Hyperscaling violation for scalar black branes in arbitrary dimensions

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We extend to black branes (BB) in arbitrary dimensions the results of Ref. [1] about hyperscaling violation and phase transition for scalar black 2-branes. We derive the analytic form of the \((d+1)\)-dimensional scalar soliton interpolating between a conformal invariant AdS\(_{d+2}\) vacuum in the infrared and a scale covariant metric in the ultraviolet. We show that the thermodynamical system undergoes a phase transition between Schwarzschild-AdS\(_{d+2}\) and a scalar-dressed BB. We calculate the critical exponent \(z\) and the hyperscaling violation parameter \(\theta\) in the two phases. We show that our scalar BB solutions generically emerge as compactifications of \(p\)-brane solutions of supergravity theories. We also derive the short distance form of the correlators for the scalar operators corresponding to an UV exponential potential supporting our black brane solution. We show that also for negative \(\theta\) these correlators have a short distance power-law behavior.

I. INTRODUCTION

Recent investigations on the application of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence to strongly interacting quantum field theories (QFT) have emphasized the importance played by non-AdS gravitational backgrounds and the related dual nonconformal QFTs [2–14].

The standard setup for this kind of holographic applications is a black brane in a AdS background endowed with non trivial scalar field configurations and finite electromagnetic charge density. It has been shown that this produces a rich phenomenology in the dual QFT, such as spontaneous symmetry breaking, phase transitions and non-trivial transport properties [2–26].

In the case of nonminimal, exponential, coupling between the scalar field and the Maxwell tensor the bulk gravity allows for extremal, near-horizon, solutions which break the conformal symmetry of the AdS vacuum [3–6, 8, 10, 12–14, 27]. It has been realized that these IR metrics belong to a general class of metrics that are not scale invariant but only scale covariant [7, 8, 28] and lead to hyperscaling violation in the dual field theory [12, 13, 29–35]. They are characterized by two parameters, the critical exponent \(z\) and the hyperscaling violation parameter \(\theta\), which characterize both the transformation weight of the infinitesimal length \(ds\) under scale transformations and the scaling behavior of the free energy as function of the temperature [13, 29].

Scale covariant metrics are a very promising framework for the holographic description of quantum phase transitions and hyperscaling violation in condensed matter critical systems (e.g Ising models) [36].

An important holographic feature of scale-covariant metrics is the emergence of a length-scale in the IR [13], which decouples in the UV when the theory has an UV fixed point. The emergence of a length-scale in the IR is crucial for the description of Fermi surfaces and for the related area-law violation of entanglement entropy [13, 29, 37–40].

The standard framework for obtaining, dynamically, scale-covariant metrics in the IR, is given by Einstein-scalar gravity, possibly coupled – minimally or non-minimally – to a \(U(1)\) field. The self-interaction potential \(V(\phi)\) for the scalar field \(\phi\) must have a negative local maximum at \(\phi = 0\), with a corresponding scalar tachyonic excitation whose mass is slightly above the Breitenlohner-Freedman (BF) bound. Under suitable conditions, usually an exponential behavior of the potential and/or scalar-Maxwell tensor coupling functions, the theory admits black brane solutions with scalar hair that in the near-extremal regime approach the scale covariant metrics.

In a recent paper [1] it has been shown that this framework is not the only possible way to produce scale covariant BB. These solutions can be also obtained from Einstein-scalar gravity with a positive squared mass for the scalar, when the potential behaves exponentially in the asymptotic region of the spacetime [28, 41]. Standard no-hair theorems for Einstein-scalar gravity model with
positive squared mass for the scalar field $[42–44]$ can be circumvented by giving up the condition that the BB solution has AdS asymptotics $[28, 41]$.

Although BB solutions with scale-covariant asymptotics have been explicitly derived for particular four-dimensional (4D) Einstein-scalar gravity models, their existence is a rather generic feature of a broad class of 4D models $[1]$. Moreover, in the extremal limit the BB solution reduces to a fully regular scalar soliton, which interpolates between an AdS$_4$ vacuum in the near-horizon region and a scale covariant solution in the asymptotic region.

These results allow to realize an alternative scenario, which exchanges IR and UV regions. In the dual QFT we have an infrared fixed point, corresponding to the AdS vacuum, whereas in the UV regime we have hyperscaling violation.

Detailed investigation of the symmetries and thermodynamics of these BB solutions revealed rather interesting and intriguing features. The thermodynamical phase diagram of the system is characterized by the presence of different phases. Above a critical temperature $T_c$ the scalar dressed BB brane becomes energetically preferred with respect to the Schwarzschild-AdS$_4$ (SAdS) solution and the thermodynamical system undergoes a first order phase transition. Moreover, for some values of the parameters characterizing the model, at low temperatures different phases may coexist. In the dual QFT the scalar dressed, stable, BB corresponds to a phase with a negative hyperscaling violation parameter $\theta$. Although negative values of $\theta$ do not have analogous in condensed matter system, they are consistent with the null energy conditions for the bulk stress-energy tensor. Moreover they also arise in string theory and supergravity constructions $[13, 23, 45–47]$.

The purpose of this paper is the generalization of the results of Ref. $[1]$ concerning 2-branes to branes of arbitrary dimensions. We will show that basically all the results of Ref. $[1]$ can be generalized to arbitrary dimensions and therefore generically hold for $d$-branes. Apart from generalizing the results of Ref. $[1]$ to arbitrary dimensions, we will also show that our scalar BB solutions can be obtained in several ways as compactifications of $p$-brane solutions of supergravity (SUGRA) theories.

Finally, we will be concerned with some holographic features of QFTs with negative hyperscaling violation parameter $\theta$. Extending the results of Ref. $[13]$, which hold for positive $\theta$ and for a massive scalar field, we will derive the short distance form of the correlators for scalar operators corresponding to an UV exponential potential supporting our black brane solution. We show that for negative $\theta$ these correlators have a short distance power-law behavior.

The paper is organized as follows. In Sect. II we present our Einstein-scalar gravity model, derive the BB solutions with scale-covariant asymptotics and discuss their solitonic extremal limit. In Sect. III we show how our BB solutions can be obtained as compactifications of $p$-brane solutions of SUGRA theories. The thermodynamics of our solutions is investigated in Sect. IV. The symmetries of the BB are discussed in Sect. V, where also critical exponent and hyperscaling violation parameters are calculated. In Sect. VI we extend our investigation to general models whose potential behaves exponentially in the asymptotic region. In Sect. VII we study holographic properties of our BB solution and in particular the two-point function of scalar operators in the dual QFT. Finally, in Sect. VIII we state our conclusions.

