On sums of generalized Ramanujan sums

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Abstract

Ramanujan sums have been studied and generalized by several authors. For example, Nowak studied these sums over quadratic number fields, and Grytczuk defined that on semigroups. In this note, we deduce some properties on sums of generalized Ramanujan sums and give examples on number fields. In particular, we have a relational expression between Ramanujan sums and residues of Dedekind zeta functions.

1 Introduction

For positive integers $m$ and $k$ the Ramanujan sum $c_k(m)$ is defined as

$$c_k(m) = \sum_{h \mod k} \exp \left( \frac{2\pi i m h}{k} \right) = \sum_{d \mid m, k} d \mu \left( \frac{k}{d} \right)$$

where $\mu$ is the Möbius function. This sum was generalized by several authors. (For example, see [1], [5], [8], [4], and so on.) In this paper, we define generalized Ramanujan sums in another way and show some properties on them. We see that $F_X$ is an abelian group with respect to addition. For $A, B \in F_X$, we denote $A \leq B$ if $A(x) \leq B(x)$ for every $x \in X$. Let $I_X = \{ A \in F_X : A \geq 0 \}$. When $X$ is the set of all prime ideals of some Dedekind domain $O$, we regard $I_X$ as the set of all non-zero ideals of $O$. Now fix a real-valued function $\mathcal{N} : I_X \to \mathbb{Z}_{>0}$ such that $\mathcal{N}(0) = 1$, $\mathcal{N}(A) > 1$ if $A \neq 0$, and $\mathcal{N}(A + B) = \mathcal{N}(A)\mathcal{N}(B)$ for all $A, B \in I_X$. The Möbius function $\mu$ on $I_X$ is defined as $\mu(A) = (-1)^{\sum_{x \in X} A(x)}$ when $A(X) \subset \{0, 1\}$ and $\mu(A) = 0$ otherwise. For $M, K \in I_X$, we put

$$C_K(M) = \sum_{D \in I_X} \mathcal{N}(D) \mu(K - D).$$
There are many expressions on Ramanujan sums. For example,

\[ \sum_{d|k} c_k(d) = k \prod_{p|k} (1 - 2/p), \]

where the product is over all prime divisors \( p \) of \( k \), and

\[ \sum_{d|m} c_d(m) = \begin{cases} n & \text{if } n|m, \\ 0 & \text{otherwise.} \end{cases} \]

It is also known that

\[ \sum_{m=1}^{\infty} \frac{c_k(m)}{m} = -\Lambda(k) \quad \text{if } k \neq 1 \]

where \( \Lambda \) is the von Mangoldt function. Firstly, we shall show these analogues. Put \( [x] = \#\{A \in I_X : N(A) \leq x\} \) for a real number \( x > 0 \) when \( X \) is at most countable. We shall show the next theorem.

**Theorem 1.**

1. For \( K \in I_X \), we have

\[ \sum_{D \leq K} C_K(D) = N(K) \prod_{p} \left( 1 - \frac{2}{N(A_p)} \right) \]

where the product is over points \( p \in X \) such that \( K(p) \neq 0 \) and \( A_p \) is the map such that \( A_p(p) = 1 \) and \( A_p(q) = 0 \) if \( p \neq q \).

2. For \( M, N \in I_X \), we have

\[ \sum_{D \leq N} C_D(M) = \begin{cases} N(N) & \text{if } N \leq M, \\ 0 & \text{otherwise.} \end{cases} \]

3. Suppose that \( X \) is at most countable and \( [x] = cx + O(x^\alpha) \) for some \( c > 0 \) and \( \alpha \in [0, 1) \). For \( K \neq 0 \in I_X \), we have

\[ \sum_{M \in I_X} \frac{C_K(M)}{N(M)} = -c\Lambda(K). \]

where \( \Lambda(A) = \sum_{D \leq A} \mu(A - D) \log N(D) \).

Chan and Kumchev [2] studied the sums

\[ \sum_{m \leq x} \left( \sum_{k \leq y} c_k(m) \right)^n \]
where \( n \) is a positive integer, \( x \) and \( y \) are large real numbers. In particular, they obtain

\[
\sum_{\substack{m \leq x \\
k \leq y}} c_k(m) = x + O(y^2).
\]

We shall show an analogue of this expression.

