Testing de Broglie’s Double Solution in the Mesoscopic Regime

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Abstract
We present here solutions of a non-linear Schrödinger equation in presence of an arbitrary linear external potential. The non-linearity expresses a self-focusing interaction. These solutions are the product of the pilot wave with peaked solitons the velocity of which obeys the guidance equation derived by Louis de Broglie in 1926. The degree of validity of our approximations increases when the size of the soliton decreases and becomes negligible compared to the typical size over which the pilot wave varies. We discuss the possibility to reveal their existence by implementing a humpty-dumpty Stern-Gerlach interferometer in the mesoscopic regime.

Keywords de Broglie–Bohm dynamics · Double solution program · Self-interaction · Gravitation

1 Introduction

The double-solution program was firstly proposed by de Broglie in the twenties. The idea of de Broglie was that the quantum corpuscles would be associated to singularities moving in accordance with the guidance equation. A simple version of this program was presented by de Broglie at the Solvay conference of 1927 [1], which constitutes the backbone of what is now called the pilot wave interpretation [2–4]. Later, in the 50’s, de Broglie came back to these ideas but he supposed by then that the corpuscle is associated to a solitonic solution of a (non-linear generalization of) Schrödinger equation [5, 6]. So far, there were several proposals aimed at accomplishing de Broglie’s program but most often, as a consequence of Ehrenfest’s theorem [7, 8], they can be shown to lead to classical trajectories which obey the guidance equation only in the case of uniform movement (free particles).

Our aim in this paper is threefold:
1. To propose a non-linear equation such that solitons move according to de Broglie’s guidance equation.
2. To consider exact solutions of the full potential such that self-focusing and de Broglie–Bohm trajectories are simultaneously guaranteed, even when the particle is not free (e.g. in presence of a harmonic potential).
3. To propose an experimental test aimed at revealing the existence of such solitonic solutions, based on the idea that they would constitute source terms for the gravitational interaction.

In continuation of a previous paper [8] we consider here modifications of the linear Schrödinger equation due to the hypothetical existence of a non-linear potential expressing the self-interaction of the quantum particle. The modified evolution equation then reads

\[ i\hbar \frac{\partial \Psi(t, \mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi(t, \mathbf{x}) + V^L(t, \mathbf{x})\Psi(t, \mathbf{x}) + V^{NL}(\Psi)\Psi(t, \mathbf{x}), \]  

(1)

where \( V^L \) represents an arbitrary linear potential, of the type commonly considered when solving the linear Schrödinger equation (for instance an electro-magnetic potential) while \( V^{NL} \) represents a non-linear self-focusing potential which supposedly concentrates the wave function of the particle over a tiny region of space, in accordance with de Broglie’s double solution program. We also impose the factorization ansatz:

\[ \Psi(t, \mathbf{x}) = \Psi_L(t, \mathbf{x}) \cdot \phi_{NL}(t, \mathbf{x}), \]  

(2)

where \( \Psi_L \), the so-called “pilot” wave, is a solution of the linear Schrödinger equation:

\[ i\hbar \frac{\partial \Psi_L(t, \mathbf{x})}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V^L(x, y, z, t) \right) \Psi_L(t, \mathbf{x}), \]  

(3)

while \( \phi_{NL}(t, \mathbf{x}) \) is supposed to be localized over a very small region of space. Our aim is to represent the particle by \( \phi_{NL}(t, \mathbf{x}) \), a soliton guided by the pilot wave according to de Broglie’s guidance equation:

\[ \mathbf{v} = \mathbf{v}_{dB-B} = \frac{\hbar}{m} \frac{\text{Im}(\Psi_L^*(t, \mathbf{x}) \nabla \Psi_L(t, \mathbf{x}))}{|\Psi_L(t, \mathbf{x})|^2}, \]  

(4)

where \( \mathbf{v} \) represents the velocity of the (barycentre of) the soliton. de Broglie presented the guidance equation (4) in the Solvay conference of 1927 [1], together with an interpretation which constitutes the so-called de Broglie–Bohm interpretation [2–4], sometimes called the pilot-wave interpretation, a simplified version of de Broglie’s double solution program. According to the pilot-wave interpretation particles are localized in tiny regions of space at any time and follow continuous trajectories in accordance with the guidance equation (4) which expresses how the linear (pilot) wave guides the particles. According to de Broglie’s double solution program [5, 6, 9], a supplementary constraint should be imposed, which is, roughly summarized:
Particles are waves, and are identified with solutions of a modified non-linear Schrödinger equation, here denoted $\Psi(t, x)$ (see equation (2)). These solutions are solitons, that is to say waves for which spread gets compensated by the (self-focusing) non-linearity.

The factorization ansatz (2) is an essential ingredient of our approach [8]. It results from the recognition that, due to the fundamental non-linearity of the wave dynamics, a linear partition of the type $\Psi(t, x) = \Psi_L(t, x) + \phi_{NL}(t, x)$ as was considered by de Broglie [5] is irrelevant. From this point of view the factorization ansatz incorporates non-linearity from the beginning.

As we shall show in section 2, a well-chosen non-linearity makes it possible to derive solutions of equations (1,2,3) whose trajectories obey the guidance equation (4). These are in general approached solutions, and the degree of validity of our approximations is shown to increase when the size of the soliton decreases. When the external potential is harmonic (quadratic) and when the linear wave represents a coherent state, our approach delivers an exact solution however, as we shall show in the same section. In section 3, we show how to generalize our approach to the many particles case. In section 4, we incorporate the gravitational interaction into our model and propose an experiment aimed at revealing the existence of solitonic solutions similar to those described here, thanks to an interferometric humpty-dumpty Stern-Gerlach experiment. We conclude in the last section.

2 Single particle case

2.1 Preliminary Results

Combining equations (1,2,3), expressing $\Psi_L(t, x)$ in function of its modulus and its phase as $R_L(t, x)e^{i\phi_L(t, x)}$, and also making use of the identity $\nabla \Psi_L(t, x) = (\nabla R_L(t, x))e^{i\phi_L(t, x)} + \Psi_L(t, x)i \nabla \phi_L(t, x)$, it is straightforward to show that $\phi_{NL}$ obeys the non-linear equation

$$i\hbar \cdot \frac{\partial \phi_{NL}(t, x)}{\partial t} = -\frac{\hbar^2}{2m} \cdot \Delta \phi_{NL}(t, x) - \frac{\hbar^2}{m} \cdot (i \nabla \phi_L(t, x) \cdot \nabla \phi_{NL}(t, x))$$

$$+ \frac{\nabla R_L(t, x)}{R_L(t, x)} \cdot \nabla \phi_{NL}(t, x)) + V_{NL}(\Psi)\phi_{NL}(t, x).$$

By doing so we replace thus equation (1) by a system of three equations (2,3,5). This replacement is one to one and can be done without loss of generality whenever $x$ is not a node of the pilot-wave $\Psi_L(t, x)$ which is the case “nearly everywhere”.

