ON THE CONCENTRATION OF SEMICLASSICAL STATES FOR NONLINEAR DIRAC EQUATIONS

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Abstract. In this paper, we study the following nonlinear Dirac equation

\[-i\varepsilon \alpha \cdot \nabla w + a\beta w + V(x)w = g(|w|)w, \quad x \in \mathbb{R}^3, \quad w \in H^1(\mathbb{R}^3, \mathbb{C}^4),\]

where \(a > 0\) is a constant, \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\), and \(\beta\) are 4 \times 4 Pauli–Dirac matrices. Under the assumptions that \(V\) and \(g\) are continuous but are not necessarily of class \(C^1\), when \(g\) is super-linear growth at infinity we obtain the existence of semiclassical solutions, which converge to the least energy solutions of its limit problem as \(\varepsilon \to 0\).

1. Introduction and main results. In this paper, we consider the existence and concentration behavior of semiclassical states of the following stationary Dirac equation

\[-i\varepsilon \sum_{k=1}^{3} \alpha_k \partial_k w + a\beta w + V(x)w = g(|w|)w, \quad x \in \mathbb{R}^3, \quad w \in H^1(\mathbb{R}^3, \mathbb{C}^4). \tag{D}\]

Here, \(\partial_k = \partial/\partial x_k\), \(a > 0\) is a constant, \(V : \mathbb{R}^3 \to \mathbb{R}\), \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta\) are 4 \times 4 Pauli–Dirac matrices:

\[\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,\]

with

\[\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\]

Equation (D) or a more general one

\[-i\hbar \sum_{k=1}^{3} \alpha_k \partial_k w + a\beta w + V(x)w = F_w(x, w), \tag{1}\]

arises when one seeks for the standing wave solutions of the nonlinear Dirac equation

\[-i\hbar \partial_t \psi = ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi - M(x)\psi + G(x, \psi). \tag{2}\]

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Assuming that $G(x, e^{i\theta} \psi) = G(x, \psi)$ for all $\theta \in [0, 2\pi]$, a standing wave solution of (2) is a solution of the form $\psi(t, x) = e^{i\pi t} w(x)$. It is clear that $\psi(t, x)$ solves (2) if and only if $w(x)$ solves (1) with $a = mc, V(x) = M(x)/c + \mu I_4$ and $F(x, w) = G(x, w)/c$.

Recent, equation (1) has been widely studied and the existence and multiplicity results for such a equation have been discussed in many papers under different assumptions on the potential and nonlinearity (see [2, 14, 16, 19, 20, 21, 29] for example).

When $\hbar$ is small, the standing waves are referred to as semiclassical states. The concentration phenomenon of semiclassical states as $\hbar \to 0$ reflects the transformation process between quantum mechanics and classical mechanics. So it possesses an important physical interest. As we know, there have been so many results that relate to the existence and concentration phenomenon of semiclassical states for nonlinear Schrödinger equations, [1, 3, 4, 6, 7, 8, 9, 24, 25, 26, 30, 32, 34, 35]. However, such results on Dirac equations are relatively few: In [11], Ding considered (1) with $V(x) \equiv 0$ and $F_w(x, w) = P(x)|w|\theta w$, $\theta \in (2, 3)$ and showed the existence of a family of ground states of the equation for small $\hbar$, which concentrates around the maximum points of $P(x)$ as $\hbar \to 0$. This result was later generalized to the case

$$V(x) \neq 0, \quad \min_{x \in \mathbb{R}^3} V(x) < \lim_{|x| \to +\infty} V(x),$$

with $F_w(x, w) = g(|w|)w$ in [12]; $F_w(x, w) = P(x)g(|w|)w$ in [13] and $F_w(x, w) = P(x)(g(|w|) + |w|)w$ in [15]. In [18], Ding and Xu weakened the assumptions on $V(x)$ to the following local version: there is a bounded domain $\Lambda \subset \mathbb{R}^3$ such that

$$\min_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x).$$

And they obtain the existence and concentration results when $F_w(x, w) = g(|w|)w$ possesses a supper linear or a asymptotically linear growth at infinity.

However, in all previous results involving semiclassical states for Dirac equations, there holds the strong differentiability conditions: $g$ is of class $C^1$. And the standard method is a reduction method in two steps: Using the differentiability conditions and the monotonicity conditions on $g$, one reduce the problem first to $E^+$ and then to the Nehari manifold on $E^+$. By corresponding the least energy to the infimum of the Nehari manifold on $E^+$, one could prove the concentration result. While, in this paper, we mainly consider the case $g \in C(\mathbb{R}^+, \mathbb{R})$ but is not necessarily of class $C^1(0, +\infty)$ in which such a standard method does not work.

To state our main results, we make the following assumptions:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$, sup $|V(x)| < a$ and there is a bounded domain $\Lambda \subset \mathbb{R}^3$, such that $\varepsilon = \min_{x \in \mathbb{R}^3} V(x) < \min_{x \in \partial \Lambda} V(x)$.

In this paper, we denote by $\mathbb{V}$ the set $\mathbb{V} := \{x \in \Lambda : V(x) = \varepsilon\}$, and according to (V), we know

$$\text{dist}(\mathbb{V}, \partial \Lambda) > 0.$$  \quad (3)

Without loss of generality, we assume that $0 \in \mathbb{V}$.

Setting $G(t) := \int_0^t g(s)sds$, we also make the following assumptions on the nonlinearity:

$$(g_1) \ g \in C(\mathbb{R}^+, \mathbb{R}), \quad \lim_{t \to 0^+} g(t) = 0 \text{ and } g \text{ is increasing on } \mathbb{R}^+;$$

$$(g_2) \text{ there exist } p \in (2, 3) \text{ and } C > 0 \text{ such that } g(t) \leq C(1 + t^{p-2}) \text{ for } t \geq 0;$$

$$(g_3) \ |g(t)| \leq C|t|^{p-2} \text{ for } t \geq 0;$$

$$(g_4) \ g \text{ is Lipchitz continuous on } [0, R],$$

$$(g_5) \ g \text{ satisfies the Caffarelli-Kohn-Nirenberg inequality: }$$

$$\int_0^R |g(t)|^q dt < \infty \quad \text{if } q \in (0, p)$$

$$\int_0^R |g(t)|^q dt > \infty \quad \text{if } q \in [p, +\infty)$$
Suppose that Theorem 1.1. holds:

\begin{enumerate}
\item[(i)] $w_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for any $q \geq 2$;
\item[(ii)] $|w_\varepsilon|$ attains its maximum at $\theta_\varepsilon$. Moreover, up to a subsequence, there hold $V(\theta_\varepsilon) \to \underline{v}$, $w_\varepsilon(\varepsilon x + \theta_\varepsilon) \to w(x)$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$, as $\varepsilon \to 0$, with $w$ a ground state solution of
\begin{equation}
-ia \cdot \nabla w + a \beta w + \underline{v} w = g(|w|)w;
\end{equation}
\item[(iii)] there holds
\begin{equation}
|w_\varepsilon(x)| \leq C \exp\left(- \frac{c}{\varepsilon} |x - \theta_\varepsilon|\right),
\end{equation}
where $c, C$ are positive constants independent of $\varepsilon$.
\end{enumerate}

Using the scaling $u(x) = w(\varepsilon x)$, it is easy to see that $w$ is a solution of (D) if and only if $u$ is a solution of
\begin{equation}
-ia \cdot \nabla u + a \beta u + V(\varepsilon x)u = g(|u|)u. \tag{D_\varepsilon}
\end{equation}

Therefore, we will mainly focus on this equivalent equation in the following.

Our strategy of proof is as follows: We first make a slight modification of the functional corresponding to (D_\varepsilon) such that it satisfies the Palais–Smale condition, and show the existence of ground states of the modified problem via a classical linking theorem. Then by relating the least energy to the infimum of the modified functional restrained on a generalized Nehari set, we prove that the least energy accumulating to the least energy of the limit problem. Lastly, using this fact, we prove that the least energy of the limit problem possesses a ground state and then by introducing the generalized Nehari set we prove the upper limit of the least energy $m_\varepsilon$ is less than or equal to the least energy of the limit problem as $\varepsilon \to 0$. In Section 4, we finish the proof of Theorem 1.1.
2. Preliminaries. To prove the main results, some preliminaries are first in order. We denote by $| \cdot |_q$ the usual $L^q$ norm, and by $\langle \cdot , \cdot \rangle$ the $L^2$ inner product. Let $H_0 = -i\alpha \cdot \nabla + a\beta$ denote the self-adjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. It is clear that $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R} \setminus (-a, a)$, where $\sigma(H_0)$ and $\sigma_c(H_0)$ denote the spectrum and the continuous spectrum of $H_0$ respectively. Thus the space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ possesses the orthogonal decomposition:

$$L^2(\mathbb{R}^3, \mathbb{C}^4) = L^+ \oplus L^-,$$

such that $H_0$ is positive definite in $L^+$ and negative definite in $L^-$. So for $u \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, we denote by $u = u^+ + u^-$ the decomposition of $u$ with $u^+ \in L^+$ and $u^- \in L^-$. Let $|H_0|$ denote the absolute value of $H_0$ and $|H_0|^{1/2}$ denote its square root. Then we define $E := \mathcal{D}(|H_0|^{1/2})$ endowed with the inner product

$$\langle u, v \rangle = \langle |H_0|^{1/2}u, |H_0|^{1/2}v \rangle_2$$

and the induced norm $\|u\|^2 = \langle u, u \rangle$. From [10] Lemma 7.4, it follows that $E = H^1(\mathbb{R}^3, \mathbb{C}^4)$ and the norm $\| \cdot \|$ is equivalent to the usual $H^1(\mathbb{R}^3, \mathbb{C}^4)$ norm. Hence, $E$ embeds into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ continuously for $q \in [2, 3]$ and into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ compactly for $q \in [1, 3]$. Moreover, since $\sigma(H_0) = \mathbb{R} \setminus (-a, a)$, we have

$$a|u|^2 \leq \|u\|^2, \quad \text{for all } u \in E. \quad (4)$$

It is clear that (see [10] Section 7) $E$ possesses the following decomposition

$$E = E^+ \oplus E^-,$$

where $E^+ = E \cap L^+$ and $E^- = E \cap L^-$ and the sum is orthogonal with respect to both $\langle \cdot , \cdot \rangle_2$ and $\langle \cdot , \cdot \rangle$. Additionally, this decomposition of $E$ induces a natural decomposition of $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in (1, +\infty)$ and we refer the readers to [18] Proposition 2.1 for the proof of the following proposition.

Proposition 1. Let $E^+ \oplus E^-$ be the decomposition of $E$ according to the positive and negative part of $\sigma(H_0)$. Then, if we set $E^\pm_q := E^\pm \cap L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in (1, +\infty)$, there holds

$$L^q(\mathbb{R}^3, \mathbb{C}^4) = \text{cl}_q E^+ + \text{cl}_q E^-,$$

where $\text{cl}_q$ denotes the closure in $L^q(\mathbb{R}^3, \mathbb{C}^4)$. Moreover, for any $q \in (1, +\infty)$, there exists positive constant $d_q$ such that

$$d_q \|u^\pm\|_q \leq \|u\|, \quad u \in E \cap L^q(\mathbb{R}^3, \mathbb{C}^4).$$

From standard arguments, the functional $J_\varepsilon : E \to \mathbb{R}$ defined by

$$J_\varepsilon(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x)|u|^2 - \int_{\mathbb{R}^3} G(|u|)$$

is of class $C^1$. For $u, v \in E$, there holds

$$J_\varepsilon'(u)v = \text{Re} \int_{\mathbb{R}^3} (H_0 u + V(\varepsilon x) u - g(|u|) u) \cdot v$$

with $u \cdot v$ denoting the usual inner product in $\mathbb{C}^4$, i.e., $u \cdot v = \sum_{j=1}^4 u_j \bar{v}_j$. And hence the critical points of $J_\varepsilon$ are weak solutions of (D$_\varepsilon$) (one can find more details in [13]).

However, we will not deal with $J_\varepsilon$ directly. Instead, we use a convenient truncation of the nonlinearity such as that used in [7, 8] to modify the function such that
the energy functional becomes coercive far from origin and satisfies the Palais–Smale condition.

