Perfectly matched layers for the stationary Schrödinger equation in a periodic structure

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Abstract

We construct a perfectly matched absorbing layer for stationary Schrödinger equation with analytic slowly decaying potential in a periodic structure. We prove the unique solvability of the problem with perfectly matched layer of finite length and show that solution to this problem approximates a solution to the original problem with an error that exponentially tends to zero as the length of perfectly matched layer tends to infinity.

Key words: radiation conditions, perfectly matched layer, PML, absorbing layers, complex scaling, schrödinger equation, slowly decaying potential

PACS:

1 Introduction

The Perfectly Matched Layer (PML) method, introduced in [1], is in common use for a numerical analysis of a wide class of problems. For some problems the convergence of the method has been proved mathematically, see e.g. [2–5]. In this paper, we introduce the PML method for the stationary Schrödinger equation in a “half-plane” with periodic boundary and Dirichlet boundary condition. We suppose that the potential \( q \) allows an analytic continuation to a cone on some distance from the boundary and \( q(z) \) uniformly tends to zero as \( z \) goes to infinity inside the cone. We include into consideration potentials decaying at infinity as slowly as \( z^\nu \), \( \nu < 0 \), or even \( 1/\ln z \). Since the potentials are not compactly supported the modal analysis employed in [4] cannot be

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used here, this leads to significant difficulties. Using the tools of complex scaling \[8,9\] we construct a PML of infinite length for the original problem supplied with some generalized radiation condition. The form of this radiation condition is similar to the pole condition \[3,10\]. The generalized radiation condition turns out to be equivalent to the classical radiation condition in the case of sufficiently rapid decay of the potential. As an approximation of a solution satisfying the original problem and the radiation condition, we take a solution to the problem with PML of finite length. We prove that the problem with PML of finite length is uniquely solvable and that the error of the approximation tends to zero with an exponential rate as the length of PML tends to infinity. The proof is based on weak statements of problems in weighted Sobolev spaces \[7\] and on a modification of the compound expansion method \[6\].

We consider the Dirichlet boundary condition as a boundary condition of the original problem and as an artificial boundary condition, however one can use the Neumann boundary condition instead. The approach is easily extended for this case, the results remain the same.

2 Statement of the problem

Let \( \mathcal{P} \) be an upper “half-plane” in \( \mathbb{R}^2 \) with smooth \( 2\pi \)-periodic boundary \( \partial \mathcal{P} \) such that \( \mathcal{P} \subset \{(y,t) \in \mathbb{R}^2 : t > c\} \) and \( \partial \mathcal{P} \subset \{(y,t) \in \mathbb{R}^2 : t < 0\} \). Let \( \mathcal{E} = \{(y,t) \in \mathcal{P} : |y| < \pi\} \) be the periodicity cell of \( \mathcal{P} \). We set \( \Upsilon^\pm = \{(y,t) \in \mathcal{P} : y = \pm \pi\} \) and \( \Upsilon^0 = \partial \mathcal{E} \setminus \{\Upsilon^+ \cup \Upsilon^\_\} \). As it usually is, the problem in \( \mathcal{P} \) reduces to a quasi-periodic boundary value problem in the periodicity cell \( \mathcal{E} \), see Fig. 1.

![Fig. 1. Geometry of the problem.](image.png)

We consider the stationary Schrödinger equation

\[
(\Delta + k^2 + q(y,t))u(y,t) = F(y,t), \ (y,t) \in \mathcal{E},
\]

with the quasi-periodicity conditions

\[
\partial^j_y u(\pi,t) = e^{2\pi i \alpha} \partial^j_y u(-\pi,t), \ j = 0,1, \ (\pm \pi,t) \in \Upsilon^\pm,
\]
and the Dirichlet boundary condition
\[ u(y, t) = 0, \ (y, t) \in \Upsilon^0. \] (3)

Here \( \alpha \in [0, 1) \), \( \partial_y = \partial/\partial y \), and the parameter \( k \) is a fixed real number that does not coincide with a threshold value, i.e. \( k^2 \neq (n + \alpha)^2 \) for all \( n \in \mathbb{Z} \). Let \( K_T^\phi \) denote the closed cone
\[ K_T^\phi = \{ z \in \mathbb{C} : z = T + e^{i\psi}t, 0 \leq \psi \leq \phi, t \geq 0 \}. \]

We assume that the potential \( q \) in the equation (1) satisfies the conditions: (i) \( q \) is a bounded real-valued function in \( \mathcal{P} \), \( q(y, t) = q(y + 2\pi, t) \) for all \( (y, t) \in \mathcal{P} \); (ii) for some \( T > 0 \) and \( \phi \in (0, \pi/2) \) the function \( F \) is analytic in \( z \) (and \( 2\pi \)-periodic in \( y \)) function \( \mathbb{R} \times K_T^\phi \ni (y, z) \mapsto q(y, z) \in \mathbb{C} \), which uniformly tends to zero as \( |z| \to +\infty \).

We also make the following assumptions on the right hand side \( F \) of the equation (1): (i) \( F \) is in the space \( L^l_2(\mathcal{E}) \) of locally square summable functions on \( \mathcal{E} \); (ii) for some \( T > 0 \) and \( \phi \in (0, \pi/2) \) the function \( F \) is analytic in \( z \) function \((-\pi, \pi) \times K_T^\phi \ni (y, z) \mapsto F(y, z) \in \mathbb{C} \) satisfying the uniform in \( \psi \in [0, \phi] \) estimate
\[ \int_0^{+\infty} \int_{-\pi}^{\pi} \exp(2\tau \sin \psi)|F(y, T + e^{i\psi}t)|^2 dy \, dt \leq \text{Const} \] (4)
with some \( \tau > 0 \).

