A SERRIN-TYPE PROBLEM WITH PARTIAL KNOWLEDGE OF THE DOMAIN

SERENA DIPIERRO, GIORGIO POGGESI, AND ENRICO VALDINOCI

To Matilde Aurora Alessi, for inspiring us with a question on heating devices

Abstract. We present a quantitative estimate for the radially symmetric configuration concerning a Serrin-type overdetermined problem for the torsional rigidity in a bounded domain \( \Omega \subset \mathbb{R}^N \), when the equation is known on \( \Omega \setminus \omega \) only, for some open subset \( \omega \subset \Omega \).

The problem has concrete motivations in optimal heating with malfunctioning, laminar flows and beams with small inhomogeneities.

1. Introduction

In this article we consider a variation of the classical Serrin’s overdetermined problem \cite{Ser71} in which the equation is only known in a subset of the domain. We will provide quantitative results showing, roughly speaking, that when the part of the domain in which we do not have information is “small”, then the domain is “close” to a ball.

1.1. Statement of the problem and main result. The precise problem that we consider may be stated as follows. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain – that is a bounded, open, connected set, whose boundary will be denoted by \( \Gamma \), and let \( \omega \subset \Omega \) be an open (not necessarily connected) subset of \( \Omega \) with boundary denoted by \( \partial \omega \). We consider the following problem:

\[
\begin{aligned}
\Delta u &= 1 \quad \text{in } \Omega \setminus \omega, \\
u
u

u &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]

(1.1)

under the overdetermined condition

\[
u
\begin{equation}
\begin{aligned}
u
\end{aligned}
\end{equation}
\]

(1.2)

for some \( c \in \mathbb{R} \). Here and in what follows \( \nu \) denotes the outward unit normal of \( \Omega \setminus \omega \) and \( u_\nu \) the derivative of \( u \) in the direction \( \nu \). Concerning the setting in (1.2), we remark that even without explicitly imposing any regularity assumptions on \( \Gamma \), \cite[Theorem 1]{Vog92} guarantees that

\[
\text{if (1.2) holds true (in the appropriate weak sense)}
\]

(1.3)

then \( \Gamma \) is of class \( C^{2,\alpha} \), with \( 0 < \alpha \leq 1 \),

therefore the notation \( u_\nu \) on \( \Gamma \) is well posed in the classical sense, being \( u \in C^{2,\alpha}((\Omega \setminus \omega) \cup \Gamma) \) by standard elliptic regularity theory.

We will further assume \( u \) to be of class \( C^2 \) up to \( \partial \omega \), and hence

\[
u
\begin{equation}
\begin{aligned}
u
\end{aligned}
\end{equation}
\]

(1.4)

(see Section 3 for details on this).

Concerning the regularity of the domain taken into account, to avoid unessential technicalities we will first assume that

\[
\partial \omega \text{ is of class } C^1
\]

(1.5)

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Ω \ \overline{\omega} satisfies the “uniform interior sphere condition”,
i.e. there exists \( r_i > 0 \) such that for each \( p \in \Gamma \cup \partial \omega \)
there exists a ball contained in \( \Omega \setminus \overline{\omega} \) of radius \( r_i \)
such that its closure intersects \( \Gamma \cup \partial \omega \) only at \( p \).

(1.6)

We recall, for instance, that domains with \( C^{1,1} \) boundaries satisfy (1.6), see e.g. [ROV16, Lemma A.1].

To state our main result we introduce some notation. For a given domain \( D \subset \mathbb{R}^N \), we
denote by \( |D| \) and \( |\partial D| \) the \( N \)-dimensional Lebesgue measure of \( D \) and the surface measure
of \( D \), respectively. Our main result aims at considering a convenient point \( z \) and at obtaining
suitable bounds on the “pseudo-distance”

(1.7)

\[ \int_\Gamma \left| \frac{|x - z|}N - c \right|^2 dS_x \]

and on the “asymmetry”

(1.8)

\[ \frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|}, \]

where \( \Omega \Delta B_{Nc}(z) \) denotes the symmetric difference of \( \Omega \) and the ball \( B_{Nc}(z) \) of radius \( Nc \)
centered at \( z \). In addition, we provide a “geometric” bound on the set by estimating the
difference between the largest ball centered at \( z \) contained in \( \Omega \) and the smallest ball centered
at \( z \) that contains \( \Omega \).

The precise result that we have here goes as follows:

**Theorem 1.1.** Let \( \Omega \setminus \overline{\omega} \subset \mathbb{R}^N \) be a bounded domain satisfying assumptions (1.5) and (1.6). Let \( u \) satisfy (1.1), (1.2) and (1.4). Assume that \( u \leq 0 \) on \( \partial \omega \). Set

(1.9)

\[ z := \frac{1}{|\Omega \setminus \overline{\omega}|} \left\{ \int_{\Omega \setminus \overline{\omega}} x \, dx - N \int_{\partial \omega} u \, \nu \, dS_x \right\}. \]

Then,

(1.10)

\[ \int_\Gamma \left| \frac{|x - z|}N - c \right|^2 dS_x \leq C |\partial \omega| \]

and

(1.11)

\[ \frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|} \leq C |\partial \omega|^{1/2}. \]

In (1.10) and (1.11), the quantity \( C \) denotes a positive constant depending only on the di-

dimension \( N \), on the internal radius \( r_i \) in (1.6), on the diameter of \( \Omega \), and on \( \|u\|_{C^2(\partial \omega)} \).

Moreover, if

(1.12)

\[ z \in \Omega, \]

then there exist \( \rho_e \geq \rho_i > 0 \) such that

(1.13)

\[ B_{\rho_i}(z) \subset \Omega \subset B_{\rho_e}(z) \]

and

(1.14)

\[ \rho_e - \rho_i \leq C |\partial \omega|^{\tau N/2}, \]

where \( C \) is a positive constant depending only on the dimension \( N \), on the internal radius \( r_i \)
in (1.6), on the diameter of \( \Omega \) and on \( \|u\|_{C^2(\partial \omega)} \), and \( \tau_N > 0 \) is a constant depending only
on \( N \).
We point out that Theorem 1.1 collects three different aspects on the problem: indeed, the statement in (1.10) is an “integral” stability result in $L^2$ (see Section 5), the estimate in (1.11) provides a stability “in measure”, and the result in (1.14) deals with a “pointwise” notion of stability.

The exponent $\tau_N$ in (1.14) can be better precised. Indeed, we can take

$$\tau_2 := 1.$$  

Moreover, $\tau_3$ can be taken arbitrarily close to one, in the sense that for any $\theta > 0$, we have that (1.14) holds with $\tau_3 := 1 - \theta$ and $C$ depending also on $\theta$.

Furthermore, for $N \geq 4$, we can take

$$\tau_N := \frac{2}{N-1}.$$  

We think that it is an interesting open problem to detect whether these choices of exponents $\tau_2$, $\tau_3$, $\tau_N$ in (1.14), as well as the exponents appearing in (1.10) and (1.11), are optimal in Theorem 1.1. It would also be very interesting to have explicit examples to check the optimality of the structural assumptions in Theorem 1.1.

We also point out that condition (1.12) is naturally satisfied in many concrete situations (see also Remark 7.2): in particular, for “small” $\omega$, the point $z$ is “close” to the baricenter of $\Omega$, hence condition (1.12) is fulfilled in this case when the baricenter of $\Omega$ lies in $\Omega$ (as it happens, for instance, for convex sets).

Of course, when $\omega = \emptyset$, we have that (1.14) reduces to $\rho_e = \rho_i$ and therefore (1.13) gives that $\Omega$ is a ball: in this sense Theorem 1.1 recovers the classical results of [Ser71, Wei71] for overdetermined problems. The main difference here is that, differently from the existing literature, the equation is supposed to hold possibly only outside a subdomain $\omega$: as a counterpart, Theorem 1.1 does not prove that the full domain is a ball, but only that it is geometrically close to a ball whenever the subdomain $\omega$ has a small Lebesgue measure.

In the present setting, Theorem 1.1 will be in fact a particular case of more general quantitative results, presented in Section 7, and relying on a number of auxiliary integral identities. For more details, see Theorems 7.1, 7.3, 7.4, and 7.6.

In this spirit, Theorem 1.1 falls within the broad stream of research aiming at obtaining quantitative rigidity results, see e.g. [ABR99, BNST08, CMV16, CV18, Fel18, Mag17, MP19, MP17, MP20a, Pog18] and the references therein. More generally, overdetermined problems have been also considered e.g. in [AB98, CV19, EPS09, FV10a, FV10b, FV12, FV13, FG08, FGK06, FGLP09, Gre01, Gre03, GL89, GS99, GX19, PS89, Pog19] and in the references therein.

1.2. Comments on the structural assumptions and generalizations. We remark that assumption (1.6) gives a lower bound on the distance not only between $\partial \omega$ and $\Gamma$ (being $\text{dist}(\partial \omega, \Gamma) \geq 2r_i$), but also between the boundaries of the different connected components of $\omega$ (if any).

Interestingly, assumptions (1.5) and (1.6) can be relaxed, as explained in Section 8.

In particular, suitable counterparts of (1.10) and (1.11) can be obtained if (1.5) and (1.6) are replaced by the weaker assumptions

\begin{align*}
\text{(1.15)} & \quad \Omega \setminus \varpi \text{ is a John domain} \\
\text{(1.16)} & \quad \Omega \setminus \varpi \text{ is of finite perimeter.}
\end{align*}
Moreover, a counterpart of the pointwise estimate (1.14) can be obtained if (1.15) and (1.16) are dropped and replaced by (1.17), (1.16), and the assumption
\[
\Omega \setminus \overline{\omega} \text{ satisfies the “uniform interior sphere condition only on } \Gamma' , \text{ i.e., there exists } r_i > 0 \text{ such that for each } p \in \Gamma \text{ there exists a ball contained in } \Omega \setminus \overline{\omega} \text{ of radius } r_i \text{ such that its closure } \Gamma \cup \partial \omega \text{ only at } p \text{ on } \Gamma. 
\]

We stress that (1.17) is equivalent to assume a lower bound only for dist(\partial \omega, \Gamma). Indeed, being } \Gamma \text{ of class } C^{2,\alpha}, \text{ the set } \Omega \text{ surely satisfies the uniform interior sphere condition on } \Gamma. 

In this situation, Theorem 1.1 remains valid, with the following structural modifications:

- The boundary measure of \partial \omega \text{ is replaced by its perimeter, namely by the } (N - 1)\text{-dimensional Hausdorff measure } \mathcal{H}^{N-1}(\partial^* \omega) \text{ of its reduced boundary } \partial^* \omega. \text{ In turn, } \nu \text{ on } \partial^* \omega \text{ has to be intended as the (measure-theoretic) outer unit normal (see Section 8).}
- The constants } C \text{ depend on } r_i \text{ defined in (1.17) and on the structural constant } b_0 \text{ of the given } b_0\text{-John domain;}
- The explicit expression of the exponents } \tau_N \text{ in the pointwise estimate (1.14) is possibly worse than the ones obtained in Theorem 1.1.}

The definition and details for } b_0\text{-John domains can be found in Subsection 8.1. Here, we just stress that the class of John domains is huge: in particular, if (1.6) is satisfied then } \Omega \setminus \overline{\omega} \text{ is surely a } b_0\text{-John domain with } b_0 \leq d_{\Omega}/r_i \text{ (see [Pog18, (iii) of Remark 3.12]).}

The precise statement that we have in this framework is stated next, and can be deduced from more general results presented in Theorems 8.3, 8.6, 8.9 and 8.10.

**Theorem 1.2.** Let } \Omega \setminus \overline{\omega} \subset \mathbb{R}^N \text{ be a bounded domain satisfying assumptions (1.15) and (1.16). Let } u \text{ satisfy (1.1), (1.2) and (1.4). Assume that } u \leq 0 \text{ on } \partial \omega. \text{ Set}
\[
z := \frac{1}{|\Omega \setminus \overline{\omega}|} \left\{ \int_{\Omega \setminus \overline{\omega}} x \, dx - N \int_{\partial \omega} u \nu \, d\mathcal{H}^{N-1} \right\}.
\]

Then, the pseudodistance defined in (1.7) and the asymmetry defined in (1.8) satisfy
\[
\int_{\Gamma} \left| \frac{x - z}{N} - c \right|^2 \, d\mathcal{H}^{N-1} \leq C \mathcal{H}^{N-1}(\partial^* \omega),
\]
\[
\frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|} \leq C \mathcal{H}^{N-1}(\partial^* \omega)^{1/2},
\]

where the constants } C > 0 \text{ appearing in (1.18) and (1.19) depend only on } N, \ b_0, \ d_{\Omega}, \ c, \ \text{ and } \|u\|_{C^2(\partial \omega)}. 

If in addition (1.17) is verified and } z \in \Omega, \text{ then there exist } \rho_c \geq \rho_i > 0 \text{ such that }
\[
B_{\rho_c}(z) \subset \Omega \subset B_{\rho_i}(z)
\]

and
\[
\rho_c - \rho_i \leq C \left( \mathcal{H}^{N-1}(\partial^* \omega) \right)^{\tau_N/2},
\]

where } C > 0 \text{ is a constant depending only on } N, \ \text{ the internal radius } r_i \text{ in (1.17), } d_{\Omega}, \ b_0, \ \text{ and } \|u\|_{C^2(\partial \omega)}. \text{ The exponents } \tau_N > 0 \text{ depend only on } N.

We point out that in Theorem 1.2 we maintained the “same” choice (1.9) for the point } z.

The dependence of the constants } C \text{ on } c \text{ in (1.18) and (1.19) could be replaced with the dependence on the surface measure } |\Gamma|, \text{ as explained in Remark 8.11.}

The explicit values of } \tau_N \text{ in (1.20) are the following ones. We have that } \tau_2 \text{ can be taken as close as we wish to } 1, \text{ namely one can fix any } \theta > 0 \text{ and take } \tau_2 := 1 - \theta \text{ (in this case, the constant } C \text{ in (1.20) will also depend on } \theta). \text{ When } N \geq 3, \text{ one can take } \tau_N := \frac{2}{N}. \]
We notice that these exponents are all smaller (i.e., “worse”) than the ones obtained in Theorem 1.1. Nevertheless, it is possible to get the pointwise estimate (1.20) with the better exponents \( \tau_N \) obtained in (1.14), and by removing the John condition (1.15), provided that we make a different choice of \( z \).

The precise statement, that can be deduced from more general results obtained in Theorems 8.15 and 8.16 is the following:

**Theorem 1.3.** Let \( \Omega \setminus \varnothing \subset \mathbb{R}^N \) be a bounded domain satisfying assumptions (1.16) and (1.17). Let \( u \) satisfy (1.1), (1.2) and (1.4). Assume that \( u \leq 0 \) on \( \partial \omega \). Set

\[
(1.21) \quad z := \frac{1}{|\Omega^c_{r_i}|} \left\{ \int_{\Omega^c_{r_i}} x \, dx - N \int_{\Gamma_{r_i}} u \, dS_x \right\},
\]

where \( \Omega^c_{r_i} \) denotes the points in \( \Omega \) which lie at distance strictly less than \( r_i \) from \( \Gamma \), and \( \Gamma_{r_i} \) denotes the points in \( \Omega \) which lie at distance \( r_i \) from \( \Gamma \).

If \( z \in \Omega \), then there exist \( \rho_e \geq \rho_i > 0 \) such that

\[
B_{\rho_i}(z) \subset \Omega \subset B_{\rho_e}(z)
\]

and

\[
\rho_e - \rho_i \leq C \left( \mathcal{H}^{N-1}(\partial^* \omega) \right)^{\tau_N/2},
\]

where \( C > 0 \) is a constant depending only on \( N \), the internal radius \( r_i \) in (1.17), \( d_\Omega \), and \( \|u\|_{C^2(\partial \omega)} \). The exponents \( \tau_N > 0 \) depend only on \( N \).

We stress that the approach used in Theorem 1.3 does not need the assumption in (1.15) that \( \Omega \setminus \varnothing \) is a John domain and hence the dependence on \( b_0 \), present in (1.20), has been dropped.

Interestingly, the values of the exponents \( \tau_N \) in Theorem 1.3 are the same as those in Theorem 1.1 (and therefore they are “better” than the ones obtained in (1.20), though they rely on a different choice of \( z \)).

We think that it would be interesting to investigate the possible optimality of these exponents also in the framework provided by Theorem 1.3.

We remark that the setting of \( z \) in (1.21) is modeled on the annular set \( \Omega^c_{r_i} \) rather than on \( \Omega \setminus \varnothing \) as in (1.9). It would be interesting to further investigate the impact of different possible choices for \( z \).

### 1.3. Organization of the paper

The rest of this paper is organized as follows. Section 2 contains some detailed motivations from shape optimization, fluid dynamics and mechanics which naturally lead to the problem considered in this paper.

In Section 3 we make some notation precise.

In Section 4 we present some integral identities of Rellich-Pohozaev-type for solutions of (1.1). In these computations, one does not need to impose the additional condition in (1.2) from the beginning, and aims at comparing a weighted “deficit” on \( \Omega \setminus \varnothing \) (measuring “how far from rotational invariant” the solution is) with suitable surface integrands on \( \Gamma \) and \( \partial \omega \). From these identities, the auxiliary information in (1.2) provides a more precise, and simpler information.

In Section 5 we collect useful estimates and we use them to obtain a suitable stability bound on the spherical pseudo-distance defined in (1.7) (see Theorem 5.7). We also put in relation this pseudo-distance with the asymmetry defined in (1.8) (see Lemma 5.1).

The estimates collected in Section 5 are then also used in Section 6 to bound the difference \( \rho_e - \rho_i \).

In Section 7, from the information obtained in Sections 5 and 6 we obtain a number of quantitative results, that also contain Theorem 1.1 as a particular case.
Finally, in Section 8 we obtain generalizations of the quantitative results presented in Section 7 by relaxing the regularity assumptions (1.5) and (1.6), and hence we establish Theorems 1.2 and 1.3 as particular cases.

