Knot Theory of Coxeter type B and its physical applications

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Abstract

Braid groups may be defined for every Coxeter diagram. Artin’s braid group is of type A. Analogs of Temperley-Lieb, Hecke and Birman-Wenzl algebras exist for B-type.

Our general hypothesis is that the braid group of B-type replaces Artin’s braid group in most physical applications if the model is equipped with a nontrivial boundary. Solutions of a Potts model with a boundary and the reflection equation illustrate this principle.

Braided tensor categories of B-type and dually Coxeter-B braided Hopf algebras are introduced. The occurrence of such categories in QFT on a half plane is discussed.

1 Introduction

To every Coxeter diagram a braid group is associated that has the same presentation as the Coxeter group but has no degree 2 relations for the generators. Artin’s braid group is the braid group $\mathbb{Z}_A^n$ of Coxeter type A.

Definition 1 The braid group $\mathbb{Z}_B^n$ of Coxeter type B is generated by $X_0, X_1, \ldots, X_{n-1}$ with relations

$$X_iX_j = X_jX_i \quad \text{if} \quad |i-j| > 1 \quad (1)$$
$$X_iX_jX_i = X_jX_iX_i \quad \text{if} \quad i, j \geq 1, |i-j| = 1 \quad (2)$$
$$X_0X_1X_0X_1 = X_1X_0X_1X_0 \quad (3)$$

Generators $X_i, i \geq 1$ satisfy the relations of Artin’s braid group.

$\mathbb{Z}_B^n$ may be graphically interpreted (cf. figure [4]) as symmetric braids or cylinder braids: The symmetric picture shows it as the group of braids with $2n$ strands (numbered $-n, \ldots, -1, 1, \ldots, n$) which are fixed under a 180 degree rotation about the middle axis. In the cylinder picture one adds a single fixed line (indexed 0) on the left and obtains $\mathbb{Z}_B^n$ as the group of braids with $n$
strands that may surround this fixed line. The generators \( X_i, i \geq 0 \) are mapped to the diagrams \( X_i^{(G)} \) given in figure 1.

The group algebra of the B braid group has finite dimensional quotients which generalize the Temperley-Lieb, Hecke and Birman-Murakami-Wenzl algebras. All these algebras support Markov traces that give rise to B-type versions of the link polynomials of Jones, HOMFLY and Kauffman.

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**Definition 2** The Temperley-Lieb Algebra \( \mathcal{T}_B \) of Coxeter type B over a ring with parameters \( c, c', d \) is generated by \( e_0, e_1, \ldots, e_{n-1} \) and relations

\[
\begin{align*}
e_1 e_0 e_1 &= c' e_1 & e_0^2 &= de_0 & e_i^2 &= c e_i \\
e_i e_j e_i &= e_i & |i-j|=1, i, j \geq 1 \\
e_i e_j &= e_j e_i & |i-j| > 1
\end{align*}
\]

The Hecke algebra \( \mathcal{H}_B \) has been studied by Dipper/James, tom Dieck, Lambropoulou, Ariki and others.

**Definition 3** The Hecke Algebra of B-type \( \mathcal{H}_B \) has generators \( X_0, X_1, \ldots, X_{n-1} \) over a ring with parameters \( Q, Q_0 \) and relations (1)-(3) and:

\[
X_0^2 = (Q_0 - 1)X_0 + Q_0 \quad X_i^2 = (Q - 1)X_i + Q \quad i \geq 0
\]

The reduced Birman-Wenzl algebra has been studied in [6].

