A Topological Criterion for Alice Strings

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Abstract

Symmetry breaking can produce “Alice” strings, which alter scattered charges and carry monopole number and charge when twisted into loops. Alice behavior arises algebraically, when a string’s untraced Wilson loop obstructs unbroken symmetries — a fragile criterion. We give a topological criterion, compelling Alice behavior or deforming it away. Our criterion, that $\pi_0(H)$ acts nontrivially on $\pi_1(H)$, links topologically Alice strings to topological monopoles. We twist Alice loops to form monopoles, and find nematic and $^3$He-A Alice strings are topologically Alice, and carry fundamental monopole charge when twisted into loops.
INTRODUCTION AND OUR CRITERION

Among the defects created when gauged symmetries break down are Alice strings. [1, 2, 3, 4] Alice strings obstruct the global extension of unbroken symmetries, making them multivalued when parallel transported around the string. This algebraic obstruction has two prominent physical consequences. First, it produces nonconservation of associated charges, when Aharonov-Bohm scattered around the string. Second, it induces monopoles, as twisted loops of Alice string.

These Alice features arise due to gauge flux on the string’s core. The gauge flux generates the condensate winding, while acting on asymptotic particles through the Wilson line $U(\varphi)$. This action fixes particles’ Aharonov-Bohm scattering around the string, to one changing both charge and monopole number. Loops of string, which leave charge and monopole number well-defined asymptotically, thus support deposited unlocalizable charge (“Cheshire charge”) and deposited monopole number. [1, 2] Alice loops carry this deposited monopole number by twisting, [4] as we probe further below.

Alice strings in condensed matter systems are global, not gauged, defects. They have no gauge flux to fix their Aharonov-Bohm scattering, and guarantee altered charge and monopole number upon string traversal. However, their Aharonov-Bohm scattering was considered in [5, 6]; with [6] showing that global Alice strings generically share all Alice behaviors. They alter both charge and monopole number on string traversal, with twisted loops supporting both Cheshire charge and deposited monopole number.

The criterion for Alice string formation was first stated algebraically, in terms of the string’s untraced Wilson loop $U(2\pi)$. When $U(2\pi)$ fails to commute with an unbroken symmetry $h$, the symmetry cannot be globally extended; when it fails to commute with unbroken generator $T_h$, the associated charge is nonconserved. Thus Alice strings arise when $U(2\pi)$ lies outside the center of the unbroken symmetry group $H$. As noted in [2], this is an inherently nontopological criterion, as topologically equivalent choices for $U(2\pi)$ can commute with different subgroups of $H$. Thus emergence of Alice behavior appeared a dynamical question. Steps toward topologizing this criterion came in [3]. They noted that when all topologically equivalent choices for $U(2\pi)$ lie outside the center of $H$, Alice strings must form. Equivalently, topological Alice strings form when the fiber bundle of $H$ parallel transported around the string is nontrivializable. Both criteria, while accurate,
seem difficult to apply.

We here establish an easily applied topological criterion which states when Alice strings must form. We take $G$ to be the simply connected cover of the initial symmetry — a connected Lie group — and $H \subset G$ its unbroken subgroup. A topologically stable string has flux $U(2\pi)$ in a disconnected component of $H$, with topology determined by $\pi_o(H)$. Similarly, the topology of the monopole is given by $\pi_1(H)$, describing loops $h(\alpha)$ of different winding in $H$. By taking seriously the change in monopole number in circumnavigating the Alice string, we construct our criterion. Note that, in Aharonov-Bohm scattering around the string, the monopole $h(\alpha)$ is conjugated by the string's Wilson loop $U(2\pi)$:

$$h(\alpha) \rightarrow \tilde{h}(\alpha) = U(2\pi) \ h(\alpha) \ U^{-1}(2\pi) \ .$$

Monopole number changes if $\tilde{h}(\alpha)$ and $h(\alpha)$ are topologically distinct loops. We represent this transformation topologically, as $\pi_o(H)$ acting naturally on $\pi_1(H)$ by conjugation. Topological Alice strings form if that action is nontrivial: that is, for $h_o$ a representative element of $\pi_o(H)$ and $h(\alpha)$ a representative loop in $\pi_1(H)$,

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} \not\sim h(\alpha) \ .$$

A string with untraced Wilson loop $U(2\pi) \sim h_o$ meeting this criterion is topologically guaranteed to change monopole number; we dub it a topologically Alice string.

