Particles within extended-spin space: Lagrangian connection

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Abstract

A spin-space extension is reviewed, which provides information on the standard model. Its defining feature is a common matrix space that describes symmetries and representations, and leads to limits on these, for given dimension. The model provides additional information on the standard model, whose interpretation requires an interactive formulation. Within this program, we compare the model’s lepton-W generated interactive Lagrangian in (5+1)-dimensions, and that of the standard model. We derive the conditions for this matching, which apply to other Lagrangian terms. We also discuss the advantages of this extension, as compared to others.

Keywords: Spin-extension, Lagrangian, electroweak, lepton, W.

1 Introduction

1.1 Background

The standard model is the theory that describes the key elements of nature, also one of the most successful theories and, at the same time, which presents the greatest enigmas in modern physics. Although the model correctly describes the elementary particles, it is phenomenological. On the one hand, the fermion representations have been established, as well as their classification in generations, and the forces acting between these particles, which define the vector bosons transmitting these interactions. On the other hand, the origin of the specific types of representations and forces that nature has chosen is not known. In particular, we do not know why matter consists of leptons and quarks, nor the reason for the interaction groups $U_Y(1) \times SU_L(2) \times SU(3)$ and related particles: the $Z$ boson carries the hypercharge, is associated to the group $U_Y(1)$, and applies to all particles; the $W$ bosons are associated to the $SU(2)_L$ group and act on left-chirality particles; gluons produce the strong interaction, derive from the color group $SU(3)$, and act only upon the quarks. The origin of this behavior is unknown. Finally, we need a more fundamental reason for the existence of
the scalar particle that is suggested in recent experiments\cite{1, 2}, the Higgs, and which gives mass to particles. This ignorance is also reflected in the relative large number of parameters required by the model, of the order of twenty, as the particle masses and charges, which are fixed experimentally. By the nature of the standard model, it is understood that by itself, it will never explain these unknowns and that, therefore, we need to investigate options beyond it.

Great insights have been reached throughout the history of Physics by the discovery of connections between phenomena. Traditional examples include Newton’s connection of the Moon’s movement with the fall of an object on Earth, through gravity, and Maxwell’s understanding of light as an electromagnetic phenomenon, obtained from wave solutions in his equations, and the speed of light built in terms of the relative permittivity and magnetic permeability of the vacuum. Furthermore, advances in the understanding of elementary particles have been obtained from a framework that assumes unification and/or symmetry of the above physical quantities that characterize them. Indeed, the successes of the past include the chiral-symmetry assumption, involved in the generation of hadron masses; supersymmetry, a hypothesis currently under investigation, explains symmetry breaking at the low-energy electroweak scale, and creates the masses of the known elementary particles.

A partial but practical description of the fundamental physical elements that participate in the modern unification ideas consists of particles, classified as bosons and fermions; spin and space as their associated attributes; and finally, their interactions, as described within general relativity and the standard model. These are the key elements to investigate. Before introducing this paper’s proposal, we briefly review some standard-model extensions:

1.2 Kaluza-Klein and grand-unification theories

A promising unification is the idea of Kaluza-Klein, who proposed extra spatial dimensions\cite{d}, beyond $3 + 1$, to be associated with gauge symmetries. In the case of grand-unification theories in their application to the standard model, there are restrictions on the standard-model $U(1)_Y \times SU(2)_L \times SU(3)$ gauge groups, as well as on the representations and coupling-constant values.

1.3 Quasi-particles

This idea, originated by Landau, suggests that it is possible to achieve an adequate description of interactive particles, if one manages to describe their effective degrees of freedom in an appropriate way. To first order, it would be possible to consider particles as free, while parameters such as mass would be modified. The search for these degrees of freedom represents one of the main objectives in studies in areas that engage many-particle systems, with quantum behavior, such as nuclear physics and superconductivity. Indeed, in the area of elementary particles, Nambu and Jona-Lasinio\cite{3} described an interactive model within the framework of field theory, inspired by superconductivity, and which leads to masses of composite particles, from an assumed interaction. The lesson
is that finding the correct degrees of freedom may hold the clue to gain insight into the standard model.

