EICHLER-SHIMURA ISOMORPHISM FOR COMPLEX HYPERBOLIC LATTICES

INKANG KIM AND GENKAI ZHANG

ABSTRACT. We consider the cohomology group $H^1(\Gamma, G, \rho)$ of a discrete subgroup $\Gamma \subset G = SU(n, 1)$ and the symmetric tensor representation $\rho$ on $S^k \mathbb{C}^{n+1}$. We give an elementary proof of the Eichler-Shimura isomorphism that harmonic forms $H^1(\Gamma, G, K, \rho)$ are $(0, 1)$-forms for the automorphic holomorphic bundle induced by the representation $S^k \mathbb{C}^n$ of $K$.

1. INTRODUCTION

Let $B$ be the unit ball in $\mathbb{C}^n$ considered as the Hermitian symmetric space $B = G/K$ of $G = SU(n, 1)$, $n > 1$. Let $\Gamma$ be a cocompact torsion free discrete subgroup of $G$ and $\rho$ a finite dimensional representation of $G$, and $X = \Gamma \backslash B$. The representation $\rho$ of $G$ defines also one for $\Gamma \subset G$. The first cohomology $H^1(\Gamma, G, \rho)$ is of substantial interests and appears naturally in the study of infinitesimal deformation of $\Gamma$ in a bigger group $G' \supset G$; see [4, 3, 1]. It is a classical result of Ragnunathan [7] that the cohomology group $H^1(\Gamma, G, \rho)$ vanishes except when $\rho = \rho_k$ is the symmetric tensor $S^k(\mathbb{C}^{n+1})$ (or $\rho'_k$ on $S^k(\mathbb{C}^{n+1})$). In a recent work [4] it is proved that realizing $H^1(\Gamma, B, \rho)$ as harmonic forms, it consists of $(0, 1)$ forms for the symmetric tensor of the holomorphic tangent bundle of $X = \Gamma \backslash B$. The proof in [4] uses a Hodge vanishing theorem and the Koszul complex. In the present paper we shall give a rather elementary proof of the result. We will prove that any harmonic form with values in $S^m(\mathbb{C}^{n+1})$ is $(0, 1)$-form taking values in $S^m(\mathbb{C}^n)$. Let $\mathcal{L}^{-1}$ be the line bundle on $X$ defined so that $\mathcal{L}^{-(n+1)}$ is the canonical line bundle $\mathcal{K}$. More precisely we shall prove the following, the notations being explained in §2,

Theorem 1.1. Let $\Gamma$ be a torsion free subgroup of $G$ acting properly discontinuously on $B$.

1. Let $\alpha \in A^1(\Gamma, B, \rho_m)$ be a harmonic form. Then $\alpha$ is a $(0, 1)$ form on $\Gamma \backslash B$ with values in the symmetric tensor $S^m TX \otimes \mathcal{L}^{-m}$ of the holomorphic tangent bundle $TX$.

2. Let $\alpha \in A^1(\Gamma, B, \rho'_m)$ be a harmonic form. Then $\alpha$ is a $(1, 0)$-form on $\Gamma \backslash B$ with values in the symmetric tensor $S^m T'X$ of the holomorphic cotangent bundle $T'X$.

Research partially supported by STINT-NRF grant (2011-0031291). Research by G. Zhang is supported partially by the Swedish Science Council (VR). I. Kim gratefully acknowledges the partial support of NRF grant (2010-0024171) and a warm support of Chalmers University of Technology during his stay.
and $\alpha$ is symmetric in all $m + 1$ variables. In particular $\alpha$ is naturally identified with a section of the bundle $S^{m+1}T'X \otimes L^m$.

**Corollary 1.2.** Let $\Gamma$ be as above and assume that $\Gamma \setminus B$ is compact then we have

$$H^1(\Gamma, \rho_m) = H^1(\Gamma \setminus B, S^mTX \otimes L^{-m}), \quad H^1(\Gamma, \rho'_m) = H^0(\Gamma \setminus B, S^{m+1}T'X \otimes L^m),$$

where the cohomology on the right hand side are the Dolbeault cohomology of $\bar{\partial}$-closed $(0, 1)$ forms of the holomorphic vector bundles.

The case $n = 1$, namely a Riemann surface $\Gamma \setminus B$, is slightly different. In that case the group cohomology $H^1(\Gamma, \rho_{2j})$ of the $2j$-th power of the defining representation of $\Gamma \subset SU(1, 1)$ will have both holomorphic and antiholomorphic components, $H^{(1,0)}(\Gamma, \rho_{2j}), H^{(0,1)}(\Gamma, \rho_{2j})$, the holomorphic part $H^{(1,0)}(\Gamma, \rho_{2j})$ corresponds to $H^{(1,0)}(\Gamma, \rho_{2j}) = H^{(1,0)}(\Gamma \setminus B, K^{j+1}) = H^0(\Gamma \setminus B, K^{j+1})$ of the tensor power of the canonical line bundle. This is known as the Eichler-Shimura correspondence; see [9, THÉORÈME 1] where a concrete construction was given. We can also follow our proof and get an elementary proof of this result; see Remark 3.8.

