“Explosive Percolation” Transition is Actually Continuous

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The basic notion of percolation in physics assumes the emergence of a giant connected (percolation) cluster in a large disordered system when the density of connections exceeds some critical value. Until recently, the percolation phase transitions were believed to be continuous, however, in 2009, a remarkably different, discontinuous phase transition was reported in a new so-called “explosive percolation” problem. Each link in this problem is established by a specific optimization process. Here, employing strict analytical arguments and numerical calculations, we find that in fact the “explosive percolation” transition is continuous though with an uniquely small critical exponent of the percolation cluster size. These transitions provide a new class of critical phenomena in irreversible systems and processes.
The modern understanding of disordered systems in statistical and condensed matter physics is essentially based on the notion of percolation \((I)\). When one increases progressively the number of connections between nodes in a network, above some critical number (percolation threshold) a giant connected (percolation) cluster emerges in addition to finite clusters. This percolation cluster contains a finite fraction of nodes and links in a network. The percolation transition was widely believed to be a typical continuous phase transition for various networks architectures and space dimensionalities \((2)\), so it shows standard scaling features, including a power-law size distribution of finite cluster sizes at the percolation threshold. Recently, however, it was reported that a remarkable percolation problem exists in which the percolation cluster emerges discontinuously and already contains a finite fraction of nodes at the percolation threshold \((3)\). This conclusion was based on numerical simulations of a model in which each new connection is made in the following way: choose at random two links that could be added to the network, but add only one of them, namely the link connecting two clusters with the smallest product of their sizes. To emphasize this surprising discontinuity, this kind of percolation was named “explosive” \((3)\). Further investigations of “explosive percolation” in this and similar systems, also mainly based on numerical simulations, supported this strong result but, in addition, surprisingly for abrupt transitions, revealed power-law critical distributions of cluster sizes \((4–10)\) resembling those found in continuous percolation transitions. This self-contradicting combination of discontinuity and scaling have made explosive percolation one of the challenging and urgent issues in the physics of disordered systems.

Here we resolve this confusion. We show that there is not actually any discontinuity at the “explosive percolation” threshold, contrary to the conclusions of the previous investigators. We consider a simple representative model demonstrating this new kind of percolation and show that the “explosive percolation” transition is a continuous, second-order phase transition but, importantly, with an uniquely small critical exponent \(\beta \approx 0.0555\) of the percolation cluster
One of the simplest systems in which classical percolation takes place is as follows. Start with \( N \) unconnected nodes, where \( N \) is large, and at each step add connection between two uniformly randomly chosen nodes. In essence, this is a simple aggregation process (II), in which at each step, a pair of clusters, to which these nodes belong, merge together (Fig. 1A). If we introduce “time” \( t \) as the ratio of the total number of added links in this system, \( L \), and its size \( N \), i.e., \( t = L/N \), then the percolation cluster of relative size \( S \) emerges at the percolation threshold \( t_c = 1/2 \) and grows with \( t \) in the following way: \( S \sim \delta^{\beta} \), where \( \delta = |t - t_c| \) and \( \beta = 1 \). At the critical point, the cluster size distribution (fraction of finite connected components of \( s \) nodes), \( n(s) \), in this classical problem is power-law: \( n(s) \sim s^{-\tau} \) with exponent \( \tau = 5/2 \), see Ref. (I).

In this report, we consider a direct generalization of this process. Namely, at each step, we sample twice:

(i) choose two nodes uniformly at random and compare the clusters to which these nodes belong; select that node of the two ones, which belongs to the smallest cluster;

(ii) choose a second pair of nodes and, again, as in (i), select the node belonging to the smallest of the two clusters;

(iii) add a link between the two selected nodes thus merging the two smallest clusters (Fig. 1B).

