ENTROPIES OF COMMUTING TRANSFORMATIONS ON HILBERT SPACES

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(Communicated by Shaobo Gan)

Abstract. By establishing Multiplicative Ergodic Theorem for commutative transformations on a separable infinite dimensional Hilbert space, in this paper, we investigate Pesin’s entropy formula and SRB measures of a finitely generated random transformations on such space via its commuting generators. Moreover, as an application, we give a formula of Friedland’s entropy for certain $C^2 \mathbb{N}^2$-actions.

1. Introduction. The significance of Pesin’s entropy formula (or Ledrappier-Young’s entropy formula for SRB measures) lies in its characterizing SRB measures by their Lyapunov exponents and entropy [10]. Pesin’s entropy formula for random transformations and stochastic flows of diffeomorphisms in finite dimensional compact spaces were established in [2, 11, 17, 9]. The extension of the above theories to infinite dimensional spaces were presented in [3, 12, 13, 19, 14, 15, 20, 21]. In this paper, we further study the Pesin’s entropy formula and SRB measures for random transformations generated by finitely commutative transformations in infinite dimensional Hilbert spaces via its generators, which can be viewed as a generalization of the work in [7, 8, 24, 25] to the infinite dimension spaces. However, the techniques and strategies are completely different due to the feature of infinite dimensional smooth dynamics. For more recent progress of SRB measures in infinite dimensional spaces, we refer to the elegant survey [23].

To obtain the relations of metric entropy of the random transformation and the Lyapunov exponents of its generators, the basic strategy is to estimate the random exponential expanding rate in a deterministic subspace by exponential expanding rates of generators in this subspace. Intuitively, random exponential expanding

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2020 Mathematics Subject Classification. Primary: 37H15, 37A35, 37C85; Secondary: 37H99, 37D20.

Key words and phrases. SRB measures, metric entropy, Pesin’s entropy formula, $\mathbb{N}^2$-action, infinite dimensional random dynamical system.

The first author is supported by NSFC (No: 11871394) and Natural Science Foundation of Shaanxi Province (2020JC-39), the second author is supported by NSFC (No: 11771118).

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rate should be the weighted combination of exponential expanding rates of generators, and the weights depend on the probability law of choosing the generators for each iteration. So we first establish Multiplicative Ergodic Theorem (Theorem A) for commutative transformations on a separable infinite dimensional Hilbert space, which is a higher rank group actions version of [16] and infinite dimensional version of [7]. By our assumptions, the deterministic subspace is the common expanding subspace of each generators. Then by comparing the dynamics of the random transformation with the dynamics of its generators, we reformulate Ruelle’s entropy inequality (Theorem B), the Pesin’s entropy formula and SRB measures (Theorem C) via the generators. Moreover, as an application, we give a formula of Friedland’s entropy (Theorem D) for certain $C^2$-actions.

This paper is organized as follows. In Section 2, basic notions such as finitely generated random transformations, Lyapunov exponents, metric entropy and Friedland’s entropy will be introduced. Then we will state the main results (Theorem A-Theorem D). Section 3 is devoted to the proofs of the main results.

2. Preliminaries and statement of main results. Let $X$ be a separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, distance function $d$ and $\sigma$-algebra $\mathcal{B}$ of Borel sets.

2.1. Deterministic infinite dimensional dynamical systems. We begin with the notion of $C^1$ map. Let $L(X, Y)$ denote the collection of bounded linear operators from Banach space $X$ to $Y$. Let $U$ be a non-empty open subset of $X$. A measurable map $g : U \to Y$ is said to be $C^1$ if there exists $\{d_x g : X \to Y\}_{x \in U}$ of $L(X, Y)$ such that i) for each $x \in U$,

$$\lim_{y \to x} \frac{\|g(x) - g(y) - d_x g(x - y)\|}{\|x - y\|} = 0;$$

ii) the map $x \to d_x g$ is continuous from $U$ to $L(X, Y)$. The map $g$ is said to be $C^2$ if its derivative $d_{(\cdot)} g$ is also $C^1$ from $U$ to $L(X, Y)$. For any bounded subset $A$ of $X$, denoted by $\alpha(A)$ the smallest nonnegative real number $r$ such that $A$ can be covered by finite many Borel balls of $X$ with radius at most $r$. (It is called the Kuratowski measure of non-compactness of the set $A$.) Define also the index of compactness of a map $g : X \to X$ as being the number

$$\|g\|_\alpha := \inf \{k > 0 : \alpha(g(A)) \leq k\alpha(A) \text{ for any bounded set } A \text{ of } X\}.$$

In case $g$ is a continuous linear operator, we have $\|g\|_\alpha = \alpha(g(B_X))$, where $B_X$ is the open unit ball of $X$. Let $h$ be another continuous linear operator of $X$, then we have

$$\|g + h\|_\alpha \leq \|g\|_\alpha + \|h\|_\alpha, \quad \|g \circ h\|_\alpha \leq \|g\|_\alpha \cdot \|h\|_\alpha.$$

Then for any $C^1$ map $g : X \to X$ and $g$-invariant compact set $K \subset X$, (1) gives the existence of the limits

$$l_\alpha(g) := \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in K} \|d_x g^n\|_\alpha$$

and

$$l_\alpha(x, g) := \lim_{n \to \infty} \frac{1}{n} \log \|d_x g^n\|_\alpha \text{ for } \mu \text{ almost every } x \in K,$$

where $\mu$ is any $g$-invariant measure.
For \((\xi_1, \cdots, \xi_p) \in X^p, p \in \mathbb{N}\), define
\[
V_p(\xi_1, \cdots, \xi_p) := \left( \prod_{i=1}^{p-1} \text{dist}(\xi_i, \text{span}\{\xi_{i+1}, \cdots, \xi_p\}) \right) \cdot ||\xi_p||,
\]
where for \(i = 1, 2, \cdots, p - 1,\)
\[
\text{dist}(\xi_i, \text{span}\{\xi_{i+1}, \cdots, \xi_p\}) = \inf \{||\xi_i - \eta|| : \eta \in \text{span}\{\xi_{i+1}, \cdots, \xi_p\}\}.
\]
For \(T \in L(X, X)\), define
\[
V_p(T) := \sup_{\|\xi_i\| = 1, 1 \leq i \leq p} V_p(T(\xi_1), \cdots, T(\xi_p)).
\]

By a detailed exploration of the asymptotic behaviors of \(\{V_p(d_x g^n)\}_{p \in \mathbb{N}}\), Lian-Lu [16] proved the following theorem concerning the existence of Lyapunov exponents, we only present the part which is adequate for our purposes. We need the following assumptions to get Multiplicative Ergodic Theorem.

**H.**  
(i) \(g\) is \(C^1\) Fréchet differentiable and injective;  
(ii) the derivative of \(g\) at \(x \in X\), denoted \(d_x g\), is also injective;  
(iii) there exists a \(g\)-invariant compact set \(K \subset X\).

**Theorem 2.1.** [16, Theorem 3.1] Suppose \(g\) satisfies (H). For any \(g\)-invariant measure \(\mu\), and \(\lambda_\alpha > l_\alpha(x, g)\), there is a measurable, \(f\)-invariant set \(\Gamma_g \subset X\) with \(\mu(\Gamma_g) = 1\) and at most finitely many real numbers
\[
\lambda_1(x, g) > \lambda_2(x, g) > \cdots > \lambda_r(x, g)(x, g)
\]
with \(\lambda_r(x, g)(x, g) > \lambda_\alpha\) for which the following properties hold. For any \(x \in \Gamma_g\), there is a splitting
\[
X = E_1(x, g) \oplus E_2(x, g) \oplus \cdots \oplus E_r(x, g)(x, g) \oplus E_\alpha(x, g)
\]
such that
(a) for each \(j = 1, 2, \cdots, r(x, g)\), \(\dim E_j(x, g) = m_j(x, g)\) is finite, \(d_x E_j(x, g) = E_j(gx, g)\), and for any \(v \in E_j(x, g) \setminus \{0\}\), we have
\[
\lambda_j(x, g) = \lim_{n \to \infty} \frac{1}{n} \log \|d_x g^nv\|;
\]
(b) the distribution \(E_\alpha(x, g)\) is closed and finite-codimensional, satisfies \(d_x E_\alpha(x, g) \subset E_\alpha(gx, g)\) and
\[
\lambda_\alpha \geq \limsup_{n \to \infty} \frac{1}{n} \log ||d_x g^n E_\alpha(x, g)||;
\]
(c) for \(p \leq \sum_{i=1}^r m_i(x, g)\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \log V_p(d_x g^n) = \sum_{k=1}^r \lambda_k(x, g),
\]
where \(\{\lambda_k(x, g)\}\) are \(\lambda_j(x, g)\)’s repeated with multiplicity \(m_j(x, g)\);  
(d) the mappings \(x \mapsto E_j(x, g), x \mapsto E_\alpha(x, g)\) are measurable,  
(e) writing \(\pi_j(x, g)\) for the projection of \(X\) onto \(E_j(x, g)\) via the splitting at \(x\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \log |\pi_j(g^n x, g)| = 0 \quad a.s.
\]
In order to get SRB measures, it is necessary to put some restrictions on \(g\). Under above setting, it is true by seeing Theorem 2.1 that \(l_\alpha(g) < 0\) implies the existence of Lyapunov exponents \(\lambda_1(x, g) > \lambda_2(x, g) > \cdots\) with multiplicities \(m_1(x, g), m_2(x, g), \cdots\) at \(\mu\)-a.e. \(x\), which can be infinitely many but only admits finitely many positive ones.

