REAL INFLECTION POINTS OF REAL HYPERELLIPTIC CURVES

INDRANIL BISWAS, ETHAN COTTERILL, AND CRISTHIAN GARAY LÓPEZ

ABSTRACT. Given a real hyperelliptic algebraic curve $X$ of genus $g \geq 2$ with non-empty real part and $[D] \in \text{Pic}^2 X$ its $g_2^1$, we study the real inflection points of the complete real linear series $|L_k(kD)|$ for $k \geq g$.

To do so we use Viro’s patchworking of real plane curves, recast in the context of some Berkovich spaces studied in [5]. This notably is an alternative to limit linear series on metrized complexes of curves, as studied in [6].

1. Introduction

Degeneration has long been a potent tool in the study of linear series on projective algebraic curves. In the 1880’s Castelnuovo used degenerations to irreducible rational nodal curves; one hundred years later Eisenbud and Harris [2] introduced a theory of limit linear series to deal systematically with degenerations to (abstract) reducible curves. Around the same time that Eisenbud and Harris developed their theory, Viro [8] introduced his patchworking degeneration for hypersurfaces embedded in toric varieties. In Viro’s construction the limit is not algebraic in the usual sense, but comes naturally equipped with a sheaf of piecewise-linear regular functions. This limit is combinatorially robust, and for small values of the deformation parameter the algebraic hypersurfaces retain (some of) its properties. Viro’s construction plays a key rôle in tropical geometry and in real algebraic geometry, and notably has been used successfully to study linear series over the real numbers.

The second author has used Viro’s patchworking construction in [1] to study real inflection points of real canonical curves of genus four in $\mathbb{P}^3$. There, the limiting object was equal to the dual graph of a necklace of elliptic curves; by combining the real inflectionary behavior of a single elliptic curve with Viro’s patchworking construction he was able to exhibit canonical curves of genus four in $\mathbb{R}\mathbb{P}^3$ with 30 real inflection points. In doing so, he built on earlier work of Brugallé and López-Medrano [7] who used Viro’s patchworking to systematically construct real algebraic plane curves with the maximal possible number of real inflection points. In this paper, we will apply an enhanced version of patchworking to exhibit real maximally-inflected linear series on hyperelliptic curves of arbitrary genus. More precisely, by combining patchworking with a Berkovich-analytic construction of Jonsson [5] we exhibit a metrized complex of curves, in the sense of [6], as the analytic limit of a family of hyperelliptic curves. As systems of linear series along the smooth curve components of a metrized complex, the corresponding limits of linear series arising from our construction are similar to those of [6]. Our construction is, however,
more explicit, as each curve component is the normalization of a curve defined by an equation. The explicit nature of our construction should be useful more generally for probing the geometry of linear series on curves embedded as complete intersections in toric varieties.

The main results of this work are as follows. Given a real hyperelliptic algebraic curve $X$ of genus $g \geq 1$ with non-empty real part, let $[D] \in \text{Pic}^2 X$ be a $g^1_2$ on $X$. Whenever $1 \leq k \leq g$, the real complete linear series $|\mathcal{L}_R(kD)|$ is a $g^1_{2k}$: let $w_\mathcal{L}(g,k) = k(k+1)(g+1)$ denote the total number of inflection points of its complexification $|\mathcal{L}(kD)|$.

In Theorem 1.1, we give a complete characterization of the distribution of the real inflection points of $|\mathcal{L}_R(kD)|$ along the connected components of the real part of $X$ for $1 \leq k \leq g$. In particular, we deduce the following result.

**Theorem 1.1.** Let $X$ be a real hyperelliptic curve of genus $g \geq 1$ with $n(X) \geq 1$ real connected components and let $1 \leq k \leq g$. The real linear series $|\mathcal{L}_R(kD)|$ has precisely $\frac{n(X)}{g+1}w_\mathcal{L}(g,k) = k(k+1)n(X)$ real inflection points.

Whenever $k > g$, the real complete linear series $|\mathcal{L}_R(kD)|$ is a $g^1_{2k-g}$. We show the following result.

**Theorem 1.2.** The real linear series $|\mathcal{L}_R(kD)|$ has at least $g(g+1)n(X)$ real inflection points.

Theorem 1.2 is explained by a more general fact that we prove via a local analysis of vanishing orders of holomorphic sections of $|\mathcal{L}_R(kD)|$, namely that each point of the ramification locus of the hyperelliptic cover $X \to \mathbb{P}^1$ has inflectionary weight $(\frac{g+1}{2})$. The remainder of our results are related to our Jonsson and Viro-based construction of limit linear series on a (marked) metrized complex of elliptic curves. In Lemma 5.2, we show how the Jonsson–Viro degeneration produces a $g^1_{2k-g}$ along each elliptic component $E_i$, $i = 1, \ldots, g$ of the metrized complex. In Theorem 5.3, we calculate the vanishing sequences for holomorphic sections in each of the four marked points along $E_i$, including those points of attachment corresponding to edges linking neighboring elliptic components. One upshot (see Corollary 5.4) of Theorem 5.3 is that our limit linear series satisfy the natural analogue of the compatibility condition for vanishing sequences in points of attachment that characterizes Eisenbud–Harris limit linear series. Another is an explicit calculation of the inflectionary weight contributed by the marked points along each $E_i$. Further, in Theorem 5.8, we prove a regeneration-type result that specifies precisely how the inflection divisor of the complete linear series $|\mathcal{L}_R(kD)|$ along the hyperelliptic curve $X$ is related to the inflection divisors of the series $g^1_{2k-g}$ along the elliptic components $E_i$. Because we have complete control over inflection in the marked points, in Theorem 6.1, we obtain the following lower bound for the number of real inflection points $\omega_\mathcal{L}(k,g)$ along each $g^1_{2k-g}$ as a function of the topology of the real locus of the underlying curve $E_i$:

$$\omega_\mathcal{L}(k,g) \geq \begin{cases} g(g+1) + 2(k-g)(g-1), & \text{if } n(E) = 1, \\ 2g(g+1) + 2(k-g)(g-1), & \text{if } n(E) = 2. \end{cases}$$
We expect that Theorem 6.1 may be improved, but this will require generalizing
the third author’s characterization [1, Theorem 3.2.5] of real inflection points of a
complete linear series on a real elliptic curve.

The roadmap of this paper is as follows. In Section 2, we recall some basic
facts about real linear series on a real algebraic curve and their inflection points,
which determine a corresponding inflection divisor. In Section 3 we recall some
basic facts about real hyperelliptic curves, and given a real hyperelliptic curve
X of genus \( g \geq 1 \) with non-empty real part and a \([D] \in \text{Pic}^2(X)\) a \( g_2 \), we give some
general properties of the real inflection points of the real linear series \(|L_R(kD)|\) on
X, whenever \( k \geq g \). In Sections 4 and 5 we use tropicalization techniques to enhance
Viro’s patchworking and associate to a family of affine plane hyperelliptic curves
a metrized complex of curves. A key technical point here is that tropicalization
in our context allows us to set up a canonical specialization process. Finally in
Section 6 we give a combinatorial method to construct real hyperelliptic curves of
genus \( g \geq 2 \) with controlled number of real inflection points of the real linear series
\(|L_R(kD)|\) for \( k \geq g + 1 \).

2. Fundamental facts and definitions

In what follows a real algebraic variety \( X \) denotes a pair \( X = (X_C, \sigma_X) \) consisting
of a complex algebraic variety \( X_C \) together with an anti-holomorphic involution
\( \sigma_X : X_C \to X_C \). Equivalently, a real algebraic variety is any complex algebraic
variety of the form \( X = X_R \otimes_R \mathbb{C} \) for some scheme \( X_R \) defined over \( \mathbb{R} \). We denote
by \( X(\mathbb{R}) \) the real part of \( X \) and by \( n(X) \) the number of connected components of
\( X(\mathbb{R}) \).

A morphism between real algebraic varieties \( (X_C, \sigma_X) \) and \( (Y_C, \sigma_Y) \) is a mor-
phism \( f : X_C \to Y_C \) of complex algebraic varieties compatible with the corre-
sponding anti-holomorphic involutions, i.e. such that \( f \circ \sigma_X = \sigma_Y \circ f \).

Hereafter we restrict our attention to smooth real algebraic curves \( X = (X_C, \sigma_X) \)
with \( X(\mathbb{R}) \neq \emptyset \). In that case, any \( \sigma_X \)-invariant divisor on \( X \) is of the form
\begin{equation}
D = \sum_{p \in X(\mathbb{R})} n_p \cdot p + \sum_{p \notin X(\mathbb{R})} n_p \cdot (p + \sigma_X(p)).
\end{equation}
The first summand in \( 1 \) is the real part of \( D \), and we denote its degree as \( \deg_{\mathbb{R}}(D) \).
We say that \( D \) is totally real if it coincides with its real part.

Moreover, as explained in [4], the real part \( \text{Pic} X(\mathbb{R}) \) of Pic \( X \) is precisely the set
of linear equivalence classes represented by a \( \sigma_X \)-invariant divisor. Let \( S_1, \ldots, S_{n(X)} \)
be the connected components of \( X(\mathbb{R}) \). There is a corresponding parity homomor-
phism
\[ c : \text{Pic} X(\mathbb{R}) \to (\mathbb{Z}/2\mathbb{Z})^{n(X)} \]
defined by
\[ [D] \mapsto (\deg(D|_{S_1}) \mod 2, \ldots, \deg(D|_{S_{n(X)}}) \mod 2). \]
An obvious but nonetheless useful fact is that the parity of \( \deg(D) \) is the parity of
the sum of the components of its parity vector \( c(D) \).
Given a $\sigma_X$-invariant divisor $D$, we denote by $\mathcal{L}_R(D)$ the real algebraic line bundle defined by $D$, and we denote its complexification $\mathcal{L}_R(D) \otimes_R \mathbb{C}$ simply by $\mathcal{L}(D)$. Then $H^0(\mathcal{L}_R(D))$ is a real vector space satisfying

$$H^0(\mathcal{L}_R(D)) \otimes_R \mathbb{C} = H^0(\mathcal{L}(D)).$$

A real linear series (of degree $d$ and rank $r$) on $X$ is a pair $(L, V_R)$ consisting of an algebraic line bundle $L_R \in \text{Pic}(X)$ of degree $d$ and a real vector subspace $V_R \subseteq H^0(L_R)$ of dimension $r$. The ramification divisor associated to $(L, V_R)$ is the divisor $\text{Inf}(L_R, V_R) = \sum_{p \in X} |p| \cdot p$, where $|p|$ is the ramification weight of $(L_R, V_R)$ at $p$; it is an effective $\sigma_X$-invariant divisor of degree $(r+1)(d+r(g-1))$, where $g$ is the genus of $X$.