Repeat this procedure again and again. Note that in (i) and (ii) a cluster can be selected several times. This is the case for the percolation cluster. These rules contain the key element of other explosive percolation models, e.g., model (3), namely, for merging, select the minimal clusters from a few possibilities. Importantly, our procedure provides even more stringent selection of small components for merging than model (3) since guarantees that the product of the sizes of two merging clusters is the smallest of the four possibilities (each of the first pair of chosen
nodes (i) may connect with any node of the second pair (ii) in contrast to selection from only two possibilities in model (3). Consequently, if we show that the transition in our model is continuous, then model (3) also must have a continuous transition. More generally, in each sampling, one can select at random $m$ nodes thus choosing the minimal cluster from $m$ clusters. Classical percolation corresponds to $m = 1$ (Fig. 1A). In this report we mostly focus on $m = 2$ (Fig. 1B). One should stress that the explosive percolation processes are irreversible in stark contrast to ordinary percolation. In the latter, one can reach any state either adding or removing connections. For explosive percolation, only adding links makes sense, and an inverse process is impossible.

We simulated this irreversible aggregation process for a large system of $2 \times 10^9$ nodes. When plotted over the full time range, the obtained dependence $S(t)$ shows what seems to be discontinuity at the critical point $t_c$ (Fig. 1C) similar to previous results, but a more thorough inspection of the critical region (log-log plot in Fig. 1D) reveals that the obtained data is definitely better fitted by the law $a\delta^\beta$, which indicates a continuous transition, than, say, by $0.3 + b\delta^\beta$. Fitting the data of our simulation by the law $S_0 + b\delta^\beta$, we find that $S_0$ is at least smaller than 0.05. This shows that for a definite conclusion, even so large a system turns out to be not sufficient, and a discontinuity can be ruled out or validated only by analytical arguments for the infinite size limit.

We address this problem analytically and numerically by considering the evolution of the size distribution $P(s)$ for a finite cluster of $s$ nodes to which a randomly chosen node belongs: $P(s) = sn(s)/\langle s \rangle$, where $\langle s \rangle$ is the average cluster size (the ratio of the number of nodes and the total number of clusters). This distribution satisfies the sum rule $\sum_s P(s) = 1 - S$. Another important characteristic in this model is the probability $Q(s)$ that if we choose at random two nodes than the smallest of the two clusters to which these nodes belong is of size $s$. $Q(s)$ provides us with the size distribution of merging clusters. Here $\sum_s Q(s) = 1 - S^2$. If we
introduce the cumulative distributions $P_{\text{cum}}(s) \equiv \sum_{u=s}^{\infty} P(u)$ and $Q_{\text{cum}} \equiv \sum_{u=s}^{\infty} Q(u)$, then probability theory gives $Q_{\text{cum}}(s) + S^2 = [P_{\text{cum}}(s) + S]^2$. Consequently

$$Q(s) = [P_{\text{cum}}(s) + P_{\text{cum}}(s+1) + 2S]P(s) = [2 - 2P(1) - \ldots - 2P(s-1) - P(s)]P(s), \quad (1)$$

that is $Q(s)$ is determined by $P(s')$ with $s' \leq s$. The evolution of these distributions in the infinite system is exactly described by the master equation:

$$\frac{\partial P(s, t)}{\partial t} = s \sum_{u+v=s} Q(u, t)Q(v, t) - 2sQ(s, t), \quad (2)$$

which generalizes the standard Smoluchowski equation. We derived Eq. (2) only assuming the infinite system size. The only difference from the classical percolation problem (II) is the presence of the distribution $Q(s, t)$ instead of $P(s, t)$ on the right-hand side of this equation. Thus we have a chain of coupled equations, which should be solved analytically or numerically. To find a numerical solution, first solve the first equation of the chain, which gives $P(1, t)$. Substitute this result into the second equation and solve it, which gives $P(2, t)$, and so on. In this way we find numerically the distributions $P(s, t)$ and $Q(s, t)$ at any $t$ for infinite $N$. Solving $10^6$ equations gives the evolution of these distributions for $1 \leq s \leq 10^6$ and $S(t) \approx 1 - \sum_{s=1}^{10^6} P(s, t)$.

