Rationality theorems for curvature invariants of 2-complexes

Henry Wilton

June 1, 2023

Abstract

Let $X$ be a finite, 2-dimensional cell complex. The curvature invariants $\rho_{\pm}(X)$ and $\sigma_{\pm}(X)$ were defined in [12], and a programme of conjectures was outlined. Here, we prove the foundational result that the quantities $\rho_{\pm}(X)$ and $\sigma_{\pm}(X)$ are the extrema of explicit rational linear-programming problems. As a result they are rational, realised, and can be computed algorithmically.

Throughout this paper, we will be interested in a finite, 2-dimensional cell complex $X$. Perhaps $X$ can be simplified in a trivial way, either because some face of $X$ has a free edge, or because some vertex of $X$ is locally separating. More generally, even if $X$ does not have these features, either feature may appear after modifying the 1-skeleton of $X$ by a homotopy equivalence. In this case $X$ is called reducible, and so we will always assume that $X$ is irreducible.

The average curvature of $X$, defined to be

$$\kappa(X) := \frac{\chi(X)}{\text{Area}(X)},$$

where $\chi(X)$ is the Euler characteristic and $\text{Area}(X)$ is the number of 2-cells, provides a crude measure of the curvature of $X$. Four more refined curvature invariants of $X$ are proposed in [12]. The idea is to probe $X$ by measuring the average curvatures of 2-complexes $Y$ that map combinatorially to $X$. In fact, it is fruitful to endow $Y$ with some extra structure – a notion of area on the 2-cells – and to allow the map $Y \to X$ to branch over the centres of the

\[\text{See Definition 3.12 for the precise definition of an irreducible 2-complex.}\]
2-cells. This leads to the notion of a branched 2-complex, and the average curvature $\kappa(Y)$ extends naturally to this setting.\footnote{See Definitions 3.3 and 5.1 for the definitions of a branched 2-complex and its average curvature.}

The maximal irreducible curvature of $X$ is then defined to be

$$\rho_{\ast}(X) := \sup_{Y \in \text{Irred}(X)} \kappa(Y),$$

where Irred($X$) consists of all finite, irreducible, branched 2-complexes $Y$ equipped with an essential map to $X$\footnote{See Definition 3.10 for the definition of an essential map.} The corresponding infimum, $\rho_{\ast}(X)$, is called the minimal irreducible curvature. Our first main theorem asserts that the extrema in the definitions of $\rho_{\ast}$ and $\rho_{\ast}$ can be computed using an explicit rational linear-programming problem.

**Theorem A** (Rationality theorem for irreducible curvatures). If $X$ is a finite, 2-dimensional cell complex and Irred($X$) is non-empty, then:

(i) the curvature invariants $\rho_{\ast}(X)$ and $\rho_{\ast}(X)$ are the maximum and minimum, respectively, of an explicit rational linear-programming problem; furthermore.

(ii) $\rho_{\ast}(X)$ is attained by some $Y_{\max} \in \text{Irred}(X)$ and $\rho_{\ast}(X)$ is attained by some $Y_{\min} \in \text{Irred}(X)$.

In particular, $\rho_{\ast}(X) \in \mathbb{Q}$ and both quantities can be computed algorithmically from $X$.

As long as $X$ itself is irreducible, Irred($X$) is non-empty because $X \in \text{Irred}(X)$. See also [5, Lemma 3.10] or [12, Lemma 3.6] for weaker conditions under which Irred($X$) is non-empty. By convention, $\rho_{\ast}(X) = -\infty$ and $\rho_{\ast}(X) = \infty$ if Irred($X$) is empty.

Theorem A plays a foundational role in a programme to study 2-complexes outlined in [12]. The reader is referred to that paper for motivation and applications of this theorem, as well as its companion Theorem C below. We describe a few applications next, although many further applications are conjectured in [12].

If $\rho_{\ast}(X) \leq 0$ then $X$ is said to have non-positive irreducible curvature, and likewise if $\rho_{\ast}(X) < 0$ then $X$ is said to have negative irreducible curvature. Both of these curvature bounds have consequences for the topology of $X$ and for the structure of its fundamental group: non-positive irreducible curvature implies that $X$ is aspherical, $\pi_1(X)$ is locally indicable, and every
finitely generated subgroup of \( \pi_1(X) \) has finite second Betti number. Negative irreducible curvature implies that \( \pi_1(X) \) is 2-free, coherent, and every one-ended finitely generated subgroup is co-Hopf and large. Thus, Theorem A demonstrates that the invariant \( \rho_+(X) \) provides an effective sufficient condition for these properties.

Upper bounds on irreducible curvature are closely related to the curvature properties for 2-complexes used by Wise \cite{2,13} and Louder–Wilton \cite{3,4,6,5} to study one-relator groups. Non-positive irreducible curvature implies Wise’s \textit{non-positive immersions} property, while negative irreducible curvature implies the \textit{(uniform) negative immersions} property studied by Louder–Wilton. In \cite[Problem 1.8]{13}, Wise asked whether non-positive immersions can be recognised algorithmically. Theorem A does not answer Wise’s question, since there are 2-complexes that have non-positive immersions but do not have non-positive irreducible curvature. Nevertheless, Theorem A can be thought of as providing a positive answer to Wise’s question for close cousins of the non-positive and negative immersions conditions.

As well as implying that the curvature bounds \( \rho_+(X) \) are computable, Theorem A also asserts that they are \textit{realised}, and this fact is expected to play a role in the deepest applications of this work. A similar realisation result is at the heart of the main theorem of \cite{5}, that one-relator groups with negative immersions are coherent. The realisation of \( \rho_-(X) \) is also used in the remarkable fact that irreducible curvature nearly characterises surfaces among branched 2-complexes. An easy argument (essentially the Riemann–Hurwitz theorem) implies that, if the realisation of \( X \) is homeomorphic to a surface, then \( X \) has \textit{constant} irreducible curvature, meaning that \( \rho_+(X) = \rho_-(X) \). The following corollary of Theorem A provides a partial converse.

\textbf{Corollary B.} \textit{If} \( X \) \textit{is an irreducible branched 2-complex and} \( \rho_+(X) = \rho_-(X) \) \textit{then} \( X \) \textit{is irrigid. In particular, there is an essential map from a surface} \( S \to X \) \textit{such that} \( \kappa(S) = \kappa(X) \).

The reader is referred to \cite{12} for the proof of Corollary B as well as the definition of an irrigid 2-complex. Besides Theorem A, the techniques of \cite{10,11} provide the main ingredients of the proof.

Since any branched 2-complex \( Y \) can be simplified to make it either empty or irreducible, \text{Irred}(X) \text{ is the largest class of complexes mapping to } X \text{ that we can usefully consider. At the other extreme, the smallest class of (branched) 2-complexes that we can usefully consider consists of closed surfaces. This motivates the definition of the} \textit{maximal surface curvature}\n
\[ \sigma_+(X) := \sup_{Y \in \text{Surf}(X)} \kappa(Y), \]
where \( \text{Surf}(X) \) consists of all branched 2-complexes \( Y \) with realisation homeomorphic to a closed surface, equipped with an essential map to \( X \). Again, the corresponding infimum, \( \sigma_-(X) \), is called the \textit{minimal surface curvature}. Our second main theorem is the analogue of Theorem \( \mathbf{A} \) for \( \sigma_+(X) \) and \( \sigma_-(X) \).

**Theorem C** (Rationality theorem for surface curvatures). \textit{If} \( X \) \textit{is a finite, 2-dimensional cell complex and} \( \text{Surf}(X) \) \textit{is non-empty, then:}

(i) the curvature invariants \( \sigma_+(X) \) and \( \sigma_-(X) \) are the maximum and minimum, respectively, of an explicit rational linear-programming problem; furthermore,

(ii) \( \sigma_+(X) \) is attained by some \( Y_{\max} \in \text{Surf}(X) \) and \( \sigma_-(X) \) is attained by some \( Y_{\min} \in \text{Surf}(X) \).

\textit{In particular,} \( \sigma_\pm(X) \in \mathbb{Q} \) \textit{and can be computed algorithmically from} \( X \).

The question of when \( \text{Surf}(X) \) is empty is more subtle than for \( \text{Irred}(X) \) since, on the face of it, there could be irreducible 2-complexes with no essential maps from surfaces. However, the main theorem of [11] implies that \( \text{Surf}(X) \) is non-empty whenever \( \text{Irred}(X) \) is, in particular, whenever \( X \) is itself irreducible. \textit{(See also Remark 0.1 below.)} Again, by convention, \( \sigma_+(X) = -\infty \) and \( \sigma_-(X) = \infty \) if \( \text{Surf}(X) \) is empty.

Theorems \( \mathbf{A} \) and \( \mathbf{C} \) are in fact both special cases of the more general Theorem 5.7, which proves rationality for any invariants defined by essential maps from branched 2-complexes defined by certain conditions on the links of vertices.

Alongside \( \rho_\pm(X) \), the invariants \( \sigma_\pm(X) \) also play central roles in the programme described in [12], and the reader is again referred to that paper for further details and motivation. We record one especially important consequence here: contrary to initial appearances, the above definitions only give three invariants rather than four.

**Corollary D.** \textit{For any finite 2-complex} \( X \), \( \rho_-(X) = \sigma_-(X) \).

See [12, Theorem A] for the proof, which is similar to Corollary \( \mathbf{D} \).

**Remark 0.1.** As a special case, Corollary [12] implies the fact mentioned above that \( \text{Surf}(X) \) is empty if and only if \( \text{Irred}(X) \) is empty: \( \text{Irred}(X) \) is empty if and only if \( \rho_-(X) = \infty \) while \( \text{Surf}(X) \) is empty if and only if \( \sigma_-(X) = \infty \). Indeed, the main theorem of [11] can be given a slightly simpler proof, modulo Theorem \( \mathbf{A} \) (which is more complicated than the rationality theorem of [11]).
Rationality theorems have played important roles in geometric group theory over the last two decades, starting with Calegari’s proof that stable commutator length is rational in free groups \(^4\) and continuing with \([11]\) and \([5]\), in which they were applied to analyse the subgroup structures of various classes of finitely presented groups. All these rationality theorems deal with families of maps \(Y \to X\), and one major difficulty is to ensure that the maps being considered are homotopically non-trivial.

Calegari’s maps in \([1]\) are automatically homotopically non-trivial because they represent a non-trivial second homology class. The rationality theorems of \([11]\) and \([5]\) apply to face-essential maps \(^5\) (in the terminology of \([5]\)).

Here, we are concerned with the more restrictive class of essential maps \(^5\), which are required to be injective on fundamental groups of 1-skeleta. We therefore need a method of locally recognising \(\pi_1\)-injective maps of graphs.

Such a method is provided by Stallings’ famous observation that immersions of graphs are \(\pi_1\)-injective \([9]\). Furthermore, Stallings’ folding lemma (Lemma 1.1 below) asserts that any map of graphs can be folded to an immersion. However, folding the 1-skeleton usually obscures the structure of a 2-complex. For an example, consider a 2-complex \(Y'\) which is a fine triangulation of a surface, and then let \(Y\) be the result of folding a pair of edges in \(Y'\) with one common endpoint. To see that \(Y\) is homotopy equivalent to a surface, we need to unfold to \(Y'\). Because of these kinds of considerations, we need a criterion to ensure that a map of graphs \(\Delta \to \Gamma\) is \(\pi_1\)-injective, which does not require the map to be an immersion.

