Experimental observation of thermalisation with noncommuting charges

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Quantum simulators have recently enabled experimental observations of quantum many-body systems’ internal thermalisation. Often, the global energy and particle number are conserved, and the system is prepared with a well-defined particle number—in a microcanonical subspace. However, quantum evolution can also conserve quantities, or charges, that fail to commute with each other. Noncommuting charges have recently emerged as a subfield at the intersection of quantum thermodynamics and quantum information. Until now, this subfield has remained theoretical. We initiate the experimental testing of its predictions, with a trapped-ion simulator. We prepare 6–15 spins in an approximate microcanonical subspace, a generalisation of the microcanonical subspace for accommodating noncommuting charges, which cannot necessarily have well-defined nontrivial values simultaneously. We simulate a Heisenberg evolution using laser-induced entangling interactions and collective spin rotations. The noncommuting charges are the three spin components. We find that small subsystems equilibrate to near a recently predicted non-Abelian thermal state. This work bridges quantum many-body simulators to the quantum thermodynamics of noncommuting charges, whose predictions can now be tested.

INTRODUCTION

Two disparate, but conceptually kindred, fields have advanced rapidly over the past few years. In many-body physics, quantum simulators have grown to tens or even hundreds of particles [1–7] that can be individually controlled and measured [8, 9]. Experiments have elucidated how closed quantum many-body systems thermalise internally [10–12]. Typically, the evolutions conserve no quantities (as in gate-based evolutions), conserve energy (in quantum simulators), or conserve energy and particle number (in nonphotonic systems).

Separately, the subdiscipline of noncommuting charges has been emerging within quantum thermodynamics [13, 14], a subject grounded in foundational and information theory. Throughout thermodynamics, a system of interest S exchanges quantities—particles, magnetization, etc.—with an environment E. As S has far fewer degrees of freedom than E, we call S “the small system” for short. As the quantities are conserved globally, we call them charges. Quantum charges are represented by Hermitian operators Qγ=1,2,…,c, which are implicitly assumed to commute with each other in common thermodynamic arguments [15–21]. Yet noncommuting operators underlie quantum physics from uncertainty relations to measurement disturbance. What happens if thermodynamic charges fail to commute with each other? Whether S can even thermalise is not obvious, as derivations of the thermal state’s form implicitly rely on charges’ commutation [18, 19, 22]. Recently, the study of noncommuting charges has swept across quantum thermodynamics [17–46]. Results have primarily been information-theoretic, although forays into many-body theory have begun recently [22, 27, 32–36, 43]..

Several questions about noncommuting thermodynamic charges have been debated during the past few years. Information-theoretic arguments suggest that S equilibrates to near the non-Abelian thermal state (NATS) [15, 18–21],

\[
\rho_{\text{NATS}} := \exp \left( -\beta \left( H - \sum_{\gamma=1}^{c} \mu_{\gamma} Q_{\gamma} \right) \right) / Z_{\text{NATS}} .
\]

\(\beta\) denotes the inverse temperature, \(H\) denotes the Hamiltonian of S, the \(\mu_{\gamma}\) denote effective chemical potentials, the \(Q_{\gamma}\) denote the c non-energy charges of S, and the partition function \(Z_{\text{NATS}}\) normalises the state. States of the form (1) are called also generalised Gibbs ensembles, especially in the absence of noncommutation and the presence of integrability [47–49]. If the charges fail to commute, arguments for Eq. (1) rely on kinematics and idealisations, such as a very large SE composite. Whether S thermalises outside these idealisations, under realistic dynamics, has remained unclear, although [22]...
presented numerical evidence that $\mathcal{S}$ does.

Whether experimentalists can observe $\rho_{\text{NATS}}$ has remained even unclearer: Suppose that $SE$ is a closed quantum many-body system, as in many recent quantum-thermalisation experiments. Experimental control is finite. Any of the many charges may leak out of the ideally isolated $SE$, breaking the conservation laws.

Another open question is the extent to which $\mathcal{S}$ thermalises, even ideally. Recall typical arguments for thermalisation in quantum many-body systems: The dynamics conserve at most one global charge, $Q^{\text{tot}}_1$, apart from the energy—often, the particle number. $SE$ begins in a $Q^{\text{tot}}_1$ sector and in a narrow energy window—in a microcanonical subspace. But suppose that the dynamics conserve multiple charges $Q^{\text{tot}}_{\gamma=1,2,\ldots,c}$ that do not all commute with each other. Due to the noncommutation, the charges share no eigenbasis, so they may share no eigenspace: They cannot necessarily have well-defined values simultaneously; no microcanonical subspace necessarily exists. Microcanonical subspaces have been generalised to approximate microcanonical (AMC) subspaces, to accommodate noncommuting charges [19]: In an AMC subspace, every $Q^{\text{tot}}_{\gamma}$ has a fairly well-defined value. The uncertainty in the global charges’ initial values has been conjectured to generate uncertainty in the long-time state of $\mathcal{S}$. $\mathcal{S}$ may remain a little farther from $\rho_{\text{NATS}}$ than it would remain from the relevant thermal state if the charges commuted [22].

We provide answers—some complete and some partial—to these and other open questions, by uniting the fields of quantum simulators and noncommuting thermodynamic charges. Using a quantum simulator of 9–15 trapped ions, we implement the experimental proposal in [22]. Each ion’s two electronic states forms a qubit. We initialise the qubits in an AMC subspace. The evolution—an effective long-range Heisenberg coupling—conserves all the global-spin components $S^{\text{tot}}_{x,y,z}$. We experimentally implement the evolution by interspersing a long-range Ising coupling with global rotations and dynamical-decoupling sequences. Trotterisation of Heisenberg dynamics has been proposed theoretically [50, 51], realised experimentally in toy examples [52, 53], and used very recently to explore many-body physics in ensembles of Rydberg atoms [54, 55]. Here, we demonstrate its effectiveness in many-body experiments with trapped ions. Two nearest-neighbour ions form the system of interest, and the other ions form an effective environment (Fig. 1). We find that $\mathcal{S}$ thermalises to near $\rho_{\text{NATS}}$, on average over copies of $\mathcal{S}$ [15, 18–21]. The long-time distance to $\rho_{\text{NATS}}$ is consistent with a bound predicted in [19]. Furthermore, we observe the thermalisation process in what is, to our knowledge, the first study of noncommuting charges’ influence on thermalisation dynamics. To begin to isolate the noncommutation’s effects, we compare our experiment with an evolution that conserves just commuting charges, $S^{\text{tot}}_z$ and the energy. We observe that $\mathcal{S}$ remains farther from the thermal state if the charges fail to commute. This observation is consistent with the conjecture that noncommuting charges hinder thermalisation [19], as well as with the expectation that, in finite-size global systems, resistance to thermalisation grows with the number of charges [56, 57]. Our experiment offers a particularly quantum counterpart to the landmark experiment [58], in which a hitherto-unobserved equilibrium state was observed but the quantum physics of charges’ noncommutation was left unexplored. The present work opens the emerging subfield of noncommuting thermodynamic charges to experiments.

**RESULTS**

We begin by explaining the experimental setup and initial state. We then observe the dynamics of thermalisation influenced by noncommuting charges, observe thermalisation to near $\rho_{\text{NATS}}$, and compare with thermalisation in the presence of two commuting charges.

**Experimental setup**

We perform the experiment on a trapped-ion quantum simulator [8]. A linear string of $N = 6–15$ $^{40}$Ca$^+$ ions is confined in a linear Paul trap (Fig. 1a). Let $\sigma_\gamma = \frac{1}{2} S_\gamma$ denote the Pauli-$\gamma$ operator, for $\gamma = x, y, z$. Let $|\pm\rangle$ denote the $\pm 1$ eigenstates of $\sigma_\gamma$. We denote by $\sigma^{(j)}_\gamma$ the site-$j$ Pauli operators; and, by $\sigma^{(\text{tot})}_\gamma$, the whole-chain operators. Each ion encodes a qubit in the Zeeman states $4^2D_{5/2}$ and $4^2S_{1/2}$, of respective magnetic quantum numbers $m = 5/2$ and 1/2. We denote the states by $|z+\rangle$ and $|z-\rangle$. A pair of nearest-neighbour qubits forms the small system of interest; the remaining qubits form the environment.

