Limit cycle bifurcations from a nilpotent focus or center of planar systems∗

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Abstract: In this paper, we study the analytical property of the Poincaré return map and the generalized focal values of an analytical planar system with a nilpotent focus or center. Then we use the focal values and the map to study the number of limit cycles of this kind of systems with parameters, and obtain some new results on the lower and upper bounds of the maximal number of limit cycles near the nilpotent focus or center.

Keywords: Nilpotent focus; nilpotent center; limit cycle; bifurcation.

1 Introduction and main result

Consider an analytic system of the form

\[ \dot{x} = y + X(x,y), \quad \dot{y} = Y(x,y), \] (1.1)

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where $X, Y = O(|x, y|^2)$ for $(x, y)$ near the origin. The following criterion for the existence of a center or a focus at the origin of (1.1) has been established in \cite{4, 13}.

**Theorem 1.1.** Let (1.1) have an isolated singular point at the origin. Let

$$Y(x, F(x)) = ax^{2n-1} + O(x^{2n}), \quad a \neq 0,$$

$$\frac{\partial X}{\partial x}(x, F(x)) + \frac{\partial Y}{\partial y}(x, F(x)) = bx^{n-1} + O(x^n),$$

where $y = F(x)$ is the solution of the equation $y + X(x, y) = 0$ satisfying $F(0) = 0$. Then the origin of (1.1) is a center or a focus if and only if $a$ is negative and $b^2 + 4an < 0$.

Lyapunov \cite{15} also introduced the generalized polar coordinates

$$x = r \cos(\theta), \quad y = r^n \sin(\theta)$$

and the return map to give a way to find focal values in solving the center-focus problem for (1.1), where $(Cs(t), Sn(t))$ is the solution of the initial problem

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1}, \quad (x(0), y(0)) = (1, 0).$$

Sadovski \cite{19} (see also \cite{3}) and Moussu \cite{16} investigated the problem using Lyapunov function (Lyapunov constants) and normal form, respectively. Then different ways of obtaining the focal values, Lyapunov constants or their equivalent values and the bifurcation method of local limit cycles were further given by Chavarriga, Giacomini, Gine & Llibre \cite{7}, Alvarez & Gasull \cite{11, 12} and Liu & Li \cite{11, 12, 13, 14}. From Takens \cite{21} we know that (1.1) can be formally transformed into a formal normal form

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x), \quad (1.2)$$

where $g(x) = ax^m + O(x^{m+1}), m \geq 2$ (the system (1.2) is a generalized Liénard system). Then, Stróżyńska & Żołnadel \cite{20} proved that this formal normal form can be achieved through an analytic change of variables. Thus, if (1.1) has a center or focus at the origin, then it can be changed into (1.2) with

$$g(x) = x^{2n-1}(a_{2n-1} + O(x)), \quad n \geq 2, \quad a_{2n-1} > 0. \quad (1.3)$$
From Alvarez & Gasull [2] we see that under (1.3) through a change of variables $x$ and $t$ of the form
\[ u = [2n \int_0^x g(x)dx]^{\frac{1}{2n}}(\text{sgn}x) \equiv u(x), \quad \frac{dt}{dt_1} = \frac{u^{2n-1}(x)}{g(x)} \]
the system (1.2) can be changed into
\[ \dot{x} = y, \quad \dot{y} = -x^{2n-1} - y \tilde{f}(x), \quad (1.4) \]
where
\[ \tilde{f}(x) = x^{2n-1}f(u^{-1}(x))/g(u^{-1}(x)), \quad n \geq 2, \quad (1.5) \]
\[ u(x) = [2n \int_0^x g(x)dx]^{\frac{1}{2n}}(\text{sgn}x) = (a_{2n-1})^{\frac{1}{2n}}(x + O(x^2)). \]

Then, by Theorem 1.1 system (1.4) has a center or a focus at the origin if and only if the function $\tilde{f}$ given in (1.5) satisfies
\[ \tilde{f}(x) = \sum_{j \geq n-1} b_j x^j, \quad b_{n-1}^2 - 4n < 0. \quad (1.6) \]

By Filippov’s theorem (see e.g. Ye et al. [22]) under (1.6) the system (1.4) has a stable (unstable) focus at the origin if there exists an integer $l$ with $2l \geq n - 1$ such that
\[ b_{2l} > 0 (< 0), \quad b_{2j} = 0 \text{ for } j < l, \quad (1.7) \]
and it has a center at the origin if $b_{2j} = 0$ for all $2j \geq n - 1$.

Passing to the generalized polar coordinate $(x, y) = (rCs(\theta), r^nSn(\theta))$ we obtain from the system (1.4) the equation
\[ \frac{dr}{d\theta} = \frac{\sum_{j \geq n-1} b_j (Sn(\theta))^2 (Cs(\theta))^{j-1} r^{-n+j}}{1 + \sum_{j \geq n-1} b_j Sn(\theta)(Cs(\theta))^{j-1} r^{-n+j}}. \quad (1.8) \]

The function on the right hand side of (1.8) is periodic of the period $T = 2\sqrt{\frac{\pi}{n}} \Gamma(\frac{1}{2n})/\Gamma(\frac{n+1}{2n})$.

Let $r(\theta, r_0)$ denote the solution of (1.8) with the initial value $r(0) = r_0$. Then
\[ r(T, r_0) = \sum_{j \geq 1} V_j r_0^j. \]

Alvarez & Gasull [2] called the constant $V_k$ the $k$th generalized Lyapunov constant of (1.8) assuming $V_1 = 1, V_2 = \cdots = V_{k-1} = 0$. They also studied the normal form (1.4) and proved the following theorem.

**Theorem 1.2.** Let (1.6) and (1.7) be satisfied. Then
For the case of $n = 2$, Liu and Li [11] introduced a different generalized polar coordinates of the form $x = r \cos \theta$, $y = r^2 \sin \theta$ to change (1.1) into the form

\[
\frac{dr}{dt} = R(\theta, r), \quad \frac{d\theta}{dt} = Q(\theta, r),
\]

assuming the origin is a center or a focus. Let $\tilde{r}(\theta, h)$ denote the solution of the $2\pi$-periodic system

\[
\frac{dr}{d\theta} = \frac{R(\theta, r)}{Q(\theta, r)}
\]

satisfying $\tilde{r}(0) = h$. Note that the initial value problem is well-defined also for negative $h$.

For analytic functions $\phi, \phi_1, \ldots, \phi_k$ defined on a domain $D$ we will write $\phi = O(|\phi_1, \ldots, \phi_k|)$ if there are analytic functions $\psi_1, \ldots, \psi_k$ on $D$, such that $\phi = \psi_1\phi_1 + \cdots + \psi_k\phi_k$ on $D$.

Liu and Li [11] found the following facts.

**Theorem 1.3.** Consider the system (1.1). Let the conditions of Theorem 1.1 be satisfied with $n = 2$ (or $m = 3$) such that the origin is a center or a focus. Then,

1. $\tilde{r}(\theta, -\tilde{r}(\pi, h)) = -\tilde{r}(\pi - \theta, h)$;
2. $\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k \geq 2} v_k h^k$, where $v_{2k+1} = O(|v_2, v_4, \ldots, v_{2k}|)$, $k \geq 1$;
3. the origin is a stable (unstable) focus if $v_{2k} < 0 (> 0)$, and $v_{2j} = 0$ for $j < k$.

In the latter case the origin is called a $k$th order weak focus of (1.1).

Liu and Li [11] also gave some new methods to compute the focus values $v_2, v_4, \ldots, v_{2k}$, or equivalent values, and studied the problem of limit cycle bifurcations near the origin, finding a new phenomenon: a node can generate a limit cycle when its stability changes.

