NONTRIVIAL SOLUTIONS FOR KIRCHHOFF TYPE EQUATIONS VIA MORSE THEORY

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Abstract. In this paper, the existence of nontrivial solutions is obtained for a class of Kirchhoff type problems with Dirichlet boundary conditions by computing the critical groups and Morse theory.

1. Introduction and main results. In this paper we consider the following Kirchhoff type problems with Dirichlet boundary conditions

\[
\begin{cases}
-a - b \int_{\Omega} |\nabla u|^2 \, dx \Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N (N = 1, 2, 3) \), \( a, b > 0 \), and \( f \in C(\Omega \times \mathbb{R}) \) satisfying the subcritical growth

\[
(f_*) \quad |f(x, t)| \leq c_0 (|t|^{q-1} + 1) \quad \text{for some } 4 < q < 2^* = \begin{cases} 6, & \text{if } N = 3, \\ \infty, & \text{if } N = 1, 2, \end{cases}
\]

some positive constant \( c_0 \) and \( (x, t) \in \Omega \times \mathbb{R} \).

The condition \( f_* \) implies that finding weak solutions of (1) in \( X := H^1_0(\Omega) \) is equivalent to finding the critical points of the \( C^1 \) functional given by

\[
I(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^2 - \int_{\Omega} F(x, u) \, dx, \quad u \in X,
\]

where \( F(x, t) = \int_0^t f(x, s) \, ds \) and \( X \) is the usual Hilbert space endowed with the norm \( \|u\| = (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2} \).

Assume that \( f(x, 0) \equiv 0 \), then \( u \equiv 0 \) is a trivial solution of problem (1). We are interested in finding nontrivial solutions for problem (1). For this purpose, we should consider the behaviors of the nonlinearity \( f(x, t) \) or its primitive \( F(x, t) \) near zero and infinity. To state our results, we recall some results on the eigenvalue problems:

\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
and
\[
\begin{cases}
-\|u\|^2 \Delta u = \mu u^3 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (3)

Denote by \(0 < \lambda_1 < \lambda_2 \leq \cdots\) the eigenvalues of the linear problem (2). Using the Yang index, Perera and Zhang [20] constructed an unbounded minimax eigenvalues sequence \(0 < \mu_1 \leq \mu_2 \leq \cdots\) of the nonlinear problem (3).

Recently, the solvability of the Kirchhoff type problem (1) has been paid much attention by various authors. Alves et al. [1], Ma and Rivera [16] and Chen et al. [5] for example, obtained positive solutions via variational methods. Zhang and Perera [28] and Mao and Zhang [17] using variational methods and invariant sets of descent flow, obtained sign changing solutions of such problem. Sun and Tang [22] obtained the existence and multiplicity of solutions for problem (1) by minimax theorems. To the best of our knowledge, the first result on resonance problems for Kirchhoff type equations is due to Sun and Tang [23]; they considered the existence of weak solutions for Kirchhoff type equations with Dirichlet boundary conditions which are resonant at an arbitrary eigenvalue under a Landesman-Lazer type condition by the minimax methods. He and Zou [7] studied the existence, multiplicity and concentration behavior of positive solutions for a singular perturbation problem of (1) in \(\mathbb{R}^3\) by using the variational methods. To our knowledge, few papers in the literature studied the existence of nontrivial solutions for problem (1) via Morse theory (see [20, 27]). In [20] Perera and Zhang obtained nontrivial solutions of problem (1) for the case that \(f\) satisfies the following assumption:

\[
\lim_{t \to 0} \frac{f(x,t)}{at} = \lambda, \quad \lim_{|t| \to \infty} \frac{f(x,t)}{bt^3} = \mu, \quad \text{uniformly in } x,
\] (4)

where \(\lambda \in (\lambda_l, \lambda_{l+1}), \mu \in (\mu_m, \mu_{m+1})\) and \(l \neq m\).

Motivated by [20], in this paper, we will consider the cases that \(\lambda\) may be equal to some \(\lambda_l\) or \(\mu\) equal to some \(\mu_m\) in (4), and obtain the existence of nontrivial solutions of the problem (1) by using Morse theory. So our results are new. Firstly, we consider the case that the nonlinearity \(f(x,t)\) is asymptotically 2-linear near zero and 4-superlinear at infinity. More precisely, we have the following result:

**Theorem 1.1.** Suppose \(f(x,t)\) satisfies \((f_\ast)\) and the following conditions:

\((f_0^1)\) There exist \(\delta > 0\) and \(\lambda_l < \tilde{\lambda} < \lambda_{l+1}\) with \(l \in \mathbb{N}\) such that \(a\tilde{\lambda}t^2 \leq 2F(x,t) \leq a\lambda_{l+1}t^2\), for \(|t| \leq \delta, x \in \Omega\).

\((f_1)\) There exist \(\theta \geq 1\) and \(c_\ast > 0\) such that \(\theta F(x,t) \geq F(x,st) - c_\ast\) for \((x,t) \in \Omega \times \mathbb{R}\) and \(s \in [0,1]\), where \(F(x,t) = f(x,t)t - \int_0^t \frac{f(x,s)}{s} ds\).

\((f_\infty^1)\) \(f(x,t)t \geq 0\) and \(\lim_{|t| \to +\infty} \frac{f(x,t)}{t^3} = +\infty\) uniformly for \(x \in \Omega\).

