CHERN FORMS OF HERMITIAN METRICS WITH ANALYTIC SINGULARITIES ON VECTOR BUNDLES

RICHARD LÄRKÄNG & Hossein RAUFi & MARTIN SERA & ELIZABETH WULCAN

Abstract. We define Chern and Segre forms, or rather currents, associated with a Griffiths positive singular hermitian metric $h$ with analytic singularities on a holomorphic vector bundle $E$. The currents are constructed as pushforwards of generalized Monge-Ampère products on the projectivization of $E$. The Chern and Segre currents represent the Chern and Segre classes of $E$, respectively, and coincide with the Chern and Segre forms of $E$ and $h$ where $h$ is smooth. Moreover, our currents coincide with the Chern and Segre forms constructed by the first three authors and Ruppenthal in the cases when these are defined.

1. Introduction

Singular metrics on line bundles were introduced by Demailly in [De4], and have since developed to be an influential analytic tool in complex algebraic geometry. In [BP] Berndtsson and Păun introduced singular metrics on vector-bundles in order to prove results about pseudo-effectivity of relative canonical bundles. These have been further studied in a series of papers including, e.g., [H,HPS,R]. In order to develop a theory for singular metrics on vector bundles it seems crucial to have Chern forms. In the line bundle case the (first) Chern form is a well-defined current, whereas any attempt to construct Chern forms of singular metrics on higher rank bundles seems to involve multiplication of currents. In [LRRS] the first three authors together with Ruppenthal defined Chern forms for positive singular metrics on vector bundles under a certain natural condition on the dimension of the degeneracy locus. In this paper we define Chern forms without this assumption but for metrics with so-called analytic singularities. To do this we develop a new formalism for generalized (mixed) Monge-Ampère operators for plurisubharmonic functions with analytic singularities extending the construction in [AW].

Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$ of dimension $n$ and let $h$ be a smooth hermitian metric on $E$. Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of lines in $E^*$. Then $h^*$ induces a metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^*E^*$; let $e^{-\varphi}$ be the dual metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. If $h$ is Griffiths positive, then $e^{-\varphi}$ is a positive metric, i.e., the local weights $\varphi$ are plurisubharmonic (psh), and the first Chern form of $e^{-\varphi}$ is given as $dd^c\varphi$, where $d^c = (1/4\pi i)(\partial - \bar{\partial})$. Note that this is a well-defined global positive $(1,1)$-form. Following the ideas in, e.g., [F] one can define the associated $k$th Segre form as

\begin{equation}
    s_k(E,h) := (-1)^k \pi_* (dd^c \varphi)^{k+r-1},
\end{equation}

cf. [M] Section 7.1. Since $\pi$ is a submersion this is a smooth form of bidegree $(k,k)$. It was proved in [M] Proposition 6], see also [Di] Proposition 1.1 and [G] Proposition 3.1, that (1.1) coincides with the classical definition of Segre forms, which means that the total Segre form $s(E,h) = 1 + s_1(E,h) + s_2(E,h) + \cdots$ is the multiplicative inverse of the total Chern form $c(E,h) = 1 + c_1(E,h) + c_2(E,h) + \cdots$. Identifying components of the same bidegree, this can be expressed as

\begin{equation}
    s_k(E,h) + s_{k-1}(E,h) \wedge c_1(E,h) + \cdots + c_k(E,h) = 0, \ k = 1,2,\ldots .
\end{equation}
In particular, \((1.2)\) holds on cohomology level, i.e., the total Segre class \(s(E) = 1 + s_1(E) + s_2(E) + \cdots\) is the multiplicative inverse of the total Chern class \(c(E) = 1 + c_1(E) + c_2(E) + \cdots\); here \(c_k(E)\) and \(s_k(E)\) are the \(k\)th Chern and Segre classes of \(E\), defined as the de Rham cohomology classes of \(c_k(E, h)\) and \(s_k(E, h)\), respectively.

The aim of this paper is to construct Chern and Segre forms, or rather currents, associated with singular metrics. Therefore let \(h\) be a Griffiths positive singular metric on \(E\) in the sense of Berndtsson-Păun, \(\text{BP}\), see Section 5 then the induced singular metric \(e^{-\varphi} \ddc\) on \(\mathcal{O}_P(E)\) is positive, cf. Proposition 5.2. Our strategy is to mimic the construction \((1.1)\) of Segre forms and use them to construct Chern and Segre currents. However, in general one cannot take products of currents and in particular \((ddc\varphi)^k\) is not always well-defined.

Recall that a psh function \(u\) has analytic singularities if it is locally of the form

\[
u = c \log |F|^2 + v,
\]

where \(c > 0\), \(F\) is a tuple of holomorphic functions \(f_j\), \(|F|^2 = \sum |f_j|^2\), and \(v\) is bounded. We say that \(h\) has analytic singularities if the weights \(\varphi\) are psh with analytic singularities; for a direct definition in terms of \(h\), see Proposition 5.4. In [H] Hosono constructed a class of examples of singular hermitian metrics on vector bundles, that in fact have analytic singularities, see Example 5.5.

Given a psh function \(u\) with analytic singularities, in [AW] Andersson and the last author defined generalized Monge-Ampère products \((ddc u)^m\) recursively as

\[
(ddc u)^k := ddc (u1_{X \setminus Z}(ddc u)^{k-1}),
\]

for \(Z\) the unbounded locus of \(u\), i.e., locally defined as \(\{ F = 0 \}\) where \(u\) is given by \((1.3)\); for \(u\) of the form \(u = \log |F|^2\) the currents \((1.4)\) were defined in [A]. The current \((ddc u)^m\) is positive and closed and of bidegree \((m, m)\). For \(m \leq \text{codim } Z\), it coincides with Bedford-Taylor-Demailly’s classically defined \((ddc u)^m\), cf. Section 2.1. If \(\alpha\) is a closed smooth \((1, 1)\)-form, inspired by [ABW] Theorem 1.2], cf. Remark 3.6, we let

\[
(ddc u)^m_{\alpha} := (ddc u)^m + \sum_{\ell=0}^{m-1} \alpha^{m-\ell}1_Z (ddc u)^{\ell},
\]

for \(m \geq 1\) and \((ddc u)^0 = 1\); see (3.11) for a recursive description. Note that if \(m \leq \text{codim } Z\), then \(1_Z (ddc u)^{\ell} = 0\) for \(\ell < m\) and thus \((ddc u)^m_{\alpha} = (ddc u)^m\).

Now assume that \(h\) has analytic singularities and let \(\theta\) be the first Chern form of a smooth metric \(e^{-\psi}\) on \(\mathcal{O}_P(E)(1)\); e.g., \(e^{-\psi}\) can be chosen as the metric on \(\mathcal{O}_P(E)(1)\) induced by a smooth metric on \(E\). Since the difference of two local weights \(\varphi\) is of the form \(\log |f|^2\), where \(f\) is a nonvanishing holomorphic function, \([ddc \varphi]_m\) is a globally defined current on \(P(E)\), see Section 4.

Inspired by \((1.1)\) we define

\[
s_k(E, h, \theta) := (-1)^k \pi_* [(ddc \varphi)^{k+r-1}].
\]

If the \(\varphi\) are smooth, then clearly \(s_k(E, h, \theta)\) coincides with \(s_k(E, h)\) defined by \((1.1)\).

To construct Chern currents we need to define products of this kind of currents. Let \(E_1, \ldots, E_t\) be disjoint copies of \(E\) and let \(\pi : Y \to X\) be the fiber product \(Y = P(E_1) \times_X \cdots \times_X P(E_t)\). Let \(\varphi_j\) and \(\theta_j\) denote the pullbacks to \(Y\) of the metric and the form on \(P(E_j)\) corresponding to \(\varphi\) and \(\theta\), respectively. By extending ideas in [AW] and [ASWY] we give meaning to products

\[
[ddc \varphi_1]^m_{\theta_1} \wedge \cdots \wedge [ddc \varphi_1]^m_{\theta_1}
\]
on \(Y\), see Sections 3 and 4.

Next for \(k_j \geq 1\) we define

\[
s_k(E, h, \theta) \wedge \cdots \wedge s_k(E, h, \theta) := (-1)^k \pi_* \left( [ddc \varphi_1]^k_{\theta_1} \wedge \cdots \wedge [ddc \varphi_1]^k_{\theta_1} \right),
\]

see Section 5.1 and throughout \(k := k_1 + \cdots + k_t\). If \(h\), and thus \(\varphi\), is smooth, then \((1.7)\) just coincides with the product \(s_k(E, h) \wedge \cdots \wedge s_k(E, h)\) of smooth Segre forms, cf. Section 6.2.
The currents \((1.7)\) are in general not commutative in the factors \(s_{k_j}(E,h,\theta)\), see Example \[8.4\] Now we can recursively define Chern currents \(c_k(E,h,\theta)\) using the identities \((1.2)\), i.e.,

\[
\begin{align*}
 c_1(E,h,\theta) &:= -s_1(E,h,\theta), \\
 c_2(E,h,\theta) &:= s_1(E,h,\theta)^2 - s_2(E,h,\theta), \\
 &\vdots \\
 c_k(E,h,\theta) &:= \sum_{k_1+\cdots+k_i=k} (-1)^i s_{k_i}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta).
\end{align*}
\]

Theorem 1.1. Let \(h\) be a Griffiths positive hermitian metric with analytic singularities on the holomorphic vector bundle \(E \to X\) over a complex manifold \(X\) of dimension \(n\) and let \(\theta\) be the first Chern form of a smooth metric on \(\mathcal{O}_{\mathbf{P}(E)}(1)\). Then for \(k = 1,2,\ldots\) \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) defined by \((1.6)\) and \((1.8)\), respectively, are closed normal \((k,k)\)-currents; more precisely they are locally differences of closed positive currents. Moreover

1. \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) represent the \(k\)th Chern and Segre classes \(c_k(E)\) and \(s_k(E)\) of \(E\), respectively, as de Rham cohomology classes of currents,
2. \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) coincide with the Chern and Segre forms \(c_k(E,h)\) and \(s_k(E,h)\), respectively, where \(h\) is smooth,
3. the Lelong numbers of \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) at each \(x \in X\) are independent of \(\theta\).

Note that Lelong numbers of \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) are well-defined since the currents are locally differences of closed positive currents, cf. Section \[6.4\].

Assume that the unbounded locus of \(\log \det h^*\) is contained in a variety \(V\) of pure codimension \(p\). Then for \(k \leq p\), \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) are independent of \(\theta\), see Section \[8\] In general, however, \(c_k(E,h,\theta)\) and \(s_k(E,h,\theta)\) do depend on \(\theta\), cf. Examples \[8.1\] \[8.2\] and \[8.4\].

In [LRRS] the first three authors together with Ruppenthal showed that if \(h\) is a singular hermitian metric (not necessarily with analytic singularities) such that the unbounded locus of \(\log \det h^*\) is contained in a variety \(V\) of pure codimension \(p\), then for \(k_1+\cdots+k_t \leq p\) one can give meaning to currents \(s_{k_1}(E,h) \wedge \cdots \wedge s_{k_1}(E,h)\) as limits of \(s_{k_1}(E,h_{\varepsilon_j}) \wedge \cdots \wedge s_{k_1}(E,h_{\varepsilon_j})\), where \(h_{\varepsilon_j}\) are smooth metrics approximating \(h\), see Section \[7\]. Analogously to \((1.8)\) one can then define Chern currents \(c_k(E,h)\) for \(k \leq p\). We should remark that this construction cannot be extended to general \(k\), see Example \[8.1\].

Theorem 1.2. Let \(h\) be a Griffiths positive hermitian metric with analytic singularities on the holomorphic vector bundle \(E \to X\) over a complex manifold \(X\) and let \(\theta\) be the first Chern form of a smooth metric on \(\mathcal{O}_{\mathbf{P}(E)}(1)\). Assume that the unbounded locus of \(\log \det h^*\) is contained in a variety \(V \subset X\). Then for \(k_1+\cdots+k_t \leq \text{codim} V\),

\[(1.9)\]

\[s_{k_1}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta) = s_{k_t}(E,h) \wedge \cdots \wedge s_{k_1}(E,h)\].

In particular \(c_k(E,h,\theta) = c_k(E,h)\) for \(k \leq \text{codim} V\).

The paper is organized as follows. In Section \[2\] we give some background on currents and classical Monge-Ampère products. In Section \[3\] we introduce mixed Monge-Ampère products of psh functions with analytic singularities generalizing \((1.4)\). Next in Sections \[4\] and \[5\] we study metrics with analytic singularities on line bundles and vector bundles, respectively. The proofs of Theorems \[(1.1)\] and \[(1.2)\] occupy Sections \[6\] and \[7\] respectively. Finally, in Section \[8\] we conclude with some examples and remarks.

2. Preliminaries

Let us first recall some results on (closed positive) currents. If \(\pi : \tilde{X} \to X\) is a proper map, \(\mu\) is a current on \(\tilde{X}\), and \(\alpha\) is a smooth form on \(X\), then we have the projection formula

\[(2.1)\]

\[\alpha \wedge \pi_* \mu = \pi_*(\pi^* \alpha \wedge \mu)\].
Moreover, if \( p \) is a proper submersion, \( \mu \) is a current on \( X \), and \( \alpha \) is a smooth form on \( \tilde{X} \), then
\[
p_* \alpha \wedge \mu = p_*(\alpha \wedge \rho^* \mu).
\]

The Poincaré-Lelong formula asserts that if \( f \) is a holomorphic function defining a divisor \( D \), then
\[
d\!d^c \log |f|^2 = |D|,
\]
where \([D]\) is the current of integration along the divisor of \( f \).

Given a subset \( A \subset X \), let \( 1_A \) denote the characteristic function of \( A \). If \( Z \subset X \) is a subvariety and \( T \) is a closed positive current on \( X \), then the Skoda-El Mir theorem asserts that \( 1_X \setminus Z \) is again positive and closed. It follows that if \( U \subset X \) is any constructible set, i.e., a set in the Boolean algebra generated by Zariski open sets in \( X \), then also \( 1_U T \) is positive and closed. Note that if \( U_1 \) and \( U_2 \) are two constructible sets in \( X \), then
\[
1_{U_1 \cap U_2} T = 1_{U_1} 1_{U_2} T.
\]

Also, note that if \( \chi_\epsilon \) is any sequence of bounded functions such that \( \chi_\epsilon \to 1_U \) pointwise, then by dominated convergence, \( 1_U T = \lim \chi_\epsilon T \). It follows that if \( \pi \) is as above, then
\[
1_U \pi_* T = \pi_* (1\! \!-\!1_{-U} T).
\]

Moreover, if \( Z \subset X \) is a subvariety (locally) defined by a holomorphic tuple \( F \), then \( 1_X \setminus Z \) equals the limit of \( |F|^{2\lambda} T \) as \( \lambda \to 0^+ \).

