ON GROUP SYMMETRIES OF THE HYDRODYNAMIC EQUATIONS FOR RAREFIED GAS

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ABSTRACT. The invariant group transformations of three-dimensional hydrodynamic equations derived from the Boltzmann equation are studied. Three levels (with respect to the Knudsen number) of hydrodynamic description are considered and compared: (a) Euler equations, (b) Navier-Stokes equations, (c) Generalized Burnett equations (GBEs), which replace the original (ill-posed) Burnett equations. The main attention is paid to group analysis of GBEs in their most general formulation because this and related questions have not been studied before in the literature. The results of group analysis of GBEs and, for comparison, of similar results for Euler and Navier-Stokes equations are presented in two theorems and discussed in detail. It is remarkable that the use of computer code greatly simplifies the proof of the results for GBEs, which are very cumbersome equations with many undetermined parameters.

1. Introduction. The paper is devoted to the study of symmetry properties of various equations of hydrodynamics derived from the Boltzmann equation by Chapman-Enskog expansion. The small parameter in the expansion is the so-called Knudsen number $Kn = l/L$, where $l$ and $L$ are, respectively, the mean free path of a gas molecule and a typical macroscopic length. The continuum limit $Kn = 0$ corresponds to the classical Euler equations of gas dynamics, whereas the terms linear in $Kn$ correspond to the contribution of viscosity and thermal conductivity in the Navier-Stokes equations for a rarefied gas. These are classical results of the kinetic theory of gases (see e.g. the well-known books [10, 16]). There is, however, a difficulty with the next step of the expansion, which introduces terms quadratic in $Kn$. Then one obtains the so-called Burnett equations [10, 16], which are ill-posed [6]. There are several ways to deal with this problem (see, in particular, [4, 3, 9] and references therein). In this paper we will use just one approach which seems to us the most natural. This approach was proposed in [4] by one of the authors. Its main idea is to use successive changes of dependent variables, similarly, to some extent, to the

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transformation of a vector ODE to its Poincare normal form \[1\]. The details of this approach are published in \[4, 3, 9\] (see also the book \[5\]) and therefore we do not discuss them here. Instead we just use the results of this approach, the so-called Generalized Burnett Equations (GBEs) introduced in \[3\]. Generally speaking, this is a family of equations which generalizes the ill-posed Burnett equations and makes them well-posed without any loss of formal accuracy with respect to the Knudsen number. These equations are described in more detail in Sections 2, 3 of the paper.

One of the methods for constructing solutions of differential equations is the group analysis method \[19, 18\]. The main tool for constructing solutions in this method is a Lie group of transformations which transform a solution of a system of differential equations into another solution of the same system. Such a Lie group is called admitted. The notion of an admitted Lie group has also been extended to equations with nonlocal terms (see, for example, \[17, 13, 12\]).

Our main goal in this paper is to perform the complete Lie group analysis \[19, 18\] of GBEs and compare the results with similar results for Euler equations and Navier-Stokes equations. For readers who are not familiar enough with methods of group analysis we briefly present all necessary information in Section 4. The main results of the paper are formulated in Theorem 4.1, which describes the admitted Lie group of GBEs under all possible values of the arbitrary elements (14 real constants and one function of absolute temperature \(T\)). For comparison, we also included in Section 4 similar results for Euler equations from \[19\] and Navier-Stokes equations (Theorem 4.2). Thus, we have obtained a general picture of all changes in group properties from the continuum media (Euler equations) to the Burnett level (GBEs). Details of the proofs of Theorems 4.1 and 4.2 are given in Section 5. Explicit formulas for all related one-parameter group transformations are also presented in Appendix.

Note that the complexity of GBEs, which include the third order derivatives and 14 undetermined real parameters (plus one arbitrary function) is much higher than the complexity of the Euler or Navier-Stokes equations. Therefore it is not surprising that we need to use a computer in order to perform all steps of the complete group analysis. The details of the amount of computer work are also given in Section 5. We believe that this is a good example of use of computers for rigorous proofs of mathematical results.

2. Burnett equations. First we define the original (ill-posed) Burnett equations (see \[10, 16\]) for 3D flows of rarefied monatomic gas. We denote by \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\) and \(t \in \mathbb{R}_+\) the space and time variables respectively. The gas flow is characterized by five scalar functions of \((x, t)\): the density \(\rho\), the bulk velocity \(u = (u_1, u_2, u_3)\) and the absolute temperature \(T\). The Burnett equations for these variables read

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho D^\alpha + \frac{\partial p}{\partial x_\alpha} + \frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} &= 0, \\
\frac{3}{2} \rho DT + p \text{div} u + \pi_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + \text{div} q &= 0,
\end{align*}
\]

(1)

where \(\alpha, \beta = 1, 2, 3\), \(p = \rho T\) denotes the pressure of a gas,

\[
\pi_{\alpha\beta} = \pi_{\alpha\beta}^{NS} + \pi_{\alpha\beta}^B, \quad q_\alpha = q_\alpha^{NS} + q_\alpha^B,
\]

(2)

and

\[
D = \frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial x_\alpha}.
\]
Here and below the standard summation rule over repeating tensor indices $\alpha$, $\beta = 1, 2, 3$ is assumed. The tensors $\pi^{NS}$, $\pi^{B}$ and vectors $q^{NS}$, $q^{B}$ denote the Navier-Stokes and Burnett terms, respectively. The exact formulas for the Navier-Stokes terms are well-known:

$$
\pi^{NS}_{\alpha\beta} = -2\mu(T) \frac{\partial u_{\alpha}}{\partial x_{\beta}}, \quad q^{NS}_{\alpha} = -\lambda(T) \frac{\partial T}{\partial x_{\alpha}}, \quad \lambda(T) = \theta_0 \mu(T),
$$

where $\theta_0$ is constant. Here and below the bar denotes the symmetric and traceless part of the tensor, i.e.,

$$
\bar{a}_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} + a_{\beta\alpha}) - \frac{1}{3} \delta_{\alpha\beta} \text{Tr} a, \quad \text{Tr} a = a_{11} + a_{22} + a_{33}.
$$

