Research Article

On Computing Techniques for Sombor Index of Some Graphs

Kiran Naz, 1 Sarfraz Ahmad 1, and Eihab Bashier 2

1Department of Mathematics, Comsats University Islamabad, Lahore Campus, Pakistan
2Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Khartoum, Khartoum, Sudan

Correspondence should be addressed to Eihab Bashier; eihabbashier@gmail.com

Received 30 June 2022; Accepted 25 September 2022; Published 10 October 2022

Abstract
In all types of topological indicators, degree-based indicators play a major role in chemical graph theory. The topological index is a fixed numeric value associated with graph isomerism. Firstly, in 1972, the concept of degree-based index was developed by Gutman and Trinajstic. These degree-based indices are divided into two ways, namely, degree and connection number. These degree-based graph indices are positive-valued for non-regular graphs and zero for regular graphs. In this article, we discussed the degree-based Sombor, reduced Sombor, and average Sombor indices for wheel graph, gear graph, helm graph, flower graph, sunflower graph, and lobster graph.

1. Introduction

Graph theory has proved to be very useful in modeling key system components with limited components. Image models are used to represent the telephone network, train network, communication problems, traffic network, etc. A graph is a simple way to represent information that connects relationships between objects. Things are represented by nodes and relationships with edges [1].

Chemical graph theory is the source of mathematical chemistry that used graph theory in the structure of composite chemical cells. The topological index [2, 3] is part of a chemical graph theory that integrates physicochemical factors such as freezing point, boiling point, melting point, infrared spectrum, electrical parameters, viscosity, and density of substrate chemical graphs.

Biological testing of chemical compounds is very expensive. It requires a very large laboratory and advanced equipment to test these compounds. This process is expensive and time consuming. Because of this feature, pharmaceutical companies are keenly interested in finding new ideas or ways to reduce costs. One can reduce costs when there is no need for a laboratory and no need for equipment, but you just need to study a specific chemical structure using topological indicators. Topological indices are of different types such as distance-based topological indices [4], spectrum-based topological indices, and degree-based topological indices.

A topological index is a number to describe the graph of a molecule. Topological indices are related to topological distances in a graph or vertex adjacency. Wiener index [5, 6] is the first topological index which is equal to the sum of all distances between the vertices.

\[ W(G) = \sum_{\{v_i,v_j\}\subseteq V(G)} d(v_i,v_j). \]  

(1)

Gutman and Trinajstic introduced the first and second Zagreb indices in [7] as

\[ M_1(G) = \sum_{v_i\in V(G)} \deg(v_i)^2, \]

(2)

\[ M_2(G) = \sum_{v_i,v_j\in E(G)} \deg(v_i)\deg(v_j). \]

The first natural degree-based topological indicator was introduced by Milan Randic. His index was defined as

\[ R(G) = \sum_{v_i\sim v_j} \frac{1}{\sqrt{d_{v_i}(G)d_{v_j}(G)}}. \]  

(3)
Let $E$ be the edge of the graph $G$, between the vertices $v_i$ and $v_j$. Later on, Ernesto Estrada derived a new topological indicator and named it atom bond connectivity index. It is defined as

$$ABC(G) = \sum_{v_i \sim v_j} \sqrt{d_{v_i}(G) + d_{v_j}(G) - 2} \frac{d_{v_i}(G)d_{v_j}(G) - 2}{d_{v_i}(G)d_{v_j}(G)} + \frac{2}{d_{v_i}(G)d_{v_j}(G)}.$$  \hspace{1cm} (4)

Furtula et al. derived the modified version of the atom bomb connectivity index, and they named it augmented Zagreb index. It is defined as

$$AZI(G) = \sum_{v_i \sim v_j} \left( \frac{d_{v_i}(G)d_{v_j}(G)}{d_{v_i}(G) + d_{v_j}(G) - 2} \right)^3.$$ \hspace{1cm} (5)

The first geometric arithmetic index was proposed by Vukicevic in [8].

$$GA(G) = \sum_{v_i \sim v_j} \sqrt{d_{v_i}(G)d_{v_j}(G)} + 3(1/2) \frac{d_{v_i}(G) + d_{v_j}(G)}{d_{v_i}(G) + d_{v_j}(G)}$$ \hspace{1cm} (6)

In 1980s, Siemion Fajtlowicz introduced a new topological indicator and named it as harmonic index. It is also a degree-based topological indicator.

