Measurement and ergodicity in quantum mechanics

Mariano Bauer and Pier A Mello

Instituto de Física, Universidad Nacional Autónoma de México, México, D.F. C.P. 04510, Mexico

E-mail: mello@fisica.unam.mx

Received 9 December 2014, revised 18 March 2015
Accepted for publication 20 March 2015
Published 14 April 2015

Abstract

The experimental realization of successive non-demolition measurements on single microscopic systems brings up the question of ergodicity in quantum mechanics (QM). We investigate whether time averages over one realization of a single system are related to QM averages over an ensemble of similarly prepared systems. We adopt a generalization of the von Neumann model of measurement, coupling the system to \( N \) ‘probes’, with a strength that is at our disposal, and detecting the latter. The model parallels the procedure followed in experiments on quantum electrodynamic cavities. The modification of the probability of the observable eigenvalues due to the coupling to the probes can be computed analytically and the results compare qualitatively well with those obtained numerically by the experimental groups. We find that the problem is not ergodic, except in the case of an eigenstate of the observable being studied.

Keywords: quantum ergodicity, von Neumann model of measurement, non-demolition measurements

(Some figures may appear in colour only in the online journal)

1. Introduction

The question of ergodicity in quantum mechanics (QM) has long been studied, a ‘quantum ergodic theorem’ (QET) having been formulated by von Neumann in 1929 [1]. Reference [2] discusses the investigations on the subject, from von Neumann’s QET up to recent publications [2]. QM ergodicity for a macroscopic (more than \( 10^{20} \) particles) quantum system means [2]...
The QM expectation values \( \langle \psi_t(A) \rangle \) and \( \langle \psi_i(A) \rangle \) are averages over an ensemble of similarly prepared systems [3], to be called a QM ensemble, they result from measurements of the dynamical variable \( \hat{A} \), independently of the invasive nature of the observations. In addition, a time average appears on the lhs of equation (1.1) and, on the rhs, an ensemble average over the \( d \) eigenstates of the microcanonical subspace. As usual, the systems described by a microcanonical ensemble have a fixed number of particles and volume, and an energy lying in an interval \( \Delta \ll E \) containing \( d \) levels [4]. However, [2] remarks that the property described in the QET is not precisely analogous to the standard notion of ergodicity as known from classical mechanics and the mathematical theory of dynamical systems. Illustrations of the time average appearing in equation (1.1) can be found in [5].

Here we pose a question of a different kind, applicable to systems not necessarily macroscopic, motivated by theoretical considerations and experimental developments. Non-demolition measurements distributed in time [6] and ensembles of measurements on single microscopic systems (few particles or field modes) [7, 8] provide a situation closer to the classical notion of ergodicity [9], as briefly described above and further explained after equation (2.2), and in relation with equations (3.1) and (3.2).

The question will then be the following. Out of a QM ensemble, consider one single system \( s \); at time \( t_1 \) we measure the observable \( \hat{A} \) and again at \( t_2 \). If the first measurement is very invasive, we disturb the system \( s \) so much that the next measurement does not find \( s \) in the original state. We thus introduce the first stage of the measurement, or ‘pre-measurement’, explicitly in the QM description, by coupling \( s \) to a ‘probe’ \( \pi_1 \) at time \( t_1 \); we control the disturbance through the system–probe coupling strength. Next, we couple the same system \( s \) to another probe \( \pi_2 \) at \( t = t_2 \) (see the model Hamiltonian of equation (2.3) below, extended to \( N \) probes), etc; each probe \( \pi_i, i = 1, \ldots, N \), interacts with the system proper at some instant \( t_i \) and evolves freely thereafter, carrying the information about the system picked up at \( t_i \) (this is an extension of von Neumann’s model (vNM) of measurement [10, 11]). We may then detect that information at a later time; we choose the detection time for all of the \( N \) probes as \( t_N \), i.e., right after the last coupling time \( t_N \); i.e., at \( t_N^{+} \) we detect an observable of each one of the \( N \)
probes, not of the system itself; through the entanglement of system and probes we obtain information on the system observable $\hat{A}$ (see figure 1).

