Massive Monopoles and Massless Monopole Clouds

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Abstract

Magnetic monopole solutions naturally arise in the context of spontaneously broken gauge theories. When the unbroken symmetry includes a non-Abelian subgroup, investigation of the low-energy monopole dynamics by means of the moduli space approximation reveals degrees of freedom that can be attributed to massless monopoles. These do not correspond to distinct solitons, but instead are manifested as a cloud of non-Abelian field surrounding one or more massive monopoles. In these talks I explain how one is led to these solutions and then describe them in some detail.

1 Introduction

One-particle states arise in the spectra of weakly coupled quantum field theories in two rather different ways. By quantizing the small oscillations about the vacuum, one finds the states, with a characteristic mass $m$, that correspond to the quanta of the fundamental fields of the theory. It may also happen that the classical field equations of the theory possess localized solutions with energies of order $m/\lambda$, where $\lambda$ is a typical small coupling of the theory; these soliton solutions also give rise to particles in the quantum theory. At first sight, these two classes of particles appear quite different: the former seem to be point particles with no internal structure, while the latter are extended objects described by a classical field profile $\phi(r)$.

However, these distinctions are not quite so clearcut. On the one hand, in an interacting theory even the fundamental point particles can be viewed as having a partonic substructure that evolves with momentum scale according to the DGLAP equations. On the other, one can analyze the behavior of the soliton states in terms of the normal modes of small fluctuations about the soliton. The modes in the continuum part of the spectra can be interpreted as scattering states of elementary

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quanta in the presence of the soliton; there may also be discrete eigenvalues corresponding to quanta bound to the soliton. This leaves only a small number of zero frequency modes whose quantization entails the introduction of the collective coordinates that may be viewed as the fundamental degrees of freedom of the soliton.

These considerations suggest that the particle states built from solitons and those based the elementary quanta do not differ in any essential way. Indeed, it can happen that states that appear as elementary quanta in one formulation of the theory correspond to solitons in another. The classic example of this is the correspondence between the sine-Gordon model and the Thirring model. Of more immediate relevance to my talks is the conjecture by Montonen and Olive that certain theories possess an exact electromagnetic duality relating magnetically charged solitons and electrically charged elementary quanta.

If there is such a duality, then one would expect the classical solutions to display particle-like properties. In particular, one would expect the classical solutions with higher charges to have a structure consistent with an interpretation in terms of component solitons of minimal charge. This is indeed found to be the case in many theories. However, in the course of studying magnetic monopoles in the context of larger gauge groups, Kimyeong Lee, Piljin Yi, and I found that in some theories there are classical solutions that do not quite fit this picture. As I will explain below, there is a sense in which these solutions can be understood as multimonopole solutions containing both massive and massless magnetic monopoles. While the massive components are quite evident when one examines the classical solutions, the massless monopoles appear to lose their individual identity and merge into a “cloud” of non-Abelian fields. Because these massless monopoles can be viewed as the duals to the massless gauge bosons of the theory, a better understanding of the nature of these unusual solutions may well provide deeper insight into the properties of non-Abelian gauge theories.

In these talks I will explain how one is led to these solutions and then describe them in some detail. I begin in the next section by reviewing some general properties of magnetic monopoles. In Sec. 3, I describe the Bogomolny-Prasad-Sommerfield, or BPS, limit and its application to multimonopole solutions in an SU(2) gauge theory. The extension of these results to larger gauge groups is discussed in Sec. 4. The treatment of low-energy monopole dynamics by means of the moduli space approximation is discussed in Sec. 5, while the actual determination of some moduli space metrics is described in Sec. 6. The theories where one actually encounters evidence of massless monopoles are those in which the unbroken gauge symmetry has a non-Abelian component. These are discussed in Sec. 7. Explicit examples of solutions in which the massless monopoles appear
to condense into a non-Abelian cloud are described in Secs. 8 and 9. Section 10 contains some concluding remarks.

2 Magnetic monopoles

In the absence of sources, Maxwell’s equations display a symmetry under the interchange of electric and magnetic fields. This suggests that there might also be a symmetry in sources, and that in addition to the familiar electric charges there might also be magnetically charged objects, usually termed magnetic monopoles, that act as sources for magnetic fields. A static monopole with magnetic charge $Q_M$ would give rise to a Coulomb magnetic field

$$B_i = \frac{Q_M \hat{r}_i}{4\pi r^2}. \quad (2.1)$$

In the canonical treatment of the behavior of charged particles in a magnetic field, either classically or quantum mechanically, it is most convenient to express the magnetic field as the curl of a vector potential $A_i$. For the magnetic field of Eq. (2.1), a suitable choice is

$$A_i = -\epsilon_{ij3} \frac{Q_M}{4\pi} \frac{\hat{r}_j (1 - \cos \theta)}{r} \sin \theta. \quad (2.2)$$

Note that this is singular along the negative $z$-axis. This “Dirac string” singularity is an inevitable consequence of trying to express a field with nonvanishing divergence as the curl of a potential; any potential leading to Eq. (2.1) will have a similar singularity along some curve running from the position of the monopole out to infinity. Physically, this singularity is a difficulty only if it actually observable. In classical physics, where only the magnetic field, and not the vector potential, is measurable, it causes no problem. However, there are quantum mechanical interference effects that are sensitive to the quantity

$$U = \exp \left[ ie \oint_C A_i dl_i \right] \quad (2.3)$$

where $e$ is the electric charge of some particle and the integration is around any closed curve. If $C$ is taken to be an infinitesimal closed curve in a region where $A_i$ is nonsingular, $U$ is clearly equal to unity. On the other hand, if the integral is taken around an infinitesimal closed curve encircling the Dirac string, $U$ is not equal to unity, and the string is thus observable, unless the magnetic charge obeys the Dirac quantization condition

$$Q_M = \frac{4\pi}{e} \left( n \right) \quad (2.4)$$

I am assuming units in which $\hbar = 1$; otherwise there is a additional factor of $\hbar$ on the right hand side.
for some integer $n$. If we want the string to be unobservable, this condition must hold for all possible electric charges. This is only possible if all electric charges are integer multiples of some minimum charge for which Eq. (2.4) is satisfied. Thus, the existence of a single monopole in the universe would be sufficient to explain the observed quantization of electric charge.

There is an alternative approach that avoids the appearance of string singularities [7]. Instead, one introduces two gauge patches, one excluding the negative $z$-axis and one excluding the positive $z$-axis, and in each one chooses a vector potential that is nonsingular in that region. In the overlap of the two regions, the two vector potentials can differ only by the gauge transformation that relates the two patches. In order that this gauge transformation be single-valued in the overlap region, Eq. (2.4) must hold.

One can always incorporate magnetic monopoles into a theory with electrically charged particles simply by postulating a new species of fundamental particles. However, it turns out [8] that monopoles are already implicit in many theories with electrically charged fundamental fields. In these theories, the classical field equations have localized finite energy solutions with magnetic charge that correspond to one-particle states of the quantized theory. Although topological arguments are usually used to demonstrate the existence of these solutions, their existence and many of their features can in fact be understood on the basis of energetic arguments alone [9].

To begin, note that the Coulomb magnetic field Eq. (2.1) has a $1/r^2$ singularity at the origin. In contrast to the Dirac string, this a true physical singularity, as can be seen by noting that it leads to a $1/r^4$ divergence in the energy density $\mathcal{E} = \frac{1}{2}B^2$. This singularity must somehow be tamed if finite energy classical solutions are to exist. One approach might be to replace the point magnetic charge by a charge distribution, but the Dirac quantization condition forbids such continuous charge distributions in theories with both electric and magnetic charges.

However, there is another possibility for removing the divergence. When placed in a magnetic field, a magnetic dipole $d$ acquires an energy $-d \cdot B$. Thus, the singular energy density in the magnetic field might be cancelled by introducing a suitable (singular) distribution of magnetic dipoles. This idea can be implemented by introducing a complex vector field $W$ with electric charge $e$ and a magnetic dipole density $d = iegW^* \times W$, with $g$ a real constant that for the moment can be taken to be arbitrary. Since we want there to be a lower bound on the energy, the energy density must also contain terms of higher order in $d$. In particular, adding a term $d^2/2$ allows the $1/r^4$ divergence of the energy density to be cancelled if $|W| \sim 1/r$ near the origin.

Since we want the $W$ field to be localized within a finite region, the energy density should contain a mass term of the form $M_W^2 |W|^2$. However, this would give a $1/r^2$ contribution to $\mathcal{E}$ near
the origin. This singularity can be eliminated by allowing the $W$ mass to be dependent on some spatially varying field $\phi$. In particular, let us assume that $M_W = G\phi$, where $G$ is a constant and the scalar field potential $V(\phi)$ is minimized by $\phi = v \neq 0$. Finiteness of the energy then implies that at large distances $\phi \approx v$ and $M_W \neq 0$, but at $r = 0$ both $\phi$ and $M_W$ can be taken to vanish. Provided that the contribution from the gradients of the fields introduce no additional singularities (which can be arranged), the energy density will then be nonsingular everywhere.

An energy density of the sort described here can be obtained from a Lagrangian density of the form

$$\mathcal{L} = -\frac{1}{4}(\mathcal{F}_{\mu\nu} - i e g W_{\mu}^{\ast} W_{\nu})^2 - \frac{1}{2} |D_{\mu} W_{\nu} - D_{\nu} W_{\mu}|^2 + G^2 \phi^2 |W_{\mu}|^2 - \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi)$$  \hspace{1cm} (2.5)

where $D_{\mu} = \partial_{\mu} + ie A_{\mu}$ and $\mathcal{F}_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ are the electromagnetic covariant derivative and electromagnetic field strength. A solution of the resulting Euler-Lagrange equations that carries unit magnetic charge (i.e., $Q_M = 4\pi/e$) can be obtained by introducing the ansatz

$$A_i = -\epsilon_{ij3} \frac{\dot{r}_j}{er} (1 - \cos \theta) \frac{\sin \theta}{i},$$

$$W_1 = -i \frac{u(r)}{\sqrt{2} er} [1 - e^{i\phi} \cos \phi (1 - \cos \theta)],$$

$$W_2 = i \frac{u(r)}{\sqrt{2} er} [1 + e^{i\phi} \sin \phi (1 - \cos \theta)],$$

$$W_3 = \frac{i \sqrt{2} u(r)}{er} e^{i\phi} \sin \theta,$$

$$\phi = h(r).$$ \hspace{1cm} (2.6)

Substitution of this ansatz into the Euler-Lagrange equations leads to a pair of coupled second order ordinary differential equations that can be solved numerically to yield a finite energy solution. However, that the Dirac string singularity still remains.