This criterion captures physical Alice behavior, is easily applied, and is topological. Its result, for any chosen $h_o$ and $h(\alpha)$, remains invariant under deformations of either flux $U(2\pi) = h_o$ or monopole loop $h(\alpha)$. By construction, the strings are topologically Alice if monopoles change topologically in traversing them. Of course, monopoles $h(\alpha)$ change because of algebraic Alice behavior: loop $h(\alpha) = e^{i\alpha T_h}$ alters only if its generator $T_h$ alters; that is, if $U(2\pi)$ fails to commute with generator $T_h$. This algebraic noncommutation creates the standard Alice constellation of behaviors: multivalued symmetry, charge-violating Aharonov-Bohm scattering, Cheshire charge on Alice loops. It is captured by our criterion only when altering generators alters the topology of the loops they generate. This misses some Alice phenomena — particularly in models with poorly distinguished loops, when $\pi_1(H) = 0$ (and all loops are trivial) or $\pi_1(H) = \mathbb{Z}_2$ (and all nontrivial loops, including a loop and its inverse, are identified). We claim that Alice behavior in these models is not robust topologically; that is, continuous deformation of such strings removes their
Alice behavior. In such cases, persistence of Alice behavior can arise only from dynamical arguments, favoring Alice strings over non-Alice strings of the same winding. Dynamically stabilized features remain interesting — for example, nontopological defects including embedded, semilocal, or electroweak strings. However, we seek here for Alice behavior the more robust motivation of topological imperative.

Note that this topological imperative comes at a cost: to fulfill our topological criterion, of $\pi_0(H)$ acting nontrivially on $\pi_1(H)$, $\pi_1(H)$ itself must be nontrivial. That is, only in a theory with topological strings and monopoles — and more pointedly, monopoles topologically distinct from antimonopoles — do topologically Alice strings arise.

Our topological criterion for Alice strings ensures that twisted loops carry monopole charge, as we see by explicit construction below. We note that topological arguments only indicate that deposited monopole charge can be carried by twisted Alice loops. In many models, Alice strings interchange monopoles with antimonopoles, depositing monopole charge in units of 2. Whether fundamental monopoles, or only those with even charge, are deformable to twisted loops of Alice string is model-dependent. We see both possibilities arise in.

We present our results as follows. We first show that Alice strings failing our topological test have topologically unstable Alice features; that is, their Alice behavior can be deformed away. We then show, by construction, that twisted Alice strings carry monopole charge. We argue topologically, using our criterion to display a twisted Alice loop, carrying monopole number deposited in the monopole scattering $h(\alpha) \rightarrow \tilde{h}(\alpha)$. We then illustrate our criterion and Alice loop twisting to form monopoles, for key models: the Schwarz Alice string, coinciding with the Alice string of liquid crystals and of non-chiral Bose condensates and a nontopologically Alice string introduced in. Key points include, for the nontopological model, a focus on how Alice candidates may fail our criterion; and for the Schwarz string, analysis of the monopole charge carried by twisted Alice loops. In both Schwarz and $^3$He-A Alice models, Alice string scattering changes monopole number by 2, yet twisted Alice loops carry a single fundamental unit of monopole charge. Thus, for Alice strings in condensed matter, even the fundamental monopole can be, in fact, a twisted Alice loop. In contrast, model Alice loops discussed elsewhere support only even, not fundamental, monopole charge. In this paper we elucidate our topological criterion and its generic consequences for twisted Alice loops; we elaborate consequences of the criterion for prominent condensed matter.
matter models in a companion paper [9].

