TAUTOLOGICAL BASES OF COHA MODULES

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ABSTRACT. Given a quiver, we consider its cohomological Hall algebra (CoHA) as well as CoHA modules built of cohomology groups of non-commutative Hilbert schemes. We investigate cell decompositions of non-commutative Hilbert schemes and the corresponding (non-canonical) bases of CoHA modules. We construct canonical bases of CoHA modules, which consist of products of Chern classes of tautological vector bundles over non-commutative Hilbert schemes. This result generalizes classical results for Grassmannians and (partial) flag varieties.

1. INTRODUCTION

Cohomological Hall algebras (abbreviated as CoHAs) were introduced in [7] as a mathematical incarnation of the algebras of BPS states proposed in string theory. The definition of CoHAs goes through the same lines as the definition of conventional Hall algebras [11]. The benefit of the former is that cohomology groups used in the definition of CoHAs are smaller and easier to describe explicitly than the spaces of functions on the groupoids of objects used in the definition of conventional Hall algebras.

In particular, for the category of representations of a quiver $Q$, the corresponding CoHA $\mathcal{K}$ was described explicitly in [7] as a shuffle algebra. More precisely, let $Q$ have the set of vertices $I$. We can represent $\mathcal{K}$ as a graded vector space

$$\mathcal{K} = \bigoplus_{d \in \mathbb{N}^I} \mathcal{K}_d,$$

where $\mathcal{K}_d$ denotes the ring of symmetric polynomials in $d_i$ variables and $\Sigma_d = \prod_{i \in I} \Sigma_{d_i}$ is the product of symmetric groups. The shuffle product on $\mathcal{K}$ is encoded by the Euler form of $Q$.

The next natural question is a study of modules over the CoHA $\mathcal{K}$ [12, 2, 3, 4, 13]. A structure of an $\mathcal{K}$-module can be introduced on the cohomology of the moduli spaces of stable framed representations of $Q$, similar to the action of quantum affine algebras on (equivariant) cohomology of Nakajima quiver varieties [8, 9]. More precisely, given vectors $f \in \mathbb{N}^I$ and $d \in \mathbb{N}^I$ (called framing and dimension vectors respectively), we consider the moduli space $N_{d,f}$ that parametrizes pairs $(M, s)$, where $M$ is a $Q$-representation having dimension vector $d$ and $s \in \bigoplus_{i \in I} \text{Hom}(\mathbb{C}^w, M_i)$ is such that its image generates $M$ as a $Q$-representation. This moduli space is called a non-commutative Hilbert scheme. For example, for the quiver having one vertex and no loops and natural numbers $d \leq w$, the moduli space $N_{d,w}$ can be identified with the Grassmannian $\text{Gr}(w-d, w)$. Similarly, for the quiver $1 \to 2 \to \cdots \to n$ and vectors $f = (w, 0, \ldots, 0)$, $d = (d_1, \ldots, d_n)$ with $w \geq d_1 \geq \cdots \geq d_n \geq 0$, the moduli space $N_{d,f}$ can be identified with the space of (partial) flags.
The graded vector space
\[ \mathcal{M}_f = \bigoplus_{d \in \mathbb{N}^f} H^*(N_{d,f}) \]
can be naturally equipped with the structure of a module over the CoHA \( \mathcal{H} \) [12, 2]. In [4, 2] one constructed an explicit epimorphism \( \mathcal{H} \to \mathcal{M}_f \) between \( \mathcal{H} \)-modules and determined its kernel in terms of the shuffle product on \( \mathcal{H} \). This, gives, in principle, a description of cohomologies \( H^*(N_{d,f}) \) of non-commutative Hilbert schemes.

The epimorphism \( \mathcal{H} \to \mathcal{M}_f \) is given for the degree \( d \) components by
\[
\mathcal{H}_d \to H^*(N_{d,f}), \quad \prod_{i \in I} e_{i,k}^{n_{i,k}} \mapsto \prod_{i \in I} c_k(U_i)^{n_{i,k}},
\]
where \( e_{i,k} \in \Lambda_{d_i} \) is the \( k \)-th elementary symmetric polynomial and \( c_k(U_i) \) is the \( k \)-th Chern class of the tautological vector bundle \( U_i \) over \( N_{d,f} \) at the vertex \( i \in I \). This implies that for appropriate choices of the powers \( (n_{i,k}) \) we can obtain a basis of \( H^*(N_{d,f}) \).

On the other hand, it was proved in [10, 1] that non-commutative Hilbert schemes \( N_{d,f} \) possess a cell decomposition, parametrized by subtrees of the tree of paths \( \mathcal{T}^f \) in the framed quiver \( Q^f \). While the set of parameters (subtrees of the tree of paths) is independent of any choices, the corresponding cells (and the classes of their closures) generally depend on a (non-canonical) choice of a total order on the tree of paths. Taking duals of the cycles of cell closures, we obtain a (non-canonical) basis of \( H^*(N_{d,f}) \).

In this paper we relate the above two approaches to cohomology of non-commutative Hilbert schemes and, as a result, we describe a canonical basis of \( H^*(N_{d,f}) \) in terms of the Chern classes of tautological vector bundles over \( N_{d,f} \). There exists a bijection between the set of subtrees of \( \mathcal{T}^f \) and the set of certain multi-partitions [1, §8], which is obtained by applying a modification of a depth-first search algorithm for plane trees. This bijection plays a fundamental role in our approach, as we parametrize products of Chern classes of tautological bundles by multi-partitions and relate them to the classes of cell closures parametrized by subtrees of \( \mathcal{T}^f \). Our main result is (see Theorem 6.4)

**Theorem 1.1.** Let \( S^f(d) \) denote the set of multi-partitions
\[
\lambda = (\lambda^{(i)})_{i \in I}, \quad \lambda^{(i)} = (\lambda_1^{(i)} \geq \cdots \geq \lambda_{d_i}^{(i)} \geq 0), \quad i \in I,
\]
satisfying condition (22). Then the classes
\[
\prod_{i \in I} \prod_{k \geq 1} c_k(U_i)^{\lambda_k^{(i)} - \lambda_{k+1}^{(i)}} \in H^{2|\lambda|}(N_{d,f}), \quad \lambda \in S^f(d),
\]
form a basis of \( H^*(N_{d,f}) \), where \( |\lambda| = \sum_{i \in I} |\lambda^{(i)}| \).

2. **Cell decomposition of non-commutative Hilbert schemes**

2.1. **Non-commutative Hilbert schemes.** Let \( Q \) be a quiver with the set of vertices \( I = Q_0 \) and the set of arrows \( Q_1 \). The bilinear form \( \chi = \chi_Q \colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z} \) defined by
\[
\chi(d, e) = \sum_{i \in I} d_i e_i - \sum_{(a \colon i \to j) \in Q_1} d_i e_j
\]
is called the *Euler form* of \( Q \). Given a vector \( f \in \mathbb{N}^I \), called a framing vector, we define a new (framed) quiver \( Q^f \) by adding a new vertex \( \infty \) and \( f_i \) arrows \( \infty \to i \), for all \( i \in I \).
We define a framed representation $M$ to be a representation of $Q^f$ such that $\dim M_\infty = 1$. We call it stable if $M$ is generated, as a $Q^f$-representation, by the subspace $M_\infty$.

For any dimension vector $d \in \mathbb{N}^f$, we define
\begin{align}
R(Q,d) &= \bigoplus_{(a : i \to j) \in Q_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}), \\
R^f(Q,d) &= R(Q,d) \oplus \bigoplus_{i \in I} \text{Hom}(\mathbb{C}^{f_i}, \mathbb{C}^{d_i}).
\end{align}

An element $M \in R^f(Q,d)$ can be interpreted as a framed representation. Let $R^{f,\text{st}}(Q,d) \subset R^f(Q,d)$ be the open subspace of stable framed representations. The space $R^f(Q,d)$ is equipped with the action of the group $G_d = \prod_{i \in I} \text{GL}_{d_i}(\mathbb{C})$. The induced action of $G_d$ on $R^{f,\text{st}}(Q,d)$ is free and there exists a smooth algebraic variety parameterizing stable framed representations up to isomorphism, called the non-commutative Hilbert scheme. If non-empty, it has dimension
\begin{align}
\dim N_d &= f \cdot d - \chi(d,d).
\end{align}

Every representation $M \in N_d$ is equipped with a nonzero element $m_* \in M_\infty$ that generates $M$ as a representation of $Q^f$. Let $A = \mathbb{C}Q^f$ be the path algebra of the quiver $Q^f$ and let $P = Ae_\infty$ be the projective $A$-module corresponding to the idempotent $e_\infty \in A$ (the trivial path at $\infty$). For any $M \in N_d$, we have an epimorphism $P \to M$, $u \mapsto um_*$. The moduli space $N_d$ parametrizes such epimorphisms up to an automorphism of $M$.

The projective module $P = Ae_\infty$ has a basis consisting of paths in $Q^f$ that start at $\infty$. This set of paths has a tree structure which will be important in our analysis of stratifications of the moduli space $N_d$. Therefore we will discuss these combinatorial structures in more detail in the next sections.

2.2. Posets and trees. Let $(\mathcal{P}, \preceq)$ be a partially ordered set (poset). We will write $u \prec v$ if $u \preceq v$ and $u \neq v$, for $u, v \in \mathcal{P}$. For any $v \in \mathcal{P}$, we define
\begin{align}
\mathcal{P}_{\preceq v} &= \{ u \in \mathcal{P} \mid u \preceq v \} \quad \mathcal{P}_{< v} = \{ u \in \mathcal{P} \mid u \prec v \}.
\end{align}

The elements of $\mathcal{P}_{< v}$ are called predecessors of $v$. A subset $S \subset \mathcal{P}$ is called a lower set if $\mathcal{P}_{\preceq v} \subset S$, for all $v \in S$.

We define a (rooted) tree to be a poset $(\mathcal{P}, \preceq)$ such that
\begin{enumerate}
\item There is a unique minimal element $\ast \in \mathcal{P}$, called the root.
\item For any $v \in \mathcal{P} \setminus \{\ast\}$, the poset $\mathcal{P}_{< v}$ is a finite chain (totally ordered set). Its unique maximal element is denoted by $p(v)$, called the parent of $v$.
\end{enumerate}

We define a subtree of a tree $\mathcal{P}$ to be a non-empty lower set $S \subset \mathcal{P}$. In what follows we will consider only finite subtrees, unless otherwise stated. For any subtree $S \subset \mathcal{P}$, we define its critical set to be
\begin{align}
C(S) &= \min_{\preceq}(\mathcal{P} \setminus S) = \{ v \in \mathcal{P} \setminus S \mid u \prec v \implies u \in S \}.
\end{align}

Note that if $v \in C(S)$, then $S \cup \{v\}$ is again a tree.
2.3. Path poset. For any path \( u \) in \( Q^f \), we denote by \( s(u) \) and \( t(u) \) its source and target vertices, respectively. Let \( \mathcal{P} = \mathcal{P}^f \) be the set of paths in \( Q^f \) that start at \( \infty \). It is a basis of the module \( P = A e_\infty \) considered earlier. We define the partial order on \( \mathcal{P} \) given by

\[
\text{For any } u \text{ and } v \text{ with } s(u) = t(v), \text{ the poset } \mathcal{P} \text{ is a tree with the root } * = e_\infty. \text{ For any non-trivial path } u = a_n \ldots a_1 \in \mathcal{P} \text{ (where } a_i \text{ are arrows with } s(a_i) = t(a_{i-1}), \text{ for } 1 < i \leq n), \text{ the parent of } u \text{ is } a_{n-1} \ldots a_1 \in \mathcal{P}.
\]

For any (finite) subset \( S \subset \mathcal{P} \), we define its dimension vector (note that we omit the vertex \( \infty \) here)

\[
\dim S = (\# S_i)_{i \in I}, \quad S_i = \{ u \in S \mid t(u) = i \}.
\]

For any \( d \in \mathbb{N}^I \), let \( \mathcal{P}(d) \) denote the set of trees \( S \subset \mathcal{P} \) with \( \dim S = d \). This set is finite. Indeed, for any \( S \in \mathcal{P}(d) \) and \( v \in S \), the length of the path \( v \) is \( \leq \# S - 1 = |d| = \sum_{i \in I} d_i \) and the number of paths in \( \mathcal{P} \) having a given length is finite.

