INTEGRABLE SYSTEMS ON $\text{Fl}_n \times \text{Fl}_n \times \text{Fl}_n/\text{SU}(n)$ AND $\text{SU}(n)$ TENSOR PRODUCT MULTIPlicITIES

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Dedicated to the memory of Andrei V. Zelevinsky

ABSTRACT. We construct a densely defined torus action on the symplectic quotient of the product of three complete flag varieties. The closure of the image of the corresponding moment map is a convex polytope. The dimension of the geometric quantization of this space gives the structure constants of the representation ring of $\text{SU}(n)$, which we show is given by counting lattice points in the image of the moment map. Such lattice point formulas for the structure constants were given by Berenstein-Zelevinsky; our results show how such formulas arise geometrically as an example of “invariance of polarization”, analogous to the description of the Gelfand-Cetlin polytopes by Guillemin-Sternberg. We outline applications to loop groups and to moduli spaces of vector bundles.

1. INTRODUCTION

1.1. Geometric Quantization and Invariance of Polarization. Let $(M, \omega)$ be an integral symplectic manifold. A prequantization of $(M, \omega)$ is a line bundle $L$ with a connection $\nabla$ of curvature $\omega$. The program of geometric quantization would assign to the quadruple $(M, \omega, L, \nabla)$ a Hilbert space $Q(M)$ with certain reasonable properties (see e.g. [14]). In practice all known constructions of $Q(M)$ involve an additional structure known as a polarization. One example of a polarization is a choice of a complex structure on $M$ compatible with $\omega$; then if $M$ is compact, $Q(M) = H^0(M, L)$.

Another example is a real polarization—a foliation of $M$ by Lagrangian submanifolds, given by a map $\pi : M \to B$ where $B$ is some space of half the dimension of $M$. One example of a map with generically Lagrangian fibres is the moment map for an effective torus action of half the dimension of $M$, when $M$ is a toric variety. This situation can be generalized to the case of abelian varieties, where the moment map is not quite globally defined, or to other situations where the torus action is densely defined and the moment map is differentiable only on a dense open set, but only continuous globally; see below. In the case where $M$ is compact, the quantization in terms of a real polarization is then given, as a vector space, by

$$Q(M) = \bigoplus_{b \in B_{\text{bs}}} \mathbb{C} \langle s_b \rangle$$

where the Bohr-Sommerfeld set $B_{\text{bs}}$ is the set of points $b \in B$ such that the line bundle $(L, \nabla)_{\pi^{-1}(b)}$ is trivial as a line bundle with connection.$^1$

It is expected that the quantization $Q(M)$, and in particular its dimension, should be independent of the choice of polarization. This is the case for toric varieties, abelian varieties, and other examples.

We focus on two examples which will be useful in this paper.

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$^1$Some care must be taken on the singular fibres of $\pi$. 

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1.2. Coadjoint orbits of $U(n)$. Let $\lambda \in \mathfrak{u}(n)^*$ and let $O_\lambda$ be the coadjoint orbit of $\lambda$ in $\mathfrak{u}(n)^*$; $O_\lambda$ is a Hamiltonian $U(n)$ space with the moment map given by the inclusion $O_\lambda \to \mathfrak{u}(n)^*$. If $\lambda$ is a weight of $U(n)$ the manifold $O_\lambda$ is equipped with a line bundle with connection whose curvature is the Kirillov-Kostant symplectic form on $O_\lambda$. Let $f_\lambda : \mathfrak{u}(n)^* \to \mathbb{R}^n$ be the map sending an element of $\mathfrak{u}(n)^*$, which from now on we identify with the space of Hermitian, $n \times n$ matrices, to its eigenvalues, listed in increasing order. The inclusion $i_n : \mathfrak{u}(n-1) \to \mathfrak{u}(n)$ induces a projection $\mathfrak{u}(n)^* \to \mathfrak{u}(n-1)^*$; by composition of repeated inclusions with the maps $f_1, \ldots, f_n$, we obtain a map

\[ f_{GS} : \mathfrak{u}(n)^* \to \mathbb{R}^{n(n+1)/2}. \]

**Theorem 1.2** (Guillemin-Sternberg [6]). The map $f_{GS}$ gives the moment map for a $(n-1)/2$-dimensional torus action defined on a dense open set of $O_\lambda$. If $\lambda \in \mathfrak{u}(n)^*$ is generic, this action is effective. The image of $f_{GS}$ is the Gelfand-Cetlin convex polytope $\Delta_{GC}(\lambda)$ (see [6]). The fibres of $f_{GS}$ are generically Lagrangian. If $\lambda$ is a weight of $U(n)$, the points in $\Delta_{GC}(\lambda) \cap \mathbb{Z}^{n(n-1)/2}$ are the Bohr-Sommerfeld set. The number of such points gives the dimension of the representation of $U(n)$ given by the weight $\lambda$.