II. EINSTEIN-SCALAR GRAVITY IN $d + 2$-DIMENSIONS

We consider $d + 2$-dimensional (with $d \geq 2$) Einstein gravity minimally coupled to a scalar field $\phi$,

$$I = \int d^{d+2}x \sqrt{-g} \left[ R - 2(\partial\phi)^2 - V(\phi) \right].$$  

(2.1)

We will focus on models for which the scalar self-interaction potential $V(\phi)$ is given by

$$V(\phi) = -\frac{d(d+1)}{\gamma L^2} \left( e^{2s\beta\phi} - \beta^2 e^{2\phi} \right), \quad \gamma = 1 - \beta^2, \quad s = \sqrt{\frac{2(d+1)}{d}}.$$  

(2.2)
where $\beta$ is a (real) parameter characterizing the model and $L$ is the AdS length. The action (2.1) is the $d+2$-dimensional generalization of the four-dimensional Einstein-scalar gravity model investigated in Refs. [1, 41]. It shares with the 4D model of Refs [1, 41] several interesting features. The potential (2.2) has a minimum at $\phi = 0$ with $V(0) = -d(d+1)/L^2$, corresponding to an AdS$_{d+2}$ vacuum and a local scalar excitation of positive squared-mass $m^2 = 2(d+1)^2/L^2$. The model is a fake SUGRA model. In fact the potential (2.2) can be derived from the superpotential

$$P(\phi) = \sqrt{2\gamma}^{-1}L^{-1}\left(e^{\beta \phi} - \beta^2 e^{\gamma \phi}\right).$$

The action (2.1) is invariant under the duality transformation

$$\beta \rightarrow \frac{1}{\beta}.$$  

\[ (2.3) \]

\section{A. Black brane solutions}

We now look for static, radially symmetric, planar solutions of the field equations stemming from the action (2.1).

The presence of the $\phi = 0$ minimum of the potential (2.2) for every value of $\beta$, implies the existence of the Schwarzschild-AdS (SAdS) solution with $\phi = 0$:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2dx_idx_i, \quad f(r) = \frac{r^2}{L^2} - \frac{2M}{r^{d-1}},$$

where $M$ is the black brane mass and $i = 1, 2 \ldots d$.

Solutions with a non-trivial scalar field can be found using the procedure used in Ref. [1, 41] for the 4D case. Using for the metric the parametrization,

$$ds^2 = -e^{2\nu}dt^2 + e^{2\nu+2d\rho}d\xi^2 + e^{2\rho}dx_idx_i,$$

the field equations can be recast in the form of the $SU(2) \times SU(2)$ Toda molecule [48]. Solutions with a regular horizon and nontrivial scalar field exist only if they do not approach asymptotically to AdS$_{d+2}$. The form of these hairy black brane solutions can be found using the procedure explained in Ref. [1, 41]. For $\beta^2 < 1$ we get the two-parameter family of solutions:

$$ds^2 = \left(\frac{r_0}{r}\right)^{2\beta} \left\{ \Delta(r) \frac{2\beta^2}{\Delta(r)} \left[ -\Gamma(r) dt^2 + dx_idx_i \right] + E \Delta(r) \frac{2\beta^2}{\Delta(r)} \Gamma(r)^{-1} dr^2 \right\},$$

$$e^{2\phi} = \left[ \frac{A}{\Delta(r)} \right]^{\frac{\beta^2}{2\beta}} \left( \frac{r}{r_0} \right)^{\frac{d\beta}{\omega}}, \quad \Gamma(r) = 1 - \mu_1 \left( \frac{r_0}{r} \right)^{\delta}, \quad \Delta(r) = 1 + \mu_2 \left( \frac{r_0}{r} \right)^{\delta},$$

$$\omega = 1 - (d+1)\beta^2, \quad \delta = -\frac{(d+1)\gamma}{\omega}, \quad A = \sqrt{\mu_2(\mu_1 + \mu_2)}, \quad E = \left( \frac{\gamma L}{r_0^2 \omega} \right)^2 A^{\frac{2\beta^2}{\omega}},$$

where $\mu_{1,2}$ are dimensionless free parameters and $r_0$ is a length scale that must be introduced in order to get the correct physical dimensions.

The asymptotic region of the spacetime (2.6) is given by $r \rightarrow 0$ for $\beta^2 < 1/(d+1)$, whereas it is given by $r = \infty$ when $\beta^2 \geq 1/(d+1)$. In both cases the asymptotic behavior of the solution (2.6) is given by

$$ds^2 = \left(\frac{r_0}{r}\right)^{2\beta} (-dt^2 + dx_idx_i + dr^2), \quad \phi = \frac{sd\beta}{2\omega} \log(r/r_0).$$

\[ (2.7) \]

This metric is not invariant under scale transformation, but still transforms with definite weight, so it is scale-covariant.

The solution (2.6) becomes singular for $\beta^2 = 1/(d+1)$. This is related to the fact that this value of $\beta$ corresponds to a divergent hyperscaling parameter $\theta$. Nevertheless a fully regular solution can
be written using a different parametrization for the radial coordinate $r$,

$$ds^2 = \left( \frac{r_0}{r} \right)^2 \left\{ \Delta(r) \frac{2}{d(d+1)} \left[ -\Gamma(r) dt^2 + dx_i dx_i \right] + E \Delta(r)^{\frac{2}{d(d+1)}} \Gamma(r)^{-1} \left( \frac{r_0}{r} \right)^2 dr^2 \right\} ,$$

$$e^{2\phi} = \left[ \frac{A}{\Delta(r)} \right]^{\frac{2}{d+1}} \left( \frac{r}{r_0} \right)^{\frac{2\delta}{d+2}} , \quad \delta = -d \quad E = \left( \frac{d^2 L}{r_0(d+1)(d+2)} \right)^2 A^{-\frac{2}{d+1}} \; (2.8)$$

whereas $\Delta, \Gamma, A$ are given as in Eq. (2.6).

The radial coordinate in the metric (2.7) gives the information about the various energy scales in the dual QFT. A proper energy $E_0$ is redshifted according to the law:

$$E(r) = r^{-\frac{1}{d}} E_0 . \quad (2.9)$$

This equation tells us that for $\omega > 0 \; (\beta^2 < 1/(d+1))$, $r \to 0 \; (\to \infty)$ corresponds to the UV (IR) region of the dual QFT, whereas for $\omega < 0 \; (\beta^2 > 1/(d+1))$ the UV (IR) corresponds to $r = \infty \; (r \to 0)$.

For $\mu_1, \mu_2 \geq 0$, the metric (2.6) exhibits a singularity at $r = \infty \; (r = 0)$ for $\beta^2 < 1/(d+1)$ ($\beta^2 \geq 1/(d+1)$) shielded by a horizon at $r/r_0 = \mu_1^{1/\delta}$, and therefore represents a regular black brane.

