A Classification of Connected $f$-factor Problems inside $NP^*$

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Abstract

Given an undirected graph $G = (V,E)$ with $|V| = n$, and a function $f : V \to \mathbb{N}$, we consider the problem of finding a connected $f$-factor in $G$. This problem is $NP$-Complete when $f(v) \geq n'$ for every $v$ in $V$ and a constant $\epsilon > 0$. We design an algorithm to check for the existence of a connected $f$-factor, for the case where $f(v) \geq n/g(n)$, for all $v$ in $V$ and $g(n)$ is polylogarithmic in $n$. The running time of our algorithm is $\tilde{O}(n^2 g(n))$. As a consequence of this algorithm, we conclude that the complexity of connected $f$-factor for the case we consider is unlikely to be $NP$-Complete unless the Exponential Time Hypothesis (ETH) is false. Secondly, under the ETH assumption, we show that the problem is also unlikely to be in $P$ for $g(n)$ in $O((\log n)^{1+\epsilon})$ for any constant $\epsilon > 0$. These results show that for each $\epsilon > 0$ and $g(n)$ in $O((\log n)^{1+\epsilon})$, connected $f$-factor problem for $f(v) \geq n/g(n)$ is in $NP$-Intermediate unless the ETH is false. Further, for any constant $c > 0$, when $g(n) = c$, our algorithm for connected $f$-factor runs in polynomial time. Finally we extend our algorithm to compute a minimum weight connected $f$-factor in edge weighted graphs in the same asymptotic time bounds.

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Introduction

Let $G = (V,E)$ be an undirected graph with $n$ vertices and $f : V \to \mathbb{N}$ be a function. An $f$-factor of $G$ is a spanning subgraph $H$ such that $d_H(v) = f(v)$, for each $v$ in $V$. The problem of deciding whether a given graph $G$ has an $f$-factor is a well studied problem over many years [1,4,8,14,15,17] and the problem is shown to be polynomial time solvable by Tutte [18]. When edges have weights, a simple modification to Tutte’s reduction solves the minimum weighted $f$-factor problem. A connected $f$-factor is an $f$-factor which is connected. For the case when $f(v) = 2$ for all $v$ in $V$, a connected $f$-factor is a Hamiltonian cycle [20] and is $NP$-Complete to decide. In fact, Cheah and Corneil [2] showed that the connected $f$-factor problem is $NP$-Complete where $f(v) = d$ for each $v$ in $V$ and an integer constant $d > 1$. For $f(v) \geq \lceil \frac{n}{2} \rceil$ for every $v$ in $V$, deciding a connected $f$-factor is same as deciding whether there exists an $f$-factor or not. This is because in this case, any $f$-factor turns out to be connected, due to Ore [13].

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and Dirac [5].

**Past Work on Connected Factors.** There has been an extensive study on connected \([a, b]-factors\) in the literature over the past twenty years. An \([a, b]-factor\) is a subgraph \(H\) of a graph \(G\) such \(a \leq d_H(v) \leq b\), for each \(v\) in \(V\). There are many results on sufficiency conditions for a graph to have a connected \([a, b]-factor\). For example, when \(\delta(G) \geq \frac{n}{2}\) the Graph is Hamiltonian, due to Ore [13] and Dirac [5]. Also, if the sum of degrees of every pair of non-adjacent vertices is at least \(n - 1\), then the graph has a Hamilton path, and this is a connected \([1, 2]-factor\). Similarly, by relating the size of the maximum independent set and the vertex connectivity of a graph, there are sufficiency conditions for the existence of connected \([a, b]-factors\). The survey article by Kouider and Vestergaard [19] and Plummer [14] present more results on connected \(f\)-factors.

**Our Work.** To the best of our knowledge, our study is the first of this kind in the area of connected factors. We are motivated by this line of study with an aim to classify functions \(f\) for which the connected \(f\)-factor problem is polynomial time solvable and those for which the problem is NP-Complete. In particular, our interest is to obtain a dichotomy for connected \(f\)-factor problem based on \(f\). To conceptualize the nature of \(f\), \(f(v)\) is taken to be at least \(n/g(n)\) for each \(v\) in \(V\) where \(g(n)\) is a function in \(o(n)\). In \([3]\) we have shown that the problem is NP-Complete when \(g(n)\) is \(n^{1-\epsilon}\) for any constant \(\epsilon\) between 0 and 1. In recent work \([12]\), we showed that the problem is polynomial time solvable if \(g(n) = 3\). While connected \([a, b]-factors\) are studied extensively from the point of view of identifying sufficient conditions, our work is on understanding how the computational complexity of connected \(f\)-factor problem varies with \(f\). We summarize our results as follows:

1. An algorithm running in time \(\tilde{O}(n^{2g(n)})\) for deciding the existence of connected \(f\)-factor in a graph \(G\) where \(f(v) \geq n/g(n)\) for each \(v\) in \(V\) and \(g(n)\) is in \(O(\text{polylog}(n))\). Clearly, the algorithm takes polynomial time when \(g(n)\) is a constant and quasi-polynomial time when \(g(n)\) is polylogarithmic in \(n\). It is interesting that connected \(\frac{n}{c}\)-factor problem is polynomial time solvable for any constant \(c\), as this refines the class of functions \(f\) for which the connected \(f\)-factor problem is NP-Complete: connected \(d\)-factor is NP-Complete for each constant \(d\), as shown by Cheah and Corneil in [2].

2. A refined characterization of graphs having connected \(f\)-factor where \(f(v) \geq n/g(n)\) for every \(v\) in \(V\) and \(g(n)\) is in \(O(\text{polylog}(n))\).

3. An extension of the above mentioned algorithm to solve the minimum weighted connected \(f\)-factor problem where \(f(v) \geq n/g(n)\) for every \(v\) in \(V\) and \(g(n)\) is in \(O(\text{polylog}(n))\), without increasing the asymptotic running time.

