A Conditional Quenched CLT for Random Walks Among Random Conductances on $\mathbb{Z}^d$

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Abstract. Consider a random walk among random conductances on $\mathbb{Z}^d$ with $d \geq 2$. We study the quenched limit law under the usual diffusive scaling of the random walk conditioned to have its first coordinate positive. We show that the conditional limit law is a linear transformation of the product law of a Brownian meander and a $(d−1)$-dimensional Brownian motion.

Keywords: random conductance model, uniform heat kernel bounds, Brownian meander, reversibility

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1. Introduction and results

In this paper we study random walks on a $d$-dimensional integer lattice with random conductances. One can briefly describe the model in the following way: initially, weights (i.e., some nonnegative numbers) are attached to the edges of the lattice at random. The transition probabilities are then defined to be proportional to the weights, thus obtaining a reversible Markov chain; due to a well-known correspondence between reversible Markov chains and electric networks, the weights are also called conductances. We refer to the collection of all conductances as “environment”. This model attracted considerable attention
recently, and, in particular, quenched (i.e., for fixed environment) functional
central limit theorems and heat kernel estimates were obtained in rather gen-
eral situations, see e.g. [2, 3, 6, 16] and references therein. We also refer to the
survey paper [5]. To prove the quenched functional CLT, one usually uses the
so-called corrector approach, described in the following way. First, one con-
structs an auxiliary random field (which depends only on the environment),
with the following property: the sum of the corrector and the random walk is
a martingale, for which it is not difficult to show the CLT. Then, using the Er-
godic Theorem, one shows that the corrector is likely to be small in comparison
to the random walk itself.

While this approach has been quite fruitful, it also has its limitations, mainly
due to the fact that the construction of the corrector is not very explicit. For
example, it was understood only quite recently how to prove the quenched CLT
for the random walk with i.i.d. conductances in half-space, see [7, 19]. It is
therefore important to go beyond the usual setup, proving other types of limit
laws. In this paper, we continue the line of research of [11] and [12] (which were,
by their turn, mainly motivated by [8,9]), where a one-dimensional model with
random conductances (but with unbounded jumps) was considered.

We now define the model formally. For \( x, y \in \mathbb{Z}^d \) with \( d \geq 2 \), we write
\( x \sim y \) if \( x \) and \( y \) are neighbors in the lattice \( \mathbb{Z}^d \) and we let \( B_d \) be the set of
unordered nearest-neighbor pairs \((x, y)\) of \( \mathbb{Z}^d \). Let \((\omega_b)_{b \in B_d}\) be non-negative
random variables; \( \mathbb{P} \) stands for the law of this family. We assume that
\( \mathbb{P} \) is stationary and ergodic with respect to the family of shifts \((\theta_x, x \in \mathbb{Z}^d)\). The
quantity \( \omega_b \) is usually called the conductance of the edge \( b \). The collection of all
conductances \( \omega = (\omega_b)_{b \in B_d} \) is called the environment. If \( x \sim y \), we will also write
\( \omega_{x,y} \) to refer to the conductance between \( x \) and \( y \). For a particular realization \( \omega \)
of our environment, we define \( \pi_x = \sum_{y \sim x} \omega_{x,y} \). Given that \( \pi_x \in (0, \infty) \) for all
\( x \in \mathbb{Z}^d \) (which is \( \mathbb{P} \)-a.s. the case by Condition UE below), the random walk \( X \)
in the environment \( \omega \) is defined through its transition probabilities

\[
p_\omega(x, y) = \begin{cases} \frac{\omega_{x,y}}{\pi_x}, & \text{if } y \sim x, \\ 0, & \text{otherwise}, \end{cases}
\]

that is, if \( \mathbb{P}_\omega^x \) is the quenched law of the random walk starting from \( x \), we have
\[
\mathbb{P}_\omega^x[X(0) = x] = 1, \quad \mathbb{P}_\omega^x[X(k+1) = z \mid X(k) = y] = p_\omega(y, z).
\]

Clearly, this random walk is \( \mathbb{P} \)-a.s. reversible with the reversible measure \( (\pi_x, x \in \mathbb{Z}^d) \). Also, we denote by \( \mathbb{E}_\omega^x \) the quenched expectation for the process starting
from \( x \). When the random walk starts from 0, we use the shorter notations
\( \mathbb{P}_\omega, \mathbb{E}_\omega \).

In order to prove our results, we need to make the uniform ellipticity as-
sumption on the environment:
Condition UE. There exists \( \kappa > 0 \) such that, \( \mathbb{P} \)-a.s., \( \kappa < \omega_{0,x} < \kappa^{-1} \) for \( x \sim 0 \).

For all \( n \geq 1 \), we define the continuous map \( (Z^n(t), t \in [0,1]) \) as the natural polygonal interpolation of the map \( k/n \mapsto n^{-1/2}X(k) \). In other words
\[
\sqrt{n}Z^n(t) = X(\lfloor nt \rfloor) + (nt - \lfloor nt \rfloor)X(\lfloor nt \rfloor + 1)
\]
with \( \lfloor \cdot \rfloor \) the integer part. Also, we denote by \( W^{(d)} = (W_1, \ldots, W_d) \) the \( d \)-dimensional standard Brownian motion. Now, let us embed the graph \( \mathbb{Z}^d \) in \( \mathbb{R}^d \). Denote by \( B = \{e_1, \ldots, e_d\} \) the canonical basis of \( \mathbb{R}^d \) and by \( x_1, \ldots, x_d \) the vector coordinates in \( \mathbb{R}^d \). By Condition UE and as our environment is stationary and ergodic there exists an invertible linear transformation \( D : \mathbb{R}^d \to \mathbb{R}^d \) letting the hyperplane \( \{x_1 = 0\} \) invariant and such that the sequence \( (DZ^n)_{n \geq 1} \) tends weakly to \( W^{(d)} \). Indeed, by Condition UE and ergodicity of the environment, it is well known (cf. [5]) that \( (Z^n)_{n \geq 1} \) tends weakly to a \( d \)-dimensional Brownian motion with a positive definite covariance matrix \( \Sigma \). This implies that \( \Sigma \) has positive eigenvalues \( \lambda_i \) and is diagonalizable in an orthonormal basis. If the law of the environment is also invariant under the symmetries of \( \mathbb{Z}^d \), it is known that \( \Sigma = \sigma^2I \) for some constant \( \sigma \), where \( I \) is the identity matrix. Thus, there exists a rotation \( T \) such that \( (TZ^n)_{n \geq 1} \) tends weakly to Brownian motion with diagonal covariance matrix \( \Sigma' = (\lambda_i)_{1 \leq i \leq d} \) in the basis \( B \). This implies that \((\Sigma')^{-1/2}TZ^n)_{n \geq 1} \) tends weakly to \( W^{(d)} \). Finally, by some unitary transformation \( R \), we can rotate the hyperplane \( (\Sigma')^{-1/2}T\{x_1 = 0\} \) to make it coincide with the hyperplane \( \{x_1 = 0\} \). Now, using the isotropy of \( W^{(d)} \) we obtain that \((R(\Sigma')^{-1/2}TZ^n)_{n \geq 1} \) tends weakly to \( W^{(d)} \). For convenience, in the rest of the paper, we will choose \( R \) such that \( DE_1 \cdot e_1 > 0 \). (\( R \) can also involve a reflection). In the case that the law of the environment is also invariant under the symmetries of \( \mathbb{Z}^d \), then the last statement is true with \( D = \sigma^{-1}I \) (where \( \sigma \) is from the quenched CLT).

Denoting \( X = (X_1, \ldots, X_d) \) in the basis \( B \), we define
\[
\hat{\tau} = \inf\{k \geq 1 : X_1(k) = 0\}
\]
and
\[
\Lambda_n = \{\hat{\tau} > n\} = \{X_1(k) > 0 \text{ for all } k = 1, \ldots, n\}.
\]
Consider the conditional quenched probability measure \( Q_{\omega}^n[\cdot] := \mathbb{P}_{\omega}[\cdot | \Lambda_n] \), for all \( n \geq 1 \). Denote by \( C([0,1]) \) the space of continuous functions from \([0,1]\) into \( \mathbb{R}^d \) and by \( B_1 \) the Borel \( \sigma \)-field on \( C([0,1]) \). For each \( n \), the random map \( DZ^n \) induces a probability measure \( \mu^n_\omega \) on \( (C[0,1], B_1) \): for any \( A \in B_1 \),
\[
\mu^n_\omega(A) := Q_{\omega}^n[DZ^n \in A].
\]
Let us next recall the formal definition of the Brownian meander \( W^+ \). For this, define \( \tau_1 = \sup\{s \in [0,1] : W_1(s) = 0\} \) and \( \Delta_1 = 1 - \tau_1 \). Then,
\[
W^+(s) := \Delta_1^{-1/2}[W_1(\tau_1 + s\Delta_1)], \quad 0 \leq s \leq 1.
\]
We denote by $P_{W^+} \otimes P_{W(d-1)}$ the product law of Brownian meander and $(d-1)$-dimensional standard Brownian motion on the time interval $[0,1]$. Now, we are ready to formulate the quenched invariance principle for the random walk conditioned to stay positive, which is the main result of this paper:

**Theorem 1.1.** Under Condition UE, we have that, $\mathbb{P}$-a.s., $\mu^n$ tends weakly to $P_{W^+} \otimes P_{W(d-1)}$ as $n \to \infty$ (as probability measures on $C[0,1]$).

The next result, referred as Uniform Central Limit Theorem (UCLT), will be useful in order to prove Theorem 1.1. Let $W_\Sigma$ be a $d$-dimensional Brownian motion with covariance matrix $\Sigma$ defined above. Denoting by $C_b(C([0,1],\mathbb{R}))$ (respectively, $C_b^u(C([0,1],\mathbb{R}))$) the space of bounded continuous (respectively, bounded uniformly continuous) functionals from $C([0,1])$ into $\mathbb{R}$, we have the following result:

**Theorem 1.2.** Under Condition UE, the following statements hold and are equivalent:

(i) we have $\mathbb{P}$-a.s., for all $H > 0$ and any $F \in C_b(C([0,1],\mathbb{R}))$,

$$
\lim_{n \to \infty} \sup_{x \in [-H\sqrt{n},H\sqrt{n}]^d} \left| E^{\theta_x}[F(Z^n)] - E[F(W_\Sigma)] \right| = 0;
$$

(ii) we have $\mathbb{P}$-a.s., for all $H > 0$ and any $F \in C_b^u(C([0,1],\mathbb{R}))$,

$$
\lim_{n \to \infty} \sup_{x \in [-H\sqrt{n},H\sqrt{n}]^d} \left| E^{\theta_x}[F(Z^n)] - E[F(W_\Sigma)] \right| = 0;
$$

(iii) we have $\mathbb{P}$-a.s., for all $H > 0$ and any closed set $B$,

$$
\limsup_{n \to \infty} \sup_{x \in [-H\sqrt{n},H\sqrt{n}]^d} P^{\theta_x}[Z^n \in B] \leq P[W_\Sigma \in B];
$$

(iv) we have $\mathbb{P}$-a.s., for all $H > 0$ and any open set $G$,

$$
\liminf_{n \to \infty} \inf_{x \in [-H\sqrt{n},H\sqrt{n}]^d} P^{\theta_x}[Z^n \in G] \geq P[W_\Sigma \in G];
$$

(v) we have $\mathbb{P}$-a.s., for all $H > 0$ and any $A \in \mathcal{B}$ such that $P[W_\Sigma \in \partial A] = 0$,

$$
\lim_{n \to \infty} \sup_{x \in [-H\sqrt{n},H\sqrt{n}]^d} \left| P^{\theta_x}[Z^n \in A] - P[W_\Sigma \in A] \right| = 0.
$$

As mentioned, our approach does not involve the corrector in a direct way (although, of course, we use the “classical” invariance principle which relies on the corrector approach). Instead, the key ingredients are the following. We
use the uniform heat kernel bounds of [10] to prove a uniform CLT (see Theorem 1.2.) In addition, we rely on several auxiliary results giving uniform bounds for exit times and hitting times of hyperplanes, leading to statements which say, roughly speaking, that the conditioning does not change the size of fluctuations (see the beginning of Section 2 for a more detailed description). These auxiliary results are shown in Section 2. They are of independent interest and might be applied to study other “fine” questions for the random conductance model than the conditional quenched CLT considered here. To prove that the conditioning does not change the size of fluctuations, we use an iteration method relying on the Markov property and the uniform bounds shown before. In Section 3, we give the proof of Theorem 1.2. Finally, in Section 4, we give the proof of Theorem 1.1.

We will denote by $C_1, C_2, \ldots$ the “global” constants, that is, those that are used all along the paper and by $\gamma, \gamma_1, \gamma_2, \ldots$ the “local” constants, that is, those that are used only in the subsection in which they appear for the first time. For the local constants, we restart the numeration in the beginning of each subsection.

Also, whenever the context is clear, to avoid heavy notations, we will not put the integer part symbol $\lfloor \cdot \rfloor$. For example, for $\delta \in (0, 1)$ we will write $X(\delta n)$ instead of $X(\lfloor \delta n \rfloor)$.

### 2. Auxiliary results

In this section, we will prove several auxiliary results that will be needed later to prove Theorem 1.1. Before going to the technical side, let us give a short description. Lemma 2.1 gives a uniform bound on the upper tail of the exit time of a strip as well as on the lower tail of the hitting time of a set (sufficiently far away from the starting point). Lemma 2.2 provides a uniform lower bound for the probability of progressing in the direction $e_1$ before backstepping to the hyperplane of the origin. Lemma 2.3 says that the probability that the hitting time of a hyperplane is larger than it should be, conditioned on the first coordinate being positive, decays fast enough. Lemma 2.4 says that the probability that the “transversal fluctuations” are larger than they should be, conditioned on the first coordinate being positive, decays fast enough.

Instead of considering the process $X$ in the canonical basis $B$ of $\mathbb{R}^d$ it is also convenient to introduce the embedded graph $\tilde{\mathbb{Z}}^d := D \mathbb{Z}^d$ with the basis $B' = \{e'_1, \ldots, e'_d\} := DB$ and consider the process $DX$ in this new basis. All the results obtained in this section concern the original random walk $X$ expressed in $B$ but they remain valid for $DX$ expressed in $B'$ with the $\| \cdot \|_1$-norm replaced by the graph distance in $\tilde{\mathbb{Z}}^d$.

Let us introduce the following notations. First, for $a, b \in \mathbb{Z}$, $a < b$, we denote by $[a, b]$ the set $[a, b] \cap \mathbb{Z}$. Vectors of $\mathbb{Z}^d$ will be denoted by $x, y$ or $z$. 
For $x \in \mathbb{Z}^d$ we denote by $x_1, \ldots, x_d$ its coordinates in $B$. For $l \in \mathbb{R}$, we denote

$$\{l\}_j = \begin{cases} 
\{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_j = \lfloor l \rfloor\}, & \text{if } l \geq 0, \\
\{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_j = \lceil l \rceil\}, & \text{if } l < 0,
\end{cases}$$

for $j \in [1, d]$. If $F \subset \mathbb{Z}^d$, let us define

$$\tau_F = \inf\{n \geq 0 : X(n) \in F\} \quad \text{and} \quad \tau_F^+ = \inf\{n \geq 1 : X(n) \in F\}.$$

At this point we mention that under Condition UE, we can apply Theorem 1.7 of [10] to the random walks $Y(n) := X(2n)$ and $Y'(n) := X(2n + 1)$, to obtain that uniform heat kernel lower and upper bounds are available for this model.

That is, there exist absolute constants $C_1$, $C_2$, $C_3$ and $C_4$ such that $\mathbb{P}$-a.s., for $n \in \mathbb{N}$,

$$p_n^\omega(x, y) \leq \frac{C_1}{n^{d/2}} \exp\left\{ - \frac{C_2 \|x - y\|_1^2}{n} \right\}$$

and if $\|x - y\|_1 \leq n$ (with $\| \cdot \|_1$ the 1-norm on $\mathbb{Z}^d$) and has the same parity as $n$,

$$p_n^\omega(x, y) \geq \frac{C_3}{n^{d/2}} \exp\left\{ - \frac{C_4 \|x - y\|_1^2}{n} \right\}. \quad (2.2)$$

We denote by $d_1$ the distance induced by the 1-norm. The heat kernel upper bound (2.1) has two simple consequences gathered in the following

**Lemma 2.1.** Estimate (2.1) implies that there exist positive constants $C_5$ and $C_6$ such that $\mathbb{P}$-a.s., for $h > 0$ and $\delta > 0$, the following holds.

