BORDISM CLASSES OF THE MULTIPLE POINTS MANIFOLDS OF SMOOTH IMMERSIONS

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ABSTRACT. Let \( f : V^n \natural M^m \) be a smooth generic immersion. Then the set of points, that have at least \( k \) preimages is an image of a (non-generic) immersion. If the manifolds \( V^n \) and \( M^m \) are oriented and \( m - n \) is even, then the manifold of \( k \)-fold points is also oriented. In this paper we compute the oriented bordism class of the manifold of \( k \)-fold points in terms of the differential \( df \), provided the tangent bundle of the manifold \( M^m \) has a nowhere zero cross-section.

1. Introduction

In this paper we will consider smooth orientable \( C^\infty \)-manifolds and \( C^\infty \)-mappings between them. Let \( V^n \) and \( M^m \) be manifolds without boundary, \( V^n \) be compact, \( f : V^n \natural M^m \) be a smooth generic immersion and \( m - n \) be even. From Thom multijet transversality theorem [1] it follows that the set of immersions \( f \) such that the map \( f^{(k)} : V^{(k)} \to M^{(k)} \) is transverse to the “thin” diagonal \( \Delta_k(M) = \{ (x, \ldots, x) \mid x \in M \} \subset M^{(k)} \) outside the “thick” diagonal \( \Delta_2(V) = \{ (x_1, \ldots, x_k) \mid \exists \ i \neq j : x_i = x_j \} \subset V^{(k)} \) is open and everywhere dense in the set of all immersions \( V^n \natural M^m \) (in \( C^\infty \) Whitney topology). Therefore, for a generic \( f \) the set \( V_k = (f^{(k)})^{-1} (\Delta_k(M)) \setminus \Delta_2(V) \subset V^{(k)} \) is an oriented submanifold. Denote by \( \Sigma_k \) the group of permutations on \( k \) elements. Permutation of factors in the product \( V^{(k)} \) induce the free action of \( \Sigma_k \) on the set \( V^{(k)} \). Denote by \( \tilde{V}_k \) and \( \tilde{M}_k \) the quotient manifolds \( V_k/\Sigma_{k-1} \) and \( \tilde{V}_k/\Sigma_k \), where the subgroup \( \Sigma_{k-1} \subset \Sigma_k \) is the stabilizer of the first element. Since \( m - n \) is even, the action of \( \Sigma_k \) preserves the orientation of the manifold \( V_k \). Therefore \( \tilde{V}_k \) and \( \tilde{M}_k \) are canonically oriented. Let us define immersions \( f_k : \tilde{V}_k \natural V^n \) and \( g_k : \tilde{M}_k \natural M^m \) by formulae \( f_k(x_1, [x_2, \ldots, x_k]) = x_1 \) and \( g_k[x_1, \ldots, x_k] = f(x_1) \) (see [2] for details).

For an integer \( k > 0 \) and an immersion \( f : V^n \natural M^m \) we assign the oriented bordism classes \( (\tilde{V}_k, f_k) \in \Omega_{m-k(m-n)}(V^n) \) and \( (\tilde{M}_k, g_k) \in \Omega_{m-k(m-n)}(M^m) \). From [3] it follows that these classes do not change under a regular homotopy of the immersion \( f \). By the fundamental Smale-Hirsch theorem [4], the set of regular homotopy classes of immersions \( V^n \natural M^m \) is in 1-1 correspondence with the set of linear monomorphism classes of the tangent bundles \( \tau V \to \tau M \). The class of the immersion \( f : V^n \natural M^m \) corresponds to the class of the differential \( df : \tau V \to \tau M \). Hence, the classes \( (\tilde{V}_k, f_k) \in \Omega_{m-k(m-n)}(V^n) \) and \( (\tilde{M}_k, g_k) \in \Omega_{m-k(m-n)}(M^m) \) have to be computable in terms of the differential \( df : \tau V \to \tau M \). In this paper (see corollary 2.3) we compute \( (\tilde{V}_k, f_k) \) and \( (\tilde{M}_k, g_k) \) up to elements of order \( (k - 1)! \) and \( k! \), respectively. This weakening is in some sense natural, for the manifolds \( \tilde{V}_k \) \( \tilde{M}_k \) were constructed as the images of \( (k - 1)! \)- and \( k! \)-fold coverings.