A very important special case is obtained by setting $g = 2$ and $G = e$. With this choice of parameters, the theory described by Eq (2.5) is in fact an $SU(2)$ gauge theory in disguise. Let us define the components of an $SU(2)$ gauge field $A_{\mu} \equiv A_{\mu}^a T^a$ and a triplet Higgs field $\Phi \equiv \Phi^a T^a$ by

$$A_{\mu}^1 + i A_{\mu}^2 = W_{\mu}, \quad A_{\mu}^3 = A_{\mu},$$

$$\Phi^a = \delta^{a3} \phi(r).$$ \hspace{1cm} (2.7)

where the $T^a$ are the generators of $SU(2)$. The Lagrangian (2.5) can then be rewritten in the form

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F_{\mu\nu}^2 + \frac{1}{2} \text{Tr} (D_{\mu} \Phi)^2 - V(\Phi).$$ \hspace{1cm} (2.8)

Here

$$D_{\mu} = \partial_{\mu} + ie A_{\mu}$$ \hspace{1cm} (2.9)
is the non-Abelian covariant derivative and
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \] (2.10)

is the field strength with magnetic components \( B_i = (1/2)\epsilon_{ijk}F_{jk} \) and electric components \( E_i = F_{oi} \).

For definiteness, let us assume that the potential is of the form
\[ V(\Phi) = -\frac{\mu^2}{2} \text{Tr} \Phi^2 + \frac{\lambda}{4} (\text{Tr} \Phi^2)^2. \] (2.11)

If \( \mu^2 < 0 \), the classical energy has a minimum at \( \Phi = 0 \) that preserves the \( SU(2) \) symmetry. The spectrum of the quantum theory includes three massless gauge bosons and three massive scalars with equal masses. If instead \( \mu^2 > 0 \), there is a family of physically equivalent degenerate minima given by \( \text{Tr} \Phi_0^2 = \mu^2/\lambda \equiv v^2 \); the “vacuum manifold” of such minima can be identified with the coset space \( SU(2)/U(1) = S^2 \). In each of these vacuum states the \( SU(2) \) symmetry is spontaneously broken to the \( U(1) \) subgroup that leaves \( \Phi_0 \) invariant; this \( U(1) \) subgroup may be identified with electromagnetism. After quantization of the theory, the small fluctuations about the vacuum lead to a spectrum of elementary particles that includes a massless photon, an electrically neutral Higgs scalar with mass \( \sqrt{2}\mu \), and a pair of vector bosons with mass \( ev \) and electric charges \( \pm e \).

In describing either the vacuum or configurations that are small perturbations about the vacuum, it is most natural to take the orientation of the Higgs field to be uniform in space; indeed, our ansatz for the monopole solution corresponds to a vacuum with \( \Phi_0^a = v\delta^a_3 \). However, the orientation of the Higgs field is gauge-dependent quantity that need not be uniform. In particular, by applying a spatially varying \( SU(2) \) gauge transformation (with a singularity along the negative \( z \)-axis), we can bring our monopole solution into the manifestly nonsingular “radial gauge” form
\[ A_j^a = \epsilon_{jak}\hat{r}_k \frac{1 - u(r)}{er}, \]
\[ \Phi^a = \hat{r}_a h(r). \] (2.12)

At large-distances the resulting magnetic field
\[ B_i^a = \frac{1}{e} \frac{\hat{r}_i \hat{r}_a}{r^2} + O(1/r^3) \] (2.13)
is parallel to \( \Phi \) in internal space, showing that it lies in the unbroken electromagnetic subgroup.

In any finite energy solution, the Higgs field must approach one of the minima of \( V(\Phi) \) as \( r \to \infty \) in any fixed direction. Hence, for any nonsingular solution the Higgs configuration gives a map from the \( S^2 \) at spatial infinity into the vacuum manifold \( SU(2)/U(1) \). Any such map corresponds to an element of the homotopy group \( \Pi_2(SU(2)/U(1)) = Z \) and can therefore be assigned an
integer “topological charge” \( n \). While the vacuum solutions correspond to the identity element with \( n = 0 \), the radial gauge monopole solution gives a topologically nontrivial map with \( n = 1 \). In fact, one can show that there is a one-to-one correspondence between the magnetic charge and the topological charge, with a nonsingular \( SU(2) \) configuration of total magnetic charge \( Q_M = 4\pi n/e \) having topological charge \( n \). Although magnetic charges that are half-integer multiples of \( 4\pi n/e \) are allowed by the Dirac quantization condition, they cannot be obtained from nonsingular field configurations.

### 3 The BPS limit

An especially interesting special case, which I will assume for the remainder of these talks, is known as the Bogomolny-Prasad-Sommerfield, or BPS, limit \[3\]. It can be motivated by considering the expression for the energy of a static configuration with magnetic, but not electric, charge. Assuming for the moment that \( A_0 \) vanishes identically, we have

\[
E = \int d^3 x \left[ \frac{1}{2} \text{Tr} B_i^2 + \frac{1}{2} \text{Tr} (D_i \Phi)^2 + V(\Phi) \right]
\]

\[
= \int d^3 x \left[ \frac{1}{2} \text{Tr} (B_i + D_i \Phi)^2 + V(\Phi) \pm \text{Tr} B_i D_i \Phi \right].
\]

(3.1)

With the aid of the Bianchi identity \( D_i B_i = 0 \) the last term on the right hand side may be rewritten as a surface integral over the sphere at spatial infinity:

\[
\int d^3 x \text{Tr} B_i D_i \Phi = \int d^3 x \partial_i (\text{Tr} B_i \Phi) = \int dS_i \text{Tr} B_i \Phi \equiv Q_M v.
\]

(3.2)

(The normalization of \( Q_M \) implied by the last equality agrees with that of Eq. (2.1).) Substituting this back into the previous equation yields the bound

\[
E = \pm Q_M v + \int d^3 x \left[ \frac{1}{2} \text{Tr} (B_i + D_i \Phi)^2 + V(\Phi) \right]
\]

\[
\geq |Q_M| v.
\]

(3.3)

The BPS limit is obtained by dropping the contribution of \( V(\Phi) \) to the energy. This can be done most simply by letting \( \mu^2 \to 0, \lambda \to 0 \), with \( v^2 = \mu^2/\lambda \) held fixed. It can also be obtained by considering the extension of this theory to a Yang-Mills theory with extended supersymmetry. The latter approach is particularly attractive from a physical point of view, and can be formulated in such a way that the BPS limit is preserved by higher order quantum corrections.

Now recall that any static configuration that is a local minimum of the energy is a stable solution of the classical equations of motion. Because the magnetic charge is quantized, any configuration
that saturates the lower bound in Eq. \(3.3\) will be such a solution. With \(V(\Phi)\) absent, the conditions for saturation of this bound are the BPS equations

\[ B_i = D_i \Phi. \tag{3.4} \]

(I have assumed here, and henceforth, that \(Q_M \geq 0\); the extension to the case \(Q_M < 0\) is obvious.) One can easily verify by direct substitution that any solution of the first-order BPS equations is indeed a solution of the second-order Euler-Lagrange equations.

This result can easily be extended to the case of dyons, solutions carrying not only a magnetic charge \(Q_M\) but also a nonzero electric charge

\[ Q_E = v^{-1} \int dS_i \text{Tr} E_i \Phi. \tag{3.5} \]

The bound on the energy is generalized to

\[ E \geq v \sqrt{Q_M^2 + Q_E^2} \tag{3.6} \]

with the minimum being achieved by configurations that satisfy

\[ B_i = \cos \beta D_i \Phi \]
\[ E_i = \sin \beta D_i \Phi \]
\[ D_0 \Phi = 0 \tag{3.7} \]

with \(\beta = \tan^{-1}(Q_E/Q_M)\).

An attractive feature, which was in fact one of the original motivations for the BPS approximations, is that it is possible to obtain a simple analytic expression for the singly charged monopole solution. By a rescaling of fields and distances the gauge coupling \(\epsilon\) can be set equal to unity; for the remainder of these talks I will assume that this has been done. The solutions can then be written as \[6\],

\[ A^a_j = \epsilon_{jak} \hat{r}_k \left[ \frac{v}{\sinh(vr)} - \frac{1}{r} \right] \]
\[ \Phi^a = \hat{r}_a \left[ v \coth(vr) - \frac{1}{r} \right]. \tag{3.8} \]

Note that the Higgs field does not approach its asymptotic value exponentially fast, but instead has a \(1/r\) tail. This is because the absence of a potential term makes the Higgs field massless. Since a massless scalar field carries a long-range force that is attractive between like objects, this raises the possibility that the magnetic repulsion between two BPS monopoles might be exactly cancelled by their mutual scalar attraction, thus allowing for the existence of static multimonopole solutions.
In fact, it turns out that such solutions — indeed continuous families of solutions — exist for all values of $Q_M$.

The actual construction of these multimonomopole solutions is a difficult, but fascinating, problem. For the moment, I will simply concentrate on the problem of counting the number of physically meaningful parameters, or “collective coordinates”, needed to specify these solutions. Each of these corresponds to a zero frequency eigenmode (a “zero mode”) in the spectrum of small fluctuations about a given solution. However, there are also an infinite number of zero modes, corresponding to local gauge transformations of the solution, that do not correspond to any physically meaningful parameter. To eliminate these, a gauge condition must be imposed on the fields.

I will start with the zero modes about the solution with unit magnetic charge. The elimination of the gauge modes is particularly transparent if we work in the singular “string gauge” of Eq. (2.7) where $\Phi^1 = \Phi^2 = 0$. This leaves only a $U(1)$ gauge freedom that can be fixed by imposing, e.g. the electromagnetic Coulomb gauge condition $\nabla \cdot A = 0$. Explicit solution of the zero mode equations then shows that there are precisely four normalizable zero modes about the solution. Three of these correspond to infinitesimal spatial translations of the monopole; the corresponding parameters are most naturally chosen to be the spatial coordinates of the center of the monopole. The fourth zero mode corresponds to a spatially constant phase rotation of the massive vector field, $W_\mu(r) \rightarrow e^{i\alpha}W_\mu(r)$. Since this mode is in fact a gauge mode that has no effect on gauge-invariant quantities, one might think that it should be discarded as unphysical. The justification for not doing so comes from considering the effect of allowing the collective coordinates to be time-dependent. In the case of the translation modes, this gives a solution with nonzero linear momentum. For the gauge mode, allowing the phase $\alpha$ to vary linearly in time produces a dyon solution that carries an electric charge proportional to $d\alpha/dt$.

Although explicit solution of the zero mode equations suffices for the case of unit magnetic charge, where the monopole solution is known explicitly, index theory methods are needed to count the zero modes about solutions with higher charges [10]. Each zero mode consists of perturbations $\delta A_j$ and $\delta \Phi$ that can be viewed as three-component vectors transforming under the adjoint representation of $SU(2)$. Since these preserve the BPS equations, they must satisfy

$$0 = \delta (B_j - D_j \Phi) = D_j \delta \Phi - \phi \delta A_j - \epsilon_{jkl} D_k \delta A_l. \quad (3.9)$$

(Here $D_j = \partial_j + A_j$ and $A_j$ and $\Phi$ are $3 \times 3$ anti-Hermitian matrices in the adjoint representation of $SU(2)$.) These must be supplemented by a gauge condition that eliminates the unwanted gauge
modes. A convenient choice is the background gauge condition

\[ 0 = D_j \delta A_j + \Phi \delta \Phi \]  

(3.10)

which is equivalent to requiring that the perturbation be orthogonal, in the functional sense, to all normalizable gauge modes. The number of collective coordinates is just equal to the number of linearly independent normalizable solutions of Eqs. (3.9) and (3.10).

