Electromagnetic radiation, motion of a particle and energy-mass relation

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Abstract. Equation of motion of an uncharged arbitrarily shaped dust particle under the effects of (stellar) electromagnetic radiation and thermal emission is derived. The resulting relativistically covariant equation of motion is expressed in terms of standard optical parameters. Relations between energy and mass of the incoming and outgoing radiation are obtained, together with relations between radiation energy and mass of the particle. The role of the diffraction nicely fits the relativistic formulation of the momentum of the outgoing radiation.

Covariant formulations yield several simple consequences. It is shown that the frequently used statement "energy equals $\gamma \times \text{mass} \times c^2$" or "energy equals mass $\times c^2$" are not correct, in general; "rest energy equals mass $\times c^2$" holds. The inequality $0 < \bar{Q}'_{pr,1}/\bar{Q}'_{ext} < 2$ is a simple relativistic consequence for the Poynting-Robertson (P-R) effect ($\bar{Q}'_{ext}$ and $\bar{Q}'_{pr,1}$ are dimensionless efficiency factors for the extinction and radial direction of the radiation pressure, integrated over stellar spectrum). The condition for the P-R effect is $p'_0 = (1 - \bar{Q}'_{pr,1}/\bar{Q}'_{ext}) p'_i$, where $p'_i$ and $p'_0$ are incoming and outgoing radiation momenta (per unit time) measured in the proper frame of reference of the particle. The case of "perfectly absorbing spherical dust particle", within geometrical optics approximation, corresponds to the condition $p'_0 = 0.5 p'_i$. The obtained results enable understanding of the physics of the P-R effect and evaluate the published explanations. As for arbitrarily shaped dust particle, the general condition $0 < C'_{pr,1} / C'_{ext} < 2 / (1 + \sum_{j=2}^{3} \bar{C}'_{pr,j}/\bar{C}'_{pr,1})$ holds for cross sections of extinction and radiation pressure components, if thermal emission force is of negligible importance. The condition can add a new information to the results obtained from observations, measurements and numerical calculations of the optical properties of the particle.

Key words. cosmic dust, electromagnetic radiation, thermal emission, relativity theory, equation of motion
1. Introduction

The Poynting-Robertson effect is used in modelling orbital evolution of dust grains under the action of electromagnetic radiation (of the central star), for many decades (e. g., Poynting 1903, Robertson 1937, Wyatt and Whipple 1950, Dohnanyi 1978, Kapišinský 1984, Jackson and Zook 1989, Leinert and Grün 1990, Gustafson 1994, Dermott et al. 1994, Reach et al. 1995, Murray and Dermott 1999, Woolfson 2000, Danby 2003, Quinn 2005, Harwit 2006, Grün 2007, Sykes 2007, Krügel 2008). It is assumed that particle with spherically distributed material can be used as a correct approximation to reality. However, real nonspherical particles interact with electromagnetic radiation in a complicated manner and particles of various optical properties exist (e. g., Mishchenko et al. 2002). Thus, it is essential to have an equation of motion sufficiently general to cover a wide range of optical parameters, not just the limited cases standardly investigated. The first presentations of the equation of motion for arbitrarily shaped particle were given by Klačka (1994), Klačka and Kocifaj (1994), Klačka (2000), Kocifaj et al. (2000), Klačka and Kocifaj (2001). Relativistically covariant derivation can be found in Klačka (2000) and Klačka (2004), where also thermal emission of the particle is taken into account – Mishchenko (2001) has formulated the radiation pressure on arbitrarily shaped particles arising from an anisotropy of thermal emission. Krauss and Wurm (2004) have published the first experimental evidence that nonspherical dust grains behave in a different way than the spherical ones.

We derive equation of motion for arbitrarily shaped particle in a physically correct way and relativistic covariant formulations are presented. A new form for the validity of the Poynting-Robertson (P-R) effect is found: the P-R effect holds only when the total momentum of the outgoing radiation per unit time is colinear with the incident radiation, in the proper/rest frame of reference of the particle, and, a new physical condition for the P-R effect is found, as for the dimensionless efficiency factors for radiation pressure and extinction (optical properties) of the particle. Moreover, relations between energy and mass of the incoming and outgoing radiation are presented, both for arbitrarily shaped particle and also, as a special case, for spherical particle for which the P-R effect holds. The relations between energy and mass for the incoming and outgoing radiation, together with relations between radiative energy and mass of the particle, present a practical contribution to the discussion on relativistic energy-mass relation (see, e. g., Ugarov 1969, Secs. 57-58, pp. 246-252; Okun 1989a, 1989b; Taylor and Wheeler 1992, Sec. 8.4, pp. 228-233; Einstein 1999). The relations are consistent with the consideration of the diffracted light within geometrical optics approximation. The results of the paper, including detail discussion also with respect to the statements presented in the original papers by Poynting (1903) and Robertson (1937), enable to understand physics of the P-R effect, which is frequently used in studies of orbital evolution of cosmic dust particles. As
for the arbitrarily shaped dust particles, a condition yielding some information on cross sections, unknown from experiments, observations or theoretical/numerical solutions is obtained as a simple relativistic consequence of the covariant form of the four-momentum of the outgoing radiation.

We begin by reviewing in Sec. 2 and 3 the basic physical processes in proper and stationary frames, for an arbitrarily shaped particle. The physical equations use cross sections for all relevant phenomena (extinction, scattering, absorption, radiation pressure components) and the equation of motion of the particle for the case of the irradiated particle is discussed at the end of Sec. 3 (Eqs. 40-41). The following Sec. 4 deals with the covariant formulation for the incoming and outgoing radiation, which are used in Sec. 5 for finding energy-mass relation for the radiation. Sec. 6 discusses the relation between the change of the particle’s mass, separately for the incoming and outgoing radiation. Application of the results from the previous sections are discussed, for special particle shapes, in Sec. 7: (i) it is shown that a simple relations holds for dimensionless efficiency factors for radiation pressure and extinction for the case of spherical particle (Eq. 79), including presentation of the value for perfectly absorbing (or perfectly reflecting) sphere (Eq. 104) which differs from the conventional statement coming back to Poynting and Robertson; (ii) an understanding of the action of the radiation on motion of spherical particles is discussed in detail in Sec. 7.4; (iii) an analogy between mechanics and electromagnetism for the case of planar surface is discussed in Sec. 7.5. Results of Secs. 2-7 are used in Sec. 8, which presents considerations on the equivalence principle of mass and energy for absorbing planar surfaces. A special Sec. 9 derives a simple condition for cross sections for radiation pressure components and extinction, using results of Sec. 5.

The most important results are shortly summarized in Sec. 10.

2. Proper reference frame of the particle – stationary particle

The term “stationary particle” will denote a particle which does not move in a given inertial frame of reference. Primed quantities will denote quantities measured in the proper reference frame of the particle – rest frame of the particle.

The flux density of photons scattered into an elementary solid angle $d\Omega' = \sin \theta' \ d\theta' \ d\phi'$ is proportional to $p'(\theta', \phi') \ d\Omega'$, where $p'(\theta', \phi')$ is the “phase function”. The phase function depends on orientation of the particle with respect to the direction of the incident radiation and on the particle characteristics; angles $\theta', \phi'$ correspond to the direction (and orientation) of travel of the scattered radiation, $\theta'$ is the polar angle which vanishes for propagation along the unit vector $e'_1$ of the incident radiation. The phase function fulfills the normalisation condition

$$\int_{4\pi} p'(\theta', \phi') \ d\Omega' = 1 \ . \quad (1)$$
The momentum of the incident beam of photons which is lost in the process of interaction with the particle is proportional to the cross section $C'_{ext}$ (extinction). The part proportional to $C'_{abs}$ (absorption) is emitted in the form of thermal radiation and the part proportional to $C'_{ext} - C'_{abs} = C'_{sca}$ is scattered. The differential scattering cross section $dC'_{sca}/d\Omega' \equiv C'_{sca} p'/(\theta', \phi')$ depends on the polarization state of the incident light as well as on the incidence and scattering directions (e. g., Mishchenko et al. 2002).

The momentum (per unit time) of the scattered photons into an elementary solid angle $d\Omega'$ is

$$dp'_{sca} = \frac{1}{c} S' C'_{sca} \int_{4\pi} p'(\theta', \phi') K' d\Omega' ,$$

where the unit vector in the direction of scattering is

$$K' = \cos \theta' e'_1 + \sin \theta' \cos \phi' e'_2 + \sin \theta' \sin \phi' e'_3 .$$

$S'$ is the flux density of radiation energy (energy flow through unit area perpendicular to the ray per unit time). The system of unit vectors used on the RHS of the last equation forms an orthogonal basis. The total momentum (per unit time) of the scattered photons is

$$p'_{sca} = \frac{1}{c} S' C'_{sca} \int_{4\pi} p'(\theta', \phi') K' d\Omega' .$$

The momentum (per unit time) obtained by the particle due to the interaction with radiation – radiation force acting on the particle – is

$$\frac{d p'}{d t'} = \frac{1}{c} S' \left\{ C'_{ext} e'_1 - C'_{sca} \int_{4\pi} p'(\theta', \phi') K' d\Omega' \right\} + F'_e(T') ,$$

where the emission component of the radiation force acting on the particle of absolute temperature $T'$ is (Mishchenko et al. 2002, pp. 63-66)

$$F'_e(T') = - \frac{1}{c} \int_0^{\infty} d\omega' \int_{4\pi} \hat{\mathbf{r}}' K'_e(\hat{\mathbf{r}}', T', \omega') d\hat{\mathbf{r}}' .$$

The unit vector $\hat{\mathbf{r}}' = \mathbf{r}'/r'$ is given by position vector $\mathbf{r}'$ of the observation point with origin inside the particle (the emitted radiation in the far-field zone of the particle propagates in the radial direction, i. e., along the unit vector $\hat{\mathbf{r}}'$), $\omega'$ is (angular) frequency of radiation,

$$K'_e(\hat{\mathbf{r}}', T', \omega') = I'_b(T', \omega') \left[ K'_{11}(\hat{\mathbf{r}}', \omega') - \int_{4\pi} Z'_{11}(\hat{\mathbf{r}}', \hat{\mathbf{r}}'', \omega') d\hat{\mathbf{r}}'' \right] ,$$

where $K'_{11}$ is the (1,1) element of the particle extinction matrix, $Z'_{11}$ is the (1,1) element of the phase matrix and the Planck blackbody energy distribution is given by the well-known relation

$$I'_b(T', \omega') = \frac{\bar{h} \omega'^3}{4 \pi^3 c^2} \left\{ \exp \left( \frac{\bar{h} \omega'}{k T'} \right) - 1 \right\}^{-1} .$$

Thermal emission has to be included in the total interaction of the particle with electromagnetic radiation: if the particle’s absolute temperature is above zero, it can emit
as well as scatter and absorb electromagnetic radiation. The particle is assumed to be isothermal, in Eqs. (6)-(8).

Equation (5) can be rewritten to the form

\[
\frac{dp'}{d\tau} = \frac{1}{c} S' \left\{ [C'_{\text{ext}} - \langle \cos \theta' \rangle C'_\text{sca}] e'_1 + \right.
\]

\[
[ - \langle \sin \theta' \cos \phi' \rangle C'_\text{sca}] e'_2 + \right.
\]

\[
[ - \langle \sin \theta' \sin \phi' \rangle C'_\text{sca}] e'_3 \left\} + \sum_{j=1}^{3} F'_{e,j} e'_j , \right.
\]

\[
(9)
\]

where \(< x' > \equiv \int_{4\pi} x' p'(\theta', \phi') d\Omega'\) and \(F'_{e,j} \equiv F'_e(T') \cdot e'_j\). As for the energy, we assume that it is conserved: the energy (per unit time) of the incoming radiation \(E'_i\), equals to the energy (per unit time) of the outgoing radiation (after interaction with the particle) \(E'_o\). We will use the fact that time \(t' = \tau\), where \(\tau\) is proper time.

Summarizing important equations, we can write them in a short form

\[
\frac{dp'}{d\tau} = \sum_{j=1}^{3} \left( \frac{S'}{c} C'_{pr,j} + F'_{e,j} \right) e'_j ; \quad \frac{dE'}{d\tau} = 0 , \right.
\]

\[
(10)
\]

where \(C'_{pr,1} \equiv C'_{\text{ext}} - \langle \cos \theta' \rangle C'_\text{sca}, C'_{pr,2} \equiv - \langle \sin \theta' \cos \phi' \rangle C'_\text{sca}, C'_{pr,3} \equiv - \langle \sin \theta' \sin \phi' \rangle C'_\text{sca}\) are cross sections for radiation pressure. We have added an assumption of equilibrium state when the particle’s mass does not change.

### 2.1. Summary of the important equations

Using the text concerning energy below Eq. (9) and the last Eq. (10), we may describe the total process of interaction in the form of the following equations (energies and momenta per unit time):

\[
E'_o = E'_i = S' C'_{\text{ext}} , \right.
\]

\[
p'_o = \left( 1 - \frac{C'_{pr,1}}{C'_{\text{ext}}} \right) p'_i - \left( \frac{C'_{pr,2}}{C'_{\text{ext}}} e'_2 + \frac{C'_{pr,3}}{C'_{\text{ext}}} e'_3 \right) \frac{E'_i}{c} - \sum_{j=1}^{3} F'_{e,j} e'_j , \right.
\]

\[
p'_i = \frac{E'_i}{c} e'_i . \right.
\]

The index ”i” represents the incoming (incident) radiation, beam of photons, the index ”o” represents the outgoing radiation.

The changes of energy and momentum of the particle due to the interaction with electromagnetic radiation are

\[
\frac{dE'}{d\tau} = E'_i - E'_o = 0 , \right.
\]

\[
\frac{dp'}{d\tau} = p'_i - p'_o . \right.
\]

### 3. Stationary frame of reference

By the term “stationary frame of reference” (laboratory frame) we mean a frame of reference in which particle moves with a velocity vector \(v = v(t)\). The physical quantities measured in the stationary frame of reference will be denoted by unprimed symbols.
Our aim is to derive equation of motion for the particle in the stationary frame of reference. We will use the fact that we know this equation in the proper frame of reference – see Eqs. (11) and (12).

If we have a four-vector \( A^\mu = (A^0, A) \), where \( A^0 \) is its time component and \( A \) is its spatial component, generalized special Lorentz transformation yields

\[
A^0' = \gamma (A^0 - v \cdot A/c) ,
\]

\[
A' = A + [(\gamma - 1) v \cdot A/v^2 - \gamma A^0/c] v ,
\]

with inverse

\[
A^0 = \gamma (A^0 + v \cdot A'/c) ,
\]

\[
A = A' + [(\gamma - 1) v \cdot A'/v^2 + \gamma A^0/c] v ,
\]

where \( \gamma = 1/\sqrt{1 - v^2/c^2} \).

As for four-vectors we immediately introduce four-momentum:

\[
p^\mu = (p^0, p) \equiv (E/c, p) .
\]

### 3.1. Incoming radiation

Applying Eqs. (14) and (15) to quantity \((E^\prime_i/c, p^\prime_i)\) (four-momentum per unit time – proper time is a scalar quantity) and taking into account also Eqs. (11), we can write

\[
E_i = E_i' \gamma (1 + v \cdot e_1'/c) ,
\]

\[
p_i = \frac{E_i'}{c} \{e_1' + [(\gamma - 1) v \cdot e_1'/v^2 + \gamma/c] v \} .
\]

Using the fact that \( p^\mu = (h \nu/c, h \nu e_1/c) \) for photons, we have

\[
\nu' = \nu w_1 ,
\]

\[
e_1' = \frac{1}{w_1} \{e_1 + [(\gamma - 1) v \cdot e_1/v^2 - \gamma/c] v \} ,
\]

where

\[
w_1 \equiv \gamma (1 - v \cdot e_1/c) .
\]

Inserting the second of Eqs. (17) into Eq. (16), one obtains

\[
E_i = (1/w_1) E_i' ,
\]

\[
p_i = (1/w_1) (E_i'/c) e_1 .
\]

We have four-vector \( p_i^\mu = (E_i/c, p_i) = (1, e_1)E_i/c = (1/w_1, e_1/w_1) \times w_1 E_i/c \equiv b_1^a w_1 E_i/c .
\]

For monochromatic radiation the flux density of radiation energy becomes

\[
S' = n' h \nu' c ; \quad S = n h \nu c ,
\]

\[
(20)
\]
where \( n \) and \( n' \) are concentrations of photons (photon number densities) in the corresponding reference frames. We also have continuity equation

\[
\partial_{\mu} j^\mu = 0, \quad j^\mu = (c n, c n e_1),
\]

with current density \( j^\mu \). Application of Eq. (13) then yields

\[
n' = w_1 n.
\]

Using Eqs. (17), (20) and (22) we finally obtain

\[
S' = w_1^2 S.
\]

Eqs. (11), (19) and (23) then together give

\[
E_i = w_1 S_{\text{ext}}', \quad p_i = w_1 S_{\text{ext}}' e_1/c.
\]

### 3.2. Outgoing radiation

The situation is analogous to that of the preceding subsection. It is only a little more algebraically complicated, since radiation may also spread out in directions given by unit vectors \( e_2, e_3 \). We need transformations \( e'_j \rightarrow e_j, \ j = 2, 3 \). The vectors \( e'_2 \) and \( e'_3 \) can be used to describe directions of propagation of radiation scattered by the particle. Thus, aberration of light also exists for each of these unit vectors. The relations between \( e'_2 \) and \( e_2 \), \( e'_3 \) and \( e_3 \), are analogous to that presented by the second of Eq. (17):

\[
e'_j = \frac{1}{w_j} \left( e_j + \left[ (\gamma - 1) \frac{v \cdot e_j}{v^2} - \frac{\gamma}{c} \right] v \right), \quad j = 1, 2, 3,
\]

where

\[
w_j \equiv \gamma (1 - v \cdot e_j/c), \quad j = 1, 2, 3.
\]

It is worth mentioning that vectors \( \{e'_j; j = 1, 2, 3\} \) form an orthonormal set of vectors, and, unit vectors \( \{e_j; j = 1, 2, 3\} \) are not orthogonal unit vectors.

