QUANTUM DILOGARITHMS AND PARTITION $q$-SERIES

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Abstract. In our previous work [9], we introduced the partition $q$-series for mutation loop $\gamma$ — a loop in exchange quiver. In this paper, we show that for certain class of mutation sequences, called reverse-ending mutation loops, a graded version of partition $q$-series essentially coincides with the ordered product of quantum dilogarithm associated with each mutation; the partition $q$-series provides a state-sum description of combinatorial Donaldson-Thomas invariants introduced by B. Keller.

1. Introduction

Kontsevich and Soibelman’s groundbreaking work [13, 14] introduced some completely new ideas and techniques into the BPS state counting problems in physics. Their work as well as [15, 16] and Reineke [18, 19] motivated Keller [10, 11, 12] to study the product of quantum dilogarithms along a quiver mutation sequence. He showed that it is independent of the choice of a reddening mutation sequence and is an important invariant of a quiver which he called the combinatorial Donaldson-Thomas (DT) invariant.

In our previous work [9], we introduced the partition $q$-series for a mutation loop. A mutation loop is a mutation sequence supplemented by a boundary condition which specifies how the vertices of the initial and the final quiver are identified. One of our motivation is to provide a solid mathematical foundation to extract an essential information of the partition function of a 3-dimensional gauge theory. In particular, we showed for a special sequence of a Dynkin quiver or square product thereof, the partition $q$-series reproduce so-called fermionic character formulas of certain modules associated with affine Lie algebras.

In this paper, we analyze the relationship between partition $q$-series and the combinatorial DT-invariants. For that purpose, we refined the definition of our partition $q$-series by introducing a (noncommutative) grading and making it sensitive to “orientation” (green or red) of each mutation.

The main result of this paper is summarized as follows (see Theorem 5.7 for more precise statement): for any reddening sequence (= a mutation sequence for which the combinatorial DT invariant is defined), the refined version of the partition $q$-series coincides with the combinatorial DT invariant (up to involution $q \leftrightarrow q^{-1}$). Therefore, the partition $q$-series provide “state-sum” description of combinatorial DT-invariants that are given in “operator formalism”.

The paper is organized as follows. In Section 2 we recall some basic concepts of quiver mutation sequences and $c$-vectors. In Section 3 we introduce the (refined version of) the partition $q$-series $Z(\gamma)$ for the mutation loop $\gamma$. In Section 4 we show that the partition $q$-series are invariant under insertion/deletion of backtracking. In Section 5 is the main part of this paper; for “reverse-ending mutation loops”, the partition $q$-series can be expressed as a product of quantum dilogarithms and
so coincides with the combinatorial DT invariant. The final section is devoted to some explicit computation of partition \(q\)-series for various type of quivers.

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2. Quivers mutation sequences

2.1. Quivers and mutation sequences. A quiver \(Q = (Q_0, Q_1)\) is an oriented graph with the set \(Q_0\) of arrows and the set \(Q_1\) of vertices. In this paper all quivers are assumed to be finite connected oriented graphs without loops or 2-cycles:

\[
\begin{array}{c}
\text{loop} & \bullet \\
\text{2-cycle} & \bullet \circ \bullet
\end{array}
\]

Throughout the paper, we identify the set of vertices \(Q_0\) with \(\{1, 2, \ldots, n\}\). By a slight abuse of notation, we denote by

\[
(2.1) \quad Q_{ij} := \#\{(i \to j) \in Q_1\}
\]

the multiplicity of the arrow, and consider them as entries of an \(n \times n\) matrix. There is a bijection

\[
\begin{align*}
\{ & \text{the quivers without loops or} \\
\text{2-cycles, } Q_0 \to \{1, \ldots, n\} & \} \\
\text{via} & \{ \text{the skew-symmetric integer } n \times n\text{-matrices } B \}
\end{align*}
\]

via

\[
(2.2) \quad B_{ij} = Q_{ij} - Q_{ji}, \quad Q_{ij} = \max(B_{ij}, 0).
\]

For a quiver \(Q\) and its vertex \(k\), the mutated quiver \(\mu_k(Q)\) is defined \[4\]: it has the same set of vertices as \(Q\); its set of arrows is obtained from that of \(Q\) as follows:

1) for each path \(i \to j \to k\) of length two, add a new arrow \(i \to k\);
2) reverse all arrows with source or target \(k\);
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles.

It is well known that \(\mu_k(\mu_k(Q)) = Q\) for any \(1 \leq k \leq n\).

A finite sequence of vertices of \(Q\), \(m = (m_1, m_2, \ldots, m_T)\) is called mutation sequence. By putting \(Q(t) := \mu_{m_t}(Q(t-1))\), \(m\) induces a (discrete) time evolution of quivers:

\[
(2.3) \quad Q(0) \xrightarrow{\mu_{m_1}} Q(1) \xrightarrow{\mu_{m_2}} \cdots \xrightarrow{\mu_{m_T}} Q(T)
\]

\(Q(0)\) and \(Q(T)\) are called the initial and the final quiver, respectively.

The quiver mutation corresponds to matrix mutation defined by Fomin-Zelevinsky \[7\]. The matrix \(B(t)\) corresponding to \(Q(t)\) is given by \[7\]

\[
(2.4) \quad B(t)_{ij} = \begin{cases} 
-B(t-1)_{ij} & \text{if } i = k \text{ or } j = k \\
B(t-1)_{ij} + \text{sgn}(B(t-1)_{ik}) \max(B(t-1)_{ik}B(t-1)_{kj}, 0) & \text{otherwise}
\end{cases}
\]

Suppose that \(Q(0)\) and \(Q(T)\) are isomorphic as oriented graphs. An isomorphism \(\varphi : Q(T) \to Q(0)\) regarded as a bijection on the set of vertices, is called boundary condition of the mutation sequence \(m\). We represent \(\varphi\) by a permutation of \(\{1, \ldots, n\}\), i.e. \(\varphi \in S_n\). The triple \(\gamma = (Q; m, \varphi)\) is called a mutation loop.
2.2. **Ice quivers and c-vectors.** We will follow the terminology in [1]. An ice quiver is a pair \((\tilde{Q}, F)\) where \(\tilde{Q}\) is a quiver and \(F \subset \tilde{Q}_0\) is a (possibly empty) subset of vertices called frozen vertices such that there are no arrows between them. Two ice quivers \((\tilde{Q}, F)\) and \((\tilde{Q}', F')\) are called frozen isomorphic if \(F = F'\) and there is an isomorphism of quivers \(\phi : \tilde{Q} \to \tilde{Q}'\) such that \(\phi|_F = \text{id}_F\).

For any quiver \(Q\), there is a standard way of constructing an ice quiver \(Q^{\wedge}\) called framed quiver. \(Q^{\wedge}\) is an ice quiver obtained from \(Q\) by adding, for each vertex \(i\), a new frozen vertex \(i'\) and a new arrow \(i \to i'\):

\[
F = \{i'|i \in Q_0\}, \quad (Q^{\wedge})_0 = Q_0 \cup F, \quad (Q^{\wedge})_1 = Q_1 \cup \{i \to i'|i \in Q_0\}.
\]

Let \(m = (m_1, m_2, \ldots, m_T)\) be a mutation sequence for \(Q\). By putting

\[
\tilde{Q}(0) = Q^{\wedge}, \quad \tilde{Q}(t) = \mu_{m_t}(\tilde{Q}(t - 1)) \quad (t = 1, 2, \ldots, T)
\]

we can construct a sequence of ice quivers

\[
\tilde{Q}(0) \xrightarrow{\mu_{m_{T}}} \tilde{Q}(1) \xrightarrow{\mu_{m_{T-1}}} \cdots \xrightarrow{\mu_{m_{1}}} \tilde{Q}(T).
\]

Note that we never mutate at frozen vertices \(F = \{i'|i \in Q_0\}\). The quiver \(\tilde{Q}(t)\) will be called the ice quiver corresponding to \(Q(t)\). Let \(B(t)\) be the antisymmetric matrix corresponding to \(\tilde{Q}(t)\). The c-vectors are defined by counting the number of arrows to/from frozen vertices:

**Definition 2.1.** A c-vector of vertex \(v\) in \(Q(t)\) is a vector in \(\mathbb{Z}^n\) defined by

\[
c_v(t) := (\tilde{B}(t)_{v'})_{i=1}^n.
\]

If the vertices of \(\tilde{Q}(t)\) are ordered as \((1, \ldots, n, 1', \ldots, n')\), the antisymmetric matrix \(\tilde{B}(t)\) has the block form

\[
\tilde{B}(t) = \begin{bmatrix}
B(t) & C(t) \\
-C(t)^\top & 0
\end{bmatrix}, \quad C(t) = \begin{bmatrix}
c_1(t) \\
c_2(t) \\
\vdots \\
c_n(t)
\end{bmatrix}
\]

where \(X^\top\) denotes the transpose of \(X\). The \(n \times n\) block \(C(t)\) is called c-matrix, which consists of row of c-vectors. By construction, \(c_i(0) = e_i\), where \(e_i\) is the standard unit vector in \(\mathbb{Z}^n\).

