A construction of $\text{Spin}(7)$-instantons

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Abstract

Joyce constructed examples of compact eight-manifolds with holonomy $\text{Spin}(7)$, starting with a Calabi–Yau four-orbifold with isolated singular points of a special kind. That construction can be seen as the gluing of ALE $\text{Spin}(7)$-manifolds to each singular point of the Calabi–Yau four-orbifold divided by an anti-holomorphic involution fixing only the singular points.

On the other hand, there are higher-dimensional analogues of anti-self-dual instantons in four dimensions on $\text{Spin}(7)$-manifolds, which are called $\text{Spin}(7)$-instantons. They are minimizers of the Yang–Mills action, and the $\text{Spin}(7)$-instanton equation together with a gauge fixing condition forms an elliptic system.

In this article, we construct $\text{Spin}(7)$-instantons on the examples of compact $\text{Spin}(7)$-manifolds above, starting with Hermitian–Einstein connections on the Calabi–Yau four-orbifolds and ALE spaces. Under some assumptions on the Hermitian–Einstein connections, we glue them together to obtain $\text{Spin}(7)$-instantons on the compact $\text{Spin}(7)$-manifolds. We also give a simple example of our construction.

1 Introduction

This article is about a construction of $\text{Spin}(7)$-instantons on examples of compact Riemannian 8-manifolds with holonomy $\text{Spin}(7)$. We construct those on Joyce’s $\text{Spin}(7)$-manifolds of the second type, namely on a resolution of the quotient of a Calabi–Yau four-orbifold by an anti-holomorphic involution fixing only the singular points.

A $\text{Spin}(7)$-manifold is an 8-dimensional Riemannian manifold with holonomy contained in the group $\text{Spin}(7)$. The holonomy group $\text{Spin}(7)$ is one of exceptional cases (the other is the group $G_2$) of Berger’s classification of Riemannian holonomy groups of simply-connected, irreducible, non-symmetric Riemannian manifolds [Ber55]. Later metrics with holonomy $\text{Spin}(7)$ (and
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$G_2$ as well) were obtained by Bryant [Bry87], Bryant–Salamon [BSS90] for non-compact cases, and by Joyce [Joy96, Joy99, Joy00] for compact cases.

There are two types in Joyce’s constructions of compact Spin(7)-manifolds, namely, the construction of the metrics on

(I) the resolution of $T^8/\Gamma$, where $T^8$ is a torus and $\Gamma$ is a finite subgroup of automorphisms of $T^8$ [Joy96, Joy00].

(II) the resolution of Calabi–Yau four-orbifolds with finitely many singular points, and an anti-holomorphic involution fixing only the singular points [Joy99, Joy00].

Spin(7)-instantons are Yang–Mills connections on a Spin(7)-manifold, which minimize the Yang–Mills action. They are higher-dimensional analogues of anti-self-dual instantons in four dimensions, discussed firstly by physicists such as Corrigan–Devchand–Fairlie–Nuyts [CDFN83], Ward [War84], and later, in the String Theory context, by Acharya–O’Loughlin–Spence [AOS97], Baulieu–Kanno–Singer [BKS98], and others. In mathematics, they were studied by Reyes Carrión [RC98], Lewis [Lew98], Donaldson–Thomas [DT98], and later by Donaldson–Segal [DS09], and several others. Analytic results concerning gauge theory in higher dimensions were obtained by Nakajima [Nak88, Nak89], Tian [Tia00], Brendle [Bre03a, Bre03b], Tao–Tian [TT04], and others.

Lewis [Lew98] constructed Spin(7)-instantons on the Spin(7)-manifolds of the first type (I). He constructed them from a family of anti-self-dual instantons on $\mathbb{R}^4$ along a Cayley submanifold and glued them together to get a Spin(7)-instanton on the Spin(7)-manifold.

In this article, we construct Spin(7)-instantons on the Spin(7) manifolds of the second type (II). The Spin(7)-manifold is obtained by gluing ALE Spin(7)-manifolds at each singular point of a Calabi–Yau four-orbifold with finitely many singular points, and an anti-holomorphic involution fixing only the singular points. In this article, assuming that there are Hermitian–Einstein connections with certain conditions on both the Calabi–Yau four-orbifold and the ALE spaces, we glue them together to obtain a Spin(7)-instanton on the manifold (Theorem 6.1).

The organization of this article is as follows. In Section 2, we outline Joyce’s construction of Spin(7)-manifolds from a Calabi–Yau four-orbifold with finitely many singular points, and an anti-holomorphic involution fixing only the singular set. In Section 3, we introduce the Spin(7)-instanton equation and describe some of its properties, such as its relation to the complex ASD equation and the Hermitian–Einstein equation, and the linearization
of them. In Section 4, we construct approximate solutions from Hermitian–Einstein connections with certain conditions on a Calabi–Yau four-orbifold and ALE spaces, and derive an estimate which we need for the construction. In Section 5, we discuss the linearization of the $Spin(7)$-instanton equation, and derive estimates coming from the Fredholm property of the linearized operator. In Section 6, we give a construction of $Spin(7)$-instantons by using the estimates in Sections 4 and 5. A simple example of the construction is given in Section 7.

Notations. Throughout this article, $C$ is a positive constant independent of $t$, where $t$ is a gluing parameter which is introduced in Section 2.3, but it can be different each time it occurs.

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2 Joyce’s second construction of compact $Spin(7)$-manifolds

We briefly describe the $Spin(7)$-manifolds constructed by Joyce in [Joy99] (see also [Joy00], Chapter 15). General references for $Spin(7)$-manifolds are Salamon [Sal89] and Joyce [Joy00].

2.1 $Spin(7)$-manifolds

The group $Spin(7) \subset SO(8)$ is a compact, connected, simply-connected, semi-simple Lie group of dimension 21, the double cover of $SO(7)$. We introduce it as a subgroup of $GL(8, \mathbb{R})$ in the following manner.

Let $(x_1, x_2, \ldots, x_8)$ be coordinates of $\mathbb{R}^8$, $g_0$ the standard metric on $\mathbb{R}^8$. The $GL(8, \mathbb{R})$-stabilizer of the four-form defined by

$$
\Omega_0 := \tau^{1256} + \tau^{1278} + \tau^{3456} + \tau^{3478} + \tau^{1357} - \tau^{1368} - \tau^{1457} + \tau^{2468} - \tau^{1458} - \tau^{1467} - \tau^{2358} - \tau^{2367} + \tau^{1234} + \tau^{5678},
$$

is
where $\tau^{ijkl}$ denotes $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$, is isomorphic to the group $Spin(7)$ ([HL82]).

The group $Spin(7)$ preserves the metric $g_0$ and an orientation on $\mathbb{R}^8$. Let $\Omega$ be a four-form on $M$ and $g$ a metric on $M$. We call a pair $(\Omega, g)$ a $Spin(7)$-structure if $(\Omega, g)$ is isomorphic to $(\Omega_0, g_0)$ at each point in $M$. We call $\nabla \Omega$ the torsion of the $Spin(7)$-structure, where $\nabla$ is the Levi-Civita connection of $g$, and $(\Omega, g)$ torsion-free if $\nabla \Omega = 0$.

**Proposition 2.1** ([Joy00] Proposition 10.5.3). Let $M$ be an eight-manifold with a $Spin(7)$-structure $(\Omega, g)$. Then the following are equivalent:

(i) $\text{Hol}(g) \subset Spin(7)$;

(ii) $\nabla \Omega = 0$; and

(iii) $d\Omega = 0$.

An eight-manifold with a $Spin(7)$-structure $(\Omega, g)$ is called a $Spin(7)$-manifold if the $Spin(7)$-structure is torsion-free. If $g$ has holonomy $\text{Hol}(g) \subset Spin(7)$, then $g$ is Ricci-flat. The following holds for compact eight-manifolds with holonomy $Spin(7)$.

**Theorem 2.2** ([Joy00] Theorem 10.6.8). Let $M$ be a compact $Spin(7)$-manifold with torsion-free $Spin(7)$-structure $(\Omega, g)$. Then $\text{Hol}(g) = Spin(7)$ if and only if $\pi_1(M) = 0$ and $\hat{A}(M) = 1$.

### 2.2 Ingredients for the construction

In the construction ([Joy99], $Spin(7)$-manifolds are constructed from the following ingredients:

(A) a Calabi–Yau four-orbifold $Y$ with only isolated singular points, and an anti-holomorphic involution $\sigma$ on $Y$ which fixes only the singular points,

(B) ALE $Spin(7)$-manifolds $X_1, X_2$.

We describe each of pieces needed for the construction in more detail below.

(A) **The Calabi–Yau four-orbifolds.** A Calabi–Yau $m$-orbifold is a Kähler orbifold $Y$ of dimension $m$ with a Kähler metric of holonomy contained in $SU(m)$.

For the construction, we take a Calabi–Yau four-orbifold $Y$ with Kähler form $\omega$ and holomorphic volume $\theta$. We assume that $Y$ has finitely many singular points $p_1, p_2, \ldots, p_k$ satisfying the following conditions:
• Each singularity is modeled on \( \mathbb{C}^4/\langle \alpha \rangle \), where \( \alpha : \mathbb{C}^4 \to \mathbb{C}^4 \) is defined by

\[
\alpha : (z_1, z_2, z_3, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4).
\]

Here \( \langle \alpha \rangle \equiv \mathbb{Z}_4 \), and \( \mathbb{C}^4/\langle \alpha \rangle \) has an isolated singularity at the origin.

• \( Y \) has an anti-holomorphic involution \( \sigma \), which fixes only the singular points \( p_1, p_2, \ldots, p_k \).

• \( Y \setminus \{p_1, p_2, \ldots, p_k\} \) is simply-connected, and \( h^{2,0}(Y) = 0 \).

Since \( SU(4) \subset Spin(7) \), and \( Y \) has holonomy \( SU(4) \), there is a torsion-free \( Spin(7) \)-structure on \( Y \), which is given by \( \Omega = \frac{1}{2} \omega^2 + \text{Re}(\theta) \). If we take a \( \sigma \)-invariant \( Spin(7) \)-structure \( (\Omega, g) \) on \( Y \), then this descends to \( Z = Y/\langle \sigma \rangle \).

Hence \( Z \) is a \( Spin(7) \)-orbifold with finitely many singular points \( p_1, \ldots, p_k \).

For each \( j = 1, 2, \ldots, k \), the tangent space \( T_{p_j}Y \) can be identified with \( \mathbb{C}^4/\langle \alpha \rangle \) so that \( g_Y \) is identified with \( |dz_1|^2 + \cdots + |dz_4|^2, \theta_Y \) is identified with \( dz_1 \wedge \cdots \wedge dz_4 \), and \( d\sigma : T_{p_j}Y \to T_{p_j}Y \) is identified with \( \beta : \mathbb{C}^4/\langle \alpha \rangle \to \mathbb{C}^4/\langle \alpha \rangle \) defined by

\[
\beta : (z_1, z_2, z_3, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3).
\]

Thus, all singularities are modeled on \( \mathbb{R}^8/\Gamma_8 \), where \( \Gamma_8 = \langle \alpha, \beta \rangle \) is a non-abelian group of order 8, and there is an isomorphism \( i_j : \mathbb{R}^8/\Gamma_8 \to T_{p_j}Z \) which identifies the \( Spin(7) \)-structure \( (\Omega_0, g_0) \) on \( \mathbb{R}^8/\Gamma_8 \) with \( (\Omega_Z, g_Z) \) on \( T_{p_j}Z \) for each \( j = 1, \ldots, k \).

Many examples of Calabi–Yau four-orbifolds satisfying the requirements in the construction are in hypersurfaces or complete intersections in the weighted projective spaces. The simplest is the following:

**Example 2.3 (Joy00) pp. 406-407.** Consider the following hypersurface of degree 12 in the weighted projective space \( \mathbb{CP}^5_{1,1,1,1,1,4,4} \) given by

\[
[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{CP}^5_{1,1,1,1,1,4,4} : z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + z_4^3 + z_5^3 = 0.
\]

Then \( c_1(Y) = 0 \), thus \( Y \) is a Calabi–Yau four-orbifold. It has three singular points \( p_1 = [0, 0, 0, 0, 1, -1], p_2 = [0, 0, 0, 0, 1, e^{i\pi/3}], p_3 = [0, 0, 0, 0, 1, e^{-i\pi/3}] \).

Define \( \sigma : Y \to Y \) by

\[
[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [\bar{z}_1 : -\bar{z}_0 : \bar{z}_3 : -\bar{z}_2 : \bar{z}_5 : \bar{z}_4].
\]

Then \( \sigma \) is an anti-holomorphic involution which fixes only the singular points \( p_1, p_2, p_3 \).
(B) The ALE $\text{Spin}(7)$-manifolds. Let $\Gamma$ be a finite subgroup of the group $\text{Spin}(7)$ which acts freely on $\mathbb{R}^8 \setminus 0$. We call a $\text{Spin}(7)$-manifold $M$ with $\text{Spin}(7)$-structure $(\Omega, g)$ an ALE $\text{Spin}(7)$-manifold asymptotic to $\mathbb{R}^8/\Gamma$ if there is a proper continuous surjective map $\pi : X \to \mathbb{R}^8/\Gamma$ such that $\pi : X \setminus \pi^{-1}(0) \to (\mathbb{R}^8/\Gamma) \setminus 0$ is a diffeomorphism, and

$$\nabla^k(\pi_* (g) - g) = O(r^{-8-k}), \quad \nabla^k(\pi_* (\Omega) - \Omega_0) = O(r^{-8-k})$$

on $\{x \in \mathbb{R}^8/\Gamma : r(x) > 1\}$ for all $k \geq 0$, where $r$ is the radius function on $\mathbb{R}^8/\Gamma$.