Our proof is a bit tricky but it is still very akin to the variation of Hodge structures; conceptually we are treating explicitly the filtration of holomorphic bundles defined by the central action of $K$. It is stated in [4] that the results can be derived from the work of Deligne and Zucker [12, 13]. We note here that results of this type that $(0, q)$-forms in the group cohomology $H^q(\Gamma, B, \rho)$ are actually $(0, q)$-forms for a corresponding automorphic bundle have been obtained much earlier by Matsushima and Murakami [5, 6], and presumably one can prove the above result by combining the results of [5, 6] and by proving certain vanishing theorem of $(r, p - s)$-forms in $H^p(\Gamma, G, \rho)$. But our method is down-to-earth hence we expect that our method can apply to various situations. We will investigate further applications in a near future.

2. Preliminaries

Let $V = \mathbb{C}^{n+1}$ be equipped with the Hermitian inner product $(Jv, v)$ of signature $(n, 1)$, where $J$ is the diagonal matrix $J = \text{diag}(1, \cdots, 1, -1)$ and $(v, v)$ the Euclidean form in $\mathbb{C}^{n+1}$. We write $V = V_1 \oplus \mathbb{C}e_{n+1}$ with $V_1$ being the Euclidean space $\mathbb{C}^n$ with an orthonormal basis $\{e_k, k = 1, \ldots, n\}$. Let $G = SU(n, 1)$ be the group of linear transformations on $V$ preserving the Hermitian form. The maximal compact subgroup of $G$ is

$$K = \left\{ \begin{bmatrix} A & 0 \\ 0 & e^{i\theta} \end{bmatrix} : A \in U(n), e^{i\theta} \det A = 1 \right\} = U(n),$$

the identification with $U(n)$ being the natural one. The Lie algebra $\mathfrak{g} = \mathfrak{su}(n, 1)$ consists of matrices $X$ such that $X^*J + JX = 0$. The symmetric space $G/K$ can be realized as
the unit ball \( B \) in \( V_1 = \mathbb{C}^n \), \( B = G/K \) with \( x_0 = 0 \) being the base point. Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be the Cartan decomposition of \( \mathfrak{g} \) and the subspace \( \mathfrak{p} = \{ \xi_v; v \in \mathbb{C}^n \} \) with
\[
\xi_v = \begin{pmatrix} 0 & v \\ \bar{v} & 0 \end{pmatrix}.
\]
The tangent space \( T_{x_0}(B) \) at \( x_0 \) will be identified with \( \mathfrak{p} = \mathbb{C}^n \) as real spaces.

The center of the maximal compact subalgebra \( \mathfrak{k} = \mathfrak{u}(n) \) is
\[
H_0 = (n+1)^{-1} \sqrt{-1} \text{diag}(1, \cdots, 1, -n),
\]
which defines the complex structure on \( B \), and we have
\[
\mathfrak{sl}(n+1) = \mathfrak{sl}(n) + \mathbb{C}H_0 + \mathfrak{p}^+ + \mathfrak{p}^-.
\]
Then the holomorphic and anti-holomorphic tangent space \( \mathfrak{p}^\pm \) consists of upper triangular, respectively lower triangular matrices. We denote
\[
(2.1) \quad \xi_v^+ = \frac{1}{2}(\xi_v - i\xi_{iv}) = \xi_v = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{p}^+, \quad \xi_v^- = \frac{1}{2}(\xi_v + i\xi_{iv}) = \begin{pmatrix} 0 & 0 \\ \bar{v} & 0 \end{pmatrix} \in \mathfrak{p}^-,
\]
the \( \mathbb{C} \)- and \( \overline{\mathbb{C}} \)-linear components of \( \xi_v \).

Let \( V_1 = \mathbb{C}^n \) be the defining representation and \( \det(A) \) the determinant representation of \( U(n) \). We take the diagonal elements as Cartan algebra of \( \mathfrak{gl}(n, \mathbb{C}) \) and the upper triangular matrices as positive root vectors. Denote \( \omega_1, \cdots, \omega_{n-1} \) the fundamental representations of \( U(n) \), so that \( \omega_1 = V_1 \) is the defining representation above and \( \omega_{n-1} \) the dual representation.

As complex representation of \( \mathfrak{u}(n) \) we have
\[
\mathfrak{p}^+ = \omega_1 \otimes \det = V_1 \otimes \det, \quad \mathfrak{p}^- = \omega_{n-1} \otimes \det^{-1}.
\]
This entails that, for \( A \in U(n) \),
\[
A(\xi_{v_1}^+ \wedge \cdots \wedge \xi_{v_n}^+) = (\det A)^n A\xi_{v_1}^+ \wedge \cdots \wedge A\xi_{v_n}^+ = (\det A)^{n+1}(\xi_{v_1}^+ \wedge \cdots \wedge \xi_{v_n}^+).
\]
Hence
\[
(2.2) \quad K_X^{-1} = \wedge^n \mathfrak{p}^+ = (\det)^{n+1}
\]
and \( \mathcal{L} = \det \).

