Marangoni convection in superposed fluid and anisotropic porous layers with throughflow

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Abstract

Marangoni convective flow of fluid layer overlying a porous layer with anisotropic permeability and thermal diffusivity is addressed. Flow analysis has been carried out in presence of throughflow. Beavers–Joseph’s slip condition is applied to the fluid-porous layer interface. The boundaries are known to be rigid, but permeable, and insulated to fluctuations in temperature. The problem of own value resulting from the stability analysis is solved through regular perturbation technique. Flow pattern with the influence of pertinent parameters namely the throughflow parameter, mechanical, thermal anisotropy parameters Prandtl number and depth ratio is investigated. Expression of critical Marangoni number is computed and analyzed. It is found that the depth of the relative layers, the direction of throughflow and mechanical and thermal anisotropy parameters deeply affect system stability. Reducing the parameter of mechanical anisotropy and increasing the parameter of thermal anisotropy contributes to process stabilization. In addition, the probability of regulating surface driven convection is discussed in detail through the appropriate choice of physical parameters.

Keywords
Composite layer, Mechanical anisotropy, Thermal anisotropy, Throughflow.

AMS Subject Classification
35Q30.

1 Introduction

The problem of Convective flows heat and species transport at the interface between a fluid and a porous region is encountered in a wide range of industrial and geophysical applications, such as cooling of electronic system, flows in fuel cells, thermal hydraulics of nuclear reactor, chemical processing equipment, filtration methods, the extraction of oil from underground reservoirs, contamination of groundwater, manufacture of composite materials and the flow of biological materials and so on (Vafai [1]; Nield and Bejan [2]; Nield and Bejan [3]). Marangoni flows most frequently occur when localized deposition of a surface tensioning agent such as a surfactant allows a gradient of surface tension to build (for examples, see references (Groebger and Gaver [4–6] and Afsar-Siddiqui et al. [7]). This gradient induces an outward stream of convective Marangoni from the deposition area. Induced Marangoni flow can be used to enhance drug delivery in patients with obstructive pulmonary conditions such as cys-
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The system under investigation consisting of an fluid layer of thickness $d$ (zone1) and saturating an underlying porous layer of thickness $d_m$ (zone2) with throughflow of constant vertical velocity $W_0$. Thus the $z$ indicating distances vertically upwards. The fluid-porous interface at $z = 0$.

### 2. Conceptual Model

The system under investigation consisting of an fluid layer of thickness $d$ (zone1) and saturating an underlying porous layer of thickness $d_m$ (zone2) with throughflow of constant vertical velocity $W_0$. Thus the $z$ indicating distances vertically upwards. The fluid-porous interface at $z = 0$.

### 3. Mathematical Formulation

The fluid-porous layers governing equations are:

**Zone1: Governing model for the layer of fluid ($0 \leq z \leq d$)**

$$\nabla \cdot \vec{V} = 0$$

$$\rho_0 \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) = -\nabla p + \mu \nabla^2 \vec{V}$$

$$\frac{\partial T}{\partial t} \left( \nabla \cdot \vec{V} \right) = \kappa \nabla^2 T$$

**Zone2: Governing model for the porous layer ($-d_m \leq z \leq 0$)**

$$\nabla \cdot \vec{V}_m = 0$$

$$\frac{\rho_0}{\phi} \frac{\partial \vec{V}_m}{\partial t} = -\nabla p_m + \mu K^{-1} \cdot \nabla \vec{V}_m$$

$$A \frac{\partial T_m}{\partial t} \left( \vec{V}_m \cdot \nabla \vec{V}_m \right) = \nabla \cdot \left\{ \kappa_m \cdot \nabla T_m \right\}$$

In these equations, $T$ is the temperature, $\vec{V}$ the velocity vector, $p$ is the pressure, $\kappa$ is the thermal diffusivity, $\rho_0$ the reference fluid density. The subscript $m$ refers to the value of the parameter in the zone2, while $K$ and $\kappa_m$ are respectively the tensors of permeability and effective thermal diffusivity which are given by

$$K = \bar{K}_h (\bar{i} \bar{i} + \bar{j} \bar{j}) + \bar{K}_v \bar{k} \bar{k}$$

$$\kappa_m = \bar{\kappa}_h (\bar{i} \bar{i} + \bar{j} \bar{j}) + \bar{\kappa}_m \bar{k} \bar{k}$$

where the subscripts $h$ and $v$ refer to the quantities in the horizontal and vertical directions respectively.

