A NOTE ON THE ANALYSIS OF ASYMPTOTIC MEAN-SQUARE STABILITY PROPERTIES FOR SYSTEMS OF LINEAR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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Abstract. The stability of equilibrium solutions of a deterministic linear system of delay differential equations can be investigated by studying the characteristic equation. For stochastic delay differential equations stability analysis is usually based on Lyapunov functional or Razumikhin type results, or Linear Matrix Inequality techniques. In [7] the authors proposed a technique based on the vectorisation of matrices and the Kronecker product to transform the mean-square stability problem of a system of linear stochastic differential equations into a stability problem for a system of deterministic linear differential equations. In this paper we extend this method to the case of stochastic delay differential equations, providing sufficient and necessary conditions for the stability of the equilibrium. We apply our results to a neuron model perturbed by multiplicative noise. We study the stochastic stability properties of the equilibrium of this system and then compare them with the same equilibrium in the deterministic case. Finally the theoretical results are illustrated by numerical simulations.

1. Introduction. Delay differential equations (DDEs) give a mathematical formulation of problems with time-lag or after-effect. In such cases the rate of change of a system depends not only on the present but also on the past states of the system. They have a wide range of applications in physics, engineering, biology and economics. Introductions to the theory of DDEs and their application to the natural sciences can be found, e.g., in [12, 14, 18, 19, 36]. Considering stochastic effects in the models, such as influence due to thermal noise or intrinsic variations, by studying Stochastic Delay Differential Equations (SDDEs), has attracted an increasing interest in many different disciplines. For a theoretical background we refer to [18, 21, 22, 28]. An important area of application problems involving SDDEs is in biology, the references [2, 3, 4, 5, 6, 8, 9, 10, 16, 23, 24, 26, 29, 30, 33] illustrate the wide variety of topics. Often, an important first step in the analysis of qualitative properties of (S)DDE models consists of a linear stability analysis and for deterministic delay differential equations the corresponding stability conditions are determined by the study of the characteristic equation, see, e.g., [13, 17, 36]. For SDDES stability analysis is usually based on Lyapunov functional or Razumikhin type results, for linear systems also on Linear Matrix Inequalities (LMI-methods).
For the case of systems of stochastic ordinary differential equations, in [7] the authors proposed a technique based on the vectorisation of matrices and the Kronecker product to transform the mean-square stability problem of a linear stochastic system into a stability problem for a deterministic system. The aim of this article is to extend this method to the case of linear systems of SDDEs. Deterministic differential equations describing the mean-square evolution of linear systems of SODEs have already been derived in [1] and for stochastic functional equations in [32]. The main advantage of the approach presented in [7] for SODEs and here for SDDEs lies in the fact that it facilitates using the background and results in linear algebra concerning matrices and their Kronecker products. Thus, in this article we consider $d$-dimensional systems with an $m$-dimensional Wiener process of the form

$$dX(t) = (AX(t) + BX(t - \tau))dt + \sum_{r=1}^{m} (C_rX(t) + D_rX(t - \tau))dW_r(t)$$

where $A, B, C_r, D_r, r = 1, \ldots, m$ are real $d \times d$ matrices.

The paper is organised as follows. In Section 2 we introduce basic notions concerning mean-square stability for stochastic systems, and in Section 3 we present two algebraic tools with their properties: the vectorisation operator and the Kronecker product of matrices. We use them in Section 4 to reformulate the stability problem of an SDDE in terms of a DDE. In Section 5 we apply our results to a neuron model perturbed by multiplicative noise. We study the stochastic stability properties of the equilibrium of this system and then compare them with the same equilibrium in the deterministic case. Finally the theoretical results are illustrated by numerical simulations. In Section 6 we discuss the results of our work.

2. Mean-square stability notions. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$. Let $W(t) = (W_1(t), \ldots, W_m(t))^T$ be an $m$-dimensional Wiener process on that probability space. Consider a $d$-dimensional stochastic delay system with constant delay of the form

$$dX(t) = F(t, X(t), X(t - \tau))dt + G(t, X(t), X(t - \tau))dW(t)$$

with initial data

$$X(s) = \phi(s) \quad s \in J := [-\tau, 0], \quad \tau > 0.$$

We denote the Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$ and the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. Let $C[-\tau, 0] = \{\phi : [-\tau, 0] \to \mathbb{R}^d\}$ be the space of continuous functions with the norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. We assume that the drift and diffusion coefficients $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, G : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are continuous and that $F(t, 0, 0) = G(t, 0, 0) = 0$ for all $t$. We denote by $X(t; t_0, \phi)$ a solution depending on the initial time $t_0$ and initial data $\phi$, where $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ is a $\mathcal{F}_{t_0}$-measurable, $C([-\tau, 0], \mathbb{R}^d)$-valued stochastic process such that $\mathbb{E}\|\phi\|^2 < \infty$.

