The BSE property for vector-valued Frechet Lipschitz algebras

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Abstract

Let \((X,d)\) be a metric space with at least two elements and \((A,p_l)\) be a commutative semisimple Frechet algebra over the scalar field \(\mathbb{C}\). The correlation between the BSE-property of the Frechet algebra \((A,p_l)\) and \(\text{Lip}_d(X,A)\) is assessed. It is found and approved that if \(\text{Lip}_d(X,A)\) is a BSE-Frechet algebra, then so is \(A\). The opposite correlation will hold if \((A,p_l)\) is unital. **Keywords:** BSE-Frechet algebra, Frechet- Lipschitz algebra, Metric space.

1 Introduction and preliminaries

The class of Frechet algebras which is an important class of locally convex algebras has been widely studied by many authors. For a full study of Frechet algebras, one may see (\textsuperscript{3,5,8}). A Frechet space is a metrizable complete locally convex vector space. The topology of Frechet algebra \(A\) can be given by a sequence \((p_n)\) of increasing submultiplicative seminorms. Algebra \(A\) is called without order if \(aA = 0\) concludes that

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Let \( A \) be commutative and without order Frechet algebra and \( \Delta(A) \), be the character space of \( A \) with the Gelfand topology. In this study, \( \Delta(A) \) represents the set of all non-zero multiplicative linear functionals over \( A \). Assume that \( C_b(\Delta(A)) \) is the space consisting of all complex-valued continuous and bounded functions on \( \Delta(A) \).

A linear operator \( T \) on \( A \) is named a multiplier if \( T(xy) = xT(y) \), for all \( x, y \in A \). The set of all multipliers on \( A \) will be expressed as \( M(A) \). The strong operator topology (briefly SOT-topology) on \( M(A) \) is generated by the family of seminorms \( \{p_{x,l}\} \) defined as

\[
p_{x,l}(T) := p_l(T(x))
\]

for all \( x \in A, l \in \mathbb{N} \) and \( T \in M(A) \). If the Frechet algebra \( A \) is semisimple, then the Gelfand map \( \Gamma : A \to \hat{A}, f \mapsto \hat{f} \), is injective, or equivalently, and the following equation holds:

\[
\bigcap_{\varphi \in \Delta(A)} \ker(\varphi) = \{0\}
\]

Note that every semisimple commutative Frechet algebra is without order. As observed in [2], if the Frechet algebra \( (A, p_l) \) is semisimple, then

\[
(M(A), SOT) \cong (\hat{M(A)}, \mathcal{T}_p)
\]

Where \( \mathcal{T}_p \) is pointwise topology on \( \hat{M(A)} \).

The Bochner-Schoenberg-Eberlein (BSE) is derived from the famous theorem proved in 1980 by Bochner and Schoenberg for the group of real numbers; [11] and [10]. The researcher in [4], revealed that if \( G \) is any locally compact abelian group, then the group algebra \( L_1(G) \) is a BSE algebra. The researcher in [10], [13], [14] assessed the commutative Banach algebras that meet the Bochner-Schoenberg-Eberlein-type theorem and explained their properties. They are introduced and assessed in [12] the first and second types of BSE algebras. This concept is expanded in [6] and [7].

The researchers are introduced and assessed in [2], the concept of BSE-Frechet algebra.

The big and little Frechet \( \alpha \)-Lipschitz vector-valued algebra of order \( \alpha \), where \( \alpha \in \mathbb{R} \) with \( \alpha > 0 \) was introduced in [9]. The researchers are provided a survey of the similarities and differences between Banach and Frechet algebras include some known results and examples. (See [3]).

That the Lipschitz algebra \( \text{Lip}_\alpha(K, A) \) is a BSE-algebra if and only if \( A \) is a BSE-algebra, where \( K \) is a compact metric space, \( A \) is a commutative unital semisimple Banach algebra, and \( 0 < \alpha \leq 1 \) is proved in [11]. In this article, this result is generalized,
for any metric space \((X,d)\) and any commutative semisimple Frechet algebra \((A,p_l)\).
That the \(C_{BSE}(\Delta(\text{Lip}_d(X,A)))\) can be embedded in \(\text{Lip}_d(X,C_{BSE}(\Delta(A)))\) will be proved
in the article first, followed by proving that \(\text{Lip}_d(X,M(A)) \subseteq M(\text{Lip}_d(X,A))\). By
proving that if \(\text{Lip}_d(X,A)\) is a BSE- Frechet algebra, so is \(A\). If \((A,p_l)\) is unital Frechet
algebra and BSE- Frechet algebra, then \(\text{Lip}_d(X,A)\) is so, is assessed in this article.

1 Some basic properties of BSE- Frechet algebra

The basic terminologies and the related information on BSE-Frechet algebras are ex-
tracted from \([2]\) and prove some primary, basic results, and properties related to them.