Finally recall that a closed positive (or normal) current of bidegree \( (k,k) \) on \( X \) that has support on a subvariety of \( X \) of codimension \( > k \) vanishes. We refer to this as the dimension principle. In particular, if \( W, Z \subset X \) are subvarieties such that \( W \) is of pure codimension \( p \) and \( \text{codim} \ (Z \cap W) > p \), then
\[
1_Z[W] = 0.
\]

2.1. Classical Bedford-Taylor-Demailly Monge-Ampère products. Let \( u_1, \ldots, u_m \) be locally bounded psh functions on a complex manifold \( X \) and let \( T \) be a closed positive current on \( X \). The classical Bedford-Taylor theory asserts that one can define a closed positive current
\[
d\!d^c u_m \wedge \cdots \wedge d\!d^c u_1 \wedge T
\]
recursively as
\[
d\!d^c u_k \wedge \cdots \wedge d\!d^c u_1 \wedge T := d\!d^c \left( u_k d\!d^c u_{k-1} \wedge \cdots \wedge d\!d^c u_1 \wedge T \right),
\]
for \( k = 1, \ldots, m \). This current satisfies the following version of the Chern-Levine-Nirenberg inequalities, see [De3, Chapter III, Propositions 3.11 and 4.6].

**Proposition 2.1.** Given compacts \( L \subset K \) there is a constant \( C_{K,L} \) such that for all closed positive currents \( T \) and psh functions \( \nu, u_1, \ldots, u_m \), where \( u_1, \ldots, u_m \) are locally bounded,
\[
||\nu d\!d^c u_m \wedge \cdots \wedge d\!d^c u_1 \wedge T||_L \leq C_{K,L} ||\nu||_{L^1(K)} ||u_1||_{L^\infty(K)} \cdots ||u_m||_{L^\infty(K)} ||T||_K.
\]

Here \( ||S||_K \) is the mass semi-norm of the order zero current \( S \) with respect to the compact set \( K \), see [De3, Chapter III.3].

Recall that the unbounded locus \( L(u) \) of a psh function \( u \) is the set of points \( x \in X \) such that \( u \) is unbounded in every neighborhood of \( x \). Note that if \( u \) has analytic singularities, then \( L(u) \) is an analytic set, locally defined by \( F = 0 \) where \( u \) is given by \( |F| \). Demailly [De1] extended the definition (2.6) to the case when the intersection of the unbounded loci of the \( u_j \) is small in a certain sense. The following is a simple corollary of [De3, Chapter III, Theorem 4.5].

**Proposition 2.2.** Let \( u_1, \ldots, u_m \) be psh functions on a complex manifold \( X \) such that the unbounded locus \( L(u_j) \) is contained in analytic set \( Z_j \subset X \) for each \( j \). Moreover, let \( T \) be a closed positive current of bidegree \( (p,p) \) with support contained in an analytic set \( W \subset X \). Assume that
\[
\text{codim} (Z_{i_1} \cap \cdots \cap Z_{i_\ell} \cap W) \geq \ell + p
\]

\(^1\)Simple examples of constructible sets are \( V \setminus W \) or \((X \setminus V) \cup W \) for analytic sets \( V \) and \( W \) in \( X \).
for all choices of $1 \leq i_1 < \cdots < i_t \leq m$. Then $u_m \ddc u_{m-1} \wedge \cdots \wedge \ddc u_1 \wedge T$ and $\ddc u_m \wedge \cdots \wedge \ddc u_1 \wedge T$ are well-defined and have locally finite mass. The latter is a closed positive current.

These products satisfy the following continuity properties, see, e.g., [De3] Chapter III, Proposition 4.9.

**Proposition 2.3.** Let $u_j$ and $T$ be as in Proposition 2.2. If $(u_j^{(i)})$ are sequences of psh functions decreasing to $u_j$, then

$$
u_m \ddc u_{m-1} \wedge \cdots \wedge \ddc u_1 \wedge T \rightarrow u_m \ddc u_{m-1} \wedge \cdots \wedge \ddc u_1 \wedge T$$

$$
\ddc u_{m} \wedge \cdots \wedge \ddc u_1 \wedge T \rightarrow \ddc u_{m} \wedge \cdots \wedge \ddc u_1 \wedge T.
$$

We have the following generalization of (2.3).

**Lemma 2.4.** Assume that $W \subset X$ is a subvariety of pure codimension $p$, and assume that $Z \subset X$ is a subvariety, such that $\text{codim } X(Z \cap W) > p$. Moreover, assume that $b_1, \ldots, b_\ell$ are locally bounded psh functions. Then

$$(2.7) \quad \mathbf{1}_Z \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = 0.$$

**Remark 2.5.** Assume that $W \subset X$ is a subvariety and $U \subset X$ is a constructible set. Then it follows from Lemma 2.4 that

$$(2.8) \quad \mathbf{1}_U \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W].$$

To prove (2.8) we may assume that $U$ is a subset of $W$ and that $W$ is irreducible. We then need to prove that

$$
\mathbf{1}_U \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W]
$$

if $U$ is dense in $W$ and $\mathbf{1}_U \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = 0$ otherwise.

To see this, note on the one hand, that if $U$ is not dense in $W$, then, since $W$ is irreducible, $\overline{U}$ is a subvariety of $W$ of positive codimension. Thus

$$
\mathbf{1}_U \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = \mathbf{1}_U \mathbf{1}_U \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = 0
$$

in view of (2.3) and Lemma 2.4. On the other hand, if $U$ is dense in $W$, i.e., $\overline{U} = W$, since $\overline{U} \setminus U$ is a subvariety of $W$ of positive codimension, then again using Lemma 2.4, note that

$$
\mathbf{1}_U \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = \mathbf{1}_W \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] - \mathbf{1}_{U \setminus U} \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W].
$$

**Proof of Lemma 2.4.** We may assume that $X$ is connected. Let us first assume that $W = X$ so that $[W] = 1$ and $Z \subset X$ is a subvariety of positive codimension. Then the lemma follows by a small modification of the proof of Corollary 3.3 in [AW]: It is enough to consider the case when $Z$ is smooth. The general case then follows by stratification. Since it is a local statement, we may choose coordinates $z$ so that $Z = \{z_1 = \cdots = z_q = 0\}$, where $q = \text{codim } Z$. In view of (2.3) it is enough to prove that $\mathbf{1}_{\{z_1 = 0\}} \ddc b_\ell \wedge \cdots \wedge \ddc b_1 = 0$. Notice that in a set $|z_1| \leq r, (|z_2, \ldots, z_n| \leq r^t$,

we have that $\mathbf{1}_{\{z_1 = 0\}} \ddc b_\ell \wedge \cdots \wedge \ddc b_1$ is the limit of

$$-(|z_1|^{2\lambda} - 1)(\ddc b_\ell \wedge \cdots \wedge \ddc b_1)$$

as $\lambda \to 0^+$, cf. the beginning of this section. Since $|z_1|^{2\lambda} - 1$ is psh, (2.7) follows from Proposition 2.1 since the total mass of $|z_1|^{2\lambda} - 1$ tends to 0 when $\lambda \to 0^+$.

Next, let us assume that $W$ is smooth. We claim that then

$$(2.9) \quad \ddc b_\ell \wedge \cdots \wedge \ddc b_1 \wedge [W] = i_* (\ddc i_\ast b_\ell \wedge \cdots \wedge \ddc i_\ast b_1),$$

where $i$ is the inclusion of $Z$ in $X$.
where \( i \) is the inclusion \( i : W \to X \). Taking this for granted, since \( \dim(Z \cap W) > p \) it follows that \( i^{-1}Z \) is a proper subvariety of \( W \) and thus \( 1_{i^{-1}Z}dd^{c}i^{*}b_{\ell} \land \cdots \land dd^{c}i^{*}b_{1} \) vanishes by the argument above, and thus in view of (2.4),

\[
1_{Z}dd^{c}b_{\ell} \land \cdots \land dd^{c}b_{1} \land [W] = i_{*} \left( 1_{i^{-1}Z}dd^{c}i^{*}b_{\ell} \land \cdots \land dd^{c}i^{*}b_{1} \right) = 0.
\]

It is clear that (2.4) holds if the \( b_{j} \) are smooth. For general locally bounded psh \( b_{j} \) let \( b_{j}^{(i)} \) be sequences of smooth psh functions decreasing to \( b_{j} \). Then (2.4) follows by approximating by sequences of smooth psh functions converging to \( b_{j} \) using Proposition 2.3.

For the general case, let \( \pi : \tilde{X} \to X \) be an embedded resolution of \( W \) in \( X \). In particular, \( \tilde{W} := \pi^{-1}W_{\text{reg}} \) is a smooth manifold of pure codimension \( p \) in \( \tilde{X} \) and \( \pi \) is a biholomorphism outside a hypersurface \( H \subset \tilde{X} \), such \( \pi(H \cap \tilde{W}) \) has codimension \( > p \) in \( X \). It follows that outside \( \pi(H \cap \tilde{W}) \), \( \pi_{*}[W] = [W] \), and by the dimension principle this holds everywhere on \( X \).

If \( b_{j} \) are smooth, then in view of (2.1)

\[
(dd^{c}b_{\ell} \land \cdots \land dd^{c}b_{1} \land [W]) = \pi_{*}(dd^{c}\pi^{*}b_{\ell} \land \cdots \land dd^{c}\pi^{*}b_{1} \land [\tilde{W}]).
\]

For general \( b_{j} \) (2.10) follows by approximating by sequences of smooth psh functions converging to \( b_{j} \) using Proposition 2.3.

Next let \( \tilde{Z} = \pi^{-1}Z \). Then \( \tilde{Z} \cap \tilde{W} \subset \tilde{X} \) is a subvariety of codimension \( > p \). Indeed, for each connected component \( \tilde{W}_{j} \) of \( \tilde{W} \), \( \tilde{Z} \cap \tilde{W}_{j} \) is a subvariety of \( \tilde{W}_{j} \). Assume that \( \tilde{Z} \cap \tilde{W}_{j} = \tilde{W}_{j} \) for some \( j \). Then

\[
Z \cap W \supset \pi(\tilde{Z} \cap \tilde{W}) \supset \pi(\tilde{W}_{j});
\]

however \( \pi(\tilde{W}_{j}) \) has codimension \( p \), which contradicts that \( Z \cap W \) has codimension \( > p \). Thus \( \tilde{Z} \cap \tilde{W}_{j} \) is a proper subvariety of \( \tilde{W}_{j} \) for each \( j \). Thus, as proved above,

\[
1_{Z}dd^{c}\pi^{*}b_{\ell} \land \cdots \land dd^{c}\pi^{*}b_{1} \land [\tilde{W}] = 0
\]

and, in view of (2.4), we conclude that

\[
1_{Z}dd^{c}b_{\ell} \land \cdots \land dd^{c}b_{1} \land [W] = \pi_{*}(1_{Z}dd^{c}\pi^{*}b_{\ell} \land \cdots \land dd^{c}\pi^{*}b_{1} \land [\tilde{W}]) = 0.
\]

\( \square \)

3. Generalized mixed Monge-Ampère products

Assume that \( u_{1}, \ldots, u_{m} \) are psh functions with analytic singularities on a complex manifold \( X \) with unbounded loci \( Z_{1}, \ldots, Z_{m} \), respectively. Moreover assume that \( U_{1}, \ldots, U_{m} \subset X \) are constructible sets contained in \( X \setminus Z_{1}, \ldots, X \setminus Z_{m} \), respectively. Inspired by [AW, Section 4] we consider currents

\[
(dd^{c}u_{m}1_{U_{m}} \land \cdots \land dd^{c}u_{1}1_{U_{1}})
\]

defined recursively as

\[
(dd^{c}u_{k}1_{U_{k}} \land \cdots \land dd^{c}u_{1}1_{U_{1}}) := dd^{c}(u_{k}1_{U_{k}}dd^{c}u_{k-1}1_{U_{k-1}} \land \cdots \land dd^{c}u_{1}1_{U_{1}})
\]

for \( k = 1, \ldots, m \). In particular, if \( u_{j} = u \) and \( U_{j} = X \setminus Z_{j} \) for all \( j \), then (3.2) coincides with (1.4). For aesthetic reasons, and to emphasize that \( U_{j} \) is associated with \( u_{j} \), we choose to write (3.1) rather than

\[
dd^{c}u_{m} \land 1_{U_{m}}dd^{c}u_{m-1} \land \cdots \land dd^{c}u_{2} \land 1_{U_{2}}dd^{c}u_{1}1_{U_{1}},
\]

We say that a current of the form \( 1_{U}dd^{c}u_{m}1_{U_{m}} \land \cdots \land dd^{c}u_{1}1_{U_{1}} \), where \( U \) is a constructible set, is a (closed positive) current with analytic singularities. We also include currents (3.1) with no factor \( dd^{c}u_{1}1_{U_{1}} \); in other words \( 1_{U} \) is also a current with analytic singularities. For \( u_{j} \) of the form \( \log |F_{j}|^{2} \) currents like (3.1) were defined in [ASWY] Section 5. Note that \( dd^{c}u_{m}1_{U_{m}} \land \cdots \land dd^{c}u_{1}1_{U_{1}} \) vanishes unless \( U_{1} \) is dense in (at least one connected component of) \( X \).
Proposition 3.2 below asserts that this definition makes sense and that the currents $ddc u_1 U_k \wedge \cdots \wedge ddcc_{1} u_1 U_1$ are positive and closed. Moreover, Proposition 3.3 asserts that

$$ddc u_m X Z_m \wedge \cdots \wedge ddcc_{1} u_1 X Z_1$$

coincides with $ddcc u_m Z_m \wedge \cdots \wedge ddcc u_1 X Z_1$, when this current is well-defined, cf. Proposition 2.2. It is therefore tempting to think of (3.3) as a generalized mixed Monge-Ampère product, and just denote it by $ddc u_m Z_m \wedge \cdots \wedge ddcc u_1 X Z_1$. The following example shows, however, that this “product” lacks some properties one would naturally ask for of a product. In particular, it is not additive in any factor except the right-most one nor commutative.

Example 3.1. Let $u_1 = \log |z_1|^2$ and $u_2 = \log |z_1 z_2|^2$ in $X = \mathbb{C}^2$. Then $u_1$ and $u_2$ are psh with analytic singularities with unbounded loci $Z_1 = \{z_1 = 0\}$ and $Z_2 = \{z_1 = 0\} \cup \{z_2 = 0\}$, respectively. In view of the Poincaré-Lelong formula it follows that

$$ddcc u_2 X Z_2 \wedge ddcc u_1 X Z_1 = ddcc (u_2 X Z_2 \{z_1 = 0\}) = 0$$

but

$$ddcc u_1 X Z_1 \wedge ddcc u_2 X Z_2 = ddcc (u_1 X Z_1 (\{z_1 = 0\} + \{z_2 = 0\})) = [z_1 = 0] \wedge [z_2 = 0] = [0],$$

so that $ddcc u_2 X Z_2 \wedge ddcc u_1 X Z_1$ is not commutative in the factors $ddcc u_j X Z_j$.