**Definition 2.1.** The trivial solution $X(t) \equiv 0$ of the SDDE (1) is termed

1. **mean-square stable**, if for each $\epsilon > 0$, there exists $\delta \geq 0$ such that $\mathbb{E}\|\phi\|^2 < \delta$ implies

$$\mathbb{E}\|X(t; t_0, \phi)\|^2 < \epsilon, \quad \text{for } t \geq t_0;$$

2. **asymptotically mean-square stable**, if it is mean-square stable and if there exists $\delta \geq 0$ such that $\mathbb{E}\|\phi\|^2 < \delta$ implies

$$\mathbb{E}\|X(t; t_0, \phi)\|^2 \to 0 \quad \text{as } t \to \infty.$$
3. Vectorisation and the Kronecker product. We collect here relevant definitions and results on the Kronecker product, the vectorisation operator and commutator matrices, a comprehensive account of these and further results can be found in [20].

(i) The vectorisation vec(A) of an \( m \times n \) matrix \( A \) transforms the matrix \( A \) into an \( mn \)-dimensional column vector obtained by stacking the columns of the matrix \( A \) on top of one another.

(ii) The Kronecker product of an \( m \times n \) matrix \( A \) and a \( p \times q \) matrix \( B \) is the \( mp \times nq \) matrix defined by

\[
A \otimes B = \begin{pmatrix}
a_{11}B & \ldots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \ldots & a_{mn}B
\end{pmatrix}.
\]

(iii) The \((mn \times mn)\)-dimensional matrix \( K_{m,n} \) is a commutation matrix if

\[
\text{vec}(A^T) = K_{m,n}\text{vec}(A)
\]

for any \((m \times n)\)-dimensional matrix \( A \). The matrix operator \( K_{m,n} \) is a real matrix and a permutation matrix, many further properties of commutation matrices can be found in [20].

Lemma 3.1. (a) \((A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D\), if the matrices \( A + B \) and \( C + D \) exist;

(b) \( \text{vec}(ABC) = (C^T \otimes A)\text{vec}(B) \), when \( A, B \) and \( C \) are three matrices, such that the matrix product \( ABC \) is defined;

(c) A special case of (b) is given by \( \text{vec}(AB) = (B^T \otimes I_m)\text{vec}(A) = (I_q \otimes A)\text{vec}(B) \), where \( A \) is an \( m \times n \) matrix, \( B \) a \( n \times q \) matrix, and \( I_s \) is the \( s \)-dimensional identity matrix for any \( s \in \mathbb{N} \).

(d) For matrices \( A \) and \( B \), where \( A \) is an \( m \times n \) matrix, \( B \) a \( p \times q \) matrix, it holds that \((A \otimes B)T_{q,n} = T_{p,m}(B \otimes A)\).

4. The mean-square process of a linear SDDE system. Consider the following linear \( d \)-dimensional system of stochastic delay differential equations

\[
dX(t) = (AX(t) + BX(t - \tau))dt + \sum_{r=1}^{m}(C_rX(t) + D_rX(t - \tau))dW_r(t),
\]

where \( A, B, C_r, D_r, r = 1, \ldots, m \) are \( d \times d \) matrices. In the following we derive a deterministic differential equations for the evolution of the second moment of Equation (4). In [1, Thm. 8.5.1] and [15, Section 6.2] an analogous approach has been used for deriving the evolution of the second moment of linear systems of SODEs, giving the implications for mean-square stability in [1, Remark 8.5.5, Section 11] and [15, Remark 6.4]. Similarly, equations for the second moments of stochastic functional differential equations have been derived in [32]. In all these approaches, the symmetry of the arising matrices has been employed to reduce the size of the resulting system. This would correspond to applying the so-called half-vectorisation operator \( \text{vech} \), see [20], to the systems below, instead of the \( \text{vec} \) operator, without taking advantage of the more practical relations between the \( \text{vec} \) operator and the Kronecker product.
By using the matrix property $(AB)^T = B^T A^T$ and the Itô formula (see [1]) we have