**FAILURE MEANS TOPOLOGICAL INSTABILITY**

Consider an Alice string with Wilson loop $U(2\pi) = h_o$. The Alice string fails our topological criterion if, for a nontrivial loop $h(\alpha) \in H$ describing a monopole, the parallel transported monopole is homotopic to the original; that is,

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} \sim h(\alpha) . \quad (0.1)$$

This occurs only if there exists some continuous map $f(x)$ deforming $\tilde{h}(\alpha)$ to $h(\alpha)$; that is

$$f(x) : \tilde{h}(\alpha) \rightarrow \begin{cases} \tilde{h}(\alpha) \text{ when } x = 0 \\ h(\alpha) \text{ when } x = 1 \end{cases} .$$

Note that the map $f(x)$ relates nontrivial loops in $H$, with basepoint $\alpha = 0$ fixed at the group identity. We write it as a continuously varying group element $f(x)$ acting on $\tilde{h}(\alpha)$ by conjugation,

$$f(x) : \tilde{h}(\alpha) \rightarrow f(x) \tilde{h}(\alpha) f(x)^{-1} .$$

Without loss of generality we take $f(0) = \mathbb{1}$.

Now consider the continuous map $h'_o(x) = f(x) \ h_o$, where $f(x)$ acts on $h_o$ by left multiplication. This interpolates between the Alice string’s Wilson loop $h_o$, and group element $h'_o(1)$ in the same disconnected component of $H$. $h'_o(1)$ thus defines a topologically equivalent string, with Wilson loop $U(2\pi) = h'_o(1)$. In circumnavigating this deformed string, the original monopole $h(\alpha)$ is unchanged: it goes to

$$\tilde{h}'(\alpha) = h'_o(1) \ h(\alpha) \ h'_o^{-1}(1) = f(1) \ h_o \ h(\alpha) \ h_o^{-1} f(1)^{-1}$$

$$\quad = f(1) \ h_o \ f(1)^{-1} = h(\alpha) ,$$

by construction of $f(x)$. Thus the monopole loop $h(\alpha)$ remains identical on circumnavigating the string. Choosing as our nontrivial loop $h(\alpha) = e^{i\alpha T_h}$, the loop (at each value of $\alpha$) remains unchanged only if the generator $T_h$ remains unchanged. Thus by continuously deforming our Alice string’s flux from $h_o$ to $h'_o(1)$, we have obtained a string flux $U(2\pi) = h'_o(1)$ which commutes with all generators; that is, we have removed all Alice behavior of the string. This renders Alice behavior for strings failing our criterion topologically unstable; it can be deformed away, and stabilized only in dynamical, model-dependent ways.
TWISTED ALICE LOOPS ARE MONOPOLES

Monopoles lie on the vacuum manifold at spatial infinity, with topology given by \( \pi_2(G/H) \). We here show that a twisted topologically Alice loop is necessarily a topological monopole; that is, an infinite sphere enclosing it has nontrivial \( \pi_2(G/H) \).

First, we construct a sensible twisted Alice loop.

Recall that our Alice string has a condensate \( \langle \phi \rangle \) which winds asymptotically over the vacuum manifold \( G/H \) according to

\[
\langle \phi(\varphi) \rangle = U(\varphi) \langle \phi \rangle_o
\]

where the Wilson line \( U(\varphi) \) acts on the vev \( \langle \phi \rangle_o \) according to its group representation. \( U(\varphi) \) varies continuously over \( G \) for \( 0 < \varphi < 2\pi \), and connects the identity at \( \varphi = 0 \) to a distinct Wilson loop \( U(2\pi) \) in \( H \). The string is topological when \( U(2\pi) = h_o \) lies in a disconnected component of \( H \), with nontrivial \( \pi_o(H) \), and is topologically Alice when it meets our criterion (0.1).