Another source of disturbance is $\hat{A}$ failing to commute with the total Hamiltonian $\hat{H}$ of the system and the probe: a system prepared in an eigenstate $|\alpha_\omega\rangle$ of $\hat{A}$ would be found, in the course of time, in other eigenstates $|\alpha'_\omega\rangle$ with non-zero probability. This remedy by requiring $[\hat{A}, \hat{H}] = 0$, so that $\hat{A}$ is a constant of motion, and thus a ‘quantum non-demolition’ (QND) observable [12–14].

From the detection at $t^+_N$ of the various probes successively coupled to one system we average the information acquired at $N$ successive times and define what can aptly be called a time average. This average is performed on one realization $\omega$ of the QM ensemble. We inquire whether, as $N \to \infty$, this result depends on the specific realization and whether it coincides—up to a set of zero measure—with the expectation value of the detected observable for one probe performed over many realizations $\omega$ of the QM ensemble, which we shall call an ensemble average. This is the question of ergodicity (see figure 1), as it is studied in the theory of stationary random processes [9]. We then inquire whether from these averages we can find information on the system observable $\hat{A}$ as applicable to a QM ensemble.

The motivations of this paper are: (i) to develop a theoretical analysis of an ergodic property—relating time averages on a single system to QM ensemble averages—which, to the best of our knowledge, has not been done in the past; (ii) to compare the theory with its actual materialization in true QND laboratory experiments. This paper is organized as follows. In section 2 we recall the vNM of measurement for the system proper and $N$ probes, and introduce the time-averaged operator $\hat{Q}$ in equation (2.1) below. The analysis of the question of ergodicity is performed in section 3, where we compute the QM average of $\hat{Q}$ and its dispersion over the ensemble. We then study how the reduced density operator of the system proper is affected by its interaction with the $N$ probes. In section 4 we analyze the probability distribution (pd) of the eigenvalues of the observable $\hat{A}$ conditioned on the $N$ detected probe positions: we give an analytical treatment of the ‘decimation’ process that has been observed —and analyzed numerically—in the experiments. We finally conclude in section 5.

2. The vNM of measurement for the system proper and the $N$ probes. The time-averaged operator

In the above gedanken experiment, an ‘extended system’ (ES) consisting of the system proper $s$ plus $N$ probes $\pi_1, \ldots, \pi_N$ is considered. Call $\omega$ one preparation of the ES, which we shall call one realization: the QM ensemble is the collection $\{\omega\} \equiv \Omega$ of such realizations (see figure 1).

Each probe $\pi_i$ will be considered one-dimensional, with canonically conjugate dynamical variables $\hat{Q}_i, \hat{P}_i$ in the Schrödinger picture. We define, for the ES, the operator

$$\hat{Q} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{Q}_i,$$  \hspace{1cm} (2.1)

which we call a ‘time-averaged’ operator, because the $N$ probes are applied at $N$ successive times. The $\hat{Q}_i$’s commute among themselves: we can thus detect the observable $\hat{Q}$ by detecting, on the realization $\omega$ of the ensemble and at the fixed time $t^+_N$, the observables $\hat{Q}_i$’s (arising from probe $\pi_i$ each), and constructing the ‘time-average’
\[
\bar{Q}_N^{(\omega)} = \frac{1}{N} \sum_{i=1}^{N} Q_i^{(\omega)},
\] (2.2)

where \(Q_1^{(\omega)} \) is the result of detecting \(Q_1^\omega \) in the realization \(\omega \) (figure 1). QM provides no way of calculating the time average \(\bar{Q}_N^{(\omega)}\), as it is the result of one preparation. However, the standard rules of QM allow to compute the statistical properties of \(Q_1^{(\omega)}\) across a QM ensemble of preparations (the set \(\Omega\) introduced above).

