A generalization of the Minkowski distance 
and a new definition of the ellipse

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Abstract
In this paper, we generalize the Minkowski distance by defining a new distance function in 
n-dimensional space, and we show that this function determines also a metric family as the 
Minkowski distance. Then, we consider three special cases of this family, which generalize 
the taxicab, Euclidean and maximum metrics respectively, and finally we determine circles 
of them with their some properties in the real plane. While we determine some properties 
of circles of the generalized Minkowski distance, we also discover a new definition for the 
ellipse.

Keywords: Minkowski distance, \( l_p \)-metric, taxicab distance, Manhattan distance, Euclidean 
distance, maximum distance, Chebyshev distance, circle, ellipse, conjugate diameter, eccentric.

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1 Introduction

Beyond the mathematics; distances, especially the well-known Minkowski distance (also known 
as \( l_p \)-metric) with its special cases taxicab (also known as \( l_1 \) or Manhattan), Euclidean (also 
known as \( l_2 \)) and maximum (also known as \( l_\infty \) or Chebyshev) distances, are very important 
keys for many application areas such as data mining, machine learning, pattern recognition and 
spatial analysis (see [1], [3], [9], [12], [13], [18], [19] and [21] for some of related studies).

Here, we generalize the Minkowski distance for \( n \)-dimensional case, and we show that this 
generalization gives a new metric family for \( p \geq 1 \) as the Minkowski distance itself. Then, we 
give some basic distance properties of this generalized Minkowski distance, and we consider the 
new metric family for cases \( p = 1 \), \( p = 2 \) and \( p \to \infty \), which we call the generalized taxicab, 
Euclidean and maximum metrics respectively. Finally, we determine circles of them in the real 
plane. We see that circles of the generalized taxicab and maximum metrics are parallelograms 
and circles of the generalized Euclidean metric are ellipses. While we determine some properties 
of circles of the generalized Euclidean distance, we also discover a new definition for the ellipse, 
which can be referenced by ”two-eccentrices” definition, as the well-known ”two-foci” and ”focus-
directrix” definitions.

Throughout this paper, symmetry about a line is used in the Euclidean sense and angle mea-
surement is in Euclidean radian. Also the terms square, rectangle, rhombus, parallelogram and 
ellipse are used in the Euclidean sense, and center of them stands for their center of symmetry.
2 A generalization of the Minkowski distance

We generalize the Minkowski distance using linearly independent \( n \) unit vectors \( v_1, \ldots, v_n \) and \( n \) positive real numbers \( \lambda_1, \ldots, \lambda_n \), as in the following definition. For the sake of shortness we use notation \( d_{p(v_1, \ldots, v_n)} \), instead of for example \( d_{p(v_1, \ldots, v_n, \lambda_1, \ldots, \lambda_n)} \), for the new distance family, supposing \( \lambda \) weights are initially determined and fixed, and we call it \((v_1, \ldots, v_n)\)-Minkowski distance.

**Definition 2.1** Let \( X = (x_1, \ldots, x_n) \) and \( Y = (y_1, \ldots, y_n) \) be two points in \( \mathbb{R}^n \). For linearly independent \( n \) unit vectors \( v_1, \ldots, v_n \) where \( v_i = (v_{i1}, \ldots, v_{in}) \), and positive real numbers \( p, \lambda_1, \ldots, \lambda_n \), the function \( d_{p(v_1, \ldots, v_n)} : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \) defined by

\[
\begin{align*}
d_{p(v_1, \ldots, v_n)}(X, Y) &= \left( \sum_{i=1}^{n} (\lambda_i |v_{i1}(x_1 - y_1) + \ldots + v_{in}(x_n - y_n)|)^p \right)^{1/p} \\
&\leq \sigma n^{1/p}
\end{align*}
\]

is called \((v_1, \ldots, v_n)\)-Minkowski (or \( l_p(v_1, \ldots, v_n) \)) distance function in \( \mathbb{R}^n \), and real number \( d_{p(v_1, \ldots, v_n)}(X, Y) \) is called \((v_1, \ldots, v_n)\)-Minkowski distance between points \( X \) and \( Y \). In addition, if \( p = 1 \), \( p = 2 \) and \( p \to \infty \), then \( d_{p(v_1, \ldots, v_n)}(X, Y) \) is called \((v_1, \ldots, v_n)\)-taxicab distance, \((v_1, \ldots, v_n)\)-Euclidean distance and \((v_1, \ldots, v_n)\)-maximum distance between points \( X \) and \( Y \) respectively, and we denote them by \( d_T(v_1, \ldots, v_n)(X, Y) \), \( d_E(v_1, \ldots, v_n)(X, Y) \) and \( d_M(v_1, \ldots, v_n)(X, Y) \) respectively.

Here, since \( \sigma \leq d_{p(v_1, \ldots, v_n)}(X, Y) \), we have that
\[
\lim_{p \to \infty} d_{p(v_1, \ldots, v_n)}(X, Y) = \max_{i \in \{1, \ldots, n\}} \{ \lambda_i |v_{i1}(x_1 - y_1) + \ldots + v_{in}(x_n - y_n)| \}
\]
and so
\[
d_{M(v_1, \ldots, v_n)}(X, Y) = \max_{i \in \{1, \ldots, n\}} \{ \lambda_i |v_{i1}(x_1 - y_1) + \ldots + v_{in}(x_n - y_n)| \}.
\]

**Remark 2.1** In \( n \)-dimensional Cartesian coordinate space, let \( \Psi_P^X \) denote hyperplane through point \( P \) and perpendicular to the vector \( v_i \) for \( i \in \{1, \ldots, n\} \). Since Euclidean distance between the point \( Y \) and hyperplane \( \Psi_P^X \) (or the point \( X \) and hyperplane \( \Psi_P^Y \)) is

\[
d_E(Y, \Psi_P^X) = |v_{i1}(x_1 - y_1) + \ldots + v_{in}(x_n - y_n)|,
\]

we have that \( d_{p(v_1, \ldots, v_n)}(X, Y) \) is the geometric interpretation of \((v_1, \ldots, v_n)\)-Minkowski distance between the points \( X \) and \( Y \) is

\[
d_{p(v_1, \ldots, v_n)}(X, Y) = \left( \sum_{i=1}^{n} (\lambda_i d_E(Y, \Psi_P^X))^p \right)^{1/p}
\]

which is the well-known Minkowski (or \( l_p \)) distance between the points \( X \) and \( Y \), that gives the well-known taxicab, Euclidean and maximum distances denoted by \( d_T(X, Y) \), \( d_E(X, Y) \) and \( d_M(X, Y) \), for \( p = 1 \), \( p = 2 \) and \( p \to \infty \) respectively (see [8] pp. 94, 301; see also [11] and [12]).

