Abstract: We consider two families of type II multiple orthogonal polynomials. Each family has orthogonality conditions with respect to a discrete vector measure. The \( r \) components of each vector measure are \( q \)-analogues of Meixner measures of the first and second kind, respectively. These polynomials have lowering and raising operators, which lead to the Rodrigues formula, difference equation of order \( r + 1 \), and explicit expressions for the coefficients of recurrence relation of order \( r + 1 \). Some limit relations are obtained.

Keywords: Hermite–Padé approximation; multiple orthogonal polynomials; discrete orthogonality; recurrence relations

MSC: 42C05; 33C47; 33E99

1. Introduction

Hermite’s proof [1] of the transcendence of the number \( e \) uses the notion of simultaneous approximation, which was subsequently studied in approximation theory and number theory [2–8]. Multiple orthogonal polynomials are polynomials that satisfy orthogonality conditions shared with respect to a set of measures [9–17]. They are related to the simultaneous rational approximation of a system of \( r \) analytic functions [18,19] and play an important role both in pure and applied mathematics (see for instance [20–22] as well as [23–27]). In this context, some families of continuous and discrete multiple orthogonal polynomials have been studied [3,28–30] as well as some multiple \( q \)-orthogonal polynomials [31–33]. The goal of the present paper is to study some multiple Meixner polynomials on a non-uniform lattice \( x(s) = q^s - 1/q - 1 \), \( s = 0, 1, \ldots \)

The paper is structured as follows. Section 2 is devoted to introduce the necessary background material. In Section 3, we consider two families of multiple \( q \)-orthogonal polynomials, namely, multiple \( q \)-Meixner polynomials of the first and second kind, respectively. They are analogous to the discrete multiple Meixner polynomials studied in [28]. We obtain the raising and lowering \( q \)-difference operators as well as the Rodrigues-type formula, which lead to an explicit expression for the multiple \( q \)-Meixner polynomials. Then, the recurrence relations as well as the \( q \)-difference equations with respect to the independent variable \( x(s) \) are obtained. In Section 4, some limit relations as the parameter \( q \) approaches 1 are studied. An appendix to the Section 3 is considered in Section 5, in which the AT-property of the involved system of \( q \)-discrete measures is addressed. We make concluding remarks in Section 6.

2. Background Material

Let \( \vec{\mu} = (\mu_1, \ldots, \mu_r) \) be a vector of \( r \) positive Borel measures supported on \( \mathbb{R} \) with finite moments. By \( \Omega_1 \) we denote the smallest interval that contains \( \text{supp}(\mu_i) \). Define a multi-index \( \vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r \), where \( \mathbb{N} \) stands for the set of nonnegative integers. For the multi-index \( \vec{n} \), a type II multiple orthogonal
polynomial $P_{\vec{n}}$ is a polynomial of degree $\leq |\vec{n}| = n_1 + \cdots + n_r$, which satisfies the orthogonality conditions [34]

$$\int_{\Omega_i} P_{\vec{n}}(x)x^k d\mu_i(x) = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r. \quad (1)$$

Special attention is paid to a unique solution of (1) (up to a multiplicative factor) with $\deg P_{\vec{n}}(x) = |\vec{n}|$ for every $\vec{n}$. In this situation the index is said to be normal [34]. In particular, if the above system of measures forms an AT system [34], then every multi-index is normal.

The polynomial $P_{\vec{n}}(z)$ is the common denominator of the simultaneous rational approximants $Q_{\vec{n}}(z)$ to Cauchy transforms

$$\hat{\mu}_i(z) = \int_{\Omega_i} \frac{d\mu_i(x)}{z - x}, \quad z \notin \Omega_i, \quad i = 1, \ldots, r, \quad (2)$$

of the vector components of $\vec{n} = (\mu_1, \ldots, \mu_r)$, i.e., for function (2) we have the following simultaneous rational approximation with prescribed order near infinity [34]

$$P_{\vec{n}}(z)\hat{\mu}_i(z) - Q_{\vec{n},i}(z) = \frac{\varepsilon_i}{z^{n_i+1}} + \cdots + O(z^{-n_i-1}), \quad i = 1, \ldots, r.$$

If the measures in (1) are discrete

$$\mu_i = \sum_{k=0}^{N_i} \omega_{i,k} \delta_{x_{i,k}}, \quad \omega_{i,k} > 0, \quad x_{i,k} \in \mathbb{R}, \quad N_i \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, 2, \ldots, r, \quad (3)$$

where $\delta_{x_{i,k}}$ denotes the Dirac delta function and $x_{i,k} \neq x_{i,\ell}, k = 0, \ldots, N_i$, whenever $i_1 \neq i_2$, the corresponding polynomial solution $P_{\vec{n}}(x)$ of the linear system of Equation (1) is called discrete multiple orthogonal polynomial (see [28] and the examples therein). In particular, the paper [28] considers discrete multiple orthogonal polynomial on the linear lattice $x(k) = k, k = 1, \ldots, N, N \in \mathbb{N} \cup \{+\infty\}$.

We will deal only with systems of discrete measures, for which $\Omega_i = \Omega \subset \mathbb{R}_+$ (the set of nonnegative reals) for each $i = 1, 2, \ldots, r$. Recall that the system of positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$, given in (3), forms an AT system if there exist $r$ continuous functions $v_1, \ldots, v_r$ on $\Omega$ with $v_i(x_k) = \omega_{i,k}, k = 0, \ldots, N_i, i = 1, 2, \ldots, r$, such that the $|\vec{n}|$ functions

$$v_1(x), xv_1(x), \ldots, x^{n_i-1}v_1(x), \ldots, v_r(x), xv_r(x), \ldots, x^{n_i-1}v_r(x),$$

form a Chebyshev system on $\Omega$ for each multi-index $\vec{n}$ with $|\vec{n}| < N + 1$, i.e., every linear combination $\sum_{i=1}^r Q_{n_i-1}(x)v_i(x)$, where $Q_{n_i-1} \in \mathbb{P}_{n_i-1} \setminus \{0\}$, has at most $|\vec{n}| - 1$ zeros on $\Omega$. Here $\mathbb{P}_m \subset \mathbb{P}$ denotes the linear subspace (of the space $\mathbb{P}$) of polynomials of degree at most $m \in \mathbb{Z}^+$.

In the sequel we will consider discrete multiple orthogonal polynomials on a non-uniform lattice $x(s) = q^s - 1/q - 1$ (see [35,36]).

**Definition 1.** A polynomial $P_{\vec{n}}(x(s))$ on the lattice $x(s) = c_1q^s + c_3, q \in \mathbb{R}^+ \setminus \{1\}, c_1, c_3 \in \mathbb{R}$, is said to be a multiple $q$-orthogonal polynomial of a multi-index $\vec{n} \in \mathbb{N}^r$ with respect to positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$ (with finite moments) such that $\text{supp } (\mu_i) \subset \Omega_i \subset \mathbb{R}, i = 1, 2, \ldots, r$, if the following conditions hold:

$$\deg P_{\vec{n}}(x(s)) \leq |\vec{n}| = n_1 + n_2 + \cdots + n_r,$$

$$\sum_{s=0}^{N_i} P_{\vec{n}}(x(s))x(s)^k d\mu_i = 0, \quad k = 0, \ldots, n_i - 1, \quad N_i \in \mathbb{N} \cup \{+\infty\}. \quad (4)$$
In Section 3 we will deal with particular measures involving the $q$-Gamma function, which is defined as follows

$$
\Gamma_q(s) = \begin{cases}
    f(s;q) = (1-q)^{1-s} \frac{\prod_{k=0}^{(s-1)/2} (1-q^{k+1})}{\prod_{k=0}^{s-1} (1-q^{k+1})}, & 0 < q < 1, \\
    q^{(s-1)/2} f(s;q^{-1}), & q > 1.
\end{cases}
$$

(5)

See also [37,38] for the definition of the $q$-Gamma function. In addition, we use the $q$-analogue of the Stirling polynomials denoted by $[s]^{(k)}_q$, which is a polynomial of degree $k$ in the variable $x(s) = (q^s - 1)/(q-1)$, i.e.,

$$
[s]^{(k)}_q = \prod_{j=0}^{k-1} \frac{q^s - j - 1}{q-1} = x(s)x(s-1) \cdots x(s-k+1) \quad \text{for} \quad k > 0, \quad \text{and} \quad [s]^{(0)}_q = 1.
$$

(6)

Hereafter, confusion should be avoided between (6) and the notation for the $q$-analogue of a complex number $z \in \mathbb{C}$,

$$
[z] = \frac{q^z - q^{-z}}{q - q^{-1}}.
$$

(7)

The relation between (6) and (7) is as follows: $[z] = q^{1-z}[2z]^{(1)}_q/(q+1)$. The term $q$-analogue means that the expression $[z]$ tends to $z$, as $q$ approaches 1. In general, we say that the function $f_q(s)$ is a $q$-analogue to the function $f(s)$ if for any sequence $(q_n)_{n \geq 0}$ approaching to 1, the corresponding sequence $(f_{q_n}(s))_{n \geq 0}$ tends to $f(s)$ (see Section 4).

The following difference operators are used throughout this paper

$$
\Delta \overset{\text{def}}{=} \frac{\nabla x(s-1/2)}{x(s-1/2)}, \quad \nabla \overset{\text{def}}{=} \frac{\nabla x(s+1/2)}{x(s+1/2)},
$$

$$
\nabla^{n_j} = \nabla \cdots \nabla \quad \text{n}_j \in \mathbb{N},
$$

(8)

(9)

where $\nabla f(x) = f(x) - f(x-1)$ and $\Delta f(x) = \nabla f(x+1)$ denote the backward and forward difference operators, respectively. When convenient, a less common notation taken from [38] will also be used: $\nabla x_1(s) \overset{\text{def}}{=} \nabla x(s+1/2) = \Delta x(s-1/2) = q^{-s+1/2}$.

Observe that

$$
\nabla^m (f(s)g(s)) = \sum_{k=0}^{m} \binom{m}{k} \left(\nabla^k f(s) \right) \left(\nabla^{m-k} g(s-k) \right), \quad m \in \mathbb{N},
$$

(10)

is a discrete analogue of the well-known Leibniz formula (product rule for derivatives). In particular,

$$
\nabla^m f(s) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(s-k).
$$

(11)

Finally, we will make use of the following notations for multi-indices: The multi-index $\vec{e}_i$ denotes the standard $r$-dimensional unit vector with the $i$-th entry equals 1 and 0 otherwise, the multi-index $\vec{e}$ with all its $r$-entries equal 1. In addition, for any vector $\vec{a} \in \mathbb{C}^r$ and number $p \in \mathbb{C}$,

$$
\vec{a}_{i,p} \overset{\text{def}}{=} \vec{a} - a_i(1-p)\vec{e}_i = (a_1, \ldots, pa_i, \ldots, a_r).
$$

(12)
Multiple Meixner Polynomials of the First and Second Kind

In [28], for multiple Meixner polynomials, it was considered two vector measures \( \vec{\mu} = (\mu_1, \ldots, \mu_r) \) and \( \vec{\nu} = (\nu_1, \ldots, \nu_r) \), where in both cases each component is a Pascal distribution (negative binomial distribution) with different parameters

\[
\mu_i = \sum_{x=0}^{\infty} \nu^{\alpha_i} v^{\beta_i}(x) \delta_x, \quad v^{\alpha_i, \beta_i}(x) = \begin{cases} \frac{\Gamma(\beta + x)}{\Gamma(\beta)} \frac{\alpha_i^x}{x!}, & x \in \mathbb{R} \setminus \{ -\beta, -\beta - 1, -\beta - 2, \ldots \}, \\ 0, & \text{otherwise}, \end{cases}
\]

Notice that \( v^{\alpha_i, \beta_i}(x) \) is a \( C^\infty \)-function on \( \mathbb{R} \setminus \{ -\beta, -\beta - 1, -\beta - 2, \ldots \} \) with simple poles at the points \( \{ -\beta, -\beta - 1, -\beta - 2, \ldots \} \). For the above measures \( 0 < a, a_i < 1 \), with all the \( a_i \) different, and \( \beta, \beta_i > 0 \) (\( \beta_i \neq \beta_j \) for all \( i \neq j \)). Under these conditions for both \( \vec{\mu} \) and \( \vec{\nu} \) the multi-index \( \vec{n} \in \mathbb{N}^r \) is normal.

For the monic multiple Meixner polynomial of the first kind \([28]\) corresponding to the multi-index \( \vec{n} \in \mathbb{N}^r \) and the vector measure \( \vec{\mu} \), define the monic polynomial \( M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x) \) of degree \( |\vec{n}| \) and different positive parameters \( a_1, \ldots, a_r \) (indexed by \( \vec{\alpha} = (a_1, \ldots, a_r) \)) and the same \( \vec{\beta} \) which satisfies the orthogonality conditions

\[
\sum_{x=0}^{\infty} M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x)(-x)_j \nu^{\alpha_i, \beta_i}(x) = 0, \quad j = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r,
\]

where \( (x)_j = (x)(x+1)\cdots(x+j-1), \quad (x)_0 = 1, \quad j \geq 1 \), denotes the Pochhammer symbol. This polynomial of degree \( j \) is used to deal more conveniently with the orthogonality conditions (1)–(3) on the linear lattice \( \{ x = 0, 1, \ldots \} \).

