HARISH-CHANDRA BIMODULES IN THE DELIGNE CATEGORY $\text{Rep}(GL_t)$

ALEXANDRA UTIRALOVA

ABSTRACT. In this paper we study the category of Harish-Chandra bimodules $HC_{\chi,\psi}$ in the Deligne category $\text{Rep}(GL_t)$. In particular, we answer Question 3.25 posed in Pavel Etingof’s paper [7] and determine for which central characters $\chi$ and $\psi$ this category is not zero.

1. Introduction

Representation theory in complex rank first started as an example in the paper by Deligne and Milne [5], where the category $\text{Rep}(GL_t)$ for $t$ not necessarily integer was introduced. It was further developed in later papers by Deligne, where he introduced the categories $\text{Rep}(O_t)$, $\text{Rep}(SP_{2n})$ and $\text{Rep}(S_t)$, interpolating the categories of representations of the groups $O_n$, $SP_{2n}$ and $S_n$ correspondingly, and also suggested the ultraproduct realization of these categories [2, 3].

The goal of this paper is to study the categories $HC_{\chi,\psi}$ of Harish-Chandra bimodules for the Lie algebra $\mathfrak{gl}_t$ in the Deligne category $\text{Rep}(GL_t)$ where $t$ is a generic complex number, which interpolate the categories of Harish-Chandra bimodules for $\mathfrak{gl}_n(C)$ with fixed central characters to non-integer values of $n$. Namely, we determine for which values of central characters this category is not zero. This answers Question 3.25 in [7].

Harish-Chandra bimodules for $GL_t$ were defined in [7] as follows. A $(\mathfrak{gl}_t, \mathfrak{gl}_t)$-bimodule $M \in \text{Ind}(\text{Rep}(GL_t))$ is a Harish-Chandra bimodule if it is finitely generated (i.e. it is a quotient of $U(\mathfrak{gl}_t) \otimes U(\mathfrak{gl}_t) \otimes X$ for some $X \in \text{Rep}(GL_t)$), the action of the diagonal copy of $\mathfrak{gl}_t \subset \mathfrak{gl}_t \oplus \mathfrak{gl}_t$ is natural and both copies of $Z(U(\mathfrak{gl}_t))$ act locally finitely on $M$. The category $HC_{\chi,\psi}$ is then the category of Harish-Chandra bimodules on which the left copy of the center $Z(U(\mathfrak{gl}_t))$ acts by a central character $\chi$ and the right one acts by $\psi$.

Our main result is Theorem 3.19 which shows that $HC_{\chi,\psi}$ is nonzero if and only if $\chi(u) - \psi(u) = \sum_{i=1}^{r} e^{b_i u} - \sum_{i=1}^{s} e^{c_i u}$ (central characters for $\mathfrak{gl}_t$ and the generating function notation for them are defined in 2.7).
Acknowledgements. I would like to thank Pavel Etingof for suggesting this problem to me and for all the valuable discussions we had about it.

2. Preliminaries

2.1. Notations. From now on let $k := \overline{\mathbb{Q}}$, let $\mathfrak{h}$, $\mathfrak{b}$ and $\mathfrak{n}_-$ denote the standard Cartan subalgebra, Borel subalgebra of upper-triangular matrices and the subalgebra of strictly lower-triangular matrices in $\mathfrak{gl}_n(k)$ correspondingly, for any $\lambda \in \mathfrak{h}^*$ let $\mathbb{C}_\lambda$ be the extension to $\mathfrak{b}$ of the one-dimensional $\mathfrak{h}$-module corresponding to $\lambda$. Let $W \simeq S_n$ be the Weyl group of $\mathfrak{gl}_n(k)$, let $\Lambda^+$ be the lattice of integral dominant weights and $\rho$ be the half sum of all positive roots of $\mathfrak{gl}_n(k)$. We denote by $V^{(n)}$ the $n$-dimensional defining representation of $GL_n$. By $\Delta$ we will always mean the coproduct map.

2.2. Basic results and definitions.

Definition 2.1. The Deligne category $\text{Rep}(GL_t)$ is the Karoubian envelope (formally adjoining images of idempotents) of the rigid symmetric $\mathbb{C}$-linear tensor category generated by a single object $V$ of dimension $t \in \mathbb{C}$, such that $\text{End}(V^\otimes m) = \mathbb{C}[S_m]$ for all $m \geq 1$.

Let us denote the object $V^\otimes r \otimes (V^*)^\otimes s \in \text{Rep}(GL_t)$ by $[r,s]$.

The following theorem is copied from [7], Theorem 2.9.

Theorem 2.2. [3] The category $\text{Rep}(GL_t)$ has the following universal property: if $\mathcal{D}$ is a symmetric tensor category then isomorphism classes of (possibly non-faithful) symmetric tensor functors $\text{Rep}(GL_t) \to \mathcal{D}$ are in bijection with isomorphism classes of objects of dimension $t$, via $F \mapsto F([1,0])$.

It turns out that for non-integer values of $t$ the categories $\text{Rep}(GL_t)$ are abelian and semisimple (see for example [5], Subsection 9.12).

We will now state the result showing that $\mathcal{C} := \text{Rep}(GL_t)$ can be constructed as a subcategory in the ultraproduct of the categories $\mathcal{C}_n$, where $\mathcal{C}_n$ is the category of finite-dimensional representations of $GL_n$. For more details on the ultrafilters and ultraproducts see [10] or [12]. A very detailed and nice explanation of the following construction can be found in [11]. The original statement for transcendental $t$ is due to Pierre Deligne [3] (but is left without proof). And the similar statement for all values of $t$ (requiring passing to positive characteristics) was proved in [9] by Nate Harman.

Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$. We will fix some isomorphism of fields $\prod_{\mathcal{F}} k \simeq \mathbb{C}$. 

2 ALEXANDRA UTIRALOVA
Theorem 2.3. \[ \mathfrak{C} \] is equivalent to the full subcategory \( \tilde{\mathfrak{C}} \) in \( \prod_F \mathfrak{C}_n \) generated by \( \tilde{V} := \prod_F V^{(n)} \) under the operations of taking duals, tensor products, direct sums, and direct summands if \( t \) is the image of \( \prod_F n \) under the isomorphism \( \prod_F \mathbb{k} \simeq \mathbb{C} \).

The proof of this statement is almost identical to the proof of Theorem 1.1 in [9] or Theorem 1.4.1 in [10].

