Conditional Results for a Class of Arithmetic Functions: a variant of H. L. Montgomery and R. C. Vaughan’s method

Xiaodong Cao and Wenguang Zhai

Abstract. Let $a, b, c$ and $k$ be positive integers such that $1 \leq a \leq b, a < c < 2(a + b), c \neq b$ and $(a, b, c) = 1$. Define the arithmetic function $f_k(a, b; c; n)$ by

$$
\sum_{n=1}^{\infty} \frac{f_k(a, b; c; n)}{n^s} = \frac{\zeta(as)\zeta(bs)}{\zeta^k(cs)}, \Re s > 1.
$$

Let $\Delta_k(a, b; c; x)$ denote the error term of the summatory function of the function $f_k(a, b; c; n)$. In this paper we shall give two expressions of $\Delta_k(a, b; c; x)$. As applications, we study the so-called $(l, r)$-integers, the generalized square-full integers, the $e - r$-free integers, the divisor problem over $r$-free integers, the $e$-square-free integers. An important tool is a generalization of a method of H. L. Montgomery and R. C. Vaughan.

1 Introduction and main results

W. G. Nowak\cite{23}, M. Küleitner and W. G. Nowak \cite{26} studied a class of very general arithmetic function $a(n)$, which possess a generating Dirichlet series

$$
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{f_1(m_1s) \cdots f_K(m_Ks)}{g_1(n_1s) \cdots g_J(n_Js)} h(s),
$$

where $f_k$ and $g_j$ are certain generalizations of Riemann zeta-function, $m_1 \leq \cdots \leq m_K$ and $n_1 \leq \cdots \leq n_J$ are natural numbers, and $h(s)$ is a good function which is regular and bounded in a sufficiently large half-plane. People are usually concerned with the summatory function $\sum_{n\leq x} a(n)$, especially sharp upper and lower bounds of its error term. The above two papers give an upper bound and a lower bound for $a(n)$ in a very general sense. Some special cases are also studied, see for example, \cite{1, 2, 27, 50}.

The aim of this paper is to study a special case of $a(n)$, in which case we can get better upper results. Let $a, b, c$ and $k$ be positive integers such that $1 \leq a \leq b, a < c < 2(a + b), c \neq b$ and $(a, b, c) = 1$. Define the arithmetic function $f_k(a, b; c; n)$ by

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Key Words: square-full integer, exponential convolution, exponential divisor, $e$-$r$-free integer, $(r, l)$-integer, exponent pair.

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it is easy to prove that respectively. We also use notations

\[ (1.1) \]

As usual, \( \Delta \) are studied for the the generalized square-full number problem(see [8, 48]).

By Theorem 2 of M. K"uleitner and W. G. Nowak [26] or Theorem 3 of W. G. Nowak [33], it is easy to prove that

\[ (1.4) \]

Hence one may conjecture that

\[ (1.5) \]

Many special cases of the function \( f_k(a, b; c; n) \) have been extensively studied in number theory. We take some examples.

(1) The case \((a, b, c, k) = (2, 3, 6, 1)\) is the well-known square-full number problem(see [5, 8, 11, 32, 56]). Suppose \( a \nmid b \), the cases \((a, b, c, k) = (a, b, 2b, 1)\) or \((a, b, c, k) = (b, a, 2b, 1)\) are studied for the the generalized square-full number problem(see [8, 48]).

(2) Suppose \( r \geq 2 \) is a fixed integer, the case \((a, b, c, k) = (1, 1, r, 1)\) is the \( r \)-free divisor problem(see [1, 2, 14, 15, 27]).

(3) Suppose \( 1 < r < l \) are fixed integers, the case \((a, b, c, k) = (1, r, l, 1)\) corresponds to the distribution of the so-called \((r, l)\)-integers(see [10, 44, 46, 47, 49, 57]).

(4) Suppose \( r \geq 1 \) is a fixed integer, the case \((a, b, c, k) = (1, 2^r + 1, 2^r, 1)\) corresponds to the the distribution of the so-called \( e-r \)-free integers(see [7, 45, 51, 52, 53]).

(5) Suppose \( r \geq 2 \) is a fixed integer, the case \((a, b, c, k) = (1, 1, r, r + 1)\) corresponds to the Dirichlet divisor problem over the set of \( r \)-free integers(see [12, 36]).

From the right-hand side of (1.1) it is easily seen that the unconditional asymptotic formula we could possibly prove at present is at most

\[ (1.6) \]

2 \((a+b), c \neq b \) and \((a, b, c) = 1\). Let \( \zeta(s) \) denote the Riemann zeta-function. The arithmetic function \( f_k(a, b; c; n) \) is defined by

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The expected asymptotic formula of \( A_k(a, b; c; x) \) is of the form

\[ (1.3) \]

when \( a \neq b \). When \( a = b \), then an appropriate limit should be taken in the above formula. As usual, \( \Delta_k(a, b; c; x) \) is called the error term of the function \( A_k(a, b; c; x) \). For convenience, we also use notations \( A(a, b; c; x), \Delta(a, b; c; x) \) to denote \( A_1(a, b; c; x), \Delta_1(a, b; c; x) \), respectively.

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\[ (1.1) \]
where $A > 0$ is some absolute constant. Now define $\theta_k(a,b;c)$ denote the infimum of $\alpha_k(a,b;c)$ such that the estimate

$$\Delta_k(a,b;c;x) \ll x^{\alpha_k(a,b;c)+\varepsilon}$$

holds.

From (1.1) we also see that the function $f_k(a,b;c;n)$ is related to the divisor function $d(a,b;n) := \sum_{m=m_1m_2=1}^{n} d(a,b;n)_m$ which satisfies

$$\sum_{n=1}^{\infty} \frac{d(a,b;n)}{n^s} = \zeta(as)\zeta(bs), \mathfrak{R}s > 1.$$

We write

$$D(a,b;x) := \sum_{n \leq x} d(a,b;n) = \zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}} + \zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}} + \Delta(a,b;x)$$

for $a \neq b$, and let $0 < \alpha(a,b) < 1/(a+b)$ be a real number such that the estimate

$$\Delta(a,b;x) \ll x^{\alpha(a,b)+\varepsilon}$$

holds.

As usual, $\Delta(a,b;x)$ is called the error term of the asymmetric two-dimensional divisor problems. For the history and classical results of $\Delta(a,b;x)$, see for example \[20, 21, 25, 29, 37, 58\].

If $\alpha(a,b) \geq 1/c$, then by the convolution approach we get easily $\theta_k(a,b;c) \leq \alpha(a,b)$. Thus the difficulty of the evaluation of the function $A_k(a,b;c;x)$ is basically the difficulty of the evaluation of the function $D(a,b;x)$. Without the loss of generality, we always suppose later that $\alpha(a,b) < 1/c$.

The exponent $1/c$ in the error term in (1.6) is closely related to the distribution of the non-trivial zeros of $\zeta(s)$. People usually assume the Riemann-hypothesis (RH) to reduce the constant $1/c$. See for example, \[1, 32, 47, 48\]. From now on, we always suppose that RH holds.

In 1981, Montgomery and Vaughan \[31\] developed a new ingenious method to treat the distribution of $r$-free integers, which was also used by many other authors, see for example, Baker \[1\], Nowak and Schmeier \[32\], Nowak \[33\] etc. However, as W. G. Nowak and M. Schmeier \[32\] observed in subsection (The Divisor Problem For $(l,r)$-Integers) that: The $r = 2, l = 3$ is some exceptional. That is, in some cases, by Montgomery-Vaughan’s method one could not get directly better estimates than the usual approach.

The main aim of this paper is to find a suitable expression of the error term in (1.3) for every case. We have to consider two different cases: $c > b$ and $c < b$. For these two cases, we have to use different convolution approaches.

Consider first $c > b$. We define the function $\mu_k$ by

$$\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta^k(s)}, \mathfrak{R}s > 1.$$
Clearly $\mu_1$ is the well-known Möbius function $\mu$.

Then Theorem 2 of Nowak\cite{33} essentially implies the following theorem.

**Theorem 1.** (W. G. Nowak) Let $x \geq 2$, $a \leq b < c < 2(a + b)$ and $\Delta_k(a, b; c; x)$ be defined by (1.3). If the RH is true, then for any $1 \leq y < x^{\epsilon}$

$$\Delta_k(a, b; c; x) = \sum_{l \leq y} \mu_k(l) \Delta(a, b; \frac{x}{l}) + O(x^{\frac{1}{2} + \epsilon} \frac{\log x}{\log \log x} + x^{\epsilon}).$$  

Taking $y = x^{\frac{1-2\alpha(a,b)}{1+2\alpha(a,b)}}$ and noting $\mu_k(l) \ll l^\epsilon$, we get

**Corollary 1.1.** Suppose RH. If $a \leq b < c < 2(a + b)$ and $\alpha(a, b) < \frac{1}{c}$, then

$$\theta_k(a, b; c) \leq \frac{1 - a\alpha(a, b)}{a + c - 2\alpha a \alpha(a, b)}.$$

**Remark 1.1.** The proof of Theorem 1 is based on a classical idea of Montgomery and Vaughan\cite{31} and the Dirichlet convolution

$$f_k(a, b; c; n) = \sum_{n=m_1m_2} d(a, b; m_1) \mu_k(m_2).$$  

**Remark 1.2.** Very fortunately, for arithmetical function $\mu_k$ we have an analogue of the well-known Vaughan’s identity of Möbius function $\mu$(see Lemma 4.1 below, in fact this is the third useful Vaughan-type’s identity except the well-known von Mangoldt function $\Lambda$ and Möbius function $\mu$). By the method of exponential sums, one could improve the result in Corollary 1.1. For the related works, we refer to papers\cite{1, 2, 5, 27, 56}. However, this is not the main aim of the present paper.

We now turn to the case $c < b$. Some examples of this type can be found in\cite{46, 47, 48, 49}. In this case, we hope to find an estimate of the form $\Delta_k(a, b; c; x) \ll x^\alpha (\alpha < 1/b)$ such that the second main term $\frac{\zeta'(x)}{\zeta(x)} x^\frac{1}{\alpha}$ becomes a real main term. We can also use the convolution (1.13) as our first choice to study $A_k(a, b; c; x)$. Actually it is easy to check that Theorem 1 also holds for $b/2 < c < b$. But when $a + c \leq b$, we have checked that it is very difficult to prove $\theta_k(a, b; c) < 1/b$ via Theorem 1 directly(Also see page 9, section 5 in\cite{7}). In order to overcome this difficulty, we choose another convolution approach.