If we define

\[ \psi = I \delta \Phi + i \sigma_j \delta A_j \]  

(3.11)

where \( I \) is the unit \( 2 \times 2 \) matrix and the \( \sigma_j \) are the Pauli matrices, Eqs. (3.9) and (3.10) can be combined into the single Dirac-type equation

\[ 0 = (-i \sigma_j D_j + i \Phi) \psi \equiv \mathcal{D} \psi. \]  

(3.12)

We must remember, however, that two solutions \( \psi \) and \( i \psi \) that are linearly dependent as solutions of Eq. (3.12) actually correspond to linearly independent solutions of the original bosonic equations (3.9) and (3.10). The number of collective coordinates is thus actually twice the number of linearly independent normalizable zero eigenmodes of \( \mathcal{D} \).

Note that if \( \psi(\mathbf{r}) \) is a solution of Eq. (3.12), then so is

\[ \psi'(\mathbf{r}) = \psi(\mathbf{r}) U \]  

(3.13)

where \( U \) is any \( 2 \times 2 \) unitary matrix. This fact, which be of importance later, implies that number of normalizable zero eigenmodes of the bosonic equation must be a multiple of four.

The next step is to define

\[ \mathcal{I}(M^2) = \text{Tr} \frac{M^2}{\mathcal{D} \mathcal{D}^\dagger + M^2} - \text{Tr} \frac{M^2}{\mathcal{D}^\dagger \mathcal{D} + M^2} \]  

(3.14)

where \( M \) is an arbitrary real number and

\[ \mathcal{D}^\dagger = -i \sigma_j D_j - i \Phi \]  

(3.15)

is the adjoint of \( \mathcal{D} \). The quantity

\[ \mathcal{I} = \lim_{M^2 \to 0} \mathcal{I}(M^2) \]  

(3.16)

is then equal to the number of zero eigenvalues of \( \mathcal{D} \mathcal{D}^\dagger \) minus the number of zero eigenvalues of \( \mathcal{D} \mathcal{D}^\dagger \). Using the fact that the unperturbed solution obeys the BPS equations, one finds that

\[ \mathcal{D} \mathcal{D}^\dagger = -D_j^2 + 2 \sigma_j B_j + \Phi^2 \]
\[ \mathcal{D} \mathcal{D}^\dagger = -D_j^2 + \Phi^2. \]  
(3.17)

The second equation shows that \( \mathcal{D} \mathcal{D}^\dagger \) is a positive operator with no normalizable zero modes. Since every normalizable zero mode of \( \mathcal{D} \) is also a normalizable zero mode of \( \mathcal{D}^\dagger \mathcal{D} \), and conversely, \( \mathcal{I} \) would clearly give the desired counting of zero modes if it were not for the fact that these operators have continuous spectra extending down to zero.

The contribution from these continuous spectra can be written as
\[
\mathcal{I}_{\text{continuum}} = \lim_{M^2 \to 0} \int \frac{d^3k}{(2\pi)^3} \frac{m^2}{k^2 + M^2} [\rho_{\mathcal{D}^\dagger \mathcal{D}}(k) - \rho_{\mathcal{D} \mathcal{D}^\dagger}(k)]
\]  
(3.18)

where \( \rho_{\mathcal{O}}(k) \) is the density of continuum eigenstates of an operator \( \mathcal{O} \). This contribution can be nonzero only if these density of states factors are singular near \( k = 0 \). For the case at hand, one can show that this is not the case. The essential idea is that such singularities are determined by the large \( r \) behavior of the potential terms in the operators. Since \( \mathcal{D} \mathcal{D}^\dagger - \mathcal{D} \mathcal{D}^\dagger = 2 \sigma_j B_j \), the potentially dangerous behavior is associated with the long-range behavior of the magnetic field. But, up to exponentially small corrections, the long-range part of the \( B_j \) lies in the unbroken \( U(1) \) subgroup and so does not act on the massless components of the fields, which also lie in this \( U(1) \). Since only these latter fields have spectra that extend down to zero, \( \mathcal{I}_{\text{continuum}} \) vanishes.

Having eliminated the continuum contribution, let us now turn to the evaluation of \( \mathcal{I} \). For this purpose it is convenient to adopt a pseudo-four-dimensional notation and define a four-vector \( V_\mu \) with components \( V_j = A_j \) for \( j = 1, 2, 3 \) and \( V_4 = \Phi \). Because this is actually a three-dimensional space, \( \partial_4 = 0 \) and so \( D_4 = \Phi \). Similarly, \( G_{ij} = F_{ij} \) while \( G_{i4} = -G_{4i} = D_i \Phi \). Finally, the Dirac matrices
\[
\gamma_k = \begin{pmatrix} 0 & -i \sigma_k \\ i \sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\]  
(3.19)

all anticommute with each other and all have square equal to unity.

With these definitions, we can write
\[
\gamma_\mu D_\mu = \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D}^\dagger & 0 \end{pmatrix}
\]  
(3.20)

and hence
\[
\mathcal{I}(M^2) = -\text{Tr} \frac{M^2}{(\gamma \cdot D)^2 + M^2} = -\int d^3x \langle x| \text{tr} \gamma_5 \gamma_i \frac{M}{\gamma \cdot D + M} |x \rangle.
\]  
(3.21)

(Here \( \text{Tr} \) indicates a functional trace, while \( \text{tr} \) denotes a trace over Dirac and \( SU(2) \) matrix indices.) The integrand in the last expression can be written as the three-dimensional divergence of the current
\[
J_i = \frac{1}{2} \int d^3x \langle x| \text{tr} \gamma_5 \gamma_i \frac{1}{\gamma \cdot D + M} |x \rangle = -\frac{1}{2} \int d^3x \langle x| \text{tr} \gamma_5 \gamma_i (\gamma \cdot D) \frac{1}{(\gamma \cdot D)^2 + M^2} |x \rangle
\]  
(3.22)
and so

\[ \mathcal{I}(M^2) = \int d^3 x \partial_i J_i(x) = \int dS_i J_i(x) \]  \hspace{1cm} (3.23)

where the surface integral in the last term is over the sphere at spatial infinity.

We now write

\[ \frac{1}{-(\gamma \cdot D)^2 + M^2} = \frac{1}{-D^2 + \Phi^2 + M^2} + \frac{1}{-D^2 + \Phi^2 + M^2} \left( \frac{i}{2} \gamma \mu \gamma \nu G_{\mu \nu} \right) - D^2 + \Phi^2 + M^2 + \cdots \]  \hspace{1cm} (3.24)

where \( D^2 = D_j D_j \) and the dots represent terms of order \( G^2 \) or higher that vanish at least as fast as \( 1/r^4 \) at spatial infinity. When this expansion is substituted into the expression for \( J_i \), the contribution from the first term vanishes after the trace over Dirac indices is performed. The remaining terms give

\[ \hat{x}_i J_i = -\frac{i}{2} \epsilon_i \lambda \mu \nu \hat{x}_i \text{tr} \langle x | D_{\lambda} - \frac{1}{D^2 + \Phi^2 + M^2} G_{\mu \nu} - \frac{1}{-D^2 + \Phi^2 + M^2} | x \rangle + O(x^{-4}) \]

\[ = \frac{i}{2} \langle x | \Phi - \nabla^2 + \Phi^2 + M^2 \hat{x} \cdot B - \nabla^2 + \Phi^2 + M^2 | x \rangle + O(|x|^{-3}) \]  \hspace{1cm} (3.25)

where the trace is now only over \( SU(2) \) indices.

The evaluation of this last expression is most transparent if we work in the singular “string gauge”. If the magnetic charge is \( Q_M = 4\pi n \), then asymptotically \( \Phi \rightarrow v T^3 \), and \( \hat{x} \cdot B \rightarrow (n/x^2) T^3 \) and one finds that

\[ \hat{x}_i J_i = \frac{n}{2\pi x^2} \frac{v}{\sqrt{v^2 + M^2}} + O(|x|^{-3}) \]  \hspace{1cm} (3.26)

It follows that

\[ \mathcal{I}(M^2) = 2n \frac{v}{\sqrt{v^2 + M^2}} \]  \hspace{1cm} (3.27)

and that the number of linearly independent normalizable zero modes of the original bosonic problem is

\[ 2\mathcal{I} = 4n \]  \hspace{1cm} (3.28)

A priori, one might have expected that classical solutions with higher charges could lead to new types of magnetically charged particles. Eq. (3.28), together with the fact that the BPS energy is strictly proportional to the magnetic charge, suggests that this is not the case. Instead, all higher charged solutions should be viewed as being multimonopole solutions composed of \( n > 1 \) unit monopoles, each with three translational and one \( U(1) \) degree of freedom.\(^3\) In the quantum theory, these solutions would thus correspond to multiparticle states.

\(^3\)As was the case with the unit monopole, there is only a single gauge mode that is not eliminated by the gauge condition; this corresponds to a simultaneous \( U(1) \) rotation of all the monopoles. The modes corresponding to relative \( U(1) \) rotations are not simply gauge transformations of the underlying solution.
It is useful at this point to review the spectrum of particles in this theory. Quantization of the small fluctuations of the fundamental fields yielded two particles, the photon and the Higgs scalar, that have neither electric nor magnetic charge and that in the BPS limit are both massless. It also gave two massive vector particles, with electric charges $\pm e$, no magnetic charge, and mass $ev$. In addition to these we have the monopole and antimonopole, with no electric charge, magnetic charges $\pm 4\pi/e$, and mass $(4\pi/e)v$. A curious feature of this spectrum is that the pattern of masses and charges remains the same under the interchanges $e \leftrightarrow 4\pi/e$ and $Q_E \leftrightarrow Q_M$. There is a mismatch in spin, since the monopole and antimonopole are spinless, while the vector bosons have spin one, but this can be remedied by enlarging the theory so that it has $N = 4$ extended supersymmetry \[12\]; once this is done, the elementary electrically charged particles and the magnetically charged BPS soliton states form supermultiplets with corresponding spins. These facts suggest that this duality symmetry, which exchanges solitons and elementary particles, and weak and strong coupling, might in fact be an exact symmetry of the theory, as was first conjectured by Montonen and Olive \[2\].

4 Monopoles in theories with larger gauge groups

This analysis can be extended to the case of a Yang-Mills theory with an arbitrary simple gauge group $G$ of rank $r$ and dimension $d$ and a Higgs field $\Phi$ transforming under the adjoint representation. To begin, recall that the generators of the Lie algebra of $G$ can be chosen to be $r$ commuting generators $H_a$ that span the Cartan subalgebra, together with a number of generators $E_\alpha$ associated with the $d - r$ root vectors $\alpha$ that are defined by the commutation relations

$$[E_\alpha, H_j] = \alpha_j E_\alpha.$$  \hspace{1cm} (4.1)

The asymptotic value of the Higgs field in some fixed reference direction can always be chosen to lie in the Cartan subalgebra. It thus defines an $r$-component vector $h$ through the relation

$$\Phi_0 = h \cdot H.$$  \hspace{1cm} (4.2)

The unbroken gauge symmetry is the subgroup $G$ that leaves $\Phi_0$ invariant. The maximal symmetry breaking occurs if $h$ has nonzero inner products with all the root vectors, in which case the unbroken subgroup is the $U(1)^r$ generated by the Cartan subalgebra. If instead some of the roots are orthogonal to $h$, then these form the root lattice for a non-Abelian group $K$ of rank $k < r$ and the unbroken symmetry is $U(1)^{r-k} \times K$.