Proof. Clearly, the categorical dimension of \( \tilde{V} \) is \( t \). Therefore, by Theorem 2.2, we get a symmetric tensor functor \( F : \mathfrak{C} \to \tilde{\mathfrak{C}} \) with \( F([1,0]) = \tilde{V} \). Now, the category \( \text{Rep}(GL_t) \) is generated by \([1,0]\) under the operations of taking duals, tensor products, direct sums, and direct summands. Thus, \( F \) is essentially surjective. It is left to show that it is fully faithful. Since both \( \mathfrak{C} \) and \( \tilde{\mathfrak{C}} \) are Karoubian envelopes of additive categories generated by \([r,s]\) and \( \tilde{V} \otimes (\tilde{V}^*)^S \) correspondingly, it is enough to check that \( F \) induces an isomorphism of algebras

\[ \text{End}_\mathfrak{C}([r,s]) \to \text{End}_\tilde{\mathfrak{C}}(\tilde{V}^r \otimes (\tilde{V}^*)^S). \]

But this is an easy consequence of the Schur-Weyl duality, since both algebras are isomorphic to the walled Brauer algebra \( B_{r,s}(t) \) (it is ensured for the ultraproduct, since it holds for \( \text{End}_{GL_n}((V^{(n)})^r \otimes ((V^{(n)})^*)^S) \) for all \( n > r + s \)).

Remark 2.4. Clearly, one can only obtain transcendental numbers \( t \) as the image of \( \prod_F n \). To get this construction for algebraic \( t \), one would need to consider the representations of \( GL_n \) over some fields of positive characteristic (see [9]).

However, applying automorphisms of \( \mathbb{C} \) over \( \mathbb{k} \), one can show that any transcendental \( t \) can be obtained in this manner.

So, from now on we assume for simplicity that \( t \) is non-algebraic. We expect, however, that similar results hold for all non-integer values of \( t \).

Let us denote by \( \text{Ind}(\mathfrak{C}) \) the category of ind-objects (i.e. filtered colimits of regular objects) of \( \mathfrak{C} \).

Definition 2.5. Let \( g = \mathfrak{gl}_t = V \otimes V^* \). It is a Lie algebra in \( \mathfrak{C} \). Let us denote the commutator map \( g \otimes g \to g \) by \( c \). We define its universal enveloping algebra \( U(g) \in \text{Ind}(\mathfrak{C}) \) as a quotient of the tensor algebra \( T(g) := \bigoplus_{k=0}^\infty g^\otimes k \) by the ideal generated by the image of the map \( r : g \otimes g \to g \oplus (g \otimes g) \subset T(g) \),

\[ r = c \oplus (\sigma_g - id_g), \]

where \( \sigma_g : g \otimes g \to g \otimes g \) is the permutation of tensor factors.
For any object $X = \prod_{\mathcal{F}} X^{(n)}$ of $\mathcal{C}$ we can define an action map
\[ a_X : g \otimes X \to X \]
as the ultraproduct $\prod_{\mathcal{F}} a_{X^{(n)}}$, where $a_{X^{(n)}} : gl_n(k) \otimes X^{(n)} \to X^{(n)}$ is the natural action of $gl_n(k)$ on $X^{(n)}$. Clearly, $a_X$ is a Lie algebra action, i.e.
\[ (a_X + (a_X \circ (id_g \otimes a_X))) \circ (r \otimes id_X) = 0 \]
as a map from $g \otimes g \otimes X$ to $X$ (where $r$ is the map $g \otimes g \to g \oplus (g \otimes g)$ defined in [2.5]). That is, $a_X$ induces the action of $U(g)$ on $X$. We will refer to this action as the natural action of $g$ (or $U(g)$).

Clearly, the natural action of $g$ on $\mathcal{C}$ extends to $\text{Ind}(\mathcal{C})$.

We define the Poincaré–Birkhoff–Witt (or PBW) filtration $F$ on $U(g)$ as the image of $T^i(g) := g^{\otimes i}$ under the quotient map $T(g) \to U(g)$. We have
\[ F_i U(g) := \prod_{\mathcal{F}} F_i U(gl_n(k)), \]
where we abuse the notation and denote by $F$ the PBW-filtration on $U(gl_n(k))$ as well.

The center of $U(g)$
\[ Z(U(g)) = U(g)^{GL_t} = \text{Hom}(\mathbb{1}, U(g)) = \bigcup_i \text{Hom}(\mathbb{1}, F_i U(g)) \]
is a filtered algebra in the category of $\mathbb{C}$-vector spaces. We have
\[ F_i Z(U(g)) := Z(U(g)) \cap F_i U(g) = \prod_{\mathcal{F}} (Z(U(gl_n(k))) \cap F_i U(gl_n(k))). \]

The Harish-Chandra isomorphism tells us that $Z(U(gl_n(k)))$ is isomorphic to the algebra of symmetric polynomials $k[h^*]^W$ with filtration given by the degree. Given a central element $C$ we recover the corresponding symmetric polynomial $p$ by looking at the action of $C$ on the Verma module $M_\lambda := U(gl_n(k)) \otimes_{U(h)} k_{\lambda - \rho}$ with the highest weight $\lambda - \rho \in h^*$. We have $p(\lambda) = C|_{M_\lambda}$.

**Remark 2.6.** The shift of the highest weight in the definition of $M_\lambda$ is needed to replace the dot-action of $W$ on $h^*$ with the usual action.

Symmetric polynomials in $n$ variables are freely generated (as an algebra) by the first $n$ power sums polynomials $p_k := \sum_i x_i^k$. For any $k$ and any $n$ we define the element $C_k \in Z(U(gl_n(k)))$ to be the image of $p_k$ under the isomorphism $k[x_1, \ldots, x_n]^S_n \to Z(U(gl_n(k)))$. I.e. $C_k$ is the central element, which acts on each $M_\lambda$ via the constant $\sum_i \lambda_i^k$.

We get that $Z(U(gl_n(k))) \simeq k[C_1, \ldots, C_n]$ with deg $C_k = k$. And therefore,
\[ Z(U(g)) \simeq \mathbb{C}[C_1, C_2, \ldots], \]
\[ C_i := \prod_{\mathcal{F}} C_i \] (by the abuse of notation).

**Definition 2.7.** A central character is an algebra homomorphism
\[ \psi : Z(U(\mathfrak{g})) \to \mathbb{C}. \]

By the result above, \( \psi \) is completely determined by the numbers \( \psi_k = \psi(C_k) \). For convenience let us adopt the (exponential) generating function notation \( \psi(u) = \frac{1}{(e^u - 1)} \sum \frac{1}{k!} \psi_k u^k \in \mathbb{C}(\langle u \rangle) \), where we put \( \psi_0 = 1 \).

**Remark 2.8.** We divide the generating function by \( (e^u - 1) \) only for the reason that it yields better looking formulas, and we treat the factor \( \frac{1}{(e^u - 1)} \) formally.