4. Connected \(f\)-factor problem for \(f(v) \geq n/g(n)\) for every \(v\) in \(V\) and \(g(n)\) is in \(O((\log n)^{1+\epsilon})\) for any constant \(\epsilon > 0\), is in NP-Intermediate under the ETH [9]. Thus, this infinite class of problems parameterized by \(\epsilon\) is similar in complexity to the LOGCLIQUE [10] problem where the goal is to decide whether there exists a clique of size \(\log n\) in an \(n\)-vertex graph.

As a consequence of this work, we have a better refined understanding of computational complexity of the connected \(f\)-factor problem based on the nature of \(f\). The main technique in this work is a natural way of converting one \(f\)-factor to another by exchanging a set of edges. This is formalized using the notion of *Alternating Circuits* that we use extensively in this work. We believe that these techniques for enforcing connectedness along with the results of Tutte [18] for finding \(f\)-factors plays an important role in understanding the nature of the connected \(f\)-factor problem for different classes of functions \(f\).
Preliminaries

2.1 Definitions and Notations

We use standard definitions and notations from West [20]. $G = (V, E)$ represents an undirected graph on $n$ vertices, $d_G(v)$ denotes the degree of a vertex $v$ in a graph $G$ and $N(v)$ denotes the open neighborhood of a vertex $v$. $g(n)$ is in $O(\text{polylog}(n))$ and $f$ is a function whose domain is the vertex set of $G$ and range is the set $\{[n/g(n)], \ldots, n-1\}$. Given two subgraphs $G_1$ and $G_2$ of a graph $G$, we use the basic definitions of binary operations $G_1 \cap G_2$, $G_1 \cup G_2$ to be subgraphs obtained by the vertex and edge set intersection and union operations respectively. We define the symmetric difference of two subgraphs $G_1$ and $G_2$ of $G$ to be the spanning subgraph whose edge set is $E(G_1) \triangle E(G_2)$.

Further, the concepts of circuit, decomposition of a graph $G$, the subgraph of $G$ induced by $S \subseteq V$ denoted by $G[S]$ are standard. We use $w(e)$ to represent weight of an edge $e$ in a weighted graph and $w(G)$ to denote the sum of weights of edges in $G$.

Given a partition $Q = \{Q_1, Q_2, \ldots, Q_r\}$ of the vertex set of $G$, a graph $G/Q$ is constructed as follows: The vertex set of $G/Q$ is $Q$. Corresponding to each edge $(u, v)$ in $G$ where $u$ in $Q_i$, $v$ in $Q_j$, $i \neq j$, there exists an edge $(Q_i, Q_j)$ in $G/Q$. $G/Q$ is a multigraph without loops.

For a spanning subgraph $G'$ of $G$, we say $G'$ connects a partition $Q$ if $G'/Q$ is connected. A refinement $Q'$ of a partition $Q$ is a partition of $V$ where each part $Q_i$ in $Q$ is a subset of some part $Q$ in $Q$. This concept of partition refinement is from Kaiser [11]. Whenever we say a spanning tree of $G/Q$, we refer to a spanning subgraph $T$ of $G$ having $|Q|$-1 edges that connects $Q$.

2.2 Colored Graphs and Alternating Circuits

A colored graph $G$ is one in which each edge is assigned a color from the set $\{\text{red, blue}\}$. In a colored graph $G$, we use $R$ and $B$ to denote subgraphs of $G$ whose edges are the set of red edges ($E(R)$) and blue edges ($E(B)$) of $G$, respectively, and $V(R) = V(B) = V(G)$.

We use this coloring in our algorithm to distinguish between edge sets of two distinct $f$-factors of the same graph $G$. A main computation step in our algorithm is to consider the symmetric difference between edge sets of two distinct $f$-factors and perform a sequence of edge exchanges preserving the degree of each vertex. The following definition is used extensively in our algorithm.

- **Definition 1.** A subgraph $S$ of a colored graph $G$ is an alternating circuit if $S$ is a circuit, and there exists an Eulerian tour of $S$ in which every pair of consecutive edges are of different colors.

Clearly, an alternating circuit has an even number of edges and is connected. Further, $d_R(v) = d_B(v)$ for each $v$ in $S$. We define a minimal alternating circuit $S$ to be an alternating circuit where each vertex $v$ in $S$ has at most two red edges and two blue edges incident on it.

- **Definition 2.** A spanning subgraph $S$ of $G$ is defined to be a switch on another spanning subgraph $H$ of $G$ if we could color edges in $S \cap H$ with color red and those in $S \setminus H$ with color blue such that each component in $S$ is an alternating circuit.

- **Definition 3.** For an $S$ which is a switch on $H$, we define Switching($H, S$) to be a subgraph $G'$ of $G$ obtained by removing all edges in $S \cap H$ from $H$ and adding all the edges in $S \setminus H$ to $H$. 


Whenever the operation Switching$(H,S)$ is used, $S$ is assumed to be a switch on $H$. Finally the weight of an alternating circuit $S$, denoted by $W(S)$, is $w(B) - w(R)$. This will be used along with switching operation. If $G' = \text{Switching}(H,S)$, then it implies that $w(G') = w(H) + W(S)$. The weight of a switch $S$ is also similarly defined to be $w(S \setminus H) - w(S \cap H)$. In our arguments we reason about an $f$-factor obtained by switching a sequence of alternating circuits, and for this we introduce the following notation. Let $S$ be a set of edge disjoint alternating circuits each of which is a switch on $H$. Let $S' = \bigcup_{S \in S} S$. Then the operation Switching$(H,S)$ is the $f$-factor that results from Switching$(H,S')$.

Unless otherwise mentioned, $g(n)$ is a function in $O(\text{polylog}(n))$. We justify why this choice of $g(n)$ is crucial for our analysis in Lemma 10. $f$ is a function $f : V \rightarrow \mathbb{N}$ such that $f(v) \geq \lceil n/g(n) \rceil$, for each $v \in V$ where $n = |V|$. A consequent fact is that, if $H$ is an $f$-factor of $G$, then the number of components in $H$ is at most $g(n) - 1$. We use two crucial subroutines from the literature: Tutte’s-Reduction$(G,f)$ is a subroutine which outputs an $f$-factor of $G$ if one exists using the reduction in [20] example 3.3.12. Modified-Tutte’s-Reduction$(G,f)$ is an extension of Tutte’s-Reduction$(G,f)$, which computes a minimum weighted $f$-factor of the input weighted graph $G$ by reducing it to the problem of finding a minimum weighted perfect matching [9]. We assume that both the above subroutines return empty graphs if they fail to compute $f$-factor.

