Statistical analysis of the Diffie-Hellman key exchange protocol in a finite group†

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Summary. This paper presents a novel methodology to test the security of the Diffie-Hellman public key exchange protocol. The security of many cryptographic schemes rely on the hardness of this problem. We are presenting a purely statistical test to compare this problem in different groups. We are using groups included in \( \mathbb{Z}_p \) with \( p \) prime as a major example, however the methods presented are not restricted to these groups. The presentation of the results is primarily intended to introduce novel applications of statistical methodologies to the area of mathematical cryptography. As such we will emphasize the cryptographical aspects of the work more than the statistical notions.

Keywords: public key cryptography, permutation testing, prime subgroups

1. Introduction.

Informally, through a key exchange protocol, two parties \( A \) and \( B \) agree on a common key \( K_{A,B} \) pooled from a set \( S \) while communicating over an insecure channel. Once the key is established, any further information shared between the parties is encoded, transmitted and decoded using the key \( K_{A,B} \). The protocol is secure if any third party \( C \) with access to the initial communication between \( A \) and \( B \) cannot tell apart \( K_{A,B} \) from any other value in the set \( S \). This guarantees that it is computationally unfeasible for an outside adversary to gain “any” partial information on \( K_{A,B} \).

The Diffie-Hellman key exchange protocol [Diffie and Hellman (1976)] is a primary example of a public key exchange protocol. In its most basic form, the protocol chooses a finite cyclic group \( (G, \cdot) \) of order \( N \), with generator \( g \), where \( \cdot \) denotes the group operation. In what follows we chose the multiplicative operation to denote the operation in the group, and thus the group \( G \) is generated by the powers of \( g \) (i.e., \( G = \{g^0, g^1, \ldots, g^{N-1}\} \)), symbolically \( G = \langle g \rangle \). Note that \( G, g \) and \( N \) are public information.

The participants in the information transfer \( A \) and \( B \) each randomly chooses an integer \( a \in [1, N] \) and \( b \in [1, N] \) independently. Then \( A \) computes \( g^a \), \( B \) computes \( g^b \) and exchange these elements of \( G \) over an insecure channel. Since each of \( A \) and \( B \) knows their respective \( a \) and \( b \) they can compute \( g^{ab} \), which or a publicly known derivation \( K_{A,B} \) of that becomes the public key.

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Any method of converting $g^{ab}$ to $K_{A,B}$ is publicly known, and the security of the key $K_{A,B}$ is directly dependent on the security of $g^{ab}$, therefore for the sake of simplicity we will consider $g^{ab}$ as the established key of the exchange for the rest of this paper.

In the present article we will be concerned with the security of this protocol. We will interpret security in a probabilistic manner and will devise a statistical test that will “assess” the security of the exchange in a given group.

In the cryptology literature there are two concepts of security – the core security and the concept of semantic security which leads to various security models. The semantic security and the related concepts come under the name of “provable security” (Koblitz and Menezes, 2004, Section 2). The core security of the Diffie-Hellman key exchange protocol depends on the discrete logarithm problem, the computational Diffie-Hellman problem and the decision Diffie-Hellman problem. In this article we are concerned with the core security of the exchange. We give a brief introduction to the discrete logarithm problem and the computational Diffie-Hellman problem, for more on these a reader can look at (Koblitz and Menezes, 2004, Section 5) or (Stinson, 2005, Chapter 6).

**Assumption 1 (DL).** For a cyclic group $G$, generated by $g$, we are given $g$ and $g^n$, $n \in \mathbb{N}$, the challenge is to compute $n$.

**Assumption 2 (CDH).** Given $g, g^a, g^b$ it is hard to compute $g^{ab}$.

Clearly, if these assumptions are not satisfied then $C$, an adversary, can gain access to the key $g^{ab}$. The relationship between these two assumptions has been extensively studied. It is clear that the CDH assumption will not be satisfied in a group where finding the discrete logarithm solution is easy. In Maurer and Wolf (1999), Boneh and Lipton (1996), the authors show that in several settings the validity of the CDH assumption and the hardness of the discrete logarithm problem are in fact equivalent.

Unfortunately, the DL and the CDH assumptions are not enough to ensure security of the Diffie-Hellman key exchange protocol. Even if these assumptions are true, the eavesdropper $C$ may still be able to gain useful information about $g^{ab}$. For example, if $C$ can predict 90% of the bits in $g^{ab}$ with high probability then for all intents and purposes the key exchange protocol is broken. Moreover, there exist protocols where the knowledge of even one bit will break its security (some Casino electronic games). With the current state of knowledge we cannot be confident that assuming only CDH, a scenario like the one described above does not exist (Boneh (1998)).

1.1. *Our main contribution.*

Lemma 2.1 states that the security of the Diffie Hellman exchange protocol is best studied from a statistical perspective. We introduce a statistical treatment of this particularly important problem in cryptography and it is our hope that many more problems will be approached in a similar fashion.

We present novel methodologies to help assess the security of the Diffie-Hellman key exchange protocol in a given group $(G, \cdot)$. In Section 2 we present the statistical criteria we use as well as the relevance and connection with the security assessment. Sections 3 and 4 present statistical tests to check the validity of the statistical criteria presented in Section 2.