3 Radiation condition and complex scaling

For all \( n \in \mathbb{Z} \) we set \( \lambda_n^\pm = \mp\sqrt{k^2 - (n + \alpha)^2} \), where we take the main branch of the square root. Let \( \mathfrak{N} = \{ n \in \mathbb{Z} : |n + \alpha| < |k| \} \). The finite set of points \( \{ \lambda : \lambda = \lambda_n^+ \text{ or } \lambda = \lambda_n^-, n \in \mathfrak{N} \} \) consists of all points \( \lambda_n^\pm \) lying on the real axis. The remaining points \( \lambda_n^\pm, n \in \mathbb{Z} \setminus \mathfrak{N} \) are on the imaginary axis. With every \( \lambda_n^\pm \) we associate the function \( w_n^\pm(y, t) = \exp(i\lambda_n^\pm t + i(n + \alpha)y) \). The functions \( w_n^\pm \) satisfy the quasi-periodicity conditions (2) and the homogeneous equation (1) with \( q \equiv 0 \). If \( n \in \mathfrak{N} \) then \( w_n^+ \) is an incoming wave and \( w_n^- \) is an outgoing wave of the problem (1)–(3) with \( q \equiv 0 \). If \( n \in \mathbb{Z} \setminus \mathfrak{N} \) then \( w_n^+ \) is a growing mode and \( w_n^- \) is an evanescent mode of the unperturbed problem (1)–(3). Let \( \phi \) be the angle for which the assumptions of Section 2 on the potential \( q \) and the right hand side \( F \) are satisfied. We introduce the open cone
\[ \mathcal{K}_\beta^\phi = \{ \lambda \in \mathbb{C} : \lambda = i\beta - e^{-i\psi}\xi, \psi \in (0, \phi), \xi > 0 \} \]
with the vertex \( i\beta \in \mathbb{C} \) and the angle \( \phi, \phi \in (0, \pi/2) \). Denote by \( H_\beta^\ell(\mathcal{E}) \), \( \ell \geq 0 \), the weighted space with the norm \( \|e_\beta; H_\beta^\ell(\mathcal{E})\| \), where \( H_\beta^\ell(\mathcal{E}) \) is the
Sobolev space, and $e^{\beta t} : (y, t) \mapsto \exp \beta t$, $\beta \in \mathbb{R}$. We say that the parameter $\beta$ is admissible if $\beta \in (\max\{\Re \lambda_n^+: n \in \mathbb{Z} \setminus \mathfrak{N}\}, 0) \cap [-\tau \sin \phi, 0)$ and the cone $\mathcal{K}_\beta^\phi$ contains all the points from the set $\{\lambda_n^+: n \in \mathfrak{N}\}$, see Fig. 2.

**Fig. 2.** Arrangement of the cone $\mathcal{K}_\beta^\phi$ for an admissible $\beta$.

**Definition 1** A solution $u$ to the problem (1)–(3) satisfies the radiation condition if for some admissible $\beta$ the solution $u$ is in the space $H_\beta^2(\mathcal{E})$ and the Fourier-Laplace transform $\hat{u}(y, \lambda) = \int_{0}^{\infty} e^{-i\lambda t} u(y, t) \, dt$ is an analytic in the cone $\mathcal{K}_\beta^\phi$ function $\lambda \mapsto \hat{u}(\cdot, \lambda)$ taking values in the Sobolev space $H^2(-\pi, \pi)$.

**Theorem 2** Let the assumptions of Section 2 be fulfilled. (i) If the homogeneous problem (1)–(3) has no nontrivial solution in the space $H_\gamma^2(\mathcal{E})$ for any $\gamma > 0$ then the problem (1)–(3) has a unique solution satisfying the radiation condition. (ii) Let a solution $u$ to the problem (1)–(3) satisfy the radiation condition for an admissible $\beta$. Then $u$ satisfies the radiation condition for every admissible $\beta$.

We briefly discuss our formulation of radiation condition, for the details as well as for the proof of Theorem 2 we refer to [8]. One can note that our radiation condition looks quite similar to the pole condition introduced in [3,10] as an equivalent and universal formulation of the classical radiation conditions for a wide class of problems. Formally, the only difference between the pole condition and our radiation condition is that we require the analyticity of the Fourier-Laplace transform in a cone instead of the half-plain. Nevertheless, the classical radiation conditions are not applicable under our assumptions on the decay of the potential. The introduced radiation condition should be considered as a generalization of the classical one. In the case of sufficiently rapid decay (say, with an exponential rate) of the potential $q$ and of the right hand side $F$ at infinity, our radiation condition is equivalent to the well known one: a solution satisfies the radiation condition if the principal term of its asymptotic at infinity is a linear combination of the outgoing waves.

