THE $p$-ADIC CLOSURE OF A SUBGROUP OF RATIONAL POINTS ON
A COMMUTATIVE ALGEBRAIC GROUP

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Abstract. Let $G$ be a commutative algebraic group over $\mathbb{Q}$. Let $\Gamma$ be a subgroup of $G(\mathbb{Q})$ contained in the union of the compact subgroups of $G(\mathbb{Q}_p)$. We formulate a guess for the dimension of the closure of $\Gamma$ in $G(\mathbb{Q}_p)$, and show that its correctness for certain tori is equivalent to Leopoldt’s conjecture.

1. Introduction

1.1. Notation. Let $\mathbb{Q}$ be the field of rational numbers. Let $p$ be a prime, and let $\mathbb{Q}_p$ be the corresponding completion of $\mathbb{Q}$. Let $\mathbb{Z}_p$ be the completion of $\mathbb{Z}$ at $p$. If $K$ is a number field, then $\mathcal{O}_K$ is the ring of the integers, and for any finite set $S$ of places, $\mathcal{O}_{K,S}$ is the ring of $S$-integers. If $G$ is a group (or group scheme), then $H \leq G$ means that $H$ is a subgroup (or subgroup scheme) of $G$. If $A$ is an integral domain with fraction field $K$, and $M$ is an $A$-module, then $\text{rk}_A M$ is the dimension of the $K$-vector space $M \otimes_A K$; we write $\text{rk}$ for $\text{rk}_\mathbb{Z}$.

1.2. The logarithm map for a $p$-adic Lie group. Let $G$ be a finite-dimensional commutative Lie group over $\mathbb{Q}_p$ (see [Bou98, III.§1] for terminology). The Lie algebra $\text{Lie} G$ is the tangent space of $G$ at the identity. So $\text{Lie} G$ is a $\mathbb{Q}_p$-vector space of dimension $\text{dim} G$. Let $G_f$ be the union of the compact subgroups of $G$. By [Bou98, III.§7.6], $G_f$ is an open subgroup of $G$, and there is a canonical homomorphism

$$\log : G_f \to \text{Lie} G,$$

defined first on a sufficiently small compact open subgroup by formally integrating translation-invariant 1-forms, and then extended by linearity. Moreover, $\log$ is a local diffeomorphism, and its kernel is the torsion subgroup of $G_f$. It behaves functorially in $G$.

Examples 1.1.

(i) If $G = \mathbb{Q}_p$ (the additive group), then $G_f = \mathbb{Q}_p$, and $\log$ is an isomorphism. In this example, $G_f$ is not compact.

(ii) If $G = \mathbb{Q}_p^\times$, then $G_f = \mathbb{Z}_p^\times$.

(iii) If $G = A(\mathbb{Q}_p)$ for an abelian variety $A$ over $\mathbb{Q}_p$, then $G_f = G$.
1.3. **Dimension of an analytic subgroup.** Let \( \Gamma \) be a finitely generated subgroup of \( G_f \). Then \( \log \Gamma \subseteq \text{Lie} \ G \) is a finitely generated abelian group of the same rank. The closure \( \overline{\log \Gamma} = \log \Gamma \) with respect to the \( p \)-adic topology equals the \( \mathbb{Z}_p \)-submodule of \( \text{Lie} \ G \) spanned by \( \log \Gamma \), so it is a finitely generated \( \mathbb{Z}_p \)-module. Define
\[
\dim \Gamma := \text{rk}_{\mathbb{Z}_p} \overline{\log \Gamma}.
\]
This agrees with the dimension of \( \Gamma \) viewed as a Lie group over \( \mathbb{Q}_p \).

1.4. **Rational points.** Now let \( G \) be a commutative group scheme of finite type over \( \mathbb{Q} \). Fix a prime \( p \). Define \( G(\mathbb{Q})_f := G(\mathbb{Q}) \cap G(\mathbb{Q}_p)_f \). We specialize the previous sections to the Lie group \( G(\mathbb{Q}_p) \) and to a finitely generated subgroup \( \Gamma \) of \( G(\mathbb{Q})_f \). Our goal is to predict the value of \( \dim \Gamma \).