For the monic multiple Meixner polynomial of the second kind \([28]\) corresponding to the multi-index \( \vec{n} \in \mathbb{N}^r \) and the vector measure \( \vec{\nu} \), define the monic polynomial \( M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x) \) of degree \( |\vec{n}| \) and \( \vec{\beta} = (\beta_1, \ldots, \beta_r) \), with different components, which satisfies the orthogonality conditions

\[
\sum_{x=0}^{\infty} M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x)(-x)_j \nu^{\alpha_i, \beta_i}(x) = 0, \quad j = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r.
\]

For both families of multiple orthogonal polynomials the following \( r \) raising operators were found

\[
\mathcal{L}^{\alpha_i, \beta_i} \left( M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x) \right) = -M_{\vec{n} + e_i}^{\vec{\alpha}, \vec{\beta} - 1}(x),
\]

\[
\mathcal{L}^{\alpha_i, \beta_i} \left( M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x) \right) = -M_{\vec{n} + e_i}^{\vec{\alpha}, \vec{\beta} - e_i}(x),
\]

where

\[
\mathcal{L}^{\sigma, \tau} \overset{\text{def}}{=} \frac{\sigma(\tau - 1)}{(1 - \sigma) \nu^{\sigma, \tau - 1}(x)} \nu^{\sigma, \tau}(x), \quad (\sigma, \tau) \in \{ (\alpha_i, \beta_i) \} \cup \{ (\alpha, \beta_i) \}, \quad i = 1, \ldots, r.
\]

As a consequence of (13) and (14), there holds the Rodrigues-type formulas

\[
M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x) = (\beta)_{|\vec{n}|} \left( \prod_{i=1}^{r} \left( \frac{\alpha_i}{\alpha_i - 1} \right)^{n_i} \frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)} M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}} \left( \frac{\Gamma(\beta + |\vec{n}| + x)}{\Gamma(\beta + |\vec{n}|)\Gamma(x+1)} \right) ,
\]

\[
M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}}(x) = \left( \frac{\alpha}{\alpha - 1} \right)^{|\vec{n}|} \left( \prod_{i=1}^{r} (\beta_i)^{n_i} \frac{\Gamma(x+1)}{\alpha^x} M_{\vec{n}}^{\vec{\alpha}, \vec{\beta}} \left( \frac{\alpha^2}{\Gamma(x+1)} \right) ,
\]
where \( \mathcal{M}_n^\alpha = \prod_{i=1}^{r} (a_i - x \nabla^i a_i) \) and \( \mathcal{N}_n^\beta = \prod_{i=1}^{r} \frac{\Gamma(\beta_i)}{\Gamma(\beta_i + x)} \nabla^i \frac{\Gamma(\beta_i + n_i + x)}{\Gamma(\beta_i + n_i + 1)}. \) Then, from (10) and (11) the above formulas (15) and (16) provide an explicit expressions for the above polynomials \( \mathcal{M}_n^{\alpha,\beta} (x) \) and \( \mathcal{M}_n^{\alpha,\beta} (x). \)

Two important algebraic properties are known for multiple Meixner polynomials \([28]\), namely the \((r + 1)\)-order linear difference equations \([39]\)

\[
\prod_{i=1}^{r} \mathcal{L}^\alpha_{i,\beta_i+1} \left( \triangle \mathcal{M}_n^{\alpha,\beta} (x) \right) = - \sum_{i=1}^{r} n_i \prod_{j=1}^{r} \mathcal{L}^{\alpha_{j},\beta_j+1} \left( \triangle \mathcal{M}_n^{\alpha,\beta} (x) \right),
\]

where

\[
d_i = \sum_{j=1}^{r} \frac{(-1)^{i+j} \prod_{k=1}^{i} (n_j + \beta_j - \beta_k)}{(n_j + \beta_j - \beta_i) \prod_{k=1}^{r-1} (n_k - n_j + \beta_k - \beta_i) \prod_{l=i+1}^{r} (n_j - n_l + \beta_j - \beta_l)},
\]

and the recurrence relations \([28]\)

\[
x \mathcal{M}_n^{\alpha,\beta} (x) = \mathcal{M}_n^{\alpha,\beta} (x) + \left( (\beta + \mid i \mid) \left( \frac{\alpha_k}{1 - \alpha_k} \right) + \sum_{i=1}^{r} \frac{n_i}{1 - \alpha_i} \right) \mathcal{M}_n^{\alpha,\beta} (x)
\]

\[
+ \sum_{i=1}^{r} \frac{n_i (\beta + \mid i \mid - 1)}{(\alpha_i - 1)^2} \mathcal{M}_n^{\alpha,\beta} (x),
\]

\[
x \mathcal{M}_n^{\alpha,\beta} (x) = \mathcal{M}_n^{\alpha,\beta} (x) + \left( (n_k + \beta_k) \left( \frac{\alpha}{1 - \alpha} \right) + \frac{\mid i \mid}{1 - \alpha} \right) \mathcal{M}_n^{\alpha,\beta} (x)
\]

\[
+ \alpha \sum_{i=1}^{r} \frac{n_i (\beta_i + n_i - 1)}{(1 - \alpha)^2} \prod_{j \neq i} \frac{n_i + \beta_j - \beta_i}{n_i - n_j + \beta_i - \beta_j} \mathcal{M}_n^{\alpha,\beta} (x).
\]

Note that each relation (19) and (20) involve \( r \) relations of nearest-neighbor polynomials. Moreover, each family of multiple Meixner polynomials \( \mathcal{M}_n^{\alpha,\beta} (x) \) and \( \mathcal{M}_n^{\alpha,\beta} (x) \) forms common eigenfunctions of the above two linear difference operators of order \((r + 1)\), namely (17)–(20), respectively.

3. Multiple Meixner Polynomials on a Non-Uniform Lattice

Some algebraic properties will be studied in this section: The Rodrigues-type formula, some recurrence relations and the difference equations with respect to the independent discrete variable \( x(s) \). For the \( q \)-difference equation (of order \( r + 1 \)) we will proceed as follows. First, we define an \( r \)-dimensional subspace \( V \) of polynomials of degree at most \( \mid i \mid - 1 \) in the variable \( x(s) \) by using some interpolation conditions. Then, we find the lowering operator and express its action on the polynomials as a linear combination of the basis vectors of \( V \). This operator depends on the specific family of multiple orthogonal polynomials, therefore some ‘ad hoc’ computations are needed. Finally, we combine the lowering and the raising operators to derive the \( q \)-difference equation. A similar procedure is given in \([31,32,36,39–41]\). Finally, the recurrence relations will be derived from some specific difference operators used in Theorems 2 and 4.
3.1. On Some $q$-Analogues of Multiple Meixner Polynomials of the First Kind

Consider the following vector measure $\bar{\mu}_q$ with positive $q$-discrete components on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k) \delta(k - s), \quad \omega_i > 0, \quad i = 1, 2, \ldots, r. \quad (21)$$

Here $\omega_i(s) = v_i^{\alpha_i, \beta_i}(s) \triangle x(s - 1/2)$, and

$$v_i^{\alpha_i, \beta_i}(s) = \begin{cases} \frac{\alpha_i^1 \Gamma_i(\beta_i + s)}{\Gamma_q(s + 1)}, & \text{if } s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise}, \end{cases} \quad (22)$$

where $0 < \alpha_i < 1, \beta > 0, i = 1, 2, \ldots, r$, and with all the $\alpha_i$ different.

The system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (21) forms an AT system on $\mathbb{R}^+$ (see Lemma 9).

**Definition 2.** A polynomial $M^{\bar{\alpha}, \bar{\beta}}_{q, \bar{n}}(s)$, with multi-index $\bar{n} \in \mathbb{N}^r$ and degree $|\bar{n}|$, that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} M^{k, \beta}_{q, \bar{n}}(s) \nabla^k v^{\alpha, \beta}_i(s) \triangle x(s - 1/2) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, \ldots, r, \quad (23)$$

is said to be the $q$-Meixner multiple orthogonal polynomial of the first kind. See also (4) with respect to measure (21).

Notice that for $r = 1$ we recover the scalar $q$-Meixner polynomials given in [35] and that the orthogonality conditions (4) have been written more conveniently as (23), in which the monomials $x(s)^k$ were replaced by $[s]_q^{(k)}$. In addition, because we have an AT-system of positive discrete measures the $q$-Meixner multiple orthogonal polynomial of the first kind $M^{\bar{\alpha}, \bar{\beta}}_{q, \bar{n}}(s)$ has exactly $|\bar{n}|$ different zeros on $\mathbb{R}^+$ (see [28], theorem 2.1, pp. 26–27). Finally, in Section 4 we will recover the multiple Meixner polynomials of the first kind given in [28] as a limiting case of $M^{\bar{\alpha}, \bar{\beta}}_{q, \bar{n}}(s)$.

Let us replace $[s]_q^{(k)}$ in (23) by

$$[s]_q^{(k)} = \frac{q^{k-1/2}}{[k+1]_q^{(1)}} \nabla [s+1]_q^{(k+1)}, \quad (24)$$

then, we have

$$\sum_{s=0}^{\infty} M^{k, \beta}_{q, \bar{n}}(s) \nabla [s+1]_q^{(k+1)} v^{\alpha, \beta}_i(s) \triangle x(s - 1/2) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, \ldots, r. \quad (25)$$

Using summation by parts and condition $v^{\alpha, \beta}_i(-1) = v^{\alpha, \beta}_i(\infty) = 0$, we have that for any two polynomials $\phi$ and $\psi$ in the variable $x(s)$,

$$\sum_{s=0}^{\infty} \Delta \phi(s) \psi(s) v^{\alpha, \beta}_i(s) \triangledown x_1(s) = - \sum_{s=0}^{\infty} \phi(s) \nabla \left( \psi(s) v^{\alpha, \beta}_i(s) \right) \triangle x(s - 1/2). \quad (25)$$
Thus, the following relation
\[
\sum_{s=0}^{\infty} \nabla \left( M_{q,\bar{\alpha}}^{\bar{\alpha}i} (s) v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \right) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = - \sum_{s=0}^{\infty} M_{q,\bar{\alpha}}^{\bar{\alpha}i} (s) v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \Delta [s]_{q}^{(k+1)} \Delta x(s - 1/2)
\]
holds. Equivalently,
\[
\sum_{s=0}^{\infty} \nabla \left( M_{q,\bar{\alpha}}^{\bar{\alpha}i} (s) v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \right) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = 0, \quad 0 \leq k \leq n_{i} - 1, \quad i = 1, \ldots, r.
\]

Observe that
\[
\nabla \left( M_{q,\bar{\alpha}}^{\bar{\alpha}i} (s) v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \right) = q^{-|\bar{\alpha}|+1/2} \frac{a_{i} \bar{\alpha}}{\alpha x(\bar{\beta} - 1)} v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) Q_{q,\bar{\alpha}+\bar{\beta}} (s),
\]
where
\[
\epsilon_{q,\bar{\alpha}}^{a_{i} \bar{\alpha}} = \left( a_{i} q |\bar{\alpha}|+\bar{\beta} - 1 \right).
\]

This coefficient will be extensively used throughout the paper and \( Q_{q,\bar{\alpha}+\bar{\beta}} (s) \) represents a monic polynomial \( x^{|\bar{\alpha}|+1} + \) lower degree terms. Consequently,
\[
\sum_{s=0}^{\infty} Q_{q,\bar{\alpha}+\bar{\beta}} (s) v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = \sum_{s=0}^{\infty} \nabla \left( M_{q,\bar{\alpha}}^{\bar{\alpha}i} (s) v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \right) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = 0.
\]

From the next Lemma 1 we will conclude that \( Q_{q,\bar{\alpha}+\bar{\beta}} (s) = M_{q,\bar{\alpha}+\bar{\beta}} (s) \).

**Lemma 1.** Let the vector subspace \( \mathcal{W} \subset \mathbb{P} \) of polynomials \( W(s) \) of degree at most \( |\bar{\alpha}| + 1 \) in the variable \( x(s) \) be defined by conditions
\[
\sum_{s=0}^{\infty} W(s) [s]_{q}^{(k)} v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \nabla x_{1}(s) = 0, \quad 0 \leq k \leq n_{j}, \quad j = 1, \ldots, r,
\]
\[
W(-1) \neq 0.
\]

Then, the spanning set of the system \( \left\{ M_{q,\bar{\alpha}+\bar{\beta}}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \right\}_{j=1}^{r} \) coincides with \( \mathcal{W} \) (see notation (12) for the index \( \bar{\alpha}_{j,1/q} \)).

**Proof.** The polynomials \( M_{q,\bar{\alpha}+\bar{\beta}}^{a_{i} / q, \bar{\alpha}^{-1}} (-1) \neq 0, j = 1, \ldots, r, \) because they have exactly \( |\bar{\alpha}| + 1 \) different zeros on \( \mathbb{R}^{+} \). Moreover, from orthogonality relations
\[
\sum_{s=0}^{\infty} M_{q,\bar{\alpha}+\bar{\beta}}^{a_{i} / q, \bar{\alpha}^{-1}} (s) [s]_{q}^{(k)} v_{q}^{a_{i} / q, \bar{\alpha}^{-1}} (s) \nabla x_{1}(s) = 0, \quad 0 \leq k \leq n_{j}, \quad j = 1, \ldots, r,
\]
we have that the system of polynomials \( M_{q,\bar{\alpha}+\bar{\beta}}^{a_{i} / q, \bar{\alpha}^{-1}} (s), j = 1, \ldots, r, \) belongs to \( \mathcal{W} \).

Assume that there exist numbers \( \lambda_{j}, j = 1, \ldots, r, \) such that
\[
\sum_{j=1}^{r} \lambda_{j} M_{q,\bar{\alpha}+\bar{\beta}}^{a_{i} / q, \bar{\alpha}^{-1}} (s) = 0, \quad \text{where} \quad \sum_{j=1}^{r} |\lambda_{j}| > 0.
\]
Multiplying the previous equation by \( [s]_{\bar{q}}^{(n_i - 1)} v_{\bar{q}}^{\alpha_i} (s) \) and then summing from \( s = 0 \) to \( \infty \), one gets

\[
\sum_{j=1}^{r} \lambda_j \sum_{s=0}^{\infty} M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) [s]_{\bar{q}}^{(n_i - 1)} v_{\bar{q}}^{\alpha_i} (s) \nabla x_1(s) = 0.
\]

Thus, from relations

\[
\sum_{s=0}^{\infty} M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) [s]_{\bar{q}}^{(n_i - 1)} v_{\bar{q}}^{\alpha_i} (s) \nabla x_1(s) = c \delta_{i,k}, \quad c \in \mathbb{R} \setminus \{0\},
\]  

one concludes that \( \lambda_k = 0 \) for \( k = 1, \ldots, r \). Here \( \delta_{i,k} \) denotes the Kronecker delta symbol. Thus, the assumption (28) is false, so the system \( \{ M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) \}_{i=1}^{r} \) is linearly independent in \( \mathbb{W} \).

Moreover, we know that any polynomial from vector subspace \( \mathbb{W} \) is determined by its \( |\bar{\beta}| + 2 \) coefficients while \( |\bar{\beta}| + 2 + r \) conditions are imposed on \( \mathbb{W} \). Consequently the dimension of \( \mathbb{W} \) is at most \( r \). Therefore, \( \text{span} \{ M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) \}_{i=1}^{r} = \mathbb{W}. \)

From Equation (27) and Lemma 1 we have

\[
\nabla \left( M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) v_{\bar{q}}^{\alpha_i} (s) \right) = q^{-|\bar{\beta}| + 1/2} \frac{\alpha_i x (\bar{\beta} - 1)}{\alpha_i x (\bar{\beta} + 1)} v_{\bar{q}}^{\alpha_i} (s) M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s).
\]

Then, for monic \( q \)-Meixner multiple orthogonal polynomials of the first kind we have \( r \) raising operators

\[
\mathcal{D}_{\bar{q}}^{\alpha_i,\beta} M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) = -q^{1/2} M_{\bar{q},\bar{\beta} + 1}(s), \quad i = 1, \ldots, r,
\]  

(30)

where

\[
\mathcal{D}_{\bar{q}}^{\alpha_i,\beta} \overset{\text{def}}{=} \frac{\alpha_i x (\bar{\beta} - 1)}{q^{-|\bar{\beta}|} v_{\bar{q}}^{\alpha_i}} \left( \frac{1}{v_{\bar{q}}^{\alpha_i} (s)} \nabla v_{\bar{q}}^{\alpha_i} (s) \right).
\]

Furthermore,

\[
\mathcal{D}_{\bar{q}}^{\alpha_i,\beta} f(s) = q^{-|\bar{\beta}| + 1/2} \frac{\alpha_i q^{\beta} - 1 (x(1 - \beta) - x(s)) + x(s)}{v_{\bar{q}}^{\alpha_i} (s)} \mathcal{I} f(s),
\]

for any function \( f(s) \) defined on the discrete variable \( s \). Here \( \mathcal{I} \) denotes the identity operator. We call \( \mathcal{D}_{\bar{q}}^{\alpha_i,\beta} \) a raising operator since the \( i \)-th component of the multi-index \( \bar{n} \) in (30) is increased by 1.