Then problem (1) has at least one nontrivial weak solution in \(X\).

**Remark 1.** Clearly, the condition \((f_0^1)\) contains the situation \(\lim_{t \to 0} 2F(x,t)/at^2 = \lambda \in (\lambda_l, \lambda_{l+1})\) and \(\lambda \to \lambda_{l+1}\) from the left. But here we do not need to assume that the limit exists. The same conclusion holds if we replace the condition \((f_0^1)\) by the following stronger condition:

\((f_0^1)'\) There exist \(\delta > 0\), and \(\lambda_l < \tilde{\lambda} < \lambda_{l+1}\) with \(l \in \mathbb{N}\) such that \(a\tilde{\lambda}t^2 \leq f(x,t)t \leq a\lambda_{l+1}t^2\), for \(|t| \leq \delta, x \in \Omega\).

There is an analogous result in our previous work [22], where the problem (1) was studied via local linking theorem under the following stronger condition for the case \(p = 4\):
There exists \( \theta \geq 1 \) such that \( \theta \mathcal{F}(x,t) \geq \mathcal{F}(x,st) \) for \((x,t) \in \Omega \times \mathbb{R} \) and \( s \in [0,1] \), where \( \mathcal{F}(x,t) = f(x,t)t - p \mathcal{F}(x,t) \).

In the case \( p = 2 \), the condition \((f_1)'\) was first introduced by Jeanjean in [8]. Later, it was used by Liu and Li [15] to deal with the superlinear \( p \)-Laplacian equations for the general case \( p > 1 \). When \( \theta = 1 \), we can easily prove that the condition \((f_1)'\) is satisfied if

\[ \frac{f(x,t)}{t} \text{ is nondecreasing in } t \geq 0, \text{ and nonincreasing in } t \leq 0 \text{ for } x \in \Omega, \]

(see Proposition 2.3 in [15]). To our knowledge, \((f_1)'\) is widely used by many authors (see for instance [6, 11, 24]) to ensure the Euler-Lagrange functional satisfies the Cerami condition (see Section 2 for details). Both \((f_1)'\) and \((f_2)\) are global condition on \( f(x,t) \). In [14], Liu showed that the functional satisfies the Cerami condition under the local condition near infinity:

\((f_2)\) There exists \( R > 0 \), such that for each \( x \in \Omega \), \( \frac{f(x,t)}{t} \) is nondecreasing in \( t \geq R \), and nonincreasing in \( t \leq -R \).

And the author also showed \((f_2)\) implies (see Lemma 2.3 in [14] also Lemma 2.4 in [10])

\((f_3)\) There exists \( c_* > 0 \), such that \( \mathcal{F}(x,t) \geq \mathcal{F}(x,st) - c_* \) for \((x,t) \in \Omega \times \mathbb{R} \) and \( s \in [0,1] \), where \( \mathcal{F}(x,t) = f(x,t)t - p \mathcal{F}(x,t) \).

It is easy to see that when \( \theta = 1 \), the condition \((f_3)\) implies \((f_1)\) when \( p = 4 \). Hence, our condition \((f_1)\) is more general and concludes the global condition and also the local condition near infinity. In Lemma 2.1 we will show that under the condition \((f_1)\), the functional \( I \) satisfies the Cerami condition.

Under our assumption \((f_1)\), the functional \( I \) has a local linking at 0 (see Lemma 2.3), then 0 is not a local minimizer of \( I \). We may pay attention to finding nontrivial solutions for the case that 0 is a local minimizer of \( I \). We have the following result.

**Theorem 1.2.** Suppose \( f(x,t) \) satisfies \((f_*)\), \((f_1)\), \((f_{\infty}^1)\) and the following conditions:

\( (f_0^3) \) There exist \( \delta > 0 \), such that \( 2F(x,t) \leq a \lambda_1 t^2 \), for \( |t| \leq \delta, x \in \Omega. \)

Then problem (1) has at least one nontrivial weak solution in \( X \).

**Remark 2.** In our previous work [22], we obtained positive solution of problem (1) via the mountain pass theorem under the assumption \((f_*)\), \((f_0^3)\), \((f_{\infty}^1)\) and the stronger condition \((f_1)'\) with \( f \in C(\Omega \times \mathbb{R}^+). \)

We now consider the dual case that the nonlinearity \( f(x,t) \) is asymptotically 4-linear at infinity and sublinear at zero. We have the following theorem.

**Theorem 1.3.** Suppose \( f(x,t) \) satisfies \((f_*)\) and the following conditions:

\( (f_0^3) \) There exist \( \delta > 0 \) and \( \nu \in (0,2) \) such that \( f(x,t)t > 0 \) for \( x \in \Omega, 0 < |t| \leq \delta \); and \( \nu F(x,t) - f(x,t)t \geq 0, \) for \( x \in \Omega, |t| \leq \delta. \)

\( (f_{\infty}^2) \) There exist \( R > 0 \) and \( \mu_m < \bar{\mu} < \mu_{m+1} \) such that \( b \delta t^4 \leq 4F(x,t) \leq b \mu_{m+1} t^4, \) for \( |t| \geq R, x \in \Omega, m \in \mathbb{N}, \) and \( \lim_{|t| \to \infty} (f(x,t)t - 4F(x,t)) = +\infty. \)

Then problem (1) has at least one nontrivial solution in \( X \).