Moreover, let $u_3 = \log |z_2|^2$. Then $u_3$ is psh with analytic singularities with unbounded locus $Z_3 = \{z_2 = 0\}$. Note that $u_2 = u_1 + u_3$. Now

$$ddcc u_1 X Z_1 \wedge ddcc u_2 X Z_2 \wedge ddcc u_3 X Z_3 \wedge ddcc u_1 X Z_1 =$$

$$0 + [z_2 = 0] \wedge [z_1 = 0] = [0] = ddcc u_2 X Z_2 \wedge ddcc u_1 X Z_1.$$

□

Proposition 3.2. Let $u_j$ and $U_j$ be as above. Assume that $ddcc u_1 U_k \wedge \cdots \wedge ddcc u_1 U_1$ is inductively defined via (2.2). Let $u^{(i)}_{k+1}$ be a sequence of smooth psh functions in $X$ decreasing to $u_{k+1}$. Then

$$u_{k+1} U_{k+1} \wedge \cdots \wedge ddcc u_{1} \U_1 := \lim_{i \to \infty} u^{(i)}_{k+1} U_{k+1} \wedge \cdots \wedge ddcc u_{1} \U_1$$

has locally finite mass and does not depend on the choice of sequence $u^{(i)}_{k+1}$. Moreover $ddcc u_{k+1} U_{k+1} \wedge \cdots \wedge ddcc u_{1} \U_1$, defined by (3.2), is positive and closed.

Remark 3.3. Note that Proposition 3.2 asserts that if $T$ is a current with analytic singularities, $u$ is a psh function with analytic singularities with unbounded locus $Z$, and $U$ is a constructible set contained in $X \setminus Z$, then

$$ddcc u \wedge 1 U T := ddcc (u \U T)$$

is a well-defined current with analytic singularities.

□

The proof is a generalization of the proof of Proposition 4.1 in [AW].

Proof. Since the statement is local we may assume that $u_j = \log |F_j|^2 + v_j$. Moreover without loss of generality we may assume that $X$ is connected and that $U_1$ is dense in $X$, and thus $1 U_1 = 1$ as a distribution. Indeed, otherwise $ddcc u_k U_k \wedge \cdots \wedge ddcc u_1 U_1 = 0$.

Let $\pi: \tilde{X} \to X$ be a smooth modification such that locally on $\tilde{X}$, $\pi^* F_j = f_j f_j'$, where $f_j$ is a holomorphic function and $f_j'$ is a nonvanishing tuple of holomorphic functions, for each $j$. It follows that $\pi^* u_j = \log |f_j|^2 + b_j$, where $b_j := \log |f_j|^2 + \pi^* v_j$ is psh and locally bounded, cf. the proof of Proposition 4.1 in [AW]. Note that for two different local representations, the $f_j$ differ by a nonvanishing holomorphic factor, and thus the $b_j$ differ by a pluriharmonic term. Therefore the local divisors $\{f_j = 0\}$ define a global divisor $D_j$ on $\tilde{X}$, such that $\pi^{-1} Z_j = |D_j|$ and moreover the currents $ddcc b_j$ define a global positive current on $\tilde{X}$. By the Poincaré-Lelong formula

$$ddcc \pi^* u_j = [D_j] + ddcc b_j.$$
Let \( u_1^{(i)} \) be a sequence of smooth psh functions decreasing to \( u_1 \). Since \( \pi \) is a modification

\[
\ddc u_1^{(i)} = \pi_* (\ddc \pi^* u_1^{(i)}) \to \pi_* (\ddc \pi^* u_1) = \pi_* ([D_1] + \ddc b_1)
\]

and it follows that

\[
\ddc u_1 = \pi_* ([D_1] + \ddc b_1).
\]

Let us now assume that we have proved that \( u_k \in U_j \ddc u_{k-1} U_{k-1} \land \cdots \land \ddc u_1 U_1 \) is well-defined with the desired properties and that \( \ddc u_k U_k \land \cdots \land \ddc u_1 U_1 \) is the pushforward of

\[
(3.4) \sum_{I=(i_1,\ldots,i_s)\subset\{1,\ldots,k\}} \ddc b_I \land [V_I],
\]

where \( \ddc b_I = \ddc b_{i_1} \land \cdots \land \ddc b_{i_s} \) and \( V_I \) are analytic cycles\(^2\) of pure codimension \( k-s \) on \( \tilde{X} \). Since the \( b_I \) are determined up to addition by a pluriharmonic term, each \( \ddc b_I \land [V_I] \) is a globally defined current on \( \tilde{X} \), cf. Section 2.1.

We will prove that:

(i) \( u_{k+1} \in U_{k+1} \ddc u_k U_k \land \cdots \land \ddc u_1 U_1 := \lim \ u^{(i)}_{k+1} \in U_{k+1} \ddc u_k U_k \land \cdots \land \ddc u_1 U_1 \) has locally finite mass and is independent of \( u^{(i)}_{k+1} \),

(ii) the current

\[
(3.5) \ddc u_{k+1} U_{k+1} \land \cdots \land \ddc u_1 U_1 := \ddc (u_{k+1} \in U_{k+1} \ddc u_k U_k \land \cdots \land \ddc u_1 U_1),
\]

is the pushforward of a current of the form (3.4).

As soon as (i) and (ii) are verified, the proposition follows by induction.

Note that \( \tilde{U}_j := \pi^{-1} U_j \) is a constructible set in \( \tilde{X} \). Let us consider one summand \( \ddc b_I \land [V_I] \) in (3.4). Let \( V_I \) be the union of the irreducible components \( V^j_I \) of \( V_I \) such that \( \tilde{U}_{k+1} \cap V^j_I \) is dense in \( V^j_I \). Then in view of Remark 2.3.

\[
1 \in U_{k+1} \ddc b_I \land [V_I] = \ddc b_I \land [V_I].
\]

We claim that \( |D_{k+1}| \cap |V_I| \) has positive codimension in \( |V_I| \). To see this let \( V^j_I \) be an irreducible component of \( |V_I| \). Then either \( V^j_I \subset |D_{k+1}| \) or \( |D_{k+1}| \cap V^j_I \) has positive codimension in \( V^j_I \). However \( V^j_I \) cannot be contained in \( |D_{k+1}| \) since \( \tilde{U}_{k+1} \cap V^j_I \subset (\tilde{X} \setminus |D_{k+1}|) \cap V^j_I \) is dense in \( V^j_I \); this proves the claim.

Since \( \text{codim} (|V_I| \cap |D_{k+1}|) > \text{codim} |V_I| \), by Proposition 2.2, \( \pi^* u_{k+1} \ddc b_I \land [V_I] \) has locally finite mass and by Proposition 2.3.

\[
\pi^* u^{(i)}_{k+1} \ddc b_I \land [V_I] \to \pi^* u_{k+1} \ddc b_I \land [V_I]
\]

if \( u^{(i)}_{k+1} \) is any sequence of psh functions decreasing to \( u_{k+1} \). If \( u^{(i)}_{k+1} \) are smooth, using (2.1), that \( \pi^{-1} U_{k+1} = \tilde{U}_{k+1} \), and (2.2), we get

\[
u^{(i)}_{k+1} U_{k+1} \ddc u_k U_k \land \cdots \land \ddc u_1 U_1 = \pi_* (\pi^* u^{(i)}_{k+1} \sum_I \ddc b_I \land [V_I]).
\]

Proposition 2.3 then yields

\[
(3.6) \ddc \pi^* u_{k+1} \land \ddc b_I \land [V_I] = \ddc \log |f_{k+1}|^2 \land \ddc b_I \land [V_I] + \ddc b_{k+1} \land \ddc b_I \land [V_I].
\]

\(^2\)formal linear combinations of irreducible analytic sets
since where the first equality follows from the definition of the first current, the second equality follows is small enough. We can thus find a sequence Proposition 2.3 because of (3.8). The proposition then follows by induction over Lemma 3.5.

Proposition 3.4. Let $u_1, \ldots, u_m$ be psh functions with analytic singularities with corresponding unbounded loci $Z_1, \ldots, Z_m$. Assume that

$$\text{codim} (Z_{i_1} \cap \cdots \cap Z_{i_r}) \geq \ell,$$

for each choice of $1 \leq i_1 < \cdots < i_r \leq m$. Then

$$dd^c u_m 1_{X \setminus Z_m} \cdots \cap dd^c u_1 1_{X \setminus Z_1} = dd^c u_m \cdots \cap dd^c u_1,$$

where the right hand side is defined in the sense of Bedford-Taylor-Demailly.

Proof. The statement is local, so it is enough to assume that $X$ is a fixed relatively compact coordinate neighborhood of any given point. We let $u_m^N := \max(u_m, -N)$, which is psh and decreases pointwise to $u_m$ when $N \to \infty$. Since $u_m$ has analytic singularities, $u_m^{N,\varepsilon}$ is smooth. We let $u_m^{N,\varepsilon}$ be obtained from $u_m^N$ through convolution with an approximate identity, so that $u_m^{N,\varepsilon}$ is smooth, psh and decreases pointwise to $u_m^{N,\varepsilon}$ when $\varepsilon \to 0$. Since $u_m^{N} \equiv -N$ in a neighborhood of $Z_m$, it follows that $u_m^{N,\varepsilon} \equiv -N$ in some smaller neighborhood of $Z_m$ when $\varepsilon$ is small enough. We can thus find a sequence $u_m^{(i)}$ of smooth psh functions decreasing pointwise to $u_m$ such that each $u_m^{(i)}$ is constant in some neighborhood of $Z_m$. We then get that

$$dd^c u_m 1_{X \setminus Z_m} \cdots \cap dd^c u_1 1_{X \setminus Z_1} = \lim_{i} dd^c u_m^{(i)} \cdots \cap dd^c u_1^{(i)} = dd^c u_m \cdots \cap dd^c u_1,$$

where the first equality follows from the definition of the first current, the second equality follows since $u_m^{(i)}$ is smooth and constant in a neighborhood of $Z_m$, and the last equality follows from Proposition 2.3 because of (3.8). The proposition then follows by induction over $m$. \hfill $\Box$

Recall that a function $q$ is quasi plurisubharmonic (qpsh) if it is of the form $q = u + a$, where $u$ is psh and $a$ is smooth. We say that the qpsh function $q$ has analytic singularities if $u$ has analytic singularities. The unbounded locus of $q$ is defined as the unbounded locus of $u$.

Lemma 3.5. Let $T$ be a current with analytic singularities, let $q$ be a qpsh function with analytic singularities with unbounded locus $Z$, and let $U \subset X \setminus Z$ be a constructible set. Then,

$$q 1_U T := u 1_U T + a 1_U T$$

is independent of the decomposition $q = u + a$, where $u$ is psh and $a$ is smooth.

Proof. Let $q = u_1 + a_1 = u_2 + a_2$ be two decompositions of $q$ such that $u_j$ are psh and $a_j$ are smooth. The statement is local. Therefore we may approximate $q$ by convoluting with a sequence of regularizing kernels $\rho^{(i)}$ so that for $j = 1, 2$, $u_j^{(i)} := u_j * \rho^{(i)}$ is a sequence of smooth psh functions decreasing to $u_j$, and $a_j^{(i)} := a_j * \rho^{(i)}$ is a sequence of smooth functions converging to $a_j$, and for each $i$, $u_1^{(i)} + a_1^{(i)} = u_2^{(i)} + a_2^{(i)}$. Thus in light of Proposition 3.2 we get

$$u_1 1_U T + a_1 1_U T = \lim_i (u_1^{(i)} 1_U T + a_1^{(i)} 1_U T) = \lim_i (u_2^{(i)} 1_U T + a_2^{(i)} 1_U T) = u_2 1_U T + a_2 1_U T.$$

\hfill $\Box$
Let \( u \) be a psh function with analytic singularities with unbounded locus \( Z \), let \( \alpha \) be a closed smooth \((1,1)\)-form, and let \( T \) be a current (locally) of the form
\[
T = \sum \beta_i \wedge T_i,
\]
where the sum is finite, \( \beta_i \) are closed smooth forms, and \( T_i \) are currents with analytic singularities. We define the operator \( T \mapsto [dd^c u]_\alpha \wedge T \) by
\[
[dd^c u]_\alpha \wedge T := dd^c u \wedge \alpha(1_Z T + \alpha \wedge 1_Z T := dd^c(u1_{\mathcal{X} \setminus Z}) + \alpha \wedge 1_Z T.
\]
By Remark 3.3 this is a well-defined current of the form (3.10). Using that \( 1_{\mathcal{X} \setminus \mathcal{Z}}(\alpha^k \wedge 1_Z (dd^c u)^k) = 0 \) for \( k, \ell \geq 0 \), we get that
\[
[dd^c u]^m_\alpha = [dd^c u]_\alpha \wedge [dd^c u]^{m-1}_\alpha,
\]
where \([dd^c u]^m_\alpha\) is defined by (1.5).

Next, for currents \( T \) of the form (3.10) we define operators \( T \mapsto [dd^c u]_\alpha \wedge T \) recursively by
\[
[dd^c u]^k_\alpha \wedge T := [dd^c u]_\alpha \wedge [dd^c u]^{k-1}\wedge T.
\]
Again by Remark 3.3 these are currents of the form (3.10). In particular, if \( u_1, \ldots, u_t \) are psh functions with analytic singularities, and \( \alpha_1, \ldots, \alpha_t \) are closed smooth \((1,1)\)-forms, the current
\[
[dd^c u]^m_\alpha \wedge \ldots \wedge [dd^c u]^{m_k}_\alpha
\]
is a well-defined current of the form (3.10).

Note that if \( \beta \) is a closed smooth form, then
\[
[dd^c u]^k_\alpha \wedge \beta \wedge T = \beta \wedge [dd^c u]^k_\alpha \wedge T.
\]

Indeed, multiplication with \( \mathcal{U} \) and \( dd^c u \) commutes with multiplication with closed smooth forms.