**Theorem 2.2.** [13, Theorem 1.1] Suppose \(g\) is \(C^2\) Fréchet differentiable, satisfies (H) and \(l_\alpha(g) < 0\). \(\mu\) is supported on \(K\) with \(\mu_K(g) < \infty\), where \(K\) is a compact invariant set. If \(\mu\) is an SRB measure, then

\[
\mu_K(g) = \int \sum_{\lambda_j(x, g) > 0} m_j(x, g) \lambda_j(x, g) d\mu.
\]

The converse is also true if \((g, \mu)\) has no zero Lyapunov exponents and the set \(K\) has finite box-counting dimension.

### 2.2. Random transformations with finite commuting generators.

#### 2.2.1. Multiplicative ergodic theorem for \(\mathbb{N}^2\)-actions

We now consider \(\mathbb{N}^N\) actions generated by commuting maps \(\mathfrak{A} = \{f_i : X \to X\}_{i=1,\ldots,N}\), in which \(f_i \circ f_j = f_j \circ f_i\) for all \(1 \leq i, j \leq N\). For simplicity of the notations, we assume \(N = 2\). We denote by \(\mathcal{M}_i\) (resp. \(\mathcal{M}_i^e\)) the set of (resp. ergodic) Borel probability measures on \(X\) which are invariant under \(f_i\), for any \(i = 1, 2\). Let \(\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2\) and \(\mathcal{M}^e = \mathcal{M}_1^e \cap \mathcal{M}_2^e\).

By [7, Proposition 1.3, 1.4], \(\mathcal{M} \neq \emptyset\) and \(\mathcal{M}^e \neq \emptyset\). Our first result of Multiplicative Ergodic Theorem for infinite dimensional \(\mathbb{N}^2\)-actions will prove extremely useful in the next section. We need the following assumptions on generators which will be needed throughout the paper.

**H0.** (i) \(f_1, f_2\) are \(C^1\) Fréchet differentiable and injective;
(ii) the derivatives of \(f_1\) and \(f_2\) are also injective;
(iii) there exists a compact set \(K \subset X\) such that \(f_1(K) = K\) and \(f_2(K) = K\).

**Theorem A.** Suppose \(f_1, f_2\) satisfy (H0) and \(\mu \in \mathcal{M}\) with \(\text{supp}\mu \subset K\). For any \(\lambda_\alpha > \max\{l_\alpha(f_1), l_\alpha(f_2)\}\), there exists a measurable set \(\Gamma \subset \Gamma_{f_1}\) with \(f_1(\Gamma) = \Gamma\) for each \(i = 1, 2\) and \(\mu(\Gamma) = 1\), such that for any \(x \in \Gamma\), there exists a decomposition of \(X\) into at most finitely many subspaces

\[
X = \bigoplus_{j_1=1}^{r(x,f_1)} \bigoplus_{j_2=1}^{r(x,f_2)} E_{j_1,j_2}(x) = \bigoplus_{j_1=1}^{r(x,f_1)} E_{j_1,r(x,f_2)+1}(x) + \bigoplus_{j_1=1}^{r(x,f_1)} E_j(x)
\]

satisfying the following properties:

(a) if \(E_{j_1,j_2}(x) \neq \{0\}\), for any \(s_1, s_2 \in \mathbb{Z}^+, 0 \neq v \in E_{j_1,j_2}(x), 1 \leq j_1 \leq r(x, f_i)\) and \(i = 1, 2\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x(f_1^{s_1} \circ f_2^{s_2})^n v\| = s_1 \lambda_{j_1}(x, f_1) + s_2 \lambda_{j_2}(x, f_2);
\]

(b) if \(E_{j_1,r(x,f_2)+1}(x) \neq \{0\}\), for \(0 \neq v \in E_{j_1,r(x,f_2)+1}(x), 1 \leq j_1 \leq r(x, f_1)\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f_1^n v\| = \lambda_{j_1}(x, f_1)
\]

and

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|d_x f_2^n v\| \leq \lambda_\alpha;
\]
(c) if $E_{r(x,f_1)+1,j_2}(x) \neq \{0\}$, for $0 \neq v \in E_{r(x,f_1)+1,j_2}(x)$, $1 \leq j_2 \leq r(x,f_2)$,
\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f_n^v\| = \lambda_{j_2}(x, f_2)
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f_n^v\| \leq \lambda_{\alpha};
\]
(d) for $0 \neq v \in E_{\alpha}(x)$ and $i = 1, 2$,
\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f_n^v\| \leq \lambda_{\alpha};
\]
(e) for $1 \leq j_i \leq r(x, f_i)$ and $i = 1, 2$, each $E_{j_1,j_2}(x)$, we have the following invariance properties:
\[
df_i(x) E_{j_1,j_2}(x) = E_{j_1,j_2}(f_i(x)) \quad \text{and} \quad \lambda_{j_i}(f_i(x), f_i) = \lambda_{j_i}(x, f_i), \quad \text{where} \quad 1 \leq i, i' \leq 2;
\]
(f) for $1 \leq j_i \leq r(x, f_i)$ and $i = 1, 2$, each $E_{r(x,f_1)+1,j_2}(x)$ and $E_{j_1,r(x,f_2)+1}(x)$ we have the following invariance properties:
\[
df_1(x) E_{r(x,f_1)+1,j_2}(x) \subset E_{r(x,f_1)+1,j_2}(f_1(x)),
\]
\[
df_2(x) E_{j_1,r(x,f_2)+1}(x) \subset E_{j_1,r(x,f_2)+1}(f_2(x)),
\]
\[
df_2(x) E_{r(x,f_1)+1,j_2}(x) = E_{r(x,f_1)+1,j_2}(f_2(x)),
\]
\[
df_1(x) E_{j_1,r(x,f_2)+1}(x) = E_{j_1,r(x,f_2)+1}(f_1(x));
\]
(g) writing $\pi_{j_1,j_2}(x)$ for the projection of $X$ onto $E_{j_1,j_2}(x)$ via the splitting at $x$,
for every $i = 1, 2$, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log |\pi_{j_1,j_2}(f_n^v x)| = 0 \quad \text{a.s.} \tag{4}
\]

2.2.2. Random transformations with finite commuting generators. Again, without loss of generality, we consider random transformations generated by two maps $\bar{f} = \{f_1, f_2\}$ with $f_1 \circ f_2 = f_2 \circ f_1$.

Let $\Omega = \bar{f}^\mathbb{N} = \prod_0^\infty \bar{f}$ be the infinite product of $\bar{f}$, endowed with the product topology and the product Borel $\sigma$-algebra $\mathcal{A}$, and let $\theta$ be the left shift operator on $\Omega$ which is defined by $(\theta \omega)_n = \omega_{n+1}$ for $\omega = (\omega_n) \in \Omega$. Given $\omega = (\omega_n) \in \Omega$, we write $f_\omega = \omega_0$ and

\[
f(n, \omega) := \begin{cases} 
  f_{\theta^{n-1} \omega} \circ \cdots \circ f_{\theta \omega} \circ f_{\omega} & n > 0 \\
  \text{id} & n = 0.
\end{cases}
\]

There is a natural skew product transformation $F : \Omega \times X \to \Omega \times X$ over $(\Omega, \theta)$ which is defined by $F(\omega, x) = (\theta \omega, f_\omega(x))$. For any probability measure $\nu$ on $\bar{f}$, we can define a probability measure $P_\nu = \nu^\mathbb{N}$ on $\Omega$ which is invariant with respect to $\theta$. By the induced finitely generated (i.i.d.) random transformation $f$ over $(\Omega, \mathcal{A}, P_\nu, \theta)$ we mean the system generated by the randomly composition of $f_i$, $i = 1, 2$ in the law of $\nu$. We are interested in dynamical behaviors of these actions for $P_\nu$-a.e. $\omega$ or on the average on $\omega$. It is clear that $f(n, \omega)$ is injective and strongly measurable for any $n \in \mathbb{N}$, (in the sense for each $x \in X$ fixed, the map $\omega \mapsto f(n, \omega)(x)$ is measurable from $\Omega$ to $X$).

A Borel probability measure $\mu$ on $X$ is called $f$-invariant if $\int \mu(f_\omega^{-1} A) \, dP_\nu(\omega) = \mu(A)$ for all Borel $A \subset X$. We denote by $\mathcal{M}_f$ (resp. $\mathcal{M}_f^\mathbb{N}$) the set of all $f$-invariant (resp. ergodic) Borel probability measures. Clearly, $\mathcal{M} \subset \mathcal{M}_f$ and $\mathcal{M}^\mathbb{N} \subset \mathcal{M}_f^\mathbb{N}$.

For each $\omega \in \Omega$, using (1), we see that for $x \in K$, $m, n \in \mathbb{N}$,
\[
\log \|d_x f(n + m, \omega)\|_\alpha \leq \log \|d_x f(n, \omega)\|_\alpha + \log \|d_{f(n,\omega)x}(f(n, \theta^n \omega))\|_\alpha.
\]
This gives the existence of the limits
\[ l_\alpha(\omega, x) := \lim_{n \to \infty} \frac{1}{n} \log \|d_x f(n, \omega)\|_\alpha \]
and
\[ l_\alpha(\omega, f) := \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in K} \|d_x f(n, \omega)\|_\alpha, \]
where \( K \subset X \) is a compact set such that \( f_1(K) = K \) and \( f_2(K) = K \). By Random Subadditive Ergodic Theorem [9, Theorem 2.2], one can show that \( l_\alpha(\omega, x) \) and \( l_\alpha(\omega, f) \) are non-random in the sense that there is a measurable, \( f \)-invariant set \( \Gamma_0 \subset X \) with \( \mu(\Gamma_0) = 1 \) such that for \( P_\nu \)-a.e. \( \omega \in \Omega \), \( l_\alpha(\omega, x) = l_\alpha(x) \) and \( l_\alpha(\omega, f) = l_\alpha(f) \) for any \( x \in \Gamma_0 \). For the Lyapunov exponents for random transformations, we only present the part which is adequate for our purposes.