### 4. Linear Stability Analysis

The temperature distributions in the basic state is specified by

$$T_b(z) = T_0 - (T_0 - T_a) \left( \frac{1 - e^{\frac{W_0 z}{\kappa}}}{e^{W_0 d_m / \kappa} - 1} \right), 0 \leq z \leq d$$

...
At the interface (i.e., z = 0), p = p₀(z) + Pₐ(z) + Pₚ(z), pₚ = pₚ(z) and p₀ = p₀(z) + Pₐ(z). The corresponding quantities for the porous region are

\[ p = p₀(z) + Pₐ(z), pₚ = pₚ(z), p₀ = p₀(z) + Pₐ(z) \]

where, \( p₀ = \frac{Pₐ(z)}{\kappa_m} \) is the thermodynamic pressure, \( pₚ = pₚ(z) \) is the extra pressure due to the presence of a porous medium, and \( p₀ = p₀(z) + Pₐ(z) \) is the total pressure. The thermal and mechanical anisotropy parameters are \( \eta = \frac{\kappa}{\kappa_m} \) and \( \xi = \frac{\kappa}{\kappa_m} \), respectively. Here, \( \eta = \frac{\kappa}{\kappa_m} \) is the ratio of thermal diffusivity and \( \xi = \frac{\kappa}{\kappa_m} \) is the ratio of mechanical anisotropy parameters, \( \epsilon = \frac{\kappa}{\kappa_m} \) is the ratio of thermal diffusivity and mechanical anisotropy parameters, \( \epsilon = \frac{\kappa}{\kappa_m} \).

The solutions to the zeroth order equations \( (4.5)-(4.9) \) become

\[ (W, \varnothing) = \sum_{i=0}^{N} \left( \frac{a_i}{\epsilon} \right) (W_i, \varnothing) \]  
\[ (W_m, \varnothing) = \sum_{i=0}^{N} \left( \frac{a_i}{\epsilon} \right) (W_m, \varnothing) \]

Substitution of equations (5.1) and (5.2) into equations (4.1)-(4.4) and the boundary conditions (4.5)-(4.9) yields a sequence of equations for the unknown functions \( W(z), \varnothing(z), W_m(z) \) for \( i = 0, 1, 2, 3, \ldots \) At the leading order in \( a \) equations (4.1)-(4.4) become respective,

\[ D^3W₀ - \eta D^3W₀ = 0 \]  
\[ D^2 \varnothing₁ - PₑD\varnothing₁ = W₀f(z) \]  
\[ D^2 \varnothing₀ = 0 \]  
\[ D^2 \varnothing₁ = 0 \]

where

\[ f(z) = \frac{Pₑ}{1 - e^{T/m}} e^{T/m} \]

and the equations (4.5)-(4.9) become

\[ W₀ = 0, \varnothing₀ = 0, D\varnothing₀ = 0 \quad \text{at} \quad z = 1 \]

\[ W_m = 0, D_m\varnothing_m = 0 \quad \text{at} \quad z_m = -1. \]

And at the interface (i.e., z = 0)

\[ W₀ = \frac{ζ}{\epsilon'T} W_m, \varnothing₀ = \frac{ζ}{\epsilon'T} \varnothing_m, D\varnothing₀ = D_m\varnothing_m \]

\[ D^3W₀ - \eta D^3W₀ = \frac{−ζ^4}{\epsilon'T Dₐ ζ} Dₐ W_m \]

\[ D^2W₀ = \frac{−ζ^3}{\epsilon'T Dₐ ζ} Dₐ W_m \]