**Definition 4** Let \( B^*B \) be the algebra generated by invertible \( Y, X_1, \ldots, X_{n-1} \) over a ring with parameters \( \lambda, q, q_0 := q^{-1}, q_1 \). The relations are: (4)-(6) and:

\[
\begin{align*}
\delta &= q - q^{-1} & x &= 1 - \frac{\lambda - \lambda^{-1}}{\delta} & e_i &= 1 - \frac{X_i - X_i^{-1}}{q - q^{-1}}
\end{align*}
\]
\[ X_i e_i = e_i X_i = \lambda e_i \quad \text{and} \quad e_i x^{\pm 1} e_i = \lambda^\pm 1 e_i \] (9)

\[ X_1 Y X_1 Y = Y X_1 Y X_1 \quad Y X_i = X_i Y \quad i > 1 \] (10)

\[ Y^2 = q_1 Y + q_0 \quad Y X_1 Y e_1 = e_1 \] (11)

**Proposition 1** \( B^*B_n \) is a semi simple algebra. The simple components are indexed by the set \( \tilde{\Gamma}_n \) pairs of Young diagrams of size \( n, n - 2, \ldots \).

\[
B^*B_n = \bigoplus_{(\mu, \lambda) \in \tilde{\Gamma}_n} B^*B_{\mu,\lambda}
\] (12)

The Bratteli rule for restrictions of modules: A simple \( B^*B_{\mu,\lambda} \) module \( V_{(\nu,\rho)} \), \((\nu,\rho) \in \tilde{\Gamma}_n \) decomposes into \( B^*B_{\mu,\lambda} \) modules such that the \( B^*B_{\mu,\lambda} \) module \((\mu, \lambda) \in \tilde{\Gamma}_n \) occurs iff \((\mu, \lambda)\) may be obtained from \((\nu, \rho)\) by adding or removing a box. There exists a nondegenerate Markov trace on \( B^*B_n \).

The dimension of \( B^*B_n \) is \( 2^n(2n - 1)!! \). In [3] a combinatorial basis was found.

\[
B^*B_0 \quad (\cdot, \cdot)
\]

\[
\begin{array}{c}
\text{B}^*\text{B}_1 \\
\quad (\emptyset, \cdot) \\
\quad (\cdot, \emptyset) \\
\end{array}
\]

\[
\begin{array}{c}
\text{B}^*\text{B}_2 \\
(\emptyset \square, \cdot) \\
(\cdot, \cdot) \\
(\emptyset, \emptyset) \\
\emptyset \square \square
\end{array}
\]

**Figure 2:** The Bratteli digram of \( B^*B_n \)

Two dimensional integrable systems are described by solutions of the spectral parameter dependent Yang-Baxter-Equation (YBE) that reads: \( R_1(t_1) R_2(t_1 t_2) R_1(t_2) = R_2(t_2) R_1(t_1 t_2) R_2(t_1) \). If the system is restricted to a half plane an additional matrix \( K(t) \in \text{End}(V) \) is needed to describe reflections. It has to fulfill Sklyanin’s reflection equation:

\[
R(t_1/t_2)(K(t_1) \otimes 1) R(t_1 t_2) (K(t_2) \otimes 1) = (K(t_2) \otimes 1) R(t_1 t_2) (K(t_1) \otimes 1) R(t_1/t_2)
\] (13)

Solutions of the Yang Baxter equation can be obtained from the standard (type A) Birman-Wenzl algebra by the following Baxterization procedure:

\[
R_i(t) = -\delta t(t + q \lambda^{-1}) + (t - 1)(t + q \lambda^{-1}) X_i + \delta t(t - 1)e_i
\] (14)

**Proposition 2** \( K(t) = (t^2 q_1 (1 - t^2)^{-1} + Y) f_1(t) \) is (for all \( f_1 \)) a solution of the reflection equation (13).

The proof of this result is given in [7].