Let $\mathcal{E}^T = \{(y, t) \in \mathcal{E} : t < T\}$. By applying the complex scaling $t \to T + e^{i\phi}(t - T)$ for $t \geq T$ (complex change of variables) to the original problem (1–3), we obtain the problem

$$
(\Delta + k^2 + q(y, t))v(y, t) = F(y, t), \ (y, t) \in \mathcal{E}^T,
$$

(5)
\[
\left( \partial_y^2 + e^{-2i\phi} \partial_t^2 + q_T^\phi(y,t) + k^2 \right)v(y,t) = F(y,t), \quad (y,t) \in \mathcal{E} \setminus \mathcal{E}^T, \quad (6)
\]
\[
\partial_y^j v|_{\mathcal{T}^+} = e^{2\pi i \alpha} \partial_y^j v|_{\mathcal{T}^-}, \quad j = 0, 1, \quad (7)
\]
\[
v = 0 \text{ on } \mathcal{T}^0 \quad (8)
\]
\[
\partial_t^j v(y,T) = e^{-i\phi_j} \partial_t^j v(y,T^-), \quad j = 0, 1, |y| \leq \tau, \quad (9)
\]
where the equation (6) is a jump condition, \(\partial_t^j v(y, T^-)\) and \(\partial_t^j v(y, T^+)\) denote the limits of \(\partial_t^j v(y, t)\) as \(t\) tends to \(T\) from the left and from the right side correspondingly. The potential \(q_T^\phi\) in (6) is defined by the equality
\[
q_T^\phi(y,t) = q(y,T + e^{i\phi}(t - T)), \quad (y,t) \in \mathcal{E} \setminus \mathcal{E}^T, \quad (10)
\]
the right hand side \(F\) in (6), (8) is given by
\[
\mathcal{F}(y,t) = \begin{cases} F(y,t), & (y,t) \in \mathcal{E}^T, \\ F(y, T + e^{i\phi}(t - T)), & (y,t) \in \mathcal{E} \setminus \mathcal{E}^T. \end{cases} \quad (11)
\]
The problem (5)–(9) is elliptic for \(\phi \in (0, \pi/2)\).

Let \(H_{\gamma}^{1,\alpha}(\mathcal{E})\) denote the closed subspace in \(H_{\gamma}^{1}(\mathcal{E})\) of all functions satisfying the quasi-periodicity condition \(u|_{\mathcal{T}^+} = e^{2\pi i \alpha} u|_{\mathcal{T}^-}\). By \(\hat{H}_{\gamma}^{1,\alpha}(\mathcal{E})\) we denote the space of all functions \(u \in H_{\gamma}^{1,\alpha}(\mathcal{E})\) such that \(u = 0\) on \(\mathcal{T}^0\). Then we introduce the space \(H_{\gamma}^{-1,\alpha}(\mathcal{E})\) as the dual space of \(\hat{H}_{\gamma}^{1,\alpha}(\mathcal{E})\) endowed with the natural norm
\[
\|\mathcal{F} ; H_{\gamma}^{-1,\alpha}(\mathcal{E})\| = \sup\{ |(\mathcal{F}, w)_{\mathcal{E}}| : w \in \hat{H}_{\gamma}^{1,\alpha}(\mathcal{E}), \|w ; \hat{H}_{\gamma}^{1,\alpha}(\mathcal{E})\| = 1\}, \quad (12)
\]
where \((\cdot, \cdot)_{\mathcal{E}}\) is the extension of the inner product in \(L_2(\mathcal{E})\) to the pairs \((\mathcal{F}, w) \in H_{\gamma}^{-1,\alpha}(\mathcal{E}) \times \hat{H}_{\gamma}^{1,\alpha}(\mathcal{E})\). Let us note that if the right hand side \(F\) of the equation (11) satisfies the assumption (ii) from Section 2 for some \(\psi, T = T_0,\) and \(\tau = \tau_0\) then \(F\) satisfies the uniform in \(\psi \in [0, \rho]\) estimate (4) for every \(T > T_0, \quad \tau \leq \tau_0,\) and \(\varphi \leq \phi,\) see [9]. Without loss of generality we can assume that the potential \(q\) and the right hand side \(F\) satisfy the assumptions of Section 2 for some \(\tau > 0, \phi \in (0, \pi/2),\) and for all sufficiently large positive \(T\). Due to (4) and (11) we have \(\mathcal{F} \in H_{\gamma}^{0}(\mathcal{E})\) for all \(\gamma \leq \tau \sin \phi,\) it is clear that \(\|\mathcal{F} ; H_{\gamma}^{-1,\alpha}(\mathcal{E})\| \leq \|\mathcal{F} ; H_{\gamma}^{0}(\mathcal{E})\|\).

Consider the variational statement of the problem (5)–(9): find a function \(v \in \hat{H}_{\gamma}^{1,\alpha}(\mathcal{E})\) satisfying the equation
\[
-e^{-i\phi} \int_{\mathcal{E}^T} \left( \partial_y v \cdot \partial_y \bar{w} + \partial_t v \cdot \partial_t \bar{w} - (q(y,t) + k^2) v \cdot \bar{w} \right) dy \, dt \\
- \int_{\mathcal{E} \setminus \mathcal{E}^T} \left( \partial_y v \cdot \partial_y \bar{w} + e^{-2i\phi} \partial_t v \cdot \partial_t \bar{w} - (q_T^\phi(y,t) + k^2) v \cdot \bar{w} \right) dy \, dt \\
= e^{-i\phi} (\mathcal{F}, w)_{\mathcal{E}} + (\mathcal{F}, w)_{\mathcal{E} \setminus \mathcal{E}^T} \quad \forall w \in \hat{H}_{\gamma}^{1,\alpha}(\mathcal{E}).
\]
The variational form of the problem (5)–(9) generates the linear continuous operator
\[ \hat{H}_1^{1,\alpha}(\mathcal{E}) \ni v \mapsto A_{\gamma}v \in \mathcal{F} \subset H_{\gamma}^{-1,\alpha}(\mathcal{E}). \] (13)

The proof of the following proposition can be found in [8].

