Four-loop large-$n_f$ contributions to the non-singlet structure functions $F_2$ and $F_L$

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ABSTRACT: We have calculated the $n_f^2$ and $n_f^3$ contributions to the flavour non-singlet structure functions $F_2$ and $F_L$ in inclusive deep-inelastic scattering at the fourth order in the strong coupling $\alpha_s$. The coefficient functions have been obtained by computing a very large number of Mellin-$N$ moments using the method of differential equations, and then determining the analytic forms in $N$ and Bjorken-$x$ from these. Our new $n_f^2$ terms are numerically much larger than the $n_f^3$ leading large-$n_f$ parts which were already known; they agree with predictions of the threshold and high-energy resummations. Furthermore our calculation confirms the earlier determination of the four-loop $n_f^2$ part of the corresponding anomalous dimension. Via the no-$\pi^2$ theorem for Euclidean physical quantities, we predict the $\zeta_4 n_f^3$ part of the fifth-order anomalous dimension for the evolution of non-singlet quark distributions.

KEYWORDS: Deep Inelastic Scattering or Small-x Physics, Higher-Order Perturbative Calculations

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1 Introduction

Inclusive deep-inelastic lepton nucleon scattering (DIS) via the exchange of an electro-weak gauge boson is an experimental and theoretical benchmark process of perturbative QCD. Data on its main structure functions provide a rather direct determination of (linear combinations of) the quark momentum distributions of the nucleon. The structure function \( F_2(x, Q^2) \), in particular, has been determined in the past decades over a wide range of the scale \( Q^2 = -q^2 \), where \( q \) is the momentum of the exchanged boson and the Bjorken variable \( x = Q^2/(2p \cdot q) \), where \( p \) is the momentum of the nucleon) in fixed-target experiments and at the electron-proton collider HERA, see ref. [1] and references therein. Further measurements of inclusive DIS are planned for future facilities, in particular the Electron Ion Collider (EIC) at Brookhaven National Lab [2, 3] and the Large Hadron Electron Collider (LHeC) [4, 5].

Precise determinations of the quark momentum distributions \( q_i(\xi, Q^2) \) (with \( \xi = x \) at the leading-order of perturbative QCD) as well as, less directly, of the gluon distribution \( g(\xi, Q^2) \) and the strong coupling \( \alpha_s \) from structure-function data require higher-order calculations of the corresponding coefficient functions (partonic structure functions). These coefficient functions are of relevance also beyond the cross sections for inclusive DIS, see, e.g., refs. [6, 7] on Higgs production in vector-boson fusion and ref. [8] on jet production in DIS.

For the quantities under consideration in this article, the flavour non-singlet contributions to \( F_2 \) and the longitudinal structure function \( F_L \), the second-order corrections have been calculated and verified long ago [9–12]. The corresponding three-loop expressions were obtained in ref. [13] and recently re-calculated in ref. [14]. At the fourth order, only the lowest five Mellin-\( N \) moments have been computed so far [15–17] using the FORCER program [18], in addition to the leading terms in the limit of a large number of flavours \( n_f \) [19, 20].

In the present article, we take the next step towards the determination of the fourth-order non-singlet coefficient functions \( c_{2,ns}^{(4)}(x) \) and \( c_{L,ns}^{(4)}(x) \) and compute their doubly fermionic \( n_f^2 \) contributions. These results are obtained by a new method which allows the determination
of their moments up to very high (even) values of $N$, beyond $N = 1000$, from which the analytic dependence on $N$, and hence on $x$, can be re-constructed in terms of harmonic sums \cite{21} and harmonic polylogarithms (HPLs) \cite{22}, respectively. As a by-product, we have checked the $n_f^2$-contributions to the four-loop non-singlet splitting function $P_{\text{ns}}^{(3)+}(x)$ of ref. \cite{23}.

The remainder of this article is organized as follows: in section 2 we briefly recall the theoretical framework for the coefficient functions in inclusive DIS and their determination to the fourth order in $\alpha_s$. In section 3 we describe our method of the calculation based on iteratively solving a system of recurrence relations which is derived via the method of differential equations \cite{24–27}. The analytic results for the coefficient functions in $N$-space are presented in section 4, which also includes a resulting partial prediction for the five-loop non-singlet splitting function $P_{\text{ns}}^{(4)}(N)$. The corresponding $x$-space coefficient functions and their threshold and high-energy limits are written down and discussed in section 5. We summarize our method and results and give a brief outlook in section 6.

## 2 Theoretical framework and notations

The subject of our computations is unpolarized inclusive lepton-nucleon DIS

$$\text{lepton}(k) + \text{nucleon}(p) \rightarrow \text{lepton}(k') + X$$

at the lowest order of QED (i.e., via the exchange of one photon with momentum $q = k - k'$). $X$ stands for all hadronic states allowed by quantum number conservation. The double-differential cross section in $Q^2 = -q^2$ and $x = Q^2/(2 p \cdot q)$ for this process can be expressed as the product of a calculable and well-known leptonic tensor and the hadronic tensor

$$W_{\mu\nu}(p, q) = \left( \frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) F_1(x, Q^2) - (q_\mu + 2xp_\mu)(q_\nu + 2xp_\nu) \frac{1}{2xq^2} F_2(x, Q^2).$$

Neglecting contributions that are suppressed at large scales by powers of $1/Q^2$, the structure functions $F_2 = 1/x F_2$ and $F_L = 1/x (F_2 - 2x F_1)$ can be expressed in terms of the universal but perturbatively incalculable quark and gluon parton distribution functions (PDFs), $q_i(\xi, Q^2)$ and $g(\xi, Q^2)$, and the perturbative coefficient functions $C_{a,p}(x, Q^2)$. In the present article, we are specifically interested in non-singlet (combinations of) structure functions $F_{a,\text{ns}}$, such as $F^\text{proton}_a - F^\text{neutron}_a$, which decouple from the gluon distribution, viz

$$F_{a,\text{ns}}(x, Q^2) = [C_{a,\text{ns}} \otimes q_{\text{ns}}](x, Q^2)$$

where $\otimes$ abbreviates the Mellin convolution. The non-singlet combinations $q_{\text{ns}}$ of quark distributions are normalized such that the expansion of the coefficient functions in powers of $a_s \equiv \alpha_s(Q^2)/(4\pi)$ is given by

$$C_{a,\text{ns}}(x, Q^2) = (1 - \delta_{aL}) \delta(1-x) + \sum_{n=1} a_n^a c^{(n)}_{a,\text{ns}}(x).$$

Here and below we identify the $\overline{\text{MS}}$ renormalization and mass-factorization scales $\mu_r^2$ and $\mu_f^2$, at which the strong coupling and the PDFs are evaluated, with $Q^2$. The dependence on $\mu_r^2$ and $\mu_f^2$ can be readily reconstructed a posteriori, see, e.g., eqs. (2.17) and (2.18) of ref. \cite{28}.
In terms of the general framework, our determination of the coefficient functions uses the method set out (and applied to the lowest moments at the third order) in refs. [29, 30], see also refs. [12–14]: the cross section for inclusive DIS for quark external states, projected onto the structure functions \( F_a \), is related by the optical theorem to the imaginary parts of the corresponding amplitudes for photon-quark forward scattering. Via a dispersion relation the coefficients of \([ (2p \cdot q) / Q^2 ]^N = 1 / x^N\) lead to the even-integer (see also ref. [31]) Mellin-\( N \) moments

\[
\tilde{F}_{a, \text{ns}}(N, Q^2) = \int_0^1 dx \ x^{N-1} \tilde{F}_{a, \text{ns}}(x, Q^2),
\]

(2.5)
of the bare partonic structure functions. These are computed from Feynman diagrams in dimensional regularization with \( D = 4 - 2 \varepsilon \) dimensions.

After the \( \overline{\text{MS}} \) renormalization of the coupling constant to the fourth order,

\[
a_{a, \text{bare}} = a_s \left( 1 - \frac{\beta_0}{\varepsilon} a_s + \left( \frac{\beta_3^0}{\varepsilon^3} - \frac{\beta_1}{2\varepsilon} a_s^2 - \left( \frac{\beta_3^0}{\varepsilon^3} - \frac{7\beta_1\beta_0}{6\varepsilon^2} + \frac{\beta_2}{3\varepsilon} \right) a_s^3 + \ldots \right) \right)
\]

(2.6)
with \( \beta_0 = 11 - 2/3 n_f \) etc in QCD [32, 33], the left-hand-side of eq. (2.5) can be written as

\[
\tilde{F}_{a, \text{ns}}(N, Q^2) = \tilde{C}_{a, \text{ns}}(N, a_s) Z_{\text{ns}}(N, a_s).
\]

(2.7)
The \( D \)-dimensional coefficient function \( \tilde{C}_{a, \text{ns}} \) includes additional terms with positive powers of \( \varepsilon \) on top of the Mellin transform of eq. (2.4), i.e.,

\[
\tilde{c}_{a, \text{ns}}^{(n)}(N) = c_{a, \text{ns}}^{(n)}(N) + \varepsilon a_{a, \text{ns}}^{(n)}(N) + \varepsilon^2 b_{a, \text{ns}}^{(n)}(N) + \varepsilon^3 d_{a, \text{ns}}^{(n)}(N) + \ldots.
\]

(2.8)
The quantity \( Z_{\text{ns}} \) which renormalizes the non-singlet quark distributions is given by

\[
Z_{\text{ns}} = 1 + a_s \frac{1}{\varepsilon} \gamma_{\text{ns}}^{(0)} + a_s^2 \left[ \frac{1}{2\varepsilon^2} \left( \gamma_{\text{ns}}^{(0)} - \beta_0 \right) \gamma_{\text{ns}}^{(0)} \right] + \frac{1}{2\varepsilon} \gamma_{\text{ns}}^{(1)}
\]

\[
+ a_s^3 \left[ \frac{1}{6\varepsilon^3} \left( \left( \gamma_{\text{ns}}^{(0)} - 2\beta_0 \right) \left( \gamma_{\text{ns}}^{(0)} - \beta_0 \right) \gamma_{\text{ns}}^{(0)} \right) \right]
\]

\[
+ \frac{1}{6\varepsilon^2} \left( \left( 3\gamma_{\text{ns}}^{(0)} - 2\beta_0 \right) \gamma_{\text{ns}}^{(1)} - 2\beta_1 \gamma_{\text{ns}}^{(0)} \right) + \frac{1}{3\varepsilon} \gamma_{\text{ns}}^{(2)}
\]

\[
+ a_s^4 \left[ \frac{1}{24\varepsilon^4} \left( \left( \gamma_{\text{ns}}^{(0)} - 3\beta_0 \right) \left( \gamma_{\text{ns}}^{(0)} - \beta_0 \right) \left( \gamma_{\text{ns}}^{(0)} - \beta_0 \right) \gamma_{\text{ns}}^{(0)} \right) \right]
\]

\[
+ \frac{1}{12\varepsilon^3} \left( \left( 3\gamma_{\text{ns}}^{(0)} - 7\beta_0 \right) \gamma_{\text{ns}}^{(0)} + 3\beta_2 \right) \gamma_{\text{ns}}^{(1)} - 2 \left( 2\gamma_{\text{ns}}^{(0)} - 3\beta_0 \right) \beta_1 \gamma_{\text{ns}}^{(0)} \right)
\]

\[
+ \frac{1}{24\varepsilon^2} \left( \left( 3\gamma_{\text{ns}}^{(0)} - 7\beta_0 \right) \gamma_{\text{ns}}^{(2)} + 3 \left( \gamma_{\text{ns}}^{(1)} - 2\beta_0 \right) \gamma_{\text{ns}}^{(1)} - 6\beta_2 \gamma_{\text{ns}}^{(0)} \right) + \frac{1}{4\varepsilon} \gamma_{\text{ns}}^{(3)}
\]

(2.9)
to the fourth order. Here \( \gamma_{\text{ns}}^{(n)}(N) \) — the arguments \( N \) have been suppressed in eq. (2.9) for brevity — are the N\( ^n \)LO non-singlet anomalous dimensions related by

\[
\gamma_{\text{ns}}^{(n)}(N) = - \int_0^1 dx x^{N-1} P_{\text{ns}}^{+(n)}(x)
\]

(2.10)
to the expansion coefficients of the \( \overline{\text{MS}} \)-scheme splitting function for the evolution of flavour differences of the sums (hence ‘+’) of quark and antiquark PDFs,

\[
P_{\text{ns}}^{+(n)}(x, a_s) = \sum_{n=0} a_s^{n+1} P_{\text{ns}}^{+(n)}(x).
\]

(2.11)
Inserting the expansions (2.4), (2.8) and (2.9) into eq. (2.7), the anomalous dimension and \((D\)-dimensional\) coefficient functions can be extracted order by order from the results of the diagram calculations. In order to obtain the fourth-order coefficient functions \(c_{a,\text{ins}}^{(4)}\), the lower-order calculations need to include terms up to \(\varepsilon^{4-n}\) at order \(\alpha_s^n\). In particular, the determination of the \(n_f^2\) contributions to \(c_{a,\text{ins}}^{(4)}\) requires the \(n_f\) parts of \(a_{a,\text{ins}}^{(3)}\) which were beyond the scope of ref. [13] — at the time only the integrals required for one simpler Lorentz projection of \(W_{\mu\nu}\) were extended to this accuracy for ref. [34].

3 Method and computations

In terms of the diagram sets and the treatment to the point at which the Feynman integrals are evaluated, our computation is closely related to that of third-order fermionic \((n_f\) and \(n_f^2\)) contributions in ref. [35] and the non-singlet part of ref. [23]. Our evaluation of the Feynman integrals is entirely different, though, from both. In ref. [35] the analytic \(N\)-dependence was determined by setting up and solving, in a far from fully automated manner, complicated systems of difference equations. In ref. [23] the even moments were computed to \(N = 22\) for the \(C_F C_A n_f^2\) terms and to \(N = 42\) for the \(C_A^2 n_f^2\) terms using FORCER [18]. From these it was possible, just, to reconstruct the analytic \(N\)-dependence of the four-loop anomalous dimensions using all available physics constraints and systems of Diophantine equations.

Below we give details on the methods used in the present computation. We believe that some of the techniques we have used here have been employed for the first time in a multi-loop calculation, and that these should be useful not only for tackling DIS at four loops but also for other multi-loop calculations. Using these techniques we have been able to generate a very large number of Mellin moments, up to \(N = 1500\), by evaluating the series expansion of the forward scattering amplitude around \(1/x \equiv \omega = 0\), recall the discussion above eq. (2.5). With that many moments, it is possible to reconstruct the analytic \(N\)-dependence of the fourth-order coefficient functions by a direct (and over-constrained) Gaussian elimination for a sufficiently general ansatz in terms of harmonic sums.

Our basic setup relies on QGRAF [36] and FORM [37–39], employing the program MINOS [40] as a diagram database tool. Many of the FORM libraries we use have been employed in a substantial number of earlier calculations, e.g., in refs. [12, 13, 15, 23, 35] and have been highly optimized for multi-loop perturbative QCD calculations. In particular, as in refs. [15, 23], the database combines diagrams whose underlying graph topology is equivalent and whose colour factors are the same. Such sets of diagrams lead to faster evaluation times as they allow to realize algebraic cancellations between the individual diagrams.

As the \(n_f\) contributions at three loops, the present \(n_f^2\) contributions at four loops are special since they do not yet involve the more difficult topologies in their respective orders. In fact, the most difficult four-loop cases derive from the hardest three-loop diagrams shown in figure 1 by simply inserting an additional quark loop into one of the gluon propagators, i.e., no ‘genuine’ (non-insertion) four-loop self-energy topologies are required [15].

The best route to determine the all-\(N\) expressions for the \(n_f^2\) four-loop contributions is via the large-\(n_c\) limits and the \(C_A^2 n_f^2\) terms. The former do not involve alternating harmonic sums, as even and odd \(N\)-values must lead to the same \(x\)-space function. For the latter one
Figure 1. The hardest diagrams contributing to the $n_f$ contributions to the third-order coefficient functions for $F_{2,ns}$ and $F_{L,ns}$. In the notation of MINCER \cite{41, 42}, the two on the left are of the BE topology and the two on the right are O4 cases. All four diagrams have the colour factor $C_F C_A n_f$.

expects a somewhat simpler form than for the $C_F C_A n_f^2$ terms, since only two-loop diagrams with two one-loop or one two-loop self-energy insertion(s) contribute. In practice, the non-$\zeta$ part of $c_{2,ns}^{(4)}$ was the most difficult case, which we solved in FORM using an ansatz with a little less than 600 even-$N$ coefficients. All other cases required fewer than 400 coefficients.

**Topologies for the non-singlet $n_f^2$ structure functions.** As discussed at the end of section 2, we also need the lower-order corrections to a sufficient power in $\varepsilon$; in particular we have to compute the $\varepsilon^1$ terms for the $n_f$ parts at three loops.

The diagrams contributing to the final result are organized into topologies, and described in terms of a linear independent set of propagators and scalar products ready to be reduced to a smaller set of master integrals \cite{43}. In order to obtain a more efficient reduction, especially at four loops, we build our topologies by keeping the number of propagators within a topology as low as possible. In practice, a four-loop DIS topology has 18 linearly independent scalar products; instead of describing them in terms of as many linearly independent propagators, we opt for at most 12 propagators and 6 scalar products that will appear only in the numerator. At three-loop, this results in a topology described by 10 propagators and 2 scalar products.

We further simplify the expressions of the diagrams by rewriting multi-loop self-energy corrections in terms of chained bubbles which has the additional benefit of keeping all the propagator powers as whole numbers more suited for the Laporta reductions,

\[
\begin{align*}
q & \quad \Pi_L(q^2) = \frac{\Pi_L(q^2)}{[\text{Bub}(q^2)]^L} \\
& \quad \quad \quad \quad q \\
& \quad \quad \quad \quad 2 \\
& \quad \quad \quad \quad \cdots \\
& \quad \quad \quad \quad \mathcal{L} \\
& \quad \quad \quad \quad q.
\end{align*}
\]

In figure 2 we show the range of topologies that are required for the non-singlet $n_f^2$ correction at the fourth order, with diagrams ranging from 12 to 9 distinct scalar propagators, one propagator short from the most complicated case one can encounter at four loops.

The number of scalar propagators in the definition of each topology is not the only factor to take into account when considering the problem of reducing our integrals to master integrals (very important is also, for example, the dimensionality of the numerator). However, it remains a key aspect since it drastically reduces the degrees of freedom one has to consider during the reductions by effectively vetoing some of the sectors. To perform the reductions to master integrals we have used both the FIRE \cite{44} and KIRA \cite{45} programs.
Figure 2. Three- and four-loop propagator topologies used for the reduction to master integrals and the expansion in $\omega$ using differential equations. The external double line represent the off-shell photon while the simple lines the parton. Each diagram allows for two independent flows of the external momenta resulting in the same topology up to a sign inversion $\omega \leftrightarrow -\omega$.

**Series expansion and differential equations.** Once all the diagrams appearing in the process are reduced to master integrals, we build a system of differential equations by acting on them with the differential operator $\partial/\partial \omega$ and further reduce the resulting integrals. This step may also involve extending the initial set of master integrals to achieve a closed system. The resulting differential equations can be written as

$$\frac{\partial}{\partial \omega} \vec{M}(\omega, \varepsilon) = A(\omega, \varepsilon) \cdot \vec{M}(\omega, \varepsilon),$$  \hspace{1cm} (3.2)

where $A \in \mathbb{Q}^{n \times n}(\omega, \varepsilon)$ is a square matrix whose elements are rational functions in $\omega$ and the dimensional regulator $\varepsilon$ over the field $\mathbb{Q}$, with $n$ being the number of master integrals. The general ansatz for this system of differential equations can be written as a linear combination of meromorphic functions,

$$\vec{M}(\omega, \varepsilon) = \omega^{\vec{\alpha}} \sum_{s \subseteq \mathbb{Z}} \sum_{k=0}^{\infty} \vec{m}_{k}^{(s)}(\omega, \varepsilon) \omega^{k+s \varepsilon}, \quad \vec{\alpha} \in \mathbb{Z}^n,$$  \hspace{1cm} (3.3)

where $\vec{\alpha}$ defines the leading power of each master and the sum over $s$ goes over a finite set of integers which multiply $\varepsilon$ in the power of $\omega$, and represent independent (not necessarily physical) solutions to the differential equations. The space of solutions is reduced by noticing that the boundary conditions for the master integrals in the limit $\omega = 0$ are finite ($\alpha \geq 0$) and support only the regular sector $s = 0$. This may in fact not be true for individual integrals, but it is well known that the forward amplitudes for DIS are regular at $p = 0$ and hence $w = 0$. We can thus safely ignore solutions with $s \neq 0$. The general problem of obtaining series expansions from differential equations is of course not new in the context...
of Feynman integrals and indeed has been employed successfully in, e.g., refs. [43, 46–52], and public implementations exist [53–55]; and its application to DIS has been addressed in refs. [14, 56, 57].

Here we present a new implementation of this method which takes advantage of the specific analytic properties of the problem and which is well suited for obtaining many coefficients in the expansion, while at the same time being simple enough to be implemented purely in FORM. We describe this procedure in the following.

We make use of the information coming from the boundary conditions to simplify the ansatz for the master integral expansion, and at the same time we consider the differential matrix to contain at most simple poles at $\omega = 0$, viz

$$M_i = \omega^{\alpha_i} \sum_{k=0}^{\infty} m_{ik} \omega^k, \quad A = \frac{A_{-1}}{\omega} + \sum_{k=0}^{\infty} A_k \omega^k. \quad (3.4)$$

In general, the matrix $A$ can contain higher order poles around $\omega = 0$, depending on the choice of master integrals. While it is known in the Mathematics literature since the 1950s that a basis transformation to remove higher poles exists [58], finding the transformation to bring the differential equations into a so-called canonical form is non-trivial in practice. By now there exist nevertheless several strategies/algorithms to solve this problem [59, 60], and some implementations are now publicly available [61–63]. For our purpose, we do not actually require the complete reduction of the system into the canonical form — as it is sufficient to simply remove higher order poles at $\omega = 0$.

We achieve this by applying a rescaling transformation $T$ to the master integrals,

$$M \to T \cdot M, \quad A \to \frac{\partial T}{\partial \omega} T^{-1} + T \cdot A \cdot T^{-1}, \quad (3.5)$$

with the transformation matrix taking the form

$$T = \text{diag} \left( \omega^\beta \right), \quad \beta \in \mathbb{N}_0^n \Rightarrow A_{ij} \to \delta_{ij} \frac{\beta_i}{\omega} + \omega^{\beta_i-\beta_j} A_{ij}. \quad (3.6)$$

In general we found that the form of $T$ required to remove all poles is not unique. We explored several algorithms allowing to construct a suitable $T$; the probably simplest was based on constructing $T$ by iteratively removing a pole of order $k$ in the $j^{th}$ row by setting $\beta_j = \omega^{k-1}$. While this particular pole would be removed, new poles could be created in other rows in the transformed matrix $A$. In all cases we found that iterating this procedure allowed to eventually remove all non-simple poles from $A$. We actually have no proof for this simple procedure to terminate and it remains to be seen whether it will also work for even more complicated cases. The advantage of the procedure in comparison to the much more involved algorithms to bring $A$ into a Fuchsian form is that it is computationally far simpler and can be applied with ease also to comparably large systems of sizes around the 100s or even 1000s. It is not clear to us that the same holds for the algorithms mentioned above.