1.4 Extended-spin model

As for the actual description of the elementary particles, we concentrate on their degrees of freedom. Particles and interactions obey Lorentz and scalar symmetries, global and local, and are described with non-trivial discrete quantum numbers. While the space degree of freedom is common to all elementary particles, the discrete degrees of freedom associated with the fundamental representations are more elementary insofar as they can be used to build the others. Spin is a physical manifestation of the representation of the Lorentz group. In relation to space, spin maintains this role since the first uses the vector representation, and can be constructed in terms of the second. Other similar investigations underscore the spin degree of freedom in the extensions of the standard model (see, e.g., Refs. [4, 5, 6]). The fact that the known fermions participate in the fundamental representation of the Lorentz and gauge groups, and that the gauge bosons, the interaction carriers, belong to the adjoint representation of these groups, suggests a common description [7]. Indeed, such similarities and the presence of symmetry suggest a unified description, i.e., an elementary space for the discrete degrees of freedom: Lorentz and scalar. In fact, there are similar common requirements that emerge from the quantum description and quantization of particles and interactions, such as the restrictions on representations from unitarity.

The extended-spin model, just as the idea of Kaluza-Klein, assumes a common space for the spin and scalar degrees of freedom. While the idea of mixing these is tempting, the Coleman and Mandula theorem [8] prohibits a non-trivial mixing. Obeying this restriction means that the resulting scalar generators commute with the Lorentz ones, which is equivalent to the requirement that these two elements be described as direct products. However, a simple classification of spaces is permitted with symmetries as the chiral one, and this leads to limitations in the elements that can be obtained within the space, which ultimately, gives information, for example, on representations and interactions. New information is derived as constraints on the chirality of the interactions and representations [7, 9, 10], the coupling constants [7, 10, 11], connections among the standard-model particle masses [11], and a fermion hierarchy effect [12].

The spin-extended model can be interpreted within the Kaluza-Klein framework, as a result that the additional spatial dimensional components are frozen. Conceptually, the construction of the model in terms of matrices comes from incremental direct products with $2 \times 2$ matrices, suggesting the discrete Hilbert space considered is made from elementary degrees of freedom (e.g., q-bits or particles of spin $1/2$).

A field theory can be equivalently formulated in terms of such degrees of freedom. Work on that direction was carried out on Ref. [12]. In this paper, after introducing the spin-extended model by presenting its landmarks, we examine in detail its formulation within a standard Lagrangian, using representation fields
and symmetries that derive from it; in particular, we look at a specific vertex and study its connection to a standard formulation. This complements Ref. [12], which also deals with this connection, with a general analysis of the fields’ construction, various vertices, and symmetry implementation. Here we examine the W-fermion interaction term derived from $(5+1)$-d, making a detailed description of its Lagrangian, with further analysis of the projection operator involved, allowing for this equivalence. In particular, we focus on its coefficients and phases, extending previous work [11, 12, 13].

The dimension $N = 4$ case was analyzed in Refs. [7, 9], $N = 6$ in [7, 9, 11], $N = 8$ partially in [12], and $N = 10$ in [10].

The paper is organized as follows: In Section 2, we review the construction of the proposed extended-spin space, based on a matrix space. For this purpose, we present as example a massless Hamiltonian. A Clifford algebra helps in the classification of both operators and states. Under the demand that the Lorentz symmetry be maintained, scalar degrees of freedom emerge, associated to global and gauge symmetries. The matrix space restrains the allowed representations. In Section 3, we use as example lepton and electroweak fields, expressed in the $(5+1)$-d space; in Section 4, their the gauge-invariant $SU(2)_L \times U(1)_Y$ interactive theory is formulated, and its Lagrangian compared with the standard one. We concentrate on the W-lepton vertex contribution; we find the correct phases and coefficients in a projection operator that allow for this equality. In Section 5, we summarize relevant points in the paper.