We shall just identify \( \mathfrak{p}^+ \) with \( V_1 \), \( \mathfrak{p}^+ = V_1 \), when the center action of \( U(n) \) is irrelevant.

The defining representation \( V \) of \( \mathfrak{l} \) under \( \mathfrak{u}(n) \) is
\[
V = V_1 + \det^{-1}
\]
We shall consider its symmetric representation \( (S^m(V), \rho_m) \) of \( G \) and \( \mathfrak{g} \). Note that we have
\[
(2.3) \quad W = S^m(V) = \bigoplus_{k=0}^m W_k = \bigoplus_{k=0}^m S^k(V_1) \otimes e_{n+1}^{m-k},
\]
and we make the identification of the spaces
\[
W_k = S^k(V_1) \otimes e_{n+1}^{m-k} = S^k(V_1)
\]
whenever the factor \( e_{n+1}^{m-k} \) is irrelevant.

Note that the Euclidean inner product on \( V \) induces one on \( W = S^m(V) \) and the above decomposition is an orthogonal decomposition. Note also that action of \( \rho_m(X) \) is Hermitian for \( X \in p \) and skew Hermitian for \( \xi \in \mathfrak{t} \).

A representation of \( G \) on a finite dimensional complex vector space defines also a vector bundle over the quotient space \( \Gamma \backslash B \) and we recall briefly its construction. Let \( (W, \rho) \) be a finite dimensional representation of \( G \) on a complex vector space \( W \). Eventually we shall only consider \( W = S^k(V) \) as above and its dual \( S^k(V') \). We fix on \( W \) a positive definite Hermitian form so that \( \Gamma \) acts unitarily. Let \( \Gamma \) be a torsion free discrete subgroup of \( G \). The restriction of \( \rho \) on \( \Gamma \) will also be written as \( \rho \). Suppose \( \Gamma \) acts properly discontinuously on \( B \). We recall the known construction of vector bundle \( E_{\rho} \) on \( \Gamma \backslash B \), following the exposition \([8]\) and also some notations there. Let \( \Gamma \times K \) acts on \( G \times W \) by \( (\gamma, k)(g, w) := (\gamma g k^{-1}, \rho(\gamma) w) \). Then \( E_{\rho} = G \times W / \Gamma \times K \) is a holomorphic vector bundle on \( \Gamma \backslash B \). The de Rham operator \( d \) is well-defined on \( E_{\rho} \) and we let \( \Delta_{\rho} = dd^* + d^* d \) be the corresponding Hodge Laplacian operator on space of \( p \)-forms on \( E_{\rho} \). We choose its standard realizations as \( W \)-valued \( p \)-forms on \( G \) as follows. Let \( A^p(\Gamma, B, \rho) \) be the space of \( W \)-valued \( p \)-forms \( \alpha \) on \( G \) satisfying

(a): \( \alpha(\gamma g) = \alpha(g), \gamma \in \Gamma \).

(b): \( \rho(k) \alpha(\gamma g k^{-1}) = \alpha(g), \quad k \in K \).

(c): \( \iota(Y)\alpha = 0, Y \in \mathfrak{t} \).

Here \( \iota(Y) \) is the pairing of \( Y \in \mathfrak{g} \) as left-invariant vector fields on \( G \) (by differentiation from right) with a \( p \)-form \( \alpha \) on \( G \), \( \iota(Y)\alpha(Z_1, \ldots, Z_{p-1}) = \alpha(Y, Z_1, \ldots, Z_{p-1}) \). Equivalently it can be realized as \( p \)-forms on \( \Gamma \backslash G \) satisfying (b) – (c) above. With some abuse of notation we denote \( \Delta_{\rho} \) the corresponding Hodge Laplacian on \( A^p(\Gamma, B, \rho) \).

We shall also need the automorphic bundle defined by representations of \( K \). So let \( (V, \tau) \) be a representation of the complexification of \( K_\mathbb{C} \) and we fix as above a Hermitian inner product on \( V \) so that \( K \) acts unitarily. The group \( \Gamma \times K \) acts on \( G \times V \) by \( (\gamma, k)(g, w) = (\gamma g k^{-1}, \tau(k) w) \). Then \( \mathcal{E}_{\rho_1} = \Gamma \times K \backslash G \times V \) defines a holomorphic vector bundle over \( \Gamma \backslash B \). The \( p \)-forms on the vector bundle can be realized as the space \( A^p(\Gamma, B, \tau) \) (again with some abuse of notation) of \( p \)-forms on \( \Gamma \backslash G \) satisfying

(b'): \( \tau(k) \alpha(\gamma g k^{-1}) = \alpha(g), \quad k \in K \).

(c'): \( \iota(Y)\alpha = 0, Y \in \mathfrak{t} \).

3. The Eichler-Shimura Isomorphism

As indicated in \([3, 4]\) the part (2) of our theorem is a consequence of (1), so we shall only prove (1).