**Theorem 2.** Suppose that \( X \) is at most countable and \([x] = cx + O(x^\alpha)\) for some \( c > 0 \) and \( \alpha \in [0, 1) \).

(1) If we fix \( K \in I_X \), then

\[
\sum_{N(M) \leq x} C_K(M) = \begin{cases} 
    cx + O(x^\alpha) & \text{when } K = 0, \\
    O(x^\alpha) & \text{otherwise.}
\end{cases}
\]

(2) Put \( S(x, y) := \sum_{N(M) \leq x} C_K(M) \). For any \( \lambda > \frac{2-\alpha}{1-\alpha} \), under the condition \( y^\lambda \ll x \), considering \( x \to \infty \),

\[
S(x, y) = cx + o(x).
\]

### 2 Preliminary

In this section, we review or construct some basic facts of arithmetical functions in a generalized situation. (See [1], [3], or [9].) Put \( \mathcal{A} := \{ f : I_X \to R \} \) where \( R \) is a commutative ring. When \( X \) is the set of prime numbers and \( R \subset \mathbb{C} \), we may regard an elements of \( \mathcal{A} \) as an arithmetical function in the usual case. Let \( f \) and \( g \in \mathcal{A} \). The Dirichlet convolution \( f \ast g \) is defined as

\[
f \ast g(A) = \sum_{\substack{D \in I_X \\
D \leq A}} f(D)g(A-D) = \sum_{\substack{B, C \in I_X \\
B + C = A}} f(B)g(C)
\]

for \( A \in I_X \). The operator \( \ast \) on \( \mathcal{A} \) is commutative, and associative. The identity element is the function \( \delta \) such that \( \delta(0) = 1 \) and \( \delta(A) = 0 \) when \( A \neq 0 \). A function \( f \in \mathcal{A} \) is invertible if and only if \( f(0) \in R^\times \). For simplicity, we suppose that \( R = \mathbb{R} \) or \( \mathbb{C} \). The function \( \mu \) is the inverse of the function \( 1 \) such that \( 1(A) = 1 \) for all \( A \in I_X \), that is, \( \mu \ast 1 = \delta \). One can see that \( f = g \ast 1 \) if and only if \( g = f \ast \mu \).

The partial summation formula is generalized as follows.
Lemma 3. Suppose \([x] < \infty\) for all \(x > 0\). Let \(F : [1, \infty) \to \mathbb{C}\) be a \(C^1\) function and \(x \geq 1\). For \(g \in A\), put \(S(x) = \sum_{N(A) \leq x} g(A)\). Then

\[
\sum_{N(A) \leq x} g(A)F(N(A)) = S(x)F(x) - \int_1^x S(t)F'(t)\,dt.
\]

This lemma is shown by the ordinary partial summation formula. So we omit the proof.

For a complex number \(s = \sigma + it\), set

\[
Z(s) = \sum_{A \in \mathcal{I}X} \frac{1}{N(A)^s}.
\]

Suppose \([x] = cx + R(x)\) and \(R(x) = O(x^\alpha)\) for some \(c > 0\) and \(\alpha \in (0, 1)\). For \(x > 0\) and \(\sigma > 1\), using Lemma 3 we see that

\[
\left| \sum_{N(A) \leq x} \frac{1}{N(A)^s} \right| = \sum_{N(A) \leq x} \frac{1}{N(A)^\sigma} = \frac{1}{x^\sigma}[x] + \sigma \int_1^x \frac{t}{t^\sigma+1}\,dt.
\]

By our assumption, the above expression is

\[
\frac{c}{x^{\sigma-1}} + O(x^{\alpha-\sigma}) + \frac{\sigma cx^{1-\sigma}}{1-\sigma} + \frac{\sigma c}{\sigma - 1} + O \left( \sigma \int_1^x t^{\alpha-\sigma-1}\,dt \right).
\]

Considering \(x \to \infty\), we have

\[
Z(\sigma) = \frac{\sigma c}{\sigma - 1} + O \left( \sigma \int_1^\infty t^{\alpha-\sigma-1}\,dt \right).
\]

Therefore, the series \(Z(s)\) is absolutely convergent for \(\sigma > 1\). Moreover, we can see \(Z(s)\) has an analytic continuation to \(\sigma > \alpha\), and the residue of \(Z(s)\) at \(s = 1\) is the constant \(c\).

