On a Logistic Differential Model. Some Applications

Nikolay Kyurkchiev¹ and Svetoslav Markov²

¹Faculty of Mathematics and Informatics,
University of Plovdiv Paisii Hilendarski,
24, Tzar Asen Str., 4000 Plovdiv, Bulgaria,
nkyurk@uni-plovdiv.bg,

²Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria,
smarkov@bio.bas.bg

Abstract. In this article we will consider the possibility of approximating the input function \( s(t) \) (the nutrient supply for cell growth) of the form
\[
s(t) = \frac{1}{1+mt}e^{-mt}
\]
where \( m > 0 \) is parameter.

We prove upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function \( h_{t^*}(t) \) by means of the general solution of the differential equation \( y'(t) = ky(t)s(t) \) with \( y(t_0) = y_0 \).

We will illustrate the evolution of the solution \( y(t) \) for approximating and modelling of three data sets: i) ”data on the development of the \textit{Drosophila melanogaster} population”, published by Pearl in 1920, ii) dataStormIdentifications (Storm worm was one of the most biggest cyber threats of 2008, and ”cancer data” [49]–[50].

Citation: Nikolay Kyurkchiev, Svetoslav Markov, On a Logistic Differential Model. Some Applications, Biomath Communications 6, pp. 34-50, https://doi.org/10.11145/bmc.2019.04.307
Numerical examples using *CAS Mathematica*, illustrating our results are given.

**AMS Subject Classification:** 41A46  
**Keywords:** Nutrient supply, Parametric input function, ”Supersaturation” of the model, Heaviside function, Hausdorff distance, Upper and lower bounds

# 1 Introduction

Sigmoidal functions find multiple applications to population dynamics, analysis of nutrient supply for cell growth in bioreactors, population survival functions, classical predator–prey models, debugging theory and others [19]–[47].

Evidently, the Verhulst model can be considered as a prototype of models used in bioreactor modelling. In batch growth, the rate of biomass production is given by $\frac{dx}{dt} = \kappa x$, where: $x =$ biomass concentration; $\kappa =$ specific growth rate; $t =$ time.

The rate $\kappa$ is a function of nutrient supply and therefore can be a function of time (i.e., if nutrient supply is changing with time).

In general, $\kappa = F(S, P, I, X, T, osmotic\ pressure);$ $S =$ substrate concentration; $P =$ product concentration; $I =$ inhibitor concentration.

There, especially in the case of continuous bioreactor, the nutrient supply is considered as an input function $s(t)$ as follows:

$$\frac{dy(t)}{dt} = ky(t)s(t) \quad (1)$$

where $s$ is additional specified.

To the role and choice of nutrient supply for cell growth in bioreactors are devoted to a number of studies [1]–[16].

Following the ideas given in [13], in this paper we consider the following differential model:
\[
\begin{cases}
\frac{dy(t)}{dt} = k y(t) \frac{1}{1 + mt} e^{-mt} \\
y(t_0) = y_0
\end{cases}
\] (2)

where \(k\) and \(m\) are parameters.

We prove upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function \(h_{t^*}(t)\) by means of the general solution of this differential equation.

We will illustrate the advances of the solution \(y(t)\) for approximating and modelling of:

- ”data on the development of the \textit{Drosophila melanogaster} population”, published by biologist Raymond Pearl in 1920 (see, also Alpatov, Pearl [17]);
- \textit{data Storm Identifications} [48], [47];
- ”cancer data” [49]–[50].

2 Preliminaries

\textbf{Definition 1.} The shifted Heaviside step function is defined by

\[
h_{t^*}(t) = \begin{cases} 
0, & \text{if } t < t^*, \\
[0, 1], & \text{if } t = t^*, \\
1, & \text{if } t > t^*.
\end{cases}
\] (3)

\textbf{Definition 2.} [18] The Hausdorff distance (the \(H\)–distance) \(\rho(f, g)\) between two interval functions \(f, g\) on \(\Omega \subseteq \mathbb{R}\), is the distance between their completed graphs \(F(f)\) and \(F(g)\) considered as closed subsets of \(\Omega \times \mathbb{R}\).