We employ two types of coherent operations using a laser at 729 nm, which drives the quadrupole transition that connects the qubit states: (i) Denoting a rotated Pauli operator by $\sigma^{(j)}_\phi = \cos \phi \sigma^{(j)}_x + \sin \phi \sigma^{(j)}_y$, we perform global qubit rotations $U(\theta, \phi) = \exp(-i \frac{\phi}{2} \sum_{j=1}^{N} \sigma^{(j)}_\phi)$. (ii) The effective long-range $x$-type Ising Hamiltonian

$$H_{xx} := \sum_{j<k} \frac{J_0}{|j-k|^{1/3}} \sigma^{(j)}_x \sigma^{(k)}_x$$

entangles qubits. We effect $H_{xx}$ by off-resonantly coupling to the lower and upper vibrational sideband transitions of the ion strings’ transverse collective modes [59]. Combining these two ingredients, we Trotter-approximate the Heisenberg Hamiltonian

$$H_{\text{Hein}} := \sum_{j<k} \frac{J_0}{3 |j-k|^{1/3}} \left( \sigma^{(j)}_x \sigma^{(k)}_x + \sigma^{(j)}_y \sigma^{(k)}_y + \sigma^{(j)}_z \sigma^{(k)}_z \right),$$

as shown in Fig. 1b and the Methods. The 1/3 appears because the Ising coupling (2) is distributed across three
FIG. 1: Experimental setup and protocol: (a) A linear ion crystal of \( N = 6 \)–15 ions is trapped in a linear Paul trap. A small system exchanges charges (local instances of quantities that are conserved globally) with the surrounding environment: energy, \( E \), and all components of angular momentum. (b) One can Trotter-approximate the Heisenberg evolution by consecutively evolving the state across short time steps under the three terms in the Hamiltonian (3). We experimentally realise two terms directly and generate the third using resonant \( \pi/2 \)-pulses (\( R_0 \) and \( R_y \)). This pulse sequence further protects the state against dephasing noise (ii). Some of the building blocks \( E_\pm \) are replaced with equivalent pulse sequences (\( E_\mp \)) to reduce pulse-length errors (iii). For further details, see the Methods. (c) Observed evolution of the initial state, \(|y^+, x^+, z^+\rangle\otimes^N \), under the Trotter-approximated Heisenberg model (wherein \( J_0 = 508 \) rad/s and \( \alpha = 1.02 \)), when \( N = 12 \). To present a complete picture of the dynamics, we derive the spin-excitation–hopping rate in App. A.

directions (\( x \), \( y \), and \( z \)). We implement a \( \sigma_y^{(j)} \sigma_y^{(k)} \) coupling similarly, as described in the Methods. The pulse sequence was designed to realise \( H_{\text{Heis}} \) while, via dynamical decoupling, mitigating dephasing and rotation errors.

At the beginning of each experimental trial, the ion string’s transverse collective modes are cooled to near their motional ground state. Then, we prepare the qubits in the product state described in the next subsection. We then evolve the global system for a time \( t \) up to \( J_0 t = (508 \) rad/s \( \times \) (15 ms) \( \approx \) 7.6 (Fig. 1c). The global system has largely equilibrated internally, and fluctuations are small, as shown two subsections below. Finally, we measure the states of all neighbouring qubit pairs (qubits 1 and 2, qubits 2 and 3, etc.) via quantum state tomography.

Initial state

Conventional thermalisation experiments begin with the global system in a microcanonical subspace. As our global charges do not all commute, they cannot have well-defined nonzero values simultaneously; no nontrivial microcanonical subspace exists. We therefore prepare the global system in an AMC subspace, where the charges have fairly well-defined values. We follow the proposal in [22] for extending the AMC subspace’s definition, devised abstractly in [19], to realistic systems: In an AMC subspace, each global charge \( Q_i^{\text{tot}} \) has a variance \( \sim O(\langle N \rangle^\nu) \), wherein \( \nu \leq 1 \). A product of single-qubit pure states meets this requirement [22].

We choose the tensor product to affirmatively answer an open question. In [22], \( \rho_{\text{NATS}} \) was found numerically to predict a small system’s long-time state best. However, other thermal states approached \( \rho_{\text{NATS}} \) in accuracy as \( N \) grew. (Accuracy was quantified with the long-time state’s relative-entropy distance to a thermal state, as detailed below.) Does the NATS’s accuracy remain greatest by a constant amount, as \( N \) grows, for any initial state? The answer is yes for all \( N \) realised in our experiment. App. B explains why the initial state,

\[
|\psi_0\rangle := |y^+, x^+, z^+\rangle\otimes^N, \tag{4}
\]

consistently distinguishes the NATS. We also prove the following under assumptions met by our experiment in App. C: Consider averaging each thermal state over the global system. The average \( \rho_{\text{NATS}} \) differs from average competitor thermal states, as measured by nonzero relative-entropy distances. A constant lower-bounds the distances at all \( N \), even in the thermodynamic limit (as \( N \to \infty \)). These empirical and analytical results establish the NATS as distinguishable even at large \( N \).

The initial state determines the inverse temperature \( \beta \) and chemical potentials \( \mu_\gamma \) in Eq. (1) [22]. Denote the global NATS by \( \rho_{\text{NATS}}^{\text{tot}} := \exp \left( -\beta \left[ H_{\text{Heis}} - \sum_{\gamma=x,y,z} \mu_\gamma S_\gamma^{\text{tot}} \right] \right) / Z_{\text{NATS}}^{\text{tot}} \), with \( Z_{\text{NATS}}^{\text{tot}} \) defined to normalise the state. We define \( \beta \) and the \( \mu_\gamma \)’s through

\[
\langle \psi_0 | H_{\text{Heis}} | \psi_0 \rangle = \text{Tr} \left( H_{\text{Heis}} \rho_{\text{NATS}}^{\text{tot}} \right) \quad \text{and} \quad \langle \psi_0 | \sigma_\gamma^{\text{tot}} | \psi_0 \rangle = \text{Tr} \left( \sigma_\gamma^{\text{tot}} \rho_{\text{NATS}}^{\text{tot}} \right) \quad \forall \gamma = x, y, z. \tag{5} \tag{6}
\]

We calculate the parameters numerically by solving a maximum-entropy problem, following [60, 61]: \( \beta = -4.2 \times 10^{-4} \text{ s rad}^{-1} \), and \( \mu_{x,y,z} = -307 \text{ rad s}^{-1} \).

Equilibration to near the non-Abelian thermal state

Figure 2 shows how accurately the NATS predicts a small system’s state, as a function of time. The
global system size is \( N = 12 \). To construct the blue dots, we measure the time-dependent state \( \rho_{j,j+1}^{(N-1)} \) of each nearest-neighbour qubit pair \((j,j+1)\), for \( j = 1,2,\ldots,N-1 \). We then calculate the state’s relative-entropy distance to the NATS (defined below) [19]. In Eq. (1), \( H \) denotes the term in \( H_{\text{Heis}} \) [Eq. (3)] that acts nontrivially just on pair \((j,j+1)\). Following [19, 22], we measure the distance with the relative entropy: If \( \chi \) and \( \xi \) denote quantum states (density operators) defined on the same Hilbert space, the relative entropy is \( D(\chi||\xi) = \text{Tr}(\chi \log \chi - \log \xi) \). All logarithms in this paper are base-e: Entropies are measured in units of nats (not to be confused with the NATS), rather than in bits. \( D(\chi||\xi) \) quantifies the optimal efficiency with which the states can be distinguished on average, in a binary hypothesis test. We average \( D(\rho_{j,j+1}^{(N-1)}||\rho_{\text{NATS}}) \) over the \( N - 1 \) qubit pairs, producing \( \langle D(\rho_{j,j+1}^{(N-1)}||\rho_{\text{NATS}}) \rangle \). To our knowledge, this is the first report on the process of thermalisation colored by noncommuting charges (e.g., begun in an AMC subspace).

As in [48, 58], we compare the small system’s state with competing predictions by other thermal states: the canonical state, \( \rho_{\text{can}} := e^{-\beta H} / Z_{\text{can}} \), and the grand canonical state, \( \rho_{\text{GC}} := \exp \left( -\beta \left\{ H - \mu z \right\} \right) / Z_{\text{GC}} \). The partition functions \( Z_{\text{can}} \) and \( Z_{\text{GC}} \) normalise the states. We denote by \( S^{(2)}_{z} \) the total spin operator on two sites. We call \( \rho_{\text{GC}} \) “grand canonical” because \( S_z \) is equivalent to a spinless-fermion particle-number operator via a Jordan-Wigner transformation. As the blue discs (distances to \( \rho_{\text{NATS}} \)) are lower than the orange triangles (\( \rho_{\text{GC}} \)) and blue squares (\( \rho_{\text{can}} \)), the NATS always predicts the state best.

The curves show results from numerical simulations. We calculate the global state’s time evolution via direct multiplication with the Hamiltonian’s sparse matrix representation. The experimental markers lie below the numerical curves, so the small system thermalises more in the experiment than theoretically. After all, the experimental ion chain is open and so has more opportunities to thermalise. By the same token, however, the ion chain can leak its four charges \( (S_{x,y,z} \text{ and energy}) \) to the environment, violating the conservation laws ideally imposed on the ions. One might expect such a violation to prevent \( \rho_{\text{NATS}} \) from predicting the long-time state accurately. However, our results show that the chain is closed enough that no competitor thermal state, defined in terms of fewer charges, bests \( \rho_{\text{NATS}} \) as a prediction.

In Fig. 3, we fix \( t_f = 15 \) ms while varying the global system size. The blue discs represent the relative-entropy distance from the final system-of-interest state, \( \rho_{j,j+1}^{(12)} \), to the NATS, averaged over qubit pairs. The average distance declines from 0.2 nats to 0.12 nats as \( N \) grows from 6 to 15. For reference, \( D(\chi||\xi) \) obeys no upper bound. We hence answer two open questions: Equilibration to the NATS occurs in realistic systems and is experimentally observable, despite the opportunity for the spin chain to leak many charges via decoherence. Furthermore, the orange triangles (distances to \( \rho_{\text{GC}} \)) lie 0.11 nats above the blue discs (distances to \( \rho_{\text{NATS}} \)), on average; and the green squares (distances to \( \rho_{\text{can}} \)) lie 0.16 nats above the blue discs, on average. Hence the NATS prediction is distinctively most accurate at all \( N \).

