat{K}$. We prove that every countable subgroup of a compact Hausdorff group is $g$-closed, and thus give a positive answer to two problems of Dikranjan, Milan and Tonolo. We also show that every $g$-closed subgroup of a compact Hausdorff group is realcompact. The techniques developed in the paper are used to construct a close relative of the closure operator $g$ that coincides with the $G_\delta$-closure on compact Hausdorff abelian groups, and thus captures realcompactness and pseudocompactness of subgroups.

1. INTRODUCTION

For a sequence $u = (u_n)$ of integers, one defines $t_u(T) = \{x \in T \mid u_n x \to 0\}$, the subgroup of topologically $u$-torsion elements in $T = \mathbb{R}/\mathbb{Z}$, and sets $t(H) = \bigcap\{t_u(T) \mid H \leq t_u(T)\}$ for every subgroup $H \leq T$ (cf. [14], [1], [18]). This enables one to speak of $t$-closed subgroups (i.e., $H \leq T$ such that $t(H) = H$). The subgroups $t_u(T)$ are called basic $t$-closed subgroups. Biro, Deshouillers and Sós proved:

**Theorem 1.1.** ([4, Theorem 2]) Every countable subgroup of $T$ is basic $t$-closed.

Although it is possible to define $t_u(G)$ for an arbitrary abelian topological group in a similar fashion, the closure $t_G$ thus obtained turned out to have worse properties than anticipated: Dikranjan and Di Santo [16] showed that if $L$ is a non-discrete locally compact Hausdorff abelian group and $L \not\cong T$, then there is $x \in L$ such that the subgroup $\langle x \rangle$ is not even $t$-closed in $L$. In particular, Theorem 1.1 fails for $L \not\cong T$.

Following Dikranjan, Milan and Tonolo [18, 2.1], given an abelian topological group $G$ and a sequence $u = (u_n)$ in its Pontryagin dual $\hat{G}$, we put $s_u(G) = \{x \in G \mid u_n(x) \to 0 \text{ in } T\}$, the subgroup of topologically $u$-annihilating elements of $G$. For every subgroup $H \leq G$, we set

$$g_G(H) = \bigcap\{s_u(G) \mid u \in \hat{G}^N, H \leq s_u(G)\}. \tag{1}$$

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One says that $H \leq G$ is $g$-closed in $G$ if $g_G(H) = H$, and $H$ is $g$-dense there if $g_G(H) = G$. Using this terminology, $g$ is the largest closure operator with respect to which each basic $g$-closed set $s_G(G)$ is closed. (For details on closure operators of topological groups, we refer the reader to the survey of Dikranjan [15].)

For a category $T$, $\text{Grp}(T)$ stands for the category of group objects in $T$ and their group homomorphisms, and $\text{Ab}(T)$ for its full subcategory consisting of the abelian group objects. Thus, $\text{Grp}(\text{Top})$, $\text{Grp}(\text{CompH})$, $\text{Ab}(\text{Top})$, and $\text{Ab}(\text{CompH})$ are the categories of topological groups (without separation axioms), compact Hausdorff topological groups, and their abelian counterparts, respectively.

For $G \in \text{Grp}(\text{Top})$, we denote by $\rho_G : G \to bG$ the Bohr-compactification of $G$, that is, the reflection of $G$ into $\text{Grp}(\text{CompH})$. One puts $G^+$ for $G$ equipped with the initial topology induced by $\rho_G$. The kernel of $\rho_G$ is denoted by $n(G)$ and called the von Neumann radical of $G$. One says that $G$ is maximally almost periodic if $n(G) = 1$—in other words, if $\rho_G$ is injective (cf. [22]).

In this paper, we study the closure operator $g$ (and its relatives that are introduced in section 2) by means of modification of precompact group topologies. Our main results in the context of the $g$-closure itself are Theorems 1.2-1.4 below, whose proofs appear in section 4.

**Theorem 1.2.** Let $H$ be a countable subgroup of a compact Hausdorff abelian topological group $K$. Then $H$ is $g$-closed in $K$.

Theorem 1.2 is a positive solution for two problems of Dikranjan, Milan and Tonolo [18, Problem 5.1, Question 5.2], and it generalizes Theorem 1.1. Independently, Dikranjan and Kunen [17, 1.9] and Mathias Beiglböck, Christian Steineder and Reinhard Winkler [3] have also obtained Theorem 1.2.

**Theorem 1.3.** Every $g$-closed subgroup of a compact Hausdorff abelian group is realcompact.

**Theorem 1.4.** For $G \in \text{Ab}(\text{Top})$, the following statements are equivalent:

(i) $G$ is maximally almost periodic;

(ii) every countable subgroup of $G$ is $g$-closed;

(iii) every cyclic subgroup of $G$ is $g$-closed.

Theorem 1.4 is a variant of the main result of [18], and so its novelty lies in its significantly simplified proof that uses techniques developed below.

One says that $G \in \text{Grp}(\text{Top})$ is precompact if for every neighborhood $U$ of the identity in $G$, there exists a finite subset $F \subseteq G$ such that $G = FU$.

In order to describe our method of investigating the closure operator $g$ (and its relatives that are introduced in section 2), we first survey some basic facts related to precompact abelian groups. Fix $K \in \text{Ab}(\text{CompH})$, and let $A = \hat{K}$ be its Pontryagin dual. It is a well-known result of Comfort and Ross [10] that every precompact Hausdorff group topology $\tau$ on $A$ coincides with the initial topology on $A$ with respect to characters $\chi$ in a dense subgroup $H \leq K$. In other words, one has $\tau = \tau_H$, where $\tau_H$ is the coarsest (group) topology on $A$ such that each $\chi : A \to \mathbb{T}$ in $H$ is continuous. There is a close relationship between properties of $(A, \tau_H)$ and $H$. To mention only
two basic ones, the identity map $(A, \tau_{H_1}) \to (A, \tau_{H_2})$ is continuous (i.e., $\tau_{H_1} \supseteq \tau_{H_2}$) if and only if \( H_1 \geq H_2 \) (cf. [21, 2.6]), and $(A, \tau_H)$ is metrizable if and only if $H$ is countable (cf. [21, 2.12]). For a detailed account of results of this nature, we refer the reader to [21] and [8].

