A THRESHOLD FOR CUTOFF IN TWO-COMMUNITY RANDOM GRAPHS

ANNA BEN-HAMOU

Abstract. In this paper, we are interested in the impact of communities on the mixing behavior of the non-backtracking random walk. We consider sequences of sparse random graphs of size $N$ generated according to a variant of the classical configuration model which incorporates a two-community structure. The strength of communities is measured by a parameter $\alpha$ which roughly corresponds to the fraction of edges that go from one community to the other. We show that if $\alpha \gg \frac{1}{\log N}$, then the non-backtracking random walk exhibits cutoff at the same time as in the one-community case, but with a larger cutoff window, and that the distance profile inside this window converges to the Gaussian tail function. On the other hand, if $\alpha \ll \frac{1}{\log N}$, then there is no cutoff.

1. Introduction

1.1. Setting. We consider an extension of the classical configuration model, designed to incorporate a two-community structure. Let $V$ be a vertex set partitioned into two non-empty communities $V_0$ and $V_1$, i.e.

$$V = V_0 \cup V_1 \quad \text{and} \quad V_0 \cap V_1 = \emptyset.$$  

Let $d : V \to \mathbb{N} \setminus \{0, 1\}$ be a fixed degree sequence such that

$$\sum_{v \in V_0} d(v) = N_0, \quad \text{and} \quad \sum_{v \in V_1} d(v) = N_1$$  

are both even. Let $N = N_0 + N_1$. Each vertex $v$ of $V$ is endowed with $d(v)$ half-edges, and, for $i = 0, 1$, we denote by $\mathcal{H}_i$ the set of half-edges attached to a vertex of $V_i$, and let $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$. By definition, $|\mathcal{H}_0| = N_0$, $|\mathcal{H}_1| = N_1$ and $|\mathcal{H}| = N$.

Now let $p$ be a fixed even integer between 2 and $\min\{N_0, N_1\}$ and choose uniformly at random $p$ distinct half-edges in $\mathcal{H}_0$ to form the random subset of outgoing half-edges of $\mathcal{H}_0$. Similarly, and independently, choose uniformly at random $p$ distinct half-edges in $\mathcal{H}_1$ to form the random subset of outgoing half-edges of $\mathcal{H}_1$. Half-edges which are not outgoing are called internal half-edges. Let $\alpha_0 = p/N_0$, $\alpha_1 = p/N_1$ and $\alpha = \alpha_0 + \alpha_1$. 


We now generate the random graph $G$ by choosing uniformly at random three independent pairings: a uniform pairing on the set of internal half-edges of $H_0$, a uniform pairing on the set of internal half-edges of $H_1$ (this is feasible since both sets have even size), and a uniform pairing between the set of outgoing half-edges of $H_0$ and the set of outgoing half-edges of $H_1$ (which have equal size $p$). We let $\eta$ be the induced pairing on $H$. If $x$ and $y$ are two distinct half-edges attached to vertices $u$ and $v$ respectively, then the pairing $\eta(x) = y$ induces an edge between $u$ and $v$ in the resulting graph.

We are interested in the mixing properties of the non-backtracking random walk (nbrw) on $G$, defined as the Markov chain with state space $H$ and transition matrix

$$P(x, y) = \begin{cases} \frac{1}{\deg(\eta(x))} & \text{if } y \text{ and } \eta(x) \text{ are neighbors}, \\ 0 & \text{otherwise}, \end{cases}$$

where two half-edges $x$ and $y$ are called neighbors if they are attached to the same vertex and are different. The degree of half-edge $x$, denoted $\deg(x)$, corresponds to the number of neighbors of $x$ (if $x$ is attached to vertex $u$, then $\deg(x) = \deg(u) - 1$). The nbrw thus moves at each step from the current state $x$ to a uniformly chosen neighbor of $\eta(x)$.

The matrix $P$ enjoys the following symmetry property with respect to $\eta$: for all $x, y \in H$,

$$P(\eta(y), \eta(x)) = P(x, y). \quad (1.1)$$

In particular, $P$ is doubly stochastic and the stationary distribution of the chain is the uniform distribution $\pi$ on $H$. The worst-case total-variation distance to equilibrium at time $t \geq 0$ is

$$D(t) = \max_{x \in H} D_x(t), \quad \text{where} \quad D_x(t) = \sum_{y \in H} \left( \frac{1}{N} - P^t(x, y) \right)_+.$$

This quantity is weakly decreasing in $t$, and the first time when it falls below a given threshold $0 < \varepsilon < 1$ is the $\varepsilon$-mixing time:

$$t_{\text{mix}}(\varepsilon) = \inf \{ t \geq 0, \ D(t) < \varepsilon \}.$$

We also let $t_{\text{mix}}^{(x)}(\varepsilon) = \inf \{ t \geq 0, \ D_x(t) < \varepsilon \}$, the $\varepsilon$-mixing time of the walk started at $x$.

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1The graph model may very well be defined with $N_0$, $N_1$, and $p$ all odd, the important thing being that $N_0 - p$ and $N_1 - p$ are both even. However, assuming that $p$ is even is quite convenient for the analysis, in particular in Section 3.
1.2. Results. Let

\[ \mu = \frac{1}{N} \sum_{x \in H} \log \deg(x) \quad \text{and} \quad \sigma^2 = \frac{1}{N} \sum_{x \in H} (\log \deg(x) - \mu)^2 \]  \hspace{1cm} (1.2)

be the mean and variance of the logarithmic degree of a uniformly chosen half-edge. For \( i \in \{0, 1\} \), let also

\[ \mu_i = \frac{1}{N_i} \sum_{x \in H_i} \log \deg(x) \]  \hspace{1cm} (1.3)

be the mean logarithmic degree within community \( H_i \).

We consider a sequence \( (G_n)_{n \geq 1} \) of graphs distributed according to this model, with \( N \to \infty \) as \( n \to \infty \) (the index \( n \) will be omitted from notation) and are interested in the following regime:

\[ \alpha_1 + \alpha_0 \leq 1 \]  \hspace{1cm} (there is a community structure) \hspace{1cm} (1.4a)

\[ N_0 \asymp N_1 \asymp N \]  \hspace{1cm} (communities have comparable size) \hspace{1cm} (1.4b)

\[ \lim \inf |\mu_0 - \mu_1| > 0 \]  \hspace{1cm} (communities are distinguishable) \hspace{1cm} (1.4c)

\[ \lim \inf \sigma^2 > 0 \]  \hspace{1cm} (non-vanishing variance) \hspace{1cm} (1.4d)

\[ \min_{v \in V} d(v) \geq 3 \]  \hspace{1cm} (branching degrees) \hspace{1cm} (1.4e)

\[ \Delta = \max_{v \in V} d(v) = O(1) \]  \hspace{1cm} (sparse regime) \hspace{1cm} (1.4f)

**Theorem 1.1.** Under assumptions (1.4), if \( \alpha \gg \frac{1}{\log N} \), then for all \( \varepsilon \in (0, 1) \),

\[ \frac{t_{\text{mix}}(\varepsilon) - \frac{\log N}{\mu}}{\sqrt{\frac{v^2 \log N}{\mu^3}}} \overset{p}{\to} \Phi^{-1}(\varepsilon), \]

where

\[ v^2 = \sigma^2 + \frac{2 \alpha_0 \alpha_1 (1 - \alpha)}{\alpha^3} (\mu_0 - \mu_1)^2. \]

**Theorem 1.2.** Under assumptions (1.4), if \( \alpha \ll \frac{1}{\log N} \), then for \( i = 0, 1 \), for all \( x \in H_i \) and \( \delta \in (0, 1) \), with probability \( 1 - o(1) \),

\[ t_{\text{mix}}^{(x)} \left( \frac{N_{1-i}}{N} (1 + \delta) \right) \leq \frac{2 \log N}{\log(2)}, \]

and

\[ t_{\text{mix}}^{(x)} \left( \frac{N_{1-i}}{N} (1 - \delta) \right) \geq \frac{\delta N_0 N_1}{2 N p}. \]

In particular, the NBRW started at \( x \) has no cutoff.

