Abstract

In 1993 Hong asked what are the best bounds on the $k$’th largest eigenvalue $\lambda_k(G)$ of a graph $G$ of order $n$. This challenging question has never been tackled for any $2 < k < n$. In the present paper tight bounds are obtained for all $k > 2$, and even tighter bounds are obtained for the $k$’th largest singular value $\lambda_k^*(G)$.

Some of these bounds are based on Taylor’s strongly regular graphs, and other on a method of Kharaghani for constructing Hadamard matrices. The same kind of constructions are applied to other open problems, like Nordhaus-Gaddum problems of the kind: How large can $\lambda_k(G) + \lambda_k(G)$ be?

These constructions are successful also in another open question: How large can the Ky Fan norm $\lambda_1^*(G) + \cdots + \lambda_k^*(G)$ be? Ky Fan norms of graphs generalize the concept of graph energy, so this question generalizes the problem for maximum energy graphs.

In the final section, several results and problems are restated for $(-1,1)$-matrices, which seem to provide a more natural ground for such research than graphs.

Many of the results in the paper are paired with open questions and problems for further study.

AMS classification: 15A42; 05C50.

Keywords: $k$’th largest eigenvalue of a graph; $k$’th largest singular eigenvalue of a graph; spectral Nordhaus-Gaddum problems; Ky Fan norms of graphs.

1 Introduction

What are the best possible lower and upper bounds on the $k$’th largest eigenvalue $\lambda_k(G)$ of a graph $G$ of order $n$?

Yuan Hong raised this fundamental question in 1993, in his paper [16]. Apparently he was unaware that five years earlier Powers [30], p. 5, had published the following result:

If $G$ is a connected graph of order $n$, then

$$\lambda_k(G) \leq \left\lfloor \frac{n}{k} \right\rfloor.$$  

(1)
It is not hard to realize that if inequality (1) were true, it would essentially answer Hong’s question. Alas, it is not. Its proof is flawed, and it fails for all \( k \geq 5 \). However, in fairness, the inequality certainly holds for \( k = 1, 2 \), while for \( k = 3, 4 \), it is a challenging open problem. For that matter, except for \( k = 2 \), connectedness is an irrelevant premise in these questions.

Other than this unsuccessful attempt, the general problem of Hong has never been tackled seriously. This is all the more inexplicable, as the problem is indeed challenging, easier for some values of \( k \), and well beyond reach for other. What is more, Hong’s problem is not a backyard puzzle that is of interest only to spectral graph theorists; it is related to other fundamental areas of combinatorics and analysis, like existence of symmetric Hadamard matrices, Ramsey’s theorem, and extremal norms of graphs. We feel that the appeal and the importance of Hong’s problem should attract the attention of many a researcher, and to this effect we take a few steps in the present paper.

We shall extend Hong’s problem to the \( k \)’th largest singular value \( \lambda_k^*(G) \) of \( G \), and shall give upper and lower bounds on \( \lambda_k(G) \) and \( \lambda_k^*(G) \), including a few exact results and asymptotics for the general cases. Many open problems and questions will be raised to outline directions for further study.

Two fundamental results underpin our constructions: first, the strongly regular graphs of Taylor [31, 32]; and second, Kharaghani’s method for constructing Hadamard matrices [20]. These two topics deserve to be known better in spectral graph theory, as their potential uses seem indeed unlimited.

We shall apply the same constructions to other open problems, like, e.g., Nordhaus-Gaddum problems of the following kind:

\[
\text{If } G \text{ is a graph of order } n \text{ and } \overline{G} \text{ is its complement, how large can } \lambda_k(G) + \lambda_k(\overline{G}) \text{ be?}
\]

Such problems have been raised in [26], and some recent progress has been given in [28]. We shall exhibit a new infinite family of solutions, and derive general asymptotics.

We make also progress with an open problem about maximum Ky Fan norms of graphs. Recall that the Ky Fan \( k \)-norm of a graph \( G \) is defined as \( \lambda_1^*(G) + \cdots + \lambda_k^*(G) \). In [27], the following problem has been raised:

\[
\text{If } G \text{ is a graph of order } n, \text{ how large can the Ky Fan } k \text{-norm of } G \text{ be?}
\]

Note that the Ky Fan \( n \)-norm is also known as the trace norm of \( G \), and has been extensively studied under the name graph energy, a concept introduced by Gutman in [12]. There is vast research on graph energy, but Ky Fan norms can open even larger horizons. Here we shall solve the above problem whenever \( k \) is an even square.

The structure of the paper is as follows: In Section 2 we present results on Hong’s problem. Section 3 is dedicated to spectral Nordhaus-Gaddum problems, and in Section 4 we present results on Ky Fan norms of graphs. Section 5 is for reader’s convenience: it contains references, notation and basics on Weyl’s inequalities, blow-ups of graphs, Taylor’s strongly regular graphs, and symmetric Latin squares. Section 6 contains the proofs of several theorems, which are either too involved of would have disrupted the
exposition. Finally, Section 7 contains a selection of the presented results and problems, translated from graphs to symmetric \((-1,1)\)-matrices. It becomes obvious that such matrices provide a more balanced and natural setup for such research. In particular, the strongly regular graphs of Taylor are translated into a \((-1,1)\)-matrix with a rather peculiar spectrum.

2 Hong’s problem and its variations

Let \(G\) be a graph of order \(n\). The eigenvalues \(\lambda_1 (G), \ldots, \lambda_n (G)\) of \(G\) are the eigenvalues of its adjacency matrix \(A (G)\), ordered as \(\lambda_1 (G) \geq \cdots \geq \lambda_n (G)\). The singular values \(\lambda^*_1 (G), \ldots, \lambda^*_n (G)\) of \(G\) are the absolute values of \(\lambda_1 (G), \ldots, \lambda_n (G)\), ordered as \(\lambda^*_1 (G) \geq \cdots \geq \lambda^*_n (G)\). In particular, \(\lambda^*_1 (G) = \lambda_1 (G)\), and

\[
\{\lambda^*_1 (G), \lambda^*_2 (G), \ldots, \lambda^*_n (G)\} = \{\lambda_1 (G), |\lambda_2 (G)|, \ldots, |\lambda_n (G)|\}.
\]

Note that, in general, graph singular values cannot be reduced to graph eigenvalues, as the two multisets may be ordered very differently.

We shall extend the original problem of Hong to the largest singular values of graphs, and shall bring to the fore the study of the smallest eigenvalues. These changes correspond to the present day interest in these spectral parameters.

Let \(n \geq k \geq 1\). Define the functions \(\lambda_k (n)\), \(\lambda_{-k} (n)\), and \(\lambda^*_k (n)\) as

\[
\lambda_k (n) = \max_{v(G) = n} \lambda_k (G), \\
\lambda_{-k} (n) = \max_{v(G) = n} |\lambda_{n-k+1} (G)|, \\
\lambda^*_k (n) = \max_{v(G) = n} \lambda^*_k (G).
\]

Note that if \(n \geq \binom{2k-1}{k-1}\), then \(\lambda_{n-k+1} (G)\) is always nonpositive (see Theorem 2.3 below), and so, \(\min_{v(G) = n} \lambda_{n-k+1} (G) = -\lambda_{-k} (n)\); thus, the use of the absolute value in the definition of \(\lambda_{-k} (n)\) is just to make the setup more uniform.

Now, we restate Hong’s problem into two separate problems:

**Problem 2.1** For any \(k \geq 1\), find \(\lambda_k (n)\), \(\lambda_{-k} (n)\), and \(\lambda^*_k (n)\).

**Problem 2.2** For any \(k \geq 1\), find \(\min_{v(G) = n} \lambda_k (G)\), \(\max_{v(G) = n} \lambda_{n-k+1} (G)\), and \(\min_{v(G) = n} \lambda^*_k (G)\).

This separation is justified, as the two problems are of incomparable difficulty: indeed, presently the full solution of Problem 2.1 is beyond reach, while we shall dispose of Problem 2.2 right away.

Indeed, if \(n \geq k\), the complete graph \(K_n\) of order \(n\) satisfies \(\lambda_k (K_n) = -1\); and likewise, the edgeless graph \(\overline{K}_n\) of order \(n\) satisfies \(\lambda_{n-k+1} (\overline{K}_n) = 0\) and \(\lambda^*_k (\overline{K}_n) = 0\). These bounds are also best possible: indeed, obviously \(\lambda^*_k (G) \geq 0\) for any graph \(G\) of order \(n \geq k\); and for \(\lambda_k\) and \(\lambda_{n-k+1}\) this fact is true in view of the following theorem:
Theorem 2.3 If \( n \geq \binom{2k-1}{k-1} \) and \( G \) is a graph of order \( n \), then
\[
\lambda_k (G) \geq -1 \quad \text{and} \quad \lambda_{n-k+1} (G) \leq 0. \tag{2}
\]

Proof We shall use Ramsey’s theorem, whose application in graph spectra has been pioneered only recently, in [28] and [36].

The classical bound of Erdős and Szekeres implies that every graph of order at least \( \binom{2k-1}{k-1} \) contains either a complete graph on \( k+1 \) vertices or an independent set on \( k \) vertices. If \( G \) contains a complete graph on \( k+1 \) vertices, then Cauchy’s interlacing theorem implies that
\[
\lambda_k (G) \geq \lambda_k (K_{k+1}) = -1 \quad \text{and} \quad \lambda_{n-k+1} (G) \leq \lambda_2 (K_{k+1}) = -1,
\]
so (2) follows. If \( G \) contains an independent set on \( k \) vertices, then Cauchy’s interlacing theorem implies that
\[
\lambda_k (G) \geq \lambda_k (\overline{K}_k) = 0 \quad \text{and} \quad \lambda_{n-k+1} (G) \leq \lambda_1 (\overline{K}_k) = 0,
\]
and (2) follows again. \(\Box\)

Now, let us turn to Problem 2.1. We start with an observation, which exhibits some dependencies between the functions \( \lambda_k (n) \), \( \lambda_{-k} (n) \), and \( \lambda^*_k (n) \).

Proposition 2.4 If \( k \geq 2 \), then
\[
\lambda_k (n) \leq \lambda^*_k (n), \quad \lambda_{-k+1} (n) \leq \lambda^*_k (n), \quad \text{and} \quad \lambda_k (n) + 1 \leq \lambda_{-k+1} (n).
\]

The first two inequalities follow from the definition of \( \lambda^*_k (n) \). For the last inequality recall that Weyl’s inequalities (see [5.1]) imply that if \( G \) is a graph of order \( n \) and \( 2 \leq k \leq n \), then \( \lambda_k (G) + \lambda_{n-k+2} (\overline{G}) \leq -1 \), and so, \( \lambda_k (G) + 1 \leq |\lambda_{-k+1} (\overline{G})| \).