The solution to the zeroth order equations \( (5.3)-(5.6) \) is given by

\[ W₀ = 0, \varnothing₀ = \frac{ζ}{\epsilon'T}, W_m = 0, \varnothing_m = 1. \]

At the first order in \( a \) equations (4.1)-(4.4) then reduce to

\[ D^4W₁ - \eta D^3W₁ = 0 \]  
\[ D^2 \varnothing₁ - PₑD\varnothing₁ = W₁f(z) + \frac{ζ}{\epsilon'T} \]  
\[ D^2 \varnothing₀ = 0 \]  
\[ D^2 \varnothing₁ = 0 \]

and the equations (4.5)-(4.9) become

\[ W₁ = 0, \varnothing₁ = 0, D\varnothing₁ = 0 \quad \text{at} \quad z = 1 \]

\[ W_m₁ = 0, D_m\varnothing_m₁ = 0 \quad \text{at} \quad z_m = -1. \]
And at the interface (i.e. \( z = 1 \))

\[
W_1 = \frac{\zeta}{\xi_T}W_{m1}, \quad \partial z = \frac{\varepsilon_T}{\xi_T} \partial m, \quad D \partial z = \frac{1}{\xi_T}D_m \partial m
\]

\[
D^3W_1 - \eta D^2W_1 = -\frac{\zeta^2}{\varepsilon_T D} D_m W_{m1}
\]

\[
D^2W_1 - \frac{\beta \zeta}{\sqrt{Da \xi}} DW_1 = -\frac{\beta \zeta}{\xi_T D_a \xi} D_m W_{m1}
\]

The general solutions of equations (5.7) and (5.9) are respectively given by

\[
W_1 = [C_1 + C_2 z + C_3 z^2 + C_4 e^{\eta z}]
\]

\[
W_{m1} = [C_5 + C_6 z^m]
\]

where

\[
\begin{align*}
C_1 &= \left( \frac{C_2}{\varepsilon T} - \frac{\zeta^2 C_4}{\xi T} \right), \\
C_2 &= \left( \frac{2 \xi T - 2\xi}{\varepsilon T} - \frac{C_1 - \varepsilon^3 C_4}{\xi T} \right), \\
C_3 &= \frac{b_1 + b_2 C_6}{b_4}, \\
C_4 &= \frac{b_3 e^T - \varepsilon T C_3 + C_3}{\xi T C_1}, \\
C_5 &= \frac{C_1}{2(1 + 2\zeta)} + C_6, \\
b_1 &= \frac{2 \zeta^2 \xi T + \beta \zeta^3}{\varepsilon T}, \\
b_2 &= \left( \frac{\eta^2 \xi T \sqrt{Da \xi} - \beta \zeta^3}{\varepsilon T} \right), \\
b_3 &= \left( \frac{\Delta_2 \xi T \sqrt{Da \xi} - \beta \zeta^3}{\varepsilon T} \right), \\
b_4 &= \frac{\varepsilon T (\beta \zeta^3 - \xi T \sqrt{Da \xi})}{6 \eta T}, \\
b_5 &= (\eta - 1)e^T + \sqrt{Da \xi}, \\
b_6 &= \left( \frac{\varepsilon T \beta \zeta^3 - \xi T \sqrt{Da \xi} - \beta \zeta^3}{\varepsilon T} \right), \\
b_7 &= \left( b_1 b_3 - \frac{b_2}{\varepsilon T} \right), \\
b_8 &= \left( b_3 b_5 - \frac{b_2}{\varepsilon T} \right), \\
b_9 &= (b_4 b_5 + 2b_7 b_6), \\
b_{10} &= \frac{b_1 b_7 - b_2 b_6}{b_1 b_7 - \frac{b_4}{\varepsilon T}}.
\end{align*}
\]

Equations (5.8) and (5.10) involving \( D^2 \partial_1 \) and \( D^2 \partial_{m1} \) provides the condition

\[
\int_0^1 f(z)W_1 dz + \frac{\chi}{\zeta} \int_{-1}^0 g(z_m)W_{m1} dz = -\frac{\varepsilon T}{\xi_T} - \frac{\chi}{\zeta}
\]

\[
(5.11)
\]