3 Proof of Theorem I

For \(f, g \in A\) and \(M, K \in \mathcal{I}X\), define the sum \(S_{f,g}(M, K)\) as

\[
S_{f,g}(M, K) = \sum_{D \leq M, K} f(D)g(K-D).
\]

Note that \(S_{\mathcal{N},\mu}(M, K) = C_K(M)\).

For \(A \in \mathcal{I}X\), the function \(\chi_A \in A\) is defined as \(\chi_A(B) = 1\) when \(B \leq A\) and \(\chi_A(B) = 0\) otherwise. Let \(v(D) = \chi_K(D)f(D)g(K-D)\). When \(K\) is fixed, we
see $S_{f,g}(M,K) = (v * 1)(M)$ and

$$\sum_{D \leq N} S_{f,g}(D,K)h(N-D) = \sum_{D \leq N} (v * 1)(D)h(N-D)$$

$$= (v * 1 * h)(N) = \sum_{D \leq N} v(D)(1 * h)(N-D)$$

$$= \sum_{D \leq N} \chi_K(D)f(D)g(K-D)(1 * h)(N-D).$$

Thus, we have that

$$\sum_{D \leq N} S_{f,g}(D,K)h(N-D) = \sum_{D \leq N,K} f(D)g(K-D)(1 * h)(N-D)$$

which is an analogue of Theorem 1 and 2 in [1]. When $f(A) = \mathcal{N}(A)$, $g(A) = \mu(A)$, and $h(A) = 1$ for all $A \in I_X$, the above equation is

$$\sum_{D \leq N} C_K(D) = \sum_{D \leq N,K} \mathcal{N}(D)\mu(K-D)(1 * 1)(N-D).$$

Put $K = N$. We obtain

$$\sum_{D \leq K} C_K(D) = \sum_{D \leq K} \mathcal{N}(D)\mu(K-D)(1 * 1)(K-D)$$

$$= \sum_{D \leq K} \mathcal{N}(K-D)\mu(D)(1 * 1)(D)$$

$$= \mathcal{N}(K) \sum_{D \leq K} \frac{\mu(D)(1 * 1)(D)}{\mathcal{N}(D)}$$

$$= \left( 1 + \frac{\mu(A_p)(1 * 1)(A_p)}{\mathcal{N}(A_p)} \right)$$

$$= \mathcal{N}(K) \prod_{p} \left( 1 - \frac{2 \mathcal{N}(A_p)}{\mathcal{N}(A_p)} \right)$$

where the product is over points $p \in X$ such that $K(p) \neq 0$ and $A_p$ is the map $A_p(p) = 1$ and $A_p(q) = 0$ if $p \neq q$. Hence (1) of Theorem 1 is proved.

Next, in order to show (2), fix $M \in I_X$. Then we see $S_{f,g}(M,K) = (w * g)(K)$
where \( w(A) = \chi_M(A) f(A) \) and

\[
\sum_{D \leq N} S_{f,g}(M, D) h(N - D) = \sum_{D \leq N} (w * g)(D) h(N - D)
= (w * g * h)(N)
= \sum_{D \leq N} w(D)(g * h)(N - D)
= \sum_{D \leq N} \chi_M(D)f(D)(g * h)(N - D)
= \sum_{D \leq N,M} f(D)(g * h)(N - D).
\]

Thus we obtain

\[
\sum_{D \leq N} S_{f,g}(M, D) h(N - D) = \sum_{D \leq N,M} f(D)(g * h)(N - D)
\]

which is an analogue of Theorem 3 and 4 in [1]. When \( f(A) = \mathcal{N}(A), g(A) = \mu(A), \) and \( h(A) = 1 \) for all \( A \in I_X \), the above equation is

\[
\sum_{D \leq N} C_D(M) = \sum_{D \leq N,M} \mathcal{N}(D) \delta(N - D)
= \begin{cases} 
\mathcal{N}(N) & \text{if } N \leq M, \\
n & \text{otherwise.} 
\end{cases}
\]

Hence (2) is proved.