**Remark 3.6.** Recall that if \((X, \omega)\) is a Kähler manifold, then a function \( \phi: X \to \mathbb{R} \cup \{-\infty\} \) is called \( \omega \)-plurisubharmonic (\( \omega \)-psh) if locally the function \( g + \phi \) is psh, where \( g \) is a local potential for \( \omega \), i.e., \( \omega = dd^c g \). Moreover \( \phi \) is said to have analytic singularities if the functions \( g + \phi \) have analytic singularities. If \( \phi \) is an \( \omega \)-psh function with analytic singularities, we can define a global positive \( C^2 \) current \((\omega + dd^c \phi)^n\), by locally defining it as \((dd^c (g + \phi))^k\), see [ABW] Lemma 5.1. Analogously to (1.5) we can define
\[
[\omega + dd^c \phi]^m_\omega := (\omega + dd^c \phi)^m + \sum_{k=0}^{m-1} \omega^{m-k} \wedge 1_Z (\omega + dd^c \phi)^k,
\]
where \( Z \) is the unbounded locus of \( \phi \), cf. [ABW]. With this notation, Theorem 1.2 in [ABW] can be formulated as:

**Let \( \phi \) be an \( \omega \)-psh function with analytic singularities on a compact Kähler manifold \((X, \omega)\) of dimension \( n \). Then**
\[
\int_X [\omega + dd^c \phi]^n_\omega = \int_X \omega^n.
\]

\[\square\]

4. **Hermitian metrics with analytic singularities on line bundles**

Let \( L \to X \) be a holomorphic line bundle. A *singular hermitian metric* on \( L \), as introduced by Demailly [De4], consists of (possibly infinite) seminorms \( || \cdot ||_{h(x)} \) on \( L_x \) for all \( x \in X \) such that if \( \vartheta: L|\mathcal{U} \to \mathcal{U} \times \mathbb{C} \) is a local trivialization of \( L \) and \( \xi \) is a local section, then \( ||\xi||^2_{h} = |\vartheta(\xi)|^2 e^{-\varphi}, \) where \( \varphi \) is a locally integrable function in \( \mathcal{U} \) called the (local) weight of \( h \) with respect to \( \vartheta \), see, e.g., [L Chapter 9.4.D]. The metric \( h \) is often denoted by \( e^{-\varphi} \) or just \( \varphi \). If \( \vartheta': L|\mathcal{U}' \to \mathcal{U}' \times \mathbb{C} \) is another local trivialization, with transition function \( g \), then in \( \mathcal{U} \cap \mathcal{U}' \) the corresponding weight \( \varphi' \) satisfies
\[
\varphi' = \varphi + \log |g|^2.
\]
From (4.1) and the local integrability of the weights it follows that the curvature current \( \Theta_h = dd^c \varphi \) is a well-defined global current on \( X \). The Chern form \( c_1(L, h) = (i/2\pi)\Theta_h = dd^c \varphi \) represents the Chern class \( c_1(L) \).

The metric \( e^{-\varphi} \) is positive if the weights \( \varphi \) are psh. We say that a positive singular metric \( e^{-\varphi} \) has analytic singularities if the weights \( \varphi \) have analytic singularities. In view of (4.1) there is a well-defined associated unbounded locus \( Z \subset X \), that is a subvariety of \( X \), locally defined as the unbounded loci of the \( \varphi \). Since the local weights are integrable it follows that \( Z \) has positive codimension in \( X \).

**Example 4.1.** Assume that \( s_1, \ldots, s_N \) are nontrivial holomorphic sections of a line bundle \( L \to X \). Then \( h = e^{-\varphi} \) with

\[
\varphi = \log \sum_{j=1}^N |s_j|^2
\]

is a positive metric with analytic singularities, cf. [Dei] Example 2.4. In other words, if \( \vartheta : U \to U \times \mathbb{C} \) is a local trivialization and \( \xi \) is a local section, then

\[
||\xi||_h^2 = \frac{|\vartheta(\xi)|^2}{\sum_{j=1}^N |\vartheta(s_j)|^2}.
\]

**Lemma 4.2.** Assume that \( \varphi_1, \ldots, \varphi_t \) are positive metrics with analytic singularities on line bundles \( L_1, \ldots, L_t \) over \( X \) with unbounded loci \( Z_1, \ldots, Z_t \), respectively. If \( U_1, \ldots, U_t \) are constructible sets contained in \( X \setminus Z_1, \ldots, X \setminus Z_t \), respectively, and \( \theta_1, \ldots, \theta_t \) are closed \((1,1)\)-forms, then the a priori locally defined currents

\[
(4.2) \quad dd^c \varphi_1|_{U_1} \wedge \cdots \wedge dd^c \varphi_1|_{U_1},
\]

\[
(4.3) \quad [dd^c \varphi_1]_{\theta_1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}
\]

are globally defined currents; (4.2) has analytic singularities and (4.3) is of the form (3.10).

**Proof.** Since the local weights of \( \varphi_j \) are psh functions with analytic singularities, (4.2) and (4.3) are locally well-defined and of the desired form in view of Section 3. Since two local weights differ by a pluriharmonic function, cf. (4.1), it follows using Lemma 5.3 that they are globally defined.

If \( \varphi_j = \varphi \), with singular set \( Z \), and \( U_j = X \setminus Z \) for all \( j \), we write \( (dd^c \varphi)^t \) for the generalized Monge-Ampère product (4.2), cf. (1.2).

For the proofs of Theorems 1.1 and 1.2 we will need the following results.

**Proposition 4.3.** Let \( \varphi_1, \ldots, \varphi_t \) be hermitian metrics with analytic singularities on holomorphic line bundles \( L_1, \ldots, L_t \), respectively, over a complex manifold \( X \). Moreover, let \( \theta_1, \ldots, \theta_t \) be first Chern forms of smooth metrics on \( L_1, \ldots, L_t \), respectively. Then

\[
(4.4) \quad [dd^c \varphi_1]_{\theta_1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1} = \theta_t^{m_1} \wedge \cdots \wedge \theta_1^{m_1} + dd^c S,
\]

where \( S \) is a current on \( X \).

**Proof.** In view of (3.12) it is enough to prove the result for \( m_j = 1, j = 1, \ldots, t \). Assume that \( \theta_j \) is the first Chern form of the smooth metric \( \psi_j \). Then note that \( \varphi_j - \psi_j \) defines a global qpsh function on \( X \) for each \( j \), cf. (4.1).

Let \( T \) be a current of the form (3.10). Then

\[
(4.5) \quad [dd^c \varphi_j]_{\theta_j} \wedge T = dd^c ( (\varphi_j 1_{X \setminus Z_j} T) + \theta_j \wedge 1_{Z_j} T) = dd^c ( (\varphi_j 1_{X \setminus Z_j} T) - \theta_j \wedge 1_{X \setminus Z_j} T + \theta_j \wedge T = dd^c ((\varphi_j - \psi_j) 1_{X \setminus Z_j} T) + \theta_j \wedge T,
\]

where we have used that \( dd^c \psi_j = \theta_j \), Lemma 3.5 and that \( 1_{X \setminus Z_j} T \) is closed for the last equality.

Assume that \( t = 1 \). Then it follows from (4.5) (applied to \( T = 1 \) and \( j = 1 \)) that (4.3) holds with \( S = (\varphi_1 - \psi_1) 1_{X \setminus Z_1} \). In fact, \( S = \varphi_1 - \psi_1 \) since \( Z_1 \) has positive codimension in \( X \). Next
assume that (4.4) holds for $t = \kappa$. Then (4.5) (applied to $T = [dd^c \varphi_\kappa]_{\theta_1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}$ and $j = \kappa + 1$) gives

$$[dd^c \varphi_{\kappa+1}]_{\theta_{\kappa+1}} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1} = dd^c U + \theta_{\kappa+1} \wedge [dd^c \varphi_\kappa]_{\theta_1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1} = dd^c U + \theta_{\kappa+1} \wedge (\theta_\kappa \wedge \cdots \wedge \theta_1 + dd^c S) = \theta_{\kappa+1} \wedge \cdots \wedge \theta_1 + dd^c (U + \theta_{\kappa+1} \wedge S),$$

where

$$U = (\varphi_{\kappa+1} - \psi_{\kappa+1})1_{X \setminus Z}, [dd^c \varphi_\kappa]_{\theta_1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}.$$

Thus (4.4) holds for $t = \kappa + 1$ and the lemma follows by induction.

Lemma 4.4. Let $\varphi$ be a positive metric with analytic singularities on a holomorphic line bundle $L$ over a complex manifold $X$, and let $\theta$ be the first Chern form of a smooth metric on $L$. Let $\varphi_\varepsilon$ be a sequence of smooth positive metrics decreasing to $\varphi$ and let $\omega_\varepsilon$ be the corresponding first Chern forms. Moreover, let $T$ be a current of the form (3.10), and let $\beta$ be a test form such that the support of $dd^c \beta$ does not intersect the unbounded locus $Z$ of $\varphi$. Then

\begin{equation}
\int_X [dd^c \varphi_\varepsilon]^m \wedge T \wedge \beta = \lim_{\varepsilon \to 0} \int_X \omega_\varepsilon^m \wedge T \wedge \beta.
\end{equation}

Proof. Assume that $\theta$ is the first Chern form of the smooth metric $\psi$. First note that for any current $S$ of the form (3.10)

\begin{equation}
[dd^c \varphi_\theta \wedge S - \omega_\varepsilon \wedge S = dd^c (\varphi 1_{X \setminus Z}) + \theta \wedge 1_{Z} S - \omega_\varepsilon \wedge S = dd^c (\varphi - \varphi_\varepsilon) 1_{X \setminus Z} S + dd^c (\psi - \varphi_\varepsilon) 1_{Z} S).
\end{equation}

Since $\varphi - \varphi_\varepsilon$ and $\psi - \varphi_\varepsilon$ are globally defined qph functions, cf. (4.1), in view of Lemma 3.5 the currents on the last line of (4.7) are globally defined currents of the form (3.10).

By applying (4.7) to $S = [dd^c \varphi]^\ell - 1 \wedge T$ for $\ell = 1, \ldots, m$, it follows that

\begin{equation}
[dd^c \varphi]^m_\theta \wedge T - \omega_\varepsilon^m \wedge T = \sum_{\ell=1}^m \omega_\varepsilon^{m-\ell} \wedge ([dd^c \varphi_\theta]^\ell \wedge T - \omega_\varepsilon \wedge [dd^c \varphi_\theta]^\ell \wedge T) = \sum_{\ell=1}^m \omega_\varepsilon^{m-\ell} \wedge dd^c ((\varphi - \varphi_\varepsilon) 1_{X \setminus Z} [dd^c \varphi_\theta]^\ell \wedge T) + \sum_{\ell=1}^m \omega_\varepsilon^{m-\ell} \wedge dd^c ((\psi - \varphi_\varepsilon) 1_{Z} [dd^c \varphi_\theta]^\ell \wedge T).
\end{equation}

Again, the currents in the last line are well-defined global currents by Lemma 3.5.

Let us integrate one of the currents in the second sum against $\beta$. Then by Stokes’ theorem

$$\int_X \omega_\varepsilon^{m-\ell} \wedge dd^c ((\psi - \varphi_\varepsilon) 1_{Z} [dd^c \varphi_\theta]^\ell \wedge T) \wedge \beta = \int_X \omega_\varepsilon^{m-\ell} \wedge (\psi - \varphi_\varepsilon) 1_{Z} [dd^c \varphi_\theta]^\ell \wedge T \wedge dd^c \beta = 0,$n

where the last equality follows since $\text{supp}(1_{Z} [dd^c \varphi_\theta]^\ell \wedge T) \subset Z$ is disjoint from $\text{supp} dd^c \beta$. Outside $Z$, the current

$$\omega_\varepsilon^{m-\ell} \wedge (\varphi - \varphi_\varepsilon) 1_{X \setminus Z} [dd^c \varphi_\theta]^\ell \wedge T = (\varphi - \varphi_\varepsilon) (dd^c \varphi_\varepsilon)^{m-\ell} \wedge 1_{X \setminus Z} [dd^c \varphi_\theta]^\ell \wedge T$$

converges weakly to 0 by Proposition 2.3. Thus, integrating one of the terms in the first sum in the last line of (4.8) against $\beta$ and taking the limit gives

$$\lim_{\varepsilon \to 0} \int_X \omega_\varepsilon^{m-\ell} \wedge dd^c ((\varphi - \varphi_\varepsilon) 1_{X \setminus Z} [dd^c \varphi_\theta]^\ell \wedge T) \wedge \beta = \lim_{\varepsilon \to 0} \int_X \omega_\varepsilon^{m-\ell} \wedge (\varphi - \varphi_\varepsilon) 1_{X \setminus Z} [dd^c \varphi_\theta]^\ell \wedge T \wedge dd^c \beta = 0.$$

Now (4.6) follows by integrating (4.8) against $\beta$ and taking the limit. □
5. Hermitian metrics with analytic singularities on vector bundles

Let $E \to X$ be a holomorphic vector bundle over a complex manifold $X$. A singular hermitian metric $h$ on $E$ in the sense of Berndtsson–Păun, [BP], is a measurable function from $X$ to the space of nonnegative hermitian forms on the fibers. The hermitian forms are allowed to take the value $\infty$ at some points in the base (i.e., the norm function $\|\xi\|^2_h$ is a measurable function with values in $[0, \infty]$), but for any fiber $E_x$, the subset $E_0 := \{ \xi \in E_x : \|\xi\|^2_h < \infty \}$ has to be a linear subspace, and the restriction of the metric to this subspace must be an ordinary hermitian form.

Every singular hermitian metric $h$ on $E$ induces a canonical dual singular hermitian metric $h^*$ on $E^*$ such that $(h^*)^* = h$ under the natural isomorphism $(E^*)^* \cong E$, see, e.g., [LRRS] Lemma 3.1. Following [BP] Section 3 we say that $h$ is Griffiths negative if the function

$$\chi_h(x, \xi) := \log \|\xi\|^2_{h(x)}$$

is psh on the total space of $E$. Moreover we say that $h$ is Griffiths positive if the dual metric $h^*$ on $E^*$ is negative.

**Proposition 5.1.** Let $h$ be a singular hermitian metric on a holomorphic vector bundle $E$. Let $0_E$ denote the zero section of $E$. Then the following conditions are equivalent.

1. $h$ is Griffiths negative, i.e., $\chi_h$ is psh on the total space of $E$,
2. $\chi_h$ is psh on $E \setminus 0_E$,
3. the function $x \mapsto \log \|u(x)\|^2_{h(x)}$ is psh for each local section $u$ of $E$,
4. $\log \|u\|^2_h$ is psh for each local nonvanishing section $u$ of $E$.

**Proof.** We first prove that 3 is equivalent to 1 and that 4 is equivalent to 2. Note that if $u$ is a local holomorphic section of $E$, then $\log \|u(x)\|^2_{h(x)} = \chi_h \circ u(x)$. Thus if $\chi_h$ is psh, then so is $\log \|u\|^2_h$ since $u$ is a holomorphic map. Hence 4 implies 3. If $u \neq 0$, then it is enough that $\chi_h$ is psh on $E \setminus 0_E$ in order for $u$ to be psh. Thus 2 implies 4.