The most important technical innovation of this paper is the notion of an origami, which provides the needed criterion. An origami \(\Omega\) on a graph \(\Delta\) can be thought of as a certain kind of topological graph of graphs, with (some of) the vertices of the vertex-spaces equal to the vertex-set of \(\Delta\) – see Definition 2.1 for a precise definition. There is a natural combinatorial \(\pi_1\)-surjection from \(\Delta\) to the underlying graph \(\Delta/\Omega\) of this graph of graphs. The idea behind the definition of an origami is that it keeps track of a sequence of folds that compose to give the map \(\Delta \to \Delta/\Omega\).

More importantly, an origami can also detect whether the associated map is \(\pi_1\)-injective. An origami \(\Omega\) is called essential if every vertex-space of \(\Omega\) is a tree, and is said to be compatible with a map of graphs \(f : \Delta \to \Gamma\) if \(f\) factors through \(\Delta/\Omega\), and the induced map \(\Delta/\Omega \to \Gamma\) is an immersion. The next theorem, which is the most important theorem about origamis, encapsulates the fact that they can be used to certify that maps of graphs are \(\pi_1\)-injective.

---

\(^4\)See \([12]\) for a discussion of the relationship between stable commutator length in free groups and the surface curvatures of 2-complexes.

\(^5\)Again, see Definition \([3.10]\) for a precise definition.
I believe it to be of independent interest.

**Theorem E.** A map of finite, connected, core graphs \( f : \Gamma \to \Delta \) is injective on fundamental groups if and only if there is an essential origami \( \Omega \) on \( \Delta \) that is compatible with \( f \).

The first two sections of the paper are concerned entirely with graphs. In §1 we recall the basic formalism of graphs, especially Serre graphs, which will be our primary concern. Origamis are introduced and studied in §2, leading up to the proof of Theorem E. The basics of 2-dimensional cell complexes, including branched 2-complexes and essential maps, are introduced in §3. The rational cone \( C(\mathbb{R}) \) that underlies the proofs of Theorems A and C is constructed in §4. Finally, in §5, the invariants \( \rho_{\pm}(X) \) and \( \sigma_{\pm}(X) \) are defined, and it is shown that they can be seen as extremal values of a projective function \( \kappa \) on \( C(\mathbb{R}) \), which completes the proof.

**Acknowledgements**

Thanks are due, as ever, to Lars Louder for useful conversations. I am also very grateful to Doron Puder and Niv Levhari for pointing out an error in an earlier version.

1 Graphs

This section introduces the graphs that will play a role in the argument. The material is all standard, but the details of the definitions will be important in the new material on origamis in the next section.

We start with the simplest kind of graph. A **directed graph** \( \Gamma \) consists of a vertex set \( V_\Gamma \), an edge set \( E_\Gamma \), an **initial map** \( \iota = \iota_\Gamma : E_\Gamma \to V_\Gamma \), and a **terminus map** \( \tau = \tau_\Gamma : E_\Gamma \to V_\Gamma \).

Most of our graphs will be Serre graphs. Following Serre \[8\] and Stallings \[9\], a Serre graph, or usually just a **graph**, is a directed graph \( \Gamma \) together with a fixed-point-free involution \( e \mapsto e^* \) on \( E_\Gamma \), such that

\[
\tau(e) = \iota(e^*)
\]

for all \( e \in E_\Gamma \). The elements \( e \) of \( E_\Gamma \) should be regarded as **oriented** edges of \( \Gamma \), while an **unoriented** or geometric edge is a pair \( \{e, e^*\} \).

The reader is referred to the paper of Stallings \[9\] for further facts, constructions and terminology, but we adapt and synthesise a few important notions here.
The link (which Stallings calls the star) of a vertex \( v \) of \( \Gamma \) is the subset
\[ \text{Lk}_\Gamma(v) = \iota^{-1}(v). \]

A morphism of graphs \( f : \Delta \to \Gamma \) naturally induces a map of links \( df_v : \text{Lk}_\Delta(v) \to \text{Lk}_\Gamma(f(v)) \). The morphism \( f \) is an immersion if \( df_v \) is injective for all \( v \), and in this case the induced map on fundamental groups is injective [9, Proposition 5.3]. A famous lemma of Stallings provides the basis for analysing morphisms of graphs in terms of their effects on fundamental groups [9, §5.4].

**Lemma 1.1** (Stallings’ folding lemma). Any morphism of finite graphs \( f : \Delta \to \Gamma \) factors as
\[ \Delta \xrightarrow{f_0} \Delta \xrightarrow{\bar{f}} \Gamma \]
where \( \bar{f} \) is an immersion. Furthermore, the map \( f_0 \) factors as a finite sequence of folds.

We recall the definition of a fold, and introduce some notation and terminology.

**Definition 1.2** (Fold). Let \( \Delta \) be a graph and \( a_1, a_2 \) a pair of edges with \( u = \iota(a_1) = \iota(a_2) \). The corresponding fold is the quotient morphism
\[ f_a : \Delta \to \Delta' \]
where \( \Delta' \) is obtained by identifying the edges \( a_1 \) and \( a_2 \) to a single edge \( a \). This forces \( a_1^* \) and \( a_2^* \) to both be identified with \( a^* \), but the remaining edges of \( \Delta \) correspond bijectively with the remaining edge of \( \Delta' \). See Figure 1. The fold \( f_a \) is called essential if \( \tau(a_1) \neq \tau(a_2) \) or, equivalently, if \( f_a \) is a homotopy equivalence.

![Figure 1: A Stallings fold \( \Delta \to \Delta' \). Note that we do not in general insist that \( v_1 \neq v_2 \).](image)

We say that \( \Delta' \) is constructed from \( \Delta \) by folding, and likewise \( \Delta' \) is constructed from \( \Delta \) by unfolding. It is useful to notice that unfolding is characterised by a certain partition of a link of a vertex.
Remark 1.3. Consider an essential fold \( f_a : \Delta \to \Delta' \) as above, and write \( v' = \tau(a) \), \( v_1 = \tau(a_1) \) and \( v_2 = \tau(a_2) \). The map \( f_a \) induces a bijection between 
\[(Lk_{\Delta}(v_1) \setminus \{a_1^*\}) \sqcup (Lk_{\Delta}(v_2) \setminus \{a_2^*\}) \text{ and } Lk_{\Delta'}(v') \setminus \{a^*\},\]
and hence induces a partition of \( Lk_{\Delta'}(v') \setminus \{a^*\} \) into two. Conversely, a partition of \( Lk_{\Delta'}(v') \setminus \{a^*\} \) into two provides exactly the data needed to construct the graph \( \Delta' \) and the fold map \( f_a \). Thus, the data of an essential unfolding of the edge \( a \) in \( \Delta' \) is equivalent to a partition of \( Lk_{\Delta'}(v') \setminus \{a^*\} \) into two sets.

One of the uses of Stallings’ folding lemma is to construct cores for covering spaces of graphs. We next recall the relevant definitions.

Definition 1.4 (Cores and core graphs). Let \( \Gamma \) be a (not necessarily finite) graph. A **core** for \( \Gamma \) is a finite subgraph \( \Gamma_0 \) such that the inclusion map \( \Gamma_0 \hookrightarrow \Gamma \) is a homotopy equivalence. The graph \( \Gamma \) itself is said to be a **core graph** if \( \Gamma \) is a core for itself. It is easy to see that a finite graph is core if and only if it has no vertices of valence 1.

The power of Stallings’ folding lemma lies in the fact that it provides local certificates – the induced maps on links – to certify that a morphism of graphs is injective on fundamental groups. However, we will need to locally certify that a graph morphism \( \Delta \to \Gamma \) is injective on fundamental groups while simultaneously remembering the structure of \( \Delta \), which presents a problem since the \( \pi_1 \)-injectivity of the folding map \( \Delta \to \overline{\Delta} \) cannot be certified on any finite ball in \( \Delta \). (Consider, for instance, the map of graphs that folds a circle \( \Delta \) with \( 2n \) edges to a path with \( n \) edges.) This problem is solved by origamis, which are the subject of the next section.

Although origamis will be defined by certain equivalence relations, it will be important to think of them as graphs of spaces, in the spirit of Scott and Wall [7]. A **graph of spaces** \( X \) consists of a (possibly disconnected) vertex space \( V_X \), a (necessarily disconnected) edge space \( E_X \) equipped with an involution \( \cdot \mapsto \cdot^* \) that induces a fixed-point free involution of \( \pi_0(E_X) \), and a continuous map \( \iota : E_X \to V_X \). The **underlying graph** \( \Gamma_X \) of \( X \) is obtained by applying the \( \pi_0 \) functor that takes a space to its set of path components. Adjointly, every graph is a graph of spaces with discrete vertex and edge sets. The **geometric realisation** of \( X \) is defined to be 
\[
(V_X \sqcup (E_X \times [-1,1])) / \sim
\]
where \( (y,-1) \sim \iota(y) \) and \( (y,t) \sim (y^*,1-t) \) for all \( y \in E_X \). We will often abuse notation and denote the geometric realisation of \( X \) by \( X \).
2 Origamis

In this section we introduce origamis, which can provide a local certificate that a morphism of graphs $\Delta \to \Gamma$, which may not be an immersion, is $\pi_1$-injective. They are called ‘origamis’ because they tell you how to fold.

**Definition 2.1** (Origami). Let $\Delta$ be a (finite, Serre) graph. An origami $\Omega$ on $\Delta$ is defined by an equivalence relation $\sim_O$ on the edge set $E_\Delta$. We shall call this relation the open relation. The open relation in turn defines the closed relation $\sim_C$ on $E_\Delta$ by

$$e_1 \sim_C e_2 \iff e_1^* \sim_O e_2^*,$$

and these two relations in turn allow us to define two auxiliary (directed) graphs. The edge graph $E_\Omega$ is a bipartite directed graph, with the following two kinds of vertices:

(i) the $\sim_O$-equivalence classes of $E_\Delta$, denoted by $[e]_O$ (called the open vertices); and

(ii) the $\sim_C$-equivalence classes of $E_\Delta$, denoted by $[e]_C$ (called the closed vertices).

The edge set of $E_\Omega$ is just $E_\Delta$, the edge set of $\Delta$. Each edge $e$ has initial vertex $[e]_O$ and terminal vertex $[e]_C$.

The vertex graph $V_\Omega$ is also a bipartite directed graph, with the following two kinds of vertices:

(i) the vertex set $V_\Delta$ (called the $\Delta$-vertices);

(ii) the $\sim_C$-equivalence classes of $E_\Delta$, denoted by $[e]_C$ (again called the closed vertices).

The edge set of $V_\Omega$ is $E_\Delta$; the initial vertex of $e \in E_\Delta$ is the $\Delta$-vertex $\iota_\Delta(e) \in V_\Delta$, and its terminal vertex is $[e]_C$.

So far, these are merely notations defined by the equivalence relation $\sim_O$. To define an origami, three conditions are required on the equivalence relation $\sim_O$.

(a) *Non-singularity:* For each $e \in E_\Delta$, the edges $e$ and $e^*$ are in different connected components of $E_\Omega$.

(b) *Global consistency:* If $e_1 \sim_O e_2$ then $\iota_\Delta(e_1)$ and $\iota_\Delta(e_2)$ are in the same component of $V_\Omega$. 


(c) **Local consistency:** If \( e_1, e_2 \in [e]_\Omega \) then \( e \) does not separate \( \iota_\Delta(e_1) \) from \( \iota_\Delta(e_2) \) in \( V_\Omega \). That is, \( \iota_\Delta(e_1) \) and \( \iota_\Delta(e_2) \) are in the same component of \( V_\Omega \setminus e \).