We introduce two types of ALE $\text{Spin}(7)$-manifolds denoted by $X_1, X_2$ for the construction as follows.

(I) Define complex coordinates $(z_1, z_2, z_3, z_4)$ on $\mathbb{R}^8$ by

$$(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8).$$

Then $g_0 = |dz_1|^2 + \cdots + |dz_4|^2$ and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \text{Re}(\theta_0)$.

Define $\alpha, \beta : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\alpha : (z_1, z_2, z_3, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4),$$
$$\beta : (z_1, z_2, z_3, z_4) \mapsto (\bar{z}_2, \bar{z}_1, \bar{z}_4, \bar{z}_3).$$

We denote by $W_1$ the crepant resolution $\pi_1 : W_1 \to \mathbb{C}^4/\langle \alpha \rangle$ of $\mathbb{C}^4/\langle \alpha \rangle$, which is the blow-up of $\mathbb{C}^4/\langle \alpha \rangle$ at $0$, with $\pi_1^{-1}(0) = \mathbb{C}P^3$. The action of $\beta$ lifts to a free anti-holomorphic involution of $W_1$. Hence $X_1 = W_1/\langle \beta \rangle$ is a resolution of $\mathbb{R}^8/\Gamma_8$, where $\Gamma_8 = \langle \alpha, \beta \rangle$.

(II) There is another complex structure on $\mathbb{R}^8$, namely, we define complex coordinates $(w_1, w_2, w_3, w_4)$ on $\mathbb{R}^8$ by

$$(w_1, w_2, w_3, w_4) = (-x_1 + ix_3, x_2 + ix_4, -x_5 + ix_7, x_6 + ix_8).$$

Then $g_0 = |dw_1|^2 + \cdots + |dw_4|^2$ and $\Omega_0 = \frac{1}{2}\omega'_0 \wedge \omega'_0 + \text{Re}(\theta'_0)$. Define $\alpha, \beta : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\alpha : (w_1, w_2, w_3, w_4) \mapsto (\bar{w}_2, -\bar{w}_1, \bar{w}_4, -\bar{w}_3),$$
$$\beta : (w_1, w_2, w_3, w_4) \mapsto (iw_1, iw_2, iw_3, iw_4).$$

We denote by $W_2$ the crepant resolution $\pi_2 : W_2 \to \mathbb{C}^4/\langle \beta \rangle$ of $\mathbb{C}^4/\langle \beta \rangle$, which is the blow-up of $\mathbb{C}^4/\langle \beta \rangle$ at $0$, with $\pi_2^{-1}(0) = \mathbb{C}P^3$. The action of $\alpha$ lifts to a free anti-holomorphic involution of $W_2$. Hence $X_2 = W_2/\langle \alpha \rangle$ is a resolution of $\mathbb{R}^8/\Gamma_8$, where $\Gamma_8 = \langle \alpha, \beta \rangle$. 

2.3 The manifolds

We glue either $X_1$ or $X_2$ to each singular point $p_j (j = 1, \ldots, k)$ of the $Spin(7)$-orbifold $Z$ to obtain a compact smooth 8-manifold $M$.

Firstly, we have the following description around the singular points of $Z$: We denote by $exp_{p_j} : T_{p_j}Z \to Z$ the exponential map. Then $exp_{p_j} \circ i_j$ maps $\mathbb{R}^8/\Gamma_8$ to $Z$. We take $\zeta$ small and define $U_j \subset Z$ by $U_j = exp_{p_j} \circ i_j (B_{\zeta}(\mathbb{R}^8/\Gamma_8))$, where $B_{\zeta}(\mathbb{R}^8/\Gamma_8)$ is the open ball of radius $\zeta$ about 0. We take $\zeta$ small enough so that $U_j$ is open in $Z$ and $\psi_j := exp_{p_j} \circ i_j : B_{\zeta}(\mathbb{R}^8/\Gamma_8) \to U_j$ is a diffeomorphism for $j = 1, \ldots, k$, and that $U_i \cap U_j = \emptyset$ for $i \neq j$.

Next, we introduce a scaling parameter $t \in (0, 1]$. For each $i = 1, 2$ we consider the rescaled ALE $Spin(7)$-manifold $X_i^t = X_i$ with a $Spin(7)$-structure $(\Omega_i^t, g_i^t)$ defined by

$$
\Omega_i^t = t^4 \Omega_i, \quad g_i^t = t^2 g_i,
$$

and the projection $\pi_i^t : X_i^t \to \mathbb{R}^8/\Gamma_8$ given by $\pi_i^t = t \pi_i$. Then each $(X_i^t, \Omega_i^t, g_i^t) (i = 1, 2)$ is an ALE $Spin(7)$-manifold asymptotic to $\mathbb{R}^8/\Gamma_8$.

We now define $M_i^t (i = 0, 1, 2, \ldots, k)$ by

$$
M_0^t = Z \setminus \bigcup_{j=1}^k \psi_j (\overline{B_{\frac{\zeta}{t} \Gamma_8}(\mathbb{R}^8/\Gamma_8)}) \subset Z,
$$

$$
M_j^t = (\pi_i^t)^{-1}(B_{\frac{\zeta}{t} \Gamma_8}(\mathbb{R}^8/\Gamma_8)) \subset X_{n_j} \quad (j = 1, 2, \ldots, k),
$$

where $n_j = 1$ or 2 for $j = 1, 2, \ldots, k$. Now we define a resolution $M = M^t$ of $Z$ by $\bigsqcup_{j=0}^k M_j^t \sim$, where the equivalence relation $\sim$ is given by $x \sim y$ if either (a) $x = y$,

(b) $x \in M_j^t$ and $y \in U_j \cap M_0^t$, and $\psi \circ \pi_{n_j}^t (x) = y$ for $j = 1, \ldots, k$, or

(c) $y \in M_j^t$ and $x \in U_j \cap M_0^t$, and $\psi \circ \pi_{n_j}^t (y) = x$ for $j = 1, \ldots, k$.

Then $M$ is a compact 8-manifold, and $\pi_1(M) = \mathbb{Z}_2$ if $n_j = 1$ for all $j = 1, \ldots, k$, or otherwise, i.e., if $n_j = 2$ for some $j$, $M$ is simply-connected.

We define a radius function on $M^t$ as follows: Firstly, define a radius function on $M^t_0 \subset Z$ to be a function $\rho_{M^t_0} : M_0^t \to [t^\frac{9}{4}, 1]$ such that $\rho_{M^t_0} \circ \pi_j = r$ for $r$ in $[t^\frac{9}{4} \zeta, \zeta]$ and $j = 1, \ldots, k$, and $\rho_{M^t_0}$ in $[\zeta, 1]$ on $Z \setminus \bigcup_{j=1}^k U_j$. Secondly, define a radius function on $M_j^t \subset X_{n_j} (j = 1, \ldots, k)$ to be a function $\rho_{M_j^t} : M_j^t \to [t, t^\frac{9}{4}]$ such that $\rho_{M_j^t} = t(r \circ \pi_j)$ for $r$ in $[t^\frac{9}{4}, 1]$ and $t$ for $r < 1$. Then these functions $\rho_{M^t_0}, \rho_{M_j^t}$ coincide on each $M^t_0 \cap M_j^t (j = 1, \ldots, k)$, hence we get a radius function $\rho : M^t \to [t, 1]$ from these.
2.4 The Spin(7)-structure

We glue together the torsion-free Spin(7)-structures \((\Omega_Z, g_Z)\) on \(M_0\) and \((\Omega_{n_j}, g_{n_j})\) on \(M_j^t (j = 1, \ldots, k)\) by a partition of unity.

Firstly, under the identification of \(B_\zeta(\mathbb{R}^8/\Gamma_8)\) with \(U_j \subset Z\) by \(\psi_j\), there is a smooth three-form \(\sigma_j\) on \(B_\zeta(\mathbb{R}^8/\Gamma_8)\) such that \(\psi_j^*(\Omega_Z) - \Omega_0 = d\sigma_j\) for each \(j = 1, \ldots, k\) with \(|\nabla^\ell \sigma_j| \leq D_1 r^{3-\ell} (\ell = 0, 1, 2)\) on \(B_\zeta(\mathbb{R}^8/\Gamma_8)\), where \(D_1 > 0\) is a constant independent of \(t\), and \(|\cdot|, \nabla\) are with respect to the metric \(g_0\) (Joy00 Proposition 15.2.6).

On the other hand, there exists a smooth three-form \(\tau_j^t\) on \((\mathbb{R}^8/\Gamma_8) \setminus B_{\zeta}(\mathbb{R}^8/\Gamma_8)\) such that \((\pi_n)_*(\Omega_{n_j}) = \Omega_0 + d\tau_{n_j}^t\) with \(|\nabla^\ell \tau_{n_j}^t| \leq D_2 r^{3-\ell}\) for \(\ell = 0, 1, 2\) on \((\mathbb{R}^8/\Gamma_8) \setminus B_{\zeta}(\mathbb{R}^8/\Gamma_8)\), where \(D_2 > 0\) is a constant independent of \(t\), and \(|\cdot|, \nabla\) are with respect to the metric \(g_0\). This can be proved by using an explicit metric by Calabi (Cal79).

Let \(\eta : [0, \infty) \to [0, 1]\) be a smooth function with \(\eta(x) = 0\) for \(x \leq 1\) and \(\eta(x) = 1\) for \(x \geq 2\). We take \(t\) small enough that \(2t^\frac{3}{5} \leq t^\frac{2}{5}\), and define a closed four-form \(\xi^t\) on \(M^t\) by \(\xi^t = \Omega_Z\) on \(M_0^t \setminus \bigcup_{j=1}^k M_j^t\), and \(\xi^t = \Omega_{n_j}\) on \(M_j^t \setminus M_j^0 (j = 1, \ldots, k)\), and

\[
\xi^t = \Omega_0 + d(\eta(t^{-\frac{3}{5}} r)\sigma_j) + d((1 - \eta(t^{-\frac{3}{5}} r))\tau_{n_j}^t)
\]
on \(M_0^t \cap M_j^t (j = 1, \ldots, k)\). Here we identify \(M_0^t \cap M_j^t\) with an annulus in \(\mathbb{R}^8/\Gamma_8\).

By using this \(\xi^t\), we can construct a family of Spin(7)-structures \((\Omega^t, g^t)\) for small \(t\) and the difference \(\phi^t = \xi^t - \Omega^t\) can be estimated by

\[
||\phi^t||_{L^2} \leq \lambda t^{\frac{13}{10}}, \quad ||d\phi^t||_{L^1} \leq \lambda t^{\frac{7}{10}},
\]

where \(\lambda\) is a constant, as well as \(\delta(g^t) \geq \mu t\) and \(||R(g^t)||_{C^0} \leq \nu t^{-2}\), where \(\delta(g^t)\) is the injective radius of \(g^t\), \(R(g^t)\) is the Riemannian curvature, and \(\mu, \nu > 0\) are constants (Joy00 Theorem 15.2.13). Here all norms are calculated by the metric \(g^t\) on \(M^t\).

Then the existence of torsion-free Spin(7)-structures follows from

**Theorem 2.4** (Joy00 Theorem 13.6.1, Proposition 13.7.1). Let \(\lambda, \mu, \nu > 0\) be constants. Then there exists constants \(\kappa, K > 0\) such that for \(0 < t \leq \kappa\) the following holds. Let \(M\) be a compact 8-manifold, and \((\Omega^t, g^t)\) a Spin(7)-structure on \(M\). Suppose that \(\phi^t\) is a four-form on \(M\) with \(d\Omega^t + d\phi^t = 0\), and

(i) \(||\phi^t||_{L^2} \leq \lambda t^{\frac{13}{10}}\) and \(||d\phi^t||_{L^1} \leq \lambda t^{\frac{7}{10}}\);
(ii) the injectivity radius $\delta(g^t)$ satisfies $\delta(g^t) \geq \mu t$; and

(iii) the Riemannian curvature $R(g^t)$ satisfies $||R(g^t)||_{C^0} \leq \nu t^{-2}$.

Then there exists a smooth torsion-free Spin(7)-structure $(\tilde{\Omega}^t, \tilde{g}^t)$ on $M$ such that $||\tilde{\Omega}^t - \Omega^t||_{C^0} \leq K t^{\frac{1}{3}}$ and $||\nabla(\tilde{\Omega}^t - \Omega^t)||_{L^{10}} \leq K t^{\frac{2}{5}}$.