Recall that all the finite order elements of \( \Omega_*(pt) \) have the order 2. Therefore, for \( M^m = \mathbb{R}^m \) we compute the classes \( (\tilde{M}_k, g_k) \) up to elements of order 2. It is another approach to [5, Theorem 5], where all the Pontrjagin numbers of the manifolds \( \tilde{M}_k \) were computed in terms of the Pontrjagin classes of the manifold \( V^n \) and the integral Euler class of the normal bundle of the immersion \( f \). Lemma 2.3 gives a formula to compute the unoriented bordism class
(\tilde{V}_2, f_2) \in \mathcal{R}_*(V^n)$, if we do not require $m - n$ to be even, and $V^n$ and $M^m$ to be oriented. For up-to-date reviews of results on the bordism classes of self-intersection manifolds see for oriented case, and for unoriented case.

2. Formulation of results

Denote by $\varSigma M$ the spherical fibration, associated to the tangent bundle $\tau M$ and by $Sdf : \varSigma V \rightarrow \varSigma M$ the fiberwise monomorphism of spherical fibrations, induced by the differential $df : \tau V \rightarrow \tau M$. Since the manifolds $V^n$, $M^m$, $\varSigma V$ and $\varSigma M$ are oriented (in usual sense), they are oriented in oriented bordism theory. Thus, there is the Poincare duality on these manifolds. Denote by $Sdf : \Omega_*(\varSigma M) \rightarrow \Omega_*(\varSigma V)$ the Gysin homomorphism, induced by the mapping $Sdf$.

Let us formulate the key lemma of this paper.

**Lemma 2.1.** Let $M^m$ be an oriented manifold without boundary such that there is a nowhere zero cross-section of the tangent bundle $\tau M$, or, in other words, a section $s_M : M^m \rightarrow \varSigma M$. Then for any generic immersion $f : V^n \simeq M^m$ of compact oriented manifold without boundary $V^n$ the bordism class $(\tilde{V}_2, f_2) \in \Omega_{2n-m}(V^n)$ is

$$((\tilde{V}_2, f_2) = (-1)^{m-1}i_*Sdf^*(M^m, s_M),$$

where $i$ is the natural projection $\varSigma V \rightarrow V^n$.

To formulate our results, it will be convenient to use the oriented cobordism classes $v_k \in \Omega^{(k-1)(m-n)}(V^n)$ and $m_k \in \Omega^{(m-n)}_{\text{comp.}}(M^m)$, the Poincare duals to $(\tilde{V}_k, f_k)$ and $(\tilde{M}_k, g_k)$, respectively. Denote by $1_V$ the identity element of the ring $\Omega^*(V^n)$, and by $f_1$ the Gysin homomorphism, induced by the map $f$.

**Corollary 2.2.** Under the conditions of lemma 2.1 and if $m - n$ is even, the Euler class $e$ of the normal bundle of the immersion $f$ is

$$e = f^*f_1(1_V) + (-1)^m\gamma i_*Sdf^*(M^m, s_M),$$

where $\gamma : \Omega_*(V^n) \rightarrow \Omega^m(V^n)$ is the Poincare duality.

**Corollary 2.3.** Under the conditions of lemma 2.1 and if $m - n$ is even,

$$(k-1)! \cdot v_k = \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1_V)

k! \cdot m_k = f_1 \circ \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1_V),$$

where $\varphi_k(a) = f^*f_1(a) - k \cdot e \cup a$, and $e$ is the Euler class of the normal bundle of the immersion $f$ (which was computed in corollary 2.2).