In the sequel we will only consider monic \( q \)-Meixner multiple orthogonal polynomials of the first kind.

**Proposition 1.** The following \( q \)-analogue of Rodrigues-type formula holds:

\[
M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) = G_{\bar{q}}^{\bar{\beta}}(\bar{q}) \Gamma_{\bar{q}}(\bar{q} + s + 1) M_{\bar{q},\bar{\beta}}^{x,1/q,\bar{\beta} - 1}(s) \left( \frac{\Gamma_{\bar{q}}(\bar{\beta} + |\bar{n}| + s)}{\Gamma_{\bar{q}}(\bar{\beta} + |\bar{n}| + s + 1)} \right),
\]  

(31)
where
\[
\mathcal{M}_{q,\vec{n}}^{\vec{\alpha},\vec{\beta}} = \prod_{i=1}^{r} \mathcal{M}_{q,\vec{n}_i}^{\alpha_i,\beta_i}, \quad \mathcal{M}_{q,\vec{n}_i}^{\alpha_i,\beta_i} = (\alpha_i)^{-\vec{\beta}} \nabla_{q^n} (\alpha_i q^{n_i})^{\vec{\alpha}_i},
\] (32)

and
\[
G_{q,\vec{n}}^{\vec{\alpha},\vec{\beta}} = (-1)^{|\vec{n}|} [\beta]_q^{(|\vec{n}|)} q^{-|\vec{n}|} \left( \prod_{i=1}^{r} \frac{\alpha_i^{n_i} \prod_{j=1}^{n_i} q^{\vec{\alpha}_i |\vec{n}| + \beta + j - 1}}{\prod_{j=1}^{n_i} (\alpha_i q^{\vec{\alpha}_i |\vec{n}| + \beta + j - 1} - 1)} \right)^{r \sum_{i=1}^{r} n_i},
\] (33)

with \(|\vec{n}| = n_1 + \cdots + n_{r-1}, |\vec{n}| = 0\).

**Proof.** For \(i = 1, \ldots, r\), applying \(k_i\)-times the raising operators (30) in a recursive way one obtains
\[
\prod_{i=1}^{r} \left( q^{k_i} \right)^{-s} \nabla^{k_i} \Gamma_{q}^{\vec{\beta} + s} \frac{\Gamma_{q}^{\vec{\beta}}}{\Gamma_{q}^{\vec{\beta} + s}} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha},\vec{\beta}}(s) = \left[ \beta - 1 \right] q^{|\vec{n}|/2} \left( \prod_{i=1}^{r} \frac{k_i^{n_i} \prod_{j=1}^{n_i} q^{\vec{\alpha}_i |\vec{n}| + \beta - j - 1}}{\prod_{j=1}^{n_i} (\alpha_i q^{\vec{\alpha}_i |\vec{n}| + \beta - j - 1} - 1)} \right)^{r \sum_{i=1}^{r} n_i} \times \prod_{i=1}^{r} q^{-n_i} \prod_{i=1}^{r} q^{\vec{\alpha}_i} \prod_{i=1}^{r} \mathcal{M}_{q,\vec{n}_i}^{\alpha_i,\beta_i}(s) \frac{\Gamma_{q}^{\vec{\beta} + |\vec{n}| + s}}{\Gamma_{q}^{\vec{\beta} - |\vec{n}|}} \Gamma_{q}^{\vec{\beta} + s}. \]

Taking \(n_1 = n_2 = \cdots = n_r = 0\) and replacing \(\beta\) by \(\beta + |\vec{n}|\), \(\alpha_i\) by \(\alpha_i q^{k_i}\), and \(k_i\) by \(n_i\), for \(i = 1, \ldots, r\), yields the Formula (31). \(\square\)

3.2. \(q\)-Difference Equation for the \(q\)-Analogue of Multiple Meixner Polynomials of the First Kind

We will find a lowering operator for the \(q\)-Meixner multiple orthogonal polynomials of the first kind. We will follow a similar strategy used in [32].

**Lemma 2.** Let \(\mathbb{V}\) be the linear subspace of polynomials \(Q(s)\) on the lattice \(x(s)\) of degree at most \(|\vec{n}| - 1\) defined by the following conditions
\[
\sum_{s=0}^{\infty} Q(s)[s]_q^{(k)} u_q^{q^{a_i} q^{n_i + 1}}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2 \quad \text{and} \quad j = 1, \ldots, r.
\]

Then, the system \(\left\{ \mathcal{M}_{q,\vec{n} - \vec{\epsilon}_i}^{\vec{\alpha},\vec{\beta} + 1}(s) \right\}_{i=1}^{r}\), where \(\vec{\alpha}, \vec{\beta} = (a_1, \ldots, qa_i, \ldots, a_r)\), is a basis for \(\mathbb{V}\).

**Proof.** From orthogonality relations
\[
\sum_{s=0}^{\infty} \mathcal{M}_{q,\vec{n} - \vec{\epsilon}_i}^{\vec{\alpha},\vec{\beta} + 1}(s)[s]_q^{(k)} u_q^{q^{a_i} q^{n_i + 1}}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, \ldots, r,
\]
we have that polynomials \(\mathcal{M}_{q,\vec{n} - \vec{\epsilon}_i}^{\vec{\alpha},\vec{\beta} + 1}(s), i = 1, \ldots, r\), belong to \(\mathbb{V}\).

Now, aimed to get a contradiction, let us assume that there exist constants \(\lambda_i, i = 1, \ldots, r\), such that
\[
\sum_{i=1}^{r} \lambda_i \mathcal{M}_{q,\vec{n} - \vec{\epsilon}_i}^{\vec{\alpha},\vec{\beta} + 1}(s) = 0, \quad \text{where} \quad \sum_{i=1}^{r} |\lambda_i| > 0.
\]
Then, multiplying the previous equation by \([s]_q^{(n_k-1)}v_q^{α,β}(s)\) and then taking summation on \(s\) from 0 to \(∞\), one gets

\[
\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{∞} M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s)[s]_q^{(n_k-1)}v_q^{α,β}(s) \nabla x_1(s) = 0.
\]

Thus, from relations

\[
\sum_{s=0}^{∞} M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s)[s]_q^{(n_k-1)}v_q^{α,β}(s) \nabla x_1(s) = c \delta_{i,k}, \quad c \in \mathbb{R} \setminus \{0\},
\]

we deduce that \(λ_k = 0\) for \(k = 1, \ldots, r\). Here \(δ_{i,k}\) represents the Kronecker delta symbol. Therefore, the vectors \(\{M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s)\}_{i=1}^{r}\) are linearly independent in \(\mathbb{V}\). Furthermore, we know that any polynomial of \(\mathbb{V}\) can be determined with \(|\vec{α}|\) coefficients while \(|\vec{α}| - r\) linear conditions are imposed on \(\mathbb{V}\). Consequently the dimension of \(\mathbb{V}\) is at most \(r\). Hence, the system \(\{M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s)\}_{i=1}^{r}\) spans \(\mathbb{V}\), which completes the proof. \(\square\)

Now we will prove that the operator (8) is indeed a lowering operator for the sequence of \(q\)-Meixner multiple orthogonal polynomials of the first kind \(M_{q,\vec{α}}^{\vec{α}}(s)\).

**Lemma 3.** The following relation holds:

\[
ΔM_{q,\vec{α}}^{\vec{α},\vec{β}}(s) = \sum_{i=1}^{r} q_{|\vec{α}| - n_i + 1/2} 1 - \alpha_i q_n^{n_i + 1} \frac{1}{1 - \alpha_i q_{|\vec{α}| + \vec{β}}} [n_i]_q^{(1)} M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s).
\]

**Proof.** Using summation by parts we have

\[
\sum_{s=0}^{∞} ΔM_{q,\vec{α}}^{\vec{α},\vec{β}}(s)[s]_q^{(k)}v_q^{α,β}(s) \nabla x_1(s) = -\sum_{s=0}^{∞} M_{q,\vec{α}}^{\vec{α},\vec{β}}(s) \nabla \left[s]_q^{(k)}v_q^{α,β}(s) \nabla x_1(s)\right]
\]

\[
= -\sum_{s=0}^{∞} M_{q,\vec{α}}^{\vec{α},\vec{β}}(s) \varphi_{j,k}(s) v_q^{α,β}(s) \nabla x_1(s),
\]

where

\[
\varphi_{j,k}(s) = q^{1/2} \left(\frac{q^{-1/2}s^{-1/2}x(s)}{\overset{β}{\overset{α}{x}}} + 1\right) [s]_q^{(k)} - q^{-1/2} \frac{x(s)}{\overset{α}{\overset{β}{x}}} [s-1]_q^{(k)},
\]

is a polynomial of degree \(≤ k + 1\) in the variable \(x(s)\). Consequently, from the orthogonality conditions (23) we get

\[
\sum_{s=0}^{∞} ΔM_{q,\vec{α}}^{\vec{α},\vec{β}}(s)[s]_q^{(k)}v_q^{α,β}(s) \nabla x_1(s) = 0, \quad 0 ≤ k ≤ n_j - 2, \quad j = 1, \ldots, r.
\]

Hence, from Lemma 2, \(ΔM_{q,\vec{α}}^{\vec{α},\vec{β}}(s) \in \mathbb{V}\). Moreover, \(ΔM_{q,\vec{α}}^{\vec{α},\vec{β}}(s)\) can be expressed as a linear combination of polynomials \(\{M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s)\}_{i=1}^{r}\), i.e.,

\[
ΔM_{q,\vec{α}}^{\vec{α},\vec{β}}(s) = \sum_{i=1}^{r} \xi_i M_{q,\vec{α} - \vec{β}}^{\vec{α},\vec{β} + 1}(s), \quad \sum_{i=1}^{r} |\xi_i| > 0.
\]
Multiplying both sides of the Equation (37) by $\left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s)$ and using relations (34) one has

$$\sum_{s=0}^{\infty} \Delta M_{q, \beta}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s) = \sum_{i=1}^{r} \xi_i \sum_{s=0}^{\infty} M_{q, \beta - \vec{e}_i}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s)$$

$$= \sum_{s=0}^{\infty} \xi_k \sum_{s=0}^{\infty} M_{q, \beta - \vec{e}_k}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s).$$

(38)

If we replace $\left[ s \right]_q^{(k)}$ by $\left[ s \right]_q^{(n_q - 1)}$ in the left-hand side of Equation (36), then Equation (38) transforms into

$$\sum_{s=0}^{\infty} \Delta M_{q, \beta}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s) = - \sum_{s=0}^{\infty} M_{q, \beta}^k(s) \varphi_{k,n_k - 1}(s) v_q^{\alpha_q, \beta}(s) \nabla x_1(s)$$

$$= q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{\alpha_k x(\beta)} \sum_{s=0}^{\infty} M_{q, \beta}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta}(s) \nabla x_1(s).$$

(39)

For this transformation we have used that $x(s) \left[ s - 1 \right]_q^{(n_q - 1)} = \left[ s \right]_q^{(n_q)}$ to get

$$\varphi_{k,n_k - 1}(s) = - \frac{q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{\alpha_k x(\beta)}}{\left[ s \right]_q^{(n_q)}} + \text{lower degree terms.}$$

On the other hand, from (30) one has that

$$q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{\alpha_k x(\beta)} v_q^{\alpha_q, \beta}(s) M_{q, \beta}^k(s) = - q^{-1/2} \nabla \left( v_q^{\alpha_q, \beta + 1}(s) M_{q, \beta - \vec{e}_k}^{k+1}(s) \right).$$

(40)

Considering (40) and using once more summation by parts on the right-hand side of Equation (39) we obtain

$$\sum_{s=0}^{\infty} \Delta M_{q, \beta}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s)$$

$$= - q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{1 - \alpha_k q^{n_k + \beta}} \sum_{s=0}^{\infty} \left[ s \right]_q^{(n_q)} \nabla \left( v_q^{\alpha_q, \beta + 1}(s) M_{q, \beta - \vec{e}_k}^{k+1}(s) \right) \nabla x_1(s)$$

$$= q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{1 - \alpha_k q^{n_k + \beta}} \sum_{s=0}^{\infty} M_{q, \beta - \vec{e}_k}^{k+1}(s) \Delta \left[ s \right]_q^{(n_q)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s).$$

Since $\Delta \left[ s \right]_q^{(n_q)} = q^{3/2 - n_k} \left[ n_k \right]_q^{(1)} \left[ s \right]_q^{(n_q - 1)}$, we have

$$\sum_{s=0}^{\infty} \Delta M_{q, \beta}^k(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s)$$

$$= q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{1 - \alpha_k q^{n_k + \beta}} \left[ n_k \right]_q^{(1)} \sum_{s=0}^{\infty} M_{q, \beta - \vec{e}_k}^{k+1}(s) \left[ s \right]_q^{(n_q - 1)} v_q^{\alpha_q, \beta + 1}(s) \nabla x_1(s).$$

Comparing this equation with (38), we obtain the coefficients in the expansion (37), i.e.,

$$\xi_k = q^{-1/2} \frac{1 - \alpha_k q^{n_k + \beta}}{1 - \alpha_k q^{n_k + \beta}} \left[ n_k \right]_q^{(1)}.$$

Therefore, relation (35) holds.
Theorem 1. The $q$-Meixner multiple orthogonal polynomial of the first kind $M_{q,\vec{n}}^{\alpha,\beta} (s)$ satisfies the following $(r+1)$-order $q$-difference equation

$$
\prod_{i=1}^{r} D_q^{\alpha_i,\beta+1} \Delta M_{q,\vec{n}}^{\alpha,\beta} (s) = - \sum_{i=1}^{r} q^{\vec{\alpha}_i - \vec{n}_i + 1} \frac{1 - q^{\vec{\alpha}_i + \vec{\beta}}}{1 - \alpha_i q^{\vec{\alpha}_i + \vec{\beta}}} [n_i]_q^{(1)} \prod_{j \neq i}^{r} D_q^{\alpha_j,\beta+1} M_{q,\vec{n}}^{\alpha,\beta} (s).
$$

Proof. Since the operators (30) commute, we write

$$
\prod_{i=1}^{r} D_q^{\alpha_i,\beta+1} = \left( \prod_{j \neq i}^{r} D_q^{\alpha_j,\beta+1} \right) D_q^{\alpha_i,\beta+1}.
$$

Using (30) when acting on Equation (35) with the product of operators (42), we obtain (41), i.e.,

$$
\prod_{i=1}^{r} D_q^{\alpha_i,\beta+1} \Delta M_{q,\vec{n},\vec{\alpha}}^{\alpha,\beta} (s) = \sum_{i=1}^{r} q^{\vec{\alpha}_i - \vec{n}_i + 1/2} \frac{1 - q^{\vec{\alpha}_i + \vec{\beta}}}{1 - \alpha_i q^{\vec{\alpha}_i + \vec{\beta}}} [n_i]_q^{(1)} \prod_{j \neq i}^{r} D_q^{\alpha_j,\beta+1} \left( D_q^{\alpha_i,\beta+1} M_{q,\vec{n},\vec{\alpha}}^{\alpha,\beta} (s) \right)
$$

$$
= - \sum_{i=1}^{r} q^{\vec{\alpha}_i - \vec{n}_i + 1} \frac{1 - q^{\vec{\alpha}_i + \vec{\beta}}}{1 - \alpha_i q^{\vec{\alpha}_i + \vec{\beta}}} [n_i]_q^{(1)} \prod_{j \neq i}^{r} D_q^{\alpha_j,\beta+1} M_{q,\vec{n},\vec{\alpha}}^{\alpha,\beta} (s).
$$

This completes the proof of the theorem. $\square$

3.3. Recurrence Relation for $q$-Meixner Multiple Orthogonal Polynomials of the First Kind

In this section we will study the nearest neighbor recurrence relation for any multi-index $\vec{n}$. The approach presented here differs from those used in [28,42]. We begin by defining the following linear difference operator

$$
F_{q,\vec{n}_i} := s^{\vec{n}_i - 1}(s) \nabla^{n_i} s_{q,\vec{\alpha}}(s),
$$

where $n_i$ is the $i$-th entry of the vector index $\vec{n}$ and $s_{q,\vec{\alpha}}$ is defined in the variable $s$ and depends on the $i$-th component of the vector orthogonality measure $\vec{\mu}$. In the case that $s_{q,\vec{\alpha}}$ depends also on the $i$-th component of $\vec{n}$, then the index $k = n_i$; otherwise $k = i$.