In this paper we study the problem of limit cycle bifurcations near the origin for the analytic system

\[
\dot{x} = y + X(x, y, \delta), \quad \dot{y} = Y(x, y, \delta),
\] (1.9)
where \( \delta = (\delta_1, \ldots, \delta_m) \in D \subset \mathbb{R}^m \) with \( D \) compact, and \( X, Y = O(|x, y|^2) \) for \( |x| \) small and \( \delta \in D \). Let \( y = F(x, \delta) \) be the solution of the equation \( y + X(x, y, \delta) = 0 \). We define the following two functions:

\[
g(x, \delta) = -Y(x, F(x, \delta), \delta), \quad f(x, \delta) = - \left[ \frac{\partial X}{\partial x}(x, F(x, \delta), \delta) + \frac{\partial Y}{\partial y}(x, F(x, \delta), \delta) \right].
\] (1.10)

By Theorem 1.1, if

\[
g(x, \delta) = x^{2n-1}(a_{2n-1}(\delta) + O(x)), \quad n \geq 2, \quad a_{2n-1}(\delta) > 0,
\] (1.11)

\[
f(x, \delta) = \sum_{j \geq n-1} b_j(\delta)x^j, \quad b_{n-1}^2(\delta) - 4na_{2n-1}(\delta) < 0,
\] (1.12)

then the origin is a center or a focus of (1.9) for all \( \delta \in D \).

Let us define a Poincaré return map for the plane system (1.9). For each \( \delta \in D \) and \( x_0 \neq 0 \) with \( |x_0| \) small consider the solution \((x(t, x_0, \delta), y(t, x_0, \delta))\) of (1.9) with the initial condition \((x(0), y(0)) = (x_0, F(x_0, \delta))\). Then there is a unique least positive number \( \tau = \tau(x_0, \delta) > 0 \) such that \( y(\tau, x_0, \delta) = F(x(\tau, x_0, \delta), \delta) \) and \( x_0x(\tau, x_0, \delta) > 0 \). See Figure 1 for \( x_0 > 0 \) small.

Figure 1. The Poincaré map of (1.9) with \( x_0 > 0 \).

Thus, the Poincaré return map is defined as

\[
P(x_0, \delta) = \begin{cases} 
  x(\tau, x_0, \delta), & 0 < |x_0| < \varepsilon_0, \\
  0, & x_0 = 0
\end{cases}
\]
where \( \varepsilon_0 \) is a small positive constant. Evidently, the function is uniquely defined, and it is continuous at \( x_0 = 0 \) under (1.11) and (1.12). Moreover, (1.9) has a periodic orbit near the origin if and only if the map has two fixed points near zero: one positive and the other one negative. For the analytical property of this function at \( x_0 = 0 \), we have the following theorem.

**Theorem 1.4.** Let (1.9) satisfy (1.11) and (1.12) for all \( \delta \in D \). Then there is a unique analytic function \( \tilde{P}(x_0, \delta) \) in \( x_0 \) at \( x_0 = 0 \), satisfying \( \frac{\partial \tilde{P}}{\partial x_0}(0, \delta) > 0 \) and having the expansion

\[
\tilde{d}(x_0, \delta) = \tilde{P}(x_0, \delta) - x_0 = \sum_{j \geq 1} v_j(\delta)x_0^j
\]

for \( |x_0| \) small, such that

1. if \( n \) is odd, then \( P(x_0, \delta) = \tilde{P}(x_0, \delta) \) for all \( |x_0| \) small;
2. if \( n \) is even, then for all \( |x_0| \) small

\[
P(x_0, \delta) = \begin{cases} 
\tilde{P}(x_0, \delta) & \text{for } x_0 > 0, \\
\tilde{P}^{-1}(x_0, \delta) & \text{for } x_0 < 0,
\end{cases}
\]

where \( \tilde{P}^{-1} \) denotes the inverse of \( \tilde{P} \) in \( x_0 \).

Hence, the system (1.9) has a periodic orbit near the origin if and only if the analytic function \( \tilde{d} \) defined in (1.13) has two zeros in \( x_0 \) near \( x_0 = 0 \), among which one is positive and the other one is negative. The function \( \tilde{d} \) is called the succession function or the bifurcation function of (1.9).

The above theorem tells us that the function \( P(x_0, \delta) \) is analytic in \( x_0 \) at \( x_0 = 0 \) as \( n \) is odd, and not analytic in \( x_0 \) at \( x_0 = 0 \) as \( n \) is even unless the origin is a center (in this case, \( P \) is the identity).

For the property of the coefficients \( v_j \) in (1.13) we have further

**Theorem 1.5.** Let (1.9) satisfy (1.11) and (1.12) for all \( \delta \in D \). Then

1. For \( n \) odd we have \( v_{2k} = O(|v_1, v_3, \cdots, v_{2k-1}|), \ k \geq 1 \).
2. For \( n \) even we have \( v_1 = 0, v_{2k+1} = O(|v_2, v_4, \cdots, v_{2k}|), \ k \geq 1 \).

Define \( p_n = [1 + (-1)^n]/2 \). Then the conclusions of the above theorem can be written uniformly as

\[
v_{2k+p_n} = O(|v_1+p_n, v_3+p_n, \cdots, v_{2k-1+p_n}|), \ k \geq 1.
\]
From the proof of the above theorem we see that $v_{2k+p_n}$ depends on $v_{1+p_n}$, $v_{3+p_n}$, \ldots, $v_{2k-1+p_n}$ smoothly. Using the theorem we derive the following two statements on limit cycle bifurcations near the origin.

**Theorem 1.6** (Bifurcation from Focus). Let (1.9) satisfy (1.11) and (1.12) for all $\delta \in D$. Denote $p_n = [1 + (-1)^n]/2$.

(1) If there is an integer $k \geq 1$ such that

$$\sum_{j=1}^{k+1} |v_{2j-1+p_n}(\delta)| > 0 \text{ for all } \delta \in D,$$

then there exists a neighborhood $U$ of the origin such that (1.9) has at most $k$ limit cycles in $U$ for all $\delta \in D$.

(2) If there is $\delta_0 \in D$ such that $v_{2j-1+p_n}(\delta_0) \neq 0$, and

$$v_{2j-1+p_n}(\delta_0) = 0, j = 1, \ldots, k;$$

$$\text{rank} \left( \frac{\partial(v_{1+p_n}, v_{3+p_n}, \ldots, v_{2k-1+p_n})}{\partial(\delta_1, \delta_2, \ldots, \delta_m)} \right)(\delta_0) = k,$$

then for an arbitrary sufficiently small neighborhood of the origin there are some $\delta \in D$ near $\delta_0$ such that (1.9) has exactly $k$ limit cycles in the neighborhood.

**Theorem 1.7** (Bifurcation from Center). Let (1.9) satisfy (1.11) and (1.12) for all $\delta \in D$. Assume that there exist $\delta_0 \in D$ and an integer $k \geq 1$ such that (1.14) is satisfied. If the origin is a center of (1.9) as $v_{2j-1+p_n}(\delta) = 0, j = 1, \ldots, k$, then there exists a neighborhood $U$ of the origin such that (1.9) has at most $k - 1$ limit cycles in $U$ for all $\delta \in D$ near $\delta_0$, and also, for an arbitrary sufficiently small neighborhood of the origin there are some $\delta \in D$ near $\delta_0$ such that (1.9) has exactly $k - 1$ limit cycles in the neighborhood.

The theorem means that the cyclicity of the system at the point $\delta_0$ is equal to $k - 1$.

Now, different from [2] and [11]–[14], we give the following new and more reasonable definition.

**Definition 1.1.** We call $v_{2k+1+p_n}(\delta)$ the generalized focal values of order $k$ of (1.9) at the origin.

By Theorem 1.6, we see that a nilpotent focus of order $k$ generates at most $k$ limit cycles under perturbations as long as the perturbations always satisfy (1.11) and (1.12).
The generalized focal values \(v_{1+p_n}, v_{3+p_n}, \ldots, v_{2k+1+p_n}, \ldots\) can be calculated using the normal form of system (1.9). We will give a method how to do it. By Stróżyna and Žoładek \[20\] we know that (1.9) has the following analytic normal form:

\[
\dot{x} = y, \quad \dot{y} = -g(x, \delta) - yf(x, \delta). \tag{1.15}
\]

We remark that here \(f\) and \(g\) in (1.15) may be different from ones given by (1.10). As before, let \(\delta \in D \subset \mathbb{R}^m\) with \(D\) compact. Also, suppose for \(|x|\) small the function \(g(x, \delta)\) satisfies (1.11). Define

\[
F(x, \delta) = \int_0^x f(x, \delta)dx, \quad G(x, \delta) = \int_0^x g(x, \delta)dx.
\]

It is easy to see that the equation \(G(x, \delta) = G(y, \delta)\) for \(xy < 0\) defines a unique analytic function \(y = \alpha(x, \delta) = -x + O(x^2)\). Introduce

\[
F(\alpha(x, \delta), \delta) - F(x, \delta) = \sum_{j \geq 1} B_j(\delta)x^j. \tag{1.16}
\]

By Theorem 1.1, if (1.15) satisfies (1.11) and (1.12) then it has a center or focus at the origin. Thus, under (1.11) and (1.12) the Poincaré return map for (1.15) is well defined near the origin.