2.4. Properties of stable framed representations. Recall that every \( M \in \mathcal{N}_d \) is equipped with a vector \( m_* \in M_\infty \) that generates \( M \) as a \( Q^f \)-representation. For any path \( u \in \mathcal{P} \), we define

\[
m_u = u m_* \in M_{t(u)}.
\]

These vectors generate \( M \) as a vector space. For any subset \( S \subset \mathcal{P} \), we define the subspace

\[
M_S = \langle m_u : u \in S \rangle \subset M.
\]

Note that if \( (m_u)_{u \in S} \) is a basis of \( M \), then \( \dim S = d \).

**Lemma 2.1.** Let \( S \subset \mathcal{P} \) be a tree such that \( m_v \in M_S \) for all \( v \in C(S) \). Then \( M_S = M \).

**Proof.** It is enough to show that \( M_S \subset M \) is a subrepresentation. Let \( u \in S \) and \( a \) be an arrow with \( s(a) = t(u) \). Then \( au \in S \) or \( au \in C(S) \). If \( au \in S \), then \( am_u = m_{au} \in M_S \) by the definition of \( M_S \). If \( au \in C(S) \), then \( am_u = m_{au} \in M_S \) by assumption. \( \square \)

**Corollary 2.2.** Let \( S \subset \mathcal{P} \) be a tree, such that \( (m_u)_{u \in S} \) are linearly independent. Then there exists a tree \( S' \subset \mathcal{P} \) such that \( (m_u)_{u \in S'} \) is a basis of \( M \).

**Proof.** If \( M_S = M \), then we are done. Otherwise, by the previous lemma, there exists \( v \in C(S) \) such that \( m_v \notin M_S \). Then \( S' = S \cup \{v\} \) is a tree and \( (m_u)_{u \in S'} \) are linearly independent. Then we proceed by induction. \( \square \)

2.5. The cells. To define a cell decomposition of the non-commutative Hilbert scheme \( \mathcal{N}_d \), we need to fix a total order on the set of paths \( \mathcal{P} \). We will say that a total order \( \leq \) on \( \mathcal{P} \) is admissible if it extends the partial order \( \leq \) on \( \mathcal{P} \). We will say that a total order \( \leq \) on \( \mathcal{P} \) is monomial if

(1) \( \leq \) extends the partial order \( \leq \) on \( \mathcal{P} \).
(2) If \( u < v \), then \( au < av \) for any compatible arrow \( a \).
(3) \( \leq \) is a well-order.
Example 2.3 (Shortlex order). Assume that the set of arrows $Q_1^f$ is totally ordered. Then, for two paths in $P$

\[ u = a_m \ldots a_1, \quad v = b_n \ldots b_1, \]

we say that $u < v$ if $m < n$ or if $m = n$ and $a_k < b_k$ for the minimal $k$ with $a_k \neq b_k$. This is a monomial order.

Example 2.4 (Weighted shortlex order). Let $Q_1^f$ be totally ordered and let $\text{wt}: Q_1^f \to \mathbb{R}_{>0}$ be a map. For any path $u = a_m \ldots a_1$, we define its weight $\text{wt}(u) = \sum_i \text{wt}(a_i)$. Given two paths $u, v$ (12), we say that $u < v$ if $\text{wt}(u) < \text{wt}(v)$ or if $\text{wt}(u) = \text{wt}(v)$ and $u < v$ with respect to the shortlex order. This is a monomial order.

Example 2.5 (Lex order). As before, let $Q_1^f$ be totally ordered. Given two paths $u, v$ (12), we say that $u < v$ if $a_k < b_k$ for the minimal $k$ with $a_k \neq b_k$ (if such $k$ exists) or if $m < n$ and $a_k = b_k$ for all $1 \leq k \leq m$. This is the order used in [10, 1]. It is admissible, but not monomial. For example, for the quiver $Q^f$ with arrows $f: \infty \to 0$ and $a, b: 0 \to 0$ and the order on arrows $f < a < b$, we have $af < a^2 f$, but $ba f > ba^2 f$.

In what follows, we fix an admissible order on $P$. For any tree $S \in P(d)$, we define the sets

1. $U_S \subset \mathbb{N}_d$ that consists of $M \in \mathbb{N}_d$ such that the tuple $(m_u)_{u \in S}$ is a basis of $M$.
2. $Z_S \subset U_S$ that consists of $M \in U_S$ such that, for every $v \in C(S)$, we have $m_v \in M_{S_{<v}}$.

Remark 2.6. Note that $U_S$ is independent of the choice of the total order on $P$, while $Z_S$ depends on it. It follows from Corollary 2.2 that $\mathbb{N}_d$ is covered by the sets $U_S$.

The following result was proved in [10, Lemma 3.4] for the $m$-loop quiver. The proof for a general quiver is analogous. We give it for completeness.

Lemma 2.7. For any tree $S \in P(d)$, the set $U_S \subset \mathbb{N}_d$ is open. It is an affine space of dimension $\# \bigcup_{i \in I} (S_i \times C(S)_i)$.

Proof. The condition $M \in U_S$ means that the vectors $(m_u)_{u \in S}$ are linearly independent and this is an open condition. We may express $m_v$ in a unique way as a linear combination $m_v = \sum_{u \in S_i} c_{u,v} m_u$ for every $v \in C(S)_i$ and every $i \in I$. The scalars $c_{u,v}$ provide a morphism $U_S \to \mathbb{A}^N$, where $N = \# \bigcup_{i \in I} S_i \times C(S)_i$.

Conversely, let us define a map $\mathbb{A}^N \to U_S$. Fix a basis $(m_u)_{u \in S}$ of each vector space $M_i$ for $i \in Q_0^f$. To a tuple $(c_{u,v})$ of scalars with $(u, v) \in \bigcup_{i \in I} S_i \times C(S)_i$, we assign a representation $M \in U_S$ as follows. To specify a representation, we need to specify the maps $a: M_i \to M_j$ for all arrows $a: i \to j$ in $Q_1^f$. This means that, for every $u \in S_i$, we need to specify $am_u \in M_j$. If $au \in S_j$, then we set $am_u = m_{au}$. If $au \in C(S)_j$, then we define $am_u = \sum_{u' \in S_i} c'_{u,u'} m_{u'}$. The thus obtained representation $M$ lies in $U_S$. The two maps are mutually inverse.

Lemma 2.8. For any tree $S \in P(d)$, the set $Z_S \subset \mathbb{N}_d$ is locally-closed. It is an affine space of dimension $\# \bigcup_{i \in I} \{(u, v) \in S_i \times C(S)_i \mid u < v\}$.

Proof. For any $M \in U_S$, the condition $m_v \in M_{S_{<v}}$ for all $v \in C(S)_i$ is a closed condition: when writing $m_v = \sum_{u \in S_i} c_{u,v} m_u$ as in the proof of Lemma 2.7, it is given by the vanishing
implies that 

\[ 2.13 \]

that the cells 

Then the same is true for all 

\[ 2.14 \]

meaning that 

\[ 2.15 \]

Lemma 2.12.

Given 

\[ \text{Proof.} \]

The statement is clear for 

\[ i \]

\[ 2.16 \]

\[ (15) \]

\[ \text{Lemma} \]

\[ \text{Proof.} \]

We will see in Lemma 

\[ \text{Properties of monomial orders.} \]

In this section we will assume that 

\[ 2.17 \]

\[ \text{Lemma} \]

\[ \text{Proof.} \]

We can assume by induction that 

\[ \text{Corollary} \]

\[ \text{Theorem} \]

\[ \text{Given} \]

\[ \text{Then} \]

\[ \text{Then} \]

\[ \text{Then} \]
Proof. We have \( m_{v_{k+1}} \notin (m_{v_i} : i < k) \), hence \( v_k < v_{k+1} \) by the minimality of \( v_k \).

Let us assume by induction that \( S' = (v_1 < \cdots < v_k) \) is a tree. If \( u \in \mathcal{P} \) is such that \( m_u \notin M_{S_{\leq u}} \), then \( S'_{\leq u} = S' \) and \( m_u \notin M_{S'} \). Indeed, otherwise \( S'_{\leq u} = (v_1 < \cdots < v_i) \) for some \( i < k \). But then \( v_{i+1} \leq u \) by the minimality of \( v_{i+1} \). Therefore \( v_{i+1} \in S'_{\leq u} \), a contradiction.

Let \( v = v_{k+1} \in \mathcal{P} \) be the minimal element such that \( m_v \notin M_{S'} \). For any \( u < v \), we have \( m_u \in M_{S'} \), hence \( m_u \in M_{S'_{\leq u}} \) by the previous statement. If \( v \notin C(S') \), then we conclude from Lemma 2.10 that \( m_v \notin M_{S'_{\leq u}} \), a contradiction. Therefore \( v \in C(S') \) and \( S' \cup \{v\} \) is a tree.

By the maximality of the sequence, we conclude that \( m_v \in M_S \) for all \( v \in \mathcal{P} \), hence \( (m_u)_{u \in S} \) is a basis of \( M \). If \( M \notin Z_S \), then there exists an element \( v \in \mathcal{P} \) such that \( m_v \notin M_{S_{\leq u}} \). But then \( m_v \notin M_S \) by the statement we proved. This is a contradiction. \( \square \)

2.7. Cell decomposition. In this section we give slightly simplified proofs of some results from \([1, 10]\). As before, we assume that \( \mathcal{P} \) is equipped with an admissible order. We define a total order on the set of trees \( \mathcal{P}(d) \) as follows. Given trees \( S, S' \in \mathcal{P}(d) \) we write them in the form

\[
S = (v_1 < \cdots < v_k), \quad S' = (v'_1 < \cdots < v'_k)
\]

and define \( S' < S \) if, for the minimal \( 1 \leq i \leq k \) with \( v'_i \neq v_i \), we have \( v'_i < v_i \).

Lemma 2.13. If \( S' < S \), then \( Z_S \cap U_{S'} = \emptyset \). In particular, if \( S \neq S' \), then \( Z_S \cap Z_{S'} = \emptyset \).

Proof. If \( M \in Z_S \cap U_{S'} \), then \( k = \dim M = \#S = \#S' \). Let

\[
S = (v_1 < \cdots < v_k), \quad S' = (v'_1 < \cdots < v'_k)
\]

and let \( 1 \leq i \leq k \) be minimal such that \( v_i \neq v'_i \). Then \( v'_i < v_i \) and \( v'_i \notin S \) (if \( v'_i \in S \), then \( v'_i = v_j \) for some \( j < i \), hence \( v'_j < v'_i = v_j \), a contradiction). Note that \( v_1 = v'_1 = \ast \), hence \( v'_i \neq \ast \). Let \( v'_i = av \), where \( a \) is an arrow. Then \( v < v'_i \) and \( v \in S' \), hence \( v = v'_j = v_j \in S \) for some \( j < i \). This implies that \( v'_i = av \in C(S) \). By the assumption \( M \in Z_S \), we have \( m_{v_i} \in M_{S_{<S'_v}} \). If \( u \in S \) and \( u < v'_i < v_i \), then \( u = v_j = v'_j \in S' \) for some \( j < i \). This implies \( m_{v_i} \in M_{S_{<S'_v}} \). A contradiction to \( M \in U_{S'} \). \( \square \)

Lemma 2.14. For any tree \( S \in \mathcal{P}(d) \), we have \( U_S \cap Z_S \subset \bigcup_{S' \subset S} U_{S'} \).

Proof. We need to show that \( U_S \backslash Z_S \subset \bigcup_{S' \subset S} U_{S'} \). Let \( M \in U_S \backslash Z_S \) and let \( v \in C(S) \) be minimal such that \( m_v \notin M_{S_{<v}} \). Consider the tree

\[
\tilde{S} = \{ u \in S \mid u < v \} \cup \{v\}.
\]

Then the vectors \( (m_u)_{u \in \tilde{S}} \) are linearly independent and there exists a tree \( \tilde{S} \subset S' \subset \mathcal{P} \) such that \( M \in U_{S'} \), by Corollary 2.2. We will prove that \( S' < S \). We can write

\[
\tilde{S} = (v_1 < \cdots < v_{i-1} < v), \quad S = (v_1 < \cdots < v_{i-1} < v_i < \ldots),
\]

where \( v < v_i \). We claim that for all \( w \in S' \backslash \tilde{S} \), we have \( w > v \), hence

\[
S' = (v_1 < \cdots < v_{i-1} < v < \ldots)
\]

and therefore \( S' < S \).

