Integer sequences and \( k \)-commuting permutations

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Abstract

Let \( \beta \) be any permutation of \( n \) symbols and let \( c(k, \beta) \) be the number of permutations that \( k \)-commute with \( \beta \). The cycle type of a permutation \( \beta \) is a vector \((c_1, \ldots, c_n)\) that means that \( \beta \) has exactly \( c_i \) cycles of length \( i \) in its disjoint cycle factorization. It is known that \( c(k, \beta) \) only depends of the cycle type of \( \beta \). In this article we obtain formulas for \( c(k, \beta) \), for some cycle types. Also we express these formulas in terms of integer sequences in “The On-line Encyclopedia of Integer Sequences” (OEIS). As an application we obtain some relations between sequences in OEIS or new interpretations for some of these sequences.

1 Introduction

Let \( S_n \) denote the group of permutations of \( n \) symbols. Let \( k \) be a nonnegative integer, we say that two permutations \( \alpha, \beta \in S_n \) \( k \)-commute (resp. \((\leq k)\)-commute) if \( H(\alpha\beta, \beta\alpha) = k \) (resp. \( H(\alpha\beta, \beta\alpha) \leq k \)), where \( H \) denotes the Hamming metric between permutations (see, e.g., the work of M. Deza and T. Huang [5] for a survey about metrics on permutations). For a given permutation \( \beta \) and a nonnegative integer \( k \), let \( c(k, \beta) \) (resp. \( c(\leq k, \beta) \)) denote the number of permutations that \( k \)-commute (resp. \((\leq k)\)-commute) with \( \beta \). It is known [10] that \( c(k, \beta) \) only depends of the cycle type of \( \beta \). Rutilo Moreno and the author of this article [10] began the study of \( k \)-commuting permutations and presented the first partial results about the problem of computing \( c(k, \beta) \). The original motivation for studying this type of questions is to develop tools to solve the open problem of determining if equation \( xy = yx \) is stable or not in permutations (see [7] for definitions), but we think that the enumerative problem of determining \( c(k, \beta) \) is also interesting by itself and in the context of integer sequences as discussed below. The problem of obtain explicit formulas for \( c(k, \beta) \), for any \( k \) and any \( \beta \), seems to be a difficult task in its generality, however we think that for some cases of \( k \) and \( \beta \) the problem is manageable. For example in [10] was proved a characterization of permutations that \( k \)-commute with a given permutation \( \beta \) and as a consequence we obtained formulas for \( c(k, \beta) \) when \( k = 3, 4 \). For the case when \( k > 4 \) the problem was solved only for the cases when \( \beta \) is any transposition, any fixed-point free involution, or any \( n \)-cycle. In this article we continue this line of research and we obtain formulas when \( \beta \) is any 3-cycle, any 4-cycle, any \((n-1)\)-cycle, and for another special cases. We also present explicit formulas

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for \(c(\leq k, \beta)\), when \(\beta\) is any transposition, any 3-cycle, and any 4-cycle, and an upper bound when \(\beta\) is any \(n\)-cycle.

Surprisingly, in [10] and in this work, we found that for some values of \(k\) and some permutations \(\beta\), \(c(k, \beta)\) is related with some integers sequences in the The On-Line Encyclopedia o Integer Sequences (OEIS) [13] that for some cases provides another interpretation for these sequences. For example:

- Sequence A208528 [4, 11] correspond to the “number of permutations of \(n > 1\) having exactly 3 points \(P\) on the boundary of their bounding square”. This sequence also counts the number of permutations of \(n\) symbols that 3-commute with a transposition. In Section 4.1 we provide an explicit relation between these two structures.

- The formula for sequence A001044 is \(a(n) = (n!)^2\) and has different interpretations [17]. In this article we show that \(a(n)\) also counts the number of permutations of \(2n\) symbols that \(2n\)-commute with any permutation of \(2n\) symbols with exactly \(n\) fixed points.

- Let \(a(n)\) denote the sequence A004320. For \(n \geq 3\), \(a(n - 2)\) is the number of permutations of \(n\) symbols that 3-commute with an \(n\)-cycle.

- Sequence A001105 is given by \(a(n) = 2n^2\) and has several interpretations [18]. We note that \(a(n)\) is also the number of permutations of \(2n\) symbols that 0-commute, i.e., that commute, with any permutation whose disjoint factorization consist of a product of two \(n\)-cycles.

- Let \(a(n)\) denote sequence A027764. For \(n \geq 3\), \(a(n)\) is also the number of permutations of \(n + 1\) symbols that 4-commute with an \((n + 1)\)-cycle.

- Sequence A000165 is the double factorial of even numbers, that is defined as \(a(n) = (2n)!! = 2^n n!\), and has different interpretations [14]. We note that \(a(n)\) also counts the number of permutations of \(2n\) symbols that 0-commute with a fixed-point free involution (a permutation whose disjoint factorization consists of \(n\) transpositions).

For another cases we find expressions like the following:

- Let \(a(n)\) denote the sequence A027765. We can show that “\(8a(n)\) is the number of permutations of \((n + 1)\) symbols that 5-commute with an \((n + 1)\)-cycle”.

- Let \(a(n)\) and \(b(n)\) denote sequences A016777 and A052560, respectively. For \(n \geq 3\), we can show that \(a(n - 3) \times b(n - 3)\) is the number of permutations of \(n\) symbols that 3-commute with any 3-cycle.

- Let \(a(n)\) and \(b(n)\) denote the sequences A134582 and A052578, respectively. For \(n \geq 5\) we can show that \(a(n - 3) \times b(n - 4)\) is the number of permutations of \(n\) symbols that 5-commute with any 4-cycle.
The work of express our formulas in terms of sequences in OEIS was motivated after the author read section Future Projects in the Wiki page of OEIS [13]. One of such project is about to search sequences in books, journals and preprints and one comment in this Wiki page saids the following: “What needs to be done: Scan these journals, books and preprints looking for new sequences or additional references for existing sequences”. Then, after we have found some formulas for $c(k, \beta)$, for some $k$ and some $\beta$, we decide to work through these results to find all the sequences that occur in these formulas in an implicit or explicit way. One way to do this was by first factoring these formula and then searching the OEIS database to see if these factors are known sequences. We show some tables with our results and with the help of this tables and by using the meaning of $k$-commuting permutations we have found some known and new relations between sequences in OEIS. The rest of the paper is organized as follows: In Section 2 we present some basic definitions and notation used through the article. In Section 3 we present formulas for $c(k, \beta)$ when $\beta$ is any 3-cycle, any 4-cycle, any $(n - 1)$-cycle, and for another special cases. We also present an upper bound when $\beta$ is any $n$-cycle. In Section 4 we present some relations of the number $c(k, \beta)$ with integer sequences in OEIS, when $\beta$ is any transposition, any 3, 4, $n$ and $(n - 1)$-cycle and for another cycle types of $\beta$. In Section 5 we present our final comments and the case when $k = 0$, for some cycle types of $\beta$.

