On the problem of mass-dependence of the two-point function of the real scalar free massive field on the light cone

Peter Ullrich

Institut für Informatik, TU München,
Boltzmannstraße 3, D-85748 Garching, Germany*

Ernst Werner

Institut für Physik, Universität Regensburg,
Universitätsstraße 31, D-93040 Regensburg, Germany†

Abstract

We investigate the generally assumed inconsistency in light cone quantum field theory that the restriction of a massive, real, scalar, free field to the nullplane $\Sigma = \{x^0 + x^3 = 0\}$ is independent of mass, but the restriction of the two-point function is mass-dependent (see, e.g.,9,16). We resolve this inconsistency by showing that the two-point function has no canonical restriction to $\Sigma$ in the sense of distribution theory. Only the so-called tame restriction of the two-point function, which we have introduced in14, exists. Furthermore, we show that this tame restriction is indeed independent of mass. Hence the inconsistency is induced by the erroneous assumption that the two-point function has a (canonical) restriction to $\Sigma$.

PACS numbers:
I. INTRODUCTION

Let \( \phi(x) \) be the real, scalar, free quantum field of mass \( m > 0 \), and let \( |0\rangle \) denote the (unique) vacuum state. The (Wightman) \( n \)-point functions (or vacuum expectation values) are defined by

\[
W_n(x_1, \ldots, x_n) = \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle \quad (n \in \mathbb{N}).
\]

Since \( \phi \) is a free field, the two-point function \( W_2(x,y) \) is explicitly given by

\[
W_2(x,y) = -iD_m^{(-)}(x-y),
\]

where \( D_m^{(-)}(x) \) is the negative frequency Pauli-Jordan function

\[
D_m^{(-)}(x) = \frac{-1}{i(2\pi)^3} \int \frac{d^3p}{2\omega(p)} e^{-i\omega(x^0+x \cdot p)}.
\]

Treating the field \( \phi \) in the framework of light cone quantization, the canonical commutator relation reads

\[
[\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{y})]_{x^+ = y^+ = 0} = \frac{1}{4i} \epsilon(x^- - y^-)\delta(x_\perp - y_\perp), \tag{I.1}
\]

where we have introduced light cone coordinates

\[
\tilde{x} = (x^+, \tilde{x}) = (x^+, x_\perp, x^-) = \kappa(x^0, x^1, x^2, x^3)
\]

by

\[
x^+ = (1/\sqrt{2})(x^0 + x^3), \quad x_\perp = (x^1, x^2), \quad x^- = (1/\sqrt{2})(x^0 - x^3).
\]

Furthermore, \( \tilde{\phi}(\tilde{x}) = \phi(\kappa^{-1}(\tilde{x})) \) denotes the transformed field. There is a generally alleged inconsistency in light cone quantum field theory (see for example \(^9,16^\)) which we explain now in detail: Using the commutator relation (I.1) one formally obtains the equation

\[
\langle 0 | \tilde{\phi}(\tilde{x})\tilde{\phi}(0) | 0 \rangle_{x^+ = 0} = \frac{1}{2\pi} \int_{p^+ > 0} dp^+ \frac{e^{-p^+x^-}}{2p^+} \delta(x_\perp), \tag{I.2}
\]

where the right-hand side obviously does not depend on the mass. Since \( W_2(x,y) = -iD_m^{(-)}(x - y) \), (I.2) should be equal to \(-i\) times the restriction of \( \tilde{D}_m^{(-)}(\tilde{x}) \) to \( x^+ = 0 \), where \( \tilde{D}_m^{(-)}(\tilde{x}) = D_m^{(-)}(\kappa^{-1}(\tilde{x})) \) denotes the negative frequency Pauli-Jordan function transformed to light cone coordinates. In \((3+1)\)-dimensional Minkowski space \( D_m^{(-)}(x) \) has the following explicit representation

\[
D_m^{(-)}(x) = \lim_{\xi \to 0} \frac{im^2}{4\pi^2} h(-m^2(x - i\xi)^2), \tag{I.3}
\]

where \( h(\zeta) = K_1(\sqrt{\zeta})/\sqrt{\zeta} \), \( K_1 \) is the modified Bessel function of second kind and the branch of \( \sqrt{\zeta} \) is taken to be positive for \( \zeta > 0 \). One seemingly obtains a contradiction by...
transforming formally the right-hand side of (I.3) to LC-coordinates and putting $x^+ = 0$, because then the right-hand side remains dependent on the mass $m$. However, as we will see later, the formal manipulations at the right hand side of (I.3) are ill-defined, since $D_m(-)(x)$ has no (canonical) restriction to \( \{ x^0 + x^3 = 0 \} \). More precisely, the operations of taking the limit $\xi \rightarrow 0$ ($\xi \in V^+$) (in $S'(\mathbb{R}^4)$ – the space of tempered distributions) and putting $x^+ = 0$ do not commute in (I.3).

