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The charge-dyon bound system in the spherical quantum well

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The spherical wave functions of charge-dyon bounded system in a rectangular spherical quantum dot of infinitely and finite height are calculated. The transcendent equations, defining the energy spectra of the systems are obtained. The dependence of the energy levels from the wall sizes is found.

It is known \cite{1}, that the charge-dyon system possesses the hidden symmetry, and the group of hidden symmetry for discrete spectrum is $SO(4)$, and for the continuous spectrum is $SO(3,1)$. Let’s remind, that the dyon is a hypothetical particle, introduced by Schwinger \cite{2}, which serves as a source of both electrical and magnetic fields. The problem of charge-dyon system due to the hidden symmetry can be factorized not only in spherical but also in parabolic coordinates.

In the present article the charge-dyon bounded system is considered in a spherical quantum dot of finite height. As a cause of such investigation on the one hand served the recently established fact (named dyon-oscillator duality). According to this fact 4-dimensional isotropic oscillator is dual to the 3-dimensional charge-dyon system \cite{3-5}. However, the property of dyon-oscillator duality is inherent not only to $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ mapping, but to $\mathbb{R}^1 \rightarrow \mathbb{R}^1$, $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\mathbb{R}^8 \rightarrow \mathbb{R}^5$, mappings as well. In the first two cases one-dimensional and two-dimensional anyons are obtained at the output \cite{6, 7}, in third case non-Abelian Yang monopole arise \cite{8}.

On the other hand, due to the Hamiltonian of the system, factorable in the same coordinates, as in the case of the hydrogen-like system, it becomes possible to state that this Hamiltonian serves as a generalization of the hydrogen-like system in case of magnetic charge. From this point of view the considered problem is the generalization of the well-known problem of the hydrogen-like system behavior inside semiconductor quantum dots with different confinement potentials \cite{9-13}. In other words it should be expected, that in some cases energy levels of charge-dyon system are analogical to hydrogen-like levels inside quantum dot. In this connection let’s mention that the behavior of the hydrogen-like system in a rectangular spherical quantum dot of finite height has been investigated in \cite{9}.

Let’s consider the charge-dyon bounded system inside potential sphere with confinement potential, which looks like

$$U(r) = \begin{cases} 0, & r < r_0; \\ U_0, & r \geq r_0. \end{cases}$$

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The Schrödinger equation of such system can be written as
\[
\left( \frac{\partial}{\partial x_j} - i \frac{e}{\hbar c} A_j \right)^2 \psi + \frac{2\mu}{\hbar^2} \left[ E + \frac{e^2}{r} - \frac{\hbar^2 s^2}{2\mu r^2} - U(r) \right] \psi = 0,
\]
with the vector potential
\[
\vec{A} = \frac{g}{r(r+z)} (y,-x,0)
\]
which corresponds to the Dirac monopole [14] with the magnetic charge \( g = \frac{n e s}{c} \) \((s = 0, \pm \frac{1}{2}, \pm 1, \ldots)\), and singularity axis at \( z > 0 \).

We look for the solution of Eq. (1) in spherical coordinates as
\[
\psi(r, \theta, \varphi) = R(r) Z(\theta) e^{im\varphi} \sqrt{\frac{2}{\pi(\ell + 1)}},
\]
where \( m = -|s|, |s| + 1, \ldots, |s| - 1, |s| \). Now, inserting Eq. (2) into Eq. (1), which is the Schrödinger equation in spherical coordinates we come to the following pair of ordinary differential equations
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dZ}{d\theta} \right) - \left[ \frac{(m-s)^2}{2(1-\cos \theta)} + \frac{(m+s)^2}{2(1+\cos \theta)} \right] Z + \ell(\ell+1)Z = 0,
\]
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{\ell(\ell+1)}{r^2} R + \frac{2\mu}{\hbar^2} \left( E + \frac{e^2}{r} - U(r) \right) R = 0,
\]
where \( \ell(\ell+1) \) is a separation constant, and \( \ell \) is a non-negative integer or a half-integer which accepts the following values
\[
\ell = \frac{|m-s| + |m+s|}{2}, \frac{|m-s| + |m+s|}{2} + 1, \ldots.
\]
Fermion values of orbital moment are conditioned by the dyon magnetic charge.

The solution of Eq. (3) normalized by the condition
\[
\int_0^\pi Z_{\ell ms}(\theta) Z_{\ell' ms}(\theta) \sin \theta d\theta = \delta_{\ell\ell'}
\]
is the form
\[
Z_{\ell ms}(\theta) = \sqrt{\frac{2\ell + 1}{4\pi}} d_{\ell ms}(\theta),
\]
where \( d_{\ell ms}(\theta) \) is the Wigner function [15].