## 2 Gamma-matrix symmetry classification

In this section, we summarize the main points in the classification of states and symmetries. More details may be found in Ref. [12]. A massless Dirac equation formulated over the matrix $\Psi$ (and corresponding conjugate equation)

$$i\gamma_\mu \partial_\mu \gamma^\rho \Psi = 0,$$

may be used as framework for the classification of states and operators in an extended space and study symmetry transformations. It also generates free-particle fermion and bosons on the extended space. Appropriate transformation operators $U$ acting on field states $\Psi$ can generically be characterized by the expression

$$\Psi \rightarrow U \Psi U^\dagger.$$

for both Lorentz and scalar symmetries. In the massive case, some symmetries are broken, leading to effects as fermion-mass hierarchy generation, treated elsewhere [12].

The dot product between the elements $\Psi_a, \Psi_b$ can be defined using the trace

$$\text{tr} \, \Psi_a \Psi_b.$$

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1We assume throughout $\hbar = c = 1$, and 4-d diagonal metric elements $g_{\mu\nu} = (1, -1, -1, -1)$. 

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4
An operator $Op$ within this space characterizes a state $\Psi$ with the eigenvalue rule

$$[Op, \Psi] = \lambda \Psi,$$  \hspace{1cm} (4)

consistent with the hole interpretation, and anticipating a second-quantization description. For example, a boson may be constructed by two fermion components with positive frequencies $\psi_1(x)$, $\psi_2(x)$ through $\psi_1(x)\bar{\psi}_2(x)$, with $\psi_2(x)$ describing an antiparticle.

Eq. 1 keeping $\mu = 0, ..., 3$, is assumed within the larger Clifford algebra, here also understood as a matrix space:

$$\{\gamma_\eta, \gamma_\sigma\} = 2g_{\eta\sigma}, \eta, \sigma = 0, ..., 3, 5, ..., N,$$

with $N$ the (assumed even) dimension, whose structure is helpful in classifying the available symmetries $U$, and solutions $\Psi$, both represented by $2^{N/2} \times 2^{N/2}$ matrices. The 4-d Lorentz symmetry is maintained, and uses the generators

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}],$$  \hspace{1cm} (5)

where $\mu, \nu = 0, ..., 3$. $U$ contains also $\gamma_a$, $a = 5, ..., N$, and their products as possible symmetry generators. The latter elements are scalars for they commute with the Poincaré generators, which contain $\sigma_{\mu\nu}$, and they are also symmetry operators of the massless Eq. 1 bilinear in $\gamma_{\mu}$, $\mu = 0, ..., 3$ which is not necessarily the case for mass terms (containing $\gamma_0$). In addition, their products with

$$\bar{\gamma}_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$$  \hspace{1cm} (6)

are Lorentz pseudoscalars, as $[\bar{\gamma}_5, \gamma_a] = 0$.

The operator algebra was described in Refs. [10] and [13]. In accordance with the above symmetry generators that emerge from the Clifford algebra $C_N$, for given dimension $N$, any matrix element representing a state is obtained by combinations of products of one or two $\gamma_{\mu}$, and elements of the algebra generated by $\gamma_a$, $a = 5, ..., N$, which define, respectively, their Lorentz (as for 4-d) and scalar-group representation $S_{N-4}$. A 4-d Clifford matrix subalgebra is obtained, implying spinor up to bi-spinor elements, thus vectors and scalar fields, can be described. There is a finite number of partitions on the matrix space for the states and symmetry operators, consistent with Lorentz symmetry. These variations are defined by a projection operators $\mathcal{P}_P$ with $[\mathcal{P}_P, \mathcal{P}_S] = 0$; $\mathcal{P}_P$ acts on the Lorentz generator

$$\mathcal{P}_P[\frac{1}{2}\sigma_{\mu\nu} + i(x_\mu \partial_\nu - x_\nu \partial_\mu)],$$  \hspace{1cm} (7)

and $\mathcal{P}_S$ on the symmetry operator space leading to projected scalar generators $I_a = \mathcal{P}_S I_a$, so that they determine, respectively, the Poincaré generators and the scalar groups.

The application of these operators follows the operator rule in Eq. 4 which assigns states to particular Lorentz and scalar group representations. For simplicity, we assume $\mathcal{P}_P = \mathcal{P}_S \neq 1$, as other possibilities are less plausible[10].
Thus, the Lorentz or scalar operators act trivially on one side of solutions of the form $\Psi = P_P \Psi (1 - P_P)$, since $(1 - P_P) P_P = 0$, leading to spin-1/2 states or states belonging to the fundamental representation of the non-Abelian symmetry groups, respectively.