**Proposition 3**

(i) Let the potential \( q \) satisfy the assumptions of Section 2, and let \( T \) be a sufficiently large positive number. We define the potential \( q_T \phi \) for the equation (6) by the equality (10). If the homogeneous problem (1)–(3) has no nontrivial solution in the space \( H_2^{1,\alpha}(\mathcal{E}) \) for any \( \gamma > 0 \) then the operator (13) of the problem (5)–(9) yields an isomorphism if and only if \( |\gamma| < \min_{n \in \mathbb{Z}} \{ \Im(e^{i\phi}\lambda_n^-) \} \).

(ii) Assume that the potential \( q \) and the right hand side \( F \) of the problem (1)–(3) satisfy the assumptions of Section 2. Let \( T \) be a sufficiently large positive number. We define the potential \( q_T \phi \) and the right hand side \( F \) of the problem (5)–(9) by the equalities (10), (11). Let \( u \) be a (unique) solution to the problem (1)–(3) satisfying the radiation conditions. Then a (unique) solution \( v \in \hat{H}_0^{1,\alpha}(\mathcal{E}) \) to the problem (5)–(9) is the analytic continuation of \( u \) in the sense that \( v = u \) on \( \mathcal{E}^T \) and \( v(y,t) = u(y,T + e^{i\phi}(t - T)) \) for \( (y,t) \in \mathcal{E} \setminus \mathcal{E}^T \).

4 PML method. Rate of convergence and error estimate

We search for an approximation in a domain \( \mathcal{E}^L \), \( 0 < L < T \), of a solution \( u \) to the problem (1)–(3) subjected to the radiation condition. Since a solution \( u \) is \( \hat{H}_0^{1,\alpha}(\mathcal{E}) \) to the problem (5)–(9) and \( u \) are coincident on \( \mathcal{E}^T \) (see Proposition 3 ii), one can search for an approximation of \( u \) instead of an approximation of \( u \). The advantage is that \( u \) is in the space \( H_1^{2}(\mathcal{E}) \), \( 0 < \gamma < \min_{n \in \mathbb{Z}} \{ \Im(e^{i\phi}\lambda_n^-) \} \), of functions “exponentially decaying” at infinity, while \( u \notin H^1(\mathcal{E}) \). It is clear that \( u \) has these properties because of the perfectly matched equation (6). In other words, the equation (6) describes a PML of infinite length.

We truncate the domain \( \mathcal{E} \) at a finite distance \( R > T \). By \( \mathcal{Y}^R \) we denote the boundary of truncation, \( \mathcal{Y}^R = \partial \mathcal{E}^R \setminus \partial \mathcal{E} \). Let us also set \( \mathcal{Y}^{\pm,R} = \{(y,t) \in \mathcal{Y}^\pm : t < R\} \). With the aim of approximating \( u \) by a solution \( v^R \) to some problem in the bounded domain \( \mathcal{E}^R \), we introduce the problem

\[ (\Delta + k^2 + q)v^R(y,t) = \mathcal{F}(y,t), \quad (y,t) \in \mathcal{E}^T, \] (14)

\[ (\partial_y^2 + e^{-2i\phi}\partial_t^2 + q_T\phi(y,t) + k^2)v^R(y,t) = \mathcal{F}(y,t), \quad (y,t) \in \mathcal{E}^R \setminus \mathcal{E}^T, \] (15)

\[ \partial_y^jv^R|_{\mathcal{Y}^{+,R}} = e^{2\pi i\alpha}\partial_y^jv^R|_{\mathcal{Y}^{-,R}}, \quad j = 0, 1, \] (16)

\[ v^R = 0 \quad \text{on} \ \mathcal{Y}^0, \] (17)
\[ \partial^j_t v^R(y, T-) = e^{-i\phi_j} \partial^j_t v^R(y, T+), \quad j = 0, 1, \quad (18) \]
\[ v^R = G \quad \text{on } \mathcal{Y}^R, \quad (19) \]
where as an artificial boundary condition on \( \mathcal{Y}^R \) we take the Dirichlet boundary condition. The equation (15) describes a PML of the finite length as an artificial boundary condition on \( \mathcal{Y}^R \).

Let \( \tilde{H}^{1,\alpha}(\mathcal{E}^R) \) denote the closed subspace in \( H^1(\mathcal{E}^R) \) of all functions satisfying the quasi-periodicity condition \( v|_{\mathcal{Y}^+, R} = e^{2\pi i\gamma} v|_{\mathcal{Y}^-, R} \) and the boundary condition \( v|_{\mathcal{Y}^0} = 0 \). By \( \hat{H}^{1,\alpha}(\mathcal{E}^R) \) we denote the space of all functions \( v \in H^{1,\alpha}(\mathcal{E}^R) \) such that \( v = 0 \) on \( \mathcal{Y}^0 \cup \mathcal{Y}^R \). Then we introduce the space \( H^{-1,\alpha}(\mathcal{E}^R) \) as the dual space of \( \hat{H}^{1,\alpha}(\mathcal{E}^R) \). Consider the variational statement of the problem (14)–(19): find a function \( v^R \in \tilde{H}^{1,\alpha}(\mathcal{E}^R) \) satisfying the equation

