STRONG SUBADDITIVITY INEQUALITY AND ENTROPIC UNCERTAINTY RELATIONS

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ABSTRACT. We prove a generalization of the strong-subadditivity (SSA) inequality in the context of finite dimensional von Neumann algebras. Our generalized SSA inequality subsumes various entropic uncertainty relations, such as the Maassen-Uffink inequality in the presence of quantum memory. Motivated by the generalization of SSA, we introduce an analog of conditional mutual information and discuss superadditivity and monotonicity under certain operations. Our proof uses complex interpolation and noncommutative mixed $L_p$ spaces.

1. Introduction

The strong subadditivity (SSA) of quantum entropy, proved by Lieb and Ruskai [19] in 1973, is one of most fundamental inequalities in Quantum Information Theory. It states that for any tripartite state $\rho^{ABC}$,

$$H(AC|\rho) + H(BC|\rho) \geq H(C|\rho) + H(ABC|\rho).$$
(1.1)

Some other important theorems in the field, such as the data processing inequality of quantum relative entropy and non-negativity of conditional mutual information, are known to be equivalent to SSA. The aim of this paper is to unify the SSA inequality with entropic uncertainty relations, an equally historical topic that recently experienced an explosion of development (see [3] for a survey). One remarkable recent advance is the uncertainty relations with quantum memory [3],

$$H(X|B) + H(Z|B) \geq \log \frac{1}{c} + H(A|B),$$
(1.2)

in which $X$ and $Z$ are two complementary measurements of quantum system $A$, allowing non-classical correlations between $A$ and a memory system $B$.

To broaden this picture, we generalize to a setting with finite dimensional von Neumann algebras. Recall that a finite dimensional von Neumann algebra $\mathcal{M}$ is a $*$-subalgebra

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of a matrix space and always isomorphic to an orthogonal sum of matrix blocks. Given a faithful trace $\tau$ on $M$, a state of $M$ is given by a density $\rho$ ($\rho \geq 0$, $\tau(\rho) = 1$), and its von Neumann entropy is given by $H(\rho) = -\tau(\rho \log \rho)$. For a subalgebra $N \subset M$, the conditional expectation $E_N$ is a map that sends every state $\rho$ to its restriction on $N$. Our first main result gives a complete characterization of SSA-type inequalities.

**Theorem A.** Let $R, S, T$ be subalgebras of $M$ satisfying the quadrilateral of inclusions

\[
\begin{array}{c}
S \subset M \\
\cup \cup \\
R \subset T
\end{array}
\]

Then the inequality

\[
H(\tau(E_S(\rho))) + H(\tau(E_T(\rho))) \geq H(\tau(\rho)) + H(\tau(E_R(\rho)) \quad (1.3)
\]

holds for all states $\rho \in M$ if and only if $E_S \circ E_T = E_T \circ E_S = E_R$.

In von Neumann algebra theory, a quadrilateral satisfying the condition $E_S \circ E_T = E_T \circ E_S = E_R$ is called a commuting square [28]. Commuting squares are useful in the study of subfactors [29]. Petz [25] studied several examples of commuting squares in finite dimensions.

To illustrate, the standard SSA (1.1) are given by overlapped subsystems $AC$ and $BC$ within $ABC$, which forms a natural commuting square $C \subset AC$, $BC \subset ABC$. On the other hand, two mutually unbiased bases $X, Z$ in a $d$-dimensional system $A$ also form a standard example of a commuting square. With quantum memory, Theorem A gives

\[
H(XB)_\rho + H(ZB)_\rho \geq H(AB)_\rho + H(B)_\rho + \log d, \quad (1.4)
\]

which translates to the uncertainty relation with memory (1.2) for mutually unbiased bases. In the tripartite case, we obtain the following uncertain relation.

**Theorem B.** Let $\left(\begin{array}{c}S \subset A \\
\cup \cup \\
C \subset T\end{array}\right)$ be a commuting square. Then for any state $\rho^{ABC}$,

\[
H(S'|C)_{\tilde{\rho}} + H(T|B)_\rho \geq \log |A| \quad (1.5)
\]

where $H(T|B)_\rho := H(E_T \otimes \text{id}(\rho^{AB})) - H(\rho^B)$ and $\tilde{\rho}$ is the image of complementary channel of $E_S$.

The complementary channel of a conditional expectation $E_S$ is in general equivalent to the conditional expectation onto the commutant $S'$ (see Section 2 for the details). When $S = X$ is commutative, this conditional expectation is the same as the complementary channel of $E_S$. So the commutant $S'$ is a noncommutative feature that is hidden when $S$ is the algebra of a measurement basis.

In our argument, we introduce for a subalgebra $N$ the asymmetry measure of relative entropy $D_N(\rho) := H(E_N(\rho)) - H(\rho)$. In particular, the inequality (1.3) can be rephrased
as an uncertainty relation of asymmetry
\[ D^S(\rho) + D^T(\rho) \geq D^R(\rho). \]
This notion \( D^N \) is motivated by two related concepts— asymmetry measures \([22]\) and relative entropy of quantum coherence \([1]\). Quantum coherence is intuitively the non-classicality in a given basis (see \([31]\) for a review). For an orthonormal basis \( \{|x\rangle\} \), the relative entropy of coherence is defined as
\[ C^X(\rho) = H(E_X(\rho)) - H(\rho), \]
and proven to have the operational meaning of distillable coherence \([35]\). It has been noted in \([30]\) that \( C^r \) relates to uncertain relations and a special example of asymmetry measures. A general asymmetry measure is a real-valued function of states that is non-increasing under quantum operations covariant with symmetry transformations (see \([23]\)). The asymmetry measure of relative entropy with respect to the invariant subalgebras of a group action was introduced in \([14, 32]\). Our notation \( D^N \) is a generalization of both concepts to von Neumann subalgebras \( N \) and also a convex, non-negative, faithful asymmetry measure that is monotone under the covariant channels. Equation (1.3) hence demonstrates a connection between asymmetry, uncertainty and strong-subadditivity.

In Section 4, we extend Theorem A and Theorem B to the case where \( S \) and \( T \) are not necessarily a commuting square, in which the inequality (1.3) holds up to a constant \( \log \frac{1}{c} \). This correction term measures how close the setting is to a commuting square, and it recovers the Maassen-Uffink inequality (1.5) for two complementary measurements. In the proof, we use complex interpolation and the connection between noncommutative \( L_p \)-spaces and Rényi asymmetry measures. Our method is partially related to recent works on uncertainty principles in quantum groups and Kac algebras \([9, 10, 20, 16]\).

Motivated by conditional mutual information, we define for a commuting square \( \left( \begin{array}{ccc} S & \subset & M \\ \cup & \cup & \cup \\ R & \subset & T \end{array} \right) \) the quantity
\[ I(S : T | R)_\rho := H_r(E_S(\rho)) + H_r(E_T(\rho)) - H_r(E_R(\rho)) - H_r(\rho). \] (1.6)
We show this quantity is monotone under a notation of “alternating” operations, which is a generalization of local operations for separated system \( A \otimes B \). Moreover we prove its squashed version
\[ I^{sq}(S : T | R)_\rho := \inf_{\text{tr}C(\rho^M) = \rho^M} I^{sq}(S^C : T^C | R^C) \] (1.7)
is superadditive and monotone under “alternating” operations with classical communication, an analog of LOCC (local operation and classical communication). The physical
interpretation of how this quantity reflects the correlations between \(S\) and \(T\) is an open question.

The paper is organized as follows: in Section 2 we prove Theorem A and B using purely information-theoretic tools. Section 3 introduces Rényi asymmetry measures and explains the connection to noncommutative \(L_p\)-spaces. Section 4 is devoted to the SSA inequality with correction terms. Section 5 discusses the quantity \(I(S : T|R)\). Finally we end with some concluding remarks.

2. Asymmetry measure and generalized SSA inequality

2.1. Notations and definitions. We denote by \(\mathbb{B}(H)\) the bounded operators on a Hilbert space \(H\). We use the capital letters \(A, B, C, \ldots\) as short notations for the quantum systems \(\mathbb{B}(H_A), \mathbb{B}(H_B), \mathbb{B}(H_C), \ldots\) and denote \(|A| = \dim H_A\). Recall that a von Neumann algebra \(\mathcal{M}\) is a weak* closed *-subalgebra of some \(\mathbb{B}(H)\). For simplicity, we restrict ourselves to finite dimensional Hilbert spaces and finite dimensional von Neumann algebras. A faithful trace \(\tau\) is positive linear functional \(\tau : \mathcal{M} \to \mathbb{C}\) satisfying

i) traciality: \(\tau(xy) = \tau(yx)\) for all \(x, y \in \mathcal{M}\).

ii) faithfulness: for \(x \geq 0\), \(\tau(x) = 0\) if and only if \(x = 0\).

A state of \(\mathcal{M}\) is given by a density operator \(\rho\) \((\tau(\rho) = 1, \rho \geq 0)\) and we denote by \(S(\mathcal{M})\) the state space of \(\mathcal{M}\). The von Neumann entropy and relative entropy for states of \(\mathcal{M}\) are defined as

\[
H_\tau(\rho) = -\tau(\rho \log \rho), \quad D(\rho||\sigma) = \begin{cases} \tau(\rho \log \rho - \rho \log \sigma) & \text{if } \text{supp}(\rho) \geq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}.
\]

The standard case is \(\mathbb{B}(H)\) with the matrix trace \(\text{tr}\). We will often omit \(\tau\) in \(H_\tau\) if it is clear from the text. Let \(\mathcal{N} \subset \mathcal{M}\) be a von Neumann subalgebra. The conditional expectation is the unique completely positive trace preserving (cptp) and unital map such that for any \(\rho \in \mathcal{M}\)

\[
\tau(a\rho) = \tau(aE_N(\rho)), \quad \forall a \in \mathcal{N}.
\] (2.1)

2.2. Asymmetry measure of a subalgebra. Viewing \(\mathcal{N}\) as a “symmetric” subalgebra, we define the asymmetry measure of relative entropy with respect to \(\mathcal{N}\).

