To the question of calculation of amendments to the Navier – Stokes equation

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Abstract. In the paper provides a detailed conclusion of the Navier-Stokes equation using the classic Boltzmann equation to the third order in the free length path of the molecules. It is noted that the biharmonic component of the Laplace’s operator will play an important role in the research of wrap flow of fluid of nanoparticles.

1. Introduction
The challenge for this article is to solve the problem of calculating additional amendments to the Navier-Stokes equation in the form of a number of Knudsen numbers. In a lot of monographs and original articles, in one way or another related to the hydrodynamic theory of liquids and gases that we have previously reviewed (as example see [1-7]), we have not found a solution of this problem. We are talking about calculating additional amendments to the right side of the Navier-Stokes equation in the form of additives by the number of Knudsen, defined in a standard way, as \(Kn = \frac{l}{L}\), where \(l\) – is a free length path of molecules of liquid, \(L\) – is some characteristic linear size.

Interest in this issue is quite objective, which is connected with the fact that recently quite fashionable objects of research are nanoparticles, the size of which lies in the range \(10^{-6} - 10^{-5}\) cm. It is quite clear that when it comes to the size of this order, the classic Navier-Stokes equation should be modified somewhat, since in this case the free length path of molecules of liquid (or gas) will be comparable to the linear size of the nanoparticle.

To solve the problem, we will use a well-tested, both theory and practice, by the method of Boltzmann's kinetic equation (see refs [8, 9]). With it, we will give a detailed conclusion of the basic equation of hydrodynamics, in which we take into account all the additional components of the accuracy of the members of the order \(\tau^3_p\) on relaxation time \(\tau_p\), where \(p\) – is a momentum of molecule.

2. The conclusion of the Navier-Stokes equation, taking into account the amendments to Knudsen's number
Let's write down Boltzmann's classic kinetic equation in the traditional form [8]
where \( f = f(t, p, r) \) is the distribution function you're in the same place, \( \mathbf{v} \) is a speed of molecules, \( \mathbf{F} \) is a force acting on molecules, which in our case is equal to zero. \( L(f) \) is a collision integral. His explicit expression for Maxwell's gas can be recorded as in [9]

\[
L(f) = n \sum [f - v_i] \sigma \left[ f(v) f(v_i) - f(v') f(v_i) \right],
\]

where \( n \) is a concentration of molecules, \( \sigma \) is a cross section of scattering. Note that we will no longer need a clear kind of this expression. As part of the task we are up to, we can take advantage of the approach of relaxation time (so-called tau-approximation [8]) and in the accordance with [8] will find the decision as a raw

\[
f = f_0 + f_1 + f_2 + ... ,
\]

where the quasi-equal distribution function

\[
f_0 = \frac{1}{Z} \exp \left( -\frac{\varepsilon(p) - p\mathbf{V}}{T} \right). \tag{3}
\]

Note that the series (2) is a decomposition by the parameter \( \frac{l}{L} \), where \( l \) is the free length path of molecules, and \( L \) is some characteristic size (if considering, for example, the wrapping of the ball, it can be considered equal to the radius of the ball \( R \)). The standard multiplier in (3) is

\[
Z = \int f_0 d\Gamma = \int \exp \left( -\frac{\varepsilon(p)}{T} \right) d\Gamma, \tag{4}
\]

where \( d\Gamma = d^3 p d\mathbf{V} \) is an element of phase volume [10], equilibrium distribution function is \( f = f_0|_{\mathbf{V}=0} \), \( \varepsilon(p) = \frac{p^2}{2m} \) - kinetic energy of molecule, \( m \) is its mass, and integration is carried out throughout the momentum space, the volume element of which is defined as \( d^3 \mathbf{p} = dp_x dp_y dp_z \).

