Reference system approach within the white-dwarfs theory

M.V. Vavrukh, D.V. Dzikovskyi, N.L. Tyshko
Department of Astrophysics, Ivan Franko National University of Lviv, 8 Kyrylo and Methodiy St., 79005 Lviv, Ukraine

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Reference system approach of non-relativistic electron fluid theory was adapted for calculation of characteristics of the electron-nuclear model at the densities typical of degenerate dwarfs. Two- and three correlation functions of degenerate relativistic electron gas have been calculated in the momentum-frequency representation in the local field approximation. Main contributions of the Coulomb interactions to the energy and equation of state of the model at $T = 0$ K have been calculated in the adiabatic approximation.

Key words: electron-nuclear model, correlation functions, local field correction function, zero-temperature energy, equation of state

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1. Introduction

A hundred years have passed since the discovery of degenerate dwarfs [1]. The theory of an internal structure of cold dwarfs was developed by Chandrasekhar in 1940s, and it was based on the equation of state of ideal relativistic electron gas at $T = 0$ K [2,3]. Generalization of this theory followed in the next decades, when in the works of many authors there were investigated effects of such important factors as axial rotation [4,5], Coulomb interactions [6], incomplete degeneration of an electron subsystem [7,8], effects of magnetic fields [9,10], effects of general relativity [5,11], processes of neutronization [12], etc. Interpretation of the whole diversity of properties of the dwarfs obtained from the observations of space [13,14], requires constructing a general theory that also takes into account the effects of the above mentioned factors, among which there are the competing ones.

The effect of interactions that play an important role in determining the structure of dwarfs at different masses and luminosities, and especially for the case of massive cold dwarfs, is the least studied. Basing on Wigner-Seitz, Thomas-Fermi approximations and non-relativistic random phase approximation corresponding to the approximate accounting of two-particle electron correlations, Salpeter [6] showed that Coulomb interactions lead to a small decrease of pressure of degenerate relativistic electron gas at $T = 0$ K, which is still considered to be the basis of Chandasekhar’s theory [10].

Due to the high density ($\sim 10^5$ g/cm$^3$), the matter in the cores of degenerate dwarfs has a metal electron structure with completely collectivized electrons, and the Fermi momentum is of the order $m_0 c$. Thus, the non-relativistic approach is not applicable, and it complicates the calculations, because the correlation functions of the reference system (of an interacting relativistic electron gas) are less investigated than the functions of the analogical non-relativistic model. On the other hand, the electron-nuclear model at “dwarfs” densities is slightly non-ideal, which allows one to use a random phase approximation in order to account for the interactions. A consistent approach that corresponds to the modern metals many-electron theory is proposed in [15]. A disadvantage of this work is the use of approximate expressions for two- and three particle correlation functions of ideal relativistic degenerate electron gas. In the present work we use an accurate static two-electron correlation function and a three-particle correlation function in
the long-wavelength approximation, which improves the reliability of the results for the energy of the ground state of the model and equation of state.

2. General relations

We consider a more realistic, compared to [15], spatially homogeneous electrically neutral model, which consists of \(N_e\) electrons, \(N_1\) nuclei of charge \(z_1\) and \(N_2\) nuclei of charge \(z_2\) in the volume \(V\) in thermodynamic limit \(N_e, V \rightarrow \infty, N_e/V = \text{const}\) at the temperatures much lower than the temperature of the electron subsystem degeneration. Generalization for a larger number of nuclei species is obvious.

For the Hamiltonian of the model

\[
\hat{H} = \hat{H}_0 + \hat{V}_{ee} + \sum_{i=1}^{2} \hat{V}_{en}^{i} + \hat{V}_{nn},
\]

we use a secondary quantization representation for the electrons and a coordinate one for the nuclear subsystem:

\[
\hat{H}_0 = \sum_{k,s} E_k a_{k,s}^\dagger a_{k,s},
\]

is the Hamiltonian of free electrons (\(E_k = [(m_0 c^2)^2 + \hbar^2 k^2 c^2]/2 - m_0 c^2\)),

\[
\hat{V}_{ee} = \frac{1}{2V} \sum_{q \neq 0} V_q \hat{I}_2(q, -q)
\]

is the operator of electron-electron interactions,

\[
\hat{V}_{en}^{i} = -\frac{1}{V} z_i \sum_{q \neq 0} V_q S_{q}^{(i)} \hat{\rho}_q
\]

is the operator of electron interactions with \(i\)-th nuclear subsystem,

\[
\hat{V}_{nn} = \frac{1}{2V} \sum_{q \neq 0} V_q \sum_{i,j} z_i z_j [S_{q}^{(i)} S_{-q}^{(j)} - N_i \delta_{i,j}]
\]

is the sum of direct nuclear interactions. Here, \(V_q = 4\pi e^2 / q^2\), \(S_{q}^{(i)} = \sum_{l=0}^{N_i} \exp[i(q, \mathbf{R}_l^i)]\) is the structure factor of \(i\)-th nuclear subsystem,

\[
\hat{I}_2(q, -q) = \sum_{k_1, k_2, \xi_1, \xi_2} a_{k_1+q, s_1}^\dagger a_{k_2-q, s_2}^\dagger a_{k_2, \xi_2} a_{k_1, \xi_1},
\]

\[
\hat{\rho}_q = \sum_{k,s} a_{k+s}^\dagger a_{k,s},
\]

\(a_{k,s}^\dagger, a_{k,s}\) are the creation and annihilation operators of electrons in quantum states with the given vector \(k\) and the spin variable \(s\), \(\mathbf{R}_l^i\) is the radius-vector of \(l\)-th nucleus with the charge \(z_i\).

For the calculation of a partition function of an electron subsystem in the fixed nuclei field in the grand canonical ensemble

\[
Z(\mu) = Sp_e \exp \left[ -\beta \left( \hat{H}_0 - \mu \hat{N}_e + \hat{V}_{ee} + \sum_{i=1}^{2} \hat{V}_{en}^{i} \right) \right],
\]

we summarize the reference system approach developed in [16, 17] for a description of non-relativistic model of electron liquid. In the formula (2.7), \(N_e\) is the operator of the number of electrons, \(\mu\) is the
variable of chemical potential. As in [16], let us move to the “frequency” representation of electron operators

\[ a_{k,s}(\nu_s) = \int_0^\beta a_{k_s}(\beta') \psi_{\nu_s}(\beta') d\beta', \]  

(2.8)

where \( \psi_{\nu_s}(\beta') = \beta^{-1/2} \exp(i \nu_s \beta'), \nu_s = (2n + 1) \pi \beta^{-1}, n = 0; \pm 1; \pm 2; \ldots \). In the “frequency” representation

\[ \exp \left[ -\beta \left( \hat{H}_0 - \mu \hat{N}_c + \hat{V}_{ee} + \sum_i \hat{V}^i_{en} \right) \right] = \exp (\beta \hat{H}_\mu) T_\beta \{ \hat{S}_{en}(\nu) \hat{S}_{en}(\nu) \}, \]  

(2.9)

and \( \hat{H}_\mu = \hat{H}_0 - \mu \hat{N}_c \). The operator \( T_\beta \) is a generalized ordering operator,

\[ \hat{S}_{en}(\nu) = \exp \left[ -(2\beta V)^{-1} \sum_{q \neq 0} \sum_{\nu} V_q \hat{\beta}_{q,\nu} \hat{\beta}_{-q,-\nu} \right], \]

\[ \hat{S}_{en}(\nu) = \exp \left[ V^{-1} \sum_{i=1}^2 \sum_{q \neq 0} V_q S_{q,\nu} \hat{\beta}_{q,\nu} \right], \]

\[ \hat{\beta}_{q,\nu} = \sum_{k,s} \sum_{\nu_s} a_{k+q,s}(\nu_s + \nu) a_{k,s}(\nu_s), \]  

(2.10)

moreover, \( \nu = 2\pi n \beta^{-1}; n = 0, \pm 1, \pm 2, \ldots \), and the calculation of the average of the product of operators \( a_{k,s}(\nu_s) \) is performed according to the rule [18, 19]

\[ -(T_\beta \{ a_{k_1,s_1}(\nu_s_1) a_{k_2,s_2}(\nu_s') \}) H_\mu = G_{k_1:s_1}^{s_2} (\nu_s_1) \delta_{\nu_s_1,\nu'_s} \delta_{s_1:s_2} \delta_{k_1,k_2}, \]  

(2.11)

where \( G_{k,s}(\nu_s) = (i \nu_s - E_k + \mu)^{-1} \exp(i \nu_s \beta) \) is the spectral image of one-electron Green’s function of an ideal system (\( \delta \to +0 \)), which is the reference system for a description of the interacting electron gas model.