**Theorem 2.2** (Sign coherence). Each c-vector is nonzero and lies in \(\mathbb{N}^n\) or \((-\mathbb{N})^n\).

This is conjectured in [7] and was proved in [3, 17]. Nagao [16] gave an alternative proof by using Donaldson-Thomas theory.

2.3. **Green and red mutations.** Following Keller [10], we call the vertex \(v\) of \(Q(t)\) is green (resp. red) if \(c_v(t) \in \mathbb{N}^n\) (resp. \(-c_v(t) \in \mathbb{N}^n\)). By definition, every vertex of the initial quiver \(Q(0)\) is green. The mutation \(\mu_{m_i} : Q(t - 1) \to Q(t)\) is green (resp. red) if the mutating vertex \(m_i\) is green (resp. red) on \(Q(t - 1)\), i.e. on
the quiver before mutation and the sign $\varepsilon_t$ of the mutation $\mu_{m_t}$ is defined as

$$\varepsilon_t = \begin{cases} +1 & \text{if } \mu_{m_t} \text{ is green,} \\ -1 & \text{if } \mu_{m_t} \text{ is red.} \end{cases}$$

A mutation sequence $m = (m_1, m_2, \ldots, m_T)$ is called reddening if all vertices of the final quiver $Q(T)$ are red. A mutation sequence $m$ is called green sequence if $m_t$ is green for all $t$, and is maximal green sequence if all of the vertex of the final quiver $Q(T)$ are red. Clearly, all maximal green sequences are reddening. In Figure 1 the two maximal green sequences (12) and (212) are shown for $A_2$ quiver.

By inspecting the matrix mutation rules for the ice quivers $Q(t)$, it is easy to see how the c-vector changes via mutations:

**Lemma 2.3.** Under the mutation $\mu_v : Q(t) \to Q(t + 1)$, c-vector changes as

$$c_i(t + 1) = \begin{cases} -c_i(t) & \text{if } i = v \\ c_i(t) + Q(t)_{i,v} \cdot c_v(t) & \text{if } i \neq v \text{ and } \mu_v \text{ is green} \\ c_i(t) + Q(t)_{v,i} \cdot c_v(t) & \text{if } i \neq v \text{ and } \mu_v \text{ is red} \end{cases}$$

**Corollary 2.4.** $\det C(t) = (-1)^t$. In particular, $C(t) \in GL_n(\mathbb{Z})$ and, c-vectors $\{c_i(t)\}_{i=1}^n$ constitutes a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ for each $t$.

2.4. Noncommutative algebra $\hat{A}_Q$. We introduce a noncommutative associative algebra in which quantum dilogarithms and combinatorial Donaldson-Thomas invariants take their values.

Let $Q$ be a quiver with vertices $\{1, 2, \ldots, n\}$. We define a skew symmetric bilinear form $(\ , \ ) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ by

$$\langle c_i, c_j \rangle = B_{ij} = -B_{ji} = Q_{ij} - Q_{ji},$$

1\text{The sign of the mutating vertex changes after the mutation. If the vertex } m_t \text{ is green on } Q(t - 1), \text{ then it is red on } Q(t).
where \(e_1, \ldots, e_n\) are the standard basis vectors in \(\mathbb{Z}^n\).

Let \(R\) be a commutative ring containing \(\mathbb{Q}(q^{1/2})\). Let \(A_Q\) be a noncommutative associative algebra over \(R\) presented as
\[
A_Q = R\langle y^\alpha, \alpha \in \mathbb{N}^n \mid y^\alpha y^\beta = q^{\frac{1}{2} \langle \alpha, \beta \rangle} y^{\alpha + \beta} \rangle.
\]

Its completion with respect to the \(\mathbb{N}^n\)-grading is denoted by \(\hat{A}_Q\). We may regard \(A_Q\) as the ring of noncommutative polynomials in \(y_i = y^{e_i} (i = 1, \ldots, n)\). Later we will frequently use the following relations (\(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n\)):
\[
y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} = q^{\frac{1}{2} \sum_{i<j} B_{ij} \alpha_i \alpha_j} y^{\alpha},
\]
\[
y^{\alpha} = q^{-\frac{1}{2} \sum_{i<j} B_{ij} \alpha_i \alpha_j} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n}.
\]

Later we will use a \(\mathbb{Q}\)-algebra anti-automorphism \(\tau: \hat{A}_Q \to \hat{A}_Q\) defined by
\[
y^\alpha \mapsto y^{\tau(\alpha)}, \quad q \mapsto q^{-1}
\]
Note that this is an involutive anti-automorphism of \(\mathbb{Q}\)-algebra, not of \(R\)-algebra.

### 3. Partition \(q\)-series

In this section, we recapitulate the notion of partition \(q\)-series introduced in [9]. As mentioned in Introduction, we refine and extend the definition, so that we can state the relationship between our partition \(q\)-series and the products of quantum dilogarithm (combinatorial DT-invariants) in full generality.

(i) We introduce noncommutative variables \(y_1, \ldots, y_n\) to keep track of the \(\mathbb{N}^n\)-grading. This is in conformity to the custom of quantum dilogarithms and DT-invariants. They are naturally associated with the \(c\)-vectors as well as \(s\)-variables. This weighted version of partition \(q\)-series now take their values in \(\hat{A}_Q\) — the (completed) ring of noncommutative polynomials in \(y_1, \ldots, y_n\), rather than \(\mathbb{Z}[[q^{1/\Delta}]]\).

(ii) We make a distinction between green and red mutations, and we add a new rule for red mutations. Although this refinement requires additional data — \(c\)-vectors, or equivalently ice quivers, we obtain perfect matching (Theorem 3.7) between the partition \(q\)-series and the combinatorial Donaldson-Thomas invariant wherever the latter invariant are defined.

As a bonus of these refinements, we can handle arbitrary non-degenerate mutation sequences. Moreover, the refined version acquire the invariance under the insertion or deletion of backtracking in mutation sequence (Theorem 4.1).

In the case of green mutation sequences, this new definition coincides with the original one [9] just by forgetting \(N^n\) gradings.

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2The coefficient ring \(R\) should be chosen in such a way that the factors \(q^{\pm \frac{1}{2} k k'}\) of mutation weight \(b_{k, k'}\) belong to \(R\). The exponent of \(q\) can have nontrivial denominator through the process of expressing \(k\)-variables in terms of \(k\)-variables. As discussed in [11], there is a positive integer \(\Delta\), depending only on the mutation loop, such that \(\frac{1}{2} k k' \in \frac{1}{\Delta} \mathbb{Z}\). Then we can choose \(R = \mathbb{Q}(q^{1/\Delta})\).

3Except the grading, all the results in [11] remains the same in our new setting; all the mutation sequence considered there are green sequences.

4In [9], the partition \(q\)-series were well-defined only for “positive” mutation loops.
3.1. The partition $q$-series. Let $Q$ be a quiver with vertices $\{1, 2, \ldots, n\}$. We consider the mutation sequence $m = (m_1, m_2, \ldots, m_T)$ of $Q$:

\[
Q(0) \xrightarrow{\mu_{m_1}} Q(1) \xrightarrow{\mu_{m_2}} \cdots \xrightarrow{\mu_{m_T}} Q(T) = Q(T) .
\]

The partition $q$-series is defined as follows. We first introduce a family of $s$-variables $\{s_i\}$, $k$-variables $\{k_t\}$, and $k^\nu$-variables $\{k_t^\nu\}$ by the following rule:

(i) An “initial” $s$-variable $s_v$ is attached to each vertex $v$ of the initial quiver $Q$.

(ii) Every time we mutate at vertex $v$, we add a “new” $s$-variable associated with $v$. We often write $s_v$, $s'_v$, $s''_v$, $s'''_v$, . . . to distinguish $s$-variables attached to the same vertex.

(iii) We associate $k_t$ and $k_t^\nu$ with each mutation at $m_t$.

(iv) If two vertices are identified by a boundary condition, then the corresponding $s$-variables are also identified.

The $s$-, $k$-, and $k^\nu$-variables are not considered independent; we impose a linear relation for each mutation step. Suppose that the quiver $Q(t - 1)$ equipped with $s$-variables $\{s_i\}$ is mutated at vertex $v = m_t$ to give $Q(t)$. Then $k$- and $s$-variables are required to satisfy

\[
k_t = \begin{cases} 
- \sum_{a \rightarrow v} s_a & \text{if } \varepsilon_t = 1 \\
- \sum_{v \rightarrow b} s_b & \text{if } \varepsilon_t = -1 
\end{cases}
\]

Here, $s'_v$ is the “new” $s$-variable attached to mutated vertex $v$, and the sum is over all the arrows of $Q(t - 1)$. Similarly, $k^\nu$- and $s$-variables are related as

\[
k_t^\nu = \begin{cases} 
- \sum_{v \rightarrow b} s_a & \text{if } \varepsilon_t = 1 \\
- \sum_{a \rightarrow v} s_b & \text{if } \varepsilon_t = -1 
\end{cases}
\]

Therefore,

\[
k_t^\nu - k_t = \sum_{a \rightarrow v} s_a - \sum_{v \rightarrow b} s_b
\]

holds at each mutation.