If all three of these conditions are satisfied, then \( \sim_\Omega \) is said to define an *origami* \( \Omega \).

We will be primarily interested in origamis that also satisfy an additional property.

(d) **Essentiality:** If the vertex graph \( V_\Omega \) and edge graph \( E_\Omega \) are forests then \( \Omega \) is called *essential*.

In fact, an origami \( \Omega \) naturally defines a graph of spaces \( X_\Omega \), with vertex space the realisation of \( V_\Omega \) and edge space the realisation of \( E_\Omega \), given by continuous maps \( \iota_\Omega : E_\Omega \to V_\Omega \) and \( \ast : E_\Omega \to E_\Omega \).

**Remark 2.2.** The initial map \( \iota_\Omega : E_\Omega \to V_\Omega \) is defined non-canonically as follows.

The closed vertices of \( E_\Omega \) are also vertices of \( V_\Omega \), so we can canonically define \( \iota_\Omega \) to be the identity on these; that is, \( \iota_\Omega([e]_C) = [e]_C \). There is no canonical choice for the image of an open vertex \( [e]_O \), but we may set \( \iota_\Omega([e]_O) \) to be the \( \Delta \)-vertex \( \iota_\Delta(e_0) \) of \( V_\Omega \) for any fixed choice of representative \( e_0 \in [e]_O \). Evidently this depends on the choice of representative, but note that the global consistency condition implies that all such choices are in the same path component of \( V_\Omega \).

It remains to define \( \iota_\Omega \) on an edge \( e \) of \( E_\Omega \). Since \( [e]_C \) is joined to \( \iota_\Delta(e) \) by the edge \( e \), and consistency tells us that \( \iota_\Delta(e) \) is in the same path component as \( \iota_\Delta(e_0) \) for the chosen representative \( e_0 \in [e]_O \), it follows that there is a choice of path \( \gamma_e \) in \( V_\Omega \) from \( \iota_\Omega([e]_C) = [e]_C \) to \( \iota_\Omega([e]_O) = \iota_\Delta(e_0) \). Therefore, we may extend \( \iota_\Omega \) continuously across \( E_\Omega \) by parametrising it as \( \gamma_e \) on \( e \).

To complete the definition of the graph of spaces structure \( X_\Omega \), we need to define the involution \( \ast \) on \( E_\Omega \). This naturally extends the orientation-reversing involution on \( E_\Delta \): thought of as an edge of \( E_\Omega \), \( e \) is sent to \( e^* \), while \( [e]_O \) is sent to \( [e^*]_C \) and \( [e]_C \) is sent to \( [e^*]_O \). (Note that the latter is well defined by the definition of the closed equivalence relation.) Finally, note that non-singularity of \( \Omega \) implies that \( \ast \) fixes no elements of \( \pi_0(E_\Omega) \), so \( X_\Omega \) is indeed a graph of spaces.

**Remark 2.3.** Let \( e \) be an edge of \( \Delta \). Since \( \iota_\Omega(e) = \iota_{V_\Omega}(e) = \iota_\Delta(e) \), we will use the notation \( \iota(e) \) to denote all of these, without fear of confusion.

For our purposes, the most important feature of this graph of spaces is the underlying graph.
**Definition 2.4** (Quotient graph and equivalence). The underlying graph of \(X_{Ω}\) is called the *quotient graph* of \(Ω\), and denoted by \(\Delta/Ω\). The quotient notation is justified by the existence of a quotient map \(q: \Delta \to \Delta/Ω\), defined by sending \(e \in E_\Delta\) to the path component of \([e]_C\) in \(E_\Omega\) and \(v \in V_\Delta\) to the path component of \(v\) in \(V_\Omega\). For any edge \(e \in E_\Delta\), \(q(e^*) = q(e)^*\) immediately from the definitions, while \(ι(q(e)) = q(ι(e))\) follows using the fact that \([e]_C\) and \(ι(e)\) are joined by an edge in \(V_Ω\); therefore, \(q\) is indeed a morphism of graphs.

The first example of an origami is the *trivial* origami, which exists on any graph \(Δ\).

**Example 2.5** (Trivial origami). For any finite graph \(Δ\), the *trivial* origami \(Ω\) is defined by taking \(∼_O\) to be equality on \(E_Ω\), meaning that \(∼_C\) is also equality. In this case, the edge graph \(E_Ω\) is just a disjoint union of intervals: each \(e \in E_Δ\) is incident at the vertices \([e]_O\) and \([e]_C\), each of valence 1. In the vertex graph \(V_Ω\), each vertex \(v\) of \(Δ\) is incident at exactly those edges \(e\) in the link \(Lk_Δ(v)\); the terminus of such an edge \(e\) is the singleton equivalence class \([e]_C\), which is not incident at any other edges. In summary, \(V_Ω\) is a disjoint union of ‘star graphs’, one for each vertex of \(Ω\). Non-singularity, together with global and local consistency, are now trivial, so \(Ω\) is indeed an origami. Even more, since the vertex and edge graphs are forests, \(Ω\) is essential.

The next lemma explains the link between origamis and (un)folding.

**Lemma 2.6** (Unfolding origamis). Let \(f = f_a : Δ \to Δ'\) be an essential fold morphism, and \(Ω'\) be an essential origami on the graph \(Δ'\). There is an essential origami \(Ω\) on \(Δ\) such that \(f_a\) descends to an isomorphism \(Δ/Ω ≅ Δ'/Ω'\).

**Proof.** First, we fix some notation: let \(u = ι(a_1) = ι(a_2)\), let \(v_i = τ(a_i)\), and for their images in \(Δ'\) we write \(w' = f(u)\) and \(v = f(v_1) = f(v_2)\). Apart from \(u, v_1\) and \(v_2\), together with \(a_1, a_2\) and their opposites, \(f\) maps the vertices and edges of \(Δ\) bijectively to \(Δ'\).

We can now define the origami \(Ω\). To avoid a proliferation of notation, we shall use the notation \(∼_O\) for the open equivalence relations on both \(Δ\) and \(Δ'\) – since they are distinct graphs, this should not cause confusion. The definition of \(∼_O\) on \(Δ\) falls into three cases.

(a) Unless \(f(e_1)\) and \(f(e_2)\) are open-equivalent to one of \(a\) or \(a^*\), then \(e_1 ∼_O e_2\) if and only if \(f(e_1) ∼_O f(e_2)\).

(b) If \(f(e) ∼_O a\) then \(e ∼_O a_1 ∼_O a_2\). In particular, \(a_1 ∼_O a_2\).
(c) Finally, suppose that \( f(e) \sim_{\Omega} a^* \). If \( f(e) = a^* \) then either \( e = a_1^* \) or \( e = a_2^* \), and we insist that \( a_1^* \sim_{\Omega} a_2^* \). To complete the definition of \( \Omega \) it remains to describe whether \( e \sim_{\Omega} a_1^* \) or \( e \sim_{\Omega} a_2^* \) when \( f(e) \in [a^*]_\Omega \setminus \{a^*\} \).

To this end, let \( \gamma \) be the (unique, since \( V_{\Omega'} \) is a forest) embedded path in \( V_{\Omega'} \) from \([f(e)]_C\) to \( v \), and let \( b \) be the edge of \( \gamma \) adjacent to \( v \). The local consistency of \( \Omega' \) implies that \( f(e) \) is the only edge of \([a^*]_\Omega \) that \( \gamma \) traverses; in particular, \( b \neq a^* \) since \( f(e) \neq a^* \). By Remark 1.3, the links of \( v_1 \) and \( v_2 \) induce a partition

\[
\text{Lk}_{\Delta}(v) \setminus \{a^*\} = L_1 \cup L_2
\]

where \( L_i = f(\text{Lk}_{\Delta}(v_i) \setminus \{a_i^*\}) \). Note that \( \text{Lk}_{\Omega'}(v') = \text{Lk}_{\Delta}(v') \) by construction, so \( b \in \text{Lk}_{\Delta'}(v) \setminus \{a^*\} \). We now set \( e \sim_{\Omega} a_i^* \) for the unique \( i = 1, 2 \) such that \( b \in L_i \).

To check that \( \Omega \) really is an origami with \( \Delta/\Omega \simeq \Delta'/\Omega' \), we first explain how the vertex and edge graphs of \( \Omega \) relate to those for \( \Omega' \). The edge graph \( E_{\Omega} \) is obtained from \( E_{\Omega'} \) by dividing \( a \) into \( a_1 \) and \( a_2 \), and likewise dividing \( a^* \) into \( a_1^* \) and \( a_2^* \). Since \( a_1 \sim_{\Omega} a_2 \) and \( a_1^* \sim_{C} a_2^* \), this has the effect of unfolding \( a \) and \( a^* \). In summary, \( E_{\Omega} \) is obtained by unfolding \( E_{\Omega'} \) twice. Furthermore, since \( a_1^* \sim_{\Omega} a_2^* \) and \( a_1 \sim_{C} a_2 \), these unfolds are essential. This discussion is illustrated in Figure 2.

The vertex graph \( V_{\Omega} \) is also obtained by unfolding \( V_{\Omega'} \). The edge \( a \) unfolds to the pair of edges \( a_1 \) and \( a_2 \), which meet at the vertex \( u = \iota(a_1) = \iota(a_2) \). The terminal vertices \([a_i]_C\) are now distinct since \( a_1^* \sim_{\Omega} a_2^* \), so this unfold is essential. Likewise, the edge \( a^* \) unfolds to the pair of edges \( a_1^* \) and \( a_2^* \), which meet at the vertex \([a_i^*]_C = [a_i^*]_C \), and their initial vertices \( v_1 \) and \( v_2 \) are distinct since \( f \) is essential. See Figure 3.

From these descriptions, it follows that \( f \) preserves connectivity in the vertex and edge graphs: edges \( e_1, e_2 \) of \( \Delta \) are in the same component of \( E_{\Omega} \) if and only if \( f(e_1) \) and \( f(e_2) \) are in the same component of \( E_{\Omega'} \), and the same is true for \( V_{\Omega} \) and \( V_{\Omega'} \). The same holds for vertices \( v_1, v_2 \): except in the trivial case where one is isolated, there are edges \( e_i \) such that \( v_i = \iota(e_i) \), and it follows from the case of edges that \( v_1 \) and \( v_2 \) are in the same component of \( V_{\Omega} \) if and only if \( f(v_1) \) and \( f(v_2) \) are in the same component of \( V_{\Omega'} \).

This connectivity information makes it possible to check that \( \Omega \) is an origami. If \( e \) and \( e^* \) were in the same component of \( E_{\Omega} \) then \( f(e) \) and \( f(e^*) \) would be in the same component of \( E_{\Omega'} \), a contradiction, which proves non-singularity. Global consistency also follows quickly: by the definition of \( \Omega \), if \( e_1 \sim_{\Omega} e_2 \) then \( f(e_1) \sim_{\Omega} f(e_2) \), so \( \iota(f(e_1)) \) is in the same component of \( V_{\Omega'} \) as \( \iota(f(e_2)) \), by the global consistency of \( \Omega' \). Since \( f \) preserves connectivity, \( \iota(e_1) \) and \( \iota(e_2) \) are in the same component of \( V_{\Omega} \), as required.
Figure 2: Given an essential origami $\Omega'$ on $\Delta'$, an essential fold $\Delta \rightarrow \Delta'$ as in Figure 1 induces an origami $\Omega$ on $\Delta$ with an edge graph $E_\Omega$ obtained by unfolding $E_{\Omega'}$, as illustrated. These unfolds are always essential.