Since by the Borel-Weil-Bott Theorem, the representation with highest weight $\lambda$ can be obtained from a complex polarization of $O_\lambda$, Theorem 1.2 is an example of invariance of polarization.

Guillemin and Sternberg also prove an extension of their result which we will use below.

**Theorem 1.3.** [7] Let $(M, \omega)$ be a Hamiltonian $U(n)$ space; denote the moment map by $\mu : M \to \mathfrak{u}(n)^*$. Then $f_{GS} \circ \mu$ is the moment map for a densely defined torus action on $M$.

1.3. The cotangent bundle of $U(n)$. A second example of invariance of polarization is related to Thimm’s theorem about integrability of geodesics on symmetric spaces. Let $M = T^*U(n)$, equipped with the standard symplectic form on a cotangent bundle. The right and left actions of $U(n)$ on $U(n)$ induce two actions on $M$, with two moment maps $\mu_R, \mu_L : M \to \mathfrak{u}(n)^*$, and hence by Theorem 1.3 a densely defined action of a torus $(S^1)^{n(n+1)/2} \times (S^1)^{n(n+1)/2}$. This action is not effective, but there is an effective action of a torus of dimension $n^2 = \dim U(n)$, which makes $M$ into an integrable system.

More explicitly, in terms of the left trivialization of the cotangent bundle $T^*U(n) = U(n) \times \mathfrak{u}(n)^*$, the moment maps for the right and left $U(n)$ actions are (see e.g. [16])

\[ \mu_R(g, \xi) = \xi \]
\[ \mu_L(g, \xi) = -g\xi g^{-1} \]

and the moment map for the torus action is $(f_{GS} \circ \mu_R, f_{GS} \circ \mu_L)$. Note that $f_n \circ \mu_R = -f_n \circ \mu_L$, which gives rise to the degeneracy of the torus action. This fact also shows that the quantization of $M$ is given by

\[ Q(T^*U(n)) = \bigoplus_\lambda Q(O_\lambda) \otimes Q(O_{-\lambda}), \]

where the sum is taken over all weights $\lambda$ of $U(n)$ lying in a Weyl chamber.

On the other hand, like any cotangent bundle, $M = T^*U(n)$ has another real polarization by the bundle projection, in terms of which $Q(M) = L^2(U(n))$. In this case the invariance of polarization,

\footnote{The eigenvalues of a matrix lying in a fixed orbit are constant, so the map $x \to f_n(x)$, which gives $n$ components of the map $f_{GS}$, is a constant on a coadjoint orbit. We will need these eigenvalues in the rest of the paper, so to avoid notational complications, we retain them in the map $f_{GS}$, even though they are a nuisance in the statement of the present theorem.}
which leads us to expect that these two real polarizations should give isomorphic quantizations, is the Peter-Weyl Theorem. This was the result of the thesis of Filippini [4].

1.4. The main theorem: An integrable system on \((O_\alpha \times O_\beta \times O_\gamma)/SU(n)\). The purpose of this paper is to construct a densely defined torus action on the reduced space

\[ M_{\alpha,\beta,\gamma} = (O_\alpha \times O_\beta \times O_\gamma)/SU(n) \]

