Duality equivalence between nonlinear self-dual and topologically massive models

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(October 30, 2018)

Abstract

In this report we study the dual equivalence between the generalized self-dual (SD) and topologically massive (TM) models. To this end we linearize the model using an auxiliary field and apply a gauge embedding procedure to construct a gauge equivalent model. We clearly show that, under the above conditions, a nonlinear SD model always has a duality equivalent TM action. The general result obtained is then particularized for a number of examples, including the Born-Infeld-Chern-Simons (BICS) model recently discussed in the literature.

I. INTRODUCTION

In this report we are interested in the duality equivalence between models which are apparently different but nevertheless describe the same physical phenomenon, keeping invariant some properties such as the number of degrees of freedom, propagator and equations of motion. The paradigm of this equivalence is the well known duality between the SD...
models in 2+1 dimensions. This is made possible by the introduction of the topological and gauge invariant Chern-Simons term (CST), also responsible for essential features manifested by three-dimensional field theories, such as parity breaking and anomalous spin.

The investigation of duality equivalence in three dimensions involving CST has had a long and fruitful history, beginning when Deser and Jackiw used the master action concept to prove the dynamical equivalence between the SD and MCS theories, in this way proving the existence of a hidden symmetry in the SD version. This approach has been extensively used thereafter, providing an invaluable tool in the study of the planar physics phenomena and in the extension of the bosonization program from two to three dimensions with important phenomenological consequences.

Led by this well-known equivalence, we ask ourselves if these dualities can be extended in an arbitrary way, i.e., given a “general” nonlinear self dual model (NSD), what is its corresponding MCS-like dual equivalent? To answer this we use the auxiliary field technique to linearize the NSD model in terms of the $A^2 = A_\mu A^\mu$ argument and employ an iterative embedding procedure to construct a gauge invariant theory out of the non-linear SD which leads to a general MCS model. As it is appropriate for a gauge embedding procedure, it produces changes in the nature of the constraints of the SD theory. However, instead of focusing on the constraints, we iteratively introduce counter-terms built with powers of the SD Euler vector. Clearly, the resulting theory is on-shell equivalent with the original nonlinear SD model but, by construction, the result is gauge invariant. Basically this involves disclosing, in the language of constraints, hidden gauge symmetries in such systems. The nonlinear SD can be considered as the gauge fixed version of the gauge theory. The latter reverts to the former under certain gauge fixing conditions, thus obtaining a deeper and more illuminating interpretation of these systems. The associated gauge theory is therefore to be considered as the “gauge embedded” version of the original second-class theory. The advantage in having a gauge theory lies in the fact that the underlying gauge symmetry allows us to establish a chain of equivalence among different models by choosing different
gauge fixing conditions.

This paper is organized as follows. In the next section we start with a discussion of the linearization procedure that allows us to reduce the NSD model into an ordinary SD model plus a function of an auxiliary field. After that the dual transformation is performed and the final effective theory is finally obtained after the removal of the auxiliary field. Some examples are discussed in Section III and in Section IV we extend the present formalism to include coupling with fermionic matter. Our results are discussed in the last section.

II. GAUGE EMBEDDING

To derive our results we will consider the following nonlinear generalization of the Townsend-Pilch-Nieuwenhuizen SD model,

\[ L_{NSD} = g(A^2) - j \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \]  

where \( m \) is a coupling constant playing the role of a mass parameter, \( \chi \) is the chirality signal, assuming the values \( \chi = \pm 1 \), and \( g \) is a generic function of the model’s basic field \( A_\mu \). Note that \( g \) depends explicitly on \( A^2 = A_\mu A^\mu \) only which, together with the existence of a linear representation (10) below, are the only restrictions we will put on this function.

It is useful to briefly clarify some properties exhibited by this model. The equations of motion derived from Eq.(1) are given by

\[ A_\mu = \frac{\chi}{2 g^\prime} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda, \]

where the prime denotes a derivative with respect to the \( A^2 \) argument. From these equations the following two relations can be verified,

\[ \partial_\mu A_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu \left( \frac{\chi}{2 g^\prime} \right) \partial^\nu A^\lambda, \]

and

\[ \left( \Box + 4 g^\prime 2 \right) A_\mu = \partial_\mu (\partial_\nu A^\nu) + g^\prime \partial^\nu \left( \frac{1}{g^\prime} \right) F_{\mu\nu}, \]
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that unless we have the linear SD model, $g(A^2) = \frac{m^2}{2} A^2$, the nonlinear SD model defined by Eq.(1) in general does not propagate a transverse massive mode. However, the nonlinear SD model possess a well-defined self-dual property in the same manner as in its linear counterpart. This can be seen as follows. Define a field dual to $A_\mu$ as