Let the arithmetic function $u_k(a; c; n)$ be defined by

$$u_k(a; c; n) := \sum_{n=a, d^c} \mu_k(d),$$

which satisfies

$$\sum_{n=1}^\infty \frac{u_k(a; c; n)}{n^s} = \frac{\zeta(as)}{\zeta^k(cs)}, \Re s > 1.$$
We note that when \( k = 1 \), the function \( u_k(a; c; n) \) is just the characteristic function of the set of the \( c \)-free integers. Hence we can write

\[
f_k(a, b; c; n) = \sum_{n=m_1 \mu_b} u_k(a; c; m).
\]

The function \( u_k(a; c; n) \) plays an important role in this case. Define

\[
\Delta_k(a; c; x) := \sum_{n \leq x} u_k(a; c; n) - \frac{x}{\zeta_k \left( \frac{x}{a} \right)} := A_k(a; c; x) - \frac{x}{\zeta_k \left( \frac{x}{a} \right)}.
\]

For \( \Delta_k(a; c; x) \), similar to Theorem 1 we also have

**Theorem 2.** Let \( x \geq 2 \), \( a < c \) and \( \Delta_k(a; c; x) \) be defined by (1.17). If the RH is true, then for any \( 1 \leq y < x^{\frac{1}{b}} \) we have

\[
\Delta_k(a; c; x) = \sum_{l \leq y} \mu_k(l) \psi \left( \left( \frac{x}{l} \right)^{\frac{1}{b}} \right) + O \left( x^{\frac{1}{2b} + \varepsilon} y^{\frac{1}{2} - \frac{1}{2b}} + y^{\frac{1}{2} + \varepsilon} \right).
\]

On taking \( y = x^{\frac{1}{b}} \) in Theorem 2 we get immediately the following

**Corollary 1.2.** Under the conditions of Theorem 2, we have

\[
\Delta_k(a; c; x) \ll x^{\frac{1}{a+b} + \varepsilon}.
\]

Now we state our main result for the case \( c < b \), which improves Theorem 1 in the case \( a < c < b < 2c \).

**Theorem 3.** Suppose RH is true. Let \( x \geq 2 \), \( a < c < b < 2c \), \( \Delta_k(a, b; c; x) \) and \( \Delta_k(a, c; x) \) be defined by (1.3) and (1.17) respectively. Suppose \( \Delta_k(a; c; x) \ll x^{\alpha_k(a; c) + \varepsilon} \) such that \( \alpha_k(a; c) < 1/b(a \text{ natural restriction}) \). Then for any \( 1 \leq y < x^{\frac{1}{b}} \) we have

\[
\Delta_k(a, b; c; x) = \sum_{d \leq y} \Delta_k(a; c; \frac{x}{d}) - \sum_{m \leq \frac{x}{a+b}} u_k(a; c; m) \psi \left( \left( \frac{x}{m} \right)^{\frac{1}{b}} \right)
+ O \left( x^{\frac{1}{b}} y^{\frac{1}{2} - \frac{1}{2b}} + (xy^{-b})^{\alpha_k(a; c)} + x^{\frac{1}{2}} y^{1 - \frac{1}{2b}} \right).
\]

**Corollary 1.3.** Under the conditions of Theorem 3, we have

\[
\Delta_k(a, b; c; x) \ll x^{\frac{1}{a+b-b\alpha_k(a; c)} + \varepsilon}.
\]

**Remark 1.3.** Since we use different convolution approaches in Theorem 1 \((b < c)\) and Theorem 3 \((b > c)\), the exponential sums appeared in these two theorems are also different. Hence we have to use different ways to estimate exponential sums in these two theorems. We note that Corollary 3 implies \( \frac{1}{a+b-b\alpha_k(a; c)} \ll \frac{1}{b} \), hence in the asymptotic formula (1.3), the second main term becomes a real main term.
Remark 1.4. All corollaries above can be further improved by more precise estimate for the exponential sums involved (e.g., see [2, 3, 6, 13, 15, 29, 38, 41, 56]).

The organization of this paper is as follows. In section 2 we shall give short proofs of Theorem 1 and Theorem 2. The proof of Theorem 3 will be given in section 3. In section 4, by the well-known Heath-Brown’s method we shall further improve Corollary 2 and obtain a non-trivial estimate for $\Delta_k(a;c;x)(k = 1, 2)$, and then give some of its applications to problems related to the exponential convolution. In section 5 we discuss some applications of Corollary 1.1 and Corollary 1.3. Finally, in section 6 we give an example to explain how to get a sharper upper bound by Theorem 3.

Notation. Throughout this paper $\varepsilon$ denotes a fixed positive constant, not necessarily the same in all occurrences. As usual, let $\tau(n)$ and $\omega(n)$ denote the divisor function and the number of prime factors of $n$, respectively. We also use $\tau_k(n)$ to denote the number of decompositions of $n$ into $k$ factors, and let $\tau_1(n) = 1$. Let $q_r(n)$ denote the characteristic function of the set of $r$-free integers. $x > 1$ is real, $L = \log x$, $\{t\}$ denotes the fractional part of $t$, $\psi(t) = \{t\} - 1/2$, $\|t\| = \min(\{t\}, 1 - \{t\})$. We let $e(t) = \exp(2\pi it)$ and $\delta(x) = \exp(-A(\log x)^3(\log \log x)^{-3} - 1/5)$ for some fixed constant $A > 0$. $m \sim M$ means that $cM < m < CM$ for some constants $0 < c < C$.

2 The proof of Theorem 1 and 2

Lemma 2.1. Let $A(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ converge absolutely for $\Re s = \sigma > \sigma_a$, and let functions $H(u)$ and $B(u)$ be monotonically increasing such that

$$|a(n)| \leq H(n), \quad (n \geq 1),$$

$$\sum_{n=1}^{\infty} |a(n)n^{-\sigma}| \leq B(\sigma), \quad \sigma > \sigma_a.$$  

If $s_0 = \sigma_0 + it_0, b_0 > \sigma_a, b_0 \geq b > 0, b_0 \geq \sigma_0 + b > \sigma_a, T \geq 1$ and $x \geq 1$, then for $x \notin \mathbb{N}$

$$\sum_{n \leq x} a(n)n^{-s_0} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} A(s_0 + s) \frac{x^s}{s} ds + O \left( \frac{x^b B(b + \sigma_0)}{T} \right) + O \left( x^{1-\sigma_0} H(2x) \min(1, \frac{\log x}{T}) \right) + O \left( x^{-\sigma_0} H(N) \min(1, x \|x\|) \right).$$

Proof. This lemma is the well-known Perron’s formula, for example, see Theorem 2 of page 98 in Pan[34].

Let $y \geq 1$ and define

$$g_y(s) := \sum_{n \geq y} \frac{\mu_k(n)}{n^s}, \quad \Re s > 1.$$  

6
Lemma 2.2. Suppose RH is true, then \( g_y(s) \) can be continued analytically to \( \Re s = \sigma > \frac{1}{2} + \varepsilon \), and we have uniformly for \( \sigma \) that

\[
g_y(s) \ll y^{\frac{1}{2} - \sigma + \varepsilon(|t| + 1)^\varepsilon}, \quad \sigma \geq \frac{1}{2} + \varepsilon.
\]

(2.2)

Proof. This lemma follows from Lemma 3 of Nowak[33] immediately. □

The Proof of Theorem 1 and 2. Let \( \delta = \frac{\varepsilon}{10} \) and \( 1 \leq y < x^{\frac{7}{10}} \). We begin the proof of Theorem 1 in the same way as that of Theorem 1 in Montgomery and Vaughan[31]. Here we only give the details of our proof for the case \( a < b \). The proof for the case \( a = b \) is similar.

Define

\[
f_{1,y}(n) := \sum_{\substack{l \leq y \mid m \mid n \mid \gcd(m,n) \mid \gcd(l,m) \mid \gcd(a,b;m) \mid \gcd(l,a,b;m) \mid \gcd(a,b;m)}} \mu_k(l)d(a,b;m),
\]

(2.3)

\[f_{2,y}(n) := \sum_{\substack{l \mid m = n \mid l > y \not\mid \gcd(m,n) \not\mid \gcd(l,m) \not\mid \gcd(a,b;m) \not\mid \gcd(l,a,b;m) \not\mid \gcd(a,b;m)}} \mu_k(l)d(a,b;m),
\]

hence

\[
f_k(a,b;c;n) = \sum_{l \mid m = n} \mu_k(l)d(a,b;m) = f_{1,y}(n) + f_{2,y}(n).
\]

(2.4)

We now write \( A_k(a,b;x) \) in the form

\[
A_k(a,b;c;x) := \sum_{n \leq x} f_k(a,b;c;n) = S_1(x) + S_2(x),
\]

(2.5)

where

\[
S_1(x) = \sum_{n \leq x} f_{1,y}(n),
\]

(2.6)

and

\[
S_2(x) = \sum_{n \leq x} f_{2,y}(n).
\]

(2.7)

We first evaluate \( S_1(x) \). From (1.8) we get
\begin{align}
S_1(x) &= \sum_{\substack{\mathfrak{g} \leq n \leq x \atop l \leq y}} \mu_k(l)d(a, b; m) = \sum_{l \leq y} \mu_k(l) D \left( a, b; \frac{x}{l} \right) \\
&= \sum_{l \leq y} \mu_k(l) \left( \frac{\zeta \left( \frac{b}{a} \right)}{l^\beta} x^\frac{1}{\beta} + \frac{\zeta \left( \frac{a}{b} \right)}{l^\beta} x^\frac{1}{\beta} + \Delta \left( a, b; \frac{x}{l} \right) \right) \\
&= \zeta \left( \frac{b}{a} \right) x^\frac{1}{\beta} \sum_{l \leq y} \frac{\mu_k(l)}{l^\beta} + \zeta \left( \frac{a}{b} \right) x^\frac{1}{\beta} \sum_{l \leq y} \frac{\mu_k(l)}{l^\beta} + \sum_{l \leq y} \mu_k(l) \Delta \left( a, b; \frac{x}{l} \right). \\
&= \text{Res} \left( \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} \sum_{l \leq y} \frac{\mu_k(l)}{l^\epsilon s}, \frac{1}{a} \right) \\
&\quad + \text{Res} \left( \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} \sum_{l \leq y} \frac{\mu_k(l)}{l^\epsilon s}, \frac{1}{b} \right) + \sum_{l \leq y} \mu_k(l) \Delta \left( a, b; \frac{x}{l^\beta} \right).
\end{align}

From (2.1) we have

\begin{align}
\sum_{n=1}^{\infty} \frac{f_{2,y}(n)}{n^s} &= g_y(cs) \zeta(\alpha s) \zeta(\beta s).
\end{align}

From (2.3), (2.7), (2.9) and Lemma 2.1 we obtain that

\begin{align}
S_2(x) &= \frac{1}{2\pi i} \int_{\frac{1}{\alpha} + \epsilon - i \infty}^{\frac{1}{\alpha} + \epsilon + i \infty} g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} ds + O(x^\delta),
\end{align}

since \( f_{2,y}(n) \ll n^\delta \) by a divisor argument.