More precisely,

\[
\rho(f, g) = \max \left\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B|| \right\},
\] (4)
| wherein \(|\cdot|\) is any norm in \(\mathbb{R}^2\), e. g. the maximum norm \(|(t, x)| = \max\{|t|, |x|\}; hence the distance between the points \(A = (t_A, x_A), B = (t_B, x_B)\) in \(\mathbb{R}^2\) is \(|A - B| = \max(|t_A - t_B|, |x_A - x_B|)).

3 Main Results

3.1 A New Model

The general solution of the differential equation (2) is of the following form:

\[ y(t) = y_0 e^{\frac{ck}{m} Ei(-1 - mt)} - e^{\frac{ck}{m} Ei(-1 - mt_0)}, \]

where \(Ei(.)\) is the exponential integral function defined by

\[ Ei(z) = - \int_{-z}^{\infty} \frac{e^{-t}}{t} \, dt \]

(for \(z > 0\)), where the principal value of the integral is taken.

It is important to study the characteristic - "supersaturation" of the model to the horizontal asymptote.

In this Section we prove upper and lower estimates for the one-sided Hausdorff approximation of the Heaviside step–function \(h_t^*(t)\) by means of families (5).

Without loss of generality, we consider the following class of this family for:

\[ t_0 = 0; \ y_0 = e^{\frac{ck}{m} Ei(-1)} \]

\[ M(t) = e^{\frac{ck}{m} Ei(-1 - mt)}. \]

The function \(M(t)\) and the "input function" \(s(t)\) are visualized on Fig. 1.

Denoting by \(t^*\) the unique positive solution of the nonlinear equation:
The one-sided Hausdorff distance $d$ between the function $h_{t^*}(t)$ and the sigmoid (6) satisfies the relation

$$M(t^* + d) = 1 - d.$$  \hspace{1cm} (8)

The following theorem gives upper and lower bounds for $d$

**Theorem 1.** Let

$$\alpha = -\frac{1}{2},$$

$$\beta = 1 + \frac{k}{2} \frac{1}{1 + mt^*} e^{-mt^*},$$

$$\gamma = 2.1 \beta.$$  \hspace{1cm} (9)

For the one-sided Hausdorff distance $d$ between $h_{t^*}(t)$ and the sigmoid (6) the following inequalities hold for the condition: $\gamma > e^{1.05}$:

$$d_l = \frac{1}{\gamma} < d < \frac{\ln \gamma}{\gamma} = d_r.$$  \hspace{1cm} (10)

**Proof.** Let us examine the function:
Figure 2: The functions $F(d)$ and $G(d)$ for $k = 100; m = 10$.

\[ F(d) = M(t^* + d) - 1 + d. \]  \hfill (11)

From $F'(d) > 0$ we conclude that function $F$ is increasing.

Consider the function

\[ G(d) = \alpha + \beta d. \]  \hfill (12)

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \rightarrow 0$ as $O(d^2)$ (see Fig. 2). In addition $G'(d) > 0$.

Further, for $\gamma > e^{1.05}$ we have

\[ G(d_l) < 0; \quad G(d_r) > 0. \]

This completes the proof of the theorem.

Approximations of the $h_{t^*}(t)$ by model (6) for various $k$, $m$ and $L$ are visualized on Fig. 3–Fig. 4.

4 Some applications

The proposed model can be successfully used to approximating data from Population Dynamics, Debugging Theory and Theory of Computer Viruses Propagation.
Figure 3: The model (6) for $k = 100; m = 10; \ t^* = 0.148284$; Hausdorff distance $d = 0.128078; \ d_l = 0.0854317; \ d_r = 0.210165$.

Figure 4: The model (6) for $k = 100; m = 15; \ t^* = 0.0786205$; Hausdorff distance $d = 0.0986342; \ d_l = 0.059118; \ d_r = 0.1672$. 

40
4.1 Approximating the “data on the development of the Drosophila melanogaster population”

We will illustrate the advances of the solution $y(t)$ for approximating and modelling of “data on the development of the Drosophila melanogaster population”, published by biologist Raymond Pearl in 1920 (see, also [17]).

We consider the following data:

\[
data_{\text{Pearl}} := \{(9, 39), (12, 105), (15, 152), (18, 225), (21, 390), (25, 547), (29, 618), (33, 791), (36, 877), (39, 938)\}.
\]

After that using the model

\[
M^*(t) = \omega e^{\frac{t}{m}} Ei(-1 - mt)
\]

for $\omega = 1040.42$, $k = 2.37757$ and $m = 0.09$ we obtain the fitted model (see, Fig. 5).

