Ideal theory of infinite directed unions of local quadratic transforms

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Abstract
Let \((R, \mathfrak{m})\) be a regular local ring of dimension at least 2. Associated to each valuation domain birationally dominating \(R\), there exists a unique sequence \(\{R_n\}\) of local quadratic transforms of \(R\) along this valuation domain. We consider the situation where the sequence \(\{R_n\}\) is infinite, and examine ideal-theoretic properties of the integrally closed local domain \(S = \bigcup_{n \geq 0} R_n\). Among the set of valuation overrings of \(R\), there exists a unique limit point \(V\) for the sequence of order valuation rings of the \(R_n\). We prove the existence of a unique minimal proper Noetherian overring \(T\) of \(S\), and establish the decomposition \(S = T \cap V\). If \(S\) is archimedian, then the complete integral closure \(S^*\) of \(S\) has the form \(S^* = W \cap T\), where \(W\) is the rank 1 valuation overring of \(V\).

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1. Introduction
Let \((R, \mathfrak{m})\) be a \(d\)-dimensional regular local ring with \(d \geq 2\). The morphism \(\phi : \text{Proj } R[\mathfrak{m}t] \to \text{Spec } R\) defines the blow-up of the maximal ideal \(\mathfrak{m}\) of \(R\). Let \((R_1, \mathfrak{m}_1)\) be the local ring of a point in the fiber of \(\mathfrak{m}\) defined by \(\phi\). Then \(R_1\), called
a local quadratic transform of $R$, is a regular local ring of dimension at most $d$ that birationally dominates $R$. Local quadratic transforms have historically played an important role in resolution of singularities and in the understanding of regular local rings. Classically, Zariski’s unique factorization theorem for ideals in a 2-dimensional regular local ring \[36\] relies on local quadratic transforms in a fundamental way. The 2-dimensional regular local rings birationally dominating $R$ are all iterated local quadratic transforms of $R$, and they are in one-to-one correspondence with the simple complete $m$-primary ideals of $R$. More recently, Lipman \[24\] uses similar methods to prove a unique factorization theorem for a special class of complete ideals in regular local rings of dimension $\geq 2$.

By taking regular local rings of dimension at least 2, iteration of the process of local quadratic transforms yields an infinite sequence $\{(R_n, m_n)\}_{n \geq 0}$. We consider the directed union of this infinite sequence of local quadratic transforms, and set $S = \bigcup_{n \geq 0} R_n$. Since the rings $R_n$ are local rings that are linearly ordered under domination, $S$ is local, and since the rings $R_n$ are integrally closed, so is $S$. However, if $S$ is not a discrete valuation ring, then $S$ is not Noetherian. On the other hand, one may consider a valuation ring $(V, \mathfrak{N})$ that dominates $R$. There is a unique local quadratic transform $R_1$ of $R$ that is dominated by $V$, called the local quadratic transform of $R$ along $V$. If $V$ is the order valuation ring of $R$, then $R_1 = V$, but otherwise, one may take the local quadratic transform $R_2$ of $R_1$ along $V$. Specifically, $R_1 = R[\frac{m}{x}]_{x \in R[\frac{m}{x}]}$, where $x \in m$ is such that $xV = mV$.

Iterating this process yields a possibly infinite sequence $\{(R_n, m_n)\}$ of local quadratic transforms, where this process terminates if and only if $V$ is the order valuation ring of some $R_n$. Abhyankar \[1, \text{Proposition 4}\] proves that this sequence is finite if and only if the transcendence degree of $V/\mathfrak{N}$ over $R/m$ is $d - 1$ (that is, by the dimension formula \[25, \text{Theorem 15.5}\], the residual transcendence degree of $V/\mathfrak{N}$ is as large as possible; in such a case $V$ is said to be a prime divisor of $R$). Otherwise, the induced sequence is infinite, and it is in this case that we are especially interested in this article.

Abhyankar \[1, \text{Lemma 12}\] proves that if $\dim R = 2$, then the union $\bigcup_{i \geq 0} R_i$ is equal to $V$. For the setting where $\dim R > 2$, Shannon \[35\] presents several examples showing that the directed union $S = \bigcup_{i \geq 0} R_i$ is properly contained in $V$, and in particular $S$ is not a valuation ring. More generally, Lipman \[24, \text{Lemma 1.21.1}\] observes that if $P$ is a nonmaximal prime ideal of the regular local ring $R$, then there exists an infinite sequence of local quadratic transforms of $R$ whose union $S$ is contained in $R_P$. Thus if we take $P$ so that $\dim P > 1$, then $S$ cannot be a valuation ring, since the overring $R_P$ of $S$ is not a valuation ring. Thus arises the question of the nature of $S$ when $S$ is not a valuation ring.

Since Shannon’s work in \[35\] has sparked the present authors’ interest in this topic, we refer to the directed union $S = \bigcup_{i \geq 0} R_i$ of the local quadratic transforms
of $R$ along a valuation ring as the Shannon extension of $R$ along this valuation ring. In this article we examine the nature of Shannon extensions, with special emphasis on the ideal theory and representation of such rings. We prove in Theorem 5.4 that if $S$ is a Shannon extension, then there exists a unique minimal proper Noetherian overring $T$ of $S$ and a valuation overring $V$ of $R$ such that $S = T \cap V$. The ring $T$ is even a localization of one of the $R_i$ and hence is itself a regular Noetherian domain (Theorem 4.1). The valuation domain $V$, which we term the boundary valuation ring of $S$, is the unique limit point in the patch topology of the order valuation rings of the regular local rings $R_i$ (Corollary 5.3). While $V$ determines the sequence of the $R_i$’s, it is not generally unique in doing so. In fact, if $S$ is not a valuation ring, then there are infinitely many valuation rings that give rise to the same Shannon extension. However, in light of its topological interpretation, the boundary valuation ring is in a sense the valuation ring that is “preferred” by the sequence.

From this representation we deduce in Corollary 5.6 that the principal $N$-primary ideals of $S$ are linearly ordered with respect to inclusion. In Theorem 6.2, we use the representation of $S$ to describe the complete integral closure $S^*$ of $S$ in the case where $S$ is archimedean, and in Theorem 6.9 we describe $S^*$ in the case where $S$ is not archimedean. Along the way in Sections 3 and 4 we also describe the maximal and nonmaximal prime ideals of Shannon extensions. To illustrate several of the ideas in the paper, we present details in Examples 7.2 and 7.4 about an example given by David Shannon that motivated our work in this paper. For these examples, we explicitly describe the Noetherian hull and the complete integral closure of the Shannon extension $S$.

As we show in Section 8, our representation of a Shannon extension $S$ also proves useful in determining when $S$ is a valuation ring, since in terms of our representation theorem, this is equivalent to asking when $S$ is equal to its boundary valuation ring. Shannon proves in [35, Prop. 4.18] that if $S$ is a Shannon extension $S = \bigcup R_i$ along a valuation ring $V$ that is nondiscrete and rank 1, then $S = V$ if and only if for every height 1 prime ideal $P$ of any $R_i$, we have $(R_i)_P \not\supseteq S$. If this condition holds, Shannon says the sequence \{\{R_i\}\} switches strongly infinitely often. More recently, in [8, 9, 10, 11, 12], interesting work has been done by A. Granja, M. C. Martinez, C. Rodriguez and T. Sánchez-Giralda, on the question of when $S = V$. In [8, Theorem 13], Granja proves that $S = V$ if and only if either

1. the sequence \{\{R_i\}\} switches strongly infinitely often, or
2. there exists a unique rank one valuation domain $W$ such that $W$ is a localization of $R_n$ for all large $n$.

In the case of item 1, the valuation ring $V$ has rank 1, while in the case of item 2, the valuation ring $V$ has rank 2 and $V$ is contained in $W$. In the case of item 2, the
sequence \( \{R_i\} \) is said to be \textit{height 1 directed}. This describes the fact that \( W \) is a localization of \( R_n \) for all large \( n \).

In this same direction, in Theorem 8.1 we prove that \( S \) is a valuation ring \( \iff \) \( S \) has only finitely many height 1 prime ideals \( \iff \) either (a) \( \dim S = 1 \) or (b) \( \dim S = 2 \), and the boundary valuation ring \( V \) of \( S \) has value group \( \mathbb{Z} \oplus G \) ordered lexicographically, where \( G \) is an ordered subgroup of \( \mathbb{Q} \). We also show how to recover some of the results of Abhyankar and Granja from our point of view.

In general, our notation is as in Matsumura [25]. Thus a local ring need not be Noetherian. An element \( x \) in the maximal ideal \( \mathfrak{m} \) of a regular local ring \( R \) is said to be a \textit{regular parameter} if \( x \notin \mathfrak{m}^2 \). It then follows that the residue class ring \( R/xR \) is again a regular local ring. We refer to an extension ring \( B \) of an integral domain \( A \) as an \textit{overring of} \( A \) if \( B \) is a subring of the quotient field of \( A \). If, in addition, \( A \) and \( B \) are local and the inclusion map \( A \hookrightarrow B \) is a local homomorphism, we say that \( B \) \textit{birationally dominates} \( A \).

2. Essential prime divisors of a sequence of quadratic transforms

Let \( \{R_i\} \) be an infinite sequence of local quadratic transforms of a regular local ring \( R \). In this section we consider the set consisting of the DVRs that are essential prime divisors of infinitely many of the \( R_i \). We see later in Proposition 3.3 that if the Shannon extension \( S = \bigcup_i R_i \) is not a rank 1 valuation ring, then this set is precisely the set of localizations of \( R \) at the height 1 prime ideals of \( S \). A key technical tool in describing these DVRs, as well as one that we use heavily throughout the rest of the paper, is that of the transform of an ideal. We first review this concept.

Let \( R \subseteq S \) be Noetherian UFDs with \( S \) an overring of \( R \), and let \( I \) be a nonzero ideal of \( R \). Then the ideal \( I \) can be written uniquely as \( I = P_1^{e_1} \cdots P_n^{e_n} J \), where the \( P_i \) are principal prime ideals of \( R \), the \( e_i \) are positive integers and \( J \) is an ideal of \( R \) not contained in a principal ideal of \( R \) [24, p. 206]. For each \( i \), set \( Q_i = P_i(R \setminus P_i)^{-1} S \cap S \). If \( S \supseteq R_P \), then \( R_P = SQ_i \), and otherwise \( Q_i = S \). The \textit{transform} of \( I \) in \( S \) is the ideal

\[
I^S = Q_1^{e_1} \cdots Q_n^{e_n} (JS)(JS)^{-1}.
\]

Alternatively, \( I^S = Q_1^{e_1} \cdots Q_n^{e_n} K \), where \( K \) is the unique ideal of \( S \) such that both \( JS = xK \) for some \( x \in S \) and \( K \) is not contained in a proper principal ideal of \( S \).

\footnote{We are following Lipman’s terminology in [24]. In the terminology of Hironaka [21], Definition 5, p. 213], our notion of the transform is the weak transform of \( I \). If \( I \) is a nonzero principal ideal of the regular local ring \( R \), then the weak and strict transforms of \( I \) coincide. Thus, in [8, p. 701], our notion of the transform coincides with the strict transform of \( I \), since Granja restricts to principal ideals.}
We recall the following useful result about transforms.