**Case (i).** \( a < b < 2a \). When we move the line of integration to \( \Re{s} = \sigma = \frac{1}{\alpha} + \delta_0 \) with \( \delta_0 = \min\{ \delta, \frac{1}{\alpha \beta}(1 - \frac{b}{a}) \} \), then by the residue theorem

\begin{align}
\frac{1}{2\pi i} \int_{\frac{1}{\alpha} + \epsilon - i \infty}^{\frac{1}{\alpha} + \epsilon + i \infty} g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} ds \\
&= \text{Res} \left( g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1}, \frac{1}{a} \right) + \text{Res} \left( g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1}, \frac{1}{b} \right) \\
&\quad + I_1 + I_2 - I_3,
\end{align}

where

\begin{align}
I_1 &= \frac{1}{2\pi i} \int_{\frac{1}{\alpha} + \delta_0 + \epsilon - i \infty}^{\frac{1}{\alpha} + \epsilon + i \infty} g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} ds, \\
I_2 &= \frac{1}{2\pi i} \int_{\frac{1}{\alpha} + \delta_0 + \epsilon - i \infty}^{\frac{1}{\alpha} + \epsilon + i \infty} g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} ds, \\
I_3 &= \frac{1}{2\pi i} \int_{\frac{1}{\alpha} + \delta_0 - \epsilon - i \infty}^{\frac{1}{\alpha} + \delta_0 + \epsilon - i \infty} g_y(cs) \zeta(\alpha s) \zeta(\beta s) x^s s^{-1} ds.
\end{align}
From Lemma 2.2, we have
\[ g_y(cs) \ll y^{\frac{1}{2} - \frac{\epsilon}{2}} (|t|^\delta + 1), (\sigma \geq \frac{1}{2a} + \delta) \]

Thus
\[ g_y(cs)\zeta(as)\zeta(bs) \ll y^{\frac{1}{2} - \frac{\epsilon}{2}} (|t|^\delta + 1), (\sigma \geq \frac{1}{2a} + \delta). \]  

From (2.12) it is not difficult to see that
\[ I_j \ll y^{\frac{1}{2} - \frac{c}{2} a} x^{\frac{1}{2} a + 8} \delta, (j = 1, 2, 3). \]  

Now combining (1.3), (2.5), (2.8), (2.10), (2.11) and (2.13) completes the proof of Theorem 1 in this case.

Case (ii). \( b \geq 2a \). In this case, moving the line of integration in (2.10) to \( \Re s = \sigma = \frac{1}{2a} + \delta \), we can treat \( S_2(x) \) as in the above case except the second residue in relation (2.11) vanishes. In addition, applying Abel summation formula and the estimate \( \sum_{n \leq x} \mu_k(n) \ll x^{\frac{1}{2} + \epsilon} \) (this can be proved in the same way as that of Theorem 14.25(C) in Titchmarsh[50], also see (2.10) in Nowak[33]), it is easy to check that
\[ x \frac{1}{b} \sum_{l > y} \frac{\mu_k(l)}{l^\sigma} \ll x^{\frac{1}{2} + \delta} y^{\frac{1}{2} - \frac{\epsilon}{2}} \ll x^{\frac{1}{2} + \delta} y^{\frac{1}{2} - \frac{\epsilon}{2}}. \]

Hence
\[ x \frac{1}{b} \sum_{l \leq y} \frac{\mu_k(l)}{l^\sigma} = x \frac{1}{b} \sum_{l=1}^\infty \frac{\mu_k(l)}{l^\sigma} - x \frac{1}{b} \sum_{l > y} \frac{\mu_k(l)}{l^\sigma} = \frac{1}{\zeta(k)(\frac{1}{b})} x^{\frac{1}{b}} + O \left( x^{\frac{1}{2a} + \delta} y^{\frac{1}{2} - \frac{\epsilon}{2}} \right). \]

Therefore, Theorem 1 also holds in this case. This completes the proof of Theorem 1.

The proof of Theorem 2 is very similar to that of Theorem 1, we omit the details here.

3 The proof of Theorem 3

Lemma 3.1. Let \( a < c < b \) and \( \Delta_k(a; c; x) \) be defined by (1.17). If \( \Delta_k(a; c; x) \ll x^{\alpha(a;c) + \epsilon} \) such that \( \alpha(a; c) < 1/b \), then for \( s > \alpha(a; c) \) we have
\[ \sum_{m \leq x} u_k(a; c; m)m^{-s} = \frac{x^{\frac{1}{2} - s}}{(1 - as)\zeta(k)(\frac{1}{a})} + \frac{\zeta(as)}{\zeta(cs)} + \Delta_k(a; c; x)x^{-s} - s \int_x^\infty \Delta_k(a; c; t)t^{-s-1} dt. \]
Proof. By partial summation formula and (1.17) we get

\begin{align}
(3.2) \quad \sum_{m \leq x} u_k(a; c; m)m^{-s} \\
&= A_k(a; c; x)x^{-s} + s \int_1^{x} A_k(a; c; t)t^{-s-1}dt \\
&= \frac{x^{\frac{1}{n} - s}}{\zeta(k(\frac{a}{n}))} + \Delta_k(a; c; x)x^{-s} + s \int_1^{x} \left( \frac{t^{\frac{1}{n}}}{\zeta(k(\frac{a}{n}))} + \Delta_k(a; c; t) \right)t^{-s-1}dt \\
&= \frac{x^{\frac{1}{n} - s}}{(1 - as)\zeta(k(\frac{a}{n}))} + \Delta_k(a; c; x)x^{-s} - \frac{as}{(1 - as)\zeta(k(\frac{a}{n}))} + s \int_1^{x} \Delta_k(a; c; t)t^{-s-1}dt.
\end{align}

Suppose that \( s > 1 \), we have from (1.15) and the condition \( \alpha(a; c) < \frac{1}{b} \), when \( x \to \infty \)

\begin{align}
(3.3) \quad \frac{\zeta(as)}{\zeta(cs)} = -\frac{as}{(1 - as)\zeta(k(\frac{a}{n}))} + s \int_1^{\infty} \Delta_k(a; c; t)t^{-s-1}dt.
\end{align}

By analytic continuation this equation also holds for \( s > \alpha(a; c) \). Substituting (3.3) into (3.2) completes the proof of Lemma 3.1. \( \square \)

**Lemma 3.2.** Let \( x \geq 2 \), \( a < c < b \), and \( \Delta_k(a, b; c; x) \) be defined by (1.3). If \( \Delta_k(a; c; x) \ll x^{\alpha(a; c) + \epsilon} \) such that \( \alpha(a; c) < 1/b \), then for any \( 1 \leq y < x^{1/b} \) we have

\[
\Delta_k(a, b; c; x) = \sum_{d \leq y} \Delta_k(a; c; \frac{x}{db}) - \sum_{m \leq \frac{x}{y^{\frac{1}{b}}}} u_k(a; c; m)\psi \left( \frac{x}{m} \right) \\
- \frac{x^{\frac{1}{b}}}{b} \int_{y^{\frac{1}{b}}}^{\infty} \Delta_k(a; c; t) \frac{dt}{t^{1+\frac{1}{b}}} + \psi(y) \Delta_k(a; c; \frac{x}{y^{\frac{1}{b}}}) + O(x^{\frac{1}{b}}y^{-1-\frac{1}{b}}).
\]

**Proof.** Let \( 1 \leq y \leq x^{1/b} \). Applying (1.1),(1.15)-(1.17) and Dirichlet’s hyperbolic argument, we get

\begin{align}
(3.4) \quad A_k(a, b; c; x) &= \sum_{n \leq x} f_k(a, b; c; n) = \sum_{md^k \leq x} u_k(a; c; m) \\
&= \sum_{d \leq y} \sum_{m \leq \frac{x}{db}} u_k(a; c; m) + \sum_{m \leq \frac{x}{y^{\frac{1}{b}}}} u_k(a; c; m) \sum_{y < d \leq \left(\frac{x}{y^{\frac{1}{b}}}\right)} 1 \\
&= \frac{x^{\frac{1}{n}}}{\zeta(k(\frac{a}{n}))} \sum_{d \leq y} \frac{1}{d^{\frac{1}{b}}} + \frac{x^{\frac{1}{b}}}{y^{\frac{1}{b}}} \sum_{m \leq \frac{x}{y^{\frac{1}{b}}}} \frac{u_k(a; c; m)}{m^{\frac{1}{b}}} - y \sum_{m \leq \frac{x}{y^{\frac{1}{b}}}} u_k(a; c; m) \\
&+ \sum_{d \leq y} \Delta_k(a; c; \frac{x}{db}) - \sum_{m \leq \frac{x}{y^{\frac{1}{b}}}} u_k(a; c; m)\psi \left( \frac{x}{m} \right) + \psi(y) \sum_{m \leq \frac{x}{y^{\frac{1}{b}}}} u_k(a; c; m).
\end{align}
Applying Lemma 3.1 with \( s = \frac{1}{b} \), we get

\[
\sum_{m \leq \frac{x}{y^b}} u_k(a; c; m) \frac{1}{m^{\frac{1}{b}}} = \left( \frac{x}{y^b} \right) \frac{1}{1 - \frac{1}{b}} \zeta^k(\frac{a}{b}) + \frac{\zeta(\frac{a}{b})}{\zeta^k(\frac{a}{b})} + \Delta_k(a; c; \frac{x}{y^b}) - \frac{1}{b} \int_{\frac{x}{y^b}}^{\infty} \Delta_k(a; c; t) \frac{dt}{t^{1 + \frac{1}{b}}}
\]

In addition, we have from the Euler-Maclaurin formula that

\[
\sum_{d \leq y} \frac{1}{d^x} = \zeta\left( \frac{b}{a} \right) + \frac{y^{1 - \frac{b}{a}}}{(1 - \frac{b}{a})} - \psi(y)y^{-\frac{b}{a}} + O(y^{-\frac{b}{a}})
\]

Applying (1.17) again we also have

\[
\sum_{m \leq \frac{x}{y^b}} u_k(a; c; m) = \left( \frac{x}{y^b} \right) \frac{1}{\zeta^k(\frac{a}{b})} + \Delta_k(a; c; \frac{x}{y^b})
\]

Substituting (3.5)-(3.7) into (3.4), we get

\[
A_k(a, b; c; x) = \zeta\left( \frac{b}{a} \right) x^\frac{1}{b} + \frac{\zeta(\frac{a}{b})}{\zeta^k(\frac{a}{b})} x^\frac{1}{b} - \sum_{m \leq \frac{x}{y^b}} u_k(a; c; m) \psi\left( \left( \frac{x}{m} \right)^{\frac{1}{b}} \right)
\]

\[
+ \sum_{d \leq y} \Delta_k \left( a; c; \frac{x}{d^x} \right) - \frac{x^{\frac{1}{b}}}{b} \int_{\frac{x}{y^b}}^{\infty} \Delta_k(a; c; t) \frac{dt}{t^{1 + \frac{1}{b}}}
\]

\[
+ \psi(y) \Delta_k \left( a; c; \frac{x}{y^b} \right) + O(x^\frac{1}{b} y^{-1 - \frac{b}{a}})
\]

Now Lemma 3.2 follows from (1.3) and (3.8) at once.