2 Basic definitions

In this section we present some of the definitions used through the article. Let $[n]$ denote the set $\{1, \ldots, n\}$ whose elements are called points. A permutation of $[n]$ is a bijection from $[n]$ onto $[n]$. We use $S_n$ to denote the group of all permutations of set $[n]$. We write $\pi = p_1 p_2 \ldots p_n$ for the one-line notation of $\pi \in S_n$, i.e., $\pi(i) = p_i$, for every $i \in [n]$, and $\tau = (a_1 a_2 \ldots a_m)$ for an $m$-cycle in $S_n$, i.e., $\tau(a_i) = a_{i+1}$, for $1 \leq i \leq m - 1$, $\tau(a_m) = a_1$ and $\tau(x) = x$ for every $x \in [n] \setminus \{a_1, \ldots, a_m\}$. The support of a permutation $\pi \in S_n$ is defined as $\text{supp}(\beta) := \{x \in [n] : \pi(x) \neq x\}$. We denote with $\text{fix}(\beta)$ the set of fixed points by $\beta$, i.e., $\text{fix}(\beta) = [n] \setminus \text{supp}(\beta)$. The product $\alpha \beta$ of permutations is computed by first applying $\beta$ and then $\alpha$. We will say that $\pi$ has cycle $\pi'$ or that $\pi'$ is a cycle of $\pi$, if $\pi'$ is a factor in the disjoint cycle factorization of $\pi$. The cycle type of a permutation $\beta$ is a vector $(c_1, \ldots, c_n)$ that indicates that $\beta$ has exactly $c_i$ cycles of length $i$ in its disjoint cycle factorization. The Hamming metric between permutations $\alpha$ and $\beta$ of $[n]$ is defined as $H(\alpha, \beta) = |\{a \in [n] : \alpha(a) \neq \beta(a)\}|$. We say that $\alpha$ and $\beta$ $k$-commute if $H(\alpha \beta, \beta \alpha) = k$. We say that $a \in [n]$ is a good commuting point (resp. bad commuting point) of $\alpha$ and $\beta$ if $\alpha \beta(a) = \beta \alpha(a)$ (resp. $\alpha \beta(a) \neq \beta \alpha(a)$). Usually, we abbreviate good commuting points (resp. bad commuting points) with g.c.p. (resp. b.c.p.), and by abuse of notation we sometimes omit the reference to $\alpha$ and $\beta$. Let $C(k, \beta) = \{\alpha \in S_n \mid H(\alpha \beta, \beta \alpha) = k\}$ and $c(k, \beta) = |C(k, \beta)|$. In this article, we use the convention $m \mod m = m$, for any positive integer $m$. 

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2.1 Blocks in cycles

The following section is based on Section 2.1 in [10]. Let $\pi \in S_n$, a block $A$ in a cycle $\pi' = (a_1 a_2 \ldots a_m)$ of $\pi$ is a consecutive nonempty substring $A = a_i a_{i+1} \ldots a_{i+l}$ of $a_i a_{i+1} \ldots a_{i-1}$ where $(a_i a_{i+1} \ldots a_{i-1})$ is one of the $m$ equivalent expressions of cycle $\pi'$ (the sums on the subindex are taken modulo $m$). This definition was motivated by the notion of block when permutation is written in one-line-notation as in [1, 3]. The length of a block $A$ is the number of elements in the string $A$, and is denoted by $|A|$. Two blocks $A$ and $B$ are said to be disjoint if they do not have points in common. The product $AB$ of two disjoint blocks, $A$ and $B$, not necessarily from the same cycle of $\pi$, is defined by the usual concatenation of strings. Notice that with this definition the product $AB$ is not necessarily a block in a cycle of $\pi$. If $(a_1 \ldots a_m)$ is a cycle of $\pi$ we write $(A_1 \ldots A_k)$ to mean that $A_1 \ldots A_k = a_i a_{i+1} \ldots a_{i-1}$ and $(a_1 \ldots a_m) = (a_i a_{i+1} \ldots a_{i-1})$. A block partition of a cycle $\pi'$ is a set $\{A_1, \ldots, A_l\}$ of pairwise disjoint blocks in $\pi'$ such that there exist a block product $A_{i_1} \ldots A_{i_l}$ with $\pi' = (A_{i_1} \ldots A_{i_l})$. If $A = P_1 P_2 \ldots P_k$ is a block product of $k$ pairwise disjoint blocks, not necessarily from the same cycles of $\pi$, and $\tau$ is a permutation in $S_k$, the block permutation $\phi_\tau$ of $\{B_1, \ldots, B_k\}$ induced by $\tau$ is defined as the block product $\phi_\tau(A) = B_{\tau(1)} B_{\tau(2)} \ldots B_{\tau(k)}$.

Example 2.1. Let $\pi = (1 \ 2 \ 3 \ 4 \ 5)(6 \ 7 \ 8 \ 9) \in S_9$. Some blocks in cycles of $\pi$ are $P_1 = 1 \ 2 \ 3$, $P_2 = 4$, $P_3 = 6$, $P_4 = 7 \ 8 \ 9$. One block in $(1 \ 2 \ 3 \ 4 \ 5)$ is $P_5 = 3 \ 4 \ 5 \ 1 \ 2$. The blocks $P_3, P_4$ form a block partition of $(6 \ 7 \ 8 \ 9)$. The product $P_1 P_2$ is a block in $(1 \ 2 \ 3 \ 4 \ 5)$. The product $P_1 P_3 = 1 \ 2 \ 3 \ 6$ is not a block in any cycle of $\pi$.

Example 2.2. Let $\pi' = (3 \ 4 \ 1 \ 2 \ 6)$ be a cycle of $\pi \in S_6$ and $P = P_1 P_2 P_3$ a block product of the block partition $\{P_1, P_2, P_3\}$ of $\pi'$ given by $P_1 = 3 \ 4$, $P_2 = 1$ and $P_3 = 2 \ 6$. Let $\alpha = (2 \ 3 \ 1) \in S_3$. The block permutation $\phi_\alpha(P)$ is $P_2 P_3 P_1 = 1 \ 2 \ 6 \ 3 \ 4$, where, for example, $P_3 P_1 = 2 \ 6 \ 3 \ 4$ is a block in $\pi'$. If $\tau = (1 \ 3)$ then $\phi_\tau(P) = P_3 P_2 P_1 = 2 \ 6 \ 1 \ 3 \ 4$, where $P_3 P_2 = 2 \ 6 \ 1$ is not a block in $\pi'$.

In $\alpha \in S_n$ and $X \subseteq [n]$, we denote by $\alpha|_X$ the restriction function of $\alpha$ to set $X$, i.e., $\alpha|_X : X \to X$ is defined as $\alpha|_X(a) = \alpha(a)$ for every $a \in X$. Let $\alpha, \beta \in S_n$. Let $\beta' = (b_1 \ldots b_m)$ be a cycle of $\beta$. As $\alpha \beta \alpha^{-1} = \alpha(b_1) \ldots \alpha(b_m)$ (see, e.g., [6, Prop. 10, p. 125]) we use the following notation for $\alpha|_{\text{supp}(\beta')}$

$$\alpha|_{\text{supp}(\beta')} = \begin{pmatrix} b_1 & b_2 & \ldots & b_m \\ \alpha(b_1) & \alpha(b_2) & \ldots & \alpha(b_m) \end{pmatrix}.$$  

(1)

If $\alpha|_{\text{supp}(\beta')}$ is written as in [1], we will write

$$\alpha|_{\text{supp}(\beta'), k} = \begin{pmatrix} B_1 B_2 \ldots B_k \\ J_1 J_2 \ldots J_k \end{pmatrix},$$