‡ There are various concepts of adversary in cryptographic literature, the power and authority they have. In this article we assume that our adversary is a passive eavesdropper.
2. Statistical analysis of the DH problem

In particular, Subsection 4.1 detail the use of the permutation testing methodology to calculate concrete values for the probability of Type I error of the tests. This section contains the important idea that the method can be used to compare the security of the DH key exchange protocol in two or more different groups. Furthermore, the groups which we use to perform the comparison do not need to have the same operational structure. Thus, in principle, it is possible to compare the security of the exchange in finite groups generated using elliptical curves versus the same order prime subgroups of $\mathbb{Z}_n$, $n \in \mathbb{N}$. We do not pursue this direction in the presented work.

Section 5 applies the methodology we develop to some examples where the security of the DH-exchange has been conjectured. It is found that the results obtained strengthen the conjectured hypotheses. Finally, in Section 6 we present general conclusions and directions of future research.

The treatment of the problem is based on the empirical distribution of the key $g^{ab}$. We mention that a better approach from the cryptographic perspective would be to look at the distribution of a collection of bits in the binary expansion of $g^{ab}$. We believe our methods could be extended and applied to this representation as well.

2. Statistical criteria to assess the security of the Diffie Hellman key exchange protocol.

In its most basic form described above the security of the Diffie-Hellman key exchange protocol relies on an approximate identification of the key $g^{ab}$ from the public information $g$, $g^a$, $g^b$. In statistical terms there exist a clear concept that answers the question of identification: statistical independence. Therefore a sufficient condition for the security of the DH key exchange is:

**Assumption 3 (DH-Independence).** Given a cyclic group $G$ of order $N$, generated by $g$, let $a$ and $b$ be chosen independently, uniformly at random from the set $\{1, 2, \ldots, N\}$. Then the random variables $(g^a, g^b)$ and $g^{ab}$ are independent.

For a given set $S$ we will use the notation $DU(S)$ to denote the discrete uniform distribution on the elements of $S$. With this notation $a$ and $b$ are independent random variables with the $DU(\{1, 2, \ldots, N\})$ distribution.

Clearly this is a sufficient condition for the security of the Diffie Hellman key exchange protocol. There is no information to be gained about $g^{ab}$ from seeing $(g^a, g^b)$. Unfortunately, as one’s intuition may indicate, this assumption is rejected for any finite group $G$ we have looked at. In the next section we construct a statistical test for this assumption which will help introduce the notations and the further testing procedures.

If the assumption presented above is not true, hope is not lost. The DH-Independence assumption is a sufficient condition. In fact, in the cryptographic literature this assumption is not even mentioned, however a weaker necessary condition is presented:

**Assumption 4 (DDH).** Given $g, g^a, g^b$ and an element $z \in G$ it is hard to decide whether or not $z = g^{ab}$.

In this form the DDH assumption constitutes a necessary condition for the security of the Diffie-Hellman key exchange protocol. Furthermore, Joux and Nguyen [2003] construct groups based on elliptic curves where the DDH assumption is not satisfied while the CDH
and the discrete logarithm problem are proven to be equivalent and hard. This fact prompts the necessity to directly check the validity of the DDH assumption for a given group.

The DDH assumption is assumed, either implicitly or explicitly in many cryptographic systems and protocols. Applications include: the many implementations of the DH key exchange itself (e.g., Diffie et al. (1992)), the El-Gamal encryption scheme El-Gamal (1984), the undeniable signatures algorithm Chaum and van Antwerpen (1989), Feldsman's verifiable secret sharing protocol Feldman (1987), Pedersen (1991), and many others; we point to Naor and Reingold (1997) for a more detailed list.

Notice that the DDH assumption in the form presented above is a little vague because of the use of the predicate, “hard to decide”. Surprisingly, attempts to make the DDH assumption explicit were not made until late after its formulation in Diffie and Hellman (1976). The first ventures (Boneh and Lipton, 1996) use standard cryptographic machinery (Yao (1982); Goldwasser and Micali (1984)), to express the assumption in terms of computational indistinguishability. Put in this traditional cryptographic form it was discovered quickly by Stadler (1996) and independently Naor and Reingold (1997) that if one assumes the existence of a polynomial time probabilistic algorithm which distinguishes the real key \( g^{ab} \) from the other possible values even with a very small probability \( \frac{1}{p(n)} \) (for all the possible inputs), then another polynomial time algorithm can be constructed from the first which will output \( g^{ab} \) with a very large (almost one) probability. The only requirement is that the size of the group is known, requirement lessened by Boneh (1998) which only requires finiteness of the group.

All this evidence points toward a more specific definition based entirely on the notion of statistical significance. Indeed, this fact materialized in a series of papers Canetti et al. (1999, 2000); Friedlander and Shparlinski (2001); Vasco et al. (2004), which call this new form of the assumption the Diffie Hellman Indistinguishability assumption (DHI). We note that Gennaro et al. (2004); Joux and Nguyen (2003) use the same form except it continues to call it DDH. We point the reader to Hastad et al. (1999) for a detailed discussion on the concept of statistical significance versus computational significance; in the context of pseudo-random number generation.

