ON BV-EXTENSION OF ASYMPTOTICALLY CONSTRAINED
CONTROL-AFFINE SYSTEMS
AND COMPLEMENTARITY PROBLEM
FOR MEASURE DIFFERENTIAL EQUATIONS

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Abstract. The goal of the paper is to design a constructive impulsive trajectory extension for a class of control-affine dynamical systems subject to an asymptotic mixed constraint of complementarity type. An inspiration for the addressed models comes from the framework of Lagrangian mechanical systems with impactively blockable degrees of freedom. The constraint formalizes the requirement that “control actions steer the system’s state from one prescribed configuration $Z^-$ to another one $Z^+$.” This issue is also closely connected with the problem of continuous trajectory approximation of hybrid systems with control switches.

1. Introduction. This study springs from an analysis of a specific complementarity problem for impulsive, measure-driven dynamical systems, first stated in [17] for the simplest scalar case and further generalized in [18] for the case of vector-valued measures, in respect of optimal control issues. Given a finite time interval $T \triangleq [0, T]$, a positive real $M$, a vector $x_0 \in \mathbb{R}^n$, closed sets $Z^{-} \subseteq \mathbb{R}^n$, and functions $f: \mathbb{R}^n \to \mathbb{R}^n$, $G: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^{m}$, such a problem takes the form

$$\begin{align*}
\frac{dx}{dt} &= f(x(t)) \, dt + G(x(t)) \, \vartheta(dt), \quad x(0^-) = x_0, \\
|\vartheta|(T) &\leq M, \\
x(t^-) &\in Z^- \text{ and } x(t) \in Z^+ \quad |\vartheta|-\text{a.e.}
\end{align*}$$

Here, $\vartheta$ is said to be impulsive control, which is, roughly, an $m$-dimensional Borel measure with the total variation $|\vartheta|$, $x(t^-)$ stands for the left one-sided limit of a function $x(\cdot)$ at a point $t \in T$, and “a.e.” abbreviates “almost everywhere” with respect to a measure. Solutions $x = x(\cdot)$ of the Cauchy problem (1) – to be defined in Section 2 – are right continuous functions $T \to \mathbb{R}^n$ of bounded variation (we write $x \in BV^+(T, \mathbb{R}^n)$ and abbreviate $BV^+(T, \mathbb{R}^n)$, if possible, as $BV$). Note that measure differential equation (1), (2) performs an impulsive trajectory relaxation.

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in the weak* topology of \( BV \) of the ordinary control system
\[
\frac{dx(t)}{dt} \triangleq \dot{x}(t) = f(x(t)) + G(x(t)) u(t), \quad x(0) = x_0, \quad \|u\|_{L_1} \leq M.
\]
(5)

Here, trajectories \( x = x(\cdot) \) are absolutely continuous functions \( T \rightarrow \mathbb{R}^n \), controls \( u = u(\cdot) \) are measurable maps \( T \rightarrow \mathbb{R}^m \); \( \|\cdot\|_{L_1} \) denotes the norm in the Lebesgue quotient space of summable functions.

Yet another important note is as follows: a solution to the complementary problem (1)–(3) — if exists — is generically non-unique [27]. Furthermore, though the trajectory tube of (1), (2) is compact in the weak* topology of \( BV \), the set of solutions to (1)–(3) is not expected to be closed.

For constitutive results on control theory with affine impulses one can review [1, 5, 11, 12–14, 19, 20, 26, 27, 29–35] and the bibliography therein.

**Impulsive control.** We recall the notion of impulsive control of a nonlinear measure differential equation (for details and motivation see, e.g., [23]).

If the matrix function \( G \) in (1) possesses the well-known Frobenius commutativity property, \( \vartheta \) is indeed a vector-valued measure. In general, \( \vartheta \) has to be thought of as a collection
\[
\vartheta \triangleq (\mu, \nu, \{u_\tau\}_{\tau \in D_\nu}),
\]

where \( \mu \) is an actual \( \mathbb{R}^m \)-valued control measure on \( T \); \( \nu \) is a scalar nonnegative measure being a majorant of the total variation \( |\mu| \) of \( \mu \), and \( \{u_\tau\}_{\tau \in D_\nu} \), \( D_\nu \triangleq \{\tau \in T : \nu(\{\tau\}) \neq 0\} \), is a family of Borel measurable functions \( u_\tau : T_\tau \triangleq [0, \nu(\{\tau\})] \rightarrow \mathbb{R}^m \), parameterized by atoms \( \tau \) of the measure \( \nu \), with the properties:
\[
|u_\tau(\theta)| = 1 \lambda\text{-a.e. on } T_\tau, \quad \int_{T_\tau} u_\tau(\theta) \, d\theta = \mu(\{\tau\}), \quad \tau \in D_\nu
\]

(\( \lambda \) denotes the usual Lebesgue measure on \( \mathbb{R} \)). We put \( |\theta| \triangleq \nu \) and call \( |\theta| \) the total variation of impulsive control; the set of impulsive controls is denoted by \( \Theta \).