In Figure 1(a), presented also in Ref. [12], we show schematically the organization of the symmetry operators, producing corresponding Lorentz and scalar generators. Fig. 1(b) also depicts the resulting solution representations, distributed according to their Lorentz classification: fermion, scalar, vector, and antisymmetric tensor. The matrices are classified according to the chiral projection operators $\frac{1}{2}(1 \pm \tilde{\gamma}_5)$, leading to $N/2 \times N/2$ matrix blocks in $C_N$. The space projected by $P_P = P_S \neq 1$ is also depicted. Specific combinations also emerge, corresponding to spin-1/2-fundamental and vector-adjoint, Lorentz and scalar groups representations, respectively; graphically, scalar-group elements and vectors occupy the same matrix spots.

In the next Section, we generalize these fields.
3 (5+1)-dimensional representations

We review the (5+1)-dimensional representations, which reproduce a standard-model lepton electroweak sector\[11\]; one of its coupling terms will be analyzed in the next Section.

3.1 Fields’ construction

As derived in Section 2, it is possible to write fundamental fields using as basis matrix products conformed of Lorentz and scalar group representations. Indeed, the commuting property of the respective degrees of freedom allows for states and operators to be written as a product of matrices belonging to the 4-d $C_4$, and matrices within $S_{N-4}$ projected by $P_S$; explicitly, $\Psi = M_1 M_2$, where

$$M_1 \in C_4 \text{ and } M_2 \in P_S S_{N-4}. \quad (8)$$

An expression with elements of each set is possible through their passage to each side, using commutation or anticommutation rules.

In the presence of interactions, free fields as generated by Eq. 1 give way to more general expressions of interactive fermion and boson fields, keeping their transformation properties:

**Vector field**

$$A_{\mu}^a(x) \gamma_\mu I_a, \quad (9)$$

where $\gamma_\mu \in C_4, I_a \in P_S S_{N-4}$ is a generator of a given unitary group, according to the projection operator $P_S$.

**Fermion field**

$$\psi^a_\alpha(x) L^\alpha P_F M^F_a, \quad (10)$$

where $M^S_a, M^F_a \in P_S S_{N-4}$ are, respectively, scalar and fermion components, and $L^\alpha$ represents a spin component; for example, $L^1 = (\gamma_1 + i \gamma_2), P_F$ is a projection operator of the type in Eq. 7 such that

$$P_F \gamma_\mu = \gamma_\mu P_F. \quad (11)$$

and we use the complement $P_F^c = 1 - P_F$, so that a Lorentz transformation with $P_F \sigma_{\mu\nu}$, will describe fermions, as argued in Section 3; the simplest example for an operator satisfying such conditions is $P_F = (1 - \tilde{\gamma}_5)/2 \quad [7, 9]$, used by the fermion doublet on Table 1 (see below.) By the argument after Eq. 4 the fundamental-representation state is derived from the trivial right-hand action of the operator within the transformation rule in Eq. 11. This means the matrix entitles spurious ket states contained in the Lorentz-scalar term $M_2$ in Eq. 8.

For the (5+1)-dimensional space, among few choices, $P_P = L$, with $L = \frac{3}{4} - \frac{1}{4}(1 + \tilde{\gamma}_5)\gamma^5 \gamma^6 - \frac{1}{4}\tilde{\gamma}_5$ is associated to the lepton number, and the resulting
symmetry generators and particle spectrum fits the standard-model electroweak sector. Specifically, the projected symmetry space also includes the SU(2)\(_L\) × U(1)\(_Y\) groups, with respective generators \(I_i\) and hypercharge \(Y\):

\[
\begin{align*}
I_1 &= \frac{i}{4}(1 - \tilde{\gamma}_5)\gamma^5 \\
I_2 &= -\frac{i}{4}(1 - \tilde{\gamma}_5)\gamma^6 \\
I_3 &= -\frac{i}{4}(1 - \tilde{\gamma}_5)\gamma^5\gamma^6 \\
Y &= -1 + \frac{i}{2}(1 + \tilde{\gamma}_5)\gamma^5\gamma^6.
\end{align*}
\] (12)

We note that the SU(2) generators correctly contain the projection operator \(\frac{1}{2}(1 - \tilde{\gamma}_5)\), confirming the interaction’s chiral nature, which also leads to chiral representations, a feature that results from nature of the matrix space under projector \(L\) and the Lorentz group.