By this theorem, we can deform the Spin(7)-structure $(\Omega^t, g^t)$ above to a torsion-free Spin(7)-structure $(\tilde{\Omega}^t, \tilde{g}^t)$ on $M$ for $t$ sufficiently small. Theorem 2.2 then shows that $\text{Hol}(\tilde{g}^t) = \text{Spin}(7)$ provided $\pi_1(M) = 0$ and $\hat{A}(M) = 1$.

3 Spin(7)-instantons

In Section 3.1, we introduce the Spin(7)-instanton equation, and describe its relation to the complex ASD equation and the Hermitian–Einstein equation. In Section 3.2, we describe the linearization of the Spin(7)-instanton equation and the Hermitian–Einstein equation.

3.1 Spin(7)-instantons

Let $M$ be a Spin(7)-manifold. Then the space of two-forms $\Lambda^2$ on $M$ splits as

$$\Lambda^2 = \Lambda^2_{21} \oplus \Lambda^2_7,$$

where $\Lambda^2_{21}$ is a rank 21 vector bundle which corresponds to the Lie algebra of Spin(7) under the identification of $\Lambda^2$ with the Lie algebra of SO(8), and $\Lambda^2_7$ is a rank 7 vector bundle which is orthogonal to $\Lambda^2_{21}$. Alternatively, if we consider the operator on $\Lambda^2$ defined by $\alpha \mapsto * (\Omega \wedge \alpha)$, then it is self-adjoint with eigenvalues $-1$ and $3$, and its eigenspaces are $\Lambda^2_{21}$ and $\Lambda^2_7$ respectively.

Let $P$ be a principal bundle on $M$ with the structure group $G$. We denote by $\text{Ad}(P)$ the adjoint vector bundle associated with $P$. The space of $\text{Ad}(P)$-valued 2-forms is also decomposed as

$$\Omega^2(\text{Ad}(P)) = \Omega^2_{21}(\text{Ad}(P)) \oplus \Omega^2_7(\text{Ad}(P)).$$

We call a connection $A$ on $P$ a Spin(7)-instanton if $A$ satisfies the following equation:

$$\pi_7^2(F_A) = 0,$$

(3.1)

where $F_A$ is the curvature of $A$, and $\pi_7^2$ is the projection to the $\Omega^2_7(\text{Ad}(P))$ component. Equation (3.1) together with a gauge fixing condition form an
elliptic system. Note that the projection $\pi^2 : \Omega^2(\text{Ad}(P)) \to \Omega^2_7(\text{Ad}(P))$ can be written as

$$\alpha \mapsto \frac{1}{4}(\ast\Omega \wedge \alpha + \alpha)$$

for $\alpha \in \Omega^2(\text{Ad}(P))$.

**Complex ASD.** Let $M$ be a compact Calabi–Yau four-fold with Kähler form $\omega$ and holomorphic $(4,0)$-form $\theta$. We assume the normalization condition $\theta \wedge \bar{\theta} = \frac{16}{3} \omega^4$ on $\omega$ and $\theta$. A Calabi–Yau four-fold is a $\text{Spin}(7)$-manifold as $SU(4) \subset \text{Spin}(7)$, and the $\text{Spin}(7)$-structure $\Omega$ is given by $\Omega = \frac{1}{2} \omega^2 + \text{Re}(\theta)$. Let $E$ be a Hermitian vector bundle over $M$.

In general, if the underlying manifold is Kähler, then we have the following decomposition of the space of complexified two forms:

$$\Lambda^2 \otimes \mathbb{C} = \Lambda^{1,1} \oplus \Lambda^{2,0} \oplus \Lambda^{0,2},$$

and $\Lambda^{1,1}$ further decomposes into $\mathbb{C}\langle \omega \rangle \oplus \Lambda^{1,1}_0$.

In the case where the underlying Kähler manifold $M$ is a Calabi–Yau four-fold, we define the complex Hodge operator $\ast_\theta : \Lambda^{0,2} \to \Lambda^{0,2}$ by

$$\phi \wedge \ast_\theta \psi = \langle \phi, \psi \rangle \bar{\theta}, \quad \phi, \psi \in \Lambda^{0,2}.$$

Then $\ast_\theta^2 = 1$, and the space of $(0,2)$-forms further decomposes into

$$\Lambda^{0,2} = \Lambda^{0,2}_+ \oplus \Lambda^{0,2}_-,$$

where

$$\Lambda^{0,2}_+ = \{ \phi \in \Lambda^{0,2} : \ast_\theta \phi = \phi \},$$

$$\Lambda^{0,2}_- = \{ \phi \in \Lambda^{0,2} : \ast_\theta \phi = -\phi \}.$$ 

Note that the operator $\ast_\theta$ is an anti-holomorphic map, hence $\Lambda^{0,2}_+$ and $\Lambda^{0,2}_-$ are real subspaces of $\Lambda^{0,2}$. We obtain

$$\Lambda^2_{21} = \Lambda^2 \cap (\Lambda^{1,1}_0 \oplus \Lambda^{-0,2}_- \oplus \Lambda^{-2,0}_-),$$

$$\Lambda^2_7 = \mathbb{R}\langle \omega \rangle \oplus (\Lambda^2 \cap (\Lambda^{0,2}_+ \oplus \Lambda^{2,0}_+)).$$

Hence, the $\text{Spin}(7)$-instanton equation on a Calabi–Yau four-fold can be written as

$$(1 + \ast_\theta)F^{0,2}_A = 0, \quad \Lambda F^{1,1}_A = 0.$$

These are called *complex anti-self-dual equations* [DT98].
Hermitian–Einstein connections. Hermitian–Einstein connections also give examples of \( \text{Spin}(7) \)-instantons.

Let \( X \) be a compact Kähler manifold of complex dimension \( n \) with Kähler form \( \omega \), and \( E \) a Hermitian vector bundle over \( X \) with Hermitian metric \( h \).

A metric-preserving connection \( A \) of \( E \) is said to be a Hermitian–Einstein connection if \( A \) satisfies the following equations:

\[
F_A^{0,2} = 0, \quad \Lambda F_A^{1,1} = \lambda(E) \text{Id}_E,
\]

where \( \Lambda := (\wedge \omega)^* \), and \( \lambda(E) \) is defined by

\[
\lambda(E) := \frac{n(c_1(E) \cdot [\omega]^{n-1})}{\text{vol}(\omega)_n}.
\]

If \( E \) is a unitary vector bundle with \( c_1(E) \cdot [\omega]^3 = 0 \) over a Kähler manifold with holonomy contained in \( SU(4) \), then \( \lambda(E) = 0 \) and \( F_A \in \Lambda_0^{1,1} \), and Hermitian–Einstein connections are \( \text{Spin}(7) \)-instantons, since \( \Omega^2 \cap \Omega_0^{1,1} \subset \Omega^2 \) as described above.

3.2 Linearizations

The infinitesimal deformation of \( \text{Spin}(7) \)-instantons was studied by Reyes Carrión [RC98], and it is given by the following 3-term complex:

\[
0 \to \Omega^0(u(E)) \xrightarrow{d_A} \Omega^1(u(E)) \xrightarrow{d_A^*} \Omega^2(u(E)) \to 0,
\]

(3.3)

where \( d_A := \pi_2^* \circ d_A \). This complex is elliptic [RC98], hence

\[
L_A := (d_A^*, d_A^2) : \Omega^1(u(E)) \to \Omega^0(u(E)) \oplus \Omega^2(u(E))
\]

(3.4)

is an elliptic operator. The local model of the moduli space of \( \text{Spin}(7) \)-instantons is described in Lewis’ thesis [Lew98]. The operator \( L_A \) is the twisted Dirac operator between the Spin bundles twisted by \( u(E) \):

\[
S^+ = \Omega^0(u(E)) \oplus \Omega^2(u(E)), \quad S^- = \Omega^1(u(E)).
\]

(3.5)

Hence, the index of the complex (3.3) can be calculated by the Atiyah–Singer Index Theorem, it is \( \langle \hat{A}(M) \text{ch}(u(E)), [M] \rangle \), and when \( M \) is a compact 8-manifold with holonomy \( \text{Spin}(7) \), \( \text{Ind}(L_A) \) turns out to be

\[
\text{Ind}(L_A) = -r^2 - \left( -\frac{p_1(M)}{24} \left( -c_1(E)^2 + r(c_1(E)^2 - 2c_2(E)) \right) \right.
\]

\[
+ \frac{r}{12} \left( c_1(E)^4 - 4c_1(E)^2 c_2(E) + 2c_2(E)^2 + 4c_1(E)c_3(E) - 4c_4(E) \right)
\]

\[
- \frac{1}{12} c_1(E)^4 - c_1(E)c_3(E) + c_2(E)^2, [M] \right).
\]

(3.6)
where we used the fact that $\langle -4p_2(M) + 7p_1(M)^2, [M] \rangle = 5760$ if $M$ has holonomy $Spin(7)$.

If $E$ is an $SU(r)$ bundle, rather than a $U(r)$ bundle, then we replace $u(E)$ by $su(E)$ in (3.3), (3.4), and (3.5), and the first term $r^2$ in (3.6) is replaced by $r^2 - 1$. In particular, if we take $E$ to be an $SU(2)$ bundle, then (3.6) becomes

$$\text{Ind}(L_A) = -3 - \frac{1}{6} \langle p_1(M)c_2(E) + 8c_2(E)^2, [M] \rangle. \quad (3.7)$$

**Infinitesimal deformation of Hermitian–Einstein connections.**

Let $X$ be a Kähler four-fold with Kähler form $\omega$, $E$ a Hermitian vector bundle over $X$ with Hermitian metric $h$.

The infinitesimal deformation of a Hermitian–Einstein connection $A$ of $E$ was studied by Kim [Kim87] (see also [Kob87]), and it is described by the following complex:

$$0 \rightarrow \Omega^0(X,u(E)) \xrightarrow{d_A} \Omega^1(X,u(E)) \xrightarrow{d_A^+} \Omega^+(X,u(E)) \xrightarrow{D_A'} A^{0,3}(X,u(E)) \xrightarrow{D_A} A^{0,4}(X,u(E)) \rightarrow 0$$

(3.8)

where

$$A^{0,q}(X,u(E)) := C^\infty(u(E) \otimes A^{0,q}),$$

$u(E) = \text{End}(E,h)$ is the bundle of skew-Hermitian endomorphisms of $E$, $A^{0,p}$ is the space of real $(0,p)$-forms (see [Sal89] pp. 32–33) over $X$, defined by

$$A^{0,p} \otimes \mathbb{C} = \Lambda^{0,p} \oplus \Lambda^{p,0},$$

$$\Omega^+(X,u(E)) := \Omega^{0,2}(X,u(E)) \oplus \Omega^0(X,u(E))\omega$$

$$= \{ \phi + i\phi + f\omega : \phi \in \Omega^{0,2}(X,u(E)), f \in \Omega^0(X,u(E)) \},$$

$$\bar{D}_A : A^{0,p}(X,u(E)) \rightarrow A^{0,p+1}(X,u(E))$$

is defined by $\bar{D}_A \alpha = \bar{\partial}_A \alpha + \partial_A \alpha$, where $\alpha^{0,p} \in \Omega^{0,p}(X,u(E))$, and

$$d_A^+ := \pi^+ \circ d_A, \quad \bar{D}_A' := \bar{D}_A \circ \pi^{0,2},$$

where $\pi^+, \pi^{0,2}$ are respectively the orthogonal projections from $\Omega^2$ to $\Omega^+, A^{0,2}$. As described in [Kim87] (see also [Kob87]), the complex (3.8) has the associated Dolbeault complex:

$$0 \rightarrow \Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{d_A^+} \Omega^+ \xrightarrow{D_A'} A^{0,3} \xrightarrow{D_A} A^{0,4} \rightarrow 0$$

$$\downarrow j_0 \quad \downarrow j_1 \quad \downarrow j_2 \quad \downarrow j_3 \quad \downarrow j_4$$

$$0 \rightarrow \Omega^{0,0} \xrightarrow{\bar{\partial}_A} \Omega^{0,1} \xrightarrow{\bar{\partial}_A} \Omega^{0,2} \xrightarrow{\bar{\partial}_A} \Omega^{0,3} \xrightarrow{\bar{\partial}_A} \Omega^{0,4} \rightarrow 0,$$
where \( j_0 \) is injective, \( j_1 \) is bijective, \( j_2 \) is surjective with the kernel \( \{ \beta \omega : \beta \in \Omega^0 \} \), and \( j_3, j_4 \) are bijective. We denote the \( i \)-th cohomology of the complex (3.8) by \( H^i \) (\( i = 0, 1, \ldots, 4 \)). Kim [Kim87] proved that

\[
H^0 \otimes \mathbb{C} \cong H^{0,0}, \quad H^1 \cong H^{0,1}, \quad H^2 \cong H^0 \oplus H^{0,2},
\]

\[
H^3 \cong H^{0,3}, \quad H^4 \cong H^{0,4}.
\]

In particular, if an \( SU(r) \) bundle \( E \) is irreducible, and \( H^{0,2} = 0 \), then the linearized operator \( L_A = (d_A^+, d_A^*) : \Omega^1(\mathfrak{su}(E)) \to \Omega^0(\mathfrak{su}(E)) \oplus \Omega^2(\mathfrak{su}(E)) \) is surjective.