**Lemma 3.3.** Let \( \Delta_k(a; c; x) \) be defined by (1.17). If RH is true, then for any fixed \( \delta > 0 \) we have

\[
\int_{1}^{T} \Delta_k(a; c; u) du \ll T^{1 + \frac{1}{b} + \delta}
\]

**Proof.** It suffices to prove that for any \( M > 2 \), we have

\[
\int_{M}^{2M} \Delta_k(a; c; u) du \ll M^{1 + \frac{1}{b} + \delta}
\]

Taking in Lemma 2.1 \( H(n) = n^\varepsilon, B(\sigma) = (\sigma - 1)^{-k}, b = 1 + 1/\log M, T = M^5 \), we get

\[
\sum_{n \leq x} u_k(a; c; u) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(as) \frac{u^s}{\zeta^k(cs)} \frac{ds}{s} + O(M^{-4}).
\]
It is well-known that if RH is true, then for any fixed $0 < \eta < 1/2$, we have

\begin{equation}
\zeta(s) \ll (|t| + 1)^\eta, \quad \zeta^{-1}(s) \ll (|t| + 1)^\eta, \quad \sigma > \frac{1}{2} + \eta.
\end{equation}

Moving the line of integration to $\Re s = \frac{1}{2} + \delta$, we have by (3.11) and the estimate $\zeta(s) \ll (1 + |t|)^{1/2} (\sigma \geq 0)$ that

$$
\Delta_k(a; c; u) = \frac{1}{2\pi i} \int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT} \frac{\zeta(as)}{\zeta^k(cs)} \frac{u^s}{s} ds + O(M^{-1}).
$$

Thus we have

$$
\int_M^{2M} \Delta_k(a; c; u) du = \frac{1}{2\pi i} \int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT} \frac{\zeta(as)}{\zeta^k(cs)} \frac{u^s}{s} ds + O(1)
$$

$$
= \frac{1}{2\pi i} \int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT} \frac{\zeta(as)}{\zeta^k(cs) s(1+s)} \left( (2M)^{1+s} - M^{1+s} \right) ds + O(1).
$$

$$
\ll M^{1+\frac{1}{10}+\delta} \int_{-T}^{T} \left| \frac{\zeta(a(\frac{1}{2c} + \delta + it))}{\zeta(c(\frac{1}{2c} + \delta + it))} \right| \frac{dt}{(1+|t|)^2} + 1
$$

$$
\ll M^{1+\frac{1}{10}+\delta}.
$$

Namely (3.10) holds. This completes the proof of Lemma 3.3. \( \square \)

The Proof of Theorem 3. Theorem 3 follows immediately from Lemma 3.2 and Lemma 3.3 with $\delta = \frac{\epsilon_0}{10}$. \( \square \)

The Proof of Corollary 1.3. It is easy to check

\begin{equation}
\sum_{d \leq y} \Delta_k \left( a; c; \frac{x}{d^x} \right) \ll \sum_{d \leq y} \left( \frac{x}{d^x} \right)^{\alpha_k(a;c)} \ll x^{\alpha_k(a;c)} y^{1-b\alpha_k(a;c)}
\end{equation}

and

\begin{equation}
\sum_{m \leq \frac{x}{y^x}} u_k(a; c; m) \psi \left( \frac{x}{m} \right)^{\frac{1}{p}} \ll \sum_{m \leq \frac{x}{y^x}} |u_k(a; c; m)|
\end{equation}

$$
\ll \sum_{m \leq \frac{x}{y^x}} d(a, c; m) \ll \left( \frac{x}{y^x} \right)^{\frac{1}{p}}.
$$

Taking $y = x^{\frac{1-b\alpha_k(a;c)}{a+b-\alpha_k(a;c)}}$ we find that Corollary 1.3 is an immediate consequence of Theorem 3 and the above two estimates. \( \square \)
Estimates for $\Delta_k(a; c; x)(k = 1, 2)$ and an application of Theorem 2

4.1 Some preliminary lemmas

To treat the exponential sums appeared in Theorem 1 and 2, for the arithmetic function $\mu_k(k \geq 2)$ one needs an analogue of the well-known Vaughan’s identity of Möbius function $\mu$. First we shall prove such an identity.

Lemma 4.1. (Vaughan’s identity). Let $1 \leq N_1 < N$. Suppose that $U, V$ be two parameters with $1 \leq U, V \leq N_1$. Then for any arithmetic function $f$ we have

$$
\sum_{N_1 < n \leq N} \mu_k(n)f(n) = \sum_1 - \sum_2 - \sum_3,
$$

where

$$
\sum_1 = \sum_{U < m \leq N/V} A(m) \sum_{N_1/m < n \leq N/m} \mu_k(n)f(mn),
$$

$$
\sum_2 = \sum_{U < m \leq UV} B(m) \sum_{N_1/m < n \leq N/m} \tau_k(n)f(mn),
$$

$$
\sum_3 = \sum_{m \leq U} B(m) \sum_{N_1/m < n \leq N/m} \tau_k(n)f(mn),
$$

$$
A(m) = \sum_{\substack{ed_1 = m \leq U \\ \text{gcd}(d_1, d_2) = m}} \mu_k(e)\tau_k(d_1), \quad B(m) = \sum_{\substack{d_1d_2 = m \\ 1 \leq d_1 \leq U, d_2 \leq V}} \mu_k(d_1)\mu_k(d_2).
$$

Proof. Let

$$
F(U, s) := \sum_{d \leq U} \frac{\mu_k(d)}{ds}, \Re s > 1,
$$

then

$$
\frac{1}{\zeta^k(s)} = \left(\frac{1}{\zeta^k(s)} - F(V, s)\right) \left(1 - \zeta^k(s)F(U, s)\right) - F(U, s)F(V, s)\zeta^k(s) + F(U, s) + F(V, s), \Re s > 1.
$$

Equating coefficients from both sides of (4.3) gives the following identity

$$
\mu_k(n) = b_1(n) + b_2(n) + b_3(n) + b_4(n),
$$
where

\[
b_1(n) = - \sum_{d_1 \geq 1} \mu_k(d) \left( \sum_{e \leq U} \mu_k(e) \tau_k(d_1) \right),
\]

\[
b_2(n) = - \sum_{d_1, d_2 \geq 1} \mu_k(d_1) \mu_k(d_2) \tau_k(m),
\]

\[
b_3(n) = \begin{cases} 
\mu_k(n) & \text{if } n \leq U, \\
0 & \text{if } n > U,
\end{cases}
\]

\[
b_4(n) = \begin{cases} 
\mu_k(n) & \text{if } n \leq V, \\
0 & \text{if } n > V.
\end{cases}
\]

From (1.10) we have

\[(4.5) \quad \sum_{m \geq 1} \tau_k(m) \mu_k(d) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } n > 1.
\end{cases}
\]

In the sum for \(b_1(n)\) we can replace the condition \(m > 1\) by \(m > U\), since the sum over \(m\) vanishes by (4.5) when \(1 < m \leq U\). Now multiplying the above identity (4.4) by \(f(n)\) we get (4.1).

We will also exploit the following several lemmas. Lemma 4.2 is Lemma 1 of Graham and Pintz [16] (also see Theorem 18 of Vaaler [54]), Lemma 4.3 is well-known, Lemma 4.4 is Lemma 6 of Fouvry and Iwaniec [13], Lemma 4.5 is Lemma 4 of the second paper in [5] (also see (2.1) in Wu [56]), Lemma 4.6 is Lemma 12 of Cao [5].

**Lemma 4.2.** Suppose \(H > 0\). There is a function \(\psi^*(x)\) such that

\[
\psi^*(x) = \sum_{1 \leq |h| \leq H} \gamma(h) e(hx), \quad \gamma(h) \ll \frac{1}{|h|},
\]

\[
|\psi^*(x) - \psi(x)| \leq \frac{1}{2H + 2} \sum_{|h| \leq H} \left( 1 - \frac{|h|}{H} \right) e(hx).
\]

**Lemma 4.3.** Let \(X \neq 0\) and \(\nu \neq 0, 1\). If \((\kappa, \lambda)\) is an exponent pair, then

\[
\sum_{n \sim N} e(Xn^\nu) \ll (XN^{\nu-1})^\kappa N^\lambda + X^{-1}N^{-\nu+1}.
\]

**Lemma 4.4.** Let \(0 < M \leq N < \gamma N \leq \lambda M\), and \(|a_n| \leq 1\). Then we have

\[
\sum_{N < n \leq \gamma N} a_n = \frac{1}{2\pi} \int_{-M}^M \left( \sum_{M < n \leq \lambda M} a_n n^{-it} \right) N^{it} (\gamma^{it} - 1) t^{-1} dt + O(\log(2 + M)).
\]
Lemma 4.5. Let $x \geq 2$, $\alpha, \beta, \gamma$ be given real numbers with $\alpha(\alpha - 1)\beta\gamma \neq 0$, $|a(m)| \leq 1$, $b(n_1, n_2) \leq 1$. Suppose $G = xM^\alpha N_1^\beta N_2^\gamma$, $(\kappa, \lambda)$ is an exponent pair and

$$T(M, N_1, N_2) = \sum_{m \sim M} \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_2} a(m)b(n_1, n_2)e(xm^\alpha n_1^\beta n_2^\gamma).$$

Then

$$T(M, N_1, N_2) \ll \left( G^n M^{1+\lambda+\kappa}(N_1 N_2)^{2+\kappa} \right)^{\frac{1}{2+\kappa}} + M^{\frac{1}{2}} N_1 N_2 + M(N_1 N_2)^{\frac{1}{2}} + G^{-\frac{1}{2}} M N_1 N_2.$$  

Lemma 4.6. Let $x \geq 2$, $\beta, \gamma$ be given real numbers with $\beta \gamma \neq 0, |a(m)| \leq 1$, $|b(n)| \leq 1$, $(\kappa, \lambda)$ is an exponent pair. Suppose $D$ is a subdomain of $\{(m, n) : m \sim M, n \sim N\}$ bounded by finite algebraic curves, $G = xM^\beta N^\gamma$ and

$$T_1(M, N) = \sum_{(m, n) \in D} a(m)b(n)\psi(xm^\beta n^\gamma).$$

Then

$$T_1(M, N) \ll \left( G^n M^{1+\lambda+\kappa} N^{2+\kappa} \right)^{\frac{1}{2+\kappa}} + M^{\frac{1}{2}} N + MN^{\frac{1}{2}} + G^{-\frac{1}{2}} MN.$$  

4.2 An estimate of $\Delta(a; c; x)$

Suppose $1 \leq a < c$ are two fixed integers. In this subsection, we shall estimate the error term $\Delta(a; c; x)$ defined by (1.17) with $k = 1$.

Let $r > 1$ be a fixed real number. The function $\Delta_k(1; r; x)$ is defined on $[1, \infty)$ such that for any $1 \leq y \leq x^{1/r}$ one has

$$\Delta_k(1; r; x) = \sum_{l \leq y} \mu_k(l)\psi\left(\frac{x}{l^r}\right) + O\left(x^{\frac{1}{2} + \varepsilon} y^{\frac{1}{2} - \frac{1}{r} - \varepsilon} + y^{\frac{1}{2} + \varepsilon}\right).$$

It follows easily from Theorem 2 that

$$\Delta_k(a; c; x) = \Delta_k\left(1; \frac{c}{a}; x^{\frac{1}{r}}\right).$$

Hence we only need to estimate $\Delta_k(1; r; x)$ for real $r > 1$.

Now we define

$$\alpha(r) = \begin{cases} 
7 & \text{if } 1 < r \leq 5 \text{ and } r \neq 2, \\
17 & \text{if } r = 2, \\
67 & \text{if } 5 < r \leq 6, \\
11(r - 4) & \text{if } 6 < r \leq 12, \\
12r^2 - 37r - 41 & \text{if } 12 < r \leq 20. 
\end{cases}$$
For \( r > 20 \), \( \alpha(r) \) is defined by the following procedure. Let \( q \geq 2 \) and \( Q = 2^q \). For every \( r > 20 \), there is a unique integer \( q \) such that \( \frac{12Q-5}{q} < r \leq \frac{24Q-6}{q} \). With this value of \( q \), define

\[
\alpha(r) = \frac{(12Q - 1)r - 12Q + 1}{12Qr^2 + (6Qq + 1)r - (6Qq + 12Q + 1)}.
\]

In this subsection we shall prove that

**Theorem 4.** Let \( \alpha(r) \) be defined by (4.8) and (4.9), respectively. If RH holds, then

\[
\Delta(1; r; x) = O\left(x^{\alpha(r) + \varepsilon}\right).
\]

**Remark 4.1.** Certainly one can improve the exponent \( \alpha(r) \) further for some special values of \( r \). For example, R. C. Baker and K. Powell\[^3\] obtained recently that \( \alpha(3) = \frac{17}{54} \), \( \alpha(4) = \frac{17}{84} \), and \( \alpha(5) = \frac{3}{20} \). In addition, for large values of \( r \), one can take \( \alpha(r) = \frac{1}{r+c^*r^{1/3}} \) for some constant \( c^* > 0 \) (see Theorem 2, \[^16\].)