Let $(L_R, V_R)$ be a real linear series of degree $d$ and rank $r$ on $X$. Associated to a basis $F = \{f_0, \ldots, f_r\}$ of $V_R$, there is a (real) section $\text{Wr}(F) \in H^0(L_R((r+1)D + \left(\frac{g+1}{2}\right)K_X))$ called the Wronskian of $F$. The divisor $\text{div} \text{Wr}(F)$ is independent of the choice of the basis for $V_R$, and coincides with the ramification divisor of $(L_R, V_R)$.

Hereafter, given a hyperelliptic curve $X$ over a field $F$, $\pi : X \to FP^1$ will denote the branched 2-sheeted cover obtained from its $g_2^1$, and $R_\pi$ will denote the ramification divisor of $\pi$.

3. Real linear series on real hyperelliptic curves

Let $X = (X_C, \sigma_X)$ be a real hyperelliptic curve of genus $g \geq 1$ and let $[D] \in \text{Pic} X$ be a $g^2_1$ on $X$. If $X(\mathbb{R}) \neq \emptyset$, then $[D] \in \text{Pic} X(\mathbb{R})$ and $\mathbb{P}(H^0(\mathcal{L}(D))) \cong (\mathbb{C}P^1, \sigma_{\mathbb{C}P^1})$, where $\sigma_{\mathbb{C}P^1}$ is the canonical real structure $[z_1 : z_2] \mapsto [\overline{z_1} : \overline{z_2}]$. It follows that the branched 2-sheeted cover $\pi : X_C \to \mathbb{C}P^1$ obtained from $[D]$ is defined over $\mathbb{R}$.

The function field $K(\mathbb{C}P^1, \sigma_{\mathbb{C}P^1})$ of $(\mathbb{C}P^1, \sigma_{\mathbb{C}P^1})$ is of the form

$$K(\mathbb{C}P^1, \sigma_{\mathbb{C}P^1}) = \mathbb{R}(x);$$

with respect to the identification $[2]$, the function field of $X$ is of the form

$$K(X) = \mathbb{R}(x)[y]/(y^2 - f(x))$$

for some separable polynomial $f(x) \in \mathbb{R}[x]$ of degree $2g+1$ or $2g+2$. Let us suppose that $2g$ branch points of the map $\pi$ lie in $\mathbb{C}^*$ and that $X$ has $1 \leq n(X) \leq g+1$ real connected components. Let $U \subset \mathbb{C}^2$ be the real affine plane curve defined by the following equation:

$$y^2 = f(x) = x \prod_{i=1}^{2n(X)-2} (x-p_i) \prod_{j=1}^{g+1-n(X)} (x-q_j)(x-\overline{q_j}),$$

where the $p_i \in \mathbb{R}^*$ and $q_j \in \mathbb{C} \setminus \mathbb{R}$ are all distinct. If $\mathcal{U}$ denotes the compactification of $U$ inside the toric variety $\mathbb{C}P^1 \times \mathbb{C}P^1$, then we have $X = \mathcal{U}$ and $\mathcal{U} \setminus U = \{\infty\}$.

**Remark 3.1.** Hereafter we will denote by $U$ the real affine plane curve $[3]$, and we will denote by $X$ its compactification $\mathcal{U} \cup \{\infty\}$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Now consider the divisor $D' = 1 \cdot \infty$ in $(\mathbb{C}P^1, \sigma_{\mathbb{C}P^1})$; we have

$$[D] = [\pi^*(D')] = [2 \cdot \infty] \text{ and } [K_X] = [(g-1)\pi^*(D')] = [(2g-2) \cdot \infty]$$

and for $k \geq g$, the space $H^0(\mathcal{L}_R(kD))$ is a real vector space of dimension $r+1 = 2k-g+1$ with a real basis $F = \{f_0, \ldots, f_{2k-g}\} \subset K(X)$ given by $f_i = x^i$ for $0 \leq i \leq g$. 
when \( k = g \), and by \( f_i = x^i \) for \( 0 \leq i \leq k \), \( f_i = x^{i-k-1}y \) for \( k + 1 \leq i \leq 2k - g \) whenever \( k > g \).

Applying the basic theory of Section 2, we see that for a certain real rational function \( h \) on \( X \), the divisor of the Wronskian associated to the basis \( \mathcal{F} \) of \( H^0(\mathcal{L}_R(kD)) \) may be realized as

\[
\text{div } \text{Wr}(\mathcal{F}) = (r + 1)kD + \frac{r(r+1)}{2}K_X + \text{div}_X(h)
\]

\[
= (r + 1)(2k + r(g - 1)) \cdot \infty + \text{div}_X(h)
\]

which is an effective \( \sigma_X \)-invariant divisor of degree

\[(r + 1)(2k + r(g - 1)) = g(2k - g + 1)^2\]

on \( X \).

Note that since \( \text{div } \text{Wr}(\mathcal{F}) \) is effective, the pole divisor \( \text{div}_\infty(h) \) of \( h \) is supported on \( \{\infty\} \). In particular, there exist a real regular function \( \alpha \) on \( U \) such that \( \text{div}_0(h) = \text{div}_U(\alpha) \); further, since \( U \subset \mathbb{C}^2 \), we can choose a representative for \( \alpha \) that is regular on \( \mathbb{C}^2 \).

In other words, there exists some \( \alpha \in \mathbb{R}[x, y] \) in terms of which the inflection divisor of the complete real linear series \( |\mathcal{L}_R(kD)| \) on \( X \) may be realized as

\[(4) \quad \text{Inf}(|\mathcal{L}_R(kD)|) = \text{div}_U(\alpha) + m \cdot \infty = [U \cap V(\alpha)] + m \cdot \infty\]

where \( m = g(2k - g + 1)^2 - \deg \text{div}_U(\alpha) \geq 0 \) and \( [U \cap V(\alpha)] \) is the divisor associated to the intersection scheme \( U \cap V(\alpha) \). In particular, the real part of \( \text{Inf}(|\mathcal{L}_R(kD)|) \) consists of the real part of \( [U \cap V(\alpha)] \) together with \( m \cdot \infty \).

Now let \( \sigma_h : X \rightarrow X \) be the hyperelliptic involution sending \((x, y) \) to \((x, -y)\), and let \( G \subset \text{Aut}(X) \) be the subgroup generated by \( \sigma_X, \sigma_h \), which is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), as \( \sigma_X \sigma_h = \sigma_h \sigma_X \).

**Proposition 3.2.** The inflection divisor \( \text{Inf}(|\mathcal{L}_R(kD)|) \) may be expressed as the sum of two effective divisors with disjoint supports

\[\text{Inf}(|\mathcal{L}_R(kD)|) = R + S\]

where \( R \) is supported on the ramification locus of \( \pi \) and \( S \) is \( G \)-invariant.

**Proof.** Let \( \alpha \) be as in (4). Since \( \frac{\partial(y^2 - f)}{\partial y} = 2y \), it follows that in the open subset \( U_y := U \setminus V(y) \) the restriction \( \alpha|_{U_y} \) may be expressed as \( \alpha|_{U_y} = \det(f^{(j)}_r)_{0 \leq i, j \leq 2k - g} \), where \( f^{(j)}_r = \frac{\partial f^{(j)}}{\partial x} \). Note that \( \text{div}_U(\alpha) = \text{div}_{U_y}(\alpha|_{U_y}) + R' \), where \( R' \) is a divisor supported on the closed subset \( V(y) \cap U \). We need to show that \( \text{div}_{U_y}(\alpha|_{U_y}) \) is \( G \)-invariant.

If \( k = g \), then \( \alpha|_{U_y} = 1 \), and it follows that\n
\[\text{Inf}(|\mathcal{L}_R(gD)|) = R' + m \cdot \infty\]

On the other hand, if \( k > g \) we have

\[
\alpha|_{U_y} = \det\begin{pmatrix}
(x^0y)^{(k+1)} & (x^0y)^{(k+2)} & \cdots & (x^0y)^{(2k-g)} \\
(x^1y)^{(k+1)} & (x^1y)^{(k+2)} & \cdots & (x^1y)^{(2k-g)} \\
(x^{k-g-1}y)^{(k+1)} & (x^{k-g-1}y)^{(k+2)} & \cdots & (x^{k-g-1}y)^{(2k-g)}
\end{pmatrix}
\]

where
which is a square \((k - g) \times (k - g)\) matrix. In the open set \(U_y\) we have

\[
y' = \frac{f'}{f} = \left(\frac{f'}{f}\right) y := P_{0,1}(x)y
\]

and it follows by induction that for \(j > 1\) we have \(y^{(j)} = P_{0,j}(x)y\) for some \(P_{0,j}(x) \in \mathbb{R}(x)\). It follows that for \(i, j \geq 0\), \((x^i y)^{(j)} = P_{i,j}(x)y\) for some \(P_{i,j}(x) \in \mathbb{R}(x)\). The upshot is that there exists some \(Q(x) \in \mathbb{R}(x)\) for which

\[
a|_{U_y} = Q(x)y^{k-g}.
\]

Since \(y \neq 0\) on \(U_y\), it follows that the divisor \(\text{div}_{U_y}(a|_{U_y})\) is determined by the divisor \(\text{div}_{U_x}(Q)\) of \(Q\), which is a real regular function on the open set \(U_f := \mathbb{C} \setminus V(f) \subset \mathbb{C}\).