The log-log plot (Fig. 2A) shows that the obtained $S(t)$ dependence is well described by the power law: $S \propto \delta^\beta$ with $\delta = t - t_c$, $t_c = 0.923207508(2)$, and small $\beta = 0.0555(1)$ (which is very close if not equal to $1/18 = 0.05555\ldots$). Here we find $t_c$ as the point at which $P(s)$ is power-law over the full range of $s$, see below. To check the correctness and precision of our calculations, we repeated them for ordinary percolation and obtained the classical results with the same precision as for our model. Although the small exponent $\beta$ makes it difficult to approach the narrow region of small $S$, fitting this data by the law $S_0 + b\delta^\beta$, we find that $S_0$ is smaller than 0.005. This supports our hypothesis that the transition is continuous, but still does not prove it. Moreover, both our extensive simulations and numerical solution results clearly demonstrate that the analysis of the $S(t)$ data cannot validate or rule out a discontinuity.
Figure 2B shows the evolution of the distribution $P(s,t)$, which we compare with the evolution of this distribution for ordinary percolation, Fig. 2C. The difference is strong at $t < t_c$, where the distribution for explosive percolation has a bump, but above $t_c$ the behaviors are similar. The distribution function $Q(s,t)$ evolves similarly to $P(s,t)$ in the full time range. At the critical point, we find power-law $P(s) \sim s^{1-\tau}$ and $Q(s) \sim s^{3-2\tau}$ in the full range of $s$ (six orders of magnitude), where $\tau = 2.04762(2)$, which is close to 2, as in $(7,8,10)$, in contrast to $\tau = 5/2$ for classical percolation. The first moments of these distributions, $\langle s \rangle_P \equiv \sum_s sP(s)$ (the mean size of a finite cluster to which a randomly chosen node belong) and $\langle s \rangle_Q \equiv \sum_s sQ(s)$, demonstrate power-law critical singularities $\langle s \rangle_P \sim |\delta|^{-\gamma_P}$ and $\langle s \rangle_Q \sim |\delta|^{-\gamma_Q}$, where exponents $\gamma_P = 1.0111(1)$ and $\gamma_Q = 1.0556(5)$ both below and above the transition. Note that $\gamma_P > 1$ in contrast to ordinary percolation, where the mean-field value of exponent $\gamma$ is 1. Figure 2D shows the set of time dependencies of $P(s,t)$ for fixed cluster sizes (the time dependencies of $Q(s,t)$ are similar). These dependencies strongly differ from those for ordinary percolation (Fig. 2E) in the following respect. The peaks in Fig. 2D for explosive percolation are below $t_c$, while the peaks in Fig. 2E for ordinary percolation are symmetrical with respect to the critical point at large $s$.

The inspection of these numerical results in the critical region ($t < t_c$) reveals a scaling behavior typical for continuous phase transitions, $P(s,t) = s^{1-\tau}f(s\delta^{1/\sigma})$ and $Q(s,t) = s^{3-2\tau}g(s\delta^{1/\sigma})$, respectively, where $f(x)$ and $g(x)$ are scaling functions, and $\sigma = (\tau - 2)/\beta$, which is a standard scaling relation. One should stress that these functions are quite unusual. Figure 3 shows the resulting scaling functions and, for comparison, the scaling function for ordinary percolation. Remarkably, $f(x)$ and, especially, $g(x)$ are well fitted by Gaussian functions. These functions differ dramatically from the monotonously decaying exact scaling function $e^{-x}/\sqrt{2x}$ for ordinary percolation. Effective elimination of the smallest clusters in this merging process results in the minima of the scaling functions at $x=0$. On the other hand, the stunted
emergence of large clusters results in the particularly rapid decay of these functions at $x \gg 1$.