The coefficients of viscosity $\mu(T)$ and heat conduction $\lambda(T)$ are considered here as given positive functions of $T > 0$. The Burnett terms are more complicated. They read as \cite{10, 16}

$$
\pi^{B}_{\alpha\beta} = \frac{\nu^2}{p} \left\{ \omega_1 \frac{\partial u_{\alpha}}{\partial x_{\beta}} \right\} \left\{ \frac{\partial^2 T}{\partial x_{\alpha} \partial x_{\beta}} + \omega_2 \left\{ \frac{1}{3} \frac{\partial T}{\partial x_{\beta}} \right\} \right\} + \omega_3 \frac{\partial T}{\partial x_{\alpha} \partial x_{\beta}} + \omega_4 \frac{\partial T}{\partial x_{\alpha}} \frac{\partial T}{\partial x_{\beta}} + \omega_5 \frac{\partial T}{\partial x_{\alpha}} \frac{\partial u_{\beta}}{\partial x_{\beta}} + \omega_6 \frac{\partial u_{\alpha}}{\partial x_{\beta}} \frac{\partial u_{\beta}}{\partial x_{\beta}},
$$

and

$$
q^{B}_{\alpha} = \frac{\nu^2}{p} \left\{ \theta_1 \frac{\partial T}{\partial x_{\alpha}} \right\} \left\{ \frac{1}{3} \frac{\partial^2 T}{\partial x_{\alpha} \partial x_{\beta}} + \theta_2 \left\{ \frac{\partial T}{\partial x_{\alpha}} \right\} \right\} \left\{ \frac{1}{3} \frac{\partial^2 T}{\partial x_{\alpha} \partial x_{\beta}} + \theta_3 \frac{\partial T}{\partial x_{\alpha}} \frac{\partial T}{\partial x_{\beta}} + \theta_4 \frac{\partial T}{\partial x_{\alpha}} \frac{\partial u_{\beta}}{\partial x_{\beta}} + \theta_5 \frac{\partial u_{\alpha}}{\partial x_{\beta}} \frac{\partial u_{\beta}}{\partial x_{\beta}} \right\},
$$

where $p = \rho T$, $\mu(T)$ is given in Eq. (3). The dimensionless parameters $\omega_i$, $i = 1, \ldots, 6$; $\theta_i, i = 0, \ldots, 5$ are in the general case known (in implicit form) functions of $T$. In the particular cases most typical for applications, namely those of molecules-hard spheres or point particles interacting with power-like repulsive pair potential $U(r) = \text{const}/r^n$, $n > 1$, where $r > 0$ stands for distance between colliding particles, these parameters are simply real numbers. In such cases the coefficients $\mu$ and $\lambda$ are also power-like functions of $T$, namely they both are proportional to $T^{k(n)}$, $k(n) = 1/2 + 2/n$. The case of hard spheres formally corresponds to the limit $n = \infty$, then $k(\infty) = 1/2$. There is only one particular case $n = 4$ (Maxwell molecules), for which all Burnett coefficients are known exactly. The exact values are \cite{10, 16}:

$$
\theta_0 = \frac{15}{4}, \quad \theta_1 = \frac{75}{8}, \quad \theta_2 = -\frac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3, \quad \theta_5 = \frac{117}{4},
$$

$$
\omega_1 = \frac{10}{3}, \quad \omega_2 = 2, \quad \omega_3 = 3, \quad \omega_4 = 0, \quad \omega_5 = 3, \quad \omega_6 = 8.
$$

The exact formulas for the Navier-Stokes terms in the case of Maxwell molecules read

$$
\mu(T) = \eta T, \quad \lambda(T) = (15/4) \eta T,
$$

where $\eta > 0$ is a known constant. Hence, everything is defined exactly in Burnett equations (1)-(6) for this particular molecular model.