(i) Let $H_1$ and $H_2$ be two parallel hyperplanes in $\mathbb{Z}^d$ orthogonal to $e_i$ for some $i \in [1, d]$ and let us denote by $S$ the strip delimited by $H_1$ and $H_2$. If $2 \leq d_1(H_1, H_2) \leq hn^{1/2}$ then there exists $n_0 = n_0(\delta, h)$ such that

$$\sup_{x \in S} \mathbb{P}_\omega[\tau_{H_1 \cup H_2} > \delta^2 n] \leq \frac{C_5 h}{\delta}$$

for all $n \geq n_0$;

(ii) Let $x \in \mathbb{Z}^d$. If $A \subset \mathbb{Z}^d$ is such that $d_1(x, A) > hn^{1/2} \geq 1$ then there exists $n_1 = n_1(\delta, h)$ such that

$$\mathbb{P}_\omega[\tau_A \leq \delta^2 n] \leq \frac{C_6 \delta}{h}$$

for all $n \geq n_1$.

**Proof.** Let us denote by $S$ the strip delimited by $H_1$ and $H_2$. To prove (i), we just notice that $\mathbb{P}_\omega[\tau_{H_1 \cup H_2} > \delta^2 n] \leq \mathbb{P}_\omega[X(\delta^2 n) \in S]$ and apply (2.1). More precisely, suppose that $H_1$ and $H_2$ are orthogonal to $e_1$. With a slight abuse of
notation, we also denote by $H_1$ and $H_2$ the coordinates where the hyperplanes $H_1$ and $H_2$ cross the first axis. We have

$$\mathbb{P}_x^\omega [X(\delta^2n) \in S] \leq \sum_{y \in S} \frac{C_1}{\delta^2n^{d/2}} \exp \left\{ -C_2 \frac{\|x - y\|^2}{\delta^2n} \right\}$$

$$\leq \frac{C_1}{\delta^2n^{d/2}} \sum_{y_1 \in [H_1, H_2]} \exp \left\{ -C_2 \frac{(y_1 - x_1)^2}{\delta^2n} \right\}$$

$$\times \prod_{i=2}^d \sum_{y_i \in \mathbb{Z}} \exp \left\{ -C_2 \frac{(y_i - x_i)^2}{\delta^2n} \right\}. \quad (2.3)$$

Using (2.3), we can see that there exist positive constants $\gamma_1$, $\gamma_2$ and $n_0 = n_0(\delta, h)$ such that

$$\mathbb{P}_x^\omega [X(\delta^2n) \in S] \leq \gamma_1 \gamma_2 \frac{h}{\delta} \int_0^{\gamma_2(h/\delta)} \exp \{-C_2 t^2\} dt$$

for all $n \geq n_0$. We deduce that there exists a constant $\gamma_3 > 0$ such that

$$\mathbb{P}_x^\omega [X(\delta^2n) \in S] \leq \gamma_3 \frac{h}{\delta}$$

for all $n \geq n_0$.

To prove (ii) we use an argument by Barlow (cf. [1] Chapter 3). First, if we denote by $B(x, r)$ the $\| \cdot \|_1$-ball of center $x$ and radius $r := \lfloor hn^{1/2} \rfloor$ we have that

$$\mathbb{P}_x^\omega [\tau_A \leq \delta^2n] \leq \mathbb{P}_x^\omega [\tau_{B^c(x, r)} \leq \delta^2n].$$

Then, we have

$$\mathbb{P}_x^\omega [\tau_{B^c(x, r)} \leq \delta^2n] \leq \mathbb{P}_x^\omega \left[ \|X(\delta^2n) - x\|_1 > \frac{r}{2} \right]$$

$$+ \mathbb{P}_x^\omega \left[ \tau_{B^c(x, r)} \leq \delta^2n, \|X(\delta^2n) - x\|_1 \leq \frac{r}{2} \right]. \quad (2.4)$$

Writing $S = \tau_{B^c(x, r)}$, by the Markov property, the second term of the right-hand side of (2.4) equals

$$\mathbb{E}_x^\omega \left[ 1_{S \leq \delta^2n} \mathbb{P}_x^X \left[ \|X(\lfloor \delta^2n \rfloor - S) - x\|_1 \leq \frac{r}{2} \right] \right]$$

$$\leq \sup_{y \in \partial B(x, r+1)} \sup_{m \leq \lfloor \delta^2n \rfloor} \mathbb{P}_x^y \left[ \|X(\lfloor \delta^2n \rfloor - m) - y\|_1 > \frac{r}{2} \right]$$

where $\partial B(x, r) := \{ y \in \mathbb{Z}^d : \|y - x\|_1 = r \}$. Combining this last inequality with (2.4) we obtain,

$$\mathbb{P}_x^\omega [\tau_{B^c(x, r)} \leq \delta^2n] \leq 2 \sup_{y \in \mathbb{Z}^d} \sup_{m \leq \lfloor \delta^2n \rfloor} \mathbb{P}_x^y \left[ \|X(\lfloor \delta^2n \rfloor - m) - y\|_1 > \frac{r}{2} \right]$$
Next, let us define $\nu$ all. We are going to show that we can choose $v > 0$ small enough in such a way that the statement of Lemma 2.2 is true for this $v$. Let $\nu$ be a constant in the proof of (i), we obtain (ii).

**Proof.** We are going to show that we can choose $v > 0$ small enough in such a way that the statement of Lemma 2.2 is true for this $v$. For the moment, let $v \in (0, 1)$ and fix $l$ such that $vl \geq 1$. Then, consider $w \in (v, 1]$. We start by writing

$$
P_w^y[\tau_{(v+1)l} < \tau_{(0)_1}] \geq P_w^y[X_1(wl^2) \geq (v + 1)l, \tau_{(0)_1} > wl^2] \geq P_w^y[X_1(wl^2) \geq (v + 1)l] - P_w^y[\tau_{(0)_1} \leq wl^2].$$

(2.5)

Next, let us define $\nu := |wl^2|$ if $|wl^2|$ is even or $\nu := |wl^2| + 1$ otherwise. In the same way, we define $\rho := |vl|$ if $|vl|$ is even or $|vl| + 1$ otherwise. Observe that in any of these cases,

$$
P_w^y[X_1(wl^2) \geq (v + 1)l] \geq P_w^y[X_1(\nu) > l + \rho].$$

(2.6)

We will bound the term of the right-hand side of (2.6) from below. For $y \in \{l\}_1$, we denote by $P(y)$ the (non-empty) set of vectors $z \in \mathbb{Z}^d$ that satisfy the following conditions: $z_i - y_i > \rho, \|y - z\|_1$ is even and $\|y - z\|_1 \leq \nu$. Applying (2.2), we obtain

$$
P_w^y[X_1(\nu) > l + \rho] \geq \frac{C_3}{\nu d/2} \sum_{u \in P(y)} \exp \left\{ - C_4 \frac{\|u - y\|_1^2}{\nu} \right\}

\geq \frac{C_3}{\nu d/2} \sum_{v_1=(\rho+2)/2}^{\nu/2} \sum_{v_2=0}^{(\nu-2v_1)/2} \cdots \sum_{v_d=0}^{(\nu-(2v_1+\cdots+2v_{d-1}))/2} \exp \left\{ - \gamma_1 \frac{v_1^2 + \cdots + v_d^2}{\nu} \right\}

\geq \frac{C_3}{\nu d/2} \int_0^{(\rho+2)/2} \int_0^{\nu/2} \cdots \int_0^{(\nu-(2v_1+\cdots+2v_{d-1}))/2} \exp \left\{ - \gamma_1 \frac{\sum_{i=1}^d v_i^2}{\nu} \right\} dv_d \cdots dv_1.$$
with \(\gamma_1\) a positive constant depending only on \(d\). Now making the change of variables \(u_i = (2v_i)/\sqrt{\nu}, i \in \{1, \ldots, d\}\), in the last multiple integral, we obtain

\[
P_y^\nu[X_1(\nu) > l + \rho] \geq \frac{C_3}{2^d} \int_{(\rho+2)/\sqrt{\nu}}^1 \int_0^{1-u_1} \cdots \int_0^{1-\sum_{i=1}^{d-1} u_i} \exp \left\{ -\frac{\gamma_1}{4} \sum_{i=1}^d u_i^2 \right\} du_d \cdots du_1.
\]

Now, by definition of \(\rho, \nu\) and the way we chose \(v, l\) and \(w\) we have that

\[
\rho + 2 \sqrt{\nu} \leq 4 \sqrt{2} vw^{-1/2}
\]

and \(\nu \geq 1\). This implies that

\[
P_y^\nu[X_1(\nu) > l + \rho] \geq \frac{C_3}{2^d} \int_{1\wedge(\rho+2)/\sqrt{\nu}}^{1} \int_0^{1-u_1} \cdots \int_0^{1-\sum_{i=1}^{d-1} u_i} \exp \left\{ -\frac{\gamma_1}{4} \sum_{i=1}^d u_i^2 \right\} du_d \cdots du_1
=: \frac{C_3}{2^d} J(vw^{-1/2}). \tag{2.7}
\]

By (ii) of Lemma (2.1) we obtain \(P_y^\nu[\tau_{(0)} \leq w^2] \leq C_6 w^{1/2}\). Combining this last inequality with (2.5), (2.6) and (2.7) we obtain

\[
P_y^\nu[\tau_{(v+1)l} < \tau_{(0)}] \geq \frac{C_3}{2^d} J(vw^{-1/2}) - C_6 w^{1/2}. \tag{2.8}
\]

Observe that for fixed \(w\), we have \(J(vw^{-1/2}) \to J(0) > 0\) as \(v \to 0\), since the integrated function is positive and the domain of integration of \(J(0)\) has Lebesgue measure equal to \(1/d!\). Let

\[
w^* = \max \left\{ s > 0 : C_6 s^{1/2} \leq \frac{1}{4} \frac{C_3}{2^d} J(0) \right\}
\]

that is,

\[
w^* = \left( \frac{C_3}{2^{d+2} C_6} J(0) \right)^2.
\]

Letting \(v < w^* \wedge (1/4)\), we can choose a sufficiently small \(w\) in such a way that the second term of the right-hand side of (2.8) is smaller than \(\frac{1}{4} \frac{C_3}{2^d} J(0)\).
Once we have chosen \( w \), we can choose \( v \) sufficiently small in such a way that \( J(vw^{-1/2}) > J(0)/2 \). We obtain that

\[
P^\omega_\omega [\tau_{\{v+1\}l_1} < \tau_{\{0\}l_1}] \geq \frac{1}{4} \frac{C_3}{2^7} J(0) > 0.
\]

This shows Lemma 2.2.

\[\square\]

For \( \varepsilon \in (0,1] \), we denote \( N := \lfloor \varepsilon \sqrt{n} \rfloor \). We next prove an upper bound for the probability that the hitting time of the hyperplane \( \{N\}_1 \) is larger than \( \varepsilon^{1/2} n \), given \( \Lambda_n \).

**Lemma 2.3.** There exists a function \( f = f(\varepsilon) \) with \( \lim_{\varepsilon \to 0} \varepsilon^{-2} f(\varepsilon) = 0 \) such that we have \( \mathbb{P} \)-a.s.

\[
\limsup_{n \to \infty} \mathbb{P}_\omega [\tau_{\{N\}_1} > \varepsilon^{1/2} n \mid \Lambda_n] \leq f(\varepsilon).
\]

**Proof.** Let us begin the proof by sketching the main argument. Consider \( \alpha \in (0,1) \), we will show that

\[
\limsup_{n \to \infty} \mathbb{P}_\omega [\tau_{\{N\}_1} > \varepsilon^{1/2} n \mid \Lambda_n] \leq \limsup_{n \to \infty} \mathbb{P}_\omega [\tau_{\{2^{-1}N\}_1} > \alpha^{1/2} n \mid \Lambda_n] + o(1) \varepsilon)
\]

when \( \varepsilon \to 0 \). Then, iterating the argument using hyperplanes of the form

![Figure 1. Iteration method.](image-url)
\( (2^{-j}N) \) (cf. Figure 1) we will have that for all \( j \geq 0 \),
\[
\limsup_{n \to \infty} P_\omega[\tau_{(2^{-j}N)}_1 > \alpha^j \varepsilon^{1/2} n \mid \Lambda_n] \leq \limsup_{n \to \infty} P_\omega[\tau_{(2^{-(j+1)}N)}_1 > \alpha^{j+1} \varepsilon^{1/2} n \mid \Lambda_n] + o_j(\varepsilon)
\]

when \( \varepsilon \to 0 \). Finally, restricting \( \alpha \) to the interval (1/4, 1), we will show that the \( o_j(\varepsilon) \) are decreasing fast enough. Now, let us start the formal argument. Fix \( \alpha \in (1/4, 1) \) and let \( A_1 := \{ \tau_{(1)}_1 < \tau_{(0)}_1 \} \). We have
\[
P_\omega[\tau_{(N)}_1 > \varepsilon^{1/2} n \mid \Lambda_n] = \frac{1}{P_\omega[\Lambda_n]} \left( P_\omega[\tau_{(N)}_1 > \varepsilon^{1/2} n, \tau_{(2^{-1}N)}_1 > \alpha \varepsilon^{1/2} n, \Lambda_n] \right.
\]
\[
+ P_\omega[\tau_{(N)}_1 > \varepsilon^{1/2} n, \tau_{(2^{-1}N)}_1 \leq \alpha \varepsilon^{1/2} n, \Lambda_n] \right) \leq P_\omega[\tau_{(2^{-1}N)}_1 > \alpha \varepsilon^{1/2} n \mid \Lambda_n]
\]
\[
+ \frac{1}{P_\omega[\Lambda_n]} P_\omega[\tau_{(N)}_1 > \varepsilon^{1/2} n, \tau_{(2^{-1}N)}_1 \leq \alpha \varepsilon^{1/2} n, A_{2^{-1}N}, \Lambda_n]. \tag{2.9}
\]

Then, we have by the Markov property
\[
P_\omega[\tau_{(N)}_1 > \varepsilon^{1/2} n, \tau_{(2^{-1}N)}_1 \leq \alpha \varepsilon^{1/2} n, A_{2^{-1}N}, \Lambda_n] = \sum_{y \in \{2^{-1}N \}, k \leq \lfloor \alpha \varepsilon^{1/2} n \rfloor} P_\omega[X(\tau_{(2^{-1}N)}_1) = y, \tau_{(2^{-1}N)}_1 = k, \tau_{(N)}_1 > \varepsilon^{1/2} n, A_{2^{-1}N}, \Lambda_n]
\]
\[
\leq \max_{y \in \{2^{-1}N \}, k \leq \lfloor \alpha \varepsilon^{1/2} n \rfloor} \max_{n \leq \varepsilon^{1/2} n - k, \Lambda_n-k} P_\omega[\tau_{(N)}_1 > \varepsilon^{1/2} n - k, \Lambda_n-k] P_\omega[A_{2^{-1}N}]. \tag{2.10}
\]

Now, let us bound from above the term \( P_\omega^y[\tau_{(N)}_1 > \varepsilon^{1/2} n - k, \Lambda_n-k] \) uniformly in \( y \in \{2^{-1}N \} \) and in \( k \leq \lfloor \alpha \varepsilon^{1/2} n \rfloor \). Observe that, since \( \varepsilon \in (0, 1) \), we have
\[
P_\omega^y[\tau_{(N)}_1 > \varepsilon^{1/2} n - k, \Lambda_n-k] \leq P_\omega^y[\tau_{(N)}_1 > (1-\alpha)\varepsilon^{1/2} n, \Lambda_{(1-\alpha)n}]
\]
\[
\leq P_\omega^y[\tau_{(0)}_1 \cup \{N \}, 1 > (1-\alpha)\varepsilon^{1/2} n]. \tag{2.11}
\]

Let \( \delta := \beta^{-1} \varepsilon \), where \( \beta \) is a positive constant to be determined later. Then, consider \( \varepsilon \) small enough in such a way that \( \delta < (1-\alpha)\varepsilon^{1/2} n \). Then, divide the time interval \([0, (1-\alpha)\varepsilon^{1/2} n]\) into intervals of size \( \lfloor \delta^2 n \rfloor \). Denoting \( S(0, N) = \bigcup_{i=1}^{N-1} \{ i \} \), we obtain by the Markov property
\[
P_\omega^y[\tau_{(0)}_1 \cup \{N \}_1 > (1-\alpha)\varepsilon^{1/2} n]
\]
Let us now bound from below both terms in the right-hand side of (2.16).

\[
\leq P^y_{\omega}[\tau_{[0], \cup(N), n} \not\in \bigcup_{i=1}^{\lfloor (1-\alpha)\frac{2\log n}{\sqrt{n}}} (i-1)[\delta^2 n, i[\delta^2 n])
\]

\[
\leq \left( \max_{z \in S(0, N)} P^y_{\omega}[\tau_{[0], \cup(N), n} > \delta^2 n] \right)^{(1-\alpha)\varepsilon^{1/2} \delta^{-2} - 2}
\]