Until now we have considered only the case $\beta^2 < 1$. The form of the solutions for $\beta^2 > 1$ can be simply found using the duality (2.3) into Eq. (2.6). All the considerations of this section can be trivially extended to the case $\beta^2 > 1$.

**B. Extremal limit and scalar soliton**

The extremal limit of the solution (2.6) is obtained setting $\mu_1 = 0$. When $\mu_2 = 0$ this extremal limit is singular, with a naked singularity at $r = \infty$ for $\beta^2 < 1/(d+1)$ (at $r = 0$ for $\beta^2 > 1/(d+1)$) with $\phi \sim \ln r$. On the other hand for $\mu_2 > 0$ the extremal BB is a regular scalar soliton that interpolates between a scale covariant solution in the UV and the AdS$_{d+1}$ in the IR:

$$ds^2 = \left( \frac{r_0}{r} \right)^{1/2} \left\{ \Delta(r)^{\frac{1}{(d+1)}} \left[ -dt^2 + dx_i dx_i \right] + E \Delta(r)^{\frac{2}{d+1}} \right\} ,$$

$$e^{2\phi} = \left[ \frac{\mu_2}{\Delta(r)} \right]^{\frac{2}{d+1}} \left( \frac{r}{r_0} \right)^{-\frac{2\delta}{d+2}} . \quad (2.10)$$

Let us now consider the UV (asymptotic) and IR (near-horizon) limit of the scalar soliton (2.10). For $\beta^2 < 1/(d+1)$ this corresponds to take, respectively, $r \to 0$ and $r \to \infty$. For $\beta^2 \geq 1/(d+1)$ these limits are reversed (the UV corresponds to $r \to \infty$ and the IR to $r \to 0$).

In the IR limit, the scalar field $\phi$ vanishes, the length scale $r_0$ decouples and the metric (2.10) becomes that of AdS$_d$. The length-scale $r_0$ is an UV scale, which decouples in the IR, where conformal invariance is restored. On the other hand, in the UV limit it is the AdS length $L$ that decouples: the metric (2.10) can be written in terms of $r_0$ only and takes the scale-covariant form given by Eq. (2.7).

**III. COMPACTIFICATIONS OF p-BRANE SOLUTIONS OF SUGRA THEORIES**

In this section we will look for string theory realizations that produce, after compactification, an Einstein-scalar model (2.1) with potential of the form (2.2). This means that we are considering our models just as as effective description, which breaks down in the far UV. The short-distance physics will be therefore described by the UV completion of our effective model.
We will show in the following that BB solutions (2.6) arise as compactifications of black \( p \)-brane solutions of SUGRA theories. We will see that they emerge from the \( p \)-brane both as simple spherical compactification or also as a more general Kaluza-Klein compactification parametrized by a parameter.

Black \( p \)-branes are classical Ramond-Ramond charged solutions of \( D \)-dimensional SUGRA theories supported by a \((p + 2)\)-form field strength \( G_{p+2} \) \cite{49, 50}. In the Einstein frame the bosonic part of the action is

\[
I = \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} (\partial \Phi)^2 - e^{\alpha \Phi} \frac{1}{2(p+2)!} G_{(p+2)} \right),
\]

where \( \Phi \) is the dilaton field and \( \alpha \) is constant, which is zero for non-dilatonic \( p \)-branes, whereas

\[
a^2 = 4 - [(p + 1)(D - p - 3)]/(D - 2),
\]

for dilatonic branes. The metric part of the \( p \)-brane solution is given in terms of two integration constants \( h_0, g_0 \) by

\[
ds_D^2 = H(r)^{-\frac{2q}{\rho}} \left( -g(r)dt^2 + \sum_{i=1}^{p} dx_i dx_i \right) + H(r)^{\frac{2q}{\rho}} \left( g(r)^{-1} dr^2 + r^2 d\Omega_{q+2}^2 \right),
\]

\[
H(r) = 1 + \left( \frac{h_0}{r} \right)^{\frac{\rho}{q}}, \quad g(r) = 1 - \left( \frac{g_0}{r} \right)^{\frac{\rho}{q}}, \quad \rho = (p + 1)\bar{d} + a^2 \frac{D - 2}{2}, \quad \bar{d} = D - p - 3,
\]

where \( d\Omega_{q+2}^2 \) is the line element of a compact space \( K^q \) with \( q = D - p - 2 \) dimensions.

Let us first consider nondilatonic \( p \)-branes. The simplest diagonal ansatz for the \( D \)-dimensional metric, which gives the \( p + 2 \)-dimensional theory in the Einstein frame, is obtained by setting \( K^q = S^q \) and

\[
ds_D^2 = e^{-\frac{2q}{\rho} \psi} ds_{p+2}^2 + e^{2\psi} d\Omega_{q+2}^2.
\]

Taking into account that for nondilatonic branes \( e^\psi = rH^{1/\rho} \) one finds after compactification the BB metric:

\[
ds_{p+2}^2 = r^{-\frac{2q}{\rho} (p+2-D)} H(r)^{\frac{2(p+2-D)}{\rho}} \left( -g(r)dt^2 + \sum_{i=1}^{p} dx_i dx_i \right) + H(r)^{\frac{2q}{\rho} (p+2-D)} g(r)^{-1} dr^2.
\]

One can easily see that the metric (3.5) matches exactly, after some trivial identification of the parameters, the metric (2.6) if we take \( d = p \) and

\[
\beta^2 = \frac{D - 2}{(p + 1)(D - p - 2)}.
\]

It is important to notice that this value of \( \beta \) always satisfy the inequality \( 1/(p + 1) < \beta^2 < 1 \). Particularly interesting cases are represented by the 2 and 5-brane in \( D = 11 \) corresponding, respectively, to \( \beta^2 = 3/7 \) and \( \beta^2 = 3/8 \).

Compactification of \( p \)-branes with the diagonal ansatz (3.4) produces BB solutions of the form (2.6) only for the particular values of the parameter \( \beta \) given in Eq. (3.6). This limitation can be removed by considering the more general diagonal ansatz of Ref. [7] for the \( D \)-dimensional metric.