Applying Eqs. (14) and (15) to the quantity \( (E'_o/c, p'_o) \) (four-momentum per unit time – proper time is a scalar quantity), we can write

\[
E_o = \gamma \left( E'_o + v \cdot p'_o \right),
\]

\[
p_o = p'_o + \left[ (\gamma - 1) \frac{v \cdot p'_o}{v^2} + \gamma \frac{E'_o}{c^2} \right] v.
\]

Using also \( p'_i = E'_i e'_i/c \) and Eqs. (11), (24), (26),

\[
E_o = \frac{C'_{\text{pr},1}}{C'_{\text{ext}}} w_1 E_i \gamma + \left( 1 - \frac{C'_{\text{pr},1}}{C'_{\text{ext}}} \right) E_i
\]

\[
+ w_1 E_i \left( \frac{C'_{\text{pr},2}}{C'_{\text{ext}}} + \frac{C'_{\text{pr},3}}{C'_{\text{ext}}} \right) \gamma - w_1 E_i \left( \frac{C'_{\text{pr},2}}{C'_{\text{ext}}} \frac{1}{w_2} + \frac{C'_{\text{pr},3}}{C'_{\text{ext}}} \frac{1}{w_3} \right)
\]

\[
- \sum_{j=1}^{3} F_{e,j}' \left( \frac{c}{w_j} - \gamma c \right),
\]

\[
p_o = \left( 1 - \frac{C'_{\text{pr},1}}{C'_{\text{ext}}} \right) \frac{E_i}{c} e_1 + \frac{C'_{\text{pr},1}}{C'_{\text{ext}}} \frac{w_1 E_i}{c^2} \gamma v
\]
3.3. Equation of motion

In analogy with Eqs. (12), we have for the changes of energy and momentum of the particle due to the interaction with electromagnetic radiation

\[
\frac{d E}{d \tau} = E_i - E_o ,
\]

\[
\frac{d p}{d \tau} = p_i - p_o .
\]  

(28)

Putting Eqs. (27) into Eqs. (28), using also \( p_i = (E_i / c)e_1 \), one easily obtains

\[
\frac{d E/c}{d \tau} = \frac{3}{c} \sum_{j=1}^{3} \left( C'_{pr,j} w_1 E_i c^2 / e_{w_j} + \frac{1}{c} F'_{e,j} \right) \left( c \frac{e_j}{w_j} - \gamma c \right) ,
\]

\[
\frac{d p}{d \tau} = \frac{3}{c} \sum_{j=1}^{3} \left( C'_{pr,j} w_1 E_i c^2 / e_{w_j} + \frac{1}{c} F'_{e,j} \right) \left( c \frac{e_j}{w_j} - \gamma v \right) .
\]

(29)

Eq. (29) may be rewritten in terms of four-vectors:

\[
\frac{d p^\mu}{d \tau} = \sum_{j=1}^{3} \left( C'_{pr,j} w_1 E_i c^2 / e_{w_j} + \frac{1}{c} F'_{e,j} \right) \left( c b^\mu_j - u^\mu \right) ,
\]

(30)

where \( p^\mu \) is four-vector of the particle of mass \( m \)

\[
p^\mu = m \, u^\mu ,
\]

(31)

four-vector of the world-velocity of the particle is

\[
u^\mu = (\gamma c, \gamma v) .
\]

(32)

We have also other four-vectors

\[
b^\mu_j = (1/w_j, e_j/w_j) , \quad j = 1, 2, 3 .
\]

(33)

It can be easily verified that:

i) the quantity \( w E_i \) is a scalar quantity – see first of Eqs. (19);

ii) Eq. (30) reduces to Eq. (10) for the case of proper inertial frame of reference of the particle;

iii) Eq. (30) yields \( d m / d \tau = 0 \).

We introduce

\[
b^0_j \equiv 1/w_j \approx 1 + v \cdot e'_j / c ,
\]

\[
b_j \equiv e_j / w_j \approx e'_j + v / c ,
\]

\[
j = 1, 2, 3
\]
for the purpose of practical calculations. Physics of these relations corresponds to aberration of light. In general case, \( b'_{\mu} = (1, e'_j) \), \( b'_{\mu} = \Lambda_{\nu}^{\mu} b'_\nu \), \( j \in \{1, 2, 3\} \), where \( \Lambda_{\nu}^{\mu} \) represents general Lorentz transformation. Eq. (34) is correct to the first order in \( v/c \), as it is presented.

We have derived an equation of motion for real dust particle under the action of electromagnetic radiation (including thermal emission). It is supposed that the equation of motion is represented by Eqs. (11) and (12) in the proper frame of reference of the particle. The final covariant form is represented by Eq. (30), or using \( E_i = w_1 S C'_{ext} \) (see Eqs. (11), (19) and (23)),

\[
\frac{d p'^\mu}{d \tau} = \sum_{j=1}^{3} \left( \frac{w^2_1}{c^2} S C'_{pr,j} + \frac{1}{c} F'_{e,j} \right) \left( c b'^\mu_j - u'^\mu \right).
\] (35)

To first order in \( v/c \), Eqs. (34)-(35) yield

\[
\frac{d \mathbf{v}}{d t} = \frac{S}{mc} \sum_{j=1}^{3} C'_{pr,j} \left[ (1 - 2 \mathbf{v} \cdot \mathbf{e}_1/c + \mathbf{v} \cdot \mathbf{e}_j/c) \mathbf{e}_j - \mathbf{v}/c \right] +
\]

\[
\frac{1}{m} \sum_{j=1}^{3} F'_{e,j} \left[ \left( 1 + \frac{\mathbf{v} \cdot \mathbf{e}_j}{c} \right) \mathbf{e}_j - \frac{\mathbf{v}}{c} \right],
\]

\[
\mathbf{e}_j = (1 - \mathbf{v} \cdot \mathbf{e}_j'/c) \mathbf{e}_j' + \mathbf{v}/c , \quad j = 1, 2, 3.
\] (36)

It is worth mentioning to stress that the values of radiation pressure cross sections \( C'_{pr,j} \), \( j = 1, 2, 3 \), depend on particle’s orientation with respect to the incident radiation – their values are time dependent.

It can be verified that Eq. (35) (or Eq. 36 within the accuracy to the first order in \( v/c \)) yields as special cases the situations discussed in Einstein (1905) and Robertson (1937). However, Eqs. (11) and (27) are not consistent with the statement presented by Poynting (1903), Robertson (1937), Wyatt and Whipple (1950) and others about the process of reemission of perfectly absorbing spherical dust particle (explanation is presented in Secs. 7.3.3 and 7.3.4).

### 3.4. Continuous distribution of density flux of energy

For a continuous frequency distribution of density flux of energy, we can write

\[
\frac{d p'^\mu}{d \tau} = \sum_{j=1}^{3} \left\{ \frac{1}{c} \int_{\nu}^{\nu'} c h \nu' \frac{\partial n'}{\partial \nu'} C'_{pr,j}(\nu') \, d\nu' + F'_{e,j} \right\} \mathbf{e}'_j =
\]

\[
= \sum_{j=1}^{3} \left( \frac{S'}{c} C'_{pr,j} + F'_{e,j} \right) \mathbf{e}'_j.
\] (37)

Taking into account that concentration of photons fulfills \( n' = w_1 n \) (Eq. (22)) and that Doppler effect yields \( \nu' = w_1 \nu \) (Eq. (17)), we have \( \partial n'/\partial \nu' = \partial n/\partial \nu \). Lorentz transformation finally yields

\[
\frac{d p'^\mu}{d \tau} = \sum_{j=1}^{3} \left\{ \frac{w^2_1}{c^2} \int_{\nu}^{\infty} c \frac{\partial n}{\partial \nu} \nu C'_{pr,j}(w_1 \nu) \, d\nu + \frac{1}{c} F'_{e,j} \right\} \times
\]
\[
\left( c b^\mu_j - u^\mu \right) = \sum_{j=1}^{3} \left( \frac{u^2}{c^2} C'_{pr,j} + \frac{1}{c} F'_{e,j} \right) \left( c b^\mu_j - u^\mu \right).
\]  

As a consequence, \( dm/d\tau = 0 \) (this corresponds to the condition \( dE'/d\tau = 0 \)). As for the accuracy to the first order in \( v/c \), equation of motion of the type Eq. (36) holds if the substitution \( C'_{pr,j} \rightarrow \bar{C}'_{pr,j} \), \( j = 1, 2, 3 \) is done:

\[
\frac{d}{d\tau} v = \frac{S}{mc} \sum_{j=1}^{3} \bar{C}'_{pr,j} \left[ (1 - 2 v \cdot e_1/c + v \cdot e_j/c) e_j - v/c \right] + \frac{1}{m} \sum_{j=1}^{3} F'_{e,j} \left[ \left( 1 + \frac{v \cdot e_j}{c} \right) e_j - \frac{v}{c} \right],
\]

\( e_j = (1 - v \cdot e'_j/c) e'_j + v/c \), \( j = 1, 2, 3 \).

The values of radiation pressure cross sections \( \bar{C}'_{pr,j} \), \( j = 1 \) to 3, depend on particle’s orientation with respect to the incident radiation.

If the particle is not irradiated for a time interval, then outgoing radiation due to the thermal emission is given by four-vector \( ( E'_o/c, p'_o ) \), where \( E'_o \) is energy per unit time and \( p'_o = - F'_e \equiv - \sum_{j=1}^{3} F'_{e,j} e'_j \) is momentum per unit time. Using transformations represented by Eq. (14), or Eq. (26), and using also Eq. (24) together with the fact that the four-force acting on the particle is \( ( d p^\mu / d \tau )_e = - p'_o \), one obtains

\[
\left( \frac{d}{d\tau} p^\mu \right)_e = - \frac{E'_o}{c} u^\mu + \sum_{j=1}^{3} F'_{e,j} \left( b^\mu_j - \frac{u^\mu}{c} \right),
\]  

instead of Eq. (38) and \( E'_o \) is energy per unit time, which is lost due to the thermal emission. Eq. (40) yields

\[
\left( \frac{d}{d\tau} u^\mu \right)_e = \frac{1}{m} \sum_{j=1}^{3} F'_{e,j} \left( b^\mu_j - \frac{u^\mu}{c} \right),
\]

\[
\left( \frac{dm}{d\tau} \right)_e = - \frac{E'_o}{c^2}.
\]

Mass of the particle decreases due to the thermal emission, alone.

4. Incoming and outgoing radiation – covariant formulation

We have treated the process of interaction between the electromagnetic radiation and a dust grain in two ways: we have considered the incoming/incident radiation and the outgoing radiation. As we are interested in the relations between the corresponding part of radiation and it’s mass, we formulate covariant forms for the two parts of radiation, in this section.
4.1. Momentum-four vector for incoming radiation

On the basis of Eq. (11) and Sec. 3.1, mainly Eqs. (18)-(23), and also using the results of Sec. 3.4, we can write four-momentum of the incoming (incident) radiation

\[ dp_{\text{in}}^\mu = \frac{w_1^2 S \bar{C}_{\text{ext}}' \bar{C}_{\text{pr},1}}{c} b_1^\mu \, d\tau , \]  

(42)

where also Eqs. (25) (or 18) and (33) have to be used.

4.2. Momentum-four vector for outgoing radiation

On the basis of Eqs. (23), (25), (27), (32), (33) and the results of Sec. 3.4, we can write for the energy and momentum of the outgoing radiation

\[ \frac{dE_{\text{out}}}{c} = X \left\{ \frac{1}{w_1} \left( \frac{\bar{C}_{\text{pr},j}}{\bar{C}_{\text{ext}}'} + X^{-1} F_{e,j}' \right) \right\} \left( \frac{1}{w_j} - \gamma \right) d\tau , \]

\[ dp_{\text{out}}^\mu = X \left\{ \frac{e_1}{w_1} - \sum_{j=1}^{3} \left( \frac{\bar{C}_{\text{pr},j}}{\bar{C}_{\text{ext}}'} + X^{-1} F_{e,j}' \right) \left( \frac{e_j}{w_j} - \gamma \frac{v}{c} \right) \right\} d\tau , \]

\[ X \equiv \frac{w_1^2 S \bar{C}_{\text{ext}}' c}{c} , \]

(43)

Eq. (43) can be written in a short relativistically covariant form

\[ dp_{\text{out}}^\mu = X \left\{ b_1^\mu - \sum_{j=1}^{3} \left( \frac{\bar{C}_{\text{pr},j}}{\bar{C}_{\text{ext}}'} + X^{-1} F_{e,j}' \right) \left( b_j^\mu - \frac{u_\mu}{c} \right) \right\} d\tau , \]

\[ X \equiv \frac{w_1^2 S \bar{C}_{\text{ext}}' c}{c} , \]

\[ p_{\text{out}}^\mu = \left( \frac{E_{\text{out}}}{c}, p_{\text{out}} \right) \]

(44)

and Eqs. (25), (32) and (33) can be used.

4.3. Relation between outgoing and incoming radiation

Eq. (44) reads

\[ dp_{\text{out}}^\mu = \frac{w_1^2 S \bar{C}_{\text{ext}}' c}{c} \left\{ b_1^\mu - \sum_{j=1}^{3} \frac{\bar{C}_{\text{pr},j}}{\bar{C}_{\text{ext}}'} \left( b_j^\mu - \frac{u_\mu}{c} \right) \right\} d\tau \]

\[ - \sum_{j=1}^{3} F_{e,j}' \left( b_j^\mu - \frac{u_\mu}{c} \right) d\tau . \]  

(45)

Eqs. (42) and (45) immediately yield

\[ dp_{\text{out}}^\mu = \left( 1 - \frac{\bar{C}_{\text{pr},1}}{\bar{C}_{\text{ext}}'} \right) dp_{\text{in}}^\mu + \frac{w_1^2 S \bar{C}_{\text{ext}}' c}{c} \frac{\bar{C}_{\text{pr},1}}{\bar{C}_{\text{ext}}'} \frac{u_\mu}{c} \, d\tau \]

\[ - \frac{w_1^2 S \bar{C}_{\text{ext}}' c}{c} \sum_{j=2}^{3} \frac{\bar{C}_{\text{pr},j}}{\bar{C}_{\text{ext}}'} \left( b_j^\mu - \frac{u_\mu}{c} \right) d\tau \]

\[ - \sum_{j=1}^{3} F_{e,j}' \left( b_j^\mu - \frac{u_\mu}{c} \right) d\tau . \]  

(46)

Eq. (46) is covariant form of Eq. (11).
4.4. Momentum-four vector for outgoing radiation – thermal emission

If the particle is not irradiated, then, in accordance with Eq. (40), we have

\[
[(d\mathbf{p}_{\text{out}})_c]_\mu = \left\{ \frac{E'_o}{c} \frac{u^\mu}{c} - \sum_{j=1}^{3} F'_{c,j} \left( b'_j - \frac{u^\mu}{c} \right) \right\} \, d\tau ,
\]

(47)

for outgoing radiation due to the thermal emission.

5. Radiation: Energy-mass relation

Now, we are interested in the relations between the corresponding part of radiation, incoming or outgoing, and its mass.

5.1. Incoming radiation

Making an invariant of Eq. (42)

\[
(c \, dM_{\text{in}})^2 = dp_{\text{in}}^\mu \, dp_{\text{in},\mu} ,
\]

(48)

one immediately obtains, on the basis of Eqs. (25) and (33) – \( b'_1 \, b_1^\mu = 0 \) –

\[
dM_{\text{in}} = 0 ;
\]

(49)

a summation over repeated upper and lower indices is always implied, e. g., \( dp_{\text{in}}^\mu \, dp_{\text{in},\mu} = dp_{\text{in},0}^0 \, dp_{\text{in},1}^1 + dp_{\text{in},1}^2 \, dp_{\text{in},2}^1 + dp_{\text{in},3}^3 \, dp_{\text{in},3}^2 = (dp_{\text{in},0}^0)^2 - (dp_{\text{in},1}^1)^2 - (dp_{\text{in},2}^2)^2 - (dp_{\text{in},3}^3)^2 \).

On the basis of Eqs. (42) and (49), we can summarize:

\[
dE_{\text{in}} = \frac{w_1^2}{c} \, S \, \tilde{C}_{\text{ext}}' \, \frac{1}{w_1} \, d\tau ,
\]

\[
dp_{\text{in}} = \frac{w_1^2}{c} \, S \, \tilde{C}_{\text{ext}}' \, e_1 \, \frac{1}{w_1} \, d\tau ,
\]

\[
dM_{\text{in}} = 0 .
\]

(50)

Eq. (49) states that energy of the incoming/incident radiation is nonzero, while (the invariant) mass of the radiation is zero. The zeroness of the mass of the incident radiation is understandable, since the radiation is produced by the parallel flux of photons moving in the same direction and orientation (see Okun 1989a, 1989b; Schröder 1990, p. 108: "For the invariant description of the inertial behaviour of a particle, only its rest mass \( m \) can be used, because it has the same value in all reference systems.").

5.2. Outgoing radiation

On the basis of Eqs. (25), (31)-(33) and (44), we can write

\[
dp_{\text{out}}^\mu = \frac{w_1^2}{c} \, S \, \tilde{C}_{\text{ext}}' \, \left\{ b_1^\mu - \sum_{j=1}^{3} \frac{C'_{\text{pr},j}}{C'_{\text{ext}}} \left( b_j^\mu - \frac{u^\mu}{c} \right) \right\} \, d\tau -
\]
\[-\sum_{j=1}^{3} F_{e,j}' \left( b_j' - \frac{u_j'}{c} \right) \, d\tau,\]

\[u_j' b_j' = c, \quad j = 1, 2, 3,\]

\[u_j' u_j' = c^2,\]

\[b_{ij}' b_{ij}' = 1 - \delta_{ij}, \quad i, j = 1 \text{ to } 3,\]

\[(51)\]

where \(\delta_{ij}\) is Kronecker delta (\(\delta_{ij} = 1\) if \(i = j\), \(\delta_{ij} = 0\) if \(i \neq j\)).

