Proof of tightness of Varshamov - Gilbert bound for binary codes

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Abstract

We prove tightness of Varshamov - Gilbert (VG) bound for binary codes

Subset $C \subset 2^n$ such that Hamming distance $d_H(x, y) \geq d$, $x \neq y \in C$ we call binary code with minimal distance $d$. The main Problem in Information Theory is to determine the asymptotic of the rate $R(d = [\delta n]) = (\log_2 |C|)/n$, $\delta \in (0, 1/2)$. Famous asymptotic lower bound is Varshamov - Gilbert bound [3], [4] which was established in 50'th and state that

$$R > 1 - H_2(d/n) + o(1),$$

when $n \to \infty$ and $H_2(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)$ is binary entropy function.

To complete this famous problem it is necessary to prove (for example) that this bound is tight. The best upper bound at this moment it MRRW bound [3]. It states that

$$R < \min_{1 \leq u \leq 2\delta} (1 + h(u^2) - h(u^2 + 2\delta u + 2\delta)),$$

where

$$h(v) = H_2\left(\frac{1 - \sqrt{1 - v}}{2}\right).$$

In this article using original Symmetrical Smoothing Method we complete the story and prove that bound (??) is tight.
We use the fact that system of equations
\begin{align*}
    x_1 + x_2 + \ldots + x_n &= p_1, \\
    x_1^2 + x_2^2 + \ldots + x_n^2 &= p_2, \\
    &\vdots \\
    x_1^n + x_2^n + \ldots + x_n^n &= p_n.
\end{align*}
(1)
has unique solution up to permutations, for the details see [4].

Define functions \((x \in \left[\frac{n}{n/2}\right], \beta_i \in R^{n+1}, \beta_{i,j} \in \{0, 1\}: |\{j : \beta_{i,j} \neq 0\}| = n/2 + 1, \sum_j \beta_{i,j} = n/2 + 1)\)
\begin{align*}
    \varphi(x, \beta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{((\beta, x)-(n/2+1))/\sigma} e^{-\xi^2/2} d\xi, \\
    \lambda(p, q, \{\beta\}, x, y, i) &= 1 - \left[ \prod_{\ell=1, \neq i}^M (1 - (2\varphi((x, \beta_\ell)))^p) (1 - 2\varphi((x, \beta_{i}\sqrt{pq}))) \\
    &+ \prod_{\ell=1, \neq i}^M (1 - (2\varphi((y, \beta_\ell)))^p) (1 - 2\varphi((y, \beta_{i}\sqrt{pq})))) \\
    &+ (1 - (2\varphi((x, \beta_{i}\sqrt{pq}))) + 2\varphi((y, \beta_{i}\sqrt{pq})))) \\
    &\times \prod_{\ell=1, \neq i}^M (1 - (2\varphi((x, \beta_\ell)))^p) (1 - (2\varphi((y, \beta_{i}\sqrt{pq}))))
\right], \\
    \Lambda(p, q, \{\beta\}, i) &= \sum_{x \in \left[\frac{n}{n/2}\right]} \prod_{\ell=1, \neq i}^M (1 - (2\varphi((x, \beta_\ell)))^p) (1 - 2\varphi((y, \beta_{i}\sqrt{pq}))), \\
    R(p, q, \{\beta\}, i) &= \sum_{x,y \in \left[\frac{n}{n/2}\right]}^{\{\beta\}, x, y, i, p \in D_1, q \in D_2,}
\end{align*}
where constants \(D_1, D_2\) is chosen to be sufficiently large.

Optimization Problem is to find
\begin{align*}
    \max \sum_{i=1}^M \sum_{p=1}^{D_1} \sum_{q=1}^{D_2} \Lambda(p, q, \{\beta\}, i) = \max \sum_{i=1}^M \sum_{p=1}^{D_1} \sum_{q=1}^{D_2} \sum_{x \in \left[\frac{n}{n/2}\right]} \lambda(p, q, \{\beta\}, x, i),
\end{align*}
(2)
when
\begin{align*}
    \sum_{i=1}^M \sum_{p=1}^{D_1} \sum_{q=1}^{D_2} R(p, q, \{\beta\}, i) \leq o(1), \ \sigma \rightarrow 0.
\end{align*}
(3)
Values $\Lambda(p,q,\{\beta\},i)$ approximate the volume of the code (if all $\beta_i$ are different) and condition (3) means that there is no codewords at distance less than $d$.

Define

$$L_j = \sum_{i=1}^M \sum_{p=1}^{D_1} \sum_{q=1}^{D_2} L_{i,p,q,j}(\{\beta\})$$

$$= \sum_{i=1}^M \sum_{p=1}^{D_1} \sum_{q=1}^{D_2} ((\Lambda(p,q,\{\beta\},i))_{\beta_{i,j}}' - \mu(R(p,q,\{\beta\},i))_{\beta_{i,j}}'), \mu \geq 0.$$

Kuhn- Tucker condition for conditional extremum of the function $\Lambda$ is as follows

$$L_j = 0, \; j \in \lfloor n/2 \rfloor.$$ (4)

Next we show that there is no solutions of system (4) which depends on $\sigma$.