(2.12)

for large enough \(n\). Using (i) of Lemma 2.1, we have for all \(z \in S(0, N)\),

\[
P^y_{\omega}[\tau_{[0], \cup(N), n} > \delta^2 n] \leq C_5 \varepsilon
\]

(2.13)

for sufficiently large \(n\). Since \(\varepsilon/\delta = \beta\), let us choose the constant \(\beta\) such that \(C_5 \beta \leq 1/2\). Thus, for \(\varepsilon\) sufficiently small such that \(\beta^{-1} \varepsilon < (1-\alpha)\varepsilon^{1/2}\), we obtain by (2.12)

\[
P^y_{\omega}[\tau_{[0], \cup(N), n} > (1-\alpha)\varepsilon^{1/2} n] \leq 4 \left( \frac{1}{2} \right)^{(1-\alpha)\varepsilon^{-2} \beta^2}
\]

From (2.9), we deduce

\[
P_{\omega}[\tau_{[0], n} > \varepsilon^{1/2} n \mid \Lambda_n] \leq P_{\omega}[\tau_{[2^{-1}N], n} > \alpha \varepsilon^{1/2} n \mid \Lambda_n]
\]

\[
+ 4 \left( \frac{1}{2} \right)^{(1-\alpha)\varepsilon^{-2} \beta^2} \frac{P_{\omega}[A_{2^{-1}N}]}{P_{\omega}[\Lambda_n]}
\]

(2.14)

Then, we will find an upper bound for the ratio in the second term of the right-hand side of (2.14). By the Markov property we have

\[
\frac{P_{\omega}[A_{2^{-1}N}]}{P_{\omega}[\Lambda_n]} \geq P_{\omega}[A_{2^{-1}N} \mid \Lambda_n] \geq \min_{y \in \{2^{-1}N\}} P_{\omega}^y[\tau_{[0], n} > n]
\]

(2.15)

Let \(K \geq 2\varepsilon\) and let \(N' = \lfloor K \sqrt{n} \rfloor\). We start by noting that for any \(y \in \{2^{-1}N\}\) we have by the Markov property

\[
P_{\omega}^y[\tau_{[0], n} > n] \geq P_{\omega}^y[\tau_{[0], n} > n, \tau_{[N'], n} > \tau_{[0], n}]
\]

\[
\geq \min_{z \in \{N'\}} P_{\omega}^z[\tau_{[0], n} > n] P_{\omega}^y[\tau_{[N'], n} < \tau_{[0], n}]
\]

(2.16)

Let us now bound from below both terms in the right-hand side of (2.16).

We first show that we can choose a sufficiently large \(K\) in such a way that \(P_{\omega}^z[\tau_{[0], n} > n] \geq 1/2\) uniformly in \(z \in \{N'\}\). Using (ii) of Lemma 2.1, we have \(P_{\omega}^z[\tau_{[0], n} \leq n] \leq C_6/K\) for sufficiently large \(n\). Choosing \(K\) sufficiently large so that \(C_6/K \leq 1/2\) we obtain

\[
P_{\omega}^z[\tau_{[0], n} > n] \geq \frac{1}{2}
\]

(2.17)
uniformly in \( z \in \{N'\}_1 \). Now going back to equation (2.16), we now show that with probability of order \( \varepsilon^\gamma \) with \( \gamma > 0 \), starting from the line \( \{2^{-1}N\}_1 \), the random walk reaches the line \( \{N'\}_1 \) before reaching the line \( \{0\}_1 \). By Lemma 2.2, there exists \( C^* > 0 \) such that for every \( l > 1 \), \( P^u_0[\tau(l) < \tau(0)] \geq C^* \), with \( u \in \{l\}_1 \). Now consider, the following sequence \( (U_j)_{j \geq 1} \) of hyperplanes defined by

\[
\begin{align*}
U_1 &= \{2, 2^{-1}N\}_1 \\
U_{j+1} &= \{2U_j\}_1.
\end{align*}
\]

Let \( j^* \) the smallest \( j \) such that \( U_j \geq K \sqrt{n} \). Using the induction relation, we obtain that for some constant \( \gamma_1 > 0 \), \( j^* \leq \gamma_1 \ln(K/\varepsilon) \) for large enough \( n \). By convention, set \( U_0 = \{2^{-1}N\}_1 \). By the Markov property, we obtain that uniformly in \( y \in \{2^{-1}N\}_1 \),

\[
P^y_\omega[\tau(N')_1 < \tau(0)_1] \geq P^y_\omega\left[ \bigcap_{i=1}^{j^*} \{\tau U_i < \tau(0)_1\} \right] \geq \prod_{i=1}^{j^*} \left( \min_{u \in U_{i-1}} P^u_\omega[\tau U_i < \tau(0)_1] \right) \geq \left( \frac{\varepsilon}{K} \right)^{\gamma_2} \tag{2.18}
\]

for some constant \( \gamma_2 > 0 \) and large enough \( n \). Combining (2.16), (2.17), and (2.18) we deduce

\[
\min_{y \in \{2^{-1}N\}_1} P^y_\omega[\tau(0)_1 > n] \geq \frac{1}{2} \left( \frac{\varepsilon}{K} \right)^{\gamma_2} \tag{2.19}
\]

for large enough \( n \). Then by (2.14), (2.15) and (2.19) we obtain

\[
P_\omega[\tau(N)_1 > \varepsilon^{1/2}n \mid \Lambda_n] \leq P_\omega[\tau(2^{-1}N)_1 > \alpha\varepsilon^{1/2}n \mid \Lambda_n] + 16K^{\gamma_2} \varepsilon^{\gamma_2} \left( \frac{1}{2} \right)^{(1-\alpha)\varepsilon^{-\beta} - \frac{3}{2} \beta^2}. \tag{2.20}
\]

By the same argument, we can deduce that for all \( j \geq 1 \) we have

\[
P_\omega[\tau(2^{-j}N)_1 > \alpha^j\varepsilon^{1/2}n \mid \Lambda_n] \leq P_\omega[\tau(2^{-j+1}N)_1 > \alpha^{j+1}\varepsilon^{1/2}n \mid \Lambda_n] + 16K^{\gamma_2} \left( \frac{\varepsilon}{2^j} \right)^{-\gamma_2} \left( \frac{1}{2} \right)^{(1-\alpha)\beta^2\varepsilon^{-\frac{3}{2}}(4\alpha)^j}. \tag{2.21}
\]

for large enough \( n \). Iterating (2.20) using (2.21), we deduce

\[
\limsup_{n \to \infty} P_\omega[\tau(N)_1 > \varepsilon^{1/2}n \mid \Lambda_n] \leq 16K^{\gamma_2} \sum_{j=0}^{\infty} \left( \frac{\varepsilon}{2^j} \right)^{-\gamma_2} \left( \frac{1}{2} \right)^{(1-\alpha)\beta^2\varepsilon^{-\frac{3}{2}}(4\alpha)^j}.
\]  

(2.22)
As $\alpha \in (1/4, 1)$, the last series is convergent. Define the function $f$ in the statement of Lemma 2.3 as

$$f(\varepsilon) := 16K^{\gamma_2} \sum_{j=0}^{\infty} \left( \frac{\varepsilon}{2} \right)^{-\gamma_2} \left( \frac{1}{2} \right)^{(1-\alpha)\beta^2\varepsilon^{-\frac{3}{2}}}.$$

Using the dominated convergence theorem, it is straightforward to show that

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} f(\varepsilon) = 0.$$ This proves Lemma 2.3.

In the next lemma, $N$ still stands for $\lfloor \varepsilon \sqrt{n} \rfloor$. However, the quantities (like $\alpha, \delta, \beta, \ldots$) defined in the proof of the lemma are not related to the corresponding quantities defined in the proof of Lemma 2.3. The next lemma controls the “transversal fluctuations” of $X_2, \ldots, X_d$, given $\Lambda_n$.

**Lemma 2.4.** We have $\mathbb{P}$-a.s.,

$$\limsup_{n \to \infty} \mathbb{P}_\omega \left[ \max_{i \in [2,d]} \sup_{j \leq \tau(N)} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_n \right] \leq g(\varepsilon)$$

with $\lim_{\varepsilon \to 0} \varepsilon^{-2} g(\varepsilon) = 0$.

**Proof.** First, observe that, by symmetry, it suffices to show that there exists $g' = g'(\varepsilon)$ such that

$$\limsup_{n \to \infty} \mathbb{P}_\omega \left[ \sup_{j \leq \tau(N)} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_n \right] \leq g'(\varepsilon) \quad (2.23)$$

with $\lim_{\varepsilon \to 0} \varepsilon^{-2} g'(\varepsilon) = 0$ for some $i \in [2,d]$. For the sake of simplicity, let us take $i = 2$ in the rest of the proof. Fix $\alpha \in (1/2, 1)$ and let

$$\tilde{\varepsilon}^{-1/2} := \frac{1-\alpha}{\alpha} \varepsilon^{-1/2} > 2.$$

We introduce the following sequence of events (cf. Figure 2),

$$G_k = \left\{ \sup_{j \in (\tau_{(2-k)N}, \tau_{(2-k+1)N})} |X_2(j) - X_2(\tau_{(2-k)N})| \leq \tilde{\varepsilon}^{-1/2} \alpha^k N \right\}$$

for $k \geq 1$, with the convention that $\sup_{j \in \emptyset} \cdot = 0$. Then, we denote

$$B_k^\delta = \{ \tau_{(2-k)N}, \leq \delta n \} \cap \{ \tau_{(2-k)N} < \tau_{(0)1} \}$$

for $\delta \in (0, 1]$ and $k \geq 1$.

Now, observe that on the event $B_0^\delta \cap (\cap_{k \geq 1} G_k)$ we have that

$$\sup_{j \leq \tau(N)} |X_2(j)| \leq \varepsilon^{-1/2} N$$
A conditional quenched CLT for random walks among random conductances

Figure 2. On the definition of $G_k$.

since $\alpha \in (1/2, 1)$. This implies that

$$P_\omega \left[ \sup_{j \leq \tau \{2^{-k}N\}} |X_2(j)| \leq \varepsilon^{-1/2}N | \Lambda_n \right] \geq P_\omega \left[ B_0^\delta \cap \left( \bigcap_{k \geq 1} G_k \right) | \Lambda_n \right].$$

In order to prove Lemma 2.4, we will show that $\liminf_{n \to \infty} P_\omega \left[ B_0^\delta \cap \left( \bigcap_{k \geq 1} G_k \right) | \Lambda_n \right]$ tends to 1 when $\varepsilon \to 0$. We start by writing

$$P_\omega \left[ B_0^\delta \cap \left( \bigcap_{k \geq 1} G_k \right) | \Lambda_n \right] = P_\omega \left[ B_0^\delta \cap \left( \bigcap_{k \geq 1} G_k \right) | \Lambda_n \right] - P_\omega \left[ B_0^\delta \cap \left( \bigcap_{k \geq 1} G_k^c \right) | \Lambda_n \right]
\geq P_\omega \left[ B_0^\delta | \Lambda_n \right] - \sum_{k=1}^{\lfloor \ln N \ln 2 \rfloor} P_\omega \left[ B_0^\delta \cap G_k^c | \Lambda_n \right]. \tag{2.24}$$

From now on, we dedicate ourselves to bounding from above the terms $P_\omega \left[ B_0^\delta \cap G_k^c | \Lambda_n \right]$ for $k \leq \lfloor \ln N \ln 2 \rfloor$. We have by the Markov property,

$$P_\omega \left[ B_0^\delta \cap G_k^c | \Lambda_n \right] \leq P_\omega \left[ B_k^\delta \cap G_k^c | \Lambda_n \right]
\leq \frac{1}{P_\omega[\Lambda_n]} \sum_{j \leq \lfloor \delta n \rfloor} \sum_{y \in \{2^{-k}N\}_1} P_\omega \left[ B_k^\delta, G_k^c, \Lambda_n, \tau(2^{-k}N)_1 = j, X(\tau(2^{-k}N)_1) = y \right]
\leq \frac{P_\omega[B_k]}{P_\omega[\Lambda_n]} \max_{j \leq \lfloor \delta n \rfloor} \max_{y \in \{2^{-k}N\}_1} \sup_{i \leq \tau(2^{-k}N)_1} \left| (X(i) - y) \cdot e_2 \right| > \varepsilon^{-1/2} \alpha k N, \Lambda_n - j \right].$$
\[
\leq \frac{\mathbb{P}_\omega[B_k^1]}{\mathbb{P}_\omega[A_n]} \max_{y \in \{2^{-k}N\}_1} \mathbb{P}_\omega^y \left[ \sup_{i \leq \tau(2^{-k+1}N)} |(X(i) - y) \cdot e_2| > \varepsilon^{-1/2}2^{k}N, \Lambda_{(1-\delta)n} \right].
\]

Using again the Markov property, we obtain
\[
\frac{\mathbb{P}_\omega[A_n]}{\mathbb{P}_\omega[B_k^1]} \geq \mathbb{P}_\omega[A_n | B_k^1] \geq \min_{y \in \{2^{-k}N\}_1} \mathbb{P}_\omega^y[\tau(0), > n].
\]
By the same argument which we used in Lemma 2.3 to treat the term
\[
\min_{y \in \{2^{-k}N\}_1} \mathbb{P}_\omega^y[\tau(0), > n]
\]
(cf. the derivation of (2.19)), we obtain, for large enough \(n\) and all \(k \leq \left\lfloor \frac{\ln N}{\ln 2} \right\rfloor\),
\[
\frac{\mathbb{P}_\omega[B_k^1]}{\mathbb{P}_\omega[A_n]} \leq \gamma_1 \left( \frac{K2^k}{\varepsilon} \right)^{\gamma_2}
\]
for some positive constants \(\gamma_1, \gamma_2\) and \(K\) from Lemma 2.3. Now, we need to bound the terms
\[
\mathbb{P}_\omega^y \left[ \sup_{i \leq \tau(2^{-k+1}N)} |(X(i) - y) \cdot e_2| > \varepsilon^{-1/2}2^{k}N, \Lambda_{(1-\delta)n} \right]
\]
from above, uniformly in \(y \in \{2^{-k}N\}_1\). In order not to carry on heavy notations we treat the case \(y_2 = 0\). However, as one can check, the bound we will obtain is uniform in \(y \in \{2^{-k}N\}_1\). Let
\[
E_k = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_2 = \pm \left\lfloor \varepsilon^{-1/2}2^{k}N \right\rfloor \}.
\]
We start by writing
\[
\mathbb{P}_\omega^y \left[ \sup_{i \leq \tau(2^{-k+1}N)} |X_2(i)| > \varepsilon^{-1/2}2^{k}N, \Lambda_{(1-\delta)n} \right]
= \mathbb{P}_\omega^y[\tau_{E_k} < \tau(2^{-k+1}N), \tau(0), > (1 - \delta)n]
\leq \mathbb{P}_\omega^y[\tau_{E_k} < \tau(2^{-k+1}N), \cup(0),] + \mathbb{P}_\omega^y[\tau_{E_k} > (1 - \delta)n].
\]
Let us bound the first term of the right-hand side of (2.26) from above. To do so, we first write
\[
\mathbb{P}_\omega^y[\tau_{E_k} < \tau(2^{-k+1}N), \cup(0),] \leq \mathbb{P}_\omega^y[\tau(-\varepsilon^{-1/2}2^{k}N), \cup(0),]
+ \mathbb{P}_\omega^y[\tau(2^{-k+1}N), \cup(0),].
\]
We treat the first term of the right-hand side of (2.27) (the method for the second term is similar). Let \(L \in (2, \varepsilon^{-1/2})\) and divide the interval \([0, \varepsilon^{-1/2}2^{k}N]\) into intervals of size \([L2^{-k}N]\). Furthermore, let
\[
F_k = \bigcup_{j=1}^{\left\lfloor 2^{-k+1}N \right\rfloor - 1} \{j\}_1.
\]
We have by the Markov property,
\[
P_\omega^y[\tau_{(j-1/2)^kN} < \tau_{(2^{-k+1}N)1\cup(0)_1}] 
\leq P_\omega^y\left[ \bigcap_{j=1}^{(L-1/2)^kN} \left\{ \tau_{(jL^{-k}N)} < \tau_{(2^{-k+1}N)1\cup(0)_1} \right\} \right] 
\leq \prod_{j=1}^{(L-1/2)^kN} \max_{z \in \{(j-1)[L^{-2}N]\}_2 \cap F_k} P_\omega^z[\tau_{(jL^{-2}N)} < \tau_{(2^{-k+1}N)1\cup(0)_1}] .
\] (2.28)