**Complementarity property. Connection with the framework of hybrid systems.** Condition (3) says that a control action is permitted only when a state belongs to the set \( \mathcal{Z}_- \), and any such action has to steer the state to the set \( \mathcal{Z}_+ \). Following the tradition of hybrid systems theory [3], the sets \( \mathcal{Z}_\pm \) are said to be the jump permitting and jump destination sets, respectively. Assumed that \( \mathcal{Z}_\pm \) are represented as \( \mathcal{Z}_\pm = \{x \in \mathbb{R}^n : W_\pm(x) = 0\} \), where \( W_\pm \) are nonnegative continuous functions \( \mathbb{R}^n \rightarrow \mathbb{R} \), one can rewrite (3) in the following equivalent form:
\[
\int_T \left[ W_-(x(t)) + W_+(x(t)) \right] |\theta|_c (dt) = 0, \quad \text{and}
\]
\[
W_-(x(\tau^-)) + W_+(x(\tau)) = 0 \quad \forall \tau \in D_{|\theta|},
\]
(6)
(7)

where \( |\theta|_c \) denotes the continuous part of \( |\theta| \). Conditions (6), (7) motivate the usage of the term “complementarity”. One observes that, if \( |\theta| \) is absolutely continuous (that is, \( |\theta|(dt) = |u(t)| \, dt \) with \( u \in L_\infty \), in respect of vectors, \( |\cdot| \) is an appropriate finite-dimensional norm), then (3) reduces to orthogonality of \( u \) with respect to the composition \( W_+ + W_- \circ x \) in \( L_2 \).

Our goal is to provide an approximation of system (1)–(3) by solutions to the ordinary counterpart (4), (5) of the measure differential equation (1), (2). Towards this, we will introduce a special concept of quasi-solution to problem (1)–(3). We will show that a closure of the set of quasi-solutions in the weak* topology of
BV absorbs all solutions to the complementarity problem (1)–(3), and therefore performs its desired compactification.

Mechanical background. Impactively blockable degrees of freedom as impulsive actions. A motivation for studying systems (1)–(3) mainly stems from two frameworks: 1) a close connection of impulsive control theory with hybrid systems (here we refer to [15,21,24]), and 2) control of mechanical systems.

A situation, where affine impulsive actions naturally appear, is due to impact mechanics and contact dynamics, say, elastic collisions with unilateral constraints (the boundary of these constraints is the set $\mathcal{Z}$ in our formalism), or dry friction. A progress in this area is made in [4,28]. Another relevant mechanical framework are Lagrangian systems with impactively blockable degrees of freedom [16,25,36], where the second condition in (3) does appear. In [36] we find the following motivating example: The dynamical system represents the motion of a double pendulum with links of unit length and mass in the vertical plane. The system has two degrees of freedom being angular positions $\varphi = (\varphi_1, \varphi_2)$ of the links. The dynamics is described by the Lagrange equation

$$
\begin{pmatrix}
1 & 1/2 \cos \Delta \varphi \\
\cos \Delta \varphi & 1
\end{pmatrix}
\begin{pmatrix}
d\dot{\varphi}_1 \\
d\dot{\varphi}_2
\end{pmatrix}
+ \begin{pmatrix}
1/2 \sin \Delta \varphi \varphi_2^2 \\
- \sin \Delta \varphi \varphi_1^2
\end{pmatrix}
dt
+ g \begin{pmatrix}
\sin \varphi_1 \\
\sin \varphi_2
\end{pmatrix}
dt = \begin{pmatrix}
0 \\
\mu(dt)
\end{pmatrix}.
$$

Here, $\Delta \varphi = \varphi_1 - \varphi_2$, and $g$ denotes the acceleration due to gravity. One can control the pendulum by instantaneously blocking/releasing the joint between the two links. Such blocking is provided by an impulsive force, formalized by a scalar signed Borel measure $\mu$. This can be formally written as

$$
\dot{\varphi}_1(t) - \dot{\varphi}_2(t) = 0 \quad |\mu|-a.e.
$$

A common difficulty faced in the control theory of mechanical systems is as follows: physical quantities (typically, forces or torques) conventionally regarded as control inputs are not technically observable signals. Practically, it is convenient to operate with “integrals” of these controls (certain state coordinates regarded as system’s inputs [9]), or design control signals based on the knowledge of their influence on the system. Impactively blockable degrees of freedom provide a good idealization of mechanical systems, where control forces produce prescribed influence on the system.