For the radial equation (4) we obtain two solutions, which satisfy standard conditions inside quantum dot \((r < r_0)\) and outside it \((r \geq r_0)\)
\[
R(r) = \begin{cases} 
  R^{(1)}_{k_1\ell}(r) &= C^{(1)}_{k_1\ell} e^{-\gamma_1 r} r^{\ell} F(\ell + 1 - k_1; 2\ell + 2; 2\gamma_1 r), & r < r_0; \\
  R^{(2)}_{k_2\ell}(r) &= C^{(2)}_{k_2\ell} e^{-\gamma_2 r} r^{\ell} U(\ell + 1 - k_2; 2\ell + 2; 2\gamma_2 r), & r \geq r_0,
\end{cases}
\]
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where

\[ \gamma_1^2 = -\frac{2\mu E}{\hbar^2}, \quad \gamma_2^2 = -\frac{2\mu(E-U_0)}{\hbar^2}, \quad k_1^2 = -\frac{\mu e^4}{2\hbar^2 E}, \quad k_2^2 = -\frac{\mu e^4}{2\hbar^2(E-U_0)}, \]

\( F(\alpha; \beta; x) \) and \( U(\alpha; \beta; x) \) are confluent hypergeometrical functions of the first and the second kind, consequently.

Radial wave function normality condition can be written as

\[
\int_0^{r_0} r^2 \left[ R_{k_1 \ell}^{(1)}(r) \right]^2 dr + \int_{r_0}^{\infty} r^2 \left[ R_{k_2 \ell}^{(2)}(r) \right]^2 dr = 1. \tag{6}
\]

From the continuity condition for the logarithmic derivative of the radial wave functions \( R_{k_1 \ell}^{(1)}(r) \) and \( R_{k_2 \ell}^{(2)}(r) \) in \( r = r_0 \) point

\[
\left( \frac{d \ln R_{k_1 \ell}^{(1)}(r)}{dr} \right)_{r=r_0} = \left( \frac{d \ln R_{k_2 \ell}^{(2)}(r)}{dr} \right)_{r=r_0}, \tag{7}
\]

we obtain, that \( C_{k_2 \ell} = C_{k_1 \ell} A_{k_1 k_2 \ell} \), where

\[
A_{k_1 k_2 \ell} = \frac{e^{(\gamma_2 - \gamma_1) r_0} F(\ell + 1 - k_1; 2 \ell + 2; 2 \gamma_1 r_0)}{U(\ell + 1 - k_2; 2 \ell + 2; 2 \gamma_2 r_0)}. \]

Then, using the normality condition (6), for \( C_{k_1 \ell}^{(1)} \) we obtain the expression

\[
C_{k_1 \ell}^{(1)} = \left[ I_{k_1 \ell}^{(1)} + A_{k_1 k_2 \ell} I_{k_2 \ell}^{(2)} \right]^{-1/2}, \]

where

\[
I_{k_1 \ell}^{(1)} = \int_0^{r_0} e^{-2 \gamma_1 r} r^{2 \ell+2} [F(\ell + 1 - k_1; 2 \ell + 2; 2 \gamma_1 r)]^2 \, dr,
\]

\[
I_{k_2 \ell}^{(2)} = \int_{r_0}^{\infty} e^{-2 \gamma_2 r} r^{2 \ell+2} [U(\ell + 1 - k_2; 2 \ell + 2; 2 \gamma_2 r)]^2 \, dr.
\]

Then wave functions (5) can be rewritten in the following form

\[
R(r) = \begin{cases} 
R_{k_1 \ell}^{(1)}(r) = C_{k_1 \ell}^{(1)} e^{-\gamma_1 r} F(\ell + 1 - k_1; 2 \ell + 2; 2 \gamma_1 r), & r < r_0; \\
R_{k_2 \ell}^{(2)}(r) = C_{k_1 \ell}^{(1)} A_{k_1 k_2 \ell} e^{-\gamma_2 r} U(\ell + 1 - k_2; 2 \ell + 2; 2 \gamma_2 r), & r \geq r_0.
\end{cases} \tag{8}
\]