Let us show that

\[
\max_{z \in \{(j-1)[L^{-2}N]\}_2 \cap F_k} P_\omega^z[\tau_{(jL^{-2}N)} < \tau_{(2^{-k+1}N)1\cup(0)_1}] \leq \frac{1}{2}
\]

for \( \varepsilon \) sufficiently small and \( L \) sufficiently large belonging to \((2, \varepsilon^{-1/2})\). Consider \( w \in (4, L^2) \), we have for \( z \in \{(j-1)[L^{-2}N]\}_2 \cap F_k \),

\[
P_\omega^z[\tau_{(jL^{-2}N)} > \tau_{(2^{-k+1}N)1\cup(0)_1}] 
\geq P_\omega^z[\tau_{(2^{-k+1}N)1\cup(0)_1} \leq w^{-2k}N^2, \tau_{(jL^{-2}N)} > w^{-2k}N^2] 
\geq P_\omega^z[\tau_{(2^{-k+1}N)1\cup(0)_1} \leq w^{-2k}N^2] 
- P_\omega^z[\tau_{(jL^{-2}N)} \leq w^{-2k}N^2].
\] (2.29)

Using (i) of Lemma 2.1, we deduce

\[
P_\omega^z[\tau_{(2^{-k+1}N)1\cup(0)_1} \leq w^{-2k}N^2] \geq 1 - C_5w^{-1/2} .
\] (2.30)

Using (ii) of Lemma 2.1, we obtain for all \( j \geq 1 \),

\[
P_\omega^z[\tau_{(jL^{-2}N)}] \leq w^{-2k}N^2] \leq C_6 \frac{w^{1/2}}{L} .
\] (2.31)

Combining (2.29), (2.30) and (2.31) we obtain for all \( j \geq 1 \),

\[
P_\omega^z[\tau_{(jL^{-2}N)} > \tau_{(2^{-k+1}N)1\cup(0)_1}] \geq 1 - C_5w^{-1/2} - C_6 \frac{w^{1/2}}{L} .
\] (2.32)

First, choose \( w \) sufficiently large such that \( C_5w^{-1/2} \leq 1/4 \) and thus choose \( L \) sufficiently large in such a way that \( C_6w^{1/2}/L \leq 1/4 \). We obtain

\[
P_\omega^z[\tau_{(jL^{-2}N)} > \tau_{(2^{-k+1}N)1\cup(0)_1}] \geq \frac{1}{2} .
\] (2.33)

Now using (2.27), (2.28) and (2.33) we have since \( \varepsilon^{-1/2} > L \),

\[
P_\omega^y[\tau_{E_k} \leq \tau_{(2^{-k+1}N)1\cup(0)_1}] \leq 16 \left( \frac{1}{2} \right)^{(L-1/2)^k(2\alpha)^k} .
\] (2.34)
Next, let us treat the term $P^y_n[\tau_{E_k} > (1 - \delta)n]$. Let $\eta = \beta^{-1}\epsilon$ where $\beta$ is a positive constant to be chosen later. Then suppose that $\epsilon$ is sufficiently small such that $\eta\epsilon^{-1/2}\alpha^k < 1 - \delta$ and divide the time interval $[0, [(1 - \delta)n]]$ into intervals of size $[(\eta^2\epsilon^{-1/2}\alpha^2)^n]$. Using the notation

$$H(E_k) = \bigcup_{j=-[\epsilon^{-1/2}\alpha^k N]}^{1} \{j\},$$

we obtain by the Markov property

$$P^y_n[\tau_{E_k} > (1 - \delta)n] \leq P^y_n[\tau_{E_k} \notin \bigcup_{i=1}^{(i-1)[\eta^2\epsilon^{-1/2}\alpha^2^n]} (i-1)[\eta^2\epsilon^{-1/2}\alpha^2^n], i[\eta^2\epsilon^{-1/2}\alpha^2^n]] \leq (\max_{z \in H(E_k)} P^z_n[\tau_{E_k} > [\eta^2\epsilon^{-1/2}\alpha^2^n]])^{(1-\delta)(\eta\epsilon^{-1/2}\alpha^2)^2 - 2},$$

for $n$ sufficiently large. We now bound the term $P^z_n[\tau_{E_k} > [\eta^2\epsilon^{-1/2}\alpha^2^n]]$ from above uniformly in $\epsilon \in H(E_k)$. Using (i) of Lemma 2.1, we have

$$P^z_n[\tau_{E_k} > [\eta^2\epsilon^{-1/2}\alpha^2^n]] \leq C_5 \frac{\epsilon}{\eta}.$$ 

Since $\epsilon\eta^{-1} = \beta$, choose $\beta$ small enough such that $C_5 \beta \leq 1/2$. For $\epsilon$ sufficiently small such that $\eta\epsilon^{-1/2}\alpha^k < 1 - \delta$, we obtain using (2.35),

$$P^y_n[\tau_{E_k} > (1 - \delta)n] \leq 4 \left(\frac{1}{2}\right)^{(1-\delta)(\beta^{-1}\epsilon^{1/2}\alpha^2)^2}.$$ (2.36)

Combining (2.26), (2.27), (2.34) and (2.36), we deduce that, $\mathbb{P}$-a.s., for all large enough $n$ and $k \leq \lfloor \frac{\ln N}{\ln 2} \rfloor$,

$$\max_{y \in (2^{-k}N)_{1}} P^y_n \left[ \sup_{i \leq \tau_{(2^{-k+1}N)_{1}}} |(X(i) - y) \cdot e_2| > \epsilon^{-1/2}\alpha^k N, \Lambda_{(1 - \delta)n} \right] \leq 16 \left(\frac{1}{2}\right) L^{-1}\epsilon^{-1/2}(2\alpha)^k + 4 \left(\frac{1}{2}\right)^{(1-\delta)(\beta^2\epsilon^{-1}\alpha^{-2k})},$$ (2.37)

Using (2.25) and (2.37), we obtain for all large enough $n$ and $k \leq \lfloor \frac{\ln N}{\ln 2} \rfloor$,

$$P_n[B^k_0 \cap G^c_k \mid \Lambda_n] \leq \gamma_1 K^{\gamma_2} 2^k \gamma_2 + \gamma_2 \left(16 \left(\frac{1}{2}\right) L^{-1}\epsilon^{-1/2}(2\alpha)^k + 4 \left(\frac{1}{2}\right)^{(1-\delta)(\beta^2\epsilon^{-1}\alpha^{-2k})}\right).$$
We finally deduce that, $\mathbb{P}$-a.s., for large enough $n$,
\[
\sum_{k=1}^{[\ln N/\ln 2]} P_{\omega}[B_0^k \cap G^c_k | \Lambda_n] \leq \sum_{k=1}^{\infty} \gamma_1 K\gamma 2^{k\gamma_2+1} \varepsilon^{-\gamma_2} (16\left(\frac{1}{2}\right)^{L^{-1} \varepsilon^{-1/2}(2\alpha)^k} + 4\left(\frac{1}{2}\right)^{(1-\delta)(\beta^2 \varepsilon^{-1} \alpha^{-2\gamma})}).
\]

Observe that since $\alpha \in (1/2, 1)$, the series above converges. Let $\delta = \varepsilon^{1/2}$, we have for $\varepsilon < 1/4$,
\[
\sum_{k=1}^{[\ln N/\ln 2]} P_{\omega}[B_0^k \cap G^c_k | \Lambda_n] \leq \sum_{k=1}^{\infty} \gamma_1 K\gamma 2^{k\gamma_2+1} \varepsilon^{-\gamma_2} (16\left(\frac{1}{2}\right)^{L^{-1} \varepsilon^{-1/2}(2\alpha)^k} + 4\left(\frac{1}{2}\right)^{(1-\delta)(\beta^2 \varepsilon^{-1} \alpha^{-2\gamma})}).
\]

Let
\[
h(\varepsilon) := \sum_{k=1}^{\infty} \gamma_1 K\gamma 2^{k\gamma_2+1} \varepsilon^{-\gamma_2} (16\left(\frac{1}{2}\right)^{L^{-1} \varepsilon^{-1/2}(2\alpha)^k} + 4\left(\frac{1}{2}\right)^{(1-\delta)(\beta^2 \varepsilon^{-1} \alpha^{-2\gamma})}).
\]

By the Lebesgue dominated convergence theorem, we have $\varepsilon^{-2} h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using (2.24) and Lemma 2.3 (since $\delta = \varepsilon^{1/2}$) we have for $\varepsilon < 1/4$,
\[
\liminf_{n \to \infty} P_{\omega}[B_0^k \cap \bigcap_{k \geq 1} G_k | \Lambda_n] \geq 1 - f(\varepsilon) - h(\varepsilon).
\]

This last term tends to 1 as $\varepsilon \to 0$. Now, take $g'(\varepsilon) := f(\varepsilon) + h(\varepsilon)$ to show (2.23) and therefore Lemma 2.4. \hfill $\Box$

### 3. Proof of the UCLT

In this section we prove Theorem 1.2. The proof is similar in spirit to the proof of Theorem 1.2 of [11], but it is greatly simplified in the present case by the use of the heat kernel upper bounds. As in [11], we will consider “good” sites (see Definition 3.1), where uniform estimates hold for the distance of $Z^n$ and $W$, and then show that the random walk hits, with high probability, a good site in small distance from its starting point.

In order to take advantage of the natural left shift on the space $C(\mathbb{R}_+)$ of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$, we will rather prove Theorem 1.2 for $Z^n$. 
assuming values in $C(\mathbb{R}^+) \text{ instead of } C([0,1])$. Then, the result for $Z^n$ assuming values in $C([0,1])$ will be easily obtained by the mapping theorem (cf. [4]). Let $C_b(C(\mathbb{R}^+),\mathbb{R})$ be the space of bounded uniformly continuous functionals from $C(\mathbb{R}^+)$ into $\mathbb{R}$. In this section, we write $W$ for the $d$-dimensional Brownian motion with covariance matrix $\Sigma$ from section 1. The first step is to prove the following

**Proposition 3.1.** For all $F \in C_b^u(C(\mathbb{R}^+),\mathbb{R})$, we have $\mathbb{P}$-a.s., for every $H > 0$,

$$
\lim_{n \to \infty} \sup_{x \in [-H\sqrt{n},H\sqrt{n}]^d} \left| \mathbb{E}_{\theta_x \omega}[F(Z^n)] - E[F(W)] \right| = 0.
$$

Fix $F \in C_b^u(C(\mathbb{R}^+),\mathbb{R})$. We will prove that, $\mathbb{P}$-a.s., for every $\tilde{\varepsilon},H > 0$,

$$
\sup_{x \in [-H\sqrt{n},H\sqrt{n}]^d} \left| \mathbb{E}_{\theta_x \omega}[F(Z^n)] - E[F(W)] \right| \leq \tilde{\varepsilon}
$$

(3.1)

for $n$ large enough. Before this, we need to introduce some definitions and prove an intermediate result. Let $d$ be the distance on the space $C_{\mathbb{R}^+}$ defined by

$$
d(f,g) = \sum_{n=1}^{\infty} 2^{-n+1} \min \left\{ 1, \sup_{s \in [0,n]} \| f(s) - g(s) \| \right\}
$$

with $\| \cdot \|$ the euclidian norm on $\mathbb{R}^d$. Now, for any given $\varepsilon > 0$, let

$$
h_{\varepsilon} := \max \left\{ h \in (0,1] : P \left[ \sup_{s \leq h} \| W(s) \| > \varepsilon \right] + P \left[ \sup_{s \leq h} d(\theta_s W, W) > \varepsilon \right] \leq \frac{\varepsilon}{2} \right\}.
$$

(3.2)

Observe that $h_{\varepsilon} > 0$ for $\varepsilon > 0$ and $h_{\varepsilon} \to 0$ when $\varepsilon \to 0$. Next, adapting section 3 of [11] we introduce the following

**Definition 3.1.** For a given realization of the environment $\omega$ and $N \in \mathbb{N}$, we say that $x \in \mathbb{Z}^d$ is $(\varepsilon,N)$-good, if

- $\min \left\{ n \geq 1 : \mathbb{E}_{\omega}[F(Z^n)] - E[F(W)] \leq \varepsilon, \text{ for all } m \geq n \right\} \leq N$;

- $P_{\theta_x \omega} \left[ \sup_{s \leq h_x} \| Z^m(s) \| \leq \varepsilon, \sup_{s \leq h_x} d(\theta_s Z^m, Z^m) \leq \varepsilon \right] \geq 1 - \varepsilon$, for all $m \geq N$.

We now show that starting from a site $x \in [-H\sqrt{n},H\sqrt{n}]^d$, with high probability, the random walk $X$ will meet a $(\varepsilon,n)$-good site at a distance at most $h'\sqrt{n}$ before time $hn$ (unlike as in [11], there is no need here to introduce the notion of a nice site since by (2.1), every point in $[-H\sqrt{n},H\sqrt{n}]^d$ is nice). We denote by $\mathcal{G}$ the set of $(\varepsilon,n)$-good sites in $\mathbb{Z}^d$. 


Proposition 3.2. Fix $h' > 0$. For any $\varepsilon_1 > 0$, we can choose $\varepsilon$ small enough in such a way that we have $P$-a.s., for all sufficiently large $n$ and all $x \in [-H\sqrt{n}, H\sqrt{n}]^d$:

(i) $P_x[\tau_\mathcal{B} > h_\varepsilon n] \leq \varepsilon_1$;

(ii) $P_x^c \left[ \sup_{j \leq h_\varepsilon n} \|X(j) - X(0)\| > h'\sqrt{n} \right] \leq \varepsilon_1$.

Proof. Fix $\varepsilon$. Then, for any $\varepsilon' > 0$ there exists $N$ such that

$$P[0 \text{ is } (\varepsilon, N)\text{-good}] > 1 - \varepsilon'.$$

By the Ergodic Theorem, we have $P$-a.s. for all $n > n_1(\omega)$,

$$\left| \{x \in [-2H\sqrt{n}, 2H\sqrt{n}]^d \text{ and } x \text{ is not } (\varepsilon, N)\text{-good}\} \right| < 5^d \varepsilon'H^d n^\frac{d}{2}. \quad (3.3)$$

Let us define

$$\text{Bad} := \{x \in [-2H\sqrt{n}, 2H\sqrt{n}]^d \text{ and } x \text{ is not } (\varepsilon, N)\text{-good}\}$$

and $\text{Cub} := [-2H\sqrt{n}, 2H\sqrt{n}]^d$.

In order to show (i) we observe that for all $x \in [-H\sqrt{n}, H\sqrt{n}]^d$,

$$P_x[\tau_\mathcal{B} > h_\varepsilon n] \leq P_x^c[X(h_\varepsilon n) \in \text{Bad}] + P_x^c[\tau_\text{Cub} \leq h_\varepsilon n]. \quad (3.4)$$

For the second term of the right-hand side of (3.4), we apply (ii) of Lemma 2.1 to obtain that $P_x^c[\tau_\text{Cub} \leq h_\varepsilon n] \leq \gamma_2(h_\varepsilon)^{1/2}$. Thus, we can choose $\varepsilon$ small enough in such a way that $P_x^c[\tau_\text{Cub} \leq h_\varepsilon n] \leq \varepsilon_1/2$. Then, using (2.1) and the fact that $|\text{Bad}| < 5^d \varepsilon'H^d n^\frac{d}{2}$ for large $n$, we can show that uniformly in $x \in \text{Bad} \cap [-H\sqrt{n}, H\sqrt{n}]^d$ we have $P_x^c[X(h_\varepsilon n) \in \text{Bad}] \leq \gamma_1 \varepsilon'/h_\varepsilon$ for $n$ sufficiently large. Thus, choosing $\varepsilon'$ sufficiently small in such a way that $\gamma_1 \varepsilon'/h_\varepsilon \leq \varepsilon_1/2$ we obtain $P_x^c[X(h_\varepsilon n) \in \text{Bad}] \leq \varepsilon_1/2$.

To show (ii), we notice that

$$P_x^c \left[ \sup_{j \leq h_\varepsilon n} \|X(j) - X(0)\| > h'\sqrt{n} \right] = P_x^c[\tau_{\mathcal{B}'}(x, h', \sqrt{n}) \leq h_\varepsilon n] \quad (3.5)$$

with $\mathcal{B}(x, r)$ the euclidian ball of center $x$ and radius $r$. Now, we can apply (ii) of Lemma 2.1 to the right-hand term of (3.5) to obtain that

$$P_x^c \left[ \sup_{j \leq h_\varepsilon n} \|X(j) - X(0)\| > h'\sqrt{n} \right] \leq \gamma_3 \frac{h_\varepsilon^{1/2}}{h'}.$$

Finally, choosing $\varepsilon$ sufficiently small such that $\gamma_3 h^{1/2}/h' \leq \varepsilon_1$ we obtain (ii).