3. The bordism group of immersions

Let us call two oriented immersions $f_0 : V_0^n \simeq M^m$ and $f_1 : V_1^n \simeq M^m$ bordant, if there exists a compact oriented manifold with boundary $W^{n+1}$ such that $\partial W^{n+1} = V_0^n \sqcup (-V_1^n)$, and an immersion $W^{n+1} \rightarrow M^m \times [0, 1]$ such that for a collar $V_0^n \times [0, \varepsilon] \sqcup (-V_1^n) \times (1-\varepsilon, 1]$ of the boundary $\partial W^{n+1}$ the restrictions $F|_{V_0^n \times [0, \varepsilon]} = f_0 \times \text{id}$ and $F|_{(-V_1^n) \times (1-\varepsilon, 1]} = f_1 \times \text{id}$. Then the set of equivalence classes of bordant oriented immersions with disjoint union operation is a group $\text{Imm}^{SO}(M^m)$. The group $\text{Imm}^{SO}_n(M^m) = [M^m, QMSO(m - n)]$, where $QX = \lim \Omega^qS^qX$ is the infinite loop space of infinite suspension and $M^O$ is the Thom spectrum.

From the results of paper [3] it follows that the map, assigning for any immersion $f : V^n \simeq M^m$ the bordism class $(\tilde{M}_k, g_k) \in \Omega_{m-k(m-n)}(M^m)$, is a well-defined homomorphism $\varepsilon_k : \text{Imm}^{SO}_n(M^m) \rightarrow \Omega_{m-k(m-n)}(M^m)$. The classes $(\tilde{M}_k, g_k)$ involve much information about the class of immersion $[f] \in \text{Imm}^{SO}_n(M^m)$. The following theorem was proved in [10], using algebraic technics.
Theorem 3.1 ([11, Corollary 1]). If $3n + 1 < 2m$ and $m - n$ is even, then the homomorphism
\[ \varepsilon_1 \oplus \varepsilon_2 : \text{Imm}^n_\tau (\mathbb{R}^m) \to \Omega_n \oplus \Omega_{2n-m} \]
is an isomorphism modulo the class $C_2$ of finite 2-primary groups.

Our calculations probably clarify the geometric core of theorem 3.1. The matter is that in these very dimensional restrictions ($3n + 1 < 2m$) any skew map $\tau V \to \tau M$ can be homotoped to a monomorphism of tangent bundles (see details in [11]). Our formula (lemma 2.1) connects the map of spherical fibrations, induced by the differential $df : \tau V \to \tau M$, and the oriented bordism class $2(\tilde{M}_2, g_2) \in \Omega_{2n-m}(M^m)$. These reasoning was the initial motivation of this paper.

Conjecture 3.2. Let $M^m$ be an oriented manifold without boundary such that there is a nowhere zero cross-section of the tangent bundle $\tau M$, $3n + 1 < 2m$, and $m - n$ be even. Then the homomorphism
\[ \varepsilon_1 \oplus \varepsilon_2 : \text{Imm}^n_\tau (M^m) \to \Omega_n(M^m) \oplus \Omega_{2n-m}(M^m) \]
is an isomorphism modulo the class $C_2$ of finite 2-primary groups.

4. PROOFS

Denote the diagonal $\Delta(V) = \{(x, x) \in V \times V | x \in V\}$. Since $f : V^n \to M^m$ is an immersion, there exists a small enough tubular neighborhood $U_V$ of the diagonal $\Delta(V)$ in $V \times V$ such that $f^2(U_V \setminus \Delta(V)) \cap \Delta(M) = \emptyset$. Note that $\partial(V^2 \setminus U_V) = \partial(U_V)$. Since $f^2(U_V \setminus \Delta(V)) \cap \Delta(M) = \emptyset$, we get the map $f^2 : (V^2 \setminus U_V, \partial(V^2 \setminus U_V)) \to (M^2, M^2 \setminus \Delta(M))$. Denote by $U_M$ a tubular neighborhood of the diagonal $\Delta(M)$ in $M^2$. Without loss of generality we may assume that $f^2(U_V) \subseteq U_M$. Denote the inclusion
\[ j : (U_M, U_M \setminus \Delta(M)) \hookrightarrow (M^2, M^2 \setminus \Delta(M)) \]
By excision axiom [8], the homomorphisms $j_* : \Omega_*(U_M, U_M \setminus \Delta(M)) \to \Omega_*(M^2, M^2 \setminus \Delta(M))$ and $j^* : \Omega^*(M^2, M^2 \setminus \Delta(M)) \to \Omega^*(U_M, U_M \setminus \Delta(M))$ are isomorphisms. Note that the pair $(U_M, \Delta(M))$ is canonically isomorphic to the pair $(\tau M, \tau_0 M)$ [12]. Since $M^m$ is oriented, there exists the Thom class $t = \Omega^m(\tau M, \tau M \setminus \tau_0 M)$ of the tangent bundle $\tau M$.