Lemma 4. Let $n_i$ be a positive integer and let $f(s)$ be a function defined on the discrete variable $s$. The following relation is valid

$$
F_{q,\vec{n}_i} x(s) f_q(s) = q^{-n_i + 1/2} x(n_i) s^{\vec{n}_i - 1}(s) \nabla^{n_i - 1} s_{q,\vec{\alpha}}(s) f_q(s) + q^{-n_i} (x(s) - x(n_i)) F_{q,\vec{n}_i} f_q(s).
$$

Proof. Let us act $n_i$-times with backward difference operators (9) on the product of functions $x(s)f(s)$. Assume that $n_i \geq N > 1$,

$$
\nabla^{n_i} x(s)f(s) = \nabla^{n_i - 1}(\nabla x(s)f(s)) = \nabla^{n_i - 1}(q^{-1/2} f(s) + x(s - 1) \nabla f(s))
$$

$$
= q^{-1/2} \nabla^{n_i - 1} f(s) + \nabla^{n_i - 1} (x(s - 1) \nabla f(s))
$$

$$
= q^{-1/2} \nabla^{n_i - 1} f(s) + \nabla^{n_i - 2} (\nabla x(s - 1) \nabla f(s)).
$$

Repeating this process, but on the second term of the right-hand side of Equation (45)

$$
\nabla^{n_i} x(s)f(s) = (q^{1/2 - n_i} + \cdots + q^{-5/2} + q^{-3/2} + q^{-1/2}) \nabla^{n_i - 1} f(s) + x(s - n_i) \nabla^{n_i} f(s)
$$

$$
= q^{1/2 - n_i} x(n_i) \nabla^{n_i - 1} f(s) + x(s - n_i) \nabla^{n_i} f(s).
$$
Thus,
\[ \nabla^n x(s)f(s) = q^{-n_i+1/2}x(n_i)\nabla^{n_i-1}f(s) + q^{-n_i}(x(s) - x(n_i))\nabla^nf(s), \quad n_i \geq 1. \] (46)

Now, to involve the difference operator \( F_{q, n_i} \) in the above equation, we multiply the Equation (46) from the left by \( g_{q, l}(s)^{-1} \) and replace \( f(s) \) by \( g_{q, k}(s)f(s) \). Therefore, the Equation (46) transforms into (44). \( \square \)

**Theorem 2.** The \( q \)-Meixner multiple orthogonal polynomials of the first kind satisfy the following \((r + 2)\)-term recurrence relation

\[ x(s)M_{q, i \ell + e_i}^{k, \beta}(s) = M_{q, i \ell + e_i}^k(s) + b_{k, j} M_{q, i \ell + e_j}^{k, \beta}(s) + \sum_{i=1}^r x(n_i)\alpha_i q^{[i]}x(\beta + [i]) - 1)B_{q, i \ell + e_i}, \] (47)

where

\[ b_{k, j} = -\alpha_k q^{[i]+n_i+1} + (q - 1) \prod_{i=1}^{r} \frac{x(n_i)}{c_{q, i \ell + n_i e_i}} \left( \frac{\alpha_i q^{n_i} - 1}{\alpha_i q^{[i]}x(\beta + [i]) - 1} \right) \]

and

\[ B_{q, i \ell + e_i} = \frac{\alpha_i q^{n_i} - 1}{\alpha_i q^{[i]+n_i+1}} \prod_{i \neq j} \frac{\alpha_j q^{n_j} - 1}{\alpha_j q^{[j]+n_j+1}} \prod_{i=1}^r \frac{c_{q, i \ell + n_i e_i}}{c_{q, i \ell + n_j e_j}}. \]

**Proof.** Let

\[ f_n(s; \beta) = \frac{\Gamma_q(\beta + n + s)}{\Gamma_q(\beta + n)\Gamma_q(s + 1)}, \quad \text{where } n = [i]. \]

We will use Lemma 4 involving this function \( f_n(s; \beta) \) as well as difference operator (32). Consider equation

\[ (\alpha_k)^{-s}\nabla^{n_k+1}(\alpha_k q^{n_k+1})f_{n+1}(s; \beta) = (\alpha_k)^{-s}\nabla^{n_k} \left( q^{-s+1/2} \nabla \left( (\alpha_k q^{n_k+1})f_{n+1}(s; \beta) \right) \right) \]

\[ = q^{1/2}(\alpha_k)^{-s}\nabla^{n_k} \left( (\alpha_k q^{n_k})^s \left( 1 + \frac{c_{q, i \ell + n_i e_i}}{(\alpha_k q^{n_k+1})x(\beta + [i])} \nabla x(s) \right) f_n(s; \beta) \right), \]

which can be rewritten in terms of difference operators (32) as follows

\[ q^{-1/2}M_{q, i \ell + e_i}^{k, \beta}f_{n+1}(s; \beta) = M_{q, i \ell + e_i}^k f_n(s; \beta) + \frac{c_{q, i \ell + n_i e_i}}{(\alpha_k q^{n_k+1})x(\beta + [i])}M_{q, i \ell + e_i}^{n_k} x(s) f_n(s; \beta). \] (48)

Since operators (32) commute, the multiplication of Equation (48) from the left-hand side by the product \( \prod_{i \neq k}^r M_{q, i \ell + e_i}^{n_k} \) yields the following relation

\[ M_{q, i \ell + e_i}^k x(s) f_n(s; \beta) = \frac{(\alpha_k q^{n_k+1})x(\beta + [i])}{c_{q, i \ell + n_k e_i}} \left( q^{-1/2}M_{q, i \ell + e_i}^k f_{n+1}(s; \beta) - M_{q, i \ell + e_i}^k f_n(s; \beta) \right). \] (49)
Let us recursively use Lemma 4 involving the product of \( r \) difference operators acting on the function \( f_n(s; \beta) \), which in this case is the operator \( \mathcal{M}_{q, s}^{\beta} \) (see expression (32)). Thus,

\[
\left( q^{\frac{1}{2}} \mathcal{M}_{q, s}^{\beta} x(s) - q^{-1/2} \sum_{i=1}^{r} \left[ \mathcal{M}_{q, s}^{\alpha_i, \beta} \prod_{i=1}^{r} \frac{\alpha_i^q q^{n_i+1}}{c_{q, s}^{\alpha_i, \beta}} \right] \frac{x(n_i)}{c_{q, s}^{\alpha_i, \beta}} \prod_{i=1}^{r} \frac{1}{c_{q, s}^{\alpha_i, \beta}} \right) f_n(s; \beta)
\]

\[
= \left( x(s) \prod_{i=1}^{r} \frac{c_{q, s}^{\alpha_i, \beta}}{c_{q, s}^{\alpha_i, \beta}} \sum_{i=1}^{r} \frac{q^{\frac{1}{2}} x(n_i)}{c_{q, s}^{\alpha_i, \beta}} \prod_{i=1}^{r} \frac{\alpha_i^q q^{n_i+1}}{c_{q, s}^{\alpha_i, \beta}} \left( \frac{1 - q^{2\frac{1}{2}}}{} \right) \prod_{i=1}^{r} \frac{x(n_i)}{c_{q, s}^{\alpha_i, \beta}} \prod_{i=1}^{r} \frac{1}{c_{q, s}^{\alpha_i, \beta}} \right) f_{n-1}(s; \beta).
\]

Using the expressions (49) and (50) one gets

\[
x(s) \mathcal{M}_{q, s}^{\beta} f_n(s; \beta) = q^{\frac{1}{2}} - \frac{1}{2} \sum_{i=1}^{r} \frac{q^{\frac{1}{2}} x(n_i)}{c_{q, s}^{\alpha_i, \beta}} \prod_{i=1}^{r} \frac{\alpha_i^q q^{n_i+1}}{c_{q, s}^{\alpha_i, \beta}} \mathcal{M}_{q, s}^{\beta} \mathcal{M}_{q, s}^{\beta} f_{n+1}(s; \beta)
\]

\[
+ \prod_{i=1}^{r} \frac{c_{q, s}^{\alpha_i, \beta}}{c_{q, s}^{\alpha_i, \beta}} \left( \sum_{i=1}^{r} \frac{q^{\frac{1}{2}} x(n_i)}{c_{q, s}^{\alpha_i, \beta}} \prod_{i=1}^{r} \frac{1}{c_{q, s}^{\alpha_i, \beta}} \frac{1}{c_{q, s}^{\alpha_i, \beta}} \frac{x(n_i)}{c_{q, s}^{\alpha_i, \beta}} \prod_{i=1}^{r} \frac{1}{c_{q, s}^{\alpha_i, \beta}} \right) f_{n-1}(s; \beta).
\]

Observe that when \( l = i \) in the above expression we have

\[
\mathcal{M}_{q, n_i}^{\beta} f_n(s; \beta) = q^{-1/2} \frac{\alpha_i^q q^{n_i+1}}{c_{q, s}^{\alpha_i, \beta}} \mathcal{M}_{q, n_i}^{\beta} f_n(s; \beta) - \frac{1}{c_{q, s}^{\alpha_i, \beta}} \mathcal{M}_{q, n_i}^{\beta} f_{n-1}(s; \beta).
\]

Therefore,

\[
x(s) \mathcal{M}_{q, s}^{\beta} f_n(s; \beta)
\]

\[
= q^{-1/2} \frac{1}{c_{q, s}^{\alpha_i, \beta}} \mathcal{M}_{q, s}^{\beta} f_n(s; \beta) + b_{n, k} \mathcal{M}_{q, s}^{\beta} f_n(s; \beta)
\]

\[
- q^{1/2} \prod_{i=1}^{r} \frac{\alpha_i^q q^{n_i+1}}{c_{q, s}^{\alpha_i, \beta}} \mathcal{M}_{q, s}^{\beta} f_{n+1}(s; \beta) + b_{n, k} \mathcal{M}_{q, s}^{\beta} f_n(s; \beta).
\]

Finally, multiplying from the left both sides of the previous expression by \( q^{\frac{1}{2}} \mathcal{M}_{q, s}^{\beta} \mathcal{M}_{q, s}^{\beta} f_{n}(s; \beta) \) and using Rodrigues-type Formula (31) we obtain (47). This completes the proof of the theorem. \( \square \)

3.4. On Some \( q \)-Analogue of Multiple Meixner Polynomials of the Second Kind

Consider the following vector measure \( \bar{v}_q \) with positive \( q \)-discrete components

\[
\bar{v}_q = \sum_{s=0}^{\infty} \bar{v}_q^a \delta_k (k) \triangle (k - 1/2) \delta (k - s), \quad i = 1, 2, \ldots, r,
\]

(51)
where $v_{q}^{\alpha,\beta}(s)$ is defined in (22), but here the domain for its non-identically zero part is
$s \in \Omega = \mathbb{R} \setminus \{ \mathbb{Z}^{-} \cup \{-\beta_{i} - \beta_{i} - 1, -\beta_{i} - 2, \ldots \} \}$, $\beta_{i} > 0, \beta_{i} - \beta_{j} \notin \mathbb{Z}$ for all $i \neq j$, and $0 < \alpha < 1$. Indeed,

$$v_{q}^{\alpha,\beta}(s) = \begin{cases} \frac{\alpha^{s} \Gamma_{q}(\beta_{1} + s)}{\Gamma_{q}(s + 1)}, & \text{if } s \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 3.** A polynomial $M_{q, \vec{n}}^{\alpha,\beta}(s)$, with multi-index $\vec{n} \in \mathbb{N}^{r}$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} M_{q, \vec{n}}^{\alpha,\beta}(s) [s]_{q}^{(k)} v_{q}^{\alpha,\beta}(s) \Delta x(s - 1/2) = 0, \quad 0 \leq k \leq n_{i} - 1, \quad i = 1, \ldots, r,$$  

(52)

is said to be the $q$-Meixner multiple orthogonal polynomial of the second kind.

The general orthogonality relations (4) have been conveniently written involving the $q$-analogue of the Stirling polynomials (6) as in relations (52). In Section 5 we will address the AT-property of the system of positive discrete measures (51). This fact guarantees that the $q$-Meixner multiple orthogonal polynomial of the second kind $M_{q, \vec{n}}^{\alpha,\beta}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^{+}$ (see [28], theorem 2.1, pp. 26–27). In Section 4, the multiple Meixner polynomials of the second kind (16) given in [28] will be recovered as $q$ approaches 1.

