Quantization of a Theory of 2d Dilaton Gravity

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ABSTRACT

We discuss the quantization of the 2d gravity theory of Callan, Giddings, Harvey, and Strominger (CGHS), following the procedure of David, and of Distler and Kawai. We find that the physics depends crucially on whether the number of matter fields is greater than or less than 24. In the latter case the singularity pointed out by several authors is absent but the physical interpretation is unclear. In the former case (the one studied by CGHS) the quantum theory which gives CGHS in the linear dilaton semi-classical limit, is different from that which gives CGHS in the extreme Liouville regime.

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Recently Callan, Giddings, Harvey, and Strominger [1] (CGHS), discussed a model for two dimensional (dilaton) gravity coupled to matter. They showed that classically the theory has solutions corresponding to collapsing matter forming a black hole. This solution is in fact a linear dilaton flat metric one, patched together with Witten’s [7] 2d black hole solution, along the infall line of a shock wave of 2d massless matter. In order to incorporate the quantum effects (in lowest order) CGHS included the contribution of the conformal anomaly coming from the conformally non-invariant measure in the matter sector path integral.

In this paper we examine the consistency of this procedure. It is argued that one way of carrying out the quantization of the theory is to follow the procedure of David, and of Distler and Kawai [2]. Then we rediscover the singularity pointed out in [4, 5] when the number $N$ of matter fields is greater than 24, and furthermore we find that the quantum theory which leads to the CGHS action in the semi-classical linear dilaton region is different from the the one which gives the CGHS action in the extreme Liouville region. For $N < 24$ there is no field space singularity but it seems to lead to an unphysical theory with a negative flux of black hole radiation. The classical CGHS action is

$$S = \frac{1}{4\pi} \int d^2\sigma \sqrt{-g} [e^{-2\phi}(R + 4(\nabla \phi)^2 - 4\lambda^2) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f^i)^2]. \quad (1)$$

where $\phi$ is the dilaton and $f^i$ are $N$ (unitary) matter fields. The corresponding quantum field theory is defined by

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† Similar methods have been used in [3]. However these works do not discuss the particular conclusions for the CGHS theory which is our main focus here. I wish to thank Dr. Chamseddine for bringing these references to my attention after an earlier version of this paper had been circulated.
‡ we use MTW[6] conventions
§ This Lagrangian comes from the low energy limit of string theory.
The measures in the above path integral are derived from the metrics,

\[ \frac{d\rho}{g} = \int d^2\sigma \sqrt{-\hat{g}} \delta \rho^2, \quad \frac{d\phi}{g} = \int d^2\sigma \sqrt{-\hat{g}} \delta \phi^2, \quad \frac{df}{g} = \int d^2\sigma \sqrt{-\hat{g}} \delta f^i \delta f^j. \tag{3} \]

To evaluate the path integral one needs to gauge fix it. We choose the conformal gauge \( g = e^{2\rho} \hat{g} \), where \( \hat{g} \) is a fiducial metric. Then the path integral becomes,

\[ Z = \int (d\rho)(d\phi) e^{iS(\phi, \rho)} \Delta_f(e^{2\rho} \hat{g}) \Delta_{FP}(e^{2\rho} \hat{g}), \tag{4} \]

where \( S(\phi, \rho) \) is the pure graviton-dilaton part of (1), the last factor is the Fadeev-Popov ghost determinant, and

\[ \Delta_f(e^{2\rho} \hat{g}) = \int [df] e^{iS(f)}, \tag{5} \]

\( S(f) \) being the matter action.

The measures in (4), (5), are again given by (3) except that we must put \( g = e^{2\rho} \hat{g} \). In particular we have (up to a constant)

\[ \frac{d\rho}{g} = \int d^2\sigma \sqrt{-\hat{g}} \delta \rho^2. \tag{6} \]

From the well known transformation properties [8] of the matter and ghost determinants,
\[
\Delta_f(e^{2\rho\hat{g}})\Delta_{F.P.}(e^{2\rho\hat{g}}) = \Delta_f(\hat{g})\Delta_{F.P.}(\hat{g}) \exp i\left[\frac{N - 26}{6} S_L(\rho, \hat{g}) + \mu \int d^2\sigma e^{2\rho} \sqrt{-\hat{g}}\right],
\]

where
\[
S_L(\rho, \hat{g}) = \frac{1}{4\pi} \int d^2\sigma \sqrt{-\hat{g}}((\hat{\nabla}\rho)^2 + \hat{R}\rho)
\]