Our first goal is to establish concise asymptotics of \( \lambda_k (n) \), \( \lambda_{-k} (n) \), and \( \lambda^*_k (n) \). To this effect, for any integer \( k \geq 1 \), define the real numbers \( c_k \), \( c_{-k} \), and \( c^*_k \) as
\[
c_k = \sup \left\{ \frac{\lambda_k (G)}{n} : G \text{ is a graph of order } n \geq k \right\},
\]
\[
c_{-k} = \sup \left\{ \frac{\lambda_{n-k+1} (G)}{n} : G \text{ is a graph of order } n \geq k \right\},
\]
\[
c^*_k = \sup \left\{ \frac{\lambda^*_k (G)}{n} : G \text{ is a graph of order } n \geq k \right\}.
\]

Clearly, these definitions imply that if \( G \) is a graph of order \( n \), then
\[
\lambda_k (G) \leq c_k n, \quad \lambda_{n-k+1} (G) \geq -c_{-k} n, \quad \text{and} \quad \lambda^*_k (G) \leq c^*_k n.
\]

The above bounds are handy, and fortunately they are also tight, as shown by the following theorem, which can be proved with the methods of [24]:

Theorem 2.5 For every \( k \geq 1 \),
\[
\lim_{n \to \infty} \frac{\lambda_k (n)}{n} = c_k, \quad \lim_{n \to \infty} \frac{\lambda_{-k} (n)}{n} = c_{-k}, \quad \text{and} \quad \lim_{n \to \infty} \frac{\lambda^*_k (n)}{n} = c^*_k.
\]

Therefore, a good deal of information about \( \lambda_k (G) \), \( \lambda_{n-k+1} (G) \), and \( \lambda^*_k (G) \) can be obtained if we knew the constants \( c_k \), \( c_{-k} \), and \( c^*_k \) or some good estimates thereof.
2.1 Upper bounds on $\lambda_k(n)$, $\lambda_{-k}(n)$, and $\lambda^*_k(n)$

Next, we give an easy upper bound on $\lambda^*_k(n)$; later, by much harder work, we shall show that this bound is almost as good as one can get.

Note that if $G$ is a graph of order $n$, with $e(G)$ edges and adjacency matrix $A$, then
\[
\lambda_1^2(G) + \lambda_2^2(G) + \cdots + \lambda_n^2(G) = \text{tr} A^2 = 2e(G).
\]
Hence, using the inequality $\lambda_1(G) \geq 2e(G)/n$ and the AM-GM inequality, one finds that
\[
\lambda_2^2(G) + \cdots + \lambda_k^2(G) \leq 2e(G) - \lambda_1^2(G) \leq 2e(G) - \left(\frac{2e(G)}{n}\right)^2 \leq \frac{n^2}{4}.
\]
Therefore, $(k - 1) \lambda_k^2(G) \leq n^2/4$, and Proposition 2.4 implies the following bounds:

**Theorem 2.6** If $n \geq k \geq 2$ and $G$ is a graph of order $n$, then
\[
\lambda_k(G) \leq \lambda^*_k(G) \leq \frac{n}{2\sqrt{k-1}},
\]
and
\[
|\lambda_{n-k+2}(G)| \leq \lambda^*_k(G) \leq \frac{n}{2\sqrt{k-1}}.
\]

Further, letting $n \to \infty$, we get the bounds
\[
c_k \leq \frac{1}{2\sqrt{k-1}}, \quad c_{-k+1} \leq \frac{1}{2\sqrt{k-1}}, \quad \text{and} \quad c^*_k \leq \frac{1}{2\sqrt{k-1}}. \tag{3}
\]
Simple as they are, bounds (3) give the correct rate of growth of $c_k$, $c_{-k}$ and $c^*_k$ in $k$; in particular, the bound on $c^*_k$ is quite tight. Note also that if $k = 2$, then equality holds in each of the bounds (3); on the other hand, if $k \geq 3$, the bound on $c^*_k$ is attained for infinitely many $k$, but the bounds on $c_k$ and $c_{-k+1}$ are never attained.

**Theorem 2.7** If $k \geq 3$, then there is an $\varepsilon_k > 0$ such that
\[
c_k < \frac{1}{2\sqrt{k-1}} - \varepsilon_k \quad \text{and} \quad c_{-k+1} < \frac{1}{2\sqrt{k-1}} - \varepsilon_k.
\]

Our proof of Theorem 2.7 is quite complicated and uses the Removal Lemma of Alon, Fischer, Krivelevich, and Szegedy [1], together with other tools of analytic graph theory. Due to its length, it will not be given in the present paper.

Although Theorem 2.7 may cast doubts as to the tightness of the bounds (3), we shall show that they can be matched by close lower bounds.
2.2 Lower bounds on $c_k$ and $c_{-k}$

In [31, 32], Taylor came up with a remarkable class of strongly regular graphs, which we shall use in several ways to give lower bounds on $\lambda_k(n)$ and $\lambda_{-k}(n)$. It seems that Taylor’s strongly regular graphs are a cornerstone in spectral graph theory, and need to be known better. For reader’s sake, in Section 5 gives a brief discussion on Taylor’s graphs and their complements.

Note that Taylor’s graphs contain roughly half of the total number of edges, and the same holds for their complements. At the same time, almost all eigenvalues of a Taylor graph are positive, and almost all eigenvalues of its complement are negative. This combination of properties makes Taylor’s graphs and their complements very suitable for lower bounds on $c_k$ and $c_{-k}$.

To begin with, in the following theorem, we shall use Taylor’s graphs to show the tightness of the bounds (3) for infinitely many, albeit handpicked values of $k$. The proof of Theorem 2.8 is in Section 6.

**Theorem 2.8** If $q$ is an odd prime power and $k = q^2 - q + 1$, then

$$c_k > \frac{1}{2\sqrt{k - 1} + 1} \quad \text{and} \quad c_{-k+1} > \frac{1}{2\sqrt{k - 1} + 1}.$$  \hfill (4)

Unfortunately, the bounds (4) seem to be close to the best ones that Taylor graphs can provide. Nevertheless, in the following theorem, we shall use Taylor graphs to provide general asymptotics of $c_k$ and $c_{-k}$ for any $k$:

**Theorem 2.9** There exists $k_0$ such that if $k > k_0$, then

$$c_k > \frac{1}{2\sqrt{k - 1} + \sqrt{k}}, \quad \text{and} \quad c_{-k+1} > \frac{1}{2\sqrt{k - 1} + \sqrt{k}}.$$  \hfill (5)

Theorem 2.9, whose proof is in Section 6, shows that the upper bounds (3) on $c_k$ and $c_{-k}$ are asymptotically tight, although there is a lot to improve. A particularly weak point of bounds (3) is the fact that $k_0$ is not known explicitly, due to a number-theoretic result used in the proof. Below we provide a weaker theorem, with explicit bounds. It also shows that the bound (3) fails for any $k \geq 5$ and $n$ sufficiently large.

**Theorem 2.10** If $5 \leq k \leq 15$, then

$$c_k \geq \frac{1}{k - 1/2} \quad \text{and} \quad c_{-k+1} \geq \frac{1}{k - 1/2}.$$  

If $k \geq 16$, then

$$c_k \geq \frac{1}{4\sqrt{k - 1}} \quad \text{and} \quad c_{-k+1} \geq \frac{1}{4\sqrt{k - 1}}.$$  

Theorem 2.10, whose proof is also in Section 6, leaves the following two questions open:

**Question 2.11** Is it true that $c_3 = 1/3$?

**Question 2.12** Is it true that $c_4 = 1/4$?
2.3 Bounds on $c_k^n$

It turns out that $\lambda_k^*(n)$ and $c_k^*$ can be estimated with greater precision than $\lambda_k(n)$ and $c_k$. We start by establishing a crucial connection between $c_k^*$ and the existence of certain symmetric $(-1,1)$-matrices. Thus, write $\mathbb{U}_n$ for the set of symmetric $(-1,1)$-matrices or order $n$.

**Theorem 2.13** If $A$ is a symmetric $(-1,1)$-matrix of order $n$, with $\lambda_k^*(A) = n/\sqrt{k}$, then

$$c_k^{*+1} = \frac{1}{2\sqrt{k}}.$$  

**Proof** Let $A = [a_{i,j}] \in \mathbb{U}_n$, with $\lambda_k^*(A) = n/\sqrt{k}$. Since

$$\sum_{i=1}^{k} (\lambda_i^*(A))^2 \geq k \left( \frac{n}{\sqrt{k}} \right)^2 = n^2 = \sum_{i=1}^{n} \sum_{l=1}^{n} a_{i,l}^2 = \sum_{i=1}^{n} (\lambda_i^*(A))^2,$$

we see that

$$\lambda_1^*(A) = \cdots = \lambda_k^*(A) = n/\sqrt{k} \quad \text{and} \quad \lambda_i^*(A) = 0 \quad \text{for} \quad k < i \leq n.$$  

Define a matrix $A' \in \mathbb{U}_{2n}$ by

$$A' = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix},$$

and note that all rowsums of $A'$ are zero; therefore 0 is an eigenvalue of $A'$ with eigenvector $j_{2n}$. Now, define a symmetric $(0,1)$-matrix $B$ by

$$B = \frac{1}{2} (A' \otimes J_t + J_{2nt}).$$

The order of $B$ is $2nt$. Obviously $\lambda_1^*(B) = nt$ and

$$\lambda_2^*(B) = \cdots = \lambda_{k+1}^*(B) = nt/\sqrt{k},$$

$$\lambda_i^*(B) = 0, \quad i = k + 1, \ldots, nt.$$  

Now, zero the diagonal of $B$ and write $A$ for the resulting matrix. Clearly $A$ is the adjacency matrix of some graph $G$ of order $2nt$. Using Weyl's inequalities (see Proposition [5.1]), we find that

$$\lambda_{k+1}^*(G) = \lambda_{k+1}^*(B - (B - A)) \geq \lambda_{k+1}^*(B) - \lambda_1^*(B - A) \geq \lambda_{k+1}^*(B) - \lambda_1^*(J_{2nt}) = \frac{2nt}{2\sqrt{k}} - 1.$$  

Letting $t \to \infty$, we get

$$c_k^{*+1} = \frac{1}{2\sqrt{k}},$$

completing the proof.  

Theorem 2.13 motivates the introduction of a class of symmetric $(-1,1)$-matrices, which extend symmetric Hadamard matrices in a natural way.
2.4 An extension of symmetric Hadamard matrices

Write \( n(A) \) for the order of a square matrix \( A \), and let \( \mathbb{S}_k \) be the set of symmetric \((-1, 1)\)-matrices with \( \lambda^*_k(A) = n(A)/\sqrt{k} \). Note that \( \mathbb{S}_k \) can contain matrices of different order; for that matter, if \( \mathbb{S}_k \) is nonempty, then it is infinite. Note also that if \( A \in \mathbb{S}_k \), and a matrix \( B \) can be obtained by permutations or negations performed simultaneously on rows and columns of \( A \), then \( B \in \mathbb{S}_k \) as well; the reason is that singular values are not affected by such operations.

As in the proof of Theorem 2.13, one can check the validity of the following statement.

**Proposition 2.14** A symmetric \((-1, 1)\)-matrix \( A \) belongs to \( \mathbb{S}_k \) if and only if

\[
\lambda^*_1(A) = \cdots = \lambda^*_k(A) = n(A)/\sqrt{k}, \quad \text{and} \quad \lambda^*_i(A) = 0 \quad \text{for} \quad k < i \leq n(A).
\]

Here is a summary of some properties of \( \mathbb{S}_k \):

1. If \( A \in \mathbb{S}_k \) then \(-A \in \mathbb{S}_k\);
2. If \( A \) is a symmetric \((-1, 1)\)-matrix of rank 1, then \( A \in \mathbb{S}_1 \); thus, \( J_n \in \mathbb{S}_1 \);
3. If \( H \) is a symmetric Hadamard matrix of order \( k \), then \( H \in \mathbb{S}_k \);
4. If \( A \in \mathbb{S}_k \) and \( B \in \mathbb{S}_l \), then \( A \otimes B \in \mathbb{S}_{kl} \); hence, if \( \mathbb{S}_k \neq \emptyset \), then \( \mathbb{S}_{2k} \neq \emptyset \);
5. If \( A \in \mathbb{S}_k \), then \( A \otimes J_n \in \mathbb{S}_k \) for any \( n \geq 1 \); hence \( \mathbb{S}_k \) is infinite;
6. If \( \mathbb{S}_k \neq \emptyset \), then \( \mathbb{S}_k \) contains matrices with all their rowsums equal to zero.