At large distances, $F_{\mu\nu}$ must commute with the Higgs field. Hence, along the same direction used to define $h$, the asymptotic magnetic field may be chosen to also lie in the Cartan subalgebra.
and to be of the form
\[ B_i = g \cdot H \frac{\dot{x}_i}{r^2} + O(r^{-3}). \] (4.3)

The generalized quantization condition on the magnetic charge then becomes
\[ e^{ig}H = I. \] (4.4)

I will begin by considering the case of maximal symmetry breaking. Because \( \Pi_2(G/U(1)^r) = \Pi_1(U(1)^r) = Z^r \), there are \( r \) topologically conserved charges. These can be identified in a particularly natural fashion by recalling that a basis for the root lattice can be chosen to be a set of \( r \) simple roots \( \beta_a \) with the property that all other roots are linear combinations of simple roots with coefficients that are either all positive or all negative. There are many possible choices for this basis. However, a unique set of simple roots can be specified by requiring that
\[ h \cdot \beta_a \geq 0 \] (4.5)
for all \( a \). If all of the fields are in the adjoint representation, the quantization condition (4.4) then reduces to the requirement that
\[ g = 4\pi \sum a n_a \beta_a^* \] (4.6)
where \( \beta_a^* = \beta_a / \beta_a^2 \) and the integers \( n_a \) are are the topological charges.

The BPS mass formula is easily extended to this case. One finds that
\[ M = g \cdot h = \sum_a n_a \left( \frac{4\pi}{e} h \cdot \beta_a \right) = \sum_a n_a m_a. \] (4.7)

The methods used to count the zero modes about \( SU(2) \) solutions can also be applied here [14]. As before, there is no continuum contribution \( I_{\text{continuum}} \) because the long-range part of the magnetic field lies in the Cartan subalgebra and so does not act on the massless fields, which also lie in the Cartan subalgebra. The calculation of \( I \) proceeds very much as before until one gets to Eq. (3.27), which is replaced by
\[ I(M^2) = \frac{1}{4\pi} \sum_{\alpha'} (\alpha \cdot h)(\alpha \cdot g) \left( (\alpha \cdot h)^2 + M^2 \right)^{1/2} = \frac{1}{2\pi} \sum_{\alpha} (\alpha \cdot h)(\alpha \cdot g) \left( (\alpha \cdot h)^2 + M^2 \right)^{1/2}. \] (4.8)
Here the first sum is over all roots \( \alpha \), while the prime on the second sum indicates that it is to be taken only over the positive roots (those that are positive linear combinations of simple roots). Taking the limit \( M^2 \to 0 \) gives
\[ I = \frac{1}{2\pi} \sum_{\alpha} \alpha \cdot g = 2 \sum a n_a \left( \sum_{\alpha} \alpha \cdot \beta_a^* \right). \] (4.9)
In the sum inside the parentheses, the contributions from the roots other than \( \beta_a \) cancel, so that the sum is just \( \beta_a \cdot \beta_a^* = 1 \). Hence, the number of normalizable zero modes is

\[
2I = 4 \sum_a n_a. \tag{4.10}
\]

It was argued above that the \( SU(2) \) solutions with higher magnetic charge should be understood as being composed of a number of unit monopoles. The mass formula and the zero mode counting suggest that the higher charged solutions in the present case should also be understood as multimonopole solutions. Now, however, there are \( r \) different species of fundamental monopoles, with the \( a \)th fundamental monopole having mass \( m_a \), topological charges \( n_b = \delta_{ab} \) and four degrees of freedom. Classical solutions corresponding to these fundamental monopoles can be constructed by appropriate embeddings of the \( SU(2) \) solution. Any root \( \alpha \) defines an \( SU(2) \) subgroup of of \( G \) with generators

\[
\begin{align*}
t^1(\alpha) &= \frac{1}{\sqrt{2\alpha^2}}(E\alpha + E_-\alpha) \\
t^2(\alpha) &= -\frac{i}{\sqrt{2\alpha^2}}(E\alpha - E_-\alpha) \\
t^3(\alpha) &= \alpha^* \cdot H.
\end{align*} \tag{4.11}
\]

If we denote by \( A^a_i(r; v) \) and \( \Phi^a(r; v) \) the unit \( SU(2) \) monopole with Higgs expectation value \( v \), then the embedded solution

\[
\begin{align*}
A_i(r) &= \sum_{s=1}^{3} A^s_i(r; h \cdot \beta_a) t^s(\beta_a) \\
\Phi(r) &= \sum_{s=1}^{3} \Phi^s(r; h \cdot \beta_a) t^s(\beta_a) + (h - h \cdot \beta_a^* \beta) \cdot H \tag{4.12}
\end{align*}
\]

gives the fundamental monopole corresponding to the root \( \beta_a \). It has the expected mass and topological charges and four zero modes, three corresponding to translational degrees of freedom and the fourth to a phase angle in the \( U(1) \) generated by \( \beta_a \cdot H \).

As an example, consider the case of \( SU(3) \) broken to \( U(1) \times U(1) \) by an adjoint representation Higgs field that can be represented by a traceless Hermitian \( 3 \times 3 \) matrix. Let \( \Phi_0 \) be diagonal, with its eigenvalues decreasing along the diagonal. With this convention, the \( SU(2) \) subgroup defined by \( \beta_1 \) lies in the upper left \( 2 \times 2 \) block. Embedding the \( SU(2) \) monopole in this block gives a solution with a mass \( m_1 \), topological charges \( n_a = (1, 0) \), and four zero modes. After quantization, there is a family of monopole and dyon one-particle states corresponding to this solution. Similarly, \( \beta_2 \) defines an \( SU(2) \) subgroup lying in the lower right \( 2 \times 2 \) block. Using this subgroup for the
embedding gives a solution with mass $m_2$, topological charges $(0, 1)$, and again four zero modes. This, too, corresponds to a particle in the spectrum of the quantum theory.

There is a third $SU(2)$ subgroup, lying in the four corner matrix elements, defined by the composite root $\beta_1 + \beta_2$. Using this subgroup to embed the $SU(2)$ monopole also gives a spherically symmetric BPS solution, with mass $m_1 + m_2$ and topological charges $(1, 1)$. However, Eq. (4.10) (as well as explicit solution of the zero mode equations) shows that there are not four, but eight zero modes. Hence, this embedding solution is just one out of a continuous family of two-monopole solutions; in contrast to the two fundamental solutions, it can be continuously deformed into a solution containing two widely separated fundamental monopoles. It does not lead to a new particle in the spectrum of the quantum theory, but instead corresponds to a two-particle state.

Let us now consider this result in the light of the Montonen-Olive duality conjecture. Although this conjecture was first motivated by the spectrum of the $SU(2)$ theory, it is natural to test it with larger gauge groups. The elementary particle sector of the theory contains a number of massless particles, carrying no $U(1)$ charges, that are presumably self-dual. There are also six massive vector bosons, one for each root of the root diagram, that carry electric-type charges in one or both of the unbroken $U(1)$’s. The duals of the $\pm \beta_1$ and $\pm \beta_2$ vector bosons are clearly the one-particle states corresponding to the $\beta_1$- and $\beta_2$-embeddings of the $SU(2)$ monopole and antimonopole solutions. One might have thought that the duals of the vector bosons corresponding to $\pm(\beta_1 + \beta_2)$ would be obtained from the $(\beta_1 + \beta_2)$-embedding solutions, but we have just seen that these do not correspond to single-particle states. Some other state must be found if the duality is to hold. The most likely candidate would be some kind of threshold bound state \cite{15}. To explore this possibility, we need to understand the interactions of low-energy BPS monopoles. This can be done by making use of the moduli space approximation, to which I now turn.

## 5 The moduli space approximation

The essential idea of the moduli space approximation \cite{16} is that, since the static multimonopole solutions are all BPS, the time-dependent solutions containing monopoles with sufficiently small velocities should in some sense also be approximately BPS.\footnote{Here velocities should be understood to include not only spatial velocities but also the time derivatives of the $U(1)$ phases. Thus, we are considering slowly moving dyons with small (and possibly zero) electric charges.}

To make this more precise, let $\{A_{i}^{BPS}(r, z), \Phi^{BPS}(r, z)\}$ be a family of static, gauge-inequivalent BPS solutions parameterized by a set of collective coordinates $z_j$. The moduli space approximation is obtained by assuming that the fields at any fixed time are gauge-equivalent to some configuration
in this family, so that they can be written as

\[ A_0(r, t) = 0 \]
\[ A_i(r, t) = U^{-1}(r, t) A^{BPS}_i(r, z(t)) U((r, t) - iU^{-1}(r, t) \partial_i U((r, t) \]  \tag{5.1} \]

Their time derivatives are then of the form

\[ \dot{A}_i = \dot{z}_j \left[ \frac{\partial A_i}{\partial z_j} + D_i \epsilon_j \right] \equiv \dot{z}_j \delta_j A_i \]
\[ \dot{\Phi} = \dot{z}_j \left[ \frac{\partial \Phi}{\partial z_j} + [\Phi, \epsilon_j] \right] \equiv \dot{z}_j \delta_j \Phi \]  \tag{5.2} \]

where the gauge function \( \epsilon_j(r, t) \) arises from the time derivative of \( U(r, t) \). These are constrained by Gauss’s law, which takes the form

\[ 0 = -D_\mu F^{\mu 0} + [\Phi, \partial_0 \Phi] = D_i \dot{A}_i + [\Phi, \dot{\Phi}] \]
\[ = \dot{z}_j (D_i \delta_j A_i + [\Phi, \delta_j \Phi]) \]  \tag{5.3} \]

Because they arise from variation of a collective coordinate, the quantities \( \delta_j A_i \) and \( \delta_j \Phi \) form a zero mode about the underlying solution BPS solution. The Gauss’s law constraint shows that they obey the background gauge condition Eq. (3.10).

With \( A_0 \) identically zero, the Lagrangian of the theory can be written as

\[ L = \frac{1}{2} \int d^3 r \text{Tr} \left[ \dot{A}_i^2 + \dot{\Phi}^2 + B_i^2 + D_i \dot{\Phi}^2 \right] \]  \tag{5.4} \]

Since for fields obeying the ansatz (5.1) the configuration at any fixed time is BPS, the contribution of the last two terms to the integral is just the BPS energy determined by the topological charge. This is a time-independent constant that has no effect on the dynamics and so can be dropped. The remaining terms then give an effective Lagrangian

\[ L_{MS} = \frac{1}{2} g_{ij}(z) \dot{z}_i \dot{z}_j \]  \tag{5.5} \]

where

\[ g_{ij}(z) = \int d^3 r [\delta_i A_k \delta_j A_k + \delta_i \Phi \delta_j \Phi] \]  \tag{5.6} \]

Thus, the full field theory dynamics for low energy monopoles has been reduced to a problem involving a finite number of degrees of freedom. If one views \( g_{ij}(z) \) as a metric for the moduli space spanned by the collective coordinates, the dynamics described by \( L_{MS} \) is simply geodesic motion on the moduli space.
6 Determining the moduli space metric

Actually determining the moduli space metric is a nontrivial matter. To apply Eq. (5.6) directly one needs to know the zero modes, whereas we do not in general even know the underlying solution. However, some more indirect approaches can sometimes be brought to bear on the problem.