The following proposition shows that \((v_1, \ldots, v_n)\)-Minkowski distance function for \( p \geq 1 \) satisfies the metric properties in \( \mathbb{R}^n \):
Theorem 2.1 For \( p \geq 1 \), \((v_1,...,v_n)\)-Minkowski distance function determines metric in \( \mathbb{R}^n \).

Proof. Let \( X = (x_1,...,x_n) \), \( Y = (y_1,...,y_n) \) and \( Z = (z_1,...,z_n) \) be three points in \( \mathbb{R}^n \).

(M1) Clearly, if \( X = Y \), then \( d_{p(v_1,...,v_n)}(X,Y) = 0 \). Conversely, if \( d_{p(v_1,...,v_n)}(X,Y) = 0 \), then we get \( Ax = 0 \) where

\[
A = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix}.
\]

Since \( v_1,...,v_n \) are linearly independent, we have \( |A| \neq 0 \). Therefore the homogeneous system \( Ax = 0 \) has only trivial solution. Thus, we have \( x_i - y_i = 0 \), and so \( X = Y \).

(M2) It is clear that \( d_{p(v_1,...,v_n)}(X,Y) = d_{p(v_1,...,v_n)}(Y,X) \).

(M3) The triangle inequality can be proven using the Minkowski inequality for \( p \geq 1 \) (see [2, p. 25]) as follows:

\[
d_{p(v_1,...,v_n)}(X,Y) = \left( \sum_{i=1}^{n} (\lambda_i |v_{1i}(x_1 - y_1) + ... + v_{ni}(x_n - y_n)|)^p \right)^{1/p} \\
= \left( \sum_{i=1}^{n} |\lambda_i|v_{1i}(x_1 - z_1 + z_1 - y_1) + ... + v_{ni}(x_n - z_n + z_n - y_n)|^p \right)^{1/p} \\
= \left( \sum_{i=1}^{n} |\lambda_i|v_{1i}(x_1 - z_1) + ... + v_{ni}(x_n - z_n)|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |\lambda_i|v_{1i}(z_1 - y_1) + ... + v_{ni}(z_n - y_n)|^p \right)^{1/p} \\
\leq \left( \sum_{i=1}^{n} |\lambda_i|v_{1i}(x_1 - z_1) + ... + v_{ni}(x_n - z_n)|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |\lambda_i|v_{1i}(z_1 - y_1) + ... + v_{ni}(z_n - y_n)|^p \right)^{1/p} \\
= d_{p(v_1,...,v_n)}(X,Z) + d_{p(v_1,...,v_n)}(Z,Y).
\]

Since the Minkowski inequality does not hold for \( 0 < p < 1 \) (see [2, pp. 26-27]), \((v_1,...,v_n)\)-Minkowski distance function family does not hold the triangle inequality for \( 0 < p < 1 \). So, for \( 0 < p < 1 \), it does not determine a metric in \( \mathbb{R}^n \) while it determines a distance. We denote by \( \mathbb{R}^2_{p(v_1,v_2)} \) the real plane \( \mathbb{R}^2 \) equipped with the \((v_1,v_2)\)-Minkowski metric.

The following theorem shows that \((v_1,...,v_n)\)-Minkowski distance and Euclidean distance between two points on any given line \( l \) are directly proportional:

Theorem 2.2 Given two points \( X \) and \( Y \) on a line \( l \) with the direction vector \( u = (u_1,...,u_n) \). Then,

\[
d_{p(v_1,...,v_n)}(X,Y) = \phi_{p(v_1,...,v_n)}(l)d_E(X,Y)
\]

where \( \phi_{p(v_1,...,v_n)}(l) = \left( \frac{\sum_{i=1}^{n} (\lambda_i |v_{1i}u_1 + ... + v_{ni}u_n|)^p}{\sqrt{u_1^2 + ... + u_n^2}} \right)^{1/p} \).

Proof. For any two points \( X = (x_1,...,x_n) \) and \( Y = (y_1,...,y_n) \) there is \( k \in \mathbb{R} \) such that \( x_1 - y_1,...,x_n - y_n = k(u_1,...,u_n) \). Then, we have \( d_E(X,Y) = |k| \sqrt{u_1^2 + ... + u_n^2} \) and

\[
d_{p(v_1,...,v_n)}(X,Y) = |k| \left( \sum_{i=1}^{n} (\lambda_i |v_{1i}u_1 + ... + v_{ni}u_n|)^p \right)^{1/p}
\]

which complete the proof.

Now, the following corollaries are trivial:

Corollary 2.1 If \( W, X \) and \( Y,Z \) are pair of distinct points such that the lines determined by them are the same or parallel, then

\[
d_{p(v_1,...,v_n)}(W,X)/d_{p(v_1,...,v_n)}(Y,Z) = d_E(W,X)/d_E(Y,Z).
\]
Corollary 2.2 Circles and spheres of \((v_1, ..., v_n)\)-Minkowski distance are symmetric about their center.

Corollary 2.3 Translation by any vector preserves \((v_1, ..., v_n)\)-Minkowski distance.

Corollary 2.4 For a vector \(x = (x_1, ..., x_n)\) in \(\mathbb{R}^n\), the induced norm is

\[
\|x\|_{p(v_1, ..., v_n)} = \left( \sum_{i=1}^{n} (v_i |v_i x_1 + ... + v_n x_n|)^p \right)^{1/p}.
\]

(8)

Remark 2.2 Instead of unit vectors, one can define \((v_1, ..., v_n)\)-Minkowski distance for any linearly independent \(n\) vectors \(v_1, ..., v_n\) as follows

\[
d'_p(x_1, ..., x_n)(X,Y) = \left( \sum_{i=1}^{n} \left( \lambda_i \frac{|v_i(x_1 - y_1) + ... + v_n(x_n - y_n)|}{(v_i^2 + ... + v_n^2)^{1/2}} \right)^p \right)^{1/p}
\]

(9)

or one can define it by unit vectors \(v_1, ..., v_n\), and positive real numbers \(\mu_1, ..., \mu_n\) as follows

\[
d''_p(x_1, ..., x_n)(X,Y) = \left( \sum_{i=1}^{n} \mu_i |v_i(x_1 - y_1) + ... + v_n(x_n - y_n)|^p \right)^{1/p}.
\]

(10)

These distance functions also determine metric families for \(p \geq 1\), generalizing the Minkowski distance. But then, we have

\[
d'_p(x_1, ..., x_n)(X,Y) = d'_p(k_1, v_1, ..., k_n, v_n)(X,Y) \text{ for any } k_i \in \mathbb{R} - \{0\},
\]

(11)

and

\[
d''_M(x_1, ..., x_n)(X,Y) = \lim_{p \to \infty} d'_p(x_1, ..., x_n)(X,Y) = \max_{i \in \{1, ..., n\}} \{ |v_i(x_1 - y_1) + ... + v_n(x_n - y_n)| \},
\]

(12)

which is independent from \(\mu_i\). However, we see that for every \(p\) values, circles of \(d_p(x_1, x_2)\) distance having the same center and radius, have four common points, and they are nested inside one another. So, it is easier to illustrate their circles in a figure (see Figure 1 for some examples of \((v_1, v_2)\)-Minkowski circles having the same center and radius).