Making an invariant of Eq. (44) or of the first equation in Eqs. (51),

\[(c \, dM_{out})^2 = dp_{out} \, dp_{out}',\]

one immediately obtains, on the basis of Eqs. (51) and (52),

\[dM_{out} = \frac{w_1^2 S \bar{C}_{ext}'}{c^2} \left[ 2 \left( \frac{C_{pr,1}'}{C_{ext}'} + X^{-1} F_{e,1}' \right) - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}' + X^{-1} F_{e,j}' \right) \right]^2 \, d\tau,\]

\[X = \frac{w_1^2 S \bar{C}_{ext}'}{c}.\]

(53)

On the basis of Eqs. (44) and (53), we can summarize:

\[dE_{out} = c \frac{w_1}{w_1} - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} + X^{-1} F_{e,j}' \right) \left( \frac{e_j}{w_j} - \frac{u_j'}{c} \right) \, d\tau,\]

\[dM_{out} = X \frac{1}{c} \left[ 2 \left( \frac{C_{pr,1}'}{C_{ext}'} + X^{-1} F_{e,1}' \right) - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} + X^{-1} F_{e,j}' \right) \right]^2 \, d\tau,\]

\[X = \frac{w_1^2 S \bar{C}_{ext}'}{c}.\]

(54)

Eq. (54) states that energy of the outgoing radiation is nonzero, and also (the invariant) mass of the radiation is nonzero. The procedure of the calculation of the mass is consistent with the procedure discussed by Okun (1989a, 1989b).

Eqs. (51) and (54) yield

\[dE_{out} = \gamma_{out} (dM_{out}) \, c^2,\]

\[dp_{out} = \gamma_{out} (dM_{out}) \, v_{out},\]

\[v_{out} = \left\{ \frac{e_1}{w_1} - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} + X^{-1} F_{e,j}' \right) \left( \frac{e_j}{w_j} - \gamma \frac{v}{c} \right) \right\} \left( ZC \right)^{-1} c,\]

\[\gamma_{out} = ZC \times ZM,\]

\[ZC = \frac{1}{w_1} - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} + X^{-1} F_{e,j}' \right) \left( \frac{1}{w_j} - \gamma \right),\]

\[ZM = \sqrt{2 \left( \frac{C_{pr,1}'}{C_{ext}'} + X^{-1} F_{e,1}' \right) - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} + X^{-1} F_{e,j}' \right)^2}.\]

\[X = \frac{w_1^2 S \bar{C}_{ext}'}{c}.\]

(55)

We have obtained the standard formulae: energy equals \(\gamma \times \text{mass} \times c^2\), momentum equals \(\gamma \times \text{mass} \times \text{velocity}\).
5.3. Outgoing radiation – thermal emission

Making an invariant of Eq. (47)

\[ [(c \Delta M_{\text{out}})_e]^2 = [(dp_{\text{out}})_e]_{\mu} \cdot [(dp_{\text{out}})_e]^\mu, \quad (56) \]

one easily obtains

\[ (\Delta M_{\text{out}})_e = \sqrt{\left(\frac{E'_o}{c^2}\right)^2 - \sum_{j=1}^3 \left(\frac{F'_{e,j}}{c}\right)^2} \, d\tau, \quad (57) \]

Eqs. (47) and (57) yield

\[ (\Delta E_{\text{out}})_e = \left\{ \gamma E'_o - \sum_{j=1}^3 c \, F'_{e,j} \left( \frac{1}{w_j} - \gamma \right) \right\} \, d\tau, \]
\[ (\Delta M_{\text{out}})_e = \sqrt{\left(\frac{E'_o}{c^2}\right)^2 - \sum_{j=1}^3 \left(\frac{F'_{e,j}}{c}\right)^2} \, d\tau, \]
\[ w_j = \gamma \left(1 - \frac{\mathbf{v} \cdot \mathbf{e}_j}{c}\right), \quad j = 1, 2, 3, \quad (58) \]

where also Eq. (25) was used.

Eqs. (47) and (58) yield

\[ (\Delta E_{\text{out}})_e = \gamma_e \cdot (\Delta M_{\text{out}})_e \cdot c^2, \]
\[ (\Delta p_{\text{out}})_e = \gamma_e \cdot (\Delta M_{\text{out}})_e \cdot \mathbf{v}_e, \]
\[ \mathbf{v}_e = \left\{ \frac{E'_o}{c} \gamma \frac{\mathbf{v}}{c} - \sum_{j=1}^3 c \, F'_{e,j} \left( \frac{\mathbf{e}_j}{w_j} - \gamma \frac{\mathbf{v}}{c}\right) \right\} \cdot (ZC)_e^{-1} \cdot c^2, \]
\[ \gamma_e = \frac{(ZC)_e}{(ZM)_e}, \]
\[ (ZC)_e = \gamma E'_o - \sum_{j=1}^3 c \, F'_{e,j} \left( \frac{1}{w_j} - \gamma \right), \]
\[ (ZM)_e = \sqrt{\left(E'_o\right)^2 - \sum_{j=1}^3 \left(c \, F'_{e,j}\right)^2}. \quad (59) \]

We have obtained standard formulae: energy equals $\gamma \times \text{mass} \times c^2$, momentum equals $\gamma \times \text{mass} \times \text{velocity}$.

6. Incoming and outgoing radiation – change of particle’s mass

As it was already pointed out just below Eq. (38), mass of the particle in the whole process of interaction with the radiation is conserved, i.e. $dm/d\tau = 0$; if the particle is not irradiated, then Eqs. (40)-(41) hold. Now, we are interested in the effect of incoming and outgoing radiation on the mass of the particle.
6.1. Incoming radiation

On the basis of Eq. (42), we can write for the change of four-momentum of the particle due to the incoming radiation

\[ \frac{dp^\mu}{d\tau} = \frac{dp^\mu_{\text{in}}}{d\tau}, \]

\[ \frac{dp^\mu_{\text{in}}}{d\tau} = \frac{w_1^2}{c} \tilde{C}'_{\text{ext}} b_1^\mu, \]  \hspace{1cm} (60)

which leads to the result

\[ \frac{dm}{d\tau}_{\text{in}} = \frac{w_1^2}{c} \tilde{C}'_{\text{ext}} \frac{1}{c}. \]  \hspace{1cm} (61)

Comparison of Eq. (61) with Eq. (50) yields

\[ \frac{dE_{\text{in}}}{d\tau} = \left( \frac{dm}{d\tau} \right)_{\text{in}} c^2 \frac{1}{w_1} \]  \hspace{1cm} (62)

and

\[ \frac{dp_{\text{in}}}{d\tau} = \left( \frac{dm}{d\tau} \right)_{\text{in}} c \frac{e_1}{w_1}. \]  \hspace{1cm} (63)

6.2. Outgoing radiation

On the basis of Eq. (51), we can write for the change of four-momentum of the particle due to the outgoing radiation

\[ \frac{dp^\mu}{d\tau} = -\frac{dp^\mu_{\text{out}}}{d\tau}, \]

\[ \frac{dp^\mu_{\text{out}}}{d\tau} = \frac{w_1^2}{c} \tilde{C}'_{\text{ext}} \left\{ b_1^\mu - \sum_{j=1}^{3} \frac{\tilde{C}'_{\text{pr},j}}{C'_{\text{ext}}} \left( b_j^\mu - \frac{u_j^\mu}{c} \right) \right\} - \sum_{j=1}^{3} F_{e,j} \left( b_j^\mu - \frac{u_j^\mu}{c} \right). \]  \hspace{1cm} (64)

This leads to the result

\[ \frac{dm}{d\tau}_{\text{out}} = -\frac{w_1^2}{c} \tilde{C}'_{\text{ext}} \frac{1}{c}. \]  \hspace{1cm} (65)

According to Eqs. (61) and (65): \((dm/d\tau)_{\text{out}} = -(dm/d\tau)_{\text{in}}.\) Comparison of Eq. (65) with Eq. (54) yields

\[ \frac{dE_{\text{out}}}{d\tau} = -\left\{ \frac{1}{w_1} - \sum_{j=1}^{3} \left( \frac{\tilde{C}'_{\text{pr},j}}{C'_{\text{ext}}} + X^{-1} F_{e,j} \right) \left( \frac{1}{w_j} - \gamma \right) \right\} \left( \frac{dm}{d\tau} \right)_{\text{out}} c^2, \]

\[ \frac{dp_{\text{out}}}{d\tau} = -\left\{ \frac{e_1}{w_1} - \sum_{j=1}^{3} \left( \frac{\tilde{C}'_{\text{pr},j}}{C'_{\text{ext}}} + X^{-1} F_{e,j} \right) \left( \frac{e_1}{w_j} - \gamma \frac{v}{c} \right) \right\} \left( \frac{dm}{d\tau} \right)_{\text{out}} c, \]

\[ X \equiv \frac{w_1^2}{c} \tilde{C}'_{\text{ext}}. \]  \hspace{1cm} (66)
6.3. Thermal emission

As for the thermal emission alone, the relevant result is given by Eqs. (40)-(41):

\[
\left( \frac{dN}{d\tau} \right)_e = -\frac{E'_o}{c^2}.
\]

Eqs. (40)-(41) immediately yield for the energy and momentum of the outgoing radiation:

\[
\frac{dE_o}{d\tau} = -\gamma \left( \frac{dN}{d\tau} \right)_e c^2 - c \sum_{j=1}^{3} F'_{e,j} \left( \frac{1}{w_j} - \gamma \right),
\]

\[
\frac{dp_o}{d\tau} = -\gamma \left( \frac{dN}{d\tau} \right)_e \mathbf{v} - \sum_{j=1}^{3} F'_{e,j} \left( \frac{e_1}{w_j} - \gamma \frac{v}{c} \right).
\]

Eq. (68) yields the standard formulae: energy equals \( \gamma \times \text{mass} \times c^2 \), momentum equals \( \gamma \times \text{mass} \times \text{velocity} \) in the case \( F'_{e,j} = 0, j = 1 \) to 3.

7. Discussion

7.1. Physics of \( p'_{\text{in}} \)

Eq. (11) is physically different from Eq. (11) in Klačka (2004) and also from Eq. (122) in Klačka (1992). In order to understand the physics behind the formulae, we will consider a simple case of a spherically symmetric particle fulfilling \( C'_{pr,2} = C'_{pr,3} = C'_{abs} = F'_{e,1} = F'_{e,2} = F'_{e,3} = 0 \), now.

For the sake of brevity, we will use dimensionless efficiency factors \( Q'_x \) instead of cross sections \( C'_x \): \( C'_x = Q'_x A' \), where \( A' \) is geometrical cross section of the spherical particle, and, we will not use primed quantities.

Let a beam of homogeneous incoming photons, interacting with the spherical particle, consists of \( N \) photons each of which carries energy \( \varepsilon \). The total incoming momentum is \( N (\varepsilon/c) \mathbf{e} \) and the elastically scattered photons are characterized by the total outgoing momentum \( (\varepsilon/c) \sum_{j=1}^{N} e_j \); momentum of an individual photon is \( (\varepsilon/c) e_j, j = 1 \) to \( N \).

The momentum of the particle is given by conservation of the total momentum: \( N (\varepsilon/c) Q'_{pr,1} \mathbf{e} = N (\varepsilon/c) \mathbf{e} - (\varepsilon/c) \sum_{j=1}^{N} e_j \), according to Klačka (1992, 2004). This yields \( Q'_{pr,1} \mathbf{e} = \mathbf{e} - \sum_{j=1}^{N} e_j / N \). Taking into account that the last term is an average which can be written as \( \langle \cos \Theta \rangle \mathbf{e} \), we finally obtain \( Q'_{pr,1} = 1 - \langle \cos \Theta \rangle \). However, this is not correct. We know that \( Q'_{pr,1} A \equiv C'_{pr,1} = C'_{ext} - \langle \cos \Theta \rangle C'_{sca} \) and \( Q'_{ext} A = Q_{abs} A + Q_{sca} A = Q_{sca} A \), since we deal with the case \( Q_{abs} = 0 \), in our example. Thus, \( Q'_{pr,1} = Q'_{ext} (1 - \langle \cos \Theta \rangle) \) and the correct form of the conservation of the total momentum is \( N (\varepsilon/c) Q'_{pr,1} \mathbf{e} = Q'_{ext} [N (\varepsilon/c) \mathbf{e} - (\varepsilon/c) \sum_{j=1}^{N} e_j] \) .... (*). This explains why the ratios of the pressure terms to the extinction term are present in Eq. (11), and, also, why the extinction cross section is present in energy in Eq. (11). The equation (*) is fully consistent with Eq. (11): \( p'_t = (C'_{ext}/A') N' (\varepsilon'/c) \mathbf{e}', p'_t = [1 - (C'_{pr,1} / C'_{ext})] p'_t \), where the dot denotes differentiation with respect to time. All is consistent with Eq. (5), too.
7.2. Center-of-momentum

Short comment on Eqs. (55) and (59). We have defined velocities $v_{\text{out}}$ and $v_e$ for radiation in Eqs. (55) and (59). This in reality corresponds to the fact that we can introduce center-of-momentum frame, where the total momentum vanishes. Let $v_c$ be the velocity of the center-of-momentum frame with respect to the reference frame in which an object (radiation, particle) of a mass $dM$ moves with velocity $v$. Then

\begin{align*}
    dE &= \gamma_c (dM) c^2, \\
    dp &= \gamma_c (dM) v_c, \\
    v_c &= c^2 \frac{dp}{dE},
\end{align*}

and center-of-inertia of the system of the outgoing photons, defined as the point with radius vector $R(t) = \left( \sum_i E_i(t) r_i(t) \right) / \left( \sum_i E_i(t) \right)$, moves with the velocity $v_c$ given by Eq. (69) (see, e.g., Landau and Lifshitz 2005: §14, pp. 42-45, or, Ferraro 2007: §6.5, pp. 146-148, and §6.7, pp. 153-155).

7.3. Special case

Now, we will deal with the special case $C_{pr,2}' = C_{pr,3}' = F_{e,j}' = 0$ for $j = 1, 2, 3$. This case corresponds to secular decrease of semi-major axis and eccentricity of neutral dust grain in the gravitational and electromagnetic fields of a central star, e.g., Sun, if the cross section $C_{pr,1}'$ is not a function of time (see Secs. 6.1 and 6.2 in Klačka 2004). Thus, we can call this effect as the (generalized) Poynting-Robertson effect. As in Sec. 6.1, we will use dimensionless efficiency factors $\bar{Q}_{pr,1}'$, $\bar{Q}_{ext}'$ instead of cross sections $C_{pr,1}'$, $C_{ext}'$: $\bar{Q}_{pr,1}' = C_{pr,1}' / A'$, $\bar{Q}_{ext}' = C_{ext}' / A'$, where $A'$ is geometrical cross section of a sphere of volume equal to the volume of the particle. In reality, the case treated in this section holds only for particles with spherically symmetric distribution of mass. Dimensionless efficiency factors can be calculated on the basis of Mie theory (Mie 1908, Bohren and Huffman 1983).

Equation of motion for the discussed case is given by reduction of Eq. (38):

\begin{align*}
    \frac{dp^\mu}{d\tau} &= \frac{w_1^2 S A'}{c^2} \bar{Q}_{pr,1}' \left( c b_1^\mu - u^\mu \right), \\
    b_1^\mu &= \left( 1/w_1, e_1/w_1 \right), \\
    w_1 &= \gamma \left( 1 - v \cdot e_1/c \right),
\end{align*}

where also Eqs. (18) and (33) were used. As a consequence, $dm/d\tau = 0$ (this corresponds to the condition $dE'/d\tau = 0$, where $E'$ is energy of the particle measured in the rest frame of the particle). If the particle is not irradiated, then only thermal emission may exist and one has to use

\begin{equation}
    \left( \frac{d u^\mu}{d\tau} \right)_e = 0,
\end{equation}

instead of Eq. (70) (see Eq. 41).

To first order in \( v/c \), Eq. (70) yields

\[
\frac{d v}{d t} = \frac{S A'}{m c} \bar{Q}_{pr,1}' \left\{ \left( 1 - \frac{v \cdot e_1}{c} \right) e_1 - \frac{v}{c} \right\}
\]

\[e_1 = \left( 1 - \frac{v \cdot e_1'}{c} \right) e_1' + \frac{v}{c} . \tag{72}\]

It is worth mentioning to stress that the value \( \bar{Q}_{pr,1}' \) does not depend on particle’s orientation with respect to the incident radiation, but it’s value may be time dependent, e. g., when optical properties change with the particle’s distance from the central star (Klačka et al. 2007).

The total process of interaction can be written in the form of the following equations (energies and momenta per unit time):

\[
E'_o = E'_i = S' A' \bar{Q}'_{ext} ,
\]

\[
p'_o = (1 - \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}}) p'_i ,
\]

\[
p'_i = (E'_i/c) e'_1 , \tag{73}\]

where the index ”i” represents the incoming (incident) radiation, beam of photons, the index ”o” represents the outgoing radiation. Eq. (73) is a fundamental condition for the validity of the Poynting-Robertson effect and it is a new condition. It differs from the condition \( p'_o = (1 - \bar{Q}'_{pr,1}) p'_i \) consistent with the statements presented in Poynting (1903), Robertson (1937), Wyatt and Whipple (1950), Burns et al. (1979; Eqs. 10 and 11) and in other papers. The condition represented by Eq. (73) does not exist in the literature and any statement about the outgoing radiation, published up to now, is not consistent with Eq. (73).