For $H \leq K$, consider $H^\perp := \bigcap_{\chi \in H} \ker \chi$. It follows from the Pontryagin duality that the dual of $A/H^\perp$ is the closure $\bar{H}$ of $H$ in $K$, and $H^\perp = \bar{H}^\perp$. Thus, although $H$ need not separate points in $A$, it always separates the points of $A/H^\perp$. In particular, $H$ separates the points of $A$ if and only if $H$ is dense in $K$ (cf. [10]). Therefore, in the context of characters and duality, the precompact (but not necessarily Hausdorff) group $(A, \tau_H)$ is equivalent to the precompact Hausdorff group $(A/H^\perp, \tau_H)$, because the latter is the maximal Hausdorff quotient of the earlier (see [21, 2.10]).

The key to our approach is our viewing $g$ as a functor that assigns to each precompact abelian group $(A, \tau_H)$ the group $(A, \tau_{gK}(H))$. We show that $gK(H)$ consists of the elements of $K$ that are sequentially continuous as characters of $(A, \tau_H)$ (i.e., $\chi(a_n) \to \chi(a_0)$ for every sequence $\{a_n\}$ in $A$ such that $a_n \to a_0$ in $\tau_H$; cf. Lemma 2.9(b)).

In the course of investigating $g$, we discovered that many of its properties remain true if sequences are replaced with a “nice” method of choosing filter-bases in $\hat{G}$. Thus, in order to achieve a greater generality, in section 2, we study such mutants of $g$. Each closure operator of this type gives rise to a full coreflective subcategory of the category of precompact abelian groups. In section 3, we present such a “mutant” $l$ of $g$ (in the sense of section 2) that coincides with the $G_\delta$-closure on compact Hausdorff abelian groups (cf. Theorem 3.1); thus, $l$-closed subgroups are the realcompact ones, and $l$-dense subgroups are the dense pseudocompact ones. Finally, in section 4, the general theory developed in section 2 is applied to the closure operator $g$, and Theorems 1.2-1.4 are proven.

## 2. CONTINUITY TYPES

In this section, we develop the filter-base analogue of the $g$-closure. Let $G \in \text{Ab(Top)}$ and let $\mathcal{F}$ be a filter-base on its Pontryagin dual $\hat{G}$. We put

$$z_{\mathcal{F}}(G) = \{ \chi \in G \mid \chi(\mathcal{F}) \to 0 \text{ in } \mathbb{T} \},$$

where elements of $G$ are seen as characters of $\hat{G}$, and thus they are denoted by Greek letters.

**Lemma 2.1.** Let $\varphi: G_1 \to G_2$ be a morphism in $\text{Ab(Top)}$, and $\mathcal{F}$ be a filter-base on $\hat{G}_2$. Then

$$z_{\varphi(F)}(G_1) = \varphi^{-1}(z_{\mathcal{F}}(G_2)).$$

**Proof.** Since $\hat{\varphi}: \hat{G}_2 \to \hat{G}_1$ is defined by $\hat{\varphi}(x) = x \circ \varphi$,

$$\chi \in z_{\hat{\varphi}(\mathcal{F})}(G_1) \iff \chi(\hat{\varphi}(\mathcal{F})) \to 0 \iff \varphi(\chi)(\mathcal{F}) \to 0 \iff \varphi(\chi) \in z_{\mathcal{F}}(G_2),$$

as desired. □
A continuity type $\mathcal{T}$ on a subcategory $A$ of $\text{Grp}(\text{Top})$ is an assignment of a collection $\mathcal{T}(G)$ of filter-bases to each $G \in A$ in a way that $\varphi(\mathcal{F}) \in \mathcal{T}(G_2)$ for every morphism $\varphi : G_1 \to G_2$ in $A$ and every $\mathcal{F} \in \mathcal{T}(G_1)$. Examples 2.6 and 2.7 below explain the motivation behind the term “continuity type.”

**Proposition 2.2.** Let $\mathcal{T}$ be a continuity type on $\text{Ab}(\text{Top})$. For every $G \in \text{Ab}(\text{Top})$ and $H \leq G$, put

\[ f_G(H) = \bigcap \{ z_\mathcal{F}(G) \mid \mathcal{F} \in \mathcal{T}(\hat{G}), H \leq z_\mathcal{F}(G) \}. \tag{5} \]

Then $f$ is an idempotent closure operator on $\text{Ab}(\text{Top})$.

We call $f$ the closure operator associated with $\mathcal{T}$.

**Proof.** Clearly, $f$ is extensive and monotone, so we prove that each continuous homomorphism $\varphi : G_1 \to G_2$ is $f$-continuous. Let $H \leq G_2$. Since $\varphi : G_2 \to \hat{G}_1$ is continuous, $\varphi(\mathcal{F}_2) \in \mathcal{T}(\hat{G}_1)$ for every $\mathcal{F}_2 \in \mathcal{T}(G_2)$. Thus,

\[ f_{G_1}(\varphi^{-1}(H)) = \bigcap \{ z_{\varphi(\mathcal{F}_1)}(G_1) \mid \mathcal{F}_1 \in \mathcal{T}(\hat{G}_1), \varphi^{-1}(H) \leq z_{\varphi(\mathcal{F}_1)}(G_1) \} \tag{6} \]

\[ \subseteq \bigcap \{ z_{\varphi(\mathcal{F}_2)}(G_1) \mid \mathcal{F}_2 \in \mathcal{T}(\hat{G}_1), \varphi^{-1}(H) \leq z_{\varphi(\mathcal{F}_2)}(G_1) \} \tag{7} \]

\[ = \bigcap \{ \varphi^{-1}(z_{\mathcal{F}_2}(G_2)) \mid \mathcal{F}_2 \in \mathcal{T}(\hat{G}_2), (H) \leq z_{\mathcal{F}_2}(G_2) \} = \varphi^{-1}(f_{G_2}(H)), \tag{8} \]

as desired. Obviously, $f$ is idempotent (cf. [13, Exercise 2.D(a)])..  