For our purposes of studying the security of the Diffie Hellman exchange we will use the following assumption:

**Assumption 5 (DHI).** Given \( g, g^a, g^b \) the distribution of \( g^{ab} \) is indistinguishable from the Discrete Uniform distribution on the elements of \( G (DU(G)) \).

The notion of indistinguishability used here is the usual statistical one. Two variables are indistinguishable if they have essentially the same distribution, or put formally, \( X_1 \) and \( X_2 \) are indistinguishable if their distribution functions \( F_i(x) = P(X_i \leq x) \) with \( i = 1, 2 \) have the property:

\[
F_1(x) = F_2(x), \quad \text{for all } x \in \mathbb{R} \setminus (A_1 \cup A_2),
\]

but not negligible. For the sake of completeness we give here the whole definition. It is presented in the footnote since it is not relevant to our approach at all. Suppose that the group \( G \) where the exchange takes place has order \( N \) and \( n = \log_2 N \). It is said that a probabilistic algorithm \( \mathcal{A} \) decides on the right key with small (non-negligible) probability if there exist a polynomial expression \( p(\cdot) \) such that for any \( r \in G \):

\[
\left| \text{Prob}(\mathcal{A} \text{ outputs } g^{ab}) - \text{Prob}(\mathcal{A} \text{ outputs } r) \right| > \frac{1}{p(n)}.
\]
where $A_1, A_2$ are the sets which contain the discontinuity points of $F_1(\cdot)$, respectively $F_2(\cdot)$. Applied to our specific case the distributions are discrete, therefore the distribution functions $F_1$ and $F_2$ are just step functions with jumps in a compact set in $\mathbb{R}$, thus using the right continuity of the distribution functions, the usual definition translates here in equality everywhere. We conclude that in our context, indistinguishability means that the variables have the same distribution.

This formulation is perfectly natural for a statistician who tries to express the DDH formulation presented above. We note that our version of the DHI assumption requires that the conditional distribution $(g^{ab}|g^a, g^b)$ is uniform while the previous articles Canetti et al. (1999, 2000); Friedlander and Shparlinski (2001); Vasco et al. (2004); Gennaro et al. (2004); Joux and Nguyen (2003) require that the distribution of the triple $(g^a, g^b, g^{ab})$ be Discrete Uniform on the elements of $G \times G \times G = G^3$ ($DU(G^3)$). Given an outcome $(x, y, z)$ we can write using the simple multiplicative rule:

$$P\left(g^a = x, g^b = y, g^{ab} = z|g\right) = P\left(g^{ab} = z|g^a = x, g^b = y, g\right)P\left(g^a = x, g^b = y|g\right)$$ (1)

Under the original condition that $a$ and $b$ are $DU(\{1, \ldots, N\})$ and using the fact that $g$ is a generator for $G$ then the distribution of $(g^a, g^b|g)$ is $DU(G^2)$, thus the two formulations are perfectly equivalent.

It is known that in general statistical indistinguishability implies computational indistinguishability, but the reverse is not in general true, (Goldreich, 2001, Section 3.2.2). The following lemma states the same result in our specific case using the assumptions presented in this section: DHI and DDH.

**Lemma 2.1.** *In a group $G$ of order $N$, if the DHI assumption is true then the DDH assumption is true as well.*

**Proof.** Assume that DHI is true in $G$, then for given $g^a, g^b$, the probability $P\left(g^{ab} = z|g^a, g^b\right) = 1/N$ for any $z \in G$. This is the hardest possible scenario in the DDH assumption and hence we claim that DDH is satisfied.

This lemma says that in any group $G$, DHI is a stronger\footnote{or at least as strong} condition than that of the DDH assumption. If we look at the proof closely then we will see that the difference between the DDH and the DHI consists in the fact that a measure of hardness has been provided in the DDH assumption via the uniform distribution.

### 3. Testing for DH-Independence.

We give general definitions, then we go to our specific case.

Let $X, Y,$ and $Z$ be three discrete random variables taking values in the sets $\{x_1, x_2, \ldots, x_n\}$, $\{y_1, y_2, \ldots, y_m\}$, $\{z_1, z_2, \ldots, z_l\}$ respectively. Denote with:

$$p(x_i, y_i, z_i) = P\{X = x_i, Y = y_i, Z = z_i\}, \ \forall i, j, k$$

the joint probability function corresponding to $(X, Y, Z)$. With usual notations we denote $p(y_j|x_i)$, $p(x_i|y_j)$, etc. the conditional probability functions of $X|Y$, $(X, Y)|Z$, etc. Furthermore, assume that for all $k \in \{1, 2, \ldots, l\}$ the marginal distribution $p(z_k) = P\{Z = z_k\} \neq 0$ to avoid complications conditioning on a set of measure zero.
**Definition 3.1 (Entropy).** We define the joint and conditional measures of uncertainty.

\[
H(X, Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} p(x_i, y_j, z_k) \log p(x_i, y_j)
\]  

(2)

\[
H(X, Y|Z) = -\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} p(x_i, y_j, z_k) \log p(x_i, y_j|z_k),
\]  

(3)

with the convention \(0(-\infty) = 0\).

In the above definition we choose to work with the natural logarithm, however any other basis will be equivalent for our purpose due to the constant in the usual definition of the entropy function (see Shannon (1948)).

