Rational homology 3-spheres and simply connected definite bounding

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Abstract

For each rational homology 3-sphere \( Y \) which bounds simply connected definite 4-manifolds of both signs, we construct an infinite family of irreducible rational homology 3-spheres which are homology cobordant to \( Y \) but cannot bound any simply connected definite 4-manifold. As a corollary, for any coprime integers \( p,q \), we obtain an infinite family of irreducible rational homology 3-spheres which are homology cobordant to the lens space \( L(p,q) \) but cannot obtained by a knot surgery.

1 Introduction

Throughout this paper, all manifolds are assumed to be smooth, compact, orientable and oriented, and diffeomorphisms are orientation-preserving unless otherwise stated.

The intersection form of a \( 4n \) dimensional manifold has been used to study the topology of its boundary. For instance, the first exotic 7-spheres discovered by Milnor [11] were distinguished by using the intersection form of 8-manifolds whose boundaries are the exotic 7-spheres. In the case of dimension 4, Donaldson’s diagonalization theorem [5] implies that if a homology 3-sphere bounds a 4-manifold with non-diagonalizable definite intersection form, then it cannot bound any rational homology 4-ball.

In light of the above results, for any 3-manifold \( Y \), it seems natural to ask which bilinear forms are realized by the intersection form of a 4-manifold with boundary \( Y \). In the case where \( Y \) is a rational homology 3-sphere, Choe and Park [2] define \( \mathcal{T}(Y) \) (resp. \( \mathcal{T}^{\text{TOP}}(Y) \)) as the set of all negative definite bilinear forms realized by the intersection form of a (resp. topological) 4-manifold with boundary \( Y \), up to stable-equivalence. They prove in [2] that \( |\mathcal{T}^{\text{TOP}}(Y)| = \infty \) for any \( Y \), while \( |\mathcal{T}(Y)| < \infty \) if \( \mathcal{T}(-Y) \) is not empty. Moreover, they show that either \( \mathcal{T}(Y) \neq \emptyset \) or \( \mathcal{T}(-Y) \neq \emptyset \) holds for any Seifert rational homology sphere \( Y \). Here we note that all 4-manifolds constructed in their proof of the above results are simply connected, and hence if we define \( \mathcal{T}_s(Y) \) (resp. \( \mathcal{T}^{\text{TOP}}_s(Y) \)) by replacing “4-manifolds” in the definition of \( \mathcal{T}(Y) \) (resp. \( \mathcal{T}^{\text{TOP}}(Y) \)) with “simply connected 4-manifolds”, then we can prove that \( |\mathcal{T}^{\text{TOP}}_s(Y)| = \infty \) and \( \mathcal{T}_s(-Y) \neq \emptyset \Rightarrow |\mathcal{T}_s(Y)| < \infty \) for any \( Y \), and either \( \mathcal{T}_s(Y) \neq \emptyset \) or \( \mathcal{T}_s(-Y) \neq \emptyset \) holds if \( Y \) is Seifert fibered. Then, how different are they? The aim of this
paper is to prove the following theorem, which shows a big gap between $\mathcal{T}(Y)$ and $\mathcal{T}_s(Y)$.

**Theorem 1.1.** For any rational homology 3-sphere $Y$ satisfying $\mathcal{T}_s(Y) \neq \emptyset$ and $\mathcal{T}_s(-Y) \neq \emptyset$, there exist infinitely many rational homology 3-spheres $\{Y_k\}_{k=1}^{\infty}$ which satisfy the following conditions.

1. $Y_k$ is homology cobordant to $Y$.
2. $\mathcal{T}(Y_k) = \mathcal{T}(Y) \neq \emptyset$ and $\mathcal{T}(-Y_k) = \mathcal{T}(-Y) \neq \emptyset$.
3. $\mathcal{T}_s(Y_k) = \emptyset$ and $\mathcal{T}_s(-Y_k) = \emptyset$.
4. If $k \neq k'$ then $Y_k$ is diffeomorphic to neither $Y_{k'}$ nor $-Y_{k'}$.
5. Each $Y_k$ is irreducible and toroidal.

