Counting faces of graphical zonotopes

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Abstract

It is a classical fact that the number of vertices of the graphical zonotope \( Z_\Gamma \) is equal to the number of acyclic orientations of a graph \( \Gamma \). We show that the \( f \)-polynomial of \( Z_\Gamma \) is obtained as the principal specialization of the \( q \)-analog of the chromatic symmetric function of \( \Gamma \).

Keywords: graphical zonotope, \( f \)-vector, graphical matroid, symmetric function

1 Introduction

The \( f \)-polynomial of an \( n \)-dimensional polytope \( P \) is defined by \( f(P,q) = \sum_{i=0}^{n} f_i(P)q^i \), where \( f_i(P) \) is the number of \( i \)-dimensional faces of \( P \). The \( f \)-polynomial \( f(Z_\Gamma, q) \) of the graphical zonotope \( Z_\Gamma \) is a combinatorial invariant of a finite, simple graph \( \Gamma \). The vertices of \( Z_\Gamma \) are in one-to-one correspondence with regions of the graphical hyperplane arrangement \( H_\Gamma \), which are enumerated by acyclic orientations of \( \Gamma \).

Stanley’s chromatic symmetric function \( \Psi(\Gamma) = \sum \text{proper } x_f \) of a graph \( \Gamma = (V,E) \), introduced in [6], is the enumerator function of proper colorings \( f : V \to \mathbb{N} \), where \( x_f = x_{f(1)} \cdots x_{f(n)} \) and \( f \) is proper if there are no monochromatic edges. The chromatic polynomial \( \chi(\Gamma, d) \) of the graph \( \Gamma \), which counts proper colorings with a finite number of colors, appears as the principal specialization

\[
\chi(\Gamma, d) = \text{ps}(\Psi(\Gamma))(d) = \Psi(\Gamma) \big|_{x_1=\cdots=x_d=1, x_{d+1}=\cdots=0}.
\]

The number of acyclic orientations of \( \Gamma \) is determined by the value of the chromatic polynomial \( \chi(\Gamma, d) \) at \( d = -1 \), [7]

\[
a(\Gamma) = (-1)^{|V|} \chi(\Gamma, -1).
\] (1)

There is a \( q \)-analog of the chromatic symmetric function \( \Psi_q(\Gamma) \) introduced in a wider context of the combinatorial Hopf algebra of simplicial complexes considered in [2]. It is a symmetric function over the field of rational functions...
in $q$. The principal specialization of $\Psi_q(\Gamma)$ is the $q$-analog of the chromatic polynomial $\chi_q(\Gamma, d)$.

The main result of this paper is the following generalization of formula (1)

**Theorem 1.1.** Let $\Gamma = (V, E)$ be a simple connected graph and $Z_\Gamma$ the corresponding graphical zonotope. Then the $f$-polynomial of $Z_\Gamma$ is given by

$$f(Z_\Gamma, q) = (-1)^{|V|} \chi_{-q}(\Gamma, -1).$$

The cancellation-free formula for the antipode in the Hopf algebra of graphs, obtained by Humpert and Martin in [3], reflects the fact that $f(Z_\Gamma, q)$ depends only on the graphical matroid $M(\Gamma)$ associated to $\Gamma$. For instance, for any tree $T_n$ the graphical matroid is the uniform matroid $M(T_n) = U_n^n$ and the corresponding graphical zonotope is the cube $Z_{T_n} = I^{n-1}$. Whitney’s theorem from 1933 describes how two graphs with the same graphical matroid are related [9]. It can be used to find more interesting nonisomorphic graphs with the same $f$-polynomials of corresponding graphical zonotopes.

The paper is organized as follows. In section 2, we review the basic facts about zonotopes. In section 3, the $q$-analog of the chromatic symmetric function $\Psi_q(\Gamma)$ of a graph $\Gamma$ is introduced. Theorem 1.1 is proved in section 4. We present some examples and calculations in section 5.

## 2 Zonotopes

A **zonotope** $Z = Z(v_1, \ldots, v_m)$ is a convex polytope determined by a collection of vectors $\{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$ as the Minkowski sum of line segments

$$Z = [-v_1, v_1] + \cdots + [-v_m, v_m].$$

It is a projection of the $m$-cube $[-1, 1]^m$ under the linear map $t \mapsto tA$, $t \in [-1, 1]^m$, where $A = [v_1 \cdots v_m]$ is an $n \times m$-matrix whose columns are vectors $v_1, \ldots, v_m$. The zonotope $Z$ is symmetric about the origin and all its faces are translations of zonotopes.

