DEFORMING GEOMETRIC TRANSITIONS

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Dedicated to Professor D. Gallarati on the occasion of his 90th birthday

Abstract. After a quick review of the wild structure of the complex moduli space of Calabi–Yau threefolds and the role of geometric transitions in this context (the Calabi–Yau web) the concept of deformation equivalence for geometric transitions is introduced to understand the arrows of the Gross-Reid Calabi–Yau web as deformation-equivalence classes of geometric transitions. Then the focus will be on some results and suitable examples to understand under which conditions it is possible to get simple geometric transitions, which are almost the only well-understood geometric transitions both in mathematics and in physics.

1. Introduction

The aim of this paper is that of extending to geometric transitions (see Definition 13) the well-known concept of deformation equivalence of complex manifolds. Geometric transitions have interesting applications both in mathematics, for the study of the wild structure of the moduli space of Calabi–Yau varieties, and in physics, describing the transition between topologically distinct super-string models of Calabi–Yau vacua. Then deformation equivalence of geometric transitions and Calabi–Yau 3-folds, seems to give a more direct relation between the string theoretic Calabi–Yau web and the mathematical Gross-Reid Calabi–Yau web (see sections 6 and 7 respectively).

A large first part of these notes (sections from 2 to 6) has a purely expository purpose. For a broader discussion of these aspects the interested reader is referred to the extensive survey on the subject [37]. A second part (sections 7 and 8) is instead devoted to giving some new ideas, partial results and examples, with the aim of shedding a brighter light on the study of geometric transitions and, more generally, on the Calabi–Yau moduli space.

Deformation equivalence of geometric transitions is introduced in Definition 20 allowing us, on the one hand, to think of the Gross-Reid Calabi–Yau web as a kind of quotient of the string theoretic Calabi–Yau web by means of def-equivalence, and, on the other hand, to isolate a class of geometric transitions (referred to as simple, see Definition 25) having the properties of being well understood both from the physical and the mathematical point of view: in particular Calabi–Yau threefolds connected by a simple geometric transition turns out to admit the same fundamental group (see Remark 26). New results in this context are then given by Proposition 22 and Theorem 29. The former gives an easier formulation of def-equivalence between geometric transitions admitting singular loci comprising at

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most isolated terminal singularities; this is the case e.g. for small geometric transitions (see Definition 30). The latter characterizes the type II geometric transitions (see Definition 27) as never simple ones. From the physical point of view, probably the more interesting property of deformation equivalence of both Calabi–Yau vacua and geometric transitions, is the fact that string theories “passes through” such an equivalence, meaning that main parameters of a physical theory must be sought among geometric def-equivalence invariants (see Remark 24).

This paper ends up by studying when a small geometric transition is actually a simple one: examples of both simple and non-simple small geometric transitions are given. The main result in this context is given by Proposition 32 giving a necessary cohomological condition for small geometric transitions to be simple.

2. Calabi–Yau varieties

Definition 1. A compact, complex, Kähler manifold \( Y \) is a Calabi–Yau variety if

1. \( \bigwedge^n \Omega_Y = K_Y \cong \mathcal{O}_Y \)
2. \( h^{p,0}(Y) = 0 \quad \forall 0 < p < \dim Y \)

A \( n \)-dimensional Calabi–Yau variety will be also called a Calabi–Yau \( n \)-fold.

Remarks 2. (1) The given definition of Calabi–Yau variety includes the following lower dimensional cases
   • smooth elliptic curves,
   • smooth K3 surfaces.

(2) Observe that although a smooth elliptic curve always admits a projective embedding, this is no longer the case for K3 surfaces. By the way, for \( \dim Y \geq 3 \), the given definition of a Calabi–Yau variety \( Y \) implies that \( Y \) is a projective variety: the embedding can be fixed by a suitable integer multiple of a rational Kähler form near enough to the Kähler metric of \( Y \).

Examples 3. (1) Smooth hypersurfaces of degree \( n + 1 \) in \( \mathbb{P}^n \) (use Adjunction Formula and the Lefschetz Hyperplane Theorem).
(2) Smooth hypersurfaces (if exist!) of a weighted projective space \( \mathbb{P}(q_0, \ldots, q_n) \) of degree \( d = \sum_{i=0}^{n} q_i \).
(3) The general element of the anti–canonical system of a sufficiently good 4–dimensional toric Fano variety (see [1]).
(4) Suitable complete intersections.... (iterate the previous examples).
(5) The double covering of \( \mathbb{P}^3 \) ramified along a smooth surface of degree 8 in \( \mathbb{P}^3 \) (otic double solid).

In dimension greater than or equal to 3 the previous examples give topologically distinct complex varieties, implying immediately that the complex moduli space of Calabi–Yau \( n \)-folds, with \( n \geq 3 \), has to be necessarily disconnected. This fact apparently clashes with the smaller dimensional cases:

• the complex moduli space of elliptic curves is given by the modular curve \( \Gamma(1) \backslash \mathbb{H} \cong \mathbb{A}^1 \) which parameterizes complex structures over the topological torus \( S^1 \times S^1 \),
• after Kodaira [24], the complex moduli space of K3 surfaces is given by a smooth, complex, irreducible space of dimension 20.

Anyway if we insist on looking at this situation from the algebraic point of view, then the moduli space of algebraic K3 surfaces turns out to be a dramatically more complicated object: the following facts were known to F. Enriques [12]:
• ∀g ≥ 3 there exists a K3 surface of degree 2g − 2 in \( \mathbb{P}^g \); hence its sectional genus is \( g \);
• ∀g ≥ 3 we can obtain a space \( M_g \) of complex projective moduli of such surfaces, by imposing a polarization: \( M_g \) is an irreducible, analytic variety with \( \dim \mathbb{C} M_g = 19 \);
• then the complex moduli space \( M^{alg} \) of algebraic K3 surfaces is a reducible analytic variety and it admits a countable number of irreducible components;
• there exist K3 surfaces belonging to more than one irreducible component of \( M^{alg} \); anyway if we restrict to K3’s admitting \( \text{Pic} \cong \mathbb{Z} \) (they give the general element of any irreducible component) then they belong to only one irreducible component.

What could appear to F. Enriques as a wildly reducible moduli space was explained by K. Kodaira [24] as an analytic codimension 1 subvariety of a smooth, irreducible, analytic variety \( M \). More precisely:

• there exist analytic non-algebraic K3 surfaces,
• the Kuranishi space of any analytic K3 surface is smooth and of dimension 20.

The latter suffices to construct a smooth, irreducible, analytic universal family of K3 surfaces: its base \( M \) is the complex analytic moduli space of K3 surfaces and \( \dim \mathbb{C} M = 20 \). Moreover \( M^{alg} \) turns out to be a dense subset of \( M \).

In other terms, Kodaira recovered an irreducible moduli space for K3’s by leaving the algebraic category and working in the bigger category of compact complex surfaces. By this observation, in the late 80’s, M.Reid [36] proposed a conjectural construction of a sort of connected moduli space for Calabi–Yau 3-folds suggesting a construction (originally due to F. Hirzebruch and later called conifold transition) to parameterize birational classes of Calabi–Yau 3-folds by means of moduli of complex structures on suitable non-Kähler complex 3-folds given by the connected sum of copies of solid hypertori \( S^3 \times S^3 \): this is the famous Reid’s fantasy.