The expressions for \( W_1 \) and \( W_{m1} \) are back substituted into equation (5.11) and integrated to yield an expression for the critical Marangoni number \( M_c \), which is given by

\[
M_c = \frac{-\left( \frac{\varepsilon T}{\xi_T} + \frac{\chi}{\zeta} \right)}{(\delta_1 C_1 + \delta_2 C_2 + \delta_3 C_3 + \delta_4 C_4 + \delta_5) + \frac{1}{\xi_T}(-C_5 + \delta_6 C_6 - \delta_7)}
\]

where

\[
\begin{align*}
\delta_1 &= \left( \frac{4N_s}{Pe} + 1 \right), \\
\delta_2 &= \left( \frac{4 \xi}{Pe} + \frac{4 \xi^3}{Pe} \right), \\
\delta_3 &= \left[ \frac{2}{3 Pe} \left( \frac{Pe}{1 - Pe} - \frac{2 e Pe}{Pe - 1} \right) \right], \\
\delta_4 &= \left[ \frac{6}{\eta Pe + \xi \left( 1 - 3 e Pe \right)} \right], \\
\delta_5 &= \left( \frac{\varepsilon T \xi T \left( 1 - Pe \right)}{Pe - 6 (e Pe - 1)} \right), \\
\delta_6 &= \left[ \frac{Pe + 2}{1 - e Pe} \left( \frac{Pe}{Pe m - 1} \right) \right], \\
\delta_7 &= \left[ \frac{2 Pe + 2}{3 Pe m - 1} + \frac{2 e Pe - 1}{Pe m} + \frac{2 Pe - 1}{Pe m} \right].
\end{align*}
\]

6. Results and Discussion

The initiation of Marangoni convection in the presence of a vertical through flow is considered in a process consisting of a liquid surface overlaid by anisotropic porous layer.

6.1 Depth ratio \( \zeta \gg 1 \)

This is the case with a pure zone1 layer and the system’s stability characteristic is measured by Marangoni number when \( Pe = 0 \), the known exact value \( M = 48 \) (Nield [19]) is obtained. Figure 2 is a plot of \( M \) as a function \( Pe \). The following conclusions can be drawn from this figure:

(i) For upward throughflow \( (Pe > 0) \) an increase in \( Pe \) is to increase \( M \), and thus upward throughflow makes the system more stable.

(ii) For downward throughflow \( (Pe > 0) \) an increase in \( Pe \) is to decrease \( M \) initially, and a further increase in \( Pe \) increases \( M \). Thus a weak downward throughflow destabilize.
6.2 Depth ratio $\zeta = 0.1$

The stability of the system is characterized by $M_c$. Figure 3 exhibit plots of $M_c$ as a function of $Pe_m$ respectively for isotropic porous layer ($\chi = 1 = \xi$) and anisotropic porous layer ($\chi = 0.5 = \xi$). The results are presented for three different values of Prandtl number $Pr = 0.1,0.5$ and 1. From Figure 4 it is seen that for all values of $Pr$ for the both case of isotropic and anisotropic porous layer stabilizes the system for upward throughflow and the system is stabilizing for downward throughflow when $-2.5 \leq Pe_m \leq -0.5$ for $Pr = 0.1$, when $-4 \leq Pe_m \leq -0.8$ and $-2.8 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-5.8 \leq Pe_m \leq -3.8$ for $Pr = 1$ respectively for the isotropic case. But the system is stabilizing for downward throughflow when $-2.5 \leq Pe_m \leq -0.5$ for $Pr = 0.1$, when $-8 \leq Pe_m \leq -5$ and $-2.8 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-4.5 \leq Pe_m \leq -2.5$ for $Pr = 1$ respectively for the anisotropic case.

