ON SOME TOUCHDOWN BEHAVIORS OF THE GENERALIZED MEMS DEVICE EQUATION

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Abstract. We study the quenching behaviors for the generalized microelectromechanical system (MEMS) equation $u_t - \Delta u = \lambda \rho(x) f(u)$, $0 < u < A$ ($A = 1$ or $+\infty$), in $\Omega \times (0, +\infty)$, $u(x, t) = 0$ on $\partial \Omega \times (0, +\infty)$, $u(x, 0) = u_0(x) \in [0, A)$ in $\Omega$, where $\lambda > 0$, $\Omega \subset R^N$ is a bounded domain, $0 \leq \rho(x)$ $C^\alpha(\Omega)$, for some constant $0 < \alpha < 1$, $0 < f \in C^2((0, A))$ such that $f'(s) \geq 0$, $f''(s) \geq 0$ for any $s \in [0, A)$ and $u_0$ is smooth, $u_0 = 0$ on $\partial \Omega$. It is well known that quenching does occur and corresponds to a touchdown phenomenon. We establish an interesting quenching rate, and based on which we then prove that touchdown cannot occur at zero points of $\rho(x)$ or at the boundary of $\Omega$, without the assumption of compactness of the touchdown set.

1. Introduction. Let $\Omega$ be a bounded domain of class $C^{2+\nu}$ for some $\nu \in (0, 1)$ in $\mathbb{R}^N$. Let

$$0 \leq \rho \in C^{\alpha}(\Omega) \quad \text{for some constant } 0 < \alpha < 1 \text{ and } \rho \not\equiv 0 \text{ in } \Omega \quad (1)$$

and let $f > 0$ satisfy one of the following two conditions:

(R): $f$ is increasing, convex, of $C^2$ class on $[0, +\infty)$ with $f(0) = 1$, $\lim_{s \to +\infty} f(s) = +\infty$, and $\lim_{s \to +\infty} \frac{f'(s)}{f(s)} = +\infty$;

(S): $f$ is increasing, convex, of $C^2$ class on $[0, 1)$ with $f(0) = 1$, and $\lim_{s \to 1^-} f(s) = +\infty$.

For convenience, we introduce

$$A = \begin{cases} 1, & \text{if } f \text{ satisfies } (S), \\ +\infty, & \text{if } f \text{ satisfies } (R). \end{cases}$$

(2)

In this paper we consider the generalized MEMS equation

$$\begin{cases} u_t - \Delta u = \lambda \rho(x) f(u) & (x, t) \in \Omega \times (0, T), \\ u(x, 0) = u_0(x) \in [0, A) & x \in \Omega, \\ u(x, t) = 0 & (x, t) \in \partial \Omega \times (0, T), \end{cases}$$

(3)

where $\lambda > 0$ and the initial data $u_0(x)$ is smooth satisfying $\Delta u_0 + \rho(x) f(u_0) \geq 0$.

When $f(u) = (1 - u)^{-2}$ and $u_0 \equiv 0$, (3) reduces to the evolution Micro-electromechanical systems (MEMS) equations which were studied extensively in [2, 3, 5, 6, 10, 11, 14]. For the details of background and derivation of MEMS model, one

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can refer to [16]. The equation (3) with \( f(u) = (1 - u)^{-p} \) \((p > 0)\) was studied in [9, 13, 17]. We know by the standard parabolic theory or by [12] that there exists a unique classical solution \( u \) of (3) in a short time interval. Also, by the strong maximum principle, we can see \( u > 0 \) in \( \Omega \) for \( t > 0 \). Moreover, \( u \) can be continued as long as \( \sup_{x \in \Omega} u(x,t) < A \). It is well known (see, e.g., [1, 18] and the references therein) that for any given \( \rho \), there exists a critical value \( \lambda^* > 0 \) such that if \( \lambda \in (0, \lambda^*) \), the solution to (3) is global with \( u_0 = 0 \); while for \( \lambda > \lambda^* \), the solution to (3) will reach the value \( A \) at finite time \( T \), i.e. the so called quenching or touchdown phenomenon occurs. The more precise definition of the quenching time \( T \) is

\[
T = \sup \{ t > 0 : \|u(\cdot, s)\|_\infty < A, \forall s \in [0, t) \}. 
\]  

(4)

The corresponding quenching set is defined as

\[
\Sigma = \{ x \in \overline{\Omega} : \exists (x_n, t_n) \in \Omega \times [0, T) s.t. \ x_n \to x, t_n \to T, u(x_n, t_n) \to 1 \}. 
\]  

(5)

A point \( x = x_0 \in \Sigma \) is called a quenching point. It is essential to understand the quenching phenomenon, such as the quenching set \( \Sigma \), the quenching points, the rate of the quenching solution. Some interesting results have been obtained in several recent works (see for example [6, 7, 10] and the references therein).

