Coisotropic deformations of associative algebras and dispersionless integrable hierarchies

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Abstract

The paper is an inquiry of the algebraic foundations of the theory of dispersionless integrable hierarchies, like the dispersionless KP and modified KP hierarchies and the universal Whitham’s hierarchy of genus zero. It stands out for the idea of interpreting these hierarchies as equations of coisotropic deformations for the structure constants of certain associative algebras. It discusses the link between the structure constants and the Hirota’s tau function, and shows that the dispersionless Hirota’s bilinear equations are, within this approach, a way of writing the associativity conditions for the structure constants in terms of the tau function. It also suggests a simple interpretation of the algebro-geometric construction of the universal Whitham’s equations of genus zero due to Krichever.
1 Introduction.

The purpose of this paper is to introduce the concept of "coisotropic deformations" of associative algebras and to show its relevance for the theory of integrable hierarchies of dispersionless PDE's.

The concept of coisotropic deformation originates from a melting of ideas borrowed from commutative algebra and Hamiltonian mechanics. From commutative algebra comes the idea of structure constants $C^i_{jk}$ defining the table of multiplication

$$p_j p_k = \sum_{l=1}^{n} C^l_{jk} p_l$$

of a commutative associative algebra with unity. They obey the commutativity conditions

$$C^i_{jk} = C^i_{kj}$$

and the associativity conditions

$$\sum_{l=1}^{n} C^l_{jk} C^p_{lm} = \sum_{l=1}^{n} C^l_{mk} C^p_{lj}.$$  

Furthermore, if the algebra is infinite-dimensional, they are assumed to vanish for $l$ sufficiently large, so that the sums are always over a finite number of terms. From deformation theory comes the idea of regarding the structure constants $C^i_{jk}$ as functions of a certain number of deformation parameters $x_j$. From Hamiltonian mechanics come the ideas of introducing a deformation parameter $x_j$ for each generator $p_j$ of the algebra, and of considering the pairs $(x_j, p_j)$ as pairs of conjugate canonical variables.

In this frame, the characteristic trait of the theory of coisotropic deformations is to associate with the structure constants $C^i_{jk}(x_1, x_2, \ldots, x_n)$ the set of quadratic Hamiltonians

$$f_{jk} = -p_j p_k + \sum_{i=1}^{n} C^i_{jk}(x_1, x_2, \ldots, x_n) p_i,$$

the polynomial ideal

$$J = \langle f_{jk} \rangle$$

generated by these Hamiltonians, and the submanifold

$$\Gamma = \{(x_j, p_j) \in R^{2n} \mid f_{jk} = 0\}$$

where these Hamiltonians vanish. It lives in $R^{2n}$ endowed with the canonical Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_i} \right).$$
Definition 1 If the ideal \( J \) is closed with respect to the Poisson bracket

\[
\{J, J\} \subset J,
\]
so that \( \Gamma \) is a coisotropic submanifold of \( \mathbb{R}^{2n} \), the functions \( C_{jk}^l \) of the deformation parameters \( x_j \) are said to define a coisotropic deformation of the associative algebra.

Owing to the identity

\[
\{f_{jk}, f_{lr}\} = \sum_{m=1}^{n} [C, C]_{jklr}^m p_m
\]

\[
+ \sum_{m=1}^{n} \frac{\partial C^{lm}}{\partial x_k} f_{jm} + \sum_{m=1}^{n} \frac{\partial C^{lm}}{\partial x_j} f_{km}
\]

\[
- \sum_{m=1}^{n} \frac{\partial C^{rm}}{\partial x_r} f_{jm} - \sum_{m=1}^{n} \frac{\partial C^{rm}}{\partial x_l} f_{lm},
\]

defining the Schouten-type bracket

\[
[C, C]_{jklr}^m = \sum_{s=1}^{n} \left( C_{sj}^m \frac{\partial C^{rs}}{\partial x_k} + C_{sk}^m \frac{\partial C^{rs}}{\partial x_j} - C_{sr}^m \frac{\partial C^{ls}}{\partial x_k} \right)
\]

\[
- C_{sl}^m \frac{\partial C^{jk}}{\partial x_r} + \frac{\partial C^{jm}}{\partial x_s} C_{lr}^s - \frac{\partial C^{rm}}{\partial x_s} C_{jk}^s \right),
\]

one readily sees that the structure constants define a coisotropic deformation if and only if they satisfy the system of partial differential equations

\[
[C, C]_{jklr}^m = 0
\]

for any value of the indices \((j, k, l, m, r)\) ranging from zero to the dimension of the algebra. This system and the associativity conditions are central in our approach. For this reason they will be afterwards referred to as the central system of the theory of coisotropic deformations.

The thesis of the present paper is that the central system conceals a lot of interesting examples of dispersionless integrable hierarchies of soliton theory, and that it provides a new interesting route to understand their integrability. This thesis is argued in six sections according to the following plan:

- Sec.2 explains the origin of the new viewpoint from the theory of generalized dispersionless KP equations. In particular it explains how to write these equations in the form of central system.

- Sec.3 is a short study of the structure constants associated with the dispersionless KP hierarchy, in search of property of these constants which make the dKP equations integrable.
• Sec. 4 presents the first noticeable outcome of the new approach. It shows that, for certain classes of algebras, the associative conditions (3) and the coisotropy conditions (10) nicely interact to produce the existence of a tau function seen as a potential for the structure constants $C_{\ell}^{jk}$. It also shows that the well-known dispersionless Hirota’s bilinear equations are nothing else than the associativity conditions (3) written in terms of this potential. This result should make easier the comprehension of the link between the Hirota’s equations and the associativity equations of Witten, Dijkgraaf, Verlinde, Verlinde.

• Sec. 5 is a brief study of the symmetry properties of the central system, which has an infinite-dimensional Abelian symmetry group. The study of the orbits of this group in the space of structure constants allows to enlarge the arsenal of interesting systems of structure constants at our disposal, and to prove the invariance of the tau function under the action of the symmetry group.

• Sec. 6 presents the second noticeable outcome of the new approach. The technique of quotient algebras is used to glue together several copies of the algebras associated with the dKP and dmKP hierarchies in such a way to obtain new interesting examples of central systems. The class of integrable equations covered by the new central systems is sufficiently large to encompass the universal Whitham’s hierarchy of genus zero studied by Krichever.

• Sec. 7 is a terse study of coisotropic deformations from the viewpoint of the geometry of the submanifolds $\Gamma$ previously introduced.

The paper ends with the indication of a few further possible developments, and with two Appendices containing the details of the computations presented along the paper.

2 Introduction to the idea of coisotropic deformations

The examples of dispersionless KP and mKP equations are well suited to illustrate the connection between coisotropic deformations and integrable hierarchies. In this section we recast these equations as a central system for the structure constants of a simple commutative associative algebra with unity.

In the early eighties, when the interest for the dispersionless limit of soliton equations increased (see e.g. [1]-[38]), it was well-known that the standard dKP and dmKP equations arise as compatibility conditions of the linear problem
\[ \frac{\partial \psi}{\partial x_2} = \frac{\partial^2 \psi}{\partial x_1^2} + u_1(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_1} + u_0(x_1, x_2, x_3) \psi, \]

\[ \frac{\partial \psi}{\partial x_3} = \frac{\partial^3 \psi}{\partial x_1^3} + v_2(x_1, x_2, x_3) \frac{\partial^2 \psi}{\partial x_1^2} + v_1(x_1, x_2, x_3) \frac{\partial \psi}{\partial x_1} + v_0(x_1, x_2, x_3) \psi. \]

It was therefore natural to assume that their dispersionless limits are the compatibility conditions of the pair of time dependent Hamilton-Jacobi equations:

\[ \frac{\partial S}{\partial x_2} = \left( \frac{\partial S}{\partial x_1} \right)^2 + u_1(x_1, x_2, x_3) \frac{\partial S}{\partial x_1} + u_0(x_1, x_2, x_3), \]

\[ \frac{\partial S}{\partial x_3} = \left( \frac{\partial S}{\partial x_1} \right)^3 + v_2(x_1, x_2, x_3) \left( \frac{\partial S}{\partial x_1} \right)^2 + v_1(x_1, x_2, x_3) \frac{\partial S}{\partial x_1} + v_0(x_1, x_2, x_3). \]

The compatibility conditions were obtained, as in the dispersive case, by imposing the equality of the second-order mixed derivatives of the function \( S(x_1, x_2, x_3) \), and the result was the following system of four partial differential equations:

\[ \frac{\partial}{\partial x_1} (2v_2 - 3u_1) = 0, \]

\[ \frac{\partial}{\partial x_1} (2v_1 - 3u_0) - \frac{\partial v_2}{\partial x_2} + u_1 \frac{\partial v_2}{\partial x_1} - 2v_2 \frac{\partial u_1}{\partial x_1} = 0, \]

\[ 2 \frac{\partial v_0}{\partial x_1} - \frac{\partial v_1}{\partial x_2} - 2v_2 \frac{\partial v_1}{\partial x_1} - v_1 \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0, \]

\[ - \frac{\partial v_0}{\partial x_2} + u_1 \frac{\partial v_1}{\partial x_1} - v_1 \frac{\partial u_0}{\partial x_1} + \frac{\partial u_0}{\partial x_3} = 0. \]

From it the dKP and dmKP equations \(^{[10]}\) are obtained by setting

\[ u_1 = 0 \quad v_2 = 0 \quad v_1 = 3/2u_0, \]

and

\[ u_0 = 0 \quad v_0 = 0 \quad v_2 = 3/2u_1 \]

respectively. This selection of particular classes of equations by means of suitable additional constraints was called "gauge fixing".

Looking critically at this procedure, one may remark that the method of compatibility conditions leaves a wide freedom in the choice of the form of the auxiliary problem, allowing to substitute the above pair of Hamilton-Jacobi equations by any equivalent system of partial differential equations, without that this change does modify the compatibility conditions and hence the equations one is interested in. This inherent freedom is poorly reflected by the above approach. For this reason we prefer to follow a different route, by replacing the above pair of Hamilton-Jacobi equations with the polynomial ideal

\[ J = \langle h_2, h_3 \rangle \]
generated by the pair of Hamiltonian functions

\begin{align*}
h_2 &= -p_2 + p_1^2 + u_1(x_1, x_2, x_3)p_1 + u_0(x_1, x_2, x_3), \\
h_3 &= -p_3 + p_1^3 + v_2(x_1, x_2, x_3)p_1^2 + v_1(x_1, x_2, x_3)p_1 + v_0(x_1, x_2, x_3).
\end{align*}

The use of this ideal allows to account simultaneously for all the possible forms of the auxiliary problems. The compatibility conditions as easily recovered as the conditions saying that the ideal \( J \) is closed with respect to the classical Poisson bracket in \( \mathbb{R}^6 \). This classical viewpoint throws us immediately into the theory of coisotropic deformations. To arrive to the central system, it is sufficient to follow the construction of the dKP and dmKP hierarchies.

