Dissipativity analysis of Markovian Switched Neural Networks using Extended Reciprocally Convex Matrix Inequality

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Abstract: This paper aims to present improved result of dissipativity analysis in Markovian switched neural networks with time-delays. By applying the novel extended reciprocally convex matrix inequality with the construction of Lyapunov-Kravsovskii functional and by using the linear matrix inequality technique, a set of sufficient conditions has been established to guarantee the existence of analysis. Numerical example with simulation is illustrated for the effectiveness of the proposed method.

Keywords: Stability analysis, extended reciprocally convex matrix, Markovian switched neural networks

1. INTRODUCTION

There has been increasing research interests in the dynamical analysis of neural networks (NNs) such as pattern recognition, associative memories, optimization and in the field of data mining, medical diagnosis and so on [1-4]. It is worthwhile to note that the majority of the research in todays’ digital world is concerned with discrete-time systems than the continuous-time systems. This is due to the reason that discretization dynamics of the continuous-time cannot be preserved when compared with the dynamics of the discrete-time systems. Therefore, more emphasis has taken to concentrate on the discrete-time switched NNs.

Moreover, time delays are inevitably encountered as the source of oscillation and instability in the feature of regulatory transmissions between the neurons. Hence, great deal of research is focussed towards the study the robust asymptotic and exponential stability of NNs with time-delay [1,2,5,7,9].

In order to achieve the stability criteria among the NN’s, the dissipativity analysis has been used. The concept of dissipativity in dynamical system has found a lot of applications in chaos, synchronization, robust control and in stability theory [8].

On the other hand, Markovian jump systems have attracted may researchers from Mathematics and control communities. These Markovian jump systems are more appropriate to model practical systems that is found in the power systems, network control systems and manufacturing systems etc [6]. A Markovian jump system is governed by a Markov process where a Transition Probability Matrix (TPM) is required. Here, we apply the concept of Markovian process to the switched systems that are also called the hybrid systems containing finite number of subsystems with divergent trajectories [4].
The concept of dissipative analysis in dynamical system has found a lot of applications in chaos, synchronization, robust control and in stability theory [5]. We apply the extended convex approach lemma so as to reduce the conservatism. It is to be noticed that this lemma holds good for discrete-time NNs.

On the basis of the above discussions, the objective of this paper is to study the problem on the dissipative analysis for Markovian switched NNs with time varying delays and generalized activation functions. By employing appropriate Lyapunov-Krasovskii functionals and using stochastic analysis method with respect to LMI technique, we obtain a new sufficient condition for checking the global dissipativity of the addressed NNs.

**Notations:** \( \mathbb{R}^n \) denotes the n-dimensional Euclidean space, \( \mathbb{R}^{m \times n} \) is the set of all \( m \times n \) real matrices. \((\Omega, F, P)\) is a probability space, \( \Omega \) is the sample space, \( F \) is the sigma algebra of subsets of the sample space and \( P \) is the probability measure on \( F \). \( \Pr \{ \cdot \} \) represents the probability. \( \mathbb{E}\{ \cdot \} \) denotes the expectational operator with respect to some probability measure \( P \) and \(| \cdot |\) refers to the Euclidean vector norm. The superscript “ T ” represents the transpose and “ * ” denotes the term that is induced by symmetry.

### 2. PROBLEM DESCRIPTION AND PRELIMINARIES

Consider a complete probabilistic metric space \((\Omega, F, P)\) define a discrete-time Markov jump NNs as follows:

\[
x(t + 1) = B(\zeta(t))x(t) + C(\zeta(t))f(x(t)) + D(\zeta(t))f(x(t - d(t))) + L(\zeta(t))\omega(t) \\
y(t) = Hx(t), \quad t \in [-d, 0]
\]

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \rightarrow \mathbb{R}^n \) is the neural state vector and the activation functions of the neuron are given as \( f(x(t)) = [f(x_1(t)), f(x_2(t)), \ldots, f(x_n(t))]^T \in \mathbb{R}^n \). For each \( \zeta(t) \) in the system (1); \( \omega(t) \in \mathbb{R}^n \) is the exogenous disturbance that belongs to \( l_2[0, \infty) \), \( y(t) \) is the output of the system.