\[ -e^{-i\phi} \int_{\mathcal{E}^T} (\partial_y v^R \cdot \partial_y \tilde{w} + \partial_t v^R \cdot \partial_t \tilde{w} - (q(y, t) + k^2)v^R \cdot \tilde{w}) \, dy \, dt \]
\[ -\int_{\mathcal{E}^R \setminus \mathcal{E}^T} (\partial_y v^R \cdot \partial_y \tilde{w} + e^{-2i\phi} \partial_t v^R \cdot \partial_t \tilde{w} - (q_T(y, t) + k^2)v^R \cdot \tilde{w}) \, dy \, dt \]
\[ = e^{-i\phi} (\mathcal{F}, w)_{\mathcal{E}^T} + (\mathcal{F}, w)_{\mathcal{E}^R \setminus \mathcal{E}^T} \quad \forall w \in \tilde{H}^{1,\alpha}(\mathcal{E}^R) \]

and the boundary condition \( v^R = G \) on \( \mathcal{Y}^R \). The variational statement generates the linear continuous operator

\[ \tilde{H}^{1,\alpha}(\mathcal{E}^R) \ni v^R \mapsto A^R v^R = \{ \mathcal{F}, G \} \in H^{-1,\alpha}(\mathcal{E}^R) \times H^{1/2,\alpha}(\mathcal{Y}^R), \quad (20) \]

where \( H^{1/2,\alpha}(\mathcal{Y}^R) \) is the space of traces on \( \mathcal{Y}^R \) of the functions from \( \tilde{H}^{1,\alpha}(\mathcal{E}^R) \).

**Proposition 4** Let \( T \) be a sufficiently large positive number and \( \phi \in (0, \pi/2) \). Assume that for all \( \gamma > 0 \) there is no nontrivial solution to the original homogeneous problem (11)–(13) in the space \( H^2(\mathcal{E}) \). Then there exists \( R_0 > T \) such that for all \( R > R_0 \) the problem (14)–(19) with right hand side \( \{ \mathcal{F}, G \} \in H^{-1,\alpha}(\mathcal{E}^R) \times H^{1/2,\alpha}(\mathcal{Y}^R) \) admits a unique variational solution \( v^R \in \tilde{H}^{1,\alpha}(\mathcal{E}^R) \).

The estimate

\[ \| v^R; H^{1,\alpha}(\mathcal{E}^R) \| \leq C(\| \mathcal{F}; H^{-1,\alpha}(\mathcal{E}^R) \| + \| G; H^{1/2,\alpha}(\mathcal{Y}^R) \|) \quad (21) \]

is valid, where the constant \( C \) does not depend on \( R > R_0 \).

**Proof.** The proof is carried out by a modification of the compound expansion method [6]. In other words, we find an approximate solution to the problem (14)–(19) compounded of solutions to first and second limit problems. As the first limit problem we take the scaled problem (3)–(9). The second limit problem is the elliptic problem with constant coefficients

\[ (\partial_y^2 + e^{-2i\phi} \partial_t^2 + k^2) U_2(y, t) = F_2(y, t), \quad (y, t) \in \Pi^R, \]
\[ \partial_y^j U_2(y, t) = e^{2\pi i\alpha} \partial_y^j U_2(-\pi, t), \quad j = 0, 1, \quad t < R, \]
\[ U_2 = G_2 \quad \text{on } \mathcal{Y}^R, \quad (22) \]
where \( \Pi^R = \{(y,t) : y \in (-\pi, \pi), t < R \} \).

Let us define the functional spaces for the problem (22). By \( H^1_\gamma(\Pi^R) \) we denote the weighted Sobolev space with the norm \( \|u; H^1_\gamma(\Pi^R)\| = e^{-\gamma R}\|e_\gamma u; H^1(\Pi^R)\| \).

The space \( H^1_\gamma(\Pi^R) \) is the closed subspace in \( H^1(\Pi^R) \) of all elements satisfying \( u(\pi, t) = e^{2\pi i \alpha} u(-\pi, t) \) for \( t < R \). The space \( \hat{H}^1_\gamma(\Pi^R) \) consists of all elements \( u \in H^1_\gamma(\Pi^R) \) having the traces \( u|_{\gamma R} = 0 \). We set \( H^{-1,\alpha}_\gamma(\Pi^R) = (\hat{H}^{-1,\alpha}_\gamma(\Pi^R))^* \), the space \( H^{-1,\alpha}_\gamma(\Pi^R) \) is provided with the natural norm, cf. (12). Consider the variational statement of the problem (22): find a function \( U_2 \in H^{-1,\alpha}_\gamma(\Pi^R) \) which satisfies the equation

\[
\int_{\Pi^R} (-\partial_y U_2 \cdot \partial_y \bar{w} - e^{-2i\phi \partial_t} \partial_t U_2 \cdot \partial_t \bar{w} + k^2 U_2 \cdot \bar{w}) \, dy \, dt = (F_2, w)_{\Pi^R} \quad \forall w \in \hat{H}^{-1,\alpha}_\gamma(\Pi^R)
\]

and the boundary condition \( U_2 = G_2 \) on \( \gamma R \). To the variational form of the problem (22) there corresponds the linear continuous operator

\[
H^{-1,\alpha}_\gamma(\Pi^R) \ni U_2 \mapsto A_{-\gamma} U_2 = \{F_2, G_2\} \in H^{-1,\alpha}_\gamma(\Pi^R) \times H^{1/2,\alpha}(\gamma R). \tag{23}
\]

As is well known \( \ref{7} \), the operator \( A_{-\gamma} \) is Fredholm (i.e. the range of the operator is closed, kernel and cokernel are finite-dimensional) if and only if there are no numbers \( e^{i\phi} \lambda_n^\pm, n \in \mathbb{Z}, \) on the line \( \{\lambda \in \mathbb{C} : \Im \lambda = \gamma\} \). Suppose that \( \gamma \in [0, \min_n \{\Im (e^{i\phi} \lambda_n^-)\}] \). Then \( e^{i\phi} \lambda_n^+ \notin \{\lambda \in \mathbb{C} : \Im \lambda = \gamma\}, n \in \mathbb{Z} \).