We omit the proofs of (1)-(5), but here is sketch of a proof of (6): the rank of the matrix

\[
K = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

is 1, so \( K \in \mathbb{S}_1 \). Now, if \( A \in \mathbb{S}_k \), then \( K \otimes A \in \mathbb{S}_k \), and the rowsums of \( K \otimes A \) are zero.

Properties (1)-(6) allow to show that \( \mathbb{S}_k \) contains matrices with some special properties, as in the following proposition:

**Proposition 2.15** If \( A \in \mathbb{S}_k \), then there is a \( B \in \mathbb{S}_{2k} \) such that:

(i) \( B \) has exactly \( k \) positive and exactly \( k \) negative eigenvalues;
(ii) the rowsums of \( B \) are equal to 0.

To check this proposition, set

\[
H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},
\]

and let \( B = K \otimes (H_2 \otimes A) \); obviously \( B \in \mathbb{S}_{2k} \), and \( B \) satisfies (i) and (ii).

The principal question about \( \mathbb{S}_k \) is the following one:

**Problem 2.16** For which \( k \) is \( \mathbb{S}_k \) nonempty?
Below we shall show that $S_k$ is empty if $k$ is odd and is not a square. On the positive side, Property (3) implies that there are infinitely many $k$ for which $S_k$ is nonempty. In particular, Paley's construction of symmetric Hadamard matrices (see, e.g., [6]) implies the following fact:

**Proposition 2.17** If $p$ is a prime power and $p = 1 \mod 4$, then $S_{2(p+1)}$ is not empty.

We shall prove more definite assertions about $S_k$ by using the fact that the singular values of a real symmetric matrix are the absolute values of its eigenvalues.

**Proposition 2.18** If $A \in S_k$, then either $k$ is an exact square, or $A$ has the same number of positive and negative eigenvalues.

**Proof** Let $A = [a_{i,j}] \in S_k$. Setting $l = \lambda^*_1 (A)$ and $n = n(A)$, we have

$$kl^2 = \sum_{\lambda_i(A) > 0} \lambda_i^2 (A) + \sum_{\lambda_i(A) < 0} \lambda_i^2 (A) = \text{tr} A^2 = \sum_{i=1}^{n} \sum_{l=1}^{n} a_{i,j}^2 = n^2.$$ 

On the other hand, writing $n_+$ and $n_-$ for the number of positive and negative eigenvalues of $A$, we see that

$$(n_+ - n_-) l = \sum_{\lambda_i(A) > 0} \lambda_i (A) + \sum_{\lambda_i(A) < 0} \lambda_i (A) = \text{tr} A.$$ 

Since $\text{tr} A$ is an integer, either $l$ is rational or $n_+ = n_-$. Since $l = n/\sqrt{s}$, it may be rational only if $\sqrt{s}$ is an integer, completing the proof. \(\square\)

**Corollary 2.19** If $k$ is odd and $S_k$ is nonempty, then $k$ is an exact square.

It is interesting to see what we can say about $S_k$ for small $k$, say for $k \leq 10$. First, the above corollary implies that $S_k$ is empty for $k = 3, 5,$ and $7$. Below we shall show that the converse of Proposition 2.18 is partially true as well, that is to say: if $k$ is an exact square, then $S_k$ is nonempty. Thus, in view of properties (1)-(5), we see that $S_k$ is nonempty for $k = 1, 2, 4, 8,$ and $9$. The first unknown cases are $k = 6$ and $k = 10$.

**Question 2.20** Is $S_6$ empty?
2.5 Constructions of matrices in $\mathbb{S}_{s^2}$

In this subsection we shall construct classes of symmetric $(-1,1)$-matrices, which show that $\mathbb{S}_k$ is nonempty if $k$ is an exact square or is the double of an exact square. Except for this primary goal, our matrices must have other specific properties, which are necessary for subsequent applications; thus the statements are somewhat involved.

The proofs of Theorems 2.21, 2.23, and 2.24 are variations of Khara ghani’s method [20] for constructing Hadamard matrices. Our approach suggests that this method is generic and can be used to construct $(-1,1)$-matrices that are more general than Hadamard matrices.

The proofs of Theorems 2.21, 2.23, and 2.24 are in Section 6.

Theorem 2.21 For any integer $s \geq 2$, there are an integer $n \geq s$ and a symmetric $(-1,1)$-matrix $B$ of order $ns$ such that:

(i) $B$ has exactly $s^2$ nonzero eigenvalues, of which $\binom{s-1}{2}$ are equal to $n$, and $\binom{s+1}{2}$ are equal to $-n$;

(ii) the rowsums of $B$ are equal to 0;

(iii) the diagonal entries of $B$ are equal to $-1$.

Note that clause (i) of Theorem 2.21 implies that $B \in \mathbb{S}_{s^2}$. Further, negating the matrix $B$, we get the following variation.

Corollary 2.22 For any integer $s \geq 2$, there are an integer $n \geq s$ and a symmetric $(-1,1)$-matrix $B$ of order $ns$ such that:

(i) $B$ has exactly $s^2$ nonzero eigenvalues, of which $\binom{s+1}{2}$ are equal to $n$, and $\binom{s-1}{2}$ are equal to $-n$;

(ii) the rowsums of $B$ are equal to 0;

(iii) the diagonal entries of $B$ are equal to 1.

Theorem 2.21 will be used to give some answers to Problem 2.1. For other purposes we shall need two other theorems.

Theorem 2.23 For any integer $s \geq 2$, there are an integer $n \geq s$ and a symmetric $(-1,1)$-matrix $B$ of order $ns$ such that:

(i) $B$ has exactly $s^2$ nonzero eigenvalues, of which $\binom{s+1}{2}$ are equal to $n$, and $\binom{s-1}{2}$ are equal to $-n$;

(ii) the vector $j_{ns}$ is an eigenvector of $B$ to the eigenvalue $-n$;

(iii) the diagonal entries of $B$ are equal to 1.

Theorem 2.24 For any integer $s \geq 2$, there are an integer $n \geq s$ and a symmetric $(-1,1)$-matrix $B$ of order $ns$ such that:

(i) $B$ has exactly $s^2$ nonzero eigenvalues, of which $\binom{s+1}{2} - 1$ are equal to $n$, and $\binom{s-1}{2} + 1$ are equal to $-n$;

(ii) all rowsums of $B$ are equal to $-n$. 

10
Note that Theorem 2.21 and Property (2) imply that for any natural number \( s \), the classes \( S_{s^2} \) and \( S_{2s^2} \) are nonempty. Before continuing, let us state some explicit solutions to Problem 2.16:

**Proposition 2.25** Let \( s \) be a natural number and \( p \) be a prime power, with \( p = 1 \mod 4 \). Then the classes \( S_{s^2} \), \( S_{2s^2} \), \( S_{2s^2(p+1)} \) and \( S_{4s^2(p+1)} \) are nonempty.

### 2.6 Finding \( \lambda_{k}^*(n) \) and \( c_{k}^* \) for infinitely many \( k \) and \( n \)

Observe that using the classes \( S_k \), Theorem 2.13 can be restated as: if \( S_k \) is nonempty, then \( c_{k+1}^* = 1/(2\sqrt{k}) \). Proposition 2.25 and properties (1)-(6) give many values of \( k \) for which this equality holds. Moreover, using the finer properties outlined in Theorem 2.21 we can determine the exact value of \( \lambda_{s^2+1}^*(n) \) for infinitely many \( n \).

**Theorem 2.26** If \( s \geq 1 \), then there is an integer \( n > s \), such that for every integer \( t \geq 1 \), there is a graph \( G \) of order \( n \) with

\[
\lambda_{s^2+1}^*(G) = \frac{nt}{2}.
\]

**Proof** Let \( n \geq s \), and let \( B \) be the matrix of order \( sn \) constructed in Theorem 2.21. Set

\[
C = \frac{1}{2} (B + J_{sn}),
\]

and note that \( C \) is a symmetric \((0,1)\)-matrix with zero diagonal. Since all rowsums of \( B \) are zero, the vector \( J_{sn} \) is an eigenvector of \( B \) to the eigenvalue 0; thus all eigenvectors to nonzero eigenvalues of \( B \) are orthogonal to \( J_{sn} \). Obviously every eigenvector of \( B \) is an eigenvector to \( C \). Thus, \( sn/2 \) is an eigenvalue of \( C \); also, \( C \) has \( \binom{s-1}{2} \) eigenvalues equal to \( n/2 \) and \( \binom{s+1}{2} \) eigenvalues equal to \( -n/2 \); the remaining eigenvalues of \( C \) are zero. Now, letting \( A = C \otimes J_t \), one sees that \( A \) is the adjacency matrix of a graph \( G \) of order \( sn \) with

\[
\lambda_{s^2+1}^*(G) = \frac{1}{2} \lambda_{s^2}^*(B) t = \frac{sn t}{2s},
\]

completing the proof of Theorem 2.26.

Theorem 2.21 helps also to find concise asymptotics of \( c_{k}^* \). Indeed, since \( c_{k}^* \) is non-increasing in \( k \), letting \( s \) to be the smallest integer such that \( s^2 + 1 \geq k \), we see that \( c_{k}^* \geq 1/(2s) \); since \( (s - 1)^2 + 1 < k \), we get the following theorem:

**Theorem 2.27** For any \( k \geq 3 \),

\[
\frac{1}{2\sqrt{k-1}} \geq c_{k}^* > \frac{1}{2\sqrt{k-1} + 2} = \frac{1}{2\sqrt{k-1}} + O(k^{-1}).
\]

It seems quite clear that the lower bound on \( c_{k}^* \) can be improved, so we raise the following problem.

**Problem 2.28** Is there a positive constant \( C \) such that for any \( k \geq 3 \),

\[
c_{k}^* > \frac{1}{2\sqrt{k + C}}?
\]
3 Spectral Nordhaus-Gaddum problems

It turns out that the classes $S_k$ help to find infinitely many solutions to a general spectral Nordhaus-Gaddum problem. Nordhaus-Gaddum problems in general, form a notable part of extremal graph theory, see, e.g., the recent survey [2] for their numerous variations. In particular, spectral Nordhaus-Gaddum problems have been studied first in 1970, by Nosal [29], and have attracted a lot of attention since then.

Thus, let $\overline{G}$ denote the complement of a graph $G$. Given $n \geq k \geq 1$, define the functions $f_k(n)$, $f_{-k}(n)$, and $f_k^*(n)$ as

$$f_k(n) = \max_{\nu(G)=n} \lambda_k(G) + \lambda_k(\overline{G}),$$
$$f_{-k}(n) = \max_{\nu(G)=n} |\lambda_{n-k+1}(G)| + |\lambda_{n-k+1}(\overline{G})|,$$
$$f_k^*(n) = \max_{\nu(G)=n} \lambda_k^*(G) + \lambda_k^*(\overline{G}).$$

Clearly, we define $f_k(n)$, $f_{-k}(n)$, and $f_k^*(n)$ similarly to $\lambda_k(n)$, $\lambda_{-k}(n)$, and $\lambda_k^*(n)$. Note that $f_k^*(n)$ is a new function, but $f_k(n)$ and $f_{-k}(n)$ have been introduced in [26] with different, albeit essentially equivalent definitions.

