Spin-Domain Formation in Antiferromagnetic Condensates

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Antiferromagnetic condensates are generally believed not to display modulational instability and subsequent spin-domain formation. Here we demonstrate that in the presence of a homogeneous magnetic field antiferromagnetic spin-1 Bose-Einstein condensates can undergo spatial modulational instability followed by the subsequent generation of spin domains. Employing numerical simulations for realistic conditions, we show how this novel effect can be observed in sodium condensates confined in an optical trap. Finally, we link this instability and spin-domain formation with stationary modes of the condensate.

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I. INTRODUCTION

The appearance of spin degrees of freedom in atomic matter waves opens up possibilities for new phenomena such as spin waves [1], spontaneous magnetization [2] and spin mixing [3]. However, perhaps the most intriguing effect is associated with complex patterns, such as spin textures [4] or domains [5], which may appear either as stationary low-energy states or emerge spontaneously due to condensate instabilities. Pattern formation is a common feature in the dynamics of extended nonlinear systems ranging from optics [6] to fluids [7]. Such patterns often develop through the exponential growth of unstable spatial modulations, known as modulational instability. In the spinor condensates we have the opportunity to examine such effects in an environment which is remarkably easy to control and manipulate, simply through the addition of an external magnetic field.

The origin of the intriguing physics of spinor condensates lies in the spin interaction between atoms, which allows for an exchange of atoms between different spin components. The parametric nature of this interaction mirrors similar effects observed in nonlinear optics, where the interaction of several optical modes may lead to the development of new frequencies [6]. Of particular interest to our case is the possibility that instabilities of an intense light beam may occur even when the wave is coupled to a spatially stable eigenmode and propagates in the normal-dispersion regime [8]. Of particular interest to our case is the possibility that instabilities of an intense light beam may occur even when the wave is coupled to a spatially stable eigenmode and propagates in the normal-dispersion regime [8]; in this case the interplay of natural and self-induced birefringence leads to nonlinear polarization symmetry breaking and polarization modulational instability. By analogy we thus might expect similar instabilities in an initially stable polar condensate subjected to additional spin component coupling through an external magnetic field.

In the absence of an external magnetic field, the development of spatial modulational instability in three-component (or spin-1) ferromagnetic condensates and the subsequent formation of spin domains has been well established both theoretically [9,10,11] and experimentally [12]. However, early work on the zero field case [9] determined that antiferromagnetic (or polar) condensates are modulationally stable. Experimental observations suggested this is also true for a weak magnetic field, however these experiments were carried out with a condensate smaller than a spin domain [13].

In this paper we reveal that in fact the presence of a weak magnetic field (∼175 mG) leads to spin domain formation in antiferromagnetic condensates, provided the condensate is larger than the spin healing length. Furthermore we show that this spin domain formation is initiated by a new type of modulational instability, reminiscent of instabilities observed in nonlinear optics [8] and not seen before in Bose-Einstein condensates. While spin-domain formation in antiferromagnetic condensates has been observed before in the presence of a magnetic field gradient [5], we show here that it occurs equally well in the presence of a homogeneous magnetic field. Furthermore we reveal that this modulational instability and spontaneous spin-domain formation is associated with stationary states which exist in the presence of the weak magnetic field, and which intrinsically break the validity of the single-mode approximation (as seen earlier in [14]). We discuss realistic experimental conditions for the observation of these novel effects.

The paper is organized as follows. Section II introduces a theoretical model of spin-1 condensate in a homogeneous magnetic field. In Sec. III we investigate homogeneous stationary states in magnetic field and analyze their stability with respect to plane wave perturbations (modulational stability). Section IV presents results of numerical simulations corresponding to experimentally relevant condensate evolution, demonstrating the possibility of observation of new instability in antiferromagnetic condensate. In Sec. V we link this instability and spin-domain formation with stationary modes of the condensate, and Sec. VI concludes the paper.