4 Approximate solution and the estimate

In this section, we construct an approximate solution to the \( \text{Spin}(7) \)-instanton equation on a vector bundle over the \( \text{Spin}(7) \)-manifold \( M = M^t \) of Section 2. The ingredients are Hermitian–Einstein connections on vector bundles over the Calabi–Yau four-orbifold \( Z \) and the ALE spaces \( W_{nj} \)'s. We also prove an estimate on the approximate solution needed in the later section.

4.1 Ingredients for the construction

We take a complex vector bundle \( E_0^j \) of rank \( r \) over \( \mathbb{R}^8/\mathbb{Z}_8 \) \( \setminus \) 0 with a flat \( U(r) \)-connection \( A_0^j \) for each \( j = 1, \ldots, k \). Then ingredients for the construction consist of

(A) A complex orbifold vector bundle \( E_Z = E_Y/\langle \sigma \rangle \) of rank \( r \) over \( Z = Y/\langle \sigma \rangle \), which is isomorphic to \( E_0^j \) near each singular point \( p_j \in Z \) (\( j = 1, 2, \ldots, k \)), where \( E_Y \) is a \( \langle \sigma \rangle \)-equivariant holomorphic orbifold vector bundle over \( Y \), equipped with a \( \langle \sigma \rangle \)-equivariant Hermitian–Einstein connection \( A_Y \), and the connection \( A_Z \) on \( E_Z \) induced by \( A_Y \) is asymptotic to the flat connection \( A_0^j \) with the decay rate \( A_Z \sim A_0^j + O(r) \) and \( \nabla^\ell(A_Z - A_0^j) \sim O(1) \) for all \( \ell > 0 \) at each \( p_j \in Z \) (\( j = 1, 2, \ldots, k \)).

(B) A complex vector bundle \( E_{X_{nj}} = E_{W_{nj}}/\langle \sigma \rangle \) of rank \( r \) over each \( X_{nj} \) (\( j = 1, \ldots, k \)), which is isomorphic to \( E_0^j \) near \( \infty \), where \( E_{W_{nj}} \) is a \( \langle \sigma \rangle \)-equivariant holomorphic vector bundle over \( W_{nj} \), equipped with a \( \langle \sigma \rangle \)-equivariant Hermitian–Einstein connection \( A_{W_{nj}} \), and the connection \( A_{X_{nj}} \) on \( E_{X_{nj}} \) induced by \( A_{W_{nj}} \) is asymptotic to the flat connection \( A_0^j \) with the decay rate \( A_{X_{nj}} \sim A_0^j + O(r^{-7}) \) and \( \nabla^\ell(A_{X_{nj}} - A_0^j) \sim O(r^{-7-\ell}) \) for all \( \ell > 0 \) at infinity.
We also assume that the cokernel of $L_{AZ}$ lies in $C^\infty(u(E_Z) \otimes \Lambda^0(Z))$, namely, the cohomology $H^2(Z, u(E))$ of the complex (3.3) vanishes, but $H^0(Z, u(E))$ of the complex (3.3) does not necessarily vanish, and $L_{AX_{nj}} : L^1_{d-1}(u(E_{X_{nj}}) \otimes \Lambda^1(X_{nj})) \to L^4_{d-1}(u(E_{X_{nj}}) \otimes (\Lambda^0(X_{nj}) \oplus \Lambda^2(X_{nj})))$ for $j = 1, 2, \ldots, k$ for $\delta \in (-7, 0)$ is surjective, where $L^1_{d-1}(u(E_{X_{nj}}) \otimes \Lambda^1(X_{nj}))$ and $L^4_{d-1}(u(E_{X_{nj}}) \otimes (\Lambda^0(X_{nj}) \oplus \Lambda^2(X_{nj})))$ are weighted Sobolev spaces with the weights $\delta, \delta - 1$ (See Section 5.1 for more detail about the weighted Sobolev spaces).

Note that we do not assume $E_Z$ or $E_{X_{nj}} (j = 1, \ldots, k)$ to be irreducible, for instance, $E_Z$ can be trivial. In fact, even if $E_Z$ and $E_{X_{nj}}$ are reducible, one can construct irreducible $Spin(7)$-instantons, provided that the intersection of the symmetry groups of $E_Z, E_{X_{nj}} (j = 1, \ldots, k)$ is the multiples of the identity.

Also notice that for both $E_Y$ and $E_{W_{nj}}$, the constant $\lambda(E)$ in Section 3.1 are zero, since $E_Y$ and $E_{W_{nj}}$ are assumed to be $\sigma$-equivariant, so $\lambda(E) = \frac{4c_1(E) \cdot [\omega]^3}{r[\omega]^4}$ changes sign under the action of $\sigma$. Therefore $A_Y$ and $A_{W_{nj}}$ are $Spin(7)$-instantons, not just Hermitian–Einstein connections, and $A_Z$ and $A_{X_{nj}}$ are $Spin(7)$-instantons.

### 4.2 Approximate solution

We identify a small ball around each $p_j (j = 1, \ldots, k)$ in $Z$ with a small ball in $\mathbb{R}^8/\Gamma_8$, and identify $E_Z$ with $E^0_j$ over the balls. Similarly, we identify the complement of a large ball around the origin of $X_{nj}$ with the complement of a large ball around the origin of $(\mathbb{R}^8/\Gamma_8) \setminus 0$, and identify $E_{X_{nj}}$ with $E^0_{X_{nj}}$ over those complements for each $j = 1, \ldots, k$. We then glue $E_Z$ and $E_{X_{nj}}$ (together) by the above identifications, namely, $E_Z\mid M^0_j$ is identified with $E_{X_{nj}}\mid M^0_j$ by $E_Z\mid M^0_j \cap M^j_j \cong E_{X_{nj}}\mid M^0_j \cap M^j_j$ on $M^0_j \cap M^j_j$ for each $j = 1, \ldots, k$. We denote by $E$ the resulting vector bundle over $M$.

Next, we consider a smooth function $\chi : \mathbb{R} \to [0, 1]$ with $\chi(x) = 0$ for $x \leq \frac{3}{4}$ and $\chi(x) = 1$ for $x \geq \frac{5}{4}$. We define $\chi^j_t(\rho)$ on $M^0_j \cap M^j_j$ by

$$\chi^j_t(\rho) := \chi \left( \frac{\log \rho}{\log t} \right),$$

where $\rho$ is the radius function defined in Section 2.3. Then we have

$$\chi^j_t(\rho) = \begin{cases} 1 & (\rho \leq t^{\frac{\rho}{4}}), \\ 0 & (\rho \geq t^{\frac{\rho}{4}}). \end{cases}$$
We now define a connection \( A_t \) on \( E \) by
\[
A_t = A_Z + t^* (A_{X_{n_j}}) \quad \text{on} \quad M_0 \setminus \bigcup_{j=1}^k M_j,
\]
and
\[
A_t = t^* (A_{X_{n_j}}) \quad \text{on} \quad M_j \setminus M_0,
\]
on \( M_0 \cap M_j \) for \( j = 1, \ldots, k \).

### 4.3 Estimate on the error

In this section, we prove an estimate on the approximate solution (Proposition 4.1). Since the \( \text{Spin}(7) \)-manifold \( M = M^t \) depends on the parameter \( t \) from the rescaling around the singular points, we use scale-invariant norms such as the \( L^8 \)-norm for one-forms and \( L^4 \)-norms for two-forms to obtain \( t \)-independent estimates.

**Proposition 4.1.** Let \( A \) be the approximate solution in Section 4.2. Then there exists a constant \( C_1 > 0 \) independent of \( t \) such that
\[
||\tilde{\pi}_7^2(F_A_t)||_{L^4} \leq C_1 t^\frac{1}{2},
\]
where \( \tilde{\pi}_7^2 \) is the projection with respect to the torsion-free \( \text{Spin}(7) \)-structure \( \tilde{\Omega}^t \).

**Proof.** From (3.2), we have
\[
||\tilde{\pi}_7^2(F_A_t)||_{L^4} \leq ||\pi_7^2(F_A_t)||_{g^t} + C||\tilde{\Omega}^t - \Omega^t||_{g^t}||F_A_t||_{g^t},
\]
where \( \pi_7^2 \) is the projection with respect to the \( \text{Spin}(7) \)-structure \( \Omega^t \) and \( ||\cdot||_{g^t} \) is a point-wise norm with respect to the metric \( g^t \). Hence, raising to the fourth power and using the Hölder inequality, we obtain
\[
||\tilde{\pi}_7^2(F_A_t)||_{L^4} \leq ||\pi_7^2(F_A_t)||_{L^4} + C||\tilde{\Omega}^t - \Omega^t||_{L^8}||F_A_t||_{L^8},
\]
where \( L^p \) norms are taken by the metric \( g^t \).

We will prove Proposition 4.1 by estimating each term in the right-hand-side of (4.1).

**Lemma 4.2.**
\[
||\tilde{\Omega}^t - \Omega^t||_{L^8} \leq C t^{\frac{1}{8}}.
\]
Proof. From Proposition 13.7.1 in Chapter 13 of [Joy00],

\[ ||\tilde{\Omega}^t - \Omega^t||_{L^2} \leq Ct^{\frac{13}{3}}, \quad (4.3) \]
\[ ||\tilde{\Omega}^t - \Omega^t||_{C^0} \leq Ct^{\frac{1}{3}}. \quad (4.4) \]

Hence, for \( p > 2 \) we have

\[ ||\tilde{\Omega}^t - \Omega^t||_{L^p} \leq ||\tilde{\Omega}^t - \Omega^t||_{C^0} \leq Ct^{\frac{p-2}{p}} ||\tilde{\Omega}^t - \Omega^t||_{L^2} \leq Ct^{\frac{24+6}{3p}}. \]

In particular, \( ||\tilde{\Omega}^t - \Omega^t||_{L^8} \leq Ct^{\frac{4}{3}}. \)

**Lemma 4.3.** Let \( \tilde{A}_t \) be the approximate solution in Section 4.2. Then,

\[ |\pi_7^2(F_{\tilde{A}_t})| = \begin{cases} O(t^6 r^{-8}) + O(1), & \rho \in (t^{\frac{5}{6}}, t^{\frac{3}{4}}), \\ 0, & \text{otherwise}, \end{cases} \quad (4.5) \]

and

\[ |F_{\tilde{A}_t}| = \begin{cases} O(t^6 r^{-8}), & \rho \leq t^{\frac{5}{6}}, \\ O(t^6 r^{-8}) + O(1), & \rho \in (t^{\frac{5}{6}}, t^{\frac{3}{4}}), \\ O(1), & \rho \geq t^{\frac{3}{4}}. \end{cases} \quad (4.6) \]

Proof. These follow from the definition of \( \tilde{A}_t \), in particular, from \( |d\chi^2| = O(r^{-1}) \) and \( |A_{\tilde{A}_t} - A_Z| = O(t^6 r^{-7}) \) on \( M'_0 \cap M'_j (j = 1, \ldots, k) \).

From (4.5), we obtain

\[ ||\pi_7^2(F_{\tilde{A}_t})||_{L^4} \sim \left( \int_{t^{\frac{5}{6}}}^{t^{\frac{3}{4}}} (t^6 r^{-8} + 1)^4 r^7 dr \right)^{\frac{1}{4}} \sim \left( \left[ t^{24} r^{-24} + r^8 \right]_{t^{\frac{5}{6}}}^{t^{\frac{3}{4}}} \right)^{\frac{1}{4}} = O(t). \]

Also, from (4.6), we obtain

\[ ||F_{\tilde{A}_t}||_{L^8} = \left[ O(t^{-16}) O(t^8) + \int_{t}^{1} (t^6 r^{-8} + 1)^8 r^7 dr + O(1) \right]^{\frac{1}{8}} = O(t^{-1}). \]

Hence Proposition [4.1] follows.
5 Linear problem

In this section, we derive an estimate (Proposition 5.8) which comes from the Fredholm property of the linearized operator of the $\text{Spin}(7)$-instanton equation.

5.1 Fredholm property of the linearized operator on ALE $\text{Spin}(7)$-manifolds

We use weighted Sobolev spaces on the ALE side in order to obtain the Fredholm property of the linearized operator from the direct use of the Lockhart-McOwen theory [LM85] (see also [Loc87], [Bar86]).

**Weighted Sobolev spaces.** Let $X$ be an ALE $\text{Spin}(7)$-manifold. We denote by $\rho$ the radius function on $X$. Let $E \to X$ be a unitary vector bundle equipped with a connection $A$ which is asymptotic to a flat connection at infinity. For $p \geq 1$, $k \geq 0$ and $\delta \in \mathbb{R}$, we define the weighted Sobolev space $L^p_{k,\delta}(E)$ by the set of locally integrable and $k$ times weakly differentiable section $f$ of $E$, for which the norm

$$||f||_{L^p_{k,\delta}} = \sum_{j=0}^{k} \left( \int_X \rho^{-8}|\rho^{-\delta+j} \nabla^j f|^p dV \right)^{\frac{1}{p}}$$

is finite. Then $L^p_{k,\delta}(E)$ is a Banach space. We remark the following relations between the scale-invariant norms mentioned above and the weighted norms:

$$||a||_{L^8} = ||a||_{L^{8}_{-1}}, \ ||\nabla A a||_{L^4} = ||\nabla A a||_{L^4_{-2}}, \ ||F_A||_{L^4} = ||F_A||_{L^4_{-2}}.$$

We have the following Sobolev embedding theorem for the weighted spaces as well.

