PERFECTOID MULTIPLIER/TEST IDEALS IN REGULAR RINGS AND BOUNDS ON SYMBOLIC POWERS

LINQUAN MA AND KARL SCHWEDE

Abstract. Using perfectoid algebras we introduce a mixed characteristic analog of the multiplier ideal, respectively test ideal, from characteristic zero, respectively $p > 0$, in the case of a regular ambient ring. We prove several properties about this ideal such as subadditivity. We then use these techniques to derive a uniform bound on the growth of symbolic powers of radical ideals in all excellent regular rings. The analogous result was shown in equal characteristic by Ein-Lazarsfeld-Smith and Hochster-Huneke.

1. Introduction

A driving force at the intersection of commutative algebra and algebraic geometry has been the relation between characteristic $p > 0$ commutative algebra (and in particular test ideals) and higher dimensional algebraic geometry (and in particular multiplier ideals). This relationship goes back at least to Fedder [Fed83] and Mehta-Ramanathan [MRS5] who both related Frobenius splittings and rational singularities. The connection between these two fields inspired a number of parallel applications in characteristic $p > 0$ and characteristic zero proved frequently using very different looking methods. One notable such application was by Ein-Lazarsfeld-Smith and Hochster-Huneke who proved the following

Theorem ([ELS01] [HH02]). Suppose that $R$ is a regular ring containing a field and that $Q \subseteq R$ is a prime ideal of height $h$, then for all $m > 0$

$$Q^{(mh)} \subseteq Q^m$$

where $Q^{(mh)}$ is the $mh$ symbolic power of $Q$.\(^1\)

Also see [Swa00] where it was first shown that there is such a linear containment relation between symbolic and ordinary powers of ideals. However, the case of an arbitrary regular ring, i.e., the case of mixed characteristic, was left open.

In this paper, building on ideas from the recent proof of the direct summand conjecture and its derived variant [And16a] [Bha16], we introduce a mixed characteristic analog of the multiplier/test ideal, prove many basic properties of it, and use those

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\(^1\) The elements of $R$ which vanish generically to order $mh$ at $Q$.
to deduce the mixed characteristic analog of the Theorem above. This answers a question of Hochster and Huneke in [HH02, Section 5].

1.1. Motivation - Multiplier and test ideals. Suppose $R$ is an equal characteristic regular domain satisfying mild geometric assumptions. Further suppose that $a \subseteq R$ is an ideal and $t \geq 0$ a formal exponent for $a$. In this setting we can form the test ideal in characteristic $p > 0$

$$\tau(R, a^t)$$

or the multiplier ideal in characteristic 0

$$\mathcal{J}(R, a^t).$$

This is an ideal of $R$ which measures the singularities of $V(a) \subseteq \text{Spec } R$, scaled by $t$. The multiplier or test ideal satisfies the following list of properties (and many more, see for example [HH90, HY03, Laz04]).

- **Measuring singularities**: If $a \subseteq b$ is a containment of ideals, then

  $$(\mathcal{J}/\tau)(R, a^t) \subseteq (\mathcal{J}/\tau)(R, b^t)$$

  and if $t < t'$ then

  $$(\mathcal{J}/\tau)(R, a^{t'}) \subseteq (\mathcal{J}/\tau)(R, a^t).$$

  In other words, the multiplier or test ideal is smaller/deeper if $V(a)$ is more singular or if $t$ is larger.

- **Unambiguity of exponent**: With notation as above, for any integer $n$

  $$(\mathcal{J}/\tau)(R, a^{nt}) = (\mathcal{J}/\tau)(R, (a^t)^n).$$

- **Not too small**: $a \subseteq (\mathcal{J}/\tau)(R, a)$.

- **Not too big**: If $a$ is prime of height $h$, $(\mathcal{J}/\tau)(R, (a^{(h)})^\ast) \subseteq a$.

- **Subadditivity**: If $b$ is another ideal and if $s \geq 0$ is another real number, then

  $$(\mathcal{J}/\tau)(a^s b^t) \subseteq (\mathcal{J}/\tau)(a^s) \cdot (\mathcal{J}/\tau)(b^t).$$

  In particular we have

  $$(\mathcal{J}/\tau)(a^{tn}) \subseteq (\mathcal{J}/\tau)(a^t)^n.$$ Combining these results, the application to growth of symbolic powers mentioned on the previous page follows from a clever asymptotic construction of the multiplier/test ideals [ELS01].

Historically, what has made multiplier and test ideals so powerful is that built into their construction is a way to kill cohomology (in particular, the sort of cohomology whose vanishing shows that a ring is Cohen-Macaulay). For example, if $\pi : Y \to X = \text{Spec } R$ is a log resolution of singularities of $(R, a)$ then $\mathcal{J}(R, a^t) := H^0(Y, \mathcal{O}_Y(\lceil K_Y/X - t \rceil))$.
tG|) where \( a \cdot \mathcal{O}_Y = \mathcal{O}_Y(\mathcal{G}) \). By Matlis duality, at least if \( R \) is \( d \)-dimensional local ring with maximal ideal \( m \),

\[
\mathcal{J}(R, a') = \text{Ann}_R \left( \ker \left( H^d_m(R) \rightarrow \mathbb{H}^d_m(\mathbb{R} \pi_* \mathcal{O}_Y([tG])) \right) \right)
\]

\[
= \text{Ann}_R \{ \eta \in H^d_m(R) \mid \eta \text{'s image in } \mathbb{H}^d_m(\mathbb{R} \pi_* \mathcal{O}_Y) \text{ is } \text{"killed" by } a' \}.
\]

Crucially for most applications, at least in characteristic zero, \( H^i(Y, \mathcal{O}_Y([K_{Y/X} - tG])) = 0 \) for \( i > 0 \), or equivalently \( \mathbb{H}^{d-1}_m(\mathbb{R} \pi_* \mathcal{O}_Y([tG])) = 0 \) by Matlis duality. This vanishing is exactly relative Kawamata-Viehweg vanishing [Kaw82, Vie82] and so it will be false in characteristic \( p > 0 \) in general [Ray78], see also [HK15, Example 3.11]. On the other hand, in characteristic \( p > 0 \), that cohomology group for any blowup is zero up to

- Frobenius, and
- small perturbations (by test elements).

essentially as a consequence of relative Fujita vanishing [Kee08] (a jazzed up version of Serre vanishing). With this in mind, the test ideal \( \tau(R, a') \) is defined by replacing the log resolution of singularities \( Y \rightarrow \text{Spec} \, R \) with \( R \subseteq R^{1/p^\infty} \) up to perturbations by \( c^{1/p^k} \) for \( k \gg 0 \), for some fixed \( c \) called a test element. In particular we define

\[
\tau(R, a') = \text{Ann}_R \{ \eta \in H^d_m(R) \mid 0 = c^{1/p^e} (a^{[1/p^e]})^{1/p^k} \eta \in H^d_m(R^{1/p^\infty}) \text{ for all } e \gg 0 \} = \text{Ann}_R \{ \eta \in H^d_m(R) \mid \eta \text{'s image in } H^d_m(R^{1/p^\infty}) \text{ is } \text{"killed" by } c^{1/p^\infty} a' \}.
\]

Our goal in this article is to develop an analogous definition of the multiplier/test ideal in mixed characteristic regular local rings.

1.2. Perfectoid multiplier/test ideals. Now let us assume that \( A \) is a complete regular local ring of mixed characteristic. In this situation, for every fixed element \( g \in A \), André constructed an \( A \)-algebra \( A_\infty \) that is an integral perfectoid algebra almost flat over \( A \) mod \( p \) and such that \( g \in A \) has a compatible system of \( p \)-power roots in \( A_\infty \) (in this case, \( g^{1/p^e} \) will be declared \textit{compatible}). This ingenious construction is crucial in the solution of the mixed characteristic case of the direct summand conjecture [And16a], [Bha16] and the existence of big Cohen-Macaulay algebras [And16a], [HM17].

In this article, we iterate André’s construction to obtain a huge extension \( A \rightarrow A_\infty \) that is almost flat over \( A \) mod \( p \) and such that all elements of \( A \) have a compatible systems of \( p \)-power roots in \( A_\infty \). We will use this \( A_\infty \) as a replacement of \( R^{1/p^\infty} \) that we used to define the test ideal in characteristic \( p > 0 \) (or as a replacement for the \( \mathbb{R} \pi_* \mathcal{O}_Y \) that we used to define the multiplier ideal in characteristic 0). Additionally, we use a special element \( \varpi \), which is a uniformizer of the integral perfectoid algebra \( A_\infty \) (and is a \( p \)-th root of \( p \) up to a unit), as a perturbation element in the same way that we used the test element in characteristic \( p > 0 \). Inspired by this, let \( a \subseteq A \) be an ideal, we define the perfectoid multiplier/test ideal to be the annihilator of the elements of \( H^d_m(A) \) whose image in \( H^d_m(A_\infty) \) are “almost” annihilated by \( a' \). There are different ways to interpret this \( a' \) and at least in some proofs, it is convenient to define our analog of the multiplier/test ideal with respect to a set of elements \( \{ f_1, \ldots, f_n \} \) that happen to generate \( a \). Thus we have two definitions, the first is the
one that depends on the choice of generators \( \{ f_i \} \) of \( a \) and a fixed compatible system of \( p \)-power roots of the \( f_i \) (this data we denote by \([f_1, \ldots, f_n]\)).

\[
\tau(A, [f_1, \ldots, f_n]) = \text{Ann}_A \left\{ \eta \in H^d_m(A) \mid \varpi^{1/p^n} (\prod_{i=1}^n f_i^{1/p}) \eta = 0 \text{ in } H^d_m(A_{\infty}) \right. \\
\text{for all } a \geq (t + \epsilon)p^e, e > 0 \text{ and } 0 < \epsilon \ll 1 \}\]

Our second definition is independent of the choice of generators and the choice of their \( p^e \)-th roots.

\[
\tau(A, a^t) = \text{Ann}_A \left\{ \eta \in H^d_m(A) \mid \varpi^{1/p^e} (a^{((t+\epsilon)p^e)})^{1/p^e} \eta = 0 \text{ in } H^d_m(A_{\infty}) \right. \\
\text{for all } e > 0 \text{ and } 0 < \epsilon \ll 1 \}\]

Here \( b^{1/p^e} \) denotes the set of all \( p^e \)-th roots of elements of an ideal \( b \) that are part of a compatible system of \( p \)-power roots. Note that in both definitions we need to build in an additional \( 0 < \epsilon \ll 1 \), this is important as it essentially gives us another perturbation (which in characteristic \( p > 0 \) is built into the \( c \) and in characteristic zero is built into the roundings).

The key difference between the two definitions is that, in the first definition, we are not taking \( p^e \)-th roots of elements like \( f_1 + f_2 \), and this is important when we establish the subadditivity property. Because of this if \( \{ f_1, \ldots, f_n \} \) generate \( a \), it is easy to see that

\[
\tau(A, [f_1, \ldots, f_n]) \subseteq \tau(A, a^t).
\]

Although the first definition depends on the choice of a generating set of the ideal \( a \), it satisfies the properties we need and thus helps lead to our final applications.

We show that the above perfectoid multiplier/test ideals satisfy the following properties.

- **Measuring singularities:** Proposition 3.4  
  If we have a containment of ideals
  
  \[
  (f_1, \ldots, f_m) = a \subseteq b = (f_1, \ldots, f_m, f_{m+1}, \ldots, f_n)
  \]
  
  then we have
  
  \[
  \tau(A, a^t) \subseteq \tau(A, b^t) \text{ and } \tau(A, [f_1, \ldots, f_m]^t) \subseteq \tau(A, [f_1, \ldots, f_n]^t),
  \]
  
  and if \( t < t' \) then
  
  \[
  \tau(A, a^{t'}) \subseteq \tau(A, a^t) \text{ and } \tau(A, [f_1, \ldots, f_m]^{t'}) \subseteq \tau(A, [f_1, \ldots, f_m]^t)
  \]

  In other words, the perfectoid multiplier/test ideal is smaller/deeper if \( V(a) \) is more singular or if \( t \) is larger.

- **Unambiguity of exponent:** Proposition 3.8  
  Proposition 3.9  
  \[
  \tau(A, a^{tm}) = \tau(A, (a^n)^t) \text{ and } \tau(A, [f]^{tn}) = \tau(A, [f^{*n}]^t)
  \]
  
  where \( f^{*n} \) denotes the set of all monomials of degree \( n \) in \( f = f_1, \ldots, f_m \).

- **Not too small:** Proposition 3.12  
  \( a \subseteq \tau(A, [f]) \subseteq \tau(A, a) \).

- **Not too big:** Lemma 5.8  
  If \( a \) is prime of height \( h \), then we have
  
  \[
  \tau(A, (a^{(h)})^t) \subseteq a.
  \]
Subadditivity: Theorem 4.3. With the notations as above, we have

\[ \tau(A, [f^n]^t) \subseteq \tau(A, [f]^t)^n. \]

Putting these together, and defining asymptotic perfectoid multiplier/test ideals similar to how asymptotic multiplier ideals were introduced in [ELS01], we obtain our main application.

Main Application (Theorem 7.4). Suppose that \( A \) is an excellent Noetherian regular ring. If \( Q \subseteq A \) is a prime ideal of height \( h \) then for all \( m > 0 \) we have

\[ Q^{(mh)} \subseteq Q^m. \]

Remark 1.1. Just as in [ELS01] or [HH02], we actually obtain a more general statement about symbolic powers of radical ideals.

Beyond this, it is also natural to try to compare our perfectoid multiplier/test ideal with those in equal characteristic. Besides the obvious analogies described above, we also obtain the following result.

Theorem (Theorem 6.1). Suppose that \( (A, a^t) \) is a pair. Since \( A[1/p] \) is characteristic zero, we can form the multiplier ideal \( J(A[1/p], (a \cdot A[1/p])^t) \). We have:

\[ \tau(A, a) \cdot A[1/p] \subseteq J(A[1/p], (a \cdot A[1/p])^t). \]

We hope that we actually obtain equality but we do not know how to show this. This difficulty is related to perhaps the biggest gap in the theory we have developed so far, how our perfectoid multiplier/test ideal behaves under localization, see Section 9 for additional discussion.

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2. Perfectoid algebras and André’s construction of \( A_\infty \)

Throughout this paper we will use similar notations as in [BMS16, Section 3] for (integral) perfectoid algebras in a generalized sense, see also [KL15], [Fon13], and [GR03]. We want to emphasize integral perfectoid to follow the language in [Bha16].