Here, rational homology 3-spheres $Y_0$ and $Y_1$ are *homology cobordant* if there exists a cobordism $W$ from $Y_0$ to $Y_1$ (i.e. $\partial W = (-Y_0) \sqcup Y_1$) such that the inclusion $Y_i \hookrightarrow W$ induces an isomorphism between $H_*(Y_i; \mathbb{Z})$ and $H_*(W; \mathbb{Z})$ for each $i \in \{0, 1\}$. (Then we call $W$ a *homology cobordism.* ) We note that since $\mathcal{T}(Y)$ is invariant under homology cobordism (more generally, rational homology cobordism), the first condition implies the second condition. Moreover, the third condition implies that any $Y_k$ is non-Seifert. We also note that there exist infinitely many rational homology 3-spheres satisfying $\mathcal{T}_s(Y) \neq \emptyset$ and $\mathcal{T}_s(-Y) \neq \emptyset$. For instance, any $p/q$ surgery of $S^3$ over any 0-negative knot (defined in [4]) with $p/q > 0$ satisfies this condition. (In this case, there is a negative definite cobordism $W$ from the lens space $L(p, q)$ to such a $p/q$ surgery such that $i_s(\pi_1(L(p, q)))$ normally generates $\pi_1(W)$, and $\mathcal{T}_s(L(p, q)) \neq \emptyset$.)

In order to prove Theorem 1.1, we first prove the following proposition, which is obtained by generalizing Auckly’s construction in [4].

**Proposition 1.2.** For any rational homology 3-spheres $Y$ and $M$, there exist a rational homology 3-sphere $Y_M$ and a homology cobordism $W_M$ from $Y \# M \# (-M)$ to $Y_M$ which satisfy

1. $i_s : \pi_1(Y_M) \to \pi_1(W_M)$ is surjective,
2. $i_s : \pi_1(Y_M \# (-M)) \to \pi_1(W_M)$ is bijective, and
3. $Y_M$ is irreducible and toroidal,

where $i_s$ denotes the induced homomorphism from the inclusion.

Then, by assuming that $\mathcal{T}_s(Y) \neq \emptyset$, $\mathcal{T}_s(-Y) \neq \emptyset$ and $|\mathcal{T}_s(M)| > 1$, and combining the first condition with Taubes’s theorem in [14] (stated as Theorem 3.2 in Section 3), we prove that $\mathcal{T}_s(Y_M) = \emptyset$ and $\mathcal{T}_s(-Y_M) = \emptyset$. Finally, we combine the second condition with the Chern-Simons invariants for 3-manifolds to find an infinite family $\{M_k\}_{k=1}^{\infty}$ of integer homology 3-spheres such that the 3-manifolds $\{Y_{M_k}\}_{k=1}^{\infty}$ are mutually distinct. (Note that if $M$ is an integer homology 3-sphere, then $Y \# M \# (-M)$ is homology cobordant to $Y$.)

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As an application of Theorem 1.1 we provide a huge number of irreducible rational homology 3-spheres that are not obtained by a knot surgery. Here we note that if $Y$ is obtained by a knot surgery, then either $T_s(Y) \neq \emptyset$ or $T_s(-Y) \neq \emptyset$ holds (see [12]). Hence the 3-manifolds $\{Y_k\}_{k=1}^{\infty}$ in Theorem 1.1 are not obtained by a knot surgery. Therefore, for instance, we have the following corollary.

**Corollary 1.3.** For any non-zero integers $p, q$, there are infinitely many irreducible rational homology 3-spheres which are homology cobordant to $L(p,q)$ but not obtained by a knot surgery.

These are the first examples of irreducible rational homology 3-spheres which have non-trivial torsion first homology and are not obtained by a knot surgery. Since infinitely many irreducible 3-manifolds with $H_1(Y) \cong \mathbb{Z}$ which are not obtained by a knot surgery are given in [8], now we have infinitely many irreducible 3-manifolds with $H_1(Y) \cong \mathbb{Z}/p\mathbb{Z}$ which are not obtained by a knot surgery for any integer $p$.

Finally we discuss some questions related to our results on knot surgery. We first mention that it remains open whether the examples given in the proof of Corollary 1.3 have weight one fundamental group.