**Lemma 2.1.** (Lipman [24, Lemma 1.2 and Proposition 1.5]) Let $R \subseteq S \subseteq T$ be Noetherian UFDs with $S$ and $T$ overrings of $R$. Then

1. $(I^S)T = I^T$ for all ideals $I$ of $R$.
2. $(IJ)^S = I^SJ^S$ for all ideals $I$ and $J$ of $R$.
3. Suppose that $P$ is a nonzero principal prime ideal of $R$. Then the following are equivalent.
   - (i) $P^S \neq S$.
   - (ii) $S \subseteq R_P$.
   - (iii) $P^S$ is the unique prime ideal $Q$ in $S$ such that $Q \cap R = P$; and $R_P = S_Q$.

If $R$ is a regular local ring and $S$ is a local quadratic transform of $R$, then the order valuation of $R$ can be used to calculate the transform of an ideal of $R$. How to do this is indicated in Remark 2.2, but first we recall the construction of the order valuation of $R$. Let $R$ be a regular local ring with maximal ideal $m$ and quotient field $F$. For each $0 \neq x \in R$, we define $\text{ord}_R(x) = \min\{i \mid x \in m^i\}$, and we extend $\text{ord}_R$ to a map from $F$ to $\mathbb{Z} \cup \{\infty\}$ by defining $\text{ord}_R(0) = \infty$ and $\text{ord}_R(x/y) = \text{ord}_R(x) - \text{ord}_R(y)$ for all $x, y \in R$ with $y \neq 0$. From the fact that $R$ is a regular local ring, it follows that $\text{ord}_R$ is a discrete rank one valuation on $F$. The valuation $\text{ord}_R$ is the *order valuation* of $R$. The valuation ring of $\text{ord}_R$ is said to be the *order valuation ring* of $R$. It follows that if $R_1$ is any local quadratic transform of $R$, then $(R_1)_mR_1$ is the order valuation ring of $R$.

**Remark 2.2.** Let $R_1$ be a local quadratic transform of a regular local ring $R$, and let $x$ be an element of $m$ such that $mR_1 = xR_1$. If $I$ is an ideal of $R$ and $e = \text{ord}_R(I)$, then $I^{R_1} = x^{-e}IR_1$ and $m^eI^{R_1} = IR_1$. In [10, p. 1349], this equation is used to define the (strict) transform of a height 1 prime ideal in $R_1$.

**Definition 2.3.** For an integral domain $A$, let

$$\text{epd}(A) = \{AP \mid P \text{ is a height 1 prime ideal of } A\}.$$ 

The notation is motivated by the fact that if $A$ is a Noetherian integrally closed domain, then $\text{epd}(A)$ is the set of essential prime divisors of $A$. With $R$ a regular local ring, let $\{(R_i, m_i)\}$ be a sequence of local quadratic transforms of $R$ (so that for each $i \geq 0$, $m_i$ is the maximal ideal of $R_i$ and $R_{i+1}$ is a local quadratic transform of $R_i$), and let $S = \bigcup R_i$. Define

$$\text{epd}(S/R) = \{W \in \bigcup_{i \geq 0} \text{epd}(R_i) \mid S \subseteq W\}.$$
Remark 2.4. The set \( \text{epd}(S/R) \) consists of the essential prime divisors of \( R \) that contain \( S \) along with the order valuation rings of any of the \( R_i \) that contain \( S \). This follows, for example, from Lemma 3.2.

Moreover, \( S \) is a rank 1 valuation domain if and only if \( \text{epd}(S/R) = \emptyset \). Proposition 4.18 and \( S \) is a rank 2 valuation domain if and only if \( \text{epd}(S/R) \) consists of a single element. If \( S \) is not a rank 1 valuation domain, then Proposition 3.3 implies that \( \text{epd}(S/R) = \text{epd}(S) \).

Notation 2.5. In the setting of Definition 2.3, there is naturally associated to the sequence \( \{(R_i, m_i)\} \) a sequence \( \{I_i\} \) of ideals of \( R \), where each \( i \geq 0 \), \( I_i = R \cap m_0 m_1 \cdots m_i R_{i+1} \).

Lemma 2.6. In the setting of Notation 2.5, let \( P \) be a height 1 prime ideal of \( R \) generated by a regular parameter of \( R \). Then for \( k \geq 1 \), \( R_k \subseteq R_P \) if and only if \( P \subseteq I_k \).

Proof. Suppose that \( R_k \subseteq R_P \). An inductive argument using Lemma 2.1(3) shows that \( P R_k \) is a prime ideal of \( R_k \), and an inductive argument using Remark 2.2 and the transitivity of the transform (Lemma 2.1(1)) shows that

\[ m_0 m_1 \cdots m_k R_{k+1} P R_k = PR_k. \]  

Therefore, since \( P R_k \subseteq m_k R_{k+1} \), we have \( P \subseteq m_0 m_1 \cdots m_k \cdot m_{k-1} m_k R_{k+1} \), from which we conclude that \( P \subseteq I_k \).

Conversely, suppose that \( P \subseteq I_k \). Along with (1), this implies

\[ m_0 m_1 \cdots m_k P R_k \subseteq m_0 m_1 \cdots m_k R_{k+1}. \]

Since \( m_i R_{i+1} \) is a principal ideal of \( R_{i+1} \) for each \( i \), we conclude that \( P R_k \subseteq m_k R_{k+1} \). Therefore, by Lemma 2.1(3), \( R_k \subseteq R_P \). \( \square \)

The following classical fact is used without proof in [9, Lemma 11]. Because it will be important in what follows, we include a proof. The proof illustrates calculations involved in local quadratic transforms.

Lemma 2.7. Let \( (R, m) \) be a regular local ring and let \( R_1 \) be a local quadratic transform of \( R \). We have \( m R_1 = z R_1 \) for some \( z \in m \setminus m^2 \). Assume that \((x_1, \ldots, x_s) R \) is a regular prime ideal of \( R \) of height \( s \) such that \( R_{i, R} \supset R_1 \) for each \( i \in \{1, \ldots, s\} \).

Then:

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2In this case, the sequence \( \{R_i\} \) switches strongly infinitely often.

3In this case, the sequence \( \{R_i\} \) is height 1 directed.
1. \((z, x_1, \ldots, x_s)R \) is a regular prime ideal of \(R \) of height \(s + 1 \), and  
2. \((z, \frac{x_1}{z}, \ldots, \frac{x_s}{z})R \) is a regular prime ideal of \(R_1 \) of height \(s + 1 \).

Proof. Since \(R_{x_iR} \supset R_1 \), the transform of \(x_iR \) in \(R_1 \), which is \(\frac{x_i}{z}R_1 \), is a regular prime of \(R_1 \). We first prove that \((z, x_1, \ldots, x_s)R \) is a regular prime ideal of \(R \) of height \(s + 1 \). Assume, by way of contradiction, that \(z + f \in (x_1, \ldots, x_s)R \) for some \(f \in m^2 \). Then \(\frac{f}{z} \in m_1 \), so \(1 + \frac{f}{z} \) is a unit in \(R_1 \), but \(1 + \frac{f}{z} \in (\frac{x_1}{z}, \ldots, \frac{x_s}{z})R_1 \), contradicting the fact that \(\frac{z}{z} \in m_1 \) for each \(i \). This proves item 1.

We may extend the ideal \((z, x_1, \ldots, x_s)R \) to a minimal generating set for \(m \), say \(m = (z, x_1, \ldots, x_1, y_1, \ldots, y_l)R \). By construction of local quadratic transform, \(R_1/zR_1 \) is isomorphic to the localized polynomial ring

\[
R_1/zR_1 \cong k\left[\frac{x_1}{z}, \ldots, \frac{x_s}{z}, \frac{y_1}{z}, \ldots, \frac{y_l}{z}\right]_p
\]

where \(k = R/m \) and \(p \) is some prime ideal containing \(\frac{x_1}{z}, \ldots, \frac{x_s}{z} \). Thus \(\frac{x_1}{z}, \ldots, \frac{x_s}{z} \) generate a regular prime of \(R_1/zR_1 \) of height \(s \). It follows that \(z, \frac{x_1}{z}, \ldots, \frac{x_s}{z} \) generate a regular prime ideal of \(R_1 \) of height \(s + 1 \).

**Proposition 2.8.** In the setting of Definition 2.3, the set \(\text{epd}(S/R) \) contains at most \(\dim R - 1 \) of the order valuation rings of the quadratic sequence \(\{R_i\} \).

Proof. For each \(i \), let \(V_i \) be the order valuation ring for \(R_i \). Let \(d = \dim R \), and suppose by way of contradiction that \(V_{i_1}, \ldots, V_{i_d} \), with \(i_1 < \cdots < i_d \), contain \(S \). For each \(k \in \{1, \ldots, d\} \), let \(P_{i_k} \) denote the center of \(V_{i_k} \) in \(R_{i_k+1} \). Let \(j = i_d + 1 \), and let \(P_{i_k}^R \) be the transform of \(P_{i_k} \) in \(R_j \). By Lemma 2.7 (3), \(P_{i_1}^R, \ldots, P_{i_d}^R \) are proper ideals of \(R_j \). For each \(k \), write \(P_{i_k}^R = x_k R_j \). By Lemma 2.7, the elements \(x_1, \ldots, x_d \) form part of a regular sequence of parameters of \(R_j \). Since \(\dim R_j \leq \dim R \) [23, Theorems 15.5, p. 118] and \(d = \dim R \), the elements \(x_1, \ldots, x_d \) generate the maximal ideal of \(R_j \). Also by Lemma 2.6, each \(P_{i_k}^R \subseteq \bigcap_{k > j} \mathcal{J}_k \), and hence \(m_j = \bigcap_{k > j} \mathcal{J}_k \), which in turn forces \(m_j \subseteq m_{j+1} R_{j+2} \). Since \(m_j R_{j+2} \) is a principal ideal of \(R_{j+2} \), we have \(R_{j+2} \subseteq m_{j+1} R_{j+2} \), a contradiction.

**Remark 2.9.** The bound of \(\dim R - 1 \) in Proposition 2.8 can be refined with the following observation. In the setting of Definition 2.3, \(\dim R_i = \dim R_{i+1} \) is equal to the transcendence degree of the residue field of \(R_{i+1} \) over that of \(R_i \) [2, (1.4.2)]. Thus, for each \(i \), \(\dim R_i \geq \dim R_{i+1} \), so that there is positive integer \(d \leq \dim R \) with \(d = \dim R_i \) for all \(i \gg 0 \). The proof of Proposition 2.8 shows that \(\text{epd}(S/R) \) contains at most \(d - 1 \) valuation rings. Moreover, since \(S \) is an overring of \(R \), we have \(\dim S \leq d \) [23, Theorem 15.5, p. 118].
3. The maximal ideal of a Shannon extension

As a directed union of local rings, a Shannon extension $S$ is local. In this section we focus on the maximal ideal $N$ of $S$ and show that either $N$ is principal or idempotent (Proposition 3.5), and that in either case, $N$ is the radical of a principal ideal (Proposition 3.8).