Now, let us reiterate and extend a problem raised in [26]:

**Problem 3.1** For any $k \geq 1$, find $f_k(n)$, $f_{-k}(n)$, and $f_k^*(n)$.

The function $f_1(n)$ has been studied by Nosal [29]: finding it have turned out to be a hard problem, which has been resolved only recently, in [7] and [33]. This case sticks out from the rest, both with its particular extremal graphs, as with its particular methods. The first "mainstream" case is $f_2(n)$, which has been determined in [26].

Recently, in [28], two tight upper bounds have been given:

If $k \geq 2$ and $n \geq 15(k-1)$, then

$$f_k(n) \leq \frac{n}{\sqrt{2(k-1)}} - 1. \quad (6)$$

Likewise, if $k \geq 1$ and $n \geq 4^k$, then

$$f_{-k}(n) \leq \frac{n}{\sqrt{2k}} + 1. \quad (7)$$

In turns out that the bounds (6) and (7) capture the rate of growth of $f_k(n)$ and $f_{-k}(n)$ pretty tightly. In [28], it has been shown that (6) and (7) are essentially best possible if $k = 2^{s-1} + 1$ and $s = 2, 3, \ldots$. What is more, for such $k$ it has proved that

$$f_k(n) \geq \frac{n}{\sqrt{2(k-1)}} - 2 \quad \text{and} \quad f_{-k}(n) \geq \frac{n}{\sqrt{2k}},$$

for infinitely many $n$. 

12
The study of \( f_k(n) \), \( f_{-k}(n) \), and \( f^*_k(n) \) can be put on the same ground as \( \lambda_k(n) \), \( \lambda_{-k}(n) \), and \( \lambda^*_k(n) \). First, using the methods of [24], one can show that the limits

\[
\begin{align*}
    f_k &= \lim_{n \to \infty} \frac{f_k(n)}{n}, \\
    f_{-k} &= \lim_{n \to \infty} \frac{f_{-k}(n)}{n}, \\
    f^*_k &= \lim_{n \to \infty} \frac{f^*_k(n)}{n}
\end{align*}
\]

exist, and the following inequalities hold for every \( n \):

\[
\begin{align*}
    f_k(n) &\leq f_k n - 1, \\
    f_{-k}(n) &\leq f_{-k} n + 1, \\
    f^*_k(n) &\leq f^*_k n + 1.
\end{align*}
\]

Now, the constructions developed in Section 2.5 allow to resolve Problem 3.1 for infinitely many values of \( k \). Indeed, it turns out that if \( S_k \) is nonempty, then

\[
\begin{align*}
    f_{k+1} &= \frac{1}{\sqrt{2k}} \\
    f_{-k} &= \frac{1}{\sqrt{2k}}.
\end{align*}
\]

These equalities follow from the theorem below, where we prove a more precise result, involving \( n \) as well.

**Theorem 3.2** If \( S_k \) is nonempty, then there is an integer \( n > k \), such that for every integer \( t \geq 1 \), there is a graph \( G \) of order \( nt \) with

\[
\begin{align*}
    f_{k+1}(n) &\geq \frac{nt}{\sqrt{2k}} - 2 \\
    f_{-k}(n) &\geq \frac{nt}{\sqrt{2k}}.
\end{align*}
\]

**Proof** If \( S_k \) is nonempty, Proposition 2.15 implies that there exists \( B \in S_{2k} \), say of order \( n \), such that,

\[
\begin{align*}
    \lambda_1(B) &= \cdots = \lambda_k(B) = \frac{n}{\sqrt{2k}}, \\
    \lambda_{n-k+1}(B) &= \cdots = \lambda_{n}(B) = -\frac{n}{\sqrt{2k}}, \\
    \lambda^*_i(B) &= 0 \text{ for } k < i \leq n - k.
\end{align*}
\]

Define a matrix \( A' \) by

\[
A' = \frac{1}{2} (B \otimes I_t) + J_{nt}.
\]

Now, zero the diagonal of \( A' \) and write \( A \) for the resulting matrix. Note that \( A \) is a symmetric \((0,1)\) matrix with zero diagonal, so it is the adjacency matrix of graph \( G \) of order \( nt \). As in the proof of Theorem 2.13, we see that

\[
\begin{align*}
    \lambda_{k+1}(G) &\geq \frac{nt}{2\sqrt{2k}} - 1.
\end{align*}
\]
On the other hand, taking the matrix 
\[ A' = \frac{1}{2} (-B \otimes I_t) + J_{nt}, \]
and zeroing its main diagonal, we obtain the matrix \( A' \), which is obviously the adjacency matrix of the complement of \( G \). Like above we have, 
\[ \lambda_{k+1}(G) \geq \frac{nt}{2\sqrt{2k}} - 1, \]
and inequality (8) follows. The proof of (9) is similar and is omitted. \( \square \)

In view of Theorem 3.2 and properties (1)-(6), we get numerous examples for which bounds (6) and (7) are essentially best; we refer to Proposition 2.25 for some explicit values.

We finish the discussion of \( f_k(n) \) and \( f_{-k}(n) \) with general asymptotics of \( f_k \) and \( f_{-k} \). Since \( f_k \) and \( f_{-k}(n) \) are nonincreasing in \( k \), letting \( s \) to be the smallest integer such that \( s^2 \geq k \), we see that 
\[ f_{k+1} \geq \frac{1}{\sqrt{2s}} \quad \text{and} \quad f_{-k} > \frac{1}{\sqrt{2s}}; \]
since \((s-1)^2 < k\), we get the following theorem:

**Theorem 3.3** For any \( k \geq 2 \),
\[ \frac{1}{\sqrt{2(k-1)}} \geq f_k > \frac{1}{\sqrt{2(k-1)} + \sqrt{2}} = \frac{1}{\sqrt{2(k-1)}} + O(k^{-1}), \]
and
\[ \frac{1}{\sqrt{2(k-1)}} \geq f_{-k+1} > \frac{1}{\sqrt{2(k-1)} + \sqrt{2}} = \frac{1}{\sqrt{2(k-1)}} + O(k^{-1}). \]

Finally, let us briefly discuss the function \( f_k^*(n) \), which is somewhat easier to deal with, and can be derived mainly from \( \lambda_k^*(n) \). First, Theorem 2.6 implies immediately that 
\[ f_k^*(n) \leq 2\lambda_k^*(n) \leq \frac{n}{\sqrt{k-1}}. \]
This easy bound is rather different from (6) and (7), whose proofs are much subtler anyway. Nonetheless, bound (10) gives the correct rate of growth of \( f_k^*(n) \). Using Theorems 2.13 and 2.27 we immediately come up with the following statements:

**Theorem 3.4** If \( S_k \) is nonempty, then there is an integer \( n > k \), such that for every integer \( t \geq 1 \), there is a graph \( G \) of order \( nt \) with 
\[ f_{k+1}^*(n) \geq \frac{nt}{\sqrt{k}} - 2. \]

**Theorem 3.5** For any \( k \geq 2 \),
\[ \frac{1}{\sqrt{k-1}} \geq f_k^* > \frac{1}{\sqrt{k-1} + 1} = \frac{1}{\sqrt{k-1}} + O(k^{-1}). \]
4 Sums of eigenvalues and sums of singular values

In addition to individual eigenvalues and singular values of graphs, it is of interest to consider certain sums thereof. In particular, let

\[ \tau_k(n) = \max_{v(G)=n} \lambda_1(G) + \cdots + \lambda_k(G), \]
\[ \xi_k(n) = \max_{v(G)=n} \lambda_1^*(G) + \cdots + \lambda_k^*(G). \]

Note that \( \xi_k(n) \) is the maximal Ky Fan \( k \)-norm of a graph of order \( n \). In particular, the Ky Fan \( n \)-norm is known as the trace norm of \( G \), and has been extensively studied under the name graph energy, a concept introduced by Gutman in [12]; see also [13] for the current state of this research. Note that, \( \xi_n(n) \) is just the maximum energy of a graph of order \( n \), which also has been studied, see, e.g., [14], [21], and [25].

The research on graph energy is truly monumental, but with the flexibility of the parameter \( k \), the Ky Fan \( k \)-norms offer a considerably vaster playground. Here we shall focus only on the following principal question, raised in [27]:

**Problem 4.1** For any \( k \geq 1 \), find \( \tau_k(n) \) and \( \xi_k(n) \).

For a start, note that the inequality \( \lambda_i(G) \leq \lambda_i^*(G) \) implies that \( \tau_k(n) \leq \xi_k(n) \) for any \( k \) and \( n \). For general \( k > 2 \), the function \( \tau_k(n) \) has been studied by Mohar in [23]; and in turn, \( \xi_k(n) \) has been studied by the author in [27]. The exact values of \( \tau_k(n) \) and \( \xi_k(n) \) are unknown for most values of \( k \); in particular, neither \( \tau_2(n) \) nor \( \xi_2(n) \) are known yet: see [9] for \( \tau_2(n) \), and [11] for \( \xi_2(n) \). However, estimating \( \tau_k(n) \) and \( \xi_k(n) \) is possible for large \( k \). Indeed, Mohar [23] proved the asymptotics

\[ \frac{1}{2} \left( \frac{1}{2} + \sqrt{k} - o(k^{-2/5}) \right) \leq \frac{\tau_k(n)}{n} \leq \frac{1}{2} \left( 1 + \sqrt{k} \right). \]

(11)

Note the gap \( 1/2 + o(1) \) between the upper and lower bounds in (11), which is very challenging to close. In general, finding \( \tau_k(n) \) seems a hard problem, a lot harder than finding \( \xi_k(n) \). In particular, it is easy to show that the limit \( \tau_k = \lim_{n \to \infty} \tau_k(n)/n \) exists for any fixed \( k \geq 1 \), but this limit is not known for any \( k \geq 2 \). Thus, we suggest the following concrete conjecture:

**Conjecture 4.2** For any \( k \geq 2 \), there is an \( \varepsilon_k > 0 \) such that

\[ \tau_k < \frac{1}{2} \left( 1 + \sqrt{k} - \varepsilon_k \right). \]

In contrast to \( \tau_k(n) \), we shall find \( \xi_k(n) \) for infinitely many values of \( k \) and \( n \). To begin with, in [27] it was shown that if \( n \geq k \geq 1 \), then

\[ \xi_k(n) \leq \frac{1}{2} \left( 1 + \sqrt{k} \right) n, \]

(12)
which strengthens the upper bound (11). In fact, unlike $\tau_k(n)$, the function $\xi_k(n)$ attains the upper bound (12) for infinitely many $k$ and $n$. Indeed, let $H$ be a symmetric regular Hadamard matrix of order $k$, with positive rowsums, and with $-1$ along the main diagonal. Then the matrix

$$A = \frac{1}{2} (H \otimes J_n) + J_{kn}$$

is the adjacency matrix of a graph $G$ of order $kn$, with

$$\lambda^*_1(G) + \cdots + \lambda^*_k(G) = \frac{1}{2} \left( 1 + \sqrt{k} \right) n,$$

and so $\xi_k(n)$ attains the upper bound (12). It is known that symmetric regular Hadamard matrix with equal rowsums and with $-1$ along the main diagonal exist for $k = 4m^4$ and any $m = 1, 2, \ldots$, see [15] for details. In fact, there are many more cases of $k$ for which the upper bound (12) is attained.