### 3. Aspects of deformation theory of Calabi–Yau varieties

Let \( \mathcal{X} \rightarrow B \) be a flat and proper, surjective map of complex spaces such that \( B \) is connected and there exists a special point \( o \in B \) whose fibre \( X = f^{-1}(o) \) is a, possibly singular, compact complex space. Then \( \mathcal{X} \) is called a deformation family of \( X \). If the fibre \( X_b = f^{-1}(b) \) is smooth, for some \( b \in B \), then \( X_b \) is called a smoothing of \( X \). Moreover \( X_b \) is also called a deformation of \( X_o \).

If the morphism \( f \) is smooth then \( \mathcal{X} \rightarrow B \) is called a smooth deformation family. In the following \( T_X^i \) will denote the global deformation object of Lichtenbaum–Schlessinger [20]. Since we will always deal with at least normal complex algebraic varieties, we can think of \( T_X^i = \text{Ext}^i(\Omega_X, \mathcal{O}_X) \), where \( \Omega_X \) is the sheaf of holomorphic differential forms on \( X \). Consider the Lichtenbaum–Schlessinger cotangent sheaves of \( X \), \( \Theta_X^i = \text{Ext}^i(\Omega_X, \mathcal{O}_X) \). Then \( \Theta_X^0 = \text{Hom}(\Omega_X, \mathcal{O}_X) =: \Theta_X \) is the “tangent” sheaf of \( X \) and \( \Theta_X^i \) is supported over \( \text{Sing}(X) \), for any \( i > 0 \). By the local to global spectral sequence relating the global Ext and sheaf \( \text{Ext} \) (see [22] and [16] II, 7.3.3) we get

\[
E_2^{p,q} = H^p(X, \Theta_X^q) \Rightarrow T^{p+q}_X
\]
giving that

\begin{align}
T_X^0 & \cong H^0(X, \Theta_X) , \\
T_X^* & \cong H^1(X, \Theta_X) ,
\end{align}

Given a deformation family $X \xrightarrow{f} B$ of $X$ for each point $b \in B$ there is a well defined linear (and functorial) map

$$D_b f : T_b B \longrightarrow T^1_{X_b}$$

(Generalized Kodaira–Spencer map)

(see e.g. [31] Theorem 5.1). Recall that $X \xrightarrow{f} B$ is called

- a versal (some authors say complete) deformation family of $X$ if for any deformation family $(Y, X) \xrightarrow{g} (C, 0)$ of $X$ there exists a map of pointed complex spaces $h : (U, 0) \rightarrow (B, o)$, defined on a neighborhood $0 \in U \subset C$, such that $Y|_U$ is the pull-back of $X$ by $h$, i.e.

$$Y|_U = U \times_B X \xrightarrow{g} X$$

\[ C \xrightarrow{h} U \xrightarrow{g} X \]

In particular the generalized Kodaira–Spencer map $\kappa(f)$ turns out to be surjective ([32], § 2.6);

- an effective versal (or miniversal) deformation family of $X$ if it is versal and the generalized Kodaira–Spencer map evaluated at $o \in B$, $D_o f : T_o B \longrightarrow T^1_X$ is injective, hence an isomorphism;

- a universal family if it is versal and the map $h : U \rightarrow B$ is uniquely determined over the neighborhood $0 \in U \subset C$. This suffices to imply that $f$ is an effective versal deformation of $X$ ([32], § 2.7.1).

**Theorem 4** (Douady–Grauert–Palamodov [11], [18], [30] and [31] Theorems 5.4-6).

Every compact complex space $X$ has an effective versal deformation $X \xrightarrow{f} B$ which is a proper map and a versal deformation of each of its fibers. Moreover the germ of analytic space $(B, o)$ is isomorphic to the germ of analytic space $(q^{-1}(0), 0)$, where $q : T^1_X \rightarrow T^1_B$ is a suitable holomorphic map (the obstruction map) such that $q(0) = 0$. In particular if $T^0_X = 0$ then the previous versal effective deformation of $X$ is actually a universal one for all the fibres close enough to $X$.

**Definition 5** (Kuranishi space). The germ of analytic space

$$\text{Def}(X) := (B, o)$$

defined in the previous Theorem, is called the *Kuranishi space* of $X$.

**Theorem 6** (Bogomolov-Tian-Todorov [3], [43], [44], [34]). Any Calabi–Yau variety $Y$ have unobstructed deformations, i.e., its Kuranishi space is smooth. In particular this means that $\text{Def}(Y) \cong T^1_Y$.

Since for a Calabi–Yau variety $Y$ we get

$$T^0(Y) \cong H^0(Y, \Theta_Y) \cong H^0(Y, \Omega^n_Y) = 0$$

then
Corollary 7. Every Calabi–Yau variety $Y$ admits a universal effective family of Calabi–Yau deformations of $Y$. In particular $h^{n-1,1}(Y)$ turns out to be the dimension of the complex moduli space of $Y$.

3.1. Deformation equivalence of Calabi–Yau varieties. In the late 80's R. Friedman and J.W. Morgan [15] introduced the following equivalence relation between complex manifolds. Here we use notation introduced by F. Catanese and M. Manetti in several subsequent discussions of related problems and conjectures [4], [6], [27], [5].

Definition 8 (Deformation equivalence, [15] pg. 10). Two complex manifolds $X_1$ and $X_2$ are direct deformation equivalent (i.e. direct def-equivalent) if there exists a smooth deformation family $X \to B$ whose base $B$ is an irreducible complex space admitting two points $b_1, b_2 \in B$ such that $X_i = f^{-1}(b_i), \ i = 1, 2$.

The equivalence relation generated by direct def-equivalence is called def-equivalence (or deformation type): this means that two complex manifolds $X$ and $Y$ are def-equivalent (we will write $X \sim Y$) if and only if there exist a positive integer $n$ and smooth manifolds $X_1, \ldots, X_n$ such that

1. $X \cong X_1$ and $X_n \cong Y$,
2. for every $1 \leq i \leq n-1$, $X_i$ and $X_{i+1}$ are direct def-equivalent.

Remark 9. Let us observe that:

1. F. Catanese observed that “in order to analyse deformation equivalence, one may restrict oneself to the case where dim($B$) = 1: since two points in a complex space $B$ belong to the same irreducible component of $B$ if and only if they belong to an irreducible curve $B' \subset B$. One may further reduce to the case where $B$ is smooth simply by taking the normalization $B^0 \to B_{\text{red}} \to B$ of the reduction $B_{\text{red}}$ of $B$, and taking the pull-back of the family to $B^0$” [5]; more generally, up to a resolution of singularities of $B$ and a base change, one can always assume $B$ to be a smooth and irreducible complex space; this is also observed by Friedman and Morgan immediately after the definition of def-equivalence: “Equivalently, deformation type is the equivalence relation generated by declaring that two complex manifolds are equivalent if they are both fibers in a proper smooth map between two connected complex manifolds” [15, pg. 10];

2. if a concept of coarse moduli space for the manifolds $X, Y$ is defined, an equivalent formulation of Definition 8 is the following: two complex manifolds $X$ and $Y$ are def-equivalent if and only if they are elements of the same irreducible component of their moduli space. This is the case e.g. of minimal compact complex surfaces [2, Def. 23], [27]. For what concerns Calabi–Yau varieties a coarse moduli space is well defined as a quasi–projective scheme [45, § 1.2], then such an equivalent definition can be applied.

Remark 10. Given a Calabi–Yau manifold $Y$, let $D_Y$ denote the def-equivalence class of $Y$. For what observed in the previous Remark 9 and recalling Corollary 7, $D_Y$ can be thought of as $h^{n-1,1}(Y)$–dimensional irreducible complex space giving an irreducible component of the coarse moduli space of Calabi–Yau manifolds. There are many ways of compactifying such an irreducible component: here we will not discuss this aspect, being beyond the scope of the present paper. Anyway in the
following we will assume that a closure $M$ of a def-equivalence class $D$ of Calabi–Yau manifolds will include any singular degeneration of elements in $D$ carrying either terminal or canonical singularities.