Definition 2.1. A ring \( S \) is integral perfectoid if and only if it is \( \varpi \)-adically complete for some element \( \varpi \in S \) such that \( \varpi^p \) divides \( p \), the Frobenius map \( \varphi: S/pS \to S/pS \) is surjective, and the kernel of \( W(S^p) \to S \) is principal, where \( S^p = \lim_{\leftarrow \varphi} S/pS \) is the tilt of \( S \).

We remark that, when \( S \) is an integral perfectoid algebra in the above generalized sense, by [BMS16, Remark 3.8 and Lemma 3.9], we can actually pick \( \varpi \) such that \( \varpi^p = up \) for some unit \( u \) in \( S \) and that \( \varpi \) admits a compatible system of \( p \)-power roots, i.e., there exists elements \( \varpi_1 = \varpi, \varpi_2, \ldots, \varpi_n, \ldots \) such that \( \varpi_{i+1}^p = \varpi_i \) for all \( i \).

Throughout the rest of the article we will make these assumptions on \( \varpi \) when we talk
about integral perfectoid algebras, and unless otherwise stated, almost mathematics will always be measured with respect to \((\varpi^{1/p^\infty})\).

We next point out that if \(S\) is in fact a \(K^0\)-algebra for some perfectoid field \(K\), one can take \(\varpi\) to be any topological nilpotent of \(K\) such that \(|p|\leq |\varpi^p|\), and then the above definition is the same as Scholze’s integral perfectoid \(K^0\)-algebra as in [Sch12] (up to almost mathematics), see also [Bha16]. Moreover, when \(\varpi\) is a nonzerodivisor, \(S[1/\varpi]\), endowed with the \(\varpi\)-adic topology on \(S\), is a perfectoid complete Tate ring in the terminology of Fontaine [Fon13], and \(S\) is almost isomorphic to the ring of powerbounded elements of \(S[1/\varpi]\) [BMS16 Lemma 3.20 and 3.21]. In particular, replacing \(S\) by \(\{x \in S[1/\varpi] \mid \varpi^{1/p^\infty}x \in S\}\), we can and we will always assume that \(S\) is integrally closed in \(S[1/\varpi]\).

Throughout this article, \((A, m)\) will always denote a mixed characteristic complete regular local ring of dimension \(d\). We want to fix an integral perfectoid \(A\)-algebra \(A_\infty\) that satisfies the following additional hypothesis:

(a) All elements of \(A\) have a compatible system of \(p\)-power roots in \(A_\infty\).
(b) If \(p^k, x_1, \ldots, x_{d-1}\) is a system of parameters of \(A\), then it is an almost regular sequence on \(A_\infty\).
(c) \(A_\infty\) is almost flat over \(A\), i.e., \(\operatorname{Tor}^A_1(M, A_\infty)\) is almost zero for all \(M\).

We claim that such \(A_\infty\) exists. Below we give one way of constructing it. One first constructs an integral perfectoid \(A\)-algebra \(A_{\infty,0}\) that is faithfully flat over \(A\) as in [And16b] Example 3.4.5 [Shi16 Section 4]. This construction is not canonical and it depends on the choice of coordinates of \(A\). For example when \((A, m)\) is unramified, i.e., \(A \cong W(k)[x_1, \ldots, x_{d-1}]\) where \(W(k)\) is the ring of Witt vectors with coefficients in a characteristic \(p > 0\) field \(k\), one can let \(A_{\infty,0}\) be the \(p\)-adic completion of

\[
A[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}]
\]

and set \(\varpi = p^{1/p}\). Next we fix such an \(A_{\infty,0}\) and \(\varpi\). For any finite sequence of elements \(g_1, \ldots, g_n\) in \(A\), we have

\[
A_{\infty,0} \to A_{\infty,1} \to \cdots \to A_{\infty,n}
\]

such that \(A_{\infty,i-1} \to A_{\infty,i}\) is André’s construction of integral perfectoid algebras with respect to \(g_i\). More explicitly, \(A_{\infty,i}\) is the \(\varpi\)-adic completion of the integral closure of \(A_{\infty,i-1}[T^{1/p^\infty}]/(T - g_i)\) in \(A_{\infty,i-1}[T^{1/p^\infty}]/(T - g_i)[1/\varpi]\). It is readily seen that \(g_i\) admits a compatible system of \(p\)-power roots in \(A_{\infty,i}\) (and thus in \(A_{\infty,n}\)) and the construction \(A_{\infty,0} \to A_{\infty,n}\) does not depends on the order of \(g_1, \ldots, g_n\), in particular all these \(A_{\infty,n}\) form a direct limit system. More importantly, Andre proved that \(A_{\infty,i}\) is almost faithfully flat over \(A_{\infty,i-1}\) mod \(\varpi\) [And16b], also see [Bha16 Theorem 2.3 and 6.1]. We next set

\[
A_\infty = \varprojlim_{g_1, \ldots, g_n} A_{\infty,n},
\]

which is the \(\varpi\)-adic completion of the colimit of \(A_{\infty,n}\) for all finite sequences of elements of \(A\). It is not hard to show that \(A_\infty\) is still an integral perfectoid algebra

\[\text{This means that } \frac{(p^k, x_1, \ldots, x_{d-1})A_{\infty,j+1}}{(p^k, x_1, \ldots, x_{d-1})A_\infty} \text{ is almost zero and } A_\infty/(p^k, x_1, \ldots, x_{d-1})A_\infty \text{ is not almost zero.}\]
that is almost faithfully flat over \( A_{\infty,0} \) mod \( \varpi \), and that all elements of \( A \) have a compatible system of \( p \)-power roots in \( A_\infty \): in this case since \( \varpi \) is a nonzerodivisor on \( A_\infty \), the last condition of being an integral perfectoid algebra is satisfied if \( \varphi: A_\infty / \varpi \to A_\infty / \varpi^p \) is an isomorphism \([BMS16 \text{ Lemma 3.10}]\), but this is true for all \( A_{\infty,n} \) appearing in the colimit. Note that this also implies \( A_\infty \) is almost faithfully flat over \( A_{\infty,0} \) mod \( \varpi^m \) for all \( m > 0 \). Now we prove that our \( A_\infty \) also satisfies properties (b) and (c) mentioned above.

**Lemma 2.2.** If \( p^k, x_1, \ldots, x_{d-1} \) is a system of parameters of \( A \), then it is an almost regular sequence on \( A_\infty \). Moreover, Tor\(_1^A(M, A_\infty)\) is almost zero for all \( A \)-modules \( M \). In particular, \( A_\infty \otimes \text{Hom}_A(N, M) \) is almost isomorphic to \( \text{Hom}_{A_\infty}(N \otimes A_\infty, M \otimes A_\infty) \) for all finitely generated \( A \)-modules \( N \) and all \( A \)-modules \( M \).

**Proof.** Clearly \( p^k, x_1, \ldots, x_{d-1} \) is a regular sequence on \( A_{\infty,0} \) since \( A_{\infty,0} \) is faithfully flat over \( A \). Because \( \varpi^p = u \) for some unit \( u \), \( A_\infty / p^k \) is almost faithfully flat over \( A_{\infty,0} / p^k \) for every \( k \) by our construction (since this is true for each \( A_{\infty,n} \)). Thus \( x_1, \ldots, x_{d-1} \) is an almost regular sequence on \( A_\infty / p^k \). This together with the fact that \( p \) is a nonzerodivisor on \( A_\infty \) proves the first assertion.

We next prove Tor\(_1^A(M, A_\infty)\) is almost zero. We first observe that since \( A \to A_{\infty,0} \) is faithfully flat and \( A_{\infty,0} / p^k \to A_\infty / p^k \) is almost faithfully flat, we know that \( A / p^k \to A_\infty / p^k \) is almost flat, i.e., Tor\(_1^{A/p^k}(N, A_\infty / p^k)\) is almost zero for all \( A/p^k \)-module \( N \). In order to show Tor\(_1^A(M, A_\infty)\) is almost zero for all \( A \)-module \( M \), we may assume \( M \) is a finitely generated \( A \)-module by taking direct limit. By looking at

\[
0 \to \bigcup_k \text{Ann}_A p^k \to M \to \overline{M} \to 0,
\]

we only need to prove Tor\(_1^A(M, A_\infty)\) is almost zero when \( M \) is killed by \( p^k \) for some \( k \) or when \( M \) is \( p \)-torsion free.

If \( M \) is killed by \( p^k \), then we have \( 0 \to K \to F \to M \to 0 \) where \( F \) is a free \( A / p^k \)-module. We have

\[
0 = \text{Tor}_1^A(F, A_\infty) \to \text{Tor}_1^A(M, A_\infty) \to K \otimes A_\infty \to F \otimes A_\infty
\]

where the leftmost 0 is because \( A_\infty \) is \( \varpi \)-torsion free and thus \( p \)-torsion free. From this sequence it is enough to prove \( K \otimes A_\infty \to F \otimes A_\infty \) is an almost injection. But since \( K \) and \( F \) are \( A / p^k \)-modules, \( K \otimes A_\infty \cong K \otimes_{A/p^k} A_\infty / p^k \) and similarly for \( F \). Therefore almost injectivity follows because \( A_\infty / p^k \) is almost flat over \( A / p^k \).

Now we assume \( p \) is a nonzerodivisor on \( M \). If \( F_\bullet \to M \to 0 \) is a finite free resolution of \( M \), then \( F_\bullet \otimes A / p^m \to M / p^m \to 0 \) is a finite free resolution of \( M / p^m \) over \( A / p^m \) for all \( m \). Since \( A_\infty \) is \( \varpi \)-adic complete and thus \( p \)-adically complete,

\[
M \otimes^L A_\infty \cong F_\bullet \otimes A_\infty \cong \lim_{\leftarrow m} (F_\bullet \otimes A_\infty) / p^m(F_\bullet \otimes A_\infty)
\]

\[
\cong \lim_{\leftarrow m} F_\bullet \otimes A / p^m \otimes_{A/p^m} A_\infty / p^m \cong \lim_{\leftarrow m} M / p^m \otimes^L_{A/p^m} A_\infty / p^m
\]

Since \( A_\infty / p^m \) is almost flat over \( A / p^m \), \( h^{-1}(M / p^m \otimes^L_{A/p^m} A_\infty / p^m) \) is almost zero for all \( m \), thus \( h^{-1}(M \otimes^L A_\infty) \) is almost zero.
Finally, the last assertion follows formally from the fact that $A_{\infty}$ is almost flat over $A$: if $A_{\infty}^{\oplus l} \rightarrow A_{\infty}^{\oplus n} \rightarrow N \rightarrow 0$ is a presentation of $N$, then we have $A_{\infty}^{\oplus l} \rightarrow A_{\infty}^{\oplus n} \rightarrow N \otimes A_{\infty} \rightarrow 0$. We look at the following commutative diagram:

\[
\begin{array}{ccc}
A_{\infty} \otimes \text{Hom}_A(A_{\infty}^{\oplus l}, M) & \cong & \text{Hom}_{A_{\infty}}(A_{\infty}^{\oplus l}, M \otimes A_{\infty}) \\
\downarrow & & \downarrow \\
A_{\infty} \otimes \text{Hom}_A(A_{\infty}^{\oplus n}, M) & \cong & \text{Hom}_{A_{\infty}}(A_{\infty}^{\oplus n}, M \otimes A_{\infty}) \\
\downarrow & & \downarrow \\
A_{\infty} \otimes \text{Hom}_A(N, M) & \phi & \text{Hom}_{A_{\infty}}(N \otimes A_{\infty}, M \otimes A_{\infty}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

The right vertical line is exact, and the left vertical line is almost exact because $A_{\infty}$ is almost flat over $A$. Now chasing the above commutative diagram, $A_{\infty} \otimes \text{Hom}_A(N, M)$ is almost isomorphic to $\text{Hom}_{A_{\infty}}(N \otimes A_{\infty}, M \otimes A_{\infty})$. \qed

Remark 2.3. In fact, very recently Bhatt [Bha17, Corollary 9.4.7] showed that one can construct an integral perfectoid $A$-algebra $A_{\infty}$ that is almost faithfully flat over $A_{\infty,0}$ mod $\varpi$ and such that $A_{\infty}$ is absolutely integrally closed, i.e., each monic polynomial has a root in $A_{\infty}$. In particular every element of $A_{\infty}$ admits a compatible system of $p$-power roots. It then follows from the proof of the above Lemma 2.2 that such $A_{\infty}$ also satisfies the required properties (a), (b), (c).

3. Perfectoid multiplier/test ideals

Our goal in this section is to introduce a mixed characteristic ideal which is analogous to the multiplier/test ideal which appeared independently in higher dimensional algebraic geometry and also in characteristic $p > 0$ commutative algebra [Laz04, HH90].

We recall the reader that we are always using the notations as in Section 2. That is, $A$ is a $d$-dimensional complete regular local ring of mixed characteristic, and $A \rightarrow A_{\infty}$ is an almost flat extension such that $A_{\infty}$ is an integral perfectoid algebra and all elements of $A$ have a compatible system of $p$-power roots in $A_{\infty}$.

Definition 3.1. We fix a set of elements $\Lambda \subseteq A_{\infty}$. We define

\[
\phi_{H_{\infty}^d(A)}^A \Lambda = \{ \eta \in H_{m}^d(A) \mid \varpi^{1/p^\infty} \lambda \eta = 0 \text{ in } H_{m}^d(A_{\infty}) \text{ for all } \lambda \in \Lambda \},
\]

and then we set $\tau_{A_{\infty}}(\Lambda) = \text{Ann}_A(\phi_{H_{\infty}^d(A)}^A \Lambda)$. Since we always work in a fixed $A_{\infty}$ in this article, we will simplify our notation and write $\tau^d(\Lambda) = \tau_{A_{\infty}}^d(\Lambda)$.

We will be mostly interested in the following two cases:
(a) A consists of all $f^{1/p^e}$ where $f \in \mathfrak{a}^{[tp^e]}$ for some $e \geq 0$, where $\mathfrak{a} \subseteq A$ is an ideal and $f^{1/p^e}$ is part of a compatible system of $p$-power roots of $f$ in $A_{\infty}$. In this case, we write $0^{\wedge}_{H_{\mathfrak{m}}(A)}\mathfrak{a}$ for $0^{\wedge}_{H_{\mathfrak{m}}(A)}\Lambda$ and we write $\tau^2(\mathfrak{a}^t)$ for $\tau^2(\Lambda)$.