**Definition 2.1.** For a state \(\rho\) of \(\mathcal{M}\), the asymmetry measure of relative entropy with respect to \(\mathcal{N}\) is defined as

\[
D^N(\rho) := \inf_{\sigma \in S(\mathcal{N})} D(\rho||\sigma) = H(E_N(\rho)) - H(\rho),
\]

where the infimum runs over all state \(\sigma \in \mathcal{N}\).
The optimal value is always attained by condition expectation because for any $\sigma \in S(\mathcal{N})$,

$$
D(\rho \| \sigma) = \tau(\rho \log \rho - \rho \log \sigma) = \tau(\rho \log \rho) - \tau(E_N(\rho) \log \sigma)
$$

$$
= \tau(\rho \log \rho - E_N(\rho) \log E_N(\rho)) - \tau(E_N(\rho) \log \sigma - E_N(\rho) \log E_N(\rho))
$$

$$
= D(\rho \| E_N(\rho)) + D(\sigma \| E_N(\rho))
$$

$$
\geq D(\rho \| E_N(\rho)),
$$

where in the last step we use the non-negativity of $D(\cdot \| \cdot)$.

**Example 2.2.** a) Let $B \subset AB$ be a subsystem. In terms of subalgebras, this means the inclusion $C_1 \otimes \mathbb{B}(H_B) \subset \mathbb{B}(H_A \otimes H_B)$. Then the conditional expectation is completely depolarization on $A$,

$$
E_A(\rho^{AB}) = \frac{1}{|A|} \otimes \rho^B,
$$

where $\rho^B = tr_A(\rho^{AB})$ is the reduced density. Then,

$$
D^B(\rho^{AB}) = H\left(\frac{1}{|A|} \otimes \rho^B\right) - H(\rho^{AB}) = \log |A| + H(B) - H(AB),
$$

which is the coherent information up to a normalization constant. This constant comes from the fact the embedding $\rho^B \to 1^A \otimes \rho^B$ is trace preserving up to a constant $|A|$.

b) Let $\{|x\rangle\}$ be an orthonormal basis of $H$. The projection measurements $\{|x\rangle\langle x|\}$ generate the commutative subalgebra $\mathcal{X} = \text{span}\{|x\rangle\langle x|\}$. The condition expectation onto $\mathcal{X}$ is the completely dephasing map $E_x(\rho) = \sum_x \langle x|\rho|x\rangle|x\rangle\langle x|$. Then

$$
D^X(\rho) = H(E_X(\rho)) - H(\rho) = H(\{|x\rangle\langle x|\}) - H(\rho).
$$

This is called relative entropy of coherence introduced in [1] (see [35] for operational meanings).

c) Let $G$ be a finite or compact group. Suppose $G$ acts on the Hilbert space $H$ as symmetry transformations via a unitary representation

$$
u : G \to \mathbb{B}(H), u_{g_1}u_{g_2} = u_{g_1g_2}. \quad (2.2)$$

The invariant subalgebra is $\mathbb{B}(H)_G := \{x \in \mathbb{B}(H) \mid uxu^* = x\}$. The conditional expectation onto $\mathbb{B}(H)_G$ is given by

$$
E_G(\rho) = \int_G u_g \rho u_g^* dg.
$$

where $dg$ is the Haar measure on $G$. Then

$$
D^{\mathbb{B}(H)_G}(\rho) := H(E_G(\rho)) - H(\rho)
$$

is the $G$-asymmetry measure of relative entropy introduced in [32]. Actually the part b) is also a special cases of this category.
From the above example, one sees that $D^N$ is a generalization of relative entropy of coherence to noncommutative algebra $\mathcal{N}$ and relative entropy of asymmetry by viewing $\mathcal{N} = \mathcal{B}(H)_G$ as the symmetric algebra. We show that $D^N$ satisfies the properties related to a resource theory framework (see [6]).

**Proposition 2.3.** $D^N$ satisfies the following properties.

i) (Non-negativity): $D^N(\rho) \geq 0$ for any state $\rho \in \mathcal{M}$.

ii) (Faithfulness): $D^N(\rho) = 0$ if and only if $\rho \in \mathcal{N}$.

iii) (Monotonicity): Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a completely positive trace preserving map. If $\Phi(\mathcal{N}) \subset \mathcal{N}$, then

$$D^N(\Phi(\rho)) \leq D^N(\rho).$$

**Proof.** The properties i) and ii) follow easily from the parallel properties of relative entropy $D(\cdot||\cdot)$ (see e.g. [33]). For iv),

$$D^N(\rho) = \inf_\sigma D(\rho||\sigma) = \inf_\sigma D(\Phi(\rho)||\Phi(\sigma)) \leq \inf_\sigma D(\Phi(\rho)||\sigma) = D^N(\Phi(\rho)).$$

The first inequality above is the data processing inequality and the second inequality follows from the fact $\Phi$ sends states of $\mathcal{N}$ to states of $\mathcal{N}$. \hfill \blacksquare

Let us introduce a short notation for subalgebra entropy that $H^N(\rho) = H(E^N(\rho))$. This notation works well for subalgebras but please note the difference with subsystem entropy $H(\mathcal{C}_1 \otimes \mathcal{B}) = H(E_{\mathcal{C}_1 \otimes \mathcal{B}}(\rho^{AB})) = H(\frac{1}{|A|} \otimes \rho^B) = H(\mathcal{B}) + \log |A|$. The difference is the normalization constant of subalgebra inclusions as mentioned in Example 2.2 a). We will make it clear whenever there is a possible confusion.

### 2.3. A characterization of SSA inequality.

For a general setting of the SSA inequality, we consider a quadrilateral of inclusions

$$\left( \begin{array}{c} S \subset \mathcal{M} \\ \cup \quad \cup \\ R \subset \mathcal{T} \end{array} \right).$$

Such a quadrilateral is called a commuting square if the conditional expectations commute, $E_S \circ E_T = E_T \circ E_S = E_R$. In this situation, $R = S \cap T$ always. We find the following correspondence between commuting squares and SSA-type inequalities.

**Theorem 2.4.** Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra with a faithful trace $\tau$. A quadrilateral of inclusions

$$\left( \begin{array}{c} S \subset \mathcal{M} \\ \cup \quad \cup \\ R \subset \mathcal{T} \end{array} \right)$$

is a commuting square if and only if for any state $\rho \in \mathcal{M}$,

$$H(S)_\rho + H(T)_\rho \geq H(\mathcal{M})_\rho + H(R)_\rho. \quad (2.3)$$

**Proof.** Suppose $E_S \circ E_T = E_T \circ E_S = E_R$. We have

$$H(S)_\rho + H(T)_\rho - H(\mathcal{M})_\rho - H(R)_\rho.
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\[
= \left( H(E_S(\rho)) - H(\rho) \right) - \left( H(E_S \circ E_T(\rho)) - H(E_T(\rho)) \right)
\]
\[
= D^S(\rho) - D^S(E_R(\rho)) \geq 0,
\]

The last step follows from \( E_T(S) = R \subset S \) and the monotonicity of \( D^S \) in Proposition 2.3. This proves the necessity. On the other hand, for any state \( \sigma \in S \), the inequality (2.3) implies that

\[
D^R(E^T(\sigma)) = H(E_R(\sigma)) - H(E_T(\sigma)) = 0,
\]

which implies \( E_T(\sigma) = E_R(\sigma) \). Thus for arbitrary \( \rho \in M \),

\[
E_T \circ E_S(\rho) = E_R \circ E_S(\rho) = E_R(\rho).
\]

The other equality follows similarly.

**Remark 2.5.** The inequality (2.3) can be rephrased as an uncertainty relation of asymmetry measure

\[
D^S(\rho) + D^T(\rho) \geq D^R(\rho).
\] (2.4)

Moreover, the inequality is saturated if and only if there exists a recovery cptp map \( \Phi : M \to M \) satisfying

\[
\Phi(E_S(\rho)) = \rho, \quad \Phi(E_R(\rho)) = E_T(\rho);
\]

or equivalently ii) there exists a recovery map

\[
\Phi(E_T(\rho)) = \rho, \quad \Phi(E_R(\rho)) = E_S(\rho).
\]

**Remark 2.6.** When \( R = \mathbb{C} \) is the trivial subalgebra, the commuting square condition means that \( S \) and \( T \) are independent, i.e. \( \tau(ab) = \tau(a)\tau(b) \) for all \( a \in S, b \in T \). Then for any state \( \rho \in M \),

\[
H(S)_\rho + H(T)_\rho \geq H(M)_\rho + \log \tau(1).
\] (2.5)

Here \( \log \tau(1) \) is a dimension constant of \( (M, \tau) \).

The SSA inequality and entropic uncertainty principle of mutually unbiased measurement are special cases of Theorem 2.4.