The weight of the mutation $\mu_{m_t} : Q(t - 1) \rightarrow Q(t)$ at $v = m_t$ is defined as

\[
W(m_t) := \frac{q^{\frac{1}{2}k_t k_t^\nu}}{(q^{\nu})_{k_t}} = \begin{cases} 
\frac{q^2 k_t k_t^\nu}{(q^{\nu})_{k_t}} & \text{if } \varepsilon_t = 1 \\
\frac{q^{\frac{1}{2}k_t k_t^\nu}}{(q^{\nu})_{k_t}} & \text{if } \varepsilon_t = -1 
\end{cases}
\]

Here, $\varepsilon_t$ is the sign of $\mu_{m_t}$ and

\[
(q)_k := \prod_{i=1}^{k} (1 - q^i)
\]

is the $q$-Pochhammer symbol.
The \(\mathbb{N}^n\)-\textit{grading} of the mutation \(\mu_{m_i}\) is \(k_i \alpha_i\) by definition, where
\begin{equation}
(3.7) \quad \alpha_i := \varepsilon e c_{m_i}(t-1) \in \mathbb{N}^n \setminus \{0\}
\end{equation}
is the (sign-corrected) \(c\)-vector of the vertex on which mutation is applied.

It is occasionally useful to regard the relation \((3.2)\) as the time evolution of \(s\)-variables with the control parameters \(\{k_i\}\). Let \(s_i(t)\) denote the value of the \(s\)-variable associated with vertex \(i\) at \(Q(t)\). Then \((3.2)\) can be written as
\begin{equation}
(3.8) \quad s_i(t) = \begin{cases} 
s_i(t-1) & \text{if } i \neq v, \\
k_t - s_v(t-1) + \sum_a Q(t)_{a,v}s_a(t-1) & \text{if } i = v \text{ and } \mu_v \text{ is green}, \\
- k_t - s_v(t-1) + \sum_b Q(t)_{v,b}s_b(t-1) & \text{if } i = v \text{ and } \mu_v \text{ is red}.
\end{cases}
\end{equation}

With this notation, \((3.4)\) reads as
\begin{equation}
(3.9) \quad k^0_t = k_t - \sum_i B(t-1)_{v,i} s_i(t-1) = k_t + \sum_i B(t-1)_{i,v} s_i(t-1).
\end{equation}

One can usually solve the linear relations \((3.2)\) for \(s\)-variables in terms of \(k\)-variables. If this is the case, the mutation loop is called \textit{non-degenerate} (see \([9]\)). Then using \((3.3)\) and \((3.5)\), we can express all the mutation weights \(\{W(m_t)\}\) as functions of \(k = (k_1, \ldots, k_T)\).

Hereafter we assume that the mutation loop \(\gamma\) is non-degenerate. Then the \((\mathbb{N}^n\text{-graded})\) \textit{partition} \(q\text{-series}\) associated with \(\gamma\) is defined as
\begin{equation}
(3.10) \quad Z(\gamma) := \sum_{k \in \mathbb{N}^T} \left( \prod_{t=1}^T W(m_t) \right) y^{\sum_{t=1}^T k_t \alpha_t}.
\end{equation}

\textbf{Remark 3.1.} For a fixed \(\beta \in \mathbb{N}^n\), there is only a finite number of \(k = (k_1, \ldots, k_T) \in \mathbb{N}^T\) satisfying \(\beta = \sum_{t=1}^T k_t \alpha_t\). Therefore \(Z(\gamma)\) is well-defined as an element of \(\hat{\mathbb{N}}_Q\).

In our previous paper \([9]\), we had no \(\mathbb{N}^n\)-grading and thus needed the additional “positive” assumption on the quadratic form in the mutation weight to guarantee this finiteness.

\section{4. Backtrack invariance of the partition \(q\)-series}

Two successive mutations at the same vertex is called \textit{backtracking}. It is well known that backtracking brings a quiver back into the same quiver: \(\mu_v(\mu_v(Q)) = Q\). All the \(c\)-vectors recover their original values since \(\mu_v(\mu_v(Q)) = Q\).

In this section, we prove that the partition series has a backtrack invariance. The original version \([9]\) of partition \(q\)-series lacks this property; this is one reason why we adopt different rules (e.g. \((3.8)\)) for different signs (red/green).

\textbf{Theorem 4.1.} The partition \(q\)-series is invariant under insertion or deletion of backtracking:
\begin{equation}
(4.1) \quad Z((Q; m_1 v v m_2, \varphi)) = Z((Q; m_1 m_2, \varphi)).
\end{equation}

\textbf{Proof.} The mutation loop \((Q; m_1 v v m_2, \varphi)\) is shown in Figure\([2]\). We concentrate on two successive mutations constituting the backtracking:
\begin{equation}
(4.2) \quad \ldots \rightarrow Q' \xrightarrow{\mu_v} Q'' \xrightarrow{\mu_v} Q''' \rightarrow \ldots
\end{equation}
respectively, and \( v \) for some \( \alpha \).

The proof is given only for the case when the signs of these two mutations are \((+, -)\) = (green, red); the other case \((-+, +)\) is left to the reader. By assumption, the \( c \)-vector \( \alpha \) of the vertex \( v \) changes as

\[
\alpha \mapsto -\alpha \mapsto \alpha
\]

for some \( \alpha \in \mathbb{N}^n \setminus \{0\} \).

Let \( x', x'', x''' \) be the \( s \)-variables associated with the vertex \( v \) of \( Q' \), \( Q'' \), \( Q''' \) respectively, and \( k_1, k_2 \) be the \( k \)-variables corresponding to the two mutations at \( v \). As in Figure 3 we collectively denote by \( i \rightarrow v, v \rightarrow j \) the arrows of \( Q' \) touching \( v \), and \( a_i, b_j \) be the corresponding \( s \)-variables; some of the vertices \( i, j \) may be missing, duplicated or identified. By \([5.2], [3.3]\), these \( s \)-variables are related with \( k \)- and \( k \)-variables as

\[
\begin{align*}
    k_1 &= x' + x'' - \sum a_i, \\
    k_2 &= \sum a_i - (x'' + x'''), \\
    k_1' &= x' + x'' - \sum b_j, \\
    k_2' &= \sum b_j - (x'' + x''').
\end{align*}
\]

The weight corresponding to the backtracking \((v v)\) is given by

\[
W((v v)) = \left( q^{\frac{1}{2}x'(x'' - \sum a_i)(x'' - \sum b_j)} \right) \times \left( q^{-\frac{1}{2}(\sum a_i - (x'' + x'''))(\sum b_j - (x'' + x'''))} \right)
\]

\[
= q^{\frac{1}{2}(k_1(k_1 + \sum a_i - \sum b_j))} q^{\frac{1}{2}(k_2 + \sum b_j - \sum a_i)}
\]

\[
= q^{\frac{1}{2}(k_1^2 - k_1k_2)} q^{\frac{1}{2}(\sum a_i - \sum b_j)} k_1 + k_2.
\]

By summing over \( k_1, k_2 \) and including the \( \mathbb{N}^n \)-degree, we have

\[
\sum_{k_1, k_2=0}^{\infty} \frac{q^{\frac{1}{2}(k_1^2 - k_1k_2)}}{(q)_{k_1}(q^{-1})_{k_2}} \left( q^{\frac{1}{2}(\sum a_i - \sum b_j)} \right)^{k_1 + k_2} y^{(k_1 + k_2)\alpha}
\]
to the results of Brüstle–Dupont–Pérotin \[1\], this class coincides with reddening sequences studied by Keller \[12\]; for any reddening sequence, he associates so-called dilogarithms. Therefore, the partition \(q\)-series and the product of quantum dilogarithms (Theorem 5.7) for a wide class of mutation loops which we call reverse-ending (see Definition 5.6 and Remark 5.10). Thanks to the results of Brüstle–Dupont–Pérotin \[1\], this class coincides with reddening sequences studied by Keller \[12\]; for any reddening sequence, he associates so-called combinatorial Donaldson-Thomas invariant defined as the product of quantum dilogarithms. Therefore, the partition \(q\)-series provide a state-sum interpretation (fermionic sum formula) for the combinatorial Donaldson-Thomas invariants.

Throughout this section, the mutation sequence we consider has the form

\[
Q(0) \xrightarrow{\mu_{m_1}} Q(1) \xrightarrow{\mu_{m_2}} \cdots \xrightarrow{\mu_{m_{t-1}}} Q(t-1) \xrightarrow{\mu_{m_t}} Q(t) \xrightarrow{\mu_{m_{t+1}}} \cdots \xrightarrow{\mu_{m_T}} Q(T).
\]

The algebra \(\hat{A}_Q\) and the skew-symmetric form \(\langle , \rangle\) are always with respect to the initial quiver \(Q = Q(0)\).

5. Partition \(q\)-series and the Product of Quantum Dilogarithms

In this section, we show that there is a precise match between partition \(q\)-series and the product of quantum dilogarithms (Theorem 5.7) for a wide class of mutation loops which we call reverse-ending (see Definition 5.6 and Remark 5.10). Thanks to the results of Brüstle–Dupont–Pérotin \[1\], this class coincides with reddening sequences studied by Keller \[12\]; for any reddening sequence, he associates so-called combinatorial Donaldson-Thomas invariant defined as the product of quantum dilogarithms. Therefore, the partition \(q\)-series provide a state-sum interpretation (fermionic sum formula) for the combinatorial Donaldson-Thomas invariants.