Figure 1. The unit \((v_1, v_2)\)-Minkowski circles; \(v_1 = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right), v_2 = \left(\frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\).
3 Circles of the generalized taxicab metric in \( \mathbb{R}^2 \)

By Definition 2.1 and Remark 2.1, \((v_1, v_2)\)-taxicab distance between points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) in \( \mathbb{R}^2 \) is

\[
d_{T(v_1, v_2)}(P_1, P_2) = \lambda_1 |v_{11}(x_1 - x_2) + v_{12}(y_1 - y_2)| + \lambda_2 |v_{21}(x_1 - x_2) + v_{22}(y_1 - y_2)|
\]

that is, the sum of weighted Euclidean distances from the point \( P_2 \) to the lines \( l_1 \) and \( l_2 \), which are passing through \( P_1 \) and perpendicular to the vectors \( v_1 \) and \( v_2 \) respectively. For vectors \( v_1 = (1, 0) \) and \( v_2 = (0, 1) \), \( d_{T(v_1, v_2)} \) in \( \mathbb{R}^2 \) is the same as the (slightly) generalized taxicab metric (also known as the weighted taxicab metric) defined in [22] (see also [5] and [6]). In addition, for unit vectors \( v_1 \) and \( v_2 \) such that \( v_1 \perp v_2 \) and \( v_{12}/v_{11} = m \) where \( v_{11} \neq 0 \), \( d_{T(v_1, v_2)} \) in \( \mathbb{R}^2 \) is the same as the \( m \)-generalized taxicab metric \( d_{T_p(m)} \) defined in [4].

The following theorem determines circles of the generalized taxicab metric \( d_{T(v_1, v_2)} \) in \( \mathbb{R}^2 \):

**Theorem 3.1** Every \((v_1, v_2)\)-taxicab circle is a parallelogram with the same center, each of whose diagonals is perpendicular to \( v_1 \) or \( v_2 \). In addition, if \( \lambda_1 = \lambda_2 \) then it is a rectangle, if \( v_1 \perp v_2 \) then it is a rhombus, and if \( \lambda_1 = \lambda_2 \) and \( v_1 \perp v_2 \) then it is a square.

**Proof.** Without loss of generality, let us consider the unit \((v_1, v_2)\)-taxicab circle. Clearly, it is the set of points \( P = (x, y) \) in \( \mathbb{R}^2 \) satisfying the equation

\[
d_{T(v_1, v_2)}(O, P) = \lambda_1 d_E(P, l_1) + \lambda_2 d_E(P, l_2) = 1
\]

where \( l_i : v_{1i}x + v_{2i}y = 0 \) for \( i = 1, 2 \), that is

\[
\lambda_1 |v_{11}x + v_{12}y| + \lambda_2 |v_{21}x + v_{22}y| = 1. \tag{14}
\]

One can see that this equation is the image of \(|x| + |y| = 1\) which is the well-known taxicab circle, under the linear transformation

\[
T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{v_{21}}{\lambda_1 \tau} & \frac{-v_{12}}{\lambda_1 \tau} \\ \frac{v_{12}}{\lambda_2 \tau} & \frac{-v_{21}}{\lambda_2 \tau} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{15}\]

where \( \tau = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \). Thus, the unit \((v_1, v_2)\)-taxicab circle is a parallelogram symmetric about the origin, having vertices \( A_1 = \left( \frac{v_{22}}{\lambda_1 \tau}, \frac{-v_{21}}{\lambda_1 \tau} \right), A_2 = \left( \frac{-v_{12}}{\lambda_2 \tau}, \frac{v_{11}}{\lambda_2 \tau} \right), A_3 = \left( \frac{-v_{22}}{\lambda_1 \tau}, \frac{v_{21}}{\lambda_1 \tau} \right), A_4 = \left( \frac{v_{12}}{\lambda_2 \tau}, \frac{-v_{11}}{\lambda_2 \tau} \right) \), and having diagonals on the lines \( l_1 \) and \( l_2 \), each of which is perpendicular to \( v_1 \) or \( v_2 \), since

\[
A_1A_2//A_3A_4, A_1A_4//A_2A_3, A_2A_4 = l_1 \text{ and } A_1A_3 = l_2.
\]

In addition, if \( \lambda_1 = \lambda_2 \) then \( d_E(O, A_1) = d_E(O, A_2) \) and since a parallelogram having diagonals of the same length is a rectangle, the unit \((v_1, v_2)\)-taxicab circle is a rectangle. Notice that sides of the rectangle are parallel to angle bisectors of the lines \( l_1 \) and \( l_2 \). If \( v_1 \perp v_2 \) then \( l_1 \perp l_2 \) and since a parallelogram having perpendicular diagonals is a rhombus, the unit \((v_1, v_2)\)-taxicab circle is a rhombus. Finally, it is clear that if \( \lambda_1 = \lambda_2 \) and \( v_1 \perp v_2 \), then the unit \((v_1, v_2)\)-taxicab circle is a square (see Figure 2 and Figure 3 for examples of the unit \((v_1, v_2)\)-taxicab circles).
Figure 2. The unit \((v_1, v_2)\)-taxicab circles for \(v_1 = \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right)\), \(v_2 = \left(-\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\).

Figure 3. The unit \((v_1, v_2)\)-taxicab circles for \(v_1 = \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right)\), \(v_2 = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\).

Let us consider the case of \(\lambda_1 = \lambda_2 = 1\): Now, we know that a \((v_1, v_2)\)-taxicab circle with center \(C\) and radius \(r\), that is the set of all points \(P\) satisfying the equation

\[d_E(P, l_1) + d_E(P, l_2) = r,\]

is a rectangle with the same center, whose diagonals are on the lines \(l_1\) and \(l_2\), and sides are parallel to angle bisectors of the lines \(l_1\) and \(l_2\). Besides, if \(v_1 \perp v_2\) then \((v_1, v_2)\)-taxicab circle is a square with the same properties. On the other hand, for a point \(Q_i\) on both line \(l_i\) and the \((v_1, v_2)\)-taxicab circle (see Figure 4), it is clear that

\[d_E(Q_1, l_2) = d_E(Q_2, l_1) = r.\]

Figure 4. \((v_1, v_2)\)-taxicab circles with center \(C\) and radius \(r\), for \(\lambda_1 = \lambda_2 = 1\).