II. MODEL

The evolution of a dilute spin-1 (F = 1) Bose-Einstein condensate (BEC) in a homogeneous magnetic field is
given by the coupled Gross-Pitaevskii equations,
\[
\begin{align*}
    i\hbar \frac{\partial \Psi_+}{\partial t} &= [\mathcal{L} + \tilde{c}_2(n_+ + n_0 - n_-)]\Psi_+ + \tilde{c}_2\Psi_0^*\Psi_+^* , \\
    i\hbar \frac{\partial \Psi_0}{\partial t} &= [\mathcal{L} - \delta E + \tilde{c}_2(n_+ + n_-)]\Psi_0 + 2\tilde{c}_2\Psi_+\Psi_-\Psi_0^*,
\end{align*}
\]
(1)
where \( \mathcal{L} = -\hbar^2\nabla^2/2m + \tilde{c}_0 n + V(r) \), \( n_j = |\Psi_j|^2 \), \( n = n_+ + n_0 + n_- \), and \( V(r) \) is an external potential. The nonlinear coefficients are: \( \tilde{c}_0 = 4\pi \hbar^2(2a_2 + a_0)/3m \) and \( \tilde{c}_2 = 4\pi \hbar^2(a_2 - a_0)/3m \). The total number of atoms \( N = \int |n(r)|^2 \text{d}r \) and the total magnetization \( M = \int [n_+(r)]^2 - [n_-(r)]^2 \text{d}r \) are conserved quantities. The Zeeman-energy shifts for each component can be calculated using the Breit-Rabi formula \( \frac{\hbar}{2} \)
\[
E_{\pm} = -\frac{1}{8}E_{\text{HFS}} \left(1 + 4\sqrt{1 \pm \alpha^2 + \alpha^2} \right) \mp g_I \mu_B B ,
\]
\[
E_0 = -\frac{1}{8}E_{\text{HFS}} \left(1 + 4\sqrt{1 + \alpha^2} \right),
\]
(2)
where \( E_{\text{HFS}} \) is the hyperfine energy splitting at zero magnetic field, \( \alpha = (g_I + g_J)\mu_B B/E_{\text{HFS}} \), \( \mu_B \) is the Bohr magneton, \( g_I \) and \( g_J \) are the gyromagnetic ratios of electron and nucleus. The linear part of the Zeeman effect does not affect the condensate evolution, except for a change in the relative phases \( \theta \) and so we remove it with the transformation \( \Psi_\pm \to \Psi_\pm \text{exp}(-i\mathbf{E}_{\pm}t) \), \( \Psi_0 \to \Psi_0 \text{exp}[-i(E_+ + E_-)t/2] \). We thus consider only the effects of the quadratic Zeeman shift, \( \delta E = (E_+ + E_- - 2E_0)/2 \approx \alpha^2E_{\text{HFS}}/16 \), which is always positive.

III. MODULATIONAL INSTABILITY OF HOMOGENEOUS STATIONARY STATES

First, we are interested in the stability analysis of the homogeneous condensate and consider the case of vanishing potential, \( V(r) = 0 \). We look for the homogeneous solutions in the form \( \psi_j = \sqrt{n_j}e^{i\mu_j t + i\theta_j} \). The “phase matching condition” for Eqs. (1) gives \( \mu_+ + \mu_- = 2\mu_0 \). We find that both in the case of \( B = 0 \) and in the case of \( M = 0 \), the steady state fulfills the stronger condition \( \mu_+ = \mu_- = \mu_0 \). However, if both magnetic field and magnetization are nonzero, which is the case in real experiments, the chemical potentials will be different, satisfying the less stringent phase matching condition.

We define the density fraction in each component as \( \rho_j = n_j/n_0 \). If we assume that all three spin components \( \rho_j \) are nonvanishing, the relative phase between them, \( \theta = 2\theta_0 - \theta_+ - \theta_- \), can take the value 0 or \( \pi \). We will describe the corresponding stationary states as phase-matched (\( \theta = 0 \)) and anti-phase-matched (\( \theta = \pi \)). Note that both types of states can exist in both ferromagnetic and polar condensates \( [16, 17, 18] \). However, phase-matched states are energetically favorable in ferromagnetic condensates, and anti-phase-matched states in polar condensates \( [14, 16] \). For that reason, they were named ferromagnetic and polar states respectively in Ref. \( [16] \). In Fig. 1 we present the existence diagram for three-component homogeneous stationary states. For generality we also include the results for ferromagnetic condensates. The ferromagnetic condensates, such as \( ^{87}\text{Rb} \), occur in the lower half (where \( c_2 \) is negative), while polar condensates, such as \( ^{23}\text{Na} \), occur in the upper half (\( c_2 \) is positive). There is clearly a region of coexistence of anti-phase-matched and phase-matched states for a polar condensate in nonzero magnetic field. In addition, a two-component solution with \( \rho_0 = 0 \), and one-component solutions with \( \rho_j = 1 \) exist. Our results are in agreement with the previous analysis of homogeneous ground states \( [14] \).

The energy density is related to the Hamiltonian of the system, from which Eqs. (1) are derived, by \( H = \int E\text{d}r \). In addition to the anti-phase-matched ground state \( [14] \), the polar condensate in the coexistence region of Fig. 1 has an excited phase-matched state corresponding to the energy maximum at \( \theta = 0 \). This state is stable with respect to spatially homogeneous spin mixing, because the possible dynamical trajectories in the \( (\rho_0, \theta) \) plane correspond to a constant energy value, hence both minima and maxima are stable.