2. Quasi-solutions of the complementarity problem for measure differential equation. We make the following assumptions (H): The functions $f$ and $G$ are uniformly Lipschitz continuous.

Recall the notion of solution to measure differential equation (1) [1, 27]. Given an impulsive control $\vartheta = (\mu, \nu, \{u_\tau\}) \in \Theta$, by a solution to the Cauchy problem (1) under the input $\vartheta$ we mean a function $x \in BV^+(\mathcal{T}, \mathbb{R}^n)$ which turns the following relation into identity:

$$
x(t) = x_0 + \int_0^t f(x(\theta)) d\theta + \int_0^t G(x(\theta)) \mu_c(d\theta) + \sum_{\tau \in D_\nu} [x(\tau) - x(\tau^-)], \quad t \in \mathcal{T}. \quad (8)
$$

The integral with respect to the continuous part $\mu_c$ of the measure $\mu$ is understood in the Lebesgue-Stieltjes sense; values $x(\tau)$ of a function $x$ at jump points $\tau \in D_\nu$.
are defined as \( x(\tau) = \sigma_\tau(T_\tau) \), where \( \sigma_\tau \) is a Carathéodory solution of the limit control system [19,20]
\[
\frac{d}{d\varsigma} x(\varsigma) = G(x(\varsigma)) u(\varsigma), \quad x(0) = x(\tau)^-, \quad \varsigma \in T_\tau.
\]

Assumptions (H) ensure the existence and uniqueness of a solution \( x[\vartheta] \) to (8), (9) for any input \( \vartheta \in \Theta \) (see, e.g., [27]).

Consider ordinary control system (4), (5). Let \( U \) denote the set of controls \( u \in L_1 \cap L_\infty \) satisfying (5), and \( x[u] \) stand for a Carathéodory solution of (4) under a control \( u \in U \). A pair \( \sigma \equiv (x,u) \) with \( u \in U \) and \( x = x[u] \) is called a control process.

Before passing to the notion of quasi-solution, we state that condition (3) cannot be trivially transferred to the framework of ordinary system (4), (5). The following simple example shows that problem (4)–(6) commonly lacks not only exact solutions, but even approximate ones.

On the interval \( [0,1] \), consider the scalar system, where control has to drive a state to the singleton \( \mathcal{Z}_+ = \{1\} \):
\[
\dot{x}(t) = u(t), \quad x(0) = 0; \quad u \geq 0, \quad \int_{[0,1]} u(t) \, dt \leq 1;
\]
\[
I \equiv \int_{[0,1]} (1 - x(t)) u(t) \, dt = 0.
\]

(Although in (4), (5) there were no conical constraints on control, here we deal with nonnegative inputs. This is not a matter of principle, see Conclusions. The function \( W_+ = 1 - x \) is nonnegative for all admissible states \( x(t) \). For any control with \( \|u\|_{L_1} = 1 \), we get
\[
I = 1 - \frac{x^2(1)}{2} = 1/2,
\]
i.e., the complementarity condition never holds, even approximately. Still, it holds in a certain weak sense. Introduce a more cautious definition of quasi-solution to problem (1)–(3):

**Definition 2.1.** Given \( \varepsilon > 0 \), a control process \( \sigma = (x,u) \), \( u \in U, x = x[u], \) is said to be an \( \varepsilon \)-solution of (1)–(3), if there exists another control process \( \tilde{\sigma} \equiv (\tilde{x},\tilde{u}), \)
\( \tilde{u} \in U, \tilde{x} \equiv x[\tilde{u}], \) with the properties:
\[
F[\tilde{u}] \leq F[u],
\]
\[
\left\| (F[u],x) - (F[\tilde{u}],\tilde{x}) \right\|_{L_1} \leq \varepsilon,
\]
\[
\left| (F[u],x)(T) - (F[\tilde{u}],\tilde{x})(T) \right| \leq \varepsilon,
\]
\[
\int_T \left[ W_-(\tilde{x}(t)) \left| u(t) \right| + W_+(x(t)) \left| \tilde{u}(t) \right| \right] \, dt \leq \varepsilon.
\]

Here, \( F_f \) stands for the primitive of a function \( f \). Hereinafter, as the norm \( \| \cdot \| \) in \( \mathbb{R}^n \), we fix the Manhattan norm \( \| \cdot \|_1 \).