We use model with the Hamiltonian \( \hat{H}_0 + \hat{V}_{ee} \) as a reference system to calculate \( Z(\mu) \):

\[ Z(\mu) = \exp \left[ -\beta \Omega_c(\mu) \right] \langle \hat{S}_{en}(\nu) \rangle_c, \]

\[ \exp \left[ -\beta \Omega_c(\mu) \right] = \exp \left[ -\beta \Omega_0(\mu) \right] \langle T_\beta \{ \hat{S}_{en}(\nu) \} \rangle_0, \]

\[ \langle \hat{S}_{en}(\nu) \rangle_c = \langle T_\beta \{ \hat{S}_{en}(\nu) \} \rangle_0 \langle \hat{S}_{en}(\nu) \rangle_0^{-1}. \]  

(2.12)

The symbol \( \langle \hat{A} \rangle_0 \) denotes a statistical averaging over the states, and \( \langle \hat{A} \rangle_c \) for the states of the reference system;

\[ \Omega_0(\mu) = -\beta^{-1} \sum_{k,s} \ln \left( 1 + \exp \left[ -\beta(E_k - \mu) \right] \right) \]  

(2.13)

is the grand potential of an ideal system of electrons, and \( \Omega_c(\mu) \) is the grand potential of the interacting electron gas.

Expanding the operator \( \hat{S}_{en}(\nu) \) in the formula (2.12) in a power series, averaging by states for the reference system and presenting the result in an exponential form, we have obtained contributions to the grand potential of electron-nuclear interactions

\[ \Omega_{en} = - \sum_{n \geq 2} (n! V^n)^{-1} \sum_{i_1, i_2, \ldots, i_n} z_{i_1} z_{i_2} \cdots z_{i_n} \sum_{q_1, \ldots, q_n \neq 0} V_{q_1} V_{q_2} \cdots V_{q_n} S_{q_1}^{(i_1)} S_{q_2}^{(i_2)} \cdots S_{q_n}^{(i_n)} \]

\[ \times \hat{\rho}_n(q_1, q_2, \ldots, q_n | 0) \delta_{q_1, q_2, \ldots, q_n, 0}, \]  

(2.14)
where
\[ \tilde{\mu}_n(q_1, \ldots, q_n) = \beta^{-1} \langle T_\beta \{ \hat{\rho}_{q_1,0} \hat{\rho}_{q_2,0} \cdots \hat{\rho}_{q_n,0} \} \rangle \]  
(2.15)
is a cumulant part of the average of the product of \( n \) operators \( \hat{\rho}_{q,0} \). This is a static limit of \( n \)-particles correlation function of the reference system
\[ \bar{\mu}_n(y_1, \ldots, y_n) = \bar{\mu}_n(q_1, \ldots, q_n | y_1, \ldots, y_n) = \beta^{-1} \langle T_\beta \{ \hat{\rho}_{q_1,y_1} \hat{\rho}_{q_2,y_2} \cdots \hat{\rho}_{q_n,y_n} \} \rangle. \]  
(2.16)
The functions (2.15), (2.16) are a generalization of “many-tails” of non-relativistic theory of the electron-ion model of metals \([16, 17]\) and have a well-defined physical meaning. In particular, the functions (2.16) are a spectral representation of \( n \)-particles correlation functions, which are given in the coordinate space.

The ratio \( \tilde{\mu}_2(q, -q|\nu, -\nu) \) of the expression
\[ F_2(r) = 1 + [\beta N_c(N_c - 1)]^{-1} \sum_{\nu \geq 0} \sum_{q \geq 0} \tilde{\mu}_2(q, -q|\nu, -\nu) \exp [i(q, r)]. \]  
(2.17)

The dynamic correlation functions of \( n \)-particles (2.16) are determined by the polarization operators \( M_n(y_1, \ldots, y_n) \) of \( n \)-particles \([17]\)
\[ \mu_2(y, -y) = M_2(y, -y) \left[ 1 + \frac{V_q}{V} M_2(y, -y) \right]^{-1}, \]
\[ \mu_n(y_1, \ldots, y_n) = M_n(y_1, \ldots, y_n) \prod_{i=1}^{n} \left[ 1 + \frac{V_q}{V} M_2(y_i, -y_i) \right]^{-1}, \quad n \geq 3. \]  
(2.18)

These ratios generalize the well-known random phase approximation, in which \( M_2(y, -y) \Rightarrow \tilde{\mu}_2(y, -y) \), \( M_n(y_1, \ldots, y_n) \Rightarrow \tilde{\mu}_n(y_1, \ldots, y_n) \),
\[ \tilde{\mu}_n(y_1, \ldots, y_n) = \beta^{-1} \langle T_\beta \{ \hat{\rho}_{q_1,y_1} \cdots \hat{\rho}_{q_n,y_n} \} \rangle \]  
(2.19)
is the spectral image of correlation functions of an ideal electron gas model,
\[ M_2(y, -y) = \tilde{\mu}_2^0(y, -y) \left[ 1 - \frac{V_q}{V} \tilde{\mu}_2^0(y, -y) G(y) \right]^{-1}, \]
\[ M_n(y_1, \ldots, y_n) = \tilde{\mu}_n^0(y_1, \ldots, y_n) \prod_{i=1}^{n} \left[ 1 - \frac{V_q}{V} \tilde{\mu}_2^0(y_i, -y_i) G(y_i) \right]^{-1} \]  
(2.20)
for the \( n \geq 3 \), so the problem of calculating the functions (2.16) is reduced to the calculation of correlation functions of the ideal relativistic gas model (2.19) and the local field correction function \( G(y) \) for the relativistic interacting electron gas.

### 3. Correlation functions of ideal degenerate relativistic electron gas

Static and dynamic correlation functions of the non-relativistic ideal electron gas are well known. The analytical expression for \( \tilde{\mu}_2^0(y, -y) \) was obtained in \([20]\). The function \( \tilde{\mu}_3^0(q_1, q_2, -q_1 - q_2|0, 0) \) and the function \( \tilde{\mu}_4^0(q_1, -q_1, q_2, -q_2|0, 0, 0, 0) \) were calculated in \([19, 21]\). The dynamic functions \( \tilde{\mu}_2^0(y_1, y_2, -y_1 - y_2) \) and \( \tilde{\mu}_4^0(y_1, -y_1, y_2, -y_2) \) were first calculated in \([16]\). The calculation of these functions for a relativistic model is a complex problem, because the electron spectrum is not a quadratic function of the wave vector.
According to the definition (2.19), the functions $\tilde{\mu}_n^0(y_1, \ldots, y_n)$ are a convolution of one-particle Green’s functions, the same as in a non-relativistic case:

$$\mu_2^0(y, -y) = -\frac{1}{\beta} \sum_{k,s} \sum_{\nu} G_{k,s}(\nu) G_{k+q,s}(\nu + \nu),$$

$$\mu_3^0(y_1, y_2, y_3) = \frac{2}{\beta} \sum_{k,s} \sum_{\nu} G_{k,s}(\nu) G_{k+q_1,s}(\nu + \nu_1) G_{k-q_2,s}(\nu - \nu_2) \delta_{q_1 + q_2 + q_3, 0} \delta_{\nu_1 + \nu_2 + \nu_3, 0},$$

$$\mu_4^0(y_1, -y_1, y_2, -y_2) = \frac{1}{\beta} \sum_{k,s} \sum_{\nu} G_{k,s}(\nu) G_{k-q_1,s}(\nu - \nu_1) \sum_{\sigma=\pm 1} G_{k-\sigma q_2,s}(\nu - \sigma \nu_2)$$

$$\times [2G_{k,s}(\nu) + G_{k+q_1,\sigma q_2,s}(\nu + \nu + \sigma \nu_2)].$$

(3.1)