The stability properties of these states change when we consider the possibility of a spatial, or modulational, instability (MI). We calculate the growth rate of the Bogoliubov modes \( [19] \),
\[
\psi_j = \sqrt{n_j} + u_j(t)e^{ikx} + v_j^*(t)e^{-ikx} \right] e^{i\mu_j t + i\theta_j}. 
\] (3)
After substituting the above to Eq. (1) we obtain a set of equations for the vector \( \mathbf{z} = (u_+, u_0, u_-, v_+^*, v_0^*, v_-^*) \).
TABLE I: Stability of spin-1 condensate states in absence and presence of the magnetic field: PM - phase-matched, APM - anti-phase-matched, X - state does not exist. 1A family of stationary states. 2Neutral stability with respect to spatially homogeneous spin mixing.

| condensate type | state type | $B = 0$ | $B = 0$ | $B \neq 0$ | $B \neq 0$ |
|-----------------|------------|---------|---------|-----------|-----------|
| ferro           | PM         | stable  | stable  | stable    | stable    |
|                 | APM        | unstable| unstable| X         | X         |
| $\rho_0 = 0$    | stable     | unstable| unstable| X         | unstable  |
| $\rho_0 = 1$    | stable     | X       | unstable| unstable  | unstable  |
| polar           | PM         | stable  | stable  | unstable  | unstable  |
|                 | APM        | stable  | X       | unstable  | unstable  |
| $\rho_0 = 0$    | stable     | unstable| stable  | X         | X         |
| $\rho_0 = 1$    | stable     | X       | X       | unstable  | unstable  |

In the form $\frac{d\mathbf{z}}{dt} = iA\mathbf{z}$, where $A$ is a 6th-rank matrix [20]. For an equilibrium state, it is possible to eliminate $\mu_1$ and $\delta E$ from $A$, expressing it in terms of $n_j$, $c_0$, $c_2$ and $k$ only for specific $\theta$. The Bogoliubov modes are the solutions of the characteristic equation $\det(A - \hbar \omega I) = 0$, with $\omega$ being the eigenfrequency of the excitation. The form of this equation is too cumbersome to present here, in contrast to the $B = 0$ case considered before [11]. In general, one has to use numerical methods to obtain a set of solutions.

The general numerical results are presented in a systematic way in Table II. These results have been calculated for the specific cases of the $^{87}$Rb condensate (upper half) and $^{23}$Na condensate (lower half) with scattering lengths given in [16] and hyperfine energy splitting given in [21]. We are considering the homogeneous case the result is applicable to one-, two- and three-dimensional condensates. We do not consider here the trivial case of $\rho_{\pm} = 1$ states, which have extremal value of magnetization $M = \pm N$. All the cases presented in this table were double checked by direct numerical integration of Eq (4) (see Sec. IV). The results cover the area of Fig. II and do not apply to the case of very high magnetic field, when the quadratic Zeeman energy dominates.

We find that our results are in agreement with the existing data for the vanishing magnetic field case. In Ref. [9] the authors found that all the condensate equilibrium states are stable, with the exception of one state in the ferromagnetic condensate, which corresponds to “ferro-APM”, $M = 0$ state in Table I. In Ref. [16], it was also found that ground states of both ferromagnetic and polar condensates (“ferro-PM” and “polar-APM”) are modulationally stable. The authors of Ref. [16] were considering mainly non-equilibrium (spin-mixing) states, but their general conclusion was that all polar condensate states are dynamically stable, and ferromagnetic are not. Here we report that in fact specific ferromagnetic condensate states are stable, while polar condensates may become unstable. Furthermore, as discussed below, we see that dynamic domain formation may occur for polar condensates, and lead to convergence to new stationary states.

As one can see in Table I, the magnetic field affects stability of polar condensates. We investigate this phenomenon in detail by calculating the growth rate $\kappa = \text{Im}(\omega)$ corresponding to unstable Bogoliubov modes. In Fig. II we present results for a “polar-PM” state of a sodium condensate as a function of the magnetic field strength. The growth rate is proportional to the square of the quadratic Zeeman shift, which is in turn proportional to the square of the magnetic field strength. Hence, one has to apply a relatively strong magnetic field to observe MI on a reasonable time scale. Another interesting feature is the range of wavevectors $k$ corresponding to unstable modes. In contrast to the typical case where this range starts from $k = 0$, here the unstable region begins at a nonzero minimum value. This type of “optical mode” branch has been reported before in the case of parametric optical solitons [20].