**Remark 3.** (i) It is easy to check that \((f_0^3)\) implies that \( \lim_{t \to 0} F(x,t)/t^2 = +\infty \) which means that the nonlinearity \( f(x,t) \) is sublinear near zero. (ii) The condition \((f_{\infty}^2)\) contains the situation \( \lim_{|t| \to \infty} 4F(x,t)/bt^4 = \mu \in (\mu_m, \mu_{m+1}) \) and \( \mu \to \mu_{m+1} \) from the left. But here we do not need to assume that the limit exists. The same conclusion holds if we replace the condition \((f_{\infty}^2)\) by the following stronger condition:
There exist \( R > 0 \) and \( \mu_m < \bar{\mu} < \mu_{m+1} \) such that \( bj\bar{\mu}^4 \leq f(x,t)t \leq b\mu_{m+1}t^4 \), for \( |t| \geq R, x \in \Omega, m \in \mathbb{N}, \) and \( \lim_{|t| \to \infty}(f(x,t)t - 4F(x,t)) = +\infty \).

Finally, we consider the case that \( \lambda \) may be equal to some \( \lambda_i \) and \( \mu \) equal to some \( \mu_m \) in (4) simultaneously. We have the following theorem.

**Theorem 1.4.** Suppose \( f(x,t) \) satisfies \((f_1), \ (f_0^2)\) and \((f_\infty^2)\) with \( l \neq m \). Then problem (1) has at least one nontrivial solution in \( X \).

**Remark 4.** Let \( \mu_0 = -\infty \) and \( m = 0 \) in \((f_\infty^2)\), similar to the Theorem 2.2 in [27], we can obtain two nontrivial solutions by using the theorem 2.1 in [13], since the functional \( I \) is coercive in \( X \). But here we consider the case that \( m \geq 1 \).

This paper is divided into three sections. In Section 2, we first recall the definition and some basic properties of the critical groups, and then discuss the compactness issue and compute the relevant critical groups of the functional \( I \). In Section 3, we give the proof of our main results.

### 2. Compactness and critical groups.

#### 2.1. Preliminaries.

We denote by \( \| \cdot \|_{L^r} \) the usual \( L^r \)-norm. The Lebesgue measure of \( \Omega \) is denoted by \( |\Omega| \). \( c_i \) will denote a positive constant unless specified, \( i = 0, 1 \cdots \). Since \( \Omega \) is a bounded domain, \( X \to L^r(\Omega) \) continuously for \( r \in [1, 2^*) \), compactly for \( r \in [1, 2^*) \), and there exists \( \gamma_r > 0 \) such that

\[
\|u\|_{L^r} \leq \gamma_r \|u\|, \quad \forall u \in X.
\]

Let \( I \) be a \( C^1 \) functional defined on a Hilbert space \( X \), then the \( k \)-th critical group of \( I \) at an isolated critical point \( u \) with \( I(u) = c \) is defined by

\[
C_k(I, u) := H_k((U \cap \{u\}) \setminus \{u\}), \quad k = 0, 1, 2 \cdots,
\]

where \( U \) is a neighborhood of \( u \), containing the unique critical point and \( H_k \) is the singular relative homology with coefficients in an Abelian group \( \mathcal{G} \). According to the excision property of the singular homology theory, the critical groups do not depend on a special choice of the neighborhood \( U \).

We denote a subsequence of a sequence \( \{u_n\} \) as \( \{u_n\} \) to simplify the notation unless specified. Now we recall the definition of some compactness conditions. We say that \( I \) satisfies the Cerami condition at the level \( c \in \mathbb{R} \) \((\text{Ce})_{c} \) for short), if any sequence \( \{u_n\} \subset X \) with \( I(u_n) \to c \) and \( (1 + \|u_n\|)\|I'(u_n)\| \to 0 \) in \( X^* \) as \( n \to \infty \) possesses a convergent subsequence in \( X \); such a sequence is then called a Cerami sequence; \( I \) satisfies the \((\text{Ce}) \) condition if \( I \) satisfies the condition \((\text{Ce})_{c} \) for all \( c \in \mathbb{R} \). The Cerami condition introduced by Cerami [3] is a weak version of the \((\text{PS}) \) condition: any sequence \( \{u_n\} \) in \( X \) such that \( \{I(u_n)\} \) is bounded and \( I'(u_n) \to 0 \) in \( X^* \) as \( n \to \infty \) has a convergent subsequence in \( X \).

If \( I \) satisfies \((\text{Ce}) \) condition or \((\text{PS}) \) condition and the critical values of \( I \) are bounded from below by some \( \alpha > -\infty \), then the critical groups of \( I \) at infinity introduced by Bartsch and Li [2] as

\[
C_k(I, \infty) := H_k(X, I^\alpha), \quad k = 0, 1, 2 \cdots,
\]

Note that by the second deformation lemma and the homotopy invariance of homology groups, \( H_k(X, I^\alpha) \) does not depend on the choice of \( \alpha \).

We refer the readers to [4, 18, 19] for more details on Morse theory. In the proof of our theorems we shall use the following results.
Proposition 1. Suppose that \( I \in C^1(X, \mathbb{R}) \) satisfies (Ce) condition and \( I \) has only finitely many critical points, then:

(i) If for some \( k \geq 0 \) we have \( C_k(I, \infty) \neq 0 \), then \( I \) has a critical point \( u \) with \( C_k(I, u) \neq 0 \);

(ii) Let 0 be an isolated critical point of \( I \). If for some \( k \geq 0 \) we have \( C_k(I, \infty) \neq C_k(I, 0) \), then \( I \) has a nontrivial critical point.