For any real linear map \( A : p \to W \) from \( p \) to any complex vector space \( W \) we let

\[
A^+(\xi_v) = \frac{1}{2}(A(\xi_v) - iA(\xi_{iv})), \quad A^-(\xi_v) = \frac{1}{2}(A(\xi_v) + iA(\xi_{iv}))
\]
be the \( \mathbb{C} \)-linear and respectively \( \overline{\mathbb{C}} \)-linear components. In particular for any complex representation \((W, \rho)\) of \( G \) and \( g \) we have

\[
\rho^\pm(\xi_v) = \rho(\xi_v^\pm),
\]

where \( \xi_v^\pm \) are defined in (2.1). Let now \( \rho = \rho_m \) be the representation \( S^m(V) \) and \( \rho^m \) the dual representation \( S^m(V') \) of \( g \). We start now with a few simple observations formulated only \( \rho = \rho_m \); the corresponding ones hold for \( \rho^m \).

Denote by

\[
P_k : W \to W_k = S^k(V_1) \otimes e_{m-k}^{n+1}
\]

the orthogonal projection onto the component \( W_k \) in (2.3), and write

\[
\alpha = \sum_{k=0}^m \alpha_k
\]

the corresponding decomposition for \( \alpha \in W = \sum_{k=0}^m W_m \).

Let \( \{X_j\} \) be an orthogonal basis of \( p \) viewed as tangent vectors on \( \Gamma \setminus G \) at a fixed point \( \Gamma g \) and \( \{e_j\} \) the corresponding orthonormal basis of \( V_1 \). An arbitrary vector in \( \{X_j\} \) will be written as \( Y_i \). Let \( T = T_\rho \) and \( T^* = T_\rho^* \) be the operator defined on \( A^1(\Gamma, B, \rho) \) as follows.

\[
T_\rho(Y_1, Y_2) = \rho(Y_1)\alpha(Y_2) - \rho(Y_2)\alpha(Y_2)
\]

\[
T^*_\rho \alpha = \sum_{j=1}^n \rho(X_j)\alpha(X_j)
\]

We recall the following result [8, Corollary 7.50]

**Proposition 3.1.** Suppose \( \alpha \in A^1_0(\Gamma, B, \rho) \) is harmonic, \( \Delta_\rho \alpha = 0 \). Then \( T_\rho \alpha = 0 \) and \( T^*_\rho \alpha = 0 \).

This can be restated as the following (which is also proved in [3] for \( k = 2 \) by using matrix computations).

**Corollary 3.2.** Suppose \( \alpha \in A^1_0(\Gamma, B, \rho) \) satisfies \( T_\rho \alpha = 0 \) and \( T^*_\rho \alpha = 0 \). Then the \( W \)-valued \( \mathbb{R} \)-bilinear form \( (X, Y) \mapsto \rho(X)\alpha(Y) \) is symmetric

\[
(3.1) \quad \rho(\xi_v)\alpha(\xi_u) = \rho(\xi_u)\alpha(\xi_v),
\]

and trace free

\[
(3.2) \quad \sum_j (\rho(\xi_{e_j})\alpha(\xi_{e_j}) + \rho(\xi_{ie_j})\alpha(\xi_{ie_j})) = 0.
\]

Our theorem will be an easy consequence of the following proposition, whose proof is based on a few elementary lemmas.

**Proposition 3.3.** (1) Suppose \( \alpha \in \text{Hom}_\mathbb{R}(p, W) \) satisfies \( T_\rho \alpha = T^*_\rho \alpha = 0 \). Then \( \alpha \) is \( \overline{\mathbb{C}} \)-linear and takes value in \( W_m = S^m V_1 \), that is, \( \alpha = \alpha_m = \alpha^-_m \in \text{Hom}_\overline{\mathbb{C}}(p^-, W_m) \).
(2) Suppose $\alpha \in \text{Hom}_{k}(p, W')$ satisfies $T_{\rho'} \alpha = T_{\rho}^{*} \alpha = 0$. Then $\alpha$ is $C$-linear and takes value in $S^{m}(V_{1}')$. Moreover as an element in $(p^{+})' \otimes S^{m}(V_{1}') = (V_{1})' \otimes S^{m}(V_{1}')$, it is symmetric in all variables, i.e., an element in $S^{m+1}(V_{1})$, the leading component in $(V_{1})' \otimes S^{m}(V_{1}')$.

Denote $u^{i}v^{j-i}$ the symmetric tensor power of $u$ and $v$ normalized by
\[
(u + v)^{j} = \otimes^{j}(u + v) = \sum_{i=0}^{j} \binom{j}{i} u^{i}v^{j-i}.
\]

Note that the representation $\rho = \rho_{m}$ is the symmetric tensor $S^{m}(\mathbb{C}^{n+1})$ throughout the paper.