A state basis is presented on Table 1, that contains lepton, as well as scalar and electroweak vector components; \(W\) and \(Z\) components are shown, where the latter normalizations require relative coupling-constant \(g\) and \(g'\) factors, respectively. Within Eq. 4 the action of these operators on choices of states \(\Psi\) produce their quantum numbers, also represented. For fermions and vectors, the second spin component may be obtained from the first by flipping the spin; e. g., \(\nu_2^L = [L(\gamma^2\gamma^3 - i\gamma^3\gamma^1), \nu_1^L]\).

Ref. [11] set thumb rules to derive some gauge-invariant terms, identifying elements between the extended-spin space and standard Lagrangian terms. Ref. [12] formally translated the field information that emerges from the extended-spin space, to derive an interactive gauge theory. Next, we show for the lepton-\(W\) vertex the workings of the equivalence between the extended-spin model and the standard Lagrangian formulation.

### 4 Fermion-W electroweak Lagrangian

The fields within the extended-spin basis can be used to construct a standradly-formulated Lagrangian. This amounts to using elements with a well-defined group structure to get Lorentz-scalar gauge-invariant combinations. Choosing scalar elements that result from the direct product in Eq. 3, one obtains an interactive theory, as the same particle content is maintained.

Indeed, a gauge-invariant fermion-vector interaction term results, constructing matrix elements containing the vector field, together with fermion, with input from Eqs. 9-10 by taking the trace. Invariant elements are obtained adding to the fermion free Lagrangian (that implies the Dirac equation) the vector contribution in Eq. 9. The latter extracts the identity-matrix coefficient, leading to the usual Lagrangian components. A general fermion-vector
| Electroweak multiplet | States \( \Psi \) | \( I_3 \) | \( Y \) | \( Q \) | \( L \gamma^1 \gamma^2 \) | \( L \tilde{\gamma}_5 \) |
|----------------------|-----------------|---------|--------|--------|------------------|------------------|
| Fermion doublet      | \( \nu_L = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 + \gamma^3) (\gamma^5 - i \gamma^6) \) | 1/2 | -1    | 0      | 1/2              | -1               |
|                      | \( \nu_R = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 + \gamma^3) (\gamma^5 + i \gamma^6) \) | 0    | -2    | -1     | 1/2              | 1                |
|                      | \( \nu_R = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 - \gamma^3) (\gamma^5 - i \gamma^6) \) | 0    | -2    | -1     | 1/2              | 1                |
|                      | \( \nu_R = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 - \gamma^3) (\gamma^5 + i \gamma^6) \) | 0    | -2    | -1     | 1/2              | 1                |
|                      | \( \epsilon_1^L = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 + \gamma^3) (\gamma^5 - i \gamma^6) \) | 1/2  | 1     | 1      | 0                | 0                |
|                      | \( \epsilon_1^R = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 - \gamma^3) (\gamma^5 - i \gamma^6) \) | 1/2  | 1     | 1      | 0                | 0                |
|                      | \( \epsilon_2^L = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 + \gamma^3) (\gamma^5 + i \gamma^6) \) | 1/2  | 1     | 1      | 0                | 0                |
|                      | \( \epsilon_2^R = \frac{1}{2} (1 - \tilde{\gamma}_5) \gamma^0 (\gamma^0 - \gamma^3) (\gamma^5 + i \gamma^6) \) | 1/2  | 1     | 1      | 0                | 0                |