The discrete-time homogeneous Markov chain is given by the parameter \( \zeta(t) (t \in \mathbb{Z}^+) \) that takes value in the finite state space \( \mathbb{S} = \{1, 2, \ldots, S\} \) with the transition probability matrix \( \hat{\Pi} = [\pi_{ij}] \) given by \( \Pr \{ \zeta(t + 1) = j | \zeta(t) = i \} = \pi_{ij} \), where \( 0 \leq \pi_{ij} \leq 1, \forall i \in \mathbb{S} \), \( B, C, D, L, H \) are known constant matrices with appropriate dimensions and \( d(t) \) is the time-varying delay satisfying

\[
d_1 \leq d(t) \leq d_2, \quad \gamma_1 \leq \Delta d(t) = d(t + 1) - d(t) \leq \gamma_2 \quad \text{where } d_1, d_2 \text{ are known integers}
\]

The activation functions \( f_i(\cdot) \) in (1) are continuous and bounded and satisfies the following condition as

\[
\theta_\gamma - \frac{f_i(l_1) - f_i(l_2)}{l_1 - l_2} \leq \theta_{\gamma_i}, \quad \forall i = 1, 2, \ldots, n
\]
\[ f_i(0) = 0, l_1, l_2 \in R, l_1 \neq l_2 \] with \( \theta_i^+ \) and \( \theta_i^- \) are known real scalars and \( \Theta = \text{diag} \{ \theta_1^-, \theta_2^-, \ldots, \theta_n^- \} \) and \( \Theta = \text{diag} \{ \theta_1^+, \theta_2^+, \ldots, \theta_n^+ \} \).

**Definition 1:** For any positive integer \( h \) and for given real symmetric matrices \( \delta_i, i = 1, 2, 3, 4 \) with \( \delta_i \leq 0, \delta_3 > 0, \delta_i \geq 0 \) and \( (||\delta_1|| + ||\delta_2|| + ||\delta_3|| + ||\delta_4|| = 0) \), system (1) is extended dissipative if the following condition holds good
\[ \langle y, \delta_1 y \rangle + 2\langle y, \delta_2 y \rangle + \langle \omega, \delta_3 y \rangle \leq -\sup_{0 \leq t \leq h} y(t) \delta_4 y(t) \geq 0. \quad (5) \]

**Lemma:** *(Wirtinger-based Inequality)*: For any sequence of discrete-time variable \( y : \mathbb{Z}[a,b] \to \mathbb{R}^n \) with the positive integers \( b > a \) and for a given matrix \( M > 0 \), the following inequality holds
\[ \sum_{j=a}^{b} \beta(j) M \beta(j) \geq \frac{1}{b - a} \left[ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} \right]^T M \left[ \begin{array}{c} b - a + 1 \\ b - a - 1 \end{array} \right] M \left[ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} \right] \quad (6) \]
where \( \beta(j) = y(j+1) - y(j) \), \( \sigma_1 = y(b) - y(a) \) with \( \sigma_2 = y(b) + y(a) - 2\sum_{j=a}^{b} \frac{y(j)}{b - a + 1} \).

**3. MAIN RESULTS**

In this section, we apply the extended reciprocally convex matrix inequality to handle the delay product type of LKF.

**Theorem 1:** Under the assumptions, the given system (1), is extended dissipative for a given set of integers \( d_1, d_2, \gamma \) and matrices \( \delta_i, i = 1, 2, 3, 4 \) satisfying Definition 1, with the given set of symmetric \( 3n \times 3n \) matrix \( P \), \( 2n \times 2n \) matrix \( P_1 \) matrix, \( n \times n \) diagonal matrix \( G_i > 0, i=1,2,3, n \times n \) matrices \( Q_1, Q_2, R_i \) such that the following LMIs hold with
\[ \psi(d_1) > 0, \quad \psi(d_2) > 0, \quad (7) \]
\[ \Xi_{1a} < 0, \Xi_{2a} < 0, \Xi_{1b} < 0, \Xi_{2b} < 0, \quad (8) \]
\[ H^T \delta 2 H \leq \psi(d_1), H^T \delta 1 H \leq \psi(d_2) \quad (9) \]
where
\[ \Xi_{1a} = \psi_0(d_2) + \psi_3(y_1') + \psi_2 + \psi_3 + \Phi - \Psi \]
\[ \Xi_{1b} = \psi_0(d_1) + \psi_3(y_2') + \psi_2 + \psi_3 + \Phi - \Psi \]
\[ \Xi_{2a} = \psi_0(d_2) + \psi_3(y_1') + \psi_2 + \psi_3 + \Phi - \Psi \]
\[ \Xi_{2b} = \psi_0(d_1) + \psi_3(y_2') + \psi_2 + \psi_3 + \Phi - \Psi \]