We next check local consistency. To this end, given edges $e_1, e_2 \in [e]_O$, our goal is to prove that $e$ does not separate $\iota(e_1)$ from $\iota(e_2)$ in $V_\Omega$. By the definition of $\Omega$, $f(e_1), f(e_2) \in [f(e)]_O$ so, by the local consistency of $\Omega'$, the embedded path $\gamma'$ in $V_{\Delta'}$ from $\iota(f(e_1))$ to $\iota(f(e_2))$ doesn’t cross $f(e)$. Since $V_\Delta$ is obtained from $V_{\Delta'}$ by unfolding the edges $a$ and $a^*$, the path $\gamma'$ lifts to a path $\gamma$ in $V_\Omega$ unless $\gamma'$ crosses either of the vertices $v$ or $[a]_C$ of $V_{\Delta'}$. If it does lift, then $\gamma$ also doesn’t cross $e$, as required.

Therefore, it remains to consider the cases in which $\gamma'$ fails to lift at either of the vertices $v$ or $[a]_C$ of $V_{\Delta'}$. Note that, since $\gamma'$ is an embedded path, it crosses each of these vertices at most once.

Suppose first that $\gamma'$ crosses $[a]_C$ but not $v$. For $i = 1, 2$, let $\gamma'_i$ be the embedded path from $\iota(f(e_i))$ to $[a]_C$, so $\gamma'$ is the concatenation $\gamma'_1 \cdot \gamma'_2$. Each $\gamma'_i$ lifts to a path $\gamma_i$ in $V_\Omega$ from $\iota(e_i)$ to $[a_{j(i)}]_C$ for some $j(i) = 1, 2$. Therefore, the concatenation

$$\gamma = \gamma_1 \cdot a_{j(1)} \cdot a_{j(2)} \cdot \gamma_2$$

is a path in $V_\Omega$ from $\iota(e_1)$ to $\iota(e_2)$. Neither $\gamma_1$ nor $\gamma_2$ cross $e$, since $\gamma'$ doesn’t cross $f(e)$, so it remains to show that $e$ is not equal to $a_1$ or $a_2$. But $a$ separates $u = \iota(a)$ from at least one of $\iota(f(e_1))$ and $\iota(f(e_2))$, so $a \not\sim_O f(e_i)$ and therefore $a_j \not\sim_O e_i$, whence $e$ is neither $a_1$ nor $a_2$, as required.
Figure 3: Under an essential fold as in Figure 1, the vertex graph $V_\Omega$ is obtained by unfolding $V_{\Omega'}$, as illustrated. These unfolds are always essential.

Next, suppose that $\gamma'$ crosses $v$ but not $[a]_C$, and let $\gamma'_i$ be the embedded path from $\iota(f(e_i))$ to $v$. Again, each $\gamma'_i$ lifts to a path $\gamma_i$ in $V_\Omega$ from $\iota(e_i)$ to some $v_k(i)$. If $f(e_i) \sim_O a^*$ then the concatenation

$$\gamma = \gamma_1 \cdot a^*_{k(1)} \cdot a^*_{k(2)} \cdot \gamma_2$$

does not cross $e$, as in the previous case. Hence, we may assume that $f(e_i) \sim_O a^*$. Let $b'_i$ be the edge of $\gamma'_i$ incident at $v$ (or take $b'_i = f(e_i)$ if $\gamma'_i$ is just a point). Local consistency of $\Omega'$ implies that $\gamma'$ does not cross $a^*$. Therefore, if $b'_i = a^*$, it follows that $\gamma'_i$ is a point and $f(e_i) = a^*$, so $e_i = a^*_{k(i)}$. On the other hand, if $b'_i \neq a^*$, then $b'_i \in L_{k(i)}$, as in item (c) of the definition of $\Omega$. If $k(1) \neq k(2)$ then item (c) of the definition of $\Omega$ implies that $e_1 \sim_O e_2$, which contradicts our hypothesis that $e_1, e_2 \in [e]_O$. Therefore, $k(1) = k(2)$, so the lifts $\gamma_1$ and $\gamma_2$ both end at the same vertex $v_k$, and the concatenation $\gamma = \gamma_1 \cdot \gamma_2$ is a path from $\iota(e_1)$ to $\iota(e_2)$ that does not traverse $e$, as required.

It remains to deal with the case when $\gamma'$ crosses both $v$ and $[a]_C$. In this case, swapping $e_1$ and $e_2$ if necessary, we may write

$$\gamma' = \gamma'_1 \cdot \delta' \cdot \gamma'_2$$

where $\gamma'_1$ is the shortest path from $\iota(f(e_1))$ to $[a]_C$, $\delta'$ is the shortest path from $[a]_C$ to $v$ and $\gamma'_2$ is the shortest path from $v$ to $\iota(f(e_2))$. The arguments
of the previous two cases now apply verbatim to prove that \( e \) does not separate \( \iota(e_1) \) from \( \iota(e_2) \). In more detail, suppose that \( \gamma_i \) is the lift of \( \gamma'_i \) to \( V_\Omega \) and \( \delta \) is the lift of \( \delta' \). As above, \( e \) is not equal to \( a_1 \) or \( a_2 \) so a concatenation

\[
\gamma = \gamma_1 \cdot a_{j(1)} \cdot a_{j(2)} \cdot \delta \cdot a_{k(1)}^* \cdot a_{k(2)}^* \cdot \gamma_2
\]

is the required path that does not cross \( e \) unless \( e = a_1^* \) or \( a_2^* \). If \( e = a_k^* \) for some \( k \) then, as above, the hypothesis that \( e_1 \sim_\mathcal{O} e_2 \) and item (c) of the definition of \( \Omega \) imply that \( \delta \) and \( \gamma_2 \) end at the same vertex \( v_k \), so

\[
\gamma = \gamma_1 \cdot a_{j(1)} \cdot a_{j(2)} \cdot \delta \cdot \gamma_2
\]

is the required path.

Finally, that \( \Delta/\Omega \cong \Delta/\Omega' \) is also an immediate consequence of the fact that \( f \) preserves connectivity in the vertex and edge graphs.

Since homotopy equivalences factor as compositions of essential folds, it follows that they can be realised by origamis.

**Proposition 2.7.** Let \( f : \Delta \to \Gamma \) be a morphism of finite, connected, core graphs. If \( f \) is a homotopy equivalence, then there is an essential origami \( \Omega \) on \( \Delta \) and an isomorphism \( \Delta/\Omega \cong \Gamma \) that identifies \( f \) with the quotient map.

**Proof.** By Lemma 1.1, \( f \) factors as

\[
\Delta \xrightarrow{f_0} \Delta \xrightarrow{\tilde{f}} \Gamma,
\]

where \( f_0 \) is a composition of folds and \( \tilde{f} \) is an immersion. Since \( f \) is itself \( \pi_1 \)-surjective, if follows that \( \tilde{f} \) is also \( \pi_1 \)-surjective and hence injective by a standard argument. Since \( \Gamma \) is core and \( \tilde{f} \) is a homotopy equivalence, it follows that \( \tilde{f} \) is an isomorphism.

Thus, it suffices to identify \( \Delta \) with a quotient \( \Delta/\Omega \), in such a way that \( f_0 \) becomes the quotient map. This is done by induction on \( n \), where \( f_0 \) factors as a composition of \( n \) folds. In the base case \( n = 0 \), the trivial origami of Example 2.5 is as required. The inductive step is provided by Lemma 2.6, noting that every fold is essential since \( f_0 \) is injective on fundamental groups.

This completes the proof that \( f \) is realised by an essential origami.

The most important theorem about origamis is a kind of converse to Proposition 2.7: if an origami is essential then the quotient map is a \( \pi_1 \)-injection. This is how origamis make it possible to certify \( \pi_1 \)-injectivity, even for morphisms of graphs that are not immersions. The next lemma provides the key step in the proof.

---

\[\text{If } f(u) = f(v) \text{ with } u \text{ and } v \text{ distinct vertices then a shortest path } \gamma \text{ from } u \text{ to } v \text{ maps to a loop in } \Gamma \text{ representing an element of } \pi_1(\Gamma) \text{ not in } f_*\pi_1(\Delta).\]
**Lemma 2.8.** Let $\Omega$ be an essential origami on a finite graph $\Delta$. If the quotient map $q : \Delta \to \Delta/\Omega$ is not an immersion then there are distinct edges $a_1$ and $a_2$ such that $\iota(a_1) = \iota(a_2)$ and $a_1 \sim_\Omega a_2$.

Proof. Since the quotient map $q$ is not an immersion, it folds some pair of edges by Stallings’ folding lemma. That is, there are distinct edges $a_1, a_2$ of $\Delta$ with $u = \iota(a_1) = \iota(a_2)$ and with $q(a_1) = q(a_2)$, which means that $a_1$ and $a_2$ are in the same component of $\Omega$. We will prove that $a_1 \sim_\Omega a_2$.

Because $E_{\Omega}$ is a forest, there is a unique shortest edge-path $e_1, e_2, \ldots, e_{2n-1}, e_{2n}$ in $E_{\Omega}$ from $[a_1]_\Omega$ to $[a_2]_\Omega$. That is, the edges $e_i$ satisfy

$$a_1 \sim_\Omega e_1 \sim_C e_2 \sim_\Omega \cdots \sim_\Omega e_{2n-1} \sim_C e_{2n} \sim_\Omega a_2$$

and $n$ is minimal.

We now use this path in $E_{\Omega}$ to construct a path $\beta$ in $V_{\Omega}$ that starts and ends at $u$. Let $\alpha_0$ be the shortest path in $V_{\Omega}$ from $u = \iota(a_1)$ to $\iota(e_1)$; for each $1 \leq i < n$, let $\alpha_i$ be the shortest path from $\iota(e_{2i})$ to $\iota(e_{2i+1})$; and finally, let $\alpha_n$ be the shortest path from $\iota(e_{2n})$ to $\iota(a_2) = u$. Note that any of the paths $\alpha_i$ might be points, and indeed this will certainly occur if $a_1 = e_1$ or $a_2 = e_{2n}$.

The required path is the concatenation

$$\beta = \alpha_0 \cdot e_1 \cdot e_2 \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-1} \cdot e_{2n-1} \cdot e_{2n} \cdot \alpha_n.$$ 

Note that $\beta$ is an immersed path in $V_{\Omega}$. Indeed, $e_{2i-1}$ and $e_{2i}$ are distinct by the minimality of $n$, while $\alpha_i$ does not traverse $e_{2i}$ or $e_{2i+1}$, by the local consistency of $\Omega$.

In summary, $\beta$ is an immersed path in the forest $V_{\Omega}$, starting and ending at $u$. Since any immersed path in a tree must be embedded, $\beta$ has length zero, which in turn implies that $n = 0$ and $a_1 \sim_\Omega a_2$, as required. \hfill \Box

Lemma 2.8 is useful because it enables us to fold $\Delta$. The next lemma shows how to push the essential origami $\Omega$ down to an essential origami on the folded graph.