### 3 Outline of the Algorithm and a refined Characterization

The following is a natural characterization of graphs that have a connected $f$-factor, and it is almost a restatement of the definition of a connected $f$-factor.

**Theorem 4.** Let $G$ be an undirected graph and $f$ be a function $f : V \rightarrow \mathbb{N}$. $G$ has a connected $f$-factor if and only if for each partition $Q$ of the vertex set $V$, there exists an $f$-factor $H$ of $G$ that connects $Q$.

The forward direction of the proof is the observation that a connected $f$-factor connects any partition of the vertex set. The converse is proved by applying the hypothesis to the partition $Q = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$. Theorem 3 sets up the foundation of our algorithm outlined below.

**Outline of the search for connected $f$-factors:** Here we set up the template to search for a connected $f$-factor in an input graph $G$ based on Theorem 4. The details are in Algorithm 4 in section 4. Our algorithm constructs a maximal sequence of pairs $(H_0, Q_0), (H_1, Q_1), \ldots, (H_k, Q_k)$ satisfying the following properties:

1. Each $Q_i, 0 \leq i \leq k$ is a partition of the vertex set $V$, and $Q_0 = \{V\}$.
2. Each $H_i, 0 \leq i \leq k$ is an $f$-factor of $G$, and $H_i$ connects $Q_i$.
3. For each $1 \leq i \leq k$, Each $Q_i$ is a refinement of $Q_{i-1}$ satisfying the following:
   a. Each part in $Q_i$ is a maximal component in $H_{i-1}[Y]$ for some $Y$ in $Q_{i-1}$.
   b. $Q_i \neq Q_{i-1}$ and hence $|Q_i| > |Q_{i-1}|$ for $1 \leq i \leq k$.

The meaning of maximality of the sequence is that the sequence we consider is not a prefix of a longer sequence satisfying the 3 conditions listed above. Since $Q_i$ is a refinement of $Q_{i-1}$, it follows that $k$ can be at most $n$. The following is an interesting and useful fact.

**Fact 1.** Let $H$ be an $f$-factor of $G$ and let $Q$ be a partitioning of the vertex set $V$. If $H/Q$ is connected and $H[Q]$ is connected for each $Q$ in $Q$, then $H$ is a connected $f$-factor.
If a refinement of $Q_k$ satisfies conditions 1 and 3(a), it can be a fixed number, and let $g(n)$ be in $O((\log n)^{1+\epsilon})$. We consider the case when $f(v) \geq n/g(n)$ for all $v$ in $V$. In this section we show that for each such $f$, the connected $f$-factor problem is in NP-Intermediate under ETH.
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Connected $n/\mathcal{O}(\text{polylog}(n))$-factor is unlikely to be NP-Complete. Assume that the connected $f$-factor problem is NP-Complete for some $f$ satisfying the condition mentioned above. This implies there exists a polynomial time reduction from 3-SAT to the connected $f$-factor problem where the reduction algorithm outputs the graph $G$ and the function $f$ on the vertex set, both of which are polynomial in size of the instance of 3-SAT. Further, we reiterate, $f$ satisfies the condition outlined above. From this reduction and the guarantee on our Algorithm in Theorem 5 for connected $f$-factor, it follows that we have an algorithm which decides 3-SAT that runs in time $\mathcal{O}(n^{\text{polylog}(n)})$, and this is impossible under the ETH. The following lemma states this observation.

**Lemma 6.** The connected $f$-factor problem where $f(v) \geq n/g(n)$ for every $v$ in $V$ and $g(n)$ is $\mathcal{O}((\log n)^{1+\epsilon})$ is not NP-Complete for any $\epsilon \geq 0$ unless the ETH is false.

Connected $(n/\mathcal{O}((\log n)^{1+\epsilon}))$-factor is unlikely to be in P. Here we assume that $f(v) \geq n/g(n)$ where $g(n)$ is $\mathcal{O}((\log n)^{1+\epsilon})$ for some $\epsilon > 0$. Note that here $\epsilon > 0$ and in claim that it is unlikely to be NP-Complete, we had assumed that $\epsilon \geq 0$. We now present a sub-exponential time reduction $R$ from the Hamiltonian cycle problem to the connected $f$-factor problem. The reduction algorithm takes $G$ on $N$ vertices and $\epsilon > 0$ as input and outputs a set $\mathcal{G}$ of $(|V(G)|^N)$ pairs. Each pair in $\mathcal{G}$ is of the form $(G', f)$ where $f$ is a function and $G'$ is a graph. The set $\mathcal{G}$ satisfies the following:

1. For each pair $(G', f)$ in $\mathcal{G}$, $G'$ is a graph having $n$ vertices and $f(v) \geq n/\log^{1+\epsilon} n$ for every $v$ in $G'$.
2. $G$ has a Hamiltonian cycle if and only if there exists a pair $(G', f)$ in $\mathcal{G}$ such that $G'$ has a connected $f$-factor.
3. For each $G'$ output by $R$, the number of edges in $G'$ is $2^{\Theta(N)}$ where $N$ is the number of vertices in $G$.

