Characterization of the minimal penalty of a convex risk measure with applications to Lévy processes.

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Abstract

The minimality of the penalty function associated with a convex risk measure is analyzed in this paper. First, in a general static framework, we provide necessary and sufficient conditions for a penalty function defined in a convex and closed subset of the absolutely continuous measures with respect to some reference measure $P$ to be minimal on this set. When the probability space supports a Lévy process, we establish results that guarantee the minimality property of a penalty function described in terms of the coefficients associated with the density processes. The set of densities processes is described and the convergence of its quadratic variation is analyzed.

Key words: Convex risk measures, Fenchel-Legendre transformation, minimal penalization, Lévy process.

Mathematical Subject Classification: 91B30, 46E30.

1 Introduction

The definition of coherent risk measure was introduced by Artzner et al. in their fundamental works [1], [2] for finite probability spaces, giving an axiomatic characterization that was extended later by Delbaen [6] to general probability spaces. In the papers mentioned above one of the fundamental axioms was the positive homogeneity, and in further works it was removed, defining the concept of convex risk measure introduced by Föllmer and Schied [8], [9], Frittelli and Rosazza Gianin [11], [12] and Heath [13].

This is a rich area that has received a lot of attention and much work has been developed. There exists by now a well established theory in the static and dynamic cases, but there are

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still many questions unanswered in the static framework that need to be analyzed carefully. The one we focus on in this paper is the characterization of the penalty functions that are minimal for the corresponding static risk measure. Up to now, there are mainly two ways to deal with minimal penalty functions, namely the definition or the biduality relation. With the results presented in this paper we can start with a penalty function, which essentially discriminate models within a convex closed subset of absolutely continuous probability measures with respect to (w.r.t.) the market measure, and then guarantee that it corresponds to the minimal penalty of the corresponding convex risk measure on this subset. This property is, as we will see, closely related with the lower semicontinuity of the penalty function, and the complications to prove this property depend on the structure of the probability space.

We first provide a general framework, within a measurable space with a reference probability measure $\mathbb{P}$, and show necessary and sufficient conditions for a penalty function defined in a convex and closed subset of the absolutely continuous measures with respect to the reference measure to be minimal within this subset. The characterization of the form of the penalty functions that are minimal when the probability space supports a Lévy process is then studied. This requires to characterize the set of absolutely continuous measures for this space, and it is done using results that describe the density process for spaces which support semimartingales with the weak predictable representation property. Roughly speaking, using the weak representation property, every density process splits in two parts, one is related with the continuous local martingale part of the decomposition and the other with the corresponding discontinuous one. It is shown some kind of continuity property for the quadratic variation of a sequence of densities converging in $L^1$. From this characterization of the densities, a family of penalty functions is proposed, which turned out to be minimal for the risk measures generated by duality.

The paper is organized as follows. Section 2 contains the description of the minimal penalty functions for a general probability space, providing necessary and sufficient conditions, the last one restricted to a subset of equivalent probability measures. Section 3 reports the structure of the densities for a probability space that supports a Lévy processes and the convergence properties needed to prove the lower semicontinuity of the set of penalty functions defined in Section 4. In this last section we show that these penalty functions are minimal.

## 2 Minimal penalty function of risk measures concentrated in $\mathcal{Q}_{\ll} (\mathbb{P})$.

Any penalty function $\psi$ induce a convex risk measure $\rho$, which in turn has a representation by means of a minimal penalty function $\psi^*$. Starting with a penalty function $\psi$ concentrated in a convex and closed subset of the set of absolutely continuous probability measures with respect to some reference measure $\mathbb{P}$, in this section we give necessary and sufficient conditions in order to guarantee that $\psi$ is the minimal penalty within this set. We begin recalling briefly some known results from the theory of static risk measures, and then a characterization for
2.1 Preliminaries from static measures of risk

Let \( X : \Omega \to \mathbb{R} \) be a mapping from a set \( \Omega \) of possible market scenarios, representing the discounted net worth of the position. Uncertainty is represented by the measurable space \((\Omega, \mathcal{F})\), and we denote by \( \mathcal{X} \) the linear space of bounded financial positions, including constant functions.

**Definition 2.1**

(i) The function \( \rho : \mathcal{X} \to \mathbb{R} \), quantifying the risk of \( X \), is a monetary risk measure if it satisfies the following properties:

\[
\text{Monotonicity: If } X \leq Y \text{ then } \rho(X) \geq \rho(Y) \quad \forall X, Y \in \mathcal{X}.
\]

\[
\text{(2.1)}
\]

\[
\text{Translation Invariance: } \rho(X + a) = \rho(X) - a \quad \forall a \in \mathbb{R} \quad \forall X \in \mathcal{X}.
\]

\[
\text{(2.2)}
\]

(ii) When this function satisfies also the convexity property

\[
\rho(\lambda X + (1-\lambda) Y) \leq \lambda \rho(X) + (1-\lambda) \rho(Y) \quad \forall \lambda \in [0,1] \quad \forall X, Y \in \mathcal{X},
\]

\[
\text{(2.3)}
\]

it is said that \( \rho \) is a convex risk measure.

(iii) The function \( \rho \) is called normalized if \( \rho(0) = 0 \), and sensitive, with respect to a measure \( P \), when for each \( X \in L_\infty^\mathcal{F}(P) \) with \( P[X > 0] > 0 \) we have that \( \rho(-X) > \rho(0) \).

We say that a set function \( Q : \mathcal{F} \to [0,1] \) is a probability content if it is finitely additive and \( Q(\Omega) = 1 \). The set of probability contents on this measurable space is denoted by \( \mathcal{Q}_{\text{cont}} \). From the general theory of static convex risk measures \[10\], we know that any map \( \psi : \mathcal{Q}_{\text{cont}} \to \mathbb{R} \cup \{+\infty\} \), with \( \inf_{Q \in \mathcal{Q}_{\text{cont}}} \psi(Q) \in \mathbb{R} \), induces a static convex measure of risk as a mapping \( \rho : \mathcal{M}_b \to \mathbb{R} \) given by

\[
\rho(X) := \sup_{Q \in \mathcal{Q}_{\text{cont}}} \{E_Q[-X] - \psi(Q)\}.
\]

\[
\text{(2.4)}
\]

Here \( \mathcal{M} \) denotes the class of measurable functions and \( \mathcal{M}_b \) the subclass of bounded measurable functions. The function \( \psi \) will be referred as a penalty function. Föllmer and Schied \[9, Theorem 3.2\] and Frittelli and Rosazza Gianin \[11, Corollary 7\] proved that any convex risk measure is essentially of this form.

More precisely, a convex risk measure \( \rho \) on the space \( \mathcal{M}_b(\Omega, \mathcal{F}) \) has the representation

\[
\rho(X) = \sup_{Q \in \mathcal{Q}_{\text{cont}}} \{E_Q[-X] - \psi^*_\rho(Q)\},
\]

\[
\text{(2.5)}
\]

where

\[
\psi^*_\rho(Q) := \sup_{X \in A_\rho} E_Q[-X],
\]

\[
\text{(2.6)}
\]
and $\mathcal{A}_\rho := \{ X \in \mathfrak{M}_b : \rho(X) \leq 0 \}$ is the acceptance set of $\rho$.

The penalty $\psi_\rho^*$ is called the minimal penalty function associated to $\rho$ because, for any other penalty function $\psi$ fulfilling (2.5), $\psi(Q) \geq \psi_\rho^*(Q)$, for all $Q \in \mathcal{Q}_{cont}$. Furthermore, for the minimal penalty function, the next biduality relation is satisfied

$$\psi_\rho^*(Q) = \sup_{X \in \mathfrak{M}_b(\Omega, \mathcal{F})} \{ E_Q[-X] - \rho(X) \}, \quad \forall Q \in \mathcal{Q}_{cont}. \tag{2.7}$$

Let $\mathcal{Q}(\Omega, \mathcal{F})$ be the family of probability measures on the measurable space $(\Omega, \mathcal{F})$. Among the measures of risk, the class of them that are concentrated on the set of probability measures $\mathcal{Q} \subset \mathcal{Q}_{cont}$ are of special interest. Recall that a function $I : E \subset \mathbb{R}^\Omega \to \mathbb{R}$ is sequentially continuous from below (above) when $\{X_n\}_{n \in \mathbb{N}} \uparrow \downarrow X \Rightarrow \lim_{n \to \infty} I(X_n) = I(X)$ (respectively $\{X_n\}_{n \in \mathbb{N}} \uparrow \downarrow X \Rightarrow \lim_{n \to \infty} I(X_n) = I(X)$). Föllmer and Schied [10] proved that any sequentially continuous from below convex measure of risk is concentrated on the set $\mathcal{Q}$. Later, Krätschmer [17, Prop. 3 p. 601] established that the sequential continuity from below is not only a sufficient but also a necessary condition in order to have a representation, by means of the minimal penalty function in terms of probability measures.