Throughout this section, the mutation sequence we consider has the form

\[
Q(0) \xrightarrow{\mu_{m_1}} Q(1) \xrightarrow{\mu_{m_2}} \cdots \xrightarrow{\mu_{m_{t-1}}} Q(t-1) \xrightarrow{\mu_{m_t}} Q(t) \xrightarrow{\mu_{m_{t+1}}} \cdots \xrightarrow{\mu_{m_T}} Q(T).
\]

The algebra \(\hat{A}_Q\) and the skew-symmetric form \(\langle , \rangle\) are always with respect to the initial quiver \(Q = Q(0)\).

5.1. Products of quantum dilogarithms. A quantum dilogarithm series is defined by

\[
E(y; q) := 1 + \frac{y^{1/2}}{q-1} y + \cdots + \frac{q^{n^{2}/2}}{(q^n-1)(q^n-q^2)\cdots(q^n-q^{n-1})} y^n + \cdots.
\]

It is also expressed as

\[
E(y; q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q)_n} y^n = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q^{-1})_n} y^n = \prod_{n=0}^{\infty} \frac{1}{1+q^{n+2}y^n}
\]

\[
= \exp \left( \sum_{k=1}^{\infty} \frac{(-y)^k}{k(q^{-k/2} - q^{k/2})} \right).
\]

We will mostly use the following form

\[
E(y; q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q)_n} y^n, \quad E(y; q) = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q^{-1})_n} y^n
\]

It is well-known that if \(u\) and \(v\) satisfies the relation \(uv = qvu\), then the pentagon identity holds \[20\] \[4\] \[5\]:

\[
E(u; q) E(v; q) = E(v; q) E(q^{-1/2}uv; q) E(u; q).
\]
Following Keller [10], we associate a quantum dilogarithm $\mathbb{E}(y^{a_1};q^{e_1})$ for each mutation $\mu_m : Q(t-1) \to Q(t)$. Here $\varepsilon_t$ is the sign of $\mu_{m_t}$ (see (2.10) and $\alpha_t = \varepsilon_t c_{m_t} (t-1) \in \mathbb{N}^n$ is the (sign-corrected) c-vector (see (5.7)). For a mutation sequence (5.1), we define its quantum dilogarithm product as the ordered product of these:

$$
(5.7) \quad \mathbb{E}(Q;m) := \mathbb{E}(y^{a_1};q^{e_1}) \mathbb{E}(y^{a_2};q^{e_2}) \cdots \mathbb{E}(y^{a_T};q^{e_T}).
$$

They enjoy the following nice properties.

**Theorem 5.1** (Keller [11], Nagao [15]). If $m$ and $m'$ are two sequences of vertices of $Q$ such that there is a frozen isomorphism between $\mu_m(Q)$ and $\mu_{m'}(Q)$, then we have $\mathbb{E}(Q;m) = \mathbb{E}(Q;m')$.

**Theorem 5.2** (Keller [11]). If $m$ and $m'$ are reddening sequences, then there is a frozen isomorphism $\mu_m(Q) \simeq \mu_{m'}(Q)$.

Theorems 5.1 and 5.2 imply that if $Q$ admits a reddening sequence $m$,

$$
(5.8) \quad \mathbb{E}_Q := \mathbb{E}(Q;m) \in \mathbb{A}_Q.
$$

is independent of the choice of the reddening sequence $m$ and defines an invariant of $Q$. Keller named this invariant as combinatorial Donaldson-Thomas (DT) invariant. The pentagon identity (5.6) is nothing but the combinatorial DT invariant of $A_2$ quiver $Q = (1 \to 2)$, whose reddening sequences are depicted in Figure 1.

### 5.2. Evolution along mutation sequence

In this subsection, we collect some results how the $s$-variables $\{s_i(t)\}$ and c-vectors $\{c_i(t)\}$ evolve along the mutation sequence (5.1). These are needed to keep track of $\mathbb{N}^n$ grading of partition $q$-series.

**Proposition 5.3.** For the mutation sequence (5.1), we have

$$
(5.9) \quad B(t)_{ij} = \langle c_i(t), c_j(t) \rangle, \quad (0 \leq t \leq T)
$$
or equivalently,

$$
(5.10) \quad C(t)B(0)C(t)^\top = B(t). \quad (0 \leq t \leq T)
$$

**Proof.** We prove (5.9) by induction on $t$. The case $t = 0$ is clear from (2.12) and $c_i(0) = c_i$. Assuming (5.9) to hold for $t$, we consider the mutation $\mu_v : Q(t) \to Q(t+1)$ where $v = m_{t+1}$. By skewness of $\langle , \rangle$, it suffices to consider the following four cases:

- **(case A-1)** $i = v$, $j \neq v$.
  
  $$
  \langle c_v(t+1), c_j(t+1) \rangle = \langle -c_v(t), c_j(t+1) \rangle \quad \text{(using (2.11))}
  $$
  
  $$
  = -\langle c_v(t), c_j(t+1) \rangle \quad \text{(skew-symmetry of $\langle , \rangle$)}
  $$
  
  $$
  = -B(t)_{v,j} \quad \text{(by induction hypothesis)}
  $$
  
  $$
  = B(t+1)_{v,j}. \quad \text{(using (2.4))}
  $$

- **(case A-2)** $i \neq v$, $j = v$. The proof closely parallels that of (case A-1).

- **(case B-1)** $i \neq v$, $j \neq v$, $\mu_v$ is green mutation

  $$
  \langle c_i(t+1), c_j(t+1) \rangle
  $$
  
  $$
  = \langle c_i(t+1), c_j(t+1) \rangle \quad \text{(using (2.11))}
  $$
  
  $$
  = \langle c_i(t), c_j(t+1) \rangle + Q(t)_{i,v} c_v(t) \quad \text{(using (2.11))}
  $$
  
  $$
  = \langle c_i(t), c_j(t) \rangle + Q(t)_{i,v} c_v(t) + Q(t)_{j,v} (c_i(t), c_v(t)).
  $$
Along the mutation sequence (5.1), the state vector changes as

Proposition 5.4.

Proof. In conclusion, (5.9) is also true for $t = 0$ of the state vector $(5.11)$.

\begin{align*}
\psi(t) := & \sum_{i=1}^{n} s_i(t)c_i(t) \in \mathbb{Z}^n \quad (0 \leq t \leq T). \\
& \text{(skew-symmetry of } \langle \ , \rangle \text{)}
\end{align*}

= $B(t)_{i,j} + Q(t)_{i,v}B(t)_{v,j} + Q(t)_{j,v}B(t)_{i,v}$ \hspace{1cm} \text{(by induction hypothesis)}

= $B(t)_{i,j} + Q(t)_{i,v}(Q(t)_{v,j} - Q(t)_{j,v}) + Q(t)_{j,v}(Q(t)_{i,v} - Q(t)_{v,i})$

= $B(t)_{i,j} + Q(t)_{i,v}Q(t)_{v,j} - Q(t)_{j,v}Q(t)_{i,v}$

= $B(t+1)_{i,j}$. \hspace{1cm} \text{(using (2.4))}

• (case B-2) $i \neq v$, $j \neq v$, $\mu_v$ is red mutation. The proof is similar to that of (case B-1).

In conclusion, (5.9) is also true for $t + 1$. \hfill $\Box$

Since $c$-vectors $\{c_i(t)\}$ form a basis of $\mathbb{Z}^n$ for each $t$, it is natural to introduce the state vector of $Q(t)$ defined by

\begin{equation}
\psi(t) := \sum_{i=1}^{n} s_i(t)c_i(t) \in \mathbb{Z}^n \quad (0 \leq t \leq T). 
\end{equation}

**Proposition 5.4.** Along the mutation sequence, the state vector changes as

\begin{equation}
\psi(t) = \psi(t-1) - k_i\alpha_i, \hspace{0.5cm} (t = 1, \ldots, T).
\end{equation}

**Proof.** There are two cases to be considered.

Case 1) $\mu_{m_i} : Q(t-1) \to Q(t)$ is green ($\varepsilon_i = +1$):

\begin{align*}
\psi(t) &= \sum_{i} s_i(t)c_i(t) \\
&= s_{m_i}(t)c_{m_i}(t) + \sum_{i \neq m_i} s_i(t)c_i(t) \\
&= \left(k_i - s_{m_i}(t-1) \sum_{a} Q(t-1)_{a,m_i}s_a(t-1)\right)(-c_{m_i}(t-1)) \hspace{1cm} \text{(by (3.8))} \\
&\quad + \sum_{i \neq m_i} s_i(t-1)(c_i(t-1) + Q(t-1)_{i,m_i}c_{m_i}(t-1)) \hspace{1cm} \text{(by (2.11))} \\
&= (k_i - s_{m_i}(t-1))(-c_{m_i}(t-1)) + \sum_{i \neq m_i} s_i(t-1)(c_i(t-1)) \\
&= -k_i c_{m_i}(t-1) + \sum_{i \neq m_i} s_i(t-1)c_i(t-1) \\
&= \psi(t-1) - k_i\varepsilon_i c_{m_i}(t-1) \\
&= \psi(t-1) - k_i\alpha_i.
\end{align*}