**Example 2.7** (SSA inequality). Two overlapped subsystems form the commuting square

\[
\left( \begin{array}{c}
\mathbb{A} \otimes \mathbb{C} \otimes C \\
\mathbb{C} \cup C \cup \mathbb{A} \otimes BC
\end{array} \right) \subset \mathbb{A} \otimes \mathbb{C} \otimes BC.
\]

Then Theorem 2.4 gives

\[
H(\rho^{AC} \otimes \frac{1}{|B|}) + H(\rho^{BC} \otimes \frac{1}{|A|}) \geq H(\rho^C \otimes \frac{1}{|A|} \otimes \frac{1}{|B|}) + H(\rho^{ABC}).
\]
Canceling the dimension constants, this is equivalent to the SSA inequality
\[ H(\rho^{AC}) + H(\rho^{BC}) \geq H(\rho^C) + H(\rho^{ABC}) . \] (2.6)

**Example 2.8** (Entropic uncertainty principle). Let \( H_A \cong \mathbb{C}^d \) be the \( d \)-dimensional Hilbert space. Two orthonormal bases \( \{|x_1\rangle, \ldots, |x_d\rangle\} \) and \( \{|z_1\rangle, \ldots, |z_d\rangle\} \) are mutually unbiased if
\[ |\langle x_j|z_k \rangle| = \frac{1}{d}, \text{ for all } j, k = 1, \ldots, d . \] (2.7)

Denote \( \mathcal{X} \) as the subalgebra generated by \( \{|x_j\rangle\langle x_j|\} \) and \( \mathcal{Z} \) the algebra generated by \( \{|z_j\rangle\langle z_j|\} \). The condition (2.7) is equivalent to that \( (\mathcal{X} \subset A \bigcup \bigcup C \subset \mathcal{Z}) \) forms a commuting square. In this situation Theorem 2.4 recovers the uncertainty relation
\[ H(\mathcal{X}) + H(\mathcal{Z}) \geq H(\rho) + \log d . \]

Moreover if \( B \) is an auxiliary system, \( ((\mathcal{X}B \subset AB) \bigcup (C1 \otimes B \subset ZB)) \) is also a commuting square, which yields
\[ H(\mathcal{X}B) + H(\mathcal{Z}B) \geq H(AB) + H(\frac{1}{|A|} \otimes \rho^B) . \]

By subtracting \( 2H(B) \) from both sides, we have the uncertainty principle with memory in [3]
\[ H(\mathcal{X}|B) + H(\mathcal{Z}|B) \geq H(A|B) + \log d . \] (2.8)

Let \( B = \mathbb{B}(H_B) \) be an auxiliary system. We introduce the notation
\[ H(\mathcal{M}|A) := H_{trxtr}(\mathcal{M}B) - H_{tr}(B) = H_{trxtr}(\rho^{MB}) - H(\rho^B) . \]

Here \( \tau \times tr \) is the product trace and \( H(B) = H_{tr}(\rho^B) \) is the subsystem entropy (not the subalgebra entropy \( H(C1 \otimes B) = H_{tr}(\frac{1}{|\tau(1)|} \otimes \rho^B) = H(B) + \log \tau(1) \)). It has been noted that the uncertainty relation with memory (2.8) is equivalent to the tripartite uncertainty relation
\[ H(\mathcal{X}|B)_{\rho} + H(\mathcal{Z}|C)_{\rho} \geq \log d \] (2.9)
via a purification argument [3]. We extend this to a commuting square \( ((\mathcal{S} \subset A) \bigcup \bigcup \mathcal{R} \subset \mathcal{T}) \). A von Neumann subalgebra \( \mathcal{N} \subset \mathbb{B}(H_A) \) is isomorphic to a diagonal sum of matrix blocks with multiplicity, i.e.,
\[ \mathcal{N} \cong \bigoplus_k M_{n_k} \otimes \mathbb{C}_{m_k} , \]
where $1_{m_k}$ stands the $m_k \times m_k$ identity matrix. Let $P_k$ be the projections onto $k$-th diagonal block $M_{n_k} \otimes \mathbb{C}1_{m_k}$. The conditional expectation $E_N$ is given by a diagonal sum of partial trace with multiplicity

$$E_N(\rho) = \bigoplus_k \left( \text{tr}_{m_k} (P_k \rho P_k) \otimes \frac{1}{m_k} \right).$$

The complementary channel $E_N^c$ is ctp map from $M_d$ to $\bigoplus_k (\mathbb{C}1_{m_k} \otimes M_{m_k})$. Note that the range of $E_N^c$ isomorphic to the commutant of $N$. We denote $E_N' := E_N^c : M_d \rightarrow \bigoplus_k (\mathbb{C}1_{m_k} \otimes M_{m_k})$

$$E_N'(\rho) = \bigoplus_k \frac{1}{m_k} \otimes \text{tr}_{m_k} (P_k \rho P_k).$$

Note that we regard the commutant $N'$ as a subalgebra in the environment not in the system $A$.

**Theorem 2.9.** Let \( \begin{pmatrix} S \subset A \\ \cup \cup \\ R \subset T \end{pmatrix} \) be a commuting square and $B, C$ be two auxiliary systems. Then for any tripartite state $\rho^{ABC}$,

$$H(S'|B)_{\tilde{\rho}} + H(T|C)_{\rho} \geq H(R|C)_{\rho},$$

where $\tilde{\rho}^{SB} = E_S' \otimes \text{id}(\rho^{AB})$. In particular, if $R = \mathbb{C}$, then $H(R|C)_{\rho} = \log |A|$ always.

**Proof.** We first consider $\rho^{ABC}$ is pure. Let $E_S(\cdot) = \text{id}_A \otimes \text{tr}_F(V^* \cdot V)$ be a Stinespring dilation of $E_S$ such that $F$ is the environment system and $E_S'(\cdot) = \text{tr}_A \otimes \text{id}_F(V^* \cdot V)$ is the complementary channel. Denote $\tilde{\rho}^{FABC} = (V \otimes 1_{BC})\rho^{ABC}(V \otimes 1_{BC})^*$. Since $\tilde{\rho}$ is also pure, we have

$$H(F|B)_{\tilde{\rho}} = -H(F|AC)$$

$$= -H(FAC)_{\tilde{\rho}} + H(AC)_{\tilde{\rho}}$$

$$= -H(AC)_{\rho} + H(SC)_{\rho}$$

$$\geq -H(TC)_{\rho} + H(RC)_{\rho}.$$

The third inequality is because $\tilde{\rho}^{FAC} = (V \otimes 1_C)\rho^{AC}(V \otimes 1_C)^*$ and

$$\tilde{\rho}^{AC} = \text{tr}_F((V \otimes 1_C)\rho^{AC}(V \otimes 1_C)^*) = E_S \otimes \text{id}_C(\rho^{AC}).$$

The last inequality is the SSA inequality for the commuting square \( \begin{pmatrix} SC \subset AC \\ \cup \cup \\ RC \subset TC \end{pmatrix} \). This inequality

$$H(F|B)_{\tilde{\rho}} \geq -H(TC)_{\rho} + H(RC)_{\rho}$$

extends to all states $\rho$ because the LHS is concave and RHS is convex of $\rho$. Rewriting it we obtained (2.10).
Remark 2.10. Let $X$ be the subalgebra generated measurement of orthonormal basis $\{|x\rangle\}$. We have discussed that the conditional expectation $E_X$ is the completely dephasing map

$$E_X(\rho) = \sum_x \langle x|\rho|x\rangle \langle x|x\rangle.$$ 

In this particular case, $X = X'$ and $F \cong A$ and the condition expectation $E_X$ equals to its complementary channel. Thus $H(X'|B)_{\tilde{\rho}} = H(X|B)_\rho$ which recovers the standard tripartite uncertainty relation. This indicated the commutant $S'$ in (2.10) is a noncommutative feature hidden in the measurement case.

The above cases are standard examples of commuting squares in which two intermediate subalgebras $S, T$ are given by subsystems or mutually unbiased bases. We provide several more examples with noncommutative features.

Example 2.11 (Regular representations of groups). Let $G$ be a finite group and $\{|g\rangle\}_{g \in G}$ forms an orthonormal basis of the Hilbert space $l_2(G)$. The left regular representation is $\lambda(g)|s\rangle = |gs\rangle$ and the right regular representation is $r(g)|s\rangle = |sg^{-1}\rangle$. Denote $L(G) \subset B(l_2(G))$ (resp. $R(G)$) the algebra generated by left (resp. right) regular representation and $l_\infty(G)$ the diagonal operator with respect to $|g\rangle$ basis. Then $L(G)' = R(G)$ and $R(G) = L(G)$. We have two commuting squares $\left(\begin{array}{c} L(G) \subset B(l_2(G)) \\ \cup \subset \cup \\ \subset \subset \ell_\infty(G) \end{array}\right)$ and $\left(\begin{array}{c} R(G) \subset B(l_2(G)) \\ \cup \subset \cup \\ \subset \subset \ell_\infty(G) \end{array}\right)$. This fact is actually the origin of the work on quantum group channels in [13], and we leave the verification to the reader. In this setting (and the generalization of quantum groups [9]), we find two uncertainty relations

$$H(L(G)|B) + H(\ell_\infty(G)|C) \geq \log |G|,$$

$$H(R(G)|B) + H(\ell_\infty(G)|C) \geq \log |G|.$$ 

Indeed, the Peter-Weyl theorem for finite groups show that the complementary channel $E^c_{L(G)}$ is exactly $E_{R(G)}$, and vice versa. These uncertainty relations have been discovered in [9], but without realizing that for quantum groups two possible formulations exists. In the case of an abelian group where $L(G) = R(G)$ is commutative, this phenomenon is easy to overlook.