The red curve in Fig. 3 shows the best polynomial fit to the blue points (the distances to \( \rho_{\text{NATS}} \)), \( 0.42N^{-0.57} + 0.02 \). The fit was calculated via weighted least squares. For comparison, the bound predicted information-theoretically is \( \langle D(\rho_{j,j+1}^{(N-1)}||\rho_{\text{NATS}}) \rangle \leq \text{(positive constant)}N^{-0.50} + \text{(positive constant)} \) [19]. The experimental scaling exponent, \( -0.57 \), lies below the information-theoretic exponent, \( -0.50 \), and above the numerically predicted exponent, \( -2.448 \) [22]. Hence the experiment obeys the theoretical bound [19]. However, the experimental best-fit parameters have significant uncertainties (0.35, 1.58, and 0.32), being inferred from four data points.
We have observed equilibration to near the NATS, but the small system does not thermalise entirely: \( \langle D(\rho_{\text{NATS}}^{(N+1)}) \rangle \neq 0 \). We expect the lingering athermality to stem partially from the global system’s finite size [63, 64]. Yet charges’ noncommutation has been conjectured to hinder thermalisation additionally [22]. We provide the first data about this conjecture, by comparing thermalisation steered by noncommuting charges with thermalisation steered by just commuting charges.

We realise the commuting case with the long-range XY Hamiltonian

\[
H_{xy} := \sum_{j<k} \frac{1}{2} |j - k|^\alpha J_0 \left( \sigma_x^{(j)} \sigma_x^{(k)} + \sigma_y^{(j)} \sigma_y^{(k)} \right),
\]

for \( N = 12 \), with \( J_0 = 508 \text{ rad/s} \) and \( \alpha = 1.02 \) (see Methods for details). The charges are the total energy and \( \sigma_z^{\text{tot}} \). We Trotter-approximate \( H_{xy} \) similarly to \( H_{\text{Heis}} \) (App. E). After preparing \( |y+, x+, z+\rangle \otimes \otimes N/3 \), we simulate \( H_{xy} \) for 10 ms.

Table I shows the results. The average small system thermalises more thoroughly when determined by commuting charges than when determined by noncommuting charges. For instance, in the commuting case, the relative entropy to \( \rho_{\text{GC}} \) descends as low as 0.070 nats, when \( N = 15 \). In the noncommuting case, when \( N = 15 \), the relative entropy to \( \rho_{\text{NATS}} \) reaches 0.105 > 0.070 nats. (The relative entropy is lower in the noncommuting case just when \( N = 6 \), at the edge of a reasonable size for a global system in a thermalisation experiment.) This result is consistent with the conjecture that charges’ noncommutation hinders thermalisation [22], as well as with the expectation that, in finite-size global systems, a small system’s long-time entanglement entropy decreases as the number of charges grows [57]. Future work will distinguish how much our charges’ noncommutation is hindering thermalisation and how much the multiplicity of charges is.

### CONCLUSIONS

We have observed the first experimental evidence of a particularly quantum equilibrium state: the non-Abelian thermal state, which depends on noncommuting conserved charges. Whereas typical many-body experiments begin in a microcanonical subspace, our experiment begins in an approximate microcanonical subspace. This generalisation accommodates the noncommuting charges’ inability to have well-defined nontrivial values simultaneously. Our experiment affirmatively answers an open question: whether, for any initial state, the NATS remains a substantially better prediction than other thermal states as the global system grows. Our trapped-ion experiment affirmatively answers two more open questions: (i) whether realistic systems exhibit the thermodynamics of noncommuting charges and (ii) whether this thermodynamics can be observed experimentally, despite the multiplicity of conservation laws that decoherence can break. Our work therefore bridges quantum simulators to the emerging subfield of noncommuting charges in quantum-information-theoretic thermodynamics. The subfield has remained theoretical until now: hence many predictions now can, and should, be tested experimentally—predictions about reference frames, second laws of thermodynamics, information storage in dynamical fixed points, and more [17–46].

In addition to answering open questions, our results open avenues for future work. First, Fig. 3 contains a line of best fit whose scaling exponent lies near the theoretical prediction in [19]. However, the paucity of data points leads to a large experimental uncertainty. One can reduce the uncertainty by scaling up the system further. Similarly reducing the constant offset’s uncertainty would provide evidence about another open question: whether

### TABLE I: Average distance from long-time system-of-interest state to thermal prediction vs. total number of qubits

| Number of qubits N | 6    | 9    | 12   | 15   |
|--------------------|------|------|------|------|
| Relative entropy D | 0.258±0.006 | 0.102±0.008 | 0.087±0.007 | 0.070±0.006 |
the small system thermalises fully as $N \to \infty$ [22]. A nonvanishing offset would suggest that noncommuting charges block thermalisation in the thermodynamic limit. Our offset, although positive, has too large an uncertainty for drawing conclusions.

Third, we observed that the small system thermalises less in the presence of noncommuting charges than in the presence of just commuting charges. This observation is consistent with the conjecture that charges’ noncommutation may hinder thermalisation at all $N$ [22]. Future study will tease apart effects of the charges’ noncommutation from effects of the charges’ multiplicity.

Finally, the quantum-simulation toolkit developed here merits application to other experiments. We combined our quantum simulator’s native interaction with rotations and dynamical decoupling to simulate a non-native Heisenberg interaction. The Trotterised long-range Hamiltonian, together with the single-qubit control used to initialise the system here, can be advantageous for studying more many-body physics with quantum simulators. These techniques can be leveraged to explore nonequilibrium Heisenberg dynamics [54, 55], topological excitations [65], and more.

METHODS

This section details the setup, the realisation of spin-spin interactions, the Trotterisation of the Heisenberg Hamiltonian, and the quantum state tomography and statistical analysis.

Experimental setup

A linear ion crystal of up to 15 $^{40}\text{Ca}^+$ ions is trapped in a linear Paul trap with trapping frequencies of $\omega_\perp = 2\pi \times 2.924$ MHz (radially) and $\omega_\parallel = 2\pi \times 0.217$ MHz (axially). The qubit states $|0\rangle$ and $|1\rangle$ are coupled by an optical quadrupole transition, which we drive with a titanium-sapphire laser, with a sub–10 Hz linewidth, at 729 nm. Collective qubit operations are implemented with a resonant beam that couples to all the qubits with approximately equal strengths. Single-qubit operations are performed with a steerable, tightly focused beam that induces AC Stark shifts. The initial product state (4) is prepared with a fidelity $|\langle \psi_{\text{exp}} | \psi_{\text{th}} \rangle|^2 = 0.959(4)$. In each experimental cycle, we cool the ions via Doppler cooling and polarisation-gradient cooling [66]. We also sideband-cool all transverse collective motional modes to near their ground states. Then, we prepare the state (4), simulate the Heisenberg evolution, and measure the state. The cycle is repeated 300–500 times per quantum-state-tomography measurement basis.

Implementing the effective Heisenberg interaction

We implement the long-range spin-spin interaction (2) with a laser beam carrying two frequencies that couple motional and electronic degrees of freedom of the ion chain. The beam’s frequency components, $\omega_k = \pm (\omega_x + \Delta)$, are symmetrically detuned by $\Delta = 2\pi \times 25$ kHz (for $N = 12$ ions) from the transverse–center-of-mass mode, which has a frequency $\omega_x = 2\pi \times 2.924$ MHz. A third frequency component, $\omega_{AC} = 2\pi \times 1.4$ MHz, is added to the bichromat beam. This component compensates for the additional AC-Stark shift caused by other electronic states [67].

The resulting spin-spin coupling takes the form of a long-range Ising model, $\sum_{j<k} J_{j,k} \sigma^x_j \sigma^x_k$. $J_{j,k}$ denotes the strength of the coupling between ions $j$ and $k$. It approximates the power law in Eq. (2), where the coupling strength equals $J_0 = 606$ rad/s and the exponent $\alpha = 1.02$. The measurements for $N = 6$, 9 and 15 ions were taken at different axial trapping frequencies $\omega_{ax}$ and different sideband detunings $\Delta$, to keep the power-law exponent approximately constant. The parameters were, for 6 ions, $\omega_{ax} = 2\pi \times 314.8$ kHz, $\Delta = 2\pi \times 29.5$ kHz, $J_0 = 604$ rad/s, $\alpha = 1.02$; for 9 ions, $\omega_{ax} = 2\pi \times 254.9$ kHz, $\Delta = 2\pi \times 26$ kHz, $J_0 = 584$ rad/s, $\alpha = 1.02$; and, for 15 ions, $\omega_{ax} = 2\pi \times 194.1$ kHz, $\Delta = 2\pi \times 23$ kHz, $J_0 = 564$ rad/s, $\alpha = 1.04$.

The desired long-range Heisenberg Hamiltonian (3) cannot easily be realised directly for trapped ions [59, 68]. Instead, we simulate $H_{\text{Heis}}$ via Trotterisation. After the first time step, we change the interaction from $H_{xx}$ to $H_{yy}$; after the second, to $H_{zz}$; and, after the third, back to $H_{xx}$. We perform this cycle, or Trotter step, $N_T$ times [69]. We can realise $H_{yy}$ by shifting the bichromat light’s phase by $\pi/2$ relative to the phase used to realise $H_{xx}$. Implementing $H_{zz}$ requires a global rotation: Denote by $R_y$ a $\pi/2$ rotation of all the qubits about the $y$-axis. We can effect $H_{zz}$ with, e.g., $R^\dagger_y H_{zz} R_y$.