**Question 1.4.** Do the examples given in the proof of Corollary 1.3 have weight one fundamental group?

Next, while we have a huge number of irreducible rational homology 3-spheres which are not obtained by a knot surgery, all of our examples are toroidal. Recently, Hom and Lidman [9] provided infinitely many hyperbolic integral homology 3-spheres which are not obtained by surgery on a knot, while the following question is still open.

**Question 1.5.** Does there exist a hyperbolic rational homology 3-sphere $Y$ such that $H_1(Y; \mathbb{Z}) \neq 0$ and $Y$ is not obtained by a knot surgery?

In addition, our examples in Corollary 1.3 are homology cobordant to $L(p,q)$, and hence their $d$-invariants (defined in [13]) satisfy

$$\{d(Y,s) \mid s \in \text{Spin}^c(Y)\} = \{d(L(p,q),s) \mid s \in \text{Spin}^c(L(p,q))\}$$

for some $p, q$. So we suggest the following question.

**Question 1.6.** Does there exist a rational homology 3-sphere $Y$ such that $\{d(Y,s) \mid s \in \text{Spin}^c(Y)\} \neq \{d(L(p,q),s) \mid s \in \text{Spin}^c(L(p,q))\}$ for any $p, q$ and $Y$ is not obtained by a knot surgery?

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### 2 Proof of Proposition 1.2

In this section, we prove Proposition 1.2.
Proposition 1.2. For any rational homology 3-spheres \( Y \) and \( M \), there exist a rational homology 3-sphere \( Y_M \) and a homology cobordism \( W_M \) from \( Y \# M \# (-M) \) to \( Y_M \) which satisfy

1. \( i_* : \pi_1(Y_M) \to \pi_1(W_M) \) is surjective,
2. \( i_* : \pi_1(Y \# M \# (-M)) \to \pi_1(W_M) \) is bijective, and
3. \( Y_M \) is irreducible and toroidal,

where \( i_* \) denotes the induced homomorphism from the inclusion.

Proof of Proposition 1.2. We first describe the construction of \( Y_M \) in the proposition. Let \( Y \) and \( M \) be rational homology 3-spheres and \( Y'_M := Y \# M \# (-M) \) . By the result of [15], there exists a null-homologous knot \( K \) in \( Y'_M \) whose complement \( Y'_M \setminus K \) has a hyperbolic structure. We denote the exterior of \( K \) in \( Y'_M \) by \( E_K \). Let \( C \) be a 3-manifold with torus boundary shown in Figure 1, where \( \mu \) and \( \lambda \) in the figure are simple closed curves in the boundary of \( C \). Define \( Y_M \) as \( C \cup \text{torus} E_K \) by identification

\[
\begin{align*}
\mu & \mapsto \text{meridian} \\
\lambda & \mapsto \text{preferred longitude}.
\end{align*}
\]

It is easy to see that \( Y_M \) is a rational homology 3-sphere.

![Figure 1: A 3-manifold C](image)

Claim 1. \( Y_M \) is irreducible and toroidal.

Proof. We make a similar argument to [1]. More precisely, we use the following lemmas.

Lemma 2.1 (See p.13 in [1]). Let \( N \) be a 3-manifold and \( F \) a properly embedded incompressible surface in \( N \). If every component of \( N - F \) is irreducible, then \( N \) is also irreducible.
Lemma 2.2 (Papakyriakopoulos’ Loop Theorem). If the boundary of a 3-manifold $N$ is incompressible in $N$, then $i_* : \pi_1(\partial N) \to \pi_1(N)$ is injective.

Lemma 2.3 (See Theorem 2.6 in [10], for example). Let

$$
\begin{array}{ccc}
C & \xrightarrow{i_A} & A \\
\downarrow i_B & & \downarrow j_A \\
B & \xrightarrow{j_B} & A \ast C \ B
\end{array}
$$

be the defining diagram of $A \ast C \ B$. If $i_A$ and $i_B$ are injective, then $j_A$ and $j_B$ are injective.