According to the assumptions in (V): $V$ is continuous and $\varphi < \min_{x \in \overline{\Lambda}} V(x)$, we can fix a small $\delta_1 > 0$ such that

$$\varphi < V(x) \text{ for any } x \in \overline{\Lambda} \setminus \Lambda$$

(5)

where $\Lambda^{\delta_1} = \{ x \in \mathbb{R}^3 : \text{dist}(x, \Lambda) := \inf_{y \in \Lambda} |x - y| < \delta_1 \}$ is the $\delta_1$-neighborhood of $\Lambda$ and $\overline{\Lambda^{\delta_1}}$ is the closure of $\Lambda^{\delta_1}$. Let $\zeta \in C(\mathbb{R}^3, \mathbb{R})$ be a function such that $0 \leq \zeta(t) \leq 1$, $\zeta(t) = 1$ if $t \leq 0$ and $\zeta(t) = 0$ if $t \geq \delta_1$. Setting $\chi(x) = \zeta(\text{dist}(x, \Lambda))$ and $\bar{g} \in C(\mathbb{R}^+, \mathbb{R})$:

$$\bar{g}(t) = \min \left\{ \frac{g(t)}{2}, \frac{a - |V|_{\infty}}{2} \right\}.$$ 

we define

$$f(x, t) = \chi(x)g(t) + (1 - \chi(x))\bar{g}(t), \quad F(x, t) = \int_0^t f(x, s)ds.$$ 

Then using $(g_1) - (g_3)$, we can verify that $f$ is a Caratheodory function and it satisfies:

$(f_1)$ $\lim_{t \to 0^+} f(x, t) = 0$ uniformly for $x \in \mathbb{R}^3$ and $f(x, t)$ is non-decreasing in $t \in \mathbb{R}^+$ for $x \in \mathbb{R}^3$;

$(f_2)$ $0 \leq f(x, t) \leq g(t)$ for every $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$;

$(f_3)$ for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$, $F(x, t) = \chi(x)G(t) + (1 - \chi(x))\bar{G}(t)$, where $\bar{G}(t) = \int_0^t \bar{g}(s)ds$. Moreover, there holds

$$\bar{g}(t)^2 - 2\bar{G}(t) \geq 0.$$ 

(6)

From $(g_1)$ and $(g_2)$, it follows that there exist $t_0 > 0$ and $c_1 > 0$ so that

$$g(t) \leq \frac{a - |V|_{\infty}}{2}, \quad t \leq t_0$$

(7)

and $g(t) \leq c_1 t^{p-2}$ for $t \geq t_0$. So for $t \geq t_0$, there holds $(g(t)t)^{\frac{1}{p-1}} \leq c_2 t$ and

$$\left(\frac{g(t)t}{t^{p-1}}\right)^{\frac{p-1}{p}} = (g(t)t)^{\frac{1}{p-1} + 1} \leq c_2 t \cdot g(t)t = c_2 g(t)t^2.$$ 

This, together with $(g_3)$: $\frac{1}{2}g(t)t^2 - G(t) \geq \frac{a^2}{4\epsilon^2}g(t)t^2$, yields that

$$\left(\frac{g(t)t}{t^{p-1}}\right)^{\frac{p-1}{p}} \leq c_3 \left(\frac{1}{2}g(t)t^2 - G(t)\right), \quad t \geq t_0.$$ 

By this and $(7)$, we get

$$g(t)t \leq \frac{a - |V|_{\infty}}{2} t + c_4 \left(\frac{1}{2}g(t)t^2 - G(t)\right)^{\frac{p-1}{p}}, \quad t \geq 0.$$ 

Since $\bar{g} \leq \frac{a - |V|_{\infty}}{2}$, from the definition of $f(x, t)$, it follows that

$$f(x, t)t \leq \frac{a - |V|_{\infty}}{2} t + c_4 \chi(x) \left(\frac{1}{2}g(t)t^2 - G(t)\right)^{\frac{p-1}{p}}, \quad t \geq 0.$$ 

(8)

Now we are ready to define the modified functional $\Phi_\epsilon : E \to \mathbb{R}$,

$$\Phi_\epsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^3} V(\epsilon x)|u|^2 - \int_{\mathbb{R}^3} F(\epsilon x, |u|).$$

Similarly, $\Phi_\epsilon$ is of class $C^1$ and the critical points correspond to weak solutions of

$$-i\alpha \cdot \nabla u + a\beta u + V(\epsilon x)u = f(\epsilon x, |u|)u.$$
For the sake of simplicity, in the following of this paper, we denote by
\[ V_\varepsilon(x) = V(\varepsilon x), \quad \chi_\varepsilon(x) = \chi(\varepsilon x), \quad f_\varepsilon(x, t) = f(\varepsilon x, t) \text{ and } F_\varepsilon(x, t) = \int_0^t f_\varepsilon(x, s)ds. \]

**Lemma 2.1.** For every \( \varepsilon > 0 \), the functional \( \Phi_\varepsilon \) satisfies the Palais–Smale condition.

**Proof.** Suppose that \( \{u_n\} \subset E \) is a sequence such that \( \Phi_\varepsilon(u_n) \) is bounded and \( \Phi'_\varepsilon(u_n) \to 0 \). Then there exists a constant \( C > 0 \) satisfying
\[
C(1 + \|u_n\|) \geq \Phi_\varepsilon(u_n) - \frac{1}{2} \Phi'_\varepsilon(u_n)u_n = \int_{\mathbb{R}^3} \frac{1}{2} f_\varepsilon(x, |u_n|)|u_n|^2 - F_\varepsilon(x, |u_n|)
\]
By \((f_3)\) and \((6)\), we have
\[
C(1 + \|u_n\|) \geq \int_{\mathbb{R}^3} \chi_\varepsilon(x)\left(\frac{1}{2}g(|u_n|)|u_n|^2 - G(|u_n|)\right)
+ \int_{\mathbb{R}^3} (1 - \chi_\varepsilon(x))\left(\frac{1}{2}g(|u_n|)|u_n|^2 - \tilde{G}(|u_n|)\right)
\geq \int_{\mathbb{R}^3} \chi_\varepsilon(x)\left(\frac{1}{2}g(|u_n|)|u_n|^2 - G(|u_n|)\right).
\]
On the other hand, \((8)\) yields that
\[
o(1)\|u_n\| = \Phi'_\varepsilon(u_n)(u_n^+ - u_n^-)
= \|u_n\|^2 + \text{Re} \int_{\mathbb{R}^3} V_\varepsilon(x)u_n \cdot (u_n^+ - u_n^-) - f_\varepsilon(x, |u_n|)u_n \cdot (u_n^+ - u_n^-)
\geq \|u_n\|^2 - \|V\|_{\infty} \int_{\mathbb{R}^3} |u_n||u_n^+ - u_n^-| - \int_{\mathbb{R}^3} f_\varepsilon(x, |u_n|)|u_n||u_n^+ - u_n^-|
\geq \|u_n\|^2 - \|V\|_{\infty} \int_{\mathbb{R}^3} |u_n||u_n^+ - u_n^-| - \frac{a - \|V\|_{\infty}}{2} \int_{\mathbb{R}^3} |u_n||u_n^+ - u_n^-|
- \int_{\mathbb{R}^3} c_4 \chi_\varepsilon(x)\left(\frac{1}{2}g(|u_n|)|u_n|^2 - G(|u_n|)\right)^{\frac{p-1}{p}} |u_n^+ - u_n^-|.
\]
Using the Hölder’s inequality, by \( \chi_\varepsilon \in [0, 1] \), we deduce that
\[
o(1)\|u_n\| \geq \|u_n\|^2 - \|V\|_{\infty} \|u_n\| \frac{1}{2} - \frac{a - \|V\|_{\infty}}{2} \|u_n\| \frac{1}{2}
- c_4 \left(\int_{\mathbb{R}^3} \chi_\varepsilon(x)\left(\frac{1}{2}g(|u_n|)|u_n|^2 - G(|u_n|)\right)^{\frac{p-1}{p}} |u_n^+ - u_n^-|\right)^{\frac{p}{p-1}}.
\]
Then from \((4)\) and \((9)\), it follows
\[
\frac{a - \|V\|_{\infty}}{2a} \|u_n\|^2 \leq C' \left(\|u_n\| + \|u_n\|^{\frac{2p}{p-1}}\right),
\]
which means that \( \{u_n\} \) is bounded in \( E \). Therefore \( u_n \rightharpoonup u \) weakly in \( E \) and \( u_n \to u \) strongly in \( L^p_\text{loc}(\mathbb{R}^3, \mathbb{C}^4) \), \( q \in [1, 3] \).
For $z_n = u_n - u$, we have $\Phi'_\varepsilon(u_n)(z_n^+ - z_n^-) = o(1)$ and $\Phi'_\varepsilon(u)(z_n^+ - z_n^-) = 0$, which imply that

$$o(1) = \text{Re}(u_n^+ z_n^+) + \text{Re}(u_n^- z_n^-) + \int_{\mathbb{R}^3} V_\varepsilon(x) u_n \cdot (z_n^+ - z_n^-)$$

$$- \text{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |u_n|) u_n \cdot (z_n^+ - z_n^-);$$

$$0 = \text{Re}(u^+ z_n^+) + \text{Re}(u^- z_n^-) + \int_{\mathbb{R}^3} V_\varepsilon(x) u \cdot (z_n^+ - z_n^-)$$

$$- \text{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |u|) u \cdot (z_n^+ - z_n^-).$$

Subtracting the left and right sides of the two equations above and using the fact that

$$\lim_{n \to \infty} \text{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |u_n|) u \cdot (z_n^+ - z_n^-) = \lim_{n \to \infty} \text{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |u_n|) u \cdot (z_n^+ - z_n^-) = 0,$$

we obtain

$$\|z_n\|^2 + \text{Re} \int_{\mathbb{R}^3} V_\varepsilon(x) z_n \cdot (z_n^+ - z_n^-) = \text{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |u_n|) z_n \cdot (z_n^+ - z_n^-) + o(1).$$

Then it follows from (4) that

$$\frac{a - |V|_\infty}{a} \|z_n\|^2 \leq \text{Re} \int_{\mathbb{R}^3} (\chi_\varepsilon(x) g(|u_n|) + (1 - \chi_\varepsilon(x)) \tilde{g}(|u_n|)) z_n \cdot (z_n^+ - z_n^-) + o(1).$$

Noting that $\tilde{g} \leq \frac{a - |V|_\infty}{2a}$ and $\chi_\varepsilon \in [0, 1]$, we have

$$\frac{a - |V|_\infty}{2a} \|z_n\|^2 \leq \text{Re} \int_{\mathbb{R}^3} \chi_\varepsilon(x) g(|u_n|) z_n \cdot (z_n^+ - z_n^-) + o(1).$$

Since the support of $\chi_\varepsilon$ is bounded for every fixed $\varepsilon > 0$ and $z_n \to 0$ strongly in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{C}^3)$, we know that $z_n \to 0$, $n \to \infty$ strongly in $E$. \hfill \Box

Next we shall verify that $\Phi_\varepsilon$ possesses the linking structure. Before that we give the following notations.

$$B_r = \{u \in E : \|u\| \leq r\}, \quad S_r = \{u \in E : \|u\| = r\};$$

$$E(e) = \{u \in E : u = se + v, \ s \geq 0 \text{ and } v \in E^-, \ e \in E^+\}.$$

When $x \in \Lambda$, we have $F(x, t) = G(t) \geq C_f t^\mu - \frac{a - |V|_\infty}{4a} t^2$ for $t \geq 0$, where $C_f$ is a positive constant depending only on $f$ and $\frac{a - |V|_\infty}{4a}$. For any $e \in E^+ \cap S_1$, there exists $R_1 > 0$ such that

$$\frac{3a + |V|_\infty}{4a} s^2 - C_f d_\mu s^\mu |e|_\mu s^\mu \leq -1, \ s \geq R_1,$$

where $d_\mu$ is the constant appeared in Proposition 1. Then setting

$$R := R(e) = \sqrt{\frac{2a}{a - |V|_\infty} R_1},$$

we have the following property.

Lemma 2.2. There exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, $\Phi_\varepsilon|_{E(e) \cap S_\varepsilon} \leq 0$. 

Proof. We only need to show
\[ \lim_{\varepsilon \to 0} \sup_{w \in E(\varepsilon) \cap S_R} \Phi_\varepsilon(w) < 0. \]

Otherwise, there exist \( \varepsilon_n \to 0 \) and \( w_n \in E(\varepsilon) \cap S_R \), such that \( \Phi_{\varepsilon_n}(w_n) \geq -\frac{1}{n} \).