1.5. **Applications.** The value of \( \dim \Gamma \) is important for a few reasons:

(i) If \( C \) is a curve of genus \( g \geq 2 \) over \( \mathbb{Q} \) embedded in its Jacobian \( J \), then the condition \( \dim J(\mathbb{Q}) < g \) is necessary for the application of Chabauty’s method, which attempts to calculate \( C(k) \) or at least bound its size [Cha41, Col85].

(ii) Leopoldt’s conjecture on \( p \)-adic independence of units in a number field predicts \( \dim \Gamma \) in a special case: see Corollary 5.3 in Section 5. Leopoldt’s conjecture is important because it governs the abelian extensions of \( K \) of \( p \)-power degree.

1.6. **Outline of the paper.** Section 2 axiomatizes some of the properties of \( \dim \Gamma \) in order to identify possible candidates for its value. Section 3 defines a “maximal” function \( d(\Gamma) \) satisfying the same axioms and Question 3.3 asks whether it always equals \( \dim \Gamma \). Section 4 shows that \( \dim \Gamma \) and \( d(\Gamma) \) share many other properties. Section 5 computes \( d(\Gamma) \) for subgroups of integer points on tori, and shows that a positive answer to Question 3.3 for certain tori would imply Leopoldt’s conjectures. We end with further open questions.

2. **Dimension functions**

Let \( \mathcal{G} \) be the set of pairs \((G, \Gamma)\) where \( G \) is a commutative group scheme of finite type over \( \mathbb{Q} \) and \( \Gamma \) is a finitely generated subgroup of \( G(\mathbb{Q})_f \).

**Definition 2.1.** A **dimension function** is a function \( \partial: \mathcal{G} \to \mathbb{Z}_{\geq 0} \) satisfying

1. If \( \Gamma \leq H(\mathbb{Q})_f \) for some subgroup scheme \( H \leq G \), then \( \partial(H, \Gamma) = \partial(G, \Gamma) \). (Because of this, we generally write \( \partial(\Gamma) \) instead of \( \partial(G, \Gamma) \).)
2. \( \partial(\Gamma) \leq \text{rk} \Gamma \).
3. If \( H \leq G \) and \( \Gamma'' \) is the image of \( \Gamma \) in \( (G/H)(\mathbb{Q})_f \), then \( \partial(\Gamma) \leq \dim H + \partial(\Gamma'') \).

**Proposition 2.2.** The expression \( \dim \Gamma \) is a dimension function.

**Proof.** Since \( H(\mathbb{Q}_p) \) is closed in \( G(\mathbb{Q}_p) \), the closure of \( \Gamma \) in \( H(\mathbb{Q}_p) \) equals the closure of \( \Gamma \) in \( G(\mathbb{Q}_p) \); therefore (1) holds. The fact that \( \overline{\log \Gamma} \) is the \( \mathbb{Z}_p \)-submodule spanned by \( \log \Gamma \) gives the middle step in
\[
\dim \Gamma = \text{rk}_{\mathbb{Z}_p} \overline{\log \Gamma} \leq \text{rk}(\log \Gamma) \leq \text{rk} \Gamma,
\]
so (2) holds. Finally, by continuity, the image of \( \Gamma \) in \( (G/H)(\mathbb{Q}_p) \) equals \( \Gamma'' \), so we have an exact sequence
\[
0 \to \Gamma \cap H \to \Gamma \to \Gamma'' \to 0.
\]
Taking dimensions of \(p\)-adic Lie groups and observing that the group on the left has dimension at most \(\dim H\) yields (3).

3. The guess

With notation as before, define

\[
d(G, \Gamma) := \inf_{H \leq G} (\dim H + \rk \Gamma - \rk(\Gamma \cap H)),
\]

where the infimum is over all subgroup schemes \(H \leq G\).

**Proposition 3.1.** The function \(d\) is a dimension function, and any dimension function \(\partial\) satisfies \(\partial \leq d\).

**Proof.** First we check that \(d\) is a dimension function:

1. Suppose \(G' \leq G\) and \(\Gamma \leq G'(\mathbb{Q})\). If \(H \leq G'\) then the subgroup \(H' := H \cap G'\) satisfies \(\dim H' \leq \dim H\) and \(\Gamma \cap H' = \Gamma \cap H\), so
   \[
   \dim H' + \rk \Gamma - \rk(\Gamma \cap H') \leq \dim H + \rk \Gamma - \rk(\Gamma \cap H).
   \]
   Therefore the infimum in the definition of \(d(G, \Gamma)\) is attained for some \(H \leq G'\), so \(d(G, \Gamma) = d(G', \Gamma)\).