Noise-robust Trotter sequence

In our experimental setup, most decoherence is dephasing relative to the $\sigma^x$ eigenbasis. This noise results from temporal fluctuations of (i) the magnetic field and (ii) the frequency of the laser that drives the qubits. Earlier experiments on this platform involved XY-interactions, which enable the quantum state to stay in a decoherence-free subspace [5, 67, 70]. Here, the dynamics must be shielded from dephasing differently. We mitigate magnetic-field noise by incorporating a dynamical-decoupling scheme into the Trotter sequence (Fig. 1b). Furthermore, we design the Trotter sequence to minimise the number of global rotations. This minimisation suppresses the error accumulated across all the rotations. We reduce this error further by alternating the rotations’ directions between Trotter steps. For further details, see App. E.
To formalise the Trotter sequence, we introduce notation: Let $U_{xx} = \exp(-iH_{xx}t)$ and $U_{yy} = \exp(-iH_{yy}t)$. For $\gamma \in \{x, y\}$, $R_\gamma := \exp(-i\frac{\pi}{2} \sigma^{\gamma}_x)$ denotes a global π/2 rotation about the $\gamma$-axis. $N_T$ denotes the number of Trotter steps. Each Trotter step consists either of the operation $E_+ = U_{yy}U_{xx}R_xU_{yy}$ or the operation $E_- = U_{yy}U_{xx}R_x^4U_{yy}$. To simulate a Heisenberg evolution for a time $t$, we implement the Trotter sequence

$$U_{\text{Heis}}(t) \approx R_y^4 R_x \left[(E_-)^4(E_+)^4\right]^{N_T/8} R_x^4 R_y,$$

(8)

This sequence protects against decoherence and over-/under-rotation errors caused by global pulses. Numerical simulations supporting this claim are presented in App. F.

The Trotter sequence lasts for 15 ms, containing ≤ 36 Trotter steps. Each Trotter step consists of three sub-steps, each lasting for approximately 139 μs. Each sub-step’s rising and falling slopes are pulse-shaped to avoid incoherent excitations of vibrational sidebands of the qubit transition. The slopes reduce the effective spin-spin coupling by a factor of 0.84, and the actual interaction time is 115 μs.

The magnetic-field variations occur predominantly at temporally stable 50-Hz harmonics. We reduce the resulting Zeeman-level shifts via feed-forward to a field-temporally stable 50-Hz harmonics. We reduce the re-part by the National Science Foundation under Grant No. 1748958, under Grant No. NSF PHY-1748958, and under an NSF grant for the Institute for Theoretical Atomic, Molecular, and Optical Physics at Harvard University and the Smithsonian Astrophysical Observatory.

Appendix A  RATE OF HOPPING DURING THE HEISENBERG EVOLUTION

In this section, we derive an expression for the Heisenberg Hamiltonian’s spin exchange rate. For simplicity, we model two qubits governed by the Heisenberg Hamiltonian

$$H = \frac{J_0}{3} \left( \sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} + \sigma_z^{(1)} \sigma_z^{(2)} \right).$$

(A1)

We relabel the $\sigma_z$ eigenstates as $|z+\rangle \equiv |\uparrow\rangle$ and $|z-\rangle \equiv |\downarrow\rangle$. Matrices are expressed relative to the basis formed from products of $|\uparrow\rangle$ and $|\downarrow\rangle$. The Hamiltonian can be expressed as

$$H = \frac{J_0}{3} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};$$

(A2)

and a pure two-qubit state, as $|\psi(t)\rangle = c_1(t)|\downarrow\downarrow\rangle + c_2(t)|\uparrow\downarrow\rangle + c_3(t)|\downarrow\uparrow\rangle + c_4(t)|\uparrow\uparrow\rangle$. The coefficients $c_k(t) \in \mathbb{C}$ depend on the time, $t$, and are normalised as $\sum_{k=1}^4 |c_k(t)|^2 = 1$. The dynamics obey the Schrödinger equation, $H |\psi(t)\rangle = i\hbar \frac{\partial |\psi(t)\rangle}{\partial t}$. Defining $\Omega := \frac{J_0}{3}$ and setting $\hbar = 1$, we express the Schrödinger equation in matrix form as

$$\begin{pmatrix}
\dot{c}_1(t) \\
\dot{c}_2(t) \\
\dot{c}_3(t) \\
\dot{c}_4(t)
\end{pmatrix} = -i\Omega \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
c_1(t) \\
c_2(t) \\
c_3(t) \\
c_4(t)
\end{pmatrix}.$$  

(A3)

Quantum state tomography

We measure each qubit pair’s state via quantum state tomography. In each measurement basis, 300–500 quantum state measurements are carried out. To reconstruct the state from the measurements, we use maximum-likelihood estimation [71]. We estimate statistical uncertainties by bootstrapping [62].

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The solution is
\[
\begin{align*}
    c_1(t) &= c_1(0)e^{-i\Omega t} \\
    c_2(t) &= \frac{1}{2}c_2(0)e^{-i\Omega t}(1 + e^{4i\Omega t}) + \frac{1}{2}c_3(0)e^{-i\Omega t}(1 - e^{4i\Omega t}) \\
    c_3(t) &= \frac{1}{2}c_3(0)e^{-i\Omega t}(1 + e^{4i\Omega t}) + \frac{1}{2}c_2(0)e^{-i\Omega t}(1 - e^{4i\Omega t}) \\
    c_4(t) &= c_4(0)e^{-i\Omega t}.
\end{align*}
\] (A4)

We aim to derive the time required for $|\uparrow\downarrow\rangle$ to transform into $|\downarrow\uparrow\rangle$. If the initial state is $|\psi(0)\rangle = |\uparrow\downarrow\rangle$, the solution reduces to
\[
\begin{align*}
    c_1(t) &= 0 \\
    c_2(t) &= \frac{1}{2}e^{-i\Omega t}(1 + e^{4i\Omega t}) \\
    c_3(t) &= \frac{1}{2}e^{-i\Omega t}(1 - e^{4i\Omega t}) \\
    c_4(t) &= 0
\end{align*}
\]
Consider measuring the $\sigma_z$ product eigenbasis at time $t$. The possible outcomes $\uparrow\downarrow$ and $\downarrow\uparrow$ result with probabilities
\[
\begin{align*}
    |c_2(t)|^2 &= \frac{1}{4}(2 + e^{4i\Omega t} + e^{-4i\Omega t}) = \frac{1}{2} + \frac{1}{2}\cos(4\Omega t) \quad \text{and} \\
    |c_3(t)|^2 &= 1 - |c_2(t)|^2.
\end{align*}
\] (A5) (A6)
Therefore, the two-qubit excitation-hopping frequency is $\Gamma_{\text{flip-flop}} = 4\Omega = 4J_0/3$. The corresponding period is defined as the hopping time:
\[
T_{\text{hop}} = \frac{2\pi}{2\Gamma_{\text{flip-flop}}} = \frac{3\pi}{4J_0}.
\] (A7)
This result agrees with our experimental results and can be extended simply to $N$ qubits. If the interaction is nearest-neighbour only, the time for hopping from site 1 to site $N$ is $T_{\text{hop}} = (N - 1)\frac{2\pi}{4J_0}$.

Appendix B INITIAL STATE

If the global system is prepared in $|\psi_0\rangle$ [Eq. (4)], $\rho_{\text{NATS}}$ models a small system’s long-time state distinctly more accurately than other thermal states ($\rho_{\text{can}}$ and $\rho_{\text{GCC}}$) do, at all the global system sizes $N$ realised (Fig. 3). Equation (4) distinguishes $\rho_{\text{NATS}}$ for two reasons.

First, suppose that the temperature is high ($\beta > 0$). All the thermal states resemble the maximally mixed (infinite-temperature) state $1^{\otimes N}/2^N$ and so resemble each other. We therefore keep the temperature low, by keeping each charge’s spatial density low: We separate the $|\gamma\rangle$’s from each other maximally, for each $\gamma = x, y, z$, to provide a sense of $\beta$’s size at $N = 12$. We compare with the bandwidth of the Heisenberg Hamiltonian (3), the greatest energy minus the least. $\beta$ equals $-1.78$ times the bandwidth’s inverse and $-4.24 \times 10^{-4}$ times the average energy gap’s inverse.

Second, noncommuting charges distinguish NATS thermodynamics from more-classical thermodynamics. If we are to observe NATS physics, therefore, $\rho_{\text{NATS}}$ [Eq. (1)] should depend significantly on $Q_\gamma^{\text{tot}}$, for all $\gamma$. Hence the $\mu_\gamma$’s should have large magnitudes—and so should the expectation values $\langle \psi_0|\sigma_\gamma^{\text{tot}}|\psi_0\rangle$, by Eq. (6). Hence, for each $\gamma$, the $\sigma_\gamma$ eigenstates in $|\psi_0\rangle$ should be identical. The ordering of the $x+$, $y+$, and $z+$ in Eq. (4) does not matter. Importantly, $|\psi_0\rangle$ is not an eigenstate of any $Q_\gamma^{\text{tot}}$; so the global system does not begin in a microcanonical subspace; so the experiment is not equivalent, by any global rotation, to any experiment that conserves just $\sigma_z^{\text{tot}}$ and that leads to $\rho_{\text{GCC}}$. When $N = 12$, $\beta \mu_z$ equals $-0.145$ times the inverse of each nonzero gap of $S_z^{\text{tot}}$.