**Remark 11.** The closures $M_1$ and $M_2$ of two def-equivalence classes $D_1$ and $D_2$, respectively, may admit a common limit point $b \in M_1 \cap M_2$; M. Gross exhibited an effective example of this fact [20]. Recalling that a 3-dimensional terminal singularity is necessarily an isolated singularity, the extension, due to Y. Namikawa, of the Bogomolov-Tian-Todorov Theorem [4] to the Kuranishi space of an isolated 3-dimensional terminal singularity ([28], Theorem 1), allows us to conclude that,

- for Calabi–Yau 3–folds, given a common point $b \in M_1 \cap M_2$ then $Y_b$ can't admit terminal singularities, since $\text{Def}(Y_b)$ is clearly reducible, hence singular.

In fact the Gross’ example in [20] exhibits the case of a Calabi–Yau 3–fold admitting canonical singularities.

**Remark 12.** By the classical theorem of Ehresmann:
- two def-equivalent complex manifolds are orientedly diffeomorphic.

Friedman and Morgan conjectured that the converse could be true: this is the so called $\text{def} = \text{diff}$ problem. Friedman actually proved this equivalence for compact complex surfaces with $b_1 = 0$ and Kodaira dimension less than or equal to 1 [14]; in particular it holds for $K3$ surfaces. Notice that
- since the $\text{def} = \text{diff}$ problem admits a positive answer for elliptic curves (obvious) and $K3$ surfaces (Friedman [14]) it makes sense to ask if does it hold for any Calabi–Yau varieties.

This problem has been negatively settled in dimension $n \geq 3$ by Y. Ruan [40] who showed that the two Calabi–Yau varieties constructed by M. Gross in [20], and belonging to different irreducible components of the moduli space, are actually diffeomorphic but not symplectomorphic, hence not def-equivalent.

Subsequently, the general problem has been negatively settled by several counter-examples, the first of which was given by M. Manetti, then followed by many others by F. Catanese, Kharlamov-Kulikov, Bauer-Catanese-Grunewald, Catanese-Wajnryb ... (see [2] and therein references).

4. **Geometric transitions**

**Definition 13.** Let $Y$ be a Calabi–Yau $n$–fold and $\phi : Y \to Y'$ be a birational contraction onto a normal variety. If there exists a complex deformation (smoothing) of $Y'$ to a Calabi–Yau $n$–fold $\tilde{Y}$, then the process of going from $Y$ to $\tilde{Y}$ is called a geometric transition (for short g.t.) and denoted by $T(Y,Y',\tilde{Y})$ or by the diagram

$$Y \xrightarrow{\phi} Y' \xrightarrow{z} \tilde{Y}.$$

A g.t. $T(Y,Y',\tilde{Y})$ is called trivial if $\tilde{Y}$ is a deformation of $Y$.

A g.t. $T(Y,Y',\tilde{Y})$ is called a conifold transition (for short c.t.) if $\tilde{Y}$ admits only ordinary double points (nodes) as singularities.

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I thank the unknown referee who pointed me out such a Ruan’s counterexample of which I was not aware.
Remarks 14.  

1. Trivial geometric transitions may occur: in fact it is not possible to realize non-trivial transitions in dimension less than or equal to 1. For a trivial g.t. in dimension 3 one may e.g. consider Example 4.6 in [48] where $\phi$ admits an elliptic scroll as exceptional divisor and contracts it down to an elliptic curve $C$.

2. The transition process was firstly (locally) observed by H. Clemens in the study of double solids $V$ admitting at worst nodal singularities [8]: in his Lemma 1.11 he pointed out “the relation of the resolution of the singularities of $V$ to the standard $S^3 \times D_4$ to $S^2 \times D_4$ surgery”.

3. Let $T(Y, \bar{Y}, \tilde{Y})$ be a geometric transition of Calabi–Yau 3–folds. Then $Y$ can be supposed to carry canonical singularities, at worst (see [48] and references therein): it is then a limit point of the def-equivalence class $\mathcal{D}_\bar{Y}$ of $\bar{Y}$.

4.1. The basic example: the conifold transition in $\mathbb{P}^4$. The following example, given in [19], shows that non-trivial (conifold) transitions occur when $\dim Y \geq 3$.

Let $\bar{Y} \subset \mathbb{P}^4$ be the singular hypersurface given by the following equation
\[ x_3g(x_0, \ldots, x_4) + x_4h(x_0, \ldots, x_4) = 0 \]
where $g$ and $h$ are generic homogeneous polynomials of degree 4. $\bar{Y}$ is then the generic quintic 3-fold containing the plane $\pi : x_3 = x_4 = 0$. Then the singular locus of $\bar{Y}$ is given by
\[ \text{Sing}(\bar{Y}) = \{ [x] \in \mathbb{P}^4 | x_3 = x_4 = g(x) = h(x) = 0 \}. \]

One can then easily prove that:
- $\text{Sing}(\bar{Y})$ is composed by 16 nodes,
- \textit{(the resolution $Y$)}: $\text{Sing}(\bar{Y})$ can be simultaneously resolved and the resolution $\phi : Y \to \bar{Y}$ is a small blow up such that $Y$ is a smooth Calabi–Yau 3–fold,
- \textit{(the smoothing $\tilde{Y}$)}: $\bar{Y}$ admits the obvious smoothing given by the generic quintic 3-fold $\tilde{Y} \subset \mathbb{P}^4$. In particular $\tilde{Y}$ cannot be a deformation of $Y$ i.e. the conifold transition $T(Y, \bar{Y}, \tilde{Y})$ is not trivial.

The latter fact can be easily shown by applying the Lefschetz Hyperplane Theorem and the Künneth Formula to get the following relations on the second Betti numbers:
\[ b_2(\bar{Y}) = b_2(\mathbb{P}^4) = 1 \]
\[ b_2(Y) = b_2(\mathbb{P}^4 \times \mathbb{P}^1) = 2 \]
Therefore $\bar{Y}$ and $Y$ cannot be smooth fibers of the same analytic family.

4.2. Local topology of a conifold transition. From now on we will restrict to consider the case $n = 3$ of Calabi–Yau 3-folds. Then we can observe the following facts (for full details the interested reader is referred to [17], §1.1).

1. Locally a 3-dimensional node can be described by the local equation
\[ \mathcal{U} := \{ z_1z_3 + z_2z_4 = 0 \} \subset \mathbb{C}^4. \]

Topologically $\mathcal{U}$ turns out to be a cone over $S^3 \times S^2$. 
2. A local resolution of $\mathcal{U}$ is described by

$$\mathring{\mathcal{U}} := \left\{ \begin{array}{l}
y_0 z_4 - y_1 z_3 = 0 \\
y_0 z_1 + y_1 z_2 = 0
d\end{array} \right\} \subset \mathbb{C}^4 \times \mathbb{P}^1.$$ 

Then there exists a diffeomorphism $\mathring{\mathcal{U}} \cong \mathbb{R}^4 \times S^2$. Moreover $\mathring{\mathcal{U}}$ can be identified with the total space of the rank 2 holomorphic vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over the exceptional fibre $\mathbb{P}^1_{\mathbb{C}} = \varphi^{-1}(0)$. In particular $\mathring{\mathcal{U}}$ admits a natural complex structure.

