COUNTING SPANNING TREES IN DOUBLE NESTED GRAPHS

FERNANDO TURA

Abstract. In this paper we give a linear time algorithm for determining the number of spanning trees of a double nested graph. This class of graphs is a bipartite graph (two color classes) which admits a partition on both color classes into cells with a nesting property imposed. The algorithm proposed here takes advantage of the structure of these graphs. In this way, it works taking the values of vertices degree with an additional increment, such that the number of spanning trees of $G$ is computed as the product of a set of values which are associated with the vertices of $G$. Our proofs are based on Kirchhoff matrix tree theorem which expresses the number of spanning trees of a graph in terms of the cofactor of its Kirchhoff matrix. We finish the paper by applying the algorithm for a special cases of bipartite graphs.

1. Introduction

A spanning tree of a connected undirected graph $G$ on $n$ vertices is a connected $(n-1)$-edge subgraph of $G$. The problem of computing the number of spanning trees on the graph $G$ is an important problem in graph theory. In this context, there are a lot of papers that derive formulas and algorithms (see [7, 8, 9, 10]). In particular, if $K_{m,n}$ is the complete bipartite graph, it is very well known that the number of spanning tree is equal to $m^{n-1}n^{m-1}$ [6].

This paper is concerned with double nested graphs, also called bipartite chain graphs. This class of graphs plays an important role in the study of extremal graphs, a branch of graph theory, well known as spectral graph theory [2]. In fact, among all connected bipartite graphs with fixed order and size, the graphs with maximal index (largest eigenvalue of adjacency matrix) are double nested graphs [11].

A double nested graph is a bipartite graph (two color classes) which admits a partition on both color classes into cells with a nesting property imposed. Another way to characterize this class of graphs is through forbidden induced subgraphs. It is known that a double nested graph is characterized as being \{2$K_2$, $C_3$, $C_5$\}-free graphs. Linear time algorithms for recognizing this class of graphs are given in [12].

The goal of this paper is to give a linear time algorithm for determining the number of spanning trees of double nested graphs. The algorithm proposed here is based on the linear time algorithms of diagonalization matrices associates to graphs. In [4] was presented a linear time algorithm for computing a diagonal matrix congruent to $A + xI$, where $A$ is the adjacency matrix of a threshold graph and $x \in \mathbb{R}$. Although the main application of the diagonalization algorithm for threshold graphs is to localize the eigenvalues, it also has been used as a theoretical tool. For example, in [5] the diagonalization algorithm was used to prove that there is no threshold graph with eigenvalues in the interval $(-1, 0)$. Recently, in [1] a similar procedure was given for localizing the adjacency eigenvalues of chain graphs.

The proofs of ours results are based on Kirchhoff Matrix Tree Theorem [3], which expresses the number of spanning trees of a graph in terms of the cofactor of its Kirchhoff matrix.
In this paper we also apply the algorithm for some double nested graphs. Then, for a special cases, we determine the number of spanning trees of some bipartite graphs that contain few cycles.

Here is an outline of the remainder of this paper. In Section 2, we present some definitions and background results. In Section 3, we present the algorithm for determining the number of spanning trees of a double nested graph, as well as its correctness. In Section 4, we finish the paper by applying the algorithm for a special cases of bipartite graphs.

2. Preliminares

We consider finite undirected graphs with non loops or multiple edges. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively. The neighborhood $N(v)$ of a vertex $v$ of $G$ is the set of all the vertices of $G$ which are adjacent to $v$. We use $d(v)$ to denote the degree of vertex $v$, that is the number of edges incident on $v$, thus, $d(v) = N(v)$.