Case 2) $\mu_{m_i} : Q(t-1) \to Q(t)$ is red ($\varepsilon_i = -1$):

\begin{align*}
\psi(t) &= \sum_{i} s_i(t)c_i(t) \\
&= s_{m_i}(t)c_{m_i}(t) + \sum_{i \neq m_i} s_i(t)c_i(t) \\
&= \left(-k_i - s_{m_i}(t-1) \sum_{b} Q(t-1)_{m_i,b}s_b(t-1)\right)(-c_{m_i}(t-1)) \hspace{1cm} \text{(by (3.8))} \\
&\quad + \sum_{i \neq m_i} s_i(t-1)(c_i(t-1) + Q(t-1)_{i,m_i}c_{m_i}(t-1)) \hspace{1cm} \text{(by (2.11))} \\
&= (-k_i - s_{m_i}(t-1))(-c_{m_i}(t-1)) + \sum_{i \neq m_i} s_i(t-1)(c_i(t-1)) \\
&= \psi(t-1) + k_i\varepsilon_i c_{m_i}(t-1) \\
&= \psi(t-1) + k_i\alpha_i.
\end{align*}
\[ = +k t c m(t-1) + \sum_i s_i(t-1)c_i(t-1) \]
\[ = \psi(t-1) - k t c m(t-1) \]
\[ = \psi(t-1) - k t \alpha t. \]

Therefore \( \mathbb{N}^n \)-grading of the partition \( q \)-series expresses the total change of the state vector around the mutation loop:

**Corollary 5.5.** The state vectors of the initial and the final quivers are related as
\[ \psi(0) - \psi(T) = \sum_{t=1}^{T} k t \alpha t. \]

### 5.3. Reverse-ending mutation loops.

**Definition 5.6.** A mutation sequence \( m = (m_1, \ldots, m_T) \) with initial quiver \( Q \) is called reverse-ending if there is a permutation \( \varphi \) of the vertices \( Q_0 = \{1, 2, \ldots, n\} \) such that

(i) \( \varphi \) represents an isomorphism of quivers \( Q(T) \simeq Q(0) \).

(ii) The \( c \)-vectors satisfy \( c_i(T) = -c_{\varphi(i)}(0) \) for all \( i \in Q_0 \).

The mutation loop \( \gamma = (Q; m, \varphi) \) is called reverse-ending mutation loop.

The each \( c \)-vector of the final quiver has a form \( c_i(T) = -e_{\varphi(i)} \) since \( c_i(0) = e_i \). This means that in the corresponding final ice quiver \( \tilde{Q}(T) \), \( i' \rightarrow \varphi(i) \) is the only arrow starting from \( i' \) and there is no arrow pointing to \( i' \).

The advantage of considering a reverse-ending loop \( m \) is that there is a canonical choice of the boundary condition \( \varphi \) to form a mutation loop \( \gamma \). It is thus natural to study its partition \( q \)-series \( Z(\gamma) \).

### 5.4. Partition \( q \)-series and product of quantum dilogarithms.

The following theorem is the main result of this paper.

**Theorem 5.7.** Let \( \gamma = (Q; m, \varphi) \) be a reverse-ending mutation loop. Then, the partition \( q \)-series and the combinatorial Donaldson-Thomas invariant are related as
\[ Z(\gamma) = \mathbb{E}(Q; m). \]

Here \( - : \hat{\mathbb{A}}_Q \to \hat{\mathbb{A}}_Q \) is a \( \mathbb{Q} \)-algebra anti-automorphism defined in (2.15).

The rest of this section is devoted to the proof of Theorem 5.7.

**Lemma 5.8.** Let \( \gamma \) be a reverse-ending mutation loop. Then,

(i) The state vectors \( \{\psi(t)\}_{t=0}^{T} \) are anti-periodic along the loop, that is,
\[ \psi(T) = -\psi(0). \]

(ii) The mutation loop \( \gamma \) is nondegenerate. In particular, the initial \( s \)-variables are expressed as
\[ s(0) := (s_1(0), \ldots, s_n(0)) = \frac{1}{2} \sum_{t=1}^{T} k t \alpha t. \]
Proof. In $\gamma$, the initial and the final $s$-variables are identified: $s_i(T) = s_{\varphi(t)}(0)$. By the reverse-ending condition, we have also $c_i(T) = -c_{\varphi(t)}(0)$. Therefore

$$\psi(T) = \sum_{i=1}^{n} s_i(T)c_i(T) = -\sum_{i=1}^{n} s_{\varphi(t)}(0)c_{\varphi(t)}(0) = -\sum_{j=1}^{n} s_j(0)c_j(0) = -\psi(0).$$

This proves (i). We have then

$$s(0) = \sum_{i=1}^{n} s_i(0)e_i = \sum_{i=1}^{n} s_i(0)c_i(0) = \psi(0) = \frac{1}{2}(\psi(0) - \psi(T)) = \frac{1}{2} \sum_{t=1}^{T} k_t\alpha_t,$$

where the last equality is by Corollary 5.5. Thus the initial $s$-variables are expressed in terms of $k$-variables alone. We can obtain similar formulas for the remaining $s$-variables by recursive use of the relation (5.8). This proves (ii). \qed

The following relation will play the key role in the proof of Theorem 5.7.

**Proposition 5.9.** For any mutation sequence, we have

$$(5.14) \quad \sum_{t=1}^{T} \varepsilon_t k_t k_t' + \langle \psi(0), \psi(T) \rangle = \sum_{t=1}^{T} \varepsilon_t k_t^2 - \sum_{1 \leq i < j \leq T} k_i k_j \langle \alpha_i, \alpha_j \rangle.$$ 

Proof.

$$\sum_{t=1}^{T} \varepsilon_t k_t k_t' - \sum_{t=1}^{T} \varepsilon_t k_t^2 = \sum_{t=1}^{T} \varepsilon_t k_t (k_t' - k_t)$$

(by 5.50)

$$= \sum_{t=1}^{T} \varepsilon_t k_t \sum_{i=1}^{n} B(t-1)_{i,m_i} s_i(t-1)$$

(by 5.51)

$$= \sum_{t=1}^{T} \varepsilon_t k_t \sum_{i=1}^{n} (c_i(t-1), c_{m_i}(t-1)) s_i(t-1)$$

(by 5.57)

$$= \sum_{t=1}^{T} k_t \langle \psi(t-1), \alpha_t \rangle$$

(by 5.11)

$$= \sum_{t=1}^{T} k_t \langle \psi(0), \alpha_t \rangle - \sum_{i=1}^{T-1} k_t \alpha_t$$

(by 5.12)

$$= \sum_{t=1}^{T} k_t \langle \psi(0), \alpha_t \rangle - \sum_{t=1}^{T-1} \sum_{i=1}^{T-1} k_t k_i \langle \alpha_i, \alpha_t \rangle$$

(by Corollary 5.5)

$$= \langle \psi(0), \sum_{t=1}^{T} k_t \alpha_t \rangle - \sum_{j=1}^{T-1} \sum_{i=1}^{j-1} k_t k_j \langle \alpha_i, \alpha_j \rangle$$

(by skewness of $\langle \, , \rangle$)

$$= -\langle \psi(0), \psi(T) \rangle - \sum_{1 \leq i < j \leq T} k_i k_j \langle \alpha_i, \alpha_j \rangle.$$ (by skewness of $\langle \, , \rangle$)
By arranging the terms, we obtain \(5.14\).

We are now ready to prove Theorem 5.7. The partition \(q\)-series associated with the loop \(\gamma\) is defined to be

\[
Z(\gamma) = \sum_{k \in \mathbb{N}^T} \prod_{i=1}^{T} W(m_i) y^{\sum_{i=1}^{T} k_i \alpha_i} = \sum_{k \in \mathbb{N}^T} \frac{q^{\frac{1}{2} \sum_{i=1}^{T} \varepsilon_i k_i^2}}{\prod_{i=1}^{T} (q^{r_i})_{k_i}} y^{\sum_{i=1}^{T} k_i \alpha_i}.
\]

On the other hand, the quantum dilogarithm product along \(\gamma\)

\[
\square
\]

By arranging the terms, we obtain \(5.14\).

Thus, all we have to show is that the exponents of \(q\) in the summands of \(5.15\) and \(5.16\) are equal for every \(k\):

\[
\sum_{i=1}^{T} \varepsilon_i k_i k_i' = \sum_{i=1}^{T} \varepsilon_i k_i^2 - \sum_{1 \leq i < j \leq T} k_i k_j \langle \alpha_i, \alpha_j \rangle.
\]

Indeed, by Lemma 5.8 (i), we have \(\psi(T) = -\psi(0)\), which implies \(\langle \psi(0), \psi(T) \rangle = -\langle \psi(0), \psi(0) \rangle = 0\) by the skewness of \(\langle , \rangle\). Then \(5.14\) follows from Proposition 5.9. This completes the proof of Theorem 5.7.