To show (3) we use \( Z(s) \) which is defined in the previous section. By an argument similar to that of Titchmarsh [9] p.10, we obtain that

\[
\sum_{M \in I_X} \frac{C_K(M)}{\mathcal{N}(M)^s} = \sum_{M \in I_X} \frac{1}{\mathcal{N}(M)^s} \sum_{D \leq M,K} \mathcal{N}(D) \mu(K - D)
= \sum_{D \leq K} \mu(K - D) \mathcal{N}(D) \sum_{C \in I_X} \frac{1}{\mathcal{N}(C + D)^s}
= Z(s) \phi_{-1}(K)
\]
where $\phi_{1-s}(A) = \sum_{D \leq A} \mu(A - D) N(D)^{1-s}$. For $A \neq 0 \in I_X$, we see

\[
\phi_s(A) = \sum_{D \leq A} \mu(A - D) N(D)^s = \sum_{D \leq A} \mu(A - D) \exp(s \log N(D)) = \sum_{D \leq A} \mu(A - D) \left( \sum_{n=0}^{\infty} \frac{(s \log N(D))^n}{n!} \right).
\]

Thus, we have

\[
\lim_{s \to 1} \frac{\phi_{1-s}(A)}{1-s} = \lim_{s \to 0} \frac{\phi_s(A)}{s} = \Lambda(A).
\]

The expression (3) is proved from this and $\lim_{s \to 1} (s-1)Z(s) = c$.

### 4 Proof of Theorem 2

Firstly, we fix $K \in I_X$. Then,

\[
\sum_{N(M) \leq x} C_K(M) = \sum_{N(M) \leq x} \sum_{D+E=M, D+E=K} N(D) \mu(E)
\]

\[
= \sum_{D+E=K} N(D) \mu(E) \left[ \frac{x}{N(D)} \right] = \sum_{D+E=K} N(D) \mu(E) \left( e^{x/N(D)} + R \left( \frac{x}{N(D)} \right) \right)
\]

where $R(x) = O(x^\alpha)$. By the assumption, the above expression is

\[
ex \sum_{D+E=K} \mu(E) + O \left( x^\alpha \sum_{D+E=K} N(D)^{1-\alpha} \right).
\]

Hence, (1) is shown.
Next, we shall show (2). We have

\[ S(x, y) = \sum_{N(K) \leq y} \sum_{D \leq M, K} N(D)\mu(K - D) \]

\[ = \sum_{N(D) \leq x} \sum_{N(K) \leq y} N(D)\mu(A) \]

\[ = \sum_{D,A \in I_X \subseteq y} N(D)\mu(A) \left[ \frac{x}{N(D)} \right]. \]

By the assumption,

\[ S(x, y) = \sum_{D,A \in I_X \subseteq y} N(D)\mu(A) \left( c \frac{x}{N(D)} + R \left( \frac{x}{N(D)} \right) \right) \]

\[ = cx \sum_{D,A \in I_X \subseteq y} \mu(A) + \sum_{D,A \in I_X \subseteq y} N(D)\mu(A)R \left( \frac{x}{N(D)} \right) \]

where \( R(x) = O(x^\alpha) \).

Since

\[ \sum_{D,A \in I_X \subseteq y} \mu(A) = \sum_{C \in I_Y \subseteq y} \sum_{A \in C} \mu(A) = 1, \]

we obtain \( S(x, y) = cx + T(x, y) \) where

\[ T(x, y) = \sum_{D,A \in I_X \subseteq y} N(D)\mu(A)R \left( \frac{x}{N(D)} \right). \]

Note that

\[ T(x, y) \ll \sum_{D,A \in I_X \subseteq y} N(D) \left( \frac{x}{N(D)} \right)^\alpha = \sum_{N(A) \leq y} \sum_{N(D) \leq y/N(A)} x^\alpha N(D)^{1-\alpha} \]

\[ \ll x^\alpha \sum_{N(A) \leq y} \left( \frac{y}{N(A)} \right)^{1-\alpha} \sum_{N(D) \leq y/N(A)} 1 \ll x^\alpha y^{2-\alpha}. \]

Hence \( S(x, y) = cx + O(x^\alpha y^{2-\alpha}) \). If \( \frac{2-\alpha}{1-\alpha} < \lambda \) and \( y^\lambda \ll x \), then \( T(x, y) = o(x) \).