Let us now briefly consider compactification of dilatonic \( p \)-branes. In this case we must use in (3.3) the value (3.2) for \( a \) giving \( \rho = 2(D - 2) \). The diagonal ansatz (3.4) produces now the BB solution

\[
ds_{p+2}^2 = r^{\frac{1}{\rho} (p+2-D)} H(r)^{\frac{1}{\rho}} \left( -g(r)dt^2 + \sum_{i=1}^{p} dx_i dx_i \right) + H(r)^{\frac{q+1}{\rho}} g(r)^{-1} dr^2.
\]

\[
(3.7)
\]
Matching this BB solution with Eq. (2.6) requires

\[ D = \frac{3p+1}{p-1} + p + 2, \quad \beta^2 = \frac{p+1}{3p+1}. \] (3.8)

These are very stringent constraints which however are satisfied by a very interesting case, the 3-brane in \( D = 10 \) which gives \( \beta^2 = 2/5 \). It is likely that also for dilatonic branes the use of the more general diagonal ansatz of Ref. [7] would allow to circumvent the constraints (3.8). In this paper we will not further investigate on this point.

\section*{IV. THERMODYNAMICS AND PHASE TRANSITIONS}

In this section we will consider the BB solutions (2.6) as a thermodynamical system. We will use the euclidean action formulation of Martinez et al. [51]. As it has been already noted in Ref. [1] for the 4D case, the two-parameter family of solutions (2.6) is not suitable for setting up a consistent BB thermodynamics. The problem is the explicit dependence of the scalar field from the parameter \( \mu_1 \), which causes divergences in the boundary action, that determines the mass of the solution. This explicit dependence can be eliminated by constraining the possible values of \( \mu_{1,2} \) in Eq. (2.6) with \( \mu_2(\mu_2 + \mu_1) = 1 \). We end up with the one-parameter family of solutions

\[ ds^2 = r^{-2 \left( \left( \frac{\gamma L}{\omega} \right)^2 \left[ -\Delta(r)^{-2\beta^2} \Gamma(r) dt^2 + \Delta(r)^{-2\beta^2} \Gamma(r)^{-1} dr^2 \right] + \Delta(r)^{-2\beta^2} dx_i dx_i \right)}, \]

\[ e^{2\phi} = \Delta(r)^{-2\beta} r^{\frac{2\beta}{d-2}}, \quad \Gamma(r) = 1 - \frac{\nu_1}{r^2}, \quad \Delta(r) = 1 + \frac{\nu_2}{r^2}, \] (4.1)

where the parameters \( \nu_{1,2} \) are constrained by

\[ \nu_1 = \frac{1}{\nu_2} - \nu_2, \quad 0 < \nu_2 \leq 1, \quad 0 \leq \nu_1 < \infty \] (4.2)

Notice that in writing Eq. (4.1) we have introduced dimensionless coordinates \( t, r \) and parameters \( \nu_{1,2} \). This is necessary because (2.10) is a global solution interpolating between the IR and the UV regimes that are characterized by two different length scales \( r_0 \) and \( L \).

Starting from Eq. (4.1) one can now calculate, using standard formulas, the temperature \( T \) and entropy \( S \) of the BB. One has

\[ T = \frac{1}{4\pi} \frac{d(d+1)\gamma}{d+2(d+1)\beta^2} \nu_2^{-\frac{(d+1)}{d+2\beta^2}} \left( 1 - \nu_2^{1/(d+1)} \right), \quad S = 4\pi V \nu_2^{-\frac{(d-1)}{d+2\beta^2}} \left( 1 - \nu_2^{1/(d+1)} \right), \] (4.3)

where \( V \) is the volume of the transverse \( d \)-dimensional space.

We construct the thermodynamics of our BB solutions using the Euclidean action formalism. Thermodynamical potentials are given by boundary terms of the action. We use the parametrization of the metric of Ref. [51]:

\[ ds^2 = N^2 f^2 dt^2 + f^{-2} dr^2 + R^2 dx_1 dx_i, \]

The variations of the boundary terms of the action evaluated for the solution (4.1) are

\[ \delta I_C = -\frac{V d^2}{T \left[ (d+2(d+1)\beta^2) \right]} \left[ \delta \nu_1 + \frac{2\beta^2}{\gamma} (2 - \beta^2) \delta \nu_2 \right], \]

\[ \delta I_\phi = \frac{2\beta^2 V d^2}{\gamma T \left[ (d+2(d+1)\beta^2) \right]} \delta \nu_2, \] (4.4)

\[ \delta I_C \big|_{r_0} = -\frac{V d^2}{T \left[ (d+2(d+1)\beta^2) \right]} \left[ (\nu_1 + \nu_2) \right]^{-1} \left[ (\nu_1 + \gamma \nu_2) \delta \nu_1 + \beta^2 \nu_1 \delta \nu_2 \right], \]

\[ \delta I_\phi \big|_{r_0} = 0. \]
where $I_G$ and $I_\phi$ are, respectively, the gravitational and scalar field part of the boundary action.

One can easily show that the BB entropy $S$ is correctly given by $S = I_G|_{r_h}$. The mass of the BB is given in terms of the free energy $F$ and the entropy $S$ by $M = F + TS = -T(I_\phi^\infty + I_G^\infty)$.

Using Eqs. (4.4) one finds

$$M = \frac{V d^2}{d + 2(d + 1)\beta^2} \left( \nu_1 + 2\beta^2 \nu_2 \right) = \frac{V d^2}{d + 2(d + 1)\beta^2} \left\{ \frac{1}{\nu_2} + \left( 2\beta^2 - 1 \right) \nu_2 \right\}. \quad (4.5)$$

Using Eqs. (4.3), (4.5) and the constraint (4.2) one can now check that the first principle $dM = T dS$ is satisfied. As usual the results can be trivially extended to the parameter region $\beta^2 > 1$ just by using the duality $\beta \to 1/\beta$ in Eqs. (4.1), (4.3) and (4.5).

### A. Phase transition

The global stability of our BB solution, considered as a thermodynamical system, can be investigated by computing the free energy and the specific heat. In particular, comparison of the free energies of different configurations at fixed temperature allows us to single out the energetically preferred phase, whereas a positive (negative) specific heat indicates local stability (instability) of a given phase. We start with the case $\beta^2 < 1/(d + 1)$, where, as we will see, we observe a phase transition.

#### 1. Free energy

In the case under consideration the two competitive phases are represented by the black brane with scalar hair (scalar BB) (4.1) and the SAdS BB (2.4). The free energy of the scalar black brane is

$$F_{SB}(T) = M - TS = \frac{V d}{d + 2(d + 1)\beta^2} \left\{ - \frac{\omega}{\nu_2(T)} + \left[ 1 + (d - 1)\beta^2 \right] \nu_2(T) \right\}, \quad (4.6)$$

where $\nu_2(T)$ is defined implicitly by the first equation in (4.3). For the free energy of the SAdS black brane we have instead,

$$F_{SAdS}(T) = -V \left( \frac{4\pi}{d + 1} \right)^{d+1} T^{d+1}. \quad (4.7)$$

The relevant quantity $\Delta F(T) = F_{SB}(T) - F_{SAdS}(T)$ cannot be computed explicitly in closed form because $\nu_2(T)$ is only implicitly defined. Nevertheless, one can show that for $\beta^2 < 1/(d + 1)$, $\Delta F(T)$ is positive for small $T$, vanishes at finite value of the temperature and becomes negative at large $T$.