**Lemma 3.4.**

(1) Let $1 \leq k \leq m - 1$. Then for any $0 \neq \xi_{v} \in p$,
\[
\rho(\xi_{v}) : W_{k} \rightarrow W_{k+1} + W_{k-1}, \quad \rho(\xi_{v}^{+}) : W_{k} \rightarrow W_{k+1}, \quad \rho(\xi_{v}^{-}) : W_{k} \rightarrow W_{k-1},
\]
and on each space it is nonzero. Moreover if $w \in W_{k}$ and $\rho(\xi_{v}^{+})w = 0$ or $\rho(\xi_{v}^{-})w = 0$ for all $\xi_{v}^{\pm} \in p^{\pm}$ then $w = 0$.

(2) The restriction $\rho(\xi_{v})|_{W_{n}} : W_{n} \rightarrow W_{n-1}$ on the top component $W_{m}$ of $W$ is $\mathbb{C}$-linear in $\xi_{v}$, $\rho(\xi_{v})|_{W_{n}} = \rho^{\bot}(\xi_{v})|_{W_{n}}$, and $\rho(\xi_{v})|_{W_{0}} = \rho^{\bot}(\xi_{v})|_{W_{0}}$ on the bottom component is $\mathbb{C}$-linear in $\xi_{v}$, $\rho(\xi_{v})|_{W_{0}} = \rho^{\bot}(\xi_{v})|_{W_{0}}$.

**Proof.** The defining representation $\rho_{1}$ is just the matrix multiplication and we have $\rho_{1}(\xi_{v})u = \langle u, v \rangle e_{n+1}$ for $u \in V_{1}$, and $\rho_{1}(\xi_{v})e_{n+1} = v$. Thus $\rho_{1}(\xi_{v}^{+})u = 0$, $\rho_{1}(\xi_{v}^{-})u = \langle u, v \rangle e_{n+1}$, $\rho_{1}(\xi_{v})e_{n+1} = v$, and $\rho_{1}(\xi_{v}^{+})e_{n+1} = v$, $\rho_{1}(\xi_{v}^{-})e_{n+1} = 0$. Taking the tensor power we find
\[
\rho(\xi_{v}^{\pm})e_{n+1}^{k} = ke_{n+1}^{k-1}, \quad \rho(\xi_{v}^{\pm})e_{j}^{k} = k\theta_{j}e_{n+1}^{k-1}, \quad 1 \leq j \leq n,
\]
which are non-zero if $v_{j} \neq 0$. First note that
\[
k\rho^{\bot}(\xi_{v})e_{k-1} = \rho^{\bot}(\xi_{kv}), \quad k \in K, v \in V_{1}.
\]
If $\rho(\xi_{v}^{\pm})w = 0$ for all $\xi_{v}^{\pm} \in p^{\pm}$ and for a fixed $w \neq 0$, then
\[
k\rho(\xi_{v}^{\pm})e_{k-1} = \rho(\xi_{kv}^{\pm})w = 0
\]
for all $k \in K$. Hence it is zero for all $\rho(k^{-1})w$, and therefore zero for $w = e_{j}^{k}, j = 1, \ldots, n$, contradicting the previous claim. \(\square\)

The space $\text{Hom}_{\mathbb{C}}(p^{-}, W_{j})$ of $\mathbb{C}$-linear forms $\beta$ on $p^{-}$, $\beta = \beta^{-}$ will be identified with the tensor product $(p^{-})' \otimes W_{j}$. Recall [11] that the tensor product is decomposed under $K$ as
\[\text{(3.3)}\]
\[
\text{Hom}_{\mathbb{C}}(p^{-}, W_{j}) = (p^{-})' \otimes S^{j}(V_{1}) \otimes e_{n+1}^{m-j} \equiv (S^{j+1}(V_{1}) \otimes e_{n+1}^{m-j}) \oplus (S^{j-1,1}(V_{1}) \otimes e_{n+1}^{m-j})
\]
with the corresponding highest weights
\[
\omega_{1} \otimes j\omega_{1} = (j + 1)\omega_{1} + ((j - 1)\omega_{1} + \omega_{2}).
\]
Lemma 3.5. Suppose $\beta = \beta^+$ is $S^m(V'_1)$-valued $\mathbb{C}$-linear form on $p^+$. If $\rho(\xi^+_v)\beta(\xi^+_u) = \rho(\xi^+_v)\beta(\xi^+_u)$ then $\beta$ as an element in $(p^+) \otimes S^m(V'_1)$ is symmetric in all $m + 1$ variables.