6.3 Depth ratio $\zeta = 0.2$

Figure 4 depicts plots of $M_c$ as a function of $Pe_m$ respectively for isotropic porous layer $\chi = 1 = \xi$ and anisotropic porous layer $\chi = 0.5 = \xi$. The results are presented for three values of Prandtl number $Pr = 0.1,0.5$ and 1. From Figure 4 it is seen that for all values of $Pr$ for the both case of isotropic and anisotropic porous layer stabilizes the system for upward throughflow and the system is stabilizing for downward throughflow when $-2.5 \leq Pe_m \leq -0.5$ for $Pr = 0.1$, when $-4 \leq Pe_m \leq -0.8$ and $-2.8 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-5.8 \leq Pe_m \leq -3.8$ for $Pr = 1$ respectively for the isotropic case. But the system is stabilizing for downward throughflow when $-2.5 \leq Pe_m \leq -0.5$ for $Pr = 0.1$, when $-8 \leq Pe_m \leq -5$ and $-2.8 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-4.5 \leq Pe_m \leq -2.5$ for $Pr = 1$ respectively for the anisotropic case.

6.4 Depth ratio $\zeta = 0.5$

Figure 5 depicts plots of $M_c$ as a function of $Pe_m$ respectively for isotropic porous layer $\chi = 1 = \xi$ and anisotropic porous layer $\chi = 0.5 = \xi$. The results are presented for three values of Prandtl number $Pr = 0.1,0.5$ and 1. From Figure 5 it is seen that for all values of $Pr$ for the both case of isotropic and anisotropic porous layer stabilizes the system for upward throughflow and the system is stabilizing for downward throughflow when $-2.5 \leq Pe_m \leq -1.5$ for $Pr = 0.1$, when $-3.5 \leq Pe_m \leq -1.8$ for $Pr = 0.5$ and when $-7.5 \leq Pe_m \leq -3.5$ for $Pr = 1$ respectively for the isotropic case. But the system is stabilizing for downward throughflow when $-2.5 \leq Pe_m \leq -0.5$ for $Pr = 0.1$, when $-8 \leq Pe_m \leq -5$ and $-2.8 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-4.5 \leq Pe_m \leq -2.5$ for $Pr = 1$ respectively for the anisotropic case.

6.5 Depth ratio $\zeta = 1$

Figure 6 depicts plots of $M_c$ as a function of $Pe_m$ respectively for isotropic porous layer $\chi = 1 = \xi$ and anisotropic porous
layer $\chi = 0.5 = \xi$. The results are presented for three values of Prandtl number $Pr = 0.1, 0.5$ and 1. From Figure 6 it is seen that for all values of Pr for the both case of isotropic and anisotropic porous layer stabilizes the system for upward throughflow and the system is destabilizing for downward throughflow for $Pr = 0.1$, and the system is stabilizing when $-2.5 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-6.5 \leq Pe_m \leq -2.5$ for $Pr = 1$ respectively for the isotropic case. But the system is destabilizing for downward throughflow for $Pr = 0.1$ and system is stabilizing when $-2.5 \leq Pe_m \leq 0$ for $Pr = 0.5$ and when $-8.5 \leq Pe_m \leq -5$ for $Pr = 1$ respectively for the anisotropic case.

![Figure 6](image-url)

Figure 6. Critical Marangoni number versus $Pe$ for different values of $Pr$ with $\xi = 1$

![Figure 7](image-url)

Figure 7. $Mc$ versus $\xi$ for different values of $\chi$ with $Pr = 1$ and $\xi = 1$

The effect of $\xi$ and $\chi$ on the onset of convection is emphasized by depicting the variation of $Mc$ and over a range of $\xi$ for different values of $\chi$ in Figure 7 for a fixed value of $Pr = 1$, $\xi = 1$. It is observed that $Mc$ increases with the decreasing $\xi$. It mechanically means the conductive solution becomes more stable in the porous medium.