By using the definitions in eq. (3.4) it is possible to rewrite eq. (3.2) into a recursive expression that allows for an efficient extraction of the expansion coefficients of the master integrals,

$$\left( (k+1) I - A_{-1} \right) \cdot \vec{m}_{k+1} = \sum_{j=0}^{k} A_j \vec{m}_{k-j} \quad (3.7)$$
where $\mathbb{I}$ is the identity matrix. Note that we can have at most $n$ cases where $\det(B_k) = 0$, corresponding to positive integer values of the eigenvalue of $A_{-1}$. In these cases the system cannot be inverted and is solved by means of Gaussian elimination. For high enough $k$ we have $\det(B_k) \neq 0$ and we can write

$$m_{k+1} = B_k^{-1} \cdot \left( \sum_{j=0}^{k} A_j \ m_{k-j} \right).$$  

(3.8)

The inversion of the matrix $B$ is performed for generic values of $k$, avoiding the expensive procedure of performing a new inversion for every step of the expansion, especially for high values of $k$.

We have implemented the Gaussian elimination for the arbitrary steps of eq. (3.7) in C++ while the case of non-singular $B_k$ matrices in eq. (3.8) has been implemented in FORM. The required boundary conditions in the limit of $\omega = 0$, where the external parton is taken to be soft, for all the master integrals naturally lead to self-energy integrals, which have been computed using FORCER [18]. The performances of eq. (3.8) can be further improved by truncating the expansion in the dimensional regulator to the required order to obtain an amplitude known up to $\varepsilon^{4-n}$ at the $n$-th order.

One can benefit from the combined expansion only when the two limits $\varepsilon, \omega = 0$ of the differential matrix $A$ commute. For example, the series expansion in $\omega$ of a denominator of the form $(\omega - \varepsilon)$ has a convergence radius that goes to zero as $\varepsilon$ vanishes. These kind of poles are unphysical and will lead to arbitrary high poles in $\varepsilon$ when expanding the differential matrix and will cancel in the final expression of the expansion coefficients $\vec{m}_{k+1}$.

To circumvent this problem one can transform to a basis of master integrals in which the reduction coefficients — and therefore also the coefficients of the differential equation matrix $A$ — have the property that their dependence on $\varepsilon$ factorizes from their dependence on $\omega$. The existence of such a basis was proven in ref. [64], and two independent implementations which construct the corresponding basis of master integrals have been published [65, 66] as complementary codes to both the FIRE and KIRA reduction programs respectively. We used the former one in our reductions with FIRE.

With this implementation, and after obtaining a factorized form for the differential equations, we were able to push the recursive algorithm in eq. (3.8) up to $O(\omega^{1500})$ within no more than a few days for all of the masters. Such a large number of coefficients was necessary to be certain to fully constrain the expression of the coefficient functions in $N$-space from an ansatz in terms of harmonic sums and rational coefficients.

**Four-loop rescaling example.** Here we expand on the method used to bring the differential system into the form with at most simple poles in $\omega$. We will illustrate this method with one of the four-loop topologies we encountered in our computation.
\[ D_1 = k_1^2, \quad D_2 = k_2^2, \quad D_3 = k_3^2, \quad D_4 = k_4^2, \quad D_5 = (k_3 - q)^2, \quad D_6 = (k_1 - k_2)^2, \quad \]
\[ D_7 = (k_2 - k_3)^2, \quad D_8 = (k_3 + p - q)^2, \quad D_9 = (k_2 - k_3 + q)^2, \quad D_{10} = (k_1 - k_3 - p + q)^2, \quad D_{11} = (k_2 - k_3 - p + q)^2, \quad D_{12} = (k_3 + k_4 + p - q)^2, \quad D_{13} = 2k_1 \cdot k_4, \quad D_{14} = 2k_2 \cdot k_4, \quad D_{15} = 2k_1 \cdot p, \quad D_{16} = 2k_4 \cdot p, \quad D_{17} = 2k_1 \cdot q, \quad D_{18} = 2k_4 \cdot q. \]  

The reduction of this topology produced a set of 55 distinct master integrals:

\[
\begin{align*}
\{ & I_{[0,0,1,0,1,1,1,0,0,0,0,0,0,0,0]}, I_{[0,0,1,0,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[0,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[0,0,1,0,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[0,0,1,0,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[0,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[0,1,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0]}, I_{[0,1,0,1,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[0,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[1,0,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0]}, I_{[1,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[1,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0]}, I_{[1,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[1,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[1,0,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0]}, I_{[1,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[1,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0]}, I_{[1,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[1,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} , \\
& I_{[1,0,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0]}, I_{[1,0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]}, I_{[1,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0]} .
\end{align*}
\]

For the purpose of the rescaling, the only information we are concerned about is the leading behaviour in \( \omega \) of each of the coefficients of the matrix \( A \) appearing in the differential equations. We represent the leading behaviour for the topology of eq. (3.9) using the following colour coding

\[
\omega^{-4,-3,-2,-1,0,1,2} = \{\textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast}, \textcolor{red}{\ast} \}.
\]

We also use a circle, instead of a square, to highlight the position of the deepest poles at each step of the transformation. The elimination is performed row-wise, where we rescale each master whose row contains at least one of the deepest poles by a factor of \( \omega \). In the current
example the iterative rescaling procedure leads to the following sequence of transformations:
where the transformation map is defined as

\[ A \xrightarrow{T_a} T_a \cdot \left( A \cdot T_a^{-1} - \frac{\partial T_a^{-1}}{\partial \omega} \right) \]  

(3.10)

with

\[(T_a)_{ij} = \delta_{ij} \times \begin{cases} 
\omega, & a = 1, \ i \in \{39, 40, 41\} \\
\omega, & a = 2, \ i \in \{25, 39, 40, 41\} \\
\omega, & a = 3, \ i \in \{23, 24, 25, 27, 34, .., 41, 47, 52, 53\} \\
\omega, & a = 4, \ i \in \{38\} \\
1. &
\end{cases} \]  

(3.11)

Here we use .. as a short-hand for sequence of consecutive integers. The complete transfor-
mation for the matrix $A$ can then be build by combining the intermediate steps

$$
T = T_4 \cdot T_3 \cdot T_2 \cdot T_1 = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \omega, \omega, \\
\omega^2, 1, \omega, 1, 1, 1, 1, \omega, \omega, \omega, 1, \omega^2, \omega^3, \omega^3, 1, 1, 1, \\
1, 1, \omega, 1, 1, 1, 1, \omega, \omega, 1, 1, 1) .
$$

(3.12)

Alternatively, one could decide remove the deepest pole by rescaling column-wise, where each master, whose column contains one of the deepest poles, is rescaled by a factor of $\omega^{-1}$ or, equivalently, all the others are rescaled by $\omega$. We use the latter because it does not affect the ability to express each master integral as a Taylor expansion.

If we were to start with the same matrix but cancel all the poles column-wise, the resulting transformation matrix would have been

$$(T_a)_{ij} = \delta_{ij} \times \begin{cases} 
\omega, & a = 1, i \notin \{9, 10\} \\
\omega, & a = 2, i \notin \{39, 40, 41\} \\
\omega, & a = 3, i \notin \{1, \ldots, 6, 9, \ldots, 19, 21, 22, 23, 24, 26, \ldots, 30, 37, 42, 44\} \\
1, & \text{otherwise}
\end{cases}
$$

(3.13)

and $T = T_3 \cdot T_2 \cdot T_1$ with

$$
T = \text{diag}(\omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^3, \omega^3, 1, 1, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega, \omega, \omega^2, \\
\omega^3, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega^3, \omega^3, \omega^3, \omega^3, \omega^3, \omega^3, \omega^3, \omega^3) .
$$

(3.14)

In general, one could combine the two approaches by alternating between column- and row-wise rescaling at any step of the procedure. Such a combination will generate different rescaling matrices $T$. We have observed that a column-wise approach is generally preferred for our computation because it requires the rescaling of fewer integrals, resulting in the ability to truncate the expansion for a larger set of master integrals.

4 Results in N-space

We are now ready to present our new analytic even-$N$ expressions for the $n_f^2$ contributions to the fourth-order coefficient functions $c_{2,ns}^{(4)}$ and $c_{L,ns}^{(4)}$ in eq. (2.4). The corresponding $n_f^3$ results were derived long ago [19, 20]. We confirm those results and include them below for completeness; for the case of $C_2$ in a more transparent form than given in ref. [20].

The quantities under consideration can be expressed in terms of harmonic sums. Following the notation of ref. [21], these sums are recursively defined by

$$
S_{\pm m}(M) = \sum_{i=1}^{M} \frac{(\pm 1)^i}{i^m}
$$

(4.1)
and
\[
S_{\pm m_1, m_2, \ldots, m_k}(M) = \sum_{i=1}^{M} \frac{(\pm 1)^i}{i_{m_1}} S_{m_2, \ldots, m_k}(i) .
\] (4.2)

The sum of the absolute values of the indices \(m_k\) defines the weight \(w\) of the harmonic sum. In the \(n\)-loop coefficient functions one encounters sums up to weights \(2n\) for \(C_2\) and \(2n - 1\) for \(C_L\). The present non-singlet \(n_f^2\) contributions only include sums up to \(w = 6\) for \(C_2\) and \(w = 5\) for \(C_L\), for the \(n_f^2\) terms the corresponding maximal weights are lower by 1. Below all harmonic sums have the argument \(N\), which is omitted in the formulae for brevity, and we use the short-hand
\[
D_a = (N + a)^{-1} .
\] (4.3)

We first present the result for the coefficient function for \(F_L\) which we write as
\[
c_{L,ns}^{(4)}(N) = n_f^0 + n_f^1 \text{ contributions}
\]
\[
+ C_F C_A n_f^2 \frac{16}{9} c_{L,ns}^{(4)L}(N) + C_F (C_F - \frac{1}{2} C_A) n_f^2 \frac{16}{9} c_{L,ns}^{(4)N}(N)
\]
\[
+ C_F n_f^3 \frac{16}{27} c_{L,ns}^{(4)F}(N)
\] (4.4)

where \(C_A = n_c\) and \(C_F = (n_c - n_c^{-1})/2\) in SU(\(n_c\)), with \(n_c = 3\) colours in QCD. For compactness, we have decomposed the \(n_f^2\) part into leading (L) and non-leading (N) contributions in the large-\(n_c\) limit. The factors \(16/9\) and \(16/27\) have been put in order to shorten, on average, the lengths of the fractions in the expressions below. The \(n_f^2\) contributions are given by
\[
c_{L,ns}^{(4)L}(N) =
\]
\[
+ S_{1,4} (-80 D_{-2} + 40 D_{-1} - 40 D_1 + 80 D_2 - 120 D_3) + S_{2,3} (-32 D_{-2} + 16 D_{-1}
\]
\[
- 16 D_1 + 32 D_2 - 48 D_3)
\]
\[
+ S_{4,1} (80 D_{-2} - 40 D_{-1} + 40 D_1 - 80 D_2 + 120 D_3) + S_{1,1,3} (40 D_{-2} - 20 D_{-1}
\]
\[
+ 20 D_1 - 40 D_2 + 60 D_3) + S_{1,2,2} (8 D_{-2} - 4 D_{-1} + 4 D_1 - 8 D_2 + 12 D_3)
\]
\[
+ S_{1,3,1} (-8 D_{-2} + 4 D_{-1} - 4 D_1 + 8 D_2 - 12 D_3) + S_{2,1,2} (8 D_{-2} - 4 D_{-1}
\]
\[
+ 4 D_1 - 8 D_2 + 12 D_3) + S_{2,2,1} (-8 D_{-2} + 4 D_{-1} - 4 D_1 + 8 D_2 - 12 D_3)
\]
\[
+ S_{3,1,1} (-40 D_{-2} + 20 D_{-1} - 20 D_1 + 40 D_2 - 60 D_3) + S_{1,1,2,1} (-18 D_1 + 72 D_2
\]
\[
- 60 D_3) + S_{1,2,1,1} (18 D_1 - 72 D_2 + 60 D_3) + S_4 (120 D_{-2} - 40 D_{-1} - 55/3 D_1
\]
\[
- 80 D_2 + 180 D_3) + S_{1,3} (-40 D_{-2} + 32 D_{-1} - 16 D_0 + 65 D_1 - 24 D_2 + 12 D_3
\]
\[
- 48 D_{-2}^2 + 24 D_{-1}^2) + S_{2,2} (-12 D_{-2} + 4 D_{-1} + 38 D_1 + 8 D_2 - 18 D_3)
\]
\[
+ S_{3,1} (-8 D_{-2} - 16 D_{-1} + 16 D_0 + 9 D_1 + 56 D_2 - 84 D_3 + 48 D_{-2}^2 - 24 D_{-1}^2)
\]
\[
+ S_{1,1,2} (-34 D_1 + 16 D_2 - 8 D_3) + S_{1,2,1} (-38 D_1 + 72 D_2 + 90 D_3)
\]
\[
+ S_{2,1,1} (-46 D_1 + 56 D_2 - 82 D_3) + 36 D_1 S_{1,1,1,1} + S_3 (-30 D_{-2} - 4 D_{-1}
\]
\[
- 26 D_0 + 1097/9 D_1 + 56 D_2 - 108 D_3 + 72 D_{-2}^2 - 24 D_{-1}^2 - 18 D_1^2)
\]
\[ + S_{1,2} (-10 D_{-2} + 4 D_{-1} + 65/3 D_0 - 895/6 D_1 + 56 D_2 - 116/3 D_3 + 29 D_1^2) \\
+ S_{2,1} (10 D_{-2} - 4 D_{-1} + 91/3 D_0 - 913/6 D_1 - 163/3 D_3 + 25 D_1^2) \\
+ S_{1,1,1} (-25 D_0 + 293/2 D_1 - 25 D_1^2) + S_3 (-30 D_{-2} + 16 D_{-1} + 410/3 D_0 \\
- 1493/4 D_1 - 16 D_2 - 11/2 D_3 - 35 D_0^2 + 263/3 D_1^2 - 27 D_1^3) + S_{1,1} (45 D_{-2} \\
- 20 D_{-1} - 140 D_0 + 14351/36 D_1 + 16 D_2 - 5 D_3 + 39 D_0^2 - 99 D_1^2 + 25 D_1^3) \\
+ S_1 (135/2 D_{-2} - 20 D_{-1} - 4045/12 D_0 + 1030465/1296 D_1 - 152 D_2 \\
+ 2431/12 D_3 - 54 D_2^2 + 24 D_1^2 + 911/6 D_0^2 - 75/2 D_0^3 - 10565/36 D_1^2 \\
+ 245/3 D_1^3 - 37/2 D_1^4) - 75/2 D_{-2} - 714425/1296 D_0 + 3332269/2592 D_1 \\
- 935/12 D_3 + 18 D_2^2 + 3355/12 D_0^2 - 3661/36 D_0^3 + 115/6 D_0^4 \\
- 785749/1296 D_1^2 + 3491/18 D_1^3 - 125/36 D_1^4 - 115/6 D_1^5 \\
+ \zeta_3 \left[ S_2 (112 D_{-2} - 56 D_{-1} + 56 D_0 - 112 D_2 + 168 D_3) + S_{1,1} (-80 D_{-2} + 40 D_{-1} \\
+ 32 D_1 - 208 D_2 + 120 D_3) + S_1 (80 D_{-2} - 64 D_{-1} + 32 D_0 - 254/3 D_1 \\
+ 368 D_2 - 400 D_3 + 96 D_2^2 - 48 D_1^2) + 32 D_{-1} - 14/3 D_0 - 253/3 D_1 \\
+ 356 D_3 - 144 D_2^2 + 48 D_1^2 + 50/3 D_1^3 \right] \\
+ \zeta_5 \left[ -240 D_{-2} + 120 D_{-1} + 240 D_1 - 1200 D_2 + 840 D_3 \right] \quad (4.5) \]

and

\[ e^{(4)N}_{L_m} (N) = \\
+ 288 D_1 S_{-5} - 288 D_1 S_5 - 256 D_1 S_{-3,-2} - 288 D_1 S_{-2,-3} - 576 D_1 S_{1,-4} \\
- 560 D_1 S_{2,-3} - 72 D_1 S_{2,3} - 368 D_1 S_{3,-2} + 112 D_1 S_{3,2} + 344 D_1 S_{4,1} \\
+ 192 D_1 S_{-2,-2,1} + 128 D_1 S_{1,-2,-2} + 254 D_1 S_{1,1,-3} + 88 D_1 S_{1,1,3} \\
+ 352 D_1 S_{1,2,-2} + 64 D_1 S_{1,2,2} - 184 D_1 S_{1,3,1} + 352 D_1 S_{2,1,-2} + 56 D_1 S_{2,1,2} \\
- 16 D_1 S_{2,2,1} - 168 D_1 S_{3,1,1} - 320 D_1 S_{1,1,1,-2} - 72 D_1 S_{1,1,1,2} \\
+ 32 D_1 S_{1,1,2,1} + 40 D_1 S_{1,2,1,1} + S_{-4} (576/5 D_{-2} + 576 D_0 - 664 D_1 \\
+ 864/5 D_3 - 432 D_1^2) + S_4 (1138/3 D_1 - 144 D_1^2) + S_{-3,1} (-224/5 D_{-2} \\
+ 112 D_{-1} - 224 D_0 + 224 D_2 - 336/5 D_3 + 224 D_1^2) + S_{-2,-2} (-64/5 D_{-2} \\
- 32 D_{-1} - 64 D_0 + 336 D_1 - 64 D_2 - 96/5 D_3 - 32 D_1^2) + S_{-2,2} (-144/5 D_{-2} \\
+ 72 D_{-1} - 144 D_0 + 144 D_2 - 216/5 D_3 + 144 D_1^2) + S_{1,-3} (-272/5 D_{-2} \\
- 136 D_{-1} - 272 D_0 + 1040 D_1 - 272 D_2 - 408/5 D_3 - 16 D_1^2) + S_{1,3} (-16/5 D_{-2} \\
- 8 D_{-1} - 16 D_0 + 126 D_1 - 72 D_2 - 724/5 D_3 + 72 D_1^2) + S_{2,-2} (-176/5 D_{-2} \\
- 88 D_{-1} - 176 D_0 + 688 D_1 - 176 D_2 - 264/5 D_3 - 16 D_1^2) + S_{2,2} (-32/5 D_{-2} \\
- 16 D_{-1} - 32 D_0 + 176 D_1 - 32 D_2 - 48/5 D_3 + 8 D_1^2) + S_{3,1} (64/5 D_{-2} \\
+ 32 D_{-1} + 64 D_0 - 370 D_1 + 120 D_2 + 796/5 D_3) + S_{-2,1,1} (128/5 D_{-2} - 64 D_{-1}) \]
\[ + 128 D_0 - 128 D_2 + 192/5 D_3 - 128 D_4^2 + S_{1,1,-2} (32 D_{-2} + 80 D_{-1} + 160 D_0 - 672 D_1 + 160 D_2 + 48 D_3 + 32 D_4^2) + S_{1,1,2} (16/5 D_{-2} + 8 D_{-1} + 16 D_0 - 188 D_1 + 48 D_2 + 544/5 D_3 - 32 D_4^2) + S_{1,2,1} (-16/5 D_{-2} - 8 D_{-1} - 16 D_0 + 8 D_{-1} + 16 D_2 - 24/5 D_3 - 8 D_4^2) + S_{2,1,1} (-56 D_1 - 32 D_2 - 104 D_3 + 40 D_4^2) + 72 D_1 S_{1,1,1} + S_{-3} (-1576/25 D_{-2} + 136 D_{-1} - 1384 D_0 + 2492/3 D_1 + 272 D_2 - 2724/5 D_3 + 576/5 D_2^2 + 576 D_{0}^2 + 824 D_{1}^2 - 240 D_{2}^3) + S_3 (24/5 D_{-2} + 8 D_{-1} - 48 D_0 - 2672/9 D_1 + 72 D_2 + 1086/5 D_3 + 160 D_4^2 - 48 D_{4}^3) + S_{-2,1} (1232/25 D_{-2} - 160 D_{-1} + 448 D_0 - 48 D_{1} - 368 D_2 + 1968/25 D_3 - 192/5 D_{2}^2 + 96 D_{2}^2 - 192 D_{0}^2 - 352 D_{1}^2 + 96 D_{4}^3) + S_{1,-2} (32/25 D_{-2} + 80 D_{-1} + 432 D_0 - 2200/3 D_1 + 208 D_2 + 168/25 D_3 - 192/5 D_{2}^2 - 96 D_{2}^2 - 192 D_{0}^2 + 16 D_{4}^2) + S_{1,2} (-24/5 D_{-2} - 8 D_{-1} + 178/3 D_0 - 319/3 D_1 + 100 D_2 - 8/15 D_3 - 34 D_{1}^2 + 24 D_{4}^3) + S_{2,1} (24/5 D_{-2} + 8 D_{-1} + 134/3 D_0 + 5/3 D_1 - 132 D_2 - 2332/15 D_3 + 106 D_{1}^2 - 24 D_{4}^3) + S_{1,1,1} (-50 D_0 + 73 D_1 - 50 D_{4}^2) + S_{-2} (5432/375 D_{-2} - 136 D_{-1} + 3856/3 D_0 - 2194/3 D_1 - 320 D_2 + 3606/125 D_3 - 944/25 D_{2}^2 + 384/5 D_{3}^2 + 96 D_{2}^2 - 928 D_{0}^2 + 384 D_{0}^2 - 1828/3 D_{1}^2 + 344 D_{4}^3 - 96 D_{4}^4) + S_{2} (-12 D_{-2} - 56 D_{-1} + 244/3 D_0 + 801/2 D_1 - 276 D_2 - 98 D_3 - 30 D_{4}^2 - 320/3 D_{1}^2 + 6 D_{4}^3) + S_{1,1} (96/6 D_{-2} + 64 D_{-1} - 82 D_0 - 3721/18 D_1 + 96 D_2 - 506/5 D_3 + 34 D_{0}^2 - 34 D_{1}^2 + 42 D_{4}^3) + S_{1} (924/25 D_{-2} + 208 D_{-1} - 499/6 D_0 - 418193/648 D_1 + 244 D_2 - 21269/150 D_3 - 144/5 D_{2}^2 - 96 D_{4}^2 + 65/3 D_{0}^2 - 15 D_{3}^2 + 4699/18 D_{1}^2 + 46/3 D_{3}^2 + 3 D_{4}^3 + 10663/125 D_{-2} - 1031/648 D_0 - 421403/1296 D_1 + 153647/750 D_3 - 2388/25 D_{2}^2 + 288/5 D_{3}^2 + 307/6 D_{0}^2 - 1285/18 D_{0}^3 + 115/3 D_{4}^4 + 256829/648 D_{1}^2 - 2269/9 D_{1}^3 + 225/18 D_{1}^4 - 115/3 D_{1}^5 + \zeta_3 \left[ - 192 D_1 S_{-2} - 432 D_1 S_2 + 288 D_1 S_{1,1} + S_1 (-48 D_{-2} - 120 D_{-1} - 240 D_0 + 1964/3 D_1 - 64 D_2 + 416 D_3 - 192 D_{4}^2) - 1776/25 D_{-2} - 40 D_{-1} - 3916/3 D_0 + 3196/3 D_1 + 584 D_2 + 3076/25 D_3 + 576/5 D_{2}^2 + 96 D_{2}^2 + 576 D_{0}^2 + 172/3 D_{4}^2 \right] + \zeta_5 \left[ 180 D_1 \right]. \] (4.6)