Factorizing the products of Green’s functions and using the ratio

$$\frac{1}{\beta} \sum_{\nu} G_{k,s}(\nu) = n_{k,s} = \{1 + \exp[\beta(E_k - \mu)]\}^{-1},$$

we obtain representation functions $\tilde{\mu}_n^0(y_1, \ldots, y_n)$:

$$\mu_2^0(y, -y) = -2 \text{Re} \sum_{k,s} n_{k,s} \frac{1}{i\nu + E_k - E_{k+q}},$$

$$\mu_3^0(y_1, y_2, y_3) = \delta_{y_1 + y_2 + y_3, 0} [\gamma_1(y_1, -y_2) + \gamma_2(y_2, -y_3) + \gamma_3(y_3, -y_1)],$$

$$\gamma_3(y_1, y_2) = 2 \text{Re} \sum_{k,s} n_{k,s} (i\nu_1 + E_k - E_{k+q_1})^{-1} (i\nu_2 + E_k - E_{k+q_2})^{-1} \ldots.$$ 

(3.2)

**3.1. Two-particle correlation function**

Rewriting the sum over vector $k$ via integral using a spherical coordinate system and integrating by the angular variables, we obtain a representation:

$$\tilde{\mu}_2^0(y, -y) = \frac{3N_e}{m_0 c^2 x^2} J_2(q_s, \tilde{\nu}|x),$$

$$J_2(q_s, \tilde{\nu}|x) = \frac{1}{2x \bar{q}_s} \sum_{s} \int_0^\infty dk_s k_s n_{k,s} A(k_s, q_s, \tilde{\nu}),$$

$$A(k_s, q_s, \tilde{\nu}) = \sum_{\sigma=\pm 1} \sigma \left[ \left( 1 + (k_s + \sigma q_s)^2 \right)^{1/2} - \tilde{\nu} \arctan \left( \tilde{\nu}^{-1} \eta_{\sigma}(k_s, q_s) \right) \right]$$

$$+ \frac{1}{2} (1 + k_s^2)^{1/2} \ln \left[ \tilde{\nu}^2 + \eta_\sigma^2(k_s, q_s) \right].$$

(3.3)

(3.4)

Here, there appear dimensionless variables

$$k_s = \frac{x|k|}{k_F}, \quad q_s = \frac{x|q|}{k_F}, \quad \tilde{\nu} = \frac{\nu}{m_0 c^2},$$

(3.5)

where $x = \hbar k_F (m_0 c)^{-1}$ is a relativistic parameter [$k_F = (3\pi^2 N_e/V)^{1/3}$ is the Fermi wave number]. In the static case,

$$q_s \bar{J}_2(q_s, 0|x) = \frac{2}{9} (R_+ - R_-) \left( 1 + \frac{7}{4} x^2 - \frac{q_s^2}{8} \right) + \frac{5q_s x (R_+ + R_-) + q_s x R_0}{72} + \frac{R_0^3}{3} \ln \left| \frac{R_+ - R_0}{R_- - R_0} \right| + \frac{q_s x}{8} \left( 1 + \frac{q_s^2}{6} \right) [2 \ln |x + R_0| - \ln |(R_+ + x + q_s)(R_- + x - q_s)|]$$

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As in the non-relativistic case, the function \( J_2(q_*, 0) \) has a weak logarithmic peculiarity of type \((x - \frac{1}{2} q_*) \ln |x - \frac{1}{2} q_*| \) in the vicinity of \( q_* = 2x \) (\(|q| = 2k_F\)). In general, the correlation functions of a relativistic model are similar to the corresponding functions of a non-relativistic model, which is clearly visible from the asymptotic of the function \( \rho_2^0(q, -q|0) \):

\[
\tilde{\rho}_2^0(q, -q|0) \Rightarrow \begin{cases} 
\frac{3N_c(1 + x^2)^{1/2}}{m_0 c^2 x} + \ldots & \text{by } q \to 0; \\
\frac{2N_c}{m_0 c^2 q_*} + \ldots & \text{by } q \to \infty
\end{cases}
\]

in the relativistic case,

\[
\tilde{\rho}_2^0(q, -q|0) \Rightarrow \begin{cases} 
\frac{3N_c}{2\varepsilon_F} + \ldots & \text{by } q \to 0; \\
\frac{2N_c}{\hbar^2 q^2 / 2m_0} + \ldots & \text{by } q \to \infty
\end{cases}
\]

in the non-relativistic approximation (\(2\varepsilon_F \equiv m_0 c^2 x^2\)). A peculiarity of the function \( \rho_2^0(q, -q|0) \) is its strong dependence on the relativistic parameter, as shown in figure [1].

In the dynamic case, \( \tilde{\rho}_2^0(y, -y) \) can be represented as the approximation

\[
\tilde{\rho}_2^0(y, -y) \approx \frac{3N_c}{m_0 c^2 x^2} J_2(q, v|x),
\]

\[
J_2(q, v|x) = \frac{1}{4} C(q_*|x) \left[ 1 - v \left( \arctan \frac{1 + q_* / 2x}{v} - \arctan \frac{1 - q_* / 2x}{v} \right) \right]
\]

Figure 1. Dependence of the function \( J_2(q_*, 0|x) \) on the wave vector \( q \) at different values of the relativistic parameter.
The expression (3.9) is here only for completeness. We used the result of numerical calculations for 3.2. Three-particle correlation function (interacting electron gas).

which makes it possible to restrict the consideration to two- and three-electron correlations (we note, that correlation energy of the reference system is of the order of which approximately corresponds to the long-wavelength asymptotic. For the magnitude of that only slightly deviates near the maximum from the numerical result, calculated by formula (3.4). In the non-relativistic limit and the long-wavelength approximation \( C(q_i|x) \rightarrow 2, v \rightarrow v(2\epsilon q/k_F)^{-1} \), and the function (3.9) coincides with the known expression for a non-relativistic function \( \tilde{\mu}_0^S(y, -y) \) \([15]\). The expression (3.9) is here only for completeness. We used the result of numerical calculations for the dynamic function \( \tilde{\mu}_0^Q(y, -y) \) to obtain the correlation energy of the reference system (the model of interacting electron gas).

### 3.2. Three-particle correlation function

The function \( \tilde{\mu}_0^Q(y, -y, 0) \), which is a partial case of three-particle dynamic functions when \( q_2 = -q_1 \), \( v_2 = -v_1 \), at a full degeneration has an exact analytical image:

\[
\tilde{\mu}_0^Q(y, -y, 0) = \frac{3N_e}{(m_0e^2)^2} (1 + x^2)^{1/2} A(x|q_s, \bar{v}).
\]

(3.10)

In the static limit,

\[
\tilde{\mu}_0^Q(q, -q, 0|0) = \frac{3N_e}{(m_0e^2 x^2)^2} J_3(q, -q, 0|x),
\]

\[ J_3(q, -q, 0|x) = \frac{R_0}{\bar{q}} \left( \frac{1}{\tilde{R}_+ - \tilde{R}_-} + R_0 \ln \left| \frac{\tilde{R}_+ - R_0}{\tilde{R}_- - R_0} \right| \right), \]

\[ \tilde{R}_\pm = [1 + x^2(1 \pm \bar{q})^2]^{1/2}, \quad R_0 = (1 + x^2)^{1/2}. \]

(3.11)

In the formula (3.11), a “non-relativistic” scale was used for the wave vector (\( \bar{q} \equiv |q|/k_F \)). In the long-wavelength limit, \( \tilde{\mu}_0^Q(q, -q, 0|0) = \frac{3N_e}{(m_0e^2 x^2)^2} (1 + 2x^2). \)

(3.12)

Dependence of the dimensionless factor \( J_3(q, -q, 0|x) \) on the wave vector and the relativistic parameter is illustrated in figure 2. As in non-relativistic case, the function (3.11) has a logarithmic peculiarity at \( q = 2k_F \).