$$H(G) = \sum_{v_i \sim v_j} \frac{2}{d_{v_i}(G) + d_{v_j}(G)}.$$ \hspace{1cm} (7)

where $\sqrt{d_{v_i}(G)d_{v_j}(G)}$ and $1/2[d_{v_i}(G) + d_{v_j}(G)]$ are the geometric and arithmetic means, respectively. The sum connectivity index is a new invention by Nenad Trinajstic and Bo Zhou. It is defined as

$$SCI(G) = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_{v_i}(G)d_{v_j}(G)}}.$$ \hspace{1cm} (8)

where the product is replaced by sun in Randic index. In this article, we will compute the degree-based indices like SO, SO$_{red}$, and SO$_{avg}$. SO, SO$_{red}$, and SO$_{avg}$ are defined in [9] as

$$SO(G) = \sum_{e_i \in E(G)} \sqrt{\deg(v_i)^2 + \deg(v_j)^2},$$

$$SO_{red}(G) = \sum_{e_i \in E(G)} \sqrt{(\deg(v_i) - 1)^2 + (\deg(v_j) - 1)^2},$$

$$SO_{avg}(G) = \sum_{e_i \in E(G)} \sqrt{\left(\frac{(\deg(v_i) - \frac{2m}{n})^2 + (\deg(v_j) - \frac{2m}{n})^2}{2}\right).}$$ \hspace{1cm} (9)

We have discussed wheel graph, gear graph, helm graph, flower graph, sunflower graph, and lobster graph with the help of all these indices. Recently, Sombor index has received a lot of attention within mathematics and chemistry. For example, the chemical function of the Sombor index, especially the ability to speculate and discriminate, has been investigated. The results show that the Sombor index [9] can be used effectively in modeling computer thermodynamic structures and confirming the validity of this new indicator in QSPR analysis.

### 2. Wheel Graph

The wheel graph $W_n$ is determined by connecting $K_1$ and $C_n$. $K_1$ is the graph of order 1 and $C_n$ is the cycle graph as shown in Figure 1. The size of wheel graph [7] is $2n$ and the order of wheel graph is $n + 1$. Apparently, $W_n$ wheel graph has every node of degree 3 other than the internal node. Internal node has degree 4. The wheel graphs are planer graphs. All dual planer graphs are isomorphic to wheel graphs. The chromatic number of wheel graph $W_n$ is 3 if $n$ is odd and 4 if $n$ is even. For $n = 3$, we have calculated three different Sombor indices of $W_3$:

$$SO(W_3) = 6 \sqrt{18},$$

$$SO_{red}(W_3) = 12 \sqrt{2},$$

$$SO_{avg}(W_3) = 0.$$ \hspace{1cm} (10)

The general representation of the wheel graph for three different Sombor indices is given in Theorems 1–3.

**Theorem 1.** Let $W_n$ be a wheel graph of order $n + 1$. Then, the Sombor index of $W_n$ is

$$SO(W_n) = n \sqrt{18} + n \sqrt{n} - n^2.$$ \hspace{1cm} (11)

**Proof.** Since $W_n$ is a wheel graph of order $n + 1$, where $n \geq 3$, by definition,

$$SO(W_n) = \sum_{e_i \in E(W_n)} \sqrt{\deg(v_i)^2 + \deg(v_j)^2},$$

$$SO(W_n) = n \sqrt{18} + n \sqrt{n} - n^2. \hspace{1cm} \Box$$ \hspace{1cm} (12)

**Theorem 2.** Let $W_n$ be a wheel graph of order $n + 1$. Then, the reduced Sombor index of $W_n$ is

$$SO_{red}(W_n) = n \sqrt{8} + n \sqrt{4 + (n - 1)^2}.$$ \hspace{1cm} (13)

**Proof.** Since $W_n$ is a wheel graph of order $n + 1$, where $n \geq 3$, by definition,

$$SO_{red}(W_n) = \sum_{e_i \in E(W_n)} \sqrt{(\deg(v_i) - 1)^2 + (\deg(v_j) - 1)^2},$$

$$SO_{red}(W_n) = n \sqrt{2} + n \sqrt{n^2 - 2n + 5} \hspace{1cm} \Box.$$ \hspace{1cm} (14)

**Theorem 3.** Let $W_n$ be a wheel graph of order $n + 1$. Then, the average Sombor index of $W_n$ is
\[ SO_{\text{avg}}(W_n) = \sqrt{2n \left( 3 - \frac{4n}{n+1} \right) + n \left( \frac{3 - 4n}{n+1} \right)^2 + \left( \frac{n - 4n}{n+1} \right)^2}. \]  

\[ (15) \]

**Proof.** Since \( W_n \) is a wheel graph of order \( n + 1 \), where \( n \geq 3 \), by definition,

\[ SO_{\text{avg}}(W_n) = \sum_{c_{i,j} \in E(W_n)} \sqrt{\left( \deg(v_i) - \frac{2m}{n} \right)^2 + \left( \deg(v_j) - \frac{2m}{n} \right)^2}, \]

\[ \text{SO}_{\text{avg}}(W_n) = \sqrt{2n \left( \frac{n-3}{n+1} \right)^2 + n \left( \frac{n-3}{n+1} \right)^2 \left( n^2 + 1 \right) \left( n+1 \right)^2}. \]  

\[ (16) \]

### 3. Comparison between SO, SO\(_{\text{red}}\), and SO\(_{\text{avg}}\) for Wheel Graph

By getting results from Theorems 1–3, we will make the comparison [10] between the values of Sombor, reduced Sombor, and average Sombor indices of a wheel graph. Table 1 and Figure 2 represent the numerical and graphical comparison of the three indices.