First, we shall show that if the bound (12) is attained, then $k$ is an exact square:

**Theorem 4.3** If $G$ is a graph of order $n$ such that

$$\lambda^*_1(G) + \cdots + \lambda^*_k(G) = \frac{1}{2} \left( 1 + \sqrt{k} \right) n,$$

then $k$ is an exact square.

**Proof** Write $\|A\|_{sk}$ the sum of the $k$ largest singular values of $A$. Suppose that $G$ is a graph that satisfies the hypothesis and write $A$ for its adjacency matrix.

Note that $J_n - 2A$ is a symmetric $(-1,1)$-matrix and so, in view of the AM-QM inequality, we find that

$$\sum_{i=1}^{k} \lambda^*_i(J_n - 2A) \leq \sqrt{\sum_{i=1}^{k} \lambda^*_i^2(J_n - 2A)} \leq \sqrt{k \sum_{i=1}^{n} \lambda^*_i^2(J_n - 2A)} = \sqrt{kn}.$$

Therefore, using the triangle inequality for the Ky Fan $k$-norm $\|X + Y\|_{sk} \leq \|X\|_{sk} + \|Y\|_{sk}$ (see [18], p.196), we find that

$$\left( 1 + \sqrt{k} \right) n = 2 \|A\|_{sk} = \|2A\|_{sk} \leq \|2A - J_n\|_{sk} + \|J_n\|_{sk} \leq \sqrt{kn} + n.$$

Thus, equalities hold throughout the above line, and so, $2A - J_n$ has $k$ nonzero singular values, which are equal. We get

$$\lambda^*_k(J_n - 2A) = n/\sqrt{k},$$

implying that $J_n - 2A \in S_k$. On the other hand, $\text{tr}(J_n - 2A) = n \neq 0$, so $J_n - 2A$ cannot have the same number of positive and negative eigenvalues, and Proposition 2.18 implies that $k$ is an exact square. \qed

The matrix built in Theorem 2.23 helps to prove that the converse of the above theorem is partially true as well.
Theorem 4.4 Let $s$ be an even positive integer. There exists a positive integer $n$, such that for every positive integer $t$, there is a graph $G$ of order $snt$, with

$$
\lambda_1^s(G) + \cdots + \lambda_s^s(G) = \frac{1}{2} (1 + s) snt.
$$

Theorem 4.4 is proved in Section 6. It is as good as one can get, but we can prove it only if $s$ is even. If $s$ is odd, we can do just slightly worse, showing that $\xi_k(n)$ is just below the upper bound. To this effect, we shall prove a more general theorem, and deduce this fact as a corollary. The proof of the theorem is in Section 6.

Theorem 4.5 Suppose that $S_k$ contains a regular matrix $B$ with nonzero rowsums, say of order $n$. Then for any positive integer $t$, there is a graph $G$ of order $nt$ with

$$
\lambda_1^s(G) + \cdots + \lambda_k^s(G) \geq \frac{1}{2} \left( 1 + \sqrt{k} \right) nt - k. \quad (13)
$$

Dividing both sides of (13) by $nt$ and letting $t \to \infty$, we obtain the following corollary.

Corollary 4.6 If $S_k$ contains a regular matrix $B$ with nonzero rowsums, then

$$
\lim_{n \to \infty} \frac{\xi_k(n)}{n} = \frac{1 + \sqrt{k}}{2}.
$$

Let us note that the premise that $S_k$ contains a regular matrix, with nonzero rowsums is not difficult to satisfy. Indeed, Theorem 2.24 implies that for any integer $s \geq 2$, the set $S_{s^2}$ contains a regular matrix with nonzero rowsums.

Theorem 4.3 does not shed any light on the case when $k$ is not an exact square, so we suggest the following concrete conjecture.

Conjecture 4.7 There exist infinitely many integers $k$ such that

$$
\lim_{n \to \infty} \frac{\xi_k(n)}{n} < \frac{1 + \sqrt{k}}{2}.
$$

We end up this section with the easy asymptotics

$$
\frac{\sqrt{k}}{2} \leq \lim_{n \to \infty} \frac{\xi_k(n)}{n} \leq \frac{1 + \sqrt{k}}{2},
$$

whose proof is omitted.
5 Notation, background, and support

For graph notation and concepts undefined here, the reader is referred to [4]. For general reference on graph spectra, see [8]; for reference on Hadamard matrices and symmetric Latin squares, see [6] and [19]; for reference on strongly regular graphs and their eigenvalues, see [10].

We write $I_n$ and $J_n$ for the identity and the all ones matrix of order $n$. The $n$-dimensional vector of all ones is denoted by $j_n$. The usual, the Kronecker product of two matrices $A$ and $B$ is denoted by $A \otimes B$. We recall that if $A$ and $B$ are square, then the spectrum of $A \otimes B$ consists are all products of eigenvalues of $A$ and eigenvalues of $B$, with multiplicities counted. Also, the Kronecker product of symmetric matrices is symmetric.

In this paper regular matrix means a matrix whose rowsums are equal.

Next, we shall give necessary details on Weyl’s inequalities, graphs blowups, Taylor strongly regular graphs, and symmetric Latin squares.

5.1 Weyl’s inequalities

If $A$ is a Hermitian matrix of order $n$, write $\lambda_1 (A), \ldots, \lambda_n (A)$ for its eigenvalues ordered as $\lambda_1 (A) \geq \cdots \geq \lambda_n (A)$. Weyl proved the following useful inequalities for the eigenvalues of sums of Hermitian matrices, (see, e.g. [17], p. 181):

Let $A$ and $B$ be Hermitian matrices of order $n$, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then

$$\lambda_i (A) + \lambda_j (B) \leq \lambda_{i+j-n} (A + B), \text{ if } i + j \geq n + 1.$$  

The following two immediate corollaries are used throughout the paper.

**Proposition 5.1** Suppose that $A'$ is a symmetric $(0,1)$-matrix of order $n$. If $A$ is the matrix obtained by zeroing the main diagonal of $A'$ and $1 \leq k \leq n$, then

$$\lambda_k (A) \geq \lambda_k (A') - 1.$$ 

Indeed, $X = A' - A$ is a $(0,1)$-diagonal matrix, and so $\lambda_1 (A) \leq 1$. Therefore,

$$\lambda_k (A) + 1 \geq \lambda_k (A) + \lambda_1 (A) \geq \lambda_k (A').$$

**Proposition 5.2** If $G$ is a graph of order $n$ and $2 \leq k \leq n$, then

$$\lambda_k (G) + \lambda_{n-k+2}( \overline{G}) \leq -1.$$ 

Indeed, if $A$ and $\overline{A}$ are the adjacency matrices of $G$ and $\overline{G}$, then $A + \overline{A}$ is the adjacency matrix of the complete graph $K_n$. Hence $\lambda_k (G) + \lambda_{n-k+2}( \overline{G}) \leq \lambda_k (K_n) = -1$. 

18
5.2 Blowups of graphs and their eigenvalues

Given a graph $G$ and an integer $t \geq 1$, replace each vertex of $G$ by an independent set on $t$ vertices and each edge of $G$ by a complete bipartite graph $K_{t,t}$. Write $G^{(t)}$ for the resulting graph and call it a blowup of $G$.

If $G$ is a graph of order $n$, then $G^{(t)}$ is a graph of order $nt$ and its adjacency matrix $A(G^{(t)})$ is given by the equation

$$A(G^{(t)}) = A(G) \otimes J_t.$$ 

This algebraic representation of $A(G^{(t)})$ gives a key to its spectrum:

**Proposition 5.3** If $t \geq 1$ and $G$ is a graph of order $n$, with eigenvalues $\lambda_1(G), \ldots, \lambda_n(G)$, then the eigenvalues of $G^{(t)}$ are $\lambda_1(G)t, \ldots, \lambda_n(G)t$, together with $(t-1)n$ additional zeros.

Most often we shall use the following variation of the blow-up operation: given a graph $G$ and an integer $t \geq 1$, replace each vertex of $G$ by a complete graph on $t$ vertices and each edge of $G$ by a complete bipartite graph $K_{t,t}$. Write $G^{[t]}$ for the resulting graph and call it a closed blowup of $G$.

If $G$ is a graph of order $n$, then $G^{[t]}$ is a graph of order $nt$ and its adjacency matrix $A(G^{[t]})$ is given by the equation

$$A(G^{[t]}) = (A(G) + I_n) \otimes J_t - I_{nt}.$$ 

This algebraic representation of $A(G^{[t]})$ can be used to find the spectrum of $G^{[t]}$:

**Proposition 5.4** If $t \geq 1$ and $G$ is a graph of order $n$, with eigenvalues $\lambda_1(G), \ldots, \lambda_n(G)$, then the eigenvalues of $G^{[t]}$ are $\lambda_1(G)t+1, \ldots, \lambda_n(G)t+1$, together with $(t-1)n$ additional $-1$’s.

5.3 Taylor’s strongly regular graphs and their complements

In [31, 32] Taylor came up with a remarkable family of strongly regular graphs $T(q)$, defined for every odd prime power $q$, and with parameters

$$v = q^3, \quad k = \frac{1}{2} (q-1)(q^2+1), \quad a = \frac{1}{4} (q-1)^3 - 1, \quad c = \frac{1}{4} (q-1)(q^2+1).$$

Following the general rules, one finds that the eigenvalues of $T(q)$ are

$$\lambda_1(T(q)) = \frac{1}{2} (q-1)(q^2+1) \text{ with multiplicity 1;}$$

$$\lambda_2(T(q)) = \frac{1}{2} (q-1), \text{ with multiplicity } (q-1)(q^2+1);$$

$$\lambda_n(T(q)) = -\frac{1}{2} (q^2+1), \text{ with multiplicity } q(q-1).$$
The complement $\overline{T(q)}$ is a strongly regular graph with parameters

$$v = q^3, \quad k = \frac{1}{2} (q + 1) (q^2 - 1), \quad a = \frac{1}{4} (q + 3) (q^2 - 3), \quad c = \frac{1}{4} (q + 1) (q^2 - 1).$$

For the eigenvalues of $\overline{T(q)}$ one finds that

$$\lambda_1(\overline{T(q)}) = \frac{1}{2} (q + 1) (q^2 - 1) \text{ with multiplicity 1;}$$
$$\lambda_2(\overline{T(q)}) = \frac{1}{2} (q^2 - 1), \text{ with multiplicity } q (q - 1);$$
$$\lambda_n(\overline{T(q)}) = -\frac{1}{2} (q + 1), \text{ with multiplicity } (q - 1) (q^2 + 1).$$

5.3.1 Some analytic properties of Taylor graphs

Below we focus on certain properties of the Taylor graphs that may be of interest to researchers in spectral and extremal graph theory, as well as in quasi-random ([5]) and pseudo-random ([22], [34], [35]) graphs. To simplify the view on the graph $T(q)$ for sufficiently large $q$, we let $q^3 = n$, and disregard low order terms when needed. Then $T(q)$ is a $(n/2)$-regular graph $G$ of order $n$, with the following properties:

1. Every two distinct vertices of $G$ have $\approx n/4$ common neighbors, and the same holds for $\overline{G}$. Therefore, both $G$ and $\overline{G}$ are quasi-random (pseudo-random) graphs of density $1/2$;

2. For the spectrum of $G$ one finds that

$$\lambda_1(G) \approx n/2 \text{ with multiplicity 1;}$$
$$\lambda_2(G) \approx n^{1/3}/2, \text{ with multiplicity } \approx n;$$
$$\lambda_n(G) \approx -n^{2/3}/2, \text{ with multiplicity } n^{2/3}.$$

Therefore, almost all eigenvalues of $G$ are positive.