3. A local smoothing $\mathring{\mathcal{U}}$ of the node $\mathcal{U}$ can be given by the 1–parameter family $f : \mathcal{U} \rightarrow \mathbb{R}$ where

$$U_t := f^{-1}(t) = \left\{ z_1 z_3 + z_2 z_4 = t \right\} \subset \mathbb{C}^4.$$ 

Setting $\mathring{U} := U_{t_0}$ for some $t_0 \in \mathbb{R}, t_0 > 0$, then $\mathring{U} \cong S^3 \times \mathbb{R}^3$ since it is diffeomorphic to the cotangent bundle $T^* S^3$ of the 3–sphere giving the vanishing cycle of the smoothing. In particular $\mathring{U}$ admits a natural symplectic structure for which the vanishing sphere turns out to be a lagrangian submanifold.

\textbf{Theorem 15} (Clemens [8] Lemma 1.11, [17] Thm. 1.6). Let $D_n \subset \mathbb{R}^n$ be the closed unit ball and consider

- $S^3 \times D_3 \subset S^3 \times \mathbb{R}^3 \cong \mathring{\mathcal{U}}$
- $D_4 \times S^2 \subset \mathbb{R}^4 \times S^2 \cong \mathring{\mathcal{U}}$

Then $\mathring{\mathcal{D}} := \Psi^{-1}\mathcal{D}$ and $\mathring{\mathcal{D}} := \Phi^{-1}\mathcal{D}$ are compact tubular neighborhoods of the vanishing cycle $\mathring{\mathcal{S}} \subset \mathring{\mathcal{U}}$ and of the exceptional cycle $\mathcal{P}_{\mathbb{C}} \subset \mathring{\mathcal{U}}$, respectively. Consider the standard diffeomorphism

$$\alpha' : (\mathbb{R}^4 \setminus \{0\}) \times S^2 \xrightarrow{\sim} S^3 \times (\mathbb{R}^3 \setminus \{0\}), \quad (u, v) \mapsto \left( \frac{u}{|u|}, |u|v \right)$$

and restrict it to $D_4 \times S^2$. Since

$$\partial(D_4 \times S^2) = S^3 \times S^2 = \partial(S^3 \times D_3)$$

observe that $\alpha'|_{\partial(D_4 \times S^2)} = \text{id}|_{S^3 \times S^2}$. Hence $\alpha'$ induces a standard surgery from $\mathbb{R}^4 \times S^2$ to $S^3 \times \mathbb{R}^3$. Then $\mathring{\mathcal{U}}$ can be obtained from $\mathring{\mathcal{U}}$ by removing $\mathring{\mathcal{D}}$ and pasting in $\mathring{\mathcal{D}}$, by means of the diffeomorphism $\alpha := \Psi^{-1} \circ \alpha' \circ \Phi$.

Let us underline a global consequence of the Theorem as a straightforward application of the Seifert-van Kampen Theorem

\textbf{Corollary 16}. A conifold transition does not change the fundamental group.

5. Reid’s fantasy

Since geometric transitions (and in particular conifold transitions) may connect topologically distinct Calabi–Yau 3-folds, M. Reid thought that they could be the right instrument to recover a sort of connectedness of the Calabi–Yau 3-folds complex moduli space. Quickly, his construction was the following.

1. **Assumption**: every projective Calabi–Yau 3–fold $Y$ is birational to a Calabi–Yau 3–fold $Y'$ such that $H^2(Y')$ is generated by rational curves.
2. Consequently if $\phi : Y' \to \overline{Y}$ is the morphism contacting all the homologically independent rational curves, then $\overline{Y}$ is always smoothable, by a Friedman result ([13], Corollary 4.7), and every smoothing $\tilde{Y}$ has $b_2(\tilde{Y}) = 0$, meaning that $\overline{Y}$ can be smoothed only to non–Kähler compact complex 3–folds.

3. By results of C. T. C. Wall [47], any such smoothing $\tilde{Y}$ has topological type completely determined by its third Betti number $b_3(\tilde{Y})$, implying that it is diffeomorphic to a connected sum $(S^3 \times S^3)^{\# r}$ of $r$ copies of the solid hypertorus $S^3 \times S^3$.

Then we get the famous:

\textit{Conjecture 17 (Reid’s fantasy).} Up to some kind of inductive limit over $r$, the birational classes of projective Calabi–Yau 3–folds can be fitted together, by means of geometric transitions, into one irreducible family parameterized by the moduli space $\mathcal{N}$ of complex structures over suitable connected sum of copies of solid hypertori.

This conjecture has the further fascinating property of recovering the idea that moduli could be described by studying complex structures over a (hyper)-torus, typical of elliptic curves.

Unfortunately, the description of the moduli space $\mathcal{N}$ turns out to be a quite hard problem.

6. The string theoretic Calabi–Yau web

Calabi–Yau 3–folds play a fundamental role in 10–dimensional string theories: locally, four dimensions give rise to the usual Minkowski space–time $M_4$ while the remaining six dimensions (the so called hidden dimensions for their microscopic extension) are compactified to a geometric model $Y$ which, essentially to preserve the required supersymmetry, turns out to be a Calabi–Yau 3–fold. Therefore the string theoretic space-time looks like

- a locally trivial 10-dimensional bundle whose base is the usual space-time of Einstein and which is locally isomorphic to $M_4 \times Y$.

6.1. The vacuum degeneracy problem. In spite of the fact that there are only five consistent 10–dimensional super–string theories, actually nearly unique via dualities,
the compactification process give rise to the problem of choosing the appropriate Calabi–Yau model: which fibers in the space-time bundle?

In fact on the physical side, there is not any prescription for making a precise choice of the vacuum model and on the mathematical side there is a multitude of topologically distinct Calabi–Yau 3–folds. By the way, the choice of two distinct models does not at all give rise to equivalent physical theories, since the physics turns out to be strictly related to the cohomology of the Calabi–Yau model.

Ideas connected with the formulation of Reid’s fantasy, suggested to physicists, such as P. Candelas, P. S. Green, T. Hubsch and others that:

- (simply connected) Calabi–Yau 3–folds could be, at least mathematically, connected with each other by means of geometric (conifold) transitions.

This is the so called Calabi–Yau web conjecture described in many insightful papers starting from 1988. The word “mathematically” in the statement above is a prelude to the further problem of understanding how physics passes through the singularities of a geometric transition process. Actually, as far as I know, conifold transitions are almost the only geometric transitions which have been understood from the physical point of view, after the work of A. Strominger [42]: the word “almost” refers to the so-called hyperconifolds transitions, which are divisorial geometric transitions, discovered by R. Davies, having the property of being mirror reverse transitions of conifold ones (see [9] and [10]). Let us here underline that the hypothesis of simply connectedness is then necessary if one would only use conifold transitions, as a consequence of Corollary 16.

In this sense, conifold transitions turn out to be very interesting both mathematically and physically: probably because they give a concrete bridge between the complex structures and the symplectic structures on a Calabi–Yau 3-fold.

7. The Gross Calabi–Yau web: nodes and arrows

A mathematically refined version of the Calabi–Yau web conjecture was presented by M. Gross in [21]: it is a sort of synthesis between Reid’s fantasy and the Calabi–Yau web.

The construction.

1. On the contrary of the K3 case for which an algebraic K3 surface can be smoothly deformed to a non–algebraic one, the deformation of a projective Calabi–Yau 3–fold, even singular, is still projective.

2. Since the hardest part of Reid’s fantasy seems to be in dealing with non–Kähler 3–folds, one could skip this part by insisting on staying within the projective category.