Now suppose that

\[
\text{div}_W(Q) = \sum_{p \neq P_i} n_p \cdot p + \sum_{q \neq Q_i} n_q \cdot (q + \overline{q})
\]

then \(p\) appears on \(\text{div}_{U_x}(a|_{U_x})\) as \((p, \sqrt{f(p)}) + \sigma_h((p, \sqrt{f(p)})\), and \(q + \overline{q}\) appears as \((q, \sqrt{f(q)}) + \sigma_X(q, \sqrt{f(q)}) + \sigma_h(q, \sqrt{f(q)})\). It follows that \(\text{div}_U(a|_U)\) is \(G\)-invariant.

\[\square\]

**Theorem 3.3.** When \(1 \leq k \leq g\), the inflection divisor \(\text{Inf}(\mathcal{L}_k(kD))\) is \((k+1)\) times the ramification divisor of \(\pi\). In particular, the linear series \(|\mathcal{L}_k(kD)|\) has \(k(k+1)n(X)\) real inflection points.

**Proof.** To simplify the exposition we focus on the case \(k = g\); the extension to the seemingly more general case \(k \leq g\) is easy, and will be described at the end. We saw in Proposition 3.2 that \(\text{Inf}(\mathcal{L}_k(gD))\) is supported along the ramification locus of \(\pi\). Let \(\alpha\) be as in 1. Since \(\frac{\partial y^2 - f}{\partial y} = f'(x)\), it follows that on the open set \(U_x = U \setminus \{(p, \pm \sqrt{f(p)}) : f'(p) = 0\}\) the restriction \(a|_{U_x}\) may be expressed as \(a|_{U_x} = \det(f^{(j)}_{i,j})\) \(0 \leq i,j \leq g\); where \(f^{(j)}_{i,j} = \frac{\partial^j f}{\partial y^j}\).

Write \(\mathcal{D}^j = \frac{\partial^j}{\partial y^j}\); then, since \(f_0 = 1\), we have \(a|_{U_x} = \det(\mathcal{D}^j(x))\) \(1 \leq i,j \leq g\). Let \(R_2 = (\mathcal{D}^j(x^2))\) \(1 \leq i,j \leq g\) be the vector corresponding to the second row of the matrix \((\mathcal{D}^j(x^2))\) \(1 \leq i,j \leq g\). It may be expressed as \(2x \cdot R_1 + R_2^{(1)}\), where \(R_2^{(1)} = (r_{2,j}^{(1)})\) \(1 \leq i,j \leq g\) is the vector with entries \(r_{2,1}^{(1)} = 0\) and \(r_{2,j}^{(1)} = 2 \sum_{k=1}^{j-1} (\binom{j-1}{k} \mathcal{D}^k(x) \mathcal{D}^{j-k}(x))\) for \(j \geq 2\).

In particular, we have \(r_{2,2}^{(1)} = 2(\mathcal{D}^1(x))^2\).

Similarly, let \(R_3 = (\mathcal{D}^j(x^3))\) \(1 \leq i,j \leq g\) be the third row. It may be expressed as \(3x^2 \cdot R_1 + R_2^{(1)}\), where \(R_3^{(1)} = (r_{3,j}^{(1)})\) \(1 \leq i,j \leq g\) is the vector with entries \(r_{3,1}^{(1)} = 0\) and \(r_{3,j}^{(1)} = 3 \sum_{k=1}^{j-1} (\binom{j-1}{k} \mathcal{D}^k(x) \mathcal{D}^{j-k}(x))\) for \(j \geq 2\). This time we have \(R_3^{(1)} = 3xR_2^{(1)} + R_3^{(2)}\), where \(r_{3,1}^{(2)} = r_{3,2}^{(2)} = 0\) and \(r_{3,j}^{(2)} = 6 \sum_{k=2}^{j-1} \sum_{\ell=1}^{k-1} (\binom{k-1}{\ell} \mathcal{D}^{k-\ell}(x) \mathcal{D}^{j-k}(x))\) for \(j \geq 3\). So \(R_3 = 3x^2 \cdot R_1 + 3xR_2^{(1)} + R_3^{(2)}\), with \(r_{3,3}^{(2)} = 6(\mathcal{D}^1(x))^3\).

In general, we may express the \(i\)-th row \(R_i\) as a linear combination \(\alpha_{i,1}R_1 + \alpha_{i,2}R_2^{(1)} + \cdots + \alpha_{i,i-1}R_{i-1}^{(i-2)} + R_i^{(i-1)}\) in which the vector \(R_i^{(i-1)} = (r_{i,j}^{(i-1)})\) satisfies \(r_{i,j}^{(i-1)} = 0\) for \(1 \leq j \leq i - 1\) and \(r_{i,i}^{(i-1)} = \alpha_{i,i}(\mathcal{D}^1(x))^i\) for some \(\alpha_{i,i} \in \mathbb{R} \setminus \{0\}\).
Consequently, we have
\[ \alpha|_{U_x} = \det((x^i)^{(j)})_{1 \leq i,j \leq g} = a(x')^{(g+1)/2} \]
for some \( a \in \mathbb{R} \setminus \{0\} \), and the desired result when \( k = g \) follows since \( x' = \frac{2}{\sqrt{k}} y \) on \( U_x \).

Finally, if \( 1 \leq k \leq g \), then \( H^0(\mathcal{L}_k(D)) \) is spanned by \( \{x^0, \ldots, x^k\} \), and \( \text{Inf}(\mathcal{L}_k(D)) \) is computed exactly as above. In particular, we have
\[ \alpha|_{U_x} = a(x')^{(g+1)/2} \]
for some \( a \in \mathbb{R} \setminus \{0\} \), and we conclude. \( \square \)

**Remark 3.4.** When \( X \) is hyperelliptic, the inflection divisor of its canonical series is \( \binom{g}{2} \) times the ramification divisor of \( \pi \) [3, p.274]. This is in agreement with our result, as \( [K_X] = (g-1)|D| \). In Theorem 5.6, we will prove a strengthening of Theorem 3.3 via a local analysis of vanishing orders of sections of \( |\mathcal{L}_R(D)| \).

**Remark 3.5.** When \( k > g \), we have
\[ \alpha|_{U_y} = \det((k + j)!\frac{y^{(k+1+j-i)}}{(k+1+j-i)!})_{1 \leq i,j \leq k-g} \]
and
\[ \alpha|_{U_x} = \det(a_{i,j})_{1 \leq i,j \leq 2k-g-1} \]
where
\[ a_{i,j} = \begin{cases} (x^i)^{(j+1)}, & \text{if } 1 \leq i \leq k, \\ (j+1)(x^{i-k})^{(j)}, & \text{if } k+1 \leq i \leq 2k-g-1. \end{cases} \]

Note that \( \alpha|_U \) is in fact the determinant of a square Toeplitz matrix. It would be interesting to identify these functions explicitly.

We end this section by commenting on the case \( g = 1 \), which was resolved in [1] (see also Section 6). In this situation, the real components of the hyperelliptic curve defined by (compactifying) (3), along with the real inflection points of \( |\mathcal{L}_R(D)| \), obey the following dichotomy.

1. \( n(X) = 1 \) if and only if \( f \) has conjugate non-real roots \( q_1, \overline{q_1} \in \mathbb{C} \setminus \mathbb{R} \). In this situation, the inflection divisor \( \text{Inf}(\mathcal{L}_R(kD)) \) has \( 2k \) (distinct) real points.
2. \( n(X) = 2 \) if and only if \( f \) has distinct roots \( p_1, p_2 \in \mathbb{R}^* \). Since the parity vector \( c(kD) \) of \( kD \) is \( (0,0) \), the inflection divisor \( \text{Inf}(\mathcal{L}_R(kD)) \) has \( 2k \) real points on each connected component of \( X(\mathbb{R}) \).

In the remainder of this work we will focus on the case \( k > g > 1 \).

4. Enhancing Viro’s Patchworking Construction

Viro’s patchworking method is a tool for constructing real plane algebraic curves with controlled topology. In this section we will apply an enhanced version of this method to construct useful one-parameter families of real affine plane hyperelliptic curves of the form (3).

Any such family may be viewed as a plane curve defined over a non-Archimedean field, and so is associated with a subdivision of a lattice triangle. More precisely, for a fixed choice of \( g \geq 2 \), let \( \Delta \subset \mathbb{R}^2 \) denote the lattice triangle \( \Delta = \ldots \)
Conv\{\emptyset, (0,2), (1,0), (2g + 1, 0)\}. Let \(\Theta\) denote the regular subdivision of \(\Delta\) whose 2-dimensional faces are the triangles
\[
\Theta_j = \text{Conv}\{\emptyset, (0,2), (2j - 1, 0), (2j + 1, 0), j = 1, \ldots, g\).
\]

For each \(i = 1, \ldots, 2g + 1\) we now choose \(\nu_i \in \mathbb{N} \cup \{\infty\}\) in such a way that the function \(\nu : \Delta \cap \mathbb{Z}^2 \to \mathbb{N} \cup \{\infty\}\) defined by
\[
\nu(i, j) = \begin{cases} 
\nu_i, & \text{if } j = 0, \\
\infty, & \text{if } j = 1, \\
0, & \text{if } j = 2 
\end{cases}
\]
induces the subdivision \(\Theta\) on \(\Delta\). We correspondingly define
\[
y^2 - f := y^2 - \sum_{i=1}^{2g+1} a_i t^{\nu_i} x^i \in \mathbb{R}[t^\pm][x,y]
\]
where \(a_i \neq 0\) if \(\nu_i \neq \infty\). In particular we have \(a_{2i-1} \neq 0\) for \(i = 1, \ldots, g + 1\).

Our present aim is to show how any family of hyperelliptic curves embedded as a hypersurface in a toric variety as in [7] naturally has an associated limit

Remark 4.1. We will work over the non-Archimedean field \(K = \mathbb{C}((t))^\text{alg}\) endowed with the \(t\)-adic norm \(v_t\) normalized to satisfy \(v_t(t) = e^{-1}\). It should be noted that in every Berkovich-analytic argument we make hereafter we work over the completion \(\hat{K}\), which remains algebraically closed.

To begin, set \(\mathcal{U} := \text{Spec}(\mathbb{R}[t^\pm][x,y]/(y^2 - f) \otimes_{\mathbb{R}} \mathbb{C})\) and \(\mathbb{G}_m := \text{Spec}(\mathbb{R}[t^\pm] \otimes_{\mathbb{R}} \mathbb{C})\).