First, Gibbons and Manton [17] showed how one could obtain the metric for the region of moduli corresponding to widely separated monopoles. They pointed out that, since the moduli space metric determines the low energy dynamics, the metric can be inferred if this dynamics is known. The only long-range interactions between widely separated monopoles are those mediated by massless fields. These are the electromagnetic interactions and an interaction due to the massless Higgs field. The Lagrangian describing the interactions between moving point electric and magnetic forces is well known, while that for the scalar force is easily worked out. To obtain the metric, these must be expanded up to terms quadratic in the velocities and the electric charges. A Legendre transformation must then be used to replace the electric charges by the time derivatives of the corresponding phase angles. Apart from a constant term, the result is a Lagrangian, of the form of Eq. (5.5), from which the metric can be read off directly. For the case of many $SU(2)$ monopoles, each with mass $m$ and magnetic charge $g$ and with positions $x_i$ and phase angles $\xi_i$, this gives

$$ds^2 = \frac{1}{2} M_{ij} dx_i \cdot dx_j + \frac{g^4}{2(4\pi)^2} (M^{-1})_{ij} (d\xi_i + W_{ik} \cdot dx_k) (d\xi_j + W_{jl} \cdot dx_l)$$

(6.1)

where

$$M_{ii} = m - \sum_{k \neq i} \frac{g^2}{4\pi r_{ik}},$$

$$M_{ij} = \frac{g^2}{4\pi r_{ij}} \quad \text{if } i \neq j,$$

(6.2)

and

$$W_{ii} = - \sum_{k \neq i} w_{ik},$$

$$W_{ij} = w_{ij} \quad \text{if } i \neq j,$$

(6.3)

with $r_{ij}$ the distance between the $i$th and $j$th monopoles and $w_{ij}$ the value at $x_i$ of the Dirac vector potential due to the $j$th monopole, defined so that

$$\nabla_i \times w_{ij}(x_i - x_j) = \frac{x_i - x_j}{r_{ij}^3}.$$  

(6.4)

The extension of this result to the case of maximal symmetry breaking of an arbitrary simple group $G$ is quite simple. For a collection of fundamental monopoles, with the $i$th monopole
corresponding to the simple root $\beta_i$, we need only replace Eqs. (6.2) and (6.3) by

$$M_{ii} = m_i - \sum_{k \neq i} g^2 \beta_i^* \cdot \beta_k^* \frac{1}{4\pi r_{ik}},$$

$$M_{ij} = \frac{g^2 \beta_i^* \cdot \beta_j^*}{4\pi r_{ij}} \quad \text{if } i \neq j,$$

(6.5)

and

$$W_{ii} = -\sum_{k \neq i} \beta_i^* \cdot \beta_k^* w_{ik},$$

$$W_{ij} = \beta_i^* \cdot \beta_j^* w_{ij} \quad \text{if } i \neq j,$$

(6.6)

with $m_i = g \beta_i^* \cdot h$.

Although the derivation of these expressions was only valid in the region of moduli space corresponding to widely separated monopoles, one might wonder whether the asymptotic metric could be exact. For the case of two $SU(2)$ monopoles, several considerations show that it cannot be. The matrix $M$ of Eq. (6.2) reduces to

$$M = \begin{pmatrix} m - \frac{g^2}{4\pi r} & \frac{g^2}{4\pi r} \\ \frac{g^2}{4\pi r} & m - \frac{g^2}{4\pi r} \end{pmatrix}.$$  

(6.7)

The determinant of this matrix vanishes at $r = \frac{g^2}{2\pi m}$, implying a singularity in the metric, despite the fact that there is no reason to expect any type of singular behavior near this value of the intermonopole distance. Furthermore, we know that there is a short-range force, carried by the massive vector bosons, that was ignored in the derivation of the metric. If one works in a singular gauge in which the Higgs field orientation is uniform in space, this interaction is proportional to the gauge-invariant quantity $\text{Re}[W^{(1)} W^{(2)}]$ where $W^{(1)}$ and $W^{(2)}$ are the massive vector fields of the two monopoles. Because these fall exponentially with distance from the center of the monopole, their overlap, and hence the interaction, falls exponentially with the intermonopole separation.

Neither of these objections apply when the two monopoles are fundamental monopoles associated with different simple roots of a large gauge group. The simple roots have the property that their mutual inner products are always negative. The resulting sign changes in $M$ eliminate the zero of the determinant and make the asymptotic metric everywhere nonsingular. In addition, the quantity characterizing the interactions carried by the massive vector fields is now $\text{Re}[\text{Tr}W^{(1)} W^{(2)}]$, which vanishes when the two monopoles arise from different simple roots.

This, of course, is not sufficient to show that the asymptotic metric is exact. To do this, we first note that the coordinates for the moduli space can always be chosen so that three specify the
position of the center-of-mass of the monopoles and a fourth is an overall $U(1)$ phase. The moduli space metric can then be written in the factorized form

$$\mathcal{M} = R^3 \times \left( \frac{R^1 \times \mathcal{M}_{\text{rel}}}{D} \right)$$

where the factors of $R^3$ and $R^1$ are associated with the center-of-mass coordinates and the overall $U(1)$ phase, while $\mathcal{M}_{\text{rel}}$ is the metric on the subspace spanned by the relative positions and phases. The factoring by the discrete group $D$ arises from difficulties in globally factoring out an overall $U(1)$ phase.

$\mathcal{M}_{\text{rel}}$ has several important properties. First, it must have a rotational isometry reflecting the fact that the interactions among an assembly of monopoles are unaffected by an overall spatial rotation of the entire assembly. Second, the $SU(2)$ relations among the zero modes shown in Eq. (3.13) imply that the moduli space metric must be hyper-Kähler\(^5\). Finally, the relative moduli space for a collection of $n$ monopoles is $4(n-1)$-dimensional. Hence, we are seeking a four-dimensional hyper-Kähler manifold with a rotational isometry. There are four such:\(^5\)

1) Flat four-dimensional Euclidean space
2) The Eguchi-Hanson manifold \(^{19}\)
3) The Atiyah-Hitchin manifold \(^{18}\)
4) Taub-NUT space

The first of these would imply that there were no interactions at all between the monopoles, and so is clearly ruled out if $\beta_1 \cdot \beta_2 \neq 0$. The Eguchi-Hanson metric has the wrong asymptotic behavior for large intermonopole separation, and so can be ruled out. At large $r$ (but not at small $r$) the Atiyah-Hitchin metric approaches the two-monopole asymptotic metric with $M$ given by Eq. (6.2). It thus describes the moduli space for two $SU(2)$ monopoles (or for two identical monopoles in a larger group), but not that for two distinct monopoles. The only remaining possibility is the Taub-NUT metric. This not only agrees at large $r$ with the asymptotic metric, but is in fact equal to it everywhere. Thus, for the case of two distinct fundamental monopoles the asymptotic metric is in fact exact \(^{3,20,21}\).

If a collection of more than two monopoles includes two corresponding to the same simple root, then the asymptotic metric develops a singularity when these approach each other. However, this metric is everywhere nonsingular if the monopoles are all distinct. It is therefore natural to conjecture that for this case also the asymptotic metric is exact \(^{4}\). Proofs of this conjecture have been given by Chalmers \(^{22}\) and by Murray \(^{23}\).

\(^{5}\)A metric is hyper-Kähler if it possess three covariantly constant complex structures that also form a quaternionic structure and if it is pointwise Hermitian with respect to each.
Let us now briefly return to the issue of duality in the theory with $SU(3)$ broken to $U(1) \times U(1)$. As noted above, duality is expected to hold only if the theory has an extended supersymmetry, which means that the low-energy fermion dynamics must be included. It turns out that these fermions will give rise to a supermultiplet of threshold bound states if and only if there is a normalizable harmonic form on the relative moduli space \[24\]. Having determined the metric for this moduli space metric, one can easily verify that such a harmonic form exists, and hence that the test of the duality conjecture is met \[3, 20\].

### 7 Nonmaximal symmetry breaking

Let us now turn to the case of non-maximal symmetry breaking, where the gauge symmetry $G$ is spontaneously broken to $K \times U(1)^{r-k}$. As in the case of maximal symmetry breaking, we can require that inner products of the simple roots with $\mathbf{h}$ be all non-negative. It is useful to distinguish between those for which this inner product is greater than zero and those for which it vanishes. I will continue to denote the former by $\beta_a$, and will label the latter, which form a set of simple roots for $K$, by $\gamma_i$. In contrast with the previous case, the condition on the inner products with $\mathbf{h}$ does not uniquely determine the set of simple roots. Instead, there can be many acceptable sets, all related by Weyl reflections of the root diagram that result from global gauge transformations by elements of $K$.

The quantization condition on the magnetic charge now takes the form

$$g = 4\pi \left[ \sum_a n_a \beta_a^* + \sum_j q_j \gamma_j^* \right]. \quad (7.1)$$

As in the case of maximal symmetry breaking, the integers $n_a$ are the topological charges, one for each $U(1)$ factor of the unbroken group. They are gauge-independent, and thus independent of the choice of the set of simple roots. The $q_j$ must also be integers, but they are neither topologically conserved nor gauge-invariant. We will see that there is an important distinction to be made between the case where

$$g \cdot \gamma_j = 0, \quad \text{all } j, \quad (7.2)$$

and that where some of the $g \cdot \gamma_j$ are nonzero. (Note that these do not in general correspond to vanishing or nonvanishing $q_j$.) In the former case, the long-range magnetic fields are purely Abelian with only $U(1)$ components, whereas in the latter the configuration has a non-Abelian magnetic charge. We will see that there are a number of pathologies associated with the latter case.

\footnote{Consider, for example, the case of $SU(3)$ broken to $SU(2) \times U(1)$. If one set of simple roots is denoted by $\beta$ and $\gamma$, with the latter being a root of the unbroken $SU(2)$, then another acceptable set is given by $\beta + \gamma$ and $-\gamma$.}
The BPS mass formula takes the same form as before,

\[ M = \sum_a n_a m_a \]  

(7.3)

but with the sum running only over the indices corresponding to simple roots that are not orthogonal to \( \mathbf{h} \).