We numerically identified many tensor products of $|\gamma\rangle$’s, as well as superpositions of energy eigenstates, that have $\beta$’s much greater than our $\beta$. These states suffer from drawbacks that render the states unsuitable for observing the NATS: Either $\mu_{x,y,z} = 0$ or only one of the three charges has a nonzero expectation value. Such states reveal nothing about noncommuting charges. Furthermore, the states are highly entangled and so are difficult to prepare experimentally. $|\psi_0\rangle$ is easy to generate, aside from having a large $\sum_\gamma \beta \mu_\gamma S_\gamma^{\text{tot}}$.

Appendix C DISTINCTION BETWEEN NON-ABELIAN THERMAL STATE AND COMPETITORS AT ALL GLOBAL SYSTEM SIZES

The main text answers a question established in [22]: Consider a global system of $N$ subsystems, which exchange noncommuting charges. Consider measuring one subsystem’s long-time state. Measure the state’s distance to $\rho_{\text{NATS}}$
and to the competitor thermal states: the canonical $\rho_{\text{can}}$ and the grand canonical $\rho_{\text{GC}}$. The NATS was found, in [22], to predict the final state most accurately. However, as $N$ grew, $\rho_{\text{can}}$ and $\rho_{\text{GC}}$ approached $\rho_{\text{NATS}}$ in accuracy. The reason was believed to be the initial global state, which had a high temperature and low chemical potentials (see Methods). Does $\rho_{\text{NATS}}$ remain substantially more accurate at all $N$, for any initial state $|\psi_0\rangle$? Or do all the thermal states’ predictions converge in the thermodynamic limit (as $N \to \infty$), for every $|\psi_0\rangle$?

Our experiment suggests the former, as explained in this paper’s Results. We constructed a $|\psi_0\rangle$ for which the NATS prediction remains more accurate than the $\rho_{\text{can}}$ and $\rho_{\text{GC}}$ predictions, by approximately constant-in-$N$ amounts, at all $N$ values realised experimentally. (Fig. 3). The Methods provides one perspective on why this $|\psi_0\rangle$ distinguishes the thermal states. We provide another perspective here.

We prove that, under conditions realised in our experiment, $\rho_{\text{NATS}}$, averaged over space, differs from the average $\rho_{\text{can}}$ and $\rho_{\text{GC}}$. This difference remains nonzero even in the thermodynamic limit. App. C 1 presents the setup, which is more general than our experiment’s. In App. C 2, we formalise and prove the result.\(^1\) App. C 3 shows how our experiment realises the general setup.

## C 1 Setup

Consider a global system of $N$ identical subsystems. Let $O^{(j)}$ denote observable $O$ of subsystem $j$. Sometimes, $O^{(j)}$ will implicitly be padded with identity operators $\mathbb{1}$ acting on the other subsystems. The corresponding global observable is $O^{\text{tot}} := \sum_{j=1}^{N} O^{(j)}$.

The Hamiltonian $H^{\text{tot}}$ is translationally invariant. $H^{\text{tot}}$ conserves global charges $Q^{\text{tot}}_\gamma := \sum_{j=1}^{N} Q^{(j)}_\gamma$, for $\gamma = 0, 1, 2, \ldots, c$. The charges do not all commute pairwise: $[Q^{(j)}_\gamma, Q^{(j')}_{\gamma'}] \neq 0$ for at least one pair $(\gamma, \gamma')$.

We assume that some global unitary $V$ satisfies two requirements. First, the unitary commutes with the Hamiltonian: $[V, H^{\text{tot}}] = 0$. Second, conjugating at least one global charge $Q^{\text{tot}}_\gamma$ with $U$ negates the charge:

$$VQ^{\text{tot}}_\gamma V^\dagger = -Q^{\text{tot}}_\gamma. \quad (C1)$$

We assume that this global charge’s initial expectation value is proportional to the global system size, as in the trapped-ion experiment:

$$\langle Q^{\text{tot}}_\gamma \rangle_0 = q_\gamma N \neq 0, \quad (C2)$$

for some constant-in-$N$ $q_\gamma$.

Let $|\psi_0\rangle$ denote the initial global state. It is invariant, we assume, under translations through $\tau$ sites, for some nonnegative integer $\tau$. More precisely, divide the chain into clumps of $\tau$ subsystems. Index the clumps with $m = 1, 2, \ldots, N/\tau$. (We assume for convenience that $N$ is an integer multiple of $\tau$.) Consider tracing out all the subsystems except the $m$th clump: $\text{Tr}_{1,2,\ldots,(m-1)\tau, \ m+\tau+1, m+\tau+2,\ldots,N}(|\psi_0\rangle\langle\psi_0|)$. This state’s form does not depend on $m$.

Let us define a state averaged over clumps of subsystems. Let $\rho$ denote any state of the global system. Consider the clump that, starting at subsystem $j$, encompasses $\tau$ subsystems. This clump occupies the state

$$\rho^{(j,j+1,\ldots,j+\tau-1)} := \text{Tr}_{1,2,\ldots,j-1, \ j+\tau,j+\tau+1,\ldots,N}(\rho). \quad (C3)$$

Let $T_j$ denote the operator that translates a state $j - 1$ sites leftward. We define the average

$$\rho^{\text{avg}} := \frac{1}{N} \sum_{j=1}^{N} T_j \left( \rho^{(j,j+1,\ldots,j+\tau-1)} \right) \quad (C4)$$

on the joint Hilbert space of subsystems $1$ through $\tau$. Addition and subtraction modulo $N$ are denoted by $\oplus$ and $\ominus$. If $\rho$ is fully translationally invariant (if $\tau = 1$), then $\rho^{\text{avg}} = \text{Tr}_{2,3,\ldots,N}(\rho)$, and this definitional step can be skipped.

Multiple thermal states will interest us. The global canonical state is defined as $\rho^{\text{can}} := \exp(-\beta H^{\text{tot}})/Z^{\text{can}}$. The partition function is $Z^{\text{can}} := \text{Tr}(e^{-\beta H^{\text{tot}}})$. The inverse temperature $\beta$ is defined through $\langle \psi_0|H^{\text{tot}}|\psi_0\rangle = \text{Tr}(H^{\text{tot}}\rho^{\text{can}})$. Define the single-site $\rho^{(j)} := \text{Tr}_j(\rho^{\text{can}})$. Denote by $\rho^{\text{avg}}_{\text{can}}$ the result of averaging $\rho^{\text{can}}$ over clumps, as in Eq. (C4).

\(^1\) We thank Ignacio Cirac for framing this argument.
The global NATS is defined as
\[ \rho_{\text{NATS}}^\text{tot} := \exp \left( -\beta \left( H_{\text{tot}} - \sum_{\gamma=1}^{c} \mu_{\gamma} Q_{\gamma}^\text{tot} \right) \right) / Z_{\text{NATS}}^\text{tot}. \tag{C5} \]
This \( \beta \) is defined analogously to the canonical \( \beta \). The temperatures’ values might differ, but we reuse the symbol \( \beta \) for convenience. The effective chemical potentials \( \mu_{\gamma} \) are defined through \[ q_{\gamma} N = \text{Tr} \left( Q_{\gamma}^\text{tot} \rho_{\text{NATS}}^\text{tot} \right). \tag{C6} \]
The partition function \( Z_{\text{NATS}}^\text{tot} := \text{Tr} \left( e^{-\beta (H_{\text{tot}} - \sum_{\gamma=1}^{c} \mu_{\gamma} Q_{\gamma}^\text{tot})} \right) \). Define \( \rho_{\text{NATS}}^{(j)} \) and \( \rho_{\text{NATS}}^{\text{avg}} \) analogously to \( \rho_{\text{can}}^{(j)} \) and \( \rho_{\text{can}}^{\text{avg}} \).

Our argument concerns multiple distance measures. Let \( O \) denote an arbitrary observable defined on an arbitrary Hilbert space. The Schatten \( p \)-norm of \( O \) is \( ||O||_p := \left[ \text{Tr} \left( |O|^p \right) \right]^{1/p} \), wherein \( |O| := \sqrt{O \dagger O} \) and \( p \in [0, \infty) \). The limit as \( p \to \infty \) yields the operator norm: \( \lim_{p \to \infty} ||O||_p := ||O||_{\text{op}} \). Let \( \rho \) and \( \sigma \) denote operators defined on an arbitrary Hilbert space. The Schatten \( p \)-distance between the states is \( ||\rho - \sigma||_p \). The trace distance is \( D_{\text{tr}}(\rho, \sigma) = \frac{1}{2} ||\rho - \sigma||_1 \).

\section*{C 2 Lower bounds on distances between thermal states}

We now formalise the result.

\textbf{Theorem 1.} Let the setup and definitions be as in the previous subsection. Consider the distance from the average NATS to the average canonical state. Measured with the Schatten 1-distance or the relative entropy, this distance obeys the lower bound
\[ D (\rho_{\text{NATS}}^{\text{avg}}||\rho_{\text{can}}^{\text{avg}}) \geq D_{\text{tr}} (\rho_{\text{NATS}}^{\text{avg}}, \rho_{\text{can}}^{\text{avg}}) \geq \frac{|q_{\gamma}|}{||Q_{\gamma}^{(j)}||_{\text{op}}} > 0, \tag{C7} \]
for an arbitrary \( j = 1, 2, \ldots, N \). \( \rho_{\text{can}}^{(j)} \) can be replaced with any grand canonical state that commutes with \( V \).

The bound does not depend on \( N \) and so holds in the thermodynamic limit.