Example 2.12 (Subgroups algebras). Let $J, K \subset G$ be subgroups. The conditional expectations from $L(G)$ to subgroup algebra $\lambda(K)$ are simply the Fourier multiplier of indicator function $E_K(\lambda(g)) = 1_{g \in K}\lambda(g)$. Then we have a natural commuting square by inclusions of subgroups $\left(\begin{array}{c} \lambda(J) \subset L(G) \\ \cup \subset \cup \\ \lambda(J \cap K) \subset \lambda(K) \end{array}\right)$. We have the following uncertainty relations

$$H(\lambda(J)) + H(\lambda(K)) \leq H(L(G)) + H(\lambda(J \cap K)).$$
3. Rényi asymmetry measure and Noncommutative $L_p$-spaces

3.1. Rényi asymmetry measure. In this section we introduce the asymmetry measure of Rényi relative entropy and explain the connection to augmented $L_p$ spaces introduced in [18]. Recall that for $1 \leq p \leq \infty$ noncommutative $L_p$-spaces $L_p(M, \tau)$ of a tracial von Neumann algebra $(M, \tau)$ is the Banach space equipped with the norm

$$\| a \|_{L_p(M, \tau)} := (\tau((a^*a)^{\frac{p}{2}}))^{1/p}.$$  

If there is no ambiguity, we simply write $L_p(M)$ and $\| \cdot \|_p$ and assume all subalgebra $L_p$-norms coincide with the large algebra (with induced trace). For $1 < p \leq \infty$ and $\frac{1}{p} + \frac{2}{r} = 1$, the sandwiched $p$-Rényi relative entropy introduced in [24, 34] is defined as

$$D_p(\rho \| \sigma) = \begin{cases} p \frac{1}{p-1} \log \| \sigma^{-\frac{1}{p}} \rho \sigma^{-\frac{1}{r}} \|_p & \text{if supp(\rho) \geq supp(\sigma)} \\ +\infty & \text{otherwise} \end{cases}.$$  

where the inverse $\sigma^{-\frac{1}{r}}$ always means the inverse on its support supp(\sigma). By Rényi relative entropy we always means the sandwiched ones (see e.g. [15] for other Rényi relative entropy). Let $\mathcal{N} \subset \mathcal{M}$ be a subalgebra. We define the asymmetry measure of Rényi relative entropy with respect to $\mathcal{N}$ as follows,

$$D^N_p(\rho) := \inf_{\sigma \in S(\mathcal{N})} D_p(\rho \| \sigma) = \frac{p}{p-1} \log \left( \inf_{\sigma \in \mathcal{N}} \| \sigma^{-\frac{1}{p}} \rho \sigma^{-\frac{1}{r}} \|_p \right),$$  

where the infimum runs over all states $\sigma \in \mathcal{N}$.

Proposition 3.1. $D^N_p$ satisfies all the properties in Proposition 2.3. Moreover, for any state $\rho$, $D^N_p(\rho)$ is monotonically increasing with $p$ and

$$\lim_{p \to 1^+} D^N_p(\rho) = D^\mathcal{N}(\rho).$$  

All the properties are direct consequences of the standard properties of $D_p$ proved in [24] (see also [4]). Later in this section we provide a short proof use noncommutative $L_p$ spaces.

3.2. Augmented $L_p$ spaces. The vector-valued noncommutative $L_p$-spaces was first introduced by Pisier in [26] and later found its connection to Rényi information measures (see e.g. [11, 13]). Our asymmetry measure of Rényi relative entropy corresponds to the augmented $L_p$-space studied in [18].

Definition 3.2. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{2}{r} = 1$. The augmented $L_{1,p}$ norm of the inclusion $\mathcal{N} \subset \mathcal{M}$ is defined as

$$\| x \|_{L_{1,p}(\mathcal{N} \subset \mathcal{M})} := \inf\{\| a \|_r, \| y \|_p, \| \beta \|_r \mid x = ayb\},$$  

(3.1)
where the infimum runs over all factorizations $x = ayb$. We denote by $L_{1,p}(N \subset M)$ for the Banach space with respect to this norm.

When $x$ is an positive, the above expression can be simplified to

$$\|x\|_{L_{1,p}(N \subset M)} = \inf_{\sigma \in S(N)} \|\sigma^{-\frac{1}{p}}x\sigma^{-\frac{1}{p}}\|_p,$$

which gives the our R'enyi asymmetry measure up to a logarithm and a normalization constant. In particular, for $p = \infty$,

$$\|x\|_{L_{1,\infty}(N \subset M)} = \inf\{\lambda \mid x \leq \lambda \sigma \text{ for some } \sigma \in S(N)\}.$$

It is a special case of [18, Theorem] that $L_{1,p}(N \subset M)$ is indeed a Banach space and satisfies the following interpolation relations:

$$[L_{1,p_0}(N \subset M), L_{1,p_1}(N \subset M)]_{\theta} = L_{1,p}(N \subset M),$$

for $\frac{1}{p} = \frac{1}{p_0} - \frac{\theta}{p_1}$. We refer to [2] for detailed information of complex interpolation.

**Example 3.3.** a) When $p = 1$, $L_{1,1}(N \subset M) \cong L_1(M)$ with equal norms. 
b) Let $M = \mathbb{B}(H_A \otimes H_B)$ and $N = \mathbb{C}1 \otimes \mathbb{B}(H_B)$. For a bipartite state $\rho^{AB}$,

$$\|\rho\|_{L_{1,p}(N \subset M)} = \inf\{(1 \otimes \sigma)^{-\frac{1}{p}}\rho(1 \otimes \sigma^{-\frac{1}{p}}) \mid \sigma \geq 0, \text{tr}(1^A \otimes \sigma) = 1\}$$

$$= \exp(\frac{p}{p-1}(\log |A| - H_p(A|B))),$$

where $H_p(A|B)_{\rho}$ is the sandwiched Rényi conditional entropy (see [24]) given by

$$H_p(A|B)_{\rho} = \frac{p}{1-p} \log \inf_{\sigma \in B} \| (1_A \otimes \sigma^{-\frac{1}{p}})\rho(1_A \otimes \sigma^{-\frac{1}{p}})\|_p.$$

Now we give a proof of Proposition 3.1 using noncommutative $L_p$-norms and complex interpolation. Proposition 2.3 also follows from Proposition 3.1.

**Proof Proposition 3.1.** Let $\rho$ be a state of $M$ and $\sigma$ be an arbitrary state of $N$. Note that $\|\sigma^\frac{1}{p}\|_1 = 1$. By Hölder inequality,

$$\|\sigma^{-\frac{1}{p}}\rho\sigma^{-\frac{1}{p}}\|_p = \|\sigma^{\frac{1}{p}}\|_p \|\sigma^{-\frac{1}{p}}\|_p \|\sigma\sigma^{-\frac{1}{p}}\|_p \|\sigma^{-\frac{1}{p}}\|_p \|\sigma\|_p \geq \|\rho\|_1 = \tau(\rho) = 1.$$

Then

$$D^N_p(\rho) = \frac{p}{p-1} \log \left(\inf_{\sigma \in S(N)} \|\sigma^{-\frac{1}{p}}\rho\sigma^{-\frac{1}{p}}\|_p\right) \geq 0.$$

Because the state space $S(N)$ is a compact set, the infimum is always attained. $D^N_p(\rho) = 0$ implies that for some $\sigma \in S(N)$ the above Hölder inequality is saturated and hence $\rho = \sigma \in N$. 
For monotonicity, we use an interpolation argument. For $p = 1$, $L_{1,\infty}(N \subset M) \cong L_1(M)$ and any ctp map $\Phi : M \to M$ is complete contractive on $L_1(M)$. For $p = \infty$, the augmented $L_{1,\infty}(N \subset M)$ norm is given by

$$\| x \|_{L_{1,\infty}(N \subset M)} = \inf \{ \| a^{-\frac{1}{\theta}} x b^{-\frac{1}{\theta}} \|_\infty | a, b \in S(N) \} = \inf \{ \lambda \left| \begin{array}{cc} a & x \\ x^* & b \end{array} \right| \geq 0, a, b \in S(N) \}.$$ 

Note that $\Phi$ is ctp and $\Phi(N) \subset N$. Then

$$\left[ \begin{array}{cc} \lambda a & x \\ x^* & \lambda b \end{array} \right] \geq 0 \Rightarrow \left[ \begin{array}{cc} \lambda \Phi(a) & \Phi(x) \\ \Phi(x^*) & \lambda \Phi(b) \end{array} \right] \geq 0,$$

and $\Phi(a), \Phi(b)$ are states in $S(N)$. Thus, $\| \Phi(x) \|_{L_{1,\infty}(N \subset M)} \leq \| x \|_{L_{1,\infty}(N \subset M)}$ and $\Phi$ is a contraction on $L_{1,\infty}(N \subset M)$. Then the case $1 < p < \infty$ follows from interpolation (3.4).

