A NOTE ON THE DISTRIBUTIONS OF THE MAXIMUM OF LINEAR BERNOULLI PROCESSES

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Abstract
We give a characterization of the family of all probability measures on the extended line $(-\infty, +\infty]$, which may be obtained as the distribution of the maximum of some linear Bernoulli process.

On a probability space $(\Omega, \mathcal{P})$ consider a linear process

$$X(t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \xi_n, \quad t \in T,$$

(1)

generated by independent, identically distributed random variables $\xi_n$ with $E\xi_n = 0$, $E\xi_n^2 = 1$. The coefficients $a_n(t)$ are assumed to be arbitrary functions on the parameter set $T$, satisfying $\sum_{n=1}^{\infty} a_n(t)^2 < +\infty$ for any $t \in T$, so that the series (1) is convergent a.s. Define

$$M = \sup_{t} X(t)$$

(2)

in the usual way as the essential supremum in the space of all random variables with values in the extended real line (identifying random variables that coincide almost surely; cf. Remark 4 below).

We consider the question on the characterization of the family $\mathcal{F}(L)$ of all possible distribution functions $F(x) = P\{M \leq x\}$ of $M$, assuming that the common law $L$ of $\xi_n$ is given. In general, $M$ may take the value $+\infty$ with positive probability, so its distribution is supported on $(-\infty, +\infty]$. Introduce also the collection $\mathcal{F}_0(L)$ of all possible distribution functions of $M$ in (2), such that in the series (1), for all $t \in T$,

$$a_n(t) = 0, \quad \text{for all sufficiently large } n.$$

(3)
When $\xi_n$ are standard normal, i.e., $L = N(0, 1)$, we deal in (1) with an arbitrary Gaussian random process. As is well-known, for the distribution function $F$ of $M$, $x_0 = \inf\{x \in \mathbb{R} : F(x) > 0\}$ may be finite, and then it is sometimes called a take-off point of the maximum of the Gaussian process. Moreover, $F$ may have an atom at it. But anyway $F$ is absolutely continuous and strictly increasing on $(x_0, +\infty)$, which follows from the log-concavity of Gaussian measures (cf. also [C], [HJ-S-D]).

A complete characterization of all possible distributions $F$ in the Gaussian case may be derived from the Brunn-Minkowski-type inequality for the standard Gaussian measure $\gamma_n$ on $\mathbb{R}^n$ due to A. Ehrhard [E]. It states that, for all convex (and in fact, for all Borell measurable, cf. [Bo2]) sets $A$ and $B$ in $\mathbb{R}^n$ of positive measure and for all $\lambda \in (0, 1),$

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)),$$

where $\Phi^{-1}$ denotes the inverse to the standard normal distribution function on the line. This inequality immediately implies that, if $F$ is non-degenerate, the function $U = \Phi^{-1}(F)$ must be concave on $\mathbb{R}$ in the generalized sense as a function with values in $[-\infty, +\infty)$. But the converse is true, as well.

Indeed, suppose $U = \Phi^{-1}(F)$ is concave on $\mathbb{R}$, and for simplicity let $F$ be non-degenerate and do not assign a positive mass to the point $+\infty$. Then $F$ is strictly increasing on $(x_0, +\infty)$, so is its inverse $F^{-1} : (F(x_0), 1) \to (x_0, +\infty)$. Moreover, the inverse function $U^{-1} = F^{-1}(\Phi)$ is convex and strictly increasing on $(U(x_0), +\infty)$. Put $M(x) = U^{-1}(x)$ for $x > U(x_0)$, and if $x_0$ is finite, $M(x) = x_0$ on $(-\infty, U(x_0)]$. Then $M$ is convex and finite on the whole real line, and therefore admits a representation

$$M(x) = \sup_{t \in T} [a_0(t) + a_1(t)x], \quad x \in \mathbb{R},$$

for some coefficients $a_0(t), a_1(t)$. By the construction, $M$ has the distribution function $F$ under the measure $\gamma_1$, as was required.

Thus, a given non-degenerate distribution function $F$ belongs to $\mathcal{F}(N(0, 1))$, if and only if the function $\Phi^{-1}(F)$ is concave. A similar characterization holds true, when $\xi_n$’s have a shifted one-sided exponential distribution with mean zero. Then, $F$ represents the distribution function of $M$ for some coefficients $a_n(t)$, if and only if the function $\log F$ is concave. This follows from the log-concavity of the multidimensional exponential distribution (which is a particular case of Prékopa’s theorem [P]; cf. also [Bo1] for a general theory of log-concave measures).