2. Models and motivations

2.1. Heating with source malfunctioning. A natural motivation for problem (1.1)-(1.2) comes from the optimal heating theory. In this setting, a region $\Omega$ is given which is in direct contact with an external environment having constant (say, zero) temperature. In this setting, at the equilibrium, the temperature on the boundary of $\Omega$ is set to be zero. The region $\Omega$ is also provided by a fine set of heating devices. All these devices are the same and produce the same heating effect with the exception of those placed in a small subregion $\omega$, see Figure 1.

In this setting, at the equilibrium, the local heat flow, normalized by the flow surface, is constant, say equal to 1, in $\Omega \setminus \omega$, but it may be different from 1 in $\omega$. In a nutshell, the mathematical description of this situation is given by

\[
\begin{aligned}
\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with $f(x) = 1$ for every $x \in \Omega \setminus \omega$.

A natural question in this setting is to optimize the shape of $\Omega$ in order to store in the domain the biggest possible amount of caloric energy. Concretely, given $u = u^{(f)}$ satisfying (2.1), one may try to select $\Omega$ in order to maximize (among the domains of fixed measure) the functional

\[
\int_\Omega |\nabla u^{(f)}(x)|^2 dx.
\]

This variational problem is well posed (see e.g. Theorem 4.5.2 in [HP18]) and it is naturally related to the boundary derivative prescription

\[
\partial_\nu u^{(f)} = \text{const on } \partial \Omega,
\]

thus leading to the problem described in (1.1)-(1.2). We refer to Proposition 6.1.10 in [HP18] and Appendix A here for a direct computation relating the shape optimization of (2.2) and the Neumann condition in (2.3).
In this framework, our result in Theorem 1.1 states that, for a small region $\omega$ of malfunctioning of the heat source, the optimal domain $\Omega$ is necessarily close to a ball (with a quantitative information on the proximity between $\Omega$ and a suitable ball). This result is close to intuition, since jagged domains end up dissipating most of the caloric energy from their boundaries.

2.2. Laminar flows with a small tube of unknown density. Another motivation of the problem in (1.1)-(1.2) comes from laminar flows, as modeled by a Navier-Stokes equation of the type

$$\partial_t (\rho v) + \text{div}(\rho v \otimes v) - \mu \Delta v + \nabla p = -\rho g,$$

where $v$ is the vectorial velocity of the fluid, $\rho$ is its density, $\mu$ is its viscosity coefficient, $p$ is the pressure, $g$ is the gravity acceleration and $\otimes$ denotes the outer product (i.e., given two vectors $a$ and $b$, $a \otimes b$ is the matrix whose $(i,j)$th entry is $a_i b_j$) see e.g. [Dan03]. The incompressibility condition

$$\text{div} v = 0$$

leads to

$$\text{div}(\rho v \otimes v) = (v \cdot \nabla)(\rho v).$$

In this way, one obtains from (2.4) that

$$\partial_t (\rho v) + (v \cdot \nabla)(\rho v) - \mu \Delta v + \nabla p = -\rho g.$$

One assumes that the flow is “vertical”, namely $v = (0,0,u)$ for some scalar function $u$, thus obtaining that

$$(v \cdot \nabla)(\rho v) = ((0,0,u) \cdot \nabla)(\rho(0,0,u)) = u \partial_3 (\rho(0,0,u)).$$

We also suppose that the laminar flow occurs in a “vertical tube” of the type $\Omega \times \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, and that the density of the fluid only depends on the horizontal position (that is, the fluid maintains the same density along its vertical flow). In this setting, we have that $\rho = \rho(x_1,x_2)$ and accordingly, by (2.5) and (2.7),

$$(v \cdot \nabla)(\rho v) = (0,0,\rho u \partial_3 u) = (0,0,\rho u \text{div} v) = 0.$$

Hence, we deduce from the third component of (2.6) that

$$\rho \partial_t u - \mu \Delta u + \partial_3 p = -\rho g.$$

The case of a “steady state” flow (that is $\partial_t u = 0$) with constant pressure (hence $\partial_3 p = 0$) reduces (2.8) to

$$\Delta u = \frac{\rho g}{\mu}.$$

If the fluid has constant (say, up to changing the inertial reference frame, zero) velocity at the boundary of the pipe, equation (2.9) is complemented by the boundary condition

$$u = 0 \quad \text{on } \partial \Omega.$$

Also, if the fluid presents a constant tangential stress on the pipe, we have that

$$\partial_\nu u = \text{constant} \quad \text{on } \partial \Omega.$$

The case described in (1.1)-(1.2) is, in this setting, a byproduct of (2.9), (2.10) and (2.11) in which the density of the fluid (as well as its viscosity and the gravity acceleration) is constant in the region $\Omega \setminus \omega$, but it is possibly unknown in $\omega$. In this framework, our result in Theorem 1.1 says that if a laminar fluid is homogeneous out of a small region and presents

\footnote{Strictly speaking, (2.1) with a source $f \geq 0$ should be described as an optimal cooling, rather than heating, problem: speaking of heating problem should require to add a minus sign in front of $\Delta$ in (2.1). Nevertheless, we preferred to keep the sign convention in accordance with (1.1) and the rest of the paper.}
a constant tangential stress on the pipe, then necessarily the pipe is close to a right circular cylinder.

2.3. Traction of beams with small inhomogeneity. In the theory of elasticity, one can consider the displacement vector $U = (U_1, U_2, U_3)$ that describes the deformation of some material. Also, it is customary to introduce the stress tensor

\begin{equation}
\sigma_{ij} := \partial_i U_j + \partial_j U_i,
\end{equation}

describing the force (per unit area) in the $i$th direction along the infinitesimal surface orthogonal to $e_j$, being $e_1 := (1, 0, 0)$, $e_2 := (0, 1, 0)$ and $e_3 := (0, 0, 1)$, see e.g. formula (129) in [Hje05] (here, we are supposing the strain and the stress to be proportional, setting the proportionality constant equal to 1 for the sake of simplicity).

In this framework, the equilibrium configurations are those for which the forces are infinitesimally balanced, that is

\begin{equation}
\sum_{j=1}^{3} \partial_j \sigma_{ij} = 0 \quad \text{for all } i \in \{1, 2, 3\},
\end{equation}

see e.g. formula (189) in [Hje05].

More specifically, we focus here on the torsion of a vertical beam $B$ (see e.g. [Sok56, pages 100-120]). The fact that the beam is vertical means, in our setting, that

\begin{equation}
B = \{(x_1, x_2, x_3) \text{ s.t. } (x_1, x_2) \in \Omega_{x_3}\},
\end{equation}

for a suitable (bounded and smooth) family of domains $\Omega_{x_3} \subset \mathbb{R}^2$. We assume that each point of the beam lying on a given horizontal plane $\{x_3 = \bar{x}_3\}$ performs a horizontal rotation of a small angle $\vartheta = \vartheta(x)$, plus a vertical movement that is the same for every height of the bar (we will denote this vertical movement by $v = v(x_1, x_2)$). In this setting, the torsion of the bar moves a point $x = (x_1, x_2, \bar{x}_3)$ to the point

\[ X = (x_1 \cos \vartheta + x_2 \sin \vartheta, -x_1 \sin \vartheta + x_2 \cos \vartheta, \bar{x}_3 + v) \approx (x_1 + x_2 \vartheta, -x_1 \vartheta + x_2, \bar{x}_3 + v), \]

and so, in this approximation, we can write the displacement vector as

\[ U = X - x = (\vartheta x_2, -\vartheta x_1, v). \]

From this and (2.12), we can write

\begin{equation}
\begin{align*}
\sigma_{31} &= \partial_3 U_1 + \partial_1 U_3 = \partial_3 \vartheta x_2 + \partial_1 v, \\
\sigma_{32} &= \partial_3 U_2 + \partial_2 U_3 = -\partial_3 \vartheta x_1 + \partial_2 v, \\
\sigma_{33} &= 2 \partial_3 U_3 = 0.
\end{align*}
\end{equation}

As a result, exploiting the latter equation and (2.13) with $i := 3$, we get

\begin{equation}
0 = \sum_{j=1}^{3} \partial_j \sigma_{3j} = \partial_1 \sigma_{31} + \partial_2 \sigma_{32}.
\end{equation}

We then fix $x_3 = 1$ and consider the 1-form on $\mathbb{R}^2$ given by

\begin{equation}
\alpha := -\sigma_{32} \, dx_1 + \sigma_{31} \, dx_2,
\end{equation}

and we deduce from (2.16) that

\[ d\alpha = (\partial_2 \sigma_{32} + \partial_1 \sigma_{31}) \, dx_1 \wedge dx_2 = 0. \]

This gives that there exists a “warping potential” $u$ such that $\alpha = du$ as 1-forms in $\mathbb{R}^2$. Accordingly, by (2.17), we have that

\begin{equation}
\partial_1 u = -\sigma_{32} \quad \text{and} \quad \partial_2 u = \sigma_{31}.
\end{equation}
Combining this with (2.15), one sees that
\[ \partial_1 u = \partial_3 \vartheta x_1 - \partial_2 v \quad \text{and} \quad \partial_2 u = \partial_3 \vartheta x_2 + \partial_1 v, \]
and therefore
\[ (2.19) \quad \Delta u = \partial_1 \left( \partial_3 \vartheta x_1 - \partial_2 v \right) + \partial_2 \left( \partial_3 \vartheta x_2 + \partial_1 v \right) = 2 \partial_3 \vartheta + \partial_1^2 \vartheta x_1 + \partial_2^2 \vartheta x_2, \]
with these functions evaluated at \( x_3 = 1 \). As a special case, one can take into account the situation in which the angle \( \vartheta \) depends linearly on the height of the beam, say \( \vartheta(x) = \theta(x_1, x_2) x_3 \) (this is physically reasonable, for instance, if the beam is constrained at \( \{ x_3 = 0 \} \) and some torque is applied from the top of the beam). In this setting, (2.19) reduces to (2.20)
\[ \Delta u = 2 \theta + \partial_1 \theta x_1 + \partial_2 \theta x_2. \]

If we suppose that the beam is built by two different materials, one occupying \( \Omega \setminus \varpi \) and the other occupying \( \varpi \) (where \( \Omega \) here represents the domain \( \Omega_{x_3} \) in (2.14) with \( x_3 = 1 \)), the expression in (2.20) takes two different forms in \( \Omega \setminus \varpi \) and \( \varpi \). In particular, if we know that the material in \( \Omega \setminus \varpi \) is homogeneous we can suppose that the horizontal rotation is uniform there and thus \( \theta \) is independent on the point (i.e., \( \partial_1 \theta = \partial_2 \theta = 0 \)), hence deducing from (2.20) that
\[ (2.21) \quad \Delta u = \text{constant} \quad \text{in} \quad \Omega \setminus \varpi. \]

Also, the surface traction, as a force per unit of area, at a boundary point of the beam is defined as the normal component of the vertical stress, that is
\[ T := (\sigma_{31}, \sigma_{32}, \sigma_{33}) \cdot \bar{\nu}, \]
being \( \bar{\nu} \in \mathbb{R}^3 \) the normal to the beam (see e.g. the third component in formula (174) of [Hje05]). Recalling (2.15), we have that
\[ T = (\sigma_{31}, \sigma_{32}, 0) \cdot \bar{\nu} = (\sigma_{31}, \sigma_{32}) \cdot \nu, \]
being \( \nu \in \mathbb{R}^2 \) normal to \( \Omega \). Thus, in view of (2.18),
\[ T = (\partial_2 u, -\partial_1 u) \cdot \nu = \nabla u \cdot \tau, \]
being \( \tau := (-\nu_2, \nu_1) \) a unit tangent vector to \( \Omega \). Therefore, if the traction vanishes, the tangential derivative of \( u \) vanishes as well, hence \( u \) is constant along \( \partial \Omega \). Since \( u \) was introduced as a potential, it is defined up to an additive constant, hence we can rephrase these considerations by saying that if the traction vanishes, then
\[ (2.22) \quad u = 0 \quad \text{on} \quad \partial \Omega. \]

We also remark that, if \( \sigma_3 = (\sigma_{31}, \sigma_{32}, \sigma_{33}) \), then
\[ |\sigma_3| = |(\sigma_{31}, \sigma_{32}, 0)| = |\nabla u|, \]
thanks to (2.15) and (2.18).

Hence, for constant stress intensity \( |\sigma_3| \), we deduce from (2.22) that
\[ (2.23) \quad \partial_\nu u = \text{constant} \quad \text{on} \quad \partial \Omega. \]

We thus observe that problem (1.1)-(1.2) arises naturally from (2.21), (2.22) and (2.23), and, in this framework, Theorem 1.1 says that if a beam is homogeneous out of a small region and presents zero traction and constant stress intensity, then necessarily the horizontal section of the beam is close to a disk.
3. Notation

Unless differently specified, we will denote by $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, a connected, bounded open set, and call $\Gamma := \partial \Omega$ its boundary. We will denote indifferently by $|\Omega|$ and $|\Gamma|$ the $N$-dimensional Lebesgue measure of $\Omega$ and the surface measure of $\Gamma$.

Moreover, we will denote by $d_\Omega$ the diameter of $\Omega$, that is

\begin{equation}
(3.1) \quad d_\Omega := \sup_{x,y \in \Omega} |x - y|.
\end{equation}

Also, we shall denote by $\omega$ an open subset of $\Omega$, such that $\overline{\omega} \subseteq \Omega$.

For all $x \in \Omega \setminus \overline{\omega}$, we consider the distance function

\begin{equation}
(3.2) \quad \delta(x) := \text{dist}(x, \Gamma \cup \partial \omega).
\end{equation}

As already mentioned in the Introduction, we first consider the case in which, being (1.5) in force and by recalling (1.3), $\Omega \setminus \overline{\omega}$ is of class $C^1$: in this setting $\nu$ denotes the (exterior) unit normal vector field to $\Omega \setminus \overline{\omega}$. Then, we will clarify in Section 8 the notation that we use when the regularity assumption in (1.5) is dropped.

Now, we clarify the notation used for the spaces $C^k$ and we discuss the regularity assumption in (1.4).

By $C^k(D)$ we denote the space of functions that are restrictions to $D$ of functions in $C^k(\mathbb{R}^N)$. The same definition has been adopted for instance in [Leo17, Appendix C, pag. 562] and also in many books of Differential Geometry.

We recall that, thanks to Whitney extension theorem (see the original paper [Whi34a] or [KP02, Theorem 2.3.6]), this definition is equivalent to [KP02, Definition 2.3.5] based on a Taylor-expansion condition.

Moreover, if $D$ satisfies property (P) of [Whi34b] – i.e., quasiconvexity, that in particular is surely satisfied if $D$ is Lipschitz (see BB12, Sections 2.5.1, 2.5.2) – then, thanks to the main theorem on page 485 in [Whi34b], our definition of $C^k(\overline{D})$ (as well as [KP02, Definition 2.3.5]) is also equivalent to the definition used in many books of PDEs (e.g., GT01), that is: $C^k(\overline{D})$ is the space of functions in $C^k(D)$ whose derivatives up to order $k$ have continuous extensions to the closure $\overline{D}$.

For more general domains, [KP02, Definition 2.3.5] (as well as our definition) is stronger than that adopted in GT01. For more details on this subject we refer to KP02, Section 2.3, KP99 Section 5.2, Kra83, BB12, and Leo17, Appendix C.

In our setting, by taking $D := \Omega \setminus \overline{\omega}$ and $k := 2$ in the definition of $C^2(\overline{D})$, we have that the assumption in (1.4) guarantees that $u$ can be extended to a $C^2$ function throughout $\mathbb{R}^N$. This will allow us to perform the generalizations described in Section 8 when the regularity assumption on $\partial \omega$ in (1.5) is dropped and replaced just by (1.16). In particular, when integrating by parts, we can still write the derivatives of $u$ up to the second order on the reduced boundary $\partial^* \omega$.

We remark that up to Section 7, we will take $\Omega \setminus \overline{\omega}$ to be of class $C^1$, and therefore assumption (1.4) has univocal meaning, no matter what definition of $C^2(\Omega \setminus \omega)$ we adopt (among the three presented above).

4. Integral identities

The goal of this section is to develop a series of integral identities which will be conveniently exploited to deduce quantitative bounds on the solution of (1.1)-(1.2) and on its domain of
definition. We start by proving a Rellich-Pohozaev-type identity and its consequences. Notice that in the next two statements we are not imposing yet the overdetermined condition in (1.2).

Lemma 4.1 (A Rellich-Pohozaev-type identity). Suppose that \( \Omega \setminus \bar{\omega} \) is of class \( C^1 \). If \( u \in C^2(\Omega \setminus \omega) \) satisfies (1.1), then the following identity holds:

\[
(N + 2) \int_{\Omega \setminus \bar{\omega}} |\nabla u|^2 \, dx = \int_{\Gamma} <x, \nu > u^2 \, dS_x
\]

(4.1)

\[
+ 2N \int_{\partial \omega} \left\{ u u_\nu - \frac{<x, \nu >}{N} u + \frac{<x, \nabla u >}{N} u_\nu - \frac{<x, \nu >}{2N} |\nabla u|^2 \right\} \, dS_x.
\]

Proof. By a direct computation, it is easy to verify the following differential identity (valid for every function \( u \)):

\[
\text{div} \left\{ <x, \nabla u > \nabla u - \frac{|\nabla u|^2}{2} x \right\} = <x, \nabla u > \Delta u - \frac{N - 2}{2} |\nabla u|^2.
\]

(4.2)

We now specialize this identity to a solution of (1.1). For this, we remark that the Dirichlet boundary condition in (1.1) gives that

\[
\nabla u = u_\nu \nu \text{ on } \Gamma.
\]

(4.3)

We also remark that

\[
\partial(\Omega \setminus \bar{\omega}) = \Gamma \cup \partial \omega.
\]

(4.4)

Hence, by integrating (4.2) over \( \Omega \setminus \bar{\omega} \), exploiting (1.1), (4.3) and (4.4), and applying the divergence theorem, we see that

\[
\int_{\Omega \setminus \bar{\omega}} \{<x, \nabla u > \Delta u - \frac{N - 2}{2} |\nabla u|^2\} \, dx
\]

\[
= \int_{\Omega \setminus \bar{\omega}} \{<x, \nabla u > \nabla u - \frac{|\nabla u|^2}{2} x\} \, dx
\]

(4.5)

\[
= \int_{\partial(\Omega \setminus \bar{\omega})} \left\{<x, \nabla u > <\nabla u, \nu > - \frac{|\nabla u|^2}{2} <x, \nu >\right\} \, dS_x
\]

\[
= \frac{1}{2} \int_{\Gamma} <x, \nu > u_\nu^2 \, dS_x + \int_{\partial \omega} \left\{<x, \nabla u > u_\nu - |\nabla u|^2 \frac{<x, \nu >}{2}\right\} \, dS_x.
\]

Now we observe that, in \( \Omega \setminus \bar{\omega} \),

\[
\text{div} \left( \frac{x}{N} u - u \nabla u \right) = u + \frac{<x, \nabla u >}{N} |\nabla u|^2 - u \Delta u
\]

\[
= \frac{<x, \nabla u >}{N} - |\nabla u|^2,
\]

where the equation in (1.1) has been used in the last step.