\[ \psi(d(t)) = \sum_{j \in S} \pi_j P_j + d(t) \left[ \begin{array}{c} \pi_j P_1 \\ \pi_j P_2 \end{array} \right], \quad (0) \]
\[\psi_0(d(t)) = \Gamma_2 \psi(d(t)) \Gamma_2^T - \Gamma_1 \psi(d(t)) \Gamma_1^T,\]

\[\psi_1(\Delta d(t)) = \Delta d(t) \Gamma_1 \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \Gamma_2^T,\]

\[\Gamma_1 = [e_1, e_5 - e_1, e_6 + e_7],\]

\[\Gamma_2 = [e_r + e_1, e_5 - e_2, e_2 - e_4 + e_6 + e_7],\]

\[\Gamma_3 = \begin{bmatrix} e_1 - e_2, & e_1 + e_2 - \frac{2}{d_1 + 1} e_5 \end{bmatrix},\]

\[W_0 = \begin{bmatrix} R_1 & 0 \\ 0 & \frac{3(d_1 + 1)}{d_1 - 1} R_1 \end{bmatrix},\]

\[\psi_2 = e_1 Q e_1^T - e_2 (Q_1 - Q_2) e_2^T - e_4 Q_2 e_4^T,\]

\[\psi_3 = d_1^2 e_r R e_r^T - \Gamma_3 W_0 \Gamma_3^T,\]

\[\Sigma = -\text{sym}\left\{e_8 - e_1 \Theta_u U_1 [e_8 - e_1 \Theta_u]^T\right\} - \text{sym}\left\{e_9 - e_3 \Theta_u U_2 [e_9 - e_3 \Theta_u]^T\right\}
- \text{sym}\left\{(e_8 - e_9) - (e_1 - e_3) \Theta_u U_3 [(e_8 - e_9) - (e_1 - e_3) \Theta_u]^T\right\},\]

\[A = e_1 H \delta_1 H e_1^T + \text{sym}(e_1 H \delta_2 e_1^T) + e_10 \delta_3 e_10^T,\]

\[e_i = [0_{(i-1) \times n}, I_n, 0_{(10-i) \times n}], i = 1, 2, \ldots, 10\]

Then, the system (1) is globally asymptotically stable if the LMI’s (9)-(11) holds good with disturbance \(\omega(t) \equiv 0\) and \(A = 0\) with vectors \(e_r, e_i\) which can be are rewritten as

\[e_r^T = (B - I_n) e_1^T + C e_9^T + D e_5^T\]

and

\[e_i = [0_{(i-1) \times n}, I_n, 0_{(9-i) \times n}], i = 1, 2, \ldots, 9\]

**Proof:** Construct the following LKF candidate as

\[V_i(t) = \sum_{j=1}^{3} V_{ij}(t),\]

\[V_{1i}(t) = \xi_1^T(t) P_{\xi_1}(t) + d(k) \xi_2^T(t) P_{\xi_2}(t),\]

\[V_{2i}(t) = \sum_{s=t-d_i}^{t-1} x^T(s) Q_1 x(s) + \sum_{s=t-d_2}^{t-1} x^T(s) Q_2 x(s),\]

\[V_{3i}(t) = d_1 \sum_{j=1}^{3} \sum_{s=t+d_j}^{t-1} \beta^T(s) R_1 \beta(s)\]
\[ \beta(s) = \beta(s + 1) - \beta(s), \quad d_{i1} = d_2 - d_i \]

with
\[ \xi_1^T(t) = \left[ \sum_{s = t - d_2}^{t - d_i - 1} X^T(s), \xi_2^T(t) = \sum_{s = t - d_i}^{t - 1} X^T(s) \right] \]

and the first LKF candidate \( V_i(t) \) can be rewritten as \( V_i(t) = \xi_1^T(t)\psi(d(t))\xi_1(t) \).