To find a raising operator we substitute $[s]_{q}^{(k)}$ in (52) for the finite-difference expression (24) and then we use summation by parts along with conditions $v_{q}^{\alpha,\beta}(1) = v_{q}^{\alpha,\beta}(\infty) = 0$. Thus,

$$\sum_{s=0}^{\infty} M_{q, \vec{n}}^{\alpha,\beta}(s) \nabla [s + 1]_{q}^{(k+1)} v_{q}^{\alpha,\beta}(s) \Delta x(s - 1/2) = 0, \quad 0 \leq k \leq n_{i} - 1, \quad i = 1, \ldots, r.$$  

Using (25), one gets

$$\sum_{s=0}^{\infty} \nabla \left( M_{q, \vec{n}}^{\alpha,\beta}(s) v_{q}^{\alpha,\beta}(s) \right) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = -\sum_{s=0}^{\infty} M_{q, \vec{n}}^{\alpha,\beta}(s) v_{q}^{\alpha,\beta}(s) [s]_{q}^{(k+1)} \Delta x(s - 1/2)$$

$$= -\sum_{s=0}^{\infty} M_{q, \vec{n}}^{\alpha,\beta}(s) v_{q}^{\alpha,\beta}(s) \nabla [s + 1]_{q}^{(k+1)} \Delta x(s - 1/2).$$

Hence

$$\sum_{s=0}^{\infty} \nabla \left( M_{q, \vec{n}}^{\alpha,\beta}(s) v_{q}^{\alpha,\beta}(s) \right) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = 0, \quad 0 \leq k \leq n_{i} - 1, \quad i = 1, \ldots, r,$$

where

$$\nabla \left( M_{q, \vec{n}}^{\alpha,\beta}(s) v_{q}^{\alpha,\beta}(s) \right) = \frac{q^{-|\vec{n}|+1/2, \alpha,\beta_{i}^{-1}}}{\alpha x(\beta_{i}^{-1})} v_{q}^{\alpha,\beta_{i}^{-1}}(s) P_{q, \vec{\alpha} + \vec{\beta}}(s).$$

$P_{q, \vec{\alpha} + \vec{\beta}}(s)$ denotes a monic polynomial of degree $|\vec{n}| + 1$. Therefore, from (52) the relation

$$\sum_{s=0}^{\infty} P_{q, \vec{n}}^{\alpha,\beta_{i}^{-1}}(s) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = \sum_{s=0}^{\infty} \nabla \left( M_{q, \vec{n}}^{\alpha,\beta}(s) v_{q}^{\alpha,\beta}(s) \right) [s]_{q}^{(k+1)} \Delta x(s - 1/2) = 0,$$
implies that \( \mathcal{P}_{q, \tilde{a} + \tilde{c}}(s) = M_{q, \tilde{a} + \tilde{c}}^{\alpha/\beta - \tilde{c}}(s) \). Therefore

\[
\nabla \left( M_{q, \tilde{a}}^{\alpha/\beta}(s) v_q^{\alpha, \beta_1}(s) \right) = \frac{q^{-|\tilde{a}| + 1/2} c_{\tilde{a}, \tilde{b}}^{-1}}{ax(\tilde{b}_i - 1)} c_q^{\alpha/\beta - \tilde{c}}(s) M_{q, \tilde{a} + \tilde{c}}^{\alpha/\beta}(s),
\]

which leads to the following \( r \) raising operators for the monic \( q \)-Meixner multiple orthogonal polynomials of the second kind

\[
\mathcal{D}_{q, \tilde{a}}^{\alpha/\beta} M_{q, \tilde{a}}^{\alpha, \beta}(s) = -q^{1/2} M_{q, \tilde{a} + \tilde{c}}^{\alpha/\beta - \tilde{c}}(s).
\] (53)

The operator \( \mathcal{D}_{q, \tilde{a}}^{\alpha/\beta} \) is given in (30) with the replacements: \( a_i \) by \( \alpha \) and \( \beta_i \) by \( \beta_i \), respectively. Indeed,

\[
\mathcal{D}_{q, \tilde{a}}^{\alpha/\beta} f(s) = \frac{q^{|\tilde{a}| + 1/2}}{c_{\tilde{a}, \tilde{b}}^{-1}} \left( c_q^{\alpha, \beta_1}(x(1 - \beta_i) - x(s)) + x(s) \right) \nabla - x(s) \nabla f(s),
\] (54)

holds for any function \( f(s) \) defined on the discrete variable \( s \).

**Proposition 2.** The following finite-difference analogue of the Rodrigues-type formula holds:

\[
M_{q, \tilde{a}}^{\alpha/\beta}(s) = \mathcal{G}_{q, \tilde{a}}^{\alpha/\beta, \alpha} \frac{\Gamma_q(s + 1)}{\alpha^s} \mathcal{N}_{q, \tilde{a}}^{\alpha/\beta} \left( \frac{(a q |\tilde{a}|)^s}{\Gamma_q(s + 1)} \right),
\] (55)

where

\[
\mathcal{N}_{q, \tilde{a}}^{\alpha/\beta} = \prod_{i=1}^r \mathcal{N}_{q, \tilde{a}}^{\alpha/\beta_i}, \quad \mathcal{N}_{q, \tilde{a}}^{\alpha/\beta_i} = \frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i + s)} \nabla_i \left( \frac{\Gamma_q(\beta_i + n_i + s)}{\Gamma_q(\beta_i + n_i)} \right),
\]

and

\[
\mathcal{G}_{q, \tilde{a}}^{\alpha/\beta, \alpha} = (-1)^{|\tilde{a}|} (a q |\tilde{a}|)^{|\tilde{a}|} \left( \frac{r}{s} \prod_{i=1}^n \frac{q^{\alpha_i + j - 1}}{c_{\tilde{a}, \tilde{b}}^{-1}} \right) \left( \frac{r}{s} \prod_{j=1}^{|\tilde{b}|} [\beta_i]^q \right). \] (57)

**Proof.** We follow the same pattern given in Proposition 1 adapted to the operator \( \mathcal{N}_{q, \tilde{a}}^{\alpha/\beta} \). For \( i = 1, \ldots, r \), by applying \( k_i \)-times the raising operators (53) in a recursive way, the following expression holds

\[
\prod_{i=1}^r \frac{\Gamma_q(\beta_i - k_i)}{\Gamma_q(\beta_i - k_i + s)} \nabla_i \left( \frac{\Gamma_q(\beta_i + s)}{\Gamma_q(\beta_i)} \right) M_{q, \tilde{a}}^{\alpha/\beta}(s) = \prod_{i=1}^r \left[ \beta_i - 1 - 1(q)_{q^{\gamma_i - 1/2}}^{(k_i)} \right]^{(-1)^{|\tilde{a}|}} \left( \frac{r}{s} \prod_{i=1}^n \frac{q^{\alpha_i + j - 1}}{c_{\tilde{a}, \tilde{b}}^{-1}} \right) M_{q, \tilde{a}}^{\alpha/\beta}(s) \frac{\Gamma_q(\beta_i + n_i + s)}{\Gamma_q(\beta_i + n_i)}.
\]

Let \( n_1 = n_2 = \cdots = n_r = 0 \) and replace \( \beta_i \) by \( \beta_i + k_i \) and \( \alpha \) by \( a q^{\gamma_i} \). Finally, if we rename the new index component \( k_i \) with the old index component \( n_i \), for \( i = 1, \ldots, r \), the expression (55) holds. □

### 3.5. \( q \)-Difference Equation for the \( q \)-Analogue of Multiple Meixner Polynomials of the Second Kind

In this section we will find the lowering operator for the \( q \)-Meixner multiple orthogonal polynomials of the second kind.

**Lemma 5.** The \( q \)-Meixner multiple orthogonal polynomials of the second kind satisfy the following property

\[
\sum_{s = 0}^{\infty} M_{q, \tilde{a} + \tilde{c}}^{\alpha/\beta - \tilde{c}}(s) \frac{(n_1 - 1)}{q} \frac{v_q^{\alpha, \beta_1} + 1}{x_1(s)} = m_{k_j} \sum_{s = 0}^{\infty} M_{q, \tilde{a} - \tilde{c}}^{\alpha/\beta - \tilde{c}}(s) \frac{(n_1 - 1)}{q} \frac{v_q^{\alpha, \beta_1 + 1}}{x_1(s)}.
\]
where
\[
m_{k,i} = \frac{1 - a q^{[\|i\| + \beta_i}}{1 - a q^{[\|i\| + \beta_i}} x(n_k + \beta_k - \beta_i), \quad k, i = 1, 2, \ldots, r,
\]
and \(\vec{e} = \sum_{i=1}^{r} \vec{e}_i\).

**Proof.** By shifting conveniently the parameters involved in (53) and (54), respectively, one has
\[
\mathbf{M}^{\alpha, \beta}(s) = -q^{-1/2} D_q^{\alpha, \beta, \vec{e}} \left( \mathbf{M}^{\alpha, \beta, \vec{e}}(s) \right)
\]
\[
\mathbf{M}^{\alpha, \beta}(s) = -\frac{q^{[\|i\| - 1]}}{1 - a q^{[\|i\| + \beta_i}} \sum_{i=0}^{\infty} \frac{[s_q]_{[n_k - 1]} q^{[\alpha, \beta, \eta, -\vec{e}_i]}(s) \nabla x_1(s)}{[s_q]_{[n_k - 1]} q^{[\alpha, \beta, \eta, -\vec{e}_i]}(s) \nabla x_1(s)}
\]
\[
\sum_{s=0}^{\infty} \mathbf{M}^{\alpha, \beta}(s) [s_q]_{[n_k - 1]} q^{[\alpha, \beta, \eta, -\vec{e}_i]}(s) \nabla x_1(s) = 0.
\]

Therefore,
\[
\sum_{s=0}^{\infty} \mathbf{M}^{\alpha, \beta}(s) [s_q]_{[n_k - 1]} q^{[\alpha, \beta, \eta, -\vec{e}_i]}(s) \nabla x_1(s) = 0.
\]

Then, by iterating recursively (59), the relation (58) holds. This completes the proof of the lemma.

**Lemma 6.** Let \(\mathbf{M} = (m_{k,i})_{k,i=1}^{r}\) be the matrix with entries given in (58). Then, \(\mathbf{M}\) is non-singular.
Proof. Let us rewrite the entries in $M$ as $m_{k,i} = c_k d_i / [n_k + \beta_k - \beta_i]$, where
\[
c_k = q^{(1 - n_k - \beta_k)/2} \sum_{j=0}^{r} a q^{[i] + \beta_j} x(n_k + \beta_k - \beta_j),
\]
\[
d_i = q^{\beta_i/2} \left( \frac{1 - a q^{[i] + \beta_i}}{a q^{[i] + \beta_i}} \right),
\]

$[n_k + \beta_k - \beta_i] = q^{(1 - n_k - \beta_k + \beta_i)/2} x(n_k + \beta_k - \beta_i)$.

The matrix $M$ is the product of three matrices; that is $M = C \cdot A \cdot D$, where $A = (1 / [n_k + \beta_k - \beta_i])_{k,i=1}^{r}$ and matrices $C, D$ are the diagonal matrices $C = \text{diag}(c_1, c_2, \ldots, c_r)$, $D = \text{diag}(d_1, d_2, \ldots, d_r)$, respectively.

In ([31], lemma 3.2, p. 7) it was proved that $A$ is nonsingular. Therefore, $M$ is also a nonsingular matrix. Indeed,
\[
\det M = q^{(r-|\mathbb{V}|)/2} \prod_{j=1}^{r-1} c_j d_j \det A,
\]
\[
= \frac{\prod_{k=1}^{r-1} \prod_{l=k+1}^{r} x(\beta_l - \beta_k) q^{n_k n_l} x(n_k - n_l + \beta_k - \beta_l)}{\prod_{k=1}^{r} \prod_{l=k+1}^{r} x(n_l + \beta_l - \beta_k)}.
\]

Lemma 7. Let $\mathbb{V}$ be the subspace of polynomials $\theta$ on the discrete variable $x(s)$, such that $\deg \theta \leq |\mathbb{V}| - 1$ and
\[
\sum_{s=0}^{\infty} \theta(s) [s]_{q}^{(k)} \nu_{q}^{\alpha, \beta_i+1} (s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, 2, \ldots, r.
\]

Then, the system $\left\{ M_{\nu_{q}^{\alpha, \beta_i+1}} (s) \right\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$.

Proof. From orthogonality relations
\[
\sum_{s=0}^{\infty} M_{\nu_{q}^{\alpha, \beta_i+1}} (s) [s]_{q}^{(k)} \nu_{q}^{\alpha, \beta_i+1} (s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, 2, \ldots, r,
\]
we have that polynomials $M_{\nu_{q}^{\alpha, \beta_i+1}} (s) \in \mathbb{V}$, for $i = 1, 2, \ldots, r$.

Suppose that there exist constants $\lambda_i$, $i = 1, \ldots, r$, such that
\[
\sum_{i=1}^{r} \lambda_i M_{\nu_{q}^{\alpha, \beta_i+1}} (s) = 0, \quad \text{where} \quad \sum_{i=1}^{r} |\lambda_i| > 0.
\]

Then, multiplying the previous equation by $[s]_{q}^{(n_k-1)} \nu_{q}^{\alpha, \beta_k+1} (s) \nabla x_1(s)$ and then taking summation on $s$ from 0 to $\infty$, one gets
\[
\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{\infty} M_{\nu_{q}^{\alpha, \beta_i+1}} (s) [s]_{q}^{(n_k-1)} \nu_{q}^{\alpha, \beta_k+1} (s) \nabla x_1(s) = 0.
\]
Using Lemma 5 and relation
\[ \sum_{s=0}^{\infty} M_{q,\lambda}^{\alpha,\beta+\xi_j}(s) \psi_q^{(n_1-1)}(s) \psi_q^{\alpha,\beta+1}(s) \nabla x_1(s) \neq 0, \]
we obtain the following homogeneous linear system of equations
\[ \sum_{i=1}^{r} m_{k,i} \lambda_i = 0, \quad k = 1, \ldots, r, \]
or equivalently, in matrix form \( M \lambda = 0 \), where \( \lambda = (\lambda_1, \ldots, \lambda_r)^T \). From Lemma 6, we have that \( M \) is nonsingular, which implies \( \lambda_i = 0 \) for \( i = 1, \ldots, r \); that is, the previous assumption (61) is false. Therefore, \( \{ M_{q,\lambda}^{\alpha,\beta+\xi_j}(s) \}_{i=1}^{r} \) is linearly independent in \( \mathbb{V} \). Furthermore, we know that any polynomial from subspace \( \mathbb{V} \) can be determined with \( |\vec{\lambda}| \) coefficients while \( (|\vec{\lambda}| - r) \) conditions are imposed on \( \mathbb{V} \), consequently the dimension of \( \mathbb{V} \) is at most \( r \). Therefore, the system \( \{ M_{q,\lambda}^{\alpha,\beta+\xi_j}(s) \}_{i=1}^{r} \) spans \( \mathbb{V} \). This completes the proof of the lemma. \( \square \)

Now we will prove that operator (8) is indeed a lowering operator for the sequence of \( q \)-Meixner multiple orthogonal polynomials of the second kind \( M_{q,\lambda}^{\alpha,\beta}(s) \).

