Factor complexity of infinite words associated with non-simple Parry numbers

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The factor complexity of the infinite word $u_\beta$ canonically associated to a non-simple Parry number $\beta$ is studied. Our approach is based on the notion of special factors introduced by Berstel and Cassaigne. At first, we give a handy method for determining infinite left special branches; this method is applicable to a broad class of infinite words which are fixed points of a primitive substitution. In the second part of the article, we focus on infinite words $u_\beta$ only. To complete the description of its special factors, we define and study $(a,b)$-maximal left special factors. This enables us to characterize non-simple Parry numbers $\beta$ for which the word $u_\beta$ has affine complexity.

1 Introduction

The aim of this work is to compute the factor complexity function $C(n)$ of the infinite word $u_\beta$ associated with $\beta$-expansions $[24]$, where $\beta$ is a non-simple Parry number. The definition of Parry numbers is connected with the Rényi expansion of unity $d_\beta(1)$. Parry numbers are those $\beta$ for which $d_\beta(1)$ is eventually periodic. Positional numerical systems with a Parry number as a base have a nice behavior. For example, if we consider $\beta$-integers, i.e., real numbers with vanishing $\beta$-fractional part in their $\beta$-expansion, then the distances between two consecutive $\beta$-integers take only finitely many values. In fact, this property can be used as an equivalent definition of Parry numbers. In this sense, positional numeration systems based on Parry numbers are a natural generalization of the classical decimal or binary systems. Let us mention that even the innocent looking rational base $\beta = \frac{3}{2}$ brings into numeration systems phenomena never observed before $[1]$.

The most prominent Parry number is the golden mean $\tau = \frac{1+\sqrt{5}}{2}$ with $d_\tau(1) = 11$. The infinite word associated to $\tau$ is the famous Fibonacci chain, i.e., the word generated by the substitution $0 \mapsto 01$ and $1 \mapsto 0$. The Fibonacci chain codes the distances between $\tau$-integers. Fabre in $[12]$ showed that for any Parry number there exists a canonical substitution over a finite alphabet such that its unique fixed point $u_\beta$ represents the distribution of $\beta$-integers on the real line.

$\beta$-integers attracted attention of physicists after the discovery of quasicrystals in 1982 $[25]$. $\tau$-integers were shown to be a suitable tool for describing atomic positions in solid materials.
with long range order and non-crystallographical five-fold symmetry [20], [3]. The knowledge
of the factor complexity of the Fibonacci chain is the first step towards the description of
variability of local configurations in quasicrystals [19].

Parry numbers are split into two groups: a Parry number \( \beta \) is called simple if the Rényi
expansion of unity \( d_\beta(1) \) has only a finite number of nonzero elements, otherwise \( \beta \) is non-
simple. The questions concerning the factor complexity of words \( u_\beta \) associated with simple
Parry numbers were discussed in [15] and [4]. Of course, since among \( u_\beta \) one can find some
Sturmian sequences and Arnoux-Rauzy words, the complexity of \( u_\beta \) for some specific values
of \( \beta \) were known earlier.

The first non-simple Parry number \( \beta \) for which the factor complexity of \( u_\beta \) was precisely
determined is such that \( d_\beta(1) = 2(01)^\omega \), i.e., \( \beta \) is a root of the polynomial \( x^3 - 2x^2 - x + 1 \).
This non-simple Parry number appears naturally when describing the model of quasicrystals
with seven-fold symmetry [14]. The first attempt to study factor complexity of \( u_\beta \) for broader
class of non-simple Parry number can be found in [16].

Since any infinite word \( u_\beta \) is the fixed point of a primitive substitution, the factor
complexity of \( u_\beta \) can be estimate from above by a linear function, see [23]. Moreover, we know
that the first difference of complexity is bounded by a constant [8] [21]. Nevertheless, in
general, it is hard to find an explicit formula for the complexity function of an infinite word
and it seems it holds also for the case of \( u_\beta \). However, we are able to find all left special
factors that, in a certain sense, completely determine the factor complexity. The notion of
(right) special factor was introduced by Berstel [5] in 1980 and considerably enhanced by
Cassaigne in his paper [9] in 1997. We introduce another slight enhancement, a tool that will
help us to identify all infinite left special branches of fixed point of substitutions satisfying
some natural assumption. Further, the knowledge of the structure of left special factors will
allow us to identify all non-simple Parry numbers \( \beta \) for which the complexity of \( u_\beta \) is affine:
The complexity of \( u_\beta \) is affine if and only if \( t_1(0\cdots0(t_1 - 1))^\omega \) (Theorem [57]).

2 Parry numbers and associated infinite words

For each \( x \in [0, 1) \) and for each \( \beta > 1 \), using a greedy algorithm, one can obtain the unique
\( \beta \)-expansion \( (x_i)_{i\geq 1}.x_i \in \mathbb{N}, \) of the number \( x \) such that
\[
x = \sum_{i\geq 1} x_i \beta^{-i} \quad \text{and} \quad \sum_{i\geq k} x_i \beta^{-i} < \beta^{-k+1}.
\]

By shifting, each non-negative number has a \( \beta \)-expansion. For \( x \in [0, 1) \), the \( \beta \)-expansion can
be computed also by using the piecewise linear map \( T_\beta : [0, 1) \rightarrow [0, 1) \) defined as
\[
T_\beta(x) = \{ \beta x \},
\]
where \( \{ \beta x \} \) is the fractional part of the real number \( \beta x \). The sequence \( d_\beta(x) = x_1x_2x_3 \cdots \) is
obtained by iterating \( T_\beta \) with
\[
x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor.
\]
The difference between $\beta$-expansion and $d_\beta(x)$ arises for $x = 1$ since the Rényi expansion of unity $d_\beta(1)$ is not a $\beta$-expansion. Parry [22] showed that $d_\beta(1)$ plays a very important role in the theory of $\beta$-numeration. Among other things, it allows us to define Parry numbers.

**Definition 1.** A real number $\beta > 1$ is said to be a Parry number if $d_\beta(1)$ is eventually periodic. In particular,

a) if $d_\beta(1) = t_1 \cdots t_m$ is finite, i.e., it ends in infinitely many zeros, then $\beta$ is a simple Parry number,

b) if it is not finite, i.e., $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+p})^\omega$, then $\beta$ is called a non-simple Parry number.

Note, that the parameters $m, p > 0$ are taken the least possible. It implies that $t_m \neq t_{m+p}$ which will be a very important fact. Another crucial property of $d_\beta(1)$ is the following Parry condition [22] valid for all $\beta > 1$

$$t_j t_{j+1} t_{j+2} \cdots < t_1 t_2 t_3 \cdots \quad \text{for every } j > 1,$$

where $<$ is the (strict) lexicographical ordering.

As the infinite word $u_\beta$ is tightly connected with a geometrical interpretation of $\beta$-integers, we first introduce $\beta$-integers along with some of their properties.

**Definition 2.** The real number $x$ is a $\beta$-integer if the $\beta$-expansion of $|x|$ is of the form $\sum_{i=0}^{k} a_i \beta^i$. The set of all $\beta$-integers is denoted by $\mathbb{Z}_\beta$.

The definition of $\beta$-integers coincides with the definition of classical integers in the case of $\beta$ in $\mathbb{Z}$. But there are several new phenomena linked with the notion of $\beta$-integers when $\beta$ is not an integer. For our purposes, the most interesting difference between classical integers and $\beta$-integers is the difference in their distribution on the real line. While the classical integers are distributed equidistantly, i.e., gaps between two consequent integers are always of the same length 1, the lengths of gaps between $\beta$-integers can take their values even in an infinite set. More precisely, Thurston [26] proved the following theorem.

**Theorem 3.** Let $\beta > 1$ be a real number and $d_\beta(1) = (t_i)_{i \geq 1}$. Then the length of gaps between neighbors in $\mathbb{Z}_\beta$ takes values in the set $\{\triangle_0, \triangle_1, \ldots\}$, where

$$\triangle_i = \sum_{k \geq 1} \frac{t_{k+i}}{\beta^k}, \quad \text{for } i \in \mathbb{N}.$$

**Corollary 4.** The set of lengths of gaps between two consecutive $\beta$-integers is finite if and only if $\beta$ is a Parry number. Moreover, if $\beta$ is a simple Parry number, i.e., $d_\beta(1) = t_1 \cdots t_m$, the set reads $\{\triangle_0, \triangle_1, \ldots \triangle_{m-1}\}$, if $\beta$ is a non-simple Parry number, i.e., $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+p})^\omega$, we obtain $\{\triangle_0, \triangle_1, \ldots \triangle_{m+p-1}\}$. 
Now, let us suppose that we have drawn $\beta$-integers on the real line and assume that $\beta$ is a Parry number. If we read the length of gaps from zero to the right, we obtain an infinite sequence, say $\{\triangle_i\}_{i \geq 0}$. Further, if we read only indices, we obtain an infinite sequence, say $\{\triangle_i\}_{k \geq 0}$. Further, if we read only indices, we obtain an infinite word over the alphabet $\{0, \ldots, m - 1\}$ in the case of simple Parry numbers, and over the alphabet $\{0, \ldots, m + p - 1\}$ in the non-simple case. The obtained infinite word is just the word $u_\beta$ we are interested in. However, there exists another way to define it. Fabre [12] proved that $u_\beta$ can be defined as the unique fixed point of a substitution $\varphi_\beta$ canonically associated with a Parry number $\beta$ and defined as follows.