4.2 Approximating the data_Storm_Identifications [48], [47]

Storm worm was one of the most biggest cyber threats of 2008. In [48] are noticed particular periods during which their Storm specimen published different IDs every 10 minutes, that behavior cannot account for the very large number of IDs.

We analyze the following data:

\[
data_{\text{Storm\_Identifications}} := \\
\{(1, 0.843), (4, 0.926), (5, 0.954), (6, 0.967), (7, 0.976), (8, 0.981), (9, 0.985), (10, 0.991), (12, 0.995), (38, 0.997), (51, 0.998), (64, 0.9985), (74, 0.999), (83, 1), (100, 1)\}.
\]
After that using the model $M^*(t)$ for $\omega = 1, k = 0.0583363$ and $m = 0.169$ we obtain the fitted model (see, Fig. 6).

4.3 Application of the new cumulative sigmoid for analysis of the ”cancer data”

We will illustrate the advances of the modified inverse Rayleigh cumulative sigmoid for approximation and modelling of ”cancer data” (for some details see, [49]–[50]).

| days | 4   | 7   | 10  | 12  | 14  | 17  | 19  | 21  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|
| $R(t)$ | 0.415 | 0.794 | 1.001 | 1.102 | 1.192 | 1.22 | 1.241 | 1.3 |

Table 1: The ”cancer data” [49]–[50]

The model $M^*(t)$ based on the data from Table 1 for the estimated parameters:

$$\omega = 1.38611; \ m = 0.13; \ k = 0.522489$$
As should be expected, the experiments conducted (see, Sections 4.1 - 4.3) show a very good approximation of data from the field of population dynamics and computer viruses propagation, with suggested in this article, modified logistic model.

5 Conclusion.

A special choice of nutrient supply for cell growth in a continuous bioreactor is introduced.

We prove upper and lower estimates for the one-sided Hausdorff approximation of the shifted Heaviside function $h_{t^*}(t)$ by means of the general solution of the differential equation $y'(t) = ky(t)s(t)$ with $y(t_0) = y_0$, where $s(t) = \frac{1}{1+mt}e^{-mt}$.

We propose a software module within the programming environment CAS Mathematica for the analysis of the considered family of functions.
The module offers the following possibilities:
- calculation of the H-distance between the \( h_{t^*} \) and the model \( M(t) \) (6);
- generation of the functions under user defined values of the parameters \( k \) and \( m \);
- numerical solution of the differential model (2) and opportunities for comparison with other logistics models;
- software tools for animation and visualization.

**Acknowledgment**
This paper is supported by the National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)", financed by the Ministry of Education and Science.
References

[1] P. Pathi, T. Ma, R. Locke, Role of nutrient supply on cell growth in bioreactor design for tissue engineering of hematopoietic cells, *Biotechnology and Bioengineering*, **89**, No. 7 (2005).

[2] N. Li, H. Y. Sun, Q. L. Zhang, The dynamics and bifurcation control of a singular biological economic model, *Int. J. of Automat. and Comp.*, **9**, No. 1 (2012), 1–7.

[3] T. Egli, M. Zinn, The concept of multiple–nutrient–limited growth of microorganisms and its application in biotechnological processes, *Biotechnol. Adv.*, **22**, 2003, 5–43.

[4] D. F. Gerson, M. M. Kole, B. Ozum, M. N. Oguztoreli, Substrate concentration control in bioreactors, *Biotechnology and Genetic Engineering Reviews*, **6** (1), 1988, 67–150.

[5] Y. Zhang, Q. L. Zhang, L. C. Zhao, Analysis and feedback control for a class of bioeconomic systems, *J. Control Eng. of China*, **14**, No. 6 (2007), 599–603.

[6] Y. Zhang, Q. L. Zhang, L. C. Zhao, Bifurcations and control in singular biological economic model with stage structure, *J. of Syst. Eng.*, **22**, No. 3 (2007), 233–238.

[7] S. Luan, B. Liu, L. X. Zhang, Dynamics on a single-species model in a polluted environment, *J. of Biomath.*, **26**, No. 4 (2011), 689–694.