**Lemma 8.** The following relation holds:
\[ \Delta M_{q,\lambda}^{\alpha,\beta}(s) = \sum_{i=1}^{r} \xi_i M_{q,\lambda}^{\alpha,\beta+\xi_i}(s), \tag{62} \]
where
\[ \xi_i = \frac{\prod_{k=1, k \neq i}^{\infty} x(n_i + \beta_i - \beta_k) \prod_{i=1}^{r} (1 - q^{n_i + \beta_i}) q^{n_i - n_i + 1/2} (-1)^{i+j} \prod_{k=1}^{\infty} x(n_k + \beta_j - \beta_i) \prod_{j=1}^{r} (1 - q^{n_j + \beta_j}) x(n_j + \beta_j - \beta_i)}{\prod_{k=1, k \neq j}^{\infty} q^{n_i} x(n_k - n_j + \beta_i - \beta_k) \prod_{i=1}^{r} q^{n_i} x(n_i - n_j + \beta_j - \beta_i)}. \tag{63} \]

**Proof.** Using summation by parts we have
\[ \sum_{s=0}^{\infty} \Delta M_{q,\lambda}^{\alpha,\beta}(s) \psi_q^{(k)}(s) \psi_q^{\alpha,\beta+1}(s) \nabla x_1(s) = - \sum_{s=0}^{\infty} M_{q,\lambda}^{\alpha,\beta}(s) \nabla (\psi_q^{(k)} \psi_q^{\alpha,\beta+1}(s)) \nabla x_1(s) = - \sum_{s=0}^{\infty} M_{q,\lambda}^{\alpha,\beta}(s) \varphi_{j,k}(s) \psi_q^{\alpha,\beta_1}(s) \nabla x_1(s), \tag{64} \]

where
\[ \varphi_{j,k}(s) = q^{1/2} \left( \frac{q^\beta x(s)}{x(\beta_j)} + 1 \right) |s|_{q}^{(k)} - q^{-1/2} \frac{x(s)}{ax(\beta_j)} |s-1|_{q}^{(k)}, \]
is a polynomial of degree \( \leq k + 1 \) in the variable \( x(s) \). Then, from the orthogonality conditions (52) we get
\[ \sum_{s=0}^{\infty} \Delta M_{q,\lambda}^{\alpha,\beta}(s) \psi_q^{(k)}(s) \psi_q^{\alpha,\beta+1}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, \ldots, r. \]
From Lemma 7, \( \Delta M_{q', \ell}^{a', \hat{\beta}}(s) \in \mathbb{V} \). Moreover, \( \Delta M_{q', \ell}^{a', \hat{\beta}}(s) \) can be expressed as a linear combination of polynomials \( \{ M_{q', \ell}^{a', \hat{\beta} + \hat{\xi}_i}(s) \}_{i=1}^r \), i.e.,

\[
\Delta M_{q', \ell}^{a', \hat{\beta}}(s) = \sum_{i=1}^r \xi_i M_{q', \ell}^{a', \hat{\beta} + \hat{\xi}_i}(s), \quad \sum_{i=1}^r |\xi_i| > 0. \tag{65}
\]

Thus, for finding explicitly \( \xi_1, \ldots, \xi_r \) one takes into account Lemma 5 and (65) to get

\[
\sum_{s=0}^\infty \Delta M_{q', \ell}^{a', \hat{\beta}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \nabla x_1(s) = \left( \sum_{k=1}^r \xi_k m_{k,i} \right) \sum_{s=0}^\infty M_{q', \ell}^{a', \hat{\beta} + \hat{\xi}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \times v_q^{a', \hat{\beta}_k+1}(s) \nabla x_1(s). \tag{66}
\]

If we replace \([s]_q^{(k)} \) by \([s]_q^{(n_k-1)} \) in the left-hand side of Equation (64), then left-hand side of Equation (66) transforms into relation

\[
\sum_{s=0}^\infty \Delta M_{q', \ell}^{a', \hat{\beta}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \nabla x_1(s) = -\sum_{s=0}^\infty M_{q', \ell}^{a', \hat{\beta}}(s) \varphi_{k,n_k-1}(s) v_q^{a, \hat{\beta}_k}(s) \nabla x_1(s) = \frac{q^{1/2} (1 - a q^{n_k+\hat{\beta}_k})}{\alpha x(\hat{\beta}_k)} \sum_{s=0}^\infty M_{q', \ell}^{a', \hat{\beta}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \nabla x_1(s).
\]

We have used that \( x(s)[s-1]_{q}^{(n_k-1)} = [s]_q^{(n_k)} \) to get

\[
\varphi_{k,n_k-1}(s) = -\frac{q^{-1/2} (1 - a q^{n_k+\hat{\beta}_k})}{\alpha x(\hat{\beta}_k)} [s]_{q}^{(n_k)} + \text{lower degree terms}.
\]

Using Lemma 5, we have that

\[
\sum_{s=0}^\infty \Delta M_{q', \ell}^{a', \hat{\beta}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \nabla x_1(s) = \frac{(1 - a q^{n_k+\hat{\beta}_k})q^{1/2} - n_k+1/2}{1 - a q^{1/2} + \hat{\beta}_k} x(n_k) \sum_{s=0}^\infty M_{q', \ell}^{a', \hat{\beta} + \hat{\xi}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \nabla x_1(s) = \tilde{b}_k \sum_{s=0}^\infty M_{q', \ell}^{a', \hat{\beta} + \hat{\xi}}(s)[s]_{q}^{(n_k-1)} v_q^{a, \hat{\beta}_k+1}(s) \nabla x_1(s), \tag{67}
\]

where

\[
\tilde{b}_k = \frac{q^{1/2}(1 - a q^{n_k+\hat{\beta}_k})}{a q^{n_k+\hat{\beta}_k}} \prod_{i=1}^r \frac{a q^{1/2} + \hat{\beta}_i}{1 - a q^{1/2} + \hat{\beta}_i} x(n_k + \hat{\beta}_k - \hat{\beta}_i).
\]

From Equations (66) and (67) we get the following linear system of equations for the unknown coefficients \( \xi_1, \ldots, \xi_r \),

\[
b_j = \sum_{i=1}^r \xi_i s_{j,i}, \quad k = 1, \ldots, r, \quad \iff S \xi = b, \quad \xi = (\xi_1, \ldots, \xi_r), \tag{68}
\]

where the entries of the vector \( b \) and matrix \( S \) are as follows

\[
b_j = \frac{(1 - a q^{n_k+\hat{\beta}_k})q^{1/2} - n_k+1/2}{(1 - a q^{1/2} + \hat{\beta}_k)}, \quad s_{j,i} = m_{j,i}.
\]
Theorem 3. The q-Meixner multiple orthogonal polynomial of the second kind $M_{q,h}^{\alpha,\beta}(s)$ satisfies the following $(r + 1)$-order q-difference equation

$$\prod_{i=1}^{r} D_{q,h}^{\alpha_i,\beta_i + 1} \Delta M_{q,h}^{\alpha,\beta}(s) = - \sum_{j=1}^{r} q^{1/2} \xi_j \prod_{j \neq i}^{r} D_{q,h}^{\alpha_j,\beta_j + 1} M_{q,h}^{\alpha,\beta}(s).$$

(69)

where $\xi_i$'s are the constants in (63).

Proof. Since the operators (53) commute, we write

$$\prod_{i=1}^{r} D_{q,h}^{\alpha_i,\beta_i + 1} = \left( \prod_{j=1}^{r} D_{q,h}^{\alpha_j,\beta_j + 1} \right) D_{q,h}^{\alpha_i,\beta_i + 1}. \quad (70)$$

3.6. Recurrence Relation for q-Meixner Multiple Orthogonal Polynomials of the Second Kind

Theorem 4. The q-Meixner multiple orthogonal polynomials of the second kind satisfy the following $(r + 2)$-term recurrence relation

$$x(s) M_{q,h}^{\alpha,\beta}(s) = M_{q,h}^{\alpha,\beta}(s) + b_{\alpha,\beta} M_{q,h}^{\alpha,\beta - 1}(s) + a q^{2|d|-1} \sum_{i=1}^{r} x(n_i) x(\beta_i + n_i - 1) \prod_{j \neq i}^{r} x(n_i + \beta_i - n_j - \beta_j) B_{\alpha,\beta} M_{q,h}^{\alpha,\beta - 1}(s), \quad (71)$$
where

\[
b_{n,k} = \prod_{i=1}^{r} c_{q,\beta_i}^{\alpha_i} \left( \sum_{j=1}^{\ell} \frac{-q^{\beta_j} x(n_i)}{q^{\beta_j + n_i}} - \frac{a q^{\beta_j + n_i} x(n_i)}{c_{q,\beta_j}^{\alpha_j + n_i}} \right)
\]

and

\[
f_{n,i} = \frac{a q^{\beta_i} - 1}{\prod_{i=1}^{r} c_{q,\beta_i}^{\alpha_i}}.
\]

**Proof.** Let

\[
g_n(s; \alpha) = \frac{(aq^n)^s}{\Gamma_q(s + 1)}, \quad \text{where} \quad n = |\beta|.
\]

We will use Lemma 4 involving this function \(g_n(s; \alpha)\) as well as difference operator (56).

Consider the following equation

\[
\left( \gamma_q(\beta_k) \gamma_q(\beta_k + n_k + 1) \gamma_q(s + 1) \right) = \frac{\gamma_q(\beta_k)}{\gamma_q(\beta_k + s)} \gamma_q(s + 1) \gamma_q(\beta_k + n_k + 1 + s) \left( (aq^{\beta_k + n_k + 1})^s \right)
\]

which can be rewritten as follows

\[
\frac{\gamma_q(\beta_k + n_k + s)}{\gamma_q(\beta_k + s)} = \frac{\gamma_q(\beta_k)}{\gamma_q(\beta_k + s)} \gamma_q(n_k) \left( (aq^{\beta_k + n_k + 1})^s \right)
\]

Since operators (56) commute, the multiplication of Equation (72) from the left-hand side by the product \(\prod_{i=1}^{r} \lambda_{q,\beta_i}^{\beta_i} \) yields

\[
\lambda_{q,\beta_i}^{\beta_i} x(s) \frac{(aq^{\beta_i})^s}{\gamma_q(s + 1)} = \frac{ax(\beta_k + n_k)}{q^{\beta_i - n_i - 1/2} c_{q,\alpha_i}^{\beta_i + n_i}} \left( \lambda_{q,\beta_i}^{\beta_i} x(s) - q^{1/2} \lambda_{q,\beta_i}^{\beta_i} x(s) \right).
\]

Let us recursively use Lemma 4 involving the product of \(r\) difference operators \(\prod_{i=1}^{r} \lambda_{q,\beta_i}^{\beta_i} \) acting on the function \(g_n(s; \alpha)\), that is, the operator \(\lambda_{q,\beta_i}^{\beta_i} \) (see expression (56)). Thus,

\[
q^{\beta_i} \lambda_{q,\beta_i}^{\beta_i} x(s) g_n(s; \alpha) = \sum_{i=1}^{r} \prod_{j=1}^{r} q^{\beta_i} x(n_i + \beta_j - \beta_i) \frac{x(n_i) c_{q,\alpha_i}^{\beta_i}}{\prod_{i=1}^{r} c_{q,\beta_i}^{\alpha_i}} \prod_{i=1}^{r} \lambda_{q,\beta_i}^{\beta_i} \prod_{i=1}^{r} \lambda_{q,\beta_i}^{\beta_i} g_n(s; \alpha) + \left( q^{\beta_i} \sum_{i=1}^{r} x(n_i) c_{q,\alpha_i}^{\beta_i} \right) \lambda_{q,\beta_i}^{\beta_i} g_n(s; \alpha) + \prod_{i=1}^{r} c_{q,\beta_i}^{\alpha_i} x(s) \lambda_{q,\beta_i}^{\beta_i} g_n(s; \alpha).
\]
Hence, using expressions (73) and (74) one gets

\[ x(s)N_{q,\beta}^{\bar{\gamma}}n_s(s;\alpha) = q^{-1/2} \prod_{i=1}^{r} \frac{\frac{\alpha_i}{\alpha_i - 1}}{\frac{\beta_i}{\beta_i - 1}} N_{q,\beta}^{\bar{\gamma}}(s;\alpha) \]

\[ + b_{i,j}N_{q,\beta}^{\bar{\gamma}}n_s(s;\alpha) - q^{-1/2}(1 - q) \sum_{i=1}^{r} \frac{\beta_i}{\beta_i - 1} \frac{\alpha_i}{\alpha_i - 1}, \]

which is used in the previous expression when the indices \( i \) and \( j \) coincide. Therefore, the following expression holds

\[ x(s)N_{q,\beta}^{\bar{\gamma}}n_s(s;\alpha) = q^{-1/2} \prod_{i=1}^{r} \frac{\frac{\alpha_i}{\alpha_i - 1}}{\frac{\beta_i}{\beta_i - 1}} N_{q,\beta}^{\bar{\gamma}}(s;\alpha) \]

\[ + b_{i,j}N_{q,\beta}^{\bar{\gamma}}n_s(s;\alpha) - q^{-1/2}(1 - q) \sum_{i=1}^{r} \frac{\beta_i}{\beta_i - 1} \frac{\alpha_i}{\alpha_i - 1}, \]

Finally, multiplying from the left both sides of the previous expression by \( G_{q,\beta}^{\bar{\gamma},\alpha} \Gamma_\beta(\beta_i) / \Gamma_\beta(\beta_i + s) \) and using Rodrigues-type Formula (55), we obtain (71). This completes the proof of the theorem. \( \square \)

4. Limit Relations as \( q \) Approaches 1

The lattice \( x(s) = (q^s - 1) / (q - 1) \) allows to transit from the non-uniform distribution of points \( (q^s - 1) / (q - 1), \) \( s = 0, 1, \ldots, \) to the uniform distribution \( s, \) as \( q \) approaches 1. Under this limiting process one expects that the \( q \)-algebraic relations studied in this paper transform into the corresponding relations for discrete multiple orthogonal polynomials [28]. Indeed, the \( q \)-analogue of Rodrigues-type Formulas (31) and (55) will be transformed into their discrete counterparts (15) and (16), respectively. As a consequence, the recurrence relations (19) and (20) can be derived from (47) and (71), respectively.

We begin by analyzing the Rodrigues-type formulas, which then can be used for addressing the limit relations involving other algebraic properties.

**Proposition 3.** The following limiting relations for \( q \)-Meixner multiple orthogonal polynomials of the first kind (31) and second kind (55) hold:

\[ \lim_{q \to 1} M_{q,\beta}^{\bar{\gamma}}(s) = (\beta_i)(\beta_i + s) \prod_{i=1}^{r} \frac{\alpha_i}{\alpha_i - 1} \frac{\Gamma(\beta_i)(s + 1)}{\Gamma(\beta_i + s)}. \]

\[ \lim_{q \to 1} M_{q,\beta}^{\bar{\gamma}}(s) = \frac{\alpha}{\alpha - 1} \prod_{i=1}^{r} \frac{\beta_i}{\beta_i + s} \frac{\Gamma(\beta_i)(s + 1)}{\Gamma(\beta_i + s)}. \]

The right-hand side limiting results are the corresponding discrete multiple orthogonal polynomials \( M_{\bar{\gamma}}^{\bar{\gamma}}(s) \) and \( M_{\bar{\gamma}}^{\bar{\gamma}}(s) \) given in (15) and (16), respectively.
Theorem. We begin by proving (75). Let us rewrite the \( m \)-th action of the difference operator \( \nabla \) on a function \( f(s) \) defined on the \( q \)-lattice \( x(s) \) as follows (see formula (3.2.29) from [38])

\[
\nabla^m f(s) = q^{(m+1)/2 - ms} \sum_{k=0}^{m} \binom{m}{k} (-1)^k q^{-k} f(s-k),
\]

where

\[
\binom{m}{k} = \frac{(q;q)_m}{(q;q)_k(q;q)_{m-k}}, \quad m = 1, 2, \ldots,
\]

\( (a;q)_k = \prod_{j=0}^{k-1} (1 - aq^j) \) for \( k > 0 \), and \( (a;q)_0 = 1 \).