If our claim is not true, choose the minimal \( w \in S' \backslash \tilde{S} \) with \( w < v \). Then \( w \in C(\tilde{S} \backslash \{v\}) \), hence either \( w \in S \) (but then \( w \in \tilde{S} \), a contradiction) or \( w \in C(S) \). We have \( m_w \notin M_{S_{<w}} \).
as \( w \in S' \) and \( M \in U_{S'} \). Note that if \( u \in S \) and \( u < w < v \), then \( u \in \bar{S} \subset S' \). Therefore \( M_{S' \downarrow w} \subset M_{S' \downarrow w} \), hence \( m_w \notin M_{S' \downarrow w} \), contradicting to the minimality of \( v \). \( \square \)

We arrive at the following result (cf. [10, Theorem 7.7]).

**Theorem 2.15.** For any tree \( S \in \mathcal{P}(d) \), define the open set \( V_S = \bigcup_{S' \leq S} U_{S'} \). Then \( V_S = \bigcup_{S' \leq S} Z_{S'} \). In particular,

\[
\mathcal{N}_d = \bigsqcup_{S \in \mathcal{P}(d)} Z_S.
\]

**Proof.** Recall that \( \mathcal{P}(d) \) is a finite set. We conclude from Lemma 2.14 by induction that \( V_S \subset \bigcup_{S' \leq S} Z_{S'} \). This union is disjoint by Lemma 2.13. Moreover, for any \( S' \leq S \), we have \( Z_{S'} \subset U_{S'} \subset V_S \). Therefore \( V_S = \bigcup_{S' \leq S} Z_{S'} \). Taking the maximal tree \( S \in \mathcal{P}(d) \), we obtain the last statement. \( \square \)

**Corollary 2.16 (Cell decomposition).** The moduli space \( \mathcal{N}_d \) admits a cell decomposition, meaning that it admits a filtration by open subvarieties

\[
\emptyset = V_0 \subset V_1 \subset \ldots \subset V_r = \mathcal{N}_d
\]

such that \( V_k \setminus V_{k-1} \) are affine spaces, for all \( 1 \leq k \leq r \).

**Proof.** Let us order the trees in \( \mathcal{P}(d) \) as \( S_1 < \cdots < S_r \). Then \( V_k = V_{S_k} = \bigcup_{i \leq k} U_{S_i} = \bigcup_{i \leq k} Z_{S_i} \) are open and satisfy \( V_k \setminus V_{k-1} = Z_{S_k} \). \( \square \)

2.8. **Examples of cell closures.**

**Example 2.17.** For any \( S \in \mathcal{P}(d) \), the subset \( \bigcup_{S' \geq S} Z_{S'} \subset \mathcal{N}_d \) is closed. Let us show that there exist trees \( S' > S \) such that \( d(S') > d(S) \). This will imply, in particular, that \( \bigcup_{S' \geq S} Z_{S'} \) is not equal to the closure \( \bar{Z}_S \). We consider the framed quiver \( Q^f \)

\[
\begin{array}{c}
\infty \\
\end{array} \xrightarrow{e} \begin{array}{c}
\infty \\
\end{array} \xrightarrow{f} \begin{array}{c}
\infty \\
\end{array} \xrightarrow{a} \\
\begin{array}{c}
\infty \\
\end{array} \xrightarrow{b}
\]

with the order of arrows \( e < f < a < b \) and the corresponding shortlex order on the set of paths \( \mathcal{P} \). Let us consider the trees in \( \mathcal{P}(3) \) (we omit the root *).

\[
S = (e, f, bf), \quad S' = (e, ae, be)
\]
satisfying \( S < S' \). Then \( d(S) = 12 \) and \( d(S') = 13 \), hence \( Z_{S'} \) is not contained in \( \bar{Z}_S \).

**Example 2.18.** Let us show that the closure of a cell is not necessary a disjoint union of cells. Let us consider the framed quiver \( Q^f \)

\[
\begin{array}{c}
\infty \\
\end{array} \xrightarrow{f} \begin{array}{c}
\infty \\
\end{array} \xrightarrow{a} \\
\begin{array}{c}
\infty \\
\end{array} \xrightarrow{b}
\]

with the order of arrows \( f < a < b \) and the corresponding shortlex order on the set of paths \( \mathcal{P} \). We consider the trees in \( \mathcal{P}(4) \) (we omit the root *).

\[
S = (f, af, bf, b^2 f), \quad S' = (f, bf, abf, b^2 f)
\]
satisfying \( S < S' \). Their critical sets are

\[
C(S) = (a^2 f, ba f, ab f, ab^2 f, b^3 f), \quad C(S') = (a f, a^2 bf, babf, ab^2 f, b^3 f).
\]
The corresponding cells have dimensions $d(S) = d(S') = 17$. We will show that $\bar{Z}_S \cap Z_{S'} \neq \emptyset$. On the other hand, $Z_{S'} \not\subseteq \bar{Z}_S$ for dimension reasons, hence $\bar{Z}_S$ is not a disjoint union of cells.

To find $\bar{Z}_S \cap Z_{S'}$, we first determine $\bar{Z}_S \cap U_{S'}$ and then intersect it with $Z_{S'}$. The intersection $\bar{Z}_S \cap U_{S'}$ is equal to the closure of $Z_S \cap U_{S'} \cap \emptyset$ in $U_{S'}$. Let us denote the elements of $S'$ by $u_1 < \cdots < u_4$ and the elements of $C(S')$ by $v_1 < \cdots < v_5$. Then the coordinates $(c_{ij})$ of $M \in U_{S'}$ satisfy $m_{v_1} = \sum_i c_{ij} m_{u_i}$. For $M \in Z_S$, we require that $(m_f, m_{af}, m_{bf}, m_{abf})$ are linearly independent and that $m_{af}, m_{abf}$ are contained in $V = \langle m_f, m_{bf}, m_{abf} \rangle$. The first condition means that for

$$m_{af} = c_{11} m_f + c_{21} m_{bf} + c_{31} m_{abf} + c_{41} m_{af}$$

we have $c_{31} \neq 0$. Then condition $m_{abf} \in V$ means that $c_{41} = 0$, hence $V = \langle m_f, m_{bf}, m_{abf} \rangle$.

We have

$$m_{af} = am_{af} = c_{11} m_f + c_{21} m_{bf} + c_{31} m_{abf} + c_{41} m_{af} \in V + c_{31} c_{42} m_{af},$$

$$m_{abf} = bm_{af} = c_{11} m_f + c_{21} m_{bf} + c_{31} m_{abf} + c_{43} m_{bf} \in V + (c_{21} + c_{31} c_{43}) m_{bf}.$$

Therefore $Z_S \cap U_{S'}$ is given by $c_{31} \neq 0$ and the equations

$$c_{41} = c_{42} = c_{21} + c_{31} c_{43} = 0$$

while $\bar{Z}_S \cap U_{S'}$ is given by the above equations. The cell $Z_{S'} \subset U_{S'}$ is given by the equations $c_{21} = c_{31} = c_{41} = 0$. We conclude that $\bar{Z}_S \cap Z_{S'}$ is given by the equations $c_{21} = c_{31} = c_{41} = c_{42} = 0$. In particular, it is non-empty.

2.9. Degeneracy loci. For any subset $S \subseteq \mathcal{P}$, we define the degeneracy locus

\begin{equation}
D(S) \subset N_d
\end{equation}

to be the set of representations $M \in N_d$ such that the vectors $(m_u)_{u \in S}$ in $M$ are linearly dependent. For any tree $S \in \mathcal{P}(d)$ and $v \in C(S)$, let

\begin{equation}
S(v) = \{ u \in S_{t(v)} \mid u < v \} \cup \{ v \}.
\end{equation}

Define the degeneracy locus of the tree $S$ to be

\begin{equation}
D_S = \bigcap_{v \in C(S)} D(S(v)).
\end{equation}

Lemma 2.19. We have $D_S \subseteq \bigcup_{S' \geq S} Z_{S'}$ and $D_S \cap U_S = Z_S$.

Proof. We note that $\bigcup_{S' \leq S} Z_{S'} = \bigcup_{S' \leq S} U_{S'}$. Therefore to show that $D_S \subseteq \bigcup_{S' \geq S} Z_{S'}$ it is enough to show that $D_S \cap U_{S'} = \emptyset$ for $S' < S$. The proof of this statement goes through the same lines as the proof of Lemma 2.13.

We have $\bar{Z}_S \subset D_S \cap U_S$ by definition. If $M \in D_S \cap U_S$, then $m_v \in M_{S_{\leq v}}$, for all $v \in C(S)$, hence $M \in Z_S$. Therefore $D_S \cap U_S = Z_S$. \qed

Remark 2.20. If $Q$ is a quiver with one vertex, then one can show that $D_S = \bigcup_{S' \geq S} Z_{S'}$ [3, Lemma 2.2]. We don’t know if this statement is true in general.
2.10. Varieties having cell decomposition. The following result is folklore.

**Theorem 2.21.** Assume that an algebraic variety $X$ admits a cell decomposition, meaning that it admits a filtration by open subvarieties

$$\emptyset = V_0 \subset V_1 \subset \ldots \subset V_r = X$$

such that $Z_k = V_k \setminus V_{k-1}$ are affine spaces, for all $1 \leq k \leq r$. Then

$$A_*(X) = \bigoplus_k \mathbb{Z}[\tilde{Z}_k].$$

Moreover, $H^BM_*(X, \mathbb{Z}) = 0$ for odd $i$ and the cycle map $A_*(X) \to H^BM_*(X, \mathbb{Z})$ is an isomorphism for $i \in \mathbb{Z}$.

**Proof.** We have $V_k = \bigsqcup_{i \leq k} Z_i$. Define closed subsets $D_k = \bigsqcup_{i \geq k} Z_i = X \setminus V_{k-1}$. Then $Z_k$ is open in $D_k$ and its complement is $D_{k+1}$. We obtain a long exact sequence of BM homology

$$\cdots \to H_i^BM(D_{k+1}) \to H_i^BM(D_k) \to H_i^BM(Z_k) \to \cdots.$$ 

Note that $H_i^BM(Z_k) = 0$ for odd $i$ and we obtain by induction (on decreasing $k$) that $H_i^BM(D_k) = 0$ for odd $i$. This implies that we have a commutative diagram with exact rows

$$
\begin{array}{cccccc}
A_i(D_{k+1}) & \longrightarrow & A_i(D_k) & \longrightarrow & A_i(Z_k) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^BM_{2i}(D_{k+1}) & \longrightarrow & H^BM_{2i}(D_k) & \longrightarrow & H^BM_{2i}(Z_k) & \longrightarrow & 0.
\end{array}
$$

The right vertical map is an isomorphism as $Z_k$ is an affine space. We can assume by induction (on decreasing $k$) that the left vertical map is also an isomorphism. Then by the snake lemma we conclude that the middle vertical arrow is also an isomorphism. Therefore we have an exact sequence

$$0 \to A_*(D_{k+1}) \to A_*(D_k) \to A_*(Z_k) \to 0.$$ 

Note that $A_*(Z_k) = \mathbb{Z}[Z_k]$ and the above exact sequence splits canonically, where we map $[Z_k] \in A_*(Z_k)$ to $[\tilde{Z}_k] \in A_*(D_k)$. Therefore we obtain

$$A_*(D_k) \cong \mathbb{Z}[\tilde{Z}_k] \oplus A_*(D_{k+1}).$$

By induction this implies $A_*(X) = A_*(D_1) \cong \bigoplus_{k=1}^r \mathbb{Z}[\tilde{Z}_k].$ \hfill $\Box$

3. Multi-partitions

3.1. Trees and multi-partitions. Let, as before, $f \in \mathbb{N}^I$ be a framing vector, $Q^f$ be the corresponding framed quiver, and $\mathcal{P} = \mathcal{P}^f$ be the poset of paths in $Q^f$ that start at $\infty$. For $d \in \mathbb{N}^I$, let $\mathcal{P}(d) = \mathcal{P}^f(d)$ be the set of trees $S \subset \mathcal{P}$ with $\dim S = d$. We define

$$c(d) = f - \chi(d, -) \in \mathbb{Z}^I, \quad c(d)_i = f_i - \chi(d, e_i), \quad i \in I,$$

so that $\dim C(S) = c(d)$, for any $S \in \mathcal{P}(d)$ (see Lemma 2.9).