The first higher equation in the dKP and dmKP hierarchies is defined by the compatibility conditions expressing the closure with respect to the classical Poisson bracket in \( \mathbb{R}^8 \) of the polynomial ideal generated by the three Hamiltonians

\begin{align*}
h_2 &= -p_2 + p_1^2 + u_1(X)p_1 + u_0(X) \\
h_3 &= -p_3 + p_1^3 + v_2(X)p_1^2 + v_1(X)p_1 + v_0(X) \\
h_4 &= -p_4 + p_1^4 + w_3(X)p_1^3 + w_2(X)p_1^2 + w_1(X)p_1 + w_0(X),
\end{align*}

where \( X = (x_1, x_2, x_3, x_4) \) is the new set of coordinates. Continuing the process, at each step one introduces a new pair of coordinates \((x_j, p_j)\) and a new Hamiltonian

\[ h_j = -p_j + P_j(p_1) \text{ for } j \geq 2, \]

where \( P_j \) is a monic polynomial of degree \( j \). The sequence of polynomials \( P_j \), defining the hierarchy, does not contain the polynomials \( P_0 = 1 \) and \( P_1 = p_1 \), and therefore does not form a basis of the ring of polynomials. To remedy this defect without changing the system of compatibility conditions, it is expedient to introduce four new coordinates \((x_0, p_0, x, p)\) and two new Hamiltonians

\[ h_0 = -p_0 + 1 \quad h_1 = -p_1 + p, \]

and to regard the polynomials \( P_j \) as polynomials in \( p \) rather than in \( p_1 \). The ideal \( K \) generated by the Hamiltonians

\[ h_j = -p_j + P_j(p) \text{ for } j \geq 0 \quad (12) \]

has the same closure conditions of the previous ideal \( J \). Indeed the conditions

\[ \{h_0, K\} \subseteq K \quad \{h_1, K\} \subseteq K \]

simply entail that \( x_0 \) is a cyclic coordinate, and that \( x \) appears always in the form \( x + x_1 \). By this property the remaining closure conditions

\[ \{h_j, K\} \subseteq K \]

\[ j \geq 2 \]
give exactly the compatibility conditions of $J$.

The advantage of the completion of the basis is that one may now consider the structure constants $C_{jk}^l(X)$ defined by

$$P_j(P)P_k(P) = \sum_{l=0}^{l=j+k} C_{jk}^l(X)P_l(P).$$

By construction they depend polynomially on the coefficients of the Hamiltonians $h_j$, and therefore they also satisfy a system of partial differential equations. The main task of this section is to identify these equations.

If one tries to attack the problem directly, by using the explicit form of the dKP and dmKP equations, one may get easily lost. The best strategy is to use the properties of the ideal $K$, and to notice that the double sequence of polynomials

$$f_{jk} = -p_jp_k + \sum_{l \geq 0} C_{jk}^l(X)p_l$$

belong to $K$ since

$$f_{jk} = h_jh_k - \sum_{l \geq 0} C_{jk}^l(X)h_l + p_jh_k + p_kh_j,$$

as one can easily check by using the definition of the Hamiltonians $h_j$. For the condition $\{K,K\} \subset K$, each Poisson bracket $\{f_{lm},f_{pq}\}$ belongs to the ideal, and therefore the structure constants $C_{jk}^l$ satisfy the equations $[C,C]_{jkklr}^m = 0$ as explained in the Introduction. Since they obviously satisfy also the associativity conditions, it is proved that the structure constants of the dKP and dmKP equations satisfy the central system.

To understand how the dKP and dmKP equations are sitting inside this system, it is worth at this point to give a closer look at the huge set of partial differential equations $[C,C]_{jkklr}^m = 0$. Not to be lost, it is convenient to fix at first the values of the indices $(j,k,l,r)$. In this way one may exploit the condition

$$C_{jk}^l = 0 \quad \text{for} \quad l > j + k$$

(15)

to reduce drastically the number of equations. For instance, for $j = k = l = 1$ and $r = 2$, the infinite sequence of equations

$$\sum_{s \geq 0} \left( C_{12}^s \frac{\partial}{\partial x_s} C_{11}^m - C_{11}^s \frac{\partial}{\partial x_s} C_{12}^m - C_{12}^m \frac{\partial}{\partial x_1} C_{11}^s - C_{11}^m \frac{\partial}{\partial x_2} C_{12}^s \right) + C_{12}^m \frac{\partial}{\partial x_1} C_{12}^s + C_{12}^m \frac{\partial}{\partial x_1} C_{12}^s = 0,$$
contracts to four equations
\[
\sum_{s=0}^{s=3} \left( C_{12s}^{m} \frac{\partial}{\partial x_s} C_{11}^{m} - C_{11s}^{m} \frac{\partial}{\partial x_s} C_{12}^{m} - C_{s2}^{m} \frac{\partial}{\partial x_1} C_{11}^{m} - C_{s1}^{m} \frac{\partial}{\partial x_2} C_{11}^{m} + C_{1s}^{m} \frac{\partial}{\partial x_1} C_{12}^{m} + C_{1s}^{m} \frac{\partial}{\partial x_1} C_{12}^{m} \right) = 0 \quad m = 0, 1, 2, 3
\]

owing to the triangularity condition \[(15)\]. In the same way the choice \(j = k = l = 1\) and \(r = 3\) gives rise to five equations, and so on.

The next question is to give these equations a sense. The way is to have recourse to the table of multiplication

\[
P_1P_1 = P_2 - u_1P_1 - u_0P_0,
\]

\[
P_1P_2 = P_3 + (u_1 - u_2)P_2 + (u_0 - u_1^2 + u_1v_2 - v_1)P_1 + (v_2u_0 - v_0 - u_1v_0)P_0,
\]

\[
P_1P_3 = P_4 + (v_2 - w_3)P_3 + (v_1 - w_2 - v_2^2 + w_3v_2)P_2
\]

\[
+ (v_2^2u_1 - u_1v_2w_3 - v_1v_2 + v_1w_3 - u_1v_1 + u_1w_2 + v_0 - w_1)P_1
\]

\[
- (w_0 + v_0v_2 + v_0w_3 + u_0v_1 - u_0w_2)P_0,
\]

\[
P_2P_2 = P_4 + (2u_1 - w_3)P_3 + (2u_0 - w_2 + u_1^2 - 2u_1v_2 + w_3v_2)P_2
\]

\[
+ (2u_1^2v_2 - 2u_1v_1 - u_1v_2w_3 + v_1w_3 - u_1^3 + u_1w_2 - w_1)P_1
\]

\[
+ (w_0 + u_0w_2 - u_0^2 - u_0u_1^2 + v_0w_3 - 2u_1v_0 - u_0v_2w_3 + u_0u_1v_2)P_0
\]

of the polynomials \(P_j\). From it one may read the structure constants \(C_{jk}^l\) as functions of the coefficients of the Hamiltonians \(h_j\). For instance

\[
C_{12}^2 = u_1 - v_2,
\]

\[
C_{12}^{-1} = u_0 - v_1 - u_1^2 + u_1v_2,
\]

\[
C_{12}^0 = u_0v_2 - v_0 - u_1v_0.
\]

The insertion of these expressions into the first fragment \[(16)\] of the central system allows to realize that these equations coincide with the set of four compatibility conditions \[(11)\] used to defined the dKP and dmKP equations. Similarly, one may check that the fragment corresponding to \(j = k = l = 1\) and \(r = 3\) coincides with the compatibility conditions defining the next members of the dKP and dmKP hierarchies. In general one may prove that for \(j = k = l = 1\) and \(r\) arbitrary one obtains the full dKP and dmKP hierarchies. Equivalently the same equations can be obtained by imposing the vanishing of the Poisson brackets \(\{f_{11}, f_{1r}\}\) on the submanifold \(f_{1k} = 0\) for \(k = 1, \ldots, r\).

At the end of this introductory discussion, it is interesting to emphasize again that the discovery of the second interpretation of the dispersionless KP hierarchy, as coisotropic deformation of the structure constants of a specific associative algebra, is the outcome of the replacement of the scheme of zero-curvature representations with the scheme of coisotropic ideals, imposed by
the desire of making the theory covariant with respect to all allowed changes of the auxiliary problems. The latter scheme encompasses the former one, and allows to understand that there is a unique mechanism behind the different representations of the dispersionless KP hierarchy. Each representation is the expression of the coisotropy of the ideal $K$ in a different system of generators. By choosing the system of generators $h_j$ one obtains the standard zero-curvature representation. By choosing instead the system of generators $f_{jk}$ one arrives to see them as equations controlling the evolution in time of the structure constants of a certain associative algebra. This freedom in the choice of the representation is a powerful tool for the study of the integrable hierarchies. Soon we shall see that it allows to account very easily for the passage from the zero-curvature representation to the Hirota’s representation of the dKP hierarchy.

3 The structure constants of dKP theory

The plan of this section is to study the form of the structure constants associated with the dispersionless KP hierarchy. According to the viewpoint of the previous section, this hierarchy is defined as the system of coisotropy conditions of the submanifold $\Gamma$ formed by the zeroes of the Hamiltonian functions

$$h_j = -p_j + \sum_{l=0}^{t=j-1} u_{jl} p_l^l,$$

and its structure constants are defined by the table of multiplication

$$(h_j + p_j)(h_k + p_k) = \sum_{l=1}^{n} C_{jk}^l (h_l + p_l).$$

The point to be noticed is that the coisotropy conditions split in two classes. Certain among them have the form of algebraic constraints on the coefficients of the Hamiltonian functions, and the question is to see the effect of these constraints on the form of the structure constants $C_{jk}^l$. To have a reasonable control of the question it is sufficient to consider the first few Hamiltonians

$$
\begin{align*}
h_2 &= -p_2 + p_1^2 + u_0(X), \\
h_3 &= -p_3 + p_1^3 + v_1(X)p_1 + v_0(X), \\
h_4 &= -p_4 + p_1^4 + w_2(X)p_1^2 + w_1(X)p_1 + w_0(X), \\
h_5 &= -p_5 + p_1^5 + z_3(X)p_1^3 + z_2(X)p_1^2 + z_1(X)p_1 + z_0(X),
\end{align*}
$$
and the algebraic constraints

\[
\begin{align*}
v_1 &= 3/2u_0, \\
w_2 &= 2u_0, \\
w_1 &= 4/3v_0, \\
z_3 &= 5/2u_0, \\
z_2 &= 5/3v_0, \\
z_1 &= 5/4w_0 + 5/8u_0^2,
\end{align*}
\]

originating from the coisotropy conditions of the ideal \( J \) generated by them. These constraints may be encoded into the definition of a special class of polynomials, henceforth called \textit{Faa’ di Bruno polynomials}. The first six polynomials are

\[
\begin{align*}
P_0(p) &= 1, \\
P_1(p) &= p, \\
P_2(p) &= p^2 + u_0, \\
P_3(p) &= p^3 + 3/2u_0p + v_0, \\
P_4(p) &= p^4 + 2u_0p^2 + 4/3v_0p + w_0, \\
P_5(p) &= p^5 + 5/2u_0p^3 + 5/3v_0p^2 + (5/4w_0 + 5/8u_0^2) + z_0.
\end{align*}
\]

Our interest is in their table of multiplication. The simplest part is

\[
\begin{align*}
P_1P_1 &= P_2 - [u_0]P_0, \\
P_1P_2 &= P_3 - [1/2u_0]P_1 - [v_0]P_0, \\
P_1P_3 &= P_4 - [1/2u_0]P_2 - [1/3v_0]P_1 - [w_0 - 1/2u_0^2]P_0, \\
P_1P_4 &= P_5 - [1/2u_0]P_3 - [1/3v_0]P_2 - [1/4w_0 - 1/8u_0^3]P_1 - [z_0 - 5/6u_0v_0]P_0.
\end{align*}
\]