2. The \(H = \{0\}\) term in the infimum is \(0 + \rk \Gamma - 0\), so \(d(\Gamma) \leq \rk \Gamma\).

3. Let \(K''\) be the subgroup of \(G/H\) realizing the infimum defining \(d(\Gamma'')\). Let \(K\) be the inverse image of \(K''\) under \(G \to G/H\). Then \(\Gamma'' \cap K''\) is a homomorphic image of \(\Gamma \cap K\), so \(\rk(\Gamma'' \cap K'') \leq \rk(\Gamma \cap K)\) and
   \[
   d(\Gamma) \leq \dim K + \rk \Gamma - \rk(\Gamma \cap K)
   = \dim H + \dim K'' + \rk \Gamma - \rk(\Gamma \cap K)
   \leq \dim H + \dim K'' + \rk \Gamma - \rk(\Gamma'' \cap K'').
   \]
   Therefore \(\dim H + d(\Gamma'')\) is also an upper bound.

Now we check that any dimension function \(\partial\) satisfies \(\partial \leq d\). If \(H \leq G\) and \(\Gamma''\) is the image of \(\Gamma\) in \((G/H)(\mathbb{Q})\), then properties (3) and (2) for \(d\) and the isomorphism \(\Gamma'' \simeq \Gamma/(\Gamma \cap H)\) yield
\[
\partial(\Gamma) \leq \dim H + \partial(\Gamma'') \leq \dim H + \rk \Gamma'' = \dim H + \rk \Gamma - \rk(\Gamma \cap H).
\]
This holds for all \(H\), so \(\partial(\Gamma) \leq d(\Gamma)\).

**Corollary 3.2.** We have \(\dim \overline{\Gamma} \leq d(\Gamma)\).

Proposition 3.1 shows that the function \(d(\Gamma)\) gives the largest guess for \(\dim \overline{\Gamma}\) compatible with the elementary inequalities based on rank and the dimension of the group. Therefore we ask:

**Question 3.3.** Does \(\dim \overline{\Gamma} = d(\Gamma)\) always hold?

In other words, are rational points \(p\)-adically independent whenever dependencies are not forced by having a subgroup of too high rank inside an algebraic subgroup?
Proposition 4.1. Let $\partial(\Gamma)$ denote either $\dim \Gamma$ or $d(\Gamma)$. Then

(i) If $\Gamma' \leq \Gamma \leq G(\mathbb{Q})_f$, then $\partial(\Gamma') \leq \partial(\Gamma)$.
(ii) If $G \to G''$ is a homomorphism and $\Gamma \leq G(\mathbb{Q})_f$, then the image $\Gamma''$ in $G''(\mathbb{Q})$ is contained in $G''(\mathbb{Q})_f$ and $\partial(\Gamma'') \leq \partial(\Gamma)$.
(iii) If $G \to G''$ has finite kernel and $\Gamma \leq G(\mathbb{Q})$, then $\Gamma$ is contained in $G(\mathbb{Q})_f$ if and only if its image $\Gamma''$ in $G''(\mathbb{Q})$ belongs to $G''(\mathbb{Q})_f$; in this case, $\partial(\Gamma) = \partial(\Gamma'')$.
(iv) Suppose that $\Gamma_1, \Gamma_2$ are commensurable subgroups of $G(\mathbb{Q})$; i.e., $\Gamma_1 \cap \Gamma_2$ has finite index in both $\Gamma_1$ and $\Gamma_2$. Then $\Gamma_1 \leq G(\mathbb{Q})_f$ if and only if $\Gamma_2 \leq G(\mathbb{Q})_f$; in this case, $\partial(\Gamma_1) = \partial(\Gamma_2)$.
(v) If $\Gamma_i \leq G_i(\mathbb{Q})_f$ for $i = 1, 2$, then $\Gamma_1 \times \Gamma_2 \leq (G_1 \times G_2)(\mathbb{Q})_f$ and $\partial(\Gamma_1 \times \Gamma_2) = \partial(\Gamma_1) + \partial(\Gamma_2)$.
(vi) If $\Gamma_1, \Gamma_2 \leq G(\mathbb{Q})_f$, then $\partial(\Gamma_1 + \Gamma_2) \leq \partial(\Gamma_1) + \partial(\Gamma_2)$.
(vii) If $\text{rk } G = 1$, then $\partial(\Gamma) = 1$.
(viii) If $G \simeq \mathbb{Q}_a^n$, then $\partial(\Gamma) = \text{rk } \Gamma$.

