GAPS IN THE MILNOR-MOORE SPECTRAL SEQUENCE AND
THE HILALI CONJECTURE

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Abstract. In his study of Halperin’s toral-rank conjecture, M. R. Hilali conjectured that for any simply connected rationally elliptic space $X$, one must have $\dim \pi_\ast (X) \otimes \mathbb{Q} \leq \dim H^\ast (X, \mathbb{Q})$. Let $(AV, d)$ denote a Sullivan minimal model of $X$ and $d_k$ the first non-zero homogeneous part of the differential $d$. In this paper, we use spectral sequence arguments to prove that if $(AV, d_k)$ is elliptic, then, there is no gaps in the $E_\infty$ term of the Milnor-Moore spectral sequence of $X$. Consequently, we confirm the Hilali conjecture when $V = V_{\text{odd}}$ or else when $k \geq 3$ and $(AV, d_k)$ is elliptic.

1. Introduction

The rational dichotomy theorem ([3]) states that a $1$-connected finite CW-complex $X$ is either $\mathbb{Q}$-elliptic or $\mathbb{Q}$-hyperbolic. The first ones are characterized by the inequalities $\dim (\pi_\ast (X) \otimes \mathbb{Q}) < \infty$ and $\dim (H^\ast (X, \mathbb{Q})) < \infty$. Although they are not generic, they are subject of many work in rational homotopy theory and several conjectures are put around them (refer to [5], Part VI for details). Among these we cite the toral-rank conjecture due to S. Halperin. Recall first that the action of an $n$-dimensional torus on $X$ is said almost-free if all ensuing isotropy groups are finite. The largest integer $n \geq 1$, denoted $rk(X)$, for which $X$ admits an almost-free $n$-torus action is called the toral-rank of $X$. In [7], S. Halperin conjectured the following relation between this rank and the rational cohomology of $X$:

**Conjecture 1.0.1.** (The toral-rank conjecture): If $X$ is a simply-connected finite type CW-complex, then $\dim (H^\ast (X, \mathbb{Q})) \geq 2^{rk(X)}$.

In [9], M. R. Hilali studied this latter and was led to pose the following conjecture:

**Conjecture 1.0.2.** (The H-conjecture): If $X$ is an elliptic simply-connected space, then $\dim (H^\ast (X, \mathbb{Q})) \geq \dim (\pi_\ast (X) \otimes \mathbb{Q})$.

This conjecture was resolved in several cases, as pure elliptic spaces ([9]), formal spaces ([8]), hyperelliptic spaces ([11]), elliptic spaces with formal dimension $\leq 16$ ([10]) and a large family of elliptic spaces who’s Sullivan minimal model has an homogeneous differential ([9]). These results are obtained after the translation of 1.0.2 in terms of the Sullivan minimal model of $X$. We state it in a general form as follow:

**Conjecture 1.0.3.** (The algebraic version H-conjecture): If $(AV, d)$ is an elliptic Sullivan minimal algebra, then $\dim H(AV, d) \geq \dim (V)$.

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The starting point in our treatment of 1.0.3 relies on its validity for any model with an homogeneous differential $d$ of length $k \geq 3$ and for $k = 2$ with some restrictions ([8]).

The main tool, we will use to do this, is the following spectral sequence introduced by the author in [18]:

$$E^p,q_k = H^{p,q}(\Lambda V, d_k) \implies H^{p+q}(\Lambda V, d)$$

When $k = 2$ this is identified to the Milnor-Moore spectral sequence of $X$ ([4], Prop. 9.1):$$\text{Ext}^{p,q}H^*(\Omega X, \mathbb{Q})(\mathbb{Q}, \mathbb{Q}) \implies H^{p+q}(X, \mathbb{Q}).$$

Further, remark that the $\infty$-terms of this later coincides with that of (1).

Our first purpose in this paper is to look for possible gaps in $E^{*,*}_\infty$, assuming that spaces in consideration are rationally elliptic. Indeed, notice that in [10], T. Kahl and L. Vandemroucq proved that the first term of the Milnor-Moore spectral sequence has no gaps and provided a non rationally elliptic finite CW-complex which presents gaps in the term $E^{*,*}_\infty$.