Note that conditions (11)–(13) can be rewritten symmetrically due to the permutation \( \sigma \leftrightarrow \tilde{\sigma} \). Condition (10) performs a certain ordering \( \tilde{\sigma} \preceq \sigma \), which serves to recognize – in the limit – jumps from \( \mathcal{Z}_- \) to \( \mathcal{Z}_+ \) compared to jumps from \( \mathcal{Z}_- \) to
condition (13) is a relaxed version of (6), (7), saying that states of the ordinary control system should shift from $\mathcal{Z}_-$ to $\mathcal{Z}_+$ “fast enough”, and such fastness is measured by $\varepsilon$.

Let us return to our example. Consider the two families of control processes:

$$
\sigma_\varepsilon(t) = \begin{cases} 
\left( \frac{1}{\varepsilon} t, 1 \varepsilon \right), & t \in [0, \varepsilon), \\
(1, 0), & t \in [\varepsilon, 1], 
\end{cases} \quad \tilde{\sigma}_\varepsilon(t) = \begin{cases} 
(0, 0), & t \in [0, \varepsilon), \\
\left( \frac{1}{\varepsilon} t - 1, 1 \varepsilon \right), & t \in [\varepsilon, 2\varepsilon), \\
(1, 0), & t \in [2\varepsilon, 1]. 
\end{cases}
$$

Clearly, these processes satisfy the assumptions of Definition 2.1, in particular, condition (13) reduces to

$$
\int_{[0,1]} (1 - x_\varepsilon(t)) \tilde{u}_\varepsilon(t) \, dt = 0.
$$

Given $\varepsilon > 0$, denote by $\mathcal{X}_\varepsilon$ the set of control processes $\sigma$ being $\varepsilon$-solutions of (1)–(3).

Now we consider sequences $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ of control processes $\sigma_\varepsilon \triangleq (x, u)_\varepsilon$, $u_\varepsilon \in U$ and $x_\varepsilon \triangleq x[u_\varepsilon]$, which “tend to fulfill constraint (3) in an asymptotic sense”, i.e., in the limit, as $\varepsilon \to 0$.

Let $\mathcal{X}$ denote the set of functions $x \in BV^+(T, \mathbb{R}^n)$ with $x(0^-) = x_0$, such that there exists a sequence $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ of control processes with the properties:

- for each $\varepsilon > 0$, $\sigma_\varepsilon$ is an $\varepsilon$-solution of (1)–(3) in the sense of Definition 2.1,
- $x_\varepsilon \rightharpoonup x$ as $\varepsilon \to 0$.

Here, $\rightharpoonup$ denotes the convergence in the weak* topology of $BV$ (i.e., at points of continuity and at $t = T$).

Our forthcoming goal is to give a constructive representation of $\mathcal{X}$ by an ordinary differential inclusion.

3. **Trajectory extension by discontinuous space-time transform.** As is conventional in the impulsive control theory, a desired representation of $\mathcal{X}$ is based on a specific singular space-time transform. The idea goes back to the so-called discontinuous time change [30, 32, 34]: one can make a certain Lipschitzian parameterization of the time variable $t$ with a consequent relaxation (convexification) of the system’s velocity set, which performs a trajectory compactification. The inverse (in general, discontinuous) reparameterization then gives the set of generalized states, which could be discontinuous functions of bounded variation.

In the presence of condition (3), a proper system transformation requires also a certain conversion of the state space, i.e., it is an actual space-time reparameterization. To give the reader an intuition, we note that, in the definition of the set $\mathcal{X}$, one, actually, operates with two families of control processes: the sequence $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ itself, and its perturbation $\{\tilde{\sigma}_\varepsilon\}_{\varepsilon > 0}$ provided by Definition 2.1. One can suspect that, in the asymptotic sense, $\tilde{x}$ plays the part of the left one-sided limit of a generalized solution, and $x$ becomes the right one-sided limit.

Consider the following auxiliary reduced ordinary control system with a new time variable $s$, acting on an interval $\mathcal{S} \triangleq [0, S]$, $S \geq T$:

$$
\frac{dx(s)}{ds} = F(x(s), u(s)), \quad x(0) = x_0, \quad u \in U,
$$

(14)

(15)
where \( x \triangleq (\xi, y \triangleq (y_+, y_-), \eta \triangleq (\eta_+, \eta_-), \zeta) \), \( u \triangleq (\alpha, \beta \triangleq (\beta_+, \beta_-)) \), the initial state \( x_0 \triangleq (0, x_0, 0, 0, 0) \), and the right-hand side is of the form

\[
F \triangleq \begin{pmatrix}
\alpha \\
\alpha f(y_+) + G(y_+)\beta_+ \\
\alpha f(y_-) + G(y_-)\beta_- \\
|\beta_+| \\
|\beta_-|
\end{pmatrix},
\]

subject to the following terminal conditions

\[\xi(S) = T, \quad \Delta^\pm(y, \eta)(S) = 0 \in \mathbb{R}^{n+1}, \quad \eta_+(S) \leq M, \quad \zeta(S) = 0, \quad (16)\]

and state constraint

\[\Delta^\mp \eta \leq 0. \quad (17)\]