The solutions to the homogeneous problem (22) are easily found in an explicit form, one can see that they do not belong to the space \( H^1_\gamma(\Pi^R) \). Analogously, we consider the formally adjoint to (22) homogeneous problem, and check that it has no solution in the space \( H^1_\gamma(\Pi^R) \). Therefore, if \( \gamma \in [0, \min_n \{\Im (e^{i\phi} \lambda_n^-)\}] \) then the operator (23) implements an isomorphism, a variational solution \( U_2 \in H^{-1,\alpha}_\gamma(\Pi^R) \) of the second limit problem (22) with right hand side \( \{F_2, G_2\} \in H^{-1,\alpha}_\gamma(\Pi^R) \times H^{1/2,\alpha}(\gamma R) \) satisfies the estimate

\[
\|U_2; H^{-1,\alpha}_\gamma(\Pi^R)\| \leq C (\|F_2; H^{-1,\alpha}_\gamma(\Pi^R)\| + \|G_2; H^{1/2,\alpha}(\gamma R)\|). \tag{24}
\]

The constant \( C \) in (24) is independent of \( R \) because the problem (22) reduces to the same problem with \( R = 0 \) by the shift \( t \mapsto t + R \), and the norms in (24) are invariant with respect to \( R \), e.g. \( \|U_2; H^1(\Pi^R)\| = \|U_2(\cdot + R); H^1(\Pi^0)\| \).

Now we are in position to construct the approximate solution. Let \( \chi \) be a smooth cut-off function on the real line, \( \chi(t) = 0 \) for \( t > 1 \) and \( \chi(t) = 1 \) for \( t < -1 \). We denote \( \chi_{R/2}(t) = \chi(t - R/2), t \in \mathbb{R} \). For a sufficiently large \( R > 0 \) we set \( F_1 = \chi_{R/2} F \) and \( F_2 = (1 - \chi_{R/2}) F \), where \( (y, t) \in \mathcal{E} R \) and \( F \in H^{-1,\alpha}(\mathcal{E} R) \) is the right hand side of the problem (14)–(19). We extend the functional \( F_1 \) (the functional \( F_2 \)) by zero to all \( t \geq R \) (to all \( t \leq 0 \)). It is clear that \( \mathcal{F} = F_1 + F_2 \), for all \( \gamma \in [0, \min_n \{\Im (e^{i\phi} \lambda_n^-)\}] \) we have

\[
\|F_1; H_{\gamma}^{-1,\alpha}(\mathcal{E} R)\| \leq e^{\gamma(R/2+1)} \|F; H_{\gamma}^{-1,\alpha}(\mathcal{E} R)\|, \tag{25}
\]
From (29) it follows that
\[ \| F_2; H^{-1,\alpha}(\Pi_R^1)\| \leq e^{\gamma(R/2+1)} \| F; H^{-1,\alpha}(\mathcal{E}^R)\|. \]  

(26)

Let \( U_1 \in \tilde{H}^{1,\alpha}(\mathcal{E}) \) be a (unique) solution to the first limit problem \((5)-(9)\) with the right hand side \( F_1 \), and let \( U_2 \in H_0^{1,\alpha}(\Pi_R^1) \) be a (unique) solution to the second limit problem \((22)\) with the right hand side \( F_2 \) and \( G_2 \equiv G \), where \( G \) is the same as in \((18)\), \( G \in H^{1/2,\alpha}(Y^R) \). Due to the first assertion of Proposition 3, the estimate
\[ \| U_1; H_0^{1,\alpha}(\mathcal{E})\| \leq C \| F_1; H^{-1,\alpha}(\mathcal{E})\| \]  

(27)

is valid. We define the approximate variational solution \( Y \in \tilde{H}^{1,\alpha}(\mathcal{E}^R) \) to the problem \((14)-(19)\) by the equality
\[ Y(y,t) = \chi(t-2R/3)U_1(y,t) + (1-\chi(t-R/3))U_2(y,t), \quad (y,t) \in \mathcal{E}^R. \]

By setting \( \gamma = 0 \) in the estimates \((24), (25), (26), \) and \((27)\), we derive
\[ \| Y; H^{1,\alpha}(\mathcal{E}^R)\| \leq \text{Const}(\| F; H^{-1,\alpha}(\mathcal{E}^R)\| + \| G; H^{1/2,\alpha}(Y^R)\|) \]  

(28)

with some constant independent of \( F, G, \) and \( R \).