### 4. Gear Graph

The gear graph \( G_n \), also known as bipartite wheel graph, is determined by adding a new node between each pair of adjacent nodes of rim as shown in Figure 3. The size of gear graph is \( 3n \) and the order of gear graph [7] is \( 2n + 1 \). The gear graphs are a special case of Jahangir graph. The gear graphs are matchsticks and a unit distance graphs. For \( n = 3 \), we have calculated three different Sombor indices of \( G_3 \):

\[ \text{SO}(G_3) = 6\sqrt{13} + 9\sqrt{2}, \]

\[ \text{SO}_{\text{red}}(G_3) = 6\sqrt{5} + 6\sqrt{2}, \]

\[ \text{SO}_{\text{avg}}(G_3) = \frac{30}{7} + \frac{9\sqrt{2}}{7}. \]  

\[ (17) \]

The general representation of the gear graph for three different Sombor indices is given in Theorems 4–6.

**Theorem 4.** Let \( G_n \) be a gear graph of order \( 2n + 1 \). Then, the Sombor index of \( G_n \) is

\[ \text{SO}(G_n) = 2n\sqrt{13} + n\sqrt{9 + n^2}. \]  

\[ (18) \]

**Proof.** Since \( G_n \) is a gear graph of order \( 2n + 1 \), where \( n \geq 3 \), by definition,

\[ \text{SO}(G_n) = \sum_{c_{i,j} \in E(G_n)} \sqrt{\deg(v_i)^2 + \deg(v_j)^2}, \]

\[ \text{SO}(G_n) = 2n\sqrt{13} + n\sqrt{9 + n^2}. \]

\[ (19) \]

**Theorem 5.** Let \( G_n \) be a gear graph of order \( 2n + 1 \). Then, the reduced Sombor index of \( G_n \) is

\[ \text{SO}_{\text{red}}(G_n) = 2n\sqrt{5} + n\sqrt{4 + (n - 1)^2}. \]  

\[ (20) \]

**Proof.** Since \( G_n \) is a gear graph of order \( 2n + 1 \), where \( n \geq 3 \), by definition,

\[ \text{SO}_{\text{red}}(G_n) = \sum_{c_{i,j} \in E(G_n)} \sqrt{(\deg(v_i) - 1)^2 + (\deg(v_j) - 1)^2}, \]

\[ \text{SO}_{\text{red}}(G_n) = 2n\sqrt{5} + n\sqrt{n^2 - 2n + 5}. \]

\[ (21) \]

**Theorem 6.** Let \( G_n \) be a gear graph of order \( 2n + 1 \). Then, the average Sombor index of \( G_n \) is
Proof. Since $G_n$ is a gear graph of order $2n + 1$, where $n \geq 3$, by definition,

$$SO_{avg}(G_n) = 2n \sqrt{\left(2 - \frac{6n}{2n + 1}\right)^2 + \left(3 - \frac{6n}{2n + 1}\right)^2} + n \sqrt{\left(3 - \frac{6n}{2n + 1}\right)^2 + \left(n - \frac{6n}{2n + 1}\right)^2}. \quad (22)$$
\[ \text{SO}_{\text{avg}}(G_n) = \sum_{e_{ij} \in E(G_n)} \sqrt{\left( \text{deg}(v_i) - \frac{2m}{n} \right)^2 + \left( \text{deg}(v_j) - \frac{2m}{n} \right)^2}, \]
\[ \text{SO}_{\text{avg}}(G_n) = 2n \sqrt{\left( 2 - \frac{6m}{2n + 1} \right)^2 + \left( 3 - \frac{6m}{2n + 1} \right)^2 + n \left( \frac{3 - 6m}{2n + 1} \right)^2 + \left( n - \frac{6m}{2n + 1} \right)^2}. \] (23)

5. Comparison between SO, SO\text{red}, and SO\text{avg} for Gear Graph

By getting results from Theorems 4–6, we will make the comparison [10] between the values of Sombor, reduced Sombor, and average Sombor indices of a gear graph. The numerical and graphical representation of indices is given in Table 2 and Figure 4, respectively.