**Definition 5.** For a simple Parry number $\beta$ the canonical substitution $\varphi_\beta$ over the alphabet $A = \{0, 1, \ldots, m - 1\}$ is defined by

\[
\begin{align*}
\varphi_\beta(0) &= 0^t 1 \\
\varphi_\beta(1) &= 0^t 2 \\
&\vdots \\
\varphi_\beta(m-2) &= 0^{tm-1}(m-1) \\
\varphi_\beta(m-1) &= 0^{tm} 
\end{align*}
\]

**Definition 6.** For a non-simple Parry number $\beta$ the canonical substitution $\varphi_\beta$ over the alphabet $A = \{0, 1, \ldots, m + p - 1\}$ is defined by

\[
\begin{align*}
\varphi_\beta(0) &= 0^t 1 \\
\varphi_\beta(1) &= 0^t 2 \\
&\vdots \\
\varphi_\beta(m-1) &= 0^{tm} m \\
\varphi_\beta(m) &= 0^{tm+1}(m+1) \\
&\vdots \\
\varphi_\beta(m+p-2) &= 0^{tm+p-1}(m+p-1) \\
\varphi_\beta(m+p-1) &= 0^{tm+p} m. 
\end{align*}
\]

We see that the definition of $\varphi_\beta$ is given by $d_\beta(1)$ and that the only difference between simple and non-simple cases lies in the images of the last letters $m - 1$ and $m + p - 1$. While in the simple case the last letters of images $\varphi_\beta(k), k = 0, 1, \ldots, m - 1$, are all distinct and so the images form a suffix-free code, in the non-simple case either $\varphi_\beta(m) = 0^{tm} m$ is a prefix of $\varphi_\beta(m + p - 1) = 0^{tm+p} m$ or vice versa. As we will see later on, this property is crucial from the point of view of computing the complexity of the infinite word $u_\beta$.

**Definition 7.** Let $\beta > 1$ be a Parry number. The unique fixed point of the canonical substitution $\varphi_\beta$ is denoted by

\[
u_\beta = \lim_{n \to \infty} \varphi_\beta^n(0) = \varphi_\beta^\infty(0).
\]

The uniqueness of $u_\beta$ follows from the definitions of $\varphi_\beta$, the letter 0 is the only admissible starting letter of a fixed point.
3 Special factors and factor complexity

In this section, we will recall the notion of special factors of an arbitrary infinite word and we will explain how the structure of special factors of an infinite word determines its factor complexity. To be able to do it, we need some usual basic notation, see [9] for more.

**Definition 8.** Let $\mathcal{A} = \{0, 1, \ldots, q - 1\}, q \geq 1$ be a finite alphabet. An infinite word over the alphabet $\mathcal{A}$ is a sequence $u = (u_i)_{i \geq 1}$ where $u_i \in \mathcal{A}$ for all $i \geq 1$. If $v = u_j u_{j+1} \cdots u_{j+n-1}$, $j, n \geq 1$, then $v$ is said to be a factor of $u$ of length $n$ and the index $j$ is an occurrence of $v$, the empty word $\epsilon$ is the factor of length 0.

By $L_n(u)$ we denote the set of all factors of $u$ of length $n \in \mathbb{N}$, the language of $u$ is then the set $L(u) = \bigcup_{n \in \mathbb{N}} L_n(u)$.

**Definition 9.** Let $u$ be an infinite word over an alphabet $\mathcal{A}$. The function $C(n) = \#L_n(u)$ is the factor complexity function of $u$. We further define the first difference of the complexity by $\triangle C(n) = C(n+1) - C(n)$.

In what follows, we shall restrict ourself to those infinite words which are fixed point of some substitution (morphism) $\varphi$ defined over a finite alphabet $\mathcal{A}$. We shall further assume that $\varphi$ is injective and primitive.

**Definition 10.** A substitution $\varphi$ is primitive if there exists $k \in \mathbb{N}$ such that for all $a, b \in \mathcal{A} \varphi^k(a)$ contains $b$.

Equivalently, $\varphi$ is primitive if the incidence matrix $M_{\varphi}$ is primitive.

There are several well-known properties of the complexity function $C$.

**Proposition 11.**

(i) For each infinite word $u$, $0 \leq C(n) \leq (#A)^n$,

(ii) if $u$ is eventually periodic then $C(n)$ is eventually constant,

(iii) $u$ is aperiodic in and only if $C(n)$ is unbounded and $C(n)$ is unbounded if and only if $\triangle C(n) \geq 1$, for all $n \in \mathbb{N}$,

(iv) if $u$ is a fixed point of a primitive substitution then $C(n)$ is a sublinear function, i.e., $C(n) \leq an + b$, for some $a, b \in \mathbb{N}$,

(v) if $u$ is a fixed point of primitive substitution then $\triangle C(n)$ is bounded.

Items (i) – (iii) are obvious, (iv) is due to [23], (v) was proved in [21] and in a more general context in [8].

It is also well known that any fixed point of a primitive substitution is uniformly recurrent, i.e., each factor occurs infinitely many times and the gaps between its two consecutive occurrences are bounded in length. It implies that each factor is extendable both to the right and to the left.
Definition 12. Let \( v \) be a factor of \( u \), the set of left extensions of \( v \) is defined as
\[
\text{Lext}(v) = \{ a \in \mathcal{A} \mid av \in L(u) \}.
\]

If \( \#\text{Lext}(v) \geq 2 \), then \( v \) is said to be a left special (LS) factor of \( u \).

In the analogous way we define the set of right extensions \( \text{Rext}(u) \) and a right special (RS) factor. If \( v \) is both left and right special, then it is called bispecial.

The connection between (left) special factors and the complexity follows from the following reasoning. Let us suppose that \( L_n(u) = \{ v_1, \ldots, v_k \} \), \( k \geq 1 \) and let \( \text{Lext}(v_i) = \{ a_{1}^{(i)}, \ldots, a_{\ell_i}^{(i)} \} \), \( \ell_i \geq 1, i = 1, \ldots, k \). Now, it is not difficult to realize that
\[
L_{n+1}(u) = \{ a_{1}^{(1)} v_1, \ldots, a_{\ell_1}^{(1)} v_1, a_{1}^{(2)} v_2, \ldots, a_{\ell_{k-1}}^{(k-1)} v_{k-1}, a_{1}^{(k)} v_k, \ldots, a_{\ell_k}^{(k)} v_k \},
\]
i.e., by concatenating all factors of length \( n \) and all their left extensions we obtain all factors of length \( n + 1 \). It implies that
\[
\#L_{n+1}(u) - \#L_n(u) = \Delta C(n) = \sum_{v \in L_n(u) \atop \text{v is LS}} (\#\text{Lext}(v) - 1).
\] (2)

Hence, if we know all LS factors along with the number of their left extensions, we are able to evaluate the complexity \( C(n) \) using this formula.

3.1 Classification of LS factors

Let \( a, b \in \text{Lext}(v) \) be left extensions of a factor \( v \) of \( u \), it means both \( av \) and \( bv \) are factors of \( u \). If there exists a letter \( c \in \text{Rext}(av) \cap \text{Rext}(bv) \), we say that \( v \) can be extended to the right such that it remains LS with left extensions \( a, b \), indeed \( a, b \in \text{Lext}(vc) \).

Definition 13. Let \( a, b \in \text{Lext}(v) \) be distinct left extensions of a LS factor \( v \) of \( u \). We say that \( v \) is an \((a, b)\)-maximal LS factor if \( \text{Rext}(av) \cap \text{Rext}(bv) = \emptyset \), in words, \( v \) cannot be extended to the right such that it remains LS with left extensions \( a, b \).

![Figure 1: Two types of \((a, b)\)-maximal LS factor \( v \).](image)

In general, there are two types of \((a, b)\)-maximal LS factors both depicted in Figure 1. In Case a), \( a \) and \( b \) are only left extensions of \( v \) and so \( v \) cannot be extended to the right and...
remain LS. In Case b), \(v\) can be prolonged by letter \(e\) such that \(ve\) is still a LS factor but it looses its left extension \(a\).

It can also happen that a factor \(v\) with left extensions \(a\) and \(b\) is extendable to the right infinitely many times. In this way we obtain an infinite LS branch.

**Definition 14.** An infinite word \(w\) is an infinite LS branch of \(u\) if each prefix of \(w\) is a LS factor of \(u\). We put

\[
\text{Lext}(w) = \bigcap_{v \text{ prefix of } w} \text{Lext}(v).
\]

**Proposition 15.**

(i) If \(u\) is eventually periodic, then there is no infinite LS branch of \(u\),

(ii) if \(u\) is aperiodic, then there exists at least one infinite LS branch of \(u\),

(iii) if \(u\) is a fixed point of a primitive substitution then the number of infinite LS branches is bounded.

(i) is obvious, (iii) is a direct consequence of [2] and Proposition [11] (v). Item (ii) is a direct consequence of the famous König’s infinity lemma [18] applied on sets \(V_1, V_2, \ldots\), where the set \(V_k\) comprises all LS factors of length \(k\) and where \(v_1 \in V_i\) is connected by an edge with \(v_2 \in V_{i+1}\) if \(v_1\) is prefix of \(v_2\).

Taking all together, our aim is to find all \((a,b)\)-maximal LS factors and also all infinite LS branches of \(u\).

**Remark 16.** The term “special factor” (for us it was RS factor) was introduced in 1980 [5] and it has been used for computing the factor complexity since then (eg. [6], [11]). The notations introduced above are based on Cassaigne’s article [9]. An \((a,b)\)-maximal factor is a new term, actually it is a special case of a weak bispecial factor proposed there. It is also shown in the article that bispecial factors determine the second difference of the complexity in a similar way as LS factors determine the first difference of the complexity.

**Remark 17.** Everything what has been (and will be) defined or showed for LS factors can be defined or showed similarly for RS factors.