\[
d(X(t)X(t)^T) = X(t) \left[ (AX(t) + BX(t - \tau))dt + \sum_{r=1}^{m} (C_rX(t) + D_rX(t - \tau))dW_r(t) \right]^T
\]

\[
+ \left[ (AX(t) + BX(t - \tau))dt + \sum_{r=1}^{m} (C_rX(t) + D_rX(t - \tau))dW_r(t) \right] X(t)^T
\]

\[
+ \sum_{r=1}^{m} (C_rX(t)X(t)^T C_r^T + C_rX(t)X(t - \tau)^T D_r^T + D_rX(t - \tau)X(t)^T C_r^T + D_rX(t - \tau)X(t - \tau)^T D_r^T) dt
\]

\[
= \left( X(t)X(t)^T A^T + X(t)X(t - \tau)^T B^T \right) dt
\]

\[
+ \sum_{r=1}^{m} (X(t)X(t)^T C_r^T + X(t)X(t - \tau)^T D_r^T) dW_r(t)
\]

\[
+ (AX(t)X(t)^T + BX(t - \tau)X(t)^T) dt
\]

\[
+ \sum_{r=1}^{m} (C_rX(t)X(t)^T + D_rX(t - \tau)X(t)^T) dW_r(t)
\]

\[
+ \sum_{r=1}^{m} (C_rX(t)X(t)^T C_r^T + C_rX(t)X(t - \tau)^T D_r^T + D_rX(t - \tau)X(t)^T C_r^T + D_rX(t - \tau)X(t - \tau)^T D_r^T) dt.
\]

Then the evolution of the second moment of the solution of (4) is given as

\[
\frac{d}{dt} \mathbb{E} \left[ X(t)X(t)^T \right] = \mathbb{E} \left[ X(t)X(t)^T \right] A^T + \mathbb{E} \left[ X(t)X(t - \tau)^T \right] B^T
\]

\[
+ \mathbb{E} \left[ X(t)X(t)^T \right] + \mathbb{E} \left[ X(t - \tau)X(t)^T \right] + \sum_{r=1}^{m} (C_r \mathbb{E} \left[ X(t)X(t)^T \right] C_r^T + C_r \mathbb{E} \left[ X(t)X(t - \tau)^T \right] D_r^T + D_r \mathbb{E} \left[ X(t - \tau)X(t)^T \right] C_r^T + D_r \mathbb{E} \left[ X(t - \tau)X(t - \tau)^T \right] D_r^T).
\]

Setting $P(t, t) = X(t)X(t)^T$, $P(t, t - \tau) = X(t)X(t - \tau)^T$ and $P(t - \tau, t - \tau) = X(t - \tau)X(t - \tau)^T$, we obtain

\[
\frac{d}{dt} \mathbb{E}P(t, t) = \mathbb{E}P(t, t)A^T + \mathbb{E}P(t, t - \tau)B^T
\]

\[
+ \mathbb{E}P(t, t) + \mathbb{E}P(t, t - \tau)^T + \sum_{r=1}^{m} (C_r \mathbb{E}P(t, t)C_r^T + C_r \mathbb{E}P(t, t - \tau)D_r^T + D_r \mathbb{E}P(t, t - \tau)^T C_r^T + D_r \mathbb{E}P(t, t - \tau - \tau)D_r^T).
\]
Applying the vectorisation operator on both sides of (5), as well as the results of Lemma 3.1, yields

\[
\frac{d}{dt} \mathbb{E} \text{vec}(P(t, t)) = \text{vec}(\mathbb{E} P(t, t) A^T) + \text{vec}(\mathbb{E} P(t, t - \tau) B^T) + \text{vec}(\mathbb{E} P(t, t)) \\
+ \text{vec}(\mathbb{E} P(t, t - \tau)^T) \\
+ \text{vec} \left( \sum_{r=1}^{m} (C_r \mathbb{E} P(t, t) C_r^T + C_r \mathbb{E} P(t, t - \tau) D_r^T) \\
+ D_r \mathbb{E} P(t, t - \tau)^T C_r^T + D_r \mathbb{E} P(t - \tau, t - \tau) D_r^T \right) \]