The coefficients of these equations are the footprints of the structure constants we are looking for. One may notice that each coefficient becomes stationary exactly one step after it appears for the first time. To account for this phenomenon it is useful to rewrite the table in the form

\[
\begin{align*}
P_1P_1 &= P_2 - [1/2u_0]P_0 - [1/2u_0]P_0, \\
P_1P_2 &= P_3 - [1/2u_0]P_1 - [1/3v_0]P_0 - [2/3v_0]P_0, \\
P_1P_3 &= P_4 - [1/2u_0]P_2 - [1/3v_0]P_1 - [1/4w_0 - 1/8u_0^2]P_0 - [3/4w_0 - 3/8u_0^2]P_0, \\
P_1P_4 &= P_5 - [1/2u_0]P_3 - [1/3v_0]P_2 - [1/4w_0 - 1/8u_0^2]P_1 - [1/5z_0 - 1/6u_0v_0]P_0 - [4/5z_0 - 2/3u_0v_0]P_0,
\end{align*}
\]
showing that

\[ P_1 P_j = P_{j+1} + \sum_{l=1}^{j} H^j_l P_{j-l} + \sum_{l=1}^{1} H^j_l P_{1-l} \]

for two suitable sequences of constants \((H^j_1, H^j_1)\). Things go similarly for the second table of multiplication

\[
\begin{align*}
P_2 P_1 & = P_3 - ([2/3v_0]P_0) - ([1/2u_0]P_1 + [1/3v_0]P_0), \\
P_2 P_2 & = P_4 - ([2/3v_0]P_1 + [1/2u_0 - 1/2u_0]P_1 - ([2/3v_0]P_1 + [1/2w_0 - 1/2u_0]P_0), \\
P_2 P_3 & = P_5 - ([2/3v_0]P_2 + [1/2u_0 - 1/2u_0]P_1 + [\ast \ast \ast]P_0) - ([\ast \ast \ast]P_1 + [\ast \ast \ast]P_0),
\end{align*}
\]

showing that

\[ P_2 P_j = P_{j+2} + \sum_{l=1}^{j} H^j_l P_{j-l} + \sum_{l=1}^{2} H^j_l P_{2-l} \]

for two new sequences of constants \((H^j_2, H^j_2)\). In general one may expect (and prove subsequently) that

\[
\begin{align*}
P_k P_j & = P_{k+j} + \sum_{l=1}^{j} H^j_k P_{j-l} + \sum_{l=1}^{k} H^j_l P_{k-l}. \quad (17)
\end{align*}
\]

By this formula we have reached our aim, discovering that the structure constants of the Faa’ di Bruno polynomials have the form

\[
C^d_{kj} = \delta^d_{k+j} + H^d_k + H^d_j. \quad (18)
\]

where \(\delta^d_{k+j}\) is the Kronecker symbol. This formula is the starting point of the algebraic analysis of the Hirota’s bilinear formulation of the dispersionless KP hierarchy.

4 Tau function and Hirota’s equations.

The important question which remains unanswered at the bottom of the algebraic approach is to understand the link of coisotropic deformations to integrability. So far we have realized that certain integrable partial differential equations may be written in the form of central system, but we do not yet know if all that has a sense. In this section we point out a second occurrence that partly answers the previous question.

We assume that the structure constants have the form suggested by the examination of the dKP hierarchy, without specifying the coefficients \(H^d_k\), and we investigate the implications of the process of coisotropic deformation for this class of algebras. Our purpose is to show that the associativity conditions \(d\) and the coisotropy conditions \(10\) cooperate to produce the existence of a tau
function, which is the first sign of the integrability of the given set of partial differential equations.

The first step is to implement the associativity conditions on the coefficients \( H_{jk} \). By direct substitution of expressions (18) into equations (3), and by the use of the identity

\[
\sum_{l=1}^{k-1} H_{k-l} H_{l-n} = \sum_{l=1}^{k-1} H_{k-l} H_{l-n},
\]

one is led to the system of equations

\[
H_{i+k-n}^m + H_{k-m-n}^i = H_{i+k-n}^m + H_{k-m-n}^i + \theta(n-m) H_{i+m-n}^k + \theta(n-m) H_{k+m-n}^i - \theta(n-i) H_{i+k-n}^i + \sum_{l=1}^{i-1} H_{i-l} H_{l-m}^i + \sum_{l=1}^{i-1} H_{l-i} H_{m-l}^i
\]

\[
+ \sum_{l=1}^{m-1} H_{k-l} H_{m-n}^l - \sum_{l=1}^{m-1} H_{m-l}^k H_{l-i}^l
= 0,
\]

where \( \theta(n) = 1 \) or \( \theta(n) = 0 \) according if \( n \geq 0 \) or \( n < 0 \). However these equations are not all independent. The analysis of Appendix A allows to reduce the number of equations, and shows that the structure constants \( C_{jk}^l \) obey the associativity conditions if and only if the bracket

\[
[H, H]_{ikm} := H_{i+k-n}^m - H_{k-m-n}^i + \sum_{l=1}^{i-1} H_{i-l} H_{l-m}^i + \sum_{l=1}^{k-1} H_{k-l} H_{m-l}^i - \sum_{l=1}^{m-1} H_{m-l} H_{l-i}^l = 0,
\]

vanishes identically for any choice of the indices \( (i, k, m) \in N \). An interesting consequence of this result can be drawn by contraction. Indeed one may check that the equations

\[
\sum_{m+k=p} [H, H]_{ikm} = 0
\]

and the use of the identities (19) lead to the contracted identities

\[
pH_p^i = H_p^i + \sum_{k=1}^{p-1} (H_{k+p-k}^i - H_{k-p-k}^i) - \sum_{l=1}^{p-1} \sum_{k=1}^{l} H_{l-p-k}^i H_{k-l}^i,
\]

entailing the useful symmetry relations

\[
pH_p^i = iH_p^i.
\]
As we shall see in a moment, this remarkable outcome of the associativity conditions is at the basis of Hirota’s formulation.

Next one has to implement the coisotropy conditions. In terms of the coefficients $H_{ij}$, the ensuing equations are, at first sight, rather complicated, and they are presented in Appendix A (as a particular case of equation (58)). However a closer scrutiny shows that they are simplified drastically on account of the associativity conditions just obtained. Indeed in Appendix A it is proved that the coisotropy conditions may be reduced to the equations

$$\frac{\partial[H,H]_{i,n,k-l}}{\partial x_i} + \frac{\partial[H,H]_{i,n,k-l}}{\partial x_l} - \frac{\partial[H,H]_{i,k,l-n}}{\partial x_n} - \frac{\partial[H,H]_{i,k,n-l}}{\partial x_l} = 0,$$

which are automatically fulfilled owing to the associativity conditions, and to the linear equations

$$\frac{\partial H^l_{ip}}{\partial x_l} = \frac{\partial H^l_{ip}}{\partial x_i}.$$  

So, to summarize, the associativity and coisotropy conditions in the case of the structure constants of the form (18) are together equivalent to the set of quadratic algebraic equations

$$[H, H]_{ikl} = 0,$$  

(23)

entailing the symmetry conditions

$$pH^l_{ip} = iH^l_{pi},$$  

(24)

and to the set of linear differential equations

$$\frac{\partial H^l_{ip}}{\partial x_l} = \frac{\partial H^l_{ip}}{\partial x_i}$$  

(25)

having the form of a system of conservation laws. The equations (23) and (25) give the specific form of the central system of the dispersionless KP hierarchy. It encodes all the informations about the hierarchy. In particular it entails that for any solution of the central system one has

$$\{f_{ik}, f_{ln}\} = \sum_{s,t \geq 1} K_{ikln}^{st} f_{st}$$  

(26)

where

$$K_{ikln}^{st} = \left(\delta_{ls} \frac{\partial}{\partial x_k} + \delta_{lt} \frac{\partial}{\partial x_l}\right) (H_{i,n-s}^n + H_{i,n-s}^l) - \left(\delta_{ns} \frac{\partial}{\partial x_l} + \delta_{nt} \frac{\partial}{\partial x_n}\right) (H_{i,s}^{l-n} + H_{i,s}^{n-l}).$$

From this formula one sees that the Hamiltonians $f_{jk}$ of the dispersionless KP hierarchy form a Poisson algebra. The above central system can be seen also as the dispersionless limit of the central system of the full dispersive KP hierarchy [41].

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There are presently two strategies to decode the informations contained into the central system. According to the first strategy, one first tackles the associativity conditions (23), noticing that they allow to compute the coefficients \((H_2^k, H_3^k, \ldots)\) as polynomial functions of \(H_1^k\). For instance, the symmetry conditions

\[
\begin{align*}
H_1^2 &= 2H_2^1, \\
H_1^3 &= 3H_3^1
\end{align*}
\]

give \((H_1^1, H_3^1)\), and then the condition (23) with \(i = k = 1, l = 2\), i.e.

\[
H_1^3 - H_1^1 - H_2^1 + H_1^1 H_1^1 = 0
\]

gives \(H_2^2\), and so forth. Renaming the free coefficients as suggested by the table of multiplication of the previous section, that is by setting

\[
\begin{align*}
H_1^1 &= -1/2u_0, \\
H_2^1 &= -1/3v_0, \\
H_3^1 &= -1/4w_0 + 1/8u_0^2,
\end{align*}
\]

one gets

\[
\begin{align*}
H_1^2 &= -2/3v_0, \\
H_2^2 &= -1/2w_0 + 1/2u_0^2, \\
H_3^3 &= -3/4w_0 + 3/8u_0^2.
\end{align*}
\]

At this point one may plug these expressions into the simplest linear coisotropy conditions

\[
\begin{align*}
\frac{\partial H_1^1}{\partial x_2} - \frac{\partial H_2^1}{\partial x_1} &= 0, \\
\frac{\partial H_3^1}{\partial x_2} - \frac{\partial H_3^3}{\partial x_1} &= 0, \\
\frac{\partial H_1^1}{\partial x_3} - \frac{\partial H_3^3}{\partial x_1} &= 0,
\end{align*}
\]

arriving to the equations

\[
\begin{align*}
\frac{\partial v_0}{\partial x_1} &= 3/4 \frac{\partial u_0}{\partial x_2}, \\
\frac{\partial v_0}{\partial x_2} &= 3/2 \frac{\partial u_0}{\partial x_1} - 3u_0 \frac{\partial u_0}{\partial x_1}, \\
\frac{\partial u_0}{\partial x_3} &= 3/2 \frac{\partial u_0}{\partial x_1} - 3/2u_0 \frac{\partial u_0}{\partial x_1}.
\end{align*}
\]

The elimination of \(\frac{\partial u_0}{\partial x_1}\) leads finally to the dispersionless KP equation and to the higher equations, if one insists enough in the computations. By this strategy one come back to the hierarchy in its standard formulation.