**Theorem 1.8.** Let (1.15) satisfy (1.11) and (1.12) for all \(\delta \in D\). Then for \(x_0 > 0\) small, the Poincaré return map \(P(x_0, \delta)\) has the form

\[
P(x_0, \delta) - x_0 = \sum_{j \geq 0} v_{2j+1+p_n}(\delta)x_0^{2j+1+p_n}(1 + P_j^*(x_0, \delta)),
\]

where \(P_j^*(x_0, \delta) = O(x_0)\),

\[
v_{1+p_n}(\delta) = K^*_1 B_{2l+1}(\delta) + (1 - p_n)O(B_{2l+1}^2), \tag{1.17}
\]

\[
v_{2j+1+p_n}(\delta) = K^*_{l+j} B_{2l+2j+1}(\delta) + O(\{B_{2l+1}, B_{2l+3}, \ldots, B_{2l+2j-1}\}), \quad j \geq 0,
\]

\(l = [n/2]\), and \(K^*_{l+j}, j \geq 0\) are positive constants. Thus, Theorems 1.6 and 1.7 hold if \(v_{2j+1+p_n}\) is replaced by \(B_{2l+2j+1}, j \geq 0\).

Let

\[
f(x, \delta) = \sum_{j \geq 0} b_j(\delta)x^j. \tag{1.18}
\]

Then, we further have for (1.15)
**Theorem 1.9.** Let \((1.15)\) satisfy \((1.11), (1.16)\) and \((1.18)\) for all \(\delta \in D\). Assume there exist \(\delta_0 \in D\) and \(k \geq \lceil n/2 \rceil\) such that

\[
B_{2k+1}(\delta_0) < 0 (> 0), \quad B_{2j-1}(\delta_0) = 0, \ j = 1, \cdots, k. \tag{1.19}
\]

Let one of the following conditions be satisfied:

(a) \(n = 2\), and

\[
b_0(\delta_0) = 0, \ b_1^2(\delta_0) - 8a_3(\delta_0) < 0; \tag{1.20}
\]

(b) \(n > 2\),\( g(-x, \delta) = -g(x, \delta), \ f(-x, \delta) = f(x, \delta), \) and

\[
b_j(\delta_0) = 0 \text{ for } j = 0, \cdots, n - 2 \text{ and } b_{n-1}^2(\delta_0) - 4na_{2n-1}(\delta_0) < 0. \tag{1.21}
\]

Then we have

1. For \(\delta = \delta_0\) \((1.15)\) has a stable (unstable) focus at the origin.
2. If further

\[
\rank \frac{\partial(B_1, B_3, \cdots, B_{2k-1})}{\partial(\delta_1, \delta_2, \cdots, \delta_m)}(\delta_0) = k,
\]

then for an arbitrary sufficiently small neighborhood of the origin there are some \(\delta \in D\) near \(\delta_0\) such that \((1.15)\) has at least \(k\) limit cycles in the neighborhood.

From Theorems 1.4–1.8, it seems that under \((1.11)\) and \((1.12)\) we have solved the problem of limit cycle bifurcation for generic systems. Theoretically it is, but in practice it is not. The reason is that in general we do not know what is the transformation from \((1.9)\) to its normal form \((1.15)\). Here we give a method to solve the problem completely both theoretically and in practice. It includes three main steps below.

First, under \((1.11)\) and \((1.12)\) by the normal form theory (see, for instance, \([21]\)), for any integer \(m > 2n - 1\) there is a change of variables of the form

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
u \\
v
\end{pmatrix} + H_m(u, v, \delta),
\]

where \(H_m(u, v, \delta) = O(|u, v|)\) is a polynomial in \(u, v\) of degree at most \(m\), such that it transforms \((1.9)\) into \((1.22)\) (called the normal form of order \(m\) of \((1.9)\), or the Takens normal form; we still use \((x, y)\) for the new variables \(u, v)\)

\[
\dot{x} = y + X_{m+1}(x, y, \delta), \quad \dot{y} = -g_m(x, \delta) - yf_{m-1}(x, \delta) + Y_{m+1}(x, y, \delta), \tag{1.22}
\]
where 
\[ g_m(x, \delta) = \sum_{j=2n-1}^m a_j(\delta)x^j, \quad f_{m-1}(x, \delta) = \sum_{j=n-1}^{m-1} b_j(\delta)x^j \]
with \(a_{2n-1}(\delta) > 0\) and \(b_{n-1}^2(\delta) - 4na_{2n-1}(\delta) < 0\), and \(X_{m+1}(x, y, \delta), Y_{m+1}(x, y, \delta)\) being analytic functions satisfying \(X_{m+1}, Y_{m+1} = O(|x, y|^{m+1})\). Here, we should mention that the functions \(g_m\) and \(f_{m-1}\) depend only on the terms of degree at most \(m\) of the expansions of the functions \(X\) and \(Y\) in (1.9) at the origin.

The Poincaré maps of (1.9) and (1.22) are essentially the same. We can suppose that the Poincaré map of (1.22) is \(P(x_0, \delta)\) having the expansion
\[ P(x_0, \delta) - x_0 = \sum_{j \geq 1} v_j(\delta)x_0^j \] (1.23)
for \(x_0 > 0\) small.

Second, truncating the higher order terms in (1.22) we obtain the following polynomial system of degree \(m\)
\[ \dot{x} = y, \quad \dot{y} = -g_m(x, \delta) - yf_{m-1}(x, \delta). \] (1.24)

In practice, for given system (1.9) it is not difficult to find the corresponding system (1.24). For (1.24) we can further use Theorem 1.8 to find its focal values at the origin up to any large order. Let \(P_m(x_0, \delta)\) denote the Poincaré map of (1.24). It has the expansion
\[ P_m(x_0, \delta) - x_0 = \sum_{j \geq 1} \bar{v}_j(\delta)x_0^j \] (1.25)
for \(x_0 > 0\) small.

Third, we want to use \(\bar{v}_j(\delta)\) for \(v_j(\delta)\). Here, a problem we would like to solve is the following: For any given \(k > 1\) find \(m > 2n - 1\) such that \(v_j(\delta) = \bar{v}_j(\delta)\) for \(1 \leq j \leq k\).

The following theorem gives an answer.

**Theorem 1.10.** Consider (1.22) and (1.24). Then for any integer \(k \geq 1\), if \(m \geq (k+2)n - 2\) then
\[ v_j(\delta) = \bar{v}_j(\delta) \text{ for } 1 \leq j \leq kn. \] (1.26)

Therefore, we have
Corollary 1.1. Under (1.11) and (1.12) for any integer $k \geq 1$ for (1.9) the coefficients $v_1, v_2, \ldots, v_{kn}$ in (1.13) depend only on the terms of degree at most $(k + 2)n - 2$ of the expansions of the functions $X$ and $Y$ at the origin.

Obviously, in the case of $n = 1$ (the elementary case), the above conclusion is a well-known result.

We organize the paper as follows. In section 2 we first give preliminary lemmas. In section 3 we prove our main results. In section 4 we provide some application examples.