3. Nonetheless, the sum of squares of the non-principal positive eigenvalues of $G$ is a vanishing proportion of the sum of squares of all eigenvalues:

$$\sum_{\lambda_i(G) > 0, i > 1} \lambda_i^2(G) \approx \frac{1}{2} n^{5/3} = o(1) \sum_{i=1}^{n} \lambda_i^2(G) = o(1) e(G).$$

4. For the spectrum of $\overline{G}$ one finds that

$$\lambda_1(\overline{G}) \approx n/2 \text{ with multiplicity 1;}$$
$$\lambda_2(\overline{G}) \approx n^{2/3}/2, \text{ with multiplicity } \approx n^{2/3};$$
$$\lambda_n(\overline{G}) \approx -n^{1/3}/2, \text{ with multiplicity } \approx n.$$

Therefore, almost all eigenvalues of $\overline{G}$ are negative.
5. Nonetheless, the sum of squares of the negative eigenvalues of $G$ is a vanishing proportion of the sum of squares of all eigenvalues:

\[
\sum_{\lambda_i(G) < 0} \lambda_i^2(G) \approx \frac{1}{2} \frac{n^{5/3}}{n} = o(1) \sum_{i=1}^{n} \lambda_i^2(G) = o(1) e(G).
\]

5.4 Some symmetric Latin squares

In the proofs of Theorems 2.21, 2.23, and 2.24 we shall use two types of symmetric Latin squares: back-circulant Latin square and symmetric Latin square with constant diagonal. These constructions are simple and well-known, but for reader’s sake we shall describe them below.

Let $s$ be a positive integer. The back-circulant Latin square of size $s$ with symbol set \{1, \ldots, s\} is an $s \times s$ square matrix $L = [l_{i,j}]$, with $l_{i,j}$ given by

\[
l_{i,j} = ((i + j) \mod s) + 1, \quad 1 \leq i, j \leq s.
\]

Obviously $L$ is a symmetric Latin square and its entries belong to \{1, \ldots, s\}.

Note that if $L$ is a symmetric Latin square of odd order with symbol set $S$, then every symbol $s \in S$ occurs above the main diagonal as many times as below it; hence, $s$ also occurs on the main diagonal, as the total number of occurrences of $s$ is odd. Therefore, the main diagonal of $L$ contains each symbol exactly once.

Next we want to construct symmetric Latin squares with constant diagonals. By the above observation, the order of such Latin square cannot be odd, and for any even $s$, we shall give a construction, which seems well-known: we borrow it from [19]. Thus, let $s$ be an even positive integer, and define an $s \times s$ square matrix $L = [l_{i,j}]$, with entries given by

\[
l_{i,j} = \begin{cases}  
s, & \text{if } 1 \leq i \leq s \text{ and } i = j; 
((i + j) \mod (s - 1)) + 1, & \text{if } 1 \leq i < s, 1 \leq j < s, \text{ and } i \neq j; 
(2j \mod (s - 1)) + 1, & \text{if } i = s \text{ and } 1 \leq j < s; 
(2i \mod (s - 1)) + 1, & \text{if } 1 \leq i < s \text{ and } j = s.
\end{cases}
\]

The matrix $L$ is a symmetric Latin square with symbol set \{1, \ldots, s\} and the symbol $s$ along the main diagonal. For example, for $s = 2, 4,$ and $6$, this construction gives

\[
L = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 4 & 1 & 2 & 3 \\
1 & 4 & 3 & 2 \\
2 & 3 & 4 & 1 \\
3 & 2 & 1 & 4 \end{bmatrix}, \quad L = \begin{bmatrix} 6 & 4 & 5 & 1 & 2 & 3 \\
4 & 6 & 1 & 2 & 3 & 5 \\
5 & 1 & 6 & 3 & 5 & 2 \\
1 & 2 & 3 & 6 & 2 & 4 \\
2 & 3 & 5 & 2 & 6 & 1 \\
3 & 5 & 2 & 4 & 1 & 6 \end{bmatrix}.
\]
6 Proofs of some theorems

6.1 Proofs of Theorems 2.8, 2.9, and 2.10

Proof of Theorem 2.8 We shall prove only the first bound, as the other one follows by Proposition 2.4. Let \( T(q) \) be the complement of the Taylor strongly regular graph \( T(q) \) of order \( q^3 \) (see 5.3 for details). Let \( G \) be a closed blowup of \( T(q) \), i.e., \( G = T(q)^[t] \), and let \( n = tq^3 = v(G) \). Then

\[
c_k \geq \sup \frac{\lambda_k(G)}{n} = \sup \frac{\lambda_{q(q-1)+1}(T(q))t + t - 1}{q^3 t} = \sup \frac{q^2 + 1}{2q^3} - \frac{1}{q^3 t} = \frac{q^2 + 1}{2q^3}.
\]

To finish the proof we need to show that

\[
\frac{q^2 + 1}{2q^3} \geq \frac{1}{2\sqrt{q^2 - q} + 1} = \frac{1}{2\sqrt{k-1} + 1}.
\]

This inequality follows from

\[
\frac{q^2 + 1}{2q^3} > \frac{2q - 1}{4q^2 + 2q - 1} > \frac{1}{2\sqrt{q^2 - q} + 1}
\]

after some simple algebra, which we omit.

Proof of Theorem 2.9 Fix a sufficiently large integer \( k \), and let \( q \) be the smallest prime such that

\[
q(q-1) + 1 \geq k.
\]

A result of Baker, Harman, and Pintz \[3\] on the distribution of primes implies that if \( k \) is sufficiently large, then

\[
q \leq \sqrt{k} + 1/2 + \left(\sqrt{k} + 1/2\right)^{21/40}.
\]

It is not hard to see that if \( k \) is sufficiently large, then \( q < \sqrt{k-1} + \sqrt{k}/2 \). Let \( T(q) \) be the complement of the Taylor graph \( T(q) \), let \( G \) be a closed blowup of \( T(q) \), i.e., \( G = T(q)^[t] \), and set \( n = tq^3 = v(G) \). We see that

\[
\lambda_k(G) = \lambda_2(T(q))t + t - 1 = \frac{1}{2} (q^2 - 1) t + t - 1 > \frac{1}{2} q^2 t.
\]

Hence,

\[
c_k \geq \frac{\lambda_k(G)}{n} > \frac{q^2 t}{2q^3 t} = \frac{1}{2q} > \frac{1}{2\sqrt{k-1} + \sqrt{k}}.
\]

Now, the bound on \( c_{k+1} \) follows in view of Proposition 2.4.

\[\square\]
Proof of Theorem 2.10  In view of Proposition 2.4 if \( k \geq 2 \), we always have \( c_{k+1} \geq c_k \), so our main goal is to prove the bounds on \( c_k \). This proof naturally splits into two cases: \( 5 \leq k \leq 15 \) and \( k \geq 16 \).

If \( 5 \leq k \leq 15 \), we just take an appropriate strongly regular graph \( H \) with parameters \((v, k, a, c)\), and let \( G \) be a closed blowup of \( H \) of order \( n \), i.e., \( G = H[v/n] \), where \( n \) is a multiple of \( v \). In view of Proposition 5.4,

\[ \lambda_k(G) = (\lambda_k(H) + 1) \frac{n}{v} - 1. \]

Using well known sources, say the home page of A. Brouwer, we obtain the following table:

| \( H(v, k, a, c) \) | Eigenvalue of \( H \) | Eigenvalue of \( G = H[v/n] \) |
|---------------------|----------------------|-----------------------------|
| (9, 4, 1, 2)        | \( \lambda_5(H) = 1 \) | \( \lambda_5(G) = \frac{2}{9}n - 1 \) |
| (10, 3, 0, 1)       | \( \lambda_6(H) = 1 \) | \( \lambda_6(G) = \frac{1}{5}n - 1 \) |
| (13, 6, 2, 3)       | \( \lambda_7(H) = \frac{\sqrt{13} - 1}{2} \) | \( \lambda_7(G) = \frac{\sqrt{13} + 1}{26}n - 1 \) |
| (15, 6, 1, 3)       | \( \lambda_8(H) = 1 \) | \( \lambda_8(G) = \frac{2}{15}n - 1 \) |
| (15, 6, 1, 3)       | \( \lambda_9(H) = 1 \) | \( \lambda_9(G) = \frac{2}{15}n - 1 \) |
| (15, 6, 1, 3)       | \( \lambda_{10}(H) = 1 \) | \( \lambda_{10}(G) = \frac{2}{15}n - 1 \) |
| (21, 10, 3, 6)      | \( \lambda_{11}(H) = 1 \) | \( \lambda_{11}(G) = \frac{2}{21}n - 1 \) |
| (21, 10, 3, 6)      | \( \lambda_{12}(H) = 1 \) | \( \lambda_{12}(G) = \frac{2}{21}n - 1 \) |
| (21, 10, 3, 6)      | \( \lambda_{13}(H) = 1 \) | \( \lambda_{13}(G) = \frac{2}{21}n - 1 \) |
| (21, 10, 3, 6)      | \( \lambda_{14}(H) = 1 \) | \( \lambda_{14}(G) = \frac{2}{21}n - 1 \) |
| (21, 10, 3, 6)      | \( \lambda_{15}(H) = 1 \) | \( \lambda_{15}(G) = \frac{2}{21}n - 1 \) |

Now, letting \( n \to \infty \), we obtain

\[ c_5 \geq \frac{2}{9}, \quad c_6 \geq \frac{1}{5}, \quad c_7 \geq \frac{\sqrt{13}/2 + 1/2}{2}, \quad c_8 \geq \frac{2}{15}, \quad c_9 \geq \frac{2}{15}, \quad c_{10} \geq \frac{2}{15}. \]

Likewise, if \( 11 \leq k \leq 15 \), we obtain \( c_k \geq \frac{2}{21} \). These inequalities obviously imply that

\[ c_k \geq \frac{1}{k - 1/2} \]

whenever \( 5 \leq k \leq 15 \).