Covariant form of Eq. (73) can be obtained from Eq. (46):

\[
\frac{dp^\mu_{out}}{d\tau} = \left( 1 - \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}} \right) \frac{dp^\mu_{in}}{d\tau} + \frac{w_1^2 S A' \bar{Q}'_{ext}}{c} \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}} \frac{w^\mu}{c} \tag{74}\]

and incoming four-momentum of the radiation is given by Eq. (42):

\[
dp^\mu_{in} = \frac{w_1^2 S A' \bar{Q}'_{ext}}{c} b^\mu_1 d\tau . \tag{75}\]

Momentum-four vector for the outgoing radiation can be written in a short relativistically covariant form, on the basis of Eq. (44), as follows:

\[
\frac{dp^\mu_{out}}{d\tau} = X \left\{ b^\mu_1 - \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}} \left( b^\mu_1 - \frac{w^\mu}{c} \right) \right\} ,
\]

\[X = \frac{w_1^2 S A' \bar{Q}'_{ext}}{c} ,
\]

\[p^\mu_{out} = \left( E_{out}/c, p_{out} \right) . \tag{76}\]

and Eqs. (18), (32) and (33) can be used.
7.3.1. Energy-mass relation for radiation

As for the incoming radiation, the relevant statements are represented by Eqs. (42) and (50), as for the four-momentum, and also, as for the energy and the mass of the radiation.

Now, we will treat the outgoing radiation. Eq. (51) reduces to (see also Eq. 76)

\[
dp_{\text{out}}^{\mu} = \frac{w_1^2 S A'}{c} \left( 1 - \frac{Q'_{\text{ext}}}{Q_{\text{ext}}'} \right) b_1^{\mu} + \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \frac{u^{\mu}}{c} \right) d\tau,
\]

\[
u_\mu b_1^\mu = c,
\]

\[
u_\mu u^\mu = c^2,
\]

\[b_1 \mu b_1^\mu = 0.\]

Definition of the invariant mass represented by Eq. (52),

\[
dM_{\text{out}} = \sqrt{dp_{\text{out}}^{\mu} dp_{\text{out}}^{\nu} / c^2},
\]

leads to

\[
dM_{\text{out}} = \frac{w_1^2 S A'}{c^2} \sqrt{\frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \left( 2 - \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \right)} d\tau,
\]

if one uses results presented in Eq. (77); compare with Eq. (53). As a consequence of Eq. (78), one obtains a simple statement:

\[
0 < \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \leq 2.
\]

Eqs. (77) and (78) yield

\[
dE_{\text{out}} = \frac{(ZC)_1}{(ZM)_1} (dM_{\text{out}}) c^2,
\]

\[
dp_{\text{out}} = \frac{1}{(ZM)_1} \left( 1 - \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \right) \frac{e_1}{w_1} + \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \frac{\gamma v}{c} \right) (dM_{\text{out}}) c,
\]

\[
(ZC)_1 = \left( 1 - \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \right) \frac{1}{w_1} + \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \gamma,
\]

\[
(ZM)_1 = \sqrt{\frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \left( 2 - \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'} \right)},
\]

\[
w_1 = \gamma \left( 1 - \frac{v \cdot e_1}{c} \right),
\]

see also Eq. (55).

As for the center-of-momentum frame, Eqs. (69) and (80) lead to

\[
v_c = \frac{Q' \gamma v + \left( 1 - Q' \right) c e_1 / w_1}{Q' \gamma + \left( 1 - Q' \right) / w_1},
\]

\[
\tilde{Q}' = \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'}.
\]

It can be verified that Eq. (81) yields

\[
\begin{align*}
\nu_c &= \left( 1 - Q' \right) c e_1, \quad \text{if} \quad \nu = 0, \\
\nu &= - \left( 1 - Q' \right) c e_1, \quad \text{if} \quad \nu_c = 0, \\
\nu_c &= \nu, \quad \text{if} \quad \tilde{Q}' = 1,
\end{align*}
\]

\[
\tilde{Q}' = \frac{Q'_{\text{pr}.1}}{Q_{\text{ext}}'}.
\]
The case \( v = - \left( 1 - Q_{\text{pr},1} / Q_{\text{ext}}' \right) c e_1 \) yields \( dE_{\text{out}} = \left( dM_{\text{out}} \right) c^2 \), as can be verified on the basis of Eq. (80).

If the particle is not irradiated, then the general Eqs. (58)-(59) reduce to

\[
\begin{align*}
(dE_{\text{out}})_e &= \gamma (dM_{\text{out}})_e c^2, \\
(dp_{\text{out}})_e &= \gamma (dM_{\text{out}})_e v, \\
(dM_{\text{out}})_e &= \frac{E'_o}{c^2} d\tau .
\end{align*}
\] (83)

Moreover, Eq. (59) or comparison of Eq. (83) with Eq. (69) immediately gives

\[ v_c = v . \] (84)

Eq. (84) states that the center-of-momentum frame for thermally emitted radiation is in the center of the spherical particle.

### 7.3.2. Energy-mass relation for the particle

At first, we will treat the incoming radiation. On the basis of Eqs. (61)-(62), we have

\[
\begin{align*}
\frac{dE_{\text{in}}}{d\tau} &= \frac{1}{w_1} \left( \frac{dm}{d\tau} \right)_{\text{in}} c^2, \\
\left( \frac{dm}{d\tau} \right)_{\text{in}} &= \frac{w_1^2 S A' Q_{\text{ext}}'}{c^2}. 
\end{align*}
\] (85)

This is not equivalent to the change of mass of the particle according to the "definition" \( c^{-2} dE_{\text{in}} / d\tau \) ("mass equals energy divided by \( c^2 \)), mainly if one takes into account that Eqs. (60)-(61) and (69) lead to \( v_c = c e_1 \): the relation between the momentum of the incoming radiation and the particle’s mass is (see Eq. 63)

\[
\begin{align*}
\frac{dp_{\text{in}}}{d\tau} &= \frac{1}{w_1} \left( \frac{dm}{d\tau} \right)_{\text{in}} c e_1, \\
\left( \frac{dm}{d\tau} \right)_{\text{in}} &= \frac{w_1^2 S A' Q_{\text{ext}}'}{c^2}. 
\end{align*}
\] (86)

We also remind of Eq. (49) for the mass of the incoming radiation: \( dM_{\text{in}} / d\tau = 0 \).

For the outgoing radiation, Eq. (66) reduces, for the spherically distributed mass, to

\[
\begin{align*}
\frac{dE_{\text{out}}}{d\tau} &= - \left\{ \left( 1 - Q_{\text{pr},1}' / Q_{\text{ext}}' \right) \frac{1}{w_1} + Q_{\text{pr},1}' / Q_{\text{ext}}' \gamma \right\} \left( \frac{dm}{d\tau} \right)_{\text{out}} c^2, \\
\left( \frac{dm}{d\tau} \right)_{\text{out}} &= - \frac{w_1^2 S A' Q_{\text{ext}}'}{c^2}. 
\end{align*}
\] (87)

This is not equivalent to the change of mass of the particle according to the "definition" \( \gamma^{-1} c^{-2} dE_{\text{out}} / d\tau \). Changes of the particle’s mass and outgoing momentum are related as follows:

\[
\begin{align*}
\frac{dp_{\text{out}}}{d\tau} &= - \left( \frac{dm}{d\tau} \right)_{\text{out}} \left\{ \left( 1 - Q_{\text{pr},1}' / Q_{\text{ext}}' \right) \frac{e_1}{w_1} + \frac{Q_{\text{pr},1}'}{Q_{\text{ext}}'} \gamma \frac{v}{c} \right\} c, \\
\left( \frac{dm}{d\tau} \right)_{\text{out}} &= - \frac{w_1^2 S A' Q_{\text{ext}}'}{c^2}. 
\end{align*}
\] (88)
Eqs. (87)-(88) yield, together with equation analogous to Eq. (69), that \( v_c \) is given by Eq. (81), and, thus, we can write

\[
\frac{dE_{\text{out}}}{d\tau} = - \gamma_c \sqrt{Q'(2 - Q')} \left( \frac{dm}{d\tau} \right)_{\text{out}} c^2,
\]

\[
\frac{dp_{\text{out}}}{d\tau} = - \gamma_c \sqrt{Q'(2 - Q')} \left( \frac{dm}{d\tau} \right)_{\text{out}} v_c,
\]

\[
v_c = \frac{Q' \gamma v + (1 - Q') e E_1/w_1}{\gamma + (1 - Q')/w_1},
\]

(89)

The relations for \( E_{\text{out}} \) and \( p_{\text{out}} \) seem to be only a little modified standard relations (invariant multiplier \( \sqrt{Q'(2 - Q')} \) is present). However, this is not true: the velocity is \( v_c \) and it is not equal to the velocity of the particle \( v \) (except for the case \( Q' = 1 \)): Eqs. (69) hold. Also, comparison of Eqs. (87)-(88) with Eq. (80) yields

\[
\frac{dM_{\text{out}}}{d\tau} = - \sqrt{\frac{Q'_{\text{pr},1}}{Q'_{\text{ext}}}} \left( 2 - \frac{Q'_{\text{pr},1}}{Q'_{\text{ext}}} \right) \left( \frac{dm}{d\tau} \right)_{\text{out}}.
\]

(90)

Eqs. (69), (89)-(90) state that the formulation "mass equals energy divided by \( c^2 \)" holds for radiation \( (dE_{\text{out}}, dM_{\text{out}}) \), but not for the mass of the particle \( (dE_{\text{out}}, dm) \), if \( Q' \neq 1 \).

It is worth mentioning that the special case \( Q'_{\text{pr},1} / Q'_{\text{ext}} = 1 \), yielding \( p'_{\text{o}} = 0 \) (generally believed to be the case of perfect absorption within geometrical optics approximation and which was treated by Robertson 1937, see also Robertson and Noonan 1968, p. 114), leads to very simple physics: \( v_c = v \), according to Eq. (89), \( dM_{\text{out}} / d\tau = - (dm/d\tau)_{\text{out}} \), according to Eq. (90), and \( (dE)_{\text{out}} = - \gamma (dm)_{\text{out}} c^2 \).

If the particle is not irradiated, then Eq. (41) holds: \( (dm/d\tau)_{\text{e}} = - E'_o / c^2 \). Comparison with Eq. (83) yields: \( (dm/d\tau)_{\text{e}} = - (dM_{\text{out}}/d\tau)_{\text{e}} \).

7.3.3. Mirror-like spherical surface and geometrical optics – specular reflection

On the basis of Eq. (79) we know that \( 0 < Q'_{\text{pr},1} / Q'_{\text{ext}} \leq 2 \). Moreover, one could say that the special case of perfect absorption within geometrical optics approximation would yield \( Q'_{\text{pr},1} / Q'_{\text{ext}} = 1 \), according to Eq. (73) and according to the conventional statement that the absorbed radiation is isotropically reemitted (in the particle’s frame of reference) – \( p'_{\text{o}} = 0 \). Moreover, a statement that \( Q'_{\text{pr},1} / Q'_{\text{ext}} = 2 \) should hold for spherical particle with mirror-like surface (totally reflecting sphere), can also be found. However, the statements presented in the last two sentences are not correct. In order to prove this conclusion, we will treat a nonrotating totally reflecting sphere, within a geometrical optics approximation. We will consider specular reflection of the electromagnetic radiation at the surface of the spherical particle, in the particle’s rest frame of reference (for the sake of brevity, we will not use primed quantities, in this subsection).

Let the center of the cartesian reference frame \( x - y - z \) be situated at the center of the spherical particle and the incoming beam of photons is characterized by unit vector
of momentum \( \mathbf{p}_{\text{inc}} = - \mathbf{\hat{x}} = - (1, 0, 0) \). Any point on the surface of the particle, which interacts with the incoming radiation, can be characterized by coordinates of the surface position vector in the following way:

\[
\mathbf{R} = R \, \mathbf{n} ,
\]

\[
\mathbf{n} = \left( \cos \alpha \, \cos \delta , \sin \alpha \, \cos \delta , \sin \delta \right) ,
\]

\[
\alpha \in \left( -\frac{\pi}{2} , +\frac{\pi}{2} \right) ,
\]

\[
\delta \in \left( -\frac{\pi}{2} , +\frac{\pi}{2} \right) ,
\]

(91)

where \( R \) is particle’s radius and \( \mathbf{n} \) is unit vector.

Now, we have to use the law of reflection: The angle of incidence equals the angle of reflection, and the incident and reflected rays are in the same plane. The angle of incidence \( \varphi \) is given by condition

\[
\cos \varphi = | \mathbf{n} \cdot (-\mathbf{\hat{x}}) | = \cos \alpha \, \cos \delta .
\]

(92)

The plane of incidence is characterized by its normal unit vector \( \mathbf{N} \):

\[
\mathbf{N} = \frac{(-\mathbf{\hat{x}}) \times \mathbf{n}}{|\mathbf{\hat{x}} \times \mathbf{n}|} =
\]

\[
= \left( 0 , \sin \delta , -\sin \alpha \, \cos \delta \right) \sqrt{\sin^2 \delta + \sin^2 \alpha \, \cos^2 \delta} .
\]

(93)

The equation of the plane of incidence is given as \( \mathbf{N} \cdot \mathbf{R} + k = 0 \), which yields \( k = -\mathbf{N} \cdot \mathbf{R} = 0 \), according to Eqs. (91) and (93). The plane of incidence is identical to the plane of reflection:

\[
\mathbf{N} \cdot \mathbf{p}_{\text{out}} = 0 ,
\]

\[
\mathbf{p}_{\text{out}} \cdot \mathbf{n} = \cos \varphi ,
\]

(94)

where also the condition ”the angle of incidence equals the angle of reflection” was added (the case \( \alpha = 0 , \delta = 0 \) yields \( \mathbf{p}_{\text{out}} = +\mathbf{\hat{x}} \)). Conditions formulated by Eq. (94) yield, on the basis of Eqs. (91)-(93):

\[
\mathbf{p}_{\text{out}} \equiv (x_{\text{out}} , y_{\text{out}} , z_{\text{out}}) ,
\]

\[
x_{\text{out}} = (\cos^2 \alpha - \sin^2 \alpha) \cos^2 \delta - \sin^2 \delta ,
\]

\[
y_{\text{out}} = (1 - x_{\text{out}}) \frac{\cos \alpha \, \sin \alpha \, \cos^2 \delta}{\sin^2 \alpha \, \cos^2 \delta + \sin^2 \delta} ,
\]

\[
z_{\text{out}} = (1 - x_{\text{out}}) \frac{\cos \alpha \, \cos \delta \, \sin \delta}{\sin^2 \alpha \, \cos^2 \delta + \sin^2 \delta} .
\]

(95)

Unit vector along the small circle parallel with the equator \( \mathbf{e}_\alpha \), and, unit vector along the great circle normal to the equator \( \mathbf{e}_\delta \), both localized at the position vector \( \mathbf{R} \), are given by formulae

\[
\mathbf{e}_\alpha = \left( \cos \left( \alpha + \frac{\pi}{2} \right) , \sin \left( \alpha + \frac{\pi}{2} \right) , 0 \right)
\]

\[
= (- \sin \alpha , \cos \alpha , 0) ,
\]
\[ e_\delta = n \times e_\alpha = (- \cos \alpha \sin \delta, \ - \sin \alpha \sin \delta, \ \cos \delta) , \]  

(96)

since the relation for \( e_\delta \) is a consequence of the relation \( e_\alpha \times e_\delta = n \). Element of the length vector \( dl \equiv dx \hat{x} + dy \hat{y} + dz \hat{z} = R \, dn \) is, on the basis of Eq. (91):

\[
dl = R (- \sin \alpha \cos \delta \, d\alpha \ - \cos \alpha \sin \delta \, d\delta) \, \hat{x} + R (\cos \alpha \cos \delta \, d\alpha \ - \sin \alpha \sin \delta \, d\delta) \, \hat{y} + R \cos \delta \, d\delta \, \hat{z} .
\]  

(97)

Components of the length vector along the vectors \( e_\alpha \) and \( e_\delta \) are:

\[
dl_\alpha = dl \cdot e_\alpha = R \cos \delta \, d\alpha ,
\]

\[
dl_\delta = dl \cdot e_\delta = R \, d\delta .
\]

(98)

Vector of the elementary/infinitesimal area is

\[
dA = (dl_\alpha e_\alpha) \times (dl_\delta e_\delta) = R^2 \cos \delta \, d\alpha \, d\delta \, n
\]

= \( R^2 \, d\Omega \, n \),

(99)

where \( d\Omega \) is the solid angle element at the position \((\alpha, \delta)\). The amount of energy (per unit time) hitting the area \( dA \) at the position \( R \) is \( S \, dA_\perp \), where \( S \) is the flux density of radiation energy (energy flow through unit area perpendicular to the ray of photons per unit time) and

\[
dA_\perp = dA \cdot \hat{x} = (R^2 \, d\Omega) \cos \alpha \cos \delta = \]

= \( R^2 \cos \alpha \cos^2 \delta \, d\alpha \, d\delta \),

(100)

if also Eq. (91) is used. The quantity \( dA_\perp \) is projection of the area \( dA \) on the direction normal to \( \hat{x} \).

On the basis of Eqs. (91) and (100) we can write for the incoming momentum per unit time

\[
\frac{dp_{\text{in}}}{dt} = \frac{1}{c} S \int dA_\perp \hat{p}_{\text{inc}} = 
\]

= \( \frac{S}{c} \hat{p}_{\text{inc}} \int_{-\pi/2}^{\pi/2} d\alpha \int_{-\pi/2}^{\pi/2} d\delta \ \{ R^2 \cos \alpha \cos^2 \delta \} = 
\]

= \( \frac{S}{c} \pi R^2 \hat{p}_{\text{inc}} \).