By Lemma 2.1, it suffices to prove

1. $\partial C$ is incompressible in $Y_M$, and

2. both $C$ and $E_K$ are irreducible

for proving Claim 1 (Note that the first condition implies that $Y_M$ is a toroidal 3-manifold with essential torus $\partial C$.) Moreover, Lemma 2.2 and Lemma 2.3 implies that $\partial C$ is incompressible in $Y_M$ if $\partial C$ is incompressible both in $C$ and in $E_K$. To prove it, suppose that $\partial C$ is incompressible both in $C$ and in $E_K$. Then it follows from Lemma 2.2 that both of the induced homomorphisms $(i_C)_* : \pi_1(\partial C) \to \pi_1(C)$ and $(i_{E_K})_* : \pi_1(\partial C) \to \pi_1(E_K)$ are injective. In addition, Since $\pi_1(Y_M) = \pi_1(C) \ast \pi_1(\partial C) \pi_1(E_K)$, Lemma 2.3 implies that both of the induced homomorphisms $(j_C)_* : \pi_1(C) \to \pi_1(Y_M)$ and $(j_{E_K})_* : \pi_1(E_K) \to \pi_1(Y_M)$ are injective. Now, assume that there exists a compressing disk for $\partial C$ in $Y_M$, and then $i_* : \pi_1(\partial C) \to \pi_1(Y_M)$ is not injective. However, since $i_* = (j_C)_* \circ (i_C)_*$ and the right hand side is injective, it leads to a contradiction.

Here, we note that the 3-manifold $C$ is exactly the same as the manifold $C$ appearing in [1], and it is proved that $\partial C$ is incompressible in $C$, and $C$ is irreducible. Now let us prove that $\partial C = \partial E_K$ is incompressible in $E_K$, and $E_K$ is irreducible. The irreducibility of $E_K$ immediately follows from the fact that $E_K$ has a hyperbolic structure. Assume that there exists a compressing disk $D$ for $\partial E_K$ in $E_K$. Then it follows from elementary arguments that $\partial D$ is a preferred longitude for $K$, and hence $K$ bounds a disk in $Y_M'$. This implies that $E_K$ is homeomorphic to $Y_M' \# (S^1 \times D^2)$, and $E_K$ does not have any hyperbolic structure. This leads to a contradiction, and hence $\partial E_K$ is incompressible in $E_K$.

Next, let $W_M$ denote a cobordism described by the relative Kirby diagram shown in Figure 2. Here, the tangle diagram $\langle D \rangle$ in Figure 2 is obtained as follows. We first take a diagram $D'$ of $K$ in $Y_M'$ (i.e. a knot diagram of $K$ in a surgery diagram of $Y_M'$) such that the linking number between $K$ and each component of the surgery link for $Y_M'$ is zero. Next, we derive a tangle diagram $D$ from $D'$ by removing a small disk whose intersection with $K$ is a small arc. Finally, by putting brackets around each surgery coefficient in $D$, we have the
Then, it follows from elementary handle theory that $W_M$ is a homology cobordism from $Y_M'$ to $Y_M$, and it admits a handle decomposition consisting of a single 1-handle $h^1$ and single 2-handle $h^2$.

![Figure 2: A cobordism $W_M$](image)

**Claim 2.** $i_* : \pi_1(Y_M) \to \pi_1(W_M)$ is surjective.

**Proof.** By considering the dual decomposition, we have a handle decomposition of $W_M$ consisting a single 2-handle and single 3-handle. This implies that $i_* : \pi_1(Y_M) \to \pi_1(W_M)$ is surjective. \hfill \square

**Claim 3.** $i_* : \pi_1(Y_M') \to \pi_1(W_M)$ is bijective.

**Proof.** Let $\langle S \mid R \rangle$ be a presentation for $\pi_1(Y_M') = \pi_1(Y_M' \times [0,1])$ and $l$ a loop shown in Figure 2. Then $\pi_1((Y_M' \times [0,1]) \cup h^1)$ is presented by $\langle S \cup \{x \} \mid R \rangle$, where $x$ corresponds to $h^1$, and $\partial h^2$ is homotopic to $l$ in $(Y_M' \times [0,1]) \cup h^1$. This implies that the homotopy class of $\partial h^2$ is a word of the form $xw$, where $w$ is a word on $S$, and hence $\pi_1(W_M) = \pi_1((Y_M' \times [0,1]) \cup h^1 \cup h^2)$ is represented by $\langle S \cup \{x \} \mid R \cup \{xw \} \rangle$. Moreover, the diagram