For the converses, since plurisubharmonicity is a local property, we may assume that $X$ is an open subset of $\mathbb{C}^n$ and that $E = X \times \mathbb{C}^r$. To prove that $\chi_h$ is psh it is then sufficient to prove that $\chi_h \circ \gamma(t)$ is subharmonic on (the restriction to $E$ of) any complex line $\gamma(t)$ in $\mathbb{C}^n \times \mathbb{C}^r$. We let $\gamma_0$ and $\gamma_1$ denote the components of $\gamma$ in $\mathbb{C}^n$ and $\mathbb{C}^r$, respectively. If $\gamma(0)$ is constant then

$$\chi_h \circ \gamma(t) = \log \|\gamma_1(t)\|^2_h(\gamma_0(t)) = \log \|\gamma_1(t)\|^2_{h_0},$$

where $h_0$ is the constant metric $h(\gamma_0)$. If $\|u\|^2_h$ is psh for all $u$, then $h_0$ has to be finite, and thus since $\gamma_1(t)$ is a holomorphic curve, it follows that $\chi_h \circ \gamma$ is subharmonic. If $\gamma(0)$ is not constant, then note that $\gamma_1(t) = u \circ \gamma_0(t)$ for some (linear) holomorphic function $u$, that can be extended to a holomorphic section on $X$. Thus

$$\chi_h \circ \gamma(t) = \log \|\gamma_1(t)\|^2_h(\gamma_0(t)) = \log \|u \circ \gamma_0(t)\|^2_{h(\gamma_0(t))} = \log \|u\|^2_h \circ \gamma_0(t).$$

Since $\gamma_0$ is holomorphic, $\chi_h \circ \gamma$ is subharmonic if $\|u\|^2_h$ is psh. Hence 4 implies 1. To show that $\chi_h$ is psh outside the zero section of $E$, $u$ can be chosen nonvanishing, and thus 2 follows from 4.

Next we prove that 1 is equivalent to 2. Clearly 1 implies 2. To prove the converse assume that $\chi_h$ is psh on $E \setminus 0_E$. Then $\|\xi\|^2_{h(x)}$ is finite on $E \setminus 0_E$ and thus by homogeneity it must vanish on $0_E$, which means that $\chi_h|_{0_E} \equiv -\infty$. It follows that $\chi_h$ trivially satisfies the sub-mean value property at each point of $0_E$.

To prove that $\chi_h$ is upper semicontinuous at $0_E$, choose $(x_0, \xi_0) \in 0_E$ and let $(x_k, \xi_k)$ be a sequence of points converging to $(x_0, \xi_0)$. We need to prove that $\lim_k \chi_h(x_k, \xi_k) = -\infty$. As above, we may assume that $X$ is an open subset of $\mathbb{C}^n_{\mathbb{R}}$ and $E = X \times \mathbb{C}^r_{\mathbb{R}}$, and moreover that $0_E = \{ \xi = 0 \}$ and $(x_0, \xi_0) = (0, 0)$. Also we may assume that $(x_k, \xi_k)$ are contained in the set $\{ |x| \leq 1, |\xi| \leq 1 \}$. Let $C$ be the compact “cylinder”

$$C = \{ |x| \leq 1, |\xi| = 1 \}.$$

The function $\chi_h$ is sometimes called the logarithmic indicatrix of the (Finsler) metric $h$, see, e.g., [De2].
Since \( \chi_h \) is psh and thus upper semi-continuous outside \( 0_E \), \( \chi_h|_C \leq M \) for some \( M < \infty \). By homogeneity it follows that
\[
\chi_h(x_k, \xi_k) \leq M + \log |\xi_k|^2 \to -\infty.
\]
Thus \( \chi_h(x, \xi) \) is upper semicontinuous at \( 0_E \) and hence it is psh in \( E \).

Let \( \pi : P(E) \to X \) denote the projectivization of \( E \), i.e., the projective bundle of lines in the dual bundle \( E^* \) of \( E \), i.e., \( P(E)_x = P(E_x^*) \). The pullback bundle \( \pi^*E^* \to P(E) \) then carries a tautological line bundle
\[
\mathcal{O}_{P(E)}(-1) = \{(x, [\xi]; v), v \in \mathbb{C}\xi \} \subset \pi^*E^*.
\]
Let \( e^{\varphi} \) denote the restriction of \( \pi^*h^* \) to \( \mathcal{O}_{P(E)}(-1) \). Then \( e^{-\varphi} \) is the dual metric on the dual line bundle \( \mathcal{O}_{P(E)}(1) \). If \( E \) is a line bundle, then \( \mathcal{O}_{P(E)}(1) \cong E \) and \( e^{-\varphi} \cong h \).

Let us describe \( \varphi \) in a local trivialization. After possibly shrinking \( X \) we may assume that \( E = X \times \mathbb{C}^r \); then \( P(E) = X \times \mathbb{P}^{r-1} \). For \( i = 1, \ldots, r \), let
\[
\mathcal{U}_i = \{(x, [\xi]) \in P(E), \xi_i \neq 0 \}.
\]
Then \( \{\mathcal{U}_i\} \) is an open cover of \( P(E) \) and \( \mathcal{O}_{P(E)}(-1) \) is defined by the trivializations
\[
\psi_i : \mathcal{O}_{P(E)}(-1)|_{\mathcal{U}_i} \to \mathcal{U}_i \times \mathbb{C}, \,(x, [\xi]; v) \mapsto (x, [\xi]; v_i).
\]
Now, on \( \mathcal{U}_i \),
\[
\|v\|_{\pi^*h^*(x,[\xi])} = \|\psi_i(v)\|^2 e^{\varphi_i(x,[\xi])} = |v_i|^2 e^{\varphi_i(x,[\xi])}.
\]
Moreover, since \( \pi^*h^* \) is a pullback metric
\[
\|v\|_{\pi^*h^*(x,[\xi])} = \|v\|_{h^*(x)}.
\]
By applying (5.1) and (5.2) to \( v = \xi \) we get
\[
\varphi_i(x,[\xi]) = \log \|\xi/\xi_i\|^2_{h^*(x)} = \chi_{h^*}(x,\xi/\xi_i).
\]
Note that this is well-defined since the second and third expressions only depend on \([\xi]\).

**Proposition 5.2.** Let \( h \) be a singular hermitian metric on a holomorphic vector bundle \( E \). Then \( h \) is Griffiths positive if and only if the induced singular metric \( e^{-\varphi} \) on \( \mathcal{O}_{P(E)}(1) \) is positive.

**Proof.** Since this is a local statement we may assume that we are in the situation above. Then \( e^{-\varphi} \) is positive if and only if \( \varphi_i \) is psh on \( \mathcal{U}_i \) for all \( i \). Moreover, by Proposition 5.1 \( h \) is Griffiths positive if and only if \( \chi_{h^*} \) is psh on \( E \) or equivalently on \( E \setminus 0_E \).

Since \( \xi_i \neq 0 \) on \( \mathcal{U}_i \), in view of (5.3), \( \varphi_i \) is psh there if \( \chi_{h^*} \) is psh. Thus \( e^{-\varphi} \) is positive if \( h \) is Griffiths positive. For the converse, if \( (x, \xi) \in E \setminus 0_E \), then \( \xi_i \neq 0 \) for some \( i \) in some neighborhood \( \mathcal{U} \) of \( (x, \xi) \). Then, by (5.3),
\[
\chi_{h^*}(x,\xi) = \chi_{h^*}(x,\xi/\xi_i) + \log |\xi_i|^2 = \varphi_i(x,[\xi]) + \log |\xi_i|^2
\]
there. Since \( \xi \to [\xi] \) is holomorphic and \( |\xi_i|^2 \) is pluriharmonic where \( \xi_i \neq 0 \), it follows that \( \chi_{h^*} \) is psh in \( \mathcal{U} \) if \( \varphi_i \) is psh. We conclude that \( h \) is Griffiths positive if \( e^{-\varphi} \) is positive.

**Definition 5.3.** We say that a Griffiths positive hermitian metric has analytic singularities if the induced positive metric \( e^{-\varphi} \) on \( \mathcal{O}_{P(E)}(1) \) has analytic singularities.

**Proposition 5.4.** Let \( h \) be a Griffiths positive hermitian metric on a holomorphic vector bundle \( E \). Then \( h \) has analytic singularities if and only if \( \chi_{h^*} \) is psh with analytic singularities on \( E \setminus 0_E \).

**Proof.** Let us assume that we are in the local situation above. Then \( h \) has analytic singularities if and only if \( \varphi_i \) are psh with analytic singularities for all \( i \). In view of the proof of Proposition 5.2 this is in turn equivalent to that \( \chi_{h^*} \) is psh with analytic singularities on \( E \setminus 0_E \).

We do not know whether it is possible to express analytic singularities of \( h \) in terms (of analytic singularities) of the functions \( \log \|u\|_{h^*}^2 \).
Example 5.5. In [HExample 3.6] Hosono constructed a family of examples of singular hermitian metrics that generalize the metrics in Example [11]. Assume that \( E \to X \) is a holomorphic vector bundle with global holomorphic sections \( s_1, \ldots, s_N \). Let \( s \) be the morphism from the dual bundle \( E^* \) to the trivial bundle \( X \times \mathbb{C}^N \) given by \((x, \xi) \mapsto (s_1(x, \xi), \ldots, s_N(x, \xi))\) and let \( h^* \) be the pullback under \( s \) of the trivial metric on \( X \times \mathbb{C}^N \), i.e.,
\[
\langle \xi, \eta \rangle_{h^*(x)} := \langle s(x, \xi), s(x, \eta) \rangle.
\]

Then
\[
\|\xi\|_{h^*(x)}^2 = |s(x, \xi)|^2 = \sum_j |s_j(x, \xi)|^2.
\]

It follows that \( \chi_{h^*}(x) = \log |s(x, \xi)|^2 \) is psh with analytic singularities on \( E^* \). Thus by Proposition [5.3] the dual metric \( h^* \) on \( E = (E^*)^* \) is Griffiths positive with analytic singularities.

Given a Griffiths positive singular metric \( h \), \( \log \det h^* \) is psh, see [R Proposition 1.3], and we can define the degeneracy locus of \( h \) as the unbounded locus of \( \log \det h^* \). The following lemma gives alternative definitions in terms of (the unbounded loci) of \( \chi_{h^*} \) and \( \varphi \).

Lemma 5.6. Assume that \( h \) is a Griffiths positive singular metric on \( E \to X \). Then, using the notation from above and denoting the projection \( E^* \to X \) by \( p \),
\[
\tag{5.5}
L(\log \det h^*) = p(L(\chi_{h^*}) \setminus 0_E) = \pi(L(\varphi)).
\]

In particular, it follows that if \( h \) has analytic singularities, then the degeneracy locus of \( h \) is a subvariety of \( X \).

Proof. In view of (5.5), \( (x, \xi) \in E \setminus 0_E \) is in \( L(\chi_{h^*}) \) if and only if \((x, [\xi]) \in P(E) \) is in \( L(\varphi) \), and thus the second equality in (5.5) follows. The inclusion \( \pi(L(\varphi)) \subset L(\log \det h^*) \) is an immediate consequence of Lemma 3.7 in [LRRS].

Thus it remains to prove that
\[
\tag{5.6}
L(\log \det h^*) \subset p(L(\chi_{h^*}) \setminus 0_E).
\]

Since the statement is local we may assume that \( E = X \times \mathbb{C}^r \). Take \( x \in L(\log \det h^*) \). Then, by definition there is a sequence \( x_k \to x \) such that \( \log \det h^*(x_k) \to -\infty \). This means that there is a sequence \( \varepsilon_k \to 0 \) such that \( h^*(x_k) < \varepsilon_k \), which implies that \( h^*(x_k) \) has at least one eigenvalue less than \( \varepsilon_k \). Thus there are \( \xi_k \in E^*_k = \mathbb{C}^r \) such that \( \|\xi_k\|_{\mathbb{C}^r} = 1 \) and \( \|\xi_k\|_{h^*(x_k)} < \varepsilon_k \). Since \( \|\xi_k\|_{\mathbb{C}^r} = 1 \), \( \{\xi_k\} \) has at least one accumulation point \( \xi \) in \( \mathbb{C}^r \) and thus we can find a subsequence \( (x_k, \xi_k) \to (x, \xi) \). Since \( \|\xi_k\|_{h^*(x_k)} \to 0 \), \( (x, \xi) \in L(\chi_{h^*}) \). Moreover, since \( \|\xi_k\|_{\mathbb{C}^r} = 1 \), \( \|\xi\|_{\mathbb{C}^r} \neq 0 \) and thus \( x \in p(L(\chi_{h^*}) \setminus 0_E) \), which proves (5.6). \( \square \)

6. Construction of Segre and Chern currents, proof of Theorem 1.1

6.1. Construction, basic properties. Assume that \( X \) is a complex manifold of dimension \( n \), that \( E \to X \) is a holomorphic vector bundle of rank \( r \), and that \( h \) is a Griffiths positive hermitian metric with analytic singularities on \( E \). Let \( \pi : P(E) \to X \) be the projectivization of \( E \) and let \( \varphi \) denote the metric on \( L := O_{P(E)}(1) \to P(E) \) induced by \( h \). Then \( \varphi \) has analytic singularities, cf. Definition [5.7] let \( Z \subset P(E) \) denote the unbounded locus of \( \varphi \). Moreover, assume that \( \psi \) is a smooth metric on \( L \) and let \( \theta \) be the corresponding first Chern form.

Next, let \( E_1, \ldots, E_t \) be \( t \) disjoint copies of \( E \), let \( \pi_i \) denote the projections \( P(E_i) \to X \), and \( p_i \) the identifications \( P(E_i) \to P(E) \). Let \( \tilde{\varphi}_i \) denote the metric \( p_i^*\varphi \) on \( \tilde{L}_i := p_i^*L \to P(E_i) \) induced by \( h \) with unbounded locus \( \tilde{Z}_i := p_i^{-1}(Z) \) and let \( \theta_i = p_i^*\theta \) and \( \psi_i = p_i^*\psi \). Moreover, let \( Y \) be the fiber product
\[
Y = P(E_1) \times_X \cdots \times_X P(E_t),
\]
with projections \( \varpi : Y \to P(E_i) \) and \( \pi : Y \to X \). Let \( \varphi_i \) denote the pullback metric \( \varpi_i^*\tilde{\varphi}_i \) on \( L_i := \varpi_i^*\tilde{L}_i \) with unbounded locus \( Z_i := \varpi_i^{-1}(\tilde{Z}_i) \) and let \( \theta_i = \varpi_i^*\theta_i \) and \( \psi_i = \varpi_i^*\psi_i \). Now, in view of Lemma [4.2] (1.7) is a well-defined \((k, k)\)-current.
Remark 6.1. Let $V$ be the degeneracy locus of $h$. Then in $Y \setminus \pi^{-1}V$, $\varphi_j$ are locally bounded by Lemma 6.1 and thus

$$[dd^c \varphi_t]^k_{\theta_1} \wedge \cdots \wedge [dd^c \varphi_t]^{k_r}_{\theta_1} = (dd^c \varphi_t)^{k_1+r-1} \wedge \cdots \wedge (dd^c \varphi_t)^{k_1+r-1},$$

where the right hand side is locally defined in the sense of Bedford-Taylor. Hence outside $V$,

$$s_{k_1}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) = (-1)^k \pi_*((dd^c \varphi_t)^{k_1+r-1} \wedge \cdots \wedge (dd^c \varphi_t)^{k_1+r-1});$$

in particular it is independent of $\theta$, and thus so are $c_k(E, h, \theta)$ and $s_k(E, h, \theta)$.