Here the expression \( (a;q)_k \) denotes the \( q \)-analogue of the Pochhammer symbol \([37,38,43,44]\). Moreover, expression (77) is a \( q \)-analogue of (11).

In (31) we have the following expression

\[
M^{\alpha,\beta}_{q,n}(s) = (a_1 q^{n_1})^s (a_2 q^{n_2})^s \left( \Gamma(q(\beta + |\vec{a}| + s)) / \Gamma(q(\beta + |\vec{a}|)) \right) \left( \Gamma(q(\beta + |\vec{a}|)) / \Gamma(q(s + 1)) \right).
\]

where the normalizing coefficient \( (a_1 q^{n_1})^s (a_2 q^{n_2})^s \) is given in (33) and it tends to the following expression, as \( q \) approaches to 1

\[
(\beta)|_{q=1} \left( \prod_{i=1}^{r} \left( \frac{a_i}{a_i - 1} \right)^{n_i} \right).
\]

Without loss of generality, let us consider a multi-index \( \vec{n} = (n_1, n_2) \) and rewrite the above expression in accordance with Formula (77); that is, we first need to express \( \nabla^{n_1}(a_1 q^{n_1})^s \Gamma(q(s + 1))/\Gamma(q(s + 1)) \) in terms of a finite sum and then compute the action of \( \nabla^{n_2} \) on the product formed by this resulting expression and \( (a_2 q^{n_2})^s \). Namely,

\[
M^{\alpha_1,\alpha_2,\beta}_{q,n_1,n_2}(s) = (a_1 q^{n_1})^s (a_2 q^{n_2})^s \left( \Gamma(q(\beta + |\vec{a}|)) / \Gamma(q(s + 1)) \right) \left( \Gamma(q(s + 1)) / \Gamma(q(\beta + |\vec{a}|)) \right)
\]

\[
\times \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} (-1)^{l+k} \binom{n_1}{l} \binom{n_2}{k} \frac{q^{(n_1+l-n_2-k)/2} \Gamma(q(\beta + n_1 + n_2 - k - l + s)) / \Gamma(q(s - k - l + 1))}{a_2^{k} a_1^{L}}.
\]

Applying limit in the above expression as \( q \) approaches to 1 yields

\[
\lim_{q \to 1} M^{\alpha_1,\alpha_2,\beta}_{q,n_1,n_2}(s) = (\beta)^{n_1+n_2} \left( \binom{\alpha_1}{1} \right)^{n_1} \left( \binom{\alpha_2}{2} \right)^{n_2} \frac{\Gamma(\beta) \Gamma(s+1)}{\Gamma(\beta + s)} \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} (-1)^{l+k} \binom{n_2}{k} \binom{n_1}{l} \frac{1}{a_2^{k} a_1^{L}} \frac{\Gamma(\beta + n_1 + n_2 - k - l + s)}{\Gamma(s - k - l + 1)}.
\]
Using (11), one rewrites Equation (79) such that it involves the product of raising operators as in (13) to obtain
\[
\lim_{q \to 1} M_{q,n_1 n_2}^{\alpha_1, \alpha_2, \alpha} (s) = (\beta)_{n_1+n_2} \left( \frac{\alpha_1}{\alpha_1 - 1} \right)^{n_1} \left( \frac{\alpha_2}{\alpha_2 - 1} \right)^{n_2} \frac{\Gamma(\beta) \Gamma(n_1+1)}{\Gamma(\beta+s)} \times (a_2^{-s} \nabla^{n_2} a_2^s) \left( a_1^{-s} \nabla^{n_1} a_1^s \right) \frac{\Gamma(\beta + n_1 + n_2 + s)}{\Gamma(\beta + n_1 + n_2) \Gamma(s+1)}
\]
\[
= M_{n_1 n_2}^{\alpha_1, \alpha_2, \alpha} (s),
\]
which coincides with (15) for \( \vec{n} = (n_1, n_2) \). Observe that repeating the aforementioned procedure for a multi-index \( \vec{n} \) of dimension \( r \), we obtain for the polynomial
\[
M_{q, \vec{n}}^{\alpha_1, \alpha_2, \alpha} (s) = G_q^{\alpha_1, \alpha_2, \alpha} \frac{\Gamma_q(\beta) \Gamma_q(n_1+1)}{\Gamma_q(\beta+s) \Gamma_q(\beta + |\vec{n}|)} \times \prod_{k_1=0}^{n_1} \sum_{k_r=0}^{r-1} (-1)^{|k|} \left[ n_r \atop k_r \right] \prod_{i=1}^{r} a_i^{-s} \nabla^{n_i} a_i^s \left( \frac{\Gamma(\beta + |\vec{n}| + s)}{\Gamma(\beta + |\vec{n}|) \Gamma(s+1)} \right),
\]
where \( \vec{k} = (k_1, \ldots, k_r) \), the following relation
\[
\lim_{q \to 1} M_{q, \vec{n}}^{\alpha_1, \alpha_2, \alpha} (s) = (\beta)_{|\vec{n}|} \left( \prod_{j=1}^{r} \frac{\alpha_j}{\alpha_j - 1} \right)^{|\vec{n}|} \frac{\Gamma(\beta) \Gamma(n_1+1)}{\Gamma(\beta+s)} \prod_{i=1}^{r} a_i^{-s} \nabla^{n_i} a_i^s \left( \frac{\Gamma(\beta + |\vec{n}| + s)}{\Gamma(\beta + |\vec{n}|) \Gamma(s+1)} \right),
\]
\[
= M_{\vec{n}}^{\alpha_1, \alpha_2, \alpha} (s).
\]
This proves the expression (75).

Next, we will prove the second limiting relation (76). Notice that the normalizing coefficient \( G_q^{\vec{n}, \beta_1, \beta} \) given in (57) has the following limit expression, as \( q \) approaches 1,
\[
\lim_{q \to 1} G_q^{\vec{n}, \beta_1, \beta} = \lim_{q \to 1} (-1)^{|\vec{n}|} (a q^{\vec{n}}) |\vec{n}|^{-1} q^{-\frac{1}{2}} \prod_{i=1}^{r} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i + n_i)} \prod_{j=1}^{n_i} \frac{q^{\beta_i+j-1}}{(a q^{\vec{n}})^{\beta_i+j-1} - 1} \left( \prod_{i=1}^{r} [\beta_i]_{n_i} \right)
\]
\[
= \left( \frac{\alpha}{\alpha - 1} \right)^{|\vec{n}|} \left( \prod_{i=1}^{r} (\beta_i)_{n_i} \right).
\]
From (55) and (77) we have
\[
M_{q, \vec{n}}^{\alpha_1, \alpha_2, \alpha} (s) = G_q^{\alpha_1, \alpha_2, \alpha} \frac{\Gamma_q(\beta_1)}{\Gamma_q(\beta_1 + n_1)} \prod_{i=1}^{r} \frac{\Gamma_q(\beta_i) \Gamma_q(n_1+1)}{\Gamma_q(\beta_i + n_i) \alpha_i^s} \times \prod_{k_1=0}^{n_1} \sum_{k_r=0}^{r-1} (-1)^{|k|} \left[ n_r \atop k_r \right] \prod_{i=1}^{r} a_i^{-s} \nabla^{n_i} a_i^s \left( \frac{\Gamma(\beta + |\vec{n}| + s)}{\Gamma(\beta + |\vec{n}|) \Gamma(s+1)} \right) \times \prod_{i=1}^{r} a_i^{n_i} \frac{\Gamma_q(\beta_i + n_i + s - k_i)}{\Gamma_q(s - |\vec{k}| + 1)}
\]
\[
= M_{q, \vec{n}}^{\alpha_1, \alpha_2, \alpha} (s).
\]
Therefore, we evaluate the following limit:

\[
\lim_{q \to 1} M_{q, n}^{\alpha, \beta}(s) = \left( \frac{\alpha}{\alpha - 1} \right)^{|\beta|} \prod_{i=1}^{r} \frac{\Gamma(s + 1)}{\Gamma((\beta_i)_{n_i})} \prod_{j=1}^{n} \frac{\Gamma(\beta_r + \sum_{k=1}^{j} n_k) \cdot \Gamma(\beta_r + 1 + s - k) \cdots \Gamma(\beta_r + 1 + s - \sum_{k=1}^{j} n_k)}{\Gamma(\beta_r + 1 + s - j)}.
\]

Finally, using (11) one rewrites the right-hand side as follows

\[
\lim_{q \to 1} M_{q, n}^{\alpha, \beta}(s) = \left( \frac{\alpha}{\alpha - 1} \right)^{|\beta|} \prod_{i=1}^{r} \frac{\Gamma(s + 1)}{\Gamma((\beta_i)_{n_i})} \prod_{j=1}^{n} \frac{\Gamma(\beta_r + \sum_{k=1}^{j} n_k) \cdot \Gamma(\beta_r + 1 + s - k) \cdots \Gamma(\beta_r + 1 + s - \sum_{k=1}^{j} n_k)}{\Gamma(\beta_r + 1 + s - j)} = M_{\alpha, \beta}^{\alpha, \beta}(s).
\]

This completes the proof of expression (76).

\[\square\]

5. Appendix: AT-Property for the Studied Discrete Measures

Lemma 9. The system of functions

\[a_{i, 1}, x(s)a_{i, 2}, \ldots, x(s)^{n_i - 1}a_{i, n_i}, \ldots, a_{r, 1}, x(s)a_{r, 2}, \ldots, x(s)^{n_r - 1}a_{r, n_r},\]

with \(a_i > 0, i = 1, 2, \ldots, r,\) with all the \(a_i\) different, and \((a_i/a_j) \not\equiv q^k, k \in \mathbb{Z}, i, j = 1, \ldots, r, i \neq j,\) forms a Chebyshev system on \(\mathbb{R}^+\) for every \(n = (n_1, \ldots, n_r) \in \mathbb{N}^r.\)

Proof. For a Chebyshev system every linear combination \(\sum_{i=1}^{r} Q_{n_i-1}(x(s))a_{i}^{n_i}\) has at most \(|\beta|\) zeros on \(\mathbb{R}^+\) for every \(Q_{n_i-1}(x(s)) \in \mathbb{P}_{n_i-1} \setminus \{0\}.\) Since \(x(s) = c_1 q^s + c_3,\) where \(c_1, c_3\) are constants, we consider \(\sum_{i=1}^{r} Q_{n_i-1}(q^s)a_{i}^{s}\) instead. Thus, the system (80) transforms into

\[a_{i,j, 0}, a_{i, 1, 1}, \ldots, a_{i, n_i - 1}, a_{r, 0, 1}, a_{r, 1, 1}, \ldots, a_{r, n_r - 1},\]

where \(a_{i,j} = (q^i a_i),\) with \(k = 0, \ldots, n_i - 1, i = 1, \ldots, r.\) Observe that \(a_{i,j} \neq a_{r,p}\) for \(j \neq l, m \neq p.\) Hence, identity \(a_{i,j} = e^{\log a_i} a_j\) yields the well-known Chebyshev system (see [34], p. 138)

\[e^{s log a_{j, 0}}, e^{s log a_{j, 1}}, \ldots, e^{s log a_{j, n_i - 1}}, \ldots, e^{s log a_{r, 0}}, e^{s log a_{r, 1}}, \ldots, e^{s log a_{r, n_r - 1}}.\]

Then, we conclude that the functions (80) form a Chebyshev system on \(\mathbb{R}^+.\)

\[\square\]

Lemma 10. Let \(\beta_i > 0\) and \(\beta_i - \beta_j \not\in \mathbb{Z}\) whenever \(i \neq j.\) Assume \(\nu(s)\) is a continuous function with no zeros on \(\mathbb{R}^+,\) then the functions

\[\nu(s) \Gamma_q (s + \beta_1), \nu(s) x(s) \Gamma_q (s + \beta_1), \ldots, \nu(s) x(s)^{n_i - 1} \Gamma_q (s + \beta_1),\]

\[\vdots\]

\[\nu(s) \Gamma_q (s + \beta_r), \nu(s) x(s) \Gamma_q (s + \beta_r), \ldots, \nu(s) x(s)^{n_i - 1} \Gamma_q (s + \beta_r),\]

form a Chebyshev system on \(\Omega\) for every \(n \in \mathbb{N}^r.\)
Proof. For the system of functions (81) we have a Chebyshev system on $\Omega$ for every $\vec{n} \in \mathbb{N}^r$ if and only if every linear combination of these functions (except the one with each coefficient equals 0) has at most $|\vec{n}| - 1$ zeros. This linear combination can be rewritten as a function of the system

$$v(s)\Gamma_q(s + \beta_1), v(s)[s + \beta_1]_q^{(1)} \Gamma_q(s + \beta_1), \ldots,$$

$$v(s)[s + \beta_1 + n_1 - 2]_q^{(n_1 - 1)} \Gamma_q(s + \beta_1),$$

$$v(s)\Gamma_q(s + \beta_r), v(s)[s + \beta_r]_q^{(1)} \Gamma_q(s + \beta_r), \ldots,$$

$$v(s)[s + \beta_1 + n_r - 2]_q^{(n_r - 1)} \Gamma_q(s + \beta_r),$$

where $[s + \beta_i]_q^{(n_i)}$, $i = 1, \ldots, r$, is given in (6).