$$
\begin{array}{ccc}
\langle S \mid R \rangle & \xrightarrow{f} & \langle S \cup \{x \} \mid R \cup \{xw \} \rangle \\
\cong \downarrow & & \downarrow \cong \\
\pi_1(Y_M') & \xrightarrow{i_*} & \pi_1(W_M)
\end{array}
$$

is commutative, where $f$ maps $y \in S$ to $y$. We construct the inverse of $f$. Define a map $g : \langle S \cup \{x \} \mid R \cup \{xw \} \rangle \to \langle S \mid R \rangle$ by

$$y \mapsto \begin{cases} y & (y \in S) \\ w^{-1} & (y = x). \end{cases}$$

Then it is easy to see that $g$ is well-defined and both $f \circ g$ and $g \circ f$ are the identity maps. (Note that $w^{-1} = x$ in $\langle S \cup \{x \} \mid R \cup \{xw \} \rangle$.) This completes the proof. \hfill \square
The above arguments complete the proof of Proposition 1.2.

3 Definite bounding and Taubes’s theorem

In this section, we prove the following proposition by using a theorem of Taubes.

**Proposition 3.1.** Let $Y$ and $M$ be rational homology 3-spheres, and $Y_M$ a rational homology 3-sphere given by Proposition 1.2. If $T_s(Y) \neq \emptyset$, $T_s(-Y) \neq \emptyset$ and $|T_s(M)| > 1$, then we have $T_s(Y_M) = T_s(-Y_M) = \emptyset$.

First we state Taubes’s theorem. This is an end-periodic version of Donaldson’s diagonalization theorem.

**Theorem 3.2.** [14] Let $Y$ be a rational homology 3-sphere and $W := K \cup_Y W_0 \cup_Y W_1 \cup_Y \cdots$ a connected non-compact 4-manifold satisfying the following conditions.

- $K$ is a simply connected negative definite 4-manifold with $\partial K = Y$.
- $W_0$ is a negative definite cobordism from $Y$ to itself.
- $W_i$ are copies of $W_0$ for any $i \in \{1, 2, \ldots\}$.

We also assume that there is no non-trivial representation from $\pi_1(W_0)$ to $SU(2)$. Then the intersection form of $K$ is diagonalizable.

The assumption about $SU(2)$ representation is essential. If the assumption is removed, then one can easily find a counterexample to the theorem. (For instance, take $Y = \Sigma(2, 3, 5)$ and $W_0 = Y \times I$.) This is an essential reason why we can claim the nonexistence of simply connected definite bounding.

As a corollary of Theorem 3.2, we have the following lemma.

**Lemma 3.3.** Let $M$ be a rational homology 3-sphere and $W$ is a negative definite cobordism from $M$ to itself. If $W$ is simply connected, then $|T_s(M)| \leq 1$. 

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Proof. Suppose that $|T_s(M)| > 1$. Then there exists a simply connected negative definite 4-manifold $K$ with $\partial K = M$ whose intersection form is not diagonalizable. However, the end-periodic manifold $K \cup_M W \cup_M W \cup_M \cdots$ satisfies all assumptions of Theorem 3.2. This leads to a contradiction. □

Now we prove Proposition 3.1.

Proof. We first assume that $T_s(Y_M) \neq \emptyset$. Then there exists a simply connected negative definite 4-manifold $X$ with $\partial X = Y_M$. Moreover, since $T_s(-Y) \neq \emptyset$, $-Y$ also bounds a simply connected negative definite 4-manifold $U$. We use $X$, $U$ and $-W_M$ to construct a simply connected negative definite cobordism from $M$ to itself.

We first glue $X$ with $-W_M$ along $Y_M$, and denote it by $X'$. (Note that $\partial(-W_M) = -Y_M \amalg (Y \amalg M \amalg (-M))$.) Then $X'$ is negative definite and $\partial X' = Y \amalg M \amalg (-M)$. Furthermore, since $\pi_1(X) = 1$ and $i_* : \pi_1(-Y_M) \to \pi_1(-W_M)$ is surjective, we have $\pi_1(X') = 1$. Next, by attaching two 3-handles to $X'$, we obtain a 4-manifold $X''$ with $\partial X'' = Y \amalg M \amalg (-M)$. Finally, by gluing $X''$ with $U$ along $Y$, we have a 4-manifold $W$ with boundary $M \amalg (-M)$. By the construction, it is easy to check that $\pi_1(W) = 1$ and $W$ is negative definite.