(2)

to mean that $B_1 B_2 \ldots B_k = b_1 \ldots b_m$ and $J_1, \ldots, J_k$ are blocks in cycles of $\beta$, where $J_i = \alpha(b_1) \alpha(b_2) \ldots \alpha(b_m)$, $|B_i| = |J_i|$, $1 \leq i \leq k$. This notation is called a block notation (with respect to $\beta$) of $\alpha|_{\text{supp}(\beta')}$. Sometimes we will omit the subindex $k$ in $\alpha|_{\text{supp}(\beta'), k}$. Notice that this notation depends of the particular selection of one of the $m$ equivalent cyclic expressions $(b_1 \ldots b_m) = (b_2 \ldots b_1) = \ldots = (b_m \ldots b_{m-1})$ of $\beta'$. 4
Example 2.3. Let $\alpha, \beta \in S_6$ with $\alpha = (1 \ 2 \ 3)(4 \ 5 \ 6)$ and $\beta = (1 \ 2 \ 3 \ 4)(5 \ 6)$. Let $\beta' = (1 \ 2 \ 3 \ 4)$, then $\alpha(1 \ 2 \ 3 \ 4)\alpha^{-1} = (2 \ 3 \ 1 \ 5)$ and $\alpha|_{\text{supp} (\beta')}$ can be expressed as

$$
\alpha|_{\text{supp} (\beta')} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \end{pmatrix},
$$

Two ways to express $\alpha|_{\text{supp} (\beta')}$ in block notation are

$$
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \end{pmatrix}.
$$

where the vertical lines denote the limits of the blocks. If we use $\beta' = (2 \ 3 \ 4 \ 1)$ then a block notation of $\alpha|_{\text{supp} (\beta')}$ is

$$
\begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 1 & 5 & 2 \end{pmatrix}.
$$

3 Formulas for $c(k, \beta)$ for some cycle types of $\beta$

In this section we show some formulas for $c(k, \beta)$ when $\beta$ is any 3-cycle, or any 4-cycle or any $(n-1)$-cycle. The following observation is easy and is also a consequence of Theorem 3.8 in [10].

Observation 3.1. Let $\beta$ be any permutation.

1. Let $x \in \text{fix} (\beta)$, then $\alpha \beta (x) = \beta \alpha (x)$ if and only if $\alpha (x) \in \text{fix} (\beta)$.

2. If $x \in \text{supp} (\beta)$ and $\alpha (x) \in \text{fix} (\beta)$ then $\alpha \beta (x) \neq \beta \alpha (x)$.

3.1 The case of $m$-cycles

Let $\beta_m = (b_1 \ldots b_m) \in S_n$ with $\text{fix} (\beta_m) = \{f_1, \ldots, f_{n-m}\}$. By Theorem 3.8 in [10], and without lost of generality, we have that any permutation $\alpha$ that $k$-commute with $\beta_m$, has the following block notation

$$
\begin{pmatrix} B_1 & \ldots & B_{k_2} \\ J_1 & \ldots & J_{k_2} \end{pmatrix} \begin{pmatrix} f_1 \ b_1' \\ f_1 & f_1' \end{pmatrix} \cdots \begin{pmatrix} f_{k_1} \ b_{k_1}' \\ f_{k_1} & f_{k_1}' \end{pmatrix} \cdots \begin{pmatrix} f_{n-m} \ b_{n-m-k_1}' \\ f_{n-m} & f_{n-m-k_1}' \end{pmatrix},
$$

(3)

where: 1) $f_i, f_j' \in \text{fix} (\beta_m), b_i' \in \text{supp} (\beta_m)$, for every $i, j, l$; and $(B_1 \ldots B_{k_2}) = (b_1 \ldots b_m); 2) k = k_1 + k_2$, with $k_1 \leq k_2; 3)$ every $J_i$ is either a block in $(b_1 \ldots b_m)$ or a block in an 1-cycle of $\beta_m; 4)$ $J_i, J_{i+1} \ (\text{mod} \ k_2)$ is not a block in $(b_1 \ldots b_m)$, for every $i$.

In order to obtain a formula for $c(k, \beta_m)$, for every $m$-cycle and every $k$, we need to work with all the solutions to equation $k = k_1 + k_2$ subject to the restriction $0 \leq k_1 \leq k_2$. Even more, for every one of these solutions we need to count all the possible ways to write $\alpha$ as in equation (3) which seem to be a difficult task in its generality, so we have worked with some specific cases. First we prove the following result for any $m$-cycle.

Proposition 3.2. Let $n, m \in \mathbb{Z}$, with $2 \leq m \leq n$. Let $\beta_m \in S_n$ be any $m$-cycle, then
1. $c(2m, \beta_m) = \binom{n-m}{m} m!(n-m)!$;

2. $c(2m - 1, \beta_m) = m!(\binom{n-m}{m-1} m(n-m))!$

Proof. Let $\beta_m = (b_1 \ldots b_m) \in S_n$ with $\text{fix}(\beta_m) = \{f_1, \ldots, f_{n-m}\}$. If $\alpha$ is any permutation that does not commute with $\beta_m$ on exactly $k_1$ (resp. $k_2$) points in $\text{fix}(\beta_m)$ (resp. in $\text{supp}(\beta_m)$) then from Observation 3.1 and Theorem 3.8 in [10] follows that if $k = 2m$ (resp. $k = 2m - 1$) then $k_1 = m$ (resp. $k_1 = m - 1$). We obtain our result by constructing all permutations $\alpha$ that $k$-commute with $\beta$.

Proof of part 1. We have that $k_1 = k_2 = m$ and hence $\alpha|_{\text{supp}(\beta_m)}$ should be a bijection from $\text{supp}(\beta_m)$ to a subset $B' \subseteq \text{fix}(\beta_m)$, with $|B'| = m$. There are $\binom{n-m}{m}$ ways to choose $B'$ and there are $m!$ bijections from $\text{supp}(\beta_m)$ onto $B'$. In order to construct $\alpha|_{\text{fix}(\beta_m)}$ we first select the $m$ points $B = \{f_{i_1}, \ldots, f_{i_m}\} \subseteq \text{fix}(\beta_m)$ that will be b.c.p. of $\alpha$ and $\beta_m$ (in $\binom{n-m}{m}$ ways) and then we construct any bijection from $B$ onto $\text{supp}(\beta_m)$ (in $m!$ ways) and any bijection from $\text{fix}(\beta) \setminus B$ onto $\text{fix}(\beta) \setminus B'$ (in $(n-2m)!$ ways). Then we obtain

$$c(2m, \beta_m) = \left(\binom{n-m}{m}\right)^2 (m!)^2 (n-2m)! = \binom{n-m}{m} m!(n-m)!$$

Proof of part 2. For this case $k_1 = m - 1$ and $k_2 = m$. We have that exactly $m-1$ points in the support of $\beta_m$ should be images under $\alpha$ of exactly $m-1$ fixed points. There are $\binom{n-m}{m-1}$ ways to select a subset $B \subseteq \text{fix}(\beta_m)$, with $|B| = m - 1$, that will be b.c.p. of $\alpha$ and $\beta_m$; there are $m$ ways to select a set $S' \subset \text{supp}(\beta_m)$ that will be the range of $\alpha|_B$, and there are $(m-1)!$ bijections from $B$ onto $S'$. Let $\{y\}' = \text{supp}(\beta_m) \setminus S'$. There are $\binom{n-m-1}{m-1}$ ways to select a set $B'$ of $m-1$ fixed points that will be the images under $\alpha$ of $m-1$ points in $\text{supp}(\beta_m)$ and there are $m!$ bijections from $\text{supp}(\beta_m)$ onto $B' \cup \{y\}'$ (notice that any of this bijection produce $m$ b.c.p. points in $\text{supp}(\beta_m)$ (by Theorem 3.8 in [10]). Finally the $(n-2m+1)!$ bijections from $\text{fix}(\beta_m) \setminus B$ onto $\text{fix}(\beta_m) \setminus B'$ produces only g.c.p. as desired. So we obtain

$$c(2m - 1, \beta_m) = \left(\binom{n-m}{m-1}\right)^2 m(m-1)!m!(n-2m+1)! = \binom{n-m}{m-1} m(m-1)! m!(n-m)!$$

We have the following expression in terms of sequences in OEIS

$$c(2m, \beta_m) = \left(\binom{n-m}{m}\right) m!(n-m)!,$$

$$= A052553(n, m) \times A098361(n, m).$$

### 3.2 The case of 3 and 4-cycles

In this subsection we present explicit formulas when $\beta$ is any 3-cycle or any 4-cycle.