6. Helm Graph

The helm graph \( H_n \) is determined by adding a single edge to every node of the rim as shown in Figure 5. The size of helm graph is \( 3n \) and the order of helm graph is \( 2n + 1 \). The chromatic number of helm graph is 3 if \( n \) is even and 4 if \( n \) is odd. The helm graph [7] contains three type of vertices, \( n \) vertices on outer rim, pendant vertices, and \( n \) vertices of degree 4. It is a node prime graph. For \( n = 3 \), we have calculated three different Sombor indices of \( H_3 \):

\[ \text{SO}(H_3) = 3\sqrt{17} + 3\sqrt{32} + 15; \]
\[ \text{SO}_{\text{red}}(H_3) = 3\sqrt{18} + 3\sqrt{13} + 6; \]
\[ \text{SO}_{\text{avg}}(H_3) = 3\sqrt{221} + 3\sqrt{109} + 30\sqrt{2}. \] (24)

The general representation of the helm graph for three different Sombor indices is given in Theorems 4–6.

**Theorem 7.** Let \( H_n \) be a gear graph of order \( 2n + 1 \). Then, the Sombor index of \( H_n \) is

\[ \text{SO}_{\text{avg}}(H_n) = n \sqrt{\left( 1 - \frac{6n}{2n + 1} \right)^2 + \left( 4 - \frac{6n}{2n + 1} \right)^2 + \sqrt{2n} \left( 4 - \frac{6n}{2n + 1} \right)^2 + \left( n - \frac{6n}{2n + 1} \right)^2}. \] (25)

**Proof.** Since \( H_n \) is a helm graph of order \( 2n + 1 \), where \( n \geq 3 \), by definition,

\[ \text{SO}(H_n) = n \sqrt{17} + n \sqrt{32} + n \sqrt{16 + n^2}. \] (26)

\[ \text{SO}_{\text{red}}(H_n) = \sum_{e_{ij} \in E(H_n)} \sqrt{\left( \text{deg}(v_i) - 1 \right)^2 + \left( \text{deg}(v_j) - 1 \right)^2}, \]
\[ \text{SO}_{\text{red}}(H_n) = 3n + n \sqrt{18} + n \sqrt{9 + (n - 1)^2}. \] (27)

**Theorem 8.** Let \( H_n \) be a helm graph of order \( 2n + 1 \). Then, the reduced Sombor index of \( H_n \) is

\[ \text{SO}_{\text{avg}}(H_n) = \sum_{e_{ij} \in E(H_n)} \sqrt{\left( \text{deg}(v_i) - 1 \right)^2 + \left( \text{deg}(v_j) - 1 \right)^2}, \]
\[ \text{SO}_{\text{avg}}(H_n) = 3n + n \sqrt{18} + n \sqrt{9 + (n - 1)^2}. \] (28)

**Theorem 9.** Let \( H_n \) be a helm graph of order \( 2n + 1 \). Then, the average Sombor index of \( H_n \) is

\[ \text{SO}_{\text{avg}}(H_n) = n \sqrt{\left( 1 - \frac{6n}{2n + 1} \right)^2 + \left( 4 - \frac{6n}{2n + 1} \right)^2 + \sqrt{2n} \left( 4 - \frac{6n}{2n + 1} \right)^2 + \left( n - \frac{6n}{2n + 1} \right)^2}. \] (29)

**Proof.** Since \( H_n \) is a helm graph of order \( 2n + 1 \), where \( n \geq 3 \), by definition,

\[ \text{SO}_{\text{avg}}(H_n) = \sum_{e_{ij} \in E(H_n)} \sqrt{\left( \text{deg}(v_i) - 1 \right)^2 + \left( \text{deg}(v_j) - 1 \right)^2}, \]
\[ \text{SO}_{\text{avg}}(H_n) = n \sqrt{\left( 1 - \frac{6n}{2n + 1} \right)^2 + \left( 4 - \frac{6n}{2n + 1} \right)^2 + \sqrt{2n} \left( 4 - \frac{6n}{2n + 1} \right)^2 + \left( n - \frac{6n}{2n + 1} \right)^2}. \] (30)
Table 2: Comparison between Sombor, red. Sombor, and avg. Sombor indices.

| n   | SO($G_n$)   | SO$_{red}$($G_n$) | SO$_{avg}$($G_n$) |
|-----|-------------|-------------------|-------------------|
| 3   | 34.3612     | 21.9017           | 6.1040            |
| 4   | 48.8444     | 32.3107           | 11.4603           |
| 5   | 65.2103     | 44.7214           | 19.2124           |
| 6   | 83.5158     | 59.1438           | 29.0712           |
| 7   | 103.7881    | 75.5768           | 40.9780           |
| 8   | 126.0409    | 94.0180           | 54.9114           |
| 9   | 150.2814    | 114.4651          | 70.8613           |

Comparison between $SO$, $SO_{red}$, and $SO_{avg}$ for Helm Graph

By getting results from Theorems 7–9, we are able to make the comparison [10] between the values of Sombor, reduced Sombor, and average Sombor indices of a helm graph. The numerical and graphical representation of indices is given in Table 3 and Figure 6, respectively.