In both above examples, for the "if" part it suffices to consider simple linear processes $X(t) = a_0(t) + a_1(t)\xi_1$. Hence, $\mathcal{F}_0(L) = \mathcal{F}(L)$. The situation is completely different, when $\xi$ have a symmetric Bernoulli distribution $L$, i.e., taking the values $\pm 1$ with probability $\frac{1}{2}$. This may be seen from:

**Theorem 1.** Any distribution function $F$, such that $F(x) = 0$, for some $x \in \mathbb{R}$, may be obtained as the distribution function of the supremum $M$ of some linear Bernoulli process $X$ in (1) with coefficients, satisfying the property (3).

In turn, the condition (3) ensures that all random variables $X(t)$ in (1) are bounded from below, so is the random variable $M$ in (2). Therefore, the distribution $F$ of $M$ must be one-sided. Thus, we have a full description of the family $\mathcal{F}_0(L)$ in the Bernoulli case. Removing the condition (3), we obtain a larger family $\mathcal{F}(L)$; however, it is not clear at all how to characterize it.
One should also mention that in the homogeneous case \( a_0(t) = 0 \), much is known about various properties of \( M \) in terms of \( L \), but the characterization problem is more delicate, and it seems no description or even conjecture are known in all above cases.

For the proof of Theorem 1 one may assume that \( \Omega = \{-1,1\}^\infty \) is the infinite dimensional discrete cube, equipped with the product Bernoulli measure \( \mathbb{P} \). An important property of \( \Omega \), which will play the crucial role, is that it represents the collection of all extreme points in the cube \( K = [-1,1]^\infty \). More precisely, we apply the following statement.

**Lemma 2.** Any lower semi-continuous function \( f : \{-1,1\}^\infty \to (-\infty, +\infty] \) is representable as
\[
f(x) = \sup_{t \in T} \left[ a_0(t) + \sum_{n=1}^{\infty} a_n(t)x_n \right], \quad x = (x_1, x_2, \ldots),
\]
for some family of the coefficient functions \( a_n(t) \), defined on a countable set \( T \) and satisfying the property (3).

Note any function of the form (4) is lower semi-continuous.

**Proof.** First, more generally, let \( K \) be a non-empty, compact convex set in a locally convex space \( E \), and denote by \( \Omega \) the collection of all extreme points of \( K \). A function \( f : \Omega \to (-\infty, +\infty] \) is representable as
\[
f(x) = \sup_{t} f_t(x), \quad x \in \Omega,
\]
for some family \( (f_t)_{t \in T} \) of continuous, affine functions on \( E \), if and only if
\begin{enumerate}
  \item \( f \) is lower semi-continuous on \( \Omega \);
  \item \( f \) is bounded from below.
\end{enumerate}

This characterization follows from a theorem, usually attributed to Hervé [H]; see E. M. Alfsen [A], Proposition 1.4.1, and historical remarks. Namely, a point \( x \) is an extreme point of \( K \), if and only if \( \bar{g}(x) = g(x) \), for any lower semi-continuous function \( g \) on \( K \), where \( \bar{g} \) denotes the lower envelope of \( g \) (i.e., the maximal convex, lower semi-continuous function on \( K \), majorized by \( g \)).

Clearly, the equality (5) defines a function with properties (a) – (b). For the opposite direction one may use an argument, contained in the proof of Corollary 1.4.2 of [A]. If \( f \) is bounded and lower semi-continuous on \( \Omega \), put \( g(x) = \liminf_{y \to x} f(y) \) for \( x \in \text{clos}(\Omega) \) and \( g = \sup_{\Omega} f \) on \( K \setminus \text{clos}(\Omega) \). Then \( g \) is lower semi-continuous on \( K \) and \( g = f \) on \( \Omega \). By Hervé’s theorem, \( \bar{g}(x) = g(x) = f(x) \), for all \( x \in \Omega \). Since \( \bar{g} \) is also convex on \( K \), one may apply to it the classical theorem on the existence of the representation
\[
\bar{g}(x) = \sup_{t} f_t(x), \quad x \in K,
\]
for some family \( (f_t)_{t \in T} \) of continuous, affine functions on \( E \) (cf. e.g. [A], Proposition 1.1.2, or [M], Chapter 11). Thus, restricting this representation to \( \Omega \), we arrive at (5). Finally, if \( f \) is unbounded from above, write \( f = \sup_n \min\{f, n\} \) and apply (5) to the sequence \( \min\{f, n\} \).