In order to solve the system of equations (3,5), it is worth noting that while the $L_2$ norm of the linear wave $\Psi_L$ is preserved throughout time, because (3) is unitary, this is
no longer true in the case of the non-linear wave $\phi_{NL}$, because the terms mixing $\Psi_L$ and $\phi_{NL}$ in (5) are not hermitian. The change of norm of $\phi_{NL}$ can be shown [8], to obey
\[
\frac{d}{dt} <\phi_{NL}|\phi_{NL}> = \frac{\hbar}{m} \Delta \varphi_L(t, x_0) <\phi_{NL}|\phi_{NL}> - 2 \frac{\nabla R_L(t, x_0)}{R_L(t, x_0)} \cdot \int d^3x (\phi_{NL}(t, x)) \cdot \frac{\hbar}{m} \nabla \varphi_L(t, x),
\]
where we introduced the barycentre $x_0$ of the soliton: $x_0 \equiv <\phi_{NL}|x|\phi_{NL}>$. Remark that the bra-ket notation introduced here should not necessarily be interpreted as a quantum statistical average in the usual sense; it rather indicates an average quantity in regard of the weight (density of stuff) $|\phi_{NL}(t, x)|^2$.

Still in reference [8], we established the following results (i,ii):

i) If we define the velocity $v_{\text{drift}}$ of the barycentre $x_0$ as follows:
\[
v_{\text{drift}} \equiv \frac{d}{dt} \left( \frac{<\phi_{NL}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>} \right)
\]
Then,
\[
v_{\text{drift}} = \frac{\hbar}{m} \nabla \varphi_L(t, x_0(t)) + \frac{<\phi_{NL}|\frac{\hbar}{m} \nabla |\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}
\]
which contains the well-known Madelung-de Broglie–Bohm contribution ($v_{dB-B} = \frac{\hbar}{m} \nabla \varphi_L(t, x_0(t))$) plus a new contribution due to the internal structure of the soliton ($v_{\text{int.}} = \frac{<\phi_{NL}|\frac{\hbar}{m} \nabla |\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$).

ii) In the limit where the soliton is peaked enough around its barycentre:
\[
\frac{<\phi_{NL}|\phi_{NL}> (t)}{<\phi_{NL}|\phi_{NL}> (t = 0)} = \frac{R^2_L(x_0, t = 0)}{R^2_L(t, x_0)},
\]
where $x_0(t)$, the barycentre of $\phi_{NL}$, moves according to the generalized dB-B guidance equation (8).

The first result (i) strongly suggests the possible existence of a purely real solitonic solution of equation (5) such that $v_{\text{int.}} = 0$ in which case the guidance equation of de Broglie (4) is satisfied: $v_{\text{drift}} = \frac{\hbar}{m} \nabla \varphi_L(t, x_0(t))$. Having in mind that all aforementioned results were derived in the limit where the width of the peaked soliton is quite smaller than the typical scales of variation of $R_L(t, x)$ and $\varphi_L(t, x)$ over space, the second result (ii) implies that the full wave function $\Psi$ solution of (5) has the form
\[ \Psi(x, y, z, t) \approx \phi'_{NL}(t, x)e^{i\phi_L(t, x)}, \]  

where \( \phi'_{NL}(t, x) \) is centered in \( x_0(t = 0) + \int_0^t dv_{\text{drift}} \) and is of constant \( L_2 \) norm.

### 2.2 A Formal Realization of de Broglie’s Double Solution Program

In order to represent the particle by \( \phi_{NL}(t, x) \), a soliton guided by the pilot wave according to de Broglie’s guidance equation (4), let us make the following choice\(^1\) for \( V^{NL} \):

\[ V^{NL}(\Psi) = \frac{\hbar^2}{2m} \Delta |\Psi(t, x)| - \frac{\hbar^2}{2m} |\Psi_L(t, x)| \]  

As has been shown by Bohm [2–4], making use of the linear Schrödinger equation (3), when the guidance condition (4) is fulfilled, the acceleration of the barycentre of the quantum particle obeys Newton’s equation:

\[ m\mathbf{a}(t, x_0) = m\frac{d}{dt}\mathbf{v}^{ab}(t, x_0) = -\nabla V_L^O(t, x_0) - \nabla V_L^O((t, x_0)) \]  

where \( V_L^O((t, x_0)) \) represents the (non-linear) quantum potential:

\[ V_L^O(\Psi_L) = -\frac{\hbar^2}{2m} \Delta |\Psi_L(t, x)| \]  

Here, we introduce a non-linear potential which is the difference between \( V_L^O(\Psi_L) \) and the “usual” quantum potential, associated to the pilot wave, with a quantum potential associated to the full wave function \( V^O(\Psi) = -\frac{\hbar^2}{2m} \Delta|\Psi(t, x)| \).

Making use of the factorisation ansatz (2) we can rewrite \( V^{NL}(\Psi) \) as follows:

\[ V^{NL}(\Psi) = V_L^O(\Psi_L) - V^O(\Psi) \]

\[ = \frac{\hbar^2}{m} \nabla |\Psi_L(t, x)| \cdot \frac{\nabla |\phi_{NL}(t, x)|}{|\phi_{NL}(t, x)|} + \frac{\hbar^2}{2m} |\phi_{NL}(t, x)|. \]  

The potential \( \frac{\hbar^2}{2m} \frac{\Delta|\phi_{NL}(t, x)|}{|\phi_{NL}(t, x)|} \) plays the role of a self-focusing potential but is non-accelerating as we shall show soon. The role of the potential \( \frac{\hbar^2}{m} \frac{\nabla |\Psi_L(t, x)|}{|\Psi_L(t, x)|} \cdot \frac{\nabla |\phi_{NL}(t, x)|}{|\phi_{NL}(t, x)|} \) is more subtle: in combination with the non-hermitian terms mixing \( \Psi_L \) and \( \phi_{NL} \) in equation (5), it contributes to the de Broglie–Bohm self-acceleration. Indeed, combining equations (5) and (13), and making use of the fact that \( |\Psi_L(t, x)| = R_L(t, x) \), we get

\(^1\) It is worth noting that, although the potential chosen here presents some analogy with previous proposals aimed at fulfilling de Broglie’s double solution program [10, 11], it differs from them by the fact that it is undissociable from the factorisation ansatz (2). This ansatz has been studied in the past, in the framework of the linear Schrödinger equation actually, by Barut [12] and Bindel [13] for instance, but, to the knowledge of the author, the system of coupled equations (1,2,11) has never been considered in the past.
\[ i\hbar \cdot \frac{\partial \phi_{NL}(t, \mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m} \left( \Delta \phi_{NL}(t, \mathbf{x}) - \Delta |\phi_{NL}(t, \mathbf{x})| \cdot \frac{\phi_{NL}(t, \mathbf{x})}{|\phi_{NL}(t, \mathbf{x})|} \right) \]

This equation grandly simplifies whenever \( \phi_{NL}(t, \mathbf{x}) \) is a real positive function, in which case equation (14) reads

\[ i\hbar \cdot \frac{\partial \phi_{NL}(t, \mathbf{x})}{\partial t} = -\frac{\hbar^2}{m} \cdot i \nabla \phi_L(t, \mathbf{x}) \cdot \nabla \phi_{NL}(t, \mathbf{x}). \]  

(15)

If the size of the soliton is quite smaller than the typical size of variation of \( \nabla \phi_L(t, \mathbf{x}) \), that is to say, if it is quite smaller than \( \|\nabla \phi_L(t, \mathbf{x})\| \), we may replace, in good approximation, \( \nabla \phi_L(t, \mathbf{x}) \) by \( \nabla \phi_L(t, \mathbf{x}_0) \). Then, it is easy to solve equation (15) for which we find a solitary wave solution of the type

\[ \phi_{NL}(t, \mathbf{x}) = \phi_{NL}(\mathbf{x} - \mathbf{x}_0(t)), \]  

(16)

with \( \mathbf{x}_0(t) = \mathbf{x}_0(t_0) + \int_{t_0}^t \mathbf{v}_{dB-B}(\mathbf{x}_0(t)) \, dt \). Remarkably, this solution is valid independently of the initial shape of \( \phi_{NL} \) at time \( t_0 \), provided it is a real positive function of the position.