A continuity type $\mathcal{T}$ is regular on $G$ if $\mathcal{T}(G) = \mathcal{T}(G_\text{d})$, where $G_\text{d}$ is the group $G$ with discrete topology; a subgroup $N$ of $G \in \text{Ab}(\text{Top})$ is $\mathcal{T}$-dually embedded into $G$ if for every $\mathcal{F}_2 \in \mathcal{T}(N)$, there is $\mathcal{F}_1 \in \mathcal{T}(\hat{G})$ such that $\hat{N}(\mathcal{F}_1) = \mathcal{F}_2$.

If $G \in \text{Ab}(\text{Top})$, then $G^+$ is the group $G$ with the topology of pointwise convergence on $\hat{G}$, and $\mathfrak{n}(G) = \bigcap_{x \in G} \ker x$. It follows from the universal property of $bG$ that $\hat{bG} = \hat{G}_d$.

**Proposition 2.3.** Let $\mathcal{T}$ be a continuity type on $\text{Ab}(\text{Top})$, $f$ its associated closure operator, and $G \in \text{Ab}(\text{Top})$.

(a) One has $\mathfrak{n}(G) \subseteq f_G(\{0\})$.

(b) If every principal ultrafilter on $\hat{G}$ is in $\mathcal{T}(\hat{G})$, then $\mathfrak{n}(G) = f_G(\{0\})$.

(c) If $\mathcal{T}$ is regular on $\hat{G}$, then $f_G(H) = f_{G^+}(H)$ for every $H \leq G$.

(d) If $\mathcal{T}$ is regular, then every dense subgroup of $G$ is $\mathcal{T}$-dually embedded into $G$.

(e) If $N$ is $\mathcal{T}$-dually embedded into $G$, then $f_N(H) = N \cap f_G(H)$ for every $H \leq N$.

**Proof.** (a) If $\chi \in \mathfrak{n}(G)$, then $\chi(\mathcal{F}) = \{0\}$ for every filter-base $\mathcal{F}$ on $\hat{G}$, and so $\chi \in z_\mathcal{F}(G)$. Thus, $\mathfrak{n}(G) \subseteq z_\mathcal{F}(G)$ for every $\mathcal{F}$, and therefore $\mathfrak{n}(G) \subseteq \bigcap_{\mathcal{F} \in \mathcal{T}(\hat{G})} z_\mathcal{F}(G) = f_G(\{0\})$.

(b) Let $x \in \hat{G}$. By the assumption, the principal ultrafilter $\hat{x} \in \mathcal{T}(\hat{G})$, and $z_{\hat{x}}(G) = \ker x$. Thus, $f_G(\{0\}) \subseteq \ker x$. This holds for every $x \in \hat{G}$, and therefore (using (a)) $\mathfrak{n}(G) = f_G(\{0\})$.

(c) The continuous homomorphisms $G \to \hat{G} \to bG$ give rise to $\hat{G}_d = b\hat{G} \to \hat{G}^+ \to \hat{G}$, and thus $\mathcal{T}(\hat{G}_d) \subseteq \mathcal{T}(\hat{G}^+) \subseteq \mathcal{T}(\hat{G})$. Since $\mathcal{T}$ is regular on $\hat{G}$, the two ends of this inclusion are
equal, and therefore $\mathcal{T}(\hat{G}^+) = \mathcal{T}(\hat{G})$. The definition of $z_\mathcal{F}$ is independent of the topology, and thus $z_\mathcal{F}(G) = z_\mathcal{F}(\hat{G}^+)$ for every filter-base $\mathcal{F}$ on the set $\hat{G} = \hat{G}^+$. Hence, the collections whose intersection give $f_G(H)$ and $f_{G^+}(H)$ coincide.

(d) Let $D \leq G$ be a dense subgroup. Then $\hat{G}$ and $\hat{D}$ have the same underlying group, and so $\mathcal{T}(\hat{D}) = \mathcal{T}(\hat{D}^d) = \mathcal{T}(\hat{G})$, because $\mathcal{T}$ is regular.

(e) Clearly, $f_N(H) \subseteq f_G(H)$, because the inclusion $N \to G$ is $f$-continuous. Let $\mathcal{F}_2 \in \mathcal{T}(\hat{G})$, and let $\mathcal{F}_1 \in \mathcal{T}(\hat{G})$ be such that $i_N(\mathcal{F}_1) = \mathcal{F}_2$ (the existence of $\mathcal{F}_1$ is guaranteed as $N$ is $\mathcal{T}$-dually embedded into $G$). By Lemma 2.1, $z_{\mathcal{F}_2}(N) = z^1_N(\mathcal{F}_1)(N) = i_N^{-1}(z_{\mathcal{F}_1}(G)) = N \cap z_{\mathcal{F}_1}(G)$, so if $H \leq z_{\mathcal{F}_2}(N)$, then $z_{\mathcal{F}_2}(N) \supseteq N \cap f_G(H)$. Therefore, $f_N(H) \supseteq N \cap f_G(H)$, as desired. \hfill $\square$

**Theorem 2.4.** Let $\mathcal{T}$ be a regular continuity type on Ab(\text{Top}) and $f$ its associated closure operator. Then $f_G(H) = \rho_G^{-1}(f_{bG}(\rho_G(H)))$ for every abelian topological group $G$ and subgroup $H \leq G$.

**Proof.** Set $b^+G = \rho_G(G)$; it is a dense subgroup of $bG$. Observe that $bG = b(G^+)$ (and so $b^+(G^+) = b^+G$), and $b^+G^+ = G^+ / n(G)$. Since $b^+(G^+)$ is dense in $b(G^+)$, by Proposition 2.3(d), $b^+(G^+)$ is $\mathcal{T}$-dually embedded into $bG$. Thus, one has

$$f_G(H) = \rho_G^{-1}(f_{bG}(\rho_G(H))) \quad (9)$$

Note that the last equality in (9) holds because every continuous character of $G^+$ factors through $\rho_G^+$ (so $\hat{G}^+ = b^+(G^+)$ as sets), and $\mathcal{T}$ is regular. \hfill $\square$

**Example 2.5.** For every topological group $G$, put $\mathcal{D}(G)$ to be the set of all filter-bases on $G$. The associated closure operator is discrete on every $K \in \text{Ab(CompH)}$, because for every (not necessary closed) subgroup $H$, there is a filter $\mathcal{F}$ on $A = \hat{K}$ such that $H = z_\mathcal{F}(K)$ (cf. [3, 2.1]). Therefore, by Theorem 2.4, for $G \in \text{Ab(Top)}$, $f_G^2(H) = \rho_G^{-1}(\rho_G(H)) = H + n(G)$. Hence, $f_G^2(H) = c_n$, the minimal closure operator associated with $n$ (cf. [15, 3.5.3-6]).