Proposition 2 (see [12]). Suppose the \( I \in C^1(X, \mathbb{R}) \) has a critical point \( u = 0 \) with \( I(0) = 0 \). If \( I \) has a local linking at 0 with respect to \( X = V \oplus W \), \( d = \dim V < \infty \), i.e., there exists \( \rho > 0 \) small such that

\[
I(u) \leq 0, \ u \in V; \ ||u|| \leq \rho; \ I(u) > 0, \ u \in W, \ 0 < ||u|| \leq \rho.
\]

Then \( C_d(I, 0) \neq 0 \); that is, 0 is a homological nontrivial critical point of \( I \).

Proposition 3 (see Proposition 3.8 in [2]). Suppose \( X \) splits as \( X = V \oplus W \) such that \( I \) is bounded from below on \( W \) and \( I(u) \to -\infty \) for \( u \in V \) as \( ||u|| \to \infty \). Then \( C_d(I, \infty) \neq 0 \) for \( d = \dim V \).

In order to apply the Morse theory to \( I \), we must show that \( I \) satisfies the compactness condition. We have the following lemmas.

Lemma 2.1. Assume that \((f_s), (f_1)\) and \((f_{\infty})\) are satisfied, then \( I(u) \) satisfies the (Ce) condition.

Proof. We suppose that \( \{u_n\} \) is the (Ce) sequence, that is for \( c \in \mathbb{R} \)

\[
I(u_n) = \frac{a}{2} ||u_n||^2 + \frac{b}{4} ||u_n||^4 - \int_{\Omega} F(x, u_n) \, dx \to c \quad (n \to \infty), \tag{6}
\]

and

\[
(1 + ||u_n||) I'(u_n) \to 0 \quad (n \to \infty). \tag{7}
\]

From (6) and (7), for \( n \) large enough, we have

\[
1 + c \geq I(u_n) - \frac{1}{4} I'(u_n), u_n = \frac{a}{4} ||u_n||^2 + \int_{\Omega} \left( \frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right) \, dx \tag{8}
\]

If \( \{u_n\} \) is unbounded, there exists a subsequence of \( \{u_n\} \) satisfying \( ||u_n|| \to \infty \) as \( n \to \infty \). Set \( w_n = u_n/||u_n|| \), then \( ||w_n|| = 1 \). Going if necessary to a subsequence, also denoted by \( \{w_n\} \), there is \( w \in X \) such that

\[
w_n \to w \quad \text{weakly in } X,
\]

\[
w_n \to w \quad \text{strongly in } L^r(\Omega) \quad (1 \leq r < 2^*),
\]

\[
w_n(x) \to w(x) \quad \text{a.e. } x \in \Omega.
\]

If \( w \neq 0 \). From (7), we obtain

\[
a(1) = I'(u_n), u_n = (a + b ||u_n||^2) \int_{\Omega} |\nabla u_n|^2 \, dx - \int_{\Omega} f(x, u_n) u_n \, dx,
\]

thus

\[
\frac{a}{||u_n||^2} + b - a(1) = \int_{\Omega} \frac{f(x, u_n) u_n}{||u_n||^4} \, dx = \left( \int_{w \neq 0} + \int_{w = 0} \right) \frac{f(x, u_n) u_n}{u_n^4} w_n^4 \, dx
\]

\[
\geq \int_{w \neq 0} \frac{f(x, u_n) u_n}{u_n^4} w_n^4 \, dx, \tag{9}
\]
since \( f(x,u)u \geq 0 \). For \( x \in \Theta := \{ x \in \Omega : w(x) \neq 0 \} \), we have \( |u_n(x)| \to +\infty \) as \( n \to \infty \). By \((f_1)\) we obtain
\[
\frac{f(x,u_n(x))u_n(x)}{w_n^4(x)}w_n^4(x) \to +\infty, \text{ as } n \to \infty.
\]
Note that \( |\Theta| > 0 \), using the Fatou’s lemma one has
\[
\int_{w \neq 0} \frac{f(x,u_n(x))u_n(x)}{w_n^4(x)}w_n^4 \, dx \to +\infty, \text{ as } n \to \infty.
\]
This contradicts with \((9)\).

If \( w = 0 \), set a sequence \( \{t_n\} \) of real numbers such that \( I(t_n u_n) = \max_{t \in [0,1]} I(t u_n) \).

For any integer \( m > 0 \), set \( w_n^m = (8m/b)^{1/4}w_n \). By \((f_\ast)\), one has \( |F(x,t)| \leq C(|t|^q + 1) \). Since \( w_n^m \to 0 \) in \( L^q(\Omega) \), we see that \( F(\cdot, w_n^m) \to 0 \) in \( L^1(\Omega) \) (see Proposition B.1 in \([21]\)). Thus
\[
\lim_{n \to \infty} \int_\Omega F(x,w_n^m) = 0.
\]

So for \( n \) large enough, one has \( 0 \leq (8m/b)^{1/4}/\|u_n\| \leq 1 \) and we obtain
\[
I(t_n u_n) \geq I(w_n^m) = \frac{a}{2} \|w_n^m\|^2 + \frac{b}{4} \|w_n^m\|^4 - \int_\Omega F(x,w_n^m) \, dx
\]
\[
\geq 2m - \int_\Omega F(x,w_n^m) \, dx
\]
\[
\geq m,
\]
which implies that
\[
I(t_n u_n) \to +\infty, \quad \text{as } n \to \infty.
\]