Thus, we set \( w_n = s_n e + v_n \) with \( s_n \geq 0 \) and \( v_n \in E^- \). Since \( \Phi_{\varepsilon_n}(w_n) \geq -\frac{1}{n} \), we have
\[ \frac{-1}{n} \leq \frac{1}{2} s_n^2 - \frac{1}{2} \|v_n\|^2 + \frac{|V|}{2} (s_n^2 e^2 + |v_n|^2) - \int_{\mathbb{R}^3} F_{\varepsilon_n}(x, |w_n|), \]
which implies that
\[ \frac{a - |V|}{2a} \|v_n\|^2 \leq \frac{a + |V|}{2a} s_n^2 + \frac{1}{n} = \frac{a + |V|}{2a} (R^2 - \|v_n\|^2) + \frac{1}{n}. \]

Therefore, \( \|v_n\|^2 \leq \frac{a + |V|}{2a} R^2 + o_n(1) \), \( R^2 \geq s_n^2 \geq \frac{a - |V|}{2a} R^2 + o_n(1) \). So up to a subsequence if necessary, \( v_n \rightharpoonup v \) weakly in \( E \) with \( v \in E^- \) and \( s_n \to s_0 \) with \( s_0 \in \left[ \sqrt{\frac{a - |V|}{2a}} R, R \right] \). Thus \( w_n \rightharpoonup s_0 e + v \) weakly in \( E \), \( w_n \to s_0 e + v \) strongly in \( L^q_{\text{loc}} \) for \( q \in [1, 3) \) and \( w_n(x) \to s_0 e(x) + v(x) \) a.e. \( x \in \mathbb{R}^3 \).

According to (3) and \( 0 \in V \), we can assume that there exists \( \delta > 0 \) such that \( B_\delta(0) \subset \Lambda \). Define \( \eta(x) \in C_0^\infty(B_\delta(0), [0, 1]) \) such that \( \eta(x) = 1 \) for \( |x| \leq \delta \), \( \eta(x) = 0 \) for \( |x| \geq \delta \) and \( |\nabla \eta| \leq \frac{\delta}{\delta} \). Then setting \( \chi_n(x) = \eta(\varepsilon_n x) \), we have
\[ \chi_n(x) w_n(x) \to s_0 e(x) + v(x), \text{ a.e. } x \in \mathbb{R}^3, \]
and
\[ \text{supp} \chi_n \subset \Lambda_{\varepsilon_n} = \{ x \in \mathbb{R}^3 : \varepsilon_n x \in \Lambda \}. \]

Using Fatou’s Lemma, we deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^3} |\chi_n w_n|^\mu \geq \int_{\mathbb{R}^3} |s_0 e + v|^\mu, \]
which implies that
\[ \int_{\mathbb{R}^3} F_{\varepsilon_n}(x, |w_n|) \geq \int_{\Lambda_{\varepsilon_n}} F_{\varepsilon_n}(x, |\chi_n w_n|) \]
\[ \geq C_f \int_{\mathbb{R}^3} |\chi_n w_n|^\mu - \frac{a - |V|}{4} \int_{\Lambda_{\varepsilon_n}} |\chi_n w_n|^2 \]
\[ = C_f \int_{\mathbb{R}^3} |\chi_n w_n|^\mu - \frac{a - |V|}{4} \int_{\mathbb{R}^3} |\chi_n w_n|^2 \]
\[ \geq C_f \int_{\mathbb{R}^3} |s_0 e + v|^\mu - \frac{a - |V|}{4} \int_{\mathbb{R}^3} |w_n|^2 + o_n(1) \]
\[ \geq C_f d_{\mu}^\varepsilon |e_\mu s_0^\mu - \frac{a - |V|}{4} \int_{\mathbb{R}^3} |w_n|^2 + o_n(1) \].

Here we have used the fact that \( F(x, t) \) is nondecreasing in \( t \), \( F(x, t) \geq C_f t^\mu - \frac{a - |V|}{4} t^2 \) for \( x \in \Lambda \), \( t \geq 0 \) and Proposition 1. Taking the above inequality into (12), we obtain
\[ -\frac{1}{n} \leq \frac{3a + |V|}{4a} s_n^2 - \frac{a - |V|}{4a} \|v_n\|^2 - C_f d_{\mu}^\varepsilon |e_\mu s_0^\mu + o_n(1). \]

Let \( n \to \infty \),
\[ \frac{3a + |V|}{4a} s_0^2 - C_f d_{\mu}^\varepsilon |e_\mu s_0^\mu \geq 0, \]
Since $s_0 \geq \sqrt{\frac{a-|V|}{2a}} R = R_1$, this is a contradiction to (10).

Then, for the fixed $e \in E^+ \cap S_1$, we set

$$Q(e) = \{ u \in E(e) : u = se + v, \ s > 0, \ v \in E^- \text{ and } \|u\| < R \},$$

with $R$ the positive constant we defined in (11). Obviously, for any $v \in E^-$ and $\varepsilon > 0$, there holds

$$\Phi_{\varepsilon}(v) = -\frac{1}{2}\|v\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon(x)|v|^2 - \int_{\mathbb{R}^3} F_\varepsilon(x,|v|) \leq -\frac{a-|V|_\infty}{2a}\|v\|^2 \leq 0.$$ Combining this with Lemma 2.2, we know

$$\sup_{u \in \partial Q(e)} \Phi_{\varepsilon}(u) \leq 0, \ \varepsilon \in (0, \varepsilon_0). \quad (13)$$ Additionally, a standard argument shows that there exist $r \in (0, R)$ and $\rho > 0$, such that

$$\inf_{u \in S, \varepsilon E^+} \Phi_{\varepsilon}(u) \geq \rho, \ \text{ for any } \varepsilon > 0. \quad (14)$$ Define

$$\Psi_{\varepsilon}(u) = \int_{\mathbb{R}^3} \left( |V|_\infty - V_\varepsilon(x) \right) |u|^2 + \int_{\mathbb{R}^3} F_\varepsilon(x,|u|).$$ Then, using the fact that $E$ embeds into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ continuously for $q \in [2,3]$ and embeds into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ compactly for $q \in [1,3]$, it is easy to see that:

**Lemma 2.3.** $\Psi_{\varepsilon}$ is weakly sequentially lower semi-continuous and $\Psi'_{\varepsilon}$ is weakly sequentially continuous.

Define $\langle \cdot, \cdot \rangle_V : E \times E \to \mathbb{C}$ by

$$\langle u, v \rangle_V = \langle u, v \rangle + |V|_\infty \langle u^+, v^+ \rangle_2 - |V|_\infty \langle u^-, v^- \rangle_2.$$

According to (4), $\langle \cdot, \cdot \rangle_V$ is an inner product on $E$, the induced norm $\| \cdot \|_V$ is equivalent to $\| \cdot \|$. Moreover, computing directly, we have

$$\Phi_{\varepsilon}(u) = \frac{1}{2}\|u^+\|_V^2 - \frac{1}{2}\|u^-\|_V^2 - \Psi_{\varepsilon}(u). \quad (15)$$

Let $S$ be a countable dense subset of $(E^-)^*$ (the dual space of $E^-$) and $D = \{ d_s : s \in S, \ d_s(u, v) = |s(u-v)| \text{ for } u, v \in E^- \}$ be the associated family of semi-metrics on $E^-$. Let $\mathcal{P}$ be the family of semi-norms on $E$ consisting of all semi-norms:

$$p_s : E \to \mathbb{R}; \ \ p_s(u) = |s(u^-)| + \|u^+\|, \ u \in E, \ s \in S.$$ Thus $\mathcal{P}$ is countable and it induces the product topology on $E$ given by the $D$-topology on $E^-$ and the norm topology on $E^+$. We denote this topology by $(E, \mathcal{T}_{\mathcal{P}})$ and denote the weak* topology on $E^*$ by $(E^*, \mathcal{T}_{w^*})$. Then using (15), Lemma 2.3 and the arguments in Theorem 4.1 of [10], we can prove that:

*(Φ)* For any $\varepsilon > 0$, $c \in \mathbb{R}$ and $\Phi_{\varepsilon,c} = \{ u \in E : \Phi_{\varepsilon}(u) \geq c \}$, there hold $\Phi_{\varepsilon} : (E, \mathcal{T}_{\mathcal{P}}) \to \mathbb{R}$ is upper semi-continuous and $\Phi'_{\varepsilon} : (\Phi_{\varepsilon,c}, \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous.

According to this, (13) and (14), we may apply the linking theorem (see Theorem 4.4 in [10] for example), and obtain a Palais–Smale sequence at level

$$c_{\varepsilon} \in \left[ \inf_{u \in S, \varepsilon E^+} \Phi_{\varepsilon}(u), \ \sup_{u \in \partial Q(e)} \Phi_{\varepsilon}(u) \right], \ \varepsilon \in (0, \varepsilon_0).$$
Combining this with Lemma 2.1, we obtain a critical point of $\Phi_\varepsilon$ at level $c_\varepsilon$. Denoting the critical set of $\Phi_\varepsilon$ by $\mathcal{K}_\varepsilon := \{ u \in E \setminus \{ 0 \} : \Phi'_\varepsilon(u) = 0 \}$ and setting $m_\varepsilon = \inf_{u \in \mathcal{K}_\varepsilon} \Phi_\varepsilon(u)$, we claim $m_\varepsilon > 0$. In fact, from

$$
\Phi_\varepsilon(u) = \Phi_\varepsilon(u) - \frac{1}{2} \Phi'_\varepsilon(u) u = \int_{\mathbb{R}^3} \frac{1}{2} f_\varepsilon(x, |u|) |u|^2 - F_\varepsilon(x, |u|) \geq 0, \quad u \in \mathcal{K}_\varepsilon,
$$

it follows that $m_\varepsilon \geq 0$. If $m_\varepsilon = 0$, there exists $\{ u_n \} \subset \mathcal{K}_\varepsilon$ such that $\Phi_\varepsilon(u_n) \to 0$. Similarly as the proof in Lemma 2.1, we can prove $\| u_n \| \to 0$. While, by (4) and the fact $f_\varepsilon(x,t)t \leq g(t)t \leq \frac{a-|V|\infty}{4\alpha} t + C t^{p-1}$ for $x \in \mathbb{R}^3$, $t \geq 0$, we know

$$
0 = \Phi'_\varepsilon(u_n)(u_n^+ - u_n^-) \geq \frac{a-|V|\infty}{4\alpha} \| u_n \|^2 - C \int_{\mathbb{R}^3} |u_n|^{p-1}|u_n^+ - u_n^-|, \quad u_n \in \mathcal{K}_\varepsilon.
$$

So $\| u_n \| \geq \eta' > 0$, which is a contradiction. Thus $m_\varepsilon > 0$, combining which with the fact that $\Phi_\varepsilon$ satisfies the Palais–Smale condition, there exists a ground state solution $u_\varepsilon \in \mathcal{K}_\varepsilon$, such that

$$
\Phi_\varepsilon(u_\varepsilon) = m_\varepsilon > 0. \quad (16)
$$

We end this section by two technical results, which play an important role in the following proof.

**Lemma 2.4.** [21]. Assume that continuous function $h: \mathbb{R}^+ \to \mathbb{R}$ satisfies

\begin{enumerate}[(h_1)]
\item $\lim_{t \to 0^+} h(t) = 0$;
\item $h$ is increasing on $\mathbb{R}^+$;
\item $\lim_{t \to +\infty} \frac{H(t)}{t^2} = +\infty$, where $H(t) = \int_0^t h(s) ds$.
\end{enumerate}

Then for $s \geq 0$ and $u, v \in \mathbb{C}^4$ such that $u \neq su + v$, there holds

$$
\text{Re} \ h(|u|) u \cdot \left( \frac{s^2}{2} u - \frac{1}{2} u + su \right) + H(|u|) - H(|su + v|) < 0.
$$

**Proof.** In the case $v \neq 0$, one can see the proof of [21] Lemma 3.3.

In the case $v = 0$, since $u \neq su + v$, we have $u \neq 0$ and $s \neq 1$. Setting

$$
\mathcal{H}(s) = \text{Re} \ h(|u|) u \cdot \left( \frac{s^2}{2} u - \frac{1}{2} u \right) + H(|u|) - H(|su|),
$$

then $\mathcal{H}'(s) = s|u|^2(h(|u|) - h(|su|))$. Thus $\mathcal{H}(s)$ attains its unique maximum at $s = 1$ and hence the conclusion follows from the fact $\mathcal{H}(1) = 0$. \hfill \Box

**Proposition 2.** Assume that $h \in C(\mathbb{R}^+, \mathbb{R})$ satisfies that

\begin{enumerate}[(h'_1)]
\item $\lim_{t \to 0^+} h(t) = 0$;
\item $h$ is non-decreasing on $\mathbb{R}^+$.
\end{enumerate}

Then for $H(z) = \int_0^z h(t) dt$, $s \geq 0$ and $u, v \in \mathbb{C}^4$ with $u \neq su + v$, we have

$$
\text{Re} \ h(|u|) u \cdot \left( \frac{s^2}{2} u - \frac{1}{2} u + su \right) + H(|u|) - H(|su + v|) \leq 0.
$$