The inclusion of rings
\[
\mathbb{R}[t^\pm] \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathbb{R}[t^\pm][x,y]/(y^2 - f) \otimes_{\mathbb{R}} \mathbb{C}
\]
induces a surjective real morphism \(p : \mathcal{U} \to \mathbb{G}_m\) of real algebraic varieties. Let
\[
\mathcal{U}_{\mathbb{C}((t))} := \mathcal{U} \times_{\mathbb{G}_m} \mathbb{C}((t)) = \text{Spec}(\mathbb{C}((t))[x,y]/(y^2 - f))
\]
denote the \(\mathbb{C}((t))-\)curve obtained from the \(\mathbb{G}_m\)-surface \(\mathcal{U}\) via the obvious base change.

Now let \(v_0, v_\infty : \mathbb{C} \to \mathbb{R}_{\geq 0}\) denote the trivial and the Archimedean norms of \(\mathbb{C}\) respectively, and consider the function \(v_{\text{hyb}} : \mathbb{C} \to \mathbb{R}_{\geq 0}\) defined by
\[
z \mapsto \max\{v_0(z), v_\infty(z)\}.
\]
This is a sub-multiplicative norm on \(\mathbb{C}\) and \((\mathbb{C}, v_{\text{hyb}})\) is a Banach ring; its Berkovich spectrum \(\mathcal{M}(\mathbb{C}, v_{\text{hyb}})\) is thus well-defined, and is in fact homeomorphic to the unit interval \([0,1]\). Further, there is an analytification functor \(\text{an}(\cdot, v_{\text{hyb}})\) from the category of complex varieties to the category of \(\mathcal{M}(\mathbb{C}, v_{\text{hyb}})\)-analytic spaces.

Let \(\text{an}(\mathcal{U}, v_{\text{hyb}})\) and \(\text{an}(\mathbb{G}_m, v_{\text{hyb}})\) be the analytification of \(\mathcal{U}\) and \(\mathbb{G}_m\) with respect to \(v_{\text{hyb}}\). Let
\[
\mathcal{U}^\# := \{\rho \in \text{an}(\mathcal{U}, v_{\text{hyb}}) : \rho(t) = e^{-1}\}
\]
and
\[ \mathbb{G}_m^\# := \{ \rho \in \text{an}(\mathbb{G}_m, v_{\text{hyb}}) : \rho(t) = e^{-1} \}. \]

Note that as a topological space, \( \mathbb{G}_m^\# \) is precisely the Archimedean closed disk \( D_{e^{-1}} = \{ z \in \mathbb{C} : v_\infty(z) \leq e^{-1} \} \).

The fiberwise behavior of the restriction \( p^\# : U^\# \rightarrow D_{e^{-1}} \) to \( U^\# \) of the hybrid analytification
\[ \text{an}(p, v_{\text{hyb}}) : \text{an}(U, v_{\text{hyb}}) \rightarrow \text{an}(\mathbb{G}_m, v_{\text{hyb}}) \]
of the projection \( p : U \rightarrow \mathbb{G}_m \) is explained by the following dichotomy.

1. The fiber \( U^\#_{\varepsilon} \) of \( p^\# \) over \( \varepsilon \in D_{e^{-1}} \cap \mathbb{R}_{>0} \) is homeomorphic to the fiber above \( t = \varepsilon \) of the morphism \( \text{an}(p, v_\infty) \), which is none other than the usual holomorphic analytification. In other words, \( U^\#_{\varepsilon} \) is the real plane curve
\[ V(y^2 - f_\varepsilon), \]
where \( f_\varepsilon = f|_{t=\varepsilon} \).

2. The fiber \( U^\#_0 \) of \( p^\# \) over \( 0 \in D_{e^{-1}} \) is \( \text{an}(\mathcal{U}_{\mathbb{C}((t))}, v_t) \), the analytification of the non-Archimedean plane curve \( V(y^2 - f) \subset \mathbb{C}((t))^2 \) with respect to \( v_t \), the \( t \)-adic norm normalized to satisfy \( v_t(e^{-1}) = 1 \).

Moreover, the map \( p^\# \) is open over the Archimedean closed disk \( D_0 \) whenever \( 0 \leq \delta \ll 1 \).

From a practical point of view, the map \( p^\# \) will allow us to relate inflection divisors of complex hyperelliptic curves with inflection divisors of non-Archimedean hyperelliptic curves over the field of Puiseux series with complex coefficients. Accordingly, let \( K = \mathbb{C}((t))^{\text{alg}} \) denote the Puiseux field. Note that the non-Archimedean plane curve \( \mathcal{U}_{\mathbb{C}((t))}(K) \subset \mathbb{K}^2 \) is smooth if and only if \( f \) has \( 2g + 1 \) distinct roots in \( K \). Suppose that this is the case, and let \( \mathcal{X}_{\mathbb{C}((t))} \) denote the compactification \( \mathcal{U}_{\mathbb{C}((t))}(K) \cup \{ \infty \} \subset \mathbb{K}P^1 \times \mathbb{K}P^1 \). It follows that \( \mathcal{X}_{\mathbb{C}((t))} \) is a hyperelliptic curve of genus \( g \) over \( K \), with associated two-sheeted cover \( \pi : \mathcal{X}_{\mathbb{C}((t))} \rightarrow \mathbb{K}P^1 \).

The analysis of (individual) complex curves carried out in Section 3 remains valid in this context. Namely, consider the divisor \( D = 2 \cdot \infty \) on \( \mathcal{X}_{\mathbb{C}((t))} \). Just as before, the complete linear series \( |\mathcal{L}(kD)| \) on \( \mathcal{X}_{\mathbb{C}((t))} \) for \( k > g \) has a basis of global sections \( \mathcal{F} = \{ \phi_0, \ldots, \phi_{2k-g} \} \subset \mathbb{R}[t^\pm 1][x, y] \) defined by
\[ \phi_i = x^i \]
for \( 0 \leq i \leq k \), and \( \phi_i = x^{i-k-1}y \) for \( k + 1 \leq i \leq 2k - g \).

By computing \( \text{div} \text{Wr}(\mathcal{F}) \), we see that there is some \( \alpha \in \mathbb{R}[t^\pm 1][x, y] \) for which
\begin{equation}
\text{Inf}(|\mathcal{L}(kD)|) = \text{div} \text{Wr}_{\mathcal{U}_{\mathbb{C}((t))}}(\alpha) + m \cdot \infty = |\mathcal{U}_{\mathbb{C}((t))} \cap V(\alpha)| + m \cdot \infty,
\end{equation}
where \( m = g(2k - g + 1)^2 - \deg \text{div} \text{Wr}_{\mathcal{U}_{\mathbb{C}((t))}}(\alpha) \geq 0 \), and \( |\mathcal{U}_{\mathbb{C}((t))} \cap V(\alpha)| \) is the divisor associated to the intersection scheme \( \mathcal{U}_{\mathbb{C}((t))} \cap V(\alpha) \) in \( \mathbb{K}^2 \).

We will now analyze the \( v_{\text{hyb}} \)-analytification of the intersection scheme \( \mathcal{U}_{\mathbb{C}((t))} \cap V(\alpha) \) on \( \mathbb{K}^2 \), which is the closed subscheme defined by the ideal \( (y^2 - f, \alpha) \). To this end, set
\[ \mathcal{Z} := \text{Spec}(\mathbb{R}[t^\pm 1][x, y]/(y^2 - f, \alpha) \otimes_{\mathbb{R}} \mathbb{C}); \]
this is a real algebraic variety, canonically equipped with a surjective real morphism \( q : \mathcal{Z} \rightarrow \mathbb{G}_m \) between real algebraic varieties. Let
\[ \mathcal{Z}_{\mathbb{C}((t))} := \mathcal{Z} \times_{\mathbb{G}_m} \mathbb{C}((t)) = \text{Spec}(\mathbb{C}((t))[x, y]/(y^2 - f, \alpha)) \]
denote the \( \mathbb{C}((t)) \)-curve associated to \( \mathcal{Z} \) by carrying out the obvious base change.
According to our discussion above, the induced map \( q^\# : Z^\# \to D_{c-1} \) has the following properties.

1. The fiber \( Z^\#_{\varepsilon} \) of \( p^\# \) over \( \varepsilon \in D_{c-1} \cap \mathbb{R}_{>0} \) is the 0-dimensional scheme \( V(y^2 - f_\varepsilon, \alpha_\varepsilon) \), where \( f_\varepsilon = f|_{t=\varepsilon} \) and \( \alpha_\varepsilon = \alpha|_{t=\varepsilon} \).
2. The fiber \( Z^0_0 \) of \( q^\# \) over \( 0 \in D_{c-1} \) is \( \text{an}(Z_{C((-1))}, v_\ell) \), the analytification of the 0-dimensional scheme \( V(y^2 - f, \alpha) \subset C((t))^2 \) over the non-Archimedean field \( (C((t)), v_t) \).
3. For \( 0 < \delta \ll 1 \), the map \( q^\# \) is open above \( D_\delta \).

Moreover, for generic values \( 0 < \varepsilon < 1 \), the real plane curve \( U^\#_\varepsilon \subset \mathbb{C}^2 \) is hyperelliptic. Let \( X^\#_\varepsilon \) denote its compactification \( U^\#_\varepsilon \cup \{ \infty_\varepsilon \} \subset \mathbb{C}P^1 \times \mathbb{C}P^1 \) and \( D = 2 \cdot \infty_\varepsilon \). The inflection divisor of the complete real linear series \( |L_{kD}| \) on \( X^\#_\varepsilon \) is given by

\[
\text{Inf}(|L_{kD}^\#(kD)|) = [Z^\#_{\varepsilon}] + m_\varepsilon \cdot \infty_\varepsilon
\]

where \( [Z^\#_{\varepsilon}] \) is the divisor associated to the 0-dimensional closed subscheme \( Z^\#_{\varepsilon} \subset U^\#_\varepsilon \). Similarly, the inflection divisor of the linear series \( |L(kD)| \) on the hyperelliptic curve \( X^\#_{C((t))} \) is given by

\[
\text{Inf}(|L(kD)|) = [Z^\#_0] + m_0 \cdot \infty,
\]

where \( [Z^\#_0] \) is the divisor associated to the 0-dimensional closed subscheme \( Z^\#_0 \subset U^\#_0 \).