**Lemma 3.2.** The following property holds for the above uncertainty measures: \(H(X, Y|Z) \leq H(X, Y)\) with equality if and only if \((X, Y)\) and \(Z\) are independent.

The proof is an easy exercise in probability, the reader is directed to Shannon (1948) or Rokhlin (1967) for more details.

Lemma 3.2 gives a clear criterion for our first test. More specifically: assume that the number of elements in \(G\) is \(N\), i.e. \(|G| = N\). As an example \(|Z_p^p| = p - 1\).

The plan is to apply the above lemma with \(X = g^a, Y = g^b\) and \(Z = g^{ab}\). Since both participants in the Diffie-Hellman protocol choose \(a\) and \(b\) at random and \(g\) is the generator of \(G\) we can assume that \(g^a\) and \(g^b\) are independent and their distribution is \(DU(G)\). Thus, the distribution of \((g^a, g^b)\) is \(DU(G \times G)\). This in turn implies that \(p(x_i, y_j) = 1/N^2\) for all \(i, j \in \{1, \ldots, N\}\), and thus the first entropy measure (2) becomes:

\[
H(g^a, g^b) = -\sum_{i,j,k} p(x_i, y_j, z_k) \log \frac{1}{N^2} = 2 \log N
\]  

(4)

At this point we can devise a test of the hypotheses:

\[
\begin{align*}
H_0 &: \quad g^{ab} \text{ is independent of } (g^a, g^b) \\
H_a &: \quad g^{ab} \text{ is NOT independent of } (g^a, g^b)
\end{align*}
\]  

(5)

using Lemma 3.2. The test in (5) is equivalent with:

\[
\begin{align*}
H_0 &: \quad H(g^a, g^b|g^{ab}) = 2 \log N \\
H_a &: \quad H(g^a, g^b|g^{ab}) < 2 \log N
\end{align*}
\]  

(6)

The question is: how do we proceed with this test? Since all the distributions are finite, in theory at least, we could calculate \(p(x_i, y_j|z_k)\) for all the possible triples in \(G \times G \times G = G^3\). If we had these quantities it would be a simple matter to calculate \(H(g^a, g^b|g^{ab})\) according to (3).

Denote this value based on the whole set \(G^3\) by \(T_N\). The test will then compare this value with \(2\log N\). If equal then the variables are independent and the DH-Independence assumption is satisfied. If smaller then we could not prove independence of the variables.

At this point let us make two important remarks.
Remark 3.3. In the definition of the entropy functions (1) and (2) we did not use the structure of the group $G$ in any way, only the relative frequency of the elements in the group. This fact make the methods based on the entropy function well suited for comparison between diverse groups. We will take advantage of this feature later in this paper.

Remark 3.4. In practice if we wish to calculate $T_N$ we have to calculate all the possible values for $(g^a, g^b)$ and this will take longer than an exhaustive search. Thus calculating $T_N$ is not practical, instead we would have to estimate it. We will detail the estimation in the next section.

Alas, as we suspected from the beginning, implementing this first test tells us that $(g^a, g^b)$ and $g^{ab}$ are not independent in every group that we tried. For example in $Z_p$ with multiplication, calculating $T_N$ the entropy in (3) for $p \in \{1193, 2131, 11093\}$ will yield values which are far apart from $2 \log(p - 1)$. In fact when looking at the values obtained we see that they are close to $\log(p - 1)$ thus the value of our first test increases with $p$. The closeness of the test to $\log(p - 1)$ is an interesting experimental fact. This fact is investigated and explained by our second test presented in the next section.

4. Testing the DHI assumption

If the DH-Independence assumption is satisfied in a given group $G$, then we could stop and decide that we found a perfect group for the Diffie Hellman key exchange. However, the experimental procedures and our intuition point out that the DH-Independence assumption is never satisfied in any finite group $G$. The next task is to obtain a statistical testing procedure to verify the validity of the DHI assumption in a given group $G$. The idea is to use the entropy function (3) in the sense of Kullback-Leibler divergence Kullback and Leibler (1951) as a measure of departure from the entropy calculated under the hypothesis of Uniform distribution. Specifically, using earlier notation, we wish to construct a statistical test that will check the validity of the following hypotheses:

$$
\begin{align*}
H_0 : \quad & \text{The distribution of } g^{ab} \mid (g^a, g^b) \text{ is } DU(G) \\
H_a : \quad & \text{The distribution of } g^{ab} \mid (g^a, g^b) \text{ is NOT } DU(G)
\end{align*}
$$

(7)