The key point of our report is the following strict analytical derivation. We start from the fact that in this problem the cluster size distributions are power-law at the percolation threshold. We observed these power-laws over 6 orders of magnitude, and they were observed in works (7,8,10) for explosive percolation though in less wide range of $s$. Now we strictly show that if the cluster size distribution is power-law at the critical point, then this phase transition is continuous. We use the fact that in the critical region above the percolation threshold, the distributions $Q(s)$ and $P(s)$ at large $s$ are proportional to each other, namely $Q(s) \approx 2SP(s)$, see Eq. (1). This relation crucially simplifies our problem, since the resulting evolution equation for the asymptotics of the distribution in this region contains only $P(s)$ and the relative size $S(t)$ of the percolation cluster. As a result, Eq. (2) becomes very similar to that for ordinary percolation (the only difference is the presence of $S(t)$ terms on the right-hand side), and so it can be easily analysed explicitly in the same way as for ordinary percolation (11). In this way, using a power-law critical distribution $P(s) \sim s^{1-\tau}$ as an initial condition for this equation at $t = t_c$, we find that the behavior of the distribution $P(s)$ and $S(t)$ in explosive percolation above the percolation threshold is qualitatively similar to that for ordinary percolation. Specifically, we show that the percolation cluster emerges continuously, and $S \propto \delta^{\beta}$, where $\tau = 2 + \beta/(1 + 3\beta)$ and so $\sigma = 1/(1 + 3\beta)$. The obtained numerical values of exponents $\tau$, $\beta$, and $\sigma$ agree with these two scaling relations and thus also confirm the continuous transition. So the results of this report are self-consistent. Our results are summarized in Table 1. Assuming a scaling form for the distributions gives $\gamma_P = 1 + 2\beta$ and $\gamma_Q = 1 + \beta$, which agree with our numerical solution of Eq. (2). Furthermore, applying standard scaling relations (1), we calculate the fractal dimension for this model, $d_f = 2/\sigma = 2(1 + 3\beta)$, and the upper critical dimension, $d_u = d_f + 2\beta = 2(1 + 4\beta)$ (see Table 1). The latter determines the finite size effect: $t_c(\infty) - t_c(N) \propto N^{-2/d_u}$, where $2/d_u = 0.818(1)$. Interestingly, the obtained fractal and upper critical dimensions for
explosive percolation are less than 3. They are much smaller than those for ordinary percolation, which are 4 and 6, respectively. Our model is infinite-dimensional, that is above the upper critical dimension, where mean-field theories must work, which makes the observed smallness of exponent $\beta$ particularly remarkable. We know no other model in statistical physics with a small $\beta$ in such a situation.

In the general case, each of the samplings in the process involves $m \geq 1$ nodes, and so the minimal cluster is selected from the $m$ possibilities. In this case, the relation between between the distributions $Q(s)$ and $P(s)$ is substituted by a more general one, but the form of the evolution equation (2) does not change. We find that with increasing $m$, $t_c$ approaches 1 and $\beta \simeq \tau - 2 \sim m^{-1} e^{-m}$, that is, $\beta$ rapidly decreases with $m$, but the transition remains continuous. Thus the critical exponents depend on $m$, and so the “explosive percolation” transitions have no universal critical behavior.

We have shown that the “explosive percolation” transition is actually continuous. It is the smallness of the $\beta$ exponent for the size of the percolation cluster that makes it virtually impossible to distinguish this phase transition from a discontinuous one even in very large systems. Indeed, suppose that $N = 10^{18}$ and $\beta \approx 1/18$. The addition of one link changes $t$ by $\Delta t = 1/N$, which is the smallest time interval in the problem. Then a single step $\Delta t = 10^{-18}$ from the percolation threshold already gives $S \sim (\Delta t)^\beta \sim 0.1$. Other critical exponents and dimensions also differ radically from classical values. Furthermore, we have derived a complete set of scaling relations between the critical exponents for this problem, which were also supported by our numerical results. The real absence of explosion topples an already established view of explosive percolation. We believe, however, that, thanks to the observed unique properties of this phase transition, our findings make this new class of irreversible systems an even more appealing subject for further extensive exploration.
References and Notes

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Table 1: Threshold values, critical exponents, and fractal and upper critical dimensions for ordinary percolation and explosive \((m = 2)\) one. Critical exponents for explosive percolation are expressed in terms of exponent \(\beta\): 
\[
\begin{align*}
\tau &= 1 + \beta/(1 + 3\beta), \\
\sigma &= 1/(1 + 3\beta), \\
\gamma_P &= (3 - \tau)/\sigma = 1 + 2\beta, \\
\gamma_Q &= (5 - 2\tau)/\sigma = 1 + \beta, \\
d_f &= 2(1 + 3\beta), \\
d_u &= 2(1 + 4\beta).
\end{align*}
\]

