Energy Density of a Dissipative Polarizable Solid by a Lagrangean Formalism

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Abstract

A Lagrangean for the dynamics of an electromagnetic field in a dispersive and dissipative material is constructed (adapting some ideas by Bekenstein and Hannay) and an expression for the energy density that is positive is obtained from it. The expression contains extra (sink) degrees of freedom that represent dissipating modes. In simplified cases the sink modes can be eliminated to yield an energy density expression in terms of the electromagnetic fields, the polarization and the magnetization only, but which contains parameters associated with the sink modes. The method of adding extra modes can be used to set up a Lagrangean formalism for dissipative systems in general, such that will reinstate time-translation invariance and will yield a unique energy density.

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3 Background

The problematic nature of the electromagnetic field energy in a dissipative material is apparent already at a graduate-teaching level: a frequently used textbook notes that in a dispersive medium the energy density lacks a thermodynamical interpretation [1]. (A dispersive material is necessarily dissipative, since by the Kramers-Kronig relations its constitutive constants, that are by definition frequency (ω) dependent, must have an imaginary part, which represents absorption or energy loss.) The expression for the energy density that was derived in [1], was valid only for fields that are nearly mono-chromatic. For other basic treatments we quote textbooks as [2]-[4], and note that the last reference labels the expression obtained under the above conditions as an effective energy density.

We write the relations between the electric field and the displacement using the frequency dependent permittivity $\epsilon(\omega)$ and between the magnetic field and the induction using the permeability $\mu(\omega)$, as

$$\vec{D}(\vec{x}, \omega) = \epsilon(\omega) \vec{E}(\vec{x}, \omega), \quad \vec{B}(\vec{x}, \omega) = \mu(\omega) \vec{H}(\vec{x}, \omega).$$  \hspace{1cm} (1)

Then according to [4] we obtain an effective energy density:

$$u_{\text{eff}} = \text{Re} \left[ \frac{d(\omega \epsilon)}{d\omega}(\omega_0) \right] \langle \vec{E}(\vec{x}, t) \cdot \vec{E}(\vec{x}, t) \rangle + \text{Re} \left[ \frac{d(\omega \mu)}{d\omega}(\omega_0) \right] \langle \vec{H}(\vec{x}, t) \cdot \vec{H}(\vec{x}, t) \rangle$$ \hspace{1cm} (2)

The brackets $\langle \rangle$ designate an averaging over a period of $\frac{2\pi}{\omega_0}$, where $\omega_0$ is the carrier frequency.

Alternative expressions that were subsequently proposed were controversial. One derivation, that required a significant departure from standard electromagnetic theory [5], postulated the independence of the expression of particular material constants. This was strongly criticized in [6], where the energy-density formula contained explicitly parameters that were present in the equation of motion for the field. Another derivation, in terms of constitutive constants [7], was found to lead to energy densities that are negative for a medium with a narrow resonance [8].

The quest for a non-controversial energy density can be regarded from a more fundamental angle. This quantity is expected to be part of a conservation equation, which (by a well-known theorem due to Noether) is tied to the time invariance of the Lagrangean. As soon as the invariance is lost (which
is the case for dissipative systems), the quantity to be conserved is undefined. Thus the need to find an appropriate energy density is expected to arise in a wider context, too. (In a dramatic account, in which the proposer of a time-varying light-velocity describes his Iliad to get his idea published, one also finds the problem of the proper Lagrangean formulation to play a key role [9].)

The issue of the electromagnetic-field energy density in a dispersive and dissipative medium has recently resurfaced in the contexts of the subluminality of light-propagation and of left handed materials ([10],[11]). The former work utilizes some analytical properties of the constitutive relations, while the departure point in the latter are the equations of motions for the macroscopic polarizabilities. As an application, the energy density for a left handed medium has been calculated in [11]. The expressions in the two papers differ.

Our approach is based on the recognition that in standard field theories (classical or quantal) the energy momentum tensor satisfies a conservation equation having the form of a "div"-equation involving all degrees of freedom. Thus, it seems, that one should be able to write out an energy-density in an unambiguous fashion starting with a Lagrangean. (The idea of a Lagrangean formulation was raised earlier [12], but only in a programmatic manner and by making an assumption that the detailed treatment worked out in this paper does not justify.)