(101)

The outgoing momentum per unit time is, on the basis of Eqs. (95) and (100),

\[
\frac{dp_{\text{out}}}{dt} = \frac{1}{c} S \int dA_\perp \hat{p}_{\text{out}} = 0 .
\]

(102)

The mirror-like spherical particle obtains the following momentum per unit time:

\[
\frac{d\mathbf{p}}{dt} = \frac{d\mathbf{p}_{\text{in}}}{dt} - \frac{d\mathbf{p}_{\text{out}}}{dt} = \frac{S}{c} \pi R^2 \hat{p}_{\text{inc}} ,
\]

(103)
which immediately follows from Eqs. (101)-(102). Comparison with Eq. (70) or (72) yields $\bar{Q}_{pr,1}' = 1$. This is consistent with the results presented in van de Hulst (van de Hulst 1981, p. 161: see Table 13 "Efficiency Factors for Totally Reflecting Spheres"). However, comparison of Eqs. (73) and (102) would yield $\bar{Q}_{pr,1}' / \bar{Q}_{ext}' = 1$, and, thus, $\bar{Q}_{ext}' = 1$. But this is not consistent with the results of the Mie theory presented in van de Hulst (1981, p. 161 – Table 13). This seems surprisingly, since van de Hulst states on p. 223: "a smooth totally reflecting sphere with radius large compared to the wavelength scatters light by reflection isotropically". Where’s the problem? In order to understand physics of the results, we have to realize that the result presented by Eq. (102) is based on geometrical optics and the result does not respect all physical interactions, only reflection is considered. In reality, "extinction paradox" exists (van de Hulst 1981, p. 107) and diffracted light plays a non-negligible role in treating the incoming and outgoing radiation, even for large particles (in comparison with a wavelength of the interacting light); "diffraction = small-angle scattering" (van de Hulst 1981, p. 107). We know that "the diffracted light gives a zero contribution" to the radiation pressure of large spheres (van de Hulst 1981, p. 225), but diffraction cannot be neglected in a separate treatment of the incoming and outgoing radiation. In order to be consistent with equations of the type of Eq. (52), i. e., Eqs. (53) and (78)-(79), we have to take into account diffraction, also. Thus, we have $\bar{Q}_{pr,1}' = 1$ and $\bar{Q}_{ext}' = 2$, both for perfectly absorbing sphere ($\bar{Q}_{sca}' = 1$ due to the diffracted light and $\bar{Q}_{abs}' = 1$) and for totally reflecting sphere ($\bar{Q}_{sca}' = 1 + 1$, 1 due to the diffracted light + 1 due to the reflected light, $\bar{Q}_{abs}' = 0$):

$$\frac{\bar{Q}_{pr,1}'}{\bar{Q}_{ext}'} = \frac{1}{2}. \quad (104)$$

The result $\bar{Q}_{pr,1}' / \bar{Q}_{ext}' = 1/2$ holds for spherical particles not only with perfect absorption, but, also with mirror-like surface characterized by specular reflection. Eq. (73) holds for the incoming and outgoing radiation. Eq. (78) yields

$$dM_{out} = \sqrt{3} \frac{w_1^2 S A'}{c^2} \, d\tau. \quad (105)$$

One would await, on the basis of a simple idea about the isotropic light reemission or scattering (diffraction is neglected), the result without the numerical factor $\sqrt{3}$ – the same result would be obtained on the basis of the following frequent idea (see also Sec. 8): "Photons at rest do not exist! It is therefore somehow artificial to speak of the photon rest mass. It is more logical to use the relation $m = h \nu / c^2$ for the definition of the photon mass." (Demtröder 2006, p. 92).

7.3.4. Summary

We have already mentioned "the extinction paradox" (van de Hulst 1981, p. 107). It states that "a large particle removes from the incident beam exactly twice the amount of light it can intercept" (van de Hulst 1981, p. 107). As van de Hulst explains, the
explanation is in the fact that "(a) all scattered light, including that at small angles, is counted as removed from the beam, and (b) that the observation is made at a very great distance, i. e., far beyond the zone where a shadow can be distinguished".

Let us take into account the values presented in Table 13 in van de Hulst (1981, p. 161). One can immediately realize that the values $Q_{pr}$ greater than 2 would be in contradiction with equations analogous to Eqs. (78)-(79), if $p'_o = (1 - Q'_{pr} / Q'_{ext}) p'_i$ would be the condition for the Poynting-Robertson effect. Thus, the case of "perfectly absorbing case" does not correspond to $p'_o = 0$ within geometrical optics approximation plus diffraction (relevant physical processes), i. e., isotropic reradiation/reemission. Instead of this we have to use Eq. (11) and the "perfectly absorbing case" yields $p'_o = (1 - Q'_{pr} / Q'_{ext}) p'_i = (1 / 2) p'_i$ for large particles $[2 \pi \times \text{radius of the particle} / (\text{wavelength of the incident radiation}) \gg 1]$. The term "isotropic reemission" does not hold, strictly speaking, neither for small particle, nor for large sphere. Similar situation holds for totally reflecting sphere. [Although, within geometrical optics approximation, the term "isotropic reradiation" is used, which is based on the fact that "for large spheres we may separate the effects of geometrical optics (reflection and refraction) and the effect of diffraction" (van de Hulst 1981, p. 225).]

We have found that the condition for the validity of the Poynting-Robertson effect is given by Eq. (73). The case of "perfectly absorbing spherical particle" is given by Eq. (104), or, by the condition $p'_o = 0.5 p'_i$ for large spherical particle. This condition differs from the condition used by Robertson (1937): it is generally accepted that the Robertson’s case corresponds to $p'_o = 0$. E. g.: "Consider a spherical particle with proper mass $m$ and world-velocity $u^\mu$ which scatters (whether by reflection or by absorption and re-radiation) electromagnetic radiation isotropically in all directions relative to its proper system. ... The proper system $\bar{x}^\alpha$ of the particle at a given event is used for writing down the equation of motion. The force on the particle is then due only to the incident radiation, because the re-emitted radiation is isotropic." (Robertson and Noonan 1968, p. 114). Similarly, in his original paper, Robertson writes on p. 427: "the particle is assumed to possess spherical symmetry and to radiate isotropically in the rest system" of the particle. On the basis of this assumption, the author deals with geometrical optics approximation and he writes equation of motion in the form corresponding to our equation \[ dp^\mu / d\tau = dp^\mu_{in} / d\tau - dp^\mu_{out} / d\tau, \] where $dp^\mu_{in} / d\tau$ is given by Eq. (75) and $dp^\mu_{out} / d\tau$ by Eq. (76). However, Robertson asserts that $\dot{Q}_{pr,1} = 1$ and $\dot{Q}_{ext} = 1$ (see Eqs. 2.7 and 2.8 on p. 427 in Robertson 1937, and, also the statement that "the first term $[(u^2 S A' / c) b^\mu_1]$ represents the total power-force four-vector due to the incident radiation, with which it coincides completely in direction, and the second $[(u^2 S A' / c) u^\mu / c]$ that due to the radiation re-emitted at the rate $[u^2 S A' c]$ "). The statements of Robertson are not correct, since $\dot{Q}_{pr,1} = 1$ and $\dot{Q}_{ext} = 2$, in reality (see Eq. 104). The Robertson’s approach would yield, on the basis of equations similar to Eqs. (77)-(78) (with the transformation $\dot{Q}_{pr,1} / \dot{Q}_{ext}$
that $0 < Q'_{pr,1} < 2$, which would be an inconsistency between the Mie theory and the relativity theory. These considerations of Robertson are equivalent to that of Poynting (1903, p. 539: "... it is receiving a stream of momentum $u_1^2 S A' / c$ directed from the sun. Its own radiation outwards being equal in all directions has zero resultant pressure." ... we have used symbols of our paper). In reality, the case $p'_o = 0$, considered by Poynting (1903), Robertson (1937), Wyatt and Whipple (1950, p. 558: "the process of absorption and re-emission produces no net force on a particle when one chooses to work with a stationary frame referred to the particle" ... the part "absorption and" should be omitted in the formulation) and others, does not hold and the correct form is $p'_o = 0.5 p'_i$. Thus, we have understood the real physics of the P-R effect for the first time, now! We are aware that the effect of diffraction must not be neglected, even in the case of geometrical optics approximation.

Finally, let us consider a large plane mirror of geometrical area $A'$ characterized by specular reflection (totally reflecting plane) and normal incident radiation. We have $C'_{abs} = 0$, $C'_{sca} = 2 A'$, $C'_{ext} = 2 A'$ and $\langle \cos \theta' \rangle = 0$, $C'_{pr,1} = 2 A'$. We can write

$$p'_o = (1 - C'_{pr,1} / C'_{ext}) p'_i = 0 \ldots (**)$$

and Eqs. (11) and (**) yield

$$E'_i = S' C'_{ext} = 2 S' A', \; p'_i = (E'_i / c) e'_1,$$

$$E'_o = E'_i = 2 S' A', \; p'_o = 0.$$  

Eqs. (50) and (53) yield, then:

$$dM_{in} = 0, \; dM_{out} = (2 S' A' / c^2) d\tau.$$  

On the basis of geometrical optics considerations without the effect of diffraction ($C'_{pr,1} = 2 A', \; C'_{ext} = A'$ would have to be used in Eqs. 50 and 54), one would await that $p'_o = - p'_i$ and $(dM_{out})_{naive} = (dM_{in})_{naive} = 0$ - the idea that photon's mass equals its energy divided by $c^2$ would yield $(dM_{in})_{ph} = (dM_{out})_{ph} = (S' A' / c^2) d\tau$. The result for $dM_{out}$ can be easily physically understood: the situation is similar to finding a mass of two photons, each of which travels in the same direction but orientations of their motions are different – reflected and diffracted photons are important, in our case.

### 7.4. Orbital evolution

In general, equation of motion for arbitrarily shaped particle is given by Eqs. (38) or (39), or, by Eqs. (40) or (41). Laboratory experiments carried out by Krauss and Wurm (2004) proved that nonspherical particles behave in a different way than spherical ones.

The Poynting-Robertson effect yields the following property, as for the particle's orbital evolution in gravitational and electromagnetic fields of a central star: semi-major axis and eccentricity secularly decrease (see Klačka 2004: Secs. 6.1 and 6.2 - mainly 6.2.3, for more details), if the optical properties of the particle do not change (an increase of the orbital elements can occur, if $\beta$ increases: see Eq. 12 in Klačka 1993). This is fulfilled
under assumptions presented in Sec. 7.3, i.e. $C'_{pr,2} = C'_{pr,3} = F'_e = 0$ and Eqs. (73) or
(74) hold. The equation of motion is given by Eqs. (70) or (71), then. As a consequence,
the equation of motion holds for particle with spherically symmetric mass distribution.

As for presentations in current literature, the statements are not consistent with the
discussed results.

### 7.4.1. Accelerations of the spherical particle

Relations for the accelerations of the spherical particle interacting with the incident
electromagnetic radiation can be obtained from Eqs. (75)-(76). The relevant relations for
the accelerations can be summarized in the following way.

Eq. (75) yields for the four-acceleration of the spherical particle due to the incoming
radiation:
\[
\left( \frac{d\mathbf{u}^\mu}{d\tau} \right)_{in} = X \left( \mathbf{b}^\mu_i - \frac{\mathbf{u}^\mu}{c} \right),
\]
where
\[
X \equiv \frac{w_1^2 S A' \bar{Q}'_{ext}}{m c}.
\]

or, to the first order in $v/c$
\[
\left( \frac{d\mathbf{v}}{d\tau} \right)_{in} = \frac{S A' \bar{Q}'_{ext}}{m c} \left\{ \left( 1 - \frac{v \cdot e_1}{c} \right) e_1 - \frac{v}{c} \right\}.
\]

The change of the particle’s mass is
\[
\left( \frac{d\mathbf{m}}{d\tau} \right)_{in} = \frac{w_1^2 S A' \bar{Q}'_{ext}}{c^2}.
\]
Eq. (76) enables to find the four-acceleration of the spherical particle due to the
outgoing radiation. Eq. (76) yields for the force acting on the particle $(dp^\mu/d\tau)_{out} = -
dp^\mu_{out}/d\tau$, which yields for the four-acceleration of the particle
\[
\left( \frac{d\mathbf{u}^\mu}{d\tau} \right)_{out} = - X \left( 1 - \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}} \right) \left( \mathbf{b}^\mu_i - \frac{\mathbf{u}^\mu}{c} \right),
\]
where
\[
X \equiv \frac{w_1^2 S A' \bar{Q}'_{ext}}{m c}.
\]

or, to the first order in $v/c$
\[
\left( \frac{d\mathbf{v}}{d\tau} \right)_{out} = - \frac{S A' \bar{Q}'_{ext}}{m c} \left( 1 - \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}} \right)
\times \left\{ \left( 1 - \frac{v \cdot e_1}{c} \right) e_1 - \frac{v}{c} \right\}.
\]

The change of the particle’s mass is
\[
\left( \frac{d\mathbf{m}}{d\tau} \right)_{out} = - \frac{w_1^2 S A' \bar{Q}'_{ext}}{c^2}.
\]

Total four-acceleration of the spherical particle is
\[
\left( \frac{d\mathbf{u}^\mu}{d\tau} \right)_{particle} = \left( \frac{d\mathbf{u}^\mu}{d\tau} \right)_{in} + \left( \frac{d\mathbf{u}^\mu}{d\tau} \right)_{out}
\]
\[
= \frac{\bar{Q}'_{pr,1}}{\bar{Q}'_{ext}} X \left( \mathbf{b}^\mu_i - \frac{\mathbf{u}^\mu}{c} \right),
\]
where
\[
X \equiv \frac{w_1^2 S A' \bar{Q}'_{ext}}{m c}.
\]
and the mass of the particle is conserved: \((dm/d\tau)_{\text{particle}} = (dm/d\tau)_{\text{in}} + (dm/d\tau)_{\text{out}} = 0\). To the first order in \(v/c\)

\[
\left(\frac{dv}{dt}\right)_{\text{particle}} = \frac{Q'_{\text{pr,1}}}{Q'_{\text{ext}}} \frac{S}{mc} \left\{ \left(1 - \frac{v \cdot e_1}{c}\right) e_1 + \frac{\vec{v}}{c} \right\}.
\]

(113)

If \(Q'_{\text{pr,1}}/Q'_{\text{ext}} = 1\), then \((dv/dt)_{\text{out}} = 0\) (see Eq. 110). Further, \(0 < Q'_{\text{pr,1}}/Q'_{\text{ext}} < 2\), according to Eq. (79). If \(Q'_{\text{pr,1}}/Q'_{\text{ext}} < 1\), then the re-radiated light yields \((dv/dt)_{\text{out}} \propto +v\). If \(Q'_{\text{pr,1}}/Q'_{\text{ext}} > 1\), then the re-radiated light yields \((dv/dt)_{\text{out}} \propto -v\). The case \((dv/dt)_{\text{out}} \propto -v\) leads to a decrease of particle’s total energy (kinetic plus potential) and to an increase of the orbital speed \(|v|\), if the optical properties of the particle do not significantly change during the particle’s motion around a star.

7.4.2. Karttunen et al. (2007)

Karttunen et al. (2007, p. 201) state: “When a small body absorbs and emits radiation, it loses its orbital angular momentum and the body spirals to the Sun.”

The statement holds only for spherical particle orbiting a star (Sun), under the assumption that the particle is characterized with spherically symmetric mass distribution and constant optical properties. Moreover, even when spherical particle does not absorb radiation, it may (secularly) lose orbital angular momentum (see also Sec. 7.3.3 as a very special case, or any elastic scattering). If optical parameters of the particle change, then the particle may not lose its (secular) orbital angular momentum. If the particle is nonspherical, there may not exist systematic secular decrease of particle’s semi-major axis and eccentricity, and, moreover, particle’s orbital plane may change.

7.4.3. Carroll and Ostlie (2007)

Carroll and Ostlie (2007, p. 806) describe the Poynting-Robertson effect in this way:

"When particles absorb sunlight, they must re-radiate that energy again if they are to remain in thermal equilibrium. The original light was emitted from the Sun isotropically, but in the Sun’s rest frame the re-radiated light is concentrated in the direction of motion of the particle. Since the re-radiated light carries away momentum as well as energy, the particle slows down and its orbit decay."

The statement does not hold for arbitrarily shaped particles. It holds, partially, only for dust grains with spherically symmetric mass distribution and constant optical properties. If such type of spherical particles is considered, then Eq. (74) holds for the outgoing radiation. Eq. (74) shows that “re-radiation” (but also scattering, in general) occurs both in directions \(dp_{\text{in}}/dt \propto e_1\) and \(v\). We can write, on the basis of Eqs. (77) and (78) (to the first order in \(v/c\)):

\[
dp_{\text{out}}/dt = (S A' Q'_{\text{ext}} / c) \times \left[ \left(1 - Q'\right) \left(1 - v \cdot e_1 / c\right) e_1 + Q' v / c\right] = (S A' Q'_{\text{ext}} / c) \times \left[ \left(1 - v \cdot e_1 / c\right) e_1 - Q' e_1 + Q' \left( v \cdot e_1 / c\right) e_1 + v / c\right].
\]

Only the direction \(v\) is considered in the explanation...
of Carroll and Ostlie ("... in the Sun’s rest frame the re-radiated light is concentrated in the direction of motion of the particle.").