**Theorem 3.3.** Let $\beta_3 \in S_n$ be any 3-cycle. Then

1. $c(0, \beta_3) = 3(n-3)!$, $n \geq 3$,
2. \( c(3, \beta_3) = (3(n-3) + 1)3(n-3)!, \ n \geq 3, \)
3. \( c(4, \beta_3) = 3(n-3)3(n-3)!, \ n \geq 4, \)
4. \( c(5, \beta_3) = 6{n-3 \choose 2}3(n-3)!, \ n \geq 5, \)
5. \( c(6, \beta_3) = 2{n-3 \choose 3}3(n-3)!, \ n \geq 6, \)
6. \( c(k, \beta_3) = 0, \ n \geq k \geq 7, \)

where \( |C_{S_n} (\beta)| = 3(n-3)! \).

Proof. The case \( c(k, \beta_3) \), for \( k \geq 7 \), follows from Proposition 6.1 in \[10\]. The cases \( c(3, \beta) \) and \( c(4, \beta) \) follows from Theorems 5.1 and 5.2 in \[10\], respectively. The cases \( c(5, \beta) \) and \( c(6, \beta) \) follows from Proposition 3.2.

Theorem 3.4. Let \( \beta_4 \in S_n \) be any 4-cycle. Then
1. \( c(0, \beta_4) = 4(n-4)!, \ n \geq 4, \)
2. \( c(3, \beta_4) = (4n-12)4(n-4)!, \ n \geq 4, \)
3. \( c(4, \beta_4) = (1 + 8(n-4))4(n-4)!, \ n \geq 4, \)
4. \( c(5, \beta_4) = (12(n-4) + 8{n-4 \choose 2})4(n-4)!, \ n \geq 5 \)
5. \( c(6, \beta_4) = (14n^2 - 126n + 280)4(n-4)!, \ n \geq 6, \)
6. \( c(7, \beta_4) = 24{n-4 \choose 3}4(n-4)!, \ n \geq 7, \)
7. \( c(8, \beta_4) = 6{n-4 \choose 4}4(n-4)!, \ n \geq 8, \)
8. \( c(k, \beta_4) = 0, \ n \geq k \geq 9. \)

Proof. The case \( c(k, \beta_4) \), for \( k \geq 9 \), follows from Proposition 6.1 in \[10\]. The cases \( k = 3, 4 \) follows from Theorems 5.1 and 5.2 in \[10\], respectively. The case \( k = 7, 8 \) follows from Proposition 3.2.

Proof of the case \( k = 5 \).

Let \( \alpha \) be any permutation that 5-commute with \( \beta_4 \). In this proof we use the notation \([s, f]\) to indicate that \( \beta_4 \) has \( s \) (resp. \( f \)) b.c.p. of \( \alpha \) and \( \beta_4 \) in \( \text{supp}(\beta_4) \) (resp, \( \text{fix}(\beta_4) \)). When \( k = 5 \) the unique possible options are either \([4, 1]\) or \([3, 2]\).

Subcase \([4, 1]\). For this case, and without lost of generality, any permutation (in block notation) that 5-commute with \( \beta_4 \) see as

\[
\begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 \\
  f_1' & a_{i_1} & a_{i_2} & a_{i_3}
\end{pmatrix}
\begin{pmatrix}
  f_1 & \ \ \\
  f_2 & \ \ \\
  \vdots & \ \ \\
  f_{n-m} & \ \ \\
\end{pmatrix}
\]

where \( f_i' \in \text{fix}(\beta_4) \), for every \( i \), and for \( 1 \leq j \leq 2 \), \( a_{i_{j+1}} \) is not a block in \( (a_1 \ldots a_4) \). There are \( n-4 \) ways to select the point \( f_1 \) and there are \( 4 \) ways to select the point \( a_{i_4} \). There are \( 4 \) ways to select point \( a_1 \in \text{supp}(\beta_4) \) and there are \( n-4 \) ways to select the fixed point.
There are 4 ways to select the subset \( \{a_i \} \) from \( \{a_1, a_2, a_3, a_4 \} \) that 5-commute with \( \beta_4 \). Finally, there are \((n - 5)!\) bijections from \( \text{fix}(\beta_4) \setminus \{f_1\} \) onto \( \text{fix}(\beta_4) \setminus \{f_1'\} \).

Subcase [3, 2]. For this case, and without lost of generality, any permutation (in block notation) that 5-commute with \( \beta_4 \) see as follows

\[
\begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 \\
  f_1 & f_2 & f_3 & f_4 \\
  f_1' & f_2' & f_3' & f_4'
\end{pmatrix}
\]

where \( f_j \in \text{fix}(\beta_4) \), for every \( j \), and \( a_{i_1}a_{i_2} \) is a block in \( (a_1 \ldots a_4) \). Notice that either \( a_{i_3}a_{i_4} \) or \( a_{i_4}a_{i_3} \) is a block in \( (a_1 \ldots a_4) \). There are \((n - 4)\) ways to select the subset \( \{f_1, f_2\} \subseteq \text{fix}(\beta_4) \).

There are 4 ways to select the subset \( \{x, y\} \subseteq \{a_1, a_2, a_3, a_4\} \) that will satisfy \( \alpha(\{f_1, f_2\}) = \{x, y\} = \{a_{i_3}, a_{i_4}\} \) (once we select a point, say \( x \), the second is uniquely determined). There are 2 ways to select \( a_{i_3} \) from \( \{x, y\} \) and one way to select \( a_{i_4} \) from \( \{x, y\} \setminus \{a_{i_3}\} \). There are 4 ways to select \( a_1 \) from \( \text{sup}(\beta_4) \). There are \((n - 4)\) ways to select the set \( \{x', y'\} \subseteq \text{fix}(\beta_4) \) that will satisfy \( \alpha(\{a_1, a_2\}) = \{x', y'\} = \{f_1, f_2\} \). There are 2 ways to select \( f_1' \) from \( \{x', y'\} \) and one way to select \( f_2' \) from \( \{x', y'\} \setminus \{f_1'\} \). Finally there are \((n - 6)!\) bijections from \( \text{fix}(\beta_4) \setminus \{f_1, f_2\} \) onto \( \text{fix}(\beta_4) \setminus \{f_1', f_2'\} \). Therefore, for this case, we have

\[
8 \binom{n - 4}{2} 4(n - 4)!
\]

permutations \( \alpha \) that 5-commute with \( \beta_4 \). Then

\[
c(5, \beta_4) = 12(n - 4)4(n - 4)! + 8 \binom{n - 4}{2} 4(n - 4)!
\]

Finally, as

\[
c(6, \beta_4) = n! - \left( c(7, \beta_4) + c(8, \beta_4) + \sum_{k=0}^{5} c(k, \beta_4) \right),
\]

the result for \( c(6, \beta_4) \) follows by direct computation.