Let us denote the elements of $G$ as $\{g_1, g_2, \ldots, g_N\}$. Suppose we can look at all the possible triples $(g^a, g^b, g^{ab})$ when $a, b \in \{1, 2, \ldots, N\}$ take all the possible values. Clearly, there are $N^2$ such possible triples and assuming that $a$ and $b$ are chosen at random, each such triple will have probability $1/N^2$. The last element in the triple $g^{ab}$ will get mapped into $N$ possible values (the elements of $G$). Thus, some values in $G$ will be repeated. For an element $g_k \in G$ denote $m_k$ the number of times $g_k$ appears in the place of $g^{ab}$ among all the $N^2$ triples. We have then $\sum_k m_k = N^2$. For any pair $(g^a, g^b)$ that corresponds to $g^{ab} = g_k$ we can then calculate the conditional probability as:

$$
p(g^a = g_i, g^b = g_j \mid g^{ab} = g_k) = \frac{1}{m_k} \mathbf{1}(g_i, g_j, g_k),
$$

where $A$ is the set of all possible $N^2$ tuples $(g^a, g^b, g^{ab})$, and we have used the notation $\mathbf{1}_A(x)$ to denote the indicator function of the set $A \subset \Omega$, i.e., $\mathbf{1}_A : \Omega \to \{0, 1\}$ is given by:

$$
\mathbf{1}_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases}
$$
We can continue:

\[ H(g^a, g^b | g^{ab}) = - \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} p(g_i, g_j, g_k) \log p(g_i, g_j | g_k) \]
\[ = - \sum_{k=1}^{N} \sum_{i,j=1}^{N} \frac{1}{N^2} \log \frac{1}{m_k} 1_{A_i}(g_i, g_j, g_k) \]
\[ = - \sum_{k=1}^{N} \frac{m_k}{N^2} \log \frac{1}{m_k} \]

(8)

Under the null hypothesis \( H_0 \), the distribution of \((g^{ab} | g^a, g^b)\) is uniform, therefore we should have the \( m_k \) multiplicities equal. This automatically implies that \( m_k = N \) for all \( k \)’s and then the entropy function in (8) is:

\[ H(g^a, g^b | g^{ab}) = - \sum_{k=1}^{N} \frac{1}{N} \log \frac{1}{N} = \log N \]

The testing statistics is:

\[ T_N = H(g^a, g^b | g^{ab}) - \log N = \sum_{k=1}^{N} \frac{m_k}{N^2} \log m_k - \log N. \]

(9)

This test is based on the whole set of values in \( G^2 \). Accordingly, if the value of the test equals zero then the null hypothesis \( H_0 \) is true, any other value of the test will support the alternative hypothesis. We summarize this result in the following:

**Lemma 4.1 (Testing Procedure).** With the previous notations if \( T_N = 0 \) then the DHI assumption is satisfied in a given group \( G \).

Both remarks 3.3 and 3.4 certainly apply for this testing procedure as well. In particular, remark 3.4 means that we have to find procedures to estimate \( T_N \) instead of calculating it. This will introduce distributions and we detail the approach next.

### 4.1. The permutation test approach.

Assume that we can obtain a sample of \( n \) pairs \( \{(a_i, b_i)\} \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, N\} \). For each pair in the sample we can calculate the triple \((g^{a_i}, g^{b_i}, g^{a_i b_i})\). Let \( A_n \) be the set of all the triplets in the sample.

Using (8) we can calculate an estimate of \( H(g^a, g^b | g^{ab}) \) using:

\[ \hat{p}_n(g_i, g_j, g_k) = \frac{k_{ijk}}{n} 1_{A_n}(g_i, g_j, g_k), \]
\[ \hat{p}_n(g_i, g_j | g_k) = \frac{k_{ijk}}{m_k} 1_{A_n}(g_i, g_j, g_k), \]

where once again \( m_k \) denotes the multiplicity of \( g_k \), but in the given sample of \( n \) observations. We took into account the possibilities of obtaining repeated observations in the
sample by multiplying with the factor $k_{ijk}$; which represents the number of times we see the same observation $(g_i, g_j, g_k)$ in our sample.

The test statistic is:

$$T_n = -\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \hat{p}_n(g_i, g_j, g_k) \log \hat{p}_n(g_i, g_j | g_k) - \log n \quad (11)$$

All that is left, is to investigate the distribution of $T_n$ under the null hypothesis $H_0$. Under the null hypothesis the $m_k$’s are the multiplicities of $g_k$’s in a sample of size $n$ drawn from the set

$$\{g_1, \ldots, g_1, g_2, \ldots, g_2, \ldots, g_N, \ldots, g_N\}$$

where each element in the group $G$ are repeated $N$ times.

Let us denote $M_1, M_2, \ldots, M_N$ the multiplicities of the elements $\{g_1, g_2, \ldots, g_N\}$ in a sample of size $n$. It is not hard to show that the joint probability distribution of $(M_1, \ldots, M_N)$ is the so called multivariate hypergeometric distribution:

$$P(M_1 = m_1, \ldots, M_N = m_N) = \frac{\binom{N}{m_1} \binom{N}{m_2} \cdots \binom{N}{m_N}}{\binom{N}{n}}$$

The test statistic under $H_0$ is:

$$\hat{T}_n = \sum_{k=1}^{N} \frac{M_k}{n} \log M_k - \log n. \quad (12)$$

If we would be able to calculate the distribution of $T_n$ knowing that $(M_1, M_2, \ldots, M_N)$ are multivariate hypergeometric then we would be in position to reach the conclusion of the test of uniformity (7) by calculating the $p$-value of the test statistic (11) using this distribution.

Finding the distribution of the test statistic under $H_0$ (12) is however not an easy task. This is the reason we propose the use of permutation testing for which knowledge of this distribution is not necessary.