[8] V. Noris, E. Maurice, D. Maurice, Modelling Biological Systems with Competitive Coherence, *J. of Appl. Math.*, (2012), 1–20.

[9] M. Borisov, N. Dimitrova, V. Beshkov, Stability analysis of a bioreactor model for biodegradation of xenobiotics, *Comp. and Math. with Appl.*, **64**, (2012).

[10] N. Dimitrova, Local bifurcations in a nonlinear model of a bioreactor, *Serdica J. Computing*, **3** (2009), 107–132.
[11] T. Ivanov, N. Dimitrova, Analysis of a Bioreactor Model with Microbial Growth Phases and Spatial Dispersal, *Biomath Communications*, 2, No. 2 (2016).

[12] N. Kyurkchiev, S. Markov, G. Velikova, The dynamics and control on a singular bio-economic model with stage structure, *Biomath Communications*, 2, No. 2 (2015).

[13] N. Kyurkchiev, Investigations on a hyper-logistic model. Some applications, *Dynamic Systems and Applications*, 28, No. 2 (2019), 351–369.

[14] E. Angelova, A. Golev, T. Terzieva, O. Rahneva, A study on a hyper–power–logistic model. Some applications, *Neural, Parallel and Scientific Computations*, 27 No. 1 (2019), 45–57.

[15] N. Kyurkchiev, A. Iliev, A. Rahnev, On a special choice of nutrient supply for cell growth in a continuous bioreactor. Some modeling and approximation aspects, *Dynamic Systems and Applications*, 2019 (to appear).

[16] N. Kyurkchiev, A. Iliev, A. Rahnev, A special choice of nutrient supply for cell growth in logistic differential model. Some applications, *Proc. of the NTADES Series of AIP*, 2019 (to appear).

[17] W. Alpatov, R. Pearl, Experimental studies on the duration of life. XII. Influence of temperature during the larval period and adult life on the duration of the life of the imago of Drosophila melanogaster, *Am. Nat.*, 69, No. 63 (1929), 37–67.

[18] F. Hausdorff, *Set Theory*, 2nd ed., Chelsea Publ., New York (1962).

[19] G. Lente, *Deterministic kinetics in chemistry and systems biology*, In: Briefs in Molecular Science, Cham Heidelberg New York Dordrecht London: Springer, (2015).

[20] J. D. Murray, *Mathematical Biology: I. An Introduction*, 3rd ed., New York Berlin Heidelberg, Springer-Verlag, (2002).
[21] V. Chellaboina, S. P. Bhat, W. M. Haddat, D. S. Bernstein, Modeling and analysis of mass-action kinetics, *IEEE Contr. Syst. Mag.*, **29** (2009), 60–78.

[22] S. Markov, Reaction Networks Reveal New Links Between Gompertz and Verhulst Growth Functions, *Biomath*, **8**, No. 1 (2019). (to appear)

[23] N. Kyurkchiev, S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function, *J. Math. Chem.*, **54**, No. 1 (2016), 109–119.

[24] N. Kyurkchiev, S. Markov, *Sigmoid functions: Some Approximation and Modelling Aspects*, LAP LAMBERT Academic Publishing, Saarbrucken (2015), ISBN 978-3-659-76045-7.

[25] S. Markov, Reaction Networks Reveal New Links Between Gompertz and Verhulst Growth Functions, *Biomath*, **8**, No. 1 (2019).

[26] N. Kyurkchiev, A. Iliev, S. Markov, *Some techniques for recurrence generating of activation functions*, LAP LAMBERT Academic Publishing (2017), ISBN: 978-3-330-33143-3.

[27] R. Anguelov, N. Kyurkchiev, S. Markov, Some properties of the Blumberg’s hyper-log-logistic curve, *Biomath*, **7** No. 1 (2018), 8 pp.

[28] A. Iliev, N. Kyurkchiev, S. Markov, On the Approximation of the step function by some sigmoid functions, *Mathematics and Computers in Simulation*, **133** (2017), 223–234.

[29] A. Iliev, N. Kyurkchiev, S. Markov, Approximation of the cut function by Stannard and Richards sigmoid functions, *IJPAM*, **109** No. 1 (2016), 119–128.