Now, let \( k \geq 16 \), and let \( q \) be the smallest prime \( q \) such that

\[ q \geq 1/2 + \sqrt{k - 3/4}. \]

(15)

Bertrand’s postulate guarantees that for any real \( x > 3 \), there is a prime \( q \) such that

\[ [x] < q \leq 2[x] - 3. \]

23
Since $2\lfloor x \rfloor - 3 < 2x - 1$, in our case this implies that

$$q < 2 \left(1/2 + \sqrt{k - 1}\right) - 1 = \sqrt{4k - 3} < 2\sqrt{k - 1}. \quad (16)$$

Let $\overline{T(q)}$ be the complement of the Taylor graph $T(q)$, and let $G = \overline{T(q)^{[t]}}$. Since inequality (15) implies that $k \leq q(q - 1) + 1$, in view of (14), we see that

$$\lambda_k(\overline{T(q)}) = \frac{1}{2} \left(q^2 - 1\right),$$
and therefore,

$$\lambda_k(G) = \lambda_k(\overline{T(q)}) + t - 1 = \frac{1}{2} \left(q^2 - 1\right) + t - 1 > \frac{1}{2} q^2 t.$$ 

Now, inequality (16) implies that

$$c_k^* \geq \frac{\lambda_k(G)}{q^2 t} > \frac{1}{2q} > \frac{1}{4\sqrt{k - 1}},$$
completing the proof of Theorem 2.10. \qed

6.2 Proofs of Theorems 2.21, 2.23, and 2.24

The proofs of Theorems 2.21, 2.23, and 2.24 are very close, but for reader’s sake we give them separately. All three proofs exploit the construction of Hadamard matrices due to Kharaghani [20], see also [19], Theorem 4.4.16. We shall vary both the blocks and the underlying Latin square, so the reader is referred to [5.4] for necessary details about Latin squares. The idea of using symmetric Latin squares with constant diagonal is borrowed from Haemers [14], Theorem 2, and Ionin and Shrikhande [19], Corollary 5.3.17.

Proof of Theorem 2.21 Suppose that $L = [l_{i,j}]$ is a back-circulant Latin square of size $s$, with symbol set $\{1, \ldots, s\}$. Let $x_1, \ldots, x_s$ be orthogonal $(-1, 1)$-vectors of dimension $n \geq s$ that are also orthogonal to the all ones vector $j_n$. An easy choice is to take the last $k$ rows of a normalized Hadamard matrix of order $n > k$. For each $s = 1, \ldots, s$, define a square matrix $A_s$ by $A_s = -x_s \otimes x_s$. Obviously $A_1, \ldots, A_s$ are symmetric $(-1, 1)$-matrices of size $n$ and rank 1, with diagonal entries equal to $-1$.

Now, let $B$ be the block matrix obtained by replacing each entry $l_{i,j}$ of $L$ by the matrix $A_{l_{i,j}}$. Note that $B$ is a symmetric $(-1, 1)$-matrix of size $sn$. Obviously the diagonal entries of $B$ are equal to $-1$, thus (iii) holds. Note also that

$$A_pA_q = \begin{cases} 0 & \text{if } q \neq p; \\ -nA_p & \text{if } q = p. \end{cases}$$

24
Hence, $B^2$ is block diagonal with each diagonal block equal to $-nA_1 - \cdots - nA_s$. But $-nA_1 - \cdots - nA_s$ is of rank $s$, so it has exactly $s$ nonzero eigenvalues, each equal to $n^2$. Thus, $B$ has $s^2$ nonzero eigenvalues, and their absolute value is equal to $n$.

Writing $n_+$ and $n_-$ for the number of positive and negative eigenvalues of $B$, we have

$$(n_+ - n_-) n = \text{tr } B = -sn,$$

and so, $n_+ - n_- = -s$, implying that

$$n_+ = \begin{pmatrix} s - 1 \\ 2 \end{pmatrix} \quad \text{and} \quad n_- = \begin{pmatrix} s + 1 \\ 2 \end{pmatrix},$$

completing the proof of (i).

To prove (ii) note that each rowsum of each matrix $A_i$ is zero, so the rowsums of $B$ are zero as well. This completes the proof of Theorem 2.21. $\square$

**Proof of Theorem 2.23** Suppose that $L = [l_{i,j}]$ is a symmetric Latin square of order $s$ with constant diagonal. Let $\{1, \ldots, s\}$ be the symbol set of $L$ and $s$ be the diagonal symbol. Next, select $s$ vectors $x_1, \ldots, x_s$ of dimension $n \geq s$ such that $x_1 = j_n$ and every two of the vectors $x_1, \ldots, x_s$ are orthogonal. An easy choice is to take the first $k$ rows of a normalized Hadamard matrix of order $n \geq k$. Let $A_1 = -J_n$, and for each $i = 2, \ldots, s$, define a square matrix $A_i$ by

$$A_i = x_i \otimes x_i.$$  

Obviously $A_1, \ldots, A_s$ are symmetric $(-1, 1)$-matrices of size $n$ and rank 1. Note also that the diagonal entries of $A_s$ are equal to 1.

Now, let $B$ be the block matrix obtained by replacing each entry $l_{i,j}$ of $L$ by the matrix $A_{i,j}$. Note that $B$ is a symmetric $(-1, 1)$-matrix of size $sn$. The diagonal entries of $B$ are equal to 1, thus (iii) holds. Note also that

$$A_pA_q = \begin{cases} 0 & \text{if } q \neq p; \\ nJ_n & \text{if } q = p = 1; \\ nA_p & \text{if } q = p \neq 1. \end{cases}$$

Hence, $B^2$ is block diagonal with each diagonal block equal to $nJ_n + \cdots + nA_s$. Since $nJ_n + \cdots + nA_s$ has exactly $s$ nonzero eigenvalues, each equal to $n^2$, we see that $B$ has $s^2$ nonzero eigenvalues, and their absolute value is equal to $n$.

Writing $n_+$ and $n_-$ for the number of positive and negative eigenvalues of $B$, we have

$$(n_+ - n_-) n = \text{tr } B = sn,$$

and so, $n_+ - n_- = s$, implying that

$$n_+ = \begin{pmatrix} s + 1 \\ 2 \end{pmatrix} \quad \text{and} \quad n_- = \begin{pmatrix} s - 1 \\ 2 \end{pmatrix},$$

25
completing the proof of \((i)\).

To prove \((ii)\) note that
\[
A_p j_n = \begin{cases} 
0 & \text{if } p \neq 1; \\
-n_j_n & \text{if } p = 1.
\end{cases}
\]
Therefore,
\[
B j_{sn} = \begin{pmatrix} A_1 j_n \\ A_1 j_n \\ \vdots \\ A_1 j_n \end{pmatrix} = -n j_{sn}.
\]
Thus, \(j_{kn}\) is an eigenvector of \(B\) to the eigenvalue \(-n\), completing the proof of Theorem 2.23.

**Proof of Theorem 2.24**

Our proof combines the proofs of Theorems 2.21 and 2.23. Suppose that \(L = [l_{i,j}]\) is a back-circulant Latin square of size \(s\), with symbol set \(\{1, \ldots, s\}\). Next, select \(s\) vectors \(x_1, \ldots, x_s\) of dimension \(n \geq s\) such that \(x_1 = j_n\) and every two of the vectors \(x_1, \ldots, x_s\) are orthogonal. Let \(A_1 = -J_n\), and for each \(i = 2, \ldots, s\), define a square matrix \(A_i\) by \(A_i = x_i \otimes x_i\). Let \(B\) be the block matrix obtained by replacing each entry \(l_{i,j}\) of \(L\) by the matrix \(A_{l_{i,j}}\). Note that \(B\) is a symmetric \((-1, 1)\)-matrix of size \(sn\), with \((s-1)n\) diagonal entries equal to 1 and \(n\) diagonal entries equal to \(-1\). Note also that
\[
A_p A_q = \begin{cases} 
0 & \text{if } q \neq p; \\
J_n & \text{if } q = p = 1; \\
A_p & \text{if } q = p \neq 1.
\end{cases}
\]
Hence, \(B\) has \(s^2\) nonzero eigenvalues, and their absolute value is equal to \(n\). Writing \(n_+\) and \(n_-\) for the number of positive and negative eigenvalues of \(B\), we have
\[
(n_+ - n_-) n = \text{tr } B = (s - 2)n,
\]
and so, \(n_+ - n_- = s - 2\), implying that
\[
n_+ = \left( \frac{s + 1}{2} \right) - 1 \quad \text{and} \quad n_- = \left( \frac{s - 1}{2} \right) + 1,
\]
completing the proof of \((i)\).

To prove \((ii)\) let us note that if \(2 \leq i \leq s\), then all rowsums of \(A_p\) are zero. So the rowsums of \(B\) are equal to the rowsums of \(-J_n\), which are equal to \(-n\). \(\square\)
6.3 Proofs of Theorems 4.4 and 4.5

Proof of Theorem 4.4 Let $B$ be a matrix constructed by Theorem 2.23 and let $C = B \otimes J_t$. The properties of $B$ given by Theorem 2.23 imply that $C$ is a symmetric $(-1, 1)$-matrix of order $snt$ such that:
- $C$ has exactly $s^2$ nonzero eigenvalues, of which $\binom{s-1}{2}$ are equal to $nt$ and $\binom{s+1}{2}$ are equal to $-nt$;
- the diagonal entries of $C$ are equal to $-1$;
- the vector $\mathbf{j}_{snt}$ is an eigenvector of $C$ to the eigenvalue $-nt$.

Now, let
$$A = \frac{1}{2} \left( J_{snt} - C \right).$$

Clearly $A$ is a symmetric $(0, 1)$-matrix of order $snt$, with zero diagonal; hence, $A$ is the adjacency matrix of some graph $G$ of order $snt$. Let $x_1 = \mathbf{j}_{snt}, x_2, \ldots, x_{snt}$ be orthogonal eigenvectors to $C$. Note that
$$\frac{1}{2} (J_{snt} - C) \mathbf{j}_{snt} = \left( \frac{snt}{2} + \frac{nt}{2} \right) \mathbf{j}_{snt},$$
so $snt/2 + nt/2$ is an eigenvalue to $G$. Also, for any $i = 2, \ldots, snt$, we see that
$$A x_i = \frac{1}{2} (J_{snt} - C) x_i = -\frac{1}{2} C x_i,$$
so $G$ has $\binom{s+1}{2}$ eigenvalues equal to $-nt/2$ and $\binom{s-1}{2} - 1$ eigenvalues equal to $nt/2$. Therefore,
$$\lambda_1^* (G) + \cdots + \lambda_s^* (G) = \frac{snt}{2} + \frac{nt}{2} + (s^2 - 1) \frac{nt}{2} = \frac{1}{2} (1 + s) \frac{snt}{2},$$
completing the proof of Theorem 4.4.

Proof of Theorem 4.5 Let $\lambda$ be the rowsum of $B$, which clearly is a nonzero eigenvalue of $B$ with eigenvector $\mathbf{j}_n$. Since $B \in S_k$, either $\lambda = n/\sqrt{k}$ or $\lambda = -n/\sqrt{k}$. We shall assume that $\lambda = -n/\sqrt{k}$, for otherwise we just take $-B$ for $B$. Now, for any positive integer $t$, define a symmetric $(0, 1)$-matrix $A'$ by
$$A' = \frac{1}{2} \left( J_{nt} - B \otimes J_t \right),$$
and note that
$$\lambda_1 (A') = \frac{nt}{2} + \frac{nt}{2\sqrt{k}}, \quad \text{and} \quad \lambda_i^* (A') = \frac{nt}{2\sqrt{k}} \quad \text{for} \quad 1 < i \leq k.$$

Next, zero the diagonal of $A'$ and write $A$ for the resulting matrix. Clearly $A$ is a symmetric $(0, 1)$-matrix with zero diagonal, so $A$ is the adjacency matrix of some graph $G$ of order $nt$. Using Weyl’s inequalities (Proposition 5.1), we see that
$$\lambda_1 (G) \geq \frac{nt}{2} + \frac{nt}{2\sqrt{k}} - 1, \quad \text{and} \quad \lambda_i^* (G) \geq \frac{nt}{2\sqrt{k}} - 1 \quad \text{for} \quad 1 < i \leq k.$$
Therefore,
\[
\lambda_1^* (G) + \cdots + \lambda_k^* (G) \geq \left( \frac{nt}{2} + k \frac{nt}{2 \sqrt{k}} \right) - k = \frac{1}{2} \left( 1 + \sqrt{k} \right) nt - k,
\]
completing the proof of Theorem 4.5. \qed

7 A recap for symmetric \((-1, 1)\)-matrices

Many solutions in this paper come from \((-1, 1)\)-matrices. This is not incidental, for if \(G\) is a regular graph, then its adjacency spectrum is linearly equivalent to the spectrum of its Seidel’s matrix, which is a \((0, -1, 1)\)-matrix. But there is more to that: if \(G\) is a \((n/2)\)-regular graph, the Seidel matrix effectively eliminates the largest eigenvalue of \(G\), which may be nuisance in certain spectral problems, like most of the problems discussed in this paper. One cannot but agree that many of the questions raised above for graphs seem more balanced and natural if translated for \((-1, 1)\)-matrices. In this section we explore such translations.