II. NOTATIONS AND CONVENTIONS

Already in the introduction we have introduced light cone coordinates $\tilde{x} = \kappa(x)$ by using the Kogut-Soper convention\(^2\), where $x = (x^\mu)$ are Minkowski coordinates. As usual in light cone physics one writes

$$\tilde{x} = (x^+, \tilde{x}) = (x^+, x_\perp, x^-), \quad x_\perp = (x^1, x^2).$$

The Minkowski bilinear form $\langle x, y \rangle_M = x^\mu x_\mu = x^\mu g_{\mu\nu} x_\nu$, where \((g_{\mu\nu}) = \text{diag}(1, -1, -1, -1)\) is the usual Minkowski metric, transforms to the so-called LC-bilinear from

$$\langle \tilde{x}, \tilde{y} \rangle_L = \langle \kappa^{-1}(\tilde{x}), \kappa^{-1}(\tilde{y}) \rangle_M = x^+ y^- + x^- y^+ - x_\perp \cdot y_\perp \quad (x_\perp \cdot y_\perp = x^1 y^1 + x^2 y^2)$$

do not commute in (I.3).

If $U \subset \mathbb{R}^m$ is an open set, we denote by $\mathcal{D}(U)$ the (complex) vector space consisting of all (complex-valued) smooth, i.e., $C^\infty$ functions on $U$ with compact support. On $\mathcal{D}(U)$
one defines a topology which makes \( D(U) \) into a complete locally convex space\(^{7,11} \), the dual space \( D'(U) \) is called the space of distributions. One canonically identifies \( D(U) \) with a subspace of \( D'(U) \), i.e., we may assume \( D(U) \subset D'(U) \), and, with respect to the weak*-topology\(^{11} \) on \( D'(U) \), \( D(U) \) is even dense in \( D'(U) \). Along with \( D(R^m) \) one introduces the Schwartz space \( S(R^m) \) of rapidly decreasing functions and defines on \( S(R^m) \) a topology which makes \( S(R^m) \) into a Fréchet space. The dual space \( S'(R^n) \) is called the space of tempered distributions (or generalized functions)\(^{1,7,11} \). As in the case of distributions we may assume \( S(R^m) \subset S'(R^m) \), and \( S(R^m) \) is dense in \( S'(R^m) \) where \( S'(R^m) \) is endowed with the weak*-topology. Notice, that \( D(R^m) \subset S(R^m) \), but the topology of \( D(R^m) \) is finer than the subspace topology induced by \( S(R^m) \). One usually identifies the subspace of distributions \((\in D'(R^m)) \) which admit a linear, continuous extension to \( S(R^m) \) with \( S'(R^m) \).

III. CANONICAL RESTRICTION AND WAVE FRONT SET

In this section we summarize some well-known results from distribution theory\(^7 \) which will be needed in the sequel. Assume \( U \subset R^m \) and \( V \subset R^n \) are open sets and \( a \in U \) is fixed. Then the restriction of a (classical) function \( \phi(x,y) \) on \( U \times V \) to \( \{x = a\} \) can be viewed as the result of a pullback operation. More precisely, the restriction \( y \mapsto \phi(a, y) \) equals the pullback \( \iota_a^*\phi = \phi \circ \iota_a \), where \( \iota_a : V \to U \times V, y \mapsto (a, y) \). Hence, the restriction operation is a special case of the pullback operation which is generally defined by \( \phi \mapsto f^*\phi = \phi \circ f \) where \( f : X \to Y \) is some (fixed) map and \( \phi \) is a function on \( Y \); \( f^*\phi \) is called the pullback of \( \phi \) with respect to \( f \). Especially if \( f \) is smooth, i.e., \( C^\infty \), and \( X, Y \) are open sets then \( f^* \) maps \( D(Y) \) into \( D(X) \); moreover, \( f^* \) is linear and continuous. From distribution theory it follows\(^7 \) that it is impossible to extend \( f^* \) (sequentially) continuously to a linear map from \( D'(Y) \) into \( D'(X) \) unless conditions are imposed on \( f \). Only if \( f \) is a submersion, i.e., the differential \( d_x f \) is surjective for every \( x \in X \), a sequentially continuous extension of \( f^* \) exists\(^7 \). However, in the situation of the restriction operation the map \( \iota_a : V \to U \times V \) is by no means a submersion – this is easily seen by comparing the dimensions of the associated tangent spaces. Hence the extension of the restriction operation from classical functions to distributions is more subtle. The most crucial ingredient in this case is the so-called wave front set which takes control on the singularities of a distribution. For details on the wave front set we refer the reader to\(^7 \). The following theorem from\(^7 \) determines the right
subspace of $D'(Y)$ to which the pullback operation $f^*$ can be extended from $C^\infty(Y)$ when $f : X \rightarrow Y$ is generally a $C^\infty$ map. Thereby Hoermander introduces for any conic subset $\Gamma$ of $Y \times (\mathbb{R}^n \setminus 0)$ the subspace

$$D'_\Gamma = \{ \phi \in D'(Y) : \text{supp}(\phi) \subset \Gamma \}$$

which, however, carries a stronger topology than the subspace topology induced by $D'(Y)$. Furthermore, one also needs to define the subspace

$$N_f = \{(f(x), \eta) \in Y \times \mathbb{R}^n : (d_x f)^t \eta = 0\}$$

which is called the set of normals of $f$.