### 3.2 How to find infinite LS branches

Before introducing a new notion, let us consider the example substitution

\[
\varphi : 1 \mapsto 1211, 2 \mapsto 311, 3 \mapsto 2412, 4 \mapsto 435, 5 \mapsto 534
\]

with \(u = \varphi^\infty(1)\). Further, let \(w\) be a LS factor (or infinite LS branch) of \(u\) with left extensions 1 and 2. Is \(\varphi(w)\) again LS factor? From Figure 2 (first line) we see that it is not since the letter 1 is its only left extension. In order to obtain a LS factor, we have to prepend the factor 11 which is the longest common suffix of \(\varphi(1) = 1211\) and \(\varphi(2) = 311\), then \(11\varphi(w)\) is
Let $\varphi$ be a substitution defined over an alphabet $A$. For each couple of distinct letters $a, b \in A$ we define $f_{L}(a, b)$ as the longest common suffix of words $\varphi(a)$ and $\varphi(b)$.

Definition 18. Let $\varphi$ be a substitution defined over an alphabet $A$. For each couple of distinct letters $a, b \in A$ we define $f_{L}(a, b)$ as the longest common suffix of words $\varphi(a)$ and $\varphi(b)$.

Definition 19. Let $v$ be a prefix of a word $w$, then $v^{-1}w$ is the word $w$ without the prefix $v$. Analogously, we define $wv^{-1}$, if $v$ is a suffix of $w$.

Definition 20. Let $\varphi$ be an injective substitution defined over an alphabet $A$ having a fixed point $u$. For each unordered couple of distinct letters $a, b \in A$ such that $Rext(a) \cap Rext(b) \neq \emptyset$ we define the set $g_{L}(a, b)$ as follows.

(i) If $f_{L}(a, b)$ is a proper suffix of both $\varphi(a)$ and $\varphi(b)$, then $g_{L}(a, b)$ contains just the last letters of factors $\varphi(a)(f_{L}(a, b))^{-1}$ and $\varphi(b)(f_{L}(a, b))^{-1}$.

(ii) If $f_{L}(a, b) = \varphi(a)$ (i.e., W.L.O.G. $|\varphi(a)| < |\varphi(b)|$), then $g_{L}(a, b)$ contains the last letter of the factor $\varphi(b)(f_{L}(a, b))^{-1}$ and all the last letters of factors $\varphi(c)$, where $c \in Lext(a)$ such that $Rext(ca) \cap Rext(b) \neq \emptyset$.

Assumption 21. A substitution $\varphi$ defined over $A$ is injective and it has a fixed point $u$ such that for all $a, b \in A$, for which $g_{L}(a, b)$ is defined, it holds that $\#g_{L}(a, b) = 2$.

Moreover, if $f_{L}(a, b) = \varphi(a)$ and $d$ is the last letter of the factor $\varphi(b)(f_{L}(a, b))^{-1}$, then for all $c \in Lext(a)$ such that $Rext(ca) \cap Rext(b) \neq \emptyset$ it holds that $d$ is not the last letter of $\varphi(c)$.

Assumption 21 is valid for all suffix-free substitutions since $g_{L}(a, b)$ from point (i) of Definition 20 contains always just two elements and the case when $f_{L}(a, b) = \varphi(a)$ never happens. If $f_{L}(a, b) = \varphi(a)$, then Assumption 21 says that if $v$ is a LS factor with $Lext(v) = \{a, b\}$, then the last letter $e$ of $\varphi(c)$ is the same for all $c \in Lext(av)$ and, moreover, $e\varphi(a)$ is not a suffix of $\varphi(b)$ – in other words, for each LS factor $v$ the factor $f_{L}(a, b)\varphi(v)$ is again LS.

We will see that this complicated assumption is satisfied for the (not suffix-free) substitution $\varphi_{\beta}$, where $\beta$ is a non-simple Parry number.
**Definition 22.** Let \( \varphi \) be a substitution satisfying Assumption 21. Then for each LS factor (or infinite LS branch) \( w \) having distinct left extensions \( a \) and \( b \) we define \( f \)-image of \( w \) as the factor \( f_L(a, b)\varphi(w) \).

With respect to the preceding discussion, Assumption 21 says that \( f \)-image is always LS factor and it has just two left extensions, namely two elements of \( g_L(a, b) \), corresponding to two original left extensions \( a \) and \( b \).

Assumption 21 along with the notation introduced above allow us to define the following graph.

**Definition 23.** Let \( \varphi \) be a substitution defined over an alphabet \( A \) satisfying Assumption 21. We define a directed labelled graph \( GL_\varphi \) as follows:

1. Vertices of \( GL_\varphi \) are couples of distinct letters \( a, b \) such that \( \text{Rext}(a) \cap \text{Rext}(b) \neq \emptyset \).
2. If \( g_L(a, b) = \{c, d\} \), then there is an edge from a vertex \((a, b)\) to a vertex \((c, d)\) labelled by \( f_L(a, b) \).

In fact, the crucial result of Assumption 21 is that out-degree of each vertex is exactly one. The graph \( GL_\varphi \) for our example substitution is drawn in Figure 3, this substitution satisfies Assumption 21 for it is suffix-free.

Now, let us consider the case when \( w \) is an infinite LS branch with \( a, b \in \text{Ext}(w), a \neq b \). Obviously, \( f \)-image of \( w \) is uniquely given. For most substitutions even a “\( f \)-preimage” of each infinite LS branch exists.

**Assumption 24.** An infinite word \( u \) is a fixed point of a substitution \( \varphi \) satisfying Assumption 21. For each infinite LS branch \( w \) of \( u \) with \( a, b \in \text{Ext}(w), a \neq b \) there exists at least one infinite LS branch \( \overline{w} \) with left extensions \( c \) and \( d \) such that \( f \)-image of \( \overline{w} \) equals \( w \) and \( g_L(c, d) = \{a, b\} \).
This assumption is very weak. Actually, we have not found any primitive substitution not satisfying it. The reason for it is the following. It is not difficult to prove (but it requires a lot of new notation) this: Disruption of Assumption 24 implies that each factor of \( u \) can be decomposed into images of letters \( \varphi(a), a \in A \), in at least two different ways. Thus, in order to prove that Assumption 24 is satisfied, it suffices to find any factor with unique decomposition to images of letters. For the example substitution (3) the factor 1211 is such a factor since it can arise only as \( \varphi(1) \varphi(1) \) and \( \varphi(3) \varphi(1) \). In the case of \( \varphi_\beta \), we will prove that Assumption 24 is satisfied differently.

**Theorem 25.** Let \( u \) be a fixed point of a primitive injective substitution \( \varphi \) satisfying Assumption 24 and let \( w \) be an infinite LS branch with \( a, b \in \text{Lex}(w), a \neq b \). Then either \( w \) is a periodic point of \( \varphi \), i.e.,

\[
    w = \varphi^\ell(w) \quad \text{for some } \ell \geq 1, \tag{4}
\]

and \( (a, b) \) is a vertex of a cycle in \( GL_{\varphi} \) labelled by \( \epsilon \) only or \( w = s\varphi^\ell(s)\varphi^{2\ell}(s)\cdots \) is the unique solution of the equation

\[
    w = s\varphi^\ell(w), \tag{5}
\]

where \( (a, b) \) is a vertex of a cycle in \( GL_{\varphi} \) containing at least one edge with non-empty label, \( \ell \) is the length of this cycle and

\[
    s = f_L(g_L^{\ell-1}(a, b))\cdots\varphi^{\ell-2}(f_L(g_L(a, b))\varphi^{\ell-1}(f_L(a, b)). \tag{6}
\]

**Proof.** Due to Assumption 24 both the \( f \)-image and the \( f \)-preimage of \( w \) exist. The \( f \)-image is unique due to Assumption 21 and the uniqueness of \( f \)-preimage follows from the fact that the number of infinite LS branches is finite. Thus, \( f \)-image is one-to-one mapping on the finite set of all ordered couples

\[
    \{(c, d), \mathcal{W}\},
\]

where \( \mathcal{W} \) is an infinite LS branch of \( u \) and \( (c, d) \) is an unordered couple of letters such that \( c, d \in \text{Lex}(\mathcal{W}) \), \( c \neq d \). The \( f \)-image can be viewed as a permutation on this finite set and so it decomposes the set to independent cycles as depicted in Figure 4. \( \ell \) is then the length of the cycle containing \( ((a, b), w) \).

As explained earlier, applying \( f \)-image on a LS factor having left extensions \( a, b \) corresponds to the movement along the edge in \( GL_{\varphi} \) which leads from \( (a, b) \). If the labels of the edges of the cycle are all \( \epsilon \), then the \( f \)-image coincides with \( \varphi \) and so we obtain the periodic point (4). If at least one edge is labelled by non-empty word, then \( \ell \) is also length of the cycle in \( GL_{\varphi} \) containing the vertex \( (a, b) \) and Equation (5) then corresponds to \( \ell \)-times applying of \( f \)-image on \( w \).

Our example substitution \( \varphi \) (see 3) has five periodic points

\[
    \varphi^\infty(1), \varphi^\infty(4), \varphi^\infty(5), (\varphi^2)^\infty(2), (\varphi^2)^\infty(3).
\]
Figure 4: Circular structure of infinite LS branches.