To show that $D_N^p(\rho)$ is monotone increasing for $p$, we use the interpolation inequality that for $1/p = (1 - \theta)/p_0 + \theta/p_1$, $p_1 \leq p \leq p_0$,

$$\| \rho \|_{L_{1,p}(N \subset M)} \leq \| \rho \|_{L_{1,p_0}(N \subset M)}^{1-\theta} \| \rho \|_{L_{1,p_1}(N \subset M)}^\theta.$$

Choosing $p_1 = 1$, we have

$$D_N^p(\rho) = \frac{p}{p-1} \log \| \rho \|_{L_{1,p}(N \subset M)} \leq \frac{(1 - \theta)p}{p-1} \log \| \rho \|_{L_{1,p_0}(N \subset M)} + \frac{\theta p}{p-1} \log \| \rho \|_{L_{1,p_1}(M)} = \frac{p_0}{p_0 - 1} \log \| \rho \|_{L_{1,p_0}(N \subset M)} = D_N^{p_0}(\rho).$$

Finally the limit $p \to 1^+$ follows from

$$\lim_{p \to 1^+} D_N^p(\rho) = \inf_{p > 1} \inf_{\sigma \in S(N)} \frac{p}{p-1} \log \| \sigma^{-\frac{1}{\theta}} \rho \sigma^{-\frac{1}{\theta}} \|_p = \inf_{\sigma} \frac{p}{p-1} \log \| \sigma^{-\frac{1}{\theta}} \rho \sigma^{-\frac{1}{\theta}} \|_p = \inf_{\sigma} \tau(\rho \log \rho - \rho \log \sigma) = \inf_{\sigma \in S(N)} D(\rho || \sigma) = D(\rho || E_N(\rho)).$$

**Remark 3.4.** It has been pointed out in [23] that asymmetry measure can be constructed from information measures. The asymmetry measure of relative entropy $D_N^p(\rho) = D(\rho || E_N(\rho))$ arise from the relative entropy $D(\cdot || \cdot)$ and the conditional expectation. Using the same idea, one can define another Rényi asymmetry measure

$$\tilde{D}_N^p(\rho) = D_p(\rho || E_N(\rho)).$$
For general $\rho$, $\tilde{D}_p^N$ is different with $D_p^N$. As in the Example 3.3 b), $D_p^N$ is more analogous to sandwiched Rényi conditional entropy. It is clear from the definition that $D_p^N(\rho) \leq \tilde{D}_p^N(\rho)$ and the limits $p \to 1^+$ coincides,
\[
\lim_{p \to 1} D_p^N(\rho) = \lim_{p \to 1} \tilde{D}_p^N(\rho) = D_N(\rho).
\]

4. SSA inequality with correction terms

In this section, we prove a general SSA inequality without commuting square condition. We consider the situation $\bigg( \mathcal{T} \subset \mathcal{M} \bigcup \bigcup_{\mathcal{R} \subset \mathcal{S}} \bigg)$ as inclusions of subalgebra in a finite dimensional von Neumann algebra $(\mathcal{M}, \tau)$. We always use the induced trace for $p$-norms and entropy on the algebra.

**Theorem 4.1.** Let $(\mathcal{M}, \tau)$ be a finite dimensional von Neumann algebra with a finite trace $\tau$. Let $\bigg( \mathcal{T} \subset \mathcal{M} \bigcup \bigcup_{\mathcal{R} \subset \mathcal{S}} \bigg)$ be inclusions of subalgebras. Then for any state $\rho \in \mathcal{M}$,

\[
H(\mathcal{S})_\rho + H(\mathcal{T})_\rho \geq H(\mathcal{M})_\rho + H(\mathcal{R})_\rho + \log \frac{1}{c},
\]

where $c$ is a constant only depending on $(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{M})$ given by

\[
c = \sup_{\sigma \in \mathcal{S}(\mathcal{T})} \inf \{ \lambda \mid E_{\mathcal{S}}(\sigma) \leq \lambda \omega \text{ for some } \omega \in \mathcal{S}(\mathcal{R}) \}, \quad \log c = \sup_{\sigma \in \mathcal{S}(\mathcal{T})} D_{\infty}^{\mathcal{R}}(E_{\mathcal{S}}(\sigma)).
\]

**Proof.** We first estimate the norm of $E_{\mathcal{S}} : L_{1,\infty}(\mathcal{T} \subset \mathcal{M}) \to L_{1,\infty}(\mathcal{R} \subset \mathcal{S})$. For any $x \in \mathcal{M}$,

\[
\| E_{\mathcal{S}}(x) \|_{L_{1,\infty}(\mathcal{R} \subset \mathcal{S})} = \inf \{ \lambda \mid \begin{bmatrix} \frac{\lambda a}{E_{\mathcal{S}}(x)} & E_{\mathcal{S}}(x) \\ E_{\mathcal{S}}(x^*) & \frac{\lambda b}{E_{\mathcal{S}}(x^*)} \end{bmatrix} \geq 0, \; a, b \in \mathcal{S}(\mathcal{R}) \}
\]
\[
\leq c \cdot \inf \{ \lambda \mid \begin{bmatrix} \lambda E_{\mathcal{S}}(a) & E_{\mathcal{S}}(x) \\ E_{\mathcal{S}}(x^*) & \lambda E_{\mathcal{S}}(b) \end{bmatrix} \geq 0 \text{ for some } a', b' \in \mathcal{S}(\mathcal{T}) \}
\]
\[
\leq c \cdot \inf \{ \lambda \mid \begin{bmatrix} \lambda a' & x \\ x^* & \lambda b' \end{bmatrix} \geq 0 \text{ for some } a', b' \in \mathcal{S}(\mathcal{T}) \}
\]
\[
= c \cdot \| x \|_{L_{1,\infty}(\mathcal{T} \subset \mathcal{M})},
\]

where the constant $c$ is given by

\[
c = \sup_{\sigma \in \mathcal{S}(\mathcal{T})} \inf \{ \lambda \mid E_{\mathcal{S}}(\sigma) \leq \lambda \omega \text{ for some } \omega \in \mathcal{S}(\mathcal{R}) \}.
\]

Thus we know

\[
\| E_{\mathcal{S}} : L_{1,\infty}(\mathcal{T} \subset \mathcal{M}) \to L_{1,\infty}(\mathcal{R} \subset \mathcal{S}) \| = \| E_{\mathcal{S}} : L_{1}(\mathcal{T}) \to L_{1,\infty}(\mathcal{R} \subset \mathcal{S}) \|.
\]
and it is further attained by positive elements of $\mathcal{T}$. On the other hand, for $p = 1$, $E_S : L_1(M) \to L_1(S)$ is a contraction because it is cptp. Using complex interpolation, we have for all $1 < p \leq \infty$,

$$\| E_S : L_{1,p}(T \subset M) \to L_{1,p}(R \subset S) \| \leq c^{1-\frac{1}{p}}.$$

This implies for any state $\rho \in M$,

$$c^{1-\frac{1}{p}} \| \rho \|_{L_{1,p}(T \subset M)} \geq \| E_S(\rho) \|_{L_{1,p}(R \subset S)} \implies \log c + D_T^p(\rho) \geq D_R^p(E_S(\rho)) \quad \forall 1 < p \leq \infty \implies \log c + D_T(\rho) \geq D_R(E_S(\rho)) \implies \log c + H(T)_\rho - H(M)_\rho \geq H(R)_\rho - H(S)_\rho.$$

Rearranging the terms we complete the proof.  

Remark 4.2. It is clear from Theorem 2.3 that for a commuting square \( \begin{array}{c}
\mathcal{T} \\
\cup \\
\mathcal{R} \\
\cup \\
\mathcal{S}
\end{array} \) \( \begin{array}{c}
\mathcal{M} \\
\cup \\
\cup \\
\mathcal{S}
\end{array} \) the constant

\[ c = \| E_S : L_1(\mathcal{T}) \to L_{1,\infty}(\mathcal{R} \subset \mathcal{S}) \| = \| E_R : L_1(\mathcal{T}) \to L_1(\mathcal{R}) \| = 1. \]

Conversely, suppose that for \( \begin{array}{c}
\mathcal{T} \\
\cup \\
\mathcal{R} \\
\cup \\
\mathcal{S}
\end{array} \) \( \begin{array}{c}
\mathcal{M} \\
\cup \\
\cup \\
\mathcal{S}
\end{array} \) the best constant in (4.1) is $c = 1$. Then for any state $\sigma \in S(\mathcal{R})$, $H(E_T(\rho)) = H(\rho)$ and hence $E_T(\sigma) = \sigma$. Thus $\mathcal{R} \subset \mathcal{T}$, we can complete the quadrilateral and have \( \begin{array}{c}
\mathcal{T} \\
\cup \\
\mathcal{R} \\
\cup \\
\mathcal{S}
\end{array} \) \( \begin{array}{c}
\mathcal{M} \\
\cup \\
\cup \\
\mathcal{S}
\end{array} \) as a commuting square.

When $\mathcal{R} = \mathbb{C}$, $H(\mathcal{R})_\rho = \log \tau(1)$ is a constant for all $\rho$. In this situation, we have the following variant of inequality.

**Corollary 4.3.** Let \( \begin{array}{c}
\mathcal{T} \\
\cup \\
\mathcal{C} \\
\cup \\
\mathcal{S}
\end{array} \) \( \begin{array}{c}
\mathcal{M} \\
\cup \\
\cup \\
\mathcal{S}
\end{array} \) be inclusions of subalgebras in \( (\mathcal{M}, \tau) \). Then for any state $\rho \in \mathcal{M}$,

\[ H(S)_\rho + H(T)_\rho \geq H(\rho) + \log \frac{1}{c'}, \quad \text{(4.2)} \]

where $c'$ is a constant only depending on \( (\mathcal{S}, \mathcal{T}, \mathcal{M}) \) given by

\[ c' = \min \{ \| E_S : L_1(\mathcal{T}) \to \mathcal{S} \|, \| E_S : L_1(\mathcal{S}) \to \mathcal{T} \| \} \]

\[ = \min \{ \sup_{\sigma \in S(\mathcal{T})} \| E_S(\sigma) \|_\infty, \sup_{\sigma \in S(\mathcal{S})} \| E_T(\sigma) \|_\infty \}. \]

Proof. Apply Theorem 4.1 for the inclusions \( \begin{array}{c}
\mathcal{T} \\
\cup \\
\mathcal{C} \\
\cup \\
\mathcal{S}
\end{array} \) and \( \begin{array}{c}
\mathcal{T} \\
\cup \\
\mathcal{C} \\
\cup \\
\mathcal{S}
\end{array} \).
Example 4.4 (Maassen-Uffink inequality). Let \( \{ |x\rangle \} \) and \( \{ |z\rangle \} \) be two arbitrary orthonormal bases of \( H_A \cong \mathbb{C}^d \). Denote \( \mathcal{X}, \mathcal{Z} \) as the subalgebra generated by the measurement operators of each basis. We have the inclusions \( \left( \mathcal{X} \subset \mathcal{M} \cup \mathcal{C} \subset \mathcal{Z} \right) \). In this setting, the constant in Corollary 4.3 is either of the terms in the maximum, 
\[
c' = \sup_{\sigma \in \mathcal{S}(\mathcal{Z})} \| E_{\mathcal{X}}(\sigma) \|_{\infty} = \max_{x,z} |\langle x|z \rangle|^2 ,
\]
which is exactly the maximum overlap of two measurements. Therefore, we recover Maassen-Uffink inequality [21] as a special case, 
\[
H(\mathcal{X}) + H(\mathcal{Z}) \geq H(\rho) + \log \frac{1}{c'} .
\]

Recall the operator-valued trace class space \( S_1(B, \mathcal{M}) \) introduced in [26] is given by 
\[
\| x \| = \inf_{x = (1 \otimes a)_y (1 \otimes b)} \| a \|_2 \| x \|_\infty \| b \|_2 ,
\]
and \( S_1(B, L_1(M)) \cong L_1(\mathcal{M}B, \tau \times \text{tr}) \).