Lemma 6.2. Let $X$, $E$, and $h$ be as above. Given $x \in X$, there is a neighborhood $x \in U \subset X$ such that in $U$,

$$s_{k_1}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) = S_+ - S_-,$$

where $S_+, S_-$ are closed positive currents.

In view of (1.3), it follows that $s_k(E, h, \theta)$ and $c_k(E, h, \theta)$ are differences of positive closed $(k, k)$-currents; this proves the first part of Theorem 1.1.

Proof. Let us use the notation from above. Since the statement is local we may assume that $X$ is an open neighborhood of $x$ in $\mathbb{C}^n$ and that $E = X \times \mathbb{C}'$ is trivial bundle. Then $P(E_j) \cong X \times Y_j$, where $Y_j \cong \mathbb{P}^{r-1}$. Let $\rho_j$ be the projection $P(E_j) \rightarrow \mathbb{P}^{r-1}$. Moreover, let $\omega_{\text{FS}}$ denote the Fubini-Study metric on $\mathbb{P}^{r-1}$, let $\omega_0$ be the standard Euclidean metric on $X$, and let $\omega_j = \rho_j^* \omega_{\text{FS}} + \pi_j^* \omega_0$.

Then for some large enough $C > 0$, there is a neighborhood $x \in U \subset X$ such that $\tilde{\alpha}_j := C \omega_j$ satisfies that

$$\tilde{\beta}_j := \alpha_j + \tilde{\theta}_j \geq 0$$

in $\pi_j^{-1}U \subset P(E_j)$ for each $j$. Let $\alpha_j = \omega_j^* \tilde{\alpha}_j$ and $\beta_j = \omega_j^* \tilde{\beta}_j$ be the corresponding closed $(1, 1)$-forms on $\pi_j^{-1}U \subset Y$, so that $\theta_j = \beta_j - \alpha_j$.

We claim that in $\pi^{-1}U$, for $m_j \geq 1$,

$$[dd^c \varphi_t]_{\theta_1}^{m_j} \wedge \cdots \wedge [dd^c \varphi_t]_{\theta_1}^{m_1} = T_+ - T_-,$$

where $T_\pm$ are closed positive currents with analytic singularities, cf. the beginning of Section 3. Then in view of (1.4), $s_k(E, h, \theta) \wedge \cdots \wedge s_k(E, h, \theta)$ is of the desired form in $U$.

To prove the claim, first note that if $m_j = 0$, then,

$$[dd^c \varphi_t]_{\theta_1}^{m_1} = [dd^c \varphi_t]_{\theta_1}^0 = 1 \geq 0.$$

Next, assume that

$$T := [dd^c \varphi_t]_{\theta_1}^{m_1} \wedge \cdots \wedge [dd^c \varphi_t]_{\theta_1}^{m_1} = T_+ - T_-,$$

where $m_\kappa \geq 1$, and where $T_\pm$ are as above. Then

$$[dd^c \varphi_t]_{\theta_1}^{m_\kappa} \wedge \cdots \wedge [dd^c \varphi_t]_{\theta_1}^{m_1} = [dd^c \varphi_t]_{\theta_1}^{m_\kappa} \wedge T = dd^c \varphi_t \wedge 1_{Y \setminus Z_\kappa} \circ T_+ \wedge T_- = dd^c \varphi_t \wedge \beta_\kappa \wedge \alpha_\kappa \wedge 1_{Z_\kappa} \circ T_+ \wedge T_- - (dd^c \varphi_t \wedge 1_{Y \setminus Z_\kappa} \circ T_+ \wedge \beta_\kappa \wedge \alpha_\kappa \wedge 1_{Z_\kappa} \circ T_-) - (dd^c \varphi_t \wedge 1_{Y \setminus Z_\kappa} \circ T_- \wedge \beta_\kappa \wedge \alpha_\kappa \wedge 1_{Z_\kappa} \circ T_+) =: T_+ - T_-.$$

Since $T_\pm$ are closed positive currents with analytic singularities and $\beta_\kappa$ and $\alpha_\kappa$ are positive $(1, 1)$-forms, $T_\pm$ are well-defined closed positive currents with analytic singularities. The claim now follows by induction.

$\square$
6.2. Comparison to the smooth case, proof of statement (2) in Theorem 1.1. Note that to prove statement (2) in Theorem 1.1, it suffices to prove that
\[ \alpha_j := (dd^c \varphi_j)^{k_j+r-1}, \]
where we use the notation from Section 6.1. Then \( s_{k_j}(E, h) = (-1)^{k_j} (\pi_j)_* \alpha_j \), cf. (6.1), and thus
\[ s_{k_1}(E, h) \wedge \cdots \wedge s_{k_1}(E, h) = (-1)^k (\pi_1)_* \alpha_1 \wedge \cdots \wedge (\pi_1)_* \alpha_1. \]

Note that in this case
\[ \varpi_j^* \alpha_j = (dd^c \varpi_j^* \varphi_j)^{k_j+r-1} = (dd^c \varphi_j)^{k_j+r-1}, \]
and thus in view of Remark 6.1
\[ \varpi_1^* \alpha_1 \wedge \cdots \wedge \varpi_1^* \alpha_1 = [dd^c \varphi_1]_{\theta_1}^{k_1+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}^{k_1+r-1}, \]
so that
\[ s_{k_1}(E, h) \wedge \cdots \wedge s_{k_1}(E, h) = (-1)^k \pi_* (\varpi_1^* \alpha_1 \wedge \cdots \wedge \varpi_1^* \alpha_1). \]

Now (6.1) follows from the following lemma (with \( Y_j = \mathbf{P}(E_j) \)).

**Lemma 6.3.** Let \( X \) be a complex manifold, let \( \pi_j : Y_j \to X, j = 1, \ldots, t \), be proper submersions, and let \( Y \) be the fiber product \( Y := Y_1 \times_X \cdots \times_X Y_t \) with projections \( \varpi_j : Y \to Y_j \) and \( \pi : Y \to X \). Let \( \alpha_1 \) be a current on \( Y_1 \), and let \( \alpha_2, \ldots, \alpha_t \) be smooth forms on \( Y_2, \ldots, Y_t \), respectively. Then
\[ \pi_* (\varpi_1^* \alpha_1 \wedge \cdots \wedge \varpi_1^* \alpha_1) = (\pi_1)_* \alpha_1 \wedge \cdots \wedge (\pi_1)_* \alpha_1. \]

**Proof.** By induction it is enough to prove the case \( t = 2 \). It is also enough to prove (6.3) locally. We may therefore assume that \( Y_j \cong X \times Z_j \), where \( Z_j \) is a manifold for \( j = 1, 2 \). It is readily verified that
\[ \pi_1^* (\varpi_2)_* \alpha_2 = (\varpi_1)_* \varpi_2^* \alpha_2 \]

since the pushforwards on both sides are just integration along \( Z_2 \). By (2.1), (2.2), (6.4), and the fact that \( \pi_1 \circ \varpi_1 = \pi \), we obtain that
\[ (\pi_2)_* \alpha_2 \wedge (\pi_1)_* \alpha_1 = (\pi_1)_* (\pi_2^* \alpha_2 \wedge \alpha_1) = (\pi_1)_* (\pi_1)_* (\varpi_2^* \alpha_2 \wedge \alpha_1) = (\pi_1)_* (\varpi_2^* \alpha_2 \wedge \varpi_1^* \alpha_1). \]

\[ \square \]

6.3. The cohomology class, proof of statement (1) in Theorem 1.1. Note in view of (1.3) that to prove statement (1) in Theorem 1.1, it is enough to prove that \( s_{k_1}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) \) is cohomologous to \( s_{k_1}(E, g) \wedge \cdots \wedge s_{k_1}(E, g) \), where \( g \) is a smooth metric on \( E \).

From Proposition 1.3, we have that
\[ s_{k_1}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) = (-1)^k \pi_* ([dd^c \varphi]_{\theta}^{k_1+r-1} \wedge \cdots \wedge [dd^c \varphi]_{\theta}^{k_1+r-1}) = (-1)^k \pi_* (\theta_1^{k_1+r-1} \wedge \cdots \wedge \theta_1^{k_1+r-1}) + (-1)^k dd^c \pi_* S, \]
for some current \( S \); here we have used the notation from Section 6.1.

By applying Lemma 6.3 to \( \theta_j^{k_j+r-1} \), noting that \( \theta_j^{k_j+r-1} = \varpi_j^* \theta_j^{k_j+r-1} \), we get that
\[ \pi_* (\theta_j^{k_j+r-1} \wedge \cdots \wedge \theta_j^{k_j+r-1}) = (\pi_j)_* \pi_* (\varpi_j^* \theta_1^{k_j+r-1} \wedge \cdots \wedge (\pi_1)_* \theta_1^{k_j+r-1}) \]

Let \( \eta \) be the metric on \( \mathcal{O}_{\mathbf{P}(E)}(1) \) associated with \( g \). Then \( \theta \) is cohomologous to \( dd^c \eta \) and since \( \pi_* \) commutes with exterior differentiation, it follows from (1.1) that \( (-1)^{k_j} (\pi_j)_* \theta_j^{k_j+r-1} \) is a form in the class of \( s_{k_j}(E, g) \). It follows that \( (-1)^k \) times right hand side of (6.5) is cohomologous to \( s_{k_1}(E, g) \wedge \cdots \wedge s_{k_1}(E, g) \), and we conclude that so is \( s_{k_1}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) \).
6.4. Lelong numbers, proof of statement (3) in Theorem 1.1. We begin by recalling the definition of the Lelong number of a closed positive current. We assume that we are on a complex manifold \( X \) of dimension \( n \) and that around a given point \( a \in X \), we have local coordinates \( z \).

If \( T \) is a closed positive \((p,p)\)-current on \( X \), then the Lelong number of \( T \) at \( a \) can be defined as

\[
\nu(T, a) := \int 1_{(a)}(dd^c \log |z-a|^2)^{n-p} \wedge T,
\]

which is independent of the local coordinate system, see for example [De3] Definition III.5.4 and Corollary III.7.2. Since \( L(\log |z-a|^2) = \{a\} \), which has codimension \( n \), the product in the integrand is indeed well-defined.

Note that the definition of Lelong numbers can be extended to currents that are locally of the form \( T_+ - T_- \), where \( T_\pm \) are closed positive currents, through \( \nu(T, a) = \nu(T_+, a) - \nu(T_-, a) \) if \( T = T_+ - T_- \) in a neighborhood of \( a \). In particular, in view of Lemma 6.2 the Lelong numbers are defined for the currents \( s_k(E, h, \theta) \wedge \cdots \wedge s_k(E, h, \theta) \), and thus in particular for \( t_k(E, h, \theta) \).

Remark 6.4. Let us consider (6.6). For simplicity, assume that \( a = 0 \). Note that by the dimension principle for any \((p,p)\)-current \( T \) that is (locally) the difference of two closed positive currents,

\[
(dd^c \log |z|^2)^{n-p} \wedge T = dd^c \log |z|^2 1_{X \setminus \{0\}} \wedge \cdots \wedge dd^c \log |z|^2 1_{X \setminus \{0\}} \wedge T,
\]

cf. Proposition 2.2.

Now assume that \( T = s_k(E, h, \theta) \wedge \cdots \wedge s_k(E, h, \theta) \), i.e., \( T = (-1)^k \pi_* \mu \), where

\[
\mu = [dd^c \varphi_1]_{\theta_1}^{k_1+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}^{k_1+r-1}
\]

and we are using the notation from Section 6.1. Notice that \( \log |\pi^* z|^2 \) is psh with analytic singularities on \( Y \) with unbounded locus \( Z := \pi^{-1}\{0\} \). Thus, in view of Lemma 6.2, arguing as in the proof of Lemma 6.2 (regarding \( \log |\pi^* z|^2 \) as a metric on the trivial line bundle over \( Y \)), one gets that

\[
dd^c \log |\pi^* z|^2 1_{Y \setminus Z} \wedge \cdots \wedge dd^c \log |\pi^* z|^2 1_{Y \setminus Z} \wedge \mu
\]

is a globally defined current that in a neighborhood of \( Z \) is the difference of two closed positive currents with analytic singularities.

Next, note that if \( u^{(i)} \) is a sequence of smooth psh functions decreasing to \( \log |z|^2 \), then \( \pi^* u^{(i)} \) is a sequence of smooth psh functions decreasing to \( \log |\pi^* z|^2 \). Using (2.1), Propositions 2.3 and 3.2 and (2.4) we conclude that

\[
1_{\{0\}}(dd^c \log |z|^2)^{n-k} \wedge T = \pi_* (1_{Z} dd^c \log |\pi^* z|^2 1_{Y \setminus Z} \wedge \cdots \wedge dd^c \log |\pi^* z|^2 1_{Y \setminus Z} \wedge \mu).
\]

Proof of statement (3) in Theorem 1.1. Let us choose coordinates so that \( a = 0 \). Since Lelong numbers are locally defined, cf. (6.6), we may assume that we are in a neighborhood of \( 0 \in U \subset X \) as in the proof of Lemma 6.2. Let \( \theta \) and \( \theta' \) be two first Chern forms on \( \mathcal{O}_{P(E)}(1) \) corresponding to smooth metrics \( \psi \) and \( \psi' \), respectively, and let \( \mu = [dd^c \varphi_1]_{\theta_1}^{k_1+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}^{k_1+r-1} \) and \( \mu' = [dd^c \varphi_1]_{\theta'_1}^{k_1+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta'_1}^{k_1+r-1} \) be the corresponding currents on \( Y \), where we use the notation from Section 6.1 and \( \theta' \) is defined analogously to \( \theta \).