As is well known, the energy density $T^0_0$ is a component of the energy momentum tensor $T^k_j$, which is uniquely derivable from the Lagrangean density $\mathcal{L}(u_k, u_{k,j})$, this being a function of the field variables $u_k$ and of their derivatives $u_{k,j}$. (The indexes take the value 0 for the time component and the numbers (1, 2, 3) for the remaining, space components. Summation is implied for repeated indexes and a symbol after the comma represents a derivative. The dot will also be used for a time derivative.) The formal definition is given by

$$T^k_j = \frac{\partial \mathcal{L}}{\partial u_{i,k}} u_{i,j} - \mathcal{L} \delta^k_j$$

Two circumstances appear to prevent one from deriving the energy-density from the conservation equation in a dissipative material. First, a Lagrangean has not been formulated for the equations of motion (but this is done below). Secondly, the energy sinks involved in the dissipative mechanisms have not been given a dynamic representation, but only a phenomenological one (see equation (1)), namely, through the appearances of complex
permittivity and permeability in the constitutive relations or, equivalently, through the presence of time-reversal non-invariant terms in the equations of motion.

The first issue was recently resolved in a short note [13], where it was shown how to formulate the Lagrangean (actually, the Hamiltonian) for a dissipative case. The second point was (indirectly) addressed in [14], following a previous publication in [15]. These works formulated the dynamics of a (conjectural) time dependent fine-structure constant by including in the Universe an additional degree of freedom. We follow up both these approaches, with changes required by the different context.

4 A Lagrangean for static polarization and magnetization

Before approaching the general problem we will first present a Lagrangean density for the electromagnetic field in a material which has a static polarization and magnetization. The following expression connects the electric displacement field $\vec{D}$ with the electric field $\vec{E}$ and polarization $\vec{P}$:

$$\vec{D} = \epsilon \vec{E} + \vec{P}$$  \hspace{1cm} (4)

($\epsilon$ is not necessarily the vacuum permittivity $\epsilon_0$). Similarly the magnetic field $\vec{H}$ is connected with the magnetic induction field $\vec{B}$ and magnetization $\vec{M}$ by:

$$\vec{H} = \frac{1}{\mu} \vec{B} - \vec{M}$$  \hspace{1cm} (5)

(again $\mu$ is not necessarily the vacuum inverse permeability $\mu_0$). The above fields satisfy both the homogeneous Maxwell’s equations:

$$\text{curl} \vec{E} + \dot{\vec{B}} = 0, \quad \text{div} \vec{B} = 0$$  \hspace{1cm} (6)

and the inhomogeneous equations:

$$\text{curl} \vec{H} - \dot{\vec{D}} = \vec{J}, \quad \text{div} \vec{D} = \rho$$  \hspace{1cm} (7)

In the following we assume that both free charges $\rho$ and currents $\vec{J}$ are zero. The above equations can not be obtained from a Lagrangean density expressed in terms of those fields. However, this situation can be amended
by representing the fields in terms of vector $\vec{A}$ and scalar $\Phi$ potentials, as follows:

$$\vec{E} = -\nabla \Phi - \dot{\vec{A}}, \quad \vec{B} = \text{curl} \vec{A}$$  \hspace{1cm} (8)

Using these definitions we see that the homogenous equations (6) are satisfied automatically. The inhomogeneous equations (7) can be obtained from the functional derivative of the Lagrangean:

$$\mathcal{L} = \mathcal{L}_{EB} + \mathcal{L}_{PE} + \mathcal{L}_{MH}$$  \hspace{1cm} (9)

$$\mathcal{L}_{EB} = \frac{1}{2} \epsilon \vec{E}^2 - \frac{\vec{B}^2}{\mu}$$  \hspace{1cm} (10)

$$\mathcal{L}_{PE} = \vec{P} \cdot \vec{E}$$  \hspace{1cm} (11)

$$\mathcal{L}_{MH} = \mu \vec{M} \cdot \vec{H}$$  \hspace{1cm} (12)

in which the reader should think of all field quantities as given in terms of the potentials of equation (8). Using equation (3) one can obtain the energy density:

$$T^0_0 = \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} \cdot \dot{\vec{A}} - \mathcal{L}.$$  \hspace{1cm} (13)

which yields for the Lagrangean of equation (12) after some manipulations the result:

$$T^0_0 = \frac{1}{2}[\epsilon \vec{E}^2 + \mu \vec{H}^2 + \mu \vec{M}^2]$$  \hspace{1cm} (14)