We denote by $\mathcal{Q}_\ll(\mathbb{P})$ the subclass of absolutely continuous probability measure with respect to $\mathbb{P}$ and by $\mathcal{Q}_\approx(\mathbb{P})$ the subclass of equivalent probability measure. Of course, $\mathcal{Q}_\approx(\mathbb{P}) \subset \mathcal{Q}_\ll(\mathbb{P}) \subset \mathcal{Q}(\Omega, \mathcal{F})$.

**Remark 2.1** When a convex risk measures in $\mathcal{X} := L^\infty(\mathbb{P})$ satisfies the property

$$\rho(X) = \rho(Y) \text{ if } X = Y \ \mathbb{P}\text{-a.s.} \tag{2.8}$$

and is represented by a penalty function $\psi$ as in (2.4), we have that

$$Q \in \mathcal{Q}_{cont} \setminus \mathcal{Q}_{cont}^\ll \implies \psi(Q) = +\infty, \tag{2.9}$$

where $\mathcal{Q}_{cont}^\ll$ is the set of contents absolutely continuous with respect to $\mathbb{P}$; see [11, Lemma 4.30 p. 172].

### 2.2 Minimal penalty functions

The minimality property of the penalty function turns out to be quite relevant, and it is a desirable property that is not easy to prove in general. For instance, in the study of robust portfolio optimization problems (see, for example, Schied [18] and Hernández-Hernández and Pérez-Hernández [15]), using techniques of duality, the minimality property is a necessary condition in order to have a well posed dual problem. More recently, the dual representations of dynamic risk measures were analyzed by Barrieu and El Karoui [3], while the connection with BSDEs and $g$–expectations have been studied by Delbaen et al. [7]. The minimality of the penalty function also plays a crucial role in the characterization of the time consistency property for dynamic risk measures (see Bion-Nadal [4], [5]).

In the next sections we will show some of the difficulties that appear to prove the minimality of the penalty function when the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supports a Lévy process.
However, to establish the results of this section we only need to fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

When we deal with a set of absolutely continuous probability measures $\mathcal{K} \subset \mathcal{Q}_{ac}(\mathbb{P})$ it is necessary to make reference to some topological concepts, meaning that we are considering the corresponding set of densities and the strong topology in $L^1(\mathbb{P})$. Recall that within a locally convex space, a convex set $\mathcal{K}$ is weakly closed if and only if $\mathcal{K}$ is closed in the original topology [10, Thm A.59].

**Lemma 2.1** Let $\psi : \mathcal{K} \subset \mathcal{Q}_{ac}(\mathbb{P}) \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function with $\inf_{Q \in \mathcal{K}} \psi(Q) \in \mathbb{R}$, and define the extension $\psi(Q) := \infty$ for each $Q \in \mathcal{Q}_{cont} \setminus \mathcal{K}$, with $\mathcal{K}$ a convex closed set. Also, define the function $\Psi$, with domain in $L^1$, as

$$\Psi(D) := \begin{cases} \psi(Q) & \text{if } D = dQ/d\mathbb{P} \text{ for } Q \in \mathcal{K} \\ \infty & \text{otherwise.} \end{cases}$$

Then, for the convex measure of risk $\rho(X) := \sup_{Q \in \mathcal{Q}_{cont}} \{\mathbb{E}_Q[-X] - \psi(Q)\}$ associated with $\psi$ the following assertions hold:

(a) If $\rho$ has as minimal penalty $\psi^*_{\rho}$ the function $\psi$ (i.e. $\psi = \psi^*_{\rho}$), then $\Psi$ is a proper convex function and lower semicontinuous w.r.t. the (strong) $L^1$-topology or equivalently w.r.t. the weak topology $\sigma(L^1, L^\infty)$.

(b) If $\Psi$ is lower semicontinuous w.r.t. the (strong) $L^1$-topology or equivalently w.r.t. the weak topology $\sigma(L^1, L^\infty)$, then

$$\psi_{1_{\mathcal{Q}_{ac}(\mathbb{P})}} = \psi^*_{\rho} 1_{\mathcal{Q}_{ac}(\mathbb{P})}.$$  

(2.10)

**Proof:** (a) Recall that $\sigma(L^1, L^\infty)$ is the coarsest topology on $L^1(\mathbb{P})$ under which every linear operator is continuous, and hence $\Psi_0^X(Z) := \mathbb{E}_\mathbb{P}[Z(-X)]$, with $Z \in L^1$, is a continuous function for each $X \in \mathcal{M}_b(\Omega, \mathcal{F})$ fixed. For $\delta(\mathcal{K}) := \{Z : Z = dQ/d\mathbb{P} \text{ with } Q \in \mathcal{K}\}$ we have that

$$\Psi^X_1(Z) := \Psi^X_0(Z) 1_{\delta(\mathcal{K})}(Z) + \infty \times 1_{L^1 \setminus \delta(\mathcal{K})}(Z)$$

is clearly lower semicontinuous on $\delta(\mathcal{K})$. For $Z' \in L^1(\mathbb{P}) \setminus \delta(\mathcal{K})$ arbitrary fixed we have from Hahn-Banach’s Theorem that there is a continuous lineal functional $l(Z)$ with $l(Z') < \inf_{Z \in \delta(\mathcal{K})} l(Z)$. Taking $\varepsilon := \frac{1}{2} \{\inf_{Z \in \delta(\mathcal{K})} l(Z) - l(Z')\}$ we have that the weak open ball $B(Z', \varepsilon) := \{Z \in L^1(\mathbb{P}) : |l(Z') - l(Z)| < \varepsilon\}$ satisfies $B(Z', \varepsilon) \cap \delta(\mathcal{K}) = \emptyset$. Therefore, $\Psi^X_1(Z)$ is weak lower semicontinuous on $L^1(\mathbb{P})$, as well as $\Psi^X_2(Z) := \Psi^X_1(Z) - \rho(X)$. If

$$\psi(Q) = \psi^*_{\rho}(Q) = \sup_{X \in \mathcal{M}_b(\Omega, \mathcal{F})} \left\{ \int Z(-X) d\mathbb{P} - \rho(X) \right\},$$

where $Z := dQ/d\mathbb{P}$, we have that $\Psi(Z) = \sup_{X \in \mathcal{M}_b(\Omega, \mathcal{F})} \{\Psi^X_2(Z)\}$ is the supremum of a family of convex lower semicontinuous functions with respect to the topology $\sigma(L^1, L^\infty)$, and $\Psi(Z)$ preserves both properties.
(b) For the Fenchel - Legendre transform (conjugate function) $\Psi^* : L^\infty(\mathbb{P}) \rightarrow \mathbb{R}$ for each $U \in L^\infty(\mathbb{P})$

$$\Psi^*(U) = \sup_{Z \in \mathcal{K}} \left\{ \int Z Ud\mathbb{P} - \Psi(Z) \right\} = \sup_{Q \in \mathcal{Q}_{cont}} \{ E_Q[U] - \psi(Q) \} \equiv \rho(-U).$$

From the lower semicontinuity of $\Psi$ w.r.t. the weak topology $\sigma(L^1, L^\infty)$ that $\Psi = \Psi^{**}$. Considering the weak*-topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $Z = d\mathcal{Q}/d\mathbb{P}$ we have that

$$\psi(Q) = \Psi(Z) = \Psi^{**}(Z) = \sup_{U \in L^\infty(\mathbb{P})} \left\{ \int Z (-U)d\mathbb{P} - \Psi^*(-U) \right\} = \psi^*_\rho(Q).$$

Remark 2.2 1. As pointed out in Remark 2.1 we have that $\mathcal{Q} \in \mathcal{Q}_{cont} \setminus \mathcal{Q}_{cont}^\infty \implies \psi^*_\rho(Q) = +\infty = \psi(Q)$.