\[
= (A \otimes I_d) \text{vec}(\mathbb{E} P(t, t)) + (B \otimes I_d) \text{vec}(\mathbb{E} P(t, t - \tau)) \\
+ (I_d \otimes A) \text{vec}(\mathbb{E} P(t, t)) + (I_d \otimes B) \text{vec}(\mathbb{E} P(t, t - \tau)^T) \\
+ \sum_{r=1}^{m} ((C_r \otimes C_r) \text{vec}(\mathbb{E} P(t, t)) + (D_r \otimes C_r) \text{vec}(\mathbb{E} P(t, t - \tau)) \\
+ (C_r \otimes D_r) \text{vec}(\mathbb{E} P(t, t - \tau)^T) + (D_r \otimes D_r) \text{vec}(\mathbb{E} P(t - \tau, t - \tau))) \]

\[
= \left( A \otimes I_d + I_d \otimes A + \sum_{r=1}^{m} C_r \otimes C_r \right) \text{vec}(\mathbb{E} P(t, t)) \\
+ \left( B \otimes I_d + \sum_{r=1}^{m} D_r \otimes C_r \right) \text{vec}(\mathbb{E} P(t, t - \tau)) \\
+ \left( I_d \otimes B + \sum_{r=1}^{m} C_r \otimes D_r \right) \text{vec}(\mathbb{E} P(t, t - \tau)^T) \\
+ \sum_{r=1}^{m} (D_r \otimes D_r) \text{vec}(\mathbb{E} P(t - \tau, t - \tau)) \]

\[
= \left( A \otimes I_d + I_d \otimes A + \sum_{r=1}^{m} C_r \otimes C_r \right) \text{vec}(\mathbb{E} P(t, t)) \\
+ \left( B \otimes I_d + \sum_{r=1}^{m} D_r \otimes C_r + (I_d \otimes B + \sum_{r=1}^{m} C_r \otimes D_r) K_{d,d} \right) \text{vec}(\mathbb{E} P(t, t - \tau)) \\
+ \sum_{r=1}^{m} (D_r \otimes D_r) \text{vec}(\mathbb{E} P(t - \tau, t - \tau)) \]

\[
= \left( A \otimes I_d + I_d \otimes A + \sum_{r=1}^{m} C_r \otimes C_r \right) \text{vec}(\mathbb{E} P(t, t)) \\
+ (K_{d,d} + I_d^2) \left( B \otimes I_d + \sum_{r=1}^{m} D_r \otimes C_r \right) \text{vec}(\mathbb{E} P(t, t - \tau)) \\
+ \sum_{r=1}^{m} (D_r \otimes D_r) \text{vec}(\mathbb{E} P(t - \tau, t - \tau)) \]

(6)
By introducing the notation \( Z(t, s) = E(\text{vec} P(t, s)) \), system (6) can be written as
\[
\frac{d}{dt} Z(t, t) = S \, Z(t, t) + T_1 \, Z(t, t - \tau) + T_2 \, Z(t - \tau, t - \tau),
\]
where \( S, T_1 \) and \( T_2 \) are \( d^2 \times d^2 \) real valued matrices given by
\[
S = A \otimes I_d + I_d \otimes A + \sum_{r=1}^{m} C_r \otimes C_r,
\]
\[
T_1 = (I_d^2 + K_{d,d}) \left( B \otimes I_d + \sum_{r=1}^{m} D_r \otimes C_r \right),
\]
\[
T_2 = \sum_{r=1}^{m} D_r \otimes D_r.
\]

Considering the solution of (7) as being of the form \( Z(t, s) = e^{\lambda t} e^{\lambda_s} v \) for \( v \neq 0 \), analogously to the standard case of deriving the characteristic equation in, e.g., [36], and as suggested already in [25], and using that \( Z(t, t) = e^{2\lambda t} v, Z(t, t - \tau) = e^{2\lambda t} e^{-\lambda \tau} v \) and \( Z(t - \tau, t - \tau) = e^{2\lambda t} e^{-2\lambda \tau} v \), we obtain a characteristic equation in the following form
\[
\det \left( \lambda I - \frac{1}{2} S - \frac{1}{2} T_1 e^{-\lambda \tau} - \frac{1}{2} T_2 e^{-2\lambda \tau} \right) = 0,
\]
where its eigenvalues are given by the roots \( \lambda \) of equation (8). The trivial solution of the system (7) is asymptotically stable if and only if all the eigenvalues \( \lambda \) have a negative real part, and it is unstable if there exists an eigenvalue with positive real part. Bifurcations occur whenever eigenvalues move through the imaginary axis as one or more parameters are changed.

Then conditions for asymptotic mean-square stability of the trivial solution of the stochastic system (4) can be deduced by studying the eigenvalues of deterministic equation (7).