The control set \( U \triangleq U(S) \) consists of Borel measurable vector functions \( u \) with components \( \alpha : S \to [0, 1], \beta_\pm : S \to \mathbb{R}^m \), such that \( \alpha(s) + |\beta_+(s)| + |\beta_-(s)| = 1 \) \( \lambda \)-a.e. on \( S \) (\( \lambda \) denotes the usual Lebesgue measure). The operation \( \Delta^\pm \) applied to a vector \( c \in \mathbb{R}^{2m} \) defines the vector \( c_+ - c_- \in \mathbb{R}^m \), and \( \Delta^\mp = -\Delta^\pm \).

Note that the dimension of the state space of the reduced system is \( 2n + 4 \) (the dimension of the original system (1)–(3) was \( n \)), and the dimension of control vector is \( 2m + 1 \) (originally, it was \( m \)).

By \( x[u] \) we denote the Carathéodory solution of system (14) on \( S \) corresponding to a control \( u \in U \).

Given an impulsive controls \( \vartheta \in \Theta \), put \( \mu \triangleq \lambda + 2|\vartheta| \) and introduce a strictly increasing function \( \Upsilon : T \to [0, \mu(T)] \), \( \Upsilon(t) = F_\mu(t), \quad t \in T \). Let \( v : [0, \mu(T)] \to T \) denote the inverse of \( \Upsilon \).

In [18], systems (1)–(3) and (14)–(17) are proved to be equivalent to each other in the following sense:

**Theorem 3.1.** 1) Let \( \vartheta \) satisfy (2) and be such that the solution \( x = x[\vartheta] \) of measure differential equation (1) meets condition (3). Then, there are a real \( S \supseteq T \) and a control \( u \in U(S) \) such that the trajectory \( x = x[u] \) of control system (14), (15) satisfies (16), (17), and

\[y_\circ \Upsilon = y_\circ \Upsilon = x, \quad v = \xi. \quad (18)\]

2) Assume that \( S \supseteq T \) and \( u \in U(S) \) are such that the respective solution \( x \triangleq (\xi, y, \eta, \zeta)[u] \) of system (14), (15) satisfies (16), (17). Define the function \( x \in BV^+(T, \mathbb{R}^n) \) by the composition

\[x = y_\circ \Xi \quad \text{on} \quad T, \quad (19)\]

where the map \( \Xi : T \to S \) is defined by

\[\Xi(t) = \inf \{ s \in S \mid \xi(s) > t \}, \quad t \in [0, T], \quad \Xi(T) = S.\]

Then, \( x \) satisfies (1) together with constraint (3).

Consider the standard relaxation of the reduced dynamics (14), (15):

\[\frac{dx(s)}{ds} \in \overline{F(x, u)} \cup \{ u \in U \}, \quad x(0) = x_0. \quad (20)\]
The main result is presented by the following

**Theorem 3.2.** The set $\mathcal{X}$ coincides with the trajectory tube of (20), (16), (17) up to a discontinuous time change. More precisely:

1) For any $x \in \mathcal{X}$, there exist $S \geq T$ and a solution $x = (\xi, y_+, y_-, \eta_+, \eta_-, \zeta)$ of the constrained differential inclusion (20), (16), (17) such that (18) holds.

2) Let $x = (\xi, y_+, y_-, \eta_+, \eta_-, \zeta)$ be a solution to the Cauchy problem for differential inclusion (20) on a time interval $S = [0, S]$, $S \geq T$, such that conditions (16), (17) hold. Define $x$ by (19). Then, $x \in \mathcal{X}$.

**Proof.**

1) Given $\varepsilon > 0$, consider ordinary control processes $u_\varepsilon = (x, u)_\varepsilon$ and $u_\varepsilon = (\bar{x}, \bar{u})_\varepsilon$ from Definition 2.1, which approximate the impulsive process $(x, \vartheta)$. Introduce the function

$$\Upsilon_\varepsilon(t) = t + F|_{u_\varepsilon}(t) + F|_{\bar{u}_\varepsilon}(t), \quad t \in T,$$

and assume that $u_\varepsilon$ is its inverse. Set $S_\varepsilon = \Upsilon_\varepsilon(T)$. On time intervals $S_\varepsilon = [0, S_\varepsilon]$, define controls $u_\varepsilon = (\alpha, \beta_+, \beta_-)_\varepsilon$ as follows:

$$\alpha_\varepsilon = (1/\Upsilon_\varepsilon) \circ u_\varepsilon, \quad \beta_+\varepsilon = (u_\varepsilon \circ u_\varepsilon) \cdot \alpha_\varepsilon, \quad \beta_-\varepsilon = (\bar{u}_\varepsilon \circ u_\varepsilon) \cdot \alpha_\varepsilon.$$

It is easily checked that $u_\varepsilon \in \mathcal{U}(S_\varepsilon)$.