Proof.

(i) For $\partial(\Gamma) := \dim \Gamma$ the result is obvious. For $d(\Gamma)$ it follows since $\text{rk } G - \text{rk } (\Gamma \cap H)$ equals the rank of the image of $\Gamma$ in $G/H$.

(ii) We have $\Gamma'' \leq G''(\mathbb{Q})_f$ by functoriality. For $\dim \Gamma$, the inequality follows since $\log \Gamma$ surjects onto $\log \Gamma''$. For $d(\Gamma)$, if $H \leq G$ and $H''$ is its image in $G''$, then the subgroup $\Gamma/(\Gamma \cap H)$ of $G/H$ surjects onto the subgroup $\Gamma''/(\Gamma'' \cap H''$) of $G''/H''$, and this implies the second inequality in

\[ d(\Gamma') \leq d H'' + \text{rk } \Gamma'' - \text{rk } (\Gamma'' \cap H'') \]
\[ \leq \text{dim } H'' + \text{rk } \Gamma - \text{rk } (\Gamma \cap H) \]
\[ \leq \text{dim } H + \text{rk } \Gamma - \text{rk } (\Gamma \cap H). \]

This holds for all $H \leq G$, so $d(\Gamma'') \leq d(\Gamma)$.

(iii) The map of topological spaces $G(\mathbb{Q}_p) \to G''(\mathbb{Q}_p)$ is proper, so the inverse image of $G''(\mathbb{Q}_p)_f$ is contained in $G(\mathbb{Q}_p)_f$; this gives the first statement. To prove $\partial(\Gamma) = \partial(\Gamma'')$, first use (1) to assume that $G \to G''$ is surjective, so $G'' = G/H$ for some finite $H \leq G$. By (ii), $\partial(\Gamma'') \leq \partial(\Gamma)$. By (3), $\partial(\Gamma) \leq \dim H + \partial(\Gamma'') = \partial(\Gamma'')$. Thus $\partial(\Gamma) = \partial(\Gamma'')$.

(iv) We may reduce to the case in which $\Gamma_1$ is a finite-index subgroup of $\Gamma_2$. Let $n = (\Gamma_2 : \Gamma_1)$, so $n\Gamma_1 \leq \Gamma_2 \leq \Gamma_1$. If $\Gamma_1 \leq G(\mathbb{Q})_f$, then $\Gamma_2 \leq G(\mathbb{Q})_f$. Conversely, if $\Gamma_2 \leq G(\mathbb{Q})_f$, then $n\Gamma_1 \leq G(\mathbb{Q})_f$, so $\Gamma_1 \leq G(\mathbb{Q})_f$ by (iii) applied to $G \to G$. In this case, (iii) gives $\partial(n\Gamma_1) = \partial(\Gamma_1)$, and (ii) implies that both equal $\partial(\Gamma_2)$.

(v) Let $\Gamma := \Gamma_1 \times \Gamma_2$. Since a product of compact open subgroups is a compact open subgroup, we have $G_1(\mathbb{Q}_p)_f \times G_2(\mathbb{Q}_p)_f \leq (G_1 \times G_2)(\mathbb{Q}_p)_f$. (In fact, equality holds.) Thus $\Gamma \leq (G_1 \times G_2)(\mathbb{Q}_p)_f$. The equality $\dim \Gamma = \dim \Gamma_1 \times \dim \Gamma_2$ follows from the definitions. To prove the corresponding equality for $d$, we must show that the infimum in the definition of $d(\Gamma)$ is realized for an $H$ of the form $H_1 \times H_2$ with $H_i \leq G_i$. Suppose instead that $K \leq G_1 \times G_2$ realizes the infimum. Let $\pi_1 : G_1 \times G_2 \to G_1$ be the first projection. Let $H_1 = \pi_1(K)$. Let $H_2 = \ker(\pi_1|_K)$; view $H_2$ as a subgroup
scheme of \(G_2\). Let \(H = H_1 \times H_2\). Thus \(\dim K = \dim H\). The exact sequence
\[0 \to \Gamma_2 \cap H_2 \to \Gamma \cap K \xrightarrow{\pi_1} \Gamma_1 \cap H_1\]
shows that \(\text{rk}(\Gamma \cap K) \leq \text{rk}(\Gamma \cap H)\), so
\[\dim H + \text{rk} \Gamma - \text{rk}(\Gamma \cap H) \leq \dim K + \text{rk} \Gamma - \text{rk}(\Gamma \cap K)\]
Thus \(H\) too realizes the infimum in the definition of \(d(\Gamma)\), as desired.