Now let \( X^\#_0 \) denote the compactification \( U^\#_0 \cup \{ \infty \} \subset \mathbb{K}P^{1,an} \times \mathbb{K}P^{1,an} \); this is just the analytification \( \text{an}(X_{C((t))}, v_\ell) \). We can easily compute the skeleton

\[
\text{Sk}(X^\#_0) \hookrightarrow X^\#_0
\]

since \( X_{C((t))} \) is smooth and proper; see Figure 1. A detailed discussion of the skeleton is given in the next section, in connection with the tropicalization technique; however, it already seems useful to give a concrete description here.

![Figure 1. Skeleton of the compactification \( X^\#_0 = \text{an}(X_{C((t))}, v_\ell) \). Here \( D = 2 \cdot \infty \).](image)

Explicitly, a model for the object \( \text{Sk}(X^\#_0) \) is the graph \( G = (V, E) \) consisting of the vertices

\[
V = \{ v_1, \ldots, v_g \} \cup \{ 0, \infty, v_{i,j} : i = 1, \ldots, g, j = 1, 2 \}
\]

and the edges

\[
E = \{ e_1, \ldots, e_{g-1} \} \cup \{ e_0, e_\infty, e_{i,j} : i = 1, \ldots, g, j = 1, 2 \}.
\]

We begin by describing the set \( V \). The \( 2g + 2 \) vertices \( \{ 0, \infty, v_{i,j} : i = 1, \ldots, g, j = 1, 2 \} \) are all type I points, and this set is precisely \( \text{Supp}(R_\pi) \), the support of the ramification divisor of the map \( \pi : X_{C((t))} \to \mathbb{K}P^1 \). We thus write

\[
V = \{ v_1, \ldots, v_g \} \cup \text{Supp}(R_\pi).
\]
The $g$ vertices $\{v_1, \ldots, v_g\}$, on the other hand, are all type II points; for every $i = 1, \ldots, g$, the corresponding residue field $H(v_i)$ of the completed residue field $\mathcal{H}(v_i)$ has transcendence degree one over $\mathbb{C}$. Let $C_{v_i}$ denote the (unique) smooth projective algebraic curve over $\mathbb{C}$ whose field of rational functions $K(C_{v_i})$ equals $\mathcal{H}(v_i)$. We will see later that $C_{v_i}$ is a real algebraic curve of genus 1.

For every $i = 1, \ldots, g$ let $N(v_i) = \{w \in V : v$ is adjacent to $w\}$ be the neighborhood of the vertex $v_i$ inside the graph $G$. We then have a bijection $w \mapsto p(w)$ between the elements $w \in N(v_i)$ and a subset $\mathcal{A}_i = \{p(w)\}_{w \in N(v_i)}$ of $C_{v_i}(\mathbb{C})$.

Finally, we have a function $\ell : E \rightarrow \mathbb{R}_{>0} \cup \{\infty\}$ satisfying $0 < \ell(e) < \infty$ whenever $e \in \{e_1, \ldots, e_{g-1}\}$ and $\ell(e) = \infty$ otherwise. We conclude that the object $Sk(X^\#_0)$ is a metrized complex of algebraic curves over $\mathbb{C}$ in the sense of [6], described as a tuple $(G = (V, E), \ell, \{(C_{v_i}, \mathcal{A}_i)\}_{i=1,\ldots,g})$.

Actually, the points of $\text{Supp}(R_\pi)$ are marked points of our metrized complex $Sk(X^\#_0)$; moreover, our metrized complex naturally carries a real structure, since $(C_{v_i}, \mathcal{A}_i)$ is a marked real elliptic curve for every $i = 1, \ldots, g$. For these reasons we will refer to a marked metrized complex of algebraic curves over $\mathbb{R}$.

Note that each neighborhood $N(v_i)$ now has two types of points. We will call $p(w) \in \mathcal{A}_i$ a point of attachment of $C_{v_i}$ if $w \in V$ is a point of type II.

Remark 4.2. Now consider the very affine curve $X^\circ_{\mathbb{C}(t)(i)} \subset (\mathbb{K}^\star)^2$ defined by

$$X^\circ_{\mathbb{C}(t)(i)} := X_{\mathbb{C}(t)(i)} \cap (\mathbb{K}^\star)^2 = X_{\mathbb{C}(t)(i)} \setminus \text{Supp}(R_\pi).$$

The marked points $\text{Supp}(R_\pi)$ of $Sk(X^\#_0)$, together with their corresponding edges $\{e_0, e_\infty, e_{i,j} : i = 1, \ldots, g, j = 1, 2\}$, emerge naturally from the tropicalization process; namely, they represent the points needed to compactify $X^\circ_{\mathbb{C}(t)(i)}$.

We construct a metrized complex $\Gamma = (G = (V', E), \ell, \{(Y_{v_i}, B_i)\}_{i=1,\ldots,g})$ representing $X^\circ_{\mathbb{C}(t)(i)}$ using our marked metrized complex $Sk(X^\#_0)$. We take the same underlying set $E$ and the same function $\ell$ as before, while setting $V' = \{v_1, \ldots, v_g\}$. The marked curves $(Y_{v_i}, B_i)$ are defined as follows: let $C_i \subset A_i$ consist of those points that are not points of attachment of $C_{v_i}$. Then $Y_{v_i} = C_{v_i} \setminus C_i$ and $B_i = A_i \setminus C_i$. Note that

$$Sk(X^\#_0) = \Gamma \coprod \text{Supp}(R_\pi),$$

and we will apply this decomposition in the specialization-based analysis of the Section 7.

5. Specialization via embedded tropicalization

In this section we will use Viro’s theorem on the convergence of amoebas of affine hypersurfaces to obtain a precise description of the marked curves $\{(C_{v_i}, \mathcal{A}_i)\}_{i=1,\ldots,g}$ and of the specialization of the inflection divisor of the complete linear series $|L(kD)|$ on the hyperelliptic curve $X^\circ_{\mathbb{C}(t)}$ to the metrized complex of curves $Sk(X^\#_0)$ from the previous section. To do so we will use the decomposition [10].

To begin, let $(\mathbb{K}, v_t)$ denote the non-Archimedean Puiseux field valued by the t-adic norm $v_t$, normalized to satisfy $v_t(t) = e^{-1}$. Let $T = (\mathbb{R} \cup \{-\infty\}, \max, +)$
denote the tropical semifield, and set
\[ \text{val} := \log \circ v_t : \mathbb{T} \to \mathbb{T}. \]

Let \( \text{Trop} : \mathcal{U}_0^\# \to \mathbb{T}^2 \) denote the tropicalization morphism. Since \( \mathbb{T} \) is algebraically closed and non-trivially valued, it follows that
\[ (11) \quad \text{Trop}(\mathcal{U}_0^\#) = \overline{\text{Val}(\mathcal{U}_C((t))(\mathbb{T}))} \]
where the bar over the right-hand side denotes Euclidean closure. On the other hand, by Kapranov’s theorem, the right-hand side of (11) is precisely the tropical curve \( V(\text{Trop}(y^2 - f)) \) associated to the tropical polynomial \( \text{Trop}(y^2 - f) = \max_{i=1, \ldots, 2g+1} \{2y, ix - v_i\} \).

Accordingly we get a map
\[ (12) \quad \text{Trop} : \text{Sk}(\mathcal{U}_0^\#) \to V(\text{Trop}(y^2 - f)) \]
that is \( n \)-to-1 along an edge \( e \subset V(\text{Trop}(y^2 - f)) \) with weight \( n \). Finally, the compactification of \( V(\text{Trop}(y^2 - f)) \) inside \( \mathbb{T}\mathbb{P}^1 \times \mathbb{T}\mathbb{P}^1 \) yields the image of \( \text{Trop}(\text{Sk}(\mathcal{U}_0^\#)) \). See Figure 2.

\[ \text{Figure 2.} \text{ The tropical curve which is the image of the map } \text{Trop} : \text{Sk}(\mathcal{U}_0^\#) \to \mathbb{T}\mathbb{P}^1 \times \mathbb{T}\mathbb{P}^1. \text{ The weight of each edge represents the local degree the map.} \]

Now let \( A = \mathbb{K}[x, y]/(y^2 - f) \), so that \( \mathcal{U}_0^\# = \text{an}(\text{Spec}(A), v_t) \). The vertex \( v_i \in \text{Sk}(\mathcal{U}_0^\#) \) is sent under the morphism (12) to the point \( u_i = (a_i, b_i) \) that induces the logarithmic valuation \( u_i : A \to \mathbb{T} \) defined by
\[ F(x, y) = \sum_{m,n} c_{m,n} x^m y^n \mapsto \text{Trop}(F)(u_i) = \max_{m,n} \{\text{val}(c_{m,n}) + ma_i + nb_i\}. \]

We now compute \( \mathcal{H}(u_i) \). Since \( \text{Ker}(u_i) = (0) \), it follows that \( \mathcal{H}(u_i) \) is the completion of \( \text{Frac}(A) \) with respect to \( u_i \). In Berkovich’s notation, we have
\[ \widehat{\mathcal{H}(u_i)} = \mathcal{H}(u_i)^\circ / \mathcal{H}(u_i)_{\text{eff}} \]
where \( \mathcal{H}(u_i)^\circ = \{ F/G \in \text{Frac}(A) : u_i(F) \leq u_i(G) \} \) and \( \mathcal{H}(u_i)_{\text{eff}} = \{ F/G \in \text{Frac}(A) : u_i(F) < u_i(G) \} \).

Now suppose that \( F(x, y) = \sum_{m,n} c_{m,n} x^m y^n \) is an element of \( A \) satisfying \( F/G \in \mathcal{H}(u_i)^\circ \). Then its residue \( \tilde{F} \) in \( \widehat{\mathcal{H}(u_i)} \) is the polynomial \( \text{in}_{u_i}(F) \in \mathbb{C}[X, Y] \) where
the parameter $Y$ satisfies the equation $Y^2 - \sum_{(j,0) \in \Theta_i \cap \mathbb{Z}^2} a_j X^j = in_{u_i}(y^2 - f) = 0$. Here $in_{u_i}(F)$ is the limit of the polynomial $y^2 - f$ under the flat degeneration defined by the weight $u_i \in \mathbb{R}^2$.