Now, by applying Lemma 3.3 to $W$, we conclude that $|T_s(M)| \leq 1$. However, since $|T_s(M)| > 1$ is assumed, this leads to a contradiction. As a consequence, we have $T_s(Y_M) = \emptyset$.

If we assume $T_s(-Y_M) \neq \emptyset$, then a similar argument gives $|T_s(M)| \leq 1$, which contradicts to the assumption $|T_s(M)| > 1$. (In this case, use $T_s(Y) \neq \emptyset$ and $W_M$ instead of $T_s(-Y) \neq \emptyset$ and $-W_M$.) This completes the proof. □

4 Chern-Simons invariants

In this section, we give a method for finding an infinite family $\{M_k\}$ such that $\{Y_{M_k}\}$ are mutually disjoint. The goal of this section is to prove the following proposition. Here we denote the $(p,q,r)$-Brieskorn sphere by $\Sigma(p,q,r)$.

Proposition 4.1. Let $Y$ be a rational homology 3-sphere, $p, q$ coprime integers, $M_n := \Sigma(p,q,pqn - 1)$ and $Y_n := Y_{M_n}$ a rational homology 3-sphere given by Proposition 1.2. Then there exists a numerical sequence $\{n_k\}_{k=1}^\infty$ such that if $k \neq k'$, then $Y_{n_k}$ is diffeomorphic to neither $Y_{n_{k'}}$ nor $-Y_{n_{k'}}$.

In [3], it is shown that $|T_s(\Sigma(p,q,pqn - 1))| > 1$. Hence we can apply Proposition 3.1 to $Y_{M_n}$ whenever $Y$ satisfies $T_s(Y) \neq \emptyset$ and $T_s(-Y) \neq \emptyset$. In order to prove Proposition 4.1, we use the Chern-Simons invariants for 3-manifolds. Here we recall the Chern-Simons invariants. For a given 3-manifold $Y$, let $P_Y$ be the product $SO(3)$ bundle. We introduce several definitions which are used for gauge theory. We denote by $\text{Map}(Y,SO(3))$ the set of smooth maps from $Y$ to $SO(3)$. The group structure on $SO(3)$ induces a group structure on $\text{Map}(Y,SO(3))$. Let $A_Y$ be the set of $SO(3)$-flat connections on $P_Y$. Since $\text{Map}(Y,SO(3))$ can be identified with the set of automorphisms on $P_Y$,
Map\((Y, SO(3))\) acts on \(A_Y^f\) by the pull-back of connections. The set of \(SO(3)\)-connections on \(P_Y\) can be identified with the \(so(3)\)-valued 1-forms on \(Y\). Therefore we regard any element of \(A_Y^f\) as an element of \(\Omega^1(Y) \otimes so(3)\). Under these identifications, the action of Map\((Y, SO(3))\) on \(A_Y^f\) is written:
\[
g^*a = g^{-1}dg + g^{-1}ag,
\]
where \(a \in \Omega^1(Y) \otimes so(3)\). This action defines the quotient
\[
R(Y) := A_Y^f/\text{Map}(Y, SO(3)).
\]
Then the Chern-Simons functional \(\tilde{cs} : A_Y^f \to \mathbb{R}\) is defined by
\[
\tilde{cs}(a) = \frac{1}{8\pi^2} \int_Y \text{Tr}(a \wedge da + \frac{2}{3}a \wedge a \wedge a),
\]
where \(a \in \Omega^1(Y) \otimes so(3)\). It is known that
\[
\tilde{cs}(g^*a) = \tilde{cs}(a) + \text{deg}(g),
\]
where \(g \in \text{Map}(Y, SO(3)), a \in \Omega^1(Y) \otimes so(3)\) and \(\text{deg}(g)\) is the mapping degree of \(g\). Therefore the map \(\tilde{cs}\) descends the map:
\[
cs : R(Y) \to \mathbb{R}/\mathbb{Z}.
\]
Since the space \(R(Y)\) is compact and the map \(cs\) is locally constant, one can show the set \(\text{Im } cs \subset \mathbb{R}/\mathbb{Z}\) is a finite set.