|          | \(t_c\) | \(\beta\) | \(\tau\) | \(\sigma\) | \(\gamma_P\) | \(\gamma_Q\) | \(d_f\) | \(d_u\) |
|----------|---------|-----------|-----------|-------------|---------------|---------------|--------|--------|
| Ordinary | 1/2     | 1         | 5/2       | 1/2         | 1             | —             | 4      | 6      |
| Explosive| 0.923207508(2) | 0.0555(1) | 2.04762(2) | 0.857(3)    | 1.111(1)      | 1.0556(5)     | 2.333(1) | 2.445(1) |
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Supporting Online Material

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4. Relation between the critical time $t_c$ and exponent $\tau$
Here we compare models of explosive percolation and present some details of our analytical calculations.

1 Comparison between explosive percolation models

Let us show that our model provides more efficient merging of small clusters than the model (3). So if the explosive percolation transition is continuous in our model, then the model (3) also has a continuous transition.

Recall the selection rule in our model. At each step, choose two pairs of nodes, \( i \) and \( j \), \( k \) and \( l \), uniformly at random. Let they belong to clusters of sizes \( s_i \) and \( s_j \), \( s_k \) and \( s_l \), respectively. Connect the smallest cluster of the pair \( s_i \) and \( s_j \), with the smallest cluster of the pair \( s_k \), and \( s_l \). One can immediately see that this rule is equivalent to the following one. Let \( f(s, s') \) be an arbitrary monotonously growing function of its arguments, e.g., \( f(s, s') = ss' \). Consider four possibilities to add a new link, \( ik \), \( il \), \( jk \), and \( jl \), and choose that one which provides the smallest value of \( f(s_i, s_k) \), \( f(s_i, s_l) \), \( f(s_j, s_k) \), and \( f(s_j, s_l) \), see Fig. 4A in the Supporting online material.

In the model (3), the product selection rule is as follows. At each step choose nodes \( i, j, k \) and \( l \) uniformly at random, and select from two possibilities, either connect \( i \) and \( k \) or connect \( j \) and \( l \), choosing the pair with the smallest of the products \( s_i s_k \) and \( s_j s_l \), see Fig. 1B.

Then the only difference between our model and the model (3) is that we select from four possibilities (comparison of \( s_i s_k \), \( s_i s_l \), \( s_j s_k \) and \( s_j s_l \)) while in the model (3) the selection is from only two possibilities (comparison of \( s_i s_k \) and \( s_j s_l \)). Therefore our choice guarantees merging of clusters with the product of sizes \( ss' \) which is equal or smaller than in the model (3). So our process should generate discontinuity even more efficiently than in the model (3). Consequently, if our model has a continuous phase transition, that the model (3) also must have a continuous transition.
2 Analysis of the evolution equation above the explosive percolation transition

Let us consider the master equation for our model

$$\frac{\partial P(s, t)}{\partial t} = s \sum_{u + v = s} Q(u, t)Q(v, t) - 2sQ(s, t), \quad (3)$$

where

$$Q(s) = [P_{\text{cum}}(s) + P_{\text{cum}}(s + 1) + 2S]P(s) = [2 - 2P(1) - \ldots - 2P(s - 1) - P(s)]P(s). \quad (4)$$

(Here we only consider the case of \( m = 2 \).) We introduce generating functions for the distributions \( P(s) \) and \( Q(s) \),

$$\rho(z) \equiv \sum_{s=1}^{\infty} P(s)z^s \quad (5)$$

and

$$\sigma(z) \equiv \sum_{s=1}^{\infty} Q(s)z^s. \quad (6)$$

Note that \( \rho(z = 1) = 1 - S \) and \( \sigma(z = 1) = 1 - S^2 \), where \( S \) is the relative size of the percolation cluster. Using these generating functions for the distributions we represent the master equation \( (3) \) in the following form:

$$\frac{\partial}{\partial t}[1 - \rho(z, t)] = -1 \frac{\partial}{\partial \ln z}[1 - \sigma(z, t)]^2. \quad (7)$$