Theorem III.1 ($^7$, Thm. 8.2.4 ). Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open subsets and let $f : X \rightarrow Y$ be a $C^\infty$ map. Then the pullback $f^* \phi$ can be defined in one and only one way for all $\phi \in D'(Y)$ with $N_f \cap WF(\phi) = \emptyset$ so that $f^* \phi = \phi \circ f$ when $\phi \in C^\infty(Y)$. Moreover, for any closed conic set $\Gamma$ of $Y \times (\mathbb{R}^n \setminus 0)$ with $\Gamma \cap N_f = \emptyset$ we have a continuous map

$$f^* : D'_\Gamma(Y) \rightarrow D'_{f^*\Gamma}(X),$$

where

$$f^*\Gamma = \{(x, (d_x f)^t \eta) : (f(x), \eta) \in \Gamma\}.$$ 

From Theorem III.1 one immediately obtains

$$WF(f^* \phi) \subset f^* WF(\phi) \quad (\text{III.1})$$

whenever $N_f \cap WF(\phi) = \emptyset$. Since the pullback operation is a (contravariant) functor, i.e., $(g \circ f)^* = f^* \circ g^*$ ($g : V \rightarrow W$), one obtains from (III.1):

Corollary III.2. Let $f : X \rightarrow Y$ ($X, Y \subset \mathbb{R}^m$) be a $C^\infty$ diffeomorphism. Then

$$WF(f^* \phi) = f^* WF(\phi)$$

for all $\phi \in D'(Y)$. (Notice that $N_\lambda = Y \times \{0\}$.)
The definition of the canonical restriction of a distribution rests on the above theorem.
One just applies the theorem to the case when \( f \) is the map \( \iota_a : V \to U \times V \). Note that,
by this, not all distributions of \( \mathcal{D}'(U \times V) \) have a canonical restriction to \( \{ x = a \} \).

**Definition III.3.** Let \( U \subset \mathbb{R}^m, V \subset \mathbb{R}^n \) be open subsets and \( a \in U \). Then we say that \( \phi(x, y) \in \mathcal{D}'(U \times V) \) has a (canonical) restriction to \( \{ x = a \} \) if

\[
N_{\iota_a} \cap \text{WF}(\phi) = \emptyset,
\]

where \( \iota_a : V \to U \times V, y \mapsto (a, y) \), and call \( \phi|_{x=a}(y) = \phi(a, y) = \iota^*_a \phi(y) \in \mathcal{D}'(V) \) the canonical restriction of \( \phi(x, y) \) to \( \{ x = a \} \).

**Remark III.4.** (a) One easily computes the set of normals of \( \iota_a \):

\[
N_{\iota_a} = (\{ a \} \times \mathbb{R}^n) \times (\mathbb{R}^m \times \{ 0 \}) \tag{III.2}
\]

Hence \( \phi(x, y) \) has a canonical restriction to \( \{ x = a \} \) if and only if

\[
((\{ a \} \times \mathbb{R}^n) \times (\mathbb{R}^m \times \{ 0 \})) \cap \text{WF}(\phi) = \emptyset.
\]

(b) If \( \phi(x, y) \in \mathcal{D}'(U \times V) \) has a restriction to \( \{ x = a \} \) then also any \( \partial^\alpha_x \partial^\beta_y \phi(x, y) \) (\( \alpha, \beta \) multi-indices) has a restriction to \( \{ x = a \} \) — by \( \text{WF}(\partial^\alpha_x \partial^\beta_y \phi) \subset \text{WF}(\phi) \).

(d) Since the wave front set is a closed set one easily finds that if \( \phi(x, y) \in \mathcal{D}'(U \times V) \) has a restriction to \( \{ x = a \} \) then there is an open neighborhood \( U' \subset U \) of \( a \) such that \( \phi(x, y) \) has a restriction to \( \{ x = a' \} \) for all \( a' \in U' \).

**Remark III.5.** Condition (III.2) in the definition of the restriction of a distribution looks a little bit artificial, however, one can show (see, e.g.,\(^{15}\)) that \( \phi(x, y) \in \mathcal{D}'(U \times V) \) has a restriction to \( \{ x = a \} \) if and only if, sufficiently close to \( x = a \), \( \phi(x, y) \) is \( C^\infty \)-dependent on \( x \) as a parameter, i.e., there is an open neighborhood \( U' \subset U \) of \( a \) and a family \( \phi_x(y) \in \mathcal{D}'(Y) \) \((x \in U') \) such that \( U' \to \mathbb{C}, x \mapsto (\phi_x, g) \) is \( C^\infty \) for every \( g \in \mathcal{D}(Y) \) and

\[
(\phi(x, y), f(x)g(y)) = \int_{U'} (\phi_x, g)f(x)dx
\]

for all \( f(x) \in \mathcal{D}(U'), g(y) \in \mathcal{D}(V) \); if this is the case \( \phi(a, y) = \phi_a(y) \).