The $C_F^2$ contribution \((4.6)\) includes, as the corresponding lower-order quantities, only the denominator $D_1 = 1/(N+1)$ at the maximal overall weight $w = 5$ of the sums and Riemann $\zeta$-values. The $C_F C_A$ part, and hence the large-$n_c$ coefficient \((4.5)\), does not have this expected property, but instead involves the linear combinations

\[ 2D_{-2} - D_{-1} + D_1 - 2D_2 + 3D_3 \quad \text{and} \quad 3D_1 - 12D_2 + 10D_3. \]
The presence of terms with $D_{-2} = 1/(N-2)$ and $D_{-1} = 1/(N-1)$ does not imply poles at $N = 2$ or $N = 1$, as the corresponding numerators also vanish at this point. This feature already occurred in the second-order coefficient functions of refs. [9–12]. At $N = 2$ these functions are given by

\begin{equation}
    c_{L,ns}^{(4)I}(N = 2) = \frac{1058755}{2916} - \frac{6713}{135} \zeta_3 - 32 \zeta_5 + 24 \zeta_3^2,
\end{equation}

\begin{equation}
    c_{L,ns}^{(4)N}(N = 2) = \frac{1720501}{29160} + \frac{247}{270} \zeta_3 + 30 \zeta_5.
\end{equation}

Note the $\zeta_3^2$ term which does not occur at higher $N$. The $n_f^2$ contribution to eq. (4.4) reads

\begin{equation}
    c_{L,ns}^{(4)F}(N) =
    - 12 D_1 \mathbf{S}_3 + 12 D_1 \mathbf{S}_{1,2} + 12 D_1 \mathbf{S}_{2,1} - 12 D_1 \mathbf{S}_{1,1,1} + \mathbf{S}_2 (-12 D_0 + 50 D_1
    - 12 D_2^2) + \mathbf{S}_{1,1} (12 D_0 - 50 D_1 + 12 D_2^2) + \mathbf{S}_1 (38 D_0 - 317/3 D_1 - 12 D_0^2
    + 50 D_1^2 - 12 D_2^2) + 203/3 D_0 - 8609/54 D_1 - 38 D_0^2 + 12 D_0^3 + 317/3 D_1^2
    - 50 D_1^3 + 12 D_1^2).
\end{equation}

The corresponding result for $C_2$ can be decomposed as

\begin{equation}
    c_{2,ns}^{(4)}(N) = n_f^0 \text{ and } n_f^1 \text{ contributions}
    \begin{array}{l}
        + C_F C_A n_f^2 \frac{16}{9} c_{2,ns}^{(4)I}(N) + C_F (C_F - \frac{1}{2} C_A) n_f^2 \frac{16}{9} c_{2,ns}^{(4)N}(N) \\
        + C_F (C_F - C_A) n_f^2 \frac{1}{3} \zeta_4 c_{2,ns}^{(4)Z}(N) + C_F n_f^2 \frac{16}{27} c_{2,ns}^{(4)F}(N).
    \end{array}
\end{equation}

In addition to the structures present in eq. (4.4), the $n_f^2$ coefficient function for $F_2$ includes a $\zeta_4$ contribution, which is proportional to $(C_F - C_A)$ and hence vanishes vanish for $C_A = C_F$ which is part of the choice of the colour factors that leads to a $\mathcal{N} = 1$ supersymmetric theory. The three $n_f^2$ coefficients in eq. (4.10) read

\begin{equation}
    c_{2,ns}^{(4)I}(N) =
    - 1951/12 \mathbf{S}_6 + 671/6 \mathbf{S}_{1,5} + 352/3 \mathbf{S}_{2,4} + 335/2 \mathbf{S}_{3,3} + 643/3 \mathbf{S}_{4,2} + 1445/6 \mathbf{S}_{5,1}
    - 265/3 \mathbf{S}_{1,1,4} - 89 \mathbf{S}_{1,2,3} - 117 \mathbf{S}_{1,3,2} - 412/3 \mathbf{S}_{1,4,1} - 119 \mathbf{S}_{2,1,3} - 125 \mathbf{S}_{2,2,2}
    - 137 \mathbf{S}_{2,3,1} - 169 \mathbf{S}_{3,1,2} - 188 \mathbf{S}_{3,2,1} - 688/3 \mathbf{S}_{4,1,1} + 71 \mathbf{S}_{1,1,1,3} + 81 \mathbf{S}_{1,1,2,2}
    + 98 \mathbf{S}_{1,1,3,1} + 103 \mathbf{S}_{1,2,1,2} + 108 \mathbf{S}_{1,2,2,1} + 135 \mathbf{S}_{1,3,1,1} + 109 \mathbf{S}_{2,1,1,2} + 122 \mathbf{S}_{2,1,2,1}
    + 139 \mathbf{S}_{2,2,1,1} + 176 \mathbf{S}_{3,1,1,1} - 70 \mathbf{S}_{1,1,1,1,2} - 82 \mathbf{S}_{1,1,1,2,1} - 94 \mathbf{S}_{1,1,2,1,1} - 114 \mathbf{S}_{1,2,1,1,1}
    - 115 \mathbf{S}_{1,1,1,1,1} + 80 \mathbf{S}_{1,1,1,1,1,1} + 85 (20567/36 - 671/12 D_0 + 671/12 D_1)
    + \mathbf{S}_{1,4} (-9995/36 - 20 D_{-2} + 265/6 D_0 - 505/6 D_1 + 120 D_2 - 180 D_3)
    + \mathbf{S}_{2,3} (-13373/36 - 8 D_{-2} + 89/2 D_0 - 121/2 D_1 + 48 D_2 - 72 D_3)
    + \mathbf{S}_{4,2} (-9817/18 + 8 D_{-2} + 117/2 D_0 - 85/2 D_1 - 48 D_2 + 72 D_3)
    + \mathbf{S}_{4,1} (-12343/18 + 20 D_{-2} + 206/3 D_0 - 86/3 D_1 - 120 D_2 + 180 D_3)
\end{equation}
\[ + S_{1,1,3} \left( \frac{8495}{36} + 10 \, D_{-2} - 71/2 \, D_0 + 111/2 \, D_1 - 60 \, D_2 + 90 \, D_3 \right) + S_{1,2,2} \left( \frac{803}{3} + 2 \, D_{-2} - 81/2 \, D_0 + 89/2 \, D_1 - 12 \, D_2 + 18 \, D_3 \right) + S_{1,3,1} \left( \frac{12263}{36} - 2 \, D_{-2} - 49 \, D_0 + 45 \, D_1 + 12 \, D_2 - 18 \, D_3 \right) + S_{2,1,2} \left( \frac{4283}{12} + 2 \, D_{-2} - 103/2 \, D_0 + 111/2 \, D_1 \right) - 12 \, D_2 + 18 \, D_3 \right) + S_{2,1,1} \left( \frac{4835}{12} - 2 \, D_{-2} - 54 \, D_0 + 50 \, D_1 + 12 \, D_2 - 18 \, D_3 \right) + S_{3,1,1} \left( \frac{19967}{36} - 10 \, D_{-2} - 135/2 \, D_0 + 95/2 \, D_1 + 60 \, D_2 - 90 \, D_3 \right) + S_{1,1,1,2} \left( \frac{-1279}{6} + 35 \, D_0 - 35 \, D_1 \right) + S_{1,1,1,1} \left( \frac{-1511}{6} + 41 \, D_0 - 71 \, D_1 + 108 \, D_2 - 90 \, D_3 \right) + S_{2,1,1,1} \left( \frac{-605}{2} + 47 \, D_0 - 17 \, D_1 - 108 \, D_2 + 90 \, D_3 \right) + S_{2,1,1,1} \left( \frac{-2135}{6} + 57 \, D_0 - 57 \, D_1 \right) + S_{1,1,1,1,1} \left( \frac{226}{-60} - 40 \, D_0 + 40 \, D_1 \right) + S_3 \left( \frac{-152383}{144} + 30 \, D_{-2} + 5929/36 \, D_0 - 1906/9 \, D_1 - 120 \, D_2 + 270 \, D_3 - 319/6 \, D_0^2 + 20 \, D_1^2 \right) + S_{1,3} \left( \frac{3800}{9} - 13 \, D_{-2} + 6 \, D_{-1} - 4369/36 \, D_0 + 7735/36 \, D_1 - 36 \, D_2 + 18 \, D_3 \right) - 12 \, D_{-2} - 71/2 \, D_0^2 - 17/4 \, D_1^2 \right) + S_{2,2,2} \left( \frac{24305}{36} - 3 \, D_{-2} - 827/6 \, D_0 + 1235/6 \, D_1 + 12 \, D_2 - 27 \, D_3 + 57 \, D_0^2 - 55/2 \, D_1^2 \right) + S_{3,1} \left( \frac{17689/18}{D_{-2}} - 6 \, D_{-1} - 6559/36 \, D_0 + 8071/36 \, D_1 + 84 \, D_2 - 126 \, D_3 + 12 \, D_0^2 + 63 \, D_1^2 \right) - 19 \, D_0^2 + 59/2 \, D_1^2 \right) + S_{1,2,2} \left( \frac{-10459}{24} + 805/6 \, D_0 - 593/3 \, D_1 - 108 \, D_2 \right) + 135 \, D_3 - 111/2 \, D_0^2 + 51/2 \, D_1^2 \right) + S_{2,1,1} \left( \frac{-23191}{36} + 613/4 \, D_0 - 1837/8 \, D_1 \right) + 84 \, D_2 - 123 \, D_3 - 64 \, D_0^2 + 47/2 \, D_1^2 \right) + S_{1,1,1,1,1} \left( \frac{25979/72}{231/2} \, D_0 - 357/2 \, D_1 + 52 \, D_0^2 - 23 \, D_1^2 \right) + S_3 \left( \frac{842039/648}{D_{-2}} - 3 \, D_{-1} - 6 \, D_{-2} - 18863/72 \, D_0 \right) + 1123/24 \, D_1 + 84 \, D_2 - 162 \, D_3 + 18 \, D_0^2 + 3107/18 \, D_1^2 - 283/4 \, D_1^3 \right) - 421/4 \, D_1^2 + 91/4 \, D_1^3 \right) + S_{1,2} \left( \frac{-147071/324}{237/3} \, D_{-2} + 2005/9 \, D_0 \right) - 34157/72 \, D_1 + 84 \, D_2 - 58 \, D_3 - 1969/12 \, D_0^2 + 143/2 \, D_1^3 + 134 \, D_1^2 - 24 \, D_1^3 \right) + S_{2,1} \left( \frac{-1178369/1296}{5/2} \, D_{-2} + 557/2 \, D_0 - 11939/24 \, D_1 - 163/2 \, D_3 \right) - 1127/6 \, D_0^2 + 81 \, D_0^3 + 323/3 \, D_1^2 - 49/2 \, D_1^3 \right) + S_{1,1,1} \left( \frac{46483/108}{D_{-2}} - 8321/36 \, D_0 + 32905/72 \, D_1 + 2009/12 \, D_0^2 - 75 \, D_0^3 - 1331/12 \, D_1^2 + 21 \, D_1^3 \right) + S_2 \left( \frac{-5764837/5184}{15/2} \, D_{-2} + 1006669/2592 \, D_0 - 222569/2592 \, D_1 - 24 \, D_2 \right) - 33/4 \, D_3 - 12665/36 \, D_0^3 + 16325/72 \, D_0^4 - 1025/12 \, D_1^4 + 875/3 \, D_1^2 \right) - 7073/72 \, D_1^3 + 263/12 \, D_1^4 \right) + S_{1,1} \left( \frac{2134163/5184}{45/4} \, D_{-2} - 356983/864 \, D_0 \right) + 780301/864 \, D_1 + 24 \, D_2 - 15/2 \, D_3 + 1090/3 \, D_0^2 - 17063/72 \, D_0^3 + 1115/12 \, D_1^4 \right) - 7363/24 \, D_1^2 + 6887/72 \, D_1^3 - 239/12 \, D_1^4 \right) + S_1 \left( \frac{33182/81}{27} \, D_{-2} + 27 \, D_{-2} + 6 \, D_{-1} \right) - 1115063/1728 \, D_0 + 243035/162 \, D_1 - 228 \, D_2 + 2431/8 \, D_3 - 27/2 \, D_1^2 \right) + 275219/432 \, D_1^2 - 22763/48 \, D_0^3 + 19123/72 \, D_0^4 - 1069/12 \, D_0^5 \right) - 1669825/2592 \, D_1^4 + 34919/144 \, D_1^3 - 604/9 \, D_1^4 + 32/3 \, D_1^5 \right) - 18199451/27648 - 33/4 \, D_{-2} - 2362801/2304 \, D_0 + 5233867/2304 \, D_1 - 935/8 \, D_3 + 9/2 \, D_1^2 \right) \]
\[ + 9889087/10368 \, D_0^2 - 1874987/2592 \, D_0^3 + 63839/144 \, D_0^4 - 14161/72 \, D_0^5 \\
+ 1951/48 \, D_0^6 - 3790549/3456 \, D_1^2 + 367649/864 \, D_1^3 - 12191/144 \, D_1^4 \\
- 41/3 \, D_1^5 + 149/16 \, D_1^6 \\
+ 3 \zeta_3 \left[ - 331/6 \, S_3 + 121/3 \, S_{1,2} + 166/3 \, S_{2,1} - 14 \, S_{1,1,1} + S_2 (215/2 + 28 \, D_{-2}) \\
- 121/6 \, D_0 + 457/6 \, D_1 - 168 \, D_2 + 252 \, D_3 + S_{1,1} (-111/2 - 20 \, D_{-2} + 7 \, D_0) \\
- 312 \, D_2 + 180 \, D_3) + S_1 (-274/3 + 26 \, D_{-2} - 12 \, D_{-1} - 29/3 \, D_0 - 497/4 \, D_1 \\
+ 73 \, D_1 + 552 \, D_2 - 600 \, D_3 + 24 \, D_{-2}^2 - 25 \, D_0^2 + 133/3 \, D_1^2) + 2083/32 - 9 \, D_{-2} \\
+ 12 \, D_{-1} - 101/24 \, D_0 - 6115/24 \, D_1 + 534 \, D_3 - 36 \, D_{-2}^2 - 122/3 \, D_0^2 \\
+ 257/12 \, D_0^3 + 401/3 \, D_1^2 - 33/4 \, D_1^3 \right] (4.11) \\
+ 3 \zeta_5 \left[ - 191/2 \, S_1 + 693/8 - 60 \, D_{-2} + 191/4 \, D_0 + 1729/4 \, D_1 - 1800 \, D_2 + 1260 \, D_3 \right] , \\
\] 
\[ c^{(4)N}_{3,ns} (N) = \\
- 150 \, S_{-6} - 1051/6 \, S_6 + 6 \, S_{-5,1} + 408 \, S_{-4,-2} + 510 \, S_{-3,-3} + 352 \, S_{-2,-4} \\
+ 450 \, S_{1,-5} + 446/3 \, S_{1,5} + 538 \, S_{2,-4} + 548/3 \, S_{2,4} + 586 \, S_{3,-3} + 273 \, S_{3,3} \\
+ 434 \, S_{4,-2} + 968/3 \, S_{4,2} + 932/3 \, S_{5,1} + 8 \, S_{-4,1,1} - 264 \, S_{3,-3,1,2} + 4 \, S_{-3,1,1,2} \\
- 216 \, S_{-2,3,1} - 136 \, S_{-2,-2,2} - 4 \, S_{1,-4,1} - 220 \, S_{1,-3,-2} + 252 \, S_{1,-2,2} - 520 \, S_{1,1,-4} \\
- 770/3 \, S_{1,1,4} - 496 \, S_{1,1,2,3} - 336 \, S_{1,3,-2} - 160 \, S_{1,3,2} + 4/3 \, S_{1,4,1} \\
- 4 \, S_{2,3,1} - 140 \, S_{2,-2,2} - 496 \, S_{2,1,-3} - 326 \, S_{2,1,3} - 320 \, S_{2,2,-2} - 314 \, S_{2,2,2} \\
- 64 \, S_{2,3,-1} - 4 \, S_{3,-1,2} - 344 \, S_{3,1,-2} - 344 \, S_{3,1,2} - 276 \, S_{3,2,1} - 1016/3 \, S_{4,1,1} \\
+ 8 \, S_{-3,1,1,1} + 112 \, S_{-2,-2,1,1} - 8 \, S_{-3,1,1,1} + 144 \, S_{1,-2,2,1} - 8 \, S_{1,-2,1,2} + 112 \, S_{1,1,-2,2} \\
+ 448 \, S_{1,1,1,3} + 274 \, S_{1,1,2,-2} + 288 \, S_{1,1,2,2} + 52 \, S_{1,1,3,1} + 288 \, S_{1,2,1,-2} \\
+ 254 \, S_{1,2,1,2} + 188 \, S_{1,2,2,1} + 88 \, S_{1,3,1,1} - 8 \, S_{2,-2,1,1} + 288 \, S_{2,1,1,-2} + 314 \, S_{2,1,1,2} \\
+ 210 \, S_{1,2,1,2} + 4168 \, S_{2,2,1,1} + 302 \, S_{3,1,1,1} - 16 \, S_{1,2,1,1,1} - 256 \, S_{1,1,1,1,2} + 274 \, S_{1,1,1,1,2} \\
+ 148 \, S_{1,1,1,2,1} - 144 \, S_{1,2,1,1,1} - 184 \, S_{2,1,1,1,1} - 230 \, S_{2,1,1,1,1} + 160 \, S_{1,1,1,1,1,1} \\
+ S_{-5} (310/3 - 297 \, D_0 + 585 \, D_1) + S_5 (3463/9 - 7/3 \, D_0 - 857/3 \, D_1) \\
+ S_{-4,1} (-40/3 + 2 \, D_0 - 2 \, D_1) + S_{-3,-2} (-1192/3 + 174 \, D_0 - 430 \, D_1) \\
+ S_{-2,-3} (-392 + 198 \, D_0 - 486 \, D_1) + S_{1,-4} (-448 + 404 \, D_0 - 980 \, D_1) \\
+ S_{1,4} (-6395/18 + 385/3 \, D_0 - 385/3 \, D_1) + S_{2,-3} (-460 + 388 \, D_0 - 948 \, D_1) \\
+ S_{2,3} (-6125/18 + 169 \, D_0 - 241 \, D_1) + S_{3,-2} (-376 + 260 \, D_0 - 628 \, D_1) \\
+ S_{3,2} (-5683/9 + 52 \, D_0 + 60 \, D_1) + S_{4,1} (-7129/9 - 260/3 \, D_0 + 1292/3 \, D_1) \\
+ S_{-3,1,1} (-40/3 + 4 \, D_0 - 4 \, D_1) + S_{-2,-2,1} (160 - 120 \, D_0 + 312 \, D_1) \\
+ S_{-2,1,2} (4 \, D_0 - 4 \, D_1) + 40/3 \, S_{1,-3,1} + S_{1,-2,-2} (512/3 - 88 \, D_0 + 216 \, D_1) \\
+ S_{1,1,-3} (1360/3 - 360 \, D_0 + 904 \, D_1) + S_{1,1,3} (7679/18 - 159 \, D_0 + 247 \, D_1) \\
}
\[ + S_{1,2,-2} (880/3 - 232 D_0 + 584 D_1) + S_{1,2,2} (1417/3 - 142 D_0 + 206 D_1) \\
+ S_{1,3,1} (5885/18 + 17 D_0 - 201 D_1) + 40/3 S_{2,-2,1} + S_{2,1,-2} (880/3 - 232 D_0 \\
+ 584 D_1) + S_{2,1,2} (1027/2 - 141 D_0 + 197 D_1) + S_{2,2,1} (1250/3 - 90 D_0 + 74 D_1) \\
+ S_{3,1,1} (6313/9 - 2 D_0 - 166 D_1) + S_{-2,1,1,1} (8 D_0 - 8 D_1) + 80/3 S_{1,-2,1,1} \\
+ S_{1,1,1,-2} (-800/3 + 208 D_0 - 528 D_1) + S_{1,1,1,2} (-1163/3 + 140 D_0 - 212 D_1) \\
+ S_{2,1,2,1} (-1015/3 + 66 D_0 - 34 D_1) + S_{1,2,1,1} (-413 + 62 D_0 - 22 D_1) \\
+ S_{2,1,1,1} (-1343/3 + 92 D_0 - 92 D_1) + S_{1,1,1,1,1} (320 - 80 D_0 + 80 D_1) \\
+ S_{-4} (-1270/9 + 144/5 D_{-2} + 1094 D_0 - 1182 D_1 + 1296/5 D_3 - 625 D_0^2 \\
- 707 D_1^2) + S_{4} (-18371/72 + 1105/18 D_0 + 2344/9 D_1 - 133/3 D_3^2 - 206 D_1^2) \\
+ S_{-3,1} (38/9 - 56/5 D_{-2} - 692/3 D_0 + 20/3 D_1 + 336 D_2 - 504/5 D_3 + 166 D_0^2 \\
+ 386 D_1^2) + S_{-2,-2} (200 - 16/5 D_{-2} - 796/3 D_0 + 1612/3 D_1 - 96 D_2 - 144/5 D_3 \\
+ 110 D_0^2 - 54 D_1^2) + S_{-2,2} (-36/5 D_{-2} - 140 D_0 - 4 D_1 + 216 D_2 - 324/5 D_3 \\
+ 104 D_0^2 + 248 D_1^2) + S_{1,-3} (2338/9 - 68/5 D_{-2} - 2624/3 D_0 + 4928/3 D_1 \\
- 408 D_2 - 612/5 D_3 + 388 D_0^2 - 44 D_1^2) + S_{1,3} (4369/9 - 4/5 D_{-2} - 5005/18 D_0 \\
+ 3524/9 D_1 - 108 D_2 - 1086/5 D_3 + 156 D_0^2 + 53 D_1^2) + S_{2,-2} (1522/9 \\
- 44/5 D_{-2} - 1712/3 D_0 + 3248/3 D_1 - 264 D_2 - 396/5 D_3 + 252 D_0^2 - 36 D_1^2) \\
+ S_{2,2} (12331/36 - 8/5 D_{-2} - 920/3 D_0 + 1541/3 D_1 - 48 D_2 - 72/5 D_3 + 160 D_0^2 \\
- 32 D_1^2) + S_{3,1} (23429/36 + 16/5 D_{-2} + 1067/18 D_0 - 2239/9 D_1 + 180 D_2 \\
+ 1194/5 D_3 - 25 D_0^2 - 14 D_1^2) + S_{2,1,1} (32/5 D_{-2} - 296/3 D_0 + 88/3 D_1 \\
- 192 D_2 + 288/5 D_3 - 84 D_0^2 - 220 D_1^2) - 76/9 S_{1,-2,1} + S_{1,1,-2} (-496/3 + 8 D_{-2} \\
+ 1600/3 D_0 - 3136/3 D_1 + 240 D_2 + 72 D_3 - 232 D_0^2 + 56 D_1^2) \\
+ S_{1,1,2} (-15511/36 + 4/5 D_{-2} + 799/3 D_0 - 1372/3 D_1 + 72 D_2 + 816/5 D_3 \\
- 156 D_0^2 + 6 D_1^2) + S_{1,2,1} (-1839/4 - 4/5 D_{-2} + 398/3 D_0 - 593/3 D_1 - 24 D_2 \\
- 36/5 D_3 - 84 D_0^2 + 32 D_1^2) + S_{2,1,1} (-3491/9 + 184 D_0 - 341 D_1 - 48 D_2 \\
- 156 D_3 - 97 D_0^2 + 97 D_1^2) + S_{1,1,1,1} (12119/36 - 165 D_0 + 291 D_1 + 104 D_0^2 \\
- 46 D_1^2) + S_{-3} (4346/27 - 214/25 D_{-2} - 16850/9 D_0 + 11870/9 D_1 + 408 D_2 \\
- 4086/25 D_3 + 144/5 D_{-2} + 1533 D_0^2 - 776 D_1^2 + 3839/3 D_1^2 - 408 D_1^3) \\
+ S_{3} (-103447/1296 + 6/5 D_{-2} - 3599/36 D_0 - 1963/12 D_1 + 108 D_2 + 1629/5 D_3 \\
+ 2131/18 D_0^2 - 197/2 D_1^2 + 530/3 D_1^2 - 171/2 D_1^3) + S_{-2,1} (248/25 D_{-2} \\
+ 24 D_{-1} + 4208/9 D_0 - 608/9 D_1 - 552 D_2 + 2952/25 D_3 - 48/5 D_0^2 \\
- 1220/3 D_1^2 + 206 D_0^3 - 1636/3 D_1^2 + 178 D_1^3) + S_{1,-2} (-2428/27 - 52/25 D_{-2} \\
- 24 D_{-1} + 2420/3 D_0 - 1108 D_1 + 312 D_2 + 252/25 D_3 - 48/5 D_{-2}^2 - 1766/3 D_0^2 \\
+ 266 D_0^3 + 106/3 D_1^2 - 10 D_1^3) + S_{1,2} (-110893/324 - 6/5 D_{-2} + 4537/18 D_0) \]
\[
\begin{align*}
&\quad - \frac{11255}{36} D_1 + 150 D_2 - 4/5 D_3 - 237 D_0^2 + 146 D_0^3 + 475/6 D_1^2 + 5 D_1^3) + S_{2,1} (-81845/648 + 6/5 D_{-2} + 146 D_0 - 3817/12 D_1 - 198 D_2 - 1166/5 D_3 \\
&\quad - 935/6 D_0^2 + 112 D_0^3 + 1411/6 D_1^2 - 58 D_1^3) + S_{1,1} (26339/108 - 2957/18 D_0 \\
&\quad + 11089/36 D_1 + 1091/6 D_0^2 - 129 D_0^3 - 1061/6 D_1^2 + 43 D_1^3) + S_{-2} (-232/3 \\
&\quad + 23/375 D_{-2} + 24 D_{-1} + 88627/54 D_0 - 58711/54 D_1 - 480 D_2 + 5409/125 D_3 \\
&\quad - 116/25 D_{-2}^2 + 96/5 D_{-2}^3 - 27859/18 D_0^2 + 3611/3 D_0^3 - 589 D_0^4 \\
&\quad - 17027/18 D_1^2 + 1609/3 D_1^3 - 177 D_1^4) + S_2 (162721/1296 - 3 D_{-2} \\
&\quad - 66817/1296 D_0 + 566587/1296 D_1 - 414 D_2 - 147 D_3 - 4849/36 D_0^2 \\
&\quad + 6167/36 D_0^3 - 707/6 D_1^3 - 1307/12 D_1^4 - 1355/36 D_1^3 - 61/6 D_1^4 \\
&\quad + S_{1,1} (239633/2592 + 24/5 D_{-2} + 26327/432 D_0 - 71081/432 D_1 + 144 D_2 \\
&\quad - 759/5 D_3 + 385/3 D_0^2 - 5999/36 D_0^3 + 731/6 D_1^2 - 1475/12 D_1^3 \\
&\quad + 5471/36 D_1^3 - 113/6 D_1^4) + S_1 (161929/1728 + 186/25 D_{-2} - 24 D_{-1} \\
&\quad + 33065/96 D_0 - 107355/1296 D_1 + 366 D_2 - 21269/100 D_3 - 36/5 D_2^2 \\
&\quad - 1819/216 D_0^2 - 687/8 D_0^3 + 4237/36 D_0^4 - 287/3 D_0^5 + 493793/1296 D_1^2 \\
&\quad + 2093/72 D_1^3 - 1285/18 D_1^4 - 43/6 D_1^5) - 61555/512 + 1807/125 D_{-2} \\
&\quad - 269059/1296 D_0 - 810827/5184 D_1 + 153647/500 D_3 - 507/25 D_2^2 \\
&\quad + 72/5 D_2^3 + 460601/2592 D_0^2 - 271195/2592 D_0^3 + 3533/72 D_0^4 - 595/9 D_0^5 \\
&\quad + 1951/24 D_0^6 + 57113/96 D_1^2 - 380303/864 D_1^3 + 17011/72 D_1^4 - 1087/12 D_1^5 \\
&\quad + 149/8 D_1^6 \\
&\quad + \zeta_3 \left[ 436 S_{-3} + 1061/3 S_3 - 200 S_{1,-2} - 496/3 S_{1,2} - 100/3 S_{2,1} + 44 S_{1,1,1} \\
&\quad + S_{-2} (-388 + 148 D_0 - 340 D_1) + S_2 (-28 + 572/3 D_0 - 1868/3 D_1) \\
&\quad + S_{1,1} (35 - 94 D_0 + 382 D_1) + S_1 (-397/6 - 12 D_{-2} - 1396/3 D_0 + 1005 D_1 \\
&\quad - 96 D_2 + 624 D_3 + 142 D_0^2 - 688/3 D_0^2) - 15883/16 - 264/25 D_{-2} + 24 D_{-1} \\
&\quad - 19655/12 D_0 + 22715/12 D_1 + 876 D_2 + 4614/25 D_3 + 144/5 D_2^2 + 3947/3 D_0^2 \\
&\quad - 3955/6 D_0^3 + 964/3 D_1^2 - 51/2 D_1^3 \\
&\quad + \zeta_5 \left[ 81 S_1 - 513/4 - 171/2 D_0 + 531/2 D_1 \right] \\
\end{align*}
\]
and
\[
\begin{align*}
c_{2,\text{mes}}^{(4)Z}(N) &= 24 S_2 - 80 S_1 + 33 + 64 D_0 - 64 D_1 - 12 D_0^2 + 12 D_1^2. \\
\end{align*}
\]
Again as for $C_L$, terms with $D_a$, $a \neq 0, 1$, are not present with $w = 5$ sums in the $C_F^2 n_f^2$ coefficient (4.12). The situation at $N = 2$ is the same as that for $C_L$ above, with
\[
\begin{align*}
\zeta_{2,ns}^{(4)L}(N = 2) &= \frac{1163533}{5832} - \frac{4613}{540} \zeta_3 - \frac{290}{3} \zeta_5 + 6 \zeta_5^2, \\
\zeta_{2,ns}^{(4)N}(N = 2) &= \frac{1720051}{29160} + \frac{247}{270} \zeta_3 + 30 \zeta_5.
\end{align*}
\] (4.14)