In ordinary percolation, \( \sigma(z, t) \) in this equation is substituted by \( \rho(z, t) \), namely

$$\frac{\partial \rho(z, t)}{\partial t} = 2[\rho(z, t) - 1] \frac{\partial \rho(z, t)}{\partial \ln z}. \quad (8)$$

In the critical region above the percolation threshold, Eq. \( (4) \) gives

$$Q(s) \approx 2SP(s) \quad (9)$$
asymptotically at large $s$, which leads to a simple relation between the generating functions $\sigma(z)$ and $\rho(z)$ in the range $z$ close to 1. Indeed,

$$1 - S^2 - \sigma(z) = \sum_s Q(s)[1 - z^s] \cong \sum_s 2SP(s)[1 - z^s] = 2S[1 - S - \rho(z)],$$

so

$$1 - \sigma(z) \cong 2S[1 - \rho(z) - S/2]$$

if $z$ is close to 1 in the critical region at $t > t_c$. One can check this relation at $z = 1$, namely,

$$1 - \sigma(1) = S^2 = 2S[1 - \rho(1) - S/2] = 2S(S - S/2).$$

Therefore, in the critical region above the percolation threshold, at $z$ close to 1, the master equation takes a convenient form,

$$\frac{\partial \rho(z,t)}{\partial t} = 8S^2(t)[\rho(z,t) - 1 + S(t)/2] \frac{\partial \rho(z,t)}{\partial \ln z}.$$  

Note that this equation essentially differs from Eq. (8) for ordinary percolation because of the terms $S(t)$ on the right-hand side. Nonetheless Eq. (13) can be analysed in the same way as for ordinary percolation.

Our numerical solution of Eq. (3) showed convincingly that at the critical point, the distribution $P(s, t_c)$ is power-law, namely, at large $s$, $P(s, t_c) \cong f(0)s^{1-\tau}$. Here $f(0)$ is the critical amplitude for this distribution. This is also the value of the scaling function for this distribution, $f(x = 0)$, see below. Let us show that, if at the critical point $P(s, t_c) \propto s^{1-\tau}$, then Eq. (13) has a solution with $1 - \rho(z = 1, t) = S(t) \propto (t - t_c)^\beta$ in the critical region, which just means that the transition is continuous. The existence of this solution can be demonstrated in the following way. Let us assume that $P(s, t_c) \cong f(0)s^{1-\tau}$ at large $s$, and $S \cong B(t - t_c)^\beta$ at small $t - t_c$ and check whether this assumption is correct or not. Here we assume that $f(0)$ and the exponent $\tau$ are known (numerical solution gave $f(0) = 0.04618(2)$ and $\tau = 2.04762(2)$), while $B$ and the exponent $\beta$ are to be found.
We use the power-law asymptotics of the distribution \( P(s, t_c) \cong f(0)s^{1-\tau} \) as the initial condition for Eq. (13). This corresponds to the following singularity of the generating function at \( z = 1 \):

\[
1 - \rho(z, t_c) = \text{analytic terms} - f(0)\Gamma(2 - \tau)(1 - z)^{\tau - 2}. \tag{14}
\]

Introducing \( \epsilon \equiv (t - t_c)^{2\beta + 1} \) and \( x \equiv \ln z \), we transform Eq. (13) to the following form:

\[
\frac{\partial \rho}{\partial \epsilon} = \frac{8B^2}{1 + 2\beta} \left( \rho - 1 + \frac{B}{2} \epsilon^{\beta/(1 + 2\beta)} \right) \frac{\partial \rho}{\partial x}. \tag{15}
\]

To solve this equation, we use the hodograph transformation approach. We pass from \( \rho = \rho(x, \epsilon) \) to \( x = x(\rho, \epsilon) \), which leads to a simple linear partial differential equation for \( x(\rho, \epsilon) \) and enables us to find the general solution

\[
\ln z = \frac{8B^2}{1 + 2\beta} \left[ 1 - \rho - \frac{B}{2} \epsilon^{\beta/(1 + 2\beta)} \right] \epsilon + F(\rho), \tag{16}
\]

where the function \( F(\rho) \) is obtained from the initial condition (14), which gives the solution

\[
\ln z = \frac{8B^2}{1 + 2\beta} \left[ 1 - \rho - \frac{B}{2} \frac{(t - t_c)^\beta}{1 + \beta/(1 + 2\beta)} \right] (t - t_c)^{1 + 2\beta} - \left[ f(0) \right]^{-1/(\tau - 2)} |\Gamma(2 - \tau)|^{-1/(\tau - 2)} [1 - \rho]^{1/(\tau - 2)}. \tag{17}
\]