**Proof.** Adopting the method in the proof of [28] Lemma 3.2, for any $\delta > 0$, we define $h_\delta: \mathbb{R}^+ \to \mathbb{R}$ by

$$
h_\delta(t) = h(t) + \delta t.
$$

It is easy to check that for any $\delta > 0$, $h_\delta$ satisfies $(h_1) - (h_3)$ of Lemma 2.4. And the desired result follows by applying Lemma 2.4 to $h_\delta$ and then letting $\delta \to 0$. \hfill \Box
3. Limit problems. For \( \lambda \in (-a,a) \), we consider such a problem

\[-i\alpha \cdot \nabla u + a\beta u + \lambda u = g(|u|)u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}^4),\]

(D\(\lambda\))

that corresponds to the \( C^1(E, \mathbb{R}) \) functional

\[J_\lambda(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) + \frac{\lambda}{2} \int_{\mathbb{R}^3} |u|^2 - \int_{\mathbb{R}^3} G(|u|).\]

Note that for any \( e \in E^+ \cap S_1 \) and \( u \in E(e) \) with \( u = se + v, \ s \geq 0 \) and \( v \in E^- \), there holds

\[J_\lambda(u) = \frac{1}{2} s^2 - \frac{1}{2} ||v||^2 + \frac{\lambda}{2} (s^2|e|^2 + |v|^2) - \int_{\mathbb{R}^3} G(|u|).\]

According to (g1) and (g3), \( G(t) \geq C_\lambda t^\mu - \frac{a-\lambda}{4} t^2 \) for all \( t \geq 0 \), with \( C_\lambda \) a positive constant depending only on \( g \) and \( \lambda \). Using this and Proposition 1, we deduce

\[J_\lambda(u) \leq \frac{3a + \lambda}{4a} s^2 - \frac{a - \lambda}{4a} ||v||^2 - C_\lambda \int_{\mathbb{R}^3} |se + v|^\mu \leq \frac{3a + \lambda}{4a} s^2 - \frac{a - \lambda}{4a} ||v||^2 - C_{d\mu}s^\mu \int_{\mathbb{R}^3} |e|^\mu.\]

This implies that there exists \( R_\lambda = R_\lambda(e) > 0 \) such that

\[\sup_{u \in \partial Q_\lambda(e)} J_\lambda(u) \leq 0,\]

where \( Q_\lambda(e) = \{ u \in E(e) : u = se + v, \ s > 0, \ v \in E^- \ and \ ||u|| < R_\lambda \} \). Obviously, there exist \( r_\lambda \in (0, R_\lambda) \) and \( \rho_\lambda > 0 \), such that

\[\inf_{u \in S_r \cap E^+} J_\lambda(u) \geq \rho_\lambda.\]

Then by the same procedure as that in Section 2, we obtain a Palais–Smale sequence for \( J_\lambda \) at level

\[c_\lambda \in \left[ \inf_{u \in S_{r_\lambda} \cap E^+} J_\lambda(u), \sup_{u \in Q_\lambda(e)} J_\lambda(u) \right].\]

Moreover, we have the following proposition.

**Proposition 3.** For \( \lambda \in (-a,a), (D\(\lambda\)) possesses a ground state solution \( u_\lambda \) such that \( J_\lambda(u_\lambda) = m_\lambda > 0 \).

**Proof.** Let \( \{u_n\} \) be a Palais–Smale sequence for \( J_\lambda \) at level \( c_\lambda \), similar to the proof in Lemma 2.1, \( \{u_n\} \) is bounded in \( E \). Moreover, there holds

\[\lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_n|^2 > 0.\]

Otherwise, Lions’ Lemma yields that \( u_n \to 0 \) in \( L^q(\mathbb{R}^3, \mathbb{C}^4) \) for \( q \in (2,3) \). Noting that \( g(t) \leq \frac{a-\lambda}{4} t + Ct^{p-1} \) for any \( t \geq 0 \) with \( C \) a positive constant depending only on \( g \) and \( \lambda \), by (4) and \( a_\lambda(1) = J'_\lambda(u_\lambda)(u_\lambda^+ - u_\lambda^-) \), we get that

\[a_\lambda(1) \geq \frac{a - \lambda}{4a} ||u_n||^2 - C \int_{\mathbb{R}^3} |u_n|^{p-1} |u_n^+ - u_n^-|.\]

So \( ||u_n|| \to 0 \) as \( n \to \infty \). However, this is impossible since \( c_\lambda \geq \rho_\lambda > 0 \).
Then, there exists \{y_n\} ⊂ \mathbb{R}^3 such that \( \int_{B_1(y_n)} |u_n|^2 \geq \eta > 0 \). Consequently, \{u_n(-y_n)\} is a bounded Palais–Smale sequence for \( J_\lambda \), which converges weakly to a nontrivial solution \( u \) of \((D_\lambda)\). Moreover, by Fatou’s Lemma,
\[
J_\lambda(u) = J_\lambda(u) - \frac{1}{2} J'_\lambda(u)u = \int_{\mathbb{R}^3} \frac{1}{2} g(|u|)|u|^2 - G(|u|)
\leq \lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{1}{2} g(|u_n|)|u_n|^2 - G(|u_n|)
= \lim_{n \to \infty} (J_\lambda(u_n) - \frac{1}{2} J'_\lambda(u_n)u_n) = c_\lambda.
\]

We denote the critical set of \( J_\lambda \) by \( K_\lambda := \{ u \in E \setminus \{0\} : J'_\lambda(u) = 0 \} \) and set \( m_\lambda = \inf_{u \in K_\lambda} J_\lambda(u) \). Since for any \( u \in K_\lambda \),
\[
J_\lambda(u) = J_\lambda(u) - \frac{1}{2} J'_\lambda(u)u = \int_{\mathbb{R}^3} \frac{1}{2} g(|u|)|u|^2 - G(|u|) \geq 0,
\]
we know \( m_\lambda \geq 0 \). Then we claim \( m_\lambda > 0 \). In fact, if \( m_\lambda = 0 \), there exists \( \{w_n\} \subset K_\lambda \) such that \( J_\lambda(w_n) \to 0 \). According to this, similarly as the proof in Lemma 2.1, we can prove \( \|w_n\| \to 0 \). While, from (4) and the fact that \( g(t)t \leq \frac{a-\lambda}{4a} t + C t^{p-1} \) for \( t \geq 0 \), it follows that
\[
0 = J'_\lambda(u)(u^+ - u^-) \geq \frac{a-\lambda}{4a} \|u\|^2 - C \int_{\mathbb{R}^3} |u|^{p-1} |u^+ - u^-|, \quad u \in K_\lambda.
\]
Hence \( \|u\| \geq \eta' > 0 \) for any \( u \in K_\lambda \), which contradicts to \( \|w_n\| \to 0 \).

Thus for \( \{w_n\} \subset K_\lambda \) such that \( J_\lambda(w_n) \to m_\lambda > 0 \), using Lions’ Lemma, we can obtain a sequence \( \{z_n\} \subset \mathbb{R}^3 \) such that \( w_n(-z_n) \rightharpoonup u_\lambda \) weakly in \( E \). Moreover, \( u_\lambda \) is a nontrivial solution of \((D_\lambda)\) with \( J_\lambda(u_\lambda) \leq m_\lambda \) (see (17)), and hence is a ground state with \( J_\lambda(u_\lambda) = m_\lambda \).

In Section 2, for \( \varepsilon \in (0, \varepsilon_0) \), we have obtained a least energy solution \( u_\varepsilon \) for \( \Phi_\varepsilon \) with \( \Phi_\varepsilon(u_\varepsilon) = m_\varepsilon \). And if we can show that there exists a uniformly decay for \( u_\varepsilon \):
\[
|u_\varepsilon(x)| \to 0, \quad \text{as} \ |x| \to +\infty,
\]
uniformly for small \( \varepsilon \), then \( u_\varepsilon \) is actually a solution of \((D_\varepsilon)\). In order to show this, as well as the convergence property and the exponential decay results in Theorem 1.1, the following property is the key:
\[
\lim_{\varepsilon \to 0} m_\varepsilon \leq m_\lambda,
\]  
(18)
where \( \nu = \min_{x \in \Lambda} V(x) \) and \( m_\lambda \) is the least energy of \((D_\lambda)\) with \( \lambda = \nu \).

As far as we know, in all the results involving semiclassical states for Dirac equations, there exists the differentiability conditions: \( g \) is of class \( C^1(0, +\infty) \). And the standard method is a reduction method in two steps, first to \( E^+ \) and then to a Nehari manifold on \( E^+ \). By corresponding the least energy to the infimum of the Nehari manifold on \( E^+ \), one could obtain (18). However, such a method does not work here, and since it is difficult to obtain the relation (18) directly, we need more characterizations on \( m_\varepsilon \) and \( m_\lambda \).

Motivated by [21], we consider the generalized Nehari set for \( \Phi_\varepsilon \) and \( J_\lambda \):
\[
\mathcal{M}_\varepsilon := \{ u \in E \setminus E^- : \Phi'_\varepsilon(u)u = 0, \Phi'_\varepsilon(v)v = 0 \text{ for all } v \in E^- \};
\]
\[
\mathcal{M}_\lambda := \{ u \in E \setminus E^- : J'_\lambda(u)u = 0, J'_\lambda(v)v = 0 \text{ for all } v \in E^- \}.
\]
Observing that for $\varepsilon \in (0, \varepsilon_0)$, $\Phi_\varepsilon$ possesses positive least energy, so for any nontrivial solution $u$ of $\Phi_\varepsilon$ there holds
\[
\frac{1}{2} ||u^+||^2 - \frac{1}{2} ||u^-||^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x)||u||^2 - \int_{\mathbb{R}^3} F_\varepsilon(x, |u|) \geq m_\varepsilon > 0.
\]
Combining this with (4) and $|V|_\infty < a$, we know $u^+ \neq 0$ and hence $u \in \mathcal{M}_\varepsilon$.
Similarly, we can prove nontrivial solutions of $J_\lambda$ for $\lambda \in (-a, a)$ belong to $\mathcal{M}_\lambda$. In summary, the nontrivial critical set of $\Phi_\varepsilon$ and $J_\lambda$ satisfy
\[
\mathcal{K}_\varepsilon \subset \mathcal{M}_\varepsilon, \quad \varepsilon \in (0, \varepsilon_0); \quad \mathcal{K}_\lambda \subset \mathcal{M}_\lambda, \quad \lambda \in (-a, a).
\]
Then, $\mathcal{M}_\varepsilon$ and $\mathcal{M}_\lambda$ are nonempty and for each $u \in \mathcal{M}_\lambda$, $J_\lambda|_{E(u^+)}$ attains its unique maximum at $u$; for $u \in \mathcal{M}_\varepsilon$, $\Phi_\varepsilon|_{E(u^+)}$ attains its maximum at $u$.

**Proposition 4.** For any $t \geq 0$ and $v \in E^-$, there holds
\[
J_\lambda(u) > J_\lambda(tu^+ + v) \text{ when } u \in \mathcal{M}_\lambda \text{ and } u \neq tu^+ + v;
\]
\[
\Phi_\varepsilon(u) \geq \Phi_\varepsilon(tu^+ + v) \text{ when } u \in \mathcal{M}_\varepsilon.
\]

**Proof.** Let $u = u^+ + u^- \in \mathcal{M}_\lambda$, $t \geq 0$ and $v \in E^-$. For $w = v - tu^- \in E^-$, we have
\[
J_\lambda(tu^+ + v) - J_\lambda(u) = J_\lambda(tu + w) - J_\lambda(u)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} H_0(tu + w) \cdot (tu + w) - H_0 u \cdot u + |tu + w|^2 - |u|^2
\]
\[
+ \int_{\mathbb{R}^3} G(|u|) - G(|tu + w|)
\]
\[
= \text{Re} \int_{\mathbb{R}^3} (H_0 u + \lambda u) \cdot (\frac{t^2}{2} u - \frac{1}{2} u + tw) - \frac{1}{2} ||w||^2 + \frac{\lambda}{2} |w|^2
\]
\[
+ \int_{\mathbb{R}^3} G(|u|) - G(|tu + w|).
\]
Since $u \in \mathcal{M}_\lambda$ and $w \in E^-$, we have $J_\lambda'(u)(\frac{t^2}{2} u - \frac{1}{2} u + tw) = 0$, which means that
\[
\text{Re} \int_{\mathbb{R}^3} (H_0 u + \lambda u) \cdot (\frac{t^2}{2} u - \frac{1}{2} u + tw) - \text{Re} \int_{\mathbb{R}^3} g(|u|)u \cdot (\frac{t^2}{2} u - \frac{1}{2} u + tw) = 0.
\]
Taking this into the above equation, using Lemma 2.4 and (4), we deduce that
\[
J_\lambda(tu^+ + v) - J_\lambda(u) = \int_{\mathbb{R}^3} \text{Re} g(|u|)u \cdot (\frac{t^2}{2} u - \frac{1}{2} u + tw) + G(|u|) - G(|tu + w|)
\]
\[
- \frac{1}{2} ||w||^2 + \frac{\lambda}{2} |w|^2 < 0.
\]
To show the second property, we set $u = u^+ + u^- \in \mathcal{M}_\varepsilon$, $t \geq 0$ and $v \in E^-$. By an argument similar to that above, for $w = v - tu^- \in E^-$, we have
\[
\Phi_\varepsilon(tu^+ + v) - \Phi_\varepsilon(u) = \Phi_\varepsilon(tu + w) - \Phi_\varepsilon(u)
\]
\[
= \int_{\mathbb{R}^3} \text{Re} f_\varepsilon(x, |u|)u \cdot (\frac{t^2}{2} u - \frac{1}{2} u + tw) + F_\varepsilon(x, |u|) - F_\varepsilon(x, |tu + w|)
\]
\[
- \frac{1}{2} ||w||^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\varepsilon(x)|w|^2.
\]
From (f1), we know that for any $x \in \mathbb{R}^3$, $f_\varepsilon(x, \cdot)$ satisfies the assumption $(h'_1)$ and $(h'_2)$ in Proposition 2. So applying Proposition 2 and (4), we get the conclusion. \qed
Proposition 5. For any \( u \in E \) with \( u^+ \neq 0 \), the following statements hold true:

(i) there exists \( \varepsilon_0 > 0 \), such that \( M_\varepsilon \cap E(u^+) \neq \emptyset \), \( \varepsilon \in (0, \varepsilon_0) \);
(ii) \( M_\lambda \cap E(u^+) \neq \emptyset \), \( \lambda \in (-a, a) \);
(iii) \( m_\lambda = \inf_{u \in M_\lambda} J_\lambda(u), \lambda \in (-a, a) \).