A bounded complex-valued continuous function \(\sigma\) on \(\Delta(A)\), is named BSE-Frechet
function, if there exists a bounded set \(M\) in \(A\) and a positive real number \(\beta_M\) in a sense
that for every finite complex-number \(c_1, \ldots, c_n\) and the same many \(\varphi_1, \ldots, \varphi_n\) in \(\Delta(A)\) the following
inequality
\[
|\sum_{j=1}^{n} c_j \sigma(\varphi_j)| \leq \beta_M P_M(\sum_{j=1}^{n} c_j \varphi_j)
\]
holds; where \(P_M\) is defined as
\[
P_M(f) := \sup\{|f(x)| : x \in M\} \quad (f \in A^*)
\]
The set of all BSE- functions is expressed by \(C_{BSE}(\Delta(A))\). The BSE- seminorm of
\(\sigma \in C_{BSE}(\Delta(A)), q_l(\sigma)\), is expressed as:
\[
q_l(\sigma) = \sup\{|\sum_{i=1}^{n} c_i \sigma(\varphi_i)| : P_M(\sum_{i=1}^{n} c_i \varphi_i) \leq 1, \varphi_i \in \Delta(A), c_i \in \mathbb{C}, n \in \mathbb{N}\}
\]
where
\[
M_l := \{a \in A : p_l(a) \leq 1\}
\]
It was shown that \((C_{BSE}(\Delta(A)), q_l)\) is a semisimple commutative Frechet subalgebra of
\(C_b(\Delta(A))\). It is easy to prove that
\[
q_l(\sigma) = \inf\{\beta_M | \sum_{j=1}^{n} c_j \sigma(\varphi_j) | \leq \beta_M P_M(\sum_{j=1}^{n} c_j \varphi_j), c_j \in \mathbb{C}, \varphi_j \in \Delta(A)\}
\]
It is obvious that if \(x \in A\) then \(\hat{x} \in C_{BSE}(\Delta(A))\) and \(q_l(\hat{x}) \leq p_l(x)\), where \(\hat{x}(\varphi) = \varphi(x)\)
for all \(\varphi \in \Delta(A)\). The set \(M(A)\) with the strong operator topology, is an unital
commutative locally convex algebra. It was shown that for each $T \in M(A)$ there exists a unique bounded continuous function $\hat{T}$ on $\Delta(A)$ expressed as:

$$\varphi(Tx) = \hat{T}(\varphi)(x),$$

for all $x \in A$ and $\varphi \in \Delta(A)$. By setting $\{\hat{T} : T \in M(A)\}$, the $\widehat{M(A)}$ is yield. A commutative Frechet algebra $A$ is called BSE- Frechet- algebra if it meets the following condition:

$$\widehat{M(A)} = C_{BSE}(\Delta(A)).$$

A bounded net $\{e_\beta\}$ in $A$ is named a bounded $\Delta$- weak approximate identity for $A$ if $\varphi(ae_\beta) \to \varphi(a)$ for all $\varphi \in \Delta(A)$ and $a \in A$, equivalently $\varphi(e_\beta) \to 1$.

**Proposition 1.** Let $(A,p_l)$ be a commutative semisimple Frechet algebra. Then $\sigma \in C_{BSE}(\Delta(A))$ if and only if there exists a bounded net $\{x_\lambda\}$ in $A$ with

$$\lim \hat{x_\lambda}(\varphi) = \sigma(\varphi)$$

for all $\varphi \in \Delta(A)$.

**Theorem 2.** Let $(A,p_l)$ be a commutative semisimple Frechet algebra. Then $A$ has a bounded $\Delta$- weak approximate identity if and only if

$$\widehat{M(A)} \subseteq C_{BSE}(\Delta(A)).$$

## 2 Some basic properties of vector-valued Frechet-Lipschitz algebra

The basic terminologies and the related information on vector-valued Frechet-Lipchitz algebras are reviewed. In the sequel, some primary, basic results, and properties related to them are proved.

Throughout this section, $(X,d)$ is a metric space with at least two elements and $(A,p_l)$ is a commutative semisimple Frechet algebra over the scaler field $\mathbb{C}$. Let $f : X \to A$ be a function. Set

$$q_{l,A}(f) = \sup_{x \in X} p_l(f(x))$$

and

$$p_{l,A}(f) = \sup_{x \neq y} \frac{p_l(f(x) - f(y))}{d(x,y)}$$
The set of all functions such $f : X \to A$ satisfies in the following conditions:

i) $q_{l,A}(f) < \infty$, for each $l \in \mathbb{N}$;

ii) $p_{l,A}(f) < \infty$, for each $l \in \mathbb{N}$. 

is named the vector-valued Fréchet Lipschitz algebra and is expressed by $\text{Lip}_d(X, A)$.

Put

$$ r_{l,A}(f) = q_{l,A}(f) + p_{l,A}(f) \quad (f \in \text{Lip}_d(X, A)) $$

$C_b(X, A)$ is the set of all bounded continuous functions from $X$ into $A$. Let $f \in C_b(X, A)$. If $f, g \in C_b(X, A)$ and $\lambda \in \mathbb{C}$, define

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x) \quad (x \in X)$$

It is obvious that $(C_b(X, A), q_{l,A})$ is a Fréchet space over $\mathbb{C}$. That $(\text{Lip}_d(X, A), r_{l,A})$ is a Fréchet subalgebra of $C_b(X, A)$ is in Lemma 3.1 proved in [9].

In this article, $f_a : X \to A \quad (a \in A)$ is the constant function on $X$, where $f_a(x) = a \quad (x \in X)$. It is obvious that these functions belong to $\text{Lip}(X, A)$ and

$$ r_{l,A}(f_a) = q_{l,A}(f_a) = p_{l}(a), $$

for each $a \in A$ and $f \in \text{Lip}_d(X, A)$.

Let $(A, p_l)$ and $(B, q_l)$ be Frechet algebras. The function $\Phi : (A, p_l) \to (B, q_l)$ is called isometric, if

$$ q_l(\Phi(a)) = p_l(a) \quad (a \in A, l \in \mathbb{N}) $$

Let $K_A : A \to \text{Lip}_d(X, A)$ such that $a \mapsto f_a$. Then $K_A$ is a continuous, linear and injective function. Furthermore, $\widehat{K}_A : \Delta(\text{Lip}_d(A)) \to \Delta(A) \cup \{0\}$ is homomorphism and $A$ can be considered as a closed subalgebra of $\text{Lip}_d(X, A)$.