Remark 5.10. Some remarks are in order on the relationship between “reverse-ending” and “reddening” sequence. Clearly all reverse-ending mutation sequences are redening. Conversely, suppose \(m\) is a redening sequence; all the non-frozen vertices of \(\tilde{Q}(T)\) are red. Thanks to the result of Brüstle–Dupont–Pérotin (Proposition 2.10 of [11]), \(\tilde{Q}(T)\) is frozen isomorphic to a “coframed” quiver \(Q'\); \(Q'\) is constructed from \(Q\) in the same way as the framed quiver \(Q^\wedge\), except that the added arrows are \(i' \rightarrow i\) instead of \(i \rightarrow i'\). This means that all redening sequence are in fact reverse-ending.

Consequently, two notions reverse-ending and redening coincide. Therefore, all combinatorial DT invariants \(E_Q\) defined by Keller are equal (up to the involution \(q \leftrightarrow q^{-1}\)) to the partition series \(Z(\gamma)\) associated with a reverse-ending mutation loop \(\gamma = (Q; m, \varphi)\). We adopt the term “reverse-ending” because it contains explicitly the boundary condition data \(\varphi\), which is hidden in the definition of redening sequence. The coincidence of the two notions is a nontrivial fact whose known proof requires categorification [11] in terms of Ginzburg dg-algebra [8].
6. Examples

In this section, we collect various examples of the reverse-ending mutation loops and the associated partition $q$-series to illustrate Theorem 5.7.

6.1. $A_2^{(1)}$-quiver. As a simplest example of quiver with an oriented cycle, let us take the $A_2^{(1)}$ quiver

$$Q = \begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
2 & \leftarrow & 3
\end{array}
\end{array}$$

By performing successive mutations on $\tilde{Q}(0) := Q^\vee$ (Figure 4), it is easy to see that the mutation sequence

$$m = (1, 2, 3, 1)$$

is maximal green, reverse-ending sequence with the boundary condition

$$\gamma := (13) = (1 \rightarrow 3, \ 2 \rightarrow 2, \ 3 \rightarrow 1 ) \in S_3.$$  \hfill (6.1)

From Figure 4 we can read off the c-vectors of the mutating vertices:

$$\alpha_1 = c_1(0) = (1, 0, 0), \quad \alpha_2 = c_2(1) = (0, 1, 0), \quad \alpha_3 = c_3(2) = (1, 0, 1), \quad \alpha_4 = c_4(3) = (0, 0, 1).$$

The $s$-variables change as follows (cf. (3.3)):

| $Q(0)$ | $1$ | $2$ | $3$ |
|-------|-----|-----|-----|
| $s_1$ | $s_2$ | $s_3$ |
| $Q(1)$ | $s_1' = -s_1 + s_3 + k_1$ | $s_2$ | $s_3$ |
| $Q(2)$ | $s_1' = -s_1 + s_3 + k_1$ | $s_2' = k_2 - s_2$ | $s_3$ |
| $Q(3)$ | $s_1' = -s_1 + s_3 + k_1$ | $s_2' = k_2 - s_2$ | $s_3' = -s_3 + s_1' + k_3$ |
| $Q(4)$ | $s_1'' = -s_1' + s_3' + k_4$ | $s_2' = k_2 - s_2$ | $s_3' = -s_3 + s_1' + k_3$ |

The boundary condition (6.1) imposes the relation

$$s_1'' = s_3, \quad s_2' = s_2, \quad s_3' = s_1.$$  \hfill (6.2)

From these relations, we can express $s$-variables in terms of $k$-variables:

$$s_1 = s_3' = \frac{1}{2} (k_1 + k_3), \quad s_1' = \frac{1}{2} (k_1 + k_4), \quad s_2 = s_2' = \frac{k_2}{2}, \quad s_3 = s_3'' = \frac{1}{2} (k_3 + k_4).$$

The $k^\vee$-variables are then

$$k_1^\vee = s_1 + s_1' - s_2 = k_1 - \frac{k_2}{2} + \frac{k_3}{2} + \frac{k_4}{2},$$

$$k_2^\vee = s_2 + s_2' - s_1 = -\frac{k_2}{2} + k_2 - \frac{k_3}{2},$$

$$k_3^\vee = s_3 + s_3' = \frac{k_2}{2} + k_3 + \frac{k_4}{2},$$

$$k_4^\vee = s_1' + s_2' - s_3 = \frac{k_1}{2} - \frac{k_2}{2} + \frac{k_3}{2} + k_4.$$  \hfill (6.3)

Plugging these into the definition of mutation weights (5.5) and summing over $k$-variables, we obtain

$$Z(\gamma) = \sum_{k \in \mathbb{N}^4} \frac{q^\frac{1}{2}(k_1^2 + k_2^2 + k_3^2 + k_4^2 - k_1 k_2 + k_1 k_3 + k_1 k_4 - k_2 k_4 + k_3 k_4)}{(q)_k (q)_2 (q)_3 (q)_4} y^{(k_1 + k_3, k_2, k_3, k_4)}.$$  \hfill (6.4)
6.2. **Square product** $A_3 \square A_2$. As an example of the quivers of square product type (see [9] for definition), consider

\[ Q = A_3 \square A_2 = \begin{array}{c}
1 \rightarrow 3 \rightarrow 5 \\
2 \rightarrow 4 \rightarrow 6
\end{array} \]

One can check that

\[ m = (1, 4, 5, 2, 3, 6, 1, 4, 5) \]

is a reverse-ending sequence with the boundary condition

\[ \varphi = (12)(34)(56) = \begin{pmatrix} 1 \ 2 \\ 2 \ 1 \end{pmatrix} \begin{pmatrix} 3 \ 4 \ 5 \ 6 \\ 3 \ 4 \ 5 \ 6 \end{pmatrix} \in S_6. \]

Let $k = (k_1, \ldots, k_9)$ be the $k$-variables corresponding to mutation sequence (6.4). The evolution of $s$-variables along the mutation loop is summarized as follows:

- $s_1 \mapsto s'_1 = k_1 - s_1 + s_2 \mapsto s''_1 = k_1 - s'_1 + s'_2 = s_2$
- $s_2 \mapsto s'_2 = k_4 - s_2 + s'_1 \mapsto s_3 = k_5 - s_3 + s'_4 = s_4$
- $s_3 \mapsto s'_3 = k_5 - s_3 + s'_4 \mapsto s'_4 = k_8 - s'_4 + s'_3 = s_3$
- $s_4 \mapsto s'_4 = k_2 - s_4 + s_3 \mapsto s'_5 = k_9 - s'_5 + s'_6 = s_6$
- $s_5 \mapsto s'_5 = k_3 - s_5 + s_6 \mapsto s'_6 = k_6 - s_6 + s'_5 = s_5$
- $s_6 \mapsto s'_6 = k_6 - s_6 + s'_5 \mapsto s'_7 = k_8 - s'_7 + s'_6 = s_6$

One can express all $s$-variables in terms of $k$-variables:

- $s_1 = s'_1 = (k_1 + k_4) / 2$
- $s_2 = s'_2 = (k_4 + k_7) / 2$
- $s_3 = s'_3 = (k_5 + k_8) / 2$
- $s_4 = s'_4 = (k_2 + k_5) / 2$
- $s_5 = s'_5 = (k_3 + k_6) / 2$
- $s_6 = s'_6 = (k_6 + k_9) / 2$
- $s'_1 = (k_1 + k_7) / 2$
- $s'_2 = (k_2 + k_8) / 2$
- $s'_3 = (k_3 + k_9) / 2$
The $c$-vectors of mutating vertices are
\[ \alpha_1 = (100000), \quad \alpha_2 = (001000), \quad \alpha_3 = (000010), \]
\[ \alpha_4 = (110000), \quad \alpha_5 = (001100), \quad \alpha_6 = (000011), \]
\[ \alpha_7 = (010000), \quad \alpha_8 = (001000), \quad \alpha_9 = (000001). \]
We obtain the partition $q$-series
\[ Z(\gamma) = \sum_{k \in \mathbb{N}} q^{\frac{1}{2} k \top A k} \prod_{i=1}^{9} (q)^{k_i} \beta(k) \]
where
\[ \beta(k) = (k_1 + k_4, k_4 + k_7, k_5 + k_8, k_2 + k_5, k_3 + k_6, k_6 + k_9) \]
and $A$ is a symmetric $9 \times 9$ matrix given by
\[
A = \begin{pmatrix}
A' & A'' & A'' \\
A'' & A' & A'' \\
A'' & A'' & A'
\end{pmatrix}, \quad A' = \begin{pmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \end{pmatrix}, \quad A'' = \begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}.
\]

Remark 6.1. The mutation loop (6.4) is different from the one considered in our previous work:
\[ \gamma' = (Q, m', \text{id}) \quad m' = (1, 4, 5, 2, 3, 6). \]
(See Theorem 6.1 of [9] and the example therein.) Although $\mu_{m'}(Q)$ is isomorphic to $Q$, $m'$ is not a reverse-ending sequence. The sequence $m$ contains $m'$ as a proper subsequence.

6.3. Octahedral quiver. Here is another example of non-alternating quiver — the octahedral quiver:

\[ Q = \begin{array}{c}
3 \\
\downarrow \\
2
\end{array} \quad 4 \\
\downarrow \\
1
\quad 5 \\
\downarrow \\
6
\]

The mutation sequence
\[ m = (1, 2, 5, 6, 3, 4, 1, 2, 5, 6, 3, 4) \]

Together with the boundary condition
\[ \varphi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 5 & 6 & 4 & 2 & 3
\end{pmatrix} \in S_6. \]

Form a reverse-ending, maximal green mutation loop $\gamma = (Q, m, \varphi)$ of length $T = 12$. Indeed, the $c$-matrix of the final quiver $Q(T)$ is given by
\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix}
\]
where is a (negative of) permutation matrix corresponding to \( \varphi \).