\( d\mathbf{V} = dx dy dz \) - volume element in Cartesian coordinates. The permanent Boltzmann \( k_B \) here and everywhere further will be believed to be equal unit, the vector \( \mathbf{V} = \mathbf{V}(t, \mathbf{r}) \) is the speed of hydrodynamic flow, which fascinates liquid molecules, functions \( f_1, f_2, f_3, ... \) are the sought-after amendments to the quasi-equal distribution function that we should find. The temperature \( T \) is considered constant throughout the area of space. To solve the problem, the right part of the equation (1) is convenient to present as a series

\[
L(f) = -\frac{f_0 + f_1 + f_2 + ...}{\tau_p}, \tag{5}
\]

where \( \tau_p \) - the time of molecules collisions. In this decision of the amendment \( f_1, f_2, ... \) as well as in (2), there are the values of the order of only even degrees, that is \( f_1 \sim \left( \frac{l}{L} \right)^2, f_2 \sim \left( \frac{l}{L} \right)^4, ... \) (see calculations below). Before looking for amendments to (2), we should write down the general
principle of obtaining motion equations for a case where the temperature \( T \neq 0 \). If the temperature were zero, the motion equation would be easy to obtain from the principle of maintaining the full power of the system, similar to how it was done, for example, in the work [7], that is, based on the condition

\[ \dot{E} + \dot{Q} = 0, \] (6)

where the full energy of the liquid flow has the appearance

\[ E = \frac{1}{Z} \int \left[ P + \varepsilon(p) + \frac{mV^2}{2} \right] f d\Gamma, \]

where \( P - \) is a pressure. Therefore, the change in energy per unit of the time will be

\[ \dot{E} = \frac{1}{Z} \int \varepsilon(p) \dot{f} d\Gamma + \frac{m}{Z} \int \nabla \dot{\mathbf{v}} d\Gamma + \frac{1}{Z} \int (\nabla \cdot \mathbf{V}) f d\Gamma. \] (7)

and dissipative function

\[ \dot{Q} = TS, \] (8)

where \( S - \) is an entropy. I.e. at \( T = 0 \) it simply disappears. In the case when \( T \neq 0 \) the equation (6) don’t “work” and we have to write down the equation in the form of

\[ \dot{F} = \frac{d}{dt}(E - TS) = 0, \] (9)

where \( F = E - TS \) is a Gibbs’s free energy. At \( T = \text{const} \) we have hence

\[ \dot{E} - \dot{Q} = 0. \] (10)

In the according with [10], entropy of the nonequilibrium classical Boltzmann’s gas can record in the following

\[ S = -\frac{1}{Z} \int f \ln \left( \frac{f}{e} \right) d\Gamma. \] (11)

Substituting definition (11) in (8), we get

\[ \dot{Q} = -\frac{T}{Z} \int \dot{f} \ln f d\Gamma. \] (12)

Lowering while the last amount in (7), which we take into account at the very end of all calculations, and substituting in account this remark (7) and (12) in (10), we find

\[ \frac{1}{Z} \int \left[ \varepsilon(p) + T \ln f \right] \dot{f} d\Gamma + \frac{m}{Z} \int \nabla \dot{\mathbf{v}} d\Gamma = 0. \] (13)

Following (5), we have

\[ \dot{f} = L(f) = -\frac{f_0 + f_1 + f_2 + ...}{\tau_p} \]

And so from (13) we get
\[
\frac{1}{Z}\int \left[ \epsilon(p) + T \ln(f_0 + f_1 + f_2 + \ldots) \right] \frac{(f_0 + f_1 + f_2 + \ldots)}{\tau_p} d\Gamma + \frac{m}{Z} \int \mathbf{V} \mathbf{V} d\Gamma = 0. \tag{14}
\]

Similar to how this was done, for example, in the work [11] recurrent formula for determining the arbitrary correction of the order \( n \) to the quasi-equilibrium distribution function can be presented in the form of

\[
f_n = (-1)^n \tau_p^n \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f_0. \tag{15}
\]

As can be seen from (15), in a stationary case \( \frac{\partial f}{\partial t} = 0 \) all even degree amendments will be of order \( \left( \frac{1}{L} \right)^2 \) and so on what we mentioned above. More detail it can be learn in the article [11]. To decompose the logarithm in (13), can use the following ratio

\[
\ln(1 + \lambda) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\lambda^k}{k}. \tag{16}
\]