Henceforth, $(\Lambda V, d)$ is a Sullivan minimal algebra, that is, its differential $d$ has the form $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$. By degree reason, $d_k$ is also a differential.

In addition, $H(\Lambda V, d_k)$ admits a second graduation given by lengths of representative cocycles; so $H^+(\Lambda V, d_k) = \bigoplus_{p \geq 1} H^+_p(\Lambda V, d_k)$ (the elements of $H^+_p(\Lambda V, d_k)$ are represented by homogeneous cocycles of length $p$). Assume that $(\Lambda V, d_k)$ is elliptic. The part (B) of Theorem 2.2 in [12] states that the subspace $H^+_p(\Lambda V, d_k)$ ($p = 1, \ldots, e$) doesn’t reduce to zero. Hence by denoting $[\omega_0] = 1_\mathbb{Q}$ we deduce that for any $p = 0, \ldots, e$, $E^{p,0}_k \neq 0$. This is expressed by saying that the spectral sequence (1) has no gaps in its first term $E^{*,*}_0$.

Recall that, $(\Lambda V, d_k)$ being elliptic (cf.§2), its cohomology $H(\Lambda V, d_k)$ satisfies Poincaré-duality ([4]) and all the representing cocycles of its fundamental class $\omega$ are homogeneous of length $e = \dim V^{odd} + (k-2)\dim V^{even}$ ([11]). A particular cocycle representing $\omega$ is given by the evaluation map:

$$e_{\omega}(\Lambda V, d) : \text{Ext}^*(\Lambda V, d)(\mathbb{Q}, (\Lambda V, d)) \to H(\Lambda V, d)$$

introduced by Y. Félix et al. in [6] (see §3.1 for more details). It will serves us to obtain an cocycle that survives to the $\infty$-term of (1) (see Remark 3.5. (2) for another eventual use of this spectral sequence).

In fact this map connects the spectral sequence (1) to the following one (see also [18]):

$$E^p,q_k = \text{Ext}^p(\Lambda V, d_k)(\mathbb{Q}, (\Lambda V, d_k)) \implies \text{Ext}^{p+q}(\Lambda V, d)(\mathbb{Q}, (\Lambda V, d)).$$

In the sequel, we shall call (1) and (2), respectively, the generalized and the Ext-version generalized Milnor-Moore spectral sequences.

Our main theorem in this paper gives a partial response to the above purpose.
Theorem 1.0.4. If $X$ is a simply connected finite type space whose Sullivan minimal model $(\Lambda V, d)$ is such that $(\Lambda V, d_k)$ is elliptic, then, at the $E_\infty$ term of the generalized Milnor-Moore spectral sequence (1) of $X$, there can’t be any gap.

As a consequence, we have:

Theorem 1.0.5. For any space $X$ with Sullivan minimal model $(\Lambda V, d)$, the H-conjecture holds if:

1. $V = V^{\text{odd}}$, or else
2. $(\Lambda V, d_k)$ is elliptic and $k \geq 3$.

We note that the first case in the last theorem gives an improvement of the corollary of theorem A in [17].

Finally, as mentioned G. Lupton in his paper, we find interesting to recall that his main motivation was the following question asked by Y. Félix:

Question 1.0.6. Can an elliptic space have $e_0$-gaps in its cohomology?

Here the cohomology $H^*(X, \mathbb{Q})$ has an $e_0$-gap if it has an element $x$ whose Toomer invariant $e_0(x) = k$ (see §2 for the definition of $e_0(x)$), but does not have any element whose Toomer invariant is $k - 1$. It follows from Theorem 1.0.4 the

Theorem 1.0.7. If $X$ is a simply connected finite type space whose Sullivan minimal model $(\Lambda V, d)$ is such that $(\Lambda V, d_k)$ is elliptic. Then $H^*(X, \mathbb{Q})$ has no $e_0$-gaps

The rest of the paper is organized as follows: In §2, we give a brief summary on ingredients of rational homotopy theory we will use in the sequel. We recall also the filtrations inducing the spectral sequences (1) and (2). §3 is reserved to the proofs our results and for some remarks.