The reduction $R$ is as follows:

1. Compute $n = \lceil 2^{N/((1+\epsilon))} \rceil$ where $N = |V(G)|$.
2. Construct an empty graph $G'$ containing $n$ vertices.
3. Define a partition $Q$ of vertices in $G'$ where each part contains at least $\lceil n/(\log^{1+\epsilon} n) \rceil + 1$ vertices and $|Q| = N - 2$. The existence of partition $Q$ is proved in lemma 4.
4. For each $Q$ in $Q$, make $G'[Q]$ a clique. For each $v \in Q$, define $d_{G'}(v) = |Q| - 1$.
5. Let $A$ be a set consisting of exactly one vertex from each $Q$ in $Q$, $|A| = N - 2$.
6. Let $f(v) = d_{G'}(v) + 2$ for each vertex $v$ in $A$ and $f(v) = d_{G'}(v)$ for each $v$ in $G' \setminus A$.
7. Let $u_1$ be a vertex in $G$.
8. For each 4-vertex path $u_0, u_1, u_2, u_3$ in $G$, do the following:
   a. Let $\sigma$ be a bijection from $V(G) \setminus \{u_1, u_2\}$ to $A$.
   b. For each edge $\{u, v\}$ in $G \setminus \{u_1, u_2\}$, add edge $\{\sigma(u), \sigma(v)\}$ to $G'$. //Make $G \setminus \{u_1, u_2\}$ isomorphic to $G'[A]$ under $\sigma$.
   c. Fix $f(\sigma(u_0)) = d_{G'}(v) + 1$ and $f(\sigma(u_3)) = d_{G'}(v) + 1$. //For a connected $f$-factor $H'$ in $G'$, the graph $H'[A]$ is a spanning path with $\sigma(u_0)$ and $\sigma(u_3)$ as end points.
   d. Output $(G', f)$.
   e. //resetting $G'$ for the next iteration.
   f. Remove all the edges in $G'[A]$ from $G'$.
   g. Fix $f(\sigma(u_0)) = d_{G'}(v) + 2$ and $f(\sigma(u_3)) = d_{G'}(v) + 2$. 

Observe that if the number of vertices $N$ in $G$ is sufficiently large (depending on $\epsilon$), the total space required to hold $G$ output by $R$ is $2^{o(N)}$.

- **Lemma 7.** Let $m$ be an integer and let $k < \sqrt{m}$. Then $[m/([m/k] + 1)] \geq k - 2$.

  *Proof in Appendix.*

  The following lemma proves the correctness of the reduction.

- **Lemma 8.** The graph $G$ has a Hamiltonian cycle $H$ if and only if $R$ outputs a pair $(G', f)$ such that $G'$ has a connected $f$-factor $H'$.

  *Proof in Appendix.*

  If we have a polynomial time algorithm for the connected $f$-factor problem for a given constant $\epsilon > 0$, then we test for the existence of a connected $f$-factor of $G'$ for each $(G', f)$ in $G$. The size of the set $G$ is $O(N^3)$ where $N = |V(G)|$. Computation of each $G'$ takes $2^{o(N)}$ time. Thus, in time $2^{o(N)}$, we check for the existence of a pair $(G', f)$ in $G$ such that $G'$ has a connected $f$-factor.

- **Lemma 9.** Let $\epsilon > 0$ and let $f(v) \geq n/g(n)$ for every $v$ in $V$ and $g(n)$ is in $O((\log n)^{1+\epsilon})$.

  Then the connected $f$-factor problem is not in $\text{NP}$ unless the ETH is false.

From lemmas 6 and 9 we come up with the following theorem.

- **Theorem 10.** Let $G$ be a graph having $n$ vertices and $f$ be a function where $f(v) \geq n/g(n)$ for each $v$ in $V$. For each $\epsilon > 0$ and each $g(n)$ in $O((\log n)^{1+\epsilon})$, the connected $f$-factor problem is in $\text{NP}$-Intermediate unless the ETH is false.

## 5 Properties of alternating circuits and $f$-factors

To start with, we present properties of alternating circuits which we use extensively. Alternating circuits are intricately related to $f$-factors as they provide a way of moving from one $f$-factor to another. We present the following lemmas from our previous work in [12] and the proofs of the lemmas in this section, which are necessary are in the appendix.

- **Lemma 11.** Let $T$ be a graph in which each edge is assigned a color from the set $\{\text{red, blue}\}$. Each component in $T$ is an alternating circuit if and only if $d_H(v) = d_B(v)$ for every $v$ in $T$.

Consider two $f$-factors $H_1$ and $H_2$ of a graph $G$. If color the edges in $H_1$ with color red and those in $H_2$ with color blue, then each component in $H_1 \triangle H_2$ is an alternating circuit. Note that if two alternating circuits $T_1$ and $T_2$ have a vertex in common, then $T_1 \cup T_2$ is an alternating circuit.

- **Lemma 12.** Let $H$ and $H'$ be two $f$-factors of $G$. If $T = H \triangle H'$ (symmetric difference of the edge sets) then $T$ is a switch on both $H$ and $H'$.

- **Lemma 13.** Let $H$ be a subgraph of $G$ and let $T$ be a switch on $H$. Assign color red to edges in $T \cap H$ and blue to those in $T \setminus H$. If $T$ is a minimal alternating circuit and $G' = \text{Switching}(H, T)$, then $|N_H(v) \cap N_{G'}(v)| \geq d(v) - 2$, for each $v$ in $V$.

- **Lemma 14.** Let $S \subseteq E(G)$. An $f$-factor $H$ containing all the edges in $S$, if one exists, can be computed in polynomial time.
Decomposing an alternating circuit into minimal alternating circuits. In our algorithm we repeatedly take an alternating circuit and decompose into a set of minimal alternating circuits containing a given set of edges. The function Min-AC-Set($U, S$) in \[12\] take an alternating circuit $U$ and a set of edges $S \subseteq U$ as input and output a set $U$ of edge disjoint minimal alternating circuits each of which is present in $U$. Further, each edge in $S$ is present in some minimal alternating circuit $C$ in $U$. Min-AC-Set($U, S$) identifies an alternating circuit $C$ having $d_E(v)$ and $d_B(v)$ at most 2 and adds it to $U$ only if some edge in $S$ is present in $C$. Further it removes the identified $C$ from $U$. This step is repeated until $U$ is empty. The crucial step in Min-AC-Set($U, S$) is to find a minimal alternating circuit in $U$. This is presented in the recursive Procedure Find-Min-AC($U$)(in Appendix).

Lemma 15. The procedure Min-AC-Set($U, S$) outputs a set $U$ of edge disjoint minimal alternating circuits each of which has at least one edge from $S$.