$$^{*}A_\mu \equiv \frac{1}{2 g'} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda, \quad (5)$$

and repeat this dual operation to find that

$$^{*} (^{*}A_\mu) = \frac{1}{2 g'} \epsilon_{\mu\nu\lambda} \partial^\nu \left( \frac{1}{2 g'} \epsilon^{\lambda\alpha\beta} \partial_\alpha A_\beta \right), \quad (6)$$

which can be rewritten as

$$^{*} (^{*}A_\mu) = \frac{1}{4 g'^2} \left[ \partial_\mu (\partial_\nu A^\nu) + 2 g' \partial^\nu \left( \frac{1}{2 g'} \right) F_{\mu\nu} - \Box A_\mu \right]. \quad (7)$$

Exploiting Eq.(4) we have

$$^{*} (^{*}A_\mu) = A_\mu. \quad (8)$$

thereby validating the definition of the dual field. Combining these results with Eq.(2), we conclude that

$$^{*}A_\mu = \chi A_\mu, \quad (9)$$

hence, depending on the signature of $\chi$, the theory will correspond to a self-dual or an anti-self-dual model, irrespective of the particular form assumed by the function $g(A^2)$.

Next, let us deal with the nonlinear term. In order to take the nonlinearity of the NSD model (1) into account, within the gauge embedding procedure, we assume that the $g(A^2)$ term possess a linear representation, in terms of an ancillary field $\lambda$ (basically a Legendre transformation), such that

$$g(A^2) \rightarrow \frac{A^2}{\lambda} + f(\lambda), \quad (10)$$
in the Lagrangian, with \( f(\lambda) \) being an auxiliary function to be determined in the case by case basis. By writing the nonlinear SD model in this form we have actually encapsulated all its former nonlinearity upon the field \( \lambda \) and all we have now is a standard SD model, in terms of the basic field \( A_\mu \),

\[
\mathcal{L}_\lambda = \frac{A^2}{\lambda} + f(\lambda) - \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda ,
\]  

(11)

with a field dependent mass parameter. The crucial point is, of course, how to find an appropriate \( f(\lambda) \) for which Eq.(10) holds true. To this end we find the variational solution \( \bar{\lambda} \) of the Lagrangian (10) for the auxiliary field \( \lambda \),

\[
\left[ f'(\lambda) - \frac{A^2}{\lambda^2} \right]_{\lambda=\bar{\lambda}} = 0 ,
\]

(12)

that can be integrated as

\[
f(\lambda) = \int^{\lambda} d\sigma \frac{1}{\sigma^2} A^2(\sigma) .
\]

(13)

where we have relabeled \( \bar{\lambda} \to \lambda \).

The next step is to find the relation of the basic field \( A_\mu \) with the auxiliary field \( \lambda \) by an inverse Legendre transform. We then find, from Eq.(10)

\[
\left[ g'(A^2) - \frac{1}{\lambda} \right]_{A^2=\bar{A}^2} = 0 ,
\]

(14)

and define, formally, a new function \( h(A^2) \equiv g'(A^2) \) such that its inverse produces the desired relation upon use of Eq.(14),

\[
A^2(\lambda) = h^{-1}\left( \frac{1}{\lambda} \right) .
\]

(15)

Bringing this result in Eq.(13), we have

\[
f(\lambda) = \int^{\lambda} d\sigma \frac{1}{\sigma^2} h^{-1}\left( \frac{1}{\sigma} \right) ,
\]

(16)

less an integration constant which is of no consequence for the equations of motion.