For each \( \mathfrak{gl}_n(\mathbb{K}) \) let us fix a central character \( \psi^{(n)} : Z(U(\mathfrak{gl}_n(\mathbb{K}))) \to \mathbb{K} \). It is determined by \( n \) numbers \( \psi_k^{(n)} = \psi^{(n)}(C_k) \) with \( 1 \leq k \leq n \). However, since the elements \( C_k \) are defined for all \( k \), we can define the (exponential) generating function \( \psi^{(n)}(u) = \frac{1}{(e^u - 1)} \sum \frac{1}{k!} \psi_k^{(n)} u^k \in \mathbb{K}(\langle u \rangle) \). Due to the algebraic independence of \( \{C_k\} \) in \( Z(U(\mathfrak{gl}_n(\mathbb{K}))) \) we can choose \( \psi_k^{(n)} \) to be an arbitrary number for \( n > k \). Thus, any central character \( \psi \) of \( \mathfrak{g} \) can be obtained as an ultraproduct of central characters \( \psi^{(n)} \) of \( \mathfrak{gl}_n(\mathbb{K}) \) (i.e. \( \psi_k = \prod_{\mathcal{F}} \psi_k^{(n)} \)). We write \( \psi = \prod_{\mathcal{F}} \psi^{(n)} \).

### 3. The category \( HC_{\chi,\psi} \)

#### 3.1. Definitions and notations.

We will henceforth consider objects with the action of \( \mathfrak{g} \oplus \mathfrak{g} \). Let us denote by \( \mathfrak{g}_l \), \( \mathfrak{g}_r \), and \( \mathfrak{g}_d \) the left, the right and the diagonal copy of \( \mathfrak{g} \) inside \( \mathfrak{g} \oplus \mathfrak{g} \) (with the diagonal copy being the image of the map \( \text{id}_\mathfrak{g} \oplus \text{id}_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \)). Moreover, we will freely switch between the left action of \( \mathfrak{g} \oplus \mathfrak{g} \) and the double action of \( \mathfrak{g} \) both on the left and on the right, the first given by the action of \( \mathfrak{g}_l \) and the second by minus the action of \( \mathfrak{g}_r \).

Clearly, any left \( \mathfrak{g} \)-module \( M \) in \( \text{Ind}(\mathbb{C}) \) with the action map \( f : \mathfrak{g} \otimes M \to M \) has a unique action of \( \mathfrak{g} \oplus \mathfrak{g} \), s.t. \( \mathfrak{g}_l \) acts by the original action and the diagonal copy \( \mathfrak{g}_d \) acts naturally, i.e. via the map \( a_M \). Indeed, one defines the new action map \( (\mathfrak{g} \oplus \mathfrak{g}) \otimes M \to M \) as \( f \oplus (a_M - f) \).

**Definition 3.1.** A Harish-Chandra bimodule for \( GL_t \) is a \( \mathfrak{g} \oplus \mathfrak{g} \)-module \( M \in \text{Ind}(\mathbb{C}) \), such that

1. \( M \) is finitely generated, i.e. it is a quotient of \( (U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes X \) for some \( X \in \mathbb{C} \);
2. \( \mathfrak{g}_d \) acts naturally (i.e. via \( a_M \)) on \( M \);
3. the center \( Z(U(\mathfrak{g}) \otimes U(\mathfrak{g})) \) acts locally finitely on \( M \), i.e. \( \text{Ann}_{Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))} M \) is an ideal of finite codimension.
Any maximal ideal in $Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$ (i.e. ideal of codimension 1) corresponds to a pair of central characters $\chi, \psi : Z(U(\mathfrak{g})) \to \mathbb{C}$. Denote by $HC_{\chi,\psi}$ the category of Harish-Chandra bimodules on which $Z(U(\mathfrak{g}))$ acts via $\chi$ and $Z(U(\mathfrak{g}))$ acts via $\psi$.

**Example.** Let $U_\chi := U(\mathfrak{g})/(z - \chi(z))$, where $z$ runs over all elements in the center. Clearly, $U_\chi \in HC_{\chi,\chi}$.

**Note.** For each $i$ we have $F_i U_\chi = \prod F_i U(\mathfrak{gl}_n(k))/(z - \chi(n)(z))$, where $\chi = \prod F \chi(n)$.\footnote{We will abuse the notation and denote by $U_\chi$ the quotient $U(\mathfrak{gl}_n(k))/(z - \chi(z))$ too, when it is clear from the context which one we refer to.}

Clearly, every irreducible Harish-Chandra bimodule must lie in one of the categories $HC_{\chi,\psi}$, so they are interesting to study.

**Remark 3.2.** Note that for regular Harish-Chandra bimodules (for $GL_n$) condition (2) of Definition 3.1 translates to the fact that one can integrate the action of the diagonal copy $(\mathfrak{gl}_n)^d$ on $M$ to the action of $GL_n$; and condition (3) is equivalent to $M$ having a finite $K$-type, which means that for any simple $GL_n$ representation $L$ the multiplicity space $\text{Hom}_{GL_n}(L, M)$ is finite-dimensional (see [1], Proposition 5.3). However, in $\text{Rep}(GL_t)$ the latter is no longer true. Harish-Chandra bimodules of finite $K$-type are studied to some extent in [13]. One can also find there some non-trivial examples of irreducible Harish-Chandra bimodules.

For any object $X \in \mathcal{C}$ the tensor product $U_\psi \otimes X$ is naturally a left $\mathfrak{g}$-module with $\mathfrak{g}$ acting on $U_\psi$ by multiplication on the left and on $X$ naturally (so it is easy to deduce that the right action of $\mathfrak{g}$ is by (minus) multiplication on the right and affects only the $U_\psi$ part of the tensor product). Let $N(\chi, \psi, X) := (U_\psi \otimes X)_{\chi}$ denote the quotient $(U_\psi \otimes X)/(z - \chi(z))(U_\psi \otimes X)$, where we factor out by the action of the left copy of $Z(U(\mathfrak{g}))$. Clearly, $N(\chi, \psi, X) \in HC_{\chi,\psi}$.

**3.2. Bimodules $N(\chi, \psi, X)$.**

**Lemma 3.3.** (see [7] Section 3.6) The category $HC_{\chi,\psi}$ is nonzero iff $N(\chi, \psi, X) \neq 0$ for some $X \in \mathcal{C}$.