### 7. Conclusions

In an anisotropic porous layer underlined by a fluid layer, an exact analysis is made to study the influence of throughflow on the onset of Marangoni convection. It is observed from the above analysis that the stability characteristics of the configuration are crucially dependent on

(i) throughflow direction

(ii) depth ratio $\zeta$

(iii) mechanical anisotropy parameter $\xi$

(iv) thermal anisotropy parameter $\chi$.

Therefore, convective instability resulting either in a porous layer or in a fluid layer by adjusting the $\zeta$ or $Pe$ or $\xi$ or $\chi$ by considering all the effects together, since both the stabilizing and the destabilizing factors can be increased more in the combined porous and fluid layer system than in the combined porous and fluid layer system.

### References

[1] K. Vafai, *Handbook of Porous Media*, Taylor/CRC Press, Francis, London/Boca Ratou, FL, 2005.

[2] D. A. Nield and A. Bejan, *Convection in Porous Media*, Second ed. Springer-Verlag, New York, 2006.

[3] D. A. Nield and A. Bejan, *Convection in Porous Media*, Springer, New York, 2013.

[4] J. B. Groberg, and D. P. Gaver, A synopsis of surfactant spreading research, *Journal of Colloidal and Interface Science*. 178(1996), 377–378.

[5] D. P. Gaver, J. B. Groberg, Droplet Spreading on a Thin Viscous Film, *J Fluid Mech.* 235(1992), 399–414.

[6] D. P. Gaver and J. B. Groberg, The dynamics of a localized surfactant on a thin film, *J. Fluid. Mech.* (1990).

[7] A. B. Afsar-Siddiqui, P. F. Luckham and O. K. Matar, The spreading of surfactant solutions on thin liquid films, *Advances in Colloid and interface science*. 03(2003), 1–8.

[8] J. B. Groberg, Respiratory fluid mechanics and transport processes, *Annu. Rev. Biomed. Eng.* 3(2001), 421–457.

[9] D. Halpern, O. E. Jensen and J. B. Groberg, A theoretical study of surfactant and liquid delivery into the lung, *J. Appl. Physiol.* 85(1998), 333–352.

[10] C. D. Bertram and D. P. Gaver, Biofluid Mechanics of the Pulmonary System, *Ann. Biomed. Eng.* 33(2005), 1681–1688.

[11] F. Chen, Throughflow effects on convective instability in superposed fluid and porous Layers, *J. Fluid. Mech.* 231(1990), 113–133.

[12] A. Khalili, I. S. Shivakumara and S. P. Suma, Convective instability in superposed fluid and porous layers with vertical throughflow, *Transp. Porous Med.* 51(2003), 1–18.

[13] S. P. Suma, Y. H. Gangadharaih, R. Indira and I. S. Shivakumara, Throughflow effects on penetrative convection...
Marangoni convection in superposed fluid and anisotropic porous layers with throughflow — 851/851

in superposed fluid and porous layers, *Transp. Porous Med.* 95(2012), 91–110.

[14] I. S. Shivakumara, S. P. Suma, R. Indira and Y. H. Gangadharraiah, Effect of internal heat generation on the onset of Marangoni convection in a fluid layer overlying a layer of an anisotropic porous medium, *Transp. Porous Med.* 92(2012), 727–743.

[15] I. S. Shivakumara, M. Venkatachalappa and S. P. Suma, Exact analysis of Marangoni convection with throughflow, *Acta Mech.* 136(1999), 109–117.

[16] S. Saravanan and T. Sivakumar, Exact solution of Marangoni convection in a binary fluid with throughflow and Soret effect, *Applied Mathematical Modelling.* 33(2009), 3674–3681.

[17] I. S. Shivakumara, Jinho Lee and K. B. Chavaraddi, Onset surface tension convection in a fluid layer overlying a layer of an anisotropic porous medium, *Int. J. Heat Mass Transf.* 54(2011), 994–1001.

[18] G. S. Beavers and D. D. Joseph, Boundary conditions at a naturally permeable wall, *J. Fluid Mech.* 30(1967), 197–207.

[19] D. A. Nield, The onset of transient convective instability, *J. Fluid Mech.* 71(1975), 441–454.

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