It moves without deformation at all times and remains thus a real positive function, moving at the velocity \( \mathbf{v}_{dB-B} \), in accordance with equation (8) because when \( \phi_{NL} \) is real, \( \mathbf{v}_{int} = \frac{<\phi_{NL}|\nabla \phi_{NL}>}{<\phi_{NL}|\phi_{NL}>} = 0. \)

2.3 Guidance Equation Associated to the Coherent State of an Harmonic Oscillator: An Exact Solution

Let us consider in particular that the linear potential is harmonic and that the pilot wave is a coherent state:

\[ \Psi_L(t, \mathbf{x}) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp \left[ -\left( \frac{m\omega}{2\hbar} \right) (\mathbf{x} - \mathbf{x}_0 \cos(\omega t))^2 \right. \]

\[ \left. + i(\mathbf{x}_0 \cdot \mathbf{x}) \sin(\omega t) \right] + i\theta(t) \]  

(17)

Here, each component of \( \mathbf{x}_0 \) represents the amplitude of the oscillation of the barycentre of the coherent state around the origin, along a Cartesian axis; at time \( t \) the location of this barycentre is equal to \( \mathbf{x}_0 \cos(\omega t) \) and its velocity is equal to \( -\omega \mathbf{x}_0 \sin(\omega t) \).
Then, equation (14) is gaussian which means that if we impose that $\phi_{NL}$ is
gaussian at time $t = t_0$ it will remain gaussian for all times which grandly simplifies
the computations.

Accordingly, let us try the following expression for $\phi_{NL}$:

$$
\phi_{NL}(t, x) = e^{-\left[A_x(t) \frac{x^2}{2} + B_x(t) x + C_x(t) - A_z(t) \frac{z^2}{2} + B_z(t) z + C_z(t)\right]}
$$

where $A(t)$, $B(t)$
and $C(t)$ are complex functions of time.

Without losing generality, the purely solitonic part of the non-linear potential
(that is to say $\frac{\hbar^2}{2m} \Delta |\phi_{NL}(t, x)|$) can be expressed in terms of the barycentre of the
soliton through

$$
\frac{\hbar^2}{2m} \Delta |\phi_{NL}(t, x)| = \frac{\hbar^2}{2m} ((Re A_x(t))^2(x - x_0)^2 + (Re A_y(t))^2(y - y_0)^2 + (Re A_z(t))^2(z - z_0)^2),
$$

up to an irrelevant additive constant, so that equation (5) becomes

$$
i\hbar \frac{\partial \phi_{NL}(t, x)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi_{NL}(t, x) - \frac{\hbar^2}{m} \nabla \phi_{NL}(t, x) \cdot \nabla \phi_{NL}(t, x)
+ \frac{\hbar^2}{2m} ((Re A_x(t))^2(x - x_0)^2 + (Re A_y(t))^2(y - y_0)^2
+ (Re A_z(t))^2(z - z_0)^2)\phi_{NL}(t, x),
$$

which puts into evidence the self-focusing nature of the purely solitonic part of the
non-linear potential. This potential is not self-accelerating as can be shown from a
straightforward application of Ehrenfest’s theorem (see e.g. Ref. [54] for a similar
result derived in the framework of the logarithmic non-linear Schrödinger equation:
due to the parity of the gaussian function under reflexions around its barycentre, the
global contribution of the non-linear potential (here of $\frac{\hbar^2}{2m} \Delta |\phi_{NL}(t, x)|$) to the self-acceleration
vanishes).

The equation (19) is also separable in cartesian coordinates because the pilot
wave is separable. Henceforth, to simplify the treatment, we shall from now on
consider the evolution of the $x$ component only and again force a gaussian solution
of the type

$$
\phi_{NL}(t, x) = \exp \left[-A_x(t) \frac{x^2}{2} + B_x(t) x + C_x(t)\right]
$$

The self-interaction potential along $X$, $\frac{\hbar^2(Re A_x(t))^2}{2m}(x - x_0)^2$, is peaked around the barycentre $x_0$ of the soliton and is equal to its Taylor development to the second order in
$x$, of the form $V_0(t) + V_1(t) x + V_2(t) x^2$.

When the pilot wave is a coherent state, $\nabla_x \phi_L(t, x)$ is exactly equal to $\nabla_x \phi_{NL}(t, x_0)$ because the phase of the pilot wave linearly depends on the position.

We get thus after straightforward computations the following system of
equations:
\[
\begin{aligned}
\begin{cases}
  i \frac{dA_x(t)}{dt} = \frac{\hbar}{m} A_x(t)^2 - 2 \frac{V_0(t)}{\hbar} = \frac{\hbar}{m} A_x(t)^2 - 2 \frac{\hbar \text{Re}(A_x(t))}{2m} \\
  i \frac{dB_y(t)}{dt} = \frac{\hbar}{m} A_x(t) B_y(t) + \frac{V_0(t)}{\hbar} + i \frac{\hbar}{m} A_x(t) \nabla \varphi_L(t, x_0) \\
  i \frac{dB_z(t)}{dt} = \frac{\hbar}{2m} [A_x(t) - B_x(t)^2] + \frac{V_0(t)}{\hbar} - i \frac{\hbar}{m} B_x(t) \nabla \varphi_L(t, x_0).
\end{cases}
\end{aligned}
\]  

(21)

It admits solitonic solutions (similar to coherent states) for which \(A_x = A_0\) (the same condition holds also for other cartesian axes of reference: \(A_y = A_z = A_0\)). Then the evolution of \(B_x(t)\) considerably simplifies and we get

\[
i \frac{dB_x(t)}{dt} = \frac{\hbar}{m} A_0 B_x(t) + \frac{V_1(t)}{\hbar} + i \frac{\hbar}{m} A_0 \nabla \varphi_L(t, x_0).
\]  

(22)

Now, \(x_0 = \frac{\text{Re}B_y}{\text{Re}A_0} = \frac{\text{Re}B_z}{A_0}\) and \(V_1(t) = -\frac{\hbar^2}{m} A_0 \text{Re}B_x\). Making use of equation (22) we also have \(-\frac{d\text{Im}B_y}{dt} = \frac{\hbar}{m} A_0 \text{Re}B_x(t) + \frac{V_0(t)}{\hbar} = 0\) so that at all times the soliton remains real \((\text{Im}B_x(t) = 0)\), provided the initial condition is a purely real gaussian function (all this up to an irrelevant global phase).

We also get \(\frac{d\text{Re}B_x}{dt} = \frac{\hbar}{m} A_0 \nabla \varphi_L(t, x_0) = -\omega \tilde{x}_0 \sin(\omega t)\).