We can now be more specific about the ergodicity question formulated above: it will be answered by inquiring whether the time average \(\bar{Q}_N^{(\omega)}\) over one realization \(\omega\) of equation (2.2), as it is the result of one preparation. However, the standard rules of QM allow to compute the statistical properties of \(Q_1^{(\omega)}\) across a QM ensemble of preparations (the set \(\Omega\) introduced above).

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\[
p_f(Q_1, Q_2) = f(\Psi) \hat{\Psi}_Q \hat{\Psi}_Q |\Psi\rangle_f
\]
\[
= \sum_n W_n^{(0)} \chi_n^{(0)}(Q_1 - e\alpha_n) \left| \chi_n^{(0)}(Q_2 - e\alpha_n) \right|^2.
\]  
(2.6)

Here, \(\hat{\Psi}_Q\) denotes an eigenprojector of \(\hat{Q}\). The scalar product in (2.6) is understood to be evaluated with respect to all the degrees of freedom of the ES. The quantity

\[
\psi = (2.7)
\]
is the Born probability for the value \(\alpha_n\) in the original system state; \(\chi_n^{(0)}(Q_1 - e\alpha_n)\) is the shifted wave function of probe \(\pi_1\) in the position representation, and similarly for \(\pi_2\).

### 2.2. The arbitrary-N case. The expectation value of probe positions

From equation (2.5) generalized to \(N\) probes, the \(j\)p amplitude for \(a_n, Q_1, \ldots, Q_N\) is

\[
\langle a_n, Q_1, \ldots, Q_N |\Psi\rangle_f = \langle a_n |\psi_s^{(0)}\rangle \prod_{i=1}^{N} \chi_i^{(0)}(Q_i - e\alpha_n).
\]  
(2.8)

For Gaussian packets with the same width \(\sigma\) (the probe resolution) for the initial probe states \([6, 11]\), the \(j\)pd’s of \(a_n, Q_1, \ldots, Q_N\) and of \(Q_1, \ldots, Q_N\) are given by

\[
p_f(a_n, Q_1, \ldots, Q_N) = W_n^{(0)} \prod_{i=1}^{N} e^{-\frac{(\alpha_i - \mu)^2}{2\sigma^2}},
\]

(2.9a)

\[
p_f(Q_1, \ldots, Q_N) = \sum_n W_n^{(0)} \prod_{i=1}^{N} e^{-\frac{(\alpha_i - \mu)^2}{2\sigma^2}}.
\]  
(2.9b)

The Gaussian assumption allows an analytical treatment. Use of equations (2.2) and (2.9b) gives (the index \(f\) indicates an expectation value evaluated with the state \(|\Psi\rangle_f\))

\[
\langle \hat{A}_i \rangle_f = \langle \hat{A}_i \rangle_f = \sum_n W_n^{(0)} \langle (e\alpha_n) \rangle_f = \langle e \hat{A} \rangle_0,
\]  
(2.10a)

\[
\langle \hat{A}_i^2 \rangle_f = \langle \hat{A}_i^2 \rangle_f = \sum_n W_n^{(0)} \left[ \langle (e\alpha_n)^2 \rangle_f + \sigma^2 \right] = e^2 \langle \hat{A}_i^2 \rangle_0 + \sigma^2.
\]  
(2.10b)

The notation \(\langle \hat{A}_i \rangle_0, \langle \hat{A}_i^2 \rangle_0\) indicates expectation values in the original system state. The first equality in equation (2.10) expresses the property of stationarity. One can also show, for \(i \neq j\)

\[
\langle \hat{A}_i \hat{A}_j \rangle_f = \langle \hat{A}_i \hat{A}_j \rangle_f = \sum_n W_n^{(0)} \langle (e\alpha_n)(e\alpha_n) \rangle_f = e^2 \langle \hat{A}_i \hat{A}_j \rangle_0.
\]  
(2.11)