Let $A$ be a subcategory of Grp(\text{Top}) and $\mathcal{T}$ be a continuity type on $A$. For $G_1, G_2 \in A$, we say that $\varphi$ (a (not necessarily continuous) homomorphism $\varphi: G_1 \to G_2$ is $\mathcal{T}$-continuous if $\varphi(\mathcal{F}) \to e$ in $G_2$ for every $\mathcal{F} \in \mathcal{T}(G_1)$ such that $\mathcal{F} \to e$ in $G_1$.

**Example 2.6.** For every topological group $G$, put $\mathcal{S}(G)$ to be the set of all filter-bases on $G$ generated by sequences (i.e., of the form $\{\{a_n \mid n \geq m\} \mid m \in \mathbb{N}\}$, where $\{a_n\} \subseteq G$). A homomorphism $\varphi: G_1 \to G_2$ is $\mathcal{S}$-continuous if and only if it is sequentially continuous. For $G \in \text{Ab(\text{Top})}$ and $\underline{u} \in \check{G}^\mathbb{N}$, one has $s_{\underline{u}}(G) = z_\mathcal{F}(G)$, where $\mathcal{F} \in \mathcal{S}(G)$ is the filter base generated by the sequence $\underline{u}$ and vice versa, to each $\mathcal{F} \in \mathcal{S}(G)$ there is a corresponding sequence that generates it. Therefore, the associated closure operator of $\mathcal{S}$ is precisely $\underline{g}$, which was defined earlier.

A map $f: X \to Y$ between topological spaces is said to be countably continuous if $f$ is continuous on every countable subset of $X$. 


Example 2.7. For every topological group \( G \), put \( C(G) \) to be the set of filter-bases on \( G \) that contain a countable subset of \( G \). In Lemma 3.4, we show that a homomorphism \( \varphi : G_1 \to G_2 \) is \( C \)-continuous if and only if it is countably continuous. Moreover, in Theorem 3.1, we prove that the closure operator \( I \) associated with \( C \) coincides with the \( G_\delta \)-closure on precompact Hausdorff abelian groups.

Convention 2.8. Until the end of the section, \( \mathcal{T} \) is a fixed regular continuity type on \( \text{Ab}(\text{Top}) \), and \( f \) is its associated closure operator.

Motivated by Theorem 2.4, we confine our attention to \( \text{Ab}(\text{CompH}) \). In the sequel, we put \( \mathcal{F} \xrightarrow{H} a_0 \) whenever a filter-base \( \mathcal{F} \) converges to a point \( a_0 \) in \( A \) in the topology \( \tau_H \).

Lemma 2.9. Let \( K \in \text{Ab}(\text{CompH}) \), \( A = \hat{K} \), and \( H \leq K \). Put \( \mathcal{T}_H(A) = \{ \mathcal{F} \in \mathcal{T}(A) \mid \mathcal{F} \xrightarrow{H} 0 \} \).

Then:
- (a) \( \mathcal{f}_K(H) = \{ \chi \in K \mid \chi(\mathcal{F}) \longrightarrow 0 \text{ for all } \mathcal{F} \in \mathcal{T}_H(A) \} \);
- (b) \( \mathcal{f}_K(H) = \{ \chi \in K \mid \chi : (A, \tau_H) \to \mathbb{T} \text{ is } \mathcal{T} \text{-continuous} \} \);
- (c) the identity homomorphism \( A, \tau_H \to (A, \tau_{\mathcal{f}_K(H)}) \) is \( \mathcal{T} \)-continuous.

**Proof.** (a) For \( \mathcal{F} \in \mathcal{T}(A) \), \( H \leq z_\mathcal{F}(K) \) if and only if \( \chi(\mathcal{F}) \longrightarrow 0 \) for every \( \chi \in H \)—in other words, \( \mathcal{F} \xrightarrow{H} 0 \), which is the same as \( \mathcal{F} \in \mathcal{T}_H(A) \). Therefore,
\[
\mathcal{f}_K(H) = \bigcap \{ z_\mathcal{F}(K) \mid \mathcal{F} \in \mathcal{T}_H(A) \} = \{ \chi \in K \mid \chi(\mathcal{F}) \longrightarrow 0 \text{ for all } \mathcal{F} \in \mathcal{T}_H(A) \}. 
\]

(b) Let \( \chi \in \mathcal{f}_K(H) \), and suppose that \( \mathcal{F} \in \mathcal{T}(A) \) is such that \( \mathcal{F} \longrightarrow 0 \) in \( \tau_H \). Then \( \mathcal{F} \in \mathcal{T}_H(A) \), and so \( \chi(\mathcal{F}) \longrightarrow 0 \) by (a). Thus, \( \chi \) is \( \mathcal{T} \)-continuous. Conversely, suppose that \( \chi : (A, \tau_H) \to \mathbb{T} \) is \( \mathcal{T} \)-continuous, and let \( \mathcal{F} \in \mathcal{T}_H(A) \). Then \( \chi(\mathcal{F}) \longrightarrow 0 \), because \( \chi \) is \( \mathcal{T} \)-continuous, and therefore \( \chi \in \mathcal{f}_K(H) \) by (a).

(c) If \( \mathcal{F} \in \mathcal{T}_H(A) \), then by (a), \( \chi(\mathcal{F}) \longrightarrow 0 \) for every \( \chi \in \mathcal{f}_K(H) \), and therefore \( \mathcal{F} \xrightarrow{f(H)} 0 \).