8. Flower Graph

The flower graph $F_{ln}$ is determined from the helm graph by joining each single node to the apex of helm graph as shown in Figure 7. The size of flower graph is $4n$ and the order of flower graph [7] is $2n + 1$. For $n = 3$, we have calculated three different Sombor indices of $F_{l3}$:

$$SO(F_{l3}) = 6\sqrt{5} + 12\sqrt{2} + 6\sqrt{10} + 6\sqrt{13},$$

$$SO_{red}(F_{l3}) = 3\sqrt{10} + 9\sqrt{2} + 3\sqrt{26} + 3\sqrt{34},$$

$$SO_{avg}(F_{l3}) = \frac{6\sqrt{29}}{7} + \frac{12\sqrt{2}}{7} + \frac{6\sqrt{106}}{7} + \frac{6\sqrt{85}}{7}.$$

The general representation of the flower graph for three different Sombor indices is given in Theorems 10–12.

Theorem 10. Let $F_{ln}$ be a flower graph of order $2n + 1$. Then, the Sombor index of $F_{ln}$ is

$$SO(F_{ln}) = n\sqrt{20 + n\sqrt{32 + n\sqrt{4 + (2n)^2 + n\sqrt{16 + (2n)^2}}}.$$  

(32)

Proof. Since $F_{ln}$ is a flower graph of order $2n + 1$, where $n \geq 3$, by definition,

$$SO(F_{ln}) = \sum_{e_{ij} \in E(F_{ln})} \left(\text{deg}(v_i) + \text{deg}(v_j)\right)^{\frac{1}{2}},$$

$$SO(F_{ln}) = n\sqrt{20 + n\sqrt{32 + n\sqrt{4 + (2n)^2} + n\sqrt{16 + (2n)^2}}.$$  

(33)

□

Theorem 11. Let $F_{ln}$ be a flower graph of order $2n + 1$. Then, the reduced Sombor index of $F_{ln}$ is

$$SO_{red}(F_{ln}) = n\sqrt{10 + n\sqrt{18 + n\sqrt{1 + (2n - 1)^2 + n\sqrt{9 + (2n - 1)^2}}}.$$  

(34)

Proof. Since $F_{ln}$ is a flower graph of order $2n + 1$, where $n \geq 3$, by definition,

$$SO_{red}(F_{ln}) = \sum_{e_{ij} \in E(F_{ln})} \left(\text{deg}(v_i) - 1\right)^{\frac{1}{2}} + \left(\text{deg}(v_j) - 1\right)^{\frac{1}{2}},$$

$$SO_{red}(F_{ln}) = n\sqrt{10 + n\sqrt{18 + n\sqrt{1 + (2n - 1)^2} + n\sqrt{9 + (2n - 1)^2}}.$$  

(35)

□

Theorem 12. Let $F_{ln}$ be a flower graph of order $2n + 1$. Then, the average Sombor index of $F_{ln}$ is
Table 3: Comparison between Sombor, red. Sombor, and avg. Sombor indices.

| n  | \( SO(H_n) \) | \( SO_{\text{red}}(H_n) \) | \( SO_{\text{avg}}(H_n) \) |
|----|----------------|----------------|----------------|
| 3  | 44.3399        | 32.5446        | 16.9065        |
| 4  | 61.7473        | 45.9411        | 23.6225        |
| 5  | 80.9154        | 61.2132        | 32.7513        |
| 6  | 101.9464       | 78.4416        | 44.1183        |
| 7  | 124.8955       | 97.6559        | 57.5992        |
| 8  | 149.7939       | 118.8673       | 73.1389        |
| 9  | 176.6594       | 142.0798       | 88.7120        |

\[
SO_{\text{avg}}(F_{1n}) = n \sqrt{\left(2 - \frac{8n}{2n + 1}\right)^2 + \left(4 - \frac{8n}{2n + 1}\right)^2} + \sqrt{2n\left(4 - \frac{8n}{2n + 1}\right) + n\sqrt{\left(2 - \frac{8n}{2n + 1}\right)^2 + \left(2n - \frac{8n}{2n + 1}\right)^2}}
\]  

(36)
Proof. Since \( F_{2n} \) is a flower graph of order \( 2n + 1 \), where \( n \geq 3 \), by definition,

\[
\text{SO}_{\text{avg}}(F_{2n}) = \sum_{e_{ij} \in E(F_{2n})} \sqrt{\left( \text{deg}(v_i) - \frac{2m}{n} \right)^2 + \left( \text{deg}(v_j) - \frac{2m}{n} \right)^2},
\]

\[
\text{SO}_{\text{avg}}(F_{2n}) = n \sqrt{\left( 2 - \frac{8n}{2n + 1} \right)^2 + \left( 4 - \frac{8n}{2n + 1} \right)^2 + 2n \left( 4 - \frac{8n}{2n + 1} \right) + n \sqrt{\left( 2 - \frac{8n}{2n + 1} \right)^2 + \left( 2n - \frac{8n}{2n + 1} \right)^2}}.
\]