Finally, making use of the previous results, we get

\[
\frac{d\text{Re}C_z(t)}{dt} = -\frac{\hbar}{m} B_x(t) \nabla \varphi_L(t, x_0) = -A_0 \cdot x_0 \cdot \frac{dx_0}{dt} = -A_0 \frac{dx_0^2}{2dt}
\]

so that \(\text{Re}(-A_x(t) \frac{dx_0^2}{2} + B_x(t)x + C_z(t)) = -A_0(x - x_0)^2\).

For solitons prepared at time \(t = 0\) in the state \(\Psi(t = 0, x) = \Psi_L(t = 0, x) \cdot N \cdot e^{-\frac{\hbar}{2m}(x-x_0)^2}\), we predict thus that at time \(t\), the state evolves to \(\Psi(t, x) = \Psi_L(t, x) \cdot N \cdot e^{-\frac{\hbar}{2m}(x-x_0(t))^2}\) with \(\frac{dx_0}{dt} = \frac{\hbar}{m} \nabla \varphi_L(t, x_0)\). The guidance condition (4) is thus satisfied. Actually the barycentres of all solitons follow trajectories of the type \(x_0(t) = x_0(t = 0) - \tilde{x}_0 + \tilde{x}_0 \cos(\omega t)\). They remain thus equidistant at all times with the peak of the pilot wave. This is due to the fact that the envelope of the pilot wave (a coherent state) moves without deformation. de Broglie–Bohm trajectories thus conspire in order to transport the initial position density as a whole. Remark that if initially the shape of the soliton is not gaussian it will also move without deformation and remain equidistant at all times from the peak of the pilot wave, because the solution (16) is an exact solution when the pilot wave is a coherent state.

Another situation for which the solution (16) is exact occurs when the linear potential vanishes; then plane pilot waves solutions of the linear Schrödinger equation (3) are associated to a uniform movement of the soliton, which is a

\footnote{Note that, to simplify the mathematical treatment we assumed here to begin with that the three cartesian components of the barycentre of the pilot wave oscillate in phase. Now, as the equation (19) is separable in cartesian coordinates, our results are still valid if we relax this assumption.}
manifestation in this case of Galilei invariance [8, 14]. Other non-linear modifications of the Schrödinger equation such as e.g. the logarithmic non-linear Schrödinger equation [54], the NLS equation [15] or the Schrödinger-Newton equation [7, 16] also possess exact solutions in the form of the product of a plane wave modulation with a soliton moving at constant speed; such potentials can be shown however to be deprived of self-acceleration [7, 8, 54] which prohibits their use for attempting to realize de Broglie’s double solution program.

3 The Many Particles Case: Guidance Equation in Configuration Space

When more than one particle is present, a formal generalization of the previous results holds as we show here. To do so, we consider a wave function of the type

\[
\Psi(t, x^1, x^2, \ldots x^i \ldots x^N) = \Psi_L(t, x^1, x^2, \ldots x^i \ldots x^N) \cdot \phi_{NL}(t, x^1) \cdot \phi_{NL}(t, x^2) \ldots \phi_{NL}(t, x^i) \ldots \phi_{NL}(t, x^N),
\]

where the linear wave is properly symmetrized in the case of identical particles, and contains actually all the physical features (such as e.g. entanglement) associated to the usual, linear quantum physics, which has been succesfully confirmed in a multitude of experiments in the last century. The non-linear part is not entangled however and does not respect in general symmetrisation (anti-symmetrisation) properties required in presence of bosons (fermions). Even bosonic symmetry is systematically broken because particles are supposedly located in tiny regions of space separated by distances quite larger than their extent. We also assume that the wave function obeys a generalization of equation (1):

\[
i \hbar \frac{\partial \Psi(t, x^1, x^2, \ldots, x^i, \ldots, x_N)}{\partial t} = -\sum_{i=1}^{N} \frac{\hbar^2 \Delta \Psi(t, x^1, x^2, \ldots, x^i, \ldots, x_N)}{2m_i} + V^L(t, x)\Psi(t, x^1, x^2, \ldots, x^i, \ldots, x_N) + V^{NL}(\Psi)\Psi(t, x^1, x^2, \ldots, x^i, \ldots, x_N),
\]

with

\[
V^{NL}(\Psi) = \sum_i \frac{\hbar^2}{2m_i} \frac{\Delta |\Psi(t, x^i)|}{|\Psi(t, x)|} - \frac{\hbar^2}{2m_i} \frac{\Delta |\Psi_L(t, x^i)|}{|\Psi_L(t, x)|},
\]

where we sum over elementary particles (typically electrons and nucleons), symbolically differentiated here via the label \(i\). Then all the results of the previous section still apply in the present case. For instance, privileging gausson type solutions, we find, in first approximation, a formal solution of equation (24) of the type
\[ \Psi(t, x^1, x^2, \ldots x^i, \ldots x^N) \]
\[ \approx N', \frac{R_L(x^1, x^2, \ldots x^i, \ldots x^N, t)}{R_L(x^1_0, x^2_0, \ldots x^i_0, \ldots x^N_0, t)} e^{i\phi(t,x^1, x^2, \ldots x^i, \ldots x^N, t)} e^{-\frac{i}{2} \sum_{i=N+1}^N (x^i_0 - x^i(t))^2} \]  \( (26) \)

with \( \frac{dx}{dt} = \frac{\hbar}{m} \nabla_i \phi_L(x^1, x^2, \ldots x^i, \ldots x^N, t) \) and \( N' = R_L(x^1_0, x^2_0, \ldots x^i_0, \ldots x^N_0, t = 0) \cdot N \), making use of equation (9). The guidance equation in configuration space originally derived by de Broglie in 1926 [17] is thus fulfilled, generalizing the results of the previous section to the many particles case.

At this level, the solution (26) remains a somewhat formal solution, among others because the sizes of the solitons \( \frac{1}{\sqrt{A_0(i)}} \) are left undetermined, excepted that we implicitly assume that they are smaller than the typical size of variation of the linear (pilot) wave function. We expect however that these sizes have something to do with the masses of the elementary particles. For instance, if we accept that \( 3\hbar \omega_0(i)/2 \), the energy of the soliton estimated on the basis of equation (19), with \( A_x = A_y = A_z = A_0(i) \), is of the order of \( m_i c^2 \), the rest-mass/energy of the particle, we find that the size of the soliton (which is of the order of \( 1/\sqrt{A_0(i)} \)) can be shown to be of the order of the Compton wavelength \( (\hbar/(mc)) \) of the associated particle, making use of the relations \( \omega_0(i) = \sqrt{k_0(i)/m_i}, \ h^2A^2_0(i)/2m_i = k_0(i)/2, \) and \( 3\hbar \omega_0(i)/2 = (3h^2/2m)A_0(i) \). Even if it is a bit surprising to introduce the relativistic relation \( E = mc^2 \) in the present discussion, we believe that, in the case where our model makes sense, the self-focusing and self-accelerating mechanisms invoked here ought to find a justification, ultimately, in the framework of relativistic quantum field theory. Although this is out of the scope of the present paper, it would be very interesting and challenging to try for instance to connect the non-linear potential (13) to fundamental non-linearities characterizing the Lagrangians of gauge theories [18]. It could be the case after all that the self-interaction considered here has something to do with the renormalisation procedure the scope of which being precisely to eliminate self-interaction and to focus on interactions between different particles. This is in a sense what we did here by factorizing the wave function into a “linear” wave function \( \Psi_L \) containing the usual physics (and interactions between different particles) and a “non-linear” wave function \( \phi_{NL} \) associated to self-interaction.