In general, for a stationary random process, \((\text{cov}(Q_i, Q_j))_f \equiv \langle \hat{A}_i \hat{A}_j \rangle_f - \langle \hat{A}_i \rangle_f \langle \hat{A}_j \rangle_f\) depends only on \(|i - j|\). In our present case, \((\text{cov}(Q_i, Q_j))_f\) is independent of \(i, j\) (for \(i \neq j\)) and does not decrease as \(|i - j|\) increases. This behavior is due to the structure of the \(j\)pd of \(Q_1, \ldots, Q_N\) of equation (2.9b).
3. The analysis of the question of ergodicity

The ensemble expectation value over the realizations $\Omega$ of the time average $\overline{Q_N^{(\omega)}}$ of equation (2.2) is given by

$$\left\langle \overline{Q_N^{(\omega)}}_f \right\rangle = \langle \hat{Q} \rangle_f = \frac{1}{N} \sum_{i=1}^{N} \langle \hat{Q}_i \rangle_f = \langle \hat{Q}_1 \rangle_f,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.1a)

$$\left\langle \overline{Q}_f \right\rangle / \epsilon = \langle \hat{\Lambda} \rangle_0.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.1b)

We remark again that the same ensemble $\Omega$ of realizations is employed in the various QM expectation values appearing here and below (figure 1). Equation (3.1a) states that the ensemble expectation value of the time average $\overline{Q_N^{(\omega)}}$, which is just the QM expectation value of the operator $\hat{Q}$, coincides with the QM expectation value of any one of the probe positions, like $\hat{Q}_1$. This is the standard result in the theory of stationary random processes [9]. Equation (3.1b) states that this result, in turn, equals, in units of $\epsilon$, $\langle \hat{\Lambda} \rangle_0$, the QM Born expectation value of the observable $\hat{\Lambda}$ in the original system state (the expectation value of $\hat{\Lambda}$ is time-independent, because $[\hat{\Lambda}, H] = 0$ in our model).

The crucial question is the dispersion of $\overline{Q_N^{(\omega)}}$ over the ensemble $\Omega$, which we calculate as

$$\text{var} \hat{Q} = \left\langle \left( \overline{Q} - \langle \overline{Q} \rangle_f \right)^2 \right\rangle_f = \left\langle \left( \overline{Q} \right)^2 \right\rangle_f - \left\langle \overline{Q} \right\rangle_f^2,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2a)

$$= \left\langle \left( \frac{1}{N} \sum_{i=1}^{N} \hat{Q}_i \right)^2 \right\rangle_f - \left( \frac{1}{N} \sum_{i=1}^{N} \langle \hat{Q}_i \rangle_f \right)^2,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2b)

$$= \frac{1}{N^2} \sum_{i,j=1}^{N} \left[ \langle \hat{Q}_i \hat{Q}_j \rangle_f - \langle \hat{Q}_i \rangle_f \langle \hat{Q}_j \rangle_f \right],$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2c)

$$= \frac{N-1}{N} \left( \text{cov} (\hat{Q}_1, \hat{Q}_2) \right)_f + \frac{1}{N} \text{var} \hat{Q}_1,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2d)

$$- \left( \text{cov} (\hat{Q}_1, \hat{Q}_2) \right)_f, \quad \text{as } N \to \infty.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2e)