Let $x_\varepsilon = (\xi, y_+, y_-, \eta_+, \eta_-, \zeta)_\varepsilon$ be the associated solution to system (14). Note that $(F|_{u_\varepsilon}, x_\varepsilon) = (\eta, y)_{+\varepsilon} \circ \Upsilon_\varepsilon$ and $(F|_{\bar{u}}, x_\varepsilon) = (\eta, y)_{-\varepsilon} \circ \Upsilon_\varepsilon$.

Put $S \triangleq \sup\{S_\varepsilon | \varepsilon > 0\}$ (clearly, $S \leq T + 2M$), $\mathcal{S} \triangleq [0, S]$, and define functions $x_\varepsilon : \mathcal{S} \to \mathcal{X}$ as follows: $x_\varepsilon(s) = x_\varepsilon(s)$ on $S_\varepsilon$ and $x_\varepsilon(s) = x_\varepsilon(S_\varepsilon)$ on $(S_\varepsilon, S]$.

Since $u_\varepsilon, \bar{u}_\varepsilon \in \mathcal{U}$, the functions $x_\varepsilon$, $\varepsilon > 0$, satisfy the third constraint in (16), and their $\xi$- and $(\eta_+, \eta_-)$-components meet the conditions: $\hat{\xi}_\varepsilon(S_\varepsilon) = T$ and $|\Delta^\pm \hat{\eta}_\varepsilon(S_\varepsilon)| \leq \varepsilon$ by (12). Furthermore, thanks to (10), their $(\eta_+, \eta_-)$-components fulfill state constraint (17). Due to property (11) and the change of variable $t = \hat{\xi}_\varepsilon(s)$ (recall that the map $s \mapsto \hat{\xi}_\varepsilon(s)$ is strictly increasing on $S_\varepsilon$ and therefore it is a bijection from $T$ to $S_\varepsilon$), we get:

$$\int_{S_\varepsilon} \alpha_\varepsilon [\Delta^\pm \hat{\eta}_\varepsilon + |\Delta^\pm \hat{\gamma}_\varepsilon|] \, ds = \| (F|_{u_\varepsilon}, x_\varepsilon) - (F|_{\bar{u}}, x_\varepsilon) \|_{L_1} \leq \varepsilon.$$

Similarly, from (13) we extract:

$$\int_{S_\varepsilon} \left[ W_-(\hat{\gamma}-\varepsilon) |\beta_-\varepsilon| + W_+(\hat{\gamma}+\varepsilon) |\beta_+\varepsilon| \right] \, ds \leq \varepsilon.$$

Combining the two latter estimates, we conclude that $\zeta(S) = \zeta(S_\varepsilon) \leq 2\varepsilon$. It remains to notice that (12) guarantees the inequality $|\Delta^\pm \hat{\gamma}_\varepsilon| \leq \varepsilon$. Thus, $x_\varepsilon$ satisfies on $S$ all the constraints (16), (17) with accuracy of order $\varepsilon$.

Thanks to hypotheses (H), $x_\varepsilon$ are equicontinuous and uniformly bounded. Then, by the Arzelà-Ascoli selection principle, there is a uniformly converging subsequence $\{x_\varepsilon\} \subset \{x_\varepsilon\}$. By the above estimates, we deduce that a limit function $\hat{x}$ satisfies constraints (16), (17) strictly. Furthermore, $\hat{\xi}_\varepsilon(S_\varepsilon) \to \hat{\xi}(S)$, that is $\hat{\xi}(S) = T$ (here $\xi$ is the $\xi$-component of the vector function $\hat{x}$).