In this perspective, the gravitational interaction, which is not renormalisable, as is well-known, could play a different role, compared to gauge interactions, as we will discuss now...

4 The Role of Gravitation: Experimental Proposal

4.1 The Role of Gravitation

If we accept the possible existence of a non-linearity of the type considered in the present paper, we are forced to consider seriously the logical possibility of the picture according to which particles are identified with solitons of extremely small size.
following the lines of flow associated to the pilot wave, in accordance with de Broglie’s guidance equation. In virtue of the H-theorem established by Valentini and coworkers in the framework of the pilot wave interpretation [19–29], the distribution of positions of these solitons converges in time to the Born distribution in $|\Psi_L|^2$. This important technical result holds that we interpret in the present context $|\Psi|^2$ as a probability distribution (which is the standard, Copenhagen interpretation) or not. This is so because we are free to normalise $\Psi$ at our convenience, which will affect neither the value of its barycentre nor the value of the non-linear potential which does not depend on rescalings of $\Psi, \Psi_L$ or $\phi_{NL}$ by construction. The quantum H-theorem [19] implies, at this level, that our model provides an ad hoc reformulation of the standard interpretation provided we treat the position as a beable (or element of reality). Now, it is worth noting that, in accordance with the spirit of the pilot wave interpretation, there is in our model no influence at all of the soliton on the pilot wave (no feedback), which implies that a priori there is no way to put into evidence the reality of the solitons. All this is not new, it was already postulated in the de Broglie–Bohm interpretation that the particle is passively guided by the pilot wave according to the guidance condition. From this point of view the de Broglie–Bohm ontology, and the double solution program as well, merely deliver an ad hoc reformulation of the standard theory.

It is our hope however that it is possible to discriminate the standard interpretation from the double solution approach in the presence of gravitational fields. Nothing forbids indeed to assume that the source of the gravitational potential is the density of stuff $|\Psi|^2$ and to treat gravity classically [30]. This is even a natural condition in the framework of semi-classical gravity [16] and in certain extensions of spontaneous localisation theories where the source of gravity is the stuff at the time of spontaneous collapse (when the ”flash” occurs) [31]. In the framework of “conventional” semi-classical gravity, a similar intuition is actually at the origin of the Schrödinger-Newton equation [16, 32–37]). This means that to the difference of say electro-magnetic interactions for which standard, linear predictions have been tested with very high accuracy in the framework of QED, we shall assume here that gravity provides a feedback from the non-linear to the linear sector. From that point of view the gravitation would be “other”. In particular it would differ from other fundamental (gauge) interactions which a priori admit a satisfactory description in the framework of linear quantum mechanics. In this perspective, linearity is, roughly speaking, a corollary of renormalizability.

### 4.2 Experimental Proposal

The presence of a feedback from the “particle” (soliton) on the pilot wave makes it possible to discriminate between, at one side, the standard interpretation (where

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3 The main difference between the present approach and “conventional” semi-classical gravity à la Schrödinger-Newton is that here the source term is supposed to be proportional to $|\Psi|^2$ and not to $|\Psi_L|^2$ as is the case in semi-classical gravity. Our approach however is reminiscent of Penrose’s ideas [16, 35, 36] according to which a fundamental non-linearity explains the collapse of the wave function, excepted that here the non-linearity is not of gravitational origin, and that the collapse of each elementary particle is permanent and occurs over regions of size of the order of the Compton wavelength.
the description of the quantum system is encapsulated in the linear wave function \( \Psi_L \) solely) and, at the other side, the approach followed here. To show this, let us consider a recent proposal aimed at testing the nature of the gravitational interaction at the level of elementary quantum systems. Roughly summarized, the idea [38, 39] is to prepare two mesoscopic systems \( A \) and \( B \) (for instance, diamond nanospheres (beads) with a spin 1/2 NV-center inside) in a pure factorisable spin state \( | \Psi(t = 0) \rangle = (\alpha_A | +_A > + \beta_A | -_A >)(\alpha_B | +_B > + \beta_B | -_B >) \) and to let each of them fall along a humpty-dumpty [40] Stern-Gerlach device, in such a way that during a long time \( \tau \) the two spheres remain parallel to each other. Each of the 4 spin states will thus accumulate a phase, due to the gravitational interaction between the two systems, so that, after recombining the wave packets at the end of the (double) Stern-Gerlach device, the state \( | \Psi(\tau) > \) of the full system is equal to

\[
\begin{align*}
\alpha_A \alpha_B e^{i\theta_{++}} | +_A +_B > + \alpha_A \beta_B e^{i\theta_{+-}} | +_A -_B > \\
+ \beta_A \alpha_B e^{i\theta_{-+}} | -_A +_B > + \alpha_A \beta_B e^{i\theta_{--}} | -_A -_B >.
\end{align*}
\]

Performing tomography of this state delivers information about \( \theta \) and thus about the nature of the gravitational interaction between the two systems. Let us denote \( d_{ij} \) (with \( i,j \in \{+, -\} \)) the distance between the vertical axes along which the \( i \) spin component of the \( A \) system and the \( j \) component of the \( B \) systems parallely move. The standard, linear approach [38, 39], predicts that

\[
\theta_{i,j}^{standard} = \tau \frac{Gm_A m_B}{\hbar} \frac{1}{d_{ij}}. 
\]  

In order to modelize the self-interaction of the soliton, let us add to the non-linearity (13) a non-linear coupling à la Schrödinger-Newton [16]: the self-gravitational potential is thus equal to \( V_{self-grav.}(x^i) = -\int d^3x^{ij} | \Psi(t, x^{ij}) |^2 \frac{Gm^2}{|x^{ij}|} \), where \( x^i \) represents the position of the \( i \)-particle (here the L2-norm of \( |\Psi(t, x^i)\rangle \) is normalised to 1). If the shape of the soliton is a Heaviside isotropic function of radius \( R \), we find by a straightforward computation that \( V_{self-grav.}(x^i) = \frac{Gm^2}{R^3} \left( -\frac{3}{2} + \frac{1}{2} \left( \frac{d}{R_i} \right)^2 \right) \) if \( d \leq R_i \) where \( d \) denotes the distance to the center of the soliton: \( d = | x^i - x_0^i | \) and \( R_i \) is the Compton wavelength of the \( i \) particle \( R_i = \hbar / (m_i \cdot c) \). Otherwise, for larger distances \( d \) larger than the radius \( R_i \), one can integrate the internal contributions using Gauss’s theorem so that: \( V_{eff}(d) = -\frac{Gm^2}{d} \) \( (d \geq R_i) \).