**Proposition 5.1** (Sobolev embedding ([Loc87], Theorems 4.8)). Let $k \geq l \geq 0$, $p, q \geq 1$. If $\frac{1}{p} \leq \frac{1}{q} + \frac{k-l}{n}$, $\delta \leq \delta'$, then $L^p_{k,\delta}(E) \to L^q_{l,\delta'}(E)$ is a continuous inclusion.

**Fredholm property.** We deduce the Fredholm property of the linearized operator on ALE spaces by using the Lockhart-McOwen theory.

Since we consider a connection asymptotic to a flat connection at infinity, the linearized operator $L_A$ reduces to the Dirac operator on $S^7/T_8 \times (R, \infty)$ with the metric $r^2g_{S^7} + dr^2$ at infinity. Then the Lockhart-McOwen theory [LM85], [Loc87] tells us that the linearized operator $L_A : L^p_{k+1,\delta}(u(E) \otimes$
Lemma 5.4. Let $X$ be an ALE Spin(7)-manifold, $E$ a unitary vector bundle over $X$, and $A$ a Spin(7)-instanton on $E$ asymptotic to a flat connection $A_0$ at infinity. Let $p \geq 1, k \geq 0$, and $\delta \in \mathbb{R} \setminus \mathcal{D}$. Then the operator

$$L_A : L^p_{k+1,\delta}(u(E) \otimes \Lambda^1) \to L^p_{k,\delta-1}(u(E) \otimes (\Lambda^0 \oplus \Lambda^2))$$

is Fredholm. Moreover, the kernel, cokernel, and index are independent of $p, k$ and $\delta$ within any connected component in $\mathbb{R} \setminus \mathcal{D}$.

**Proof.** Firstly, the following is a consequence of Proposition 5.2.

**Proposition 5.3.** Let $X$ be an ALE Spin(7)-manifold, $E$ a unitary vector bundle over $X$, and $A_0$ a connection asymptotic at rate $\lambda$ to a flat connection at infinity. Assume that $a \in L^8_{k+1,\mu}(u(E) \otimes \Lambda^1), \mu < -1$, and $A_0 + a$ satisfies the Spin(7)-instanton equation with $d^*_{A_0} a = 0$. Then $a \in L^8_{k+1,\mu'}(u(E) \otimes \Lambda^1)$ for any $\mu$ with $\lambda \leq \mu' < \mu$, which satisfies $\{\mu', \mu\} \cap \mathcal{D} = \emptyset$.

**Proof.** Since $a \in L^8_{k+1,\delta}(u(E) \otimes \Lambda^1)$, thus $\pi^2_2(a \wedge a) \in L^4_{k+1,2\delta}(u(E) \otimes \Lambda^2) \subset L^4_{k+1,\epsilon-1}(u(E) \otimes \Lambda^3)$. From Proposition 5.2, $L_{A_0} : L^4_{k+2,\epsilon}(u(E) \otimes \Lambda^1) \to L^4_{k+1,\epsilon-1}(u(E) \otimes (\Lambda^0 \oplus \Lambda^2))$ is Fredholm. Hence the cokernel of $L_{A_0}$ is finite-dimensional, of dimension $n$ say. We take compactly supported sections $\varphi_1, \ldots, \varphi_n$ such that

$$L^4_{k+1,\epsilon-1}(u(E) \otimes (\Lambda^0 \oplus \Lambda^2)) = L_{A_0} \left( L^4_{k+2,\epsilon}(u(E) \otimes \Lambda^1) \right) \otimes \langle \varphi_1, \ldots, \varphi_n \rangle.$$

Improvement of decay rates. By using the Fredholm property of the operator $L_A$ on ALE Spin(7)-manifolds, we prove the following:

**Proposition 5.5.** Let $x \in L^8_{k+1,\delta}(u(E) \otimes \Lambda^1), \delta < -1$. Then for $\varepsilon \geq 2\delta + 1$ with $\varepsilon \in \mathbb{R} \setminus \mathcal{D}$ there exists $\beta \in L^4_{k+1,\varepsilon}(u(E) \otimes \Lambda^1)$ such that $L_{A_0} \beta = (0, \pi^2_2(a \wedge a))$ holds near infinity.
Then there exist $\beta \in L_{k+2\varepsilon}^4(u(E) \otimes \Lambda^1)$ and unique constants $u_1, \ldots, u_n$ such that

$$(0, \pi_2^\varepsilon(a \wedge a)) = L_{A_0}\beta + u_1\varphi_1 + \cdots + u_n\varphi_n.$$ 

Thus, we have $L_{A_0}\beta = (0, \pi_2^\varepsilon(a \wedge a))$ outside the support of $\varphi_1, \ldots, \varphi_n$ in $X$. By Sobolev embedding $L_{k+2\varepsilon}^4(u(E) \otimes \Lambda^1) \to L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$, this $\beta$ lies in $L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$.

We take $\delta = \mu$ and $\varepsilon = \max\{2\mu + 1, \mu'\}$ in Lemma 5.3. Then we have $[\varepsilon, \delta] \cap \mathcal{D} = \emptyset$ and $\varepsilon \in \mathbb{R} \setminus \mathcal{D}$, since $[\mu', \mu] \cap \mathcal{D} = \emptyset$. Lemma 5.3 gives $\beta \in L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$ with $L_{A_0}\beta = (0, \pi_2^\varepsilon(a \wedge a))$ near infinity. Thus, we get $L_{A_0}(a + \beta) = 0$ near infinity. Since the cokernel of $L_{A_0}$ is independent of the choice of weights within any component in $\mathbb{R} \setminus \mathcal{D}$, $L_{A_0}(a + \beta) \perp (\text{coker } L_{A_0})_{\delta}$ implies $L_{A_0}(a + \beta) \perp (\text{coker } L_{A_0})_{\varepsilon}$. Thus, $L_{A_0}(a + \beta) \in (\text{Im} L_{A_0})_{\varepsilon}$, and there exists $\gamma \in L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$ such that $L_{A_0}\gamma = L_{A_0}(a + \beta)$. As the kernel of $L_{A_0}$ is also independent of the choice of weights within any component in $\mathbb{R} \setminus \mathcal{D}$, we obtain, $a + \beta - \gamma \in (\text{ker } L_{A_0})_{\delta} = (\text{ker } L_{A_0})_{\varepsilon}$. Since $\beta, \gamma \in L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$, thus $a \in L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$. Therefore, starting with $a \in L_{k+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$ with $\mu < -1$, we see that $a \in L_{k+2\mu+1,\varepsilon}^8(u(E) \otimes \Lambda^1)$, provided $[2\mu + 1, \mu] \cap \mathcal{D} = \emptyset$. We then put $\mu_0 = \mu = (\mu + 1) - 1$, $\mu_1 = 2\mu + 1 = 2(\mu + 1) - 1$, $\ldots$, $\mu_k = 2(\mu + 1) - 1$, and let $\ell$ be the least satisfying $2\ell(\mu + 1) - 1 \leq \mu'$, and say $\mu_\ell = \mu'$ for simplicity. Since $[\mu', \mu] \cap \mathcal{D} = \emptyset$, thus $\mu_0, \ldots, \mu_\ell \in \mathbb{R} \setminus \mathcal{D}$. Hence, we inductively obtain $a \in L_{k+1,\mu_\ell}^8(u(E) \otimes \Lambda^1)$ ($i = 1, \ldots, \ell$). Thus, $a \in L_{k+1,\mu'}^8(u(E) \otimes \Lambda^1)$. 

### 5.2 Estimates

We choose a finite dimensional vector space $K_Z$ in $\Omega^1(Z, u(E_Z))$, whose elements are supported away from $p_j$ ($j = 1, \ldots, k$) with the following properties:

- $\dim K_Z = \dim \ker(L_{A_Z})$ and
- $\Omega^1(Z, u(E_Z)) = K_Z^\perp \oplus \ker(L_{A_Z}),$ 

where $K_Z^\perp$ is the $L^2$-orthogonal complement of $K_Z$ in $\Omega^1(Z, u(E_Z))$. Since all elements in $K_Z$ are supported on the region $M^t_0$ of $Z \subset M$ for small $t$, we can think of $K_Z$ as lying in $\Omega^1(M, u(E))$. We will use $K_Z$ as a substitute for the kernel of $L_{A_Z}$, which also makes sense on $M$.

In a similar way to $K_Z$ above, we choose $K_{X_{n,j}}$ in $\Omega^1(X_{n,j}, u(E_{X_{n,j}}))$ for each $j = 1, \ldots, k$, whose elements are compactly supported away from
infinity, and think of $K_{X_{n_j}}$ in $\Omega^1(M, \Omega^1)$. These $K_Z, K_{X_{n_j}} (j = 1, \ldots, k)$ are substitutes for the kernels of $L_Z, L_{X_{n_j}} (j = 1, \ldots, k)$. We then put

$$K = K_Z \oplus \bigoplus_{j=1}^k K_{X_{n_j}} \subset \Omega^1(M, \Omega^1).$$

Firstly, we prove the following:

**Lemma 5.5.** There exists a constant $C_2 > 0$ such that the following holds for any $a_{X_{n_j}} \in L^4_{-1} (u(E_{X_{n_j}}) \otimes \Lambda^1)$, which is $L^2$-orthogonal to $K_{X_{n_j}}$:

$$||a_{X_{n_j}}||_{L^8} + ||\nabla a_{X_{n_j}}||_{L^4} \leq C_2 ||L_{A_{X_{n_j}}}a_{X_{n_j}}||_{L^4}. \quad (5.1)$$

**Proof.** From Proposition [5.2]

$$L_{A_{X_{n_j}}} : L^4_{1,-1} (u(E_{X_{n_j}}) \otimes \Lambda^1) \to L^4_{-2} (u(E_{X_{n_j}}) \otimes (\Lambda^0 \oplus \Lambda^2))$$

is Fredholm, thus, if $a_{X_{n_j}} \perp K_{X_{n_j}}$, we have

$$||a_{X_{n_j}}||_{L^4_{1,-1}} \leq C ||L_{A_{X_{n_j}}}a_{X_{n_j}}||_{L^4_{-2}}.$$

From the Sobolev embedding $L^4_{1,-1} \to L^8_{-1}$, we obtain

$$||a_{X_{n_j}}||_{L^8_{-1}} \leq C \left(||a_{X_{n_j}}||_{L^4_{1,-1}} + ||\nabla a_{X_{n_j}}||_{L^4_{-2}}\right).$$

Thus, we get

$$||a_{X_{n_j}}||_{L^8_{-1}} + ||\nabla a_{X_{n_j}}||_{L^4_{-2}} \leq C ||L_{A_{X_{n_j}}}a_{X_{n_j}}||_{L^4_{-2}}.$$

Since $|| \cdot ||_{L^8_{-1}} = || \cdot ||_{L^8}, || \cdot ||_{L^4_{-2}} = || \cdot ||_{L^4}$, hence, (5.1) follows. \qed

Similarly we have the following:

**Lemma 5.6.** There exists a constant $C_3 > 0$ such that the following holds for any $a_Z \in L^4_{1} (u(E_Z) \otimes \Lambda^1)$, which is $L^2$-orthogonal to $K_Z$:

$$||a_Z||_{L^8} + ||\nabla a_Z||_{L^4} \leq C_3 ||L_{A_{Z}}a_Z||_{L^4}. \quad (5.2)$$

With these lemmas above in mind, we prove the following:
Proposition 5.7. There exists a constant $C_4 > 0$ independent of $t$ such that if $t$ is sufficiently small and $a \in L^4_1(u(E) \otimes \Lambda^1)$ is $L^2$-orthogonal to $K$, then

$$\|a\|_{L^8} + \|\nabla a\|_{L^4} \leq C_4 \|L_{A_t} a\|_{L^4},$$

where $L_{A_t}$ is the linearized operator with respect to the $\text{Spin}(7)$-structure $\Omega^t$ on $M$.

Proof. We decompose $a \in \Omega^1(M, u(E))$ as

$$a = \sum_{j=1}^k \chi_j^t a + \left(1 - \sum_{j=1}^k \chi_j^t\right) a,$$

where $\chi_j^t$ is the cut-off function around each $p_j$ ($j = 1, \ldots, k$), defined in Section 4.2.