On the next step we estimate the discrepancy that \( Y \) leaves in the right hand side of the problem \((14)-(19)\), in other words, we estimate the value
\[ \| A^R Y - \{F, G\}; H^{-1,\alpha}(\mathcal{E}^R) \times H^{1/2,\alpha}(Y^R)\|; \]

here \( A^R \) is the operator \((20)\). Recall that by \( A_\gamma \) and \( A_{-\gamma} \) we denote the operators \((13)\) and \((23)\) of the first and second limit problems. It is clear that the mappings
\[ \tilde{H}^{1,\alpha}(\mathcal{E}) \ni U_1 \mapsto A_0\chi_{2/3R}U_1 \in H^{-1,\alpha}(\mathcal{E}^R), \]
\[ H_0^{1,\alpha}(\Pi_R^1) \ni U_2 \mapsto A_0(1-\chi_{R/3})U_2 \in H^{-1,\alpha}(\mathcal{E}^R) \times H^{1/2,\alpha}(Y^R) \]

are continuous. We have
\[ A^R Y = \{ A_0\chi_{2/3R}U_1, 0 \} + \{ q_t^\phi(1-\chi_{R/3})U_2, 0 \} + A_0(1-\chi_{R/3})U_2 \equiv \{ \tilde{F}, \tilde{G} \}, \]

(29)

where we assume that the function \( q_t^\phi(1-\chi_{R/3}) \) is extended to \( Y^R \) by zero. From \((29)\) it follows that
\[ \{ \tilde{F}, \tilde{G} \} - \{ F, G \} = \{ F_1 + [A_0, \chi_{2R/3}]U_1, 0 \} + \{ q_t^\phi(1-\chi_{R/3})U_2, 0 \} \]
\[ + \{ F_2 - [A_0, \chi_{R/3}]U_2 - \{ F, G \} \} \]
\[ = \{ [A_0, \chi_{2R/3}]U_1, 0 \} + \{ q_t^\phi(1-\chi_{R/3})U_2, 0 \} - [A_0, \chi_{R/3}]U_2; \]

(30)

here \( [a, b] = ab - ba \). The term \([A_0, \chi_{2R/3}]U_1\) is equal to zero outside of the set
\{ (y, t) : y \in [-\pi, \pi], t \in [2R/3 - 1, 2R/3 + 1] \}. We get

\[
\| [A_0, \chi_{2R/3}] U_1; H^{-1, \alpha}(E^R) \| \leq C e^{-2\gamma R/3} \| U_1; \hat{H}^{1, \alpha}(\mathcal{E}) \|
\leq C e^{-\gamma R/6} \| \mathcal{F}; H^{-1, \alpha}(E^R) \|, \tag{31}
\]

where the last estimate is a consequence of the estimates \([25]\) and \([27]\), the constant \(C\) does not depend on \(R\). A similar reasoning together with \([24]\) and \([26]\) leads to the estimates

\[
\| ([A_0, \chi_{R/3}] U_2)_1; H^{-1, \alpha}(E^R) \| \leq C e^{-2\gamma R/3} \| U_2; H^{1, \alpha}(\Pi^R) \|
\leq C e^{-\gamma R/6} \left( \| \mathcal{F}_2; H^{-1, \alpha}(\Pi^R) \| + \| G; H^{1/2, \alpha}(\gamma^R) \| \right), \tag{32}
\]

for the first component of the pair \([A_0, \chi_{R/3}] U_2 = \{ ([A_0, \chi_{R/3}] U_2)_1, 0 \}. At last, due to our assumptions on the potential \(q\) (see Section \(2\)), we have

\[
\| \mathcal{q}_T^\phi (1 - \chi_{R/3}) U_2; H^{-1, \alpha}(E^R) \| \leq c(R) \| U_2; H^{1, \alpha}(\Pi^R) \|, \tag{33}
\]

where \(c(R)\) tends to zero as \(R \to +\infty\). From \((33)\) and the estimates \([24]\), \([26]\) with \(\gamma = 0\), we see that

\[
\| \mathcal{q}_T^\phi (1 - \chi_{R/3}) U_2; H^{-1, \alpha}(E^R) \| \leq C(R) \left( \| \mathcal{F}; H^{-1, \alpha}(E^R) \| + \| G; H^{1/2, \alpha}(\gamma^R) \| \right). \tag{34}
\]

Taking into account the equalities \((30)\) and the estimates \((31), (32), \) and \((34)\), we arrive at the estimate

\[
\| \hat{\mathcal{F}} - \mathcal{F}; H^{-1, \alpha}(E^R) \| \leq C(R) \left( \| \mathcal{F}; H^{-1, \alpha}(E^R) \| + \| G; H^{1/2, \alpha}(\gamma^R) \| \right), \tag{35}
\]

where \(C(R)\) does not depend on \(\{ \mathcal{F}, G \}\) and \(C(R) \to 0\) as \(R \to +\infty\).