(b) Fix a sequence of elements $\{f_1, \ldots, f_n\}$ of $A$, and fix a compatible system of $p$-power roots $\{f_i^{1/p^e}\}_{i=1}^n$ for each $f_i$. Let $\Lambda = \{g \mid g = \prod_{i=1}^a f_i^{1/p^e} \text{ where } a \geq tp^e\}$. In this case, we write $0^{\wedge}_{H_{\mathfrak{m}}(A)}[f_1, \ldots, f_n]^t$ for $0^{\wedge}_{H_{\mathfrak{m}}(A)}\Lambda$ and we write $\tau^2([f_1, \ldots, f_n]^t)$ or $\tau^2([f]^t)$ for $\tau^2(\Lambda)$. In practice, we always assume $f_1, \ldots, f_n$ generate the ideal $\mathfrak{a} \subseteq A$.

Because of the importance of the above two cases, we also give their direct definitions below.

**Definition 3.2.** Fix an ideal $\mathfrak{a} \subseteq A$ and a real number $t \geq 0$. We define

$$0^{\wedge}_{H_{\mathfrak{m}}(A)}\mathfrak{a}^t = \{\eta \in H_{\mathfrak{m}}^d(A) \mid \varpi^{1/p^\infty} f^{1/p^e}\eta = 0 \text{ in } H_{\mathfrak{m}}^d(A_{\infty}) \text{ for all } f \in \mathfrak{a}^{[tp^e]}\},$$

where $f^{1/p^e}$ denotes all possible $p^e$-th roots of $f$ in $A_{\infty}$ that are part of a compatible system of $p$-power roots. If we also fix a sequence of elements $\{f_1, \ldots, f_n\}$ generating $\mathfrak{a}$ and for each $i$, one fixed compatible system of $p$-power roots $\{f_i^{1/p^e}\}_{i=1}^n$ for $f_i$, then we set

$$0^{\wedge}_{H_{\mathfrak{m}}(A)}[f_1, \ldots, f_n]^t = \{\eta \in H_{\mathfrak{m}}^d(A) \mid \varpi^{1/p^\infty} g\eta = 0 \text{ in } H_{\mathfrak{m}}^d(A_{\infty}) \text{ for all } g = \prod_{i=1}^a f_i^{1/p^e} \text{ where } a \geq tp^e\}.$$

We then define

$$\tau^2(\mathfrak{a}^t) = \tau^2(A, \mathfrak{a}^t) = \text{Ann}_A(0^{\wedge}_{H_{\mathfrak{m}}(A)}\mathfrak{a}^t)$$

and

$$\tau^2([f_1, \ldots, f_n]^t) = \tau^2(A, [f_1, \ldots, f_n]^t) = \text{Ann}_A(0^{\wedge}_{H_{\mathfrak{m}}(A)}[f_1, \ldots, f_n]^t).$$

We will usually omit the subscript and write $0^{\wedge}_{H_{\mathfrak{m}}(A)}\mathfrak{a}^t$ or $0^{\wedge}_{H_{\mathfrak{m}}(A)}[f_1, \ldots, f_n]^t$. We will usually write $\tau^2([f]^t)$ for $\tau^2([f_1, \ldots, f_n]^t)$ when $f_1, \ldots, f_n$ is clear from the context. In the case that $\mathfrak{a} = \langle f \rangle$ is principal, we will write $\tau^2(f^t)$ for $\tau^2(\mathfrak{a}^t)$. We emphasize that a priori this is different from $\tau^2([f]^t)$.

**Remark 3.3.** A key part of the definition of $0^{\wedge}_{H_{\mathfrak{m}}(A)}[f_1, \ldots, f_n]^t$ and $\tau^2([f_1, \ldots, f_n]^t)$ is that we are not taking $p^e$-th roots of elements like $f_1 + f_2$, and this is important, because taking $p^e$-th roots is not additive, as in characteristic $p > 0$.

The following properties are straightforward from the definition.

**Proposition 3.4.** Fix $\{f_1, \ldots, f_n\}$ a sequence of generators of an ideal $\mathfrak{a} \subseteq A$ and for each $f_i$ fix a compatible system of $p$-power roots of $f_i$ (in order to define $\tau^2([f_1, \ldots, f_n]^t)$). Then we have

4We call this a sequence because we allow repeats.
(a) For every $t \geq 0$, $\tau^d([f_1, \ldots, f_n]^t) \subseteq \tau^d(a^t)$.

(b) If $t' > t$, then $\tau^d(a^{t'}) \subseteq \tau^d(a^t)$ and $\tau^d([f_1, \ldots, f_n]^{t'}) \subseteq \tau^d([f_1, \ldots, f_n]^t)$.

(c) If $a \subseteq b$, and $f_1, \ldots, f_n, f_{n+1}, \ldots, f_m$ is a fixed set of generators of $b$ (we should also fix a compatible system of $p$-power roots of each $f_i$), then for every $t \geq 0$, $\tau^d(a^t) \subseteq \tau^d(b^t)$ and $\tau^d([f_1, \ldots, f_n]^t) \subseteq \tau^d([f_1, \ldots, f_n]^t)$.

One of the key properties of multiplier and test ideals is the fact that small positive perturbations of the exponent do not change the ideal. We do not know if this is true for $\tau^d$.

**Question 3.5.** Is it true that $\tau^d(a^{t'}) = \tau^d(a^{t+\epsilon})$ or $\tau^d([f_1, \ldots, f_n]^{t'}) = \tau^d([f_1, \ldots, f_n]^{t+\epsilon})$ for all $\epsilon \ll 1$?

Because of this, we make the following definition. This would be our real definition of perfectoid multiplier/test ideals.

**Definition 3.6.** Fix $\{f_1, \ldots, f_n\}$ a sequence of generators of an ideal $a \subseteq A$ and for each $f_i$ fix a compatible system of $p$-power roots of $f_i$ (in order to define $\tau^d([f_1, \ldots, f_n]^t)$).

We define $\tau^d(a^t) = \tau(A, a^t)$ to be the union or sum of $\{\tau^d(a^t')\}$ for all $t' > t$. Since $\tau^d(a^{t'}) \subseteq \tau^d(a^t)$ for all $t' > t$, by the Noetherian property of $A$, this is $\tau^d(a^{t+\epsilon})$ when $\epsilon \ll 1$. Similarly, we define $\tau([f_1, \ldots, f_n]^{t'}) = \tau(A, [f_1, \ldots, f_n]^t)$ to be $\tau^d([f_1, \ldots, f_n]^{t+\epsilon})$ for $\epsilon \ll 1$.

It is clear from the definition and Proposition 3.4 that we have:

(1) $\tau^d(a^t) \supseteq \tau^d([f_1, \ldots, f_n]^t) \supseteq \tau([f_1, \ldots, f_n]^t) \subseteq \tau(a^t) \subseteq \tau^d(a^t)$.

As with multiplier ideals and test ideals, in the notation $\tau(a^t)$, $a^t$ is a formal object. This can cause confusion when $t$ is an integer since $a^2$ for instance makes sense on its own. Thus it is natural to ask whether $\tau^d((a^n)^t) = \tau^d(a^{nt})$ and whether $\tau((a^n)^t) = \tau(a^{nt})$. We do not know how to do this with our definition of $\tau^d$, but it is not difficult to show this for $\tau(a^t)$ and $\tau([f_1, \ldots, f_n]^t)$. This will be crucial for our later purposes (and suggests that $\tau$ is a better definition than $\tau^d$).

**Lemma 3.7.** In the definition of $0^{\Diamond_{A_{\infty}}}a^n_{H_{m}(A)}$, one may restrict to $e \gg 0$.

**Proof.** Indeed, obviously restricting the $e$ to those $e \gg 0$ results in fewer conditions and hence a larger subset of $H^d_{m}(A)$. On the other hand suppose that $\eta$’s image in $H^d_{m}(A_{\infty})$ is killed by $\varpi ^{1/p} f^{1/p^e}$ for all $f \in a^{[tp^n]}$ where $f^{1/p^e}$ is part of a compatible system of $p$-power roots, for $e \geq e_0$. Fix now some $n \geq 0$ and $f \in a^{[tp^n]}$. Then for $e \geq e_0$,

$$\varpi ^{1/p} f^{1/p^e} \varpi ^{1/p^{\eta}} = \varpi ^{1/p^e} (f^{p^e})^{1/p^{\eta^e}} \eta = 0$$

since $f^{p^e} \in a^{[tp^n]} \subseteq a^{[tp^{n+e}]}$. This shows that $\eta \in 0^{\Diamond_{A_{\infty}}}a^n_{H_{m}(A)}$.

**Proposition 3.8.** For all positive integers $n$, $\tau((a^n)^t) = \tau(a^{nt})$.

**Proof.** Fix an actual $\epsilon > 0$ so that $\tau((a^n)^{t'}) = \tau^d((a^n)^{t+\epsilon})$ and $\tau(a^{nt}) = \tau^d(a^{nt+\epsilon})$. By definition, any $\tau^d((a^n)^{t'+\epsilon})$ where $\epsilon' \leq \epsilon$ also computes $\tau((a^n)^t)$. We first show that $0^{\Diamond_{A_{\infty}}}a^{nt+\epsilon} \supseteq 0^{\Diamond_{A_{\infty}}}a^{nt+\epsilon}$. Note that $0^{\Diamond_{A_{\infty}}}a^{nt+\epsilon}$ consists of $\eta \in H^d_{m}(A)$ whose
images in $H^d_m(A_\infty)$ are annihilated by $\varpi^{1/p^\infty} f^{1/p^e}$ with $f \in a^{[(nt+\varepsilon)p^e]}$ and $f^{1/p^e}$ part of a compatible system of $p$-power roots of $f$, while $0^{\bigotimes A_\infty(a^n)^{t+\varepsilon}}$ consists of $\eta$ that are annihilated by $\varpi^{1/p^\infty} g^{1/p^e}$, where $g \in (a^n)^{(t+\varepsilon)p^e}$ and $g^{1/p^e}$ part of a compatible system of $p$-power roots of $g$. One sees immediately that $(a^n)^{(t+\varepsilon)p^e} \subseteq a^{[(nt+\varepsilon)p^e]}$ and so $0^{\bigotimes A_\infty(a^n)^{t+\varepsilon}} \supseteq 0^{\bigotimes A_\infty(a^n)^{t+\varepsilon}}$ follows and thus $\tau((a^n)^{t+\varepsilon}) \subseteq \tau(a^{nt+\varepsilon})$.

Conversely, note that

$$[(nt+\varepsilon)p^e] \geq n[(t+\varepsilon/(2n))p^e]$$

for $e \gg 0$. Thus

$$a^{[(nt+\varepsilon)p^e]} \subseteq a^{n[(t+\varepsilon/(2n))p^e]}$$

and so arguing as above, and using Lemma 3.7, we see that

$$0^{\bigotimes A_\infty(a^n)^{t+\varepsilon}} \supseteq 0^{\bigotimes A_\infty(a^n)^{t+\varepsilon}} = 0^{\bigotimes A_\infty(a^n)^{t+\varepsilon}}$$

where the last equality follows from our choice of $\varepsilon$. This finishes the proof. \[\Box\]

We next prove the analogous result for $\tau((f_1, \ldots, f_n)^t) = \tau([f]^t)$.

**Proposition 3.9.** Fix $\{f\} = \{f_1, \ldots, f_n\}$ a sequence of generators of an ideal $a \subseteq A$ and for each $f_i$ fix a compatible system of $p$-power roots of $f_i$ to define $\tau_i([f_i]^t)$. Set $f^{\bullet n}$ to be the set of degree $n$ monomials in the $f_i$, and we use the product of the fixed compatible system of $p$-power roots of $f_i$ to build compatible system of $p$-power roots for $f^{\bullet n}$. Then for all positive integers $n$ and real numbers $t \geq 0$, $\tau([f^{\bullet n}]^t) = \tau([f]^n)$.

**Proof.** Fix an actual $\varepsilon > 0$ so that $\tau((f^{\bullet n})^t) = \tau_i((f^{\bullet n})^t)$ and $\tau((f^n)^t) = \tau_i((f^n)^t)$. By definition, any $\tau_i((f^{\bullet n})^{t+\varepsilon'})$ where $\varepsilon' \leq \varepsilon$ also computes $\tau((f^{\bullet n})^t)$.

As in the proof of Proposition 3.8, the containment $0^{\bigotimes A_\infty(f^{\bullet n})^{t+\varepsilon}} \subseteq 0^{\bigotimes A_\infty(f^{\bullet n})^{nt+\varepsilon}}$ is straightforward. To see $0^{\bigotimes A_\infty(f^{\bullet n})^{t+\varepsilon}} \subseteq 0^{\bigotimes A_\infty(f^{\bullet n})^{nt+\varepsilon}}$, we note that if $a/p^e \geq nt + \varepsilon$, then we can write $a = bn + c$ such that $b = [a/n]$ and $0 \leq c \leq n - 1$. Pick $e \gg 0$ such that $c/p^e \leq \varepsilon/2$, since $bn + c/p^e \geq nt + \varepsilon$ and we must have $bn/p^e \geq nt + \varepsilon/2$ and thus $b/p^e \geq t + \varepsilon/2n$. Therefore if $\eta$ is killed by all $\varpi^{1/p^\infty} g$, where $g = \prod b_i g_i^{1/p^e}$ with $g_i^{1/p^e} \equiv \prod f_j^{1/p^e}$ and $b/p^e \geq t + \varepsilon/2n$, then it is killed by $\varpi^{1/p^\infty} f$ with $f = \prod a_j^{1/p^e}$ and $a/p^e \geq nt + \varepsilon$ in $H^d_m(A_\infty)$. This proves that

$$0^{\bigotimes A_\infty(f^{\bullet n})^{nt+\varepsilon}} \supseteq 0^{\bigotimes A_\infty(f^{\bullet n})^{nt+\varepsilon}} = 0^{\bigotimes A_\infty(f^{\bullet n})^{nt+\varepsilon}}$$

where again the last equality follows from our choice of $\varepsilon$. This finishes the proof. \[\Box\]

Our next goal is to show that $a \subseteq \tau((f_1, \ldots, f_n))$ for any set of generators $\{f_1, \ldots, f_n\}$ of $a$ (and any fixed set of compatible system of $p$-power roots $\{f_1^{1/p^e}\}_{e=1}^\infty$). It would follow that $a \subseteq \tau(a)$ by (11). To establish this result, we need the following lemma on the construction of big Cohen-Macaulay $A_\infty$-algebras.