It is an easy exercise to show that

\[ \text{Lext}(1) = \{1, 2, 3, 4, 5\}, \text{Lext}(2) = \{1, 4, 5\}, \text{Lext}(3) = \{1, 4, 5\}, \]
\[ \text{Lext}(4) = \{1, 2, 3\}, \text{Lext}(5) = \{1, 2, 3\}. \]

Looking at the graph \( GL_\phi \) depicted in Figure 3 we see that \( \varphi^\infty(4), \varphi^\infty(5) \) are not infinite LS branches as none of the vertices (1, 2), (2, 3) and (1, 3) is a vertex of a cycle labelled by \( \epsilon \) only.

Hence, only \( \varphi^\infty(1), (\varphi^2)^\infty(2), (\varphi^2)^\infty(3) \) are infinite LS branches with left extensions 1, 4, 5.

As for infinite LS branches corresponding to Equation (5), in the case of our example, there is only one cycle not labelled by the empty word only between vertices (1, 2) and (2, 3).

There are two (= the length of the cycle) equations corresponding to this cycle

\[ w = \varphi(11)\varphi^2(w) \quad \text{and} \quad w = 11\varphi^2(w). \]

They give us two infinite LS branches

\[ \varphi(11)\varphi^3(11)\varphi^5(11)\cdots, \]
\[ 11\varphi^2(11)\varphi^4(11)\cdots, \]

the former having left extensions 1 and 2 and the latter 2 and 3.

\textbf{Remark 26.} Assumption 21 can be reformulated into a weaker form but to do so, it would require to introduce rather complicated notation. The important fact is that the canonical substitution \( \varphi_\beta \) satisfies Assumption 21.

\section{Infinite LS branches of \( u_\beta \)}

At first, let us recall known results for simple Parry numbers. The substitution \( \varphi_\beta \) from Definition 5 is suffix-free and it implies that it satisfies Assumption 21. One can easily prove that even Assumption 24 is satisfied. As mentioned earlier, the last letters of images of letters
are all distinct and so \( f_L(a,b) = \epsilon \) for all couples \( a,b \in A \). The graph \( GL_{\varphi_\beta} \) then looks as in Figure 5. It contains \( m - 1 \) cycles labelled by \( \epsilon \) only and hence the only candidate for being an infinite LS branch is the unique fixed (and periodic) point of \( \varphi_{\beta} \), namely \( u_\beta \) with \( \text{Lext}(u_\beta) = A \). The same result is proved in [15] using different techniques.

![Figure 5: \( GL_{\varphi_\beta} \) for simple Parry \( \beta \).](image)

**Theorem 27** ([4], [15]). Let \( \beta > 1 \) be a simple Parry number with \( d_\beta(1) = t_1 \cdots t_m \) and let \( u_\beta \) be the fixed point of the canonical substitution \( \varphi_\beta \) ([5]). Then

(i) if \( t_1 = t_2 = \cdots = t_{m-1} \) or \( t_1 > \max\{t_2, \ldots, t_{m-1}\} \), the exact value of \( C(n) \) is known [13],

(ii) in particular, \((m - 1)n + 1 \leq C(n) \leq mn\), for all \( n \geq 1 \),

(iii) \( C(n) \) is affine \( \iff \) the following two conditions are fulfilled

1) \( t_m = 1 \)

2) for all \( i = 2, 3, \ldots, m-1 \) we have

\[
t_i t_{i+1} \cdots t_{m-1} t_1 \cdots t_{i-1} \leq t_1 t_2 \cdots t_{m-1}.
\]

Then \( C(n) = (m - 1)n + 1 \).

In this paper, we will find the necessary and sufficient condition for the complexity being affine in the case of non-simple Parry numbers. We will see that it is more restrictive than the one from point \( (iii) \).

### 4.1 Infinite LS branches in case of non-simple Parry numbers

In this section, we will apply hitherto introduced theory on the fixed point \( u_\beta \) of the substitution \( \varphi_\beta \), where \( \beta \) is a non-simple Parry number. To be able to do so, we need some more notation and simple but useful technical lemmas.

**Definition 28.** For all \( k, \ell \in \mathbb{N} \), we define the addition \( \oplus : \mathbb{N} \times \mathbb{N} \to A \) as follows.

\[
k \oplus \ell := \begin{cases} k + \ell & \text{if } k + \ell < m + p, \\ m + (k + \ell - m \mod p) & \text{otherwise.} \end{cases}
\]
Similarly, if used with parameters $t_i$, we define for all $k, \ell \in \mathbb{N}, k + \ell > 0$

$$t_{k+\ell} := \begin{cases} t_{k+\ell} & \text{if } 0 < k + \ell < m + p + 1, \\ t_{m+1+(k+\ell-m-1 \mod p)} & \text{otherwise}. \end{cases}$$

In fact, the addition $\oplus$ tracks the last letters of the words $\varphi^n(0), n = 0, 1, \ldots$. Therefore, we can rewrite the definition of the substitution $\varphi$ in a simpler form

$$\varphi(k) = 0^{t_{k+1}}(k \oplus 1), \quad \forall k \in A.$$ 

Further, employing the new notation and the definition of the substitution $\varphi$, one can easily prove the following simple observations.

**Lemma 29.** For the substitution $\varphi$ it holds

(i) for all $n \in \mathbb{N}$ and for all $k \in A$

$$\varphi^n(k) = (\varphi^{n-1}(0))^{t_{k+1}}(\varphi^{n-2}(0))^{t_{k+2}} \ldots (\varphi(0))^{t_{k+(n-1)}}0^{t_{k+n}}(k \oplus n),$$

(ii) if $avb$ is a factor of $u$, $v \in A^*$ and $a, b \neq 0$, then there exists unique factor $v'$ such that $\varphi(v') = vb$.

Our aim is to obtain the graph $GL\varphi$, thus, we need to know left extensions of letters and also all $g_L(a, b)$.

**Definition 30.** Let us define for all $k \in A$, $k \neq 0$, a function $z : \{1, \ldots, m + p - 1\} \rightarrow \{0, 1, \ldots, m + p - 2\}$ by

$$z(k) = \max\{j \in \mathbb{N} \mid 0^j \text{ is a suffix of } t_1t_2 \cdots t_k\}.$$ 

For $k \in \{m, \ldots, m + p - 1\}$ we also define a function $y : \{m, \ldots, m + p - 1\} \rightarrow \{0, 1, \ldots, p - 1\}$ by

$$y(k) = \begin{cases} \max\{j \in \mathbb{N} \mid 0^j \text{ is a suffix of } t_{m+1}t_{m+2} \cdots t_{m+p} \cdots t_k\} & \text{if } k > m, \\ \max\{j \in \mathbb{N} \mid 0^j \text{ is a suffix of } t_{m+1}t_{m+2} \cdots t_{m+p}\} & \text{if } k = m. \end{cases}$$

Further, we define

$$\ell_0 = \begin{cases} 0 & \text{if } t_1 > 1, \\ 1 + \max\{j \in \mathbb{N} \mid 0^j \text{ is a prefix of } t_2t_3 \cdots t_m\} & \text{otherwise}. \end{cases}$$

and finally we put $t = \min\{t_m, t_{m+p}\}$.

Note that $z(k)$ and $y(k)$ can return the same value for $k \geq m$, a necessary condition for $z(k) \neq y(k)$ is that $t = 0$ and $z(\ell) \neq y(\ell)$ for all $m \leq \ell < k$. Due to Parry condition \[1\] we must have $1 \leq \ell_0 \leq m - 1$ as the case $d_\beta(1) = 10 \cdots 0(t_{m+1} \cdots t_{m+p-1})^\omega$ is not admissible.

**Lemma 31.** For $u_\beta$ the fixed point of $\varphi$ it holds
(i) \( \text{Lex}(0) = \{ \ell_0, \ldots, m + p - 1 \} \).

(ii) \( \text{Lex}(k) = \{ z(k) \} \), for \( k \in \{ 2, 3, \ldots, m - 1 \} \).

(iii) \( \text{Lex}(k) = \{ z(k), y(k) \} \), for \( k \in \{ m, m + 1, \ldots, m + p - 1 \} \).

Proof. (ii) Each letter \( k > 0 \) can appear in \( u_\beta \) as the image of \( k - 1 \), namely \( \varphi_\beta(k - 1) = 0^{t_k}k \). If \( t_k > 0 \), then \( 0 \in \text{Lex}(k) \), if \( t_k = 0 \) we consider \( \varphi_\beta^2(k - 2) = \varphi_\beta(0^{t_{k-1}})k = (0^1)0^{t_{k-1}}k \). Again, if \( t_{k-1} > 0 \), then \( 1 \in \text{Lex}(k) \), otherwise we continue in the same way. Since \( t_1 > 0 \), this process is finite.

(iii) The letter \( m \) can appear in \( u_\beta \) not only as an image \( \varphi_\beta(m - 1) \) (i.e., case (ii)) but as well as \( \varphi_\beta(m + p - 1) = 0^{t_{m+p}}m \). If we realize this second possible origin of the letters \( m, m + 1, \ldots, m + p - 1 \), then the proof is the same as for (ii).

(i) If \( t_1 > 1 \), then \( 00 \) is a factor of \( u \). Hence, for all \( n \in \mathbb{N} \) the word \( \varphi_\beta^n(0)0 = \cdots (0 \oplus n)0 \) is a factor as well and so \( \text{Lex}(0) = \mathcal{A} \).

Let \( t_1 = 1 \), it implies \( t_i \in \{ 0, 1 \} \) for \( i = 1, \ldots, m + p \). It holds \( \varphi_\beta^0(01) = \varphi_\beta((\ell_0 - 1)\ell_0) = \ell_00(\ell_0 + 1) \), hence, \( \ell_0, \ell_0 + 1, \ldots, m + p - 1 \in \text{Lex}(0) \). But \( d_\beta(1) \) cannot contain a sequence of consecutive 0’s shorter than \( \ell_0 \) due to Parry condition (1) and so \( \ell_0 \) is the least letter in \( \text{Lex}(0) \).