**Proof.** For any $X \in \mathcal{C}$ we have that as $(\mathfrak{g}, \mathfrak{g})$-bimodules

$$N(\chi, \psi, X) := U_\chi \otimes_{U(\mathfrak{g}_d)} (U_\psi \otimes X) \simeq (U_\chi \otimes U_{\psi}^{op})_{U(\mathfrak{g}_d)} \otimes X,$$

where $\mathfrak{g}_d$ acts on the right on $U_\chi \otimes U_{\psi}^{op}$.
Now suppose $M \in HC_{\chi,\psi}$, then $M$ is finitely generated, i.e. it is generated by some subobject $X$ (that lies in $C$). There is a natural morphism $N(\chi, \psi, X) \to M$ whose image is the subbimodule of $M$ generated by $X$ (since $X$ generates $M$, it is surjective). Thus, if $M$ is nonzero then so is $N(\chi, \psi, X)$. ■

**Corollary 3.4.** The category $HC_{\chi,\psi}$ is nonzero iff $N(\chi, \psi, [r, s]) \neq 0$ for some $[r, s]$.

So, we want to understand for which $\chi, \psi$ the bimodule $(U_\psi \otimes [r, s])_\chi$ is not zero. For this purpose let us first look at $(U_\psi \otimes V)_\chi$ and understand for which $\chi$ and $\psi$ it is not zero.

### 3.3. The basic case for $\mathfrak{gl}_n(\mathbb{k})$

Let us look at the finite-dimensional case, namely $\mathfrak{gl}_n(\mathbb{k})$. We have a homomorphism $U_\psi \to \text{End}_\mathbb{k}(M_\lambda)$ for some $\lambda = (\lambda_1, \ldots, \lambda_n)$ (determined up to permutation of $\lambda_i$) which is injective by Duflo’s theorem (which states that the annihilator of a Verma module is the ideal generated by the kernel of the corresponding central character, see [6], Theorem 8.4.3). So there is an injective morphism of $\mathfrak{gl}_n(\mathbb{k})$-bimodules

$$U_\psi \otimes V^{(n)} \hookrightarrow \text{Hom}_\mathbb{k}(M_\lambda, M_\lambda \otimes V^{(n)}),$$

where the right action of $\mathfrak{gl}_n(\mathbb{k})$ is on the source and the left action is on the target. Thus, it is enough to consider the action of the center on $M_\lambda \otimes V^{(n)}$ (instead of considering the left action on $U_\psi \otimes V^{(n)}$). For generic (without nontrivial stabilizers in $W$) weight $\lambda$ we have $M_\lambda \otimes V^{(n)} = \bigoplus_{i=1}^n M_{\lambda_i+e_i}$, where $\lambda + e_i = (\lambda_1, \ldots, \lambda_i-1, \lambda_i + 1, \lambda_{i+1}, \ldots, \lambda_n)$. When restricted to the direct summand $M_{\lambda + e_i}$, each $C_k$ acts by $\sum_{i=1}^n \lambda_i^k + (\lambda_i + 1)^k - \lambda_i^k$.

Let $\Omega := \frac{1}{2}(\Delta(C_2) - C_2 \otimes 1 - 1 \otimes 1)$. Then

$$(2\Omega + 1)|_{M_{\lambda+e_i}} = 2\lambda_i + 1$$

and thus,

$$(\Delta(C_k) - C_k \otimes 1)|_{U_\psi \otimes V^{(n)}} = (\Omega|_{U_\psi \otimes V^{(n)}} + 1)^k - \Omega^k|_{U_\psi \otimes V^{(n)}}.$$

Let us denote $P_k(b) := (b+1)^k - b^k$. Then, we have proved the following lemma:

**Lemma 3.5.** For generic central character $\psi$, i.e. such that $U_\psi$ acts on $M_\lambda$ with generic $\lambda$, we have:

$$\Delta(C_k)|_{U_\psi \otimes V^{(n)}} - C_k \otimes 1|_{U_\psi \otimes V^{(n)}} = P_k(\Omega|_{U_\psi \otimes V^{(n)}}).$$

Now let us identify the $\mathbb{k}$-algebra $\text{End}_\mathbb{k}(M_\lambda)$ with $A := \text{End}_\mathbb{k}(U(n_-))$ via the natural isomorphism of vector spaces $M_\lambda$ and $U(n_-)$. The
representations \( U(\mathfrak{gl}_n(\mathbb{k})) \rightarrow \text{End}_k(M_\lambda) \) thus give us a family of algebra homomorphisms to \( A \) depending on \( \lambda \) polynomially. That is to say, we have a map of algebras \( \varphi: U(\mathfrak{gl}_n(\mathbb{k})) \rightarrow A \otimes \mathbb{k}[x_1, \ldots, x_n] \), which composed with the quotient map \( \mathbb{k}[x_1, \ldots, x_n] \rightarrow \mathbb{k} \) sending \( x_i \) to \( \lambda_i \), and the natural isomorphism between \( A \) and \( \text{End}_k(M_\lambda) \), gives us precisely the representation of \( \mathfrak{gl}_n(\mathbb{k}) \) on \( M_\lambda \). Let us denote by \( m_\lambda \) the maximal ideal generated by \( x_i - \lambda_i, 1 \leq i \leq n \).

The family of representations of \( \mathfrak{gl}_n(\mathbb{k}) \oplus \mathfrak{gl}_n(\mathbb{k}) \) on \( M_\lambda \otimes V(n) \) for varying \( \lambda \) (with the left copy acting on \( M_\lambda \) and the right copy acting on \( V(n) \)) produces a map \( \varphi \otimes \rho_{V(n)} : U(\mathfrak{gl}_n(\mathbb{k})) \otimes U(\mathfrak{gl}_n(\mathbb{k})) \rightarrow A \otimes \mathbb{k}[x_1, \ldots, x_n] \otimes \text{End}_k(V(n)) \). We have proved above that the image of \( \Delta(C_k) - C_k \otimes 1 - P_k(\Omega) \) under \( \varphi \otimes \rho_{V(n)} \) has to lie in \( A \otimes (\bigcap_{\lambda \text{generic}} m_\lambda) \otimes \text{End}_k(V(n)) \). Since the set of \( \lambda \in \mathbb{k}^n \) such that some \( \lambda_i = \lambda_j \) for \( i \neq j \) (i.e. \( \lambda \) has a nontrivial stabilizer in \( W \) and hence is non-generic by our definition) is closed in \( \mathbb{k}^n \), the intersection \( \bigcap_{\lambda \text{generic}} m_\lambda \) is zero. Thus, \( \Delta(C_k) - C_k \otimes 1 - P_k(\Omega) \) acts as zero on \( M_\lambda \otimes V(n) \) for any (not necessary generic) \( \lambda \).