The following theorem shows that every rectangle is a \((v_1, v_2)\)-taxicab circle with the same center for a proper generalized taxicab metric \(d_T(v_1, v_2)\) with \(\lambda_1 = \lambda_2 = 1\):
Theorem 3.2 Every rectangle with sides of lengths $2a$ and $2b$, is a $(v_1, v_2)$-taxicab circle with the same center and the radius $\frac{2ab}{\sqrt{a^2 + b^2}}$, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors $v_1$ and $v_2$, each of which is perpendicular to a diagonal of the rectangle.

Proof. Without loss of generality, let us consider a rectangle with center $C$ and sides of lengths $2a$ and $2b$, as in Figure 5. Denote the diagonal lines of the rectangle by $d_1$ and $d_2$. Clearly, $C$ is the intersection point of $d_1$ and $d_2$. Draw two lines $d'_1$ and $d''_1$, each of them is passing through a vertex on $d_1$ and parallel to $d_2$. Since sides of the rectangle are angle bisectors of pair of lines $d_1, d'_1$ and $d_2, d''_1$, we have

$$d_E(P, d_1) + d_E(P, d_2) = d_E(d_2, d'_2) = d_E(d_2, d''_2). \quad (16)$$

On the other hand, for the area of the rectangle we have

$$4ab = 2\sqrt{a^2 + b^2}d_E(d_2, d'_2), \quad (17)$$

so we get

$$d_E(d_2, d'_2) = \frac{2ab}{\sqrt{a^2 + b^2}} \quad (18)$$

Then, for every point $P$ on the rectangle, we have

$$d_E(P, d_1) + d_E(P, d_2) = \frac{2ab}{\sqrt{a^2 + b^2}}. \quad (19)$$

Thus, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors $v_1$ and $v_2$, each of which is perpendicular to a diagonal, the rectangle is a $(v_1, v_2)$-taxicab circle with center $C$ and radius $2ab/\sqrt{a^2 + b^2}$. \hfill \blacksquare

![Figure 5](image_url)

**Figure 5.** A rectangle with center $C$ and sides of lengths $2a$ and $2b$.

4 Circles of the generalized maximum metric in $\mathbb{R}^2$

By Definition 2.1 and Remark 2.1, $(v_1, v_2)$-maximum distance between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is

$$d_{M(v_1,v_2)}(P_1, P_2) = \max\{\lambda_1 |v_11(x_1 - x_2) + v_12(y_1 - y_2)|, \lambda_2 |v_21(x_1 - x_2) + v_22(y_1 - y_2)|\}$$

$$= \max\{\lambda_1 d_E(P_2, l_1), \lambda_2 d_E(P_2, l_2)\}$$

that is the maximum of weighted Euclidean distances from the point $P_2$ to the lines $l_1$ and $l_2$, which are passing through $P_1$ and perpendicular to the vectors $v_1$ and $v_2$ respectively.

The following theorem determines circles of the generalized maximum metric $d_{M(v_1,v_2)}$ in $\mathbb{R}^2$:
Theorem 4.1 Every \((v_1, v_2)\)-maximum circle is a parallelogram with the same center, each of whose sides is perpendicular to \(v_1\) or \(v_2\). In addition, if \(\lambda_1 = \lambda_2\) then it is a rhombus, if \(v_1 \perp v_2\) then it is a rectangle, and if \(\lambda_1 = \lambda_2\) and \(v_1 \perp v_2\) then it is a square.

Proof. Without loss of generality, let us consider the unit \((v_1, v_2)\)-maximum circle. Clearly, it is the set of points \(P = (x, y)\) in \(\mathbb{R}^2\) satisfying the equation

\[ d_M(v_1,v_2)(O,P) = \max \{\lambda_1 d_E(P,l_1), \lambda_2 d_E(P,l_2)\} = 1 \tag{20} \]

where \(l_i : v_{1i}x + v_{2i}y = 0\) for \(i = 1, 2\), that is

\[ \max \{\lambda_1 |v_{11}x_1 + v_{12}x_2|, \lambda_2 |v_{21}x_1 + v_{22}x_2|\} = 1. \tag{21} \]

One can see that this equation is the image of \(\max \{|x|, |y|\} = 1\) which is the well-known maximum circle, under the linear transformation

\[ T\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{v_{22}}{\lambda_1} & -\frac{v_{12}}{\lambda_1} \\ \frac{v_{12}}{\lambda_2} & \frac{v_{22}}{\lambda_2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{22} \]

where \(\tau = \left| \begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array} \right|\). Thus, the unit \((v_1, v_2)\)-maximum circle is a parallelogram symmetric about the origin, having vertices \(B_1 = \left(\frac{-v_{12}\lambda_1 + v_{22}\lambda_2}{\lambda_1\lambda_2\tau}, \frac{v_{11}\lambda_2 - v_{21}\lambda_2}{\lambda_1\lambda_2\tau}\right),\)

\(B_2 = \left(\frac{-v_{12}\lambda_1 - v_{22}\lambda_2}{\lambda_1\lambda_2\tau}, \frac{v_{11}\lambda_2 + v_{21}\lambda_2}{\lambda_1\lambda_2\tau}\right)\),

\(B_3 = \left(\frac{v_{12}\lambda_1 - v_{22}\lambda_2}{\lambda_1\lambda_2\tau}, \frac{-v_{11}\lambda_1 + v_{21}\lambda_2}{\lambda_1\lambda_2\tau}\right),\)

\(B_4 = \left(\frac{v_{12}\lambda_1 + v_{22}\lambda_2}{\lambda_1\lambda_2\tau}, \frac{-v_{11}\lambda_1 - v_{21}\lambda_2}{\lambda_1\lambda_2\tau}\right)\),

and having sides parallel to the lines \(l_1\) and \(l_2\), each of which is perpendicular to \(v_1\) or \(v_2\), since

\[ B_1B_2//B_3B_4//l_2 \text{ and } B_1B_4//B_2B_3//l_1. \]