The quantum theory is then given by
\[
Z = \int ([d\rho][d\phi]) e^{2\rho\hat{g}} [df][\hat{g}] [db][dc] e^{iS(\rho, \phi, f, \hat{g}) + iS(b, c, \hat{g})}.
\]

In the above equation \(S(b, c, \hat{g})\) is the ghost action and
\[
S(\rho, \phi, f, \hat{g}) = \frac{1}{4\pi} \int d^2\sigma \sqrt{-\hat{g}} e^{-2\phi}(4(\hat{\nabla}\phi)^2 - 4\hat{\nabla}\phi.\hat{\nabla}\rho) - \kappa\hat{\nabla}\rho.\hat{\nabla}\rho
- \frac{1}{2} \sum_{i=1}^{N} \hat{\nabla} f^i \hat{\nabla} f^i + \hat{R}(e^{-2\phi} - \kappa\rho) - 4\lambda^2 e^{2(\rho - \phi)}
\]

where \(\kappa = \frac{26 - N}{6}\). For \(\hat{g} = \eta\) the Minkowski metric, this reduces to the CGHS action with conformal anomaly term.

There is however something strange about the path integral (9). The measures for matter and ghost are defined relative to the fiducial metric \(\hat{g}\) while the \(\rho\) and \(\phi\) measures are still defined in terms of the original metric \(g = e^{2\rho}\hat{g}\). In particular this means that the \(\rho\) measure is not translationally invariant, and therefore that for example the (Dyson-Schwinger) quantum equation of motion gets modified from the equation derived from (10). In order to formulate the quantum theory in a

\[\text{equation (23) of [1] except that the ghost contribution is ignored there.}\]
manner which yields a systematic semiclassical (or $1/N$) expansion it is necessary to rewrite all measures in terms of the fiducial metric $\hat{g}$. Thus we need to do what David and Distler and Kawai [2] did for coformal field theory coupled to $2d$ gravity.

Assume (as in [2]) that the jacobian which arises in transforming to the measures defined in terms of $\hat{g}$ is of the form $e^{iJ}$ where $J$ is a local renormalizable action in $\rho$ and $\phi$. Putting $X^\mu = (\phi, \rho)$ we may write,

$$Z = \int [dX^\mu]_\hat{g} [df]_\hat{g}([db][dc])_\hat{g} e^{iI(X,\hat{g})+iS(f,\hat{g})+iS(b,c,\hat{g})}$$

(11)

where

$$I[X, \hat{g}] = -\frac{1}{4\pi} \int \sqrt{-\hat{g}} [\frac{1}{2} \hat{g}^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \hat{R} \Phi(X) + T(X)].$$

(12)

In the above $G_{\mu\nu}, \Phi$ and $T$ are functions of $X$ which are to be determined and the measure $[dX^\mu]$ is derived from the natural metric on the space $||\delta X_\mu||^2 = \int d^2 \sigma \sqrt{-\hat{g}} G_{\mu\nu} \delta X^\mu \delta X^\nu$.

The only a priori restriction on the functions $G, \Phi$, and $T$, come from the fact that $Z$ must be independent of the fiducial metric $\hat{g}$, i.e. the theory defined by the action $I + S_f + S_{b,c}$ with the standard translationally invariant measures is a conformal field theory with zero central charge. So we must satisfy the $\beta$-function equations,

$$\beta_{\mu\nu} = \mathcal{R}_{\mu\nu} + 2 \nabla_\mu G \partial_\nu \Phi - \partial_\mu T \partial_\nu T + \ldots$$

$$\beta_\Phi = -\mathcal{R} + 4 G_{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 4 \nabla_\mu G \Phi + \frac{(N + 2) - 26}{6} + G_{\mu\nu} \partial_\mu T \partial_\nu T - 2T^2 + \ldots$$

$$\beta_T = -2 \nabla^2 G \Phi + 4 G_{\mu\nu} \partial_\mu \Phi \partial_\nu T - 4T + \ldots$$

(13)