**Lemma 2.9** (Folding origamis). Let $\Omega$ be an essential origami on a finite graph $\Delta$. Consider two edges $a_1, a_2$ of $\Delta$ with $\iota(a_1) = \iota(a_2)$, and let $f = f_a : \Delta \to \Delta'/\Omega'$ be the associated fold. If $a_1 \sim_\Omega a_2$ then there is an essential origami $\Omega'$ on $\Delta'$ such that $f$ descends to an isomorphism $\Delta/\Omega \cong \Delta'/\Omega'$. Furthermore, the fold $f$ is also essential.
Proof. We will use the same notation as in the proof of Lemma 2.6: \( u = \iota(a_1) = \iota(a_2), v_i = \tau(a_i) \) and for their images in \( \Delta' \) we write \( u' = f(u) \) and \( v = f(v_1) = f(v_2) \). As in the earlier lemma, apart from \( u, v_1 \) and \( v_2, \) together with \( a_1, a_2 \) and their opposites, \( f \) maps the vertices and edges of \( \Delta \) bijectively to \( \Delta' \).

To define the origami \( \Omega' \) we need to specify the open equivalence relation \( \sim_O \) on the edges of \( \Delta' \), and we do this by setting it to be the finest equivalence relation on the edges of \( \Delta' \) with the property that \( e_1 \sim_O e_2 \) implies that \( f(e_1) \sim_O f(e_2) \). This relation can be described exactly using a brief case analysis: if neither \( e_1 \) nor \( e_2 \) is open-equivalent to any of \( a_1, a_2, a_1^* \) or \( a_2^* \), then \( f(e_1) \sim_O f(e_2) \) if and only if \( e_1 \sim_O e_2 \); if \( e_1 \sim_O a_i \) then \( f(e_2) \sim_O f(e_1) \) if and only if \( e_2 \sim_O a_i^* \); and if \( e_1 \sim_O a_i \) then \( f(e_2) \sim_O f(e_1) \) if and only if \( e_2 \sim_O a_i^* \) or \( e_2 \sim_O a_2^* \).

To check that this defines an origami \( \Omega' \) on \( \Delta' \), we need to describe the vertex graph \( V_{\Omega'} \) and the edge graph \( E_{\Omega'} \). These are both obtained by the reverse procedure to that of the proof of Lemma 2.6: that is, they are obtained by folding the pairs \( \{a_1, a_2\} \) and \( \{a_1^*, a_2^*\} \) in \( V_\Omega \) and \( E_\Omega \), respectively. In particular, connectivity in the vertex and edge graphs is preserved, and the fact that \( \Omega' \) is an essential origami with the desired properties follows quickly, except for the claim that \( \Omega' \) is locally consistent, which is more delicate. We will therefore check local consistency in detail, and leave the reader to fill in the remaining arguments, which are straightforward.

Suppose, therefore, that \( e'_i, e'_2 \in [e']_O \) for edges \( e'_i = f(e_i) \) and \( e' = f(e) \). To check local consistency, we need to find a path in \( V_{\Omega'} \) from \( \iota(e'_i) \) to \( \iota(e'_2) \) that does not cross \( e' \). By the definition of \( \Omega' \), unless \( e' \sim_O a^* \), we have that \( e_1, e_2 \in [e]_O \). Therefore, by the local consistency of \( \Omega \), the shortest path \( \gamma \) in \( V_\Omega \) from \( \iota(e_1) \) to \( \iota(e_2) \) does not cross \( e \), and so \( f(\gamma) \) is the required path from \( \iota(e'_i) \) to \( \iota(e'_2) \).

It remains to consider the case in which \( e' \sim_O a^* \). By the definition of \( \Omega' \), \( e_1 \sim_O a_i^* \) and \( e_2 \sim_O a_j^* \), for some \( i, j \in \{1, 2\} \). We claim that \( \gamma_1 \), the shortest path in \( V_\Omega \) from \( \iota(e_1) \) to \( v_1 \), crosses no preimage of \( e' \). Indeed, let \( \epsilon \) be any edge crossed by \( \gamma_1 \). The local consistency of \( \Omega \) implies that \( \epsilon \sim_O a_i^* \). Also, some subpath \( \beta \) of \( \gamma_1 \) is an embedded path from \( \iota(\epsilon) \) to \( v_1 \), so the concatenation of \( \beta \) with \( a_i^* \) and \( a_j^* \) (suitably oriented) is the unique embedded path from \( \iota(\epsilon) \) to \( v_{3-1} \). Since this embedded path traverses \( a_{3-1}^* \), it follows that \( \epsilon \sim_O a_{3-1}^* \) by the local consistency of \( \Omega \). Thus \( \epsilon \) is open-equivalent to neither \( a_1^* \) nor \( a_2^* \), so \( f(\epsilon) \sim_O a^* \), and in particular \( f(\epsilon) \neq e' \), as required.

Similarly, the shortest path \( \gamma_2 \) from \( \iota(e_2) \) to \( v_j \) also crosses no preimage of \( e' \). The images \( f(\gamma_1) \) and \( f(\gamma_2) \) both end at \( v \), and so their concatenation is a path in \( V_{\Omega'} \) from \( \iota(e'_1) \) to \( \iota(e'_2) \) that does not cross \( e' \). This proves the local consistency of \( \Omega' \), which completes the proof. \( \square \)
With these lemmas in hand, we can now prove the aforementioned theorem: essential origamis induce $\pi_1$-injective maps.

**Theorem 2.10.** Let $\Omega$ be an origami on a finite, connected graph $\Delta$. If $\Omega$ is essential then the quotient map $q : \Delta \to \Delta/\Omega$ induces an injective homomorphism of fundamental groups.

**Proof.** The proof is by induction on the difference between the number of edges of $\Delta$ and the number of edges of $\Delta/\Omega$. In the base case, that difference is zero, meaning that every component of $E_{\Omega}$ contains a unique edge of $E_{\Delta}$. This implies that $\Omega$ is the trivial origami of Example 2.5, so the quotient map is an isomorphism and the result is immediate.

For the inductive step, either $q$ is an immersion and hence $\pi_1$-injective, so there is nothing to prove, or Lemma 2.8 provides a pair of edges $a_1$ and $a_2$. By Lemma 2.9 the quotient map $q$ factors through the essential fold $f = f_{a} : \Delta \to \Delta'$ that identifies $a_1$ and $a_2$, and $\Omega$ descends to an essential origami $\Omega'$ on $\Delta'$.

The folded graph $\Delta'$ has one fewer edges than $\Delta$, so the quotient map $q' : \Delta' \to \Delta'/\Omega'$ is $\pi_1$-injective by induction. Thus

$$q = q' \circ f$$

is also $\pi_1$-injective, as required, because $f$ is an essential fold. \qed

The theorem tells us that essential origamis can be used to certify the $\pi_1$-injectivity of a quotient map. More generally, origamis can also provide certificates of $\pi_1$-injectivity for morphisms that factor through their quotient maps.

**Definition 2.11.** Let $f : \Delta \to \Gamma$ be a morphism of graphs. An origami $\Omega$ on $\Delta$ is said to be compatible with $f$ if $f$ factors through the quotient map $\Delta \to \Delta/\Omega$, and the factor map $\Delta/\Omega \to \Gamma$ is an immersion.

**Remark 2.12.** Note that compatibility is easy to certify: $f$ is compatible with $\Omega$ if and only if all of the following conditions hold.

(i) If edges $e_1, e_2$ of $\Delta$ are in the same component of $E_{\Omega}$ then $f(e_1) = f(e_2)$.

(ii) If vertices $v_1, v_2$ of $\Delta$ are in the same component of $V_{\Omega}$ then $f(v_1) = f(v_2)$.

(iii) If $e_1, e_2$ are edges of $\Delta$ with $f(e_1) = f(e_2)$ and $\iota(e_1), \iota(e_2)$ are in the same component of $V_{\Omega}$ then $e_1, e_2$ are in the same component of $E_{\Omega}$. 

18
Putting together the results of this section, we can prove that origamis can be used to certify \( \pi_1 \)-injectivity for morphisms of graphs.

**Proof of Theorem 3.** If there is an essential origami \( \Omega \) compatible with \( f \) then, by definition, \( f \) factors as \( \bar{f} \circ q \), where \( \bar{f} \) is an immersion and \( q \) is the quotient map \( \Delta \to \Delta/\Omega \). By Theorem 2.10, \( q \) is \( \pi_1 \)-injective because \( \Omega \) is essential, while \( \bar{f} \) is \( \pi_1 \)-injective because it is an immersion, so \( f \) is also \( \pi_1 \)-injective.

For the other direction, by Stallings’ folding lemma \( f \) factors as \( \bar{f} \circ f_0 \), where \( \bar{f} \) is an immersion and \( f_0 \) is \( \pi_1 \)-surjective. Since \( f \) is \( \pi_1 \)-injective, it follows that \( f_0 \) is in fact a homotopy equivalence and so, by Proposition 2.7, \( \Delta \) admits an essential origami for which \( f_0 \) is the quotient map. Thus, \( \Omega \) is indeed compatible with \( f \), as required.

3 Branched 2-complexes and their morphisms

Intuitively, a 2-dimensional cell complex, or 2-complex, is the result of gluing a set of discs to a graph along their boundaries. This is a fundamental notion in topology, but our setting requires a few unusual modifications of the standard definition, and the precise combinatorial model that we use will also be important in proving the rationality theorem. We therefore develop the definitions carefully here.

3.1 Branched 2-complexes

**Definition 3.1 (2-complex).** A 2-complex \( X \) consists of a pair of (Serre) graphs \( \Gamma = \Gamma_X \) and \( S = S_X \), together with an attaching map \( w = w_X : S_X \to \Gamma_X \) subject to two conditions:

(i) the realisation of \( S \) is a disjoint union of circles; and

(ii) the morphism \( w \) is required to be an immersion.

The graph \( \Gamma \) is the 1-skeleton of \( X \). The *vertices* of \( X \) are the vertices of \( \Gamma \), so we write \( V_X \) for \( V_\Gamma \), and likewise the *edges* of \( X \) are the edges of \( \Gamma \), so we write \( E_X \) for \( E_\Gamma \). The path components \( f \in \pi_0 S \) are called the *2-cells* or *faces* of \( X \), so we write \( F_X \) for \( \pi_0 S \). The *realisation* of \( X \), also denoted by \( X \), is obtained by coning off each face \( f \) to produce a disc \( D_f \), and then gluing the discs \( D_f \) to the 1-skeleton \( \Gamma \) along the map \( w \).

A *morphism* of 2-complexes \( \phi : Y \to X \) consists of a morphism of graphs \( \phi : \Gamma_Y \to \Gamma_X \) and a morphism of graphs \( \phi : S_Y \to S_X \), injective on path components, such that \( \phi \circ w_Y = w_X \circ \phi \). Note that a morphism induces a
continuous map of realisations (also denoted by \( \phi : Y \to X \)) which sends cells of \( Y \) homeomorphically to cells of \( X \).

**Remark 3.2.** The second hypothesis, that \( w_X \) should be an immersion, differentiates this definition from the standard one. However, note that it can always be achieved by modifying \( w_X \) by a homotopy, except in the trivial case where some face is attached along a loop that is null-homotopic in the 1-skeleton. After discarding such trivial faces, there is therefore no harm in imposing the second hypothesis.

For a wealth of finite examples, the reader should have in mind the standard presentation complex of any finite group presentation

\[
G = \langle a_1, \ldots, a_m \mid w_1, \ldots, w_n \rangle.
\]

The second hypothesis in the definition corresponds to assuming that each relator \( w_j \) is a non-trivial element of the free group on the generators.

In order to achieve our goal of ‘linearising’ the study of 2-complexes, it is fruitful to enrich them by equipping them with some extra information – a notion of area. This leads to the definition of a branched 2-complex. A branched 2-complex is just a 2-complex with a little extra structure, but some care is needed about the morphisms allowed between them.