Noting that \( I(0) = 0 \) and \( I(u_n) \to c \), so \( 0 < t_n < 1 \) when \( n \) is large enough. It follows that
\[
\left( a + b \int_\Omega |\nabla t_n u_n|^2 \, dx \right) \int_\Omega |\nabla t_n u_n|^2 \, dx - \int_\Omega f(x,t_n u_n) t_n u_n \, dx
\]
\[
= \langle I'(t_n u_n), t_n u_n \rangle = t_n \frac{dI(t_n u_n)}{dt} \bigg|_{t=t_n} = 0.
\]

Therefore, by \((f_1)\), we have
\[
\frac{a}{4} \|u_n\|^2 + \frac{1}{4} \int_\Omega G(x,u_n) \, dx
\]
\[
\geq \frac{a}{4\theta} \|t_n u_n\|^2 + \frac{1}{4\theta} \int_\Omega \left( G(x,t_n u_n) - C_* \right) \, dx
\]
\[
= \frac{1}{\theta} \int_\Omega \left( \frac{a}{4} |\nabla t_n u_n|^2 + \frac{1}{4} f(x,t_n u_n) t_n u_n - F(x,t_n u_n) \right) \, dx - \frac{C_* |\Omega|}{4\theta}
\]
\[
= \frac{1}{\theta} \int_\Omega \left( \left( \frac{a}{2} + \frac{b}{4} \right) |\nabla t_n u_n|^2 dx \right) |\nabla t_n u_n|^2 - F(x,t_n u_n) \right) \, dx - \frac{C_* |\Omega|}{4\theta}
\]
\[
\to +\infty
\]
as \( n \to \infty \), which contradicts \((8)\). Hence \( \{u_n\} \) is bounded, that is, there exists a positive constant \( M \) such that
\[
\|u_n\| \leq M, \quad \text{for all } n \in \mathbb{N}.
\]
Now, we prove that \( \{ u_n \} \) has a convergent subsequence in \( X \). Indeed, since \( \{ u_n \} \) is bounded in \( X \), by the reflexivity of \( X \) we can assume that there exists \( u \in X \) such that
\[
{\begin{array}{c}
u_n \rightharpoonup u \text{ weakly in } X, \\
u_n \to u \text{ strongly in } L^r(\Omega) \quad (1 \leq r < 2^*).
{\end{array}}
\] (11) (12)

From Hölder’s inequality, \((f_\ast)\) and (10), we obtain
\[
\left| \int_{\Omega} f(x, u_n)(u_n - u) \, dx \right| \leq \int_{\Omega} |f(x, u_n)||u_n - u| \, dx
\leq \int_{\Omega} C(|u_n|^{q-1} + 1)|u_n - u| \, dx
\leq C|u_n|^{q-1}_{L^q} |u_n - u|_{L^q} + C|\Omega|^\frac{q-1}{q} |u_n - u|_{L^q}
\leq C\gamma_q^{q-1} |u_n|^{q-1} |u_n - u|_{L^q} + C|\Omega|^\frac{q-1}{q} |u_n - u|_{L^q}
\leq \left( C\gamma_q^{q-1} M^{q-1} + C|\Omega|^\frac{q-1}{q} \right) |u_n - u|_{L^q}
\]
for all \( n \in \mathbb{N} \), which implies that
\[
\int_{\Omega} f(x, u_n)(u_n - u) \, dx \to 0
\]
as \( n \to \infty \) by (12). Then, it follows from (7) that
\[
(a + b\|u_n\|^2) \int_{\Omega} \nabla u_n \nabla (u_n - u) \, dx = \langle I'(u_n), u_n - u \rangle + \int_{\Omega} f(x, u_n)(u_n - u) \, dx
\to 0
\]
as \( n \to \infty \), which implies
\[
\int_{\Omega} \nabla u_n \nabla (u_n - u) \, dx \to 0.
\]
as \( n \to \infty \). Thus by (11), we have
\[
\int_{\Omega} |\nabla (u_n - u)|^2 \, dx = \int_{\Omega} (\nabla u_n - \nabla u, \nabla (u_n - u)) \, dx \to 0
\]
as \( n \to \infty \). Hence \( u_n \to u \) in \( X \) as \( n \to \infty \). The proof is completed. \( \square \)

**Lemma 2.2.** Assume that \((f_\ast)\) and \((f_\infty^2)\) are satisfied, then \( I(u) \) satisfies the \((Ce)\) condition.