As a consequence,

\[
<x, \nabla u > = N |\nabla u|^2 + N \text{ div} \left( \frac{x}{N} u - u \nabla u \right).
\]

By integrating this identity over \( \Omega \setminus \bar{\omega} \), and using the boundary condition in (1.1), we deduce that

\[
\int_{\Omega \setminus \bar{\omega}} <x, \nabla u > \, dx = N \int_{\Omega \setminus \bar{\omega}} |\nabla u|^2 \, dx + N \int_{\partial \omega} \left\{\frac{<x, \nu >}{N} u - u u_\nu\right\} \, dS_x.
\]

By putting together the last identity and (4.5), we obtain (4.1), as desired. \( \square \)
We now obtain another useful integral identity, which is based on (4.1) and a suitable $P$-function computation.

**Theorem 4.2.** Suppose that $\Omega \setminus \omega$ is of class $C^1$. If $u \in C^2(\overline{\Omega} \setminus \omega)$ satisfies (1.1), then the following identity holds:

\[
\int_{\Omega \setminus \omega} (-u)^2 \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} \, dx = \int_{\Gamma} u \nu \left\{ \frac{|\nabla u|^2}{N} \right. \left. - \frac{2 < x, \nabla u >}{N} \right\} \, dS_x + \int_{\partial \omega} \left\{ u \nu |\nabla u|^2 - 2 \frac{x, \nabla u >}{N} + |\nabla u|^2 \frac{x, \nu >}{N} + 2 \frac{N uu \nu}{N} - 2 \frac{\nabla^2 u \langle \nabla u, \nu \rangle}{N} u \right\} \, dS_x.
\]

We point out that the term in the braces in the left-hand side of (4.6) could be written as $\left\{ |\nabla^2 u|^2 - \frac{1}{N} \right\}$. Nevertheless, we preferred to use the notation (4.7)

\[|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\]

to emphasize that this quantity plays the role of a Cauchy-Schwarz deficit. In fact, by Cauchy-Schwarz inequality we have that (4.7) is nonnegative, and equals 0 in $\Omega \setminus \omega$ if and only if $\Omega$ is a ball of radius $R = N|\Omega|/|\Gamma|$ and $u(x) = \frac{|x|^2 - R^2}{2N}$ in $\Omega \setminus \omega$, up to translations (see, e.g., [Pog19, Lemma 1.9]).

**Proof of Theorem 4.2.** If we set

\[P := |\nabla u|^2 - \frac{2}{N} u,
\]

a direct computation, valid for any function, informs us that

\[\Delta P = 2 |\nabla^2 u|^2 + 2 < \nabla u, \nabla \Delta u > - \frac{2}{N} \Delta u,
\]

where $\nabla^2 u$ denotes the Hessian matrix of $u$ and $|\nabla^2 u|$ denotes its Frobenius norm, that is

\[|\nabla^2 u|_{i,j} = \partial^2_{ij} u \quad \text{and} \quad |\nabla^2 u|^2 = \sum_{i,j=1}^n (\partial^2_{ij} u)^2.
\]

Then, using (4.9) and the equation in (1.1), we conclude that

\[\Delta P = 2 \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} \quad \text{in} \ \Omega \setminus \omega.
\]

On the other hand, by (1.1) and the Green identity,

\[
\int_{\Omega \setminus \omega} (-u) \Delta P \, dx = - \int_{\Omega \setminus \omega} P \, dx + \int_{\Gamma} u \nu P \, dS_x + \int_{\partial \omega} \{ u \nu P - u P \nu \} \, dS_x.
\]

Let us work on the first integral on the right-hand side of (4.11). For this, integrating over $\Omega \setminus \omega$ the differential identity

\[\text{div}(u \nabla u) = u \Delta u + |\nabla u|^2,
\]

and recalling (1.1), we get that

\[
\int_{\Omega \setminus \omega} u \, dx = \int_{\Omega \setminus \omega} u \Delta u \, dx = - \int_{\Omega \setminus \omega} |\nabla u|^2 \, dx + \int_{\partial \omega} u \nu \, dS_x.
\]
Thus, in light of (4.8),
\[- \int_{\Omega_\omega} P \, dx = - \int_{\Omega_\omega} \left\{ \frac{2}{N} u - |\nabla u|^2 \right\} \, dx = - \frac{N+2}{N} \int_{\Omega_\omega} |\nabla u|^2 \, dx + \frac{2}{N} \int_{\partial \omega} u u_\nu \, dS_x.\]

Consequently, using (4.1),
\[- \int_{\Omega_\omega} P \, dx = - \int_{\Gamma} \frac{\langle x, \nu \rangle}{N} u^2 \, dS_x.
\]

(4.12)

Moreover, to deal with the second integral in the right-hand side of (4.11), by recalling (1.1) and (4.8), we have
\[
\int_{\Gamma} u \nu P \, dS_x = \int_{\Gamma} u^3 \nu \, dS_x.
\]

(4.13)

Furthermore, using (4.8), we see that
\[
P_\nu = 2 < \nabla^2 u \nabla u, \nu > - \frac{2}{N} u_\nu
\]

and accordingly
\[
u P - u P_\nu = u_\nu \left\{ |\nabla u|^2 - \frac{2}{N} u \right\} - 2u < \nabla^2 u \nabla u, \nu > + \frac{2}{N} uu_\nu
\]

As a result, the third integral in the right-hand side of (4.11) becomes
\[
\int_{\partial \omega} \{ u_\nu P - u P_\nu \} \, dS_x = \int_{\partial \omega} \{ u_\nu |\nabla u|^2 - 2u < \nabla^2 u \nabla u, \nu > \} \, dS_x.
\]

(4.14)

Thus, (4.6) follows by putting together (4.10), (4.11), (4.12), (4.13) and (4.14). □

We now impose the overdetermined condition (1.2), and we obtain from (4.6) the following integral identity:

**Corollary 4.3.** Suppose that \( \Omega \setminus \omega \) is of class \( C^1 \). If \( u \in C^2(\Omega \setminus \omega) \) satisfies (1.1) and (1.2), then the following identity holds:

\[
\int_{\Omega_\omega} (-u)2 \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} \, dx
\]

\[
= c^2 \int_{\partial \omega} \left\{ \frac{\langle x, \nu \rangle}{N} - u_\nu \right\} \, dS_x + \int_{\partial \omega} 2u \left\{ \frac{\langle x, \nu \rangle}{N} - u_\nu \right\} \, dS_x
\]

\[
+ \int_{\partial \omega} \left\{ u_\nu |\nabla u|^2 - 2\frac{\langle x, \nabla u \rangle}{N} u_\nu + |\nabla u|^2 \frac{\langle x, \nu \rangle}{N} + \frac{2}{N} uu_\nu - 2 < \nabla^2 u \nabla u, \nu > u \right\} \, dS_x.
\]

**Proof.** We observe that the constant \( c \) in (1.2) can be determined explicitly in terms of \( |\Gamma| \), \( |\Omega| \), \( |\omega| \) and the values of \( u_\nu \) along \( \partial \omega \). Indeed, by using (1.1) and (1.2) together with the divergence theorem, we deduce that

\[
c|\Gamma| = \int_{\Gamma} u_\nu \, dS_x = \int_{\Omega_\omega} \Delta u \, dx - \int_{\partial \omega} u_\nu \, dS_x = |\Omega| - |\omega| - \int_{\partial \omega} u_\nu \, dS_x.
\]
In particular, we will use that

\[ \int_{\Gamma} u_\nu \, dS_x = |\Omega| - |\omega| - \int_{\partial \omega} u_\nu \, dS_x. \]

On the other hand, by applying again the divergence theorem,

\[ |\Omega| - |\omega| = \int_{\Omega} \frac{\text{div} \, x}{N} \, dx = \int_{\Omega} \frac{< x, \nu >}{N} \, dx. \]

From this and (4.17), we conclude that

\[ \int_{\Gamma} u_\nu^2 \left( u_\nu - \frac{< x, \nu >}{N} \right) \, dS_x = c^2 \int_{\Gamma} \left\{ u_\nu - \frac{< x, \nu >}{N} \right\} \, dS_x \]

\[ = c^2 \left( |\Omega| - |\omega| - \int_{\partial \omega} u_\nu \, dS_x - \int_{\Gamma} \frac{< x, \nu >}{N} \, dS_x \right) \]

\[ = c^2 \left( - \int_{\partial \omega} u_\nu \, dS_x + \int_{\partial \omega} \frac{< x, \nu >}{N} \, dS_x \right). \]

Plugging this information into (4.6) we obtain the desired result in (4.15). \( \square \)

5. Some estimates on a spherical pseudo-distance

In this section, we will obtain a suitable bound on the following pseudo-distance

\[ \int_{\Gamma} \left| \frac{|x - z|}{N} - c \right|^2 \, dS_x, \]

for a suitable \( z \in \mathbb{R}^N \).

We point out that the quantity in (5.1) plays the role of an “integral distance” of \( \Gamma \) from the sphere centered at a point \( z \in \Omega \) of radius \( Nc \): indeed, when \( \Gamma = \partial B_{Nc}(z) \), the quantity in (5.1) vanishes, and, in general, this quantity can be considered an \( L^2 \)-measure on how far points on \( \Gamma \) are from points on \( \partial B_{Nc}(z) \).

We also notice that the pseudo-distance in (5.1) can be put in relation with the following asymmetry:

\[ |\Omega \triangle B_{Nc}(z)| / |B_{Nc}(z)|, \]

where \( \Omega \triangle B_{Nc}(z) \) denotes the symmetric difference of \( \Omega \) and the ball \( B_{Nc}(z) \) of radius \( Nc \) centered at \( z \).

In particular, the asymmetry in (5.2) is bounded from above by the pseudo-distance in (5.1), as stated in the following result:

**Lemma 5.1.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with Lipschitz boundary \( \Gamma \), satisfying the uniform interior sphere condition with radius \( r_i \). Then, there exists a positive constant \( C \), only depending on \( N \), \( r_i \) and \( c \), such that

\[ \frac{|\Omega \triangle B_{Nc}(z)|}{|B_{Nc}(z)|} \leq C \left[ \int_{\Gamma} \left( \left| \frac{|x - z|}{N} - c \right| \right)^2 \, dS_x \right]^{\frac{1}{2}}. \]
Proof. The desired result follows by applying [Fel18, Lemma 11] with
\[ K := \max \left\{ \frac{Nc}{r_i}, \left( \frac{d_\Omega}{2Nc} \right)^N \right\} \quad \text{and} \quad r := Nc. \]
Notice that [Fel18, Lemma 11] can be applied with these choices for \(K\) and \(r\) because the following two relations hold true: the first is
\[ K |B_{Nc}| \geq \left( \frac{d_\Omega}{2Nc} \right)^N |B_1|(Nc)^N \geq |\Omega|, \]
where in the last inequality we used that \( |B_1|(\frac{d_\Omega}{2Nc})^N \geq |\Omega| \); the second is
\[ Kr_{in}(\Omega) \geq Nc \frac{r_{in}(\Omega)}{r_i} \geq Nc, \]
where \(r_{in}(\Omega) := \max_{\Gamma \cup \partial \omega} \delta_r(x)\) denotes the inradius of \(\Omega\) and in the last inequality we used that, by definition, \(r_{in}(\Omega) \geq r_i\). \(\square\)

To obtain our bounds for the pseudo-distance introduced in (5.1), we recall the notation in (3.2) and we detect an optimal growth of the solution from the boundary, by adapting an idea from [MP17, Lemma 3.1]:

Lemma 5.2. Let \(u\) satisfy (1.1). Assume that \(u \leq 0\) on \(\partial \omega\). Then,
\[ -u(x) \geq \frac{1}{2N} \delta(x)^2 \quad \text{for every} \quad x \in \Omega \setminus \overline{\omega}. \]

Moreover, if \(\Omega \setminus \overline{\omega}\) is of class \(C^1\) and satisfies the uniform interior sphere condition with radius \(r_i\), that is (1.6), then it holds that
\[ -u(x) \geq \frac{r_i}{2N} \delta(x) \quad \text{for every} \quad x \in \Omega \setminus \overline{\omega}. \]

Proof. Let \(x \in \Omega \setminus \overline{\omega}\) and set \(r := \delta(x)\). We consider
\[ w(y) := \frac{|y - x|^2 - r^2}{2N}. \]
We remark that \(w\) is the solution of the classical torsion problem in \(B_r(x)\), namely
\[ \begin{cases} \Delta w = 1 & \text{in } B_r(x), \\ w = 0 & \text{on } \partial B_r(x). \end{cases} \]
By comparison we have that \(w \geq u\) on \(\overline{B}_r(x)\). In particular,
\[ -\frac{1}{2N} \delta(x)^2 = w(x) \geq u(x), \]
and (5.5) follows.
We point out that (5.6) follows from (5.5) if \(\delta(x) \geq r_i\). Hence, from now on, we can suppose that
\[ \delta(x) < r_i. \]
Let \(\bar{x}\) be the closest point in \(\Gamma \cup \partial \omega\) to \(x\) and call \(\bar{B} \subset \Omega \setminus \overline{\omega}\) the ball of radius \(r_i\) touching \(\Gamma \cup \partial \omega\) at \(\bar{x}\) and containing \(x\). Up to a translation, we can always suppose that
\[ \text{the center of the ball } \bar{B} \text{ is the origin.} \]
Now, we let \(\tilde{w}\) be the solution of (5.7) in \(\tilde{B}\), that is \(\tilde{w}(y) := (|y|^2 - r_i^2)/(2N)\). By comparison, we have that \(w \geq u\) in \(\tilde{B}\), and hence, being \(x \in \tilde{B}\),
\[ -u(x) \geq \frac{1}{2N} \left( r_i^2 - |x|^2 \right) = \frac{1}{2N} \left( r_i + |x| \right) (r_i - |x|) \geq \frac{r_i}{2N} (r_i - |x|). \]
Moreover, from (5.9),
\[ r_i - |x| = \delta(x). \]
This and (5.10) give (5.6), as desired. \[\square\]

We recall now some Hardy-Poincaré-type inequalities that have been proved in [Pog18, Section 3.2] by exploiting the works of Hurri-Syrjänen [Hur88, HS94]. In what follows, for a set \( D \) and a function \( v : D \to \mathbb{R} \), \( v_D \) denotes the mean value of \( v \) in \( D \), that is
\[ (5.11) \quad v_D := \frac{1}{|D|} \int_D v \, dx. \]
Also, for a function \( v : D \to \mathbb{R} \) we define by \( \|v\|_{p,D} \) its \( L^p \)-norm in \( D \), that is
\[ (5.12) \quad \|v\|_{p,D} := \left( \int_D |v(x)|^p \, dx \right)^{1/p}, \]
and
\[ \|\delta^\alpha \nabla v\|_{p,D} := \left( \sum_{i=1}^N \|\delta^\alpha v_i\|_{p,D}^p \right)^{1/p} \quad \text{and} \quad \|\delta^\alpha \nabla^2 v\|_{p,D} := \left( \sum_{i,j=1}^N \|\delta^\alpha v_{ij}\|_{p,D}^p \right)^{1/p}, \]
for \( 0 \leq \alpha \leq 1 \) and \( p \in [1, \infty) \). Here and whenever no confusion is possible, we will use the abbreviated notation
\[ \delta(x) = \text{dist}(x, \partial D), \]
that agrees with (3.2) when \( D = \Omega \setminus \overline{\Omega} \).

Lemma 5.3. Let \( D \subset \mathbb{R}^N \) be a bounded domain satisfying the uniform interior sphere condition with radius \( r_i \), and consider three real numbers \( r, p, \) and \( \alpha \) such that either
\[ (5.13) \quad 1 \leq p \leq r \leq \frac{Np}{N - p(1 - \alpha)}, \quad p(1 - \alpha) < N \quad \text{and} \quad 0 \leq \alpha \leq 1, \]
or
\[ (5.14) \quad r = p \in [1, \infty) \quad \text{and} \quad \alpha = 0. \]
Then,
(i) given \( x_0 \in D \), there exists a positive constant \( \mu_{r,p,\alpha}(D, x_0) \) such that
\[ (5.15) \quad \|v\|_{r,D} \leq \mu_{r,p,\alpha}(D, x_0)^{-1}\|\delta^\alpha \nabla v\|_{p,D}, \]
for every function \( v \) which is harmonic in \( D \) and such that \( v(x_0) = 0 \);
(ii) there exists a positive constant \( \overline{\mu}_{r,p,\alpha}(D) \) such that
\[ (5.16) \quad \|v - v_D\|_{r,D} \leq \overline{\mu}_{r,p,\alpha}(D)^{-1}\|\delta^\alpha \nabla v\|_{p,D}, \]
for every function \( v \) which is harmonic in \( D \).