The zero mode counting proceeds as before, but with some complications \[25\]. First, the continuum contribution cannot be immediately discarded, since the massless fields cannot all be brought into the Cartan subalgebra. Because of this, there can be a singularity in the density of states factor that is strong enough to give a nonzero \( I_{\text{continuum}} \) if \( \mathcal{D}^\dagger \mathcal{D} - \mathcal{D} \mathcal{D}^\dagger \) contains order \( 1/r^2 \) terms that act on fields lying in the unbroken non-Abelian subgroup \( K \). This will be the case whenever there is a net non-Abelian magnetic charge. Explicit solution of the zero mode equations in some simple cases shows that the number of zero modes is not equal to the expression for \( 2I \) given below, implying that there is indeed a nonvanishing continuum contribution. This difficulty does not arise when Eq. (7.2) is satisfied.

Second, the expression for \( I \) is more complicated. The same procedures as used before again lead to

\[ 2I = \lim_{M^2 \to 0} \frac{1}{\pi} \sum_{\alpha}^\prime \frac{(\alpha \cdot \mathbf{h})(\alpha \cdot \mathbf{g})}{[\alpha \cdot \mathbf{h}]^2 + M^2}^{1/2} \]  

(7.4)

with the prime indicating that the sum is only over positive roots. Now, however, the contribution from the roots orthogonal to \( \mathbf{h} \) (i.e., those of the subgroup \( K \)) vanishes even for finite \( M^2 \) and so gives no contribution to the limit. As a result, the expression for \( 2I \) is in general much less simple than before. But, again, matters simplify if the asymptotic magnetic field is purely Abelian. Because the roots of \( K \) are now all orthogonal to \( \mathbf{g} \), they would not have contributed in any case, and the methods used for the case with maximal symmetry breaking yield

\[ 2I = 4 \left[ \sum_a n_a + \sum_j q_j \right] . \]  

(7.5)

As was noted earlier, the \( q_j \) are not gauge-invariant. However, when \( \mathbf{g} \) is orthogonal to all of the \( \gamma_k \), the sum appearing on the right-hand side of Eq. (7.5), and hence \( 2I \), is gauge-invariant.

The difficulties with applying index theory when there is a non-Abelian magnetic charge are related to other known difficulties with such solutions. Since the unbroken gauge group acts non-trivially on these, one would expect to find gauge zero modes, analogous to the \( U(1) \) modes of the maximally symmetric case, whose excitation would lead to “chromodyons”, objects with non-Abelian electric-type charge. Instead, one finds that these modes are non-normalizable and that
the expected chromodyon states are absent \cite{26}. This can be traced to the fact that the existence of the non-Abelian magnetic charge creates an obstruction to the smooth definition of a set of generators for $K$ over the sphere at spatial infinity; i.e., one cannot define “global color” \cite{27}.

It is instructive to return to the $SU(3)$ example considered in Sec. 4, but with the last two eigenvalues of $\Phi_0$ taken to be equal so that the unbroken group is $SU(2) \times U(1)$. While $n_1$ remains a topological charge, $n_2$ must be replaced by the nontopological integer $q_1$. The first fundamental monopole solution of the maximally broken case, obtained by embedding in the upper left $2 \times 2$ block, is still present with a nonzero mass. As before, it has three translational zero modes and a $U(1)$ phase mode. There are no other normalizable zero modes, even though the solution is not invariant under the unbroken $SU(2)$, and even though Eq. (7.4) gives $2I = 6$. Embedding in the lower right $2 \times 2$ block, which previously gave a second fundamental monopole, is no longer possible. Indeed, if one starts with the maximally broken case, and follows the behavior of the second fundamental monopole as the last two eigenvalues of $\Phi_0$ approach one another, one finds that its mass tends to zero, its core radius tends to infinity, and the fields at any fixed point approach their vacuum value. Finally, the embedding in the corner matrix elements, which previously gave a solution with eight zero modes that was naturally understood to be a two-monopole solution, now gives a solution that is gauge-equivalent to the first fundamental monopole and hence has only four zero modes. In all three of these cases the magnetic charge has a non-Abelian component.

Eqs. (7.3) and (7.4) are consistent with the idea that even for non-maximal symmetry breaking one should interpret all solutions — or at least those with purely Abelian magnetic charges — in terms of a number of component fundamental monopoles. However, there are clearly two quite different kinds of fundamental monopoles. The massive monopoles corresponding to the $\beta_a$ carry $U(1)$ magnetic charges and appear to have four associated degrees of freedom. They can be realized as classical solitons, even though the latter may not be unique, as the $SU(3)$ example shows. The remaining fundamental monopoles, corresponding to the $\gamma_j$, would have to be massless. Indeed, the duality conjecture would lead us to expect to find massless magnetically charged states that would be the duals of the massless gauge bosons of the unbroken non-Abelian subgroup. The difficulty is that, precisely because they are massless, these monopoles cannot be associated with any localized classical solutions. To learn more about them, we must examine multimonopole solutions containing both massive and massless components.

The pathologies associated with non-Abelian magnetic charges suggest that this is best done by concentrating on configurations that obey Eq. (7.2). This should not impose any real physical restriction, since the additional monopoles needed to cancel any non-Abelian charge can be placed
at an arbitrarily large distance. It also turns out to be useful to treat non-maximal symmetry breaking as a limit of maximal symmetry breaking in which one or more of the $h \cdot \beta^*_a$ tend to zero. As we will see, it appears that the moduli space for the maximally broken case behaves smoothly in this limit, with the limit of its metric being the metric for the non-maximally case. Although some of the fundamental monopoles become massless in this limit and no longer have corresponding soliton solutions, their degrees of freedom of these massless monopoles are still evident in the low-energy moduli space Lagrangian.

8 An $SO(5)$ example

A particularly simple example [5] for illustrating this arises with the gauge group $SO(5)$, whose root diagram is shown in Fig. 1 with the simple roots labeled $\beta$ and $\gamma$. Consider the solutions whose magnetic charge is such that

$$g = 4\pi (\beta^* + \gamma^*) .$$

(8.1)

With $h$ as in Fig. 1a, the symmetry breaking is maximal and there is an eight-parameter family of solutions composed of two monopoles, of masses $m_\beta$ and $m_\gamma$ respectively. Because the two monopoles correspond to different simple roots, the moduli space metric is known from the results of Sec. 6. If instead $h$ is perpendicular to $\gamma$, as in Fig. 1b, the unbroken gauge group is $SU(2) \times U(1)$, with the roots of the $SU(2)$ being $\pm \gamma$. These are both orthogonal to $g$, so Eq. (7.2) is satisfied and

Figure 1: The root diagram of $SO(5)$. With the Higgs vector $h$ oriented as in (a) the gauge symmetry is broken to $U(1) \times U(1)$, while with the orientation in (b) the breaking is to $SU(2) \times U(1)$.
Eq. (7.5) tells us that there is again an eight-parameter family of solutions. It turns out that these solutions, which are spherically symmetric, can be found explicitly [28]. This makes it possible to determine the background gauge zero modes and then use Eq. (5.6) to obtain the moduli space metric directly. The result can then be compared with the $m_\gamma \to 0$ limit of the first case.

I will begin by describing the $SU(2) \times U(1)$ solutions. Three of its eight parameters give the location of the center-of-mass. Four others are phase angles that specify the $SU(2) \times U(1)$ orientation. (All elements of the unbroken group act nontrivially on the solution.) This leaves only a single parameter, which I will denote by $b$, whose significance can be found by examining the detailed form of the solutions. To write these we need some notation. Let $\mathbf{t}(\alpha)$ and $\mathbf{t}(\gamma)$ be defined as in Eq. (4.11) and let

$$M = \frac{i}{\sqrt{\beta^2}} \begin{pmatrix} E_\beta & -E_\mu \\ E_\mu & E_\beta \end{pmatrix}.$$  

(8.2)

Any adjoint representation $SO(5)$ field $P$ can then be decomposed into parts that are respectively singlets, triplets, and doublets under the unbroken $SU(2)$ by writing

$$P = P_{(1)} \cdot \mathbf{t}(\alpha) + P_{(2)} \cdot \mathbf{t}(\gamma) + \text{tr} P_{(3)} M.$$  

(8.3)

With this notation, the solutions can be written as

$$A_{i(1)}^a = \epsilon_{aim} \hat{r}_m A(r) \quad \quad \phi_{(1)}^a = \hat{r}_a H(r)$$

$$A_{i(2)}^a = \epsilon_{aim} \hat{r}_m G(r,b) \quad \quad \phi_{(2)}^a = \hat{r}_a G(r,b)$$

$$A_{i(3)} = \tau_i F(r,b) \quad \quad \phi_{(3)} = -i F(r,b)$$  

(8.4)

where $A(r)$ and $H(r)$ are the same as the coefficient functions in the $SU(2)$ BPS monopole solution given in Eq. (3.8) and

$$F(r,b) = \frac{v}{\sqrt{8 \cosh(vr/2)}} L(r,b)^{1/2}$$

$$G(r,b) = A(r) L(r,b)$$  

(8.5)

with

$$L(r,b) = [1 + (r/b) \coth(vr/2)]^{-1}$$  

(8.6)

and $v = \mathbf{h} \cdot \alpha$.

The parameter $b$, which has dimensions of length, can take on any positive real value. It only enters into the doublet and triplet components of the fields, and then only through the function $L(r,b)$. While the doublet fields decrease exponentially fast outside the monopole core, the triplet fields have long-range components whose character is determined by $b$. For $1/v \lesssim r \lesssim b$ these...
fall as $1/r$, resulting in a Coulomb magnetic field appropriate to a non-Abelian magnetic charge. At larger distances, however, the vector potential falls as $1/r^2$, implying a field strength falling as $1/r^3$ and thus showing that the magnetic charge is purely Abelian. Thus, one might view these solutions as being composed of a massive monopole, with a core of radius $\sim 1/v$, surrounded by a “non-Abelian cloud” of radius $\sim b$ that cancels the non-Abelian part of its charge.

In order to obtain the moduli space metric from Eq. (5.6), we need the background gauge zero modes about these solutions. An infinitesimal variation with respect to $b$ gives one zero mode, which turns out to already be in background gauge. The three $SU(2)$ modes can then be obtained from this by a transformation of the type shown in Eq. (3.13). The translational and $U(1)$ modes could also be obtained in the usual fashion. However, we do not need to do so, since the corresponding parts of the metric can be inferred from the BPS mass formulas for monopoles and dyons. The result of all this is

$$ds^2_{SU(2) \times U(1)} = Mdx^2 + \frac{16\pi^2}{M}d\chi^2 + k \left[ \frac{db^2}{b} + b \left( da^2 + \sin^2 \alpha d\beta^2 + (d\gamma + \cos \alpha d\beta)^2 \right) \right]$$

(8.7)

where $M$ is the monopole mass, $x$ is the location of the center of the monopole, $\chi$ is the $U(1)$ phase, and $\alpha$, $\beta$, and $\gamma$ are the three angles specifying the $SU(2)$ orientation of the solution. The coefficient $k$ is a constant whose value is unimportant for our purposes.