From (4.7) and Theorem 4 we get

**Corollary 4.1.** Let \( 1 \leq a < c \) be two fixed integers. If RH holds, then

\[
\Delta(a; c; x) = O\left(x^{\frac{1}{a}\alpha(\frac{c}{a}) + \varepsilon}\right).
\]

**Proof.** Theorem 4 is proved in Jia\[^22\] for the case \( r = 2 \). S. W. Graham and J. Pintz\[^16\] showed that Theorem 4 holds for any integer \( r > 3 \). However it is easily seen that the argument of \[^16\] can be applied to any \( r \geq 2 \). So we only give a proof of Theorem 4 for \( 1 < r < 2 \).

Taking \( y = x^{\frac{4}{a+c^*}} \), by Theorem 2, (4.6) and a simple splitting argument, an estimate for \( x^{\alpha(r) + \varepsilon} \ll Y \leq y \)

\[
\sum_{Y < l \leq 2Y} \mu(l) \psi \left( \frac{x}{l^r} \right) \ll x^{\alpha(r) + \varepsilon}
\]

would suffice to complete the proof of Theorem 4.

Choose \( U = Y^{\frac{1}{r}} \), \( V = Y^{\frac{1}{2r}} \). Let \( |a(m)| \leq 1 \) and \( |b(n)| \leq 1 \) be any complex-valued arithmetic functions. If we can show the estimates

\[
\sum_{U < m \leq Y/V} a(m) \sum_{Y < mn \leq 2Y} b(n) \psi \left( \frac{x}{mn^{1/r}} \right) \ll x^{\alpha(r) + \varepsilon}
\]

and

\[
\sum_{m \leq U} a(m) \sum_{Y < mn \leq 2Y} \psi \left( \frac{x}{mn^{1/r}} \right) \ll x^{\alpha(r) + \varepsilon},
\]

then (4.11) follows from Lemma 4.1.
We first estimate the type II sum (4.12). Assume $N \ll M$, applying Lemma 4.6 with $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$, we get

\begin{equation}
\mathcal{L}^{-6} \sum_{m \sim M} a(m) \sum_{Y \ll mn \leq 2Y} b(n) \psi \left( \frac{x}{mn^r} \right) \ll \left( \frac{x}{(MN)^{\frac{r}{2}}} \right)^{\frac{1}{2}} M^{\frac{3}{2}} N^{\frac{5}{2}} + x^{\frac{1}{2}} (MN)^{1 + \frac{1}{2}}
\end{equation}

\begin{equation}
\ll (x(MN)^{4-r}(MN)^{\frac{1}{2}})^{\frac{1}{2}} + (MN)N^{\frac{1}{2}} + x^{-\frac{1}{2}} (MN)^{1 + \frac{1}{2}}
\end{equation}

\begin{equation}
\ll x^{\frac{1}{2}} Y^{\frac{1}{2} - \alpha(r)} + YN^{\frac{1}{2}} + x^{-\frac{1}{2}} Y^{1 + \frac{1}{2}} \ll x^{\alpha(r)}.
\end{equation}

Hence (4.14) holds under the condition $N \ll M$. If $N \gg M$, using Lemma 4.4 to separate the dependence between the variables $n$ and $m$, then interchanging the roles of $m$ and $n$, we can show that (4.14) also holds in this case. The estimate (4.12) follows from (4.14) by a simple splitting argument.

Now we turn to estimate the type I sum (4.13). If $M \geq V$, by the same the argument as that of (4.14), we get

\begin{equation}
\mathcal{L}^{-6} \sum_{m \sim M} a(m) \sum_{Y \ll mn \leq 2Y} \psi \left( \frac{x}{mn^r} \right) \ll x^{\alpha(r)}.
\end{equation}

If $M \leq V$, applying Lemma 4.2 with $H = Y^{1-\alpha(r)}$ and Lemma 4.3 with $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$, we get that

\begin{equation}
\sum_{m \leq V} a(m) \sum_{Y \ll mn \leq 2Y} \psi \left( \frac{x}{mn^r} \right) \ll \left( \sum_{m \leq V} \left( \frac{1}{Hm} + \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \left| \sum_{Y/m \ll n \leq 2Y/m} e \left( \frac{hx}{m^r n^r} \right) \right| \right) \right)
\end{equation}

\begin{equation}
\ll x^{\alpha(r)} \mathcal{L} + \sum_{m \leq V} \sum_{1 \leq h \leq H} \frac{1}{h} (hx)^{\frac{1}{2}} Y^{-\frac{1}{2}} + (hx)^{-1} Y^{1+r} \mathcal{L}
\end{equation}

\begin{equation}
\ll x^{\alpha(r)} \mathcal{L} + x^{\frac{1}{2}} Y^{-\frac{1}{2}} H^{\frac{1}{2}} V + x^{-1} Y^{1+r} \mathcal{L}
\end{equation}

\begin{equation}
\ll x^{\alpha(r)} \mathcal{L} + x^{\frac{1}{2} - \alpha(r)} Y^{\frac{1}{2} - \frac{1}{2}} + x^{-1} y^{1+r} \mathcal{L}
\end{equation}

\begin{equation}
\ll x^{\alpha(r)} \mathcal{L}.
\end{equation}

(Here note that if $1.5 \leq r < 2$, we use the bound $Y \gg x^{\alpha(r)}$, otherwise we use $Y \leq y$)

Finally, it follows from (4.15) and (4.16) that (4.13) always holds. This completes the proof of Theorem 4. □
4.3 An estimate of $\Delta_2(a; c; x)$

**Theorem 5.** Let $a, c$ be two fixed integers such that $1 \leq a < c \leq 9a/2$ and $\Delta_2(a; c; x)$ be defined by (1.17) with $k = 2$. Assume that RH holds, then

\[
\Delta_2(a; c; x) \ll x^{7/8 + \varepsilon}.
\]

**Proof.** Similar to the proof of Theorem 4, we only need to show that for $1 < r \leq 9/2$ one has

\[
\Delta_2(1; r; x) = O \left( x^{\beta(r) + \varepsilon} \right), \beta(r) = 7/(8r + 6).
\]

Taking $y = x^{1/r + \varepsilon}$, by Theorem 2, (4.6) and a simple splitting argument, an estimate

\[
\sum_{Y < l \leq 2Y} \mu_2(l)\psi \left( \frac{x}{l} \right) \ll x^{\beta(r) + \varepsilon} \quad (x^{\beta(r) + \varepsilon} \ll Y \leq y)
\]

would suffice to complete the proof of Theorem 5.

Choose $U = Y^{1/2}, V = Y^{1/4}$. Let $|a_1(m)| \leq 1$ and $|b_1(n)| \leq 1$ be any complex-valued arithmetic functions. If we can show

\[
\sum_{U < m \leq Y/V} a_1(m) \sum_{Y < mn \leq 2Y} b_1(n)\psi \left( \frac{x}{mn} \right) \ll x^{\beta(r) + \varepsilon}
\]

and

\[
\sum_{m \leq U} a_1(m) \sum_{Y < mn \leq 2Y} \tau(n)\psi \left( \frac{x}{mn} \right) \ll x^{\beta(r) + \varepsilon},
\]

then (4.19) follows from Lemma 4.1.

The estimate (4.20) can be proved by the same approach of (4.12), so we omit its details. Hence we only need to prove (4.21). From (4.20) we get easily that

\[
\sum_{Y^{1/4} < m \leq U} a_1(m) \sum_{Y < mn \leq 2Y} \tau(n)\psi \left( \frac{x}{mn} \right) \ll x^{\beta(r) + \varepsilon}.
\]

So it suffices for us to prove

\[
\sum_{m \leq V} a_1(m) \sum_{Y < mn \leq 2Y} \tau(n)\psi \left( \frac{x}{mn} \right) \ll x^{\beta(r) + \varepsilon}.
\]

Let $1 \leq M \leq V$. Now we are in a position to estimate the exponential sum

\[
S_r(M, Y) := \sum_{M < m \leq 2M} a_1(m) \sum_{Y < mn_1n_2 \leq 2Y} \psi \left( \frac{x}{mn_1n_2} \right).
\]
Without the loss of generality, we suppose \( n_1 \ll n_2 \), hence \( n_1 \ll (YM^{-1})^{\frac{3}{2}} \). Applying a simple splitting argument, we have for some \( N_1 \ll (YM^{-1})^{\frac{3}{2}} \)

\[
\mathcal{L}^{-1} S_r(M, Y) \ll \sum_{M \leq m \leq 2M} a_1(m) \sum_{N_1 < n_1 \leq 2N_1} \sum_{Y < mn_1n_2 \leq 2Y} \psi \left( \frac{x}{mn_1n_2} \right) .
\]

Now we consider two cases.

(Case i): \( \frac{1}{2} \leq N_1 \ll VM^{-1} \). In this case, applying Lemma 4.2 with \( H = Yx^{-\beta(r)} \) and Lemma 4.3 with \( (\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2}) \), similar to the estimate of (4.16), we can obtain

\[
S_r(M, Y) \ll x^{\beta(r)} \mathcal{L}^3 .
\]

(Case ii): \( VM^{-1} \leq N_1 \ll (YM^{-1})^{\frac{3}{2}} \). Applying Lemma 4.4 to separate the dependence between the variable \( n_2 \) and the variables \( m, n_1 \), we get for \( N_2 = \frac{Y}{MN_1} \) that

\[
\sum_{M \leq m \leq 2M} a_1(m) \sum_{N_1 < n_1 \leq 2N_1} \sum_{Y < mn_1n_2 \leq 2Y} \psi \left( \frac{x}{mn_1n_2} \right) \\
= \frac{1}{2\pi} \left( \int_{-N_2}^{N_2} \sum_{M \leq m \leq 2M} a_1(m) \sum_{N_1 < n_1 \leq 2N_1} \sum_{n_2 < n_2 \leq 2N_2} n_2^{-it} \psi \left( \frac{x}{mn_1n_2} \right) N_2^t(s^t - 1)t^{-1} \ dt + O(MN_1\mathcal{L}) \right) \\
= \frac{1}{2\pi} \left( \int_{MN_1 < d \leq 2MN_1} \sum_{N_1 < n_1 \leq 2N_1} \sum_{N_2 < n_2 \leq 2N_2} c(d)n_2^{-it} \psi \left( \frac{x}{d^2n_2^2} \right) N_2^t(s^t - 1)t^{-1} \ dt + O(MN_1\mathcal{L}) \right),
\]

where

\[
c(d) = \sum_{d = mn_1, M < m \leq 2M, N_1 < n_1 \leq 2N_1} a_1(m) \ll d^\varepsilon.
\]