We set \( z = 1 \), and taking into account \( 1 - \rho(1) = S = B(t - t_c)^\beta \) and comparing resulting powers and coefficients in Eq. (17), we obtain a relation between the critical exponents,

\[
\tau = 2 + \frac{\beta}{1 + 3\beta}, \tag{18}
\]

and express the critical amplitude \( B \) for the relative size of the percolation cluster in terms of the critical amplitude \( f(0) \) for the distribution \( P(s, t_c) \),

\[
B = \left[ \frac{4(\tau - 1)(7 - 3\tau)}{3 - \tau} \right]^{(\tau - 2)/(7 - 3\tau)} \left[ f(0) \right]^{1/(7 - 3\tau)} |\Gamma(2 - \tau)|^{1/(7 - 3\tau)}. \tag{19}
\]

The relation (18) precisely agrees with our numerical results, \( \tau = 2.04762(2) \) and \( \beta = 0.0555(1) \). Substituting \( f(0) = 0.04618(2) \) (our numerical result) into Eq. (19) gives \( B = 1.075 \), which agrees with the corresponding value 1.080 obtained by solving the muster equation numerically.
Furthermore, the power-law distribution at the critical point can be justified strictly by using an equation for scaling functions. In the normal phase \((t < t_c)\), we derived an equation for the scaling functions in this problem. This is a nonlinear integral-differential eigenfunction equation, where eigenfunctions are the scaling functions for \(P(s, t)\) and \(Q(s, t)\), see Eq. (20), and a critical exponent, e.g., \(\tau\), plays the role of an eigenvalue. This equation can be solved numerically, which is, however, a difficult task. We verified that the scaling functions and \(\tau\), which we found numerically by solving Eq. (3), satisfy this equation. This shows that the distributions are power-law at the critical point.

3 Relations between critical exponents

Here we present a list of relations for critical exponents for explosive percolation \((m = 2)\).

In the critical region, the distributions \(P(s, t)\) and \(Q(s, t)\) have a scaling form:

\[
P(s, t) = s^{1-\tau} f(s^{\delta^{1/\sigma}}) \\
Q(s, t) = s^{3-2\tau} g(s^{\delta^{1/\sigma}}),
\]

(20)

where \(\delta = |t - t_c|\). Note that the critical exponents below and above the transition are equal, while the scaling functions below \(t_c\) differ dramatically from those above the transition. The exponent \(\beta\) of the size of the percolation cluster is expressed in terms of \(\tau\) and \(\sigma\) as follows:

\[
\beta = \frac{\tau - 2}{\sigma}.
\]

(21)

For the first moments of the distributions \(P(s)\) and \(Q(s)\), critical exponents are \(\gamma_P\) and \(\gamma_Q\), respectively. Namely, \(\langle s \rangle_P \propto |\delta|^{-\gamma_P}\) and \(\langle s \rangle_Q \propto |\delta|^{-\gamma_Q}\). These exponents are expressed in terms of \(\tau\) and \(\sigma\) as follows:

\[
\gamma_P = \frac{3 - \tau}{\sigma},
\]

(22)\n
\[
\gamma_Q = \frac{5 - 2\tau}{\sigma}.
\]

(23)
Note the relation between $\gamma_P$ and $\gamma_Q$:

$$\gamma_P + 1 = 2\gamma_Q.$$  \hfill (24)