3.3 The case of any \((n - 1)\)-cycle

Let \( \pi = p_1p_2 \ldots p_n \) be a permutation of \([n]\) in its one line notation. A substring \( p_ip_{i+1}, \) \( 1 \leq i \leq n - 1, \) is called a succession of \( \pi \) if \( p_{i+1} = p_i + 1 \). Let \( S(n) \) be the number of permutations of the set \([n]\) without a succession. The following formulas for \( S(n) \) are taken from [2 Sec. 5.4]

\[
S(n) = (n - 1)! \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{k!}, \quad n \geq 1,
\]

and

\[
S(n) = (n - 1)S(n-1) + (n - 2)S(n-2), \quad n = 3, 4, \ldots, \text{with } S(1) = S(2) = 1.
\]
The sequence A000255 satisfies the recurrence relation
\[ A000255(n) = nA000255(n - 1) + (n - 1)A000255(n - 2), \]
\[ A000255(0) = 1, \ A000255(1) = 1. \]

Then it is easy to see that \( S(n) = A000255(n - 1), \ n \geq 1 \). Let \( C(k) \) be the number of cyclic permutations of \( \{1, \ldots, k\} \) with no \( i \mapsto i + 1 \bmod k \) (see, e.g., [2] Sec. 5.5). The number \( C(k) \) is sequence A000757 in [13].

**Theorem 3.5.** Let \( \beta_{n-1} \in S_n \) be any \((n - 1)\)-cycle, \( n \geq 4 \). Then
\[
c(k, \beta_{n-1}) = (n - 1)\left(\frac{n - 1}{k}\right)C(k) + (n - 1)^2\left(\frac{n - 3}{k - 3}\right)S(k - 2),
\]

**Proof.** In this proof we will denote \( \beta_{n-1} \) simply by \( \beta \). Let \( \beta = (b_1, \ldots, b_{n-1})(b_n) \) be any \((n - 1)\)-cycle. Let \( C = \{ \alpha \in C(k, \beta) \mid \alpha(b_n) = b_n \} \). Notice that \( \alpha \in C \) if and only if \( \alpha \beta(b_n) = \beta \alpha(b_n) \), and this implies that for any \( \alpha \in C \), \( \alpha|_{\text{supp}(\beta)} \) is any permutation of \( \{b_1, \ldots, b_{n-1}\} \) that \( k \)-commutes with the cycle permutation \( (b_1, \ldots, b_{n-1}) \) of \( \{b_1, \ldots, b_{n-1}\} \), and then we can use Theorem 4.13 in [10] to show that \( |C| = (n - 1)^2\left(\frac{n - 1}{k}\right)C(k) \).

Let \( \overline{C} := C(k, \beta) \setminus C \). We will compute \( |\overline{C}| \) by constructing all permutations in \( \overline{C} \), i.e., permutations \( \alpha \) such that \( \alpha(b_n) \neq b_n \) and that does not commute with \( \beta \) on exactly \( k - 1 \) points in \( \text{supp}(\beta) \). First, \( \alpha \in \overline{C} \) if and only if \( \alpha(b_n) = x \in \text{supp}(\beta) \), and from Theorem 3.8 in [10], \( \alpha \) restricted to \( \text{supp}(\beta) \) see as

\[
\alpha|_{\text{supp}(\beta)} = \left(\begin{array}{cccccc}
B_1 & B_2 & \ldots & B_j & b_r & B_{j+1} & \ldots & B_{k-2} \\
B'_{i_1} & B'_{i_2} & \ldots & B'_{i_j} & b_n & B'_{i_{j+1}} & \ldots & B'_{i_{k-2}}
\end{array}\right),
\]

(4)

where

1. the set \( \{B_1, B_2, \ldots, B_j, b_r, B_{j+1}, \ldots, B_{k-2}\} \) is a block partition of \( \{b_1 \ldots b_{n-1}\} \) such that
\[
(b_1 \ldots b_{n-1}) = (B_1B_2 \ldots B_j b_r B_{j+1} \ldots B_{k-2}),
\]

2. The string \( B'_{i_1} B'_{i_2} \ldots B'_{i_j} B'_{i_{j+1}} \ldots B'_{i_{k-2}} \) is a block permutation of \( B'_1 B'_2 \ldots B'_{k-2} x \), where \( \{B'_1, B'_2, \ldots, B'_{k-2}, x\} \) is a block partition of \( \{b_1 \ldots b_{n-1}\} \) and
\[
(b_1 \ldots b_{n-1}) = (B'_1 B'_2 \ldots B'_{k-2} x),
\]

3. \( |B_j| = |B'_{i_j}| \), for every \( j \), \( B'_{i_s} B'_{i_{s+1}} \) is not a block in \( \{b_1 \ldots b_{n-1}\} \) for \( s \in \{1, \ldots, k-3\} \setminus \{j\} \) and \( B_{i_j} B_{i_{j+1}} \) can be or not a block in \( \{b_1 \ldots b_{n-1}\} \).

Then any \( \alpha \) in \( \overline{C} \) can be constructed by first selecting \( x \in \text{supp}(\beta) \) that will satisfy \( x = \alpha(b_n) \) (in \( (n - 1) \) ways). Now we can choose the block partition \( \{B'_1, B'_2, \ldots, B'_{k-2}, x\} \) of \( \{b_1, \ldots, b_{n-1}\} \) by selecting the first element in each block. As the point \( x \) was previously selected then the block \( x \) is unique determined and also the first element in the block \( B'_1 \), so we only need to choose the first element of \( k - 3 \) blocks between \( n - 3 \) possibilities (in \( (n - 3) \) ways). There are \( n - 1 \) ways to select the first element of block \( B_1 \) and the rest of the blocks are uniquely determined. Then we have
\[
(n - 1)^2\left(\frac{n - 3}{k - 3}\right)R,
\]

9
ways to construct $\alpha|_{\text{supp}(\beta)}$, where $R$ is the number of ways to construct the second row in matrix (4) in such a way that we obtain exactly $k - 1$ b.c.p. of $\alpha$ and $\beta$. Now, matrix (4) can be rewritten as

$$\alpha|_{\text{supp}(\beta)} = \begin{pmatrix} B_{j+1} & \ldots & B_{k-2} & B_1 & B_2 & \ldots & B_j & b_r \\ B'_{i_{j+1}} & \ldots & B'_{i_{k-2}} & B'_{i_1} & B'_{i_2} & \ldots & B'_{i_j} & b_n \end{pmatrix}. \quad (5)$$

As $b_r$ is necessarily a b.c.p. of $\alpha$ and $\beta$ then in order to obtain exactly $k - 1$ b.c.p. it is enough that in

$$B'_{i_{j+1}} \ldots B'_{i_{k-2}} B'_{i_1} B'_{i_2} \ldots B'_{i_j}$$

not appears any string of the form $B'_r B'_{r+1}$, for $1 \leq r \leq k - 3$ (in this way we obtain a b.c.p. per block) and this condition is fulfilled if and only if $B'_{i_{j+1}} \ldots B'_{i_{k-2}} B'_{i_1} B'_{i_2} \ldots B'_{i_j}$ is equal to a block permutation $B'_{\tau(1)} \ldots B'_{\tau(k-2)}$ of $B_1' \ldots B'_{k-2}$, where $\tau$ is a permutation without a succession. As there are $S(k-2)$ such permutations we have that $R = S(k-2)$ and the result follows.

