Collective excitation frequencies and damping rates of a two-dimensional deformed trapped Bose gas above the critical temperature

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We derive a equation of motion for the velocity fluctuations of a 2D deformed trapped Bose gas just above the critical temperature in the hydrodynamical regime. From this equation, we calculate the eigenfrequencies and the corresponding density fluctuations for a few low-lying excitation modes. Using the method of averages, we derive a dispersion relation in a deformed trap at very high temperature that interpolates between the collisionless and hydrodynamic regimes. We make use of this dispersion relation to calculate the frequencies and the damping rates for monopole and quadrupole mode in both the regimes. We also discuss the possibility of creating a quasi-2D trapped Bose gas

It is well known that the excitation frequencies for monopole and quadrupole modes are 2ω0 and √2ω0 respectively in a 2D isotropic trapped Bose gas. Using the approximation, ωz ≫ ω, the dispersion relation of the excitation frequencies does not produce the correct frequencies for monopole and quadrupole modes in a 2D trapped Bose system. There has been no systematic study on the collective excitations of a 2D deformed trapped Bose gas above the critical temperature.

The purpose of this paper is to give analytic results for the dispersion law of low-lying collective modes in 2D deformed trapped Bose gas and their damping rates in both regime, hydrodynamic and collisionless.

Above the critical temperature, one can distinguish two regimes, the hydrodynamic(collisional) one where collisions ensure the local thermal equilibrium and collisionless where the motion is described by the single particle hamiltonian. In the hydrodynamic region, the characteristic mode frequency is small compared to the collision frequency (ωτ << 1). In the collisionless region (ωτ >> 1), the collision are not important.

The paper is organised as follows. We derive in sec[II] a closed equation of motion for the velocity fluctuations of a 2D deformed trapped Bose gas just above the critical temperature (T > Tc) using the kinetic theory. We make use of this equation in sec[III] to calculate the excitation frequencies for a few low-lying collective modes and the corresponding density fluctuations. In sec[IV], we derive a dispersion relation in a 2D deformed trap at very high temperature using the method of averages that interpolates between the collisionless and hydrodynamic regimes. From this dispersion relation, we calculate the eigenfrequencies and damping rates for monopole and quadrupole mode. We discuss the evolution of the wave packet width of a Bose gas in a time independent as well as time dependent trap.

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approximation for the dynamics of a Bose gas, using the
following Boltzmann equation \[13\] for the phase space
distribution function \(f(\vec{r}, \vec{p}, t)\)
\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \vec{F} \cdot \nabla f = I_{\text{coll}}(f)
\] (1)
where \(I_{\text{coll}}\) is the collisional integral and \(\vec{F} = -\nabla U_0(\vec{r})\).
The trap potential is \(U_0(\vec{r}) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2)\).

In the hydrodynamic regime, collisions ensure the local thermodynamic equilibrium. To lowest order, the per-
momentum \(\vec{p}\) distribution function \(\eta(\vec{r},\vec{p},t)\) is
\[
\eta(\vec{r},\vec{p},t) = \left[\exp(\beta(\vec{r},t)\vec{p}) - 1\right]^{-1}
\] (2)
\(
\beta(\vec{r},t) = \frac{1}{k_B T} \left(\frac{1}{g(\vec{r})} - \frac{\mu(\vec{r},t)}{k_B T} \right)
\) (3)
\(
\mu(\vec{r},t)\) is the chemical potential. The inverse temperature \(\beta(\vec{r})\) is.
In classical limit, the static density profile is
\[
\rho_0(\vec{r}) = \frac{n(\vec{r})}{\pi^{1/2} \Lambda^3}
\] (4)
\[
\Lambda = \left(\frac{2\pi^2 m}{\hbar^2} \right)^{1/3}
\] (5)
where \(g_n(z) = \sum_{n=1}^{\infty} (\frac{1}{n!})^2\) are the Bose-Einstein functions.