Therefore, under the conditions of Lemma 2.1 (b) the penalty function $\psi$ might differ from $\psi^*_\rho$ on $\mathcal{Q}_{cont}^\infty \setminus \mathcal{Q}_{cont}$. For instance, the penalty function defined as $\psi(Q) := \infty \times 1_{\mathcal{Q}_{cont} \setminus \mathcal{Q}_{cont}(\mathbb{P})}(Q)$ leads to the worst case risk measure $\rho(X) := \sup_{Q \in \mathcal{Q}_{cont}(\mathbb{P})} E_Q[-X]$, which has as minimal penalty the function $\psi^*_\rho(Q) = \infty \times 1_{\mathcal{Q}_{cont} \setminus \mathcal{Q}_{cont}^\infty}(Q)$.

2. Note that the total variation distance $d_{TV}(Q^1, Q^2) := \sup_{A \in \mathcal{F}} |Q^1[A] - Q^2[A]|$, with $Q^1, Q^2 \in \mathcal{Q}_{cont}$, fulfills that $d_{TV}(Q^1, Q^2) \leq ||dQ^1/d\mathbb{P} - Q^2/d\mathbb{P}||_1$. Therefore, the minimal penalty function is lower semicontinuous in the total variation topology; see Remark 4.16 (b) p. 163 in [10].

3 Preliminaries from stochastic calculus

Within a probability space which supports a semimartingale with the weak predictable representation property, there is a representation of the density processes of the absolutely continuous probability measures by means of two coefficients. Roughly speaking, this means that the “dimension” of the linear space of local martingales is two. Throughout these coefficients we can represent every local martingale as a combination of two components, namely as a stochastic integral with respect to the continuous part of the semimartingale and an integral with respect to its compensated jump measure. This is of course the case for local martingales, and with more reason this observation about the dimensionality holds for the martingales associated with the corresponding densities processes. In this section we review those concepts of stochastic calculus that are relevant to understand this representation properties, and prove some kind of continuity property for the quadratic variation of a sequence of uniformly integrable martingales converging in $L^1$. This result is one of the contributions of this paper.
3.1 Fundamentals of Lévy and semimartingales processes

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. We say that \(L := \{L_t\}_{t \in \mathbb{R}_+}\) is a Lévy process for this probability space if it is an adapted càdlàg process with independent stationary increments starting at zero. The filtration considered is \(\mathbb{F} := \{\mathcal{F}_t^L\}_{t \in \mathbb{R}_+}\), the completion of its natural filtration, i.e. \(\mathcal{F}_0^L := \sigma\{L_s : s \leq t\} \vee \mathcal{N}\) where \(\mathcal{N}\) is the \(\sigma\)-algebra generated by all \(\mathbb{P}\)-null sets. The jump measure of \(L\) is denoted by \(\mu : \Omega \times (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0)) \to \mathbb{N}\) where \(\mathbb{R}_0 := \mathbb{R} \setminus \{0\}\). The dual predictable projection of this measure, also known as its Lévy system, satisfies the relation \(P \subset F \otimes B\) respectively. Further, we denote by \(P := \text{intensity or Lévy measure of } L\).

It implies that \(L^c = W\) is the Wiener process, and hence \([L^c]_t = t\), where \((\cdot)^c\) and \([\cdot]\) denote the continuous martingale part and the process of quadratic variation of any semimartingale, respectively. For the predictable quadratic variation we use the notation \(\langle \cdot \rangle\).

Denote by \(\mathcal{V}\) the set of càdlàg, adapted processes with finite variation, and let \(\mathcal{V}^+ \subset \mathcal{V}\) be the subset of non-decreasing processes in \(\mathcal{V}\) starting at zero. Let \(\mathcal{A} \subset \mathcal{V}\) be the class of processes with integrable variation, i.e. \(A \in \mathcal{A}\) if and only if \(\bigvee_{t=0}^\infty A \in L^1(\mathbb{P})\), where \(\bigvee_{t=0}^t A\) denotes the variation of \(A\) over the finite interval \([0, t]\). The subset \(\mathcal{A}^+ = \mathcal{A} \cap \mathcal{V}^+\) represents those processes which are also increasing i.e. with non-negative right-continuous increasing trajectories. Furthermore, \(\mathcal{A}_{loc}\) (resp. \(\mathcal{A}^+_{loc}\)) is the collection of adapted processes with locally integrable variation (resp. adapted locally integrable increasing processes). For a càdlàg process \(X\) we denote by \(X^- := (X_-)\) the left hand limit process, where \(X_{0-} := X_0\) by convention, and by \(\Delta X = (\Delta X_t)\) the jump process \(\Delta X_t := X_t - X_{t-}\).

Given an adapted càdlàg semimartingale \(U\), the jump measure and its dual predictable projection (or compensator) are denoted by \(\mu_U([0, t] \times A) := \sum_{s \leq t} 1_A(\Delta U_s)\) and \(\mu_U^\circ\), respectively. Further, we denote by \(\mathcal{P} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)\) the predictable \(\sigma\)-algebra and by \(\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0)\). With some abuse of notation, we write \(\theta_1 \in \widetilde{\mathcal{P}}\) when the function \(\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \to \mathbb{R}\) is \(\widetilde{\mathcal{P}}\)-measurable and \(\theta \in \mathcal{P}\) for predictable processes.

Let

\[
\mathcal{L}(U^c) := \left\{ \theta \in \mathcal{P} : \exists \{\tau_n\}_{n \in \mathbb{N}} \text{ sequence of stopping times with } \tau_n \uparrow \infty \text{ and } \mathbb{E} \left[ \int_0^\tau \theta^2 d[U^c] \right] < \infty \forall n \in \mathbb{N} \right\}
\]

be the class of predictable processes \(\theta \in \mathcal{P}\) integrable with respect to \(U^c\) in the sense of local martingale, and by

\[
\Lambda(U^c) := \left\{ \int \theta_0 dU^c : \theta_0 \in \mathcal{L}(U^c) \right\}
\]
the linear space of processes which admits a representation as the stochastic integral with respect to $U^c$. For an integer valued random measure $\mu'$ we denote by $\mathcal{G}(\mu')$ the class of $\bar{\mathcal{P}}$-measurable processes $\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \to \mathbb{R}$ satisfying the following conditions:

1. $\theta_1 \in \bar{\mathcal{P}},$
2. $\int_{\mathbb{R}_0} |\theta_1(t, x)| (\mu')^\mathcal{P} (\{t\}, dx) < \infty \ \forall t > 0,$
3. The process
   \[
   \left\{ \sqrt{\sum_{s \leq t} \left\{ \int_{\mathbb{R}_0} \theta_1(s, x) \mu' (\{s\}, dx) - \int_{\mathbb{R}_0} \theta_1(s, x) (\mu')^\mathcal{P} (\{s\}, dx) \right\}^2} \right\}_{t \in \mathbb{R}_+} \in \mathcal{A}_{loc}^+.
   \]

The set $\mathcal{G}(\mu')$ represents the domain of the functional $\theta_1 \to \int \theta_1 d \left( \mu' - (\mu')^\mathcal{P} \right),$ which assign to $\theta_1$ the unique purely discontinuous local martingale $M$ with

\[
\Delta M_t = \int_{\mathbb{R}_0} \theta_1(t, x) (\mu')^\mathcal{P} (\{t\}, dx) - \int_{\mathbb{R}_0} \theta_1(t, x) \mu' (\{t\}, dx).
\]

We use the notation $\int \theta_1 d \left( \mu' - (\mu')^\mathcal{P} \right)$ to write the value of this functional in $\theta_1$. It is important to point out that this functional is not, in general, the integral with respect to the difference of two measures. For a detailed exposition on these topics see He, Wang and Yan \cite{14} or Jacod and Shiryaev \cite{16}, which are our basic references.