Moreover it is observed that \( \psi(d(t)) > 0 \) when the LMI's given in based on the convex combination technique,
\[ \chi^T(t) = \left[ x^T(t), x^T(t - d_i), x^T(t - d_2), \sum_{s = t - d_i}^{t - d_2 - 1} x^T(s), \sum_{s = t - d_2}^{t - 1} x^T(s) \right]. \]

By differencing the above inequalities based on the expectation, we get
\[
\begin{align*}
E\{\Delta V_i(t)\} &= E\{V(t + 1) - V(t)\} \\
E\{\Delta V_1(t)\} &= \xi_1^T(t + 1)\psi(d(t + 1)) - \xi_1^T(t)\psi(d(t))\xi_1(t) \\
&= \chi^T(t)\psi_1(t)\chi(t) \\
E\{\Delta V_2(t)\} &= E\{x^T(t)Q_1x(t) - x^T(t - d_1)Q_1x(t - d_1) - x^T(t - d_2)Q_2x(t - d_2)\} \\
&= \chi^T(t)\psi_2(t)\chi(t) \\
E\{\Delta V_3(t)\} &= d_i^2\beta^T(s)R_i\beta(t) - d_i \sum_{s = t - d_i}^{t - 1} \beta^T(s)Q_1\beta(s) \\
&= \chi^T(t)\psi_3(t)\chi(t)
\end{align*}
\]

Then by applying Lemma 1, the \( R_i \) dependent term is estimated as,
\[ d_i \sum_{s = t - d_i}^{t - 1} \beta^T(s)Q_1\beta(s) \geq \chi^T(t) \Gamma_i \begin{bmatrix} R_i & 0 \\ 0 & (d_i - 1)R_i \end{bmatrix} \Gamma_i^T \chi(t) \]

Combining the right side of equations after evaluations,
\[ E\{\Delta V_3(t)\} = \chi^T(t)\psi_3(t)\chi(t) \]

Considering the activation functions which satisfies (4), and
\[ U_i = \text{diag}\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{in}\} > 0, \quad i = 1, 2, 3 \]

We have
\[
\begin{align*}
-2[f(x(t)) - \Theta_x(x(t))U_1[f(x(t)) - \Theta_x(x(t))] &\geq 0 \\
-2[f(x(t) - d(t)) - \Theta_x(x(t) - d(t))U_1[f(x(t) - d(t)) - \Theta_x(x(t) - d(t))] &\geq 0 \\
-2[f(x(t)) - f(x(t) - d(t)) - \Theta_x(x(t) - x(t - d(t))U_1[f(x(t)) - f(x(t) - d(t)) - \Theta_x(x(t) - x(t - d(t)))] &\geq 0
\end{align*}
\]
For the system (1), the rate function of the quadratic supply is given as
\[ J(i) = y^T(i)\delta_1 y(i) + 2y^T(i)\delta_2 \omega(i) + \omega^T(i)\delta_3 \omega(i) = \chi^T(i)\Lambda \chi(i) \]  \hspace{1cm} (18)
where matrices $\delta_1, \delta_2, \delta_3$ are represented in the Definition 1 and matrix $\Lambda$ is defined in the Theorem 1.

Combining all the inequalities from (10) to (18), we get,
\[ \Delta V(x(t)) = \chi^T(t)\Omega(d(t))\chi(t) + J(t) \]  \hspace{1cm} (19)
\[ \Omega(d(t)) = \psi_0(d(t)) + \psi_1(\Delta d(t)) + \psi_2 + \psi_3 + \Sigma - \Lambda \]
The time-varying delay $d(t)$ is satisfied with equations (1) and (2) and the inequality $\Omega(d(t)) < 0$ holds good which results in $\Delta V(x(t)) < J(t)$, provided all the equations in (8) are feasible.

Also, by the Definition 1, we have the extended dissipativity of the system (1), with $w(t) = 0$ provided
\[ \sup_{0 \leq \tau \leq h} y^T(t)\delta_4 y(t) \leq \sum_{i=0}^{h} J(l) \] \hspace{1cm} (20)
By non-zero conditions we have,
\[ \sum_{i=0}^{n-1} (\Delta V(x(t)) - J(i)) = V(x(t)) - \sum_{i=0}^{n-1} J(i) \leq 0 \] \hspace{1cm} (21)
The following inequality holds good if LMI (9) is true,
\[ Z_1^T(t)[H^T \delta_4 H]Z_1(t) \leq Z_1^T(t)\psi(d(t))Z_1(t) \]
We observe that from the equation (21), it follows that, $V_i(x(t)) \geq y^T(t)\delta_4 y(t)$ together with the inequality $V(x(t)) \geq V_i(x(t))$ yields to
\[ y^T(t)\delta_4 y(t) \leq \sum_{i=0}^{n-1} J(l) \] \hspace{1cm} (22)
We consider two cases, so as to prove (20), and they are $\delta_4 = 0$ and $\delta_4 > 0$.