Proof. We only give the proof of (i) and (iii) since the proof of (ii) is similar to (i) and is simpler. For \( u^+ \neq 0 \) and \( R = R(\| u^+ \|) \), the positive number we defined in (11), there exists \( \varepsilon_0 > 0 \) such that (see Lemma 2.2 and (13))

\[
\Phi_\varepsilon|_{\partial Q(u^+)} \leq 0, \quad \varepsilon \in (0, \varepsilon_0).
\]

And there exists \( v \in Q(u^+) \) such that \( \Phi_\varepsilon(v) \geq \rho > 0 \) (see (14)). So \( \sup_{Q(u^+)} \Phi_\varepsilon \geq \rho > 0 \) and there exists sequence \( \{ u_n \} \subset Q(u^+) \) such that

\[
\Phi_\varepsilon(u_n) \to \sup_{Q(u^+)} \Phi_\varepsilon.
\]

From Lemma 2.3 and (15), it follows that \( \Phi_\varepsilon \) is weakly upper semi-continuous on \( E(u^+) \), which implies the existence of \( u_0 \in Q(u^+) \) such that \( \Phi_\varepsilon(u_0) = \sup_{Q(u^+)} \Phi_\varepsilon \). According to (20), \( u_0 \in Q(u^+) \) and hence to be a local maximum of \( \Phi_\varepsilon \) on \( E(u^+) \). Therefore \( u_0 \) is a critical point of \( \Phi_\varepsilon|_{E(u^+)} \) and \( u_0 \in M_\varepsilon \cap E(u^+) \).

To prove (iii), let \( u_3 \) be the ground state solution we obtained in Proposition 3. Then (19) implies that \( u_3 \in M_\lambda \) and consequently \( m_\lambda = J_\lambda(u_3) \geq \inf_{u \in M_\lambda} J_\lambda(u) \). If we assume that \( m_\lambda > \inf_{u \in M_\lambda} J_\lambda(u) \), then there exists \( w \in M_\lambda \) satisfying \( m_\lambda > J_\lambda(w) \). From \( w \in M_\lambda \) we know \( w^+ \neq 0 \). Hence we may apply the linking theorem in \( Q_\lambda(w^+) \) to obtain a critical value \( c_\lambda \) of \( (D_\lambda) \) such that \( c_\lambda \leq \sup_{u \in \partial Q_\lambda(w^+)} J_\lambda(u) \) (see the proof at the beginning of this section and (17)). Then by Proposition 4 and \( w \in M_\lambda \), we deduce \( c_\lambda \leq \sup_{u \in \partial Q_\lambda(w^+)} J_\lambda(u) \leq J_\lambda(w) < m_\lambda \), which is a contradiction. \( \square \)

Proposition 6. \( \lim_{\varepsilon \to 0} m_\varepsilon \leq m_\varepsilon \).

Proof. Let \( w \) be the ground state solution of \( (D_\lambda) \) with \( \lambda = \gamma \). By (19), \( w^+ \neq 0 \). Replacing \( u^+ \) to \( w^+ \) in the proof of Proposition 5 (i), we obtain the existence of \( \varepsilon_0 > 0 \) and \( R(\| w^+ \|) \) such that

\[
\Phi_\varepsilon|_{\partial Q(w^+)} \leq 0, \quad \varepsilon \in (0, \varepsilon_0).
\]

Moreover, there exist \( t_\varepsilon > 0 \) and \( v_\varepsilon \in E^- \) such that \( w_\varepsilon = t_\varepsilon w^+ + v_\varepsilon \in M_\varepsilon \cap Q(w^+) \). Note that there exists \( 0 < r(w) < R(\| w^+ \|) \) such that \( \inf_{u \in S_{r(w)} \cap E^+} \Phi_\varepsilon(u) > 0 \). Thus, we can apply the linking theorem in \( Q_\varepsilon(w^+) \) for \( \Phi_\varepsilon \) (see Section 2), and obtain a critical value of \( \Phi_\varepsilon \): \( c_\varepsilon \in \left[ \inf_{u \in S_{r(w)} \cap E^+} \Phi_\varepsilon(u), \sup_{u \in Q(w^+)} \Phi_\varepsilon(u) \right], \varepsilon \in (0, \varepsilon_0) \). By Proposition 4 and \( w_\varepsilon \in M_\varepsilon \), we have \( \sup_{u \in Q(w^+)} \Phi_\varepsilon(u) \leq \Phi_\varepsilon(w_\varepsilon) \). This, together with the fact that \( m_\varepsilon \) is the least energy of \( \Phi_\varepsilon \), leads us to

\[
m_\varepsilon \leq c_\varepsilon \leq \sup_{u \in Q(w^+)} \Phi_\varepsilon(u) \leq \Phi_\varepsilon(w_\varepsilon), \quad \varepsilon \in (0, \varepsilon_0).
\]

Since \( w_\varepsilon \in Q(w^+) \), we know \( \| w_\varepsilon \| \leq R(\| w^+ \|) \). Therefore, up to a subsequence if necessary, \( t_\varepsilon \to t \) and \( v_\varepsilon \to v \) with \( t \geq 0 \) and \( v \in E^- \). According to (4), Proposition 4 and (14), there holds

\[
\frac{a + |V|_\infty}{2a} \| w_\varepsilon^+ \|^2 - \frac{a - |V|_\infty}{2a} \| w_\varepsilon^- \|^2 \geq \Phi_\varepsilon(w_\varepsilon) \geq \Phi_\varepsilon(\frac{r}{\| w_\varepsilon \|} w_\varepsilon^+) \geq \rho,
\]
where $r$ is the positive constant in (14). So $\frac{a + |V|}{\rho_0} \parallel w^+ \parallel^2 \geq \rho$ and hence $t > 0$. Noting that $w^+ \in M$, we have

$$\Phi_t'(w^+)w^+ = 0, \quad \Phi_t'(w^+)\varphi = 0,$$

combining which with the fact $w^+ \to tw^+ + v$ weakly in $E$, we get

$$J_{\frac{t}{2}}(tw^+ + v)w^+ = 0, \quad J_{\frac{t}{2}}(tw^+ + v)\varphi = 0,$$

for any $\varphi \in E^\perp$. Thus by $t > 0$, we know $tw^+ + v \in M \cap E(w^+)$. By (19), $w$ is also in $M \cap E(w^+)$. So by Proposition 4, we know $tw^+ + v = w$ and hence $t = 1$, $v = w^-$ and $w^+ \to w$ weakly in $E$.

We claim that $w^+ \to w$ in $E$. Setting $z^+ = w - w^-$, $l_z(t) = \Phi_z(w + tz^+)$ and $\mathcal{F}_z(u) = \int_{\mathbb{R}^3} F_z(x, |u|)$, we deduce that

$$\Phi_z(w) - \Phi_z(w^+) = l_z(1) - l_z(0) = \int_0^1 l'_z(s)ds = \int_0^1 \Phi'_z(w^+ + sz^+)z^+ds.$$

Writing $z^+$ as $z^+ = w^+ - w^-$ with $(w^+ - w^-) \in E^\perp$, since $\{w^\pm\}$ is bounded and $w^+ \in M$, there holds

$$\Re \int_{\mathbb{R}^3} (H_0w^+ + V(z)w) \cdot z^+ - \mathcal{F}_z(w)z^+ = \Phi'_z(w^+)z^+ = (1 - t_z)\Phi'_z(w^+)w^+ + \Phi'_z(w^+)(w^+ - v^+) = o_z(1).$$

Taking (22) and the following fact

$$\int_0^1 \left(\Re \int_{\mathbb{R}^3} (H_0sz^+ + V(z)sz^+) \cdot z^+\right)ds = \frac{1}{2} \int_{\mathbb{R}^3} H_0z^+ \cdot z^+ + V(z)|z^+|^2$$

into (21), we obtain that

$$\mathcal{F}_z(w) - \mathcal{F}_z(w^+) = \int_0^1 \mathcal{F}_z(w^+ + sz^+)z^+ds = \Phi_z(w) - \Phi_z(w^+) + \frac{1}{2} \int_{\mathbb{R}^3} H_0z^+ \cdot z^+ + V(z)|z^+|^2 + \mathcal{F}_z(w^+)z^+ + o_z(1).$$

Similarly, if we define $\mathcal{G}(u) = \int_{\mathbb{R}^3} G(|u|)$, we will deduce that

$$\mathcal{G}(w^+) - \mathcal{G}(w) = J_{\frac{t}{2}}(w) - J_{\frac{t}{2}}(w^+) + \frac{1}{2} \int_{\mathbb{R}^3} H_0z^+ \cdot z^+ + V(z)|z^+|^2 - \mathcal{F}_z'(w)z^+.$$  \hspace{1cm} (24)

Computing directly, we have

$$\Phi_z(u) = J_{\frac{t}{2}}(u) + \frac{1}{2} \int_{\mathbb{R}^3} (V_z(x) - v^+)|u|^2 + \mathcal{G}(u) - \mathcal{F}_z(u).$$

Therefore,

$$\left(\Phi_z(w^+) - J_{\frac{t}{2}}(w^+)\right) - \left(\Phi_z(w) - J_{\frac{t}{2}}(w)\right) = \frac{1}{2} \int_{\mathbb{R}^3} (V_z(x) - v^+) (|w^+|^2 - |w|^2) + \mathcal{G}(w^+) - \mathcal{F}_z(w^+) - \mathcal{G}(w) + \mathcal{F}_z(w).$$
Taking (23) and (24) into the right hand side of the above equality, we deduce that
\[
\int_{\mathbb{R}^3} H_0 z_\varepsilon \cdot z_\varepsilon + \frac{1}{2} \int_{\mathbb{R}^3} (V_\varepsilon(x) + \nu) |z_\varepsilon|^2 + \frac{1}{2} \int_{\mathbb{R}^3} (V_\varepsilon(x) - \nu)(|w_\varepsilon|^2 - |w|^2) + \mathcal{F}'_\varepsilon(w_\varepsilon) z_\varepsilon - \mathcal{G}'(w) z_\varepsilon = o_\varepsilon(1).
\]
Clearly, there hold
\[
\int_{\mathbb{R}^3} (V_\varepsilon(x) - \nu)(|w_\varepsilon|^2 - |w|^2) = \int_{\mathbb{R}^3} (V_\varepsilon(x) - \nu)|z_\varepsilon|^2 - 2\text{Re} \int_{\mathbb{R}^3} (V_\varepsilon(x) - \nu) w \cdot z_\varepsilon
\]
\[
= \int_{\mathbb{R}^3} (V_\varepsilon(x) - \nu)|z_\varepsilon|^2 + o_\varepsilon(1),
\]
and \(\mathcal{G}'(w) z_\varepsilon = o_\varepsilon(1)\), which lead us to
\[
\int_{\mathbb{R}^3} H_0 z_\varepsilon \cdot z_\varepsilon + \int_{\mathbb{R}^3} V_\varepsilon(x)|z_\varepsilon|^2 + \mathcal{F}'_\varepsilon(w_\varepsilon) z_\varepsilon = o_\varepsilon(1).
\]
On one hand,
\[
\int_{\mathbb{R}^3} H_0 z_\varepsilon \cdot z_\varepsilon + \int_{\mathbb{R}^3} V_\varepsilon(x)|z_\varepsilon|^2 \leq \|z_\varepsilon^+\|^2 - \|z_\varepsilon^-\|^2 + |V|_\infty (\|z_\varepsilon^+\|^2 + \|z_\varepsilon^-\|^2)
\]
\[
\leq \frac{a + |V|_\infty}{a} \|z_\varepsilon^+\|^2 - \frac{a - |V|_\infty}{a} \|z_\varepsilon^-\|^2
\]
\[
= - \frac{a - |V|_\infty}{a} \|z_\varepsilon^-\|^2 + o_\varepsilon(1).
\]
On the other hand, since \(f_\varepsilon(x, |w_\varepsilon|)|w_\varepsilon|^2 \to g(|w|)|w|^2\) a.e. \(x \in \mathbb{R}^3\), using Fatou’s Lemma, we have
\[
\mathcal{F}'_\varepsilon(w_\varepsilon) z_\varepsilon = \text{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |w_\varepsilon|) w_\varepsilon \cdot (w - w_\varepsilon)
\]
\[
= \int_{\mathbb{R}^3} g(|w|)|w|^2 - \int_{\mathbb{R}^3} f_\varepsilon(x, |w_\varepsilon|)|w_\varepsilon|^2 + o_\varepsilon(1) \leq o_\varepsilon(1).
\]
Consequently, \(\|z_\varepsilon^-\| \leq o_\varepsilon(1)\) and \(z_\varepsilon \to 0\) in \(E\), which imply that the claim is true. So we have
\[
\lim_{\varepsilon \to 0} m_\varepsilon \leq \lim_{\varepsilon \to 0} \Phi_\varepsilon(w_\varepsilon) = J_\varepsilon(w) = m_\omega.
\]