Now twist the Alice string: continuously rotate its Wilson line within \( G \) by the angle-dependent \( H \)-group rotation \( h^{-1}(\alpha) \):

\[
U(\varphi, \alpha) = h^{-1}(\alpha) U(\varphi) h(\alpha)
\]

as shown in Figure 1a. This, of course, rotates our condensate among the degenerate vacua on \( G/H \):

\[
\langle \phi(\varphi, \alpha) \rangle = U(\varphi, \alpha) \langle \phi \rangle_o
\]

Under what conditions may we identify string ends at \( \alpha = 0 \) and \( \alpha = 2\pi \) to form a string loop, as pictured in Figure 1b? First, we require the string configurations to match at the junction. This is assured if \( h(2\pi) = h(0) \), that is, if \( h(\alpha) \) is a loop. Second, the twisted condensate \( \langle \phi(\varphi, \alpha) \rangle \) must be single-valued. Note that the Wilson line \( U(\varphi, \alpha) \) itself need not be single-valued: indeed, for a monopole configuration, \( U(\varphi, \alpha) \) interpolates from the identity at \( \varphi = 0 \) to a nontrivial loop in \( H \) at \( \varphi = 2\pi \).

First we check singlevaluedness of \( \langle \phi(\varphi, \alpha) \rangle \) at the loop’s origin. Here \( \varphi = 0 \) (or \( 2\pi \)) while \( \alpha \) is indeterminate. Note that \( U(0, \alpha) \) is the identity, manifestly single-valued. At \( \varphi = 2\pi \),

\[
U(2\pi, \alpha) = h^{-1}(\alpha) U(2\pi) h(\alpha)
\]
This generally does vary with $\alpha$; however, it is a loop in $H$, with basepoint $U(2\pi) = h_o \in H$. It thus leaves the condensate invariant, assuming the single value $\langle \phi \rangle_o$ at loop origin.

Elsewhere, we need only show first, that $\langle \phi (\varphi, \alpha + 2\pi) \rangle = \langle \phi (\varphi, \alpha) \rangle$; and second, that $\langle \phi (\varphi + 2\pi, \alpha) \rangle = \langle \phi (\varphi, \alpha) \rangle$. The first is trivial: since $h(\alpha)$ is a loop, $h(\alpha) = h(\alpha + 2\pi)$ and both $U(\varphi, \alpha)$ and $\langle \phi (\varphi, \alpha) \rangle$ are single-valued in $\alpha$.

To show singlevaluedness in $\varphi$, let us, without loss of generality, diagonalize our string Wilson line $U(\varphi)$, taking it to be generated by a fixed generator so that $U(\varphi + 2\pi) = U(\varphi) U(2\pi)$. Then our twisted Wilson line obeys

$$U(\varphi + 2\pi, \alpha) = h^{-1}(\alpha) U(\varphi) U(2\pi) h(\alpha) = U(\varphi, \alpha) U(2\pi, \alpha).$$

As noted above, $U(2\pi, \alpha)$ is a loop in $H$, leaving $\langle \phi \rangle_o$ invariant. Thus

$$\langle \phi (\varphi, \alpha + 2\pi) \rangle = \langle \phi (\varphi, \alpha) \rangle = U(\varphi, \alpha) \langle \phi \rangle_o$$

and our twisted Alice loop is fully single-valued.

By the exact sequence for $\pi_2(G/H)$, our twisted Alice loop is a monopole when $U(\varphi, \alpha)$ interpolates between an element of $H$ at $\varphi = 0$ and a nontrivial loop in $H$ at $\varphi = 2\pi$. For convenience, right multiply $U(\varphi, \alpha)$ by $h_o^{-1}$:

$$U(\varphi, \alpha) = h^{-1}(\alpha) U(\varphi) h(\alpha) h_o^{-1}. \hspace{1cm} (0.2)$$

Since $h_o^{-1} \in H$, this right multiplication does not change the physical condensate $\langle \phi (\varphi, \alpha) \rangle$. However, it makes the topology of $U(\varphi, \alpha)$ clear, for

$$U(\varphi, \alpha) = \begin{cases} 
    h_o^{-1} & \text{for } \varphi = 0 \\
    h^{-1}(\alpha) \tilde{h}(\alpha) & \text{for } \varphi = 2\pi
\end{cases} .$$

By definition, if the string is topologically Alice, $\tilde{h}(\alpha) \not\sim h(\alpha)$ so that $h^{-1}(\alpha) \tilde{h}(\alpha)$ is a nontrivial loop in $H$, and the twisted Alice loop carries nontrivial monopole charge. If the string is not topologically Alice, the loop $h^{-1}(\alpha) \tilde{h}(\alpha)$ is trivial in $H$ and the twisted Alice loop carries no monopole charge.