Once the linear representation is found, we may return to the discussion of duality equivalence. Turning back to the Eq.(11), our solution then may follow from the iterative
embedding procedure \cite{7,8} or any other approach such as the master action of Deser and Jackiw \cite{5}. Since we are interested also in the coupling with dynamical matter, here we follow Refs. \cite{7,8} where the basic idea is to modify the original SD model (11) with counter-terms built with powers of the Euler vector of the SD model which automatically guarantees the on-shell equivalence. Besides, we look for the special form of the counter-terms that allows one to lift a global symmetry of the SD model into its local form as,

\[ A_\mu \rightarrow A_\mu + \partial_\mu \epsilon, \]  

(17)

with the lift of the global parameter \( \epsilon \). To this end we compute the Euler vector

\[ K^\mu = \frac{2}{\lambda} A_\mu - \chi m \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda, \]

(18)

and treat it as a global Noether charge, bringing it back into the SD action, with the help of an auxiliary gauge field \( B_\mu \) and define a first-iterated Lagrangian as

\[ \mathcal{L}^{(1)} = \mathcal{L}_\lambda - B_\mu K^\mu. \]

(19)

To cancel the variation of the original action \( \mathcal{L}_\lambda \) it is convenient to choose the transformation property of the auxiliary field \( B_\mu \) such that

\[ \delta B_\mu = \delta A_\mu = \partial_\mu \epsilon. \]

(20)

Under this transformation the action (19) changes as

\[ \delta \mathcal{L}^{(1)} = - \delta \left( \frac{1}{\lambda} B_\mu B^\mu \right) + m B_\mu \epsilon^{\mu\nu\lambda} \partial \delta A_\lambda. \]

(21)

Under the vector gauge transformations Eq.(20) the second term in the r.h.s. of Eq.(21) vanishes identically leading to a second iterated Lagrangian

\[ \mathcal{L}^{(2)} = \mathcal{L}^{(1)} + \frac{1}{\lambda} B_\mu B^\mu, \]

(22)

that is gauge invariant under the combined action of \( A_\mu \) and \( B_\mu \), Eq.(21). We have therefore succeeded in transforming the global SD theory (11) into a locally invariant gauge theory.
We may now take advantage of the Gaussian character of the auxiliary field $B_\mu$ to rewrite Eq. (22) as an effective action depending only on the fields $A_\mu$ and $\lambda$,

$$ L_{\text{eff}} = L_\lambda - \frac{\lambda}{4} K_\mu K^\mu. \quad (23) $$

It is straightforward to see, using the structures of the Euler vector Eq. (18), that this effective model corresponds to

$$ L_{\text{eff}} = \chi m^2 e^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{m^2}{8} \lambda F_{\mu\nu} F^{\mu\nu} + f(\lambda), \quad (24) $$

which is clearly gauge invariant. After solving for the auxiliary field $\lambda$, we will restore the nonlinearity inherent in the model, getting a functional form $H(F_{\mu\nu} F^{\mu\nu})$ dependent solely on the quantity $F_{\mu\nu} F^{\mu\nu}$,

$$ L_{\text{TM}} = \chi m^2 e^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - H(F_{\mu\nu} F^{\mu\nu}), \quad (25) $$

which is the general topologically massive theory dual to the nonlinear SD model Eq. (1). Note how $H(F_{\mu\nu} F^{\mu\nu})$ is directly related to $g(A_\mu A^\mu)$ through the auxiliary function Eq. (13). In fact, in the examples which we will discuss below we will find that the same functional form is present in both SD-TM models, which can provide powerful insights in more complicated cases.

Although we have started from a known nonlinear SD model and then determined its dual TM counterpart, we could have done it backwards as well. Starting from a known TM model, given by Eq. (25) we must find the corresponding auxiliary function $f(\lambda)$ such that

$$ f(\lambda) - \frac{m^2}{8} \lambda F_{\mu\nu} F^{\mu\nu} = H(F_{\mu\nu} F^{\mu\nu}), \quad (26) $$

holds true. The function found is thus used in Eqs. (22) and (23), so that $g(A^2)$ can be determined. This procedure would then be alternative to the gauge-fixing.

III. EXAMPLES

In this section we illustrate the procedures outlined above by giving some examples which show the power and generality of the method by investigating the dual correspondence
between some nonlinear SD and TM models. These include a rational-power generalization of the usual SD model, the recently discussed Born-Infeld-Chern-Simons (BICS) model and a logarithmic SD model. Although we will start from the gauge non-invariant model towards the gauge invariant one, i.e., by unfixing the gauge freedom as we proceed, as mentioned above, this could also be done backwards, starting from the gauge-invariant TM model and breaking the gauge freedom towards the gauge non-invariant SD model.