**Corollary 3.6.** For any central character \( \psi \) of \( U(\mathfrak{gl}_n(\mathbb{k})) \)

\[
\Delta(C_k)|_{U_\psi \otimes V(n)} - C_k \otimes 1|_{U_\psi \otimes V(n)} = P_k(\Omega)|_{U_\psi \otimes V(n)}.
\]

**Remark 3.7.** We have \( \frac{1}{(e^n-1)} \sum \frac{1}{k!} P_k(b) u^k = e^{bu} \).

Recall that \( \Omega \) was defined as \( \frac{1}{2}(\Delta(C_2) - C_2 \otimes 1 - 1 \otimes 1) \). The element \( C_k \in Z(U((\mathfrak{gl}_n)_r)) \) acts on \( U_\psi \otimes V(n) \) as \( C_k \otimes 1 \), and \( C_k \in Z(U((\mathfrak{gl}_n)_l)) \) acts on \( U_\psi \otimes V(n) \) as \( \Delta(C_k) \). Thus, \( \Omega \) acts on \( (U_\psi \otimes V(n))_\chi \) as a constant \( b = \frac{1}{2}(\chi_2 - \psi_2 - 1) \). So, by Corollary 3.6, \( (U_\psi \otimes V(n))_\chi \neq 0 \) only when \( \chi(u) - \psi(u) = e^{bu} \).

Thus, for each \( \mathfrak{gl}_n(\mathbb{k}) \) we must have \( \chi^{(n)}(u) - \psi^{(n)}(u) = e^{bu} \) for some \( b_n \in \mathbb{k} \). So, in the case of \( \mathfrak{gl}_1 \), \( \chi(u) - \psi(u) = e^{bu} \) for \( b = \prod x b_n \).

### 3.4. The basic case for \( \mathfrak{gl}_1 \)

We have just proved the following statement.

**Lemma 3.8.** For any central characters \( \psi, \chi \) of \( U(\mathfrak{g}) \) the bimodule \( (U_\psi \otimes V)_\chi \) is nonzero only if

\[
\chi(u) - \psi(u) = e^{bu}
\]

for some \( b \in \mathbb{C} \).

**Lemma 3.9.** For any finitely generated \( (\mathfrak{g}, \mathfrak{g}) \)-bimodule \( M \) in \( \text{Ind}(\mathfrak{c}) \)

\[
(\Delta(C_k) - C_k \otimes 1)|_{M \otimes V} = P_k(\Omega)|_{M \otimes V}.
\]
We also denote by $A \ldots \otimes Z$ notation.

Note. It is easy to see that the same proof works for the module $M = U_\psi \otimes U(\mathfrak{g}) \otimes X$, where $X \in \mathcal{C}$ and $U(\mathfrak{g})$ acts only on the $U_\psi$ part of the tensor product. $X$ is represented by some sequence of $GL_n$-modules $(X_1, X_2, \ldots)$, so we can pass to a finite dimensional case. There $U_\psi \otimes U(\mathfrak{g}(\kappa)) \otimes X_n \otimes V \mapsto \bigoplus_{j=1}^n \text{Hom}_{k}(M_\lambda, M_{\lambda+v_j}) \otimes U(\mathfrak{g}_n(\kappa)) \otimes X_n$ and $\mathfrak{g}_n(\kappa)_l$ acts only on the Hom-part of the tensor product. This ends the proof. Thus, we have obtained the following:

$$\left(\Delta(C_k) - C_k \otimes 1\right)\left|_{(U_\psi \otimes U(\mathfrak{g}))^y \otimes X} = P_k(\Omega_{(U_\psi \otimes U(\mathfrak{g}))^y \otimes X \otimes V}.\right.$$}

For any $n \in \mathbb{Z}_{>0}$ the universal enveloping algebra $U(\mathfrak{g}_n(\kappa))$ embeds into $\bigoplus_{\chi^{(n)}} U(\chi^{(n)})$, where the sum runs over all central characters $\chi^{(n)}: Z(U(\mathfrak{g}_n(\kappa))) \to \kappa$. Therefore, the same holds for $U(\mathfrak{g})$. Hence, the statement of the lemma holds for $M = U(\mathfrak{g}) \otimes U(\mathfrak{g})^y \otimes X$, where $X \in \mathcal{C}$.

Finally, by definition, any finitely generated $U(\mathfrak{g})$-module is a quotient of $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes X$ with $X \in \mathcal{C}$ and $U(\mathfrak{g})$ acting only on the leftmost part of this tensor product. This ends the proof. 

Remark 3.10. A similar reasoning shows that for any finitely generated $U(\mathfrak{g})$-module $M$ in $\text{Ind}(\mathcal{C})$

$$(\Delta(C_k) - C_k \otimes 1)|_{M \otimes V^*} = P_k(\Omega_M|_{M \otimes V^*}),$$

where $P_k(c) = c - 1)^k - c^k$.

Note. It is easy to see that $\frac{1}{e^{c-1}} \sum P_k(c)u^k = -e^{(c-1)u}$.

3.5. The general case. Now we are ready to consider the action of the central elements on $U_\psi \otimes [r, s]$. Note that the central element $C$ of $Z(U(\mathfrak{g}_r))$ acts on this module as $(C \otimes 1)_{U_\psi,[r,s]}$, and if it is considered as an element of $Z(U(\mathfrak{g}_r))$ then it acts as $\Delta(C)_{U_\psi,[r,s]}$ meaning that the first tensor factor acts on $U_\psi$ and the second acts on $[r, s]$.

Let us adopt some convenient notation. Let $A$ be any element in $Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$. Denote by $A_j$ its image under the following map $\tau_j : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})^\otimes (m+1)$:

$$\tau_j = (\Delta^{j-1} \otimes \text{id}) \otimes 1 \otimes \ldots \otimes 1.$$

We also denote by $A_j$ the corresponding operator acting on $X_0 \otimes X_1 \otimes \ldots \otimes X_m$, where each $X_i$ is a $U(\mathfrak{g})$-module.
Let us view $U_\psi \otimes [r,s]$ as the string of tensor factors

$$U_\psi \otimes V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*.$$ 

Then

$$(\Delta(C_k) - C_k \otimes 1)_{U_\psi [r,s]} = \Delta^{r+s}(C_k) - (C_k \otimes 1)_1 =$$

$$(\Delta^{r+s-1} \otimes \text{id})(\Delta(C_k) - C_k \otimes 1) + \Delta^{r+s-1}(C_k) \otimes 1 - (C_k \otimes 1)_1 =$$

$$(\Delta(C_k) - C_k \otimes 1)_{r+s} + \Delta^{r+s-1}(C_k) \otimes 1 - (C_k \otimes 1)_1 =$$

$$= \ldots =$$

$$(\Delta(C_k) - C_k \otimes 1)_{r+s} + (\Delta(C_k) - C_k \otimes 1)_{r+s-1} + \ldots + (\Delta(C_k) - C_k \otimes 1)_1.$$ 