(vi) Apply (ii) to the addition homomorphism \(G \times G \to G\) and \(\Gamma := \Gamma_1 \times \Gamma_2\), and use (v).

(vii) We have \(\partial(\Gamma) \leq 1\) by (2). If \(\dim \Gamma = 0\), then the finitely generated torsion-free \(\mathbb{Z}_p\)-module \(\log \Gamma\) is of rank 0, so it is 0; therefore \(\Gamma \subseteq \ker \log\), so \(\Gamma\) is torsion, contradicting the hypothesis \(\text{rk} \Gamma = 1\). If \(d(\Gamma) = 0\), then there exists \(H \leq G\) with \(\dim H = 0\) and \(\text{rk} \Gamma = \text{rk}(\Gamma \cap H)\); then \(H\) is finite, so \(\text{rk}(\Gamma \cap H) = 0\) and \(\text{rk}(\Gamma) = 0\), contradicting the hypothesis.

(viii) By applying an element of \(\text{GL}_n(\mathbb{Q}) = \text{Aut} \mathbb{G}_a^n\), we may assume that \(\Gamma = \mathbb{G}_r \times \{0\}^{n-r} \leq \mathbb{Q}^n = \mathbb{G}_a^n(\mathbb{Q})\), where \(r := \text{rk} \Gamma\). Using (v), we reduce to the case \(n = 1\). If \(r = 0\), then the result is trivial. If \(r = 1\), use (vii).

\[\square\]

5. Tori

Lemma 5.1. Let \(K\) be a Galois extension of \(\mathbb{Q}\). Let \(G := \text{Gal}(K/\mathbb{Q})\). Then the representation \(O_K^* \otimes \mathbb{C}\) of \(G\) is a subquotient of the regular representation.

Proof. Define the \(G\)-set \(E := \text{Hom}_{\mathbb{Q}}\text{-algebras}(K, \mathbb{C})\) of embeddings and the \(G\)-set \(P\) of archimedean places of \(K\), the difference being that conjugate complex embeddings are identified in \(P\).

Then \(E\) is a principal homogeneous space of \(G\), and there is a natural surjection \(E \to P\).

Therefore \(E\) is the regular representation and the permutation representation \(C_E\) is a quotient of \(C^P\). The proof of the Dirichlet unit theorem gives a \(G\)-equivariant exact sequence
\[0 \to O_K^* \otimes \mathbb{R} \xrightarrow{\log} \mathbb{R}^* \to \mathbb{R} \to 0\]
so \(O_K^* \otimes \mathbb{C}\) is a subrepresentation of \(C^P\). \(\square\)

Proposition 5.2. Let \(T\) be a group scheme of finite type over \(\mathbb{Z}\) whose generic fiber \(T := T \times \mathbb{Q}\) is a torus. Then
(a) \(T(\mathbb{Z})\) is a finitely generated abelian group.
(b) \(\text{rk} \ T(\mathbb{Z}) \leq \dim T\).
(c) If \(\Gamma \leq T(\mathbb{Z})\), then \(d(T, \Gamma) = \text{rk} \Gamma\).

Proof.
(a) For some number field \(K\) and set \(S\) of places of \(K\), we have \(T \times O_{K,S} \simeq (\mathbb{G}_m)^n_{O_{K,S}}\), so \(T(O_{K,S})\) is finitely generated by the Dirichlet \(S\)-unit theorem. Therefore the subgroup \(T(\mathbb{Z})\) is finitely generated.