Since we use the conformally-invariant norms, the same inequalities as (5.1) and (5.2) hold on the regions in $M^t$, which are isomorphic to $X_{n_j} (j = 1, \ldots, k)$ and $Z$, namely, we have

$$\|\chi_j^t a\|_{L^8} + \|\nabla (\chi_j^t a)\|_{L^4} \leq C_2 \|L_{A_{X_{n_j}}} (\chi_j^t a)\|_{L^4}$$

for each $j = 1, 2, \ldots, k$, and

$$\|(1 - \sum_{j=1}^k \chi_j^t) a\|_{L^8} + \|\nabla (1 - \sum_{j=1}^k \chi_j^t) a\|_{L^4} \leq C_3 \|L_{A_Z} (1 - \sum_{j=1}^k \chi_j^t) a\|_{L^4}.$$

Therefore,

$$\|a\|_{L^8} + \|\nabla a\|_{L^4}$$

$$\leq C_2 \sum_{j=1}^k \|L_{A_{X_{n_j}}} (\chi_j^t a)\|_{L^4} + C_3 \|L_{A_Z} (1 - \sum_{j=1}^k \chi_j^t) a\|_{L^4}$$

$$\leq C_2 \sum_{j=1}^k \|\chi_j^t \left(L_{A_{X_{n_j}}} a\right)\|_{L^4} + C \sum_{j=1}^k \|d\chi_j^t \wedge a\|_{L^4}$$

$$\quad + C_3 \|(1 - \sum_{j=1}^k \chi_j^t) L_{A_Z} a\|_{L^4} + C \sum_{j=1}^k \|d\chi_j^t \wedge a\|_{L^4}. \quad (5.3)$$

In order to prove Proposition 5.7, we estimate each term of the final two lines of (5.3).
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From the Hölder inequality
\[ \|d\chi_t^j \wedge a\|_{L^4} \leq \|d\chi_t^j\|_{L^8} \|a\|_{L^8}. \]

Since \( |d\chi_t^j| \sim \frac{1}{r|\log t|} \) on \( r \in (t^{\frac{5}{6}}, t^{\frac{3}{4}}) \),
\[ \|d\chi_t^j\|_{L^8} \leq C \left( \int_{t^{\frac{5}{6}}}^{t^{\frac{3}{4}}} \left( \frac{1}{r|\log t|} \right)^{\frac{3}{2}} r^7 dr \right)^{\frac{1}{3}} \leq C \left( \frac{1}{|\log t|} \right)^{\frac{7}{8}}. \]

Hence \( \|d\chi_t^j \wedge a\|_{L^4} \) has the order of \( O\left( |\log t|^{-\frac{7}{8}} \right) \).

On the other hand, we have
\[ L_{A_t} a = L_{A_{X_{nj}}} a - (1 - \chi_t^j)^2 \pi_7^2 \left( (A_Z - A_{X_{nj}}) \wedge a \right). \]

Thus,
\[ \chi_t^j (L_{A_t} a) = \chi_t^j \left( L_{A_{X_{nj}}} a \right) - \chi_t^j (1 - \chi_t^j)^2 \pi_7^2 \left( (A_Z - A_{X_{nj}}) \wedge a \right). \]

Therefore,
\[ \|\chi_t^j (L_{A_{X_{nj}}} a)\|_{L^4} \leq \|\chi_t^j (L_{A_t} a)\|_{L^4} + \|\chi_t^j (1 - \chi_t^j)^2 \pi_7^2 ((A_Z - A_{X_{nj}}) \wedge a)\|_{L^4}. \]

Here, \( \|\chi_t^j (1 - \chi_t^j)(A_Z - A_{X_{nj}})\| \) has the order of \( O(t^6 r^{-7}) \), thus, \( \|\chi_t^j (1 - \chi_t^j)^2 \pi_7^2 ((A_Z - A_{X_{nj}}) \wedge a)\|_{L^4} = O(t) \|a\|_{L^8}. \)

Similarly,
\[ \| (1 - \sum_{j=1}^{k} \chi_t^j) L_{A_Z} a \|_{L^4} \]
\[ \leq \| (1 - \sum_{j=1}^{k} \chi_t^j) L_{A_t} a \|_{L^4} + \| (1 - \sum_{j=1}^{k} \chi_t^j) \sum_{j=1}^{k} (\chi_t^j \pi_7^2 ((A_Z - A_{X_{nj}}) \wedge a))\|_{L^4}. \]

Again, the last term of the right-hand side of (5.4) has the order of \( O(t) \|a\|_{L^8}. \)

Hence Proposition 5.7 follows.

We now prove the following:
Proposition 5.8. There exists a constant $C_5 > 0$ independent of $t$ such that if $a \in L^4_1(u(E) \otimes \Lambda^1)$ is $L^2$-orthogonal to $K$, then the following holds for $t$ sufficiently small:

$$\|a\|_{L^8} + \|\nabla a\|_{L^4} \leq C_5 \|\tilde{L}_{A_t} a\|_{L^4},$$

where $\tilde{L}_{A_t}$ is the linearized operator with respect to the torsion-free $\text{Spin}(7)$-structure $\tilde{\Omega}^t$ on $M$.

Proof. $\tilde{L}_{A_t} a$ may be written by

$$\tilde{L}_{A_t} a = L_{A_t} a + S \cdot a + T \cdot \nabla a,$$

where $S$ and $T$ are tensor fields with

$$|S| \sim |\tilde{\Omega}^t - \Omega^t| + |\nabla (\tilde{\Omega}^t - \Omega^t)|,$$

$$|T| \sim |\tilde{\Omega}^t - \Omega^t|.$$

Hence,

$$C_4 \|\tilde{L}_{A_t} a\|_{L^4} = C_4 \|L_{A_t} a + S \cdot a + T \cdot \nabla a\|_{L^4}$$

$$\geq C_4 \|L_{A_t} a\|_{L^4} - C_4 \|S \cdot a\|_{L^4} - C_4 \|T \cdot \nabla a\|_{L^4}$$

$$\geq (\|a\|_{L^8} + \|\nabla a\|_{L^4}) - C_4 \|S\|_{L^8} \|a\|_{L^8} - C_4 \|T\|_{C^0} \|\nabla a\|_{L^4}$$

$$= (1 - C_4 \|S\|_{L^8}) \|a\|_{L^8} + (1 - C_4 \|T\|_{C^0}) \|\nabla a\|_{L^4}.$$

From Theorem 2.4 we have

$$\|\nabla (\tilde{\Omega}^t - \Omega^t)\|_{L^8} \leq C t^{3\frac{2}{3}}. \quad (5.5)$$

Thus, from (4.2), (4.4) and (5.5), we have

$$\|S\|_{L^8} \leq C t^{3\frac{2}{3}}, \quad \|T\|_{C^0} \leq C t^{\frac{1}{2}}.$$

Therefore, if we take $t$ small enough so that $C_4 \|S\|_{L^8} \leq \frac{1}{2}, C_4 \|T\|_{C^0} \leq \frac{1}{2}$ hold, then we obtain

$$\|a\|_{L^8} + \|\nabla a\|_{L^4} \leq 2C_4 \|\tilde{L}_{A_t} a\|_{L^4}.$$

This completes the proof of Proposition 5.8. \qed

The following is the direct corollary of Proposition 5.8.

Corollary 5.9. $\ker \tilde{L}_{A_t} \cap K^\perp = \{0\}$. Hence, $\dim \ker \tilde{L}_{A_t} \leq \dim K = \dim \ker L_Z + \sum_{j=1}^k \dim \ker L_{X_n j}.$
We now assume that the linearized operators $L_{A}$, $L_{A_{X_{nj}}}$ ($j = 1, \ldots, k$) satisfy the condition in Section 4.1, namely, the cohomology $H^2(Z, u(E))$ of the complex (3.3) vanishes, but $H^0(Z, u(E))$ of the complex (3.3) does not necessarily vanish, and $L_{A_{X_{nj}}}$: $L^{4}_{1, \delta}(u(E_{X_{nj}}) \otimes \Lambda^{1}(X_{nj})) \to L^{4}_{\delta-1}(u(E_{X_{nj}}) \otimes (\Lambda^{0}(X_{nj}) \oplus \Lambda^{2}_{7}(X_{nj})))$ ($j = 1, 2, \ldots, k$) for $\delta \in (-7, 0)$ is surjective.

We then consider a finite dimensional vector space $C_{Z}$ in $\Omega^{0}(Z, u(E_{Z}))$, whose elements are supported away from $p_{j}$ ($j = 1, \ldots, k$) with the following properties:

- $\dim C_{Z} = \dim \ker(L_{A_{Z}}^{*})$ and
- $\Omega^{0}(Z, u(E_{Z})) = C_{Z}^{\perp} \oplus \ker(L_{A_{Z}}^{*})$,

where $C_{Z}^{\perp}$ is the $L^{2}$-orthogonal complement of $C_{Z}$ in $\Omega^{0}(Z, u(E_{Z}))$. Since all elements in $C_{Z}$ are supported on the region $M_{0}$ of $Z \subset M$ for small $t$, we can think of $C_{Z}$ as lying in $\Omega^{0}(M, u(E))$. We choose this $C_{Z}$ in the following way. Firstly, by using the method of Proposition 5.8, one can show that there exists a constant $C_{6} > 0$ such that if $a \in L^{4}_{1}(u(E) \otimes \Lambda^{1})$ with $a \perp K$, then the following holds for $t$ sufficiently small:

$$||a||_{L^{2}} \leq C_{6}||\tilde{L}_{A_{t}}a||_{L^{2}}. \quad (5.6)$$

We then choose $C_{Z}$ such that the following holds for all $c \in C_{Z}$:

$$||L_{A_{Z}}^{*}c||_{L^{2}} \leq \frac{1}{4C_{6}}||c||_{L^{2}}. \quad (5.7)$$

This holds, provided $C_{Z}$ is sufficiently close to $\ker L_{A_{Z}}^{*}$ in $L^{2}_{1}$. Note that, by taking $t$ sufficiently small, we obtain

$$||\tilde{L}_{A_{t}}^{*}c||_{L^{2}} \leq \frac{1}{2C_{6}}||c||_{L^{2}} \quad (5.7)$$

for all $c \in C_{Z}$. We will use $C_{Z}$ as a substitute for the kernel of $L_{A_{Z}}^{*}$, which also makes sense on $M$.

Now, as for Lemmas 5.5 and 5.6, we have:

**Lemma 5.10.** There exists a constant $C > 0$ such that the following holds for any $(a_{X_{nj}}, b_{X_{nj}}) \in L_{4, -1}^{4} \left( u(E_{X_{nj}}) \otimes (\Lambda^{0}(X_{nj}) \oplus \Lambda^{2}_{7}(X_{nj})) \right)$:

$$||a_{X_{nj}}||_{L^{8}} + ||\nabla a_{X_{nj}}||_{L^{4}} + ||b_{X_{nj}}||_{L^{8}} + ||\nabla b_{X_{nj}}||_{L^{4}} \leq C||L_{A_{X_{nj}}}^{*} (a_{X_{nj}}, b_{X_{nj}})||_{L^{4}}.$$
Lemma 5.11. There exists a constant $C > 0$ such that the following holds for any $(a_Z, b_Z) \in L_4^1(u(E_Z) \otimes (\Lambda^0(Z) \oplus \Lambda^2_7(Z)))$ with $a_Z \perp C_Z$:

$$||a_Z||_{L^8} + ||\nabla a_Z||_{L^4} + ||b_Z||_{L^8} + ||\nabla b_Z||_{L^4} \leq C||L^*_A(a_Z, b_Z)||_{L^4}.$$ 

Since the argument for Proposition 5.8 also works for the formal adjoints $L^*_A, L^*_{A_{X,n_j}} (j = 1, \ldots, k)$, and $\tilde{L}^*_A$, therefore we obtain the following:

Proposition 5.12. There exists a constant $C_7 > 0$ independent of $t$ such that if $(a, b) \in L_4^1(u(E) \otimes (\Lambda^0(M) \oplus \Lambda^2_7(M)))$ with $a \perp C_Z$, then

$$||a||_{L^8} + ||\nabla a||_{L^4} + ||b||_{L^8} + ||\nabla b||_{L^4} \leq C_7||\tilde{L}^*_A(a, b)||_{L^4}.$$ 

We also have the following:

Proposition 5.13.

$$L^4(u(E) \otimes (\Lambda^0 \oplus \Lambda^2_7)) = C_Z \oplus \tilde{L}_A \left( K^\perp \cap L^4_1(u(E) \otimes \Lambda^1) \right). \tag{5.8}$$

Proof. Firstly, we prove $C_Z \cap \tilde{L}_A \left( K^\perp \cap L^4_1(u(E) \otimes \Lambda^1) \right) = \{0\}$. Suppose for a contradiction that there exists $a \in K^\perp \cap L^4_1(u(E) \otimes \Lambda^1)$ such that $\tilde{L}_A a = c$ for some $c \in C_Z$ with $c \neq 0$. Then we have

$$||c||^2_{L^2} = \langle c, c \rangle = \langle c, \tilde{L}_A, a \rangle = \langle \tilde{L}^*_A, a, c \rangle \leq ||\tilde{L}^*_A, c||_{L^2}||a||_{L^2}.$$ 

Thus, from (5.6) and (5.7), we get

$$||c||^2_{L^2} \leq \frac{1}{2}||c||^2_{L^2}.$$ 

This is a contradiction. Hence $C_Z \cap \tilde{L}_A \left( K^\perp \cap L^4_1(u(E) \otimes \Lambda^1) \right) = \{0\}$.