**Proposition 3.** Let $(X, d)$ be a metric space, $(A, p_l)$ be a commutative Frechet algebra over the scalar field $\mathbb{C}$. Then $\text{Lip}_d(X, A)$ is a semisimple Frechet algebra if and only if $A$ is so.

**Proof.** First, Assume that $A$ is semisimple and take $f, g \in \text{Lip}_d(X, A)$ such that $f \neq g$. So there exists $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Because $A$ is a semisimple algebra, there exists $\varphi \in \Delta(A)$ where

$$ \varphi(f(x_0)) \neq \varphi(g(x_0)). $$
$x_0 \otimes \varphi$ defined by $x_0 \otimes \varphi(f) = \varphi(f(x_0))$; for $f \in \text{Lip}_d(X, A)$. It is obvious that $x_0 \otimes \varphi \in \Delta(\text{Lip}_d(X, A))$ and

$$x_0 \otimes \varphi(f) \neq x_0 \otimes \varphi(g)$$

This implies that $\text{Lip}_d(X, A)$ is semisimple.

Now assume that $\text{Lip}_d(X, A)$ is a semisimple Frechet algebra. Let $a, b \in A$ where $a \neq b$, so $f_a \neq f_b$. Because $\text{Lip}_d(X, A)$ is a semisimple, there exists $\psi \in \Delta(\text{Lip}_d(X, A))$ where $\psi(f_a) \neq \psi(f_b)$. Which yield:

$$\hat{K}_A(\psi)(a) = \psi(K_A(a)) = \psi(f_a) \neq \psi(f_b) = \psi(K_A(b)) = \hat{K}_A(\psi)(b).$$

Consequently $\hat{K}_A(\psi)(a) \neq \hat{K}_A(\psi)(b)$ and $\hat{K}_A(\psi) \in \Delta(A)$. Then $\Delta(A)$ separates the points of $A$, this implies that $A$ is a semisimple Frechet algebra. \hfill \Box

**Proposition 4.** Let $(X, d)$ be a metric space, $(A, p_l)$ be a commutative Frechet algebra over the scaler field $\mathbb{C}$. Then $\text{Lip}_d(X, A)$ is without order if and only if $A$ is so.

**Proof.** Let $A$ be without order Frechet algebra and Assume that $f \in \text{Lip}_d(X, A)$ be non-zero. So there exists $x_0 \in X$ where $f(x_0) \neq 0$. Because $A$ is without order, there exists $b \in A$ where

$$f(x_0)b \neq 0.$$ 

Which yield:

$$(ff_b)(x_0) = f(x_0)f_b(x_0) = f(x_0)b \neq 0.$$ 

So $ff_b \neq 0$, therefore $\text{Lip}_d(X, A)$ is without order.

Conversely, assume that $\text{Lip}_d(X, A)$ be without order and take $a \in A$ where $a \neq 0$, so $f_a \neq 0$. Because $\text{Lip}_d(X, A)$ is without order, there exists $g \in \text{Lip}_d(X, A)$ where $f_ag \neq 0$. This follows that there exists $x_0 \in X$ where $(f_ag)(x_0) \neq 0$, thus $ag(x_0) \neq 0$ and consequently $A$ is without order. \hfill \Box

**Lemma 5.** Let $(A, p_l)$ be a commutative semisimple Frechet algebra and $(X, d)$ be a metric space. If $\text{Lip}_d(X, A)$ has a bounded $\Delta-$ weak approximate identity, then $A$ has a bounded $\Delta-$ weak approximate identity.

**Proof.** Assume that $\text{Lip}_d(X, A)$ has a bounded $\Delta-$ weak approximate identity and $(f_\beta)$ is a bounded $\Delta-$ weak approximate identity for $\text{Lip}_d(X, A)$. By allowing $\varphi \in \Delta(A)$ the following is yield:

$$\lim_{\beta} \varphi(f_\beta(x)) = \lim_{\beta} (x \otimes \varphi)(f_\beta) = 1.$$ 

because $x \otimes \varphi \in \Delta(\text{Lip}_d(X, A))$, for each $x \in X$ and $\varphi \in \Delta(A)$, thus, the net $(f_\beta(x))$ is a bounded $\Delta-$ weak approximate identity for $A$. This completes the proof. \hfill \Box
Lemma 6. Let \((A,p_l)\) and \((B,q_l)\) be a commutative Frechet algebra and \((X,d)\) be a metric space. If \(A \cong B\), as two Frechet algebras, then \(\text{Lip}_d(X,A) \cong \text{Lip}_d(X,B)\). These two as Frechet algebras are isometric.

Proof. Assume that \(\Theta : A \rightarrow B\) is an isomorphism map. Define

\[
\tilde{\Theta} : \text{Lip}_d(X,A) \rightarrow \text{Lip}_d(X,B)
\]

Where \(\tilde{\Theta}(f)(x) = \Theta(f(x))\), for all \(x \in X\) and \(f \in \text{Lip}_d(X,A)\). If \(f_1, f_2 \in \text{Lip}_d(X,A)\) and \(f_1 = f_2\), so \(f_1(x) = f_2(x)\) for each \(x \in X\). Thus \(\Theta(f_1(x)) = \Theta(f_2(x))\), then \(\tilde{\Theta}(f_1) = \tilde{\Theta}(f_2)\). This implies that \(\tilde{\Theta}\) is well-defined. At this stage, \(\tilde{\Theta}(f) \in \text{Lip}_d(X,B)\), for each \(f \in \text{Lip}_d(X,A)\) is assessed. For all \(x, y \in X\) with \(x \neq y\) the following is yield:

\[
\frac{q_l(\tilde{\Theta}(f)(x) - \tilde{\Theta}(f)(y))}{d(x,y)} = \frac{q_l(\Theta(f(x)) - \Theta(f(y)))}{d(x,y)} \leq \frac{p_l(f(x) - f(y))}{d(x,y)} \leq Kp_l(f(x) - f(y))
\]