We define $S(d) = S^f(d)$ to be the set of multi-partitions

$$\lambda = (\lambda^{(i)})_{i \in I}, \quad \lambda^{(i)} = (\lambda^{(i)}_1 \geq \cdots \geq \lambda^{(i)}_{d^{(i)}} \geq 0), \quad i \in I,$$
such that, for all $0 \leq e < d$ (meaning that $e \neq d$ and $0 \leq e_i \leq d_i$ for all $i \in I$), there exists $i \in I$ satisfying (we define $\lambda_0^{(i)} = +\infty$)

$$\lambda_{d_i - e_i}^{(i)} < c(e)_i. \quad (22)$$

We assume that $P$ is equipped with an admissible total order. The following result was formulated in [1], although some details of the proof were omitted.

**Theorem 3.1.** There is a bijection $P(d) \to S(d)$ that sends a tree $S \in P(d)$ with

$$S_i = (u_{i,1} < \cdots < u_{i,d_i}), \quad i \in I,$$  

(23)

to the multi-partition $\lambda$ with

$$\lambda_{d_i - k}^{(i)} = \# \{v \in C(S)_i \mid v < u_{i,k+1}\}, \quad i \in I, \quad 0 \leq k < d_i. \quad (24)$$

**Remark 3.2.** Given a tree $S \in P(d)$ and $v \in C(S)_i$, we define (13)

$$k_v = \# \{u \in S_i \mid u < v\} \leq d_i. \quad (25)$$

Then $k_v \leq k$ if and only if $v < u_{i,k+1}$. Therefore

$$\lambda_{d_i - k}^{(i)} = \# \{v \in C(S)_i \mid k_v \leq k\}, \quad i \in I, \quad 0 \leq k < d_i. \quad (26)$$

**Proof of Theorem 3.1.** First, let us show that the multi-partition $\lambda$ defined above satisfies the required condition (22). For convenience, we define (using $\lambda_0^{(i)} = +\infty$)

$$m_k^{(i)} = \lambda_{d_i - k}^{(i)}, \quad 0 \leq k \leq d_i. \quad (27)$$

Then condition (22) on $\lambda$ means that for all $0 \leq e < d$, there exists $i \in I$ such that

$$m_e^{(i)} < c(e)_i.$$

Assume that there exists $0 \leq e < d$ such that $m_e^{(i)} \geq c(e)_i$ for all $i \in I$. This means that

$$\# \{v \in C(S)_i \mid v < u_{i,e_i+1}\} \geq c(e)_i, \quad (28)$$

whenever $e_i < d_i$. Let $i \in I$ be such that $e_i < d_i$ and $u_{i,e_i+1}$ is minimal. We have

$$\# \{v \in C(S)_i \mid v < u_{i,e_i+1}\} = \# \{v \in S_i \cup C(S)_i \mid v \leq u_{i,e_i+1}\} - e_i - 1.$$

Every $v \in S_i \cup C(S)_i$ satisfying $v \leq u_{i,e_i+1}$ can be written in the form $v = a\bar{v}$ for some $\bar{v} \in S_j$, $j \in I \cup \{\infty\}$, and some arrow $a: j \rightarrow i$ in $Q_f$.

If $j \in I$, then $\bar{v} < v \leq u_{j,e_j+1} \leq u_{j,e_j+1}$, whenever $e_j < d_j$. The number of such $\bar{v} \in S_j$ is bounded by $e_j$ (this also includes the case $e_j = d_j$). Therefore the number of $v$ of the required form is bounded by $\sum_j a_{ji} e_j$, where $a_{ji}$ is the number of arrows $j \rightarrow i$ in $Q_f$. If $j = \infty$, then $\bar{v} = e_\infty$ and the number of $v$ of the required form is bounded by $f_i$. We conclude that

$$\# \{v \in C(S)_i \mid v < u_{i,e_i+1}\} \leq f_i + \sum_j a_{ji} e_j - e_i - 1 = f_i - \chi(e,e_i) - 1 < c(e)_i$$

and this contradicts assumption (28).

Now let us construct the inverse map $S(d) \to P(d)$. Given a multi-partition $\lambda \in S(d)$, we define the numbers $m_k^{(i)}$ as in (27). We need to construct a tree $S = (u_0 < u_1 < \ldots)$ in $P(d)$ with

$$S_i = (u_{i,1} < \cdots < u_{i,d_i}), \quad i \in I,$$
such that
\[
\# \{ v \in C(S)_i \mid v < u_{i,k+1} \} = m^{(i)}_k, \quad 0 \leq k < d_i.
\]
We construct this tree inductively as follows. We set \( u_0 = e_\infty \). Assume that a tree \( S' = (u_0 \cdots < u_r) \) is constructed and satisfies the conditions
\[
\# \{ v \in C(S')_i \mid v < u_{i,k+1} \} = m^{(i)}_k, \quad i \in I, \ 0 \leq k < e_i.
\]
(29)
\[
\# \{ v \in C(S')_i \mid v < u \} \leq m^{(i)}_{e_i}, \quad i \in I, \ u \in S'.
\]
(30)
where \( e = \dim S' \leq d \) and
\[
S'_i = (u_{i,1} < \cdots < u_{i,e_i}), \quad i \in I.
\]
Assuming that \( e < d \), we will construct a tree \( S'' = S' \cup \{ u_{r+1} \} \) satisfying the same conditions.

By the assumption on \( \lambda \), there exists \( i \in I \) such that
\[
m^{(i)}_{e_i} < c(e)_i.
\]
We order the elements of \( C(S')_i \) and denote the \( m^{(i)}_{e_i} \) + 1-st element by \( v_i \). We claim that \( v_i \) is the maximal element in \( S' \cup \{ v_i \} \). Indeed, we have
\[
\# \{ v \in C(S')_i \mid v \leq v_i \} = m^{(i)}_{e_i} + 1
\]
while by condition (30)
\[
\# \{ v \in C(S')_i \mid v \leq u \} \leq m^{(i)}_{e_i}, \quad u \in S'.
\]
This implies that \( u < v_i \) for all \( u \in S' \).

Let \( j \in I \) be such that \( v_j \in C(S')_j \) is minimal and let \( S'' = S' \cup \{ v_j \} \). Then
\[
C(S'') = C(S') \setminus \{ v_j \} \cup \text{ch}(v_j),
\]
where \( \text{ch}(v) = \min_z(\mathcal{P}_{\mathcal{P}_v}) \) denotes the set of children of \( v \in \mathcal{P} \). As \( v_j \) is the maximal element in \( S'' \), we have \( S''_i = S'_i \) for \( i \neq j \) and
\[
S''_j = (u_{j,1} < \cdots < u_{j,e_j} < v_j =: u_{j,e_j+1}).
\]
We will show that conditions (29) and (30) are satisfied for the tree \( S'' \). For \( i \in I \) and \( 0 \leq k < e_i \), we have
\[
\# \{ v \in C(S'')_i \mid v < u_{i,k+1} \} = \# \{ v \in C(S')_i \mid v < u_{i,k+1} \} = m^{(i)}_k
\]
by condition (29) for \( S' \). On the other hand
\[
\# \{ v \in C(S'')_j \mid v < v_j \} = \# \{ v \in C(S')_j \mid v < v_j \} = m^{(j)}_{e_j+1}.
\]
Therefore condition (29) for the tree \( S'' \) is satisfied. To prove condition (30) for \( S'' \) it is enough to consider the maximal element \( u = v_j \) of the tree.

If \( i \in I \) is such that \( m^{(i)}_{e_i} < c(e)_i \), then \( v_j \leq v_i \) by the minimality of \( v_j \), hence
\[
\# \{ v \in C(S')_i \mid v < v_j \} \leq m^{(i)}_{e_i}.
\]
If \( i \in I \) is such that \( m^{(i)}_{e_i} \geq c(e)_i \), then we also obtain
\[
\# \{ v \in C(S')_i \mid v < v_j \} \leq \# C(S')_i = c(e)_i \leq m^{(i)}_{e_i}.
\]
We conclude that for all \( i \in I \)
\[
\# \{ v \in C(S'')_i \mid v < v_j \} = \# \{ v \in C(S')_i \mid v < v_j \} \leq m^{(i)}_{e_i}
\]
which implies condition (30) for the tree $S''$. □

3.2. Total order on the set of multi-partitions. In the previous section we constructed a bijection $\mathcal{P}(d) \simeq S(d)$ which depends on an admissible order on $\mathcal{P}$. On the other hand, in §2.7 we constructed a total order on the set of trees $\mathcal{P}(d)$. This implies that we have the induced total order on the set of multi-partitions $S(d)$, which a priori may depend on an admissible total order on $\mathcal{P}$. In this section we will show that, for a quiver $Q$ having only one vertex, the induced total order on $S(d)$ is independent of an admissible order on $\mathcal{P}$.

Let us assume that $Q$ has one vertex. Recall that, for $d \in \mathbb{N} \simeq N^I$ and a tree $S = (\ast < u_1 < \cdots < u_d)$ in $\mathcal{P}(d)$, the corresponding partition $\lambda = (\lambda_1, \ldots, \lambda_d) \in S(d)$ is defined by (24)

$$\lambda_{d-k} = \# \{ v \in C(S) \mid v < u_{k+1} \}, \quad 0 \leq k < d. \tag{31}$$

**Lemma 3.3.** Let $S, S' \in \mathcal{P}(d)$ and $\lambda, \mu \in S(d)$ be the corresponding partitions. Then $S < S'$ if and only if $\lambda_k < \mu_k$ for the maximal $1 \leq k \leq d$ with $\lambda_k \neq \mu_k$. In particular, the induced total order on $S(d)$ is independent of the choice of an admissible order on $\mathcal{P}$.

**Proof.** It is enough to show that if partitions $\lambda, \mu$ satisfy the above property, then $S < S'$. Let $m_k = \lambda_{d-k}$ and $m'_k = \mu_{d-k}$ for $0 \leq k < d$. By the proof of Theorem 3.1 the tree $S = (\ast < u_1 < \cdots < u_d)$ has the property that $u_{k+1}$ is the $(m_k + 1)$-st element in $C(S)$, where $S$ is the tree $S = (\ast < u_1 < \cdots < u_k)$. Similarly for the tree $S' = (\ast < u'_1 < \cdots < u'_d)$.