The permutation testing procedure generates samples $(M_1, M_2, \ldots, M_N)$ from the Multivariate hypergeometric distribution. For each sample, it calculates the corresponding value of the test statistic under the null hypothesis as in (12). These values are obtained from the assumption that $H_0$ is true; this allow us to calculate the empirical distribution of our sample statistic $T_n$ under the null hypothesis. The $p$-value of our test is given by the proportion of values as extreme or more than the one calculated in (11) using the group $G$.

A small $p$-value is an evidence against the null hypothesis in (7), that the sample comes from a uniform distribution. We summarize the procedure bellow:

**Testing procedure to determine validity of DHI for a group $G$**

(i) We take a sample of size $n$ and we calculate the test statistic as in (11).

(ii) We generate many test statistic values under the hypothesis $H_0$ is true using (12), then construct their empirical distribution.

(iii) We calculate the $p$-value of the test as the proportion of values in the empirical distribution found in (ii) lower than the test value found using $G$ in (i).

(iv) If the $p$-value is small we reject the DHI assumption. If the $p$-value is big we did not find evidence that the DHI is not satisfied in the given group $G$. 
4.2. How to compare two or more groups?

We will note at this point that the absolute value of the test $|T_N|$ and its estimate $|T_n|$ represent a measure of departure from the Discrete Uniform distribution. The bigger the estimate the further is the distance from the uniform distribution and the weaker is the validity of the DHI assumption. Remark 3.3 also tells us that the nature of the group operation is irrelevant for the testing procedure. Therefore, we can use the test as a tool to compare the strength of the Diffie-Hellman key exchange protocol in two or more groups. To be able to do so we need the order of the groups compared to be similar and, more importantly, the sample size on the basis of which we calculate the permutation test to be the same. We take advantage of the ability to compare different groups in the next section.

5. Testing the DHI assumption in $\mathbb{Z}_p^*$

We are going to check the efficiency of the testing procedure for the most useful finite groups, those included in $\mathbb{Z}_p^*$ with the multiplicative operation. We present the following examples as a way for checking the validity of the testing procedure.

**Example 5.1 (A group where the DDH assumption does not hold.).** Consider $G = \mathbb{Z}_p^*$ with $p$ prime. It is known that computing Legendre symbol in this group gives a distinguisher against DDH [Gennaro et al (2004)].

**Example 5.2 (A group where the DDH assumption is conjectured to hold).** We currently do not know any DDH distinguisher for a prime order subgroup of $\mathbb{Z}_p^*$. Therefore, given $p$ and $q$ prime with $q$ divisor of $p-1$ it is conjectured that in a subgroup of order $q$ of $\mathbb{Z}_p^*$ the DDH assumption holds.

We start with a given group $G$ and using the test presented in the previous section we will test for the validity of the DHI assumption in that group $G$. This should provide a strong indication towards the security of the Diffie-Hellman key exchange protocol in that group.

5.1. The rate of convergence of the testing procedure

The first thing we investigate is the rate of convergence for our test. To do this we need to calculate the true value of $T_N$ and thus we have to look at small groups.

For space consideration we are presenting only results obtained for $p = 1193$ in Table 1 in the Appendix. The sample sizes are presented in the first column of the table and the corresponding sample entropy value $T_n$ in column two. Column 3 presents the proportion of values lower than $T_n$ – an entry equal to 1 corresponds to a $p$-value 0 of the test. The fourth value in the table represents the distance from $T_n$ to the center of the distribution of entropy values calculated under $H_0$. Finally, the last value represents the ratio of the distance in column four, to the distance from the sample entropy $T_n$ to the furthest away point in the distribution. It is an indication on how many standard deviations away $T_n$ is from the distribution.

There are two remarkable features of these values – one, we see that the test rejects the null hypothesis that the distribution of $(g^{a^2} | g^a, g^b)$ is uniform on the elements of $G$; the other remarkable feature is that we determine this fact based on a sample of 354 values or about one third of the value of $N = 1192$. In the second place if we wish to determine
the actual entropy distance from the two distributions – a feature that will be useful when comparing two or more groups; we can see that starting with a sample size of \( n = 3304 \) (or about 3 times \( N \)) we start to obtain accurate results.

To illustrate better the rate of convergence for some other groups we plot in Figure 1 the evolution of the test values with the size of the sample. This figure suggest that to get a good estimate for \( T_N \) the sample size will depend on the size of the group, for example we need a larger sample size for \( Z_{11903}^* \) than we need for \( Z_{1193}^* \). In addition, the same figure points out another interesting fact.

Following example 5.1 we know that \( Z_{2131}^* \) is not secure. It is also conjectured that some groups are more secure than others. Looking at the problem from that perspective, for which groups are more easily broken using the Legendre symbol, it is also assumed that by increasing the size of the group one can make the group more secure.

We can see from the figure that the second assertion is not true. Just increasing the size of the group does not make it more secure. Remembering that a smaller relative distance corresponds to closeness to the Discrete uniform distribution on the elements of \( G \), we see from the Figure 1 that while \( Z_{11903}^* \) the largest group is the most secure of the three, the situation between the other two groups is not what we would have expected looking at the size of the group alone. Even though \( Z_{2131}^* \) is the larger group (almost twice the
size), it is also less secure from the DHI assumption perspective than $\mathbb{Z}_{1193}^*$. This indicates that the choice of the group $G$ rather than the size of it is essential for the security of the Diffie-Hellman key exchange protocol.