This follows that

\[
p_l,B(\tilde{\Theta}(f)) \leq Kp_l,A(f)
\]

Moreover, for all \(x \in X\) Which yield:

\[
q_l(\tilde{\Theta}(f)(x)) = q_l(\Theta(f(x))) \leq Kp_l(f(x))
\]

This implies that \(q_l,B(\tilde{\Theta}(f)) \leq Kq_l,A(f)\), which \(K\) is an upper bound for \(\Theta\). Therefore \(\tilde{\Theta}(f) \in \text{Lip}_d(X,B)\).

In the sequel, it will be concluded that \(\tilde{\Theta}\) is injective. To that end, take \(f, g \in \text{Lip}_d(X,A)\), such that \(\tilde{\Theta}(f) = \tilde{\Theta}(g)\). So \(\Theta(f(x)) = \Theta(g(x))\), for all \(x \in X\), thus \(f(x) = g(x)\) for all \(x \in X\), because \(\Theta\) is injective. Then \(\tilde{\Theta}\) is injective. It remains to prove that \(\tilde{\Theta}\) is surjective. Assume that \(g \in \text{Lip}_d(X,B)\) and define \(f(x) = \Theta^{-1}(g(x))\), for all \(x \in X\). Which yield:

\[
\frac{p_l(f(x) - f(y))}{d(x,y)} = \frac{p_l(\Theta^{-1}(g(x)) - \Theta^{-1}(g(y)))}{d(x,y)} \leq M\frac{q_l(g(x) - g(y))}{d(x,y)}
\]

This follows that

\[
p_l,A(f) \leq Mp_l,B(g)
\]

In the same way, It will be concluded that \(q_l,A(f) \leq Mq_l,B(g)\), for some \(M > 0\). At the result \(f \in \text{Lip}_d(X,A)\) and \(\tilde{\Theta}(f)(x) = \Theta(f(x)) = \Theta(\Theta^{-1}(g(x))) = g(x)\) and thus \(\tilde{\Theta}(f) = g\). This completes the proof. \(\square\)
Lemma 7. Let \((X, d)\) be a metric space, \((A, p)\) be a commutative Frechet algebra over the scaler field \(\mathbb{C}\). Assume that \(M\) is a bounded set in \(A\), \(x \in X\), \(\varphi \in \Delta(A)\), \(c_1, \ldots, c_n \in \mathbb{C}\) and the same number \(\varphi_1, \ldots, \varphi_n \in \Delta(A)\), then the following is yield:

\[
P_{M'}\left(\sum_{i=1}^{n} c_i (x \otimes \varphi_i)\right) = P_M\left(\sum_{i=1}^{n} c_i \varphi_i\right).
\]

where \(M' = \{K_A(a) | a \in M\}\).

Proof. By allowing \(c_1, \ldots, c_n \in \mathbb{C}\) and the same number \(\varphi_1, \ldots, \varphi_n \in \Delta(A)\), the following is yield:

\[
P_{M'}\left(\sum_{i=1}^{n} c_i (x \otimes \varphi_i)\right) = \sup \left\{ \sum_{i=1}^{n} c_i (x \otimes \varphi_i)(f) \mid f \in M' \right\}
= \sup \left\{ \sum_{i=1}^{n} c_i \varphi_i(f(x)) \mid f(x) \in M \right\}
\leq \sup \left\{ \sum_{i=1}^{n} c_i \varphi_i(a) \mid a \in M \right\}
= P_M\left(\sum_{i=1}^{n} c_i \varphi_i\right).
\]

For the reverse inclusion, which yield:

\[
P_M\left(\sum_{i=1}^{n} c_i \varphi_i\right) = \sup \left\{ \sum_{i=1}^{n} c_i \varphi_i(a) \mid a \in M \right\}
= \sup \left\{ \sum_{i=1}^{n} c_i \varphi_i(f_a(x)) \mid a \in M \right\}
= \sup \left\{ \sum_{i=1}^{n} c_i (x \otimes \varphi_i)(f_a) \mid f_a \in M' \right\}
\leq \sup \left\{ \sum_{i=1}^{n} c_i (x \otimes \varphi_i)(f) \mid f \in M' \right\}
= P_{M'}\left(\sum_{i=1}^{n} c_i (x \otimes \varphi_i)\right)
\]

Consequently,

\[
P_{M'}\left(\sum_{i=1}^{n} c_i (x \otimes \varphi_i)\right) = P_M\left(\sum_{i=1}^{n} c_i \varphi_i\right).
\]

\(\square\)
3 Main results

The structure of the BSE functions on $\Delta(\text{Lip}_d(X, A))$ is characterization and the correlations between the BSE property of $A$ and $\text{Lip}_d(X, A)$ are assessed.

Let $f \in \text{Lip}_d(X, A)$, define $r'_{l,A}(f) = \max\{p_{l,A}(f), q_{l,A}(f)\}$. It is obvious that $r'_{d,A}$ is a seminorm on $\text{Lip}_d(X, A)$. Clearly

\[(\text{Lip}_d(X, A), r_{l,A}) \cong (\text{Lip}_d(X, A), r'_{l,A})\]

**Proposition 8.** Let $(X, d)$ be a metric space and $(A, p_l)$ be a commutative semisimple Fréchet algebra. Assume that $\text{Lip}_d(X, A)$ is a BSE- Frechet- algebra. Then $A$ is so.