By Lemma 3.9 and Remark 3.10 this is equal to

$$\mathcal{P}_k(\Omega_{r+s}) + \ldots + \mathcal{P}_k(\Omega_{r+1}) + P_k(\Omega_r) + \ldots P_k(\Omega_1).$$

**Theorem 3.11.** $(U_\psi \otimes [r,s])_\chi \neq 0$ only if there exist numbers $b_1, \ldots, b_r,$ $c_1, \ldots, c_s \in \mathbb{C}$ such that

$$\chi(u) - \psi(u) = \sum_{i=1}^{r} e^{b_i u} - \sum_{i=1}^{s} e^{(c_i-1) u}.$$ 

**Proof.** Let us consider $\operatorname{End}_k( (U_\psi \otimes [r,s])_\chi)$. It is a nonzero algebra and there is a homomorphism from $\mathbb{C}[\Omega_1, \ldots, \Omega_{r+s}]$ to it. Its image is a nonzero finitely generated commutative algebra where

$$\mathcal{P}_k(\Omega_{r+s}) + \ldots + \mathcal{P}_k(\Omega_{r+1}) + P_k(\Omega_r) + \ldots P_k(\Omega_1) = \chi_k - \psi_k.$$ 

Thus, by the Nullstellensatz, there exists a maximal ideal in this algebra, i.e. numbers $b_1, \ldots, b_r, c_1, \ldots, c_s$, such that

$$\mathcal{P}_k(c_s) + \ldots + \mathcal{P}_k(c_1) + P_k(b_r) + \ldots P_k(b_1) = \chi_k - \psi_k.$$ 

This ends the proof. $\blacksquare$

**Corollary 3.12.** If $HC_{\chi,\psi} \neq 0$ then there exist numbers $b_1, \ldots, b_r,$ $c_1, \ldots, c_s \in \mathbb{C}$ such that

$$\chi(u) - \psi(u) = \sum_{i=1}^{r} e^{b_i u} - \sum_{i=1}^{s} e^{(c_i-1) u}.$$
3.6. **The reverse direction: constructing a nonzero object in** \( HC_{\chi,\psi} \). Now we want to prove the converse, i.e. if such numbers exist, then the category is nonzero. The remainder of this paper will be devoted to proving the following theorem.

**Theorem 3.13.** Suppose there exist numbers \( b_1, \ldots, b_r, c_1, \ldots, c_s \in \mathbb{C} \) such that

\[
\chi(u) - \psi(u) = \sum_{i=1}^{r} e^{b_i u} - \sum_{i=1}^{s} e^{(c_i - 1) u}.
\]

Then \( HC_{\chi,\psi} \neq 0 \).

To prove this we are going to show that \((U_\psi \otimes X)_\chi\) is nonzero for some \( X \in \mathcal{C} \).

**Lemma 3.14.** Let \( b_i = \prod_F b_i^{(n)} \), for \( 1 \leq i \leq r \) and \( c_i = \prod_F c_i^{(n)} \), for \( 1 \leq i \leq s \), with \( b_i^{(n)}, c_i^{(n)} \in \mathbb{k} \).

Then for any \( \psi(u) \) there exists a presentation of it as an ultraproduct of \( \psi^{(n)}(u) \) - central characters of \( gl_n(\mathbb{k}) \) - so that \( \psi^{(n)}(u) = 0 \) for \( n \leq r + s \) and when \( n > r + s \) then \( gl_n(\mathbb{k}) \) acts with central character \( \psi^{(n)} \) on some \( M_{\mu^{(n)}} \), where

\[
\mu_i^{(n)} = b_i^{(n)}, \quad 1 \leq i \leq r
\]

\[
\mu_{n-j+1}^{(n)} = c_j^{(n)}, \quad 1 \leq j \leq s
\]

**Proof.** Let us take an arbitrary presentation of the numbers \( \psi_k \) as an ultraproduct: \( \psi_k = \prod_F \phi_k^{(n)} \).

We will prove that for a fixed \( k \) we can change finitely many of the numbers \( \phi_k^{(n)} \), namely those with \( n < k + s + r \) so that the resulting central characters satisfy the condition above.

For a fixed \( n \) consider the following equations on \( m := n - r - s \) variables \( x_i \) for \( k = 1, \ldots, m \):

\[
(3.14.1) \ (b_1^{(n)})^k + \ldots + (b_r^{(n)})^k + (c_1^{(n)})^k + \ldots + (c_s^{(n)})^k + x_1^k + \ldots + x_m^k = \phi_k^{(n)}.
\]

They are equations on the first \( m \) power sums of \( x_i \). The ring of symmetric polynomials in \( m \) variables is freely generated by the first \( m \) power sums, so these equations determine a point in the maximal spectrum of \( \mathbb{k}[x_1, \ldots, x_m]^S_m \). The inclusion \( \mathbb{k}[x_1, \ldots, x_m]^S_m \hookrightarrow \mathbb{k}[x_1, \ldots, x_m] \) induces a surjective map on the maximal spectra (the quotient map by the \( S_m \)-action), thus, there exists a solution \( x_i = a_i^{(n)} \) for \( 1 \leq i \leq m \).

Put

\[
\mu_1^{(n)} = b_1^{(n)}, \ldots, \mu_r^{(n)} = b_r^{(n)},
\]

\[
\mu_{r+1}^{(n)} = a_1^{(n)}, \ldots, \mu_{n-s}^{(n)} = a_{n-r-s}^{(n)}.
\]
Consider the central character $\psi^{(n)}$ corresponding to the $\mu^{(n)}$ above. Then for $1 \leq k \leq n - r - s$ we have

$$\psi_k^{(n)} := \sum_{i=1}^{n} (\mu_i^{(n)})^k = \sum_{i=1}^{r} (\mu_i^{(n)})^k + \sum_{i=1}^{m} (a_i^{(n)})^k + \sum_{i=1}^{s} (c_i^{(n)})^k = \phi_k^{(n)} \quad \text{by (3.14.1).}$$

Putting $\psi^{(n)} = 0$ for $n \leq r + s$, we see that for a fixed $k$ we have $\psi_k^{(n)} = \phi_k^{(n)}$ for all $n \geq k + r + s$ and hence, $\prod \psi_k^{(n)} = \prod \phi_k^{(n)} = \psi_k$.