By assumption, we have $m_{k+1} < m'_{k+1}$ for the minimal $k$ with $m_{k+1} \neq m'_{k+1}$. This implies that $S = (\ast < u_1 < \cdots < u_k) = (\ast < u'_1 < \cdots < u'_k)$ and $u_{k+1} < u'_{k+1}$. Therefore $S < S'$. □

3.3. Cells parametrized by multi-partitions. Theorem 3.1 implies that we can parametrize the cells $Z_S \subset N_d$, for $S \in \mathcal{P}(d)$, by multi-partitions. Note that the number of $v \in C(S)_i$ with

$$k_v = \# \{ u \in S_i \mid u < v \} = d_i - k$$

is equal to $\lambda_k^{(i)} - \lambda_{k+1}^{(i)}$ for $k \geq 0$, where we use now $\lambda_0^{(i)} = \# C(S)_i = f_i - \chi(d, i)$. Therefore, the dimension of the cell $Z_S$ is

$$\dim Z_S = \sum_{v \in C(S)} k_v = \sum_{i \in I} \sum_{k \geq 0} (d_i - k)(\lambda_k^{(i)} - \lambda_{k+1}^{(i)}) = f \cdot d - \chi(d, d) - |\lambda|, \tag{32}$$

where $|\lambda| = \sum_{i \in I} |\lambda^{(i)}| = \sum_{i \in I} \sum_{k \geq 1} \chi_k^{(i)}$. This implies that the motivic class of $N_d$ can be written in the form [1, Theorem 6.2]

$$[N_d] = L^{f-d-\chi(d, d)} \sum_{\lambda \in S(d)} L^{-|\lambda|}. \tag{33}$$

It follows from Theorem 3.1 that, for an admissible total order on $\mathcal{P}$, we have a bijection $S(d) \to \mathcal{P}(d)$, $\lambda \mapsto S_\lambda$. We will see later that the class $[Z_{S_\lambda}] \in A_*(N_d)$ generally depends on the choice of an admissible order on $\mathcal{P}$.
4. Grassmannians and non-commutative Hilbert schemes

Let \( 0 \leq d \leq w \) be two integers, \( r = w - d \) and \( W = \mathbb{C}^w \). We define the Grassmannian
\[
\text{Gr}^d(W) = \text{Gr}_r(W) = \text{Gr}(r, w)
\]
to be the algebraic variety parameterizing surjective linear maps \( M : W \to \mathbb{C}^d \) (up to the action of \( \text{GL}_d(\mathbb{C}) \)) or, equivalently, subspaces \( U = \ker(M) \subset W \) of dimension \( r \).

We can interpret it as a non-commutative Hilbert scheme as follows. Let \( Q \) be the quiver having one vertex 0 and no arrows, and let \( Q_w \) be the framed quiver obtained from \( Q \) by adding one vertex \( \infty \) and \( w \) arrows \( \infty \to 0 \). The non-commutative Hilbert scheme \( N_d = N_{d,w} \) parametrizes representations \( M \) of \( Q_w \) having dimension vector \( (d, 1) \) (with the coordinate 1 at the vertex \( \infty \)), generated by \( M_\infty \). Such representations can be identified with surjective linear maps \( M : \mathbb{C}^w \to \mathbb{C}^d \) (up to the action of \( \text{GL}_d(\mathbb{C}) \)), hence
\[
N_d = \text{Gr}^d(W) = \text{Gr}_r(W).
\]
This is a smooth projective variety of dimension \( d(w - d) \).

In the previous sections we established a cell decomposition of \( N_d \) parametrized by trees or partitions. On the other hand, Grassmannians possess a decomposition by Schubert cells parametrized by partitions. In this section we will establish a precise relationship between these decompositions. We will also discuss if the properties satisfied by cell decompositions of Grassmannians can be generalized to other quivers.

4.1. Trees and cells. Recall that the framed quiver \( Q^w \) consists of two vertices \( \infty, 0 \) and \( w \) arrows \( \infty \to 0 \). Let us define \([1, w] = \{1, \ldots, w\}\). Then
\[
\begin{align*}
(1) \text{ The path poset } \mathcal{P} \text{ can be identified with the set } \{\ast\} \cup [1, w]. \\
(2) \text{ A subtree } S \in \mathcal{P}(d) \text{ can be identified with a subset } S_0 \subset [1, w] \text{ having } d \text{ elements.} \\
(3) \text{ The critical set } C(S) \text{ can be identified with } S' = [1, w] \setminus S_0 \text{ having } r = w - d \text{ elements.}
\end{align*}
\]
In what follows we will write \( S \) for \( S_0 \subset [1, w] \) and still call it a tree. We order the elements of \( \mathcal{P} \) as
\[
\ast < 1 < \cdots < w.
\]
A framed representation \( M \in N_d \) can be identified with a surjective linear map \( M : \mathbb{C}^w \to \mathbb{C}^d \). We define
\[
m_i = M(e_i) \in \mathbb{C}^d, \quad i \in [1, w].
\]
Then the cell \( Z_S \), for \( S \subset [1, w] \), consists of such \( M \) that
\[
(1) (m_i)_{i \in S} \text{ are linearly independent.} \\
(2) m_j \in \langle m_i : i \in S_j \rangle \text{ for all } j \in S' = [1, w] \setminus S.
\]
Earlier we defined the set \( S(d) = S^w(d) \) to be the set of partitions \((21)\)
\[
\mu = (\mu_1, \ldots, \mu_d), \quad \mu_1 \leq w - d.
\]
In Theorem 3.1 we described a bijection \( \mathcal{P}(d) \to S(d) \). For every tree
\[
S = (u_1 < \cdots < u_d) \subset [1, w]
\]
we define the corresponding partition \( \mu \in S(d) \) by
\[
\mu_{d-k} = \# \{ j \in [1, w] \setminus S \mid j < u_{k+1} \}, \quad 0 \leq k < d.
\]
We denote the cell \( Z_S \) also by \( Z_\mu \).
4.2. **Schubert cells.** As before, we consider $0 \leq d \leq w$ and $r = w - d$. Consider a full flag

$$0 = F_0 \subset F_1 \subset \ldots \subset F_w = W = \mathbb{C}^w.$$ 

Given a partition

(38) $\lambda = (\lambda_1, \ldots, \lambda_r), \quad \lambda_1 \leq \lambda_0 := d,$

we define the Schubert variety [5, §9.4]

(39) $\Omega_\lambda = \{ U \in \text{Gr}^d(W) \mid \dim(U \cap F_{d+i-\lambda_i}) \geq i, 1 \leq i \leq r \}$

and the Schubert cell

(40) $\Omega^o_\lambda = \{ U \in \text{Gr}^d(W) \mid \dim(U \cap F_k) = i \text{ for } d + i - \lambda_k \leq k \leq d + i - \lambda_{i+1}, \ 0 \leq i \leq r \}.$

It is known that

1. $\Omega^o_\lambda$ is an affine space.
2. $\text{Gr}^d(W) = \bigsqcup_{\lambda} \Omega^o_\lambda$.
3. $\Omega_\lambda$ is the closure of $\Omega^o_\lambda$ in $\text{Gr}_r(W)$.
4. $\Omega_\lambda = \bigsqcup_{\nu \supseteq \lambda} \Omega^o_{\nu}$, where $\nu \supseteq \lambda$ if $\nu_i \geq \lambda_i$ for all $i$.

**Proposition 4.1.** Assume that the flag on $\mathbb{C}^w$ is given by $F_i = \langle e_1, \ldots, e_i \rangle$, for $1 \leq i \leq w$. Then, for any partition $\lambda$ as in (38), we have

(41) $\Omega^o_\lambda = Z_S,$

where $S = [1, w] \setminus S'$ with

(42) $S' = \{ v_1, \ldots, v_r \}, \quad v_i = d + i - \lambda_i, \quad 1 \leq i \leq r.$

**Proof.** We have

$$0 = v_0 < v_1 < \cdots < v_r \leq d + r = w$$

so that $S'$ consists of $r$ elements and $S$ consists of $d$ elements.

Given $U \subset \text{Gr}_r(\mathbb{C}^w)$, we consider the quotient

$$M : \mathbb{C}^w \to \mathbb{C}^w / U \cong \mathbb{C}^d.$$

Then the condition $\dim U \cap F_k = i$, means that $\dim \ker(M : F_k \to \mathbb{C}^d) = i$, hence

(43) $\text{rk} \langle m_1, \ldots, m_k \rangle = k - i, \quad m_i = M(e_i) \in \mathbb{C}^d.$

The condition that $U \in \Omega^o_\lambda$ means that we have the above equality for $v_i \leq k < v_{i+1}$. This means that the rank in (43) increases by 1 if $v_i < k < v_{i+1}$ (hence $k \in S'$) and doesn’t increase if $k = v_{i+1}$ (hence $k \in S'$). This can be reformulated as the condition that $(m_i)_{i \in S}$ are linearly independent and that $m_j \in \langle m_i : i \in S_{<j} \rangle$ for all $j \in S'$. This is exactly the definition of the cell $Z_S$. □
4.3. **Relation between partitions.** In Prop. 4.1, for any partition $\lambda$ as in (38), we constructed the sets

$$S = (u_1 < \cdots < u_d) \subset [1, w], \quad S' = [1, w] \setminus S = (v_1 < \cdots < v_r),$$

with $v_i = d + i - \lambda_i$. Conversely, we can reconstruct the partition $\lambda$ from $S$ by

$$\lambda_i = d + i - v_i, \quad 1 \leq i \leq r.$$

On the other hand, we associated with $S$ a partition $\mu$ with (37)

$$\mu_k = \# \{ i \geq 1 \mid v_i < u_{d-k+1} \}, \quad 1 \leq k \leq d,$$

**Lemma 4.2.** The partitions $\mu$ and $\lambda$ are conjugate to each other, meaning that

$$\mu_k = \# \{ i \geq 1 \mid \lambda_i \geq k \}, \quad k \geq 1.$$

**Proof.** We need to show that

$$\lambda_i \geq k \iff v_i < u_{d-k+1}.$$

The first inequality can be written in the form $v_i \leq d - k + i$. In this case the number of $j$ with $u_j < v_i$ is $\leq d - k$, implying that $v_i < u_{d-k+1}$.

If $v_i < u_{d-k+1}$, then the number of $j$ with $u_j < v_i$ is $\leq d - k$. Therefore $v_i$ is bounded by $d - k$ (the number of $u_j$ on the left of $v_i$) plus $i$ (the number of $v_j$ on the left of $v_i$), hence $v_i \leq d - k + i$. $\square$

We conclude from the previous lemma that

$$Z_\mu = \Omega_{\mu'}, \quad \bar{Z}_\mu = \Omega_{\mu'},$$

where $\mu'$ denotes the conjugate partition. Schubert varieties have a decomposition

$$\Omega_{\lambda} = \bigsqcup_{\nu \geq \lambda} \Omega_{\nu},$$

where $\nu \supseteq \lambda$ if $\nu_i \geq \lambda_i$ for all $i$. This implies that

$$\bar{Z}_\mu = \bigsqcup_{\nu \geq \mu} Z_{\nu}.$$

On the other hand, we have seen in §2.8 that the closure of a cell is not a disjoint union of cells for more general quivers. The class of a Schubert variety $[\Omega_{\mu'}] \in A_\ast(Gr^d(W))$ is independent of a flag. Therefore the class $[Z_\mu]$ is independent of the choice of a total order on $P$. On the other hand, we will see in §6.5 that this is not the case for more general quivers.

5. **Coha modules**

5.1. **Cohomological Hall algebras.** Cohomological Hall algebras were introduced in [7]. They are defined for a quiver with a potential and use cohomology of vanishing cycle complexes associated with the trace of the potential. We restrict in this paper to the case where the potential is trivial. In this case, we may work with ordinary equivariant cohomology groups.
Let $Q$ be a quiver with the set of vertices $I$ and the set of arrows $Q_1$. We call elements $d \in \mathbb{N}^I \subset \Gamma := \mathbb{Z}^I$ the dimension vectors. As before, we define

$$R_d = R(Q, d) = \bigoplus_{(a : i \to j) \in Q_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

It is a finite-dimensional complex vector space. We regard an element of $R_d$ as a representation of $Q$ of dimension vector $d$. We consider the complex linear algebraic group $G_d = \prod_{i \in I} \text{GL}_{d_i}(\mathbb{C})$ and let an element $g = (g_i)_{i \in I} \in G_d$ act on $M = (M_a)_{a \in Q_1} \in R_d$ by

$$g \cdot M := (g_j M_a g_i^{-1})(a : i \to j) \in Q_1.$$ 

Two elements of $R_d$ are isomorphic as representations of $Q$ if and only if they lie in the same $G_d$-orbit. The $\mathbb{C}$-valued points of the quotient stack $[R_d/G_d]$ therefore correspond to isomorphism classes of $Q$-representations of dimension vector $d$.