In particular, for the Lévy process $L$ with jump measure $\mu$,

\[
\mathcal{G}(\mu) \equiv \left\{ \theta_1 \in \bar{\mathcal{P}} : \left\{ \sum_{s \leq t} \{|\theta_1(s, \Delta L_s)|^2 1_{\mathbb{R}_0}(\Delta L_s)\} \right\}_{t \in \mathbb{R}_+} \in \mathcal{A}_{loc}^+ \right\}, \tag{3.3}
\]

since $\mu^\mathcal{P} (\{t\} \times A) = 0$, for any Borel set $A$ of $\mathbb{R}_0$.

We say that the semimartingale $U$ has the **weak property of predictable representation** when

\[
\mathcal{M}_{loc,0} = \Lambda(U^c) + \left\{ \int \theta_1 d \left( \mu_U - \mu_U^\mathcal{P} \right) : \theta_1 \in \mathcal{G}(\mu_U) \right\}, \tag{3.4}
\]

where the previous sum is the linear sum of the vector spaces, and $\mathcal{M}_{loc,0}$ is the linear space of local martingales starting at zero.

Let $\mathcal{M}$ and $\mathcal{M}_\infty$ denote the class of càdlàg and càdlàg uniformly integrable martingale respectively. The following lemma is interesting by itself to understand the continuity properties of the quadratic variation for a given convergent sequence of uniformly integrable martingale. It will play a central role in the proof of the lower semicontinuity of the penalization function introduced in section \cite{4}. Observe that the assertion of this lemma is valid in a general filtered probability space and not only for the completed natural filtration of the Lévy process introduced above.
Lemma 3.1 For \( \{M^{(n)}\}_{n \in \mathbb{N}} \subset \mathcal{M}_\infty \) and \( M \in \mathcal{M}_\infty \) the following implication holds

\[
M^{(n)}_\infty \xrightarrow{L^1} M_\infty \implies [M^{(n)} - M]_\infty \xrightarrow{P} 0.
\]

Moreover,

\[
M^{(n)}_\infty \xrightarrow{L^1} M_\infty \implies [M^{(n)} - M]_t \xrightarrow{P} 0 \ \forall t.
\]

Proof. From the \( L^1 \) convergence of \( M^{(n)}_\infty \) to \( M_\infty \), we have that \( \{M^{(n)}_\infty\}_{n \in \mathbb{N}} \cup \{M_\infty\} \) is uniformly integrable, which is equivalent to the existence of a convex and increasing function \( G : [0, +\infty) \to [0, +\infty) \) such that

\[
(i) \quad \lim_{x \to \infty} \frac{G(x)}{x} = \infty,
\]

and

\[
(ii) \quad \sup_{n \in \mathbb{N}} \mathbb{E} \left[ G \left( |M^{(n)}_\infty| \right) \right] \vee \mathbb{E} \left[ G \left( |M_\infty| \right) \right] < \infty.
\]

Now, define the stopping times

\[
\tau^n_k := \inf \left\{ u > 0 : \sup_{t \leq u} |M^{(n)}_t - M_t| \geq k \right\}.
\]

Observe that the estimation \( \sup_{n \in \mathbb{N}} \mathbb{E} \left[ G \left( \left| M^{(n)}_\tau^k \right| \right) \right] \leq \sup_{n \in \mathbb{N}} \mathbb{E} \left[ G \left( \left| M^{(n)}_\infty \right| \right) \right] \) implies the uniformly integrability of \( \{M^{(n)}_\tau^k\}_{n \in \mathbb{N}} \) for each \( k \) fixed. Since any uniformly integrable càdlàg martingale is of class \( D \), follows the uniform integrability of \( \{M^{(n)}_\tau^k\}_{n \in \mathbb{N}} \) for all \( k \in \mathbb{N} \), and hence \( \left\{ \sup_{t \leq \tau^k_n} |M^{(n)}_t - M_t| \right\}_{n \in \mathbb{N}} \) is uniformly integrable. This and the maximal inequality for supermartingales

\[
\mathbb{P} \left\{ \sup_{t \in \mathbb{R}_+} \left| M^{(n)}_t - M_t \right| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon} \left\{ \sup_{t \in \mathbb{R}_+} \mathbb{E} \left[ \left| M^{(n)}_t - M_t \right| \right] \right\}
\]

\[
\leq \frac{1}{\varepsilon} \mathbb{E} \left[ \left| M^{(n)}_\infty - M_\infty \right| \right] \rightarrow 0,
\]

yields the convergence of \( \left\{ \sup_{t \leq \tau^k_n} \left| M^{(n)}_t - M_t \right| \right\}_{n \in \mathbb{N}} \) in \( L^1 \) to 0. The second Davis’ inequality \cite[Thm. 10.28]{Davis} guarantees that, for some constant \( C \),

\[
\mathbb{E} \left[ \sqrt{\left| M^{(n)} - M \right|_{\tau^k_n}} \right] \leq C \mathbb{E} \left[ \sup_{t \leq \tau^k_n} \left| M^{(n)}_t - M_t \right| \right] \xrightarrow{n \to \infty} 0 \ \forall k \in \mathbb{N},
\]

and hence \( [M^{(n)} - M]_{\tau^k_n} \xrightarrow{P} 0 \) for all \( k \in \mathbb{N} \).
Finally, to prove that $[M^{(n)} - M]_\infty \overset{P}{\to} 0$ we assume that it is not true, and then $[M^{(n)} - M]_\infty \overset{P}{\to} 0$ implies that there exist $\varepsilon > 0$ and $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ with
\[
d\left([M^{(n_k)} - M]_\infty, 0\right) \geq \varepsilon
\]
for all $k \in \mathbb{N}$, where $d(X, Y) := \inf \{\varepsilon > 0 : \mathbb{P}[|X - Y| > \varepsilon] \leq \varepsilon\}$ is the Ky Fan metric. We shall denote the subsequence as the original sequence, trying to keep the notation as simple as possible. Using a diagonal argument, a subsequence $\{n_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ can be chosen, with the property that $d\left([M^{(n_i)} - M]_{\tau_i}, 0\right) < \frac{1}{k}$ for all $i \geq k$. Since
\[
\lim_{k \to \infty} [M^{(n_i)} - M]_{\tau_i} = [M^{(n_0)} - M]_\infty \quad \mathbb{P} - a.s.,
\]
we can find some $k (n_i) \geq i$ such that
\[
d\left([M^{(n_i)} - M]_{\tau_{k(n_i)}}, [M^{(n_i)} - M]_\infty\right) < \frac{1}{k}.
\]

Then, using the estimation
\[
\mathbb{P}\left[\left|[M^{(n_k)} - M]_{\tau_{k(n_k)}} - [M^{(n_k)} - M]_{\tau_{k(n_k)}}\right| > \varepsilon\right] \leq \mathbb{P}\left[\sup_{t \in \mathbb{R}_+} |M^{(n_k)}_t - M_t| \geq k\right],
\]
it follows that
\[
d\left([M^{(n_k)} - M]_{\tau_{k(n_k)}}, [M^{(n_k)} - M]_{\tau_{k(n_k)}}\right) \rightarrow 0,
\]
which yields a contradiction with $\varepsilon \leq d\left([M^{(n_k)} - M]_\infty, 0\right)$. Thus, $[M^{(n)} - M]_\infty \overset{P}{\to} 0$. The last part of the this lemma follows immediately from the first statement. \hfill \Box

Using the Doob’s stopping theorem we can conclude that for $M \in \mathcal{M}_\infty$ and an stopping time $\tau$, that $M^\tau \in \mathcal{M}_\infty$, and therefore it follows as a corollary the following result.