**Remark 1.3.** Let us briefly comment on the results. It is natural to expect that the presence of communities has an influence on the mixing behavior of the NBRW. If \( \alpha \) is very small, i.e. if there are only few edges that go from one community to the other, then the graph has a very narrow bottleneck
and the walk will take a long time to cross this bottleneck. Intuitively, the mixing time in this case is determined by the geometric time needed to hit one of those crossing edges, and the distance then decreases smoothly, as the tail function of a Geometric variable: there is no cutoff. Consider for instance the following toy-example in which two cliques $C_0$ and $C_1$ of size $n$ are joined by a single edge. A simple computation shows that the total-variation distance at time $t$ starting from some vertex $x$ in the interior of $C_0$ is roughly equal to $\mathbb{P}_x(X_t \in C_0) - \frac{1}{2}$. Now, $X_t \in C_0$ if and only if the walk has crossed the linking edge an even number of times. The time it takes for the walk to reach $C_1$ is approximately distributed according to a Geometric random variable with parameter $1/n^2$ and $\mathbb{P}_x(X_t \in C_0)$ can be approximated by the probability that a Binomial random variable with parameter $t$ and $1/n^2$ is even, i.e. $\frac{1}{2} \left( 1 + \left( 1 - 2/n^2 \right)^t \right)$. The $\varepsilon$-mixing time is thus asymptotic to $\frac{n^2 \log(1/2\varepsilon)}{2}$ and there is no cutoff.

On the other hand, if $\alpha$ is large, then the walk can easily go from one community to the other, and the mixing behavior is very similar to the case where there is no community structure, as studied by B. and Salez [6]. In this paper, the authors considered the configuration model with $\eta$ uniformly chosen among all possible pairings on $\mathcal{H}$. They showed, under much weaker degree assumptions, that the NBRW has cutoff at time $\frac{\log N}{\mu}$, with window \( \sqrt{\frac{\sigma^2}{\mu^3}} \log N \) and that the distance profile inside the window is Gaussian.

The contribution of the present paper is to determine quite precisely the threshold between those two regimes, the one with no community structure and the one with two communities connected by very few edges. As it turns out, cutoff can still occur with a strong community structure, even in a regime where the proportion $\alpha$ of crossing edges vanishes to 0, provided it decays more slowly than $1/\log N$. This threshold arises as the result of a competition between the mixing time in each community, which is of order $\log N$, and the time it takes to switch community, which is approximately Geometric with expectation of order $1/\alpha$. This result can be interpreted in light of a series of powerful results that relate mixing and hitting times [13, 23, 25] and that characterize cutoff in terms of concentration of hitting times of “worst” sets [5, 14].

An other interesting fact is the impact of communities on the cutoff window (in the regime $\alpha \gg 1/\log N$). In the case of no community structure, the window is of order $\sqrt{\frac{\sigma^2}{\mu^3}} \log N$, which, under assumptions (1.4), has order $\sqrt{\log N}$. The introduction of a community structure can significantly increase the cutoff window. Under our assumptions, this window is of order $\sqrt{\frac{\log N}{\alpha}}$, which is still much smaller than $\log N$, the first order of the mixing
time, but can be much larger than $\sqrt{\log N}$. Let us also note that, for some fixed value of $\alpha$, the window is maximized for $\alpha_0 = \alpha_1$, i.e. for $N_0 = N_1$, when the two communities have equal size.

1.3. Related work. A sequence of chains $(P_n)$ is said to exhibit the cutoff phenomenon if for all $\varepsilon \in (0, 1)$, $t^{(n)}_{\text{mix}}(\varepsilon) \sim t^{(n)}_{\text{mix}}(1 - \varepsilon)$ as $n \to \infty$. In other words, convergence to equilibrium occurs very abruptly: the total-variation distance drops from near 1 to near 0 at the mixing time, over a much shorter time known as the cutoff window. It was first observed for random walks on finite groups, such as random transpositions on the symmetric group [11], or the lazy random walk on the hypercube [2]. It was then observed in various other contexts, such as the Glauber dynamics on the Ising model at high temperature [21], or the simple exclusion process [16]. This phenomenon was quickly conjectured to be a widespread phenomenon, satisfied by a large class of finite Markov chains. However, finding simple sufficient conditions for cutoff appeared to be a very challenging task and several conditions that appeared to be “natural” have been disproved by counter-examples. For instance, regular expanders of bounded degree have remarkable mixing properties and one could reasonably expect that the (lazy) random walk on such graphs has cutoff, but this was disproved in [20]. However, one can rather ask: what is the mixing behavior of the random walk on a “typical” graph? This led to study random walks on random graphs, uniformly chosen in a given class. In this line of work, the article of Lubetzky and Sly [19] was a breakthrough: they showed that, with high probability, simple and non-backtracking random walks on random $d$-regular graphs have cutoff. Cutoff for NBRW was then established on sparse random graphs with given degrees, by B. and Salez [6], and independently by Berestycki et al. [7], and Bordenave et al. [9] established the cutoff phenomenon for the random walk on sparse random directed graphs. Recently, [3] [4] studied NBRW on dynamical random graphs, and established three different mixing behavior according to the rate at which the graph is re-randomized.

Those random graph models are “homogeneous” in the sense that with high probability, they do not give rise to a community structure within vertices. However, various real networks, such as social or biological networks [12], exhibit a community structure: there is a partition of vertices such that vertices in the same group are more likely to be connected than vertices in different groups. Probably one of the most famous random graph model with a community structure is the stochastic block model. This model was first introduced by [15], and was then studied in a wide variety of contexts, in particular in the very rich research area of community detection.
(see [1] for a survey of recent results). Fixed-degree variants of the stochastic block model, often referred to as hierarchical configuration models, were introduced and investigated by [27], [29] and [28], with a particular focus on epidemic propagation. The model considered here can be seen as a variant of the hierarchical configuration model with two communities, where randomization is used first to determine which half-edges are outgoing, and then to choose the pairings of internal and outgoing half-edges (degrees, however, are fixed). In his master thesis, Poirée [26] studied NBRW on such random graphs, in the particular case of regular degrees and communities of equal size.

1.4. Open questions. Several extensions of the model would be interesting to investigate and a lot of related questions could be raised. Let us briefly mention some of them:

- the regime in (1.4) is quite restrictive, it would be interesting to see how far those assumptions can be relaxed;
- instead of choosing the outgoing half-edges at random, it would be interesting to consider the model where each vertex initially has a fixed number of outgoing and internal half-edges;
- in the no-cutoff regime, our result in Theorem 1.2 is quite weak and could probably be improved in several ways, in particular by showing a corresponding upper bound of order $1/\alpha$, even when the walk starts from the worst possible point. Some non-rigorous computation seem to indicate that the $\varepsilon$-mixing time is asymptotic to

$$\frac{1}{\alpha} \log \left( \frac{\max\{N_0,N_1\}}{N\varepsilon} \right).$$

- what happens with more than two communities?
- what happens for the simple random walk?