**Lemma 3.15.** Let $n > r + s$ and

$$\lambda = \mu + e_1 + \ldots + e_r - e_{n-s+1} - \ldots - e_n.$$ 

Suppose $Z(U(\mathfrak{gl}_n(k)))$ acts on $M_{\lambda}$ with character $\chi$ and on $M_\mu$ with character $\psi$. Then

$$\chi(u) - \psi(u) = e^{\mu_1}u + \ldots + e^{\mu_r}u - e^{(\mu_{n-s+1}-1)}u - \ldots - e^{(\mu_{n}-1)}u.$$ 

**Proof.** The element $C_k$ acts on $M_{\mu+\epsilon_1}$ as $\sum_j \mu_j + P_k(\mu_t)$ and on $M_{\mu-\epsilon_1}$ as $\sum_j \mu_j + P_k(\mu_l)$. Thus,

$$C_k|_{M_{\mu+\epsilon_1}} - C_k|_{M_{\mu}} = P_k(\mu_t)$$

and

$$C_k|_{M_{\mu-\epsilon_1}} - C_k|_{M_{\mu}} = P_k(\mu_l).$$

We put $\lambda^{[0]} := \lambda, \lambda^{[i]} := \lambda^{[i-1]} - e_i, 1 \leq i \leq r$ and $\mu^{[i]} := \mu^{[i-1]} - e_{n-i+1}, 1 \leq i \leq s$ with $\mu^{[0]} := \mu$ (thus $\mu^{[s]} = \lambda^{[r]}$):

$$\mu^{[0]} = (\mu_1, \ldots, \mu_n),$$

$$\mu^{[1]} = (\mu_1, \ldots, \mu_{n-1}, \mu_n - 1),$$

$$\mu^{[2]} = (\mu_1, \ldots, \mu_{n-2}, \mu_{n-1} - 1, \mu_n - 1),$$

$$\ldots$$

$$\mu^{[s]} = \lambda^{[r]} = (\mu_1, \ldots, \mu_{n-s}, \mu_{n-s+1} - 1, \ldots, \mu_n - 1),$$

$$\lambda^{[r-1]} = (\mu_1, \ldots, \mu_{r-1}, \mu_r + 1, \ldots, \mu_{n-s}, \mu_{n-s+1} - 1, \ldots, \mu_n - 1),$$

$$\lambda^{[r]} = (\mu_1, \ldots, \mu_{r+1}, \ldots, \mu_{n-s}, \mu_{n-s+1} - 1, \ldots, \mu_n - 1).$$

Now let $\lambda^{[i]}$ be the central character corresponding to $\lambda^{[i]}$ and $\psi^{[i]}$ be the central character corresponding to $\mu^{[i]}$. Thus, $\lambda_k^{[i]} - \lambda_k^{[i+1]} = P_k(\lambda_k^{[i+1]}) = P_k(\mu_{i+1})$ and $\psi_k^{[i+1]} - \psi_k^{[i]} = P_k(\psi_k^{[i+1]}) = P_k(\mu_{i+1})$. Summing up these equations for all $\chi$’s and $\psi$’s we obtain

$$\chi_k - \psi_k = P_k(\mu_1) + \ldots + P_k(\mu_r) + P_k(\mu_{n-s+1}) + \ldots + P_k(\mu_n),$$
which leads to the desired relation for generating functions $\chi(u)$ and $\psi(u)$. ■

**Corollary 3.16.** (of Lemmas 3.14 and 3.15)

Suppose

$$\chi(u) - \psi(u) = \sum_{i=1}^{r} e^{b_i u} - \sum_{i=1}^{s} e^{(c_i-1)u}$$

for some $b_i \in \mathbb{C}, 1 \leq i \leq r, c_j \in \mathbb{C}, 1 \leq j \leq s$. Then there exist presentations of $\chi$ and $\psi$ as ultraproducts of $\chi^{(n)}$ and $\psi^{(n)}$ correspondingly, so that for any $n > r + s$ the algebra $U(\mathfrak{gl}_n(k))$ acts on some $M_{\chi^{(n)}}$ with central character $\chi^{(n)}$ and if $\mu^{(n)} = \lambda^{(n)} - e_1 - \ldots - e_r + e_{n-s+1} + \ldots + e_n$, then $U(\mathfrak{gl}_n(k))$ acts on $M_{\mu^{(n)}}$ with central character $\psi^{(n)}$.

**Proof.** We apply Lemma 3.14 to $\psi(u), b_i, c_j$ to obtain central characters $\psi^{(n)}$ and weights $\mu^{(n)}$, satisfying the conditions of the lemma. Then for $n > r + s$, we put $\lambda^{(n)} = \mu^{(n)} + e_1 + \ldots + e_r - e_{n-s+1} - \ldots - e_n$ and denote by $\chi^{(n)}$ the central character corresponding to $\lambda^{(n)}$.

We use Lemma 3.15 for $\lambda^{(n)}$ and $\mu^{(n)}$ to see that

$$\chi^{(n)}(u) - \psi^{(n)}(u) = e^{b_1^{(n)}u} + \ldots + e^{b_r^{(n)}u} - e^{(c_1^{(n)} - 1)u} - \ldots - e^{(c_s^{(n)} - 1)u}.$$ 

We put $\chi^{(n)} = 0$ for $n \leq r + s$. If now $\bar{\chi}(u) = \prod_{F} \chi^{(n)}(u)$ then

$$\bar{\chi}(u) - \psi(u) = e^{b_1 u} + \ldots + e^{b_r u} - e^{(c_1 - 1)u} - \ldots - e^{(c_s - 1)u}$$

and thus $\bar{\chi}(u) = \chi(u)$ and we are done. ■

**Lemma 3.17.** Let $\lambda, \mu, \chi$ be such that $\lambda - \mu \in \Lambda^+$ and suppose $X$ is a finite-dimensional $\mathfrak{gl}_n(k)$-module, with maximal weight $\lambda - \mu$, let $\chi$ be the central character corresponding to $\lambda$ and $\psi$ be the central character corresponding to $\mu$. Then we have

$$(U_{\psi} \otimes X)_{\chi} \neq 0.$$ 

Moreover, if $F_0 U_{\psi}$ is the zeroth filtered component, that is the span of $1$, the quotient $(F_0 U_{\psi} \otimes X)_{\chi}$ is nonzero.

**Proof.** Consider a natural surjective map of $U(\mathfrak{gl}_n(k))$-modules

$$U_{\psi} \otimes X \to M_{\mu} \otimes X \to 0,$$

which sends $u \otimes x$ to $w_{\mu} \otimes x$, where $v_{\mu}$ is the highest weight vector in $M_{\mu}$.

Taking tensor product with $U_{\chi}$ over $U(\mathfrak{gl}_n(k))$ for some $\chi$,

$$\chi : Z(U(\mathfrak{gl}_n(k))) \to k,$$

is right exact, thus we have
\[(U_\psi \otimes X)_\chi \rightarrow (M_\mu \otimes X)_\chi \rightarrow 0.\]

So it is left to prove that \((M_\mu \otimes X)_\chi \neq 0\) and that the image of \(v_\mu \otimes X\) is nonzero.