where \(\alpha, \beta, \gamma \in su(n)^* \subset u(n)^*\), and the action of \(SU(n)\) on \(O_\alpha \times O_\beta \times O_\gamma\) is the diagonal action. For simplicity we consider only the case where \(\alpha, \beta, \gamma\) are generic in the sense that each has minimal stabilizer (given by the maximal torus) under the coadjoint action. When \(\alpha, \beta, \gamma\) are weights of \(SU(n)\), the dimension of the quantization of \(M_{\alpha,\beta,\gamma}\) in a holomorphic polarization gives the Clebsch-Gordan coefficient for the multiplicity of the dual of the representation with highest weight \(\gamma\) in the tensor product of the representations given by \(\alpha\) and \(\beta\). Berenstein and Zelevinsky [3] showed that these coefficients could be obtained by counting lattice points in convex polytopes bearing a resemblance to Gelfand-Cetlin polytopes, by combinatorial methods. This was the topic of intensive study, for example, in the work of Knutson and Tao [15]. It is natural to ask whether there is an integrable system on \(M_{\alpha,\beta,\gamma}\) which would make these formulas into examples of invariance of polarization, and in some sense the results of Harada and Kaveh [9] show such a system much exist. We give an explicit construction which is contained in the main theorem of this paper, Theorem 2 below.

The main idea of the proof is to show that on a dense open set, the space \(M_{\alpha,\beta,\gamma}\) comes equipped with a Hamiltonian action of \(U(n-1)\), so that we can use Theorem 1.3. This \(U(n-1)\) action comes from presenting \(M_{\alpha,\beta,\gamma}\) as a symplectic quotient of Hamiltonian \(U(n-1)\) space \(M_{\beta,\gamma}\) having dimension twice that of \(U(n-1)\) and such that the action of the complexified group \(GL(n-1, \mathbb{C})\) on \(M_{\beta,\gamma}\) is free up to a finite stabilizer. Since \(GL(n-1, \mathbb{C}) = T^*(U(n-1))\), the equivariant Moser theorem shows that a neighborhood of a \(U(n-1)\) orbit in \(M_{\beta,\gamma}\) is equivariantly symplectomorphic (again, up to a finite stabilizer) to a neighborhood of a \(U(n-1)\) orbit in \(T^*(U(n-1))\), and hence the \(U(n-1)\) action in such a neighborhood extends to an action of \(U(n-1) \times U(n-1)\) arising from the right and left actions on the cotangent bundle. This gives us the desired \(U(n-1)\) action on the quotient \(M_{\alpha,\beta,\gamma}\).

Note that in the case where \(n = 2\), the space \(M_{\alpha,\beta,\gamma}\) is zero dimensional—empty or a point—so there is nothing to prove. Where \(n = 3\), the space \(M_{\alpha,\beta,\gamma}\), when nonempty, is generically symplectomorphic to the two sphere \(S^2\), and therefore is equipped with a global torus action. This case was considered in [10],[12].

1.5. Remarks and extensions. Our construction should also extend to the quasi-Hamiltonian quotients

\[ (C_a \times C_b \times C_c)/SU(n) \]

where \(a, b, c \in SU(n)\) and \(C_f\) denotes the conjugacy class of \(f \in SU(n)\). The quantization of these spaces should give the structure constants of ring of positive-energy representations of the loop group \(LSU(n)\), in other words, of the fusion ring in the corresponding conformal field theory. Formulas of this type have been considered in [2]. In combination with the Goldman flows on the moduli space of flat connections on a Riemann surface, [5], the resulting densely defined torus actions should give densely defined torus actions on the moduli space of flat connections of any rank, completing the work of [13].

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3I would like to thank Alan Weinstein for bringing this reference to my attention.
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2. Coadjoint orbits of $U(n-1)$ and coadjoint orbits of $U(n)$

For $k, l$ positive integers, let $M_{k,l}$ denote the space of $k \times l$ matrices with complex entries. Since $M_{k,l} = \mathbb{C}^{kl}$, $M_{k,l}$ is a symplectic manifold; in fact it is a Hamiltonian $U(k) \times U(l)$ space with moment map $\mu : M_{k,l} \to u(k)^* \oplus u(l)^*$ given by

$$\mu(z) = (zz^*, -z^*z).$$

Let $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ be an element in the dual of the torus of $u(n-1)$, thought of as a matrix with $(\lambda_1, \ldots, \lambda_{n-1})$ as its diagonal entries, and let $O_\lambda$ be the coadjoint orbit of $\lambda$. Given such an element of $u(n-1)^*$, we denote by $\lambda'$ the element of $u(n)^*$ given by the matrix with diagonal entries $(\lambda_1, \ldots, \lambda_{n-1}, 0)$; by a slight abuse of notation we denote by $O_{\lambda'}$ the coadjoint orbit in $u(n)^*$ through $\lambda'$. Then the following is an elementary result in linear algebra.