Table 1: Massless fermion and boson states in (5+1)-d extension, momentum along \( \pm \hat{z} \), with projection given by the lepton number \( P_L = L \), under the operators \( SU(2)_L \), \( I_3 \) component, hypercharge \( Y \), charge \( Q = I_3 + \frac{1}{2} Y \), lepton operator \( L \), spin projection \( \frac{1}{2} L \gamma^1 \gamma^2 \), and chirality \( L \tilde{\gamma}_5 \) (the coordinate dependence is omitted.)
component is

\[
\frac{1}{N_f} \text{tr} \Psi \{[i g A_{\mu}(x)] \gamma_0 \gamma^\mu - M \gamma_0 \} \Psi P_f, \tag{13}
\]

where \( \Psi \) is a field representing in this case spin-1/2 particles. \( I_a \) is the group generator in a given representation, \( g \) is the coupling constant, \( N_f \) contains the normalization (and similar terms below), and \( I_{den} \) the identity scalar group operator in the same representation (which will be omitted hence). \( M \) is generally a mass operator whose restrictions provide information on fermion masses\[11,\] \[12\]. The operator \( P_f \) is introduced to avoid cancelation of non-diagonal fermion elements. Such an operator is necessary because of spurious left ket components of fermions in \( \Psi \). For example,

\[
P_f = \frac{1}{\sqrt{2}} (\tilde{\gamma}_5 - \gamma^0 \gamma^1) \tag{14}
\]

as \([P_f, L] = [P_f, (1 - \tilde{\gamma}_5)L] = 0\), provides a non-trivial combination with the correct quantum numbers for the fermion pair \( \Psi_a P_f \Psi_b^\dagger \) (with \( \Psi_a, \Psi_b \) either doublet or singlet fermions, on Table 1), and maintains their normalization, spin, lepton and electroweak representation.

The invariance under transformations in Eq. 2 can be verified independently, using the separation in Eq. 8 into Lorentz and scalar symmetries; under Lorentz and gauge-group transformations of the extended-spin space\[12\]. Eq. 13 is invariant under the Lorentz transformation, provided the vector field transforms as

\[
A_\mu^a(x) I_a \to \Delta^a_\mu \gamma^\nu A_\nu^a(x) I_a, \tag{15}
\]

where we use the identity relating the spin representation of the Lorentz group in

\[
U \gamma^\mu U^{-1} = (\Delta^{-1})^\mu_\nu \gamma^\nu, \tag{16}
\]

and \( \Delta^a_\mu \) is a 4 × 4 Lorentz matrix transforming coordinates as \( x^\mu \to \Delta^a_\mu x^\nu \).

The equation is also invariant under the local transformation, under the condition the vector field transforms as

\[
A_\mu^a(x) I_a \to UA_\mu^a(x) I_a U^\dagger - \frac{i}{g} (\partial_\mu U) U^\dagger, \tag{17}
\]

Thus, the fermion-vector Lagrangian in Eq. 13 with the fields on Table 1, leads to the fermion electroweak standard-model Lagrangian contribution\[14,\] \[15\], also derived heuristically in Refs. \[11\] and \[13\],

\[
\bar{\Psi}_l \left[ i \partial_\mu + \frac{1}{2} g [W^a_\mu(x) - \frac{1}{2} g' g B_\mu(x)] \right] \gamma^\mu \Psi_l + \bar{\psi}_r \left[ i \partial_\mu - g' g B_\mu(x) \right] \gamma^\mu \psi_r, \tag{18}
\]
which contains a left-handed hypercharge $Y_l = -1$ $SU(2)$ doublet

$$\Psi_l(x) = \left( \begin{array}{c} \nu_L(x) \\ e_L(x) \end{array} \right),$$

(19)

with two polarization components as, e. g.,

$$\nu_L(x) = \left( \begin{array}{c} \psi^1_{\nu L}(x)e^{ip_{\nu L,1}(x)} \\ \psi^2_{\nu L}(x)e^{ip_{\nu L,2}(x)} \end{array} \right),$$

(20)

and we choose polar coordinates; a right-handed $Y_r = -2$ singlet $\psi_r$, with likewise notation, and the corresponding gauge-group vector bosons and coupling constants, $B_\mu(x)$, $W^\alpha_\mu(x)$, and $g$, $g'$, respectively.

In the following, we justify $P_f$ in Eq. (13), showing the equivalence of the spin-extended $W$-fermion vertex containing the operator $W^i_\mu(x)\gamma_0\gamma^\mu$ (the $\frac{1}{2}g$ factor hence omitted), within Eq. (13), in comparison to the conventional expression in Eq. ??.