Note that a long-standing open problem is to decide how to describe the quenching points or quenching set. In [6, 10], under the assumption that the quenching set is a compact subset of \( \Omega \), it is shown that \( x_0 \) is not a quenching point if \( \rho(x_0) = 0 \). On the other hand, the compactness assumption was proved in [10] by adapting a moving plane argument from [4, 8] when \( \rho \) is constant, or more generally, when \( \rho \) is nonincreasing as one approaches the boundary. In this paper, inspired by [9], we study the problem (3) and give some results about the quenching points and the quenching set, as well as for the case of general \( \rho \), in any space dimension. For simplicity, we denote \( d(x) = \text{dist}(x, \partial \Omega) \), \( x \in \overline{\Omega} \), the function distance to the boundary.

Our main conclusions is the following.

**Theorem 1.1.** Assume (1), (R) or (S) and let the solution \( u \) of problem (3) be such that \( T < +\infty \). Then there exist \( c_1 > 0 \), \( \varepsilon > 0 \) (independent of \( x, t \)) and the function \( h(z) \), \( C(y) \) such that

\[
h(0) = 0, h' \geq 0, \inf_{0 \leq z \leq 1} \frac{h(z)h''(z)}{(h'(z))^2} > 0, \inf_{y \geq z \geq 0} \frac{z}{h(z)} \geq C(y) > 0, \text{ for any } y \in (0, +\infty),
\]

and

\[
\int_u^A \frac{ds}{f(s)} \geq \varepsilon h(c_1 d(x))(T - t), \quad x \in \Omega, \ 0 < t < T.
\]  

(6)

(7)

**Theorem 1.2.** Assume (1), (R) or (S) and let the solution \( u \) of problem (3) be such that \( T < +\infty \). If \( x_0 \in \Omega \) is such that \( \rho(x_0) = 0 \), then \( x_0 \) is not a quenching point.

As a consequence of Theorem 1.1 and of suitable comparison arguments, we are able to obtain two further criteria for the quenching set to be compact. We note that we do not require any convexity of the domain.

**Theorem 1.3.** Assume (1), (S) and let the solution \( u \) of problem (3) be such that \( T < +\infty \). Assume that \( \int_0^1 f(s)ds < +\infty \). Then quenching does not occur near the boundary, i.e., \( \Sigma \subset \Omega \).
Theorem 1.4. Assume (1), (R)(or (S)) and let the solution $u$ of problem (3) be such that $T < +\infty$. Assume that $p(x) = o(h(d(x)))$ as $d(x) \to 0$, where $h$ is defined as in Theorem 1.1. Then quenching does not occur near the boundary, i.e., $\Sigma \subset \Omega$.

As a consequence of our results, we see that the latter has to be part of the positive set of the function $\rho$. In fact it would be desirable to gain further information about the structure of the quenching set, but this seems a difficult mathematical problem for nonconstant $\rho$, even in one space dimension.

2. Proof of Theorem 1.1. Theorem 1.1 will be proved via a nontrivial modification of the method in [4, 8]. Consider the function

$$J(x, t) = u_t - \varepsilon a(x)g(u),$$

where $\varepsilon > 0$ and $a(x) \in C^2(\Omega)$ is an auxiliary function such that $a \geq 0$, $a|_{\partial\Omega} = 0$, hence $J = 0$ on $\partial\Omega \times (0, T)$. The function $g(u) \geq 0$ will be decided later. Setting $v = u_t$, we see that $v$ satisfies

$$\begin{cases}
v_t - \Delta v = \lambda \rho(x)f'(u)v & (x, t) \in \Omega \times (0, T), \\
v(x, 0) = \Delta u_0 + f(u_0) \geq 0 & x \in \Omega, \\
v(x, t) = 0 & (x, t) \in \partial\Omega \times (0, T).
\end{cases}$$