If \( N_2 \gg MN_1 \), applying Lemma 4.6 with \( (M, N) = (N_2, MN_1) \) and \( (\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2}) \) to estimate the inner sum in the above expression, we get(similar to (4.14))

\[
x^{-\frac{4}{3}} \sum_{MN_1 < d \leq 4MN_1} \sum_{N_2 < n_2 \leq 2N_2} c(d)n_2^{-it} \psi \left( \frac{x}{d^2n_2^2} \right) \\
\ll \left( \frac{x}{Y} \right)^{\frac{1}{3}} N_2^{\frac{1}{2}} (MN_1)^{\frac{1}{2}} + N_2^{\frac{1}{2}} (MN_1) + N_2 (MN_1)^{\frac{1}{3}} + \left( \frac{x}{Y} \right)^{-\frac{1}{3}} MN_1 N_2 \\
\ll \left( x^{\frac{1}{3}} Y^{-\frac{1}{2}} (MN_1 N_2)^{\frac{1}{2}} \right)^{\frac{1}{2}} + (MN_1 N_2)^{\frac{1}{2}} (YM)^{\frac{1}{3}} + (MN_1 N_2)(MN_1)^{-\frac{1}{3}} + x^{-\frac{1}{3}} Y^{\frac{1}{2}+1} \\
\ll \left( x^{\frac{1}{3}} Y^{\frac{1}{2}+\varepsilon} \right)^{\frac{1}{2}} + Y^{\frac{1}{2}} (YV)^{\frac{1}{3}} + YV^{-\frac{1}{3}} + x^{-\frac{1}{3}} Y^{\frac{1}{2}+1} \ll x^{\beta(r)}.
\]
If $N_2 \ll MN_1$, using the same approach but with $(M, N) = (MN_1, N_2)$ in Lemma 4.6 we get that (4.27) still holds.

Combining (4.26) and (4.27), we obtain that

$$S_r(M, Y) \ll x^{\beta(r) + \varepsilon}$$

holds in the Case ii.

The estimate (4.22) now follows from the proofs of the above two cases. 

4.4 An application of Theorem 2

The exponential convolution (e-convolution) was introduced by M. V. Subbarao [45]. Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} \cdots p_s^{a_s}$. An integer $d$ is called an exponential divisor (e-divisor) of $n$ if $d = p_1^{b_1} \cdots p_s^{b_s}$, where $b_1 | a_1, \ldots, b_s | a_s$. Let $\tau^{(e)}(n)$ denote the number of exponential divisors of $n$, which is called the exponential divisor function. Let $r \geq 2$ be a fixed integer. The integer $n > 1$ is called exponentially $r$-free (e-$r$-free) if all the exponents $a_1, \ldots, a_s$ are $r$-free. Let $q^{(e)}_r(n)$ denote the characteristic function of the set of e-$r$-free integers. The e-unitary convolution was introduced by N. Minculete and L. Tóth [30]. The function $I(n) = 1(n \geq 1)$ has inverses with respect to e-convolution and e-unitary convolution denoted by $\mu^{(e)}(n)$ and $\mu^{(e)*}(n)$, respectively. These are the unitary and exponential analogues of the Möbius function. These arithmetic functions attract the interests of many authors, see for example [7, 17, 23, 24, 28, 35, 39, 40, 42, 43, 51, 52, 53].

L. Tóth [52] showed that the Dirichlet series of $\mu^{(e)}(n)$ is of the form

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} W_1(s), \Re s > 1,$$

where $W_1(s) := \sum_{n=1}^{\infty} \frac{w_1(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{2}$.

Let

$$\Delta_{\mu^{(e)}}(x) := \sum_{n \leq x} \mu^{(e)}(n) - A_1 x,$$

where

$$A_1 := m(\mu^{(e)}) = \frac{W_1(1)}{\zeta^2(2)} = \prod_p \left( 1 + \sum_{n=2}^{\infty} \frac{\mu(n) - \mu(n-1)}{p^n} \right).$$

L. Tóth [52] showed $\Delta_{\mu^{(e)}}(x) = O \left( x^{\frac{91}{202} + \varepsilon} \right)$ under RH. The exponent $\frac{91}{202}$ was improved to $\frac{37}{94}$ by X. Cao and W. Zhai [7].

Similarly, N. Minculete and L. Tóth [30] showed that the Dirichlet series of $\mu^{(e)*}$ is of the form

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)*}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} W_2(s), \Re s > 1,$$

where

$$W_2(s) := \sum_{n=1}^{\infty} \frac{w_2(n)}{n^s}$$

is absolutely convergent for $\Re s > \frac{1}{2}$. 

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where $W_2(s) := \sum_{n=1}^{\infty} \frac{w_2(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$. Let
\begin{equation}
\Delta_{\mu(e)^*}(x) := \sum_{n \leq x} \mu(e)^*(n) - A_2 x,
\end{equation}
where
\begin{equation}
A_2 := m(\mu(e)^*) = \frac{W_2(1)}{\zeta^2(2)} = \prod_p \left( 1 + \sum_{n=2}^{\infty} \frac{(-1)^{\omega(n)} - (-1)^{\omega(n-1)}}{p^n} \right).
\end{equation}

Under RH, the estimate $\Delta_{\mu(e)^*}(x) = O \left( x^{\frac{7}{22} + \varepsilon} \right)$ was proved in [30].

As an application of Theorem 2 and Theorem 5, from (4.29) and (4.31) we get the following

**Theorem 6.** Let $\Delta_{\mu(e)}(x)$ and $\Delta_{\mu(e)^*}(x)$ be defined by (4.30) and (4.32), respectively. If RH is true, then we have
\begin{equation}
\Delta_{\mu(e)}(x) = O \left( x^{\frac{7}{22} + \varepsilon} \right)
\end{equation}
and
\begin{equation}
\Delta_{\mu(e)^*}(x) = O \left( x^{\frac{7}{22} + \varepsilon} \right).
\end{equation}

For comparison, we have numerically that
\begin{align*}
\frac{91}{202} &= 0.45049 \ldots, \quad \frac{37}{94} = 0.39361 \ldots, \quad \frac{7}{22} = 0.31818 \ldots.
\end{align*}

5 Some applications of Corollary 1.1 and Corollary 1.3

5.1 The distribution of generalized square-full integers

In 1963, E. Cohen [8] generalized square-full integers in the following way: Let $a$ and $b$ are fixed positive integers. Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} \cdots p_r^{a_r}$ and $R_{a,b}$ denote the set of all $n$ such that each exponent $a_i (1 \leq i \leq r)$ is either a multiple of $a$ or is contained in the progression $at + b(t \geq 0)$. Obviously $R_{2,3}$ is the set of square-full integers. Let $a \nmid b$, $f_{(a,b)}(n)$ denote the characteristic function of the set $R_{a,b}$. By Lemma 2.1 in E. Cohen [8] one has
\begin{equation}
\sum_{n=1}^{\infty} f_{(a,b)}(n) n^{-s} = \frac{\zeta(as)\zeta(bs)}{\zeta(2bs)}, \Re s > 1.
\end{equation}

We are interested in the summatory function of $f_{a,b}(n)$.

First consider the case $a < b$. Suppose also that $a \nmid b$. In this case the problem is closely related to the estimate of $\Delta(a, b; x)$. We take $a = a, b = b, c = 2b$ and $k = 1$ in (1.1),
then the estimate (1.4) implies \( \Delta(a, b; 2b; x) = \Omega(x^{\frac{\alpha(a, b)}{2}}) \). Suppose \( \Delta(a, b; x) \ll x^{\alpha(a, b) + \varepsilon} \) such that \( \alpha(a, b) < 1/2b \). By Corollary 1.1 with \( (a, b, c, k) = (a, b, 2b, 1) \), under the RH we have the asymptotic formula

\[
(5.2) \quad \sum_{n \leq x} f(a, b)(n) = \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + O \left( x^{\frac{1 - \alpha(a, b)}{2a - 4b + \varepsilon}} \right),
\]

which improves Theorem 3.2 of [48].

The distribution of square-full integers (the case \( a = 2, b = 3 \)) has received special attention. In this special case, the error term in (5.2) becomes \( (x^{11/72 + \varepsilon}) \), which was first proved in [32]. The exponent 11/72 was improved by several authors. The best known result is due to Wu [56], who obtained exponent \( \frac{12}{58} = 0.2105 \) (also subject to the RH).

Now we suppose \( b < a \). From (1.4) we get \( \Delta(b, a; 2b; x) = \Omega(x^{\frac{b}{a}}) \). So without the loss of generality, we always suppose \( b < a < 4b \).

If \( b < a < 2b \). Applying Corollary 1.1 with \( (a, b, c, k) = (b, a, 2b, 1) \), under the RH one has

\[
(5.3) \quad \sum_{n \leq x} f(a, b)(n) = \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + O \left( x^{\frac{1 - \alpha(b, a)}{2b - 4a + \varepsilon}} \right),
\]

which improves Theorem 3.4 of [48].

Finally look at the case \( 2b < a < 4b \). In this case, D. Suryanarayana [48] proved that (see Remark 3.3 therein)

\[
(5.4) \quad \sum_{n \leq x} f(a, b)(n) = \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + O \left( x^{\frac{\varepsilon}{2b}} \right)
\]

unconditionally. If RH is true, then Remark 3.4 of D. Suryanarayana [48] claimed that

\[
(5.5) \quad \sum_{n \leq x} f(a, b)(n) = \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + O \left( x^{\frac{2a - b}{5ab - 4b^2} \omega(x)} \right),
\]

where \( \omega(x) = \exp \{ 4 \log x \log \log x \} \), \( A \) is a positive absolute constant. Here we note that on the right-hand side of (5.5) the second main term \( \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} \) is absorbed into the error term.

Applying Corollary 1.3 with \( (a, b, c, k) = (2, a, 2b, 1) \) and \( \Delta(b; 2b; x) \ll x^{\frac{17}{54} + \varepsilon} \) in Corollary 4.1, we have under RH that

\[
(5.6) \quad \sum_{n \leq x} f(a, b)(n) = \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} + O \left( x^{\frac{54}{37a + 54b + \varepsilon}} \right),
\]

which took the second main term \( \frac{\zeta(b)}{\zeta(2)} x^{\frac{b}{a}} \) out of the error term in (5.4) when \( 2b < a < 54b/17 = 3.176 \cdots b \).
5.2 On the order of the error function of the \((l, r)-\)integers

For given integers \(l, r\) with \(1 < r < l\), we say an integer \(n\) is a \((l, r)-\)integers if it has the form \(m'n\) where \(m, n\) are integers and \(n\) is \(r\)-free. The definition of the \((l, r)-\)integers was introduced by M. V. Subbarao and V. C. Harris\[44\]. Let \(g_{l,r}(n)\) denote the characteristic function of the set of \((l, r)-\)integers. By Lemma 2.6 in M. V. Subbarao and D. Suryanarayana\[46\] we have

\[
\sum_{n=1}^{\infty} \frac{g_{l,r}(n)}{n^s} = \frac{\zeta(s)\zeta(ls)}{\zeta(rs)}, \Re s > 1.
\]

(5.7)

Hence \(g_{l,r}(n) = f_{1,l,r}(n)\). We define the error term by

\[
\Delta(1, l; r; x) := \sum_{n \leq x} g_{l,r}(n) - \frac{\zeta(l)}{\zeta(r)} x - \frac{\zeta(\frac{l}{r})}{\zeta(\frac{l}{r})} x^{\frac{l}{r}}.
\]

From (1.4) we have \(\Delta(1, l; r; x) = \Omega(x^{\frac{l}{rs}})\).