Let $e \in \mathbb{N}^I$ be another dimension vector. For every $i \in I$, we identify $\mathbb{C}^{d_i}$ with the subspace of $\mathbb{C}^{d_i+e_i}$ spanned by the first $d_i$ coordinate vectors. Let

$$R_{d,e} = \{ M \in R_{d+e} \mid M_a(\mathbb{C}^{d_i}) \subset \mathbb{C}^{d_j} \forall a : i \to j \}.$$ 

It is a linear subspace of $R_{d+e}$. Elements of $R_{d,e}$ are representations $M \in R_{d+e}$ for which $M_a$, $a : i \to j$, has the following block upper triangular shape:

$$\begin{pmatrix} \mathbb{C}^{d_i} & \mathbb{C}^e \\ \mathbb{C}^e & \begin{pmatrix} M'_a & * \\ 0 & M''_a \end{pmatrix} \end{pmatrix}$$

For such $M \in R_{d,e}$, we obtain representations $M' \in R_d$ and $M'' \in R_e$. This gives rise to the maps

$$R_d \times R_e \xrightarrow{p} R_{d,e} \xrightarrow{i} R_{d+e}.$$ 

Note that $i$ is a closed embedding of codimension $r_1 := \sum_{a : i \to j} d_i e_j$. Let

$$G_{d,e} = \{ g \in G_{d+e} \mid g_i(\mathbb{C}^{d_i}) \subset \mathbb{C}^{d_i} \forall i \in I \},$$

i.e. for $g \in G_{d,e}$, the matrix $g_i$ has a similar block upper triangular form to the one from above. The group $G_{d,e}$ is a parabolic subgroup of $G_{d+e}$ and the dimension of the homogeneous space $G_{d+e}/G_{d,e}$ is $r_2 = \sum_{i \in I} d_i e_i$. The subvariety $R_{d,e}$ is $G_{d,e}$-invariant. We obtain morphisms of algebraic groups

$$G_d \times G_e \xrightarrow{p} G_{d,e} \xrightarrow{i} G_{d+e}.$$ 

The map $p$ is $G_{d,e}$-equivariant with respect to $q$ and $i$ is $G_{d,e}$-equivariant with respect to $j$.

We now pass to equivariant cohomology. For a smooth complex variety $X$, equipped with an action of a complex algebraic group $G$, we consider both $X$ and $G$ with the euclidean topology. We form its equivariant singular cohomology $H^*_G(X; \mathbb{Q})$ and regard it as a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space. In the following, we will always work with rational coefficients and therefore write $H^*_G(X)$ for $H^*_G(X; \mathbb{Q})$. 

The maps \( p, i, q, \) and \( j \) give rise to linear maps in equivariant cohomology as follows:

\[
H^*_{G_d}(R_d) \otimes H^*_{G_e}(R_e) \xrightarrow{\delta} H^*_{G_{d+e}}(R_{d+e})[2r_1 - 2r_2]
\]

Note that under the identification \( H^*_{G_{d+e}}(R_{d+e}) \simeq H^*_{G_{d+e}}(G_{d+e} \times G_{d+e}) \), the map \( j^* \) is the push-forward along \( G_{d+e} \times G_{d+e} R_{d,e} \to R_{d+e} \). Note also that \( r_1 - r_2 = -\chi(d,e) \).

We define the \( \Gamma \)-graded vector space

\[
\mathcal{H} = \bigoplus_{d \in \mathbb{N}^I} \mathcal{H}_d, \quad \mathcal{H}_d = H^*_{G_d}(R_d).
\]

The composition of the above maps gives a morphism \( \ast : \mathcal{H}_d \otimes \mathcal{H}_e \to \mathcal{H}_{d+e} \) of \( \mathbb{Q} \)-vector spaces and the induced morphism

\[
\ast : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}
\]

of \( \Gamma \)-graded vector spaces.

**Theorem 5.1** ([7, §2.2]). *The map \( \ast : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \) defines the structure of an associative (unital) \( \Gamma \)-graded algebra on \( \mathcal{H} \).*

The algebra \( \mathcal{H} \) is called the cohomological Hall algebra (CoHA) of the quiver \( Q \).

**Remark 5.2.** Assume that \( Q \) is a symmetric quiver, meaning that \( \chi = \chi_Q \) is a symmetric bilinear form. In this case, we may refine the construction as follows. Define the \( \mathbb{Z} \)-graded vector space

\[
\mathcal{H}_d := H^*_{G_d}(R_d)[-\chi(d,d)].
\]

With this degree shift, the linear map \( \ast : \mathcal{H}_d \otimes \mathcal{H}_e \to \mathcal{H}_{d+e} \) is homogenous with respect to the \( \mathbb{Z} \)-grading; here symmetry of the Euler form is crucial. This means that \( \mathcal{H} \) becomes a \( \Gamma \)-graded algebra in the category of \( \mathbb{Z} \)-graded vector spaces.

### 5.2. Shuffle algebra description.

Let us recall an explicit shuffle product description of the multiplication of CoHA from [7]. For any \( d \in \mathbb{N}^I \), the group \( G_d \) is linearly reductive. The subgroup \( T_d \subset G_d \) consisting of tuples of invertible diagonal matrices is a maximal torus. The corresponding Weyl group is isomorphic to the product of symmetric groups \( \Sigma_d = \prod_{i \in I} \Sigma_d^i \). As the space \( R_d \) is \( G_d \)-equivariantly contractible, we obtain

\[
\mathcal{H}_d = H^*_{G_d}(R_d) \simeq H^*_{G_d}(\mathfrak{pt}) \simeq H^*_{T_d}(\mathfrak{pt})_{\Sigma_d} \simeq \bigotimes_{i \in I} \mathbb{Q}[x_{i,1}, \ldots, x_{i,d_i}]^{\Sigma_d_i}.
\]

The last isomorphism is provided by the isomorphism \( H^*_{T_d}(\mathfrak{pt}) \simeq S(X^*(T_d) \otimes \mathbb{Q}) \), where \( X^*(T_d) \) is the group of characters of \( T_d \). This is the free abelian group generated by the characters \( x_{i,k} : T_d \to \mathbb{C}^\times \) which selects the \( k \)-th diagonal entry from the \( i \)-th matrix.

**Theorem 5.3** ([7, Thm. 2]). *Let \( f \in \mathcal{H}_d \) and \( g \in \mathcal{H}_e \). The product \( f \ast g \in \mathcal{H}_{d+e} \) is given by

\[
f \ast g = \sum_{\sigma \in \text{Sh}(d,e)} \sigma \left( f(x_{i,r})_{(i,r) \in \text{Id}_d} \cdot g(x_{j,d_j+s})_{(j,s) \in \text{Id}_e} \cdot \prod_{i,j \in I} \prod_{r=1}^{d_i} \prod_{s=1}^{e_j} (x_{j,d_j+s} - x_{i,r})^{-\chi(e_j,e_j)} \right)
\]
where the sum runs over all \((d, e)\)-shuffles, meaning \(\sigma \in \Sigma_{d+e}\) satisfying
\[
\sigma_i(1) < \cdots < \sigma_i(d), \quad \sigma_i(d+1) < \cdots < \sigma_i(d+e) \quad \forall i \in I,
\]
and the set \(I_d\) is given by \(I_d = \{(i, k) \mid i \in I, 1 \leq k \leq d_i\}\).

Note that in the above theorem, \(e_i \in \mathbb{N}^I\) is the simple root at the vertex \(i\), while \(e_i\) is the \(i\)th entry of the dimension vector \(e\).

### 5.3. Modules over the Cohomological Hall algebra

Modules over the Cohomological Hall algebra that come from framed stable representations were introduced in [12].

We briefly recall the construction. Given a framing vector \(f \in \mathbb{N}^I\), we defined earlier the non-commutative Hilbert scheme
\[
N_{d,f} = R_{d}^{f_{st}}/G_d, \quad d \in \mathbb{N}^I,
\]
where \(R_{d}^{f_{st}} \subset R_d\) is the open subvariety of stable framed representations in
\[
R_{d} = R(Q, d) = R(Q, d) \oplus \bigoplus_{i \in I} \text{Hom}(\mathbb{C}^i, \mathbb{C}^{d_i}).
\]

We can write \(M \in R_d\) (uniquely) as \(M = (N, f)\) with \(N \in R(Q, d)\) and \(f \in F_{r,d}\); sometimes we call \(f\) the framing datum.

Let \(d, e\) be two dimension vectors of \(Q\). We consider
\[
R_{d,e}^{f_{st}} := R_{d,e} \oplus F_{r,d+e}, \quad R_{d,e}^{f_{st}} := R_{d,e} \cap R_{d+e}^{f_{st}}.
\]
That means, \(R_{d,e}^{f_{st}}\) consists of framed stable representations \(M = (N, f)\) of dimension vector \(d + e\), such that \(N\) lies in \(R_{d,e}\). We consider the maps
\[
R_d \times R_e \xleftarrow{\rho} R_{d,e} \xrightarrow{i} R_{d+e}
\]
from \(\S 5.1\). Let \(M = (N, f) \in R_{d,e}^{f_{st}}\). We have \(p(N) = (N', N'')\). As \(F_{d,e} = F_{r,d} \oplus F_{r,e}\), we may decompose \(f = (f', f'')\) accordingly. Since \(N\) is generated by \(im f\) as a \(\mathbb{C}Q\)-module, the quotient \(N''\) is generated by \(im f''\). So \((N'', f'') \in R_{d,e}^{f_{st}}\). Note that the submodule \(N'\) need not be generated by \(f'\). We hence obtain maps
\[
R_d \times R_{e}^{f_{st}} \xleftarrow{\rho} R_{d,e}^{f_{st}} \xrightarrow{i} R_{d+e}^{f_{st}}
\]
We again pass to equivariant cohomology. As the action of \(G_d\) on \(R_d^{f_{st}}\) is free and \(N_{d,f} = R_d^{f_{st}}/G_d\) is a geometric \(G_d\)-quotient, we obtain the isomorphism
\[
H^*_G(R_d^{f_{st}}) \simeq H^*(N_{d,f}).
\]
Under this identification, we get a map
\[
*: H^*_G(R_d) \otimes H^*(N_{e,f}) \to H^*(N_{d+e,f})[-\chi(d, e)]
\]
similarly to the definition of the product in the CoHA. We now define the \(\Gamma\)-graded vector space
\[
M_{f} = \bigoplus_{d \in \Gamma} M_{f,d}, \quad M_{f,d} = H^*(N_{d,f}).
\]
The map \(*\) induces a structure of a \(\mathcal{K}\)-module on \(M_{f}\).
Remark 5.4. If the quiver $Q$ is symmetric, we define
\[ M_{r,d} = H^*(N_{d,f})[-\chi(d,d)] \]
and obtain a $\Gamma$-graded $\mathcal{H}$-module $M_r$ in the category of $\mathbb{Z}$-graded vector spaces.

Consider the composition
\[ R^f_{d} \hookrightarrow R^f_{d} = R_d \times F_{r,d} \to R_d \]
of the open inclusion and the projection which forgets the framing vector. Pulling back gives rise to the map $\phi: \mathcal{H}_d \to M_{r,d}$.

Theorem 5.5 ([4, Thm. 3.2]). The map
\[ \phi: \mathcal{H} = \bigoplus_d \mathcal{H}_d \to M = \bigoplus_d M_{r,d} \]
is an epimorphism of $\Gamma$-graded $\mathcal{H}$-modules and its kernel is given by
\[ \ker \phi = \sum_{d' \neq 0,d} \mathcal{H}_d \ast (e'_d \cup \mathcal{H}_{d'}), \quad e'_d = \prod_{i \in I}(x_{i,1} \ldots x_{i,d_i})^{f_i}, \]
where the cup product $\cup$ corresponds to the usual product in $\mathcal{H}_d = \bigotimes_{i \in I} \mathbb{Q}[x_{i,1}, \ldots, x_{i,d_i}]_{\Sigma_d}^\star$.