By maximum principle, $v = u_t > 0$ for $(x, t) \in \Omega \times (0, T)$. Let $z(x, t)$ be the solution of

$$\begin{cases}
z_t - \Delta z = 0 & (x, t) \in \Omega \times (0, T), \\
z(x, 0) = \Delta u_0 + \lambda \rho(x)f(u_0) \geq 0 & x \in \Omega, \\
z(x, t) = 0 & (x, t) \in \partial\Omega \times (0, T).
\end{cases}$$

It follows from the Hopf lemma and the strong maximum principle that for any $t_0 \in (0, T)$ there exists a constant $c(t_0) > 0$ such that $z(x, t) \geq c(t_0)d(x)$ for all $(x, t) \in \Omega \times [t_0, T)$. Then the comparison principle gives that

$$u_t \geq z \geq c(t_0)d(x) \quad \text{on} \quad \Omega \times [t_0, T).$$

(9)

For any $0 < t_0 < T$, we next claim that there exist $\varepsilon > 0$, suitable $a(x)$ and $g$, such that

$$J(x, t) = u_t - \varepsilon a(x)g(u) \geq 0 \quad \text{for all} \quad (x, t) \in \Omega \times [t_0, T).$$

(10)

Indeed, direct calculations imply that

$$J_t - \Delta J - \lambda \rho(x)f(u)J = \varepsilon \left( \lambda a(x)\rho(x) \left( f'(u)g(u) - f(u)g'(u) \right) + g(u)\Delta a \right) + a(x)g''(u) \left( \frac{g'(u)}{a(x)g''(u)} \right)^2$$

$$+ \left( \Delta u \right) \left( \left( f'(u)g(u) - f(u)g'(u) \right) + g(u)\Delta a \right) + a(x)g''(u)\left( \frac{g'(u)}{a(x)g''(u)} \right)^2$$

$$= \varepsilon \left( \lambda a(x)\rho(x) \left( f'(u)g(u) - f(u)g'(u) \right) + g(u)\Delta a \right)$$

$$+ a(x)g''(u) \left( \frac{g'(u)}{a(x)g''(u)} \right)^2$$

$$:= \varepsilon R,$$
Let $\Omega_1 \subset \Omega$ such that $\Omega_1$ is smooth. We next introduce a suitable harmonic function $\phi$ as follows

$$
\begin{aligned}
-\Delta \phi &= 0 \quad x \in \Omega \setminus \Omega_1, \\
\phi &= 0 \quad x \in \partial \Omega, \\
\phi &= 1 \quad x \in \partial \Omega_1.
\end{aligned}
$$

Hence the strong maximum principle and the Hopf lemma give $0 < \phi < 1$ in $\Omega \setminus \Omega_1$ and $c_1 d(x) \leq \phi(x) \leq c_2 d(x)$ in $\Omega \setminus \Omega_1$, for some constants $c_1, c_2$. Moreover $\phi$ is smooth. Set $a(x) = h(\phi)$ for $x \in \Omega \setminus \Omega_1$, where $h \geq 0$ satisfies

$$
\begin{aligned}
h(0) &= 0, h' \geq 0, \inf_{0 \leq z \leq 1} \frac{h(z)h''(z)}{(h'(z))^2} > 0, \inf_{M \geq z \geq 0} \frac{z}{h(z)} > 0, \quad (12)
\end{aligned}
$$

where $M := c_2 \sup_{x \in \Omega} d(x)$. We now choose

$$
\begin{aligned}
g(u) &= f(u) + \alpha(f(u)) \quad (13)
\end{aligned}
$$

such that

$$
\begin{aligned}
\alpha(1) &= 1, 1 \leq \alpha(f) \leq f, \alpha'(f) \geq 0, \alpha''(f) \not\equiv 0 \\
\text{and} \\
\inf_{1 \leq y < + \infty} \frac{y \alpha''(y)}{1 + \alpha'(y)} + \inf_{0 \leq z < A} \frac{f''(s)f(s)}{(f'(s))^2} &\geq 2 \left( \inf_{z \in [0,1]} \frac{h(z)h''(z)}{(h'(z))^2} \right)^{-1}. \quad (14)
\end{aligned}
$$