We first assume that \(G = 0\). In this case we have \(\hat{\mathcal{F}} = \mathcal{F} + \mathcal{Q}(R) \mathcal{F}\) with some operator \(\mathcal{Q}(R)\) in \(H^{-1, \alpha}(E^R)\), whose norm tends to zero as \(R \to +\infty\). Hence for a sufficiently large \(R_0\) and for all \(R > R_0\) we have \(\| \mathcal{Q}(R) \| \leq \| \mathcal{Q}(R_0) \| < 1\), where \(\| \cdot \|_R\) stands for the operator norm in \(H^{-1, \alpha}(E^R)\). There exists the operator \((I + \mathcal{Q}(R))^{-1} : H^{-1, \alpha}(E^R) \to H^{-1, \alpha}(E^R)\), the norm of this operator is bounded by the constant \(1/(1 - \| \mathcal{Q}(R_0) \|)\) uniformly in \(R, R > R_0\). We set \(\mathcal{F}' = (I + \mathcal{Q}(R))^{-1} \mathcal{F}\). In the same way as before we construct the approximate solution \(Y\) to the problem \((14)-\)\((19)\), where \(\mathcal{F}\) is replaced by \(\mathcal{F}'\) and \(G = 0\). Then \(A^R Y = \{ \mathcal{F}, 0 \}\), the estimate \((28)\) holds with \(G = 0\). This proves that in the case \(\mathcal{G} = 0\) the problem \((14)-\)\((19)\) has a solution \(v^R \in \hat{H}^{1, \alpha}(E^R)\) satisfying the estimate \((21)\). In the case \(G \neq 0\) we find an exact solution \(v^R\) to the problem \((14)-\)\((19)\) in the form \(Y_1 - Y_2\). Here \(Y_1\) is the approximation solution of the problem \((14)-\)\((19)\) with the right hand side \(\{ \mathcal{F}, G \}\), and \(Y_2\) is the approximation solution of the problem \((14)-\)\((19)\) with \(\mathcal{G} = 0\) and \(\mathcal{F}\) replaced by \((I + \mathcal{Q}(R))^{-1} (\hat{\mathcal{F}} - \mathcal{F})\). We have \(A^R Y_1 = \{ \mathcal{F}, G \}\) and \(A^R Y_2 = \{ \hat{\mathcal{F}} - \mathcal{F}, 0 \}\). Now we see that in the case \(G \neq 0\) the problem \((14)-\)\((19)\)
also has a solution \( v^R \in \tilde{H}^{1,\alpha}(\mathcal{E}^R) \) satisfying the estimate (21). Indeed, by the proved case the estimate (21) is valid for \( v^R = Y_2, G = 0 \), and \( \mathcal{F} \) replaced by \( \mathcal{F} - \mathcal{F} \). This together with the estimate (35), and the estimate (28) for \( Y = Y_1 \), leads to (21). To prove the uniqueness of the solution \( v^R \) it suffices to apply the same argument to the formally adjoint problem. Proposition 4 is proved.

**Theorem 5** Let \( T \) be a sufficiently large positive number. Assume that the potential \( q \) and the right hand side \( F \) satisfy all the assumptions of Section II, the homogeneous problem (1)–(3) has no nontrivial solution in the space \( H^2_\epsilon(\mathcal{E}) \), \( \epsilon > 0 \). We define the potential \( q^T_\epsilon \) in (15) and the right hand side \( \mathcal{F} \) of the equations (14), (15) by the equalities (10) and (11). Let \( u \) denote a solution to the original problem (1)–(3) with radiation conditions. Then there exists \( R_0 > T \) such that for \( R > R_0 \) a (unique) variational solution \( v^R \) to the problem (14)–(19), where \( G \equiv 0 \), converges to the solution \( u \) in the domain \( \mathcal{E}^T \) in the following sense

\[
\|u - v^R; H^1(\mathcal{E}^T)\| \leq C e^{-\gamma R} \|\mathcal{F}; H^0_\gamma(\mathcal{E})\|, \quad R > R_0,
\]

where the constant \( C \) is independent of \( R \) and \( \mathcal{F} \), and

\[
\gamma \in (0, \tau \sin \phi] \cap (0, \min_n \{\Re(e^{i\phi} \lambda_n^-)\})
\]

with the same \( \tau > 0 \) as in (4).

**PROOF.** Due to Proposition 3 a unique solution \( u \) to the problem (1)–(3) with radiation conditions and a unique variational solution \( v \in \tilde{H}^{1,\alpha}(\mathcal{E}) \) to the scaled problem (5)–(9) are coincident on \( \mathcal{E}^T \). Thus the estimate (36) is valid if and only if it is valid with \( u \) replaced by \( v \). The difference \( v - v^R \in \tilde{H}^{1,\alpha}(\mathcal{E}^R) \) satisfies the problem (14)–(19) with the right hand side \( \mathcal{F} \equiv 0 \) and \( \mathcal{G} = v|_{\mathcal{Y}^R} \). It is clear that

\[
\|v|_{\mathcal{Y}^R}; H^{1/2,\alpha}(\mathcal{Y}^R)\| \leq e^{-\gamma R} \|v; \tilde{H}^{1,\alpha}_\gamma(\mathcal{E})\|.
\]

This together with the first assertion of Proposition 3 and the assumption (4) leads to the estimate

\[
\|\mathcal{G}; H^{1/2,\alpha}(\mathcal{Y}^R)\| \leq C e^{-\gamma R} \|\mathcal{F}; H^0_\gamma(\mathcal{E})\|
\]

with the same restrictions on \( \gamma \) as in (37). Then by Proposition 4 we have

\[
\|v - v^R; H^1(\mathcal{E}^R)\| \leq C \|\mathcal{G}; H^{1/2,\alpha}(\mathcal{Y}^R)\| \leq C e^{-\gamma R} \|\mathcal{F}; H^0_\gamma(\mathcal{E})\|, \quad R > R_0.
\]

Theorem 5 is proved.
Remark 6 It is quite possible that Theorem 5 (as well as Propositions 3 and 4) remains valid without the assumption on the largeness of the parameter T. But we suppose it all the same because our proof of Proposition 3 is essentially based on this assumption; see [8].

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