3. Think the nodes of the giant web predicted by the web conjecture as consisting of suitable closures of def-equivalence classes of Calabi–Yau 3–folds, as described in Remarks 9 and 10 below.

4. Two such nodes, say \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), are connected by an arrow \( \mathcal{M}_1 \to \mathcal{M}_2 \) if there exist Calabi–Yau 3–folds \( Y \in \mathcal{M}_1 \) and \( \tilde{Y} \in \mathcal{M}_2 \) which are each other connected by means of a geometric transition. More precisely there exists:

- a birational contraction to a normal 3–fold \( \phi : Y \to \tilde{Y} \)
- a deformation family \( (\mathcal{Y}, \mathcal{\tilde{Y}}) \to (\Delta, 0) \) such that \( \mathcal{Y}_t \cong \tilde{Y} \) for some \( t \in \Delta, t \neq 0 \).

Example 18 (See also [21]). Let
• $\mathcal{M}_Q$ be the family of quintic 3-folds in $\mathbb{P}^4$,
• $\mathcal{M}_D$ be the family of double solids (i.e. double covers of $\mathbb{P}^3$) branching along an octic surface of $\mathbb{P}^3$,
• $\mathcal{M}_T$ be (a closure of) the def-class of a smooth blow-up of a quintic 3-fold having a triple point.

Then these deformation families are nodes of the following connected graph

\[
\begin{array}{c}
\mathcal{M}_T \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\mathcal{M}_Q \\
\leftarrow \quad \leftarrow
\end{array}
\begin{array}{c}
\mathcal{M}_D \\
\end{array}
\]

where the two arrows are obtained as follows:

• let $Z$ be a smooth element in $\mathcal{M}_T$ and $\phi : Z \to \overline{Y}$ be the contraction of the exceptional divisor of $Z$. Then $\overline{Y}$ is a quintic 3-fold in $\mathbb{P}^4$ with a triple point. Since $\overline{Y}$ can be smoothed to a quintic 3-fold we have $\mathcal{M}_T \to \mathcal{M}_Q$.

• if we project $\overline{Y}$ from its triple point then we get a rational morphism $\psi : \overline{Y} \to \mathbb{P}^3$ which can be lifted to the blow up $Z$ of $\overline{Y}$, giving rise to a generically finite morphism $\widehat{\psi} : Z \to \mathbb{P}^3$. Consider its Stein factorization $\widehat{\psi} = f \circ \varphi$. Then we get the following commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\varphi} & \overline{X} \\
\downarrow \quad \downarrow \varphi & & \downarrow \quad \downarrow f \\
\overline{Y} & \xrightarrow{\psi} & \mathbb{P}^3
\end{array}
\]

where $f$ gives to $\overline{X}$ the structure of a double solid branched along a singular octic surface $S \subset \mathbb{P}^3$. Since $\overline{X}$ can immediately be smoothed by smoothing the branching locus $S \subset \mathbb{P}^3$ this gives the arrow $\mathcal{M}_T \to \mathcal{M}_D$.

**Conjecture 19 (of Connectedness).** The graph of (simply connected) Calabi–Yau 3-folds is connected. Then their moduli can be described by starting from the primitive nodes given by def-classes of Calabi–Yau 3-folds which do not admits any birational contraction landing to a projective normal 3-fold (in general those having Picard number 1).

A major evidence for this conjecture is given by Chiang-Greene-Gross-Kanter in [7] where the authors announced that, by computer procedure, it is possible to settle in a connected graph all known examples of Calabi–Yau hypersurfaces in a 4-dimensional weighted projective space (7555 Calabi–Yau 3-folds). Actually this web is even bigger since many weighted hypersurfaces has been connected passing through hypersurfaces of more general toric, Fano 4-dimensional varieties. Moreover, lot of arrows in the previous big connected graph are not generated by conifold transitions but they are represented by very general geometric transitions. In fact many nodes of such a big connected graph are def-classes of non simply connected Calabi–Yau 3-folds, allowing us to drop the simply connectedness hypothesis in the statement of the Connectedness Conjecture [19].

8. Deformation of a Morphism

Let $\phi : Y \to X$ be a morphism of complex spaces and let $B$ be a connected complex space with a special point $o \in B$ such that $g : (Y, Y) \to (B, o)$ and
$f : (\mathcal{X}, X) \to (B, o)$ are deformation families of $Y$ and $X$, respectively. Then a deformation family of the morphism $\phi$ is a morphism $\Phi : Y \to \mathcal{X}$ such that the following diagram commutes

$\begin{array}{ccc}
Y & \xrightarrow{\phi} & \mathcal{X} \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & B
\end{array}$

with $Y = \overline{g}^{-1}(o)$, $X = f^{-1}(o)$ and $\phi = \Phi \vert_{g^{-1}(o)}$.

Given two distinct points $b_1, b_2 \in B$ the morphism $\phi_2 := \Phi \vert_{g^{-1}(b_2)}$ is called a deformation of the morphism $\phi_1 := \Phi \vert_{g^{-1}(b_1)}$ and vice versa.

Let us now introduce a non-standard notation: the morphism deformation family $Y \xrightarrow{\Phi} X$ is called smooth if the deformation family $Y \xrightarrow{g} B$ is smooth. In this case $\phi_2$ will be called a smooth deformation of $\phi_1$ and vice versa.

8.1. Deformation equivalence of geometric transitions.

**Definition 20.** Two geometric transitions $T_1(Y_1, \overline{Y}_1, \overline{\mathcal{Y}}_1)$ and $T_2(Y_2, \overline{Y}_2, \overline{\mathcal{Y}}_2)$ are direct deformation equivalent (i.e. direct def-equivalent) if

1. $Y_1$ and $Y_2$ are both fibers of a same smooth deformation family $\mathcal{Y} \xrightarrow{f} B$ over an irreducible base $B$,
2. $\overline{Y}_1$ and $\overline{Y}_2$ are both fibers of a same smooth deformation family $\overline{\mathcal{Y}} \xrightarrow{f} \overline{B}$ over an irreducible base $\overline{B}$,
3. $\overline{Y}_1$ and $\overline{Y}_2$ are both fibers of a same deformation family $\overline{\mathcal{Y}} \xrightarrow{f} \overline{B}$ and, up to shrink $\overline{B}$, there exist a map $\varphi : \overline{B} \to B$ and a morphism deformation family to the pull-back family $\varphi^* \overline{\mathcal{Y}} = B \times_\overline{B} \overline{\mathcal{Y}}$

$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\Phi} & \varphi^* \overline{\mathcal{Y}} \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & \varphi^* \overline{\mathcal{Y}}
\end{array}$

such that $\Phi \vert_{Y_i} = \phi_i$, for $i = 1, 2$. In particular the birational contractions $\phi_1 : Y_1 \to \overline{Y}_1$ and $\phi_2 : Y_2 \to \overline{Y}_2$ are smooth deformations of each other.

With a slight abuse of notation, in the following we will denote the pull-back family $\varphi^* \overline{\mathcal{Y}}$ by $\overline{\mathcal{Y}} \xrightarrow{f} B$.

The equivalence relation of geometric transitions generated by direct def-equivalence is called def-equivalence (or deformation type) of geometric transitions. We will write $T_1 \sim T_2$ for def-equivalent geometric transitions.