**Lemma 3.10.** Let $c \neq 0$ be an element of $A_\infty$ and $I \subseteq A$ be an ideal. Let $p, x_1, \ldots, x_{d-1}$ be a system of parameters on $A$. Suppose whenever $x_{s+1}t_{s+1} = pt_0 + x_1t_1 + \cdots + x_st_s$ with $t_j \in A_\infty$, we have $c^{1/p^e}t_{s+1} \in (p, x_1, \ldots, x_s)A_\infty$. Then for every $u \in A$ such that
Proposition 3.12. If \( c^{1/p^\infty} u \in IA_\infty \), there exists a balanced big Cohen-Macaulay \( A_\infty \)-algebra \( B \) such that \( u \in IB \).

Proof. Let \( I = (y_1, \ldots, y_t) \). We consider the following sequence:

\[
2 \quad A_\infty \to T_0 = \left( \frac{A_\infty[Y_1, \ldots, Y_t]}{u - y_1 Y_1 - \cdots - y_t Y_t} \right)_{\leq N_1} \to T_1 \to T_2 \to \cdots \to T_r
\]

where \( T_{i+1} \) is a partial algebra modification of \( T_i \) for \( i \geq 0 \) in the sense of [Hoc02, Section 4]. Following [Hoc02, Theorem 4.2] or [HM17, Theorem 1.6], in order to show that there exists a balanced big Cohen-Macaulay \( A_\infty \)-algebra \( B \) such that \( u \in IB \), it suffices to show that there is no such sequence such that the image of \( 1 \) in \( T_r \) is in \( m T_r \).

Now suppose \( 1 \in m T_r \) in \( 2 \) and suppose \( c^{1/p^e} u = z_1 y_1 + \cdots + z_t y_t \). We look at the following commutative diagram:

\[
\begin{array}{cccccccc}
A_\infty & \to & T_0 & \to & T_1 & \to & T_2 & \to & \cdots & \to & T_r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_\infty & \to & A_\infty[1/(c^{1/p^e})^N] & \to & A_\infty[1/(c^{1/p^e})^{N_1}] & \to & A_\infty[1/(c^{1/p^e})^{N_2}] & \to & \cdots & \to & A_\infty[1/(c^{1/p^e})^{N_r}]
\end{array}
\]

We briefly explain why such diagram exists. The second vertical map sends \( Y_i \) to \( z_i/c^{1/p^e} \), one can easily check that this is well defined. The other vertical maps exist because \( c^{1/p^e} \) trivializes all bad relations on \( (p, x_1, \ldots, x_{d-1}) \) by our hypothesis, so we can define these maps as in [Hoc02]. The key point is that, the numbers \( N_1, \ldots, N_r \) depend only on \( N \) and the sequence \( 2 \) but not on \( e! \).

Tracing the image of \( 1 \in A_\infty \) in the above commutative diagram in two different ways, we find that \( 1 \in A_\infty[1/(c^{1/p^e})^{N_r}] \) is inside \( mA_\infty[1/(c^{1/p^e})^{N_r}] \). This means \( (c^{1/p^e})^{N_r} \in mA_\infty \). Since \( N_r \) does not depend on \( e \), this is a contradiction because \( mA_\infty \) is a finitely generated ideal of \( A_\infty \).

Remark 3.11. Since \( p, x_1, \ldots, x_{d-1} \) is an almost regular sequence on \( A_\infty \) by Lemma 2.2, one can take \( c = \varpi \) or any multiple of \( \varpi \) in Lemma 3.10 and in particular, the conditions of Lemma 3.10 are satisfied.

Proposition 3.12. Fix \( \{ f \} = \{ f_1, \ldots, f_n \} \) a sequence of generators of an ideal \( a \subseteq A \), and for each \( f_i \) fix a compatible system of \( p \)-power roots of \( f_i \) to define \( \tau([f]) \). Then we have \( a \subseteq \tau([f]) = \tau([f]^1) \).

Proof. We pick \( \varepsilon \ll 1 \) such that \( \tau([f]) = \tau^\sharp([f^1])^{1+\varepsilon} \). It is enough to show that \( a = (f_1, \ldots, f_n) \) annihilates \( 0^{1+\varepsilon} A_\infty[f^1] \). Fix an \( \eta \in 0^{1+\varepsilon} A_\infty[f^1] \), it is enough to show that \( f_i \eta = 0 \) for each \( i \). Since \( 0^{1+\varepsilon} A_\infty[f^1] = 0^{1+\varepsilon} A_\infty[f^1]^{1/p} \) for all \( e \gg 0 \) by our definition of \( \tau([f]) \) and our choice of \( \varepsilon \), we know that \( \varpi^{1/p^e} f_i^{1/p^e} f_i \eta = 0 \) in \( H_{mA}^d(A_\infty) \) for all \( e \gg 0 \) by the definition of \( 0^{1+\varepsilon} A_\infty[f^1]^{1/p} \). Therefore we have \( (\varpi f_i)^{1/p^e} (f_i \eta) = 0 \) in \( H_{mA}^d(A_\infty) \).

Let \( p, x_1, \ldots, x_{d-1} \) be a system of parameters of \( A \). We can write \( f_i \eta = \frac{u^{p^t} x_1^{p^t} \cdots x_{d-1}^{p^t}}{p x_1^{p^t} \cdots x_{d-1}^{p^t}} \) for some \( u \in A \) and some \( t > 0 \). The condition \( (\varpi f_i)^{1/p^e} (f_i \eta) = 0 \) in \( H_{mA}^d(A_\infty) \) means
that for every $e$, there exists some $t_e > 0$ such that

$$(\omega f)^{1/p^e} (px_1 \ldots x_{d-1})^{t_e} u \in (p^{t+t_e}, x_1^{t+t_e}, \ldots, x_{d-1}^{t+t_e}) A_\infty.$$  

Since $p, x_1, \ldots, x_{d-1}$ is an almost regular sequence on $A_\infty$ by Lemma 2.2, it follows that

$$(\omega^{1/p^e}) (\omega f)^{1/p^e} u \in (p^t, x_1^t, \ldots, x_{d-1}^t) A_\infty$$

for every $e$. Thus $(\omega f)^{1/p^e} u \in (p^t, x_1^t, \ldots, x_{d-1}^t) A_\infty$. Now apply Lemma 3.10 to $u$ and $c = \omega f$ (the conditions are satisfied by Remark 3.11), we know that there exists a balanced big Cohen-Macaulay $A_\infty$-algebra $B$ such that $u \in (p^t, x_1^t, \ldots, x_{d-1}^t) B$. But since $A$ is regular and $B$ is balanced big Cohen-Macaulay, $B$ is faithfully flat over $A$ and hence $u \in (p^t, x_1^t, \ldots, x_{d-1}^t) B \cap A = (p^t, x_1^t, \ldots, x_{d-1}^t)$. This clearly implies that $f_i \eta = \frac{u}{p^{x_1 - x_{d-1}}} = 0$ in $H^d_m(A)$, i.e., $f_i$ annihilates $\eta$.  

Corollary 3.13. For $0 \neq f \in A$, we have $\tau([f]) = \tau^2([f]) = \langle f \rangle$.

Proof. We know $\tau([f]) \subseteq \tau^2([f])$ by (1) and $\langle f \rangle \subseteq \tau([f])$ by Proposition 3.12 thus it suffices to show that $\tau^2([f]) = \tau^3([f]) \subseteq \langle f \rangle$. For this it is enough to observe that $0 : H^d_m(A) f \subseteq 0^{\omega A_\infty} \omega f$. But this is clear since if $f \eta = 0$ then clearly $\omega^{1/p^e} f \eta = 0$ in $H^d_m(A_\infty)$ (note that $f = (f^{1/p^e})^{\omega^e}$ for every choice of $f^{1/p^e}$).

Corollary 3.14. Fix $\{f\} = \{f_1, \ldots, f_n\}$ a sequence of generators of an ideal $a \subseteq A$, and for each $f_i$ fix a compatible system of $p$-power roots of $f_i$ to define $\tau([f])$. Then we have $\tau([f]) = \tau^2([f]) = A$.

Proof. The argument is very similar to the one we used in Proposition 3.12. Since $\tau([f]) \subseteq \tau(a^0) \subseteq A$, it suffices to prove $\tau([f]) = A$, that is, $0^{\omega A_\infty} \omega f = 0$ when $\epsilon \ll 1$.

Suppose $\eta \in 0^{\omega A_\infty} \omega f = 0^{\omega A_\infty} \omega f^{1/p^e}$ for all $e > 0$, then we have $\omega^{1/p^e} f_i^{1/p^e} \eta = 0$ in $H^d_m(A_\infty)$ for all $e > 0$ and all $i$. Write $\eta = \frac{u}{p^{x_1 - x_{d-1}}}$, we have for every $e$, there exists $t_e$ such that

$$(\omega f)^{1/p^e} (px_1 \ldots x_{d-1})^{t_e} u \in (p^{t+t_e}, x_1^{t+t_e}, \ldots, x_{d-1}^{t+t_e}) A_\infty.$$  

Since $p, x_1, \ldots, x_{d-1}$ is an almost regular sequence on $A_\infty$ by Lemma 2.2, this implies $(\omega^{1/p^e}) (\omega f)^{1/p^e} u \in (p^t, x_1^t, \ldots, x_{d-1}^t)$ for every $e$. Now apply Lemma 3.10 to $u$ and $c = \omega f$, we know that there exists a big Cohen-Macaulay algebra $B$ such that $u \in (p^t, x_1^t, \ldots, x_{d-1}^t) B$. Since $B$ is faithfully flat over $A$, we have $u \in (p^t, x_1^t, \ldots, x_{d-1}^t) A$ and thus $\eta = 0$.  

4. The subadditivity theorem

The goal in this section is to prove the subadditivity for $\tau([f]_i)$. This is in analogous to [DEL00] and [HY03]. We do not know how to prove the subadditivity property for $\tau(a^i)$. This is the main reason that we need to work with $\tau([f]_i)$ in our later applications.

We first introduce the mixed perfectoid multiplier/test ideals.
Lemma 2.2 that Ann most isomorphic to b with proof of subadditivity for test ideals in characteristic τ t, s a
Definition 4.1. Let a, b ⊆ A be two ideals. Let \{f_1, \ldots, f_n\} be a fixed set of generators of a and \{g_1, \ldots, g_m\} be a fixed set of generators of b. We also fix a compatible system of p-power roots \{f_i^{1/p^e}\}_{i=1}^\infty, \{g_j^{1/p^e}\}_{j=1}^\infty for all f_i and g_j.

\[ 0^\wedge A_{\infty}\mu[t^i]s^* = \{ \eta \in H_m^d(A) \mid \omega^{1/p^\infty} f g \cdot \eta = 0 in H_m^d(A_{\infty}) \}
\]
for all f = \prod_{i=1}^b f_{j_i}^{1/p^e} and all g = \prod_{i=1}^b g_{k_i}^{1/p^e},
where \( a \geq t p^e \) and \( b \geq s p^e \}.

We then define \( \tau^2([t^i]s^* = \text{Ann}_{A} 0^\wedge A_{\infty}\mu[t^i]s^* \), and \( \tau^2([t^i]t^s) = \tau^2([t^i]t^e s^* + e) for \( e \ll 1 \).

Remark 4.2. Fix \( \{ f \} = \{ f_1, \ldots, f_n \} \) a sequence of generators of an ideal a ⊆ A and for each \( f_i \) fix a compatible system of p-power roots of \( f_i \) to define \( \tau^2([t^i]t^s) \). Moreover, fix two real number \( t, s \geq 0 \). One can check that for \( e \ll 1 \),

\( \tau([t^i]t^s) = \tau^2([t^i]t^{1+e}s^*) = \tau^2([t^i]t^{1+e+s^*} = \tau([t^i]t^{1+s^*}) \).

The proof that \( \tau^2([t^i]t^{1+e}s^*) = \tau^2([t^i]t^{1+s^*+e}) \) is similar to the proof of Proposition 3.9 and we omit the details.

We are ready to prove our subadditivity theorem. Our proof is inspired from the proof of subadditivity for test ideals in characteristic \( p > 0 \) given by S. Takagi from [Tak06, Theorem 2.4]. The essential reason that the theorem holds is because \( A_{\infty} \) is almost flat over A by Lemma 2.2.

Theorem 4.3 (Subadditivity). With notation as in Definition 4.1 and Remark 4.2, we have \( \tau^2([t^i]t^s) \subseteq \tau^2([t^i]t^{1+e}s^*) \). It follows that

\[ \tau([t^i]t^a^n) = \tau([t^i]t^a^n) \subseteq \tau([t^i]t^a) \]
for all \( t \) and all \( n \in \mathbb{N} \), where we define \( \tau([t^i]t^a^n) \) as in Proposition 3.9.

Proof. We first claim that it is enough to show that

\[ 0^\wedge A_{\infty}\mu[t^i]s^* \supseteq 0^\wedge A_{\infty}\mu[t^i]s^* : H_m^d(A) \tau^2([t^i]t^a). \]

To see this claim, if \( z \in \text{Ann}_{H_m^d(A)} \tau^2([t^i]t^a)[s^*] \) then \( \tau([t^i]t^a)z \subseteq \text{Ann}_{H_m^d(A)} \tau^2([t^i]s^*) = 0^\wedge A_{\infty}\mu[t^i]s^* \). Therefore

\[ \text{Ann}_{H_m^d(A)} \tau^2([t^i]t^a) \tau^2([t^i]s^*) \subseteq 0^\wedge A_{\infty}\mu[t^i]s^* \text{Ann}_{H_m^d(A)} \tau^2([t^i]s^*) \]
and thus \( \tau^2([t^i]t^a s^*) \subseteq \tau([t^i]t^a) \tau([t^i]t^a) s^* \) as desired.

Next we prove \( 0^\wedge A_{\infty}\mu[t^i]s^* : H_m^d(A) \tau^2([t^i]t^a) \). Suppose \( \eta \in 0^\wedge A_{\infty}\mu[t^i]s^* : H_m^d(A) \tau^2([t^i]t^a) \), then \( \tau([t^i]t^a) \eta \subseteq 0^\wedge A_{\infty}\mu[t^i]s^* \).