A simple inversion in the order in which the equations are considered leads instead to the Hirota’s formulation. It is enough to remark that equations (25) entail the existence of a sequence of potentials \(S_m\) such that

\[
H_m^i = \frac{\partial S_m}{\partial x_i}.
\]

Then the symmetry conditions (24), oblige the potentials \(S_m\) to obey the constraints

\[
\frac{\partial S_i}{\partial x_1} = l \frac{\partial S_l}{\partial x_1},
\]

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which in turn entail the existence of a superpotential $F(x_1, x_2, \ldots)$ such that

$$S_i = -1/i \frac{\partial F}{\partial x_i}, \quad H_m^i = -1/m \frac{\partial^2 F}{\partial x_i \partial x_m}.$$ 

This result provides a second parametrization of the structure constants, after that described before. The insertion of the new parametrization into the full set of associativity conditions finally leads to the system of equations

$$- \frac{1}{m} F_{i+k,m} + \frac{1}{m+k} F_{i,k+m} + \frac{1}{i+m} F_{k,i+m}$$

$$+ \sum_{l=1}^{i-1} \frac{1}{m(i-l)} F_{k,i-l} F_{1,m} + \sum_{l=1}^{k-1} \frac{1}{m(k-l)} F_{i,k-l} F_{1,m} - \sum_{l=1}^{m-1} \frac{1}{i(m-l)} F_{k,m-l} F_{1,t} = 0,$$

(27)

where $F_{i,k}$ stands for the second-order derivative of $F$ with respect to $x_i$ and $x_k$. They are equivalent to the celebrated Hirota’s bilinear equations for the tau function of the dispersionless KP hierarchy (see e.g. [12, 13, 17, 29]). For instance, for $(i = k = 1, m = 2)$ or $(i = m = 1, k = 2)$ or $(m = k = 1, i = 2)$ one obtains directly the first Hirota’s equation

$$-1/2 F_{2,2} + 2/3 F_{1,3} - (F_{1,1})^2 = 0.$$ 

For $(i = 1, k = 2, m = 2)$ or $(i = 2, k = 1, m = 2)$ or $(i = 1, k = 1, m = 3)$ it gives instead the second Hirota’s equation

$$1/2 F_{1,4} - 1/3 F_{2,3} - F_{1,1} F_{1,2} = 0.$$ 

The choices $(i = 1, k = 1, m = 4)$ and $(i = 1, k = 2, m = 3)$ lead to the equations

$$-1/4 F_{2,4} + 2/5 F_{1,5} - 2/3 F_{1,1} F_{1,3} - 1/4 (F_{1,1})^2 = 0$$ 

and

$$-1/3 F_{3,3} + 1/5 F_{1,5} + 1/4 F_{2,4} + 1/3 F_{1,1} F_{1,3} - 1/2 F_{1,1} F_{2,2} - 1/2 (F_{1,2})^2 = 0$$ 

respectively, which do not separately coincide with Hirota’s bilinear equations, but which are together equivalent to a pair of standard Hirota’s equations of the same weight. The process of identification may be continued indefinitely.

The above result lends itself to several comments. The discovery that the Hirota’s equations of the dispersionless KP hierarchy are the associativity equations for the structure constants [13] immediately reminds the associativity equations of Witten, Dijkgraaf, Verlinde, Verlinde [42, 43], first derived in the frame of two-dimensional topological field theories and subsequently interpreted by Dubrovin [44] in the frame of his theory of Frobenius’s manifolds. In spite of the ideological resemblance there are, however, essential differences between the present approach and that of Frobenius manifolds developed by
Dubrovin. In the latter approach the structure constants are derived from solutions of WDVV equations and verify the associativity conditions but not the full central system. They define deformations of Frobenius algebras, which are associative algebras endowed with a nondegenerate symmetric form, and are attached to the tangent spaces of a Frobenius manifold whose points are parametrized by the deformation coordinates $x_j$. In the present approach the structure constants are derived directly from the hierarchy and do not presuppose the preliminary knowledge of the tau function. Instead they serve to introduce the tau function, and not vice versa as happens in the theory of Frobenius manifolds. Furthermore they are used to define a coisotropic submanifold in the cotangent bundle. So the two approaches live in different spaces. Furthermore, the Riemannian structure of the base space, which is fundamental for Dubrovin, seems untied with the present approach. Vice versa the symplectic structure of the cotangent bundle, which is fundamental for us, seems do not play any role in the Dubrovin’s approach. The exact relation between these different vision is therefore a complicated problem. We note, however, that the functions $K_{ikln}$ in formula (26) are linear combinations of third-order derivatives of the tau function $F$. For example,

$$\{f_{11}, f_{1n}\} = -2 \sum_{s=1}^{n-1} \left( \frac{1}{n-s} \frac{\partial^3 F}{\partial x_1^2 \partial x_{n-s}} \right) f_1 f_{s} \quad n = 2, 3, ...$$

It is, probably, not just a coincidence that third order derivative appear in both WDVV equations and as structure constants in equations (26).

From another point of view, one may also consider the central system as an interesting linearization of the dKP hierarchy. In terms of the coefficients of the Hamiltonians $h_j$ this system is just the dKP hierarchy of nonlinear PDE’s. In terms of variables $H_{ik}$ the dKP hierarchy is represented by linear exactness conditions on the family of quadrics (21). A linearization of nonlinear PDE’s integrable by the inverse scattering transform method in terms of appropriate variables (inverse scattering data) is a common property of such equations (see e.g. [39, 40]). In particular, in the papers [45, 46] it was shown that the Korteweg-de-Vries, nonlinear Schrodinger, sine-Gordon, and Toda lattice equations are linearizable (become systems of free oscillators) on the intersection of quadrics in phase space. The peculiarity of the dispersionless hierarchies is that the family of quadrics is defined by the associativity conditions for certain algebras. In this vein, the associativity conditions also remind the relations for Plucker’s coordinates within the Sato’s universal Grassmannian approach to the full dispersive KP hierarchy [47].

5 The symmetry action

So far we have considered a relatively narrow class of algebras and of dispersionless integrable equations. To enlarge this class we shall follow two different
routes, in this and the next section, having recourse to the action of a symmetry group and to a process of gluing.

The transformation of the structure constants

$$\tilde{C}^d_{ab}(X) = \sum_{a,b,l>0} C^l_{jk}(X) A^j_a(X) A^k_b(X) A^l_d(X),$$

(28)

induced by a change of basis

$$\tilde{p}_a = \sum_{l>0} A^l_a(X) p_l$$

(29)

depending on deformation parameters \(X = (x_1, x_2, \ldots)\), obviously preserves the symmetry conditions (2) and the associativity conditions (3), but violates in general the coisotropy conditions (9).

The class of transformations preserving the last conditions can be worked out by turning to the correspondence between deformation parameters and generators, on one side, and coordinates and momenta, on the other side, which has been used to establish the Hamiltonian interpretation of the process of coisotropic deformation. Shifting from commutative algebra to Hamiltonian mechanics, one then substitutes the above change of basis in the associative algebra with the transformation of coordinates

$$\tilde{x}_a = \sum_{l>0} \delta^l_a x_l
\tilde{p}_a = \sum_{l>0} A^l_a(X) p_l$$

(30)

on the symplectic manifold related to the algebra. This change of coordinates acts on the Hamiltonians \(f_{jk}\) according to the transformation law

$$\tilde{f}_{ab}(X) = \sum_{j,k>0} A^j_a(X) A^k_b(X) f_{jk}(X),$$

(31)

which shows that the functions \(\tilde{f}_{ab}\) still belong to the ideal \(J\) generated by the functions \(f_{jk}\). This ideal is thus left invariant by the transformation (30), but it looses in general the property of being closed with respect to the classical Poisson bracket since the bracket is modified by the coordinate transformation. So to preserve the coisotropy conditions we need to restrict the change of basis in such a way that the transformation (30) be canonical. This is a severe restriction and the unique solution is

$$\tilde{p}_a = p_a + \frac{\partial \phi}{\partial x_l} p_0$$

(32)

where \(\phi\) is an arbitrary function of the deformation parameters. The conclusion is that there exists an infinite-dimensional Abelian symmetry group of
the equations defining the coisotropic deformations of associative algebras depending on an arbitrary function \( \phi \). This group is the subject of this section.

The action of the group on the space of structure constants is defined by

\[
\tilde{C}_{jk}^l = C_{jk}^l + \delta_k^l \frac{\partial \phi}{\partial x_j} + \delta_j^l \frac{\partial \phi}{\partial x_k} + \sum_{m>0} \tilde{C}_{jkm} \frac{\partial \phi}{\partial x_m},
\]

\[
\tilde{C}_{jk}^0 = C_{jk}^0 + \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} - \sum_{m>0} C_{jkm} \frac{\partial \phi}{\partial x_m},
\]

for \( j, k, l > 0 \). Therefore the equations of the orbit passing through the point corresponding to the structure constants of the dispersionless KP hierarchy are

\[
\tilde{C}_{jk}^l = \delta_{j+k}^l + \tilde{H}_{k-l}^j + \tilde{H}_{j-l}^k + \delta_{j-k}^l \tilde{D}_{jk},
\]

with the understanding that the coefficients are given by

\[
\tilde{H}_{k}^i = H_{k}^i, \\
\tilde{H}_{0}^i = \frac{\partial \phi}{\partial x_i}, \\
\tilde{H}_{k}^0 = \delta_{k}^0, \\
\tilde{D}_{ik} = -\frac{\partial \phi}{\partial x_i} - \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_j} - \sum_{m=1}^{i-1} H_{i-m}^k \frac{\partial \phi}{\partial x_m} - \sum_{m=1}^{k-1} H_{k-m}^i \frac{\partial \phi}{\partial x_m},
\]

\[
\tilde{D}_{ik} = -\delta_{k}^0
\]

for \( i, k > 0 \). One may notice that the new structure constants \( \tilde{C}_{jk}^l \) closely resemble the original ones, and one may infer from this fact that they also derive from a tau function. The first problem we are interested in is to inquiry how the symmetry group acts at the level of tau functions.

The proof of the existence of a tau function \( \tilde{F}(x_1, x_2, \ldots) \) for the new structure constants \( \tilde{C}_{jk}^l \) strictly follows the pattern of the previous section. The strategy is always to work out the implications on the coefficients \( \tilde{H}_{k}^i \) and \( \tilde{D}_{ik} \) of the conditions of coisotropy and associativity. In Appendix A it is shown that the conditions of associativity give rise to two sets of constraints, affecting separately the two families of coefficients. One set fixes the form of the coefficients \( \tilde{D}_{ik} \) according to

\[
\tilde{D}_{ik} = -\tilde{H}_{i+k}^0 + \tilde{H}_{0}^i \tilde{H}_{i}^k - \sum_{l=1}^{i} \tilde{H}_{i-l}^k \tilde{H}_{l}^0 - \sum_{l=1}^{k} \tilde{H}_{k-l}^i \tilde{H}_{l}^0,
\]

while the second set requires that the coefficients \( \tilde{H}_{k}^i \) verify the already known conditions

\[
\tilde{H}_{i+k}^l - \tilde{H}_{m+k}^l - \tilde{H}_{i+m}^l + \sum_{i=1}^{i-1} \tilde{H}_{i-l}^k \tilde{H}_{l}^m + \sum_{i=1}^{k-1} \tilde{H}_{k-l}^i \tilde{H}_{l}^m - \sum_{i=1}^{m-1} \tilde{H}_{m-l}^i \tilde{H}_{l}^m = 0.
\]
A closer scrutiny of these conditions shows that they do not contain terms of the form \( \tilde{H}_0^i \), since these terms cancel in pair, proving that the coefficients \( \tilde{H}_i^k \), for \( i, k > 0 \), satisfy again the symmetry conditions

\[
p \tilde{H}_p^i = i \tilde{H}_i^p.
\]

Similarly the coisotropy conditions also split in two sets that, owing to the associativity conditions, take the form

\[
\begin{align*}
\frac{\partial \tilde{H}_0^i}{\partial x_i} &= \frac{\partial \tilde{H}_0^i}{\partial x_i}, \\
\frac{\partial \tilde{H}_p^i}{\partial x_i} &= \frac{\partial \tilde{H}_p^i}{\partial x_i}
\end{align*}
\]

for \( i, l, p > 0 \). One infers from the last conditions the existence of a sequence of potentials \( \phi = \tilde{S}_0 \) and \( \tilde{S}_k \), such that

\[
\begin{align*}
\tilde{H}_0^i &= \frac{\partial \phi}{\partial x_i}, \\
\tilde{H}_i^k &= \frac{\partial \tilde{S}_k}{\partial x_i}.
\end{align*}
\]

By the same argument discussed in Sec.4, the potentials \( \tilde{S}_k \) lead to the existence of a single function \( \tilde{F}(x_1, x_2, \ldots) \) such that

\[
\tilde{H}_i^k = - \frac{1}{k} \frac{\partial^2 \tilde{F}}{\partial x_i \partial x_k}.
\]

The existence of the potential \( \phi \) shows instead that the new structure constants \( \tilde{C}_{jk}^i \) belong to the orbit passing through the point representing the KP hierarchy. Due to the invariance of the coefficients \( \tilde{H}_k^i \) along the orbit \( (i, h > 0) \), it turns out that the tau function is invariant along the orbit. Thus this function is a property of the orbit rather than of the points of the orbit.