The next theorem is an immediate consequence of Lemma 2.9(b).

Theorem 2.10. Let \( K \in \text{Ab}(\text{CompH}) \) and \( A = \hat{K} \). A subgroup \( H \leq K \) is \( f \)-closed in \( K \) if and only if every \( \mathcal{T} \)-continuous character of \( (A, \tau_H) \) is continuous.

Proposition 2.11. Let \( A \) and \( B \) be discrete abelian groups, \( K = \hat{A} \) and \( C = \hat{B} \) their Pontryagin duals, \( H \leq K \), and \( L \leq C \). For every homomorphism \( \varphi : A \to B \), the following statements are equivalent:
- (i) \( \varphi : (A, \tau_H) \to (B, \tau_L) \) is \( \mathcal{T} \)-continuous;
- (ii) \( \chi \circ \varphi \) is \( \mathcal{T} \)-continuous on \( (A, \tau_H) \) for every \( \mathcal{T} \)-continuous character \( \chi \) of \( (B, \tau_L) \);
- (iii) \( \mathcal{f}_\varphi : (A, \tau_{\mathcal{f}_K(H)}) \to (B, \tau_{\mathcal{f}_C(L)}) \) is continuous.

**Proof.** Clearly, (i) \( \Rightarrow \) (ii), because the composite of two \( \mathcal{T} \)-continuous homomorphisms is \( \mathcal{T} \)-continuous. The implication (ii) \( \Rightarrow \) (iii) follows from Lemma 2.9(b), because (ii) states that \( \chi \circ \varphi \in \mathcal{f}_K(H) \) for every \( \chi \in \mathcal{f}_C(L) \). To show (iii) \( \Rightarrow \) (i), notice that \( \varphi \) is the composite of \( (A, \tau_H) \to (A, \tau_{\mathcal{f}_K(H)}), \mathcal{f}_\varphi, \) and \( (B, \tau_{\mathcal{f}_C(L)}) \to (B, \tau_L) \). Since these three maps are all \( \mathcal{T} \)-continuous (cf. Lemma 2.9(c)), their composite is also \( \mathcal{T} \)-continuous.
Inspired by the notions of \( k \)-group and \( s \)-group introduced by Noble \([23] \) \& \([24]\), and the term “\( kk \)-group” used by Deaconu \([12]\), we say that a topological group \( G \) is a \( Tk \)-group if every \( \mathcal{T} \)-continuous homomorphism of \( G \) into a compact Hausdorff group is continuous.

**Proposition 2.12.** Let \( K \in \text{Ab}(\text{CompH}) \), \( A = \hat{K} \), and \( H \leq K \). The following are equivalent:

(i) every \( \mathcal{T} \)-continuous homomorphism \( \varphi: (A, \tau_H) \to (B, \tau_L) \) into a precompact abelian group (where \( B \) is a discrete group and \( L \leq B \)) is continuous;

(ii) \( (A, \tau_H) \) is an \( Tk \)-group;

(iii) every \( \mathcal{T} \)-continuous character \( \chi: (A, \tau_H) \to \mathbb{T} \) is continuous.

**Proof.** (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) is obvious, because every compact group is precompact, and \( \mathbb{T} \) is compact. In order to show (iii) \( \Rightarrow \) (i), let \( \varphi: (A, \tau_H) \to (B, \tau_L) \) be an \( \mathcal{T} \)-continuous homomorphism. By Proposition 2.11, \( \chi \circ \varphi \) is \( \mathcal{T} \)-continuous on \( (A, \tau_H) \) for every character \( \chi \in L \) of \( (B, \tau_L) \). Applying (iii) to the character \( \chi \circ \varphi \) yields that it is continuous, in other words, \( \chi \circ \varphi \in H \). Therefore, \( \chi \circ \varphi \in H \) for every \( \chi \in L \), and hence \( \varphi \) is continuous.

A combination of Theorem 2.10 and Proposition 2.12 yields:

**Theorem 2.13.** Let \( K \in \text{Ab}(\text{CompH}) \) and \( A = \hat{K} \). A subgroup \( H \leq K \) is \( \hat{\tau} \)-closed if and only if \( (A, \tau_H) \) is an \( Tk \)-group. In particular, \( (A, \tau_{\hat{k}(H)}) \) is an \( Tk \)-group for every \( H \leq K \).

Theorem 2.14 below is modeled on the relationship between the category of Hausdorff topological spaces with \( k \)-continuous maps, Hausdorff topological spaces with continuous maps, and Hausdorff \( k \)-spaces (cf. \([7, 7.2]\)). In what follows, \( \text{Pr}_{\mathcal{T}} \text{Ab} \) is the category of precompact abelian groups and their \( \mathcal{T} \)-continuous homomorphisms, \( \text{PrAb} \) stands for the category of precompact abelian groups and their continuous homomorphisms, and \( Tk\text{PrAb} \) is the full subcategory of \( \text{PrAb} \) consisting of the \( Tk \)-groups.

**Theorem 2.14.** The assignment

\[
\hat{f}: \text{Pr}_{\mathcal{T}} \text{Ab} \longrightarrow Tk\text{PrAb}
\]

\[
(A, \tau_H) \longmapsto (A, \tau_{\hat{k}(H)}), \quad (K = \hat{A}, H \leq K)
\]

is a functor. Furthermore,

(a) \( \hat{f}: \text{Pr}_{\mathcal{T}} \text{Ab} \longrightarrow Tk\text{PrAb} \) is an equivalence of categories;

(b) \( \hat{f}: \text{PrAb} \longrightarrow Tk\text{PrAb} \) is a coreflection.

**Proof.** It was explained earlier that every precompact abelian group has the form \( (A, \tau_H) \), where \( A \) is a discrete abelian group and \( H \leq \hat{A} \) (\( A \) and \( H \) are uniquely determined). By Theorem 2.13, \( \hat{f}(A, \tau_H) = (A, \tau_{\hat{k}(H)}) \in Tk\text{PrAb} \), and by Proposition 2.11, \( \hat{f}\varphi \) is continuous for every \( \mathcal{T} \)-continuous homomorphism \( \varphi: (A, \tau_H) \to (B, \tau_L) \). Therefore, \( \hat{f} \) is well-defined, and it is a functor.