In case of the infinite dimensional discrete cube, the right-hand side of (5) may further be specified. Indeed, any continuous, affine function \( g \) on \( E = \mathbb{R}^\infty \) has the form \( g(x_1, x_2, \ldots) = a_0 + \sum_{n=1}^{\infty} a_n x_n \) with finitely many non-zero coefficients. Therefore, (5) is reduced to the
relation (4) with some coefficient functions \( a_n = a_n(t) \), that are defined on non-empty, perhaps, uncountable set \( T \) and satisfy the property (3). The latter implies that the sets \( T_N = \{ t \in T : a_n(t) = 0, \text{ for all } n > N \} \) are non-empty for all \( N \geq N_0 \) with a sufficiently large \( N_0 \). Define

\[
 f_N(x) = \sup_{t \in T_N} \left[ a_0(t) + \sum_{n=1}^{\infty} a_n(t)x_n \right] = \sup_{t \in T_N} \left[ a_0(t) + \sum_{n=1}^{N} a_n(t)x_n \right],
\]

so that \( f = \sup_{N \geq N_0} f_N \). Since for each point \( v = (x_1, \ldots, x_N) \) in the finite dimensional discrete cube \( \{-1, 1\}^N \), the second supremum in (6) is asymptotically attained for some sequence of indices in \( T_N \), one may choose a countable subset \( T_N(v) \) of \( T_N \), such that

\[
 \sup_{t \in T_N} \left[ a_0(t) + \sum_{n=1}^{N} a_n(t)x_n \right] = \sup_{t \in T_N(v)} \left[ a_0(t) + \sum_{n=1}^{N} a_n(t)x_n \right].
\]

Therefore, the set \( T'_N = \bigcup_{v \in \{-1, 1\}^N} T_N(v) \) is also countable, is contained in \( T_N \), and by (6),

\[
 f_N(x) = \sup_{t \in T'_N} \left[ a_0(t) + \sum_{n=1}^{\infty} a_n(t)x_n \right], \quad \text{for all } x \in \{-1, 1\}^\infty.
\]

As a result, the supremum in (4) may be restricted to the countable set \( \bigcup_N T'_N \).

Finally, let us note \( \Omega \) is compact, so the property \( b) \) is automatically satisfied, when \( a) \) holds. This yields Lemma 2.

**Proof of Theorem 1.** According to Lemma 2, we need to show that distributions of lower semi-continuous functions \( f \) on \( \{-1, 1\}^\infty \) under the Bernoulli measure \( \mathbf{P} \) fill the family of all one-sided distributions on \( (-\infty, +\infty) \). In fact, it is enough to consider the functions of the special form \( f(x) = \varphi(Q(x)) \), where

\[
 Q(x) = \sum_{n=1}^{\infty} x_n + \frac{1}{2^{n+1}}, \quad x = (x_1, x_2, \ldots) \in \{-1, 1\}^\infty,
\]

and where \( \varphi : [0, 1] \to (-\infty, +\infty] \) is an arbitrary non-decreasing, left (or, equivalently, lower semi-) continuous function. It is allowed that for some point \( p \in [0, 1] \), \( \varphi \) jumps to the value \( +\infty \), and then we require that \( \lim_{s \to p} \varphi(s) = +\infty \), as part of the lower semi-continuity assumption.

The map \( Q \) is continuous and pushes forward \( \mathbf{P} \) to the normalized Lebesgue measure \( \lambda \) on the unit interval \([0, 1]\). Hence, \( f \) is lower semi-continuous, and its distribution under \( \mathbf{P} \) coincides with the distribution of \( \varphi \) under \( \lambda \).