Corollary 4.5. Let \( \left( \mathcal{T} \subset \mathcal{M} \cup \mathcal{C} \subset \mathcal{S} \right) \) be inclusions of subalgebras and let \( B \) be a quantum system. Then for any state \( \rho \in \mathcal{M}B \), 
\[
H(\mathcal{S}|B)_\rho + H(\mathcal{T}|B)_\rho \geq H(\mathcal{M}|B)_\rho + \log \frac{1}{c_B} ,
\]
where \( c_B \) is a constant depending on \( (\mathcal{T}, \mathcal{S}, \mathcal{M}, B) \) given by 
\[
c_B = \min\left\{ \| id_B \otimes E_S : S_1(B, L_1(\mathcal{T})) \to S_1(B, \mathcal{S}) \| , \| id_B \otimes E_T : S_1(B, L_1(\mathcal{S})) \to S_1(B, \mathcal{T}) \| \right\} .
\]

Proof. We have a quadrilateral of inclusions \( \left( \mathcal{T}B \subset \mathcal{M}B \cup \mathcal{C}B \subset \mathcal{S}B \right) \). By argument of Theorem 4.1
\[
H(\mathcal{S}B)_\rho + H(\mathcal{T}B)_\rho + \log c \geq H(\mathcal{M}B)_\rho + H(\mathcal{C}B)_\rho = H(\mathcal{M}B)_\rho + H(B)_\rho + \log \tau(1) ,
\]
where
\[
c = \min\left\{ \| E_S \otimes id_B : L_1(\mathcal{T}B) \to L_{1,\infty}(\mathcal{C} \otimes B \subset \mathcal{S}B) \| , \| E_S \otimes id_B : L_1(\mathcal{S}B) \to L_{1,\infty}(\mathcal{C} \otimes B \subset \mathcal{T}B) \| , \right\} .
\]
It can be easily verified that for any \( x \in \mathcal{M}B \),
\[
\| x \|_{L_{1,\infty}(\mathcal{C} \otimes B \subset \mathcal{S}B)} = \tau(1) \| x \|_{S_1(B, \mathcal{S})} .
\]
Therefore, by cancelling the dimension constant \( \log \tau(1) \) we have \( \frac{c}{\tau(1)} = c_B \) is exactly the constant given in the statement. \( \blacksquare \)
Remark 4.6. When the subalgebra $S$ is commutative, we know (see [27])

$$\| E_S : L_1(T) \to S \| = \| id_B \otimes E_S : S_1(B, L_1(T)) \to S_1(B, S) \| .$$

For noncommutative $S$, this fails dramatically. Take a very non-commuting square example that $S = T = M_n$ are $n \times n$ matrix algebras, we have

$$\| id_B \otimes E_S : S_1(B, L_1(T)) \to S_1(B, S) \| = \min \{ |B|, n \} .$$

In general, if the dimension $|B| = d$, we know $c_B \leq d \cdot c'$ where $c'$ is the constant in Corollary 4.3. For arbitrary $B$, $c_B$ is bounded by the completely bounded norm

$$\| E_S : L_1(T) \to S \|_{cb} := \sup_B \| id_B \otimes E_S : S_1(B, L_1(T)) \to S_1(B, S) \| .$$

Example 4.7 (Uncertainty principles with quantum memory). Let us revisit the Maassen-Uffink in the presence of quantum memory. Let $\{ |x \rangle \}$ and $\{ |z \rangle \}$ be two arbitrary orthonormal bases of $H_A \cong \mathbb{C}^d$ and $B$ is the memory system. Note that the measurement subalgebras $X, Z$ are both commutative. By the Remark 4.6 the correction term with memory remains the same,

$$H(X | B)_{\rho} + H(Z | B)_{\rho} \geq H(A | B)_{\rho} + \log \frac{1}{c'} , \quad (4.3)$$

which recovers the uncertainty principle proved in [3].

For the tripartite uncertainty relations, we have an analog of Theorem 2.9

Corollary 4.8. Let \( \left( \begin{array}{ccc} S & \subset & A \\ \cup & \cup & \cup \\ C & \subset & T \end{array} \right) \) be a quadrilateral of inclusions and $B, C$ be two auxiliary systems. Then for any tripartite state $\rho^{ABC}$,

$$H(S' | B)_{\tilde{\rho}} + H(T | C)_{\rho} \geq \log \frac{1}{c} ,$$

where $\tilde{\rho}^{S'B} = E_{S'} \otimes id(\rho^{AB})$ and

$$c = \min \{ \| E_S : S_1(C, L_1(T)) \to S_1(C, S) \|, \| E_T : S_1(C, L_1(S)) \to S_1(C, T) \| \} .$$

Proof. Combine the argument of Theorem 2.9 with Theorem 4.1. \( \blacksquare \)

Remark 4.9. We have seen in above discussion that our constant (hence also the constant in Maassen-Uffink) are from the estimate at $L_{\infty}$-norms. The similar argument at $L_p$-level can provide potentially better constant. Consider the setting \( \left( \begin{array}{ccc} T & \subset & M \\ \cup & \cup & \cup \\ C & \subset & S \end{array} \right) \) of Corollary 4.3 for example. We know by complex interpolation

$$c_p := \left( \| E_S : L_1(T) \to L_p(S) \| \right)^{\frac{1}{p}} \leq \| E_S : L_1(T) \to S \| = c' ,$$
and the inequality satisfies with constant $c_p$,\hspace{1cm}(4.5)\hspace{1cm}H(S)_\rho + H(T)_\rho \geq H(\rho) + \log 1 \frac{1}{c_p}.

However, in general the $L_p$ constants are hard to compute.

5. Commuting square and alternating operations

In this section we investigate an analogue of conditional mutual information

\[ I_\tau(S : T | R) = H_\tau(S)_\rho + H_\tau(T)_\rho - H_\tau(R)_\rho - H_\tau(M)_\rho \]

for a commuting square \( S \subset M \cup \cup R \subset T \). We will always assume that the trace on \( R, S, T \) are the induced trace and \( \tau \) will be often omitted. It is clear from Theorem 2.4 that \( I(S : T | R) \) is always positive. In the following, we assume \( 1 \leq p \leq \infty \) and fixed the relation \( \frac{1}{p} + \frac{1}{p'} = 1 \).

**Lemma 5.1.** Let \( \rho \) be a state of \( M \). Define the function \( f : [1, \infty] \to \mathbb{R}^+ \) as

\[ f(p) = \sup_{\gamma \in R} \inf_{\sigma \in S, \omega \in T} \| \sigma^{-1/2p'} \omega^{-1/2p'} \rho \gamma^{1/2p'} \omega^{-1/2p'} \sigma^{-1/2p'} \|_p \]

where the infimum and supremum are taken over states \( \gamma, \sigma, \omega \) of corresponding subalgebras. Then

\[ f(1) = 1, \lim_{p \to 1^+} \frac{f(p) - 1}{p - 1} = I(T : S | R). \]

**Proof.** The proof follows by a standard argument (see [17], [13]). Note that \( f(1) = \tau(\rho) = 1 \) and hence

\[ \lim_{p \to 1^+} \frac{f(p) - 1}{p - 1} = \lim_{p \to 1^+} \frac{f(p)^p - 1}{p - 1}. \]

For states \( \rho \in M, \sigma \in S, \)

\[ \frac{\| \sigma^{-1/2p'} \rho \sigma^{-1/2p'} \|_p - 1}{p - 1} = \frac{\tau(\sigma^{-1/2p'} \rho \sigma^{-1/2p'}) - \tau(\sigma^{-1/2p'} \rho \sigma^{-1/2p'})}{p - 1} + \frac{\tau(\sigma^{-1/2p'} \rho \sigma^{-1/2p'}) - 1}{p - 1} \]