2.1. Double nested graphs. Recall first that a bipartite graph $G$ with bipartition $(U, V)$ is a double nested graph if there exist partitions $U = U_1 \cup U_2 \cup \ldots \cup U_h$ and $V = V_1 \cup V_2 \cup \ldots \cup V_h$, such that $U_i$ and $V_i$ are non-empty sets, and the neighborhood of each vertex in $U_i$ is $V_1 \cup V_2 \cup \ldots \cup V_{h+1-i}$ for $1 \leq i \leq h$. If $|U_i| = m_i$ ($i = 1, 2, \ldots, h$) and $|V_i| = n_i$ ($i = 1, 2, \ldots, h$), then $G$ is denoted by $G(m_1, \ldots, m_h; n_1, \ldots, n_h)$. Figure 1 shows the structure of a double nested graph.

2.2. Kirchhoff Matrix. Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. If $|V| = n$, the adjacency matrix $A(G) = (a_{ij})$, is the $n \times n$ matrix of zeros and ones such that $a_{ij} = 1$ if there is an edge between $v_i$ and $v_j$, and 0 otherwise. Let $D = D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. Then the Kirchhoff (laplacian) matrix of $G$ is $K = K(G) = D(G) - A(G)$.

For an $n \times n$ matrix $A$, the $ij$th minor is the determinant of the $(n - 1) \times (n - 1)$ matrix $M_{ij}$ obtained from $A$ deleting row $i$ and column $j$. The $i$th cofactor denoted $A_i$
equals \( \text{det}(M_{ii}) \). The Kirchhoff Matrix Tree Theorem is one of the most important results in graph theory. It provides a formula for the number of spanning trees of a graph \( G \), in terms of the cofactors of its Kirchhoff Matrix.

**Theorem 1** (Kirchhoff Matrix Tree Theorem \[3\]). For any graph \( G \) with its Kirchhoff matrix \( K(G) \), the cofactors of \( K(G) \) have the same value, and this value gives the number of spanning trees of \( G \).

### 3. The Algorithm

In this section we present a linear time algorithm for determining the number of spanning trees of a double nested graph \( G \). We denote by \( \tau(G) \) the number of spanning trees of \( G \).

Recall that matrices are congruent if one can obtain the other by a sequence of pairs of elementary operations, each pair consisting of a row operation followed by the same column operation.

Let \( K(G) \) be the Kirchhoff matrix of a double nested graph \( G(m_1, \ldots, m_h; n_1, \ldots, n_h) \), \( N = \sum_{k=1}^{h} n_k \), \( M = \sum_{k=1}^{h} m_k \) and \( n = N + M \). We assume that \( K(G) \) is \( n \times n \) matrix represented in the form

\[
K(G) = \begin{bmatrix}
d_{m_1} & \cdots & H \\
\vdots & \ddots & \vdots \\
H^T & \cdots & d_{m_h}
\end{bmatrix},
\]

where \( H = (H_{ij}) \) for \( 1 \leq i, j \leq h \) is the block matrix defined by

\[
H_{ij} = \begin{cases}
-J_{m_i \times n_j} & \text{if } i + j \leq h + 1 \\
O_{m_i \times n_j} & \text{otherwise}.
\end{cases}
\] (1)

\( J_{m_i \times n_j} \) is the \( m_i \times n_j \) all 1- matrix and \( O_{m_i \times n_j} \) is the \( m_i \times n_j \) 0-matrix.

According the Kirchhoff Matrix Tree Theorem, we need to compute a cofactor of \( K(G) \). Let \( K(G') \) be the matrix obtained from \( K(G) \) deleting the last row and column. First, we show that \( K(G') \) may be reduced to a certain tridiagonal matrix congruent, by row and column operations. After this, we use the \( LU \) decomposition for computing a cofactor of \( K(G') \).

The process for computing a cofactor of \( K(G') \) will be divided into four steps.