It follows that $\overline{\mathbb{R}}(v_i)$ is precisely the field $\mathbb{R}([X]/(\text{in}_{u_i}(y^2 - f)))$ whenever the polynomial $\sum_{(j,0) \in \Theta_i \cap \mathbb{Z}^2} a_j X^j$ is separable. Suppose that this is the case, and let $U_i$ be the restriction of real curve $V(Y^2 - \sum_{(j,0) \in \Theta_i \cap \mathbb{Z}^2} a_j X^j)$ to $(\mathbb{C}^*)^2$. In particular, we deduce that $C_{v_i} \setminus A_i$ and $U_i$ are isomorphic.

Let $\overline{U_i}$ be the compactification of $U_i$ inside $\mathbb{C}P^1 \times \mathbb{C}P^1$, then $\overline{U_i}$ has geometric genus 1, because the triangle $\Theta_i$ has a single interior lattice point. Note that $\overline{U_i}$ is singular along the boundary; its normalization is precisely $C_{v_i}$. Separability ensures that for $i = 1, \ldots, g$ we may write

$$\sum_{(j,0) \in \Theta_i \cap \mathbb{Z}^2} a_j X^j = a_{2i+1} X^{2i+1}(X - x_{i,1})(X - x_{i,2})$$

for some $x_{i,j} \in \mathbb{C}^*$.

The marked points $A_i = \{p(w)\}_{w \in \mathcal{N}(v_i)}$ of $C_{v_i}$ are in correspondence with the points of $\overline{U_i} \setminus U_i$. Note that $\overline{U_i}$ is singular at $p(w)$ if and only if $w$ corresponds to point of attachment of $C_{v_i}$.

**Remark 5.1.** The upshot of the preceding discussion is that the skeleton $\text{Sk}(X^g_0)$ is a combinatorial object which simultaneously contains global information from $X_{C((t))}$ and local information from the various elliptic curves $U_i \subset (\mathbb{C}^*)^2$, and it refines the tropical curve $V(T\text{rop}(y^2 - f))$. Since it carries all the relevant information of $X^g_0$, it may be regarded as the limit object of the family $\{X^g_\varepsilon\}_{0 < \varepsilon \ll 1}$ of real hyperelliptic curves.

Now say $k > g$. Our limit construction also works at the level of divisors; we will apply it to the inflection divisor $\text{Inf}((L(kD)))$ associated with $kD = 2k \cdot \infty$ over $X_{C((t))}$. Accordingly, we define the specialization map

$$\tau_* = \tau_{\text{Sk}(X^g_0)} : \text{Div}(X_{C((t))}) \to \text{Div}(\text{Sk}(X^g_0))$$

to be the composition of the inclusion $X_{C((t))} \hookrightarrow X^g_0$ with the retraction $\tau : X^g_0 \to \text{Sk}(X^g_0)$.

We will show that $\tau_*(\text{Inf}((L(kD))))$ is of the form $E_{\text{Sk}(X^g_0)} \oplus \sum_{i=1}^g E_{v_i}$, where

- $E_{\text{Sk}(X^g_0)}$ is a divisor of degree $g(2k - g + 1)^2$ on the metric graph $(G = (V,E),\ell)$ underlying $\text{Sk}(X^g_0)$; and
- $E_{v_i} \in \text{Div}(C_{v_i} \setminus A_i)$ for $i = 1, \ldots, g$, subject to

$$E_{\text{Sk}(X^g_0)}(v_i) = \deg(E_{v_i})$$

where $E_{\text{Sk}(X^g_0)}(v)$ is the coefficient of $E_{\text{Sk}(X^g_0)}$ at $v$.

Recall that the meromorphic functions $\mathcal{F} = \{\phi_0, \ldots, \phi_{2k-g}\}$ as in [4] determine a basis for $H^0(L(kD))$. According to Proposition 3.2 the divisor $\text{Inf}((L(kD)))$ on $X_{C((t))}$ admits a decomposition

$$\text{Inf}((L(kD))) = R + S$$
where $R$ is supported on $R_\pi$ and $S = \text{div}_X^\alpha (\alpha|_{X_{C(t)}^\circ})$, where $\alpha|_{X_{C(t)}^\circ}$ is a regular function on $X_{C(t)}^\circ = X_{C(t)} \setminus R_\pi$ computed as in Equation (5). It follows that

$$
\tau_*(\text{Inf}(|L^{kD}^\text{R}|)) = R + \tau_*(\text{div}_X^\alpha (\alpha|_{X_{C(t)}^\circ})).
$$

indeed, $R$ already belongs to the skeleton, so is invariant under the specialization process. It then remains to compute

$$
(14) \quad \tau_*(\text{div}_X^\alpha (\alpha|_{X_{C(t)}^\circ})).
$$

In other words, in order to determine the specialization of the inflection divisor to the skeleton we may ignore all of the marked points (i.e., the type I points) of $\text{Sk}(X^0)$ and compute (14) along the metrized complex $\Gamma$ from Remark 4.2.

The specialization (14) may in fact be realized explicitly as follows. For every $i = 1, \ldots, g$, we have an initial coefficient map

$$
(15) \quad \text{ic} : \text{Val}^{-1}(u_i) \longrightarrow U_i
$$

given by $\text{ic}(\alpha t^{a_i} + \ldots + \beta t^{b_i} + \ldots) = (\alpha, \beta)$. We also have a diagram

$$
(16) \quad \tau^{-1}(v_i) \xrightarrow{\text{sp}} C_{vi} \setminus A_i \xrightarrow{\approx} \text{Val}^{-1}(u_i) \xrightarrow{\text{ic}} U_i
$$

which is clearly commutative. It follows that the specialization map for divisors on models coincides with the initial coefficient map (15) on points, extended by linearity to a map on divisors.

Going forward, remember that $C_{vi}$ refers to the normalization of the singular elliptic curve $U_i$ obtained from our $i$th initial degeneration, as above. Our construction in fact specifies a specialization of linear series to each curve $C_{vi}$, $1 \leq i \leq g$, which is both similar to, yet apparently distinct from, limit linear series in the sense of Eisenbud–Harris and Amini–Baker.

Namely, for each $i$, consider the collection of meromorphic functions $F(i) := \{\tilde{\phi}_0(i), \ldots, \tilde{\phi}_{2k-g}(i)\} \subset \mathcal{H}(v_i)$ canonically induced from $F$ via the initial degeneration defined by the weight $u_i$. That is,

$$
\tilde{\phi}_j(i) = X^j \text{ for } 0 \leq j \leq k, \text{ and } \tilde{\phi}_j(i) = X^{j-k-1}Y \text{ for } k + 1 \leq j \leq 2k - g.
$$

Let $H_i \subset \mathcal{H}(v_i)$ be the vector space generated by $F(i)$. This is a real vector space of dimension $2k - g + 1$, so it is reasonable to surmise that there exists a $\sigma_{C_{vi}}$-invariant divisor $D_i$ of degree $2k$ on $C_{vi}$ such that $H_i \subseteq H^0(L^R(D_i))$. The following result shows that this is indeed the case.

**Lemma 5.2.** Let $H_i$ be as above, $1 \leq i \leq g$. We have

$$
H_i \subset H^0(C_{vi}, L^R(2k \cdot \infty))
$$

where $\infty$ abusively denotes the support of the pullback of $\infty \in \mathbb{P}^1$ by the (hyperelliptic) structure morphism $C_{vi} \rightarrow \mathbb{P}^1$. 

Proof of Lemma 5.2. The key to the proof, which follows easily from our construction, is the fact that
\[
\text{div}_{C_{v_j}}(x) = 2 \cdot 0 - 2 \cdot \infty \quad \text{and} \quad \text{div}_{C_{v_j}}(y) = (2i - 1) \cdot 0 + 1 \cdot (\alpha_1, 0) + 1 \cdot (\alpha_2, 0) - (2i + 1) \cdot \infty \]
for all \( i = 1, \ldots, g \). From the equations (17) we deduce that
\[
\text{div}_{C_{v_j}}(x') = 2j \cdot 0 - 2j \cdot \infty \quad \text{for all} \quad j = 0, \ldots, k \quad \text{and} \quad \text{div}_{C_{v_j}}(x'y) = (2j + 2i - 1) \cdot 0 + 1 \cdot (\alpha_1, 0) + 1 \cdot (\alpha_2, 0) - (2j + 2i - 1) \cdot \infty \quad \text{for all} \quad j = 0, \ldots, k - g - 1.
\]
The desired result is now clear. \( \Box \)

We will now relate the inflection of the limit linear series \((L_\mathbb{R}(2k \cdot \infty), H_j)\), \( j = 1, \ldots, g \) along the elliptic curves \( C_{v_j} \) to the inflection of the original series \( |L_\mathbb{R}(kD)| \) along the hyperelliptic curve \( X \). In the proofs of Theorems 5.3 and 5.6 below we assume \( k \geq g + 1 \), but a trivial modification of the arguments settles the case \( k = g \), with the statements of the theorems unchanged.

More precisely, for \( j = 1, \ldots, g \), begin by compactifying the curve \( V(y^2 - \beta x^{2j-1}(x - \alpha_1)(x - \alpha_2)) \) inside \( \mathbb{P}^1 \times \mathbb{P}^1 \). Assume the complex numbers \( \beta, \alpha_1 \) and \( \alpha_2 \) are nonzero, and that \( \alpha_1 \neq \alpha_2 \). Suppose further that the polynomial \( \beta x^{2j-1}(x - \alpha_1)(x - \alpha_2) \) is real.

Let \( C_j = C_{v_j} \) denote the normalization of the curve above; thus \( C_j \) is a real elliptic curve with non-empty real part. The number \( n = n(C_j) \) of components of \( C_j(\mathbb{R}) \) is characterized by the following dichotomy: \( n(X_j) = 1 \) if and only if \( \alpha_1 = \overline{\alpha_2} \) and \( n(C_j) = 2 \) if and only if \( \alpha_1, \alpha_2 \in \mathbb{R}^* \). The function field \( K(C_j) \) is equal to \( \mathbb{R}(x)[y]/(y^2 - \beta x^{2j-1}(x - \alpha_1)(x - \alpha_2)) \).

We will compute the inflectionary weight of \((L_\mathbb{R}(2k \cdot \infty), H_j)\) at each of the four marked points \( P \in \{0, (\alpha_1, 0), (\alpha_2, 0), \infty\} \) of \( C_j \). Recall that the inflectionary weight \( |P| \) of a linear series \( V \) of rank \( r \) in a point \( P \) is the total difference between the sequence of vanishing orders of a local basis of holomorphic sections for \( V \) and the generic sequence \((0, 1, \ldots, r)\).