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2. Preliminary

Let $K$ a field of characteristic zero, \( V = \bigoplus_{i=0}^{\infty} V^i \) a graded $K$-vector space and $\Lambda V = \text{Exterior}(V^{\text{odd}}) \otimes \text{Symmetric}(V^{\text{even}})$ ($V^i$ is the subspace of elements of degree $i$).

A Sullivan algebra is a free commutative differential graded algebra $(\Lambda V, d)$ (cdga for short) such that $V$ admits a basis $\{x_\alpha\}$ indexed by a well-ordered set such that $dx_\alpha \in \Lambda V_{<\alpha}$ where $V_{<\alpha} = \{v_\beta \mid \beta < \alpha\}$. Such algebra is said minimal if $\deg(x_\alpha) < \deg(x_\beta)$ implies $\alpha < \beta$. If $V^0 = V^1 = 0$, this is equivalent to saying that $d(V^i) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^2 V^i$.

A Sullivan model for a commutative differential graded algebra $(A, d)$ is a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\sim} (A, d)$ (morphism inducing an isomorphism in cohomology) with source, a Sullivan algebra.

When $K = \mathbb{Q}$ and $X$ is any simply connected space, let $A(X)$ denote the algebra of polynomial differential forms associated to it ([19]). The minimal model $(\Lambda V, d)$
of this later is called the \textit{Sullivan minimal model} of $X$. It is related to rational homotopy groups of $X$ by $V^i \cong \text{Hom}_\mathbb{Z}(\pi_i(X), \mathbb{Q})$: \forall i \geq 2$, that is, in the case where $X$ is a finite type CW-complex, the generators of $V$ corresponds to those of $\pi_*(X) \otimes \mathbb{Q}$.

Recall in passing that $(\Lambda V, d)$ is an elliptic Sullivan algebra if and only if $\text{dim} V$ and $\text{dim} H(\Lambda V, d)$ are both finite dimensional. In this case, $X$ is said rationally elliptic. In addition, its cohomology is a Poincaré-duality algebra (\cite{5} Prop. 38. 3) with the formal dimension $N = \sup\{p \mid H^p(\Lambda V, d_k) \neq 0\}$ given by the formula (\cite{5}): $N = \text{dim} V_{\text{even}} - \sum_{i=1}^{\text{dim} V} (-1)^{|z_i|}|x_i|$, where $\{x_1, x_2, \ldots, x_n\}$ designates a basis of $V$. A complete reference about such algebras is (\cite{5} §32).

Now, let $(A, d)$ be an augmented $\mathbb{K}$-differential graded algebra and choose an $(A, d)$-semifree resolution (\cite{6}) $\rho : (P, d) \xrightarrow{\cong} (\mathbb{K}, 0)$ of $\mathbb{K}$. Providing $\mathbb{K}$ with the $(A, d)$-module structure induced by the augmentation, we define a chain map: $ev : \text{Hom}_{(A,d)}((P, d), (A, d)) \longrightarrow (A, d)$ by $f \mapsto f(z)$, where $z \in P$ is a cycle representing $1_{\mathbb{K}}$. Passing to homology we obtain the \textit{evaluation map} of $(A, d)$:

$$ev_{(A,d)} : \text{Ext}_{(A,d)}(\mathbb{K}, (A, d)) \longrightarrow H(A, d),$$

where $\text{Ext}$ is the differential $\text{Ext}$ of Eilenberg and Moore (\cite{15}). Note that this definition is independent of the choice of $P$ and $z$ and it is natural with respect to $(A, d)$.

The authors of \cite{6} defined also the concept of a \textit{Gorenstein algebra} over any field $\mathbb{K}$. If $(A, d)$ is as above, it is of Gorenstein, provided $\text{dim} \text{Ext}_{(A,d)}(\mathbb{K}, (A, d)) = 1$.