Using the quantum statistical mechanics, pressure and
density can be written as
\[
P = \frac{\rho_0(\vec{r})}{k_B T} = \frac{g_2(z)}{\Lambda^2}
\] (6)
\[
n(\vec{r}) = \frac{g_1(z)}{\Lambda^2}
\] (7)
where \(g_1(z) = \sum_{n=1}^{\infty} (\frac{1}{n!})\) are the Bose-Einstein functions.
\(z(r,t) = e^{\beta(r,t)\mu(r,t)}\) is the local thermodynamic fugacity
which is always less than one. \(\Lambda = \sqrt[3]{\frac{2\pi^2 m k_B T}{\hbar^2}}\) is the thermal
de-Broglie wave length. One can easily get the relation,
\[
P(\vec{r}, t) = E(\vec{r}, t)
\] (8)
in 2D. Using Eq. (6), Eq. (7) can be written as
\[
\frac{\partial P(\vec{r}, t)}{\partial t} = -2\nabla \cdot \left[ (P_0(\vec{r}) \delta \vec{v}(\vec{r}, t) - \nabla U_0(\vec{r})) \right]
\] (9)
Taking the time derivative of Eq. (6) and using Eqs. (4) and (13), we get
\[
m \frac{\partial^2 \delta \vec{v}}{\partial t^2} = 2 \frac{P_0(\vec{r})}{n_0(\vec{r})} \nabla [\nabla \cdot \delta \vec{v}] - \nabla \cdot \nabla U_0(\vec{r}) - \nabla \delta \vec{v} \nabla U_0(\vec{r})
\] (10)
The closed equation of motion for the velocity fluctuations
has been derived by Griffin et.al \[3\] for 3D trapped
Bose system. The term \(\frac{P_0(\vec{r})}{n_0(\vec{r})}\) of (11) is associated with
the Bose statistics.
Without any external potential \(U_0 = 0\), the Eq. (11)
becomes
\[
m \frac{\partial^2 \delta \vec{v}}{\partial t^2} = 2 \frac{P_0(\vec{r})}{n_0(\vec{r})} \nabla [\nabla \cdot \delta \vec{v}]
\] (12)
It has the plane wave solution with the dispersion relation
\(\omega^2 = c^2 k^2\). The sound velocity is \(c^2 = \frac{2k_B T}{m n_0(\vec{r})}\) or \(c^2 = \frac{2k_B T \rho_0(z_0)}{\rho_0(z_0)}\) where \(z_0 = e^{\mu_0(r)}/m\) and \(\mu_0(r) = \mu - U_0(r)\).
At high temperature \((z << 1)\), the sound velocity becomes
\(c^2 = \frac{2k_B T}{m}\). This sound velocity exactly matches with known result.
From the continuity Eq. (8) we have,
\[
\frac{\partial \delta n(\vec{r}, t)}{\partial t} = -\nabla \cdot \left( \delta \vec{v} \nabla n_0(\vec{r}) \right)
\] (13)
The density fluctuation is given by \(\delta n(\vec{r}, t) = \delta n(\vec{r}) e^{-i \omega t}\).
In classical limit, the static density profile is \(n_0(\vec{r}) = n_0(\vec{r}) = n_0(\vec{r} = 0) e^{-\frac{m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)}{2}}\), where \(\theta = k_B T\).

III. EIGENFREQUENCIES AND THE
CORRESPONDING DENSITY FLUCTUATIONS
IN THE HYDRODYNAMIC REGIME

1) The normal mode solution of (11) is \(\delta \vec{v}(\vec{r}) = \nabla (z')\),
here \(z = (x + iy)\) and \(l > 0\). The excitation frequencies
and the associated density fluctuations are \(\omega_x^2 = \omega_y^2\), \(\delta n_x \sim \omega_x^2 x^2 (l-1)n_0(\vec{r})\) and \(\omega^2 = \omega_x^2 + (l-1)\omega_y^2\), \(\delta n_y \sim \omega_y^2 y^2 (l-1)n_0(\vec{r})\).
For isotropic trap, the frequency is \(\omega = \omega_0 \sqrt{1}\). The corresponding density fluctuation is \(\delta n(\vec{r}) \sim n_0(\vec{r}) e^{i \theta}\).
At \(r = 0\) there is no density fluctuation. There is a maximum density fluctuation at \(r = \sqrt{\frac{\omega_0}{\omega_0^2}}\).

2) The other solution of Eq. (11) is \(\delta \vec{v}(\vec{r}) = \nabla [\alpha x^2 \pm \beta y^2]\). The positive sign is for the monopole mode and
the negative sign is for quadrupole mode. In a deformed trap, the excitation frequencies are
\[
\omega^2 = \frac{1}{2} [3(\omega_x^2 + \omega_y^2) \pm \sqrt{9(\omega_x^2 + \omega_y^2)^2 - 32\omega_x^2\omega_y^2}] \quad (14)
\]
For an isotropic trap, it becomes \( \omega = 2\omega_0 \) or \( \omega = \sqrt{2}\omega_0 \). Hence in the anisotropic trap the monopole mode is coupled to the quadrupole mode. If \( \omega_x << \omega_y \), the lowest excitation frequency is \( \omega = \sqrt{\frac{2}{3}}\omega_x \). If \( \omega_x >> \omega_y \), the lowest excitation frequency is \( \omega = \sqrt{\frac{2}{3}}\omega_y \). The density fluctuation for the monopole mode is \( \delta n(\vec{r}) \sim [2 - (m\omega_y^2 + \omega_y^2)]n_0(\vec{r}) \) where as the density fluctuation for quadrupole mode is \( \delta n(\vec{r}) \sim (\omega_x^2\omega_y^2 - \omega_x^2\omega_y^2)n_0(\vec{r}) \).