For our particular case, we will be interested only in the solution to the accuracy \( n = 2 \) in the formula (15). Hence

\[
f_1 = -\tau_p \left( \dot{f}_0 + \mathbf{v} \cdot \nabla f_0 \right), \tag{17}
\]

\[
f_2 = \tau_p^2 \left( \dot{f}_0 + 2\mathbf{v} \cdot \nabla f_0 + (\mathbf{v} \cdot \nabla)^2 f_0 \right), \tag{18}
\]

where the points under function \( f_0 \) mean private derivatives at the time of the appropriate order. By compiling relative values \( \frac{f_1}{f_0} \) and \( \frac{f_2}{f_0} \) with account of (17), (18) and (3), after trivial actions we find

\[
\frac{f_1}{f_0} = \frac{\tau_p}{T} \left( \mathbf{p} \cdot \mathbf{V} + \mathbf{v} \cdot \nabla (\mathbf{p} \cdot \mathbf{V}) \right), \tag{19}
\]

As part of our task, the expression (19) must be recorded with precision to the members of the order \( \mathbf{V}^2 \). Therefore, according to (3) approximately we have

\[
f_0 = \frac{1}{Z} \exp\left( -\frac{\epsilon(p) - \mathbf{p} \mathbf{V}}{T} \right) = \frac{1}{Z} \exp\left( -\frac{\epsilon(p) - \mathbf{V}}{T} \right) \cdot \left( 1 + \frac{\mathbf{p} \cdot \mathbf{V}}{T} \right) = \tilde{f} \left( 1 + \frac{\mathbf{p} \cdot \mathbf{V}}{T} \right). \tag{20}
\]

Indeed, because the flow rate is relatively low, the fractional expression \( \frac{\mathbf{p} \mathbf{V}}{T} \) in order of magnitude should be the value of \( \frac{\mathbf{p} \mathbf{V}}{T} \sim \frac{m \mathbf{V}^2}{T} \), but since \( T \sim m \mathbf{v}_T^2 \), and \( \mathbf{V} \ll \mathbf{v}_T \), where \( \mathbf{v}_T \) is an average thermal speed of the molecule that's where the condition comes from \( \frac{\mathbf{p} \mathbf{V}}{T} \ll 1 \), which is the justification for the possibility of species decomposition (20). Let's emphasize that the condition is caused by the Gaussian distribution. So,
\[ f_1 = \frac{\tau_p}{T} \left( (p \cdot \dot{V}) + v \cdot \nabla (p \cdot V) \right) \left( 1 + \frac{(p \cdot V)}{T} \right) = \]
\[ = \frac{\tau_p}{T} \left( p \cdot \dot{V} + (v \cdot \nabla (p \cdot V)) \frac{(p \cdot V)}{T} + \frac{(p \cdot V)^2}{T} + v \cdot \nabla (p \cdot V) \right). \] (21)

It is quite similar to the finding that
\[ \frac{f_2}{f_0} = \frac{\tau_p}{T} \left[ \left( (p \cdot V)^2 + 2v \cdot (p \cdot V) \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right) - \right. \]
\[ -p \cdot \dot{V} - 2v \cdot \nabla (p \cdot V) - v_{ij} \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \]
\[ - \left( p \cdot \dot{V} + 2v \cdot \nabla (p \cdot V) + v_{ij} \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right) \left( p \cdot V \right) \frac{T}{T} \]. \] (22)

And hence with account (20) we have
\[ f_2 = \frac{\tau_p}{T} \left[ \left( (p \cdot V)^2 + 2v \cdot (p \cdot V) \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right) - \right. \]
\[ -p \cdot \dot{V} - 2v \cdot \nabla (p \cdot V) - v_{ij} \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \]
\[ - \left( p \cdot \dot{V} + 2v \cdot \nabla (p \cdot V) + v_{ij} \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right) \left( p \cdot V \right) \frac{T}{T} \]. \] (23)