To a collection of vectors $\{v_1, \ldots, v_m\}$ is associated a central arrangement of hyperplanes $\mathcal{H} = \{H_{v_1}, \ldots, H_{v_m}\}$, where $H_v$ denotes the hyperplane perpendicular to a vector $v \in \mathbb{R}^n$. The zonotope $Z$ and the corresponding arrangement of hyperplanes $\mathcal{H}$ are closely related. In fact the associated fan $\mathcal{F}_H$ of the arrangement $\mathcal{H}$ is the normal fan $\mathcal{N}(Z)$ of the zonotope $Z$ (see [10, Theorem 7.16]). It follows that the face lattice of $\mathcal{F}_H$ and the reverse face lattice of $Z$ are isomorphic. In particular, vertices of $Z$ correspond to regions of $\mathcal{H}$ and their total numbers coincide

$$f_0(Z) = r(\mathcal{H}).$$

The faces of the zonotope $Z$ are encoded by covectors of the oriented matroid $\mathcal{M}$ associated to the collection of vectors $\{v_1, \ldots, v_m\}$. The covectors are sign vectors.
\[ Y^* = \{ \text{sign}(v) \in \{+, -, 0\}^m \mid v \in \mathbb{R}^n \}, \]

where \( \text{sign}(v)_i = \begin{cases} +, & \langle v, v_i \rangle > 0 \\ 0, & \langle v, v_i \rangle = 0 \\ -, & \langle v, v_i \rangle < 0 \end{cases} , \]
i = 1, \ldots, m. The face lattice of the zonotope \( Z \) is isomorphic to the lattice of covectors componentwise induced by \( +, -, \) \( < 0 \) on \( Y^* \).

A special class of zonotopes is determined by simple graphs. To a connected graph \( \Gamma = (V, E) \), whose vertices are enumerated by integers \( V = \{1, \ldots, n\} \), are associated the **graphical zonotope**

\[ Z_\Gamma = Z(e_i - e_j \mid i < j, \{i, j\} \in E) \]

and the **graphical arrangement** in \( \mathbb{R}^n \)

\[ \mathcal{H}_\Gamma = \{ H_{e_i - e_j} \mid i < j, \{i, j\} \in E \}. \]

There is a bijective correspondence between regions of \( \mathcal{H}_\Gamma \) and acyclic orientations of \( \Gamma \), [8, Proposition 2.5], which by (2) implies

\[ f_0(Z_\Gamma) = r(\mathcal{H}_\Gamma) = a(\Gamma). \tag{3} \]

The arrangement \( \mathcal{H}_\Gamma \) is refined by the braid arrangement \( \mathcal{A}_{n-1} \) consisting of all hyperplanes \( H_{e_i - e_j} \), \( 1 \leq i < j \leq n \). Thus \( Z_\Gamma \) belongs to a wider class of convex polytopes called generalized permutohedra introduced in [4]. Since arrangements \( \mathcal{H}_\Gamma \) and \( \mathcal{A}_{n-1} \) are not essential we take their quotients by the line \( l : x_1 = \cdots = x_n \) and without confusing retain the same notation. Consequently \( \dim Z_\Gamma = n - 1 \).

![Figure 1: Permutohedron Pe3 and cube I3](image)

**Example 2.1.** (i) The permutohedron \( Pe^{n-1} \) is represented as the graphical zonotope \( Z_{K_n} \) corresponding to the complete graph \( K_n \) on \( n \) vertices (Figure 1).

(ii) The cube \( I^{n-1} \) is represented as the graphical zonotope \( Z_{T_n} \) corresponding to an arbitrary tree \( T_n \) on \( n \) vertices. This shows that the graph \( \Gamma \) is not determined by the combinatorial type of the zonotope \( Z_\Gamma \).
3 q-analog of chromatic symmetric function of graph

Stanley’s chromatic symmetric function $\Psi(\Gamma)$ can be obtained in a purely algebraic way. A combinatorial Hopf algebra $H$ is a graded, connected Hopf algebra equipped with the multiplicative linear functional $\zeta : H \to k$ to the ground field $k$. For the theory of combinatorial Hopf algebras see [1]. Consider the combinatorial Hopf algebra of graphs $G$ which is linearly generated over a field $k$ by simple finite graphs with the product defined by disjoint union $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ and the coproduct

$$\Delta(\Gamma) = \sum_{I \subseteq V} \Gamma|_I \otimes \Gamma|_{V \setminus I},$$

where $\Gamma|_I$ denotes the induced subgraph on $I \subseteq V$. The structure of $G$ is completed by the character $\zeta : G \to k$ defined to be $\zeta(\Gamma) = 1$ for $\Gamma$ with no edges and $\zeta(\Gamma) = 0$ otherwise. Then it turns out that $\Psi(\Gamma)$ is the image of the unique morphism of combinatorial Hopf algebras to symmetric functions $\Psi : G \to \text{Sym}$, ([1, Example 4.5]).