Finally the $n_f^2$ coefficient in eq. (4.10) is given by
\[
\begin{align*}
c_{2,ns}^{(4)F}(N) &= -119/2 S_5 + 12 S_{1,4} + 24 S_{2,3} + 36 S_{3,2} + 48 S_{4,1} - 12 S_{1,1,3} - 12 S_{1,2,2} - 12 S_{1,3,1} \\
&- 24 S_{2,1,2} - 24 S_{2,2,1} - 36 S_{3,1,1} + 12 S_{1,1,1,2} + 12 S_{1,2,1,1} + 12 S_{1,2,1,1} + 24 S_{2,1,1,1} \\
&- 12 S_{1,1,1,1,1} + S_4 (853/6 - 6 D_0 - 6 D_1) + S_{1,3} (-29 + 6 D_0 - 6 D_1) \\
&+ S_{2,2} (-27 + 6 D_0 - 6 D_1) + S_{3,1} (-105 + 6 D_0 - 6 D_1) + S_{1,1,2} (29 - 6 D_0 + 6 D_1) \\
&+ S_{2,1,2} (29 - 6 D_0 + 6 D_1) + S_{2,1,1,1} (67 - 6 D_0 + 6 D_1) + S_{1,1,1,1} (-29 + 6 D_0 \\
&- 6 D_1) + S_3 (-524/3 + 13 D_0 - 34 D_1 - 12 D_0^2 + 6 D_1^2) + S_{1,2} (235/6 - 13 D_0 \\
&+ 34 D_1 + 12 D_0^2 - 6 D_1^2) + S_{2,1} (641/6 - 13 D_0 + 34 D_1 + 12 D_0^2 - 6 D_1^2) \\
&+ S_{1,1,1} (-235/6 + 13 D_0 - 34 D_1 - 12 D_0^2 + 6 D_1^2) + S_2 (14321/108 - 161/6 D_0 \\
&+ 280/3 D_1 + 29 D_0^2 - 18 D_0^3 - 34 D_1^2 + 6 D_1^3) + S_{1,1} (-4429/108 + 161/6 D_0 \\
&- 280/3 D_1 - 29 D_0^2 + 18 D_0^3 + 34 D_1^2 - 6 D_1^3) + S_3 (-25279/648 + 5729/108 D_0 \\
&- 9259/54 D_1 - 325/6 D_0^2 + 45 D_0^3 - 24 D_1^3 + 280/3 D_1^2 - 34 D_1^3 + 6 D_1^4) \\
&+ 281971/3456 + 55157/648 D_0 - 39803/162 D_1 - 9847/108 D_0^2 + 161/2 D_0^3 \\
&- 721/12 D_1^2 + 119/4 D_0^4 + 9241/54 D_1^2 + 277/3 D_1^3 + 397/12 D_1^4 - 23/4 D_1^5 \\
&+ 3/2 S_2 - 5/3 S_1 + 1/8 + 11/6 D_0 - 11/6 D_1 - 1/2 D_0^2 + 1/2 D_1^2 \\
&+ \zeta_3 \left[ S_2 - 5/3 S_1 + 1/8 + 11/6 D_0 - 11/6 D_1 - 1/2 D_0^2 + 1/2 D_1^2 \right] \\
&+ \zeta_4 \left[ 3/2 S_1 - 9/8 - 3/4 D_0 + 3/4 D_1 \right].
\end{align*}
\] (4.16)

The numerical size of the above fourth-order results is shown in figures 3 and 4 for QCD, i.e., $C_A = 3$ and $C_F = 4/3$, with $n_f = 4$ light flavours, together with the corresponding third-order contributions for the physically very wide range $2 \leq N \leq 50$. The coefficient functions $c_{L, ns}^{(n)}$ vanish for $N \to \infty$, yet only slowly: the size of the $n_f^2$ parts of $c_{L, ns}^{(4)}$ in the right part of figure 3 decreases only by a factor of 2 from $N = 10$ to $N = 50$. A further reduction by another factor of 2 and 4 is only reached at $N = 175$ and $N = 540$, respectively.

The shape of the leading large-$n_f$ contributions ($\sim n_f^{n-1}$ at order $\alpha_s^n$) in figure 3 is similar to that of the subleading large-$n_f$ contributions ($\sim n_f^{n-2}$ at order $\alpha_s^n$) in the $N$-range of the figure; its relative size is decreasing for $n_f = 4$ from about 1/10 at $n = 3$ to about 1/15 at $n = 4$. This pattern of a reduced numerical significance of the leading large-$n_f$ term at the fourth order is also seen for $c_{L, ns}^{(n)}$ in figure 4. Here the subleading large-$n_f$ contributions are larger by factors between 10 and 15 at $4 \leq N \leq 50$ for $n = 3$; the corresponding range for $n = 4$ is 19 to 24.

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Figure 3. The leading (dashed) and sub-leading (solid) large-$n_f$ contributions to the three-loop (left panel) and four-loop (right panel) coefficient functions for the structure function $F_{L,ns}$ for QCD with $n_f = 4$ flavours. The results in eqs. (4.4)–(4.16) have been converted to an expansion in $\alpha_s$, and the leading large-$n_f$ curves have been scaled up by a factor of 10 for better visibility.

Also shown in figure 4 are the dominant contributions to $C_{2,ns}$ in the large-$N$ threshold limit, where our new contributions (4.11)–(4.13) to $c_{2,ns}^{(4)}$ lead to the numerical expansion

$$ c_{2,ns}^{(4)}(N) \bigg|_{n_f^2} = 0.7233196 \ln^6 \tilde{N} + 12.339095 \ln^5 \tilde{N} 
+ 87.721224 \ln^4 \tilde{N} + 293.04552 \ln^3 \tilde{N} 
+ 233.48456 \ln^2 \tilde{N} + 65.035706 \ln \tilde{N} - 12175.412 
+ N^{-1} \left(2.1069959 \ln^5 \tilde{N} + 67.193416 \ln^4 \tilde{N} 
+ 691.16782 \ln^3 \tilde{N} + 3429.9787 \ln^2 \tilde{N} 
+ 8755.5832 \ln \tilde{N} + 12282.167 \right) + O \left( N^{-2} \right) \quad (4.17) $$

with $\ln \tilde{N} = \ln N + \gamma_e$, where $\gamma_e$ is the Euler-Mascheroni constant. Keeping only the $\ln^6 \tilde{N}$ terms in the first three lines, and the corresponding contributions at the third order, one arrives at the upper dotted curves in the figure. Adding to these the constant-$N$ contributions yields the lower dotted curves. Neither of these results can quantitatively replace the exact expressions at physically interesting moderate values of $N$.

Analytic $x$-space expression corresponding to eq. (4.17) and a further discussion of the threshold and high-energy limits can be found in section 5 below.
Figure 4. As figure 3, but for the structure function $F_{2,ns}$. In addition the $\ln^3 N$ and $\ln^4 N + \text{const.}$ threshold contributions are shown here, respectively, by the upper and lower dotted curves.

A five-loop prediction. It is possible to predict a small part, the $\zeta_4$ coefficients of $n_f^3$ and $n_f^4$, of the $n = 4$ five-loop non-singlet anomalous dimension in eq. (2.10), from our result (4.13) and eq. (4.16) for $c_{2,ns}^{(4)}(N)$. This possibility arises from the no-$\pi^2$ theorem [67, 68] for Euclidean (space-like) physical quantities in a suitable renormalization scheme, which was investigated in the context of inclusive DIS in ref. [69]. For this one considers the physical evolution kernels which were expressed in terms of the splitting functions and coefficient functions for the non-singlet case to the fifth order in eqs. (2.7)–(2.9) of ref. [70].

Keeping only the terms with $\zeta_4$ simplifies the fifth-order $\overline{\text{MS}}$ kernel for $F_{2,ns}$ to

$$\bar{K}_{2,ns}^{(4)}(N) = -\tilde{\gamma}_{ns}^{(4)}(N) - 3\beta_1 \tilde{c}_a^{(3)}(N) - 4\beta_0 \left(\tilde{c}_{2,ns}^{(4)}(N) - \tilde{c}_{2,ns}^{(1)}(N) \tilde{c}_{2,ns}^{(3)}(N)\right),$$

(4.18)

where the tilde indicates the contribution with $\zeta_4 = 1/90 \pi^4$. At this order, a scheme transformation removing the $\zeta_4$ term of the five-loop beta function [71–73] needs to be performed, and the prediction of the no-$\pi^2$ theorem becomes [69]

$$\beta_0 \bar{K}_a^{(4)} + \frac{1}{3} \tilde{\beta}_4 K_a^{(0)} = 0.$$  

(4.19)

Since $\beta_0$ includes $n_f$ and $K_{2,ns}^{(0)} \sim C_F$, only the $\zeta_4 n_f^4$ terms in $\beta_4$ can contribute, but there is no such term. Hence already the quantity (4.18) has to vanish for the $n_f^3$ and $n_f^4$ terms.

The three-loop coefficient function $c_{2,ns}^{(3)}(N)$ does not include a $\zeta_4$ term with $n_f^2$. Since the prefactors $\beta_1$ and $\beta_0 c_{2,ns}^{(1)}$ include no more than one power in $n_f$, the two terms with
\( \zeta_{2,ns}^{(3)} \) do not contribute to the \( n_f^3 \) and \( n_f^4 \) parts of eq. (4.18). So we end up with a simple relation for the \( \zeta_4 n_f^3 \) and \( \zeta_4 n_f^4 \) contributions,

\[
\tilde{P}_{ns}^{(4)}(N) = -\tilde{\gamma}_{ns}^{(4)}(N) = 4\beta_0 \tilde{c}_{2,ns}^{(4)}(N),
\]

which leads to

\[
P_{ns}^{(4)}(N)|_{\zeta_4} = n_f^0, n_f^1 \text{ and } n_f^2 \text{ contributions}
\]

\[
+ C_F C_A n_f^3 \left( \frac{64}{3} S_2 - \frac{1568}{27} \frac{S_1 + 176}{9} + \frac{1360}{27} D_0 - \frac{1360}{27} D_1 - \frac{32}{3} D_0^2 + \frac{32}{3} D_1^2 \right)
\]

\[
+ C_F^2 n_f^3 \left( -\frac{64}{3} S_2 + \frac{640}{9} \frac{S_1 - 88}{3} - \frac{512}{9} D_0 + \frac{512}{9} D_1 + \frac{32}{3} D_0^2 - \frac{32}{3} D_1^2 \right)
\]

\[
+ C_F n_f^4 \left( -\frac{64}{27} S_1 + \frac{16}{9} + \frac{32}{27} D_0 - \frac{32}{27} D_1 \right). 
\]

At \( N = 2 \) and \( N = 3 \) this result agrees with those obtained by diagram calculations in ref. [74] using the program of ref. [75] for the \( R^* \) operation. Its last line was derived long ago as part of the complete leading large-\( n_f \) result [76]. Due to eqs. (4.10) and (4.16), the whole of (4.21) is proportional to \( \beta_0 \) and the lowest-order anomalous dimension \( \gamma_{qq}^{(0)}(N) \) for \( C_F = C_A \).

5 The x-space coefficient functions

The coefficient functions \( c_{a,ns}^{(n)}(x) \) are obtained from the \( N \)-space results of the previous section by an inverse Mellin transformation, which expresses these functions in terms of harmonic polylogarithms (HPLs) \( H_{m_1,...,m_w}(x) \), \( m_j = 0, \pm 1 \). Following ref. [22], to which the reader is referred for a detailed discussion, the lowest-weight (\( w = 1 \)) functions \( H_m(x) \) are given by

\[
H_0(x) = \ln x, \quad H_{\pm 1}(x) = \mp \ln(1 \mp x), \tag{5.1}
\]

and the higher-weight (\( w \geq 2 \)) functions are recursively defined as

\[
H_{m_1,...,m_w}(x) = \begin{cases} \frac{1}{w!} \ln^w x, & \text{if } m_1, \ldots, m_w = 0, \ldots, 0 \\ \int_0^x dz f_{m_1}(z) H_{m_2,...,m_w}(z), & \text{else} \end{cases} \tag{5.2}
\]

with

\[
f_0(x) = \frac{1}{x}, \quad f_{\pm 1}(x) = \frac{1}{1 \mp x}. \tag{5.3}
\]

The inverse Mellin transformation exploits an isomorphism between the set of harmonic sums for even or odd \( N \) and the set of HPLs. Hence it can be performed by a completely algebraic procedure [12, 22], based on the fact that harmonic sums occur as coefficients of the Taylor expansion of harmonic polylogarithms. A FORTRAN program for the HPLs up to weight \( w = 4 \) has been provided in ref. [77], later this was extended to \( w = 5 \) and \( w = 6 \).
In our results below, the argument $x$ of the HPLs is suppressed for brevity, and we use the abbreviations

$$x_m = 1 - x \quad \text{and} \quad x_p = 1 + x.$$  \hspace{1cm} (5.4)

Analogous to eq. (5.5) above, the $x$-space coefficient function for $F_{L,ns}$ is decomposed as

$$c_{L,ns}^{(4)}(x) = n_f^0 \text{ and } n_f^1 \text{ contributions} \hspace{1cm} (5.5)$$

$$+ C_F C_A n_f^2 \frac{16}{9} c_{L,ns}^{(4)A}(x) + C_F (C_F - \frac{1}{2} C_A) n_f^2 \frac{16}{9} c_{L,ns}^{(4)N}(x) + C_F n_f^3 \frac{16}{27} c_{L,ns}^{(4)F}(x).$$