IV. DYNAMICS OF THE CONDENSATE IN A CIGAR-SHAPED TRAP

We now consider the implications of these results in the experimentally relevant case of a $^{23}$Na condensate localized in a cigar-shaped harmonic trap, $V(r) = \frac{1}{2}m\omega_y^2(y^2 + z^2) + \frac{1}{2}m\omega_z^2z^2$. Specifically we consider the case $\omega_z >\omega_y$ in which the Fermi radius of the transverse trapping potential is smaller than the spin healing length, and nonlinear energy scale is much smaller than the transverse trap energy scale, which allows us to reduce the problem to one spatial dimension [16, 22]. Following standard dimensionality reduction procedure we...
obtain the one-dimensional model,

\[ i\hbar \frac{\partial \psi_\pm}{\partial t} = \left(\hat{\mathcal{L}} + c_2(n_\pm + n_0 - n_\mp)\right) \psi_\pm + c_2 \psi_\mp^* \psi_\pm^*, \]

\[ i\hbar \frac{\partial \psi_0}{\partial t} = \left(\hat{\mathcal{L}} - \delta E + c_2(n_+ + n_-)\right) \psi_0 + 2c_2 \psi_+ \psi_- \psi_0^*, \]

(4)

where \( \hat{\mathcal{L}} = -(\hbar^2/2m)\partial^2/\partial x^2 + c_0 n + \frac{1}{2}m\omega_\parallel^2 x^2 \) and the interaction coefficients have been rescaled and now include the transverse trap frequency, \( c_0 = 4\hbar\omega_\perp (2a_2 + a_0)/3 \) and \( c_2 = 4\hbar\omega_\perp (a_2 - a_0)/3 \).

The experimental scenario we consider here consists of several phases. Initially, the condensate is prepared in the \( m = -1 \) ground state [12]. Next, short microwave field pulses are applied to transfer the atomic population to the desired state [23]. We consider two cases, a phase-matched state and an anti-phase-matched state. Initially, the condensate is prepared in the initial phase-matched state, \( \rho_\parallel,0,\pm = 0.351, 0.3, 0.349 \), and an anti-phase-matched state, \( \rho_\parallel,0,\mp = 0.4, 0.01, 0.59 \). Simultaneously, the magnetic field is set to the value of 175 mG [12]. The results of the corresponding numerical simulations of Eqs. (4) are presented in Fig. 3. The MI develops after tens of milliseconds, and leads rapidly to spin-domain formation. We see what appears to be initial oscillations, followed by instability dynamics leading to an apparent oscillating state. In the case of the phase-matched initial condition and absence of zero magnetization we ultimately see conversion to the \( \rho_\parallel,0,\mp = 0.351, 0.3, 0.349 \), and an anti-phase-matched state, \( \rho_\parallel,0,\pm = 0.4, 0.01, 0.59 \). The thin dotted line corresponds to the total condensate density \( n \), and the dashed, solid, and dotted lines correspond to \( n_+, n_0 \), and \( n_- \) respectively. Parameters are \( N = 3.7 \times 10^5 \), \( B = 175 \) mG, \( \omega_\perp = 2\pi \times 10^3 \) Hz, \( \omega_\parallel = 2\pi \times 32 \) Hz.

**V. STATIONARY SPINOR STATES**

The existence of MI suggests that domain-type stationary states may be expected in the trap, just as plane wave instability and solitons are typically found together in optics. Indeed, as can be seen in Fig. 4, we find that in the stationary picture the profiles always break the single-mode approximation for both the phase-matched and anti-phase-matched states (as found in Ref. [14] for the case of a anti-phase-matched state in a polar condensate). As predicted by the MI analysis, an initially smooth profile will therefore become modulated with the ensuing instability dynamics reflecting the nearby stationary state profiles. For instance comparing Fig. 3(b) and Fig. 4(a) we see that profiles similar to the stationary states appear in the evolution. Significantly, but unsurprisingly in a polar condensate, we find that the phase-matched state is generally unstable, while the anti-phase-matched state is stable. The stability of these states appears to reflect the ultimate dynamics of the condensate with the phase-matched state breaking up while the anti-phase-matched state has highly persistent spin domains. A complete analysis of the stability of the stationary states will be presented elsewhere.
VI. CONCLUSIONS

We have demonstrated that an antiferromagnetic spin-1 condensate can undergo a novel type of spatial modulational instability followed by subsequent spin-domain formation in the presence of a homogeneous magnetic field. We have employed realistic conditions to demonstrate, with the help of numerical simulations, that this novel modulational instability can be observed in a sodium condensate confined in an optical trap potential and that the ensuing instability dynamics connect with the stationary states in the trap.

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