A qualitative way to see this change of sign of $\Delta F(T)$ is to consider the small-$T$ ($\nu_2 \sim 1$) and the large-$T$ ($\nu_2 \sim 0$) behavior of $F_{SB}$. At small temperatures we have

$$F_{SB}(T) = V \left\{ \frac{2\beta^2 d^2}{d + 2(d + 1)\beta^2} - \frac{(4\pi)^{d+1}}{(d + 1)^{d+1}} \left[ \frac{d + 2(d + 1)\beta^2}{d + 1} \right] T^{d+1} \right\}. \quad (4.7)$$

The small-T behavior is determined by the $T = 0$, AdS$_{d+2}$ extremal limit and is pertinent to a holographically dual $(d + 1)$-dimensional CFT. Conversely for the large-T ($\nu_2 \sim 0$) behavior we have

$$F_{SB} = -\frac{\omega V d}{d + 2(d + 1)\beta^2} \left\{ \frac{4\pi}{d + 2(d + 1)\beta^2} \right\}^{\frac{(d+1)\nu}{\omega}} T^{\frac{(d+1)\nu}{\omega}}. \quad (4.8)$$
The free energy for the hairy BB is positive at small $T$ implying $\Delta F > 0$. For $\beta^2 < 1/(d + 1)$, $\Delta F$ becomes negative at large $T$. This shows the existence of a critical temperature $T_c$ such that $\Delta F(T_c) = 0$.

In general the critical temperature cannot be determined analytically. However, we can show the existence of $T_c$ graphically. By setting $y = \nu^2_2$ the equation $F_{SB} = F_{SAdS}$ gives:

$$g(y) = \frac{1 - (d + 1)\beta^2 - \left[1 + (d - 1)\beta^2\right] y}{1 - y} = f(y) = \left[\frac{d}{d + 2(d + 1)\beta^2}\right]^d \gamma^{d+1} \frac{\delta^2}{\pi^2}, \quad 0 \leq y \leq 1.$$ 

While the curve $f(y)$ is always positive, the behavior of $g(y)$ depends on the value of $\beta$. For $\beta^2 > \frac{1}{d+1}$, the curve starts from a negative point and is always negative; for $\beta^2 = \frac{1}{d+1}$, the curve starts from $y = 0$ and is always negative; for $\beta^2 < \frac{1}{d+1}$, the curve starts from a positive point and decreases monotonically to $-\infty$. Then the two curves $f(y)$ and $g(y)$ do not intersect for $\beta^2 \geq \frac{1}{d+1}$, while they intersect at a finite critical value of the temperature for $\beta^2 < \frac{1}{d+1}$.

In figure 1 we show the behavior of the free energy density for $d = 3$ and for $\beta^2 = 1/8$, a value in the range $0 \leq \beta^2 < 1/(d + 1)$. The critical temperature can also be determined numerically. For the case described in Fig. 1 we have $T_c = 0.20917$.

![FIG. 1: The free energy density $F/V$, as function of $T^4$, of the scalar black brane for $d = 3$ and $\beta^2 = 1/8$ (blue, thick line) and of the SAdS black brane for $d = 3$ (red, thin line).](image-url)

We have therefore discovered, in the $\beta^2 < 1/(d + 1)$ case, a cross-over behaviour for the free energies of the SAdS and scalar black brane solution. The relevant question is now the following: can we interpret this behaviour as a phase transition between two different configuration of the same bulk gravity theory? This question can be answered only if one clarifies the role played by boundary conditions in the definition of canonical thermodynamical ensembles. In fact, the two classical configurations - the SAdS and the scalar brane - actually are two different solutions of the same bulk theory defined by the action (2.1). On the other hand, these solutions correspond to different asymptotic values of the scalar field in the UV ($\phi = 0$ and $\phi = -\infty$ respectively for the SAdS solution (2.4) and the hairy black brane solution (2.6)) and to different asymptotic geometries. However, there is no obstruction in considering solutions of the same bulk theory with different boundary conditions as belonging to the same canonical ensemble. Although this is not an usual situation in the AdS/CFT correspondence, where one refers to fixed boundary conditions, one can define a canonical partition function just by evaluating the euclidean action on the particular bulk solution, without any reference to the asymptotics of the solutions. This is exactly the way we have calculated the free energy using Eq. (4.4). This is a strong argument supporting the
interpretation of the free energy cross-over described in this section as a truly first-order phase transition (the phase transition is first-order because at $T = T_c$, $dF_{SB}/dT \neq dF_{SAdS}/dT$).

A definitive answer about the existence of the phase transition could be obtained by showing that for $T > T_c$, the SAdS solution decays with finite half-life in the scalar brane solution, but such a calculation is beyond the scope of this paper.

Because of the change in asymptotics in the two competitive bulk solutions, the holographic interpretation of the phase transition is rather involved. In the usual gravity/gauge theory correspondence dictionary the sources $J$ in the dual QFT are related to small perturbations of the UV boundary conditions. The gravity/gauge theory correspondence rules allow then to compute the $n$-point functions for dual operators differentiating the bulk partition function with respect to $J$.

The dramatic change in the boundary conditions for the scalar field $\phi$ we have in our case seems to suggest that the two different phases we have on the gravity side correspond to different sources in the dual QFT. Because different sources generally lead to different Lagrangians, we are led to the conclusion that the two phases of the gravity theory - the SAdS and the scalar brane phase - correspond to two distinct dual QFTs, not to two distinct phases of a single QFT\(^1\).

An other argument supporting this interpretation is the analogy with what happens in bulk theories allowing for a flow between an AdS in the IR and an other AdS in the UV. Such solutions are known in the literature. Analogously to the case discussed in this paper, we have also here three different boundary QFTs. We have two CFTs with no flow, corresponding to fixed IR or UV fixed values for the scalar field and a QFT describing the flow between the IR and the UV fixed point, corresponding to a $r$-dependent scalar field.

\(2. \text{ Specific heat} \)

It is easy to check, using Eqs. (4.3) and (4.5), that for $\beta^2 < 1/(d + 1)$ the function $M(T)$ is a monotonic increasing function of the temperature $T$. Then the specific heat $c = \partial M/\partial T$ is positive for all values of $T$. Similarly, the specific heat of the SAdS black brane is: $c_{SAdS}(T) = \pi^{d+1} V^d T^{-d} > 0$.