Now, using the relationship (8) from the condition (7) we obtain a transcendental equation, defining the spectrum of the charge-dyon bounded system in the spherical quantum dot

\[
\gamma_1 - \frac{2 \gamma_1 (\ell + 1 - k_1) F(\ell + 2 - k_1; 2 \ell + 3; 2 \gamma_1 r_0)}{(2\ell + 2) F(\ell + 1 - k_1; 2 \ell + 2; 2 \gamma_1 r_0)} = \gamma_2 + \frac{2 \gamma_2 (\ell + 1 - k_2) U(\ell + 2 - k_2; 2 \ell + 3; 2 \gamma_2 r_0)}{U(\ell + 1 - k_2; 2 \ell + 2; 2 \gamma_2 r_0)}. \]
Let us note that the spectrum of the charge-dyon system is

\[ E_n = -\frac{\mu e^4}{2\hbar^2 n^2}, \]

where \( n = 1, 3/2, 2 \ldots \) is a principal quantum number, and the multiplicity of degeneration of the energy levels at fixed \( n \) and \( s \) is [5]

\[ g_n^s = (n - s)(n + s). \]

The curves of energy dependencies of the charge-dyon system in the spherical infinitely high potential well via well radius \( r_0 \) (in \( E_R \) and \( a_B \)) for \( \ell = 0; 0.5; 1; 1.5 \) cases are shown on Fig. 1. As it follows from this figure, at small \( r_0 \) the arrangement of the levels corresponds to the one, which takes place when single particle falls into infinitely potential well. The only difference is that besides the well-known levels, which correspond to the integer values of \( \ell \), intermediate levels with fermion values of \( \ell = 0.5; 1; 1.5 \) exist in this case. As is visible from Fig. 1 at small \( r_0 \) for given \( n_r = n - \ell - 1 \) they are arranged between levels with integer \( \ell \) as \( \ell \) increases. At the increase in \( r_0 \) charge-dyon interaction becomes more essential and levels pass to ones, which correspond to the free charge-dyon system. In other words the level degeneracy which corresponds to the charge-dyon free system is step by step reestablished at the increase in \( r_0 \). In particular, \( 2S \) level crosses the lowest level with \( \ell = 1.5 \) and joins \( 2P \) level on the same curve. In its turn the first level with \( \ell = 1 \) joins the second level \( \ell = 0.5 \). At that all these levels go down at increasing \( r_0 \).

On Fig. 2 the same \( E(r_0) \) dependence is shown for the case of the well of finite height \( (U_0 = 5E_R) \). In the case of large \( r_0 \) the behavior of curves is analogous with the one, which takes place at consideration of infinitely high well. As \( r_0 \) decreases at first the influence of walls rise in analogy with the upper considered case, that is why the levels are arranged as in infinitely high spherical well. However, at too small values of \( r_0 \) the role of the "first medium" \( (r < r_0) \) becomes inessential. In accordance with this fact the energy levels of the system pass to those, which take place for the charge-dyon system, but shifted up by the values \( U_0 \), i.e. \( E \to U_0 - E_R/n^2 \). It explains the joining of curves (e.g. \( 2P \) and \( 2S \)) at \( r_0 \to 0 \).

Thus, the presence of competition between the energy of the charge interaction with the well walls and energy of charge-dyon interaction is typical for both cases. At comparatively small radii of the well (we don’t mean the values \( r_0 \to 0 \)) the main contribution in the energy of the system is conditioned by repulsing potential of the well walls, that’s why the levels are positive. The charge-dyon interaction begins to play the main role at increasing \( r_0 \) and the levels become negative.

On Fig. 3 the curves of the observed system’s energy dependence via the height of the well \( E(U_0) \) are presented (in \( E_R \) units). As it should be expected, at increasing \( U_0 \) the levels go up. At that the lower levels, beginning from some \( U_0 \) almost don’t feel the enlargement of the well height.

Now let’s consider the point, the analogical to which plays an important role in the theory of impurity states in quantum dots. Usually, in solid state problems along with the energy
of impurity – quantum dot system one has to calculate the binding energy of the impurity inside quantum dot as well. This energy is defined as the difference between the energies of the free and impurity electron inside the quantum dot [10]. Translating it to the language of the problem under the consideration, it is necessary to find the difference between the energies, in one case defined by the Eq.(1), in other case defined by the analogical equation, but without Coulomb interaction term $e^2/r$. For that we find wave functions and energy spectrum of the equation

$$
\left( \frac{\partial}{\partial x_j} - i \frac{e}{\hbar c} A_j \right)^2 \psi^{(0)} + \frac{2\mu}{\hbar^2} \left[ E_0 - \frac{\hbar^2 s^2}{2\mu r^2} - U(r) \right] \psi^{(0)} = 0.
$$

(9)

The angular part of the Eq.(10) coincides with Eq.(3). For radial part we come to the equation