In the particular case where $(A, d) = (\Lambda V, d)$ is elliptic, its cohomology $H(\Lambda V, d)$ satisfies Poincaré-duality property over $\mathbb{K}$ with the formal dimension (\cite{6}, Prop. 5.1):

$$(3) \quad N = \sup\{p \mid H^p(\Lambda V, d) \neq 0\}.$$  

If $\mathbb{K} = \mathbb{Q}$ and $\{x_1, x_2, \ldots, x_n\}$ designates a basis of $V$, this one has an explicit expression (\cite{6}, Prop. 5.2):

$$(4) \quad N = \text{dim} V_{\text{even}} - \sum_{i=1}^{\text{dim} V} (-1)^{|z_i|}|x_i|$$

(where the elliptic nature of $(\Lambda V, d)$ is implicit).

In addition, if $h$ represents a generator of $\text{Ext}_{(\Lambda V,d)}(\mathbb{K}, (\Lambda V, d))$, the fundamental class of $(\Lambda V, d)$ is precisely $ev_{(A,d)}([h]) = [h(1)]$ (\cite{15}).

Another invariant in connection with the last ones is the Toomer invariant. It is defined by more than one way. Here we recall its definition in the context of minimal models.

Let $(\Lambda V, d)$ any Sullivan minimal algebra and denote

$$p_n : \Lambda V \rightarrow (\Lambda V / \Lambda^{\geq n+1} V),$$

the projection onto the quotient differential graded algebra obtained by factoring out the differential graded ideal generated by monomials of length at least $n+1$. The \textit{Toomer invariant} $e_\mathbb{K}(\Lambda V, d)$ of $(\Lambda V, d)$ is the smallest integer $n$ such that $p_n$ induces an injection in cohomology or $\infty$ if there is no such integer. For $\mathbb{K} = \mathbb{Q}$, we shall denote $e(\Lambda V, d)$ instead of $e_\mathbb{Q}(\Lambda V, d)$.
In fact (cf. [4]), $e(\Lambda V, d)$ is also expressed in terms of the Milnor-Moore spectral sequence (which coincide with (1) for $k = 2$) by: $e(\Lambda V, d) = \sup\{p \mid E_p^{s,q} \neq 0\} = \infty$ if such maximum doesn’t exists.

More explicitly, whenever $H(\Lambda V, d)$ has Poincaré-duality, we have

$$e(\Lambda V, d) = \sup\{k \mid \omega \text{ can be represented by a cocycle in } \Lambda^{2k}V\},$$

where $\omega$ represents the fundamental class of $(\Lambda V, d)$.

Consider now $x \in H(\Lambda V, d)$ an arbitrary non zero cohomology class. G. Lupton defines ([12]) its Toomer invariant $e_o(x)$ to be the smallest integer $n$ for which $p^n(x) \neq 0$. If the set $\{e_0(x) \mid 0 \neq x \in H(\Lambda V, d)\}$ has a maximum, then $e(\Lambda V, d)$ is this maximum. In fact, in this case, we have $e(\Lambda V, d) = e_0(\omega)$.

To finish this preliminary, we recall the filtrations that induce the spectral sequences (1) and (2) mentioned in the introduction. This passes by introducing a semifree resolution of $\mathbb{K}$ endowed with a $(\Lambda V, d)$-module structure.

Recall that the suspension $sV$ of $V$ is given by the identification $(sV)^i = V^{i+1}$, $i \geq 0$. On the graded algebra $\Lambda V \otimes \Lambda sV$, let $S$ the derivation on $\Lambda(V \otimes sV)$ specified by $S(v) = sv$ and $S(sv) = 0$, for all $v \in V$. Define $(\Lambda V \otimes \Lambda(sV), D)$ by putting $D(sv) = -S(dv)$ and $D_V = d$. It is an acyclic commutative differential graded algebra called an acyclic closure of $(\Lambda V, d)$. So this is an $(\Lambda V, d)$-semifree module and therefore, the projection $(\Lambda V \otimes \Lambda(sV), D) \overset{\simeq}{\longrightarrow} \mathbb{K}$ is a semifree resolution of $\mathbb{K}$. So on $\text{Hom}_{\Lambda V}(\Lambda V \otimes \Lambda sV, \Lambda V)$ a differential $D$ is defined by

$$D(f) = d \circ f + (-1)^{|f|+1} f \circ D.$$

The filtrations in question are defined respectively as follow:

$$F^p(\Lambda V) = \Lambda^{2p}V = \bigoplus_{i=p}^{\infty} \Lambda^iV$$

$$F^p = \{f \in \text{Hom}_{\Lambda V}(\Lambda V \otimes \Lambda(sV), \Lambda V) \mid f(\Lambda(sV)) \subseteq \Lambda^{2p}V\}$$