\(\text{B. } 1/(d + 1) \leq \beta^2 < 1 \)

For $\beta^2 = 1/(d + 1)$, the scalar black brane solution exists only for temperatures below the critical value $T = T_c = \frac{d^2}{4\pi(d+2)}$, while for $T > T_c$ only the SAdS solution (2.4) exists. The free energy is:

$$F_{SB} = \frac{2V d^2}{(d + 1)(d + 2)} \sqrt{1 - \left(\frac{T}{T_c}\right)^{d+1}}.$$

The free energy is positive definite and vanishes for $T = T_c$, while $F_{SAdS}$ is always negative. Then we have $F_{SAdS} < F_{SB}$ in the whole range $T \leq T_c$, that is the SAdS solution is always energetically favored. The specific heat of the black brane solution is always positive and diverges at the critical temperature.

For $\beta^2 > 1/(d + 1)$, the function $T(\nu_2)$ is not monotonic. It has a maximum at $\nu_2 = \nu_0 = \sqrt{[(d + 1)\beta^2 - 1]/[(d - 1)\beta^2 + 1]}$. Also in this case the black brane solution exists only below a maximum, critical temperature $T = T_c$. For what concerns the free energy, from Eq. (4.6) it is easy to realize that, for $\beta^2 > 1/(d + 1)$, $F_{SB}$ is always positive. Hence also in this case $F_{SAdS} < F_{SB}$ and the SAdS solution is energetically favored.

\(^1\) We thank the anonymous referee of JHEP for suggesting to us this interpretation.
preferred with respect to the scalar-dressed black brane. However, the non-monotonicity of the function $T(\nu_2)$ implies the existence of two different branches of the SB phase for $T \leq T_c$, as it has been already observed in Ref. [1] for the 4D case. The first branch (obtained for $\nu_0 \leq \nu_2 \leq 1$) is the analogue of the $AdS_{d+2}$ phase obtained for $\beta^2 < 1/(d+1)$ at small $T$, while the second branch (obtained for $0 < \nu_2 \leq \nu_0$) has no analogue for $\beta^2 < 1/(d+1)$. In this case the free energy scales at small temperature as $F \sim T^\alpha$, with $\alpha = (d+1)\gamma/\omega$. But $\alpha$ is negative, hence $F_{SB}$ has a singularity at $T = 0$.

For what concerns the specific heat we have an interesting peculiarity: in the first branch the specific heat is positive and hence it corresponds to a locally stable phase (although this phase is not energetically preferred with respect to the SAdS solution), while in the second branch $c(T)$ is always negative, corresponding to an unstable phase.

C. Dual solution

As already observed, using the duality (2.3) into the (4.1) we obtain the solution for $\beta^2 > 1$. The thermodynamical properties of these solutions follow easily from the case $\beta^2 < 1$ by duality. We note that in this case the phase transition between the scalar-dressed black brane solution and the SAdS solution is present for $\beta^2 > (d + 1)$, while for $\beta^2 \leq (d + 1)$ the SAdS solution is always energetically favored respect to the SB solution. The behaviors of the free energy and the specific heat in the three cases are qualitatively analogous to those discussed for $\beta^2 < 1$.

V. HYPERSCALING VIOLATION

The thermodynamical behavior of our scalar BB described in the previous sections is strongly related to the symmetries of the solutions in the UV and IR regimes.

The UV regime, where the solution takes the form (2.7), is characterized by violation of the scale symmetry, whereas in the IR regime we have the conformal invariant $AdS_{d+2}$ extremal solution. For the dual QFT this corresponds to a hyperscaling-violating phase in the UV and to a scaling-preserving phase in the IR.

To describe holographic hyperscaling violation in $d + 2$ dimensions we use the parametrization of the the scale covariant metric of Ref. [13]:

$$ds^2 = r^{-2(d-\theta)/d} \left( -r^{-2(z-1)} dt^2 + dx_i dx_i + dr^2 \right),$$

(5.1)

where $\theta$ is the hyperscaling violation parameter and $z$ is the dynamic critical exponent (it describes anisotropic scaling, hence violation of Poincaré symmetry, in the $(d + 1)$-dimensional spacetime). The transformation law under rescaling of the coordinates is

$$t \rightarrow \lambda^{z} t, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow \lambda r, \quad ds \rightarrow \lambda^{\theta/d} ds.$$

(5.2)

The scaling transformation (5.2) determines the following scaling behavior for the free energy:

$$F \sim T^{(d-\theta)\hat{\nu}/z}.$$ 

(5.3)

This relation allows a simple physical interpretation of the hyperscaling violation parameter $\theta$ in terms of the hyperscaling relation between specific heat exponent $\hat{\alpha}$ and critical exponent $\hat{\nu}$. The relation $2 - \hat{\alpha} = d \hat{\nu}$ is modified by “lowering” the dimensionality of the system from $d$ to $d - \theta$, namely $2 - \hat{\alpha} = (d - \theta)\hat{\nu}$. 
Comparing Eq. (5.1) with Eq. (2.7) one can easily read off the parameters $\theta, z$ for our BB solution:

$$z = 1, \quad \theta = \frac{d(d + 1)\beta^2}{(d + 1)\beta^2 - 1}. \quad (5.4)$$

As usual the case $\beta^2 > 1$ is covered just by using the duality (2.3). We have

$$z = 1, \quad \theta = \frac{d(d + 1)}{(d + 1) - \beta^2}. \quad (5.5)$$

As expected, we have $z = 1, \theta \neq 0$ in the scalar black brane phase, whereas we get $z = 1, \theta = 0$ in the SAdS phase. This gives the deviation from the conformal scaling of the free energy of a $d + 1$ conformal field theory.

One can easily check from Eq. (5.4) that $\theta < 0$ for $\beta^2 < \frac{d}{d + 1}$ and $\theta > d$ for $\frac{d}{d + 1} < \beta^2 < 1$, while $\theta$ diverges for $\beta^2 = \frac{1}{d - 1}$ (for the dual case (5.5) we have $\theta < 0$ for $\beta^2 > d + 1$ and $\theta > d$ for $\beta^2 < d + 1$). The null energy conditions for the bulk stress-energy tensor are satisfied: in fact for $z = 1$ these conditions require either $\theta \leq 0$ or $\theta \geq d$ [13].

A negative value of $\theta$ is not common in condensed matter critical system, for which $\theta$ is positive. However in our solutions the case $\theta < 0$ is physically more interesting (in particular for the possible holographic applications) because in this case we observe a phase transition between the scalar black brane solution and the SAdS solution, and the specific heat of the BB solution is always positive.