By the classical Filippov-Ważewski relaxation theorem [2], $\hat{x}$ is a solution to differential inclusion (20) on $S$. Let $\hat{\xi} = \hat{\xi}^{-1}$, and $\hat{\xi}(t) \triangleq \inf\{s \in S : \hat{\xi}(s) > t\}$, $t \in (0, T)$, $\hat{\xi}(T) = S$. It is simply observed that the $y_+$-part of the vector function $\hat{x}$ satisfies $\hat{y}_+ = \hat{\xi} = x_\varepsilon$. By definition, $x_\varepsilon \to x$, in particular, $x_\varepsilon \to x$. On the other
hand, $\hat{y}+ \circ \hat{\Xi} \to \hat{y}+ \circ \Xi$ (see, e.g., [27, Theorem 2.13]), where $\hat{y}+$ is the $y+$-part of $\hat{x}$. Thus, we conclude that $\hat{y}+ \circ \hat{\Xi} = x$, i.e., (19) does hold.

2) By the advanced density theorem [22], any solution to the state constrained differential inclusion (20), (17) on the interval $S$ can be uniformly approximated on it by trajectories of state constrained control system (14), (15), (17). In other words, for any $\varepsilon > 0$, there exists $u_\varepsilon \hat{=} (\alpha, \beta, \gamma) \in U(S)$ such that $\|x - x_\varepsilon\|_{C} \leq \varepsilon$ with $x_\varepsilon \hat{=} (\xi, y_\varepsilon, y_{-\varepsilon}, \eta_{-\varepsilon}, \xi, t) = x[u_\varepsilon] (\| \cdot \|_C$ denotes the uniform norm in the space of continuous functions). Assumed that $x$ satisfies (16), the following estimations hold for any $\varepsilon > 0$:

$$\|\xi_\varepsilon(S) - T\| \leq \varepsilon,$$

$$\|\Delta^\pm(\eta, y_\varepsilon(S))\| \leq 4 \varepsilon,$$  

$$\zeta_\varepsilon(S) \leq \varepsilon,$$  

$$\|\eta_\varepsilon(S) - M\| \leq \varepsilon,$$

$$\Delta^\pm \eta_\varepsilon \leq 0.$$

By a certain perturbation of $u_\varepsilon$, we define a control $\hat{u}_\varepsilon \hat{=} (\hat{\alpha}, \hat{\beta}_+, \hat{\beta}_-) \in U(S)$, such that, together with the respective solution $\hat{x}_\varepsilon \hat{=} (\hat{\xi}, \hat{y}_\varepsilon, \hat{y}_{-\varepsilon}, \hat{\eta}_{-\varepsilon}, \hat{\xi}, \hat{t})$, it enjoys the properties:

(i) $\hat{\alpha}_\varepsilon > 0$ $\lambda$-a.e. over $S$, and $\hat{\xi}_\varepsilon(S) = T$ and $|\hat{\xi}_\varepsilon(S) - T| \leq 2\varepsilon$;

(ii) $\hat{x}_\varepsilon$ satisfies the remaining terminal constraints (16) accurately within $\varepsilon$, and $\|x - \hat{x}_\varepsilon\|_{C} \sim \varepsilon$ (we abbreviate “has order $\varepsilon$” as “$\sim \varepsilon$”).

For instance, the $\alpha$-part of the desired perturbed control can be defined as $\hat{\alpha}_\varepsilon(s) = \frac{T - \varepsilon \lambda(S^\varepsilon)}{\lambda(S^\varepsilon)} \alpha_\varepsilon(s) + \varepsilon \lambda(S^\varepsilon) \lambda$-a.e. on $S$, where $S^\varepsilon \hat{=} S \setminus \text{supp} \approx \varepsilon_\varepsilon$. As is easily checked, $\hat{\alpha}_\varepsilon$ possesses property (i) and $\|\alpha_\varepsilon - \hat{\alpha}_\varepsilon\|_{L_\infty} \sim \varepsilon$. Let a Borel measurable function $(\hat{\beta}_+, \hat{\beta}_-)$ satisfy the equality $|\hat{\alpha}_\varepsilon + |\hat{\beta}_+\varepsilon| + |\hat{\beta}_-\varepsilon| = 1$ $\lambda$-a.e on $S$. Then, clearly, $\|\hat{\beta}_+\varepsilon - (\hat{\beta}_+\varepsilon, \hat{\beta}_-\varepsilon)\|_{L_\infty} \sim \varepsilon$, as well.