This should be compared with the two-monopole moduli space metric when the symmetry is broken to $U(1) \times U(1)$. Let $M = m_\beta + m_\gamma$ and $\mu = m_\beta m_\gamma/M$ denote the total mass and reduced mass of the system. After transformation into center-of-mass and relative variables, the metric given by Eq. (6.1) takes the form

$$ds^2_{U(1) \times U(1)} = Md_x^2 + \frac{16\pi^2}{M}d\chi_{\text{tot}}^2 + \left( \mu + \frac{k}{r} \right) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

$$+ k^2 \left( \mu + \frac{k}{r} \right)^{-1} (d\psi + d\cos \theta d\phi)^2.$$  

(8.8)

Here $x$ specifies the position of the center-of-mass, $r$, $\theta$ and $\phi$ are the spherical coordinates specifying the relative positions of the two monopoles, $\chi_{\text{tot}}$ and $\psi$ are overall and relative $U(1)$ phases, and $k$ is the same constant as in Eq. (8.7). We are interested in the limit where $m_\gamma \rightarrow 0$ with $M$ held fixed. In this limit the reduced mass $\mu$ vanishes, and the metric becomes

$$ds^2_{U(1) \times U(1)} = Md_x^2 + \frac{16\pi^2}{M}d\chi_{\text{tot}}^2 + k \left[ \frac{dr^2}{r} + r (d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2) \right].$$

(8.9)

This is exactly the same metric as in Eq. (8.7), but with a different notation: $b$ replaced by $r$, and $\alpha$, $\beta$, and $\gamma$ replaced by $\theta$, $\phi$, and $\psi$, respectively. Thus, the moduli space metric behaves smoothly in the limit where the unbroken symmetry becomes non-Abelian, with the number of
degrees of freedom being conserved. However, the interpretation of these coordinates undergoes a curious change. In particular, as one of the monopoles becomes massless, its position becomes somewhat ambiguous. While the separation \( r \) goes over into the cloud radius \( b \), which has a definite gauge-invariant meaning, the directional angles \( \theta \) and \( \phi \) are replaced by two global \( SU(2) \) gauge phases. Hence, two solutions with the same intermonopole separation but different values for \( \theta \) and \( \phi \) are physically distinct as long as the \( \gamma \)-monopole remains massive, but become gauge-equivalent when \( m_\gamma = 0 \).

9 More complex examples

Further insight into the nature of the massless monopoles and the non-Abelian cloud can be obtained by considering some more complex solutions that arise in \( SU(N) \) gauge theories. The asymptotic value of the adjoint Higgs field in some fixed direction can be brought into the form

\[
\Phi_0 = \text{diag} \left( t_N, t_{N-1}, \ldots, t_1 \right) \tag{9.1}
\]

with \( t_1 \leq t_2 \leq \ldots \leq t_N \). The set of simple roots picked out by Eq. (4.5) then generate the \( SU(2) \) subgroups that lie in \( 2 \times 2 \) blocks along the diagonal and the magnetic charge is given by

\[
g \cdot H = 4\pi \text{diag} \left( n_{N-1}, n_{N-2} - n_{N-1}, \ldots, n_1 - n_2, -n_1 \right). \tag{9.2}
\]

If the \( t_j \) are all unequal, the symmetry breaking is maximal, to \( U(1)^{N-1} \), and the \( n_j \) are the topological charges. Here I will be primarily interested instead in the case where the middle \( N - 2 \) eigenvalues of \( \Phi_0 \) are equal and the unbroken group is \( U(1) \times SU(N - 2) \times U(1) \). As explained previously, I will focus on configurations in which the asymptotic magnetic field is purely Abelian and commutes with all elements of the unbroken \( SU(N - 2) \); i.e., configurations for which the middle \( N - 2 \) eigenvalues of \( g \cdot H \) are all equal.

All choices for the \( \{n_j\} \) that satisfy this condition can be written as combinations of three irreducible solutions:\[
7\text{The existence three types of irreducible solutions can be understood by noting that the states corresponding to the two species of massive monopoles in this theory transform under the} \ (N - 2) \text{-dimensional fundamental and antifundamental representations of} \ SU(N - 2) \text{. An} \ SU(N - 2) \text{ singlet can be formed from} \ N - 2 \text{ fundamentals,} \ N - 2 \text{ antifundamentals, or from a fundamental and an antifundamental.}\]

1) \( n_j = j - 1 \), so that

\[
g \cdot H = 4\pi \text{diag} \left( (N - 2), -1, -1, \ldots, -1, 0 \right). \tag{9.3}
\]

2) \( n_j = N - j - 1 \), so that

\[
g \cdot H = 4\pi \text{diag} \left( 0, 1, 1, \ldots, 1, -(N - 2) \right). \tag{9.4}
\]
3) $n_j = 1$ for all $j$, leading to

$$g \cdot H = 4\pi \text{diag} (1, 0, 0, \ldots, 0, -1).$$  \hspace{1cm} (9.5)

(Note that the moduli space metric for this case can be obtained from the results of Sec. 6; the metrics for the first two cases are not known.)

Configurations of the first type can be viewed as containing $N - 2$ massive and $(N - 2)(N - 3)/2$ massless monopoles, with the massive monopoles all corresponding to the last simple root. Eq. (7.3) shows that they depend on $2(N - 1)(N - 2)$ parameters. Of these, $4(N - 2)$ presumably specify the positions and $U(1)$ phases of the massive monopoles. Specifying the $SU(N - 2)$ orientation of the configuration requires another dim $[SU(N - 2)] = (N - 2)^2 - 1$ parameters. Hence, the remaining $(N - 3)^2$ parameters describe gauge-invariant aspects of the non-Abelian cloud, showing that it is possible for this cloud to have considerably more structure than it did in the $SO(5)$ example of the previous section.

Configurations of the second type also contain $(N - 2)(N - 3)/2$ massless and $N - 2$ massive monopoles, but now with the latter corresponding to the first simple root.

Finally, configurations of the third type contain two massive monopoles (one of each massive species), together with $N - 3$ massless monopoles. There are $4(N - 1)$ parameters in all, 8 of which specify the positions and $U(1)$ phases of the massive monopoles. One might have expected to find an additional $(N - 2)^2 - 1$ parameters associated with the unbroken $SU(N - 2)$, as in the previous cases. Except for the simplest nontrivial case, $SU(4)$ broken to $U(1) \times SU(2) \times U(1)$, there are clearly not enough parameters. The explanation is that, as we will see more explicitly below, any configuration of this type for gauge group $SU(N)$ with $N > 4$ can be obtained by an embedding of an $SU(4)$ solution. As a result, there are only dim $[SU(N - 2)/U(N - 4)] = 4N - 13$ global gauge parameters. There is but a single remaining parameter, which is associated with the non-Abelian cloud.

Having explicit expressions for the solutions in these cases would clearly be quite helpful for understanding the nature and characteristics of the non-Abelian cloud. Such expressions are not known for the first two cases. However, solutions for the third case can be obtained explicitly, as I will now describe, by making use of Nahm’s construction of the BPS monopole solutions [29].

The fundamental elements in Nahm’s approach [30] are a triplet of matrices $T_a(t)$ that satisfy a set of nonlinear ordinary differential equations. These then define a set of linear differential equations for another set of matrices, $v(t, r)$, from which the spacetime fields $A(r)$ and $\Phi(r)$ can be readily obtained. I will now describe the details of this construction for the case of a gauge group
The eigenvalues $t_j$ of $\Phi_0$ divide the range $t_1 \leq t \leq t_N$ into $N - 1$ intervals. On the $j$th interval, $t_j < t < t_{j+1}$, let $k(t) \equiv n_j$, where $n_j$ is given by Eq. (9.2). The matrices $T_a(t)$ are required to have dimension $k(t) \times k(t)$. In addition, whenever two adjacent intervals have the same value for $k(t)$, there are three matrices $\alpha_j$, of dimension $k(t_j) \times k(t_j)$, defined at the interval boundary $t_j$. These matrices are required to obey the Nahm equation,

$$\frac{dT_a}{dt} = \frac{i}{2} \epsilon_{abc}[T_b, T_c] + \sum_j (\alpha_j) a \delta(t - t_j).$$

(9.6)

where the sum in the last term is understood to only run over those values of $j$ for which the $\alpha_j$ are defined. Having solved this equation, one must next find a $2k(t) \times N$ matrix function $v(t, r)$ and $N$-component row vectors $S_j(r)$ obeying the linear equation

$$0 = \left[ -\frac{d}{dt} + (T_a + r_a) \otimes \sigma_a \right] v + \sum_j a_j^\dagger S_j \delta(t - t_j)$$

(9.7)

together with the orthogonality condition

$$I = \int dt \, v^\dagger v + \sum_j S_j^\dagger S_j.$$  

(9.8)

In Eq. (9.7), $a_j$ is a $2k(t_j)$-component row vector obeying

$$a_j^\dagger a_j = \alpha_j \cdot \sigma - i(\alpha_j)_0 I$$

(9.9)

with $(\alpha_j)_0$ chosen so that the above matrix has rank 1. Finally, spacetime fields obeying the BPS equations are given by

$$\Phi = \int dt \, t v^\dagger v + t_j S_j^\dagger S_j$$

(9.10)

$$A = -\frac{i}{2} \int dt \left[ v^\dagger \nabla v - \nabla v^\dagger v \right] - \frac{i}{2} \sum_j \left[ S_j^\dagger \nabla S_j - \nabla S_j^\dagger S_j \right].$$

(9.11)

I will consider the case where the $n_j$ are all equal to unity, so $k(t) = 1$ over the entire range and there are an $\alpha_j$ and an $S_j$ for each value of $j$ from 2 through $N - 1$. To begin, I will assume that the $t_j$ are all different, so that there are $N - 1$ distinct massive monopoles, although I will soon turn to the case with unbroken $U(1) \times SU(N - 2) \times U(1)$ symmetry. Eq. (9.6) is solved by the piecewise constant solution

$$T(t) = -x_j, \quad t_j < t < t_{j+1},$$

(9.12)
where the $x_a$ have a natural interpretation as the positions of the individual monopoles. The $a_j$ of Eq. (9.9) are simply two-component row vectors that may be taken to be

$$a_j = \sqrt{2|x_j - x_{j-1}|} \left( \cos(\theta/2)e^{-i\phi/2}, \sin(\theta/2)e^{i\phi/2} \right)$$

(9.13)

where $\theta$ and $\phi$ specify the direction of the vector $\alpha_j = x_{j-1} - x_j$.