**Definition 3.3** (Branched 2-complex). A branched 2-complex is a 2-complex \( X \) and an area function \( \text{Area} : F_X \to \mathbb{R}_{\geq 0} \). That is, the area function assigns a non-negative area \( \text{Area}(f) \) to each face \( f \) of \( X \).

Morphisms of branched complexes are called branched morphisms, to distinguish them from the standard morphisms of 2-complexes in Definition 3.1 and are defined as follows.

A branched morphism \( \phi : Y \to X \) consists of a morphism of graphs \( \phi : \Gamma_Y \to \Gamma_X \) and an immersion \( \phi : S_Y \to S_X \) such that \( \phi \circ w_Y = w_X \circ \phi \), and also subject to the following condition. For each face \( f \) of \( Y \), the immersion \( \phi \) necessarily defines a covering map of circles \( \phi_f : f \to \phi(f) \). The degree of this covering map is denoted by \( m_\phi(f) \) and is called the multiplicity of \( f \). The extra condition requires that

\[
\text{Area}(f) = m_\phi(f) \text{Area}(\phi(f))
\]

for each face \( f \) of \( Y \).

As in the case of unbranched 2-complexes, a branched morphism induces a continuous map of realisations. However, while this continuous map sends vertices to vertices and edges homeomorphically to edges, the induced maps between faces may no longer be homeomorphic. Rather, on the realisations
of faces $D_f \to D_{\phi(f)}$, the realisation of $\phi$ can be modelled by the polynomial map $z \mapsto z^{m_\phi(f)}$ on the unit disc in the complex plane. This justifies the terminology ‘branched complex’ and ‘branched morphism’.

Since branched 2-complexes are in particular 2-complexes, many definitions can be transported directly from 2-complexes to branched 2-complexes. For instance, the fundamental group of a branched 2-complex is the fundamental group of its underlying 2-complex. However, branched 2-complexes can also be thought of as a generalisation of 2-complexes, since we allow more morphisms between them.

**Remark 3.4.** A 2-complex $X$ admits a canonical structure as a branched complex, by setting $\text{Area}(f) = 1$ for each face $f$ of $X$. Such branched 2-complexes will sometimes be called *standard*. Note that branched morphisms between standard 2-complexes are exactly the usual morphisms specified in Definition 3.1.

Each vertex $v$ of a (branched) 2-complex $X$ has a *link* $\text{Lk}_X(v)$, a naturally defined graph whose realisation can be thought of as a small sphere around $v$. It can also be defined formally as follows.

**Definition 3.5** (Links in 2-complexes). Let $X$ be a 2-complex defined by an immersion of graphs $w : S \to \Gamma$ as above. Both vertices and edges of $X$ have links. The *link of an edge* $e$ of $X$ is the set

$$\text{Lk}_X(e) = w^{-1}(e).$$

The *link of a vertex* $v$ of $X$ is the graph $L = \text{Lk}_X(v)$, defined as follows. The vertex set of $L$ is $\text{Lk}_\Gamma(v)$, i.e. the set of edges of $\Gamma$ with initial vertex $v$, and the edge set of $L$ is exactly $w^{-1}(\text{Lk}_\Gamma(v))$, i.e. the edges of $S$ whose initial vertex maps to $v$. The terminus map $\tau_L$ of $L$ is defined to coincide with $w$ (so the terminal vertex of $e$ is $w(e)$) and so, to complete the definition of $L$, it remains to specify the orientation-reversing involution on edges, which is defined as follows: for an edge $e$, the opposite edge $e^*$ is the unique edge of $S$ not equal to $e$ such that $\iota_S(e^*) = \iota_S(e)$. (Note that $e^*$ exists and is unique precisely because $S$ is a disjoint union of circles.)

For each vertex $v$ of $X$, a branched morphism $\phi : X \to Y$ induces a morphism of graphs $d\phi_v : \text{Lk}_X(v) \to \text{Lk}_Y(\phi(v))$.

Links of vertices and edges come with some important extra structure.

**Remark 3.6.** The orientation-reversing involution on the edges of $S$ induces a canonical bijection $\text{Lk}_X(e) \to \text{Lk}_X(e^*)$, for any edge $e$ of $X$. Likewise, the origin map $\iota$ canonically identifies $\text{Lk}_X(e)$ with a set of edges in $\text{Lk}_X(\iota(e))$:
indeed, they are exactly the edges of $\text{Lk}_X(\iota(e))$ that adjoin (the vertex) $e$ in $\text{Lk}_X(\iota(e))$. Formally, this says that $\iota$ induces a bijection

$$\text{Lk}_X(e) \to \text{Lk}_{\text{Lk}_X(\iota(e))}(e)$$

where, confusingly, $e$ is thought of as a vertex of the graph $\text{Lk}_X(\iota(e))$. Equipped with the extra data of the orientation-reversing involutions on links of edges and the inclusions of links of edges into links of vertices, the entire complex $X$ can be reconstructed from the set of links of edges and vertices.

### 3.2 Essential maps

Just as Stallings’ folding lemma explains how to put maps of graphs into a useful form, so we may likewise fold maps of branched 2-complexes. Here, the role of immersions is played by branched immersions.

**Definition 3.7.** A branched morphism $\phi : Y \to X$ is a branched immersion if its realisation is locally injective away from the centres of 2-cells. Equivalently, the induced map on links $d\phi_v$ is injective for each vertex $v$ of $Y$. In the case where $X$ and $Y$ are standard, such a map is just called an immersion, and the realisation is locally injective everywhere.

The next result is the analogue of Stallings’ folding lemma (Lemma 1.1) for 2-complexes. Cousins of this result have already been used extensively (see [3, 4, 6, 5]), but we give a careful proof here for completeness.

**Lemma 3.8 (Folding 2-complexes).** Every branched morphism of finite, connected, branched 2-complexes $\phi : Y \to X$ factors as

$$Y \xrightarrow{\phi_0} \tilde{Y} \xrightarrow{\tilde{\phi}} X$$

where $\phi_0$ is a surjection on fundamental groups and $\tilde{\phi}$ is a branched immersion. Furthermore, $\tilde{Y}$ enjoys a universal property: whenever $\phi$ factors through a branched immersion $\psi : Z \to X$, $\tilde{\phi}$ also uniquely factors through $Z$.

The idea of the proof of Lemma 3.8 is straightforward: fold the map of 1-skeleta using Lemma 1.1, and then identify any 2-cells whose images coincide. The only subtlety is to make precise sense of the idea of 2-cells coinciding, especially in the broader setting of branched complexes. This can be done using fibre products of graphs, as described by Stallings [9, §1.3]. The next remark relates branched immersions of 2-complexes to fibre products.

**Remark 3.9.** A branched morphism of 2-complexes $\phi : Y \to X$ can be thought of as a commutative diagram
Recall that we always assume that $w_X$ and $w_Y$ are immersions of graphs. One can check that $\phi$ is a branched immersion if and only if:

(i) the map of 1-skeleta $\phi : \Gamma_Y \to \Gamma_X$ is an immersion of graphs; and

(ii) the canonical map to the fibre product

$$S_Y \to \Gamma_Y \times_{\Gamma_X} S_X$$

is an embedding.

With this remark in hand, the proof of the folding lemma becomes routine.

**Proof of Lemma 3.8** Applying Stallings’ folding to the map of 1-skeleta $\phi : \Gamma_Y \to \Gamma_X$, one obtains a factorisation of maps of graphs

$$\Gamma_Y \to \bar{\Gamma}_Y \to \Gamma_X$$

where $\Gamma_Y \to \bar{\Gamma}_Y$ is a $\pi_1$-surjection and $\bar{\Gamma}_Y \to \Gamma_X$ is an immersion. The natural map $S_Y \to S_X$ factors through the fibre product of graphs $S_X \times_{\Gamma_X} \bar{\Gamma}_Y$, and setting $S_Y$ to be the image of $S_Y$ in $S_X \times_{\Gamma_X} \bar{\Gamma}_Y$ defines the required complex $\bar{Y}$.

It is clear that $\phi_0$ is a $\pi_1$-surjection, because it is true on the 1-skeleta by the result for graphs. The map of 1-skeleta $\bar{\Gamma}_Y \to \Gamma_X$ is an immersion and $S_Y$ embeds in $\bar{\Gamma}_Y \times_{\Gamma_X} S_X$ by construction, so $\phi$ is a branched immersion by Remark 3.9.

It remains to prove the universal property. To that end, suppose that $Y \to X$ factors through a branched immersion $Z \to X$. By Stallings’ folding lemma for graphs, the map of 1-skeleta $\bar{\Gamma}_Y \to \Gamma_X$ factors uniquely through $\Gamma_Z \to \Gamma_X$. Since $Z \to X$ is a branched immersion, $S_Z$ can be identified with a subset of the fibre product $\Gamma_Z \times_{\Gamma_X} S_X$. The diagram

$$\begin{CD}
S_Y @>>> \bar{\Gamma}_Y @>>> \Gamma_Z \\
@VVV @VVV @VVV \\
S_X @>>> \Gamma_X
\end{CD}$$

commutes so, by the universal property of fibre products, there is a canonical map $S_Y \to \Gamma_Z \times_{\Gamma_X} S_X$. Since $S_Y \to S_Z \subseteq \Gamma_Z \times_{\Gamma_X} S_X$ factors through $S_Y$, the image of $S_Y$ in $\Gamma_Z \times_{\Gamma_X} S_X$ is contained in $S_Z$, which completes the proof.
Since branched immersion can be recognised locally, purely by looking at links, we would like to work with them whenever possible. However, we cannot restrict our attention solely to branched immersions, since we sometimes need to unfold in order analyse the structure of a 2-complex \(X\). Maps for which the folded representative is similar to the domain are therefore especially useful to work with. This motivates the definition of an essential map.

**Definition 3.10** (Essential maps). A branched morphism \(\phi : Y \to X\) is an essential equivalence if it satisfies the following two conditions:

(i) the map of 1-skeleta \(\phi : \Gamma_Y \to \Gamma_X\) is a homotopy equivalence;

(ii) the map \(\phi : S_Y \to S_X\) is an isomorphism.

More generally, a branched morphism \(\phi : Y \to X\) is essential if the folded map \(\phi_0 : Y \to \bar{Y}\) provided by Lemma 3.8 is an essential equivalence.

### 3.3 Irreducible 2-complexes

With the definition of an essential map in hand, we are ready to define the sets \(\text{Irred}(X)\) and \(\text{Surf}(X)\) mentioned in the introduction. We start with \(\text{Surf}(X)\), which is slightly easier to characterise.

**Definition 3.11.** A branched 2-complex \(Y\) is said to be a *surface* if its realisation is homeomorphic to a surface, or equivalently if \(\text{Lk}_Y(v)\) is a circle for every vertex \(v\). For any 2-branched complex \(X\), the set \(\text{Surf}(X)\) consists of all essential maps \(Y \to X\), where \(Y\) is a finite (but not necessarily connected) surface.

The set \(\text{Irred}(X)\) is slightly more complicated to define.

**Definition 3.12.** A branched 2-complex \(Y\) is said to be visibly irreducible if it satisfies all of the following conditions.

(i) The 1-skeleton \(\Gamma_Y\) is a core graph, meaning that it has no vertices of valence 0 or 1.

(ii) There are no free faces, i.e. every edge of \(\Gamma_Y\) has at least two preimages in \(S_Y\).