Thus twisted Alice loops carry monopole charge if and only if they obey our topological Alice criterion. We note that the monopole charge displayed, with winding $h^{-1}(\alpha) \tilde{h}(\alpha)$, is exactly that deposited on an initially untwisted Alice loop, when a monopole of winding $h^{-1}(\alpha) \tilde{h}(\alpha)$ circumnavigates the string and emerges with winding $\tilde{h}^{-1}(\alpha)$. [The inverse twisted
Alice loop, generated by $U^{-1}(\varphi, \alpha)$, instead carries monopole charge $h(\alpha) \tilde{h}^{-1}(\alpha)$, deposited in the monopole circumnavigation $h(\alpha) \rightarrow \tilde{h}(\alpha)$.

We note that our final map $U(2\pi, \alpha)$ for the twisted Alice loop coincides with the flux loop (4.1) and paths $C'_\alpha$ defined in [4]. They show that this map coincides with the Lubkin classification of monopole charge for the twisted Alice loop. This reinforces our classification, as identifying twisted topologically Alice loops with physical gauged magnetic monopoles.

**THE SCHWARZ, OR NEMATIC, ALICE STRING**

The simplest Alice string is that of Schwarz, [1] whose symmetry structure coincides with Alice strings in nematic liquid crystals and non-chiral Bose condensates. [8] Here $G$ is $SO(3)$, with Higgs $\phi$ transforming in the adjoint. $\phi$ develops the vev $\langle \phi \rangle = \text{diag} (1, 1, -2)$, breaking $SO(3)$ to the residual symmetry $H = O(2)$, containing z-rotations $R_z(\alpha)$ and the discrete symmetry element $h_o = R_z(\pi) = \text{diag} (1, -1, -1)$. Here $\pi_o(H) = Z_2$ and $\pi_1(H) = Z$ so we have topological strings and monopoles. The Alice string has Wilson line $U(\varphi) = R_x(\varphi/2)$ with $U(2\pi) = h_o$. $U(2\pi)$ fails to commute with unbroken symmetry generator $T_z$; in fact, on parallel transport around the string,

$$T_z \rightarrow U(2\pi) \quad T_z \quad U^{-1}(2\pi) = -T_z \quad \text{(0.3)}$$

This Schwarz Alice string meets our topological criterion, of changing topological monopole charge on circumnavigation. By the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $O(2)$ which can be unwound in $SO(3)$. Since only even winding loops in $O(2)$ can be unwound in $SO(3)$, the fundamental monopole in this canonical Alice model has a loop in $O(2)$ of winding 2.

To apply our topological criterion, we represent the string by $h_o$, a nontrivial element of $\pi_o(H)$, and the fundamental monopole by $h(\alpha) = R_z(2\alpha)$, a winding 2 element of $\pi_1(H)$. This gives

$$\tilde{h}(\alpha) = h_o \quad h(\alpha) \quad h^{-1}_o = h^{-1}(\alpha),$$

from equation [0.3]. Note that $h^{-1}(\alpha)$ has $O(2)$ winding -2, topologically distinct from $h(\alpha)$ of $O(2)$ winding 2. Thus $\tilde{h}(\alpha) \not\sim h(\alpha)$ and our topological criterion is met.

We now construct a monopole as a twisted Alice loop. From Eq. (0.2), the twisted Wilson
line

\[ U(\varphi, \alpha) = h^{-1}(\alpha/2) \ U(\varphi) \ h(\alpha/2) \ h_o^{-1} \]

gives an Alice loop with single-valued condensate. (We take \( h(\alpha/2) \) because \( h \) need only be single-valued in \( \alpha \), and \( h(\alpha/2) \), the winding 1 loop in \( O(2) \), first achieves this.) \( U(\varphi, \alpha) \) interpolates between \( h_o^{-1} \) at \( \varphi = 0 \) and \( h^{-1}(\alpha) \) at \( \varphi = 2\pi \). It is thus the fundamental antimonopole in the model, winding \(-2\) in \( O(2) \). The inverse twisted Alice loop, with Wilson line \( U^{-1}(\varphi, \alpha) \), creates the fundamental monopole.