The two $n_f^2$ contributions in the second line are given by

$$c_{L,ns}^{(4)A}(x) =$$

$$+ H_{0,0,0,1} (-40 x + 80 x^2 - 120 x^3) + H_{0,0,1,0} (-16 x + 32 x^2 - 48 x^3) +$$

$$+ H_{0,1,0,0} (-20 x + 40 x^2 - 60 x^3) + H_{0,1,0,1} (16 x - 32 x^2 + 48 x^3) + H_{0,0,0,1} (4 x - 8 x^2 - 12 x^3) + H_{0,0,1,0} (40 x - 80 x^2 + 120 x^3) +$$

$$+ H_{0,1,0,1} (-48 x^2 - 4 x - 8 x^2 - 12 x^3) + H_{0,1,0,0} (4 x - 8 x^2 + 12 x^3) +$$

$$+ H_{0,1,0,1} (-18 x + 72 x^2 - 60 x^3) + H_{0,1,0,0} (48 x^2 - 24 x - 20 x - 40 x^2 + 60 x^3) +$$

$$+ H_{1,0,0,0} (-80 x^2 - 40 x - 10 x + 80 x^2 - 120 x^3) + H_{1,0,0,1} (-32 x^2 - 16 x - 16 x + 32 x^2 - 48 x^3) + H_{1,0,0,1} (-40 x^2 - 20 x - 20 x + 40 x^2 - 60 x^3) +$$

$$+ H_{1,0,1,0} (32 x^2 - 16 x - 16 x - 32 x^2 + 48 x^3) + H_{1,1,0,0} (8 x^2 - 4 x - 8 x^2 - 12 x^3) + H_{1,1,0,0} (8 x^2 - 4 x - 8 x^2 + 12 x^3) +$$

$$+ H_{1,1,0,1} (-18 x + 72 x^2 - 60 x^3) + H_{1,1,1,0} (40 x^2 - 20 x - 20 x - 40 x^2 + 60 x^3) +$$

$$+ H_{1,1,1,1} (18 x - 72 x^2 + 60 x^3) + H_{0,0,0,0} (80 x^2 + 115/2 x) +$$

$$+ 184 x^2 - 102 x^3) + H_{0,0,1,0} (32 x^2 + 86 x + 48 x^2) + H_{0,0,1,1} (40 x^2 + 86 x + 76 x^2 - 8 x^3) + H_{0,1,0,0} (-32 x^2 + 91 x - 112 x^2 + 102 x^3) + H_{0,1,0,1} (8 x^2 - 4 x - 8 x^2 - 12 x) +$$

$$+ H_{0,1,1,0} (-8 x^2 + 65 x + 28 x^2 + 82 x^3) + 61 x H_{0,0,1,1} + H_{0,0,0} (-80 x^2 + 115/3 x) - 120 x^2) + H_{0,0,0,1} (16 - 20 x^2 - 4 x^2 + 17 x + 7 x + 76 x^2 - 102 x^3) +$$

$$+ H_{1,0,1,0} (-8 x^2 + 40 x - 12 x^2) + H_{1,0,1,1} (4 x + 76 x^2 - 8 x^3) + H_{1,1,0,1} (-16 + 20 x^2 - 28 x^2 + 75 x - 124 x^2 + 102 x^3) + H_{1,1,0,1} (80 x - 60 x^2) + H_{1,1,1,1} (34 x - 16 x^2 + 8 x^3) +$$

$$+ 36 x H_{1,1,1,1} + H_{0,0,0} (125/6 + 10969/36 x + 120 x^2 + 40 x^2 - 80 x^2 + 120 x^2 + 3^2) + H_{0,0,1} (-23/2 - 12 x^2 + 1613/6 x + 290 x^2 - 197/3 x^3 + 4 x^2 - 8 x^2 + 12 x^2 + 3^2) +$$

$$+ H_{0,1,0} (-23 - 32 x^2 + 1307/6 x - 20 x^2 + 197/3 x^3 + 48 x^2 - 24 x^2 + 12 x^2 + 4 x^2 - 8 x^2 + 12 x^2 + 3^2) + H_{0,1,1} (-29 - 40 x^2 + 471/2 x + 8 x^2 - 18 x^2 + 72 x^2 x^2 - 60 x^2 x^3) + H_{1,0,0} (-24 - 36 x^2 + 1439/9 x - 90 x^2 + 80 x^2 x^2 - 40 x^2 x^2)$$
\[ + 40 \zeta_2 x - 80 \zeta_2 x^2 + 120 \zeta_2 x^3 \] 
\[ + 197/3 x^3 + 8 \zeta_2 x^2 - 4 \zeta_2 x - 8 \zeta_2 x^2 + 12 \zeta_2 x^3 \] 
\[ + 8 x^{-1} + 997/6 x - 100 x^2 + 197/3 x^3 + 8 \zeta_2 x^2 - 4 \zeta_2 x^2 + 4 \zeta_2 x - 8 \zeta_2 x^2 \] 
\[ + 12 \zeta_2 x^3 \] 
\[ + H_{1,1,0} (-25 + 293/2 x - 18 \zeta_2 x + 72 \zeta_2 x^2 - 60 \zeta_2 x^3) \] 
\[ + H_{0,0} (-3301/36 + 2471/3 x + 210 x^2 - 80 \zeta_2 x^2 - 115/2 \zeta_2 x - 184 \zeta_2 x^2 \] 
\[ + 102 \zeta_2 x^2 - 28 \zeta_3 x + 56 \zeta_3 x^2 - 84 \zeta_3 x^3 \] 
\[ + H_{0,1} (-941/6 - 36 x^{-1} + 10169/18 x \] 
\[ + 611/3 x^2 - 8 \zeta_2 x - 12 \zeta_2 x^2 - 48 \zeta_3 x - 24 \zeta_3 x^2 + 62 \zeta_3 x \] 
\[ - 304 \zeta_3 x^2 + 216 \zeta_3 x^3 + H_{1,0} (-464/3 + 32 x^{-1} + 4565/12 x + 43/3 x^2 + 20 \zeta_2 x^2 \] 
\[ + 4 \zeta_2 x^2 - 16 \zeta_2 - 7 \zeta_2 x - 76 \zeta_2 x^2 + 102 \zeta_3 x^2 - 56 \zeta_3 x^2 - 28 \zeta_3 x^4 - 28 \zeta_3 x^4 \] 
\[ + 56 \zeta_3 x^2 - 84 \zeta_3 x^3 \] 
\[ + H_{1,1} (-155 + 40 x^{-1} + 14459/36 x + 8 x^2 - 80 \zeta_2 x + 60 \zeta_2 x^2 \] 
\[ - 56 \zeta_3 x^2 + 28 \zeta_3 x^2 + 62 \zeta_3 x - 304 \zeta_3 x^2 + 216 \zeta_3 x^3 \] 
\[ + H_{0} (-3679/12 \] 
\[ + 839905/648 x + 587/3 x^2 + 12 \zeta_2 x^{-1} + 23/2 \zeta_2 - 1613/6 \zeta_2 x - 290 \zeta_2 x^2 \] 
\[ + 197/3 \zeta_3 x^2 - 605/6 \zeta_3 x + 164 \zeta_3 x^2 - 110 \zeta_3 x^3 - 53 \zeta_3 x \] 
\[ + 106 \zeta_4 x^2 - 159 \zeta_4 x^3 \] 
\[ + H_{1} (-4483/12 + 84 x^{-1} + 842437/1296 x + 587/3 x^2 \] 
\[ + 8 \zeta_2 x^{-1} + 85/3 \zeta_2 - 721/6 \zeta_2 x - 152 \zeta_2 x^2 + 197/3 \zeta_2 x^3 - 20 \zeta_3 x^2 - 44 \zeta_3 x^{-1} \] 
\[ + 16 \zeta_3 + 397/3 \zeta_3 x - 136 \zeta_3 x^2 - 110 \zeta_3 x^3 - 106 \zeta_4 x^2 + 53 \zeta_4 x^{-1} - 53 \zeta_4 x \] 
\[ + 106 \zeta_4 x^{-1} - 159 \zeta_4 x^3 \] 
\[ - 763025/1296 + 3130309/2592 x + 36 \zeta_2 x^{-1} + 941/6 \zeta_2 \] 
\[ - 10169/18 \zeta_2 x - 611/3 \zeta_2 x^2 + 12 \zeta_3 x^{-1} - 13/6 \zeta_3 - 391 \zeta_3 x + 650 \zeta_3 x^2 \] 
\[ - 197 \zeta_3 x^3 + 22 \zeta_3 x^2 - 80 \zeta_3 \zeta_2 x^2 + 72 \zeta_3 \zeta_2 x^3 + 106 \zeta_4 x^{-1} + 33 \zeta_4 x \] 
\[ + 263 \zeta_4 x^2 - 192 \zeta_4 x^3 + 140 \zeta_5 x - 640 \zeta_5 x^2 + 480 \zeta_5 x^3 \] 
\[ (5.6) \] 

and

\[ e^{(4)N}_{L,ss}(x) = \]
\[ - 320 x H_{-1,-1,-1,0,0} + 544 x H_{-1,-1,-1,0,0} + 352 x H_{-1,-1,0,0,0,0} + 352 x H_{-1,-1,0,0,0,0} + 352 x H_{0,0,-1,0,0,0} - 368 x H_{0,0,-1,0,0,0} + 288 x H_{0,0,0,0,0,0} \]
\[ - 352 x H_{0,0,-1,0,0,0} - 368 x H_{0,0,-1,0,0,0} + 288 x H_{0,0,0,0,0,0} + 48 x H_{0,0,0,0,0,0} - 288 x H_{0,0,-1,0,0,0} - 160 x H_{0,0,0,0,0,0} - 164 x H_{0,0,0,0,0,0} \]
\[ - 144 x H_{0,0,0,0,0,0} + 156 x H_{0,0,0,0,0,0} + 16 x H_{0,0,0,0,0,0} - 40 x H_{0,0,0,0,0,0} + 40 x H_{0,0,0,0,0,0} - 288 x H_{0,0,0,0,0,0} - 192 x H_{0,0,0,0,0,0} \]
\[ - 288 x H_{0,0,0,0,0,0} - 344 x H_{0,0,0,0,0,0} - 112 x H_{0,0,0,0,0,0} - 168 x H_{0,0,0,0,0,0} + 72 x H_{0,0,0,0,0,0} - 16 x H_{0,0,0,0,0,0} + 56 x H_{0,0,0,0,0,0} - 128 x H_{0,0,0,0,0,0} - 184 x H_{0,0,0,0,0,0} + 64 x H_{0,0,0,0,0,0} \]
\[ - 40 x H_{0,0,0,0,0,0} + 88 x H_{0,0,0,0,0,0} - 32 x H_{0,0,0,0,0,0} + 72 x H_{0,0,0,0,0,0} + H_{-1,-1,0,0,0,0} - 32 x^{-2} + 80 x^{-1} + 672 x + 160 x^2 + 48 x^3 \]
\[ + H_{-1,-1,0,0,0,0} - 272 - 272/5 x^{-2} + 136 x^{-1} \]
\[ - 1040 x - 272 x^2 + 508/5 x^3 \] 
\[ + H_{-1,-1,0,0,0,0} \] 
\[ - 176 - 176/5 x^{-2} + 88 x^{-1} - 688 x \]
\[-176 x^2 + 264/5 x^3 \] + \[H_{-1,0,0,0} + 576/5 x^{-2} + 664 x - 864/5 x^3 \] + \[H_{-1,0,0,1} (224 + 224/5 x^{-2} + 112 x^{-1} - 224 x^2 - 336/5 x^3) + H_{-1,0,1,0} (144 + 144/5 x^{-2} + 72 x^{-1} - 144 x^2 - 216/5 x^3) + H_{-1,0,1,1} (128 + 128/5 x^{-2} + 64 x^{-1} - 128 x^2 - 192/5 x^3) + H_{0,-1,0,0} (-192 - 192/5 x^{-2} + 96 x^{-1} - 688 x - 160 x^2 + 48 x^3) + H_{0,-1,0,1} (576 + 576/5 x^{-2} + 216 x + 272 x^2 - 408/5 x^3) + H_{0,-1,1,0} (192 + 192/5 x^{-2} + 96 x^{-1}) - 352 x + H_{0,0,-1,0} (384 + 384/5 x^{-2} + 24 x + 240 x^2 - 168/5 x^3) + H_{0,0,0,0} (-248/3 x + 864/5 x^3) + H_{0,0,0,1} (127 x + 344 x^2 + 1132/5 x^3) + H_{0,0,1,0} (136 x + 112 x^2 + 168/5 x^3) + H_{0,0,1,1} (148 x + 160 x^2 + 712/5 x^3) + H_{0,1,0,0} (-34 x - 72 x^2 - 724/5 x^3) + H_{0,1,0,1} (98 x + 16 x^2 + 24/5 x^3) + H_{0,1,1,0} (154 x - 48 x^2 - 544/5 x^3) + 122 x H_{0,1,1,1} + H_{1,0,-1,0} (64 + 64/5 x^{-2} + 32 x^{-1} - 336 x + 64 x^2 + 96/5 x^3) - 1138/3 x H_{1,0,0,0} + H_{1,0,0,1} (64 + 64/5 x^{-2} + 32 x^{-1} - 370 x + 120 x^2 + 796/5 x^3) + H_{1,0,1,0} (-32 - 32/5 x^{-2} - 16 x^{-1} + 176 x - 32 x^2 - 48/5 x^3) + H_{1,0,1,1} (56 x + 32 x^2 + 104 x^3) + H_{1,1,0,0} (-16 - 16/5 x^{-2} - 8 x^{-1} + 126 x - 72 x^2 - 724/5 x^3) + H_{1,1,0,1} (16 + 16/5 x^{-2} - 8 x^{-1} - 8 x + 16 x^2 + 24/5 x^3) + H_{1,1,1,0} (-16 - 16/5 x^{-2} - 8 x^{-1} + 188 x - 48 x^2 - 544/5 x^3) + 72 x H_{1,1,1,1} - 160 \zeta_2 x H_{-1,-1,-1} + H_{-1,-1,0} (-336 - 1232/25 x^{-2} + 128 x^{-1} - 2752/3 x - 320 x^2 + 1968/25 x^3 + 112 \zeta_2 x) + 176 \zeta_2 x H_{-1,0,-1} + H_{-1,0,0} (6104/5 + 3616/25 x^{-2} + 272/5 x^{-1} + 17152/15 x - 408/5 x^{-2} - 5784/25 x^3 - 56 \zeta_2 x) + H_{-1,0,1} (448 + 1232/25 x^{-2} + 160 x^{-1} + 48 x - 368 x^2 - 1968/25 x^3) + 176 \zeta_2 x H_{0,-1,-1} + H_{0,-1,0} (784 + 2384/25 x^{-2} + 112/5 x^{-1} + 1264/5 x + 296 x^2 - 1968/25 x^3 + 120 \zeta_2 x) + 96 \zeta_2 x H_{0,0,-1} + H_{0,0,0} (289/15 - 576/5 x^{-1} - 73891/90 x + 864/5 x^2 + 5784/25 x^3) + H_{0,0,1} (-143 - 288/5 x^{-1} - 143/15 x + 424 x^2 + 18104/75 x^3 + 192 \zeta_2 x) + H_{0,1,0,0} (-342/5 - 112/5 x^{-1} + 2383/15 x - 476/5 x^{-2} - 488/3 x^3 + 288 \zeta_2 x) + H_{0,1,1,1} (216 \zeta_2 x - 426/5 - 128/5 x^{-1} + 659/5 x - 328/5 x^2) + 192 \zeta_2 x H_{1,0,-1} + H_{1,0,0} (-192/5 + 16/5 x^{-1} - 6862/45 x + 724/5 x^2 + 400 \zeta_2 x) + H_{1,0,1} (192 \zeta_2 x - 814/15 - 16/5 x^{-1} - 301/15 x + 716/5 x^2 + 488/3 x^3) + H_{1,1,0,0} (-746/15 + 16/5 x^{-1} + 3131/15 x - 196/5 x^2 - 488/3 x^3 + 296 \zeta_2 x) + H_{1,1,1,1} (-50 + 73 x + 192 \zeta_2 x) + H_{-1,-1} (16 \zeta_2 x^{-2} - 40 \zeta_2 x^{-1} + 80 \zeta_2 + 336 \zeta_2 x + 80 \zeta_2 x^2 - 24 \zeta_2 x^3 + 160 \zeta_3 x) + H_{-1,0} (85732/75 + 32252/375 x^{-2} + 1632/25 x^{-1} + 79732/75 x - 2568/25 x^2 - 17916/125 x^3 - 56 \zeta_2 x^{-2} - 84 \zeta_2 x^{-1} - 280 \zeta_2 x - 176 \zeta_2 x + 168 \zeta_2 x^2 + 84 \zeta_2 x^3 - 80 \zeta_3 x) + H_{0,-1} (-288/5 \zeta_2 x^{-2} - 48 \zeta_2 x^{-1} - 288 \zeta_2 x + 8 \zeta_2 x - 80 \zeta_2 x^2 + 24 \zeta_2 x^3 - 112 \zeta_3 x) + H_{0,0} (-12541/450 - 3616/25 x^{-1} - 41282/25 x + 4444/25 x^2 + 17916/125 x^3 - 103 \zeta_2 x - 344 \zeta_2 x^2 - 260 \zeta_2 x^3 + 96 \zeta_3 x) + H_{0,1} (-7937/75 - 752/25 x^{-1} - 67831/225 x - 12836/75 x^2
The corresponding results for \(c_{2}^{2} + 203 - 304 - 336 - 96 \zeta_{2} x - 246 \zeta_{2} x - 96 \zeta_{2} x^2 - 144/5 \zeta_{2} x^3 + 160 \zeta_{3} x) + H_{1,0} (-536/15 + 112/5 x^{-1} - 799/6 x + 320/3 x^2 - 24 \zeta_{2} x^2 - 60 \zeta_{2} x^{-1} - 120 \zeta_{2} + 546 \zeta_{2} x - 176 \zeta_{2} x^2 - 176 \zeta_{2} x^3 + 56 \zeta_{3} x) + H_{1,1} (-122/5 + 128/5 x^{-1}) - 13169/90 x - 328/5 x^2 - 96/5 \zeta_{2} x^2 - 48 \zeta_{2} x^{-1} - 96 \zeta_{2} + 344 \zeta_{2} x - 96 \zeta_{2} x^2 - 144/5 \zeta_{2} x^3 + 80 \zeta_{3} x) + H_{-1} (-1848/25 \zeta_{2} x^2 - 96 \zeta_{2} x^{-1} - 616 \zeta_{2} - 1520/3 \zeta_{2} x + 208 \zeta_{2} x^2 + 2952/25 \zeta_{2} x^3 - 144/5 \zeta_{3} x^2 - 8 \zeta_{3} x^{-1} - 144 \zeta_{3} - 336 \zeta_{3} x - 16 \zeta_{3} x^2 + 216/5 \zeta_{3} x^3 - 100 \zeta_{4} x) + H_{0} (-1608/250 + 628/375 x^{-1} - 39052717/40500 x + 41888/375 x^2 + 80 \zeta_{2} x^{-1} + 143 \zeta_{2} + 787/3 \zeta_{2} x - 424 \zeta_{2} x^2 - 24008/75 \zeta_{2} x^3 + 115/3 \zeta_{3} x + 16 \zeta_{3} x^2 + 872/5 \zeta_{3} x^3 + 12 \zeta_{4} x) + H_{1} (13997/150 + 1712/25 x^{-1} - 7089221/16200 x - 7916/75 x^2 - 616/25 \zeta_{2} x^{-2} - 304/5 \zeta_{2} x^{-1} - 1706/15 \zeta_{2} + 7181/15 \zeta_{2} x - 1516/5 \zeta_{2} x^2 - 15152/75 \zeta_{2} x^3 - 88/5 \zeta_{3} x^{-2} - 44 \zeta_{3} x^{-1} - 88 \zeta_{3} + 362/3 \zeta_{3} x + 1088/5 \zeta_{3} x^3 - 190 \zeta_{4} x) - 3918163/81000 + 49532/375 x^{-1} - 54681799/162000 x + 27156/125 x^2 + 2384/25 \zeta_{2} x^{-1} + 7937/75 \zeta_{2} + 307027/225 \zeta_{2} x + 12836/75 \zeta_{2} x^2 - 17916/125 \zeta_{2} x^3 + 232/5 \zeta_{3} x^{-1} + 1591/15 \zeta_{3} + 1052/15 \zeta_{3} x - 4/5 \zeta_{3} x^2 + 1456/5 \zeta_{3} x^3 - 136 \zeta_{3} x + 301 \zeta_{4} x + 302 \zeta_{4} x^2 + 677/5 \zeta_{4} x^3 + 296 \zeta_{5} x . \tag{5.7}

The \(n_{f}^{3}\) coefficient in the last line of eq. (5.5) reads

\[ c_{L,ns}^{(4)F} = \]

\[- 48 x H_{0,0,0} - 36 x H_{0,0,1} - 24 x H_{0,1,0} - 24 x H_{0,1,1} - 12 x H_{1,0,0} - 12 x H_{1,0,1} - 12 x H_{1,1,0} - 12 x H_{1,1,1} + H_{0,0} (12 - 150 x) + H_{0,1} (12 - 100 x) + H_{1,0} (12 - 50 x) + H_{1,1} (12 - 50 x) + H_{0} (38 - 634/3 x + 36 \zeta_{2} x) + H_{1} (38 - 317/3 x + 12 \zeta_{2} x) + 203/3 - 8609/54 x - 12 \zeta_{2} + 100 \zeta_{2} x + 12 \zeta_{3} x . \tag{5.8}

The corresponding results for \(F_{2,ns}^{(4)F}\) are written in a very similar manner as

\[ c_{2,ns}^{(4)}(x) = n_{0}^{2} \text{ and } n_{1}^{2} \text{ contributions} + C_{F} C_{A} n_{f}^{2} \frac{4}{5} c_{2,ns}^{(4)L}(x) + C_{F} (C_{F} - \frac{1}{2} C_{A}) n_{f}^{2} \frac{4}{5} c_{2,ns}^{(4)N}(x) + C_{F} n_{f}^{4} \frac{4}{27} c_{2,ns}^{(4)F}(x) \tag{5.9}

with

\[ c_{2,ns}^{(4)L}(x) = \]

\[ + H_{0,0,0,0,0} (1951/4 - 1951/3 x^{-1} + 1951/4 x) + H_{0,0,0,0,1} (607 - 2890/3 x^{-1} + 411 x + 480 x^2 - 720 x^3) + H_{0,0,0,1,0} (1547/3 - 2572/3 x^{-1} + 1391/3 x + 192 x^2 - 288 x^3) + H_{0,0,0,1,1} (1367/3 - 2752/3 x^{-1} + 1343/3 x + 240 x^2 - 360 x^3) + H_{0,0,1,0,0} (387 - 670 x^{-1} + 475 x - 192 x^2 + 288 x^3) + H_{0,0,1,0,1} (428 - 752 x^{-1} + 400 x + 48 x^2} \]