2 Preliminaries

Consider (1.9). In this section we will always suppose that (1.11) and (1.12) are satisfied. Introducing a new variable $v = y - F(x, \delta)$ we can obtain from (1.9) (reusing $y$ for $v$)

\[
\begin{align*}
\dot{x} &= y(1 + Z_1(x, y, \delta)), \\
\dot{y} &= -g(x, \delta) - yf(x, \delta) + y^2Z_2(x, y, \delta),
\end{align*}
\]

where the functions $f$ and $g$ are given by (1.10), and $Z_1$ and $Z_2$ are analytic functions near the origin with $Z_1(x, y, \delta) = O(|x, y|)$. In the discussion below we will often omit $\delta$ for convenience. As in Liu and Li [14] we will make a change of variables to (2.1) using the generalized polar coordinates

\[
x = r \cos \theta, \quad y = r^n \sin \theta, \quad r > 0.
\]

Lemma 2.1. Let (1.11) and (1.12) be satisfied. Then the transformation (2.2) carries (2.1) into the form

\[
\begin{align*}
\dot{\theta} &= S(\theta, r) = \frac{r^{n-1}}{H(\theta)}[S_0(\theta) + O(r)], \\
\dot{r} &= R(\theta, r) = \frac{r^n}{H(\theta)}[R_0(\theta) + O(r)],
\end{align*}
\]

where $S$ and $R$ are $2\pi$-periodic in $\theta$, and satisfy

\[
S(\pi + (-1)^{n-1}\theta, -r) = (-1)^{n-1}S(\theta, r), \quad R(\pi + (-1)^{n-1}\theta, -r) = -R(\theta, r),
\]

and $H(\theta) = \cos^2 \theta + n \sin^2 \theta > 0$,

\[
S_0(\theta) = -[n \sin^2 \theta + b_{n-1} \cos^n \theta \sin \theta + a_{2n-1} \cos^{2n} \theta] < 0,
\]
\[ R_0(\theta) = \cos \theta \sin \theta (1 - a_{2n-1} \cos^{2n-2} \theta - b_{n-1} \sin \theta \cos^{n-2} \theta). \]

**Proof.** From (2.2) we have
\[
\dot{x} = \cos \theta \dot{r} - r \sin \theta \dot{\theta}, \quad \dot{y} = nr^{n-1} \sin \theta \dot{r} + r^n \cos \theta \dot{\theta}.
\]
We solve the above equations for \( \dot{\theta} \) and \( \dot{r} \), and obtain (2.3) with
\[
S(\theta, r) = \frac{\cos \theta \dot{y} - nr^{n-1} \sin \theta \dot{x}}{r^n (\cos^2 \theta + n \sin^2 \theta)},
\]
\[
R(\theta, r) = \frac{\sin \theta \dot{y} + r^{n-1} \cos \theta \dot{x}}{r^{n-1} (\cos^2 \theta + n \sin^2 \theta)}.
\]
Then noting that
\[
\cos(\pi \pm \theta) = -\cos \theta, \quad \sin(\pi \pm \theta) = \mp \sin \theta
\]
and that (2.2) is invariant as \((\theta, r)\) is replaced by \((\pi + (-1)^{n-1} \theta, -r)\) one can prove (2.4) easily. The other conclusions are direct. This ends the proof.

By (2.3) and (2.4) we obtain the following analytic \(2\pi\)-periodic equation
\[
\frac{dr}{d\theta} = \tilde{R}(\theta, r), \tag{2.5}
\]
where
\[
\tilde{R}(\theta, r) = r \frac{\sin \theta \dot{y} + r^{n-1} \cos \theta \dot{x}}{\cos \theta \dot{y} - nr^{n-1} \sin \theta \dot{x}} = r [R_0(\theta) / S_0(\theta) + O(r)], \tag{2.6}
\]
\[\tilde{R}(\pi + (-1)^{n-1} \theta, -r) = (-1)^n R(\theta, r).\]

Let \(r(\theta, h)\) denote the solution of (2.5) with the initial value \(r(0) = h\). For properties of the solution we have

**Lemma 2.2.** The solution \(r(\theta, h) = O(h)\) is analytic in \((\theta, h)\) for \(|h|\) small, and satisfies
\[
(1) \ r(\theta, -r(\pi, h)) = -r(\pi + (-1)^{n-1} \theta, h);
\]
\[
(2) \ r(\theta \pm 2\pi, h) = r(\theta, r(\pm 2\pi, h)).
\]

**Proof.** Let \(\tilde{r}(\theta) = -r(\pi + (-1)^{n-1} \theta, h)\). Then by (2.5) and (2.6) we have
\[
\frac{d\tilde{r}}{d\theta} = (-1)^n \tilde{R}(\pi + (-1)^{n-1} \theta, r(\pi + (-1)^{n-1} \theta, h))
\]
\[= (-1)^n \tilde{R}(\pi + (-1)^{n-1} \theta, -\tilde{r}(\theta))
\]
\[= \tilde{R}(\theta, \tilde{r}(\theta)).\]
This means that \( \tilde{r}(\theta) \) is also a solution of (2.5). Then the first conclusion follows by the uniqueness of initial problem. The second one follows in the same way. This completes the proof.

Further we have

**Lemma 2.3.** Let \( P(x_0, \delta) \) be the Poincaré return map of (1.9) defined in section 1. Then for \( |x_0| > 0 \) small we have \( P(x_0, \delta) = r(-2\pi, x_0) \) for \( x_0 > 0 \), and \( P(x_0, \delta) = r((-1)^n2\pi, x_0) \) for \( x_0 < 0 \).

**Proof.** First, it is easy to see that (1.9) and (2.1) have the same Poincaré return map \( P(x_0, \delta) \). Then, noting that \( \dot{\theta} < 0 \) for \( r > 0 \) small by (2.3), by the definition of \( P \) and (2.2) we can see that

\[
P(x_0, \delta) = x(\tau, x_0) = r(-2\pi, x_0)
\]

for \( x_0 > 0 \) small. Now consider the case of \( x_0 < 0 \). Let \( r^*(\theta, h) \) denote the solution of (2.5) satisfying \( r^*(\pi) = h \). Then we have similarly

\[
P(x_0, \delta) = x(\tau, x_0) = -r^*(-\pi, -x_0),
\]

since under (2.2) the points \((x_0, 0)\) and \((P(x_0, \delta), 0)\) on the \((x, y)\)-plane correspond to the points \((\pi, -x_0)\) and \((-\pi, -P(x_0, \delta))\) on the \((\theta, r)\)-plane respectively.

Further, by Lemma 2.2(1) we have

\[
r^*(\theta, -h) = -r(\pi - \theta, h) \text{ for } n \text{ even}, \tag{2.7}
\]

and

\[
r^*(\theta, -r(2\pi, h)) = -r(\pi + \theta, h) \text{ for } n \text{ odd}. \tag{2.8}
\]

Noting that by Lemma 2.2(2), \( x_0 = r(2\pi, h) \) if and only if \( h = r(-2\pi, x_0) \), we see that (2.8) becomes

\[
r^*(\theta, -x_0) = -r(\pi + \theta, r(-2\pi, x_0)) \text{ for } n \text{ odd}. \tag{2.9}
\]

Therefore, for \( x_0 < 0 \) by (2.7) and (2.9)

\[
P(x_0, \delta) = \begin{cases} 
    r(2\pi, x_0) & \text{for } n \text{ even,} \\
    r(-2\pi, x_0) & \text{for } n \text{ odd.}
\end{cases}
\]
This ends the proof.

**Lemma 2.4.** Let \( d(x_0, \delta) = P(x_0, \delta) - x_0 \). Then there exists an analytic function \( K(h, \delta) \) for \(|h|\) small with \( K(0, \delta) = \frac{\partial r}{\partial x_0}(\pi, 0) > 0 \) such that

\[
d(\tilde{x}_0, \delta) = -K(x_0, \delta)d(x_0, \delta)
\]

for \( x_0 > 0 \) small, where \( \tilde{x}_0 = -r(\pi, x_0) \).

**Proof.** By Lemma 2.2, we have

\[
r((-1)^n 2\pi, \tilde{x}_0) = -r(-\pi, x_0) = -r(\pi, r(-2\pi, x_0)).
\]

Hence, by Lemma 2.3 for \( x_0 > 0 \)

\[
d(\tilde{x}_0, \delta) = r((-1)^n 2\pi, \tilde{x}_0) - \tilde{x}_0
\]

\[
= -r(\pi, r(-2\pi, x_0)) + r(\pi, x_0)
\]

\[
= -K(x_0, \delta)[r(-2\pi, x_0) - x_0]
\]

\[
= -K(x_0, \delta)d(x_0, \delta),
\]

where

\[
K(x_0, \delta) = \int_0^1 \frac{\partial r}{\partial x_0}(\pi, x_0 + s(r(-2\pi, x_0) - x_0))ds.
\]

It is obvious that \( K \) is analytic for \(|x_0|\) small and \( K(0, \delta) = \frac{\partial r}{\partial x_0}(\pi, 0) > 0 \). This completes the proof.

## 3 Proof of the main results

In this section we prove our main results presented in Theorems 1.4–1.10.