4. Proof of main result.

**Lemma 4.1.** Let \(u_\varepsilon\) be the ground state solutions of \(\Phi_\varepsilon\) which we obtained in Section 2 with \(\varepsilon \in (0, \varepsilon_0)\), then for any \(q \geq 2\), \(u_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)\) and \(\|u_\varepsilon\|_{W^{1,q}} \leq C_q\), where \(C_q\) depends only on \(q\). In particular, for \(q > 3\) and \(y \in \mathbb{R}^3\), there exists positive constant \(C\) depending only on \(q\) such that,
\[
|u_\varepsilon(y)| \leq C\|u_\varepsilon\|_{W^{1,q}(B_1(y))}.
\]

**Proof.** According to Proposition 6, we have \(\Phi_\varepsilon(u_\varepsilon) = m_\varepsilon\) is uniformly bounded for \(\varepsilon \in (0, \varepsilon_0)\). By a proof similar to that of Lemma 2.1, we deduce that \(\{u_\varepsilon\}\) is bounded in \(E\). Then using the same iterative argument as that in [20] Proposition 3.2 and [17] Lemma 3.19, we obtain that \(u_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)\) with \(\|u_\varepsilon\|_{W^{1,q}} \leq C_q\), where \(C_q\) depends only on \(q\) for any \(q \geq 2\). In particular, \(u_\varepsilon\) is uniformly bounded in \(L^\infty(\mathbb{R}^3, \mathbb{C}^4)\) and \(|u_\varepsilon(y)| \leq C\|u_\varepsilon\|_{W^{1,q}(B_1(y))}\) for \(\varepsilon \in (0, \varepsilon_0)\), where \(C\) is independent of \(\varepsilon\) and the choice of \(y\). \(\square\)
**Proposition 7.** Let $u_\epsilon$ be ground state solutions which we obtained in (16). Then $|u_\epsilon|$ attains its maximum at $x_\epsilon$. Moreover, if we set $v_\epsilon(x) = u_\epsilon(x + x_\epsilon)$, up to a subsequence, as $\epsilon \to 0$, there hold

\[ \epsilon x_\epsilon \to V, \]

\[ v_\epsilon \to v \text{ in } W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \] for $q \in [2, +\infty),$

where $v$ is a ground state solution of $(D_\lambda)$ with $\lambda = v$.

**Proof.** **Step 1.** We claim that there exists $\delta > 0$, such that

\[ \lim_{\epsilon \to 0} \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} |u_\epsilon|^2 \geq \delta. \]

Arguing indirectly, we assume

\[ \lim_{\epsilon \to 0} \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} |u_\epsilon|^2 = 0. \]

From Lemma 4.1, we know that $u_\epsilon$ is bounded in $E$. Then according to Lions’ Lemma [27], up to a subsequence if necessary, $u_\epsilon \to 0$ in $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in (2, 3)$. Note that for any small $\epsilon > 0$, there exist $C_\epsilon > 0$ and $\rho \in (2, 3)$ such that, $f_\epsilon(x, t) \leq \epsilon + C_\epsilon t^{\rho - 2}$ uniformly holds for $\epsilon > 0$ (see $(f_2)$, $(g_1)$ and $(g_2)$). Then a standard argument shows that

\[ \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} f_\epsilon(x, |u_\epsilon|)|u_\epsilon|^2 \to 0, \]

and hence

\[ m_\epsilon = \Phi_\epsilon(u_\epsilon) - \frac{1}{2} \Phi_\epsilon'(u_\epsilon)u_\epsilon = \int_{\mathbb{R}^3} \frac{1}{2} f_\epsilon(x, |u_\epsilon|)|u_\epsilon|^2 - F_\epsilon(x, |u_\epsilon|) = o_\epsilon(1). \]

While according to (19), $u_\epsilon \in \mathcal{M}_\epsilon$. This, together with Proposition 4 and (14), leads us to

\[ m_\epsilon = \Phi_\epsilon(u_\epsilon) \geq \Phi_\epsilon(\|u_\epsilon^+\| u_\epsilon^+) \geq \rho, \]

which is impossible.

**Step 2.** Let $\{y_\epsilon\} \subset \mathbb{R}^3$ be points such that

\[ \int_{B_1(y_\epsilon)} |u_\epsilon|^2 \geq \frac{\delta}{2}, \]

up to a subsequence, there holds $\epsilon y_\epsilon \to V$ as $\epsilon \to 0$.

Setting $\bar{u}_\epsilon(x) = u_\epsilon(x + y_\epsilon)$, up to a subsequence, $\bar{u}_\epsilon \to \bar{u}$ in $E$ with $\bar{u} \neq 0$. Obviously, $\bar{u}_\epsilon$ satisfies

\[ -i\alpha \cdot \nabla \bar{u}_\epsilon + a\beta \bar{u}_\epsilon + V_\epsilon(x + y_\epsilon)\bar{u}_\epsilon = f_\epsilon(x + y_\epsilon, |\bar{u}_\epsilon|)\bar{u}_\epsilon. \]

Testing this equation by $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, we assume by contradiction that $|\epsilon y_\epsilon| \to \infty$ or $\epsilon y_\epsilon \to y_0 \notin \Lambda^1$, as $\epsilon \to 0$. Then $V_\epsilon(x + y_\epsilon) \to V_\infty$ with $|V_\infty| \leq |V|_\infty$ and $\chi_\epsilon(x + y_\epsilon) \to 0$ as $\epsilon \to 0$ uniformly hold for $x \in \text{supp}\varphi$. So we have

\[ 0 = \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} \left( H_0 \bar{u}_\epsilon + V_\epsilon(x + y_\epsilon)\bar{u}_\epsilon - f_\epsilon(x + y_\epsilon, |\bar{u}_\epsilon|)\bar{u}_\epsilon \right) \cdot \varphi \]

\[ = \int_{\mathbb{R}^3} \left( H_0 \bar{u} + V_\infty \bar{u} - \tilde{g}(|\bar{u}|) \bar{u} \right) \cdot \varphi, \]

and hence $\bar{u}$ satisfies

\[ -i\alpha \cdot \nabla \bar{u} + a\beta \bar{u} + V_\infty \bar{u} = \tilde{g}(|\bar{u}|) \bar{u}. \]
Taking the scalar product of this equation with \( \bar{u}^+ - \bar{u}^- \), since \( \bar{g}(t) \leq \frac{a - |V|_{\infty}}{2} \), we know

\[
0 = ||\bar{u}||^2 + V_\infty \int_{\mathbb{R}^3} \bar{u} \cdot (\bar{u}^+ - \bar{u}^-) - \int_{\mathbb{R}^3} \bar{g}(||\bar{u}||)\bar{u} \cdot (\bar{u}^+ - \bar{u}^-) \\
\geq ||\bar{u}||^2 - |V|_{\infty}||\bar{u}||^2 - \frac{a - |V|_{\infty}}{2} ||\bar{u}||^2 \\
\geq \frac{a - |V|_{\infty}}{2} ||\bar{u}||^2.
\]

This is a contradiction and hence as \( \varepsilon \to 0 \), \( \varepsilon y_e \to y_0 \in \Lambda^4 \).

Let \( f_\infty(t) = \chi(y_0)g(t) + (1 - \chi(y_0))\bar{g}(t) \), where \( \chi \) is the cut-off function we defined in Section 2. Then similarly, \( \bar{u} \) satisfies

\[-i\alpha \cdot \nabla \bar{u} + a\beta \bar{u} + V(y_0)\bar{u} = f_\infty(\bar{u}^\dagger)\bar{u} .\]

Testing this equation by \( \bar{u} \), we have

\[||\bar{u}^+||^2 - ||\bar{u}^-||^2 + V(y_0)||\bar{u}||^2 = \int_{\mathbb{R}^3} f_\infty(\bar{u}^\dagger)||\bar{u}||^2 \geq 0.\]

This yields that

\[\frac{a + |V|_{\infty}}{a} ||\bar{u}^+||^2 - \frac{a - |V|_{\infty}}{a} ||\bar{u}^-||^2 \geq 0,\]

combining which with the fact \( \bar{u} \neq 0 \), we have \( \bar{u}^+ \neq 0 \). Denoting by \( \Phi_\infty \) the associate energy functional corresponding to the above equation

\[\Phi_\infty(u) = \frac{1}{2} (||u^+||^2 - ||u^-||^2) + \frac{1}{2} \int_{\mathbb{R}^3} V(y_0)||u||^2 - \int_{\mathbb{R}^3} F_\infty(||u||),\]

with \( F_\infty(t) = \int_0^t f_\infty(s)ds \) and setting

\[\mathcal{M}_\infty := \{u \in E \setminus E^- : \Phi_\infty'(u)u = 0, \Phi_\infty(u)v = 0 \text{ for all } v \in E^- \}.\]

Obviously, \( f_\infty \) satisfies the assumptions of Proposition 2 and hence by the same procedure as the proof in Proposition 4, we know that if \( u \in \mathcal{M}_\infty \), then \( \Phi_\infty|_{E(\bar{u}^+)} \) attains its maximum at \( u \). Thus by \( \bar{u}^+ \neq 0 \) and \( \Phi_\infty(\bar{u}) = 0 \), we know \( \bar{u}^+ \in \mathcal{M}_\infty \) and hence

\[\Phi_\infty(\bar{u}) \geq \Phi_\infty(w), \text{ for any } w \in E(\bar{u}^+).\]

Since \( \bar{u}^+ \neq 0 \), by Proposition 5 (ii), there exist \( t > 0 \) and \( v \in E^- \) such that \( t\bar{u}^+ + v \in \mathcal{M}_\infty \). Using Proposition 5 (iii) and \( F_\infty(t) \leq G(t) \), we get

\[
\Phi_\infty(\bar{u}) \geq \Phi_\infty(t\bar{u}^+ + v) \\
\geq J_{\infty}(t\bar{u}^+ + v) + \frac{V(y_0) - V}{2} |t\bar{u}^+ + v|^2 \\
\geq m_\infty + \frac{V(y_0) - V}{2} |t\bar{u}^+ + v|^2 . \tag{26}
\]

On the other hand, since \( f_\varepsilon(x + y_e, |\bar{u}_\varepsilon(x)|) \to f_\infty(\bar{u}(x)) \) a.e. \( x \in \mathbb{R}^3 \), using Fatou’s Lemma, we deduce that

\[
m_\varepsilon = \Phi_\varepsilon(u_\varepsilon) - \frac{1}{2} \Phi_\varepsilon'(u_\varepsilon)u_\varepsilon \\
= \int_{\mathbb{R}^3} \frac{1}{2} f_\varepsilon(x + y_e, |\bar{u}_\varepsilon|)|\bar{u}_\varepsilon|^2 - F_\varepsilon(x + y_e, |\bar{u}_\varepsilon|) .
\]
\[
\begin{align*}
&\geq \int f(\xi)\frac{1}{2}|\vec{u}|^2 + F_\infty(|\vec{u}|) + o(1) \\
&= \Phi_\infty(\vec{u}) + \frac{1}{2} \Phi'_\infty(\vec{u}) + o(1) \\
&= \Phi_\infty(\vec{u}) + o(1). 
\end{align*}
\]  

(27)

Taking this into (26), we obtain \( \lim_{\varepsilon \to 0} m_\varepsilon \geq m_\infty + \frac{V(y_0) - \varepsilon}{2} |t \vec{u}^+ + v|_2^2 \), together with \( t > 0, \vec{u}^+ \neq 0 \) and Proposition 6, we know \( V(y_0) - \varepsilon \leq 0 \). Since \( y_0 \in A^4 \), according to (5), there hold \( V(y_0) = \overline{\mathcal{V}} \) and \( y_0 \in \mathcal{V} \).