**Corollary 3.1** For $\{M^{(n)}\}_{n \in \mathbb{N}} \subset \mathcal{M}_\infty$, $M \in \mathcal{M}_\infty$ and $\tau$ any stopping time holds
\[
M^{(n)}_\tau \overset{L^1}{\to} M_\tau \implies [M^{(n)} - M]_\tau \overset{P}{\to} 0.
\]

**Proof.** $\left((M^{(n)})^\tau - M^\tau\right)_\infty = \left[M^{(n)} - M\right]^\tau = \left[M^{(n)} - M\right]_\tau \overset{P}{\to} 0$. \hfill \Box

### 3.2 Density processes

Given an absolutely continuous probability measure $\mathbb{Q} \ll \mathbb{P}$ in a filtered probability space, where a semimartingale with the weak predictable representation property is defined, the structure of the density process has been studied extensively by several authors; see Theorem 14.41 in He, Wang and Yan [14] or Theorem III.5.19 in Jacod and Shiryaev [16].
Denote by $D_t := \mathbb{E} \left[ \frac{dQ_t}{dP_t} \mid \mathcal{F}_t \right]$ the càdlàg version of the density process. For the increasing sequence of stopping times $\tau_n := \inf \{ t \geq 0 : D_t < \frac{1}{n} \}$ $n \geq 1$ and $\tau_0 := \sup_n \tau_n$ we have $D_t(\omega) = 0 \forall t \geq \tau_0(\omega)$ and $D_t(\omega) > 0 \forall t < \tau_0(\omega)$, i.e.

$$D = D1_{[0,\tau_0]}.$$  \hspace{1cm} (3.5)

and the process

$$\frac{1}{D_s}1_{[D_\omega \neq 0]}$$

is integrable w.r.t. $D$, \hspace{1cm} (3.6)

where we abuse of the notation by setting $\{ D_\omega \neq 0 \} := \{ (\omega,t) \in \Omega \times \mathbb{R}_+ : D_{t-}(\omega) \neq 0 \}$. Both conditions (3.5) and (3.6) are necessary and sufficient in order that a semimartingale to be an exponential semimartingale \cite{14}, Thm. 9.41, i.e. $D = \mathcal{E}(Z)$ the Doléans-Dade exponential of another semimartingale $Z$. In that case we have

$$\tau_0 = \inf \{ t > 0 : D_{t-} = 0 \text{ or } D_t = 0 \} = \inf \{ t > 0 : \Delta Z_t = -1 \}. \hspace{1cm} (3.7)$$

It is well known that the Lévy-processes satisfy the weak property of predictable representation \cite{14}, when the completed natural filtration is considered. In the following lemma we present the characterization of the density processes for the case of these processes.

**Lemma 3.2** Given an absolutely continuous probability measure $Q \ll P$, there exist coefficients $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ such that

$$\frac{dQ_t}{dP_t} = \frac{dQ_t}{dP_t}1_{[0,\tau_0]} = \mathcal{E}(Z^\theta)(t), \hspace{1cm} (3.8)$$

where $Z^\theta_t \in \mathcal{M}_{loc}$ is the local martingale given by

$$Z^\theta_t := \int_{[0,t]} \theta_0 dW + \int_{[0,t] \times \mathbb{R}_0} \theta_1(s,x) (\mu(ds,dx) - ds \nu(dx)), \hspace{1cm} (3.9)$$

and $\mathcal{E}$ represents the Doléans-Dade exponential of a semimartingale. The coefficients $\theta_0$ and $\theta_1$ are $dt$-a.s and $\mu^P_t$ $(ds, dx)$-a.s. unique on $[[0,\tau_0]]$ and $[[0,\tau_0]] \times \mathbb{R}_0$ respectively for $P$-almost all $\omega$. Furthermore, the coefficients can be chosen with $\theta_0 = 0$ on $\|\tau_0, \infty\|$ and $\theta_1 = 0$ on $\|\tau_0, \infty\| \times \mathbb{R}$.\hspace{1cm}

**Proof.** We only address the uniqueness of the coefficients $\theta_0$ and $\theta_1$, because the representation follows from (3.5) and (3.6). Let assume, that we have two possible vectors $\theta := (\theta_0, \theta_1)$ and $\theta' := (\theta'_0, \theta'_1)$ satisfying the representation, i.e.

$$D_t 1_{[0,\tau_0]} = \int_{[0,t]} \theta_0(s) dW_s + \int_{[0,t] \times \mathbb{R}_0} \theta_1(s,x) (\mu(ds,dx) - ds \nu(dx)),$$

$$\int_{[0,t]} \theta'_0(s) dW_s + \int_{[0,t] \times \mathbb{R}_0} \theta'_1(s,x) (\mu(ds,dx) - ds \nu(dx)),$$

$$\int_{[0,t]} \theta'_0(s) dW_s + \int_{[0,t] \times \mathbb{R}_0} \theta'_1(s,x) (\mu(ds,dx) - ds \nu(dx)),$$
and thus

\[ \triangle D_t = D_{t-} \triangle \left( \int_{[0,t] \times \mathbb{R}_0} \theta_1(s,x) \left( \mu(ds,dx) - ds \nu(dx) \right) \right) \]

\[ = D_{t-} \triangle \left( \int_{[0,t] \times \mathbb{R}_0} \theta'_1(s,x) \left( \mu(ds,dx) - ds \nu(dx) \right) \right). \]

Since \( D_{t-} > 0 \) on \([0, \tau_0]\), it follows that

\[ \triangle \left( \int_{[0,t] \times \mathbb{R}_0} \theta_1(s,x) \left( \mu(ds,dx) - ds \nu(dx) \right) \right) = \triangle \left( \int_{[0,t] \times \mathbb{R}_0} \theta'_1(s,x) \left( \mu(ds,dx) - ds \nu(dx) \right) \right). \]

Since two purely discontinuous local martingales with the same jumps are equal, it follows

\[ \int_{[0,t] \times \mathbb{R}_0} \theta_1(s,x) \left( \mu(ds,dx) - ds \nu(dx) \right) = \int_{[0,t] \times \mathbb{R}_0} \theta'_1(s,x) \left( \mu(ds,dx) - ds \nu(dx) \right) \]

and thus

\[ \int D_{t-} d \left\{ \int_{[0,t]} \theta_0(s) dW_s \right\} = \int D_{t-} d \left\{ \int_{[0,t]} \theta'_0(s) dW_s \right\}. \]

Then,

\[ 0 = \left[ \int D_{s-} d \left\{ \int_{[0,s]} \left( \theta'_0(u) - \theta_0(u) \right) dW_u \right\} \right]_t = \int \left( D_{s-} \right)^2 \left( \theta'_0(s) - \theta_0(s) \right)^2 ds \]

and thus \( \theta'_0 = \theta_0 dt\text{-}a.s \) on \([0, \tau_0]\) for \( \mathbb{P} \)-almost all \( \omega \).

On the other hand,

\[ 0 = \left\langle \left\{ \theta'_1(s,x) - \theta_1(s,x) \right\} \left( \mu(ds,dx) - ds \nu(dx) \right) \right\rangle_t \]

\[ = \int_{[0,t] \times \mathbb{R}_0} \left\{ \theta'_1(s,x) - \theta_1(s,x) \right\}^2 \nu(dx) ds, \]

implies that \( \theta_1(s,x) = \theta'_1(s,x) \mu^\mathbb{P}_x(ds,dx)\text{-}a.s. \) on \([0, \tau_0] \times \mathbb{R}_0\) for \( \mathbb{P} \)-almost all \( \omega \). \( \square \)

For \( \mathbb{Q} \ll \mathbb{P} \) the function \( \theta_1(\omega, t, x) \) described in Lemma 3.2 determines the density of the predictable projection \( \mu^\mathbb{P}_x(dt, dx) \) with respect to \( \mu^\mathbb{P}_x(dt, dx) \) (see He,Wang and Yan [14] or Jacod and Shiryaev [16]). More precisely, for \( B \in (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0)) \) we have

\[ \mu^\mathbb{P}_x(\omega, B) = \int_B (1 + \theta_1(\omega, t, x)) \mu^\mathbb{P}_x(dt, dx). \] (3.10)
In what follows we restrict ourself to the time interval \([0, T]\), for some \(T > 0\) fixed, and we take \(\mathcal{F} = \mathcal{F}_T\). The corresponding classes of density processes associated to \(Q \lesssim (\mathbb{P})\) and \(Q \approx (\mathbb{P})\) are denoted by \(D \lesssim (\mathbb{P})\) and \(D \approx (\mathbb{P})\), respectively. For instance, in the former case

\[
D \lesssim (\mathbb{P}) := \left\{ D = \{D_t\}_{t \in [0, T]} : \exists Q \in Q \lesssim (\mathbb{P}) \text{ with } D_t = \frac{dQ}{d\mathbb{P}} \mid_{\mathcal{F}_t} \right\},
\]

and the processes in this set are of the form

\[
D_t = \exp \left\{ \int_{[0,t]} \theta_0 dW + \int_{[0,t] \times \mathbb{R}_0} \theta_1(s, x) \left( \mu(ds, dx) - \nu(dx)ds \right) - \frac{1}{2} \int_{[0,t]} (\theta_0)^2 ds \right\} \times \exp \left\{ \int_{[0,t] \times \mathbb{R}_0} \left\{ \ln \left( 1 + \theta_1(s, x) \right) - \theta_1(s, x) \right\} \mu(ds, dx) \right\}
\]

for \(\theta_0 \in L(W)\) and \(\theta_1 \in G(\mu)\).