In addition, if \(\lambda_1 = \lambda_2\) then \(OB_1 \perp OB_2\) and since a parallelogram having perpendicular diagonals is a rhombus, the unit \((v_1, v_2)\)-maximum circle is a rhombus. Notice that diagonals of the rhombus are on angle bisectors of the lines \(l_1\) and \(l_2\). If \(v_1 \perp v_2\) then \(l_1 \perp l_2\) and since a parallelogram having perpendicular sides is a rectangle, the unit \((v_1, v_2)\)-maximum circle is a rectangle. Finally, it is clear that if \(\lambda_1 = \lambda_2\) and \(v_1 \perp v_2\), then the unit \((v_1, v_2)\)-maximum circle is a square (see Figure 6 and Figure 7 for examples of the unit \((v_1, v_2)\)-maximum circles). \[\blacksquare\]

**Figure 6.** The unit \((v_1, v_2)\)-maximum circles; \(v_1 = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right), v_2 = \left(\frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\).
Now let us consider the case of $\lambda_1 = \lambda_2 = 1$: Now, we know that a $(v_1, v_2)$-maximum circle with center $C$ and radius $r$, that is the set of all points $P$ satisfying the equation

$$\max\{d_E(P, l_1), d_E(P, l_2)\} = r,$$

is a rhombus with the same center, whose sides are parallel to the lines $l_1$ and $l_2$, and whose diagonals are on angle bisectors of the lines $l_1$ and $l_2$. Besides, if $v_1 \perp v_2$ then the $(v_1, v_2)$-maximum circle is a square with the same properties. On the other hand, for a point $Q_i$ on both line $l_i$ and the $(v_1, v_2)$-maximum circle (see Figure 8), it is clear that

$$d_E(Q_1, l_2) = d_E(Q_2, l_1) = r.$$

The following theorem shows that every rhombus is a $(v_1, v_2)$-maximum circle with the same center for a proper generalized maximum metric $d_M(v_1, v_2)$ with $\lambda_1 = \lambda_2 = 1$:

**Theorem 4.2** Every rhombus with diagonals of lengths $2e$ and $2f$, is a $(v_1, v_2)$-maximum circle with the same center and the radius $\frac{ef}{\sqrt{e^2+f^2}}$, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors $v_1$ and $v_2$, each of which is perpendicular to a side of the rhombus.

**Proof.** Without loss of generality, let us consider a rhombus with center $C$ and diagonals of lengths $2e$ and $2f$, as in Figure 9. Denote by $d_1$ and $d_2$, the two distinct lines each through $C$ and parallel to a side of the rhombus. Since diagonals are angle bisectors of consecutive sides, we have

$$\max\{d_E(P, d_1), d_E(P, d_2)\} = d_E(V, d_1) = d_E(V, d_2)$$

for any vertex $V$ of the rhombus. On the other hand, for the area of the rhombus we have

$$2ef = 2\sqrt{e^2+f^2}d_E(V, d_1),$$

so, we get

$$d_E(V, d_1) = \frac{ef}{\sqrt{e^2+f^2}}.$$
Then, for every point \( P \) on the rhombus we have

\[
\max\{d_E(P, d_1), d_E(P, d_2)\} = \frac{ef}{\sqrt{e^2 + f^2}}. \tag{26}
\]

Thus, for \( \lambda_1 = \lambda_2 = 1 \) and linearly independent unit vectors \( v_1 \) and \( v_2 \), each of which is perpendicular to a side, the rhombus is a \((v_1, v_2)\)-maximum circle having center \( C \) and radius \( ef/\sqrt{e^2 + f^2} \). \( \blacksquare \)

![Figure 9. A rhombus with center \( C \) and diagonals of lengths \( 2e \) and \( 2f \).](image)

5 Circles of the generalized Euclidean metric in \( \mathbb{R}^2 \)

By Definition 2.1 and Remark 2.1, \((v_1, v_2)\)-Euclidean distance between points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) in \( \mathbb{R}^2 \) is

\[
d_{E(v_1,v_2)}(P_1, P_2) = \left[ (\lambda_1 |v_{11}(x_1 - x_2) + v_{12}(y_1 - y_2)|)^2 + (\lambda_2 |v_{21}(x_1 - x_2) + v_{22}(y_1 - y_2)|)^2 \right]^{1/2}
\]

that is the square root of the sum of square of weighted Euclidean distances from the points \( P_2 \) to the lines \( l_1 \) and \( l_2 \), which are passing through \( P_1 \) and perpendicular to the vectors \( v_1 \) and \( v_2 \) respectively. Notice that by Pythagorean theorem, for \( \lambda_1 = \lambda_2 = 1 \) and perpendicular unit vectors \( v_1 \) and \( v_2 \), we have

\[
d_{E(v_1,v_2)}(P_1, P_2) = d_E(P_1, P_2). \tag{27}
\]

The following theorem determines circles of the generalized Euclidean metric \( d_{E(v_1,v_2)} \) in \( \mathbb{R}^2 \):

**Theorem 5.1** Every \((v_1, v_2)\)-Euclidean circle is an ellipse with the same center. In addition, if \( \lambda_1 = \lambda_2 \) then its axes are angle bisectors of the lines \( l_1 \) and \( l_2 \), if \( v_1 \perp v_2 \) then its axes are the lines \( l_1 \) and \( l_2 \), and if \( \lambda_1 = \lambda_2 \) and \( v_1 \perp v_2 \) then it is a Euclidean circle with the same center.

**Proof.** Without loss of generality, we consider the unit \((v_1, v_2)\)-Euclidean circle with center at the origin. Clearly, it is the set of points \( P = (x, y) \) in \( \mathbb{R}^2 \) satisfying the equation

\[
d_{E(v_1,v_2)}(O, P) = \left[ (\lambda_1 d_E(P, l_1))^2 + (\lambda_2 d_E(P, l_2))^2 \right]^{1/2} = 1 \tag{28}
\]

where \( l_i : v_{i1}x + v_{i2}y = 0 \) for \( i = 1, 2 \), that is

\[
\lambda_1^2 (v_{11}x + v_{12}y)^2 + \lambda_2^2 (v_{21}x + v_{22}y)^2 = 1. \tag{29}
\]