The statement that "the re-radiated light carries away momentum as well as energy" is correct. Mathematical description of this physical process is given by Eq. (76). But the complete sentence "Since the re-radiated light carries away momentum as well as energy, the particle slows down and its orbit decay." is physically incorrect, in general. Really, Eq. (76) yields for the force acting on the particle \( (dp'/d\tau)_{\text{out}} = - dp'_{\text{out}}/d\tau \), which yields for the four-acceleration of the particle results given by Eqs. (109)-(110). If \( Q_{\text{pr}}/Q_{\text{ext}} = 1 \), then \( (dv/dt)_{\text{out}} = 0 \) (see Eq. 110) and the particle does not slow down. If \( Q_{\text{pr}}/Q_{\text{ext}} < 1 \), then the re-radiated light yields \( (dv/dt)_{\text{out}} \propto v \). If \( Q_{\text{pr}}/Q_{\text{ext}} > 1 \), then the re-radiated light yields \( (dv/dt)_{\text{out}} \propto -v \). The case of "perfectly absorbing sphere", treated by Robertson (1937), corresponds to \( Q_{\text{pr}}/Q_{\text{ext}} = 1/2 \) within geometrical optics approximation, and, thus, the re-radiated light yields \( (dv/dt)_{\text{out}} \propto v \), according to Eq. (110).

### 7.4.4. Lissauer and Murray (2007)

Lissauer and Murray (2007, p. 806) explain:

"A small particle in orbit around the Sun absorbs solar radiation and reradiates the energy isotropically in its own frame. The particle thereby preferentially radiates (and loses momentum) in the forward direction in the inertial frame of the Sun. This leads to a decrease in the particle’s energy and angular momentum and causes dust in bound orbits to spiral sunward. This effect is called the Poynting-Robertson drag.

The net force on a rapidly rotating dust grains is given by

\[
F_{\text{rad}} \approx \left[ \frac{L Q_{\text{pr}} A}{(4 \pi c r^2)} \right] \left[ (1 - 2 v_r/c) \hat{r} - \left( \frac{v_\Theta}{c} \right) \hat{\Theta} \right].
\]

(50-LM)

The first term in Eq. (50-LM) is that due to radiation pressure and the second and the third terms (those involving the velocity of the particle) represent the Poynting-Robertson drag.

From this discussion, it is clear that small-sized dust grains in the interplanetary medium are removed: (sub)-micronsized grains are blown out of the solar system, whereas larger particles spiral inward toward the Sun."

(Explanation to symbols in Eq. 50-LM are given also on p. 806 of the paper: "\( A \) is the particle’s geometric cross section, \( L \) is the solar luminosity, \( c \) is the speed of light, \( r \) is the heliocentric distance, and \( Q_{\text{pr}} \) is the radiation pressure coefficient, which is equal to unity for a perfectly absorbing particle and is of order unity unless the particle is small compared to the wavelength of the radiation.")

Now, we will treat the real physics.

i) "A small particle ... ." (Lissauer and Murray 2007, p. 806)
The equation of motion corresponding to the Poynting-Robertson effect holds for (nonrotating) spherical particles.

ii) "A small particle ... reradiates the energy isotropically in its own frame." (Lissauer and Murray 2007, p. 806)

General assumption/condition for the validity of the Poynting-Robertson effect is formulated by Eq. (73). This equation shows that \( p'_o = 0 \) if and only if \( \bar{Q}'_{pr,1}/\bar{Q}'_{ext} = 1 \).

Thus, the statement of the authors, presented above, corresponds only to a special combination of optical properties of the spherical particle and the wavelength of the incident electromagnetic radiation. The authors probably assume perfectly absorbing spherical particles within geometrical optics approximation (moreover, diffraction is ignored by the authors), but this case is characterized by the condition \( \bar{Q}'_{pr,1}/\bar{Q}'_{ext} = 1/2 \), as it is given by Eq. (104), and \( p'_o = (1/2) p'_i \).

iii) "The particle therefore preferentially radiates in the forward direction in the inertial frame of the Sun." (Lissauer and Murray 2007, p. 806)

According to Eq. (76), we can write for the outgoing radiation for the P-R effect (first order in \( v/c \) is considered):

\[
dp_{out}/dt = ( S A'/Q'_{ext}/c ) \left\{ (1 - Q'_{pr,1}/Q'_{ext})(1 - v \cdot e_1/c) e_1 + (Q'_{pr,1}/Q'_{ext}) v/c \right\}.
\]

Since \( Q'_{pr,1}/Q'_{ext} \neq 1 \) in general, also radial term is present in \( dp_{out}/dt \), not only "forward direction", in the inertial frame of the Sun.

As for the perfectly absorbing spherical particle: the standard approach \( Q'_{pr,1} = Q'_{ext} = 1 \) would yield \( dp_{out}/dt = ( S A'/c ) v/c = ( S A'/c ) \{ (v_R/c) e_R + (v_T/c) e_T \} \), while the correct result is \( Q'_{pr,1} = 1, Q'_{ext} = 2 \), and, \( dp_{out}/dt = ( S A'/c ) \{ e_R + (v_T/c) e_T \} \); the velocity vector \( v \) is decomposed into radial component \( v_R e_R \) and transversal component \( v_T e_T \), \( e_R \equiv e_1 \).

iv) "The particle thereby preferentially radiates in the forward direction in the inertial frame of the Sun. This causes dust in bound orbits to spiral sunward." (Lissauer and Murray 2007, p. 806)

As it is evident from Eq. (110), the special case \( Q'_{pr,1}/Q'_{ext} = 1 \) yields \( (dv/dt)_{out} = 0 \) and the reradiated energy does not cause dust to spiral sunward. Dealing with more realistic case \( Q'_{pr,1}/Q'_{ext} \neq 1 \), one obtains:

a) if \( Q'_{pr,1}/Q'_{ext} < 1 \), then the re-radiated light yields \( (dv/dt)_{out} \propto + v \),

b) if \( Q'_{pr,1}/Q'_{ext} > 1 \), then the re-radiated light yields \( (dv/dt)_{out} \propto - v \).

See also Sec. 7.4.1. Moreover, if optical properties of the dust change, then the P-R effect can cause spiralling outward from the Sun.
v) 
"The net force on a rapidly rotating dust grains is given by ... (Eq. 50-LM)" (Lissauer and Murray 2007, p. 806)

In reality, Eq. (50-LM) holds for nonrotating spherical particle, as it is evident from derivations presented in our paper (see also Sec. 7.3.3). If we would like to consider rapidly rotating arbitrarily shaped dust particle, we have to consider general equation of motion (see Eq. 39) and the corresponding averaging over rotational motion has to be done (see, e. g., Klačka and Kocifaj 2001). Krauss and Wurm (2004) have given an experimental evidence that arbitrarily shaped and rapidly rotating dust grain moves in a different way than it corresponds to the Poynting-Robertson effect.

vi) 
Equation (50-LM) contains general factor $Q_{pr}$ ($\equiv \bar{Q}_{pr,1}'$). However, this is in contradiction with the statement "A small particle in orbit around the Sun absorbs solar radiation and reradiates the energy isotropically in its own frame." (Lissauer and Murray 2007, p. 806). The isotropically reradiated energy within geometrical optics approximation (diffraction is not considered in this formulation) corresponds to the condition $\bar{Q}_{pr,1}'/\bar{Q}_{ext}' = 1/2$ (see Eq. 104 in Sec. 7.3.3), in reality. Thus, if the authors state that the essence of the Poynting-Robertson effect is "particle reradiates the energy isotropically in its own frame", then they have to use $Q_{pr} = 1$.

It is also important to stress that the process "particle reradiates the energy isotropically in its own frame" (Lissauer and Murray 2007, p. 806) is only an approximation within geometrical optics and that, in reality, the fundamental condition for the case is given by the equation $p_{o}' = (1/2) p_{i}'$ and not $p_{o}' = 0$. The last equation considers the effect of diffraction (calculation of mass of the outgoing radiation within relativity approach requires consideration of the effect of diffraction), while the phrase "particle reradiates the energy isotropically in its own frame" does not consider the effect of diffraction.

vii) 
"The first term in Eq. (50-LM) is that due to radiation pressure and the second and the third terms (those involving the velocity of the particle) represent the Poynting-Robertson drag." (Lissauer and Murray 2007, p. 806)

The effect of electromagnetic radiation on spherical dust particle is given by Eq. (70). It corresponds to radiation pressure. Eq. (70) contains all the terms present in Eq. (50-LM). Thus, Eq. (50-LM) corresponds to the radiation pressure – not only its first term, as the authors state. Physics does not allow to divide Eq. (70) into several parts which can be treated as separate physical phenomena.

viii) "From this discussion, it is clear that small-sized dust grains in the interplanetary medium are removed: (sub-)micron-sized grains are blown out of the solar system, whereas larger particles spiral inward toward the Sun." (Lissauer and Murray 2007, p. 806)
The conclusion made by the authors does not follow from the discussion presented by the authors. Even if we take into account all the corrections presented above and that the correct and fundamental result is given by Eq. (50-LM). The authors’ conclusion may be violated if one takes into account that optical properties of the particles may change.

7.4.5. Aberration (of light)

It is often stated that the aberration of the incoming light is responsible for the term $-v/c$ in the Poynting-Robertson effect (see, e. g., Dohnanyi 1978, p. 562; Leinert and Grün 1990, p. 226; Grün 2007, p. 632). Is it true? If the incoming radiation would be responsible for the term $-v/c$ in the P-R effect (“radiation falls preferentially on the leading edge of the orbiting particle and acts as a drag force” – Festou et al. 2004, p. 729), then this term should be present in $p_i = dp_{in}/d\tau$. However, Eqs. (42) and (75) show that $dp_{in}/d\tau$ is proportional to $b_1 = e_1/w_1$. Thus, the term $-v/c$ does not come from the incoming radiation. In reality, the term $-v/c$ is present in $dp_{out}/d\tau$ (Eq. 76). But it’s presence is not due to the aberration of light. It’s existence is caused by conservation of particle’s mass, when considering both the incoming and the outgoing radiation. Really, if we would require, e. g., $dE'/d\tau = (w_1^2 S \bar{Q}_{pr} A'/c) \alpha'$ and $dp'/d\tau = (w_1^2 S \bar{Q}_{pr} A'/c) e_1'$, then Eq. (14) would yield $dp/d\tau = (w_1^2 S \bar{Q}_{pr} A'/c) [ b_1 + (\alpha' - 1) u/c ]$, where $b_1 = e_1/w_1$, $u = \gamma v$, $w_1 = \gamma (1 - v \cdot e_1/c)$, and, $\alpha'$ is a constant. The term $(\alpha' - 1) \gamma v/c$ does not correspond to the aberration of light. As a similar example, we can mention Eq. (40).

7.5. Analogy between mechanics and electromagnetism?

We have discussed the effect of electromagnetic radiation (photons) on perfectly absorbing and totally reflecting spherical and planar particles. We have stressed that even within the geometrical optics approximation the effect of diffraction is important, in the process of interaction between the particle and the electromagnetic radiation. As a consequence, the relation between the outgoing and incoming momenta per unit time (Eq. 73) reduces to $p'_{o} = 0.5 p'_{i}$ for the above mentioned spherical particles (Eq. 104). The numerical factor 0.5 considers also the effect of diffraction which is conventionally neglected (e. g., Lissauer and Murray 2007).

Thus, we know that "for large spheres we may separate the effects of geometrical optics (reflection and refraction) and the effect of diffraction" (van de Hulst 1981, p. 225). But even in the case of geometrical optics approach, the simultaneous action of these effects cannot be neglected, if one wants to be consistent with the relativity theory. If we take into account a plane particle (arbitrarily shaped particle, in general), all these effects are also important.
The effect of reflection of the electromagnetic radiation on a planar object is often explained in an analogy to mechanical process. As an example, we can mention two well-known textbooks on fundamental physics. Halliday et al. (2008, p. 900) write about the comparison between the radiation effect on perfectly absorbing and totally reflecting object: "Instead of being absorbed, the radiation can be reflected by the object: that is, the radiation can be sent off in a new direction as if it bounced off the object. If the radiation is entirely reflected back along its original path, the magnitude of the momentum change of the object is twice that given above ... (the case for total absorption)." In the same way, an object undergoes twice as much momentum change when a perfectly elastic tennis ball is bounced from it as when it is struck by a perfectly inelastic ball (a lump of wet putty, say) of the same mass and velocity." Similarly, Jewett and Serway (2008, p. 963) write: "Electromagnetic waves transport linear momentum as well as energy. ... In this discussion, let’s assume the electromagnetic wave strikes the surface at normal incidence ..." "Momentum transported to the perfectly absorbing surface has a magnitude \( p \)." The authors continue (Jewett and Serway 2008, p. 964): "If the surface is a perfect reflector (such as a mirror) and incidence is normal, the momentum transported to the surface ... is twice that given by (the magnitude \( p \)). That is, the momentum transferred to the surface by the incoming light is \( p \) and that transferred to the surface by the reflected light also is \( p \). In other words, the momentum transferred to the surface by the incoming light equals the momentum transferred by the reflected light. If the mechanical process would be a good analogue to the electromagnetic process, then the relation \( p'_{o} = (1 - \frac{C'_{pr,1}}{C'_{ext}}) p'_{i} \) would reduce to \( p'_{o} = - p'_{i} \), for the perfect (both mechanical and electromagnetic) reflection. But the reality is different. The mechanical process is characterized by the values \( C'_{pr,1} = 2 A' \), \( C'_{ext} = A' \), while the electromagnetic process, including the effect of diffraction, yields cross sections \( C'_{pr,1} = 2 A' \), \( C'_{ext} = 2 A' \), where \( A' \) is geometrical cross section of the object.

Summarization of the incident and outgoing momenta per unit time can be done in the following way (\( A' \) is geometrical cross section of the planar surface, \( u' \) and \( c \) are the speeds of the incident mechanical particle and photon with respect to the surface, and, \( S' \) is the rate of flow of energy, i.e., the rate at which the energy flows through a unit surface area perpendicular to the direction of the particle or photon propagation), for normal incident energy:

i) perfectly absorbing planar surface:
- mechanics: \( p'_i = (\frac{S'}{A'} c) (\frac{u'}{c}) e'_1 \), \( p'_o = 0 \)
- electromagnetism: \( p'_i = \frac{2}{S'} (\frac{A'}{c}) e'_1 \), \( p'_o = - (\frac{S'}{A'} c) (\frac{u'}{c}) e'_1 \)

\( [1 \times (\frac{S'}{A'} c) e'_1 \) comes from the effect of diffraction, simultaneously in \( p'_i \) and \( p'_o \)]

ii) perfectly reflecting planar surface:
- mechanics: \( p'_i = (\frac{S'}{A'} c) (\frac{u'}{c}) e'_1 \), \( p'_o = - (\frac{S'}{A'} c) (\frac{u'}{c}) e'_1 \)
- electromagnetism: \( p'_i = \frac{2}{S'} (\frac{A'}{c}) e'_1 \), \( p'_o = 0 \)
[ as for the $p'_0$: the effect of diffraction ( $S' A'/c$ ) $e'_1$ is compensated by the effect of reflection − ( $S' A'/c$ ) $e'_1$ ].

Finally, we will present a short explanation to the results of mechanics. Four-momentum of the incident (classical) mechanical particle is $p'_\mu = ( E'_M P'_\mu, p'_M )$. Four-momentum per unit time is $p'_\mu = ( E'_i, p'_i ) = n' u' A'/c$ $p'_i = ( n' u' A'/ E'_M P'_i/c ) \times ( 1, u'/c )$, since $p'_M = E'_M P' u'/ c^2, u' = u' e'_1$. Defining $S' = n' u' E'_M P'$, we obtain $p'_\mu = ( S' A' / c ) \times ( 1, u' e'_1 / c )$.

8. On the equivalence principle of mass and energy

One of the well-known results of the special relativity theory is the equivalence of mass and energy (more properly: the relation between mass and energy). This "universal equivalence principle of mass and energy" is often formulated in the following way (or a little different, but equivalent form): "there does not exist a mass without energy (and vice versa) and every change in energy is connected with a corresponding change of inertial mass" (Schröder 1990, p. 114). It is stated that it holds also for a photon (light) and an inertial mass of the photon equals $m_{\text{photon}} = E/c^2$, where $E$ is energy of the photon. A simple "proof" can be found in Kittel et al. (1962; Sec. 12.6, Eqs. 58-59, p. 400): $p = m_{\text{photon}} c$ and $E/c = p$, which yields $E = m_{\text{photon}} c^2$, $p$ is momentum of the photon. (Serway et al. 2005, p. 95: "The photon has zero mass, but its effective inertial mass $m_i$ may reasonably be taken to be the mass equivalent of the photon energy $E$, or $m_i = E/c^2 = h f/c^2$. The same result is obtained if we divide the photon momentum by the photon speed $c$: $m_i = p/c = h f/c^2."$ Analogously, Gasiorowicz 1974, p. 466 states: "A photon of energy $E$ has gravitational mass $E/c^2$."") The principle is usually proved by the thought experiment, coming back to Einstein (see, e.g., Einstein 1906, p. 633; Beiser 1969, Sec. 2.6; Bernstein et al. 2000, pp. 81-82; Chow 2008, pp. 272-273; compare also Okun 1989b, p. 34). The idea is following: "Classical electromagnetism tells us that light carries both momentum and energy. Because of this momentum, light exerts a pressure. We can think of the radiation as being composed of massless particles – any object that moves with the speed of light must have no mass – carrying momentum. .... A box on a frictionless surface has a radiation emitter on one end and an absorbing surface on the other. When a burst of radiation is emitted, the box recoils, stopping only when the radiation is absorbed at the other end. The center of mass will not move if there is mass-energy equivalence." (Bernstein et al. 2000, p. 81). As a result, the relation for a beam of parallelly spreading photons is presented: $E = m c^2$, where $E$ is energy of the photons (electromagnetic radiation), $c$ is the speed of light and $m$ is the mass corresponding to the energy $E$ (the result of the thought experiment is: "The energy of the radiation is equivalent to a mass according to the Einstein formula: $E = m c^2."$ Bernstein et al. 2000, p. 82).
However, the following question arises: What is the physical sense of the mass for a beam of massless photons? According to Schröder (1990, p. 106): "The inertia of the particle is described by an invariant quantity, the mass \( m \) of the particle, more properly called its rest mass". On the other hand, the author writes: "According to the universal equivalence principle of mass and energy, there does not exist a mass without energy (or vice versa) and every change in energy is connected with a corresponding change of inertial mass." (Schröder 1990, p. 114). Thus, an inconsistency emerges: the inertia of a beam of massless particles, photons, is described by an invariant quantity, the mass of the beam \( m_{\text{beam}} = 0 \), and, the beam possesses and energy \( E_{\text{beam}} \) which should be equivalent to a mass according to the Einstein formula \( m_{\text{beam}} = E_{\text{beam}} / c^2 \).