5. Application to a neuron model. To illustrate our result, we consider the following nonlinear FitzHugh-Nagumo delay system [31]
\[
\begin{align*}
\varepsilon_1 dx_1(t) &= \left( x_1(t) - \frac{x_1(t)^3}{3} - y_1(t) + C[x_2(t - \tau) - x_1(t)] \right) \, dt \\
\, dy_1(t) &= (x_1(t) + a) \, dt \\
\varepsilon_2 dx_2(t) &= \left( x_2(t) - \frac{x_2(t)^3}{3} - y_2(t) + C[x_1(t - \tau) - x_2(t)] \right) \, dt \\
\, dy_2(t) &= (x_2(t) + a) \, dt.
\end{align*}
\]
(9)
The mutual transmission delay between the two neurons \( x_1 \) and \( x_2 \) is denoted by \( \tau \). The constant \( C \) denotes the coupling strength, \( a \) is an excitability parameter such that for \( a > 1 \) we have an excitable regime and \( a < 1 \) defines an oscillatory regime. This system has a unique fixed point given by
\[
x^* = (x_1^*, y_1^*, x_2^*, y_2^*) = (-a, \frac{a^3}{3} - a, -a, \frac{a^3}{3} - a).
\]
(10)
Following the approach suggested in [3, 33] and subsequently employed in, e.g., [2, 5, 8, 9, 24, 26, 27, 29, 34, 35], we perturb the system (9) by multiplicative noise, considering stochastic perturbations of white noise type directly proportional to
the deviations of the variables from values of the equilibrium point \( x^* \). We choose to perturb only the second and fourth equation of the system (9) in order to be consistent with the FitzHugh-Nagumo system studied in [31]. The system (9) is extended to the following form

\[
\begin{align*}
\varepsilon_1 dX_1(t) &= \left( X_1(t) - \frac{X_1(t)^3}{3} - Y_1(t) + C[X_2(t - \tau) - X_1(t)] \right) dt \\
\mu dY_1(t) &= (X_1(t) + a) dt + q_1(Y_1(t) - y_1^*) dW_1(t) \\
\varepsilon_2 dX_2(t) &= \left( X_2(t) - \frac{X_2(t)^3}{3} - Y_2(t) + C[X_1(t - \tau) - X_2(t)] \right) dt \\
\mu dY_2(t) &= (X_2(t) + a) dt + q_2(Y_2(t) - y_2^*) dW_2(t),
\end{align*}
\]

where \( q_1, q_2 \) are positive constants and \( W_1, W_2 \) are independent Wiener processes.

Let us centre the system on the equilibrium point by the change of variables

\[
U_1 = X_1 - x_1^*, \quad U_2 = Y_1 - y_1^*, \quad U_3 = X_2 - x_2^*, \quad U_4 = Y_2 - y_2^*.
\]

Then we obtain the following system

\[
\begin{align*}
\varepsilon_1 dU_1(t) &= \left( U_1(t)(1 - a^2 - C) - U_2(t) + CU_3(t - \tau) - \frac{U_1(t)^3}{3} + U_1(t)^2 a \right) dt \\
\mu dU_2(t) &= U_1(t) dt + q_1 U_2(t) dW_1(t) \\
\varepsilon_2 dU_3(t) &= \left( U_3(t)(1 - a^2 - C) - U_4(t) + CU_1(t - \tau) - \frac{U_3(t)^3}{3} + U_3(t)^2 a \right) dt \\
\mu dU_4(t) &= U_3(t) dt + q_2 U_4(t) dW_2(t).
\end{align*}
\]

This is a stochastic system of four equations in four variables \( U_i, i = 1, \ldots, 4 \). It is easy to see that the stability of the equilibrium point of (11) is equivalent to the stability of zero solution of the centred system (12).

After centring the system around the equilibrium, we linearise the centred system around \( U = (U_1, U_2, U_3, U_4) = 0 \), which yields

\[
\begin{align*}
\varepsilon_1 dU_1(t) &= (U_1(t)(1 - a^2 - C) - U_2(t) + CU_3(t - \tau)) dt \\
\mu dU_2(t) &= U_1(t) dt + q_1 U_2(t) dW_1(t) \\
\varepsilon_2 dU_3(t) &= (U_3(t)(1 - a^2 - C) - U_4(t) + CU_1(t - \tau)) dt \\
\mu dU_4(t) &= U_3(t) dt + q_2 U_4(t) dW_2(t).
\end{align*}
\]