This is actually a generic behaviour. Similar predictions can be made if for instance we assume that the shape of the soliton is not a Heaviside but a gaussian function (gausson [54]): up to a constant of the order of \( -\frac{Gm^2}{R} \) which can be considered as a correction to the rest mass energy of the particle [44], the gravitational potential is an harmonic self-focusing, potential at short scales and outside from the tiny zone where the “stuff” is located we find the usual Newton potential making use of Gauss’s theorem. In general, the potential is anharmonic in-between but this does not really matter because the gravitational interaction can be considered as a small perturbation compared to the non-linear potential (13). Generically the spring constant of the harmonic self-gravitational potential, in the vicinity of the centre, is of the order of \( G \cdot \rho_i \) where \( \rho_i \) is the density of stuff evaluated at \( x_0^i \), the center of the soliton.
In the present case, \( \rho_i \) is of the order of the mass of the particle divided by the cube of its Compton wavelength: 
\[
\rho_i \approx m_i / (\hbar / m_i \cdot c)^3 = m_i^4 c^3 / \hbar^3
\]
and one can check that if we consider a gaussian representing an electron or a nucleon, the self-gravitational spring constant is quite smaller than the spring constant associated to the self-focusing potential
\[
\frac{\hbar^2 \Delta |\psi_{NL}|}{2m_i |\psi_{NL}|} \quad (19),
\]
because their ratio is equal to \( G \cdot m_i^2 / \hbar \cdot c \) which is extremely small when the particle is an electron or a nucleon. We find the same ratio between the gravitational self-energy (which is of the order of \(-Gm_i^2 / R_i = -Gm_i^2 / (\hbar / m_i c)\)) and the self-energy \((\hbar^2 / (m_i R_i^2)) = m_i c^2\) associated to the self-focusing potential \(\frac{\hbar^2 \Delta |\psi_{NL}|}{2m_i |\psi_{NL}|}\). Making use of Ehrenfest’s theorem, it is easy to show that the self-gravitational potential is a non-accelerating potential [8], because each cartesian component of the global force
\[
\int d^3 x_i \int d^3 x'_i \ | \Psi(t, x_i) \rangle \langle \Psi(t, x'_i) | \langle \Psi(t, x'_i) | \Psi(t, x_i) \rangle^2 Gm^2 \frac{1}{|x_i - x'_i|} \text{ nullifies. This is so because it is the integral of an odd function over an even domain.}
\]
If we consider its effect on the solitons, the self-gravitational interaction is thus at short distance and at the level of elementary particles/solitons a small and non-accelerating potential [8] which will reinforce the self-focusing character of the nonlinear potential (13). As it is non accelerating, its presence does not modify our previous analysis; in particular the guidance equation (4) is still satisfied and the linear Schrödinger equation (3) remains unaltered at this level (no feedback from the nonlinear to the linear sector). However, if we consider regions of the pilot wave where the soliton is not located, and separated of the center of the soliton by a distance larger than its size, Gauss’s theorem applies and we expect the soliton to interact with the pilot wave via an effective Newtonian potential. It is at this level that the feedback from the non-linear to the linear sector appears. In this approach, we also predict that the mass-energy of the elementary particles of the A and B systems is concentrated along the axes followed by the \( k, l \) components, with a probability equal to \( |\Psi_{k,l}|^2\), according to the Born rule, in virtue of the aforementioned quantum H-theorem. In this case everything happens as if the wave function associated to the center of mass of the \( i, j \) component was moving inside an external gravitational potential created by the mass concentrated along the \( k, l \) trajectory. This results in the appearance of a self-gravitational potential equal\(^4\) to \(-Gm_i^2 A_{3/2R_i} (-Gm_j^2 B_{3/2R_j})\) along the trajectory of the \( k \) \((l)\) spin-component, and of the usual Newtonian potential along trajectories of spin components where the mass is not concentrated \((i \neq k \text{ and } l \neq j)\). Note that there also appears here a gravitational potential between the spin up (down) component and the spin down (up) component of a same nanosphere, to the difference with the standard approach. Henceforth, we predict that with a probability equal to \( |\Psi_{k,l}|^2\) (for \( k, l \in \{+,-\} \)), the state is a pure state such that

\[\text{In first approximation, the self-gravitational potential can be estimated making use of } V_{\text{eff}}(d) = -Gm_i^2 \frac{1}{2R_i} (\frac{2R_i - d^2}{2R_i}), \text{ which expresses the gravitational potential inside a homogeneous nanosphere of radius } R.\]
which obviously differs from the standard prediction \((27)\). Moreover, the resulting state is a mixture while in the standard approach we get a pure state. Tomography of the final state would thus make it possible to falsify\(^5\) one of the two models, if not both because other alternative approaches exist such as the “conventional” semi-classical approach [37] where the source term is the pilot-wave density and not the solitonic density or or such as the aforementioned model [31] based on the flash ontology.

5 Scrutinizing the Consistency of Our Approach

It is often claimed that semiclassical theories of gravity are inconsistent. In particular, the Schrödinger-Newton approach gets regularly anathemized for several reasons: it is a deterministic equation\(^6\) and moreover it can be shown [31] to violate the no signaling condition, as a special case of a general no-go theorem linking faster than light communication and non-linear Schrödinger equations (see e.g. Ref. [14] and references therein, in particular Refs. [41, 42]). Without going in all details, we show below that this no-go theorem can be circumvented in our approach. Another criticism involving gravitation has been formulated relatively to no collapse theories [43] which also applies, at first sight, to the de Broglie–Bohm interpretation. The idea is that when there is no collapse, empty waves act as a source term for gravitation, which leads to experimental predictions incompatible with observations, as has been shown in Ref. [43]. The dBB interpretation is a no collapse interpretation, which should be ruled out by the aforementioned experiment, but as we shall show an effective collapse is present in the dBB approach which nullifies the relevance of this “no-go experiment” [43].

\[
\theta_{i,j}^{\text{soliton}} = \frac{\tau}{\hbar} G \left( \delta_{k,i} \frac{3m_i^2}{2R_A} + (1 - \delta_{k,i}) \frac{m_i^2}{d_{i,j}} + \delta_{i,j} \frac{3m_j^2}{2R_B} + (1 - \delta_{i,j}) \frac{m_j^2}{d_{j,i}} \right) + m_A m_B \left( \frac{1}{d_{j,i}} - \delta_{k,i} \delta_{i,j} \frac{1}{d_{k,j}} \right) \tag{28}
\]

\(^5\) Actually, a single humpty dumpty device suffices to falsify our model, because the standard theory predicts that no dephasing appears between the two branches of the humpty dumpty device when only one nanosphere is present. This is not the case with our approach, which predicts a dephasing equal to \(\pm \frac{G \tau}{\hbar} \left( \frac{3}{d^2} - \frac{1}{d} \right)\) with \(d\) the distance between the spin up and spin down paths and \(R\) the radius of the nanosphere. The value of the sign of the dephasing depends on where the (mass/energy of the) nanosphere is located. The localisation of the mass/energy of the nanosphere obeys in turn the Born rule, in virtue of the quantum H-theorem.

\(^6\) For what concerns the criticism according to which a deterministic equation cannot mimic the indeterministic nature of the measurement process, it is nullified by Valentini’s H-theorem. Deterministic chaos leads to indeterminism as is well-known [27–29].
5.1 Circumventing Gisin’s No-go Theorem

In Ref. [14] one can read the following: ...Roughly summarized, Gisin’s argument [41] goes as follows: nonlinear corrections to the linear Schrödinger equation make it possible, in principle, to distinguish different realizations of the same density matrix (...). By performing a local measurement on a system A that is entangled with a distant system B, one is able, by collapsing the full wave function, to obtain realizations of the reduced density matrix of the system B which differ according to the choice of the measurement basis made in the region A. Therefore, in principle, nonlinearity can be a tool for sending classical information faster than light, contradicting the no-signaling property valid in the framework of linear quantum mechanics [41]. ...

... Gisin’s argument relies on a nonlinear modification of the Schrödinger equation due to Weinberg (...). Now, it could be that there exist specific nonlinear modifications of the Schrödinger equation that do not belong to the class considered by Gisin, but no such modification is known. ...