The $(5 + 1)$-$d$ space allows for charge 0 and $-1$ components, associated to lepton (neutrino and electron) fields

$$\Psi^f_L(x) = \sum_\alpha \left( \psi^\alpha_{\nu L}(x)e^{ip_{\nu L,\alpha}(x)}\nu^\alpha_L \\ \psi^\alpha_{e L}(x)e^{ip_{e L,\alpha}(x)}e^\alpha_L \right),$$

(21)

$$\Psi^f_R(x) = \sum_\alpha \psi^\alpha_{e R}(x)e^{ip_{e R,\alpha}(x)}e^\alpha_R,$$

(22)

and the spinor-lepton components shown in Table 1 (see notation). The conventional states are assumed real and obtained within the $\gamma^\mu$-Dirac representation. We also choose polar coordinates for the components to pinpoint phase effects.

The relevant $(5 + 1)$-$d$ projection terms are

$$P_f = g_5\gamma_5 + g_I I + g_{01}\gamma^{01} + g_{02}\gamma^{02} + g_{03}\gamma^{03} + g_{12}\gamma^{12} + g_{13}\gamma^{13} + g_{23}\gamma^{23} + (g_{556}\gamma_5 + g_{156}\gamma_{156} + g_{0256}\gamma_{0256} + g_{0356}\gamma_{0356} + g_{1256}\gamma_{1256} + g_{1356}\gamma_{1356} + g_{2356}\gamma_{2356})\gamma^5\gamma^0,$$

(23)

For the extended-spin model with $\Psi^f_L(x)$, the coefficient of the $e^1_L$ associated term ($\psi^1_L(x)$)$^2$ is

$$(A - B)|W^0_0(x) - W^3_0(x)|$$

$A = \frac{1}{2}(g_I + g_5 - ig_{556} - ig_{56})$

$B = -\frac{1}{2}(g_{03} - ig_{12} - i g_{0356} - g_{1256})$

For the conventional term with $\Psi^i(x)$, the $e^1_L$ coefficient is

$\frac{1}{2}|W^0_0(x) - W^3_0(x)|$.

For the extended-spin model with $\Psi^f_L(x)$, the coefficient of the $e^2_L$ associated term ($\psi^2_L(x)$)$^2$ is

$$(A + B)|W^0_0(x) + W^3_0(x)|$$
For the conventional term with $\Psi_l(x)$, the $\epsilon_L^2$ coefficient is
\[ \frac{1}{2} [W_0^2(x) + W_1^2(x)]. \]

We conclude the choice $A = 1/2, B = 0$ equates the two expressions. Given the expressions for $A, B$, there is some freedom in the coefficients $g_i$ choice. These terms do not provide phase information, unlike cross terms:

Indeed, for the extended-spin model with $\Psi_L(x)$, the $\nu_L^1, \epsilon_L^2$ coefficients of the associated term $\psi_L^1(x)\psi_L^e(x)$ are presented for each $W_L^1(x)$:

\[
\begin{align*}
W_L^1(x) &= -\frac{1}{2}ie^{-i(p_{Lz}(x) + p_{Ll}(x))} \left[ C \left( e^{2ip_{Lz}(x)} + e^{2ip_{Ll}(x)} \right) \right. \\
W_L^2(x) &= -\frac{1}{2}ie^{-i(p_{Lz}(x) + p_{Ll}(x))} \left[ D \left( e^{2ip_{Lz}(x)} - e^{2ip_{Ll}(x)} \right) \right. \\
W_L^3(x) &= \frac{1}{2}e^{-i(p_{Lz}(x) + p_{Ll}(x))} \left[ C \left( e^{2ip_{Lz}(x)} - e^{2ip_{Ll}(x)} \right) \right. \\
W_L^4(x) &= \frac{1}{2}e^{-i(p_{Lz}(x) + p_{Ll}(x))} \left[ -C \left( e^{2ip_{Lz}(x)} - e^{2ip_{Ll}(x)} \right) \right. \\
\end{align*}
\]

where $C = g_{01} - ig_{23} - ig_{0156} - g_{2356}$
$D = -ig_{02} + g_{13} - g_{0256} - ig_{1356}$.