This concludes the proof of Proposition 3.2. $\square$
Proof of Proposition 3.1. Let us prove (3.1). Consider \( x \in [-H \sqrt{n}, H \sqrt{n}]^d \). We start by writing
\[
\left| \mathbb{E}_{\theta_x}[F(Z^n)] - E[F(W)] \right| \leq \left| \mathbb{E}_{\theta_x}[F(Z^n) - \mathbb{E}_{\delta_X, \omega}[F(Z^n)]] \right| + \left| \mathbb{E}_{\theta_x}[\mathbb{E}_{\delta_X, \omega}[F(Z^n)] - E[F(W)]] \right| \\
:= U + V. \tag{3.6}
\]
First, taking \( \varepsilon \leq \frac{\bar{\varepsilon}}{2} \) we obtain \( V \leq \bar{\varepsilon}/2 \) by definition of a \((\varepsilon, n)\)-good site. It remains to treat the first term of the right-hand side of (3.6). Denote \( X' := X - x \). Now, observe that by the Markov property
\[
\left| \mathbb{E}_{\theta_x}[F(Z^n) - F \circ \theta_{n^{-1}\tau_0}(Z^n - n^{-1/2}X'_0)] \right|.
\tag{3.7}
\]
We are going to show that for \( n \) sufficiently large we have uniformly in \( x \in [-H \sqrt{n}, H \sqrt{n}]^d \),
\[
\mathbb{E}_{\theta_x}[F \circ Z^n - F \circ \theta_{n^{-1}\tau_0}(Z^n - n^{-1/2}X'_0)] \leq \frac{\bar{\varepsilon}}{2}
\]
for small enough \( \varepsilon \). Let \( M^n := Z^n - n^{-1/2}X'_0 \). Since \( F \) is uniformly continuous, we can choose \( \eta > 0 \) in such a way that if \( d(f, g) \leq \eta \) then \( |F(f) - F(g)| \leq \frac{\bar{\varepsilon}}{2} \). Then, we have
\[
\mathbb{E}_{\theta_x}[F \circ Z^n - F \circ \theta_{n^{-1}\tau_0}M^n] = \mathbb{E}_{\theta_x} \left[ F \circ Z^n - F \circ \theta_{n^{-1}\tau_0}M^n \mathbb{1}\{d(Z^n, \theta_{n^{-1}\tau_0}M^n) \leq \eta\} \right] + \mathbb{E}_{\theta_x} \left[ F \circ Z^n - F \circ \theta_{n^{-1}\tau_0}M^n \mathbb{1}\{d(Z^n, \theta_{n^{-1}\tau_0}M^n) > \eta\} \right] \leq \frac{\bar{\varepsilon}}{4} + 2\|F\|_\infty \mathbb{P}_{\theta_x}\{d(Z^n, \theta_{n^{-1}\tau_0}M^n) > \eta\}. \tag{3.8}
\]
Since \( h_\varepsilon \leq 1 \), we have
\[
\mathbb{P}_{\theta_x}\{d(Z^n, \theta_{n^{-1}\tau_0}M^n) > \eta\} \leq \mathbb{P}_{\theta_x}\{d(Z^n, \theta_{n^{-1}\tau_0}M^n) > \eta, \tau_G \leq h_\varepsilon n\} + \mathbb{P}_{\theta_x}\{\tau_G \geq h_\varepsilon n\} \leq \mathbb{P}_{\theta_x}\left[ \sup_{t \in [0, n^{-1}\tau_0]} \|Z^n - \theta_{n^{-1}\tau_0}M^n\| > \frac{\eta}{2}, \tau_G \leq h_\varepsilon n \right] + \mathbb{P}_{\theta_x}\{\tau_G \geq h_\varepsilon n\}. \tag{3.9}
\]
Let $\mathcal{F}_{\tau_G}$ be the $\sigma$-field generated by $X$ until time $\tau_G$. We first decompose the first term of the right-hand side of (3.9) in the following way:

\[
P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n - \theta_{n^{-1}\tau_G}M^n\| > \frac{\eta}{4}, \tau_G \leq h_x n \right]
\]

\[
\leq P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right] + P_{\theta_x,\omega}\left[ \sup_{t \in [0,h_x]} \|\theta_{n^{-1}\tau_G}M^n\| > \frac{\eta}{4} \right]
\]

\[
= P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right] + \mathbb{E}_{\theta_x,\omega}\left( P_{\theta_x,\omega}\left[ \sup_{t \in [0,h_x]} \|\theta_{n^{-1}\tau_G}M^n\| > \frac{\eta}{4} \mid \mathcal{F}_{\tau_G} \right] \right)
\]

\[
= P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right] + \mathbb{E}_{\theta_x,\omega}\left( P_{\theta_{X^n},\omega}\left[ \sup_{t \in [0,h_x]} \|Z^n\| > \frac{\eta}{4} \right] \right). \tag{3.10}
\]

We now deal with the second term of the right-hand side of (3.9):

\[
P_{\theta_x,\omega}\left[ d(\theta_{n^{-1}\tau_G}Z^n, \theta_{n^{-1}\tau_G}^2 M^n) > \frac{\eta}{4}, \tau_G \leq h_x n \right]
\]

\[
\leq P_{\theta_x,\omega}\left[ \|X'_{\tau_G}\| > \frac{\eta}{4} \right] + P_{\theta_x,\omega}\left[ d(\theta_{n^{-1}\tau_G}M^n, \theta_{n^{-1}\tau_G}^2 M^n) > \frac{\eta}{4}, \tau_G \leq h_x n \right]
\]

\[
\leq P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right] + \mathbb{E}_{\theta_x,\omega}\left( \mathbb{1}\{\tau_G \leq h_x n\} P_{\theta_x,\omega}\left[ d(\theta_{n^{-1}\tau_G}M^n, \theta_{n^{-1}\tau_G}^2 M^n) > \frac{\eta}{4} \mid \mathcal{F}_{\tau_G} \right] \right)
\]

\[
= P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right] + \mathbb{E}_{\theta_x,\omega}\left( \mathbb{1}\{\tau_G \leq h_x n\} P_{\theta_{X^n},\omega}\left[ d(Z^n, \theta_{n^{-1}\tau_G}Z^n) > \frac{\eta}{4} \right] \right). \tag{3.11}
\]

Combining (3.9), (3.10) and (3.11), we obtain

\[
P_{\theta_x,\omega}\left[ d(Z^n, \theta_{n^{-1}\tau_G}M^n) > \eta \right] \leq P_{\theta_x,\omega}[\tau_G > h_x n]
\]

\[
+ 2P_{\theta_x,\omega}\left[ \sup_{t \in [0,n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right]
\]

\[
+ \mathbb{E}_{\theta_x,\omega}\left( P_{\theta_{X^n},\omega}\left[ \sup_{t \in [0,h_x]} \|Z^n\| > \frac{\eta}{4} \right] \right) \tag{3.12}
\]

On one hand, by definition of a $(\varepsilon,n)$-good point, choosing small enough $\varepsilon > 0$, we have uniformly in $x \in [-H\sqrt{n}, H\sqrt{n}]^d$,

\[
\mathbb{E}_{\theta_x,\omega}\left( P_{\theta_{X^n},\omega}\left[ \sup_{t \in [0,h_x]} \|Z^n\| > \frac{\eta}{4} \right] + \mathbb{1}\{\tau_G \leq h_x n\} P_{\theta_{X^n},\omega}\left[ d(Z^n, \theta_{n^{-1}\tau_G}Z^n) > \frac{\eta}{4} \right] \right)
\]

\[
\leq \frac{\varepsilon}{32\|F\|_{\infty}} \tag{3.13}
\]
for all sufficiently large \( n \). On the other hand, by Proposition 3.2, for sufficiently small \( \varepsilon \), we have uniformly in \( x \in [-H\sqrt{n}, H\sqrt{n}]^d \),

\[
P_{\theta x}\omega [\tau > \eta n] \leq \frac{\tilde{\varepsilon}}{32\|F\|_{\infty}} \quad \text{and} \quad P_{\theta x}\omega \left[ \sup_{t \in [0, n^{-1}\tau_G]} \|Z^n\| > \frac{\eta}{4} \right] \leq \frac{\tilde{\varepsilon}}{32\|F\|_{\infty}}
\]

(3.14)

for sufficiently large \( n \). Combining (3.13), (3.14) with (3.12) and (3.7)–(3.9), we have \( U \leq \tilde{\varepsilon}/2 \). Together with \( V \leq \tilde{\varepsilon}/2 \), this leads to the desired result. \( \square \)

Denote by \( \mathcal{C}_b(C(\mathbb{R}_+), \mathbb{R}) \) the space of bounded continuous functionals from \( C(\mathbb{R}_+) \) into \( \mathbb{R} \) and by \( \mathcal{B} \) the Borel \( \sigma \)-field on \( C(\mathbb{R}_+) \). The next step is the following proposition, its proof follows essentially the proof of Theorem 2.1 of [4] (cf. also Proposition 3.7 of [11]).

**Proposition 3.3.** The first statement implies the second one:

(i) for any \( F \in \mathcal{C}_u^b(C(\mathbb{R}_+), \mathbb{R}) \), we have \( \mathbb{P}\)-a.s.,

\[
\lim_{n \to \infty} \sup_{x \in [-H\sqrt{n}, H\sqrt{n}]^d} \left| E_{\theta x}\omega [F(Z^n)] - E[F(W)] \right| = 0;
\]

(ii) for any open set \( G \), we have \( \mathbb{P}\)-a.s.,

\[
\liminf_{n \to \infty} \inf_{x \in [-H\sqrt{n}, H\sqrt{n}]^d} P_{\theta x}\omega [Z^n \in G] \geq P[W \in G].
\]

Finally, we have Proposition 3.4, which is similar to Proposition 3.8 of [11].

**Proposition 3.4.** The following statements are equivalent:

(i) we have \( \mathbb{P}\)-a.s., for every open set \( G \),

\[
\liminf_{n \to \infty} \inf_{x \in [-H\sqrt{n}, H\sqrt{n}]^d} P_{\theta x}\omega [Z^n \in G] \geq P[W \in G];
\]

(ii) for every open set \( G \), we have \( \mathbb{P}\)-a.s.,

\[
\liminf_{n \to \infty} \inf_{x \in [-H\sqrt{n}, H\sqrt{n}]^d} P_{\theta x}\omega [Z^n \in G] \geq P[W \in G].
\]

**Proof.** (i) \( \Rightarrow \) (ii) is trivial. Let us show that (ii) \( \Rightarrow \) (i). Suppose that there exists a countable family \( \mathcal{H} \) of open sets such that for every open set \( G \) there exists a sequence \( (O_n)_{n=1,2,...} \subset \mathcal{H} \) such that \( 1_{O_n} \uparrow 1_G \) pointwise as \( n \to \infty \). By (ii), since the family \( \mathcal{H} \) is countable we would have, \( \mathbb{P}\)-a.s., for all \( O \in \mathcal{H} \),

\[
\liminf_{n \to \infty} \inf_{x \in [-H\sqrt{n}, H\sqrt{n}]^d} P_{\theta x}\omega [Z^n \in O] \geq P[W \in O].
\]

(3.15)
Then, the same kind of reasoning as that used in the proof of Proposition 3.3 to prove \((i) \Rightarrow (ii)\) would provide the desired result. The fact that \(\mathcal{H}\) exists, follows from the fact that the space \(C(\mathbb{R}_+)^2\) is second-countable. 

**Proof of Theorem 1.2.** One can check that it is straightforward (using the same arguments as in the proof of Proposition 3.3) to deduce that \((i), (ii), (iii) \text{ and } (v)\) of Theorem 1.2 are equivalent to statement \((i)\) of Proposition 3.4. That is, one can prove the equivalence of items \((i)-(v)\) of Theorem 1.2. To conclude the proof of Theorem 1.2, it remains to show that \((ii)\) of Proposition 3.4 holds. By Proposition 3.3, \((ii)\) of Proposition 3.4 is equivalent to \((i)\) of Proposition 3.3. Since by Proposition 3.1, \((i)\) of Proposition 3.3 holds, the proof of Theorem 1.2 is complete. \(\square\)

4. **Proof of Theorem 1.1**

For the sake of brevity, let us denote in this section, the process \(DZ^n\) (resp. \(DX\)) by \(Z\) (resp. \(X\)). We also recall that \(W^{(d)} = (W_1, \ldots, W_d)\) is a \(d\)-dimensional standard Brownian motion. In order to prove Theorem 1.1, we first show convergence of the finite-dimensional distributions and then, in Section 4.2, we prove the tightness of the sequence \((P_{\omega}[Z^n \in \cdot | \Lambda_n])_{n \geq 1}\). For \(\varepsilon \in (0, 1)\), we recall that \(N := [\varepsilon\sqrt{n}]\). In this section for any set \(F \subset \mathbb{R}^d\) we denote 

\[
\beta_F = \inf\{n \geq 0 : \mathcal{X}(n) \in F\} \quad \text{and} \quad \beta_F^+ = \inf\{n \geq 1 : \mathcal{X}(n) \in F\}.
\]

We start by recalling the transition density function of the Brownian meander (see [14]) from \((0, 0)\) to \((t, x_1)\)

\[
q(0, 0; t, x_1) = t^{-3/2}x_1 \exp \left( -\frac{x_1^2}{2t} \right) \tilde{N}(x_1(1-t)^{-1/2})
\]

for \(x_1 > 0, 0 < t \leq 1\) and from \((t_1, x_1)\) to \((t_2, x_2)\)

\[
q(t_1, x_1; t_2, x_2) = g(t_2 - t_1) \frac{\tilde{N}(x_2(1-t_2)^{-1/2})}{\tilde{N}(x_1(1-t_1)^{-1/2})}
\]

for \(x_1, x_2 > 0, 0 < t_1 < t_2 \leq 1\), where

\[
\tilde{N}(v) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^v e^{-\frac{u^2}{2}} du
\]

for \(v \geq 0\) and

\[
g(t, x_1, x_2) = (2\pi)^{-1/2} \left( \exp \left( -\frac{(x_2 - x_1)^2}{2t} \right) - \exp \left( -\frac{(x_1 + x_2)^2}{2t} \right) \right)
\]

for \(x_1, x_2 > 0\) and \(0 < t \leq 1\).
4.1. Convergence of finite-dimensional distributions

In this subsection, we prove the convergence of the finite-dimensional distributions. A key ingredient is the decomposition according to the event $A_0 \rightarrow R$, see (4.3), which says that the random walk progresses enough in the desired direction, without big fluctuations of the other coordinates, before it returns to the hyperplane of the origin. In order to show that this event has large probability, Lemma 2.3 and Lemma 2.4 come into play, see (4.10) and (4.29).

First, let us consider the marginal for $t = 1$.

**Proposition 4.1.** We have $\mathbb{P}$-a.s.,

$$\lim_{n \to \infty} P_\omega[Z_1^n(1) > u_1, \ldots, Z_d^n(1) > u_d | \Lambda_n] = \exp(-u_1^2/2) \prod_{i=2}^d \int_{u_i}^{\infty} e^{-t^2/2} dt / \sqrt{2\pi},$$

for all $u = (u_1, \ldots, u_d) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$.