Let us observe that the statement 4 at the beginning of section 7 defines what means that two nodes are connected by an arrow, but it does not give a concrete definition of what an arrow is. Actually an arrow is defined by a geometric transition connecting smooth elements of two nodes, meaning that an arrow and its defining geometric transition have to be thought of as the same object. On the other hand,
Deforming Geometric Transitions

Notice that def-equivalent transitions connect the same def-equivalence classes of Calabi–Yau 3-folds i.e. the same nodes of the Gross Calabi–Yau web. It seems then natural to redefine an arrow as follows:

**Definition 21.** An arrow of the Gross Calabi–Yau web is a def-equivalence class of geometric transitions.

As a consequence we get that

- the Gross Calabi–Yau web is a sort of quotient, up to def-equivalence, of the string theoretic Calabi–Yau web restricted to the algebraic category.

The previous Definition [20] can be simplified by putting hypothesis on the singular loci of def-equivalent geometric transitions:

**Proposition 22.** Let $T_1(Y_1, \overline{Y}_1, \tilde{Y}_1)$ and $T_2(Y_2, \overline{Y}_2, \tilde{Y}_2)$ be geometric transitions such that both $\text{Sing}(\overline{Y}_1)$ and $\text{Sing}(\overline{Y}_2)$ comprise at most isolated terminal singularities. Then $T_1$ and $T_2$ are direct def-equivalent if and only if their associated birational contractions $\phi_i : Y_i \to \overline{Y}_i$ are smooth deformations of each other.

**Proof.** Assume that $T_1$ and $T_2$ are direct def-equivalent. Then there exist two commutative diagrams

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{\phi} & Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\phi} & Y
\end{array}
$$

meaning that $\phi_1$ and $\phi_2$ are deformations of each other. Moreover $Y_1$ and $Y_2$ are direct def-equivalent Calabi–Yau 3-folds, meaning that the morphism deformation family $\Phi$ has to be smooth.

Viceversa if $\phi_1$ and $\phi_2$ are smooth deformations of each other then the deformation family $\mathcal{Y} \xrightarrow{\varphi} B$ in the previous diagrams is necessarily smooth, giving the direct def-equivalence of $Y_1$ and $Y_2$. To prove that $\overline{Y}_1$ and $\overline{Y}_2$ are direct def-equivalent Calabi–Yau 3-folds observe that, under the notation of Definition [20], the generic fibre of the pull-back family $\mathcal{Y} \xrightarrow{\varphi} B$ admits at most isolated terminal singularities. Recalling Remark [11] we are able to assume that the image $\varphi(B)$ may live over an irreducible component of $B$ belonging to the closure of a unique def-equivalence class of Calabi–Yau threefolds. Since $\overline{Y}_i$ is a smoothing of $\overline{Y}_i$ this suffices to prove that both $\overline{Y}_1$ and $\overline{Y}_2$ have to belong to the same def-equivalence class. 

Since def-equivalence is the equivalence relation generated by direct def-equivalence, the previous Proposition [22] gives immediately the following

**Corollary 23.** Let $T_1(Y_1, \overline{Y}_1, \tilde{Y}_1)$ and $T_2(Y_2, \overline{Y}_2, \tilde{Y}_2)$ be geometric transitions such that both $\text{Sing}(\overline{Y}_1)$ and $\text{Sing}(\overline{Y}_2)$ comprise at most isolated terminal singularities. Then $T_1 \sim T_2$ if and only if their associated birational contractions $\phi_i : Y_i \to \overline{Y}_i$ are connected by a finite chain of smooth morphism deformations.
Let us say that hypothesis of the previous Proposition 22 and Corollary 23 are satisfied e.g. by small geometric transitions (see the following Definition 30).

\textbf{Remark 24.} The physics “passes through” def equivalence, in the sense that it is def-equivariant. In fact a conformal field theory on a Calabi–Yau 3-fold $Y$ is the datum of a point $\chi$ on the complexified Kähler cone

$$K_C(Y) := \{\chi \in H^2(Y, \mathbb{C}) \mid \Im(\chi) \in K(Y)\} / H^2(Y, \mathbb{Z})$$

where $K(Y)$ is the Kähler cone of $Y$ and the action of integral cohomology is additive on the real part $\Re(\chi)$. In fact the imaginary part $\omega := \Im(\chi)$ gives the Kähler, Ricci-flat metric of $Y$, which is the geometric properties of the supersymmetric string theoretic vacuum, while the real part $b := \Re(\chi)$ gives the so called $b$-field describing the strings’ charge properties of the theory.

If $Y_1 \sim Y_2$ are two def equivalent Calabi–Yau 3-folds, then there exists an orientation preserving diffeomorphism $f : Y_1 \cong Y_2$. In general $f$ induces a contravariant isomorphism $f^* : K_C(Y_2) \cong K_C(Y_1)$ ([48], Main Thm.). Then the physical theories $(Y_2, \chi)$ and $(Y_1, f^*(\chi))$ are isomorphic. In a special case, it can happen that we are dealing with a couple $(Y, \chi)$ where $Y$ contains a conic bundle over an elliptic curve: in this case, if $Y'$ is a general smooth deformation of $Y$ then the associated orientation preserving diffeomorphism $f : Y' \cong Y$ gives rise to a strict inclusion $f^* : K_C(Y) \hookrightarrow K_C(Y')$ (see [49]) meaning that one can always reduce the physical theory $(Y, \chi)$ to the isomorphic general theory $(Y', f^*(\chi))$.

9. Simple geometric transitions

Since both in mathematics and in physics the conifold transitions are the most understood geometric transitions between Calabi–Yau 3–folds, it makes sense to ask \textit{when a geometric transition is def-equivalent to a conifold one}. Let us then set the following

\textbf{Definition 25} (Simple geometric transitions and arrows). A g.t. is called \textbf{simple} if it is def-equivalent to a conifold transition. Therefore an arrow is called simple if it is the def-equivalence class of a conifold transition.

\textbf{Remark 26} (The importance of being simple). If $T(Y, \overline{Y}, \tilde{Y})$ is a simple g.t. then, for what observed above:

- it is physically well understood by the Remark 24;
- there exist finite open coverings $\{U_i\}_i$ and $\{\tilde{U}_i\}_i$ of $Y$ and $\tilde{Y}$, respectively, and almost everywhere defined diffeomorphisms from $U_i$ to $\tilde{U}_i$

$$\alpha_i : U_i \setminus \text{Exc}(\phi) \cong \tilde{U}_i \setminus \mathcal{V} \,, \quad 1 \leq i \leq N \,,$$

where $\mathcal{V} \subset \tilde{Y}$ is the vanishing locus. In particular $Y$ and $\tilde{Y}$ have the same fundamental group.

In fact if $T'(X, \overline{X}, \tilde{X})$ is a conifold t. with $T \sim T'$, then there are orientation preserving diffeomorphisms $Y \cong X$ and $\tilde{Y} \cong \tilde{X}$. Then end up by applying Theorem 15 and Corollary 16. In particular the cardinality $N$ of the open coverings is that of the singular locus of $\overline{X}$, i.e. $N = |\text{Sing}(\overline{X})|$.

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9.1. **Type II geometric transitions are not simple.**

**Definition 27.** A *type II geometric transition* $T(Y, \overline{Y}, \widetilde{Y})$ is a g.t. such that

- the associated birational morphism $\phi : Y \to \overline{Y}$ is *primitive*, i.e. it cannot be factored into birational morphisms of normal varieties,
- $\phi$ contracts a divisor down to a point; in this case the exceptional divisor $E$ is irreducible and in particular it is a (generalized) *del Pezzo surface* (see [35]).

**Example 28.** The g.t. $T(Z, \overline{Y}, \widetilde{Y})$ representing the arrow $\mathcal{M}_T \to \mathcal{M}_Q$ in the Example 18, is a type II g.t.