Furthermore, the following explicit bounds hold. Recalling the notation in (3.1), when \( r, p \) and \( \alpha \) are as in (5.13), we have that
\[ (5.17) \quad \overline{\mu}_{r,p,\alpha}(D)^{-1} \leq k_{N,r,p,\alpha} \left( \frac{d_D}{r_i} \right)^N |D|^{\frac{1}{N} + \frac{1}{p} + \frac{1}{r}} \]
and
\[ (5.18) \quad \mu_{r,p,\alpha}(D, x_0)^{-1} \leq k_{N,r,p,\alpha} \left( \frac{d_D}{\min\{r_i, \delta(x_0)\}} \right)^N |D|^{\frac{1}{N} + \frac{1}{p} + \frac{1}{r}}, \]
for some positive constant \( k_{N,r,p,\alpha} \). When instead \( r, p \) and \( \alpha \) are as in (5.14), we have that
\[ (5.19) \quad \overline{\mu}_{p,p,0}(D)^{-1} \leq k_{N,p} \frac{d_D^{3N(1 + \frac{p}{r}) - 1}}{r_i^{3N(1 + \frac{p}{r})}}, \]
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\[ \mu_{p,p,0}(D, x_0)^{-1} \leq k_{N,p} \frac{d^N}{\min[r_i, \delta(x_0)]^{3N(1+\frac{N}{p})+1}}, \]

for some positive constant \( k_{N,p} \).

Lemma 5.3 follows from [MP20a, item(i) of Lemma 2.1 and items (i) and (ii) of Remark 2.4].

From Lemma 5.3 we can derive estimates for the derivatives of harmonic functions, as stated in the next result (a proof of this can be found in [MP20a, Corollary 2.3]).

**Corollary 5.4.** Let \( D \subset \mathbb{R}^N \) be a bounded domain satisfying the uniform interior sphere condition with radius \( r_i \), and let \( v \) be a harmonic function in \( D \). Consider three real numbers \( r, p \) and \( \alpha \) satisfying either (5.13) or (5.14).

(i) If \( x_0 \) is a critical point of \( v \) in \( D \), then it holds that

\[ \| \nabla v \|_{r,D} \leq \mu_{r,p,\alpha}(D, x_0)^{-1} \| \delta^\alpha \nabla^2 v \|_{p,D}. \]

(ii) If

\[ \int_D \nabla v(x) \, dx = 0, \]

then it holds that

\[ \| \nabla v \|_{r,D} \leq \mu_{r,p,\alpha}(D)^{-1} \| \delta^\alpha \nabla^2 v \|_{p,D}. \]

**Remark 5.5.** For later use, we mention that Lemma 5.3 and Corollary 5.4 hold true more in general if the assumption of the uniform interior sphere condition is dropped and replaced by the assumption that \( D \) is a John domain (see [Pog18, Section 3.2] or [MP20a, Lemma 2.1 and Corollary 2.3]): in this case explicit estimates of the relevant constants now depending on the John parameter can be found in [MP20a, Remark 2.4].

With the aid of Corollary 5.4 we now prove the following lemma, which, together with the forthcoming Theorem 5.7, leads to a stability estimate in terms of the pseudo-distance introduced in (5.1).

**Lemma 5.6.** Let \( \Omega \setminus \omega \subset \mathbb{R}^N \) be a bounded domain of class \( C^1 \) satisfying the uniform interior sphere condition with radius \( r_i \), that is (1.6), and let \( v \in C^2(\overline{\Omega} \setminus \omega) \) be a harmonic function in \( \Omega \setminus \omega \). Let \( u \in C^1(\overline{\Omega} \setminus \omega) \) satisfy (1.1) and assume that \( u \leq 0 \) on \( \partial \omega \).

(i) If \( x_0 \) is a critical point of \( v \) in \( \Omega \), then it holds that

\[ \int_{\Gamma} |\nabla v|^2 \, dS_x \leq \frac{2N}{r_i} \left( 1 + \frac{N}{r_i \mu_{2,\frac{1}{2}}(\Omega \setminus \omega, x_0)^2} \right) \int_{\Omega \setminus \omega} (-u) |\nabla^2 v|^2 \, dx \]

\[ - \frac{N}{r_i} \int_{\partial \omega} \{ |\nabla v|^2 u_\nu - 2u < \nabla^2 v \nabla v, \nu > \} \, dS_x. \]

(ii) If

\[ \int_{\Omega \setminus \omega} \nabla v \, dx = 0, \]

then it holds that

\[ \int_{\Gamma} |\nabla v|^2 \, dS_x \leq \frac{2N}{r_i} \left( 1 + \frac{N}{r_i \mu_{2,\frac{1}{2}}(\Omega \setminus \omega)^2} \right) \int_{\Omega \setminus \omega} (-u) |\nabla^2 v|^2 \, dx \]

\[ - \frac{N}{r_i} \int_{\partial \omega} \{ |\nabla v|^2 u_\nu - 2u < \nabla^2 v \nabla v, \nu > \} \, dS_x. \]
Proof. We begin with the following differential identity:

\[
\text{div} \left\{ v^2 \nabla u - u \nabla (v^2) \right\} = v^2 \Delta u - u \Delta (v^2) = v^2 - 2u |\nabla v|^2,
\]

that holds in \( \Omega \setminus \mathcal{W} \) for any harmonic function \( v \) in \( \Omega \setminus \mathcal{W} \), if \( u \) is satisfies \( (1.1) \).

Next, we integrate \((5.22)\) on \( \Omega \setminus \mathcal{W} \) and, by the divergence theorem, we get

\[
\int_{\Gamma} v^2 u_\nu \, dS_x = \int_{\Omega \setminus \mathcal{W}} v^2 \, dx + 2 \int_{\Omega \setminus \mathcal{W}} (-u) |\nabla v|^2 \, dx - \int_{\partial \omega} \{ v^2 u_\nu - 2uvu_\nu \} \, dS_x.
\]

We use this identity replacing the harmonic function \((5.28)\) with its derivative \( v_i \), and then we sum up over \( i = 1, \ldots, N \). In this way, we obtain

\[
\int_{\Gamma} |\nabla v|^2 u_\nu \, dS_x = \int_{\Omega \setminus \mathcal{W}} |\nabla v|^2 \, dx + 2 \int_{\Omega \setminus \mathcal{W}} (-u) |\nabla v|^2 \, dx - \int_{\partial \omega} \{ |\nabla v|^2 u_\nu - 2uv < \nabla^2 v \nabla v, \nu > \} \, dS_x.
\]

We observe that, by an adaptation of Hopf’s lemma (see \cite{MP19} Theorem 3.10), the term \( u_\nu \) in the left-hand side of \((5.22)\) can be bounded from below by \( r_i/N \), namely

\[
u u_\nu \geq \frac{r_i}{N} \quad \text{on } \Gamma.
\]

Hence, we obtain from \((5.22)\) that

\[
\frac{r_i}{N} \int_{\Gamma} |\nabla v|^2 \, dS_x \leq \int_{\Omega \setminus \mathcal{W}} |\nabla v|^2 \, dx + 2 \int_{\Omega \setminus \mathcal{W}} (-u) |\nabla v|^2 \, dx - \int_{\partial \omega} \{ |\nabla v|^2 u_\nu - 2uv < \nabla^2 v \nabla v, \nu > \} \, dS_x.
\]

Now we suppose that \( x_0 \) is a critical point of \( v \) in \( \Omega \) and we use item (i) in Corollary 5.4 applied here with \( D := \Omega \setminus \mathcal{W} \), \( r := p := 2 \) and \( \alpha := 1/2 \), and we deduce from \((5.24)\) that

\[
\int_{\Gamma} |\nabla v|^2 \, dS_x \leq \frac{N}{r_i \mu_{2,2}^{\frac{1}{2}} (\Omega \setminus \mathcal{W}, x_0)^{\frac{3}{2}}} \int_{\Omega \setminus \mathcal{W}} \delta |\nabla^2 v|^2 \, dx + \frac{2N}{r_i} \int_{\Omega \setminus \mathcal{W}} (-u) |\nabla^2 v|^2 \, dx - \frac{N}{r_i} \int_{\partial \omega} \{ |\nabla v|^2 u_\nu - 2uv < \nabla^2 v \nabla v, \nu > \} \, dS_x.
\]

From this and \((5.6)\), one obtains the desired estimate in item (i). In a similar way, using item (ii) in Corollary 5.4 one shows item (ii) here, thus completing the proof.

Now, we turn our attention to the harmonic function

\[
h := q - u,
\]

where

\[
q(x) := \frac{1}{2N} (|x - z|^2 - a),
\]

for some choice of \( z \in \mathbb{R}^N \) and \( a \in \mathbb{R} \).

We remark that, by a direct computation, it is easy to check that the Cauchy-Schwarz deficit appearing in the left-hand side of \((4.15)\) can be written in terms of \( h \) as

\[
|\nabla^2 h|^2 = |\nabla^2 u|^2 - \frac{1}{N} = |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}.
\]

Now we specify the choice of the point \( z \) in \((5.26)\) as follows

\[
z := \frac{1}{|\Omega \setminus \mathcal{W}|} \left\{ \int_{\Omega \setminus \mathcal{W}} x \, dx - N \int_{\partial \omega} u \, \nu \, dS_x \right\}.
\]
We notice that as \( \omega \) tends to the empty set and \( \int_{\partial \omega} (-u) \, dS_x \) tends to 0, \( z \) tends to the baricenter of \( \Omega \) (however, \( z \) is not the baricenter of \( \Omega \backslash \omega \)).

With this choice of \( z \) we have that

\[
(5.29) \quad \int_{\Omega \backslash \omega} \nabla h \, dx = 0.
\]

Indeed, by a direct computation we get that

\[
(5.30) \quad \nabla h = \frac{(x - z)}{N} - \nabla u,
\]

and therefore, using Green’s identity and the fact that \( u = 0 \) on \( \Gamma \),

\[
\int_{\Omega \backslash \omega} \nabla h \, dx = \int_{\Omega \backslash \omega} \frac{(x - z)}{N} \, dx - \int_{\Omega \backslash \omega} \nabla u \, dx
\]

\[
= \int_{\Omega \backslash \omega} \frac{x}{N} \, dx - \frac{z}{N} |\Omega \backslash \omega| - \int_{\partial \omega} u \nu \, dS_x = 0,
\]

thus proving \((5.29)\).

Gathering the previous results, we thus obtain the desired estimate on the pseudo-distance:

**Theorem 5.7.** Let \( u \in C^2(\Omega \backslash \omega) \) satisfy \((1.1)\) and \((1.2)\), and assume that \( u \leq 0 \) on \( \partial \omega \). Let assumptions \((1.5)\) and \((1.6)\) be verified.

Then, with the notation of \((5.26)\) and \((5.28)\), we have that

\[
\int_{\Gamma} \left| \frac{x - z}{N} - c \right|^2 \, dS_x
\]

\[
\leq \frac{N}{r_i} \left( 1 + \frac{N}{r_i \bar{r}_{2,2}, \frac{1}{2}(\Omega \backslash \omega)^2} \right) \left\{ \int_{\partial \omega} \left[ c^2 \left( \frac{\langle x, \nu \rangle}{N} - u_v \right) + 2u \left( \frac{\langle x, \nu \rangle}{N} - u_v \right) \right.ight.
\]

\[
\left. + u_v |\nabla u|^2 - 2 \frac{\langle x, \nabla u \rangle}{N} u_v + |\nabla u|^2 - 2 \frac{\langle x, \nabla u \rangle}{N} u_v + \frac{2}{N} u_v u \right] \, dS_x \right\}
\]

\[- \frac{N}{r_i} \int_{\partial \omega} \left\{ |\nabla h|^2 u_v - 2u < \nabla^2 h \nabla h, \nu > \right\} \, dS_x.
\]

**Proof.** By \((5.30)\) and the Cauchy-Schwarz inequality,

\[
\left| \frac{x - z}{N} - |\nabla u| \right| \leq |\nabla h|.
\]

Hence, by using \((1.2)\), we get that

\[
(5.31) \quad \int_{\Gamma} \left| \frac{x - z}{N} - c \right|^2 \, dS_x \leq \int_{\Gamma} |\nabla h|^2 \, dS_x.
\]

Now we remark that we are in the position of using point (ii) of Lemma 5.6 with \( v := h \), thanks to \((5.29)\). Hence, putting together \((5.31)\), Lemma 5.6 \((5.27)\) and \((4.15)\) we get the desired result.

\[\Box\]

### 6. Some estimates on \( \rho_e - \rho_i \)

If \( z \) is as in \((5.28)\), we set

\[
(6.1) \quad \rho_e := \max_{x \in \Gamma} |x - z| \quad \text{and} \quad \rho_i := \min_{x \in \Gamma} |x - z|.
\]

In this way, whenever \( z \in \Omega \), we have that

\[
(6.2) \quad B_{\rho_e}(z) \subset \Omega \subset B_{\rho_e}(z) \quad \text{and} \quad \Gamma \subset \overline{B}_{\rho_e}(z) \setminus B_{\rho_i}(z).
\]

The aim of the present section is to obtain quantitative estimates for the difference \( \rho_e - \rho_i \).
We remark that, recalling the notation in (3.1),
\[ (6.3) \quad \rho_e - \rho_i \leq d_{\Omega}. \]
Indeed, for every \( w_1, w_2 \in \Gamma \), we have that
\[ |w_1 - z| \leq |w_1 - w_2| + |w_2 - z| \leq d_{\Omega} + |w_2 - z|. \]
As a result, taking \( w_1 \) maximizing the distance to \( z \), and \( w_2 \) minimizing the distance to \( z \), we obtain that \( \rho_e \leq d_{\Omega} + \rho_i \), that is (6.3).

Also, by using \( \delta_\Gamma(x) \) to denote the distance of a point \( x \in \Omega \) to the boundary \( \Gamma \) we define the complementary parallel set as
\[ (6.4) \quad \Omega^\sigma_\Gamma := \{ y \in \Omega : \delta_\Gamma(y) < \sigma \} \quad \text{for } 0 < \sigma \leq r_i. \]
Notice that, since \( \Omega \cap \overline{\omega} \) satisfies the uniform interior sphere condition of radius \( r_i \), it holds that
\[ \Omega^\sigma_\Gamma \subset \Omega \setminus \overline{\omega} \quad \text{for every } 0 < \sigma \leq r_i. \]

Lemma 6.1 below contains an inequality for the oscillation of a harmonic function \( v \) in terms of its \( L^p \)-norm in \( \Omega \setminus \overline{\omega} \) and of a bound for its gradient in \( \Omega^\sigma_\Gamma \). More precisely, recalling the notation in (5.11) and (5.12), we have:

**Lemma 6.1.** Let \( \Omega \setminus \overline{\omega} \subset \mathbb{R}^N \) satisfy the uniform interior sphere condition of radius \( r_i \) on \( \Gamma \), that is (1.17), and suppose that \( \Gamma \) is of class \( C^1 \). Let \( v \) be a harmonic function in \( \Omega \setminus \overline{\omega} \) of class \( C^1(\Omega^\sigma_\Gamma) \), and let \( G \) be an upper bound for the gradient of \( v \) on \( \Omega^\sigma_\Gamma \).

Then, given \( p \geq 1 \), there exist two positive constants \( a_{N,p} \) and \( \alpha_{N,p} \) depending only on \( N \) and \( p \) such that if
\[ (6.5) \quad \| v - v_{\Omega \setminus \overline{\omega}} \|_{p,\Omega \setminus \overline{\omega}} \leq \alpha_{N,p} r_i^{N+p} G, \]
then we have that
\[ (6.6) \quad \max_{\Gamma} v - \min_{\Gamma} v \leq a_{N,p} G^{N+p} \| v - v_{\Omega \setminus \overline{\omega}} \|_{p,\Omega \setminus \overline{\omega}}^{p/(N+p)}. \]

Lemmata 6.1 and 6.3 and Theorem 6.4 here adapt to the present situation ideas originating from [Pog18] and [MP20a] – see also [MP20b] for generalizations in other directions of those ideas. Here, we obtain Lemma 6.1 as an immediate consequence of the following refined estimate, that will be crucial in Subsection 8.2.

**Lemma 6.2.** Let \( \Omega \setminus \overline{\omega} \subset \mathbb{R}^N \) satisfy the uniform interior sphere condition of radius \( r_i \) on \( \Gamma \), that is (1.17), and suppose that \( \Gamma \) is of class \( C^1 \). Let \( v \) be a harmonic function in \( \Omega \setminus \overline{\omega} \) of class \( C^1(\Omega^\sigma_\Gamma) \), and let \( G \) be an upper bound for the gradient of \( v \) on \( \Omega^\sigma_\Gamma \).

Given \( \lambda \in \mathbb{R} \), we choose \( \overline{x} \in \Gamma \) for which
\[ (6.7) \quad |v(\overline{x}) - \lambda| = \max_{\Gamma} |v(x) - \lambda| \]
and set
\[ (6.8) \quad x_0 := \overline{x} - r_i v(\overline{x}). \]

Then, given \( p \geq 1 \), there exist two positive constants \( a_{N,p} \) and \( \alpha_{N,p} \) depending only on \( N \) and \( p \) such that if
\[ (6.9) \quad \| v - \lambda \|_{p,B_{r_i}(x_0)} \leq \alpha_{N,p} r_i^{N+p} G, \]
then we have that
\[ (6.10) \quad \max_{\Gamma} v - \min_{\Gamma} v \leq a_{N,p} G^{N+p} \| v - \lambda \|_{p,B_{r_i}(x_0)}^{p/(N+p)}. \]
Proof. By (6.7), it holds that

\[(6.11) \quad \max_{\Gamma} v - \min_{\Gamma} v \leq 2 |v(\bar{x}) - \lambda|\]

For $0 < \sigma \leq r_i$, we define

\[\bar{y} := \bar{x} - \sigma \nu(\bar{x}).\]

Notice that, in light of (6.8) and (1.17) – and being $\Gamma$ of class $C^1$ – we have that

\[B_1(\bar{y}) \subset B_{r_i}(x_0) \subset \Omega \setminus \varpi.\]

By the fundamental theorem of calculus we have that

\[(6.13) \quad v(\bar{x}) = v(\bar{y}) + \int_{0}^{\sigma} \langle \nabla v(\bar{x} - t\nu(\bar{x})), \nu(\bar{x}) \rangle \, dt.\]

Furthermore, since $v$ is harmonic in $\Omega \setminus \varpi$, we can use the mean value property for the balls with radius $\sigma$ centered at $\bar{y}$, thanks to (6.12), and find that

\[|v(\bar{y}) - \lambda| = \frac{1}{|B_1|} \int_{B_1(\bar{y})} v(y) \, dy - \lambda \]

\[\leq \frac{1}{|B_1|} \int_{B_1(\bar{y})} |v - \lambda| \, dy \]

\[\leq \frac{1}{|B_1|} \left[ \int_{B_1(\bar{y})} |v - \lambda|^p \, dy \right]^{1/p} \]

\[\leq \frac{1}{|B_1|} \left[ \int_{B_1(x_0)} |v - \lambda|^p \, dy \right]^{1/p},\]

where we used an application of Hölder’s inequality and (6.12) once again.