An important part of the structure of the Hopf algebra $G$ is the antipode $S : G \to G$. The cancellation-free formula for the antipode in terms of acyclic orientations of a graph $\Gamma$ is obtained in [3]. We recall some basic definitions. Terminology comes from matroid theory. Given a graph $\Gamma = (V,E)$, for a collection of edges $F \subseteq E$ denote by $\Gamma_{V,F}$ the graph on $V$ with the edge set $F$. A flat $F$ of the graph $\Gamma$ is a collection of its edges such that components of $\Gamma_{V,F}$ are induced subgraphs. The rank $\text{rk}(F)$ is the size of spanning forests of $\Gamma_{V,F}$.

We have that $|V| = \text{rk}(F) + c(F)$, where $c(F)$ is the number of components of $\Gamma_{V,F}$. By contracting edges from a flat $F$ we obtain the graph $\Gamma/F$. Finally, let $a(\Gamma)$ be the number of acyclic orientations of $\Gamma$. The formula of Humpert and Martin is as follows

$$S(\Gamma) = \sum_{F \in \mathcal{F}(\Gamma)} (-1)^{c(F)} a(\Gamma/F) \Gamma_{V,F}, \quad (4)$$

where the sum is over the set of flats $\mathcal{F}(\Gamma)$.

The following modification of the character $\zeta$ is considered in [2] in a wider context of the combinatorial Hopf algebra of simplicial complexes. Define $\zeta_q(\Gamma) = q^{\text{rk}(\Gamma)}$, which determines the algebra morphism $\zeta_q : G \to k(q)$, where $k(q)$ is the field of rational functions in $q$. This character produces the unique morphism $\Psi_q : G \to \mathcal{QSym}$ to quasisymmetric functions over $k(q)$. The expansion of $\Psi_q(\Gamma)$ in the monomial basis of quasisymmetric functions is determined by the universal formula [1, Theorem 4.1]

$$\Psi_q(\Gamma) = \sum_{\alpha \vdash n} (\zeta_q)_\alpha(\Gamma) M_\alpha.$$

The sum above is over all compositions of the integer $n = |V|$ and the coefficient of the expansion corresponding to the composition $\alpha = (a_1, \ldots, a_k) \vdash n$ is given by
(\zeta_q)_\alpha(\Gamma) = \sum_{I_1 \cup \ldots \cup I_k = V} q^{rk(\Gamma|_{I_1}) + \cdots + rk(\Gamma|_{I_k})},

where the sum is over all set compositions of \( V \) of the type \( \alpha \). The coefficients \((\zeta_q)_\alpha(\Gamma)\) depend only on the partition corresponding to a composition \( \alpha \), so the function \( \Psi_q(\Gamma) \) is actually symmetric and it can be expressed in the monomial basis of symmetric functions.

The invariant \( \Psi_q(\Gamma) \) is more subtle than \( \Psi(\Gamma) \). Obviously \( \Psi_q(\Gamma) = \Psi_q(\Gamma) \). It remains open to find two non-isomorphic graphs \( \Gamma_1 \) and \( \Gamma_2 \) with the same \( q \)-chromatic symmetric functions \( \Psi_q(\Gamma_1) = \Psi_q(\Gamma_2) \). Let

\[ \chi_q(\Gamma, d) = \text{ps}(\Psi_q(\Gamma))(d) \]

be the \( q \)-analog of the chromatic polynomial \( \chi(\Gamma, d) \). It is a consequence of a general fact for combinatorial Hopf algebras (see [1]) that

\[ \chi_q(\Gamma, -1) = (\zeta_q \circ S)(\Gamma). \]  

**Example 3.1.** Consider the graph \( \Gamma \) on four vertices with the edge set \( E = \{12, 13, 23, 34\} \). We find that

\[ \Psi_q(\Gamma) = 24m_{1,1,1,1} + (8q + 4)m_{2,1,1,1} + (2q^2 + 4q)m_{2,2,1} + (3q^2 + q)m_{3,1} + q^2m_4. \]