**Theorem 5.3** (Inflection in marked points of elliptic curves). For every \( j = 1, \ldots, g \), let \( C_j \) and \( H_j \subset H^0(C_j, L_\mathbb{R}(2k \cdot \infty)) \) denote the smooth elliptic curve and linear series, respectively, constructed above. The inflectionary weights of \( H_j \) in the marked points \( 0, \infty, (\alpha_1, 0) \) and \((\alpha_2, 0)\) are given by
\[
|0| = \binom{g+1}{2} + 2(k-g)(j-1);
|\infty| = \binom{g+1}{2} + 2(k-g)(g-j); \quad \text{and} \quad
|((\alpha_1, 0))| = |((\alpha_2, 0))| = \binom{g+1}{2}.
\]
In particular, we have \( |0| + |\infty| = 2\binom{g+1}{2} + 2(k-g)(g-1) \), irrespective of \( j \).

**Proof.** We begin by analyzing the case \( P = (\alpha, 0) := (\alpha_j, 0) \), where \( j = 1, 2 \). Note that \( (x - \alpha)^i = \sum_{t=0}^{i} \binom{i}{t}(-\alpha)^i-tx^t \) belongs to \( H^0(L_\mathbb{R}(2k \cdot \infty)) \) whenever \( 0 \leq i \leq k \) and has vanishing order \( \text{ord}((x - \alpha)^i, (\alpha, 0)) = 2i \). Similarly, \( (x - \alpha)^iy \) belongs to
It follows the vanishing sequence of \( H_j \) in \((\alpha, 0)\) is
\[
\text{ord}(H_j, (\alpha, 0)) = (0, 1, \ldots, 2(k - g) - 1, 2(k - g); 2(k - g) + 2, 2(k - g) + 4, \ldots, 2k)
\]
and the inflectionary weight in zero is \(|(\alpha, 0)| = \binom{g+1}{2}\).

In a similar vein, the inflection of \( H_j \) in 0 is determined by the vanishing orders of the functions \( F(j) \) in zero, namely
\[
\text{ord}(x^i, 0) = 2i, 0 \leq i \leq k \text{ and } \text{ord}(x^i y, 0) = (2i + 2j - 1), 0 \leq i \leq k - g - 1.
\]

It follows that the inflectionary weight in zero is given by
\[
|0| = 2 \left( \binom{k+1}{2} + \binom{k-g}{2} \right) + (2j-1)(k-g) - \left( \binom{2k-g+1}{2} - \binom{2k-g}{2} \right) = \left( \binom{g+1}{2} + 2(k-g)(j-1) \right).
\]

Finally, when \( P = \infty \), we proceed much as in the \( P = 0 \) case. Indeed, the inflection of \( H_i \) in \( \infty \) is determined by the pole orders in \( \infty \) of the meromorphic functions \( F(j) \), normalized by the generic pole order 2k. Namely, we have
\[
\text{ord}(x^i, \infty) = 2k - 2i, 0 \leq i \leq k \text{ and } \text{ord}(x^i y, \infty) = 2k - (2i + 2j + 1), 0 \leq i \leq k - g - 1.
\]

It follows that
\[
|\infty| = -2 \left( \binom{k+1}{2} - (2j+1)(k-g) - \binom{k-g}{2} + \frac{1}{2} (-2k)(2k-g+1) + \binom{2k-g+1}{2} \right)
\]
\[
= \left( \binom{g+1}{2} + 2(k-g)(g-j) \right).
\]

It is now easy to see that our collection of linear series \( \{H_j\} \) satisfies a compatibility relation in the points of attachment analogous to the defining condition for (refined) Eisenbud–Harris limit linear series \([2]\).

**Corollary 5.4.** The set of linear series \( \{H_j : 1 \leq j \leq g\} \) satisfies the compatibility relation
\[
o_i(H_j, \infty) + o_{2k-g-i}(H_{j+1}, 0) = 2k
\]
for all \( i = 0, \ldots, g \) and for all \( j = 0, \ldots, 2k-g \), where \( o(H_j, P) = (o_1(P), \ldots, o_{2k-g}(P)) \) denotes the set of vanishing orders of \( H_j \) in the point \( P \) of the \( j \)-th elliptic component, placed in strictly increasing order.

**Proof.** Immediate, given the proof of Theorem 5.3. \( \square \)

**Remark 5.5.** According to our construction, points of the inflection divisor \( \text{Inf}(|L_k(kD)|) \) of the complete linear series along the hyperelliptic curve specialize unambiguously to particular elliptic components \( C_{v_i} \), and never to the edge common to two adjacent type-II vertices of the skeleton or to an infinite length edge. These edges are dual to an edge in the subdivision of the Newton polygon of the hyperelliptic curve linking either the vertices labeled \( y^2 \) and \( x^{2i-1} \) for \( i = 1, \ldots, g + 1 \), or the vertices \( x^{2i-1} \) and \( x^{2i+1} \) for \( i = 1, \ldots, g \).

Further, any curve of the form \( y^2 - \nu x^\mu = 0 \) with \( \nu \in \mathbb{C} \) and \( \mu \geq 3 \) is unramified away from 0 or \( \infty \), e.g. because it admits a parametrization by monomials in a single auxiliary variable.
For $i = 1, \ldots, g$, let $E_i \subset A_i$ denote the collection of marked points on $C_{v_i}$ which are not points of attachment of the curve $C_{v_i}$, and let $E_\ast = \bigcup_i E_i$. Note that points of ramification divisor $R_\pi$ of the hyperelliptic curve specialize to points of $E_\ast$. Further, given Remark 5.5 it is clear that the analogous statement holds at the level of inflection divisors. Namely, let $R_i$ denote the restriction to $E_i$ of the inflection divisor of $H_i$ along $C_{v_i}$, $i = 1, \ldots, g$. The contribution $R$ of the $\pi$-ramification locus to $\text{Inf}(|L_\pi(kD)|)$ then specializes to the sum of inflectionary loci $\sum_{i=1}^g R_i$ along the elliptic components supported along $E_\ast$. The following result implies that the specialization $R \mapsto \sum_{i=1}^g R_i$ is in fact bijective.

**Theorem 5.6 (Contribution of $R_\pi$ to $\text{Inf}(|L_\pi(kD)|)$).** Write $\text{Inf}(|L_\pi(kD)|) = R + S$ as in Proposition 3.2. We have $R = \left(\frac{g+1}{2}\right) R_\pi$, where $R_\pi$ denotes the ramification divisor of $\pi$.

**Proof.** Much as in the proof of Theorem 5.3, we proceed by calculating the inflectionary weight in each point $P \in \text{Supp}(R_\pi)$. If $P \notin \{0, \infty\}$, essentially the same argument used in proving Theorem 5.3 yields $P = \left(\frac{g+1}{2}\right)$. It remains to compute $|0|$ and $|\infty|$. For this purpose, we use the vanishing orders of the basis $F$ of $H^0(X, L(kD))$, which in turn are prescribed by $\text{div}_X(x)$ and $\text{div}_X(y)$, much as in Lemma 5.2. This time, we have

$$\text{div}_X(x) = 2 \cdot 0 - 2 \cdot \infty$$

$$\text{div}_X(y) = 1 \cdot 0 + R_\pi^\circ - (2g + 1) \cdot \infty$$

where $R_\pi^\circ$ denotes the sum of the $2g$ simple ramification points of $\pi$ that lie inside $\mathbb{C}^*$. It follows that

$$\text{ord}(x^i, 0) = 2i, 0 \leq i \leq k$$

and

$$\text{ord}(x^i y, 0) = 2i + 1, 0 \leq i \leq k - g - 1;$$

$$\text{ord}(x^i, \infty) = 2k - 2i, 0 \leq i \leq k$$

and

$$\text{ord}(x^i y, \infty) = 2k - 2i - 2g - 1, 0 \leq i \leq k - g - 1.$$

The fact that $|0| = |\infty| = \left(\frac{g+1}{2}\right)$ now follows easily.

The following result is an immediate consequence of Theorem 5.6.

**Corollary 5.7.** Assume that $k > g$. The real linear series $|L_\pi(kD)|$ then has at least $g(g + 1)n(X)$ real inflection points.

Finally, the following regeneration-type result sums up how $\text{Inf}(|L_\pi(kD)|)$ compares to the inflection divisors $\text{Inf}_\pi(H_i)$ associated with the linear series $H_i$ along the elliptic components $C_{v_i}$, $i = 1, \ldots, g$.

**Theorem 5.8.** Fix $g \geq 2$ and $k \geq g$. Let $f = \sum_{j=0}^{2g+1} a_j x^j$ be a polynomial of degree $2g + 1$ in $\mathbb{R}[x]$ and let $\nu : \Delta \cap \mathbb{Z}^2 \rightarrow \mathbb{N}$ be a function inducing the triangulation $\Theta$. Suppose that

- for every $i = 1, \ldots, g$, the polynomial $\sum_{(j, 0) \in \Theta_i \cap \mathbb{Z}^2} a_j x^j$ is separable; and
- the polynomial $\sum_{j=0}^{2g+1} a_j t^{\nu(j, 0)} x^j$ is separable, and vanishes in $x = 0$.