This, (6.11) and (6.13) yield that

\[(6.14) \quad \max_{\Gamma} v - \min_{\Gamma} v \leq 2 |v(\bar{x}) - \lambda| \]

\[= 2 \left| v(\bar{y}) - \lambda + \int_{0}^{\sigma} \langle \nabla v(\bar{x} - t\nu(\bar{x})), \nu(\bar{x}) \rangle \, dt \right| \leq 2 (|v(\bar{y}) - \lambda| + \sigma G) \]

\[\leq 2 \left[ \frac{\|v - \lambda\|_{p, B_{r_i}(x_0)}}{|B_1|^{1/p} \sigma^{N/p}} + \sigma G \right],\]

for every $0 < \sigma \leq r_i$.

Therefore, by minimizing the right-hand side of the last inequality, we can conveniently choose

\[(6.15) \quad \sigma := \left( \frac{N \|v - \lambda\|_{p, B_{r_i}(x_0)}}{p |B_1|^{1/p} G} \right)^{p/(N+p)}\]

and obtain (6.10) if $\sigma \leq r_i$; (6.9) follows.

The computations show that

\[(6.16) \quad a_{N,p} := \frac{2(N + p)}{N^{N+p} \sigma^{N+p} |B_1|^{1/N+p}} \quad \text{and} \quad \alpha_{N,p} := \frac{p}{N} |B_1|^{1/p}.\]

From Lemma 6.2, we immediately get the proof of Lemma 6.1 as follows:

Proof of Lemma 6.1. Since, by (6.12)

\[\|v - \lambda\|_{p, B_{r_i}(x_0)} \leq \|v - \lambda\|_{p, \Omega \setminus \varpi};\]

\[\Box\]
the desired result easily follows from Lemma 6.2, by choosing \( \lambda := v_{\Omega, \Sigma} \). The constants \( a_{N,p} \) and \( \alpha_{N,p} \) are still those defined in (6.16). \( \square \)

We now turn our attention to the harmonic function \( h \) introduced in (5.25), and we modify Lemma 6.1 to link \( \rho_e - \rho_i \) to the \( L^p \)-norm of \( h \). Since \( h = q \) on \( \Gamma \), we have that

\[
\max_{\Gamma} h - \min_{\Gamma} h = \max_{\Gamma} q - \min_{\Gamma} q = \frac{1}{2N} \left( \max_{x \in \Gamma} |x - z|^2 - \min_{x \in \Gamma} |x - z|^2 \right)
\]

(6.17)

due to (5.26) and (6.1).

We also observe that, by definition of \( \rho_e \), it follows that

\[
\rho_e \geq \frac{d_\Omega}{2}.
\]

(6.18)

Then, from (6.17) and (6.18) we obtain that

\[
\max_{\Gamma} h - \min_{\Gamma} h \geq \frac{d_\Omega}{4N} (\rho_e - \rho_i).
\]

The next result gives an explicit bound on the difference \( \rho_e - \rho_i \):

**Lemma 6.3.** Let \( \Omega \setminus \omega \subset \mathbb{R}^N \) satisfy the uniform interior sphere condition of radius \( r_i \) on \( \Gamma \), that is \( (1.17) \), and suppose that \( \Gamma \) is of class \( C^1 \). Let \( u \) satisfy (1.1) and \( u \in C^1 (\Omega \setminus \omega) \cup \Gamma) \), let \( q \) be as in (5.26) with \( z \in \Omega \), and let \( h \) be as in (5.25).

Then, there exists a positive constant \( C \) such that

\[
\rho_e - \rho_i \leq C \| h - h_{\Omega \setminus \omega} \|_{p, \Omega \setminus \omega}^{p/(N + p)}.
\]

(6.20)

The constant \( C \) depends on \( N, p, d_\Omega, r_i, M \), where

\[
M := \max_{\Omega \setminus \omega} |\nabla u|.
\]

(6.21)

**Proof.** By direct computations (see e.g. (5.30)) it is easy to check that

\[
|\nabla h| \leq M + \frac{d_\Omega}{N} \quad \text{on} \ \overline{\Omega_{r_i}},
\]

where \( M \) is defined in (6.21).

We now consider the constants \( a_{N,p} \) and \( \alpha_{N,p} \) defined in (6.16) and we distinguish two cases, according on whether

\[
\| h - h_{\Omega \setminus \omega} \|_{p, \Omega \setminus \omega} \leq \alpha_{N,p} \left( M + \frac{d_\Omega}{N} \right)^{\frac{N + p}{p}} r_i
\]

or

\[
\| h - h_{\Omega \setminus \omega} \|_{p, \Omega \setminus \omega} > \alpha_{N,p} \left( M + \frac{d_\Omega}{N} \right)^{\frac{N + p}{p}} r_i
\]

(6.23)

If (6.22) holds true, we can apply Lemma 6.1 with \( v := h \) and \( G := M + \frac{d_\Omega}{N} \). Thus, by means of (6.19) we deduce that (6.20) holds with

\[
C := 4N a_{N,p} \left( M + \frac{d_\Omega}{N} \right)^{\frac{N}{N + p}} d_\Omega.
\]

(6.24)

On the other hand, if (6.23) holds true, it is trivial to check that (6.20) is verified with

\[
C := \frac{d_\Omega}{\left[ \alpha_{N,p} \left( M + \frac{d_\Omega}{N} \right) \right]^{\frac{p}{N + p}} r_i},
\]

thanks to (6.3).
Thus, (6.20) always holds true if we choose the maximum between this constant and that in (6.24).

**Theorem 6.4.** Let $\Omega \setminus \mathcal{W} \subset \mathbb{R}^N$ satisfy the uniform interior sphere condition of radius $r_i$, that is (1.6), and suppose that $\Gamma$ is of class $C^1$. Let $u$ satisfy (1.1), $u \in C^1((\Omega \setminus \mathcal{W}) \cup \Gamma)$, and suppose that $u \leq 0$ on $\partial \omega$. Let $q$ be as in (5.26) with $z$ chosen as in (5.28), and assume that $z$ belongs to $\Omega$. Let $h$ be as in (5.25).

Then, there exists a positive constant $C$ such that

\[
\rho_e - \rho_i \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|^\tau {_{2,\Omega \setminus \mathcal{W}}},
\]

with the following specifications:

(i) $\tau_2 = 1$;

(ii) $\tau_3$ is arbitrarily close to 1, in the sense that, for any $\theta \in (0, 1)$ sufficiently small, there exists a positive constant $C$ such that (6.25) holds with $\tau_3 = 1 - \theta$;

(iii) $\tau_N = 2/(N - 1)$ for $N \geq 4$.

The constant $C$ depends on $N$, $r_i$, $d_\Omega$, $M$ (as defined in (6.21)), and $\theta$ (the latter, only in the case $N = 3$).

**Proof.** For the sake of clarity, we will always use the letter $C$ to denote the constants in all the inequalities appearing in the proof. Their explicit computation will be clear by following the steps of the proof (see the forthcoming Remark 6.5).

(i) Let $N = 2$. By the Sobolev immersion theorem (for instance we apply [Fri76, Theorem 9.1] to the function $h - h_\Omega$), we deduce that there exists a positive constant $C$ such that

\[
\max_{\Omega \setminus \omega} |h - h_\Omega| \leq C \| h - h_\Omega \|^4 {_{W^{1,4}(\Omega \setminus \mathcal{W})}}.
\]

As noticed in [MP20a, Remark 2.9], the immersion constant in (6.26) depends on $N$ and $r_i$ only.

Applying (5.16) with $D := \Omega \setminus \mathcal{W}$, $v := h$, $r := p := 4$, and $\alpha := 0$ leads to

\[
\| h - h_\Omega \|^4 {_{W^{1,4}(\Omega \setminus \mathcal{W})}} \leq C \| \nabla h \|^4 {_{4,\Omega \setminus \mathcal{W}}},
\]

Also, since (5.29) holds true, we can apply item (ii) of Corollary 5.4 with $v := h$, $D := \Omega \setminus \mathcal{W}$, $r := 4$, $p := 2$, and $\alpha := 1/2$ and obtain that

\[
\| \nabla h \|^4 {_{4,\Omega \setminus \mathcal{W}}} \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|^\tau {_{2,\Omega \setminus \mathcal{W}}}.
\]

From this and (6.27), we get that

\[
\| h - h_\Omega \|^4 {_{W^{1,4}(\Omega \setminus \mathcal{W})}} \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|^\tau {_{2,\Omega \setminus \mathcal{W}}}.
\]

This inequality, together with (6.26), gives that

\[
\max_{\Gamma} h - \min_{\Gamma} h \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|^\tau {_{2,\Omega \setminus \mathcal{W}}}.
\]

Thus, by recalling (6.19) we get that (6.25) holds with $\tau_2 = 1$.

(ii) Let $N = 3$. For any $\theta \in (0, 1)$ sufficiently small, we notice that

\[
r := \frac{3(1 - \theta)}{\theta}, \quad p := 3(1 - \theta) \quad \text{and} \quad \alpha := 0
\]

satisfy (5.13) in Lemma 5.3. Hence, we can apply the estimate in (5.16) with $D := \Omega \setminus \mathcal{W}$ and $v := h$, obtaining that

\[
\| h - h_\Omega \|^4 {_{W^{1,4}(\Omega \setminus \mathcal{W})}} \leq C \| \nabla h \|^4 {_{2,\Omega \setminus \mathcal{W}}}.
\]
Furthermore,
\[ r := 3(1 - \theta), \quad p := 2, \quad \text{and} \quad \alpha := \frac{1}{2} \]
satisfy (5.13), and therefore item (ii) of Corollary 5.4 applied again with \( D := \Omega \setminus \varnothing \) and \( v := h \), yields that
\[ \| \nabla h \|_{2, \Omega \setminus \varnothing} \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|_{2, \Omega \setminus \varnothing}. \]

This and (6.28) give that
\[ \| h - h_{\Omega \setminus \varnothing} \|_{\frac{N+1}{2N-1}, \Omega \setminus \varnothing} \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|_{2, \Omega \setminus \varnothing}. \]

Thus, by using Lemma 6.3 with \( p := \frac{3(1-\theta)}{\theta} \), we obtain that
\[ \rho_c - \rho_i \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|_{2, \Omega \setminus \varnothing}, \]
which is (6.25) with \( \tau_3 = 1 - \theta \).

(iii) Let \( N \geq 4 \). In light of (5.29), we can apply to \( h \) item (ii) of Corollary 5.4 with \( D := \Omega \setminus \varnothing \), \( r := \frac{2N}{N-1} \), \( p := 2 \), and \( \alpha := 1/2 \) (noticing that they satisfy (5.13)), and obtain that
\[ \| \nabla h \|_{\frac{2N}{N-1}, \Omega \setminus \varnothing} \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|_{2, \Omega \setminus \varnothing}. \]

Being \( N \geq 4 \), we can also apply (5.16) with \( D := \Omega \setminus \varnothing \), \( v := h \), \( r := \frac{2N}{N-3} \), \( p := \frac{2N}{N-1} \), and \( \alpha := 0 \), and get
\[ \| h - h_{\Omega \setminus \varnothing} \|_{\frac{2N}{N-3}, \Omega \setminus \varnothing} \leq C \| \nabla h \|_{\frac{2N}{N-3}, \Omega \setminus \varnothing}. \]

Thus, from (6.29) and (6.30) we conclude that
\[ \| h - h_{\Omega \setminus \varnothing} \|_{\frac{2N}{N-3}, \Omega \setminus \varnothing} \leq C \| \delta^{\frac{1}{2}} \nabla^2 h \|_{2, \Omega \setminus \varnothing}. \]

Then, Lemma 6.3 applied with \( p := \frac{2N}{N-3} \), gives that (6.25) holds with \( \tau_N = 2/(N - 1) \). \( \square \)

**Remark 6.5** (On the constant \( C \) in (6.25)). The constant \( C \) in (6.25) can be explicitly computed by following the steps of the proof of Theorem 6.4, and it can be shown to depend only on the parameters mentioned in the statement of Theorem 6.4. Indeed, the parameters \( \overline{\mu}_{r,p,\alpha}(\Omega \setminus \varnothing) \) and \( \mu_{r,p,\alpha}(\Omega \setminus \varnothing) \), can be estimated by means of (5.17) and (5.19). Then, we notice that \( d_{\Omega \setminus \varnothing} \leq d_{\Omega} \) and
\[ (6.31) \quad |\Omega \setminus \varnothing| \leq |\Omega|. \]

Finally, to remove the dependence on the volume, we use the trivial bound
\[ (6.32) \quad |\Omega|^{1/N} \leq |B_1|^{1/N} d_{\Omega}/2. \]

**Remark 6.6** (Another choice for the point \( z \) in (6.1)). Another possible way to choose \( z \) in (6.1) (different from (5.28)) is
\[ z = x_0 - \nabla u(x_0), \]
where \( x_0 \in \Omega \setminus \varnothing \) is any point such that \( \delta(x_0) \geq r_i \). In fact, with this choice we obtain that \( \nabla h(x_0) = 0 \) and we can thus use item (i) of Corollary 5.4 instead of item (ii).

Then, we can estimate the parameters \( \mu_{r,p,\alpha}(\Omega \setminus \varnothing, x_0) \) in terms of \( r_i \) and \( d_{\Omega} \) by using (5.18), the fact that \( \delta(x_0) \geq r_i \), and (6.32).

Also with this choice, in order to obtain an analogue of Theorem 6.4 we should additionally require that \( z \in \Omega \), to be sure that the ball \( B_{\rho_i}(z) \) is contained in \( \Omega \).
7. Stability results and proof of Theorem 1.1

For the sake of clarity, we state here some notation that will be used throughout the rest of this paper.

We denote by \( \bar{d}_\omega \) the supremum of the diameters of all the connected components of \( \omega \) (of course, if \( \omega \) is connected, then \( \bar{d}_\omega \) coincides with the diameter of \( \omega \), and, in this case, according to the notation in (3.1), it holds that \( \bar{d}_\omega = d_\omega \)). Then, we have:

**Theorem 7.1** (General stability result for \( \rho_e - \rho_i \)). Let \( u \in C^2(\Omega \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let assumptions (1.5) and (1.6) be verified. Assume also that the point \( z \) chosen in (5.28) belongs to \( \Omega \).

If \( \psi : [0, \infty) \rightarrow [0, \infty) \) is a continuous function vanishing at 0 such that

\[
\begin{align*}
\int_{\partial \omega} (-u) \, dS_x &= \int_{\partial \omega} |u| \, dS_x \\
\int_{\partial \omega} |u| \, dS_x &\leq \psi(\eta) \quad \text{with} \quad \eta = |\partial \omega| \quad \text{or} \quad \eta = \bar{d}_\omega, \\
\int_{\partial \omega} |\nabla u|^2 \, dS_x &\leq \psi(\eta) \quad \text{with} \quad \eta = |\partial \omega| \quad \text{or} \quad \eta = \bar{d}_\omega,
\end{align*}
\]

then

\[
\rho_e - \rho_i \leq C \psi(\eta)^{\tau_N/2},
\]
where \( \tau_N \) is as in Theorem 6.4 and \( C \) is a positive constant depending on \( N \), \( r_i \), \( d_\Omega \), and \( M \) (as defined in 6.21).

**Proof.** Up to a translation, we can suppose that the origin lies in \( \Omega \), and therefore

\[ |< x, \nu >| \leq |x| \leq d_\Omega. \]

Hence, the desired result easily follows by putting together Theorem 6.4 and formulas (5.6), (5.27) and (4.15). \( \square \)

**Remark 7.2.** We notice that with the choice of \( z \) as in (5.28), (if \( \eta \) is small enough) the assumption \( z \in \Omega \) is satisfied, at least if the baricenter of \( \Omega \) lies in \( \Omega \) (in particular, if \( \Omega \) is convex).

With this preliminary work, we are now in the position of obtaining a quantitative rigidity result bounding the averaged squared pseudodistance of the form

\[
\int_F \left| \frac{|x - z|}{N} - c \right|^2 \, dS_x,
\]
where \( z \) is as in (5.28) and \( c \) is that in (1.2). The precise result goes as follows:

**Theorem 7.3** (General stability result for a pseudodistance). Let \( u \in C^2(\Omega \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let assumptions (1.5) and (1.6) be verified, and \( z \) be as in (5.28).

If \( \psi : [0, \infty) \rightarrow [0, \infty) \) is a continuous function vanishing at 0 such that (7.1) holds true together with

\[
\int_{\partial \omega} u < \nabla^2 u (x - z), \nu > \, dS_x \leq \psi(\eta) \quad \text{with} \quad \eta = |\partial \omega| \quad \text{or} \quad \eta = \bar{d}_\omega,
\]

then

\[
\int_{\partial \omega} u \, dS_x \leq C \psi(\eta)^{\tau_N/2},
\]
where \( \tau_N \) is as in Theorem 6.4 and \( C \) is a positive constant depending on \( N \), \( r_i \), \( d_\Omega \), and \( M \) (as defined in 6.21).

In 6.4 and formulas (5.6), (5.27) and (4.15).
then
\begin{equation}
\int_{\Gamma} \left| \frac{x - z}{N} - c \right|^2 dS_x \leq C \psi(\eta),
\end{equation}
where \( C \) is a constant depending on \( N, r_i, d_\Omega \).