2. Proof of Theorem 1.1

2.1. A useful coupling. Before entering into the proof, we describe a useful coupling for typical non-backtracking trajectories. This coupling takes advantage of the fact that the NBRW started at a given $x \in \mathcal{H}$ and the graph along its trajectory can be generated simultaneously as follows: initially, $X_0 = x \in \mathcal{H}$, all half-edges are unpaired and no type has been allocated yet (the property of a half-edge to be outgoing or internal will be referred to as its type); then at each time $k \geq 0$,

(1) (a) if the type of $X_k$ has not been fixed yet and if $X_k$ belongs to $\mathcal{H}_i$ for $i = 0, 1$, we make $X_k$ outgoing with probability $\alpha_i^{(k)}$ corresponding to the conditional probability that $X_k$ is outgoing given the past. With probability $1 - \alpha_i^{(k)}$, we make $X_k$ internal;
(i) if $X_k$ is outgoing, we pair it with a uniformly chosen unpaired half-edge of $\mathcal{H}_{1-i}$ and declare that this chosen half-edge is outgoing;

(ii) if $X_k$ is internal, we pair it with a uniformly chosen other unpaired half-edge of $\mathcal{H}_i$ and declare that this chosen half-edge is internal;

(b) if the type of $X_k$ has already been set, then $\eta(X_k)$ is already defined and no new pair is formed;

(2) in both cases, we let $X_{k+1}$ be a uniformly chosen neighbor of $\eta(X_k)$.

The sequence $\{X_k\}_{k \geq 0}$ is then exactly distributed according to the annealed law. Now, consider a sequence $\{X^*_k\}_{k \geq 0}$ generated in the following way: initially $X^*_0 = x \in \mathcal{H}$; then at each time $k \geq 0$,

(1) if $X^*_k$ belongs to $\mathcal{H}_i$ for $i = 0, 1$, draw a Bernoulli random variable $B_k$ with parameter $\alpha_i = p/N_i$;

(a) if $B_k = 1$, let $\eta(X^*_k)$ be a uniformly chosen half-edge in $\mathcal{H}_{1-i}$;

(b) if $B_k = 0$, let $\eta(X^*_k)$ be a uniformly chosen half-edge in $\mathcal{H}_i$;

(2) in both cases, let $X^*_{k+1}$ be a uniformly chosen neighbor of $\eta(X^*_k)$.

The process $\{X_k\}_{k \geq 1}$ and the simple Markov chain $\{X^*_k\}_{k \geq 1}$ can be coupled in such a way that the two sequences are equal up to the first time $k$ where either the types of $X_k$ and $X^*_k$ differ, or the two types are equal but the uniformly chosen half-edge $\eta(X^*_k)$ is already paired. The total-variation distance between the type indicators at time $k$ is smaller than $\max_{i=0,1} |\alpha_i^{(k)} - \alpha_i|$, Using the facts that at least $p - k$ half-edges remain to be made outgoing in each community, that there are at least $N_i - 2k$ unpaired half-edges in $\mathcal{H}_i$, and that $p \leq \min\{N_0, N_1\}$, we have, for all $k < \min\{N_0, N_1\}/2$,

$$\frac{k}{N_i} \leq \alpha_i^{(k)} - \alpha_i \leq \frac{2k}{N_i - 2k}.$$ 

Also, as there are less than $2k$ paired half-edges by step $k$, the probability that $\eta(X^*_k)$ is already paired is less than $\max_{i=0,1} 2k/N_i$. Letting $T$ be the first time where the two coupled sequence differ and using a crude union-bound yields

$$\mathbb{P}(T \leq t) = O\left(\frac{t^2}{N}\right),$$

by (1.4b). The distribution of $\{X^*_k\}_{k \geq 1}$ is much simpler than that of $\{X_k\}_{k \geq 1}$: at each step, draw a Bernoulli random variable whose parameter depends on the current community. If it is equal to 1, move to a uniform half-edge from the other community; if it is equal to 0, move to a uniform half-edge from the same community. It is not hard to check that the stationary distribution of this Markov chain is uniform over $\mathcal{H}$. 


Letting $S_t = \sum_{k=1}^{t} \log \deg(X_k^*)$, we have the following Central Limit Theorem: for all $x \in \mathcal{H}$ and $\lambda \in \mathbb{R}$,
\[
\mathbb{P}_x \left( \frac{S_t - t \mu}{\sqrt{t}} \leq \lambda \right) \xrightarrow{t \to \infty} \Phi(\lambda),
\]
where
\[
v^2 = \lim_{t \to \infty} \frac{1}{t} \operatorname{Var}_x(S_t) = \sigma^2 + 2 \sum_{s=1}^{+\infty} \operatorname{Cov}_\pi(\log \deg(X_0^*), \log \deg(X_s^*)).
\]
In the definition above, the subscript $\pi$ means that $X_0^* \sim \pi$. We have
\[
v^2 = \sigma^2 + 2 \sum_{x,y \in \mathcal{H}} \frac{1}{N} \sum_{s=1}^{+\infty} \left( \mathbb{P}_x(X_s^* = y) - \frac{1}{N} \right) \log \deg(x) \log \deg(y).
\]
Note that for all $i, j \in \{0, 1\}$, for all $x \in \mathcal{H}_i$ and $y \in \mathcal{H}_j$, we have
\[
\mathbb{P}_x(X_s^* = y) = \frac{\mathbb{P}_{\pi_i}(X_s^* \in \mathcal{H}_j)}{N_j},
\]
where $\pi_i$ is the uniform distribution over $\mathcal{H}_i$, hence
\[
v^2 = \sigma^2 + \frac{2N_0 \mu_0}{N} (\mu_0 - \mu_1) \sum_{s=1}^{+\infty} \left( \mathbb{P}_{\pi_0}(X_s^* \in \mathcal{H}_0) - \frac{N_0}{N} \right)
+ \frac{2N_1 \mu_1}{N} (\mu_1 - \mu_0) \sum_{s=1}^{+\infty} \left( \mathbb{P}_{\pi_1}(X_s^* \in \mathcal{H}_1) - \frac{N_1}{N} \right).
\]
Noticing that the sequences $(\mathbb{P}_{\pi_0}(X_s^* \in \mathcal{H}_0))_{s \geq 0}$ and $(\mathbb{P}_{\pi_1}(X_s^* \in \mathcal{H}_1))_{s \geq 0}$ obey the following induction relations
\[
\left\{ \begin{array}{ll}
\mathbb{P}_{\pi_0}(X_s^* \in \mathcal{H}_0) &= (1 - \alpha_0) \mathbb{P}_{\pi_0}(X_{s-1}^* \in \mathcal{H}_0) + \alpha_0 \left( 1 - \mathbb{P}_{\pi_1}(X_{s-1}^* \in \mathcal{H}_1) \right), \\
\mathbb{P}_{\pi_1}(X_s^* \in \mathcal{H}_1) &= (1 - \alpha_1) \mathbb{P}_{\pi_1}(X_{s-1}^* \in \mathcal{H}_1) + \alpha_1 \left( 1 - \mathbb{P}_{\pi_0}(X_{s-1}^* \in \mathcal{H}_0) \right),
\end{array} \right.
\]
we obtain
\[
\left\{ \begin{array}{ll}
\mathbb{P}_{\pi_0}(X_s^* \in \mathcal{H}_0) &= \frac{\alpha_0 (1 - \alpha_0 - \alpha_1)^s + \alpha_1}{\alpha_0 + \alpha_1}, \\
\mathbb{P}_{\pi_1}(X_s^* \in \mathcal{H}_1) &= \frac{\alpha_1 (1 - \alpha_0 - \alpha_1)^s + \alpha_0}{\alpha_0 + \alpha_1},
\end{array} \right. \tag{2.2}
\]
which yields
\[
v^2 = \sigma^2 + \frac{2\alpha_0 \alpha_1 (1 - \alpha)}{\alpha^3} (\mu_0 - \mu_1)^2, \tag{2.3}
\]
where we have used that $\frac{N_0}{N} = \frac{\alpha_1}{\alpha_0 + \alpha_1} = \frac{\alpha_1}{\alpha}$. We will also need a quantitative control on the CLT normal approximation, in the form of Berry-Esseen type bound.