The formulae (3.6), (3.9) and (3.11), and figures reveal the general property of correlation functions \( \tilde{\mu}^Q_n(y_1, \ldots, y_n) \) — a steep decrease for large wave vectors \( |q_i| > 2k_F \), providing a convergence of integrals in the series (2.14).

In general, for a rough estimation of convergence of series (2.14) we consider a chemically homogeneous model \( (z_1 = z_2 = z, N_1 + N_2 = N_n) \), constraining the integration for each independent vector \( q_i \) of the area \( |q_i| < 2k_F \), neglecting the screening interactions, replacing the product of structural factors \( S_{q_1} S_{q_2} \ldots S_{q_n} \) with \( N_n \), we replace the functions \( \tilde{\mu}_0^Q(q_1, \ldots, q_n|0) \) with \( 3N_e (m_0e^2 x^2)^{1-n} (1 + x^2)^{n+1} \), which approximately corresponds to the long-wavelength asymptotic. For the magnitude of \( n \)-th member of series (2.14) it can be estimated as

\[
N_e m_0 e^2 x^{2n-1} a_0^Q x^{2-n} (1 + x^2)^{3/2} \]

(3.13)

where \( a_0 = \alpha^2/\hbar c \) is the fine structure parameter. Hence, the series (2.14) is an expansion for a dimensionless parameter \( z_0/N \), which varies from 0.014 (helium dwarf) to 0.19 (iron dwarf). For the typical dwarfs, mainly consisting of nitrogen and oxygen, \( z_0 \approx 0.1 \). That expansion parameter is a small value, which makes it possible to restrict the consideration to two- and three-electron correlations (we note, that correlation energy of the reference system is of the order of \( a_0^Q \)). Moreover, for the three-electron function \( \tilde{\mu}_0^Q(q_1, q_2, -q_1 - q_2|0) \), there can be used an approximate analytical expression, because the main
integration over the angular variables, reducing \( \gamma \)

Rewriting the sum over vector \( k \) and the spherical coordinate system, the \( O_z \) axis of which coincides with the vector \( \rho_{ij} \), we perform integration over the angular variables, reducing \( \gamma_3(q_i, q_j) \) to one-dimensional integral:

\[
\gamma_3(q_i, q_j) = \frac{3N_e}{4(m_0c^2)^2} \int_0^1 dk \, C(|q_i| \hat{k}) C(|q_j| \hat{k}) f_{ij}(k),
\]

where \( F(q_i, q_j | \k) = \Omega_{ij} + 2(\k, \rho_{ij}) \), \( \Omega_{ij} \equiv q_i^2 + z(q_i^2 - q_j^2); \rho_{ij} = \frac{q_i^2 + q_j^2}{2} + (1 - z)\rho_{ij} \).
allows one to rewrite \( \tilde{\mu} \) the triangle constructed on the vectors \( \nu \). We have calculated the LFCF, using the correlation function \( s \) of the relativistic ideal electron model. Summarizing over the frequencies \( \nu \). In this formula, \( \delta(k) \) is the invariant of the problem \( \delta_{ij}(k) = \delta_{iz}(k) = \delta_{iz}(k) \), \( q_R \) is the radius of the circle, circumscribing the triangle constructed on the vectors \( \nu, q_2, -q_1 - q_2 \); \( t_{ij} \) is a cosine of the angle between the vectors \( \nu, q_j \). In the formulae (3.17), (3.18), the variables \( k \) and \( q_i, q_j \) are dimensionless (in unit \( k_F \)).

The substitution \( C(q_j | k) \rightarrow 2(1 + x^2 k^2)^{1/2} \) corresponds to long-wavelength approximation, which allows one to rewrite \( \tilde{\mu} \) in a compact form:

\[
\tilde{\mu}^0(q_1, q_2, q_3 | 0) = \frac{3 N_c}{(m_0 e^2 x^2)^2} \int_0^{2\pi} \lambda(k) \Phi(k) |q_1, q_2, q_3, \rangle \langle q_1, q_2, q_3| \]

\[
\Phi(k) = \frac{k}{a_1(k)} \frac{1}{q_1 q_2 q_3} \ln \left| \frac{1 + a_1(k) D(k)}{1 - a_1(k) D(k)} \right| \quad \text{by } k < q_R;
\]

\[
\Phi(k) = \frac{2k}{a_2(k)} \frac{1}{q_1 q_2 q_3} \text{arctan}[a_2(k) D(k)] \quad \text{by } k > q_R.
\]

In this formula,

\[
a_1(k) = \left( 1 - \frac{k^2}{q_R^2} \right)^{1/2}; \quad a_2(k) = \left( \frac{k^2}{q_R^2} - 1 \right)^{1/2};
\]

\[
D(k) = \frac{q_1 q_2 q_3}{4 k^3} \left[ 1 - \frac{(q_1^2 + q_2^2 + q_3^2)}{8 k^2} \right] \frac{1}{P(k)};
\]

\[
P(k) = 1 - \frac{q_1^2 + q_2^2 + q_3^2}{4 k^2} + \frac{q_1^4 + q_2^4 + q_3^4}{32 k^4} + \frac{(q_1 q_2 q_3)^2}{64 k^6}.
\]

Dependence of function \( \tilde{\mu}^0(q_1, q_2, -q_1 - q_2 | 0)(3 N_c)^{-1}(m_0 e^2 x^2)^2 \) on wave vectors for different \( t \) are illustrated in figures [3][4].

3.3. The local field correction function

It is well known from the non-relativistic electron fluid theory [16] that the local field correction function (LFCF) in the weak non-ideal model is a universal function of the variable \( y = (q, \nu) \). It does not depend on any parameters and corresponds to the approximation

\[
G_{id}(y) = -\frac{\sum y_i V_{q_i} \tilde{\mu}^0_i(y_i, -y_i)}{2 \beta V_q |\tilde{\mu}^0_q(y, -y)|^2}.
\]

We have calculated the LFCF, using the correlation functions of the relativistic ideal electron model. Summarizing over the frequencies \( \nu_1 \) and \( \nu_2 \) (which appears in \( \tilde{\mu}^0 \)) and using the formula (3.2), we obtain a representation:

\[
G_{id}(y) = -V_q^{-1} |\tilde{\mu}^0_q(y, -y)|^2 \sum_{n_{k_1}, k_2} \text{Re} \left\{ \sum_{k_{1s}, k_{2s}} V(k_{1s} - k_{1s}) f_{k_{1s}, k_{2s}}^+(q, \nu) - V(k_{1s} + k_{2s} + q) f_{k_{1s}, k_{2s}}^+(q, \nu) \right\}.
\]

\[
f_{k_{1s}, k_{2s}}^+(q, \nu) = \text{Re} \left\{ (iv + E_{k_{1s}} - E_{k_{1s} + q})^{-1} \mp (iv + E_{k_{1s}} - E_{k_{1s} + q})^{-1} \right\}^2.
\]
The function $J_3(q_1, q_2, t) = \tilde{\mu}_3(q_1, q_2, q_1 - q_2)(3N_e)^{-1}(m_0c^2x^2)^2$ at $t = 0$ and the relativistic parameter $x = 1.0$.