This equation can be written as

\[
Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0 \tag{30}
\]

where \( A = \lambda_1^2 v_{11}^2 + \lambda_2^2 v_{21}^2, B = \lambda_1^2 v_{12}^2 + \lambda_2^2 v_{22}^2, C = \lambda_1^2 v_{11}v_{12} + \lambda_2^2 v_{21}v_{22}, D = E = 0 \) and \( F = -1 \).
If we use the classification conditions for the general quadratic equations in two variables (see [23, pp. 232-233]), we have

\[
\delta = \begin{vmatrix} A & C \\ C & B \end{vmatrix} = \lambda_1^2 \lambda_2^2 \tau^2 \quad \text{and} \quad \Delta = \begin{vmatrix} A & C & 0 \\ C & B & 0 \\ 0 & 0 & -1 \end{vmatrix} = -\delta
\]

where \( \tau = \begin{vmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{vmatrix} \), and since \( v_1 \) and \( v_2 \) are linearly independent, we get \( \tau \neq 0 \), \( \delta > 0 \) and \( \Delta < 0 \). In addition, since \( A > 0 \) and \( B > 0 \), we get \( \Delta/(A + B) < 0 \). So, since

\[
\Delta \neq 0, \delta > 0 \quad \text{and} \quad \Delta/(A + B) < 0,
\]

the quadratic equation determines an ellipse with center at the origin. If \( \lambda_1 = \lambda_2 \), concerning the equation (28) geometrically, one can see that the unit \((v_1, v_2)\)-Euclidean circle is symmetric about angle bisectors of the lines \( l_1 \) and \( l_2 \), since \( l_1 \) and \( l_2 \) are symmetric about the angle bisectors of themselves. Notice that the major axis of the ellipse is the angle bisector of the non-obtuse angle between \( l_1 \) and \( l_2 \). If \( v_1 \perp v_2 \) then \( l_1 \perp l_2 \), and concerning the equation (28) geometrically again, one can see that the unit \((v_1, v_2)\)-Euclidean circle is symmetric about the lines \( l_1 \) and \( l_2 \). Finally, it is clear that if \( \lambda_1 = \lambda_2 \) and \( v_1 \perp v_2 \), then the unit \((v_1, v_2)\)-Euclidean circle is Euclidean circle with the same center, since an ellipse which is symmetric about four different lines \((l_1, l_2 \text{ and angle bisectors of them})\), is a Euclidean circle. One can also see that if \( \lambda_1 = \lambda_2 \) and \( v_1 \perp v_2 \), then \( A = B > 0 \) and \( C = 0 \), so the quadratic equation above gives an equation of a Euclidean circle with center at the origin (see Figure 10 and Figure 11 for examples of the unit \((v_1, v_2)\)-Euclidean circles). 

![Figure 10](image1.png)

**Figure 10.** The unit \((v_1, v_2)\)-Euclidean circles for \( v_1 = \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right), \quad v_2 = \left(-\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right) \). 

![Figure 11](image2.png)

**Figure 11.** The unit \((v_1, v_2)\)-Euclidean circles for \( v_1 = \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right), \quad v_2 = \left(-\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right) \).
Let us consider the case of $\lambda_1 = \lambda_2 = 1$: Now, we know that a $(v_1, v_2)$-Euclidean circle with center $C$ and radius $r$, that is the set of all points $P$ satisfying the equation

$$[(d_E(P, l_1))^2 + (d_E(P, l_2))^2]^{1/2} = r,$$

is an ellipse with the same center, whose axes are angle bisectors of the lines $l_1$ and $l_2$, such that the major axis is the angle bisector of the non-obtuse angle between $l_1$ and $l_2$. In addition, if $v_1 \perp v_2$ then $(v_1, v_2)$-Euclidean circle is a Euclidean circle having the same center and the radius. In addition, for a point $Q_i$ on both line $l_i$ and the $(v_1, v_2)$-Euclidean circle (see Figure 12), it is clear that

$$d_E(Q_1, l_2) = d_E(Q_2, l_1) = r.$$

![Figure 12. $(v_1, v_2)$-Euclidean circles with center $C$ and radius $r$, for $\lambda_1 = \lambda_2 = 1$.](image)

The following theorem determines some relations between parameters of a $(v_1, v_2)$-Euclidean circle and the ellipse related to it:

**Theorem 5.2** If a $(v_1, v_2)$-Euclidean circle with radius $r$ for $\lambda_1 = \lambda_2 = 1$, is an ellipse with the same center, having semi-major axis $a$ and semi-minor axis $b$, then

$$r = \frac{\sqrt{2ab}}{\sqrt{a^2 + b^2}}, \quad a = \frac{r}{\sqrt{1 - \cos \theta}} \quad \text{and} \quad b = \frac{r}{\sqrt{1 + \cos \theta}}$$

where $\theta$ is the non-obtuse angle between $v_1$ and $v_2$, and $\cos \theta = |v_1 v_{21} + v_2 v_{22}|$.

**Proof.** Let a $(v_1, v_2)$-Euclidean circle with radius $r$ for $\lambda_1 = \lambda_2 = 1$, be an ellipse with the same center, having semi-major axis $a$ and semi-minor axis $b$, and let $\theta$ be the non-obtuse angle between $v_1$ and $v_2$. Then non-obtuse angle between the lines $l_1$ and $l_2$ is equal to $\theta$, and the axes of the ellipse is angle bisectors of the lines $l_1$ and $l_2$, such that the major axis of is the angle bisector of the non-obtuse angle between $l_1$ and $l_2$. Using similar right triangles whose hypotenuses are $a$ and $b$ (see Figure 13), one gets $\sin \theta = \frac{r}{a \sqrt{2}}$, $\cos \theta = \frac{r}{b \sqrt{2}}$, $\tan \theta = \frac{\sqrt{2b^2 - r^2}}{r}$, and so

$$\tan \frac{\theta}{2} = \frac{b}{a}, \quad \sin \theta = \frac{r^2}{ab} \quad \text{and} \quad \cos \theta = 1 - \frac{r^2}{a^2} = \frac{r^2}{b^2} - 1.$$

Then, we have

$$r = \frac{\sqrt{2ab}}{\sqrt{a^2 + b^2}}, \quad a = \frac{r}{\sqrt{1 - \cos \theta}} \quad \text{and} \quad b = \frac{r}{\sqrt{1 + \cos \theta}}.$$

Besides, one can derive that

$$\sin \theta = \frac{2ab}{\sqrt{a^2 + b^2}}, \quad \cos \theta = \frac{a^2 - b^2}{a^2 + b^2} \quad \text{and} \quad \tan \theta = \frac{2ab}{a^2 - b^2}.$$

In addition, by $|\langle v_1, v_2 \rangle| = \|v_1\| \|v_2\| \cos \theta$ it follows immediately that

$$\cos \theta = |v_1 v_{21} + v_2 v_{22}| \quad \text{and} \quad \sin \theta = |v_1 v_{22} - v_2 v_{21}|. \quad \square$$
Figure 13. A \((v_1, v_2)\)-Euclidean circle and an ellipse that are the same.

Figure 14. The rectangle and the rhombus determined by an ellipse.