**Lemma 2.1.** For all $x \in \mathcal{H}$ and all $t \geq 1$,
\[
\sup_{\lambda \in \mathbb{R}} \left| \mathbb{P}_x \left( \frac{S_t - t \mu}{\sqrt{t}} \leq \lambda \right) - \Phi(\lambda) \right| = O \left( \frac{1}{\sqrt{\alpha t}} \right).
\]
Proof of Lemma 2.1. By Lezaud [18, Part I, Chapter 3, Theorem 3.1] (see also Mann [22]), we have

\[ \sup_{\lambda \in \mathbb{R}} \left| \mathbb{P}_x \left( \frac{S_t - \mu t}{v \sqrt{t}} \leq \lambda \right) - \Phi(\lambda) \right| \leq \frac{159 \log(\Delta) \sigma^2 a_x}{v^3 \gamma^2 \sqrt{t}} , \]

where

\[ a_x = \sqrt{N} \sum_{y \in \mathcal{H}} \mathbb{P}_x(X_1^* = y)^2 , \]

(the initial distribution here being the distribution of \( X_1^* \) given \( X_0^* = x \)), and \( \gamma^* \) is the spectral gap of the chain \( (X_k^*) \). By assumption 1.4f, \( \Delta = O(1) \) and \( \sigma^2 = O(1) \). Moreover, if \( x \in \mathcal{H}_i \) for \( i = 0, 1 \),

\[ a_x = \sqrt{N} \left( \frac{(1 - \alpha_i)^2}{N_i} + \frac{\alpha_i^2}{N_{1-i}} \right) \leq \sqrt{N} \left( \frac{1}{N_0} + \frac{1}{N_1} \right) = O(1) , \]

since \( N_0 \asymp N_1 \asymp N \) by assumption 1.4b. The second largest eigenvalue of the transition matrix of \( (X_k^*) \) is equal to \( 1 - \alpha \), i.e. \( \gamma^* = \alpha \). Using that \( \alpha \asymp \alpha_0 \asymp \alpha_1 \), we obtain

\[ \frac{1}{v^3 \gamma^2 \sqrt{t}} \leq \frac{(at)^{-1/2}}{(\alpha v^2)^{3/2}} \sim \frac{(at)^{-1/2}}{(\alpha \sigma^2 + (1 - \alpha)(\mu_0 - \mu_1)^2)^{3/2}} , \]

and the proof is concluded by assumptions 1.4d and 1.4c. ■

2.2. Lower bound. Let \( x \in \mathcal{H} \) be a fixed starting point and let

\[ t = \frac{\log N}{\mu} + \left( \lambda + o(1) \right) \sqrt{\frac{v^2}{\mu^3 \log N}} , \]

with \( v^2 \) as in (2.3). For \( \theta = \frac{\log N}{N} \), let \( A_\theta \) be the set of \( y \in \mathcal{H} \) such that there exists a path from \( x \) to \( y \) which has probability larger than \( \theta \) to be seen by a NBRW of length \( t \). Since, for all \( y \in A_\theta \), we have \( P^t(x, y) \geq \theta \), and since \( P^t(x, \cdot) \) is a probability, the set \( A_\theta \) has size less than \( 1/\theta \), hence

\[ \mathcal{D}_x(t) \geq P^t(x, A_\theta) - \pi(A_\theta) \geq P^t(x, A_\theta) - \frac{1}{\theta N} . \]

Taking expectation with respect to the pairing, we have

\[ \mathbb{E}P^t(x, A_\theta) \geq \mathbb{P}_x \left( \prod_{s=1}^t \frac{1}{\deg(X_s^*)} > \theta \right) = \mathbb{P}_x \left( \prod_{s=1}^t \frac{1}{\deg(X_s^*)} > \theta \right) + o(1) , \]

where the last equality is by (2.1). Using Lemma 2.1, we have

\[ \mathbb{P}_x \left( \prod_{s=1}^t \frac{1}{\deg(X_s^*)} > \theta \right) = \mathbb{P}_x \left( \frac{S_t - \mu t}{v \sqrt{t}} < -\lambda + o(1) \right) \geq \Phi(\lambda) + O \left( \frac{1}{\sqrt{\alpha t}} \right) . \]
Since $\alpha \gg \frac{1}{\log N}$ by assumption, we get
$$\min_{x \in H} \mathbb{E}D_x(t) \geq \Phi(\lambda) + o(1).$$

2.3. Upper bound. As in [6], the first step is to reduce the maximization over all starting points to reasonably nice starting points, namely, to points whose neighborhood up to some level is a tree.

We call $x \in \mathcal{H}$ a root if the ball of radius $R$ centered at $x$ (denoted by $B_x$) is a tree, where
$$R = \left\lceil \frac{\log N}{6 \log \Delta} \right\rceil. \quad (2.4)$$

We denote by $\mathcal{R}$ the set of roots. The following lemma shows that we may restrict our attention to starting points in $\mathcal{R}$. Its proof is very similar to the one of Proposition 4.1 in [6], the introduction of communities only slightly changes the argument.

**Lemma 2.2.** Let $K = \lfloor \log \log N \rfloor$. Then
$$\max_{x \in \mathcal{H}} P^K(x, \mathcal{H} \setminus \mathcal{R}) \overset{p}{\to} 0.$$

**Proof of Lemma 2.2.** Define $\ell = \left\lceil \frac{\log N}{3 \log \Delta} \right\rceil$ and fix $x \in \mathcal{H}$. The ball of radius $\ell$ around $x$ can be generated sequentially, its half-edges being given a type and then paired with a uniformly chosen other hitherto unpaired half-edge from the same or the other community depending on the type, until the entire ball is generated. Observe that at most $k = \frac{\Delta((\Delta-1)^{\ell-1})}{\Delta-2}$ pairs are formed, and that, for each of them, the number of unpaired half-edges having an already paired neighbor is at most $\Delta(\Delta-1)\ell$. Hence, if the half-edge that is to be paired is in $\mathcal{H}_i$, the conditional chance to pair it with a half-edge that has an already paired neighbor (thereby creating a cycle) is at most $\frac{\Delta(\Delta-1)^{\ell-1}}{N_{i-2k-1}}$ if it has been given an internal type, or $\frac{\Delta(\Delta-1)^{\ell-1}}{N_{i-2k-1}}$ if it has been given an outgoing type. Thus, letting $q$ be the minimum of those two ratios, the probability that more than one cycle is found is at most
$$(kq)^2 = O\left(\frac{\Delta^{4\ell}}{N^2}\right) = o\left(\frac{1}{N}\right),$$
by the definition of $\ell$ and assumption (1.4b). Summing over all $x \in \mathcal{H}$ (union bound), we obtain that with probability $1 - o(1)$, no ball of radius $\ell$ in $G$ contains more than one cycle. Now fix a graph $G$ with the above property. Then the NBRW on $G$ starting from any $x \in \mathcal{H}$ will very quickly reach a root, namely it satisfies
$$\mathbb{P}(X_K \notin \mathcal{R}) \leq 2^{1-K} = o(1), \quad (2.5)$$
by exactly the same argument as for the proof of equation (21) in [6].
We have
\[ D(t + K) \leq \max_{x \in H} P^K(x, H \setminus R) + \max_{x \in R} D_x(t). \]

By Lemma 2.2, the first term is \( o_P(1) \), and, for all \( x \in R \), bounding the summands corresponding to \( y \in (H \setminus R) \cup B_x \) by \( 1/N \),
\[ D_x(t) \leq \sum_{y \in R \setminus B_x} \left( \frac{1}{N} - P^t(x, \eta(y)) \right) + \frac{|(H \setminus R) \cup B_x|}{N}. \]

Observe that Lemma 2.2 together with the fact that \( P^K \) is doubly stochastic (since \( P \) is) imply that
\[ |H \setminus R| = \sum_{x \in H} P^K(x, |H \setminus R|) = o_P(N). \]

And for all \( x \in R \), we have (deterministically) \( |B_x| \leq \Delta^R \leq N^{1/6} \). Hence
\[ \max_{x \in R} \frac{|(H \setminus R) \cup B_x|}{N} = o_P(1). \]

The following proposition will therefore conclude the proof of the upper bound.