\textbf{Proof.} The proof has the following outline. First, we calculate the expectation value of \( Q_{\gamma}^{(1)} \) in \( \rho_{\text{NATS}}^{\text{avg}} \); the result is \( q_{\gamma} \). Second, the expectation value in \( \rho_{\text{can}}^{\text{avg}} \) vanishes, we show using \( V \). Because the two expectation values differ, a nonzero Schatten 1-distance separates the states. The Schatten 1-distance lower-bounds the relative entropy via Pinsker’s inequality. \( Q_{\gamma}^{(1)} \) has an expectation value, in the average NATS state, of
\[ \text{Tr} (Q_{\gamma}^{(1)} \rho_{\text{NATS}}^{\text{avg}}) = \frac{1}{N} \sum_{j=1}^{N} \text{Tr} \left( Q_{\gamma}^{(1)} T_{\gamma} (\rho_{\text{NATS}}^{(j) \otimes (j-1) \otimes (N-j)}) \right) \tag{C8} \]
\[ = \frac{1}{N} \sum_{j=1}^{N} \text{Tr} \left( Q_{\gamma}^{(j)} \rho_{\text{NATS}}^{(j)} \right) \tag{C9} \]
\[ = \frac{1}{N} \text{Tr} \left( \sum_{j=1}^{N} I_{(j-1)} \otimes Q_{\gamma}^{(j)} \otimes I_{(N-j)} \right) \left[ \bigotimes_{k=1}^{N} \rho_{\text{NATS}}^{(k)} \right] \tag{C10} \]
\[ = \frac{1}{N} \text{Tr} \left( Q_{\gamma}^{\text{tot}} \rho_{\text{NATS}}^{\text{tot}} \right) \tag{C11} \]
\[ = q_{\gamma} \tag{C12} \]
Equation (C9) follows from the definition of \( T_{\gamma} \). Equation (C10) follows Eq. (C6).

The analogous canonical expectation value vanishes, we show next. We begin with the global expectation value \( \text{Tr} (Q_{\gamma}^{\text{tot}} e^{-\beta H_{\text{tot}}}) / Z_{\text{can}}^{\text{tot}} \). By Eq. (C1), we can replace the \( Q_{\gamma}^{\text{tot}} \) with \(-V Q_{\gamma}^{\text{tot}} V^\dagger \). We then invoke the trace’s cyclicity:
\[ \text{Tr} \left( Q_{\gamma}^{\text{tot}} e^{-\beta H_{\text{tot}}} \right) / Z_{\text{can}}^{\text{tot}} = -\text{Tr} \left( [V Q_{\gamma}^{\text{tot}} V^\dagger] e^{-\beta H_{\text{tot}}} \right) / Z_{\text{can}}^{\text{tot}} = -\text{Tr} \left( Q_{\gamma}^{\text{tot}} \left[ V^\dagger e^{-\beta H_{\text{tot}}} V \right] \right) / Z_{\text{can}}^{\text{tot}} \tag{C13} \]
\[ = -\text{Tr} \left( Q_{\gamma}^{\text{tot}} e^{-\beta H_{\text{tot}}} \right) / Z_{\text{can}}^{\text{tot}} \tag{C14} \]
Equation (C14) follows from $[V, H^{tot}] = 0$. Let us compare the beginning and end of Eqs. (C13)–(C14). The expectation value $\text{Tr} \left( Q^{tot}_\gamma e^{-\beta H^{tot}} \right) / Z^{tot}_{\text{can}}$ equals its negative and so vanishes. We can re-express the null expectation value in terms of the average canonical state:

$$0 = \sum_{j=1}^{N} \text{Tr} \left( Q^{tot}_j e^{-\beta H^{tot}} \right) / Z^{tot}_{\text{can}}$$

$$= \sum_{j=1}^{N} \text{Tr} \left( \left( \mathbb{1} \otimes (j-1) \otimes Q^{(j)} \otimes \mathbb{1} \otimes (N-j) \right) e^{-\beta H^{tot}} \right) / Z^{tot}_{\text{can}} = \sum_{j=1}^{N} \text{Tr} \left( Q^{(j)}_{j} \text{Tr}_{j} \left( e^{-\beta H^{tot}} \right) \right) / Z^{tot}_{\text{can}}$$

$$= \sum_{j=1}^{N} \text{Tr} \left( Q^{(1)}_{j} \rho^{(j)}_{\text{can}} \right)$$

$$= \sum_{j=1}^{N} \text{Tr} \left( Q^{(1)}_{j} \right) \rho^{(j)}_{\text{can}}\right)$$

$$= \text{Tr} \left( Q^{(1)}_{\rho^{\text{avg}}}_{\text{can}} \right).$$

Equations (C17) and (C18) are analogous to Eqs. (C9) and (C8).

We have calculated two expectation values of $Q^{1}_{\gamma}$, one in $\rho^{\text{avg}}_{\text{NATS}}$ and one in $\rho^{\text{avg}}_{\text{can}}$. The two expectation values differ, by Eqs. (C15), (C19), and (C12):

$$\left| \text{Tr} \left( Q^{(1)}_{\rho^{\text{avg}}}_{\text{NATS}} \right) - \text{Tr} \left( Q^{(1)}_{\rho^{\text{avg}}}_{\text{can}} \right) \right| = |q_{\gamma}| > 0.$$  (C20)

The absolute difference (C20), we can relate to the trace distance. Let $\rho$ and $\sigma$ denote quantum states defined on an arbitrary Hilbert space. The interstate distance equals a supremum over observables $O$ defined on the same space [72, Lemma 9.1.1]:

$$D_{tr}(\rho, \sigma) = \sup_{O:||O||_{op}} \left\{ \left| \text{Tr}(\rho O) - \text{Tr}(\sigma O) \right| \right\}.$$  (C21)

Let $\rho = \rho^{\text{avg}}_{\text{NATS}}$ and $\sigma = \rho^{\text{avg}}_{\text{can}}$. The operator $Q^{(1)}_{\gamma} / ||Q^{(1)}_{\gamma}||_{op}$ is one normalized $O$. Therefore, by Eq. (C20), $\frac{|q_{\gamma}|}{||Q^{(1)}_{\gamma}||_{op}}$ lower-bounds the supremum in (C21). The superscript $(1)$ can be replaced with $(j)$, due to translation invariance in the $T_{j}$ argument. Hence $D_{tr}(\rho^{\text{avg}}_{\text{NATS}}, \rho^{\text{avg}}_{\text{can}}) \geq \frac{|q_{\gamma}|}{||Q^{(1)}_{\gamma}||_{op}} > 0$. The final inequality follows from (i) the assumption (C2) and (ii) the finiteness of the single-subsystem $||Q^{(1)}_{\gamma}||_{op}$. The first inequality in (C7) follows via Pinsker’s inequality: For states $\rho$ and $\sigma$, $D(\rho||\sigma) \geq D_{tr}(\rho, \sigma)$. This proof remains true if $\rho_{\text{GC}}$ replaces $\rho_{\text{can}}$ and $[\rho_{\text{GC}}, V] = 0$. $
$

C 3 Realisation in trapped-ion experiment

The general setup of App. C 1 can be realised in the main text’s trapped-ion experiment. In the simplest realisation, $Q_{\gamma} = \sigma_{x}$. The unitary $V = \sigma^{\otimes 2N}_{x}$:

$$V \sigma^{tot}_{x} V^{\dagger} = \sigma^{\otimes 2N}_{x} \left( \sum_{j=1}^{N} \sigma^{(j)}_{x} \right) \sigma^{\otimes 2N}_{x} = \sum_{j=1}^{N} \left( -\sigma^{(j)}_{x} \right) = -\sigma^{tot}_{x}.$$  (C22)

The initial state is $|\psi_{0}\rangle = |x^{+}, y^{+}, z^{+}\rangle^{\otimes 2N/3}$, so $\langle \psi_{0}| \sigma^{tot}_{x} |\psi_{0}\rangle \propto N$, and the state is invariant under translations through $\tau = 3$ sites. Define $\rho^{\text{tot}}_{\text{GC}} := \exp \left( -\beta \left[ H^{tot} - \mu \sigma^{tot}_{z} \right] \right) / Z^{tot}_{\text{GC}}$. The effective chemical potential $\mu_{z}$ is defined as in the main text, and $Z^{tot}_{\text{GC}}$ normalises the state. $\rho^{\text{tot}}_{\text{GC}}$ can replace the canonical state in Ineq. (C7).

The mapping just described is conceptually simple. However, we find analytically, another mapping achieves the tightest bound (C7): $\frac{1}{\sqrt{3}} (\sigma_{x} + \sigma_{y} + \sigma_{z})$, and $V = \left[ \frac{1}{\sqrt{6}} (2\sigma_{x} - \sigma_{y} - \sigma_{z}) \right]^{\otimes N}$. (Alternatively, the $\sigma_{z}$’s in $V$ can be permuted in any way.)
Appendix D  SPATIO-TEMPORAL FLUCTUATIONS IN STATES’ DISTANCES TO THE NON-ABELIAN THERMAL STATE

Figure 4 shows the experimentally observed fluctuations, across space and time, of the relative entropy to the NATS. The chain consists of \( N = 12 \) ions. Every ion pair’s state approaches the NATS in time. However, nonuniformity remains; edge pairs thermalise more slowly due to edge effects, while the central pairs thermalise more quickly.