Hence $\alpha'(f) \leq 1$ and then

$$
\begin{aligned}
\inf_{s \in [0, A]} \frac{f(s) + \alpha(f(s))}{f(s) + f(s)\alpha'(f(s))} &\geq \left( \inf_{s \in [0,A]} \frac{f(s)\alpha''(f(s))}{1 + \alpha'(f(s))} \right) \left( \inf_{s \in [0,A]} \frac{f(s)f''(s)}{(f'(s))^2} \right) := \mathcal{A} \geq \left( \inf_{z \in [0,1]} \frac{h(z)h''(z)}{(h'(z))^2} \right)^{-1}. \quad (15)
\end{aligned}
$$

Then

$$
\begin{aligned}
g'(u) &= f'(u) + \alpha'(f(u))f'(u), g''(u) = \alpha''(f(u))(f'(u))^2 + (1 + \alpha'(f(u)))f''(u). \quad (16)
\end{aligned}
$$

Omitting the variables $x, u$ without any confusion, we then have

$$
\begin{aligned}
f'g - fg' &= (\alpha(f) - \alpha'(f)f)f'.
\end{aligned}
$$

Since $\frac{d}{df}(\alpha(f) - \alpha'(f)f) = -\alpha''(f)f \not\equiv 0$, we get

$$
\begin{aligned}
\alpha - \alpha'f &\geq 0 \quad (17)
\end{aligned}
$$

and hence

$$
\begin{aligned}
f'g - fg' &\geq 0 \quad \text{in } \Omega.
\end{aligned}
$$

Combined with $\nabla a = h'\nabla \phi$, $\Delta a = h''|\nabla \phi|^2 + h'\Delta \phi$, it hence follows that

$$
\begin{aligned}
R &\geq (g(u)\Delta a - \frac{(g'(u))^2|\nabla a|^2}{a(x)g''(u)}) \geq 0 \quad \text{in } \Omega \setminus \Omega_1. \quad (18)
\end{aligned}
$$

On the other hand, since $\phi, a \in C^2(\Omega \setminus \Omega_1)$, $\partial \Omega_1$ is smooth, and $\phi = 1, a = h(1)$ on $\partial \Omega_1$, the function $\phi, a$ can be extended in $\Omega_1$ such that $a \in C^1(\Omega) \cap C^2(\Omega)$, $h(1) \geq a > 0$ in $\Omega$ and furthermore $a|_{\Omega_1} > 0$,

$$
\begin{aligned}
h(c_1 d(x)) \leq a(x) \leq h(c_2 d(x)) \quad \text{for all } x \in \Omega. \quad (19)
\end{aligned}
$$

Since $\rho \geq 0$ and $\rho \not\equiv 0$ is continuous, without loss of generality, we may pick a subset $\Omega_1 \subset \Omega$ such that

$$
\begin{aligned}
\sigma_1 := \inf_{x \in \Omega_1} \rho(x) > 0, \\
\sigma_2 := \inf_{x \in \Omega_1} a(x) > 0.
\end{aligned}
$$


We denote now $\Sigma := \Omega \times [t_0, T)$. Consider the three subregions of $\Sigma$ as follows:

$$
\Sigma_1 := (\Omega \setminus \overline{\Omega}_1) \times [t_0, T),
\Sigma_2 := \{(x, t) \in \overline{\Omega}_1 \times [t_0, T) : u \geq \beta\},
\Sigma_3 := \{(x, t) \in \overline{\Omega}_1 \times [t_0, T) : u \leq \beta\},
$$

for some $\beta < A$. It is clear that (18) indicates that

$$
J_t - \Delta J - \lambda \rho(x) f'(u) J \geq 0 \quad \text{in} \quad \Sigma.
$$

Next it follows from (13) and (16) that

$$
|g\Delta a| \leq Cg, |g'|\nabla a| \leq Cg',
$$

for some positive constant $C$. Consequently, from (11), (13), (16), (20) and (21), we obtain that there exist $C_1 > 0$ and $C_2 \geq 0$ such that in $\Sigma_2$

$$
J_t - \Delta J - \lambda \rho(x) f'(u) J \geq \varepsilon \left(\lambda \sigma_1 \sigma_2 (f' g - fg') - C_1 f - \frac{C_2 (g')^2}{g''} \right) = \varepsilon (\alpha - \alpha') (\lambda \sigma_1 \sigma_2 - \frac{C_1}{f (\alpha - \alpha' f)} - \frac{C_2}{g''}) f',
$$

where $I = \frac{t+a}{1+t^a} \left( \frac{\alpha''}{f'} + \frac{f''}{f'} \right) \frac{(\alpha - \alpha') f'}{f + a}$.