5.2. Comparison of the DHI assumption across groups.

Next we wished to give an indication of groups that are more secure than others. It is known that considering only the Legendre symbol criterion the safest groups among $\mathbb{Z}_p^*$ are the ones obtained when $p$ is a safe prime i.e., of the form $p = 2q + 1$ where $q$ is another prime [Menezes et al. 1996].

![Histogram of Distances for Safe Primes](image)

![Histogram of Distances for Other Primes](image)

**Fig. 2.** Histogram of all the test values for $\mathbb{Z}_p^*$ with $2000 < p < 4000$. Values closer to zero represent safer groups for DH exchange.

We wished to test this theory for a large set of $\mathbb{Z}_p^*$ groups with varying $p$’s. We looked at all primes between 2000 and 4000, and again for primes between 9000 and 11000. The reason for the two separate segments of primes is that we expect some sort of consistency between them. We show the distribution of the test values for these groups separated into safe and not safe primes in Figures 2 and 3.

First, we notice that the behavior of primes in the range 2000 to 4000 is very similar with the primes for the higher range 9000 to 11000. Second, in both ranges we see the same
conclusion applies, the safe prime groups are more secure than any other groups. However, the test estimate obtained for each of the safe prime groups is significantly different from zero therefore there is no safe group in the ranges given for which the DHI assumption is verified. This seem to confirm the assertion in the Example 5.1.

Next, we will look to Example 5.2. We will use our test for the prime subgroups of each of the safe primes in the range 9000 to 11000. More specifically, we look at each $\mathbb{Z}_p^*$ with $p$ a safe prime, and we construct the prime subgroup of order $q$ in each such group. Then we test the DHI assumption in each subgroup thus constructed. The values obtained for the distances are plotted in the upper histogram of Figure 4. We mention that the behavior of the test values for primes between 2000 and 4000 was very similar, for space consideration we omit the corresponding plot. All the values are obtained using the same sample size $n = 8 \times 10^6$. The reason for this particular value is that while the groups themselves are in the range 9000 to 11000, the subgroups are of order 4500 to 5500.

It is remarkable to see that these subgroups are clearly safer for the DH exchange than any other groups plotted in the picture. The results seem to confirm the conjecture in the Example 5.2. However, the actual test of uniformity was rejected, but we needed a very large sample size almost equal to the maximum value $N^2$. 

**Fig. 3.** Histograms of test values obtained for $\mathbb{Z}_p^*$ with $9000 < p < 11000$. Values closer to zero represent safer groups for DH exchange.
Fig. 4. Comparing values of the test for different type of groups when $9000 < p < 11000$. On top, we plot values for prime subgroups of $\Z_p^*$ when $p$ is a safe prime. Middle, we plot values for safe prime $\Z_p^*$’s. On bottom, values for all the other groups $\Z_p^*$ in the range given. Values closer to zero represent better groups for DH exchange.

For a better comparison we plotted in Figure 5 on page 15 only the histogram of the values obtained for the prime subgroups of the $\Z_p^*$ with $p$ a safe prime (top) and the histogram of the values obtained for the $\Z_p^*$ groups, $p$ a safe prime between 9000 and 11000 (bottom).

It is remarkable the closeness of these values to each other considering that the order of the group varies between 9000 and 11000 a 20% variation in size. This is an encouraging fact, which suggests that for even larger $p$’s we will see the same sort of consistency in the values. This will imply that groups with the same operational structure will have similar behavior from the point of view of the Diffie-Hellman security. However, there is a variation in the values as illustrated in the Figure 4 on page 14 which represent the histogram of the values obtained for the prime subgroup of $\Z_p^*$ groups, with $p$ a safe prime varying between 9000 and 11000.
Fig. 5. A more detailed comparison of the previous image (Fig. [4]). We are comparing the prime subgroups with the corresponding safe groups. Values closer to zero represent safer groups for DH exchange.

6. Conclusion and future work.

In this article we present a novel statistical testing procedure to help assess the security of the Diffie-Hellman key exchange protocol. The methods presented are quite general and to our knowledge represent the first systematic pure applied statistical approach to a cryptographic problem. The article is intended to open a way for methods coming from statistical world to the cryptographic domain. We do not claim to solve the security of the Diffie-Hellman exchange protocol. What we have presented are primarily sufficient conditions for the security. We also presented a way to compare the strength of these conditions in different groups. In Section 5 we show that among the groups we looked at, only the prime subgroups of a large group are close to fulfilling the conditions considered.