**Theorem 2.3.** [16, Theorem 3.1] Let \( f \) be a finitely generated random transformation on an infinite dimensional Banach space \( X \) over \((\Omega, A, P_\nu, \theta)\) by \( f_1 \) and \( f_2 \). Suppose \( \mu \in M \) and \( f_1, f_2 \) satisfy \( (H0) \). For any \( \lambda_\alpha > l_\alpha(f) \), there is a measurable, \( f \)-invariant set \( \Gamma_f \subset X \) with \( \mu(\Gamma_f) = 1 \). For any \( x \in \Gamma_f \), there are at most finitely many real numbers
\[ \lambda_1(x, f) > \lambda_2(x, f) > \cdots > \lambda_r(x, f)(x, f) \]
with \( \lambda_r(x, f)(x, f) > \lambda_\alpha \) for which the following properties hold. For \( P_\nu \)-a.e. \( \omega \in \Omega \), there is a splitting
\[ X = E_1(\omega, x, f) \oplus E_2(\omega, x, f) \oplus \cdots \oplus E_r(x, f)(\omega, x, f) \oplus E_\alpha(\omega, x, f) \]
such that
(a) for each \( j = 1, 2, \ldots, r(x, f), \dim E_j(\omega, x, f) = m_j(x, f) \) is finite, \( d_x f_\omega E_j(\omega, x, f) = E_j(F(\omega, x, f)) \), and for any \( v \in E_j(\omega, x, f) \setminus \{0\} \), we have
\[ \lambda_j(x, f) = \lim_{n \to \infty} \frac{1}{n} \log \|d_x f(n, \omega)v\|; \]
(b) the distribution \( E_\alpha \) is closed and finite-codimensional, satisfies \( d_x f_\omega E_\alpha(\omega, x, f) \subset E_\alpha(F(\omega, x, f)) \) and
\[ \lambda_\alpha \geq \limsup_{n \to \infty} \frac{1}{n} \log \|d_x f(n, \omega)\|_{E_\alpha(\omega, x, f)}; \]
(c) for \( p \leq \sum_{j=1}^{r(x, f)} m_j(x, f) \), we have
\[ \lim_{n \to \infty} \frac{1}{n} \log V_p(d_x f(n, \omega)) = \sum_{k=1}^{p} \tilde{\lambda}_k(x, f), \]
where \( \{\tilde{\lambda}_k(x, f)\} \) are \( \lambda_j(x, f) \)'s repeated with multiplicity \( m_j(x, f) \); (d) the mappings \( (\omega, x) \mapsto E_j(\omega, x, f), (\omega, x) \mapsto E_\alpha(\omega, x, f) \) are measurable, (e) writing \( \pi_j(\omega, x, f) \) for the projection of \( X \) onto \( E_j(\omega, x, f) \) via the splitting at \( x \), we have
\[ \lim_{n \to \infty} \frac{1}{n} \log |\pi_j(F^n(\omega, x, f))| = 0 \quad a.s. \]

By [13, Theorem 2.7], for \( P_\nu \times \mu \)-a.e. \( (\omega, x) \) the unstable set
\[ W^u(\omega, x) := \{ y \in X : \limsup_{n \to +\infty} \frac{1}{n} \log d(f(-n, \omega)x, f(-n, \omega)y) < 0 \} \]
is a $C^{1,1}$ immersed Hilbert manifold of $X$, the so called unstable manifold at $(\omega, x)$. A measurable partition $\eta$ of $\Omega \times X$ is subordinate to $W^u$ manifolds of $(f, \mu)$, if for $P_\nu \times \mu$-a.e. $(\omega, x)$, denote by $\eta(\omega, x)$ the element of $\eta$ that contains $(\omega, x)$, then

$$\eta^u(x) := \{ y : (w, y) \in \eta(\omega, x) \} \subset W^u(\omega, x)$$

and $\eta^u(x)$ contains an open neighborhood of $x$ in $W^u(\omega, x)$, this neighborhood being taken in the submanifold topology of $W^u(\omega, x)$. A Borel probability measure $\mu$ is said to have absolutely continuous conditional measures on $W^u$-manifolds of $(f, \mu)$, if for any measurable partition $\eta$ subordinate to $W^u$-manifolds of the system, one has $\mu^{\eta^u(x)} \ll \text{Leb}_{(\omega, x)}^u$, $P_\nu \times \mu$-a.e., where $\{ \mu^{\eta^u(x)} \}_{x \in K}$ is a canonical system of the conditional measures of $\mu$ associated with the partition $\{ \eta^u(x) \}_{x \in K}$ of $X$ and $\text{Leb}_{(\omega, x)}^u$ is the Lebesgue measure on $W^u(\omega, x)$ induced by its inherited Riemannian metric as a submanifold of $X$. We call such measure an SRB measure. Similarly, we denote by $W^u(x, f_i)$ the unstable manifold of $(f_i, \mu_i), i = 1, 2$.

Now we give more assumptions on the generators.

**H1.** $l_a(f_1) < 0$ and $l_a(f_2) < 0$.

We are in a situation to state the main results of this paper.

**Theorem B.** Let $f$ be a finitely generated random transformation of an infinite dimensional Banach space $X$ over $(\Omega, \mathcal{A}, P_\nu, \theta)$. Suppose $\mu \in \mathcal{M}$ and $(f_i, \mu)$ satisfies (H0-H1) above, then Ruelle's inequality

$$h_\mu(f) \leq \int \sum_{i=1}^2 \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i)\lambda_{jk}(x, f_i)m_{jk}(x, f_i) \, d\mu$$

holds, where $h_\mu(f)$ is the metric entropy of $f$.

For Pesin’s entropy formula, we need more smooth condition of the maps, thus we replace (H0) with the following conditions:

**H2.** (i) $f_1, f_2$ are $C^2$ Fréchet differentiable and injective;
(ii) the derivatives of $f_1$ and $f_2$ are also injective;
(iii) there exists a compact set $K \subset X$ such that $f_1(K) = K$ and $f_2(K) = K$.

**Theorem C.** Let $f$ be a finitely generated random transformation of an infinite dimensional Hilbert space $X$ over $(\Omega, \mathcal{A}, P_\nu, \theta)$. Suppose $\mu \in \mathcal{M}$ with supp$\mu \subset K$, $(f_i, \mu)$ satisfies (H1-H2) and $h_\mu(f) < +\infty$. If $\mu$ is an SRB measure, then

$$h_\mu(f) \geq \int \sum_{i=1}^2 \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i)\lambda_{jk}(x, f_i)d_{jk}(x, f_i) \, d\mu,$$

where $d_{j_1}(x, f_1) = m_{j_1}(x, f_1) - \dim E_{j_1, r(x, f_2)+1}(x)$,

$d_{j_2}(x, f_2) = m_{j_2}(x, f_2) - \dim E_{r(x, f_1)+1, j_2}(x)$.

If the following assumption (H3) on the generators are made, we will get Pesin’s entropy formula and look more closely at SRB measures. The main purpose in making such assumption lies in the fact that we lose control of the random transformation when the stable and unstable directions of the generators mixes together with an infinite dimensional freedom. A trivial motivating example is the random transformations generated by hyperbolic torus automorphisms $f_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and
$f_2 = f_1^{-1}$ with $\nu(f_1) = \nu(f_2) = \frac{1}{2}$. It is easy to see that Corollary 2.1 and Corollary 2.2 fail without (H3).

More precisely, let $\lambda_\alpha = 0$ in Theorem A and denote by

$$E^u(x, f_1) := \bigoplus_{j=1}^n E_j(x, f_1), \quad E^u(x, f_2) := \bigoplus_{j=1}^r E_j(x, f_2).$$

**H3.** $E^u(x, f_1) = E^u(x, f_2)$ for any $x \in K$.

**Corollary 1.** Let $f$ be a finitely generated random transformation of an infinite dimensional Hilbert space $X$ over $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$. Suppose $\mu \in \mathcal{M}$ with supp $\mu \subset K$ and $(f, \mu)$ satisfies (H1-H3) and $h_\mu(f) < +\infty$. Then Pesin’s entropy formula

$$h_\mu(f) = \int \sum_{i=1}^n \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i) \lambda_{jk}(x, f_i) m_{jk}(x, f_i) d\mu$$

holds if $\mu$ is an SRB measure.

**Corollary 2.** Let $f$ be a finitely generated $C^2$ random transformation of an infinite dimensional Hilbert space $X$ over $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$. Suppose $\mu \in \mathcal{M}$ with supp $\mu \subset K$ and $(f, \mu)$ satisfies (H1-H3) above and $h_\mu(f) < +\infty$. Then

(a) $h_\mu(f) \geq \sum_{i=1}^n \nu_i h_\mu(f_i)$ if $\mu$ is an SRB measure of $f$;

(b) $h_\mu(f) \leq \sum_{i=1}^n \nu_i h_\mu(f_i)$ if $\mu$ is an SRB measure of $f_1$ and $f_2$;

(c) $h_\mu(f) = \sum_{i=1}^n \nu_i h_\mu(f_i)$ if $\mu$ is an SRB measure of $f, f_1$ and $f_2$.

To date, to the best of our knowledge, there has been little discussion of relation of SRB measures of finitely generated smooth random transformation and the SRB measures of its generators. This paper only serves as a first attempt towards this direction, and the results are still far from satisfaction. The assumption (H3) in this setting seems artificial and redundant, but we can not remove it for technical reasons. We believe that if the generators have common SRB measures, then they could be SRB measures of the random transformation, and if we add some mild conditions (for example condition H3) on the generators the converse could hold true. We leave them as further questions.

**Further Questions.**

(a) Does equality (5) imply that $\mu$ is an SRB measure by adding assumption that $(f_i, \mu)$ has no zero Lyapunov exponents and the set $K$ has finite box-counting dimension?

