D-Branes on Toric Calabi–Yau Varieties

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Abstract

We analyze B-type D-branes on noncompact toric Calabi–Yau spaces. A general program is presented to find a set of tilting line bundles that yields the associated quiver and its relations. In many cases, this set remains fixed as one moves between phases in the Kähler moduli space. This gives a particularly simple picture of how the derived category remains invariant across all phases. The combinatorial problems involving local cohomology used to determine the tilting set are also related to questions of Π-stability as one moves between phases. As a result, in some cases precisely those line bundles in the tilting set remain stable over the whole moduli space in some sense.
1 Introduction

Toric varieties form a wonderful playground in providing a large class of algebraic varieties in which difficult questions in algebraic geometry can be reduced to combinatorics. In the physics of string theory, toric geometry appears in the form of gauged linear $\sigma$-models with an abelian gauge group $[1, 2]$. While one can argue that toric geometry certainly does not represent truly generic algebraic varieties, and thus generic string theory vacua, one can still learn valuable lessons by understanding this “easy” case first.

A Calabi–Yau variety cannot be both compact and toric. Here we restrict attention to noncompact toric Calabi–Yau varieties. B-type topological D-branes on a Calabi–Yau variety are represented by the bounded derived category of coherent sheaves $D(X)$ (see [3] for a review and references). It is well-known that D-branes on noncompact Calabi–Yau spaces (and thus the derived category of noncompact Calabi–Yau spaces) are best understood in terms of quivers. This paper is yet another in the vast literature of this topic to address the interplay of D-branes, derived categories and quivers.

The basic tool of the analysis in this paper is to make use of “tilting line bundles” which are line bundles supported over the whole noncompact space. These are the analogues of tautological line bundles in the McKay correspondence [4]. These bundles have also played an important rôle in the recent work [5] where D-branes are analyzed directly in terms of the gauged linear sigma model. Tilting bundles are very similar to “exceptional collections” of bundles on a compact Fano variety which have been used in this context many times (see, for example, [6–9]). Tilting collections have also previously been used to analyze D-branes in some examples [10, 11]. The use of tilting bundles allows one to avoid the assumption that the noncompact Calabi–Yau is the total space of a bundle over some compact irreducible variety. This, in turn, leads to a picture of $D(X)$ that is not particularly tied to any phase.

The goal of this paper was to reduce the main questions of $D(X)$ and D-branes on a toric Calabi–Yau to purely combinatorial questions and thus solve them. We have not been completely successful in this regard as we are unable to solve the general combinatorial problems. However, given an analysis of many examples, a general picture which is very pretty appears to emerge.

The basic claim is that one can have a tilting set of line bundles, which describes a quiver and thus $D(X)$, which is globally defined over the Kähler moduli space (if certain cuts are made to avoid arbitrary monodromy). This immediately yields a very direct picture of why $D(X)$ is invariant between phases. This property, which we call “wholesomeness” is defined more carefully below and we demonstrate its validity for some classes and specific examples of toric Calabi–Yaus. It is tempting to conjecture that wholesomeness is true in all cases.

The use of tilting objects as described in this paper gives yet another way of deriving the quiver gauge theory, complete with superpotential, from toric data describing a singularity. Other methods include resolving orbifolds [12] and dimers [13]. The connection between dimers and the case where a tilting object can be derived from an exceptional collection was discussed in [14]. We believe the tilting object method described in this paper is the quickest and mathematically most direct way of computing the quiver from the toric data but this could be a subjective statement.
As well as describing $D(X)$, one would like to understand $\Pi$-stability over the Kähler moduli space. This turns out to be very closely related to the mathematics associated to finding the tilting set. Thus, we again arrive at combinatorial problems associated with toric geometry when addressing these questions.

In section 2 we review the basic ideas of tilting sheaves and how the derived category of coherent sheaves is written in terms of a quiver with relations. In section 3 we review the algebraic geometry and commutative algebra we require from toric geometry.

The main part of the paper is section 4 which analyzes how one might go about finding a tilting set of line bundles for a given Calabi–Yau. The notion of wholesomeness is introduced and various classes and examples are discussed. In section 5 the relationship to $\Pi$-stability is discussed and finally we present some concluding remarks.

## 2 Quivers and Tilting Sheaves

Let $X$ be a smooth Calabi–Yau variety over $\mathbb{C}$ which may be noncompact and let $D(X)$ denote the bounded derived category of coherent sheaves on $X$. As usual, if $a$ denotes a complex in $D(X)$ then $a[n]$ denotes the same complex shifted $n$ places to the left. A single coherent sheaf $\mathcal{F}$ is regarded as an object in $D(X)$ in terms of a complex which is zero in every position except position zero, where it is $\mathcal{F}$. Suppose we can find a tilting sheaf $M$ which is a coherent sheaf on $X$ such that

1. $M$ decomposes as a finite direct sum of simple sheaves
   \[ M = P_1 \oplus P_2 \oplus \ldots \oplus P_k. \]  
2. Each $P_i$ satisfies
   \[ \operatorname{Ext}^n(P_i, P_j) = 0 \quad \text{for all } n > 0 \text{ and all } i, j. \]  
3. The collection of $P_i$’s generates the whole of $D(X)$. In other words, the smallest triangulated full subcategory of $D(X)$ containing $\{P_1, \ldots, P_k\}$ is $D(X)$ itself.

Let $A = \operatorname{End}(M)$ be the endomorphism algebra of $M$. The product rule in $A$ is simply composition of maps of $M$ to itself. We may view elements of $A$ as matrices whose $(i, j)$th entry is an element of $\operatorname{Hom}(P_j, P_i)$. It is then clear that the product is not, in general, commutative. $M$ has the structure of a bimodule with a left action from $\mathcal{O}_X$, as it is a sheaf, and a right action by $A$. This leads to a well-known equivalence [15–17]
\[ D(X) \cong D(A-\text{mod}), \]  
where $D(A-\text{mod})$ is the bounded derived category of finitely generated left $A$-modules.\(^2\)

This equivalence is induced by an adjoint pair of functors
\[ \operatorname{Hom}(M, -) : D(X) \to D(A-\text{mod}) \]
\[ M \otimes_A - : D(A-\text{mod}) \to D(X). \]  

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\(^1\)We also impose the technical requirement that $A$ (defined below) has finite global dimension.

\(^2\)There are related statements such as the equivalences concerning bounded derived categories of sheaves with compactly supported cohomology. See, for example, [18, 19].
The noncommutative algebra $A$ can be written in terms of the path algebra of a quiver $Q$ with relations. We may associate a node to each summand $P_i$. Then $\text{Hom}(P_i, P_j)$ is generated, as a vector space, by paths from node $j$ to node $i$. That is, arrows represent indecomposable maps between the sheaves $P_i$. Note that $\text{End}(T)$ is the path algebra of $Q^{op}$, the quiver $Q$ with all arrows reversed. An $A$-module may be identified with a quiver representation as discussed at length in [8, 11, 20, 21] for example.

Under (4) the sheaf $P_i$ in $D(X)$ is mapped to $\text{Hom}(M, P_i)$. This has the interpretation of the space of all paths starting at node $i$. With a slight abuse of notation we will also use $P_i$ to refer to this representation of $Q$. Let $e_i$ denote the trivial path of length zero beginning and ending at node $i$. The representation of $Q$ given by $P_i$ may then be written $A e_i$.

**Example 1** The classic example is $X = \mathbb{P}^n$ due to Beilinson [22]. For example, if $X$ is $\mathbb{P}^2$, with homogeneous coordinates $[x_0, x_1, x_2]$, we may put $P_i = \mathcal{O}(i)$ for $i = 0, 1, 2$. This yields a quiver

\[
\begin{array}{ccc}
    v_0 & \overset{a_0}{\longrightarrow} & v_2 \\
    v_1 & \overset{b_0}{\longrightarrow} & v_2
\end{array}
\]

Both $a_i$ and $b_i$ correspond to multiplication by $x_i$. This yields relations $a_i b_j = a_j b_i$ for all $i, j$.

In the above example we have a directed quiver without loops and $A$ is finite-dimensional. In this case, the tilting set $\{P_0, P_1, \ldots\}$ form an exceptional collection. In this paper we will be more concerned with cases where the quiver has loops and thus $A$ is infinite-dimensional.