3) There is another quadrupole mode which has velocity field \( \delta \vec{v}(\vec{r}) = \nabla (xy) \). This is also called scissor mode [16]. The excitation frequency is \( \omega^2 = \omega_x^2 + \omega_y^2 \) and the corresponding density fluctuation is \( \delta n(\vec{r}) \sim (\omega_x^2 + \omega_y^2)xy n_0(\vec{r}) \). In isotropic trap \( \omega^2 = 2\omega_0^2 \), which agrees with that for the scissors mode in hydrodynamic regime above \( T_c \) [14].

**IV. METHOD OF AVERAGES**

At very high temperature, the dynamical behaviour of a dilute gas is described by the Boltzmann transport equation. Here we include the collisional term in the Boltzmann transport equation and study the eigenfrequencies for monopole and quadrupole mode using the method of averages [13]. These two modes are coupled in a deformed trap.

From Eq. (13), one can get the equations for the average of a dynamical quantity \( \chi(\vec{r}, \vec{v}) \) is [13], [14]
\[
\frac{d < \chi >}{dt} = < \vec{v} \cdot \nabla \chi > = < \frac{\vec{F}}{m} \cdot \nabla \chi > = < I_{coll} \chi > \quad (15)
\]
where the average is taken in phase space and \( < \chi > \) can be written as
\[
< \chi > = \frac{1}{N} \int d^2r d^2v f(\vec{r}, \vec{v}, t) \chi(\vec{r}, \vec{v}) \quad (16)
\]
\[
< \chi I_{coll} > \text{ can be defined as}
\]
\[
< \chi I_{coll} > = \frac{1}{4N} \int d^2r d^2v [\chi_1 + \chi_2 - \chi'_1 - \chi'_2] I_{coll}(f) \quad (17)
\]
If \( \chi = a(\vec{r}) + b(\vec{r}).\vec{v} + c(\vec{r})\vec{v}^2 \), for elastic collision the collisional term is zero [3], [13]. \( a, b, \) and \( c \) are arbitrary functions of the position.

Now we define the following quantities,
\[
\chi_1 = x^2 + y^2 \quad (18)
\]
\[
\chi_2 = y^2 - x^2 \quad (19)
\]
\[
\chi_3 = xv + yv \quad (20)
\]
\[
\chi_4 = yv - xv \quad (21)
\]
\[
\chi_5 = v_x^2 + v_y^2 \quad (22)
\]
\[
\chi_6 = v_y^2 - v_x^2 \quad (23)
\]
Using the Boltzmann kinetic equation [13], we get the following closed set of equations.
\[
< \ddot{\chi}_1 > = 2 < \chi_5 > - t < \chi_1 > + \epsilon < \chi_2 > \quad (24)
\]
\[
< \ddot{\chi}_2 > = 2 < \chi_6 > - t < \chi_2 > + \epsilon < \chi_1 > \quad (25)
\]
\[
< \ddot{\chi}_3 > = 2 < \epsilon > < \chi_4 > - 2t < \chi_3 > \quad (26)
\]
\[
< \ddot{\chi}_4 > = 2 < \epsilon > < \chi_3 > - 2t < \chi_4 > - < \frac{\chi_6}{\tau} > \quad (27)
\]
\[
< \ddot{\chi}_5 > = < \epsilon > < \chi_6 > - t < \chi_5 > - \epsilon t < \chi_2 > \quad (28)
\]
\[
+ < \frac{\epsilon^2 + t^2}{2} > < \chi_1 > \quad (29)
\]
\[
< \ddot{\chi}_6 > = - \epsilon t < \chi_1 > + < \frac{\epsilon^2 + t^2}{2} > < \chi_2 > \quad (29)
\]
\[
+ \epsilon < \chi_5 > - < \frac{\chi_6}{\tau} > - t < \chi_6 > \quad (29)
\]
where double dot indicates the double derivative with respect to time. \( t = \omega_x^2 + \omega_y^2 \) and \( \epsilon = \omega_x^2 - \omega_y^2 \). The \( \chi_6 \) is not a conserved quantity, so the collisional contribution comes only through the \( \chi_6 \) term. We have used the fact that \( < \chi_6 I_{coll} > = - \frac{\epsilon}{\tau} \), where \( \tau \) is the relaxation time. This relaxation time \( \tau \) can be computed by a gaussian ansatz for the distribution function. The relaxation time \( \tau \) is order of the inverse of the collision rate \( \gamma_{coll} \sim n(0)v_{th}\sigma_0 \), where \( v_{th} = \sqrt{\frac{k_B T}{2m}} \) is the mean thermal velocity and \( n(0) = \frac{Nn_0\omega_0^2}{2\pi k_B T} \) is the central density for a quasi-2D system. \( \sigma_0 \) is the oscillator length in the z-direction. Hence \( \tau \sim \frac{Nn_0\omega_0^2}{2\pi k_B T} \frac{1}{m\pi^2\sqrt{\xi} (N\lambda)^\pi < \frac{a_z}{2\omega_0^3/2} > \frac{\sqrt{T}}{T_c} \quad (30) \)