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\[ -72x^3 + H_{0,0,1,1,0} \left( 390 - 676x_m^{-1} + 436x - 48x^2 + 72x^3 \right) + H_{0,0,1,1,1} \left( 404 - 704x_m^{-1} 
+ 404x \right) + H_{0,1,0,0,0} \left( 770/3 - 1408/3x_m^{-1} + 1250/3x - 480x^2 + 720x^3 \right) 
+ H_{0,1,0,1,0} \left( 296 - 48x^{-2} - 548x_m^{-1} + 256x + 48x^2 - 72x^3 \right) + H_{0,1,0,1,1} \left( 272 - 500x_m^{-1} 
+ 288x - 48x^2 + 72x^3 \right) + H_{0,1,1,0,0} \left( 260 + 48x^{-2} - 476x_m^{-1} + 364x - 240x^2 + 360x^3 \right) + H_{0,1,1,0,1} \left( 266 
- 488x_m^{-1} + 386x - 432x^2 + 360x^3 \right) + H_{0,1,1,1,0} \left( 240 - 436x_m^{-1} + 258x \right) 
+ H_{0,1,1,1,1} \left( 252 - 460x_m^{-1} + 252x \right) + H_{1,0,0,0,0} \left( 671/3 - 1342/3x_m^{-1} + 671/3x \right) 
+ H_{1,0,0,0,1} \left( 824/3 - 80x^{-2} - 1648/3x_m^{-1} + 344/3x + 480x^2 - 720x^3 \right) + H_{1,0,0,1,0} \left( 234 
- 32x^{-2} - 468x_m^{-1} + 170x + 192x^2 - 288x^3 \right) + H_{1,0,0,1,1} \left( 270 - 40x^{-2} - 540x_m^{-1} 
+ 190x + 240x^2 - 360x^3 \right) + H_{1,0,1,0,0} \left( 178 + 32x^{-2} - 356x_m^{-1} + 242x - 192x^2 
+ 288x^3 \right) + H_{1,0,1,0,1} \left( 218 - 8x^{-2} - 432x_m^{-1} + 200x + 48x^2 - 72x^3 \right) + H_{1,0,1,1,0} \left( 206 
+ 8x^{-2} - 412x_m^{-1} + 222x - 48x^2 + 72x^3 \right) + H_{1,0,1,1,1} \left( 228 - 456x_m^{-1} + 228x \right) 
+ H_{1,1,0,0,0} \left( 530/3 + 80x^{-2} - 1060/3x_m^{-1} + 1010/3x - 480x^2 + 720x^3 \right) 
+ H_{1,1,0,0,1} \left( 196 - 8x^{-2} - 392x_m^{-1} + 180x + 48x^2 - 72x^3 \right) + H_{1,1,0,1,0} \left( 162 + 8x^{-2} 
- 324x_m^{-1} + 178x - 48x^2 + 72x^3 \right) + H_{1,1,0,1,1} \left( 188 - 376x_m^{-1} + 68x + 432x^2 - 360x^3 \right) 
+ H_{1,1,1,0,0} \left( 142 + 40x^{-2} - 284x_m^{-1} + 222x - 240x^2 + 360x^3 \right) + H_{1,1,1,0,1} \left( 164 
- 328x_m^{-1} + 284x - 432x^2 + 360x^3 \right) + H_{1,1,1,1,0} \left( 140 - 280x_m^{-1} + 140x \right) 
+ H_{1,1,1,1,1} \left( 160 - 320x_m^{-1} + 160x \right) + H_{0,0,0,0,0} \left( 2997/2 - 20567/9x_m^{-1} + 11249/6x \right) 
+ H_{0,0,0,1,0} \left( 3441/2 + 80x^{-1} - 24686/9x_m^{-1} + 3717/2x + 1104x^2 - 612x^3 \right) 
+ H_{0,0,1,0,0} \left( 23231/18 + 32x^{-1} - 19634/9x_m^{-1} + 30677/18x + 288x^2 \right) 
+ H_{0,0,1,0,1} \left( 23231/18 + 40x^{-1} - 19967/9x_m^{-1} + 15163/9x + 456x^2 - 48x^3 \right) 
+ H_{0,0,1,1,0} \left( 7015/9 - 32x^{-1} - 13373/9x_m^{-1} + 11956/9x - 672x^2 + 612x^3 \right) 
+ H_{0,1,0,0,0} \left( 2593/3 + 8x^{-1} - 4835/3x_m^{-1} + 3628/3x + 72x^2 \right) + H_{0,1,0,0,1} \left( 2302/3 
- 8x^{-1} - 4283/3x_m^{-1} + 7361/6x - 168x^2 + 48x^3 \right) + H_{0,1,0,1,0} \left( 2261/3 - 4270/3x_m^{-1} 
+ 3473/3x \right) + H_{1,0,0,0,0} \left( 3706/9 - 80x^{-1} - 9995/9x_m^{-1} + 8704/9x - 720x^2 \right) 
+ H_{1,0,0,0,1} \left( 5740/9 - 8x^{-2} + 16x^{-1} - 12263/9x_m^{-1} + 7963/9x + 456x^2 - 612x^3 \right) 
+ H_{1,0,0,1,0} \left( 1546/3 - 8x^{-1} - 3212/3x_m^{-1} + 2506/3x - 72x^2 \right) + H_{1,0,1,0,0} \left( 597 
- 1210x_m^{-1} + 1333/2x + 456x^2 - 48x^3 \right) + H_{1,0,1,0,1} \left( 3946/9 + 8x^{-2} - 16x^{-1} 
- 8495/9x_m^{-1} + 8275/9x - 744x^2 + 612x^3 \right) + H_{1,1,0,0,0} \left( 1412/3 - 3022/3x_m^{-1} 
+ 3128/3x - 360x^2 \right) + H_{1,1,0,0,1} \left( 1321/3 - 2558/3x_m^{-1} + 4073/6x - 96x^2 + 48x^3 \right) 
+ H_{1,1,0,1,0} \left( 1442 - 904x_m^{-1} + 714x \right) + H_{0,0,0,0,0} \left( 22496/9 - 152383/36x_m^{-1} + 39667/9x 
+ 720x^2 + 2890/3x_m^{-1} + 607\zeta_2 - 411\zeta_2x - 480\zeta_2x^2 + 720\zeta_2x^3 \right) 
+ H_{0,0,1,0,0} \left( 73799/36 - 24x^{-1} - 35378/9x_m^{-1} + 136907/36x + 1740x^2 - 394x^3 \right) \]
+ 752 \zeta_2 x_{m}^{-1} - 428 \zeta_2 - 400 \zeta_2 x - 48 \zeta_2 x^2 + 72 \zeta_2 x^3 + H_{0,1,0} \left(\frac{3868}{3} - 32 x^{-1}\right)
- 24305/9 x_{m}^{-1} + 53105/18 x - 120 x^2 + 394 x^3 + 48 \zeta_2 x^2 - 548 \zeta_2 x_{m}^{-1} - 296 \zeta_2
- 256 \zeta_2 x - 48 \zeta_2 x^2 + 72 \zeta_2 x^3 + H_{0,1,1} \left(\frac{10021}{9} - 40 x^{-1} - 23191/9 x_{m}^{-1}\right)
+ 26957/9 x + 48 x^2 + 488 \zeta_2 x_{m}^{-1} - 266 \zeta_2 - 386 \zeta_2 x + 432 \zeta_2 x^2 - 360 \zeta_2 x^3
+ H_{1,0,0} \left(11321/18 - 24 x^{-1} - 15200/9 x_{m}^{-1} + 12601/6 x - 540 x^2 + 80 \zeta_2 x^{-2}\right)
+ 1648/3 \zeta_2 x_{m}^{-1} - 824/3 \zeta_2 - 344/3 \zeta_2 x - 480 \zeta_2 x^2 + 720 \zeta_2 x^3 + H_{1,0,1} \left(3763/6 - 8 x^{-1} - 10459/6 x_{m}^{-1} + 10787/6 x + 912 x^2 - 394 x^3 + 8 \zeta_2 x^{-2} + 432 \zeta_2 x_{m}^{-1}\right)
- 216 \zeta_2 - 200 \zeta_2 x - 48 \zeta_2 x^2 + 72 \zeta_2 x^3 + H_{1,1,0} \left(11303/18 + 8 x^{-1}\right)
- 27307/18 x_{m}^{-1} + 35993/18 x - 600 x^2 + 394 x^3 + 8 \zeta_2 x^{-2} + 392 \zeta_2 x_{m}^{-1} - 196 \zeta_2
- 180 \zeta_2 x - 48 \zeta_2 x^2 + 72 \zeta_2 x^3 + H_{1,1,1} \left(9337/18 - 25979/18 x_{m}^{-1} + 32905/18 x\right)
+ 328 \zeta_2 x_{m}^{-1} - 164 \zeta_2 - 284 \zeta_2 x + 432 \zeta_2 x^2 - 360 \zeta_2 x^3 + H_{0,0} \left(498587/216 - 842039/162 x_{m}^{-1} + 1618205/216 x + 1260 x^2 - 80 \zeta_2 x^{-1} + 24686/9 \zeta_2 x_{m}^{-1}\right)
- 3441/2 C_2 - 3717/2 \zeta_2 x - 1104 \zeta_2 x^2 + 612 \zeta_2 x^3 + 1160/3 \zeta_3 x_{m}^{-1} - 769/3 \zeta_3
- 1339/3 \zeta_3 x + 336 \zeta_3 x^2 - 504 \zeta_3 x^3 + H_{0,1} \left(86396/81 - 24 x^{-1}\right)
- 1178369/324 x_{m}^{-1} + 439538/81 x + 1222 x^2 - 8 \zeta_2 x^{-1} + 4835/3 \zeta_2 x_{m}^{-1}
- 2593/3 \zeta_2 - 3628/3 \zeta_2 x - 72 \zeta_2 x^2 - 48 \zeta_3 x^2 - 976/3 \zeta_3 x_{m}^{-1} - 520/3 \zeta_3
+ 818/3 \zeta_3 x - 1824 \zeta_3 x^2 + 1296 \zeta_3 x^3 + H_{1,0} \left(168623/648 + 32 x^{-1}\right)
- 147071/81 x_{m}^{-1} + 2253563/648 x + 86 x^2 + 8 \zeta_2 x^2 + 16 \zeta_2 x^{-1} + 12263/9 \zeta_2 x_{m}^{-1}
- 5740/9 \zeta_2 - 7963/9 \zeta_2 x - 456 \zeta_2 x^2 + 612 \zeta_2 x^3 - 56 \zeta_3 x^2 + 944/3 \zeta_3 x_{m}^{-1}
- 472/3 \zeta_3 - 808/3 \zeta_3 x + 336 \zeta_3 x^2 - 504 \zeta_3 x^3 + H_{1,1} \left(15961/216 + 40 x^{-1}\right)
- 46483/27 x_{m}^{-1} + 784189/216 x + 48 x^2 + 3022/3 \zeta_2 x_{m}^{-1} - 1412/3 \zeta_2 - 3128/3 \zeta_2 x
+ 360 \zeta_2 x^2 + 56 \zeta_3 x^2 + 176 \zeta_3 x^3 + 88 \zeta_3 x + 400 \zeta_3 x - 1824 \zeta_3 x^2 + 1296 \zeta_3 x^3)
+ H_{0} \left(1578379/2592 - 5764837/1296 x_{m}^{-1} + 25289039/2592 x + 1174 x^2\right)
+ 24 \zeta_2 x^{-1} + 35378/9 \zeta_2 x_{m}^{-1} - 73799/36 \zeta_2 - 136907/36 \zeta_2 x - 1740 \zeta_2 x^2
+ 394 \zeta_2 x^3 + 56 \zeta_3 x^2 - 3085/3 \zeta_3 x_{m}^{-1} - 5771/9 \zeta_3 - 24019/18 \zeta_3 x + 984 \zeta_3 x^2
- 660 \zeta_3 x^3 - 2506/3 \zeta_3 x_{m}^{-1} + 5755/12 \zeta_4 + 3463/12 \zeta_4 x + 636 \zeta_4 x^2 - 954 \zeta_4 x^3)
+ H_{1} \left(-601625/648 + 72 x^{-1} - 2134163/1296 x_{m}^{-1} + 831119/162 x + 1174 x^2\right)
+ 8 \zeta_2 x^{-1} + 10459/6 \zeta_2 x_{m}^{-1} - 3763/6 \zeta_2 - 10787/6 \zeta_2 x - 912 \zeta_2 x^2 + 394 \zeta_2 x^3
- 8 \zeta_3 x^2 + 32 \zeta_3 x^3 + 5879/9 \zeta_3 x_{m}^{-1} - 3823/9 \zeta_3 x + 9187/18 \zeta_3 x - 816 \zeta_3 x^2
- 660 \zeta_3 x^3 + 106 \zeta_4 x^2 - 1400/3 \zeta_4 x_{m}^{-1} + 700/3 \zeta_4 x + 64/3 \zeta_4 x + 636 \zeta_4 x^2
- 954 \zeta_4 x^3) - 12941689/5184 - 132728/81 x_{m}^{-1} + 4964587/576 x + 24 \zeta_2 x^{-1}
+ 1178369/324 \zeta_2 x_{m}^{-1} - 86396/81 \zeta_2 - 439538/81 \zeta_2 x - 1222 \zeta_2 x^2
+ 13247/9 \zeta_3 x_{m}^{-1} - 11459/12 \zeta_3 - 120413/36 \zeta_3 x + 3900 \zeta_3 x^2 - 1182 \zeta_3 x^3
\[ -\frac{980}{3} \zeta_3 \zeta_2 x^{-1} + 209 \zeta_3 \zeta_2 + 405 \zeta_3 \zeta_2 x - 480 \zeta_3 \zeta_2 x^2 + 432 \zeta_3 \zeta_2 x^3 + 106 \zeta_4 x^{-1} - 7204/36 \zeta_4 x^{-1} + 87439/72 \zeta_4 + 98209/72 \zeta_4 x + 1578 \zeta_4 x^2 - 1152 \zeta_4 x^3 + 1036/3 \zeta_5 x_m^{-1} - 206 \zeta_5 + 834 \zeta_5 x - 3840 \zeta_5 x^2 + 2880 \zeta_5 x^3 + \delta(1 - x)(-18199451/6912 - 5764837/1296 \zeta_2 - 29/648 \zeta_3 + 68705/72 \zeta_4 + 4300/9 \zeta_3 \zeta_2 + 3091/6 \zeta_5 + 35/3 \zeta_3^2 + 521/2 \zeta_6), \]  

\[ c_{2,ns}^{(4)N}(x) = \]

\[ + H_{-1,-1,-1,-1,0}(192 - 1024 x_p^{-1} - 2112 x) + H_{-1,-1,-1,-1,0}(352 + 1792 x_p^{-1} + 3616 x) \]

\[ + H_{-1,-1,0,-1,0}(-224 + 1152 x_p^{-1} + 2336 x) + H_{-1,-1,0,0,0}(646 - 2080 x_p^{-1} - 3920 x) \]

\[ + H_{-1,0,-1,0,0}(-224 + 1152 x_p^{-1} + 2336 x) + H_{-1,0,-1,0,0}(432 - 1984 x_p^{-1} - 3792 x) \]

\[ + H_{0,-1,0,-1,0}(304 - 1344 x_p^{-1} - 2512 x) + H_{0,0,0,0,0}(-612 + 1800 x_p^{-1} + 2340 x) \]

\[ + H_{1,0,0,0,0,0}(8 + 16 x_p^{-1} + 8 x) + H_{1,0,0,0,1,1}(16 - 32 x_p^{-1} - 16 x) + H_{1,0,1,1,1,0}(-32 + 64 x_p^{-1} + 32 x) + H_{0,-1,0,1,0}(272 - 1280 x_p^{-1} - 2480 x) + H_{0,0,0,0,0}(-1060 + 1408 x_m^{-1} + 2152 x_p^{-1} + 1092 x) + H_{0,0,0,0,0}(164 - 216 + 864 x_m^{-1} + 16 x_p^{-1} - 1544 x) \]

\[ + H_{0,-1,0,1,0}(-128 + 544 x_m^{-1} - 992 x) + H_{0,-1,0,1,1,1}(-80 + 448 x_m^{-1} - 32 x_p^{-1} - 880 x) \]

\[ + H_{0,0,0,1,1}(296 + 16 x_m^{-1} - 1376 x_p^{-1} - 2616 x) + H_{0,0,0,1,1}(1280 + 2040 x_m^{-1} + 2344 x_p^{-1} + 392 x) + H_{0,0,0,0,0}(2566/3 - 4064/3 x_m^{-1} + 32 x_p^{-1} + 2602/3 x) + H_{0,0,0,0,0,0}(698 - 1092 x_m^{-1} + 410 x) + H_{0,0,0,0,1}(656 - 1104 x_m^{-1} + 656 x) + H_{0,0,0,1,0}(792 - 1376 x_m^{-1} + 792 x) + H_{0,0,0,1,1}(724 + 1208 x_m^{-1} - 32 x_p^{-1} + 692 x) + H_{0,0,1,0,1,0}(-120 + 560 x_m^{-1} - 1080 x) + H_{0,0,1,0,0,0}(1660/3 - 2192/3 x_m^{-1} - 932/3 x) + H_{0,1,0,0,0,0}(356 - 256 x_m^{-1} - 748 x) + H_{0,1,0,1,0,0}(616 - 1256 x_m^{-1} + 952 x) + H_{0,1,0,1,0,1}(476 - 864 x_m^{-1} + 476 x) + H_{0,1,0,1,0,1}(680 - 1304 x_m^{-1} + 776 x) + H_{0,1,1,0,0,1}(504 - 840 x_m^{-1} + 264 x) + H_{0,1,1,1,0,0}(632 - 1256 x_m^{-1} + 872 x) + H_{0,1,1,1,1,0}(504 - 920 x_m^{-1} + 504 x) + H_{1,0,-1,-1,0}(-16 + 32 x_m^{-1} - 16 x) + H_{1,0,-1,0,0}(-216 + 1008 x_m^{-1} - 1944 x) + H_{1,0,-1,0,1}(-96 + 576 x_m^{-1} - 1248 x) + H_{1,0,0,-1,0}(-184 + 880 x_m^{-1} - 1720 x) + H_{1,0,0,0,0}(1756/3 - 1784/3 x_m^{-1} - 3428/3 x) + H_{1,0,0,0,1,1}(1024/3 + 16/3 x_m^{-1} - 5168/3 x) + H_{1,0,0,1,0,1}(432 - 640 x_m^{-1} - 240 x) + H_{1,0,0,1,1,1}(344 - 352 x_m^{-1} - 664 x) + H_{1,0,1,0,0,0}(532 - 1208 x_m^{-1} + 964 x) + H_{1,0,1,0,1,1}(452 - 1016 x_m^{-1} + 788 x) \]
\[\begin{align*}
\text{+ 321856/45} x - 2448/5 x^2 - 34704/25 x^3 - 248 \zeta_2 x^{-1} + 68 \zeta_2 - 404 \zeta_2 x \\
\text{+ H}_{-1,1,0} (16528/9 + 992/25 x^{-2} - 96 x^{-1} + 304/9 x_p^{-1} + 2432/9 x - 2208 x^2} \\
\text{- 11808/25 x^3 - 16 \zeta_2 x^{-1} + 8 \zeta_2 - 8 \zeta_2 x) + H_{0,-1,1} (576 \zeta_2 x^{-1} - 112 \zeta_2} \\
\text{+ 1168 \zeta_2 x) + H_{0,-1,0} (42214/9 + 1904/25 x^{-2} + 112/5 x^{-1} + 800 x_m^{-1}} \\
\text{+ 6088/9 x_p^{-1} + 63946/45 x + 1776 x^2 - 11808/25 x^3 - 864 \zeta_2 x^{-1} - 376 \zeta_2 x_p^{-1}} \\
\text{+ 292 \zeta_2 + 844 \zeta_2 x) + H_{0,0,-1} (-1048 \zeta_2 x^{-1} - 704 \zeta_2 x_p^{-1} + 396 \zeta_2 + 652 \zeta_2 x) \\
\text{+ H_{0,0,0} (65087/45 - 576/5 x^{-1} - 18371/18 x_m^{-1} - 5080/9 x_p^{-1} - 176803/45 x} \\
\text{+ 5184/5 x^2 + 34704/25 x^3 + 3572/3 \zeta_2 x_m^{-1} + 76 \zeta_2 x_p^{-1} - 884 \zeta_2 x} \\
\text{+ H_{0,0,1} (41267/18 - 288/5 x^{-1} - 23429/9 x_m^{-1} + 152/9 x_p^{-1} + 139937/90 x + 2544 x^2} \\
\text{+ 36208/25 x^3 + 416 \zeta_2 x_m^{-1} + 8 \zeta_2 x_p^{-1} - 508 \zeta_2 + 652 \zeta_2 x) + H_{0,1,0} (12734/15} \\
\text{+ 112/5 x^{-1} - 12331/9 x_m^{-1} + 79654/45 x - 2856/5 x^2 - 976 x^3 - 104 \zeta_2 x^{-1}} \\
\text{- 280 \zeta_2 + 1448 \zeta_2 x) + H_{0,1,1} (47296/45 - 128/5 x^{-1} - 13964/9 x_m^{-1} + 84266/45 x} \\
\text{- 1968/5 x^2 + 264 \zeta_2 x_m^{-1} - 392 \zeta_2 + 904 \zeta_2 x) + H_{1,0,0} (-560 \zeta_2 x_m^{-1} + 88 \zeta_2} \\
\text{+ 1240 \zeta_2 x) + H_{1,0,1} (69457/45 + 16/5 x^{-1} - 17476/9 x_m^{-1} + 3181/15 x + 4344/5 x^2} \\
\text{- 712/3 \zeta_2 x_m^{-1} - 844/3 \zeta_2 + 6356/3 \zeta_2 x) + H_{1,0,1} (6267/5 - 16/5 x^{-1} - 1839 x_m^{-1}} \\
\text{+ 17429/15 x + 4296/5 x^2 + 976 x^3 + 176 \zeta_2 x_m^{-1} - 280 \zeta_2 + 872 \zeta_2 x) \\
\text{+ H_{1,1,0} (32257/45 + 16/5 x^{-1} - 15511/9 x_m^{-1} + 83923/45 x - 1176/5 x^2 - 976 x^3} \\
\text{- 120 \zeta_2 x_m^{-1} - 236 \zeta_2 + 1540 \zeta_2 x) + H_{1,1,1} (6205/9 - 12119/9 x_m^{-1} + 11089/9 x} \\
\text{+ 80 \zeta_2 x_m^{-1} - 232 \zeta_2 + 920 \zeta_2 x) + H_{-1,-1} (16 \zeta_2 x^{-2} + 1600/3 \zeta_2 x_p^{-1}} \\
\text{+ 1600/3 \zeta_2 + 6272/3 \zeta_2 x + 480 \zeta_2 x^2 - 144 \zeta_2 x^3 + 512 \zeta_3 x_p^{-1} - 96 \zeta_3} \\
\text{+ 1056 \zeta_3 x) + H_{-1,0} (4252378/675 + 21512/375 x^{-2} + 1392/25 x^{-1} + 9712/27 x_p^{-1}} \\
\text{+ 4279178/675 x - 15408/25 x^2 - 107496/125 x^3 - 56 \zeta_2 x^{-2} - 896/3 \zeta_2 x_p^{-1}} \\
\text{- 3704/3 \zeta_2 - 3352/3 \zeta_2 + 1008 \zeta_2 x^2 + 504 \zeta_2 x^3 - 256 \zeta_3 x_p^{-1} + 48 \zeta_3} \\
\text{- 528 \zeta_3 x) + H_{0,-1} (-288/5 \zeta_2 x^{-2} - 640 \zeta_2 x_m^{-1} - 640 \zeta_2 x_p^{-1} - 1524 \zeta_2} \\
\text{+ 20 \zeta_2 x - 480 \zeta_2 x^2 + 144 \zeta_3 x^3 - 224 \zeta_3 x_m^{-1} - 560 \zeta_3 x_p^{-1} + 152 \zeta_3 - 728 \zeta_3 x) \\
\text{+ H_{0,0} (-350027/5400 - 2896/25 x^{-1} + 103447/324 x_m^{-1} - 17384/27 x_p^{-1}} \\
\text{- 16956149/1800 x + 26664/25 x^2 + 107496/125 x^3 + 28900/9 \zeta_2 x_m^{-1}} \\
\text{+ 32/3 \zeta_2 x_p^{-1} - 2751 \zeta_2 - 8449/3 \zeta_2 x - 2064 \zeta_2 x^2 - 1560 \zeta_2 x_m^{-3} + 5260/3 \zeta_3 x_m^{-3}} \\
\text{+ 148 \zeta_3 x_m^{-1} - 3230/3 \zeta_3 - 1346/3 \zeta_3 x) + H_{0,1} (1332331/2025 - 512/25 x^{-1}} \\
\text{- 81845/162 x_m^{-1} + 2398349/2025 x - 25672/25 x^2 - 96/5 \zeta_2 x^{-2} + 1080 \zeta_2 x_m^{-1}} \\
\text{- 1634 \zeta_2 + 430 \zeta_2 x - 576 \zeta_2 x^2 - 864/5 \zeta_2 x^3 + 1976/3 \zeta_3 x_m^{-1} - 1532/3 \zeta_3} \\
\text{+ 1348/3 \zeta_3 x) + H_{1,0} (2535097/1620 + 112/5 x^{-1} - 110893/81 x_m^{-1} - 46891/324 x} \\
\text{+ 640 x^2 - 24 \zeta_2 x^2 - 9562/9 \zeta_2 x_m^{-1} - 17192/9 \zeta_2 + 18772/9 \zeta_2 x - 1056 \zeta_2 x^2} 
\end{align*}\]
\[-1056 \zeta_2 x^3 + 2008/3 \zeta_3 x_m^{-1} - 1172/3 \zeta_3 - 164/3 \zeta_4 x) + H_{1,1} (654959/540 \\
+ 128/5 x^{-1} - 26339/27 x_m^{-1} - 159709/540 x - 1968/5 x^2 - 96/5 \zeta_2 x^{-2} \\
+ 820 \zeta_3 x_m^{-1} - 1356 \zeta_2 + 1300 \zeta_2 x - 576 \zeta_2 x^2 - 864/5 \zeta_2 x^3 + 504 \zeta_3 x_m^{-1} \\
- 332 \zeta_3 + 148 \zeta_3 x + \ H_{-1} (-1488/25 \zeta_2 x^{-2} + 32 \zeta_2 x^{-1} - 3280/9 \zeta_2 x^{-1} \\
- 28000/9 \zeta_2 - 27344/9 \zeta_2 x + 1248 \zeta_2 x^2 + 17712/25 \zeta_2 x^3 - 144/5 \zeta_2 x^{-2} \\
- 480 \zeta_3 x_p^{-1} - 784 \zeta_3 - 2032 \zeta_3 x - 96 \zeta_3 x^2 + 1296/5 \zeta_3 x^3 - 516 \zeta_4 x_p^{-1} + 158 \zeta_4 \\
- 758 \zeta_4 x) + \ H_0 (-74841839/81000 + 7768/375 x^{-1} + 162721/324 x_m^{-1} \\
- 928/3 x_p^{-1} - 423561229/81000 x + 83776/125 x^2 + 80 \zeta_2 x^{-1} + 2665 \zeta_2 x_m^{-1} \\
- 404/9 \zeta_2 x_p^{-1} - 41267/18 \zeta_2 - 803/6 \zeta_2 x - 2544/25 \zeta_2 x^2 - 48016/25 \zeta_2 x^3 \\
+ 10856/3 \zeta_3 x_m^{-1} + 2890 \zeta_3 x_p^{-1} - 19552/9 \zeta_3 - 13840/9 \zeta_3 x + 96 \zeta_3 x^2 \\
+ 5232/5 \zeta_3 x^3 - 3287/3 \zeta_4 x_m^{-1} + 300 \zeta_4 x_p^{-1} + 1961/3 \zeta_4 + 2645/3 \zeta_4 x) \\
+ \ H_1 (6570689/4050 + 1472/25 x^{-1} - 239633/648 x_m^{-1} - 16747603/8100 x \\
- 15832/25 x^2 - 496/25 \zeta_2 x^{-2} + 336/5 \zeta_2 x^{-1} + 4525/3 \zeta_2 x_m^{-1} - 37921/15 \zeta_2 \\
+ 24091/15 \zeta_2 x - 9096/5 \zeta_2 x^2 - 30304/25 \zeta_2 x^3 - 88/5 \zeta_3 x^{-2} + 11626/9 \zeta_3 x_m^{-1} \\
- 10424/9 \zeta_3 + 1180/9 \zeta_3 x + 6528/5 \zeta_3 x^3 - 484/3 \zeta_4 x_m^{-1} + 812/3 \zeta_4 \\
- 2608/3 \zeta_4 x) - 79641581/162000 + 35192/375 x^{-1} - 161929/432 x_m^{-1} \\
- 113391919/162000 x + 162936/125 x^2 + 1904/25 \zeta_2 x^{-2} + 52709/162 \zeta_2 x_m^{-1} \\
+ 4856/27 \zeta_2 x_p^{-1} - 1332331/2025 \zeta_2 + 15235883/2025 \zeta_2 x + 25672/25 \zeta_2 x^2 \\
- 107496/125 \zeta_2 x^3 + 232/5 \zeta_3 x^{-1} + 31993/9 \zeta_3 x_m^{-1} + 356 \zeta_3 x_p^{-1} - 42073/30 \zeta_3 \\
- 4933/90 \zeta_3 x - 24/5 \zeta_3 x^2 + 8736/5 \zeta_3 x^3 - 3628/3 \zeta_3 \zeta_2 x_m^{-1} - 52 \zeta_3 \zeta_2 x_p^{-1} \\
+ 842 \zeta_3 \zeta_2 - 10 \zeta_3 \zeta_2 x - 47453/18 \zeta_4 x_m^{-1} + 3488/3 \zeta_4 x_p^{-1} + 69301/36 \zeta_4 \\
+ 144733/36 \zeta_4 x + 1812 \zeta_4 x^2 + 4062/5 \zeta_4 x^3 + 44/3 \zeta_5 x_m^{-1} + 508 \zeta_5 x_p^{-1} \\
- 448 \zeta_5 + 1660 \zeta_5 x + 203 \zeta_5 (1 - x) (-61555/128 + 212833/324 \zeta_2 x - 707539/162 \zeta_3 \\
+ 123449/36 \zeta_4 + 7322/9 \zeta_3 \zeta_2 + 1394/3 \zeta_5 + 976/3 \zeta_3^2 + 1124/3 \zeta_6) \]