**Example 2** Now consider the total space of the line bundle with $c_1 = -3$ over $\mathbb{P}^2$. This is a noncompact Calabi–Yau threefold. Again we put $P_i = \mathcal{O}(i)$ for $i = 0, 1, 2$, but now these line bundles have noncompact support. The quiver looks like

\[
\begin{array}{ccc}
    v_1 & \overset{a_1}{\longrightarrow} & v_2 \\
    v_0 & \overset{a_0}{\longrightarrow} & v_2 \\
    v_0 & \overset{a_2}{\longrightarrow} & v_1
\end{array}
\]

The extra maps $c_i$ are given by multiplication by $px_i$ where $p$ is the coordinate in the fibre direction. This gives relations $a_i b_j = a_j b_i$, $b_i c_j = b_j c_i$, $c_i a_j = c_j a_i$ for all $i, j$.

Consider the center $Z(A)$ of the path algebra $A$ of (6). The rings $\text{Hom}(P_i, P_i)$ are isomorphic for all $i$ to some ring which we denote $R$. Elements of $Z(A)$ are then given by matrices of the form $\text{diag}(r, r, r)$ for $r \in R$ and thus $Z(A)$ is isomorphic to $R$. $R \cong \text{Hom}(P_0, P_0)$ is then generated as a ring by

\[
x_{ijk} = a_i b_j c_k, \quad \text{where } i \leq j \leq k.
\]
Now let $G = \mathbb{Z}_3$, generated by $g$, act on $(u_0, u_1, u_2)$ by
\[
g : (u_0, u_1, u_2) \mapsto (\omega u_0, \omega u_1, \omega u_2),
\]
where $\omega$ is a nontrivial cube root of unity. It is easy to see that $R$ is isomorphic to the $G$-invariant part of the polynomial ring $\mathbb{C}[u_0, u_1, u_2]$ by putting $x_{ijk} = u_1 u_2 u_3$. That is,
\[
\text{Spec } \mathbb{Z}(A) = \mathbb{C}^3/\mathbb{Z}_3.
\]
This is a typical example of a noncommutative resolution in the sense of [23–25]. The singular variety $\mathbb{C}^3/\mathbb{Z}_3$ has a crepant “resolution” by the noncommutative algebra $A$.

3 Toric Calabi–Yau

First we review a standard construction in toric geometry. Let $N$ be a lattice of rank $d$. Let $\mathcal{P}$ be a convex polytope in $N \otimes \mathbb{R}$ such that the vertices of the convex hull lie in $N$. Furthermore, we demand that $\mathcal{P}$ lies in a hyperplane of $N \otimes \mathbb{R}$ such that the coordinates of any point in $\mathcal{P}$ may be written $(1, \ldots)$. Let $\mathcal{A}$ denote the set of points $\mathcal{P} \cap N$ and let $n$ denote the number of elements of $\mathcal{A}$.

The coordinates of the points of $\mathcal{A}$ form a $d \times n$ matrix defining a map $A : \mathbb{Z}^\oplus n \rightarrow N$ which we assume is surjective. We form an exact sequence
\[
0 \rightarrow L \xrightarrow{A} \mathbb{Z}^\oplus n \xrightarrow{N} 0,
\]
where $L$ is the “lattice of relations” of rank $r = n - d$. Dual to this we write
\[
0 \rightarrow M \xrightarrow{\Phi} \mathbb{Z}^\oplus n \xrightarrow{D} 0,
\]
where $\Phi$ is the $r \times n$ matrix of “charges” of the points in $\mathcal{A}$. By our hyperplane condition, each row of $\Phi$ sum to zero.

Let
\[
S = \mathbb{C}[x_1, \ldots, x_n].
\]
The matrix $\Phi$ gives an $r$-fold multi-grading to this ring. In other words, we have a $(\mathbb{C}^*)^r$ torus action:
\[
x_i \mapsto \lambda_1^{\Phi_1} \lambda_2^{\Phi_2} \ldots \lambda_r^{\Phi_r} x_i,
\]
where $\lambda_j \in \mathbb{C}^*$. Let $R$ be the $(\mathbb{C}^*)^r$-invariant subalgebra of $S$. The algebra $S$ then decomposes into a sum of $R$-modules labeled by their $r$-fold grading:
\[
S = \bigoplus_{\alpha \in D} S_{\alpha},
\]
where $D \cong \mathbb{Z}^r$ from \[\text{(11)}\] and $R = S_0$. As usual we denote a shift in grading by parentheses, i.e., $S(\alpha)_\beta = S_{\alpha + \beta}.$
Let $X_0 = \text{Spec } R$. That is, $X_0$ is the toric variety associated to the fan consisting of the single cone over $\mathcal{P}$. $X_0$ is then a noncompact (typically) singular Calabi–Yau variety. We would like to find a non-commutative crepant resolution of $X_0$. This problem was solved completely in the last section of [24] for the case $r = 1$. We would like to examine the general case.

It is well-known in toric geometry that a (partial) crepant desingularization of $X_0$ is given by a simplicial decomposition of the point set $\mathcal{A}$. In order that this desingularization be Kähler we also impose that the simplicial decomposition be “regular” [26]. This simplicial decomposition may, or may not, include points in the interior of the convex hull of $\mathcal{A}$. We refer to a choice of simplicial decomposition as a “phase". $X_0$ corresponds to a phase itself if and only if the convex hull of $\mathcal{A}$ is a simplex. A phase corresponds to a complete resolution if each simplex has volume one (in the natural normalization by $(d-1)!$). Otherwise a phase has orbifold singularities.

To each phase we associate the “Cox ideal” defined in [27] as follows.

**Definition 1** Let $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ denote the set of simplices. If $\sigma$ is a simplex, we say $i \in \sigma$ if the $i$th element of $\mathcal{A}$ is a vertex of $\sigma$. Then

$$B_\Sigma = \left( \prod_{i \notin \sigma_1} x_i, \prod_{i \notin \sigma_2} x_i, \ldots \right).$$

(15)

Clearly $B_\Sigma$ is a square-free monomial ideal in $S$.

**Definition 2** Let $V(B_\Sigma)$ denote the subvariety of $\mathbb{C}^n$ given by $B_\Sigma$. Then

$$X_\Sigma = \mathbb{C}^n - V(B_\Sigma)/(\mathbb{C}^*)^r.$$  

(16)

Cox [27] shows that there is a correspondence between finitely-generated graded $S$-modules and coherent sheaves on a smooth $X_\Sigma$ which follows the usual correspondence between sheaves and projective varieties as in chapter II.5 of [28]. If $U$ is an $S$-module, we denote $\tilde{U}$ as the corresponding sheaf. $\tilde{U}$ is zero as a sheaf if and only if $U$ is killed by some power of $B_\Sigma$. This yields

**Proposition 1** Assume $X_\Sigma$ is a smooth toric variety. Then

$$D(X_\Sigma) = \frac{D(\text{gr} - S)}{T_\Sigma},$$

(17)

where $D(\text{gr} - S)$ is the bounded derived category of finitely-generated multigraded $S$-modules and $T_\Sigma$ is the full subcategory generated by modules killed by a power of $B_\Sigma$. This quotient of triangulated categories is as in [29].
If $\Sigma$ does not consist of simplices of volume one, $X_\Sigma$ will have orbifold singularities and proposition 1 will not hold. However, if we view the resulting $X_\Sigma$ as a smooth stack\footnote{There is also a notion of a “toric stack” where extra data is added to denote a lattice point in $N$ lying on each one-dimensional ray in the fan (see, for example, [30]). We are not using this technology here. There is also a related notion of boundary divisor that has been analyzed in the context of the derived category and toric geometry in [31].} then the proposition is valid (by an argument essentially given in [32]). So, the question we need to address is whether D-branes on an orbifold are described by the derived category of a variety or a stack. It has been argued in [33] that stacks are the correct language for D-branes. Indeed, one may view [5] as a linear $\sigma$-model demonstration of this idea as that paper shows that the D-branes are given by the quotient (17). So, from now on we will assume that the above proposition holds in any phase and we no longer need assume that $X_\Sigma$ is smooth.

4 $D(X)$ Generated by Line Bundles

4.1 Tilting Line Bundles

To a bounded derived category $D(X_\Sigma)$ we may associate the more crude “Grothendieck group”. This is simply the abelian group generated by all objects in $D(X)$ modulo relations generated by distinguished triangles. This is also the K-theory group of $X_\Sigma$ if $X_\Sigma$ is a smooth manifold and so measures “D-brane charge”. We denote the rank of the Grothendieck group by $T$. Clearly at least $T$ objects are needed to generate the derived category.

If $X_\Sigma$ is a crepant smooth resolution, can we find a module of the form

$$M = S(\alpha_1) \oplus S(\alpha_2) \oplus \ldots \oplus S(\alpha_T),$$

(18)

playing the rôle of (11) to provide a tilting sheaf? That is, can we find a tilting sheaf that is a sum of line bundles? This is closely related to the question of whether we can always find strong exceptional collections of line bundles for toric varieties as proposed by King [34]. Even though this is known not to be the case in general [35], it may well still be true in the “nef-Fano” case which corresponds to a fan over a convex set [36] which is the case at hand in this paper.