(b) If $\mu$ is an SRB measure of every generators, then is $\mu$ an SRB measure of $f$?

(c) If $\mu$ is an SRB measure of $f$, and assumption (H3) is satisfied, then is $\mu$ an SRB measure of every generators?

### 2.3. Friedland’s entropy of $\mathbb{N}^2$-actions

Friedland’s entropy of $\mathbb{N}^k$-actions was introduced by Friedland [5] via the topological entropy of the shift map on the induced orbit space. More precisely, let $f : \mathbb{N}^2 \to C^r(K, K)$ $(r \geq 0)$ be a $\mathbb{N}^2$-action on $X$ with the generators $\{f_i\}_{i=1}^n$. Define the orbit space of $f$ by

$$K_f = \{ \bar{x} = \{x_n\}_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{Z}} K : \text{for any } n \in \mathbb{N}, f_{n+1}(x_n) = x_{n+1} \text{ for some } f_{n+1} \in \{f_i\}_{i=1}^n \}.$$ 

This is a closed subset of the compact space $\prod_{n \in \mathbb{Z}} K$ and so is again compact. A natural metric $\bar{d}$ on $K_f$ is defined by

$$\bar{d}(\bar{x}, \bar{y}) = \sum_{n=0}^\infty \frac{d(x_n, y_n)}{2^n}.$$
for $\bar{x} = \{x_n\}_{n \in \mathbb{N}}, \bar{y} = \{y_n\}_{n \in \mathbb{N}} \in K_f$. We can define a shift map

$$\sigma_f : K_f \to K_f, \sigma_f(\{x_n\}_{n \in \mathbb{N}}) = \{x_{n+1}\}_{n \in \mathbb{N}}.$$  

Thus we have associated an $\mathbb{N}$-action $\sigma_f$ with the $\mathbb{N}^2$-action $f$.

**Definition 2.4.** Friedland’s entropy of an $\mathbb{N}^2$-action $f$ is defined by the topological entropy of the shift map $\sigma_f : K_f \to K_f$, i.e.,

$$h(\sigma_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{\varepsilon}(\sigma_f, n, \varepsilon, X_f),$$

(6)

where $s_{\varepsilon}(\sigma_f, n, \varepsilon, K_f)$ is the largest cardinality of any $(\sigma_f, n, \varepsilon)$-separated sets of $K_f$.

Unlike the classical entropy for $\mathbb{N}^2$-actions, Friedland’s entropy is positive when the generators have finite entropy as single transformations. From the known results about Friedland’s entropy, we can see that it is not an easy task to compute it, even for some “simple” examples (see for example [5, 6, 4]).

However, applying the entropy formula (5) for finitely generated random transformation, we give some formulas and bounds of Friedland’s entropy for smooth $\mathbb{N}^2$-actions in a infinite dimensional Hilbert space.

**Theorem D.** Let $f : \mathbb{N}^2 \to C^2(X, X)$ be a $C^2$ $\mathbb{N}^2$-action on an infinite dimensional Hilbert space $X$. Suppose $\mu \in M_f$ with supp$\mu \subset K$ and $(f_1, \mu)$ satisfies (H1-H3) above and $h_\mu(f) < +\infty$, where $f$ is a random transformation generated by $\{f_1, f_2\}$ over $(\Omega, A, \mathbf{P}_\nu, \theta)$. If $\mathbf{P}_\nu \times \mu$ is a measure with maximal entropy of $F$ and $\mu$ is an SRB measure, then

$$h(\sigma_f) \leq -\sum_{i=1}^2 \int \nu(f_i) \log \nu(f_i) + \int \sum_{i=1}^2 \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i) \lambda_{jk}(x, f_i) m_{jk}(x, f_i) d\mu.$$  

(7)

Furthermore, if $\mu \in M_f$ and $\mu(\{x \in X : f_1(x) = f_2(x)\}) = 0$, then we get the following formula of Friedland’s entropy

$$h(\sigma_f) = \log \left( \sum_{i=1}^2 \exp \left( \sum_{\lambda_{jk}(x, f_i) > 0} \lambda_{jk}(x, f_i) m_{jk}(x, f_i) \right) \right).$$

(8)

**Remark 1.** In Theorem D, we require that the invariant measure of $F$ is in the form of $\mathbf{P}_\nu \times \mu$, we can see [18, section 3.4] for the existence of such a measure for certain systems.

3. Proofs of Theorem A-Theorem D.

3.1. **Proof of Theorem A.** Recall that $\Gamma_{f_i} \subset X$ is a full measure set such that $f_i \Gamma_{f_i} = \Gamma_{f_i}$, for any $x \in \Gamma_{f_i}$, $\lambda(x, v, f_i) = \limsup_{n \to \infty} \frac{1}{n} \log \|dx f_i^n v\|$, for any $v \in X$.

**Lemma 3.1.** For all $i, j = 1, 2$, $i \neq j$, we have

$$\lambda(f_j x, dx f_i v, f_j) = \lambda(x, v, f_j) \text{ and } l_\alpha(x, f_i) = l_\alpha(f_j x, f_i).$$

**Proof.** By symmetry, we only prove the case for $i = 1, j = 2$. There exists $C > 0$ such that for any $x \in K, v \in X, C^{-1} \|v\| \leq \|dx f_1 v\| \leq C \|v\|$. Thus $C^{-1} \|dx f_2^n v\| \leq \|dx f_2 f_1 dx f_1^n v\| \leq C \|dx f_2^n v\|$. So

$$\limsup_{n \to \infty} \frac{1}{n} \log \|dx f_2 f_1 dx f_1^n v\| = \limsup_{n \to \infty} \frac{1}{n} \log \|dx f_2 f_1 dx f_1^n v\| = \limsup_{n \to \infty} \frac{1}{n} \log \|dx f_2^n v\|.$$

Similarly, there exists $C > 1$ such that for any $x \in K, C^{-1} \|dx f_1\| \leq C \|dx f_2^n v\|$. Thus $\|dx f_1^n\| \leq \|dx f_2 f_1 dx f_1^n\| \leq C \|dx f_2^n v\|$. Hence, $l_\alpha(x, f_1) = l_\alpha(f_2 x, f_1)$. 

\[\square\]
Corollary 3. For $i, j = 1, 2$,

(a) $\Gamma_{f_i}$ are $f_j$-invariant;
(b) $\lambda_k(x, f_i), m_k(x, f_i), \pi_k(x, f_i), k = 1, \ldots, r(x, f_i)$ are $f_j$-invariant;
(c) $d_x f_i E_k(x, f_i) = E_k(f_i x, f_i), k = 1, \ldots, r(x, f_i)$;
(d) $d_x f_j E_0(x, f_i) \subset E_0(f_j x, f_i)$.

Proof of Theorem A. For any point $x \in \Gamma_{f_1}$, let

$$X = E_1(x, f_1) \oplus E_2(x, f_1) \oplus \cdots \oplus E_{r(x, f_1)}(x, f_1) \oplus E_0(x, f_1)$$

be the decomposition for $f_1$. By Corollary 3,

$$d_x f_2 E_k(x, f_1) = E_k(f_2 x, f_1), k = 1, \ldots, r(x, f_1)$$

and $d_x f_2 E_0(x, f_1) \subset E_0(f_2 x, f_1)$. Restricted on $E_k(x, f_1)$ and $E_0(x, f_1)$, $\{d_x f_2^n\}$ is a cocycle on $K$ with respect to $f_2$. Now we use Multiplicative Ergodic Theorem (Theorem 2.1) for $E_k(x, f_1)$ and $E_0(x, f_1)$ to get subsets $\Gamma^k \subset \Gamma_{f_1}$ and $\Gamma^0 \subset \Gamma_{f_1}$, such that $\mu(\Gamma^0) = \mu(\Gamma^k) = 1$ for any $\mu \in \mathcal{M}$. Then for any $x \in \Gamma^k$ (resp. $x \in \Gamma^0$), after relabeling the subscript, if necessary, $E_{k, j_2}(x)$ and $E_{k, r(x, f_2) + 1}$ (resp. $E_{r(x, f_1) + 1, j_2}(x)$) have desired properties. We take $\Gamma = \cap_{k=1}^{\infty} \Gamma^k \cap \Gamma^0 \cap \Gamma_0$, then $f_1 \Gamma = \Gamma$ for each $i = 1, 2$. Then for any $x \in \Gamma$, $E_{j_1, j_2}(x), E_{r(x, f_1) + 1, j_2}(x), E_{j_1, r(x, f_2) + 1}$ and $E_0(x)$ have desired properties and

$$X = \bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} E_{j_1, j_2}(x) \bigoplus_{j_1=1}^{r(x, f_1)} E_{r(x, f_1) + 1, j_2}(x) \bigoplus_{j_1=1}^{r(x, f_1)} E_{j_1, r(x, f_2) + 1} \bigoplus E_0(x).$$