8.1. Thought experiment

In trying to better understand the situation, we will present detailed calculations for a thought experiment. The thought experiment will be similar to that discussed above, but instead of a box of mass \( M \) we will consider two surfaces of masses \( M/2 \) on a frictionless surface. The surfaces are perpendicular to the frictionless surface. They are at rest in an inertial frame of reference \( K \), at the beginning, with initial positions \( x_L = -L/2 \) and \( x_R = +L/2 \). The right surface emits a particle with energy \( E \) (measured in the reference frame \( K \)) towards the left surface, i.e., direction and orientation of the motion of the particle is characterized by the unit vector \( -\hat{x} \). Let the emission of the energy \( E \) corresponds to a mass decrease \( m_R \) of the right surface. The absorption of the energy \( E \) produces a mass increase \( m_L \) of the left surface.

8.1.1. Emission of a massless particle – particle with zero mass

After emission of a massless particle (photon) by the right surface and its absorption by the left surface, the center of inertia of the system does not change (see, e.g., Eq. 14.6 in Landau and Lifshitz 2005):

\[
E_L r_L + E_R r_R = 0.
\]  

(114)

We can rewrite Eq. (114) in detail:

\[
\gamma_L \left( \frac{M}{2} + m_L \right) \left( \frac{L}{2} + v_L t \right) = \gamma_R \left( \frac{M}{2} - m_R \right) \left( \frac{L}{2} + v_R \frac{L}{c} + v_R t \right),
\]

(115)

where \( t = 0 \) holds for the moment of the photon absorption by the left surface, \( \gamma_L = 1/\sqrt{1 - (v_L/c)^2} \), \( \gamma_R = 1/\sqrt{1 - (v_R/c)^2} \); the multiplicative factors \( c^2 \) present in energies \( E_L = \gamma_L (M/2 + m_L) c^2 \) and \( E_R = \gamma_R (M/2 - m_R) c^2 \) are omitted. Eq. (109) yields

\[
\gamma_L \left( \frac{M}{2} + m_L \right) = \gamma_R \left( \frac{M}{2} - m_R \right) \left( 1 + 2 \frac{v_R}{c} \right)
\]

(116)

for \( t = 0 \). Since the center of inertia does not change also for \( t > 0 \), Eq. (115) yields

\[
\gamma_L \left( \frac{M}{2} + m_L \right) v_L = \gamma_R \left( \frac{M}{2} - m_R \right) v_R.
\]

(117)
Eq. (117) corresponds to the laws of conservation of momentum for the moments of emission and absorption:

\[ \gamma_R \left( \frac{M}{2} - m_R \right) v_R = \frac{E}{c}, \]  

(118)

\[ \gamma_L \left( \frac{M}{2} + m_L \right) v_L = \frac{E}{c}. \]  

(119)

The laws of conservation of energy for the emission and the absorption are of the following forms in the inertial frame of reference \( K \):

\[ \gamma_R \left( \frac{M}{2} - m_R \right) c^2 + E = \frac{M}{2} c^2, \]  

(120)

\[ \gamma_L \left( \frac{M}{2} + m_L \right) c^2 = \frac{M}{2} c^2 + E. \]  

(121)

Eqs. (118) and (120) yield

\[ m_R = \frac{M}{2} \left\{ 1 - \sqrt{1 - \frac{2E/c^2}{M/2}} \right\}, \]  

(122)

or,

\[ E = \left( 1 - \frac{m_R}{M} \right) m_R c^2. \]  

(123)

During the process of the emission of the photon(s), the mass \( m_R \) (more correctly: the energy \( m_R c^2 \)) has been changed into the energy \( E \) and kinetic energy of the surface of mass \( M/2 - m_R \) [kinetic energy = \( (\gamma_R - 1) (M/2 - m_R) c^2 \); \( m_R \approx \left[ 1 + (E/c^2) / M \right] E/c^2 \). This is in agreement with Eq. (120) and this is the physical interpretation of the Eqs. (118), (120) and (122)-(123). Mass of the photon(s) with the energy \( E \) is zero.

Eqs. (119) and (121) yield

\[ m_L = \frac{M}{2} \left\{ \sqrt{1 + \frac{2E/c^2}{M/2}} - 1 \right\}, \]  

(124)

or,

\[ E = \left( 1 + \frac{m_L}{M} \right) m_L c^2. \]  

(125)

During the process of the absorption of the photon(s), the energy \( E \) has been changed into the mass \( m_L \) (more correctly: rest energy \( m_L c^2 \)) and motion of the surface of mass \( M/2 + m_L \) [kinetic energy = \( (\gamma_L - 1) (M/2 + m_L) c^2 \); \( m_L \approx \left[ 1 - (E/c^2) / M \right] E/c^2 \). This is in agreement with Eq. (121) and this is the physical interpretation of Eqs. (119), (121) and (124)-(125).

8.1.2. Summary

We can make the following statements:

i) Photon(s) (light) with the energy \( E \) transferred mass \( m_L \).

i) Mass of the photon(s) (light) is zero.
Proof of the statements:
i) Initial mass of the left surface was $M/2$, it’s mass after absorbing the photon(s) is
\[ \sqrt{\left(\frac{\text{energy}}{c^2}\right)^2 - \left(\frac{\text{momentum}}{c}\right)^2} = \sqrt{\gamma_L(M/2 + m_L)^2 - \gamma_L(M/2 + m_L)v_L/c^2} = M/2 + m_L. \]
The mass $m_L$ is given by Eq. (117), or, approximately, $m_L \approx |1 - (E/c^2)/M|E/c^2$.

ii) Mass of the photon(s) equals
\[ \sqrt{\left(\frac{\text{energy}}{c^2}\right)^2 - \left(\frac{\text{momentum}}{c}\right)^2} = \sqrt{\left(\frac{E}{c^2}\right)^2 - \left(\frac{p}{c}\right)^2} = \sqrt{\left(\frac{E}{c^2}\right)^2 - \left(\frac{E}{c^2}\right)^2} = 0, \]
where also the fact that the photon(s) moves with the speed $c$ in the direction and orientation $-\hat{x}$, was used.

The statements are consistent with the statements presented in Taylor and Wheeler (1992, p. 228): “Energy without mass: photon”, ”photon moves with zero mass”.

8.1.3. Emission of a particle with non-zero mass

After emission of a particle with non-zero mass by the right surface and its absorption by the left surface, the center of inertia of the system does not change, see Eq. (114). We can rewrite Eq. (114) in detail:
\[ \gamma_L \left( \frac{M}{2} + m_L \right) \left( \frac{L}{2} + v_L t \right) = \gamma_R \left( \frac{M}{2} - m_R \right) \left( \frac{L}{2} + v_R \frac{L}{u} + v_R t \right), \]
where $t = 0$ holds for the moment of the absorption of the particle by the left surface, $\gamma_L = 1/\sqrt{1 - (v_L/c)^2}$, $\gamma_R = 1/\sqrt{1 - (v_R/c)^2}$, $u$ is the speed of the particle in the frame of reference $K$; the multiplicative factors $c^2$ present in energies $E_L = \gamma_L(M/2 + m_L)c^2$ and $E_R = \gamma_R(M/2 - m_R)c^2$ are omitted. Eq. (126) yields
\[ \gamma_L \left( \frac{M}{2} + m_L \right) = \gamma_R \left( \frac{M}{2} - m_R \right) \left( 1 + 2 \frac{v_R}{u} \right) \]
for $t = 0$. Since the center of inertia does not change also for $t > 0$, Eq. (126) yields
\[ \gamma_L \left( \frac{M}{2} + m_L \right) v_L = \gamma_R \left( \frac{M}{2} - m_R \right) v_R . \]

Eq. (128) corresponds to the laws of conservation of momentum for the moments of emission and absorption:
\[ \gamma_R \left( \frac{M}{2} - m_R \right) v_R = \gamma_u m u, \]
\[ \gamma_L \left( \frac{M}{2} + m_L \right) v_L = \gamma_u m u, \]
where $m$ is mass of the particle and $\gamma_u = 1/\sqrt{1 - (u/c)^2}$.

The laws of conservation of energy for the emission and the absorption are of the following forms in the inertial frame of reference $K$:
\[ \gamma_R \left( \frac{M}{2} - m_R \right) c^2 + \gamma_u m c^2 = \frac{M}{2} c^2, \]
\[ \gamma_L \left( \frac{M}{2} + m_L \right) c^2 = \frac{M}{2} c^2 + \gamma_u m c^2. \]
Eqs. (129) and (131) yield

\[ m_R = \frac{M}{2} \left\{ 1 - \sqrt{1 - \frac{2E}{Mc^2} + \left( \frac{m}{M/2} \right)^2} \right\}, \]

\[ E = \gamma_u m c^2, \] (133)

or,

\[ E = \left(1 - \frac{m_R}{M}\right) m_R c^2 + \frac{m}{M} m c^2 \]
\[ = m_R c^2 - \frac{m^2}{M} c^2, \quad m < m_R. \] (134)

During the process of the emission of the massive particle, the energy \( m_R c^2 \) has been changed into the energy \( E \) and kinetic energy of the surface of mass \( M/2 - m_R \) [kinetic energy = \((\gamma_R - 1) \ (M/2 - m_R) \ c^2\)]. This is in agreement with Eq. (131) and this is the physical interpretation of Eqs. (129), (131) and (133)-(134). Mass of the massive particle with the energy \( E \) is \( m \).

Eqs. (130) and (132) yield

\[ m_L = \frac{M}{2} \left\{ \sqrt{1 + \frac{2E}{Mc^2} + \left( \frac{m}{M/2} \right)^2} - 1 \right\}, \]

\[ E = \gamma_u m c^2, \] (135)

or,

\[ E = \left(1 + \frac{m_L}{M}\right) m_L c^2 - \frac{m}{M} m c^2 \]
\[ = m_L c^2 + \frac{m_L^2 - m^2}{M} c^2, \quad m < m_L. \] (136)

During the process of the absorption of the massive particle, the energy \( E \) has been changed into the mass \( m_L \) (more correctly: rest energy \( m_L c^2 \)) and motion of the surface of mass \( M/2 + m_L \) [kinetic energy = \((\gamma_L - 1) \ (M/2 + m_L) \ c^2\)]. This is in agreement with Eq. (130) and this is the physical interpretation of the Eqs. (130), (132) and (135)-(136).

### 8.2. Summary

The important statements, as for massive particles, can be summarized as follows: "The theory of relativity leads to the important conclusion that the energy of a particle (with \( m \neq 0 \)) at rest is \( mc^2 \). The relations \( E_0 = mc^2, \ E = mc^2/\sqrt{1-(u/c)^2} \) for a particle at rest and a particle in motion, respectively, are the famous Einstein formulae. They express the equivalence of mass and energy." (Schröder 1990, p. 113). As for the massless particle, it moves with the speed \( c \) and, thus, it cannot be at rest with respect to any frame of reference: rest energy of the particle is \( E_0 = 0 \). The inertia of the particle is described by an invariant quantity, the mass of the particle. The mass of a photon is zero.
The term "inertial mass of a photon" is of no physical sense: the speed of the photon is constant and the photon cannot be accelerated or decelerated, in a vacuum. Similarly, the term "gravitational mass of the photon" is of no physical sense – no (thought) experiment exists which would prove the correctness of the gravitational character of the photon. (See also Ugarov 1969, p. 248).

Finally, we will reproduce several sentences on the meaning of \( E_0 = m c^2 \) from Sachs (2007, p. 84): "It has been said by many physicists and philosophers that the formula \( E_0 = m c^2 \) [we have added the subscript 0] means that ‘mass is equivalent to energy’. This is philosophically false. It is not what Einstein said when he derived this relation. What he said was that ‘the inertial mass of matter is a measure of its energy content’. In physics, as Newton originally postulated, the inertial mass of matter is, by definition, a measure of its resistance to a change of its state of rest or constant motion. The energy of matter, on the other hand, is by definition, the capability of this matter to do work. Thus, mass and energy are totally different concepts! What should be said, instead of saying that mass is equivalent to energy, is that mass (the inertia of matter) is a measure of the capability of this matter to do work (its intrinsic energy)."

Results obtained in Secs. 4-7 are consistent with the previous statements (see also Eq. 105 with comment).

### 8.3. Comparison of the results

We want to compare the results obtained in Sec. 8.1 with the results of Secs. 2-7.

#### 8.3.1. Relativity and optics from Secs. 2-7

Let us consider an incoming electromagnetic radiation which interacts with arbitrarily shaped particle. The incoming radiation is characterized by its energy and momentum. The relevant relations are given by Eqs. (42), (50), (60)-(63), or, by Eqs. (85)-(86) for spherical particles. In order to be the results easily comparable to the considerations presented in Sec. 8.1.1, we will discuss a totally absorbing planar surface of geometrical cross section \( A' \) and normal incident radiation, within geometrical optics approximation. On the basis of Secs. 7.3.3 (Eqs. 104-105 and the text above and below them) and 7.5, we can immediately write

\[
C_{\text{ext}}' = 2 \ A', \\
C_{\text{sca}}' = A', \\
E_i' = S' \ C_{\text{ext}}' = 2 \ S' \ A', \\
p_i' = \frac{E_i'}{c} \ e_i', \\
dM_{\text{in}} = 0 ,
\]

(137)
where \( S' \) is the flux of radiation energy and four-momentum per unit time is \( p'_{\mu} = ( E'_i/c, p'_i) \); \( C'_{\text{ext}} \) and \( C'_{\text{sca}} \) are cross sections for extinction and scattering.

If the absorption and thermal emission are present, then the outgoing radiation is characterized by the following relations

\[
E'_o = E'_i \\
C'_{\text{pr}, \,1} = A' \\
p'_o = \left(1 - \frac{C'_{\text{pr}, \,1}}{C'_{\text{ext}}} \right) p'_i = \frac{1}{2} p'_i ,
\]

\[
dM_{\text{out}} = \sqrt{3} S' A' c^2 d\tau ,
\]

(138)

if the results from Eq. (137) are used.

Now, let the thermal emission does not exist (thought experiment). Then

\[
E'_o = 0 \\
p'_o = 0 \\
C'_{\text{pr}, \,1} = 2 A' \\
dM_{\text{out}} = 0 ,
\]

(139)

since \( p'_{\mu} p'_o \mu \geq 0 \), \( p'_o = ( E'_o/c, p'_o) \), \( p'_o = ( 1 - C'_{\text{pr}, \,1} / C'_{\text{ext}}) p'_i \). Eqs. (50) and (61)-(63)

yield

\[
\frac{dM_{\text{in}}}{d\tau} = 0 , \\
( \frac{dm}{d\tau} )_{\text{in}} = \frac{w_1^2 S C'_{\text{ext}}}{c^2} , \\
\frac{dE_{\text{in}}}{d\tau} = \left( \frac{dm}{d\tau} \right)_{\text{in}} c^2 \frac{1}{w_1} , \\
\frac{dp_{\text{in}}}{d\tau} = \left( \frac{dm}{d\tau} \right)_{\text{in}} c \frac{e_1}{w_1} ,
\]

(140)

where also some results of Eq. (137) have been added.

8.3.2. Relativity from Sec. 8.1.1

Sec. 8.1.1 presents result for an increase of mass of a planar absorbing surface if it’s mass is \( M/2 \) and the increase of it’s mass is generated by an absorption of a photon of energy \( E \) with respect to the surface. In order to be able to compare the results of Sec. 8.1.1 with the results presented in Sec. 8.3.1, we have to generalize the considerations of Sec. 8.1.1 for the case of a moving absorbing surface.

Let us consider a photon of energy \( E \) in the laboratory frame (inertial frame of reference \( K \)) in which the absorbing surface is moving with the speed \( v \), in the same direction and orientation as the photon is travelling (also negative orientation may be considered:
it is sufficient to put $-v$ into the formulae presented below, in this subsection). We are interested in the increase of mass of the absorbing surface. The law of conservation of momentum for the absorption is of the following form in the inertial frame of reference $K$ (compare with Eq. 119):

$$
\gamma L \left( M/2 + m_L \right) v_L = E/c + \gamma v \frac{M}{2} v .
$$

(141)

Similarly, the law of conservation of energy yields (compare with Eq. 121):

$$
\gamma L \left( M/2 + m_L \right) c^2 = E + \gamma v \frac{M}{2} c^2 .
$$

(142)

Eqs. (141)-(142) yield

$$
m_L = \frac{M}{2} \left\{ \sqrt{1 + 2 w_1 \frac{E/c^2}{M/2} - 1} \right\} ,
$$

$$
w_1 = \gamma_v \left( 1 - \frac{v}{c} \right) .
$$

(143)

During the process of the absorption of the photon(s), the energy $E$ has been changed into the mass $m_L$ (more correctly: rest energy $m_L c^2$) and motion of it, and, part of the energy $E$ was changed into the change of the kinetic energy of the surface of mass $M/2$. This can be written as: $E = m_L c^2 + (\gamma_L - 1) (M/2 + m_L) c^2 - (\gamma_v - 1) (M/2) c^2$, or, $E = \gamma L m_L c^2 + (\gamma_L - \gamma_v) (M/2) c^2$.