By exhibiting a suitable weighted blow down, one can easily produce a type II g.t. $T(Y, \overline{Y}, \widetilde{Y})$ such that $Y$ and $\widetilde{Y}$ do not admit the same fundamental group. Then $T$ cannot be a simple g.t. due to the previous Remark 26 and Corollary 16. Actually a much stronger result can be established:

**Theorem 29.** A type II g.t. is never simple.

**Proof.** Let us first of all show that a type II g.t. $T(Y_1, \overline{Y}_1, \widetilde{Y}_1)$ cannot be direct def-equivalent to a conifold $T_2(Y_2, \overline{Y}_2, \widetilde{Y}_2)$. On the contrary, let us assume the existence of a deformation family of morphisms

$$
\begin{array}{c}
Y \\
\phi \\
Y \\
\downarrow f \\
\downarrow \phi \\
B
\end{array}
$$

over an irreducible base $B$ and such that $Y \xrightarrow{f} B$ is a smooth family realizing the direct def-equivalence of $Y_1$ and $Y_2$. In particular there exist two distinct points $b_1, b_2 \in B$ such that $\Phi_{b_i} := \Phi_{f^{-1}(b_i)} = \phi_i : Y_i \to \overline{Y}_i$. Since $T_2$ is conifold then $\text{Exc}(\phi_2)$ is composed by a finite number of disjoint smooth rational curves whose normal bundle is given by $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (so called $(-1,-1)$-curves). Then any $(-1,-1)$-curve is a *stable submanifold* of $Y_2$ in the sense of Kodaira [23]. For this reason, up to shrink the irreducible base $B$, we may now assume $B$ to be a suitable neighborhood $B_0$ of $b_2$ with the addition of a closure point given by $b_1$, such that $\Phi_b : Y_b \to \overline{Y}_b$ turns out to be a c.t. for any $b \in B$ with $b \neq b_1$. The contradiction is then reached by observing that $\overline{Y}_1$ is $\mathbb{Q}$-factorial while $\overline{Y}_b$ can never be $\mathbb{Q}$-factorial for any $b \neq b_1$: this fact is against a result of J. Kollár and S. Mori ([25] Thm. (12.1.10)) guaranteeing that $\mathbb{Q}$-factoriality of the fibers has to be an *open condition* for the pull-back deformation family $\mathcal{f} : \overline{Y} \to B$ i.e. that there should exist an open neighborhood of $b_1 \in B$ over which all fibers should be $\mathbb{Q}$-factorial.

Let us now assume that $T_1 \sim T_2$, meaning that there exist a finite sequence of smooth morphism deformation families connecting the birational contractions $\phi_1$ and $\phi_2$. Starting from the last family, the previous argument shows that this family cannot admit a type II birational contraction as a morphism fiber. In particular this holds for the common morphisms to the last and the penultimate families. Hence the same arguments shows that the penultimate family cannot admit a type II birational contraction as a morphism fiber, and so on until we land at the first family giving an absurd.
9.2. An example of a non-simple small geometric transition.

**Definition 30 (Small g.t.).** A g.t. $T(Y, \overline{Y}, \tilde{Y})$ is called _small_ if the associated birational morphism $\phi : Y \to \overline{Y}$ is a _small birational contraction_, i.e. its exceptional locus $\text{Exc}(\phi)$ has codimension greater than 1 in $Y$.

Possible exceptional and singular loci occurring in a small g.t. are completely classified (see [33], Thm. 6 and references therein):

- Sing($\overline{Y}$) turns out to be composed by a finite number of isolated _compound Du Val (cDV) singular points_, which in particular are terminal singularities,
- Exc($\phi$) is then composed by a finite number of trees of transversally intersecting rational curves, dually represented by ADE Dynkin graphs.

Due to the particular geometry of the exceptional locus Exc($\phi$) it is quite natural to ask for the _simplicity of any small geometric transitions_. Unfortunately this is not the case, as the following example shows.

**Example 31.** The following example is essentially due to Y. Namikawa ([29], Example 1.11).

Let $S$ be the rational elliptic surface with sections obtained as the Weierstrass fibration associated with the bundles homomorphism

$$
(0, B) : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(6) \quad (x, y, z) \quad \longrightarrow \quad -x^2 z + y^3 + B(\lambda) \ z^3
$$

for a generic $B \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))$ i.e. $S$ is the zero locus of $(0, B)$ in the projectivized bundle $\mathbb{P}(\mathcal{E})$. Then:

1. _the natural fibration_ $S \to \mathbb{P}^1$ _has generic smooth fibre and 6 distinct cuspidal fibres_,
2. _the fiber product_ $X := S \times_{\mathbb{P}^1} S$ _is a threefold admitting 6 singularities of type $\text{II} \times \text{II}$, in the standard Kodaira notation [24],
3. $X$ _admits a small resolution_ $\tilde{X} \xrightarrow{\phi} X$ _whose exceptional locus is composed by 6 disjoint couples of rational curves intersecting in one point i.e. 6 disjoint $A_2$ exceptional trees,
4. _by results of C. Schoen, X is a special fibre of the family of fiber products_ $S_1 \times_{\mathbb{P}^1} S_2$ _of rational elliptic surfaces with sections: in particular for_ $S_1$ _and_ $S_2$ _sufficiently general_ $\tilde{X} = S_1 \times_{\mathbb{P}^1} S_2$ _is a Calabi–Yau threefold giving a smoothing of_ $X$ _([41] §2).

Since $\phi$ is a small, crepant resolution, $\tilde{X}$ turns out to be a Calabi–Yau threefold and $T(\tilde{X}, X, \tilde{X})$ is a small _non–conifold_ g.t.. Let $p$ be one of the six singular points of $X$, locally defined as a germ of singularity by the polynomial

$$F := x^2 - z^2 - y^3 + w^3 \in \mathbb{C}[x, y, z, w].$$

Consider the _localization near to p_

$$
(10) \quad \xymatrix{ \hat{U}_p := \phi^{-1}(U_p) \ar@{^{(}->}[r] \ar[d]_{\phi} & \tilde{X} \ar[d]_{\phi} \ar@{^{(}->}[r] & X \\
U_p := \text{Spec} \mathcal{O}_{F, p} \ar@{^{(}->}[r] & X }
$$
which induces, since \( p \) is a rational singularity, the following commutative diagram of maps between Kuranishi spaces

\[
\begin{array}{ccc}
\text{Def}(\hat{X}) & \overset{\hat{t}_p}{\longrightarrow} & \text{Def}(\hat{U}_p) \\
\downarrow D & & \downarrow D_{\text{loc}} \\
\text{Def}(X) & \overset{t_p}{\longrightarrow} & \text{Def}(U_p) \cong T^1_{\hat{U}_p}
\end{array}
\]

where the horizontal maps are the natural localization maps while the vertical maps are injective maps induced by the resolution \( \phi \) (see [46] Propositions 1.8 and 1.12, [25] Proposition (11.4)). Then, by explicit calculations (see [39], Thm. 4), it turns out that

\[
\dim \text{Def}(\hat{U}_p) = 1 \\
\Im(l_p) \cap \Im(D_{\text{loc}}) = 0 \implies \Im(\hat{l}_p) = 0
\]

meaning that

9.3. **A necessary condition for simplicity of small transitions.** The previous example allows us to understand some further necessary condition that a small g.t. should satisfy to be a simple g.t.:

**Proposition 32.** Recall the definition of \( \Theta_* \) as the ‘tangent’ sheaf. Then if \( T(Y, Y', \tilde{Y}) \) is a simple small geometric transition of Calabi–Yau 3-folds then

\[
h^1(Y, \Theta_Y) < h^1(Y, \Theta_Y')
\]