We now show (3) by claiming that for any $\epsilon > 0$, $s_1, s_2 \in \mathbb{Z}^+$, $1 \leq j_1 \leq r(x, f_1)$ and $i = 1, 2$, the set

$$A_\epsilon = \{x \in \Gamma : \exists v_{x} \in E_{j_1, j_2}(x) \text{ such that } \lambda(x, v_x, f_1^{s_1} \circ f_2^{s_2}) - s_1 \lambda_{j_1}(x, f_1) - s_2 \lambda_{j_2}(x, f_2) > 4\epsilon\}$$

satisfies $\mu(A_\epsilon) = 0$ for all $\mu \in \mathcal{M}$. Suppose it is not true. Then there exists a $\mu \in \mathcal{M}$ with $\mu(A_\epsilon) > 0$. Choose $C > 0$ such that the sets

$$A_1 = \{x \in A_\epsilon : \|d_x f_1^{s_1} v_x\| \geq C^{-1} \|v_x\| \exp(\lambda(x, v_x, f_1^{s_1} \circ f_2^{s_2}) - \epsilon), \forall n \in \mathbb{Z}^+\},$$

$$A_2 = \{x \in A_\epsilon : \|d_x f_2^{s_2} v_x\| \leq C \|v_x\| \exp(n(s_2 \lambda(x, v_x, f_2) + \epsilon), \forall n \in E_{j_1, j_2}(x), n \in \mathbb{Z}^+)$$

have measures larger than $\frac{1}{2} \mu(A_\epsilon)$. Then $\mu(A_1 \cap A_2) > 0$. By Poincaré Recurrence Theorem we can take $x \in A_1 \cap A_2$ such that there exists a sufficient large integer $n > \frac{2 \log C}{\epsilon}$ with $f_1^{s_1} x \in A_1 \cap A_2$ and

$$\|d_x f_1^{s_1} v_x\| \leq C \|v_x\| \exp(n(s_1 \lambda_{j_1}(x, f_1) + \epsilon), \forall v_x \in E_{j_1, j_2}(x).$$

Since $d_x f_1^{s_1} v_x \in E_{j_1, j_2}(f_1^{s_1} x)$ and $f_1^{s_1} x \in A_2$, $\forall v_x \in E_{j_1, j_2}(x),

\|d_x (f_1^{s_1} \circ f_2^{s_2}) v_x\| = \|d_x f_1^{s_1} f_2^{s_2} d_x f_1^{s_1} v_x\|

\leq C \|d_x f_1^{s_1} v_x\| \exp(n(s_2 \lambda(x, v_x, f_1) + \epsilon)

\leq C \|v_x\| \exp(n(s_1 \lambda(x, v_x, f_1) + s_2 \lambda(x, v_x, f_1) + 2\epsilon),$$

In particular, take $v = v_x$, then

$$\|d_x (f_1^{s_1} \circ f_2^{s_2}) v_x\| \leq C^{-1} \|v_x\| \exp(n(\lambda(x, v_x, f_1) - c) - \epsilon),$$

which contradicts the fact $x \in A_1$. Similar claim for the set

$$B_\epsilon = \{x \in \Gamma : \exists v_x \in E_{j_1, j_2}(x) \text{ such that } \lambda(x, v_x, f_1^{s_1} \circ f_2^{s_2}) - s_1 \lambda_{j_1}(x, f_1) - s_2 \lambda_{j_2}(x, f_2) < 4\epsilon\}$$
is also true. Then (3) follows by these two claims. Using the same idea, with some modification, we can prove (4). □

3.2. Proof of Theorem B.

Lemma 3.2. \( l_\alpha(f) \leq \nu(f_1)l_\alpha(f_1) + \nu(f_2)l_\alpha(f_2) \).

Proof. For any \( j = 1, 2 \), let \( n(\omega, f_j) = \sum_{m=0}^{n-1} \chi_{f_j, f_{\theta^m}} \), by Birkhoff Ergodic Theorem \( \int \lim_{n \to \infty} \frac{1}{n} n(\omega, f_j) dP_\nu = \nu(f_j) \), where \( \chi_{f_j} \) is the character function of \( f_j \).

For any \( x \in \Gamma_f, \omega \in \Omega \),
\[
\frac{1}{n} \log \sup_{x \in K} \|d_x f(n, \omega)\|_\alpha
\leq \frac{n(\omega, f_1)}{n} \frac{1}{n(\omega, f_1)} \log \sup_{x \in K} \|d_x f^{n(\omega, f_1)}_1\|_\alpha + \frac{n(\omega, f_2)}{n} \frac{1}{n(\omega, f_2)} \log \sup_{x \in K} \|d_x f^{n(\omega, f_2)}_2\|_\alpha.
\]

Let \( n \to \infty \), and take integral with respect to \( P_\nu \), by Random Subadditive Ergodic Theorem [9, Theorem 2.2], we get the desired result. □

Lemma 3.3. [7, Lemma 3.2] For any \( \epsilon > 0 \), there exists a measurable function \( Q : \Gamma \to [1, \infty) \) such that for any \( t_1, t_2 \in \mathbb{N}, x \in \Gamma, 0 \neq \nu \in E_{j_1, j_2} \), \( 1 \leq j_1 \leq l(x, f_1) \), \( 1 \leq j_2 \leq l(x, f_2) \),
\[
Q(x)^{-1} \|v\| \exp(t_1 \lambda_{j_1}(x, f_1) + t_2 \lambda_{j_2}(x, f_2) - 3(t_1 + t_2)\epsilon)
\leq \|d_x f^{t_1}_{f_1} \circ f^{t_2}_{f_2} v\|
\leq Q(x)^{-1} \|v\| \exp(t_1 \lambda_{j_1}(x, f_1) + t_2 \lambda_{j_2}(x, f_2) + 3(t_1 + t_2)\epsilon).
\]

Lemma 3.4. For any \( x \in \Gamma_f \cap \Gamma, 1 \leq j_1 \leq l(x, f_1), 1 \leq j_2 \leq l(x, f_2), P_\nu \text{-a.e.} \omega \in \Omega \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f(n, \omega)|_{E_{j_1, j_2}(x)}\| = \nu(f_1)\lambda_{j_1}(x, f_1) + \nu(f_2)\lambda_{j_2}(x, f_2).
\]

Proof. For any \( 0 \leq j_1 \leq l(x, f_1), 1 \leq j_2 \leq l(x, f_2) \), let \( n(\omega, f_j) = \sum_{m=0}^{n-1} \chi_{f_j, f_{\theta^m}} \), by Birkhoff Ergodic Theorem \( \int \lim_{n \to \infty} \frac{1}{n} n(\omega, f_j) dP_\nu = \nu(f_j) \).

For any \( \epsilon > 0, x \in \Gamma_f, \omega \in \Omega, v \in E_{j_1, j_2} \),
\[
\frac{1}{n} (\log Q(x)^{-1} + \log \|v\|) + \frac{n(\omega, f_1)}{n} \lambda_{j_1}(x, f_1) + \frac{n(\omega, f_2)}{n} \lambda_{j_2}(x, f_2) - 3\epsilon
\leq \frac{1}{n} \log \|d_x f(n, \omega) v\|
\leq \frac{1}{n} (\log Q(x) + \log \|v\|) + \frac{n(\omega, f_1)}{n} \lambda_{j_1}(x, f_1) + \frac{n(\omega, f_2)}{n} \lambda_{j_2}(x, f_2) + 3\epsilon.
\]

Let \( n \to \infty \), and take integral with respect to \( P_\nu \), by Random Subadditive Ergodic Theorem [9, Theorem 2.2], we get the desired result. □

By a similar argument, we have the following corollary.

Corollary 4. For any \( x \in \Gamma_f \cap \Gamma, 1 \leq j_1 \leq l(x, f_1), 1 \leq j_2 \leq l(x, f_2), P_\nu \text{-a.e.} \omega \in \Omega \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f(n, \omega)|_{E_{(x, f_1)}+1, j_2}\| \leq \nu(f_1)\lambda_{j_1}(x, f_1) + \nu(f_2)\lambda_{j_2}(x, f_2)
\]

and
\[
\lim_{n \to \infty} \frac{1}{n} \log \|d_x f(n, \omega)|_{E_{j_1, r(x, f_2)+1}}\| \leq \nu(f_1)\lambda_{j_1}(x, f_1) + \nu(f_2)\lambda_{j_2}.
Since $X$ is a separable Hilbert space, we can characterize $V_p(\cdot)$ in (2) by exterior power. Denote by $X^{(p)}$ the $p$-th exterior power space of $X$, i.e., the collection of completely antisymmetric elements of the Hilbert space of tensor product of $p$ copies of $X$. Let $\{\xi_i\}_{i=1}^\infty$ be a countable orthonormal base of $X$. Then
\[
\{\xi_{i_1} \wedge \cdots \wedge \xi_{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p < \infty\}
\]
is a basis of $X^{(p)}$. Define an inner product $(\cdot, \cdot)$ on $X^{(p)}$ by letting
\[
\langle \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}, \xi_{j_1} \wedge \cdots \wedge \xi_{j_p} \rangle = \begin{cases} 1, & \text{if } (i_1, \cdots, i_p) = (j_1, \cdots, j_p); \\ 0, & \text{otherwise.} \end{cases}
\]
This inner product is independent of the choice of the orthonormal basis $\{\xi_i\}_{i=1}^\infty$. Denote by $|\cdot|$ the norm on $X^{(p)}$ induced by this inner product. For any vectors $\xi_1, \cdots, \xi_p$ of $X$, $|\xi_1 \wedge \cdots \wedge \xi_p|$ is just the volume of the parallelootope formed by these vectors and its square is the classical Gram determinant of the vectors. So we have

**Lemma 3.5.** [13, Lemma 2.3] $V_p(\xi_1, \cdots, \xi_p) = |\xi_1 \wedge \cdots \wedge \xi_p|$, for $\xi_1 \wedge \cdots \wedge \xi_p \in X^{(p)}$.