We claim that in \( \pi^{-1} U \)

\[
\mu - \mu' = \sum d\beta_i \wedge \mu_i,
\]

where \( \beta_i \) are smooth forms and \( \mu_i \) are closed positive currents with analytic singularities. Using the notation from Remark 6.4 let \( S \wedge T \) denote the operator \( T \mapsto 1_Z dd^c \log |\pi^* z|^2 1_{Y \setminus Z} \wedge \cdots \wedge dd^c \log |\pi^* z|^2 1_{Y \setminus Z} \wedge T \). Since \( \beta_i \) is smooth, applying \( S \) commutes with multiplication with \( d \beta_i \), and thus, since \( S \wedge \mu_i \) is closed, we get

\[
S \wedge d\beta_i \wedge \mu_i = d\beta_i \wedge S \wedge \mu_i = d(\beta_i \wedge S \wedge \mu_i) =: d\tau_i,
\]

where \( \tau_i = \beta_i \wedge S \wedge \mu_i \) has support on \( Z \).


Hence, in view of Remark 6.4 taking the claim for granted,
\[
(-1)^k \left( \nu(s_{k_i}(E, h, \theta) \wedge \cdots \wedge s_{k}(E, h, \theta), 0) - \nu(s_{k_i}(E, h, \theta') \wedge \cdots \wedge s_{k}(E, h, \theta'), 0) \right) = \nu(\pi_*\mu, 0) - \nu(\pi_*\mu', 0) = \int \pi_*(S \wedge (\mu - \mu')) = \int d\pi_*\tau_i = 0,
\]
where the last equality follows by Stokes' theorem since the \(\pi_*\tau_i\) have support on \(\pi(Z) = \{0\}\). This proves (3) in Theorem 1.1.

It remains to prove the claim. First note that, since \([dd^c\varphi_1]_0 = [dd^c\varphi_1]_0 = 1\), \([dd^c\varphi_1]_0 - \cdots \) vanishes and is in particular of the form (6.7). Next assume that we have proven that
\[
T - T' := [dd^c\varphi_{\kappa}]_{\theta_\kappa - \cdots} - [dd^c\varphi_1]_{\theta_1} - \cdots - [dd^c\varphi_1]_{\theta_1} = \sum i \Delta \gamma_i \wedge T_i
\]
for some smooth forms \(\gamma_i\) and closed positive currents with analytic singularities \(T_i\), where \(m_\kappa \geq 1\).

By the assumption on \(U\), \(T = T_+ \wedge T_-\) in \(\pi^{-1}U\), where \(T_\pm\) are closed positive currents with analytic singularities. Now
\[
[dd^c\varphi_{\kappa}]_{\theta_\kappa - \cdots} - [dd^c\varphi_1]_{\theta_1} - \cdots - [dd^c\varphi_1]_{\theta_1} = [dd^c\varphi_\kappa]_{\theta_\kappa} \wedge T - [dd^c\varphi_1]_{\theta_1} \wedge T =
\]
\[
\sum i \Delta \gamma_i \wedge dd^c\varphi_\kappa \wedge 1_{Y \wedge Z} T_i + dd^c(\psi - \psi') \wedge 1_{Z}(T_+ - T_-) + \sum i \Delta \gamma_i \wedge dd^c\varphi_\kappa \wedge 1_{Z} T_i,
\]
which is of the form in the right hand side of (6.7) since \(\psi_\kappa\) and \(\psi'_\kappa\) are smooth. Here we have used that set of closed positive currents with analytic singularities is closed under multiplication with \(1_U\), where \(U\) is an constructible set and Remark 6.3. The claim now follows by induction. \(\square\)

7. Comparison with [LRRS], Proof of Theorem 1.2

Assume that \(h\) is a singular Griffiths positive (negative) metric on a holomorphic vector bundle \(E \rightarrow X\) over a complex manifold \(X\) of dimension \(n\), such that that the degeneracy locus of \(h\) is contained in a variety \(V \subset X\). In [LRRS] the first three authors together with Rupenthal defined Chern and Segre currents, \(c_k(E, h)\) and \(s_k(E, h)\), for \(k \leq \text{codim} V\). Let us briefly recall the construction. Locally, \(h\) can be approximated by an increasing (decreasing) sequence \(h_\varepsilon\) of Griffiths positive (negative) smooth metrics, see, e.g., [BP Proposition 3.1] or [R] Proposition 1.3. Theorem 1.11 in [LRRS] asserts that the iterated limit
\[
\lim_{\varepsilon_1 \rightarrow 0} \cdots \lim_{\varepsilon_k \rightarrow 0} s_{k_1}(E, h_{\varepsilon_1}) \wedge \cdots \wedge s_{k}(E, h_{\varepsilon_k})
\]
exists as a current and is independent of the choice of \(h_\varepsilon\) for \(k_1 + \cdots + k_\ell \leq \text{codim} V\); in particular, it follows that \(s_{k_1}(E, h) \wedge \cdots \wedge s_{k}(E, h)\), locally given as (7.1), defines a global current on \(X\). Moreover, the Chern currents \(c_k(E, h)\), defined from \(s_{k_1}(E, h) \wedge \cdots \wedge s_{k}(E, h)\) analogously to (1.6), and the Segre currents \(s_k(E, h)\) coincide with the corresponding Chern and Segre forms where \(h\) is smooth, and are in the classes \(c_k(E)\) and \(s_k(E)\), respectively, when \(X\) is compact.

Assume that \(h\) is Griffiths positive and let \(\varphi_\varepsilon\) be the smooth metric on \(\mathcal{P}(E) \rightarrow X\) induced by \(h_\varepsilon\). Then \(\varphi_\varepsilon\) is a sequence of smooth positive metrics on \(\mathcal{O}(\mathcal{P}(E))(1)\) decreasing to \(\varphi\). Let \(\omega_\varepsilon\) be the first Chern form of \(\varphi_\varepsilon\). Then \(s_{k_1}(E, h) \wedge \cdots \wedge s_{k}(E, h)\) satisfies the following recursion for \(t > 0\):
\[
s_{k_1}(E, h) \wedge \cdots \wedge s_{k}(E, h) = \lim_{\varepsilon \rightarrow 0} (-1)^{k_1} \pi_*(\omega_\varepsilon^{k_1 + r - 1}) \wedge s_{k_1}(E, h) \wedge \cdots \wedge s_{k}(E, h)
\]
\footnote{In [LRRS] the limit is taken over certain subsequences of \(h_\varepsilon\), but this is in fact not necessary; see the end of the proof of Proposition 4.6 in [LRRS].}
Remark 7.1. Assume that \( h \) is Griffiths positive. Let us use the notation from Section 6.1 and denote the sequences of positive metrics on \( Y \) induced by \( h_x \) by \( \varphi_{j,e} \). Moreover, assume that we are outside the degeneracy locus of \( h \). Then the induced \( \varphi_j \) are locally bounded and thus, by Proposition 2.3

\[
(-1)^k s_{k_1}(E, h) \wedge \cdots \wedge s_{k_1}(E, h) = \lim_{\varepsilon_1 \to 0} \cdots \lim_{\varepsilon_r \to 0} \pi_* ((dd^c \varphi_{t,\varepsilon})^{k_1+r-1} \wedge \cdots \wedge (dd^c \varphi_{1,\varepsilon})^{k_1+r-1})
\]

\[
= \pi_* ((dd^c \varphi_t)^{k_1+r-1} \wedge \cdots \wedge (dd^c \varphi_1)^{k_1+r-1}).
\]

\[\square\]

To prove Theorem 1.2 we need to recall some auxiliary results from [LRRS]. First, following [LRRS] we say that a smooth \((n-k, n-k)\)-form \( \beta \) is a bump form at a point \( x \in X \) if it is strongly positive, and such that for some (or equivalently for any) Kähler form \( \omega \) defined near \( x \), there exists a constant \( C > 0 \) such that \( C\omega^{n-k} \leq \beta \) as strongly positive forms in a neighborhood of \( x \).

Lemma 7.2. Let \( V \subset X \) be a subvariety. Then for each \( k \leq \text{codim} V \) and each point \( x \in V \), there exists a bump form \( \beta \) at \( x \in \text{bidegree} (n-k, n-k) \) with arbitrarily small support such that \( dd^c \beta \) has support in \( X \setminus V \).

Proof. We construct the bump form \( \beta \) as in the proof of Lemma 4.3 in [LRRS] (with \( k \) equal to \( k + q \) in that proof). By that proof, one may write \( \beta \) as a sum of terms, such that each term in some local coordinate system \((z', z'') \in \mathbb{C}^{n-k} \times \mathbb{C}^k\) is of the form \( \chi_1 \chi_2 \beta_0 \), where \( \beta_0 = idz'_1 \wedge dz'_2 \wedge \cdots \wedge idz'_{n-k} \wedge dz''_1 \wedge \cdots \wedge dz''_{n-k} \) and \( \chi_1 \) and \( \chi_2 \) are cutoff functions in the variables \( z' \) and \( z'' \), respectively, such that \( \chi_2 \) is constant in a neighborhood of \( \text{supp} \chi_1 \chi_2 \cap V \). It then suffices to prove that \( d(\chi_1 \chi_2 \beta_0) \) has support in \( X \setminus V \). This holds since \( \beta_0 \) has full degree in the \( z' \)-variables so that \( d(\chi_1 \chi_2 \beta_0) = \chi_1 d\chi_2 \wedge \beta_0 \), and \( \chi_1 d\chi_2 \) has support in the set where \( \chi_1 \chi_2 \neq 0 \) and \( \chi_2 \) is not constant, which is contained in \( X \setminus V \). \[\square\]

The next result is Lemma 4.5 in [LRRS].

Lemma 7.3. Let \( S \) and \( T \) be two closed, positive \((k, k)\)-currents on \( X \) such that \( S = T \) outside a subvariety \( V \) with \( \text{codim} V \geq k \), and assume that for each point of \( x \in V \), there exists an \((n-k, n-k)\) bump form \( \beta \) at \( x \) with arbitrarily small support such that

\[
\int_X S \wedge \beta = \int_X T \wedge \beta.
\]

Then \( S = T \) everywhere.

Proof of Theorem 1.2. We will proceed by induction. Let us first choose \( t \geq 2 \) and assume that we have proved that

\[
(7.3) \quad s_{k_{t-1}}(E, h, \theta) \wedge \cdots \wedge s_{k_1}(E, h, \theta) = s_{k_{t-1}}(E, h) \wedge \cdots \wedge s_{k_1}(E, h).
\]

Let us use the notation from Section 6.1. Moreover, let \( Y' \) be the fiber product

\[
Y' = P(E_{t-1}) \times_X \cdots \times_X P(E_1),
\]

with projections \( \varphi'_j : Y' \to P(E_j) \), \( \varphi'_i : Y' \to X \), and \( p : Y \to Y' \). Then \( Y = P(E_t) \times_X Y' \).

Let \( \varphi_j \) denote the pullback metric \( (\varphi'_j)^* \varphi_j \) on \( L'_j := (\varphi'_j)^* L_j \) and let \( \theta'_j = (\varphi'_j)^* \theta_j \). Let \( \bar{\varphi}_t \) denote the metric on \( \bar{L}_t \) induced by \( h_{\bar{e}} \), let \( \bar{\varphi}_e \) denote the pullback \( \varphi'_e \bar{\varphi}_e \) to \( Y \), and let \( \bar{\omega}_e \) and \( \omega_e \) denote the corresponding first Chern forms. Let

\[
\mu' = [dd^c \varphi'_{t-1}]_{\theta'_{t-1}}^{k_{t-1}+r-1} \wedge \cdots \wedge [dd^c \varphi'_1]_{\theta'_1}^{k_{t-1}+r-1}
\]

and let \( \mu = p^* \mu' \); by regularization

\[
\mu = [dd^c \varphi_{t-1}]_{\theta_{t-1}}^{k_{t-1}+r-1} \wedge \cdots \wedge [dd^c \varphi_1]_{\theta_1}^{k_{t-1}+r-1}.
\]
Now, using the induction hypothesis \((7.3)\) and Lemma \(6.3\) we can rewrite \((7.2)\) as

\[ s_{k_i}(E, h) \wedge \cdots \wedge s_{k_i}(E, h) = (-1)^k \lim_{\varepsilon \to 0} \pi_{t} \omega_{e_{k_i}^{1+r-1}} \wedge \pi_{t}' \mu' = (-1)^k \lim_{\varepsilon \to 0} \pi_{s} (\omega_{e_{k_i}^{1+r-1}} \wedge \mu). \]

Moreover,

\[ s_{k_i}(E, h, \theta) \wedge \cdots \wedge s_{k_i}(E, h, \theta) = (-1)^k \pi^{s} (\frac{dd^c \varphi_{t}}{\theta} \omega_{e_{k_i}^{1+r-1}} \wedge \mu). \]

Since \(k \leq \text{codim} V\), by Lemma \(7.2\) for each \(x \in V\) there is a bump form \(\beta\) at \(x\) of bidegree \((n - k, n - k)\) with arbitrarily small support such that \(dd^c \beta\) vanishes in a neighborhood of \(V\). Note that \(\pi_{t}(L(\tilde{\varphi}_{t})) \subset V\) in view of Lemma \(5.6\). It follows that

\[ Z_{t} = c_{t}^{-1} L(\tilde{\varphi}_{t}) \subset c_{t}^{-1} \pi_{t}^{-1}V = \pi_{t}^{-1}V \]

and thus \(dd^c \pi^{s} \beta\) vanishes in a neighborhood of \(Z_{t} \subset Y\). Hence, by Lemma \(4.4\) (applied to \(T = \mu\))

\[ \int_{X} s_{k_i}(E, h, \theta) \wedge \cdots \wedge s_{k_i}(E, h, \theta) \wedge \beta = (-1)^k \int_{Y} [dd^c \varphi_{t}]_{\theta}^{1+r-1} \wedge \mu \wedge \pi^{s} \beta = (-1)^k \lim_{\varepsilon \to 0} \int_{Y} \omega_{e_{k_i}^{1+r-1}} \wedge \mu \wedge \pi^{s} \beta = \int_{X} s_{k_i}(E, h) \wedge \cdots \wedge s_{k_i}(E, h) \wedge \beta. \]

In view of Remarks \(6.1\) and \(7.1\) \((1.9)\) holds outside \(V\), and thus by Lemma \(7.3\) it holds everywhere.

It remains to prove \((1.9)\) for \(t = 1\). This follows, in fact, by an easier version of the argument above. If \(\beta\) is a bump form as above, then by Lemma \(4.4\) (with \(T = 1\))

\[ \int_{X} s_{k_i}(E, h, \theta) \wedge \beta = (-1)^k \int_{P(E_{1})} [dd^c \varphi_{1}]_{\theta}^{1+r-1} \wedge \pi^{s}_{1} \beta = (-1)^k \lim_{\varepsilon \to 0} \int_{P(E_{1})} \omega_{e_{k_i}^{1+r-1}} \wedge \pi^{s}_{1} \beta = \int_{X} s_{k_i}(E, h) \wedge \beta \]

and again \((1.9)\) follows from Lemma \(7.3\).

\[ \square \]

8. Remarks and examples

Let us start by discussing the uniqueness of the Chern and Segre currents. Assume that \(X\) is a complex manifold and that \(V \subset X\) is a subvariety of pure codimension \(p\). Moreover assume that \(T_{1}\) and \(T_{2}\) are closed positive \((p, p)\)-currents on \(X\) that coincide outside \(V\), and that the Lelong numbers of \(T_{1}\) and \(T_{2}\) coincide at each \(x \in V\). We claim that then \(T_{1} = T_{2}\). Indeed, since \(T := T_{1} - T_{2}\) is a closed normal \((p, p)\)-current with support on \(V\) it follows that \(T = \sum a_{j}[V_{j}]\), where \(V_{j}\) are the irreducible components of \(V\), see, e.g., [De3 Corollary III.2.14]. Next, by assumption the Lelong number of \(T\) at each point in \(V\) is zero and therefore \(a_{j} = 0\) for each \(j\).