In the above $\mathcal{R}$ is the curvature of the metric $G$. These conditions are not sufficient to determine the functions uniquely, but clearly they are necessary. If no further restrictions are imposed, they define a class of quantum $2d$ dilaton-graviton
theories. The analysis of CGHS and others [1,4,5] will be valid provided that the functions $G, \Phi, \text{and } T$, defined by (10) satisfy (13) at least in the semiclassical regime. Because of what happens in the corresponding case studied in [2] we will make $\kappa$ (see (10)) a parameter to be determined by (13). Comparing (12) with (10) we have,

$$G_{\phi\phi} = -8e^{-2\phi}, \quad G_{\phi\rho} = 4e^{-2\phi}, \quad G_{\rho\rho} = 2\kappa,$$

(14)

$$\Phi = -e^{-2\phi} + \kappa \rho, \quad T = -4\lambda^2 e^{2(\rho - \phi)}.$$  

(15)

It is easy to see that the curvature $\mathcal{R} = 0$. So we may transform to a field space coordinate system which is Euclidean (or Minkowski). The transformation

$$\rho = \kappa^{-1} e^{-2\phi} + y$$

(16)

gives for the metric in field space

$$ds^2 = -8e^{-2\phi}(d\phi^2 - d\phi d\rho) + 2\kappa d\rho^2$$

$$= -\frac{8}{\kappa} e^{-4\phi}(1 + \kappa e^{2\phi}) d\phi^2 + 2\kappa dy^2$$

(17)

In the latter form we see (for $\kappa < 0$) the singularity pointed out in [4, 5]. Now let us introduce a field space coordinate

$$x = \int e^{-2\phi}(1 + \kappa e^{2\phi})^{\frac{1}{2}}. $$

(18)

Note that if $\kappa < 0$ $x$ is real only in the ”linear dilaton” region $\kappa e^{2\phi} < 1$. It is also convenient to introduce two more coordinates
\[ X = 2\sqrt{\frac{2}{|\kappa|}} x, \quad Y = \sqrt{2|\kappa|} y. \]

Then we have,

\[ ds^2 = -\frac{8}{\kappa} dx^2 + 2\kappa dy^2 = \mp dX^2 \pm dY, \]  

(19)

where the upper or lower signs are to be taken depending on whether \( \kappa \) is positive or negative respectively.* In the Liouville region \( \frac{e^{-2\phi}}{|\kappa|} \), we define the coordinate,

\[ \bar{X} = 2\sqrt{2} \int dx e^{-x} \left(1 + \frac{e^{-2x}}{\kappa}\right)^{\frac{1}{2}} dx \]  

(20)

we get

\[ ds^2 = -d\bar{X}^2 \pm dY^2 \]  

(21)

As before the upper or lower signs are to be taken depending on whether \( \kappa \) is positive or negative. Note that \( X \) is real in the linear dilaton region and imaginary in the Liouville region while the converse is true for \( \bar{X} \). From (15) and the above we also have the form of the dilaton in the new coordinates,

\[ \Phi = \kappa y = \pm \sqrt{\frac{\pm\kappa}{2}} Y \]  

(22)

Thus in these new coordinates we have a Euclidean (Minkowski) metric linear dilaton theory in field space, and the first beta function equation (13) is satisfied if we ignore quadratic terms in \( T \). From the second equation in (13) we then get

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* One may consider the coordinates \( x, y \) (or \( X, Y \)) as the field space analog of Kruskal-Szekeres coordinates! Of course the physical interpretation in terms of the original physical coordinates is valid only outside the field space coordinate singularity.
Thus the metric signature on field space as well as the absence or presence of a singularity depends on whether $N < 24$ or $N > 24$. As is well known the linear dilaton Euclidean (Minkowski) metric theory is an exact solution of the beta function equations, i.e. our solution (with $T=0$) exactly satisfies the sufficiency criterion discussed in the sentence before (13).

Let us now discuss the tachyon $T$. We do not know how to incorporate the contribution of the tachyon exactly. All we can do is to work to linear order in $T$. Thus our discussion is valid only for $\lambda^2 << e^{2\phi}$. By going to the $x,y$ coordinate system (18),(19), we can solve the tachyon equation exactly. In these coordinates we have,

$$\frac{\kappa}{4} \partial_x^2 T - \frac{1}{k} \partial_y^2 T + 2\partial_y T - 4T = 0.$$  \hspace{1cm} (24)

This has solutions of the form $e^{\beta x + \alpha y}$, with $\frac{\kappa}{4} \beta^2 - \frac{1}{k} \alpha^2 + 2\alpha - 4 = 0$. Now we have to impose the boundary condition that this solution goes over to the CGHS form given in (15) in the semi-classical limit appropriate to the linear dilaton regime $e^{-2\phi} >> |\kappa|$. Using the expansion of (18) and the expression for $y$ (16), we find

$$T = -4\lambda^2 e^{-\frac{4}{\kappa} x + 2y} = -4\lambda^2 e^{2\rho -2\phi} h(\kappa e^{2\phi}),$$  \hspace{1cm} (25)

where $h(\kappa e^{2\phi}) = 1 + O(\kappa e^{2\phi})$ and indeed can be written out exactly.