**Proposition 2.1.** Suppose the eigenvalues of $\lambda$ are nonnegative. Then the symplectic quotient $(O_\lambda \times M_{n-1,n})//U(n-1)$ by the diagonal $U(n-1)$ action on $O_\lambda \times M_{n-1,n}$ is the coadjoint orbit $O_{\lambda'}$ of $U(n)$ through $\lambda'$.

A key element of our main construction will be the following innocuous-looking corollary.

**Corollary 2.2.** Let $\beta, \gamma \in u(n)^*$ and let

$$\mathcal{M}_{\beta, \gamma} = (M_{n-1,n} \times O_\beta \times O_\gamma)//SU(n)$$

be the quotient of $M_{n-1,n} \times O_\beta \times O_\gamma$ by the diagonal action of $SU(n)$.

Then for any diagonal element $\alpha \in u(n-1)^*$ with nonnegative eigenvalues,

$$M_{\alpha', \beta, \gamma} = (\mathcal{M}_{\beta, \gamma} \times O_\alpha)//U(n-1).$$

where the quotient is again by the diagonal action of $U(n-1)$.

The following is immediate:

**Proposition 2.3.** The space $\mathcal{M}_{\beta, \gamma}$ is a manifold for generic $\beta, \gamma$, and is a Kahler manifold of real dimension $2 \dim U(n-1)$. The action of $U(n-1)$ on $\mathcal{M}_{\beta, \gamma}$ extends to an action of the complex group $GL(n-1, \mathbb{C})$.

We will see in the following section that this action of the complex group is essentially free in the sense that the stabilizer of any point is a discrete subgroup of the center of $GL(n-1, \mathbb{C})$.

**Remark 2.4.** Coadjoint orbits of $U(n)$ and of $SU(n)$: To effectively use Proposition 2.1 we consider coadjoint orbits of $U(n)$ rather than of $SU(n)$. Since any $\lambda \in u(n)^*$ can be written $\lambda = \alpha + \rho I_n$, where $\alpha \in su(n)^*$ and $\rho \in \mathbb{R}$, and where $I_n$ is the $n \times n$ identity matrix, any coadjoint orbit of $U(n)$ is a translate of a coadjoint orbit of $SU(n)$ by a multiple of the identity matrix; and the reduced space of a product of three coadjoint orbits of $SU(n)$ is symplectomorphic to the reduced space by the diagonal action of $SU(n)$ of the product of any such translates of these orbits. Since any element of $su(n)^*$ is conjugate to a translate by some multiple of the identity matrix of an element of $u(n)^*$ with eigenvalues $\{\alpha_1, \ldots, \alpha_n, 0\}$ where all the $\alpha_i$ are nonnegative, the construction of Proposition 2.1 gives all coadjoint orbits of $SU(n)$, up to an irrelevant translation. In the rest of the paper, we suppress this issue, and assume that we can write any coadjoint orbit as a quotient as in Proposition 2.1 and any reduced space $M_{\alpha, \beta, \gamma}$ as a quotient as in Corollary 2.2.
Remark 2.5. Note that by iterating a slight generalization of the construction of Proposition 2.1 one can construct any coadjoint orbit of \( \mathcal{U}(n) \) as a quotient of \( M_{k_1,k_2} \times M_{k_2,k_3} \times \cdots \times M_{k_p,n} \) by the diagonal action of \( \mathcal{U}(k_1) \times \mathcal{U}(k_2) \times \cdots \times \mathcal{U}(k_p) \) at a reduction point given by appropriate multiples of the \( k_j \times k_j \) identity matrices. This is a generalization of the well known construction of the Grassmannian \( Gr(k, l) \) (where \( k < l \)) as the quotient of \( M_{k,l} \) by the left action of \( \mathcal{U}(k) \). I learned this construction from Shawn Martin, but have not been able to find it in the literature.

3. The symplectic geometry of \( M_{\beta, \gamma} \)

Let \( \Xi \subset \mathcal{U}(n-1) \) denote the subgroup of the center of \( \mathcal{U}(n-1) \) given by the \( n \)-th roots of unity; i.e. \( \Xi = \exp(2\pi ik/n)I_{(n-1)\times(n-1)} \) where \( k \in \mathbb{Z} \), and where we write (from now on) \( I_{k\times k} \) for the \( k \times k \) identity matrix.