6 Algorithm for computing a Connected $f$-factor

In this section we complete the algorithm outlined in Section 3. The algorithm takes an unweighted graph $G$ and a function $f$ as input and outputs a connected $f$-factor of $G$ if it exists. When the function $f(v) \geq n/g(n)$ for each $v$ in $V$ and $g(n)$ is polylogarithmic in $n$, the algorithm runs in time $\tilde{O}(n^2 g(n))$. We start with a justification of why $g(n)$ being polylogarithmic in $n$ is crucial for our analysis.

Lemma 16. Let $g$ be a function on the set of positive integers. Let there be a positive constant $b$ such that for each positive integer $n$, $g(n) \leq b \log^4 n$. Then for each $n \geq n_0$, $2(g(n))^4$ is at most $n$ for a sufficiently large constant $n_0$.

The proof of the above lemma is easy as $n/2b^4$ is asymptotically larger than $(\log n)^4 b$ for any constant $b$. Lemma 16 is crucial in the analysis of our algorithm. Algorithm 1 processes the input graph based on $n$. If $n$ is smaller than $n_0$, it exhaustively checks for a connected $f$-factor. If $n$ is at least $n_0$, then by Lemma 16 $n \geq 2(g(n))^4$ and we use this to bound the running time of our algorithm.

Description of the Algorithm The idea is to start with an arbitrary $f$-factor $H_0$ of $G$ and compute a connected $f$-factor using the template in Section 3. We use a recursive subroutine Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$) which returns a connected $f$-factor if it exists or it returns an empty graph otherwise.
Algorithm 1: The Algorithm for deciding connected $f$-factor when $g(n)$ is in $O(\text{polylog}(n))$

The following lemma plays a critical role in the correctness of our subroutine Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$).

Lemma 17. Let $G$ be a graph having a connected $f$-factor. Let $Q$ be a partition of the vertex set $V$. There exists a spanning tree $T$ of $G/Q$ and an $f$-factor $H$ of $G$ such that $E(T) \subseteq E(H)$. Further given $T$, $H$ can be computed in polynomial time.

Proof. Let $G'$ be a connected $f$-factor of $G$. For any partition $Q$ of the vertex set, $G'/Q$ is connected. Consider a spanning tree $T$ of $G'/Q$. Clearly, there exists at least one $f$-factor $H$ containing $E(T)$ and hence $H/Q$ is connected. Once we have $E(T)$, $H$ can be computed in polynomial time using Lemma 14.

We now present the recursive procedure Restricted-$f$-Factor() which expands the outline in Section 3.
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| Procedure Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$) |
|---------------------------------|
| **Q**$_0$=empty. |
| for each $X \in Q_{i-1}$ do |
| | $Q_1$= $Q_1 \cup \{Y | Y$ is vertex set of a maximal component in $H[X]\}$. |
| end |
| if $Q_i = Q_{i-1}$ then |
| | return $H_{i-1}$ and exit. // $H_{i-1}$ is a connected $f$-factor |
| end |
| $G'$=empty. |
| $H'_i$=empty. |
| // BEGIN Partition-Connector($Q_i$) |
| for each spanning tree $T$ of $G/Q_i$ do |
| | $T$= $T \setminus H_{i-1}$. // Ignore edges that are already there in $H_{i-1}$ |
| if an $f$-factor $H'_i$ containing $E(T)$ exists then |
| | exit Loop 2. // steps 12...18 |
| end |
| if $H'_i$=empty then |
| | exit and return empty. // There does not exist an $f$-factor that connects $Q_i$ |
| end |
| // END Partition-Connector($Q_i$) |
| // BEGIN Next-Factor($H_{i-1}, H'_i, Q_i$) |
| $S$= $E(H_{i-1}) \triangle E(H'_i)$. |
| $S$=empty. // Set of minimal alternating circuits containing $E(T)$ |
| for each component $U \in S$ do |
| | $S$= $S \cup$ Min-AC-Set($U, T$) |
| end |
| $H_i$=Switching($H_{i-1}, S$). // $H_i$ is an $f$-factor containing $E(T)$ |
| // END Next-Factor($H_{i-1}, H'_i, Q_i$) |
| $G'$=Restricted-$f$-Factor($H_i, Q_i$). |
| return $G'$. |

Algorithm 2: The procedure Restricted-$f$-Factor() recursively compute a connected $f$-factor of $G$, if it exists.

We use the following lemma in arguing the correctness of our algorithm.

**Lemma 18.** If $G$ has at least $2(g(n))^4$ vertices, then in each recursive call to Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$), the number of parts in $Q_{i-1}$ is at most $g(n)$.