Now due to (21) and (23) the equation (14) can be written as
\[ \frac{1}{Z} \int \frac{\tau_p}{T} \left\{ p \cdot \dot{V} + (v \cdot \nabla (p \cdot V)) \frac{(p \cdot V)}{T} + \frac{(p \cdot V)(p \cdot V)}{T} + v \cdot \nabla (p \cdot V) + \right. \]
\[ + \tau_p \left[ \left( (p \cdot V)^2 + 2v \cdot (p \cdot V) \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right) - p \cdot \dot{V} - 2v \cdot \nabla (p \cdot V) - v_{ij} \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right. \]
\[ - \left( p \cdot \dot{V} + 2v \cdot \nabla (p \cdot V) + v_{ij} \frac{\partial^2 (p \cdot V)}{\partial x_i \partial x_k} \right) \left( p \cdot V \right) \frac{T}{T} \] \[ \left. \right\} \right|_{d\Gamma + \frac{m}{Z} \int \dot{V} d\Gamma} = 0. \] (24)

After the square of expression in curly brackets (24), with precision to the members of the order \( V^2 \)
we get
\[
\frac{1}{Z} \int \frac{\mathcal{F}}{T} \left\{ (p \cdot \mathbf{V}) + \left( \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) \right) \left( \frac{p \cdot \mathbf{V}}{T} \right) + \left( p \cdot \mathbf{V} \right) \left( \frac{p \cdot \mathbf{V}}{T} \right) + \nabla \cdot (p \cdot \mathbf{V}) + \right. \\
\left. + \tau_p \left[ \frac{(p \cdot \mathbf{V})^2 + 2 \mathbf{v} \cdot (p \cdot \mathbf{V}) \nabla (p \cdot \mathbf{V}) + (\mathbf{v} \nabla (p \cdot \mathbf{V}))^2}{T} \right] - (p \cdot \mathbf{V}) - 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) - v \nabla \cdot (p \cdot \mathbf{V}) - \right. \\
\left. - \left( p \cdot \mathbf{V} + 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) + v \nabla \cdot (p \cdot \mathbf{V}) \right) \frac{\partial^2 (p \cdot \mathbf{V})}{\partial x_1 \partial x_k} \right\} \left( \frac{p \cdot \mathbf{V}}{T} \right) \right]^2 \\
+ \left. \frac{m}{Z} \int \mathbf{V} \cdot \mathbf{f} d\Gamma \right) = \\
= \frac{1}{Z} \int \frac{\mathcal{F}}{T} \left\{ (p \cdot \mathbf{V})^2 + (\mathbf{v} \cdot \nabla (p \cdot \mathbf{V}))^2 \right. \\
\left. + \tau_p \left[ (p \cdot \mathbf{V})^2 + 4 \left( \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) \right)^2 \right] + \right. \\
\left. + 2(p \cdot \mathbf{V}) \left[ \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) - \tau_p \left( p \cdot \mathbf{V} + 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) + v \nabla \cdot (p \cdot \mathbf{V}) \right) \right] - \right. \\
\left. - 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) \tau_p \left[ (p \cdot \mathbf{V}) + 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) + v \nabla \cdot (p \cdot \mathbf{V}) \right] + 2 \tau_p \left( p \cdot \mathbf{V} \right) \left( 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) + v \nabla \cdot (p \cdot \mathbf{V}) \right) \right. \\
\left. + 4 \tau_p \left[ \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) \right] \frac{\partial^2 (p \cdot \mathbf{V})}{\partial x_1 \partial x_k} \right\} d\Gamma + \frac{m}{Z} \int \mathbf{V} \cdot \mathbf{f} d\Gamma = 0.
\]

Leaving in this equation only square in momentum components (comment on this matter see a little bit below), we will have

\[
\frac{1}{Z} \int \frac{\mathcal{F}}{T} \left\{ m \mathbf{V} \mathbf{V} + (p \cdot \mathbf{V})^2 + (\mathbf{v} \cdot \nabla (p \cdot \mathbf{V}))^2 + \right. \\
\left. + \tau_p \left[ (p \cdot \mathbf{V})^2 + 4 \left( \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) \right)^2 \right] - \right. \\
\left. - 2 \tau_p \left( p \cdot \mathbf{V} \right) \left( p \cdot \mathbf{V} + 2 \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) + v \nabla \cdot (p \cdot \mathbf{V}) \right) \right. \\
\left. - 4 \tau_p \left( \mathbf{v} \cdot \mathbf{V} \left( \mathbf{v} \cdot \nabla (p \cdot \mathbf{V}) \right) + 2 \tau_p \left( p \cdot \mathbf{V} \right) v \nabla \cdot (p \cdot \mathbf{V}) \right) \right\} d\Gamma = 0.
\]
In the equation (25) the integration of momentum is convenient to conduct in the spherical coordinate system. As a result, we have integration on the bodily corner $dO = \sin \theta d\theta d\varphi$, where the angular variables change as $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. This means that when the sub-integral expression is squared in (21), we have averaging of multiplications both even degrees of impulse and the odd. At the same time, which is quite obvious, the result of averaging from any odd multiplication of impulses will give zero. Only average species are different from zero