The relaxation time \( \tau \) varies as \( \sqrt{T} \) in a quasi-2D where as in 3D it varies as \( \frac{T}{T_c} \). Now we are looking for a solutions of Eqs. (24), (25) as \( e^{-\omega t} \). We have the following dispersion relation
\[(\omega^2 - 4\omega_0^2)(\omega^2 - 4\omega_0^2) + \frac{i}{\omega_T}[\omega^4 - 3\omega^2(\omega_0^2 + \omega^2)] + 8\omega_0^2\omega^2 = 0\]

This dispersion relation interpolates between the collisionless and hydrodynamic regimes. In the hydrodynamical regime (\(\omega_T \rightarrow 0\)), the first term does not contribute. It gives \(\omega^2 = \frac{1}{4}(3(\omega_0^2 + \omega^2) \pm \sqrt{9(\omega_0^2 + \omega^2)^2 - 32\omega_0^2\omega^2})\). This eigen frequencies exactly matches with Eq. (14), a result we found using the equation of motion for the velocity fluctuations even in deformed trap also. We have considered a few low energy excitation modes for which \(\nabla \delta n\) is constant. The first term of the right-hand side of Eq. (14) does not contribute in the excitation spectrum. That why the frequencies of these normal modes are same for a Bose gas just above \(T_c\) and at very high temperature. In pure collisionless regime (\(\omega_T \rightarrow \infty\)), it gives \(\omega_C = 2\omega_0\) and \(\omega_C = 2\omega_0\).

We can write phenomenological interpolation formula for the frequency and the damping rate of the modes in the following form \([6, 7]\),

\[\omega^2 = \omega_C^2 + \omega_H^2 - \omega_C^2 \frac{1}{1 - i\omega_T}\]

The imaginary part of the above equation gives for the damping rate

\[\Gamma = \frac{\tau}{2} \frac{d}{1 + (\omega_T)^2}\]

where \(d = (\omega_C^2 - \omega_H^2)\). In the hydrodynamic limit (\(\omega_T \rightarrow 0\)), the damping rate is

\[\Gamma_{HD} = \frac{\tau}{2} d\]

where as in the collisionless region (\(\omega_T \rightarrow \infty\)),

\[\Gamma_{CL} = \frac{d}{2\omega_C^2 \tau}\]

The damping rate depends on the difference between the square of the frequencies in the collisional and hydrodynamical regime. The damping rates can be calculated for different values of temperature, number of trapped atoms as well as of the trapping parameters and scattering length through the relaxation time \(\tau\) \([20]\). For monopole mode in an isotropic trap the difference \(d\) is zero. So there is no damping in the monopole mode in a 2D isotropic trapped Bose system when the temperature is very high. It was first shown by Boltzmann \([4]\) and later Odelin et.al \([8]\) in a 3D trapped Bose system at very high temperature.

For isotropic harmonic trap, Eqs. \([24] - [29]\) decouples into two subsystem, one is for monopole and other one for quadrupole mode. The closed set of equations for monopole mode are

\[\dot{\chi}_1 = 2 < \chi_5 > - 2\omega_0^2 < \chi_1 >\]

\[\dot{\chi}_3 = -4\omega_0^2 < \chi_3 >\]

\[\dot{\chi}_5 = 2\omega_0^4 < \chi_1 > + 2\omega_0^2 < \chi_3 >\]

There is no collisional term in the above equations. So there is no damping for the monopole mode of a classical dilute gas confined in an isotropic trap. We are looking for solutions of Eqs. \([36] - [38]\) as \(e^{-\omega t}\), we get \(\omega = 2\omega_0\).