The second problem of our concern is to investigate the properties of a second remarkable point belonging to the orbit, whose definition is suggested by the equations of the orbit. They entail, in particular, that

\[
\tilde{C}_{jk}^0 = \tilde{H}_j^k + \tilde{H}_j^k + \tilde{D}_{jk}.
\]

This formula shows that the structure constants \( \tilde{C}_{jk}^0 \) vary along the orbit, and therefore one can ask if there exists a point on the orbit where these structure constants vanish. The answer is affirmative owing to the first half of the associativity conditions. Indeed to demand that \( \tilde{C}_{jk}^0 = 0 \) is equivalent to search a function \( \phi \) such that

\[
\begin{align*}
\frac{\partial \phi}{\partial x_{i+k}} + \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_k} + \sum_{m=1}^{m=i-1} H_{i-m}^k \frac{\partial \phi}{\partial x_m} + \sum_{m=1}^{m=k-1} H_{k-m}^i \frac{\partial \phi}{\partial x_m} = H_k^i + H_i^k,
\end{align*}
\]
if one takes into account the invariance of the coefficients $H_k^i$ along the orbit. The system just written is an overdetermined system of nonlinear partial differential equations on the single unknown function $\phi$ whose compatibility conditions are exactly the first half of the associativity conditions. Accordingly there exists a function $\phi$ which allows to reach the desired point from the point corresponding to dispersionless KP. It is readily seen that the new point correspond to the structure constants of the dispersionless mKP hierarchy. It is sufficient to interpret the last equations as the hierarchy of dispersionless Miura transformations relating the dispersionless KP and mKP hierarchies. This interpretation is motivated by the remark that, at the lowest level $i = k = 1$, the equation of the symmetry action has the form

$$\frac{\partial \phi}{\partial x_2} + (\frac{\partial \phi}{\partial x_1})^2 = 2H_1^1,$$

and therefore becomes

$$u_0 = \frac{1}{2}\frac{\partial^{-1}}{\partial x_1} (\frac{\partial u_1}{\partial x_2}) - \frac{1}{4}u_1^2$$

upon the insertion of the standard parametrization

$$u_0 = -2H_1^1, \quad u_1 = -2H_0^1 = -2\frac{\partial \phi}{\partial x_1}.$$ 

The above equation is the dispersionless limit of the well-known Miura transformation between the KP and mKP equations (see e.g. [23, 24]). The lesson is that the dispersionless Miura transformation is just a particular instance of the symmetry action generated by the special changes of basis (29) in the associative algebra. At the level of structure constants the Miura transformations have therefore a very simple meaning.

It remains to identify the meaning of the other points of the orbits. A possible way is to carefully study the parametrization of the structure constants $\tilde{C}_{jk}^l$ obeying the equations of the orbit. We omit this study and we limit ourselves to give the final answer. As the reader may expect at this point, one may prove that the structure constants of the orbit passing through the dKP hierarchy correspond to the dispersionless generalized dKP hierarchy briefly discussed in Sec.2. This remark closes the study of the orbit defined by the symmetry action.

6 The process of gluing.

An elementary way of gluing together two algebras of polynomials, one in the variable $p$ and the other in the variable $q$, is to add the relation

$$pq = ap + bq + c$$
which allows to write the product of a polynomial in $p$ by a polynomial in $q$ as sum of two polynomials of the same variables. The new associative algebra is known as the quotient of the algebra of polynomials in two variables $p$ and $q$ with respect to the ideal generated by the polynomial $(-pq + ap + bq + c)$.

The leading idea of this section is to apply this procedure to two copies of the algebra of Faa’ di Bruno polynomials encountered in Sec.3 and slightly generalized in Sec.5. In practice this means that we complete the tables of multiplication

\[ p_j p_k = p_{j+k} + \sum_{l=0}^{j} H_l^j p_{k-l} + \sum_{l=0}^{k} H_l^k p_{j-l} + D_{jk}, \]

\[ q_j q_k = q_{j+k} + \sum_{l=0}^{j} H_l^j q_{k-l} + \sum_{l=0}^{k} H_l^k q_{j-l} + \tilde{D}_{jk} \]

by adding the relation

\[ p_1 q_1 = ap_1 + bq_1 + c, \]

and consistently computing the products $p_j q_k$. The aim is to show that the structure constants of the enlarged table of multiplication admit interesting coisotropic deformations reproducing the universal Whitham equations of genus zero, obtained by Krichever by his technique of meromorphic functions on the Riemann sphere with punctures [14]. The process of gluing is thus the algebraic implementation of the pasting of meromorphic functions on the Riemann sphere (or of meromorphic differentials on Riemann surfaces) characteristic of the algebro-geometric approach of Krichever. What for him is the introduction of a new pole, for us is simply the gluing of an additional copy of the same algebra.

To give an idea of the potentialities of the new procedure, let us consider the fragment of the above table of multiplication consisting of the three simplest equations

\[ p_1 p_1 = p_2 - (vp_1 + u), \]
\[ q_1 q_1 = q_2 - (\tilde{v}p_1 + \tilde{u}), \]
\[ p_1 q_1 = ap_1 + bq_1 + c, \]

where new names have been used for the structure constants to simplify the notation. According to the scheme of coisotropic deformations, we introduce two sets of space coordinates $x_j$ and $y_j$ and their conjugate momenta $p_j$ and $q_j$, and we transform the partial table of multiplication (38) into the definition of three Hamiltonians

\[ f = -p_2 + p_1^2 + vp_1 + u, \]
\[ \tilde{f} = -q_2 + q_1^2 + \tilde{v}p_1 + \tilde{u}, \]
\[ g = -p_1 q_1 + ap_1 + bq_1 + c \]
on the symplectic manifold $R^8$ endowed with the classical Poisson bracket for canonical coordinates $(x_1, x_2, y_1, y_2, p_1, p_2, q_1, q_2)$. Finally we demand that the ideal $J$ generated by these Hamiltonians be closed with respect to the Poisson bracket, and we obtain three sets of equations. From the study of $\{f, g\}$ we obtain the equations

\[
\begin{align*}
    v_{y_1} + 2a_{x_1} &= 0, \\
    a_{x_2} - 2(ab + c)_{x_1} - (1/4v^2 + u)_{y_1} &= 0, \\
    b_{x_2} - (bv + b^2 - u)_{x_1} &= 0, \\
    c_{x_2} - (cv)_{x_1} - 2cb_{x_1} + au_{x_1} + bu_{y_1} &= 0,
\end{align*}
\]

while the study of $\{\tilde{f}, g\}$ leads to

\[
\begin{align*}
    2b_{y_1} + \tilde{v}_{x_1} &= 0, \\
    a_{y_2} + (a\tilde{v} - a^2 - \tilde{u})_{y_1} &= 0, \\
    b_{y_2} - 2(ab + c)_{y_1} - (\tilde{u} + 1/4\tilde{v}^2)_{x_1} &= 0, \\
    c_{y_2} - (c\tilde{v})_{y_1} - 2ca_{y_1} + a\tilde{u}_{x_1} + b\tilde{u}_{y_1} &= 0,
\end{align*}
\]

and the study of $\{f, \tilde{f}\}$ gives

\[
\begin{align*}
    2av_{y_1} - 2a\tilde{v}_{x_1} + \tilde{v}v_{y_1} - v_{y_2} - 2\tilde{u}_{x_1} &= 0, \\
    2bu_{y_1} - 2b\tilde{v}_{x_1} - v\tilde{v}_{x_1} - \tilde{v}_{x_2} - 2\tilde{u}_{y_1} &= 0, \\
    2cv_{y_1} - 2c\tilde{v}_{x_1} + \tilde{v}u_{y_1} - u_{y_2} - v\tilde{u}_{x_1} + \tilde{u}_{x_2} - u_{y_2} &= 0.
\end{align*}
\]

If these equations are satisfied then

\[
\begin{align*}
    \{f, g\} &= (v + 2b)_{x_1} g, \\
    \{\tilde{f}, g\} &= (\tilde{v} + 2a)_{y_1} g, \\
    \{f, \tilde{f}\} &= 2(\tilde{v}_{x_1} - v_{y_1}) g
\end{align*}
\]

These equations admit several interesting reductions. For instance, by setting $v = 0, \tilde{u} = 0, a = 0$ the first system becomes

\[
\begin{align*}
    u_{y_1} + 2c_{x_1} &= 0, \\
    b_{x_2} - (b^2 + u)_{x_1} &= 0, \\
    c_{x_2} - 2(bc)_{x_1} &= 0
\end{align*}
\]

which is the simplest example of generalized Benney’s system introduced in [11] [14] [16]. At the same time the second system becomes the modified Benney’s system [11]

\[
\begin{align*}
    \tilde{v}_{x_1} + 2b_{y_1} &= 0, \\
    b_{y_2} - 1/4(\tilde{v}^2)_{x_1} - 2c_{y_1} &= 0, \\
    c_{y_2} - (c\tilde{v})_{y_1} &= 0,
\end{align*}
\]

and the third system becomes the Miura type transformation

\[
\begin{align*}
    \tilde{v}_{x_2} + 2(u + b^2)_{y_1} &= 0, \\
    u_{y_2} + 2(c\tilde{v})_{x_1} &= 0
\end{align*}
\]
between them \[11\]. If instead one sets \( u = 0, \tilde{u} = 0, c = 0 \) from the first two systems of coisotropy conditions one obtains

\[
\begin{align*}
    a_{x_2} - 2(ab)x_1 - 1/4(v^2)y_1 &= 0, \\
    b_{x_2} - (b^2 + bv)x_1 &= 0, \\
    v_{y_1} + 2a_{x_1} &= 0, \\
    a_{y_2} - (a^2 + a\tilde{v})y_1 &= 0, \\
    b_{y_2} - 2(ab)y_1 - 1/4(\tilde{v}^2)x_1 &= 0, \\
    \tilde{v}_{x_1} + 2b_{y_1} &= 0
\end{align*}
\]

which is, in fact, the dispersionless limit of the equations for the components of the wave function of the Dawey-Stewartson equation discussed in \[48\].

It is remarkable that the new associative algebra, defined by the process of gluing, is isomorphic to the algebra of meromorphic functions on the Riemann sphere with two punctures. Indeed resolving the coupling relation with respect to \( q_1 \), one gets

\[
q_1 = \frac{ab + c}{p_1 - b} + a.
\]

Hence any polynomial in \( q_1 \) becomes a rational function in \( p_1 \). So the algebra \[35\], \[36\] may be identified with the algebra of rational functions with poles in \( b \) and in the point at infinity. This algebra is just the algebra of meromorphic functions on the Riemann sphere with two punctures used by Krivechev \[14\].

The general case of \( n \) punctures is equivalent to the gluing of \( n \) copies of algebras of the initial type.