Setting 3.1. We make the following assumptions throughout the rest of the paper.

1. $(R, m)$ is a regular local ring with quotient field $F$ such that $\dim R \geq 2$.
2. $\{ (R_i, m_i) \}$ is an infinite sequence of local quadratic transforms of regular local rings starting from $R_0 = R$. That is, for each $i > 0$, $R_i$ is a local quadratic transform of $R_{i-1}$, so $R_i$ is a regular local ring, $R_{i-1} \subseteq R_i$, and, by Remark 2.9, $\dim R_{i-1} \geq \dim R_i \geq 2$.
3. $S = \bigcup_{i=0}^{\infty} R_i$ is the Shannon extension of $R$ along $\{ R_i \}$ and $N = \bigcup_{i=0}^{\infty} m_i$ is the maximal ideal of $S$.
4. For each $i \geq 0$, $\text{ord}_i : F \to \mathbb{Z} \cup \{ \infty \}$ represents the order valuation of $R_i$ and $V_i = \{ q \in F \mid \text{ord}_i(q) \geq 0 \}$ is the corresponding valuation ring.

Lemma 3.2 is well known. The geometric content of the lemma is that blowing up the maximal ideal $m$ is an isomorphism outside of the fiber over $m$.

Lemma 3.2. Assume Setting 3.1. If $P$ is a prime ideal of $R_1$ such that $P := P_1 \cap R$ is a nonmaximal prime ideal of $R$, then $R_P = (R_1)_{P_1}$.

Proof. There exists a regular parameter $x$ of $R$ such that $R_1$ is a localization of $R[m/x]$ at a prime ideal $Q_1$. If $x \in P$, then $mR_1 = xR_1 \subseteq P_1$, and hence $m = P_1 \cap R$, a contradiction to the assumption that $P = P_1 \cap R$ is a nonmaximal prime ideal of $R$. Thus $x \not\in P$. It follows that both $R_P$ and $(R_1)_{P_1}$ are localizations of $R[1/x]$. Since $(R_1)_{P_1}$ birationally dominates $R_P$, we have that $R_P = (R_1)_{P_1}$. 

We see in the next proposition that a Shannon extension has an isolated singularity, in the sense that every non-closed point of Spec $S$ is nonsingular. A stronger version of the proposition is proved in Theorem 4.1(1), which asserts that the punctured spectrum of $S$ is a localization of $R_i$ for sufficiently large $i$.

Proposition 3.3. Assume Setting 3.1. If $P$ is a nonzero nonmaximal prime ideal of $S$, then $S_P = (R_i)_{P \cap R_i}$ for $i \gg 0$, and hence $S_P$ is a regular local ring.

Proof. Let $P$ be a nonmaximal prime ideal of $S$ and denote $P_n = P \cap R_n$, so set-theoretically $P = \bigcup P_n$ and $S_P = \bigcup_{n \geq 0} (R_n)_{P_n}$. Since $P$ is nonmaximal, $P_n \subseteq m_n$ for some fixed large $n$. An inductive argument with Lemma 3.2 yields that for $m \geq n$, $(R_m)_{P_m} = (R_n)_{P_n}$. It follows that $S_P = (R_n)_{P_n}$ is a regular local ring.
It follows from Theorem 4.1(1) that the positive integer $i$ in Proposition 3.3 can be chosen independently of $P$.

In light of Proposition 3.3 the ideals of the Shannon extension $S$ that are primary for the maximal ideal play an important role in our treatment of the structure of $S$. We characterize in the next lemma and proposition when the maximal ideal of $S$ is principal.

**Lemma 3.4.** Assuming Setting 3.1, the following are equivalent for $x \in S$.

1. $N = xS$.
2. $P := \bigcap_{i > 0} N^i$ is a prime ideal, $S/P$ is a DVR with maximal ideal the image of $xS$ and $P = PS_P$.
3. For every valuation ring $V$ that birationally dominates $S$, $m_i V = xV$ for all $i \gg 0$.
4. The element $x$ is a regular parameter in $R_j$ for all $i \gg 0$.

**Proof.** The equivalence of (1) and (2) is a standard argument involving only the fact that $S$ is a local domain; see [23, Exercise 1.5, p. 7]

(1) $\Rightarrow$ (3) Let $V$ be a valuation ring that birationally dominates $S$. If $i$ is such that $x \in m_i$, then $xV \subseteq m_i V \subseteq NV = xV$, and hence $xV = m_j V$ for all $j \geq i$.

(3) $\Rightarrow$ (4) Let $i$ be such that both $x \in m_i$ and $m_j V = xV$ for all $j \geq i$. Then since $m_j^2 V \subseteq m_j V$, it follows that $x \in m_j \setminus m_j^2$. Hence $x$ is a regular parameter in $R_j$.

(4) $\Rightarrow$ (1) Let $i$ be such that $x$ is a regular parameter for all $j \geq i$. Let $j \geq i$. Then since $x$ is a regular parameter in $R_{j+1}$ and $xR_{j+1}$ is contained in the height 1 prime ideal $m_jR_{j+1}$ of $R_{j+1}$, it follows that $xR_{j+1} = m_jR_{j+1}$. Since this holds for all $j \geq i$, we conclude that $N = \bigcup_{j \geq i} m_j R_{j+1} = \bigcup_{j \geq i} x R_{j+1} = xS$. \qed

Following [15], we say there is no *change of direction* for the quadratic sequence $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n$ if $m_0 \not\subseteq m_{n+1}^2$; otherwise, if $m_0 \subseteq m_{n+1}^2$, there is a *change of direction* between $R_0$ and $R_n$. We say that the quadratic sequence $\{R_i\}$ *changes direction infinitely many times* if there exist infinitely many positive integers $i$ such that there is a change in direction between $R_i$ and $R_n$ for some $n > i$.

**Proposition 3.5.** Assuming Setting 3.1, the following statements are equivalent.

1. $N$ is not a principal ideal of $S$.
2. $N = N^2$.
3. $\{R_i\}$ changes directions infinitely many times.
(4) For every nonzero element \( x \) of \( N \) and every \( n > 0 \), \( \text{ord}_i(x) > n \) for \( i \gg 0 \).

Proof. (1) \( \Rightarrow \) (2) If \( N \) is not a principal ideal of \( S \), then by Lemma 3.4, for each \( x \in N \), there exists \( i \geq 0 \) such that \( x \in R_i \) but \( x \) is not a regular parameter in \( R_i \). Hence \( x \in m_i^2 \subseteq N^2 \), which shows that \( N = N^2 \).

(2) \( \Rightarrow \) (3) For each \( i \), let \( x_i \) be a regular parameter for \( R_i \) such that \( x_iR_{i+1} = m_iR_{i+1} \). If the maximal ideal \( N \) of \( S \) is idempotent, then Lemma 3.4 implies that for each \( i \), there exists \( n_i > i \) such that \( x_i \) is not a regular parameter in \( R_{n_i} \). Thus \( m_iR_{i+1} = x_iR_{i+1} \subseteq m_n^2 \), and hence there is a change of direction between \( R_i \) and \( R_{n_i} \). Since this holds for each choice of \( i \), we conclude that \( \{R_i\} \) changes directions infinitely many times.

(3) \( \Rightarrow \) (4) Let \( x \in N \). Since \( \{R_i\} \) changes directions infinitely many times, it follows that for each \( i \) there exists \( j > i \) such that \( m_i \subseteq m_j^2 \) and hence \( \text{ord}_i(x) < \text{ord}_j(x) \). Hence for each \( n > 0 \) there exists \( i \) such that \( \text{ord}_j(x) > n \) for all \( j \gg 0 \).

(4) \( \Rightarrow \) (1) Suppose that \( N = xS \) for some \( x \in N \). Then by Lemma 3.4, \( x \) is a regular parameter in \( R_j \) for \( j \gg 0 \). This implies that \( \text{ord}_j(x) = 1 \) for \( j \gg 0 \), so that \( \{i \mid \text{ord}_j(x) > 1\} \) is a finite set. \( \square \)

**Corollary 3.6.** Assume Setting 3.1. If \( N \) is not principal, then every valuation ring between \( S \) and its field of fractions has rank at most \( \dim R - 1 \), and hence \( \dim S < \dim R \).

Proof. Let \( U \) be a valuation ring between \( S \) and its field of fractions \( F \). By Proposition 3.5, \( NU \) is an idempotent ideal of \( U \). Since \( U \) is a valuation ring, this implies \( NU \) is a prime ideal of \( U \) [2, Theorem 17.1, p. 187]. A rank \( d \) valuation ring between a \( d \)-dimensional Noetherian ring and its field of fractions is discrete [1, Theorem 1], and hence has no nonzero idempotent prime ideals. Thus the rank of \( U \) is at most \( \dim R - 1 \). Since the rank of every valuation ring between \( S \) and \( F \) is at most \( \dim R - 1 \), it follows that \( \dim S < \dim R \) [27, (11.9), p. 37]. \( \square \)

**Remark 3.7.** In contrast to Corollary 3.6, if the maximal ideal of \( S \) is principal it need not be true that \( \dim S < \dim R \). If \( \dim R = 2 \), then every rank 2 valuation ring that birationally dominates \( R \) is a Shannon extension \( S \) with \( \dim S = \dim R = 2 \); see Corollary 3.4. There also exist examples with \( \dim S = \dim R \) in which \( S \) is not a valuation ring: the Shannon extension \( S \) in Example 7.2 is not a valuation ring and \( \dim S = \dim R = 3 \).

**Proposition 3.8.** Assuming Setting 3.1, there exists a regular parameter \( x \) in one of the \( R_i \)'s such that \( xR_{i+1} = m_iR_{i+1} \) and \( xS \) is an \( N \)-primary ideal of \( S \).

Proof. By Proposition 2.8 there exists \( i \geq 0 \) such that no order valuation ring \( V_j \), \( j \geq i \), is in epd(\( S/R_i \)), so epd(\( S/R \)) \( \subseteq \) epd(\( R_i \)). Let \( x \in m_i \) be such that \( xR_{i+1} = \text{ord}_i(x) > n \).
m_i R_{i+1}. Note that m_i \subseteq xS and xS \cap R_i = m_i. Assume by way of contradiction that a non-maximal prime ideal q of S contains x. Then S_q is Noetherian by Proposition 3.3 so there exists a height 1 prime ideal p of S such that x \in p. Then

m_i = xS \cap R_i \subseteq p \cap R_i \subseteq N \cap R_i = m_i,

so p \cap R_i = m_i. However, S_p \in \text{epd}(S/R) \subseteq \text{epd}(R_i). Thus S_p = (R_i)_{p \cap R_i} = R_i, contradicting the assumption that \dim R_i \geq 2. We conclude that \sqrt{xS} = N. \hfill \Box

**Corollary 3.9.** Assuming Setting 3.1, the following are equivalent for the Shannon extension S of R.

1. S is dominated by a DVR.
2. S is a DVR.
3. S is a Noetherian ring.

**Proof.** (1) \Rightarrow (2) Suppose V is a DVR that dominates S. Replacing V by V \cap F, we may assume that V birationally dominates S. We claim that S = V. Let f \in V. Since R is a UFD, we can write f = a/b, where a, b \in R are relatively prime. Let v denote the valuation associated to V with value group the integers. We have v(f) = v(a) - v(b) \geq 0, and v(b) = 0 if and only if f \in R. Assume that v(b) = n > 0. Let x \in m be such that mV = xV. Then x is part of a regular system of parameters for R and xR_1 = mR_1. Hence there exist c, d \in R_1 such that a = xc and b = xd. If follows that f = c/d and v(d) < n. Writing c/d in lowest terms in R_1 will not increase the v-value of the denominator. Hence repeating this process at most n times gives f \in R_i, with i \leq n. Thus S = V.