and

\[
\zeta_{2,n}^{(4)} F (x) = \\
+ H_{0,0,0,0} (-119 + 238 x_m^{-1} - 119 x) + H_{0,0,0,1} (-96 + 192 x_m^{-1} - 96 x) + H_{0,0,1,0} (-72 \\
+ 144 x_m^{-1} - 72 x) + H_{0,0,1,1} (-72 + 144 x_m^{-1} - 72 x) + H_{0,1,0,0} (-48 + 96 x_m^{-1} - 48 x) \\
+ H_{0,1,0,1} (-48 + 96 x_m^{-1} - 48 x) + H_{0,1,1,0} (-48 + 96 x_m^{-1} - 48 x) + H_{0,1,1,1} (-48 \\
+ 96 x_m^{-1} - 48 x) + H_{1,0,0,0} (-24 + 48 x_m^{-1} - 24 x) + H_{1,0,0,1} (-24 + 48 x_m^{-1} - 24 x) \\
+ H_{1,0,1,0} (-24 + 48 x_m^{-1} - 24 x) + H_{1,0,1,1} (-24 + 48 x_m^{-1} - 24 x) + H_{1,1,0,0} (-24 \\
+ 48 x_m^{-1} - 24 x) + H_{1,1,0,1} (-24 + 48 x_m^{-1} - 24 x) + H_{1,1,1,0} (-24 + 48 x_m^{-1} - 24 x) \\
+ H_{1,1,1,1} (-24 + 48 x_m^{-1} - 24 x) + H_{0,0,0} (-985/3 + 1706/3 x_m^{-1} - 1621/3 x) \\
+ H_{0,0,1} (-240 + 420 x_m^{-1} - 408 x) + H_{0,1,0} (-152 + 268 x_m^{-1} - 272 x) + H_{0,1,1} (-152 \\
- 34 -
\[+ 268 x_m^{-1} - 272 x + H_{1,0,0} (-64 + 116 x_m^{-1} - 136 x) + H_{1,0,1} (-64 + 116 x_m^{-1} - 136 x) + H_{1,1,0} (-64 + 116 x_m^{-1} - 136 x) + H_{1,1,1} (-64 + 116 x_m^{-1} - 136 x) + H_{0,0,0} (-1130/3 + 2096/3 x_m^{-1} - 1116 x - 192 \zeta_2 x_m^{-1} + 96 \zeta_2 + 96 \zeta_2 x) + H_{0,0,1} (-632/3 + 1282/3 x_m^{-1} - 2240/3 x - 96 \zeta_2 x_m^{-1} + 48 \zeta_2 + 48 \zeta_2 x) + H_{1,0} (-148/3 + 470/3 x_m^{-1} - 1120/3 x - 48 \zeta_2 x_m^{-1} + 24 \zeta_2 + 24 \zeta_2 x) + H_{1,1} (-148/3 + 470/3 x_m^{-1} - 1120/3 x - 48 \zeta_2 x_m^{-1} + 24 \zeta_2 + 24 \zeta_2 x) + H_{0} (-4474/27 + 14321/27 x_m^{-1} - 37000/27 x - 420 \zeta_2 x_m^{-1} + 240 \zeta_2 + 408 \zeta_2 x - 44 \zeta_3 x_m^{-1} + 22 \zeta_3 + 22 \zeta_3 x) + H_{1} (1300/27 + 4429/27 x_m^{-1} - 18518/27 x - 116 \zeta_2 x_m^{-1} + 64 \zeta_2 + 136 \zeta_2 x) + 14939/81 + 25279/162 x_m^{-1} - 79606/81 x - 1282/3 \zeta_2 x_m^{-1} + 632/3 \zeta_2 + 2240/3 \zeta_2 x - 436/3 \zeta_3 x_m^{-1} + 266/3 \zeta_3 + 386/3 \zeta_4 x + 102 \zeta_4 x_m^{-1} - 51 \zeta_4 (1 + x) + \delta(1 - x) (281971/864 + 14321/27 \zeta_2 - 1/6 \zeta_3 + 4 \zeta_3 \zeta_2 + 1027/6 \zeta_4 + 2 \zeta_5). \quad (5.12)\]

The structure of these expressions reflects the \(N\)-space results presented above. The large-\(n_c\) coefficients (5.6), (5.8), (5.10) and (5.12) include only HPLs with non-negative indices corresponding to the non-alternating harmonic sums in eqs. (4.5), (4.9), (4.11) and (4.16). At the highest weight, \(w = 5\), the \(n_f^2\) coefficients (5.6) and (5.10) do not only appear with factors \(x\) for \(C_L\) and \(x_m^{-1} = 1/(1 - x)\), 1 and \(x\) for \(C_2\), but also include additional fixed combinations of \(x a\) with \(a = -2, -1, 2, 3\). These do not occur with \(w = 5\) HPLs in the \(C_F^2 n_f^2\) coefficients (5.7) and (5.11) corresponding to eqs. (4.6) and (4.12), which however include HPLs with negative indices and terms with \(x_p^{-1} = 1/(1 + x)\).

Up to terms that vanish in the corresponding limit, the large-\(x\) and small-\(x\) behaviour of the coefficient functions is given by plus-distributions,

\[D_n = \left[ \frac{\ln^n (1 - x)}{1 - x} \right]_+, \tag{5.13}\]

the \(\delta\)-function \(\delta(1 - x)\) and powers of the logarithms

\[L_1 = \ln(1 - x), \quad L_0 = \ln x. \tag{5.14}\]

The \(n_f^2\) and \(n_f^3\) coefficients of \(D_n\) and \(\delta(1 - x)\) for \(C_2\) are

\[c_{2,ns}^{(4)}(x) \bigg|_{D_6} = \frac{64}{27} C_F^2 n_f^2, \tag{5.15}\]

\[c_{2,ns}^{(4)}(x) \bigg|_{D_4} = -\frac{44}{9} C_F C_A n_f^2 - \frac{640}{27} C_F^2 n_f^2 + \frac{8}{27} C_F n_f^3, \tag{5.16}\]

\[c_{2,ns}^{(4)}(x) \bigg|_{D_3} = C_F C_A n_f^2 \left( \frac{1540}{27} - \frac{32}{9} \zeta_2 \right) + C_F^2 n_f^2 \left( \frac{24238}{243} - \frac{928}{27} \zeta_2 \right) - \frac{232}{81} C_F n_f^3, \tag{5.17}\]

\[c_{2,ns}^{(4)}(x) \bigg|_{D_2} = C_F C_A n_f^2 \left( -\frac{7403}{27} + \frac{688}{9} \zeta_2 + 16 \zeta_3 \right) + C_F^2 n_f^2 \left( -\frac{52678}{243} + \frac{6104}{27} \zeta_2 + \frac{304}{9} \zeta_3 \right) + C_F n_f^3 \left( \frac{940}{81} - \frac{32}{9} \zeta_2 \right).\]
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{D_1} = C_F C_A n_f^2 \left( \frac{315755}{486} - \frac{9848}{27} \zeta_2 + \frac{64}{5} \zeta_2^2 - \frac{688}{9} \zeta_3 \right) + C_F^2 n_f^2 \left( \frac{239633}{1458} - \frac{50140}{81} \zeta_2 + \frac{1312}{27} \zeta_2^2 - \frac{19304}{81} \zeta_3 \right) + C_F n_f^3 \left( - \frac{17716}{729} + \frac{464}{27} \zeta_2 \right), \quad (5.18) \]
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{D_0} = C_F C_A n_f^2 \left( - \frac{3761509}{5832} + \frac{131878}{243} \zeta_2 - \frac{616}{9} \zeta_2^2 + \frac{6092}{81} \zeta_3 - \frac{400}{9} \zeta_3 \zeta_2 + \frac{1192}{9} \zeta_5 \right) + C_F^2 n_f^2 \left( - \frac{161929}{972} + \frac{385300}{729} \zeta_2 - \frac{19004}{135} \zeta_2^2 + \frac{3812}{9} \zeta_3 \right) - \frac{1376}{27} \zeta_3 \zeta_2 - \frac{64}{9} \zeta_5 \right) + C_F n_f^3 \left( \frac{50558}{2187} - \frac{1880}{81} \zeta_2 + \frac{16}{9} \zeta_2^2 + \frac{80}{81} \zeta_3 \right), \quad (5.19) \]
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{\delta} = C_F C_A n_f^2 \left( - \frac{8268733}{7776} - \frac{2063501}{972} \zeta_2 - \frac{18248}{135} \zeta_2^2 \right) + \frac{17477}{18} \zeta_3 + \frac{284}{9} \zeta_3 \zeta_2 - \frac{604}{9} \zeta_3 \zeta_2^2 + \frac{3394}{27} \zeta_5 + \frac{878}{27} \zeta_6 \right) + C_F^2 n_f^2 \left( - \frac{61555}{288} + \frac{212833}{729} \zeta_2 + \frac{246898}{405} \zeta_2^2 - \frac{1415666}{729} \zeta_3 \right) + \frac{29288}{81} \zeta_3 + \frac{3904}{27} \zeta_3^2 + \frac{5576}{27} \zeta_5 + \frac{4496}{27} \zeta_6 \right) + C_F n_f^3 \left( \frac{281971}{5832} + \frac{57284}{729} \zeta_2 + \frac{4108}{405} \zeta_2^2 - \frac{2}{81} \zeta_3 + \frac{16}{27} \zeta_3 \zeta_2 + \frac{8}{27} \zeta_5 \right). \quad (5.20) \]

The coefficients of \( D_n \) have been obtained from the soft-gluon exponentiation in eqs. (5.6)–(5.9) of ref. [78] for \( n = 2, \ldots, 5 \), and in eq. (A.4) of ref. [79] for \( n = 1 \) and \( n = 0 \); our results in eqs. (5.15)–(5.19) agree with these predictions. On the other hand, the coefficient of \( \delta(1-x) \) in eq. (5.20) is a new result of the present article.

The corresponding terms with powers of \( \ln(1-x) \), which are subleading for \( C_L \) but leading for \( C_2 \) in the threshold limit, are given by
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{L_1^1} = - \frac{64}{27} C_F^2 n_f^2, \quad (5.21) \]
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{L_1^1} = \frac{44}{9} C_F C_A n_f^2 + \frac{1352}{27} C_F^2 n_f^2 - \frac{8}{27} C_F n_f^3, \quad (5.22) \]
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{L_1^1} = C_F C_A n_f^2 \left( - \frac{3652}{27} + \frac{448}{27} \zeta_2 \right) + C_F^2 n_f^2 \left( - \frac{70132}{243} + \frac{224}{27} \zeta_2 \right) + \frac{592}{81} C_F n_f^3, \quad (5.23) \]
\[ c_{2,\text{ns}}^{(4)}(x) \bigg|_{L_1^1} = C_F C_A n_f^2 \left( \frac{32249}{27} - 209 \zeta_2 - 112 \zeta_3 \right) + C_F^2 n_f^2 \left( \frac{122221}{243} - \frac{6800}{27} \zeta_2 + \frac{848}{9} \zeta_3 \right) + C_F n_f^3 \left( - \frac{4432}{81} + \frac{32}{9} \zeta_2 \right), \quad (5.24) \]
\[
\begin{align*}
\left. c_{2,ns}^{(4)}(x) \right|_{L_1} &= C_F C_A n_f^2 \left( -\frac{1120828}{243} + \frac{30136}{27} \zeta_2 - \frac{512}{45} \zeta_2^2 + \frac{6734}{9} \zeta_3 \right) \\
&+ C_F^2 n_f^2 \left( \frac{360863}{729} + \frac{98632}{81} \zeta_2 + \frac{800}{27} \zeta_2^2 - \frac{35056}{81} \zeta_3 \right) \\
&+ C_F n_f^3 \left( \frac{136624}{729} - \frac{1184}{27} \zeta_2 \right), \\
\left. c_{2,ns}^{(4)}(x) \right|_{L_0} &= C_F C_A n_f^2 \left( \frac{36059503}{5832} - \frac{544594}{243} \zeta_2 + \frac{572}{9} \zeta_2^2 - \frac{37004}{81} \zeta_3 \right) \\
&+ \frac{2704}{9} \zeta_3 \zeta_2 - \frac{1912}{9} \zeta_5 \right) + C_F^2 n_f^2 \left( -\frac{706090}{729} - \frac{1170910}{729} \zeta_2 \right) \\
&+ \frac{6496}{45} \zeta_2^2 - \frac{19336}{81} \zeta_3 - \frac{6688}{27} \zeta_3 \zeta_2 + \frac{1504}{9} \zeta_5 \right) \\
&+ C_F n_f^3 \left( -\frac{583016}{2187} + \frac{8864}{81} \zeta_2 - \frac{16}{9} \zeta_2^2 - \frac{128}{81} \zeta_3 \right), \\
\end{align*}
\]

and
\[
\begin{align*}
\left. c_{L,ns}^{(4)}(x) \right|_{L_1} &= \frac{16}{3} C_F^2 n_f^2 , \\
\left. c_{L,ns}^{(4)}(x) \right|_{L_1} &= C_F C_A n_f^2 \left( -\frac{880}{27} + \frac{352}{27} \zeta_2 \right) + C_F^2 n_f^2 \left( -\frac{184}{27} - \frac{704}{27} \zeta_2 \right) \\
&+ \frac{32}{27} C_F n_f^3 , \\
\left. c_{L,ns}^{(4)}(x) \right|_{L_1} &= C_F C_A n_f^2 \left( \frac{3200}{9} - 64 \zeta_2 - \frac{320}{3} \zeta_3 \right) \\
&+ C_F^2 n_f^2 \left( -\frac{15172}{81} + \frac{736}{9} \zeta_2 + 128 \zeta_3 \right) - \frac{304}{27} C_F n_f^3 , \\
\left. c_{L,ns}^{(4)}(x) \right|_{L_1} &= C_F C_A n_f^2 \left( -\frac{35846}{27} + \frac{6592}{27} \zeta_2 + \frac{1216}{45} \zeta_2^2 + \frac{2944}{9} \zeta_3 \right) \\
&+ C_F^2 n_f^2 \left( \frac{494242}{729} - \frac{2032}{27} \zeta_2 + \frac{704}{9} \zeta_2^2 - \frac{12512}{27} \zeta_3 \right) \\
&+ C_F n_f^3 \left( \frac{3248}{81} - \frac{64}{9} \zeta_2 \right) , \\
\left. c_{L,ns}^{(4)}(x) \right|_{L_0} &= C_F C_A n_f^2 \left( \frac{275728}{243} - \frac{12320}{27} \zeta_2 + \frac{704}{15} \zeta_2^2 + \frac{304}{3} \zeta_3 + 256 \zeta_3 \zeta_2 \\
&- \frac{560}{3} \zeta_5 \right) + C_F^2 n_f^2 \left( -\frac{15803}{243} - \frac{1480}{27} \zeta_2 - \frac{17728}{135} \zeta_2^2 + \frac{7984}{81} \zeta_3 \right) \\
&- \frac{896}{3} \zeta_3 \zeta_2 + 160 \zeta_5 \right) + C_F n_f^3 \left( -\frac{39640}{729} + \frac{608}{27} \zeta_2 \right) .
\end{align*}
\]

The coefficients (5.21) and (5.22) have been predicted in ref. [80] up to one number that was determined as $\xi_{\text{DIS}} = 100/3$ a little later [81, 82]; eq. (5.22) provides the first check of this result by a four-loop diagram calculation. Also the coefficient (5.27) has been predicted before [80, 83]; all other results in eqs. (5.21)–(5.31) are new.
The small-$x$ limit of the non-singlet splitting functions and coefficient function is given
by the terms with $x^0 \ln^4 x = x^0 L_0^4$. Their $n_f^2$ and $n_f^3$ contributions read

$$c_{2,ns}^{(4)} (x) \bigg|_{L_0^4} = -\frac{1951}{1620} C_F^2 n_f^2, \quad (5.32)$$

$$c_{2,ns}^{(4)} (x) \bigg|_{L_0^4} = -\frac{1309}{108} C_F C_A n_f^2 - \frac{1190}{243} C_F^2 n_f^2 + \frac{119}{162} C_F n_f^3, \quad (5.33)$$

$$c_{2,ns}^{(4)} (x) \bigg|_{L_0^4} = C_F C_A n_f^2 \left( -\frac{10051}{81} + \frac{110}{9} \zeta_2 \right)$$
$$+ C_F^2 n_f^2 \left( -\frac{3533}{243} + \frac{2296}{81} \zeta_2 \right) + \frac{1442}{243} C_F n_f^3, \quad (5.34)$$

$$c_{2,ns}^{(4)} (x) \bigg|_{L_0^4} = C_F C_A n_f^2 \left( -\frac{386531}{648} + \frac{1654}{9} \zeta_2 - \frac{188}{3} \zeta_3 \right)$$
$$+ C_F^2 n_f^2 \left( -\frac{271195}{2916} + \frac{8474}{81} \zeta_2 + \frac{4948}{27} \zeta_3 \right)$$
$$+ C_F n_f^3 \left( \frac{644}{27} - \frac{64}{9} \zeta_2 \right), \quad (5.35)$$

$$c_{2,ns}^{(4)} (x) \bigg|_{L_0^4} = C_F C_A n_f^2 \left( -\frac{2989295}{1944} + \frac{20702}{27} \zeta_2 - 224 \zeta_3 - \frac{1139}{9} \zeta_4 \right)$$
$$+ C_F^2 n_f^2 \left( -\frac{460601}{1458} + \frac{458}{3} \zeta_2 + 62144 \zeta_3 - \frac{568}{9} \zeta_4 \right)$$
$$+ C_F n_f^3 \left( \frac{39388}{729} - \frac{80}{3} \zeta_2 - \frac{88}{27} \zeta_3 \right), \quad (5.36)$$

$$c_{2,ns}^{(4)} (x) \bigg|_{L_0^4} = C_F C_A n_f^2 \left( -\frac{19112737}{11664} + \frac{30782}{27} \zeta_2 - 328 \zeta_3 + \frac{368}{9} \zeta_3 \zeta_2 \right)$$
$$- \frac{12785}{27} \zeta_4 + \frac{136}{3} \zeta_5 \right) + C_F^2 n_f^2 \left( -\frac{269059}{729} - \frac{3638}{243} \zeta_2 \right)$$
$$+ C_F n_f^3 \left( \frac{90502}{81} \zeta_3 - \frac{5032}{27} \zeta_3 \zeta_2 + \frac{5417}{27} \zeta_4 + \frac{896}{27} \zeta_5 \right)$$
$$+ C_F n_f^3 \left( \frac{110314}{2187} - \frac{2600}{81} \zeta_2 - \frac{680}{81} \zeta_3 + \frac{68}{9} \zeta_4 \right) \quad (5.37)$$

and

$$c_{L,ns}^{(4)} (x) \bigg|_{L_0^4} = -\frac{920}{81} C_F^2 n_f^2, \quad (5.38)$$

$$c_{L,ns}^{(4)} (x) \bigg|_{L_0^4} = -\frac{176}{3} C_F C_A n_f^2 - \frac{5140}{81} C_F^2 n_f^2 + \frac{32}{9} C_F n_f^3, \quad (5.39)$$

$$c_{L,ns}^{(4)} (x) \bigg|_{L_0^4} = C_F C_A n_f^2 \left( -\frac{4064}{9} + \frac{160}{3} \zeta_2 \right)$$
$$+ C_F^2 n_f^2 \left( -\frac{2456}{27} + \frac{80}{3} \zeta_2 \right) + \frac{608}{27} C_F n_f^3, \quad (5.40)$$
As a step towards the determination of the fourth-order QCD coefficient functions which yield the successive approximations shown in figure 5. The pattern seen in this parts of ref. [76].