The purpose of this section is to prove the following result.

Theorem 1. For generic \( \beta, \gamma \in u^*(n) \), a neighborhood of a generic orbit of \( \mathcal{U}(n-1) \) in \( M_{\beta, \gamma} \) mapping under the moment map for the \( \mathcal{U}(n-1) \) action to a coadjoint orbit \( O \) of \( \mathcal{U}(n-1) \) is \( \mathcal{U}(n-1) \)-equivariantly symplectomorphic to a neighborhood of a \( \mathcal{U}(n-1) \) orbit in \( T^*(\mathcal{U}(n-1)/\Xi) \) mapping to the same orbit \( O \) under the moment map for the left action of \( \mathcal{U}(n-1) \) on \( T^*(\mathcal{U}(n-1)/\Xi) \). Under this symplectomorphism, the moment maps for the \( \mathcal{U}(n-1) \) actions differ by a locally constant multiple \( s(\beta, \gamma) \) of the identity matrix \( I_{(n-1)\times(n-1)} \).

This theorem has the following remarkable Corollary.

Corollary 3.1. For generic \( \beta, \gamma \in u^*(n) \), a neighborhood of a generic orbit of \( \mathcal{U}(n-1) \) in \( M_{\beta, \gamma} \) is equipped with a Hamiltonian action of \( \mathcal{U}(n-1) \times \mathcal{U}(n-1) \) arising from the left and right actions of \( \mathcal{U}(n-1) \) on \( T^*(\mathcal{U}(n-1)/\Xi) \).

Let \( T \) denote the diagonal subgroup of \( SL(n, \mathbb{C}) \). Let \( \Xi' \subset GL(n-1, \mathbb{C}) \times T \) denote the subgroup of \( GL(n-1, \mathbb{C}) \times T \) given by

\[
\Xi' = \{ (\exp(2\pi ik/n)I_{(n-1)\times(n-1)}, \exp(-2\pi ik/n)I_{n\times n}) \}_{k \in \mathbb{Z}}.
\]

The main computation in the proof of Theorem 1 is the following.

Proposition 3.2. The group \( GL(n-1, \mathbb{C}) \times T \subset GL(n-1, \mathbb{C}) \times GL(n, \mathbb{C}) \) acts on \( M_{n-1,n} \) with generic stabilizer \( \Xi' \).

Proof. The columns of a generic element of \( z \in M_{n-1,n} \) consist of \( n \) elements \( v_1, \ldots, v_n \in \mathbb{C}^{n-1} \), the first \( n-1 \) of which are linearly independent. Write \( S \) for the matrix consisting of the first \( n-1 \) columns of \( z \). An element \( (M, D) \in GL(n-1, \mathbb{C}) \times T \) stabilizes \( z \) if, writing \( \Lambda \) for the upper left hand \( (n-1) \times (n-1) \) submatrix of \( D \), we have

\[
MS\Lambda = S
\]

and

\[
Mv_n = (\det \Lambda)v_n.
\]

Thus \( M = S\Lambda^{-1}S^{-1} \) and \( v_n \) is an eigenvector of \( M \) with eigenvalue \( \det \Lambda \). Since \( z \) is generic, the only way an arbitrarily chosen generic vector \( v_n \) can be an eigenvalue of \( M \) is if \( M \) is a multiple of the identity matrix; in other words, if \( M = \Lambda^{-1} = \Lambda^{-1}I_{(n-1)\times(n-1)} \) for some \( \lambda \). Then by equation (3.4) also \( Mv_n = \lambda^{-1}v_n = \lambda^{n-1}v_n \), so \( \lambda^n = 1 \), as needed. \( \square \)