**Example III.6.** The Pauli-Jordan function

\[
D_m(x) = \frac{1}{i(2\pi)^3} \int d^4 p \epsilon(p^0) \delta(p^2 - m^2) e^{i(p \cdot x)} u \in \mathcal{S}'(\mathbb{R}^4)
\]
FIG. 1: The wave front set of $D_m^{(-)}$

has a canonical restriction to $\{x^0 = 0\}$. Moreover, $D_m(x)$ is even a fundamental solution of the Klein-Gordon operator, i.e.,

$$D_m(0, x) = 0, \quad (\partial_{x^0} D_m)(0, x) = \delta(x).$$

That $D_m(x)$ has a restriction to $\{x^0 = 0\}$ can either be seen by considering the wave front set of $D_m$ or, more explicitly, by showing that $D_m(x^0, x)$ is $C^\infty$-dependent on $x^0$ as a parameter, where

$$(D_m)_{x^0}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{\omega(p)} \sin(\omega(p)x^0)e^{-ip\cdot x}$$

Also the positive- and negative-frequency parts $D_m^{(\pm)}(x)$ have restrictions to $\{x^0 = \tau\}$ ($\tau \in \mathbb{R}$). In the wave front set of $D_m^{(-)}(x)$ is explicitly determined:

$$\text{WF}(D_m^{(-)}(x)) = W_0^{(-)} \cup W_+^{(-)} \cup W_-^{(-)},$$

with

$$W_0^{(-)} = \{(0, \xi) : \xi \in \Gamma^- \setminus \{0\}, \quad W_+^{(-)} = \{(\xi, +\lambda \xi) : \xi \in \Gamma^+ \setminus 0, \lambda > 0\}.$$

Thus $N_{\tau} \cap \text{WF}(D_m^{(-)}) = \emptyset$ for all $\tau \in \mathbb{R}$. Since $D_m^{(+)}(x) = -D_m^{(-)}(-x)$, and hence $\text{WF}(D_m^{(+)}) = -\text{WF}(D_m^{(-)})$, the same holds true for $D_m^{(+)}$. In Figure 1 we have illustrated the wave front set of $D_m^{(-)}$. Each element $(x, \xi)$ of $\text{WF}(D_m^{(-)})$ is represented by a pointed vector with base point $x$ and unit vector in direction of $\xi$.

So far we have only defined the restriction of a distribution $\phi(x, y)$ to a hyperplane of the form $\{x = a\}$ ($a \in \mathbb{R}^m$). However, any smooth submanifold of $\mathbb{R}^m$ can be described locally
in such a manner by using appropriate charts. Let $\Sigma_\tau = \{(1/\sqrt{2})(x^0 + x^3) = \tau \} \ (\tau \in \mathbb{R})$ and $\kappa$ the linear transformation to light cone coordinates, then $\Sigma_\tau = \{\kappa^{-1}(\bar{x}) : x^+ = \tau \}$. Hence we define:

**Definition III.7.** A distribution $\phi(x) \in \mathcal{D}'(\mathbb{R}^4)$ has a (canonical) restriction to $\Sigma_\tau \ (\tau \in \mathbb{R})$ if $\kappa_* \phi(\bar{x}) = (\phi \circ \kappa^{-1})(\bar{x})$ has a (canonical) restriction to $\{x^+ = \tau \}$. In this case we call $\phi|_{\Sigma_\tau} = \kappa_* \phi(0, \bar{x}) \in \mathcal{D}'(\mathbb{R}^3)$ the (canonical) restriction of $\phi$ to $\Sigma_\tau$.

**Remark III.8.** More generally, one can define the restriction of a distribution $\phi(x_1, \ldots, x_r) \in \mathcal{D}'(\mathbb{R}^4 \times \cdots \times \mathbb{R}^4)$ to $\Sigma_{\tau_1} \times \cdots \times \Sigma_{\tau_r}$ as the restriction of $\phi(\kappa^{-1}(\bar{x}_1), \ldots, \kappa^{-1}(\bar{x}_r))$ to $\{x_1^+ = \tau_1, \ldots, x_r^+ = \tau_r \}$, where $\bar{x}_i = (x_i^+, \bar{x}_i)$ and $\tau_i \in \mathbb{R} \ (i = 1, \ldots, r)$.

**Remark III.9.** If we denote by $\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}^4$, $\tilde{x} = (x_1, x^2, x^-) \mapsto (x^-/\sqrt{2}, x^1, x^2, -x^-/\sqrt{2})$ then $\lambda(\mathbb{R}^3) = \Sigma = \{x^0 + x^3 = 0 \}$ and $\lambda = \kappa^{-1} \circ \tilde{i}_0$, where $\tilde{i}_0(\tilde{x}) = (0, \tilde{x})$. Hence

$$\lambda^* \phi = (\kappa^{-1} \circ \tilde{i}_0)^* \phi = \tilde{i}^* (\kappa_* \phi) = \phi|_{\Sigma},$$

i.e., $\phi|_{\Sigma}$ is the pullback of $\phi$ with respect to $\lambda$. Notice that $\lambda$ is a smooth parametrization of $\Sigma$, but $\Sigma$ has infinitely many. However, if $\mu$ is another smooth parametrization of $\Sigma$ then $\lambda = \mu \circ (\mu^{-1} \circ \lambda)$, where $\mu^{-1} \circ \lambda$ is a $C^\infty$ diffeomorphism from $\mathbb{R}^3$ by $\mathbb{R}^3$. Hence, by Corollary III.2, $\lambda^* \phi = \exists$ if and only $\mu^* \phi$ exists, and in this case $\lambda^* \phi$ and $\mu^* \phi$ differ only by multiplication of a smooth function – the determinant of the Jacobi matrix of $\mu^{-1} \circ \lambda$.