**Proof.** In light of (5.25), (5.26), and (5.30), we see that
\[
|\nabla h|^2 u_\nu = \left\{ \left( \frac{|x - z|}{N} \right)^2 - \frac{2}{N} < (x - z), \nabla u > + |\nabla u|^2 \right\} u_\nu
\]
and
\[
< \nabla^2 h \nabla h, \nu > = < \left( \frac{1}{N} I - \nabla^2 u \right) \left( \frac{x - z}{N} - \nabla u \right), \nu >
\]
\[
= \frac{1}{N} < (x - z) - \nabla u, \nu > - < \nabla^2 u \left( \frac{x - z}{N} - \nabla u \right), \nu >
\]
\[
= \frac{1}{N^2} < (x - z), \nu > - \frac{1}{N} < \nabla u, \nu > - \frac{1}{N} < \nabla^2 u (x - z), \nu > + < \nabla^2 u \nabla u, \nu >.
\]
As a consequence,
\[
\left| \int_{\partial \omega} |\nabla h|^2 u_\nu dS_x \right| \leq \frac{d_\Omega^2}{N^2} \int_{\partial \omega} |\nabla u| dS_x + \frac{2d_\Omega}{N} \int_{\partial \omega} |\nabla u|^2 dS_x + \int_{\partial \omega} |\nabla u|^2 u_\nu dS_x
\]
and
\[
\left| \int_{\partial \omega} u < \nabla^2 h \nabla h, \nu > dS_x \right|
\leq \frac{d_\Omega}{N^2} \int_{\partial \omega} |u| dS_x + \frac{1}{N} \left| \int_{\partial \omega} u u_\nu dS_x \right|
\]
\[
+ \frac{1}{N} \left| \int_{\partial \omega} u < \nabla^2 u (x - z), \nu > dS_x \right| + \left| \int_{\partial \omega} u < \nabla^2 u \nabla u, \nu > dS_x \right|.
\]
Hence, the proof follows from the last two formulas, Theorem 5.7, (7.1), and (7.3). \( \square \)

By means of Lemma 5.1, from Theorem 7.3 we also obtain a quantitative rigidity result for the asymmetry (5.2):

**Theorem 7.4** (General stability result for an asymmetry). Let \( u \in C^2(\overline{\Omega} \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let assumptions (1.5) and (1.6) be verified, and \( z \) be as in (5.28).

If \( \psi : [0, \infty) \to [0, \infty) \) is a continuous function vanishing at 0 such that (7.1) and (7.3) hold true, then the asymmetry defined in (5.2) satisfies
\begin{equation}
\frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|} \leq C \psi(\eta)^{1/2},
\end{equation}
where \( C \) is a constant depending on \( N, r_i, d_\Omega, \) and \( c \).

**Remark 7.5.** The dependence on \( c \) of the constant in (7.5) can be dropped by exploiting suitable bounds for it. Indeed, on the one hand, putting together (1.2) and (5.23) one obtains the lower bound
\begin{equation}
c \geq \frac{r_i}{N}.
\end{equation}
On the other hand, from the expression in (4.16) one can obtain an upper bound for $c$ in terms of $N$ and $d_\Omega$, when $\eta$ is small enough. More precisely, formula (4.16) implies that

$$(7.7) \quad c |\Gamma| = |\Omega| - |\omega| - \int_{\partial\omega} u_\nu \, dS_x \leq \frac{|\Omega| - |\omega|}{2}, \quad \text{if } \eta \text{ is small enough},$$
	hanks{thanks to (7.1).}

Moreover, exploiting the classical isoperimetric inequality and (6.32), one sees that

$$(7.8) \quad \frac{|\Omega| - |\omega|}{|\Gamma|} \leq \frac{1}{N} \left( \frac{|\Omega|}{|B_1|} \right)^{\frac{1}{2}} \leq \frac{d_\Omega}{2N},$$

Putting together (7.7) and (7.8) one obtains that

$$(7.9) \quad c \leq \frac{d_\Omega}{4N}, \quad \text{if } \eta \text{ is small enough}.$$}

We can now obtain a quantitative symmetry result by assuming a $C^2$-bound of the solution along $\partial\omega$:

**Theorem 7.6.** Let $u \in C^2(\Omega \setminus \omega)$ satisfy (1.1) and (1.2), and suppose that $u \leq 0$ on $\partial\omega$. Let assumptions (1.5) and (1.6) be verified, and $z$ be as in (5.28).

If there exists $K > 0$ such that

$$(7.10) \quad \|u\|_{C^2(\partial\omega)} \leq K,$$

then

$$(7.11) \quad \int_{\Gamma} \left| \frac{|x - z|}{N} - c \right|^2 \, dS_x \leq C|\partial\omega|,$$

where $C$ is a positive constant depending on $N$, $r_i$, $d_\Omega$, and $K$.

Also, it holds that

$$(7.12) \quad \frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|} \leq C|\partial\omega|^{1/2},$$

where $C$ is a positive constant depending on $N$, $r_i$, $d_\Omega$, and $K$.

Moreover, if $z \in \Omega$, we have that

$$(7.13) \quad \rho_e - \rho_i \leq C|\partial\omega|^{\tau_N/2},$$

where $\tau_N$ are as in Theorem 6.4 and $C$ is a positive constant depending on $N$, $r_i$, $d_\Omega$, and $K$.

**Proof.** We notice that, by (7.10), we have that the assumptions in (7.1) and (7.3) are satisfied with

$$\psi(|\partial\omega|) := \max \{ K, K^3 \} |\partial\omega|,$$

and so we are in the position of applying Theorems 7.1, 7.3, and 7.4, thus obtaining the desired estimates.

Notice that the constant in (7.13) does not depend on $M$ (as defined in (6.21)), differently from that appearing in (7.2). Indeed, we claim that, when $|\partial\omega| < 1$,

$$(7.14) \quad \max_{\Omega \setminus \omega} |\nabla u| \leq C,$$

for some positive constant $C$ depending on $N$, $d_\Omega$, $r_i$, and $K$. We point out that, since $M \leq \max_{\Omega \setminus \omega} |\nabla u|$, the estimate in (7.14) provides a bound for $M$ in terms of $N$, $d_\Omega$, $r_i$, and $K$. Hence, we now focus on the proof of (7.14).

For this, we observe that, since $|\nabla u|$ attains its maximum on $\Gamma \cup \partial\omega$, recalling (1.2) and (7.10), we have that

$$(7.15) \quad \max_{\Omega \setminus \omega} |\nabla u| \leq \max \{ c, K \}.$$
As a result, to obtain the desired estimate in (7.14), it remains to find an upper bound for \( c \) depending on \( N, d_\Omega, r_i, \) and \( K \).

For this, we notice that, by (4.16) and (7.10),
\[
c = \frac{|\Omega \setminus \omega|}{|\Gamma|} - \frac{1}{|\Gamma|} \int_{\partial \omega} u_\nu \, dS_x \leq \frac{|\Omega|}{|\Gamma|} + K|\partial \omega|,
\]
and hence
\[
(7.16) \quad c \leq \frac{|\Omega|}{|\Gamma|} + K \quad \text{if} \quad |\partial \omega| < 1.
\]

On the one hand, by (7.8) we already know that
\[
(7.17) \quad \frac{|\Omega|}{|\Gamma|} \leq \frac{d_\Omega}{2N}.
\]

On the other hand, by combining the inequality
\[
|\Omega| \geq |B_1| r_i^N,
\]
that holds true since a ball of radius \( r_i \) is surely contained in \( \Omega \), with the classical isoperimetric inequality \(|\Gamma| \geq N|B_1|^{1/N} |\Omega|^{(N-1)/N} \), we get that
\[
(7.18) \quad \frac{K}{|\Gamma|} \leq \frac{1}{N|B_1|} \frac{K}{r_i^{N-1}}.
\]

Putting together (7.16), (7.17), and (7.18), we find the desired explicit upper bound for \( c \):
\[
(7.19) \quad c \leq \frac{d_\Omega}{2N} + \frac{1}{N|B_1|} \frac{K}{r_i^{N-1}} \quad \text{if} \quad |\partial \omega| < 1.
\]

In turn, this and (7.15) give that
\[
\max_{\Omega \setminus \omega} |\nabla u| \leq \max \left\{ \frac{d_\Omega}{2N} + \frac{1}{N|B_1|} \frac{K}{r_i^{N-1}}, K \right\} \quad \text{if} \quad |\partial \omega| < 1,
\]
that is the desired estimate in (7.14).

The estimate in (7.14) proves that, if \( |\partial \omega| < 1 \), (7.13) holds true (with \( C \) not depending on \( M \)). On the other hand, if \( |\partial \omega| \geq 1 \), (7.13) trivially holds true with \( C := d_\Omega \), being \( \rho_e - \rho_i \leq \rho_e \leq d_\Omega \).

Notice also that (7.12) is a global estimate in which the constant does not depend on \( c \), differently from that appearing in (7.5).

Indeed, if \( |\partial \omega| < 1 \), we can remove the dependence of the constant on \( c \) thanks to the bounds in (7.6) and (7.19).

This proves that, if \( |\partial \omega| < 1 \), (7.12) holds true with \( C \) not depending on \( c \). On the other hand, when \( |\partial \omega| \geq 1 \), (7.12) trivially holds true with \( C := \left( \frac{d_\Omega}{2r_i} \right)^N + 1 \), being
\[
\frac{|\Omega \Delta B_{Nc}|}{|B_{Nc}|} \leq \frac{|\Omega|}{|B_{Nc}|} + 1 \leq \left( \frac{d_\Omega}{2r_i} \right)^N + 1,
\]
where we have used (7.6) and (6.32) in the last inequality.

These observations complete the proof of Theorem 7.6. \( \square \)

We observe that our main result in Theorem 1.1 is now a simple consequence of Theorem 7.6.

Another instance in which the assumptions of Theorems 7.1, 7.3, and 7.4, are surely satisfied is the following. Notice that in high dimensions (i.e., when \( N > 4 \)) the following result allows \( u \) to blow up on \( \partial \omega \) as either the perimeter or the diameter tends to zero.
Theorem 7.7. Let \( u \in C^2(\Omega \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let assumption (1.6) be verified, and \( z \) be as in (5.28). Let \( \omega \) be the union of finitely many disjoint balls of radius \( \varepsilon \) and assume that
\[
\|u\|_{L^\infty(\partial \omega)} + \varepsilon \|\nabla u\|_{L^\infty(\partial \omega)} + \varepsilon^2 \|\nabla^2 u\|_{L^\infty(\partial \omega)} = o(\varepsilon^{4-N}).
\]
Then,
\[
\int_{\Gamma} \left| \frac{|x-z|}{N} - c \right|^2 dS_x \quad \text{and} \quad \frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|}
\]
are as small as we wish for small \( \varepsilon \).

Furthermore, if in addition \( z \in \Omega \), then also \( \rho_e - \rho_i \) is as small as we wish for small \( \varepsilon \).

8. The case of general domains and proofs of Theorems 1.2 and 1.3

As mentioned in the Introduction, analogous stability results can be obtained by weakening the assumptions in (1.5) and (1.6).

Subsection 8.1 is devoted to the case in which \( \Omega \setminus \omega \) is a John domain. We extend the stability estimates for the spherical pseudodistance defined in (5.1) and for the asymmetry defined in (5.2) – i.e., Theorems 7.3, 7.4, and their corresponding consequences in Theorem 7.6 –, when (1.5) and (1.6) are dropped and replaced by the weaker assumptions (1.15), (1.16); that is,
\[
\text{(8.1) when } \Omega \setminus \omega \text{ is a bounded John domain of finite perimeter.}
\]
We also show that the pointwise results of Theorem 7.1 and its corresponding consequences in Theorem 7.6 can be obtained when (1.5) and (1.6) are replaced with the weaker assumptions (1.15), (1.16), (1.17), that is,
\[
\text{(8.2) when } \Omega \setminus \omega \text{ is a bounded John domain of finite perimeter}
\]
which satisfies the uniform interior sphere condition on the external boundary, at the cost of getting a worse stability exponent \( \tau_N \).

All the generalizations presented in Subsection 8.1 are obtained by using the same choice (5.28) for the point \( z \). As a particular case of those generalizations, we obtain Theorem 1.2.

In Subsection 8.2 we show how a different choice of the point \( z \) allows to obtain Theorem 7.1 and its corresponding consequences in Theorem 7.6 in their full power – i.e., with \( \tau_N \) given in Theorem 6.4 – under the weaker assumptions (1.16) and (1.17). We stress that this approach does not need the assumption (1.15) that \( \Omega \setminus \omega \) is a John domain, requested in the generalizations of Subsection 8.1. In fact, the set of assumptions on \( \Omega \setminus \omega \) in Subsection 8.2 is
\[
\text{(8.3) \quad \Omega \setminus \omega \text{ is a bounded domain of finite perimeter which satisfies}}
\]
the uniform interior sphere condition on the external boundary.

We recall that whenever (1.2) is in force, thanks to (1.3), the external boundary \( \Gamma \) is of class \( C^{2,\alpha} \) and \( u \in C^{2,\alpha}((\Omega \setminus \omega) \cup \Gamma) \).

To deal with sets of finite perimeter, which is common both in (8.2), (8.3), and (8.1), we recall that, thanks to De Giorgi’s structure theorem (see [Mag12, Theorem 15.9] or [Giu84]), the assumptions in (1.4) and (1.16) (in the sense explained in Section 3) guarantee that the integral identities proved in Section 4 still hold true, provided that one replaces \( \partial \omega \) with the reduced boundary \( \partial^* \omega \) and agrees to still use \( \nu \) to denote the (measure-theoretic) outer unit normal (see e.g., [Mag12, Chapter 15]).

We observe that when in particular \( \Omega \setminus \omega \) is of class \( C^1 \), that is when (1.5) is in force, then \( \partial^* \omega = \partial \omega \) and the (measure-theoretic) outer unit normal coincides with the classical notion of outer unit normal (see [Mag12, Remark 15.1]).
Moreover, in the setting of assumption (1.16), the surface measure $|\partial \omega|$ has to be replaced with the perimeter of $\omega$. We recall indeed that the perimeter of $\omega$ equals the $N-1$-dimensional Hausdorff measure of $\partial^* \omega$, denoted by $H^{N-1}(\partial^* \omega)$ (see [Giu84, Chapter 4] or [Mag12, Chapter 15]). Of course, when $\partial \omega$ is of class $C^1$, as given by (1.5), those notions agree (since in that case we have $\partial^* \omega = \partial \omega$).

8.1. Generalizations for John domains and proof of Theorem 1.2. We first deal with the setting in (8.1) and we obtain the generalizations of Theorems 7.3 and 7.4. Then, by further assuming (1.17) (and hence in the setting (8.2)), we establish the generalizations of Theorem 7.1. As a consequence, we thus obtain Theorem 1.2.

The formal framework in which we work is the following. A domain $D$ in $\mathbb{R}^N$ is a $b_0$-John domain, with $b_0 \geq 1$, if each pair of distinct points $a$ and $b$ in $D$ can be joined by a curve $\gamma : [0, 1] \to D$ such that $\gamma(0) = a$, $\gamma(1) = b$, and

$$\delta(\gamma(t)) \geq b_0^{-1} \min \{|\gamma(t) - a|, |\gamma(t) - b|\}.$$  

(8.4)

We emphasize that the class of John domains is huge: it contains Lipschitz domains, but also very irregular domains with fractal boundaries such as, e.g., the Koch snowflake. For more details on John’s domains, see [Pog18, Section 3.2] or [Aik12, MS79, NV91], and references therein. Here, we just notice that, if (1.6) is satisfied then $\Omega \setminus \omega$ is surely a $b_0$-John domain with $b_0 \leq d_{\Omega}/r_i$ (see [Pog18, (iii) of Remark 3.12]).

As already mentioned in Remark 5.5, Lemma 5.3 and Corollary 5.4 still hold true without the assumption of the uniform interior sphere condition, if $D := \Omega \setminus \omega$ is a John domain (see also [Pog18, Section 3.2]). In this case, explicit estimates (now depending on the John parameter $b_0$ instead that on $r_i$) of the relevant constants of Lemma 5.3 and Corollary 5.4 can be found in [MP20a, Remark 2.4]. For the reader’s convenience we report here the only estimate that we need to conclude our reasoning, that is,

$$\mu_{r,p,\alpha}(D)^{-1} \leq k_{N, r, p, \alpha} b_0^N |D|^{\frac{1}{N} + \frac{1}{p} + \frac{\alpha}{2}},$$  

(8.5)

where $r$, $p$ and $\alpha$ are as in (5.13).

Now we point out the main changes to perform in this situation in order to get Theorem 7.3 and its corresponding consequences in Theorems 7.6 when (1.5) and (1.6) are dropped and replaced just by (1.16).

We notice that assumption (1.5) had been exploited only to allow the use of (5.6) in the proof of Lemma 5.6. However, we notice that (5.5) still holds true without that assumption.

Thus, in order to generalize the stability result of Theorem 7.3, we replace item (ii) of Lemma 5.6 with the following result:

**Lemma 8.1.** Let $\Omega \setminus \omega \subset \mathbb{R}^N$ be a bounded $b_0$-John domain satisfying (1.16). Let $u \in C^1(\overline{\Omega} \setminus \omega)$ satisfy (1.1) and (1.2), and assume that $u \leq 0$ on $\partial \omega$. Let $v \in C^2(\Omega \setminus \omega)$ be a harmonic function in $\Omega \setminus \omega$. If

$$\int_{\Omega \setminus \omega} \nabla v \, dx = 0,$$

(8.6)
then it holds that

$$\int_{\Gamma} |\nabla v|^2 \, dH^{N-1} \leq \frac{2}{c} \left(1 + \frac{N}{\overline{p}_{2,1}(\Omega \setminus \omega)^2}\right) \int_{\Omega \setminus \omega} (-u) |\nabla^2 v|^2 \, dx$$

$$- \frac{1}{c} \int_{\partial^* \omega} \left\{ |\nabla v|^2 u_{\nu} - 2u < \nabla^2 v, \nu > \right\} \, dH^{N-1}.$$
Proof. We follow the proof of Lemma 5.6 until (5.22). Then, by using (1.2), formula (5.22) becomes
\[
\int_{\Gamma} c |\nabla v|^2 dH^{N-1} \leq \int_{\Omega\setminus\overline{\omega}} |\nabla v|^2 dx + 2 \int_{\Omega\setminus\overline{\omega}} (-u)|\nabla v|^2 dx \\
- \int_{\partial^* \omega} \{ |\nabla v|^2 u_\nu - 2 u < \nabla^2 v \nabla v, \nu > \} dH^{N-1}. \tag{8.7}
\]
Now, in light of (8.6) and Remark 5.5, we can use item (ii) in Corollary 5.4, applied here with 
\[D := \Omega \setminus \overline{\omega},\quad r := p := 2 \quad \text{and} \quad \alpha := 1,\]
and we deduce from (8.7) that
\[
\int_{\Gamma} c |\nabla v|^2 dH^{N-1} \leq \frac{1}{\mu_{2,2,1}(\Omega \setminus \overline{\omega})^2} \int_{\Omega\setminus\overline{\omega}} \delta^2 |\nabla v|^2 dx + 2 \int_{\Omega\setminus\overline{\omega}} (-u)|\nabla v|^2 dx \\
- \int_{\partial^* \omega} \{ |\nabla v|^2 u_\nu - 2 u < \nabla^2 v \nabla v, \nu > \} dH^{N-1}. \tag{8.8}
\]
From this and (5.5), one obtains the desired estimate. □

We remark that the same generalization could be applied – using item (i) in Corollary 5.4 – also to item (i) of Lemma 5.6, that could be useful for other choices of \(z\).