Proof. The statement is equivalent to that $\beta(\xi^+_v)(\xi^+_u, \cdots, \xi^+_u)$ is symmetric in all $m + 1$ variables. However the equality $\rho(\xi^+_v)\beta(\xi^+_u) = \rho(\xi^+_v)\beta(\xi^+_u)$ implies that it is symmetric in the first two variables and thus is symmetric in all $m + 1$ variables. More precisely, viewing $\rho(\xi^+_v)\beta(\xi^+_u)$ and $\rho(\xi^+_v)\beta(\xi^+_u)$ as elements in $S^m(V'_1)$,

$$\rho(\xi^+_v)\beta(\xi^+_u)(e_{n+1}, \cdots, e_{n+1}) = \beta(\xi^+_v)(\rho(\xi^+_v)e_{n+1}, \cdots, \rho(\xi^+_v)e_{n+1}) = \rho(\xi^+_v)\beta(\xi^+_u)(e_{n+1}, \cdots, e_{n+1}) = \beta(\xi^+_v)(\rho(\xi^+_v)e_{n+1}, \cdots, \rho(\xi^+_v)e_{n+1}).$$

Hence from $\rho(\xi^+_v)e_{n+1} = u$ and $\rho(\xi^+_v)e_{n+1} = v$ and identifying $p^+ = V_1$, we get

$$\beta(\xi^+_u, \cdots, \xi^+_u) = \beta(\xi^+_u, \cdots, \xi^+_u).$$

□

Lemma 3.6. If $\rho^-(\xi_u)\beta^-(\xi_v) = \rho^-(\xi_v)\beta^-(\xi_u)$ then $\beta$ is in the first component $S^{j+1}(V_1)$ in the above decomposition (3.3).

Proof. Note that the relation $\rho^-(\xi_u)\beta^-(\xi_v) = \rho^-(\xi_v)\beta^-(\xi_u)$ is invariant under the $K$-action, since

$$\rho(k)\rho^-(\xi_v)\rho(k^{-1}) = \rho^-(\xi_{kv}), \quad k \in K, v \in V_1$$

and

$$\rho(k)\beta(gk^{-1}) = \beta(g)$$

for all $k \in K$, which results in

$$\rho(k)\rho^-(\xi_v)\beta(gk^{-1}) = \rho^-(\xi_{kv})\beta(g).$$

Thus if $\beta^-$ satisfies the relation so is its component in $((j - 1)\omega_1 + \omega_2)$. We prove any element in $((j - 1)\omega_1 + \omega_2)$ satisfying the relation must be zero. This space is an irreducible representation of $K$ we need only to check the relation for its highest weight vector. The highest weight vector of $((j - 1)\omega_1 + \omega_2)$ in $V_1 \otimes S^j(V_1)$ is

$$\beta = \epsilon_2 \otimes \epsilon_1^j - \epsilon_1 \otimes (\epsilon_1^{j-1} \epsilon_2)$$

where $\epsilon_i$ is a dual vector to $\xi^-_{i}$ in $p^-$. We check the relation

$$\rho^-(\epsilon_{e_2})\beta(\epsilon_1) = \rho^-(\epsilon_{e_1})\beta(\epsilon_{e_2}).$$

The left hand side is $-\epsilon_1^{j-1} e_{n+1}$ whereas the right hand side is $j \epsilon_1^{j-1} e_{n+1}$, and the relation is not satisfied. □

For simplicity we denote $\xi_j = \epsilon_{e_j}$ where $\{e_k\}$ is an orthogonal basis of $V_1$. Observe that for any $\beta \in \text{Hom}_{\bar{p}}(p^-, W_j)$ we have

$$\rho(\xi^+_v)\beta \in \text{Hom}_{\bar{p}}(p^-, W_{j+1}).$$
Lemma 3.7. Suppose $1 \leq j < m$. The map

$$T : \text{Hom}(\mathfrak{p}^-, W_j) \equiv (j + 1)\omega_1 \oplus ((j - 1)\omega_1 + \omega_2) \to W_{j+1}, \quad \beta \mapsto \sum_{k=1}^{n} \rho(\xi_k^+)\beta(\xi_k^-)$$

is up to non-zero constant an isometry on the space $(j + 1)\omega_1$.

Proof. It is clear that $T$ is a $K$-intertwining map from $\text{Hom}(\mathfrak{p}^-, S^j(V_1))$ into $W_{j+1}$. By Schur’s lemma it’s either zero or an isometry up to non-zero constant on the irreducible space $(j + 1)\omega_1$. To find the constant we take $\beta = \varepsilon_1 \otimes e_1^m - j$ where $\varepsilon_1$ is the dual form of $\xi_{e_j}^-$. It is indeed in the first component $(j + 1)\omega_1$ and is actually the highest weight vector. Then by direct computation we find

$$T\beta = (m - j)\varepsilon_1^j e_1^{m-j-1},$$

which is nonzero. 

We prove now Proposition 3.3.

Proof. We shall prove by induction that all $\alpha_j = 0$ for $k \leq m - 1$. Let $1 \leq k \leq m - 1$. Taking the $k$-th component of (3.1) we get

$$\rho^+(\xi_u)\alpha_{k-1}^+(\xi_u) = \rho^+(\xi_v)\alpha_{k-1}^+(\xi_u),$$

(3.4)

$$\rho^-(\xi_u)\alpha_{k+1}^-(\xi_u) = \rho^-(\xi_v)\alpha_{k+1}^-(\xi_u),$$

(3.5)

$$\rho^+(\xi_u)\alpha_{k-1}^-(\xi_u) = \rho^+(\xi_v)\alpha_{k-1}^-(\xi_u).$$

(3.6)