Now let us discuss the theory in the "Liouville region" $\kappa > e^{-2\phi}$. The appropriate coordinates are $\bar{X}, \bar{Y}$ defined in (20). Solving the tachyon equation in these coordinates and imposing the boundary condition that the CGHS expression (15) is reproduced in the extreme Liouville regime $\kappa >> e^{-2\phi} >> 1$ we find

$\dagger$ This is valid for $\kappa < 0$. For $\kappa < 0$ there is a similar expression with $\cos \to \cosh$.  

\hspace{1cm} $\dagger$
\[ T = -\frac{\lambda^2}{2}\kappa \left[ \cos(X)e^{\sqrt{2|\kappa|Y}} - e^{\alpha_+Y} \right] \]  

(26)

where \( \alpha_+ = -\frac{1}{2}\sqrt{-2\kappa} + \frac{1}{2}\sqrt{-2\kappa + 8} \). By using the transformations (20) in the large \(|\kappa|\) limit it is easily seen that \( T \) goes over to the expression given in (15).

For \( \kappa < 0 \), as in the case studied by DKD the semi-classical expression for \( T \) in the Liouville region is obtained in the limit \( N \to -\infty \).

The solution (26) is obviously quite different from the solution which goes over to the CGHS value in the extreme linear dilaton region (25). What we have found is that (for \( \kappa < 0 \)) we cannot have a quantum theory which has the CGHS theory as its semi-classical limit in both the extreme linear dilaton regime as well as the extreme Liouville region. This is already obvious from the fact that the appropriate coordinates \((X, \bar{X})\) are real in different regions (see discussion after (21)). The quantum theory (defined with translationally invariant measures) which goes over to the CGHS theory in the linear dilaton regime is given by,

\[ Z = \int [dX][dY][df][db][dc]e^{iS[X,Y,f] + iS_{\text{ghost}}}, \]

where,

\[ S = \frac{1}{4\pi} \int d^2\sigma [\mp \partial_+X\partial_-X \pm \partial_+Y\partial_-Y - \sum_i \partial_+f_i \partial_-f^i - T(X,Y)]. \]  

(27)

with \( T \) given by (25) whilst the theory which goes over to CGHS in the extreme Liouville region is given by the above with \( X \) replaced by \( \bar{X} \) and \( T \) given by (26).

\[ \dagger \] The theory given by (27) is of the Liouville type and in fact can be solved. Also given that the Liouville theory is supposed to be a conformal theory it is likely that the same is true of (27) (i.e. to all orders in \( \lambda^2 \)). This solution and its physical implications are currently under investigation.
What about the case $\kappa > 0$. With the CGHS values for $G$ this would result in a negative flux of Hawking radiation and therefore this solution is unphysical. However it is possible that other solutions to $G$ exist which make this case physical though the large $N$ analysis may remain problematic. Finally we note that the theory which goes over to CGHS one in the semi-classical linear dilaton region has one wrong sign kinetic term (19) (for either sign of $\kappa$) but the theory has sufficient gauge invariance (conformal invariance - Virasoro algebra) to gauge it away. On the other hand in the theory (with $N > 24$) which gives CGHS in the large $N$ limit we have two wrong sign fields and the conformal symmetry is not sufficient to gauge them both away. However since neither the graviton nor the dilaton are propagating modes, the above probably does not mean that the theory is non-unitary.

Note added While this work was being prepared for publication, a preprint by A. Strominger [9] was received, in which the $N < 24$ case with what is effectively a modified $G$, to avoid the problem of a negative flux of Hawking radiation, is discussed in some detail. This theory can in fact again be written in the form (27) with $T$ given by (25) except that the relation between $x, y$ and $\phi, \rho$ is modified from (16) and (18) to

$$\rho = \kappa^1(e^{-2\phi} + 4) + y, \quad dx = [(e^{-2\phi} - 2)^2 + \kappa(e^{-2\phi} - 1)]^{\frac{1}{2}} d\phi.$$  

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