Corollary 3.3. The group \( GL(n-1, \mathbb{C}) \) acts on \( M_{\beta, \gamma} \) with generic stabilizer \( \Xi \).
Proof. The space \( \mathcal{M}_{\beta,\gamma} \) is the symplectic quotient of the Kahler manifold \( M_{n-1,n} \times O_\beta \times O_\gamma \) by the diagonal action of the compact group \( SU(n) \). It is therefore the geometric invariant theory quotient of \( M_{n-1,n} \times O_\beta \times O_\gamma \) by the diagonal action of \( SL(n,\mathbb{C}) \) by the Kirwan-Kempf-Ness theorem. This quotient is obtained by removing some subspace of nonzero codimension, which we need not consider when computing generic stabilizers. Since \( \beta \) and \( \gamma \) are generic, the stabilizer of a generic point of \( O_\beta \times O_\gamma \) under the diagonal action of \( SL(n,\mathbb{C}) \) is the intersection of two generic Borels, which is conjugate to the maximal torus \( T \) of \( SL(n,\mathbb{C}) \). The result now follows by Proposition 3.2.

We can now prove Theorem 1.

Proof. Consider the symplectic manifolds \( \mathcal{M}_{\beta,\gamma} \) and \( T^*(U(n-1)/\Xi) \). These manifolds have the same dimension and both are equipped with Hamiltonian actions of \( U(n-1)/\Xi \). These actions extend to actions of \( GL(n-1,\mathbb{C})/\Xi \). Near a generic orbit of \( U(n-1)/\Xi \), these \( GL(n-1,\mathbb{C})/\Xi \) actions are both free, and since \( T^*(U(n-1)/\Xi) = GL(n-1,\mathbb{C})/\Xi \), it follows that a neighborhood of a generic \( U(n-1) \) orbit in \( \mathcal{M}_{\beta,\gamma} \) is equivariantly diffeomorphic to a neighborhood of a generic \( U(n-1) \) orbit in \( T^*(U(n-1)/\Xi) = GL(n-1,\mathbb{C})/\Xi \). By the equivariant Moser theorem (see e.g. [8], p. 324, Section 41), if these orbits are sent by the moment map to the same orbit \( O \subset \mathfrak{u}(n)^* \), this diffeomorphism can be taken to be an equivariant symplectomorphism.

The statement about moment maps follows from the fact that two equivariant moment maps vary.

Problem 3.6. Compute \( s(\beta,\gamma) \).

Remark 3.7. Just as the space \( M_{\alpha,\beta,\gamma} \) can be presented as a quotient of \( O_\lambda \times M_{n-1,n} \times O_\beta \times O_\gamma \), where \( \alpha = \lambda' \), the weight variety \( O_\alpha/T \) can be presented as a quotient of \( O_\lambda \times M_{n-1,n}/T \) by the action of \( U(n-1) \), and hence, by a similar argument, is also, up to the action of a finite group and some genericity condition, given by some coadjoint orbit \( O \) of \( su(n-1) \).

4. INTEGRABLE SYSTEMS ON \( M_{\alpha,\beta,\gamma} \)

Theorem 4.1 shows that a neighborhood of a generic orbit of \( U(n-1) \) in \( \mathcal{M}_{\beta,\gamma} \) is equivariantly symplectomorphic to a neighborhood of a \( U(n-1) \) orbit in \( T^*(U(n-1)/\Xi) \), and by Corollary 3.1 is equipped with a Hamiltonian action of \( U(n-1) \times U(n-1) \) arising from the right and left \( U(n-1) \) actions on \( T^*(U(n-1)/\Xi) \). Since the space \( M_{\lambda',\beta,\gamma} \) is given by the symplectic quotient

\[
M_{\lambda',\beta,\gamma} = (O_\lambda \times M_{\beta,\gamma})/U(n-1)
\]

by the diagonal \( U(n-1) \) action, it follows that for generic \( \lambda,\beta,\gamma \), the space \( M_{\lambda',\beta,\gamma} \) has a residual Hamiltonian \( U(n-1) \) action, with locally effective action of the subgroup \( SU(n) \). In fact,

\[
M_{(\lambda + s(\beta,\gamma)I_{(n-1)\times(n-1)})',\beta,\gamma} \simeq O_\lambda
\]

where \( s(\beta,\gamma) \) is a locally constant function on \( \mathfrak{u}(n)^* \times \mathfrak{u}(n)^* \).