It remains to see that, for any one-sided probability measure \( \mu \) on \( (-\infty, +\infty] \), there is an admissible \( \varphi \) with the distribution \( \mu \) under \( \lambda \). Let us recall the standard argument (cf. e.g. [Bi], Theorem 14.1). Introduce the distribution function \( F(u) = \mu((-\infty, u]) \), \( -\infty < u \leq +\infty \), and define its "inverse"

\[
 \varphi(s) = \min\{u : F(u) \geq s\}, \quad 0 < s \leq 1.
\]

Also put \( \varphi(0) = \lim_{s \downarrow 0} \varphi(s) \). Clearly, \( \varphi \) is non-decreasing. Given a sequence \( s_n \uparrow s \), \( 0 < s_n < s \leq 1 \), take minimal values \( u_n, u \), such that \( F(u_n) \geq s_n \), \( F(u) \geq s \). We have \( u_n \uparrow u' \), for some
u' \leq u. Since F(u') \geq s_n, for all n, we get F(u') \geq s and hence u' \geq u. This shows that \varphi is left continuous. Finally, given s \in (0, 1] and \alpha > \varphi(0), by the definition, \varphi(s) \leq \alpha \Leftrightarrow F(u) \geq s, for some u \leq \alpha. Hence,

\{ s \in (0, 1] : \varphi(s) \leq \alpha \} = \{ s \in (0, 1] : F(u) \geq s, for some u \leq \alpha \} = (0, F(\alpha)].

Thus, \varphi has the distribution function F under \lambda. The proof is now complete.

**Remark 3.** The statement of Theorem 1 remains to hold in case of arbitrary independent random variables \xi_n, taking two values, say, a_n and b_n with probabilities p_n and q_n, satisfying

\prod_{n=1}^{\infty} \max\{p_n, q_n\} = 0.

In this case, the joint distribution P of \xi_n’s represents a product probability measure on \prod_{n=1}^{\infty} \{a_n, b_n\} without atoms. Let a_n = -1 and b_n = 1 (without loss of generality). Then, the map Q in the proof of Theorem 1 pushes P forward to a non-atomic probability measure \lambda on [0, 1], and a similar argument works.

**Remark 4.** The set S = S(\Omega, P) of all random variables with values in the extended line (-\infty, +\infty] represents a lattice with ordering X \leq Y a.s. Given an arbitrary non-empty collection \{X(t)\}_{t \in T} in S, there is a unique element M in S, called the essential (or structural) supremum of the family \{X(t)\}_{t \in T}, with the properties that

a) X(t) \leq M (a.s.), for all t \in T;
b) If for all t \in T we have X(t) \leq M' (a.s.), M' \in S, then M \leq M' (a.s.)

It is a well-known general fact that M can be represented as a pointwise supremum M = \sup_{n} X(t_n) a.s., for some sequence t_n in T (cf. e.g. [K-A]). In particular, the supremum in (2) may always be taken over all t’s from a countable subset of T.

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**References**

[A] E. M. Alfsen (1971), Compact convex sets and boundary integrals. Springer-Verlag, New York, Heidelberg, Berlin.

[Bi] P. Billingsley (1979), Probability and measure. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York-Chichester-Brisbane, xiv+515 pp.

[Bo1] C. Borell (1974), Convex measures on locally convex spaces. *Ark. Mat.*, 12, 239–252.

[Bo2] C. Borell (2003), The Ehrhard inequality. *C. R. Math. Acad. Sci. Paris*, 337, No. 10, 663–666.

[C] B. S. Cirel'son, Density of the distribution of the maximum of a Gaussian process. *Teor. Veroyatnost. i Primenen.*, 20 (1975), No. 4, 865–873 (Russian).

[E] A. Ehrhard (1983), Symetrisation dans l'espace de Gauss. *Math. Scand.*, 53, No. 2, 281–301.
[H] M. Hervé (1961), Sur les représentations intégrales à l’aide des points extrémaux dans un ensemble compact convexe métrisable. (French) C. R. Acad. Sci. Paris, 253, 366–368.

[HJ-S-D] J. Hoffmann-Jorgensen, L. A. Shepp, and R. M. Dudley (1979), On the lower tail of Gaussian seminorms. Ann. Probab., 7, No. 2, 319–342.

[K-A] L. V. Kantorovich, and G. P. Akilov (1982), Functional analysis. Translated from the Russian by Howard L. Silcock. Second edition. Pergamon Press, Oxford-Elmsford, N.Y., xiv+589 pp.

[M] P. A. Meyer (1966), Probability and potentials. Blaisdell Publ. Co., Toronto.

[P] A. Prékopa (1971), Logarithmic concave measures with application to stochastic programming. Acta Sci. Math. (Szeged), 32, 301–316.