**Proof.** Because $\text{Lip}_d(X, A)$ is a BSE- algebra, by referring to Theorem 2, $\text{Lip}_d(X, A)$ has a bounded $\Delta-$ weak approximate identity. Lemma 5 and Theorem 2 implies that $\hat{\mathcal{M}}(A) \subseteq \mathcal{C}_{\text{BSE}}(\Delta(\text{Lip}_d(X, A))).$

For the reverse inclusion, take $\sigma \in \mathcal{C}_{\text{BSE}}(\Delta(\text{Lip}_d(X, A)))$. There exist a bounded set $M$ in $A$ and a positive real number $\beta_M$ where by allowing $\psi_1, \ldots, \psi_n$ of $\Delta(\text{Lip}_d(X, A))$ and the same number of complex numbers $c_1, \ldots, c_n$, the following is yield:

\[
\left| \sum_{i=1}^{n} c_i \sigma o \hat{\mathcal{K}}_A(\psi_i) \right| = \left| \sum_{i=1}^{n} c_i \sigma(\psi_i o \mathcal{K}_A) \right| \\
\leq \beta_M P_M \left( \sum_{i=1}^{n} c_i(\psi_i o \mathcal{K}_A) \right) \\
\leq \beta_M K P_M' \left( \sum_{i=1}^{n} c_i \psi_i \right)
\]

for some $K > 0$, where $M' = \{K_A(a) | a \in M\}$. It follows that $\sigma o \hat{\mathcal{K}}_A \in \mathcal{C}_{\text{BSE}}(\Delta(\text{Lip}_d(X, A)))$.

By applying the BSE- property of $\text{Lip}_d(X, A)$, there exists $T \in M(\text{Lip}_d(X, A))$ where $\hat{T} = \sigma o \hat{\mathcal{K}}_A$. Now define $T' \in M(A)$ as follows:

\[T'(a) = T(K_A(a))(x_0), \quad (a \in A)\]

where $x_0 \in X$ is an arbitrary member of $X$. If $a_1, a_2 \in A$;

\[T'(a_1 a_2) = T(K_A(a_1 a_2))(x_0) = T(K_A(a_1)K_A(a_2))(x_0) = T(K_A(a_1))(x_0)K_A(a_2)(x_0) = T'(a_1).a_2\]
Hence $T' \in M(A)$. Let $\varphi \in \Delta(A)$; It is easy to prove that $\hat{K}_A(x_0 \otimes \varphi) = \varphi$ and the following is yield:

$$
\hat{T}'(\varphi) = \frac{\varphi(T'(a))}{\varphi(a)} = \frac{\varphi(T(K_A(a))(x_0))}{\varphi(a)} = \frac{(x_0 \otimes \varphi)(T(K_A(a)))}{\varphi(f_a(x_0))} = \frac{(x_0 \otimes \varphi)(T(K_A(a)))}{(x_0 \otimes \varphi)(K_A(a))} = \hat{T}(x_0 \otimes \varphi) = \sigma\hat{K}_A(x_0 \otimes \varphi) = \sigma(\hat{K}_A(x_0 \otimes \varphi)) = \sigma(\varphi)
$$

Therefore $\hat{T}' = \sigma$ and consequently $C_{BSE}(\Delta(A)) \subseteq \widehat{M}(A)$. Thus $A$ is a Frechet- BSE-algebra.

The correlation between the $C_{BSE}(\Delta(\operatorname{Lip}_d(X,A)))$ and $\operatorname{Lip}_d(X,C_{BSE}(\Delta(A)))$ is assessed as follows:

**Theorem 9.** Let $(X,d)$ be a metric space, $(A,p_l)$ be a commutative semisimple Frechet algebra. Then $C_{BSE}(\Delta(\operatorname{Lip}_d(X,A)))$ can be embedded in $\operatorname{Lip}_d(X,C_{BSE}(\Delta(A)))$. These two as Frechet algebras are isometric; 

**Proof.** Let 

$$
\phi : C_{BSE}(\Delta(\operatorname{Lip}_d(X,A))) \to \operatorname{Lip}_d(X,C_{BSE}(\Delta(A))),
$$

defined by 

$$
\phi(\Sigma) = \phi_\Sigma \quad (\Sigma \in C_{BSE}(\Delta(\operatorname{Lip}_d(X,A))),
$$

Where 

$$
\phi_\Sigma(x)(\varphi) = \Sigma(x \otimes \varphi), \quad (x \in X, \varphi \in \Delta(A))
$$