The set \(D \lesssim (\mathbb{P})\) is characterized as follow.

**Corollary 3.2** \(D\) belongs to \(D \lesssim (\mathbb{P})\) if and only if there are \(\theta_0 \in L(W)\) and \(\theta_1 \in G(\mu)\) with \(\theta_1 \geq -1\) such that \(D_t = \mathcal{E} \left( Z^\theta \right)(t)\) \(\mathbb{P}\)-a.s. \(\forall t \in [0, T]\) and \(\mathbb{E}_\mathbb{P} \left[ \mathcal{E} \left( Z^\theta \right)(t) \right] = 1 \forall t \geq 0\), where \(Z^\theta(t)\) is defined by (3.9).

**Proof.** The necessity follows from Lemma 3.2. Conversely, let \(\theta_0 \in L(W)\) and \(\theta_1 \in G(\mu)\) be arbitrarily chosen. Since \(D_t = \int D_s dZ^\theta_s \in \mathcal{M}_{\text{loc}}\) is a nonnegative local martingale, it is a supermartingale, with constant expectation from our assumptions. Therefore, it is a martingale, and hence the density process of an absolutely continuous probability measure. \(\square\)

Since density processes are essentially uniformly integrable martingales, using Lemma 3.1 and Corollary 3.1 the following proposition follows immediately.

**Proposition 3.1** Let \(\{Q^{(n)}\}_{n \in \mathbb{N}}\) be a sequence in \(Q \lesssim (\mathbb{P})\), with \(D^{(n)}_T := \frac{dQ^{(n)}}{d\mathbb{P}} \mid_{\mathcal{F}_T}\) converging to \(D_T := \frac{dQ}{d\mathbb{P}} \mid_{\mathcal{F}_T}\) in \(L^1(\mathbb{P})\). For the corresponding density processes \(D^{(n)}_t := \mathbb{E}_\mathbb{P} \left[ D^{(n)}_T \mid \mathcal{F}_t \right]\) and \(D_t := \mathbb{E}_\mathbb{P} \left[ D_T \mid \mathcal{F}_t \right]\), for \(t \in [0, T]\), we have

\[
\left[ D^{(n)} - D \right]_T \overset{\mathbb{P}}{\to} 0.
\]

4 **Penalty functions for densities**

Now, we shall introduce a family of penalty functions for the density processes described in Section 3.2 for the absolutely continuous measures \(Q \in Q \lesssim (\mathbb{P})\).
Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $h_0, h_1 : \mathbb{R} \rightarrow \mathbb{R}_+$ be convex functions with $0 = h (0) = h_0 (0) = h_1 (0)$. Define the penalty function, with $\tau_0$ as in (3.7), by

$$\vartheta (Q) := \mathbb{E}_Q \left[ T^{\wedge \tau_0} \int_0^T h \left( h_0 (\theta_0 (t)) + \int_{\tau_0}^t \delta (t, x) h_1 (\theta_1 (t, x)) \nu (dx) \right) dt \right] 1_{Q_{\ll}} (Q) + \infty \times 1_{Q_{\cont \setminus Q_{\ll}} (Q)} ,$$

where $\theta_0, \theta_1$ are the processes associated to $Q$ from Lemma 3.2 and $\delta (t, x) : \mathbb{R}_+ \times \mathbb{R}_0 \rightarrow \mathbb{R}_+$ is an arbitrary fixed nonnegative function $\delta (t, x) \in \mathcal{G} (\mu)$. Since $\theta_0 \equiv 0$ on $\left[ \tau_0, \infty \right]$ and $\theta_1 \equiv 0$ on $\left[ \tau_0, \infty \right] \times \mathbb{R}_0$ we have from the conditions imposed to $h, h_0,$ and $h_1$

$$\vartheta (Q) = \mathbb{E}_Q \left[ T \int_0^T h \left( h_0 (\theta_0 (t)) + \int_{\tau_0}^t \delta (t, x) h_1 (\theta_1 (t, x)) \nu (dx) \right) dt \right] 1_{Q_{\ll}} (Q) + \infty \times 1_{Q_{\cont \setminus Q_{\ll}} (Q)} .$$

Further, define the convex measure of risk

$$\rho (X) := \sup_{Q \in Q_{\ll} (\mathbb{P})} \{ \mathbb{E}_Q [-X] - \vartheta (Q) \} .$$

Notice that $\rho$ is a normalized and sensitive measure of risk. For each class of probability measures introduced so far, the subclass of those measures with a finite penalization is considered. We will denote by $Q^\theta$, $Q^\theta_{\ll} (\mathbb{P})$ and $Q^\theta_{\ll} (\mathbb{P})$ the respective subclasses, i.e.

$$Q^\theta := \{ Q \in Q : \vartheta (Q) < \infty \}, \ Q^\theta_{\ll} (\mathbb{P}) := Q^\theta \cap Q_{\ll} (\mathbb{P}) \text{ and } Q^\theta_{\ll} (\mathbb{P}) := Q^\theta \cap Q_{\ll} (\mathbb{P}).$$

Notice that $Q^\theta_{\ll} (\mathbb{P}) \neq \emptyset$.

The next theorem establishes the minimality on $Q_{\ll} (\mathbb{P})$ of the penalty function introduced above for the risk measure $\rho$. Its proof is based on the sufficient conditions given in Theorem 2.1.

**Theorem 4.1** The penalty function $\vartheta$ defined in (4.13) is equal to the minimal penalty function of the convex risk measure $\rho$, given by (4.15), on $Q_{\ll} (\mathbb{P})$, i.e.

$$\vartheta 1_{Q_{\ll} (\mathbb{P})} = \psi^*_\rho 1_{Q_{\ll} (\mathbb{P})}.$$  

**Proof:** From Lemma 2.1 (b), we need to show that the penalization $\vartheta$ is proper, convex and that the corresponding identification, defined as $\Theta (Z) := \vartheta (Q)$ if $Z \in \delta (Q_{\ll} (\mathbb{P})) := \{ Z \in L^1 (\mathbb{P}) : Z = dQ/d\mathbb{P} \text{ with } Q \in Q_{\ll} (\mathbb{P}) \}$ and $\Theta (Z) := \infty$ on $L^1 \setminus \delta (Q_{\ll} (\mathbb{P}))$, is lower semicontinuous with respect to the strong topology.

First, observe that the function $\vartheta$ is proper, since $\vartheta (\mathbb{P}) = 0$. To verify the convexity of $\vartheta$, choose $Q, Q \in Q^\theta_{\ll}$ and define $Q^\lambda := \lambda Q + (1 - \lambda) Q$, for $\lambda \in [0, 1]$. Notice that the corresponding density process can be written as $D^\lambda := \frac{dQ^\lambda}{d\mathbb{P}} = \lambda D + (1 - \lambda) \tilde{D}$ $\mathbb{P}$-a.s. .