### 9. Comparison between SO, SO\text{red} and SO\text{avg} for Flower Graph

By getting results from Theorems 10–12, we will make the comparison between the values of Sombor, reduced Sombor, and average Sombor indices of a flower graph. The numerical and graphical representation of indices is given in Table 4 and Figure 8, respectively.

### 10. Sunflower Graph

The sunflower graph \( S_{fn} \) is determined from the flower graph by expanding \( n \) single edges to the apex of flower graph as shown in Figure 9. The size of sunflower graph [7] is \( 5n \) and the order of flower graph is \( 3n + 1 \). For \( n = 3 \), we have calculated three different Sombor indices of \( S_{f3} \):

\[
\begin{align*}
\text{SO}(S_{f3}) &= 6\sqrt{5} + 12\sqrt{2} + 3\sqrt{85} + 3\sqrt{97} + 3\sqrt{82}, \\
\text{SO}_{\text{red}}(S_{f3}) &= 3\sqrt{10} + 9\sqrt{2} + 3\sqrt{65} + 3\sqrt{73} + 24, \\
\text{SO}_{\text{avg}}(S_{f3}) &= 6\sqrt{2} + 6\sqrt{37} + 6\sqrt{10}.
\end{align*}
\]

The general representation of the sunflower graph for three different Sombor indices is given in Theorems 13–15.

**Theorem 13.** Let \( S_{fn} \) be a sunflower graph of order \( 3n + 1 \). Then, the Sombor index of \( S_{fn} \) is

\[
\text{SO}(S_{fn}) = n\sqrt{20} + n\sqrt{32} + n\sqrt{4 + (3n)^2} + n\sqrt{16 + (3n)^2} + n\sqrt{1 + (3n)^2}.
\]

*Proof. Since \( S_{fn} \) is a sunflower graph of order \( 3n + 1 \), where \( n \geq 3 \), by definition,

\[
\text{SO}(S_{fn}) = \sum_{e_{ij} \in E(S_{fn})} \sqrt{\text{deg}(v_i)^2 + \text{deg}(v_j)^2},
\]

\[
\text{SO}(S_{fn}) = n\sqrt{20} + n\sqrt{32} + n\sqrt{4 + (3n)^2} + n\sqrt{16 + (3n)^2} + n\sqrt{1 + (3n)^2}.
\]

**Theorem 14.** Let \( S_{fn} \) be a sunflower graph of order \( 3n + 1 \). Then, the reduced Sombor index of \( S_{fn} \) is

\[
\text{SO}_{\text{red}}(S_{fn}) = n\sqrt{10} + n\sqrt{18} + n\sqrt{1 + (3n - 1)^2} + n\sqrt{9 + (3n - 1)^2} + n(3n - 1).
\]

*Proof. Since \( S_{fn} \) is a sunflower graph of order \( 3n + 1 \), where \( n \geq 3 \), by definition,
Theorem 15. Let $S_{fn}$ be a sunflower graph of order $3n + 1$. Then, the average Sombor index of $S_{fn}$ is
\[
\text{SO}_{\text{avg}}(F_{n}) = n \left( \frac{2 - 10n}{3n + 1} \right)^2 + \left( 4 - \frac{10n}{3n + 1} \right)^2 + \sqrt{2n} \left( 4 - \frac{10n}{3n + 1} \right) + n \left( \frac{2 - 10n}{3n + 1} \right)^2 + \left( 3 - \frac{10n}{3n + 1} \right)^2 \]

(43)

**Proof.** Since \( S_{f_n} \) is a sunflower graph of order \( 3n + 1 \), where \( n \geq 3 \), by definition,

\[
\text{SO}_{\text{avg}}(S_{f_n}) = \sum_{c_{ij} \in E(S_{f_n})} \left( \frac{\text{deg}(v_i) - 2m}{n} \right)^2 + \left( \frac{\text{deg}(v_j) - 2m}{n} \right)^2,
\]

\[
\text{SO}_{\text{avg}}(F_{n}) = n \left( \frac{2 - 10n}{3n + 1} \right)^2 + \left( 4 - \frac{10n}{3n + 1} \right)^2 + \sqrt{2n} \left( 4 - \frac{10n}{3n + 1} \right) + n \left( \frac{2 - 10n}{3n + 1} \right)^2 + \left( 3 - \frac{10n}{3n + 1} \right)^2 \]

(44)

**11. Comparison between SO, SO\text{red}, and SO\text{avg} for Sunflower Graph**

By getting results from Theorems 13–15, we are able to develop the comparison between the values of Sombor, reduced Sombor, and average Sombor indices \([10]\) of a sunflower graph. The numerical and graphical representation of indices is given in Table 5 and Figure 10, respectively.