Assume that $\Sigma_1, \Sigma_2 \in C_{BSE}(\Delta(\operatorname{Lip}_d(X,A)))$, where $\Sigma_1 = \Sigma_2$. So $\Sigma_1(x \otimes \varphi) = \Sigma_2(x \otimes \varphi)$, at the result $\phi_{\Sigma_1}(x)(\varphi) = \phi_{\Sigma_2}(x)(\varphi)$, for all $x \in X$ and $\varphi \in \Delta(A)$. Then $\phi_{\Sigma_1} = \phi_{\Sigma_2}$ and, therefore, $\phi$ is well defined. It is obvious that $\phi$ is
linear. Let $\Sigma_1, \Sigma_2 \in C_{BSE}(\Delta(\text{Lip}_d(X, A)))$, so $\phi(\Sigma_1, \Sigma_2) = \phi_{\Sigma_1, \Sigma_2}$, By allowing $x \in X$ and $\varphi \in \Delta(A)$, the following is yield:

$$
\phi_{\Sigma_1, \Sigma_2}(x)(\varphi) = \Sigma_1(x \otimes \varphi).
$$

Thus $\phi_{\Sigma_1, \Sigma_2}(x) = \phi_{\Sigma_1}(x) \cdot \phi_{\Sigma_2}(x)$, so $\phi_{\Sigma_1, \Sigma_2} = \phi_{\Sigma, \Sigma}$. Then $\phi$ is homomorphism. First of all, $\phi_{\Sigma}(x) \in C_{BSE}(\Delta(A))$, for each $x \in X$ and $\Sigma \in C_{BSE}(\Delta(\text{Lip}_d(X, A)))$ is assessed. In fact, Since $\Sigma \in C_{BSE}(\Delta(\text{Lip}_d(X, A)))$, so there exists a bounded set $M$ in $\text{Lip}_d(X, A)$ such that for every complex number $c_1, \cdots, c_n$ and the same number $\varphi_1, \cdots, \varphi_n \in \Delta(A)$, we have

$$
P_M\left(\sum_{i=1}^{n} c_i(x \otimes \varphi_i)\right) = \sup\left\{ |\sum_{i=1}^{n} c_i(x \otimes \varphi_i)(f)| : f \in M \right\}
\leq \sup\left\{ |\sum_{i=1}^{n} c_i\varphi_i| : a \in M' \right\}
= P_{M'}\left(\sum_{i=1}^{n} c_i\varphi_i\right).
$$

Where $M' := \hat{x}(M)$. This implies that

$$
|\sum_{i=1}^{n} c_i\phi_{\Sigma}(x)(\varphi_i)| = |\sum_{i=1}^{n} c_i\Sigma(x \otimes \varphi_i)|
\leq q_l(\Sigma)P_M\left(\sum_{i=1}^{n} c_i \otimes \varphi_i\right)
= q_l(\Sigma)P_{M'}\left(\sum_{i=1}^{n} c_i\varphi_i\right).
$$

Hence $\phi_{\Sigma}(x) \in C_{BSE}(\Delta(A))$ and $q_l(\Sigma) \leq q_l(\phi_{\Sigma}(x))$, for each $x \in X$, since $q_l(\Sigma) = \inf\left\{ \beta_M \mid |\sum_{j=1}^{n} c_j \Sigma(\varphi_j)| \leq \beta_M P_M(\sum_{j=1}^{n} c_j \varphi_j) \right\}$. In the other hand

$$
q_l(\phi_{\Sigma}(x)) = \sup\left\{ |\sum_{i=1}^{n} c_i(\phi_{\Sigma}(x))(\varphi_i)| : P_M\left(\sum_{i=1}^{n} c_i\varphi_i\right) \leq 1, \varphi_i \in \Delta(A) \right\}
= \sup\left\{ |\sum_{i=1}^{n} c_i\Sigma(x \otimes \varphi_i)| : P_M\left(\sum_{i=1}^{n} c_i(x \otimes \varphi_i)\right) \leq 1, \varphi_i \in \Delta(A) \right\}
\leq q_l(\Sigma)
$$

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Therefore
\[ q_l(\phi_\Sigma(x)) \leq q_l(\Sigma). \]
Consequently, for all \( x \in X, \Sigma \in C_{BSE}(\Delta(\text{Lip}_d(X, A))) \) and \( l \in \mathbb{N} \), we have
\[ q_l(\phi_\Sigma(x)) = q_l(\Sigma). \tag{1} \]
Note that
\[
q_l(\phi_\Sigma(x) - \phi_\Sigma(y)) = \sup \left\{ \left| \sum_{i=1}^n c_i (\phi_\Sigma(x) - \phi_\Sigma(y)) (\varphi_i) \right| : P_{M_l} \left( \sum_{i=1}^n c_i \varphi_i \right) \leq 1, \varphi_i \in \Delta(A) \right\}
\]
which implies that \( \phi_\Sigma \) is isometry. This completes the proof.

Let \( T \in M(A) \). Define
\[ q'_l(T) = \sup \{ p_l(T(a)) : a \in A, p_l(a) \leq 1 \}. \]
It is obvious that \( q'_l \) is a seminorm on \( M(A) \). In the following theorem, it will be shown that \( \text{Lip}_\alpha(X, A) \) can be embedded in \( M(\text{Lip}_d(X, A)) \), isometrically as two locally convex algebras which are isometric.
Theorem 10. Let \((X, d)\) be a metric space and \((A, p_t)\) be a commutative semisimple Frechet algebra. Then

\[ \text{Lip}_d(X, M(A)) \subseteq M(\text{Lip}_d(X, A)), \]

As two locally convex algebras which are isometric.

Proof. Let

\[ \phi : \text{Lip}_d(X, M(A)) \rightarrow M(\text{Lip}_d(X, A)) \]

Where

\[ \phi(F) = \phi_F \quad (F \in \text{Lip}_d(X, M(A))). \]

Defined by

\[ \phi_F(g) = F \odot g \quad (g \in \text{Lip}_d(X, A)) \]
\[ F \odot g(x) = F(x)(g(x)) \quad (x \in X). \]

It will be concluded that \(\phi\) is an isomorphism map. Assume that \(F_1, F_2 \in \text{Lip}_d(X, M(A))\) where \(F_1 = F_2\), so \(F_1(x) = F_2(x)\), for each \(x \in X\). Thus \(F_1(x)(g(x)) = F_2(x)(g(x))\), for all \(g \in \text{Lip}_d(X, A)\), then \(F_1 \odot g(x) = F_2 \odot g(x)\), for all \(x \in X\) and \(g \in \text{Lip}_d(X, A)\). Therefore \(\phi(F_1) = \phi(F_2)\) and so \(\phi\) is well-defined.