**Proposition 2.3.** For \( t = \frac{\log N}{\mu} + (\lambda + o(1)) \sqrt{\frac{\mu^2 \log N}{v^2}} \), we have
\[ \max_{x \in R} \sum_{y \in R \setminus B_x} \left( \frac{1}{N} - P^t(x, \eta(y)) \right) \leq \Phi(\lambda) + o_P(1). \]

To prove this proposition, we consider an exploration process which generates the pairing \( \eta \) along with two disjoint trees \( T_x \) and \( T_y \), rooted at \( x \) and \( y \) respectively. Initially, all half-edges are unpaired and no type has been revealed. Tree \( T_x \) is reduced to \( x \) and tree \( T_y \) is reduced to \( y \). Then at each time step,

1. An unpaired half-edge \( z \) of \( T_x \cup T_y \) is chosen, provided it satisfies
   \[ w(z) \geq w_{\text{MIN}} = N^{-\frac{1}{2} - \frac{\log(2)}{10 \log(\Delta)}} \quad \text{and} \quad h(z) < t/2, \]
   \[ \text{(2.6)} \]
   where \( w(z) \) and \( h(z) \) correspond to the weight and height of \( z \), defined as follows: if \( z \in T_r \) for \( r \in \{x, y\} \), there is a unique path \( (z_0, \ldots, z_h) \) from \( r \) to \( z \), with \( z_0 = r \) and \( z_h = z \). The value \( h \) is then called the height of \( z \), denoted \( h(z) \), and its weight is
   \[ w(z) = \prod_{i=1}^{h} \frac{1}{\deg(z_i)}. \]
(2) If \( z \in H_i \) for \( i \in \{0, 1\} \), the type of \( z \) is set to outgoing with probability proportional to \( p \) minus the number of paired outgoing half-edges of \( H_i \), and internal with probability proportional to \( N_i - p \) minus the number of paired internal half-edges of \( H_i \).

(3) If \( z \) is internal, it is paired with \( z' \), uniformly chosen among the unpaired half-edges of \( H_i \), and the type of \( z' \) is set to internal. If \( z \) is outgoing, it is paired with \( z' \), uniformly chosen among the unpaired half-edges of \( H_{1-i} \), and the type of \( z' \) is set to outgoing.

(4) If \( z' \) was not already in \( T_x \cup T_y \) and is not a neighbor of either \( x \) or \( y \), then the neighbors of \( z' \) are added to \( T_x \cup T_y \) as children of \( z \). Otherwise, both \( z \) and \( z' \) are marked with the color RED.

This exploration process continues until no unpaired half-edge in \( T_x \cup T_y \) satisfies (2.6). The pairing \( \eta \) is then completed to form the graph \( G \). We denote by \( \partial T_x \) (resp. \( \partial T_y \)) the set of leaves of \( T_x \) (resp. \( T_y \)), and by \( F_x \) (resp. \( F_y \)) the subset of leaves of \( \partial T_x \) (resp. \( \partial T_y \)) which are at distance \( t/2 \) of \( x \) (resp. \( y \)).

Note that, by (2.6), for \( r \in \{x, y\} \),

\[
\frac{t}{2} = \sum_{k=1}^{t/2} \sum_{z \in T_r} 1_{\{h(z)=k\}} w(z) \geq \left( \frac{|T_r|}{2} - 1 \right) \frac{w_{\text{MIN}}}{\Delta},
\]

which, together with (1.4f), implies

\[
|T_x \cup T_y| = O\left( N^{1/2 + \frac{\log(2)}{18 \log(\Delta)}} \log N \right) = O\left( N^{1/2 + \frac{\log(2)}{15 \log(\Delta)}} \right). \tag{2.7}
\]

In particular,

\[
|T_x \cup T_y| = O\left( N^{5/8} \right). \tag{2.8}
\]

**Lemma 2.4.** For all \( \varepsilon > 0 \), with probability \( 1 - o(1) \), for all \( x \in \mathcal{R} \) and \( y \in \mathcal{R} \setminus B_x \), we have

\[
\sum_{u \in \partial T_x \setminus F_x} w(u) + \sum_{v \in \partial T_y \setminus F_y} w(v) \leq \varepsilon.
\]

**Proof of Lemma 2.4.** The trees’ exploration can be stopped before height \( t/2 \) for two reasons: either the weight of the half-edge is too small, or it has been colored RED, namely, for \( r \in \{x, y\} \),

\[
\sum_{u \in \partial T_r \setminus F_r} w(u) = \sum_{u \in \partial T_r} w(u) 1_{\{w(u)<w_{\text{MIN}}\}} + \sum_{u \in \partial T_r} w(u) 1_{\{u \text{ is RED}\}}.
\]

Let us first control the weight of RED half-edges. For \( x \in \mathcal{R} \) and \( y \in \mathcal{R} \setminus B_x \), all RED half-edges are at distance at least \( R \) from \( r \), and thus have weight smaller than \( 2^{-R} \leq N^{-\frac{\log(2)}{6 \log(\Delta)}} \) by assumption (1.4e). Moreover, by the same arguments as in the proof of Lemma 2.2, and using the upper bound (2.7),
the total number of red half-edges in $T_r$ is stochastically dominated by twice a binomial random variable $B(k, q)$ where $k = O(N^{1/2 + \log(2) / 15 \log(\Delta)})$ and $q = O(N^{-1/2 + \log(2) / 15 \log(\Delta)})$. By Bennett’s Inequality,

$$
\mathbb{P} \left( \sum_{u \in \partial T_r} 1 \{ u \text{ is RED} \} > N \frac{\log(2)}{7 \log(\Delta)} \right) \leq \exp \left( -\Omega \left( N \frac{\log(2)}{7 \log(\Delta)} \right) \right).
$$

Hence, for all $\varepsilon > 0$,

$$
\mathbb{P} \left( \exists x \in \mathcal{R}, y \in \mathcal{R} \setminus B_r, r \in \{x, y\}, \sum_{u \in \partial T_r} w(u) 1 \{ u \text{ is RED} \} > \varepsilon \right) = o(1).
$$

Let us now control the weight of paths with weight smaller than $w_{\text{MIN}}$. To this end, consider $m = \lfloor \log N \rfloor$ independent nbrws on $G$ starting at $r$, each being stopped as soon as its weight falls below $w_{\text{MIN}}$, and let $A$ be the event that their trajectories form a tree of height less than $t/2$. Clearly,

$$
\mathbb{P} (A \mid G) \geq \left( \sum_{u \in \partial T_r} w(u) 1 \{ w(u) < w_{\text{MIN}} \} \right)^m.
$$

Taking expectation and using Markov inequality, we deduce that

$$
\mathbb{P} \left( \sum_{u \in \partial T_r} w(u) 1 \{ w(u) < w_{\text{MIN}} \} > \varepsilon \right) \leq \frac{\mathbb{P} (A)}{\varepsilon^m},
$$

where the average is now taken over both the walks and the graph. To prove that the above probability is $o(1/N^2)$, it is enough to show that $\mathbb{P}(A) = o(1)^m$. To do so, we generate the $m$ stopped nbrws one after the other, revealing types and pairs along the way, as described in Section 2.1. Given that the first $\ell - 1$ walks form a tree of height less than $t/2$, the conditional probability that the $\ell$th walk also fulfills the requirement is $o(1)$, uniformly in $1 \leq \ell \leq m$. Indeed,

- either it attains length $s = \lceil 4 \log \log N \rceil$ before leaving the graph spanned by the first $\ell - 1$ trajectories and reaching an unpaired half-edge: thanks to the tree structure, there are at most $\ell - 1 < m$ possible trajectories to follow, each having weight at most $2^{-s}$ by (1.4e), so the conditional probability is at most $m2^{-s} = o(1)$.
- or the remainder of its trajectory after the first unpaired half-edge $z$ has weight less than $\Delta^s w_{\text{MIN}}$: this part consists of at most $t/2$ half-edges which can be coupled with $(X_{k})_{k=1}^{t/2}$ for a total-variation
cost of $O(mt^2/N)$, and for $N$ large enough
\[
\mathbb{P}_z \left( \prod_{k=1}^{t/2} \frac{1}{\deg(X_k)} \leq \Delta^s w_{\text{MIN}} \right) \leq \mathbb{P}_z \left( S_{t/2} - \frac{\mu t}{2} \geq \frac{\log(2)}{18 \log(\Delta)} \log N \right),
\]
which is $o(1)$ by Lemma 2.1.