The last term $\mu \vec{M}^2$ can be usually discarded for static magnetization since it represents nothing but a time-independent constant. However, we include it for future sections in which the polarization and magnetization will become dynamic degrees of freedom.

**5 Equations of motion for polarization and magnetization**

In cases that the polarization $\vec{P}$ is induced by an electric field $\vec{E}$, and the magnetization $\vec{M}$ is induced by a magnetic field $\vec{H}$, one needs a set of equations to describe these processes. The equations are given, e.g., in [11] as:

$$\ddot{\vec{P}} + \Gamma_e \dot{\vec{P}} + \omega_e^2 \vec{P} = \epsilon_0 \omega_p^2 \vec{E}$$  \hspace{1cm} (15)

$$\ddot{\vec{M}} + \Gamma_h \dot{\vec{M}} + \omega_0^2 \vec{M} = F \omega_0^2 \vec{H}$$  \hspace{1cm} (16)
with all symbols and notation as defined in [11]. In addition, one has the Maxwell-equations given in the previous sections and derived from the Lagrangean of equation (12) using the contravariant four-vectors $A^\alpha = (\vec{A}, \Phi)$.

Turning now to dissipative cases, following [13] -[15] we introduce a set of new fields, here named canonical fields, which are formally distinguished from the previous, physical fields by writing them in lower case symbols. These fields (assumed to be real) are made to be part of a Hamiltonian or Lagrangean formulation, in contrast to the preceding physical quantities (written in upper case symbols), with which this cannot be done [13]. Explicitly, we shall work with the scaled polarization $p^\alpha$ and magnetization $m^\alpha$; the electromagnetic fields $e^\alpha$, $h^\alpha$, the (contravariant) vector-potentials $a^\alpha$, and (again following [13]) also the fields $r^\alpha$, $n^\alpha$ conjugate to $p^\alpha$ and to $m^\alpha$ (as defined below). The choice of the fields (and of the various constant, scaling factors) is guided by the requirement that we recapture the form of the equations of motion and the energy density currently widely employed in solid state optics in the appropriate limiting case of no dissipation. (This requirement is at times detrimental to the formal symmetry between the modes linked with the polarization and the magnetization, respectively.) The physical meaning of the canonical fields will be made clear later by comparing their equations of motion with those of the physical fields. (For a similar procedure see [13].) In addition, we shall introduce two new scalar fields: first $\Psi$, representing a degree of freedom associated with the dissipation of the polarization (analogous to the degree of freedom denoted with the same symbol in [15]) and, secondly, the ”sink-field” $\Phi$ for the dissipation in the magnetic mode.

6 The dissipative Lagrangean

This has the following parts:

$$\mathcal{L} = \mathcal{L}_{eb} + \mathcal{L}_{pe} + \mathcal{L}_{mh} + \mathcal{L}_{pr} + \mathcal{L}_{mn} + \mathcal{L}_{\Psi} + \mathcal{L}_{\Phi} + \mathcal{L}_{pr-\Psi} + \mathcal{L}_{mn-\Phi} \quad (17)$$

This Lagrangean contains first the Lagrangean of equation (12), there written in terms of the electromagnetic fields, and now re-expressed in the new lower-case variables as $\mathcal{L}_{eb} + \mathcal{L}_{pe} + \mathcal{L}_{mh}$. In addition, the above expression contains the Lagrangeans in the polarization modes, the magnetization modes, the electromagnetic fields, the $\Psi$ and $\Phi$ sink fields and then the interaction-Lagrangeans between polarization and electric field, magnetization and magnetic field and, finally, the polarization and magnetization with
having chosen the polarization and magnetization vectors to be along the $x$ (or 1) axis. Physically, the new degrees of freedom $\Psi$ and $\Phi$ can be associated with some sort of relaxation mechanism for the polarization and the magnetization, respectively. In the above, we have chosen for $L_\Psi$ and $L_\Phi$ what appear to us the simplest type of expressions that can represent unspecified (and, so far, arbitrary) degrees of freedom. The symbols $\kappa, \rho, \lambda, \nu$ stand for constant, non-negative coefficients, whose values depend on the nature of the relaxation mechanisms.