Define

$$f_j = e^{1/g} \left( 1 - \left( 1 - (L_{1,1,1,j}(\{\beta\}) + \hat{L}_j(\{\beta\})) e^{-1/g} \right) \right)$$

$$\times \prod_{i=2}^M \prod_{p=1}^{D_1} \prod_{q=1}^{D_2} \left( \text{excluding } i = p = q = 1 \right) \left( 1 - (L_{i,p,q,j}(\{\beta\})) e^{-1/g} \right).$$

It is easy to see that for some

$$\hat{L}_j(\{\beta\}) = \frac{\mu_{1,j}(\{\beta\}) e^{-1/g}}{1 + \mu_{2,j}(\{\beta\}) e^{-1/g}} \to 0, \; g \to 0$$

(here $\mu_{1,j}, \mu_{2,j}$ are functions with values uniformly bounded by constant), we have

$$f_j = \sum_{i=1}^M \sum_{p=1}^{D_1} \sum_{q=1}^{D_2} L_{i,p,q,j}(\{\beta\}).$$

Condition $f_j = 0$ is Kuhn- Tucker necessary condition for $\{\beta\}$ to be optimal. Last equality is equivalent to the system of equations

$$L_{i,p,q,j}(\{\beta\}) = 0$$ (5)

except case $i = p = q = 1$. In this case we have equation

$$L_{1,1,1,j}(\{\beta\}) + \hat{L}_j(\{\omega\}) = 0.$$ (6)
As it follows $\hat{L}_j(\{\omega\})$ can be chosen arbitrary small and we can skip it. We assume that $i = 1$ and $\beta_j = \beta_{1,j}$ (other cases are similar), assume that $\beta_1, \beta_2,$ and consider the subsystem obtaining from system (5) with these fixed values. First we assume that $\mu \neq 0$. Substituting $\mu$ which we obtain from equality (5) to all other equalities we obtain new system, whose equations do not contain $\mu$ (we skip index $i$ everywhere in the next formulas assuming $i = 1$:

$$(\Lambda(p, q, \{\beta\}, i))'_{\beta_1} (R(p, q, \{\beta\}, i))'_{\beta_2} = (R(p, q, \{\beta\}, i))'_{\beta_1} (\Lambda(p, q, \{\beta\}, i))'_{\beta_2}, \quad p \in [D_1], q \in [D_2].$$

(7)

It is easy to see that system (7) is the set of equalities of the sums of terms which are $p$'th powers. We transpose the terms of these equations in such a way that all terms in l.h.s. and r.h.s. of these equations become positive. Fixing r.h.s. of these equations we obtain for large enough $D_1$ exactly system (11). As we know this system has unique solution and as follows the same number of terms in the l.h.s. and r.h.s. of these equations, hence these terms pairwise coincide (in some order).

Varying $q$ we obtain from system (7) new system of equalities which terms are exponents

Remark. Let $\phi_i$ be the sum (with signs) of exponent $\phi_i = \sum_j \alpha_{i,j} e^{b_{i,j}}, i = 1, 2, 3, 4$ and

$$\phi_1 = \phi_2.$$  

(8)

Denote

$$\text{Exp}(\phi_i) = \sum_j \alpha_{i,j} b_{i,j}.$$  

From equality

$$\phi_1 = \phi_2$$

it follows

$$\text{Exp}(\phi_1) = \text{Exp}(\phi_2).$$

We can choose $\beta_1$ such that only $n/2+1$ coordinates $\beta_{1,j}$ are nonzero and they are positive, $\sum_j \beta_{1,j} = n/2 + 1$ (hence $\beta_{1,j} > \delta > 0$).

Now consider equations (5) for $p = q = 1$, and take from each term only multiple which is exponent (not integral). We can assume that there is position, say $j = 1$ where zero coordinate have at least $M2^{-\epsilon_2 n}$ codewords, where $\epsilon_2 >$ small positive constant. Otherwise the number of codewords would be exponentially small in compare with $M$. We choose the codeword, say $c_1$ which have units at position 1.

Varying $p, q$ we settle that actually from equality (5) follows equality

$$(\tilde{\Lambda}(1, 1, \{\beta\}, 1))'_{\beta_1} = \mu(\tilde{R}(1, 1, \{\beta\}, 1))'_{\beta_1}, \quad \mu = 1,$$

(9)
where $\tilde{G}$ means that we take only exponent multiple from each term of $G$.

It is easy to see from the expression for $\Lambda, R$ that equality (9) can be rewritten following

$\left( \sum_{x \in \binom{n}{n/2} : 1 \leq x} \left( \left( \beta, x \right) - \frac{n}{2} \right)^2 - \sum_{x \in \binom{n}{n/2}} \left( \left( \beta, x \right) - \frac{n}{2} \right)^2 \right)$

$\begin{equation}
\left( \sum_{x \in \binom{n}{n/2} : 1 \leq x, 0 < d_H(y,x) < d, y \in C} \left( \left( \beta, x \right) - \frac{n}{2} \right)^2 - \sum_{x \in \binom{n}{n/2} : 1 \leq x, 0 < d_H(y,x) < d, y \in C} \left( \left( \beta, x \right) - \frac{n}{2} \right)^2 \right) \right)
\end{equation}$

for some constant $C > 0$. Easy calculations show that form (10) follows the inequality

$2^n 2^{2\epsilon n} > M \left( \frac{n}{d} \right)^2 2^{-2\epsilon n}$, $n > n(\epsilon)$.

From here it follows that

$M < 2^{n+\epsilon n} \left( \frac{n}{d} \right)$.

Using Jonson bound (5) we obtain the bound

$A \leq \frac{2^n}{\binom{n}{n/2}} M < nM < 2^{\epsilon n} 2^n / \left( \frac{n}{d} \right)$.

This deliver us to the conclusion that VG bound for binary codes is tight in rough logarithmic asymptotic.

Remark Tightness of binary VG bound has several important consequences. One such consequence is (now) established fact that Expurgation bound settled by Shannon, Gallager and Berlecamp (see [4], [5]) for reliability function of discrete memoryless channel is tight at small rates. Because it is known that it is tight at large rates we come to the conclusion that this reliability function for BSC is known now.

References

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