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR^{(0)}}{dr} \right) - \frac{\ell(\ell + 1)}{r^2} R^{(0)} + \frac{2\mu}{\hbar^2} \left( E_0 - U(r) \right) R^{(0)} = 0.
$$

(10)

The solutions of the Eq.(10) inside and outside the well are given by the following expressions

$$
R^{(0)}(r) = \begin{cases} 
R_{k_0\ell}^{(1)}(r) = C_{k_0\ell} j_{\ell}(k_0 r), & r < r_0; \\
R_{k\ell}^{(2)}(r) = A_{k\ell}C_{k\ell} \frac{1}{\sqrt{r}} K_{\ell + \frac{1}{2}}(kr), & r \geq r_0,
\end{cases}
$$

where $k_0 = \sqrt{2\mu E/\hbar^2}$, $k = \sqrt{2\mu(U_0 - E)/\hbar^2}$, $j_{\ell}(x)$ is Bessel spherical function, $K_{\ell + \frac{1}{2}}(x)$ is McDonald function, and

$$
A_{k_0\ell} = \frac{\sqrt{r_0} j_{\ell}(k_0 r_0)}{K_{\ell + \frac{1}{2}}(k r_0)}, \quad C_{k_0\ell}^2 = \frac{1}{I_{k_0\ell} + A_{k\ell}^2 I_{k\ell}},
$$

$$
I_{k_0\ell} = \int_0^{r_0} r^2 \left[ j_{\ell}(k_0 r) \right]^2 dr, \quad I_{k\ell} = \int_{r_0}^{\infty} r^2 \left[ K_{\ell + \frac{1}{2}}(kr) \right]^2 dr.
$$

From the continuity condition of logarithmical derivatives of $R_{k_0\ell}^{(1)}$ and $R_{k\ell}^{(2)}$ in $r = r_0$ point we come to the transcendent equation, which defines the energy spectrum $E_0$ of an electron, described by the Eq.(9).

$$
\frac{j_{\ell}(k_0 r_0)}{j_{\ell}(k_0 r_0)} = \frac{\left( \frac{1}{\sqrt{r}} K_{\ell + \frac{1}{2}}(k r_0) \right)'}{\frac{1}{\sqrt{r}} K_{\ell + \frac{1}{2}}(k r_0)}.
$$

On Fig.4 the curves of electron energy $E_0$ dependencies via well radius $r_0$ are shown for the values $\ell = 0; 0.5; 1; 1.5$. Let’s mention at once, that the appearance of levels has threshold character, i.e. at increasing well height $U_0$ other new levels arise. The arrangement of levels is analogical to the one, which takes place for single particle in spherical well of finite depth. As $r_0$ increases all levels come closer to each other and go lower. Here due to the presence
of magnetic charge besides the well-known levels with integer values of $\ell$ intermediate levels with half-integer values of $\ell$ arise as well.

Now, defining the analogy to the binding energy of charge-dyon system in quantum dot as the $E_b = E_0 - E$ difference, one can plot $E_b(r_0)$ for different values of $\ell$. This dependence is presented on Fig.5. According to this figure, at large $r_0$, when the influence of well walls is inessential, $E_b$ levels by absolute value tend to ones, which are typical for the free system of charge-dyon. In particular the ground level (continuous curve) tends to $E_R$ value. At decreasing $r_0$ the influence of walls increases, which results in increasing levels. After the levels achieved their maximal value, the levels begin to go down, since the influence of well walls at to small radii begins to weaken. It looks like the particle jumps out from the well. Let’s mention, that the analogical situation is observed in solid state problems (see, e.g. [16, 17]).

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![Figure 1](image1.png)

Figure 1: The dependence of the full energy of the charge-dyon system via the radius of infinitely high quantum dot for $\ell = 0; 0.5; 1; 1.5$ cases (left figure).

![Figure 2](image2.png)

Figure 2: The dependence of the full energy of the charge-dyon system via the radius of quantum dot of finite height for $\ell = 0; 0.5; 1; 1.5$, $U_0 = 5E_R$ cases (right figure).
Figure 3: The dependence of the full energy of the charge-dyon system via the height of quantum dot $U_0$ for $\ell = 0; 0.5; 1; 1.5$, $r_0 = 3a_B$ cases (left figure).

Figure 4: The dyon energy dependence via the radius of quantum dot for $\ell = 0; 0.5; 1; 1.5$, $U_0 = 5E_R$ cases (right figure).

Figure 5: The charge-dyon system’s binding energy dependence via the radius of quantum dot for $\ell = 0; 0.5; 1; 1.5$, $U_0 = 5E_R$, $n = 1$ cases.
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