For the conventional term with $\Psi_l(x)$, the $\nu_L^1, \epsilon_L^2$ coefficient is

\[
\frac{1}{2}e^{-i(p_{Lz}(x) + p_{Ll}(x))} \left[ i \left( e^{2ip_{Lz}(x)} + e^{2ip_{Ll}(x)} \right) \left( W_L^1(x) + W_L^2(x) \right) + \left( e^{2ip_{Lz}(x)} - e^{2ip_{Ll}(x)} \right) \left( W_L^1(x) - W_L^2(x) \right) \right].
\]

We conclude the choice $C = -1, D = 0$ matches both terms.

While the expression in Eq. 14 is consistent with the above $A, B, C, D$ values (with overall factor linked to the normalization $N_f$ in Eq. 13), we highlight that a freedom exists for other $P_f$ choices.

Finally, this comparison was carried out under a $\gamma$-matrix choice that leads to a basis as in Eqs. [19] [20]. This required fixing the phases, to complete the identification of states. The phases are given as (to be put on Table 1 states): $\epsilon_L^1 \to -i\epsilon_L^1$ and $\epsilon_L^2 \to -i\epsilon_L^2$. One can check that this solution fits all other terms.

5 Conclusions

This paper dealt with translating a previously proposed standard-model extension, the spin-extended model, to a Lagrangian formalism, showing the correspondence of its generated Lagrangian with that of the standard model, making a specific comparison with one of its components. The final objective is to use the model’s restrictions to obtain standard-model information.
We first made a brief introduction to the model, highlighting its main features, and quoting relevant information it generated in previous references. A matrix space is used in which both symmetry generators and fields are formulated. For given dimension, a chosen non-trivial projection operator $P_P$ constrains the matrix space, determining the symmetry groups, and the arrangement of fermion and boson representations. In particular, spin-$1/2$, and $0$ states are obtained in the fundamental representation of scalar groups and spin-$1$ states in the adjoint representation. After expressing fields within this basis, a gauge-invariant field theory is constructed, based on the Lorentz and obtained scalar symmetries.

In comparison with Ref. [12], in which formal steps were carried out that relate the spin-extended model with the standard model, here we examine in detail two associated Lagrangian expressions, and extract information on the conditions for which they match. The term-by-term comparison shows special features: one is a need to fix phases, and the second is the freedom in the choice of the projection operator, all of which teaches how to match the two formalisms.

As it turns out, the Lagrangian fitting of the projection operator and the phases, done for the W-lepton term in (5+1)-d, is enough to show the equivalence of the rest of the other components, as the kinetic term, and other vertices.

Given the formalization level achieved by the spin-extended model, it is relevant to mention other of its advantages, as compared with other extensions. In particular, the chiral property of the model’s fermion representations contrasts with the difficulty to reproduce it in traditional extensions as the Kaluza-Klein theory. Moreover, while a grand-unified group limits the representations among which particles are chosen, in our case, the representations are determined by the chosen dimension and projection operator over the space. In fact, the specific combinations (spin-$1/2$)-fundamental and vector-adjoint are derived, matching the Lorentz scalar groups representations, respectively; graphically, vectors and scalars group elements occupy the same places in the array of extended space of spin, as shown in Fig 1.

The question about what sets the dimension of this extension to derive groups and representations of the standard model, equally applies to strings, as there is an infinite number of possible groups that contain the standard model. The answer for both extensions depends on whether low dimension numbers give relevant information, and on predictability, as in our case, in which derived features such as chiral $SU_L(2)$ representations.

Although the extensions of the standard model provide additional information about it, many mysteries remain unsolved. With its bottom-up approach, this model reduces the possibilities of groups and representations to describe the particles and their quantum numbers, in contrast, e. g., with those available in string theory, with its multiplicity of representation and compactification choices.

The paper’s standard-model extension satisfies basic requirement of correct symmetries, including Lorentz and gauge ones, description of standard-model particles, and field-theory formulation, in addition to its standard-model pre-
diction provision (the latter two is what the paper deals with.) This supports
the view that it is an extension worth considering.

The spin-extended model throws light on some standard model enigmas. To
the extent that this extension can be translated to the conventional field-theory
formulation of the standard model, which we show in this paper is possible, it
becomes more relevant.

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