**Remark 2.0.8.** Let $(\Lambda V, d)$ an elliptic Sullivan minimal algebra. The chain map $ev : (\text{Hom}_{\Lambda V}(\Lambda V \otimes \Lambda sV, \Lambda V), D) \longrightarrow (\Lambda V, d)$ inducing the evaluation map, is clearly filtration preserving. Remark also that (5) is a filtration of a filtered graded algebra. Hence (1) is a spectral sequence of a graded algebra.

3. Proofs of our results:

Denote by $(\Lambda V, d)$ a Sullivan minimal model of $X$. We remind in what follow, some facts about this model to clarify our proofs and about the spectral sequence of a filtered complex, essentially to fix notations and terminology.

3.1. Some facts about $(\Lambda V, d)$.

With notations as in the introduction, assume that $(\Lambda V, d_k)$ is elliptic. So, in one hand, by the convergence of (1), $(\Lambda V, d)$ is so and the two have the same formal dimension $N$ given by (3).

On the other hand, $(\Lambda V, d_k)$ is a Gorenstein algebra implying that the spectral sequence (2) collapse at its first term $E_1^{*,*}(\Lambda V, d_k)(\mathbb{Q}, (\Lambda V, d_k))$. Denote $[h_k]$ any
It is given by the generalized Milnor-Moore spectral sequence (1), that referring again to notations of [12], we will denote \( h_k(1) \) := \( \omega_e \) and \( N := N_e \).

3.2. Spectral sequence terminology of a filtered complex. (see for instance [13] §6):

It should be noted that the first term \( H(\Lambda V, d_k) \) of (1) corresponds effectively to the \( k \)-th term \( E_k^{p,q} \) arising in the construction process of the spectral sequence of the filtered complex \( (\Lambda V, d) \). It is given by the formula:

\[
E_k^{p,q} = Z_k^{p,q}/Z_{k-1}^{p+1,q-1} + B_{k-1}^{p,q},
\]

where

\[
Z_k^{p,q} = \{ x \in F^p(\Lambda V)^{p+q} \mid dx \in [F^{p+k}(\Lambda V)]^{p+q+1} \}
\]

and \( B_k^{p,q} = d([F^{p+k}(\Lambda V)]^{p+q+1}) \cap F^p(\Lambda V) = d(Z_{k-1}^{p+1,q+2}). \)

Recall also that the differential \( \delta_k : E_k^{p,q} \to E_k^{p+q-k,k+1} \) in \( E_k^{*,*} \) is induced from the differential \( d \) in \( (\Lambda V, d) \) by the formula \( \delta_k[v]_k = [dv]_k \), \( v \) being any representative in \( Z_k^{p,q} \) of the class \( [v]_k \) in \( E_k^{p,q} \).

Put \( Z(E_k^{p,q}) := Ker(\delta_k) \) and \( B(E_k^{p,q}) := Im(\delta_k) \). A straightforward argument permit to construct a natural monomorphism

\[
I_k^{p,q} : Z_k^{p,q} + Z_{k+1}^{p+1,q-1}/Z_{k-1}^{p+1,q-1} + dZ_{k-1}^{p-k+1,q+k-2} \to E_k^{p,q}
\]

It is given by \( I_k^{p,q}(\bar{v}_1 + \bar{v}_2) = \bar{v}_1 + \bar{v}_2 \) and it verifies the relation \( \delta_k \circ I_k^{p,q} = 0 \). Thus, \( Im(I_k^{p,q}) \subseteq Z(E_k^{p,q}) \). With a little more analysis, one shows that \( I_k^{p,q} \) is a surjection and then we have the isomorphism:

\[
I_k^{p,q} : Z_{k+1}^{p+1,q-1}/Z_{k-1}^{p+1,q-1} + dZ_{k-1}^{p-k+1,q+k-2} \cong Z(E_k^{p,q})
\]

Analogous arguments give the proof of the isomorphism:

\[
J_k^{p,q} : dZ_{k-1}^{p-k,q+k-1} + Z_{k-1}^{p+1,q-1}/Z_{k-1}^{p+1,q-1} + dZ_{k-1}^{p-k+1,q+k-2} \cong B(E_k^{p,q})
\]

3.3. Proof of Theorem 1.0.4.

Proof. As mentioned in the introduction, there is no gaps in the first term of the generalized Milnor-Moore spectral sequence [11], that is, referring again to notations of [12], for any \( p = 1, \ldots, e \), there exists a non zero class \( [\omega_p] \in H_p^e(\Lambda V, d_k) \).