Equation (3.2a) is the familiar definition of the variance; in equation (3.2b) we used the definition (2.1) to write the first and second moments of $\overline{Q}$; in equation (3.2d) we used the first equality in equations (2.10) and (2.11). Notice that $\text{var} \hat{Q}$ does not vanish in the limit $N \to \infty$. This is not surprising due to the behavior of the correlation function described right after equation (2.11) [9]. We can be more explicit using the last equalities appearing in equations (2.10) and (2.11), which give

$$\left\langle \text{var} \overline{Q} \rightangle / \epsilon^2 = \left( \text{var} \hat{\Lambda} \right)_0 + \frac{N\epsilon}{N}$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2f)

$$\geq \left( \text{var} \hat{\Lambda} \right)_0, \quad \text{if } N \gg N_c,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (3.2g)
where we have defined a 'critical $N''$, $N_c = (\sigma / \epsilon)^2$. If $N \ll N_c$, $\text{var} \overline{Q} / \epsilon^2 \gg (\text{var} \hat{A})_0$; this situation could be achieved with a very large probe resolution $\sigma$, measured in units of $\epsilon / \sqrt{N}$. If $N \gg N_c$, $\text{var} \overline{Q} / \epsilon^2 \gg (\text{var} \hat{A})_0$. Given the resolution $\sigma$ of the probe, the coupling strength $\epsilon$ defines how quickly $N_c$ is attained.

The conclusion is that the time average $\overline{Q}_N$, measured in units of $\epsilon$, has, in general, a finite dispersion over the ensemble $\Omega$ of realizations $\omega$, even when $N \rightarrow \infty$. Therefore, we do not have ergodicity and we cannot verify the ensemble predictions of QM by means of a sequence of measurements on a single system. An exception is when the original QM system state is an eigenstate of the observable $\hat{A}$, as then $(\text{var} \hat{A})_0 = 0$ and $\text{var} \overline{Q} / \epsilon^2 \rightarrow 0$ as $N \rightarrow \infty$.

3.1. The pdf of $\overline{Q}_N$

From equation (2.9b) we find the pdf of $\overline{Q}_N$ of equation (2.2) (sampled over the ensemble $\Omega$)—whose first moment and variance were computed above—as

$$p_f \left( \overline{Q}_N \right) = \sum_n W^{(0)}_{\Delta} \frac{e^{-\frac{(\sigma N_n^2 - \epsilon_n^2)^2}{2n^2 \sigma^2}}}{\sqrt{2\pi n^2 \sigma^2}},$$

(3.3)

which is shown schematically in figure 2. From equation (3.3) we see the effect of not having ergodicity. The peaks in $p_f \left( \overline{Q}_N \right)$ are centered at the various $\epsilon_n$'s. If $N \gg 1$, the peaks do not overlap and the area under the peak centered at $\epsilon_n$ gives $W^{(0)}_{\Delta}$. From one realization to another, $\overline{Q}_N$ jumps at random from one very narrow peak to another, $W^{(0)}_{\Delta}$ being the fraction of realizations whose $Q_n \approx \epsilon_n$. If the original system state is an eigenstate of $\hat{A}$, only one peak occurs and eventually we have ergodicity as $N \rightarrow \infty$.

The process leading to one limiting value (as $N \rightarrow \infty$) of $\overline{Q}_N$ for a specific realization is most clearly seen in the extreme case $\sigma \ll \epsilon N_a$, when the original probe pdf's are narrower than the separation of the $\epsilon_n$'s. We first detect $Q_1$; from (2.9b) for $N = 1$, a result for $Q_1$ between two values $\epsilon_n$ is extremely unlikely to occur. If $Q_1 \approx \epsilon_n$, the pdf of $Q_2, \ldots, Q_N$, conditioned on $Q_1 = \epsilon_n$, is, from equation (2.9b), reduced approximately to one term

$$p_f \left( Q_2, \ldots, Q_N \mid Q_1 = \epsilon_n \right) \approx \prod_{i=2}^{N} \frac{e^{-\frac{(\sigma - \epsilon_i)^2}{2\sigma^2}}}{\sqrt{2\pi \sigma}}.$$  

(3.4)