**12. Lobster Graph**

The lobster \( L_n(2, r) \) is a graph formed from a path on \( n \) nodes as a backbone, each node in the backbone is adjacent to two different node hands, and each node hand is adjacent to \( r \) different node fingers each of which has degree 1. The lobster graph or lobster tree is a tree \([1]\) in which the removal of leaf node leaves a caterpillar graph as shown in Figure 11. The size of lobster graph is \( 2n(2r + 3) - 2 \) and the order of lobster graph is \( n(2r + 3) \). We consider a special case \( p = 2 \) of a regular lobster graph \( L_n(p, r) \). For \( n, r = 2 \), we have calculated three different Sombor indices of \( L_n(2, r) \):

\[
\text{SO}_{\text{L}}(2, 2) = 12 \sqrt{2} + 10 + 8 \sqrt{10} - \sqrt{32} + 3 \sqrt{97} + 3 \sqrt{82},
\]

\[
\text{SO}_{\text{red}}L_{2}(2, 2) = 8 \sqrt{2} + 16 + 2 \sqrt{13} - \sqrt{18},
\]

\[
\text{SO}_{\text{avg}}L_{2}(2, 2) = \frac{8 \sqrt{641}}{7} + \frac{34}{7} + \frac{80}{7} - \frac{15 \sqrt{5}}{7}.
\]

(45)

The general representation of the lobster graph for three different Sombor indices is given in Theorems 16–18.

**Theorem 16.** Let \( L_n(2, r) \) be a lobster graph of order \( n(2r + 3) \). Then, the Sombor index of \( L_n(2, r) \) is

\[
\text{SO}[L_{n}(2, r)] = 4 \sqrt{9 + (r + 1)^2} + 10 + 2nr \sqrt{1 + (r + 1)^2} + (n - 3) \sqrt{32} + 2(n - 2) \sqrt{16 + (r + 1)^2}. \]

(46)

**Proof.** Since \( L_n(2, r) \) is a lobster graph of order \( n(2r + 3) \), where \( n \geq 2 \) and \( r \geq 2 \), by definition,

\[
\text{SO}(L_{n}(2, r)) = \sum_{c_{ij} \in E(L_{n}(2, r))} \sqrt{\text{deg}(v_i)^2 + \text{deg}(v_j)^2},
\]

\[
\text{SO}[L_{n}(2, r)] = 4 \sqrt{9 + (r + 1)^2} + 10 + 2nr \sqrt{1 + (r + 1)^2} + (n - 3) \sqrt{32} + 2(n - 2) \sqrt{16 + (r + 1)^2}.
\]

(47)
Theorem 17. Let \( L_n(2, r) \) be a lobster graph of order \( n(2r + 3) \). Then, the reduced Sombor index of \( L_n(2, r) \) is

\[
\text{SO}_{\text{red}}[L_n(2, r)] = 4\sqrt{4 + r^2 + 2nr^2 + 2(n - 2)\sqrt{9 + r^2 + 2\sqrt{13} + (n - 3)\sqrt{18}}.}
\]

Proof. Since \( L_n(2, r) \) is a lobster graph of order \( n(2r + 3) \), where \( n \geq 2 \) and \( r \geq 2 \), by definition,
\[ \text{SO}_{\text{red}}(L_n(2, r)) = \sum_{e_{ij} \in E(L_n(2, r))} \sqrt{[\deg(v_i) - 1]^2 + [\deg(v_j) - 1]^2}, \]
\[ \text{SO}_{\text{red}}[L_n(2, r)] = 4\sqrt{4 + r^2 + 2nr^2 + 2(n - 2)\sqrt{9 + r^2} + 2\sqrt{13} + (n - 3)\sqrt{18}}. \] (49)

**Theorem 18.** Let \( L_n(2, r) \) be a lobster graph of order \( n(2r + 3) \). Then, the average Sombo index of \( L_n(2, r) \) is

\[ \text{SO}_{\text{avg}}[L_n(2, r)] = 4 \left[ \left( 9 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right) \right)^2 + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right) \right)^2 \right] + \]
\[ \sqrt{2(n - 3)} \left[ \left( 4 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right) \right)^2 + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right) \right)^2 \right] + \]
\[ 2nr \left[ 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right] + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right) + \]
\[ 2(n - 2) \left[ 16 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right] + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right). \] (50)