Given a $C^1$ map $T : U \to X$ for some open subset $U$ of $X$, define for $x \in U$, $p \in \mathbb{N}$,
\[
(d_x T)^{(p)} : X^{(p)} \to X^{(p)}
\]
\[
\xi_1 \wedge \cdots \wedge \xi_p \mapsto (d_x T)\xi_1 \wedge \cdots \wedge (d_x T)\xi_p.
\]
It is true by Lemma 3.5 that
\[
|| (d_x T)^{(p)} ||= \sup_{||\xi||=1, 1 \leq i \leq p} V_p((d_x T)\xi_1, \cdots, (d_x T)\xi_p) = V_p(d_x T).
\]
As a consequence, we have for $f$ as in Theorem 2.3, if $X$ is a separable Hilbert space, then for $\mathcal{P}_p$-a.e $\omega \in \Omega$, $x \in \Gamma_f$ and $p \leq \sum_{j=1}^{r(x, f)} m_j(x, f)$,
\[
\lim_{n \to \infty} \frac{1}{n} \log ||(d_x f(n, \omega))^{(p)}|| = \sum_{k=1}^{p} \tilde{\lambda}_k(x, f),
\]
(9)
where $\{\tilde{\lambda}_k(x, f)\}$ are $\lambda_j(x, f)$’s repeated with multiplicity $m_j(x, f)$. Put
\[
\lambda_a = 0 > \max\{l_a(f_1), l_a(f_2)\},
\]
\[
E^{u}(x, f_1) := \oplus_{j=1}^{r(x, f_1)} E_j(x, f_1), \ E^{u}(x, f_2) := \oplus_{j=1}^{r(x, f_2)} E_j(x, f_2),
\]
and
\[
M(x, f_1) := \sum_{j=1}^{r(x, f_1)} m_j(x, f_1), \ M(x, f_2) := \sum_{j=1}^{r(x, f_2)} m_j(x, f_2).
\]
Similarly, let
\[
E^{u}(\omega, x, f) := \oplus_{j=1}^{r(x, f)} E_j(\omega, x, f) \text{ and } M(x, f) := \sum_{j=1}^{r(x, f)} m_j(x, f).
\]
Next, let $E^{u}(x) := \bigoplus_{j_1=1}^{r(x, f_1)} E_{j_1, j_2}^{(x, f_2)}(x)$, $M(x) := \dim E^{u}(x)$,
\[
E^{u}(x) := \bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} E_{j_1, j_2}(x) + E_{r(x, f_1)+1, j_2}(x) + E_{j_1, r(x, f_2)+1}(x)
\]
and $\overline{M}(x) := \dim E^{u}(x)$. Moreover, $d_{j_1}(x, f_1) = m_{j_1}(x, f_1) - \dim E_{j_1, r(x, f_2)+1}(x)$,
\[
d_{j_2}(x, f_2) = m_{j_2}(x, f_2) - \dim E_{r(x, f_1)+1, j_2}(x).
\]
Lemma 3.6. Let $f$ be a finitely generated random transformation of an infinite dimensional Banach space $X$ over $(\Omega, \mathcal{A}, P_\nu, \theta)$. Suppose $\mu \in \mathcal{M}$ and $(f, \mu)$ satisfies (H0-H1), then

$$\int \sum_{i=1}^{2} \sum_{\lambda_{jk}(x,f_i)>0} \nu(f_i)\lambda_{jk}(x,f_i)d\mu(x,f_i) \leq$$

$$\int \log |\det(D_x f|_{E^u(n,x)})| dP_\nu \times \mu \leq \int \sum_{i=1}^{2} \sum_{\lambda_{jk}(x,f_i)>0} \nu(f_i)\lambda_{jk}(x,f_i)m_{jk}(x,f_i) d\mu.$$

Proof. By Lemma 3.4 and Corollary 4, it is clear that $E^u(x) \subset E^u(\omega, x, f) \subset \bar{E}^u(x)$, in particular, $M(x) \leq M(x, f) \leq \overline{M}(x)$, for any $x \in \Gamma \cap \Gamma_f$, $P_\nu$-a.s. Therefore, combining with Birkhoff ergodic theory, we have

$$\int \log |\det(D_x f|_{E^u(n,x)})| dP_\nu \times \mu = \int \lim_{n \to \infty} \frac{1}{n} \log |(d_x f(n,\omega))^{\lambda_{M}(n,f)}| dP_\nu \times \mu \leq \int \lim_{n \to \infty} \frac{1}{n} \log |(d_x f(n,\omega))^{\lambda_{M}(n,f)}| dP_\nu \times \mu = \int \sum_{i=1}^{2} \sum_{\lambda_{jk}(x,f_i)>0} \nu(f_i)\lambda_{jk}(x,f_i)m_{jk}(x,f_i) d\mu.$$

Similarly,

$$\int \log |\det(D_x f|_{E^u(n,x)})| dP_\nu \times \mu = \int \lim_{n \to \infty} \frac{1}{n} \log |(d_x f(n,\omega))^{\lambda_{M}(n,f)}| dP_\nu \times \mu \geq \int \lim_{n \to \infty} \frac{1}{n} \log |(d_x f(n,\omega))^{\lambda_{M}(n,f)}| dP_\nu \times \mu = \int \sum_{i=1}^{2} \sum_{\lambda_{jk}(x,f_i)>0} \nu(f_i)\lambda_{jk}(x,f_i)d_{jk}(x,f_i) d\mu.$$

To prove Theorem B, we need to establish the relation between local covering numbers of tangent maps of $f$ and Lyapunov exponents of generators. For $A \subset \chi$, $\epsilon > 0$, define

$$r(A, \epsilon, d) = \inf \{ n \geq 1 : \text{ there exist } (x_1, \cdots, x_n) \in X^n \text{ and } (\epsilon_1, \cdots, \epsilon_n) \in \mathbb{R}^+$

such that $A \subset \cup_{i=1}^{n} B(x_i, \epsilon_i, \epsilon_i < \epsilon) \}.$$

For $T \in L(X)$, $\epsilon > 0$, let

$$R(T, \epsilon) := r(T(B_X), \epsilon, d),$$

where $B_X$ denotes the unit ball in $X$. Let $\beta > 0$. For $\omega \in \Omega$ and $x \in K$, define

$$\Delta_\omega(x, f) := \lim_{n \to \infty} \frac{1}{n} \log R(d_x f(n,\omega), e^{-n\beta})$$

whenever the limit exists. By [12, Proposition 3.4], the limit exists $P_\nu \times \mu$-a.s.
Lemma 3.7. Let $f$ be a finitely generated random transformation of an infinite dimensional Hilbert space $X$ over $(\Omega, \mathcal{A}, \mathbf{P}_\nu, \theta)$. Suppose $\mu \in \mathcal{M}$ and $(f_1, \mu)$ satisfies (H0-H1), and $0 < \beta < -\max\{l_\alpha(f_1), l_\alpha(f_2)\}$. Then for $\mathbf{P}_\nu \times \mu$-a.e. $(\omega, x)$,

$$
\Delta^\beta(x, f) \\
\leq \sum_{j_1=1}^{r(x, f_1)} \sum_{j_2=1}^{r(x, f_2)} [\nu(f_1)(\lambda_{j_1}(x, f_1) + \beta) + \nu(f_1)(\lambda_{j_1}(x, f_1) + \beta)]^+ m_{j_1, j_2}(x) \\
\quad + \sum_{j_2=1}^{r(x, f_2)} \nu(f_1)(\lambda_{j_1}(x, f_1) + \beta)^+ m_{r(x, f_1)+1, j_2} \\
\quad + \sum_{j_2=1}^{r(x, f_2)} \nu(f_2)(\lambda_{j_2}(x, f_2) + \beta)^+ m_{j_1, r(x, f_2)+1}.
$$

and

$$
\Delta^\beta(x, f) \geq \sum_{i=1}^{2} \sum_{k=1}^{r(x, f_i)} \nu(f_i)(\lambda_{j_k}(x, f_i) + \beta)^+ d_{j_k}(x, f_i). \quad (10)
$$

Proof. Let $0 < \beta < -\max\{l_\alpha(f_1), l_\alpha(f_2)\}$. For $x \in \Gamma_f \cap \Gamma_i$, $i = 1, 2$, let $r(x, f_i)$ be the maximal number such that $\lambda_{r(x, f_i)}(x, f_i) \geq -\beta$. By (H1) and Theorem A, consider the decomposition

$$
X = \bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} E_{j_1, j_2}(x) \bigoplus_{j_2=1}^{r(x, f_2)} E_{r(x, f_1)+1, j_2}(x) \bigoplus_{j_1=1}^{r(x, f_1)} E_{j_1, r(x, f_2)+1} \bigoplus E_\alpha(x)
$$

with $\pi_{j_1, j_2}, \pi_{r(x, f_1)+1, j_2}, \pi_{j_1, r(x, f_2)+1}, \pi_\alpha$ being the family of associated projections. Then

$$
B_E \subset \bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} \pi_{j_1, j_2} B_{j_1, j_2} \bigoplus_{j_2=1}^{r(x, f_2)} \pi_{r(x, f_1)+1, j_2} B_{r(x, f_1)+1, j_2} \bigoplus_{j_1=1}^{r(x, f_1)} \pi_{j_1, r(x, f_2)+1} B_{j_1, r(x, f_2)+1} \bigoplus \pi_\alpha B_{E_\alpha}.
$$

Consider $d_x f(n, \omega)(B_E)$. On the one hand,

$$
d_x f(n, \omega)(B_E) \subset \bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} \pi_{j_1, j_2} d_x f(n, \omega)(B_{j_1, j_2}) \bigoplus_{j_2=1}^{r(x, f_2)} \pi_{r(x, f_1)+1, j_2} d_x f(n, \omega)(B_{r(x, f_1)+1, j_2}) \bigoplus \pi_\alpha d_x f(n, \omega)(B_{E_\alpha}).
$$