(a) Clearly, \( \hat{f} \) and the inclusion \( I: Tk\text{PrAb} \longrightarrow \text{Pr}_{\mathcal{T}} \text{Ab} \) are faithful, and by Proposition 2.11, they are also full. Since \( \hat{f} \) is idempotent, \( \hat{f}I = \text{Id}_{Tk\text{PrAb}} \). By Lemma 2.9(c), \((A, \tau_H) \) and \((A, \tau_{\hat{k}(H)}) \) are isomorphic in \( \text{Pr}_{\mathcal{T}} \text{Ab} \), and therefore \( I\hat{f} \cong \text{Id}_{\text{Pr}_{\mathcal{T}} \text{Ab}} \). Hence, \( \hat{f} \) is an equivalence of categories.
(b) For \((B, \tau_L) \in \PrAb\) (where \(B\) is a discrete abelian group and \(L \leq C := \hat{B}\)), the counit \(\gamma_{(B, \tau_L)} : (\hat{B}, \tau_{\hat{L}}(L)) \to (B, \tau_L)\) is continuous (because \(L \leq \hat{f}_C(L)\)). By Proposition 2.11, every continuous homomorphism \(\varphi : (A, \tau_H) \to (B, \tau_L)\) gives rise to a continuous homomorphism \(\hat{\varphi} : (\hat{A}, \tau_{\hat{H}}(H)) \to (\hat{B}, \tau_{\hat{L}}(L))\). Whenever \((A, \tau_H)\) is an \(\mathcal{T}k\)-group, \(\hat{f}_K(H) = H\) (cf. Theorem 2.13), and therefore \(\varphi\) factors through \(\gamma_{(B, \tau_L)}\) uniquely. \(\square\)

3. APPLICATION I: THE \(G_\delta\)-CLOSURE

Recall that for a topological space \(X\), the \(G_\delta\)-closure \(\bar{A}^\delta\) of \(A \subseteq X\) consists of all points \(x \in X\) such that each \(G_\delta\)-set that contains \(x\) meets \(A\); \(A\) is \(G_\delta\)-dense in \(X\) if \(\bar{A}^\delta = X\).

Let \(\mathcal{C}\) be the continuity type described in Example 2.7, and set \(l\) to be its associated closure operator. In this section, we study the closure operator \(l\) on precompact Hausdorff abelian groups.

**Theorem 3.1.** Let \(P\) be a precompact Hausdorff abelian group. Then \(l_P(H) = \bar{H}^\delta\) for every subgroup \(H \leq P\). In other words, \(l\) coincides with the \(G_\delta\)-closure on precompact Hausdorff abelian groups.

Before proving Theorem 3.1, we present some basic properties of \(l\). Recall that a subgroup \(N\) of \(G \in \Ab(\Top)\) is said to be dually embedded if \(\hat{i}_N : \hat{G} \to \hat{N}\) is surjective, where \(i_N\) is the inclusion.

**Proposition 3.2.** For every \(G \in \Ab(\Top)\):

(a) \(l_G(\{0\}) = n(G)\);

(b) \(l_G(H) = l_{G^+}(H)\) for every \(H \leq G\);

(c) if \(N \leq G\) is dually embedded in \(G\), then \(l_N(H) = l_G(H) \cap N\) for every \(H \leq N\).

**Proof.** (a) and (b) follow from Proposition 2.3 (b) and (c), respectively. By Proposition 2.3(e), in order to show (c), it suffices to show that \(N\) is \(\mathcal{C}\)-dually embedded in \(G\). To that end, let \(\mathcal{F}_2 \in \mathcal{C}(\hat{N})\), and let \(D_2 \in \mathcal{F}_2\) be a countable subset. Since \(\hat{i}_N : \hat{G} \to \hat{N}\) is surjective, there exists a countable subset \(D_1\) of \(\hat{G}\) such that \(\hat{i}(D_1) = D_2\). It is easy to see that for

\[
\mathcal{F}_1 = \{\hat{i}_N^{-1}(F) \mid F \in \mathcal{F}_2\} \cup \{\hat{i}_N^{-1}(F) \cap D_1 \mid F \in \mathcal{F}_2\},
\]

\(\mathcal{F}_1 \in \mathcal{C}(\hat{G})\), and \(\hat{i}_N(\mathcal{F}_1) = \mathcal{F}_2\). \(\square\)

It follows by Pontryagin duality that every subgroup of a compact Hausdorff abelian group is dually embedded, and thus Proposition 3.2(c) yields:

**Corollary 3.3.** For every \(K \in \Ab(\CompH)\) and subgroups \(H \leq N \leq K\), \(l_N(H) = l_K(H) \cap N\) holds. In other words, \(l\) is hereditary on \(\Ab(\CompH)\), and thus \(l\) is hereditary on every precompact Hausdorff abelian group. \(\square\)

**Lemma 3.4.** A homomorphism \(\varphi : G_1 \to G_2\) is \(C\)-continuous if and only if it is countably continuous.
**Proof.** Let $D$ be a countable subset of $G_1$, and set $S = \langle D \rangle$, the (countable) subgroup generated by $D$. Every filter-base $F$ on $S$ belongs to $\mathcal{C}(G_1)$. Thus, if $F \to e$ in $S$, then $\varphi(F) \to e$ in $G_2$, because $\varphi$ is $\mathcal{T}$-continuous. Therefore, $\varphi|_S$ is continuous, and in particular $\varphi|_D$ is continuous.

Conversely, suppose that $\varphi$ is countably continuous, and let $F \in \mathcal{C}(G_1)$ such that $F \to e$ in $G_1$. Let $D \in \mathcal{F}$ be countable. By replacing $F$ with the filter-base $\{F \in \mathcal{F} \mid F \subseteq D\}$, we may assume that every set in $F$ is contained in $D$. Set $D_0 = D \cup \{e\}$. Since $\varphi$ is countably continuous, $\varphi|_{D_0}$ is continuous. Thus, $\varphi|_{D_0}(F) \to e$, because $F \to e$ in $D_0$. 