**Proof of Theorem 1.4.** We take \( \bar{P}(x_0, \delta) = r(-2\pi, x_0) \) for \(|x_0|\) small. Then by Lemma 2.2 \( \bar{P} \) is analytic. Note that by Lemma 2.2, \( r(2\pi, x_0) \) is the inverse of \( r(-2\pi, x_0) \) in \( x_0 \). Then Theorem 1.4 follows directly from Lemma 2.3. The proof is complete.

**Proof of Theorem 1.5.** There are two cases to consider separately.

**Case A:** \( n \) odd. By (1.13) and Theorem 1.4(1), we have

\[
d(x_0, \delta) = \bar{d}(x_0, \delta) = \sum_{j \geq 1} v_j(\delta)x_0^j
\]

(3.1)
for all \( |x_0| \) small.

By Lemma 2.4, we can suppose

\[
K(x_0, \delta) = \sum_{j \geq 0} k_j x_0^j, \quad \bar{x}_0 = -r(\pi, x_0) = \sum_{j \geq 1} l_j x_0^j, \tag{3.2}
\]

where \( k_0 > 0, l_1 = -k_0 \). Substituting (3.1) and (3.2) into (2.10), we obtain

\[
\sum_{j \geq 1} v_j \left( \sum_{i \geq 1} l_i x_0^i \right)^j = - \sum_{i \geq 0, j \geq 1} k_i v_j x_0^{i+j}.
\]

Comparing the coefficients of the terms \( x_0^2, x_0^4 \) and \( x_0^{2j} \) on both sides yields

\[
v_2 l_1^2 + v_1 l_2 = -(k_0 v_2 + k_1 v_1),
\]

\[
v_4 l_1^4 + 3 v_3 l_1^3 l_2 + v_2 (l_2^2 + 2 l_1 l_3) + v_1 l_4 = -(k_0 v_4 + k_1 v_3 + k_2 v_2 + k_3 v_1),
\]

\[
\cdots
does not display.
\]

\[
v_{2j} l_1^{2j} + v_{2j-1} L_{1,j}(l_1, l_2) + \cdots + v_2 L_{2j-2,j}(l_1, l_2, \ldots, l_{2j-1}) + v_1 l_{2j} = - \sum_{i=0}^{2j-1} k_i v_{2j-i},
\]

\[
\cdots
does not display.
\]

where \( L_{i,j}(l_1, l_2, \ldots, l_{i+1}), \ i = 1, 2, \ldots, 2j - 2, \) are all polynomials. Then from the above equations we obtain

\[
v_{2k} = O(|v_1, v_3, \ldots, v_{2k-1}|), \ k \geq 1.
\]

**Case B:** \( n \) even. By (1.13) and \( x_0 = \bar{P}^{-1}(\bar{P}(x_0, \delta), \delta) \) we can find

\[
\bar{P}^{-1}(x_0, \delta) = \bar{v}_1 x_0 + \bar{v}_2 x_0^2 + \bar{v}_3 x_0^3 + \cdots, \tag{3.3}
\]

where

\[
\bar{v}_1 = (v_1 + 1)^{-1},
\]

\[
\bar{v}_2 = -v_2 (v_1 + 1)^{-3},
\]

\[
\cdots
\]

\[
\bar{v}_j = -v_j (v_1 + 1)^{-(j+1)} + L_j(v_2, v_3, \ldots, v_{j-1}),
\]

\[
\cdots
\]

where each \( L_j \) is a polynomial of degree at least 2. Now we suppose \( x_0 > 0 \). Then (3.1) holds by Theorem 1.4. Further, noting that \( \bar{x}_0 < 0 \) by Theorem 1.4 again

\[
d(\bar{x}_0, \delta) = P(\bar{x}_0, \delta) - \bar{x}_0 = \bar{P}^{-1}(\bar{x}_0, \delta) - \bar{x}_0. \tag{3.5}
\]
Then, inserting (3.1), (3.2), (3.3) and (3.5) into (2.10) we obtain

\[(\bar{v}_1 - 1) l_1 = -k_0 v_1,\]
\[(\bar{v}_1 - 1) l_2 + \bar{v}_2 l_1^2 = -(k_0 v_2 + k_1 v_1),\]
\[\cdots\]
\[(\bar{v}_1 - 1) l_j + L_j(\bar{v}_2, \bar{v}_3, \cdots, \bar{v}_{j-1}) + \bar{v}_j l_1^j = -(k_0 v_j + k_1 v_{j-1} + \cdots + k_{j-1} v_1),\]
\[\cdots\]

where

\[L_j(\bar{v}_2, \bar{v}_3, \cdots, \bar{v}_{j-1}) \in \langle \bar{v}_2, \bar{v}_3, \cdots, \bar{v}_{j-1} \rangle.\]

Finally, noting that \(l_1 = -k_0\) and substituting (3.4) into (3.6) we easily see that

\[v_{2j+1} = O(|v_2, v_4, \cdots, v_{2j}|), \quad j \geq 1.\]

This ends the proof.

**Proof of Theorem 1.6.** For the first part, suppose the conclusion is not true. Then there exists a sequence \(\{\delta_m\}\) in \(D\) such that for \(\delta = \delta_m\) (1.9) has \(k + 1\) limit cycles \(L_{m,1}, L_{m,2}, \cdots, L_{m,k+1}\) which approach the origin as \(m \to \infty\). Then by Theorem 1.5, the function \(\bar{d}(x_0, \delta_m)\) has \(2k + 2\) non-zero roots in \(x_0\) which approach zero as \(m \to \infty\).

Since \(D\) is compact, we can assume \(\delta_m \to \delta_0 \in D\) as \(m \to \infty\). By our assumption, \(\sum_{j=1}^{k+1} |v_{2j-1+p_n}(\delta_0)| > 0\). Thus, for some \(1 \leq l \leq k + 1\),

\[v_{2l-1+p_n}(\delta_0) \neq 0, \quad v_{2j-1+p_n}(\delta_0) = 0 \quad \text{for} \quad 1 \leq j \leq l - 1.\]

Therefore, by (1.13) and Theorem 1.5, we have

\[\bar{d}(x_0, \delta_0) = v_{2l-1+p_n}(\delta_0)x_0^{2l-1+p_n} + O(x_0^{2l+p_n}).\]

Note that \(\bar{d}(0, \delta) = 0\). It follows from Rolle’s theorem that for some \(\varepsilon_0 > 0\) the function \(\bar{d}(x_0, \delta)\) has at most \(2l - 2 + p_n\) non-zero roots in \((-\varepsilon_0, \varepsilon_0)\) for all \(|\delta - \delta_0| < \varepsilon_0\). We have proved that the function \(\bar{d}(x_0, \delta_m)\) has \(2k + 2\) non-zero roots which approach zero as \(m \to \infty\). It then follows that \(2k + 2 \leq 2l - 2 + p_n\), contradicting to \(2l - 2 + p_n \leq 2k + p_n \leq 2k + 1\). The first conclusion follows.
For the second one, by Theorem 1.5, the function $\bar{d}$ can be written as

$$\bar{d}(x_0, \delta) = \sum_{j \geq 1} v_{2j-1+p_n}(\delta)x_0^{2j-1+p_n}(1 + P_j(x_0, \delta)),$$

(3.7)

where $P_j(0, \delta) = 0$. Like in [8] one can show that $P_j$ are series convergent in a neighborhood of $\delta_0$ (see also e.g. [18, 17]). Further, by (1.14), we can take $v_{1+p_n}, v_{3+p_n}, \cdots, v_{2k-1+p_n}$ as free parameters, varying near zero. Precisely, if we change them such that

$$0 < |v_{1+p_n}| \ll |v_{3+p_n}| \ll \cdots \ll |v_{2k-1+p_n}| \ll 1, \quad v_{1+p_n}v_{3+p_n} < 0, \cdots, v_{2k-1+p_n}v_{2k+1+p_n} < 0,$$

then by (3.7) the function $\bar{d}$ has exactly $k$ positive zeros in $x_0$ near $x_0 = 0$, which give $k$ limit cycles. This finishes the proof.