VI. GENERAL MODELS

In the previous sections we have investigated the Einstein-scalar gravity model defined by the potential (2.2). However, the main features of our models are dictated not by the full form of the potential but only by the behavior of the potential at $\phi = 0$ and $\phi = -\infty$. We will show that the two main features of the model (hyperscaling violation and the SAdS $\rightarrow$ scalar BB phase transition) are pertinent to all models satisfying the conditions: 1) $V(\phi)$ has a local minimum for $\phi = 0$ with $V(0) < 0$; 2) The potential approaches zero exponentially as $\phi \rightarrow -\infty$. The previous conditions ensure the existence of an AdS$_{d+1}$ vacuum and of a Schwarzschild-AdS (SAdS) black brane solution with $\phi = 0$.

In Ref. [28] has been derived the general BB solution of a model with an exponential potential in $d + 2$ dimensions. In particular, for the metric parametrization we are using in this paper, the asymptotic behavior of the solutions for the exponential potential $V = -\lambda e^{2\beta\phi}$ is given by

$$\phi = -\frac{dh}{dh^2 - 2} \log r + \frac{1}{2h} \ln C_1, \quad ds^2 = r^{\frac{1}{2h^2 - 2}} (-dt^2 + dx_1 dx_2 + dr^2), \quad (6.1)$$

where $h > 0$ and $C_1 = \{2d[2(d+1) - dh^2]\}/[\lambda^2(dh^2 - 2)^2]$.

The case $\beta^2 < 1$ described in the previous section for the model (2.1) is covered by setting $h^2 < 2(d+1)/d$, whereas the two cases $\beta^2 < 1/(d+1)$ and $\beta^2 > 1/(d+1)$ correspond, respectively, to $h^2 < 2/d$ and $h^2 > 2/d$.

For a generic model, the existence of a global scalar black brane solution interpolating between the AdS$_{d+2}$ vacuum and the asymptotic scale covariant solution has to be shown numerically. If we can prove that such a solution exists, the thermodynamical system for $h^2 < 2/d$ must have a scalar black brane $\rightarrow$ SAdS phase transition.

The derivation follows closely that used in Sect. IV. At small $T$ the free energy of the scalar black brane must have a behavior similar to that of Eq. (4.7), i.e. $F_{SB} = C_2 - C_3 T^{d+1}$, with $C_{2,3}$ positive constants. This implies that at small $T$, $F_{SB} - F_{SAdS} > 0$. On the other hand, at large $T$, the free energy scales as $F_{SB} \sim -T^{(2+2d-dh^2)/(2-dh^2)}$. For $h^2 < 2/d$ we have $T^{(2+2d-dh^2)/(2-dh^2)} > T^{d+1}$, from which follows that at large $T$, $F_{SB} - F_{SAdS} < 0$. 

The null energy conditions for the bulk stress-energy tensor are satisfied: in fact for $z = 1$ these conditions require either $\theta \leq 0$ or $\theta \geq d$ [13].
Comparing Eq. (6.1) with Eq. (5.1), one can read off the hyperscaling violation parameter and the dynamic critical exponent:

$$\theta = \frac{d^2 h^2}{dh^2 - 2}.$$  

Notice that $\theta$ is negative for $h^2 < 2/d$, whereas $\theta > d$ for $h^2 > 2/d$.

VII. HOLOGRAPHIC PROPERTIES AND TWO-POINT FUNCTIONS FOR SCALAR OPERATORS

Holographic features of theories with hyperscaling violation have been discussed in Ref. [13]. Most of the results derived in Ref. [13] for general scale-covariant metrics apply directly to the model discussed in this paper. Imposing on the gravity side the null energy conditions on the stress-energy tensor constrains the range of the possible values of the parameters $z, \theta$. In our case, $z = 1$, the conditions of Ref. [13] become simply $\theta \leq 0$ or $\theta > d$.

In Ref. [13] it has been also calculated the short distance form of the two point function of a scalar operator $O$ dual to a scalar field with a potential $2m^2 \phi^2$. It has been shown that it has a power-law form and for $z = 1, 0 < \theta < d$ is given by [13]

$$\langle O(x)O(x') \rangle = \frac{1}{|x - x'|^{2(d+1) - \theta}}.$$  

The problem is that the derivation of Ref. [13] does not hold for $\theta < 0$, which is the most interesting case for the models considered in this paper. Moreover, for $\theta > d$, $r \to 0$ corresponds to the IR regime of the dual QFT. This means that for $\theta > d$, Eq. (7.1) gives the large distance behavior of the two-point function instead of the short distance behavior.

Let us now first observe that for $\theta < 0$ Eq. (7.1) gives the IR behavior of the two-point function. This means that for $\theta < 0$ the mass term is irrelevant in the IR and dominates in the UV. Conversely, for $\theta > 0$ we have the opposite behavior: the mass term is irrelevant in the UV and becomes relevant in the IR. It is exactly this feature that allows one to use scaling arguments to determine the form (7.1) for the two-point function.

Obviously, if the theory whose solution is given by the metric (5.1) has an UV (or IR) completion with an UV (or IR) fixed point, the far short (far large) behavior of the two-point function (7.1) will be modified accordingly. This is for instance the case of the models discussed in this paper, which have an IR fixed point.

We are therefore left with the problem of finding a short distance form for two point functions of scalar operators in the case $\theta < 0$. A strong hint for tackling the problem can be obtained by looking at the gravitational dynamics that produces solution (5.1). One can easily realize that, at least in the context of Einstein-scalar gravity, what is needed is an exponential potential and a ln $r$ short distance behavior for the scalar (see Eq. (6.1)). We will therefore look for the UV behavior of two point function of a scalar operator $\mathcal{O}$ dual to a scalar field that supports our black brane solution and therefore has near the UV a potential $-\lambda^2 e^{2h\phi}$. The equation of motion for $\phi$ in the background (6.1) are

$$\left(\partial_r^2 - \frac{d - \theta}{r} \partial_r + \partial_t^2 - \partial_t^2\right) \phi + \frac{h \lambda^2}{2} e^{2h\phi} r^{-2 + \frac{\theta}{2}} = 0,$$  

where $h$ has to be expressed as a function of $\theta$ using Eq. (6.2). Eq. (7.2) can be solved perturbatively for $\theta < 0$ ($h^2 < 2/d$) by expanding $\phi$ around the background solution $\phi_0$ given by Eq. (6.1): $\phi = \phi_0 + \delta \phi$. Using Eqs. (6.1) and (6.2) one gets for the perturbation $\delta \phi$, the equation of motion
satisfied by a massive scalar field in AdS in \( d + 2 - \theta \) "bulk dimensions":

\[
\left( \partial_r^2 - \frac{d - \theta}{r} \partial_r + \partial_i^2 - \partial_t^2 \right) \delta \phi - \frac{m^2}{r^2} \delta \phi = 0
\]