**Proof.** First, we introduce some notations. Let

$$D_u = \{ x \in \mathbb{R}^d : x_1 > u_1, \ldots, x_d > u_d \}$$

and

$$R_{\varepsilon,n} = \{ x \in \mathbb{R}^d : x_1 = N, x_i \in [-\varepsilon^{-1/2}N, \varepsilon^{-1/2}N], i \in [2, d] \}.$$ 

Let us denote $R_{\varepsilon,n} = DR_{\varepsilon,n}$, and define the event

$$A_0 \rightarrow R = \{ \beta_{R_{\varepsilon,n}} < \beta^{+}_{(0)} \}.$$  

(4.3)

We start by bounding the term $P_\omega[Z^n(1) \in D_u | \Lambda_n]$ from above. Fix $\varepsilon \in (0, u_1 \wedge 1)$ and consider the following decomposition

$$P_\omega[Z^n(1) \in D_u | \Lambda_n] \leq \frac{1}{P_\omega[\Lambda_n]} \left( P_\omega[Z^n(1) \in D_u, A_0 \rightarrow R, \Lambda_n] + P_\omega[A^c_0 \rightarrow R, \Lambda_n] \right)$$

$$= \frac{1}{P_\omega[\Lambda_n]} \left( P_\omega[Z^n(1) \in D_u, A_0 \rightarrow R, \Lambda_n, \beta_{R_{\varepsilon,n}} \leq \varepsilon^{1/2} n] + P_\omega[Z^n(1) \in D_u, A_0 \rightarrow R, \Lambda_n, \beta_{R_{\varepsilon,n}} > \varepsilon^{1/2} n] \right) + P_\omega[A^c_0 \rightarrow R | \Lambda_n]$$

$$\leq (P_\omega[\Lambda_n])^{-1} P_\omega[Z^n(1) \in D_u, A_0 \rightarrow R, \Lambda_n, \beta_{R_{\varepsilon,n}} \leq \varepsilon^{1/2} n] + P_\omega[A^c_0 \rightarrow R | \Lambda_n]$$

$$+ P_\omega[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2} n | \Lambda_n] + P_\omega[A^c_0 \rightarrow R | \Lambda_n].$$

(4.4)

Since $\varepsilon^{1/2} \in (0, 1)$, we have

$$P_\omega[A^c_0 \rightarrow R | \Lambda_n] = P_\omega[\beta_{R_{\varepsilon,n}} > \beta^{+}_{(0)} | \Lambda_n].$$
By Lemmas 2.3 and 2.4 we have
\[ Z \leq P_\omega[\beta_{R_{\varepsilon,n}} > n \mid \Lambda_n] \leq P_\omega[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2}n \mid \Lambda_n]. \tag{4.5} \]

Then, using the Markov property at time $\beta_{R_{\varepsilon,n}}$ we deduce
\[ \frac{1}{P_\omega[\Lambda_n]} P_\omega[Z^n(1) \in D_u, A_{0\to R}, \Lambda_n, \beta_{R_{\varepsilon,n}} \leq \varepsilon^{1/2}n] \leq \frac{P_\omega[A_{0\to R}]}{P_\omega[\Lambda_n]} \max_{y \in R_{\varepsilon,n}} \max_{j \leq [\varepsilon^{1/2}n]} P_\omega^n[yX(n-j) / \sqrt{n}] \in D_u, \Lambda_{n-j}]. \tag{4.6} \]

Again, using the Markov property at time $\beta_{R_{\varepsilon,n}}$ we obtain
\[ \frac{P_\omega[\Lambda_n]}{P_\omega[A_{0\to R}]} \geq \min_{y \in R_{\varepsilon,n}} P_\omega^n[\Lambda_n]. \tag{4.7} \]

Combining (4.4), (4.5), (4.6) and (4.7) we obtain
\[ P_\omega[Z^n(1) \in D_u \mid \Lambda_n] \leq \max_{y \in R_{\varepsilon,n}} \max_{j \leq [\varepsilon^{1/2}n]} \frac{P_\omega^n[yX(n-j) \in D_u \sqrt{n}, \Lambda_{n-j}]}{\min_{y \in R_{\varepsilon,n}} P_\omega^n[\Lambda_n]} + 2P_\omega[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2}n \mid \Lambda_n]. \tag{4.8} \]

Now, to bound the term $P_\omega[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2}n \mid \Lambda_n]$ from above we notice that
\[ P_\omega[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2}n \mid \Lambda_n] = P_\omega[\tau_{R_{\varepsilon,n}} > \varepsilon^{1/2}n \mid \Lambda_n] \leq P_\omega[\max_{i \in [2,d]} \sup_{j \leq \tau_{(N)}(1)} |X_i(j)| > \varepsilon^{-1/2}N \mid \Lambda_n] + P_\omega[\tau_{(N)}(1) > \varepsilon^{1/2}n \mid \Lambda_n]. \tag{4.9} \]

By Lemmas 2.3 and 2.4 we have
\[ \lim_{n \to \infty} \sup_{y \in R_{\varepsilon,n}} P_\omega[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2}n \mid \Lambda_n] \leq f(\varepsilon) + g(\varepsilon). \tag{4.10} \]

By definition of $Z^n$, we have $P_\omega^n[\Lambda_n] = P_\omega^n[Z^n(1) > 0, t \in [0,1]]$. Thus, from Theorem 1.2 we obtain, recalling that $W_1$ is the first component of $W(0, t) = \max_{n \to \infty} \min_{y \in R_{\varepsilon,n}} P_\omega^n[Z^n(t) > 0, t \in [0,1]] = P^{x1}[\min_{0 \leq t \leq 1} W_1(t) > 0] = P[|W_1(1)| < \varepsilon \sigma_1] = \frac{2\varepsilon \sigma_1}{\sqrt{2\pi}} + o(\varepsilon)
\]
as $\varepsilon \to 0$, where $P^x$ is law of $W(0, t)$ starting at $x$ and $\sigma_1 := D e_1 \cdot e_1 > 0$ (cf. Section 1). Now, let us treat the term
\[ \max_{y \in R_{\varepsilon,n}} \max_{j \leq [\varepsilon^{1/2}n]} P_\omega^n[yX(n-j) \in D_u \sqrt{n}, \Lambda_{n-j}]. \]
Fix $\delta' > 0$ and let $\text{sgn}(x) = -1$ if $x \leq 0$ and $1$ if $x > 0$. Denote
\[ U_i := \left\{ X_i(n - \lfloor \varepsilon^{1/2} n \rfloor) > (u_i - \text{sgn}(u_i) \delta') \sqrt{n} \right\} \] (4.12)
and
\[ V_i := \left\{ \max_{j \leq \lfloor \varepsilon^{1/2} n \rfloor} |X_i(n - \lfloor \varepsilon^{1/2} n \rfloor) - X_i(n - j)| \geq \delta' \sqrt{n} \right\} \] (4.13)
for $i = 1, \ldots, d$. Observe that we have for $y \in R_{\varepsilon, n}$ and $j \leq \lfloor \varepsilon^{1/2} n \rfloor$
\[ P_y^\varepsilon[X(n - j) \in D_u \sqrt{n}, \Lambda_{n-j}] \leq P_y^\varepsilon \left[ \bigcap_{i=1}^d (U_i \cup V_i) \cap \Lambda_{n-\lfloor \varepsilon^{1/2} n \rfloor} \right]. \] Let us consider the set $I = \{U_1, \ldots, U_d, V_1, \ldots, V_d\}$ and denote by $J$ the set formed by all intersections of $d$ distinct elements of $I$: $J$ contains $\binom{2d}{d}$ elements. Let us denote by $J_1, \ldots, J_{\binom{d}{2}}$ all the elements of $J$. Therefore, we obtain
\[ \max_{j \leq \lfloor \varepsilon^{1/2} n \rfloor} P_y^\varepsilon[X(n - j) \in D_u \sqrt{n}, \Lambda_{n-j}] \leq \sum_{i=1}^{\binom{d}{2}} P_y^\varepsilon \left[ J_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n \rfloor} \right]. \] (4.14)
Let us treat the term $P_y^\varepsilon[\cap_{i=1}^d U_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n \rfloor}]$. We have by definition of $Z^n$
\[ P_y^\varepsilon \left[ \bigcap_{i=1}^d U_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n \rfloor} \right] \leq P_y^\varepsilon \left[ \bigcap_{i=1}^d \left\{ Z^n_{1-\lfloor \varepsilon^{1/2} n \rfloor} \cup (u_i - \text{sgn}(u_i) \delta') \right\} \cup \Lambda_{n-\lfloor \varepsilon^{1/2} n \rfloor} \times (0, t) \right]. \]
By Theorem 1.2 we deduce
\[ \lim_{n \to \infty} \sup_{y \in R_{\varepsilon, n}} \max_{j \leq \lfloor \varepsilon^{1/2} n \rfloor} P_y^\varepsilon \left[ \bigcap_{i=1}^d U_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n \rfloor} \right] \leq P^{\varepsilon_1, \varepsilon} \left[ W_1(1) > (u_1 - \text{sgn}(u_1) \delta'), \min_{0 \leq t \leq 1} W_1(t) > 0 \right] \times \prod_{i=2}^d P^{\varepsilon_1, \varepsilon} \left[ W_i(1) > (u_i - \text{sgn}(u_i) \delta') \right] \] (4.15)
for some constant $\gamma_1$. Abbreviate $\varepsilon' := \sigma_1 \varepsilon (1 - \varepsilon^{1/2})^{-1/2}$ and let us compute the first term of the right-hand side of (4.15) for sufficiently small $\varepsilon$. By the reflection principle for the Brownian motion, we have
\[ P^{\varepsilon'} \left[ W_1(1) > (u_1 - \text{sgn}(u_1) \delta'), \min_{0 \leq t \leq 1} W_1(t) > 0 \right] \]
Therefore, we obtain, as $\varepsilon \to 0$

$$
\limsup_{n \to \infty} \max_{y \in R_{\varepsilon,n}} P^y_{\omega} \left[ \bigcap_{i=1}^{d} U_i, \Lambda_{n-\lfloor \varepsilon^{1/2}n \rfloor} \right] \\
\leq \left( \frac{2\varepsilon \sigma_1 e^{-\left(\varepsilon - \frac{\varepsilon^2}{2}\right)^2}}{\sqrt{2\pi}(1-\varepsilon^{1/2})} + o(\varepsilon) \right) \prod_{i=2}^{d} \int_{(u_i-\varepsilon\sigma_1)(\delta')-\varepsilon'^2}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt. \quad (4.16)
$$

The other terms $P^y_{\omega}[J_i, \Lambda_{n-\lfloor \varepsilon^{1/2}n \rfloor}]$ necessarily contain a term $V_j$ for some $j \in [1,d]$. Thus, we have for $J_i \neq \cap_{i=1}^{d} U_i$,

$$
P^y_{\omega}[J_i, \Lambda_{n-\lfloor \varepsilon^{1/2}n \rfloor}] \leq \sum_{j=1}^{d} P^y_{\omega}[V_j]. \quad (4.17)
$$

Let us bound the terms $\limsup_{n \to \infty} \max_{y \in R_{\varepsilon,n}} P^y_{\omega}[V_j]$ for $j \in [1,d]$. We start by writing

$$
P^y_{\omega}[V_j] = P^y_{\omega} \left[ \max_{i \leq \lfloor \varepsilon^{1/2}n \rfloor} |X_j(n - \lfloor \varepsilon^{1/2}n \rfloor) - X_j(n - i)| \geq \delta' \sqrt{n} \right] \\
= P^y_{\omega} \left[ \max_{n - \lfloor \varepsilon^{1/2}n \rfloor \leq k \leq n} |X_j(k) - X_j(n - \lfloor \varepsilon^{1/2}n \rfloor)| \geq \delta' \sqrt{n} \right] \\
\leq P^y_{\omega} \left[ \max_{1-\varepsilon^{1/2} \leq t \leq 1} \left( Z_j^n(t) - \min_{1-\varepsilon^{1/2} \leq s \leq t} Z_j^n(s) \right) \geq \delta' \right] \\
+ P^y_{\omega} \left[ \min_{1-\varepsilon^{1/2} \leq t \leq 1} \left( Z_j^n(t) - \max_{1-\varepsilon^{1/2} \leq s \leq t} Z_j^n(s) \right) \leq -\delta' \right].
$$

By Theorem 1.2, we obtain

$$
\lim_{n \to \infty} \max_{y \in R_{\varepsilon,n}} P^y_{\omega} \left[ \max_{1-\varepsilon^{1/2} \leq t \leq 1} \left( Z_j^n(t) - \min_{1-\varepsilon^{1/2} \leq s \leq t} Z_j^n(s) \right) \geq \delta' \right] \\
= P \left[ \max_{1-\varepsilon^{1/2} \leq t \leq 1} \left( W_j(t) - \min_{1-\varepsilon^{1/2} \leq s \leq t} W_j(s) \right) \geq \delta' \right]. \quad (4.18)
$$
we obtain

\[ \lim_{n \to \infty} \max_{y \in R^n} P^y \left[ \min_{1 - \varepsilon^{1/2} \leq t \leq 1} \left( Z^n_j(t) - \max_{1 - \varepsilon^{1/2} \leq s \leq t} Z^n_j(s) \right) \leq -\delta' \right] \]

\[ = P \left[ \min_{1 - \varepsilon^{1/2} \leq t \leq 1} \left( W_j(t) - \max_{1 - \varepsilon^{1/2} \leq s \leq t} W_j(s) \right) \leq -\delta' \right]. \tag{4.19} \]

Observe that the right-hand sides of (4.18) and (4.19) are equal since \((-W_j)\) is a Brownian motion. Thus, let us compute for example the right-hand side term of (4.18). By Lévy’s Theorem (cf. [18], Chapter VI, Theorem 2.3), we have

\[ P \left[ \max_{0 \leq t \leq \varepsilon^{1/2}} \left( W_j(t) - \min_{0 \leq s \leq t} W_j(s) \right) \geq \delta' \right] = P \left[ \max_{0 \leq t \leq \varepsilon^{1/2}} |W_j(t)| \geq \delta' \right]. \]

Then,

\[ P \left[ \max_{0 \leq t \leq \varepsilon^{1/2}} |W_j(t)| \geq \delta' \right] \leq 2P \left[ \max_{0 \leq t \leq \varepsilon^{1/2}} W_j(t) \geq \delta' \right] = 4P[W_j(\varepsilon^{1/2}) \geq \delta']. \]

Using an estimate on the tail of the Gaussian law ([17], Appendix B, Lemma 12.9) we obtain

\[ P \left[ \max_{0 \leq t \leq \varepsilon^{1/2}} |W_j(t)| \geq \delta' \right] \leq \frac{4\varepsilon^{1/4}}{\delta' \sqrt{2\pi}} \exp \left\{ -\frac{(\delta')^2}{2\varepsilon^{1/2}} \right\}. \]

We finally obtain

\[ \lim_{n \to \infty} \max_{y \in R^n} \sum_{i=1}^d P^y_\omega[V_i] \leq \frac{8d\varepsilon^{1/4}}{\delta' \sqrt{2\pi}} \exp \left\{ -\frac{(\delta')^2}{2\varepsilon^{1/2}} \right\}. \tag{4.20} \]

To sum up, combining (4.11), (4.14), (4.16), (4.17), and (4.20), we have \(P\)-a.s.

\[ \lim_{n \to \infty} P_\omega[\mathcal{F}_n(1) \in \mathcal{D}_u \mid \Lambda_n] \]

\[ \leq \left( \frac{2\varepsilon \sigma_1}{\sqrt{2\pi}} + o(\varepsilon) \right)^{-1} \left( \frac{2\varepsilon \sigma_1 e^{-\frac{f(\varepsilon) + g(\varepsilon)}{2\varepsilon^{1/2}}}}{\sqrt{2\pi} \varepsilon (1 - \varepsilon^{1/2})} + o(\varepsilon) \right) \]

\[ \times \prod_{k=2}^d \int_{(u_k - \text{sgn}(u_k)\delta' - \varepsilon^{1/2} \sqrt{1 - \varepsilon^{1/2}})}^{(u_k - \text{sgn}(u_k)\delta' + \varepsilon^{1/2} \sqrt{1 - \varepsilon^{1/2}})} e^{-\frac{t^2}{2\pi}} dt \]

\[ + \left( \frac{2d}{d} \right) 8d\varepsilon^{1/4} \exp \left\{ -\frac{(\delta')^2}{2\varepsilon^{1/2}} \right\} + 2(\varepsilon + g(\varepsilon)). \tag{4.21} \]

Let us now bound the term \(P_\omega[\mathcal{F}_n(1) \in \mathcal{D}_u \mid \Lambda_n]\) from below. We have by the Markov property

\[ P_\omega[\mathcal{F}_n(1) \in \mathcal{D}_u \mid \Lambda_n] \]
Hence, by (4.26) we obtain

$$\geq \frac{P_{\omega}[A_{0-R}, \beta_{\mathcal{R}_{\mathcal{R}, n}} \leq \varepsilon^{1/2} n]}{P_{\omega}[\Lambda_n]} \min_{y \in R_{\mathcal{R}, n}} \min_{j \leq |\varepsilon^{1/2} n|} P_{\omega}[X(n-j) \in \mathcal{D}_{u \sqrt{n}, \Lambda_{n-j}}].$$

(4.22)

We first decompose the term \((P_{\omega}[\Lambda_n])^{-1}P_{\omega}[A_{0-R}, \beta_{\mathcal{R}_{\mathcal{R}, n}} \leq \varepsilon^{1/2} n]\) in the following way

$$\frac{P_{\omega}[A_{0-R}, \beta_{\mathcal{R}_{\mathcal{R}, n}} \leq \varepsilon^{1/2} n]}{P_{\omega}[\Lambda_n]} = \frac{P_{\omega}[A_{0-R}, \beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n]}{P_{\omega}[\Lambda_n]} = \frac{P_{\omega}[A_{0-R}]}{P_{\omega}[\Lambda_n]} (1 - P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid A_{0-R}]).$$