In the remainder, \( H_p^e(\Lambda V, d_k) \) will be identified with

\[
E_k^{p,q} = (Z_k^{p,q}/Z_{k-1}^{p+1,q-1} + B_{k-1}^{p,q})^{p,q}.
\]
With this identification, we have $[\omega_p] \in E_k^{p,n_p-p} \text{ and } [\omega_{e-p}] \in E_k^{e-p,N_{e-p}-e+p}$. We can then take $\hat{\omega}_p \in Z_k^{p,n_p-p}$ and $\hat{\omega}_{e-p} \in Z_k^{e-p,N_{e-p}-e+p}$.

Thereafter, we denote $[\omega_p] =: \hat{\omega}_p$ and $[\omega_{e-p}] =: \hat{\omega}_{e-p}$. By Theorem 2.2. (C) and Lemma 2.1. in [12] the integers $n_p$ and $N_p$ $(0 \leq p \leq e)$ satisfy the tow equivalent relations:

$$n_2 \geq 2n_1, n_3 \geq n_2 + n_1, \ldots, n_{p+1} \geq n_p + n_1, \ldots, n_e \geq n_{e-1} + n_1$$

and

$$N_e = N_{e-1} + n_1, N_{e-1} \geq N_{e-2} + n_1, \ldots, N_{p+1} \geq N_p + n_1, \ldots, N_1 \geq n_1.$$ 

Now since $H_1(\Lambda V, d_k) \neq 0$, we have $n_1 > 0$. Therefore, $\forall 1 \leq p \leq e-1$,

$$n_p - p \geq n_{p-1} - (p - 1), \quad N_p - p \geq N_{p-1} - (p - 1) \quad \text{and} \quad N_e = n_e.$$ 

Regarding to bi-degrees $(p, n_p - p)$ and $(e - p, N_e - (e - p))$ of $\omega_p$ and $\omega_{e-p}$ respectively, it results that, $\forall p = 1, \ldots, e - 1$, we have necessarily:

a) $\delta_k(\hat{\omega}_p) = 0$, so that $\hat{\omega}_p \in Z(E_k^{p,n_p-p})$, but we don’t know if it is a $\delta_k$-coboundary. So by the isomorphism (7) and due to its homogeneity of length $p$, it is obligatory an element of $Z_k^{p,n_p-p}$. Hence $\hat{\omega}_p \in F_{p+k+1}(\Lambda V)$.

b) $\hat{\omega}_{e-p}$ can’t be a $\delta_k$-coboundary i.e. $\hat{\omega}_{e-p} \notin B(E_k^{e-p,N_{e-p}-e+p})$.

c) $\hat{\omega}_e$ is a $\delta_k$-cocycle that survives to the $\infty$-term $E_{\infty}^{e,N_e-e}$. In particular we have $\omega_e \in Z_{k+1}^{e,N_e-e}$.

Using Remark 2.0.8. and the identification made later, we have $\hat{\omega}_e = \omega_p \otimes \omega_{e-p}$.