Let \( k = (k_1, \ldots, k_{12}) \) be the \( k \)-variables corresponding to mutation sequence \( m \). In this example, every vertex is mutated twice. The evolution of \( s \)-variables along the mutation loop is summarized as follows:

\[
\begin{align*}
    s_1 & \mapsto s'_1 = k_1 - s_1 + s_3 + s_6 & \implies s''_1 = k_7 - s'_1 + s'_3 + s'_6 &= s_1 \\
    s_2 & \mapsto s'_2 = k_2 - s_2 + s_4 & \implies s''_2 = k_8 - s'_2 + s'_4 &= s_5 \\
    s_3 & \mapsto s'_3 = k_5 - s_3 + s'_1 & \implies s''_3 = k_{11} - s'_3 + s'_1 &= s_6 \\
    s_4 & \mapsto s'_4 = k_6 - s_4 + s'_2 + s'_5 & \implies s''_4 = k_{12} - s'_4 + s''_2 + s''_5 &= s_4 \\
    s_5 & \mapsto s'_5 = k_3 - s_5 + s_4 & \implies s''_5 = k_9 - s'_5 + s'_4 &= s_2 \\
    s_6 & \mapsto s'_6 = k_4 - s_6 + s'_1 & \implies s''_6 = k_{10} - s'_6 + s'_1 &= s_3
\end{align*}
\]

Solving these, we can express all \( s \)-variables in terms of \( k \)-variables:

\[
\begin{align*}
    s_1'' &= (k_1 + k_4 + k_5 + k_7) / 2, & s_2'' &= (k_2 + k_6 + k_9) / 2, \\
    s_3'' &= (k_5 + k_7 + k_{10}) / 2, & s_4'' &= (k_6 + k_8 + k_9 + k_{12}) / 2, \\
    s_5'' &= (k_3 + k_6 + k_8) / 2, & s_6'' &= (k_4 + k_7 + k_{11}) / 2, \\
    s'_1 &= (k_1 + k_7 + k_{10} + k_{11}) / 2, & s'_2 &= (k_2 + k_8 + k_{12}) / 2, \\
    s'_3 &= (k_3 + k_5 + k_{11}) / 2, & s'_4 &= (k_2 + k_3 + k_6 + k_{12}) / 2, \\
    s'_5 &= (k_3 + k_9 + k_{12}) / 2, & s'_6 &= (k_1 + k_4 + k_{10}) / 2.
\end{align*}
\]

The partition \( q \)-series is now given by

\[
Z(\gamma) = \sum_{k \in \mathbb{N}^{12}} \frac{q^{\frac{1}{2}k^TAk}}{\prod_i(q)_{k_i}} y^{\beta(k)},
\]

where

\[
\begin{align*}
\beta(k) &= (k_1 + k_4 + k_5 + k_7, k_2 + k_6 + k_9, k_5 + k_7 + k_{10}, \nonumber \\
&\phantom{=} k_6 + k_8 + k_9 + k_{12}, k_3 + k_6 + k_8, k_4 + k_7 + k_{11}) \in \mathbb{N}^6
\end{align*}
\]

and \( A \) is the \( 12 \times 12 \) symmetric matrix of the following form:

\[
A = \begin{pmatrix}
A' & A'' \\
A'' & A'
\end{pmatrix},
\]

\[
A' = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & -1 & 2 \\
-1 & 2 & 0 & 0 & 0 & 1 \\
-1 & 0 & 2 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 & -1 \\
1 & 0 & 0 & 0 & 2 & -1 \\
-2 & 1 & 1 & -1 & -1 & 2
\end{pmatrix}, \quad A'' = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 \\
0 & 1 & 1 & -1 & -1 & 2
\end{pmatrix}.
\]

By Theorem 5.7, the partition \( q \)-series \[(6.6)\] is equal to \( E(Q; m) \), where

\[
E(Q; m) = E\left( y^{(1000000)} \right) E\left( y^{(0100000)} \right) E\left( y^{(0001000)} \right) E\left( y^{(0000100)} \right) E\left( y^{(1010000)} \right) E\left( y^{(1001000)} \right) E\left( y^{(0101000)} \right) E\left( y^{(0101010)} \right)
\times E\left( y^{(0101011)} \right) E\left( y^{(0001010)} \right) E\left( y^{(0101000)} \right) E\left( y^{(0001000)} \right) E\left( y^{(0000100)} \right) E\left( y^{(0000010)} \right) E\left( y^{(0000001)} \right) E\left( y^{(0000000)} \right).
\]

is the product of quantum dilogarithms, \( E(y^\alpha) = E(y^\alpha; q) \).
6.4. **Alternating quivers.** A vertex $i$ of a quiver is a *source* (respectively, a *sink*) if there are no arrows $\alpha$ with target $i$ (respectively, with source $i$). A quiver is *alternating* if each of its vertices is a source or a sink. Denote by $Q^+_0$ ($Q^-_0$) the set of all sources (sinks) of the alternating quiver $Q$, respectively. Since $Q_0 = Q^+_0 \cup Q^-_0$, the underlying graph $Q_0$, a graph obtained by forgetting the orientation of arrows, is bipartite.

For an alternating quiver $Q$, there is a simple recipe for constructing a reverse-ending mutation loop.

**Proposition 6.2.** Suppose $Q$ is an alternating quiver, and $m_\pm$ be arbitrary permutations of $Q^\pm_0$, respectively. Let $m = m_+ m_-$ be their concatenation, considered as a mutation sequence of length $n = |Q_0|$. Then, the $c$-vectors of $Q(t)$ are given by

$$c_i(t) = \begin{cases} e_i & \text{if } i \notin \{m_1, \ldots, m_t\}, \\ -e_i & \text{if } i \in \{m_1, \ldots, m_t\}, \end{cases} \quad (0 \leq t \leq n).$$

In particular, the sequence $m$ is maximal green and $\gamma = (Q; m_+, m_-, \text{id})$ is a reverse-ending mutation loop.

**Proof.** First note that $m$ is a *source sequence*, that is, each mutating vertex $m_t$ is a source of $Q(t - 1)$ for all $1 \leq t \leq n$. To see this, it is helpful to consider $m_+$ and $m_-$ separately. The claim is clear for the mutation sequence $m_+$ applied on $Q$. When the mutation sequence $m_+$ is over, we have $\mu_{m_+}(Q) = Q^{op}$; here $Q^{op}$ is the quiver obtained by reversing all the arrows in $Q$. Now all the vertices in $m_-$ are sources of $\mu_+(Q) = Q^{op}$, so $m_-$ is also a source sequence.

Since only source vertices are mutated, mutation rules 1) and 3) are never used; mutations change only the orientations of arrows. The underlying graph $Q$ remains the same.

Let $M(t) := \{m_1, \ldots, m_t\} \subset Q_0$ be the set of mutated vertices during the first $t$ mutations. We prove (6.7) by induction on $t$. The claim holds for $t = 0$, since $M(0) = \emptyset$ and $c_i(0) = e_i$ for all $i$. Suppose the claim is true for $0, 1, \ldots, t - 1$. Then the mutation $\mu_{m_t} : Q(t - 1) \to Q(t)$ is green because $m_t \notin \{m_1, \ldots, m_{t-1}\}$ and thus $c_{m_t}(t - 1) = e_{m_t} \in \mathbb{N}^n$ by induction hypothesis. Moreover, $Q(t - 1)|_{m_t} = 0$ since $m_t$ is a source of $Q(t - 1)$, as we have seen above. Thus by (2.11), the $c$-vectors change as

$$c_i(t) = \begin{cases} -c_i(t - 1) & \text{if } i = m_t, \\ c_i(t - 1) & \text{if } i \neq m_t. \end{cases}$$

With $M(t) = M(t - 1) \cup \{m_t\}$, this shows that the claim is also true for $t$. The rest of the proposition follows immediately from (6.7).

Let us compute $Z(\gamma)$ for the reverse-ending loop $\gamma = (Q; m_+ m_-, \text{id})$. Note that the sequence $m = (m_1, \ldots, m_n)$ is a permutation of $(1, \ldots, n)$. Every vertex $i$ is mutated exactly once, and the initial and final $s$-variables $s_i, s'_i$ are identified by the boundary condition $\varphi = \text{id}$. As we will soon see, it is convenient to label $k$-variables not by the mutation time but by the vertex label. From now on, $k_i$ will denote the $k$-variable associated with the mutation at vertex $i$, rather than $i$-th mutation.