1) \(\phi_F\) is a continuous linear multiplier on \(\text{Lip}_\alpha(X, A)\) is assessed in the following:
Assume that \(g_1, g_2 \in \text{Lip}_d(X, M(A))\), so

\[ \phi_F(g_1 g_2) = F \odot g_1 g_2. \]

for any \(x \in X\), the following is yield:

\[ F \odot g_1 g_2(x) = F(x)(g_1 g_2(x)) \]
\[ = F(x)(g_1(x)g_2(x)) \]
\[ = g_1(x)F(x)(g_2(x)) \]
\[ = g_1(x)F \odot g_2(x) \]

This implies that

\[ F \odot g_1 g_2 = g_1. (F \odot g_2). \]

Then \(\phi_F(g_1 g_2) = g_1. \phi_F(g_2)\), for all \(g_1, g_2 \in \text{Lip}_d(X, M(A))\), at the result \(\phi_F\) is a multiplier. It is obvious that for each \(F \in \text{Lip}_d(X, M(A))\), the map \(\phi_F\) is linear. In the
sequel, that $\varphi_F$ is a continuous map will be proved. Let $(g_n)$ be a sequence in $\text{Lip}_d(X, A)$ converges to $g \in \text{Lip}_d(X, A)$. Then $r_{l,A}(g_n) \to r_{l,A}(g)$, thus $q_{l,A}(g_n) \to q_{l,A}(g)$ and so $p_{l,A}(g_n) \to p_{l,A}(g)$. Which yield:

$$q_{l,A}(F \circ g_n - F \circ g) = \sup \{ p_l(F(x)(g_n(x) - g(x))) : x \in X \} \leq K \sup \{ p_l(g_n(x) - g(x)) : x \in X \} = Kq_{l,A}(g_n - g) \to 0,$$

Which

$$K := q_{l,M(A)}(F) = \sup \left\{ q'_l(F(x)) \mid x \in X \right\}$$

Note that

$$q'_l(F(x)) = \sup \{ p_l(F(x))(a) \mid a \in A, p_l(a) \leq 1 \}.$$  

This shows that

$$q_{l,A}(F \circ g_n - F \circ g) \to 0.$$

Also:

$$p_{l,A}(F \circ g_n - F \circ g) = \sup \left\{ \frac{p_l(F(x)(g_n(x) - g(x)) - F(y)(g_n(y) - g(y)))}{d(x, y)} : x \neq y \right\} \leq \sup \left\{ \frac{p_l(F(x)(g_n(x) - g(x)) - (g_n(y) - g(y)))}{d(x, y)} : x \neq y \right\} + \sup \left\{ \frac{p_l(F(x) - F(y))}{d(x, y)} (g_n(y) - g(y)) : x \neq y \right\} \leq q_{l,M(A)}(F)p_{l,A}(g_n - g) + p_{l,M(A)}(F)q_{l,A}(g_n - g) \to 0$$

Which yield:

$$p_{l,A}(F \circ g_n - F \circ g) \to 0.$$

Therefore $r_{l,A}(F \circ g_n - F \circ g) \to 0$. Hence $\varphi_F$ is a continuous map. This follows that $\varphi_F \in M(\text{Lip}_d(X, A))$.

2) In following, it will be concluded that $\varphi$ is an isomorphism map.

It is obvious that $\varphi$ is a linear map. Assume that $(F_n)$ is a sequence in $\text{Lip}_d(X, M(A))$ which converges to some $F \in \text{Lip}_d(X, M(A))$, so $p_{l,M(A)}(F_n - F) \to 0$ and $q_{l,M(A)}(F_n - F) \to 0$. yielding the following:

$$q'_l(\phi(F_n) - \phi(F)) = \sup \{ r_{\alpha,A}(F_n \circ g - F \circ g) : g \in \text{Lip}_d(X, A) \}$$
Hence
\[ q_{t,A}(F_n \circ g - F \circ g) = \sup \{ p_t((F_n(x) - F(x))g(x)) : x \in X \} \leq \sup_s \sup \{ p_t((F_n - F)(x)(a)) : a \in A, x \in X \} = \sup \{ q'_t((F_n - F)(x)) : x \in X \} = q_{t,M(A)}(F_n - F) \to 0 \]

Thus
\[ q_{t,A}(F_n \circ g - F \circ g) \to 0. \]

Moreover
\[ p_{t,A}(F_n \circ g - F \circ g) = \sup \{ \frac{p_t((F_n - F)(x)(g(x)) - ((F_n - F)(y)(g(y))))}{d(x,y)} : x \neq y \} \leq \sup \{ \frac{p_t((F_n - F)(x) - (F_n - F)(y))(g(x))}{d(x,y)} : x \neq y \} + \sup \{ \frac{(p_t(F_n - F)(y))(g(x) - g(y))}{d(x,y)} : x \neq y \} \leq p_{t,M(A)}(F_n - F).q_{t,A}(g) + q_{t,M(A)}(F_n - F).p_{t,A}(g) \to 0, \]
and so
\[ p_{t,A}(F_n \circ g - F \circ g) \to 0. \]

It follows that \( r_{t,A}(\phi_{F_n}(g) - \phi_F(g)) \to 0 \), for all \( g \in \text{Lip}_d(X,A) \). Consequently \( q''_{t}(\phi(F_n) - \phi(F)) \to 0 \). Therefore \( \phi \) is a continuous map.