If \(l \geq 2\), the distribution of \((l, r)-\)integers is almost the same as the distribution of \(r\)-free numbers. From Theorem 4 we get under RH that

\[
\sum_{n \leq x} g_{l,r}(n) = \zeta(l) \zeta(r)x + \zeta(\frac{l}{r}) x^{\frac{l}{r}} + O\left(x^{\alpha(r)} + \varepsilon\right),
\]

(5.8)

where \(\alpha(r)\) is defined by (4.8) and (4.9). In particular, if \(l \geq 4\) we have \(\Delta(1, l; 2; x) = O(x^{\frac{l}{rs}} + \varepsilon}\).

If \(r < l < 2r\) and \(\alpha(r) \geq 1/l\) we get that (5.8) holds too. However, if \(\alpha(r) < 1/l\), by Corollary 1.3 with \((a, b, c, k) = (1, l, r, 1)\), we get under RH that

\[
\sum_{n \leq x} g_{l,r}(n) = \frac{\zeta(l)}{\zeta(r)} x + \frac{\zeta(\frac{l}{r})}{\zeta(\frac{l}{r})} x^{\frac{l}{r}} + O\left(x^{\frac{1}{l\alpha(r)}} + \varepsilon\right).
\]

(5.9)

In particular, \(\Delta(1, 3; 2; x) = O(x^{\frac{15}{35}} + \varepsilon}\).

The previously best known error term is due to M. V. Subbarao and D. Suryanarayana\[47\].

Since \(\frac{15}{35} = 0.32727 \cdots < \frac{1}{3}\), (5.8) and (5.9) answer a conjecture of M. V. Subbarao and D. Suryanarayana in \[16\] (see page 123) for the special case \(r = 2\). It should be noted that if \(2 \leq r \leq 10, l = r + 1\), we get the second main term; if \(r \geq 11, r < l = r + 2\), we also get the second main term. Hence (5.9) is an substantial improvement to theirs.

5.3 The distribution of e-\(r\)-free integers

In this subsection we consider the distribution of e-\(r\)-free integers. For the distribution of e-square-free integers, J. Wu\[55\] showed that \(\Delta_{q_e}(x) = O\left(x^{\frac{1}{7}}\delta(x)\right)\), improving an earlier result of M. V. Subbarao\[15\]. In the general case, L. Tóth\[71\] obtained that \(\Delta_{q_e}(x) = \ldots \)
where \( \frac{1}{\alpha} \). Under RH, X. Cao and W. Zhai\(^7\) showed that \( \Delta_{q_2}(x) = O(x^{\frac{38}{193} + \varepsilon}) \), improving the exponent \( \frac{1}{8} \) of L. Tóth\(^52\). In this subsection we shall study this topic more carefully.

**Theorem 7.** (i) The Dirichlet series of \( q_r^{(e)} \) is of form

\[
Q_r^{(e)}(s) := \sum_{n=1}^{\infty} \frac{q_r^{(e)}(n)}{n^s} = \frac{\zeta(s) \zeta((2^r + 1)s)}{\zeta(2^r s) \zeta(2^r + 1 s)} U_r(s), \quad \Re s > 1,
\]

where \( U_r(s) \) is absolutely convergent for \( \Re s > \frac{1}{2^r + 1} \).

(ii) Let \( \Delta_{q_r^{(e)}}(x) := \sum_{n \leq x} q_r^{(e)}(n) - C_1(r) x - C_2(r) x^{\frac{1}{2^r + 1}} \), where

\[
C_1(r) := \frac{\zeta(2^r + 1) U_r(1)}{\zeta(2^r) \zeta(2^r + 1)}, \quad C_2(r) := \frac{\zeta(\frac{1}{2^r + 1}) U_r(\frac{1}{2^r + 1})}{\zeta(\frac{1}{2^r + 1}) \zeta(\frac{2^r + 1}{2^r + 1})}.
\]

Then

\[
\Delta_{q_r^{(e)}}(x) = O \left( x^{\frac{1}{2^r + 1}} \right).
\]

(iii) If the RH is true, we have

\[
\Delta_{q_r^{(e)}}(x) = O \left( x^{\frac{38}{193} + \varepsilon} \right),
\]

where \( \alpha(r) \) is defined by (4.8) and (4.9). In particular, \( \Delta_{q_2^{(e)}}(x) = O(x^{\frac{38}{193} + \varepsilon}) \), here \( \frac{38}{193} = 0.1968 \cdots < \frac{1}{8} \).

**Remark 5.1.** Note that we always have \( \frac{1}{2^r + 2 - (2^r + 1)\alpha(2^r)} < \frac{1}{2^r + 1} \). In addition when \( r = 2 \), if we use the new estimate \( \alpha(4) = \frac{17}{94} \) proved by R. C. Baker and K. Powell\(^3\) recently, one can slightly improve the above result to \( \Delta_{q_2^{(e)}}(x) = O(x^{\frac{34}{193} + \varepsilon}) \). In section 6 we shall improve the exponent \( \frac{34}{193} \) further by the exponential sum method.

**Proof.** Since the function \( q_r^{(e)} \) is multiplicative and \( q_r^{(e)}(p^a) = q_r(\alpha) \) for every prime power \( p^a \). For \( r \geq 2 \), it is easy to verify that \( q_r^{(e)}(p) = q_r^{(e)}(p^2) = \cdots = q_r^{(e)}(p^{2r-1}) = 1, q_r^{(e)}(p^{2r}) = 0, q_r^{(e)}(p^{2r+1}) = \cdots = q_r^{(e)}(p^{2r+1}) = 1, \) and \( q_r^{(e)}(p^{2r+1}) = 0 \). Hence for \( \Re s > 1 \)

\[
\sum_{n=1}^{\infty} \frac{q_r^{(e)}(n)}{n^s} = \prod_p \left( 1 + \sum_{m=1}^{\infty} \frac{q_r(m)}{p^{ms}} \right)
\]
Applying the product representation of Riemann zeta-function
\[
\zeta(s) = \prod_p (1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots) = \prod_p (1 - p^{-s})^{-1}, \Re s > 1,
\]
we have for \(\Re s > 1\)
\[
(5.16) \quad \zeta(s)\zeta((2^r + 1)s) = \prod_p \left( (1 - p^{-s})(1 - p^{-(2^r+1)s}) \right)^{-1}.
\]

Let
\[
(5.17) \quad f_{q^{(s)}}(z) = 1 + \sum_{m=1}^{\infty} q_r(m)z^m
\]
\[
= 1 + z + \cdots + z^{2^r-1} + z^{2^r+1} + \cdots + z^{2^r+1} + \sum_{m=2^{r+1}+1}^{\infty} q_r(m)z^m.
\]

By a simple calculation one get for \(|z| < 1\)
\[
(1 - z)(1 - z^{2^r+1}) = 1 - z - z^{2^r+1} + z^{2^r+2},
\]
\[
f_{q^{(s)}}(z)(1 - z)(1 - z^{2^r+1}) = 1 - z^{2^r} - z^{2^r+1} + \sum_{m=2^{r+1}+1}^{\infty} c_m z^m,
\]
and
\[
\left(1 + z^{2^r} + z^{3(2^r)} + z^{4(2^r)} + \cdots\right)\left(1 + z^{2^r+1} + z^{2(2^r+1)} + z^{3(2^{r+1})} + \cdots\right) = 1 + z^{2^r} + 2z^{2(2^r)} + 2z^{3(2^r)} + \cdots.
\]

From the above two relations, we easily obtain for \(|z| < 1\)
\[
(5.18) \quad f_{q^{(s)}}(z)(1 - z)(1 - z^{2^r+1})(1 - z^{2^r})^{-1}(1 - z^{2^r+1})^{-1}
\]
\[
= 1 + \sum_{m=2^{r+1}+1}^{\infty} C_m z^m.
\]

Taking \(z = p^{-s}\) in (5.18), then combining (5.15), (5.16) and (5.17) completes the proof of (5.10) in Theorem 7.

Applying Theorem 2 of M. Küleitner and W. G. Nowak[26] and (5.10), we immediately get (5.13). By Corollary 1.3 with \((a, b, c, k) = (1, 2^r + 1, 2^r, 1)\) and Theorem 4, we obtain
\[
(5.19) \quad \Delta(1, 2^r + 1; 2^r; x) = O \left( x^{\frac{1}{2r+2-(2^r+1)(2^r)+\varepsilon}} \right).
\]

Now (5.14) follows from (5.10) , (5.19) and a simple convolution argument at once, and this completes the proof of Theorem 7. \(\square\)
5.4 The divisor problem over the set of \( r \)-free numbers

Let \( r \geq 2 \) be a fixed integer. Winfried Recknagel\[36\] and Hailiang Fen\[12\] investigated the divisor problem over the set of \( r \)-free numbers. Hailiang Fen\[12\] showed that

\[
Ψ_r(s) := \sum_{n=1}^{\infty} \frac{\tau(n)q_r(n)}{n^s} = \frac{\zeta^2(s)}{\zeta^{r+1}(rs)} V_r(s), \quad \Re s > 1,
\]

where \( V_r(s) \) is absolutely convergent for \( \Re s > \frac{1}{r+1} \). In particular, \( V_2(s) = \zeta^2(3s)W_2(s) \), \( W_2(s) \) is absolutely convergent for \( \Re s > \frac{1}{4} \).

Let

\[
\sum_{n \leq x} \tau(n)q_r(n) = \text{Res}_{s=1} \Psi_r(s)x^s + \Delta^r(x).
\]

Hence this problem is reduced to estimate the error term \( \Delta_{r+1}(1, 1, r; x) \). Hailiang Fen\[12\] showed that

\[
\Delta_{r+1}(1, 1, r; x) = \begin{cases} 
O(x^{1/r\delta(x)}), & \text{if } r = 2, 3 \\
\Delta_{r+1}(1, 1, r; x) = O(x^{\frac{111}{418}} (\log x)^{\frac{26947}{8320}}), & \text{if } r \geq 4,
\end{cases}
\]

where the second estimate in (5.22) follows from M. N. Huxley’s bound (see \[19\]).

Applying Corollary 1.1 with \((a, b, c, k) = (1, 1, 2, 3)\) and \((a, b, c, k) = (1, 1, 3, 4)\) respectively, and with the help of (5.23), we obtain under RH that

\[
\Delta_3(1, 1; 2; x) = O(x^{\frac{285}{724} + \varepsilon}), \quad \Delta_4(1, 1; 3; x) = O(x^{\frac{285}{878} + \varepsilon}).
\]

Finally, from (5.20) and (5.24) we get immediately the following

**Theorem 8.** If RH is true, then

\[
\Delta_{\tau(2)}(x) = O(x^{\frac{285}{724} + \varepsilon}), \quad \Delta_{\tau(3)}(x) = O(x^{\frac{285}{878} + \varepsilon}).
\]

6 The distribution of e-square-free integers

In this section we shall use the method of exponential sums, and \( \alpha(4) = \frac{17}{96} \) (recall Remark 4.1 in Section 4) proved by R. C. Baker and K. Powell\[3\] to prove

**Theorem 9.** Let \( \Delta_{q_2}(x) \) be defined by (5.11). Assume that RH holds, then

\[
\Delta_{q_2}(x) = O\left(x^{\frac{29}{724} + \varepsilon}\right).
\]
Remark 6.1. For comparison, we have $\frac{23}{124} = 0.18548\cdots$ and $\frac{94}{479} = 0.19624\cdots$.