(2) \Rightarrow (3) This is clear.

(3) \Rightarrow (1) By Proposition 3.8 N is the radical of a principal ideal of S, and hence since S is Noetherian, \dim S = 1. Moreover, since N is finitely generated, N is not idempotent, and hence by Proposition 3.5 N is a principal ideal. Thus S is a DVR. \hfill \Box

**Remark 3.10.** Another condition that characterizes when a Shannon extension is a DVR is given in Corollary 8.5.

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The argument given here is related to classical results of Abhyankar in [1] and Zariski in [37, pp. 27-28].
4. The Noetherian hull of a Shannon extension

In this section we continue to assume Setting 3.1 and we show that there is a smallest Noetherian overring $T$ that properly contains the Shannon extension $S$ and this ring is a regular ring that is a localization of $R_i$ for sufficiently large $i$. In the next section $T$ is used to decompose $S$ as an intersection of a regular ring and a valuation ring.

**Theorem 4.1.** Assuming Setting 3.1, let $T$ be the intersection of all the DVRs with quotient field $F$ that properly contain $S$, where an empty intersection equals $F$. Then the following statements hold for $T$.

1. $T = S[1/x]$ for any $x \in N$ such that $xS$ is $N$-primary. Furthermore, $T$ is a localization of $R_i$ for $i \gg 0$. In particular, $T$ is a UFD.

2. The ring $T$ is the intersection of all the $R_P$, $P$ a height 1 prime ideal of $R$, that contain $S$, along with the at most $\dim R-1$ order valuation rings $V_i$ that contain $S$.

3. The ring $T$ is a Noetherian regular ring that is the unique minimal proper Noetherian overring of $S$ in $F$.

**Proof.** If $S$ is a DVR, then $T$ is the quotient field of $S$ and the assertions (1), (2) and (3) hold, so we assume that $S$ is not a DVR.

(1) Let $x \in S$ such that $xS$ is $N$-primary. By Proposition 2.8 there exists $i \geq 0$ such that no order valuation ring $V_j$, $j \geq i$, is in $\text{epd}(S/R_i)$. Thus no order valuation ring of the sequence $\{R_j\}_{j \geq i}$ is in $\text{epd}(S/R_i)$, and hence no order valuation ring of this sequence contains $S$. Therefore, by replacing $R$ with $R_i$ we may assume that $\text{epd}(S/R)$ contains none of the $V_i$.

By Proposition 3.8 there exists a regular parameter $x_i$ in one of the $R_i$’s such that $x_iR_{i+1} = m_iR_{i+1}$ and $x_iS$ is $N$-primary. We may assume without loss of generality that $i = 0$ and that $x = x_i$; in particular, $N = \sqrt{xS}$ and $xR_1 = mR_1$. We claim that $S[1/x]$ is a flat extension of $R$. To prove this, it is enough to show that for each prime ideal $P$ of $S$ that survives in $S[1/x]$, $S_P = R_P \cap R$. Let $P$ be such a prime ideal. Then since $x \notin P$ and $x \in m_i$ for all $i \geq 0$, it must be that for each $i$, $P \cap R_i$ is a nonmaximal ideal prime ideal of $R_i$. Therefore, by Lemma 3.2 for each $i \geq 0$, we have $R_P \cap R = (R_i)_{P \cap R_i}$, and hence $R_P \cap R = \bigcup_{i \geq 0}(R_i)_{P \cap R_i} = S_P$, which proves the claim.

If $\dim S = 1$, then $T$ is the quotient field of $S$ by Proposition 3.3 and statement (2). We regard a field to be a zero-dimensional regular local ring.
Next, since $S[1/x]$ is a flat extension of the UFD $R$, then $S[1/x]$ is a localization of $R$ at a multiplicatively closed set [21, Theorem 2.5]. To complete the proof of (1), we claim that $T = S[1/x]$. Since $S[1/x]$, as a localization of $R$, is an integrally closed Noetherian domain, it is an intersection of DVRs and hence $T \subseteq S[1/x]$. It remains to show that every DVR that contains $S$ contains $S[1/x]$. Let $V$ be a DVR that contains $S$. If $V$ dominates $S$, then by Corollary 3.9 $S$ is a DVR, contrary to our assumption. Thus $V$ does not dominate $S$ and since $xS$ is $N$-primary, it follows that $S[1/x] \subseteq V$, which proves that $S[1/x] = T$.

(2) By (1), there is $i \geq 0$ such that $T$ is a localization of $R_i$, and thus $T$ is an intersection of the $(R_i)_P$ that contain $S$, where $P$ is a height 1 prime ideal of $R_i$. It follows from Lemma 3.2 that each $(R_i)_P$ is a localization of $R$ at a height 1 prime ideal of $R$ or $(R_i)_P$ is an order valuation of some $R_j$, $j < i$. By Proposition 2.8 there are at most $\dim R - 1$ such order valuation rings.

(3) By (1), $T$ is a localization of a regular local ring and hence is a Noetherian regular ring. Suppose that $A$ is an Noetherian overring of $S$. Let $M$ be a maximal ideal of $A$. If $M \cap S = N$, then since there is a DVR that dominates the Noetherian ring $A_M$, this DVR dominates also $S$, which by Corollary 3.9 implies that $S$ is a DVR, contrary to our assumption at the beginning of the proof. Thus $T = S[1/x] \subseteq S_{M \cap S} \subseteq A_M$. Since this is true for every maximal ideal $M$ of $A$, it follows that $T \subseteq A$, which verifies (3).

**Definition 4.2.** In light of Theorem 4.1(3), we define the Noetherian regular UFD $T$ of Theorem 4.1 to be the Noetherian hull of the Shannon extension $S$.

**Remark 4.3.** Assume the notation of Setting 3.1 and assume $\dim S > 1$. Proposition 2.8 implies that for $n \gg 0$, $S$ does not contain the order valuation ring $V_n$. The proof of Theorem 4.1(1) shows that for every height 1 prime ideal $P$ of $S$ we have $S_P = (R_n)_P = T_{p^\infty}$ for some prime element $p$ of $R_n$. Notice, however, that for $i > n$, the ideal $pR_i$ is not a prime ideal.

**Proposition 4.4.** With notation as in Theorem 4.1, fix $n \gg 0$ such that $T$ is a localization of $R_n$. Let $a \in R_n$ be nonzero, and for $i \geq n$, consider the transform $(aR_n)^{R_i}$. Then $(aR_n)^{R_i} = R_i$ for $i \gg 0$ if and only if $a \in T^\infty$.

**Proof.** Since $m_i T = T$ for all $i \geq n$, we have $aT = (aR_n)^{R_i}$. If $(aR_n)^{R_i} = R_i$ for some $i \geq n$, then $aT = T$.

To see the converse, assume that $(aR_n)^{R_i} \subseteq R_i$ for all $i \geq n$. By construction of transform, the height 1 primes of $(aR_n)^{R_i}$ lie over height 1 primes of $(aR_n)^{R_i}$. This yields an ascending sequence $\{p_i\}_{i \geq n}$, where $p_i$ is a height 1 prime of $R_i$. We have $(aR_n)^{R_i} \subseteq p_i$ and $p_{i+1} \cap R_i = p_i$ for all $i \geq n$. It follows that $P = \bigcup_{i \geq n} p_i$ is a height 1 prime in the directed union $S$, and $a \in P$, so $a \notin T^\infty$. 




5. The boundary valuation of a Shannon extension

Let \( \mathcal{X} \) denote the set of all valuation overrings of \( R \). The Zariski topology on \( \mathcal{X} \) has as a basis of open sets the sets of the form \( \{ V \in \mathcal{X} \mid E \subseteq V \} \), where \( E \) ranges over the finite subsets of the quotient field \( F \) of \( R \). For our purposes we need a finer topology: The \textit{patch topology} on \( \mathcal{X} \) has a basis of open sets of the form \( \{ V \in \mathcal{X} \mid G \subseteq V \text{ and } H \subseteq \mathcal{M}_V \} \), where \( G \) and \( H \) range over all finite subsets of \( F \).

**Definition 5.1.** Assume Setting 3.1. A valuation overring \( V \) of \( R \) is a \textit{boundary valuation ring} of \( S \) if in the patch topology \( V \) is a limit point of the order valuation rings \( V_i \).

The terminology is explained by the fact that the subspace \( \{ V_i \mid i \geq 0 \} \) is discrete in the patch topology and hence a valuation ring \( V \in \mathcal{X} \) is a boundary valuation ring of \( S \) if and only if \( V \) is a boundary point in \( \mathcal{X} \) of the set \( \{ V_i \mid i \geq 0 \} \) with respect to the patch topology. Equivalently, \( V \in \mathcal{X} \) is a boundary valuation ring of \( S \) if and only if for each pair of finite subsets \( G \subseteq V \) and \( H \subseteq \mathcal{M}_V \), there exist infinitely many \( i \) such that \( G \subseteq V_i \) and \( H \subseteq \mathcal{M}_V \).

In Corollary 5.3, we show that \( S \) has a unique boundary valuation ring, and we use this valuation ring in Theorem 5.4 to give an intersection decomposition of \( S \) in terms of its boundary valuation ring and Noetherian hull.

**Lemma 5.2.** Assuming Setting 3.7, let \( q \in F \) be nonzero. Then either \( \text{ord}_n(q) > 0 \) for \( n \gg 0 \), \( \text{ord}_n(q) = 0 \) for \( n \gg 0 \), or \( \text{ord}_n(q) < 0 \) for \( n \gg 0 \).

**Proof.** If \( q \in R_n \) or \( q^{-1} \in R_n \) for some \( n \geq 0 \), then Lemma 5.2 is clear. Assume that \( q \notin R_n \) and \( q^{-1} \notin R_n \) for all \( n \geq 0 \).

We may write \( q = a_0/b_0 \), where \( a_0, b_0 \in R \) are relatively prime. Since \( q \notin R \) and \( q^{-1} \notin R \), we must have that \( a_0, b_0 \in \mathfrak{m} \). Let \( Q_0 = (a_0, b_0)R_i \), so \( Q_0 \) is an ideal of \( R \) of height 2. Let \( \{Q_i\}_{i=0}^\infty \) be the sequence of transforms of \( Q_0 \) in the sequence \( \{R_i\} \); i.e., for each \( i \geq 0 \), \( Q_{i+1} \) is the transform of \( Q_i \) in \( R_{i+1} \). For each \( i \), let \( x_i \in \mathfrak{m} \) such that \( x_i R_{i+1} = m_i R_{i+1} \). Then by Remark 2.2, \( Q_i R_{i+1} = x_i^{e_i} Q_{i+1} \), where \( e_i = \text{ord}_i(Q_i) \).