If \(T_{1}\) and \(T_{2}\) are closed positive \((k, k)\)-currents that coincide outside \(V\), where \(k < p\), then \(1_{V}T_{j}\) vanishes for \(j = 1, 2\) by the dimension principle, and hence \(T_{1} = T_{2}\).

Now assume that we are in the situation of Theorem \(1.1\) and that \(L(\log \det h^{*}) \subset V\). Then by Remark \(6.3\) \(c_{k}(E, h, \theta)\) and \(s_{k}(E, h, \theta)\) are independent of \(\theta\) outside \(V\). Since they are of bidegree \((k, k)\) and (locally) differences of closed positive currents it follows in view of \((3)\) that they are independent of \(\theta\) for \(k \leq p\). Note that if \(h\) is smooth outside \(V\) then \(c_{k}(E, h, \theta)\) and \(s_{k}(E, h, \theta)\) are uniquely determined by the condition \((2)\) for \(k < p\).

On the other hand if \(k > p\), let \(\alpha\) and \(\beta\) be real smooth forms of bidegree \((k - p, k - p)\) such that \(\alpha - \beta\) is exact. Then \((\alpha - \beta) \wedge [V] \neq 0\) has zero Lelong numbers everywhere, is cohomologous to zero, and vanishes outside \(V\). Thus there is no reason to expect \(c_{k}(E, h, \theta)\) and \(s_{k}(E, h, \theta)\) to be independent of \(\theta\) for \(k > p\) in general.

Let us consider some simple examples, where we can compute the Segre and Chern currents explicitly.
Example 8.1. Let \( L \to X \) be a line bundle and \( e^{-\varphi} \) a Griffiths positive metric with analytic singularities. Then \( \mathcal{O}_{P(L)}(1) = L \) and \( e^{-\varphi} = h \), and thus
\[
s_k(L,h,\theta) = [dd^c \varphi]_0^k = (dd^c \varphi)^k + \sum_{\ell=0}^{k-1} \theta^{k-\ell} \wedge 1_Z (dd^c \varphi)^\ell,
\]
where \( Z \) is the unbounded locus of \( \varphi \). Classically, by the Bedford-Taylor-Demailly theory, for a general \( \varphi \), \( (dd^c \varphi)^k \) is well-defined only for \( k = 1 \); if \( \varphi \) has analytic singularities it is well-defined for \( k \leq \text{codim} Z = p \).

In fact, it is not hard to find examples of psh functions \( u \) with analytic singularities and sequences \( u^{(i)} \) of psh functions decreasing to \( u \) where the corresponding sequences \( (dd^c u^{(i)})^k \) converge to different positive currents for \( k > p \), see, e.g., [ABW], Example 3.2]. In particular, this implies that the construction in [LRRS] cannot extend to \( k > p \) in general.

---

Example 8.2. Let \( X = \mathbb{P}^n \), \( L = \mathcal{O}_{\mathbb{P}^n}(1) \), and \( h = e^{-\varphi} \), where \( \varphi = \log |s|^2 \) and \( s \) is a non-trivial global holomorphic section of \( L \), cf. Example 4.1. Then the unbounded locus of \( \varphi \) is the hyperplane \( Z = \{ s = 0 \} \subset \mathbb{P}^n \) and thus \( (dd^c \varphi)^m \) is defined classically by the Bedford-Taylor-Demailly theory only for \( m = 1 \), cf. Example 8.1. By the Poincaré-Lelong formula, \( dd^c \varphi = [s = 0] = [Z] \), cf. [De4], Example 2.2. It follows that
\[
(dd^c \varphi)^2 = dd^c (\varphi 1_{X \setminus Z} dd^c \varphi) = 0
\]
and thus \( (dd^c \varphi)^m = 0 \) for all \( m > 1 \). Hence, if \( \theta \) is the first Chern form of a smooth metric on \( L \), then
\[
[dd^c \varphi]_0^m = \theta^{m-1} \wedge [s = 0].
\]

Since \( L \) is a line bundle, \( \mathbb{P}(L) = X \) and \( \mathcal{O}_{\mathbb{P}(L)}(1), h) = (L, e^{-\varphi}) \). Moreover, \( Y = X \) and \( \varphi_j = \varphi \) and \( \theta_j = \theta \) for each \( j \). Thus
\[
(8.1) \quad s_k(L,h,\theta) \wedge \cdots \wedge s_k(L,h,\theta) = (-1)^k [dd^c \varphi]_0^k = (-1)^k \theta^{k-1} \wedge [s = 0].
\]
In particular, \( (8.1) \) depends on \( \theta \) as soon as \( k > 1 \). In this case it is easy to see that the Lelong numbers are independent of \( \theta \), since \( \theta \) is smooth.

Note that \( s_k(E,h) \) and \( c_k(E,h) \) are well-defined in the classical or [LRRS] sense only for \( k \leq 1 \); it holds that
\[
c_1(E,h) = -s_1(E,h) = dd^c \varphi = [s = 0].
\]

Example 8.3. Let \( E \) be a trivial rank 2 bundle over \( X = \mathbb{C}^2 \) with coordinates \( x = (x_1, x_2) \) and let \( h \) be the singular metric \( h = e^{-\log |x|^2} \text{Id} \). In view of Section 5 in the open set \( U \) the induced metric \( \varphi \) on \( \mathcal{O}_{\mathbb{P}(E)}(1) \) is given by
\[
\varphi[(x, [\xi])] = \log |x|^2 + \log |\xi|/|x|^2.
\]
In particular, it follows that \( h \) is Griffiths positive with analytic singularities.

Since the unbounded locus of \( \varphi \), \( Z = \{ x = 0 \} \), has codimension 2 in \( \mathbb{P}(E) \), \( (dd^c \varphi)^m \) is classically well-defined for \( m \leq 2 \). Note that \( dd^c \varphi = dd^c \log |x|^2 + \omega_{FS} \), where \( \omega_{FS} \) is the Fubini-Study metric on the fibers \( \pi^{-1}(x) \cong \mathbb{P}_x^1 \), and
\[
(8.2) \quad (dd^c \varphi)^2 = (dd^c \log |x|^2)^2 + 2 \omega_{FS} \wedge dd^c \log |x|^2 = [x = 0] + 2 \omega_{FS} \wedge dd^c \log |x|^2
\]
since \( \omega_{FS}^2 \) vanishes for degree reasons. It follows that
\[
(dd^c \varphi)^3 = dd^c (\varphi 1_{\mathbb{P}(E) \setminus Z} (dd^c \varphi)^2) = dd^c (\varphi 2 \omega_{FS} \wedge dd^c \log |x|^2) = (\omega_{FS} + dd^c \log |x|^2) \wedge (2 \omega_{FS} \wedge dd^c \log |x|^2) = 2 \omega_{FS} \wedge [x = 0],
\]
where we have again used that \( \omega_{FS}^2 = 0 \). Thus if \( \theta \) is the first Chern form of a smooth metric on \( \mathcal{O}_{\mathbb{P}(E)}(1) \),
\[
[dd^c \varphi]_0^3 = (dd^c \varphi)^3 + \theta + 1_Z (dd^c \varphi)^2 + \theta^2 \wedge 1_Z dd^c \varphi = 2 \omega_{FS} \wedge [x = 0] + \theta \wedge [x = 0],
\]
where the last term in the middle expression vanishes by the dimension principle since codim $Z = 2$. Therefore

$$s_2(E, h, \theta) = \pi_4[dd^c\varphi]_0^3 = 3[0].$$

In view of (8.2), $c_1(E, h, \theta) = -s_1(E, h, \theta) = 2dd^c\log |x|^2$, and thus by (1.8) we get that $c_2(E, h, \theta) = [0]$. \hfill \Box

A naive attempt would be to define Segre currents as the pushforward of $(dd^c\varphi)^{k+r-1}$ instead of $[dd^c\varphi]_0^{k+r-1}$. Since $(dd^c\varphi)^m$ coincides with the classical Bedford-Taylor-Demailly Monge-Ampère product where $\varphi$ is locally bounded, $\pi_4(dd^c\varphi)^{k+r-1}$ coincides with $s_k(E, h)$ where $h$ is smooth, cf. Lemma 5.6. Example 8.2, however, shows that the current $\pi_4(dd^c\varphi)^{k+r-1}$ is not in $s_k(E)$ in general; in that example $(dd^c\varphi)^m = 0$ for $m > 1$, whereas $s_k(E) \neq 0$ for $0 \leq k \leq n$. Moreover, Example 8.3 shows that the Lelong number of $s_k(\pi_4(dd^c\varphi)^{k+r-1})$ is not equal to the Lelong number of $s_k(E, h, \theta)$ in general. Indeed, note that in that example $\pi_4(dd^c\varphi)^3$ equals $2[0]$ and thus has Lelong number 2 at the origin, whereas the Lelong number at the origin of $s_2(E, h, \theta)$ is 3.

The following example shows that the products (1.7) of Segre currents are not commutative in general.

**Example 8.4.** Let $X$ be the unit ball in $\mathbb{C}^3$ with coordinates $x = (z, \zeta_1, \zeta_2)$, and let $E = X \times \mathbb{C}^2 \rightarrow X$ be the trivial vector bundle of rank 2. Let $h$ be the singular hermitian metric on $E$ whose dual metric $h^*$ on $E^*$ is given by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & |z|^2 \end{bmatrix}$. Then in view of Section 5, the induced metric $\varphi$ on $O_{\mathbb{P}(E)}(1)$ is given by $\varphi(x, [\xi]) = \log |z|^2 + \log |\zeta_2/\zeta_1|^2$ and $\varphi_2 = \log |z|^2$ in $U_1$ and $U_2$, respectively. It follows that

$$dd^c \varphi = dd^c(\log |z|^2 + \log |\zeta_2|^2) = [Z] + [W],$$

where $Z = \{ z = 0 \}$ and $W = \{ \zeta_2 = 0 \}$.

Moreover let $g$ be the smooth metric on $E$ given by the matrix $\begin{bmatrix} 1 & |\zeta|^2 \\ |\zeta|^2 & 1 \end{bmatrix}$. A computation yields that the curvature form at $\zeta = 0$ is $\Theta^g|_{\zeta = 0} = \begin{bmatrix} 0 & \partial\bar{\partial}|\zeta|^2 \\ \bar{\partial}\zeta \zeta \partial |\zeta|^2 & 0 \end{bmatrix}$ so that $\frac{1}{2\pi} \Theta^g|_{\zeta = 0} = -\begin{bmatrix} d\bar{d}\zeta |\zeta|^2 \\ 0 \end{bmatrix}$. Thus at $\zeta = 0$, in view of (1.2),

$$s_1(E, g) = -c_1(E, g) = 0, \quad c_2(E, g) = -(dd^c|\zeta|^2)^2, \quad s_2(E, g) = c_1(E, g)^2 - c_2(E, g) = (dd^c|\zeta|^2)^2.$$

Let $\theta$ be the first Chern form of the smooth metric $\psi$ on $O_{\mathbb{P}(E)}(1)$ induced by $g$. Then at $(x, [\xi]) \in \mathbb{P}(E)$

$$\theta = dd^c \psi = \omega^\varphi_{FS} - \frac{i}{2\pi|\xi|_g} \Theta^\varphi|_{\xi \xi},$$

where $\Theta^\varphi$ is the curvature form on $E_2$ and $\omega^\varphi_{FS}$ is the induced Fubini-Study metric on the fiber $\pi^{-1}(x) = \mathbb{P}(E_x) \cong \mathbb{P}^1$, see, e.g., the beginning of the proof of Proposition 3.1 in [C] or the beginning of Section 2 in [D].

Note that at $\zeta = 0$, $g$ is just the standard Euclidean metric on $\mathbb{C}^2$, so that $\omega^\varphi_{FS}$ is just the standard Fubini-Study metric $\omega_{FS}$ on $\mathbb{P}^1$. Moreover, $\Theta^\varphi|_{\zeta = 0} = - (\Theta^\psi)^T|_{\zeta = 0} = -\begin{bmatrix} 0 & \partial\bar{\partial}|\zeta|^2 \\ \bar{\partial}\zeta \zeta \partial |\zeta|^2 & 0 \end{bmatrix}$, where $T$ denotes transpose. In particular, for $(x, [\xi])$ such that $\zeta = 0$ and $\xi_2 = 0$, $\Theta^\varphi|_{\xi \xi} = 0$. Hence at $\zeta = 0$,

$$\theta \wedge [W] = \omega_{FS} \wedge [W] = 0,$$

where the last equality follows for degree reasons. Therefore, for $m > 1$, noting that $(dd^c\varphi)^m = 0$, $(dd^c\varphi)^m = \theta^{m-1} \wedge ([Z] + [W]) = \theta^{m-1} \wedge [Z]$ at $\zeta = 0$. More generally, let $E_1$ and $E_2$ be copies of
In view of (2.2), (6.2), and (6.3) it follows that
\[ \pi_s^*(\theta_2^{k_2+1} \wedge \theta_1^{k_1+1} \wedge [Z]) = (-1)^{k_1+k_2}s_k(E, g) \wedge s_k(E, g) \wedge [Z]. \]
Hence at \( \zeta = 0 \)
\[ s_1(E, h, \theta) \wedge s_2(E, h, \theta) = -\pi_s^*([dd^c \varphi_2]_\theta^2 \wedge [dd^c \varphi_1]_\theta^3) = -\pi_s^*(\theta_2^3 \wedge \theta_1^2 \wedge [Z]) = -s_1(E, g) \wedge s_1(E, g) \wedge [Z] = 0, \]
and similarly
\[ s_2(E, h, \theta) \wedge s_1(E, h, \theta) = -s_2(E, g) \wedge s_0(E, g) \wedge [Z] = -(dd^c|\zeta|^2)^2 \wedge [Z] \neq 0. \]
Thus \( s_1(E, h, \theta) \wedge s_2(E, h, \theta) \neq s_2(E, h, \theta) \wedge s_1(E, h, \theta) \) in this case. \( \square \)

References