Previously, in a joint work with Rutilo Moreno (that form part of his PhD Thesis [9]), we obtained

$$c(k, \beta_{n-1}) = (n-1) \binom{n-1}{k} C(k) + (n-1)^2 \binom{n-3}{k-3} T(k),$$

where

$$T(k) = (k-2)C(k-2) + (2k-5)C(k-3) + (k-3)C(k-4).$$

Then, by Theorem 3.5 we have the following relation between sequences A000255 and A000757.

$$A000255(k-3) = (k-2)A000757(k-2)+(2k-5)A000757(k-3)+(k-3)A000757(k-4), k \geq 4.$$  

For the case when $k = 3$ we have

$$c(3, \beta_{n-1}) = (n-1) \binom{n-1}{3} + (n-1)^2$$

$$= (n-1) \times \left( \binom{n-1}{3} + n - 1 \right)$$

$$= (n-1) \times A000125(n-2), n \geq 3,$$

where

$$A000125(m) = \binom{m+1}{3} + m + 1, m \geq 0,$$

are the Cake numbers. No other relationships with sequences in OEIS were found for the cases $4 \leq k \leq 10$, and it is possible that no such relations exist, until now, for $k \geq 11$. By direct computation we obtain

$$c(\leq 3, \beta_{n-1}) = (n-1) \times \left( \binom{n-1}{3} + n \right).$$
The sequence $A011826(m)$ is equal to $\binom{m}{3} + (m + 1)$, for $1 \leq m \leq 1000$, as was noted by John W. Layman in [8], then for this case we have that
\[
c(\leq 3, \beta_{n-1}) = (n - 1) \times A011826(n - 1), \ 2 \leq n \leq 1000.
\]

### 3.4 Another cases

Now we present three more cases

**Proposition 3.6.** Let $\beta \in S_{2m}$ be any permutation with exactly $m$ fixed points. Then

1. $c(2m, \beta) = (m!)^2$,
2. $c(2m - 1, \beta) = m^2(m!)^2$

**Proof.** Part 1. If $\alpha$ is any permutation that $2m$-commute with $\beta \in S_{2m}$ then by Observation 3.1 we have that $\alpha(\text{fix}(\beta)) \subseteq \text{supp}(\beta)$ and as $|\text{fix}(\beta)| = |\text{supp}(\beta)| = m$ then $\alpha(\text{fix}(\beta)) = \text{supp}(\beta)$ which implies that $\alpha(\text{supp}(\beta)) = \text{fix}(\beta)$. Then $\alpha|_{\text{fix}(\beta)}$ (resp. $\alpha|_{\text{supp}(\beta)}$) is any bijection from $\text{fix}(\beta)$ (resp. $\text{supp}(\beta)$) onto $\text{supp}(\beta)$ (resp. $\text{fix}(\beta)$). So we have $(m!)^2$ ways to construct $\alpha$.

**Proof of part 2.** Let $x$ be the unique g.c.p. of $\alpha$ and $\beta$. It is easy to see that necessarily $x, \alpha(x) \in \text{fix}(\beta)$. Then $\alpha|_{\text{fix}(\beta)}$ is any bijection from $\text{fix}(\beta)$ onto $\text{supp}(\beta) \setminus \{a\} \cup \{x\}$ (there are $m!$ such bijections), where $x' \in \text{fix}(\beta)$, and $\alpha|_{\text{supp}(\beta)}$ is any bijection from $\text{supp}(\beta)$ onto $\text{fix}(\beta) \setminus \{x'\} \cup \{a\}$ (there are $m!$ such bijections). Now, as we have $m$ ways to select $x'$ and $m$ ways to select the point $a$ then $c(2m - 1, \beta) = m^2(m!)^2$.

For $\beta$ as in previous proposition we have
\[
c(2m, \beta) = (m!)^2 = A001044(m), \ m \geq 0,
\]
and
\[
c(2m - 1, \beta) = m^2 \times (m!)^2 = A000290(m) \times A001044(m),
\]
\[
= m \times m(m!)^2 = m \times A084915(m).
\]

From this we obtain the following relation
\[
m \times A084915(m) = A000290(m) \times A001044(m).
\]

**Proposition 3.7.** Let $\beta \in S_{2m-1}$ be any permutation with exactly $m - 1$ fixed points. Then $c(2m - 1, \beta) = (m!)^2$.

**Proof.** As all the fixed points are b.c.p, then $\alpha(\text{fix}(\beta)) \subset \text{supp}(\beta)$ and $\alpha(\text{supp}(\beta)) = \text{fix}(\beta) \cup \{a\}$, for some $a \in \text{supp}(\beta)$. Then $\alpha|_{\text{fix}(\beta)}$ should be any bijection from $\text{fix}(\beta)$ to an $(m - 1)$-subset $B$ of $\text{supp}(\beta)$ and $\alpha|_{\text{supp}(\beta)}$ should be any bijection from $\text{supp}(\beta)$ to $\text{fix}(\beta) \cup (\text{supp}(\beta) \setminus B)$. There are $\binom{m}{m-1}(m - 1)! = m(m - 1)!$ ways to construct $\alpha|_{\text{fix}(\beta)}$ and there are $m!$ ways to construct $\alpha|_{\text{supp}(\beta)}$. Then we have that $c(2m - 1, \beta) = m(m - 1)!m! = (m!)^2$. \qed
3.5 Upper bound for $c(\leq k, \beta)$ when $\beta$ is any $n$-cycle

Let $C(\leq k, \beta) = \{\alpha \in S_n \mid H(\alpha\beta, \beta\alpha) \leq k\}$ and $c(\leq k, \beta)$ the cardinality of this set.

**Theorem 3.8.** Let $\beta$ be any $n$-cycle. Then

$$c(\leq k, \beta) \leq n \binom{n}{k} (k-1)! - n \binom{n}{k} + n, \quad k \geq 0,$$

with equality for $0 \leq k \leq 3$ and $k = n$, with the convention that $(-1)! = 1$.

**Proof.** Let $\beta = (b_1, \ldots, b_n)$. To select a block partition of $b_1 \ldots b_n$ into $k$ blocks, $B_1, \ldots, B_n$ (in $\binom{n}{k}$ ways) such that $B_1B_2\ldots B_k = b_1\ldots b_n$. To construct permutation

$$\alpha = \left( \begin{array}{cccc} P_1 & P_2 & \ldots & P_k \\ B_{i_1} & B_{i_2} & \ldots & B_{i_k} \end{array} \right),$$

where $(P_1P_2\ldots P_k) = (b_1, \ldots, b_n)$, $B_{i_1}B_{i_2}\ldots B_{i_k}$ is any block permutation of $B_1B_2\ldots B_k$ and $|P_i| = |B_i|$, for every $i$. We have $n$ ways to select the first element in block $P_1$ and the rest of blocks in first row are uniquely determined. As we are considering the $n$ cyclic permutations of $b_1, \ldots, b_n$ in the first row in this matrix then for each selection of partition $B_1 \ldots B_k$, we will have $k$ repeated permutations (the $k$ cyclic permutations of $B_1 \ldots B_k$). Therefore there are at most $n \binom{n}{k} (k-1)!$ permutations that $(\leq k)$-commute with $\beta$. We can reduce this bound a little more. Notice that for any of the $n$ possibilities of the first row of the matrix, vector $< b_1, \ldots, b_n >$ appears $\binom{n}{k}$ times in the second row (each of the $B_1B_2\ldots B_k$ partitions of $b_1 \ldots b_n$ appears once in the second row). Then we have

$$c(\leq k, \beta) \leq n \binom{n}{k} (k-1)! - n \binom{n}{k} + n.$$

By direct computation we can check that equality is reached when $0 \leq k \leq 3$ and $k = n$. \(\square\)

4 Relations with integer sequences

In this section we will express some formulas for the numbers $c(k, \beta)$ and $c(\leq k, \beta)$ in terms of sequences in OEIS. These formulas are obtained by simple inspection or with the help of a computer algebra system and, as far as possible, an exhaustive search on the OEIS. These formulas have the following notation: we write a formula as $a \times b$ and its corresponding expression with sequences in OEIS as $A_i \times A_j \ldots$ to emphasize that $a$ (resp. $b$) is the formula for sequence $A_i \ldots$ (resp. $A_j \ldots$).