Appendix E  DERIVATIONS OF TROTTER SEQUENCES

The evolution implemented differs from evolution under the Heisenberg Hamiltonian (3) for three reasons. First, the Heisenberg Hamiltonian is Trotter-approximated. Second, parts of the Trotter approximation are simulated via native interactions dressed with rotations. Third, we reduce decoherence via dynamical decoupling. Here, we derive the experimental pulse sequence. We review parts of the setup and introduce notation in App. E 1. In App. E 2, we detail the two errors against which the pulse sequence protects. We derive the pulse sequence in App. E 3. App. E 4 extends the derivation from the Heisenberg evolution to the \( XY \) model (7).

E 1  Quick review of setup and notation

We break a length-\( t \) time interval into \( N_T \) steps of length \( t/N_T =: \Delta t \) each. We aim to simulate the Heisenberg Hamiltonian (3), whose \( J_0/|j - k|^\alpha \) we sometimes denote by \( J_{j,k} \) here. \( H_{\text{Heis}} \) generates the family of unitaries

\[
U_{\text{Heis}}(t) := e^{-iH_{\text{Heis}}t}.
\]

To effect this family, we leverage single-axis Hamiltonians

\[
H_{\gamma \gamma} := \sum_{j=1}^{N} \sum_{k>j} J_{j,k} \sigma_\gamma^{(j)} \sigma_\gamma^{(k)}.
\]

(E1)

\( H_{xx} \) and \( H_{yy} \) are native to the experimental platform. The Hamiltonians (E1) generate the unitaries \( e^{-iH_{\gamma \gamma} \Delta t} =: U_{\gamma \gamma} \). We interleave the interaction with rotations \( R_\gamma := \exp \left( -i \frac{\pi}{4} \sigma_\gamma^{\text{tot}} \right) \), for \( \gamma = x, y, z \). We denote the single-qubit identity operator by \( 1 \).

E 2  Two sources of error

Our pulse sequence combats detuning and rotation errors. The detuning error manifests as an undesired term that creeps into the Hamiltonian (3). Proportional to \( \sigma_\gamma^{\text{tot}} \), the term represents an external magnetic field. We protect against the detuning error with dynamical decoupling: The detuning error undesirably rotates each ion’s state about
the $z$-axis. We apply a $\pi$-pulse about the $x$-axis, reflecting the state through the $xy$-plane. The state then precesses about the $z$-axis oppositely, undoing the earlier precession. Another $\pi$-pulse undoes the reflection.

The second error plagues the engineered rotations: A qubit may rotate too little or too much, because the ion string is illuminated not quite uniformly. We therefore replace certain rotations $R_x$ with $R_y^\dagger$'s. An ion may rotate too much while undergoing $R_x$, but, while undergoing $R_y^\dagger$, rotates through the same angle oppositely. The excess rotations cancel.

### E 3 Derivation of Trotter sequence

First, we divvy up the Heisenberg evolution into steps. Then, we introduce rotations that enable dynamical decoupling. We Trotter-approximate a Heisenberg step in two ways. Alternating the two Trotter approximations across a pulse sequence mitigates rotation errors.

To simulate the Heisenberg Hamiltonian for a time $t$, we evolve for $N_T$ length-$\Delta t$ time steps: $U_{\text{Heis}}(t) = [U_{\text{Heis}}(\Delta t)]^{N_T}$. To facilitate dynamical decoupling, we insert an identity operator on the left: $U_{\text{Heis}}(t) = \mathbb{1}^{\otimes N}U_{\text{Heis}}(\Delta t)^{N_T}$. We decompose the $\mathbb{1}^{\otimes N}$ into rotations about the $z$-axis. How this decomposition facilitates dynamical decoupling is not yet obvious, as the rotations commute with the detuning expression. Later, though, we will commute some of the rotations across interaction unitaries. The commutation will transform the $z$-rotations into $R_x$'s. For now, we decompose the $\mathbb{1}^{\otimes N}$ in two ways:

$$U_{\text{Heis}}(t) = (R_z^\dagger)^{N_T}(R_z)^{N_T}[U_{\text{Heis}}(\Delta t)]^{N_T}$$

(E2)

$$= (R_z)^{N_T}(R_y)^{N_T}[U_{\text{Heis}}(\Delta t)]^{N_T}.$$  

(E3)

We will implement the right-hand side of (E2) during half the protocol and, during the other half, implement (E3). This alternation will mitigate rotation errors.

Let us analyze (E2), then (E3). $R_z$ commutes with $U_{\text{Heis}}(\Delta t)$ because the Heisenberg Hamiltonian conserves $\sigma_z^{\text{tot}}$: $[H_{\text{Heis}}, \sigma_z^{\text{tot}}] = 0$ implies that $[U_{\text{Heis}}(\Delta t), R_z] = 0$. The $R_y$'s of Eq. (E2) can therefore move inside the square brackets:

$$U_{\text{Heis}}(t) = (R_z^\dagger)^{N_T}[R_zU_{\text{Heis}}(\Delta t)]^{N_T}.$$  

(E4)

We Trotter-approximate the short Heisenberg evolution as

$$U_{\text{Heis}}(\Delta t) \approx U_{yy}U_{zz}U_{xx}.$$  

(E5)

The ordering of the directions is arbitrary.

We substitute into (E4) and rewrite the bracketed factor, pursuing three goals. First, the $U_{zz}$ is not native to our platform. We therefore simulate it with $R_y^\dagger U_{xx}R_y$. Second, one $R_x$ must end up amidst the $U_{\gamma\gamma}$'s. Two blocks of $U_{\gamma\gamma}$'s, each containing an $R_x$, will consequently effect one $\pi$ pulse. Composing these $\pi$ pulses will effect dynamical decoupling. Third, any other, stray $R_x$'s must be arranged symmetrically on either side of the $U_{\gamma\gamma}$'s, as explained below.

Let us replace the $U_{zz}$ in (E5) with $R_y^\dagger U_{xx}R_y$. The $R_y^\dagger$ commutes across the $U_{yy}$:

$$R_y U_{yy} U_{zz} U_{xx} = R_y U_{yy}(R_y^\dagger U_{xx} R_y) U_{xx} = R_y R_y^\dagger U_{yy} U_{xx} R_y U_{xx}. \tag{E6}$$

We have eliminated the $U_{zz}$. Similarly eliminating the $R_z$ will prove useful, so we invoke $R_z = R_y^\dagger R_x R_y$:

$$R_x U_{yy} U_{xx} = (R_y^\dagger R_x R_y) R_y U_{yy} U_{xx} R_y U_{xx} = (R_y^\dagger R_x) U_{yy} U_{xx} R_y U_{xx}. \tag{E7}$$

We will benefit from complementing the $R_y^\dagger R_x$ with a mirror image ($R_y^\dagger R_x)^\dagger = R_y R_y^\dagger$ on the right: We will implement $R_y U_{yy} U_{zz} U_{xx}$ many times, and instances of the left-hand $R_y^\dagger R_x$ will cancel instances of the right-hand $R_y R_y^\dagger$. Therefore, we insert $\mathbb{1}^{\otimes N} = R_y^\dagger R_x R_y^\dagger R_y$ into the right-hand side of (E7):

$$R_y U_{yy} U_{zz} U_{xx} = (R_y^\dagger R_x) U_{yy} U_{xx} R_y U_{xx} R_y (R_y^\dagger R_x R_y^\dagger R_y). \tag{E8}$$

2 One can prove the expressions’ equality by writing out the Taylor series for $U_{xx} = \exp(i\Delta t H_{xx})$, conjugating each term with the rotations, invoking the Euler decomposition $R_y = \cos(\pi/4)\mathbb{1}^{\otimes N} - i \sin(\pi/4)\sigma_y^{\text{tot}}$, multiplying out, and invoking $\sigma_\lambda \sigma_\nu = i \epsilon_{\lambda\nu\epsilon} \sigma_\epsilon$. The result is the Taylor series for $U_{zz}$.
Again, $U_{zz}$ is not native to our platform. We therefore commute the $R_x$ across the $U_{zz}$, invoking $R_x^1 U_{zz} R_x = U_{yy}$:

$$R_z U_{yy} U_{zz} U_{xx} = (R_y^1 R_x) U_{yy} U_{zz} R_x U_{yy}(R_x^1 R_y).$$

(E9)

The final expression has the sought-after form. We substitute into Eq. (E5), then into Eq. (E4), and then cancel rotations: $U_{\text{Heis}}(t) \approx (R_x^1)^{N_{\text{T}}} (R_y^1 R_x) (E_+)^{N_{\text{T}}} (R_y^1 R_y)$.

Suppose that $N_{\text{T}} = 4$. The $E_+$’s, containing four $R_x$’s total, implement two $\pi$ pulses—one round of dynamical decoupling. Furthermore, $(R_x^1)^4 = (-1)^{N_{\text{T}}} 1^{\otimes N}$, so

$$U_{\text{Heis}}(4\Delta t) \approx (R_x^1)^4 (R_y^1 R_x) (E_+)^4 (R_y^1 R_y) = (-1)^N (R_x^1 R_y) (E_+)^4 (R_y^1 R_y).$$

(E10)

Now, let $N_{\text{T}} \gg 4$, as in the experiment. After one round of dynamical decoupling, to mitigate the detuning error, we mitigate rotation errors. We effect four time steps with an alternative operator derived from Eq. (E3). Then, we continue alternating.