3. **Generalized Burnett Equations (GBEs).** It was proven long time ago \cite{6} that the constant in space and time solution ($\rho_0 > 0, \mathbf{u}_0 = 0, T_0 > 0$) of Eqs. (1)-(6) in the notation of Eqs. (7), (8), which describes the thermodynamical equilibrium of Maxwell gas, is unstable with respect to short-wave periodic perturbations. Moreover, these equations are ill-posed. The same property was later proved for more general molecular models including hard spheres \cite{4}. It was proved in \cite{4} that we
can regularize Burnett equations without loss of their accuracy (with respect to
Knudsen number, see [4] for details) by using changes of variables \((\rho, u, T)\). We
will consider below the regularized version of these equations called the General-
ized Burnett Equations (GBEs). GBEs were introduced in [3]. These are equations
based on the following transformations changing the variable \(T(x, t)\) to new variable
\(T'(x, t)\) in Eqs. (1)-(6) (other dependent variables \(\rho\) and \(u\) remain unchanged):
\[
T = T' - (1/\rho) \text{div}(S/\rho),
\]
where
\[
S = a(T')\nabla \log \rho + b(T')\nabla \log T'.
\]
It is assumed that \(a(T')\) and \(b(T')\) have the same order of magnitude \(O(\mu^2)\) as
the Burnett terms (5), (6). The functions \(a(T')\) and \(b(T')\) are to be defined from
the condition that the equations in new variables \((\rho, u, T')\) must be stable and
well-posed. Note that the right hand side of Eqs. (2) is defined from the Chapman-
Enskog series with formal error \(O(\mu^3)\) (or, equivalently, \(O(\lambda^3)\)). Therefore we can
neglect all terms of similar orders in the transformed equations for \((\rho, u, T')\). It is
shown in [3] that the resulting equations read as
\[
\rho_t + \text{div}(\rho u) = 0, \quad \rho Du_\alpha + \frac{\partial \rho}{\partial x_\alpha} + \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = 0,
\]
\[
\frac{3}{2} \rho D_T + p \text{div} u + \Pi_{\alpha\beta} \frac{\partial u_\beta}{\partial x_\alpha} + \text{div} Q = 0,
\]
where \(T\) is replaced by \(T'\) for simplicity of notation and
\[
\begin{align*}
p &= \rho T, \quad \Pi_{\alpha\beta} = \pi_{\alpha\beta}^{NS} + \pi_{\alpha\beta}^B - \delta_{\alpha\beta} \text{div}(S/\rho), \\
Q_\alpha &= q_\alpha^{NS} + q_\alpha^B + \frac{1}{\rho} \left\{ 3S_\beta \frac{\partial u_\alpha}{\partial x_\beta} + \left( \frac{2}{3} + b \right) \frac{\partial}{\partial x_\alpha} \text{div} u \right\} + (\text{div} u) \left[ \left( a'T - 2a \right) \frac{\partial}{\partial x_\alpha} \log \rho + (b'T - 2b) \frac{\partial}{\partial x_\alpha} \log T \right],
\end{align*}
\]
\[
S = a(T)\nabla \log \rho + b(T)\nabla \log T.
\]
The primes in (12) denote derivatives with respect to \(T\). The equations (11),
(12) with arbitrary functions \(a(T)\) and \(b(T)\) are called GBFs in a wide sense. Note
that the original Burnett equations (1) - (6) obviously coincide with Eqs. (11), (12)
in the particular case \(a = b = 0\). We shall use the term ‘optimal’ GBEs for some
particular choice of functions \(a(T)\) and \(b(T)\). The reasons for this choice will be
explained below. In fact, the functions \(a(T)\) and \(b(T)\) depend only on terms with
higher derivatives in tensor \(\Pi\) and vector \(Q\) given in Eqs. (5), (6), (12). Simple
calculations lead to equalities (see [9] for details)
\[
\frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = - \frac{1}{\rho} \left\{ \frac{T}{3} \left( \frac{2}{3} \mu^2 \omega_2 + a \right) \frac{\partial}{\partial x_\alpha} \Delta \rho + \left( \frac{2}{3} \mu^2 (\omega_2 - \omega_3) + b \right) \frac{\partial}{\partial x_\alpha} \Delta T \right\} + \ldots,
\]
\[
\text{div} Q = \frac{1}{\rho} \left[ \frac{2}{3} \mu^2 (\theta_2 + \theta_4) + \frac{3a}{2} + b \right] \Delta \text{div} u + \ldots,
\]
where the dots stand for terms with lower order derivatives. It is easy to see that
almost all these terms are quadratic in the first-order derivatives (except for the
Navier-Stokes terms in (12)) and therefore they do not contribute in the equations
linearized around the equilibrium constant solution. It is clear from Eqs. (13), (14)
that we can choose the coefficients \(a(T)\) and \(b(T)\) in such a way that third-order
derivatives of two from three (five scalars) dependent variables disappear from the
system. Some arguments related to the half-space problem (see [2] for details) show
that the best choice is to get rid of third derivatives of $u$ and $T$. According to Eqs. (13), (14), this is possible if and only if

$$a(T) = -4\mu^2(\theta_2 + \theta_4 + \omega_3 - \omega_2)/9,$$

$$b(T) = 2\mu^2(\omega_3 - \omega_2)/3, \quad \mu = \mu(T).$$

This completes the definition of ‘optimal’ GBEs for arbitrary given Navier-Stokes coefficients $\{\mu(T), \lambda(T)\}$ and Burnett coefficients $\{\omega_i, i = 1, \ldots, 6; \theta_i, i = 1, \ldots, 5\}$ through Eqs. (10) – (12), (15), (16). In case of Maxwell molecules we obtain

$$a(T) = 13\mu^2/18, \quad b(T) = 2\mu^2/3.$$  

We note that the Burnett coefficients satisfy the identity (see e.g. [9])

$$\theta_4 = \omega_3,$$

which allows to eliminate $\theta_4$ from the above formulas. The optimal GBEs are convenient for practical use, but we will consider below a more general class of GBEs which includes all natural extensions of Burnett equations for a wide class of intermolecular potentials.

To this purpose we introduce the general form of GBEs for hard spheres and power-like potentials. These are the above equations (11), (12) in the notation of Eqs. (3), (4), where

$$\mu(T) = \gamma T^k, \quad \lambda(T) = \theta_0 \mu(T), \quad a(T) = c_a \mu^2(T), \quad b(T) = c_b \mu^2(T),$$

with arbitrary constants $\theta_0 > 0$, $c_a$ and $c_b$. Note that $k = k(n) = 1/2 + 2/n$ for the intermolecular potential $U(r) = \text{const}/r^n$, $n > 1$ in accordance with above considerations. Moreover, all Burnett coefficients $\omega_i, (i = 1, \ldots, 6); \theta_i, (i = 1, \ldots, 5)$ are in that case some fixed real numbers obtained from the corresponding linearized Boltzmann equation [10, 16].