Since we know that $\lim_{s \to A} \frac{f(s)}{f(s)} = +\infty$, due to (14) and (17), we see that

$$
\lim_{s \to A} \frac{f'(s)}{f(s)} (\alpha(f(s)) - \alpha'(f(s)) f(s)) = +\infty.
$$

Similarly, by using (14) and (15), we get that

$$
I \geq A \cdot (\alpha(f(s)) - \alpha'(f(s)) f(s)) \frac{f'(s)}{f(s)} \to +\infty \quad \text{as} \quad s \to A.
$$

Therefore $\lim_{s \to A} \left( \lambda \sigma_1 \sigma_2 - \frac{C_1}{f (\alpha(f(s)) - \alpha'(f(s)) f(s))} - \frac{C_2}{g''} \right) = \lambda \sigma_1 \sigma_2 > 0$, which implies that

$$
J_t - \Delta J - \lambda \rho(x) f'(u) J \geq 0 \quad \text{in} \quad \Sigma_2,
$$

for some $\beta \in (0, A)$.

We next control $J$ on $\Sigma_3$. Since (9) and (19) give

$$
J = u_t - \varepsilon a(x) g(u) \geq c(t_0) d(x) - \varepsilon h(c_2 d(x)) g(\beta),
$$

we have from (12) that $J \geq 0$ on $\Sigma_3$ and

$$
J(x, t_0) \geq c(t_0) d(x) - \varepsilon h(c_2 d(x)) g(\|u(x, t_0)\|_\infty)
\geq 0 \quad \text{in} \quad \overline{\Omega},
$$

provided

$$
0 < \varepsilon \leq \min \left\{ \frac{c(t_0)}{g(\beta) c_2}, \frac{c(t_0)}{g(\|u(x, t_0)\|_\infty) c_2} \right\} \inf_{0 \leq z \leq M} \frac{z}{h(z)}
$$

is sufficiently small. Notice from $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ that

$$
\{(x, t) \in \Sigma : J(x, t) < 0\} \subset \Sigma_1 \cup \Sigma_2.
$$
Since the standard parabolic regularity theory reads $J \in C^{2,1}(\Sigma) \cap C(\Omega \times [t_0, T])$, it then follows from the maximum principle (c.f. [15]) that $J \geq 0$ in $\Sigma$. This indicates that

$$u_t \geq \varepsilon a(x)g(u) \quad \text{on} \quad \Omega \times (t_0, T).$$

Then by (13), (14) and (19),

$$u_t \geq \varepsilon h(c_1 d(x)) f(u),$$

for $(x, t) \in \Omega \times (t_0, T)$. Therefore integrating in time yields that (7) holds on $\Omega \times (t_0, T)$. For $(x, t) \in \Omega \times [0, t_0)$, we may choose furthermore

$$0 < \varepsilon \leq \inf_{x \in \Omega} \frac{1}{\inf_{x \in \Omega} u_0} \int_0^T \frac{ds}{f(s)}.$$  \hspace{1cm} \text{(23)}

Hence (7) holds for $(x, t) \in \Omega \times (0, t_0)$, and then we conclude that (7) holds on $\Omega \times (0, T)$. \hfill \Box

3. Proof of Theorem 1.2. With Theorem 1.1 at hand, Theorem 1.2 will be deduced by constructing a suitable local supersolution. Let $x_0 \in \Omega$ be a point such that $\rho(x_0) = 0$ and take $b_0 \in (0, 1)$ such that $B(x_0, 2b_0) \subset \Omega$.