By using the Chern-Simons functional, Furuta \(\text{[7]}\) defines a numerical invariant \(\epsilon\) as follows. (In \(\text{[6]}\), Fintushel and Stern also consider such an invariant.) Here we identify \((0, 1]\) with \(\mathbb{R}/\mathbb{Z}\) via the quotient map \(\mathbb{R} \to \mathbb{R}/\mathbb{Z}\) and regard \(cs\) as a map from \(R(Y)\) to \((0, 1]\).

**Definition 4.2.** For a 3-manifold \(Y\), we define
\[
\epsilon(Y) := \begin{cases} 
\min_{a \in cs^{-1}(0,1]} cs(a) & (cs^{-1}(0,1) \neq \emptyset) \\
1 & (cs^{-1}(0,1) = \emptyset)
\end{cases}
\]
There is a connected sum inequality for \(\epsilon\) stated as follows.

**Lemma 4.3.** For any two 3-manifolds \(Y_1\) and \(Y_2\), we have
\[
\epsilon(Y_1 \# Y_2) \leq \min\{\epsilon(Y_1), \epsilon(Y_2)\}.
\]

**Proof.** For proving the lemma, it suffices to prove that \(\epsilon(Y_1 \# Y_2) \leq \epsilon(Y_1)\). Let \(\rho\) be an \(SO(3)\) flat connection on \(Y_1\) satisfying \(cs(\rho) = \epsilon(Y_1)\) and \(\theta\) the product connection on \(Y_2\). By taking the connected sum of \(\rho_M\) and \(\theta\), we get an \(SO(3)\) flat connection \(\rho \# \theta\) over \(Y_1 \# Y_2\). Then it follows from the definitions of \(cs\) and \(\epsilon\) that
\[
\epsilon(Y_1 \# Y_2) \leq cs(\rho \# \theta) = cs(\rho) = \epsilon(Y_1).
\]

\(\square\)
Next, we prove the following lemma. This lemma says that if we have a nice cobordism, then we can estimate the value of $\epsilon$.

**Lemma 4.4.** Let $Y_1$ and $Y_2$ be 3-manifolds. Suppose that there is a cobordism $W$ from $Y_1$ to $Y_2$ such that $i_* : \pi_1(Y_1) \to \pi_1(W)$ is bijective. Then the inequality

$$
\epsilon(Y_2) \leq \epsilon(Y_1)
$$

holds.

**Proof.** Suppose that $\rho$ is a $SO(3)$ flat connection satisfying $cs(\rho) = \epsilon(Y_1)$. Since $\pi_1(Y_1) \to \pi_1(W)$ is bijective, we can extend $\rho$ over $W$ using the holonomy correspondence. We denote the extended connection by $\tilde{\rho}$. Then the equalities

$$
0 = \frac{1}{8\pi^2} \int_W \text{Tr}(F(\rho) \wedge F(\rho)) = cs(\rho) - cs(\tilde{\rho}|_{Y_2})
$$

hold. Therefore, we have

$$
\epsilon(Y_2) \leq cs(\tilde{\rho}|_{Y_2}) = cs(\rho) = \epsilon(Y_1).
$$

In our situation, we have the following estimate for $\epsilon(Y_M)$.

**Corollary 4.5.** For any $Y$ and $M$, we have

$$
\epsilon(Y_M) \leq \epsilon(M).
$$

**Proof.** By applying Lemma 4.4 to $W_M$, we have

$$
\epsilon(Y_M) \leq \epsilon(Y \# M \# (-M)) \leq \epsilon(M),
$$

where the second inequality follows from Lemma 4.3.

Now, let us prove Proposition 4.1.