(4.23)

Then, we write

$$P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid A_{0-R}] = \frac{P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n, A_{0-R}]}{P_{\omega}[A_{0-R}]} \leq \frac{P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n, \Lambda_{\varepsilon^{1/2} n}]}{P_{\omega}[A_{0-R}, \Lambda_{\varepsilon^{1/2} n}]} = \frac{P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n, \Lambda_{\varepsilon^{1/2} n}]}{1 - P_{\omega}[A_{0-R} \mid \Lambda_{\varepsilon^{1/2} n}]}.$$ 

(4.24)

For the term \(P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}]\), we have, recalling that \(N = \lceil \varepsilon \sqrt{n} \rceil\),

$$P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}] = P_{\omega}[\tau_{R_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}] \leq P_{\omega}\left[\max_{i \in [2, d]} \sup_{j \leq \tau_{(N)}_i} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_{\varepsilon^{1/2} n} \right] + P_{\omega}[\tau_{(N)}_i > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}].$$

(4.25)

By Lemmas 2.3 and 2.4 we deduce

$$\limsup_{n \to \infty} P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}] \leq g(\varepsilon^{3/4}) + f(\varepsilon^{3/4}).$$

(4.26)

For the term \(P_{\omega}[A_{0-R} \mid \Lambda_{\varepsilon^{1/2} n}]\), we write

$$P_{\omega}[A_{0-R} \mid \Lambda_{\varepsilon^{1/2} n}] = P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \beta_{(0)}^+, \Lambda_{\varepsilon^{1/2} n}] \leq P_{\omega}[\beta_{\mathcal{R}_{\mathcal{R}, n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}].$$

Hence, by (4.26) we obtain

$$\limsup_{n \to \infty} P_{\omega}[A_{0-R} \mid \Lambda_{\varepsilon^{1/2} n}] \leq f(\varepsilon^{3/4}) + g(\varepsilon^{3/4}).$$

(4.27)

Going back to the term \((P_{\omega}[\Lambda_n])^{-1}P_{\omega}[A_{0-R}]\) in (4.23), we write

$$\frac{P_{\omega}[A_{0-R}]}{P_{\omega}[\Lambda_n]} = \frac{P_{\omega}[A_{0-R}]}{P_{\omega}[\Lambda_n, A_{0-R}] + P_{\omega}[\Lambda_n, A_{0-R}^c]}.$$
Thus, by (4.28), (4.30), (4.24), (4.26), (4.27), and (4.29), we deduce
\[ \lim_{n \to \infty} \mathbb{P}_n[A_{0-R} \mid \Lambda_n] \leq \limsup_{n \to \infty} \mathbb{P}_n[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] \leq f(\varepsilon) + g(\varepsilon). \]  
(4.29)

By (4.10), we have
\[ \mathbb{P}_n[\Lambda_n \mid A_{0-R}] \leq \max_{y \in \mathbb{R}_{\varepsilon,n}} \mathbb{P}_n^y[\Lambda_n - [\varepsilon^{1/2} n]] + \mathbb{P}_n[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2} n \mid A_{0-R}]. \]  
(4.30)

Then, we have by the Markov property
\[ \mathbb{P}_n[\Lambda_n \mid A_{0-R}] \leq \mathbb{P}_n[A_{0-R} \mid \Lambda_n] \mathbb{P}_n[\Lambda_n \mid A_{0-R}] + \mathbb{P}_n[\beta_{R_{\varepsilon,n}} > \varepsilon^{1/2} n \mid A_{0-R}]. \]  
(4.32)

Thus, by (4.28), (4.30), (4.24), (4.26), (4.27), and (4.29), we deduce
\[ \liminf_{n \to \infty} \frac{\mathbb{P}_n[A_{0-R}]}{\mathbb{P}_n[\Lambda_n]} \geq \left( \limsup_{n \to \infty} \max_{y \in \mathbb{R}_{\varepsilon,n}} \mathbb{P}_n^y[\Lambda_n - [\varepsilon^{1/2} n]] + \frac{f(\varepsilon^{3/4}) + g(\varepsilon^{3/4})}{1 - f(\varepsilon^{3/4}) - g(\varepsilon^{3/4})} \right)^{-1}. \]  
(4.31)

Combining (4.22), (4.23), (4.26), (4.27), and (4.31), we obtain \( \mathbb{P} \)-a.s.
\[ \lim_{n \to \infty} \mathbb{P}_n[\mathcal{Z}^n(1) \in \mathcal{D}_u \mid \Lambda_n] \geq \left( \limsup_{n \to \infty} \max_{y \in \mathbb{R}_{\varepsilon,n}} \mathbb{P}_n^y[\Lambda_n - [\varepsilon^{1/2} n]] + \frac{f(\varepsilon^{3/4}) + g(\varepsilon^{3/4})}{1 - f(\varepsilon^{3/4}) - g(\varepsilon^{3/4})} \right)^{-1} \times \left( 1 - \frac{f(\varepsilon^{3/4}) + g(\varepsilon^{3/4})}{1 - f(\varepsilon^{3/4}) - g(\varepsilon^{3/4})} \right) \times \liminf_{n \to \infty} \min_{y \in \mathbb{R}_{\varepsilon,n}} \min_{j \leq \varepsilon^{1/2} n} \mathbb{P}_n^y[\mathcal{X}(n - j) \in \mathcal{D}_u \sqrt{n}, \Lambda_n - j]. \]  
(4.32)

Analogously to (4.11) we have
\[ \lim_{n \to \infty} \max_{y \in \mathbb{R}_{\varepsilon,n}} \mathbb{P}_n^y[\Lambda_n - [\varepsilon^{1/2} n]] = \frac{2\varepsilon\sigma_1}{\sqrt{2\pi(1 - \varepsilon^{1/2})}} + o(\varepsilon). \]  
(4.33)

At this point, let us introduce more notations. Let \( \delta' > 0 \) be the constant used in the definitions of \( V_i \) and \( U_i \) (cf. (4.12) and (4.13)) and introduce
\[ E_i = \left\{ \mathcal{X}_i(n) > (u_i + \text{sgn}(u_i)\delta')\sqrt{n} \right\}. \]
and

\[ F_i = \left\{ \max_{j \leq \lfloor \varepsilon^{1/2} n \rfloor} |X_i(n) - X_i(n - j)| \leq \delta' \sqrt{n} \right\} \]

for \( i \in [1, d] \). Observe that for all \( y \in R_{\varepsilon, n} \) and \( j \leq \lfloor \varepsilon^{1/2} n \rfloor \) we have

\[
P_y^\omega \left[ X(n - j) \in D_u \Lambda_{n - j} \right] \geq P_y^\omega \left[ \bigcap_{i=1}^{d} (E_i \cap F_i), \Lambda_n \right] \]

\[
\geq P_y^\omega \left[ \bigcap_{i=1}^{d} E_i, \Lambda_n \right] - \sum_{i=1}^{d} P_y^\omega [F_i^c]. \quad (4.34)
\]

By Theorem 1.2 and similar computations as those to derive equations (4.16) and (4.20), we obtain for some constant \( \gamma_2 \),

\[
\lim_{n \to \infty} \min_{y \in R_{\varepsilon, n}} P_y^\omega \left[ \bigcap_{i=1}^{d} E_i, \Lambda_n \right] = \left( \frac{2 \varepsilon \sigma_1}{\sqrt{2\pi}} \exp \left\{ - \frac{(u_1 + \text{sgn}(u_1) \delta')^2}{2} + o(\varepsilon) \right\} \right) \times \prod_{i=2}^{d} \int_{u_i + \text{sgn}(u_i) \delta'}^{\infty} \exp \left\{ - \frac{t^2}{2} \right\} \frac{dt}{\sqrt{2\pi}} \quad (4.35)
\]

as \( \varepsilon \to 0 \) and

\[
\limsup_{n \to \infty} \max_{y \in R_{\varepsilon, n}} \sum_{i=1}^{d} P_y^\omega [F_i^c] \leq \frac{8d \varepsilon^{1/4}}{\delta' \sqrt{2\pi}} \exp \left\{ - \frac{(\delta')^2}{2 \varepsilon^{1/2}} \right\}. \quad (4.36)
\]

Combining (4.32), (4.33), (4.35), and (4.36), we obtain \( \mathbb{P} \)-a.s.

\[
\liminf_{n \to \infty} P_\omega [Z^n(1) \in D_u \mid \Lambda_n] \geq \left( \frac{2 \varepsilon \sigma_1}{\sqrt{2\pi}(1 - \varepsilon^{1/2})} + o(\varepsilon) \right) + \frac{f(\varepsilon^{3/4})}{1 - f(\varepsilon^{3/4}) - \varepsilon^{3/4}} + \frac{f(\varepsilon) + g(\varepsilon)}{1 - f(\varepsilon) - g(\varepsilon)}
\]

\[
\times \left( 1 - \frac{f(\varepsilon^{3/4}) + g(\varepsilon^{3/4})}{1 - \varepsilon^{3/4} - \varepsilon^{3/4}} \right)
\]

\[
\times \left( \frac{2 \varepsilon \sigma_1}{\sqrt{2\pi}} e^{-\frac{(u_i + \text{sgn}(u_i) \delta')^2}{2}} + o(\varepsilon) \right) \prod_{i=2}^{d} \int_{u_i + \text{sgn}(u_i) \delta'}^{\infty} \exp \left\{ - \frac{t^2}{2} \right\} \frac{dt}{\sqrt{2\pi}}
\]

\[
- \frac{8d \varepsilon^{1/4}}{\delta' \sqrt{2\pi}} \exp \left\{ - \frac{1}{2 \varepsilon^{1/2}} \right\} \quad (4.37)
\]

Finally, take \( \delta' = \varepsilon^{1/8} \) and let \( \varepsilon \to 0 \) in (4.21) and (4.37) to prove (4.2). \( \square \)
The next steps in showing that the f.d.d.’s converge are standard and we follow [14] and [11].
First, we will prove the following

**Proposition 4.2.** We have \( P \)-a.s., for \( u_1 > 0, -\infty < a_i < b_i < \infty, i \in \llbracket 2, d \rrbracket \) and \( 0 < t < 1 \),

\[
\lim_{n \to \infty} P_\omega \left[ Z^n_1(t) \leq u_1, \bigcap_{i=2}^d \left\{ Z^n_i(t) \in (a_i, b_i) \right\} \mid \Lambda_n \right] = \int_0^{u_1} q(0, 0; t, v) d\prod_{i=2}^d \int_{a_i}^{b_i} \frac{\exp\left\{ -\frac{v^2}{2t} \right\}}{\sqrt{2\pi t}} dv. \tag{4.38}
\]

**Proof.** For \( \varepsilon > 0 \) we have

\[
P_\omega \left[ Z^n_1(n^{-1} \lfloor nt \rfloor) \leq u_1 - \varepsilon, \bigcap_{i=2}^d \left\{ Z^n_i(n^{-1} \lfloor nt \rfloor) \in (a_i - \varepsilon, b_i + \varepsilon) \right\} \mid \Lambda_n \right] \leq P_\omega \left[ Z^n_1(t) \leq u_1, \bigcap_{i=2}^d \left\{ Z^n_i(t) \in (a_i, b_i) \right\} \mid \Lambda_n \right] \tag{4.39}
\]

\[
\leq P_\omega \left[ Z^n_1(n^{-1} \lfloor nt \rfloor) \leq u_1 + \varepsilon, \bigcap_{i=2}^d \left\{ Z^n_i(n^{-1} \lfloor nt \rfloor) \in (a_i + \varepsilon, b_i - \varepsilon) \right\} \mid \Lambda_n \right].
\]

for all sufficiently large \( n \). Now, suppose that we have for all \( u_1 \geq 0, a_i < b_i \) and \( 0 < t < 1 \),

\[
\lim_{n \to \infty} P_\omega \left[ Z^n_1(n^{-1} \lfloor nt \rfloor) \leq u_1, \bigcap_{i=2}^d \left\{ Z^n_i(n^{-1} \lfloor nt \rfloor) \in (a_i, b_i) \right\} \mid \Lambda_n \right] = \int_0^{u_1} q(0, 0; t, v) d\prod_{i=2}^d \int_{a_i}^{b_i} \frac{\exp\left\{ -\frac{v^2}{2t} \right\}}{\sqrt{2\pi t}} dv. \tag{4.40}
\]

Combining (4.39) and (4.40) yields (4.38) since the limit distribution \( q(0, 0; t, x_1) \) is absolutely continuous. Let us denote by \( l = l(t, n) \) the quantity \( (n^{-1} |nt|)^{1/2} \).

We recall that \( x \) is the vector of coordinates \( (x_1, \ldots, x_d) \). Then, observe that

\[
P_\omega \left[ Z^n_1(n^{-1} \lfloor nt \rfloor) \leq u_1, \bigcap_{i=2}^d \left\{ Z^n_i(n^{-1} \lfloor nt \rfloor) \in (a_i, b_i) \right\} \mid \Lambda_n \right] = \frac{1}{P_\omega[\Lambda_n]}
\]
\[ \times P_\omega \left[ Z_1^{[nt]}(1) \leq lu_1, \bigcap_{i=2}^d \left\{ Z_i^{[nt]}(1) \in (la_i, lb_i) \right\}, \Lambda_{nt}, X_1(k) > 0, [nt] < k \leq n \right] \]

\[ = \frac{1}{P_\omega[\Lambda_n]} \int_{la_1}^{lu_1} \int_{lb_1}^{ld_1} \cdots \int_{lb_d}^{ld_d} \]

\[ P_\omega \left[ Z_1^{[nt]}(1) \in dx_1, \bigcap_{i=2}^d \left\{ Z_i^{[nt]}(1) \in dx_i \right\}, \Lambda_{nt}, X_1(k) > 0, [nt] < k \leq n \right] \]

\[ = \frac{P_\omega[\Lambda_{nt}]}{P_\omega[\Lambda_n]} \int_{la_2}^{lb_2} \cdots \int_{lb_d}^{ld_d} \]

\[ P_\omega \left[ X_1(k) > 0, [nt] < k \leq n \left| Z_1^{[nt]}(1) \in dx_1, \bigcap_{i=2}^d \left\{ Z_i^{[nt]}(1) \in dx_i \right\} \right. \right] \]

\[ \times P_\omega \left[ Z_1^{[nt]}(1) \in dx_1, \bigcap_{i=2}^d \left\{ Z_i^{[nt]}(1) \in dx_i \right\} \left| \Lambda_{nt} \right. \right] \]

\[ = \frac{P_\omega[\Lambda_{nt}]}{P_\omega[\Lambda_n]} \int_{la_2}^{lb_2} \cdots \int_{lb_d}^{ld_d} P_\omega^{\sqrt{|nt|}} \left[ Z_1^n(s) > 0, 0 \leq s \leq 1 - n^{-1}|nt| \right] \]

\[ \times P_\omega \left[ Z_1^{[nt]}(1) \in dx_1, \bigcap_{i=2}^d \left\{ Z_i^{[nt]}(1) \in dx_i \right\} \left| \Lambda_{nt} \right. \right]. \quad (4.41) \]

By (4.7), (4.11), (4.31), and (4.33) we have \( P \)-a.s.

\[ \lim_{n \to \infty} \frac{P_\omega[\Lambda_{nt}]}{P_\omega[\Lambda_n]} = t^{-1/2}. \quad (4.42) \]

Using Theorem 1.2 and Dini’s theorem on uniform convergence of non-decreasing sequences of continuous functions, we obtain

\[ \lim_{n \to \infty} P_\omega^{\sqrt{|nt|}} \left[ Z_1^n(s) > 0, 0 \leq s \leq 1 - n^{-1}|nt| \right] = \tilde{N} \left( z_1 \left( \frac{t}{1-t} \right)^{1/2} \right) \]

uniformly in \( z \) on every compact set of the form \( [0, K] \times [-K, K]^{d-1} \). By Proposition 4.1, we have

\[ \lim_{n \to \infty} P_\omega \left[ Z_1^{[nt]}(1) \leq x_1, \bigcap_{i=2}^d \left\{ Z_i^{[nt]}(1) \leq x_i \right\} \left| \Lambda_{nt} \right. \right] \]

\[ = \exp \left( -\frac{x_1^2}{2} \right) \prod_{i=2}^d \int_{-\infty}^{x_i} \frac{\exp \left( -\frac{v^2}{2} \right)}{\sqrt{2\pi}} dv. \]
Now, applying Lemma 2.18 of [14] to (4.41), we obtain

\[
\lim_{n \to \infty} P_\omega \left[ Z_l^n(n^{-1}[nt]) \leq u, \bigcap_{i=2}^d \left\{ Z_i^n(n^{-1}[nt]) \in (a_i, b_i] \right\} \mid \Lambda_n \right] = \\
= \int_0^{u_1 t^{-1/2} b_2 t^{-1/2}} \cdots \int_0^{u_d t^{-1/2}} \sqrt{t^{-1/2} N} \left( x_1 \left( \frac{t}{1-t} \right)^{1/2} \right) \\
\times x_1 \exp \left\{ -\frac{x_1^2}{2} \right\} \prod_{i=2}^d \frac{1}{\sqrt{2\pi}} \, dx_1 \cdots dx_d.
\]