$$p, p_k, p, p_k, p_n, p, p_k, p_n p_m p_s,$$

where the line from above we stressed that the averaging is only on angular variables. It's easy to check with direct calculations that follow the following averaging rules

$$p, p_k = \frac{1}{3} \delta_k^p p^2,$$

$$p, p_k p, p_n = \frac{p^4}{15} \left( \delta_k^p \delta_{ln} + \delta_k^l \delta_{mn} + \delta_m^l \delta_{kn} \right),$$

$$p, p_k p, p_n p, p_s = \frac{p^6}{105} \left[ \delta_k^p \left( \delta_{lm} \delta_{ms} + \delta_{ln} \delta_{ms} + \delta_{ls} \delta_{mn} \right) + \delta_l^p \left( \delta_{kn} \delta_{ms} + \delta_{ln} \delta_{ms} + \delta_{ks} \delta_{mn} \right) + \delta_m^l \left( \delta_{ls} \delta_{mn} + \delta_{ls} \delta_{mk} + \delta_{lm} \delta_{nk} \right) \right].$$

where $\delta_k$ is a Kronecker’s symbol. As a result of simple but rather cumbersome transformations, taking into account the rules of averaging (26), the equation (25) can lead to the next expression

$$\frac{1}{Z} \frac{\tau_p}{T} \left[ m V \nabla \cdot V + \frac{p^2}{3} (\nabla - \tau_p V)^2 + \frac{p^4}{15 m^2} \left( \nabla V \right)^2 + \left( \frac{\partial V}{\partial x_k} \right)^2 + \left( \frac{\partial V}{\partial x_k} \right)_x + \frac{2 \tau_p p^4}{15 m^2} \left( \nabla \cdot \nabla + 2 \nabla \cdot \nabla \right) \right] +$$

$$+ \frac{4 \tau_p^2 p^4}{15 m^2} \left( \nabla V \right)^2 + \left( \frac{\partial V}{\partial x_k} \right)^2 + \frac{4 \tau_p^2 p^4}{15 m^2} \left( \nabla V \right)^2 + \left( \frac{\partial V}{\partial x_k} \right)^2 + \frac{2 \tau_p p^4}{15 m^2} \left( \nabla \cdot \nabla V + 2 \nabla \cdot \nabla \right) +$$

$$+ \frac{4 \tau_p^2 p^4}{15 m^2} \left( \nabla V \right)^2 + \left( \frac{\partial V}{\partial x_k} \right)^2 + \frac{4 \tau_p^2 p^4}{15 m^2} \left( \nabla V \right)^2 + \left( \frac{\partial V}{\partial x_k} \right)^2 + \frac{2 \tau_p p^4}{15 m^2} \left( \nabla \cdot \nabla V + 2 \nabla \cdot \nabla \right) +$$

$$+ \frac{4 \tau_p^2 p^4}{105 m^2} \left( \nabla \cdot \nabla V + \left( \nabla \cdot \nabla \right) \nabla V \right) +$$

$$+ \frac{\partial^3 V}{\partial x_k \partial x_k} \frac{\partial^3 V}{\partial x_k \partial x_k} + \frac{1}{2} \frac{\partial^2 V}{\partial x_k \partial x_k} + \frac{1}{2} \frac{\partial^3 V}{\partial x_k \partial x_k} + \frac{1}{4} \left( \nabla \right)^2 \right] p^2 d\Omega = 0,$$