When we wrote ref. [14], we ignored equation (24) which has been shown to predict in fine that the positions of quantum systems obey the Born rule, in virtue of the H-theorem established by Valentini and coworkers in the framework of the pilot wave interpretation [19] as we discussed in a previous section. Now, once quantum equilibrium has been achieved, that is to say, once the Born rule is obeyed, no signaling is guaranteed. This is made explicit in the following quotation (still from Ref. [14]):

... Despite the fact that dB-B theory is a no-collapse theory (...)– it is well-known that it does respect the no-signaling requirement, provided equilibrium has been achieved. Once again, no-signaling results in this case from of a perfect balance between stochasticity (à la Born) and signaling. (...) All these results suggest a possible route for circumventing Gisin’s theorem: if, in accordance with the de Broglie double solution program, particles are identified with solitonic solutions of a nonlinear modification of Schrödinger’s equation such that the trajectories of these solitons obey dB-B dynamics, then, (...), the no-signaling condition is respected once the process of relaxation is achieved.

The full proof that quantum equilibrium prohibits no-signaling and restores causality is given in section 4.1 and appendix D of Ref. [14]. We invite the interested reader to consult this reference.

5.2 Circumventing Page and Geilker’s “No-Go Experiment”

Here is an idealized and simplified version of the experiment described in Ref. [43]. A quantum system is submitted to a dichotomic experiment with two equiprobable results, say Yes and No. If the answer is Yes a massive object is displaced from a macroscopic distance to position (A); otherwise it stays at its initial position (B).
According to no-collapse interpretations, when the object is displaced to position $A$ (resp. $B$), the empty wave associated to the massive object is located in $B$ (resp. $A$). If the source of gravity is the density of stuff averaged over the non-empty and the empty waves, the gravitational field measured in the middle of these two positions ought to be equal to 0 by symmetry. However this field has been measured and (of course) it points towards the effective position of the massive object. Despite the dB-B interpretation is a no-collapse interpretation, and despite our predictions are equivalent to those of the dB-B interpretation, we also predict that the gravitational field measured in the middle of the two positions $A$ and $B$ points towards the effective position of the massive object. This is so because in our case the quantum system is in a sense self-collapsed in position to begin with along either the Yes subspace or the No subspace (this happens with probability 50-50), and the measurement process transfers this property to the massive object. As we have shown, empty waves are not a source of gravity in our approach, and our predictions are thus equivalent to those of collapse interpretations in the present situation.

This argument also helps to understand why no signaling is guaranteed: the state of the reduced system of Alice in an EPR like situation (see Ref. [31] for more details about this gedanken experiment) is already the mixture of self-collapsed states before Bob carries out his measurement, so that the reduction process resulting from Bob’s measurements does not change the statistical properties of the subsystem of Alice. This explains how superluminal signaling gets prohibited in our approach.

6 Conclusions

Where is mass/energy localised? This question is as old as the quantum theory. When Einstein was struggling with the theory of the photon he already faced the following problem: (Maxwell) waves tend to spread, but particles (photons) behave as a localised indivisible whole. He even mentioned in a correspondence with H. Lorentz [45] that maybe a non-linear generalization of Maxwell’s equation would be necessary in order to solve this paradox. The problems raised by wave-particle duality in quantum wave mechanics are exactly the same. Born understood that a way to solve them was the statistical, probabilistic interpretation. There exists however another tradition in theoretical physics, that can be traced back to Poincaré [46, 47], according to which particles are concentrations of force fields [48–50]. The present paper fits to this realistic approach. It could be that, just like special relativity meant the end of aether’s theories, the quantum theory means the end of realism and that all realistic interpretations are condemned to disappear soon or late. Nevertheless they have the merit to push us to explore new mechanisms and to question the dominating orthodoxy [51]. From this point of view they let advance science because they suggest new physics. New experiments are indeed not very exciting if we know in advance their results. Testing gravitation at the microscopic and mesoscopic scales is very challenging because today no fully satisfactory quantum theoretical description of the gravitational interaction is available. The scope of the present paper is to
suggest that, in this frontier domain, direct observations can provide an answer to these old questions. Previous observations like the famous COW experiment [53] are actually not conclusive for what concerns the problems discussed here because they consider the gravitational influence of macroscopic bodies like the Earth on individual quantum systems (neutrons in Ref. [53]) which are so tiny that, in a semi-classical approach à la Bohm (our approach) or à la Schrödinger-Newton [16, 37], their self-interaction can consistently be neglected so that the effective field created by a macroscopic distribution of bodies reduces to the Newtonian potential, a result that also prevails in the linear, standard approach [38, 39] which can be shown to fit with the classical limit of quantum gravity [31]. It also fits with predictions made if we assume that the source of gravity is the density of stuff associated to GRW-Diosi flashes [31]. Henceforth, COW like experiments do not allow us to discriminate between these various interpretations. Letting interfere a mesoscopic body with itself would make it possible, on the contrary, to discriminate between them. For instance, the S-G experiment described in the present paper makes it possible to discriminate between all these approaches and it is feasible with today’s technology [52]. If one day it gets effectively implemented in the lab., it will play the role of a crucial experiment regarding the role of gravity at the quantum scale...