Lemma 4.1. The class $(\tilde{V}_2, f_2) \in \Omega_{2n-m}(V^n)$ for an immersion $f : V^n \to M^m$ can be calculated in the following way
\[ (\tilde{V}_2, f_2) = (\pi_1)_* \left( (f^2)^* (j^*)^{-1}t \cap \left[ V^2 \setminus U_V, \partial(V^2 \setminus U_V) \right] \right), \tag{2} \]
where $\pi_1 : V^2 \setminus U_V \to V$ is the projection on the first factor, and $[V^2 \setminus U_V, \partial(V^2 \setminus U_V)]$ is the fundamental class.

Proof of lemma 4.1. Let us recall the construction of the class $(\tilde{V}_2, f_2)$. Since $f$ is an immersion, $\Delta(V)$ is a closed subset in $(f^2)^{-1}(\Delta(M))$. Since $f$ is a generic immersion, $(f^2)$ is transversal to $\Delta(M)$ outside $\Delta(V)$. Therefore $(f^2)^{-1}(\Delta(M)) \setminus \Delta(V)$ is a compact oriented submanifold without boundary $f_2 : \tilde{V}_2 \to V^2 \setminus \Delta(V)$. Then the composition $\pi_1 \circ f_2$ is $f_2 : \tilde{V}_2 \to V^n$ (see details in [2]). By definition 3 of Lefschetz duality $\gamma : \Omega^*(V^2 \setminus U_V, \partial(V^2 \setminus U_V)) \to \Omega_*(V^2 \setminus U_V)$
\[ (f^2)^* (j^*)^{-1}t \cap \left[ V^2 \setminus U_V, \partial(V^2 \setminus U_V) \right] = (-1)^{2n-m} \gamma \left( (f^2)^* (j^*)^{-1}t \right) \]
\[ = \gamma \left( (f^2)^* (j^*)^{-1}t \right) \]

\footnote{a skew map $\tau V \to \tau M$ can be understood as the “fiberwise cone” over a fiber map $h : S\tau V \to S\tau M$ such that $h(-x) = -h(x)$ in each fiber}
Since $f^{(2)}$ is transversal to $\Delta(M)$ outside $\Delta(V)$, we have
\[(\pi_1)_* \left( \gamma \left( (f^{(2)})^*((j^*)^{-1} t) \right) \right) = (\pi_1)_* \left( (\tilde{f}_2)_* \left[ (f^{(2)})^{-1}(\Delta(M)) \backslash \Delta(V) \right] \right)
= (\pi_1)_* \left( \tilde{V}_2, \tilde{f}_2 \right) = (\tilde{V}_2, f_2) \]

**Proof of lemma 2.1.** To prove lemma 2.1, it suffices to interpret the right hand side of formula (2) in terms of the differential $df$. Since $\partial(V^{(2)} \backslash U_V) = \partial(U_V)$, we have
\[\partial_* \left[ V^{(2)} \backslash U_V, \partial(V^{(2)} \backslash U_V) \right] = \left[ \partial(V^{(2)} \backslash U_V) \right], \]
where $\partial_* : \Omega_2n(\partial(V^{(2)} \backslash U_V), \partial(V^{(2)} \backslash U_V)) \rightarrow \Omega_{2n-1}(\partial(V^{(2)} \backslash U_V))$ is the differential in the exact bordism sequence of pair. Denote by $j_1$ the inclusion $S\tau M \hookrightarrow \tau M \backslash \tau_0 M$. Obviously, the map $j_1$ is a homotopy equivalence. Since $j_1 \circ s_M : \tau M \rightarrow \tau M \backslash \tau_0 M$ is a nowhere zero cross-section of $\tau M$, we have
\[\delta^* \left( (j_1^*)^{-1} \gamma(M^m, s_M) \right) = t, \]
where $\delta^* : \Omega^{m-1}(\tau M \backslash \tau_0 M) \rightarrow \Omega^m(\tau M \backslash \tau_0 M)$ is the differential in the exact cobordism sequence of pair, and $\gamma$ is the Poincare duality on the total manifold $S\tau M$. Denote by $j_2$ the isomorphism $\partial(V^{(2)} \backslash U_V) \cong S\tau V$. From the explicit formula (3) for the $\mathbb{Z}_2$-equivariant isomorphism, that identify a small neighborhood of zero section of $\tau V$ with the neighborhood $U_V$
\[\tau V \ni (x, \vec{v}_x) \mapsto (\exp_{x}(-\vec{v}_x), \exp_{x}(-\vec{v}_x)) \in V \times V, \]
it follows that the following diagram commutes (double arrows here denote isomorphisms).