**Step 3.** Up to a subsequence, \( \vec{u}_\varepsilon \to \vec{u} \) in \( E \) and \( \vec{u}_\varepsilon \to \vec{u} \) in \( W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \) for \( q \in [2, +\infty) \).

It sufficient to prove that there is a subsequence \( \{\vec{u}_{\varepsilon_j}\} \) such that the above convergence holds.

Since \( y_0 \in \mathcal{V} \), we know \( f_\varepsilon(t) = g(t) \). So \( \Phi_\infty(\vec{u}) = J_\varepsilon(\vec{u}) \), and hence \( \vec{u} \) is a nontrivial solution of \((\mathcal{D}_\lambda)\) with \( \lambda = \overline{\mathcal{V}} \). Moreover, from (26), (27) and Proposition 6, it follows that \( J_\varepsilon(\vec{u}) = m_\varepsilon \) and \( \vec{u} \) is a ground state.

Let \( \eta : [0, +\infty) \to [0, 1] \) be a smooth function such that \( \eta(t) = 1 \) for \( t \leq 1 \), \( \eta(t) = 0 \) for \( s \geq 2 \). Set

\[
\vec{u}_j(x) = \eta\left(\frac{2|x|}{j}\right)\vec{u}(x),
\]

then we have

\[
\vec{u}_j \to \vec{u} \text{ strongly in } E, \text{ as } j \to \infty.
\]  

(28)

From an argument in [10] Lemma 5.7, it follows that there exists a subsequence \( \{\vec{u}_{\varepsilon_j}\} \) such that, for any \( \delta > 0 \), there exists \( r_\delta > 0 \) satisfying

\[
\limsup_{j \to \infty} \int_{B_j(0) \setminus B_{r_\delta}(0)} |\vec{u}_{\varepsilon_j}|^p \leq \delta \text{ for } r \geq r_\delta,
\]

with \( p \in (2, 3) \) the constant appeared in \((g_2)\). Setting \( z_j = \vec{u}_{\varepsilon_j} - \vec{u}_j \) and using the same arguments as that in [10] Lemma 7.10 (see also [18] (4.11)–(4.12)), we have

\[
\int_{\mathbb{R}^3} F_{\varepsilon_j}(x + y_{\varepsilon_j}, |\vec{u}_{\varepsilon_j}|) - F_{\varepsilon_j}(x + y_{\varepsilon_j}, |z_j|) - F_{\varepsilon_j}(x + y_{\varepsilon_j}, |\vec{u}_j|) = o_\varepsilon(1); \quad (29)
\]

and

\[
\int_{\mathbb{R}^3} [f_{\varepsilon_j}(x + y_{\varepsilon_j}, |\vec{u}_{\varepsilon_j}|)\vec{u}_{\varepsilon_j} - f_{\varepsilon_j}(x + y_{\varepsilon_j}, |z_j|)z_j - f_{\varepsilon_j}(x + y_{\varepsilon_j}, |\vec{u}_j|)\vec{u}_j] \cdot \varphi = o_\varepsilon(1)\|\varphi\|, \quad (30)
\]

uniformly for \( \varphi \in E \), as \( j \to \infty \). Moreover, according to (28) and the fact \( \varepsilon_j y_{\varepsilon_j} \to \mathcal{V} \) as \( j \to \infty \), we obtain that

\[
\lim_{j \to \infty} \operatorname{Re} \int_{\mathbb{R}^3} V_{\varepsilon_j}(x + y_{\varepsilon_j})\vec{u}_{\varepsilon_j} \cdot \vec{u}_j = \lim_{j \to \infty} \int_{\mathbb{R}^3} V_{\varepsilon_j}(x + y_{\varepsilon_j})|\vec{u}_{\varepsilon_j}|^2 = \frac{\overline{\mathcal{V}}}{2} |\vec{u}|^2;
\]

\[
\lim_{j \to \infty} \int_{\mathbb{R}^3} F_{\varepsilon_j}(x + y_{\varepsilon_j}, |\vec{u}_{\varepsilon_j}|) = \int_{\mathbb{R}^3} G(|\vec{u}|).
\]  

(31)
Therefore, if we denote by $\Phi_\epsilon$ the functional corresponding to (25), it follows from (29), (31) and Proposition 6 that

$$
\Phi_\epsilon(z_j) = \frac{1}{2} \int_{\mathbb{R}^3} H_0 \tilde{u}_\epsilon \cdot \tilde{u}_\epsilon + V_{\epsilon_j}(x + y_{\epsilon_j})|\tilde{u}_\epsilon|^2 - \int_{\mathbb{R}^3} F_{\epsilon_j}(x + y_{\epsilon_j}, |\tilde{u}_\epsilon|)
+ \int_{\mathbb{R}^3} F_{\epsilon_j}(x + y_{\epsilon_j}, |\tilde{u}_\epsilon|) - F_{\epsilon_j}(x + y_{\epsilon_j}, |z_j|) - F_{\epsilon_j}(x + y_{\epsilon_j}, |\tilde{u}_\epsilon|)
- \frac{1}{2} \int_{\mathbb{R}^3} H_0 \tilde{u} \cdot \tilde{u} + \epsilon \tilde{u}^2 + \int_{\mathbb{R}^3} G(|\tilde{u}|) + o_j(1)
\Phi_\epsilon(u_{\epsilon_j}) - J_\epsilon(u) + o_j(1) = m_{\epsilon_j} - m_+ + o_j(1) \leq o_j(1).
$$

Similarly, using (30), we deduce that

$$
\Phi_\epsilon'(z_j)\phi = \Phi_\epsilon'(u_{\epsilon_j})\phi(-y_{\epsilon_j}) - J_\epsilon'(u)\phi + o_j(1)\|\phi\| = o_j(1)\|\phi\|
$$

uniformly holds for $\phi \in E$. Then we have

$$
o_j(1) \geq \Phi_\epsilon(z_j) - \frac{1}{2} \Phi_\epsilon'(z_j)z_j = \int_{\mathbb{R}^3} \frac{1}{2} F_{\epsilon_j}(x + y_{\epsilon_j}, |z_j|)^2 - F_{\epsilon_j}(x + y_{\epsilon_j}, |z_j|).
$$

By (f_3) and (6) (similar to (9)), we have

$$
o_j(1) \geq \int_{\mathbb{R}^3} \chi_{\epsilon_j}(x) \left( \frac{1}{2} g(|z_j|)|z_j|^2 - G(|z_j|) \right).
$$

On the other hand, using (8), we obtain

$$
o_j(1) = \Phi_\epsilon(z_j)(z_j^+ - z_j^-)
= \|z_j\|^2 + \text{Re} \int_{\mathbb{R}^3} V_{\epsilon}(x + y_{\epsilon_j})z_j \cdot (z_j^+ - z_j^-) - F_{\epsilon_j}(x + y_{\epsilon_j}, |z_j|)z_j \cdot (z_j^+ - z_j^-)
\geq \|z_j\|^2 - |V|\infty \int_{\mathbb{R}^3} |z_j||z_j^+ - z_j^-| - \frac{a - |V|\infty}{2} \int_{\mathbb{R}^3} |z_j||z_j^+ - z_j^-|
- c_4 \int_{\mathbb{R}^3} \chi_{\epsilon_j}(x) \left( \frac{1}{2} g(|z_j|)|z_j|^2 - G(|z_j|) \right) \frac{p-1}{p} |z_j^+ - z_j^-|.
$$

Using (4), the H"older's inequality and (34), we deduce that

$$
\frac{a - |V|\infty}{2a} \|z_j\|^2
\leq c_4 \left( \int_{\mathbb{R}^3} \chi_{\epsilon_j}(x) \left( \frac{1}{2} g(|z_j|)|z_j|^2 - G(|z_j|) \right) \right) \frac{p-1}{p} |z_j^+ - z_j^-| + o_j(1)
= o_j(1).
$$

Therefore $z_j = \tilde{u}_{\epsilon_j} - \bar{u}_j \to 0$ in $E$, combining which with (28), we get $\tilde{u}_{\epsilon_j} \to \bar{u}$ in $E$. Noting that $\bar{u}$ is a solution of (D_\lambda) with $\lambda = \nu$, we can apply the same iterative argument as that in Lemma 4.1 and obtain that $\bar{u} \in W^{1,4}(\mathbb{R}^3, C^4)$ for $q \geq 2$. Then by the uniform boundedness of $\tilde{u}_{\epsilon_j}$ in $W^{1,4}(\mathbb{R}^3, C^4)$ for $q \geq 2$ and in $L^{\infty}(\mathbb{R}^3, C^4)$, and the fact

$$
H_0(\tilde{u}_{\epsilon_j} - \bar{u}) = f_{\epsilon_j}(x + y_{\epsilon_j}, |\tilde{u}_{\epsilon_j}|)\tilde{u}_{\epsilon_j} - g(|\tilde{u}|)\tilde{u} + \nu \tilde{u} - V_{\epsilon_j}(x + y_{\epsilon_j})\tilde{u}_{\epsilon_j},
$$

we know $|H_0(\tilde{u}_{\epsilon_j} - \bar{u})|_2 \to 0$ as $\epsilon \to 0$. Noting that $H_0$ is self-adjoint on $L^2(\mathbb{R}^3, C^4)$ and $H_0^* = -\Delta + a^2$, (see [2] Section 3 for example), we have $\|\bar{u}_{\epsilon_j} - \bar{u}\|_{H^1(\mathbb{R}^3, C^4)} \to 0$. 

Combining this with the bounded-ness of $\tilde{u}_{\varepsilon_j}$ in $W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for $q \geq 2$ (see Lemma 4.1) and the Hölder’s inequality, we know

$$\tilde{u}_{\varepsilon_j} \to \tilde{u} \quad \text{in} \quad W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \quad \text{for} \quad q \in [2, +\infty).$$

(35)

**Step 4.** $u_\varepsilon$ attains its maximum at $x_\varepsilon$. Moreover, up to a subsequence, as $\varepsilon \to 0$

$$\varepsilon x_\varepsilon \to \mathcal{V},$$

$$v_\varepsilon := u_\varepsilon(\cdot + x_\varepsilon) \to v \quad \text{in} \quad W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \quad \text{for} \quad q \in [2, +\infty),$$

where $v$ is a ground state solution of $(D_\lambda)$ with $\lambda = \nu$.

According to Step 1:

$$\lim_{\varepsilon \to 0} \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} |u_\varepsilon|^2 \geq \delta,$$

we know that there exists $\varepsilon_0 > 0$, such that $|\tilde{u}_\varepsilon|_\infty$ is uniformly bound away from zero by $c(\delta) > 0$ for some small $\varepsilon_0$ and $\varepsilon \in (0, \varepsilon_0)$. While, for any fixed $\varepsilon \in (0, \varepsilon_0)$, by Lemma 4.1,

$$|u_\varepsilon(x)| \leq \frac{1}{2} c(\delta), \quad |x| \geq R.$$  

So $u_\varepsilon$ can attain its maximum at $x_\varepsilon$.