A. The rational self dual model

Consider the following self-dual model, given by

\[ \mathcal{L}_{\text{NSD}} = \frac{q \beta^2}{p} \left( \frac{1}{\beta^2} A_\mu A^\mu \right)^{p/q} - \frac{m^2}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \]  

(27)

where \( p, q \in \mathbb{Z} \) but \( p/q \neq \{1, 1/2\} \). Here the constant \( \beta \) was inserted for dimensional reasons. When \( p = q \) this of course reduces to the usual SD model while \( q = 2p \) is a troublesome one and will be discussed separately. Using the definitions from the preceding section, we have

\[ g(A^2) = \frac{q \beta^2}{p} \left( \frac{1}{\beta^2} A_\mu A^\mu \right)^{p/q} = \frac{1}{\lambda} (A_\mu A^\mu) + f(\lambda). \]  

(28)

From this expression we can relate the basic field \( A_\mu \) with the auxiliary field \( \lambda \) through Eq.(14),

\[ A_\mu A^\mu = \beta^2 \left( \frac{1}{\lambda} \right)^{q/p}, \]  

(29)

and use this relation in Eq.(13) to find the expression for the auxiliary function \( f(\lambda) \),

\[ f(\lambda) = -\beta^2 \left( \frac{q-p}{p} \right) (\lambda)^{q/p}. \]  

(30)

We can now use this expression to write down the effective model. It is given by

\[ \mathcal{L}_{\text{eff}} = \chi m^2 \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{m^2}{8} \lambda F_{\mu\nu} F^{\mu\nu} - \beta^2 \left( \frac{q-p}{p} \right) (\lambda)^{q/p}. \]  

(31)

Solving for the auxiliary field \( \lambda \), we will have finally
\[ L_{TM} = \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \beta \frac{q}{2p-q} \left( \frac{2p - q}{q} \right) \left( \frac{m^2}{8} F_{\mu\nu} F^{\mu\nu} \right)^{\frac{q}{2p-q}} \]
\[ = \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{\lambda}{2} \left( \frac{s}{2r-s} \right) \left( \frac{m^2}{8} F_{\mu\nu} F^{\mu\nu} \right)^{\frac{r}{s}}, \]

which is the TM theory dual to the NSD model Eq.(27) after the relabeling \( p \to r \) and \( q \to 2r - s \). Notice that the rational SD model is mapped under duality to a rational TM model. It is clear that the case \( p = q \) gives us back the usual SD-MCS duality, as it should but the case \( q = 2p \) becomes ill defined in (32). It is valid though both in (30) and (31) where the auxiliary function \( f(\lambda) \) becomes linear in \( \lambda \). We discuss this case below. However, it is interesting to note that the rational SD model Eq.(27), when \( p = Nq \) for large value of \( N \) gives rise to a square-rooted TM model. Similarly, if we let \( p \to r \) and \( q \to 2r - s \) such that \( r = Ns \) then, for large values of \( N \) we obtain a square-rooted SD model. Also interesting to observe is that self-duality in the sense that the ratios before and after dualization are the same \( p/q = r/s \) are only satisfied if \( p = q \) or \( p = 0 \), i.e., the SD model or the pure Chern-Simons model.

Let us consider the case \( q = 2p \) explicitly. The NSD model is,

\[ L_{NSD} = 2 \beta^2 \left( \frac{1}{\beta^2} A_\mu A^\mu \right)^{1/2} - \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \]

while the effective dual equivalent theory is,

\[ L_{eff} = \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \lambda \left[ \frac{m^2}{8} F_{\mu\nu} F^{\mu\nu} - \beta^2 \right]. \]

We see that in this case the auxiliary function \( \lambda \) becomes a Lagrange multiplier imposing the condition \( F^2 \sim const. \) as a constraint. In fact, this constraint is also present in Eq.(33).

To see this, let us use the equations of motion of both Eqs.(33) and (34),

\[ A_\mu = \frac{1}{2} \chi m \sqrt{\frac{1}{\beta^2} A_\sigma A^\sigma \epsilon_{\mu\nu\lambda} F^{\nu\lambda}}, \]

and

\[ \chi \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda + \frac{m}{2} \lambda \partial^\sigma F_{\sigma\mu} = 0, \]
respectively. Multiplying both sides of Eq. (35) by $\epsilon_{\mu \alpha \beta} F_{\alpha \beta}$, we have

$$\chi m \sqrt{\frac{1}{\beta^2}} A_\sigma A^\sigma F_{\mu \nu} F^{\mu \nu} = 2 \epsilon_{\mu \alpha \beta} A_\mu F_{\alpha \beta}.$$  \hfill (37)