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Now, from Lemma 3.2, let \((\theta_0, \theta_1)\) and \((\tilde{\theta}_0, \tilde{\theta}_1)\) be the processes associated to \(Q\) and \(\tilde{Q}\), respectively, and observe that from

\[
D_t = 1 + \int_{[0,t]} D_{s-} \theta_0(s) \, dW_s + \int_{[0,t] \times \mathbb{R}_0} D_{s-} \theta_1(s, x) \, d(\mu(ds, dx) - ds \nu(dx))
\]

and the corresponding expression for \(\tilde{D}\) we have for \(\tau^\lambda_n := \inf \{ t \geq 0 : D^\lambda_t \leq \frac{1}{n} \}\)

\[
\int_{0}^{\tau^\lambda_n} (D^\lambda_s)^{-1} dD^\lambda_s = \int_{0}^{\tau^\lambda_n} \frac{\lambda D_{s-} \theta_0(s) + (1 - \lambda) \tilde{D}_{s-} \tilde{\theta}_0(s)}{\lambda D_{s-} + (1 - \lambda) \tilde{D}_{s-}} \, dW_s + \int_{[0, \tau^\lambda_n] \times \mathbb{R}_0} \frac{\lambda D_{s-} \theta_1(s, x) + (1 - \lambda) \tilde{D}_{s-} \tilde{\theta}_1(s, x)}{\lambda D_{s-} + (1 - \lambda) \tilde{D}_{s-}} \, d(\mu - \mu^P) .
\]

The weak predictable representation property of the local martingale \(\int_{0}^{\tau^\lambda_n} (D^\lambda_s)^{-1} dD^\lambda_s\), yield on the other hand

\[
\int_{0}^{\tau^\lambda_n} (D^\lambda_s)^{-1} dD^\lambda_s = \int_{0}^{\tau^\lambda_n} \theta^\lambda_0(s) \, dW_s + \int_{[0, \tau^\lambda_n] \times \mathbb{R}_0} \theta^\lambda_1(s, x) \, d(\mu - \mu^P) ,
\]

where identification

\[
\theta^\lambda_0(s) = \frac{\lambda D_{s-} \theta_0(s) + (1 - \lambda) \tilde{D}_{s-} \tilde{\theta}_0(s)}{\lambda D_{s-} + (1 - \lambda) \tilde{D}_{s-}} ,
\]

and

\[
\theta^\lambda_1(s, x) = \frac{\lambda D_{s-} \theta_1(s, x) + (1 - \lambda) \tilde{D}_{s-} \tilde{\theta}_1(s, x)}{\lambda D_{s-} + (1 - \lambda) \tilde{D}_{s-}} .
\]

is possible thanks to the uniqueness of the representation in Lemma 3.2. The convexity follows now from the convexity of \(h, h_0\) and \(h_1\), using the fact that any convex function is
continuous in the interior of its domain. More specifically,

\[ \vartheta (Q) \leq \mathbb{E}_{Q^\lambda} \left[ \int_{[0,T]} \frac{\lambda D_s}{(\lambda D_s + (1-\lambda) \bar{D}_s)} \left( h_0 (\theta_0 (s)) + \int_{\mathbb{R}_0} \delta (s, x) h_1 (\theta_1 (s, x)) \nu (dx) \right) ds \right] 
+ \mathbb{E}_{Q^\lambda} \left[ \int_{[0,T]} \frac{(1-\lambda) \bar{D}_s}{(\lambda D_s + (1-\lambda) \bar{D}_s + (1-\lambda) \tilde{D}_s)} \left( h_0 (\tilde{\theta}_0 (s)) + \int_{\mathbb{R}_0} \delta (s, x) h_1 (\tilde{\theta}_1 (s, x)) \nu (dx) \right) ds \right] 
= \int_{[0,T]} \lambda D_s \left( h_0 (\theta_0 (s)) + \int_{\mathbb{R}_0} \delta (s, x) h_1 (\theta_1 (s, x)) \nu (dx) \right) ds 
+ \int_{[0,T]} (1-\lambda) \bar{D}_s \left( h_0 (\tilde{\theta}_0 (s)) + \int_{\mathbb{R}_0} \delta (s, x) h_1 (\tilde{\theta}_1 (s, x)) \nu (dx) \right) ds 
= \lambda \vartheta (\bar{Q}) + (1-\lambda) \vartheta (\tilde{Q}), \]

where we used that \( \left\{ \int_{\mathbb{R}_0} \delta (t, x) h_1 (\theta_1 (t, x)) \nu (dx) \right\}_{t \in \mathbb{R}_+} \) and \( \left\{ \int_{\mathbb{R}_0} \delta (t, x) h_1 (\tilde{\theta}_1 (t, x)) \nu (dx) \right\}_{t \in \mathbb{R}_+} \) are predictable processes.

It remains to prove the lower semicontinuity of \( \Theta \). As pointed out earlier, it is enough to consider a sequence of densities \( Z^{(n)} := \frac{\delta (Q^{(n)})}{dQ^{(n)}} \in \delta (Q_{\mathbb{R}_+} (P)) \) converging in \( L^1 (P) \) to \( Z := \frac{dQ}{dP} \). Denote the corresponding density processes by \( D^{(n)} \) and \( D \), respectively. In Proposition 3.1 it was verified the convergence in probability to zero of the quadratic variation process

\[ [D^{(n)} - D]_T = \int_0^T \left\{ D^{(n)}_{s-} \theta^{(n)}_0 (s) - D_{s-} \theta_0 (s) \right\}^2 ds 
+ \int_{[0,T] \times \mathbb{R}_0} \left\{ D^{(n)}_{s-} \theta^{(n)}_1 (s, x) - D_{s-} \theta_1 (s, x) \right\}^2 \mu (ds, dx). \]

This implies that

\[ \int_0^T \left\{ D^{(n)}_{s-} \theta^{(n)}_0 (s) - D_{s-} \theta_0 (s) \right\}^2 ds \xrightarrow{P} 0, \]

and

\[ \int_{[0,T] \times \mathbb{R}_0} \left\{ D^{(n)}_{s-} \theta^{(n)}_1 (s, x) - D_{s-} \theta_1 (s, x) \right\}^2 \mu (ds, dx) \xrightarrow{P} 0. \]
and

\[ \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 \overset{L^1(\mu)}{\longrightarrow} 0, \]

where for simplicity we have denoted the sub-subsequence as the original sequence. Now, we claim that for the former sub-subsequence it also holds that

\[ D_{s-}^{(n)}\theta_{0}^{(n)}(s) \overset{\lambda \times \mathbb{P}\text{-a.s.}}{\longrightarrow} D_{s-}\theta_{0}(s), \]

\[ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) \overset{\mu \times \mathbb{P}\text{-a.s.}}{\longrightarrow} D_{s-}\theta_{1}(s, x). \]

(4.18)

We present first the arguments for the proof of the second assertion in (4.18). Assuming the opposite, there exists \( C \in \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}_0) \otimes \mathcal{F}_T \), with \( \mu \times \mathbb{P}[C] > 0 \), and such that for each \((s, x, \omega) \in C\)

\[ \lim_{n \to \infty} \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 = c \neq 0, \]

or the limit does not exist.

Let \( C(\omega) := \{(t, x) \in [0, T] \times \mathbb{R}_0 : (t, x, \omega) \in C\} \) be the \( \omega \)-section of \( C \). Observe that \( B := \{\omega \in \Omega : \mu[C(\omega)] > 0\} \) has positive probability: \( \mathbb{P}[B] > 0 \).