**Step 1** If \( m_1 = 1 \) then the first block is done. If \( m_1 > 1 \) then the first \( m_1 \) rows (and columns) in \( H \) (and \( H^T \) ) are pairwise equal. Then we perform the following row and column operations

\[
R_{m_1} \leftarrow R_{m_1} - R_{m_1-1}
\]
\[
C_{m_1} \leftarrow C_{m_1} - C_{m_1-1}
\]

giving the rows \( R_{m_1}, R_{m_1-1} \):

\[
\begin{pmatrix}
0 & \cdots & 0 & d_{m_1} & -d_{m_1} & 0 & \cdots & 0 & -1 & \cdots & -1 \\
0 & \cdots & 0 & -d_{m_1} & 2d_{m_1} & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Next we perform

\[
R_{m_1-1} \leftarrow R_{m_1-1} + \frac{1}{2} R_{m_1}
\]
\[
C_{m_1-1} \leftarrow C_{m_1-1} + \frac{1}{2} C_{m_1}
\]
giving
\[
\begin{pmatrix}
0 & \cdots & 0 & \frac{d_{m_1}}{2} & 0 & 0 & \cdots & 0 & -1 & \cdots & -1 \\
0 & \cdots & 0 & 0 & 2d_{m_1} & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Now consider \(R_{m_1-2}\) and \(R_{m_1-1}\), as well as its respective columns. We perform
\[
R_{m_1-1} \leftarrow R_{m_1-1} - R_{m_1-2}
\]
\[
C_{m_1-1} \leftarrow C_{m_1-1} - C_{m_1-2}
\]
and
\[
R_{m_1-2} \leftarrow R_{m_1-2} + \frac{2}{3}R_{m_1-1}
\]
\[
C_{m_1-2} \leftarrow C_{m_1-2} + \frac{2}{3}C_{m_1-1}
\]
obtaining \(R_{m_1-1}\) and \(R_{m_1-2}\) equal to
\[
\begin{pmatrix}
0 & \cdots & 0 & \frac{1}{3}d_{m_1} & 0 & 0 & \cdots & 0 & -1 & \cdots & -1 \\
0 & \cdots & 0 & 0 & \frac{2}{3}d_{m_1} & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

We continue this process until we annihilate the \(-1\) in the first \(m_1\) rows and columns except for the first row and column. Then the first \(m_1\) diagonal entries become
\[
\frac{d_{m_1}}{m_1}, \frac{m_1 d_{m_1}}{m_1 - 1}, \frac{(m_1 - 1) d_{m_1}}{m_1 - 2}, \ldots, \frac{3d_{m_1}}{2}, \frac{2d_{m_1}}{1}
\]
(2)
We repeat the same procedure to all pairwise equal rows and columns in the range of \(M_{i-1} + 1\) to \(M_{i-1} + m_i\) for \(i = 2, \ldots, h\), obtaining the matrix with rows \(R_1, R_{M_1+1}, \ldots, R_{M_{h-1}+1}\) of form
\[
R_{M_i+1} = \begin{pmatrix}
0 & \cdots & 0 & \frac{d_{m_i}}{m_i} & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \frac{2}{3}d_{m_i} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
for \(i = 1, \ldots, h\). The remaining rows if any for some \(i\) contain all entries equal to zero except the diagonal ones. For each \(m_i > 1\) we obtain \(m_i - 1\) diagonal entries equal to \((-1)\). Next by permuting the rows and columns we move rows \(R_1, R_{M_{i}+1}, \ldots, R_{M_{h-1}+1}\) to be in the last \(h\) positions among the first \(M_{h}\) rows. At the end of this Step 1, the partially tridiagonal matrix has the following form
\[
\begin{bmatrix}
D_1 & & & \\
& \ddots & & \\
& & D_h & \frac{m_1 d_{m_1}}{m_1 - 1} & \frac{2d_{m_1}}{1} \\
& & & B \\
& & & \frac{m_1 d_{m_1}}{m_1 - 1} & \frac{2d_{m_1}}{1} \\
& & & & \ddots \\
& & & & & \frac{m_1 d_{m_1}}{m_1 - 1} & \frac{2d_{m_1}}{1} \\
& & & & & & D_{N_h}
\end{bmatrix},
\]
where
\[
D_i = \text{diag}\left(\frac{m_1 d_{m_1}}{m_i - 1}, \ldots, \frac{2d_{m_1}}{1}\right) \quad (i = 1, \ldots, h)
\]
and
\[
B = \begin{pmatrix}
-1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & \cdots & -1
\end{pmatrix},
\]
is \( h \times N_h \) matrix with exactly \( N_{h+1-i} \) entries \(-1\) in the \( i\)-th row.