**Proof.** We suppose that \( \{ u_n \} \) is the \((Ce)\) sequence, that is for \( c \in \mathbb{R} \)
\[
I(u_n) = \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \int_{\Omega} F(x, u_n) \, dx \to c \quad (n \to \infty),
\] (13)
and
\[
1 + \|u_n\‖I'(u_n) \to 0 \quad (n \to \infty).
\] (14)

From (13) and (14), for \( n \) large enough, we have
\[
1 + c \geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle = \frac{a}{4}\|u_n\|^2 + \int_{\Omega} \left( \frac{1}{4} f(x, u_n)u_n - F(x, u_n) \right) \, dx
\]
Assume by contradiction that \( \{ u_n \} \) is unbounded. There exists a subsequence of \( \{ u_n \} \) satisfying \( \|u_n\| \to \infty \) as \( n \to \infty \). Set \( w_n = u_n/\|u_n\| \), then \( \|w_n\| = 1 \). Going if necessary to a subsequence, also denoted by \( \{ w_n \} \), there is \( w \in X \) such that
\[
w_n \rightharpoonup w \text{ weakly in } X,
\]
\[ w_n \to w \quad \text{strongly in } L^r(\Omega) \quad (1 \leq r < 2^*), \]
\[ w_n(x) \to w(x) \quad \text{a.e. } x \in \Omega. \]

By \((f_2^\infty)\), there exists \(R > 0\), such that \(b|t|^4 \leq 4F(x,t) \leq b\mu_{m+1}|t|^4\), for \(|t| \geq R\) and \(x \in \Omega\). Then there exists \(M_R > 0\), such that \(b\mu|t|^4 - M_R \leq 4F(x,t) \leq b\mu_{m+1}|t|^4 + M_R\), for \(t \in \mathbb{R}\) and \(x \in \Omega\). Hence from (13), we have for \(n\) large enough,
\[
\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 \leq c + 1 + \frac{1}{4} \int_{\Omega} F(x,u_n) \, dx \leq c + 1 + \frac{1}{4} \int_{\Omega} (b\mu_{m+1}|u_n|^4 + M_R) \, dx \\
\leq c' + \frac{1}{4} b\mu_{m+1} \int_{\Omega} |u_n|^4 \, dx
\]
Dividing the above inequality by \(\|u_n\|^4\), and taking \(n \to \infty\), we conclude that
\[ 1 \leq \mu_{m+1}\|w\|^4_{L^4}. \]
Therefore the set \(\Theta := \{x \in \Omega : w(x) \neq 0\}\) has positive Lebesgue measure. For \(x \in \Theta\), we have \(|u_n(x)| \to +\infty\). Hence by \((f_2^\infty)\), we deduce
\[ f(x,u_n(x))u_n(x) - 4F(x,u_n(x)) \to +\infty \]
as \(n \to \infty\). Therefore via Fatou lemma we have
\[
\frac{a}{4}\|u_n\|^2 + \int_{\Omega} \left( \frac{1}{4} f(x,u_n)u_n - F(x,u_n) \right) \, dx \\
= \frac{a}{4}\|u_n\|^2 + \left( \int_{u \neq 0} + \int_{u = 0} \right) \left( \frac{1}{4} f(x,u_n)u_n - F(x,u_n) \right) \, dx \\
\to +\infty,
\]
as \(n \to \infty\). This contradicts with (15). Hence \(\{u_n\}\) is bounded in \(X\). The proof that \(\{u_n\}\) has a convergent subsequence in \(X\) is similar to lemma 2.1, we omit it here. The proof is completed. \(\square\)

Now, we compute the critical groups \(C_k(I,0)\) and \(C_k(I,\infty)\).

2.2. Critical groups at zero. Assume that the problem (1) has finitely many solutions. Then 0 is an isolated critical point of \(I\) and the critical group of \(I\) at zero is defined.

Lemma 2.3. If \(f\) satisfies \((f_s)\) and \((f_1^0)\), then \(C_1(I,0) \neq 0\).

Proof. Let \(V = \oplus_{l \leq l} \ker(-\Delta - \lambda_l)\) and \(W = V^\perp\). Then \(X = V \oplus W\), and \(\dim V = l < \infty\). If \(u \in V\), one has \(\|u\|^2 \leq \lambda_l\|u\|^2_{L^2}\), and if \(u \in W\), we have \(\|u\|^2 \geq \lambda_{l+1}\|u\|^2_{L^2}\).

Using the conditions \((f_s)\) and \((f_1^0)\), we show that the functional \(I\) has a local linking at 0 with respect to \(X = V \oplus W\).

For \(u \in V\) with \(\|u\| \leq \rho := \delta/c_1\), we have \(|u| \leq c_1\|u\| \leq \delta\) since \(V\) is finite dimensional. Thus by \((f_1^0)\) and (5), for \(u \in V\), we have
\[ I(u) \leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a\lambda}{2}\|u\|^2_{L^2} \leq a(1 - \frac{\lambda}{\lambda_l})\|u\|^2 + \frac{b}{4}\|u\|^4 \leq 0. \]
for \(\rho > 0\) small enough.

By \((f_s)\) and \((f_1^0)\), we have \(F(x,t) \leq \frac{a}{2}\lambda_{l+1}|t|^2 + c_2|t|^q\), for \((x,t) \in \Omega \times \mathbb{R}\). Then for \(u \in W\), by (5), one has
\[ I(u) \geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a\lambda}{2}\|u\|^2_{L^2} - c_2\|u\|^q_{L^q} \geq \frac{b}{4}\|u\|^4 - c_3\|u\|^q > 0, \]
for $0 < \|u\| \leq \rho$ where $\rho > 0$ small enough, since $q > 4$. Thus $I$ has a local linking at 0 with respect to $X = V \oplus W$. By Proposition 2, we know $C(I, 0) \neq 0$. The proof is completed.

Lemma 2.4. If $f$ satisfies $(f_*)$ and $(f^3_0)$, then $\delta_k(I, 0) = \delta_k,0G$.