Finally, we give the full set of critical exponents, the fractal dimension $d_f = 2/\sigma$ and the upper critical dimension $d_u = d_f + 2\beta$ in terms of the exponent $\beta$ in the case of $m = 2$:

$$\tau = 2 + \frac{\beta}{1 + 3\beta},$$  \hfill (25)

$$\sigma = \frac{1}{1 + 3\beta},$$  \hfill (26)

$$\gamma_P = 1 + 2\beta,$$  \hfill (27)

$$\gamma_Q = 1 + \beta,$$  \hfill (28)

$$d_f = 2(1 + 3\beta),$$  \hfill (29)

$$d_u = 2(1 + 4\beta).$$  \hfill (30)

### 4 Relation between the critical time $t_c$ and exponent $\tau$

We can obtain approximate relations between $t_c$ and $\tau$ or between the critical amplitude $f(0)$ and $\tau$ by applying the sum rule $\sum_s P(s) = 1$ at the critical point. Two estimates are possible. One can estimate $P(s, t_c)$ by its asymptotics $f(0)s^{1-\tau}$, which gives

$$f(0)\zeta(\tau - 1) \approx 1,$$  \hfill (31)

where $\zeta(x) = \sum_{s=1}^{\infty} s^{-x}$ is the Riemann zeta function. If $\tau$ is close to 2, then $\zeta(\tau - 1) \approx 1/(\tau - 2)$, so we have $\tau - 2 \approx f(0)$. This estimate shows that the small values of $\tau - 2$ and $f(0)$ are interrelated. Recall that we obtained numerically $\tau = 2.04762(2)$ and $f(0) = 0.04618(2)$.

In the second estimate we use the following approximation: $P(s, t_c) \approx P(1, t_c)s^{1-\tau}$. We find $P(1, t)$ explicitly by solving the master equation (3), which gives

$$P(1, t) = \frac{2}{1 + e^{4\tau}},$$  \hfill (32)
so we have

\[
\frac{2}{1 + e^{4t_c}} \zeta(\tau - 1) \approx 1. \tag{33}
\]

Using \( t_c = 0.923207508(2) \) obtained numerically, we find approximately \( \tau - 2 \approx 0.05 \), that is, the exponent \( \tau \) is close to 2 when \( t_c \) is close to 1.
Figure 1: Percolation processes. (A) In classical percolation, at each step, two randomly chosen nodes are connected by a new link. If these nodes belong to different clusters, these clusters merge. (B) In the “explosive percolation” model, at each step, two pairs of nodes are chosen at random, and for each of the pairs, the node belonging to the minimal cluster is chosen. These two nodes (and so their clusters) are connected by a new link. (C) The relative size of the percolation cluster $S$ versus $t$ (the ration of the number of links $L$ and nodes $N$) obtained from the simulation of our model with $N = 2 \times 10^9$ nodes (1000 runs). The starting configuration is $N$ bare nodes. For comparison, the corresponding result for classical percolation is shown. (D) Log-log plot $S$ versus $t - t_c$. The data $S$ of our simulation (black curve) is better fitted by the $a(t - t_c)^\beta$ law (dashed blue curve) than by the $S_0 + b(t - t_c)^\beta$ law with $S_0 = 0.31$. In the latter case, $\ln(S - S_0)$ (red curve) does not follow a linear law (dash-dotted line).
Figure 2: Numerical solutions of the evolution equations (2) for the infinite system. (A) Log-log plot $S$ vs. $t - t_c$. The slope of the dashed line is 0.0555. (B) The evolution of the distribution $P(s)$ below (black lines) and above (red dashed lines) the percolation threshold for explosive percolation. The distribution at the critical point is show by the blue line. (C) The evolution of $P(s)$ for ordinary (classical) percolation. (D) Dependence of $P(s, t)$ on $t$ for a set of cluster sizes $s$ for explosive percolation. Numbers on curves indicate $s$. (E) $P(s, t)$ versus for normal percolation. The insets show the $P(s, t)$ curves for large values of $s$. 
Figure 3: Scaling functions $f(x)$ and $g(x)$ for explosive percolation ($t < t_c$) and, for comparison, the exact scaling function $f_{CP}(x) = e^{-2x}/\sqrt{2\pi}$ for ordinary (classical) percolation. The blue dashed lines show the Gaussian fittings of the explosive percolation scaling functions.

Figure 4: Comparison of the linking rules in our model (A) and in the model (J) (B).