To give an example of equations coming from the gluing of \( n \) copies of the algebra of Faa’ di Bruno polynomials, let us consider the fragment of the table of multiplication consisting simply of the \( 1/2n(n-1) \) relations

\[
p_{\alpha}p_{\beta} = H_{\alpha\beta}p_\alpha + H_{\beta\alpha}p_\beta \quad \alpha \neq \beta
\]

which serve to glue together the \( n \) copies of the algebra. In this example the indices \( \alpha, \beta = 1, 2, \ldots, n \) identify the copies of the algebra, and the symbol \( p_{\alpha} \) stands for \( p_{\alpha,1} \).

If \( n \) is greater than two, the coefficients \( H_{\alpha\beta} \) must verify an appropriate set of associativity conditions. In Appendix B it is shown that the part of the associativity conditions pertaining to the coefficients \( H_{\alpha\beta} \) is

\[
H_{\alpha\gamma}H_{\beta\gamma} - H_{\alpha\gamma}H_{\gamma\beta} - H_{\gamma\beta}H_{\alpha\gamma} = 0, \tag{40}
\]

where the indices \( \alpha, \beta, \gamma \) take distinct values. In the same Appendix it is also shown that the coisotropy conditions for the Hamiltonians

\[
g_{\alpha\beta} = -p_\alpha p_\beta + H_{\alpha\beta}p_\alpha + H_{\beta\alpha}p_\beta,
\]

\( \alpha \neq \beta \), contains a subset of conditions pertaining only to the coupling coefficients \( H_{\alpha\beta} \), and that this subset simplifies into the by now familiar form

\[
\frac{\partial H_{\alpha\gamma}}{\partial x_\beta} - \frac{\partial H_{\beta\gamma}}{\partial x_\alpha} = 0 \quad \alpha \neq \beta \neq \gamma \neq \alpha
\]
owing to the previous associativity conditions. The equations just written describe the coisotropic deformations of the coupling coefficients $H_\alpha^\beta$ alone.

As in the case of the dispersionless KP hierarchy, one may treat these equations in two alternative ways. If one solves first the associativity conditions by setting, for instance,

$$H_\alpha^\beta = \frac{u_\alpha}{v_\alpha - v_\beta}, \quad \alpha \neq \beta,$$

where $u_\alpha$ and $v_\alpha$ are arbitrary functions of the coordinates $x_\alpha$, one obtains from the coisotropy conditions the final system

$$\frac{\partial}{\partial x_\beta}(\frac{u_\alpha}{v_\alpha - v_\gamma}) - \frac{\partial}{\partial x_\alpha}(\frac{u_\beta}{v_\beta - v_\gamma}) = 0.$$

If instead one chooses the second strategy, and solves first the coisotropy conditions by setting

$$H_\alpha^\beta = \frac{\partial F_\beta}{\partial x_\alpha}, \quad \alpha \neq \beta,$$

one obtains from the associativity conditions the dispersionless Darboux system

$$\frac{\partial F_\alpha}{\partial x_\beta} \frac{\partial F_\alpha}{\partial x_\gamma} - \frac{\partial F_\alpha}{\partial x_\gamma} \frac{\partial F_\alpha}{\partial x_\beta} - \frac{\partial F_\alpha}{\partial x_\beta} \frac{\partial F_\beta}{\partial x_\gamma} = 0$$

introduced in [32]. It represents the dispersionless limit of the well-known Darboux system describing conjugate nets of curves in $\mathbb{R}^n$ (see e.g. [49]).

The second example is the n-component (2+1)-dimensional Benney system introduced by Zakharov [16]. It can be recovered by gluing one copy of the algebra associated with the dispersionless KP hierarchy to $(n + 1)$ copies of the algebras associated with the dispersionless mKP hierarchy. The Benney systems can then obtained as coisotropy conditions of the ideal $J$ generated by the first Hamiltonian of the dispersionless KP hierarchy

$$f = -p_2 + p_1^2 - u,$$

by the $n$ functions

$$k_\alpha = p_1 q_\alpha - a_\alpha q_\alpha - v_\alpha, \quad \alpha = 1, \ldots, n$$

which serve to glue the algebra of the dispersionless KP hierarchy to $n$ copies of the algebra of the dispersionless mKP hierarchy, and by the function

$$g = q_{n+1} - \sum_{\alpha=1}^{n} q_\alpha,$$

which serves to glue the $(n + 1)$th copy of this algebra to the previous ones.

To apply the technique of coisotropic deformations we have to simultaneously introduce new deformation parameters. Let us denote by $(x_1, x_2, y_\alpha, \tau)$ the coordinates conjugate to the momenta $(p_1, p_2, q_\alpha, q_{n+1})$ respectively, and let us
work out explicitly the coisotropy conditions of the ideal $J$. One may easily find that the condition $\{f, k_\alpha\} = 0$ on $\Gamma$ gives

\begin{align*}
u_{\alpha x_1} + 2\nu_\alpha x_1 &= 0, \quad (41) \\
u_{x_1} + (a^2_\alpha)x_1 - a_{\alpha x_2} &= 0, \quad \alpha = 1, \ldots, n \quad (42) \\
u_{\alpha x_2} + a_\alpha u_{y_\alpha} - 2\nu_\alpha a_{\alpha x_1} &= 0 \quad (43)
\end{align*}

Similarly the condition $\{f, g\} = 0$ on $\Gamma$ implies

\begin{equation*}
u_{\alpha x_2} + 2(\sum_{\alpha=1}^n \nu_\alpha) x_1 = 0. \quad (44)
\end{equation*}

while the conditions $\{k_\alpha, g\} = 0$ on $\Gamma$ give

\begin{align*}a_{\alpha x_1} - \sum_{\beta=1}^n \frac{\partial a_\alpha}{\partial y_\beta} &= 0 \\
u_{\alpha x_1} - \sum_{\beta=1}^n \frac{\partial \nu_\alpha}{\partial y_\beta} &= 0.
\end{align*}

From equations (41) and (44) one gets

\begin{equation*}u_{y_\alpha} + 2(\sum_{\alpha=1}^n \nu_\alpha) x_1 = 0.
\end{equation*}

Then from equations (41) and equations (43) one obtains

\begin{equation*}\nu_{\alpha x_2} - 2(\nu_\alpha a_\alpha) x_1 = 0.
\end{equation*}

The last two equations together with (42) are just the (2+1)-dimensional $n$-component Benney’s system [16].

If one wants to glue $n$ copies of the full algebras

\begin{equation*}p_k p_j = p_{k+j} + \sum_{l=1}^j H^k_{l} p_{j-l} + \sum_{l=1}^k H^l_{j} p_{k-l} + D_{jk},
\end{equation*}

by means of the gluing relations

\begin{equation*}p_{\alpha,i} p_{\beta,k} = \sum_{\gamma=1}^n \sum_{l \geq 1} C^{\gamma l}_{\alpha \beta ik} p_{\gamma l} + C^{0}_{\alpha \beta ik} p_0,
\end{equation*}

where $p_0$ is the unit element, $\alpha, \beta = 1, \ldots, n$ and $i, k, l = 1, 2, \ldots$, one is obliged to consider the full system of structure constants

\begin{equation*}C^{\gamma l}_{\alpha \beta ik} = \delta_{\gamma \alpha} \delta_{\beta \delta_{ik}} + \delta_{\gamma \alpha} H_{\alpha,i-l}^\beta + \delta_{\gamma \beta} H_{\beta,k}^{\alpha,i}.
\end{equation*}

Coisotropic deformations for this algebra can be constructed according to our general scheme, but they will be studied elsewhere.
7 Coisotropic and Lagrangian submanifolds.

Our purpose in this section is to stress an important geometrical difference between the usual approach to integrable dispersionless equations, based on the study of compatibility conditions of a system of Hamilton-Jacobi equations, and the present approach of coisotropic deformations.

A characteristic trait of the usual approach is to assume \( p_j = \frac{\partial S}{\partial x_j} \) since the beginning, and therefore one is unwittingly confined to a Lagrangian submanifold inside the symplectic manifold \( M^{2n} \). As a consequence one looses the canonical symplectic 2-form

\[
\omega = \sum_{i=1}^{n} dp_i \wedge dx_i
\]

which vanishes on the Lagrangian submanifold. In the present approach, instead, the main role is given the coisotropic submanifold \( \Gamma \) (for the definition of coisotropic submanifolds see e.g. [50]-[52]). It is interesting to see some examples of them. Any solution of the integrable hierarchies discussed in the paper provides us with a coisotropic submanifold. A simple example corresponds to the following solution of the dKP equation ([6])

\[
\begin{align*}
    u_0 &= -2/3 \frac{x_1}{(1 + 2x_3)} + 4/9 \frac{x_2^2}{(1 + 2x_3)^2}, \\
    v_0 &= -2/3 \frac{x_1 x_2}{(1 + 2x_3)^2} + 4/9 \frac{x_2^3}{(1 + 2x_3)^3}.
\end{align*}
\]

The associated coisotropic submanifold of dimension 4 in the six dimensional symplectic space \( R^6 \), with coordinates \((x_1, x_2, x_3, p_1, p_2, p_3)\) is defined by the equations

\[
\begin{align*}
    f_{11} &= -p_1^2 + p_2 + 2/3 \frac{x_1}{(1 + 2x_3)} - 4/9 \frac{x_2^2}{(1 + 2x_3)^2} = 0, \\
    f_{12} &= -p_1 p_2 + p_3 - p_1(-1/3 \frac{x_1}{(1 + 2x_3)} + 2/9 \frac{x_2^2}{(1 + 2x_3)^2}) - 2/3 \frac{x_1 x_2}{(1 + 2x_3)^2} + 4/9 \frac{x_2^3}{(1 + 2x_3)^3} = 0.
\end{align*}
\]

If instead one takes the zero-set of the first three Hamiltonians \((f_{11}, f_{12}, f_{13})\) of the dKP hierarchy, and one considers the common solution \(u_0(x_1, x_2, x_3, x_4)\) of the dKP and first higher dKP equations, one gets a five dimensional coisotropic submanifold in \( R^8 \). Continuing in this process, one obtains an infinite tower on coisotropic submanifolds associated with dKP hierarchy. In general they have dimension \((n + 1)\) in a symplectic space of dimension \(2n\). In this sense they are minimal since they are the most close to Lagrangian submanifolds which
have dimension \( n \). Furthermore, the restriction of the symplectic 2-form \( \omega \) to \( \Gamma \) does not vanish, but is a presymplectic 2-form. As is well-known, its kernel is spanned by the \((n-1)\) Hamiltonian vector fields associated with the functions \( f_{jk} \) defining the coisotropic submanifold. These vector fields define the so-called characteristic foliation of \( \Gamma \), whose space of leaves is called the reduced phase space \( \Gamma_{\text{red}} \) (see e.g. [50]-[52]). In our case the restriction of \( \omega \) to \( \Gamma \) has rank two and hence

\[
\omega_{\Gamma} = dL \wedge dM. \tag{45}
\]

The canonical variables \( L \) and \( M \) play an important role in the theory of dispersionless hierarchies.

Various Lagrangian submanifolds are obtained by setting a constraint on these variables. An obvious example is provided by the constraint

\[
L = z = \text{const}. \tag{46}
\]

As usual the Lagrangianity implies the existence of a generating function \( S(z, x) \) such that (see e.g. [53, 54, 55])

\[
p_j(x) = \frac{\partial S(z, x)}{\partial x_j} \quad \text{for} \quad z = \text{const}. \tag{47}
\]

One also readily concludes that the restriction of \( M \) to the Lagrangian submanifold is given by

\[
M|_z = M(z) = \frac{\partial S(z, x)}{\partial z}. \tag{48}
\]

With this identification the formula (45) coincides with that obtained in [12, 14]. Thus one may say that the foliation of the coisotropic submanifold \( \Gamma \) by Lagrangian submanifolds parametrized by \( z \) corresponds to the “algebraic orbits” of the Whitham’s equations of genus zero considered in [14]. A more general class of Lagrangian submanifolds are provided by the constraint

\[
g(L, M) = 0 \tag{49}
\]

where \( g \) is an arbitrary function.