For each $(i, j) \in \{0, 1\}^2$, define
\[
W_{i,j} = \sum_{u \in F_x \cap H_i} \sum_{v \in F_y \cap H_j} w(u)w(v),
\]
and, for $\theta = (N(\log N)^3)^{-1}$,
\[
W^\theta_{i,j} = \sum_{u \in F_x \cap H_i} \sum_{v \in F_y \cap H_j} w(u)w(v) \mathbb{1}_{\{w(u)w(v) \leq \theta\}}.
\]

**Lemma 2.5.** For all $\varepsilon > 0$, with probability $1 - o(1)$, for all $x \in \mathcal{R}$ and $y \in \mathcal{R} \setminus \mathcal{B}_x$,
\[
1 \leq \alpha \left( \frac{1 - \alpha_0}{\alpha_1} W_{0,0} + \frac{1 - \alpha_1}{\alpha_0} W_{1,1} + W_{0,1} + W_{1,0} \right) + \varepsilon.
\]

**Proof of Lemma 2.5.** Note that
\[
1 = \alpha \left( \frac{1 - \alpha_0}{\alpha_1} \frac{N_0^2}{N^2} + \frac{1 - \alpha_1}{\alpha_0} \frac{N_1^2}{N^2} + \frac{2N_0N_1}{N^2} \right),
\]
so that to prove the lemma, it is enough to establish that for all $\varepsilon$, with probability $1 - o(1)$, for all $x \in \mathcal{R}$ and $y \in \mathcal{R} \setminus \mathcal{B}_x$, for all $i, j \in \{0, 1\}$, $W_{i,j} \geq \frac{N_iN_j}{N} - \varepsilon$. Now, the event $\{W_{i,j} < \frac{N_iN_j}{N} - \varepsilon\}$ is included in
\[
\left\{ \sum_{u \in F_x \cap H_i} w(u) < \frac{N_i}{N} - \frac{\varepsilon}{2} \right\} \cup \left\{ \sum_{v \in F_y \cap H_j} w(u) < \frac{N_j}{N} - \frac{\varepsilon}{2} \right\}.
\]
By Lemma 2.4, with probability $1 - o(1)$, for all $x \in \mathcal{R}$, $y \in \mathcal{R} \setminus \mathcal{B}_x$ and $r \in \{x, y\}$, we have $\sum_{u \in F_r} w(u) \geq 1 - \varepsilon/4$, so that it remains to show that for all $\varepsilon > 0$, $r \in \{x, y\}$, and $i \in \{0, 1\}$,
\[
\mathbb{P} \left( \sum_{u \in F_r \cap H_i} w(u) > \frac{N_i}{N} + \varepsilon \right) = o \left( \frac{1}{N^2} \right).
\]
To do so, we proceed as in the proof of Lemma 2.4. Consider $m = \lfloor (\log N)^2 \rfloor$ independent NBRWS on $G$ starting at $r$, each of length $t/2$, and let $B$ be the
event that their trajectories form a tree and that they all end in $H_i$. We have
\[
\mathbb{P}\left( \sum_{u \in \mathcal{F}_r \cap H_i} w(u) > \frac{N_i}{N} + \varepsilon \right) \leq \frac{\mathbb{P}(B)}{(N_i/N + \varepsilon)^m}.
\]
To prove that the above probability is $o(1/N^2)$, it is enough to show that $\mathbb{P}(B) = (N_i/N + \varepsilon/2)^m$. Generate the $m$ killed NBRW's one after the other, revealing types and pairs along the way, as described in Section 2.1. Given that the first $\ell - 1$ walks form a tree and all end in $H_i$, the conditional probability that the $\ell$th walk also does is smaller than $N_i/N + \varepsilon/2$, uniformly in $1 \leq \ell \leq m$. Indeed,

- either it attains length $s = \lceil 4 \log \log N \rceil$ before leaving the graph spanned by the first $\ell - 1$ trajectories and reaching an unpaired half-edge: thanks to the tree structure, there are at most $\ell - 1 < m$ possible trajectories to follow, each having weight at most $2^{-s}$ by (1.4e), so the conditional probability is at most $m2^{-s} = o(1)$.
- or it encounters an unpaired half-edge $z$ at some time $s' < s$ and the remainder of its trajectory can be coupled with $(X^*_k)_{k=s'+1}$ for a total-variation cost of $O(mt^2/N)$. By (2.2), and since $t \gg 1/\alpha$,
\[
\mathbb{P}_z\left( X^*_{t/2-s'} \in H_i \right) \leq N_i/N + \varepsilon/2.
\]

\[\blacksquare\]

**Lemma 2.6.** For all $\varepsilon > 0$, with probability $1 - o(1)$, for all $x \in \mathcal{R}$ and $y \in \mathcal{R} \setminus B_x$,
\[
NP^t(x, \eta(y)) \geq \alpha \left( \frac{1 - \alpha_0}{\alpha_1} W^{\theta}_{0,0} + \frac{1 - \alpha_1}{\alpha_0} W^{\theta}_{1,1} + W^{\theta}_{0,1} + W^{\theta}_{1,0} \right) - \varepsilon.
\]

**Proof of Lemma 2.6.** By property (1.1), we can write
\[
P^t(x, \eta(y)) = \sum_{u,v} P^t/2(x,u)P^t/2(y,v)1_{\{\eta(u)=v\}}.
\]
Retaining only those paths that stay in $\mathcal{T}_x \cup \mathcal{T}_y$ and that have weight less than $\theta$, we have
\[
NP^t(x, \eta(y)) \geq N \sum_{u \in \mathcal{F}_x} \sum_{v \in \mathcal{F}_y} \omega_{uv} 1_{\{\eta(u)=v\}},
\]
where $\omega_{uv} = w(u)w(v)1_{\{w(u)w(v) \leq \theta\}}$. Let us first condition on the types of the unpaired half-edges at the end of the exploration stage, and average over the remaining pairing. For $i \in \{0,1\}$, let $I_i$ (resp. $O_i$) be the set of unpaired internal (resp. outgoing) half-edges of $H_i$ at the end of the exploration stage.
By applying [6, Lemma 6.1] to the sets of unpaired internal half-edges, we have, for \( i \in \{0, 1\} \) and for all \( \varepsilon > 0 \),

\[
\mathbb{P} \left( N \sum_{\substack{u \in F_x \cap I_i \atop v \in F_y \cap I_i}} \omega_{uv} \left( 1_{\{\eta(u) = v\}} - \frac{1}{N_i - p} \right) < -\varepsilon \left| I_i \right| \right) \leq \exp \left( -\frac{\varepsilon^2 \left( |I_i| - 1 \right)}{4\theta N^2} \right).
\]

(2.9)

Combining (2.8), (1.4b) and (1.4a), we have \( |I_i| \asymp N_i - p \asymp N \), entailing that the right-hand side in (2.9) is \( o(1/N^2) \). Now, applying [10, Proposition 1.1] (or rather its refinement for the left tail given in Theorem 1.5 of the same paper), we have for \( i \in \{0, 1\} \) and for all \( \varepsilon > 0 \),

\[
\mathbb{P} \left( N \sum_{\substack{u \in F_x \cap O_i \atop v \in F_y \cap O_{1-i}}} \omega_{uv} \left( 1_{\{\eta(u) = v\}} - \frac{1}{p} \right) < -\varepsilon \left| O_i \right| \right) \leq \exp \left( -\frac{\varepsilon^2 \left| O_i \right|}{4\theta N^2} \right).
\]