By definition we know that \( \omega^{1/p^\infty} \eta \cdot \tau([t^i]t^a) = 0 in H_m^d(A_{\infty}) \) for all \( g = \prod_{i=1}^b g_{k_i}^{1/p^e} \) with \( b \geq s p^e \). This means \( \omega^{1/p^\infty} \eta \in \text{Ann}_{H_m^d(A_{\infty})}(\tau^2([t^i]t^a) A_{\infty}) \). But we know from Lemma 2.2 that \( \text{Ann}_{H_m^d(A_{\infty})}(\tau^2([t^i]t^a) A_{\infty}) = \text{Hom}_{A_{\infty}}(A\tau^2([t^i]t^a) A_{\infty}, H_m^d(A_{\infty})) \) is almost isomorphic to

\[ A_{\infty} \otimes \text{Hom}_{A}(A/\tau^2([t^i]t^a), H_m^d(A)) = A_{\infty} \otimes \text{Ann}_{H_m^d(A)} \tau^2([t^i]t^a) A_{\infty} = A_{\infty} \otimes 0^\wedge A_{\infty}\mu[t^i]t^a. \]
Therefore $\omega^{1/p^\infty} g\eta \in A_\infty \otimes 0^\vee_{A_\infty}[I]$, which means for every $k$ we can write

$$\omega^{1/p^k} g\eta = a_1\eta_1 + \cdots + a_l\eta_l$$

where $\eta_i \in 0^\vee_{A_\infty}[I]$ and $a_i \in A_\infty$. So for all $f = \prod_{i=1}^a f_{j_i}^{1/p^e}$ with $a \geq tp^e$,

$$\omega^{1/p^{k'}} \omega^{1/p^k} fg \cdot \eta = a_1(\omega^{1/p^{k'}} f\eta_1) + \cdots + a_l(\omega^{1/p^{k'}} f\eta_l) = 0,$$

for all $k, k'$. Thus we know $\omega^{1/p^\infty} fg \cdot \eta = 0$ for all $f = \prod_{i=1}^a f_{j_i}^{1/p^e}$ and all $g = \prod_{i=1}^b g_{k_i}^{1/p^e}$ such that $a \geq tp^e$ and $b \geq sp^e$. Hence $\eta \in 0^\vee_{A_\infty}[I]^{[g]^e}$ as desired.

Finally, (3) follows from Proposition 3.9 and the inclusion $\tau^2([f]^{[g]^e}) \subseteq \tau^2([f]^{[g]^e})\tau^2([g]^e)$ we just proved applied to $f = g$, and $t, s$ both equal to $t + \epsilon$ for $\epsilon \ll 1$, plus a straightforward induction on $n$. □

We could also define the mixed perfectoid multiplier/test ideal for $\tau^2(a^t)$ in an analogue way:

$$0^\vee_{H^d_m(A)} = \{ \eta \in H^d_m(A) \mid \omega^{1/p^\infty} f^{1/p^e} g^{1/p^e} \eta = 0 \text{ in } H^d_m(A) \text{ for all } f \in a^{[p^e]} \text{ and all } g \in b^{[sp^e]} \},$$

where $f^{1/p^e}$ and $g^{1/p^e}$ denote all possible part of a compatible system of $p$-power roots of $f$ and $g$ respectively. We then define $\tau^2(a^t b^s) = \text{Ann}_A 0^\vee_{H^d_m(A)}$ and $\tau^2(a^t b^s) = \tau^2(a^{t+s})$ for $\epsilon \ll 1$. In fact, working with this definition, one can also prove $\tau^2(a^t b^s) \subseteq \tau^2(a^t)\tau^2(b^s)$ following a very similar argument as in Theorem 4.3. The problem is that, it is not clear to us whether $\tau(a^t a^s) = \tau(a^{t+s})$, and hence the second conclusion of the subadditivity theorem does not seem to work for $\tau(a^t)$.

5. Comparison with blowup

The goal of this section is to prove Lemma 5.8 that is, $\tau((I^{(h)}))^{1/l} \subseteq I$ for every radical ideal $I \subseteq A$ of height $h$ and every positive integer $l$. This follows from our core result Lemma 5.6 Lemma 5.8 implies immediately that $\tau([f]^{1/l}) \subseteq I$ for every fixed generating set $\{f\}$ of $I^{(h)}$ (and every fixed compatible system of $p$-power roots of $f$) since we always have $\tau([f]^{1/l}) \subseteq \tau((I^{(h)}))^{1/l}$ by (1).

Our key idea is to study how information about our perfectoid multiplier/test ideal can be obtained by blowup a finitely generated ideal $J$. The situation is easier if $\sqrt{J}$ contains $p$ as then the blowup of $JA_\infty$ is admissible since it is trivial outside of $V(p)$. This allows us to use Scholze’s vanishing theorem for perfectoid spaces [Sch12, Proposition 6.14] (see also [GR04, 16.6.41] in the generalized set up) which tells us that passing to the blowup is essentially harmless, up to almost mathematics and factoring. In the case that $J$ does not contain a power of $p$, however, we use a similar strategy to the one in [Bha16, section 6]: we need to pass to certain enlargement of $A_\infty$ where a multiple of $p$ is contained in $JB$.

Because of the complications involved in doing this, we start by introducing some notations which allow us to get finer control of $\tau(a^t)$. Throughout this section, we
will mostly work with $\tau(a^i)$, but as discussed earlier, the final result [Lemma 5.8] holds for $\tau([f]^t)$ as well simply because of [1].

**Definition 5.1.**

$$0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t = \left\{ \eta \in H_m^d(A) \mid \varpi_{1/p^l}^{1/p^h} \eta = 0 \text{ in } H_m^d(A_{\infty}) \text{ for all } f \in a^{[p^h]} \right\},$$

where as usual $f^{1/p^h}$ denotes all $p^h$-th roots of $f$ that are part of a compatible system of $p$-power roots of $f$. We use $0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t$ (resp. $0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t$) if we allow $h$ (resp. $l$) to range over all positive integers. By definition we have $0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t = 0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t$.

**Lemma 5.2.** We have $0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t = 0_{\left[\frac{l}{k}, \infty\right]}^{\diamond A_{\infty}} a^t$ for all $k \gg 0$.

**Proof.** It is easy to see that $\cdots \subseteq 0_{\left[\frac{l+1}{h+1}, \infty\right]}^{\diamond A_{\infty}} a^t \subseteq 0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t \subseteq \cdots$. Since $H_m^d(A)$ is Artinian, $0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t = 0_{\left[\frac{l}{h+1}, \infty\right]}^{\diamond A_{\infty}} a^t = 0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t$ for all $l \gg 0$. Now we fix such an $l \gg 0$, it follows from the proof of Lemma 3.7 that $\cdots \subseteq 0_{\left[\frac{l+1}{h+1}, \infty\right]}^{\diamond A_{\infty}} a^t \subseteq 0_{\left[\frac{l}{h+1}, \infty\right]}^{\diamond A_{\infty}} a^t \subseteq \cdots$. Thus by the Artinian property of $H_m^d(A)$ again, $0_{\left[\frac{l}{h+1}, \infty\right]}^{\diamond A_{\infty}} a^t = 0_{\left[\frac{l}{h}, \infty\right]}^{\diamond A_{\infty}} a^t$ for all $h \gg 0$. Now take $k \geq \max\{l, h\}$. We have $0_{\left[\frac{l}{k}, \infty\right]}^{\diamond A_{\infty}} a^t = 0_{\left[\frac{l}{k}, \infty\right]}^{\diamond A_{\infty}} a^t$. 

We next prove a crucial lemma. This is a slight generalization of [HMT17 Lemma 3.2], and can be also obtained using [Bha16 Theorem 4.2]. We first recall that, for any element $g \in A_{\infty}, A_{\infty}(\varpi^m)$ will denote the integral perfectoid algebra, which is the ring of bounded functions on the rational subset $\{ x \in X \mid |\varpi^n| \leq |g(x)| \}$ where $X = \text{Spa}(A_{\infty}[1/\varpi], A_{\infty})$.

Now if $g \in A$, then for any fixed compatible system of $p$-power roots of $g$ in $A_{\infty}, A_{\infty}(\varpi^m)$ can be described almost explicitly as the $\varpi$-adic completion of $A_{\infty}[1/\varpi^m]$ [Sch12 Lemma 6.4].

**Lemma 5.3.** Let $I = (p^s, y_1, \ldots, y_s)$ be an ideal of $A$ (that contains a power of $p$). Let $g = p^m g_0 \in A$ where $p \nmid g_0$, and consider the extension $A \to A_{\infty} \to A_{\infty}[\varpi^b/g]$ for every positive integer $b$. Suppose $z \in IA_{\infty}(\varpi^b/g) \cap A_{\infty}$. If $p(cp^s + m) < b$, then for every $g^{1/p^s}$ that is part of a compatible system of $p$-power roots of $g$ in $A_{\infty}$, we have $\varpi^{1/p^t} g^{1/p^t} z \in IA_{\infty}$.

**Proof.** We fixed a compatible system of $p$-power roots of $g$, call it $\{g^{1/p^s}\}_{s=1}^\infty$ that contains the particular $g^{1/p^s}$. Using the almost explicit description of $A_{\infty}(\varpi^b/g)$ [Sch12 Lemma 6.4], we have

$$\varpi^{1/p^t} z \in IA_{\infty}(\varpi^b/g)^{1/p^\infty}$$

for some $t > a$. Because $\varpi^b = u^p$, the image of $\varpi^{1/p^t} z$ inside $A_{\infty}[1/\varpi^{1/p^\infty}] / \varpi^b = A_{\infty}[1/\varpi^{1/p^\infty}] / \varpi^b$ is contained in the ideal $(y_1, \ldots, y_s)$. Therefore we can write

$$\varpi^{1/p^t} z = \varpi^b f_0 + y_1 f_1 + \cdots + y_s f_s$$

(5)
where \( f_0, f_1, \ldots, f_s \in A_\infty[(\frac{g^k}{g})^{1/p^\infty}] \). Since this is a finite sum, there exist integers \( k \) and \( h \) such that \( f_0, f_1, \ldots, f_s \) are elements in \( A_\infty[(\frac{g^k}{g})^{1/p^k}] \) of degree in \( (\frac{g^k}{g})^{1/p^k} \) bounded by \( p^k h \).

Next we claim that multiplying by \( g_0^h \) in \([5]\) will clear all the denominators of the \( f_i \), this is not a problem if our fixed compatible system of \( p \)-power roots of \( g \) has the form \( p^{m/p^e} g_0^{1/p^e} \) for a certain compatible system of \( p \)-power roots of \( g_0 \).

In general, we will show this is true up to a choice of unit in \( A_\infty \). We know that
\[
(g^{1/p^e})^{p^e} = g = p^m g_0 = \varpi^m u^m g_0.
\]
Thus we have
\[
\left( \frac{g^{1/p^e}}{(\varpi^{1/p^e})^{p^e}} \right)^{p^e} = u^m g_0 \in A_\infty.
\]
Since \( A_\infty \) is integrally closed in \( A_\infty[1/\varpi] \), we know that \( \left( \frac{g^{1/p^e}}{(\varpi^{1/p^e})^{p^e}} \right)^{p^e} \in A_\infty \) and thus \( g^{1/p^e} = (\varpi^{1/p^e})^{p^m} v_e \) for some \( v_e \in A_\infty \). A direct computation shows \( v_e^{p^e} = u^m g_0 \).

Now we abuse notation and write \( g_0^{1/p^e} \) for \( v_e \) (so \( g_0^{1/p^e} \) is a \( p^e \)-th root of \( g_0 \) up to multiplication by \( u^m \)). One checks that after multiplying by \( g_0^h \) to \([5]\) we get:
\[
\varpi^{1/p^e} g_0^h z = (g_0^{h - (1/p^e)}, \varpi^{(b - pm)/p^e}) \cdot (\varpi^{pc}, y_1, \ldots, y_s) A_\infty.
\]
From this we know:
\[
\varpi^{1/p^e} g_0^h z = g_0^{h - (1/p^e)}(\varpi^{pc} h_0 + y_1 h_1 + \cdots + y_s h_s) \text{ in } A_\infty/\varpi^{(b - pm)/p^e},
\]
where \( h_0, h_1, \ldots, h_s \in A_\infty \). Rewriting this we have
\[
g_0^{h - (1/p^e)}(\varpi^{1/p^e} g_0^{1/p^e} z - \varpi^{pc} h_0 - y_1 h_1 - \cdots - y_s h_s) = 0 \text{ in } A_\infty/\varpi^{(b - pm)/p^e}.
\]
Since \( p \nmid g_0 \), \( g_0 \) is a nonzerodivisor on \( A/p \). This implies \( g_0^{h - (1/p^e)} \) (which is really \( v_0^{(p^e - 1)} \)) is an almost nonzerodivisor on \( A_\infty/\varpi^{(b - pm)/p^e} \) since \( A \to A_\infty \) is almost flat by \textbf{Lemma 2.2}. Hence \( \varpi^{1/p^e} g_0^{1/p^e} z - \varpi^{pc} h_0 - y_1 h_1 - \cdots - y_s h_s \) is killed by \( (\varpi^{1/p^e}) \).

In particular, since \( t > a \), we know
\[
\varpi^{1/p^e} g_0^{1/p^e} z \in (\varpi^{pc}, y_1, \ldots, y_s) \text{ in } A_\infty/\varpi^{(b - pm)/p^e}.
\]
Finally, note that \( b > p(cp^a + m) \) and thus \( \varpi^{(b - pm)/p^e} \) is a multiple of \( \varpi^{pc} = u^c p^e \), and \( g^{1/p^e} \) is a multiple of \( g_0^{1/p^e} = v_a \). Therefore we have
\[
\varpi^{1/p^e} g^{1/p^e} z \in (p^c, y_1, \ldots, y_s) A_\infty.
\]
This finishes the proof.

The main technical statement which allows us to pass to the enlargement of \( A_\infty \) is contained below.

\textbf{Lemma 5.4.} Let \( p, x_1, \ldots, x_{d-1} \) be a system of parameters of \( A \), for \( \epsilon \ll 1 \) we have
\[
0^{\otimes A_\infty \Delta^t + \epsilon}_{H^d_m(A)} = \left\{ \frac{z}{p^e x_1 \cdots x_{d-1}} \in H^d_m(A) \mid \varpi^{1/p^\infty} f^{1/p^e} z \in (p^c, x_1^c, \ldots, x_{d-1}^c) A_\infty(\varpi^k/g) \text{ for all } f \in a^{[(t+\epsilon)p^e]}, \text{ all } 0 \neq g \in a, \text{ and all integers } b > 0 \right\},
\]
where $f^{1/p^e} \in A_\infty$ runs over all possible $p^e$-th roots of $f$ that are part of a compatible system of $p$-power roots of $f$.