In the present subsection we maintain the same choice (5.28) for the point \(z\), that is,
\[
z := \frac{1}{|\Omega \setminus \overline{\omega}|} \left\{ \int_{\Omega\setminus\overline{\omega}} x dx - N \int_{\partial^* \omega} u \nu dH^{N-1} \right\}. \tag{8.9}
\]
In this setting, thanks to Lemma 8.1, Theorem 5.7 is replaced by the following statement:

**Theorem 8.2.** Let \(\Omega \setminus \overline{\omega} \subset \mathbb{R}^N\) be a bounded \(b_0\)-John domain satisfying (1.16). Let \(u \in C^2(\Omega \setminus \overline{\omega})\) satisfy (1.1) and (1.2), and assume that \(u \leq 0\) on \(\partial\omega\).

Then, with the notation of (5.26) and (8.8), we have that
\[
\int_{\Gamma} \left| \frac{x - z}{N} \right|^2 dH^{N-1} \leq \frac{1}{c} \left( 1 + \frac{N}{\mu_{2,2,1}(\Omega \setminus \overline{\omega})^2} \right) \int_{\partial^* \omega} \left[ c^2 \left( \frac{x, \nu}{N} - u_\nu \right) + 2 u \left( \frac{x, \nu}{N} - u_\nu \right) \\
+ u_\nu |\nabla u|^2 - 2 \frac{x, \nabla u}{N} u_\nu + |\nabla u|^2 \frac{x, \nu}{N} + \frac{2}{N} u u_\nu - 2 \nabla^2 u \nabla u, \nu > u \right] dH^{N-1} \right) \\
- \frac{1}{c} \int_{\partial^* \omega} \{ \nabla h|^2 u_\nu - 2 u < \nabla^2 h \nabla h, \nu > \} dH^{N-1}. \tag{8.10}
\]
In the same way in which Theorem 5.7 led to Theorem 7.3, now Theorem 8.2 easily leads to a general stability result for John domains of finite perimeter. In this setting, using Theorem 8.2 in place of Theorem 5.7, one obtains a statement analogous to Theorem 7.3 with \(\partial\omega\) replaced with \(\partial^* \omega\) in (7.1) and (7.3), even if assumptions (1.5) and (1.6) are replaced by (1.16). The precise result is the following:

**Theorem 8.3 (General stability result for a pseudodistance under relaxed assumptions).** Let \(\Omega \setminus \overline{\omega} \subset \mathbb{R}^N\) a \(b_0\)-John domain. Let \(u \in C^2(\Omega \setminus \overline{\omega})\) satisfy (1.1) and (1.2), and suppose that \(u \leq 0\) on \(\partial\omega\). Let assumption (1.16) be verified, and \(z\) be as in (8.8).
If $\psi : [0, \infty) \to [0, \infty)$ is a continuous function vanishing at 0 such that
\[
\int_{\partial^* \omega} (-u) \, d\mathcal{H}^{N-1} = \int_{\partial^* \omega} |u| \, d\mathcal{H}^{N-1} - \int_{\partial^* \omega} u u_\nu \, d\mathcal{H}^{N-1} - \int_{\partial^* \omega} |\nabla u|^2 u_\nu \, d\mathcal{H}^{N-1} - \int_{\partial^* \omega} |\nabla u|^2 \, d\mathcal{H}^{N-1} - \int_{\partial^* \omega} \nabla^2 u \nabla u, \nu > u \, d\mathcal{H}^{N-1}
\]
(8.10)
and
\[
\int_{\partial^* \omega} \nabla^2 u (x-z), \nu > d\mathcal{H}^{N-1} \leq \psi(\eta) \quad \text{with} \quad \eta = \mathcal{H}^{N-1}(\partial^* \omega) \quad \text{or} \quad \eta = \tilde{d}_\omega,
\]
(8.11)
then
\[
\int_{\Gamma} \left| \frac{x-z}{N} - c \right|^2 \, d\mathcal{H}^{N-1} \leq C \psi(\eta),
\]
(8.12)
where $C$ is a constant depending on $N$, $b_0$, $d_\Omega$, and $c$.

**Remark 8.4.** Concerning the constant in formula (8.12) of Theorem 8.3, we observe that the dependence from $b_0$ comes from the estimate of $\rho_{2,2,1}(\Omega \setminus \overline{\omega})$ in (8.5). We also remark that the volume appearing in (8.5) can be estimated from above in terms of $d_\Omega$, by means of (6.31) and (6.32).

The dependence of $C$ in (8.12) on $c$ comes from the fact that the quantity $1/c$ plays a role in the estimates, as it can be seen from formula (8.9) in Theorem 8.2. We point out that such a dependence can in fact be replaced with the dependence on $|\Gamma|$, by means of a suitable lower bound for $c$. Namely, we claim that
\[
c \geq k_N \frac{1}{|\Gamma|} \left( \frac{d_\Omega}{b_0} \right)^N,
\]
if $\eta$ is small enough, where $k_N$ is a positive constant depending on $N$. To prove it, we first observe that
\[
\frac{|a-b|}{2b_0} \quad \text{is contained in} \quad \Omega \setminus \overline{\omega}.
\]
Indeed, since $\Omega \setminus \overline{\omega}$ is a $b_0$-John domain, we have that there exists a curve $\gamma : [0, 1] \to \Omega \setminus \overline{\omega}$ such that $\gamma(0) = a$, $\gamma(1) = b$ and (8.4) holds true. Moreover, by inspection one sees that there exists $t^* \in (0, 1)$ such that $|\gamma(t^*) - a| = |\gamma(t^*) - b|$, and so, by (8.4),
\[
\delta(\gamma(t^*)) \geq b_0^{-1}|\gamma(t^*) - a|,
\]
which implies that
\[
B_{b_0^{-1}|\gamma(t^*) - a|}(\gamma(t^*)) \subset \Omega \setminus \overline{\omega}.
\]
Furthermore, by the triangle inequality,
\[
|a - b| \leq |\gamma(t^*) - a| + |\gamma(t^*) - b| = 2|\gamma(t^*) - a|.
\]
This and (8.15) imply (8.14). As a consequence of (8.14), we have that, for any $a, b \in \Omega \setminus \overline{\omega}$,
\[
|\Omega \setminus \overline{\omega}| \geq |B_1| \left( \frac{|a - b|}{2b_0} \right)^N.
\]
(8.16)
Now we choose $a$ and $b$ in such a way that $|a - b|$ is arbitrarily close to $d_\Omega$, say $|a - b| \geq d_\Omega/2$, and we obtain from (8.16) that

$$
|\Omega| - |\omega| \geq |B_1| \left( \frac{d_\Omega}{4b_0} \right)^N.
$$

Then, by exploiting (4.16), (8.10) and (8.17), we see that

$$
c|\Gamma| = |\Omega| - |\omega| - \int_{\partial^* \omega} u_\nu d\mathcal{H}^{N-1} \geq |\Omega| - |\omega| \geq \frac{|B_1|}{2} \left( \frac{d_\Omega}{4b_0} \right)^N,
$$

as long as $\eta$ is sufficiently small. This proves (8.13).

Theorem 8.3 also leads to a stability bound for the asymmetry defined in (5.2) by means of the following generalization of Lemma 5.1 to the case of John domains:

**Lemma 8.5.** Let $\Omega \subset \mathbb{R}^N$ be a bounded $b_0$-John domain with Lipschitz boundary $\Gamma$. Then, there exists a positive constant $C$ only depending on $N$, $b_0$, $c$, such that

$$
\frac{|\Omega \Delta B_{NC(z)}|}{|B_{NC(z)}|} \leq C \left[ \int_{\Gamma} \frac{|x - z|}{N} - c \right]^2 d\mathcal{H}^{N-1}.
$$

**Proof.** The desired result follows by applying [Fel18, Lemma 11] with

$$
K := \max \left\{ \frac{4b_0Nc}{d_\Omega}, \left( \frac{d_\Omega}{2Nc} \right)^N \right\} \quad \text{and} \quad r := Nc.
$$

Notice that [Fel18, Lemma 11] can be applied with these choices for $K$ and $r$. Indeed, formula (5.3) is still satisfied. On the other hand, to deduce formula (5.4) in this setting, we observe that

$$
r_{in}(\Omega) \geq \frac{d_\Omega}{4b_0},
$$

where $r_{in}(\Omega) := \max_{\Omega} \delta_\Gamma(x)$ denotes the inradius of $\Omega$. To prove (8.18), one can use (8.14) and choose $a$ and $b$ in $\Omega$ such that $|a - b| \geq d_\Omega/2$.

Then, from (8.18) one deduce that

$$
Kr_{in}(\Omega) \geq Nc \frac{4b_0r_{in}(\Omega)}{d_\Omega} \geq Nc. \quad \Box
$$

In light of Lemma 8.5, we also deduce from Theorem 8.3 a stability result for an asymmetry in this setting:

**Theorem 8.6** (General stability result for an asymmetry under relaxed assumptions). Let $\Omega \setminus \overline{\omega}$ a $b_0$-John domain. Let $u \in C^2(\Omega \setminus \omega)$ satisfy (1.1) and (1.2), and suppose that $u \leq 0$ on $\partial\omega$. Let assumption (1.16) be verified, and $z$ be as in (8.8).

Let $\psi : [0, \infty) \to [0, \infty)$ be a continuous function vanishing at 0 such that (8.10) and (8.11) hold true.

Then, it holds that

$$
\frac{|\Omega \Delta B_{NC(z)}|}{|B_{NC(z)}|} \leq C \psi(\eta)^{1/2},
$$

where $C$ is a constant depending on $N$, $b_0$, $d_\Omega$, and $c$.

**Remark 8.7.** The dependence of the constant $C$ in (8.19) on $c$ can be replaced with the dependence on $|\Gamma|$, when $\eta$ is small enough. In fact, a lower bound for $c$ in terms of $N$, $b_0$, $d_\Omega$, $1/|\Gamma|$ has been obtained in (8.13); the upper bound for $c$ in terms of $N$ and $d_\Omega$ obtained in (7.9) still holds true.
Our next objective is to provide a stability estimate for \( \rho_e - \rho_i \), as given in formula (7.2), in the more general framework of John domains. This will lead to a general version of Theorem 7.1 which in turn will produce a general version of Theorem 7.6.

Notice that, when assumption (1.5) is replaced by (1.16), we have to use (5.5) instead of (5.6) to bound from below the left-hand side of (4.15). Thus, in this case, the quantity that has to be put in relation with \( \rho_e - \rho_i \) is \( \| \delta \nabla^2 h \|_{2,\Omega|\omega} \) instead of \( \| \delta^{1/2} \nabla^2 h \|_{2,\Omega|\omega} \). For this reason, the exponents \( \tau_N \) in Theorem 8.8 stated next, become worse with respect to those of Theorem 7.1.

More precisely, the counterpart of Theorem 6.4 in this more general setting is the following:

**Theorem 8.8.** Let \( \Omega \setminus \overline{\omega} \subset \mathbb{R}^N \) be a bounded \( b_0 \)-John domain, satisfying (1.17), and suppose that \( \Gamma \) is of class \( C^1 \). Let \( u \) satisfy (1.1), \( u \in C^1((\Omega \setminus \overline{\omega}) \cup \Gamma) \), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let \( q \) be as in (5.26) with \( z \) chosen as in (8.8), and assume that \( z \) belongs to \( \Omega \). Let \( h \) be as in (5.25).

Then, there exists a positive constant \( C \) such that

\[
(8.20) \quad \rho_e - \rho_i \leq C \| \delta \nabla^2 h \|_{2,\Omega|\omega}^{\tau_N} \quad \text{with} \quad \tau_N = \begin{cases} 1 - \theta, & \text{for any } \theta > 0, \text{ when } N = 2 \\ 2/N, & \text{when } N \geq 3. \end{cases}
\]

The constant \( C \) depends on \( N, r_i, b_0, d_\Omega, M \) (as defined in (6.21)).

**Proof.** The desired estimate easily follows by reasoning as in the proof of items (ii) and (iii) of Theorem 6.4. The only difference is that now we apply Poincaré inequalities of item (ii) of Corollary 5.4 with \( \alpha := 1 \) (instead that \( 1/2 \)).

More precisely, when \( N = 2 \) we apply, with \( v := h \) and \( D := \Omega \setminus \overline{\omega} \), item (ii) of Corollary 5.4 with \( r := 2(1 - \theta), \, p := 2, \, \alpha := 1 \) and (5.16) with \( r := 2(1 - \theta)/\theta, \, p := 2(1 - \theta), \, \alpha := 0 \). When \( N \geq 3 \) we apply, with \( v := h \) and \( D := \Omega \setminus \overline{\omega} \), item (ii) of Corollary 5.4 with \( r := p := 2, \, \alpha := 1 \) and (5.16) with \( r := 2N/(N - 2), \, p := 2, \, \alpha := 1 \).

We stress that assumption (1.6) in items (ii) and (iii) of Theorem 6.4 was assumed only to guarantee that the Poincaré inequalities of Lemma 5.3 and Corollary 5.4 could be applied with \( D := \Omega \setminus \overline{\omega} \); in fact, notice that Lemma 6.3 that was the main ingredient of those proofs, was already stated under the weaker assumption (1.17). Being now \( \Omega \setminus \overline{\omega} \) a \( b_0 \)-John domain, in light of Remark 5.5 and (8.5) it is clear that we can still apply those Poincaré inequalities with \( D := \Omega \setminus \overline{\omega} \), even if (1.6) has been dropped. \( \square \)

By applying Theorem 8.8 in the place of Theorem 6.4, we thus derive a counterpart of Theorem 7.1 for John domains. Namely, in this setting, the result in Theorem 7.1 still holds true if assumptions (1.5), (1.6) are replaced with (1.16), (1.17) – with \( \partial \omega \) replaced with \( \partial^* \omega \) in (7.1) – but in this situation the exponents \( \tau_N \) are those in (8.20). The precise result is indeed the following one:

**Theorem 8.9.** Let \( \Omega \setminus \overline{\omega} \subset \mathbb{R}^N \) be a bounded \( b_0 \)-John domain. Let \( u \in C^2(\overline{\Omega}) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let assumptions (1.16) and (1.17) be verified.

Assume also that the point \( z \) chosen in (8.8) belongs to \( \Omega \).

If \( \psi : [0, \infty) \to [0, \infty) \) is a continuous function vanishing at 0 such that (8.10) holds true, then

\[
(8.21) \quad \rho_e - \rho_i \leq C \psi(\eta)^{\tau_N/2},
\]

where \( \tau_N \) is as in (8.20) and \( C \) is a positive constant that depends on \( N, r_i, b_0, d_\Omega, M \) (as defined in (6.21)).

**Proof.** The desired result easily follows by putting together Theorem 8.8 and formulas (5.5), (5.27) and (1.15). \( \square \)

As can be deduced from the discussion before Theorem 8.8 if (1.5) holds true, then the statement in Theorem 8.9 can be strengthen, since in this case one can obtain \( \tau_N \) as in...
Theorem 6.4 (we do not enter into these details, since this special statement will not be used in what follows).

The corresponding generalization of Theorem 7.6 easily follows from Theorems 8.3, 8.6 and 8.9 and it can be stated as follows:

**Theorem 8.10.** Let \( \Omega \setminus \bar{\omega} \subset \mathbb{R}^N \) be a bounded \( b_0 \)-John domain. Let \( u \in C^2(\Omega \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \). Let assumption (1.16) be verified, and \( z \) be as in (8.8).

If there exists \( K > 0 \) such that

\[
\|u\|_{C^2(\partial \omega)} \leq K,
\]

then

\[
\int_\Gamma \left| \frac{x - z}{N} - c \right|^2 dH^{N-1} \leq C \mathcal{H}^{N-1}(\partial^* \omega),
\]

where \( C \) is a positive constant depending on \( N, b_0, d_\Omega, c \) and \( K \).

Also, it holds that

\[
\frac{|\Omega \Delta B_{Nc}(z)|}{|B_{Nc}(z)|} \leq C \mathcal{H}^{N-1}(\partial^* \omega)^{1/2},
\]

where \( C \) is a positive constant depending on \( N, b_0, d_\Omega, c \), and \( K \).

If in addition (1.17) is verified and \( z \in \Omega \), we have that

\[
\rho_e - \rho_i \leq C(\mathcal{H}^{N-1}(\partial^* \omega))^{\tau_N/2},
\]

where \( \tau_N \) are as in (8.20), and \( C \) is a positive constant depending on \( N, r_i, b_0, d_\Omega, \) and \( K \).

The proof of Theorem 1.2 now plainly follows from Theorem 8.10.

**Remark 8.11.** As in Theorem 7.6 the constant \( C \) in (8.25) does not depend on \( M \) (as defined in (6.21)), differently from that appearing in (8.21).