4.1 Transpositions

When $\beta$ is a transposition, the following result was presented in [10].

**Proposition 4.1.** Let $\beta_2 \in S_n$ be any transposition. Then
1. \( c(0, \beta_2) = 2(n - 2)! \), \( n \geq 2 \).

2. \( c(3, \beta_2) = 4(n - 2)(n - 2)! \), \( n \geq 3 \).

3. \( c(4, \beta_2) = (n - 2)(n - 3)(n - 2)! \), \( n \geq 4 \).

4. \( c(k, \beta_2) = 0 \), \( 5 \leq k \leq n \).

R. Moreno and the author of this article [10] noted that \( c(0, \beta_2) \), \( c(3, \beta_2) \) and \( c(4, \beta_2) \) coincide with the number of permutations of \([n]\), \( n \geq 2 \), having exactly 2, 3 and 4 points, respectively, on the boundary of their bounding square [4] (A208529, A208528 and A098916, respectively). Here we provide and explicit relation between these numbers. The following definition is taken from problem 1861 in [4]. A permutation \( \alpha \in S_n \) can be represented in the plane by the set of \( n \) points \( P_\alpha = \{(i, \sigma(i)) \mid 1 \leq i \leq n\} \). The bounding square of \( P_\alpha \) is the smallest square bounding \( P_\alpha \), i.e., as cited in OEIS, “is the square with sides parallel to the coordinate axis containing \((1,1)\) and \((n,n)\)”.

**Proposition 4.2.**

1. A permutation \( \alpha \) has only two points on the boundary of their bounding square if and only if \( \alpha \) commutes with transposition \((1, n)\).

2. A permutation \( \alpha \) has only \( m \) points on the boundary of their bounding square if and only if \( \alpha_m \)-commutes with transposition \((1, n)\), for \( m = 3, 4 \).

**Proof.** In this proof we use some paragraphs in [4].

**Proof of 1:** A permutation \( \alpha \) commutes with \((1, n)\) if and only if either \( \alpha(1) = 1 \) and \( \alpha(n) = n \) or \( \alpha(1) = n \) and \( \alpha(n) = 1 \), i.e., if and only if \( P_\alpha \) contains both \((1,1)\) and \((n,n)\) or both \((1,n)\) and \((1,1)\) which is true [4] if and only if \( P_\alpha \) has exactly two points on the boundary of its bounding square.

**Proof of 2:** A permutation \( \alpha \) \( 4 \)-commutes with \((1, n)\) if and only if \( \alpha(1) \neq 1, n \) and \( \alpha(n) \neq 1, n \). \( \{\alpha(1), \alpha(n)\} \cap \{1, n\} = \emptyset \), i.e., if and only if \( P_\alpha \) does not contain any of the points \((1,1), (n,n), (1,n), \) and \((1,1)\) which is true [4] if and only if \( P_\alpha \) has exactly four points on the boundary of its bounding square. The case \( m = 3 \) follows because both \( \{C(0, \beta), C(3, \beta), C(4, \beta)\} \) and the collection of sets \( B_m \) of permutations \( \alpha \) that have \( m \) points of \( P \) on the boundary of the bonding square of \( P_\alpha \), for \( m = 2, 3, 4 \), are partitions of \( S_n \).

By direct computation we obtain

\[
c(\leq 3, \beta_2) = 2(n - 2)! + 4(n - 2)(n - 2)! \\
= 2 \times (2(n - 2) + 1)(n - 2)!. \\
= 2 \times A007680(n - 2), n \geq 2
\]

\[
c(\leq 4, \beta_2) = n!, \\
= A000142(n), n \geq 2.
\]

From these formulas we obtain the following relation that was announced by the author in the FORMULA section of sequence A007680

\[
A007680(n - 2) = \frac{A208529(n) + A208528(n)}{2}, n \geq 2.
\]
4.2 Some cases for 3 and 4-cycles

In tables 1 and 2 we express formulas for $c(k, \beta_3)$ (Theorem 3.3) and $c(\leq k, \beta_3)$, respectively, in terms of sequences in OEIS. The formulas for $c(\leq k, \beta_3) = \sum_{i=0}^{k} c(k, \beta_3)$ are obtained by direct computation.

Another relations that can be deduced are

\[
A000142(n) = A208529(n) + A208528(n) + A098916(n), \ n \geq 3, \\
A000142(n) = 2 \times A007680(n-2) + A098916(n), \ n \geq 3.
\]
From this tables we obtain the following relations

\[
A016789(n) = 1 + A016777(n), n \geq 0.
\]
\[
A016933(n) = A016789(n) + A008585(n), n \geq 0,
\]
\[
= 1 + A016777(n) + A008585(n), n \geq 0.
\]
\[
A077588(n - 2) = A016933(n - 3) + A028896(n - 4), n \geq 4,
\]
\[
= A016789(n - 3) + A008585(n - 3) + A028896(n - 4), n \geq 4,
\]
\[
= 1 + A016777(n - 3) + A008585(n - 3) + A028896(n - 4), n \geq 4.
\]
\[
A007290(n) = A077588(n - 2) + A007290(n - 3), n \geq 3,
\]
\[
= A016933(n - 3) + A028896(n - 4) + A007290(n - 3), n \geq 4,
\]
\[
= A016789(n - 3) + A008585(n - 3) + A028896(n - 4) + A007290(n - 3), n \geq 4.
\]

In tables 3 and 4 we express formulas for \(c(k, \beta_4)\) and \(c(\leq k, \beta_4)\), respectively, in terms of sequences in OEIS. From this tables we deduce the following identities between sequences in OEIS

\[
A052578(n) = A000142(n - 1) \times (A016813(n) - 1), n \geq 1.
\]
\[
A052578(n) = A000142(n - 1) \times A008586(n), n \geq 1.
\]
\[
A008598(n) = 4 (A016813(n) - 1), n \geq 0.
\]
\[
A016813(n) = A017593(n - 1) - A017077(n - 1), n \geq 1.
\]

We encourage the interested reader to obtain more identities from this tables.