We shall see below that the main group properties of general GBEs defined by equations (3), (4), (11), (12), (19) with constants

$$k, \quad c_a, \quad c_b; \quad \omega_i, (i = 1, \ldots, 6); \quad \theta_i, (i = 0, \ldots, 5)$$

are independent of specific values of these constant parameters unless $k \neq 3/2$. The case $k = 3/2$ should be considered separately. In addition, we assume below that $\omega_2 > 0, \theta_2 < 0$. These inequalities follow from Proposition 1 of the paper [9].

4. Statement of the problem and main results. Below we use the abbreviation GBEs for the general class of equations (11), (12) written in the notation of Eqs. (3) - (6). In addition we assume that

$$a = c_a \mu^2, \quad b = c_b \mu^2,$$

and that all the constant parameters

$$c_a, \quad c_b, \quad \theta_0, \quad \theta_1, \quad \theta_2, \quad \theta_3, \quad \theta_4, \quad \theta_5,$$

$$\omega_1, \quad \omega_2, \quad \omega_3, \quad \omega_4, \quad \omega_5, \quad \omega_6.$$

are given real numbers. The function $\mu(T)$ is assumed to be sufficiently smooth.

Our goal is to perform the standard group analysis of GBEs. Generally speaking, they are five nonlinear PDEs (11), (12) of third-order for scalar functions $\rho(x,t), \quad T(x,t),$ and 3D-vector function $u(x,t)$, where $x \in R^3, \quad t \in R$. The coefficients (near derivatives) of these PDEs depend on $(\rho, u, T)$ and also on the given function $\mu(T)$ (the viscosity coefficient) and the given constants (21).
We briefly remind the reader of some key notions from group analysis. Consider a smooth function (mapping) \( \xi : \mathbb{R}^m \to \mathbb{R}^m \), \( m = 1, 2, \ldots \) and the Cauchy problem for ODE
\[
\frac{dy}{da} = \xi(y), \quad y(0) = z,
\]
where \( z \in \mathbb{R}^m \), \( a \in (-r, r) \), and \( r > 0 \).

Usually we consider not the whole space \( \mathbb{R}^m \), but a small neighbourhood of some point in \( \mathbb{R}^m \), but for brevity we ignore such details. The solution \( y(z, a) \) of this problem can be written as a one-parameter transformation \( T_a : \mathbb{R}^m \to \mathbb{R}^m \), where
\[
T_a z = y(z, a).
\]
Obviously the transformation \( T_a \) satisfies the basic group properties
\[
T_0 = 1, \quad T_a T_b = T_{a+b}, \quad (T_a)^{-1} = T_{-a},
\]
at least for sufficiently small absolute values of \( a \) and \( b \). The whole family of operators \( T_a \) for all admissible \( a \in \mathbb{R} \) form the so-called Lie group \( G \) of transformations of \( \mathbb{R}^m \) to itself. The tangential vector field \( \xi : \mathbb{R}^m \to \mathbb{R}^m \) of the group \( G \) can be written in the form of linear differential operator of the first order
\[
X = \xi(y) \cdot \frac{\partial}{\partial y},
\]
where dot denotes the scalar product in \( \mathbb{R}^m \). The operator \( X \) is called an infinitesimal generator or, equivalently, a generator of the one-parameter Lie group \( G \).

Returning to PDEs (11), (12) we can easily construct some Lie groups that leave these equations invariant. The simplest example is the group of translations of time-variable \( t' = t + t_0 \) with the parameter \( t_0 \). It obviously has the generator \( X = \frac{\partial}{\partial t} \).

The aim of group analysis of any given PDE or a set of PDEs is to find all such Lie groups acting on the ‘combined’ space (direct sum) of the independent and dependent variables that leave these PDEs invariant. In the case of GBEs (11), (12) we need to consider all Lie groups of transformations of the space \( \mathbb{R}^9 \) of the variables \((x_1, x_2, x_3, t, \rho, u_1, u_2, u_3, T)\) such that GBEs are invariant under these transformations.

There exist well-developed methods of group analysis (see, in particular, books [19, 18]) that allow in principle to construct all these transformations or, equivalently, to construct the admitted Lie group of a given set of PDEs. Our aim is to do it for GBEs and then to compare the results for three levels of hydrodynamic equations for monatomic rarefied gases: (1) the Euler equations or, equivalently, GBEs with \( \mu(T) = 0 \); (2) the Navier-Stokes equations or, equivalently, GBEs with nonzero \( \mu(T) \) and with the only nonzero constant \( \theta_0 \) in the list of given constants (21); (3) the general case of GBEs with arbitrary function \( \mu(T) \) and arbitrary constants (21).

The results of group analysis are traditionally expressed in the language of generators \( X^i \), \( (i = 1, 2, \ldots, N) \) of admitted one-parameter groups. It follows from the general theory (see e.g. [19]) of group analysis that the set of linearly independent generators \( X^i \), \( (i = 1, 2, \ldots, N) \) coincides with the set of solutions of certain linear equations, which are called determining equations. Therefore these generators form a vector space. Moreover, if \( X \) and \( Y \) belong to this space then their commutator \([X, Y] = XY - YX\) also belongs to this space [19]. Hence, the whole space of admitted generators has the structure of a Lie algebra. Therefore we can use the
equivalent notions of an admitted Lie group or admitted Lie algebra interchangeably (below our results are formulated in terms of groups).

The main results of the paper can be formulated as follows.