We prove first that $\alpha_0 = 0$. Consider the 1-component of the identity

$$T^*_\rho \alpha = \sum_j \left( \rho(\xi_{e_j})\alpha(\xi_{e_j}) + \rho(\xi_{e_{e_j}})\alpha(\xi_{e_{e_j}}) \right) = 0$$

(3.7)

and write each term in terms of their $\mathbb{C}$-linear and $\overline{\mathbb{C}}$-linear parts. Note that bilinear $\mathbb{C}$-linear and bilinear $\overline{\mathbb{C}}$-linear terms have their sum zero. Also on the component $W_0$ the action $\rho(\xi_u) = \rho(\xi_u^+) = \mathbb{C}$-linear, by Lemma 3.4. Thus

$$\sum_j \left( \rho^+(\xi_{e_j})\alpha_{0}^-(\xi_{e_j}) + \rho^-(\xi_{e_j})\alpha_{2}^+(\xi_{e_j}) \right) = 0.$$ 

But by the equality of (3.6) for $k = 1$ we have $\rho(\xi_{e_j})\alpha_{2}^+(\xi_{e_j}) = \rho(\xi_{e_j})\alpha_{0}^-(\xi_{e_j})$. Namely

$$2 \sum_j \rho(\xi_{e_j})\alpha_{0}^-(\xi_{e_j}) = 0.$$ 

(3.8)

Taking inner product with $e_1 e_{n+1}^{m-1} \in W_1$, and using the fact that

$$\langle \rho(\xi_{e_j})\alpha_{0}^-(\xi_{e_j}), e_1 e_{n+1}^{m-1} \rangle = \langle \alpha_{0}^-(\xi_{e_j}), \rho(\xi_{e_j})(e_1 e_{n+1}^{m-1}) \rangle = \langle \alpha_{0}^-(\xi_{e_j}), e_{n+1}^m \rangle$$

and

$$\langle \rho(\xi_{e_j})\alpha_{0}^-(\xi_{e_j}), e_1 e_{n+1}^{m-1} \rangle = \langle \alpha_{0}^-(\xi_{e_j}), \rho(\xi_{e_j})(e_1 e_{n+1}^{m-1}) \rangle = 0, j \neq 1,$$

we see that $\langle \alpha_{0}^-(\xi_{e_1}), e_{n+1}^m \rangle = 0$, namely $\alpha_{0}^-(\xi_{e_1}) = 0$. By the $K$-invariance of above relation (3.8) we may replace $e_1$ by any $e_j$, and get $\alpha_{0}^-(\xi_{e_j}) = 0$, i.e., $\alpha_{0}^- = 0$ and $\alpha_0$
is $\mathbb{C}$-linear, $\alpha_0 = \alpha_0^*$. Now $W_0 = \mathbb{C}e_{m+1}^n$ is one-dimensional and $\alpha_0$ is thus of the form $\alpha_0(\xi_u) = \langle u, u_0 \rangle e_{m+1}^n$ for some $u_0 \in V_1$. Now the relation (3.4) implies that 

$$\langle u, u_0 \rangle ve_{m-1}^{n+1} = \langle v, u_0 \rangle ue_{m-1}^{n+1}$$

for all $u, v \in V_1$. This is impossible unless $u_0 = 0$ since $\dim V_1 > 1$, i.e., $\alpha_0 = 0$.

Taking the 0-th component of the equality $\rho(\xi_u)\alpha(\xi_v) = \rho(\xi_v)\alpha(\xi_u)$ we get

$$\rho^-(\xi_u)\alpha_1(\xi_v) = \rho^-(\xi_v)\alpha_1(\xi_u).$$

Changing $v$ to $iv$ we find

$$\rho^-(\xi_u)\alpha_1(\xi_{iv}) = -i\rho^-(\xi_v)\alpha_1(\xi_u).$$

Summing the two results in

$$\rho^-(\xi_u)(\alpha_1(\xi_{iv}) + i\alpha_1(\xi_v)) = 0.$$

Taking inner product with $e_{m+1}^n \in W_0$ we have

$$0 = \langle (\rho^-(\xi_u)(\alpha_1(\xi_{iv}) + i\alpha_1(\xi_v)), e_{m+1}^n \rangle = \langle \alpha_1(\xi_{iv}) + i\alpha_1(\xi_v), \rho^+(\xi_u)e_{m+1}^n \rangle$$

for all $u$. Thus $\alpha_1(\xi_{iv}) + i\alpha_1(\xi_v) = 0$, namely $\alpha_1$ is $\mathbb{C}$-linear, $\alpha_1 = \alpha_1^*$. Furthermore it follows from Lemma 3.6 that $\alpha_1$ is an element in the component $S^2(V_1)$ in $(p^-) \otimes S^1(V_1)$.

We take now the 0-component of the identity (3.7) using again the fact that $\alpha_1$ is $\mathbb{C}$-linear, and find

$$0 = \sum_j (\rho^+(\xi_{e_j})\alpha_1(\xi_{e_j}) + \rho^+(\xi_{ie_j})\alpha_1(\xi_{ie_j})) = 2 \sum_j \left(\rho^+(\xi_{e_j})\alpha_1(\xi_{e_j})\right).$$

But $\alpha_1$ is in the component $2\omega_1 = S^2(V_1)$ and Lemma 3.7 implies that $\alpha_1 = 0$.