Let \( \{a_i\} \) and \( \{b_i\} \) be sequences of elements of \( S \) defined inductively for \( i \geq 0 \) by \( a_{i+1} = x_i^{-e_i} a_i \) and \( b_{i+1} = x_i^{-e_i} b_i \). It follows that \( Q_i = (a_i, b_i) R_i \) and \( q = a_i/b_i \) for all \( i \geq 0 \). If \( e_i = 0 \) for any \( i \geq 0 \), then one of either \( a_i \) or \( b_i \) is a unit in \( R_i \), so \( q \in R_i \) or \( q^{-1} \in R_i \). Thus we may assume that \( e_i > 0 \) for all \( i \geq 0 \).

By [10], Lemma 3.6 and Remark 3.7, \( \text{ord}_i(Q_i) \geq \text{ord}_{i+1}(Q_{i+1}) \) for all \( i \geq 0 \), so the sequence \( \{e_i\} \) is a nonincreasing sequence of non-negative integers. Thus \( \{e_i\} \) stabilizes to some value \( e > 0 \), say \( \text{ord}_i(Q_i) = e \) for all \( i \geq N \).

Notice that if \( \text{ord}_i(a_i) = \text{ord}_i(Q_i) \), then \( a_{i+1} R_{i+1} \) is the transform of the principal ideal \( a_i R_{i+1} \) in \( R_{i+1} \), so again by [10], Lemma 3.6 and Remark 3.7, \( \text{ord}_{i+1}(a_{i+1}) \leq \)
ord_i(a_i). Therefore if ord_j(a_j) = e for any j, then ord_i(a_i) = e ≤ ord_i(b_i) for all i ≥ j. Similarly if ord_j(b_j) = e for any j, then ord_i(b_i) = e ≤ ord_i(a_i) for all i ≥ j. Since either ord_N(a_N) = e or ord_N(b_N) = e, the lemma follows. □

**Corollary 5.3.** Assume Setting 3.1. The Shannon extension S has a unique boundary valuation ring V, and

\[ V = \bigcup_{n \geq 0} \bigcap_{i \geq n} V_i = \{ q \in F \mid \text{ord}_i(q) \geq 0 \text{ for } i > 0 \}. \]  

**Proof.** Let \( V' = \{ q \in F \mid \text{ord}_i(q) \geq 0 \text{ for } i > 0 \} \). As a directed union of rings, \( V' \) is a ring, and in view of Lemma 5.2 \( V' \) is in fact a valuation ring. Let \( V \) be a boundary valuation ring of \( S \). Then for \( q \in V \), \( q \in V_i \) for infinitely many \( i \), so by Lemma 5.2 \( q \in V_i \) for all \( i > 0 \). Thus \( V \subseteq V' \). Furthermore, for \( q \in \mathfrak{M}_V \), \( q \in \mathfrak{M}_V \) for infinitely many \( i \), so similarly, \( q \in \mathfrak{M}_V \) for all \( i > 0 \). Thus \( \mathfrak{M}_V = \mathfrak{M}_V \cap V \), so \( V = V' \). □

**Theorem 5.4.** Assume Setting 3.1. Then \( S = V \cap T \), where \( V \) is the unique boundary valuation ring of \( S \) and \( T \) is the Noetherian hull of \( S \).

**Proof.** First we observe that \( S \subseteq V \cap T \). For this it is clearly enough to verify that \( S \subseteq V \). Let \( s \in S \). Then there exists \( i \) such that \( s \in R_i \). Since \( R_i \), hence \( s \), is contained in every \( V_j \), \( j \geq i \), we have by Corollary 5.3 that \( s \in V \). Thus \( S \subseteq V \cap T \). It remains to prove that \( V \cap T \subseteq S \). If \( S \) is a DVR, then since \( S \subseteq V \cap T \) and the only proper overring of \( S \) is the quotient field of \( S \), we have \( S = V \cap T \). (Note that \( V \neq F \), since by Corollary 5.3 \( V \) dominates \( R \).) Thus we assume for the rest of the proof that \( S \) is not a DVR.

By Proposition 2.3 there exists \( k > 0 \) such that none of the \( \{ V_i \mid i \geq k \} \) contain \( S \). Thus by replacing \( R \) with \( R_k \) we may assume without loss of generality that none of the \( V_i \) contain \( S \). By Theorems 3.8 and 4.1(1) there exist \( i > 0 \) and a regular parameter \( x \) in \( R_i \) such that \( m_i R_{i+1} = x R_{i+1}, xS \) is \( N \)-primary and \( T = S[1/x] \). By replacing \( R \) with \( R_i \) we may assume without loss of generality that \( i = 0 \), so that \( m R_1 = x R_1 \).

Let \( q \in V \cap T \), and write \( q = s/x^e \) for some \( s \in S \) and \( e > 0 \). By Corollary 5.3 there exists \( n > 0 \) such that \( q \in V_i \) for all \( i > n \). Since \( s \in S \) and none of the order valuation rings \( V_i \) contain \( S \), we may choose \( k > n \) such that

\[ s \in R_k \quad \text{and} \quad R_k \not\subseteq V_i \quad \forall i = 0, 1, 2, \ldots, n. \]

We claim \( q \in R_k \). Since \( R_k \) is a Krull domain and hence an intersection of its localizations at height 1 prime ideals, it suffices to show that \( q \in (R_k)_Q \) for each height 1 prime ideal \( Q \) of \( R_k \). Let \( Q \) be a height 1 prime ideal of \( R_k \). If \( x \not\in Q \),
then clearly $q = s/x^e \in (R_k)_Q$. Suppose $x \in Q$. Since $\mathfrak{m}R_1 = xR_1 \subseteq xR_k \subseteq Q$, it follows that $Q \cap R = \mathfrak{m}$. Thus $(R_k)_Q \notin \text{epd}(R)$, so $(R_k)_Q = V_i$ for some $i \geq 0$. Since $R_k \subseteq V_i$, it follows that $i > n$, so $q \in V_i$.

**Corollary 5.5.** Assume Setting 3.1. If $\dim S = 1$, then $S$ is the boundary valuation ring $V$.

**Proof.** Let $0 \neq x \in N$. By Theorem 4.1, $T = S[1/x]$, so, since $\dim S = 1$, we have $T = S[1/x] = F$. By Theorem 5.4, $S$ is the boundary valuation ring $V$. □

**Corollary 5.6.** Assume Setting 3.1. Then the principal $N$-primary ideals of $S$ are linearly ordered with respect to inclusion.

**Proof.** Let $y, z \in S$ be such that $yS$ and $zS$ are $N$-primary ideals. By Theorem 5.4, we have $S = V \cap T$, where $V$ is the boundary valuation ring of $S$ and $T$ is the Noetherian hull of $S$. By Theorem 4.1(1), $T = S[1/y] = S[1/z]$. It follows that $yS = yV \cap T$ and $zS = zV \cap T$. Since $V$ is a valuation ring, the ideals $yV$ and $zV$ are comparable, and thus so are $yS$ and $zS$. □

**Remark 5.7.** Using different techniques from the current paper, we prove in [18] that the boundary valuation ring of a Shannon extension always has rank at most 2, and we give constraints on its value group. This is done through an analysis of asymptotic properties of the sequence of local quadratic transforms that defines the Shannon extension. In particular, we recover a result due to Granja [8, Proposition 7]: if $S$ is a valuation ring, then $S$ is the boundary valuation ring of $S$ and, in this case, $S$ has at most rank 2.

6. The complete integral closure of a Shannon extension

An element $\theta$ in the field of fractions of an integral domain $A$ is almost integral over $A$ if $A[\theta]$ is a fractional ideal of $A$. The integral domain $A$ is completely integrally closed if it contains all almost integral elements in its field of fractions. The complete integral closure of a domain is the ring of almost integral elements in its field of fractions. In general, the complete integral closure of a domain may fail to be completely integrally closed.

In this section we describe the complete integral closure of a Shannon extension. To do so, we distinguish between two classes of Shannon extensions, those that are archimedean and those that are not. Recall that an integral domain $A$ is archimedean if for each nonunit $a \in A$, we have $\bigcap_{n > 0} a^n A = 0$. A Shannon extension $S$ is archimedean if and only if $\bigcap_{n > 0} x^n S = 0$, where $x$ is as in Setting 6.1.

To simplify hypotheses, we fix some notation for this section.
Setting 6.1. In addition to Setting 3.1 assume:

1. \( x \in S \) is such that \( xS \) is \( N \)-primary (see Proposition 3.8);
2. \( T \) is the Noetherian hull of \( S \);
3. \( T \) is a localization of \( R \) (see Theorem 4.1);
4. \( S^* \) denotes the complete integral closure of \( S \);
5. \( W \) is the rank one valuation overring of the boundary valuation ring \( V \).

In Setting 6.1, in the special case in which \( S \) is a rank one valuation ring, then 
\[ S = S^* = V = W = (N :_F N). \]

Theorem 6.2. Assume notation as in Setting 3.1 and 6.1. Also assume that \( S \) is archimedean and not a DVR. Let \( V \) be the boundary valuation ring of \( S \). Then \( NV \) is the height 1 prime of \( V \) and hence \( W = V_{NV} \). Furthermore,

1. \( S^* = (N :_F N) = W \cap T \), and
2. \( W \) is the unique rank 1 valuation overring of \( S \) with this property.

Proof. If \( N = xS \) is a principal ideal, then Lemma 3.4 implies that \( \cap_{n>0} N^n = \cap_{n>0} x^nS \) is a nonzero ideal, a contradiction to the assumption that \( S \) is archimedean. Thus \( N \) is not a principal ideal of \( S \). By Proposition 3.5 \( N \) is an idempotent ideal of \( S \), thus \( NV \) is an idempotent ideal of \( V \). It follows that \( NV \) is a prime ideal [6, Theorem 17.1, p. 187]. For an \( N \)-primary element \( x \), since \( S \) is archimedean, it must be that \( \cap_{n=0}^\infty x^nS = (0) \). From Theorem 5.4 we have \( S = V \cap T \), and since by Corollary 3.9 \( V \) is not a DVR, we have from Theorem 4.1(1) that \( NT = T \), and in particular \( x^nT = T \). It follows that \( \cap_{n=0}^\infty x^nV = (0) \). Thus \( NV \) is the height 1 prime of \( V \), and hence \( W = V_{NV} \).

We show that \( (N :_F N) = W \cap T \). We have,
\[ NV \cap T = NV \cap (V \cap T) = NV \cap S = N \]
so we have the equality
\[ (N :_F N) = ((NV \cap T) :_F N). \]

By properties of colon ideals, it follows that
\[ ((NV \cap T) :_F N) = (NV :_F N) \cap (T :_F N). \]

The fact that \( NT = T \) implies that \( (T :_F N) = T \). Since \( NV \) is idempotent, \( W = (NV :_F N) \) [5, Lemma 4.4, p. 69]. We conclude that
\[ (NV :_F N) \cap (T :_F N) = W \cap T. \]
Therefore we have established that $(N : F N) = W \cap T$.