**Theorem 4.1.** Consider GBEs (11), (12) written in the notation of Eqs. (3) - (6), (20), where \( \mu(T) \) is any function having bounded derivatives up to the second order in \( R^{\infty} \) and the constants (25) are any real numbers. In addition it is assumed that

\[
\mu(T) \neq 0, \quad \omega_2 \neq 0.
\]

Then the admitted Lie group of these PDEs is defined by the generators

\[
X_i = \partial_{x_i}, \quad X_{3+i} = t \partial_{x_i} + \partial_{u_i},
\]

\[
X_{\kappa(i,j)} = x_i \partial_{x_j} - x_j \partial_{x_i} + u_i \partial_{u_j} - u_j \partial_{u_i}, \quad (i, j = 1, 2, 3, \ i < j),
\]

\[
X_{10} = \partial_t, \quad X_{11} = t \partial_t + x_i \partial_{x_i} - \rho \partial_{\rho}.
\]

The extension of the Lie algebra \( L_{11} \), consisting of generators (23), occurs for \( \mu = \gamma T^k, \gamma, k \in R \), by the generator

\[
X_{12} = -t \partial_t + (2k - 1) \rho \partial_{\rho} + u_i \partial_{u_i} + 2T \partial_T.
\]

If \( k = 3/2 \) and

\[
\omega_1 = 2 \omega_2, \quad c_a = 0, \quad c_b = \frac{1}{3} \theta_1 + \frac{4}{9} \theta_2,
\]

then the extension of the group is given by the generator \( X_{12} \) and by the generator

\[
X_{13} = t(t \partial_t + x_i \partial_{x_i} - 3 \rho \partial_{\rho} - 2T \partial_T) + (x_i - tu_i) \partial_{u_i}.
\]

Because of assumptions (22), Theorem 4.1 does not automatically include the simpler cases of the Euler equations and the Navier-Stokes equations. Their group properties are formulated in the following theorem.

**Theorem 4.2.** (A) The Euler equations

\[
\rho_t + \text{div}(\rho u) = 0, \quad \rho Du + \frac{\partial p}{\partial x_\alpha} = 0, \quad \frac{3}{2} \rho DT + p \text{div} u = 0, \quad p = \rho T,
\]

admit the Lie group defined by the generators \( X_1, \ldots, X_{11}, X_{13} \) in the notation of Eqs. (23), (26), and the two additional generators [19]

\[
X_{12} = -t \partial_t + u_i \partial_{u_i} + 2T \partial_T, \quad X_{14} = \rho \partial_\rho.
\]

(B) The Navier-Stokes equations

\[
\rho_t + \text{div}(\rho u) = 0, \quad \rho Du + \frac{\partial p}{\partial x_\alpha} + \frac{\partial \pi^{NS}_{\alpha\beta}}{\partial x_\beta} = 0, \quad \frac{3}{2} \rho DT + p \text{div} u + \pi^{NS}_{\alpha\beta} \partial_{u_\alpha} + \text{div} q^{NS} = 0,
\]

where \( \pi^{NS} \) and \( q^{NS} \) given by Eqs. (3), (4) with arbitrary constant \( \theta_0 \) and function \( \mu(T) \), admit the Lie group defined by the generators \( X_1, X_2, \ldots, X_{11} \) in the notation of Eqs. (23). The extension of this group occurs for \( \mu = \gamma T^k, \gamma, k \in R \), by the generator \( X_{12} \) (24). If \( k = 3/2 \), then the extension of the group is given by the generator \( X_{12} \) and by the generator \( X_{13} \) (26).

Proofs of the theorems are discussed in the next section.

Note that the Navier-Stokes equations admit exactly the same group as the group for GBEs from Theorem 4.1 (the conditions (25) are irrelevant for NSEs and therefore can be ignored).
The corresponding one-parameter groups of transformations of solutions \( \{ \rho(x, t), u(x, t), T(x, t) \} \) are also presented in explicit form in the Appendix for the reader’s convenience. We note that the results of Theorem 4.2 for the Euler equations follow from the more general results of L.V. Ovsiannikov for group analysis of the whole class of the gas dynamics equations [19]. It is pointed out in [19] that most of the symmetries of the Euler equations can be guessed from simple ‘physical considerations’, but this is not so for the so-called projective transformations with generator \( X_{13} \) given in Eq. (26). It was shown later [7] that this symmetry is in some sense ‘inherited’ by the Euler equations from the projective symmetry of the Hamiltonian system of N-particles interacting with pair potential \( U(r) = \alpha/r^2, \alpha > 0 \). The connection goes through the BBGKY hierarchy and the Boltzmann equation [7]. Note that the Euler equations can be considered as the continuum limit of the Boltzmann equation, when the Knudsen number tends to zero. They do not depend on the intermolecular potential and are universal in that sense. However, the dependence on the potential is back at the next level of approximation in the Knudsen number, i.e. for the Navier-Stokes equations. It is not surprising that the transition to the Navier-Stokes equations brings back the connection with intermolecular potential. It follows from Theorem 4.2 (B) that the projective generator \( X_{13} \) is admitted for NSEs only in the case when the viscosity coefficient \( \mu(T) \) is proportional to \( T^{3/2} \), which precisely corresponds to the potential \( U(r) = \alpha/r^2 \) (the connection of \( \mu(T) \) with potential is explained above, right after Eq. (19)).

Remark 1. It should be noted that the Lie algebra \( L_{11} \) of generators (23) has been completely studied in [20]. The analysis of Lie algebra \( L_{14} \) for Euler equations (27) was also a part of the program SUBMODELS [20]. Classification of all subalgebras of \( L_{14} \) is given in [11].