From (4.17), any arbitrary but fixed subsequence has a sub-subsequence converging \( \mathbb{P}\)-a.s.. Denoting such a sub-subsequence simply by \( n \), we can fix \( \omega \in B \) with

\[ \int_{C(\omega)} \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 d\mu(s, x) \leq \int_{[0, T] \times \mathbb{R}_0} \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 d\mu(s, x) \to 0, \]

and hence \( \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 \) converges in \( \mu \)-measure to 0 on \( C(\omega) \). Again, for any subsequence there is a sub-subsequence converging \( \mu\)-a.s. to 0. Furthermore, for an arbitrary but fixed \((s, x) \in C(\omega)\), when the limit does not exist

\[ a := \liminf_{n \to \infty} \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 \neq \limsup_{n \to \infty} \left\{ D_{s-}^{(n)}\theta_{1}^{(n)}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 =: b, \]

and we can choose converging subsequences \( n(i) \) and \( n(j) \) with

\[ \lim_{i \to \infty} \left\{ D_{s-}^{(n(i))}\theta_{1}^{(n(i))}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 = a \]

\[ \lim_{j \to \infty} \left\{ D_{s-}^{(n(j))}\theta_{1}^{(n(j))}(s, x) - D_{s-}\theta_{1}(s, x) \right\}^2 = b. \]
From the above argument, there are sub-subsequences \( n(i(k)) \) and \( n(j(k)) \) such that

\[
a = \lim_{k \to \infty} \left\{ D_{s^{-}}^{n(i(k))} \theta_{1}^{n(i(k))} (s, x) - D_{s^{-}} \theta_{1} (s, x) \right\}^2 = 0
\]

\[
b = \lim_{k \to \infty} \left\{ D_{s^{-}}^{n(j(k))} \theta_{1}^{n(j(k))} (s, x) - D_{s^{-}} \theta_{1} (s, x) \right\}^2 = 0,
\]

which is clearly a contradiction.

For the case when

\[
\lim_{n \to \infty} \left\{ D_{s^{-}}^{(n)} \theta_{1}^{(n)} (s, x) - D_{s^{-}} \theta_{1} (s, x) \right\}^2 = c \neq 0,
\]

the same argument can be used, and get a subsequence converging to 0, having a contradiction again. Therefore, the second part of our claim in (4.19) holds.

Since \( D_{s^{-}}^{n} \theta_{1}^{(n)} (s, x) \), \( D_{s^{-}} \theta_{1} (s, x) \in \mathcal{G}(\mu) \), we have, in particular, that \( D_{s^{-}}^{n} \theta_{1}^{(n)} (s, x) \in \widetilde{\mathcal{P}} \) and \( D_{s^{-}} \theta_{1} (s, x) \in \widetilde{\mathcal{P}} \) and hence \( C \in \widetilde{\mathcal{P}} \). From the definition of the predictable projection it follows that

\[
0 = \mu \times \mathbb{P} [C] = \int_{\Omega \times [0,T] \times \mathbb{R}_0} 1_{C} (s, \omega, t) \, d\mu \, d\mathbb{P} = \int_{\Omega \times [0,T]} \int_{\mathbb{R}_0} 1_{C} (s, \omega) \, d\mu_{\mathbb{P}} \, d\mathbb{P}
\]

\[
= \int_{\Omega \times [0,T]} \int_{\mathbb{R}_0} 1_{C} (s, \omega) \, ds \, d\nu \, d\mathbb{P} = \lambda \times \nu \times \mathbb{P} [C],
\]

and thus

\[
D_{s^{-}}^{(n)} \theta_{1}^{(n)} (s, x) \overset{\lambda \times \nu \times \mathbb{P} \text{-a.s.}}{\longrightarrow} D_{s^{-}} \theta_{1} (s, x).
\]

Since

\[
\left| \int_{\Omega \times [0,T]} D_{t^{-}}^{(n)} - D_{t^{-}} \right| \, d\mathbb{P} \times dt = \int_{\Omega \times [0,T]} \left| D_{t^{-}}^{(n)} - D_{t^{-}} \right| \, d\mathbb{P} \times dt \longrightarrow 0,
\]

we have that

\[
\left\{ D_{t^{-}}^{(n)} \right\}_{t \in [0,T]} \overset{L^{1}(\lambda \times \mathbb{P})}{\longrightarrow} \left\{ D_{t^{-}} \right\}_{t \in [0,T]} \text{ and } \left\{ D_{t}^{(n)} \right\}_{t \in [0,T]} \overset{L^{1}(\lambda \times \mathbb{P})}{\longrightarrow} \left\{ D_{t} \right\}_{t \in [0,T]}.
\]

Then, for an arbitrary but fixed subsequence \( \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \), there is a sub-subsequence \( \{n_{k_{i}}\}_{i \in \mathbb{N}} \subset \mathbb{N} \) such that

\[
D_{t^{-}}^{(n_{k_{i}})} \theta_{1}^{(n_{k_{i}})} (t, x) \overset{\lambda \times \nu \times \mathbb{P} \text{-a.s.}}{\longrightarrow} D_{t^{-}} \theta_{1} (t, x),
\]

\[
D_{t}^{(n_{k_{i}})} \overset{\lambda \times \mathbb{P} \text{-q.s.}}{\longrightarrow} D_{t},
\]

Furthermore, \( \mathbb{Q} \ll \mathbb{P} \) implies that \( \lambda \times \nu \times \mathbb{Q} \ll \lambda \times \nu \times \mathbb{P} \), and then

\[
D_{t^{-}}^{(n_{k_{i}})} \theta_{1}^{(n_{k_{i}})} (t, x) \overset{\lambda \times \nu \times \mathbb{Q} \text{-a.s.}}{\longrightarrow} D_{t^{-}} \theta_{1} (t, x),
\]

\[
D_{t}^{(n_{k_{i}})} \overset{\lambda \times \nu \times \mathbb{Q} \text{-a.s.}}{\longrightarrow} D_{t},
\]

and

\[
D_{t}^{(n_{k_{i}})} \overset{\lambda \times \nu \times \mathbb{Q} \text{-a.s.}}{\longrightarrow} D_{t}.
\] (4.19)

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Finally, noting that \( \inf D_t > 0 \) \( Q \)-a.s.

\[
\theta_1^{(n_k)}(t, x) \xrightarrow{\lambda \times Q \text{-a.s.}} \theta_1(t, x). \tag{4.20}
\]

The first assertion in (4.18) can be proved using essentially the same kind of ideas used above for the proof of the second part, concluding that for an arbitrary but fixed subsequence \( \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \), there is a sub-subsequence \( \{n_{k_i}\}_{i \in \mathbb{N}} \subset \mathbb{N} \) such that

\[
\left\{ D_t^{(n_{k_i})} \right\}_{t \in [0,T]} \xrightarrow{\lambda \times Q \text{-a.s.}} \{D_t\}_{t \in [0,T]} \tag{4.21}
\]

and

\[
\left\{ \theta_0^{(n_{k_i})}(t) \right\}_{t \in [0,T]} \xrightarrow{\lambda \times Q \text{-a.s.}} \{\theta_0(t)\}_{t \in [0,T]} \tag{4.22}
\]

We are now ready to finish the proof of the theorem, observing that

\[
\liminf_{n \to \infty} \vartheta \left( \mathcal{Q}^{(n)} \right) = \liminf_{n \to \infty} \int_{\Omega \times [0,T]} \left\{ h \left( h_0 \left( \theta_0^{(n)}(t) \right) + \int_{\mathbb{R}_0} \delta(t, x) h_1 \left( \theta_1^{(n)}(t, x) \right) \nu(dx) \right) \right\} \frac{D_t^{(n)}}{D_t} d(\lambda \times Q).
\]

Let \( \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \) be a subsequence for which the limit inferior is realized. Using (4.19), (4.20), (4.21), and (4.22) we can pass to a sub-subsequence \( \{n_{k_i}\}_{i \in \mathbb{N}} \subset \mathbb{N} \) and, from the continuity of \( h, h_0 \) and \( h_1 \), it follows

\[
\liminf_{n \to \infty} \vartheta \left( \mathcal{Q}^{(n)} \right) \geq \int_{\Omega \times [0,T]} \liminf_{i \to \infty} \left\{ h \left( h_0 \left( \theta_0^{(n_{k_i})}(t) \right) + \int_{\mathbb{R}_0} \delta(t, x) h_1 \left( \theta_1^{(n_{k_i})}(t, x) \right) \nu(dx) \right) \right\} \frac{D_t^{(n_{k_i})}}{D_t} d(\lambda \times Q)
\]

\[
\geq \int_{\Omega \times [0,T]} h \left( h_0 \left( \theta_0(t) \right) + \int_{\mathbb{R}_0} h_1 \left( \theta_1(t, x) \right) \nu(dx) \right) d(\lambda \times Q)
\]

\[
= \vartheta \left( \mathcal{Q} \right).
\]

\[\square\]
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