By Theorem 1.4 and (3.7) we immediately have

**Corollary 3.1.** Let (1.9) satisfy (1.11) and (1.12) for a fixed $\delta \in D$. Then, if

$$v_{2k+1+p_n}(\delta) < 0(> 0), \quad v_{2j-1+p_n}(\delta) = 0 \text{ for } j = 1, \cdots, k$$

the origin is a stable (unstable) focus of order $k$ of (1.9). If

$$v_{2j-1+p_n}(\delta) = 0 \text{ for all } j \geq 1$$

the origin is a center of (1.9).

**Proof of Theorem 1.7.** Under (1.14) $v_{1+p_n}, v_{3+p_n}, \cdots, v_{2k-1+p_n}$ can be taken as free parameters. Further, by our assumption, the origin is a center of (1.9) as $v_{2j-1+p_n}(\delta) = 0, j = 1, \cdots, k$. It then follows

$$v_{2j-1+p_n}(\delta) = O(|v_{1+p_n}, v_{3+p_n}, \cdots, v_{2k-1+p_n}|) \text{ for all } j \geq k + 1.$$

Therefore, (3.7) can be further written in the form

$$\bar{d}(x_0, \delta) = \sum_{j=1}^{k} v_{2j-1+p_n}(\delta)x_0^{2j-1+p_n}(1 + \bar{P}_j(x_0, \delta)),$$

where $\bar{P}_j(0, \delta) = 0$ and $P_j$ are series convergent in a neighborhood of $\delta_0$ ([8]). Using the reasoning of Bautin [6] (see also e.g. [9, 17, 18]) one can easily see that the conclusion of the theorem holds. The proof is completed.
Proof of Theorem 1.8. Now we consider (1.15), where \( g \) satisfies (1.11). Let
\[
F(x, \delta) = \int_0^x f(x, \delta) dx, \quad G(x, \delta) = \int_0^x g(x, \delta) dx.
\]
If \( f \) satisfies (1.12), then the origin is a center or focus of (1.15), and
\[
F(\alpha(x, \delta), \delta) - F(x, \delta) = \sum_{j \geq n} B_j(\delta) x^j = \sum_{j \geq n_1} B_j(\delta) x^j,
\]
where
\[
B_n = \frac{(-1)^n - 1}{n} b_{n-1}, \quad n_1 = 2l + 1, \quad l = \left\lfloor \frac{n}{2} \right\rfloor.
\]
and \( \alpha(x, \delta) = -x + O(x^2) \) satisfies \( G(\alpha(x, \delta), \delta) = G(x, \delta) \) for \(|x| \) small. Note that (1.15) is equivalent to the following system
\[
\dot{x} = y - F(x, \delta), \quad \dot{y} = -g(x, \delta)
\]
which has the same Poincaré return map \( P(x_0, \delta) \) as (1.15). Introducing the change of variables \( x \) and \( t \)
\[
u = [2nG(x, \delta)]^\frac{1}{2n} (\text{sgn} x) = (a_{2n-1})^\frac{1}{2n} (x + O(x^2)) \equiv \varphi(x), \quad \frac{dt}{dt_1} = \frac{\varphi^{2n-1}(x)}{g(x, \delta)}
\]
the system (3.9) becomes
\[
\dot{u} = y - \bar{F}(u, \delta), \quad \dot{y} = -u^{2n-1},
\]
which is equivalent to
\[
\dot{u} = y, \quad \dot{y} = -u^{2n-1} - y \bar{f}(u, \delta),
\]
where
\[
\bar{F}(u, \delta) = F(\varphi^{-1}(u), \delta), \quad \bar{f}(u, \delta) = \frac{\partial \bar{F}}{\partial u}(u, \delta).
\]
The systems (3.10) and (3.11) have the same Poincaré return map, denoted by \( P_1(u_0, \delta) \).
One can see that the maps \( P \) and \( P_1 \) have the relation \( P_1 \circ \varphi = \varphi \circ P \). Hence,
\[
P(x_0, \delta) - x_0 = K(u_0)(P_1(u_0, \delta) - u_0),
\]
where \( K(u_0) = (a_{2n-1})^{-\frac{1}{2n}} + O(u_0) \) is analytic. By (1.13) and (3.7), for \( u_0 > 0 \) small we have
\[
P_1(u_0, \delta) - u_0 = \sum_{j \geq 1} v_0(j) u_0^j = \sum_{j \geq 1} v_{2j-1+p_n}(\delta) u_0^{2j-1+p_n} (1 + P_j(u_0, \delta)).
\]
Hence,

$$P(x_0, \delta) - x_0 = \sum_{j \geq 1} v_{2j-1+p_n}(\delta)(a_{2n-1})^{-\frac{1}{2n}} u_0^{2j-1+p_n}(1 + \tilde{P}_j(u_0, \delta))$$

$$= \sum_{j \geq 1} v_{2j-1+p_n}(\delta)(a_{2n-1})^{\frac{2j-2p_n}{2n}} x_0^{2j-1+p_n}(1 + P_j^*(x_0, \delta)), \quad \text{(3.12)}$$

where $\tilde{P}_j(u_0, \delta) = O(u_0)$, $P_j^*(x_0, \delta) = O(x_0)$.

Since $\alpha$ satisfies $G(\alpha(x, \delta), \delta) = G(x, \delta)$ and $x\alpha < 0$ for $|x|$ small, we have $\varphi(\alpha) = -\varphi(x)$ or $\alpha = \varphi^{-1}(-\varphi(x)) = \varphi^{-1}(-u)$, where $u = \varphi(x)$. Thus, we have

$$F(\alpha(x, \delta), \delta) - F(x, \delta) = F(\varphi^{-1}(-u), \delta) - F(\varphi^{-1}(u), \delta) = \tilde{F}(-u, \delta) - \tilde{F}(u, \delta). \quad \text{(3.13)}$$

Let

$$\tilde{f}(u, \delta) = \sum_{j \geq n-1} \tilde{b}_j(\delta) u^j.$$

Then

$$\tilde{F}(u, \delta) = \sum_{j \geq n} \frac{\tilde{b}_{j-1}(\delta)}{j} u^j.$$

Thus, by (3.13) we have

$$F(\alpha(x, \delta), \delta) - F(x, \delta) = -2 \sum_{j \geq [n/2]} \frac{\tilde{b}_{2j}(\delta)}{2j + 1} u^{2j+1}.$$

Substituting $u = \varphi(x) = (a_{2n-1})^{\frac{1}{2n}} (x + O(x^2))$ into the equality above and comparing with (3.8) we obtain

$$B_{2l+1} = -\tilde{K} \tilde{b}_{2l}, \quad B_{2l+2} = O(\tilde{b}_{2l}),$$

$$B_{2l+2j+1} = -\tilde{K}_{l+j} \tilde{b}_{2l+2j} + O(\tilde{b}_{2l}, \tilde{b}_{2l+2}, \ldots, \tilde{b}_{2l+2j-2}), \quad \text{(3.14)}$$

$$B_{2l+2j+2} = O(\tilde{b}_{2l}, \tilde{b}_{2l+2}, \ldots, \tilde{b}_{2l+2j}), \quad j \geq 1,$$

where $\tilde{K}_l, \tilde{K}_{l+1}, \ldots$ are positive constants.

Then by Theorem 1.4 for $u_0 > 0$ small we clearly have

$$P_1(u_0, \delta) = u_0 + \sum_{j \geq 1} v_j(\delta) u_0^j = \sum_{j \geq 1} V_j(\delta) u_0^j,$$

where $V_j$ are introduced before Theorem 1.2. Thus, by Theorem 1.2, we have

$$v_1 = -\tilde{K}_l \tilde{b}_{2l} + O(\tilde{b}_{2l}^2), \quad v_{2j+1}|_{n_1 = \ldots = n_{2j-1} = 0} = -\tilde{K}_{l+j} \tilde{b}_{2l+2j}, \quad j \geq 1$$
for $n = 2l + 1$ odd, and

$$v_2 = -K_l \bar{b}_{2l}, \quad v_{2j+2}|_{v_2=\cdots=v_{2j}=0} = -K_{l+j} \bar{b}_{2l+2j}, \quad j \geq 1$$

for $n = 2l$ even, where $K_{l+j}, \ j \geq 0$ are positive constants. Hence,

$$v_{1+p_n} = -K_l \bar{b}_{2l} + (1 - p_n)O(\bar{b}_{2l}^2),$$

$$v_{2j+1+p_n} = -K_{l+j} \bar{b}_{2l+2j} + \varphi(\bar{b}_{2l}, \bar{b}_{2l+2}, \cdots, \bar{b}_{2l+2j-2}), \quad j \geq 1,$$

where $\varphi(0, 0, \cdots, 0) = 0$. Note that (1.15) is analytic in each $\bar{b}_j$. It follows from Theorem 1.4 that $\varphi$ is analytic in $(\bar{b}_{2l}, \bar{b}_{2l+2}, \cdots, \bar{b}_{2l+2j-2})$, which yields $\varphi = O(|\bar{b}_{2l}, \bar{b}_{2l+2}, \cdots, \bar{b}_{2l+2j-2}|)$. Then (3.14) and (3.15) together give

$$v_{1+p_n} = \frac{K_l}{K_i} B_{2l+1} + (1 - p_n)O(B_{2l+1}^2),$$

$$v_{2j+1+p_n} = \frac{K_{l+j}}{K_{l+j}} B_{2l+2j+1} + O(|B_{2l+1}, B_{2l+3}, \cdots, B_{2l+2j-1}|), \quad j \geq 1,$$

Then (1.17) follows from (3.12) and (3.16). This finishes the proof.