**Proof.** The proof is an application of R. Friedman techniques presented in [13]. In fact by the Leray spectral sequence applied to the birational small contraction \( \phi : Y \to \tilde{Y} \) and the local to global spectral sequence relating the global \( \mathcal{T}_\varphi := \text{Ext}^*(\Omega_{\tilde{Y}}, \mathcal{O}_{\tilde{Y}}) \) with the sheaves \( \Theta_\varphi := \mathcal{E}x t^*(\Omega_{\tilde{Y}}, \mathcal{O}_{\tilde{Y}}) \), one gets the following commutative diagram

\[
\begin{array}{cccc}
0 & \to & H^1(Y, R^0\phi_*\Theta_Y) & \overset{\lambda}{\longrightarrow} & H^0(\tilde{Y}, R^1\phi_*\Theta_Y) & \longrightarrow & \cdots \\
& & \downarrow \delta & & \downarrow \delta_{\text{loc}} & & \\
0 & \to & H^1(\tilde{Y}, \Theta_{\tilde{Y}}) & \overset{\lambda}{\longrightarrow} & H^0(\tilde{Y}, \Theta^1_{\tilde{Y}}) & \longrightarrow & \cdots
\end{array}
\]

where

- the vertical equality comes from an application of Hartogs Theorem giving \( R^0\phi_*\Theta_Y \cong \Theta_{\tilde{Y}} \) ([13] Lemma (3.1)),
- the vertical morphism \( \delta \) is the differential of an injective map between Kuranishi spaces \( \text{Def}(Y) \hookrightarrow \text{Def}(\tilde{Y}) \), constructed by J.M. Wahl [46] §1 (see also [13] Prop. (2.1)) and [25] Prop. (11.4)), which turns out to be still injective since \( \delta_{\text{loc}} \) is injective,
- \( \delta_{\text{loc}} \) is the localization of \( \delta \) near to \( \text{Sing}(\tilde{Y}) \), which is injective by a result of Friedman ([13], Prop. (2.1)).
By (2), $T^{1}_{Y} \cong H^{1}(Y, \Theta_{Y})$ and the fact that $T$ is a simple g.t. guarantees the existence of a global deformation of $Y$ inducing a first order deformation $\xi \in H^{1}(Y, \Theta_{Y})$ of $Y$ such that $\delta_{loc} := \delta_{loc} \circ \lambda(\xi)$ gives non trivial first order deformations of any singularity $p \in \text{Sing}(\overline{Y})$ to ordinary double points. Since $\delta_{loc}$ is injective, this means that $\text{Im} \lambda \neq 0$ and the exactness of the upper sequence in (13) gives necessarily the cohomological condition (13). □

**Remark 33.** Back to the Namikawa's example [31] let us observe that for the g.t. $T(\tilde{X}, X, \tilde{X})$, where $X = S \times_{p1} S$ is the fibred self-product of a cuspidal elliptic surface, one gets $h^{1}(TM, \Theta_{M}) = h^{1}(Y, \Theta_{Y}) = 3$.

**9.4. An example of a simple small geometric transition.** Let us consider the singular quintic threefold $Q \subset \mathbb{P}^{4}$ given by

$$u(u - 2x)(u - 3y)(x^{2} - y^{2}) - (z^{5} - w^{5}) = 0 \ .$$

The singular locus $\text{Sing}(Q)$ is composed by 10 isolated hypersurface singularities, each of them analytically equivalent to the one described by the local equation

$$x^{2} - y^{2} = z^{5} - w^{5}$$

which is a $cA_{4}$ singular point whose Milnor and Tyurina numbers are equal to 16.

A resolution of this singular point is obtained by a successive blow up of the planes

$$\pi_{i} : x - y = z - \epsilon^{i}w = 0 \ , \ 0 \leq i \leq 3 \ , \ \epsilon^{5} = 1 \ .$$

More precisely: blow up $\mathbb{C}^{4}$ along $\pi_{0}$, then blow up again along the strict transform of $\pi_{1}$ and so on. At the end look at the strict transform of the singularity, which carries an exceptional locus composed by a tree of 4 lines dually represented by the Dynkin graph $A_{4}$.

We are now in a position to construct a non-conifold geometric transition as follows:

- **the resolution:** the quintic threefold $Q$ admits a global resolution $\hat{Q}$ which can be obtained by the successive blow up of 16 planes

  $$\pi_{i}^{j} : l_{j} = z - \epsilon^{i}w = 0 \quad 0 \leq i \leq 3 \ , \ 1 \leq j \leq 4$$

  where $\{l_{1}, \ldots, l_{4}\} \subset \{u, u - 2x, u - 3y, x - y, x + y\}$.

- **the smoothing:** it is obviously given by a smooth quintic threefold $Q \subset \mathbb{P}^{4}$.

This gives the g.t. $T(\hat{Q}, Q, Q)$. To deform $T$ to a conifold transition consider the following deformation $\overline{Q}_{(a,b,c)}$ of $Q$

$$u(u - 2x)(u - 3y)(x^{2} - y^{2}) - (z - w)(z - \epsilon w)(z - \epsilon^{2}w + a)(z - \epsilon^{3}w + b)(z - \epsilon^{4}w + c) = 0$$

which, for a general $\alpha := (a, b, c) \in \mathbb{C}^{3}$, splits up each singular point of $Q$ into 10 nodes, hence giving 100 nodes. Since the deformation $\overline{Q}_{\alpha}$ respects the factorization in the equation of $Q$, it lifts to a deformation $\hat{Q}_{\alpha}$ of the resolution $\hat{Q}$ splitting up every exceptional $A_{4}$ tree into 10 disjoint lines. This gives a deformation family of morphisms

$$\hat{Q} \xrightarrow{\phi} \overline{Q} \xrightarrow{f} \mathbb{C}^{3} \xrightarrow{g} \overline{Q}$$

hence a def-equivalence $T \sim T_{\alpha}(\hat{Q}_{\alpha}, \overline{Q}_{\alpha}, Q)$. Let us further observe that the deformations $\overline{Q}_{\alpha}$ are not all distinct up to isomorphisms: if we consider the Kuranishi
space $T^1$ of any singularity of $\overline{Q}$, there is a well defined map $\mathbb{C}^3 \rightarrow T^1$ whose image is 1-dimensional. This is enough to show that

$$\dim \left( \text{Im} \left( \lambda : T^1_{\overline{Q}} \rightarrow H^0(\overline{Q}, R^1\phi_\ast\Theta_{\overline{Q}}) \right) \right) = 1$$

giving $h^1(\Theta_{\overline{Q}}) = 17 < 18 = h^1(\Theta_{\overline{Q}})$, which is consistent with Proposition 32.

The further main invariants of the g.t. $T$ and the conifold t. $T_\alpha$ are listed in the following table:

| Variety | $h^1(\Theta_\cdot)$ | $b_2$ | $\rho$ | $b_3$ | $b_4$ | $\chi$ |
|---------|------------------|-------|-------|-------|-------|-------|
| $\overline{Q}, \overline{Q}_\alpha$ | 18 | 17 | 17 | 36 | 17 | 0 |
| $\overline{Q}$ | 17 | 1 | 1 | 60 | 17 | -40 |
| $\overline{Q}_\alpha$ | 18 | 1 | 1 | 120 | 17 | -100 |
| $Q$ | 101 | 1 | 1 | 204 | 1 | -200 |

They can be computed from the well known invariants of the smooth quintic three-fold $Q$ by means of relations given in [38], Thm. 7.

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