Finally we would like to note that the nonlinear \( \bar{\partial} \) equation, which is the basic equation for the quasiclassical \( \bar{\partial} \) method ([30, 31, 32]), also has a simple geometric meaning within the present approach, at least if one considers the complexified version of the scheme, where all the variables are complex. In this case the rank of \( \omega|_\Gamma \) is equal to four (complex two), and

\[
\omega|_\Gamma = dL \wedge dM + d\bar{L} \wedge d\bar{M} \tag{50}
\]

where the bar denotes, as usual, the complex conjugation. In this case the class of Lagrangian submanifolds is defined by the constraint

\[
W(L, \bar{L}, M, \bar{M}) = 0 \tag{51}
\]
where $W$ is an arbitrary complex function. Using the parametrization $L = z, \bar{L} = \bar{z}$ and the formulae $M = \frac{\partial S}{\partial z}, \bar{M} = \frac{\partial S}{\partial \bar{z}}$, one writes equation (51) as

$$W(z, \bar{z}, \frac{\partial S}{\partial z}, \frac{\partial S}{\partial \bar{z}}) = 0$$

which is exactly the nonlinear $\bar{\partial}$ equation used in the papers [30, 31, 32, 34, 35]. So many of the known methods to solve dispersionless integrable equations deal, actually, with different classes of Lagrangian submanifolds contained inside the coisotropic submanifold $\Gamma$.

8 Final remarks.

The aim of this paper was to point out a new phenomenon (the appearance of structure constants inside the theory of dispersionless integrable hierarchies), without pretention of completeness or systematicity. The paper is, indeed, a first exploration of a territory which remains largely unknown. A few other directions of explorations are known. Some are routine work, consisting in encompassing different system of structure constants and therefore different classes of integrable hierarchies, such as, for instance, the dispersionless Harry Dym hierarchy. Some have the theoretical aim of probing more deeply the basis of the Hirota’s bilinear formulation, by understanding for what kind of associative algebras the associativity conditions allow to reduce the coisotropy conditions to the form of a system of conservation laws. In the same vain, the other interesting question is to understand systematically if the concept of coisotropy has nothing to do with a possible Hamiltonian (or bihamiltonian) structure of the integrable hierarchy defined by the central system on the structure constants. A last exciting direction, finally, points towards more general type of algebras. We know that to encompass, for instance, the dispersionless Veselov-Novikov hierarchy in the present scheme, one has to abandon the associative commutative algebra with unit and to consider Jordan’s triple systems. So of the scheme presented in this paper one should take mainly the spirit rather than the form, and to consider it as a possible point of departure for new interesting investigations in a field which has not yet exhausted is source of surprises, despite intensive investigations during the last twenty years or more.

9 Appendix A.

Here we will derive equations (23), (24) and (25) for the generalised dKP hierarchy.

Thus we consider an algebra with the structure constants of the form

$$C_{jk}^l = \delta_{j+k}^l + H_{j-l}^k + H_{k-l}^j + \delta_{0}^l D_{jk}$$

(53)
where $H_0^i \neq 0$, $H_k^i = 0 \; k \leq -1$.

For such the structure constants the associativity condition takes the form

\[
R_{ikmn} + H_{ikmn}^m D_{ik} - H_{ikmn}^i D_{mk} + \delta_{mn}^i D_{ik} - \delta_{mk}^i D_{mk} \\
+ \delta_{0}^m (D_{i+k,m} - D_{i,m+k} + \delta_{0}^i D_{0m} - \delta_{0}^m D_{0i}) \\
+ \delta_{0}^i \left( \sum_{l=0}^{k} H_{k-l}^i D_{lm} + \sum_{l=0}^{i} H_{l-i}^k D_{tm} - \sum_{l=0}^{k} H_{k-l}^m D_{li} - \sum_{l=0}^{m} H_{m-l}^k D_{li} \right) = 0
\]

(54)

where $R_{ikmn}$ is given by l.h.s of equation (20) with $H_0^i \neq 0$ and substitution $\sum_{l=1}^{p} \rightarrow \sum_{0}$. Equation (54) in the case when all indices $i, k, m, n$ are distinct and different from zero is of the form

\[
R_{ikmn} = 0.
\]

(55)

These equations are easily seen to be equivalent to the system

\[
\sum_{m+k=p} [H, H]_{ikm} = 0,
\]

(56)

where

\[
[H, H]_{ikm} := H_{m+k}^{i} - H_{m+k}^{i} - H_{i+k}^{m} + \sum_{l=0}^{i} H_{k-l}^{m} H_{l}^{i} + \sum_{l=0}^{k} H_{k-l}^{i} H_{l}^{m} - \sum_{l=0}^{m} H_{m-l}^{k} H_{l}^{i}.
\]

Note that these equations do not contain $H_0^i$.

Equations (54) with $n = m \neq 0$ and all other indices distinct are equivalent to

\[
D_{ik} + H_{0}^{i+k} - H_{0}^{i} H_{0}^{k} + \sum_{l=1}^{k} H_{k-l}^{i} H_{l}^{0} + \sum_{l=1}^{i} H_{l-i}^{k} H_{0}^{l} = 0.
\]

(57)

At $n = i \neq 0$ one gets equations which are equivalent to these ones.

Finally at $n = 0$ equation (54) is reduced to

\[
D_{i+k,m} - D_{i,m+k} + \sum_{l=0}^{k} H_{k-l}^{m} D_{lm} + \sum_{l=0}^{i} H_{k-l}^{i} D_{lm} \\
- \sum_{l=0}^{k} H_{m-l}^{m} D_{li} - \sum_{l=0}^{m} H_{m-l}^{i} D_{li} = 0.
\]

It is a straightforward but cumbersome check that these equations are verified in virtue of the previous ones. So the associativity conditions for the structure constants (35) are given by equations (56) and (57).

The coisotropy conditions $\{f_{ik}, f_{ln}\} = 0$ on $\Gamma$ for the Hamiltonians defined by the above structure constants are equivalent to the system

\[
T_{ikln} + \delta_{0}^m Q_{ikln} - \delta_{0}^n \frac{\partial C_{0k}^{0}}{\partial x_{l}} - \delta_{0}^i \frac{\partial C_{0k}^{0}}{\partial x_{n}} + \delta_{0}^m \frac{\partial C_{0l}^{0}}{\partial x_{i}} + \delta_{0}^n \frac{\partial C_{0l}^{0}}{\partial x_{k}} = 0,
\]

(58)
where
\[ C_{ik}^0 = H_i^k + H_i^l + D_{ik} \]
and
\[
T_{ikln} = \sum_{s=1}^{m>l} (\delta_{i+n}^s + H_{n-s}^l + H_{l-s}^n) \left( \frac{\partial H_{k-m}^l}{\partial x_s} + \frac{\partial H_{i-m}^k}{\partial x_s} \right) \\
- \sum_{s=1}^{m>l} (\delta_{i+k}^s + H_{k-s}^l + H_{l-s}^k) \left( \frac{\partial H_{i-m}^l}{\partial x_s} + \frac{\partial H_{i-m}^k}{\partial x_s} \right) \\
- \sum_{s=1}^{m>l} (\delta_{s+n}^m + H_{n-m}^l + H_{s-m}^n) \left( \frac{\partial H_{k-s}^l}{\partial x_l} + \frac{\partial H_{i-s}^k}{\partial x_l} \right) \\
- \sum_{s=1}^{m>l} (\delta_{s+k}^n + H_{k-m}^l + H_{k-m}^k) \left( \frac{\partial H_{n-s}^l}{\partial x_k} + \frac{\partial H_{l-s}^n}{\partial x_k} \right) \\
+ \sum_{s=1}^{m>l} (\delta_{s+i}^m + H_{s-m}^l + H_{i-m}^n) \left( \frac{\partial H_{k-s}^l}{\partial x_i} + \frac{\partial H_{i-s}^k}{\partial x_i} \right) \\
+ \sum_{s=1}^{m>l} (\delta_{i+s}^l + H_{s-m}^l + H_{i-m}^n) \left( \frac{\partial H_{k-s}^l}{\partial x_k} + \frac{\partial H_{i-s}^k}{\partial x_k} \right)
\]

and
\[
Q_{ikln} = \frac{\partial D_{ik}}{\partial x_{i+n}} - \frac{\partial D_{ln}}{\partial x_{i+k}} + \sum_{s=0}^{m>l} [(H_{n-s}^l + H_{l-s}^n) \left( \frac{\partial D_{ik}}{\partial x_s} + \frac{\partial D_{ln}}{\partial x_s} \right) - (H_{k-s}^l + H_{i-s}^k) \left( \frac{\partial D_{ik}}{\partial x_s} - \frac{\partial D_{ln}}{\partial x_s} \right)]
\]

- \frac{\partial H_{k+n-m}^l}{\partial x_l} + \frac{\partial H_{i+n-m}^l}{\partial x_i} = 0.

For arbitrary indices \( i, k, l, n \) and for \( m > i, k, l, n \); \( m > n + i \); \( m > l + i \); \( m > l + k \); \( m \leq n + k \) equations are reduced to
\[
(59)
\]

. Since \( k + n - m \geq 0 \) one has therefore
\[
\frac{\partial H_{i}^l}{\partial x_l} = \frac{\partial H_{i}^l}{\partial x_i}
\]

for \( p \geq 0 \) and \( i, l \geq 1 \).

Using this condition, one can show that \( T_{iklnm} \) can be represented as
\[
T_{iklnm} = \frac{\partial [H, H]_{l,n,k-m}}{\partial x_i} + \frac{\partial [H, H]_{l,n,i-m}}{\partial x_k} - \frac{\partial [H, H]_{i,k,l-m}}{\partial x_n} - \frac{\partial [H, H]_{i,k,n-m}}{\partial x_l}.
\]

30
So equations (58) take the form

$$\frac{\partial}{\partial x_i} \left( [H,H]_{l,n,k-m} + \delta^m_k C^0_{ln} \right) + \frac{\partial}{\partial x_k} \left( [H,H]_{l,n,i-m} + \delta^m_i C^0_{ln} \right) - \frac{\partial}{\partial x_n} \left( [H,H]_{i,k,l-m} + \delta^m_l C^0_{ik} \right) - \frac{\partial}{\partial x_l} \left( [H,H]_{i,k,n-m} + \delta^m_n C^0_{ik} \right) + \delta^m_0 Q_{ikln} = 0.$$  

(60)

For arbitrary indices $i, k, l, n$ and for $m < i, k, l, n$; \(m \neq 0\) these equations are reduced to

$$\frac{\partial [H,H]_{l,n,k-m}}{\partial x_i} + \frac{\partial [H,H]_{l,n,i-m}}{\partial x_k} - \frac{\partial [H,H]_{i,k,l-m}}{\partial x_n} - \frac{\partial [H,H]_{i,k,n-m}}{\partial x_l} = 0.$$  

These equations are satisfied in virtue of the associativity conditions.