where $d\Omega = dx dy dz$ is an ordinary volume element. After applying in the equation (27) the reception of integration into parts with the Gauss theorem, which aims to allocate in an explicit form of speed $V$ (for example as $\int_{\Omega} (\nabla V)^2 d\Omega = \int_{\Omega} \nabla V \cdot \nabla V d\Omega - \int_{\Omega} \nabla \cdot \nabla \nabla V d\Omega + \int_{\Omega} \nabla \cdot \nabla \nabla V d\Omega$, and
assuming that integral under surface disappears due to the boundary condition \( \mathbf{V} \cdot \mathbf{n} |_{\Sigma_\ell} = 0 \), where \( \mathbf{n} \) — is a normal to the surface), get out (27)

\[
\frac{1}{Z^*} \int \frac{\mathbf{F}}{T} \left\{ \frac{m}{\tau_p} \mathbf{V} + p^2 \left( \mathbf{V} - \mathbf{\tau}_p \mathbf{\bar{V}} \right)^2 - \frac{p^4}{15m^2} \mathbf{V} \cdot (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) \right\} -
\frac{4 \tau_p p^4}{15m^2} \mathbf{V} \cdot (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) + \frac{2 \tau_p p^4}{15m^2} \mathbf{V} \cdot (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) + \frac{2 \tau_p p^4}{15m^2} \mathbf{V} \cdot (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) +
\frac{4 \tau_p p^6}{35m^4} \mathbf{V} \cdot \left( \text{grad div} \mathbf{V} + \frac{1}{4} \Delta^2 \mathbf{V} \right) \right\} p^2 dpd\Omega = 0.
\]

The second and fourth in brackets can also be converted by part-by-piece integration, but now under time (despite the fact that there is the time integration is absence in (28)). This technique becomes quite understandable when you remember that any equation of movement is obtained due to the application of the classic action of Lagrange. As a result, the equation (28) boils down to this

\[
\frac{1}{Z^*} \int \frac{\mathbf{F}}{T} \left\{ \frac{m}{\tau_p} \mathbf{V} + p^2 \left( \mathbf{V} - \mathbf{\tau}_p \mathbf{\bar{V}} \right)^2 - \frac{p^4}{15m^2} (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) +
\frac{2 \tau_p p^4}{15m^2} (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) + \frac{4 \tau_p p^6}{35m^4} \left( \text{grad div} \mathbf{V} + \frac{1}{4} \Delta^2 \mathbf{V} \right) \right\} p^2 dpd\Omega = 0.
\]

Putting here an expression in curly braces equal to zero, we come to a generalized equation Navier - Stokes

\[
\mathbf{\dot{V}} + \frac{p^2 \tau_p}{3mT} \left( \mathbf{\dddot{V}} + 2 \tau_p \mathbf{\ddot{V}} + \tau_p^2 \mathbf{\dot{V}} \right) - \frac{p^4 \tau_p}{15mT} (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) +
\frac{2 \tau_p p^4}{15m^2 T} (\Delta \mathbf{V} + 2 \text{grad div} \mathbf{V}) + \frac{4 \tau_p p^6}{35m^4 T} \left( \text{grad div} \mathbf{V} + \frac{1}{4} \Delta^2 \mathbf{V} \right) = 0,
\]

where the trait from above means averaging on the momentums of molecules. Considering the liquid incompressible, that is, believing \( \text{div} \mathbf{V} = 0 \) and neglecting the higher derivatives in time, we find the equation we're going to find

\[
\mathbf{\dot{V}} = \nu \Delta \mathbf{V} - \nu^2 \tau^* \Delta^2 \mathbf{V},
\]

where the cinematic viscosity \( \nu \) and the time collision \( \tau^* \) determined by formulas
\[ v = \frac{p^4 \tau_p}{15m^2T} = \frac{1}{15m^2TZ_0} \int_0^\infty \tau_p p^6 e^{-\frac{p^2}{2mT}} dp, \]  
\[ v^2 \tau = \frac{\tau_p p^6}{35m^2T} = \frac{1}{35m^2TZ_0} \int_0^\infty \tau_p p^6 e^{-\frac{p^2}{2mT}} dp, \]

where a rationing multiplier is introduced for convenience.

\[ Z_0 = \int_0^\infty p^2 e^{-\frac{p^2}{2mT}} dp. \]

Given, finally the pressure gradient according to (7), which for the sake of simplicity we have previously omitted, the equation (30) at which leads us to the next generalized equation of Navier – Stokes

\[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{\nabla P}{\rho} + \nu \Delta V - v^2 \tau \Delta^2 V. \]

### 3. Conclusion

In conclusion of the work let’s pay attention to a few important points.