**Proof of Theorem 1.9.** Let $|\delta - \delta_0|$ be small. For $n = 2$ we have $p_n = 1$. Then the first conclusion is direct from Corollary 3.1 and (1.17)-(1.20). In fact, we have by Theorem 1.8

$$v_{2k}(\delta_0) = K_k^* B_{2k+1}(\delta_0), \quad K_k^* > 0, \quad v_{2j}(\delta_0) = 0 \text{ for } j = 1, \cdots, k - 1.$$

For the second conclusion, we first keep $B_1(\delta) = 0$, and vary $B_3(\delta), \cdots, B_{2k-1}(\delta)$ near zero to obtain exactly $k - 1$ simple limit cycles near the origin. These limit cycles are bifurcated by changing the stability of the focus at the origin $k - 1$ times. Then we vary $B_1$ such that $0 < |B_1| \ll |B_3|$, and $B_1B_3 < 0$. This step produces one more limit cycle bifurcated from the origin by changing the stability of the origin which is a node now by [10]. The theorem is proved for the case of $n = 2$.

For $n > 2$ since $g(-x, \delta) = -g(x, \delta), \ f(-x, \delta) = f(x, \delta)$ we have

$$b_j(\delta) = 0 \text{ for } j = 0, \cdots, n - 2 \text{ and } b_{n-1}^2(\delta) - 4na_{2n-1}(\delta) < 0$$

if

$$B_{2l+1}(\delta) < 0(> 0), \quad B_{2j-1}(\delta) = 0, \ j = 1, \cdots, l$$

(3.17)
for some \([n/2] \leq l \leq k\). In this case the origin is a stable (unstable) focus of (3.9) by Theorem 1.8. If (3.17) holds for some \(0 \leq l < [n/2]\), then by [10] again the origin is a stable (unstable) node of (3.9). Then the proof in this case is just similar to the above. This finishes the proof.

We remark that if \(g(-x, \delta) = -g(x, \delta)\) then \(\alpha(x, \delta) = -x\).

**Proof of Theorem 1.10.** Consider (1.22). Without loss of generality, we can assume \(X_{m+1} = 0\) in (1.22). Otherwise, it needs only to introduce a change of variables \(v = y + X_{m+1}(x, y)\). In this case, we can write (1.22) into the form

\[
\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y + y^2 \sum_{j \geq 0} \varphi_j(x)y^j, \quad (3.18)
\]

where

\[
g(x) = g_m(x) + O(|x|^{m+1}), \quad f(x) = f_{m-1}(x) + O(|x|^m), \quad \varphi_j(x) = O(|x|^{m-j}). \quad (3.19)
\]

For the sake of convenience below, we rewrite the functions \(g, f\) and \(\varphi_j\) as follows:

\[
g(x) = x^{2n-1}[g_0(x) + x^n g_1(x) + x^{2n} g_2(x) + \cdots],
\]

\[
f(x) = x^{n-1}[f_0(x) + x^n f_1(x) + x^{2n} f_2(x) + \cdots], \quad (3.20)
\]

\[
\varphi_j(x) = \varphi_{j0}(x) + x^{n-1} \varphi_{j1}(x) + x^{2n-1} \varphi_{j2}(x) + \cdots,
\]

where \(f_j, g_j\) and \(\varphi_{ji}, j \geq 0, l \geq 1\), are polynomials in \(x\) of degree at most \(n - 1\), and \(\varphi_{j0}, j \geq 0\), are polynomials in \(x\) with degree at most \(n - 2\).

Now we change (3.18) by using (2.2) to obtain (2.5) satisfying (2.6) where

\[
\dot{x} = r^n \sin \theta, \quad \dot{y} = r^{2n-1} \sum_{j \geq 0} V_j(\theta, r)r^jn;
\]

and by (3.20)

\[
V_0(\theta, r) = -\cos^{2n-1} \theta g_0(r \cos \theta) - \sin^n \theta f_0(r \cos \theta) + r \sin^2 \theta \varphi_{00}(r \cos \theta),
\]

\[
V_j(\theta, r) = -\cos^{2n-1+j} \theta g_j(r \cos \theta) - \sin^n \theta \cos^j \theta f_j(r \cos \theta) + r \sin^{2+j} \theta \varphi_{j0}(r \cos \theta)
\]

\[
+ \sum_{k=0}^{j-1} \sin^{2+k} \theta \cos^{(j-k)n-1} \theta \varphi_{k,j-k}(r \cos \theta), \quad j \geq 1. \quad (3.21)
\]

Hence, we obtain from (2.5)

\[
\frac{dr}{d\theta} = r \sum_{j \geq 0} R_j(\theta, r)r^jn, \quad \frac{d\theta}{dr} = \sum_{j \geq 0} S_j(\theta, r)r^jn.
\]
where

\[
S_0(\theta, r) = -n \sin^2 \theta + \cos \theta V_0(\theta, r), \quad R_0(\theta, r) = \sin \theta \cos \theta + \sin \theta V_0(\theta, r),
\]

\[
S_j(\theta, r) = \cos \theta V_j(\theta, r), \quad R_j(\theta, r) = \sin \theta V_j(\theta, r), \quad j \geq 1.
\]

(3.22)

By (3.21) and (3.22) we can further expand \(S_j\) and \(R_j\) in \(r\) to obtain for \(j \geq 0\)

\[
S_j(\theta, r) = \sum_{l=0}^{n-1} \bar{S}_{l+jn}(\theta)r^l, \quad R_j(\theta, r) = \sum_{l=0}^{n-1} \bar{R}_{l+jn}(\theta)r^l
\]

(3.23)

so that the above differential equation can be written as

\[
\frac{dr}{d\theta} = r \sum_{j \geq 0} \tilde{R}_j(\theta)r^j.
\]

Further, letting

\[
\frac{1}{\sum_{j \geq 0} \tilde{S}_j(\theta)r^j} = \sum_{j \geq 0} \tilde{S}_j(\theta)r^j
\]

and

\[
\tilde{R}_j(\theta) = \sum_{k+l=j} \bar{R}_k(\theta)\tilde{S}_l(\theta), \quad j \geq 0
\]

(3.24)

we obtain

\[
\frac{dr}{d\theta} = r \sum_{j \geq 0} \tilde{R}_j(\theta)r^j.
\]

(3.25)

Note that for any \(j \geq 0\), \(\tilde{S}_j\) depends only on \(\tilde{S}_k\) with \(0 \leq k \leq j\). Then by (3.24) one can see that

For any \(j \geq 0\), \(\tilde{R}_j\) depends only on \(\tilde{R}_k\) and \(\tilde{S}_k\) with \(0 \leq k \leq j\).  