Let us derive the alternative to $E_+$. We shift the $R_x^1$’s of Eq. (E3) inside the square brackets:

$$U_{\text{Heis}}(t) = (R_x)^{N_{\text{T}}} \left[ R_x^1 U_{\text{Heis}}(\Delta t) \right]^{N_{\text{T}}} \approx (R_x)^{N_{\text{T}}} \left[ R_x^1 U_{yy} U_{zz} U_{xx} \right]^{N_{\text{T}}}.$$  

(E11)

The final expression follows from Eq. (E5). The bracketed factor must end up with the $(R_x^1 R_y)^{\text{st}} \ldots (R_x^1 R_y)$ structure of Eq. (E9), so that rotations cancel between instances of (E9) and instances of the new bracketed factor. We therefore ensure that $R_x^1 R_x$ is on the factor’s left-hand side, then propagate extraneous rotations leftward:

$$
\begin{align*}
R_x^1 U_{yy} U_{zz} U_{xx} &= R_x^1 R_y^1 R_y = R_x^1 (R_y R_x) R_y^1 R_x^1 R_y^1 R_y, \\
&= (R_x^1 R_x) (R_y^1) R_x^2 R_y U_{yy} U_{zz} U_{xx} = (R_x^1 R_x) R_x^1 R_y U_{yy} U_{zz} U_{xx} \\
&= (R_y^1 R_x) R_x^1 U_{yy} U_{zz} U_{xx} R_y U_{yy} U_{zz} U_{xx} = (R_y^1 R_x) U_{yy} R_x^1 U_{xx} R_y U_{yy} U_{zz} U_{xx} R_y \\
&= (R_y^1 R_x) R_x^1 U_{yy} U_{xx} R_y U_{yy} U_{zz} R_y = (R_y^1 R_x) U_{yy} R_x^1 U_{xx} R_y U_{yy} U_{zz} R_y \\
&= (R_y^1 R_x) U_{yy} U_{xx} R_y U_{yy} R_x^1 R_y = (R_y^1 R_x) U_{yy} U_{xx} R_x^1 U_{yy} R_y \\
&= (R_y^1 R_x) U_{yy} U_{xx} R_y U_{yy} R_x^1 R_y = (R_y^1 R_x) U_{yy} U_{xx} R_x^1 U_{yy} R_y \\
&= (R_y^1 R_x) U_{yy} U_{xx} R_y U_{yy} R_x^1 R_y = (R_y^1 R_x) U_{yy} U_{xx} R_x^1 U_{yy} R_y.
\end{align*}
$$

(E12)

By Eq. (E11), $U_{\text{Heis}}(t) \approx (R_x)^{N_{\text{T}}} (R_x^1 R_y) (E_-)^{N_{\text{T}}} (R_y^1 R_y)$. Analogously to Eq. (E10),

$$U_{\text{Heis}}(4\Delta t) \approx (-1)^N (R_x^1 R_y) (E_-)^4 (R_y^1 R_y).$$  

(E16)

We alternate instances of (E10) with instances of (E16) to simulate long Heisenberg evolutions. Many rotations cancel. If $N_{\text{T}}$ equals an integer multiple of eight,

$$U_{\text{Heis}}(t) \approx R_y^1 R_x [(E_-)^4 (E_+)^4]^{N_{\text{T}}/8} R_x^1 R_y.$$  

(E17)

### E 4 Extension from Heisenberg model to XY model

In the Results, we experimentally compared the Heisenberg evolution with evolution under the XY model, Eq. (7). $H_{xy}$ generates the unitaries $U_{xy}(t) := \exp(-itH_{xy})$. We can more easily Trotterise $U_{xy}(t)$ while mitigating errors than Trotterise $U_{\text{Heis}}(t)$.

As before, we divvy up the evolution into steps. Then, we Trotter-approximate the steps and insert $1^{\otimes N} = \left[ (R_x^1)^2 \right]^{N_{\text{T}}} (R_y^2)^{N_{\text{T}}}$:

$$
\begin{align*}
U_{xy}(t) &= \left[ U_{xy}(\Delta t) \right]^{N_{\text{T}}} \approx (U_{xy} U_{xx})^{N_{\text{T}}} \\
&= \left[ R_y^1 \right]^{N_{\text{T}}} (R_y^2)^{N_{\text{T}}} (U_{yy} U_{xx})^{N_{\text{T}}}.
\end{align*}
$$

(E18)
Due to the square, $R_z^2$ commutes with $U_{yy}U_{xx}$:

$$R_z^2 U_{yy}U_{xx} = R_z R_z U_{yy}U_{xx} = R_z U_{zz} R_z U_{xx} = U_{yy} R_z^2 U_{xx} = U_{yy} U_{xx} R_z^2 .$$  \hspace{1cm} (E20)

Therefore, in Eq. (E19), we can pull the $(R_z^2)^{N_T}$ into the parentheses:

$$U_{xy}(t) = \left[ (R_z^2) \right]^2 \left( R_z U_{yy}U_{xx} \right)^{N_T} .$$  \hspace{1cm} (E21)

We could commute the $R_z^2$ into the center of the $U_{rr}$’s, to improve the dynamical decoupling. However, Eq. (E21) suffices; errors accumulate in only a couple of gates.

The operator $F_+ := R_z^2 U_{yy}U_{xx}$ contains a $\pi$-pulse. Therefore, we need perform $F_-$ only twice before implementing $F_- := (R_z^2)^2$. Furthermore, $\left[ (R_z^2)^2 \right]^2 = (-1)^N$. If $N_T$ is a multiple of four, then $U_{xy}(t) = \left[ (F_-)^2 (F_+)^2 \right]^{N_T/4}$.

**Appendix F ASSESSMENT OF NOISE-ROBUST TROTTER SEQUENCE**

App. E describes the Trotter sequence that we engineered to alleviate errors. Here, we demonstrate the sequence’s effectiveness in numerical simulations and in the experiment. Figure 5 shows the dynamical decoupling’s effects in the parameter regime used experimentally. Constant detuning errors of up to several hundred Hertz do not significantly reduce the time-evolved state’s fidelity to the ideal state ($< 10\%$ change in fidelity up to 500 Hz detuning error). Similarly, systematic rotation errors of $\pm 10\%$ affect the fidelity little ($4\%$ drop in fidelity). If the detunings oscillate temporally, the dynamical decoupling’s robustness depends heavily on the oscillation frequency, $f$: Recall that $t_\ell = 15$ ms denotes the experiment’s temporal length and $N_T$ denotes the number of Trotter steps. Consider an individual qubit state expressed as a combination of outer products of $\sigma_z$ eigenstates. If $f$ is an integer multiple of $f_1 = \frac{1}{2}(4t_\ell/N_T)^{-1} = 300$ Hz, the qubit’s state acquires a relative phase, reducing the fidelity to the ideal state.

![FIG. 5: Dynamical decoupling: The simulation was performed with 12 ions, a power-law approximation to the coupling, $J_0 = 510$ rad/s, $\alpha = 1.02$, and 10 ms of evolution. The fidelity compares the simulated Trotter-approximated state with the exact ideal state: $\langle \text{Tr} \sqrt{\rho_{\text{exact}} \rho_{\text{Trotter}} \sqrt{\rho_{\text{exact}}}} \rangle^2$. (a) Introducing $\pi/2$ rotations into the Trotter sequence guards against detuning errors. The right-hand side (RHS) of Eq. (E5) defines the sequence $U^{(0)}$, the RHS of Eq. (E16) defines $U^{(1)}$, and the RHS of Eq. (E17) defines $U^{(1\text{alt})}$. (b) Alternating the rotation’s direction guards against systematic rotation errors. (c) Response to oscillations of a time-varying magnetic field $B = B_{\text{amp}} \cos(2\pi ft)z$, wherein $B_{\text{amp}} \geq 0$ (15 ms evolution). The dynamically decoupled Trotter sequence $U^{(1\text{alt})}$ allows the fidelity to drop. The drops occur when the field’s frequency, $f$, is an integer multiple of $f_1 = \frac{1}{2}(4t_\ell/N_T)^{-1} = 300$ Hz. We can understand this behavior most simply when $J_0 = 0$ (top curves): The qubits do not interact, so each qubit remains in a superposition, whose relative phase undesirably changes under $B$. The experimentally observed two-qubit fidelities ($\langle \text{Tr} \sqrt{\rho_{\text{exact}} \rho_{\text{exp}} \sqrt{\rho_{\text{exact}}}} \rangle^2$ range from 0.995(4), at $t = 0$, to 0.97(1), at $t = t_\ell$ (Fig. 6). The fidelity is lower for pairs near the ion crystal’s edges than for pairs near the center. This contrast results, at $t = 0$, from imperfections in the state preparation caused by the inhomogeneous profile of the global rotations. At $t = t_\ell$, the lower fidelities result from global rotation errors in the Trotter sequence. The total number of Trotter steps, 36, is not an integer multiple of 8. Consequently, a period of the alternating Trotter sequence (E17) is not completed, and rotation errors remain.
FIG. 6: Measured fidelity of the state resulting from the Trotter approximation: The experimentally observed fidelity, averaged over the ion pairs, is shown across the evolution. Error bars indicate the standard deviation over the pairs. The smaller subplots show the fidelity, at the evolution’s start and end, from individual ion pairs’ actual states to their ideal states. The measurements were carried out (a) for the Trotter-approximated Heisenberg Hamiltonian and (b) for the Trotter-approximated XY Hamiltonian.

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