To compute the weight for $\gamma$, it suffices to know the underlying graph $Q$, because we can recover arrow orientations from the fact that “every mutation occurs at a
source”. All the information of $Q$ is encoded in the generalized Cartan matrix

$$
(C)_{ij} = \begin{cases} 
2 & \text{if } i = j, \\
-(Q_{ij} + Q_{ji}) & \text{if } i \neq j.
\end{cases}
$$

Before stating the general result for $Z(\gamma)$ (Theorem 6.3), let us take an example — an alternating quiver of affine $D_5$ type:

$$Q = \begin{array}{c}
1 \\
\downarrow \\
2 \\
\rightarrow \\
3 \\
\downarrow \\
4 \\
\rightarrow \\
5 \\
\downarrow \\
6
\end{array}$$

The generalized Cartan matrix of $Q$ is given by

$$C = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 \\
-1 & 1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}.
$$

Put $m_+ = (1, 2, 4)$ and $m_- = (3, 5, 6)$. By Proposition 6.2, the mutation sequence

$$m = m_+ m_- = (1, 2, 4, 3, 5, 6)$$

is maximal green, reverse-ending sequence with the boundary condition $\varphi = \text{id}$ (see Figure 5).

The $s$-variables change as follows:

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Q(0)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ |
| $Q(1)$ | $s_1'$ = $k_1 - s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ |
| $Q(2)$ | $s_1'$ = $k_1 - s_1$ | $s_2'$ = $k_2 - s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ |
| $Q(3)$ | $s_1'$ = $k_1 - s_1$ | $s_2'$ = $k_2 - s_2$ | $s_3$ | $s_4'$ = $k_4 - s_4$ | $s_5$ | $s_6$ |
| $Q(4)$ | $s_1'$ = $k_1 - s_1$ | $s_2'$ = $k_2 - s_2$ | $s_3'$ = $k_3 - s_3$ | $s_4'$ = $k_4 - s_4$ | $s_5$ | $s_6$ |
| $Q(5)$ | $s_1'$ = $k_1 - s_1$ | $s_2'$ = $k_2 - s_2$ | $s_3'$ = $k_3 - s_3$ | $s_4'$ = $k_4 - s_4$ | $s_5'$ = $k_5 - s_5$ | $s_6$ |
| $Q(6)$ | $s_1'$ = $k_1 - s_1$ | $s_2'$ = $k_2 - s_2$ | $s_3'$ = $k_3 - s_3$ | $s_4'$ = $k_4 - s_4$ | $s_5'$ = $k_5 - s_5$ | $s_6' = k_6 - s_6$

The boundary condition $\varphi = \text{id}$ imposes $s_i = s_i' = k_i - s_i$ for all $i$, so we have

$$s_i = s_i' = \frac{1}{2} k_i \quad (i = 1, \ldots, 6).$$

The $k^\gamma$-variables (also labeled by mutated vertices) are then given by

- $k_1' = s_1 + s_1' - s_3 = k_1 - \frac{k_1}{2}$,
- $k_2' = s_2 + s_2' - s_3 = k_2 - \frac{k_2}{2}$,
- $k_3' = s_3 + s_3' - (s_4 + s_4') = -\frac{k_3}{2} + k_4 = \frac{k_3}{2} - \frac{k_4}{2}$,
- $k_4' = s_4 + s_4' - (s_3 + s_5 + s_6) = -\frac{k_4}{2} + k_3 - \frac{k_5}{2}$,
- $k_5' = s_5 + s_5' - s_4' = -\frac{k_5}{2} + k_6$, $k_6' = s_6 + s_6' - s_4' = -\frac{k_6}{2} + k_5$. 

Exercise 6.4
Every mutating vertex $m_t$ is green with $c$-vector $\alpha_t = e_{m_t}$. The $\mathbb{N}^6$-grading of the mutation sequence $m$ is then

$$\sum_{t=1}^6 k_{m_t} e_{m_t} = \sum_{i=1}^6 k_i e_i = (k_1, k_2, k_3, k_4, k_5, k_6) = k \in \mathbb{N}^6.$$ 

Combining all these, we obtain a neat expression for the partition $q$-series:

$$(6.11) \quad \mathcal{Z}(\gamma) = \sum_{k \in \mathbb{N}^6} q^{\frac{1}{2} e^T C k} \prod_{i=1}^6 (q)_{k_i} y^k,$$

where $C$ is nothing but the generalized Cartan matrix $(6.10)$.

In fact, this generalize to all alternating quivers:

**Theorem 6.3.** Suppose $Q$ is an alternating quiver, and $\gamma = (Q; m, \text{id})$ be the reverse-ending mutation loop constructed as in Proposition 6.2. Let $k = (k_1, \ldots, k_n)$ be the vector of $k$-variables indexed by the vertices. Then the partition $q$-series is
given by

\[ Z(\gamma) = \mathbb{E}(m; q) = \sum_{k \in \mathbb{N}^{n}} q^{\frac{1}{2} \sum_{i=1}^{n} q_{i} C k_{i}} y^{k}, \]

where \( C \) is the generalized Cartan matrix of \( Q \).

**Proof.** From Proposition 6.2, \( \varepsilon_{t} = 1 \) and \( c_{m_{t}}(t - 1) = e_{m_{t}} \) for all mutation time \( 1 \leq t \leq n \). Thus we have \( \alpha_{t} = e_{m_{t}} \) in (3.7). The \( \mathbb{N}^{n} \)-grading is therefore given by

\[ \sum_{n=1}^{t} k_{m_{t}} e_{m_{t}} = \sum_{i=1}^{n} k_{i} e_{i} = k. \]

Consider a mutation at vertex \( i \). As we have seen, \( i \) is a source and there is no arrow ending on \( i \). The initial (=final) \( s \)-variable and the \( k \)-variable are thus related as \( 2s_{i} = k_{i} \), so we have

\[ s_{i} = \frac{k_{i}}{2} \quad (1 \leq i \leq n). \]

The \( k^{\gamma} \)-variables are then expressed as

\[ k_{i}^{\gamma} = 2s_{i} - \sum_{i \rightarrow j} s_{j} = 2s_{i} - \sum_{i \rightarrow j} s_{j} = k_{i} - \frac{1}{2} \sum_{i \sim j} k_{j}, \]

Here \( i \sim j \) means the vertices \( i \) and \( j \) are adjacent in the underlying graph \( Q \). Using the generalized Cartan matrix (6.8), the relation (6.14) is concisely written as

\[ k^{\gamma} = \frac{1}{2} C k. \]

Thus the partition \( q \)-series is given by

\[ Z(\gamma) = \sum_{k_{1}, \ldots, k_{n} \geq 0} \left( \prod_{i=1}^{n} q_{i}^{\frac{1}{2} k_{i}^{\gamma}} \right) y^{k} = \sum_{k \in \mathbb{N}^{n}} q^{\frac{1}{2} \sum_{i=1}^{n} q_{i} C k_{i}} y^{k}. \]

The equality \( Z(\gamma) = \mathbb{E}(m; q) \) follows from Theorem 5.7. \( \square \)

**Remark 6.4.** In our previous work, we computed the partition \( q \)-series for the product of Dynkin quivers and observed that they are fermionic character formulas of certain conformal field theories (Theorem 6.1 of [9]). The case considered here are different from those because (i) \( Q \square Q' \) is not alternating in general, and (ii) the sequences given in [9] are not reverse-ending. However, \( Q \square Q' = X \square A_{1} \) with \( C_{Q'} = (2) \) are exceptional cases to which Theorem 5.7 is applicable.

**Remark 6.5.** The paper [2] propose a relation between four-dimensional gauge theories and parafermionic conformal field theories. In particular, they claim that the \( L^{2} \)-trace of a special product of quantum dilogarithms associated with a Dynkin diagram is written in terms of characters. It would be interesting to find a precise relation with their work.

**Appendix A. Some identities related with quantum dilogarithm**

**Proposition A.1.**

\[ \mathbb{E}(y; q) \mathbb{E}(y; q^{-1}) = 1 \]

**Proof.** This follows from, for example by exchanging \( q \leftrightarrow q^{-1} \) in the expression (5.4). \( \square \)
Corollary A.2.

\[ \sum_{r,s \geq 0 \atop r+s=n} q^{\frac{1}{2}r^2} q^{-\frac{1}{2}s^2} \frac{1}{(q)_r (q^{-1})_s} = \delta_{n,0} \quad (n = 0, 1, 2, \ldots). \]  

Proof. This is proved by expanding (A.1) as a series in \( y \), and taking the coefficient of \( y^n \). An alternative proof goes as follows. We begin by the \( q \)-binomial formula

\[ n - 1 \prod_{k=0}^{n-1} (1 + q^k x) = \sum_{r=0}^{n} q^{r(r+1)/2} (q)_r (q^{-1})_{n-r} x^r. \]  
The right hand side of (A.3) can be written as

\[ \sum_{r,s \geq 0 \atop r+s=n} q^{r(r-1)/2} (q)_r (q^{-1})_s x^r = \sum_{r,s \geq 0 \atop r+s=n} \frac{(q)_n}{(q)_r (q^{-1})_s} q^{r(r-1)/2} x^r. \]  

By putting \( x = -1 \) into (A.3), we have

\[ \prod_{k=0}^{n-1} (1 - q^k) = (-1)^n q^{n^2/2} (q)_n \sum_{r,s \geq 0 \atop r+s=n} \frac{1}{(q)_r (q^{-1})_s} q^{r^2-s^2}. \]  
The left hand side of (A.4) is 1 if \( n = 0 \), and 0 otherwise. \( \square \)

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