Notice that \(r^2\) is the harmonic mean of \(b^2\) and \(a^2\), so we have \(b \leq r \leq a\) where the equality holds only for the case \(a = b = r\). Another fact is the chords derived by the lines \(l_1\) and \(l_2\) have the same length, and if \(d_E(C, Q_i) = R\) then \(\sin \theta = \frac{r}{R}\), and we get

\[
R = \sqrt{a^2 + b^2} / \sqrt{2} \quad \text{and} \quad Rr = ab.
\]

Since chords derived by the lines \(l_1\) and \(l_2\) are conjugate diameters by the following theorem, the last two equalities can also be derived by the first and the second theorems of Appollonius; which are

1. The sum of the squares of any two conjugate semi-diameters is equal to \(a^2 + b^2\),
2. The area of the parallelogram determined by two coterminous conjugate semi-diameters is equal to \(ab\) (see [14, pp. 1800-1803]).

**Theorem 5.3** The chords derived \(l_1\) and \(l_2\) are conjugate diameters of the ellipse.

**Proof.** We know that the diameters parallel to any pair of supplemental chords (which are formed by joining the extremities of any diameter to a point lying on the ellipse) are conjugate (see [14, p. 1805]). Since \(l_1\) and \(l_2\) are parallel to a pair of supplemental chords formed by joining the extremities of the minor axis to one of the extremities of the major axis, the chords derived by the lines \(l_1\) and \(l_2\) are conjugate diameters of the ellipse. 

**Remark 5.1** Since the chords derived by the lines \(l_1\) and \(l_2\) are conjugate, \(l_1\) is parallel to the tangent lines through the extremities of the chord determined by \(l_2\), and vice versa. It is clear that the tangent lines through the extremities of these conjugate diameters determine a rhombus with sides of length \(2R\), circumscribed the ellipse. Since \(l_1\) and \(l_2\) are symmetric about the axes of the ellipse, the diagonals of the rhombus are on the axes of the ellipse, and they have lengths \(2\sqrt{2}a\) and \(2\sqrt{2}b\). Similarly, the chords determined by the axes of the ellipse are also conjugate, and the tangent lines through the extremities of these conjugate diameters determine a rectangle with sides of lengths \(2a\) and \(2b\), circumscribed the ellipse. Since \(\tan \frac{\theta}{2} = \frac{b}{a}\), the diagonals of this rectangle are on the lines \(l_1\) and \(l_2\), and they have length \(2\sqrt{2}R\) (see Figure 14). Notice that there is an ellipse similar to the prior, through eight vertices of these rectangle and rhombus, whose semi-major and semi-minor axes are equal to \(\sqrt{2}a\) and \(\sqrt{2}b\) respectively (see Figure 14).

**Remark 5.2** Observe that, the rhombus derived by the tangent lines through the extremities of the conjugate diameters determined by the lines \(l_1\) and \(l_2\), is the \((v_1, v_2)\)-maximum circle, and the
let us consider the ellipse with the equation $v_1$ and $v_2$. Figure 15 illustrates $(v_1, v_2)$-taxicab, $(v_1, v_2)$-Euclidean and $(v_1, v_2)$-maximum circles with the same center and radius, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors $v_1$ and $v_2$, each of which is perpendicular to one of the lines $l_1$ and $l_2$.

Figure 15. $(v_1, v_2)$-Minkowski circles with the same center and radius.

The following theorem shows that every ellipse is a $(v_1, v_2)$-Euclidean circle with the same center for a proper generalized Euclidean metric $d_{E(v_1, v_2)}$ with $\lambda_1 = \lambda_2 = 1$:

**Theorem 5.4**  Every ellipse with semi-major axis $a$ and semi-minor axis $b$, is a $(v_1, v_2)$-Euclidean circle with the same center and the radius $\frac{\sqrt{2ab}}{\sqrt{a^2 + b^2}}$, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors $v_1$ and $v_2$, each of which is perpendicular to a diagonal lines of the rectangle circumscribed the ellipse, whose sides are parallel to the axes of the ellipse.

**Proof.** Since Euclidean distances are preserved under rigid motions, without loss of generality, let us consider the ellipse with the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (31)$$

Take the lines $l_1 : bx - ay = 0$ and $l_2 : bx + ay = 0$ passing through the origin, since they are diagonal lines of the rectangle circumscribed the ellipse, whose sides are parallel to the axes of the ellipse. So, for every point $P = (x_0, y_0)$ on the ellipse, we have

$$[(d_{E(P, l_1)})^2 + (d_{E(P, l_2)})^2]^{1/2} = \left[\frac{2(bx_0^2 + ay_0^2)}{a^2 + b^2}\right]^{1/2} = \frac{\sqrt{2ab}}{\sqrt{a^2 + b^2}} \quad (32)$$

which is a constant. Thus, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors $v_1$ and $v_2$, each of which is perpendicular to one of the lines $l_1$ and $l_2$, the ellipse is a $(v_1, v_2)$-Euclidean circle with center $O$ and radius $\frac{\sqrt{2ab}}{\sqrt{a^2 + b^2}}$. 

Now, by Theorem 5.1 and Theorem 5.2, for any positive real number $r$ and two distinct lines $l_1$ and $l_2$ intersecting at a point $C$, every point $P$ satisfying the equation

$$(d_{E(P, l_1)})^2 + (d_{E(P, l_2)})^2 = r^2 \quad (33)$$

is on an ellipse with the center $C$, semi-major axis $a = \frac{r}{\sqrt{1 - \cos \theta}}$ and semi-minor axes $b = \frac{r}{\sqrt{1 + \cos \theta}}$, where $\theta$ is the non-obtuse angle between the lines $l_1$ and $l_2$, and the lines $l_1$ and $l_2$ are diagonal lines of the rectangle circumscribed the ellipse, whose sides are parallel to the axes of the ellipse. Conversely, by the Theorem 5.4, any point $P$ on this ellipse satisfies the equation

$$(d_{E(P, l_1)})^2 + (d_{E(P, l_2)})^2 = \frac{2 \frac{r^2}{1 - \cos \theta} \frac{r^2}{1 + \cos \theta}}{\frac{r^2}{1 - \cos \theta} + \frac{r^2}{1 + \cos \theta}} = r^2. \quad (34)$$

Clearly, for every ellipse, there are unique pair of lines $l_1$ and $l_2$, and there is unique constant $r^2$ which is the square of the distance from an intersection point of the ellipse and one of the lines $l_1$ and $l_2$ to the other one of them. Notice that we discover a new definition of the ellipse:
**Definition 5.1** In the Euclidean plane, an ellipse is a set of all points for each of which sum of squares of its distances to two intersecting fixed lines is constant. We call each such fixed line an eccentric of the ellipse, and call the chord determined by an eccentric diameter of the ellipse, and half of an eccentric diameter eccentric radius of the ellipse.