Equating to zero the variation of $L$ with respect to each degree of freedom leads to the equations of motion, in accordance with the Euler-Lagrange equations. For the variables $r$ and $n$ the equations obtained are:

$$r = \dot{p} - p\dot{\Psi}$$  
$$n = \dot{m} - m\dot{\Phi}$$

which can be inserted into the equations for $p$ and $m$. These take now the following form:

$$\ddot{p} - (\dot{\Psi} + \dot{\Psi}^2 - \omega_r^2)p = \epsilon_0 \omega_p^2 e_1$$  
$$\ddot{m} - (\dot{\Phi} + \dot{\Phi}^2 - \omega_0^2) m = F \omega_0^2 h_1$$
The electromagnetic equations of motion are identical to Maxwell’s equations in the new (small letter) variables. Finally, for the sink coordinates one has

\[ \ddot{\psi} + \frac{\rho}{\kappa} \psi = \frac{1}{\kappa \epsilon_0 \omega_p^2} \frac{\partial (\rho r)}{\partial t} \]  
\[ \ddot{\Phi} + \frac{\nu}{\lambda} \Phi = \frac{\mu_0}{\lambda \omega_0^2} \frac{\partial (m \phi)}{\partial t} \]  
\[ \text{(31)} \]

From equation (3) we obtain the expression for the energy density \( T_0 \), namely,

\[ T_0 = \sum_k \frac{\partial L}{\partial \dot{u}_k} \dot{u}_k - \mathcal{L} \]  
\[ \text{(33)} \]

where the sum is over all degrees of freedom. We separate the electromagnetic, the polarization and the magnetization parts

\[ T_0 = (T_0^e,h) + (T_0^p,\Psi) + (T_0^m,\Phi) \]  
\[ \text{(34)} \]

where the first term has the well known form of the electromagnetic energy density given in equation (14) and repeated here for completeness:

\[ (T_0^e,h) = \frac{\epsilon_0}{2} \epsilon^2 + \frac{\mu_0}{2} \hbar^2 + \frac{\mu_0}{2} m^2 \]  
\[ \text{(35)} \]

For the others we obtain

\[ (T_0^p,\Psi) = \frac{1}{2 \epsilon_0 \omega_p^2} [\dot{\rho}^2 + \omega_r^2 \rho^2] + \frac{\kappa}{2} \dot{\psi}^2 + \frac{\rho}{2} \psi^2 \]  
\[ = \frac{1}{2 \epsilon_0 \omega_p^2} (\dot{\rho}^2 + \omega_r^2 \rho^2) + \frac{\kappa}{2} \dot{\psi}^2 + \frac{\rho}{2} \psi^2 + \frac{1}{2 \epsilon_0 \omega_p^2} \rho^2 \dot{\psi}^2 \]  
\[ - \frac{1}{2 \epsilon_0 \omega_p^2} \frac{\partial \rho^2}{\partial t} \dot{\psi} \]  
\[ \text{(36)} \]

and

\[ (T_0^m,\Phi) = \frac{\mu_0}{2 F \omega_0^2} [(\dot{m} - m \dot{\Phi})^2 + \omega_0^2 m^2] + \frac{\lambda}{2} \dot{\phi}^2 + \frac{\nu}{2} \Phi^2 \]  
\[ = \frac{\mu_0}{2 F \omega_0^2} (\dot{m}^2 + \omega_0^2 m^2) + \frac{\lambda}{2} \dot{\phi}^2 + \frac{\nu}{2} \Phi^2 + \frac{\mu_0}{2 F \omega_0^2} m^2 \dot{\phi}^2 \]  
\[ - \frac{\mu_0}{2 F \omega_0^2} \frac{\partial m^2 \Phi}{\partial t} \]  
\[ \text{(38)} \]
The first-written form (=sum of squares with non-negative coefficients) of either quantity guarantees that each part of the energy density is positive (non-negative). The usual expressions for the energy density, e.g. in [4] or [11], differ from the above by the presence of the terms in $\Phi$ and $\Psi$ and their time derivatives. In the following section it is our purpose to eliminate these variables by making use of the equations of motion, equation (31) and equation (32).