**Proof of Proposition 4.1.** It is proved by Furuta [7] and Fintushel-Stern [6] that

$$
\epsilon(M_n) = \frac{1}{pq(pqm - 1)}.
$$

Therefore, it follows from Corollary 4.5 that for any $n$, we have

$$
\epsilon(Y_n) \leq \epsilon(M_n) = \frac{1}{pq(pqm - 1)}.
$$

We construct a numerical sequence $\{n_k\}_{k=1}^\infty$ by induction. First, we define $n_1 := 1$. Next, suppose that $\{n_k\}_{k=1}^m$ is defined for some $m$. Since $\frac{1}{pq(pqm - 1)} \to 0$ ($n \to \infty$), there exists an integer $n$ such that

$$
\frac{1}{pq(pqm - 1)} < \min_{1 \leq k \leq m} \{\epsilon(Y_{n_k}), \epsilon(-Y_{n_k})\}.
$$
Then we define \( n_{m+1} := n \).

Now, let us prove that \( \{n_k\}_k^\infty \) is the desired sequence. Suppose that \( k \neq k' \). Without loss of generality, we may assume that \( k > k' \). Then, by the definition of \( \{n_k\}_k^\infty \), the inequalities

\[
\epsilon(Y_{n_k}) \leq \frac{1}{pq(pqn_k - 1)} < \min_{i < k} \{\epsilon(Y_{n_i}), \epsilon(-Y_{n_i})\}
\]

hold. In particular, since \( k' < k \), we have

\[
\epsilon(Y_{n_k}) < \min\{\epsilon(Y_{n_{k'}}), \epsilon(-Y_{n_{k'}})\}.
\]

This proves that \( Y_{n_k} \) is diffeomorphic to neither \( Y_{n_{k'}} \) nor \(-Y_{n_{k'}}\). \qed

5 Proof of Main Theorem

In this section, we prove Theorem 1.1, which is stated as follows.

\textbf{Theorem 1.1.} For any rational homology 3-sphere \( Y \) satisfying \( \mathcal{T}_s(Y) \neq \emptyset \) and \( \mathcal{T}_s(-Y) \neq \emptyset \), there exist infinitely many rational homology 3-spheres \( \{Y_k\}_k^\infty \) which satisfy the following conditions.

1. \( Y_k \) is homology cobordant to \( Y \).
2. \( \mathcal{T}(Y_k) = \mathcal{T}(Y) \neq \emptyset \) and \( \mathcal{T}(-Y_k) = \mathcal{T}(-Y) \neq \emptyset \).
3. \( \mathcal{T}_s(Y_k) = \emptyset \) and \( \mathcal{T}_s(-Y_k) = \emptyset \).
4. If \( k \neq k' \) then \( Y_k \) is diffeomorphic to neither \( Y_{k'} \) nor \(-Y_{k'}\).
5. Each \( Y_k \) is irreducible and toroidal.

\textit{Proof.} Let \( M_n := \Sigma(2,3,6n-1) \), \( Y_{M_n} \) be a rational homology 3-sphere given by Proposition 1.2 and \( \{n_k\}_k^\infty \) a numerical sequence given by Proposition 4.1. Then we define \( Y_k := Y_{M_{n_k}} \). Let us prove that \( \{Y_k\}_k^\infty \) is the desired family.

First, since \( Y_k \) is homology cobordant to \( Y \# M_{n_k} \# (-M_{n_k}) \) and \( M_{n_k} \) is an integer homology 3-sphere, \( Y_k \) is homology cobordant to \( Y \) for any \( k \).

Second, since \( \mathcal{T} \) is a homology cobordism invariant, \( \mathcal{T}(Y) \supset \mathcal{T}_s(Y) \), and we assume that \( \mathcal{T}_s(Y) \neq \emptyset \) and \( \mathcal{T}_s(-Y) \neq \emptyset \), both \( \mathcal{T}(Y_k) = \mathcal{T}(Y) \neq \emptyset \) and \( \mathcal{T}(-Y_k) = \mathcal{T}(-Y) \neq \emptyset \) hold for any \( k \).

Third, since \( |M_{n_k}| > 1 \) holds for any \( k \), it follows from Proposition 3.1 that \( \mathcal{T}_s(Y_k) = \emptyset \) and \( \mathcal{T}_s(-Y_k) = \emptyset \).

Fourth, it follows from Proposition 4.1 that if \( k \neq k' \) then \( Y_k \) is diffeomorphic to neither \( Y_{k'} \) nor \(-Y_{k'}\).

Finally, it follows from Proposition 1.2 that each \( Y_k \) is irreducible and toroidal. This completes the proof. \qed
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