Next, by using the index theory, one can obtain that

$$\text{Ind}(\tilde{L}_A) = \text{Ind}(L_Z) + \sum_{j=1}^k \text{Ind}(L_{X,n_j}).$$

Hence,

$$\text{Ind}(\tilde{L}_A) = (\dim K_Z - \dim C_Z) + \sum_{j=1}^k \dim K_{X,n_j} = \dim K - \dim C_Z. \tag{5.9}$$
On the other hand, $\tilde{L}_{A_t}(K^\perp) \subset \text{Im} \tilde{L}_{A_t}$ has the codimension
\[ \dim K - \dim \ker \tilde{L}_{A_t}, \]
and $\text{Im} \tilde{L}_{A_t} \subset L^4 (u(E) \otimes (\Lambda^0 \oplus \Lambda^2))$ has the codimension $\dim \ker \tilde{L}_{A_t}^*$. Thus, the codimension of $\tilde{L}_{A_t}(K^\perp) \subset L^4 (u(E) \otimes (\Lambda^0 \oplus \Lambda^2))$ is
\[ (\dim K - \dim \ker \tilde{L}_{A_t}) + \dim \ker \tilde{L}_{A_t}^*. \]
This is $\dim C_Z$ by (5.9). Hence, (5.8) holds.

\[ \Box \]

6 Construction

In this section, we prove the following:

**Theorem 6.1.** Let $M = M^t$ be the torsion-free $\text{Spin}(7)$-manifold in Section 2, that is, $M$ is a desingularization of a Calabi–Yau four-orbifold $Y$ with finitely many singular points and an anti-holomorphic involution fixing only the singular set by gluing $\text{ALE} \text{Spin}(7)$-manifold $X_{n_j}$ ($j = 1, 2, \ldots, k$) at each singular points $p_j$ ($j = 1, 2, \ldots, k$). Assume that there are Hermitian–Einstein connections on $Y$ and $W_{n_j}$’s satisfying the conditions in Section 4.1. Then there exists a $\text{Spin}(7)$-instanton on a vector bundle $E$ over $M = M^t$ for $t$ sufficiently small.

In Section 6.1, we find a $\text{Spin}(7)$-instanton $A_t + a_t$ in $L^4_1$ by an iterative method, using the estimates in Section 4 and 5. The regularity of the solution is given in Section 6.2.

6.1 Inductive construction

The equation we would like to solve is
\[ \tilde{L}_{A_t} a_t = (c_t, -\tilde{\pi}_7^2(F_{A_t}) - \tilde{\pi}_7^2(a_t \wedge a_t)) \] (6.1)
with $c_t \in C_Z$. From (5.8), for a given $e \in L^4 (u(E) \otimes \Lambda^2)$, there exists a unique $c \in C_Z$ such that $(c, e) \in \tilde{L}_{A_t}(K^\perp)$. We then inductively define a sequence $\{a^k_t\}$ and $\{c^k_t\}$ ($k = 0, 1, 2, \ldots$) by
\[ \tilde{L}_{A_t} a^{k+1}_t = (c^k_t, -\tilde{\pi}_7^2(F_{A_t}) - \tilde{\pi}_7^2(a^k_t \wedge a^k_t)), \] (6.2)
with $a^0_t = 0$, and each $a^k_t$ ($k = 0, 1, 2, \ldots$) is uniquely determined by the condition $a^k_t \perp K$, and each $c^k_t$ is uniquely determined by the right-hand-side of (6.2) lying in $\tilde{L}_{A_t}(K^\perp)$. 
Lemma 6.2. Assume that $e \in L^4 (u(E) \otimes \Lambda_t^2)$ and $c \in C_Z$ satisfy $(c, e) \in \tilde{L}_{L^*}(K^*)$. Then the following holds:

$$||c||_{L^4} \leq C_{\delta} ||e||_{L^4}. \quad (6.3)$$

Proof. Each $c \in C_Z$ uniquely extends to $(c + a', b') \in \left(\tilde{L}_{L^*}(K^*)\right)^\perp$, where $a' \in C_{\frac{1}{2}} \cap L^4 (u(E) \otimes \Lambda_t^0)$ and $b' \in L^4 (u(E) \otimes \Lambda_t^2)$. Since $(c, e) \in \tilde{L}_{L^*}(K^*)$, we obtain $\langle (c, e), (c + a', b') \rangle_{L^2} = 0$. Thus we get

$$||c||_{L^2}^2 = -\langle b', e \rangle_{L^2} \leq ||b'||_{L^2} ||c||_{L^2}. \quad (6.4)$$

Since $a' \perp C_Z$, from Proposition 5.12 we get

$$||b'||_{L^8} \leq C_7 ||\tilde{L}_{L^*}^*(a', b')||_{L^4} = C_7 ||\tilde{L}_{L^*}^*(-c, 0)||_{L^4} \leq C ||c||_{L^4}. \quad (6.5)$$

As $C_Z$ is a finite dimensional vector space, $||c||_{L^4} \leq C ||c||_{L^2}$ for all $c \in C_Z$. Hence

$$||b'||_{L^8} \leq C ||c||_{L^2}. \quad (6.6)$$

Therefore, from (6.4) and (6.5), we obtain

$$||c||_{L^2}^2 \leq C ||c||_{L^2} ||e||_{L^2}.$$ 

Thus, $||c||_{L^2} \leq C ||e||_{L^2}$. Again, using $||c||_{L^4} \leq C ||c||_{L^2}$ and $||e||_{L^2} \leq C ||e||_{L^4}$, we obtain (6.3).

Proposition 5.8 and Lemma 6.2 together with equation (6.2) give us

$$||a_t^{k+1} - a_t^k||_{L^8} + ||\nabla (a_t^{k+1} - a_t^k)||_{L^4} \leq C_5 ||c_t^k - a_t^{k-1}||_{L^4} + C_5 ||h_2^2(a_t^k \wedge a_t^k - a_t^{k-1} \wedge a_t^{k-1})||_{L^4} \leq C_5 (C_8 + 1) ||\tilde{h}_2^2(a_t^k \wedge a_t^k - a_t^{k-1} \wedge a_t^{k-1})||_{L^4} \leq C_5 (C_8 + 1) ||a_t^k - a_t^{k-1}||_{L^4} (||a_t^k||_{L^8} + ||a_t^{k-1}||_{L^8}). \quad (6.6)$$

We now prove the following:
Lemma 6.3. There exists a constant $C_9 > 0$ independent of $t$ such that the following hold for all $k$ and $t$ sufficiently small:

\[ ||a_t^k||_{L^8} \leq C_9 t^{\frac{k}{2}}, \]  

(6.7)

\[ ||a_t^k - a_t^{k-1}||_{L^8} \leq C_9 t^{\frac{k}{2} - k}. \]  

(6.8)

Proof. The proof goes by induction. For $k = 1$,

\[ ||a_t^1||_{L^8} \leq C_5 ||\nabla a_t^1||_{L^8} \leq Ct^{\frac{1}{2}}. \]

Suppose that (6.8) holds for $1, 2, \ldots, k$. Then we obtain

\[ ||a_t^k||_{L^8} \leq ||a_t^1||_{L^8} + ||a_t^1 - a_t^2||_{L^8} + \cdots + ||a_t^k - a_t^{k-1}||_{L^8} \]

\[ \leq C_9 t^{\frac{k}{2}} \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} \right) \]

\[ \leq C_9 t^{\frac{k}{2}}. \]

Hence if we assume that (6.8) holds for $1, 2, \ldots, k$, then (6.7) holds for $k$.

Now we suppose that (6.7) and (6.8) hold for $1, 2, \ldots, k$. Then, by (6.6)

\[ ||a_t^{k+1} - a_t^k||_{L^8} \leq C||a_t^k - a_t^{k-1}||_{L^8} (||a_t^k||_{L^8} + ||a_t^{k-1}||_{L^8}) \]

\[ \leq C(C_9 t^{\frac{k}{2}} - 2^{k})(C_9 t^{\frac{k}{2}} + C_9 t^{\frac{k}{2}}). \]

Therefore, if we take $t$ small enough so that $2CC_9 t^{\frac{k}{2}} \leq \frac{1}{2}$, then

\[ ||a_t^{k+1} - a_t^k||_{L^8} \leq C_9 t^{\frac{k}{2} - 2^{k-1}}. \]

Lemma 6.3 and (6.6) imply that \{a_t^k\} and \{\nabla a_t^k\} are Cauchy sequences in $L^8$ and $L^4$ respectively, thus \{a_t^k\} and \{\nabla a_t^k\} converge to $a_t$ and $\nabla a_t$ in $L^8$ and $L^4$ respectively for some unique $a_t \in L^4_1(u(E) \otimes \Lambda^4)$. In addition, Lemmas 6.2, 6.3 and (6.6) imply that there exists a constant $C_{10} > 0$ such that the following holds for all $k$ and $t$ sufficiently small:

\[ ||c_t^k||_{L^4_1} \leq C_{10} t^{\frac{k}{2}}, \]

\[ ||c_t^k - c_t^{k-1}||_{L^4_1} \leq C_{10} t^{\frac{k}{2} - 2^{k-1}}. \]

Thus, \{c_t^k\} converges in $L^4_1(u(E) \otimes \Lambda^4)$, and hence in $C_Z$.

Therefore, we obtain

Proposition 6.4. For $t$ sufficiently small there exists $a_t \in L^4_1(u(E) \otimes \Lambda^4)$ with $||a_t||_{L^8} \leq C_9 t^{\frac{k}{2}}$ and $c_t \in C_Z$ with $||c_t||_{L^4_1} \leq C_{10} t^{\frac{k}{2}}$ such that $A_t + a_t$ satisfies the Spin(7)-instanton equation and $d_{A_t} a_t = c_t$. 
6.2 Regularity

We use the elliptic theory for $L^p$ spaces \[\text{GT83}\]. If $D$ is an elliptic operator of order $\ell$, then for each $k \geq 0$

$$||s||_{L^{k+\ell}} \leq C \left(||Ds||_{L^k} + ||s||_{L^p}\right).$$

**Lemma 6.5.** If $A + a \in L^4_1(u(E) \otimes \Lambda^1)$ is a Spin(7)-instanton with $||a||_{L^8}$ sufficiently small and $||d_A^*a||_{L^4_1}$ bounded, then $a \in L^2_4(u(E) \otimes \Lambda^1)$.

**Proof.** This follows from the standard argument (see for example \[\text{DK90}\] pp. 61–62). From the elliptic regularity,

$$||a||_{L^4_2} \leq C \left(||LAa||_{L^4_1} + ||a||_{L^4}\right)$$

$$\leq C \left(||d^*_Aa||_{L^4_1} + ||a||_{L^8} ||a||_{L^4} + ||a||_{L^4}\right)$$

$$\leq C \left(||d^*_Aa||_{L^4_1} + ||a||_{L^8} ||a||_{L^2_2} + ||a||_{L^4}\right).$$

Hence,

$$(1 - C||a||_{L^8}) ||a||_{L^4_2} \leq C \left(||d^*_Aa||_{L^4_1} + ||a||_{L^4}\right).$$

Therefore, if $||a||_{L^8}$ is small enough, and $||d^*_Aa||_{L^4_1}$ is bounded, then $||a||_{L^2_2}$ is bounded.

**Remark 6.6.** From the Sobolev embedding theorem, if $a \in L^4_2$, then $a \in L^8$. Then one can use the argument in Section 8 of \[\text{Lew98}\] to obtain the smoothness of $a$.

**Remark 6.6.** From the Sobolev embedding theorem, if $a \in L^4_2$, then $a \in L^8$. Since Spin(7)-instantons are Yang–Mills connections, thus we can use results on Yang–Mills connections such as in \[\text{Uhl82a}, \text{Uhl82b}, \text{Weh04}\]. For example, use Theorem 9.4 in \[\text{Weh04}\] to find a gauge transformation $g$ such that $g^*(a)$ is smooth.

7 Example

We consider an example from \[\text{Joy00}\] (Example 15.7.3). Let $Y$ be a complete intersection in the weighted projective space $\mathbb{C}P^{6}_{3,3,3,3,4,4}$ defined by

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 + P(z_4, z_5, z_6) = 0,$$

$$iz_0^4 - iz_1^4 + 2iz_2^4 - 2iz_3^4 + Q(z_4, z_5, z_6) = 0,$$
where \(P(z_4, z_5, z_6), Q(z_4, z_5, z_6)\) are generic homogeneous cubic polynomials with real coefficients. This is a Calabi–Yau four-orbifold, and the singular set consists of the 9 points defined by

\[
\{[0, 0, 0, z_4, z_5, z_6] \in \mathbb{CP}^6_{3,3,3,4,4,4} : P(z_4, z_5, z_6) = Q(z_4, z_5, z_6) = 0\},
\]

and the curve \(\Sigma\) defined by

\[
\Sigma = \{[z_0, z_1, z_2, z_3, 0, 0, 0] \in \mathbb{CP}^6_{3,3,3,3,4,4,4} : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0,
\]
\[
i z_0^4 - i z_1^4 + 2 i z_2^4 - 2 i z_3^4 = 0\}.
\]

We consider an anti-holomorphic involution \(\sigma : \mathbb{C}^4 \to \mathbb{C}^4\) defined by

\[
\sigma : [z_0, z_1, \cdots, z_6] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5, \bar{z}_6].
\]

This fixes some points of the singular sets, which depend on the choice of \(P\) and \(Q\). We take these \(P\) and \(Q\) so that there are five fixed points, say \(p_1, p_2, p_3, p_4, p_5\), of \(\sigma\) in the singular set, and two pairs of the singular points, say \(p_6\) and \(p_7, p_8\) and \(p_9\), which are swapped with each other.