Observe that

$$[s + k - 1]_q^{(k)} \Gamma_q(s) = \Gamma_q(s + k),$$

holds. Therefore, the above system transforms into

$$v(s)\Gamma_q(s + \beta_1), v(s)\Gamma_q(s + \beta_1 + 1), \ldots, v(s)\Gamma_q(s + \beta_1 + n_1 - 1),$$

$$v(s)\Gamma_q(s + \beta_r), v(s)\Gamma_q(s + \beta_r + 1), \ldots, v(s)\Gamma_q(s + \beta_r + n_r - 1).$$

Thus, it is sufficient to prove that these systems (82) form a Chebyshev system on $\Omega$ for every $\vec{n} \in \mathbb{N}^r$. If we define the matrix $\mathcal{A} \left( \vec{n}, s_1, \ldots, s_{|\vec{n}|} \right)$ by

$$\begin{pmatrix}
\Gamma_q(s_1 + \beta_1) & \Gamma_q(s_1 + \beta_1 + 1) & \cdots & \Gamma_q(s_{|\vec{n}|} + \beta_1) \\
\Gamma_q(s_1 + \beta_1 + n_1 - 1) & \Gamma_q(s_2 + \beta_1 + n_1 - 1) & \cdots & \Gamma_q(s_{|\vec{n}|} + \beta_1 + n_1 - 1) \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_q(s_1 + \beta_r) & \Gamma_q(s_1 + \beta_r + n_1 - 1) & \cdots & \Gamma_q(s_{|\vec{n}|} + \beta_r + n_1 - 1) \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_q(s_1 + \beta_r + n_1 - 1) & \Gamma_q(s_2 + \beta_r + n_1 - 1) & \cdots & \Gamma_q(s_{|\vec{n}|} + \beta_r + n_1 - 1)
\end{pmatrix},$$

the proof is reduced to showing that $\det \mathcal{A} \left( \vec{n}, s_1, \ldots, s_{|\vec{n}|} \right) \neq 0$, for every $|\vec{n}|$, and different points $s_1, \ldots, s_{|\vec{n}|}$ in $\Omega$, because $|v| > 0$ on $\Omega$. Now we replace the $q$-gamma function in $\mathcal{A} \left( \vec{n}, s_1, \ldots, s_{|\vec{n}|} \right)$ by the integral representation

$$\Gamma_q(s) = \int_0^1 t^{s-1} E_q^{-q t} d_q t = \int_0^{\lambda(\infty)} t^{s-1} E_q^{-q t} d_q t, \quad s > 0,$$

(83)

where

$$E_q^z = q \varphi_0 (-; -; q, -(1 - q) z)$$
denotes the $q$-analogue of the exponential function. From multilinearity of the determinant we take $|\vec{n}|$ integrations out of $|\vec{n}|$ rows to obtain

$$
\det A \left( \vec{n}, s_1, \ldots, s_{|\vec{n}|} \right) = \int_0^{x(\infty)} \ldots \int_0^{x(\infty)} \prod_{1 \leq i \leq |\vec{n}|} E_{q}^{-q t_i} f_i^{s_i - 1} \\
\times \det B \left( \vec{n}, t_1, \ldots, t_{|\vec{n}|} \right) d_q t_1 \ldots d_q t_{|\vec{n}|}.
$$

(84)

where

$$
B \left( \vec{n}, t_1, \ldots, t_{|\vec{n}|} \right) = \left( \begin{array}{cccc}
1^{\beta_1} & 1^{\beta_1} & \ldots & 1^{\beta_1} \\
\vdots & \vdots & \ddots & \vdots \\
1^{\beta_1 + n_1 - 1} & 1^{\beta_2 + n_1 - 1} & \ldots & 1^{\beta_1 + n_1 - 1} \\
\vdots & \vdots & \ddots & \vdots \\
1^{\beta_1} & 1^{\beta_1} & \ldots & 1^{\beta_1 + n_1 - 1} \\
\vdots & \vdots & \ddots & \vdots \\
1^{\beta_1 + n_1 - 1} & 1^{\beta_2 + n_1 - 1} & \ldots & 1^{\beta_1 + n_1 - 1}
\end{array} \right).
$$

Notice that, from ([34], p. 138, example 4) we know that the functions

$$f^{\beta_1}, \ldots, f^{\beta_1 + n_1 - 1}, \ldots, f^{\beta_r}, \ldots, f^{\beta_1 + n_1 - 1},$$

form a Chebyshev system on $\mathbb{R}^+$ if all the exponents are different, which is in accordance with our choice $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$. Moreover, if all $n_i < N + 1$, then the exponents involved in the above matrix are different for $\beta_i - \beta_j \notin \{0, 1, \ldots, N\}$ whenever $i \neq j$. Hence, $\det B \left( \vec{n}, t_1, \ldots, t_{|\vec{n}|} \right)$ does not vanish for distinct $t_1, \ldots, t_{|\vec{n}|}$. Now, for a permutation $\sigma$ of $\{1, \ldots, |\vec{n}| \}$ we make a change of variables $t_i \mapsto t_{\sigma(i)}$ in the integral (84). Thus, we have

$$
\det A \left( \vec{n}, t_1, \ldots, t_{|\vec{n}|} \right) = \int_0^{x(\infty)} \ldots \int_0^{x(\infty)} \prod_{1 \leq i \leq |\vec{n}|} E_{q}^{-q t_i} f_i^{s_i - 1} \det B \left( \vec{n}, t_1, \ldots, t_{|\vec{n}|} \right) \\
\times \sgn(\sigma) \prod_{1 \leq i \leq |\vec{n}|} f_{\sigma(j)}^{s_{\sigma(j)} - 1} d_q t_1 \ldots d_q t_{|\vec{n}|}.
$$

(85)

We average (85) over all permutation $\sigma$, i.e.,

$$
\det A \left( \vec{n}, s_1, \ldots, s_{|\vec{n}|} \right) = \frac{1}{n!} \sum_{\sigma \in S_{|\vec{n}|}} \int_0^{x(\infty)} \ldots \int_0^{x(\infty)} \prod_{1 \leq i \leq |\vec{n}|} E_{q}^{-q t_i} f_i^{s_i - 1} \\
\times \det B \left( \vec{n}, t_1, \ldots, t_{|\vec{n}|} \right) \sgn(\sigma) \prod_{1 \leq i \leq |\vec{n}|} f_{\sigma(j)}^{s_{\sigma(j)} - 1} d_q t_1 \ldots d_q t_{|\vec{n}|}.
$$
being $S_{|\vec{n}|}$ the permutation group. Now, relabeling the choice of points, i.e., $t_1, \ldots, t_{|\vec{n}|}$, where $0 < t_1 < \cdots < t_{|\vec{n}|}$, we have

$$\det \mathcal{A} (\vec{n}, t_1, \ldots, t_{|\vec{n}|}) = \frac{1}{n!} \prod_{0 < t_1 < \cdots < t_{|\vec{n}|}} E^{- \eta(t)} \det B (\vec{n}, t_1, \ldots, t_{|\vec{n}|})$$

$$\times \sum_{\sigma \in S_{|\vec{n}|}} \text{sgn } (\sigma) \prod_{1 \leq j \leq |\vec{n}|} s_{\sigma(j)} - 1 \sigma t_1 \ldots d_q t_{|\vec{n}|}. \quad (86)$$

As a result, from the definition of determinant we have

$$\sum_{\sigma \in S_{|\vec{n}|}} \text{sgn } (\sigma) \prod_{1 \leq j \leq |\vec{n}|} s_{\sigma(j)} - 1 \sigma t_1 \ldots d_q t_{|\vec{n}|} = \begin{vmatrix} t_1^{s_1-1} & t_2^{s_2-1} & \cdots & t_{|\vec{n}|}^{s_{|\vec{n}|}-1} \\ t_1^{s_1} & t_2^{s_2} & \cdots & t_{|\vec{n}|} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{s_1} & t_2^{s_2} & \cdots & t_{|\vec{n}|} \\ \end{vmatrix}. \quad (87)$$

Taking into account that $t_1, \ldots, t_{|\vec{n}|}$ are strictly positive and different, then using the result in ([34], p. 138, example 3) with multi-index $(1, \ldots, 1)$, will imply that (87) is different from zero if all the $s_1, \ldots, s_{|\vec{n}|}$ are different. Accordingly, for distinct $s_1, \ldots, s_{|\vec{n}|}$, the integrand of Equation (86) has a constant sign in the region of integration and hence $\det \mathcal{A} (\vec{n}, s_1, \ldots, s_{|\vec{n}|})$ does not vanish. \qed

As a consequence of Lemma 10 the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (51) forms an AT system on $\Omega$.

6. Concluding Remarks

We have studied two families of multiple orthogonal polynomials on a non-uniform lattice, i.e., $q$-Meixner multiple orthogonal polynomials of the first and second kind, respectively. They are derived from two systems of $q$-discrete measures. Each system forms an AT-system. For these families of multiple $q$-orthogonal polynomials we have obtained the Rodrigues-type Formulas (31) and (55) as well as the recurrence relations (47) and (71), and the $q$-difference equations (41) and (69). The use of some $q$-difference operators has played an important role in deriving the aforementioned algebraic properties. Finally, in the limit situation $q \to 1$, we have obtained the multiple Meixner polynomials given in [28].

In closing, we address some research directions and open problems:

**Problem 1.** A description of the main term of the logarithm asymptotics of the $q$-analogues of multiple Meixner polynomials deserves special attention. For such a purpose, we will use an algebraic function formulation for the solution of the equilibrium problem with constraints [45–47] to describe the zero distribution of multiple orthogonal polynomials [48]. This approach has been recently developed for multiple Meixner polynomials in [21] (see [49] as well as [17,50] for other approaches). Moreover, by analyzing the limiting behavior of the coefficients of the recurrence relations for such polynomials we expect to obtain the main term of their asymptotics.

**Problem 2.** In [51] the authors use the annihilation and creation operators $a_i, a_i^*$ ($i = 1, \ldots, r$) satisfying the commutation relations

$$[a_i, a_j^*] = \delta_{i,j}, \quad [a_i^*, a_j] = [a_i, a_j] = 0, \quad i, j = 1, \ldots, r.$$

The generated Lie algebra is formed by $r$ copies of the Heisenberg–Weyl algebra $W_r = \text{span} \{a_i, a_i^*, 1 \}$. For a more detailed and technical information about orthogonal polynomials in the Lie algebras see [52] as well as [53] for quantum mechanics and polynomials of a discrete variable.
The normalized simultaneous eigenvectors of the \( r \) number operators \( N_i = a_i^\dagger a_i \) are denoted by

\[
|n_1, n_2, \ldots, n_r\rangle = |n_1\rangle |n_2\rangle \cdots |n_r\rangle,
\]

Indeed,

\[
N_i |n_1, n_2, \ldots, n_r\rangle = n_i |n_1, n_2, \ldots, n_r\rangle,
\]

\[
\langle m_1, m_2, \ldots, m_r|n_1, n_2, \ldots, n_r\rangle = \delta_{m_1,n_1} \cdots \delta_{m_r,n_r}.
\]

Moreover,

\[
a_i^\dagger |n_1, n_2, \ldots, n_r\rangle = \sqrt{n_i + 1} |n_1, \ldots, n_i + 1, \ldots, n_r\rangle,
\]

\[
a_i |n_1, n_2, \ldots, n_r\rangle = \sqrt{n_i} |n_1, \ldots, n_i - 1, \ldots, n_r\rangle.
\]

The Bargmann realization in terms of coordinates \( z_i, i = 1, \ldots, r \), in \( \mathbb{C}^r \) has

\[
a_i = \frac{\partial}{\partial z_i}, \quad a_i^\dagger = z_i,
\]

\[
\langle z_1, z_2, \ldots, z_r|n_1, n_2, \ldots, n_r\rangle = \frac{z_1^{n_1} \cdots z_r^{n_r}}{\sqrt{n_1! \cdots n_r!}}.
\]

For the model in [51]

\[
H_i^{\vec{\alpha}, \vec{\beta}} = a_i + \sum_{k=1}^r \frac{N_k}{1 - \alpha_k} + \left( \frac{\alpha_i}{1 - a_i} + \sum_{j=1}^r \frac{a_j}{1 - a_j} \right) \left( \sum_{k=1}^r N_k + \beta \right), \quad i = 1, \ldots, r,
\]

represent the set of non-Hermitian operators defined in the universal enveloping algebra formed by the \( r \) copies \( W_i \).

The operators making up the \( H_i \) generate an isomorphic Lie algebra to that of the diffeomorphisms in \( \mathbb{C}^r \) spanned by vector fields of the form

\[
Z = \sum_{i=1}^r f_i(\vec{z}) \frac{\partial}{\partial z_i} + g(\vec{z}), \quad \vec{z} = (z_1, \ldots, z_r).
\]

The authors indicated that although in the coordinate realization

\[
a_i = \frac{1}{\sqrt{2}} \left( x_i + \frac{\partial}{\partial x_i} \right), \quad a_i^\dagger = \frac{1}{\sqrt{2}} \left( x_i - \frac{\partial}{\partial x_i} \right),
\]

the operators \( H_i \) are third order differential operators, they can be considered as Hamiltonians and are simultaneously diagonalized by the multiple Meixner polynomials of the first kind.

Consider the states \( |x, \vec{\alpha}, \vec{\beta}\rangle \) defined by means of the combination of states \( |n_1, \ldots, n_r\rangle \) as:

\[
|x, \vec{\alpha}, \vec{\beta}\rangle = N_{x, \vec{\alpha}, \vec{\beta}} \sum \frac{M_{\vec{\alpha}, \vec{\beta}}(x)}{\sqrt{n_1! \cdots n_r!}} |n_1, n_2, \ldots, n_r\rangle, \quad x \in \mathbb{N}.
\]
Thus,

\[
H_{\vec{\alpha}, \vec{\beta}}^{\vec{\alpha}, \vec{\beta}} |x, \vec{\alpha}, \vec{\beta}\rangle = N_{\vec{\alpha}, \vec{\beta}} \sum_{\vec{n}} \frac{1}{\sqrt{n_1! \cdots n_r!}} M_{\vec{n}, \vec{n} + \vec{\beta}}^{\vec{\alpha}, \vec{\beta}} (x) \\
+ \left( (\beta + |\vec{n}|) \left( \frac{\alpha_i}{1 - \alpha_i} \right) + \sum_{k=1}^{r} \frac{n_k}{1 - n_k} \right) M_{\vec{n}, \vec{n}}^{\vec{\alpha}, \vec{\beta}} (x) \\
+ \sum_{j=1}^{r} \alpha_j n_j (\beta + |\vec{n}| - 1) \frac{(\alpha_j \beta + r \sum_{k=1}^{r} N_k) n_j}{(\alpha_j - 1)^2} M_{\vec{n}, \vec{n} - \vec{\beta}}^{\vec{\alpha}, \vec{\beta}} (x) |n_1, n_2, \ldots, n_r\rangle.
\]

In [51], by using the recurrence relation (19) for multiple Meixner polynomials of the first kind, the following relation

\[
H_{\vec{\alpha}, \vec{\beta}}^{\vec{\alpha}, \vec{\beta}} |x, \vec{\alpha}, \vec{\beta}\rangle = x |x, \vec{\alpha}, \vec{\beta}\rangle,
\]
holds.

Despite the fact the operators are non-Hermitian, they have a real spectrum given by the lattice, i.e., the non-negative integers. The states $|x, \vec{\alpha}, \vec{\beta}\rangle$ are uniquely defined as the joint eigenstates of the Hamiltonian operators with eigenvalues equal to $x$. Moreover,

\[
[H_{\vec{\alpha}, \vec{\beta}}^{\vec{\alpha}, \vec{\beta}}, H_{\vec{\alpha}, \vec{\beta}}^{\vec{\alpha}, \vec{\beta}}] |x, \vec{\alpha}, \vec{\beta}\rangle = 0.
\]

However, these Hamiltonians do not commute pairwise. Indeed,

\[
[H_{\vec{\alpha}, \vec{\beta}}^{\vec{\alpha}, \vec{\beta}}, H_{\vec{\alpha}, \vec{\beta}}^{\vec{\alpha}, \vec{\beta}}] = a_i - a_j + \frac{\alpha_i - \alpha_j}{(1 - \alpha_i)(1 - \alpha_j)} (\beta + \sum_{k=1}^{r} N_k).
\]

Finally, because they do not commute and yet have common eigenvectors, the authors in [51] say that they form a ‘weakly’ integrable system.

The physical model described above motivates the study of a q-deformed model, which is currently being considered by using the results of the present paper involving the q-analogue of multiple Meixner polynomials of the first kind. In particular, the recurrence relation (47).

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