We first introduce an auxiliary function $H$ such that $H \leq 0$, $H' \leq 0$, $H(0) = 0$ and

$$\limsup_{s \to A} f'(s) H^2\left(\frac{1}{\gamma} \int_s^A \frac{ds}{f(s)}\right) < +\infty.$$  \hspace{1cm} \text{(24)}

Therefore by defining a $C^2$ function $\varphi$

$$\left\{ \begin{array}{l}
\int_0^{\varphi(\chi)} \frac{d\tau}{H(\tau)} = \chi \in [0, b^2], \\
\varphi(b^2) = 0,
\end{array} \right.$$  \hspace{1cm} \text{(25)}

and $\phi(x) := \varphi(|x - x_0|^2) \geq 0$, we may consider the following function $w(x, t)$.

$$\int_{w(x,t)}^A \frac{ds}{f(s)} = \gamma(T - t + \phi(x)).$$  \hspace{1cm} \text{(26)}

Here, $\theta > 0$, $b \in (0, b_0)$, and $\gamma > 0$ is a positive constant such that

$$0 < \gamma \leq \min\left\{ \frac{1}{T + \theta} \int_{\inf u_0}^A \frac{ds}{f(s)}, \varepsilon h(c_1 b_0) \right\},$$  \hspace{1cm} \text{(27)}

where $\varepsilon, c_1, h$ are given in Theorem 1.1. Let $v(x)$ be such that

$$\int_{v(x)}^A \frac{ds}{f(s)} = \gamma \phi(x).$$  \hspace{1cm} \text{(28)}
then we have that \( w \leq v \) in \( B(x_0, b) \times (0, T) \). Moreover by the increasing of \( f' \), \( w \) satisfies

\[
\begin{align*}
  w_t - \Delta w - \lambda \rho(x)f(w) & = \gamma f(w)(1 + \Delta \phi - \gamma f'(w)|\nabla \phi|^2 - \frac{\lambda \rho(x)}{\gamma}) \\
  & \geq \gamma f(w)(1 + \Delta \phi - \gamma f'(w)|\nabla \phi|^2 - \frac{\lambda \rho(x)}{\gamma}) \\
  & \geq \gamma f(w)(1 + 4\phi'\|x - x_0\|^2 + 2N\phi' - 4\gamma f'(v)(\phi')^2|x - x_0|^2 - \frac{\lambda \rho(x)}{\gamma}) \\
  & \geq \gamma f(w)\left(1 + 4\phi'\|x - x_0\|^2 + 2N\phi' - 4\gamma f'(v)(\phi')^2\rho^2 - \frac{\lambda \rho(x)}{\gamma}\right) \\
  & = \gamma f(w)\left(1 + 4\phi'\|x - x_0\|^2 + 2N\phi' - 4\gamma f'(v)H^2\left(\frac{1}{\gamma} \int_v^A \frac{\partial \phi}{\partial f(s)} b^2 - \frac{\lambda \rho(x)}{\gamma}\right)\right) \\
  & \geq \gamma f(w)\left(1 + 2NH(\theta) - 4\gamma f'(v)H^2\left(\frac{1}{\gamma} \int_v^A \frac{\partial \phi}{\partial f(s)} b^2 - \frac{\lambda \rho(x)}{\gamma}\right)\right),
\end{align*}
\]

for all \((x, t) \in B(x_0, b) \times (0, T)\). Since \( \rho(x_0) = 0 \) implies \( \lim_{b \to 0} \sup_{B(x_0, b)} \rho(x) = 0 \), then we can deduce that

\[
  w_t - \Delta w - \lambda \rho(x)f(w) \geq 0, \quad \text{in } B(x_0, b) \times (0, T),
\]

by taking sufficiently small \( b \) and \( \theta \).

Notice that

\[
\int_{w(x,0)}^A \frac{\partial \phi}{\partial f(s)} \, ds = \gamma (T + \phi(x)).
\]

It then follows from (27) that \( w(x, 0) \geq u_0(x) \) for all \( x \in \overline{B}(x_0, b) \). On the other hand since

\[
\int_{w|_{\partial B(x_0, b) \times (0,T)}}^A \frac{\partial \phi}{\partial f(s)} \, ds = \gamma (T - t),
\]

(27) gives \( w|_{\partial B(x_0, b) \times (0,T)} \geq u|_{\partial B(x_0, b) \times (0,T)} \). Hence the comparison principle yields that \( w \leq u \) in \( B(x_0, b) \times (0, T) \). Note also that \( \min_{B(x_0, b/2)} \phi > 0 \). This shows that when \( t = T, x \in B(x_0, b/2), \gamma (T - t + \phi) = \gamma \phi \geq \gamma \min_{B(x_0, b/2)} \phi > 0 \). We then conclude that

\[
\int_u^A \frac{\partial \phi}{\partial f(s)} \, ds \geq \int_u^A \frac{\partial \phi}{\partial f(s)} \, ds \geq \gamma \min_{B(x_0, b/2)} \phi > 0,
\]

which implies that \( \sup_{B(x_0, b/2)} u < A \) and hence \( x = x_0 \) is not a quenching point. So we are done. \( \square \)