**Proof of Theorem 3.1.** Since both $I$ and the $G_\delta$-closure are hereditary on precompact Hausdorff groups (cf. Corollary 3.3), it suffices to show the theorem for the case where $P = K$ is compact. Set $A = \tilde{K}$. By Lemma 2.9(b), $\chi \in \mathcal{I}_K(H)$ if and only if $\chi$ is $\mathcal{C}$-continuous with respect to $\tau_H$. By Lemma 3.4, this amounts to $\chi$ being countably continuous on $(A, \tau_H)$. We follow the idea of [8, 3.1] in order to show that the last statement is equivalent to $\chi \in \mathcal{H}_K$. Since $K \in \text{Ab} \left( \text{CompH} \right)$, $K$ embeds into the product $T^A$. For each $(c_a)_{a \in A} \in T^A$ and countable subset $C \subseteq A$, the set $\{ (s_a)_{a \in A} \mid s_c = c_e \text{ for all } c \in C \}$ is $G_\delta$ in $T^A$ (because $T$ has a countable pseudocharacter), and these sets form a base for the $G_\delta$-topology on $T^A$. Thus, for every countable subset $C \subseteq A$,

$$N(\chi, C) = \{ \psi \in K \mid \psi|_C = \chi|_C \}$$

(13) is a basic $G_\delta$-neighborhood of $\chi$ in $K$, and these sets form a base for the $G_\delta$-topology on $K$.

If $\chi \in \mathcal{H}_K$, then $N(\chi, C) \cap H \neq \emptyset$ for every countable subset $C$ of $A$, and therefore for every $C$ there is $\psi_C \in H$ such that $\psi_C|_C = \chi|_C$. In particular, $\chi|_C$ is continuous with respect to the induced topology $\tau_H|_C$, and hence $\chi$ is countably continuous.

Conversely, suppose that $\chi$ is countably continuous, and let $C \subseteq A$ be countable. By replacing $C$ with the subgroup that it generates, we may assume that $C$ is a countable subgroup of $A$, because

$$N(\chi, C) = N(\chi, C).$$

One has $\chi|_C \in H|_C$, because $\chi|_C$ is continuous with respect to $\tau_H|_C$ and $(C, \tau_H|_C)$ is a precompact. Therefore, $H \cap N(\chi, C) \neq \emptyset$, as desired.

We proceed with presenting a characterization of $I$-closed and $I$-dense subgroups of compact Hausdorff abelian groups. To that end, we first formulate explicitly an argument that was used by Comfort, Hernández and Trigos-Arrieta [8]. In order to do so, the following terminology is needed: For a topological space $X$, a zero-set in $X$ is a set of the form $Z(f) = f^{-1}(0)$, where $f$ is a real-valued continuous function on $X$. A subset $Y \subseteq X$ is $z$-embedded in $X$ if for every zero-set $Z$ in $Y$ there is a zero-set $W$ in $X$ such that $Z = W \cap Y$; $Y$ is $C$-embedded in $X$ if every continuous real-valued function on $Y$ extends continuously to $X$. One says that $X$ is an $Oz$-space if every open subset of $X$ is $z$-embedded.

**Proposition 3.5.** Let $G$ be a Hausdorff group whose (Raïkov-)completion $\tilde{G}$ is locally compact (in other words, $G$ is locally precompact), and let $H$ be its subgroup. Then:

(a) $H$ is $z$-embedded in $\tilde{G}$;
(b) $H$ is $C$-embedded in $\mathcal{H}_\delta$;
(c) $\nu H \subseteq \nu G$ and $\mathcal{H}_\delta = \nu H \cap G$, where $\nu$ stands for the Hewitt-realcompactification;
(d) if $G$ is realcompact, then $\mathcal{H}_\delta = \nu H$. 

Note that in contrast to the rest of this paper, in this proposition, the group \( G \) is not assumed to be abelian. We claim no novelty for the proof of Proposition 3.5 (which heavily relies on results of Blair and Hager), and it is included here for the sake of completeness.

**Lemma 3.6.** ([6, 1.1(a)-(b)], [5, 2.1, 5.1])

(a) Every \( z \)-embedded \( G_\delta \)-dense subset of a topological space \( X \) is \( C \)-embedded in \( X \).

(b) If \( Y \) is \( z \)-embedded in \( X \), then \( \nu Y \subseteq \nu X \) and \( \nu Y \) is the \( G_\delta \)-closure of \( Y \) in \( \nu X \).

(c) A topological space \( X \) is normal if and only if every closed subset of \( X \) is \( z \)-embedded.

(d) A topological space \( X \) is an \( Oz \)-space if and only if every dense subset of \( X \) is \( z \)-embedded.

**Proof.**

(a) The completion \( \tilde{G} \) is locally compact, and thus normal. By Lemma 3.6(c), the closed subgroup \( \tilde{H} \) of \( \tilde{G} \) is \( z \)-embedded in \( \tilde{G} \). By a result of Ross and Stromberg [26], \( \tilde{H} \) is an \( Oz \)-space, because it is locally compact (as a closed subgroup of \( \tilde{G} \)). Thus, the dense subgroup \( H \) is \( z \)-embedded in \( \tilde{H} \) (cf. Lemma 3.6(d)). Therefore, \( H \) is \( z \)-embedded in \( \tilde{G} \).

(b) It follows from (a) that \( H \) is \( z \)-embedded in \( \tilde{H}^\delta \), and therefore it is \( C \)-embedded in it, by Lemma 3.6(a).

(c) It follows from (a) that \( H \) is \( z \)-embedded in \( G \), and therefore the statement follows from Lemma 3.6(b).

(d) follows from (c).

**Corollary 3.7.** Let \( K \in \text{Ab}(\text{Comp}H) \) and \( H \leq K \).

(a) If \( H \) is closed, then it is \( l \)-closed in \( K \).

(b) \( H \) is \( l \)-closed in \( K \) if and only if \( H \) is realcompact.

(c) \( H \) is \( l \)-dense in \( K \) if and only if it is dense in \( K \) and pseudocompact.

(d) \( K \) is metrizable if and only if every subgroup of \( K \) is \( l \)-closed.