**IV. NONEXISTENCE OF THE RESTRICTION OF THE TWO-POINT FUNCTION TO THE NULLPLANE**

Since we have explicit knowledge of the wave front set of $D_m^{(-)}$ it it easy now to show that the two-point function $W_2(x, y)$ has no (canonical) restriction to $\Sigma \times \Sigma$.

**Theorem IV.1.** Let $W_2(x, y) \in \mathcal{D}'(\mathbb{R}^4 \times \mathbb{R}^4)$ denote the two-point function of the real scalar free massive field. Then $W_2(x, y)$ has no canonical restriction to $\Sigma \times \Sigma = \{x^0 + x^3 = y^0 + y^3 = 0 \}$.

**Proof.** Since $W_2(x, y) = -i D_m^{(-)}(x - y)$ it is enough to show that $D_m^{(-)}(x)$ has no canonical restriction to $\Sigma = \{x^0 + x^3 = 0 \}$. By Remark III.9 we have to show that $N_\lambda \cap \text{WF}(D_m^{(-)}) \neq \emptyset$
where $N_\lambda$ is the set of normals of $\lambda : \mathbb{R}^3 \to \mathbb{R}^4$, $(x^1, x^2, x^-) \mapsto (x^-/\sqrt{2}, x^1, x^2, -x^-/\sqrt{2})$. One easily verifies that

$$N_\lambda = \{(x, \xi) \in \Sigma \times \mathbb{R}^4 : \xi^0 = \xi^3\}$$

and hence $N_\lambda \cap \text{WF}(D_m^{(-)}) \neq \emptyset$, which is easily seen by considering Figure 1 and Figure 2.

So far we have shown that $D_m^{(+)}(x)$ and $D_m^{(-)}(x)$ have no canonical restriction to $\Sigma = \{x^0 + x^3 = 0\}$. Since $\text{WF}(D_m^{(-)}) = -\text{WF}(D_m^{(+)})$, Supplementary we will show that this also holds true for the Pauli-Jordan function $D_m$ which is the sum of $D_m^{(+)}$ and $D_m^{(-)}$.

**Proposition IV.2.** The Pauli-Jordan function $D_m(x)$ has no canonical restriction to $\Sigma$.

**Proof.** We will show that $\text{WF}(D_m) = \text{WF}(D_m^{(+)}) \cup \text{WF}(D_m^{(-)})$; the assertion follows then from the proof of Theorem IV.1. Since $D_m = D_m^{(+)} + D_m^{(-)}$ we get only one direction, namely $\text{WF}(D_m) \subset \text{WF}(D_m^{(+)}) \cup \text{WF}(D_m^{(-)})$. To prove the other inclusion we may assume w.l.o.g. that $\text{WF}(D_m) \cap \text{WF}(D_m^{(-)}) \neq \emptyset$. Since $\mathcal{L}^+_+$, the restricted Lorentz group, operates transitively on $\text{WF}(D_m^{(-)})$ and since $D_m$ and hence $\text{WF}(D_m)$, is invariant under $\mathcal{L}^+_+$, one obtains $\text{WF}(D_m^{(-)}) \subset \text{WF}(D_m)$ (see also the proof of Thm. IX.48 in [10]). Furthermore, $\text{WF}(D_m^{(+)} \subset \text{WF}(D_m)$ since $\text{WF}(D_m^{(+)}(x)) = -\text{WF}(D_m^{(-)}(x)) \subset -\text{WF}(D_m)$ and $-\text{WF}(D_m(x)) = \text{WF}(D_m(-x)) = \text{WF}(D_m(x))$. 

\hfill $\Box$
V. THE TAME RESTRICTION OF THE TWO-POINT FUNCTION

The nonexistence of the restriction of the Pauli-Jordan function to \( \Sigma = \{x^0 + x^3\} \) is related to a fundamental problem in light cone quantum field theory where one describes the dynamics of a quantum field by using \( x^+ = (1/\sqrt{2})(x^0 + x^3) \) as “time”-evolution parameter. In this context it is essential to have well-defined fields for fixed \( x^+ = \text{const}. \). However, to carry out the standard construction of a free field for fixed time, one has to remain in a proper subspace of \( \mathcal{S} (\mathbb{R}^3)^8 \) which was considered as a fault of the theory\(^{12}\). In\(^{13}\) this problem was solved by introducing a new test function space \( \mathcal{S}_{\theta, -}(\mathbb{R}^3) \) on which the “restriction” of the free field can be defined and which determines the covariant field uniquely – we called this the “tame restriction” of the free field to \( \Sigma \). Now, since the covariant commutator relation of a free field \( \phi \) reads

\[
[\phi(x), \phi(y)] = -i D_m(x - y),
\]  