The previous lemma allows us to partially determine all LS factors of \( u_\beta \).

**Corollary 32.** Let \( v \) be a LS factor of \( u_\beta \) containing at least one nonzero letter, then one of the following factors is a prefix of \( v \).

(i) \( 0^{t_1}1 \),

(ii) \( 0^tm \),

(iii) \( 0^tk \), if \( k > m \) and \( t = t_{m+1} = t_{m+2} = \cdots = t_{k-1} = 0 \).

Note, that the factors from the last point are images of the factor \( 0^tm \) in the case when \( t = 0 \).

**Lemma 33.** For \( u_\beta \) the fixed point of \( \varphi_\beta \) it holds

(i) if \( (k, \ell) \) is unordered couple of distinct letters of \( \mathcal{A} \) such that \( \text{Rext}(k) \cap \text{Rext}(\ell) \neq \emptyset \), and \( (k, \ell) \neq (m - 1, m + p - 1) \), then \( f_L(k, \ell) = \epsilon \) and \( g_L(k, \ell) = \{ k \oplus 1, \ell \oplus 1 \} \);

(ii) \( f_L(m - 1, m + p - 1) = 0^tm \) and \( g_L(m - 1, m + p - 1) = \{ 0, z \} \), where

\[
z = \begin{cases} 
1 + z(m - 1) & \text{if } t_m < t_{m+p}, \\
1 + z(m + p - 1) & \text{if } t_{m+p} < t_m.
\end{cases}
\]  

(7)

Proof. (i) follows directly from the definitions of \( g_L, f_L \) and \( \varphi_\beta \). (ii) is a simple consequence of Lemma 31. Remark that if \( t_m > t_{m+p} \geq 0 \), then \( z(m + p - 1) = y(m + p - 1) \).

\[ \square \]
Now we have the knowledge necessary to complete the graph $GL_{\varphi_\beta}$ but still we have to prove that the substitution $\varphi_\beta$ satisfies Assumptions 21 and 24.

**Lemma 34.** The substitution $\varphi_\beta$ from Definition 6 satisfies Assumptions 21 and 24.

**Proof.** The fact that Assumptions 21 is fulfilled follows from Lemmas 31 and 33.

To construct an $f$-preimage for an arbitrary infinite LS branch is easy due to Lemma 29 part (ii), Corollary 32 and Lemma 33.

Now, we know all we need to be able to construct the graph $GL_{\varphi_\beta}$. For the case when $t_1 > 1$, the graph is depicted in Figure 6. Since $\text{Lext}(0) = \mathcal{A}$, all possible unordered couples of letters are vertices of the graph. If $z$ is not a multiple of $p$ (i.e., the decision condition $z = sp$ in Figure 6 returns no), then the graph contains only cycles with edges labelled by $\epsilon$ only. If $z = sp$ for certain positive integer $s$, then there is the cycle on vertices $(0, z), (1, z \oplus 1), \ldots, (m - 1, z \oplus m - 1)$, where the edge from the vertex $(m - 1, z \oplus m - 1)$ to the vertex $g_L(m - 1, z \oplus m - 1) = (0, z)$ is labelled by $f_L(m - 1, z \oplus m - 1) = 0^t m$.

If $t_1 = 1$ the graph $GL_{\varphi_\beta}$ is the same as in Figure 6 but we have to remove vertices $(k, \ell)$, where $k < \ell_0$ or $\ell < \ell_0$ and $(k, \ell) \neq (0 \oplus n, z \oplus n)$ for any $n \in \mathbb{N}$. What is important for our purpose is that the structure of cycles is the same for arbitrary value of $t_1$.

![Figure 6: $GL_{\varphi_\beta}$ for non-simple Parry $\beta$, $s$ is a positive integer.](image_url)

Since the fact whether $z$ is or is not a multiple of $p$ is crucial for the structure of cycles in $GL_{\varphi_\beta}$, we introduce the following set.

**Definition 35.** A non-simple Parry number $\beta > 1$ is an element of a set $S$ if and only if there exists a positive integer $s$ such that $z = sp$, where $z$ is the non-zero left extension of $0^t m$.

Employing Lemmas 31 and 33 one can easily prove the following.
Lemma 36. A non-simple Parry number $\beta > 1$ belongs to $S$ if and only if one of the following conditions is satisfied

a) $d_\beta(1) = t_1 \cdots t_m (0 \cdots 0 t_{m+p})^\omega$ and $t_m > t_{m+p}$,

b) $d_\beta(1) = t_1 \cdots t_{m-qp} 0 \cdots 0 t_m (t_{m+1} \cdots t_{m+p})^\omega$, $q \geq 1$, and $t_m < t_{m+p}$.

Putting it all together, we obtain a proof of the following proposition which gives us the complete list of infinite LS branches of $u_\beta$ for all non-simple Parry numbers.

Proposition 37. Let $\beta > 1$ be a non-simple Parry number and let $u_\beta$ be the fixed point of the canonical substitution $\varphi_\beta$. Then

(i) if $p > 1$, then $u_\beta$ is an infinite LS branch with left extensions $\{m, m+1, \ldots, m+p-1\}$,

(ii) if $\beta \notin S$, then $u_\beta$ is the unique infinite LS branch,

(iii) if $\beta \in S$, then there are $m$ infinite LS branches

$$0^p m \varphi^m (0^p m) \varphi^{2m} (0^p m) \cdots$$

$$\vdots$$

$$\varphi^{m-1} (0^p m) \varphi^{2m-1} (0^p m) \varphi^{3m-1} (0^p m) \cdots .$$

There are no other infinite LS branches of $u_\beta$.

5 Maximal LS factors

As explained earlier, in order to determine the complexity of an infinite word, we need to find all infinite LS branches as well as all $(a, b)$-maximal LS factors. The structure of $(a, b)$-maximal LS factors is not so simple as the one of infinite LS branches but still it can be described using the notion of $f$-image. To define an $f$-image for $(a, b)$-maximal LS factors, we need Assumption 21 to be satisfied also for $g_R$ – we will say that the right version of Assumption 21 is satisfied.

Lemma 38. For the substitution $\varphi_\beta$ and for all distinct $a, b \in A$ we have $f_R(a, b) = 0^{t_{a,b}}$, where

$$t_{a,b} = \min \{t_a, t_b\} . \quad (8)$$

Thus, the right version of Assumption 21 is satisfied for $\varphi_\beta$ is prefix-free.

Definition 39. A factor $v \in A^+$ is an $(a - c, b - d)$-bispecial factor of an infinite word $u$ defined over a finite alphabet $A$ if both $avc$ and $bvd$ are factors of $u$.

Definition 40. Let a substitution $\varphi$ defined over a finite alphabet $A$ satisfy the left and right version of Assumption 21 and let $v$ be an $(a - c, b - d)$-bispecial factor of a fixed point of $\varphi$. Then $f_L(a, b) \varphi(v) f_R(c, d)$ is said to be the $f$-image of $v$. 
Obviously, the $f$-image of $v$ is $(\tilde{a} - \tilde{c}, \tilde{b} - \tilde{d})$-bispecial, where $g_L(a, b) = \{\tilde{a}, \tilde{b}\}$ and $g_R(c, d) = \{\tilde{c}, \tilde{d}\}$.

Now, consider again the particular case of $u_\beta$. A LS factor $v$ having $a, b \in \text{Ext}(v)$ is $(a, b)$-maximal if $\text{Rext}(av) \cap \text{Rext}(bv) = \emptyset$ and so it is as well an $(a - c, b - d)$-bispecial for all $c \in \text{Rext}(av)$ and $d \in \text{Rext}(bv)$. Are $f$-images of $v$ again $(g_L(a, b))$-maximal? Not all of them as states the following simple lemma.

**Lemma 41.** Let $v$ be a bispecial factor of $u_\beta$ having left extensions $a$ and $b$. If its $f$-image

$$f_L(a, b)\varphi_\beta(v)f_R(c, d) = f_L(a, b)\varphi_\beta(v)0^{t_{c\oplus1},d\oplus1},$$

is $(g_L(a, b))$-maximal, then $c \in \text{Rext}(av), d \in \text{Rext}(bv)$ satisfy

$$t_{c\oplus1} \geq \max\{ t_{e\oplus1},f_{\oplus1} | e \in \text{Rext}(av), f \in \text{Rext}(bv) \}$$

$$t_{d\oplus1} \geq \max\{ t_{e\oplus1},f_{\oplus1} | e \in \text{Rext}(bv), f \in \text{Rext}(bv) \}. \quad (9)$$

**Definition 42.** An $f$-image of a bispecial factor $v$ having left extensions $a$ and $b$

$$f_L(a, b)\varphi_\beta(v)f_R(c, d),$$

where $c \in \text{Rext}(av), d \in \text{Rext}(bv)$ satisfy [9], is said to be the max-$f$-image of $v$.

The following lemma is crucial for understanding the structure of the max-$f$-images of $(a, b)$-maximal factors.

**Lemma 43.** If $\ell, k \in A, \ell \neq k$ and $t_{\ell\oplus1}t_{\ell\oplus2} \cdots \geq t_{k\oplus1}t_{k\oplus2} \cdots$, then for all $n \in \mathbb{N}$ the longest common prefix of the factors $\varphi_\beta^n(k)$ and $\varphi_\beta^n(\ell)$, denoted by $\text{lcp}(\varphi_\beta^n(k), \varphi_\beta^n(\ell))$, equals

$$\text{lcp}(\varphi_\beta^n(k), \varphi_\beta^n(\ell)) = \varphi_\beta^n(k)(k \oplus n)^{-1},$$

i.e., $\varphi_\beta^n(k)$ without the last letter $k \oplus n$.