(7.3)

with \( m^2 = -C_1 h^2 \lambda^2 = \frac{2 \theta(d + 1 - \theta)}{d} \). Eq. (7.3) can be solved with the usual power-law ansatz \( \delta \phi \propto r^\alpha (1 + O(r^2)) \), with \( \alpha \) given by the standard AdS formula in \( d + 1 - \theta \) dimensions:

\[
\alpha_{12} = \frac{1}{2} \left( d + 1 - \theta \pm \sqrt{(d + 1 - \theta)^2 + 4m^2} \right) = \frac{1}{2} (d + 1 - \theta) \left( 1 \pm \sqrt{1 + \frac{8 \theta}{d(d + 1 - \theta)}} \right).
\]

(7.4)

The two solutions for \( \alpha \), corresponding to a faster and slower falloff mode of the scalar for \( r \to 0 \), always exist for \( d \geq 8 \), whereas for \( d < 8 \) we must require \( \theta \geq -\frac{d}{d + 1} \).

The general solution to Eq. (7.3) is given by a superposition of the slowest and fastest fall off modes:

\[
\delta \phi = a(kr)^{\alpha_2} (1 + O(r^2)) + b(kr)^{\alpha_1} (1 + O(r^2)),
\]

(7.5)

where \( a, b \) are \( O(1) \) constants determined by the boundary conditions, we have taken the \((t, x_i)\)-Fourier transform and \( k^2 = -k_0^2 + k_i k_i \). The Green’s function \( G(k) \) for the scalar operator dual to the bulk scalar field is given by the ratio of the coefficients of the \( r^{\alpha_1} \) and \( r^{\alpha_2} \) terms in Eq. (7.5) (see for instance [52]),

\[
G(k) \sim k^{\alpha_1 - \alpha_2},
\]

(7.6)

where \( \alpha_{12} \) are given by Eq. (7.4). Taking the Fourier transform, in the coordinate space we get the power-law form for the two-point function for the scalar operator dual to a bulk scalar field with exponential potential:

\[
\langle O(x)O(x') \rangle = \frac{1}{|x - x'|^{d+1+\alpha_1 - \alpha_2}}.
\]

(7.7)

It is also of interest to compute the two-point function (7.7) for small negative values of \( \theta \):

\[
\langle O(x)O(x') \rangle = \frac{1}{|x - x'|^{2(d+1)-(d-4)\theta/d'}}.
\]

(7.8)

**VIII. CONCLUSIONS**

In this paper we have analyzed the thermodynamics and the scaling symmetries of BB solutions of AdS Einstein-scalar gravity in arbitrary dimensions for models with positive scalar squared mass and a potential that has an exponential asymptotic behavior. We have generalized the results of Ref. [1], which hold for two-dimensional scalar branes, to branes of arbitrary spacetime dimensions.

We have been mainly concerned with an integrable model, which also arises as compactification of black \( p \)-brane solutions of SUGRA theories. However, the relevant features of this model can easily be extended to a broad class of Einstein-scalar gravity models.

The striking features of these \( d \)-dimensional scalar BB solutions are an unexpected phase diagram and non-trivial behavior in the ultraviolet regime of the holographically dual QFT, which is characterized by hyperscaling violation. This generates an UV length scale which decouples in

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2 We thank the anonymous referee of JHEP for pointing out to us this fact and for finding an error in the calculations leading to Eq. (7.4) of the previous version of this paper.
the IR, where conformal invariance is restored. At high temperatures, when $\beta^2 < 1/(d + 1)$ or $\beta^2 > d + 1$ the scalar-dressed BB solution, with scale-covariant asymptotical behavior, becomes energetically preferred.

The hyperscaling violating phase is characterized by the two parameters normally used for critical systems with hyperscaling violation, namely the dynamical critical exponent $\varepsilon$ and the hyperscaling violation parameter $\theta$.

The most important peculiarity of our models is that for scalar black branes that are stable at high temperatures, the hyperscaling parameter $\theta$ is always negative. In QFTs with hyperscaling violation the scaling law for the free energy is that pertinent to a CFT in $d - \theta$ dimensions. For positive $\theta$ we have therefore a lowering of the effective dimensions. This is an important feature of the small temperature behavior of traditional hyperscaling-violating critical systems [36]. On the other hand, the scalar BB brane solutions investigated in this paper are characterized by a negative hyperscaling-violation parameter $\theta$, producing a raising of the “effective dimensions”.

It is important to notice in this context that the most general compactification of $p$-brane solutions of SUGRA theories produces hyperscaling violation in the dual QFT with both $\theta < 0$ or $\theta > d$. Both cases are consistent with the null energy condition for the bulk stress energy tensor, but for $\theta > d$ the SAdS phase is always energetically preferred (see Sect. IV). On the other hand the simplest diagonal ansatz (3.4) for the D-dimensional metric leads to BB solutions with $\theta > d$.

We have also determined, for the case of negative $\theta$, the short distance behavior of two-point functions for scalar operators of the QFT dual to a bulk scalar field with an exponential potential. We have shown that it has a power-law behavior. Our calculation completes the derivation of Ref. ([13]). In that paper the short distance, power-law, form of the two-point functions for scalar operators dual to a scalar field with a mass term potential was determined only for positive $\theta$.

A puzzling point which still remains to be clarified is the holographic interpretation of the phase transition between the two bulk phases - the SAdS and the scalar brane phase. The cross-over of the free energies for SAdS and scalar branes observed in Sect. IV seems to have a very different interpretation than a conventional phase transition in the gravity/gauge theory correspondence, such as for instance the Hawking-Page phase transition.

Usually, in the gravity/gauge theory correspondence, we fix the boundary conditions for the fields and consider two distinct extensions into the bulk. The corresponding dual solutions contribute to the same canonical ensemble of the QFT. In the large-N limit the solution with lower free energy is energetically preferred. On the other hand the two competing phases of the QFT holographically dual to the SAdS- scalar brane phases seem to correspond to different boundary QFTs. Therefore they do not contribute to the same canonical ensemble.

This is obviously related to the unusual feature that the scalar black brane solutions discussed in this paper exhibit hyperscaling violation in the UV and conformal symmetry in the IR. In the conventional setting where the solution has a UV fixed point and an emergent nonzero $\theta$ in the IR, the holographic interpretation of the phase transition is not problematic. In this latter case the SAdS and the hyperscaling violating phase contribute to the same canonical ensemble.

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