For arbitrary indices $i, k, l, n$ and for $m \neq k$; $m < i, l, n$; \(m \neq 0\) equation (60) is

$$\frac{\partial}{\partial x_i} \left( [H,H]_{l,n,0} + C^0_{ln} \right) = 0.$$  

It is satisfied if

$$[H,H]_{l,n,0} + C^0_{ln} = 0.$$  

Since

$$[H,H]_{l,n,0} = H^0_{l+n} - H^0_{l} - H^0_{n} + \sum_{s=0}^{l} H^s_{l-s} H^0_{n} + \sum_{s=0}^{n} H^s_{n-s} H^0_{l}$$

the above equations just follow from the associativity conditions. In the same manner one can show that in the cases \(m \neq 0\) and \(m = i\) or \(m = l\) or \(m = n\) equations (60) are satisfied owing to the associativity conditions.

At $m = 0$ and arbitrary $i, k, l, n \neq 0$ equations (60) take the form

$$\frac{\partial [H,H]_{l,n,k}}{\partial x_i} + \frac{\partial [H,H]_{l,n,i}}{\partial x_k} - \frac{\partial [H,H]_{i,k,l}}{\partial x_n} - \frac{\partial [H,H]_{i,k,n}}{\partial x_l} + Q_{ikln} = 0$$

and, consequently, they reduce to

$$Q_{ikln} = 0.$$  

It is a direct but quite cumbersome check that these equations also are satisfied in virtue of equations (56), (57), and (59).

Thus, the coisotropic deformations of the generalized dKP algebra defined by the structure constants (35) are given by the associativity conditions (56), (57), and the exactness conditions (59). The dKP case corresponds to the particular choice of $H^0_0 = 0$ and $D_{jk} = 0$. 

31
10 Appendix B.

For the algebra given by

\[ p_\alpha p_\beta = H_\alpha^\beta p_\alpha + H_\beta^\alpha p_\beta + C_{\alpha\beta}^0 \quad \alpha \neq \beta \]  

(61)

the associativity conditions give rise to the system

\[ C_{\alpha\beta}^0 = H_\gamma^\alpha H_\gamma^\beta + H_\gamma^\alpha H_\gamma^\beta + H_\gamma^\beta H_\gamma^\alpha = 0 \quad \alpha \neq \beta \neq \gamma \neq \alpha \]  

(62)

and

\[ H_\alpha^\beta c_{\alpha\gamma}^0 + H_\beta^\alpha c_{\beta\gamma}^0 - H_\gamma^\alpha c_{\alpha\beta}^0 - H_\gamma^\beta c_{\beta\alpha}^0 = 0. \]  

(63)

Equations (62) imply \( 1/2n(n-1)(n-2) \) equations

\[ H_\gamma^\alpha H_\gamma^\beta - H_\gamma^\beta H_\gamma^\alpha = H_\gamma^\alpha H_\gamma^\beta - H_\gamma^\beta H_\gamma^\alpha \]  

(64)

where \( \alpha, \beta, \gamma, \bar{\gamma} \) are all distinct. Then it is a simple check that (63) is satisfied due to (62) and (64). Thus the associativity conditions for the algebra (61) are given by equations (62) and (64). In the particular case when all \( C_{\alpha\beta}^0 \) vanish they are reduced to equations (40).

In order to find the coisotropy conditions, let us first compute the Poisson brackets \( \{ f_\alpha, f_\beta \} \) for pairs of Hamiltonians corresponding to indices \( \alpha, \beta, \gamma, \delta \) all distinct. Since \( x_\alpha, p_\alpha \), for \( \alpha = 1, 2, \ldots, n \) are pairs of conjugate canonical variables, we have

\[ \{ f_\alpha, f_\beta \} = \left( \frac{\partial H_\alpha}{\partial x_\beta} - \frac{\partial H_\beta}{\partial x_\alpha} \right) f_\alpha \delta + \left( \frac{\partial H_\delta}{\partial x_\alpha} - \frac{\partial H_\alpha}{\partial x_\delta} \right) f_\beta \delta \]

\[ + \left( \frac{\partial H_\gamma}{\partial x_\delta} - \frac{\partial H_\delta}{\partial x_\gamma} \right) f_\alpha \gamma + \left( \frac{\partial H_\delta}{\partial x_\alpha} - \frac{\partial H_\alpha}{\partial x_\delta} \right) \bar{f}_{\beta \gamma} \]

\[ + \left( -H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\gamma} - H_\delta^\alpha \frac{\partial H_\beta}{\partial x_\gamma} + H_\delta^\alpha \frac{\partial H_\beta}{\partial x_\delta} + H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\delta} + H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\gamma} \right) p_\alpha \]

\[ + \left( -H_\beta^\delta \frac{\partial H_\alpha}{\partial x_\gamma} - H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\gamma} + H_\delta^\beta \frac{\partial H_\alpha}{\partial x_\delta} + H_\delta^\beta \frac{\partial H_\alpha}{\partial x_\delta} + H_\beta^\delta \frac{\partial H_\alpha}{\partial x_\delta} \right) p_\beta \]

\[ - \left( -H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\delta} - H_\beta^\delta \frac{\partial H_\alpha}{\partial x_\delta} + H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\delta} + H_\delta^\alpha \frac{\partial H_\beta}{\partial x_\delta} + H_\delta^\alpha \frac{\partial H_\beta}{\partial x_\delta} \right) \bar{p}_\alpha \]

\[ + \left( -H_\beta^\delta \frac{\partial H_\alpha}{\partial x_\delta} - H_\alpha^\delta \frac{\partial H_\beta}{\partial x_\delta} + H_\delta^\beta \frac{\partial H_\alpha}{\partial x_\delta} + H_\delta^\beta \frac{\partial H_\alpha}{\partial x_\delta} + H_\beta^\delta \frac{\partial H_\alpha}{\partial x_\delta} \right) \bar{p}_\beta \]

\[ \text{where there is not summation over repeated indices. The Poisson brackets of Hamiltonians having equal indices, like } \{ f_\alpha, f_\beta \}, \text{may be obtained from this formula by means of the following substitutions: } H_\beta^\delta = 0 \text{ and } f_\beta \delta \rightarrow p_\beta + \]
where \( f_{11\beta} = p_{1\beta}^2 - p_{2\beta} - v_{\beta} p_{\beta} - u_{\beta} \) is the lowest Hamiltonian for the \( \beta - th \) dKP hierarchy.

The coisotropy condition requires that the r.h.s of (65) vanishes for the values of \( (x_\alpha, p_\alpha) \) for which \( f_{\alpha \beta} = 0 \) for all \( \alpha \neq \beta \). In the case when all indices in (65) are distinct it is satisfied if the coefficients in front of \( p_{\alpha}, p_{\beta}, p_{\gamma}, p_{\delta} \) all vanish. It is not difficult to check that all these equations are satisfied in virtue of the system

\[
\begin{align*}
H_\alpha^\delta & \frac{\partial H_\beta^\gamma}{\partial x_\gamma} - H_\alpha^\gamma \frac{\partial H_\beta^\delta}{\partial x_\delta} - H_\alpha^\delta \frac{\partial H_\beta^\gamma}{\partial x_\delta} = 0
\end{align*}
\]  
(66)

The coisotropy conditions corresponding to pair of Hamiltonians with coinciding indices, say \( \{f_{\alpha \beta}, f_{\beta \gamma}\} \), require that the coefficients in front of \( p_{\beta \gamma} \) of \( p_{\alpha} \) and \( p_{\delta} \), and of \( p_{\beta} \) vanish. The first requirement leads to

\[
\frac{\partial H_\alpha^\delta}{\partial x_\alpha} - \frac{\partial H_\beta^\gamma}{\partial x_\delta} = 0 \quad \alpha \neq \beta \neq \delta \neq \alpha.
\]  
(67)

The second requirement gives equation (66) with \( \beta \neq \gamma \), and finally the third requirement gives the equation relating \( H_\beta^\delta \) with the functions \( v_{\beta}, u_{\beta} \) for the \( \beta - th \) KP hierarchy.

So the coisotropy conditions contain the subset of equations (66), (67), containing only the functions \( H_\beta^\delta \) for \( \alpha \neq \beta \). Using (67), one shows that equations (66) can be recast in the form

\[
\frac{\partial}{\partial x_\beta} (H_\alpha^\gamma H_\delta^\beta - H_\beta^\gamma H_\alpha^\delta - H_\alpha^\delta H_\beta^\gamma) = 0.
\]  
(68)

Thus this subset of coisotropy conditions is equivalent to the associativity conditions (40) and to the exactness conditions (67).

Note that in terms of the functions \( F_\alpha \) introduced in Sec.4 one has

\[
\{f_{\alpha \beta}, f_{\gamma \delta}\} = \frac{\partial^2(F_\delta - F_\alpha)}{\partial x_\beta \partial x_\gamma} f_{\alpha \delta} + \frac{\partial^2(F_\gamma - F_\beta)}{\partial x_\alpha \partial x_\delta} f_{\beta \delta} + \frac{\partial^2(F_\gamma - F_\alpha)}{\partial x_\beta \partial x_\delta} f_{\alpha \gamma} + \frac{\partial^2(F_\gamma - F_\beta)}{\partial x_\alpha \partial x_\delta} f_{\beta \gamma}.
\]  
(69)

This representation of the dDarboux system is the quasiclassical limit of the operator representation of the original Darboux system.

The coisotropic submanifold \( \Gamma \) defined by the equations

\[
f_{\alpha \beta} = p_{\alpha} p_{\beta} - H_\alpha^\beta p_{\alpha} - H_\beta^\alpha p_{\beta} - C_{\alpha \beta}^0 = 0 \quad \alpha \neq \beta
\]  
(70)
and the quadrics defined by the associativity conditions have quite remarkable properties. First, the quadrics (70) are transformed into quadrics of the same type under a Cremona transformation $p_\alpha \to \xi_\alpha = \frac{1}{p_\alpha}$. Indeed one gets

$$\xi_\alpha \xi_\beta - \tilde{H}_\alpha^\beta \xi_\alpha - \tilde{H}_\beta^\alpha \xi_\beta - \tilde{C}_0^{\alpha\beta} = 0$$

where

$$\tilde{H}_\alpha^\beta = -\frac{H_\alpha^\beta}{C_0^{\alpha\beta}}, \quad \tilde{C}_0^{\alpha\beta} = \frac{1}{C_0^{\alpha\beta}}.$$ 

In the particular case $C_0^{\alpha\beta} = 0$ the Cremona’s transformation linearises the quadric (70), since in this case

$$H_\alpha^\beta \xi_\beta + H_\beta^\alpha \xi_\alpha - 1 = 0 \quad \text{for} \quad \alpha, \beta = 1, 2, \ldots, n \quad \alpha \neq \beta. \quad (71)$$

So for the dDarboux system any section of the coisotropic submanifold with $x_\alpha = \text{const}$ is, in fact, the intersection of the planes (71).

Secondly, the equations defining the associativity quadrics can be rewritten as

$$\frac{H_\gamma^\beta}{H_\alpha^\beta} + \frac{H_\beta^\gamma}{H_\alpha^\gamma} - 1 = 0 \quad \alpha \neq \beta \neq \gamma \neq \alpha.$$ 

In the case $n = 3$, in terms of the variables $z_1, z_2, z_3$ defined by

$$z_1 = \frac{H_1^3}{H_3^3}, \quad z_2 = \frac{H_2^3}{H_3^3}, \quad z_3 = \frac{H_3^3}{H_2^3},$$

the above equations become

$$z_1 + \frac{1}{z_2} = 1 \quad z_2 + \frac{1}{z_3} = 1 \quad z_3 + \frac{1}{z_1} = 1$$

with the obvious constraint

$$z_1 z_2 z_3 = -1.$$ 

These equations define a curve in $R^3$ which is the intersection of three cylinders which are generated by the above hyperbolas.

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