(iii) There are no local cut vertices, the punctured neighbourhood of each vertex is connected.
Note that each of these can be phrased in terms of the links: each link \( \text{Lk}_Y(v) \) should have at least two vertices and be connected, with no vertices of valence 1.

A branched 2-complex \( Y \) is said to be \textit{irreducible} if there is a visibly irreducible 2-complex \( Y' \) and an essential equivalence \( Y' \to Y \).

The set \( \text{Irred}(X) \) can then be characterised in the same way as \( \text{Surf}(X) \).

**Definition 3.13.** Let \( X \) be a branched 2-complex. The set \( \text{Irred}(X) \) consists of all essential maps \( Y \to X \), where \( Y \) is finite and visibly irreducible, but not necessarily connected.

### 3.4 \( \Pi \)-complexes and origamis

To handle the two definitions of \( \text{Surf}(X) \) and \( \text{Irred}(X) \) uniformly, we can make the following common generalisation.

**Definition 3.14.** A set \( \Pi \) of graphs is called \textit{suitable} if \( \Pi \) is closed under isomorphism and every graph in \( \Pi \) is finite, connected and contains at least one edge. For any suitable set of graphs \( \Pi \), a \( \Pi \)-\textit{complex} is a finite branched 2-complex \( Y \) such that the link of every vertex of \( Y \) is in \( \Pi \). The set \( \text{Ess}_\Pi(X) \) consists of all essential maps \( Y \to X \), where \( Y \) is a finite, but not necessarily connected, \( \Pi \)-complex.

Note that both \( \text{Surf}(X) \) and \( \text{Irred}(X) \) are instances of \( \text{Ess}_\Pi(X) \) for appropriate choices of suitable set \( \Pi \).

The idea of our main theorem, Theorem 5.7, is to encode the elements of \( \text{Ess}_\Pi(X) \) as the integer points of a suitable linear system. In fact, the linear system encodes not just \( \text{Ess}_\Pi(X) \), but elements of \( \text{Ess}_\Pi(X) \) equipped with origamis, which in turn certify that the relevant map is indeed essential. To use origamis to certify that a map of branched 2-complexes is essential, we use the following definition, which adapts the context of origamis from graphs to complexes.

**Definition 3.15.** Let \( Y \) be a branched 2-complex. An \textit{origami} \( \Omega \) on \( Y \) is an origami on the 1-skeleton \( \Gamma_Y \). The \textit{quotient branched 2-complex} \( Y/\Omega \) is now defined naturally as follows:

(i) the 1-skeleton is the quotient graph \( \Gamma_Y/\Omega \);

(ii) the faces are the same as the faces of \( Y \), attached using the composition \( S_Y \to \Gamma_Y \to \Gamma_Y/\Omega \).
There is a natural quotient map $q: Y \to Y/\Omega$. Since it is face-essential by construction, $q$ is essential if the origami $\Omega$ is essential, by Theorem \[\text{E}].

As in the case of graphs, we want to use origamis to certify that a given map is essential, using the notion of a compatible origami by extending Definition \[\text{2.11}].

**Definition 3.16.** Let $\phi: Y \to X$ be a morphism of branched 2-complexes. An origami $\Omega$ on $Y$ is said to be compatible with $\phi$ if $\phi$ factors through the quotient map $Y \to Y/\Omega$, and the factor map $Y/\Omega \to X$ is a branched immersion.

By Theorem \[E], a morphism is essential if and only if there is a compatible essential origami. Just as in the case of graphs, it is easy to certify that an origami is compatible with a given morphism.

**Remark 3.17.** A morphism of branched 2-complexes $\phi: Y \to X$ is compatible with an origami $\Omega$ if and only if items (i)-(iii) of Remark \[\text{2.12} \] are satisfied, together with the following additional condition:

(iv) if $u_1, u_2$ are vertices of $S_Y$ with $\phi(u_1) = \phi(u_2)$ and $w_Y(u_1), w_Y(u_2)$ are in the same component of $V_\Omega$ then $u_1 = u_2$.

Since we will use essential origamis to certify that our morphisms are essential, we next refine $\text{Ess}_\Pi(X)$ to include this extra information.

**Definition 3.18.** Fix a branched 2-complex $X$. The notation $\text{Orig}_\Pi(X)$ denotes the set of morphisms $\phi: Y \to X$, where $Y$ is a finite $\Pi$-complex equipped with an essential origami compatible with $\phi$. There is a natural notion of isomorphism on $\text{Orig}_\Pi(X)$: two morphisms $\phi_1: Y_1 \to X$ and $\phi_2: Y_2 \to X$, equipped with compatible essential origamis, are isomorphic if there is an isomorphism of 2-complexes $\theta: Y_1 \to Y_2$ such that $\phi_1 = \phi_2 \circ \theta$ and such that $e \sim_\Omega e'$ if and only if $\theta(e) \sim_\Omega \theta(e')$, for each pair of edges $e, e'$ of $Y_1$. We will always consider the elements of $\text{Orig}_\Pi(X)$ up to this notion of isomorphism.

The relationship between $\text{Orig}_\Pi(X)$ and $\text{Ess}_\Pi(X)$ can be conveniently summarised in the following proposition.

**Proposition 3.19.** The map $\text{Orig}_\Pi(X) \to \text{Ess}_\Pi(X)$ given by forgetting the origami on $Y$ is surjective and finite-to-one.

**Proof.** The forgetful map is well defined and surjective by Theorem \[\text{E} \]. It is finite-to-one because there are only finitely many origamis on any finite graph. \[\square\]
4 A linear system

Fix a finite branched 2-complex $X$. Our goal is now to define a rational cone $C(\mathbb{R})$ in some real vector space $\mathbb{R}^N$ such that the integer points $C(\mathbb{Z}) = C(\mathbb{R}) \cap \mathbb{Z}^N$ correspond to the elements of $\text{Orig}_{\Pi}(X)$. The construction can be summarised in the following theorem.

**Theorem 4.1.** Let $X$ be a finite branched 2-complex and $\Pi$ a set of connected finite graphs, each with at least one edge. There is an explicitly defined rational cone $C(\mathbb{R})$ and a surjection

$$\Phi : \text{Orig}_{\Pi}(X) \to C(\mathbb{Z})$$

such that the preimage of any vector in $C(\mathbb{Z})$ is a finite set of isomorphism classes.

We will usually abuse notation and let $Y$ denote an element of $\text{Orig}_{\Pi}(X)$, i.e. a morphism $Y \to X$ equipped with an essential origami $\Omega$.

The rest of this section is devoted to the construction of the cone $C(\mathbb{R})$ and the proof of Theorem 4.1.

4.1 Vertex and edge blocks

The idea is to decompose an element of $\text{Orig}_{\Pi}(X)$ into vertex blocks, of which there should only be finitely many combinatorial types. The map $\Phi$ then counts the number of vertex blocks. We next give the definition of a vertex block, which is extremely involved. Intuitively, a vertex block encodes the vertex-space of the graph of graphs associated to an origami. The construction is phrased in terms of the link graphs $\text{Lk}_X(x)$. Since the path components of a link correspond to vertices of the complex, the $\pi_0$ functor will play a role, and we will use the notation $\pi_0(v)$ to denote the path component of a vertex $v$.

**Definition 4.2** (Vertex block). A vertex block over $X$, denoted by $\beta$, consists of the following data:

(a) a vertex $x_\beta$ of $X$;

(b) two finite graphs $L(\beta)$ and $\bar{L}(\beta)$, together with morphisms $L(\beta) \to \bar{L}(\beta) \to \text{Lk}_X(x_\beta)$; and

(c) equivalence relations $\sim_O$ and $\sim_C$ on the vertices of $L(\beta)$.
The equivalence relations $\sim_O$ and $\sim_C$ together define two graphs – a vertex graph $V_\beta$ and an edge graph $E_\beta$ – as in the definition of an origami.

The vertices of the edge graph $E_\beta$ are of two kinds: the $\sim_O$-equivalence classes $[v]_O$ and the $\sim_C$-equivalence classes $[v]_C$; the edges of $E_\beta$ are the vertices of $L(\beta)$, and each edge $v$ joins $[v]_O$ to $[v]_C$.

The vertex graph $V_\beta$ also has two kinds of vertices: the path components of $L(\beta)$ and the $\sim_C$-equivalence classes $[v]_C$. The edges are again the vertices of $L(\beta)$, with an edge $v$ joining a path component $\pi_0(v)$ to the equivalence class $[v]_C$.

The data of the vertex block is then subject to the following conditions:

(i) $L(\beta)$ is a disjoint union of graphs in $\Pi$;

(ii) $L(\beta) \to \tilde{L}(\beta)$ is bijective on edges;

(iii) $\tilde{L}(\beta) \to \text{Lk}_X(x_\beta)$ is injective;

(iv) the vertex graph $V_\beta$ is a tree and the edge graph $E_\beta$ is a forest;

(v) if three vertices $v_1, v_2, v$ of $L(\beta)$ satisfy $v_1, v_2 \in [v]_O$ then $v$ does not separate $\pi_0(v_1)$ from $\pi_0(v_2)$ in $V_\beta$;

(vi) if two vertices $v_1$ and $v_2$ of $L(\beta)$ have the same image in $\text{Lk}_X(x_\beta)$ then they are in the same path component of $E_\beta$.

Let $V = V(X)$ be the set of all vertex blocks up to the natural combinatorial notion of isomorphism. Although the definition of $V$ is extremely complicated, it is not hard to see that $V$ is finite, as long as $X$ is also finite.

Remark 4.3. If $X$ is a finite branched 2-complex and graphs in $\Pi$ have no isolated vertices then $V(X)$ is finite. Indeed, since $X$ is finite there are only finitely many links $\text{Lk}_X(x_\beta)$. Since $\tilde{L}(\beta)$ injects into $\text{Lk}_X(x_\beta)$, it follows that there are only finitely many combinatorial types of maps $\tilde{L}(\beta) \to \text{Lk}_X(x_\beta)$.

Being in $\Pi$, the graph $L(\beta)$ has no isolated vertices, so the number of vertices is at most twice the number of edges, which in turn is bounded since $L(\beta) \to \tilde{L}(\beta)$ is bijective on edges. Therefore $L(\beta)$ is of bounded size, and so there are only finitely many maps $L(\beta) \to \tilde{L}(\beta)$ up to combinatorial equivalence. Finally, because $L(\beta)$ is a finite graph, there are only finitely many equivalence relations $\sim_O$ and $\sim_C$ on the vertices of $L(\beta)$.

The vertex blocks in $V$ are the variables of our linear-programming problem. To this end, we will work in the finite-dimensional vector space $\mathbb{R}^V$, in which a typical vector is denoted $(t_\beta) = (t_\beta)_{\beta \in V}$. The idea is that an essential map $Y \to X$ (equipped with an origami) is built out of blocks, and the vector $(t_\beta)$ counts the number of such blocks of isomorphism type $\beta$. This idea can be made precise by the notion of an induced vertex block.
Definition 4.4. Let $Y$ be a $\Pi$-complex, let $\phi : Y \to X$ be a morphism and let $\Omega$ be an essential origami on $Y$, compatible with $\phi$. That is, the map $\phi$ factors as

$$Y \xrightarrow{\phi_0} \bar{Y} \xrightarrow{\bar{\phi}} X$$

where $\bar{Y} = Y/\Omega$ and $\phi_0$ is the quotient map. For any vertex $\bar{u}$ of $\bar{Y}$, the induced vertex block $\beta = \beta(\bar{u})$ is defined as follows:

(a) $x_\beta := \bar{\phi}(\bar{u})$;

(b) $\bar{L}(\beta) := \text{Lk}_Y(\bar{u})$, while $L(\beta) := \bigsqcup_{u \in \phi^{-1}(\bar{u})} \text{Lk}_Y(u)$, and the map $L(\beta) \to \bar{L}(\beta)$ is the coproduct of the induced maps $(d\phi_0)_u$ on links;

(c) the equivalence relations $\sim_O$ and $\sim_C$ on the vertices of $L(\beta)$ are restricted from $\Omega$ in the natural way.