So for $\beta < \gamma \leq -\max\{l_\alpha(f_1), l_\alpha(f_2)\}$, if we choose $n$ large such that

$$
\|d_x f(n, \omega)\|_\alpha < e^{-n\gamma}
$$

$$
< \left( \sum_{j_1=1}^{r(x, f_1)} \sum_{j_2=1}^{r(x, f_2)} |\pi_{j_1, j_2}| + \sum_{j_1=1}^{r(x, f_1)} |\pi_{j_1, r(x, f_2)+1}| + \sum_{j_2=1}^{r(x, f_2)} |\pi_{r(x, f_1)+1, j_2}| + |\pi_\alpha| \right)^{-1} e^{-n\beta},
$$

$$
r(d_x f(n, \omega)(B_E), e^{-n\gamma}) \leq \prod_{j_1=1}^{r(x, f_1)} \prod_{j_2=1}^{r(x, f_2)} r(d_x f(n, \omega)(B_{E_{j_1, j_2}}), e^{-n\gamma}).
$$

$$
\prod_{j_1=1}^{r(x, f_1)} r(d_x f(n, \omega)(B_{E_{j_1, r(x, f_2)+1}}), e^{-n\gamma}) \cdot \prod_{j_2=1}^{r(x, f_2)} r(d_x f(n, \omega)(B_{E_{r(x, f_1)+1, j_2}}), e^{-n\gamma}).
$$
For each $1 \leq j_1 \leq r(x, f_1)$, $1 \leq j_2 \leq r(x, f_2)$, we have
\[
r(d_x f(n, \omega)(B_{E_{j_1, j_2}}), e^{-n\gamma})
\leq \{(m_{j_1, j_2} \cdot \|d_x f(n, \omega)|_{E_{j_1, j_2}}\| \cdot e^n) + 1\}^{m_{j_1, j_2}},
\]
\[
r(d_x f(n, \omega)(B_{E_{j_1, r(x, f_2)+1}}), e^{-n\gamma})
\leq \{(m_{j_1, r(x, f_2)+1} \cdot \|d_x f(n, \omega)|_{E_{j_1, r(x, f_2)+1}}\| \cdot e^n) + 1\}^{m_{j_1, r(x, f_2)+1}},
\]
\[
r(d_x f(n, \omega)(B_{E_{r(x, f_1)+1, j_2}}), e^{-n\gamma})
\leq \{(m_{r(x, f_1)+1, j_2} \cdot \|d_x f(n, \omega)|_{E_{r(x, f_1)+1, j_2}}\| \cdot e^n) + 1\}^{m_{r(x, f_1)+1, j_2}},
\]
where $[a]$ denotes the integer part of the number $a$, $m_{j_1, j_2} = \dim(E_{j_1, j_2})$, $m_{r(x, f_1)+1, j_2} = \dim(E_{r(x, f_1)+1, j_2})$ and $m_{j_1, r(x, f_2)+1} = \dim(E_{j_1, r(x, f_2)+1})$. From this, we deduce that
\[
\lim_{n \to \infty} \frac{1}{n} \log R(d_x f(n, \omega), e^{-n\beta})
\leq \sum_{j_1=1}^{r(x, f_1)} \sum_{j_2=1}^{r(x, f_2)} \left[\nu(f_1)\left(\lambda_{j_1}(x, f_1) + \gamma\right) + \nu(f_1)\left(\lambda_{j_2}(x, f_2) + \gamma\right)\right]\cdot m_{j_1, j_2}(x)
\leq \sum_{j_2=1}^{r(x, f_2)} \nu(f_1)\left(\lambda_{j_1}(x, f_1) + \gamma\right)\cdot m_{r(x, f_1)+1, j_2}
\leq \sum_{j_1=1}^{r(x, f_1)} \nu(f_2)\left(\lambda_{j_2}(x, f_2) + \gamma\right)\cdot m_{j_1, r(x, f_2)+1}.
\]
Since $\gamma > \beta$ is arbitrary, we have the first inequality.

For the other inequality, let $\gamma$ be such that $\max\{-\lambda_{r(x, f_1)}, -\lambda_{r(x, f_2)}\} < \gamma < \beta$. Let $n$ be large such that
\[
\|d_x f(n, \omega)\|_\alpha < e^{-n\beta} < (2r(x, f_1) \times r(x, f_2) \sum_{j_1=1}^{r(x, f_1)} \sum_{j_2=1}^{r(x, f_2)} |\pi_{j_1, j_2}|)^{-1} e^{-n\gamma}.
\]
Since $\frac{1}{r(x, f_1) \times r(x, f_2)} (\bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} B_{j_1, j_2}) \subset B_E$, we have
\[
r(d_x f(n, \omega)(B_E), e^{-n\beta}) \geq \prod_{j_1=1}^{r(x, f_1)} \prod_{j_2=1}^{r(x, f_2)} S(d_x f(n, \omega)(B_{E_{j_1, j_2}}), e^{-n\gamma}),
\]
where $S(A, \delta)$ is the maximal number of subcollection of $A$ such that any two points of it has distance at least $\delta$. Now for $1 \leq j_1 \leq r(x, f_1)$, $1 \leq j_2 \leq r(x, f_2)$,
\[
S(d_x f(n, \omega)(B_{E_{j_1, j_2}}), e^{-n\gamma}) \geq \max\{(2e^{n\gamma} m_{j_1, j_2}^{-1} \|d_x f(n, \omega)|_{E_{j_1, j_2}}\|^{-1})^{m_{j_1, j_2}}, 1\}.
\]
From this we deduce that
\[
\lim_{n \to \infty} \frac{1}{n} \log R(d_x f(n, \omega), e^{-n\beta}) \geq \sum_{i=1}^{2} \sum_{k=1}^{r(x, f_i)} \nu(f_i)\left(\lambda_{j_k}(x, f_i) + \gamma\right)\cdot d_{j_k}(x, f_i).
\]
Since $\gamma < \beta$ is arbitrary, (10) is also proved. We are done.

\textit{Proof of Theorem B.} The main strategy of the proofs are making necessary modifications by comparing the dynamics of $f$ and the generators. We indicate that Lemma 3.7 is the key step to establish the relation between Lyapunov exponents of $f$ and those of its generators $f_1$ and $f_2$. So we will concentrate upon the necessary
modifications and omit most of the parallel arguments, for which we refer the reader to [12]. By ergodic decomposition theorem [17, Theorem 1.1], we restrict ourselves to the case \( \mu \in \mathcal{M}_f \). Let \( 0 < \beta < -\max\{l_\alpha(f_1), l_\alpha(f_2)\} \). For each \( k \in \mathbb{N} \), let

\[
A_k := \{ \omega \in \Omega : \frac{1}{n} \log \sup_{x \in K} \|d_x f(n, \omega)\|_\alpha < \frac{1}{2}(l_\alpha - \beta) \text{ for } n \geq k \}. \tag{11}
\]

\( \mathbf{P}_\nu(A_k) \) increases to 1 as \( k \) goes to infinity. For \( k \in \mathbb{N} \), define

\[
f_k(\omega, x) = \begin{cases} 
\log R(d_x f(k, \omega), e^{-k\beta}), & \text{if } \omega \in A_k, \ x \in K; \\
0, & \text{otherwise.}
\end{cases}
\]

Then by Lemma 3.7 and similar argument in [12, Lemma 3.5], for \( \mathbf{P}_\nu \times \mu \text{-a.e. } (\omega, x) \),

\[
\lim_{k \to \infty} \frac{1}{k} f_k(\omega, x) \leq \sum_{j_1=1}^{r(x,f_1)} \sum_{j_2=1}^{r(x,f_2)} |\nu(f_1)(\lambda_{j_1}(x, f_1) + \beta) + \nu(f_1)(\lambda_{j_1}(x, f_1) + \beta)| + m_{j_1,j_2}(x) \\
+ \sum_{j_2=1}^{r(x,f_2)} \nu(f_1)(\lambda_{j_1}(x, f_1) + \beta) + m_{r(x,f_1)+1,j_2} \\
+ \sum_{j_2=1}^{r(x,f_2)} \nu(f_2)(\lambda_{j_2}(x, f_2) + \beta) + m_{j_1,r(x,f_2)+1}.
\]

Let \( \mathcal{P} \) be a finite measurable partition of \( \Omega \times X \). For \( k \in \mathbb{N} \), define a function

\[
A(f_k, \mathcal{P}) : \Omega \times X \to \mathbb{R}
\]

by letting

\[
A(f_k, \mathcal{P})(\omega, x) = \sum_{P \in \mathcal{P}} \left( \sup_{x \in P} f_k(\omega, x) \right) \cdot \chi_P(x),
\]

where \( \chi_P \) is the characteristic function of the set \( P \). We have the following relation between \( f_k \) and \( A(f_k, \mathcal{P}) \) [12, Proposition 3.7]:

\[
\lim_{n \to \infty} \frac{1}{n} f_n(\omega, x) = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{n} A(f_n, \mathcal{P}_m^m)(\omega, x), \quad \mathbf{P}_\nu \times \mu \text{-a.e.} \tag{12}
\]

We begin with the selection of a sequence of “good” sets \( A_{k,i}^m \). Let \( 0 < \beta < -\frac{1}{5} \max\{l_\alpha(f_1), l_\alpha(f_2)\} \) be fixed. Let \( \{\mathcal{P}_m^m\}_{m \in \mathbb{N}} \) be as above so that (12) holds. Let \( \Delta \) be

\[
\Delta = \sum_{j_1=1}^{r(x,f_1)} \sum_{j_2=1}^{r(x,f_2)} |\nu(f_1)(\lambda_{j_1}(x, f_1) + \beta) + \nu(f_1)(\lambda_{j_1}(x, f_1) + \beta)| + m_{j_1,j_2}(x) \\
+ \sum_{j_2=1}^{r(x,f_2)} \nu(f_1)(\lambda_{j_1}(x, f_1) + \beta) + m_{r(x,f_1)+1,j_2} \\
+ \sum_{j_2=1}^{r(x,f_2)} \nu(f_2)(\lambda_{j_2}(x, f_2) + \beta) + m_{j_1,r(x,f_2)+1}.
\]