By Hölder inequality, the second term

\[ \frac{\tau(\sigma^{-1/2p'} \rho \sigma^{-1/2p'}) - 1}{p - 1} \geq \frac{\| E_S(\rho) \|_q - 1}{p - 1} \xrightarrow{p \to 1^+} H(E_S(\rho)). \]
The mean value theorem for function \( p \mapsto \tau(x^p) \) implies that for some \( p_1 \in [1, p] \),

\[
\frac{\tau\left((\sigma^{-1/2^{p'}} \rho \sigma^{-1/2^{p'}})^p\right) - \tau(\sigma^{-1/2^{p'}} \rho \sigma^{-1/2^{p'}})}{p - 1} = \tau((\sigma^{-1/2^{p'}} \rho \sigma^{-1/2^{p'}})^{p_1} \log(\sigma^{-1/2^{p'}} \rho \sigma^{-1/2^{p'}})^{p_1})
\]

which converges to \(-H(\rho)\) as \( p \to 1^+ \). In our triple situation, we set

\[
\rho_p(1) = \gamma^{1/2^{p'}} \rho \gamma^{1/2^{p'}} \, , \, \rho_p(2) = \omega^{-1/2^{p'}} \rho_p(1) \omega^{-1/2^{p'}} \, , \, \rho_p(3) = \sigma^{-1/2^{p'}} \rho(2) \sigma^{-1/2^{p'}} \,.
\]

Then by repeating the previous argument,

\[
\frac{\tau(\rho(3)^p) - 1}{p - 1} = \frac{\tau(\rho(3)^p) - \tau(\rho(3))}{p - 1} + \frac{\tau(\rho(3)) - \tau(\rho(2))}{p - 1} + \frac{\tau(\rho(2)) - \tau(\rho(1))}{p - 1} + \frac{\tau(\rho(1)) - 1}{p - 1}
\]

The last term is independent of \( \sigma \) and \( \omega \), and the supremum over \( \gamma \in \mathcal{R} \) can be actually performed

\[
\sup_{\gamma \in \mathcal{R}} \frac{\tau(\gamma^{\frac{1}{2^{p'}}} \rho \gamma^{\frac{1}{2^{p'}}}) - 1}{p - 1} = \left\| E_{\mathcal{R}}(\rho) \right\|_p - 1 \xrightarrow{p \to 1^+} -H(E_{\mathcal{R}}(\rho))
\]

Moreover \( \rho(1), \rho(2) \) and \( \rho(3) \) converge to \( \rho \) when \( p \to 1^+ \), we have

\[
\lim_{p \to 1} \frac{f(p) - 1}{p - 1} \geq -H(\rho) + H(E_{\mathcal{S}}(\rho)) + H(E_{\mathcal{T}}(\rho)) - H(E_{\mathcal{R}}(\rho)) = I(\mathcal{S} : \mathcal{T} | \mathcal{R})_{\rho} \quad (5.1)
\]

For the upper bound, we choose

\[
\omega_p = E_{\mathcal{T}}(\rho_p(1)) \, , \, \sigma_p = E_{\mathcal{S}}(\rho_p(2)) \,.
\]

Then we achieve equalities in all Hölder inequalities we have used and attain the lower bound of \( (5.1) \). \[ \blacksquare \]

The quantity \( f(p) \) above is analogous to the Rényi conditional mutual information discussed in \([5]\). In the following we discuss the monotonicity of \( I \) under completely positive trace preserving (CPTP) maps.

**Theorem 5.2.** Let \( \left( \begin{array}{c} \mathcal{S} \subset \mathcal{M} \\ \cup \cup \cup \mathcal{R} \subset \mathcal{T} \end{array} \right) \) and \( \left( \begin{array}{c} \mathcal{\tilde{S}} \subset \mathcal{\tilde{M}} \\ \cup \cup \cup \mathcal{\tilde{R}} \subset \mathcal{T} \end{array} \right) \) be two commuting squares and \( \Phi : (\mathcal{M}, \tau) \to (\mathcal{\tilde{M}}, \tilde{\tau}) \) be a CPTP map. Suppose that \( \Phi(\mathcal{S}) \subset \mathcal{\tilde{S}} \) and

\[
\Phi(axb) = a \Phi(x)b \, , \, \forall \, a, b \in \mathcal{T}, x \in \mathcal{M} \,.
\]

Then for any \( \rho \in \mathcal{M} \)

\[
I_{\tilde{\tau}}(\tilde{\mathcal{S}} : \mathcal{T} | \mathcal{R})_{\Phi(\rho)} \leq I_{\tau}(\mathcal{S} : \mathcal{T} | \mathcal{R})_{\rho} \,.
\]
Proof. Fix states \( \omega \in T, \gamma \in R \) and set the density \( \rho_{\omega, \gamma} = \omega^{-1/2p'} \gamma^{1/2p'} \rho_{\gamma}^{1/2p'} \omega^{-1/2p'} \). Using data processing inequality of \( p \)-Rényi relative entropy and the assumption of \( \Phi \), we have

\[
\inf_{\sigma} \| \sigma^{-1/2p'} \rho_{\omega, \gamma} \sigma^{-1/2p'} \|_p \geq \inf_{\sigma} \| \Phi(\sigma)^{-1/2p'} \Phi(\rho_{\omega, \gamma}) \Phi(\sigma)^{-1/2p'} \|_p \\
= \inf_{\sigma} \| \Phi(\rho_{\omega, \gamma}) \Phi(\sigma)^{-1/2p'} \|_p \\
= \inf_{\sigma} \| \sigma^{-1/2p'} \omega^{-1/2p'} \gamma^{1/2p'} \Phi(\sigma)^{-1/2p'} \|_p .
\]

Taking infimum over \( \omega \in T \) and supremum over \( \gamma \in R \), we obtain \( f_{\Phi(\rho)}(p) \leq f_{\rho}(p) \) for all \( p > 1 \). Then Lemma 5.1 implies

\[
I_{\tau}(S : T | R \rho) = \lim_{p \to 1^+} \frac{f_{\rho}(p) - 1}{p - 1} \leq \lim_{p \to 1^+} \frac{f_{\Phi(\rho)}(p) - 1}{p - 1} = I_{\tilde{\tau}}(\tilde{S} : \tilde{T} | R \Phi(\rho)) ,
\]

which completes the proof.

One way to interpret theorem 5.2 would be as follows: The large algebra \( \mathcal{M} \) is a system controlled by a referee and can be observed/measured by both Alice and Bob. Alice can measure the observables from the subalgebra \( S \), and Bob can measure the observables from \( T \). In particular, they share knowledge and control of the intersection subalgebra \( R = S \cap T \). The map \( \Phi \) discussed in the above theorem is a quantum operation that changes the reduced density for Alice, but the change can not be observed by Bob. Indeed, the condition (5.2), which is called \( T \)-bimodule property, implies that \( E_T(\rho) = E_T(\Phi(\rho)) \),

\[
\tilde{\tau}(a E_T(\Phi(\rho))) = \tilde{\tau}(a \Phi(\rho)) = \tilde{\tau}(\Phi(a \rho)) = \tau(a \rho) = \tau(a E_T(\rho)) .
\]

In view of Theorem 5.2 we introduce the notation of alternating operations.

Definition 5.3. Let \( \left( \begin{array}{c} S \subset \mathcal{M} \\ U \cup U \\ R \subset T \end{array} \right) \) be a commuting square. We define alternating operations (AO) as compositions of the following two kinds of quantum operations

i) a CPTP map \( \Phi : \mathcal{M} \to \tilde{\mathcal{M}} \) with commuting square \( \left( \begin{array}{c} \tilde{S} \subset \tilde{\mathcal{M}} \\ U \cup U \\ \tilde{R} \subset \tilde{T} \end{array} \right) \) such that

\[
\Phi(S) \subset \tilde{S} \quad \text{and} \quad \Phi(axb) = a \Phi(x)b \quad \text{for all } a, b \in T, x \in \mathcal{M} . \tag{5.3}
\]

ii) a CPTP map \( \Phi : \mathcal{M} \to \tilde{\mathcal{M}} \) with commuting square \( \left( \begin{array}{c} S \subset \tilde{\mathcal{M}} \\ U \cup U \\ R \subset \tilde{T} \end{array} \right) \) such that

\[
\Phi(T) \subset \tilde{T} \quad \text{and} \quad \Phi(axb) = a \Phi(x)b \quad \text{for all } a, b \in S, x \in \mathcal{M} . \tag{5.4}
\]
We call the first kind (5.3) $S$-operations (which are $T$-bimodules, hence leaving $T$ invariant), and the second $T$-operations (5.4). Basically, each one is a bi-module map for one intermediate subalgebra ($S$ or $T$) and transforms the other subalgebra. On separated systems $\left(\begin{array}{c} AC \subset ABC \\ \cup \cup \cup C \subset BC \end{array}\right)$, alternating operations are the local operations as $\Phi_A \otimes id_{BC}$ and $\Psi_B \otimes id_{AC}$. Theorem 5.2 implies the following monotonicity of our generalized quantity $I(S : T|R)$.

**Corollary 5.4.** $I$ is non-increasing under alternating operations.

**Remark 5.5.** In above definition (5.4) and (5.3), one of the intermediate subalgebras, say $S$, is fixed during the operation. Actually we can allow the subalgebra $S$ enlarge into a larger $\tilde{S}$. We say $\Phi : M \rightarrow \tilde{M}$ is a CPTP map into the commuting square $\left(\begin{array}{c} S \subset \tilde{M} \\ \cup \cup \cup R \subset T \end{array}\right)$ with $S \subset \tilde{S}$. It is clear from Theorem 5.2 that the monotonicity remains valid under the assumption of alternating operations.