In this stage, that \( \phi \) is injective is assessed. Assume that \( F \in \text{Lip}_d(X,M(A)) \) where \( \phi_F = \phi(F) = 0 \). If \( F \neq 0 \), then there exists \( x_0 \in X \) where \( F(x_0) \neq 0 \), so there exists \( a_0 \in A \) where \( (F(x_0))(a_0) \neq 0 \). Put \( g = f_{a_0} \), thus \( g \in \text{Lip}_d(X,A) \) and
\[ F \circ g(x_0) = F(x_0)(g(x_0)) = (F(x_0))(a_0) \neq 0 \]
i.e.
\[ \phi_F(g)(x_0) = F \circ g(x_0) \neq 0 \]
This is a contradiction. Therefore \( \phi \) is injective.

At this stage, based on the established prerequisite the primary Theorem, is expressed as follows:
Theorem 11. Let \((X,d)\) be a metric space and \((A,p)\) be a commutative semisimple Frechet algebra. Then \(\text{Lip}_d(X,A)\) is a Frechet-BSE-algebra if and only if \(A\) is a Frechet-BSE algebra. Then

1) If \(\text{Lip}_d(X,A)\) is a Frechet-BSE-algebra, then \(A\) is a Frechet-BSE-algebra.
2) If \(A\) is unital and Frechet-BSE-algebra, then \(\text{Lip}_d(X,A)\) is a Frechet-BSE-algebra.

Proof. 1) If \(\text{Lip}_d(X,A)\) is a Frechet-BSE-algebra, then by Proposition 8, \(A\) is a Frechet-BSE-algebra.

2) Assume that \(A\) is a BSE-algebra. Since \(A\) is semisimple, then by applying Proposition 3, \(\text{Lip}_d(X,A)\) is semisimple. By applying Proposition 8 imply that

\[
(M(\text{Lip}_d(X,A)) \subseteq C_{\text{BSE}}(\Delta(\text{Lip}_d(X,A))).
\]

For the reverse inclusion, according to Theorem 9 and Theorem 10, the following is yield:

\[
C_{\text{BSE}}(\Delta(\text{Lip}_d(X,A))) \subseteq \text{Lip}_d(X,C_{\text{BSE}}(\Delta(A)))
\]

\[
\cong \text{Lip}_d(X,\hat{M}(A))
\]

\[
= \text{Lip}_d(X,M(A))
\]

\[
\subseteq M(\text{Lip}_d(X,A))
\]

\[
\cong (M(\text{Lip}_d(X,A))
\]

Thus

\[
C_{\text{BSE}}(\Delta(\text{Lip}_d(X,A))) \cong (M(\text{Lip}_d(X,A))).
\]

\[\square\]

Because every commutative \(C^*\)-Banach algebra is BSE algebra, [13], so by using Theorem [11] the following example is immediate:

Example 1. Let \((X,d)\) be a metric space and \(A\) be a commutative \(C^*\)-Banach algebra. Then \(\text{Lip}_d(X,A)\) is a BSE-Frechet algebra

References

[1] F. Abtahi, Z. Kamali, M. Toutounchi, The Bochner-Schoenberg-Eberline type property for vector-valued Lipschitz algebras. J. Math. Anal. Appl. (2019).
[2] M. Amiri and A. Rejali, The Bochner · Schoenberg · Eberline property for commutative Frechet algebras. arXiv:2012.06388 [math.FA].

[3] Z. Alimohammadi and A. Rejali, Frechet algebras in abstract harmonic analysis. arXiv: 1811.10981v1 [math.FA] 27 Nov (2018)

[4] W. F. Eberlein, Characterizations of Fourier- Stieltjes transforms, Duke Math. J., 22(1955), 465- 468.

[5] H. Goldmann, Uniform Frechet Algebras, North- Holland Mathematics Studies, 162. Northholland, Amsterdam- New York, 1990.

[6] J. Inoue and S-E. Takahasi, On characterization of image of the Gelfand transform of commutative Banach algebras, Math. Nachr, 280 (2007), 105- 129.

[7] E. Kaniuth and A. Ulger, The Boghner · Schoenberg · Eberline property for commutative Banach algebras, especially Fourier- Stieltjes algebras, Trans. Amer. Math. Soc., 362(2010), 4331- 4356.

[8] A. Ya. Helemskii, The homology of Banach and topological algebras (Moscow University Press, English transl: Kluwer Academic Publishers, Dordrecht 1989).

[9] A. Ranjbari, A. Rejali, Frechet α -Lipschitz vector- valued operator algebra, U.P.B. Sci. Bull., series A, Vol 80, Iss.4,(2018), 141-152

[10] I.J. Schoenberg, A remark on the preceding note by Boghner, Bull.Amer Math. Soc., 40(1934),277-278.

[11] S. Bochner, A theorem on Fourier- Stieltjes integrals, Bull.Amer Math. Soc., 40(1934),271-276.

[12] S.E.Takahasi and O.Hatori, Commutative Banach algebras Which satisfy a Boghner- Schonberg- Eberlein- type theorem, Proc. Amer. Math. Soc., 110(1990), 149- 158.

[13] S.E.Takahasi and O.Hatori, Commutative Banach algebras and BSE- inequalities, Math. Japonica, 37(1992), 47- 52.

[14] S.E.Takahasi and O.Hatori and K. Tanahashi, Commutative Banach algebras and BSE- norm, Math. Japonica, 46(1997), 59- 80.