Since $\hat{\omega}_e$ survives to the term $E_{k+1}^{*,+}$, we have $\delta_k(\hat{\omega}_e) = \delta_k(\omega_p \otimes \omega_{e-p}) = 0$ and then by homogeneity and the isomorphism (7), $\omega_p \otimes \omega_{e-p} \in Z_{k+1}^{e,N_e-e}$. Hence $d(\omega_e) = d(\omega_p) \otimes \omega_{e-p} \pm \omega_p \otimes d(\omega_{e-p}) \in F_{e+k+1}^{e+p}$. Thus $d(\omega_p) \otimes \omega_{e-p} \in F_{e+k+1}^{e+p}$ imply that $\omega_p \otimes d(\omega_{e-p}) \in F_{e+k+1}^{e+p}$ also. It results that $d(\omega_{e-p}) \in F_{e+k+1}^{e+p}$ and then $\omega_{e-p} \in Z_{k+1}^{e,p,N_{e-p}-e+p}$. Equivalently by (7), we obtain $\delta_k(\hat{\omega}_{e-p}) = 0$. That is, $\hat{\omega}_{e-p}$ is a $\delta_k$-cocycle. Finally, using (a), (b) and (c) below, we conclude that the formula $[\omega_e] = [\omega_p] \otimes [\omega_{e-p}]$ is still valid in $E_{k+1}^{e,N_e-e}$. The proof is completed by recursion. 

\section*{3.4. Proof of Theorem 1.0.5.}

\textbf{Proof.} By Theorem 1.0.4. we have $E_k^{p,n_p-p} \neq 0$ for $p = 1, \ldots, e$. Hence using the convergence of (1) it results that $\text{dim}H_p^{\text{odd}}(\Lambda V, d) \geq 1$ for $p = 1, \ldots, e$, where the lower grading in $H_p^{\text{odd}}(\Lambda V, d)$ comes from that of $H_p^{\text{odd}}(\Lambda V, d_k)$. As $\text{dim}H_0^{\text{odd}}(\Lambda V, d) = 1$, it follows that $\text{dim}H(\Lambda V, d) \geq e = \text{dim}V^{\text{odd}} = (k - 2)\text{dim}V^{\text{even}} \geq \text{dim}V$. Hence, if $V = V^{\text{odd}}$, $(\Lambda V, d_k)$ is always elliptic and $e = \text{dim}V^{(\text{odd})}$, so the inequality holds. Now, if $k = 3$, we have immediately $\text{dim}H(\Lambda V, d) \geq \text{dim}V$. \hfill \qed

\section*{3.5. Remark.}

(1) Let $X$ be as in Theorem 1.0.4. and $(\Lambda V, d)$ its Sullivan minimal model, where $d = \sum_{i \geq 2} d_i$ and $d_2 \neq 0$.

We would like to use Theorem 1.0.4. to answer the $H$-conjecture for $(\Lambda V, d)$, assuming that it is valid for $(\Lambda V, d_2)$. Unfortunately, in this case, we can only deduce from Theorem 1.0.4. that $\text{dim}H(\Lambda V, d) \geq e = \text{dim}V^{\text{odd}}$. Indeed even though we assume that $\text{dim}H_1^\ast(\Lambda V, d_2) \geq 2$, $(\forall 1 \leq p \leq e - 1)$ we can’t be sure that more than one basis element in $H_1^\ast(\Lambda V, d_2)$ survives
to the $E_\infty$-term. This fact is illustrated under the following hypothesis for which the $H$-conjecture for $(\Lambda V, d_2)$ is resolved (see [2] and [8] respectively):

(a) The Quillen model of $(\Lambda V, d_2)$ is nilpotent of degree one or two.

(b) $\ker(d_2 : V^{odd} \to \Lambda V)$ is non zero. That is, if the rational Hurewicz homomorphism is non-zero in some odd degree.

Obviously, in both cases, by Theorem 1.0.4. (when $k = 2$), their cohomologies can’t have $e_0$-gaps. Hence one can ask for a possible relation between the $H$-conjecture, the conjecture 3.4 posed by G. Lupton in [12] and the question 1.0.6 asked by Y. Félix.

(2) The spectral sequence (2) collapses at its first term whenever $\dim(V) < \infty$, due to the fact that in this case $(\Lambda V, d_k)$ is a Gorenstein graded algebra. Nevertheless if $H(\Lambda V, d_k) = \infty$, then $[h_k(1)] = 0$ ([14]). But assuming that $(\Lambda V, d)$ is elliptic, the cocycle $h_k(1)$ gives a non zero class of the $\infty$-term of (1). Thus, a possible method to use to verify the conjecture H in the general case is looking for a certain term in the spectral sequence (1) that is without gaps. This is our project in the future.

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