Let us write \( j : su(n) \rightarrow u(n) \) for the inclusion map and \( j^* : u(n)^* \rightarrow su(n)^* \) for its adjoint. Denote the moment map for our \( U(n-1) \) action on \( M_{\lambda',\beta,\gamma} \) by \( \mu \). By Theorem 4.3, the space \( M_{\lambda',\beta,\gamma} \) has a densely defined torus action; the image of the map \( f_{GS} \circ \mu \) is the Gelfand-Cetlin polytope of the coadjoint orbit \( O_\lambda \). We summarize these results in the following Theorem, which is the main result of this paper.

Theorem 2. For generic \( \alpha,\beta,\gamma \in su(n)^* \), the space \( M_{\alpha,\beta,\gamma} \) is equipped with a densely defined torus action. The moment map for this torus action extends to \( M_{\alpha,\beta,\gamma} \) as a continuous function. The image of this function is the Gelfand-Cetlin polytope \( \Delta_{GC}(\lambda) \), where we write \( \alpha = j^*((\lambda + s(\beta,\gamma)I_{(n-1)\times(n-1)})') \).
where $\lambda$ is a generic element of the torus of $u(n-1)^*$ with nonnegative eigenvalues lying in the image of the moment map for the $U(n-1)$ action on $M_{3,\gamma}$, and $s(\beta, \gamma)$ is a locally constant function on $u(n)^* \times u(n)^*$.

The principle of “Quantization commutes with Reduction” shows that where $\alpha, \beta, \gamma$ are weights of $SU(n)$, the dimension of the quantization of the space $M_{\alpha,\beta,\gamma}$ gives the Clebsch Gordan coefficient $C_{\alpha,\beta,\gamma}$ describing the multiplicity of the dual of the representation corresponding to $\gamma$ in the product of the representations corresponding to $\alpha$ and $\beta$. Thus by Theorem 1.2 $C_{\alpha,\beta,\gamma}$ is given by the number of lattice points in the Gelfand-Cetlin polytope $\Delta_{GC}(\lambda)$, as above.

It would be interesting to compare the lattice point formulas arising from our construction with those of [3].

5. Remarks

We close with a few remarks about extensions of the methods in this paper.

First, the same result shows that generically, a weight variety of $SU(n)$ is a coadjoint orbit of $SU(n-1)$.

Second, the construction of the torus action on $M_{\alpha,\beta,\gamma}$ required presenting $M_{\alpha,\beta,\gamma}$ as the quotient of the product of two Hamiltonian $U(k)$ spaces by the diagonal action of $U(k)$. In this situation a straightforward extension of the flows discovered by Goldman shows that there exists a densely defined action of the maximal torus of $U(k)$. In this paper, we consider the case where the manifolds are Kahler and the resulting action of $GL(k,\mathbb{C})$ is free$^4$ or at least free up to a finite stabilizer. We are also particularly lucky in that the dimension of one of the spaces is equal to that of $GL(k,\mathbb{C})$, which means the space is locally symplectomorphic to $T^*(U(k))$ (up to some finite quotient). This gives an additional $U(k)$ action, and by the results of Guillemin-Sternberg, a densely defined torus action.

While the condition that the complexified group act freely is essential, the dimension condition is not: If the dimension of the spaces in question is greater than that of the contangent bundle $T^*(U(k))$, the normal form theorem will show that (again, up to finite stabilizers) the neighborhood of a generic $U(k)$ orbit is symplectomorphic to some symplectic bundle over a generic $U(k)$ orbit in $T^*(U(k))$, and everything else will go through.

This is the case for moduli spaces of vector bundles of arbitrary rank on Riemann surfaces, which can be presented as symplectic quotients by diagonal $U(k)$ actions. In the case of rank two bundles, there are enough Goldman flows resulting from this construction to give a densely defined torus action on the moduli space. For higher rank, the additional flows arising from our construction should suffice to give such a torus action.

More specifically, an open dense subset of the moduli space of flat connections on a Riemann surface can be presented as a symplectic quotient of a manifold of the form $M \times T^*U(n) \times T^*U(n)$ (where $M$ is some Hamiltonian $U(n)$ manifold described in detail in [11]) by the diagonal action of $U(n)$, where the action of $U(n)$ on $T^*U(n)$ is the left action. The quotient therefore has an action of $U(n) \times U(n)$ arising from the right action of $U(n)$ on $T^*U(n)$. This gives rise to the desired torus actions (using the moment map for either one of the right $U(n)$ actions). We plan to present the details elsewhere.