**Step 2** We consider the \((h + N_h) \times (h + N_h)\) matrix

\[
\begin{pmatrix}
\pmatrix{m_1 \vdots m_N} \\
B \\
B^T \\
\pmatrix{n_1 \vdots n_N}
\end{pmatrix}
\]

The first \( n_1 \) columns of \( B \) are pairwise equal as well as \( n_2 \) and so on up to \( n_h \). We perform similar row and column operations as in Step 1 in order to remove all \(-1\) from pairwise equal columns of \( B \) except the first one in the sequence. For any \( i \), the remaining \( n_i - 1 \) columns if any have all entries equal to \( d_i \) except to the diagonal ones equal to

\[
\frac{n_i d_i}{n_i - 1}, \frac{(n_i - 1)d_i}{n_i - 2}, \ldots, \frac{3d_i}{2}, 1
\]

Again we may permute the rows and columns by pushing to the end the rows and columns with \(-1\).

**Step 3** In this stage we consider the \(2h \times 2h\) matrix

\[
\begin{pmatrix}
\pmatrix{m_1 \vdots m_N} \\
B \\
B^T \\
\pmatrix{n_1 \vdots n_N}
\end{pmatrix}
\]

First we note that

\[
n_h^* = \begin{cases} 
  n_h - 1 & \text{if } n_h > 1 \\
  n_{h-1} & \text{otherwise}
\end{cases}
\]  

since that we have eliminated the last row and column.

We perform

\[
R_{h+1} \leftarrow R_{h+1} - R_{h+2} \\
C_{h+1} \leftarrow C_{h+1} - C_{h+2}
\]

most of \(-1\) are removed from \( R_{h+1} \) and \( C_{h+1} \). We repeat this process to the others rows and columns. At the end we obtain the \(2h \times 2h\) matrix
Step 4 Since that \( \frac{d_{m_i}}{m_i} \neq 0 \), we can perform for \( i = 1, \ldots, h \)
\[
R_{2h-i+1} \leftarrow R_{2h-i+1} - \frac{m_i}{d_{m_i}} R_{2h-i+1} \\
C_{2h-i+1} \leftarrow C_{2h-i+1} - \frac{m_i}{d_{m_i}} C_{2h-i+1}
\]
and obtaining the matrix
\[
\begin{bmatrix}
\frac{d_{m_1}}{m_1} & \cdots & \frac{d_{m_h}}{m_h} \\
& \frac{d_{n_1}}{n_1} + \frac{d_{n_2}}{n_2} - \frac{m_h}{d_{m_h}} & -\frac{d_{n_2}}{n_2} \\
& & \ddots \\
& & & \frac{d_{n_h-1}}{n_h-1} + \frac{d_{n_h}}{n_h} - \frac{m_1}{d_{m_1}} \\
& & & & -\frac{d_{n_h}}{n_h}
\end{bmatrix}
\]