The first condition that we require is $\text{Ext}^k_{X_\Sigma}(\tilde{S}(\alpha_i), \tilde{S}(\alpha_j)) = 0$ for all $k > 0$ and all $i, j = 1, \ldots, T$. That is,

$$H^k(X_\Sigma, \tilde{S}(\alpha_j - \alpha_i)) = 0.$$  

(19)

The cohomology of line bundles on a toric variety is easily computed via local cohomology [37, 38]. See also [9] for an account in the physics literature. First define

$$H^i_*(\tilde{S}) = \bigoplus_{\delta \in D} H^i(\tilde{S}(\delta)).$$

(20)

Now $H^i_*(\tilde{S})$ has the structure of a graded $S$-module as can be seen as follows. The direct sum in (20) decomposes $H^i_*(\tilde{S})$ into its graded parts. Suppose $s \in S_\beta$. Then we have a degree
zero map \( S(\delta) \to S(\delta + \beta) \) given by multiplication by \( s \). Then, applying the corresponding functors, this extends to a map \( H^1(\delta(\delta)) \to H^1(\delta(\delta + \beta)) \).

If \( I \) is an ideal in \( S \) then we denote local cohomology by \( H^i_I \). For more information on local cohomology we refer to [39].

Then we have

**Proposition 2**

\[
0 \longrightarrow H^0_{B\Sigma}(S) \longrightarrow S \longrightarrow H^0(\tilde{S}) \longrightarrow H^1_{B\Sigma}(S) \longrightarrow 0, \tag{21}
\]

and

\[
H^k(\tilde{S}) \cong H^{k+1}_{B\Sigma}(S) \quad \text{for } k > 0. \tag{22}
\]

So, following [19] we want to find elements \( \delta \in D \) such that \( H^{k}_{B\Sigma}(S)_\delta = 0 \) for all \( k \geq 2 \). Actually we will impose a slightly stronger condition to include \( k = 0 \) and 1.\footnote{Actually \( H^0_{B\Sigma}(S) \) is always zero.}

**Definition 3** A vector \( \delta \in D \) is called "\( B\Sigma \)-acyclic" if the local cohomology groups \( H^k_{B\Sigma}(S)_\delta \) vanish for all \( k \).

Therefore a \( B\Sigma \)-acyclic vector \( \delta \) yields \( H^0(\tilde{S}(\delta)) = S_\delta \).

### 4.2 Computing Local Cohomology

Computing local cohomology is particularly easy when the ideal is a monomial ideal. We review the details of the construction of Mustață [40] as we will need them later in this paper.

\( S \) has an \( r \)-fold grading given by the matrix of charges \( \Phi \). It also has an \( n \)-fold "fine" grading where we simply assign \( x_i \) a grading of \((0, 0, \ldots, 0, 1, 0, \ldots, 0) \) where the 1 appears in the \( i \)th position. The matrix \( \Phi \) can then be viewed as a map from the lattice of fine grading to the \( D \)-lattice. We will use non-bold letters \( \alpha, \ldots \) for fine grading vectors.

Given a square-free monomial ideal \( B \), we denote the Alexander dual of \( B \) by \( B^\vee \). We refer to chapter one of [41] for a nice account of Alexander duality. Note that the Alexander dual of the Cox ideal of a toric variety is the Stanley–Reisner ideal \( I_\Sigma \). That is, \( I_\Sigma = B^\vee_\Sigma \) is generated by monomials of the form \( x_ix_jx_k \ldots \) where \( i, j, k \ldots \) are not the vertices of any simplex in the triangulation specified by \( \Sigma \).

Consider a minimal finely-graded free resolution of \( B^\vee \):\footnote{Actually \( H^0_{B\Sigma}(S) \) is always zero.}

\[
\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B^\vee \longrightarrow 0, \tag{23}
\]

where

\[
F_i = \bigoplus_{\alpha \in \mathbb{N}^n} S(-\alpha)^{\oplus b_i,\alpha}. \tag{24}
\]
The numbers $b_{i,\alpha}$ are known as graded Betti numbers and can also be written

$$b_{i,\alpha} = \dim \text{Tor}^i(B^\vee, C)_{\alpha}, \quad (25)$$

where $C$ is an $S$-module annihilated by any $x_i$. One may similarly define $D$-graded Betti numbers $b_{i,\alpha}$ for $\alpha \in D$.

Note that the $\alpha$'s giving rise to nonzero finely-graded Betti numbers are “binary” vectors in the sense that they are lists of 0’s and 1’s. Let $\Xi$ be a map from $\mathbb{Z}^{\oplus n}$ to $\{0, 1\}^n$ which replaces non-negative numbers by 0 and negative numbers by 1. One then has [40]

**Theorem 1** $H^i_B(S)_{\delta}$ is nonzero for some $i$ if and only if there is a nonzero Betti number $b_{k,\alpha}$ for some $k$ such that $\Xi(\delta) = \alpha$.

We therefore have an algorithm for finding valid vectors $\delta \in D$ such that the local cohomology groups $H^i_B(S)_{\delta}$ vanish for all $i$:

- For each nonzero Betti number $b_{k,\alpha}$, take the corresponding orthant of $\mathbb{Z}^{\oplus n}$ that maps via $\Xi$ to $\alpha$. Project this orthant to $D$ via $\Phi$ and remove the resulting vectors from consideration.

- The remaining vectors in $D$ satisfy the desired acyclic condition.

Once we have found the set of acyclic $\delta$ vectors we may then try to find a choice of $T$ vectors $\{\alpha_i\}$ such that $\alpha_i - \alpha_j$ is an acyclic vector for all $i, j$. Clearly given such of a choice of $\{\alpha_i\}$ one may find another valid set by shifting all the gradings by some fixed vector or by permuting the $\alpha$’s. We will refer to such a change in $\{\alpha_i\}$ as trivial.

### 4.3 Wholesomeness

Let $\{\alpha_i\}$ denote a set of $T$ vectors in $D$ such that all pairwise differences are acyclic.

**Definition 4** A given $X_\Sigma$ (or the associated point set $\mathcal{A}$) will be said to be “wholesome” if all the following conditions are met:

1. The number of acyclic $\delta$’s need not be finite but the number of acyclic $\delta$’s such that $-\delta$ is also acyclic is finite. It follows that the number of choices (up to trivial transformations) of $\{\alpha_i\}$ is finite.

2. $\{\alpha_i\}$ is maximal in the sense that no further vectors may be added such that all pairwise differences are acyclic.

3. There are no nontrivial relations (in the form of a distinguished triangle of complexes) between the $\tilde{S}(\alpha_i)$’s in $D(X_\Sigma)$.

4. The $\tilde{S}(\alpha_i)$’s generate $D(X_\Sigma)$ and so the sum of the $\tilde{S}(\alpha_i)$’s is a tilting sheaf.
5. \{α_i\} can be chosen to be identical in all phases. That is, it depends only on the choice of \mathcal{A} and not the triangulation Σ.

At first sight one might consider these conditions to be rather stringent, especially the last one. Surprisingly, however, all the examples we have considered in dimension three appear to be wholesome and it is fairly tempting to speculate that wholesomeness is guaranteed for any point set \mathcal{A} in this case. We will give a counterexample in dimension 5 later.

Note that our stronger condition that \(H^1_{B_2}(S)_δ\) vanish, in addition to the higher cohomologies, is necessary in many examples for wholesomeness to be true.

**Theorem 2** Wholesomeness condition 1 is always true.

To prove this it is useful to describe the choice of triangulations Σ in terms of the toric ideal \(I_{\mathcal{A}}\) introduced by Sturmfels [42]. Let \(v = (v_1, \ldots, v_n)\) be a vector in the kernel of \(A\) in (10). Let \(v = v_+ - v_-\) where \(v_+\) has only non-negative coordinates and let \(p_+\) be the subset of \(\{1, \ldots, n\}\) such that \(i \in p_+\) when \(v_i > 0\). Similarly let \(p_-\) be the subset for which \(v_i\) is negative. We then associate to \(v\) the binomial

\[x^{v_+} - x^{v_-} \in S.\] (26)

Here we have used the standard notation \(x^v = \prod_i x_i^{v_i}\). The ideal \(I_{\mathcal{A}}\) is then defined as the ideal in \(S\) generated by such binomials for all choices of vectors in the kernel of \(A\).