Set $x_\varepsilon = (\xi, y_\varepsilon, y_{-\varepsilon}, \eta_{-\varepsilon}, \xi, \varepsilon) \hat{=} x[u_\varepsilon]$. By vicinity of the respective controls in $L_\infty$ and standard arguments, based on the Gronwall’s inequality, we obtain: $\|x_\varepsilon - \hat{x}_\varepsilon\|_{C} \sim \varepsilon$, and therefore $\|\hat{x}_\varepsilon - x\|_{C} \sim \varepsilon$, which gives property (ii). In its turn, (ii) implies that $\hat{x}_\varepsilon$ converges uniformly to $x$ as $\varepsilon \to 0$. Then,

$$\hat{y}_\varepsilon+ \circ \hat{\Xi} \to y_+ \circ \Xi \hat{=} x,$$

where $\hat{\Xi} \hat{=} \Xi^{-1}$, $\Xi(t) \hat{=} \inf\{s \in S : \xi(s) > t\}$, $t \in [0, T]$, $\Xi(T) = S$, and $(\hat{\xi}, \hat{y}_+\varepsilon, \hat{\xi}, y_{-\varepsilon})$ are the respective parts of the vector functions $\hat{x}_\varepsilon$ and $x$.  

Recall that $\hat{\Xi}$ is designed to be absolutely continuous functions on $T$. Therefore, by setting $x_\varepsilon \hat{=} \hat{y}_\varepsilon+ \circ \Xi$, $\hat{x}_\varepsilon \hat{=} \hat{y}_\varepsilon- \circ \Xi$, $u_\varepsilon \hat{=} \hat{\beta}_+\varepsilon \circ \hat{\Xi}$, and $\hat{u}_\varepsilon \hat{=} \hat{\beta}_-\varepsilon \circ \hat{\Xi}$, we obtain that $x_\varepsilon = x[u_\varepsilon]$ and $\hat{x}_\varepsilon = x[\hat{u}_\varepsilon]$.  

Finally, consider the estimate $\zeta_\varepsilon(S) \sim \varepsilon$, which implies that

$$\varepsilon \sim \int_S \hat{\alpha}_\varepsilon \left[ \Delta^\Delta \hat{\eta}_\varepsilon + |\Delta^\Delta \hat{y}_\varepsilon\varepsilon| \right] ds = \|F[u], x\varepsilon \varepsilon - (F[u], x\varepsilon\varepsilon)\|_{L_1},$$

and

$$\varepsilon \sim \int_S \left[ W_-(\hat{y}_\varepsilon-\varepsilon)|\hat{\beta}_+\varepsilon\varepsilon| + W_+(\hat{y}_\varepsilon+\varepsilon)|\hat{\beta}_-\varepsilon\varepsilon| \right] ds = \int_T \left[ W_- (\hat{x}_\varepsilon) |u_\varepsilon| + W_+(x_\varepsilon) |\hat{u}_\varepsilon| \right] dt$$

(here we use the change of variable $s = \hat{\Xi}(t)$).

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1 In fact, we employ Corollary 3.2 from [22] which is an analog of the Filippov-Ważewski theorem for differential inclusions with states, constrained in a closure of an open subset of the state space. Sufficient hypotheses, which let us apply this result, are $C^2$-regularity of the boundary of the constraint and certain Soner’s type assumption.

2 By Lemma 1, this would imply that $\xi_\varepsilon$ has the inverse $\hat{\Xi}_\varepsilon$, which is absolutely continuous (in particular, $\xi_\varepsilon$ strictly increases).
Thus, we have designed a sequence of ordinary control processes fitting the requirements of Definition 2.1, and approximating $x$ in the weak* topology of $BV$. By definition, $x \in \overline{X}$.

4. Conclusions: Further extension of the result. Theorem 3.2 is straightforwardly extended to systems driven by measures, ranged in a given closed convex cone in $\mathbb{R}^m$, when the dynamics is defined by functions $f = f(t, x(t), w(t))$ and $G = G(t, x(t), w(t))$ with an explicit measurable dependence on $t$ and an extra compact-valued ("ordinary") control $w(\cdot)$. We are confined with the addressed simpler case for the ease of exposition.

A quite more interesting generalization is an extension to systems with quadratic impulses [6–10]. As a motivation, we exhibit the following academic example of a Lagrangian system representing the same double pendulum as in Introduction, but controlled in two ways: (i) by assigning an angle $\varphi_1$ to the first link, and (b) instantaneous blocking/releasing the joint between the links by the force $\mu$, to switch between the double and single pendulum modes. Equations of motion reduce to

$$\left(1 + \sin^2 \Delta \varphi\right) d\dot{\varphi}_2 - \left[2 \sin \Delta \varphi \varphi_2^2 + 1/2 \sin(2\Delta \varphi) \varphi_2^2 - g (\sin \varphi_2 - \sin \varphi_1 \cos \Delta \varphi)\right]dt = \mu(dt).$$

Assume that the two types of control do not act simultaneously. Then the events of blocking/releasing are subject to the condition:

$$\dot{\varphi}_2(t) = 0 \quad |\mu|-a.e.$$  

Now the mapping $t \mapsto \varphi_1(t)$ is a system’s input, but not a state trajectory anymore.

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