By principal specialization and taking into account that

\[ \text{ps}(m_{\lambda_1, \ldots, \lambda_k})(d) = \frac{(i_1 + \cdots + i_k)!}{i_1! \cdots i_k!} \binom{d}{i_1 + \cdots + i_k}, \]

we obtain

\[ \chi_q(\Gamma, d) = d(d-1)^2(d-2) + qd(d-1)(4d-5) + 4q^2d(d-1) + q^3d, \]

which by Theorem 1.1 gives

\[ f(Z_\Gamma, q) = 12 + 18q + 8q^2 + q^3. \]

### 4 Proof of Theorem 1.1

By applying (5) and the formula for antipode (4) we obtain

\[ (-1)^{|V|}\chi_q(\Gamma, -1) = (-1)^{|V|} \sum_{F \in \mathcal{F}(\Gamma)} (-1)^{c(\Gamma)} a(\Gamma/F)(-q)^{rk(F)}. \]

It follows that the statement of the theorem is equivalent to the following expression of the \( f \)-polynomial

\[ f(Z_\Gamma, q) = \sum_{F \in \mathcal{F}(\Gamma)} a(\Gamma/F)q^{rk(F)}. \]
Therefore it should be shown that components of $f$-vectors are determined by
\[ f_k(\mathcal{Z}_\Gamma) = \sum_{F \in \mathcal{F}(\Gamma) \atop \text{rk}(F) = k} a(\Gamma/F), \quad 0 \leq k \leq n - 1. \quad (7) \]

By duality between the face lattice of $\mathcal{Z}_\Gamma$ and the face lattice of the fan $\mathcal{F}_{\mathcal{H}_\Gamma}$, we have
\[ f_k(\mathcal{Z}_\Gamma) = f_{n-k-1}(\mathcal{F}_{\mathcal{H}_\Gamma}). \]

Let $L(\mathcal{H}_\Gamma)$ be the intersection lattice of the graphical arrangement $\mathcal{H}_\Gamma$. For a subspace $X \in L(\mathcal{H}_\Gamma)$ there is an arrangement of hyperplanes
\[ \mathcal{H}_\Gamma^X = \{ X \cap H \mid X \nsubseteq H, H \in \mathcal{H}_\Gamma \} \]
whose intersection lattice $L(\mathcal{H}_\Gamma^X)$ is isomorphic to the upper cone of $X$ in $L(\mathcal{H}_\Gamma)$. Since $\mathcal{H}_\Gamma$ is central and essential we have
\[ f_{n-k-1}(\mathcal{F}_{\mathcal{H}_\Gamma}) = \sum_{X \in L(\mathcal{H}_\Gamma) \atop \text{dim}(X) = n - k - 1} r(\mathcal{H}_\Gamma^X), \quad (8) \]
where $r(\mathcal{H}_\Gamma^X)$ is the number of regions of the arrangement $\mathcal{H}_\Gamma^X$, see [8, Theorem 2.6].

The intersection lattice $L(\mathcal{H}_\Gamma)$ is isomorphic to the lattice of flats of the graphical matroid $M(\Gamma)$. By this isomorphism to a flat $F$ of rank $k$ corresponds the intersection subspace $X^F = \cap_{\{i,j\} \in F} H_{e_i-e_j}$ of dimension $n-k-1$. It is easy to see that arrangements $\mathcal{H}_\Gamma^X$ and $\mathcal{H}_\Gamma/F$ coincide, which by (3) and comparing formulas (7) and (8) proves theorem.

5 Examples

By applying Theorem 1.1 we obtain the following interpretation of identities elaborated in [2, Propositions 17, 19].

Example 5.1. (i) For the permutohedron $Pe^{n-1} = \mathcal{Z}_{K_n}$, the $f$-polynomial is given by
\[ f(\mathcal{Z}_{K_n}, q) = A_n(q + 1), \]
where $A_n(q) = \sum_{\pi \in S_n} q^{\text{des}(\pi)}$ is the Euler polynomial. Recall that $\text{des}(\pi)$ is the number of descents of a permutation $\pi \in S_n$. It recovers the fact that the $h$-polynomial of the permutohedron $Pe^{n-1}$ is the Euler polynomial $A_n(q)$.