Finally, make the change of variables \( y = t^{1/2} x \) to obtain the desired result. \( \square \)

The final step in showing convergence of the f.d.d.'s is

**Proposition 4.3.** We have \( \mathbb{P}\text{-a.s.,} \) for all \( k \geq 1, u_i > 0, -\infty < a_i^j < b_i^j < \infty, \) \( i \in [1, k], \) \( j \in [2, d] \) and \( 0 < t_1 < t_2 < \cdots < t_k \leq 1, \)

\[
\lim_{n \to \infty} P_\omega \left[ \bigcap_{i=1}^k \left\{ Z_i^n(t_i) \leq u_i, Z_i^n(t_i) \in (a_i^j, b_i^j], Z_j^n(t_i) \in (a_j^j, b_j^j] \right\} \mid \Lambda_n \right] = \\
= \prod_{j=2}^d b_j^j \prod_{j=2}^d b_j^j \exp \left\{ -\frac{x_1^2}{2t_1} \right\} \exp \left\{ -\frac{(x_2-x_1)^2}{2(t_2-t_1)} \right\} \cdots \exp \left\{ -\frac{(x_k-x_{k-1})^2}{2(t_k-t_{k-1})} \right\} \\
\times \int_0^{u_1} \cdots \int_0^{u_k} q(0, 0; t_1, x_1)q(t_1, x_1; t_2, x_2) \cdots q(t_{k-1}, x_{k-1}; t_k, x_k) \, dx_k \cdots dx_1.
\]

(4.43)

**Proof.** The proof is by induction in \( k. \) This result holds for \( k = 1 \) by virtue of (4.38). Suppose (4.43) is true for \( k = m - 1, \) we show that it can be extended to \( k = m. \) Let \( t_i' = n^{-1}[t_i n] \) and let

\[ D_i = \{ x \in \mathbb{R}^d : x_1 \leq u_1, a_i^j < x_i \leq b_i^j, j \in [2, d] \} \]

for \( i \in [1, m]. \) We mention here that in this proof, \( y^i \) for \( i \in [1, m] \) are all elements of \( \mathbb{R}^d \) while \( y_i \) for \( i \in [1, m] \) belong to \( \mathbb{R}. \) By the same argument as in the beginning of the proof of Proposition 4.2, observe that

\[
\lim_{n \to \infty} P_\omega \left[ \bigcap_{i=1}^m \{ Z_i^n(t_i) \in D_i \} \mid \Lambda_n \right]
\]
provided that the limits exist. Then, we write for sufficiently large $n$

$$
\begin{align*}
\mathbb{P}_{\omega} & \left[ \bigcap_{i=1}^{m-2} \{ Z^n(t_i) \in D_1 \}, \bigcap_{i=m-1}^{m} \{ Z^n(t'_i) \in D_1 \} \mid \Lambda_n \right] \\
& = \frac{1}{\mathbb{P}_{\omega}[\Lambda_n]} \int_{D_{m-1}} \int_{D_m} \mathbb{P}_{\omega} \left[ \left. Z^n(t_1), \ldots, Z^n(t_{m-2}) \in D_{m-2}, Z^n(t'_{m-1}) \in dy^{m-1}, Z^n(t'_m) \in dy^m, X_1(1) > 0, \ldots, X_1(n) > 0 \right. \right] \\
& \times \mathbb{P}_{\omega}^{y_m^{-1} \sqrt{n}} \left[ Z^n(s) > 0, 0 \leq s \leq t'_m - t'_{m-1}, Z^n(t'_m - t'_{m-1}) \in dy^m \right] \\
& \times \mathbb{P}_{\omega}^{y_m^{-1} \sqrt{n}} \left[ Z^n(s) > 0, 0 \leq s \leq 1 - t'_m \right].
\end{align*}
$$

By the induction hypothesis we have

$$
\begin{align*}
\lim_{n \to \infty} \mathbb{P}_{\omega} & \left[ Z^n(t_1) \in D_1, \ldots, Z^n(t_{m-2}) \in D_{m-2}, Z^n(t'_{m-1}) \in D_{m-1} \mid \Lambda_{n t_{m-1}} \right] \\
& = \prod_{j=2}^{d} \int_{a_{j-1}}^{b_{j-1}} \int_{a_{j-1}}^{b_{j-1}} \frac{\exp \left\{ -\frac{(y_{m-1} - y_{m-2})^2}{2(t_{m-1} - t_{m-2})} \right\}}{\sqrt{2\pi(t_{m-1} - t_{m-2})}} dy_{m-1} \cdots dy_1 \\
& \times \frac{\exp \left\{ -\frac{(y_{m-1} - y_{m-2})^2}{2(t_{m-1} - t_{m-2})} \right\}}{\sqrt{2\pi(t_{m-1} - t_{m-2})}} dy_{m-1} \cdots dy_1 \\
& \times q(t_{m-2}/t_{m-1}, y_{m-2}; 1, y_{m-1}) dy_{m-1} \cdots dy_1.
\end{align*}
$$

On the other hand, by (4.42) we have P.a.s.

$$
\lim_{n \to \infty} \frac{\mathbb{P}_{\omega}[\Lambda_{n t_{m-1}}]}{\mathbb{P}_{\omega}[\Lambda_n]} = t_{m-1}^{1/2}.
$$
Using Theorem 1.2 and Dini’s theorem on uniform convergence of non-decreasing sequences of continuous functions, we obtain
\[
\lim_{n \to \infty} \mathbb{P}^{y_{m-1}}_\omega \left[ Z^n_1(s) > 0, 0 \leq s \leq t'_m - t'_{m-1}, Z^n(t'_m - t'_{m-1}) \in D_m \right] = \prod_{j=2}^{b_{m}} \int_{a_{j}^{m}} x_{j}^{m} \exp \left( \frac{(y_{m} - x_{j}^{m})^2}{\sqrt{2\pi t_{j}}} \right) dx_{j} \int_{0}^{u_{m}} g(t_m - t_{m-1}, y_{m-1}) dy_{m} \tag{4.48}
\]
uniformly in \(y_{m-1}\) on every compact set of the form \([0, K] \times [-K, K]^d\), and
\[
\lim_{n \to \infty} \mathbb{P}^{y_{m}}_\omega \left[ Z^n_1(s) > 0, 0 \leq s \leq 1 - t'_m \right] = \tilde{N}(y_{m}(1 - t_m)^{-1/2}) \tag{4.49}
\]
uniformly in \(y_{m}\) on every compact set of the form \([0, K] \times [-K, K]^d\). Combining (4.44), (4.45), (4.46), (4.47), (4.48), (4.49), and using Lemma 2.18 of [14] twice, we obtain
\[
\lim_{n \to \infty} \mathbb{P}_\omega \left[ \bigcap_{i=1}^{m} \{ Z^n(t_i) \in D_i \} \mid \Lambda_n \right] = \prod_{j=2}^{b_{1}} \int_{a_{j}^{1}} x_{j}^{1} \exp \left( \frac{-x_{j}^{2}}{\sqrt{2\pi t_{j}}} \right) \exp \left( \frac{-x_{j}^{2}}{2(t_{j} - t_{1})} \right) \sqrt{2\pi t_{j} - t_{1}} \exp \left( \frac{(x_{j} - x_{j}^{1})^2}{2(t_{j} - t_{1})} \right) dx_{j} \tag{4.50}
\]
Now, make the change of variables \(t_1^{1/2}y_1 = x_1, \ldots, t_{m-1}^{1/2}y_{m-2} = x_{m-2}\) in (4.50) to obtain (4.43) for \(k = m\).

### 4.2. Tightness

In this section, to finish the proof of Theorem 1.1, we prove that the sequence of measures \(\{\mathbb{P}_\omega[Z^n \in \cdot \mid \Lambda_n]\}_{n \geq 1}\) is tight \(\mathbb{P}\)-a.s. The proof is standard: we consider the modulus of continuity, divide the time interval into small subintervals,
and have to control the probability of fluctuations of our conditioned process over these small time intervals, where we use the results of the last subsection. First, we define the modulus of continuity for functions $f \in C[0,1]$:

$$w_f(\delta') = \sup_{|t-s| \leq \delta'} \left\{ \| f(s) - f(t) \|_\infty \right\}$$

where $s,t \in [0,1]$ and $\| \cdot \|_\infty$ is the $\infty$-norm on $\mathbb{R}^d$. By Theorem 14.5 of [15] it suffices to show that $\mathbb{P}$-a.s., for every $\hat{\varepsilon} > 0$

$$\lim \limsup_{n \to \infty} \mathbb{P}_\omega[w_{Z^n}(\delta') \geq \hat{\varepsilon} | \Lambda_n] = 0 \quad (4.51)$$

since $Z^n(0) = 0$. Now observe that

$$\mathbb{P}_\omega[w_{Z^n}(\delta') \geq \hat{\varepsilon} | \Lambda_n] = \mathbb{P}_\omega\left[ \sup_{|t-s| \leq \delta'} \| Z^n(t) - Z^n(s) \|_\infty \geq \hat{\varepsilon} | \Lambda_n \right]$$

$$\leq \mathbb{P}_\omega\left[ \sup_{|t-s| \leq \delta'} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} | \Lambda_n \right] \quad (4.52)$$

for $n \geq 2/\delta'$. Let $m := [1/4\delta']$ and divide the interval $[0,1]$ into intervals $I_k := [k/m, (k+1)/m)$, for $0 \leq k \leq m-1$. Additionally, consider the intervals $J_l := [(2l+1)/2m, (2l+3)/2m]$, for $0 \leq l \leq m-2$ and $J_{m-1} := \emptyset$. Observe that

$$\mathbb{P}_\omega\left[ \sup_{|t-s| \leq \delta'} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} | \Lambda_n \right]$$

$$\leq \mathbb{P}_\omega\left[ \left\{ \max_{k \leq m-1} \sup_{s,t \in I_k} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} \right\} \right.$$

$$\cup \left\{ \max_{l \leq m-2} \sup_{s,t \in J_l} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} \right\} | \Lambda_n \right]$$

$$\leq m \left( \max_{k \leq m-1} \mathbb{P}_\omega\left[ \sup_{s,t \in I_k} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} | \Lambda_n \right] \right.$$ 

$$\left. + \max_{l \leq m-1} \mathbb{P}_\omega\left[ \sup_{s,t \in J_l} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} | \Lambda_n \right] \right) \quad (4.53)$$

with the convention that $\sup_{s,t \in \emptyset} \{ \} = 0$. Our next step is to bound from above the $\limsup_{n \to \infty}$ of both terms in parentheses in the right-hand side of (4.53). As an example, let us treat the terms indexed by $I_k$ for $k \in [1,m-1]$. The term indexed by $I_0$ and those indexed by $J_k$, $k \in [1,m-1]$ can be treated in a similar way. To do that, we will use the same approach as in the proof of Proposition 4.1. Analogously to (4.4) we have for $\varepsilon \in (0,1)$ and $\delta \in (0,1)$,

$$\mathbb{P}_\omega\left[ \sup_{s,t \in I_k} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n} | \Lambda_n \right]$$

$$\leq (\mathbb{P}_\omega[\Lambda_n])^{-1} \mathbb{P}_\omega\left[ \sup_{s,t \in I_k} \| X(nt) - X(ns) \|_\infty \geq \hat{\varepsilon} \sqrt{n}, A_{\theta \to R}, \Lambda_n, \beta_{\theta,\varepsilon} \leq \delta nm^{-1} \right]$$
\[
\begin{align*}
+ P_\omega[\beta_{R,n} > \delta nm^{-1} | \Lambda_n] + P_\omega[\beta_{R,n} > \varepsilon^{1/2}n | \Lambda_n].
\end{align*}
\] (4.54)

Analogously to (4.6), we obtain
\[
(P_\omega[A_0])^{-1} P_\omega[\sup_{s,t \in I_k} \|X(nt) - X(ns)\|_\infty \geq \hat{\varepsilon} \sqrt{n}, A_0 \rightarrow R, A_n, \beta_{R,n} \leq \delta nm^{-1}] \leq P_\omega[A_0 \rightarrow R] \leq P_\omega[\Lambda_n] \max_{y \in \mathbb{R}} P_y[\sup_{s,t \in I_k} \|X(nt - j) - X(ns - j)\|_\infty \geq \hat{\varepsilon} \sqrt{n}].
\]
(4.55)

Now, observe that for all sufficiently large \(n\)
\[
\max_{j \leq \lfloor \frac{\delta n}{m} \rfloor} P_y[\sup_{s,t \in I_k} \|X(nt - j) - X(ns - j)\|_\infty \geq \hat{\varepsilon} \sqrt{n}] \leq P_y[\sup_{s,t \in I_k'} W_1(t) - W_1(s) \geq \hat{\varepsilon}] \leq 8d \cdot P[W_1 \left( \frac{1 + 3\delta}{m} \right) \geq \hat{\varepsilon}] \leq \frac{16d}{\hat{\varepsilon} \sqrt{2\pi m}} \exp \left\{ - \frac{\hat{\varepsilon}^2 m}{8} \right\}.
\] (4.56)

since \(\delta < 1\). We obtain
\[
\limsup_{n \to \infty} \max_{y \in \mathbb{R}, n} P_y[\sup_{s,t \in I_k} \|X(nt - j) - X(ns - j)\|_\infty \geq \hat{\varepsilon} \sqrt{n}] \leq \frac{16d}{\hat{\varepsilon} \sqrt{2\pi m}} \exp \left\{ - \frac{\hat{\varepsilon}^2 m}{8} \right\}.
\] (4.57)

Thus, we have by (4.7), (4.10), (4.54), and (4.57)
\[
\limsup_{n \to \infty} P_\omega[\sup_{s,t \in I_k} \|X(nt) - X(ns)\|_\infty \geq \hat{\varepsilon} \sqrt{n} | \Lambda_n] \leq \left( \frac{2\varepsilon \sigma_1}{\sqrt{2\pi}} + o(\varepsilon) \right)^{-1} \left( \frac{16d}{\hat{\varepsilon} \sqrt{2\pi m}} \exp \left\{ - \frac{\hat{\varepsilon}^2 m}{8} \right\} \right) + f(\varepsilon) + g(\varepsilon) + \limsup_{n \to \infty} P_\omega[\beta_{R,n} > \delta nm^{-1} | \Lambda_n].
\] (4.58)
Combining (4.53) and (4.58) we find
\[
\limsup_{n \to \infty} P_\omega \left[ \sup_{|t-s| \leq 2}\|X'(nt) - X'(ns)\|_\infty \geq \hat{\epsilon}\sqrt{n} | \Lambda_n \right]
\leq 2m \left( \frac{16d}{\hat{\epsilon}\sqrt{2\pi m}} \exp \left\{ -\frac{\hat{\epsilon}^2 m}{8} \right\} \frac{2\epsilon \sigma_1}{\sqrt{2\pi}} + o(\epsilon) \right)^{-1}
+ f(\epsilon) + g(\epsilon) + \limsup_{n \to \infty} P_\omega [\beta_{R_{\hat{\epsilon}n}} > \delta nm^{-1} | \Lambda_n].
\]
(4.59)

Then, let \( \epsilon = m^{-3} \) and \( \delta = m^{-1/2} \) in (4.59). We have by (4.10)
\[
\limsup_{n \to \infty} P_\omega [\beta_{R_{\hat{\epsilon}n}} > \delta nm^{-1} | \Lambda_n]
= \limsup_{n \to \infty} P_\omega [\beta_{R_{\hat{\epsilon}n}} > \epsilon^{1/2}n | \Lambda_n] \leq f(m^{-3}) + g(m^{-3}).
\]

Therefore, we obtain
\[
\lim_{m \to \infty} \limsup_{n \to \infty} P_\omega \left[ \sup_{s,t \in \hat{I}_k} \|X'(nt) - X'(ns)\|_\infty \geq \hat{\epsilon}\sqrt{n} | \Lambda_n \right] = 0.
\]

As \( \hat{\epsilon} \) is arbitrary and \( m = \lfloor 1/4\delta' \rfloor \), using (4.52), this last expression proves (4.51) and consequently the tightness of the sequence \( (P_\omega [Z^n \in \cdot | \Lambda_n])_{n \geq 1} \).

\[\Box\]

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