7 A solvable case

We take a simplified case when the coefficients $\rho$ and $\nu$ are both zero. Then the equation of motion for the polarization sink-mode equation (31) can be integrated. (A similar procedure applies to the magnetization sink-variable.) We assume the following initial conditions for $\Psi(t)$

$$\Psi(0) = 0, \quad \dot{\Psi}(0) = \Gamma_e / 2$$  \hfill (40)

The reason for these choices is that with them for short times, $t << \frac{2}{\Gamma_e}$, the differential equation (15) is regained. (This will be shown presently. Cf. also [13].) Then from equation (31)

$$\ddot{\Psi} = \frac{1}{2}\kappa_0\omega_p^2 \left( \frac{\partial^2}{\partial t^2} p^2(t) - \frac{1}{\kappa_0\omega_p^2} \frac{\partial (p^2(t) \dot{\Psi})}{\partial t} \right)$$  \hfill (41)

Integrating once

$$\dot{\Psi} = \frac{1}{2}\kappa_0\omega_p^2 \left( \frac{\partial p^2(t)}{\partial t} - \frac{p^2(t) \dot{\Psi}}{\kappa_0\omega_p^2} \right) + C$$  \hfill (42)

leading to

$$(1 + \frac{p^2(t)}{\kappa_0\omega_p^2}) \dot{\Psi} = \frac{1}{2}\kappa_0\omega_p^2 \left( \frac{\partial p^2(t)}{\partial t} \right) + C$$  \hfill (43)

with the (first) integration constant given by

$$C = \frac{\Gamma_e}{2} \left( 1 + \frac{p^2(0)}{\kappa_0\omega_p^2} \right) - \frac{p(0) \dot{p}(0)}{\kappa_0\omega_p^2}$$  \hfill (44)

Integrating once more and arranging for satisfaction of the first initial condition in equation (40), we finally obtain:

$$\Psi(t) = \frac{1}{2} \ln \left[ \frac{1 + \frac{p^2(t)}{\kappa_0\omega_p^2}}{1 + \frac{p^2(0)}{\kappa_0\omega_p^2}} \right]$$
When this expression and equation (42) are substituted into equation (37), one obtains after considerable simplification the following expression for the energy density arising from the time varying "canonical" polarization \( p(t) \):

\[
(T_0^0)_{p,\psi} = \frac{1}{2\epsilon_0\omega_p^2} \left( \frac{\dot{p}^2(t) + \kappa\epsilon_0\omega_p^2C^2}{1 + \frac{p^2(t)}{\kappa\epsilon_0\omega_p^2}} + \omega_p^2p^2(t) \right) \quad (46)
\]

When we assume that \( \Gamma_e \) is small (precisely, \( \Gamma_e\sqrt{\kappa} \ll 1 \)), and so is \( \dot{p}(0) \) (this will be confirmed in the next section), then \( C^2 \) will be a second order correction which may be neglected:

\[
(T_0^0)_{p,\psi} = \frac{1}{2\epsilon_0\omega_p^2} \left( \frac{\dot{p}^2(t)}{1 + \frac{p^2(t)}{\kappa\epsilon_0\omega_p^2}} + \omega_p^2p^2(t) \right) \quad (47)
\]

A similar expression is obtained for the part of the energy density involving the "canonical" magnetization \( m(t) \), the variables \( n(t) \) and \( \Phi(t) \) having been eliminated through their equations of motion,

\[
(T_0^0)_{m,\Phi} = \frac{\mu_0}{2F\omega_0^2} \left( \frac{\dot{m}^2(t)}{1 + \frac{\mu_0m^2(t)}{xF\omega_0^2}} + \omega_0^2m^2(t) \right) \quad (48)
\]

The above expressions, equation (47) and equation (48), are quite similar to those in equation (11) of Ruppin [11], except that they are written in the canonical (small letter) variables, rather than in the physical variables (for relations between these, see immediately below), and that they contain time dependent denominators.