For simplicity of notation, in the following we consider the alternating operations $\phi : M \rightarrow M$ that remain in the the same commuting square $\left(\begin{array}{c} S \subset M \\ \cup \cup \cup R \subset T \end{array}\right)$. Nevertheless, all discussion holds for operations that change the subalgebras $S, T$ to $\tilde{S}, \tilde{T}$. We define the squashed version (in analogy to the squashed entanglement defined in [7])

$$I^{sq}(T : S|R)_{\rho^M} = \inf_{tr_{C}(\rho^{MC}) = \rho^M} I(TC : SC|R_C)_{\rho^{MC}},$$

where $C = B(H_C)$ is an auxiliary system, and the infimum runs over all extensions $tr_C(\rho^{MC}) = \rho^M$. Note that here, adding an environment $C$ to each subalgebras stays in the framework of the commuting square $\left(\begin{array}{c} TC \subset MC \\ \cup \cup \cup RC \subset SC \end{array}\right)$. In this sense our notation (but not the quantity denoted) differs from the usual notation of mutual information. Note that the squashed version is automatically convex, because

$$\tilde{\rho} = (1 - \lambda)\rho_1 \otimes |0\rangle\langle 0| + \lambda\rho_2 \otimes |1\rangle\langle 1|$$

is an extension of the convex combination $(1 - \lambda)\rho_1 + \lambda\rho_2$. Furthermore, on physically separated systems $A$ and $B$ with commuting square $\left(\begin{array}{c} A \subset AB \\ \cup \cup \cup C \subset B \end{array}\right)$, $I^{sq}$ reduces to twice the usual squashed entanglement between $A$ and $B$. 

Let $X$ be a classical system. We consider a one round alternating operation assisted classical communication described by the following 4 steps. 

$$(S \subset M \cup \cup R \subset T) \xrightarrow{(I)} (S \subset MX \cup \cup RX \subset TX) \xrightarrow{(II)} (S \subset MX \cup \cup RX \subset TX) \xrightarrow{(III)} (SX \subset MX \cup \cup RX \subset TX) \xrightarrow{(IV)} (S \subset M \cup \cup R \subset T).$$

(5.5)

Here at every step we have a commuting square.

i) The first operation (I) is a quantum instrument $(\Phi_x)_{x \in X}$ (i.e. $\sum_x \Phi_x$ is CPTP)

$$\rho \in M \mapsto \sum_x \Phi_x(\rho) \otimes |x\rangle \langle x| \in MX,$$

such that for each $x$, $\Phi_x$ is a $S$-bimodule map and $\Phi_x(T) \subset T$. This is a $T$-alternating operation from $\left(\begin{array}{c} S \subset M \\ \cup \cup \cup \cup R \subset T \end{array}\right)$ to $\left(\begin{array}{c} S \subset MX \\ \cup \cup \cup \cup RX \subset TX \end{array}\right)$ where $S$ is identified with $S \otimes 1 \subset MX$.

ii) The classical information $X$ obtained from quantum instrument is sent to $S$ system. The state $\rho_{MX}$ is not changing at step (II).

iii) The third step (III)

$$\sum_x \rho_x \otimes |x\rangle \langle x| \mapsto \sum_x \Psi_x(\rho_x) \otimes |x\rangle \langle x|,$$

is $SX$-alternating operation $(\Psi_x)$ with respect to the enlarged commuting square $\left(\begin{array}{c} SX \subset MX \\ \cup \cup \cup \cup RX \subset TX \end{array}\right)$. Here each $\Psi_x : M \rightarrow M$ is an $S$-alternating operation that is $T$-bimodule and transform $S$.

iv) The last step (IV) is the expectation over the classical system $X$,

$$\sum_x \rho_x^M \otimes |x\rangle \langle x| \in MX \mapsto \sum_x \rho_x \in M.$$

The total operation is given by

$$\rho \mapsto \sum_x \Psi_x \circ \Phi_x(\rho).$$

(5.6)

For subsystems $B(H_A), B(H_B) \subset B(H_A \otimes H_B)$, this corresponds to a LOCC operation $\sum_x \Psi_x \otimes \Phi_x$ where $\Psi_x : B(H_A) \rightarrow B(H_A)$ are CPTP maps and $\Psi_x : B(H_B) \rightarrow B(H_B)$ forms a quantum instrument. In (5.6), $\Psi_x$ and $\Phi_x$ are not necessarily commuting or in a tensor product form $\Psi_x \circ \Phi_x$. 
Definition 5.6. We say a CPTP map $\Phi$ is alternating operation and classical communication (AOCC) if it is a composition of the operations defined in (5.5).

To discuss the monotonicity and super-additivity of $I^{sq}$, we first prove the following lemma.

Lemma 5.7. Let \[
\left( \begin{array}{c}
S \subset \mathcal{M} \\
\mathcal{R} \subset \mathcal{T}
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{c}
S' \subset \mathcal{M}' \\
\mathcal{R}' \subset \mathcal{T}'
\end{array} \right)
\] be two commuting squares. Then \[
\left( \begin{array}{c}
SS' \subset \mathcal{M}\mathcal{M}' \\
RR' \subset \mathcal{T}\mathcal{T}'
\end{array} \right)
\] is a commuting square, and we have the chain rule \[
I(SS' : TT'|RR') = I(SS' : TM'|RS') + I(RS' : TT'|RR').
\]

Proof. Note that \[
\left( \begin{array}{c}
\mathcal{R} \subset \mathcal{T} \\
\mathcal{S} \subset \mathcal{M}
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{c}
\mathcal{R} \subset \mathcal{T} \\
\mathcal{S} \subset \mathcal{M}
\end{array} \right)
\] are trivial commuting squares. Then \[
I(SS' : TT'|RR') = H(SS') + H(TM') - H(SS' - TM') - H(RS') - H(RR') + H(RS') - H(TM') - H(RR') + H(TM')
\]
\[
= I(SS' : TM'|RS') + I(RS' : TT'|RR').
\]

Theorem 5.8. $I^{sq}$ is monotone non-increasing under AOCC operations and superadditive.

Proof. Let $\Phi : \mathcal{M} \to \tilde{\mathcal{M}}$ be an alternating operation. Then $\Phi \otimes id_C$ is an alternating operation from \[
\left( \begin{array}{c}
S \subset \mathcal{M} \\
\mathcal{R} \subset \mathcal{T}
\end{array} \right) \quad \text{to} \quad \left( \begin{array}{c}
S \subset \tilde{\mathcal{M}} \\
\mathcal{R} \subset \mathcal{T}
\end{array} \right)
\]. Then for monotonicity, it suffices to consider the step (II) and (IV) above. For (II), it is clear that for any extension $\rho^{MXC}$ of $\rho^{MX}$, \[
I(SC : TXC|RC) = I(SXC : TXC|RCX) + I(SC : RXC|RC) \geq I(SXC : TXC|RCX).
\]

Here we have used Lemma 5.7 for \[
\left( \begin{array}{c}
S \subset \mathcal{M} \\
\mathcal{R} \subset \mathcal{T}
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{c}
X \subset X \\
\mathcal{C} \subset \mathcal{C}
\end{array} \right)
\]. The last inequality is because \[
\left( \begin{array}{c}
S \subset SXC \\
\mathcal{R} \subset \mathcal{RCX}
\end{array} \right)
\] forms a commuting square and hence $I(SC : RXC|RC)$ is non-negative. The step (IV) follows from the definition of squashed version \[
I^{sq}(SX : TX|RX)_{\rho^{MX}} = \inf_{\rho^{MXC}} I(SXC : TXC|RCX)_{\rho^{MXC}} \geq \inf_{\rho^{MC}} I(SC' : TC'|RC')_{\rho^{MC}}
\]
= I^{sq}(S : T|R).

For super-additivity, we use Lemma 5.7 repeatedly. First, observe that

\[ I(SS'C : TT'C|RR'C) = I(SS'C : TM'C|RS'C) + I(RS'C : TT'C|RR'C), \]

for two commuting squares \( SS' \subset MM' \cup RS' \subset TM' \) and \( RS' \subset TM' \cup RR' \subset TT' \). Then for each one we obtain

\[ I(SS'C : TM'C|RS'C) = I(SM'C : TM'C|RM'C) + I(SS'C : RM'C|RS'C) \geq I^{sq}(S : T|R), \]

\[ I(RS'C : TT'C|RR'C) = I(S'TE : T'TE|R'TE) + I(RS'C : RR'C|RR'C) \geq I^{sq}(S' : T'|R'). \]

Therefore the super-additivity follows from the infimum over all systems \( C \).

We consider the classically squashed version analogous to [36],

\[ I^{cl}(S : T|R)_\rho^M = \inf_{\rho^{MX}} I(SX : TX|RX)_\rho^M, \]

where the infimum runs over all quantum-classical states \( \rho^{MX} = \sum_x \rho^M_x \otimes |x><x| \) such that \( \sum_x \rho^M_x = \rho^M \). \( I^{cl} \) is the biggest convex functional dominated by \( I \). It is clear from the argument above that \( I^{cl} \) is also monotone under AOCC.

**Corollary 5.9.** \( I^{cl} \) is non-increasing under AOCC.

## 6. Concluding discussion

We proved that for a quadrilateral \( T \subset M \cup S \subset C \) of finite dimensional von Neumann algebras, the SSA inequality exists if and only if it forms a commuting square. Using complex interpolation and augmented \( L_p \) spaces, we obtained for two arbitrary subalgebras the SSA inequality with correction term. We use this as a universal approach to various entropic uncertainty relations. In the tripartite case, we found new noncommutative features hidden in the measurement situations.

Beyond the scope of this paper, there are infinite-dimensional entropic uncertainty relations for continuous momentum and position operators [12]. We believe that our approach, with additional assumptions providing finite entropy, generalize to semifinite von Neumann algebras. This is also related to recent work on uncertainty principles for quantum groups [9, 16]. Finally, \( I(T : S|R) \) retains many properties of and generalizes conditional mutual information. \( I^{sq} \) bears a similar analogy to squashed entanglement. We conjecture...
that $I^{sq}$ has a physical interpretation in terms of non-classical correlations between subalgebras, but the complete physical and operational interpretation of both these quantities is now an open problem.

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