4. **Proof of Theorem 1.3.** Since \( \Omega \) is of class \( C^{2+\nu}, \partial \Omega \) satisfies exterior ball condition. Take any point \( x_0 \in \partial \Omega \), and without loss of generality assume that \( x_0 \neq 0 \) with \( B(0, |x_0|) \cap \Omega = \emptyset \). We next look for a suitable supersolution. Consider the function \( z(x) := K(d - r), r = |x|, \) with \( d > |x_0| \) to be chosen, and \( 0 < r < d \).

Here \( K \) satisfies

\[
\begin{align*}
  K'' + 2\lambda f(K)||\rho||_{\infty} & = 0, \quad \text{in } [0, d], \\
  K(0) & = 1, \quad K'(0) = 0.
\end{align*}
\]

It is easy to see that \( z_r = -K'(d - r), z_{rr} = K''(d - r) \). Define now

\[
\Theta := \Omega \cap \{x \in \mathbb{R}^N : |x_0| < |x| < d\}.
\]
Then we have in \( \Theta \) that
\[
-\Delta z - \lambda \rho(x)f(z) = -K'' + \frac{N - 1}{r} K' - \lambda \rho(x)f(z) \\
= f(z) \left( -\frac{K''}{f(K)} + \frac{(N - 1)K'}{rf(K)} - \lambda \rho(x) \right) \\
\geq f(z) \left( \lambda \rho \|z\|_{\infty} + \frac{(N - 1)K'}{|x_0|f(K(d - x_0))} \right). 
\]
Because (34) gives that \( K'' \leq 0, K' \leq 0 \), we compute \( \frac{(K')'}{f(K')} < 0 \), and hence we obtain from (36) that
\[
-\Delta z - \lambda \rho(x)f(z) = -K'' + \frac{N - 1}{r} K' - \lambda \rho(x)f(z) \\
\geq f(z) \left( \lambda \rho \|z\|_{\infty} + \frac{(N - 1)K'(d - x_0)}{|x_0|f(K(d - x_0))} \right). 
\]
Letting \( d > |x_0| \) close to \( x_0 \) then yields that \( -\Delta z - \lambda \rho(x)f(z) \geq 0 \) in \( \Theta \). Similarly, by taking \( d \) possibly closer to \( |x_0| \), we get that \( z(x) = K(d - r) \geq K(d - |x_0|) \geq u_0(x) \) in \( \bar{\Theta} \), and also for \( x \in \partial \Omega \cap \partial \Theta \), \( z(x) \geq K(d - |x_0|) \geq 0 \). On the other hand, \( z(x) = K(0) = 1 > u(x, t) \) for \( x \in \{ |x| = d \} \cap \partial \Theta \) and \( t \in (0, T) \). Therefore it follows from the comparison principle that \( z \geq u \) on \( \Theta \times (0, T) \) and hence \( x_0 \) is not a quenching point. So the theorem follows.