The main results of this work, equation (46) and its analogue for the magnetization energy density, are exact and contain nonperturbative corrections to the energy density, due to the presence of the sink degrees of freedom. While exact, they are model dependent in the sense that sinks represented by different Lagrangeans would lead to different energy densities. This is clear, due to the presence in the energy densities of the parameters \( \kappa \) and \( \lambda \) that were introduced in the Lagrangean in equation (23) and equation (24). It is of interest to note that the non-dissipative limit is not regained when \( \Gamma_e,\Gamma_h \to 0 \), but only when also \( \kappa,\lambda \to \infty \).
8 The physical fields

To recapture the basic equations of motion equation (15) and equation (16) for the physical polarization variable, we proceed as follows: We postulate

\[ p(t) = e^{\Psi(t)} P(t) \] (49)
\[ m(t) = e^{\Phi(t)} M(t) \] (50)

Then, from equation (40), for short times \( 0 < t << \frac{2}{\Gamma_e} \),

\[ \Psi \approx \Psi(0) + \dot{\Psi}(0)t = \Gamma_e t/2 \] (51)

This turns (the vector form of) equation (29) into the following:

\[ \ddot{\vec{P}} + \Gamma_e \dot{\vec{P}} + \omega_e^2 \vec{P} = \epsilon \omega_p^2 e^{-\frac{\Gamma_p t}{\Gamma_e}} \vec{e} \] (52)

and likewise for the magnetization variables. Recalling equation (15) and equation (16), we can thus extrapolate to later times so as to identify

\[ \vec{e}(t) = e^{\Psi(t)} \vec{E}(t) \] (53)
\[ \vec{m}(t) = e^{\Phi(t)} \vec{H}(t) \] (54)

Thus, all the "canonical" variables are the fields in which the decay of the physical field variables has been reinstated. On the other hand, the decay is itself dependent on the fields. (Cf. [12].) Furthermore, Maxwell’s equations for the physical fields are also modified, just as in [14] and [15] (eq. (14) and eq. (12), respectively).

9 Conclusion

Using standard Lagrangean formalism, we have obtained a unique energy density for a dissipative medium (capable also of sustaining an electric polarization and/or a magnetic one). It has been found necessary to introduce two additional degrees of freedom ("sink-modes"), associated with decay mechanisms in the electrical and magnetic modes. The derived contributions to the energy density, shown in equation (46) - equation (48), are all positive and are similar in their form to the corresponding results in [11]. They contain particular physical parameters, as anticipated in [6]. The energy density is part of a conservation equation involving also an energy-current (momentum) density (the Poynting vector) formally given by the
tensor-components $T^a_0(a = 1, 2, 3)$ shown in equation (3). These properties appear to be true in general, namely, for a large variety of systems that are dissipative.

A solvable model has been worked out in this paper. Both in this case and in a general one, the energy density has corrections not appearing in other approaches not based on a Lagrangean. On the other hand, in a spatially homogeneous infinite medium (such as treated in this paper) the momentum density is still the Poynting vector. This can be seen from equation (3), since the second term in that expression is absent (for $\alpha$ differing from $\beta$) and there are no space derivatives in the Lagrangean, other than in the electromagnetic part. (As a consequence, the Lagrangean is not fully Lorentz-invariant.)

For a medium possessing space varying properties (including, under certain circumstances, a finiteness of size) the procedure outlined in this paper can be extended to space-coordinates. This means the introduction of new, space-dependent "sink" degrees of motion (e.g., at the boundaries). As a consequence, additional terms would appear also in the momentum density. Apart from an interesting extension of the Poynting vector concept, the suggested approach can also have the practical use of solving electromagnetic problems involving dissipation, such as laser gain and loss calculations, microwave losses in wave guides and cavities and transformer performance. The approach can be further extended to other fields of macroscopic physics such as viscous fluid dynamics. Work in these directions is under progress.

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