Similarly, twisted Alice loops in \( {}^3 \text{He-A} \) and amorphous chiral superconductors support fundamental monopole charge, as again only even \( U(1) \) loops unwind inside \( SO(3) \). Embedding \( H = O(2) \) in a different \( G \), however, can result in Alice loops unable to support fundamental monopole charge.

A NONTOPOLOGICALLY ALICE STRING

Consider the nontopologically Alice string introduced in [2]: a Higgs \( \phi \), transforming in the adjoint under \( G = SO(6) \), acquires the vev \( \langle \phi \rangle = \text{diag}(1^3, -1^3) \). This condensate leaves unbroken an \( SO(3) \times SO(3) \) subgroup of \( SO(6) \) and a discrete \( Z_2 \) transformation \( h_1 = -\mathbb{1}_6 \), so \( H = SO(3) \times SO(3) \times Z_2 \). Here \( \pi_o(H) = Z_2 \) and \( \pi_1(H) = Z_2 \times Z_2 \), so topological strings and monopoles form, with monopoles and antimonopoles identified. Alice characteristics of the string depend on \( U(2\pi) \). For \( U(2\pi) = h_1 \), all unbroken generators \( T_{ij} \) are single-valued under parallel transport around the string, and the string is not Alice. However, for the topologically equivalent choice \( U(2\pi) = \text{diag}(1^2, (-1)^4) = -R_{12}(\pi) \), the string is Alice, making generators \( T_{13} \) and \( T_{23} \) of rotations in the 13- and 23-planes double-valued. Since this Alice behavior is removable by deforming to the topologically equivalent string with \( U(2\pi) = h_1 \), it must be nontopological. However, it is instructive to see how the two strings fail our topological criterion. Taking as our nontrivial monopole loop \( h(\alpha) = R_{13} (\alpha) \), we obtain, for \( h_o = h_1, \ h(\alpha) = h_o \ h(\alpha) \ h_o^{-1} = h(\alpha) \). That is, our topological criterion fails, as the monopole remains unchanged in circumnavigating the Alice string. Instead, for \( h_o = -R_{12}(\pi), \ h(\alpha) = h_o \ h(\alpha) \ h_o^{-1} = h^{-1}(\alpha) \), since \( T_{13} \rightarrow -T_{13} \). Here a monopole transforms into an antimonopole on traversing the string. However, that transformation is nontopological, as monopoles and antimonopoles are topologically equivalent. So, despite algebraic Alice behavior, this string is not topologically Alice. Loops \( h(\alpha) \) alter on string
traversals, but in a topologically trivial way.

CONCLUSIONS

We have established a topological criterion for strings to display Alice behavior. This criterion, that $\pi_0(H)$ acts nontrivially on $\pi_1(H)$, depends only on the residual symmetry group $H$. Alice strings must form in models obeying this criterion, while Alice behavior can be deformed away for strings failing the criterion. Particularly, the criterion requires that topological monopoles always accompany Alice strings; and furthermore, that Alice strings alter the topological charge of monopoles that circumnavigate them. We construct monopoles as twisted loops of Alice string, and show that such twisted loops can always support deposited monopole charge. Whether twisted Alice loops can support fundamental monopole charge depends on the symmetry-breaking pattern more closely, as we discuss in §. For the Schwarz Alice string, we construct a twisted Alice loop that supports fundamental monopole charge.

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\[
U(\alpha, \varphi) = h^{-\alpha} U(\varphi) h(\alpha)
\]

\[\alpha = 0 \quad U(\varphi)\]

FIG. 1: a) A twisted Alice string. b) Identifying twisted Alice string ends to form a twisted Alice loop.
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