Proof. We first prove the containment “$\subseteq$”. So suppose that $p^{\frac{z}{p^e x_1^c \cdots x_{d-1}^c}} \in 0^{H_m^d(A)}$, then $p^{\frac{z}{p^e x_1^c \cdots x_{d-1}^c}} = 0$ in $H_m^d(A_\infty)$ for all $f \in a^{[(t+\varepsilon_0) p^e]}$ and all $f^{1/p^e}$ that are part of a compatible system of $p$-power roots. This means for every $l$,

$$\varpi^{1/p^l} f^{1/p^e} z(p x_1 \cdots x_{d-1})^w \in (p^{e+w}, x_1^{c+w}, \ldots, x_{d-1}^{c+w}) A_\infty$$

for some $w$ (which depends on $c$, $e$ and $l$). But since $p, x_1, \ldots, x_{d-1}$ is an almost regular sequence on $A_\infty$ by Lemma 2.2, this implies

$$\varpi^{1/p^l} f^{1/p^e} z \in (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty \subseteq (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty \langle \varpi^b \rangle$$

for all $f \in a^{[(t+\varepsilon_0) p^e]}$, all $0 \neq g \in a$, all $b > 0$ and all $l > 0$. Thus $\varpi^{1/p^0} f^{1/p^e} z \in (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty \subseteq (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty \langle \varpi^b \rangle$.

Next we will prove the other containment “$\supseteq$”. We fix $\varepsilon_0 \ll 1$ such that $0^{H_m^d(A)}$ computes $0^{\varphi A_\infty^{\frac{t}{p^e}}}$, and for every $\varepsilon_1 < \varepsilon_0$, $0^{\varphi A_\infty^{\frac{t}{p^e}}}$ also computes $0^{\varphi A_\infty^{\frac{t}{p^e}}}$. Next we pick $k > 0$ such that $0^{\varphi A_\infty^{\frac{t}{p^e}}} = 0^{\varphi A_\infty^{\frac{t}{p^e}}}$ by Lemma 5.2 and $\varpi^p k \geq t + \varepsilon_0$. We note that $k$ depends only on $a$, $t$, $\varepsilon_0$, and for every $k_0 / k$, then $0^{\varphi A_\infty^{\frac{t}{p^e}}}$ contains

$$\{ \frac{z}{p^e x_1^c \cdots x_{d-1}^c} \in H_m^d(A) \mid \varpi^{1/p^e} f^{1/p^e} z \in (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty \langle \varpi^b \rangle \text{ for all } f \in a^{[(t+\varepsilon_0/2) p^{2k}]} \}$$

for all $b > 0$. By Lemma 5.3, when $b > p(c p^{2k} + m)$, where $m$ depends on $g$ as in Lemma 5.3 we get

$$\varpi^{1/p^e} f^{1/p^e} z \in (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty \langle \varpi^b \rangle \cap A_\infty$$

for all $b > 0$. By Lemma 5.3, when $b > p(c p^{2k} + m)$, where $m$ depends on $g$ as in Lemma 5.3 we get

$$\varpi^{1/p^e} f^{1/p^e} z \in (p^t, x_1^c, \ldots, x_{d-1}^c) A_\infty$$

for all $f \in a^{[(t+\varepsilon_0/2) p^{2k}]}$, all $f^{1/p^e}$ part of a compatible system of $p$-power roots of $f$, all $0 \neq g \in a$, and all $g^{1/p^e}$ part of a compatible system of $p$-power roots of $f$.

Finally, for every $f \in a^{[(t+\varepsilon_0) p^k]}$, and every $f^{1/p^k}$ part of a compatible system of $p$-power roots, we can write $f^{1/p^k} = f^{1/p^k} \cdot f^{1/p^k}$, where $f^{1/p^k}$ is the $p^k$-root of $f^{1/p^k}$ in the compatible system. We claim that $f^{1/p^k} \in a^{[(t+\varepsilon_0/2) p^{2k}]}$, this is because

$$[(t+\varepsilon_0) p^r] (p^k - 1) \geq [(t+\varepsilon_0) p^{2k} - (t+\varepsilon_0) p^k] = [(t+\varepsilon_0/2) p^{2k} + (\varepsilon_0/2) p^k - (t+\varepsilon_0) p^k] \geq [(t+\varepsilon_0/2) p^{2k}]$$
by our choice of $k$. Now apply (7) to $g = \tilde{f} \in a$ (and we use $\tilde{f}^{1/p^k}$ as part of the compatible system of $p$-power roots of $g$) and $f = \tilde{f}^{p^k-1}$, we find that $\omega^{1/p^k} \tilde{f}^{1/p^k} z \in (p^c, x_1^c, \ldots, x_{d-1}^c)A_\infty$. Thus $z = \frac{z}{p^c x_1^c \cdots x_{d-1}^c}$ is killed by $\omega^{1/p^k} \tilde{f}^{1/p^k}$ in $H_m^d(A_\infty)$ for every $\tilde{f}^{1/p^k}$ part of a compatible system of $p$-power roots of $\tilde{f}$ with $\tilde{f} \in a^{[(t+\varepsilon)p^k]}$. Hence
\[
\frac{z}{p^c x_1^c \cdots x_{d-1}^c} \in 0_{[k,k]}^{\Diamond A_\infty a^{t+\varepsilon}} = 0_{H_m^d(A)}^{\Diamond A_\infty a^{t+\varepsilon}} = 0_{H_m^d(A)}^{\Diamond A_\infty a^{t+\varepsilon}}
\]
as desired. \hfill \box

**Lemma 5.5.** With the notations as in Lemma 5.4, we have
\[
0_{H_m^d(A)}^{\Diamond A_\infty a^{t+\varepsilon}} = \left\{ \frac{z}{p^c x_1^c \cdots x_{d-1}^c} \in H_m^d(A) \mid \omega^{1/p^k} \tilde{f}^{1/p^k} z = 0 \right\}
\]
for all $f \in a^{[(t+\varepsilon)p^k]}$, all $0 \neq g \in a$, and all $b > 0$.

Here we set $X^{b_g} = \text{Spa}(A_\infty(\frac{\omega^b}{g}), A_\infty(\frac{\omega^b}{g}))$ to be the perfectoid space associated to $(A_\infty(\frac{\omega^b}{g})[1/\omega], A_\infty(\frac{\omega^b}{g}))$.

**Proof.** This is true by Lemma 5.4 and utilizing the fact that $A_\infty(\frac{\omega^b}{g})$ is almost isomorphic to $\text{R}\Gamma(X^{b_g}, A_\infty(\frac{\omega^b}{g}))$ with respect to $(\omega^{1/p^k})$ by Scholze’s vanishing theorem of perfectoid spaces [Sch12, Proposition 6.14] (see also [GR04, 16.6.41]). \hfill \box

We are ready to prove our core lemma in this section.

**Lemma 5.6.** Suppose that $\pi : Y \to X = \text{Spec} \ A$ is the blowup of some ideal $J \subseteq A$ such that $Y$ is normal and that $a \subseteq \sqrt{J}$. Suppose that $E$ on $Y$ is a Weil divisor with $\pi(E) \subseteq V(J)$, and such that for each $f \in a^{[(t+\varepsilon)p^k]}$,
\[
\text{div}_Y(f) \geq p^k E.
\]
Then $\tau(a^t) \subseteq \Gamma(Y, \omega_{Y/J}(−E)) \subseteq A$.

**Proof.** Let $J = (z_1, \ldots, z_m)$, write $Y = \text{Proj} \ R \oplus J \cdot T \oplus J^2 \cdot T^2 \oplus \ldots$ and let $U_1, \ldots, U_m$ be an affine cover of $Y$ with $U_j = Y \setminus V(z_j T) \cong \text{Spec} \ A[\frac{a}{z_j}, \frac{b}{z_j}, \ldots, \frac{am}{z_j}]$. We fix an $e$ and an element $f^{1/p^k} \in A_\infty$ such that $f^{1/p^k}$ is part of a compatible system of $p$-power roots of $f$ with $f \in a^{[(t+\varepsilon)p^k]} \subseteq a^{[(t+\varepsilon)p^k]}$. Let $h$ be an element in the total quotient ring of $A_\infty$ such that $h \in f^{1/p^k} \mathcal{O}_Y(E)(U_j)$. Since $\text{div}_Y(f) \geq p^k E$, we have $h^{p^k} \in \mathcal{O}_Y(U_j)$. For any such fixed $h$, we observe the following:

**Claim 5.7.** There exists $h' \in (z_1, \ldots, z_m)A_\infty$ such that, if $Y' = \text{Spec} \ A_\infty$, then we have $h \in \mathcal{O}_{Y'_j}(U_j)$.

**Proof of Claim.** Because the image of $E$ lies in $V(J)$, we have
\[
h \in f^{1/p^k} \cdot \mathcal{O}_Y(E)(U_j) \subseteq f^{1/p^k} \cdot \mathcal{O}_Y(U_j)[z_j^{-1}] = f^{1/p^k} \cdot A[z_j^{-1}],
\]
and so we can write \( h = \frac{f^{1/p^r} w}{z_j^{p^r}} \) for some integer \( l \) and some \( w \in A \). Since
\[
\frac{f w^{p^r}}{z_j^{p^r+1}} = h^{p^r} \in \mathcal{O}_Y(U_j),
\]
we know that there exists some \( d \gg 0 \) such that \( f w^{p^r} z_j^{p^d-p^{r+l}} \in (z_1, \ldots, z_m)^{p^d} \). Fixing a compatible system of \( p \)-power roots of \( w \) and \( z_j \) in \( \mathbf{A}_\infty \), we note that
\[
(f^{1/p^r} w^{1/p^d-e} z_j^{(p^d-p^{r+l})/p^d})^{p^d} = f w^{p^r} z_j^{p^d-p^{r+l}} \in (z_1, \ldots, z_m)^{p^d}.
\]
Thus \( f^{1/p^r} w^{1/p^d-e} z_j^{(p^d-p^{r+l})/p^d} \in (z_1, \ldots, z_m)\mathbf{A}_\infty \). We set \( h' = f^{1/p^r} w^{1/p^d-e} z_j^{(p^d-p^{r+l})/p^d} \), and let \( Y'_\infty \) be the blow up. We have
\[
h = \frac{f^{1/p^r} w}{z_j^{p^r}} = \frac{(f^{1/p^r} w^{1/p^d-e} z_j^{(p^d-p^{r+l})/p^d})^{p^d-e}}{z_j^{p^d-e}} \in \mathcal{O}_{Y'_\infty}(U_j).
\]
This finishes the proof of the Claim.

We next note that, since the module \( f^{1/p^r} \mathcal{O}_Y(E)(U_j) \) is finitely generated over \( \mathcal{O}_Y(U_j) \) for every \( j \), we can collect the generators for all \( 1 \leq j \leq m \) and we call them \( h_1, \ldots, h_k \). For each \( h_i \) we can construct \( h'_i \) as in the Claim. Let \( Y_\infty \) be the blow up of \( (z_1, \ldots, z_m, h'_1, \ldots, h'_k) \) of \( \operatorname{Spec} A_\infty \). Since each \( h'_i \) is in the integral closure of \( (z_1, \ldots, z_m)\mathbf{A}_\infty \), the inverse image of the \( \{U_j\} \) forms an affine cover of \( Y_\infty \) by Lemma A.3 we have a factorization \( Y_\infty \overset{\rho}{\rightarrow} Y \rightarrow X \) with \( \rho \) affine, and for each \( j \) we have a natural map \( \mathcal{O}_Y(U_j) \rightarrow \mathcal{O}_{Y_\infty}(\rho^{-1} U_j) \).

By Claim 5.7 we know that
\[
f^{1/p^r} \mathcal{O}_Y(E)(U_j) \subseteq \mathcal{O}_{Y_\infty}(\rho^{-1} U_j)
\]
for every \( 1 \leq j \leq m \). In other words, the natural map \( \mathcal{O}_Y \rightarrow \rho_* \mathcal{O}_{Y_\infty} \overset{f^{1/p^r}}{\longrightarrow} \rho_* \mathcal{O}_{Y_\infty} \) factors through \( \mathcal{O}_Y(E) \).

Next we let \( Y^b_\infty \) be the blowup of \( (z_1, \ldots, z_m, h'_1, \ldots, h'_k) A_\infty(\overline{\omega}^b_g) \) of \( \operatorname{Spec} A_\infty(\overline{\omega}^b_g) \), where \( 0 \neq g \in \mathfrak{a} \). So we have a commutative diagram:
\[
\begin{array}{ccc}
Y^b_\infty & \rightarrow & \underbrace{Y_\infty} & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} A_\infty(\overline{\omega}^b_g) & \rightarrow & \operatorname{Spec} A_\infty & \rightarrow & \operatorname{Spec} A.
\end{array}
\]
Now, for each \( 0 \neq g \in \mathfrak{a} \subseteq \sqrt{J} \), for some \( c > 0 \) we have \( g^c \in J \) and so \( \overline{\omega}^{cb} \) is contained inside \( J \cdot A_\infty(\overline{\omega}^b_g) \subseteq (z_1, \ldots, z_m, h'_1, \ldots, h'_k) A_\infty(\overline{\omega}^b_g) \). Therefore \( Y^b_\infty \) is an admissible blowup of \( \operatorname{Spec} A_\infty(\overline{\omega}^b_g) \) for every \( 0 \neq g \in \mathfrak{a} \) and hence we have a factorization
\[
(X^b_\infty, \mathcal{O}_{X^b_\infty}^+ = A_\infty(\overline{\omega}^b_g)) \rightarrow Y^b_\infty \rightarrow \operatorname{Spec} A_\infty(\overline{\omega}^b_g)
\]
where again $X^{b,g} = \text{Spa}(A_\infty(\frac{\alpha^b}{g})[1/\overline{\nu}], A_\infty(\frac{\alpha^b}{g}))$ is defined to be the perfectoid space associated to $(A_\infty(\frac{\alpha^b}{g})[1/\overline{\nu}], A_\infty(\frac{\alpha^b}{g}))$, see [Bha17, Chapter 8] and also [Bha16, Proof of Proposition 6.2].