To our knowledge, the problem of group symmetries of hydrodynamic equations at the Burnett level has not been considered before in the literature. The only exception is a recent paper [8] by these authors, related to the very particular case of the one-dimensional version of ‘optimal’ GBEs for Maxwell molecules. It was explained above in Section 3 why we choose to consider the Generalized Burnett Equations (GBEs). Theorem 4.1 presents results of the complete group analysis of GBEs. It shows that (a) the original (ill-posed) Burnett equations have the same Lie group as the Navier-Stokes equations (with the only exception, see below), as expected, and (b) the same is true for the GBEs with any choice of regularizing constants \( c_a \) and \( c_b \). Hence, it is proved that the specific way of regularization of the ill-posed Burnett equations proposed in [3] does not change their basic symmetry properties. The exceptional case is related with the projective generator \( X_{13} \) for GBEs with the viscosity coefficient \( \mu(T) \) proportional to \( T^{3/2} \). Theorem 4.1 shows that the generator \( X_{13} \) is admitted by GBEs if and only if the constants in (21) satisfy conditions (25). The conditions for \( c_a \) and \( c_b \) can be easily fulfilled as these constants can be chosen arbitrarily (if we ignore the question of well - posedness of GBEs, which is not related to group properties). The first equality in Eqs. (19) relates to the unknown coefficients \( \omega_i, (i = 1, 2) \), that can be obtained only by calculation of exact values of these coefficients by the Chapman-Enskog method. Since the intermolecular potential \( U(r) = \alpha/r^2, \alpha > 0 \), does not play any important role in applications of the Boltzmann equation, these values can hardly be found in the literature. Similarly, the well-posedness of GBEs depends on the constant \( c_b \) from (19), expressed in terms of the unknown constants \( \theta_i, (i = 1, 2) \). Therefore
the question of the projective invariance of GBEs for this particular case remains open. It is of some interest from the theoretical point of view, though it does not seem important for applications to rarefied gas dynamics.

5. Details of the proofs. The proof is based on the standard procedure of the group analysis method [19, 18], which consists of constructing determining equations and then solving them. As a rule, after transition on the manifold, defined by the equations studied, the determining equations become polynomials with respect to parametric derivatives. Splitting these polynomials with respect to these derivatives, one obtains an overdetermined system of partial differential equations. Due to over-determinedness of the system one can find the general solution of this system. There are well-developed algorithms, which allow to perform the main part of the calculations by a computer. In particular, the calculations used for analysis of GBEs are made by using the code [17] written in the symbolic manipulation system Reduce [14]. For interested readers we should notice that for solving this and similar problems there are several softwares. Some reviews of softwares for finding admitted Lie group are presented in [15]. One of the authors developed his own code written in Reduce [14]. Some descriptions of this code are presented in [17]. This code is prepared for deriving determining equations of admitted and equivalence groups. The process of solving the determining equations is iterative: after first iteration, one chooses simple equations, and uses solutions of these equations for the next iteration. This process leads to effective algorithm of solving the determining equations.

Before presenting details of the proofs we clarify the problem studied.

Equations (11) contain the constants $c_a, c_b, \omega_i, \theta_i$, and the function $\mu(T)$ which are arbitrary elements. The particular choice of them specifies a process studied. A Lie group of transformations on the space of the independent and dependent variables and arbitrary elements, which do not change the structure of the studied equations is called an equivalence group. The latter means that the form of the equations is not changed under a transformation, but the arbitrary elements can be. Equivalence transformations separate all equations into equivalent classes, giving rise to the classification problem [19]. This problem is called group classification. The group classification problem involves finding a Lie group admitted by system (11) in the case of arbitrary elements of general form and all possible specializations of them leading to extensions of this Lie algebra.

5.1. Equivalence group of eqs. (11). As system of equations (11) contains a large number of arbitrary parameters and because of the complexity of equations (11), in the present study for finding equivalence group we only include the function $\mu(T)$ in the set of arbitrary elements: the constants $\omega_i, \theta_j, c_a, c_b$ are assumed unchangeable in the analysis.

An infinitesimal approach is used for seeking the equivalence group [19]. The generator of the equivalence group is considered in the form

$$X^e = 4 \sum_{i=1}^4 \zeta_{x_i} \frac{\partial}{\partial x_i} + 5 \sum_{i=1}^5 \zeta_{v_i} \frac{\partial}{\partial v_i} + \zeta_\mu \frac{\partial}{\partial \mu},$$

where $v = (\rho, u, T)$ is a vector, $x_4 = t$, the coefficient $\zeta^\mu$ depends on $(t, x, \rho, u, T, \mu)$
and all other coefficients depend on \(^1(t, x, \rho, u, T, \mu)\). As the function \(\mu(T)\) only depends on \(T\), for finding equivalence group, equations (11) should be extended by the auxiliary equations

\[
\mu_t = 0, \ \mu_{x_i} = 0, \ \mu_\rho = 0, \ \mu_{u_i} = 0, \ (i = 1, 2, 3). \tag{28}
\]

Eqs. (11) also contain derivatives of the function \(\mu(T)\) up to second-order and derivatives of the dependent variables \((\rho, u, T, \mu)\) up to third-order. Hence, the generator \(X^e\) has to be prolonged up to third-order

\[
\dot{X}^e = X^e + \sum_{1<|\alpha|<3} \zeta^\alpha \frac{\partial}{\partial p_\alpha} + \sum_{i=1}^4 \zeta^{\mu_{x_i}} \frac{\partial}{\partial \mu_{x_i}} + \sum_{i=1}^5 \zeta^{\mu_{u_i}} \frac{\partial}{\partial \mu_{u_i}} + \zeta^{\mu'} \frac{\partial}{\partial \mu'} + \zeta^{\mu''} \frac{\partial}{\partial \mu''}.
\]