\[
\begin{array}{ccc}
\Omega^{m-1}(\tau V) & \stackrel{j_2^*}{\longrightarrow} & \Omega^{m-1}(\partial(V^{(2)} \backslash U_V)) \\
\downarrow Sd f^* & & \downarrow (f^{(2)})^* \\
\Omega^{m-1}(\tau M \backslash \tau_0 M) & \stackrel{j_1^*}{\longleftarrow} & \Omega^{m}(\tau M \backslash \tau_0 M)
\end{array}
\]

Figure 1

Therefore, $\delta^*(j_2^*Sd f^*\gamma(M^m, s_M)) = (f^{(2)})^*((j^*)^{-1} t)$. From the naturality of the $\cap$-product $\otimes$, we have
\[(j_3)_* \left( j_2^*Sd f^*\gamma(M^m, s_M) \cap \left[ \partial(V^{(2)} \backslash U_V) \right] \right) = (f^{(2)})^*((j^*)^{-1} t) \cap \left[ V^{(2)} \backslash U_V, \partial(V^{(2)} \backslash U_V) \right], \]
where $j_3 : \partial(V^{(2)} \backslash U_V) \rightarrow V^{(2)} \backslash U_V$ is the inclusion. Then, by lemma 4.3
\[(\tilde{V}_2, f_2) = (\pi_1)_* \left( ((f^{(2)})^*((j^*)^{-1} t) \cap \left[ V^{(2)} \backslash U_V, \partial(V^{(2)} \backslash U_V) \right] \right)
= (\pi_1 \circ j_3)_* \left( j_2^*Sd f^*\gamma(M^m, s_M) \cap \left[ \partial(V^{(2)} \backslash U_V) \right] \right)
= (-1)^{(2n-1)\cdot(m-1)}(\pi_1 \circ j_3)_* \gamma(j_2^*Sd f^*\gamma(M^m, s_M))
= (-1)^m(\pi_1 \circ j_3)_* \gamma(f_2)_*^{-1}Sd f^*\gamma(M^m, s_M)
= (-1)^{m-1}(\pi_1 \circ j_3)_* \gamma(f_2)_*^{-1}Sd f^*(M^m, s_M)
\]
It remains only to note that $i_* = (\pi_1 \circ j_3)_* \circ (j_2)_*^{-1}$. \qed
To prove the corollaries 2.2 and 2.3 we will need the following results from the paper [2].

**Theorem 4.2** ([2]). For any smooth generic immersion \( f : V^n \hookrightarrow M^m \) of the compact oriented manifold without boundary \( V^n \) to the oriented manifold without boundary \( M^m \) such that \( m - n \) is even, we have

\[
v_k = f^*(m_{k-1}) - e \cup v_{k-1},
\]

where \( e \) is the Euler class of the normal bundle of the immersion \( f \) over \( V^n \).

**Corollary 4.3** ([2]). Under the conditions of theorem 4.2, we have

\[(k - 1)! \cdot v_k = \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1_V),\]

where \( \varphi_k(a) = f^*f_1(a) - k \cdot e \cup a; e \) is the Euler class of the normal bundle of the immersion \( f \) over \( V^n \).

**Proof of corollary 2.2.** It suffices to substitute (1) into formula (3) with \( k = 2 \). \( \square \)

**Proof of corollary 2.3.** follows immediately from corollaries 4.3 and 2.2. \( \square \)

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