The above discussion shows that for every $f^{1/p^e}$ part of a compatible system of $p$-power roots of $f$ with $f \in a^{(t+1)p^e}$, every $0 \neq g \in a$ and every positive integer $b$, we have the following commutative diagram:

$$
\begin{array}{c}
\xymatrix{
A \ar[r]^{f^{1/p^e}} & A_\infty \ar[r]^{f^{1/p^e}} & A_\infty \ar[r]^{f^{1/p^e}} & A_\infty \ar[r]^{f^{1/p^e}} & A_\infty \\
\text{R} \Gamma(Y, \mathcal{O}_Y) \ar[r] & \text{R} \Gamma(Y_\infty, \mathcal{O}_{Y_\infty}) \ar[r] & \text{R} \Gamma(Y_\infty, \mathcal{O}_{Y_\infty}) \ar[r] & \text{R} \Gamma(Y_\infty, \mathcal{O}_{Y_\infty}) \ar[r] & \text{R} \Gamma(X^{b,g}, A_\infty(\frac{\alpha^b}{g}))
}
\end{array}
$$

The key point is that we have the dotted line in the above diagram, because we proved that the map $\mathcal{O}_Y \rightarrow \mathcal{O}_{Y_\infty}$ factors through $\mathcal{O}_Y(E)$ (up to pushforward by affine morphisms that we omit from the notation).

By [Lemma 5.3] $\frac{z^{e}}{p^e x_1 \cdots x_{d-1}} \in 0^{\bigotimes}_{H^d_m(A)}$ if and only if the image of $z$ is killed by $(\frac{\alpha^b}{g})$ under the natural map induced from the top left to the right bottom of the above diagram

$$
\begin{aligned}
\xymatrix{
A^{(p^e, x_1^{1}, \ldots, x_{d-1}^{1})} \ar[r] & h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{1}, \ldots, x_{d-1}^{1})} \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{aligned}
$$

for every $f^{1/p^e}$, every $0 \neq g \in a$ and every $b > 0$. But this is clearly the case if $z$ has trivial image in $h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{aligned}\right)$.

Thus we have

$$
\bigcup_c \ker\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{aligned}\right)
$$

is contained in $0^{\bigotimes}_{H^d_m(A)}$. But it is easy to see that the above union is precisely

$$
\ker\left(H^d_m(A) \rightarrow h^d\text{R} \Gamma_m(Y, \mathcal{O}_Y(E))\right)
$$

since

$$
\lim_{c} h^0\left(\left(\begin{array}{c}
A^{(p^e, x_1^{c}, \ldots, x_{d-1}^{c})} \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{array}\right)_c \ar[r] & \text{R} \Gamma(Y, \mathcal{O}_Y(E))
\end{aligned}\right) \cong h^d\text{R} \Gamma_m(Y, \mathcal{O}_Y(E)).
$$

By local duality, $\text{Ann}_A \ker\left(H^d_m(A) \rightarrow h^d\text{R} \Gamma_m(Y, \mathcal{O}_Y(E))\right) = \Gamma(Y, \mathcal{O}_Y/X(-E)) \subseteq A$. Therefore we have

$$
\tau(a^t) = \tau^\ell(a^{t+\ell}) = \text{Ann}_A(0^{\bigotimes}_{H^d_m(A)}) \subseteq \Gamma(Y, \mathcal{O}_Y/X(-E))
$$

which proves the lemma.

\[\square\]

**Lemma 5.8.** If $I \subseteq A$ is a radical ideal such that each prime component has height $\leq h$ then we have

$$
\tau((I^{(l)})^{1/l}) \subseteq I
$$
for every $l$.

**Proof.** Let $\pi: Y \to \text{Spec } A$ be the normalization of the blowup of $I \subseteq A$. In particular, $\pi$ is the blowup of some $J = \overline{t^n}$. Since $A$ is regular, we let $D = \sum_i D_i$ denote the union of components of the inverse image of $V(I)$ which dominate components of $V(I)$. Apply Lemma 5.6 with $E := hD$ (the conditions are satisfied by the definition of symbolic power), it is enough to show that

$$\Gamma(Y, \omega_{Y/X}(-E)) \subseteq I.$$ 

Thus we must compute the exceptional divisor $G$ such that $\omega_{Y/X} = \mathcal{O}_Y(G)$. Since regular local rings are pseudo rational by [LT81], we see that

$$G = \sum a_i D_i + \text{other effective terms}$$

and compute the integers $a_i$. Since $D_i$ is the only exceptional divisor dominating a component $V(Q_i) \subseteq V(I)$, this can be done after localizing at $Q_i$ and so the statement reduces to computing the relative canonical divisor of the blowup of a regular local ring of dimension $h_i \leq h$ at its maximal ideal. At that point we see that $a_i = h_i - 1 \leq h - 1$ by Lemma A.4. It follows immediately that $\Gamma(Y, \omega_{Y/X}(-E))_{Q_i} \subseteq \Gamma(Y, \mathcal{O}_Y(-D))_{Q_i} = Q_i$ and so $\Gamma(Y, \omega_{Y/X}(-E)) \subseteq I$ as desired. $\square$

6. Relation with multiplier ideals

We have defined $\tau(a^t) = \tau(A, a^t) \subseteq A$ and have shown it satisfies at least some formal properties similar to those of the multiplier ideal [Laz04] (in this section we will always write $\tau(A, a^t)$ to clarify which ring we are working with). On the other hand, $A[1/p]$ is a ring of equal characteristic 0 and so there exists a log resolution of $(\text{Spec } A[1/p], a \cdot A[1/p])$ and so we can compute its multiplier ideal. Thus it is natural to compare $\tau(A, a^t) \cdot A[1/p]$ with $J(A[1/p], (a \cdot A[1/p])^t)$.

**Theorem 6.1.** Suppose that $(A, a^t)$ is a pair, then

$$\tau(A, a^t) \cdot A[1/p] \subseteq J(A[1/p], (a \cdot A[1/p])^t).$$

**Proof.** First let $J \subseteq A[1/p]$ be an ideal whose blowup produces a log resolution of $(A[1/p], (a \cdot A[1/p])^t)$ [Tem08]. Because a log resolution principalizes $a$, the blowup of $a \cdot J$ also produces the same log resolution of $(A[1/p], (a \cdot A[1/p])^t)$. Since we may choose this log resolution to be an isomorphism outside of $a \cdot A[1/p]$ (since $A[1/p]$ is regular), we may assume that $a \cdot A[1/p] \subseteq \sqrt{J}$. Consider $J' = J \cap A$ and notice that $a \subseteq (a \cdot A[1/p]) \cap A \subseteq \sqrt{J} \cap A$. Now we claim that $\sqrt{J} \cap A \subseteq \sqrt{J'}$. Indeed, if $x \in \sqrt{J} \cap A$, then $x^n \in J$ and also $x^n \in A$, so $x^n \in J'$ and thus $x \in \sqrt{J'}$. Putting this together $a \subseteq \sqrt{J'}$. Let $\pi: Y \to X = \text{Spec } A$ be the normalized blowup of $a \cdot J$ (the blowup of $(a \cdot J)^n$ for some $n > 0$). Write $a \cdot \mathcal{O}_Y = \mathcal{O}_Y(-G)$ and let $E = [tG]$. Finally write $U = \text{Spec } A[1/p] \subseteq \text{Spec } A$ and $V = \pi^{-1}(U)$.

**Claim 6.2.** $\pi|_V: V \to U$ is the blowup of $J$ and hence the log resolution we started with.
Proof. Note that the blowup of \( J \) already principalizes \( a \cdot A[1/p] \) and so the blowup of \( a \cdot J \) is the same as the blowup \( J \). That blowup on the other hand is already normal and the claim is proved.

We now see that the hypotheses of Lemma 5.6 are satisfied and so \( \tau(A, a^t) \subseteq \Gamma(Y, \omega_{Y/X}(-\lfloor tG \rfloor)) \). But now by definition, \( \Gamma(V, \omega_{Y/X}(-\lfloor tG \rfloor)) = \mathcal{J}(A[1/p], (a \cdot A[1/p])^t) \) and the result follows.

We expect that the other containment should hold as well, namely:

**Conjecture 6.3.** \( \tau(A, a^t) \cdot A[1/p] = \mathcal{J}(A[1/p], (a \cdot A[1/p])^t) \)

We do not know how to prove it unfortunately, and it is certainly related to the question of localizing \( \tau \) which we also do not know how to handle.

Alternatively, in mixed characteristic one can define the multiplier ideal \( \mathcal{J}(A, a^t) \) valuatively. This is equivalent to defining

\[
\mathcal{J}(A, a^t) = \bigcap_{Y \to \text{Spec } A} \Gamma(Y, \mathcal{O}_Y(K_{Y/X} - [tG_Y]))
\]

where \( Y \to \text{Spec } A \) runs over all proper birational maps with \( Y \) normal and such that \( a \cdot \mathcal{O}_Y = \mathcal{O}_Y(-G_Y) \). Note it is also not clear whether this definition commutes with localization (since the intersection is infinite). With this definition, we have the following.

**Theorem 6.4.** Suppose \( (A, a^t) \) is a pair, then

\[
\tau(A, a^t) \subseteq \mathcal{J}(A, a^t).
\]

Proof. The hypotheses of Lemma 5.6 are satisfied for any such \( Y \to X \) with respect to \( [tG_Y] \). The result follows.

7. **Asymptotic Perfectoid Multiplier/Test Ideals**

We let \( \{a_n\}_{n=1}^{\infty} \) be a graded sequence of ideals. By analogy with for instance [ELS01], it would be natural to define the \( n \)-th asymptotic perfectoid multiplier/test ideal of the graded sequence \( \{a_n\} \) as the ideal

\[
\tau_{\infty}(a_n) = \sum \tau(a_{i_n}^{\frac{1}{n}}).
\]

However, since we don’t know the subadditivity theorem for \( \tau(a^t) \), this version of asymptotic test ideal will not give us the desired result on symbolic powers of ideals. Instead we have to build the definition using \( \tau([f]^t) \). Now since everything depends on the choice of the generating set \( \{f\} \) and a choice of compatible system of \( p \)-power roots of \( \{f\} \), we must proceed carefully.

Let \( \{a_n\}_{n=1}^{\infty} \) be a graded sequence of ideals of \( A \), i.e., \( a_na_m \subseteq a_{m+n} \) for all \( m, n \). We will be mostly interested in the situation that \( a_n = I^{(n)} \), the \( n \)-th symbolic power of a radical ideal \( I \subseteq A \). We define a generating set of each \( a_n \) inductively as follows:

First, we let \( \{f_{(1)}\} = \{f_{(1),1}, \ldots, f_{(1),n_1}\} \) be a fixed generating set of \( a_1 \), and we also fix a compatible system of \( p \)-power roots for each \( f_{(1),i} \) (that we will use to define
\[\tau([f_{(1)}^\tau]) \]. Now, suppose a generating set \( \{f_{(s)}\} \) of \( \mathfrak{a}_s \) and a compatible system of \( p \)-power roots of \( f_{(s)}^s \) have been chosen for all \( s < m \). We let
\[ \{f_{(m)}\} = \{f_{(m),1}, \ldots, f_{(m),n_m}\} \]
be a generating set of \( \mathfrak{a}_m \) satisfying the condition that it contains all possible \( f_{(s),i} f_{(t),j} \), where \( s + t = m \) and \( f_{(s),i}, f_{(t),j} \) are part of the chosen generating set of \( \mathfrak{a}_s \) and \( \mathfrak{a}_t \). Moreover, we fix a compatible system of \( p \)-power roots for each \( f_{(m),k} \), such that, if \( f_{(m),k} = f_{(s),i} f_{(t),j} \), then we use the product of the compatible system of \( p \)-power roots of \( f_{(s),i} \) and \( f_{(t),j} \), i.e., we let
\[ f_{(m),k}^{1/p^e} = f_{(s),i}^{1/p^e} f_{(t),j}^{1/p^e}. \]
We note that, it might happen that \( f_{(s),i} f_{(t),j} = f_{(m),k} = f_{(s'),i'} f_{(t'),j'} \) for \( s + t = m = s' + t' \), but the product of the chosen compatible system of \( p \)-power roots of \( f_{(s),i} \) and \( f_{(t),j} \) is not the same as the product of the chosen compatible system of \( p \)-power roots of \( f_{(s'),i'} \) and \( f_{(t'),j'} \). In this case, we simply allow \( f_{(m),k} \) to appear multiple times in the generating set, but we use different compatible system of \( p \)-power roots, one coming from \( f_{(s),i} f_{(t),j} \) and the other coming from \( f_{(s'),i'} f_{(t'),j'} \).

We have defined a generating set \( \{f_{(m)}\} \) for each \( \mathfrak{a}_m \) in the graded sequence, as well as a compatible system of \( p \)-power roots for each element \( f_{(m),i} \) appearing in the generating set. Now we give our definition of asymptotic perfectoid multiplier/test ideal.

**Definition 7.1** (Asymptotic perfectoid multiplier/test ideals). We define
\[ \tau_\infty([f_{(n)}]) = \sum_{l=1}^{\infty} \tau([f_{(ln)}^{1/l}]). \]
We note that by Proposition 3.9
\[ \tau([f_{(ln)}^{1/l}]) = \tau([f_{(mln)}^{1/ml}]) \]
where as in the notation of Proposition 3.9, \( f_{(ln)}^{*m} \) denotes the set of all degree \( m \) monomials in \( \{f_{(ln)}\} = \{f_{(ln),1}, \ldots, f_{(ln),n_{ln}}\} \). More importantly, by our choice of the generating set \( \{f_{(mln)}\} \) (and the way we fix the compatible system of \( p \)-power roots of elements in the generating set), we have
\[ \tau([f_{(mln)}^{1/ml}]) \subseteq \tau([f_{(mln)}^{1/ml}]) \]
by Proposition 3.4. Therefore we have
\[ \tau_\infty([f_{(n)}]) = \tau([f_{(ln)}^{1/l}]) \text{ for all sufficiently large and divisible } l. \]

**Proposition 7.2.** We have \( \tau_\infty([f_{(mn)}]) \subseteq \tau_\infty([f_{(n)}])^m \) for all \( n, m \in \mathbb{N} \).
we may assume that $A$ has mixed characteristic, then we are done by Theorem 7.3. This completes the proof. □

**Theorem 7.3.** Let $I \subseteq A$ be a radical ideal such that each prime component has height $\leq h$. Then we have $I^{(hn)} \subseteq I^n$ for all $n \in \mathbb{N}$.