An obvious lack in our results is a statistical analysis for very large primes. Typically the groups used in cryptography are of the order at least $2^{1024}$. The use of our testing procedure, ad-literam as presented in section 4 prevents us from such an analysis, however currently we are investigating directions of circumventing the permutation testing approach. One direction is to approximate the distribution of the test in (12) with a multinomial distrib-
Histogram of Distances for Prime Subgroups

| Distance | Frequency |
|----------|-----------|
| 0.00017  | 0         |
| 0.00018  | 1         |
| 0.00019  | 2         |
| 0.00020  | 3         |
| 0.00021  | 4         |
| 0.00022  | 5         |
| 0.00023  | 6         |

Fig. 6. A blowup of the histogram of the values for the prime subgroups in the safe primes. Note the values are close to zero but not equal to zero.

tion, then use a multivariate normal distribution for a second approximation. This should allow us to calculate the p-value of the test directly without the need of the permutation testing. Another direction is to put together outcomes into coarser groups and look at the distribution of these groups of outcomes. This idea is similar in result with the approach of Canetti et al. (1999), Banks et al. (2006), and should allow us to speed up the procedure in order to apply it to much larger groups. It will also allow us to look at the distribution of the binary representation of prime subgroups of a large group, and extend the methodology to finite groups defined using elliptical curves.

7. Appendix

We present the actual values obtained in \( \mathbb{Z}_p \) when \( p = 1193 \) in Table 1.
Table 1: Results for $Z_{1193}^*$

| Sample size n | Sample entropy value | p-value | Distance to center | Relative distance |
|---------------|----------------------|---------|--------------------|------------------|
| 59            | 0.046993             | 0.556   | 0                  | 0                |
| 118           | 0.105734             | 0.904   | 0                  | 0                |
| 354           | 0.280115             | 1       | 0.0867205869841293 | 0.6027661994104  |
| 885           | 0.532425             | 1       | 0.088729342259792  | 0.59962439145006 |
| 1289          | 0.96382              | 1       | 0.15839336513686   | 0.75821029638124 |
| 3304          | 1.40654              | 1       | 0.18791829961140   | 0.89042990342397 |
| 5428          | 1.82531              | 1       | 0.194266177572582  | 0.935988768705612 |
| 8319          | 2.19741              | 1       | 0.181355528931107  | 0.95241145654936 |
| 12095         | 2.55884              | 1       | 0.192004761885750  | 0.966808090255525 |
| 16874         | 2.87286              | 1       | 0.187373211259202  | 0.979981576295216 |
| 22774         | 3.1674               | 1       | 0.191522958630831  | 0.98157705830948 |
| 29913         | 3.43077              | 1       | 0.188465589635031  | 0.98875796589123 |
| 38409         | 3.67754              | 1       | 0.189706561218385  | 0.98943516631953 |
| 48380         | 3.90781              | 1       | 0.19241600875197   | 0.99165124060656 |
| 59944         | 4.11938              | 1       | 0.19204137533093   | 0.9947810437756  |
| 73219         | 4.31302              | 1       | 0.187468331653386  | 0.99475817256502 |
| 88323         | 4.50025              | 1       | 0.188526799093478  | 0.996569090251708 |
| 105374        | 4.67745              | 1       | 0.19031690687346   | 0.996854516374416 |
| 124490        | 4.84304              | 1       | 0.190069869416784  | 0.997275184401866 |
| 145789        | 5.00357              | 1       | 0.19382446593161   | 0.997349334367475 |
| 169389        | 5.14947              | 1       | 0.18908592541566   | 0.99793183476464 |
| 195408        | 5.29352              | 1       | 0.191440055575543  | 0.998590870156603 |
| 223964        | 5.42925              | 1       | 0.19118605525655   | 0.998411314116565 |
| 255175        | 5.55893              | 1       | 0.190698438642096  | 0.99884534816062 |
| 289159        | 5.68315              | 1       | 0.190158376706143  | 0.998956167303273 |
| 326034        | 5.80322              | 1       | 0.189542921352024  | 0.99921427003345 |
| 365918        | 5.91821              | 1       | 0.190224685898690  | 0.99909568272852 |
| 408929        | 6.03153              | 1       | 0.192585342387550  | 0.99931590521473 |
| 455185        | 6.13611              | 1       | 0.190147697214377  | 0.99941366702933 |
| 504804        | 6.2378               | 1       | 0.188502018190133  | 0.999534588235375 |
| 557904        | 6.34038              | 1       | 0.191174900210519  | 0.999622308526143 |
| 614603        | 6.43583              | 1       | 0.189835678716927  | 0.999625957882701 |
| 675019        | 6.53032              | 1       | 0.190747529220258  | 0.99976572991958 |
| 739270        | 6.62041              | 1       | 0.18999301424682   | 0.99970537957273 |
| 807474        | 6.70913              | 1       | 0.19053231900746   | 0.999813379231167 |
| 879749        | 6.79349              | 1       | 0.189228458019329  | 0.99981490993474 |
| 956213        | 6.87841              | 1       | 0.190858661034748  | 0.99984125150255 |
| 1036984       | 6.95729              | 1       | 0.188695781636576  | 0.99991486731075 |
| 1122180       | 7.03801              | 1       | 0.190502664400493  | 0.99992616752106 |
| 1211919       | 7.11461              | 1       | 0.19020994699298   | 0.99994757464427 |
| 1306319       | 7.18871              | 1       | 0.189373283972479  | 0.999979948549426 |
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Histogram of Distances for Prime Subgroups

Histogram of Distances for Safe Primes

Histogram of Distances for Other Primes
Histogram of Distances for Safe Primes

Distance

Frequency

0.1305 0.1306 0.1307 0.1308 0.1309 0.1310 0.1311 0.1312

Distance

Frequency