The dependence of the constants \( C \) on \( c \) in (8.23) and (8.24) could be replaced with the dependence on \( |\Gamma| \), at least when \( \mathcal{H}^{N-1}(\partial^* \omega) \) is small enough. In fact, when, e.g.,

\[
\mathcal{H}^{N-1}(\partial^* \omega) < \frac{1}{2K} \left( \frac{d_\Omega}{4 b_0} \right)^N,
\]

by exploiting (4.16), (8.22), and (8.17) we have that

\[
c |\Gamma| = |\Omega| - |\omega| - \int_{\partial^* \omega} u_\nu d\mathcal{H}^{N-1} \geq |\Omega| - |\omega| - \mathcal{K} \mathcal{H}^{N-1}(\partial^* \omega)
\]

\[
\geq |B_1| \left( \frac{d_\Omega}{4 b_0} \right)^N - \mathcal{K} \mathcal{H}^{N-1}(\partial^* \omega) \geq \frac{|B_1|}{2} \left( \frac{d_\Omega}{4 b_0} \right)^N.
\]

As a result,

\[
c \geq \frac{|B_1|}{2 |\Gamma|} \left( \frac{d_\Omega}{4 b_0} \right)^N.
\]

On the other hand, in this case, by (4.16), (7.8), and (8.26), we also get a suitable upper bound for \( c \), that is

\[
c \leq \frac{1}{|\Gamma|} \left( |\Omega| - |\omega| - \int_{\partial^* \omega} u_\nu d\mathcal{H}^{N-1} \right) \leq \frac{d_\Omega}{2N} + \frac{1}{|\Gamma|} \mathcal{K} \mathcal{H}^{N-1}(\partial^* \omega) \leq \frac{d_\Omega}{2N} + \frac{1}{2 |\Gamma|} \left( \frac{d_\Omega}{4 b_0} \right)^N.
\]
8.2. A new different choice for $z$ and proof of Theorem 1.3. Our next goal is to show that it is possible to obtain Theorem 7.1 and its consequences in Theorem 7.6 with $r_N$ given in Theorem 6.4 and in the more general setting (8.3), provided that we make a different choice of $z$.

The main difficulty in this setting is that no regularity information is available on $\partial \omega$ (not even being $\Omega \setminus \overline{\omega}$ a John domain), and therefore we cannot apply Poincaré inequalities on all $\Omega \setminus \overline{\omega}$, making it difficult to establish an appropriate variant of Theorem 6.4.

The key idea to overcome this difficulty is to perform the necessary Poincaré inequalities on a suitable subset of $\Omega \setminus \overline{\omega}$. Such a suitable subset is $\Omega_{r_i}^c$, where we are using the notation introduced in (1.17) and (6.4). Notice that, by (1.17), it holds that

$$\Omega_{r_i}^c \subset \Omega \setminus \overline{\omega}.$$ 

With this setting, we start with the following estimate:

**Lemma 8.12.** Let $\Omega \setminus \overline{\omega} \subset \mathbb{R}^N$ satisfy (1.17), and suppose that $\Gamma$ is of class $C^1$. Let $v$ be a harmonic function in $\Omega \setminus \overline{\omega}$ of class $C^1(\Omega_{r_i}^c)$, and let $G$ be an upper bound for the gradient of $v$ on $\Omega_{r_i}^c$.

Then, given $p \geq 1$, there exist two positive constants $\tilde{\alpha}_{N,p}$ and $\hat{\alpha}_{N,p}$ depending only on $N$ and $p$ such that if

$$\|v - v_{\Omega_i^c}\|_{p,\Omega_i^c} \leq \hat{\alpha}_{N,p} r_i^{\frac{N+p}{p}} G,$$

then we have that

$$\max_i v - \min_i v \leq \tilde{\alpha}_{N,p} G^\frac{N}{p(N+p)} \|v - v_{\Omega_i^c}\|_{p,\Omega_i^c}^{(N+p)/p}.$$

**Proof.** The desired result follows by applying Lemma 6.2 with $r_i := r_i/2$ and $\lambda := v_{\Omega_i^c}$. Indeed, since

$$B_{r_i/2}(x_0) \subset \Omega_{r_i}^c,$$

it holds that

$$\|v - v_{\Omega_i^c}\|_{p,B_{r_i/2}(x_0)} \leq \|v - v_{\Omega_i^c}\|_{p,\Omega_i^c},$$

and (8.27), (8.28) follow from (6.9), (6.10). □

We recall that, thanks to (1.17), $\Gamma_{r_i}$ inherits the same regularity of $\Gamma$. More precisely, we have that:

$$\Gamma \in C^k \implies \Gamma_{r_i} \in C^k, \quad \text{for } k \geq 1.$$

This fact relies on the regularity of the distance function. The case $k \geq 2$ has been proved in Appendix of [GT01] (see also [KP81, Theorem 3]). The case $k = 1$ can be deduced from [KP81, Theorem 2], but we do not need this refinement here.

Moreover, we have that:

**Lemma 8.13.** Assume that (1.17) holds true. Then, the domain $\Omega_{r_i}^c$ in (6.4) satisfies the uniform interior sphere condition with radius $r_i/2$.

**Proof.** For any $y \in \Gamma_{r_i}$ and $x \in \Gamma$ such that $y = x - r_i \nu(x)$, by (1.17) and definition (6.4), we have that $B_{r_i/2} \left( \frac{x + y}{2} \right) \subset \Omega_{r_i}^c$. It follows that $B_{r_i/2} \left( \frac{x + y}{2} \right)$ is an interior touching ball in $\Omega_{r_i}^c$ (at $x$ and $y$). □

In order to use the Poincaré inequality of item (ii) of Corollary 5.4 (with $v = h$ and $D = \Omega_{r_i}^c$), we have to make a new appropriate choice of $z$.

To this aim, a possible choice of $z$ is:

$$z := \frac{1}{|\Omega_{r_i}|} \left\{ \int_{\Omega_{r_i}} x \, dx - N \int_{\Gamma_{r_i}} \nu \nu \, d\mathcal{H}^{N-1} \right\},$$

(8.30)
Theorem 8.14. Let $\Omega \setminus \overline{\omega} \subset \mathbb{R}^N$ satisfy (1.17), and suppose that $\Gamma$ is of class $C^1$. Let $u$ satisfy (1.1), $u \in C^1((\Omega \setminus \overline{\omega}) \cup \Gamma)$, and suppose that $u \leq 0$ on $\partial \omega$. Let $q$ be as in (5.26) with $z$ chosen as in (8.30), and assume that $z$ belongs to $\Omega$. Let $h$ be as in (5.25).

Then, there exists a positive constant $C$ such that

$$(8.33) \quad \rho_e - \rho_i \leq C \| (x, \partial \Omega_{r_i}^c) \|^2 \| \nabla^2 h \|_{L^\infty(\Omega_{r_i}^c)},$$

with the following specifications:

(i) $\tau_2 = 1$;

(ii) $\tau_3$ is arbitrarily close to 1, in the sense that, for any $\theta \in (0, 1)$ sufficiently small, there exists a positive constant $C$ such that (8.33) holds with $\tau_3 = 1 - \theta$;

(iii) $\tau_4 = 2/(N - 1)$ for $N \geq 4$.

The constant $C$ depends on $N$, $r_i$, $d_\Omega$, $\max_{\Omega_{r_i}^c} |\nabla u|$, and $\theta$ (the latter, only in the case $N = 3$).

Proof. By using Lemma 8.12 with $v = h$ and reasoning as in Lemma 6.3 we see that

$$\rho_e - \rho_i \leq C \| h - h_{\Omega_{r_i}^c} \|_{L^p(\Omega_{r_i}^c)} \| \nabla^2 h \|_{L^\infty(\Omega_{r_i}^c)},$$

Then, we modify the proof of Theorem 6.4 by using the Poincaré inequalities of Lemma 5.3 and Corollary 5.4 with $v = h$ on $D = \Omega_{r_i}^c$ (instead of taking $D = \Omega \setminus \overline{\omega}$).

By Lemma 8.13 we have that the domain $\Omega_{r_i}^c$ satisfies the uniform interior sphere condition with radius $r_i/2$, and hence Lemma 5.3 and Corollary 5.4 can be applied with $D = \Omega_{r_i}^c$. Also, in light of (8.32), we have that the choice in (8.30) for $z$ guarantees that the Poincaré inequalities of Corollary 5.4 can be applied with $v = h$ and $D = \Omega_{r_i}^c$.

With these modifications and proceeding as in the proof of Theorem 6.4, instead of (6.25) we obtain the refined estimate (8.33) (in the case $N = 2$, also the Sobolev inequality (6.26) has to be performed here with $\Omega \setminus \overline{\omega}$ replaced by $\Omega_{r_i}^c$).

From the previous work, we can now obtain the counterpart of Theorem 7.1 under the relaxed assumptions (1.16), (1.17):

Theorem 8.15 (General stability result for $\rho_e - \rho_i$ under relaxed assumptions). Let $u \in C^2(\Omega \setminus \overline{\omega})$ satisfy (1.1) and (1.2), and suppose that $u \leq 0$ on $\partial \omega$. Let assumptions (1.16) and (1.17) be verified. Assume also that the point $z$ chosen as in (8.30) belongs to $\Omega$.

If $\psi : [0, \infty) \to [0, \infty)$ is a continuous function vanishing at 0 and satisfying (8.10), then

$$\rho_e - \rho_i \leq C \psi(\eta)^{\tau_N/2},$$

with $\tau_N$ as in Theorem 8.14 and $C$ depending on $N$, $r_i$, $d_\Omega$, and $\max_{\Omega_{r_i}^c} |\nabla u|$.

Proof. In the notation of (8.31) we have that $\partial \Omega_{r_i}^c = \Gamma \sqcup \Gamma_{r_i}$, and, by recalling (1.3) and (8.29), $\Omega_{r_i}^c$ is of class $C^2$. Moreover, since $\Omega_{r_i}^c$ satisfies the uniform interior sphere condition with radius $r_i/2$, Lemma 5.2 can be applied to $\Omega_{r_i}^c$ in place of $\Omega \setminus \overline{\omega}$. Hence, in this setting, formula (5.6) can be rephrased as

$$(8.34) \quad -u(x) \geq \frac{r_i}{4N} \text{dist}(x, \partial \Omega_{r_i}^c) \quad \text{for any} \quad x \in \Omega_{r_i}^c.$$
Moreover, by the maximum principle,
\[ (8.35) \quad -u \geq 0 \quad \text{on} \quad \Omega \setminus \overline{\omega}. \]
Hence, since 
\[ \Omega_{r_i}^e \subset \Omega \setminus \overline{\omega}, \]
we deduce from (8.33), (8.34) and (8.35) that
\[ \rho_e - \rho_i \leq C \left( \int_{\Omega_{r_i}^e} \text{dist} \ (x, \partial \Omega_{r_i}^e) \ |\nabla^2 h|^2 \, dx \right)^{\tau_N/2} \]
\[ \leq C \left( \int_{\Omega_{r_i}} (-u) |\nabla^2 h|^2 \, dx \right)^{\tau_N/2} \leq C \left( \int_{\Omega \setminus \omega} (-u) |\nabla^2 h|^2 \, dx \right)^{\tau_N/2}. \]
Consequently, the desired result follows from (4.15) (with \( \partial \omega \) replaced by \( \partial^* \omega \)) and (5.27). \( \square \)

As a consequence of Theorem 8.15, we thus have the following generalization of (7.13):

**Theorem 8.16.** Let \( u \in C^2(\Omega \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \).

Let assumptions (1.16) and (1.17) be verified, and \( z \) be as in (8.30). Assume that there exists \( K > 0 \) such that
\[ \|u\|_{C^2(\partial \omega)} \leq K, \]
and that \( z \in \Omega \).

Then,
\[ \rho_e - \rho_i \leq C \left( \mathcal{H}^{N-1}(\partial^* \omega) \right)^{\tau_N/2}, \]
where \( \tau_N \) are as in Theorem 8.14 and \( C \) is a positive constant depending on \( N, r_i, d_{\Omega}, \) and \( K \).

The proof of Theorem 1.3 is now a plain consequence of Theorem 8.16.

Of course, from Theorems 8.3, 8.6, 8.9, and 8.15, we can also deduce the corresponding generalizations of (7.13):

**Theorem 8.17.** Let \( u \in C^2(\Omega \setminus \omega) \) satisfy (1.1) and (1.2), and suppose that \( u \leq 0 \) on \( \partial \omega \).

Let assumption (1.17) be verified, and \( z \) be as in (8.30). Let \( \omega \) be the union of finitely many disjoint balls of radius \( \varepsilon \) and assume that
\[ \|u\|_{L^\infty(\partial \omega)} + \varepsilon \|\nabla u\|_{L^\infty(\partial \omega)} + \varepsilon^2 \|\nabla^2 u\|_{L^\infty(\partial \omega)} = o(\varepsilon^{4-3N}). \]
Suppose that \( z \in \Omega \). Then \( \rho_e - \rho_i \) is as small as we wish for small \( \varepsilon \).

**Appendix A. Motivation from an optimal heating problem (with possible malfunctioning of the source)**

In this section we briefly recall how the simple model from optimal heating described in (2.2) directly produces the overdetermined condition in (2.3). For this, we consider a divergence free vector field \( v \). Also, for small \( t \geq 0 \), we introduce the diffeomorphism given by
\[ \Phi^t(x) := x + tv(x). \]
We set \( \Omega^t := \Phi^t(\Omega) \) and, given a source \( f \geq 0 \), we let \( u^t \) be the solution of
\[ \begin{cases} \Delta u^t = f \quad \text{in} \ \Omega^t, \\ u^t = 0 \quad \text{on} \ \partial \Omega^t. \end{cases} \]
We consider the energy functional
\[ I(t) := \frac{1}{2} \int_{\Omega^t} |\nabla u^t(x)|^2 \, dx. \]
We also define
\[ \psi(x, t) := \frac{1}{2} |\nabla u'(x)|^2. \]
In this way, we have that
\[ \partial_t \psi(x, t) = \nabla u'(x) \cdot \nabla \partial_t u'(x) \]
and
\[ I(t) = \int_\Omega \psi(x, t) \, dx. \]
By the Hadamard’s Differentiation Formula (see Theorem 5.2.2 in [HP18]), we know that
\[ I'(0) = \int_\Omega \partial_t \psi(x, 0) \, dx + \int_{\partial \Omega} \psi(x, 0) < \nu(x), v(x) > \, dS_x \]
\[ = \int_\Omega \nabla u^0(x) \cdot \nabla \partial_t u^0(x) \, dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u^0(x)|^2 < \nu(x), v(x) > \, dS_x. \]
Since \( \Phi^t(x) \in \partial \Omega^t \) for all \( x \in \partial \Omega \), we have that
\[ u'(\Phi^t(x)) = 0 \]
for all \( x \in \partial \Omega \), and so, taking derivatives in \( t \),
\[ \partial_t u^0(x) + < \nabla u^0(x), v(x) > = \partial_t u'(x) + < \nabla u'(\Phi^t(x)), \partial_t \Phi^t(x) > \bigg|_{t=0} = 0. \]
As a consequence,
\[ \int_\Omega \nabla u^0(x) \cdot \nabla \partial_t u^0(x) \, dx = \int_\Omega \text{div}(\partial_t u^0(x) \nabla u^0(x)) \, dx - \int_\Omega \partial_t u^0(x) f(x) \, dx \]
\[ = \int_{\partial \Omega} \partial_t u^0(x) < \nabla u^0(x), \nu(x) > \, dS_x - \int_\Omega \partial_t u^0(x) f(x) \, dx \]
\[ = - \int_{\partial \Omega} < \nabla u^0(x), \nu(x) > < \nabla u^0(x), v(x) > \, dS_x - \int_\Omega \partial_t u^0(x) f(x) \, dx, \]
and therefore
\[ I'(0) = - \int_{\partial \Omega} < \nabla u^0(x), \nu(x) > < \nabla u^0(x), v(x) > \, dS_x \]
\[ - \int_\Omega \partial_t u^0(x) f(x) \, dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u^0(x)|^2 < \nu(x), v(x) > \, dS_x. \]
We also remark that, since \( u^0 = u = 0 \) on \( \partial \Omega \), we have that
\[ \nu = \frac{\nabla u}{|\nabla u|}. \]
Thus,
\[ < \nabla u^0, \nu > < \nabla u^0, v > = |\nabla u|^2 < \nu, v > \]
\[ = ( < \nabla u, \nu >)^2 < \nu, v > = u^2_\nu < \nu, v >. \]
In this way, \( A.2 \) can be written as
\[ I'(0) = - \int_\Omega \partial_t u^0 f \, dx - \frac{1}{2} \int_{\partial \Omega} u^2_\nu < \nu, v > \, dS_x. \]
We also observe that $\Delta \partial_t u^0 = \partial_t \Delta u^0 = \partial_t f = 0$ in $\Omega$, hence
\[
\int_{\Omega} \partial_t u^0 f \, dx = \int_{\Omega} \partial_t u^0 \Delta u \, dx = \int_{\Omega} (\partial_t u^0 \Delta u - \Delta \partial_t u^0 u) \, dx \\
= \int_{\Omega} \text{div} (\partial_t u^0 \nabla u - \nabla \partial_t u^0 u) \, dx \\
= \int_{\partial \Omega} \langle \partial_t u^0 \nabla u - \nabla \partial_t u^0 u, \nu \rangle \, dS_x \\
= \int_{\partial \Omega} \partial_t u^0 u \nu \, dS_x.
\]
This, (A.1) and (A.3) entail that
\[
\int_{\Omega} \partial_t u^0 f \, dx = -\int_{\partial \Omega} \langle \nabla u^0, v \rangle u \nu \, dS_x = -\int_{\partial \Omega} u^2 \nu < \nu, v > \, dS_x.
\]
Consequently, (A.4) becomes
\[
I'(0) = \frac{1}{2} \int_{\partial \Omega} u^2 \nu < \nu, v > \, dS_x.
\]
That is, being $I$ stationary for all divergence free vector fields is equivalent to the constancy of $u_\nu$, that is (2.3).

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Serena Dipierro: Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia

E-mail address: serena.dipierro@uwa.edu.au

Giorgio Poggesi: Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia

E-mail address: giorgio.poggesi@uwa.edu.au

Enrico Valdinoci: Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia

E-mail address: enrico.valdinoci@uwa.edu.au