If the matrix $\delta_4 = 0$, $\sum_{i=0}^{h} J(l) \geq 0$ is achieved by (21) and (20).

Also, when the matrix $\delta_4 > 0$, with $\delta_1 = 0$ and $\delta_2 = 0$, then $J(i) = \omega^T(i)\delta_3 \omega(i) > 0$ holds good for $\delta_3 > 0$. Furthermore, $y^T(t)\delta_4 y(t) \leq \sum_{i=0}^{n-1} J(l) \leq \sum_{i=0}^{h} J(l)$ for any positive integers $t \leq h$. Therefore, by taking the supremum of $y^T(t)\delta_4 y(t)$ with respect to $k \in [0,h]$ results in (20).

Hence, system (1) achieves the extended dissipative, if the LMI’s (7) to (9) holds good.

Moreover, when the disturbance $\omega(t) = 0, J(i) = y^T(i)\delta_3 y(i) \leq 0$ is attained with $\delta_1 \leq 0$ then for a small scalar $c$, we obtain $\Delta V(x(t)) < J(t), \Delta V(x(t)) < -c\|x(t)\|^2$, which therefore implies that system (1) is globally asymptotically stable. Hence, the proof is complete.
Remark: It is worthwhile to note that the stability analysis criteria for neural networks with free-weighting matrix is given in [2], study of passivity and stability analysis of neural networks with time-varying delays via extended free-weighting matrices integral inequality is given in [5], extended dissipative state estimation for Markov jump neural networks with unreliable links [3], extended dissipative analysis for discrete-time delayed neural networks based on an extended reciprocally convex matrix inequality [10]. But in this paper, the dissipative analysis of Markovian switched neural networks is done using the extended reciprocally convex matrix inequality and this gives less conservative results.

4 Numerical Simulation

Example 1: Consider the following parameters for the uncertain NNs (1) with the subsystem and two neurons as:

\[
\begin{align*}
B &= \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}, \\
C &= \begin{bmatrix} 0.002 & 0 \\ 0.7 & 0.003 \end{bmatrix}, \\
D &= \begin{bmatrix} -0.2 & 0.02 \\ -0.1 & -0.2 \end{bmatrix}, \\
H &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
K_m &= \text{diag}\{0.0\}, \\
K_m &= \text{diag}\{1.1\}
\end{align*}
\]

\[f_1(s) = -\tanh(0.4s), \quad f_2(s) = 0.2 \tanh(s), \quad f_3(s) = \tanh(0.6s)\]

It is seen that for various values of ascending \(d_i\), the maximum dissipativity \(\gamma\) tends to decrease. This proves that there is a decline in the dynamic behavior of the system with the time delay and has better fault tolerance capacity.

Also, due to fact of using the delay product terms in the LKF, applying extended reciprocally convex matrix inequality together with the Wirtinger-based Inequality has received less conservative results than
the Jensen’s Inequality. Therefore, the complex computation is reduced in Theorem 1 and thereby it achieves the desired dynamic behavior.

Moreover, Fig. 1 reveals that the curve responses are converging asymptotically to zero which means that the discrete NN with the given parameters is stable at its equilibrium. Hence, the effectiveness of the given result is thus verified. Extended dissipative analysis for discrete-time delayed Neural Networks based on an extended reciprocally convex matrix inequality

5 CONCLUSIONS

This paper investigates the problem of dissipative analysis of Markovian switched neural networks using the extended reciprocally convex matrix inequality. By using a newly constructed Lyapunov functional, an effective LMI based condition has been proposed to find the asymptotic stability analysis of the discrete-time switched neural networks. An extended dissipative for non-linear process has been developed using the reciprocally convex matrix inequality and Wirtinger based inequality. A numerical example is given to demonstrate the potential and effectiveness of the results obtained.

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