Having found $Q_1 = \epsilon_n$, it is as if $Q_2, \ldots, Q_N$ were statistically independent variables, their pdf's all centered at $\epsilon_n$ and with a width $\sigma$; i.e., the first detected value $Q_1$ makes $Q_2, \ldots, Q_N$ get 'stuck' around $\epsilon_n$. As a result, $\overline{Q}_N$ tends to the limiting value $\epsilon_n$ as $N \rightarrow \infty$. Had we found $Q_1 = \epsilon_{n1}$, $Q_2, \ldots, Q_N$ would be stuck around $\epsilon_{n1}$ and $\overline{Q}_N$ would tend to the limiting value $\epsilon_{n1}$.
value $εa_i$. As a matter of fact, the pd of $\tilde{Q}_N$, conditioned on $Q_l = εa_{n_l}$, is found to be

$$p_f(\tilde{Q}_N|Q_l = εa_{n_l}) = \frac{e^{-\frac{(\frac{Q_l-a_l}{\sigma})^2}{2\sigma^2}}}{\sqrt{2\pi N\frac{1}{2}\sigma^2}}.$$ (3.5)

A corresponding analysis can be carried out in the opposite extreme case $σ ≫ εΔa_i$, when the original probe pd’s are wider than the separation of the $a_i$’s.

3.2. The reduced density operator of the system proper

To clarify to what extent has the system proper been altered due to its interaction with the $N$ probes, we calculate the final reduced density operator of the system. Tracing over $\pi_1, \cdots, \pi_N$ the density operator $|Ψ\rangle_f \langle Ψ|$ from equation (2.5) generalized to $N$ Gaussian probes, we find

$$\rho_s^{(f)} = \sum_{nm} e^{-\frac{(a_n-a_m)^2}{2\sigma^2}} \left( P_{a_n} |Ψ_{f_s}^{(0)}(0)\rangle \langle Ψ| P_{a_m} \right).$$ (3.6)

The non-demolition property is clear: the diagonal matrix elements of $\rho_s^{(f)}$ are unchanged by the interaction with the probes; the off-diagonal ones are changed, depending on the interaction strength $ε$. For $N$ probes and $N ≪ N_c$,

$$\rho_s^{(f)} \approx |Ψ_{f_s}^{(0)}(0)\rangle \langle Ψ|,$$ (3.7a)

and the system state is not altered appreciably by the detections. For $N ≫ N_c$,

$$\rho_s^{(f)} \approx \sum_n P_{a_n} |Ψ_{f_s}^{(0)}(0)\rangle \langle Ψ| P_{a_n},$$ (3.7b)

and the final state is a mixture like the one found after a non-selective projective measurement on the original pure state [17], a result eventually attained as $N$ increases, no matter how small—but fixed—is $ε/σ$. However, the final system state can be kept close to the original one for $N$ as large as we please, if $N_c$ is large enough.

The $N$ dependence of the transition between equations (3.7a) and (3.7b) exhibits the progressive modification of the density operator for the system proper resulting from the process. We complement this discussion in the next section where we study the mechanism behind what has been called by experimental groups [8, 18] the ‘progressive collapse’ of the system state.

4. Probability distribution of the $a_i$’s, conditioned on the detected values for the probes

[7, 8, 18] analyze the change suffered by the pd of the photon number in the cavity, conditioned on the detection of the $N$ probes. Here we use our model, in which the probes interact with the system at $N$ successive times and the detection of the $N$ probes takes place at the single time $t_N$. Starting from equation (2.9a): (i) integration over $Q_1, \cdots, Q_N$ gives $W_{a_s}^{(f)} = W_{a_s}^{(0)}$, as a consequence of the non-demolition character of the vNM Hamiltonian (see below equation (3.6)); (ii) the $a_n$ pd conditioned on the detected $Q_l, \cdots, Q_N$ for each preparation of the ES is
\[ p_f \left( a_n \left| Q_1, \ldots, Q_N \right. \right) = W^{(0)}_{a_n} \frac{e^{-\frac{(\omega_n-Q_N)^2}{2\sigma^2}}}{\sum_{n'} W^{(0)}_{a_n'} e^{-\frac{(\omega_{n'}-Q_N)^2}{2\sigma^2}}}. \tag{4.1} \]

Gaussian probe functions make \( p_f \left( a_n \left| Q_1, \ldots, Q_N \right. \right) \) depend on the probe positions only through \( Q_N \).