Now, given any term ordering \(<\) (see, for example, [43]) we may compute the initial ideal \(\text{in}_<(I_{\mathcal{A}})\). Sturmfels [42] then argued that the set of possible initial ideals obtained by varying \(<\) maps surjectively to regular triangulations \(\Sigma\) of \(\mathcal{A}\). This map is given simply by

\[\sqrt{\text{in}_<(I_{\mathcal{A}})} = I_\Sigma,\] (27)

where \(I_\Sigma\) is the Stanley–Reisner ideal of the triangulation.

Fix a term-ordering \(<\) and thus a triangulation \(\Sigma\). Let \(m\) be one of the monomial generators of \(I_\Sigma\). There is then a primitive binomial of the form (26) where the support of \(m\) equals \(p_+\). Let \(v = (v_1, \ldots, v_n)\) denote the associated vector in the kernel of \(A\). We know from the resolution (23) that we have a corresponding nonzero Betti number \(b_{0,\alpha}\). The location of the 1’s in \(\alpha\) is precisely \(p_+\) which, in turn, is precisely the location of the positive numbers in \(v\).

for \(v = v_+ - v_-\), we define \(N_v\) as the sum of the coordinates of \(v_+\), which is equal to negative the sum of the coordinates of \(v_-\) by our assumption that the rows of \(Φ\) sum to zero.

Since \(v\) corresponds to a vector in the kernel of \(A\), it is the image of a vector \(v_{m,Σ}\) in \(L\) from (10). It follows that the set to be excluded from consideration

\[H_{m,Σ} = \Phi(\Xi^{-1}(α)) \subset D,\] (28)

will satisfy \((δ, v_{m,Σ}) \leq -N_v\) for all \(δ \in H_{m,Σ}\), where \((,\) is the natural pairing between \(L\) and \(D\).

\(^5\)That is, the set of elements \(i \in \{1, \ldots, n\}\) such that \(x_i\) divides \(m\).
We may now choose other generators, \( m \), of \( I_\Sigma \) to remove further regions from consideration for acyclicity. These \( m \)'s produce vectors \( v_{m,\Sigma} \) that span all of \( D \otimes \mathbb{R} \). This latter statement follows from the fact that \( I_{sf} \) defines a variety of dimension \( d \) and that the deformation of \( I_{sf} \) to \( I_\Sigma \) is flat \([44]\) and thus not dimension-changing. Therefore, the space of allowed acyclic vectors in \( D \) does not contain a complete line passing through the origin. ■

4.4 \( r=1 \)

A particularly easy case is when \( r = 1 \) which was analyzed in \([24, 45]\) which we essentially follow. It was also studied in terms of the gauged linear sigma model in \([5]\).

**Theorem 3** *Wholesomeness is always true for \( r = 1 \).*

In this case the toric ideal \( I_{sf} \) has a single generator \( m_+ - m_- \). We therefore have two phases \( \Sigma_\pm \) given by an initial ideal \((m_+)\) or \((m_-)\). Suppose \( \Sigma_+ \) corresponds to a vector \( v = (v_1, \ldots, v_n) \) which generates the one-dimensional kernel of \( A \). We know the \( v_i \)'s sum to zero and so

\[
\sum_{i \in p_+} v_i = -\sum_{i \in p_-} v_i = N,
\]

for some positive integer \( N \).

It follows that the range of allowed elements of \( D \) for which we have nontrivial local cohomology is given by \( \delta \in D \) for which

\[
(\delta, v) < -N,
\]

where \( v \) generates \( L \). Since \( r = 1 \), the vector \( \delta \in D \) is specified by a single integer.

Obviously, therefore, the set of tilting objects can be chosen to be

\[
S, S(1), S(2), \ldots, S(N-1).
\]

The same result is true for \( \Sigma_- \). Thus property 5 is satisfied.

Now consider the Koszul resolution of \( S/B_{\Sigma+} \), where \( B_{\Sigma+} \) is the ideal \((x_{i_1}, x_{i_2}, \ldots)\) with \( p_+ = \{i_1, i_2, \ldots\} \).

\[
S(-N) \longrightarrow \bigoplus_{i \in p_+} S(-N + \Phi_{1i}) \longrightarrow \ldots \longrightarrow \bigoplus_{i \in p_+} S(-\Phi_{1i}) \longrightarrow S \longrightarrow \frac{S}{B_{\Sigma+}}.
\]

In the quotient triangulated category \( D(X_{\Sigma+}) \) in \([17]\), the object \( S/B_{\Sigma+} \) is obviously in \( T_{\Sigma+} \). It follows that we have two isomorphisms in \( D(X_{\Sigma+}) \):

\[
\tilde{S}(-N) \cong \left( \bigoplus_{i \in p_+} \tilde{S}(-N + \Phi_{1i}) \longrightarrow \ldots \longrightarrow \bigoplus_{i \in p_+} \tilde{S}(-\Phi_{1i}) \longrightarrow \tilde{S} \right)
\]

\[
\tilde{S} \cong \left( \tilde{S}(-N) \longrightarrow \bigoplus_{i \in p_+} \tilde{S}(-N + \Phi_{1i}) \longrightarrow \ldots \longrightarrow \bigoplus_{i \in p_+} \tilde{S}(-\Phi_{1i}) \right)
\]
In the above, the dotted line represents position 0 in the complex. By using these isomorphisms and their grade-shifted counterparts, any sheaf of the form \( \tilde{S}(k) \) where \( k < 0 \) or \( k \geq N \) can be rewritten in terms of bounded complexes using the basic set \( \tilde{S}, \tilde{S}(1), \tilde{S}(2), \ldots, \tilde{S}(N-1) \). Since any finitely-generated \( S \)-module has a finite free resolution, it follows that any such module can be written in terms of this basic set. That is, \( \mathbf{D}(X_{\Sigma^+}) \) is generated by these \( N \) sheaves. So

\[
\tilde{S} \oplus \tilde{S}(1) \oplus \tilde{S}(2) \oplus \ldots \oplus \tilde{S}(N-1),
\]

is a tilting sheaf and we have proven property 4.

Now we shall prove property 3. First we need

**Proposition 3** Let \( B \) be an ideal of \( S \) such that \( S/B \) is a regular ring. Let \( M \) be a finitely-generated graded \( S \)-module that is annihilated by some power of \( B \). Then \( M \) is in the full triangulated subcategory of \( \mathbf{D}(\text{gr}-S) \) generated by \( S/B \) (and its grade shifts).

Suppose \( M \) is annihilated by \( B^N \). Consider the following short exact sequence:

\[
0 \longrightarrow BM \longrightarrow M \longrightarrow M' \longrightarrow 0.
\]

Now \( M' \) is annihilated by \( B \) and is therefore an \( (S/B) \)-module. The regularity condition then guarantees that \( M' \) has a finite free resolution in terms of sums of \( S/B(r) \) for any grade shift \( r \). The module \( BM \) is annihilated by \( B^{N-1} \). Thus we prove the proposition by induction. \( \blacksquare \)

So we arrive at the conclusion that \( T_{\Sigma^+} \) is generated by \( S/B_{\Sigma^+} \). It follows that when performing the quotient \( [17] \), we need only consider triangles involving \( S/B_{\Sigma^+} \) (and its translations and shifts in grading). The *only* relations between the \( S(n) \)'s are then given by the triangles coming from the Koszul resolution \( [32] \). The Grothendieck group of \( \mathbf{D}(X_{\Sigma^+}) \) is therefore \( \mathbb{Z}^{\oplus N} \). That is, \( T = N \), proving property 2, and, therefore, property 3.

Obviously the analysis for \( X_{\Sigma^-} \) is identical to \( X_{\Sigma^+} \). The concludes the proof of theorem 3. \( \blacksquare \)

### 4.5 The conifold and suspended pinch point

One of the simplest examples is the conifold which was the principal example of a noncommutative resolution studied by Van der Bergh \([24, 46]\). Here \( n = 4 \) and \( d = 3 \) (putting us in the \( r = 1 \) case) and the 4 points in \( \mathcal{A} \) form a square. Put \( S = \mathbb{C}[x, y, z, w] \) where the respective charges of these 4 variables are given by

\[
\Phi = (1 \ 1 \ -1 \ -1).
\]

The resulting toric variety is the conifold. The two resolutions, related by a flop, are given by dividing the square \( \mathcal{A} \) into 2 triangles in two different ways.
Applying the results of the previous section, we have $N = 2$ and a tilting collection \( \{ S, S(1) \} \). The quiver is given by

\[
\begin{array}{c}
\circ & \circ \\
\circ & \circ \\
\end{array}
\]

The relations can be immediately read from this diagram given that $S$ is a \textit{commutative} algebra, even if the path algebra of the quiver isn’t. In this case the relations are given by $xyz = yzx, xwy = ywx$, etc. It follows that the superpotential for this theory is given by $\text{Tr}(xyzw - yzwx)$ [47].