1. The equation (33) is a generalized Navier-Stokes equation that takes into account the additional biharmonic component by Knudsen’s number. Let’s emphasize that this biharmonic component comes with a sign of "minus", which from a physical point of view is absolutely correct.

2. If we were to ask the question of writing the Navier-Stokes equation, taking into account the following amendments on the Knudsen’s number, then following the abovementioned algorithm and in accordance with eq. (2) and (5) we would simply come to the equation of the species

\[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{\nabla P}{\rho} + \nu \Delta V - v^2 \tau \Delta^2 V + v^3 \tau^2 \Delta^3 V + ... \]

where relaxation times $\tau$, $\tau^2$, $\tau^3$, ... easy calculate due to the algorithm (31). As can be seen from (34), the heat transfer ratios that we calculated using the tau approximation have a very specific analytical view, due to the specifics of the approach described above in detail. Although the goal of Maxwell et al. the authors was to conclude these coefficients directly from the integral of collisions, we took advantage here precisely the smallness of the parameter $(L/\ell)^2$ (see above), and thanks to which the Navier-Stokes equation can be written down in the form (34). It is worth noting that if you use other approximations and calculations (for example, the Wavelet - decomposition), numerical coefficients will differ from the odds in (31). However, let’s emphasize that the main purpose of our article was to refine the Navier-Stokes equation when taking into account the additional components of the free length path of molecules in the "tau-approximation", which leads to an additional biharmonic member on the operator Laplace in the right side of the equation. This is the main result to which our work was devoted. Once again, we did not set out to compare the transfer rates derived from our calculations with those of the other authors who calculated them, because, as we put it a little higher, our goal was completely different.

3. When it comes to ordinary bodies, the number of Knudsen is small and the last component on the right side of the equation (33) can be omitted, which leads us to the traditional equation of hydrodynamics. However, when it comes to movement in the viscous liquid of nanoobjects, it is necessary to use not equations Havier-Stokes but equations in the form (33) or (34).
4. And finally. In the next article, we will calculate the resistance strength of the spherical nanoparticle based on the equation (33), and we will find additional amendments to Stokes' formula by the number of Knudsen.

References
[1] Prandtl L, Titens O 1935 Hydro and Aeromechanics (Moscow) p. 456
[2] Lamb G 1947 Hydrodynamics (Moscow) p 675
[3] Birkhoff G 1963 Hydrodynamics Methods. Facts. Like (Moscow) p 542
[4] Cochin N E, Kibel I A, Rose N V Theoretical Hydromechanics (Moscow: Fizmatlit) p 584
[5] Sedov L I 1970 Mechanics solid environment (Moscow: Science) p 492
[6] Gladkov S O 2018 Russian Physics Journal On one proof of the uniqueness of the Stokes hydrodynamic solution 61(6) 1117 – 1120
[7] Gladkov S O 2018 Technical Physics To the question of calculating the time of stop rotating in the viscous continuum of the cylindrical body and the time of fascination with the co-axis with it external cylinder 59(3) 377 – 341
[8] Lifshitz E M, Pitaevsky L P 1979 Physical kinetics (Moscow: Science) p 536
[9] Resibois P, De Lerner M 1977 Classical kinetic theory of Fluids. (New York - London - Sydney – Toronto: John Wiley and Sons) p 423
[10] Gladkov S O, Bogdanova S B 2018 Russian Physics Journal On analytical solutions of the quasi-classical kinetic equation of the highest-order perturbations theory in the approximation of the relaxation time 61(5) 833-842
[11] Landau L D, Lifshitz E M 2003 Statistical physics (Moscow: Science) p 616