Our final task is to transform a tridiagonal matrix into a upper triangular matrix. For this, we appeal to \( LU \) decomposition. Let
\[
T = \begin{bmatrix}
a_1 & b_1 \\
b_1 & \ddots & \ddots \\
& \ddots & \ddots & b_{h-1} \\
& & b_{h-1} & a_h
\end{bmatrix},
\]
where
\[
a_i = \frac{d_{n_i}}{n_i} + \frac{d_{n_{i+1}}}{n_{i+1}} - \frac{m_{h-i+1}}{d_{m_{h-i+1}}}, \quad b_i = -\frac{d_{n_{i+1}}}{n_{i+1}}, \quad \text{and} \quad a_h = \frac{d_{n_h}}{n_h} - \frac{m_1}{d_{m_1}}.
\]
The \( LU \) decomposition can be described as follows: for a given non singular matrix \( T \) of order \( h \), we have that \( L \) and \( U \) are given, for \( i = 1, \ldots, h - 1 \):
\[
L = \begin{bmatrix}
1 & f_1 & \cdots & f_{h-1} \\
f_1 & 1 & \ddots & \ddots \\
& \ddots & \ddots & 1 \\
& & f_{h-2} & 1
\end{bmatrix} \quad U = \begin{bmatrix}
g_1 & h_1 & \cdots & \cdots \\
g_2 & h_2 & \ddots & \ddots \\
& g_3 & h_3 & \ddots \\
& & \ddots & g_{h-1} & h_{h-2} \\
& & & g_h
\end{bmatrix},
\]
where \( g_1 = a_1, h_1 = b_1 \), and for \( i = 2, \ldots, h - 1 \):
\[
f_i = \frac{b_i}{g_i} - \frac{b_{i-1}}{g_{i-1}}, \quad g_i = a_i - \frac{b_i}{g_i}.
\]
From this, we can compute the determinant of \( T \), in linear time, taking the product of \( g_i \), since that \( \det(T) = \det(LU) = \det(L) \cdot \det(U) = \prod_{i=1}^h g_i \).

We have proven the following result:

**Theorem 2.** Let \( G(m_1, \ldots, m_h; n_1, \ldots, n_h) \) be a double nested graph of order \( n \), and \( K(G) \) its Kirchhoff matrix. Then the number of spanning trees \( \tau(G) \) can be computed
in $O(n)$ time by taking the product of

$$
\frac{m_i d_{m_i}}{m_i - 1}, \frac{(m_i - 1) d_{m_i}}{m_i - 2}, \ldots, \frac{3d_{m_i}}{2}, \frac{2d_{m_i}}{1}, \frac{n_i d_{n_i}}{n_i - 1}, \frac{(n_i - 1) d_{n_i}}{n_i - 2}, \ldots, \frac{3d_{n_i}}{2}, \frac{2d_{n_i}}{1},
$$

where $g_1, g_2, \ldots, g_h$ are obtained from equation (4).

4. Bipartite with few cycles

In this section we apply the algorithm to get the number of spanning trees $\tau(G)$, for a some cases of bipartite graphs $G$. Here we consider the double nested unicyclic, bicyclic, tricyclic graphs and the double nested quasi-tree graphs.

4.1. Double nested unicyclic graphs. A double nested unicyclic graph of order $n$, is a bipartite graph $G(1, 1; 2, n − 4)$ with $n \geq 5$. The Figure 2 shows a double nested unicyclic graph.

![Double nested graph](image)

**Figure 2.** Double nested graph $G(1, 1; 2, n − 4)$.

**Theorem 3.** Let $G(1, 1; 2, n − 4)$ be a double nested unicyclic graph of order $n \geq 5$. Then the number of spanning trees $\tau(G)$ is equal to 4.

**Proof.** Let $G(1, 1; 2, n − 4)$ be a double nested unicyclic graph of order $n \geq 5$. First note that for any positive integer $p$, we have that

$$
\tau(G)(1, 1; 2, n - 4 + p) = \tau(G)(1, 1; 2, n - 4), \quad n \geq 5
$$

(5)

From this, taking $n = 5$ and $p = 1$, follows that the vertex set $[d_{m_1}, d_{m_2}; d_{n_1}, d_{n_2}] = [4, 2; 2, 1]$ and $[m_1, m_2; n_1, n_2] = [1, 1; 2, 1]$. From Theorem 2 we have to multiply the following values

$$
\frac{d_{m_2}}{m_1}, \frac{d_{m_2}}{m_2}, \frac{n_1 d_{n_1}}{n_1 - 1}, g_1, g_2,
$$

where $g_2 = \frac{g_1 \left(\frac{d_{m_2}}{n_1} - \frac{d_{m_2}}{n_2}\right)^2}{g_1}$ and $g_1 = \frac{d_{m_1}}{n_1} + \frac{d_{n_2}}{n_2^2} - \frac{m_2}{d_{m_2}}$. Since that $g_1 \cdot g_2 = \frac{1}{8}$, by direct calculation from the algorithm, the number of spanning trees is given by

$$
\tau(G)(1, 1; 2, 2) = 4 \cdot 2 \cdot 4 \cdot 1 = 4.
$$

Then the result follows. □
4.2. **Double nested bicyclic graphs.** Here we consider two types of double nested bicyclic graphs of order \( n \geq 6 \). They are the following bipartite graphs \( G(1, 1; 3; n - 5) \) and \( H(1, 2; 2, n - 5) \), as the Figure 3 has shown.