**Proof.** In light of Theorem 3.1, (a) is obvious, (b) follows from Proposition 3.5(d), and (c) is a reformulation of a famous theorem of Comfort and Ross [11, 1.2].

(d) If \( K \) is metrizable, then every subgroup of \( K \) is realcompact, and thus, by (c), \( l \)-closed. Conversely, if every subgroup of \( K \) is \( l \)-closed, then \( K \) is hereditarily realcompact (by (b)), and therefore metrizable (cf. [21, 3.3]).

We conclude this section by observing that a result of Hernández and Macario can be obtained as an easy consequence of the foregoing discussion:

**Corollary 3.8.** ([21, 3.2]) Let \( K \in \text{Ab}(\text{Comp}H) \) and set \( A = \hat{K} \). A subgroup \( H \leq K \) is realcompact if and only if every countably continuous character of \( (A, \tau_H) \) is continuous.

**Proof.** As we explained in the proof of Theorem 3.1, \( I_K(H) \) is the set of countably continuous characters of \( (A, \tau_H) \) (cf. Lemma 2.9(b) and Lemma 3.4). On the other hand, by Corollary 3.7(b), \( H \) is realcompact if and only if it is \( l \)-closed in \( K \).
4. APPLICATION II: THE $g$-CLOSURE

In this section, we apply the results of section 2 to $g$, and prove Theorems 1.2-1.4. As shown in Example 2.6, $g$ is the closure operator associated with the continuity type $S$ (assigning to a group the filter-bases generated by sequences). Furthermore, $S$-continuous homomorphisms are precisely the sequentially continuous ones.

**Theorem 4.1.** Let $K \in \text{Ab}(\text{Comp}\mathcal{H})$ and $A = \hat{K}$. A subgroup $H \leq K$ is:
(a) $g$-closed in $K$ if and only if every sequentially continuous character of $(A, \tau_H)$ is continuous;
(b) $g$-dense in $K$ if and only if $(A, \tau_H)$ has no non-trivial convergent sequences.

Note that part (b) of Theorem 2.10 appeared earlier in [2, 5.11].

**Proof.** (a) is Theorem 2.10 applied to $g$.

(b) If $(A, \tau_H)$ has no non-trivial convergent sequences, then all its characters are sequentially continuous, and thus, by Lemma 2.9(b), $g_K(H) = K$. Conversely, suppose that $H$ is $g$-dense in $K$—in other words, $g_K(H) = K$. Then, by Lemma 2.9(c), the identity map $(A, \tau_H) \rightarrow (A, \tau_K)$ is sequentially continuous. Thus, if $\{a_n\}$ is a convergent sequence with respect to $\tau_H$, then it must converge in $\tau_K$. On the other hand, one has $(A, \tau_K) = A^+ \subseteq bA$, and it is well known that $A^+$ has no convergent sequences, because $A$ is discrete abelian (cf. [19] and [20]).

It follows from the definition of $S$ that it is regular, and every dually embedded subgroup $N$ of a group $G \in \text{Ab}(\text{Top})$ is $S$-dually embedded (because if $\{u_n\}$ is a sequence in $\hat{N}$, then there is a sequence $\{v_n\}$ in $\hat{G}$ such that $i_N(v_n) = u_n$). Therefore, the following well-known and basic properties of $g$ are immediate consequences of Proposition 2.3 (cf. [18, 2.9, 2.11]):

**Proposition 4.2.** For every $G \in \text{Ab}(\text{Top})$:
(a) $g_G(\{0\}) = \text{n}(G)$;
(b) $g_G(H) = g_{G^+}(H)$ for every $H \leq G$;
(c) if $N \leq G$ is dually embedded in $G$, then $g_N(H) = g_G(H) \cap N$ for every $H \leq N$.

Now we are ready to prove the theorems that were formulated in section 1:

**Proof of Theorem 1.2.** Let $H = \{h_1, h_2, \ldots\}$ be countable. Since $K$ is compact, it follows from the Pontryagin duality that its subgroups are dually embedded, and so $g_H(H) = g_K(H) \cap \hat{H}$ (by Proposition 4.2). Every closed subgroup of $K$ is $g$-closed (cf. [18, 2.12]), and thus one has $g_K(H) \leq \hat{H}$. Therefore, $g_H(H) = g_K(H)$, and without loss of generality, we may assume that $H$ is dense in $K$. In particular, $(A, \tau_H)$ is Hausdorff. The topology $\tau_H$ is the initial one with respect to the homomorphism $(h_i): A \rightarrow \mathbb{T}^\omega$. Thus, $(A, \tau_H)$ is metrizable, and in particular, every sequentially continuous map on $(A, \tau_H)$ is continuous. Therefore, by Theorem 4.1(a), $H$ is $g$-closed.

**Proof of Theorem 1.3.** Let $H$ be a $g$-closed subgroup of $K$. Since $S(\hat{K}) \subseteq C(\hat{K})$ (where $S$ is as in Example 2.6), one has $H \subseteq l_K(H) \leq g_K(H) = H$, and thus $H$ is $l$-closed in $K$. Therefore, by Corollary 3.7(b), $H$ is realcompact.
PROOF OF THEOREM 1.4. (i) ⇒ (ii): Let \( H \) be a countable subgroup of \( G \). By Theorem 1.2, \( \rho_G(H) \) is \( g \)-closed in the Bohr-compactification \( bG \), because it is countable—in other words, \( g_{bG}(\rho_G(H)) = \rho_G(H) \). Since \( G \) is maximally almost periodic (i.e., \( \rho_G \) is injective), one has \( \rho_G^{-1}(g_{bG}(\rho_G(H))) = H \). Hence, by Theorem 2.4, \( H \) is \( g \)-closed in \( G \).

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i): By (iii), the (cyclic) subgroup \( \{0\} \) is \( g \)-closed in \( G \). Therefore, by Proposition 4.2(a), \( n(G) = g_G(\{0\}) = \{0\} \), as desired. \( \square \)

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Department of Mathematics and Statistics
Dalhousie University
Halifax, B3H 3J5, Nova Scotia
Canada

e-mail: lukacs@mathstat.dal.ca