Next we observe that the complete integral closure of $S$ is $(N : F N)$. Indeed, since $(N : F N)$ is the intersection of the completely integrally closed rings $W$ and $T$, the ring $(N : F N)$ is completely integrally closed. Since also $(N : F N)$ is a fractional ideal of $S$, $(N : F N)$ is almost integral over $S$, proving (1).

Since $S^* = W \cap T$ and dim $W = 1$, to prove (2), by [19, Corollary 1.4], it suffices to show that $W$ cannot be omitted from this representation of $S^*$, or equivalently, that $(N : F N) \subseteq T$. To see this, let $x \in S$ be any $N$-primary element. Then $\frac{1}{x^2} \in T = S[\frac{1}{x}]$, but $\frac{1}{x^2}x = \frac{1}{2} \notin N$, so $\frac{1}{x^2} \in T \setminus (N : F N)$. Thus $W$ cannot be omitted from $S^* = W \cap T$, and hence $W$ is the unique valuation overring of $S$ of Krull dimension one such that $S^* = W \cap T$.

**Corollary 6.3.** With notation as in Theorem 6.2, $N$ is the center of $W$ on $S^*$. In particular, $N$ is a prime ideal of $S^*$.

**Proof.** By Theorem 6.2, $S^* = W \cap T$, and by Theorem 5.4, $S = V \cap T$. Using the fact that $NV = NV_N$ set-theoretically [27, 11.2, p. 35], it follows that $N$ is the center of $W$ on $S^*$. \qed

**Corollary 6.4.** With notation as in Theorem 6.2, the units of $S^*$ are equal to the units of $S$.

**Proof.** Since the maximal ideal $N$ of $S$ is a proper ideal of $S^*$, if $u \in S$ is not a unit of $S$, then $u$ is also not a unit of $S^*$. Thus it suffices to show that if $u, u^{-1} \in S^*$, then $u \in S$ or $u^{-1} \in S$. Since $S^* = W \cap T$ by Theorem 6.2 it follows that $u, u^{-1} \in T$, and at least one of $u, u^{-1}$ is in $V$, say $u \in V$. But $S = V \cap T$ by Theorem 5.4 so $u \in S$, completing the proof. \qed

**Corollary 6.5.** Assume notation as in Theorem 6.2. The subrings $A$ of $S^*$ that contain $S$ are in a one-to-one inclusion preserving correspondence with the subrings of $S^*/N$ that contain the field $k = S/N$. In particular, $S^*$ is a finitely generated $S$-algebra if and only if $S^*/N$ is a finitely generated $k$-algebra.

**Corollary 6.6.** With notation as in Theorem 6.2, assume $S$ is not completely integrally closed and let $\theta \in S^* \setminus S$. Then $\theta^{-1}S \cap S = N$.

**Proof.** By Theorem 6.2, $(N : F N) = S^*$, so $\theta N \subseteq N$, hence $N \subseteq \theta^{-1}N \cap S \subseteq \theta^{-1}S \cap S$. Since also $\theta \notin S$, we have $S \nsubseteq \theta^{-1}S$, hence $\theta^{-1}S \cap S \nsubseteq S$. Therefore, since $N$ is the maximal ideal of $S$, we have $\theta^{-1}S \cap S = N$. \qed

**Remark 6.7.** McAdam [26] defines an integral domain $A$ to be a finite conductor domain if for elements $a, b$ in the field of fractions of $A$, the $A$-module $aA \cap bA$ is finitely generated. A ring is said to be coherent if every finitely generated ideal
is finitely presented. Chase [3, Theorem 2.2] proves that an integral domain $A$ is coherent if and only if the intersection of two finitely generated ideals of $A$ is finitely generated. Thus a coherent domain is a finite conductor domain. Examples of finite conductor domains that are not coherent are given by Glaz in [7, Example 4.4] and by Olberding and Saydam in [29, Prop. 3.7]. If $S$ is archimedean but not completely integrally closed, then $S$ is not finite conductor and thus not coherent. Indeed, if $S$ is archimedean and coherent, then Corollary 6.6 implies that $N$ is a finitely generated ideal of $S$, which by Proposition 3.5 implies that $N$ is a principal ideal. However, Lemma 3.4 and Theorem 4.1 then imply that the Noetherian hull of $S$ is a fractional ideal of $S$, a contradiction to Theorem 6.9.

**Remark 6.8.** Following [6, page 524], an integral domain $A$ with field of fractions $K$ is a generalized Krull domain if there is a set $\mathcal{F}$ of rank 1 valuation overrings of $A$ such that: (i) $A = \bigcap_{V \in \mathcal{F}} V$; (ii) for each $(V, \mathfrak{m}_V) \in \mathcal{F}$, we have $V = A_{\mathfrak{m}_V \cap A}$; and (iii) $\mathcal{F}$ has finite character; that is, if $x \in K$ is nonzero, then $x$ is a nonunit in only finitely many valuation rings of $\mathcal{F}$. This class of rings has been studied by a number of authors; see for example [13, 14, 19, 30, 31, 32]. In our setting, when $S \subset S^*$ the ring $S^*$ is a generalized Krull domain whose defining family $\mathcal{F}$ consists of rank 1 valuation rings such that all but at most one member (namely, $W$) is a DVR.

In light of Theorem 6.2, which describes the complete integral closure $S^*$ of $S$ in the archimedean case, it remains to describe $S^*$ when $S$ is not archimedean. We do this in Theorem 6.9.

**Theorem 6.9.** Assuming Setting 6.1, the following statements are equivalent.

1. $S$ is not archimedean.
2. $T$ is a fractional ideal of $S$.
3. $S^* = T$.
4. The boundary valuation of $S$ has a nonzero nonmaximal prime ideal that does not lie over $N$.
5. The ideal $\bigcap_{i > 0} x^i S$ is a nonzero prime ideal of $S$.
6. There exists $0 \neq y \in R$ such that for each $n > 0$, $\text{ord}_i(y) \geq n \cdot \text{ord}_i(x)$ for all $i \gg 0$.

**Proof.** (1) $\Rightarrow$ (2) Since $S$ is not archimedean and $xS$ is $N$-primary, it follows that $\bigcap_{i > 0} x^i S \neq 0$. By Theorem 4.1(1), $T = S[1/x]$, so that $0 \neq \bigcap_{i > 0} x^i S = (S :_S T)$, and hence $T$ is a fractional ideal of $S$. 

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(2) ⇒ (3) Since $T$ is a normal Noetherian domain, $T$ is completely integrally closed. Thus since $T$ is a fractional ideal of $S$, it follows that $T$ is the complete integral closure of $S$.

(3) ⇒ (4) Since $T$ is the complete integral closure of $S$, $T$ is contained in every rank one valuation overring of $S$. For a rank one valuation overring $U$ of $S$, it follows that $S \subseteq T \subseteq U$. Since the maximal ideal of $U$ lies over a nonzero prime ideal of $T$ and $NT = T$, we conclude that it lies over a nonzero nonmaximal prime ideal of $S$. Statement (4) follows.

(4) ⇒ (5) Since $\dim S > 1$, Theorem 6.11 implies that $T = S[1/x]$. Let $V$ be the boundary valuation ring for $S$. By statement (3) there exists a nonzero prime ideal $Q$ of $V$ such that $Q \cap S$ is a nonmaximal prime ideal of $S$. Hence $x \notin Q$ and $Q \subseteq \bigcap_i x^iV$. Since $V$ is a valuation ring, $\bigcap_{i \geq 1} x^iV$ is a nonzero prime ideal of $V$. Using the fact that $x$ is a unit in $T$, we have $P = \bigcap_{i \geq 1} x^iV \cap T = \bigcap_{i \geq 1} x^iS$ is a nonzero prime ideal of $S$.

(5) ⇒ (6) Let $0 \neq y \in \bigcap_i x^iS \cap R_i$, and let $n > 0$. Then $y/x^n \in S$. Since $S = \bigcup_{i > 0} R_i$, where $\{R_i\}$ is the sequence of quadratic transforms determined by $S$, it follows that $\ord_i(y) - n \cdot \ord_i(x) = \ord_i(y/x^n) \geq 0$ for all but finitely many $i$.

(6) ⇒ (1) Let $y$ be as in (6), and let $n > 0$. Then (6) implies that for all but finitely many $i$, $y/x^n \in V_i$. Let $V$ be the boundary valuation ring for $S$. Since $V$ is a limit point for the $V_i$ in the patch topology, $y/x^n \in V$. Since the choice of $n$ was arbitrary, we have $y \in \bigcap_{n > 0} x^nV$, so that $V$ is not archimedean. By Theorem 4.11, $T = S[1/x]$, and by Theorem 5.4, $S = V \cap T$. Thus $y/x^n \in S$ for all $n > 0$, and hence $y \in \bigcap_{n > 0} x^nS$, which shows that $S$ is not archimedean.

**Corollary 6.10.** Assume Setting 6.1. Then there is a prime ideal $P$ of $S$ such that $S^* = (P :_FP)$. The ideal $P$ is maximal if and only if $S$ is archimedean.

**Proof.** If $S$ is archimedean, then by Theorem 6.2, $S^* = (N :_FN)$. If $S$ is not archimedean, then by Theorem 6.3, $P := \bigcap_{n > 0} x^nS$ is a nonmaximal prime ideal of $S$. Also by Theorem 6.9, $S^* = T$. Since $T = S[1/x]$, it follows that $S^* = T \subseteq (P :_FP)$. Moreover, since $T$ is completely integrally closed and $P$ is an ideal of $T$, we have $S^* = T = (P :_FP)$.

7. Shannon’s examples

Two examples [35, Examples 4.7 and 4.17] of David Shannon motivated our work in this paper. In Examples 7.2 and 7.4 we present details of these examples and their relation to concepts developed in this paper. The first, Example 7.2 involves a nonarchimedean Shannon extension that is not a valuation ring, while the second, Example 7.4 deals with an archimedean Shannon extension that is not a valuation ring. In this section we make use of the following elementary lemma.
Lemma 7.1. (cf. [4, Theorem 2.4] and [23, Exercise 1.5, p. 7]) Let $A$ be a local domain with principal maximal ideal $m = xA$ and let $p = \bigcap_{n \geq 0} m^n A$.

1. $p = xp$ is a prime ideal, and every prime ideal properly contained in $m$ is contained in $p$.

2. $A$ is a valuation domain if and only if $A_p$ is a valuation domain.

Example 7.2 is based on [35, Example 4.7] of Shannon.

Example 7.2. Let $(R, m)$ be a 3-dimensional regular local ring with $m = (x, y, z)R$. Let $U$ be a valuation ring that birationally dominates $R$ such that, with $u$ the valuation of $U$, we have $nu(x) < u(y)$ and $nu(x) < u(z)$ for each positive integer $n$, that is, $u(x)$ is infinitely smaller than both $u(y)$ and $u(z)$. Let $\{R_i\}_{i=0}^{\infty}$ be the sequence of local quadratic transforms of $R$ along $U$. The maximal ideal $m_i$ of $R_i$ is $m_i = (x, y, z)^i R_i$. For each $i$, we have $z/y \not\in R_i$ and $y/z \not\in R_i$. Hence $S = \bigcup_i R_i$ is not a valuation ring.