**Theorem 4.** Let be \( G(1, 1; 3; n - 5) \) and \( H(1, 2; 2, n - 5) \), the double nested bicyclic graph of order \( n \geq 6 \). Then the number of spanning trees \( \tau(G) \) and \( \tau(H) \) is equal to 12.

**Proof.** If \( G(1, 1; 3; n - 5) \) the proof is similar to Theorem 3. Now let \( H(1, 2; 2, n - 5) \) be a double nested bicyclic graph of order \( n \geq 6 \). Since that for any positive integer \( p \)

\[
\tau(H)(1, 2; 2, n - 5 + p) = \tau(H)(1, 2; 2, n - 5), \quad n \geq 6
\]

(6)

From this, taking \( n = 6 \) and \( p = 1 \), follows that the vertex set \([d_{m_1}, d_{m_2}, d_{n_1}, d_{n_2}] = [4, 2, 3, 1] \) and \([m_1, m_2; n_1, n_2] = [1, 2, 2, 1] \). From Theorem 2 we have to multiply the following values

\[
\frac{d_{m_1}}{m_1}, \frac{d_{m_2}}{m_2}, \frac{d_{n_1}}{n_1}, \frac{d_{n_2}}{n_2}, g_1, g_2,
\]

where \( g_2 = \frac{g_1 (\frac{d_{n_2}}{n_2} - \frac{d_{m_2}}{m_2}) - (\frac{d_{n_2}}{n_2})^2}{g_1} \) and \( g_1 = \frac{d_{m_1}}{m_1} + \frac{d_{n_1}}{n_1} - \frac{d_{m_2}}{m_2} \). Since that \( g_1 \cdot g_2 = \frac{1}{8} \), by direct calculation from the algorithm, the number of spanning trees is given by

\[
\tau(H)(1, 2; 2, 2) = 4 \cdot 1 \cdot 4 \cdot 6 \cdot \frac{1}{8} = 12.
\]

Then the result follows. \( \square \)

4.3. **Double nested tricyclic graphs.** Here we consider three types of double nested tricyclic graphs of order \( n \geq 7 \). They are the following bipartite graphs \( G(1, 1; 4, n - 6) \), \( H(1, 1, 1; 2, 1, n - 6) \) and \( S(1, 3; 2, n - 6) \), as the Figure 4 has shown.

**Theorem 5.** Let be \( G(1, 1; 4, n - 6) \), \( H(1, 1, 1; 2, 1, n - 6) \) and \( S(1, 3; 2, n - 6) \), the double nested tricyclic graph of order \( n \geq 7 \). Then the number of spanning trees \( \tau(G) \), \( \tau(H) \) and \( \tau(S) \) is equal to 32, 36 and 32, respectively.

**Proof.** The proof for \( G(1, 1; 4, n - 6) \) and \( S(1, 3; 2, n - 6) \) is similar to Theorem 3 and Theorem 4. Now let be \( H(1, 1, 1; 2, 1, n - 6) \) of order \( n \geq 7 \). Since that for any positive integer \( p \)

\[
\tau(H)(1, 1, 1; 2, 1, n - 6 + p) = \tau(H)(1, 1, 1; 2, 1, n - 6), \quad n \geq 7
\]

(7)