We define \(Y'\) to be the blow-up of \(Y\) along \(\Sigma\), and lift \(\sigma\) to \(Y'\) to give an anti-holomorphic involution \(\sigma : Y' \to Y'\) with fixed points \(p_1, \ldots, p_5\). The singular points of \(Y'/\sigma\) are \(p_1, \ldots, p_5, p_6 = p_7, p_8 = p_9\). We put \(X_{n_j} (j = 1 \text{ or } 2)\) at \(p_1, \ldots, p_5\), and \(X\) at \(p_6 = p_7, p_8 = p_9\), where \(X\) is the blow-up of \(\mathbb{C}^4/\mathbb{Z}_4\) at the origin, and then apply the construction described in Section 2 (some modifications about gluing \(X\) at \(p_6 = p_7, p_8 = p_9\) are needed, but they are trivial) to get a compact \(\text{Spin}(7)\)-manifold \(M\).

**Ingredient bundles.** We consider a line bundle \(L_D\) over \(X\), which is determined by the exceptional divisor \(D = \mathbb{CP}^3\). We equip \(L_D\) with a Hermitian metric. Note that \(L_D\) has trivial holonomy at infinity.

We put \(E_{X,k} = L_D^k \oplus L_D^{-k} \ (k \in \mathbb{Z})\) at \(p_6\) and \(p_7\), \(E_{X,\ell} = L_D^\ell \oplus L_D^{-\ell} \ (\ell \in \mathbb{Z})\) at \(p_8\) and \(p_9\), and the rank two trivial bundle \(\mathbb{C}^2\) at each \(p_1 \ldots, p_5\), then glue them together to \(\mathbb{C}^2\) over \(Y'\) to get a vector bundle \(E\) over \(M\). Since

\[
\text{Aut}(L_D^m \oplus L_D^{-m}) = H^0(X, \mathcal{O}_X) \oplus H^0(X, L_D^{2m}) \oplus H^0(X, L_D^{-2m}) \oplus H^0(X, \mathcal{O}_X),
\]

the holomorphic automorphism group of \(E_{X,m}\) consists of upper triangular matrices in \(SU(2)\). Therefore, the automorphism group of \(E\) is the intersection of contributions from the stabilizer groups of \(E_{X,k}\) and \(E_{X,\ell}\) at each
\( p_i \) (\( i = 6, 7, 8, 9 \)), schematically it is

\[
\begin{pmatrix}
A_6^{-1}
&\ast &\ast \\
0 & \ast & \ast \\
\end{pmatrix}
\cap
\begin{pmatrix}
A_7^{-1}
&\ast &\ast \\
0 & \ast & \ast \\
\end{pmatrix}
\cap
\begin{pmatrix}
A_8^{-1}
&\ast &\ast \\
0 & \ast & \ast \\
\end{pmatrix}
\cap
\begin{pmatrix}
A_9^{-1}
&\ast &\ast \\
0 & \ast & \ast \\
\end{pmatrix},
\]

where \( A_6, A_7, A_8, A_9 \in SU(2) \), and \( A_6 \) and \( A_7, A_8 \) and \( A_9 \) are conjugate. This becomes \( c \cdot \text{id} \) (\( c \in \mathbb{C}^* \)) for generic \( A_6 \) and \( A_8 \). Thus, we can make the resulting vector bundle \( E \) irreducible.

**Hermitian–Einstein connections.** We equip \( L_D^m \) with a Hermitian–Einstein connection. Since \( c_1(L_D^m) \) lies in the image of \( H^2_{\text{cs}}(X) \) \( \rightarrow \) \( H^2(X) \), thus, from Lockhart (\[Loc87\], Section 8), there exists a unique harmonic 2-form \( \alpha \) in \( X \) such that \( [\alpha] = 2\pi c_1(L_D^m) \) and \( \alpha = O(r^{-7}) \). We then decompose \( \alpha \) into

\[
\alpha = \alpha^{2,0} + \alpha^{0,2} + \alpha^{1,1} + (\alpha \cdot \omega)\omega.
\]

Since \( H^{0,2}(X) = 0 \), \( \alpha^{0,2} = \alpha^{2,0} = 0 \). Moreover, since \( (\alpha \cdot \omega) \) is harmonic, vanishing at infinity, thus, \( (\alpha \cdot \omega) = 0 \) by the maximum principle. Hence \( \alpha = \alpha^{1,1} \). We now take a connection \( A \) of \( L_D^m \) with \( F_A = \alpha \), then this \( A \) is a Hermitian–Einstein connection on \( L_D^m \).

**The conditions for the linearized operators.** We examine the conditions for the linearized operators in Section 4.1. For the \( Spin(7) \)-orbifold side, we have \( H^2(Z) = 0 \), since \( Z \) has holonomy \( SU(4) \times \mathbb{Z}_2 \). Thus, the cohomology \( H^2(Z, \mathfrak{su}(E)) \) of the complex \([3,3]\) vanishes, as \( H^2(Z, \mathfrak{su}(\mathbb{C}^2)) = H^2(Z) \otimes \mathfrak{su}(\mathbb{C}^2) \). Hence, \( \ker L^m_{\alpha, \delta} \) lies in \( \Omega^0(Z, \mathfrak{su}(\mathbb{C}^2)) \).

For the ALE side, we introduce a sheaf cohomology on \( X \) with the decay rate \( \delta \) at infinity as follows. Let \( (L_D^m)_{\delta} \) be a sheaf of holomorphic sections of \( L_D^m \) with the decay rate \( \delta \). We consider the following injective resolution of \( (L_D^m)_{\delta} \):

\[
0 \rightarrow (L_D^m)_{\delta} \rightarrow i \Omega^0(L_D^m)_{\delta} \rightarrow \Omega^{0,1}(L_D^m)_{\delta-1} \rightarrow \Omega^{0,2}(L_D^m)_{\delta-2} \rightarrow \cdots \rightarrow \Omega^{0,3}(L_D^m)_{\delta-3} \rightarrow \Omega^{0,4}(L_D^m)_{\delta-4} \rightarrow 0.
\]

We then have a complex induced by \([7,1]\):

\[
0 \rightarrow C^\infty(X, \Omega^0(L_D^m)_{\delta}) \rightarrow \delta_X \rightarrow C^\infty(X, \Omega^{0,1}(L_D^m)_{\delta-1}) \rightarrow \delta_X \rightarrow \cdots \rightarrow C^\infty(X, \Omega^{0,3}(L_D^m)_{\delta-3}) \rightarrow \delta_X \rightarrow C^\infty(X, \Omega^{0,4}(L_D^m)_{\delta-4}) \rightarrow 0.
\]
We denote by $H^i_\delta(X, L_D^m)$ the $i$-th cohomology of the complex (7.2) for each $i = 0, \ldots, 4$.

**Lemma 7.1.**

$$H^0_\delta(X, L_D^m) = 0$$

for all $m \in \mathbb{Z}$ and $\delta < 0$.

**Proof.** We take the standard holomorphic section $s_D$ of $L_D$, and write $t \in H^0_\delta(L^m)$ as $t = f s_D$, where $f$ is a meromorphic function on $X$. Since $t$ is holomorphic, $f$ has a pole of the order $\leq m$ if $m \geq 0$, or $f$ has a zero of the order $\geq -m$ if $m \leq 0$ at $D$. By Hartogs’ theorem, a holomorphic function on $\mathbb{C}^4 \setminus 0$ extends to $\mathbb{C}^4$, hence $f$ has no pole at $D$, that is, $f$ is holomorphic. Since we impose growth condition at infinity, $|f| \to 0$ as $r \to \infty$. Hence $f \equiv 0$ by the maximum principle. \hfill \Box

**Lemma 7.2.**

$$H^2_\delta(X, L_D^m) = 0$$

for all $m \in \mathbb{Z}$ and $\delta < 0$.

**Proof.** We consider the following exact sequence,

$$0 \to (L^{-1})_D \to (O_X)_\delta \to O_D \to 0.$$  

Twisting by $L^m_D$ ($m \in \mathbb{Z}_{\geq 0}$), we obtain

$$0 \to (L^{-1})_D \to (L^m_D)_\delta \to O_D \to 0. \quad (7.3)$$

From this, we get a long exact sequence:

$$\cdots \to H^1(D, O_D(-4m)) \to H^2_\delta(X, L_D^{m-1}) \to \to H^2_\delta(X, L_D^m) \to H^2(D, O_D(-4m)) \to \cdots,$$

where we used $H^i(X, O_D \otimes L_D^m) \cong H^i(D, O_D(-4m))$. As $H^1(D, O_D(-4m)) = H^2(D, O_D(-4m)) = 0$, we get an isomorphism $H^2_\delta(X, L_D^{m-1}) \to H^2_\delta(X, L_D^m)$. In addition, we have $H^2_\delta(X, O_X) = 0$ for $\delta < 0$ by Theorem 5.3 in [Joy01]. Hence $H^2_\delta(X, L_D^m) = 0$ for $m \in \mathbb{Z}_{\geq 0}$ by induction. The dual argument yields $H^2_\delta(X, L_D^m) = 0$ for $m \in \mathbb{Z}_{\leq 0}$ as well. \hfill \Box

Hence,

$$H^2_\delta(X, \text{End}(E_{X,m})) = H^2_\delta(X, O_X) \oplus H^2_\delta(X, L_D^{2m}) \oplus H^2_\delta(X, L_D^{-2m}) \oplus H^2_\delta(X, O_X) = 0$$
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for $\delta < 0$. Thus, the linearized operator $L_{AX} : L_{1,\delta}^4(u(E_{X,m}) \otimes \Lambda^1(X)) \to L_{\delta-1}^4(u(E_{X,m}) \otimes (\Lambda^0(X) \oplus \Lambda^2_\delta(X)))$ with $\delta \in (-7,0)$ is surjective, thus the condition in Section 4.1 is satisfied. Therefore, we obtain a $\text{Spin}(7)$-instanton on this $E$ by Theorem 6.1.

Furthermore, in this example, $\dim C_Z = 3$ and $\dim K_Z = 0$, and we also have the following for the ALE side:

**Lemma 7.3.**

\[
\begin{align*}
H^1_\delta(X, L^m_D) &= \begin{cases} \mathbb{C} \cdot m^2(8m^2 - 5) & (m \leq 0), \\ 0 & (m > 0). \end{cases} \tag{7.4} \\
H^3_\delta(X, L^m_D) &= \begin{cases} \mathbb{C} \cdot m^2(8m^2 - 5) & (m > 0), \\ 0 & (m \leq 0). \end{cases} \tag{7.5}
\end{align*}
\]

**Proof.** We again use the long exact sequence induced by (7.3):

\[
\cdots \rightarrow H^0_\delta(X, L^m_D) \rightarrow H^0(D, \mathcal{O}_D(-4m)) \rightarrow H^1_\delta(X, L^{m-1}_D) \\
\rightarrow H^1_\delta(X, L^m_D) \rightarrow H^1(D, \mathcal{O}_D(-4m)) \rightarrow \cdots.
\]

Since $H^0_\delta(X, L^m_D) = H^1(D, \mathcal{O}_D(-4m)) = 0$, and

\[
\dim H^0(D, \mathcal{O}_D(-4m)) = \begin{cases} \frac{(3-4m)(2-4m)(1-4m)}{6} & (m \leq 0), \\ 0 & (m > 0), \end{cases}
\]

we get

\[
\dim H^1_\delta(X, L^m_D) = \dim H^1_\delta(X, L^{m-1}_D) - \begin{cases} \frac{(3-4m)(2-4m)(1-4m)}{6} & (m \leq 0), \\ 0 & (m > 0). \end{cases}
\]

From this with $\dim H^1_\delta(X, \mathcal{O}_X) = 0$, we obtain (7.4) by induction, and (7.5) follows from (7.4) either by Serre duality, or by the same method of proof using $\dim H^3(D, \mathcal{O}_D(-4m)) = -\frac{(3-4m)(2-4m)(1-4m)}{6}$ if $m > 0$ and 0 if $m \leq 0$.

Therefore, the real dimension of $K_{X_{n,j}}$ is $\frac{4k^2}{3}(32k^2 - 5)$ at $p_6 = p_7$, and $\frac{4\ell^2}{3}(32\ell^2 - 5)$ at $p_8 = p_9$. We also have $\dim C_Z = 3$ and the dimensions of all the other spaces in $K$ are zero. Hence, (5.9) shows that the virtual dimension of the moduli space in this example is given by

\[
-3 + \frac{4k^2}{3}(32k^2 - 5) + \frac{4\ell^2}{3}(32\ell^2 - 5).
\]
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When $k = \ell = 0$, we obtain the trivial, flat $SU(2)$ instanton, which is rigid with automorphism group $SU(2)$ of dimension 3, and the virtual dimension is $-3$. For $k, \ell$ not both zero, we get a positive dimensional moduli space, and for the generic gluing data, the solution given by Theorem 6.1 is unobstructed and irreducible, and the moduli space is smooth of the given dimensions near the solution.

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