For \( m \in \mathbb{N} \), consider

\[
D^m := \{ (\omega, x) : \lim_{n \to \infty} \frac{1}{n} A(f_n, \mathcal{P}_m^m)(\omega, x) \leq \Delta + \frac{1}{2} \beta \}. 
\]
It is clear that $P_\nu \times \mu(D^m)$ tends to 1 as $m$ tends to infinity. For $k \in \mathbb{N}$, let
\[
D^m_k := \{(\omega, x) : \frac{1}{n}A(f^n, \mu^m_n)(\omega, x) \leq \Delta + \beta \text{ for } n \geq k\}.
\]
Then $P_\nu \times \mu(D^m_k)$ increases to $P_\nu \times \mu(D_m)$ as $k$ increases to infinity. For $k \in \mathbb{N}$, let $A_k$ be as in (11). For $l \in \mathbb{N}$, define
\[
A_{k,l} := \{\omega \in A_k : \|f(k, \omega)x - f(k, \omega)y - d_x f(\omega, k)(x - y)\| \leq e^{-k\beta} \epsilon \text{ for } x, y \in K, \|x - y\| \leq \epsilon, \epsilon < e_0/|l|\}.
\]
Since $f(k, \omega)$ is $C^1$ in a neighbourhood of $K$ and $K$ is compact, we see that $P_\nu(A_{k,l})$ increases to $P_\nu(A_k)$ as $l$ goes to infinity and hence
\[
\lim_{l \to \infty} \lim_{k \to \infty} \int_{\Omega \setminus A_{k,l}} \log^+ \sup_{x \in B(K, \epsilon_0)} \|d_x f_\omega\| \, dP_\nu(\omega) = 0.
\]
Fix $A_{k,l}$ and for $m \in \mathbb{N}$, define
\[
A^m_{k,l} := \{\omega, x) \in D^m_k : \omega \in A_{k,l}\}.
\]
We have
\[
\lim_{m \to \infty} \lim_{k \to \infty} \lim_{l \to \infty} \frac{1}{n} \log \mu\left(B^\omega_{A_{k,l}^m, n}(x, \epsilon)\right), \quad P_\nu \times \mu \text{-a.e.}
\]
where
\[
B^\omega_{A,l,n}(x, \epsilon) := \{y \in K : \text{ for } 0 \leq i \leq n - 1, F^i(\omega, x) \in A \text{ if and only if } F^i(\omega, y) \in A\}.
\]
The proof is thus finished by [12, Proposition 3.9], since there exists $N_0 \in \mathbb{N}$ such that $P_\nu \times \mu \text{-a.e.}$ $(\omega, x)$, the inequality
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mu\left(B^\omega_{A_{k,l}^m, n}(x, \epsilon)\right) \leq \Delta + 3\beta
\]
holds for any $k, l, m \geq N_0$. \hfill \Box

3.3. **Proof of Theorem C.**

*Proof of Theorem C. Again we will concentrate upon the necessary modifications and omit most of the parallel arguments, for which we refer the reader to [13].

Let $\eta$ be a partition subordinate to the unstable manifolds as in [13, Proposition 2.9]. Since, for any $n \in \mathbb{N}$, $\frac{1}{n} H_{P_\nu \times \mu}(F^{-n}\eta|\eta) = H_{P_\nu \times \mu}(F^{-1}\eta|\eta)$, we have
\[
H_{P_\nu \times \mu}(F^{-1}\eta|\eta) = \lim_{n \to \infty} \frac{1}{n} H_{P_\nu \times \mu}(F^{-n}\eta|\eta) \leq h_\mu(f, \eta) \leq h_\mu(f).
\]
Notice that
\[
H_{P_\nu \times \mu}(F^{-1}\eta|\eta) = -\int \log \mu^{\eta^n(x)}((F^{-1}\eta)^\omega(x)) \, dP_\nu \times \mu.
\]
Assume $\mu$ satisfies SRB property, then
\[
-\int \log \mu^{\eta^n(x)}((F^{-1}\eta)^\omega(x)) \, dP_\nu \times \mu = \int \log |\det(d_x f_\omega|F^{\nu}(\omega, x))| \, dP_\nu \times \mu.
\]
By Lemma 3.6 and Theorem A, we obtain the desired inequality.

Proof of Corollary 1. By Lemma 3.2, Lemma 3.6 and assumption (H3),

\[ E^u(x) := E^u(x, f_1) = E^u(x, f_2) = \bigoplus_{j_1=1}^{r(x, f_1)} \bigoplus_{j_2=1}^{r(x, f_2)} E_{j_1, j_2}(x). \]

Thus,

\[ \int \log |\det(D_x f_w|E^u(x, w, x))| dP \times \mu \geq \int \sum_{i=1}^{2} \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i) \lambda_{jk}(x, f_i) m_{jk}(x, f_i) d\mu. \]

Therefore, combining with Theorem C, we get the desired result.

3.4. Proof of Theorem D.

Proof of Theorem D. By [6], F is an extension of \( \sigma_F \) since we can define a map

\[ \hat{\pi} : \Omega \times K \rightarrow K, \hat{\pi}(\omega, x) = \{ f(n, \omega)(x) \}_{n \in \mathbb{N}} \]

such that \( \hat{\pi} \circ F = \sigma_F \circ \hat{\pi} \). Therefore, \( h(\sigma_F) \leq h(F) \).

By Abromov-Rohklin formula [1], for any invariant measure of \( F \) in the form of \( P_{\nu} \times \mu \), we have

\[ h_{P_{\nu} \times \mu}(F) = h_{P_{\nu}}(\theta) + h_{\mu}(f). \]

Since \( h_{P_{\nu}}(\theta) = -\sum_{i=1}^{2} \nu(f_i) \log \nu(f_i) \), by (5),

\[ h_{P_{\nu} \times \mu}(F) = -\sum_{i=1}^{2} \nu(f_i) \log \nu(f_i) + \int \sum_{i=1}^{2} \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i) \lambda_{jk}(x, f_i) m_{jk}(x, f_i) d\mu. \]

(13)

By the assumption on the measure with maximal entropy of \( F \) and the fact \( h(\sigma_F) \leq h(F) \), we obtain (7).

When \( \mu \) is ergodic, (13) becomes

\[ h_{P_{\nu} \times \mu}(F) = -\sum_{i=1}^{2} \nu(f_i) \log \nu(f_i) + \sum_{i=1}^{2} \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i) \lambda_{jk}(x, f_i) m_{jk}(x, f_i). \]

(14)

Define a function

\[ J : \Omega \rightarrow \mathbb{R}^+, J(\omega) = \sum_{i=1}^{2} \sum_{\lambda_{jk}(x, f_i) > 0} \nu(f_i) \lambda_{jk}(x, f_i) m_{jk}(x, f_i). \]

Then (14) becomes

\[ h_{P_{\nu} \times \mu}(F) = -\sum_{i=1}^{2} \nu(f_i) \log \nu(f_i) + \int_{\Omega} J dP_{\nu}(\omega). \]

(15)

Since \( P_{\nu} \times \mu \) is a measure with maximal entropy of \( F \), we can apply the variational principle of \( F \) as follows

\[ h(F) = \sup_{P_{\nu'}} \left\{ h_{P_{\nu'}}(\theta) + \int_{\Omega} J dP_{\nu'}(\omega) \right\} \]

(16)

where the supremum is taken over all \( P_{\nu'} = \nu''N \) is the product measure of some Borel probability measure \( \nu'' \) on \( \mathcal{F} \) with \( \nu'' = \nu''(f_i) \), and in the last line we use the
variational principle for the topological pressure $P(\theta, \eta_J)$ of $J$ with respect to $\theta$. From [22, Chapter 9], we get that,

$$P(\theta, J) = \log \left( \sum_{i=1}^{2} \exp \left( \sum_{\lambda_{jk}(x,f_i)>0} \lambda_{jk}(x,f_i)m_{jk}(x,f_i) \right) \right).$$

(17)

Therefore, by (16) and (17), $h(\sigma_f) \leq \log \left( \sum_{i=1}^{2} \exp \left( \sum_{\lambda_{jk}(x,f_i)>0} \lambda_{jk}(x,f_i)m_{jk}(x,f_i) \right) \right)$. Moreover, by [22, Theorem 9.16], $J$ has a unique equilibrium state which is the product measure defined by the measure on $\mathfrak{F}$ which gives the element $f_i$, $i = 1, 2$, measure

$$\nu_i = \frac{\sum_{\lambda_{jk}(x,f_i)>0} \exp(\lambda_{jk}(x,f_i)m_{jk}(x,f_i))}{\sum_{i=1}^{2} \sum_{\lambda_{jk}(x,f_i)>0} \exp(\lambda_{jk}(x,f_i)m_{jk}(x,f_i))}.$$ 

So any measure $\nu$ defined by above $\nu_i$ satisfies that the product measure $\mathbf{P}_\nu \times \mu$ is a measure with maximal entropy of $F$.

Furthermore, if $\mu$ is ergodic and $\mu(\{x \in X : f_1(x) = f_2(x)\}) = 0$, then we conclude that $\tilde{\pi}$ is one-to-one on a set of full $\mathbf{P}_\nu \times \mu$ measure. So

$$h_{\mathbf{P}_\nu \times \mu}(F) = h_{\tilde{\pi}(\mathbf{P}_\nu \times \mu)}(\sigma_f).$$

(18)

Moreover, by the variational principle for $\sigma_f$ we have that

$$h_{\tilde{\pi}(\mathbf{P}_\nu \times \mu)}(\sigma_f) \leq h(\sigma_f).$$

(19)

By (16), (18) and (19), $h(F) \leq h(\sigma_f)$. Together with the previous inequality $h(\sigma_f) \leq h(F)$ we have $h(F) = h(\sigma_f)$, and hence by (16) and (17), formula (8) holds. □

Acknowledgments. The first author would like to thank Professor Jon Aaronson and School of Mathematical Sciences of Tel Aviv University for hospitality during his visit there.

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Received September 2019; revised May 2020.

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