The limiting case $M \to \infty$ of Eq. (143) yields

$$
\lim_{M \to \infty} m_L = w_1 \frac{E}{c^2} ,
$$

$$
w_1 = \gamma_v \left( 1 - \frac{v}{c} \right) .
$$

(144)

Eq. (144) can be written as

$$
\lim_{M \to \infty} m_L = \frac{E'}{c^2} ,
$$

(145)

where $E'$ is energy of the photon in the frame of reference of the absorbing surface.

### 8.3.3. Relativity from Sec. 8.1.3

Sec. 8.1.3 presents result for an increase of mass of a planar absorbing surface if it’s mass is $M/2$ and the increase of it’s mass is generated by an absorption of a massive particle of energy $E = \gamma_u m c^2$ with respect to the surface. In order to be able to compare the results of Sec. 8.1.3 with with the results presented in Sec. 8.3.2, we have to generalize the considerations of Sec. 8.1.3 for the case of a moving absorbing surface.

Let us consider a massive particle of mass $m$ and energy $E$ in the laboratory frame (inertial frame of reference $K$) in which the absorbing surface is moving with the speed $v$, in the same direction and orientation as the massive particle is travelling, $u > v$ (also negative orientation may be considered: it is sufficient to put $-v$ into the formulae presented below, in this subsection: since $u > 0$ also $u > -v$). We are interested in the
increase of mass of the absorbing surface. The law of conservation of momentum for the absorption gets the following form in the inertial frame of reference $K$ (compare with Eq. 130):

$$\gamma_L \left( \frac{M}{2} + m_L \right) v_L = \gamma_u m u + \gamma_v \frac{M}{2} v.$$  \hfill (146)

Similarly, the law of conservation of energy yields (compare with Eq. 132):

$$\gamma_L \left( \frac{M}{2} + m_L \right) c^2 = \gamma_u m c^2 + \gamma_v \frac{M}{2} c^2.$$  \hfill (147)

Eqs. (146)-(147) yield

$$m_L = \frac{M}{2} \left\{ \sqrt{1 + 2 \gamma_u \gamma_v \left( 1 - \frac{u v}{c^2} \right) \frac{m}{M/2} + \left( \frac{m}{M/2} \right)^2 - 1} \right\}.  \hfill (148)$$

Eq. (148) may be rewritten also in the following forms:

$$m_L = \frac{M}{2} \left\{ \sqrt{1 + 2 w_u \frac{E/c^2}{M/2} + \left( \frac{m}{M/2} \right)^2 - 1} \right\},$$

$$w_u = \gamma_v \left( 1 - \frac{u v}{c^2} \right),$$

$$E = \gamma_u m c^2,$$

$$\gamma_u = \gamma_u \gamma_v \frac{m}{M/2} + \left( \frac{m}{M/2} \right)^2 - 1 \right\},$$

$$u' = \frac{u - v}{1 - u v / c^2}.  \hfill (149)$$

The limiting case $M \to \infty$ of Eqs. (148)-(149) yields

$$\lim_{M \to \infty} m_L = \gamma_u' m,$$

$$w_u E/c^2,$$

$$\gamma_u' = \gamma_u \gamma_v \left( 1 - \frac{u v}{c^2} \right),$$

$$w_u = \gamma_v \left( 1 - \frac{u v}{c^2} \right),$$

$$E = \gamma_u' m c^2.  \hfill (150)$$

Eq. (150) can be written as

$$\lim_{M \to \infty} m_L = \frac{E'}{c^2},  \hfill (151)$$

where $E'$ is energy of the massive particle in the frame of reference of the absorbing surface.

8.3.4. Comparison of Secs. 8.3.1 and 8.3.2

Let us consider a continuous beam of photons, each of which is bearing the energy $E$ in the laboratory frame $K$. Let the absorbing surface is moving with the speed $v$ in the...
laboratory frame, in the same direction and orientation as the photons are travelling. The increase of mass of the absorbing surface, per unit time, is

\[ \left( \frac{dm}{d\tau} \right)_{in} = m_L \times \frac{S' C'_{ext}}{E'}. \]  

(152)

Now, we have to use Eqs. (19), (23) and (144). Eq. (152) leads to

\[ \left( \frac{dm}{d\tau} \right)_{in} = w_1 \frac{E}{c^2} \times \frac{w_1 S C'_{ext}}{w_1 E} = \frac{w_1^2 S C'_{ext}}{c^2}. \]  

(153)

The result obtained in Eq. (153) is identical to the result presented in Eq. (140) (see also Eq. 61). Thus, we have found a consistency between considerations presented in Secs. 2-7 and Sec. 8.1.1. The term \( w_1 \) present in Eqs. (143)-(144) is due to the Doppler effect; the same holds for the relations \( dE_{in}/d\tau \) and \( dp_{in}/d\tau \) in Eq. (140).

8.3.5. Comparison of Secs. 8.3.2 and 8.3.3

Absorption of a photon bearing the energy \( E \) in the laboratory frame \( K \) leads to the increase of mass of the absorbing surface moving with the speed \( v \) in the laboratory frame. The results are given by Eqs. (143)-(145). Similarly, absorption of a massive particle (the particle with a non-zero mass) bearing the energy \( E \) in the laboratory frame \( K \) also leads to the increase of mass of the absorbing surface moving with the speed \( v \) in the laboratory frame. The results are given by Eqs. (149)-(151). Eqs. (145) and (151) are equivalent. Moreover, comparison of Eq. (144) with Eq. (145) yields

\[ E' = \gamma_v \left( 1 - \frac{v}{c} \right) E, \]  

(154)

for the photon, and, comparison of Eq. (150) with Eq. (151) leads to

\[ E' = \gamma_v \left( 1 - \frac{u v}{c^2} \right) E, \]  

(155)

for the massive particle moving with the speed \( u \) in the laboratory frame of reference \( K \). In both cases, \( E \) is the energy of the incident photon / massive particle in the laboratory frame of reference and \( E' \) is the energy of the photon / massive particle measured in the reference frame of the absorbing surface. Eq. (154), which holds for the photon, corresponds to the Doppler effect. Analogous equation for the massive particle is Eq. (155) and it, formally, reduces to Eq. (154) in the limiting case \( u \rightarrow c \).

According to Eq. (150), we have \( \lim_{M \rightarrow \infty} m_L = \gamma_u m \), or, \( \lim_{M \rightarrow \infty} m_L = w_u E / c^2 \). The quantity \( m \) is the mass of the massive particle and it is a relativistic invariant \( (\sqrt{p_\mu p^\mu} / c) \). The quantity \( E / c^2 \) is not equal to \( m \): \( m < \lim_{M \rightarrow \infty} m_L < E / c^2 \). Thus, it is of no sense to use \( E / c^2 \) as a mass of the particle. As a consequence, also \( E / c^2 \) in Eq. (144) does not represent a mass of a photon. These statements are consistent with Sec. 8.2.
9. Cross sections for arbitrarily shaped particles

We have already discussed, in Secs. 5, 6, 7.3 and 8, the relations between four-momentum of the radiation and the mass of the radiation or the change of the mass of the objects interacting with the radiation. Application of these physical considerations to astrophysics was shown mainly in Sec. 7.4 for the case of dust grains with spherically symmetric mass distribution. Discussion in Sec. 7.4 explained the physical access to the Poynting-Robertson effect, which is standardly considered in evolution of interplanetary dust grains. This section will discuss the relation between four-momentum of the outgoing radiation and its mass for arbitrarily shaped dust grains.

Let us consider that an arbitrarily shaped dust particle is irradiated by the incoming radiation of a source of radiation. The relevant relations for the four-momentum of the outgoing radiation and its mass are given by equations presented in Sec. 5.2. In what follows, we want to obtain more information on cross sections (for pressure components, extinction, scattering) than can be obtained from observations and measurements.

As follows from Eqs. (53)-(54), the following inequality holds:

\[
0 \leq 2 \left( \frac{C_{pr,1}'}{C_{ext}'} + X^{-1} F_{e,1}' \right)^2 - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} + X^{-1} F_{e,j}' \right)^2, \tag{156}
\]

\[X \equiv \frac{w^2}{c} S \frac{C_{ext}'}{c}.\]

If we neglect the thermal emission force \( F_e' \), then Eq. (156) reduces to

\[
0 \leq 2 \frac{C_{pr,1}'}{C_{ext}'} - \sum_{j=1}^{3} \left( \frac{C_{pr,j}'}{C_{ext}'} \right)^2. \tag{157}
\]

Current situation is that we do not know simultaneous values of the cross sections \( C_{pr,j}' \) (\( j = 1, 2, 3 \)), \( C_{ext}' \), \( C_{sca}' \) and \( C_{abs}' \) from observations or measurements (Krauss and Wurm 2004). Moreover, if the size of the particle is larger than about 5 micrometers, then it is not possible to make numerical calculations for the above presented cross sections of the particle. Thus, there do not exist methods which determine, simultaneously, the values of cross sections \( C_{pr,j}' \) (\( j = 1, 2, 3 \)), \( C_{ext}' \), \( C_{sca}' \) and \( C_{abs}' \), at present. However, we will show that some information on the cross sections can be obtained on the basis of equations presented in Secs. 2, 3.4 and 5.2, or, Eq. (157).

Let us suppose that measurements can yield ratios of the cross sections for radiation pressure, i.e., the ratios \( \frac{C_{pr,j}'}{C_{pr,1}'} \) (\( j = 2, 3 \)) are in disposal. It is useful to rewrite Eq. (157) into the form

\[
0 < \frac{C_{pr,1}'}{C_{ext}'} \leq \frac{2}{1 + \sum_{j=2}^{3} \left( \frac{C_{pr,j}'}{C_{pr,1}'} \right)^2}. \tag{158}
\]

Eq. (158) reduces to Eq. (79) for the case \( C_{pr,2}' = C_{pr,3}' = 0 \). On the basis of Eqs. (9)-(10) and (38), we can write \( C_{pr,1}' = C_{ext}' - \langle < \cos \theta' > C_{sca}' \rangle \) (the outer symbol for the mean denotes weighting over stellar spectrum and it corresponds to the ”bar” symbol used
above the letters, standardly used in this paper). Since \(< \cos \theta' \geq 1\), we immediately obtain
\[
\frac{C_{\text{sca}}'}{C_{\text{pr},1}'} \geq \frac{C_{\text{ext}}'}{C_{\text{pr},1}'} - 1.
\]  
(159)

The equation \(\bar{C}_{\text{ext}}' = \bar{C}_{\text{abs}}' + \bar{C}_{\text{sca}}'\) can be rewritten into the form \(\bar{C}_{\text{ext}}' / \bar{C}_{\text{pr},1}' = \bar{C}_{\text{abs}}' / \bar{C}_{\text{pr},1}' + \bar{C}_{\text{sca}}' / \bar{C}_{\text{pr},1}'\), which, together with Eq. (159) yields
\[
0 \leq \frac{\bar{C}_{\text{abs}}'}{\bar{C}_{\text{pr},1}'} \leq 1.
\]  
(160)

Similarly, the relation \(\bar{C}_{\text{ext}}' / \bar{C}_{\text{pr},1}' = \bar{C}_{\text{abs}}' / \bar{C}_{\text{pr},1}' + \bar{C}_{\text{sca}}' / \bar{C}_{\text{pr},1}'\) and Eq. (160) yield
\[
0 < \frac{\bar{C}_{\text{sca}}'}{\bar{C}_{\text{pr},1}'} \leq \frac{\bar{C}_{\text{ext}}'}{\bar{C}_{\text{pr},1}'}.
\]  
(161)

Eqs. (158)-(161) offer some information on the cross sections of extinction and scattering.

10. Summary and conclusions

The paper derives and presents relativistically covariant equation of motion for dust particle under the action of electromagnetic radiation – see Eqs. (38) and (40). It yields, as special cases, the results obtained by Einstein (1905) and Robertson (1937). As for most frequent applications to systems in the universe (e.g., meteoroids in the Solar System, dust particles in circumstellar disks), equation of motion in the form of Eq. (39) is sufficient.

The general equation of motion reduces to the Poynting-Robertson effect for irradiated particle with spherically symmetric mass distribution. This yields linear relation between the incoming and outgoing momenta (per unit time) of the radiation, in the proper reference frame of the particle: \(\mathbf{p}_o' = (1 - \bar{Q}_{\text{pr},1}' / \bar{Q}_{\text{ext}}') \mathbf{p}_i'\), where \(\bar{Q}_{\text{pr},1}'\) and \(\bar{Q}_{\text{ext}}'\) are dimensionless efficiency factors for the radial direction radiation pressure and extinction, integrated over the stellar spectrum (see Eq. 73, or Eq. 74 for covariant formulation). A simple condition \(0 < \bar{Q}_{\text{pr},1}' / \bar{Q}_{\text{ext}}' \leq 2\) is obtained from the invariant mass of the outgoing radiation (see Eqs. 77-79). The case of ”perfectly absorbing” spherical dust particle corresponds to \(\bar{Q}_{\text{pr},1}' / \bar{Q}_{\text{ext}}' = 1/2\), if radius of the particle is much larger than the wavelength(s) of the incident radiation: the effect of diffraction cannot be neglected within geometrical optics approximation. While the condition \(\mathbf{p}_o' = (1 - \bar{Q}_{\text{pr},1}' / \bar{Q}_{\text{ext}}') \mathbf{p}_i'\) is consistent with the condition \(dM_{\text{out}} = \sqrt{dp_{\text{out}} \mu dp_{\text{out}}' / c^2}\), these fundamental relations are not consistent with the explanations of the Poynting-Robertson effect in the literature.

The Poynting-Robertson effect is generated by simultaneous action of the Doppler effect, change of concentration of photons (the corresponding terms \(w_1 \times w_1\) are present at flux density of radiation energy \(S\) in equation of motion of spherical particle, in Eq. 70) and the aberration of light, together with the laws of relativity theory. However, the
term \(-\frac{v}{c}\) does not correspond to the aberration of light. Its existence is caused by conservation of mass of the particle. The term \(1/w_1\) in the four-vector \(b_1^\mu\) (see Eq. 70) comes from the Doppler effect (see Eqs. 17-18) or from the change of concentration of photons (see Eqs. 21-22): Lorentz transformation of the zero-th component of a four-vector is relevant, in relation between \(e'_1\) and \(e_1\).

Covariant four-accelerations for the incoming and outgoing radiation are given by Eqs. (106) and (109). Moreover, the outgoing radiation accelerates the particle if \(\frac{Q'_{pr,1}}{Q'_{ext}} < 1\) (see Eq. 110), i.e., also the perfectly absorbing sphere treated by Poynting (1903), Robertson (1937) and others.

As for the incoming radiation, the formulae for the energy and the mass of the radiation are given in Eq. (50). The statement that the mass corresponds to the energy through the relation "mass equals energy / \(c^2\)" is not correct, because the photon is massless (see also Sec. 8). The relation between the incoming energy and the increase of the particle’s mass is presented in Eq. (62) (Eq. 63 presents the result for momentum). As a consequence, the formula "energy equals \(\gamma \times \text{mass} \times c^2\)" does not hold. The Einstein’s principle of equivalence of inertial mass and rest-energy holds (Einstein 1999, p. 43).

As for the outgoing radiation: if the particle is under the action of incident radiation, then the formulae for the energy and mass of the radiation are given by Eq. (54), see also Eq. (55): the simple formula "energy equals \(\gamma \times \text{mass} \times c^2\)" holds (see also Eq. 55). The relation between the outgoing energy and decrease of the particle’s mass is presented in Eq. (66) for general case, and in Eq. (89) for particle with spherically symmetric mass distribution. As a consequence, the formula "energy equals \(\gamma \times \text{mass} \times c^2\)" does not hold, in general. The Einstein’s principle of equivalence of inertial mass and rest-energy holds.

Eqs. (40)-(41) present effect of thermal emission alone, if the particle is not irradiated. As a consequence, the isotropic thermal emission does not influence acceleration of the particle. Eq. (41) (Eq. 67) also shows that the formulation "energy equals mass times \(c^2\)" holds between the decrease of particle’s mass and the thermally emitted energy in the rest frame of the particle. General equation is given by Eq. (68) and the thermal force disturbs the saying "energy equals \(\gamma \times \text{mass} \times c^2\)". The relation between energy and mass of the radiation is given by Eqs. (58) and (59): the standard simple formula "energy equals \(\gamma \times \text{mass} \times c^2\)" holds.

On the basis of the covariant formulations presented in the paper, we were able to show that three of the current statements on the essence of the Poynting-Robertson effect are not correct. We have explained the important points in the statements.

The relativistically covariant formulations presented in the paper enable to understand: (i) physics of the Poynting-Robertson effect (see also Secs. 7.3.4 and 7.4), which is frequently used in studies of orbital evolution of cosmic dust particles, and, (ii) why
the analogy on momentum transfer to the surface of an object in mechanics and electromagnetism does not hold (see Sec. 7.5).

We have also shown an application of four-momentum (and mass) of the outgoing radiation for arbitrarily shaped dust particle on obtaining some information on cross sections, unknown from experiments, observations or theoretical/numerical solutions. The condition is presented in Eq. (158). As a result, the non-radial components of radiation pressure cross sections decrease the ratio $\overline{C}_{pr,1} / \overline{C}_{ext}$ in comparison with the spherical particles. A more general condition is given by Eq. (156). The condition can help in better understanding of optical properties of cosmic dust particles (special cases of the condition, applied to the spherical and planar particles, enabled us to understand physics of the interaction between the incident electromagnetic radiation and the particles, see Eqs. 79, 104, Secs. 7.3.4, 7.4, 7.5).

The physics presented in the paper has direct implications for understanding of distribution and evolution of cosmic dust grains in various astrophysical systems.

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