Moreover, denote by $c$ the letter such that $\text{lcp}(\varphi_\beta^n(k), \varphi_\beta^n(\ell))c$ is a prefix of $\varphi_\beta^n(\ell)$. Then, $t_{c\oplus1}t_{c\oplus2} \cdots \geq t_{k\oplus(n+1)}t_{k\oplus(n+2)} \cdots$ for all $n \in \mathbb{N}$.

**Proof.** The case $n = 0$ is trivial.

The rest of the proof is carried on by induction on $n$.

$$\varphi_\beta^{n+1}(k) = (\varphi_\beta^n(0))^{t_{k\oplus1}}\varphi_\beta^n(k \oplus 1),$$

$$\varphi_\beta^{n+1}(\ell) = (\varphi_\beta^n(0))^{t_{k\oplus1}}(\varphi_\beta^n(0))^{t_{\ell\oplus1} - t_{k\oplus1}}\varphi_\beta^n(\ell \oplus 1), \quad (10)$$

if $t_{\ell\oplus1} = t_{k\oplus1}$, we apply the assumption of induction on $\text{lcp}(\varphi_\beta^n(k \oplus 1), \varphi_\beta^n(\ell \oplus 1))$ and if $t_{\ell\oplus1} > t_{k\oplus1}$, then on $\text{lcp}(\varphi_\beta^n(k \oplus 1), \varphi_\beta^n(0))$ (see Parry condition [11]).

As for the second part of the statement, the letter $c$ is given by [10] and this along with the Parry condition concludes the proof. 

\[\square\]
Lemma 44. Let $n \in \mathbb{N}$. The $n$-th max-$f$-image of a bispecial factor $v$ with left extensions $a$ and $b$, i.e., the factor we obtain if we apply $n$ times the mapping max-$f$-image on $v$, equals

$$
\overline{v} = s \varphi^n_\beta(v) \text{lcp}(\varphi^n_\beta(c), \varphi^n_\beta(d)),
$$

where $c \in \text{Rext}(av)$, $d \in \text{Rext}(bv)$, $s$ is given by (cf. (6))

$$
s = f_L(g_L^{n-1}(a,b)) \cdots \varphi^{n-2}(f_L(g_L(a,b))) \varphi^{n-1}(f_L(a,b)).
$$

and

$$
t_{c \oplus 1}t_{c \oplus 2} \cdots \geq t_{d \oplus 1}t_{d \oplus 2} \cdots,
$$

for all $c' \in \text{Rext}(av)$ and $d' \in \text{Rext}(bv)$.

Proof. The case $n = 0$ is obvious, we carry on by induction on $n$. Let us assume W.L.O.G. that

$$
t_{c \oplus 1}t_{c \oplus 2} \cdots \geq t_{d \oplus 1}t_{d \oplus 2} \cdots
$$

and that $g^n_L(a,b) = \{\tilde{a}, \tilde{b}\}$. Hence

$$
\overline{v} = s \varphi^n_\beta(v) \varphi^n_\beta(d)(d \oplus n)^{-1}
$$

and

$$
\text{Rext}(\tilde{b}v) = \{d' \oplus n \mid t_{d' \oplus 1} \cdots t_{d' \oplus n} = t_{d \oplus 1} \cdots t_{d \oplus n}\}.
$$

Further, let $c' \in \text{Rext}(\tilde{a}v)$, then due to Lemma 43

$$
t_{c' \oplus 1}t_{c' \oplus 2} \cdots \geq t_{d' \oplus (n+1)}t_{d' \oplus (n+2)} \cdots
$$

for all $d' \oplus n \in \text{Rext}(\tilde{b}v)$. But $t_{d \oplus (n+1)} \geq t_{d'\oplus(n+1)}$ for all $d' \oplus n \in \text{Rext}(\tilde{b}v)$ and so the max-$f$-image of $v$ equals

$$
f_L(g^n_L(a,b))\varphi^n_\beta(\overline{v})0^{\delta_{\beta}^{d \oplus (n+1)}} = f_L(g^n_L(a,b))\varphi^n_\beta(s)\varphi^{n+1}(v)\text{lcp}(\varphi^{n+1}(c), \varphi^{n+1}(d)).
$$

Each bispecial factor $v$ having left extensions $a$ and $b$ has the unique max-$f$-image. Since the substitution $\varphi_\beta$ is injective, the structure of max-$f$-images cannot be circular as it is for $f$-images of infinite LS branches – $v$ cannot be the $k$-th max-$f$-image of its own for any $k$. However, the notion of a max-$f$-image allows us to describe all $(a, b)$-maximal factors of $u_\beta$ for all $a, b \in A$. We will prove that each $(a, b)$-maximal factor is the $k$-th max-$f$-image either of $0^{t_1-1}$ if $t_1 > 1$ or of $0$ if $t_1 = 1$, for some $k \in \mathbb{N}$. A sketch of the proof is as follows. Let $v$ be an $(a, b)$-maximal factor containing at least two nonzero letters. Employing Lemma 29 part (ii), one can find a bispecial factor $\overline{v}$ such that its max-$f$-image is $v$. Again, if $\overline{v}$ contains at least two nonzero letters, we find a bispecial factor $\overline{\overline{v}}$ such that its max-$f$-image is $\overline{v}$. In this way, we
obtain a bispecial factor containing at most one nonzero letter such that its $k$-th max-$f$-image equals $v$. According to Corollary \[ \text{Lemma 45.} \]

**Lemma 45.** Let $t_1 > 1$ and $k \in \mathbb{N}$. Then the $k$-th max-$f$-image of factors $0^{t_1}$, $0^s$ and $0^t m^q$, where $1 \leq s < t_1 - 1$ and $0 \leq q \leq t_1$, are not $(a, b)$-maximal for any distinct letters $a$ and $b$.

**Proof.** First, consider $0^{t_1}$ with distinct left extensions $a$ and $b$. It holds that $\text{Lex}(0^{t_1}) = \text{Lex}(0^{t_1})$ and $\text{Rext}(0^{t_1}) \subset \{ k \in \mathcal{A} \setminus \{ 0 \} \mid t_k = t_1 \}$. For each $k \in \text{Rext}(0^{t_1})$, we must have $t_{k \oplus 1} t_{k \oplus 2} \cdots \prec t_2 t_3 \cdots$ (see Parry condition (1)) and, due to Lemma 44, $k$-th max-$f$-image of $0^{t_1}$ is a prefix of the $k$-th $f$-image of the LS factor $\varphi_{\beta}^k(0^{t_1})$, both having the same left extensions.

Similar arguments can be used in order to prove that $k$-th max-$f$-image of $0^s$ is always a prefix of $k$-th $f$-image of the LS factor $0^{t_1-1}$. Again, $\text{Lex}(0^s) = \text{Lex}(0^{t_1-1})$ and the rest is implied directly by the Parry condition.

Finally, consider the LS factor $0^t m^q$, having just two left extensions 0 and $z$ (see (7)). In accord with Lemma 44 the $m$-th max-$f$-image of $0^{t_1-1}$ with left extensions 0 and $p$ equals

$$0^t m \varphi_{\beta}^m(0^{t_1-1}) \varphi_{\beta}^m(1)(m + 1)^{-1} = 0^t m 0^{t_1} 1 \cdots.$$ (12)

Indeed, $\text{Rext}(00^{t_1-1}) = \{ k \in \mathcal{A} \setminus \{ 0 \} \mid t_k = t_1 \}$ and $0 \in \text{Rext}(p0^{t_1-1})$ and so the fact that $t_{k \oplus 1} t_{k \oplus 2} \cdots \prec t_2 t_3 \cdots$ and the Parry condition imply that the $m$-th max-$f$-image is $0^t m \varphi_{\beta}^m(0^{t_1-1}) \text{lcp}(\varphi_{\beta}^m(0), \varphi_{\beta}^m(1))$. Thus, $0^t m 0^q$, as a prefix of (12), is not a $(0, z)$-maximal.

**Lemma 46.** Let $t_1 = 1$ and $k \in \mathbb{N}$. Then $t = 0$ and the $k$-th max-$f$-image of the factor $m^q$, where $0 \leq q \leq 1$, is not $(a, b)$-maximal for any distinct letters $a$ and $b$.

**Proof.** As in the proof of the previous lemma, we can prove that the $(m - \ell_0)$-th max-$f$-image of 0 with left extensions $\ell_0$ and $\ell_0 + p$ is the factor

$$m \varphi_{\beta}^m(1)(m + 1)^{-1},$$ (13)

where, according to Lemma 29 part (i),

$$\varphi_{\beta}^m(1) = (\varphi_{\beta}^{m-1}(0))^{t_2} (\varphi_{\beta}^{m-2}(0))^{t_3} \cdots (\varphi_{\beta}(0))^{t_m} 0^{t_{m+1}} (m + 1).$$

In order that $m 0$ may be $(0, z)$-maximal, it must be $\varphi_{\beta}^m(1) = m + 1$ and so $t_2 = \cdots = t_{m+1} = 0$. But is is not possible due to the Parry condition since then $t_1 t_2 \cdots \prec (t_{m+p} = 1) t_m \cdots t_{m+p} l_{(m+p)\Theta 1} \cdots$. 


Proposition 47. Let $v$ be an $(a, b)$-maximal factor of $u_\beta$. Then there exists $k \in \mathbb{N}$ such that $v$ is the $k$-th max-$f$-image of

(i) $0^{t_1-1}$ if $t_1 > 1$,

(ii) $0$ if $t_1 = 1$.