5.4. Tautological vector bundles. We can describe the epimorphism $\phi: \mathcal{H} \to M$ explicitly using tautological vector bundles. For any vertex $i \in I$, we define the $G_d$-linearized vector bundle $\mathcal{V}_{d,i}$ of rank $d_i$ over $R_d = R(Q, d)$ to be the trivial vector bundle with the fiber over $M \in R_d$ equal to $M_i = \mathbb{C}^{d_i}$, equipped with the $G_d$-linearization induced by the projection $G_d \to \text{GL}_{d_i}$. Under the isomorphism (49)
\[ \mathcal{H}_d = H^*_{G_d}(R_d) \simeq \bigotimes_{i \in I} \mathbb{Q}[x_{i,1}, \ldots, x_{i,d_i}]_{\Sigma_d}^\star, \]
the $G_d$-equivariant Chern class $c_k(\mathcal{V}_{d,i}) \in H^*_{G_d}(R_d)$ corresponds to the $k$th elementary symmetric function
\[ c_k(\mathcal{V}_{d,i}) = e_k(x_{i,1}, \ldots, x_{i,d_i}). \]

Similarly, for any vertex $i \in I$, we define the $i$-th tautological vector bundle $\mathcal{U}_i = \mathcal{U}_{d,i}$ to be the rank $d_i$ vector bundle over $N_{d,f} = R^f_{d}\langle G_d \rangle$, with the fiber over $M \in N_{d,f}$ equal to $M_i$. More precisely, consider the trivial vector bundle of rank $d_i$ over $R^f_{d}$ with the $G_d$-linearization induced by the projection $G_d \to \text{GL}_{d_i}$. This bundle descends to the vector bundle $\mathcal{U}_{d,i}$ over the quotient $N_{d,f}$.

The epimorphism $\phi$ considered earlier is given by
\[ \phi: \mathcal{H}_d \simeq \bigotimes_{i \in I} \mathbb{Q}[x_{i,1}, \ldots, x_{i,d_i}]_{\Sigma_d}^\star \to M_{r,d} = H^*(N_{d,f}), \]
\[ e_k(x_{i,1}, \ldots, x_{i,d_i}) = c_k(\mathcal{V}_{d,i}) \mapsto c_k(\mathcal{U}_{d,i}), \quad i \in I, \; k \geq 1. \]

Indeed, the vector bundle $\mathcal{U}_{d,i}$ over $N_{d,f}$ arises as the descent of the trivial rank $d_i$ vector bundle over $R^f_{d}$, with the $G_d$-linearization given by the $i$th factor of $G_d$. This is precisely the pull-back of $\mathcal{V}_{d,i}$ along the forgetful map $R^f_{d} \to R_d$. 

Because of the surjectivity of $\phi$, we conclude that $H^*(N_{d,f})$ is spanned by the vectors of the form
\begin{equation}
\prod_{i \in I} \prod_{k \geq 1} c_k(U_{d_i})^{n_{i,k}}
\end{equation}
with $n_{i,k} \geq 0$. Later we will find a basis of $H^*(N_{d,f})$ consisting of vectors of this form.

6. Degeneracy loci and tautological classes

6.1. Products of degeneracy classes. Let $X$ be a smooth (irreducible) complex algebraic variety of dimension $N$. Let $E_1, \ldots, E_n$ be vector bundles over $X$ of ranks $r_1, \ldots, r_n$. For $1 \leq i \leq n$, let $0 \leq k_i \leq r_i$ and let
\[ \sigma_i : O^{k_i+1}_X \to E_i \]
be a morphism of vector bundles. Consider the degeneracy locus
\begin{equation}
D(\sigma_i) = D_k(\sigma_i) = \{ x \in X \mid \text{rk} \sigma_i(x) \leq k_i \}
\end{equation}
with the structure of a closed subscheme of $X$ which comes from viewing it as the zero locus $Z(\bigwedge^{k_i+1} \sigma_i)$. All irreducible components of $D(\sigma_i)$ have dimension at least $N - r_i + k_i$ (see [6, Thm. 14.4]). In [6, §14.4] one defines a degeneracy class
\begin{equation}
\mathcal{D}(\sigma_i) = \mathcal{D}_{k_i}(\sigma_i) \in A_{N-r_i+k_i}(D(\sigma_i)).
\end{equation}
such that its image under the the push-forward along the closed embedding $D(\sigma_i) \hookrightarrow X$ is equal to $c_{r_i-k_i}(E_i) \cap [X]$ (see [6, Ex. 14.4.2]). Consider the scheme-theoretic intersection
\begin{equation}
D = D(\sigma_1) \cap \ldots \cap D(\sigma_n).
\end{equation}
All of its irreducible components have dimension at least
\begin{equation}
d = N - \sum_i (r_i - k_i).
\end{equation}
Our goal is to get a better understanding of the refined product [6, §8.1]
\begin{equation}
\mathcal{D} = \mathcal{D}(\sigma_1) \ldots \mathcal{D}(\sigma_n) \in A_d(D).
\end{equation}

6.2. Refined products. Given a morphism $f : X \to Y$ with smooth $Y$, subvarieties $X' \subset X$, $Y' \subset Y$ and classes $x \in A_*(X')$, $y \in A_*(Y')$, we consider the Cartesian diagram
\[
\begin{array}{ccc}
X' \times_Y Y' & \longrightarrow & X' \times Y' \\
\downarrow & & \downarrow \\
X & \overset{\gamma_f}{\longrightarrow} & X \times Y
\end{array}
\]
where $\gamma_f$ is the graph morphism of $f$ (note that $\gamma_f$ is a regular embedding), and we define the refined product [6, §8.1]
\[ x \cdot_f y = \gamma_f^*(x \times y) \in A_*(X' \times_Y Y'), \]
where $\gamma_f^*$ is the refined Gysin morphism corresponding to the above diagram. If $X = Y$ and $f = \text{id}$, then we denote the above product by $x \cdot y \in A_*(X' \cap Y')$. 
Lemma 6.1. Let \( f : X \to Y \) be a morphism of smooth algebraic varieties (note that \( f \) is automatically an l.c.i. morphism), \( Y_i \subset Y \) be a subvariety and \( y_i \in A_*(Y_i) \), for \( i = 1, 2 \). Then we have
\[
f'(y_1 \cdot y_2) = f'(y_1) \cdot f'(y_2) \in A_*(X \times_Y (Y_1 \cap Y_2)),
\]
where \( y_1 \cdot y_2 \in A_*(Y_1 \cap Y_2) \) and \( f'(y_i) \in A_*(X \times_Y Y_i) \), for \( i = 1, 2 \).

Proof. We obtain from associativity [6, Prop. 8.1.1(a)] applied to \( X \xrightarrow{f} Y \xrightarrow{id} Y \) that
\[
[X] \cdot f(y_1 \cdot y_2) = ([X] \cdot f y_1) \cdot f y_2.
\]
By [6, Prop. 8.1.2] we have \( f'(y_i) = [X] \cdot f y_i \) for \( y_i \in A_*(Y_i) \). Therefore
\[
f'(y_1 \cdot y_2) = f'(y_1) \cdot f y_2
\]
and we just need to show that \( x \cdot f y_2 = x \cdot f'(y_2) \) for \( x = f'(y_1) \). But we have \( x \cdot f'(y_2) = x \cdot ([X] \cdot f y_2) = (x \cdot [X]) \cdot f y_2 = x \cdot f y_2 \).

6.3. Coefficient lemma. Let \( Y \) be a separated algebraic scheme over \( \mathbb{C} \). Let \( Z \subset Y \) be an irreducible component of dimension \( d \), equipped with the reduced subscheme structure, and let \( Y' \) be the closure of \( Y \setminus Z \) (it is the union of all other irreducible components of \( Y \)), also equipped with the reduced subscheme structure. There is an exact sequence
\[
A_d(Z \cap Y') \to A_d(Z) \oplus A_d(Y') \to A_d(Y) \to 0
\]
(see [6, Ex. 1.3.1]). As \( \dim Z \cap Y' < \dim Z = d \), we obtain
\[
A_d(Y) \simeq A_d(Z) \oplus A_d(Y') = \mathbb{Z}[Z] \oplus A_d(Y').
\]
Therefore every cycle \( y \in A_d(Y) \) can be written in a unique way as
\[
y = m \cdot [Z] + y', \quad m \in \mathbb{Z}, \, y' \in A_d(Y').
\]

We call \( m \) the coefficient of \([Z]\) in \( y\).

Lemma 6.2. Using notation from §6.1, let \( U \subset X \) be an open subset such that

1. \( Z := D \cap U \) is irreducible and reduced of dimension \( d \),
2. \( D(\sigma_i) \cap U \) has pure dimension \( N - r_i + k_i \), for all \( 1 \leq i \leq n \).

Then the closure \( \bar{Z} \subset D \) is an irreducible component of \( D \) and, if we equip \( \bar{Z} \) with the reduced subscheme structure, the coefficient of \([\bar{Z}]\) in the refined product \( \mathbb{D} = \prod_i D(\sigma_i) \in A_d(D) \) is 1.

Proof. We first show that \( \bar{Z} \) is an irreducible component of \( D \). It is irreducible being the closure of an irreducible set. Therefore it is contained in some irreducible component \( C \) of \( D \). If \( \bar{Z} \neq C \) then \( \dim C > d \). But \( C \) intersects \( U \) and thus \( \dim C \cap U = \dim C > d = \dim Z \), while \( Z = D \cap U \supset C \cap U \). A contradiction.

Now we will show that the coefficient of \([\bar{Z}]\) in \( \mathbb{D} \) is one. We can write \( D = \bar{Z} \cup D' \), where \( D' \) is the union of all other irreducible components of \( D \). Then
\[
\mathbb{D} = m \cdot [\bar{Z}] + x, \quad m \in \mathbb{Z}, \, x \in A_d(D').
\]

Consider the open embedding \( j : U \hookrightarrow X \). Then
\[
j^*\mathbb{D} = m \cdot j^*[\bar{Z}] + j^*x = m \cdot [Z] + j^*x.
\]
(Note that \(j^*[\bar{Z}] = [Z]\) as both \(Z\) and \(\bar{Z}\) are reduced, so \(\bar{Z} \cap U = Z\) also scheme-theoretically.) But \(D \cap U = Z \subset \bar{Z}\), hence \(D' \cap U \subset D' \cap \bar{Z}\). Therefore \(\dim D' \cap U \leq \dim D' \cap \bar{Z} < d\) and thus \(A_d(D' \cap U) = 0\) and \(j^*x = 0\). We conclude that

\[j^*D = m \cdot [Z]\]

and we need to show that the coefficient of \([Z]\) in \(j^*D\) is one. The degeneracy locus of the homomorphism \(j^*\sigma_i: \mathcal{O}^{r_i+1}_{U}|_U \to \mathcal{E}_i|_U\) is \(D(j^*\sigma_i) = D(\sigma_i) \cap U\). As degeneracy classes are compatible with flat pull-backs [6, Thm. 14.4(d)], we obtain

\[j^*D(\sigma_i) = D(j^*\sigma_i) \in A_{n-r_i+k_i}(D(\sigma_i) \cap U),\]

where, by abuse of notation, we use the letter \(j\) also for the embedding \(D(\sigma_i) \cap U \hookrightarrow D(\sigma_i)\). As refined products are compatible with pull-backs by Lemma 6.1, we obtain

\[j^*D = \prod_i j^*D(\sigma_i) = \prod_i D(j^*\sigma_i) \in A_d(D \cap U) = A_d(Z) = Z[Z].\]

By assumption (2), the scheme

\[V_i = D(\sigma_i) \cap U = D(j^*\sigma_i), \quad 1 \leq i \leq n,\]

has pure dimension \(N - r_i + k_i\). Therefore by [6, Thm. 14.4(c)], we get

\[D(j^*\sigma_i) = [D(j^*\sigma_i)] = [V_i], \quad 1 \leq i \leq n.\]

This implies that (see [6, Ex. 8.2.1])

\[j^*D = \prod_i D(j^*\sigma_i) = \prod_i [V_i] = i(Z, V_1 \cdot \ldots \cdot V_n, U) \cdot [Z] \in A_d(Z).\]

We know by [6, Prop. 8.2] that \(m = i(Z, V_1 \cdot \ldots \cdot V_n, U)\) satisfies \(1 \leq m \leq l(\mathcal{O}_{D \cap U}, Z)\), where \(\mathcal{O}_{D \cap U, Z}\) is the local ring of \(D \cap U = \cap_i V_i\) at the generic point of the irreducible component \(Z\). But as \(D \cap U = Z\) (which is reduced and irreducible), the length of this artinian local ring is one, hence \(m = 1\).