[30] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A note on the Log-logistic and transmuted Log-logistic models. Some applications, *Dynamic Systems and Applications*, **27** No. 3 (2018), 593–607.
[31] S. Markov, N. Kyurkchiev, A. Iliev, A. Rahnev, On the approximation of the cut functions by hyper-log-logistic function, *Neural, Parallel and Scientific Computations*, **26** No. 2 (2018), 169–182.

[32] N. Kyurkchiev, A Note on the Volmer’s Activation (VA) Function, *C. R. Acad. Bulg. Sci.*, **70**, No. 6 (2017), 769–776.

[33] N. Kyurkchiev, A note on the new geometric representation for the parameters in the fibril elongation process. *C. R. Acad. Bulg. Sci.*, **69**, No. 8 (2016), 963–972.

[34] N. Kyurkchiev, S. Markov, On the numerical solution of the general kinetic ”K-angle” reaction system, *Journal of Mathematical Chemistry*, **54**, No. 3 (2016), 792–805.

[35] N. Guliyev, V. Ismailov, A single hidden layer feedforward network with only one neuron in the hidden layer san approximate any univariate function, *Neural Computation*, **28** (2016), 1289–1304.

[36] D. Costarelli, R. Spigler, Constructive Approximation by Superposition of Sigmoidal Functions, *Anal. Theory Appl.*, **29** (2013), 169–196.

[37] B. I. Yun, A Neural Network Approximation Based on a Parametric Sigmoidal Function, *Mathematics*, **7** (2019), 262.

[38] N. Kyurkchiev, Comments on the Yun’s algebraic activation function. Some extensions in the trigonometric case, *Dynamic Systems and Applications*, (2019) (to appear).

[39] N. Kyurkchiev, A. Iliev, *Extension of Gompertz-type Equation in Modern Science: 240 Anniversary of the birth of B. Gompertz*, LAP LAMBERT Academic Publishing (2018), ISBN: 978-613-9-90569-0.
[40] S. Markov, N. Kyurkchiev, A. Iliev, A. Rahnev, On the approximation of the generalized cut functions of degree $p+1$ by smooth hyper-log-logistic function, Dynamic Systems and Applications, 27 No. 4 (2018), 715–728.

[41] A. Malinova, O. Rahneva, A. Golev, V. Kyurkchiev, Investigations on the Odd-Burr-III-Weibull cumulative sigmoid. Some applications, Neural, Parallel, and Scientific Computations, 27, No. 1 (2019), 35–44.

[42] N. Kyurkchiev, A. Iliev, A. Rahnev, A new class of activation functions based on the correcting amendments of Gompertz-Makeham type, Dynamic Systems and Applications, 28, No. 2 (2019), 243–257.

[43] R. Anguelov, M. Borisov, A. Iliev, N. Kyurkchiev, S. Markov, On the chemical meaning of some growth models possessing Gompertzian-type property, Math. Meth. Appl. Sci., (2017), 12 pp.

[44] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, Some software reliability models: Approximation and modeling aspects, LAP LAMBERT Academic Publishing, (2018), ISBN: 978-613-9-82805-0.

[45] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, Nontrivial Models in Debugging Theory (Part 2), LAP LAMBERT Academic Publishing, (2018), ISBN: 978-613-9-87794-2.

[46] N. Kyurkchiev, A. Iliev, A. Rahnev, Some Families of Sigmoid Functions: Applications to Growth Theory, LAP LAMBERT Academic Publishing, (2019), ISBN: 978-613-9-45608-6.

[47] A. Iliev, N. Kyurkchiev, A. Rahnev, T. Terzieva, Some models in the theory of computer viruses propagation, LAP LAMBERT Academic Publishing, (2019), ISBN: 978-620-0-00826-8.
[48] S. Sarat, A. Terzis, HiNRG Technical Report: 01-10-2007 Measuring the Storm Worm Network, (2007).

[49] M. Vinci, S. Gowan, F. Boxall, L. Patterson, M. Zimmermann, W. Court, C. Lomas, M. Mendila, D. Hardisson, S. Eccles, Advances in establishment and analysis of three-dimensional tumor spheroid-based functional assays for target validation and drug evaluation, *BMC Biology*, **10** (2012).

[50] A. Antonov, S. Nenov, T. Tsvetkov, Impulsive controllability of tumor growth, *Dynamic Systems and Appl.*, **28**, No. 1 (2019), 93–109.