For every $i$, let $\overline{U}_i$ be the compactification of the curve $V(y^2 - \sum_{(j, 0) \in \Theta_i \cap \mathbb{Z}^2} a_j x^j)$ inside $\mathbb{CP}^1 \times \mathbb{CP}^1$ and let $C_{v_i}$ be its normalization. Let $H_i$ be the real $g_{2k-2}^{2k-2}$ on $C_{v_i}$ spanned by the functions $1, \ldots, x^g; y, xy, \ldots, x^{k-g-1}y$. Then for generic values of $0 < \epsilon \ll 1$, the linear series $|L_\pi(kD)|$ on the real hyperelliptic curve $V(y^2 - f)|_{t=\epsilon}$ satisfies

$$\deg_{\pi} \text{Inf}(|L_\pi(kD)|) = \sum_{i=1}^g \deg_{\pi} (\text{Inf}(H_i)) - g(g - 1)(2k - g + 1).$$
Proof. Since the polynomial \( \sum_{j=0}^{2g+2} a_j t^{(j,0)} x^j \) is separable, the curve

\[
X_{\mathcal{C}(t)}(\mathbb{K}) = V(y^2 - \sum_{j=0}^{2g+2} a_j t^{(j,0)} x^j)
\]

is hyperelliptic. Since \( \sum_{(j,0) \in \Theta, r \in \mathbb{Z}_2} a_j x^j \) is separable for each \( i = 1, \ldots, g \), it follows that the linear series \( |\mathcal{L}(kD)| \) on \( X_{\mathcal{C}(t)}(\mathbb{K}) \) specializes to \( H_i \) on (the normalization of) \( V(y^2 - \sum_{(j,0) \in \Theta, r \in \mathbb{Z}_2} a_j x^j) \). It follows that

\[
\deg \text{Inf}(|\mathcal{L}(kD)|) = \sum_{i=1}^{g} \deg \text{Inf}(H_i) - \text{(total inflectionary weight in points of attachment)}.
\]

On the other hand, Corollary 5.4 implies that the inflectionary weight contributed by each of the \( (g-1) \) pairs of neighboring points of attachment is computed by

\[
2k(2k - g + 1) - 2 \cdot \binom{2k - g}{2} = g(2k - g + 1)
\]

which yields (18).

Further, for generic \( 0 < \varepsilon \ll 1 \), the divisor \( \text{Inf}(|\mathcal{L}(kD)|) \) on \( X_{\mathcal{C}(t)}(\mathbb{K}) \) deforms to the divisor \( \text{Inf}(|\mathcal{L}_\varepsilon(kD)|) \) of the real hyperelliptic curve \( V(y^2 - f_{l=\varepsilon}) \), as the projection \( q^\# \) is open above the Archimedean disk \( D_\delta \). Finally the deformation \( q^\# \) respects the real part, as it is a real deformation.

It is worth emphasizing that in our construction, our specialization morphism is defined by the initial coefficient morphism (16), which in turn is induced by the embedded tropicalization morphism (12). We close this section by pointing out a further important difference between our method and that of [6].

Let \( \Omega \) be the metrized complex of curves obtained from \( \text{Sk}(X_{0}^\#) \) by forgetting the marked points \( \text{Supp}(R_\pi) \). Namely, \( \Omega \) is given by the tuple \( (G = (V_\Omega, E_\Omega), \ell_\Omega, \{((C_{v_i}, \mathcal{G}_i)\})_i) \), where \( V_\Omega = \{v_1, \ldots, v_g\} \), \( E_\Omega = \{e_1, \ldots, e_{g-1}\} \), the function \( \ell_\Omega \) the restriction of \( \ell \), and \( \mathcal{G}_i \subset \mathcal{A}_i \) the set of attachment points of \( C_{v_i} \).

Since \( V_\Omega \) is a semistable vertex set for \( X_{0}^\# \), there is a semistable model \( \mathfrak{X} \) for \( X_{\mathcal{C}(t)} \) over the valuation ring \( \mathbb{K}^\circ \) for which the corresponding metrized complex \( \mathfrak{CX} \) is the metrized complex \( \Omega \).

The Amini–Baker specialization map

\[
\tau^\mathfrak{CX}_{\ast} : \text{Div}(X_{\mathcal{C}(t)}) \rightarrow \text{Div}(\mathfrak{CX})
\]

is constructed as the composition of the reduction morphism \( \text{red} : X_{\mathcal{C}(t)} \rightarrow X_s(\mathbb{C}) \) with the retraction morphism \( \tau : X_{\mathcal{C}(t)} \rightarrow \Gamma \), applying the fundamental identification

\[
\tau^{-1}(v_i) = \text{red}^{-1}(C_{v_i} \setminus \mathcal{G}_i).
\]

The two specialization maps \( \tau^\text{Sk}(X_{0}^\#) \) and \( \tau^\mathfrak{CX} \) differ since in the algebraic setting, the metrized complex \( \mathfrak{CX} \) represents an actual semistable curve \( X_s(\mathbb{C}) \), while \( \text{Sk}(X_{0}^\#) \) does not. Finally, note \( X_{\mathcal{C}(t)} \) is defined over the valuation ring \( \mathbb{K}^\circ \), so it is equipped with a natural model over \( \mathbb{K}^\circ \), but the latter is not semistable.
6. Combinatorial construction of curves

Fix \( k > g > 1 \), and let \( \Theta \) denote the regular subdivision of the triangle \( \Delta \) from the preceding sections. If follows from Theorem 5.8 that in order to construct real algebraic hyperelliptic plane curves such that \( |L| \) has many real inflection points, we start by considering collections \( \{Q_1, \ldots, Q_g\} \) of polynomials in \( \mathbb{R}[x] \) of the form

\[
Q_i = a_i x^{2i-1}(x-x_{i,1})(x-x_{i,2})
\]

for which

1. \( x_{i,1} \neq x_{i,2} \neq 0 \) for \( 1 \leq i \leq g \); and
2. the \( a_i \in \mathbb{R}^* \) satisfy the patchworking conditions \( a_{i+1} x_{i,1} x_{i,2} = a_i \) for \( i = 1, \ldots, g - 1 \).

In particular it follows that the family \( \{Q_1, \ldots, Q_g\} \) is uniquely prescribed by the \( 2g + 1 \) parameters \( \{a_1, x_{i,j}\} \). Let \( E_i \) denote the normalization of (the compactification of) \( V(y^2 - Q_i) \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \); as explained at the end of Section 3, the fact that \( Q_i \in \mathbb{R}[x] \) implies the following dichotomy:

- either \( x_{i,j} \in \mathbb{C} \setminus \mathbb{R} \), in which case \( n(E_i) = 1 \);
- otherwise, \( n(E_i) = 2 \).

The distribution of real inflection points of complete real linear series of degree \( d \geq 2 \) on a real elliptic curve \( E \) was completely characterized in [1, Thm 3.2.5]; it is predicated on the fact that inflection points are in bijection with \( d \)-torsion points, which may be visualized on the universal cover of \( E \). A natural question is how this result generalizes to a description of real inflection points of an incomplete real series along \( E \). For the sake of completeness, we recall the explicit classification of loc. cit., and finish by commenting on how it might be generalized.

Accordingly, let \( E = (E_C, \sigma_E) \) be a real algebraic curve of genus 1 with \( E(\mathbb{R}) \neq \emptyset \), and let \( V \) be a real complete linear series of degree \( d \geq 2 \). Then \( V \) has always \( d \) real inflection points when \( E(\mathbb{R}) \) is connected. On the other hand, when \( n(E) = 2 \), the distribution of real inflection points of \( V \) is determined by the parity vector \( c(V) \in (\mathbb{Z}/2\mathbb{Z})^2 \) according to the following trichotomy:

1. if \( c(V) = (1,0) \) or \( c(V) = (0,1) \), then \( V \) has \( d \) real inflection points located on the connected component of \( E(\mathbb{R}) \) on which \( V \) has odd degree;
2. if \( c(V) = (0,0) \), then \( V \) has \( d \) real inflection points on each component;
3. if \( c(V) = (1,1) \), then \( V \) has no real inflection point.

In our situation, \( V \) is not itself complete, but rather embeds in a complete linear series \( |L| = |L(2k \cdot \infty)| \) of even degree \( 2k \). So because the parity of the series is equal to the sum of the parities of its restrictions to individual real components of \( E \), we may disregard possibility (i) when \( n(E) = 2 \) above. Similarly, if \( n(E) = 2 \), then \( c(|L|) = (0,0) \). So either \( E(\mathbb{R}) \) is connected or \( n(E) = 2 \) and \( c(|L|) = (0,0) \).

We know then that \( |L| \) has \( d \) real inflection points on each component of \( E(\mathbb{R}) \), that \( |L| \) is spanned by sections

\[
\tilde{\mathcal{F}} = \{1, x, \ldots, x^k, y, yx, \ldots, yx^{k-2}\}
\]

and the question is how these relate, if at all, to the inflection points of the subseries spanned by

\[
\mathcal{F} = \{1, x, \ldots, x^k; y, yx, \ldots, yx^{k-g-1}\}.
\]
Theorem 6.1. With notation as above, we have

\[ \omega_L(k; g) \geq \begin{cases} 
g(g + 1) + 2(k - g)(g - 1), & \text{if } n(E) = 1, \\
2g(g + 1) + 2(k - g)(g - 1), & \text{if } n(E) = 2. 
\end{cases} \]

Note, in particular, that if \( n(E) = 2 \) and \( k = g \), we have \( \omega_C(k, g) = \omega_R(k, k) \).

Proof. The result follows from the fact that the functions of \( k \) and \( g \) that appear in (20) represent the inflection multiplicities of \( V(g) \) at the ramification points. \( \square \)

In order to improve upon the lower bounds in Theorem 6.1, the key issue is to understand how inflection points of the restricted basis \([19]\) lift to the universal cover of \( E \). Once this (together with the appropriate generalization of \([11\) Thm 3.2.5]) has been achieved, select polynomials \( Q_1, \ldots, Q_g \in \mathbb{R}[x] \) as above and set \( E_i := V(y^2 - Q_i) \). Let \( \nu : \Delta \cap \mathbb{Z}^2 \rightarrow \mathbb{Z} \cup \{\infty\} \) be a function that induces \( \Theta \). We then patchwork the polynomials \( Q_i \) to form the larger polynomial \( Q = y^2 - \sum_{i=0}^{2g+2} a_{ij} \nu(i,0) x^i \). For \( 0 < \epsilon \ll 1 \), the number of real inflection points of the real hyperelliptic curve defined by \( Q_{|\epsilon} \) is then dictated by Theorem 5.8.

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School of Mathematics, Tata Institute for Fundamental Research, Homi Bhabha Road, Mumbai 400005, India  
E-mail address: indranil@math.tifr.res.in

Instituto de Matemática, UFF, Rua Mário Santos Braga, S/N, 24020-140 Niterói RJ, Brazil.  
E-mail address: cotterill.ethan@gmail.com

Instituto de Matemática, UFF, Rua Mário Santos Braga, S/N, 24020-140 Niterói RJ, Brazil.  
E-mail address: cgaray@impa.br