6.4. Tautological vector bundles and their sections. As before, let \(Q\) be a quiver with the set of vertices \(I\). Let \(f \in N^{I}\) and \(Q^f\) be the corresponding framed quiver. Let \(\mathcal{P} = \mathcal{P}^f\) be the set of paths in \(Q^f\) that start at \(\infty\). Given a dimension vector \(d\), we define \(N_d = N_{d,f}\) to be the non-commutative Hilbert scheme parameterizing stable framed representations of dimension \(d\). Recall that its dimension is equal to

\[(69) \quad N = \dim N_d = f \cdot d - \chi(d, d).\]

For any vertex \(i \in I\), we have the \(i\)-th tautological vector bundle \(\mathcal{U}_i = \mathcal{U}_{d, i}\) over \(N_d\), with the fiber over \(M \in N_d\) equal to \(M_i\). This vector bundle has rank \(d_i\). For any path \(u \in \mathcal{P}_i\), we have a vector \(m_u \in M_i\). The family of such vectors over all \(M \in N_d\) induces a section

\[(70) \quad m_u \in \Gamma(N_d, \mathcal{U}_i).\]

Given a tree \(S \in \mathcal{P}(d)\) and \(v \in C(S)_i\), we define (18)

\[(71) \quad S(v) = \{ u \in S_i \mid u < v \} \cup \{ v \}, \quad k_v = \#S(v) - 1.\]

Then the above sections induce a morphism of vector bundles

\[(72) \quad \sigma_v = (m_u)_{u \in S(v)}: \mathcal{O}^{k_v+1} \to \mathcal{U}_i.\]
Note that \( k_v \leq d_v = \text{rk} \mathcal{U}_i \). The degeneracy locus \( D(S(v)) \subset \mathcal{N}_d \) considered in (17) is equal to the degeneracy locus \( D(\sigma_v) \) considered in §6.1. By the discussion in §6.1, we have a degeneracy class
\[
\mathcal{D}(\sigma_v) \in A_{N-d_v+k_v}(D(\sigma_v))
\]
such that its image in \( A_{N-d_v+k_v}(\mathcal{N}_d) \) is equal to
\[
c_{d_v-k_v}(\mathcal{U}_i) \cap [\mathcal{N}_d].
\]

Let us consider the degeneracy locus of the tree \( S \) (19)
\[
D_S = \bigcap_{v \in C(S)} D(S(v)) = \bigcap_{v \in C(S)} D(\sigma_v) \subset \mathcal{N}_d
\]
and the class
\[
\mathcal{D}_S = \prod_{v \in C(S)} \mathcal{D}(\sigma_v) \in A_m(D_S), \quad m = N - \sum_{v \in C(S)} (d_{\sigma(v)} - k_v).
\]
We obtain from (13) and (14) that
\[
d(S) = \sum_{v \in C(S)} k_v, \quad N = \text{dim} \mathcal{C}(S) \cdot d = \sum_{v \in C(S)} d_{\sigma(v)}.
\]
Therefore
\[
m = \sum_{v \in C(S)} k_v = d(S) = \text{dim} Z_S.
\]
The image of the class \( \mathcal{D}_S \in A_{d(S)}(D_S) \) in \( A_{d(S)}(\mathcal{N}_d) \) is equal to
\[
\mathcal{D}'_S = \left( \prod_{i \in I} \prod_{v \in C(S)_i} c_{d_v-k_v}(\mathcal{U}_i) \right) \cap [\mathcal{N}_d].
\]

We know that \( A_{d(S)}(\mathcal{N}_d) \) has a basis consisting of \( [\bar{Z}_S'] \) with \( d(S') = d(S) \). By Lemma 2.19, we have \( D_S \subset \bigcup_{S' \supseteq S} Z_{S'} \). Therefore we have an expression in \( A_{d(S)}(\mathcal{N}_d) \)
\[
\mathcal{D}'_S = \sum_{S' \supseteq S \atop d(S') = d(S)} n_{S',S} [\bar{Z}_S']
\]
with \( n_{S',S} \in \mathbb{Z} \). Applying Lemma 6.2 to the degeneracy loci \( D(\sigma_v) \) and the open subset \( U_S \subset \mathcal{N}_d \) (satisfying \( D_S \cap U_S = Z_S \) by Lemma 2.19), we obtain \( n_{S,S} = 1 \). This implies

**Theorem 6.3.** The classes
\[
\prod_{i \in I} \prod_{v \in C(S)_i} c_{d_v-k_v}(\mathcal{U}_i) \in A^{N-d(S)}(\mathcal{N}_d), \quad S \in \mathcal{P}(d),
\]
form a basis of \( A^*(\mathcal{N}_d) \).

Note that the numbers \( k_v \) (hence the above description of a basis of \( A^*(\mathcal{N}_d) \)) depend on the admissible total order on \( \mathcal{P} \).

**Theorem 6.4.** The classes
\[
\prod_{i \in I} \prod_{k \geq 1} c_k(\mathcal{U}_i)^{\lambda^{(i)}_k - \lambda^{(i)}_{k+1}} \in A^{\lambda}(\mathcal{N}_d), \quad \lambda \in \mathcal{S}(d),
\]
form a basis of \( A^*(\mathcal{N}_d) \), where \( |\lambda| = \sum_{i \in I} |\lambda^{(i)}| \).
Proof. Recall that for any admissible order on \( \mathcal{P} \) we have a bijection between the set of trees \( \mathcal{P}(d) \) and the set of multipartitions \( \mathcal{S}(d) \) (see Theorem 3.1). If \( \lambda \in \mathcal{S}(d) \) corresponds to the tree \( S \in \mathcal{P}(d) \) under this bijection, then
\[
\lambda_k^{(i)} - \lambda_{k+1}^{(i)} = \# \{ v \in C(S)_i \mid d_v = k \}.
\]
Therefore
\[
\prod_i \prod_{v \in C(S)_i} c_{d_v - k_v}(U_i) = \prod_i \prod_{k \geq 1} c_k(U_i) \lambda_k^{(i)} - \lambda_{k+1}^{(i)}
\]
and we apply Theorem 6.3.

Note that by Theorem 2.21, we have \( H_{BM}^i(N_d, \mathbb{Z}) = 0 \) for odd \( i \) and \( H_{BM}^{2i}(N_d, \mathbb{Z}) \cong A_i(N_d) \). Similarly, \( H^i(N_d, \mathbb{Q}) = 0 \) for odd \( i \) and \( H^{2i}(N_d, \mathbb{Q}) \cong A^i(N_d) \otimes \mathbb{Q} \). Therefore the above theorem gives a canonical basis of \( H^*(N_d, \mathbb{Q}) \).

6.5. Dependence of classes on the admissible order. Given an admissible order on \( \mathcal{P} \), we have a bijection \( \mathcal{S}(d) \to \mathcal{P}(d), \lambda \mapsto S_\lambda \). In the previous section we have seen that the classes \( \mathcal{D}'_\lambda = \mathcal{D}'_{S_\lambda} \) forming a basis of \( A_\ast(N_d) \) are independent of the choice of an admissible order on \( \mathcal{P} \). One may ask if the classes \( [\mathcal{Z}_{S_\lambda}] \) (forming another basis of \( A_\ast(N_d) \)) are independent of the choice of an admissible order. We will see in this section that this is not always the case.

Consider the quiver \( Q^f \)

\[
\infty \xrightarrow{f} 0 \xrightarrow{a} b
\]

with arrows ordered as \( f < a < b \). Let \( \mathcal{P} \) be the set of paths in \( Q^f \) starting at \( \infty \). Assume that \( \mathcal{P} \) is equipped with the shortlex order. Then the trees in \( \mathcal{P}(3) \) written in the increasing order (see §2.7) are (we omit the root *)

| Tree \( S \) | \( d(S) \) | Partition |
|-----------------|-------|-----------|
| \( S_1 = (f, af, bf) \) | 12 | () |
| \( S_2 = (f, af, a^2 f) \) | 11 | (1) |
| \( S_3 = (f, af, ba f) \) | 10 | (2) |
| \( S_4 = (f, bf, af) \) | 10 | (1,1) |
| \( S_5 = (f, bf, b^2 f) \) | 9 | (2,1) |

For a tree \( S \in \mathcal{P}(3) \) with the corresponding partition \( \lambda \), let
\[
\mathcal{D}'_S = \mathcal{D}'_\lambda = \left( \prod_{k \geq 1} c_k(U_0)^{\lambda_k - \lambda_{k+1}} \right) \cap [N_3] \in A_{d(S)}(N_3),
\]
where \( d(S) = \dim N_3 - |\lambda| = 12 - |\lambda| \). The group \( A_{10}(N_3) \) has a basis consisting of \( [\mathcal{Z}_{S_3}] \) and \( [\mathcal{Z}_{S_4}] \) and a basis consisting of \( \mathcal{D}'_{S_3} = c_1(U_0)^2 \cap [N_3] \) and \( \mathcal{D}'_{S_4} = c_2(U_0) \cap [N_3] \). By (76) we have
\[
\mathcal{D}'_{S_3} = [\mathcal{Z}_{S_3}] + n[\mathcal{Z}_{S_4}], \quad \mathcal{D}'_{S_4} = [\mathcal{Z}_{S_4}],
\]
for some \( n \in \mathbb{Z} \). In order to find the multiplicity \( n \) of \( [\mathcal{Z}_{S_4}] \) in \( \mathcal{D}'_{S_3} \), we will study the intersection \( U_{S_3} \cap D_{S_4} \). We have
\[
C(S_3) = (bf, a^2 f, abaf, b^2 af), \quad C(S_4) = (af, b f^2, a^2 bf, babf).
\]
Let us denote the elements of $S_4$ by $u_1 < u_2 < u_3$ and the elements of $C(S_4)$ by $v_1 < v_2 < v_3 < v_4$. Then the coordinates $(c_{ij})$ of $M \in U_{S_4}$ are given by $m_{ij} = \sum_i c_{ij} m_{ui}$ (cf. Lemma 2.7). For $M \in D_{S_4}$ we require that $(m_f, m_{af}, m_{a^2f})$ are linearly dependent and $(m_f, m_{af}, m_{a^2f})$ are linearly dependent. The first condition means that $c_{31} = 0$ and $m_{af} = c_{11} m_f + c_{21} m_{bf}$. For the second condition, we note that
\[
m_{a^2f} = a m_{af} = c_{11} m_{af} + c_{21} m_{abf} = c_{11} (c_{11} m_f + c_{21} m_{bf}) + c_{21} m_{abf}.
\]
Therefore we require that
\[
\det \begin{pmatrix}
1 & c_{11} & c_{11}^2 \\
0 & c_{21} & c_{11} c_{21} \\
0 & 0 & c_{21}
\end{pmatrix} = c_{21}^2 = 0.
\]
For $M \in Z_{S_4}$, we require that $m_{af}$ is contained in $\langle m_f \rangle$, hence $c_{21} = c_{31} = 0$. We conclude that the multiplicity of $\overline{Z_{S_4}}$ in $D'_{S_3}$ is $n = 2$, hence
\[
\overline{Z_{S_3}} = D'_{S_3} - 2 D'_{S_4}, \quad \overline{Z_{S_4}} = D'_{S_4}.
\]
On the other hand, let us equip $\mathcal{P}$ with the lex order. Then the trees in $\mathcal{P}(3)$ written in the increasing order are (we omit the root *)

| Tree $S$ | $d(S)$ | Partition |
|---------|--------|-----------|
| $S_1 = (f, af, a^2f)$ | 12 | () |
| $S_2 = (f, af, bf)$ | 11 | (1) |
| $S_3 = (f, af, a^2f)$ | 10 | (2) |
| $S_4 = (f, bf, abf)$ | 10 | (1,1) |
| $S_5 = (f, bf, b^2f)$ | 9 | (2,1) |

Note that the order of partitions is the same as before by Lemma 3.3. Using the same method as before one can show that (see [3, §2.2])
\[
\overline{Z_{S_3}} = D'_{S_3} - 4 D'_{S_4}, \quad \overline{Z_{S_4}} = D'_{S_4}.
\]
This implies that while the classes $D'_{S_i}$ are independent of the choice of an admissible order, the classes $\overline{Z_{S_i}}$ may depend on it.

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