We note that \(\lambda\) is maximal among weights of \(M_\mu \otimes X\). Using the standard argument, i.e. taking the sum of all submodules of \(M_\mu \otimes X\) (that are naturally objects of the category \(\mathcal{O}\)), that don’t contain weight \(\lambda\), one can show that \(M_\mu \otimes X\) has a simple quotient-module \(L\) with highest weight \(\lambda\). And if \(x_{\lambda-\mu}\) is a vector of maximal weight in \(X\) then the highest weight vector of \(L\) is the image of \(v_\mu \otimes x_{\lambda-\mu}\).

Since \(L\) is invariant under taking tensor product with \(U_\chi\) over \(U(gl_n(k))\), there is a surjective map
\[\left(M_\mu \otimes X\right)_\chi \rightarrow L \rightarrow 0.\]
And thus \((M_\mu \otimes X)_\chi\) and, consequently, \((U_\psi \otimes X)_\chi\) is nonzero. Moreover, the image of \(1 \otimes x_{\lambda-\mu}\) is nonzero in \(L\). Thus, \((F_0U_\psi \otimes X)_\chi\) is nonzero.

Theorem 3.18. Suppose there exist numbers \(b_1, \ldots, b_r, c_1, \ldots, c_s \in \mathbb{C}\) such that
\[\chi(u) - \psi(u) = \sum_{i=1}^{r} e^{b_i u} - \sum_{i=1}^{s} e^{(c_i - 1) u}.\]

Then \((U_\psi \otimes S^r V \otimes S^s V^*)_\chi \neq 0\).

Proof. By Corollary 3.16, there exist weights \(\lambda^{(n)}\) and \(\mu^{(n)}\) of \(gl_n(k)\) with
\[\mu^{(n)} = \lambda^{(n)} - e_1 - \ldots - e_r + e_{n-s+1} + \ldots + e_n,\]
and central characters \(\chi^{(n)}, \psi^{(n)}\) corresponding to these weights, such that
\[\chi = \prod_{\mathcal{F}} \chi^{(n)}, \psi = \prod_{\mathcal{F}} \psi^{(n)}.\]

Now, for every \(n > r + s\) the module \(S^r V^{(n)} \otimes S^s (V^{(n)})^*\) has maximal weight
\[\lambda^{(n)} - \mu^{(n)} = e_1 + \ldots + e_r - e_{n-s+1} - \ldots - e_n\]
Thus, by Lemma 3.17,
\[\left(U_{\psi^{(n)}} \otimes S^r V^{(n)} \otimes S^s (V^{(n)})^*\right)_{\chi^{(n)}} \neq 0,\]
when \(n > s + r\). Moreover,
\[\left(F_0U_{\psi^{(n)}} \otimes S^r V^{(n)} \otimes S^s (V^{(n)})^*\right)_{\chi^{(n)}} \neq 0,\]
We have
\[(F_0 \psi \otimes R^V \otimes S^s V^*) \chi = \prod_{\mathcal{F}} (F_0 \psi(n) \otimes R^V(n) \otimes S^s (V(n))^*) \chi(n) \neq 0.\]
And therefore,
\[(U \psi \otimes R^V \otimes S^s V^*) \chi = \operatorname{colim} \prod_{\mathcal{F}} (F_i \psi(n) \otimes R^V(n) \otimes S^s(V(n))^*) \chi(n) \neq 0.\]

We have now constructed a nonzero object in the category $HC_{\chi, \psi}$ with
\[\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{j=1}^s e^{(c_j - 1) u}\]
for any complex numbers $b_i$, $1 \leq i \leq r$, $c_j$, $1 \leq j \leq s$, and thus, we have proved Theorem 3.13.

### 3.7. Summary

The following theorem summarizes Corollary 3.12 and Theorem 3.13 and is the main result of this paper:

**Theorem 3.19.** The category $HC_{\chi, \psi}$ is nonzero if and only if there exist complex numbers $b_1, \ldots, b_r, c_1, \ldots, c_s$ such that
\[\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{c_i u} \]

Note. We replaced $c_i - 1$ with $c_i$ (as compared to Theorem 3.13) to simplify the formula.

**Examples.** Theorem 3.18 provides us with an example of a nonzero bimodule in $HC_{\chi, \psi}$ if $\chi, \psi$ satisfy the conditions of Theorem 3.19. We have
\[0 \neq (U \psi \otimes R^V \otimes S^s V^*) \chi \in HC_{\chi, \psi}.\]

Other examples can be found in [13], where we describe a construction of a family of irreducible Harish-Chandra bimodules. These bimodules are a generalization of finite-dimensional bimodules in the classical case and have finite K-type (see Remark 3.2). The corresponding central characters are computed in Section 5 of [13].

### References

[1] J.N. Bernstein and S.I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Compositio Mathematica 41 (1980), no. 2, 245–285.

[2] P. Deligne, *La Catégorie des représentations du groupe symétrique $S_t$, lorsque $t$ n’est pas un entier naturel*, Algebraic groups and homogeneous spaces (2007), 209–273.
[3] P. Deligne, *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, Progr. Math 87 (1990), 111–195.

[4] P. Deligne, *Catégories tensorielles*, Moscow Math. J 2 (2002), no. 2, 227–248.

[5] P. Deligne and J. Milne, *Tannakian categories*, Lecture notes in mathematics 900 (1982), http://www.jmilne.org/math/xnotes/tc.pdf.

[6] Jacques Dixmier, *Enveloping Algebras*, Vol. 11, Graduate Studies in Mathematics, 1996.

[7] P. Etingof, *Representation theory in complex rank, II*, Advances in Mathematics 300 (2016), 473–504.

[8] P. Etingof, S. Gelaki, Nikshych D, and V. Ostrik, *Tensor Categories*, Vol. 205, Mathematical Surveys and Monographs, 2015.

[9] N. Harman, *Deligne categories as limits in rank and characteristic*, arXiv:1601.03426.

[10] N.Harman and D.Kalinov, *Classification of simple algebras in the Deligne category Rep(S_l)*, Journal of Algebra 549 (2020), 215–248.

[11] D. Kalinov, *Finite-dimensional representations of Yangians in complex rank*, International Mathematics Research Notices 20 (2020), 6967–6998.

[12] L. Sciarappa, *Simple commutative algebras in Deligne’s categories Rep(S_l)*, arXiv:1506.07565 (2015).

[13] A. Utiralova, *Harish-Chandra bimodules of finite K-type in Deligne categories*, with an appendix by S. Hu, arXiv:2107.03173 (2021).