On the other hand, by Eq. (35) we have also

$$A_\mu A^\mu = \frac{1}{4} \chi m \sqrt{\frac{1}{\beta^2}} A_\sigma A^\sigma \epsilon_{\mu \alpha \beta} A_\mu F_{\alpha \beta}.$$  \hfill (38)

Using the above equation in Eq. (37) we have finally

$$\frac{m^2}{8} F_{\mu \nu} F^{\mu \nu} - \beta^2 = 0,$$  \hfill (39)

which is the constraint implemented in Eq. (34). We can also explicitly show the equivalence between the equations of motion Eqs. (35) and (36). Note that we can rewrite the later as

$$\epsilon^{\mu \nu \sigma} \partial_\nu \left[ \chi A_\sigma - \lambda \frac{m}{2} \epsilon_{\sigma \alpha \beta} \partial^\alpha A^\beta \right] = 0.$$  \hfill (40)

Since this expression is gauge invariant, it implies that

$$A_\sigma - \lambda \frac{m}{2} \epsilon_{\sigma \alpha \beta} \partial^\alpha A^\beta = \partial_\sigma \Omega,$$  \hfill (41)

where $\Omega$ is an arbitrary function. Now we can fix the gauge in such a manner that $\partial_\sigma \Omega = 0$. Moreover, using the relation between $\lambda$ and $A_\mu$, Eq. (29), we get

$$A_\sigma - \lambda \frac{m}{2} \sqrt{\frac{1}{\beta^2}} A_\rho A^\rho \epsilon_{\sigma \alpha \beta} \partial^\alpha A^\beta = 0,$$  \hfill (42)

which is exactly Eq. (35). In other words, the selfdual equation of motion is a gauge fixed, canonically equivalent to the topologically massive equation Eq. (36).

**B. the logarithmic SD model**

Another interesting possibility is given by the logarithmic SD model. Here we consider the following action

$$\mathcal{L}_{NSD} = \beta^2 \ln \left( \frac{1}{\beta^2} A_\mu A^\mu \right) - \chi \frac{m}{2} \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda,$$  \hfill (43)
where $\beta$ is a parameter inserted for dimensional reasons. This model has a linear representation as

$$\beta^2 \ln \left( \frac{1}{\beta^2} A_\mu A^\mu \right) = \frac{1}{\chi} (A_\mu A^\mu) + f(\lambda).$$  \hspace{1cm} (44)

with

$$f(\lambda) = \beta^2 \ln(\lambda).$$  \hspace{1cm} (45)

The effective theory resulting from this procedure is given by

$$L_{\text{eff}} = \chi m^2 \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \lambda \frac{m^2}{8} F_{\mu\nu} F^{\mu\nu} + \beta^2 \ln(\lambda).$$  \hspace{1cm} (46)

Solving this model for the auxiliary field $\lambda$, we have

$$L_{\text{TM}} = \chi m^2 \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \beta^2 \ln \left[ \frac{1}{\beta^2} \left( \frac{m^2}{8} F_{\mu\nu} F^{\mu\nu} \right) \right],$$  \hspace{1cm} (47)

less a constant that can safely be set to zero since it will give no dynamical contribution. This is the TM model dual to the NSD model Eq.(43). Notice in particular the logarithmic dependence of the TM model, exactly the same as the NSD model.

C. the BICS model

As our final example let us study the dual correspondence between the nonlinear SD model proposed in [9] and BICS model. This relationship was found in an indirect way by means of Hamiltonian techniques. Here we employ the gauge embedding procedure, starting by the following model

$$L_{\text{NSD}} = \beta^2 \sqrt{1 + \frac{1}{m^2 \beta^2} (A_\mu A^\mu)} - \chi m^2 \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda.$$  \hspace{1cm} (48)

As before, $\beta$ is a parameter inserted for dimensional reasons. Note that the above model has in the limit $\beta \to \infty$ the usual SD model. Therefore, using our notation we have

$$g(A^2) = \beta^2 \sqrt{1 + \frac{1}{m^2 \beta^2} (A_\mu A^\mu)} = \frac{1}{\chi} (A_\mu A^\mu) + f(\lambda),$$  \hspace{1cm} (49)
which, using Eq.(14) gives
\[ A_\mu A^\mu = \beta^2 \left( \frac{\lambda^2}{4 m^2} - m^2 \right). \] (50)

This identity can be used to evaluate the auxiliary function \( f(\lambda) \) by means of Eq.(12),
\[ f(\lambda) = \beta^2 \left( \frac{\lambda}{4 m^2} + \frac{m^2}{\lambda} \right), \] (51)