Another way of looking at this is that the quotient of the moduli space by the Goldman flows gives a product of quasi-Hamiltonian quotients

$$(C_a \times C_b \times C_c)//SU(n)$$

$^4$Note that freeness of a Hamiltonian action of a compact Lie group $G$ on a Kahler manifold does not necessarily imply that the action of the complexified group $G^\mathbb{C}$ is free; a counterexample is the linear action of $SU(2)$ on $\mathbb{C}^2$. So the assumption of freeness of the complex group action cannot be expressed in terms of the compact group action. This fact, which seems like an annoyingly minor detail, is probably why this torus action lay undiscovered for so many years.
where $a, b, c \in SU(n)$ and $C_f$ denotes the conjugacy class of $f \in SU(n)$. The construction of densely defined torus actions on such quotients can be carried out by using the fact that for generic $a, b, c$, the conjugacy classes are symplectic and the moment map for the quasi-Hamiltonian action can be written in terms of the moment map for the Hamiltonian $SU(n)$ action $[1]$. This should give a proof of the formulas conjectured in $[2]$. We plan to present these details elsewhere, also.

REFERENCES

[1] A. Alexeev, A. Malkin, and E. Meinrenken, Lie group valued moment maps. J. Diff. Geom. 48 445-495 (1998)
[2] Begin, L., Kirillov, A. N., Mathieu, P, Walton, M. A. Berenstein-Zelevinsky triangles, elementary couplings, and fusion rules. Lett. Math. Phys. 28 (1993), no. 4, 257-268.
[3] A. Berenstein, A. Zelevinsky, Tensor product multiplicities and convex polytopes in partition space. J. Geom. Phys. 5 (1988), no. 3, 453-472
[4] Filippini, R. The Symplectic Geometry of the Theorems of Borel-Weil and Peter-Weyl. UC Berkeley Thesis, 1995.
[5] Goldman, William M. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math. 85 (1986), no. 2, 263-302.
[6] V. Guillemin, S. Sternberg. The Gelfand-Cetlin system and quantization of the complex flag manifolds. J. Funct. Anal. 52 (1983), no. 1, 106-128.
[7] V. Guillemin, S. Sternberg. On collective complete integrability according to the method of Thimm. Ergodic Theory Dynam. Systems 3 (1983), no. 2, 219-230.
[8] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics. Cambridge University Press, 1990.
[9] M. Harada, K. Kaveh, Integrable systems, toric degenerations and Okounkov bodies. Invent. Math. 202 (2015), no. 3, 927-985.
[10] J. Hurtubise, L. Jeffrey, S. Rayan, P. Selick, J. Weitsman. Spectral curves for the triple reduced product of coadjoint orbits for $SU(3)$. Preprint arXiv:1708.00752; Proceedings of the Conference for Nigel Hitchin’s 70th Birthday. Oscar Garcia Prada, ed., to appear.
[11] L. Jeffrey, Extended moduli spaces of flat connections on Riemann surfaces. Math. Ann. 298 (1994), no. 4, 667-692.
[12] L. Jeffrey, S. Rayan, G. Seal, P. Selick, J. Weitsman. The triple reduced product and Hamiltonian flows. In Geometric Methods in Physics: XXXV Workshop and Summer School, Bialowieza, Poland. P. Kielanowski, A. Odzijewicz, E. Previato, eds. Springer Verlag, 2018, 35-50
[13] Jeffrey, Lisa C.; Weitsman, Jonathan. Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula. Comm. Math. Phys. 150 (1992), no. 3, 593-630.
[14] Kostant, B. On the Definition of Quantization. In Géométrie Symplectique et Physique Mathématique. Souriau, J. M., ed., CNRS, Paris, 1974.
—Souriau, J. M. Structure des Systèmes Dynamiques. Dunod, Paris, 1970.
—Woodhouse, N. Geometric Quantization. Clarendon Press, Oxford, 1980
[15] A. Knutson, T. Tao, The honeycomb model of $GL_n(C)$ tensor products. I. Proof of the saturation conjecture. J. Amer. Math. Soc. 12 (1999), no. 4, 1055-1090
[16] S. Sternberg, http://www.math.harvard.edu/archive/139spring05/sympgeoml3.pdf

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