The Eqs. \([36] - [38]\) can be re-written as

\[< \ddot{\chi}_1 > = \frac{1}{2} < \chi_1 >^2 + 2\omega_0^2 \chi_1 = \frac{Q}{\lambda_1}\]

where \(Q = 2(< \chi_1 > < \chi_3 > - < \chi_3 >^2)\) is invariant quantity under time evolution. Define \(X(t) = \sqrt{< \chi_1 >}\) which is the wave packet width and substituting it into Eq. \([33]\) gives,

\[\ddot{X} + \omega_0^2 X = \frac{Q}{X_3}\]

This is a nonlinear singular Hill equation. The same equation is obtained at \(T = 0\) in 2D by Garcia Ripoll et.al \([13]\). At equilibrium, \(X_0^2 = \frac{Q}{\omega_0^2}\). We linearized the Eq. \([40]\) around the equilibrium point \(X_0\), we get

\[\delta\ddot{X} + 4\omega_0^2 \delta X = 0\]

One obtains an oscillation frequency of the gas is \(\omega = 2\omega_0\), corresponding to the frequency of a single particle excitation in the gas.

For time dependent trap, the equation of motion for the width of the wave packet is

\[\ddot{X} + \omega_0^2(t)X = \frac{Q}{X_3}\]

The general solution \([7]\) is \(X(t) = \sqrt{u^2(t) + \frac{Q}{\omega_0^2}} v^2(t)\) where \(u(t)\) and \(v(t)\) are two linearly independent solutions of the equation \(\ddot{u} + \omega_0^2(t)u = 0\) which satisfy \(u(t_0) = X(t_0), \dot{u}(t_0) = X'(t_0), v(t_0) = 0, \dot{v}(t_0) \neq 0\). W is the Wronskian. This time dependent Hill equation \([12]\) can be solved explicitly only for a paricular choices of \(\omega_0(t)\).

The closed set of equation for quadrupole mode in isotropic trap are

\[< \ddot{\chi}_2 > = 2 < \chi_6 > - 2\omega_0^2 < \chi_2 >\]

\[< \ddot{\chi}_4 > = -4\omega_0^2 < \chi_4 > - \frac{\chi_6}{\tau}\]

\[< \ddot{\chi}_6 > = 2\omega_0^4 < \chi_2 > - \frac{\chi_6}{\tau} - 2\omega_0^2 < \chi_6 >\]
Solving this set of equation, we get damped quadrupole mode,

\[ (\omega^2 - 4\omega_0^2) + \frac{i}{\omega}\omega_\tau (\omega^2 - 2\omega_0^2) = 0 \]  

(46)

In the hydrodynamic regime, the oscillation frequency is \( \omega^2 = 2\omega_0^2 \) whereas in the collisionless region, the frequency is just a single particle oscillator frequency. The damping rate can be calculated by using the Eqs. (34) and (35).

V. SUMMARY

In this work, we derived the equations of motion for velocity fluctuations of a Bose gas in a 2D deformed trap potential just above the critical temperature. When \( U_0 = 0 \), it becomes a wave equation, from which we found the exact sound velocity at high temperature. We have also computed the frequency of the scissors mode in hydrodynamic regime above \( T_c \) which agrees with the result obtained by Odelin et al [16]. We have also calculated the frequencies for monopole and quadrupole mode and the corresponding density fluctuations in a deformed trap above \( T_c \).

Using the method of averages, we obtained a dispersion relation that interpolates between the collisionless and hydrodynamic regimes at very high temperature. In a deformed trap as well as an isotropic trap, we have found the frequencies and the damping rates (in terms of the relaxation time) for monopole and quadrupole mode in both the regimes. In hydrodynamical regime, the excitation frequencies for monopole and quadrupole mode exactly matches with the previous result that we have found from equation (11). We have also shown that the relaxation time \( \tau \) varies as \( \sqrt{T} \) in quasi-2D Bose gas.

We have shown that there is no damping for monopole mode in a 2D isotropic trapped Bose gas when the temperature is very high. It was shown by Boltzmann [14], later Odelin et al [6] for 3D isotropic trap.

We have discussed about the time evolution of the wave packet width of a Bose gas in a time independent as well as time dependent isotropic trap. It can be described by the solution of the Hill equation.

VI. ACKNOWLEDGEMENT

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