The behavior of relativistic LFCF at small and large values of vector $q$ is the same as that of non-relativistic case [23]:

$$G_{id}(y) \Rightarrow \begin{cases} \frac{(q/k_F)^2}{4} + \ldots \text{ by } \nu = 0; \ q \ll k_F; \\ \frac{3(q/k_F)^2}{20} + \ldots \text{ by } \nu \gg \epsilon_F; \ q \ll k_F; \\ \frac{1}{3} + \ldots \text{ by } q \gg k_F \text{ and any } \nu. \end{cases} \tag{3.23}$$

For the numerical calculation at the absolute zero temperature, we have used a cylindrical coordinate system for the vectors $k_1, k_2$ [$k_j = (z_j, \rho_j, \varphi_j)$, $k_j^2 = c_j^2 + \rho_j^2$], in which $(k_j, q) = qz_j$, $(k_1 - k_2)^2 = \rho_1^2 + \rho_2^2 + (z_1 - z_2)^2 - 2\rho_1\rho_2\cos(\varphi_1 - \varphi_2)$, $(k_1, k_2) = z_1z_2 + \rho_1\rho_2\cos(\varphi_1 - \varphi_2)$. Integrating over the
Here, the following notations are used:

\[
\begin{align*}
G_{id}(q, \nu) &= \frac{q^4 x^4}{8} f^2(q x, \nu x^2 |x|) \int_{x}^{1} \int_{-1}^{1} \int_{0}^{1} \rho_1 d\rho_1 \int_{0}^{1} \rho_2 d\rho_2 \\
& \quad \times \left[ \frac{f_{q, \nu}(z_1, z_2, \rho_1, \rho_2)}{W_q(z_1, z_2, \rho_1, \rho_2)} - \frac{f_{q, \nu}(z_1, z_2, \rho_1, \rho_2)}{W_q(z_1, z_2, \rho_1, \rho_2)} \right].
\end{align*}
\]

Figure 5. Dependence of the LFCF \(G_{id}(q, \nu)\) on the wave vector \(q\) at frequency \(\nu = 0.01m_0c^2x^2\) (curve 1 — \(x = 0.05\); 2 — \(x = 0.2\); 3 — \(x = 0.5\); 4 — \(x = 1.0\); 5 — \(x = 2.0\); 6 — \(x = 5.0\)).

angular variables \(\varphi_1, \varphi_2\), we reduce \(G_{id}(y)\) to the following 4-dimensional integral:

\[
G_{id}(q, \nu) = \frac{q^4 x^4}{8} f^2(q x, \nu x^2 |x|) \int_{x}^{1} \int_{-1}^{1} \int_{0}^{1} \rho_1 d\rho_1 \int_{0}^{1} \rho_2 d\rho_2 \\
\times \left[ \frac{f_{q, \nu}(z_1, z_2, \rho_1, \rho_2)}{W_q(z_1, z_2, \rho_1, \rho_2)} - \frac{f_{q, \nu}(z_1, z_2, \rho_1, \rho_2)}{W_q(z_1, z_2, \rho_1, \rho_2)} \right].
\]

Here, the following notations are used:

\[
\begin{align*}
f_{q, \nu}(z_1, z_2, \rho_1, \rho_2) &= \left( \frac{\eta_1}{\eta_1^2 + \nu^2} \pm \frac{\eta_2}{\eta_2^2 + \nu^2} \right)^2 - \nu^2 \left( \frac{1}{\eta_1^2 + \nu^2} - \frac{1}{\eta_2^2 + \nu^2} \right)^2, \\
\eta_1 &= \left[ 1 + x^2(z_1^2 + \rho_1^2) \right]^{1/2} - \left[ 1 + x^2(\rho_1^2 + (z_1 + q)^2) \right]^{1/2}, \\
W_q(z_1, z_2, \rho_1, \rho_2) &= \left[ (z_1 + z_2 + q)^2 + (\rho_1^2 - \rho_2^2)^2 + 2(\rho_1^2 + \rho_2^2)(z_1 + z_2 + q)^2 \right]^{1/2}, \\
W_q(z_1, z_2, \rho_1, \rho_2) &= \left[ (z_1 - z_2)^2 + (\rho_1^2 - \rho_2^2)^2 + 2(\rho_1^2 + \rho_2^2)(z_1 - z_2)^2 \right]^{1/2}.
\end{align*}
\]

For a comparison \(G_{id}(q, \nu)\) with LFCF of non-relativistic theory, there were used the variables \(q = |q|x_F^{-1}\), \(\nu = x^2\bar{y}, \bar{y} = \nu(h^2 k_F^2/m)^{-1}\). In figure there is shown a function \(G_{id}(q, \nu)\), calculated by the formula (3.24) at \(\bar{y} = 0.01\), which is very close to the static limit. It is obvious that the asymptotic behavior \(G_{id}(q, \nu)\) for small and large values of the \(|q|\) almost does not depend on the relativistic parameter. The deviation of a relativistic correction from non-relativistic one is significant near its maximum, which monotonously decreases with an increase of relativistic parameter. Figure illustrates the behavior of \(G_{id}(q, \nu)\) at a very high value of frequency (\(\bar{y} = 0.5\)). In general, the behavior of the correction corresponds to the one of non-relativistic theory, and some deviations are caused by frequency renormalization at \(x \gg 1\).

4. The energy of ground state

As we know from the theory of non-relativistic electron fluid, the transition from thermodynamic description to the quantum mechanics is performed by the following procedure: instead of the chemical
can be calculated using the expression

\[ E_{\text{id}} = \lambda_0 \sum_{\mathbf{q}} G_{\text{id}}(\mathbf{q}, \omega) \]

Here, \( \lambda_0 \) is a coupling constant. In the traditional form, this was first discovered in [24] and in the reference system approach — in [17] within the perturbation theory.

Taking into account a weak non-ideal matter of dwarfs, we will consider only two- and three-particle electron correlations. In this approach, the ground state energy of the model is given as follows:

\[
E \approx E_0 + \frac{1}{2V} \sum_{\mathbf{q} \neq 0} \sum_{i,j} z_i z_j \left[ G^{S\delta}(\mathbf{q}) - N_i \delta_{ij} \right] - \frac{3}{(n!)^2} \sum_{i_1, i_2, \ldots, i_n} z_{i_1} z_{i_2} \cdots z_{i_n} \]

\[
\times \sum_{\mathbf{q}_1, \ldots, \mathbf{q}_n \neq 0} \mathcal{V}_{\mathbf{q}_1} \cdots \mathcal{V}_{\mathbf{q}_n}(\mathbf{q}_1^{(i_1)} \cdots \mathbf{q}_n^{(i_n)}) \delta_{\mathbf{q}_1 + \cdots + \mathbf{q}_n = 0} \tilde{\mu}_n(\mathbf{q}_1 \ldots \mathbf{q}_n|0). \tag{4.1}
\]

Here, \( E_0 \) is the ground state energy of the reference system (an interacting relativistic electron gas), which can be calculated using the expression

\[
E_0 = \frac{1}{2V} \sum_{\mathbf{q} \neq 0} \sum_{\nu} \int_0^1 \tilde{\mu}_2^{(\nu)}(y, -y) d\nu, \tag{4.2}
\]

where \( \tilde{\mu}_2^{(\nu)}(y, -y) \) is the two-particle dynamic correlation function of the auxiliary model of electrons with the potential of interactions \( \lambda V_0 \), and \( E_0 \) is the energy of the ideal electron system

\[
E_0 = \sum_{\mathbf{k}_{\perp}, s} n_{\mathbf{k}_{\perp}, s}(\mu_0) E_{\mathbf{k}_{\perp}, s}. \tag{4.3}
\]