**Proof.** Since \( L_n(2, r) \) is a lobster graph of order \( n(2r + 3) \), where \( n \geq 2 \) and \( r \geq 2 \), by definition,

\[ \text{SO}_{\text{avg}}(L_n(2, r)) = \sum_{e_{ij} \in E(L_n(2, r))} \sqrt{\deg(v_i) - \frac{2m}{n} + \deg(v_j) - \frac{2m}{n}}, \]
\[ \text{SO}_{\text{avg}}[L_n(2, r)] = 4 \left[ \left( 9 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right) + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right) \right] + \]
\[ \sqrt{2(n - 3)} \left[ \left( 4 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right) + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right) \right] + \]
\[ 2nr \left[ 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right] + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right) + \]
\[ 2(n - 2) \left[ 16 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right] + \left( r + 1 - \left( \frac{2n(2r + 3) - 2}{n(2r + 3)} \right)^2 \right). \] (51)
Comparison between SO, SOred, and SOavg for Lobster Graph

By getting results from Theorems 16–18, we will make the comparison between the values of Sombor, reduced Sombor, and average Sombor indices [10] of a lobster graph. The numerical and graphical representation of indices is given in Table 6 and Figure 12, respectively.

Table 6: Comparison between Sombor, red. Sombor, and avg. Sombor indices.

| (n, r) | SO \( L_n(2, r) \) | SOred \( L_n(2, r) \) | SOavg \( L_n(2, r) \) |
|-------|-----------------|-------------------|-----------------|
| (2, 3) | 73.8204 | 53.3907 | 58.9446 |
| (3, 4) | 168.5065 | 131.0996 | 141.0331 |
| (4, 5) | 314.6446 | 256.3182 | 264.4026 |
| (3, 2) | 74.9179 | 49.7359 | 78.7685 |
| (4, 3) | 157.2388 | 114.8465 | 148.0358 |
| (5, 4) | 287.0170 | 223.5849 | 254.5741 |
| (2, 2) | 46.6119 | 30.2822 | 42.1900 |
| (3, 3) | 115.5296 | 84.1186 | 103.4957 |
| (4, 4) | 227.7618 | 177.3423 | 197.8057 |

Figure 12: Comparison between Sombor, red. Sombor, and avg. Sombor indices for lobster graph.

13. Comparison between SO, SOred, and SOavg for Lobster Graph

By getting results from Theorems 16–18, we will make the comparison between the values of Sombor, reduced Sombor, and average Sombor indices [10] of a lobster graph. The numerical and graphical representation of indices is given in Table 6 and Figure 12, respectively.

14. Conclusion

Topological indicator is a mathematical coding of the graphs. We have discussed the particular cases of SO, SOred, and SOavg to elaborate the wheel graph, gear graph, helm graph, flower graph, sunflower graph, and lobster graph, and then we find the general representation of these graphs. We have also calculated the comparison in both numerical and graphical forms between these three indices for each graph.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] H. Bian and F. Zhang, “Tree-like polyphenyl systems with extremal Wiener indices,” Match, vol. 61, no. 3, pp. 631–641, 2009.
[2] H. Deng and Z. Tang, “Kirchhoff indices of spiro and polyphenyl hexagonal chains,” Utilitas Mathematica, vol. 95, pp. 113–128, 2014.
[3] Z. Raza, “The harmonic and second Zagreb indices in random polyphenyl and spiro chains,” Polycyclic Aromatic Compounds, vol. 42, no. 3, pp. 671–680, 2020.
[4] B. Zhou, I. Gutman, B. Furtula, and Z. Du, “On two types of geometric-arithmetic index,” Chemical Physics Letters, vol. 482, no. 1–3, pp. 153–155, 2009.
[5] A. Chen and F. Zhang, “Wiener index and perfect matchings in random phenylene chains,” Match, vol. 61, no. 3, pp. 623–633, 2009.
[6] H. Deng, “Wiener indices of spiro and polyphenyl hexagonal chains,” Mathematical and Computer Modelling, vol. 55, no. 3–4, pp. 634–644, 2012.
[7] M. Javaid, U. Ali, and J. B. Liu, “Computing analysis for first Zagreb connection index and coindex of resultant graphs,” Mathematical Problems in Engineering, vol. 2021, pp. 1–19, Article ID 6019517, 2021.
[8] Z. Raza, “The expected values of arithmetic bond connectivity and geometric indices in random phenylene chains,” Helixyon, vol. 6, no. 7, pp. 044799–e4514, 2020.
[9] I. Redzepovic, “Chemical applicability of Sombor indices,” Journal of the Serbian Chemical Society, vol. 86, no. 5, pp. 445–457, 2021.
[10] Z. Raza, K. Naz, and S. Ahmad, “Expected values of molecular descriptors in random polyphenyl chains,” Emerging Science Journal, vol. 6, no. 1, pp. 151–165, 2022.