**Proof.** Let $\{a_n\} = \{I^{(n)}\}$ be the graded sequence of ideals. We pick a generating set $\{f_{(h)}\}$ for each $a_n$ in this graded sequence as well as a compatible system of $p$-power roots for each element $f_{(n),i}$ appearing in the generating set as in the discussion before Definition 7.1, and we form the asymptotic perfectoid multiplier/test ideal $\tau_{\infty}(I^{(n)})$ as in Definition 7.1.

Since $\{f_{(hn)}\}$ is a generating set of $a_{hn} = I^{(hn)}$, by Proposition 3.12 we have

$$I^{(hn)} = a_{hn} \subseteq \tau(I^{(hn)}) \subseteq \tau_{\infty}(I^{(hn)})$$

where the last containment follows from Definition 7.1. But by Proposition 7.2 we know that

$$I^{(hn)} \subseteq \tau_{\infty}(I^{(h)}) \subseteq \tau_{\infty}(I^{(hn)})^n$$

for all $n$. Therefore we are done if we can show $\tau_{\infty}(I^{(h)}) \subseteq I$. However,

$$\tau_{\infty}(I^{(h)}) = \tau(I^{(h)})^{1/l}$$

for some sufficiently large and divisible $l$, and since $\{f_{(lh)}\}$ is a generating set for $a_{lh} = I^{(lh)}$, we know from (1) and Lemma 5.8 that

$$\tau(I^{(h)})^{1/l} \subseteq \tau(I^{(lh)})^{1/l} \subseteq I$$

This finishes the proof. □

**Theorem 7.4.** Let $R$ be a regular excellent Noetherian domain and let $I \subseteq R$ be a radical ideal such that each minimal prime of $I$ has height $\leq h$. Then for every integer $n \geq 0$,

$$I^{(hn)} \subseteq I^n.$$

**Proof.** Since the formation of symbolic powers commutes with localization, it is enough to prove $I^{(hn)} \subseteq I^n$ after localizing at each prime ideal of $R$ and so we may assume $R$ is local. Quite obviously $I^{(hn)} \hat{R} \subseteq (I\hat{R})^{(hn)}$ and $I\hat{R}$ is still a radical ideal with each minimal prime has height $\leq h$ because $\hat{R}$ is excellent. Therefore if we can show $(I\hat{R})^{(hn)} \subseteq (I\hat{R})^n = I^n \hat{R}$, then it follows that $I^{(hn)} \subseteq I^n \hat{R} \cap R = I^n$. Hence we may assume that $A := \hat{R}$ is a complete regular local ring. In the case that $A$ is of equal characteristic, the result is already known by [HH02], also see [ELS01]. If $A$ has mixed characteristic, then we are done by Theorem 7.3. This completes the proof. □
8. An example

We can compute this perfectoid multiplier/test ideal in a simple case.

Example 8.1 (SNC pair). Consider $A = W(k)[x_1, \ldots, x_{d-1}]$ for $k$ some perfect field of characteristic $p > 0$ and let $f = p^{a_0} x_1^{a_1} x_2^{a_2} \cdots x_{d-1}^{a_{d-1}}$ for some integers $a_i$. We use the construction of $A_\infty$ (and $A_{\infty,0}$) as introduced in Section 2. In particular for our compatible system of $p$-power roots of $f$, we fix the ones in $A_{\infty,0}$.

We claim

$$\tau(A, [f]^t) = \langle p^{a_0 t} x_1^{a_1 t} \cdots x_{d-1}^{a_{d-1} t} \rangle.$$  

Since in the definition of $\tau([f]^t)$, we are building the $+\epsilon$ variant of the perfectoid multiplier/test ideal, we may work with a fixed $t + \epsilon = b/p^e$. Consider $A' = A[p^{1/p^e}, x_1^{1/p^e}, \ldots, x_{d-1}^{1/p^e}]$ and observe it is also regular and contains $f^{b/p^e}$.

Since $A' \subseteq A_{\infty,0} = A'_{\infty,0}$, we have a factorization

$$A \to A' \to A_\infty.$$  

Also note that we can factor the map $A \xrightarrow{f^{b/p^e}} A_\infty$ as

$$A \to A' \xrightarrow{f^{b/p^e}} A' \to A_\infty.$$  

While it is not necessarily the case that $A'_{\infty} = A_{\infty}, A_{\infty}$ does have arbitrary $p$-power roots of all the terms we will work with (in fact, so does $A_{\infty,0}$). It is then not difficult to see that

$$\ker (H^d_{m}(A') \xrightarrow{f^{b/p^e}} H^d_{m}(A_{\infty})) = 0 : H^d_{m}(A') f^{b/p^e}$$

via a slight modification of the proof of Corollary 3.13. It then follows from local duality that

$$\tau(A, [f]^t) = \tau(A, [f]^{t+\epsilon}) = \Phi(f^{t+\epsilon} A')$$

where $\Phi$ is the generator of $\Hom_{A}(A', A)$ as an $A'$-module (for example, it can be taken to be the map which sends the monomial basis $p^{a_0/p^e} x_1^{a_1/p^e} \cdots x_{d-1}^{a_{d-1}/p^e} \mapsto p^{(a_0 - p^e)} x_1^{a_1 - p^e} \cdots x_{d-1}^{a_{d-1} - p^e}$ if that term makes sense in $A$ and zero otherwise). But this image is clearly $\langle p^{a_0 t} x_1^{a_1 t} \cdots x_{d-1}^{a_{d-1} t} \rangle$ as desired (at this point, it is the same computation as the one for the test ideal).

9. Further questions

The first obvious open question is:

Question 9.1. Fix $\{f_1, \ldots, f_n\}$ a sequence of generators of an ideal $a \subseteq A$ and for each $f_i$ fix a compatible system of $p$-power roots of $f_i$ in $A_\infty$ (in order to define $\tau^i([f_1, \ldots, f_n]^t)$). Is any inclusion in (1)

$$\tau^i(a') \supseteq \tau^i([f_1, \ldots, f_n]^t) \supseteq \tau([f_1, \ldots, f_n]^t) \subseteq \tau(a') \subseteq \tau(a')$$

an equality?
Among all the definitions of $\tau^\sharp$ and $\tau$, the easiest for us to work with (that leads to the application in Section 7) seems to be $\tau([f_1, \ldots, f_n]^t)$. But $\tau([f_1, \ldots, f_n]^t)$ a priori depends on the collection $\{f_1, \ldots, f_n\}$ (and even more, it depends on the fixed choice of a compatible system of $p$-power roots of $f_i$ in $A_\infty$). Therefore it would be good to give a definition that is independent of the generators yet will still establish the results we obtained in Section 7.

Another very fundamental question left open in this paper is:

**Question 9.2.** If $\tau([f]^t)$ or $\tau(a^t)$ independent of the choice of $A_\infty$? Are their additional assumptions one could make on $A_\infty$-like objects that give us the same $\tau([f]^t)$ or $\tau(a^t)$ or perhaps better multiplier/test ideal like objects?

It is an important open question in positive characteristic whether or not the test ideal $\tau(R)$ formed, in an appropriate generalization of the analogy with our introduction, with respect to $R^+$ is the same as as the test ideal formed with respect to some variant of almost mathematics in $R^{1/p^n}$, where “almost” is taken with respect to $c^{1/p^n}$, a tower of $p$-power roots of a test element. These are known to coincide when $R$ is Gorenstein (or various generalizations of Gorenstein).

In characteristic zero, the analogous statement is that the multiplier ideal is independent of the resolution. A key difference is that in mixed or positive characteristic, we don’t know independence in as large a case as we might desire, but we do have an analog of a resolution which works for all choices of ideals.

We could also ask how our object behaves with respect to localization. Note that it is still an open question whether or not the formation of the classical characteristic $p > 0$ test ideal commutes with localization.

**Question 9.3.** If $Q \in \text{Spec} A$ is a prime containing $p$ and $a$, is it true that

$$\tau(A, a^t) \cdot \widehat{A}_Q = \tau(\widehat{A}_Q, (a\widehat{A}_Q)^t)?$$

$$\tau(A, [f]^t) \cdot \widehat{A}_Q = \tau(\widehat{A}_Q, [f]^t)?$$

It would also be quite reasonable to try to compute some examples.

**Question 9.4.** Can we compute $\tau(a^t)$ or $\tau([f]^t)$ for some other examples? For instance if $a = \langle f \rangle$ where $f = y^2 - x^3$ or $p^2 - x^3$? It is not hard to obtain bounds for such examples using blowup, via our Lemma 5.6.

One could define the following.

**Definition 9.5 (PPT).** The **Perfectoid pure threshold** of a pair $(A, a)$ (relative to a choice of $A_\infty$) is the supremum over all $t \geq 0$ such that $A = \tau(a^t)$.

It follows from Corollary 3.14 that the perfectoid pure threshold is well defined and is always greater than zero. It would then be natural to ask whether the perfectoid pure threshold is always a rational number, this is known but not trivial for an analogous notion in equal characteristic $p > 0$ [BMS08]. It follows from Example 8.1 that the perfectoid pure threshold of $(A, [p^{\alpha_1}x_1^{a_1} \cdots x_{d-1}^{a_{d-1}}])$ is $1/\max\{a_i\}$ just like for the log canonical or $F$-pure threshold.
It would be natural to write down a Skoda-type theorem or a restriction theorem in this setting. We believe we can do this for \( \tau([f]^n) \). It would also be natural to define such ideals for singular rings \( R \). We will address these issues in a future work.

### Appendix A. Blowups

In this appendix we briefly recall (and in some cases prove) facts about blowups of ideals. These are well known but we record them here for ease of the reader. Note, we are working with potentially non-Noetherian rings in most cases.

**Setting A.1.** Throughout this section, \( R \) will be a (potentially non-Noetherian) reduced ring and \( J \subseteq R \) will be a finitely generated ideal. We let \( X = \operatorname{Spec} R \) and let \( Y \to X \) be the blowup of \( J \) in \( X \). In particular, set \( S = R \oplus JT \oplus JT^2 \oplus \cdots \) where the \( T \) serve as dummy variables to help distinguish degree, and thus \( Y = \operatorname{Proj} S \).

**Lemma A.2.** If \( J = (z_1, \ldots, z_m) \), then the complements \( U_i \) of \( V(z_iT) \subseteq Y \) form an affine cover of \( Y \) with \( U = \operatorname{Spec} R[z_1/z_i, \ldots, z_m/z_i] \).

In the above \( R[z_1/z_i, \ldots, z_m/z_i] \) is viewed as the subring of elements of \( S[(z_iT)^{-1}] \) of the form \( gT^n/(z_iT)^n \) as in [Sta, Tag 00JM].

**Proof.** Note any homogeneous prime of \( S \) does not contain some \( z_i \) and so this follows from for instance [Sta, Tag 0804]. \( \square \)

**Lemma A.3.** Suppose \( f \in R \) is integral over \( J \). Define \( J' = J + (f) \) and let \( Y' \to X \) be the blowup of \( J' \). Then \( Y' \to X \) factors through \( Y \) and \( Y' \) is a partial normalization of \( Y \) generated locally by adding a single integral element to the rings defining the affine charts \( U_i \).

**Proof.** Write \( f^n + a_1 f^{n-1} + \cdots + a_n = 0 \) with \( a_i \in J' \). Now write \( J = (z_1, \ldots, z_m) \) and form the Rees algebra \( S \) as above. Let \( S' = R \oplus JT \oplus JT^2 \oplus \cdots \supseteq S \). We will first prove that the \( U'_i = Y' \setminus V(z_iT) \) form an open cover of \( Y' \) (in particular, we do not need \( V(fT) \)). Suppose that \( Q \subseteq S' \) is a homogeneous prime ideal containing all of the \( z_iT \) but not \( fT \). Obviously \( Q \) contains \( 0 = f^n T^n + a_1 f^{n-1} T^{n-1} + \cdots + a_n T^n \) also note that \( Q \) contains \( a_n T^n \) since \( a_n T^n \in (z_1, \ldots, z_m)^n T^n \). But then since \( Q \) does not contain \( fT \), \( Q \) must contain 

\[
f^{n-1} T^{n-1} + a_1 f^{n-2} T^{n-1} + \cdots + a_n T^{n-1}.
\]

But \( Q \) also contains \( a_{n-1} T^{n-1} \) as before and so continuing in this way, we eventually deduce that \( fT \in Q \), a contradiction. Thus we have shown that \( \{U'_i\} \) form an open cover of \( \operatorname{Proj} S' = Y' \).

On the other hand, each \( U'_i = \operatorname{Spec} R[z_1/z_i, \ldots, z_m/z_i, f/z_i] \) and \( y = f/z_i \) satisfies the monic polynomial equation 

\[
(f/z_i)^n + (a_1/z_i)(f/z_i)^{n-1} + \cdots + (a_n/z_i)^n
\]

where each \( a_j/z_i \in R[z_1/z_i, \ldots, z_m/z_i] \) by construction. The lemma follows. \( \square \)

Finally, we also recall a statement about blowups in regular local Noetherian rings.

\( ^9 \)This is all we need, so we do not complicate our situation further.
Lemma A.4. Suppose that \((R, \mathfrak{m}, k)\) is a regular local Noetherian ring of dimension \(d\) and that \(Y \to X = \text{Spec} \, R\) is the blowup of \(\mathfrak{m}\). Then \(Y\) is regular, has prime exceptional divisor \(E\) with \(\mathfrak{m}\mathcal{O}_Y = \mathcal{O}_Y(-E)\) and \(K_{Y/X} = K_Y - \pi^*K_X = (d-1)E\).

Proof. This is well known, but because we do not know of a reference where it is phrased in this language outside of the context of varieties over a field, we include a quick geometric proof. Equivalent commutative algebra statements can be found for example in [HV85, HSV87, TW89].

A direct computation shows that the exceptional divisor \(E \cong \mathbb{P}^{d-1}\) lives in the regular scheme \(Y\). The same computation also shows that \(\mathcal{O}_X(-E)|_E = \mathcal{O}_E(1)\). Because we know that \((K_Y + E)|_E = K_E\) and that \(\mathcal{O}_E(K_E) = \mathcal{O}_E(-d)\), if we write \(K_Y = nE\), then \((K_Y + E)|_E = (nE + E)|_E = K_E\) and so \(-(n+1) = -d\) and thus \(n = d - 1\) as claimed. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112
E-mail address: lquanma@math.utah.edu
E-mail address: schwede@math.utah.edu