In the proof of Theorem 9 we need the following lemma (see Lemma 6.9 of Krätzel[25]).

**Lemma 6.1.** Let $1 \leq \frac{1}{Z} < Z_1 \leq \frac{1}{x}$, $a > 0, b > 0$. If $(\kappa, \lambda)$ is any exponent pair and if $$(2\lambda - 1)a > 2\kappa b, \quad (2\lambda - 1)b > 2\kappa a,$$

then

$$\sum_{Z \leq n \leq Z_1} \psi \left( \left( \frac{x}{n^a} \right)^{\frac{1}{b}} \right) \ll x^{\frac{2(\kappa + \lambda - 1)}{a + b}} \log x + Z_1 \left( \frac{Z_1^n}{x} \right)^{\frac{1}{b}} \log x.$$ 

**Proof of Theorem 9.** Similar to the proof of Theorem 7, we need only to prove

$$(6.2) \quad \Delta(1, 5; 4; x) = O\left( x^{\frac{23}{124} + \varepsilon} \right).$$

Let $\alpha = \frac{23}{124}$. Applying Theorem 3 with $(a, b, c, k) = (1, 5, 4, 1)$ and $y = x^{\frac{101}{544}}$, the following two estimates

$$(6.3) \quad \sum_{d \leq y} \Delta(1, 4; \frac{x}{d^5}) \ll x^{\alpha + \varepsilon}$$

$$(6.4) \quad \sum_{m \leq \frac{x}{y}} u_1(1; 4; m) \psi \left( \left( \frac{x}{m} \right)^{\frac{1}{4}} \right) \ll x^{\alpha + \varepsilon}$$

would suffice to finish the proof of Theorem 9.

We first estimate the sum in (6.3). Let $y_1 = x^{\frac{1}{157}}$, we split the sum in (6.3) into two parts and write

$$(6.5) \quad \sum_{d \leq y} \Delta(1, 4; \frac{x}{d^5}) = \sum_{d \leq y_1} \Delta(1, 4; \frac{x}{d^5}) + \sum_{y_1 < d \leq y} \Delta(1, 4; \frac{x}{d^5}) := S_1 + S_2.$$ 

Clearly, it follows from $\alpha(4) = \frac{17}{94}$ that

$$(6.6) \quad S_1 \ll \sum_{d \leq y_1} \left( \frac{x}{d^5} \right)^{\frac{17}{94} + \varepsilon} \ll x^{\frac{17}{94} + \varepsilon} y_1^{\frac{94}{17}} \ll x^{\alpha + \varepsilon}.$$ 

To estimate $S_2$, we discuss two cases.

**Case (i)** $y_1 \ll D \leq y_2 = y_1^{\frac{2}{3} D^{-\frac{3}{4}}}$. Applying Theorem 2 with $Y_\ast = x^{\frac{1}{157} D^{-\frac{3}{4}}}$ and a simple splitting argument, we have for some $1 \ll N \ll Y_\ast$

$$(6.7) \quad \sum_{d \sim D} \Delta(1, 4; \frac{x}{d^5}) = \sum_{d \sim D} \left( \sum_{n \leq Y_\ast} \mu(n) \psi \left( \frac{x}{d^5 n^4} \right) + O \left( \left( \frac{x}{d^5} \right)^{\frac{1}{2} + \varepsilon} Y_\ast^{-\frac{3}{8}} + Y_\ast^{\frac{3}{8} + \varepsilon} \right) \right)$$

$$\ll L \left| \sum_{d \sim D} \sum_{n \sim N} \mu(n) \psi \left( \frac{x}{d^5 n^4} \right) \right| + x^{\frac{1}{2} + \varepsilon} D^{-\frac{3}{4}} + x^{\frac{3}{8} + \varepsilon} D^\frac{3}{8}$$

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Applying Lemma 4.6 with \((\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})\) and \((M, N) = (N, D)\), one get

\[
\mathcal{L}^{-6} \sum_{d \sim D} \sum_{n \sim N} \mu(n) \psi \left( \frac{x}{d^n n^4} \right) \\
\ll \left( \left( \frac{x}{D^5 N^4} \right)^{\kappa} N^{1+\kappa+\lambda} D^{2+\kappa} \right)^{\frac{1}{1+2\kappa}} + D^{\frac{7}{2}} N + DN^{\frac{1}{2}} + x^{-\frac{1}{2}} D^2 N^3 \\
\ll x^\frac{1}{2} + x^\frac{1}{2} D^{-\frac{3}{10}} + x^\frac{1}{10} D^\frac{3}{2} + x^\frac{1}{10} D^\frac{11}{20}
\ll x^\frac{3}{10} y_1 - \frac{1}{10} + x^\frac{1}{10} y_2^{\frac{11}{20}} \ll x^{\alpha+\varepsilon}.
\]

Combining (6.7) and (6.8), in this case one obtain

\[
\sum_{d \sim D} \Delta(1, 4; \frac{x}{d^{30}}) \ll x^{\alpha+\varepsilon}.
\]

**Case (ii)** \(y_2 = y_1^{\frac{34}{109}} \ll D \leq y\). Applying Theorem 2 with \(Y_* = x^{\frac{13}{109}} D^{-1}\), and a simple splitting argument, we have for some \(1 \ll N \ll Y_*\)

\[
\sum_{d \sim D} \Delta(1, 4; \frac{x}{d^{30}}) = \sum_{d \sim D} \sum_{n \sim N} \mu(n) \psi \left( \frac{x}{d^n n^4} \right) + O \left( \left( \frac{x}{D^5} \right)^{\frac{1}{2}+\varepsilon} Y_* N^{\frac{1}{2}} + Y_* \frac{1}{2}+\varepsilon \right)
\ll \mathcal{L} \left| \sum_{d \sim D} \sum_{n \sim N} \mu(n) \psi \left( \frac{x}{d^n n^4} \right) \right| + x^{\alpha+\varepsilon} + x^{\frac{13}{109}+\varepsilon} D^\frac{11}{20}.
\]

Now applying lemma 4.2 with \(H = DN x^{-\alpha}\) and Lemma 4.3 with \((\kappa, \lambda) = BA^2 BA(\frac{1}{5}, \frac{4}{15}) = (\frac{13}{50}, \frac{22}{40})\), we easily obtain that

\[
\sum_{d \sim D} \sum_{n \sim N} \mu(n) \psi \left( \frac{x}{d^n n^4} \right) \ll \sum_{n \sim N} \left| \sum_{d \sim D} \psi \left( \frac{x}{d^n n^4} \right) \right|
\ll \sum_{n \sim N} \left( \frac{D}{H} + \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \left| \sum_{d \sim D} e \left( \frac{hx}{d^n n^4} \right) \right| \right)
\ll x^{\alpha} + \sum_{n \sim N} \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \left( \left( \frac{hx}{n^4 D^6} \right)^{\kappa} D^\lambda + \left( \frac{hx}{n^4 D^6} \right)^{-1} \right)
\ll x^{\alpha} + \left( \frac{x^{1-\alpha}}{N^4 D^5} \right)^{\kappa} D^\lambda N + x^{-1} N^5 D^6
\ll x^{\alpha} + x^{\frac{1}{15} + (\frac{22}{40} - \alpha) \kappa} D^{\lambda-2\kappa-1} + x^\frac{3}{10}
\ll x^{\alpha} + x^{\frac{1339}{9900} y_2^{\frac{11}{20}} + \frac{7}{10} y} \ll x^{\alpha}.
\]

(Here we use \(1 - 3\kappa > 0\).) Combining (6.10) and (6.11), one also has

\[
\sum_{d \sim D} \Delta(1, 4; \frac{x}{d^{30}}) \ll x^{\alpha+\varepsilon}.
\]

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Combining the above two cases, then using a simple splitting argument we obtain

\[(6.13)\]

\[S_2 \ll x^{\alpha + \varepsilon} \cdot \]

Hence (6.3) follows from (6.5), (6.6) and (6.13).

Now we turn to prove (6.4). From (1.14), applying the Drichlet’s hyperbolic argument, we have for any \(1 < Z < \frac{x}{y^3} = Y\)

\[(6.14)\]

\[\sum_{m \leq \frac{x}{y^3}} u_1(1; 4; m) \psi \left( \frac{x}{m} \right)^{\frac{1}{2}} = \sum_{m_1 m_2 \leq Y} \mu(m_2) \psi \left( \frac{x}{m_1 m_2} \right)^{\frac{1}{2}} \]

\[= \sum_{m_1 \leq Z} \sum_{m_2 \leq \frac{Y}{m_1}} \mu(m_2) \psi \left( \frac{x}{m_1 m_2} \right)^{\frac{1}{2}} + \sum_{m_2 \leq \left( \frac{Y}{Z} \right)^{\frac{1}{2}}} \mu(m_2) \sum_{Z < m_1 \leq \frac{Y}{m_2}} \psi \left( \frac{x}{m_1 m_2} \right)^{\frac{1}{2}} \cdot \]

Applying Lemma 6.1 with \((\kappa, \lambda) = A^2 BA \left( \frac{1}{3}, \frac{4}{5} \right) = (\frac{2}{3}, \frac{33}{40})\) and \((a, b, Z_1) = (1, 5, \frac{Y}{m_2})\), we have

\[(6.15)\]

\[\sum_{Z < m_1 \leq \frac{Y}{m_2}} \psi \left( \frac{x}{m_1 m_2} \right)^{\frac{1}{2}} \ll \left( \frac{x}{m_2} \right)^{\frac{1}{2}} L + \frac{Y}{m_2} \left( \frac{Y}{x} \right)^{\frac{1}{2}} L \cdot \]

On taking \(Z = x^{\frac{1}{3}} Y^{-\frac{1}{3}}\), it follows from (6.14) and (6.15)

\[(6.16)\]

\[\sum_{m \leq \frac{x}{y^3}} u_1(1; 4; m) \psi \left( \frac{x}{m} \right)^{\frac{1}{2}} \ll \sum_{m_1 \leq Z} \left( \frac{Y}{m_1} \right)^{\frac{1}{2}} + \sum_{m_2 \leq \left( \frac{Y}{Z} \right)^{\frac{1}{2}}} \left( \frac{x}{m_2} \right)^{\frac{1}{2}} + \frac{Y}{m_2} \left( \frac{Y}{x} \right)^{\frac{1}{2}} L \]

\[\ll Y^{\frac{1}{3}} Z^{\frac{1}{3}} + x^{\frac{1}{3}} Y^{\frac{1}{3}} Z^{-\frac{1}{3}} L + Y^{\frac{1}{3}} x^{-\frac{1}{3}} L \ll x^{\frac{1}{3}} Y^{y^{\frac{1}{3}}} L + x y^{-6} L \ll x^{\alpha} L^2. \]

This completes the proof of Theorem 9. \(\square\)

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