The system (13) can be written in the following vectorial form

\[
\frac{dU(t)}{dt} = (AU(t) + BU(t - \tau)) + QU(t) dW(t),
\]

where

\[
U(t) = (U_1(t), U_2(t), U_3(t), U_4(t)),
\]

and \( A, B, Q \) are \( 4 \times 4 \) matrices given by

\[
A = \frac{1}{\varepsilon_1} \begin{pmatrix}
1 - a^2 - C & -1 & 0 & 0 \\
\varepsilon_1 & 0 & 0 & 0 \\
0 & 0 & 1 - a^2 - C & -1 \\
0 & 0 & \varepsilon_1 & 0
\end{pmatrix}
\]
The system (14) is a 4-dimensional SDDE system of the form of (4). Then we can employ the results of Section 4. Set \( P(t, t) = U(t)U(t)^T \) and \( P(t, t - \tau) = U(t)U(t - \tau)^T \) and \( Z(t, s) = \text{Evec}P(t, s) \). Applying the vectorisation operator to the second moment of the solution of (14) we obtain the following linear DDE system for the 16-dimensional vectors \( Z(t, t) \) and \( Z(t, t - \tau) \)

\[
\frac{d}{dt} Z(t, t) = S \ Z(t, t) + T \ Z(t, t - \tau),
\]

(15)

where \( S \) and \( T \) are \( 16 \times 16 \) real valued matrices given by

\[
S = A \otimes I_4 + I_4 \otimes A + Q \otimes Q,
\]

\[
T = (I_{4^2} + K_{4,4})(B \otimes I_4).
\]

The characteristic equation of (15), as illustrated in Section 4, has the following form

\[
\det \left( \lambda I_{16} - \frac{1}{2} S - \frac{1}{2} T e^{-\tau \lambda} \right) = 0.
\]

(16)

In order to analyse the eigenvalues of (16), we perform a bifurcation analysis in Matlab by using DDE-BIFTOOL [11]. For the parameters we choose the same values as adopted in [31], since our aim is to compare the stability results obtained for the stochastic model with the deterministic counterpart. Thus we fix the parameters \( a = 1.05, C = 0.5 \), we choose the symmetric time scales \( \varepsilon_1 = \varepsilon_2 = 0.01 \) and we fix the delay \( \tau = 3 \). We compute the rightmost roots of the characteristic equation (16) at the trivial solution.

As can be seen in Figure 1 the real parts of all the eigenvalues are negative. This implies that the trivial solution is stable for this combination of parameters, and consequently it implies also the stability of the equilibrium point of (11). This result is in agreement with the deterministic one obtained in [31]. The parameter \( a \) is an excitability parameter and the stability of the equilibrium is then affected by different choices of \( a \). Figure 2 depicts the result of choosing the parameter \( a = 0.85 \). The trivial solution loses its stability since all the real parts of the eigenvalues are positive. This result is in agreement with [31], where the system exhibits a Hopf bifurcation at \( a = 1 \) and the fixed point becomes unstable for \( a < 1 \).

6. Conclusions. We developed an approach to study asymptotic mean-square stability properties of systems of linear stochastic delay differential equations. We propose the use of the vectorisation of matrices and the Kronecker product to treat the linear multi-dimensional systems by extending a technique developed for the SDEs in [7] to the case of SDDEs. We obtained necessary and sufficient stability conditions for linear SDDEs systems. We introduced a stochastic delay neuron model in which the stochastic perturbations are assumed to be proportional to the distances of the populations from their equilibrium values. We then analysed the asymptotic stability in probability of the equilibrium point of this nonlinear SDDEs neuron...
Figure 1. Stable case: $a = 1.05$. Real parts $\mathcal{R}(\lambda)$ of the eigenvalues $\lambda$ of the characteristic equation (16) at the trivial solution versus the imaginary parts $\mathcal{I}(\lambda)$. The coefficients of the noise are $q_1 = q_2 = 0.01$.

Figure 2. Unstable case: $a = 0.85$. Real parts $\mathcal{R}(\lambda)$ of the eigenvalues $\lambda$ of the characteristic equation (16) at the trivial solution versus the imaginary parts $\mathcal{I}(\lambda)$. The coefficients noise are $q_1 = q_2 = 0.01$.

model by applying our method to its linearisation. By studying the characteristic equation and analysing the eigenvalues numerically with DDE-biftool we obtained results in agreement with the deterministic model studied in [31].

We thank the reviewer and the editor for the careful reading of our manuscript and their helpful comments.

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Received December 2011; revised April 2012.

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