**Proof.** Consider the computation of $H_i$ from $H_{i-1}$ in the first call to the subroutine with parameters $H_0$ and $Q_0$ from Algorithm 1. In each iteration of loop 2, the number of edges in $T$ is at most $g(n) – 2$. This is because the number of components in $H_0$ is at most $g(n) – 1$. Consider $S$ computed in step 24. We color edges in $H_{i-1} \cap S$ with color red and those in $H'_i \cap S$ with color blue. From Lemma 15, minimality of each $s$ in $S$ computed in step 27 and Lemma 13, $|N_{H_{i-1}}(v) \cap N_{H_i}(v)|$ is at least $n/g(n) – 2(g(n) – 2)$ for each vertex $v$ in $V$. Assume that there exists a recursive call in which the number of parts in $Q_{i-1}$ is more than $g(n)$. We prove that this contradicts our premise that $G$ has at least $2(g(n))^4$ vertices. Let the pairs $(H_0, Q_0), (H_1, Q_1), \ldots, (H_k, Q_k), \ldots$ be the sequence of arguments to Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$). Let the number of parts in $Q_k$ be larger than $g(n)$ and for each $0 \leq i \leq k - 1$, $|Q_i| \leq g(n)$. As discussed above, $k$ can not be 0. Observe that for each
1 \leq i \leq k$, $|Q_i| > |Q_{i-1}|$ and $|Q_0| = 1$. This implies $|Q_i| \geq i + 1$ for every $i$. Thus the level number $k - 1$ is at most $g(n) - 1$ and hence $k$ is at most $g(n)$. Let $T$ be a spanning tree in $G/Q_i$ and let $H'_i$ be the $f$-factor in the $i$-th recursion that contains $T$ (connects $Q_i$). Let $S$ be the symmetric difference $E(H_{i-1}) \triangle E(H'_i)$, and let $S$ be the subset of decomposition of $S$ into minimal alternating circuits which we use for switching in step 29. For each $0 \leq i \leq k - 1$, $Q_i$ has at most $g(n)$ parts. The number of parts in $Q_{k-1}$ computed in the recursive call Restricted-$f$-Factor($H_{k-2}, Q_{k-2}$) is at most $g(n)$. This implies in each of those recursive calls Restricted-$f$-Factor($H_i, Q_i$) where $0 \leq i \leq k - 2$, the number of edges in $T$ computed in step 13 is at most $g(n) - 1$. Consequently, from Lemma 15, the number of minimal alternating circuits in $S$ is at most $g(n) - 1$ for each recursive call with parameters ($H_i, Q_i$) for $0 \leq i < k - 1$. Thus, from Lemma 13, the size of $\cap_{0 \leq i < k - 1} N_{H_i}(v)$ is at least $n/g(n) - 2(g(n) - 1)(k - 1)$. Further, $|N_{H_0}(v) \cap N_{H_{k-1}}(v)| \geq n/g(n) - 2(g(n) - 1)(k - 1)$ where $H_{k-1}$ computed at the end of the call Restricted-$f$-Factor($H_{k-2}, Q_{k-2}$). This means for each vertex $v$ in $V$, at least $n/g(n) - 2(g(n) - 1)(k - 1)$ edges incident on $v$ in $H_{k-1}$ were also present in $H_0$. Since $k \leq g(n)$, we get the size of $N_{H_0}(v) \cap N_{H_{k-1}}(v)$ to be at least $n/g(n) - 2(g(n) - 1)(g(n) - 1)$. Further, each part in $Q_k$ computed in the call Restricted-$f$-Factor($H_{k-1}, Q_{k-1}$) has more than $n/g(n) - 2(g(n) - 1)(g(n) - 1)$ vertices. This implies that the total number of vertices counted in the parts of $Q_k$ is more than $(g(n) + 1)(n/g(n) - 2(g(n) - 1)(g(n) - 1))$. Clearly, the total number of vertices $n$ should be larger than $2^{\frac{k(k+1)}{2}} (n - 2(g(n))^3)$. By rearranging the terms we get $n < 2(g(n))^3$. This contradicts our premise that $G$ has at least $2(g(n))^3$ vertices. Therefore, our assumption that $|Q_k| > g(n)$ is wrong. Hence the lemma.

Lemma 19. Let $n$ be at least $2(g(n))^3$. The number of times the subroutine Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$) gets invoked recursively is at most $g(n) - 1$.

Proof. This follows immediately from Lemma 17 as the number of parts in $Q_i$ is at most $g(n)$, and between two consecutive recursive calls to Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$), either a connected $f$-factor is found or we have a $Q_i$ larger than $Q_{i-1}$ to work with.

Rest of the section contains the proof of the results discussed in Section 3.

Proof of Running time in Theorem 5. If $G$ has an $f$-factor, then step 7 of Algorithm 1 computes an arbitrary $f$-factor $H_0$. The first call to Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$) is made with arguments $H_0$ and $Q_0 = \{V(G)\}$. In the $i$-th recursive call: if $Q_i$ and $Q_{i-1}$ are the same, then a connected $f$-factor is found, and this check can be done in polynomial time, and the correctness is by Fact 1. If there does not exist a $f$-factor $H'_i$ that connects $Q_i$, then the algorithm exits at this step. This check is done by loop 2, and since the number of parts is upper bounded by $g(n)$, the time taken is at most $(n^m)$, where $m$ is the number of edges in $G$. This is the time taken to enumerate all spanning trees $T$ of $G/Q_i$, and to check if there is a $f$-factor that contains the edges of $T$. Thus Algorithm 1 completes in time $O(n^{2g(n)})$. If $G$ has a connected $f$-factor, then in at most $g(n)$-recursive calls, the Restricted-$f$-Factor($H_{i-1}, Q_{i-1}$) will succeed due to Theorem 4. Further, by the same theorem, if $G$ does not have a connected $f$-factor, the procedure will terminate by identifying a partition $Q$ that cannot be connected by an $f$-factor of $G$. Hence the theorem.

Proof of Characterization in Theorem 5. The forward direction of the proof is implied by Theorem 4 as a connected $f$-factor connects any partition, independent of the size of the partition. The reverse direction of the proof is from Algorithm 1. For each partition $Q$
of size at most \( g(n) \), if there exists an \( f \)-factor that connects \( Q \) then clearly the algorithm computes a connected \( f \)-factor with in at most \( g(n) - 1 \) recursions.

\section{Computing a minimum weighted connected \( f \)-factor}

In this section we consider a variant where the input graph \( G \) has positive weights on the edges, and the objective is to compute a minimum weighted connected \( f \)-factor. We consider the case where \( f(v) \geq n/g(n) \) for some function \( g(n) \) in \( \mathcal{O}(\text{polylog}(n)) \). We extend Algorithm \( \text{1} \) to solve this minimization problem as follows:

1. In step 7 of Algorithm \( \text{1} \) instead of initializing \( H_0 \) with an arbitrary \( f \)-factor, we use \( \text{Modified-Tutte's-Reduction}(G,f) \) to initialize \( H_0 \) with a minimum weighted \( f \)-factor of \( G \).
2. Further in loop 2 in \( \text{Restricted-f-Factor}(H_{i-1},Q_{i-1}) \), \( H'_i \) is the minimum weighted \( f \)-factor that connects \( Q_i \), if there exists one.