As another application we give the suspended pinch point of [12] which has $r = 2$. This has $S = \mathbb{C}[x, y, z, u, v]$ with charge matrix given by

\[
\Phi = \begin{pmatrix}
1 & -2 & 1 & 0 & 0 \\
0 & -1 & 1 & 1 & -1
\end{pmatrix}.
\]

A tilting set is given by \( \{ S, S(0, 1), S(1, 1) \} \). The resulting quiver is

\[
\begin{array}{c}
S(0, 1) \\
\circ & \circ & \circ \\
S(1, 1) \\
\circ & \circ & \circ \\
S(0, 0)
\end{array}
\]

and the relations (and thus superpotential) can be easily deduced from the expressions on the arrows as in the conifold above. This example has 5 phases and is wholesome.

Amongst the numerous ways of computing quivers and superpotentials (see, for example [12, 13, 48–50], and [9] to which it is closest) this method seems to be the mathematically most direct.

\section{4.6 An orbifold example with $r > 1$}

To try and go systematically beyond the case $r = 1$ we consider the relatively simple situation of an orbifold. Suppose the convex hull of the point set $\mathcal{A}$ is a simplex. This simplest phase to address is the “unresolved phase” which refers to the triangulation of $\mathcal{A}$ that has just one simplex and all points other than the vertices of the convex hull are ignored. It is interesting to ask if such a phase is wholesome (omitting, of course, property 5).

The single simplex in $\Sigma$ has $d$ vertices which we associate to $x_{n-d+1}, \ldots, x_n$ to simplify notation. Geometrically this phase corresponds to a orbifold $\mathbb{C}^d/G$, where $G$ is the finite abelian group given by $N$ divided by the lattice generated by the columns of $A$ associated to $x_{n-d+1}, \ldots, x_n$. 

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The combinatorics of this phase is straight-forward. The Cox ideal and Stanley–Reisner ideal are respectively:

\[
B_\Sigma = (x_1 x_2 \ldots x_r) \\
I_\Sigma = (x_1, x_2, \ldots, x_r).
\]

It follows that we have nonzero $D$-graded Betti numbers $b_{0, \beta}$ for $\beta$ given by any of the first $r$ columns of $\Phi$. Let us denote these vectors by $\beta_1, \ldots, \beta_r \in D$.

The exact sequences (10) and (11) are split and so we have an isomorphism of lattices

\[
L \oplus N \cong D \oplus M.
\]

Let us view this isomorphism as given by a \( n \times n \) unimodular integral matrix $C$. The first $r$ columns of $C$ are given by $^t \Phi$ and so the upper left $r \times r$ block of $C$ is given by the coordinates of $x_{a-d+1}, \ldots, x_n$. The determinant of this matrix is equal to $|G|$. By the Schur complement this is also equal to the determinant of the $r \times r$ matrix of vectors $\beta_1, \ldots, \beta_r$.

So, any $\alpha \in D$ may be written uniquely as

\[
\alpha = \sum_{i=0}^r t_k \beta_k,
\]

for rational numbers $t_k$.

**Proposition 4** The “fundamental parallelepiped” $0 \leq t_k < 1$ in $D$ contains $|G|$ vectors $\alpha_1, \alpha_2, \ldots$ which may be used to generate $D(X_\Sigma)$.

The fact that there are $|G|$ vectors follows from the statement above about the determinant.

Clearly $S/(x_j)$ is annihilated by $B_\Sigma$ for $j = 1, \ldots, r$. The short exact sequence

\[
0 \longrightarrow S(-\beta_j + \delta) \xrightarrow{x_j} S(\delta) \xrightarrow{\delta} \frac{S}{x_j(\delta)} \longrightarrow 0,
\]

(43)

gives an equivalence $\tilde{S}(-\beta_j + \delta) \cong \tilde{S}(\delta)$ in $D(X_\Sigma)$. Hence we generate the whole of $D(X_\Sigma)$ from the fundamental parallelepiped. $\blacksquare$

Next we show that the local cohomology groups $H^*_B(S)_\delta$ vanish for any $-\delta$ in the fundamental parallelepiped. The resolution (23) is the Koszul resolution of $(x_1, x_2, \ldots, x_r)$. Hence nonzero Betti numbers $b_{i, \alpha}$ appear with vectors $\alpha$ with any combination of 0’s and 1’s in the first $r$ positions and 0’s in the final $d$ positions. For any such $\alpha$ we may find a $v$ in the kernel of $A$, as above, such that the positive entries of $v$ coincide with the 1’s in $\alpha$ and again define $N_v$ as the sum of the positive entries. Let $v$ be the image of $v_{m, \Sigma}$ in $L$. So, as before, the excluded region associated to $\alpha$ is

\[
(-\delta, v_{m, \Sigma}) = \sum_i -t_i(\beta_i, v_{m, \Sigma}) \\
\leq -N_v,
\]

(44)
But \((\beta_i, v_{m, \Sigma})\) is simply the \(i\)th entry in the vector \(v\). The inequality is therefore violated for \(0 \leq t_i < 1\).

It would be nice to show that the set of line bundles \(\tilde{S}(\alpha)\) for all \(\alpha\) in the fundamental parallelepiped form a tilting collection. This requires checking that the local cohomology groups vanish for \(\delta = \alpha_i - \alpha_j\). The combinatorics of this is a little messy so we will content ourselves with examples.

Suppose \(d = 3\) and the \(n = 7\) points of \(\mathcal{A}\) lie in the plane as:

\[
\begin{array}{c}
\bullet & & \bullet & & \bullet & & \bullet \\
& x_1 & & x_2 & & & \\
& & x_3 & & x_4 & & x_5, x_7 \\
\end{array}
\]

This may be written as

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -1 & 0 & -1 & 1 & 0 & -2 \\
0 & -1 & -1 & -2 & 0 & 1 & -3
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
0 & 0 & -2 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 1 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

and \(X\) corresponds to an orbifold \(\mathbb{C}^3/\mathbb{Z}_6\), where the \(\mathbb{Z}_6\) action is generated by \((x_5, x_6, x_7) \mapsto (e^{\frac{2\pi i}{6}}x_5, -x_6, e^{\frac{2\pi i}{6}}x_7)\).

The fundamental parallelepiped then contains the 6 points

\[
\begin{align*}
\alpha_1 &= (0, 0, 0, 0) \\
\alpha_2 &= (0, 1, 0, 0) \\
\alpha_3 &= (1, 0, 0, 0) \\
\alpha_4 &= (0, 0, 0, 1) \\
\alpha_5 &= (0, 1, -1, 1) \\
\alpha_6 &= (1, 0, 0, 1).
\end{align*}
\]

In this case one can explicitly check that the sum of the corresponding six line bundles is a tilting sheaf.

The quiver associated to this tilting sheaf is, of course, nothing other than the McKay
Returning to the general orbifold for a moment, coherent sheaves on the stack $X_\Sigma = \mathbb{C}^d/G$ correspond to $G$-equivariant coherent sheaves on $\mathbb{C}^d$. These in turn have resolutions by $G$-equivariant bundles on $\mathbb{C}^d$ which are classified by finite-dimensional representations of $G$ over $\mathbb{C}$. Obviously the latter are generated by the $|G|$ one-dimensional irreducible representations of the abelian group $G$. Indeed, the sheaves $S(\alpha_i)$ we have found above correspond to these one-dimensional representations. It follows that there can be no equivalences between these generators in the derived category as then the rank of the Grothendieck group would be wrong. So this unresolved phase is wholesome.

What about the other phases? For this $\mathbb{C}^3/\mathbb{Z}_6$ orbifold there are a total of 32 phases. That is, there are 32 triangulations of the point set $\mathcal{A}$ which all happen to be regular. The secondary polytope (see, for example, [51]) has 32 vertices each of which corresponds to a phase. 5 of the phases are smooth resolutions and there are 26 partial resolutions corresponding to the remaining phases.

With a combination of Macaulay 2 (for computing the Betti numbers) and Maple (for checking the required inequalities with the linear programming package) it is not hard to show that in all 32 phases the differences $\alpha_i - \alpha_j$ in (46) are $B_\Sigma$-acyclic.

We now need to check if the $S(\alpha_i)$’s generate the whole derived category in every phase. This turns out to be a combinatorially tricky question. Let us return to the general situation. We have a prime decomposition

$$B_\Sigma = \bigcap_{k=1}^t m_k,$$

where each $m_k$ is a linearly generated monomial ideal in $S$.