Here \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) is a multi-index. For the multi-index the notations \(|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\) and \(\alpha_i = (\alpha_1 + \delta_1, \alpha_2 + \delta_2, \alpha_3 + \delta_3, \alpha_4 + \delta_4)\) are used. The variable \(p_\alpha\) plays the role of the derivative \(\frac{\partial \mu}{\partial x_i}\), if \(|\alpha| = 0\), then \(p^i_\alpha = v_i\). The coefficients of the prolonged generator are found by the recurrent prolongation formulas \([19]\)

\[
\zeta^{p_\alpha} = D_1 \zeta^{p_\alpha} - p^i_{\alpha,k} D_i \zeta^{x_k},
\]

\[
\zeta^{\mu_{x_i}} = \frac{\partial \zeta}{\partial v_i} - \mu \frac{\partial \zeta^T}{\partial v_i}, \quad \zeta^{\mu_{u_i}} = \frac{\partial \zeta}{\partial u_i} - \mu \frac{\partial \zeta^T}{\partial u_i}, \quad \zeta^{\mu'} = \frac{\partial \zeta^T}{\partial T} + \mu \frac{\partial \zeta^T}{\partial \mu}, \quad \zeta^{\mu''} = \frac{\partial \zeta^T}{\partial \mu'} + \mu \frac{\partial \zeta^T}{\partial \mu''},
\]

where \(D_i\) is the total derivative with respect to \(x_i\).

In the infinitesimal approach for finding a Lie group one has to solve the determining equations. For the equivalence group the determining equations are

\[
\dot{X}^e F_{(S_1)} = 0, \tag{30}
\]

where \(F\) is the left-hand side of equations (11) and (28), the sign \(|(S_1)|\) means that the expression \(\dot{X}^e F\) should be considered on the manifold defined by equations (11), (28), and the prolongations up to the third-order of the first equation of (11).

Solving Eqs. (30) with the left-hand side \(F\) from Eqs. (28) gives that

\[
\zeta^T = \zeta^T(T), \quad \zeta^\mu = \zeta^\mu(T, \mu),
\]

which essentially simplify the calculations.

Calculations of the equivalence group and admitted Lie group are rather laborious tasks, as even the writing of equations (11) is very cumbersome. As mentioned above, the calculations were made by using a code written in the symbolic manipulation system Reduce \([14]\), where the computer program provides the set of equations for the coefficients of the generator \(X\) after splitting the determining equations. The calculations are made by iterations. First, the prolonged generator was applied to the first equation of (11). After splitting we obtained 4473 equations. One of the features of the analysis of these equations was the following. Many of the equations had the form of a linear combination with respect to the constants \(\omega_i\) and \(\theta_j\), in particular, of the form:

\[
F (a_{i1} \omega_1 + a_{i2} \omega_2 + a_{i3} \omega_3) = 0.
\]

\(^1\)Seeking for equivalence group, where all coefficients depend on \((t, x, \rho, u, T, \mu)\) \([17]\), is very cumbersome for system (11).
As for the coefficients \( \omega_i \) and \( \theta_j \) it is only known that \( \omega_2 \neq 0 \), then among the equations we had to find \( i, j, \) and \( k \) such that \( \det A \neq 0 \), where the matrix \( A \) is

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} \neq 0.
\]

Then it can be concluded that \( F = 0 \).

After solving the first determining equation, the prolonged generator was applied to other equations of (11). The most resource-consuming part was the second iteration when the prolonged generator was applied to the first equation of the conservation law of momentum: this iteration required more than 44 hours of CPU time and more than 2 GB of RAM.

Finally, the calculations show that a basis of the Lie algebra corresponding to the equivalence group consists of the generators

\[
X^e_i = \partial_{x_i}, \quad X^e_{3+i} = t\partial_{x_i} + \partial_{u_i}, \quad X^e_{\kappa(i,j)} = x_i\partial_{x_j} - x_j\partial_{x_i} + u_i\partial_{u_j} - u_j\partial_{u_i},
\]

\((i, j) = 1, 2, 3, i < j\),

\[
X^e_{10} = \partial_t, \quad X^e_{11} = t\partial_t + x_i\partial_{x_i} - \rho\partial_\rho, \quad X^e_{12} = -t\partial_t + u_i\partial_{u_i} - \rho\partial_\rho + 2T\partial_T, \quad X^e_{13} = \rho\partial_\rho + \mu\partial_\mu,
\]

where \( \kappa(2,3) = 7, \ \kappa(1,3) = 8, \ \kappa(1,2) = 9 \).

If the constants \( c_a, c_b, \omega_1, \omega_2, \theta_1, \) and \( \theta_2 \) satisfy the relations

\[
\omega_1 = 2\omega_2, \quad c_a = 0, \quad c_b = \frac{1}{3}\theta_1 + \frac{4}{9}\theta_2,
\]

then there is one more generator of equivalence group

\[
X^e_p = t(t\partial_t + x_i\partial_{x_i} - 3\rho\partial_\rho - 2T\partial_T - 3\mu\partial_\mu) + (x_i - tu_i)\partial_{u_i}.
\]

Here the generators \( X^e_1, X^e_2, X^e_3, \) and \( X^e_{10} \) determine shifts with respect to the independent variables; the generators \( X^e_i, X^e_{3+i} \), and \( X^e_{\kappa(i,j)} \) correspond to Galilean transformations; the transformations corresponding to the generators \( X^e_1, X^e_2, X^e_3 \), and \( X^e_p \) are rotations; \( X^e_{11}, X^e_{12}, \) and \( X^e_{13} \) define scaling of corresponding variables. The generators \( X^e_{12} \) and \( X^e_{11} \) allow scaling the function \( \mu \) and its independent variable \( T \), respectively.