Next, we must find a $2 \times N$ matrix $v(t)$ and a set of $N$-component row vectors $S_k \ (k = 2, 3, \ldots, N-1)$ that satisfy Eq. (9.7). To do this, let us first define a function $f_k(t)$ for each interval $t_k \leq t \leq t_{k+1}$, with

$$f_1(t) = e^{(t-t_2)(r-x_1)\cdot \sigma}$$

$$f_k(t) = e^{(t-t_k)(r-x_k)\cdot \sigma}f_{k-1}(t_k), \quad k > 1.$$  

(9.14)

These are defined so that at the boundaries between intervals $f_k(t_k) = f_{k-1}(t_k)$. An arbitrary solution of Eq. (9.7) can then be written as

$$v^a(t) = f_k(t)\eta^a_k, \quad t_k < t < t_{k+1},$$

(9.15)

where the $\eta_k \ (1 \leq k \leq N-1)$ are a set of $N$-component row vectors. The discontinuities at the interval boundaries must be such that

$$\eta_k = \eta_{k-1} + [f_k(t_k)]_k^{-1}a^\dagger_k S_k.$$  

(9.16)

The normalization condition Eq. (9.8), takes the form

$$I = \sum_{j=2}^{N-1} S_j^\dagger S_j + \sum_{k=1}^{N-1} \eta_k^\dagger N_k \eta_k$$

(9.17)

with

$$N_k = \int_{t_k}^{t_{k+1}} dt f_k^\dagger(t)f_k(t).$$

(9.18)

These equations do not completely determine the $\eta_k$. This indeterminacy reflects the fact that Eq. (9.7) is preserved if $v$ and the $S_k$ are multiplied on the right by any $N \times N$ unitary matrix function of $r$; this corresponds to an ordinary spacetime gauge transformation. A convenient choice is to take two columns of $v$, say $v^1$ and $v^2$, to be continuous. This can be done by setting $S_k^a = 0$ for $a = 1, 2$ and choosing

$$\eta_k^a = N^{-1/2}\theta^a, \quad a = 1, 2,$$

(9.19)

with $N = \sum_k N_k$ and the $\theta^a$ being the two-component objects $\theta^1 = (1,0)^t$ and $\theta^2 = (0,1)^t$. Orthogonality of the other columns of $v$ with the first two, as required by Eq. (9.8), then implies
that
\[ 0 = \sum_{k=1}^{N-1} N_k \eta_k^\mu \]  
(9.20)

where here and below Greek indices are assumed to run from 3 to \( N \). Together with the discontinuity Eq. (9.16), this uniquely determines the \( \eta_k^\mu \). Substituting the result back into Eq. (9.17) then gives an equation for the \( S_k^\mu \),

\[ \delta^{\mu\nu} = \sum_{i,j=2}^{N-1} S_i^\mu [\delta_{ij} + a_i M_{ij} a_j^\dagger] S_j^\nu \]  
(9.21)

where the \( M_{ij} \) are matrices, constructed from the \( N_k \) and the \( f_k(t_k) \), whose precise form is not important for our purpose. After solving this equation for the \( S_k^\mu \), one can then work back to obtain the \( \eta_k^\mu \) and thus \( v \), and then substitute into Eqs. (9.10) and (9.11) to obtain the spacetime fields.

Now let us specialize to the case of unbroken \( U(1) \times SU(2) \times U(1) \) symmetry. The middle \( N - 2 \) eigenvalues of \( \Phi_0 \) are now degenerate, and so all but the first and last intervals in \( t \) vanish. Because the \( f_k(t_k) \) are all equal to unity, the discontinuity equation for the \( \eta_k \) becomes

\[ \eta_k = \eta_{k-1} + a_k^\dagger S_k . \]  
(9.22)

In addition, the matrices \( M_{ij} \) in Eq. (9.21) no longer depend on \( i \) and \( j \), but instead are all equal to a single matrix \( M \).

When the symmetry breaking was maximal, the monopole positions entered both through the functions \( f_k(t) \) and through the \( a_j \). Now however, with the middle intervals having zero width, the positions associated with the massless monopoles enter only through the \( a_j \). But these now appear in Eqs. (9.21) and (9.22) only in the combination \( \sum_j a_j^\dagger S_j^\mu \). This fact has a striking consequence. Consider two sets of monopole positions \( x_k \) and \( \tilde{x}_k \) with identical positions for the massive monopoles, but with the massless monopoles constrained only by the requirement that \( \tilde{a}_j = W_{jk} a_k \), where \( W \) is any \((N - 2) \times (N - 2)\) unitary matrix. If \( S_j^\mu \) is a solution of Eq. (9.21) for the first set of positions, then \( \tilde{S_j^\mu} = W_{jk} S_k^\mu \) is a solution for the transformed set.

This implies that the positions of the massless monopoles are not all physically meaningful quantities. This result was anticipated by the parameter counting done earlier in this section, which indicated that there should be a single gauge-invariant quantity characterizing the non-Abelian cloud. This quantity can be identified by noting that these transformations leave invariant

\[ \sum_j a_j^\dagger a_j = \sum_j \[ a_j \cdot \sigma - i a_j \sigma_0 \] = (x_1 - x_{N-1}) \cdot \sigma + \sum_{j=2}^{N-1} |x_j - x_{j-1}| . \]  
(9.23)

The first term on the right hand side is determined by the positions of the massive monopoles, while the second is just the sum of the distances between successive massless monopoles. The latter can
be used to define a cloud parameter $b$ by

$$2b + R = \sum_{j=2}^{N-1} |x_j - x_{j-1}| \quad (9.24)$$

where $R$ is the distance between the massive monopoles.

The subsequent analysis can be simplified by using a transformation of this type to choose a canonical set of massless monopole positions in which $x_2$ is located on the straight line defined by $x_1$ and $x_{N-1}$ at a distance $b$ from $x_1$, while the remaining $N - 3$ massless monopoles are located at $x_{N-1}$. Once this choice is made, one is rather naturally led to choose a solution for the $S_k$, and hence for the $\eta_k$ and $v$, such that the resulting expressions for the spacetime fields have nontrivial components only in a $4 \times 4$ block. Thus, as promised earlier, the solutions can all be obtained by embeddings of $SU(4) \to U(1) \times SU(2) \times U(1)$ solutions.

The fact that the solutions for arbitrary $SU(N)$ can be obtained from the $SU(4)$ solution underscores the difficulties in pinning down the massless monopoles. When viewed as an $SU(4)$ solution, the configuration contains a single massless monopole, but when it is interpreted as an $SU(N)$ solution there are $N - 3$ massless monopoles. Thus, not only the positions, but even the number of massless components is ambiguous.

These $SU(4)$ solutions have some features that are reminiscent of the $SO(5)$ solutions discussed in the previous section. The fields can be decomposed into pieces that transform as triplets, doublets, and singlets under the unbroken $SU(2)$. Only the first two depend on $b$, and then only through a single function $L$, which is now a $2 \times 2$ matrix. Also, the triplet and doublet components of the Higgs field are given in terms of the same spacetime functions as the corresponding gauge field components, just as was the case with the $SO(5)$ solution.

The detailed form of these solutions [29] is rather complex. However, some insight into the nature of the non-Abelian cloud can be obtained by examining the asymptotic behavior of the fields well outside the cores of the massive monopoles. Consider first the case $b \gg R$. If the distances $y_L$ and $y_R$ from a point $r$ to the two massive monopoles are both much less than $b$, the Higgs field and magnetic field can be written in the form

$$\Phi(r) = U_1^{-1}(r) \begin{pmatrix} t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\ 0 & t_2 + \frac{1}{2y_R} & 0 & 0 \\ 0 & 0 & t_2 - \frac{1}{2y_L} & 0 \\ 0 & 0 & 0 & t_1 + \frac{1}{2y_L} \end{pmatrix} U_1(r) + \cdots \quad (9.25)$$
\[
B(r) = U_1^{-1}(r) \left( \begin{array}{cccc}
\frac{\hat{\psi}_R}{2y_R} & 0 & 0 & 0 \\
0 & -\frac{\hat{\psi}_R}{2y_R} & 0 & 0 \\
0 & 0 & \frac{\hat{\psi}_L}{2y_L} & 0 \\
0 & 0 & 0 & -\frac{\hat{\psi}_L}{2y_L}
\end{array} \right) U_1(r) + \ldots
\]  
(9.26)

where \(U_1(r)\) is an element of \(SU(4)\) and the dots represent terms that are suppressed by powers of \(R/b, y_L/b,\) or \(y_R/b.\) These are the fields that one would expect for two massive monopoles, each of whose magnetic charges has both a \(U(1)\) component and a component in the unbroken \(SU(2)\) that corresponds to the middle \(2 \times 2\) block. If instead \(y \equiv (y_L + y_R)/2 \gg b,\)

\[
\Phi(r) = U_2^{-1}(r) \left( \begin{array}{cccc}
t_4 - \frac{1}{2y} & 0 & 0 & 0 \\
0 & t_2 & 0 & 0 \\
0 & 0 & t_2 & 0 \\
0 & 0 & 0 & t_1 + \frac{1}{2y}
\end{array} \right) U_2(r) + O(b/y^2) 
\]  
(9.27)

\[
B(r) = U_2^{-1}(r) \left( \begin{array}{cccc}
\frac{\hat{\psi}_R}{2y_R} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\hat{\psi}_L}{2y_L}
\end{array} \right) U_2(r) + O(b/y^3) 
\]  
(9.28)

Thus, at distances large compared to \(b\) the non-Abelian part of the Coulomb magnetic field is cancelled by the cloud, in a manner similar to that which we saw for the \(SO(5)\) case.

In the opposite limit, \(b = 0,\) the solutions are essentially embeddings of \(SU(3) \to U(1) \times U(1)\) solutions. At large distances, one finds that

\[
\Phi(r) = U_3^{-1}(r) \left( \begin{array}{cccc}
t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\
0 & t_2 - \frac{1}{2y_L} + \frac{1}{2y_R} & 0 & 0 \\
0 & 0 & t_2 & 0 \\
0 & 0 & 0 & t_1 + \frac{1}{2y_L}
\end{array} \right) U_3(r) 
\]  
(9.29)

\[
B(r) = U_3^{-1}(r) \left( \begin{array}{cccc}
\frac{\hat{\psi}_R}{2y_R} & 0 & 0 & 0 \\
0 & \frac{\hat{\psi}_L}{2y_L} - \frac{\hat{\psi}_R}{2y_R} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\hat{\psi}_L}{2y_L}
\end{array} \right) U_3(r) 
\]  
(9.30)
Viewed as $SU(3)$ solutions, the long-range fields are purely Abelian. Viewed as $SU(4)$ solutions, the long-range part is non-Abelian in the sense that the unbroken $SU(2)$ acts nontrivially on the fields. However, because of the alignment of the fields of the two massive monopoles, the non-Abelian part of the field is a purely dipole field that falls as $R/y^3$ at large distances.

10 Concluding remarks

I have shown in these talks how one is naturally led to a class of multimonopole solutions that contain one or more massive monopoles, similar to those found in the $SU(2)$ gauge theory, surrounded by a cloud, of arbitrary size, in which there are nontrivial non-Abelian fields. Analysis of the moduli space Lagrangian that governs the low-energy monopole dynamics suggests that these clouds can be understood in term of the degrees of freedom of massless monopoles carrying purely non-Abelian magnetic charges.

There remain many open questions relating to these massless monopoles. First, it would clearly be desirable to obtain additional solutions containing non-Abelian clouds. Particularly useful would be solutions with charges such that the cloud depends on more than a single gauge-invariant parameter, and solutions containing more than a single cloud. Experience with the solutions described in Sec. 9 suggests that, as a first step, it might be feasible to attack the simplified problem of determining the cloud structure for a given set of massive monopole positions. From the more physical viewpoint, one would like to use these solutions to gain further insight in the properties of non-Abelian gauge theories. The massless monopoles clearly seem to be the duals of the massless gauge bosons. Hence, one should be able to find some kind of correspondence between the behavior of the non-Abelian clouds and that of the gauge bosons. Understanding this correspondence in detail remains an important challenge.

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