(2.10)

Again, (2.8) yields \( |O_i| \asymp p \), and since by assumption \( p/N \gg 1/\log N \), the right-hand side in (2.10) is also \( o(1/N^2) \). Our second task is to average over the types of half-edges in \( F_x \cup F_y \). To this end, for \( i \in \{0, 1\} \), let \( U_i \) be the set unpaired half-edges of \( H_i \) at the end of the exploration stage and write

\[
Y = \frac{N}{N_i - p} \sum_{u \in F_x \cap I_i \atop v \in F_y \cap I_i} \omega_{uv} = \sum_{u, v \in (F_x \cup F_y) \cap U_i} q_{uv} B_u B_v,
\]

where \( q_{uv} = \frac{N}{N_i - p} \omega_{uv} 1_{\{u \in F_x\}} 1_{\{v \in F_y\}} \) and \( B_u = 1_{\{u \in I_i\}} \). Conditionally on \( T_x \) and \( T_y \), the sequence \( (B_u)_{u \in (F_x \cup F_y) \cap U_i} \) enjoys a strong negative dependence property known as the strong Rayleigh property [8] (the sequence \( (B_u)_{u \in U_i} \) enjoys it as a sequence of Bernoulli variables conditioned on its sum, and any subsequence of a strong Rayleigh sequence is also strong Rayleigh). Observing that \( Y \) is a Lipschitz function of \( (B_u) \) with constant

\[
\frac{N}{N_i - p} \theta|F_x \cup F_y| = O(N^{-3/8})
\]

by (2.8). Applying [24, Theorem 3.2], we have, for all \( \varepsilon > 0 \),

\[
\mathbb{P} \left( Y - \mathbb{E} Y < -\varepsilon \right) \leq \exp \left( -\Omega(N^{1/8}) \right),
\]

where \( \mathbb{P} \) and \( \mathbb{E} \) are the probability law and expectation given \( T_x \cup T_y \). Similarly, let

\[
Z = \frac{N}{p} \sum_{\substack{u \in F_x \cap O_i \atop v \in F_y \cap O_{1-i}}} \omega_{uv} = \sum_{u, v \in F_x \cup F_y} q'_{uv} B'_u B'_v,
\]
where now $d_{uv} = \frac{N}{p} \omega_{uv} \mathbb{1}_{\{u \in \mathcal{F}_x \cap H_i\}} \mathbb{1}_{\{v \in \mathcal{F}_y \cap H_{1-i}\}}$ and $B'_u = \mathbb{1}_{\{u \in O \cup O_{1-i}\}}$. The sequence $(B'_u)_{u \in \mathcal{U}}$ still enjoys the strong Rayleigh property (the sequences $(B'_u)_{u \in \mathcal{U}}$ and $(B'_u)_{u \in \mathcal{U}_{1-i}}$ both enjoy it as sequences of Bernoulli conditioned on their sum and the concatenation of two independent strong Rayleigh sequences is also strong Rayleigh; and, as already mentioned, if a sequence is strong Rayleigh, any of its subsequences is too). The variable $Z$ is a Lipschitz function with constant $Np \theta |F_x \cup F_y|= O(N^{-3/8})$ by (2.8) and our assumption that $p \gg \frac{N}{\log N}$. Hence another application of [24, Theorem 3.2] yields

$$P(Z - E Z < -\varepsilon) \leq \exp \left(-\Omega\left(N^{1/8}\right)\right).$$

The proof is then concluded by noticing that

$$E Y = (1 + o(1)) \frac{\alpha(1 - \alpha_i)}{\alpha_{1-i}} W_{i,i}^\theta,$$

and

$$E Z = (1 + o(1)) \alpha W_{i,1-i}^\theta,$$

Combining Lemma 2.5 and 2.6, we obtain that for all $\varepsilon > 0$, with probability $1 - o(1)$, for all $x \in \mathcal{R}$,

$$\sum_{y \in \mathcal{R} \setminus B_x} \left(\frac{1}{N} - P^t(x, \eta(y))\right) \leq \alpha \left(\frac{1 - \alpha_0}{\alpha_1} W_{0,0}^\theta + \frac{1 - \alpha_1}{\alpha_0} W_{1,1}^\theta + \bar{W}_{0,1} + \bar{W}_{1,0}\right) + \varepsilon,$$

where $\bar{W}_{i,j} = \frac{1}{N} \sum_{y \in \mathcal{H}} (W_{i,j} - W_{i,j}^\theta)$. The proof of Proposition 2.3 will then be concluded by the following lemma.

**Lemma 2.7.** For all $\varepsilon > 0$, with probability $1 - o(1)$, for all $x \in \mathcal{R}$, for all $i, j \in \{0, 1\}$,

$$\bar{W}_{i,j} \leq \frac{N_i N_j}{N^2} \bar{\Phi}(\lambda) + \varepsilon.$$

**Proof of Lemma 2.7.** Set $m = \lceil (\log N)^2 \rceil$ and let $X^{(1)}, \ldots, X^{(m)}$ be $m$ independent NBRWS of length $t/2$ started at $x$, and $Y^{(1)}, \ldots, Y^{(m)}$ be $m$ independent NBRWS of length $t/2$ started independently from the uniform distribution $\pi$ over $\mathcal{H}$, independently from $X^{(1)}, \ldots, X^{(m)}$. Let $C$ denote the event that their trajectories form a cycle-free graph and that for all $1 \leq k \leq m$, $X^{(k)}_{t/2} \in \mathcal{H}_i$, $Y^{(k)}_{t/2} \in \mathcal{H}_j$, and

$$\prod_{\ell=1}^{t/2-\Lambda \alpha^{-1}} \frac{1}{\deg(X^{(k)}_{\ell})} \prod_{\ell=1}^{t/2-\Lambda \alpha^{-1}} \frac{1}{\deg(Y^{(k)}_{\ell})} > \theta,$$
for some constant $\Lambda > 0$ to be specified later (note that by our assumption on $\alpha$, the term $\alpha^{-1}$ grows much more slowly than the window of order $\sqrt{\log N/\alpha}$). Then, $P(C \mid G) \geq \left(\prod_{i,j}^\theta \right)^m$, and

$$\mathbb{P} \left( \prod_{i,j}^\theta > \frac{N_i N_j}{N^2} \Phi(\lambda) + \varepsilon \right) \leq \frac{P(C)}{\left( \frac{N_i N_j}{N^2} \Phi(\lambda) + \varepsilon \right)^m}.$$ 

Generate the $2m$ walks $X^{(1)}, Y^{(1)}, \ldots, X^{(m)}, Y^{(m)}$ one after the other along with the underlying types and pairs, as above. Given that the first $\ell-1$ pairs already satisfy the desired property, the conditional chance that $X^{(\ell)}, Y^{(\ell)}$ also does is at most $\frac{N_i N_j}{N^2} \Phi(\lambda) + \varepsilon/2$, uniformly in $1 \leq \ell \leq m$. Indeed,