It is clear, from the arguments in Theorem \( \text{1} \) that these modifications still guarantee that the output will be a connected \( f \)-factor if one exists, and we have to show that the procedure computes an \( f \)-factor of minimum cost. We refer to the above extension of \( \text{Restricted-f-Factor}(H_{i-1},Q_{i-1}) \) as \( \text{Restricted-Min-f-Factor}(H_{i-1},Q_{i-1}) \).

To understand the behavior of the modification, recall that \( H_0 \) is a minimum weight \( f \)-factor of \( G \). Secondly, let \( H'_i \) be a minimum weighted \( f \)-factor that connects \( Q_i \) identified in the \( i \)-th recursion. The procedure builds \( H_i \) from \( H_{i-1} \) and \( H'_i \). The following lemma is used in bounding the cost of \( H_i \) in the \( i \)-th recursive call to \( \text{Restricted-Min-f-Factor}(H_{i-1},Q_{i-1}) \).

\begin{lemma}
Let \( H \) be a minimum weighted \( f \)-factor of \( G \) and let \( T \subseteq E(G) \setminus E(H) \). Let \( H' \) be a minimum weighted \( f \)-factor among all \( f \)-factors containing \( T \). Let \( S = E(H) \setminus E(H') \) and let the edges of \( S \cap H \) be colored red and the edges of \( S \cap H' \) be colored blue. Let \( S \) be a partition of \( E(S) \) into minimal alternating circuits. The following are true:

1. For each \( s \in S \), if \( W(s) > 0 \) then \( s \cap T \neq \emptyset \).
2. For any \( S' \subseteq S \) satisfying \( T \subseteq \bigcup_{s \in S'} s \), Switching\((H,T')\) is an \( f \)-factor of weight exactly equal to \( w(H') \).
\end{lemma}

The proof of Theorem 20, which is also in our recent paper \( \text{12} \), is in the Appendix. The following lemma highlights an invariant which plays a critical role in arguing the correctness of the subroutine \( \text{Restricted-Min-f-Factor}() \).

\begin{lemma}
Consider the sequence of arguments \( (H_0, Q_0), (H_1, Q_1), \ldots, (H_k, Q_k) \) to the procedure \( \text{Restricted-Min-f-Factor}() \). Let the input graph \( G \) has a connected \( f \)-factor. The cost of \( H_k \) is equal to that of \( H'_k \) in the \( k \)-th recursive call to \( \text{Restricted-Min-f-Factor}() \).
\end{lemma}

\begin{proof}
For \( k = 1 \), Theorem 20 directly completes the proof. For \( k > 1 \), we prove by induction on the recursion level \( i \). We assume the claim to be true for \( k-1 \) and we show this to be true for \( k \). We use \( S_i \) and \( S_i \) to address the value of variables \( S \) and \( S \) respectively, computed in step 28 of the function \( \text{Restricted-Min-f-Factor}(H_{i-1},Q_{i-1}) \). Assume that there exists a minimal alternating circuit \( s \) in \( H'_k \setminus H_k \) which is a switch on \( H_k \) of negative weight.

We now derive a contradiction to the optimality of \( H'_k \). Note that \( s \) is edge disjoint from \( \bigcup_{s' \in S_i} E(s') \). This is because, \( H'_k \setminus H_k \) is \( S_k \setminus \bigcup_{s' \in S_i} E(s') \). Thus \( s \) is a switch on Switching\((H_k,S_k)\) of negative weight. Further, Switching\((H_k,S_k)\) is \( H_{k-1} \). By induction, \( H_{k-1} \) is of weight \( H'_{k-1} \) computed in the previous level of recursion. From the algorithm,
Switching$(H_{k-1}, s)$ connects $Q_k$. From the fact that $Q_k$ is a refinement of $Q_{k-1}$, it follows that Switching$(H_{k-1}, s)$ is an $f$-factor that connects $Q_{k-1}$ and is of weight less than that of $H'_{k-1}$, and this contradicts the optimality of $H'_{k-1}$. We now assume that there does not exist a minimal alternating circuit $s$ of negative weight in $H_k$ $\triangle$ $H'_k$, which is a switch on $H_k$. We show that such a switch $s$ of positive weight also cannot occur. Note that such a positive weighted switch $s$ is a switch on $H'_k$ of negative cost. As described above, $s$ is edge disjoint from $\bigcup_{s' \in S_k} E(s')$. From the algorithm, Switching$(H'_k, s)$ connects $Q_k$ and is of lower cost than $H'_{k-1}$. Thus, we have a contradiction to the optimality of $H'_{k-1}$. Therefore, the cost of $H_k$ is equal to that of $H'_k$ in the $k$-th recursive call to Restricted-Min-$f$-Factor().

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Appendix

**Proof of Lemma 7** Let \( l = \lceil m/k \rceil + 1 \). Then \((l-2)k \leq m\). From \( k < \sqrt{m} \), \((l-2) \geq (k-2)\).

We have,

\[
\begin{align*}
(l-2)k & \leq m \\
(l-2)k - 2(l-2) + 2(k-2) & \leq m \\
l(k-2) & \leq m
\end{align*}
\]

\( \triangledown \)

**Proof of Lemma 8** Consider the vertex \( u_1 \) selected in step 7 of \( R \). Corresponding to each Hamiltonian cycle \( H \) in \( G \) there exists a unique 4-vertex path \( u_0, u_1, u_2, u_3 \) in \( G \) and an associated iteration of step 8. Note that \( H \setminus \{u_1, u_2\} \) is a path \( L \) of length \( N-2 \) whose end vertices are \( u_0 \) and \( u_3 \). We have a corresponding spanning path \( L' \) of length \( N-2 \) in \( G'[A] \). Removing all the edges in \( G'[A] \setminus L' \) from \( G'[A] \) gives a connected \( f \)-factor \( H' \) of \( G' \). Conversely, given a connected \( f \)-factor \( H' \), we pick the edge \( \{B^{-1}(u), B^{-1}(v)\} \) for each edge \( \{u, v\} \) in \( H'[A] \). This gives a path \( L \) of length \( N-2 \) whose end vertices are some \( u_0 \) and \( u_3 \). From the reduction algorithm, there exists another path \( u_0, u_1, u_2, u_3 \) where \( u_1 \) and \( u_2 \) are the vertices that are not present in \( V(L) \) and we have a Hamiltonian cycle in \( G \). \( \triangledown \)