\( B^*B_n \) supports a Markov trace that can be used to define a link invariant for links of B-type which are links in a solid torus. There is an analog of Markov’s theorem for type B links found by S. Lambrodopoulou in [4]. It implies that there exists an extension of the Kauffman polynomial to braids of B-type. For a B-type link \( \beta \) that is the closure of a B-braid \( \beta \in ZB_n \) we define:
Definition 5 The B-type Kauffman polynomial of a B-link $\hat{\beta}$ is defined to be
\[
L(\hat{\beta}, n) := x^{n-1} \lambda^{e(\hat{\beta})} \text{tr}(\hat{\beta}) \quad \hat{\beta} \in \mathbb{ZB}_n
\] (15)
e : \mathbb{ZB}_n \to \mathbb{Z} \mathbb{Z}
is the exponential sum with $e(X_i) = 1, e(Y) = 0$.

2 The Potts-Model with a boundary

We investigate a generalization of the ordinary Potts model \[10\] by including a reflecting boundary. Hence we have besides the usual lattice of sites (denoted by \(V\)) a wall interacting with it. An example is shown in figure 3. The dotted lines indicate interaction bonds with the wall while the solid lines are usual bonds between sites. Each site supports a 'spin' which may occupy one of \(f\) states.

The set of all states is \(S = \{S : V \to \{0, 1, \ldots, f - 1\}\}\). The partition function (with \(k\) being the Boltzmann constant and \(T\) the temperature) is
\[
Z_G = \sum_{S \in S} \exp \left( \frac{-E(S)}{kT} \right)
\] (16)
with the Hamiltonian \((\delta(x, y)\) is the Kronecker symbol with values 0, 1)
\[
E(S) = \sum_{(i, j) \in B_1} \delta(S_i, S_j) + \kappa \sum_{i \in B_0} (1 - \delta(0, S_i))
\] (17)

Here \(B_0 \subset V\) is the set of sites which have boundary bonds, and \(B_1\) is the set of inner bonds. The first term in \(E(S)\) is the usual Hamiltonian of the Potts model. The second term introduces the boundary condition.

In analogy with Kauffman’s treatment of the ordinary Potts model [10] we associate a link diagram of type B with the boundary lattice. The partition function may then be expressed as a normalization of the Markov trace of this link.

Figure 3: A lattice with boundary together with its graph
3 Coxeter-B braided categories

The language of braided tensor categories (BTCs) turned out to be the right framework for braid group applications from knot theory to QFT. In this section we generalize it to Coxeter type B. Details are given in [9].

**Definition 6** Let \( \mathcal{C} \) be a rigid ribbon BTC. A Coxeter-B braided category over \( \mathcal{C} \) is an embedding of \( \mathcal{C} \) in a rigid monoidal category \( \hat{\mathcal{C}} \) which has the same objects and obeys the following list of axioms hold. Morphisms \( \text{Mor}(X,Y) := \text{Mor}_\mathcal{C}(X,Y) \) are said to be local and morphisms \( \text{Mor}(G)(X,Y) := \text{Mor}_{\hat{\mathcal{C}}}(X,Y) \) are said to be global.

\[
\forall X \exists b_X \in \text{Mor}^{(G)}(X,X) \quad (18)
\]

\[
b_Y f = fb_X \quad \forall f \in \text{Mor}(X,Y) \quad (19)
\]

\[
id_X \otimes b_Y = \Psi_{Y,X}(b_Y \otimes \text{id}_X)\Psi_{X,Y} \quad (20)
\]

\[
b_X \otimes b_Y = b_{X \otimes Y}\Psi_{Y,X}\Psi_{X,Y} = \Psi_{Y,X}\Psi_{X,Y}b_{X \otimes Y} \quad (21)
\]

\[
b_X^* = \sigma(X^*)^2b_X^{-1} \quad (22)
\]

Note that \( \Psi \) is a braiding of \( \mathcal{C} \), not of \( \hat{\mathcal{C}} \). This makes (20) possible which otherwise would give a contradiction to naturality of \( \Psi \). The axioms imply:

\[
\Psi_{Y,X}(b_Y \otimes \text{id}_X)\Psi_{X,Y}(b_X \otimes \text{id}_Y) = (b_X \otimes \text{id}_Y)\Psi_{Y,X}(b_Y \otimes \text{id}_X)\Psi_{X,Y} \quad (23)
\]

\[
ev_X(b_{X^*} \otimes \text{id}_X)\Psi_{X^*,X}(b_X \otimes \text{id}_{X^*}) = ev_X\Psi_{X,X^*} \quad (24)
\]

A restricted Coxeter-B braided tensor category is defined similarly, but with (20) replaced by the two relations above. The graphical origin of this axioms becomes clear by drawing pictures. A nice example of a Coxeter-B braided category is given by the category of Aplimorphisms of a quasitriangular Hopf algebra.