Setting $z_\varepsilon = x_\varepsilon - y_\varepsilon$, then $z_\varepsilon$ is the maximum point of $\tilde{u}_\varepsilon$ and hence

$$|\tilde{u}_\varepsilon(z_\varepsilon)| \geq c(\delta) > 0, \quad \varepsilon \in (0, \varepsilon_0).$$

We claim $\{z_\varepsilon\}$ is bounded. Arguing indirectly, we assume that there exists subsequence $\lim_{\varepsilon \to 0} |z_\varepsilon| = \infty$. By Step 3, up to a subsequence, there holds $\tilde{u}_\varepsilon \to \tilde{u}$ in $W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ as $\varepsilon \to 0$. Together with Lemma 4.1, for any fixed $q > 3$, there exists constant $C > 0$ independent of the choice of $\varepsilon$ and $z_\varepsilon$ such that

$$|\tilde{u}_\varepsilon(z_\varepsilon)| \leq C||\tilde{u}_\varepsilon||_{W^{1,q}(B_1(z_\varepsilon))} \leq C||\tilde{u}_\varepsilon - \tilde{u}||_{W^{1,q}} + C||\tilde{u}||_{W^{1,q}(B_1(z_\varepsilon))} = o_\varepsilon(1),$$

which is a contradiction. So for $x_\varepsilon = y_\varepsilon + z_\varepsilon$, by Step 2, up to a subsequence

$$\varepsilon x_\varepsilon = \varepsilon y_\varepsilon + \varepsilon z_\varepsilon \to \mathcal{V} \quad \text{as} \quad \varepsilon \to 0.$$

Moreover, for $\varepsilon \in (0, \varepsilon_0)$, since $z_\varepsilon$ is bounded and $\int_{B_1(y_\varepsilon)} |u_\varepsilon|^2 \geq \frac{\delta}{2}$, we can choose an $R > 0$ such that

$$\int_{B_R(x_\varepsilon)} |u_\varepsilon|^2 = \int_{B_R(y_\varepsilon + z_\varepsilon)} |u_\varepsilon|^2 \geq \int_{B_1(y_\varepsilon)} |u_\varepsilon|^2 \geq \frac{\delta}{2}.$$  

Note that we have shown that, up to a subsequence, $\varepsilon x_\varepsilon \to \mathcal{V}$ as $\varepsilon \to 0$. So we can repeat the process of proof in Step 3, and obtain that, up to a subsequence, $v_\varepsilon(x) = u_\varepsilon(x + x_\varepsilon)$ converges strongly in $E$ to a ground state solution $v(x)$ of $(D_\lambda)$ with $\lambda = \nu$. Similarly to (35), this convergence also holds in $W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$, $q \in [2, +\infty)$.

Now we are in a position to finish the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We first prove $u_\varepsilon$ is actually a solution of $(D_\varepsilon)$ for small $\varepsilon_0$ and $\varepsilon \in (0, \varepsilon_0)$.

By (7), we know $\tilde{g}(t) = g(t)$ for $t \in [0, t_0]$. Therefore, if there holds $|u_\varepsilon(x)| < t_0$ for $x \notin \Lambda_\varepsilon$ when $\varepsilon$ is small, we will have $f_\varepsilon(x, |u_\varepsilon|) = g(|u_\varepsilon|)$ and $u_\varepsilon$ will be a solution of $(D_\varepsilon)$. Assume by contradiction that there exists sequence $\varepsilon \to 0$ and $p_\varepsilon \notin \Lambda_\varepsilon$ such that $|u_\varepsilon(p_\varepsilon)| \geq t_0$. Let $x_\varepsilon$ be a global maximum of $|u_\varepsilon|$, $v_\varepsilon(x) = u_\varepsilon(x + x_\varepsilon)$, then

$$|v_\varepsilon(p_\varepsilon - x_\varepsilon)| = |u_\varepsilon(p_\varepsilon)| \geq t_0.$$  


Applying Proposition 7, up to a subsequence, \( v_\varepsilon \to v \) in \( W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \), \( q \in [2, +\infty) \) and \( \varepsilon x_\varepsilon \to \mathcal{V} \). While \( \varepsilon p_\varepsilon \notin \Lambda \), so it follows from (3) that \( |p_\varepsilon - x_\varepsilon| \to +\infty \). Then similarly as that in (36), since \( |p_\varepsilon - x_\varepsilon| \to +\infty \) as \( \varepsilon \to 0 \), we know
\[
|v_\varepsilon(p_\varepsilon - x_\varepsilon)| = o_\varepsilon(1),
\]
which is impossible.

Let
\[
w_\varepsilon(x) = u_\varepsilon \left( \frac{x}{\varepsilon} \right).
\]
Then \( w_\varepsilon, \varepsilon \in (0, \varepsilon_0) \) is a solution of (D) and \( w_\varepsilon \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \) for any \( q \geq 2 \) (see Lemma 4.1). Obviously,
\[
\theta_\varepsilon := \varepsilon x_\varepsilon,
\]
is a global maximum point of \( w_\varepsilon \). According to Proposition 7, up to a subsequence as \( \varepsilon \to 0 \), there hold \( \theta_\varepsilon \to \mathcal{V} \) and
\[
w_\varepsilon(x + \theta_\varepsilon) = u_\varepsilon(x + x_\varepsilon) \to w(x) \quad \text{in} \quad W^{1,q}(\mathbb{R}^3, \mathbb{C}^4), \quad q \in [2, +\infty),
\]
with \( w(x) \) a ground state solution of
\[
-i \alpha \cdot \nabla w + a \beta w + \psi w = g(|w|)w.
\]
Thus, the only we need to prove is the exponential decay of \( w_\varepsilon \). Denoting by \( v_\varepsilon = u_\varepsilon(x + x_\varepsilon) \), then we claim that there exist a large \( M_0 > 0 \) and a small \( \varepsilon_0 > 0 \) such that, for \( \varepsilon \in (0, \varepsilon_0) \) and \( |x| \geq \frac{M_0}{2} \), there holds \( |v_\varepsilon(x)| < t_0 \), where \( t_0 \) is the positive number appeared in (7). Assuming by contradiction that there exist sequence \( \varepsilon \to 0 \) and \( |p_\varepsilon| \to +\infty \) such that \( |v_\varepsilon(p_\varepsilon)| \geq t_0 \). Then applying Proposition 7 (see (36)), we will get a contradiction.

Therefore, for \( \varepsilon \in (0, \varepsilon_0) \), \( v_\varepsilon \) satisfies \( |v_\varepsilon(x)| < t_0, |x| \geq \frac{M_0}{2} \) and
\[
H_0v_\varepsilon = -V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon, \quad (37)
\]
where \( H_0 = -i \alpha \cdot \nabla + a \beta \). According to (7),
\[
g(|v_\varepsilon(x)|) \leq \frac{a - |V|_{\infty}}{2}, \quad |x| \geq \frac{M_0}{2}.
\]
Assuming without loss of generality that \( M_0 \geq 12 \), then \( \lceil \frac{M}{2} \rceil - 1 \geq \frac{M}{4} \) if \( M \geq M_0 \). For \( M \geq M_0 \) and \( m \in \mathbb{N} \), we set
\[
D_m = \{ x \in \mathbb{R}^3 : |x| \geq \frac{M}{2} + m \},
\]
and set \( \eta_m \) be a cut-off function satisfying that \( 0 \leq \eta_m(t) \leq 1, |\eta'_m(t)| \leq 4 \) for all \( t \) and
\[
\eta_m(t) = \begin{cases}
0, & t \leq \frac{M}{2} + m; \\
1, & t \geq \frac{M}{2} + m + 1.
\end{cases}
\]
For \( x \in \mathbb{R}^3 \), let \( \phi_m(x) = \eta_m(|x|) \). Taking the scalar product of \( H_0(v_\varepsilon \phi_m) \) with (37), we deduce
\[
\text{Re} \langle H_0v_\varepsilon, H_0(v_\varepsilon \phi_m) \rangle_2 = \text{Re} \langle -V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon, H_0(v_\varepsilon \phi_m) \rangle_2. \quad (38)
\]
Claim
\[
\text{Re} \langle H_0v_\varepsilon, H_0(v_\varepsilon \phi_m) \rangle_2 = \text{Re} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \phi_m + a^2 |v_\varepsilon|^2 \phi_m + \sum_{k=1}^3 \partial_k v_\varepsilon \cdot \partial_k \phi_m v_\varepsilon. \quad (39)
\]
In fact, since $H_0$ is self-adjoint on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $H_0^2 = -\Delta + a^2$, (see [2] Section 3) for $v \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, there holds
\[
\text{Re}(H_0v, H_0(v\phi_m))_2 = \text{Re}(H_0^2v, v\phi_m)_2 = \text{Re}(-\Delta v + a^2 v, v\phi_m)_2
\]
\[= \text{Re} \int_{\mathbb{R}^3} |\nabla v|^2 \phi_m + a^2 |v|^2 \phi_m + \sum_{k=1}^3 \partial_k v \cdot \partial_k \phi_m v.
\]
Then by choosing $\{v_n\} \subset C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ such that $v_n \to v_\varepsilon$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$ as $n \to \infty$, we may prove the claim is true. On the other hand, by the form of $H_0$, (37) and the fact
\[
\text{Re}(-V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon \cdot (-i \sum_{k=1}^3 \partial_k \phi_m \alpha_k v_\varepsilon) = 0,
\]
we have
\[
\text{Re}(-V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon, H_0(v_\varepsilon\phi_m))_2
\]
\[= \text{Re} \int_{\mathbb{R}^3} (-V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon) \cdot (\phi_m H_0v_\varepsilon - i \sum_{k=1}^3 \partial_k \phi_m \alpha_k v_\varepsilon)
\]
\[= \text{Re} \int_{\mathbb{R}^3} (-V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon) \cdot (-V_\varepsilon(x + x_\varepsilon)v_\varepsilon + g(|v_\varepsilon|)v_\varepsilon)\phi_m
\]
\[= \int_{\mathbb{R}^3} \left| V_\varepsilon(x + x_\varepsilon) - g(|v_\varepsilon|) \right|^2 |v_\varepsilon|^2 \phi_m.
\]
Taking (40) and (39) into (38), it follows that
\[
\text{Re} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \phi_m + a^2 |v_\varepsilon|^2 \phi_m - (V_\varepsilon(x + x_\varepsilon) - g(|v_\varepsilon|) \phi_m)
\]
\[= - \text{Re} \int_{\mathbb{R}^3} \sum_{k=1}^3 \partial_k v_\varepsilon \cdot \partial_k \phi_m v_\varepsilon \leq \int_{\mathbb{R}^3} |\nabla v_\varepsilon| |v_\varepsilon||\nabla \phi_m|.
\]
This, together with the definition of $\phi_m$ and the fact
\[
g(|v_\varepsilon(x)|) \leq \frac{a - |V|}{2}, \quad |x| \geq \frac{M_0}{2}, \quad \varepsilon \in (0, \varepsilon_0),
\]
yields that
\[
\int_{D_{m+1}} |\nabla v_\varepsilon|^2 + \frac{(a - |V|)^2}{4} |v_\varepsilon|^2 \leq 8 \int_{D_m \setminus D_{m+1}} |\nabla v_\varepsilon|^2 + |v_\varepsilon|^2,
\]
which implies that
\[
\int_{D_{m+1}} |\nabla v_\varepsilon|^2 + |v_\varepsilon|^2 \leq C' \int_{D_m \setminus D_{m+1}} |\nabla v_\varepsilon|^2 + |v_\varepsilon|^2,
\]
with $C' = 8/\min\{1, (a - |V|)/4\}$. So $a_{m+1} \leq \varrho a_m$, with $\varrho = \frac{C'}{1+a^2} < 1$ and $a_m = \int_{D_m} |\nabla v_\varepsilon|^2 + |v_\varepsilon|^2$. Then from the bounded-ness of $v_\varepsilon$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$, we know
\[
a_m \leq a_0 \varrho^m \leq C \varrho^m = Ce^{m\ln \varrho}
\]
Let $m = \lfloor \frac{M}{2} \rfloor - 1$, by the choice of $M_0$, we have $m = \lfloor \frac{M}{2} \rfloor - 1 \geq \frac{M}{4}$ and hence
\[
\int_{|x| \geq M-1} |\nabla v_\varepsilon|^2 + |v_\varepsilon|^2 \leq a_m \leq Ce^{m\ln \varrho} \leq Ce^{M\ln \varrho}.
\]
From this, Lemma 4.1 and Hölder’s inequality, it follows that
\[
\|v_\varepsilon\|_{W^{1,q}(|x| \geq M-1)} \leq C_q e^{-c_q M}, \quad q > 3,
\]
with \(c_q, C_q\) independent of \(\varepsilon\) and \(M\). Thus by Lemma 4.1, for any \(M \geq M_0\) and \(x \in \mathbb{R}^3\) with \(|x| = M\), there holds
\[
|v_\varepsilon(x)| \leq C'_q e^{-c_q |x|}.
\]
While, since \(v_\varepsilon\) is uniformly bounded in \(L^\infty\), so \(|v_\varepsilon(x)| \leq C e^{-c |x|}\) naturally holds for \(|x| \leq M_0\). Consequently, \(|w_\varepsilon(x)| \leq C e^{-c |x-x_z|}\) and
\[
|w_\varepsilon(x)| \leq C \exp \left( - \frac{c}{\varepsilon} |x - \theta_z| \right),
\]
with \(\theta_z = \varepsilon x_z\), uniformly holds for \(\varepsilon \in (0, \varepsilon_0)\).

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