Extracting from the second term of formula (4.2) the contribution of ideal correlation

\[
E_{\text{HF}} = -\frac{1}{2V} \sum_{\mathbf{q} \neq 0} \sum_{\nu} \tilde{\mu}_2^{(\nu)}(y, -y) = -\frac{1}{2V} \sum_{\mathbf{q} \neq 0} \sum_{\mathbf{k}_{\perp}, s} V_0 n_{\mathbf{k}_{\perp} + \mathbf{q}_{\perp}/2, s} n_{\mathbf{k}_{\perp} - \mathbf{q}_{\perp}/2, s}, \tag{4.4}
\]

and considering, that \( G_{\text{id}}(\mathbf{q}) \) does not depend on the “coupling constant” \( \lambda \), we have represented \( E_0 \) in a traditional form

\[
E_0 = E_0 + E_{\text{HF}} + E_c, \tag{4.5}
\]
where
\[
E_c = \frac{1}{2 \beta} \sum_{q \neq 0} \sum_{\nu} \ln \left[ \frac{1 + L(y)}{1 - G_{\nu}(y)} \right], \quad L(y) = \frac{V_0 a_{\nu}^2(y, y - y)[1 - G_{\nu}(y)]}{V_2},
\]
(4.6)
is the so-called correlation energy (the contribution of non-ideal correlations). In units \(m_0 c^2\) we obtain
\[
E_c = N_e m_0 c^2 \left[ \varepsilon_0(x) - \frac{3}{4 \pi} a_0 x + a_0^2 \varepsilon_c(x) \right],
\]
(4.7)
where
\[
\varepsilon_0(x) = (2x)^{-3} \left\{ 3x(1 + x^2)^{1/2}(1 + 2x^2) - 8x^3 - 3 \ln \left[ x + (1 + x^2)^{1/2} \right] \right\}
\]
(4.8)
is the contribution of an ideal system per one electron, \(-3 a_0 x (4\pi)^{-1}\) is the contribution of interactions of the Hartree-Fock approximation, \(a_0^2 \varepsilon_c(x)\) is the correlation contribution. According to our calculations, the \(\varepsilon_c(x)\) can be approximated with the following expression:
\[
\varepsilon_c(x) = -\frac{b_0}{2} \int_0^x dr \frac{b_1 a + 1/2}{r^{3/2} + t b_1 a + b_2 a^{1/2} t^2 + b_3 a^2} \frac{1 + a_1 t + a_2 t^2}{1 + t d_0},
\]
(4.9)
\[
a = (a_0 t_0)^{1/2}; \quad a_1 = 2.25328; \quad a_2 = 4.87991; \quad d_0 = 0.924022;
\]
\[
b_0 = 0.0621814; \quad b_1 = 9.81379; \quad b_2 = 2.82214; \quad b_3 = 0.73701.
\]
At \(a_1 = a_2 = d_0 = 0\), the expression matches the approximation for the correlation energy \(25\), which is calculated using the Monte-Carlo method \(26\) \(\varepsilon_c^{MC}(x)\). As we can see in figure \(7\) in the range \(x < 1\), the expression \(4.9\) is close to \(\varepsilon_c^{MC}(x)\), and the deviation \(\varepsilon_c(x)\) from \(\varepsilon_c^{MC}(x)\) in the region \(x > 1\) is caused by different asymptotics of these functions \(\varepsilon_c(x) \rightarrow -\frac{b_0}{2} a_0^2 x + \ldots, \varepsilon_c^{MC}(x) \rightarrow -\frac{b_0}{2} \ln x + \ldots \) at \(x \gg 1\).

In order to calculate the contributions of electron-nuclear interactions in the products of structure factors in the formula \(4.1\), we have selected one-particle and two-particle sums by the coordinates of nuclei, and the three-nuclear effective interactions were neglected. In this approach,
\[
E \approx E_c + E_{pol} + E_{conf},
\]
(4.10)

\[\text{Figure 7. Dependence of functions } \varepsilon_{pol}^{(2)}(x), 10 \cdot \varepsilon_{pol}^{(3)}(x) \text{ and } \varepsilon_c(x) \text{ on the relativistic parameter.}\]
where $E_{\text{pol}}$ is the polarization energy of electron fluid by nuclei, which does not depend on the structure of the nuclear subsystem,

$$E_{\text{pol}} = E_{\text{pol}}^{(2)} + E_{\text{pol}}^{(3)},$$

$$E_{\text{pol}}^{(2)} = -\frac{z_1^2 N_1 + z_2^2 N_2}{2! V^2} \sum_q V_q^2 \tilde{\mu}_2(q, -q|0),$$

$$E_{\text{pol}}^{(3)} = -\frac{z_1^3 N_1 + z_2^3 N_2}{3! V^3} \sum_{q_1, q_2} V_{q_1} V_{q_2} V_{-q_1 - q_2} \tilde{\mu}_3(q_1, q_2, -q_1 - q_2|0).$$

(4.11)

The configuration energy is determined by the structure of a nuclear subsystem, and expressed through an effective two-particle potential of interactions of the nuclei, which is formed by two- and three-electron correlations:

$$E_{\text{conf}} = \sum_{\ell=1,2} \frac{z_\ell^2}{2!} \sum_q V_q \left[ 1 - \frac{V_q}{V} \tilde{\mu}_2(q, -q|0) - \frac{z_\ell}{V^2} \sum_{q_1} V_{q_1} V_{-q_1} \tilde{\mu}_3(q, q_1, -q - q_1|0) \right] S^{(\ell)}_2(q, -q)$$

$$+ \frac{z_1 z_2}{V} \sum_q V_q \left[ 1 - \frac{V_q}{V} \tilde{\mu}_2(q, -q|0) - \frac{z_1 + z_2}{2V^2} \sum_{q_1} V_{q_1} V_{-q_1} \tilde{\mu}_3(q, q_1, -q - q_1|0) \right] S^{(1)}_q S^{(2)}_q,$$

$$S^{(\ell)}_2(q, -q) = \sum_{j_1 \neq j_2} \exp \left[ i(q, R_{j_1} - R_{j_2}) \right].$$

(4.12)

### 4.1. The approximation of two-electron correlations

Let us rewrite the component $E_{\text{pol}}^{(2)}$, calculated in the local field approximation, in the form

$$E_{\text{pol}}^{(2)} = N_e m_0 c^2 \left( \frac{Z^2}{\langle z \rangle} \right) \alpha_0^{3/2} E_{\text{pol}}^{(2)}(x),$$

(4.13)

where the dimensionless function $E_{\text{pol}}^{(2)}(x)$ is of the same order as $\epsilon_c(x)$, and $\langle z^\mu \rangle = (z_1^N N_1 + z_2^N N_2)(N_1 + N_2)^{-1}$. The function $E_{\text{pol}}^{(2)}(x)$ can be approximated by the following expression:

$$E_{\text{pol}}^{(2)}(x) = -\frac{1}{x} \left[ c_0 + c_1 t + c_2 t^2 + c_3 t^3 \right] + d_1 t + d_2 t^2 + d_3 t^3,$$

$$c_0 = 4.06151; \quad c_1 = 32.6118; \quad c_2 = -43.6587; \quad c_3 = 104.13;$$

$$d_1 = 73.8252; \quad d_2 = -67.1028; \quad d_3 = 189.781.$$

(4.14)

As it is shown in figure, $E_{\text{pol}}^{(2)}(x)$ has linear asymptotics at $x \gg 1$, as well as $\epsilon_c(x)$ However, the polarization energy $E_{\text{pol}}^{(2)}$ exceeds the correlation energy of the reference system by about $\langle z \rangle \alpha_0^{-1/2} \approx 10 \langle z \rangle$ times, and for $\langle z \rangle \approx 10$ it is comparable with $E_{\text{HF}}$.