Thus, for any \(k \geq 1\), let us introduce the functions
\[
\Lambda_k (n) = \max_{A \in \mathcal{U}_n} \lambda_k (A) \quad \text{and} \quad \Lambda_k^* (n) = \max_{A \in \mathcal{U}_n} \lambda_k^* (A).
\]

Obviously \(\Lambda_k (n)\) and \(\Lambda_k^* (n)\) are the matrix analogs of \(\lambda_k (n)\) and \(\lambda_k^* (n)\); we do not need an analog to \(\lambda_{-k} (n)\), as \(\mathcal{U}_n\) is closed under negation. Next, in the general spirit of the paper, we raise the problem:

**Problem 7.1** For any \(k \geq 2\), find \(\Lambda_k (n)\) and \(\Lambda_k^* (n)\).

Much of what we have achieved for graphs applies to symmetric \((-1, 1)\)-matrices as well. First, obviously
\[
\Lambda_k (n) \leq \Lambda_k^* (n) \leq n / \sqrt{k}. \tag{17}
\]

Note that for \(\Lambda_k^* (n)\), bound (17) is precise for infinitely many \(k\) and \(n\). Indeed, if \(S_k \neq \emptyset\), for arbitrary large \(n\), we have \(\Lambda_k^* (n) = n / \sqrt{k}\).

Further, in analogy to \(c_k\) and \(c_k^*\), let
\[
d_k = \sup_{n \geq 1} \frac{\Lambda_k (n)}{n} \quad \text{and} \quad d_k^* = \sup_{n \geq 1} \frac{\Lambda_k^* (n)}{n}.
\]

The constants \(d_k\) and \(d_k^*\) are handy, as for any \(n\) and any matrix \(A \in \mathcal{U}_n\), we have
\[
\lambda_k (A) \leq d_k n \quad \text{and} \quad \lambda_k^* (A) \leq d_k^* n.
\]

In turns out that these inequalities are best possible, for one can show that
\[
\lim_{n \to \infty} \frac{\Lambda_k (n)}{n} = d_k \quad \text{and} \quad \lim_{n \to \infty} \frac{\Lambda_k^* (n)}{n} = d_k^*.
\]
Thus, much about $\Lambda_k(n)$ and $\Lambda_k^*(n)$ would be known if we knew $d_k$ and $d_k^*$ or estimates thereof. First, from (17) we immediately get an upper bound

$$d_k \leq d_k^* \leq 1/\sqrt{k}$$

so the difficulty is to find matching lower bounds.

As one may expect, $d_k^*$ is easier to tackle than $d_k$. Indeed, since $d_k^*$ is nonincreasing in $k$, letting $s$ to be the smallest positive integer such that $s^2 \geq k$, in view of $S_{s^2} \neq \emptyset$, we see that $d_k^* \geq 1/s$. But $(s - 1)^2 < k$, and so,

$$\frac{1}{\sqrt{k}} \geq d_k^* \geq \frac{1}{s} > \frac{1}{\sqrt{k} + 1} = \frac{1}{\sqrt{k}} + O \left( k^{-1} \right).$$

This argument does not fit to bound $d_k$, so we need another idea. Since Taylor’s graphs have been useful for $c_k$, we can hope to use them for $d_k$ as well. Thus, let $A(T(q))$ be the adjacency matrix of the complement of the Taylor graph $T(q)$ of order $q^3$. Define the matrix $T \in \mathbb{U}_{q^3}$, by setting

$$T = 2A(T(q)) - J_{q^3}.$$ 

It is not hard to see that the matrix $T$ has three distinct eigenvalues:

$$\lambda_1(T) = q^2 - 1, \quad \text{with multiplicity } q(q - 1);$$
$$\lambda_2(T) = q^2 - q - 1, \quad \text{with multiplicity } 1;$$
$$\lambda_3(T) = -q - 1, \quad \text{with multiplicity } (q - 1)(q^2 + 1).$$

Note in passing that the mapping $A(T(q)) \rightarrow T$ preserves all eigenvalues except $\lambda_1(T(q))$, whose magnitude is reduced essentially to $\lambda_2(T(q))$.

Now, using the Baker, Harman, and Pintz result [3] again, for sufficiently large $k$, we get the asymptotics

$$\frac{1}{\sqrt{k}} \geq d_k \geq \frac{1}{\sqrt{k} + \sqrt[3]{k}} = \frac{1}{\sqrt{k}} + O \left( k^{-2/3} \right).$$

We end up with a question about the maximum Ky Fan $k$-norm of matrices in $\mathbb{U}_n$.

**Problem 7.2** For any $k \geq 2$, find

$$\max_{A \in \mathbb{U}_n} \|A\|_{*k}.$$ 

Without a proof, let us mention the bounds

$$(\sqrt{k} - 1)n \leq \max_{A \in \mathbb{U}_n} \|A\|_{*k} \leq n\sqrt{k}.$$ 

**Acknowledgement.** Part of this paper has been prepared for a talk at the Algebraic Combinatorics Workshop held in the Fall of 2014, at the University of Science and Technology of China, Hefei. I am grateful for the hospitality of the organizers, in particular to prof. Jack Koolen.
References

[1] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient testing of large graphs, *Combinatorica*, **20** (2000) 451–476.

[2] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Appl. Math.* **161** (2013), 466–546.

[3] R. C. Baker, G. Harman, J. Pintz, The difference between consecutive primes II, *Proc. London Math. Soc.* **83** (2001), 532–562.

[4] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[5] F. Chung, R. Graham, R. M. Wilson, Quasi-random graphs. *Combinatorica* **9** (1989), 345–362.

[6] R. Craigen and H. Kharaghani, Hadamard matrices and Hadamard designs, in Handbook of combinatorial designs, 2ed., *C. Colbourn, J.H. Dinitz ed.*, Chapman & Hall/CRC press, Boca Raton, (2006), pp. 273–280.

[7] P. Csikvári, On a conjecture of V. Nikiforov, *Disc. Math.* **309** (2009), 4522-4526.

[8] D. Cvetković, P. Rowlinson, and S. Simić, An Introduction to the theory of graph spectra, *LMS Student Texts 75*, Cambridge, 2010, pp. vii+364.

[9] J. Ebrahimi, B. Mohar, V. Nikiforov, and A.S. Ahmady, On the sum of two largest eigenvalues of a symmetric matrix, *Linear Algebra Appl.* **429** (2008), 2781–2787.

[10] C. D. Godsil, G. F. Royle. Algebraic Graph Theory, Springer-Verlag, New York), 2001, xi+439 pp.

[11] D. Gregory, D. Hershkowitz, and S. Kirkland, The spread of the spectrum of a graph, *Linear Algebra Appl.***332** (2001), 23–35.

[12] I. Gutman, The energy of a graph, *Ber. Math.-Stat. Sekt. Forschungszent. Graz* **103** (1978), 1–22.

[13] I. Gutman, X. Li, and Y. Shi, *Graph Energy*, New York, Springer, 2012, 266 pp.

[14] W. Haemers, Strongly regular graphs with maximal energy, *Linear Algebra Appl.* **429** (2008) 2719–2723.

[15] W. Haemers and Q. Xiang, Strongly regular graphs with parameters \((4m^4, 2m^4+m^2, m^4+m^2, m^{4}+m^{2})\) exist for all \(m > 1\), *Eur. J. Combin.* **31** (2010), 1553–1559.

[16] Y. Hong, Bounds of eigenvalues of graphs, *Discrete Math.* **123** (1993), 65–74.
[17] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. xiii+561 pp.

[18] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1994, viii+607 pp.

[19] Y.J. Ionin and M. Shrikhande, Combinatorics of Symmetric Designs, *Cambridge University Press*, Cambridge, 2006, xiii+534 pp.

[20] H. Kharaghani, New classes of weighing matrices, *Ars Combin.* 19 (1985) 69–72.

[21] J.H. Koolen and V. Moulton, Maximal energy graphs, *Adv. Appl. Math.* 26 (2001), 47–52.

[22] M. Krivelevich and B. Sudakov, Pseudo-random graphs, in: *More Sets, Graphs and Numbers*, Bolyai Society Mathematical Studies 15, Springer, 2006, pp. 199-262.

[23] B. Mohar, On the sum of $k$ largest eigenvalues of graphs and symmetric matrices, *J. Combin. Theory, Ser. B* 99 (2009), 306–313.

[24] V. Nikiforov, Linear combinations of graph eigenvalues, *Electron. J. Linear Algebra* 15 (2006), 329–336.

[25] V. Nikiforov, Graphs and matrices maximal energy, *J. Math. Anal. Appl.* 327 (2007), 735-738.

[26] V. Nikiforov, Eigenvalue problems of Nordhaus-Gaddum type, *Discrete Math.* 307 (2007), 774–780.

[27] V. Nikiforov, On the sum of $k$ largest singular values of graphs and matrices, *Linear Algebra Appl.* 435 (2011), 2394–2401.

[28] V. Nikiforov and X.Y. Yuan, More eigenvalue problems of Nordhaus-Gaddum type, *Linear Algebra Appl.* 451 (2014), 231–245.

[29] E. Nosal, Eigenvalues of Graphs, Master’s thesis, University of Calgary, 1970.

[30] D.L. Powers, Bounds on graph eigenvalues, *Linear Algebra Appl.* 117 (1979), 1–6.

[31] D.E. Taylor, Some topics in the theory of finite groups, *PhD Thesis*, University of Oxford, 1971.

[32] D.E. Taylor, Regular 2-graphs, *Proc. London Math. Soc.* 35 (1977) 257–274.

[33] T. Terpai, Proof of a conjecture of V. Nikiforov, *Combinatorica*, 31 (2011), 739-754.

[34] A. Thomason, Pseudorandom graphs, *Proceedings in Random graphs, Pozna\'n, 1985, North-Holland Math. Stud.*, 144, North-Holland, Amsterdam, 1987, pp. 307–331.
[35] A. Thomason, Random graphs, strongly regular graphs and pseudorandom graphs, *Surveys in combinatorics 1987*, LMS Lecture Note Ser., 123, Cambridge University Press, Cambridge (1987), pp. 173–195.

[36] F. Zhang and Z. Chen, Ramsey numbers, graph eigenvalues, and a conjecture of Cao and Yuan, *Linear Algebra Appl.* 458 (2014), 526–533.