**Proposition 5** The subcategory $T_\Sigma$ of $\mathbf{D}(\text{gr}-S)$ is generated as a triangulated subcategory by $S/m_k$ (and its shifts) for all $k$.

This proposition is very similar to proposition 3. Recall that $T_\Sigma$ is generated by modules annihilated by $B_\Sigma^N$ for some $N$. Following the proof of proposition 3 we may immediately see that $T_\Sigma$ is generated by modules annihilated by $B_\Sigma$. So, suppose $M$ is annihilated by $B_\Sigma$.
and define

\[ M_m = m_1 m_2 \ldots m_M, \]

and \( M = M_0 \). Then \( M_t = 0 \) and we have a short exact sequence

\[ 0 \longrightarrow m_m M_{m-1} \longrightarrow M_{m-1} \longrightarrow M' \longrightarrow 0. \]

Assume, by decreasing induction on \( m \), that \( M_m = m_m M_{m-1} \) is in the subcategory generated by \( S/m_k \) (and its shifts) for all \( k \). Now \( M' \) is annihilated by \( m_m \) and so is an \((S/m_m)\)-module. Since \((S/m_m)\) is a regular ring, we have a finite free resolution of \( M' \) in terms of \((S/m_m)\) and its grade-shifts. Therefore \( M_{m-1} \) is in the subcategory generated by \( S/m_k \) (and its shifts) for all \( k \).

Let us write \( m = (x_{i_1}, x_{i_2}, \ldots, x_{i_p}) \). We have a Koszul resolution

\[ S(-\beta) \longrightarrow \ldots \longrightarrow \bigoplus_j S(-\Phi_{i_j}) \longrightarrow S \longrightarrow \frac{S}{m}. \]

where \( \Phi_{i_j} \) is the \( i_j \)-th column of \( \Phi \), and \( \beta \) is the sum of the columns over the index set \( \{i_1, i_2, \ldots, i_p\} \). Therefore, all of the relations between line bundles in the derived category \( D(X_S) \) are generated by triviality of complexes of the form

\[ \tilde{S}(-\beta) \longrightarrow \ldots \longrightarrow \bigoplus_j \tilde{S}(-\Phi_{i_j}) \longrightarrow \tilde{S}, \]

obtained from (51) for \( m = m_1, \ldots, m_t \).

Now, to \( m = (x_{i_1}, x_{i_2}, \ldots, x_{i_p}) \) appearing as a prime factor of \( B_S \) we may associate the corresponding generator \( x_{i_1} x_{i_2} \ldots x_{i_p} \) of the Stanley–Reisner ideal \( I_S \). Therefore we have a nonzero Betti number \( b_{0, \alpha} \) in (25) where the 1’s in \( \alpha \) correspond to the locations \( \{i_1, i_2, \ldots, i_p\} \). It immediately follows that the vector \( -\beta \) in (51) is not \( B_{S^\Sigma} \)-acyclic.

The general idea of generating the whole derived category is to take a vector \( \beta \) which is not in our tilting set \( \{\alpha_i\} \) and use the relations (52) (and their grade shifts) to replace \( \tilde{S}(\beta) \) by an isomorphic object in the derived category represented by a complex of tilting objects. Combinatorially this is messy and we will not try to confront this process directly.

Note also that the fact that \( -\beta \) in (51) is guaranteed to not be \( B_{S^\Sigma} \)-acyclic seems to indicate that there can be no relations in \( D(X_S) \) within our tilting set.

Anyway, rather than attempting to prove that \( \{\tilde{S}(\alpha_i)\} \) are independent and generate \( D(X_S) \) directly, we will resort to a string theory argument. We know that \( \{\tilde{S}(\alpha_i)\} \) are independent objects in the orbifold phase. Therefore, these \(|G| \) D-branes have central charges \( Z(\tilde{S}(\alpha_i)) \) which vary holomorphically and independently over the orbifold phase. So, therefore, they vary independently over the whole Kähler moduli space. It follows that these D-branes are independent objects in K-theory, and thus the Grothendieck group, and thus the derived category in any phase. Furthermore, the K-theory group remains fixed between phases (since it is the group of topological B-brane charges) and so our set \( \{\tilde{S}(\alpha_i)\} \) generates the whole derived category.

So we arrive at the conclusion that \( \{\tilde{S}(\alpha_i)\} \) are independent and generate \( D(X_S) \) in all 32 phases and so this \( \mathbb{C}^3/\mathbb{Z}_6 \) example is wholesome.
4.7 An unwholesome example in dimension 5

Consider $S = \mathbb{C}[x_0, \ldots, x_6]$ with charge matrix

$$
\Phi = \begin{pmatrix}
-6 & 1 & 2 & 1 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 1
\end{pmatrix}.
$$

This yields a noncompact Calabi–Yau fivefold with six phases.

There is no tilting collection of free $S$-modules that works simultaneously in all six phases but one may show that

$$
T = S(0,0) \oplus S(1,0) \oplus S(2,0) \oplus S(3,0) \oplus S(4,0) \oplus S(5,0) \oplus \\
S(0,1) \oplus S(1,1) \oplus S(2,1) \oplus S(3,1) \oplus S(4,1) \oplus S(5,1),
$$

satisfies $\text{Ext}^n_X(T, T) = 0$ for $n \neq 0$ everywhere except in one phase. Similarly

$$
T = S(0,0) \oplus S(1,0) \oplus S(2,0) \oplus S(3,0) \oplus S(4,0) \oplus S(5,0) \oplus \\
S(-1,1) \oplus S(0,1) \oplus S(1,1) \oplus S(2,1) \oplus S(3,1) \oplus S(4,1),
$$

satisfies $\text{Ext}^n_X(T, T) = 0$ for $n \neq 0$ everywhere except in one other phase.

Actually this model has a $\mathbb{Z}_2$ symmetry $(x_0, \ldots, x_5) \rightarrow (x_0, x_2, x_1, x_3, x_4, x_6, x_5)$. Dividing the moduli space out by this symmetry we have only three inequivalent phases and then we do have candidate tilting collections from above for all three phases. In this sense this example can still actually be wholesome. However, the existence of this example shows that the combinatorics of toric geometry do not enforce wholesomeness.

5 Relationship to $\Pi$-Stability

5.1 The $\mathbb{Z}_3$-orbifold

So far we have been concerned with a simple description of the derived category in terms of line bundles $\{\tilde{S}(\alpha_i)\}$ and the resulting quiver. It appears, at least in many example, that the description is constant over the whole Kähler moduli space in that the same tilting set and quiver can be used in every phase. This reflects the constancy of the B-model over the Kähler moduli space.

On the other hand, the way that our tilting collection generates the derived category changes as we move from one phase to another since all the combinatorics depend on the Stanley–Reisner ideal $I_\Sigma$. Since whether a given object is $\Pi$-stable changes as one moves around the moduli space, one might suspect that this is also related to the same combinatorics. We will see in this section that this is indeed the case.

It is perhaps easiest to begin with an example and then try to make generalizations. Consider $\mathbb{C}^3/\mathbb{Z}_3$ for which $S = \mathbb{C}[p, x, y, z]$, $d = 3$, $r = 1$ and the matrix $\Phi$ is given by

$$
(-3 1 1 1). \text{ Following section 4.4, the toric ideal is } (xyz - p^3). \text{ Note that we will restrict attention to the “physicists” notion of } \Pi\text{-stability where one studies how stability varies over}
$$
the moduli space of complexified Kähler forms. The more general notion of the full space of stability conditions was studied by Bridgeland in this same example in [18, 52].

The phase $\Sigma_-$ corresponds to a Stanley–Reisner ideal $(p)$ and yields the orbifold phase. The phase $\Sigma_+$ corresponds to a Stanley–Reisner ideal $(xyz)$ and corresponds to the geometry of $\mathcal{O}_{\mathbb{P}^2}(-3)$ which is the “large radius” resolved phase.

The tilting set in both phases is

$$\{\tilde{S}, \tilde{S}(1), \tilde{S}(2)\}$$

and the quiver is given in (6).

Inspired by the observation in [5] that the three D-branes in this set are somehow “globally defined” over the whole Kähler moduli space in terms of the gauged linear sigma model we would like to propose that this set of three D-branes is everywhere $\Pi$-stable.