The configuration energy was calculated in the coordinate representation and, introducing the effective two-nuclear interactions, yielded

$$V^{i_{j_1}, j_2}_{i_{j_2}}(R_{j_1}^{(i_1)} - R_{j_2}^{(i_2)}) = \frac{1}{V} \sum_q V_2(q) \exp \left[ i(q, R_{j_1}^{(i_1)} - R_{j_2}^{(i_2)}) \right],$$

$$V_2(q) = V_q \left[ 1 - \frac{V_q}{V} \tilde{\mu}_2(q, -q|0) \right].$$

(4.15)
In the formula (4.15) the sum of the vector $q$ includes a component with $q = 0$. Therefore,

\[
E_{\text{conf}}^{(2)} = \frac{1}{V} \sum_{q \neq 0} V_2(q) \left[ \frac{1}{2} \sum_{i=1,2} \zeta_{i(q)}(q_z - q_z)^2 + z_1z_2 \zeta_{(1)}(q_z - q_z)^2 \right]
\]

\[
= \frac{1}{2} \sum_{i=1,2} \zeta_{i}^2 \sum_{j \neq j=1} V_2(R_{ij}^{(i)} - R_{ij}^{(j)}) + z_1z_2 \sum_{j=1}^{N_1} \sum_{j=1}^{N_2} V_2(R_{ij}^{(i)} - R_{ij}^{(j)}) - \frac{1}{2} N_2^2 \lim_{q \to 0} \left[ \frac{V_2(q)}{V} \right].
\] (4.16)

To simplify the calculation of the lattice sum, we adopt a simple model of nuclei distribution:

\[
N_j^{(1)} = \frac{N_1}{N_1 + N_2} N_j, \quad N_j^{(2)} = \frac{N_2}{N_1 + N_2} N_j,
\] (4.17)

where $N_j$ is the number of all knots on the $j$-th coordination sphere, and $N_j^{(i)}$ is the number of the knots occupied by nuclei with charge $z_i$ ($i = 1, 2$). In this model,

\[
E_{\text{conf}}^{(2)} = \frac{N_e}{2} \langle z \rangle \sum_{j=1}^{N_j} N_j V_2(R_j) - \frac{N_e m_0 c^2}{6} \left[ \frac{x^2}{(1 + x^2)^{3/2}} - \frac{x a_0}{\pi} \right],
\] (4.18)

where $R_j$ is the radius of the $j$-th coordination sphere.

The effective two-particle potential is screened, and at small and medium distances between nuclei it is close to the expression

\[
V(R) = \frac{e^2}{R} \exp(-R/R_0),
\] (4.19)

and the screening radius

\[
R_0 = \frac{\sqrt{\pi}}{2} a_B^{1/2} a_B [x^{1/2}(1 + x^2)^{1/4}]^{-1}
\] (4.20)

is of the order $0.1a_B$ (where $a_B = \hbar^2/m_0 c^2$ is the Bohr radius). At large distances, $V_2(R)$ oscillates, but with a small amplitude,

\[
V_2(R) \approx \frac{e^2}{a_B} \left( \frac{R_0}{2 \pi R} \right)^3 \cos(2\pi R/R_0).
\] (4.21)

The configuration energy for a simple cubic lattice of nuclei is calculated numerically and can be represented as

\[
E_{\text{conf}}^{(2)} = N_e m_0 c^2 \langle z \rangle^{2/3} a_0 \varepsilon_L^{(2)}(\langle z \rangle),
\] (4.22)

with a dimensionless factor approximated by the expression

\[
\varepsilon_L^{(2)}(\langle z \rangle) = - \int_0^x \frac{a_1 + t a_2 + t^2 a_3 + t^3 a_4 + t^4 a_5 + t^5 a_6}{1 + t a_4 + t^2 a_5 + t^3 a_6} dt.
\] (4.23)

where all the coefficients $a_1, \ldots, a_6$ are the functions of $\langle z \rangle$, that is

\[
a_i(\langle z \rangle) = \frac{a_i(\langle z \rangle) a_i(\langle z \rangle) + (\langle z \rangle)^2 a_i(\langle z \rangle)}{a_i(\langle z \rangle) a_i(\langle z \rangle) + (\langle z \rangle)^2 a_i(\langle z \rangle)}.
\] (4.24)

The coefficients $a_i$ of the formula (4.24) are listed in table 1. In figure there is shown a dependence of the configuration energy on the relativistic parameter.
### Table 1. Coefficients $a_{ij}$ from (4.24).

| $i$ \ $j$ | 0     | 1     | 2     | 3     | 4     | 5     |
|-----------|-------|-------|-------|-------|-------|-------|
| 0         | −128.112 | −138.098 | −3.30915 | 0 | 3.74936 | 0.882489 |
| 1         | −633.899 | 297.304 | −19.5138 | 1 | −0.707632 | 1.01638 |
| 2         | −1691 | 216.967 | −8.71667 | 0 | 1.48694 | 0.12998 |
| 3         | 2.37539 | 1.74513 | 0.0417739 | 0 | 0.0212554 | 0.0056168 |
| 4         | 913.016 | −452.217 | 30.3618 | 1 | −0.750844 | 0.702212 |
| 5         | 5.68901 | −0.704184 | 0.0277872 | 0 | 0.00203554 | 0.000234399 |

Figure 8. Dependence of the lattice energy $\varepsilon_L^{(2)}(x|\langle z \rangle)$ on the nuclear charge and the relativistic parameter.

### 4.2. The effect of three-electron correlations

According to the formula (4.11), the contribution of three-particle correlations is represented as

$$E^{(3)}_{\text{pol}} = N_{e m o e} c^2 \langle z \rangle^3 \varepsilon_0^{(3)}(x). \tag{4.25}$$

The result of numerical calculations is illustrated in figure 7, which shows that for $x > 1$ the ratio $\varepsilon^{(3)}_{\text{pol}}(x) \approx 0.1\varepsilon^{(2)}_{\text{pol}}(x)$ is satisfied. Therefore, at sufficiently large values of nuclei charges, $E^{(3)}_{\text{pol}}$ is not less than the correlation energy of the reference system: at $\langle z \rangle \geq 6$, the contribution $E^{(3)}_{\text{pol}}$ is close to the correlation energy, at $\langle z \rangle \geq 12$, it exceeds the correlation energy by 5 times, and at $\langle z \rangle = 26$ — more than by 20 times. The result of numerical calculation $\varepsilon^{(3)}_{\text{pol}}(x)$ is approximated by the expression

$$\varepsilon^{(3)}_{\text{pol}}(x) = -a x - c_0 \int_x^\infty \frac{1 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} \, dt,$$

$$a = 0.0450; \quad c_0 = 0.12607; \quad c_1 = -0.93695; \quad c_2 = 78.8552;$$
$$d_1 = -23.2602; \quad d_2 = 114.5030; \quad d_3 = 164.060. \tag{4.26}$$

From the formulae (4.13), (4.25) it follows that $E^{(3)}_{\text{pol}}/E^{(2)}_{\text{pol}} \sim 0.12\varepsilon_0$, it determines the order of three-electron correlations contribution value.
Similar to the formula \(4.15\), we have introduced a correction to the effective two-nuclear potential by the three-electron correlations

\[
V_2^{(3)}(R) = -V^3 \sum_q V_q \sum_{q_1} V_{q-q_1} |\tilde{\mu}_3(q, q_1, -q - q_1)| - |0\rangle \exp[i(q, R)].
\]  

(4.27)

where the sum over the vector \(q\) includes the component \(q = 0\). As shown in figure \(4.2\), \(V_2^{(3)}(R)\) is a weak attracting potential of the type of quantum package screening potential, which is close to the expression

\[
V_2^{(3)}(R) = -\frac{e^2}{R} a_0^2 A(x) \left\{ 1 - \exp \left[ -\frac{R}{R_0} \gamma(x) \right] \right\} \exp(-R/R_0).
\]  

(4.28)