Proof. We will prove that if $v$ contains at least two nonzero letters, then it is the $k$-th max-$f$-image of a bispecial factor of the form $0^s$ or $0^q m 0^q$, where $1 \leq s \leq t_1$ and $0 \leq q \leq t_1$. The rest of the proof then follows from the previous two lemmas.

Let us assume that $v$ contains at least two nonzero letters. Then, due to Lemma 29 part (ii), $v = f_L(a', b') \varphi_\beta(\overline{v}) f_R(c', d')$, where $\overline{v}$ is a $(a' - c', b' - d')$-bispecial factor such that $v$ is the max-$f$-image of $\overline{v}$ and $g_L(a', b') = \{a, b\}$. Analogously, if $\overline{v}$ contains at least two nonzero letters, there exists an $(a'' - c'', b'' - d'')$-bispecial factor $\overline{\overline{v}}$ which is an $f$-preimage of $\overline{v}$. But it must be also a max-$f$-preimage, if it is not, then $\overline{\overline{v}}$ is also the $f$-image of $\overline{v}$ having the left extensions $a'$ and $b'$ for some $q' > 0$ and so $v$ cannot be $(a, b)$-maximal as it is a proper prefix of the max-$f$-image of LS factor $\overline{\overline{v}}$ with the left extensions $a$ and $b$. Using this argument iteratively, we will obtain a bispecial factor of the form $0^s$ or $0^q m 0^q$ such that $v$ is its $k$-th max-$f$-image.

□

In fact, the previous proposition along with Lemma 44 provides us with the complete list of $(a, b)$-maximal factors. However, in the last section of this paper we will need to know some details to be able to determine under which conditions the complexity of $u_\beta$ is affine.

Corollary 48. If $d_\beta(1) \neq t_1 (0 \cdots 0 (t_1 - 1))^\omega$, then the $k$-th max-$f$-image of the factor $[12]$ is $(g_\beta^k (0, z))$-maximal for all $k \in \mathbb{N}$.

If $\beta \notin S$, then the $k$-th max-$f$-image reads

$$\varphi_\beta^k(0^t m) \varphi_\beta^{m+1}(0^{t_1-1}) \varphi_\beta^{m+1}(1)(m \oplus k)^{-1}.$$ 

Proof. The factor $[12]$ is always LS with just two left extensions 0 and $z$. Therefore it is $(0, z)$-maximal if it is neither a prefix of any infinite LS branch or a proper prefix of the $k$-th max-$f$-image of its own for certain $k > 0$.

In the case when $\beta \notin S$, the longest common prefix of the $k$-th max-$f$-image of the factor $[12]$ and of the unique infinite LS branch $u_\beta$ equals

$$\varphi_\beta^k(0^t m)(m \oplus k)^{-1}.$$ 

Hence, either it is non-empty and shorter than the longest common prefix of the $(k + 1)$-th max-$f$-image of $[12]$ and of $u_\beta$ or it is empty, $k < p$ and $t = t_{m+1} = \cdots = t_{m+k} = 0$ (or only $t = 0$ for $k = 0$). In the latter case, the $k$-th max-$f$-image of $[12]$ begins in letter $m + k$ which is different from the first letters of $u_\beta$ and of all other max-$f$-images of $[12]$. Putting all together, the $k$-th max-$f$-image of $[12]$ is neither a prefix of $u_\beta$ or of the $\ell$-th max-$f$-image of $[12]$ for any $\ell \neq k$. 

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If $\beta \in \mathcal{S}$, then $u_\beta$ is not the only one infinite LS branch, there are $m$ other branches
\[ u_1 = 0^t m \varphi_\beta^m (0^t m) \varphi_\beta^{2m} (0^t m) \cdots \] (14)
and $u_t = \varphi_\beta^{t-1} (u_1), t = 2, \ldots, m$. To finish the proof, we have to foreclose the possibility that the factor (12) is prefix of $u_1$. Looking at (14) and (12), we see that it happens only if $t = t_1 - 1$ and $m = 1$, in other words, if $d_\beta(1) = t_1 (0 \cdots 0(t_1 - 1))^\omega$. The proof of that the factor (12) is not a prefix of any max-f-image of its own is analogous to the one above.

\[ k_0 = \begin{cases} -1 & \text{if } t \neq t_1 - 1, \\ 0 & \text{if } t = t_1 - 1 \text{ and } t_2 \neq t_{m+1} \\ \max\{\ell \in \mathbb{N} \mid t_{\ell+1} \neq t_{m+\ell}\} & \text{otherwise} \end{cases} \] (15)
then the $k$-th max-f-image of $0^{t_1-1}$ is also $(g_L^k(0, z))$-maximal factor for all $k_0 < k < m$.

\[ \text{Proof. It holds } \text{Rext}(00^{t_1-1}) = \{k \in \mathcal{A} \setminus \{0\} \mid t_k = t_1 \text{ or } k = m \text{ and } t_{m+p} = t_1\} \]
and for all $a \in \mathcal{A} \setminus \{0\}$ we have $k \in \text{Rext}(a0^{t_1-1})$ if and only if $k = 0$ or the both following conditions are satisfied:

(i) $z(k) = a - 1$ or $y(k) = a - 1$,

(ii) $t_k = t_1 - 1$ or $k = m$ and $t_{m+p} = t_1 - 1$.

The intersection of $\text{Rext}(00^{t_1-1})$ and $\text{Rext}(a0^{t_1-1})$ is not empty if and only if $a = z$ and $t = t_1 - 1$, in other words, if and only if $0^{t_1-1}$ is a prefix of $0^t m$ what is a LS factor having just two left extensions 0 and $z$.

Similarly, we can prove that the $k$-th max-f-image of $0^{t_1-1}$ is $(g_L^k(0, a))$-maximal factor for all $a \in \mathcal{A} \setminus \{0, z\}$. Also similarly, the $k$-th max-f-image of $0^{t_1-1}$, namely

\[ \varphi_\beta^k (0^{t_1-1}) \varphi_\beta^k (1)(k + 1)^{-1}, \]

is $(g_L^k(0, z))$-maximal if it is not prefix of the LS factor

\[ \varphi_\beta^k (0^t m) = \varphi_\beta^k (0^t) \varphi_\beta^k (m) \]
having the left extensions $g_L^k(0, z)$. The proof then follows from Lemma 29 part (i) and Lemma 43 applied on $\varphi_\beta^k (1)$ and $\varphi_\beta^k (m)$.
Taking into account Lemmas 31 and 47, one can prove the following corollary using analogous techniques as in the proof of the previous one. Note that $\text{Rext}(\ell_0) = \{k \in \mathcal{A} \mid z(k - 1) = \ell_0 - 1 \text{ or } y(k - 1) = \ell_0 - 1\}$ and $\text{Rext}(a) = \{1\}$ for all $a > \ell_0$, i.e., 0 is $(\ell_0, \ell_0 + z)$-maximal if it is not a prefix of the $\ell_0$-th max-f-image of the factor which reads
\[
\varphi_{\ell_0}^k(m)\varphi_{\beta}^m(1 + m + \ell_0))^{-1} = \varphi_{\ell_0}^k(m)\varphi_{\beta}^m(\ell_0 + 1) + (m + \ell_0))^{-1}.
\]

**Corollary 50.** If $t_1 = 1$, then the $k$-th max-f-image of 0 is $(g_L^k(\ell_0, a + \ell_0))$-maximal factor for all letters $a > \ell_0$, $a \neq z$ and for all $0 \leq k < m - \ell_0$.

Moreover, $k$-th max-f-image of 0 is $(g_L^k(\ell_0, z + \ell_0))$-maximal if $k_0 \geq \ell_0$ and $k = k_0 - \ell_0, k_0 - \ell_0 + 1, \ldots, m - \ell_0$, where $k_0$ is defined by (15).

6 **Affine complexity**

The aim of the present section is to find the necessary and sufficient condition for the factor complexity of $u_\beta$ being affine. In order the complexity to be affine, the first difference of complexity $\Delta C(n)$ must be constant. The following lemma says when $\Delta C(n)$ can change its value. The proof is an immediate consequence of (2).

**Lemma 51.** Let $u$ be an infinite word over a finite alphabet.

(i) If $\Delta C(n + 1) > \Delta C(n)$, then the number of LS factor of length $n + 1$ is greater then the number of LS factor of length $n$.

(ii) If $\Delta C(n + 1) < \Delta C(n)$, then $u$ contains $(a, b)$-maximal factor of length $n$ for some letters $a$ and $b$.

That is, the complexity is affine if either $u$ does not contain any $(a, b)$-maximal factor and all infinite LS branches have empty common prefix or if each $(a, b)$-maximal factor of length $n$ is “compensated” by appearing of a “new” LS factor of length $n + 1$. Examples of the first case are Arnoux-Rauzy words whose all LS factors are prefixes of unique infinite LS branch. As for the latter case, appearing of a “new” LS factor of length $n + 1$ means there is a LS factor $v$ of length $n$ and its right extensions $c$ and $d$ such that $vc$ and $vd$ are both LS, i.e $v$ is the longest common prefix of two different LS factors – Cassaigne call such LS factors *strong bispecial*.

Since $u_\beta$ comprises always at least one $(a, b)$-maximal factor, each such $(a, b)$-maximal must be as long as the longest common prefix of two different LS factors in order that the complexity may be affine. We will prove that it is possible only if the number of $(a, b)$-maximal factors is finite, thus in the case of $d_{\beta}(1) = t_1(0 \cdots 0(t_1 - 1))^\omega$.