Therefore, Theorem 2 is proved.

5 Examples

Let \( F \) be a number field of degree \( d \), \( O_F \) the integer ring of \( F \), and \( I \) is the set of all non-zero ideals of \( O_F \). The Möbius function \( \mu : I \to \mathbb{C} \) for \( O_F \) is defined
as

\[ \mu(a) = \begin{cases} (-1)^{\omega(a)} & a \text{ is square free,} \\ 0 & \text{otherwise,} \end{cases} \]

where \( \omega(a) \) is the number of distinct prime factors of \( a \) and one can define the Ramanujan sum as

\[ C_a(b) = \sum_{d|a,b} \mathcal{N}(d) \mu \left( \frac{a}{d} \right) \]

where \( a, b \in \mathcal{I} \) and \( \mathcal{N}(a) = [O_F : a] \). By (1) of Theorem 1 one have

\[ \sum_{d|a} C_a(d) = \mathcal{N}(a) \prod_{p|a} \left( 1 - \frac{2}{\mathcal{N}(p)} \right) \]

for \( a \in \mathcal{I} \). By (2) of Theorem 1 we have

\[ \sum_{d|a} C_b(d) = \begin{cases} \mathcal{N}(a) & \text{if } a \mid b, \\ 0 & \text{otherwise} \end{cases} \]

for \( a, b \in \mathcal{I} \).

The following fact is well-known.

**Lemma 4.** (cf. Lang\(^1\), Chap.VI Theorem 3, or Murty and Order \(^7\).) The number of ideals of \( O_F \) whose norms are less than or equal to \( x \) is

\[ c_F x + R_F(x) \]

where \( c_F \) is the residue of the Dedekind zeta function \( \zeta_F(s) \) of \( F \) at \( s = 1 \) and \( R_F(x) = O(x^{1 - \frac{1}{2}}) \).

It is well-known that the invariant \( c_F \) in the above lemma is given by

\[ c_F = \frac{2^{r_1}(2\pi)^{r_2}Rh}{W \sqrt{D}} \]

where \( r_1 \) is the number of real primes, \( r_2 \) is the number of complex primes, \( R \) is the regulator, \( h \) is the class number, \( W \) is the number of roots of unity, and \( D \) is the absolute value of the discriminant of \( F \). The von Mangoldt function \( \Lambda \) for \( F \) is the function such that \( \Lambda(a) = \log \mathcal{N}(a) \) if \( a \) is a power of a prime ideal \( p \), and \( \Lambda(a) = 0 \) otherwise. Using Lemma 4 and (3) of Theorem 1 we have

\[ c_F = -\frac{1}{\Lambda(a)} \sum_b \frac{C_a(b)}{\mathcal{N}(b)} \]
unless $\Lambda(a) = 0$.

In addition, if $\lambda > d + 1$ and $y^\lambda \ll x$, then

$$\sum_{\mathcal{N}(b) \leq x} C_a(b) = c_p x + o(x).$$

by Theorem 2.

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References

[1] T. M. Apostol, *Arithmetical properties of generalized Ramanujan sums*, Pacific J. Math. 41, no. 2, (1972), 281-293.

[2] T. H. Chan and A.V. Kumchev, *On sums of Ramanujan sums*, Acta arithm. 152 (2012), 1–10.

[3] H. Cohen, *Number theory. Vol. II. Analytic and modern tools*, Graduate Texts in Mathematics, 240. Springer, New York, 2007.

[4] A. Grytczuk, *On Ramanujan sums on arithmetical semigroup*, Tsukuba. J. Math. 16 (1992), no. 2, 315–319.

[5] I. Kiuchi and Y. Tanigawa, *On arithmetic functions related to the Ramanujan sum*, Period. Math. Hungar. 45 (2002), no. 1–2, 87–99.

[6] S. Lang, *Algebraic number theory*, 2nd edition, Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994.

[7] R. Murty and J. V. Order, *Counting integral ideals in a number field*, Expo. Math. 25 (2007), 53–66.

[8] W. G. Nowak, *The average size of Ramanujan sums over quadratic number fields*, Arch. Math. 99 (2012), 433–442.

[9] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edition, revised by D. R. Heath-Brown, Oxford University Press, 1986.

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