Recall that $\Pi$-stability is governed by the phase $\xi$ of a D-brane $\mathcal{F}$:

$$\xi(\mathcal{F}) = \frac{1}{\pi} \arg Z(\mathcal{F}),$$

where $Z$ is the central charge. Of course, one must be careful about defining the mod 2 ambiguity in (57). In our case the D-branes $\tilde{S}(a)$ have noncompact support and thus infinite central charge. In this case, the phase is defined purely by the dimension of the support (see, for example, section 6.2.5 of [3]):

$$\xi(\tilde{S}(a)) = -\frac{1}{2} \dim(X_\Sigma), \quad \text{for all } a \in \mathbb{Z}.$$

Now consider $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}$, the structure sheaf of the exceptional $\mathbb{P}^2$. The locus of the exceptional divisor in $X_{\Sigma_+}$ is given by $p = 0$ and so we have an exact sequence:

$$0 \longrightarrow \tilde{S}(3) \overset{p}{\longrightarrow} \tilde{S} \overset{}{\longrightarrow} \mathcal{E} \longrightarrow 0,$$

or, to write in terms of a triangle:

$$\mathcal{E} \begin{cases} \tilde{S}(3) \overset{p}{\longrightarrow} \tilde{S} \begin{cases} \begin{array}{c} \text{[1]} \end{array} \end{cases} \end{cases}$$

It is known that $\mathcal{E}$ is massless at the “conifold point” where the CFT becomes singular [53]. This implies that near this conifold point, the phase of $\mathcal{E}$ can take on any value. This, in turn, implies that the grade difference on the left or right edge of (60) can exceed one causing a decay of the opposite vertex. What actually happens is sketched in figure (61) obtained by numerical integration of the Picard–Fuchs equation as in section 7.3 of [3].

\footnote{Sometimes the word “grade” is used instead but we already have another notion of grade here.}
Figure 1: Lines of marginal stability for $\mathbb{C}^3/\mathbb{Z}_3$.

The figure is read as follows. The sketch is of the $(B + iJ)$-plane where $J$ is the Kähler form. The solid lines denote the boundaries of “fundamental regions” of the moduli space viewing the $(B + iJ)$-plane as a Teichmüller space, roughly speaking. We have a copy of the moduli space between $B = -\frac{1}{2}$ and $\frac{1}{2}$ where the region is squeezed to width less than one as one approaches the orbifold point at $B = J = 0$. At the conifold point we have our massless D-brane $\mathcal{E}$. Actually whether one considers this to be $\mathcal{E}$ or $\mathcal{E}(-1)$ depends upon paths taken in the moduli space. We have two points in the $B + iJ$ plane denoted by dots in the figure corresponding the where these D-branes become massless. Similarly $\mathcal{E}(-2)$ and $\mathcal{E}(1)$ become massless if one follows paths to other copies of the moduli space.

The dashed lines in the figure represent lines of marginal stability which are relevant to this discussion. Asymptotically, for large $|B|$, these lines become straight and at an angle of $45^\circ$ to the $B$-axis.

Consider starting at large radius limit, high on the $J$-axis. Since $\tilde{S}(a)$ is a $\mu$-stable line bundle for any $a \in \mathbb{Z}$, we expect this to correspond to a $\Pi$-stable D-brane for sufficiently large $J$. As one moves down, one eventually reaches the point labeled $P_0$ in the figure. At this point one hits the line of marginal stability radiating leftwards out of the massless $\mathcal{E}$ conifold point. At this instant, the grade of $\mathcal{E}$ rises above $-\frac{1}{2}$. This causes $\tilde{S}(3)$ to decay in (60). Thus $\tilde{S}(3)$ is unstable as one nears the orbifold phase.

Similarly consider the triangle:

\[
\begin{align*}
\tilde{S}(4) \quad p \quad \tilde{S}(1)
\end{align*}
\]
As we see from figure 1, the marginal line coming left out of the $\mathcal{E}(1)$ conifold point also crosses the $J$-axis which causes the decay of $\tilde{S}(4)$ at $P_1$. Similarly all D-branes of the form $\tilde{S}(a)$ for $a \geq 3$ decay as one moves into the orbifold phase. The larger the value of $a$, the larger the value of $J$ at which the decay takes place.

Now consider the triangle

$$
\begin{align*}
\mathcal{E}(-1) & \rightarrow \\
\tilde{S}(2) & \rightarrow \\
\tilde{S}(-1) & \rightarrow 
\end{align*}
$$

As one moves down the $J$-axis and hits $P_0$, the grade of $\mathcal{E}(-1)$ falls below $-\frac{3}{2}$. This causes a decay of $\tilde{S}(-1)$. Similarly all D-branes of the form $\tilde{S}(a)$ for $a < 0$ decay as one moves into the orbifold phase.

This yields the result that the only D-branes of the form $\tilde{S}(a)$ which remain stable as one moves from the resolved phase to the orbifold phase are those in the tilting set. Furthermore, we have a rather explicit picture of how this happens. For large positive $a$, $\tilde{S}(a)$ iteratively decays into $\mathcal{E}(a') + \tilde{S}(a')$ with $a' = a - 3$ until $a$ falls below 3. Similarly for negative $a$, the decay increases $a$ by 3 until it is positive.

It is instructive to describe the D-brane, $\mathcal{E}$, which is massless at the conifold point in terms of a quiver representation. The sequence (59) expresses $\mathcal{E}$ in terms of $\tilde{S}$ and $\tilde{S}(3)$. But we know in the resolved phase that $\tilde{S}$ can be expressed in terms of the tilting set $\{\tilde{S}, \tilde{S}(1), \tilde{S}(2)\}$. We may rewrite (59) as

$$
0 \rightarrow \tilde{S} \rightarrow \tilde{S}(1)^{\oplus 3} \rightarrow \tilde{S}(2)^{\oplus 3} \rightarrow \tilde{S} \rightarrow \mathcal{E} \rightarrow 0.
$$

In other words, $\mathcal{E}$ is the cokernel of the map $(px, py, pz)$. In terms of quivers, we know that the free module $S$ corresponds to the infinite set of paths ending at the node $v_0$ in (4). Any such path that is not of length zero must end in $px$, $py$ or $pz$. Thus, the module corresponding to the cokernel of the map $(px, py, pz)$ is precisely the one-dimensional quiver representation with the single dimension associated to vertex $v_0$. This “simple” representation of the path algebra is familiar as the fractional brane which becomes massless at the conifold point [53].

One may try to picture a similar effect starting in the orbifold phase and moving into the resolved phase. However, the result is not as pretty since we cannot begin with the assumption that the $\tilde{S}(a)$’s are all stable at the orbifold point.

There is one “symmetry” between the resolved phase and the orbifold phase which is worth emphasizing. In the resolved phase corresponding to the triangulation $\Sigma^+$, we have $B_{\Sigma^+} = (x, y, z)$. In the orbifold phase $X_{\Sigma^-}$ we have $B_{\Sigma^-} = (p)$. In the resolved phase we may write the D-brane, $\mathcal{E}$, which becomes massless at the conifold point as $S/B_{\Sigma^+}$. In the orbifold phase we have $\tilde{S}(3) \cong \tilde{S}$, which, when applied to (63) yields $\mathcal{E} \cong S/B_{\Sigma^+}(3)$.

So, in the resolved phase, $S/B_{\Sigma^+}$ (and its grade-shifts) play the role of a “no-brane” while $S/B_{\Sigma^-}$ is the brane massless at the conifold. In the orbifold phase the rôles are reversed.
5.2 The conifold

The conifold of section 4.4 is a little less satisfying. Assume we are in the phase given by $B = (x, y)$. The massless D-brane of interest is therefore given by the module $S/(z, w)$. This corresponds to the structure sheaf $\mathcal{O}_C$ of the exceptional curve $C$ in the small resolution. We have a resolution:

\[ 0 \rightarrow \tilde{S}(2) \xrightarrow{(-w)} \tilde{S}(1) \oplus \tilde{S}(1) \xrightarrow{(z,w)} \tilde{S} \rightarrow \mathcal{O}_C \rightarrow 0. \quad (64) \]

Write this as a triangle

\[ \mathcal{O}_C[-1] \quad (65) \]

\[ \begin{array}{c}
\tilde{S}(2) \\
\mathcal{X}
\end{array} \]

where

\[ \mathcal{X} = \tilde{S}(1) \oplus \tilde{S}(1) \rightarrow \tilde{S}. \quad (66) \]

Clearly the phase of $\tilde{S}(a)$ is always $-\frac{3}{2}$ for any $a \in \mathbb{Z}$ as in the previous section. Similarly $\mathcal{X}$ has phase $-\frac{3}{2}$ everywhere in the moduli space. The phase of $\mathcal{O}_C[-1]$ at large radius is also $-\frac{3}{2}$ which renders $\tilde{S}(2)$ stable, as one would expect. Now as we follow the $J$-axis down the grade of $\mathcal{O}_C[-1]$ starts to rise and so we might hope that it eventually increases above $-\frac{1}{2}$ to destabilize $\tilde{S}(2)$. What actually happens is shown in figure 2.