Using the above procedure successively we prove then that $\alpha_j = 0$ for $j \leq m - 2$. Consequently we have $\alpha_{m-1}^+ = 0$ and $\alpha_{m-1} = \alpha_{m-1}^-$. Taking the trace of $(m-2)$-th component of (3.2) we have again $\sum_j \rho^+(\xi_{e_j})\alpha_{m-1}^- (\xi_{e_j}) = 0$ and $\alpha_{m-1} = 0$ by the same arguments.

Finally we consider the $(m-1)$-th component of the equality $\rho(\xi_u)\alpha(\xi_v) = \rho(\xi_v)\alpha(\xi_u)$ we get

$$\rho^-(\xi_u)\alpha_{m}(\xi_v) = \rho^-(\xi_v)\alpha_{m}(\xi_u)$$

Replacing $u$ by $iu$ gives

$$-i\rho^-(\xi_u)\alpha_{m}(\xi_v) = \rho^-(\xi_v)\alpha_{m}(\xi_{iu}).$$

Thus

$$\rho^-(\xi_u)\alpha_{m}(\xi_u) = \frac{1}{2} \rho^-(\xi_v) \left(\alpha_{m}(\xi_u) - i\alpha_{m}(\xi_{iu})\right) = 0.$$

This holds for all $\xi_v \in p$. Thus $\alpha_{m}(\xi_u) = 0$ by Lemma 3.4 and $\alpha_m$ is $\mathbb{C}$-linear. Finally $\alpha_m \in S^{m+1}V_1$ is a consequence of Lemma 3.5.

We prove now Theorem 1.1 and Corollary 1.2.
Proof. The statements in Theorem 1.1 follows from Proposition 3.3. Indeed if \( \alpha \in A^1(\Gamma, B, \rho_m) \) then by the conditions in \( \S 2 \) it can be represented locally as a differential harmonic form. But then it will have values in \( S^m\mathbb{C}^n \) by Proposition [3.3] By the relation \( \mathbb{C}^n = p^+ \otimes \det^{-1} \) we have

\[
S^m(\mathbb{C}^n) = (p^+)^m \otimes (\det)^{-m} = S^mTX \otimes L^{-m},
\]

proving that \( \alpha \) is a \((0, 1)\)-section of \( S^mTX \otimes L^{-m} \). The proof of the second one is similar. The claim that \( \alpha \) is \( \mathbb{C} \)-linear is precisely that \( \alpha \) is a \((0, 1)\)-form. This proves the first part, and the second part follows similarly from Proposition 3.3 (2).

Let \( \alpha \) be a harmonic form representing an element \( H^1(\Gamma, \rho) \). Write \( \alpha = \sum_{k=0}^m \alpha_k \) according to the decomposition \((2.3)\). It follows then from above that \( \alpha_k = 0 \) for \( k < m \), i.e. \( \alpha = \alpha_m \). The isomorphism of \( H^1(\Gamma, \rho) \) and \( H^1(\Gamma \backslash B, S^mTX \otimes L^{-m}) \) is then a consequence of [5 Proposition 4.2 and Theorem 6.1]. The second isomorphism is proved similarly.

\[\square\]

Remark 3.8. In the case of \( n = 1 \) with Riemann surface \( \Sigma = \Gamma \backslash B \), the group cohomology \( H^1(\Gamma, \rho_m) \) will not descend to \((0, 1)\)-form on \( \Gamma \backslash B \). The line bundle \( L = K^{-\frac{j}{2}} \) is the square root (constructed using the action of \( K, B = G/K \)) of the tangent bundle. Consider for simplicity \( m = 2j \). It has a decomposition as \( H^1(\Gamma, \rho_m) = H^{(1,0)}(\Gamma, \rho_m) + H^{(0,1)}(\Gamma, \rho_m) \), and two components are dual to each other with \( H^{(0,1)}(\Sigma, K^{-j}) \).

This can also be derived from our computations above. Indeed in the proof of Proposition 3.3 we take \( \alpha \) an element in \( H^{(0,1)}(\Gamma, \rho_m) \), i.e., \( \mathbb{C} \)-linear form, and we can then derive from the same arguments that all components except \( \alpha_{2j} \) are zero, which is equivalent to that \( \alpha = \alpha_{2j} \) is a \((0, 1)\)-section of \( K^{-j} \). That is \( H^{(0,1)}(\Gamma, \rho_m) = H^{(0,1)}(\Gamma \backslash B, K^{-j}) \) with is dual to \( H^0(\Gamma \backslash B, K^{j+1}) \) by Serre duality.

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School of Mathematics, KIAS, Heogiro 85, Dongdaemun-gu Seoul, 130-722, Republic of Korea

E-mail address: inkang@kias.re.kr

Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden

E-mail address: genkai@chalmers.se