Apart from the equivalence group, there is also the obvious involution

\[
x \rightarrow -x, \quad u \rightarrow -u.
\]

### 5.2. Admitted Lie group.

The admitted Lie group is derived by solving the determining equations

\[
\tilde{X}F|_{(S_2)} = 0,
\]

where \( F \) is the left-hand side of Eqs. (11), \( \tilde{X} \) is the prolongation of the generator

\[
X = \zeta^t \frac{\partial}{\partial t} + \zeta^x \frac{\partial}{\partial x_i} + \zeta^u \frac{\partial}{\partial u_i}.
\]

Here all coefficients of the generator \( X \) depend on \( t, x, \rho, u, T \), the coefficients of the prolonged generator \( \tilde{X} \) are defined by the formulas (29), the sign \( |(S_2) \) means that the expression \( \tilde{X}F \) should be considered on the manifold defined by equations (11) and the prolongations are up to the third-order of the first equation of (11).

The method of solving the determining equations of the admitted Lie group was similar to that explained above. In particular, splitting of the determining equation,
where $F$ is the left-hand side of the second equation of (11), led to the analysis of 22358 equations. In particular, the classifying equation
\[ G \left( \frac{\mu'}{T\mu} \right)' = 0, \]
was derived, where $G$ is some expression. This leads to the analysis of the cases (a) $\left( \frac{\mu'}{T\mu} \right)' \neq 0$, and (b) $\mu = \gamma T^k$, where $\gamma$ and $k$ are constant. Further analysis of case (b) requires to study the cases (b.1) $k \neq 3/2$ and (b.2) $k = 3/2$.

Finally, calculations prove that the Theorem 4.1 and Theorem 4.2 are valid.

6. Conclusions. In this paper we have studied the invariant Lie group transformations of various sets of hydrodynamic equations derived from the Boltzmann equation. The importance of such transformations in any field of mathematical physics is well-known. The invariant transformations are related to the most basic properties of PDEs and other equations [19, 18, 13, 12]. They allow one to construct the new solutions from the known ones. They also allow introducing specific classes of solutions that are invariant under one or several admitted one-parameter groups of transformations. These facts are well-known and widely used in mathematical physics and its applications to fluid mechanics.

The classical group analysis of non-linear PDEs allows to find ALL Lie group transformations admitted by given equation. This fact is very important for our study of equations of hydrodynamics at the Burnett level. To our knowledge, the group properties of full three-dimensional hydrodynamic equations at that level have not been studied before in the literature. There are obvious reasons for this, they are explained in Section 3. It was also explained in Section 3 why we choose to consider the Generalized Burnett Equations (GBEs). Theorem 4.1 can be considered as the main new result of our paper. The theorem presents the results of the group analysis of GBEs in its most general form. It shows that (a) the original (ill-posed) Burnett equations have the same Lie group as the Navier–Stokes equations (with the only exception, see below), as expected, and (b) the same is true for the GBEs with any choice of regularizing parameters . Hence, it is proved that the specific way of regularization of the ill-posed Burnett equations, proposed in [3], does not change their basic symmetry properties. The exceptional case is connected with the projective generator for GBEs with the viscosity coefficient $\mu(T)$ proportional to $T^{3/2}$. It corresponds to the intermolecular potential $U(r) = \alpha/r^2$, $\alpha > 0$, where $r > 0$ denotes the distance between colliding molecules. This special case is discussed in more detail at the end of Section 3. The question of the projective invariance of GBEs for this case remains open. It is of some interest from theoretical point of view, but some lengthy calculations related to the linearized Boltzmann collision operator are needed to clarify it. All other questions related to Lie group symmetries of GBEs seem to be clear. The next step would be to study various families of corresponding invariant solutions. We hope to consider this question in near future.

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Appendix. Explicit formulas of transformations corresponding to the generators \( X_i, \ (i = 1, 2, \ldots, 13) \) are:

\[
X_i : \quad \tilde{x}_i = x_i + a_i, \quad (i = 1, 2, 3);
\]

\[
X_{i+3} : \quad \tilde{x}_i = x_i + ta_i, \quad \tilde{u}_i = u_i + a_i, \quad (i = 1, 2, 3);
\]

\[
X_{k(i,j)} : \quad \tilde{x}_i = x_i \cos(a) - x_j \sin(a), \quad \tilde{x}_j = x_i \sin(a) + x_j \cos(a),
\]

\[
\tilde{u}_i = u_i \cos(a) - u_j \sin(a), \quad \tilde{u}_j = u_i \sin(a) + u_j \cos(a)
\]

\((i, j = 1, 2, 3, \ i < j)\);

\[
X_{10} : \quad \tilde{t} = t + a;
\]

\[
X_{11} : \quad \tilde{t} = te^a, \quad \tilde{x}_i = x_i e^a, \quad \tilde{\rho} = \rho e^{-a};
\]

\[
X_{12} : \quad \tilde{t} = te^{-a}, \quad \tilde{u}_i = u_i e^a, \quad \tilde{\rho} = \rho e^{(2k-1)a}, \quad \tilde{T} = Te^{2a};
\]

\[
X_{13} : \quad \tilde{t} = \frac{t}{1-at}, \quad \tilde{x}_i = \frac{x_i}{1-at}, \quad \tilde{u}_i = x_i a + (1-at)u_i, \quad (i = 1, 2, 3)
\]

\[
\tilde{\rho} = \frac{\rho}{(1-at)^3}, \quad \tilde{T} = \frac{T}{(1-at)^2};
\]

\[
X_{14} : \quad \tilde{\rho} = \rho e^a,
\]

where \( a \) is a group parameter, and only changeable variables are written.

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