(ii) For the cube $I^n = \mathcal{Z}_{T_n}$, where $T_n$ is a tree on $n$ vertices, the $f$-polynomial is given by
\[ f(\mathcal{Z}_{T_n}, q) = (q + 2)^{n-1}. \]
Proposition 5.2. The f-polynomial of the graphical zonotope $Z_{C_n}$ associated to the cycle graph $C_n$ on $n$ vertices is given by

$$f(Z_{C_n}, q) = q^n + q^{n-1} + (q + 2)^n - 2(q + 1)^n.$$  

Proof. A flat $F \in F(C_n)$ is determined by the complementary set of edges. If $rk(F) = n - k, k > 1$ then the complementary set has $k$ edges and $C_n/F = C_k$. Since $a(C_k) = 2^k - 2, k > 1$, by formula (7), we obtain

$$f_{n-k}(Z_{C_n}) = (2^k - 2) \binom{n}{k}, 2 \leq k \leq n,$$

which leads to the required formula. 

Proposition 5.3. Let $\Gamma = \Gamma_1 \lor_{v} \Gamma_2$ be the wedge of two connected graphs $\Gamma_1$ and $\Gamma_2$ at the common vertex $v$. Then

$$f(Z_{\Gamma}, q) = f(Z_{\Gamma_1}, q)f(Z_{\Gamma_2}, q).$$

Proof. The graphical matroids of involving graphs are related by $M(\Gamma) = M(\Gamma_1) \oplus M(\Gamma_2)$. For the sets of flats it holds $F(\Gamma) = \{ F_1 \cup F_2 \mid F_i \in F(\Gamma_i), i = 1,2 \}$. For $F = F_1 \cup F_2$ we have $\Gamma/F = \Gamma_1/F_1 \lor [v] \Gamma_2/F_2$, where $[v]$ is the component of the vertex $v$ in $\Gamma_{\Gamma/F}$. Obviously $a(\Gamma/F) = a(\Gamma_1/F_1)a(\Gamma_2/F_2)$ and $rk(F) = rk(F_1) + rk(F_2)$. The proposition follows from formula (6).  

The formula for cubes in Example 5.1 (ii) follows from Proposition 5.3 since any tree is a consecutive wedge of edges and $f(I^1, q) = q + 2$. It also allows us to restrict ourselves only to biconnected graphs. For a biconnected graph $\Gamma$ with a disconnecting pair of vertices $\{u, v\}$ Whitney introduced the transformation
called the *twist* around the pair \{u, v\}. This transformation does not have an affect on the graphical matroid \(M(\Gamma)\) [9].

![Figure 3: Biconnected graphs related by twist transformation](image)

**Example 5.4.** Figure 3 shows the pair of biconnected graphs on six vertices obtained one from another by the twist transformation. The corresponding zonotopes have the same \(f\)-polynomial

\[
f(\mathcal{Z}_{\Gamma_1}, q) = f(\mathcal{Z}_{\Gamma_2}, q) = 126 + 348q + 358q^2 + 164q^3 + 30q^4 + q^5.
\]

On the other hand their \(q\)-chromatic symmetric functions are different. One can check that corresponding coefficients by \(m_{3,1^3}\) are different

\[
[m_{3,1^3}]\Psi_q(\Gamma_1) = (11q^2 + 8q + 1) \cdot 3!,
\]
\[
[m_{3,1^3}]\Psi_q(\Gamma_2) = (10q^2 + 10q) \cdot 3!.
\]

This shows that the \(q\)-analog of the chromatic symmetric function of a graph is not determined by the corresponding graphical matroid. By taking \(q = 0\) we obtain that even the chromatic symmetric functions are different since

\[
[m_{3,1^3}]\Psi(\Gamma_1) = 6 \quad \text{and} \quad [m_{3,1^3}]\Psi(\Gamma_2) = 0.
\]

Let us now consider Stanley’s example of nonisomorphic graphs with the same chromatic symmetric functions, see [6]. We find that the \(f\)-polynomials of the corresponding graphical zonotopes differ for those graphs. From these examples we conclude that chromatic properties of a graph and the \(f\)-vector of the corresponding graphical zonotope are not related.

We have already noted that graphical zonotopes are generalized permutohedra. The \(h\)-polynomials of simple generalized permutohedra are determined in [5, Theorem 4.2]. The only simple graphical zonotopes are products of permutohedra [5, Proposition 5.2]. They are characterized by graphs whose biconnected components are complete subgraphs. Therefore Proposition 5.3 together with Example 5.1 (i) prove that the \(h\)-polynomial of a simple graphical zonotope is the product of Eulerian polynomials, the fact obtained in [5, Corollary 5.4]. Example 3.1 is of this sort and represents the hexagonal prism which is the product \(Z_{K_3} \times Z_{K_2}\).
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