(3.26)

Let \(r(\theta, r_0)\) denote the solution of (3.25) with the initial value \(r_0\). The for \(r_0\) small we have

\[
r(\theta, r_0) = \sum_{j \geq 1} r_j(\theta)r_0^j
\]

where \(r_1, r_2, r_3, \ldots\) satisfy \(r_1(0) = 1, r_2(0) = r_3(0) = \cdots = 0\), and

\[
r'_1 = \tilde{R}_0r_1, \\
r'_2 = \tilde{R}_0r_2 + \tilde{R}_1r_1^2, \\
r'_3 = \tilde{R}_0r_3 + 2\tilde{R}_1r_1r_2 + \tilde{R}_2r_1^3, \\
\cdots
\]
Hence, by Lemma 2.3, (1.23) and (3.26) we come to the following conclusion:

For any \( j \geq 1 \), \( v_j(\delta) \) depends only on \( \bar{R}_k \) and \( \bar{S}_k \) with \( 0 \leq k \leq j - 1 \).

Further, by (3.21)–(3.23), one can observe that for \( 0 \leq l \leq n - 1 \), \( \bar{S}_l \) and \( \bar{R}_l \) depend only on the coefficients of degree \( l \) of the polynomials \( g_0 \), \( f_0 \) and \( x\varphi_{00} \) in \( x \). Hence, by (3.27) we see that for \( 1 \leq j \leq n \), \( v_j \) depends only on the coefficients of degree at most \( j - 1 \) of the polynomials \( g_0 \), \( f_0 \) and \( x\varphi_{00} \) in \( x \).

Similarly, for \( j \geq 1 \) and \( 0 \leq l \leq n - 1 \) or \( jn \leq l + jn \leq (j + 1)n - 1 \), \( \bar{S}_{l+jn} \) and \( \bar{R}_{l+jn} \) depend only on the coefficients of degree \( l \) of the polynomials \( g_j \), \( f_j \), \( x\varphi_{j0} \) and \( \varphi_{l,i-j} \) with \( i = 0, \ldots, j - 1 \) in \( x \). In other words, for \( jn + 1 \leq u \leq (j + 1)n \), \( S_{u-1} \) and \( R_{u-1} \) depend only on the coefficients of degree \( u - 1 - jn \) of the polynomials \( g_j \), \( f_j \), \( x\varphi_{j0} \) and \( \varphi_{l,i-j} \) with \( i = 0, \ldots, j - 1 \) in \( x \). Let \( N_{[a,b]} \) denote the set of integers in the interval \([a, b] \). Then for \( jn + 1 \leq u \leq (j + 1)n \), we have

\[
N_{[0,u-1]} = \bigcup_{i=0}^{j-1} N_{[in,(i+1)n-1]} \cup N_{[jn,u-1]}.
\]

Thus, for all \( k \in N_{[in,(i+1)n-1]} \), \( \bar{S}_k \) and \( \bar{R}_k \) depend only on \( g_i \), \( f_i \), \( x\varphi_{i0} \) and \( \varphi_{l,i-l} \) with \( l = 0, \ldots, i - 1 \). And for \( k \in N_{[jn,u-1]} \), \( \bar{S}_k \) and \( \bar{R}_k \) depend only on the coefficients of degree \( k - jn \) of the polynomials \( g_j \), \( f_j \), \( x\varphi_{j0} \) and \( \varphi_{l,j-l} \) with \( l = 0, \ldots, j - 1 \) in \( x \).

Therefore, by (3.27) for \( jn + 1 \leq u \leq (j + 1)n \), \( v_u(\delta) \) depends only on the functions \( g_i \), \( f_i \), \( x\varphi_{i0} \) and \( \varphi_{l,i-l} \) with \( l = 0, \ldots, i - 1 \), \( i = 0, \ldots, j - 1 \) and the coefficients of degree at most \( u - 1 - jn \) of the polynomials \( g_j \), \( f_j \), \( x\varphi_{j0} \) and \( \varphi_{l,j-l} \) with \( l = 0, \ldots, j - 1 \) in \( x \).

We claim that if \( j \geq 0 \), \( m \geq (j + 1)n \), then for \( jn + 1 \leq u \leq (j + 1)n \), \( v_u(\delta) \) depends only on the functions \( g_i \), \( f_i \), with \( i = 0, \ldots, j - 1 \) and the coefficients of degree at most \( u - 1 - jn \) of the polynomials \( g_j \), \( f_j \) in \( x \).

In fact, by the above discussion, we need only to prove \( \varphi_{00} = 0 \) in the case \( j = 0 \) and \( \varphi_{ls} = 0 \) for \( l + s \leq j \) and \( 0 \leq l \leq j - 1 \) in the case \( j > 0 \). This can be shown easily since

\[
\varphi_{j0} = O(|x|^{m-1-j}), \quad \varphi_{js} = O(|x|^{m-j-sn}) \text{ for } s \geq 1
\]

and

\[
deg \varphi_{j0} \leq n - 2, \quad \deg \varphi_{js} \leq n - 1 \text{ for } s \geq 1
\]
by (3.19) and (3.20).

By (3.20) again, the above claim can be restated that if \( j \geq 0, m \geq (j + 1)n \), then for \( jn + 1 \leq u \leq (j + 1)n \), \( v_u(\delta) \) depends only on the coefficients of degree at most \( 2n + u - 2 \) of \( g \) and the coefficients of degree at most \( n + u - 2 \) of \( f \) in \( x \). Thus, for any integers \( k \) and \( m \) satisfying \( k \geq 1 \) and \( m \geq (k + 1)n \), by taking \( j = 0, \ldots, k \) we know that for all \( 1 \leq u \leq (k + 1)n \), \( v_u(\delta) \) depends only on the coefficients of degree at most \( 2n + u - 2 \) of \( g \) and the coefficients of degree at most \( n + u - 2 \) of \( f \) in \( x \).

Finally, by (3.19), if \( m \geq (k + 3)n - 2 \) then

\[
2n + u - 2 \leq m, \quad n + u - 2 \leq m - 1 \quad \text{for } u \leq (k + 1)n.
\]

In this case, for all \( 1 \leq u \leq (k + 1)n \), \( v_u(\delta) \) depends only on \( g_m \) and \( f_{m-1} \) in (3.19). Then the conclusion of Theorem 1.10 follows.

### 4 Application examples

In this section we give some application examples based on the examples given in [2].

Consider a Kukles type system of the form

\[
\dot{x} = y, \quad \dot{y} = -(a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3).
\] (4.1)

The authors [2] proved that if \( a_{30} > 0 \) and \( a_{11}^2 - 8a_{30} < 0 \) then for (4.1) \( v_2 = v_4 = v_6 = v_8 = 0 \) if and only if \( a_{21} = a_{03} = a_{11}a_{02} = 0 \), which implies that the origin is a center. Moreover, there can be 3 limit cycles near the origin. See Theorem 4.1 in [2] and its proof.

Based on this conclusion and by Theorem 1.6 we have immediately

**Proposition 4.1.** Let \( a_{11}, a_{02}, a_{30}, a_{21}, a_{12} \) and \( a_{03} \) be bounded parameters satisfying

\[
a_{30} > 0, \quad a_{11}^2 - 8a_{30} < 0, \quad |a_{21}| + |a_{03}| + |a_{11}a_{02}| > 0.
\]

Then there exists a neighborhood \( V \) of the origin such that the system (4.1) has at most 3 limit cycles in \( V \).

Then consider

\[
\dot{x} = -y + Ax^2 + Bxy + C y^2, \quad \dot{y} = x^3 + xy^2 + y^3.
\] (4.2)
By Theorem 4.2 in [2] and its proof if \( A^2 < 2 \) then the origin of (4.2) is always a focus with \(|v_2| + |v_4| + |v_6| + |v_8| > 0\). Moreover, there are systems inside (4.2) with at least 3 limit cycles around the origin. Then by Theorem 1.6 again we have

**Proposition 4.2.** Let \( A, B \) and \( C \) be bounded parameters with \( A^2 < 2 \). Then there exists a neighborhood \( V \) of the origin such that the system (4.2) has at most 3 limit cycles in \( V \).

Finally, consider

\[
\dot{x} = y, \quad \dot{y} = -(x^3 + x^5) - \sum_{j=0}^{k} b_{2j} x^{2j} y,
\]

(4.3)

where \( k \geq 2 \). By Theorems 1.7-1.9, we obtain

**Proposition 4.3.** Let \( b_{2j} \) be bounded parameters. Then

1. If \( b_0 = 0 \), the system (4.3) has at most \( k - 1 \) limit cycles near the origin; and \( k - 1 \) limit cycles can appear.

2. If \( b_0 \neq 0 \), there are systems inside (4.3) which have at least \( k \) limit cycles near the origin.

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