References

1. Bacciagaluppi, G., Valentini, A.: Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference. Cambridge University Press, Cambridge (2010). arXiv:quant-ph/0609184
2. Bohm, D.: A suggested interpretation of the quantum theory in terms of “Hidden” variables. I. Phys. Rev. 85(2), 166–179 (1952)
3. Bohm, D.: A suggested interpretation of the quantum theory in terms of “hidden” variables. II. Phys. Rev. 85(2), 180–193 (1952)
4. Holland, P.R.: The Quantum Theory of Motion. Cambridge University Press, Cambridge (1993)
5. de Broglie, L.: Une tentative d’interprétation causale et non linéaire de la mécanique ondulatoire la théorie de la double solution. Paris: Gauthier-Villars, English translation: Nonlinear wave mechanics: A causal interpretation, p. 1960. Elsevier, Amsterdam (1956)
6. de Broglie, L.: Interpretation of quantum mechanics by the double solution theory. Annales de la Fondation Louis de Broglie, 12, 4, 1987. English translation from a paper originally published in the book Foundations of Quantum Mechanics- Rendiconti della Scuola Internazionale di Fisica Enrico Fermi, IL Corso, B. d’ Espagnat ed. Academic Press N.Y. (1972)
7. Hatifi, M., Lopez-Fortin, C., de Durt, T.: Broglie’s double solution: limitations of the self-gravity approach. Ann. Fond. Louis Broglie 43, 63–90 (2018)
8. Durt, T.: L. de Broglie’s double solution and self-gravitation. Ann. Fond. Louis de Broglie 42, 73 (2017)
9. Fargue, D.: Louis de Broglie’s double solution: a promising but unfinished theory. Ann. Fond. Louis Broglie 42, 19 (2017)
10. Guerret, P., Vigier, J.P.: De Broglie’s wave particle duality in the stochastic interpretation of quantum mechanics: a testable physical assumption. Found. Phys. 12, 1057–1083 (1982)
11. Croca, J.R.: Towards a Nonlinear Quantum Physics. World Scientific, London (2003)
12. Barut, A.: Diffraction and interference of single de Broglie wavelets - Deterministic wave mechanics. In Courants, Amers, Ecueils en Microphysique, Fondation L. de Broglie (1993)
13. Bindel, L.: Mécanique quantique non-relativiste d’une particule individuelle. Ann. Fond. Louis Broglie 37, 143–171 (2012)
14. Colin, S., Durt, T., Willox, R.: L. de Broglie’s double solution program: 90 years later. Ann. Fond. Louis Broglie 42, 19 (2017)
15. Fargue, D.: Permanence of the corpuscular appearance and non linearity of the wave equation. In S. Diner et al., editor, The wave-particle dualism, pp. 149–172. Reidel (1984)
16. Colin, S., Durt, T., Willox, R.: Can quantum systems succumb to their own (gravitational) attraction? Class. Quantum Grav. 31, 245003 (2014)
17. de Broglie, L.: La mécanique ondulatoire et la structure atomique de la matière et du rayonnement. Comptes rendus de l’ académie des sciences, 183, n° 447 (1926)
18. Zloshchastiev, K.G.: Spontaneous symmetry breaking and mass generation as built-in phenomena in logarithmic nonlinear quantum theory. Acta Physica Polonica B 42(2), 261 (2011)
19. Norsen, T.: On the explanation of Born-rule statistics in the de Broglie-Bohm pilot-wave theory. Entropy 20(6), 422 (2018)
20. Valentini, A.: On the pilot-wave theory of classical, quantum and subquantum physics. PhD Thesis, SISSA (1992)
21. Valentini, A., Westman, H.: Dynamical origin of quantum probabilities. Proc. R. Soc. A 461, 253–272 (2005)
22. Colin, S., Struyve, W.: Quantum non-equilibrium and relaxation to quantum equilibrium for a class of de Broglie-Bohm-type theories. New J. Phys. 12, 043008 (2010)
23. Towler, M.D., Russell, N.J., Valentini, Antony: Time scales for dynamical relaxation to the Born rule. Proc. R. Soc. A 468(2140), 990–1013 (2011)
24. Colin, S.: Relaxation to quantum equilibrium for Dirac fermions in the de Broglie-Bohm pilot-wave theory. Proc. R. Soc. A 468(2140), 1116–1135 (2012)
25. Abraham, E., Colin, S., Valentini, A.: Long-time relaxation in the pilot-wave theory. J. Phys. A 47, 395306 (2014)
26. Contopoulos, G., Delis, N., Efthymiopoulos, C.: Order in de Broglie - Bohm quantum mechanics. J. Phys. A 45(16), (2012)
27. Efthymiopoulos, C., Kalapotharakos, C., Contopoulos, G.: Origin of chaos near critical points of quantum flow. Phys. Rev. E 79(3), (2009)
28. Tzemos, A.C., Contopoulos, G., Efthymiopoulos, C.: Origin of chaos in 3-d Bohmian trajectories. arXiv:1609.07069 (2016)
29. Efthymiopoulos, C., Contopoulos, G., Tzemos, A.C.: Chaos in de Broglie—Bohm quantum mechanics and the dynamics of quantum relaxation. Ann. Fond. Louis Broglie 42, 73 (2017)
30. Struyve, W.: Towards a novel approach to semi-classical gravity. In: The Philosophy of Cosmology, Chap. 18. Cambridge University Press, Cambridge, p. 356 (2017)
31. Tilloy, A.: Binding quantum matter and space-time, without romanticism. Founds. Phys. 48, 1753–1769 (2018)
32. Möller, C.: The energy-momentum complex in general relativity and related problems. In A. Lichnerowicz and M.-A. Tonnelat, editor, Les Théories Relativistes de la Gravitation - Colloques Internationaux CNRS 91. CNRS (1962)
33. Rosenfeld, L.: On quantization of fields. Nucl. Phys. 40, 353–356 (1963)
34. Diósi, L.: Gravitation and quantum-mechanical localization of macro-objects. Phys. Lett. A 105, 199–202 (1984)
35. Penrose, R.: On gravity’s role in quantum state reduction. Gen. Relat. Gravit. 28(5), 581–600 (1996)
36. Penrose, R.: On the Gravitization of Quantum Mechanics 1: Quantum State Reduction. Foundations of Physics, Vol. 44, Issue 5 (2014)
37. Hatifi, M., Durt, T.: Revealing self-gravity in a Stern-Gerlach Humpty-Dumpty experiment. arxiv:quant-ph 200607420 (2019)
38. Marletto, C., Vedral, B.: Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. Phys. Rev. Lett. 119(24), 240402 (2017)
39. Bose, S., Mazumdar, A., Morley, G., Ulbricht, H., Toro, M., Paternostro, M., Geraci, A., Andrew, A., Barker, P., Kim, M.S., Milburn, G.: Spin entanglement witness for quantum gravity. Phys. Rev. Lett. 119(24), 240402 (2017)
40. Scully, M., Englert, B.-G., Schwinger, J.: Spin coherence and Humpty-Dumpty. III. The effects of observation. Phys. Rev. A 40(4), 1775 (1989)
41. Gisin, N.: Weinberg’s non-linear quantum mechanics and superluminal communications. Phys. Lett. A 143(1,2), 1–2 (1990)
42. Polchinski, J.: Weinberg’s nonlinear quantum mechanics and the Einstein-Podolsky-Rosen paradox. Phys. Rev. Lett. 66(4), 397–400 (1991)
43. Page, D.N., Geiler, C.D.: Indirect evidence for quantum gravity. Phys. Rev. Lett. 47, 979–982 (1981)
44. Lucas, R.: Sur la répartition de la masse équivalente à l’énergie potentielle et ses conséquences (par L. de Broglie), Note de M. René Lucas, Comptes rendus de l’ académie des sciences, 282 (1976)
45. Einstein, A.: Letter from A. Einstein to H. Lorentz. Collected papers of A. Einstein: The swiss years: correspondence 1902-1914, 5 (2004)
46. Poincaré, H.: La fin de la matière. Athenæum 4086, 201–202 (1906)
47. Poincaré, H.: Sur la dynamique de l’électron. Rendiconti del Circolo matematico di Palermo 21, 129–176 (1906)
48. Fer, F.: L’irréversibilité, fondement de la stabilité du monde physique. Gauthier- Villars, Paris (1977)
49. Fargue, D.: États stationnaires en symétrie sphérique d’une famille d’équation de Schrödinger non-linéaires. Annales de la Fondation Louis de Broglie 12, 203 (1987)
50. Visser, M.: A classical model for the electron. Phys. Lett. A 139(3), 4 (1989)
51. Anastopoulos, C., Hu, B.-L.: Problems with the Newton-Schrödinger equations. New J. Phys. 16, 085007 (2014)
52. Margalit, Y., Dobkowski, O., Zhou, Z., Amit, O., Japha, Y., Moukouri, S., Rohrlich, D., Mazumdar, A., Bose, S., Henkel, C., Folman, R.: Realization of a complete Stern-Gerlach interferometer, Science Advances, 7(22) (2020)
53. Colella, R., Overhauser, A.W., Werner, S.A.: Observation of gravitationally induced quantum interference. Phys. Rev. Lett. 34, 1472–1474 (1975)
54. Bialynicki-Birula, I., Mycielski, J.: Nonlinear wave mechanics. Ann. Phys. 100, 62–93 (1976)

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