(b) We may assume that \(K\) is Galois over \(\mathbb{Q}\). Let \(G = \text{Gal}(K/\mathbb{Q})\). Let \(X\) be the character group \(\text{Hom}(T_K, (\mathbb{G}_m)_K)\) of \(T\). Let \(\chi_X\) be the character of the representation \(X \otimes \mathbb{C}\) of \(G\). Let \(\chi_K\) be the character of the representation \(O_K^* \otimes \mathbb{C}\) of \(G\). By Theorem 6.7 and Corollary 6.9 of \cite{Eis03}, \(\text{rk} T(\mathbb{Z}) = (\chi_X, \chi_K)\). On the other hand, \(\dim T = \text{rk} X = (\chi_X, \chi_{\text{reg}})\), where \(\chi_{\text{reg}}\) is the character of the regular representation of \(G\). The result now follows from Lemma 5.1.
(c) First, \( T(\mathbb{Z}) \) is contained in the compact open subgroup \( T(\mathbb{Z}_p) \) of \( T(\mathbb{Q}_p) \), so \( d(T, \Gamma) \) is defined. By (2), \( d(T, \Gamma) \leq \text{rk}\, \Gamma \). To prove the opposite inequality, we must show that for every subgroup scheme \( H \leq T \), we have \( \text{rk}(\Gamma \cap H) \leq \dim H \). By replacing \( H \) by its connected component of the identity, we may assume that \( H \) is a subtorus of \( T \). Let \( \mathcal{H} \) be the Zariski closure of \( H \) in \( T \). Then \( \Gamma \cap H \leq \mathcal{H}(\mathbb{Z}) \), so \( \text{rk}(\Gamma \cap H) \leq \text{rk}\, \mathcal{H}(\mathbb{Z}) \leq \dim H \) by (b).

Corollary 5.3. Let \( K \) be a number field. Let \( T \) be the restriction of scalars \( \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m \). Let \( \Gamma \leq T(\mathbb{Q}) \simeq K^\times \) correspond to \( O_K^\times \). Then Leopoldt’s conjecture is equivalent to a positive answer to Question 5.2 for \( \Gamma \).

Proof. Leopoldt’s conjecture is the statement \( \dim \Gamma = \text{rk}\, \Gamma \). Let \( T = \text{Res}_{O_K/\mathbb{Z}} \mathbb{G}_m \). By Proposition 5.2(c) applied to \( T \), \( d(T, \Gamma) = \text{rk}\, \Gamma \). So Leopoldt’s conjecture is equivalent to \( \dim \Gamma = d(T, \Gamma) \).

Remark 5.4. In effect, we have shown that Leopoldt’s conjecture cannot be disproved simply by finding a subtorus \( H \) of \( \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m \) containing a subgroup of integer points of rank greater than \( \dim H \). This seems to have been known to experts, but we could not find a published proof.

6. Further Questions

Question 6.1. Is \( d(\Gamma) \) computable in terms of \( G \) and generators for \( \Gamma \)?

Question 6.2. If the answer to Question 6.1 is positive, can \( \dim \Gamma = d(\Gamma) \) be verified in each instance where it is true?

Question 6.3. Can one define a plausible generalization of \( d(\Gamma) \) for the analogous situation where \( \mathbb{Q} \) and \( \mathbb{Q}_p \) are replaced a number field \( k \) and some nonarchimedean completion \( k_v \)?

Remark 6.4. Applying restriction of scalars from \( k \) to \( \mathbb{Q} \) and then applying \( d \) does not answer Question 6.3: it would instead predict the dimension of the closure of \( \Gamma \) in the product \( \prod_{v|p} G(k_v) \) instead of in a single \( G(k_v) \).

Remark 6.5. If \( G \) be a commutative group scheme of finite type over \( \mathbb{Q} \), we can consider also \( G(\mathbb{R}) \), and define \( G(\mathbb{R})_f \) and \( G(\mathbb{Q})_f \). The closure \( \overline{\Gamma} \) of any subgroup \( \Gamma \leq G(\mathbb{Q})_f \) in \( G(\mathbb{R}) \) is a real Lie group. The natural guess for \( \dim \overline{\Gamma} \) seems now to be that it equals the dimension of the Zariski closure \( H \) of \( \Gamma \) in \( G \); in other words, \( \overline{\Gamma} \) should be open in \( H(\mathbb{R}) \). See [Maz92, §7] for a discussion of the abelian variety case.

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