**Definition 7** A restricted Coxeter-B braided ((weak) quasi) Hopf algebra \( H \) is a ((weak) quasi) quasi triangular ribbon Hopf algebra with an element \( v \in H \) such that

\[
R_{2,1}v_2Rv_1 = v_1R_{2,1}v_2 R \quad \epsilon(v) = 1 \quad (25)
\]

\[
\Delta(v) = R^{-1}(1 \otimes v)R(v \otimes 1) \quad (26)
\]

Such algebras can be constructed explicitly from the quantum Weyl group \[3].

We have the following Tannaka-Krein style duality between Coxeter B-type braided Hopf algebras and B-type tensor categories.

**Proposition 3** The representation category \( \text{Rep}(H) \) of a restricted Coxeter-\( B \) braided ((weak) quasi) Hopf algebra algebra is a restricted \( B \)-braided tensor category.

Contrary, if \( \hat{\mathcal{C}} \) is a Coxeter-B braided tensor category and \( F: \hat{\mathcal{C}} \to \text{Vec} \) is a ((weak) quasi) tensor functor in the restricted sense that the naturality equation
Consider a quantum field theory specified by a net of local observables \( \mathcal{A}(\mathcal{O}) \) living on the half plane \( \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \} \). We assume that boundary conditions are imposed in such a way that we have reflection at the line \( (0, \mathbb{R}) \) by letting the full translation group \( \mathbb{R}^2 \) act on the half plane. This action is not free and this will lead to global morphisms and hence to the occurrence of a Coxeter-B braided tensor category.

Fields shall be localized in double cones. Here we extend the usual notion of a double cone to include all translations of double cones. Thus we also have regions like those in the left of figure 4. This figure also shows the causal complement \( \mathcal{O}' \) of a double cone. A double cone that does not touch the boundary shall be called regular. Further we assume isotony and locality and the existence of a vacuum representation \( \pi_0 \) which is translation invariant and faithful for all local algebras \( \mathcal{A}(\mathcal{O}) \) of regular double cones \( \mathcal{O} \).

Now, let \( \mathcal{O}_1, \mathcal{O}_2 \) be two causally disjoint double cones of equal size as shown in the right half of figure 4. Further let \( \rho_1 \) be a transportable morphisms localized in \( \mathcal{O}_1 \) and let \( \rho_2 \sim \rho_1 \) be localized in \( \mathcal{O}_2 \). Assume that \( \rho_1 \) (and thus \( \rho_2 \)) is irreducible in the sense that \( \pi_0 \circ \rho_1 \) is an irreducible representation of \( \mathcal{A} \). There are two translations that map \( \mathcal{O}_1 \) onto \( \mathcal{O}_2 \): A direct one and one that passes the reflecting boundary. Thus we have two charge transporters \( U, V \in \text{Mor}^G(\rho_1, \rho_2) := \{ T \in \mathcal{A} \mid T\rho_1(A) = \rho_2(A)T\forall A \in \mathcal{A} \} \). Hence we have a self intertwiner \( Y := U^{-1}V \in \text{Mor}^G(\rho_1, \rho_1) \). We see that the vacuum representation may not be faithful in the presence of a boundary. The localized and transportable morphisms form a Coxeter-B braided tensor category.

![Figure 4: The half plane with various double cones (on the left) and with the reflecting transportation that leads to global intertwiners (on the right)](image-url)

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