- either one of the two walks attains length $s = \lceil 4 \log \log N \rceil$ before leaving the graph spanned by the first $2(\ell - 1)$ trajectories and reaching an unpaired half-edge: the conditional chance is at most $2m2^{-s} = o(1)$.
- or they both leave the graph before $s$: the remainder of their trajectory can then be coupled with $(X^*_k)$ and $(Y^*_k)$ for a total-variation cost of $O(\frac{m\ell^2}{N})$. Thus, it is enough to bound, uniformly in $x, y, z \in \mathcal{H}$,

$$\mathbb{P}_{x,\pi} \left( \prod_{k=1}^{t_*/2} \deg(X^*_k) \deg(Y^*_k) < \frac{1}{\theta} \right) \mathbb{P}_{y} \left( X^*_{\Lambda\alpha-1} \in \mathcal{H}_i \right) \mathbb{P}_{z} \left( X^*_{\Lambda\alpha-1} \in \mathcal{H}_j \right),$$

where $t_* = t/2 - s - \Lambda\alpha^{-1}$. By (2.2), the constant $\Lambda$ can be chosen large enough so that for all $z \in \mathcal{H}$, $\mathbb{P}_{z} \left( X^*_{\Lambda\alpha-1} \in \mathcal{H}_i \right) \leq \frac{N_i}{N} + \varepsilon/8$. Also, it is not hard to check that the mixing time of $(X^*_k)$ is of order $1/\alpha$, and, since $1/\alpha \ll t_*$, the total-variation distance between the law of $X^*_{t_*/2+1}$ and the law $Y^*_1$ (which is stationary) is $o(1)$. Hence

$$\mathbb{P}_{x,\pi} \left( \prod_{k=1}^{t_*/2} \deg(X^*_k) \deg(Y^*_k) < \frac{1}{\theta} \right) \leq \mathbb{P}_{x} \left( \prod_{k=1}^{t_*} \deg(X^*_k) < \frac{1}{\theta} \right) + o(1).$$

Finally, by Lemma 2.1,

$$\mathbb{P}_{x} \left( \prod_{k=1}^{t_*} \deg(X^*_k) < \frac{1}{\theta} \right) = \Phi(\lambda) + o(1).$$

3. Proof of Theorem 1.2

We now assume that $\alpha \ll \frac{1}{\log N}$. Without loss of generality, we also assume that $N_1 \geq N_0$. Let us define the two probability measures $\pi_0$ and
\[ \pi_1 \text{ on } \mathcal{H} \text{ by} \]
\[ \pi_0(x) = \begin{cases} \frac{1}{N_0} & \text{if } x \in \mathcal{H}_0, \\ 0 & \text{otherwise}, \end{cases} \]
and
\[ \pi_1(x) = \begin{cases} \frac{1}{N_1} & \text{if } x \in \mathcal{H}_1, \\ 0 & \text{otherwise}. \end{cases} \]

Let
\[ s = \frac{2 \log N}{\log 2}. \] (3.1)

**Lemma 3.1.** For \( i = 0, 1 \) and for all \( x \in \mathcal{H}_i \),

\[ \|P^s(x, \cdot) - \pi_i\|_{TV} = o_P(1). \]

**Proof.** For \( i = 0, 1 \), let us consider the random graph \( \tilde{G}_i \) formed with the half-edges of \( \mathcal{H}_i \) as follows: the internal half-edges of \( \mathcal{H}_i \) are paired exactly as in \( G \), and the outgoing half-edges of \( \mathcal{H}_i \) are paired uniformly at random within each other (recall that \( p \) is even). Since outgoing half-edges are chosen uniformly at random, the graph \( \tilde{G}_i \) is exactly distributed according to the classical configuration model on \( \mathcal{H}_i \). Let \( Q \) be the transition matrix of the NBRW on \( \tilde{G}_i \). By the triangle inequality, for all \( x \in \mathcal{H}_i \),

\[ \|P^s(x, \cdot) - \pi_i\|_{TV} \leq \|P^s(x, \cdot) - Q^s(x, \cdot)\|_{TV} + \|Q^s(x, \cdot) - \pi_i\|_{TV}. \]

By Theorem 1.1 of [6], with high probability, the NBRW on \( \tilde{G}_i \) has cutoff at time \( \frac{\log N_i}{p_i} \), which is smaller than \( \frac{\log N}{\log 2} \) by (1.4). Hence

\[ \|Q^s(x, \cdot) - \pi_i\|_{TV} = o_P(1). \]

On the other hand, observe that

\[ \|P^s(x, \cdot) - Q^s(x, \cdot)\|_{TV} \leq \mathbb{P}_x(\tau < s \mid G), \]

where \( \tau \) is the first time when the walk is on an outgoing half-edge. By Markov’s Inequality, for all \( \varepsilon > 0 \),

\[ \mathbb{P}(\|P^s(x, \cdot) - Q^s(x, \cdot)\|_{TV} > \varepsilon) \leq \frac{\mathbb{P}_x(\tau < s)}{\varepsilon}. \]

It is thus enough to show that the annealed probability \( \mathbb{P}_x(\tau < s) \) is \( o(1) \). Generating the walk and the graph along the way as in section 2.1, and using a union bound, we have

\[ \mathbb{P}_x(\tau < s) \leq \sum_{k=0}^{s-1} \frac{p}{N_0 - 2k} \leq \frac{sp}{N_0 - 2s} = o(1), \]

which concludes the proof of Lemma 3.1. \( \blacksquare \)
3.1. **Lower bound on the mixing time.** The lower bound uses a conductance argument. Let us recall that the conductance \( \Phi(S) \) of a set \( S \subset \mathcal{H} \) is defined as
\[
\Phi(S) = \frac{\sum_{x \in S} \sum_{y \in S^c} \pi(x)P(x,y)}{\sum_{x \in S} \pi(x)}.
\]
Observe that \( \Phi(\mathcal{H}_0) = \frac{p}{N_0} \), and \( \Phi(\mathcal{H}_1) = \frac{p}{N_1} \).

By the triangle inequality, we have
\[
N_1 \pi_0 - \pi_0 P^t \leq \| \pi_0 P^t - \pi_0 \|_{TV} + \| \pi_0 - \pi \|_{TV}.
\]
By [17, equation (7.15)], \( \| \pi_0 - \pi_0 P^t \|_{TV} \leq t \Phi(\mathcal{H}_0) \). On the other hand, for \( x \in \mathcal{H}_0 \),
\[
\| \pi_0 P^t - \pi \|_{TV} \leq \| \pi_0 P^t - P^{t+s}(x,\cdot) \|_{TV} + \| P^{t+s}(x,\cdot) - \pi \|_{TV},
\]
By Lemma 3.1,
\[
\| \pi_0 P^t - P^{t+s}(x,\cdot) \|_{TV} \leq \| \pi_0 - P^s(x,\cdot) \|_{TV} = o_p(1).
\]
Hence, for all \( \varepsilon > 0 \), with probability \( 1 - o(1) \),
\[
D_x(t+s) \geq \frac{N_1}{N} - \frac{tp}{N_0} - \varepsilon.
\]
For \( \delta \in (0,1) \), we have, with probability tending to 1,
\[
D_x \left( \frac{\delta N_0 N_1}{2Np} \right) \geq \frac{N_1}{N}(1 - \delta).
\]
The exact same argument may be used for \( x \in \mathcal{H}_1 \).

3.2. **No cutoff.** To see that there is no cutoff, we just observe that for \( x \in \mathcal{H}_0 \), by the triangle inequality,
\[
\| P^s(x,\cdot) - \pi \|_{TV} \leq \| P^s(x,\cdot) - \pi_0 \|_{TV} + \| \pi_0 - \pi \|_{TV}.
\]
We have \( \| \pi_0 - \pi \|_{TV} = N_1/N \) and, by Lemma 3.1, \( \| P^s(x,\cdot) - \pi_0 \|_{TV} = o_p(1) \).
Hence, for all \( x \in \mathcal{H}_0 \) and \( \delta \in (0,1) \), with probability \( 1 - o(1) \),
\[
t_{\text{mix}}(x) \left( \frac{N_1}{N}(1 + \delta) \right) \leq \frac{2\log N}{\log 2}.
\]
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A. Ben-Hamou
Sorbonne Université, LPSM
4, place Jussieu
75005 Paris, France.

E-mail address: anna.ben-hamou@upmc.fr