Proof. By $(f_*)$ and $(f^3_0)$, one has $F(x, t) \leq \frac{6}{q} \lambda_1 |t|^2 + c_4 |t|^q$, for $(x, t) \in \Omega \times \mathbb{R}$, which implies

$$I(u) \geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{a \lambda_1}{2} \|u\|_{L^2}^2 - c_4 \|u\|_{L^q}^q \geq \frac{b}{4} \|u\|^4 - c_5 \|u\|^q.$$

Since $q > 4$, it is easy to see that 0 is a local minimizer of $I$ and so $\delta_k(I, 0) = \delta_k,0G$. The proof is completed.

Lemma 2.5. Assume that $f$ satisfies $(f_*)$ and $(f^3_0)$. Then $\delta_k(I, 0) \cong 0$ for all $k \in \mathbb{Z}$.

Proof. By $(f_*)$ and $(f^3_0)$, there exists a constant $c_6, c_7 > 0$ such that

$$F(x, t) \geq c_6 |t|^\nu - c_7 |t|^q, \quad (x, t) \in \Omega \times \mathbb{R}.$$

Therefore for $u \in X \setminus \{0\}$ and $s > 0$, we have

$$I(su) = \frac{as^2}{2} \|u\|^2 + \frac{bs^4}{4} \|u\|^4 - \int_{\Omega} F(x, su) \, dx$$

$$\leq \frac{as^2}{2} \|u\|^2 + \frac{bs^4}{4} \|u\|^4 - c_6 s^\nu \|u\|_{L^\nu}^\nu + c_7 s^q \|u\|_{L^q}^q.$$

Since $0 < \nu < 2$ and $4 < q$, we see that for given $u \in X \setminus \{0\}$, there exists $s_0 = s_0(u) \in (0, 1)$ such that

$$I(su) < 0, \quad \forall \ 0 < s < s_0. \quad (16)$$

Assume that $I(u) \geq 0$, i.e., \(\frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 \geq \int_{\Omega} F(x, u) \, dx\) which implies

$$\frac{d}{ds} I(su)\big|_{s=1} = \langle I'(su), u \rangle_{s=1} = a \|u\|^2 + b \|u\|^4 - \int_{\Omega} f(x, u)u \, dx$$

$$\geq a(1 - \frac{\nu}{2}) \|u\|^2 + b(1 - \frac{\nu}{4}) \|u\|^4 + \int_{\Omega} \nu (F(x, u) - f(x, u)u) \, dx$$

$$\geq a(1 - \frac{\nu}{2}) \|u\|^2 + b(1 - \frac{\nu}{4}) \|u\|^4 - c_8 \int_{\Omega} |u|^q \, dx$$

$$\geq a(1 - \frac{\nu}{2}) \|u\|^2 + b(1 - \frac{\nu}{4}) \|u\|^4 - c_9 \|u\|^q,$$

by $(f_*)$, $(f^3_0)$ and (5). Since $0 < \nu < 2$ and $4 < q$, one concludes that there exists $\rho > 0$ such that

$$\frac{d}{ds} I(su)\big|_{s=1} > 0, \quad \forall \ u \in X \text{ with } I(u) \geq 0 \text{ and } 0 < \|u\| \leq \rho. \quad (17)$$

Let $B_\rho(0) = \{ u \in X : \|u\| \leq \rho \}$. It follows from (16) and (17) that for any $u \in I^{-1}(0, \infty) \cap B_\rho(0)$, there exists a unique $s \in (0, 1)$ such that $I(su) = 0$. Define a mapping $T : B_\rho(0) \to [0, 1]$ as

$$T(u) = \begin{cases} 1, & \text{for } u \in B_\rho(0) \text{ with } I(u) \leq 0 \\ s, & \text{for } u \in B_\rho(0) \text{ with } I(u) > 0, \ I(su) = 0, \ 0 < s < 1. \end{cases}$$

Thus the mapping $T$ is well-defined and by the implicit function theorem we see that the mapping $T$ is continuous in $u$. Similarly to the proof of the Proposition 2.1
in [9], we can construct a strong deformation retract \( \eta : [0, 1] \times B_\rho(0) \to B_\rho(0) \) by \( \eta(s, u) = (1 - s)u + sT(u)u \), for \( s \in [0, 1] \) and \( u \in B_\rho(0) \) which satisfies \( \eta(0, u) = u \) and \( \eta(1, u) \in I^0 \) for any \( u \in B_\rho(0) \). By the homotopy invariance of homology group, we have \( C_k(I, 0) = H_k(B_\rho(0) \cap I^0, B_\rho(0) \cap I^0 \setminus \{0\}) \cong H_k(B_\rho(0), B_\rho(0) \setminus \{0\}) \cong 0 \), \( \forall k \in \mathbb{Z} \), since \( B_\rho(0) \setminus \{0\} \) is contractible. The proof is completed.

2.3. Critical groups at infinity. Assume that the problem (1) has finitely many solutions. Since \( I \) satisfies the \((Ce)\) condition, the critical group of \( I \) at infinity make sense.

Lemma 2.6. Assume that \( f \) satisfies \((f_\ast)\), \((f_1)\) and \((f_\infty)\). Then \( C_k(I, \infty) \cong 0 \) for all \( k \in \mathbb{Z} \).