From this, taking \( n = 7 \) and \( p = 1 \), follows that the vertex set \([d_{m_1}, d_{m_2}, d_{m_3}; d_{n_1}, d_{n_2}, d_{n_3}] = [5, 3, 2, 3, 2, 1] \) and \([m_1, m_2, m_3; n_1, n_2, n_3] = [1, 1, 1; 2, 1, 1] \). From Theorem 2 we have to multiply the following values

\[
\frac{d_{m_1}}{m_1}, \frac{d_{m_2}}{m_2}, \frac{d_{m_3}}{m_3}, \frac{n_1d_{n_1}}{n_1}, \frac{n_2d_{n_2}}{n_2}, \frac{n_3d_{n_3}}{n_3}, g_1, g_2, g_3,
\]
where \( g_1 = 3, g_2 = \frac{4}{3}, \) and \( g_3 = \frac{1}{20}. \) Since that \( g_1 \cdot g_2 \cdot g_3 = \frac{1}{5}, \) by direct calculation from the algorithm, the number of spanning trees is given by

\[
\tau(H)(1, 1, 1; 2, 1, 2) = 5 \cdot 3 \cdot 2 \cdot 6 \cdot \frac{1}{5} = 36.
\]

Then the result follows. \( \square \)

4.4. **Double nested quasi-tree graphs.** A connected graph \( G \) is called a quasi-tree graph if there exist \( v_0 \in V(G) \) such that \( G - v_0 \) is a tree. Let be the quasi-tree graph \( G(1, 1; d_0, n - d_0 - 2) \) with \( 2 \leq d_0 < n - 2. \) The Figure 5 shows a quasi-tree graph.

![Quasi-tree graph](image)

**Figure 5.** Quasi-tree graph \( G(1, 1; d_0, n - d_0 - 2). \)

**Theorem 6.** Let \( G(1, 1; d_0, n - d_0 - 2) \) be the quasi-tree graph with \( 2 \leq d_0 < n. \) Then the number of spanning trees \( \tau(G) \) is equal to \( 2^{d_0 - 1} d_0. \)

Since that the proof is similar to the results above, we omit them.

**References**

[1] A. Alazemi, M. Andelić, S. K. Simić, Eigenvalue location for chain graphs, Linear Algebra and its Applications 504 (2016) 194–216.
[2] A. E. Brouwer, W. H. Haemers, Spectra of Graphs, Springer, Berlin, 2012.
[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
[4] D. P. Jacobs, V. Trevisan, F. Tura, Eigenvalue location in threshold graphs, Linear Algebra and its Applications 439 (2013) 2762–2773.
[5] D. P. Jacobs, V. Trevisan, F. Tura, Eigenvalues and energy in threshold graphs, Linear Algebra and its Applications 465 (2015) 412–425.
[6] Z. Mod’h, Abu-Sbeih, On the number of spanning trees of $K_n$ and $K_{m,n}$, Discrete Math. 84 (1990) 205–207.
[7] W.M. Yan, W. Myrnold and K.L. Chung, A formula for the number of spanning trees of a multi-star related graph, Inform. Process. Lett. 68 (1998) 295–298.
[8] Y. Zang, X. Yong and M.J. Golin, The number of spanning trees in circulant graphs, Discrete Applied Mathematics, 223 (2000) 337–350.
[9] S. D. Nikolopoulos, and C. Papadopoulos, On the number of spanning trees in circulant of multi-star related graphs, Inform. Process. Lett. 65 (1998) 183–188.
[10] P. L. Hammer and A.K. Kelmans, Laplacian spectra and spanning trees of threshold graphs, Discrete Applied Mathematics, 65 (1996) 255–273.
[11] A. Bhattacharya, S. Friesland, U.N. Peled, On the first eigenvalue of bipartite graphs, Electron. J. Combin. 15 (1) (2008), 144.
[12] P. Heggernes, D. Kratsch, Linear-time certifying recognition algorithms and forbidden induced subgraphs, Nordic. J. Comput. 14 (1-2) (2007), 87-108.

Departamento de Matemática, UFSM, 97105–900 Santa Maria, RS, Brazil
E-mail address: ftura@smail.ufsm.br