The effective model is thus given by
\[ L_{\text{eff}} = \chi m^2 \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda - \lambda \left[ \frac{m^2}{8} F_{\mu \nu} F^{\mu \nu} - \frac{\beta^2}{4 m^2} \right] + \frac{\beta^2 m^2}{\lambda}, \] (52)

which can be solved for the auxiliary field \( \lambda \) to produce the BICS model
\[ L_{\text{TM}} = \beta^2 \sqrt{1 - \frac{1}{2 \beta^2} F_{\mu \nu} F^{\mu \nu} + \chi \frac{m^2}{2} \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda}. \] (53)

Notice that in the limit \( \beta \to \infty \) we recover the usual TM model. Therefore, as expected, the duality relationship is also respected in this limit. It is interesting to observe, once again, the same functional relation for the general SD model and the general TM model.

**IV. COUPLING WITH DYNAMICAL MATTER**

In this section we apply the iterative procedure to construct a gauge invariant theory out of the NSD model coupled to dynamical matter fields, generalizing the treatment proposed in \[14\]. As in the free case, to guarantee equivalence with the starting non-invariant theory, we only use counter-terms vanishing in the space of solutions of the model. To be specific let us consider the minimal coupling of the NSD vector field to dynamical fermions, so that the Lagrangian becomes
\[ L^{(0)} = L_{\text{NSD}} - e A_\mu J^\mu + L_D, \] (54)

where \( J_\mu = \bar{\psi} \gamma_\mu \psi \), and the superscript index is the iterative counter. Here the Dirac Lagrangian is
\[ \mathcal{L}_D = \bar{\psi} \left( i \gamma - M \right) \psi, \] (55)

where \( M \) is the fermion mass. To implement the Noether embedding we follow the usual track and compute the Euler vector for the Lagrangian \( \mathcal{L}_{NSD} \) given by Eq. (43), showing the presence of the fermionic current,

\[ K^\mu = \frac{2}{\lambda} A^\mu - m \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda - e J^\mu. \] (56)

The effective theory that comes out after the dualization procedure is implemented as before, except with the Euler vector replaced by Eq. (56), yielding

\[ \mathcal{L}_{eff} = f(\lambda) + \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \mathcal{L}_D - \frac{\lambda}{2} \left[ \frac{m^2}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{2} J_\mu J^\mu + \chi e m \epsilon^{\mu\nu\lambda} J_\mu \partial_\nu A_\lambda \right]. \] (57)

A simple inspection shows that the minimal coupling of the nonlinear SD model was replaced by a nonminimal magnetic Pauli type coupling and the appearance of a Thirring like current-current term, which are characteristic features of Chern-Simons dualities involving matter couplings.

The auxiliary function \( f(\lambda) \) can be computed by using Eqs. (12) and (14) as before. By solving the equations of motion for the auxiliary field \( \lambda \), we will get the topologically massive model

\[ \mathcal{L}_{TM} = \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + H(F_{\mu\nu}, J_\mu) + \mathcal{L}_D, \] (58)

where \( H(F_{\mu\nu}, J_\mu) \) is a functional form depending on the field strength \( F_{\mu\nu} \) and the current couplings.

As an example consider the rational SD model treated above. Using (30) in (57) above gives

\[ \mathcal{L}_{TM} = \mathcal{L}_D + \chi \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \beta \frac{s}{2r - s} \left[ \frac{m^2}{8} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{4} J_\mu J^\mu + \frac{\chi e m}{2} \epsilon^{\mu\nu\lambda} J_\mu \partial_\nu A_\lambda \right]. \] (59)

All the other examples follow straightforwardly.
V. CONCLUSIONS

In this paper we studied the dual equivalence between the nonlinear generalization of the self dual and the topologically massive models in 2+1 dimensions, in the context of the Noether embedding procedure, which provides a clear physical meaning of the duality equivalence. This is accomplished by linearizing the nonlinear terms by means of a auxiliary field, which can be eliminated later on in order to restore the full nonlinearity of the NSD and the generalized MCS models. The usual SD-MCS dual equivalence are naturally contained in this results, including the couplings with dynamical matter. Some examples are discussed that both clarify and prove the power of the gauge embedding technique to deal with duality equivalence.

ACKNOWLEDGMENTS: This work is partially supported by CNPq, CAPES, FAPERJ and FUJB, Brazilian Research Agencies.
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