A fundamental region now looks like a vertical strip of infinite length between $B = 0$ (where $\mathcal{O}_C$ is massless) and $B = 1$ (where $\mathcal{O}_C(1)$ is massless). The phase “boundary” is the line $J = 0$ which separates the two Calabi–Yau phases related by a flop. It is not hard to show (see section 7.2 of [3] for example) that

\[ \xi(\mathcal{O}_C[-1]) = -1 + \frac{\theta}{\pi}, \quad (67) \]

where $\theta$ is the angle shown in figure 2. It follows that so long as we stay in this fundamental strip, the D-brane $\tilde{S}(2)$ never decays via the triangle (65). Only when one reaches the large radius limit of the flopped Calabi–Yau when $\theta = \pi/2$ does $\tilde{S}(2)$ become marginally unstable. Alternatively, if one ventures into the neighboring phase to the left, as shown by the dotted path in the figure, the bundle $\tilde{S}(2)$ does decay.

We have therefore demonstrated that $\tilde{S}(2)$ is unstable “in a way” when one ventures into the flopped phase, but only when one actually reaches the large radius limit of the flop, or if one winds sufficiently far around the conifold point in the moduli space to enter another fundamental region.

Similar remarks also apply to all the other line bundles $\tilde{S}(a)$ for $a$ anything other than 0 or 1. So again we have the result that the tilting set $\{\tilde{S}, \tilde{S}(1)\}$ is somehow globally stable (not crossing the walls of the fundamental region for the phase $J > 0$) in the moduli space while the other line bundles $\tilde{S}(a)$ are not.
The essential difference between the orbifold of section 5.1 and this conifold is the codimension of the exceptional set as we will see in the next section.

5.3 A general picture

Let us try to make some general comments about Π-stability based on the above examples. The idea is that we will assume that all “interesting” decays are based on triangles coming from the kinds of resolutions we have seen so far. In particular, in any phase Σ, we have the Cox ideal $B_\Sigma$ with its primary decomposition (48). The Alexander dual to this statement is that the Stanley–Reisner ideal can be written

$$I_\Sigma = m_1^\vee + m_2^\vee + \ldots + m_t^\vee,$$

where each $m_i^\vee$ is a principal ideal. The only triangles we concern ourselves with are the Koszul resolutions of $m_i$ where we consider all such primary ideals from all phases $\Sigma$. So all our statements about Π-stability will be limited in the sense that only a subset of all distinguished triangles are used.

Suppose we have a point set $\mathcal{A}$ which is wholesome and we choose a tilting set $\{S(\alpha_i)\}$. No line bundle is globally stable over the whole Kähler moduli space for arbitrary paths but we may make the situation more manageable by making cuts. That is, we fix a fundamental region of the Teichmüller space much as in figure 1. We remove from consideration paths
that cross the walls of this fundamental region. Now let us boldly assert that *there is a choice of cuts such that every object in the tilting set is stable over the whole moduli space.*

This is very similar in spirit to the picture in [5]. There they showed that only D-branes that lived within a certain grade-restricted window could be “globally defined” over the whole moduli space of gauged linear sigma models. Actually our assertion does not quite coincide with the analysis of [5]. Using the simplest ansatz for A-branes on the Coulomb branch, the authors of [5] were able to give an example where the tilting set was *not* globally defined. Hopefully this discrepancy can be avoided by using more subtle A-branes.

In general there may be many large radius Calabi–Yau phases. In any such large radius limit we expect line bundles $\tilde{S}(\delta)$ to be stable for all possible $\delta \in D$. Choose such a Calabi–Yau phase and denote it by $\Sigma_\infty$ and the associated Cox ideal by $B_\infty$. Consider some other phase $\Sigma$ with a prime decomposition of $B_\Sigma$ given by (48). To each prime ideal $m$ in this decomposition there is a Koszul resolution given by (51) which we write again for convenience:

$$S(-\beta) \rightarrow \ldots \rightarrow \bigoplus_j S(-\Phi_{i_j}) \rightarrow S \rightarrow \frac{S}{m}. \quad (69)$$

To this we can associate two distinguished triangles:

$$\begin{align*}
\frac{S}{m} \ar[b] & \ar[ll] \ar[ll] S(-\beta) \rightarrow \ldots \rightarrow \bigoplus_j S(-\Phi_{i_j}) \ar[ll] \\
\ar[b] & \ar[ll] \ar[ll] S \ar[ll] \end{align*} \quad (70)$$

and

$$\begin{align*}
S(-\beta) \ar[b] & \ar[ll] \ar[ll] \bigoplus_j S(-\Phi_{i_j}) \rightarrow S \ar[ll] \\
\ar[b] & \ar[ll] \ar[ll] \frac{S}{m} [1-c] \ar[ll] \end{align*} \quad (71)$$

where $c$ is the codimension of the ideal $m$ in $S$.

Suppose $m \supset B_\infty$. In this case $S/m$ is annihilated by $B_\infty$ and so corresponds to a no-brane in the large radius phase. The triangles above express relations between $S(\beta)$’s.

Now suppose $m \not\supset B_\infty$. In this case $S/m$ is *not* annihilated by $B_\infty$ and so corresponds to a non-trivial brane in the large radius phase $X_\Sigma$. Now the above triangles (and their grade shifts) express possible decay paths which can destabilize various $\tilde{S}(\beta)$’s. Note first that our assumption that the bundles in the tilting set $\{\tilde{S}(\alpha_i)\}$ are always stable is entirely consistent with these triangles. The statement that the tilting set is wholesome means that the elements of the tilting set are independent and therefore we can never write a triangle of the forms (70) or (71) (or their grade shifts) expressing a decay of one tilting element into a combination of the others and modules of the form $S/m$. 

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Whether or not we actually have a decay of one line bundle into other line bundles depends on the analysis of the phases. In triangles (70) or (71) the objects at the bottom left and bottom right of the triangle both always have phase $-d/2$ since they correspond to bundles supported over the entire space. The only phase which varies is therefore that of $S/m$. If $S/m$ becomes massless somewhere in the moduli space then we are in the situation similar to sections 5.1 and 5.2.

At large radius limit, the sheaf associated to $S/m$ has phase $-\frac{1}{2}(d-c)$. So, for example, the phase difference on the right edge of the triangle (71) is $1 - c/2$. In order to cause a decay of $S(\beta)$, this difference must rise to 1. Let us assume we follow a path that runs very close to the “conifold point” where $S/m$ becomes massless. We also assume that $Z(S/m)$ has a simple zero at this point. Then for decay, the path from the large radius limit needs to subtend an angle of $\pi c/2$ with respect to this point. For section 5.1 we had $c = 1$ and so we only needed to pass through an angle of $\pi/2$ which happened as we passed from one phase to another. In section 5.2, for the flop, we had an angle of $\pi$ which required going all the way to the limit of the other phase. Clearly, for higher-dimensional examples where $c > 2$ we need to start to loop around the conifold point to get the $\tilde{S}(a)$’s to decay.

So, if $m$ has codimension one (i.e., is a principal ideal) one would expect the analysis of section 5.1 to follow and some line bundles outside the tilting set will decay as we move from phase $\Sigma\infty$ to phase $\Sigma$. If the codimension of $m$ is greater than one then this need not happen and one would need to work harder, by looping around in the moduli space, to see the non-tilting line bundles decay.

6 Discussion

We have seen how, in some examples, one may define a tilting set of line bundles which works globally over the whole moduli space. Thus, the derived category is given by the same quiver in each phase and we recover the result that $D(X)$ is invariant in a very explicit way. The example of section 4.7 shows that we cannot always expect such a global set to exist but this wholesomeness does seem surprisingly ubiquitous in examples studied. Given the usefulness of such wholesome tilting sets it would be nice to find the precise combinatorics of when they exist.

The combinatorial problems we encounter are classic in combinatorial commutative algebra, such as analysis of Stanley–Reisner ideals and local cohomology. One might therefore hope that this well-developed branch of mathematics might offer some tools and techniques that can extend the results of this paper.

We also have a rather paltry understanding of $\Pi$-stability at present. What one would really like to know is, given a point in the Kähler moduli space, what is the precise set of $\Pi$-stable objects in $D(X)$. Viewing the derived category in terms of tilting line bundles seems to offer some handle on this difficult problem, as we saw in section 9 but obviously much remains to be understood.
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