Since $y, z \in \bigcap_i x^i S$, the ring $S$ is not archimedean. By Theorem 6.9, the complete integral closure $S^*$ of $S$ is the Noetherian hull $T = S[1/x]$. Observe that $xS$ is the maximal ideal of $S$, and so by Lemma 7.1, $P = \bigcap_{i>0} x^i S$ is the unique largest nonmaximal prime ideal of $S$. It follows that $T = S[1/x] = S_P$. Since $(y, z)R \subseteq P \cap R \subseteq m$, and there are no prime ideals strictly between $(y, z)R$ and $m$, we conclude that $T = R_{(y, z)R}$.

Since $S$ has principal maximal ideal $xS$, we have $P = PS_P$ as sets. Hence there are no rings properly between $S$ and $T = S[1/x]$. Since $U$ dominates $S$, we have $T \not\subseteq U$. Therefore $S = U \cap T$. However, $U$ need not be the boundary valuation ring of $S$. The boundary valuation ring is unique, and there are many possibilities for $U$; all we require of $U$ is that $U$ birationally dominates $R$ and its valuation $u$ has the property that $u(x)$ is infinitely smaller than both $u(y)$ and $u(z)$.

Example 7.4 is based on [35, Example 4.17] of Shannon. The calculation of the complete integral closure of the archimedean Shannon extension $S$ in this example relies on the following theorem, which gives a criterion for the complete integral closure of $S$ to be a simple ring extension of $S$.

Theorem 7.3. Assume notation as in Theorem 6.2. If there exists $\theta \in S^* \setminus S$ such that $S[\theta]_N = W$, then

1. $S^* = S[\theta]$;
2. $\theta^{-1} S[\theta^{-1}]$ is a maximal ideal of $S[\theta^{-1}]$, and
3. $V = S[\theta^{-1}]_{\theta^{-1} S[\theta^{-1}]}$. 

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Proof. To show that \( S[\theta] = S^* \), it suffices by Corollary [6.5] to show \( S[\theta]/N = S^*/N \).

Let \( k = S/N \) and let \((-)\) denote image modulo \( N \). Since \( \theta \notin S \), Corollary [6.4] implies \( \theta^{-1} \notin S \), so by Seidenberg’s Lemma [34], Theorem 7 it follows that \( S[\theta]/N = k[\theta] \) is a polynomial ring in one variable over the field \( k \). Thus from \( S[\theta]_N = W \), it follows by permutability of localization and residue class formation that \( W/NW = k(\theta) \) is a simple transcendental field extension. Thus \( k[\theta] \subseteq S^*/N \subseteq k(\overline{\theta}) \), so \( S^*/N \) is a localization of \( k[\theta] \). By Corollary [6.3] the units of \( S^*/N \) are the units of \( k \), so we conclude that \( S^*/N = k[\theta] \).

Let \( A = S[\theta^{-1}] \), so \( A \subseteq V \). Then \( NA \) is a prime ideal of \( A \), \( A_{NA} = V_{NV} \), and again by Seidenberg’s Lemma, \( A/NA = k[\theta^{-1}] \) is a polynomial ring in one variable over the field \( k \). Now by Theorem [6.2] \( \theta \in (N : N) \). Thus \( NA \subseteq \theta^{-1}A \), so that \( (N, \theta^{-1})A = \theta^{-1}A \) is a principal maximal ideal of \( A \). Moreover, since \( A/NA \cong k[\theta^{-1}] \), \( NA \) is a prime ideal of \( A \) just below \( \theta^{-1}A \).

Let \( \tilde{V} = A_{\theta^{-1}A} \), so \( \tilde{V} \) is a local domain with principal maximal ideal. Since \( \theta \in S^* \setminus S = (W \setminus V) \cap T \), it follows that \( \theta^{-1} \in M_V \), so \( V \) birationally dominates \( \tilde{V} \). Since \( \tilde{V} \) is a local domain with principal maximal ideal \( \theta^{-1}\tilde{V} \), it follows from Lemma [7.1(1)] that \( N\tilde{V} \) is the unique prime ideal just below \( \theta^{-1}\tilde{V} \) and that \( N\tilde{V} = N\tilde{V}_{N\tilde{V}} \). Furthermore, \( \tilde{V}[\theta] = \tilde{V}_{N\tilde{V}} = W \), so that \( \tilde{V}_{N\tilde{V}} \) is a valuation ring. By Lemma [7.1(2)], \( \tilde{V} \) is valuation domain. Since \( \tilde{V} \) is a valuation domain birationally dominated by \( V \), we have \( \tilde{V} = V \).

Example 7.4. Let \((R, m)\) be a 3-dimensional regular local ring with \( m = (x, y, z)R \). Let \( u \) be a valuation of the quotient field of \( R \) with the property that \( u(x) = a, u(y) = b, u(z) = c \) are rationally independent positive real numbers such that \( c > a + b \). Let \( \{(R_n, m_n)\}_{n \geq 0} \) be the sequence of local quadratic transforms of \( R = R_0 \) along the valuation ring determined by \( u \) and let \( S = \bigcup_{n \geq 0} R_n \). Shannon proves that \( S \) is not a valuation ring. Indeed, for each integer \( i \geq 0 \), Shannon proves that \( m_i = (x_i, y_i, z_i)R \), where \( a_i = u(x_i), b_i = u(y_i), c_i = u(z_i) \) are distinct rationally independent real numbers and \( c_i \neq \min\{a_i, b_i, c_i\} \). Thus the local quadratic transform from \( R_n \) to \( R_{n+1} \) is obtained either

1. by dividing by \( x_n \), in which case \( x_{n+1} = x_n, y_{n+1} = \frac{y_n}{x_n} \) and \( z_{n+1} = \frac{z_n}{x_n} \), or
2. by dividing by \( y_n \), in which case \( x_{n+1} = \frac{x_n}{y_n}, y_{n+1} = y_n \) and \( z_{n+1} = \frac{z_n}{y_n} \).

The valuation \( u \) defines a rank one valuation domain that birationally dominates \( S \). By varying the value of the real number \( u(z) = c \), subject only to the condition that \( c > a + b \), we conclude that there exist infinitely many rank one valuation domains that birationally dominate \( S \).

For each \( i \), the elements \( x_i \) and \( y_i \) each generate an \( N \)-primary ideal of \( S \). Consider the element \( \theta = \frac{a}{xy} = \frac{x_l}{x_ly_l} \). We show that \( \theta \in S^* \setminus S \) and that \( S^* = S[\theta] \).
Let $T$ be the Noetherian hull of $S$, and let $V$ be the boundary valuation ring for $S$. Since $x, y$ are units in $T$, $\theta \in T$. For each $i \geq 0$, it follows that $\text{ord}_i(\theta) = \text{ord}_i(\frac{\theta^i}{x^i y^i}) = -1$. Thus $\theta \not\in V$, so $\theta \not\in S$ and $\theta^{-1} \in V$. By Proposition 3.5, for each element $f \in N$, it follows that $\lim_{n \to \infty} \text{ord}_n(f) = \infty$. Thus $\text{ord}_n(\theta f) > 0$ for $n \gg 0$.

Denote $A = S[\theta]$. Since $\theta, \theta^{-1} \not\in S$, Seidenberg’s Lemma \cite[Theorem 7]{34} implies $N = NA$ is a prime ideal of $A$ and $A/N \cong k[\theta]$ is a polynomial ring in one variable over $k = S/N$. In particular, $N$ is a nonmaximal prime ideal of $A$.

We show that $A_N = W$. The ring $A_N$ is an integrally closed dimension 1 local domain that birationally dominates $R$ and has residual transcendence degree 1 over $S$. The valuation ring $U$ has rational rank three and hence is residually algebraic over $R$ \cite[Theorem 1]{1}. Therefore, $A_N$ has residual transcendence degree 1 over $R$. If $A_N$ is not a valuation domain, then it is birationally dominated by a valuation domain $B$ that has positive residual transcendence degree over $A_N$ \cite[Theorem 10, p. 19]{38}. Therefore, since $R$ has dimension 3 and $A_N$ has residual transcendence degree 1 over $R$, it must be that $B$ has residual transcendence degree 2 over $R$; cf. \cite[Theorem 1]{1}. This implies $B$ is a prime divisor of $R$ that dominates $S$. However, a Shannon extension of $R$ that is birationally dominated by a prime divisor of $R$ is necessarily equal to $R_i$ for one of the local quadratic transforms along $S$ \cite[Proposition 4]{1}, so we obtain a contradiction to the fact that in our case $\{R_i\}$ is an infinite sequence. Thus $A_N$ is a valuation domain that is birationally dominated by $W$, which forces $S[\theta]_N = A_N = W$. Thus by Theorem 7.3, we have $S^* = S[\theta]$ and $S[\theta^{-1}]_{\theta^{-1}S[\theta^{-1}]} = V$.

Finally, we note that the rank 1 valuation ring $U$ of $u$ along which $S$ was defined is not the rank 1 valuation overring $W$ of the boundary valuation ring $V$ of $S$, simply because $U$ has rational rank 3 and no such valuation overring of a 3-dimensional regular local ring can properly contain a valuation ring that contains $R$ (cf. \cite[Theorem 1]{1}).

**Remark 7.5.** Example 7.4 exhibits an archimedean Shannon extension that is neither completely integrally closed nor a valuation ring, and whose boundary valuation ring has rank 2. We prove in \cite{18} that there exist examples of archimedian Shannon extensions $S$ that are completely integrally closed and whose boundary valuation ring $V$ has rank 1 yet $S$ is not a valuation ring; i.e., $S \not\subseteq V$. 

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8. When a Shannon extension is a valuation ring

If $R$ is a regular local ring of dimension 2, then a valuation ring $V$ of $R$ that birationally dominates $R$ is the union of the sequence of local quadratic transforms of $R$ along $V$ [1, Lemma 12]. Moreover $V$ is either zero-dimensional or a prime divisor of $R$ [1, Theorem 1]. Since the sequence of local quadratic transforms along a prime divisor is finite [1, Proposition 4], it follows that the Shannon extensions of the two-dimensional regular local ring $R$ are precisely the zero-dimensional valuation rings that birationally dominate $R$. In higher dimensions, while a Shannon extension need not be a valuation ring, and a zero-dimensional valuation overring need not be a Shannon extension, much is known about when these extensions are valuation rings; cf. [8, 9, 10, 11, 12, 17, 35]. In this section we give additional characterizations of when a Shannon extension is a valuation ring and recover some of the previously known characterizations from a different point of view.