Proof. Let \( S^\infty = \{ u \in X : \|u\| = 1 \} \) be the unit sphere in \( X \) and \( B^\infty = \{ u \in X : \|u\| \leq 1 \} \). By \((f_\infty)\), for any \( M > 0 \), there exists \( c_{10} > 0 \), such that \( F(x, t) \geq Mt^4 - c_{10} \), for \( (x, t) \in \Omega \times \mathbb{R} \), which implies

\[
I(tu) \to -\infty, \quad \text{as } t \to +\infty,
\]

for any \( u \in S^\infty \). Let \( s = 0 \) in \((f_1)\), one has

\[
\mathcal{F}(x, t) \geq -\frac{c_\ast}{\theta}, \quad \text{for } (x, t) \in \Omega \times \mathbb{R}.
\] (18)

Choose

\[
\alpha < \min \left\{ \inf_{u \in B^\infty} I(u) + \frac{c_\ast}{4\theta} |\Omega| \right\}.
\]

Then for any \( u \in S^\infty \), there exists \( t > 1 \) such that \( I(tu) \leq \alpha \), that is

\[
I(tu) = \frac{at^2}{2} + \frac{bt^4}{4} - \int_\Omega F(x, tu) \, dx \leq \alpha,
\]

which implies

\[
\frac{d}{dt} I(tu) = at + bt^3 - \int_\Omega uf(x, tu) \, dx
\]

\[
\leq \frac{1}{t} \left\{ 4\alpha + \int_\Omega F(x, tu) \, dx - \int_\Omega tu f(x, tu) \, dx \right\}
\]

\[
= \frac{1}{t} \left\{ 4\alpha - \int_\Omega \mathcal{F}(x, tu) \, dx \right\}
\]

\[
\leq \frac{1}{t} \left\{ 4\alpha + \frac{c_\ast}{\theta} |\Omega| \right\} < 0,
\]

by (18). Therefore, by Implicit Function Theorem, there exists a unique \( T \in C(S^\infty, \mathbb{R}) \) such that \( I(T(u)) = \alpha \). For \( u \in X \setminus B^\infty \), we set \( \tilde{T}(u) = \frac{1}{\|u\|} T\left( \frac{u}{\|u\|} \right) \). Then \( \tilde{T} \in C(X \setminus B^\infty, \mathbb{R}) \). Similarly to the proof of the Lemma 3.2 in [25], we can construct a strong deformation retract \( \tau : [0, 1] \times (X \setminus B^\infty) \to X \setminus B^\infty \) by \( \tau(s, u) = (1 - s)u + s\tilde{T}(u)u \) which satisfies \( \tau(0, u) = u \) and \( \tau(1, u) \in I^0 \) for any \( u \in X \setminus B^\infty \). By the homotopy invariance of homology group, we have \( C_k(I, \infty) = H_k(X, I^0) \cong H_k(X, X \setminus B^\infty) \cong H_k(X, S^\infty) \cong 0 \), \( \forall k \in \mathbb{Z} \), since \( S^\infty \) is contractible. The proof is completed.

Lemma 2.7. Assume that \( f \) satisfies \((f_\infty)\). Then \( C_m(I, \infty) \neq 0 \).
Proof. Let $\varphi_i$ be the normalized eigenfunction corresponding to the eigenvalue $\mu_i$. Define $V = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ and $W = V^\perp$, then we can split the Hilbert $X$ with $X = V \oplus W$. And we have $\|u\|^4 \leq \mu_m \|u\|^4_{L^4}$ for $u \in V; \|u\|_4^4 \geq \mu_{m+1} \|u\|_{L^4}^4$ for $u \in W$.

For $u \in V$ with $\|u\| \geq M := R/c_{11}$, we have $|u| \geq c_{11} \|u\| \geq R$ since $V$ is finite dimensional. Thus by $(f_2^m)$ and (5), for $u \in V$, we have

$$I(u) \leq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{b\mu}{4} \|u\|_4^4 \leq \frac{a}{2} \|u\|^2 + b(1 - \frac{\mu}{\mu_m}) \|u\|^4.$$ 

Since $\mu_m < \mu$, one has $I(u) \to -\infty$ for $u \in V$ as $\|u\| \to \infty$.

By $(f_2^m)$, there exists $M_R > 0$ such that $F(x, t) \leq \frac{b}{4} \mu_{m+1} t^4 + M_R$, for $(x, t) \in \Omega \times \mathbb{R}$. Then for $u \in W$, by (5), one has

$$I(u) \geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{b\mu_{m+1}}{4} \|u\|_4^4 - M_R |\Omega| \geq \frac{a}{2} \|u\|^2 - c_{12},$$

which implies $I$ is bounded from below on $W$. Now the desired result follows from Proposition 3. The proof is completed. □

3. Proof of the theorems. In this section we prove our main results.

Proof of Theorem 1.1. By $(f_1^4)$, we know $0$ is a trivial critical point of $I$. From Lemma 2.3, $C(I, 0) \neq \emptyset$. While according to Lemma 2.6, $C(I, \infty) = \emptyset$. Now the desired result follows from Proposition 1. The proof is completed. □

Proof of Theorem 1.2. The result follows from Lemma 2.4, Lemma 2.6 and Proposition 1. The proof is completed. □

Proof of Theorem 1.3. The result follows from Lemma 2.5, Lemma 2.7 and Proposition 1. The proof is completed. □

Proof of Theorem 1.4. Since $l \neq m$, the result follows from Lemma 2.3, Lemma 2.7 and Proposition 1. The proof is completed. □

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