Finally we have predicted the analytic form of the contributions to the overall next-to-next-to-leading logarithms; also here we find full agreement. Our results for both structure functions make contact, just, with the resummations of logarithmic small-\(x\) expansions — the leading terms of the non-singlet contributions to the structure functions at any physically relevant small values of \(x\).

\[c_{L,ns}(x)\bigg|_{L_0} = C_F C_A n_f^2 \left( -\frac{26422}{27} + \frac{752}{3} \zeta_2 - 64 \zeta_3 \right) + C_F^2 n_f^2 \left( -\frac{2062}{729} + \frac{1040}{27} \zeta_2 + \frac{3632}{27} \zeta_3 \right) + C_F n_f^2 \left( \frac{3248}{81} - \frac{64}{9} \zeta_2 \right). \] (5.41)

Eqs. (5.32) and (5.38) agree with the resummation predictions (4.5) and (4.6) of ref. [84] after using that \(\ln^\ell x\) transforms to \((-1)^\ell \ell! N^{-\ell-1}\). All other small-\(x\) coefficients are new.

In the case of QCD, eqs. (5.32)–(5.41) lead to the numerical small-\(x\) expansions

\[c_{2,ns}(x)\big|_{n_f^2} = -2.1410151 \ln^5 x - 57.187471 \ln^4 x - 358.88192 \ln^3 x - 945.87933 \ln^2 x - 1327.8897 \ln x - 688.88341 + \mathcal{O}(x) \] (5.42)

and

\[c_{L,ns}(x)\big|_{n_f^2} = -20.192044 \ln^3 x - 347.47874 \ln^2 x - 1539.0328 \ln x - 2177.6994 + \mathcal{O}(x) \] (5.43)

which yield the successive approximations shown in figure 5. The pattern seen in this figure is the same seen for the complete third-order non-singlet coefficient functions in the right parts of figure 2 and figure 7 in ref. [13] and for the four- and five-loop resummation predictions in figures 1–3 of ref. [84]: the leading, next-to-leading and next-to-next-to-leading logarithmic small-\(x\) approximations wildly oscillate and thus, in general, cannot be used to obtain any reliable prediction for even the rough shape of high-order coefficient functions at any physically relevant small values of \(x\).

6 Summary and outlook

As a step towards the determination of the fourth-order QCD coefficient functions \(c_{\alpha}(x)\) in inclusive deep-inelastic scattering (DIS), we have computed the double-fermionic \((n_f^2)\) non-singlet contributions to the structure functions \(F_2\) and \(F_L\). Our results are applicable to electromagnetic DIS (where we have ignored the very small [13, 30] contributions in which the photon couples to different quark lines) and to charged-current \((W^+ + W^-)\) exchange. The analytic dependence of these coefficient functions on Mellin-\(N\), and hence Bjorken-\(x\), has been reconstructed from a very large number of even values of \(N\), up to \(N \simeq 1200\).

Our calculations verify the corresponding contributions to the next-to-next-to-leading order splitting function \(F_{ns}^{(4)}(x)\) obtained in ref. [23], and the much simpler (and much smaller) \(C_F n_f^2\) leading large-\(n_f\) contributions to \(c_{L}^{(4)}(x)\) and \(c_2^{(4)}(x)\) [19, 20]. The coefficients of all large-\(x\) plus-distributions in \(C_2\) at this order, \([\ln^{1-x} \ln^\ell(1-x)] \), with \(0 \leq \ell \leq 5\), have been predicted by the soft-gluon exponentiation [78, 79]; we agree with these predictions. Our results for both structure functions make contact, just, with the resummations of \(\ln(1-x)\) and \(\ln x\) double logarithms in refs. [80–84], as the leading terms of the \(n_f^2\) parts contribute to the overall next-to-next-to-leading logarithms; also here we find full agreement. Finally we have predicted the analytic form of the contributions to the Mellin-\(N\) moments of the five-loop splitting function \(F_{ns}^{(4)}(x)\), and verified the corresponding \(\zeta_4 n_f^2\) parts of ref. [76].
Figure 5. Successive small-$x$ approximations for the $n_f^2$ contributions to the fourth-order coefficient functions for $F_{2,ns}$ (left panel) and $F_{L,ns}$ (right panel). The coefficients in eqs. (5.42) and (5.43) have been converted to an expansion in $\alpha_s$.

To compute the large number of moments required for the reconstruction of the all-$N$ coefficient functions, we have developed a new approach based on the method of integration by parts, differential equations and the optical theorem. By employing the knowledge that the Mellin moments of the forward scattering amplitude correspond to the coefficients of the Taylor series around $\omega = 1/x = 0$, we derive a system of recurrence relations which holds for any linear system of differential equations, whose matrix contains no higher-order poles in $\omega$. While such higher poles are generically present, we find a simple algorithm which allows to change the basis of master integrals into a form where only simple poles appear in the differential matrix. The thereby obtained recurrence relations for the master integrals allow to obtain the Mellin moments at, in principle, arbitrary high $N$ from the knowledge of the boundary conditions, which we compute using the FORCER program [18].

In practice, we find that the algorithm starts to become computationally demanding for the problem under consideration at values of $N$ around about 1000. One element which allowed us to speed up the calculation considerably was to convert the basis of master integrals into one where the $\varepsilon$-dependence of the reduction coefficients factorizes from the dependence on $x$. An important feature of the method is the simplicity with which the recurrences can be solved — a procedure which we have implemented in FORM [37–39].

In spirit the method is not dissimilar from the ‘method of arbitrary high moments’ [56], which was used recently to recompute the 3-loop DIS coefficient functions [14]. Their method also derives recurrence relations from differential equations, but differs from our method in several ways, e.g., it relies on more advanced combinatorial algorithms to
construct and recursively solve to high $N$ the differential equations, and is implemented largely in Mathematica. While it is possible that their approach leads to a faster algorithm at very large $N$, it is difficult at this stage to comment on timings without explicit performance benchmarks.

We plan to present the $n_f^2$ fourth-order contributions to the structure function $F_3$ in $(W^+ + W^-)$ charged-current DIS in a forthcoming publication. The computation of its 'standard' $C_F C_A n_f^2$ and $C_F^2 n_f^2$ parts is not much more difficult than that presented in this article, but some additional contributions with the group invariant $d^{abc} d_{abc}$, for the third-order results see refs. [14, 85, 86], prove to be more challenging. We have also started to work on a first set of $n_f^1$ contributions, the $C_F^3 n_f$ terms, where we hope to be able to provide the first exact results of $n_f^1$ parts of $F_{ns}^{+(3)}(x)$ beyond the large-$n_c$ limit of ref. [87].

FORM files with our results can be obtained from the supplementary material attached to this paper. They are also available from the authors upon request. These files include also the analytic $N$-dependence of the third-order quantities $a_{2,ns}^{(n)}(N)$ and $a_{L,ns}^{(n)}(N)$ in eq. (2.8) which we have not included in section 4 above.

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References

[1] Particle Data Group collaboration, Review of Particle Physics, PTEP 2022 (2022) 083C01 [arXiv:2204.08012] [SPIRE].

[2] A. Accardi et al., Electron Ion Collider: The Next QCD Frontier: Understanding the glue that binds us all, Eur. Phys. J. A 52 (2016) 268 [arXiv:1212.1701] [SPIRE].

[3] R. Abdul Khalek et al., Science Requirements and Detector Concepts for the Electron-Ion Collider: EIC Yellow Report, Nucl. Phys. A 1026 (2022) 122447 [arXiv:2103.05419] [SPIRE].

[4] LHec Study Group collaboration, A Large Hadron Electron Collider at CERN: Report on the Physics and Design Concepts for Machine and Detector, J. Phys. G 39 (2012) 075001 [arXiv:1206.2913] [SPIRE].

[5] LHec and FCC-he Study Group collaborations, The Large Hadron-Electron Collider at the HL-LHC, J. Phys. G 48 (2021) 110501 [arXiv:2007.14491] [SPIRE].
[6] P. Bolzoni, F. Maltoni, S.-O. Moch and M. Zaro, Higgs production via vector-boson fusion at NNLO in QCD, *Phys. Rev. Lett.* 105 (2010) 011801 [arXiv:1003.4451] [inSPIRE].

[7] F.A. Dreyer and A. Karlberg, Vector-Boson Fusion Higgs Production at Three Loops in QCD, *Phys. Rev. Lett.* 117 (2016) 072001 [arXiv:1606.00840] [inSPIRE].

[8] J. Currie et al., $N^3$LO corrections to jet production in deep inelastic scattering using the Projection-to-Born method, *JHEP* 05 (2018) 209 [arXiv:1803.09973] [inSPIRE].

[9] J. Sanchez Guillen et al., Next-to-leading order analysis of the deep inelastic $R = \sigma_L/\sigma_T$, *Nucl. Phys. B* 353 (1991) 337 [inSPIRE].

[10] W.L. van Neerven and E.B. Zijlstra, Order $\alpha^2$ contributions to the deep inelastic Wilson coefficient, *Phys. Lett. B* 272 (1991) 127 [inSPIRE].

[11] E.B. Zijlstra and W.L. van Neerven, Order $\alpha^2$ QCD corrections to the deep inelastic proton structure functions $F_2$ and $F_L$, *Nucl. Phys. B* 383 (1992) 525 [inSPIRE].

[12] S. Moch and J.A.M. Vermaseren, Deep inelastic structure functions at two loops, *Nucl. Phys. B* 573 (2000) 853 [hep-ph/9912355] [inSPIRE].

[13] J.A.M. Vermaseren, A. Vogt and S. Moch, The Third-order QCD corrections to deep-inelastic scattering by photon exchange, *Nucl. Phys. B* 724 (2005) 3 [hep-ph/0504242] [inSPIRE].

[14] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, The massless three-loop Wilson coefficients for the deep-inelastic structure functions $F_2$, $F_L$, $xF_3$ and $g_1$, *JHEP* 11 (2022) 156 [arXiv:2208.14325] [inSPIRE].

[15] B. Ruijl et al., First Forcer results on deep-inelastic scattering and related quantities, *PoS LL2016* (2016) 071 [arXiv:1605.08408] [inSPIRE].

[16] S.-O. Moch et al., DIS coefficient functions at four loops in QCD and beyond, *PoS LL2022* (2022) 047 [arXiv:2208.11067] [inSPIRE].

[17] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, to appear.

[18] B. Ruijl, T. Ueda and J.A.M. Vermaseren, Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams, *Comput. Phys. Commun.* 253 (2020) 107198 [arXiv:1704.06650] [inSPIRE].

[19] J.A. Gracey, Large N(f) methods for computing the perturbative structure of deep inelastic scattering, in the proceedings of 4th International Workshop on Software Engineering and Artificial Intelligence for High-energy and Nuclear Physics, Pisa Italy, April 3-8 1995 [hep-ph/9509276] [inSPIRE].

[20] L. Mankiewicz, M. Maul and E. Stein, Perturbative part of the nonsinglet structure function $F_2$ in the large-Nf limit, *Phys. Lett. B* 404 (1997) 345 [hep-ph/9703356] [inSPIRE].

[21] J.A.M. Vermaseren, Harmonic sums, Mellin transforms and integrals, *Int. J. Mod. Phys. A* 14 (1999) 2037 [hep-ph/9806280] [inSPIRE].

[22] E. Remiddi and J.A.M. Vermaseren, Harmonic polylogarithms, *Int. J. Mod. Phys. A* 15 (2000) 725 [hep-ph/9905237] [inSPIRE].

[23] J. Davies et al., Large-nf contributions to the four-loop splitting functions in QCD, *Nucl. Phys. B* 915 (2017) 335 [arXiv:1610.07477] [inSPIRE].

[24] A.V. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, *Phys. Lett. B* 254 (1991) 158 [inSPIRE].
[25] A.V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, Phys. Lett. B 267 (1991) 123 [insPIRE].

[26] E. Remiddi, *Differential equations for Feynman graph amplitudes*, Nuovo Cim. A 110 (1997) 1435 [hep-th/9711188] [insPIRE].

[27] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, Nucl. Phys. B 580 (2000) 485 [hep-ph/9912329] [insPIRE].

[28] W.L. van Neerven and A. Vogt, *NNLO evolution of deep inelastic structure functions: The Singlet case*, Nucl. Phys. B 588 (2000) 345 [hep-ph/0006154] [insPIRE].

[29] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, *The Next next-to-leading QCD approximation for nonsinglet moments of deep inelastic structure functions*, Nucl. Phys. B 427 (1994) 41 [insPIRE].

[30] S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, *The Three loop QCD calculation of the moments of deep inelastic structure functions*, Nucl. Phys. B 492 (1997) 338 [hep-ph/9605317] [insPIRE].

[31] S. Moch and M. Rogal, *Charged current deep-inelastic scattering at three loops*, Nucl. Phys. B 782 (2007) 51 [arXiv:0704.1740] [insPIRE].

[32] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, *The Gell-Mann-Low Function of QCD in the Three Loop Approximation*, Phys. Lett. B 93 (1980) 429 [insPIRE].

[33] S.A. Larin and J.A.M. Vermaseren, *The Three loop QCD Beta function and anomalous dimensions*, Phys. Lett. B 303 (1993) 334 [hep-ph/9302208] [insPIRE].

[34] S. Moch, J.A.M. Vermaseren and A. Vogt, *Three-loop results for quark and gluon form-factors*, Phys. Lett. B 625 (2005) 245 [hep-ph/0508055] [insPIRE].

[35] S. Moch, J.A.M. Vermaseren and A. Vogt, *Nonsinglet structure functions at three loops: Fermionic contributions*, Nucl. Phys. B 646 (2002) 181 [hep-ph/0209100] [insPIRE].

[36] P. Nogueira, *Automatic Feynman graph generation*, J. Comput. Phys. 105 (1993) 279 [insPIRE].

[37] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025 [insPIRE].

[38] J. Kuipers, T. Ueda, J.A.M. Vermaseren and J. Vollinga, *FORM version 4.0*, Comput. Phys. Commun. 184 (2013) 1453 [arXiv:1203.6543] [insPIRE].

[39] B. Ruijl, T. Ueda and J. Vermaseren, *FORM version 4.2*, arXiv:1707.06453 [insPIRE].

[40] J.A.M. Vermaseren, *The minos database facility*, https://www.nikhef.nl/~form/maindir/others/minos/minos.html.

[41] S.G. Gorishnii, S.A. Larin, L.R. Surguladze and F.V. Tkachov, *Mincer: Program for Multiloop Calculations in Quantum Field Theory for the Schoonschip System*, Comput. Phys. Commun. 55 (1989) 381 [insPIRE].

[42] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, *The FORM version of MINCER*, NIKHEF-H-91-18 (1991) [insPIRE].

[43] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, Int. J. Mod. Phys. A 15 (2000) 5087 [hep-ph/0102033] [insPIRE].

[44] A.V. Smirnov and F.S. Chuharev, *FIRE6: Feynman Integral REDuction with Modular Arithmetic*, Comput. Phys. Commun. 247 (2020) 106877 [arXiv:1901.07808] [insPIRE].
[45] P. Maierhöfer, J. Usovitsch and P. Uwer, Kira — A Feynman integral reduction program, Comput. Phys. Commun. 230 (2018) 99 [arXiv:1705.05610] [nSPIRE].

[46] R. Boughezal, M. Czakon and T. Schutzmeier, NNLO fermionic corrections to the charm quark mass dependent matrix elements in $B \to X_s \gamma$, JHEP 09 (2007) 072 [arXiv:0707.3090] [nSPIRE].

[47] R.N. Lee, A.V. Smirnov and V.A. Smirnov, Solving differential equations for Feynman integrals by expansions near singular points, JHEP 03 (2018) 008 [arXiv:1705.05610] [nSPIRE].

[48] R. Boughezal, M. Czakon and T. Schutzmeier, NNLO fermionic corrections to the charm quark mass dependent matrix elements in $\bar{B} \to X_s \gamma$, JHEP 09 (2007) 072 [arXiv:0707.3090] [nSPIRE].

[49] R.N. Lee, A.V. Smirnov and V.A. Smirnov, Solving differential equations for Feynman integrals by expansions near singular points, JHEP 03 (2018) 008 [arXiv:1705.05610] [nSPIRE].

[50] R.N. Lee, A.V. Smirnov and V.A. Smirnov, Solving differential equations for Feynman integrals by expansions near singular points, JHEP 03 (2018) 008 [arXiv:1705.05610] [nSPIRE].

[51] I. Dubovyk et al., Evaluation of multiloop multiscale Feynman integrals for precision physics, Phys. Rev. D 106 (2022) L111301 [arXiv:2201.02576] [nSPIRE].

[52] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, Singlet and nonsinglet three-loop massive form factors, Phys. Rev. D 106 (2022) 034029 [arXiv:2207.00027] [nSPIRE].

[53] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, Singlet and nonsinglet three-loop massive form factors, Phys. Rev. D 106 (2022) 034029 [arXiv:2207.00027] [nSPIRE].

[54] X. Liu and Y.-Q. Ma, AMFlow: A Mathematica package for Feynman integrals computation via auxiliary mass flow, Comput. Phys. Commun. 283 (2023) 108565 [arXiv:2201.11669] [nSPIRE].

[55] T. Armadillo et al., Evaluation of Feynman integrals with arbitrary complex masses via series expansions, Comput. Phys. Commun. 282 (2023) 108545 [arXiv:2205.03345] [nSPIRE].

[56] J. Blümlein and C. Schneider, The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory, Phys. Lett. B 771 (2017) 31 [arXiv:1701.04614] [nSPIRE].

[57] J. Blümlein and C. Schneider, The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory, Phys. Lett. B 771 (2017) 31 [arXiv:1701.04614] [nSPIRE].

[58] J. Blümlein and C. Schneider, The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory, Phys. Lett. B 771 (2017) 31 [arXiv:1701.04614] [nSPIRE].

[59] J. Moser, The order of a singularity in Fuchs’ theory, Math. Z. 72 (1959) 379.

[60] J.M. Henn, Multiloop integrals in dimensional regularization made simple, Phys. Rev. Lett. 110 (2013) 251601 [arXiv:1304.1806] [nSPIRE].

[61] J. Ablinger et al., Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable, Nucl. Phys. B 939 (2019) 253 [arXiv:1810.12261] [nSPIRE].

[62] J. Ablinger et al., Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable, Nucl. Phys. B 939 (2019) 253 [arXiv:1810.12261] [nSPIRE].

[63] M. Prausa, epsilon: A tool to find a canonical basis of master integrals, Comput. Phys. Commun. 219 (2017) 361 [arXiv:1701.00725] [nSPIRE].
[64] C. Sabbah, *Lieu des pôles d’un système holonome d’équations aux différences finies*, Bull. Soc. Math. Fr. 120 (1992) 371.

[65] A.V. Smirnov and V.A. Smirnov, *How to choose master integrals*, Nucl. Phys. B 960 (2020) 115213 [arXiv:2002.08042] [SPIRE].

[66] J. Usovitsch, *Factorization of denominators in integration-by-parts reductions*, arXiv:2002.08173 [SPIRE].

[67] M. Jamin and R. Miravitllas, *Absence of even-integer $\zeta$-function values in Euclidean physical quantities in QCD*, Phys. Lett. B 779 (2018) 452 [arXiv:1711.00787] [SPIRE].

[68] P.A. Baikov and K.G. Chetyrkin, *The structure of generic anomalous dimensions and no-$\pi$ theorem for massless propagators*, JHEP 06 (2018) 141 [arXiv:1804.10088] [SPIRE].

[69] J. Davies and A. Vogt, *Absence of $\pi^2$ terms in physical anomalous dimensions in DIS: Verification and resulting predictions*, Phys. Lett. B 776 (2018) 189 [arXiv:1711.05267] [SPIRE].

[70] W.L. van Neerven and A. Vogt, *Nonsinglet structure functions beyond the next-to-next-to-leading order*, Nucl. Phys. B 603 (2001) 42 [hep-ph/0103123] [SPIRE].

[71] S. Moch, J.A.M. Vermaseren and A. Vogt, *Higher-order corrections in threshold resummation*, Nucl. Phys. B 726 (2005) 317 [hep-ph/0506288] [SPIRE].
[83] S. Moch and A. Vogt, Threshold Resummation of the Structure Function $F_L$, *JHEP* 04 (2009) 081 [arXiv:0902.2342] [inSPIRE].

[84] J. Davies, C.-H. Kom, S. Moch and A. Vogt, Resummation of small-$x$ double logarithms in QCD: inclusive deep-inelastic scattering, *JHEP* 08 (2022) 135 [arXiv:2202.10362] [inSPIRE].

[85] A. Retey and J.A.M. Vermaseren, Some higher moments of deep inelastic structure functions at next-to-next-to-leading order of perturbative QCD, *Nucl. Phys. B* 604 (2001) 281 [hep-ph/0007294] [inSPIRE].

[86] S. Moch, J.A.M. Vermaseren and A. Vogt, Third-order QCD corrections to the charged-current structure function $F_3$, *Nucl. Phys. B* 813 (2009) 220 [arXiv:0812.4168] [inSPIRE].

[87] S. Moch et al., Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond, *JHEP* 10 (2017) 041 [arXiv:1707.08315] [inSPIRE].