**Proof of Lemma 11** In the forward direction, consider a component \( C \) in \( T \). Since there is an Eulerian tour in \( C \) in which consecutive edges are of different colors, it follows that \( d_R(v) = d_B(v) \) for all \( v \) in \( T \). In the reverse direction, let \( d_R(v) = d_B(v) \) for all \( v \) in \( T \). To complete the proof, we point to an exercise in [20] exercise 1.2.35 which considers the formulation of Tucker’s algorithm for computing an Eulerian circuit in [16]. First, at each vertex \( v \) in a component \( C \) in \( T \), we pair each red edge incident on \( v \) to a distinct blue edge incident on \( v \). Since \( d_R(v) = d_B(v) \), such a pairing is guaranteed to exist. Secondly, using this pairing, the required alternating circuit is the Eulerian circuit constructed by Tucker’s algorithm. Hence the lemma. \( \triangledown \)

**Proof of Lemma 13** Since \( T \) is minimal, by definition, we know that \( d_R(v) = d_B(v) \) for each \( v \) in \( V \). Consequently, not more than 2 edges incident on a vertex will be removed from \( H \) as a result of applying Switching\((H,T)\). Therefore, the number of common edges incident on \( v \) in both \( G' \) and \( H \) is at least \( d(v) - 2 \). \( \triangledown \)

**Proof of Lemma 14** Observe that removing the set of edges \( S \) from an \( f \)-factor \( H \) containing \( S \), reduces the degree of each vertex \( v \) in \( H \) by \( |\{(v, u) \in S\}| \). This is exactly an \( f' \)-factor of \( G(V, E \setminus S) \) where \( f'(v) = f(v) - |\{(v, u) \in S\}| \), for each \( v \) in \( V \). Computing \( f' \) and then computing an \( f' \)-factor \( H' \) of \( G(V, E \setminus S) \) is easy. Recall that in polynomial time we can compute an \( f' \)-factor, if one exists, see West [20]. Further adding the edges in \( S \) to \( H' \) gives an \( f \)-factor \( H \) of \( G \) containing \( S \). \( \triangledown \)
Proof of Lemma \[17\] Let \(G'\) be a connected \(f\)-factor of \(G\). For any partition \(Q\) of the vertex set, from Theorem \[4\], \(G'/Q\) is connected. Consider a spanning tree \(T\) of \(G'/Q\). Clearly, there exists at least one \(f\)-factor \(H\) containing \(E(T)\) and hence \(H/Q\) is connected. Once we have \(E(T), H\), \(H\) can be computed in polynomial time using Lemma \[14\].

Proof of Theorem \[20\] For any minimal alternating circuit \(t\) which is a switch on \(H\), recall that \(W(t)\), the weight of \(t\), is \(w(Switching(H,t)) - w(H)\). Since \(H\) is optimum, for each \(t\) in \(T\), \(W(t) = w(Switching(H,t)) - w(H) \geq 0\). Suppose there exists a minimal alternating circuit \(t\) in \(T\) such that \(W(t) > 0\), and \(t\) does not contain any of the edges in \(S\). Let us consider \(T' = T \setminus t\), that is \(T'\) is an alternating circuit obtained by removing the edges of \(t\) from \(T\), then \(W(T') = W(T) - W(t)\). Then \(Switching(H,T')\) is an \(f\)-factor containing \(S\), and \(w(Switching(H,T')) = w(H) + W(T') = w(H) + W(T) - W(t) = w(H') - W(t) < w(H')\). This contradicts the optimality of \(H'\). Therefore, \(t \cap S \neq \emptyset\). This implies for any subset \(T' \subseteq T\) such that \(S \subseteq \bigcup_{t \in T'} E(t)\), \(w(Switching(H,T')) = w(H')\).

\[
\begin{align*}
1 & \textbf{Procedure Find-Min-AC}(U) \\
2 & \text{if } d_R(u) = d_B(u) \leq 2 \text{ for each } u \text{ in } U \text{ then} \\
3 & \quad \text{Exit and return } U. \text{ } U \text{ is a minimal alternating circuit} \\
4 & \text{end} \\
5 & \\
6 & \text{For each } u \text{ in } U, \text{ pair each blue edge incident on } u \text{ to a distinct red edge incident on } u. \\
7 & \text{Run Tucker’s algorithm } [20] \text{ exercise 1.2.35} \text{ on } U \text{ using the pairing defined in the} \\
8 & \text{previous step to get an Euler tour } T \text{ in which consecutive edges are of different} \\
9 & \text{colors.} \\
10 & \text{Let } v \text{ be a vertex with } d_R(v) > 2 \text{ in } U. \text{ } v \text{ such a } v \text{ exists in } U \\
11 & \text{Start the tour } T \text{ from } v \text{ and let } e_1 \text{ be the edge through which } T \text{ leaves } v \text{ for the} \\
12 & \text{first time and let } e_2 \text{ be the edge through which } T \text{ makes the first return to } v. \text{ Let} \\
13 & e_3 \text{ be the edge through which } T \text{ continues the tour and let } e_4 \text{ be the edge through} \\
14 & \text{which } T \text{ makes the next return to } v \\
15 & \text{if } \text{color}(e_2) \neq \text{color}(e_1) \text{ then} \\
16 & \quad U' = v, e_1, \ldots, e_2, v. \\
17 & \text{else if } \text{color}(e_4) \neq \text{color}(e_3) \text{ then} \\
18 & \quad U' = v, e_3, \ldots, e_4, v. \\
19 & \text{else} \\
20 & \quad U' = v, e_1, \ldots, e_4, v. \text{ } \text{color}(e_1) \neq \text{color}(e_4) \\
21 & \text{end} \\
22 & \text{end} \\
23 & \text{Return Find-Min-AC}(U').
\end{align*}
\]

Algorithm 3: The procedure Find-Min-AC\((U)\) returns a minimal alternating circuit in \(U\)