**Lemma 52.** If $k_0 < m - 1$, where $k_0$ is defined by (15), then the factor complexity of $u_\beta$ is not affine.
Proof. If \( k_0 < m - 1 \), then the \((k_0 + 1)\)-th max-f-image, if \( t_1 > 1 \), (resp. \((k_0 - \ell_0 + 1)\)-th if \( t_1 = 1 \)) of \( 0^{t_1 - 1} \) (resp. 0) is \( g_{L}^{k_0}(0, z) \)-maximal. Consider the longest common prefix of the LS factor \( \varphi_{\beta}^{k_0}(0^t m) \) having left extensions \( g_{L}^{k_0}(0, z) \) and of the infinite LS branch \( u_{\beta} \), if \( p > 1 \), or of the LS factor \( \varphi_{\beta}^{m-1}(0^{t_1 - 1}) \) with left extensions \( m - 1 \) and \( m \), if \( p = 1 \) (and so \( u_{\beta} \) is not an infinite LS branch). This factor equals \( \varphi_{\beta}^{k_0}(0^t m)(m \oplus k_0)^{-1} \) which is a prefix of the \((k_0 + 1)\)-th max-f-image (resp. \((k_0 - \ell_0 + 1)\)-th if \( t_1 = 1 \)) of \( 0^{t_1 - 1} \) (resp. 0) and hence it is not \((a, b)\)-maximal for any distinct \( a, b \in A \). Overall, \( \triangle \mathcal{C}(n_0) < \triangle \mathcal{C}(n_0 + 1) \), where \( n_0 \) is the length of the factor \( \varphi_{\beta}^{k_0}(0^t m)(m \oplus k_0)^{-1} \).

\[ \square \]

**Lemma 53.** If \( d_{\beta}(1) = t_1(0 \cdots 0(t_1 - 1))^\omega \), then the factor complexity of \( u_{\beta} \) is affine, namely \( \mathcal{C}(n) = pn + 1, n \in \mathbb{N} \).

*Proof.* In this case, \( t = t_1 - 1 \) and so \( k_0 = 0 = m - 1 \). Hence, the \((0, a)\)-maximal factor \( 0^{t_1 - 1} \) is at the same time the longest common prefix of the only infinite LS branches \( u_{\beta} \) and \( 0^t m \varphi_{\beta}(0^t m) \varphi_{\beta}^2(0^t m) \cdots \). But \( 0^{t_1 - 1} \) is the only \((a, b)\)-maximal and prefixes of these two infinite LS branches are the only LS factors of \( u_{\beta} \), thus, the proof is complete.

\[ \square \]

**Lemma 54.** If \( \beta \in S \) and \( d_{\beta}(1) \neq t_1(0 \cdots 0(t_1 - 1))^\omega \), then the factor complexity of \( u_{\beta} \) is not affine.

*Proof.* In the case when \( p > 1 \), there are \( m + 1 \) infinite LS branches given by Proposition \( 37 \). Let us denote them by \( u_0 = u_{\beta}, u_1, \ldots, u_m \) and put

\[ n_0 = \max \{|v| \mid v = \text{lcp}(u_i, u_j), i \neq j, i, j = 0, 1, \ldots, m\}. \]

We have \( \triangle \mathcal{C}(n) \geq \#\text{Lex}(u_0) - 1 + \sum_{k=1}^{m} \#\text{Lex}(u_k) - 1 \geq p - 1 + m \) for all \( n > n_0 \). Due to Corollary \( 48 \) we know that there exist infinitely many \( (g_{L}^{k}(0, z)) \)-maximal factors, \( k = 0, 1, \ldots \), and hence there must exist a LS factor of length \( n_1 > n_0 \) which is not a prefix of any LS branch and so \( \triangle \mathcal{C}(n_1) > m + p - 1 = \triangle \mathcal{C}(1) \).

In the case of \( p = 1 \), the proof is analogous. Only difference is that there are only \( m \) infinite LS branches since \( u_{\beta} \) is not.

\[ \square \]

**Remark 55.** For the word \( u_{\beta} \) with \( d_{\beta}(1) = t_1(0 \cdots 0(t_1 - 1))^\omega \) we may easily describe all left special factors. If the length of the period \( p > 1 \), each LS factor is a prefix of one of two infinite LS branches \( u_{\beta} \) and \( 0^{-1}u_{\beta} \). If \( p = 1 \), then \( u_{\beta} \) is not an infinite LS branch and so each LS factor is prefix of the unique infinite LS branch \( 0^{-1}u_{\beta} \). Hence, we obtain the known result that \( u_{\beta} \) is Sturmian if and only if \( d_{\beta}(1) = t_1(t_1 - 1)^\omega \). We were pointed out by Christiane Frougny that numbers \( \beta \) satisfying \( d_{\beta}(1) = t_1(0 \cdots 0(t_1 - 1))^\omega \) are Pisot units. Such Parry number \( \beta \) is a root of the polynomial \( x^{p+1} - t_1x^p - x + 1 \).

**Lemma 56.** Let \( \beta \notin S \) and let \( k_0 \geq m - 1 \). Then the factor complexity of \( u_{\beta} \) is not affine.
Proof. As shown in the proof of Lemma 52, the $k$-th max-$f$-image of $0^{t_1-1}$ (resp. $0$ if $t_1 = 1$), $k = 0, 1, \ldots, m - 1$, is not beginning in $0^m$ and it is equal to the longest prefixes of some two LS factors. In order that the complexity is affine, also all max-$f$-images of the factor $f$ must be as long as the longest prefixes of some two LS factors.

Let $t_1 > 1$. Then the factor $f$ must be of the same length as the longest common prefix of $u_\beta$ and $m$-th max-$f$-image of its own – remember that the longest common prefix of $u_\beta$ and $k$-th max-$f$-image of $f$ is the $k$-th max-$f$-image of $0^{t_1-1}$ for $k = 0, 1, \ldots, m - 1$. Formally,

$$|0^t m \varphi^m_\beta (0^{t_1-1})(1 + m)^{-1}| = |\text{lcp}(u_\beta, \varphi^m_\beta (0^t m) \varphi^2 m (0^{t_1-1})(1 \oplus (2m))^{-1})| = |\varphi^m_\beta (0^t m)(m \oplus m)^{-1}|$$

which is never satisfied for $|\varphi^m_\beta (0^t m)| \leq |\varphi^m_\beta (0^{t_1-1})|$. 

Let $t_1 = 1$. Following the same reasoning as for the case $t_1 > 1$, a necessary condition for the complexity to be affine is that the factor $f$ must be of the same length as the longest common prefix of the $(m - \ell_0)$-th max-$f$-image of its own and $u_\beta$, namely

$$|\text{lcp}(u_\beta, \varphi^{m-\ell_0}_\beta (m) \varphi^{2m-\ell_0}_\beta (1 \oplus (2m - \ell_0))^{-1})| = |\varphi^{m-\ell_0}_\beta (m)(m \oplus (m - \ell_0))^{-1}|$$

which is never satisfied for $|\varphi^{m-\ell_0}_\beta (m)| \leq |\varphi^m_\beta (1)|$. \qed

Putting all lemmas of this section together, we obtain the main theorem of this paper.

**Theorem 57.** Let $\beta$ be a non-simple Parry number. The factor complexity of $u_\beta$ is affine if and only if $d_\beta(1) = t_1(0 \cdots 0(t_1 - 1))^\omega$.

## 7 Conclusion

Among infinite words $u_\beta$ associated with Parry numbers we may identify Arnoux-Rauzy words. An infinite word is said to be Arnoux-Rauzy of order $\ell$, if for any length $n \in \mathbb{N}$ there exists exactly one left special factor and one right special factor both of length $n$ and, moreover, these special factors have just $\ell$ left and $\ell$ right extensions respectively. Arnoux-Rauzy words can be considered as a natural generalization of Sturmian words to more letter alphabets.

Is is easy to see that only Sturmian words among $u_\beta$ correspond to $\beta$ with $d_\beta(1) = t_1 1$ or $d_\beta(1) = t_1(t_1 - 1)^\omega$. The word $u_\beta$ is an Arnoux-Rauzy word of order $m \geq 3$ if and only if $d_\beta(1) = t_1^{m-1}$. It means that there is no Arnoux-Rauzy word over more letter alphabet associated with non-simple Parry number. A direct consequence of the definition of Arnoux-Rauzy words is that the complexity of Arnoux-Rauzy word is affine and that any left (right) special factor is a prefix of an infinite left (right) special branch.
In the previous section, we have proved that the infinite word \( u_\beta \) associated with a non-simple Parry number \( \beta \) has the affine complexity if and only if \( d_\beta(1) = t_1(0 \cdots 0(t_1 - 1))^\omega \). In fact, we have proved that the complexity is affine if and only if any left special factor of \( u_\beta \) is a prefix of an infinite left special branch. The validity of the same statement for infinite words associated with simple Parry numbers is proven in [4]. However, this equivalency is not a general rule for the factor complexity of fixed points of primitive morphisms. For a counter example see [10] and [13].

It is known that Sturmian words have many equivalent definitions, see [7] for more. In 2001 Vuillon [28] showed that a binary infinite word is Sturmian if and only if each its factor has exactly two return words. In the article [27] Vuillon introduced the property \( R_\ell \): an infinite word satisfies the property \( R_\ell \) if each its factor has exactly \( \ell \) return words. Therefore, words with \( R_\ell \) can be considered as another generalization of Sturmian words. In [17] Justin and Vuillon proved that Arnoux-Rauzy words of order \( \ell \) have the property \( R_\ell \). Applying Theorem 4.5 of [2], we see that all \( u_\beta \) with affine complexity have also the property \( R_\ell \).

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