| \(k\) | \(c(k, \beta_4)\) |
|-------|----------------|
| 0     | \(4 \times (n - 4)!\) |
|       | \(4(n - 4)!\) |
|       | \(2 \times 2(n - 4)\)! |
| 3     | \(4 \times (n - 3)!\) |
|       | \(8 \times 2(n - 3)\)! |
|       | \(16(n - 3) \times (n - 4)!\) |
|       | \(4(n - 3) \times 4(n - 4)!\) |
| 4     | \((1 + 8(n - 4)) \times 4(n - 4)!\) |
| 5     | \((2(n - 3)^2 - 4) \times 4(n - 4)!\), \(n \geq 4\) |
| 6     | \(14 \times (n - 5)(n - 4) \times 4(n - 4)!\) |
|       | \(28 \times (n - 4) \times 4(n - 4)!\) |
| 7     | \(24 \times (\binom{n - 4}{3} \times 4(n - 4)\) |
|       | \(4 \times 6(\binom{n - 4}{3} \times 4(n - 4)!\) |
|       | \(3 \times 8(\binom{n - 4}{3} \times 4(n - 4)!\) |
| 8     | \(24 \times (\binom{n - 4}{4} \times (n - 4)!\) |
|       | \(24(n - 4) \times (n - 4)!\) |
|       | \(6 \times (\binom{n - 4}{4} \times 4(n - 4)!\) |
|       | \(6(n - 4) \times 4(n - 4)!\) |
| \(\geq 9\) | 0 |
|       | \(A000004(n)\) |

Table 3: Formulas for \(c(k, \beta_4)\) expressed in terms of sequences in OEIS
Table 4: Formulas for $c(\leq k, \beta_4)$ expressed in terms of sequences in OEIS

4.3 Some cases for $n$-cycles
The following proposition was presented in [10].

Proposition 4.3. Let $n$ and $k$ be nonnegative integers, $n \geq k \geq 0$. Let $\beta_n \in S_n$ be any $n$-cycle. Then

$$c(k, \beta_n) = n \binom{n}{k} C(k),$$

where $C(k) = A000757(k)$.

Let $C_{n,k} := n \binom{n}{k} C(k)$, triangle $\{C_{n,k}\}$ is now sequence A233440 in [13]. In Table 5 we show some expressions for $c(k, \beta_n)$ in terms of sequences in OEIS, for $0 \leq k \leq 16$, this table was announced by the author in [12]. We search for some values $k \geq 17$ but no relationships with sequences in OEIS were found.

Table 5: Formulas for $c(k, \beta_n)$, $0 \leq k \leq 16$, in terms of sequences in OEIS

By direct computation we obtain the following formulas for $c(\leq k, \beta)$, $k = 3, 4$. For the
case of $k = 5, 6, 7$ no relation with sequences in OEIS was found, and it is possible that for greatest values of $k$ such relation does not exists until now.

1. $c(\leq 3, \beta_n) = n \left( 1 + \binom{n}{3} \right) = n \times A050407(n + 1), n \geq 0.$

2. $c(\leq 4, \beta_n) = n \left( 1 + \binom{n+1}{4} \right) = n \times A145126(n - 2), n \geq 2.$

From this we can deduce the following identities

$$n \times A050407(n + 1) = A233440(n, 0) + A233440(n, 3), n \geq 0.$$ 
$$n \times A145126(n - 2) = A233440(n, 0) + A233440(n, 3) + A233440(n, 4), n \geq 2.$$

From these last relations and from the relations in Table 5 we have

$$A050407(n + 1) = 1 + A004320(n - 2)/n, n \geq 2.$$ 
$$A233440(n, 4) = n \left( A145126(n - 2) - A050407(n + 1) \right), n \geq 2.$$ 
$$A027764(n - 1) = n \left( A145126(n - 2) - A050407(n + 1) \right), n \geq 4.$$

4.4 Some cases for $k = 3$ and $k = 4$

R. Moreno and the author of this article presented in [10] formulas for $c(3, \beta)$ and $c(4, \beta)$, where $\beta$ is any permutation. We use these results to obtain formulas for some type of permutations. Let $m$ be a positive integer. Let $\beta = (1 \ldots m)(m + 1 \ldots 2m) \in S_{2m}$, then

$$c(3, \beta) = 2m^2 \times 2 \binom{m}{3},$$ 
$$= A001105(m) \times A007290(m), m \geq 2.$$ 

and

$$c(4, \beta) = \left( 2 \binom{m}{4} + m \binom{m}{2} \right) \times 2m^2,$$ 
$$= \left( 2 \binom{m}{4} + (m - 1)m^2/2 \right) \times 2m^2,$$ 
$$= (A034827(m) + A006002(m - 1)) \times A001105(m), m \geq 2.$$ 

Now, for $n \geq 3$, the case when $\beta_m \in S_n$ is any $m$-cycle, $1 < m \leq n$, by Theorem 5.1 in [10] we have

$$c(3, \beta_m) = \left( \binom{m}{3} + m(n - m) \right) m(n - m)!.$$ 

In table 6 we express formulas for $c(3, \beta_m)$ for some values of $m$ in terms of sequences in OEIS.
of some is a permutation of cycle type \((c_1, \ldots, c_n)\), that is, if \(\beta\) is a permutation of cycle type \((c_1, \ldots, c_n)\), then \(c(\beta) = \prod_{i=1}^{n} i^{c_i}!\). For the case when \(\beta\) is any \(m\)-cycle we have that \(c(\beta) = m(n-m)! = m \times A000142(n-m)\) that we consider as a trivial relation with sequences in OEIS. In Table 7 we present formulas for the centralizer of some \(m\)-cycles, where we have omitted the trivial cases for \(m\)-cycles.

### Table 6: Formulas for \(c(3, \beta_m)\), for some \(m\)-cycles, in terms of sequences in OEIS

Let \(\beta_m \in S_n\) be any \(m\)-cycle, \(n \geq m \geq 4\). By Theorem 5.2 in [10] we have

\[
c(4, \beta_m) = m(n-m)! \left( \binom{m}{4} + m(m-2)(n-m) \right)
\]

In this case we found relations with sequences in OEIS only for \(4 \leq k \leq 7\) and these are showed in Table 7.

### Table 7: Formulas for \(c(4, \beta_m)\) for some \(m\)-cycles

#### 5 Final results and comments

We finish this paper by consider the case of the size of the centralizer for some cycle types of permutations. For the case when \(k = 0\), \(c(k, \beta)\) is the size of the centralizer of \(\beta\), that is, if \(\beta\) is a permutation of cycle type \((c_1, \ldots, c_n)\), then \(c(k, \beta) = \prod_{i=1}^{n} i^{c_i}!\). For the case when \(\beta\) is any \(m\)-cycle we have that \(c(0, \beta) = m(n-m)! = m \times A000142(n-m)\) that we consider as a trivial relation with sequences in OEIS. In Table 8 we present formulas for the centralizer of some \(m\)-cycles, where we have omitted the trivial cases for \(m\)-cycles.
Finally we consider two more cases. Let $\beta_{m(2)}$ denote a permutation of $2m$ symbols whose cycle factorization consists of a product of two $m$-cycles and $\beta_{2(m)}$ a fixed-point free involution of $2m$ symbols. In this cases we have

$$c(0, \beta_{m(2)}) = 2m^2,$$
$$= A001105(m), m \geq 0.$$

$$c(0, \beta_{2(m)}) = 2^m m!,$$
$$= A000165(m); m \geq 0,$$

where A000165 is the double factorial of even numbers.

It is possible that there exists more sequences related with the formulas presented in this and previous sections and also we believe that for some special cases we can obtain formulas for $c(k, \beta)$. However, we believe that in general to compute in a explicit way the number $c(k, \beta)$ is a difficult task. We have worked, without a success, to obtain a “general” formula for the case of any $m$-cycle so we leave this as a possible future project. Another future project is to found explicit relations (as in the case of transpositions) between $k$-commuting permutations and another structures counted by the same integer sequences.

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