5. **Proof of Theorem 1.4.** We shall apply Theorem 1.1 to verify Theorem 1.4. Define \( \Omega_\eta := \{ x \in \Omega : d(x) < \eta \} \). Then there exists \( \eta_0 > 0 \) such that \( \Omega_\eta \) is a smooth bounded domain for all \( \eta \in (0, \eta_0) \), due to \( \Omega \) being a smooth domain, and we have \( \partial \Omega_\eta = \partial \Omega \cup \Gamma_\eta \), where
\[
\Gamma_\eta := \{ x \in \Omega : d(x) = \eta \}. 
\]
Consider the function \( w(x, t) \) as follows,
\[
F(w) := \int_{w(x, t)}^A \frac{ds}{f(s)} = m(\eta)(T - t + \phi(x)) \quad \text{in } \Omega_\eta \times [0, T). 
\]
Here the function \( m \) satisfies
\[
m(0) = 0, m' > 0, 
\]
and \( \phi(x) \) is to be decided later. Let \( \psi_\eta \) be the unique solution of the problem
\[
\begin{cases} 
-\Delta \psi_\eta = 0, & \text{in } \Omega_\eta, \\
\psi_\eta = 1, & \text{on } \partial \Omega, \\
\psi_\eta = 0, & \text{on } \Gamma_\eta. 
\end{cases} 
\]
Then the strong maximum principle and the standard elliptic regularity theory imply that \( \psi_\eta \) is smooth and \( 0 < \psi_\eta < 1 \) in \( \Omega_\eta \). We introduce now an auxiliary function \( R(y) \) such that
\[
\int_0^{R(y)} \sqrt{km(\eta)f'(F^{-1}(m(\eta)\tau))} \, d\tau = y, 
\]
and set \( \phi(x) = \kappa R(\psi_\eta(x)) \), where \( \kappa > 0 \) and \( \eta \in (0, \eta_0) \) are constants to be chosen later. Then one can see
\[
R(0) = 0, R' \geq 0, R'' \geq 0. 
\]
Consequently, \( 0 \leq \phi \leq \kappa R(1) \).
Notice that since \( \int_{w(x,0)} A \frac{ds}{f(s)} = m(\eta)(T + \phi) \), choosing
\[
0 < \eta \leq m^{-1}\left( \frac{1}{T + \kappa R(1)} \int_{\Omega} A \frac{ds}{f(s)} \right)
\] (44)
gives
\[
w(x,0) \geq u_0(x) \quad \text{in} \quad \Omega_{\eta}. \quad (45)
\]
Similarly, owing to \( \int_{w(0)} A \frac{ds}{f(s)} = m(\eta)(T - t + \kappa R(1)) \), letting
\[
0 < \eta \leq m^{-1}\left( \frac{1}{T + \kappa R(1)} \int_{\Omega} A \frac{ds}{f(s)} \right)
\] (46)
leads to \( w(x,t) \geq 0 \) on \( \partial \Omega \times (0,T) \). If we take \( \eta = m^{-1}(\varepsilon h(c_1 d(x))) \), where \( c_1 \) and \( h \) are defined as in Theorem 1.1, then by (7), (39) and (43), we arrive at
\[
\int_{\Omega} A \frac{ds}{f(s)} = \varepsilon h(c_1 d(x))(T - t) \leq \int_{\Omega} A \frac{ds}{f(s)}, \quad \text{for} \ (x,t) \in \Gamma_{\eta} \times (0,T),
\] (47)
which implies that \( w \geq u \) on \( \Gamma_{\eta} \times (0,T) \). Using a few similar arguments in the proof of Theorem 1.2, together with \( v(x) := F^{-1}(m(\eta)\phi(x)) \), some calculations can be done, yielding that on \( \Omega_{\eta} \times (0,T) \)
\[
w_t - \Delta w - \lambda \rho(x) f(w) 
\geq m(\eta) f(w) \left( 1 + (R^\alpha - m(\eta) f'(v)(R^\alpha)^2 \kappa) |\nabla \psi_\eta|^2 \kappa - \frac{\lambda \rho(x)}{m(\eta)} \right)
\geq m(\eta) f(w) \left( 1 - |\nabla \psi_\eta|^2 \kappa - \frac{\lambda \rho(x)}{\varepsilon h(c_1 d(x))} \right).
\] (48)
Now, by the assumption on \( \rho \) in Theorem 1.4, we may choose furthermore \( \eta \) small enough so that
\[
\sup_{x \in \Omega_{\eta}} \frac{\lambda \rho(x)}{\varepsilon h(c_1 d(x))} \leq \frac{1}{2}.
\] (49)
Therefore by taking \( \kappa = \kappa(\eta) > 0 \) small enough so that \( |\nabla \psi_\eta|^2 \kappa \leq \frac{1}{2} \) in \( \Omega_{\eta} \), we get that \( w_t - \Delta w - \lambda \rho(x) f(w) \geq 0 \) in \( \Omega_{\eta} \times (0,T) \). It now follows from the comparison principle that \( w \geq u \) in \( \Omega_{\eta} \times (0,T) \). However, since \( \min_{\Omega_{\eta}} \psi_\eta > 0 \), this guarantees that \( \min_{\Omega_{\eta}} \phi > 0 \) and furthermore we conclude that no quenching occurs near the boundary.

\[\square\]

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