Clearly, eccentrices of an ellipse determine the eccentricity -so, the shape- of the ellipse, and vice versa, since the eccentricity is

\[
e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \tan^2 \frac{\theta}{2}}
\]  

where \( \theta \) is the non-obtuse angle between the eccentrices. Notice that ellipses with the same eccentrices -more generally, ellipses having the same angle between their eccentrices- are similar, since they have the same eccentricity (see Figure 16).

![Figure 16. Ellipses with the same eccentrices.](image)

Related to this new ”two-eccentrices” definition of the ellipse, we immediately have the following fundamental conclusions:

**Corollary 5.1** Given a constant \( c \in \mathbb{R}^+ \) and two fixed lines \( l_1 \) and \( l_2 \) intersecting at a point \( C \), having the non-obtuse angle \( \theta \) between them. Then the ellipse with constant \( c \) and eccentrices \( l_1 \) and \( l_2 \) is the ellipse with the center \( C \), having semi-major axis \( a = \frac{\sqrt{c}}{\sqrt{1 - \cos \theta}} \) and semi-minor axis \( b = \frac{\sqrt{c}}{\sqrt{1 + \cos \theta}} \), such that the major axis of the ellipse is the angle bisector of \( \theta \). In addition, this ellipse is a \((v_1, v_2)\)-Euclidean circle with respect to the \((v_1, v_2)\)-Euclidean metric, for \( \lambda_1 = \lambda_2 = 1 \) and linearly independent unit vectors \( v_1 \) and \( v_2 \), each of which is perpendicular to one of the eccentrices of the ellipse, having the center \( C \) and radius \( \sqrt{c} \), which is the Euclidean distance from the intersection point of an eccentric and the ellipse to the other eccentric.

**Corollary 5.2** Given an ellipse with center \( C \), semi-major axis \( a \) and semi-minor axis \( b \). Then, the eccentrices of this ellipse are diagonal lines of the rectangle circumscribed the ellipse whose sides are parallel to the axes of the ellipse, and the constant of this ellipse is \( \frac{2ab}{a^2 + b^2} \), which is the square of the Euclidean distance from the intersection point of an eccentric and the ellipse to the other eccentric. In addition, this ellipse is a \((v_1, v_2)\)-Euclidean circle with center \( C \) and radius \( \frac{\sqrt{ab}}{\sqrt{a^2 + b^2}} \), with respect to the \((v_1, v_2)\)-Euclidean metric, for \( \lambda_1 = \lambda_2 = 1 \) and linearly independent unit vectors \( v_1 \) and \( v_2 \), each of which is perpendicular to one of the eccentrices of the ellipse.

**Corollary 5.3** A \((v_1, v_2)\)-Euclidean circle with center \( C \) and radius \( r \), is an ellipse whose constant is \( r^2 \) and eccentrices are the lines through \( C \) and perpendicular to \( v_1 \) and \( v_2 \). In addition, semi-major and semi-minor axes of this ellipse are \( a = \frac{r}{\sqrt{1 - \cos \theta}} \) and \( b = \frac{r}{\sqrt{1 + \cos \theta}} \) where \( \cos \theta = |v_1^T v_2 + v_2^T v_1| \).
Remark 5.3 Clearly, an ellipse can be determined uniquely by its axes, semi-major axis $2a$ and minor axis $2b$, or simply a rectangle with sides of length $2a$ and $2b$. Here, we see that it can also be determined uniquely by its eccentricities with the angle between them and eccentric diameter $2R$, or simply a rhombus with sides of length $2R$; having the relation $\sqrt{2R} = \sqrt{a^2 + b^2}$. While diagonals of the rectangle (whose length is equal to $2\sqrt{2R}$) give eccentricities of the ellipse, diagonals of the rhombus (whose lengths are equal to $2\sqrt{2}a$ and $2\sqrt{2}b$) give axes of the ellipse (see Figure 17). Obviously, when such a rectangle is given, one can construct the related rhombus, and vice versa. So, for an ellipse whose center and four points of tangency to its rectangle are known, one can construct four points on the eccentricities of the ellipse: they are intersection points of diagonals of the rectangle and sides of the rhombus. Similarly, for an ellipse whose center and four points of tangency to its rhombus are known, one can construct four more points on the ellipse: they are intersection points of diagonals of the rhombus and sides of the rectangle (see also [15] and [10] for construction of an ellipse from a pair of conjugate diameters).

![Figure 17. Axes and eccentricities of an ellipse.](image)

Remark 5.4 Clearly, the eccentricities of ellipse with the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the asymptotes of conjugate hyperbolas with the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are the same (see Figure 18).

![Figure 18. Conics determined by the same semi-axes $a$ and $b$.](image)

One can naturally wonder the answer of the following question: So, what is the set of all points for each of which difference of squares of its distances to two intersecting fixed lines is constant,
and is it a definition for hyperbola? It can easily be seen that for any two intersecting fixed lines this set determines a hyperbola having perpendicular asymptotes: Consider the same lines \( l_i : v_{i1}x + v_{i2}y = 0 \) for \( i = 1, 2 \), and the set of points satisfying the equation

\[
(d_E(P, l_1))^2 - (d_E(P, l_2))^2 = k
\]

for \( k \in \mathbb{R} - \{0\} \), which gives the equation

\[
Ax^2 + By^2 + 2Cx + 2Ey + F = 0
\]

where \( A = (v_{11}^2 - v_{21}^2) \), \( B = (v_{12}^2 - v_{22}^2) \), \( C = (v_{11}v_{12} - v_{21}v_{22}) \), \( D = E = 0 \) and \( F = -k \). Since \( \delta < 0 \) and \( \Delta \neq 0 \), this equation determines a hyperbola (see [23, pp. 232-233]). Moreover, by the theorem given in [20], if \( v_{12} \neq v_{22} \) then slopes \( m_1 \) and \( m_2 \) of the asymptotes are the distinct real roots of the quadratic equation

\[
(v_{12}^2 - v_{22}^2)m^2 + 2(v_{11}v_{12} - v_{21}v_{22})m + (v_{11}^2 - v_{21}^2) = 0,
\]

so the asymptotes are perpendicular since the multiplication of the roots is -1, and if \( v_{12} = v_{22} \) then the hyperbola has perpendicular asymptotes one of them is vertical the other one is horizontal. On the other hand, if we use absolute value for the difference notion in the question, then clearly we get two conjugate hyperbolas having the same perpendicular asymptotes. So, this set does not give a definition for hyperbola (see [16] for the hyperbolas determined by two fixed lines using the distances of a point to the fixed lines).

Notice that, in the section 3 and 4 one can give definitions for rectangle and rhombus using two distinct lines and determine some properties of them, in a similar way.

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