(V.1)

where \( D_m \) is the Pauli-Jordan function, we see that the problem of nonexistence of the real scalar field on \( \Sigma \) results in the nonexistence of the restriction of the Pauli-Jordan function to \( \Sigma \). In\(^{14}\) we introduced the tame restriction of a generalized function and computed it for the Pauli-Jordan function, where we obtained \((1/4)\delta(x_{\perp}) \otimes \epsilon(x^-)\). Hence, if we take the tame restrictions (to \( \Sigma \)) on both sides of (V.1) we arrive at the well-known commutator relation of light cone quantum field theory\(^3\). The same happens with the two-point function

\[
W_2(x, y) = \langle 0 | \phi(x)\phi(y) | 0 \rangle \text{ since}
\]

\[
\langle 0 | \phi(x)\phi(y) | 0 \rangle = -i D_m^{-}(x - y),
\]  

(V.2)

and \( D_m^{-} \) does not have a canonical restriction to \( \Sigma \). However, since \( D_m^{-} \) is a solution of the Klein-Gordon equation \((\Box + m^2)D_m^{-} = 0\) we know from\(^{14}\) that \( D_m^{-} \) admits a tame restriction to \( \Sigma \). In the sequel we will compute this tame restriction explicitly and show that it is independent of mass. Since the tame restriction of the free field to \( \Sigma \) is also independent of mass\(^{13,15}\) no inconsistency appears if we take the tame restrictions (to \( \Sigma \)) on both sides of (V.2). First of all we have to recall the definition of the tame restriction of a generalized function to \( \Sigma \) – for details see\(^{13,14}\).

**Definition V.1.** (a) Let \( \mathcal{S}_{\theta, +}(\mathbb{R}^n) = \bigcap_{k \geq 0} \{(p^+)^k g : g \in \mathcal{S}(\mathbb{R}^n)\} \) be the topological vector space endowed with the subspace topology induced by \( \mathcal{S}(\mathbb{R}^n) \); the dual space \( \mathcal{S}'_{\theta, +}(\mathbb{R}^n) \) is called the space of *squeezed generalized functions*. 

10
(b) Let \( S_{\partial_{\phi}}(\mathbb{R}^n) = \bigcap_{k \geq 0} \{ \partial^k x \cdot g : g \in S(\mathbb{R}^n) \} \) be the topological vector space endowed with the subspace topology induced by \( S(\mathbb{R}^n) \); the dual space \( S'_{\partial_{\phi}}(\mathbb{R}^n) \) is called the space of tame generalized functions.

The spaces \( S_{\phi}(\mathbb{R}^n) \) and \( S_{\partial_{\phi}}(\mathbb{R}^n) \) are Fréchet spaces. Furthermore, the Fourier transform, which is an isomorphism from \( S(\mathbb{R}^n) \) onto \( S(\mathbb{R}^n) \), maps \( S_{\partial_{\phi}}(\mathbb{R}^n) \) onto \( S_{\phi}(\mathbb{R}^n) \). Since we are using light cone coordinates to represent \( \Sigma \) (as \( \{ x^+ = 0 \} \) we also have to use the so-called \( \mathbb{L} \)-Fourier transformation \( \mathcal{F}_L : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n) \) defined by

\[
\mathcal{F}_L(f)(\vec{p}) = \int f(\vec{x}) e^{i(\vec{x} \cdot \vec{p})} d\vec{x},
\]

where \( (\vec{x}, \vec{p})_L = x^p - x^- p^+ - x_\perp \cdot p_\perp \). Since \( x^+ \) is the time variable in light cone physics we also introduce the spatial part of the \( \mathbb{L} \)-Fourier transformation

\[
\mathcal{F}_{L_{\perp}}^{\vec{x} \rightarrow \vec{p}}(f)(\vec{p}) = \int f(\vec{x}) e^{i(x^- p^+ - x_\perp \cdot p_\perp)} d\vec{x},
\]

and, in the special case of only one dimension,

\[
\mathcal{F}_L^{\rightarrow - p^+}(f)(p^+) = \int f(x^-) e^{ix^- p^+} dx^-.
\]

Clearly, \( \mathcal{F}_L \), \( \mathcal{F}_{L_{\perp}}^{\vec{x} \rightarrow \vec{p}} \) and \( \mathcal{F}_L^{\rightarrow - p^+} \) are isomorphisms from \( S(\mathbb{R}^x) \) onto \( S(\mathbb{R}^x) \) which map \( S_{\partial_{\phi}}(\mathbb{R}^x) \) onto \( S_{\phi}(\mathbb{R}^x) \) \( (x \) appropriately chosen) which extend canonically to sequentially continuous maps from \( S'(\mathbb{R}^x) \) onto \( S'(\mathbb{R}^x) \) respectively from \( S'_{\partial_{\phi}}(\mathbb{R}^x) \) onto \( S'_{\phi}(\mathbb{R}^x) \).

**Definition V.2 (Tame Restriction).** (a) A generalized function \( \phi(y, z, x^-) \in S'(\mathbb{R}^{m+n+1}) \) admits a tame restriction to \( \{ y = y_0 \} \) \( (y_0 \in \mathbb{R}^m) \) if there is an open neighborhood \( \Omega \subset \mathbb{R}^m \) of \( y_0 \) and a family \( (\phi_y)_{y \in \Omega} \) with \( \phi_y \in S_{\partial_{\phi}}(\mathbb{R}^{n+1}) \) \( (y \in \Omega) \) such that

\[
\phi(y, z, x^-) = \int \phi(y, \bar{\phi}_y, g(y, z, x^-)) = \int \phi(y, \bar{\phi}_y, g(y, z, x^-)) d\Omega \text{ for all } f(y) \in \mathcal{D}(\Omega) \text{ and } g(z, x^-) \in S_{\partial_{\phi}}(\mathbb{R}^{n+1}).
\]

In this case we call \( \phi|_{y=y_0}^* = \phi_{y_0} \in S_{\partial_{\phi}}(\mathbb{R}^{n+1}) \) the tame restriction of \( \phi \) to \( \{ y = y_0 \} \).