The functions \(A(x)\) and \(\gamma(x)\) are determined as follows:

\[
A(x) = \frac{8 I_2(x)}{\pi x^2}, \quad \gamma(x) = \frac{a_0^{3/2} x^{1/2} (1 + x^2)^{3/4}}{2 \sqrt{\pi}} \cdot \frac{I_1(x)}{I_2(x)};
\]

\[
I_1(x) = \int_0^\infty \frac{dq_1}{\varepsilon(q_1)} \int_0^\infty \frac{dq_2}{\varepsilon(q_2)} \int_0^\infty \frac{dr}{q_1^2 \varepsilon(q_3)} f_3(q_1, q_2, 1), \quad I_2(x) = \int_0^\infty \frac{dq}{q^2 \varepsilon^2(q)} J_3(q);
\]

\[
J_3(q) = \frac{(m_0 c^2 x^2)^2}{3} \left\{ \mu_3^0(q, -q, 0) |0\rangle; \quad f_3(q_1, q_2, 1) = \frac{(m_0 c^2 x^2)^2}{3 |0\rangle (1 + x^2)^{3/4} \tilde{\mu}_3^0(q_1, q_2, -q_1 - q_1 |0\rangle);
\]

(4.29)

The contribution to the configuration energy of the model by the three-particle correlations in the model \(4.17\) takes the form, similar to the formula \(4.18\):

\[
E_{\text{conf}}^{(3)} = \frac{1}{2} N_e \langle z \rangle \sum_{j > 1} N_j V_2^{(3)}(R_j) + \frac{4}{3 \pi^2} N_e \alpha_0^2(\langle z \rangle) m_0 c^2 (1 + x^2)^{-1/2} I_2(x).
\]  

(4.30)

This contribution is calculated for a simple cubic lattice of nuclei and is represented by

\[
E_{\text{conf}}^{(3)} = N_e m_0 c^2 \alpha_0^2(\langle z \rangle) \tilde{e}_L^{(3)}(\langle z \rangle).
\]  

(4.31)
The dependence of a dimensionless factor $\varepsilon_L^{(3)}(x|\langle z \rangle)$ is illustrated in figure 10. At sufficiently large nuclei charges $\langle z \rangle$ and $x \geq 2$, the function $\varepsilon_L^{(3)}(x|\langle z \rangle) \sim 0.1 \varepsilon_L^{(2)}(x|\langle z \rangle)$, but it has a positive sign. It is approximated by the expression

$$\varepsilon_L^{(3)}(x|\langle z \rangle) = -a_0(\langle z \rangle) + a_1(\langle z \rangle)x + a_2(\langle z \rangle)x^2.$$  \hspace{1cm} (4.32)

The coefficients of the formula (4.32) are shown in table 2.

### Table 2. Coefficients from (4.32).

| $\langle z \rangle$ | $a_0$       | $a_1$       | $a_2$       |
|---------------------|-------------|-------------|-------------|
| 2                   | 0.160403    | 0.422066    | -0.707095   |
| 4                   | 0.118181    | 0.351743    | -0.320001   |
| 6                   | 0.0978784   | 0.321462    | -0.203819   |
| 8                   | 0.08553532  | 0.302602    | -0.150039   |
| 12                  | 0.0701315   | 0.278365    | -0.100583   |
| 26                  | 0.04788678  | 0.236761    | -0.0549487  |

5. **The equation of state of the model at $T = 0$ K**

For the well-known dependence of the model energy on the relativistic parameter we calculate the equation of state of cold degenerate matter using the expression

$$P(x) = \frac{dE}{dV} = \frac{x^4}{N_e \left( \frac{m_0c}{\hbar} \right)^3 (3\pi^2)^{-1}} \frac{dE}{dx}.$$ \hspace{1cm} (5.1)

Within the accepted approximation

$$P(x) = \frac{\pi m_0^5 r_3}{3\hbar^3} [F(x) + f_2(x) + f_3(x)].$$ \hspace{1cm} (5.2)
Table 3. Dependence of the functions $F(x) \cdot x^{-4}$, $f_2(x) \cdot x^{-4}$ and $f_3(x) \cdot x^{-4}$ on the relativistic parameter $x$ according to the formulae (5.2)-(5.4) and (5.5).

| $x$ | $F(x) \cdot x^{-4}$ | $f_2(x) \cdot x^{-4}$ | $f_3(x) \cdot x^{-4}$ |
|-----|---------------------|---------------------|---------------------|
|     | $z = 2$             | $z = 3$             | $z = 4$             |
| 0.5 | 0.74785             | 0.01538             | 0.0005869           |
| 0.6 | 0.83948             | 0.02563             | 0.0039354           |
| 0.7 | 0.92754             | 0.03375             | 0.0070115           |
| 0.8 | 1.01041             | 0.04204             | 0.0099522           |
| 0.9 | 1.09406             | 0.05463             | 0.0120150           |
| 1.0 | 1.17991             | 0.06679             | 0.0131648           |

Here,

$$F(x) = \ln(1 + x^2)$$

is the contribution of the ideal degenerate relativistic spatially homogeneous electron gas;

$$f_2(x) = -2\alpha_0 x^4 \left\{ \frac{1}{\pi} \frac{4}{3} \frac{d}{dx} \left[ \langle z \rangle^{2/3} \varepsilon_L^{(2)}(x | z) \right] + \frac{\langle z \rangle}{\langle z \rangle} \alpha_0^{1/2} \varepsilon_p^{(2)}(x) \right\}$$

is the contribution of the Coulomb interactions in the two-electron correlations approximation;

$$f_3(x) = 8\alpha_0 x^4 \frac{d}{dx} \left[ \langle z \rangle \varepsilon_L^{(3)}(x | z) \right] + \frac{\langle z \rangle}{\langle z \rangle} \alpha_0^{1/2} \varepsilon_p^{(3)}(x)$$

is the contribution of the three-particle electron correlations.

In the region $x \gg 1$, all contributions [with the exception of $\varepsilon_L^{(3)}(x | z)$] to the model energy caused by interactions are negative monotonously decreasing functions of the relativistic parameter. In the two-electron correlations approximation, the equation of state at $x \gg 1$ numerically is very close to the result of Salpeter [8].

In table 3 there is shown a dependence of terms $F(x)$, $f_2(x)$, $f_3(x)$ on the relativistic parameter for the helium ($z_1 = z_2 = 2$), carbon ($z_1 = z_2 = 6$), nitrogen-oxygen ($z_1 = 7$, $z_2 = 8$; $N_1 = N_2$) and magnesium ($z_1 = z_2 = 12$) dwarf models. A relative decrease of pressure caused by the interactions $[F(x) + f_2(x) + f_3(x)]F^{-1}(x)$ is illustrated in figure 1.

6. Conclusions

Reference system approach was adapted for the description of a degenerate relativistic electron subsystem. There were investigated the features of two- and three-particle correlation functions in a wide domain of a relativistic parameter, as well as there was obtained an exact expression for the static two-particle correlation function. For the first time, the expressions for the three-particle correlation function were approximated and the local field correction of interacting relativistic electron gas was studied, which is the basis for the calculation of the energy and structural characteristics of degenerated dwarfs. The energy of the ground state of the electron-nuclear model, as well as the equation of state of the model, was calculated in a wide range of the relativistic parameter at absolute zero temperature. As it was
shown in our calculation, the contributions of Coulomb interactions to the energy of the ground state and pressure, caused by two-electron correlations are important and increase with an increase of nuclear charge. The contributions caused by three-electron correlations are much smaller but they exceed the contribution of correlation energy of the electron fluid, especially at large values of the nuclear charge.

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Базисний підхід в теорії вироджених карликів

М.В. Ваврух, Д.В. Дзіковський, Н.Л. Тищко
Львівський національний університет імені Івана Франка, кафедра астрофізики, вул. Кирила і Мефодія, 8, 79005 Львів, Україна

Базисний підхід нерелятивістської теорії електронної рідини адаптовано до розрахунку характеристики електрон-ядерної моделі при густинах, що відповідають виродженим карликам. Розраховано дво- та тричастинкові кореляційні функції виродженої релятивістської електронної частинки у імпульсо-частотному представленні у наближенні локального поля. В адіабатичному наближенні обчислено основні внески кулонівських взаємодій в енергію та рівняння стану моделі при \( T = 0 \) К.

Ключові слова: електрон-ядрен модель, кореляційні функції, поправка на локальне поле, енергія моделі при \( T = 0 \) К, рівняння стану