**Theorem 8.1.** The following are equivalent for a Shannon extension $S$ of a regular local ring $R$.

1. $S$ is a valuation ring.
2. $S$ has only finitely many height 1 prime ideals.
3. Either (a) $\dim S = 1$ or (b) $\dim S = 2$ and the boundary valuation ring $V$ of $S$ has value group $\mathbb{Z} \oplus G$, where $G$ is a subgroup of $\mathbb{Q}$ and the direct sum is ordered lexicographically.

**Proof.** We use in the proof that by Theorem 5.4 we have $S = V \cap T$, where $V$ is the boundary valuation ring of $S$ and $T$ is the Noetherian hull of $S$. In particular, there is $x \in S$ such that $xS$ is primary for the maximal ideal $N$ of $S$ and $T = S[1/x]$.

(1) $\iff$ (2) It is clear that (1) implies (2) since the ideals of a valuation ring are totally ordered by inclusion. Conversely, suppose that $S$ has only finitely many height 1 prime ideals. If $P$ is a nonmaximal prime ideal of $S$ such that $P$ has height $> 1$, then since $SP$ is a localization of the Noetherian ring $T$, there exist infinitely many height 1 prime ideals of $S$ that are contained in $P$, contrary to (2). Therefore, every nonmaximal prime ideal $P$ of $S$ has height 1, and $T = S[1/x]$ is the intersection of the rings $SP$, where $P$ ranges over the height 1 prime ideals of $S$. By assumption there are only finitely many such prime ideals $P$. Moreover, since $T$ is an integrally closed Noetherian domain, $SP$ is a DVR for each each height 1 prime

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*The dimension of a valuation ring $V$ that birationally dominates $R$ is the transcendence degree of the residue field of $V$ over the residue field of $R$; cf. [38, p. 34].*
ideal $P$ of $S$. Therefore, since $S = V \cap T$, $S$ is an intersection of $V$ and finitely many DVRs. Since $S$ is local, this implies that $S$ is a valuation domain \cite[11.11]{27}.

(1) $\Rightarrow$ (3) Suppose that $S$ is a valuation ring. If $S$ is a DVR, the claim is clear, so suppose that $S$ is not a DVR. As an overring of the valuation ring $S$, $T$ is a valuation ring, and hence the Noetherian ring $T$ is either a DVR or the quotient field of $S$. If $\dim S > 1$, then necessarily $T$ is a DVR, and since every nonmaximal prime ideal of $S$ survives in $T = S[1/x]$, this forces $\dim S = 2$. Furthermore, by \cite[Proposition 14]{8}, $S/P$ has value group isomorphic to a subgroup of $\mathbb{Q}$.

(3) $\Rightarrow$ (1) If $\dim S = 1$, then $T = S[1/x]$ is the quotient field of $S$, so that $S = V \cap T = V$, and hence $S$ is a valuation ring. Suppose that $\dim S = 2$ and the value group of $V$ is $\mathbb{Z} \oplus G$, where $G$ is a subgroup of $\mathbb{Q}$. If the nonzero nonmaximal prime ideal $P$ of $V$ lies over the maximal ideal $N$ of $S$, then $S$ is dominated by a DVR (namely, the localization of $V$ at $P$), but then by Corollary 3.9, $S$ is a DVR, contrary to $\dim S = 2$. Thus $P \cap S$ is a height 1 prime ideal of $S$ and hence $S_{P \cap S}$ is a localization of $T = S[1/x]$. Since by Theorem 11, $T$ is a localization of some $R_i$, it follows that $S_{P \cap S}$ is a localization of $R_i$ at a height 1 prime. In particular, $S_{P \cap S}$ is a DVR, which forces $S_{P \cap S} = V_P$. Since $V \subseteq S_{P \cap S}$ and $V/P$ has rational value group with $V$ irredundant in the intersection $S = V \cap T$, it follows that $V$ is a localization of $S$ \cite[Lemma 3.1]{28}. Since $V$ dominates $S$, this forces $S = V$, which verifies (1).

\begin{remark}
If the Shannon extension $S$ of $R$ is a valuation ring with $\dim S = 2$, then by Theorem 8.1, the value group of $S$ has rational rank 2. There is no such bound on the rational rank of a valuation ring obtainable as a Shannon extension $S$ when $\dim S = 1$. Granja \cite[Proposition 16]{8} has shown that if $R$ is a regular local ring of dimension $d \geq 2$, then there exists a Shannon extension of $R$ that is a valuation ring and whose corresponding valuation has rational rank $d$.

\end{remark}

\begin{corollary}
Let $S$ be a Shannon extension of the regular local ring $R$. Then $S$ is a valuation domain with discrete value group if and only if $\dim S \leq 2$ and $S$ has a principal maximal ideal.

\end{corollary}

\begin{proof}
If $S$ is a valuation ring with discrete value group, then by Theorem 8.1, $\dim S \leq 2$, and since the value group of $S$ is discrete, $S$ has a principal maximal ideal. Conversely, suppose that $\dim S \leq 2$ and $S$ has a principal maximal ideal. If $\dim S = 1$, then $S$ is necessarily a DVR, so suppose that $\dim S = 2$. With the notation of Lemma 3.4, $S/P$ is a DVR and $PS_P = P$. Moreover, by Theorem 5.4, there is $x \in S$ such that $xS$ is primary for the maximal ideal and $S[1/x]$ is a regular Noetherian domain. Since $S_P$ is a localization of $S[1/x]$ and $\dim S_P = 1$, it follows that $S_P$ is a DVR. This and the fact that $S/P$ is a DVR and $PS_P = P$ imply that $S$ is a valuation domain with discrete value group.

\end{proof}
Corollary 8.4. (Abhyankar [1, Lemma 12]) If $R$ is a regular local ring with $\dim R = 2$, then the set of Shannon extensions of $R$ is precisely the set of zero-dimensional valuation overrings of $R$ that dominate $R$. 

Proof. Let $V$ be a zero-dimensional valuation overring of $R$ that dominates $R$. Then $V$ determines an infinite sequence of local quadratic transforms $\{R_i\}$ and hence there is a a Shannon extension $S$ of $R$ with $S \subseteq V$. Since $\dim R = 2$, then $\dim R_i = 2$ for all $i \geq 0$. If $\dim S = 1$, then $S$ is a valuation ring by Theorem 8.1. Suppose that $\dim S = 2$, and suppose by way of contradiction that $S$ is not a valuation ring. Then there exists $u$ in the quotient field of $S$ such that $u, u^{-1} \not\in S$. Hence, with $N$ the maximal ideal of $S$, Seidenberg’s Lemma [34, Theorem 7] implies $NS[u]$ is a nonzero nonmaximal prime of $S[u]$. Therefore $S[u]$ is contained in a valuation ring $U$ with $\dim U = 2$ such that the nonzero nonmaximal prime ideal $P$ of $U$ is centered on $NS[u]$. Since $\dim U = 2$ and $U$ is an overring of the two-dimensional Noetherian domain $R$, the value group of $U$ is discrete [1, Theorem 1], and hence $U_P$ is a DVR that dominates $S$. However, by Corollary 3.9, this implies $S$ is a DVR, contrary to assumption. Thus $S$ is a valuation ring, and since $V$ dominates $S$, $S = V$, which proves the corollary.

In general, it is not enough that the maximal ideal of a Shannon extension $S$ is principal to guarantee that $S$ is a valuation ring; see Example 7.2. However, with an additional assumption, $S$ must be a valuation ring:

Corollary 8.5. A Shannon extension $S$ of a regular local ring $R$ is a DVR if and only if the maximal ideal of $S$ is principal and $S$ is dominated by a rank 1 valuation ring.

Proof. Suppose that the maximal ideal of $S$ is principal and $S$ is dominated by a rank 1 valuation ring. The latter property implies that $S$ is archimedean, and hence since $S$ has a principal maximal ideal, Lemma 8.4(2) forces $\dim S = 1$. Therefore, by Corollary 8.3, $S$ is a DVR. The converse is clear.

Following Shannon [35], the quadratic sequence $\{R_i\}$ determined by the Shannon extension $S$ switches strongly infinitely often if epd($S/R$) is empty, and following Granja [8], the sequence $\{R_i\}$ is height 1 directed if epd($S/R$) has exactly one element. In Proposition 8.7 we show how to recover some results of Granja from our point of view. We use the notion of an essential prime divisor from Definition 2.3.

Lemma 8.6. If $S$ is a Shannon extension of the regular local ring $R$ with $\dim S > 1$, then epd($S/R$) = epd($S$).
Proof. Let $V \in \text{epd}(S/R)$. Then there exists a height 1 prime ideal $P_i$ of $R_i$ for some $i > 0$ such that $S \subseteq V = (R_i)_{P_i}$. Let $P$ be the contraction of the maximal ideal of $V$ to $S$. Then $S_P = V$, and hence $P$ has height 1 and $V \in \text{epd}(S)$. Conversely, suppose that $P$ is a height 1 prime ideal of $S$. Since dim $S > 1$, we have $T \subseteq S_P$, where $T$ is as in Theorem 5.4 and $T$ is a localization of $R_i$ for some $i$. Therefore, $S_P$ is a one-dimensional localization of $R_i$, which forces $(R_i)_{P \cap R_i} = S_P$ and $P \cap R_i$ to be a height 1 prime ideal of $R_i$. Consequently, $S_P \in \text{epd}(S/R)$.

\begin{proposition}(cf. Granja \cite{granja}, Props. 7 and 14, Thm. 13) Assume Setting 6.1.

1. \{\(R_i\)\} switches strongly infinitely often if and only if $S$ is a valuation ring with dim $S = 1$.

2. \{\(R_i\)\} is height 1 directed if and only if $S$ is a valuation ring with dim $S = 2$ and value group $G \oplus \mathbb{Z}$, where $G$ is a subgroup of $\mathbb{Q}$ and the sum is ordered lexicographically.

3. $S$ is a valuation ring if and only if \{\(R_i\)\} switches strongly infinitely often or \{\(R_i\)\} is height 1 directed.

\end{proposition}

Proof. (1) Suppose $|\text{epd}(S/R)| = 0$. Then by Lemma 8.6 there does not exist a height 1 prime ideal $P$ of $S$ such that $S_P$ is a DVR. Since $S[1/x]$ is a regular Noetherian domain and $x$ is primary for the maximal ideal of $S$ (with notation as in Setting 6.1), it follows that dim $S = 1$. Hence by Theorem 8.1, $S$ is a valuation ring. Conversely, if $S$ is a valuation ring with dim $S = 1$, then there exist no overrings properly between $S$ and its quotient field. Consequently, since $S$ is not a DVR (this possibility is ruled out by Setting 5.1(3)), epd($S/R$) is empty.

(2) Suppose that $|\text{epd}(S/R)| = 1$. Then by (1), dim $S > 1$, and hence by Lemma 8.6, epd($S/R$) = epd($S$). Therefore, $S$ has only one height 1 prime ideal, and hence by Theorem 8.1, $S$ is a valuation ring with value group $\mathbb{Z} \oplus G$, where $G$ is a subgroup of $\mathbb{Q}$. The converse follows from Lemma 8.6.

(3) In light of (1) and (2), it only needs to be observed that if $S$ is a valuation ring, then $|\text{epd}(S/R)| = |\text{epd}(S)| \leq 1$. \hfill \Box

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