(b) A generalized function \( \phi(x_1, \ldots, x_r) \in S'(\mathbb{R}^{4r}) \) admits a tame restriction to \( \Sigma_{r_1} \times \cdots \times \Sigma_{r_r} \) \( (\Sigma_{r_i} = \{ x_i \in \mathbb{R}^4 : (1/\sqrt{2})(x_i^0 + x_i^3) = \tau_i \}, i = 1, \ldots, r) \) if \( \phi(\kappa^{-1}(\vec{x}_1), \ldots, \kappa^{-1}(\vec{x}_r)) \) admits a tame restriction to \( \{ x_i^+ = \tau_i, x_i^- = \tau_i \} \); in this case we call \( \phi|_{\Sigma_{r_1} \times \cdots \times \Sigma_{r_r}} = \phi(\kappa^{-1}(\vec{x}_1), \ldots, \kappa^{-1}(\vec{x}_r))|_{x_i^+=\tau_i, x_i^- = \tau_i} \) the tame restriction of \( \phi(x_1, \ldots, x_r) \) to \( \Sigma_{r_1} \times \cdots \times \Sigma_{r_r} \).
Proposition V.3. Let \( D_m^{(-)}(x) \in S'(\mathbb{R}^4) \) denote the negative-frequency Pauli-Jordan function. Then \( D_m^{(-)}(x) \) admits a tame restriction to \( \Sigma_\tau (\tau \in \mathbb{R}) \) and

\[
(D_m^{(-)}|_{\Sigma_\tau}, g) = \frac{-1}{i(2\pi)^3} \int_{p^+<0} \frac{d^3\hat{p}}{2|p^+|} (\mathcal{F}_{\mathbb{R}}(\xi \rightarrow \hat{p})g)(\hat{p})e^{i\omega(\hat{p})\tau}.
\]

for all \( g(\tilde{x}) \in S_{\partial_-}(\mathbb{R}^3) \)

Proof. Let \( f(x^+) \in S(\mathbb{R}) \) and \( g(\tilde{x}) \in S_{\partial_-}(\mathbb{R}^3) \). By definition

\[
((D_m^{(-)} \circ \kappa^{-1})(x^+, \tilde{x}), f(x^+)g(\tilde{x})) = \frac{-1}{i(2\pi)^3} (\delta_- \langle \hat{p}^2 - m^2, \hat{f}(\hat{p}^+)g(\hat{p}) \rangle) = \frac{-1}{i(2\pi)^3} \int_{p^+<0} \frac{d^3\hat{p}}{2|p^+|} \hat{f}(\omega(\hat{p}))g(\hat{p}) \quad (V.3)
\]

Since \( g \in S_{\partial_-}(\mathbb{R}^3) \) we have \( f(x^+)g(\hat{p}) \in L^1(\mathbb{R} \times \mathbb{R}^3, dx^+ \otimes d\hat{p}) \). Hence we can put \( \hat{f}(\omega(\hat{p})) = \int dx^+ f(x^+)e^{i\omega(\hat{p})x^+} \) in (V.3), and obtain

\[
((D_m^{(-)}(x^+), g) = \frac{-1}{i(2\pi)^n} \int_{p^+<0} \frac{d^n\hat{p}}{2|p^+|} g(\hat{p})e^{i\omega(\hat{p})x^+}.
\]

Thus the assertion follows since \( (D_m^{(-)}|_{\Sigma_\tau}, g) = ((D_m^{(-)}||_{\Sigma_\tau}, g) \).

Remark V.4. Notice that \( D_m^{(-)} \) is uniquely determined by its tame restriction to \( \Sigma_0 \).\(^{14}\)

Remark V.5. One can easily verify that if a generalized function \( \psi(x,y) \in S'(\mathbb{R}^4 \times \mathbb{R}^4) \) is of the form \( \psi(x,y) = \phi(x-y) \), where \( \phi \in S'(\mathbb{R}^4) \), and \( \phi \) has a tame restriction to \( \tau_1 - \tau_2 \) then \( \psi \) has a tame restriction to \( \Sigma_{\tau_1} \times \Sigma_{\tau_2} \) and \( \psi|_{\Sigma_{\tau_1} \times \Sigma_{\tau_2}} = \phi|_{\Sigma_{\tau_1} - \tau_2}(\tilde{x} - \tilde{y}) \). Notice that \( (\phi(x-y), f(x)g(y)) = (\phi, f * g') \), where \( * \) means convolution and \( g'(x) = g(-x) \).