The pdf of the system \( a_n \)'s, which is originally \( W^{(0)}_{a_n} \), after its interaction with the probes and conditioned on a specific \( N \)-tuple \( Q_1, \ldots, Q_N \) of probe positions, has become modulated by the second factor in equation (4.1), which ‘dissects’ it into a slice centered at \( a_n \sim Q_N/e \). From the above discussion, equations (3.3)–(3.5), the centroid of the dissecting factor eventually tends to a limiting value as \( N \) increases; at the same time, its width, \( \sigma(eN) \), becomes thinner the larger is \( N \). This is the ‘decimation process’ of [7], figure 2 of [8] and [18], where probe functions arise from a Ramsey-interferometer-type experimental setup and decimation is exhibited numerically; the present model allows an analytical treatment. We think it is of interest to have pointed out explicitly the above mechanism applied to the experiments we have been referring to, because, to the best of our knowledge, it has not been put in the language of ergodicity.

Equation (4.1) gives the pdf of \( a_n \) conditioned on a given set of probe positions \( Q_1, \ldots, Q_N \); the dissecting factor is centered at \( a_n \sim Q_N/e \). For a different value of \( Q_N \), say \( Q'_N \), it is centered at \( Q'_N/e \). Running through the ensemble, the dissections in equation (4.1) appear with the frequency of occurrence of \( Q_N/e \), i.e. \( p_f \left( Q_N/e \right) \), obtainable from equation (3.3). Accumulating all the \( Q_N/e \), i.e. constructing \( \int p_f \left( a_n | Q_N/e \right) p_f \left( Q_N/e \right) dQ_N/e \), we recover the original pdf \( W^{(0)}_{a_n} \) of the \( a_n \)'s, just as observed in [8], figure 3.

5. Summary

In summary, we investigated whether ergodicity is realized in QM. To control the disturbance produced by the measurement, we required the observable \( \hat{A} \) to be a non-demolition one, and we introduced \( N \) probes which interact with the system at successive times \( t_i \) with a coupling strength \( \epsilon_i \), and we detect the probes. This scheme has been materialized in QED-cavity experiments, where the probes are atoms that traverse the cavity at successive times and are then detected. In general, the system is not ergodic: thus, from the time average over one realization of the system plus \( N \) probes, we cannot infer the QM ensemble average.

The reduced density operator for the system is not appreciably altered by the \( N \) detections if \( N \ll N_c \). If \( N \gg N_c \), an initially pure state eventually becomes a complete mixture.

The probability of the eigenvalues \( a_n \), conditioned on the detected positions of the \( N \) probes, is the original Born probability modulated by a factor that depends on \( Q_N \) for the detected values (decimation process). Probe Gaussian functions allow an analytical treatment.

The statistical distribution over the QM ensemble of \( Q_N \) consists of a series of peaks centered at the eigenvalues \( ea_n \); the presence of more than one peak is a consequence of not having ergodicity.

Finally, we wish to comment that the analysis carried out in this paper is fully quantum mechanical. It is, however, interesting to mention that the classical counterpart of our Hamiltonian, equation (2.3), (see, e.g., [14], pages 378–380), gives a joint pdf of the probe
positions $p_f(Q_1, Q_2)$ with a similar structure of our equation (2.6) for $N = 2$ probes, or its generalization (2.9b) for an arbitrary number of probes. The complementary part of our study, the density operator $\rho^{(f)}$ for the system proper, has clearly a fully QM structure, as can be seen from equation (3.6).

Acknowledgments

One of the authors (PAM) acknowledges financial support by DGAPA, Mexico, under grant IN109014.

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