Corollary V.6. Let \( \phi(x) \) be the real scalar free field of mass \( m > 0 \), and \( W_2(x,y) = \langle 0|\phi(x)\phi(y)|0 \rangle \) the associated two-point function. Then \( W_2(x,y) \) admits a tame restriction to \( \Sigma_\tau \times \Sigma_\tau = \{ x^+ = y^+ = \tau \} \) \( \tau \in \mathbb{R} \) and

\[
W_2(x,y)|_{\Sigma_\tau \times \Sigma_\tau} = \delta(\xi_\tau - \eta_\tau) \otimes G(x^- - y^-) \in S'_{\partial_-}(\mathbb{R}^3 \times \mathbb{R}^3),
\]

where \( G = (\mathcal{F}_{\mathbb{R}}(\xi \rightarrow \xi^+))^{-1}(\Theta(p^+)/p^+) \in S_{\partial_-}'(\mathbb{R}) \). In particular, the tame restriction of \( W_2(x,y) \) to \( \Sigma_\tau \times \Sigma_\tau \) is independent of mass.

Proof. Since \( W_2(x,y) = -iD_m^{(-)}(x-y) \) it is enough to show that \( D_m^{(-)} \) admits a tame restriction to \( \Sigma = \{ x^+ = 0 \} \) and \( D_m^{(-)}|_{\Sigma} = i\delta(\xi_\tau) \otimes G(x^-) \) (cf. Remark V.5); however, this follows immediately from Proposition V.3. \[\square\]
VI. CONCLUSION

To get rid of the (perturbative) zero-mode and restriction problem in light cone quantum field theory, we have introduced in\textsuperscript{13} the function space $\mathcal{S}_{\partial^{-}}(\mathbb{R}^{3})$ and its dual space – the space of tame generalized functions. The restriction problem, i.e., the problem that the real scalar free massive field has no canonical restriction to $\Sigma = \{x^{0} + x^{3} = 0\}$, manifests itself in the problem that the (positive-/ negative-frequency) Pauli-Jordan has no canonical restriction to $\Sigma$ in the sense of distribution theory. By using the so-called tame restriction of a tempered distribution, which we have already introduced in\textsuperscript{14}, we have seen that also the assumed inconsistency of the mass-dependence of the two-point function on $\Sigma$ can be resolved. Thus the result of this paper contributes to the philosophy (introduced in\textsuperscript{13}) that $\mathcal{S}_{\partial^{-}}(\mathbb{R}^{3})$ – instead of $\mathcal{S}(\mathbb{R}^{3})$ – is the right test function space when treating quantum fields on the null plane $\Sigma$.

\* Also at Institut für Physik, Universität Regensburg.; Electronic address: ullrichp@in.tum.de
\dag Electronic address: ernst.werner@physik.uni-regensburg.de

1 Bogolubov, N.N. et al.: General principles of quantum field theory. Kluwer Academic Publisher (1990).
2 Brodsky, S.J., Pauli, H.-C.: Quantum Chromodynamics and Other Field Theories on the Light Cone. Phys. Lett. C 301, 299 (1998), hep-ph/9705477.
3 Chang, S., Root, R.G. and Yan, T.: Quantum field theories in the infinite-momentum frame. I. Quantization of scalar and Dirac fields. Phys. Rev. D 7, 1133 (1973).
4 Ehrenpreis, L.: Solution of some problems of division, Part IV. American Jour. of Math. 82, 522 (1962).
5 Gårding, L., Malgrange, B.: Opérateurs différentiels partiellement hypoelliptiques. C. R. Acad. Sci. 247, 2083 (1958).
6 Gårding, L., Malgrange, B.: Opérateurs différentiels partiellement hypoelliptiques et partiellement elliptiques. Math. Scand. 9, 5 (1961).
7 Hörmander, L.: The analysis of linear partial differential operators I. Springer-Verlag, Berlin (1990).
8 Leutwyler, H., Klauder, J.R. and Streit L.: Quantum field theory on lightlike slabs. Nuovo Cimento A 66, 536 (1970).
9 Nakanishi, N., Yamawaki, K.: A consistent formulation of the null-plane quantum field theory, Nucl. Phys. B122, 15 (1977).
10 Reed, M. and Simon, B.: Methods of modern mathematical physics II. Academic Press, New York (1975).
11 Rudin, W.: Functional Analysis. McGraw Hill, Reprint (1990).
12 Schlieder, S. and Seiler E.: Some Remarks on the Null Plane Development of a Relativistic Quantum Field Theory. Commun. Math. Phys. 25, 62 (1972).
13 Ullrich, P.: On the restriction of quantum fields to a lightlike surface. J. Math. Phys. 45, 3109 (2004).
14 Ullrich, P.: Uniqueness in the characteristic Cauchy problem of the Klein-Gordon equation and tame restrictions of generalized functions. Submitted (2004), math-ph/0408022.
15 Ullrich, P.: A Wightman approach to light cone quantum field theory. Preprint (2004).
16 Yamawaki, K.: Zero Mode and Symmetry Breaking on the Light Front, Proc. of Int. Workshop New Nonperturbative Methods and Quantization on the Light Cone, Les Houches, France (1997), hep-th/9707141.

This is indeed true for every solution of the Klein-Gordon equation since the Klein-Gordon operator is hypoelliptic with respect to $x^0$, see$^4$–$^6$. 