We need to check that the conditions of Definition 4.2 are satisfied:

(i) the fact that $Y$ is a $\Pi$-complex implies that every path component of $L(\beta)$ is in $\Pi$;

(ii) $L(\beta) \to \bar{L}(\beta)$ is bijective on edges by the definition of the quotient complex $Y/\Omega$;

(iii) $\bar{\phi}$ is a branched immersion, and so the induced map on links $d\bar{\phi}_u : L(\beta) \to \text{Lk}_X(x_\beta)$ is injective;

(iv) the vertex graph $V_\beta$ is a component of $V_\Omega$ and the edge graph $E_\beta$ is a union of components of $E_\Omega$, so $V_\beta$ is a tree and $E_\beta$ is a forest because $\Omega$ is essential;

(v) the local consistency of $\Omega$ is equivalent to item (v) of Definition 4.2;

(vi) the final condition, that two vertices $v_1$ and $v_2$ of $L(\beta)$ with the same image in $\text{Lk}_X(x_\beta)$ are in the same path component of $E_\beta$, follows immediately from the fact that $\Omega$ is compatible with $\phi$, by item (iii) of Remark 2.12.

The equations that cut out the subspace $V$ from the vector space $\mathbb{R}^V$ make use of edge blocks.

Definition 4.5 (Edge block). An edge block over $X$, denoted by $\gamma$, consists of the following data:

(a) an edge $e_\gamma$ of $X$;
(b) a (possibly empty) finite subset $L(\gamma)$ of $\text{Lk}_X(e_{\gamma})$;

(c) a partition $V(\gamma)$ of $L(\gamma)$; and

(d) a pair of equivalence relations $\sim_O$ and $\sim_C$ on $V(\gamma)$.

The data of an edge block would enable us to define an edge graph, analogously to the edge graphs that appear in origamis and vertex blocks, and we could impose the condition that this edge graph should be a tree. However, we will not need to do that here. Let $E \equiv E(X)$ denote the set of edge blocks up to the natural notion of combinatorial isomorphism. Similarly to the case of vertex blocks, it is not hard to see that $E$ is finite when $X$ is, although again we will not need that here.

Edge blocks encode the part of a vertex block that lies above an edge. The next definition makes this precise.

**Definition 4.6.** For each vertex block $\beta \in \mathcal{V}$ and each edge $e$ such that $\iota(e) = x_{\beta}$, the *induced edge block* $\gamma_{\beta}(e)$ is defined in the following way.

(a) The edge $e_{\gamma_{\beta}(e)}$ is defined to be $e$.

(b) The set $L(\gamma_{\beta}(e))$ is the intersection of the image of $L(\beta)$ with $\text{Lk}_X(x_{\beta})$.

(c) The partition $V(\gamma_{\beta}(e))$ on $L(\gamma_{\beta}(e))$ is induced by the links of vertices in $L(\beta)$, so each element is of the form $L(\gamma_{\beta}(e)) \cap \text{Lk}_{L(\beta)}(v)$, for $v$ a unique vertex of $L(\beta)$.

(d) The equivalence relation $\sim_O$ and $\sim_C$ on $V(\gamma_{\beta}(e))$ are defined as follows. The open equivalence relation is now pulled back from $\beta$, so

\[ L(\gamma_{\beta}(e))) \cap \text{Lk}_{L(\beta)}(v_1) \sim_O L(\gamma_{\beta}(e))) \cap \text{Lk}_{L(\beta)}(v_2) \]

if and only if $v_1 \sim_O v_2$ in $L(\beta)$. The closed equivalence relation is pulled back in the same way, so

\[ L(\gamma_{\beta}(e))) \cap \text{Lk}_{L(\beta)}(v_1) \sim_C L(\gamma_{\beta}(e))) \cap \text{Lk}_{L(\beta)}(v_2) \]

if and only if $v_1 \sim_C v_2$ in $L(\beta)$.

Finally, to define the cone $C(\mathbb{R})$, we use the fact that the orientation-reversing involution on edges extends to edge blocks defined over them.

**Definition 4.7.** Let $\gamma \in \mathcal{E}$ be an edge block. The *opposite edge block* $\gamma^*$ is defined as follows:
(a) $e_{\gamma^*} := e_{\gamma}^*$;

(b) $L(\gamma^*)$ is the image of $L(\gamma)$ under the canonical orientation-reversing bijection $\text{Lk}_X(e) \to \text{Lk}_X(e^*)$;

(c) $V(\gamma^*)$ is the induced image of $V(\gamma)$ under the orientation-reversing bijection $L(\gamma^*) \to L(\gamma)$, and indeed we write $v^*$ for the induced image of a subset $v$ of $L(\gamma)$.

(d) finally, the roles of $\sim_O$ and $\sim_C$ are reversed, so $u^* \sim_O v^*$ in $V(\gamma^*)$ if and only if $u \sim_C v$ in $V(\gamma)$, and likewise $u^* \sim_C v^*$ in $V(\gamma^*)$ if and only if $u \sim_O v$ in $V(\gamma)$.

4.2 The rational cone

The rational cone $C(\mathbb{R})$ can be constructed in a way that is now fairly standard. We work in the real vector spaces $\mathbb{R}^V$. For each edge block $\gamma_0$ over an edge $e_0$ of $X$, the corresponding gluing equation relates vectors $(s_\beta), (t_\beta) \in \mathbb{R}^V$ by insisting that

$$\sum_{\gamma_\beta(e_0)=b_0} s_\beta = \sum_{\gamma_\beta(e_0^*)=b_0^*} t_\beta,$$

and $C(\mathbb{R})$ is the set of vectors in the positive orthant $\mathbb{R}_{\geq 0}^V$ that satisfy all the gluing equations. Note that the gluing equations are linear with integer coefficients, so $C(\mathbb{R})$ is indeed a rational cone.

The map $\Phi$ from $\text{Orig}_\Pi(X)$ to the set of integer points $C(\mathbb{Z})$ can be easily defined using induced vertex blocks. A typical element of $\text{Orig}_\Pi(X)$ consists of a morphism $\phi : Y \to X$ and a compatible essential origami $\Omega$. Now $\Phi$ counts the number of vertices of the folded representative $\bar{Y}$ for which the induced vertex block is of each isomorphism type. More precisely, $\Phi$ sends the given element to the vector $(t_\beta)$, where

$$t_\beta = \# \{ \bar{u} \in \bar{Y} \mid \beta(\bar{u}) = \beta \}.$$

The proof of Theorem 4.1 is now tedious but routine.

**Proof of Theorem 4.1.** We need to check that the image $\Phi$ satisfies the gluing equations, that $\Phi$ is surjective, and that the each vector has only finitely many isomorphism types in its preimage under $\Phi$.

To see that $\Phi$ satisfies the gluing equations, consider an edge $e$ of $\bar{Y}$, let $\bar{u} = \iota(e)$ and let $\bar{v} = \tau(e)$. By construction, the opposite block of $\gamma_{\beta(\bar{u})}(e)$ is precisely $\gamma_{\beta(\bar{v})}(e^*)$. Thus, there is a bijection between the set of vertices of $\bar{Y}$ whose vertex block induces a given edge block $\gamma$ and the set of vertices of $\bar{Y}$...
whose vertex block induces the opposite edge block $\gamma^*$. From this, it follows that the image of $\Phi$ satisfies the gluing equations.

To see that $\Phi$ is surjective, consider an arbitrary vector $(t_\beta)$ in $C(\mathbb{Z})$; we need to construct a morphism $\phi: Y \to X$ and a compatible essential origami $\Omega$ that $\Phi$ sends to $(t_\beta)$. To this end, take $t_\beta$ copies of each vertex block $\beta \in \mathcal{V}$. Since $(t_\beta)$ satisfies the gluing equations, for each pair of edges $\{e, e^*\}$ of $X$ and for each edge block $\gamma$ over $e$, we may choose a bijection between the vertex blocks inducing $\gamma$ over $e$ and the vertex blocks inducing $\gamma^*$ over $e^*$. For each such pair of vertex blocks $\beta$ and $\beta'$, choose an isomorphism between $\gamma_\beta(e)$ and $\gamma_{\beta'}(e^*)$. This provides the data to construct $Y$ uniquely, as follows: the disjoint union of the links of vertices in $Y$ is the union of the graphs $L(\beta)$ across our chosen set of vertex blocks, and the chosen isomorphisms provide bijections between the link of each edge of $Y$ and the link of its opposite. By Remark 3.6, this determines $Y$. The data of the blocks determines $\phi: Y \to X$, and the equivalence relation $\sim_O$ on the vertex blocks defines an origami $\Omega$ compatible with $\phi$. Note that $\Omega$ is essential by construction. This completes the proof that $\Phi$ is surjective.

Finally, preimages under $\Phi$ consist of finitely many isomorphism types because, in the above construction, the isomorphism type was determined by the finitely many choices of isomorphisms between between $\gamma_\beta(e)$ and $\gamma_{\beta'}(e^*)$.

5 Curvature invariants

In the next section we give the true definitions of our curvature invariants. In the subsequent section, we observe that these can be expressed in terms of linear functions on the cone $C(\mathbb{R})$.

5.1 Curvature invariants

We start with a natural extension of the notion of Euler characteristic to the context of branched complexes.

Definition 5.1 (Total and average curvatures). The area of a branched 2-complex $X$ is the sum of the areas of its faces. The total curvature of a 2-complex $X$ is the quantity

$$\tau(X) := \text{Area}(X) + \chi(\Gamma_X)$$

where $\chi(\Gamma_X)$ denotes the usual Euler characteristic of the 1-skeleton of $X$, i.e the the number of vertices minus the number of edges.
The \textit{average curvature} of $X$ is the quantity

$$\kappa(X) := \frac{\tau(X)}{\text{Area}(X)}.$$ 

In particular, if $X$ is standard then $\tau(X)$ is the usual Euler characteristic. The definition of $\kappa(X)$ is motivated by the Gauss–Bonnet theorem, which implies that for a Riemannian metric on a closed surface $S$, the average Gaussian curvature is $2\pi \chi(S)/\text{Area}(S)$.

As explained in the introduction, the invariants $\rho_\pm(X)$ are defined by extremising $\kappa$ over $\text{Irred}(X)$.

\textbf{Definition 5.2} (Irreducible curvature bounds). For any branched 2-complex, 

$$\rho_+(X) := \sup_{Y \in \text{Irred}(X)} \kappa(Y),$$

and 

$$\rho_-(X) := \inf_{Y \in \text{Irred}(X)} \kappa(Y).$$

The invariants $\sigma_\pm(X)$ are defined similarly, with $\text{Surf}(X)$ replacing $\text{Irred}(X)$.

\textbf{Definition 5.3} (Surface curvature bounds). For any branched 2-complex, 

$$\sigma_+(X) := \sup_{Y \in \text{Surf}(X)} \kappa(Y),$$

and 

$$\sigma_-(X) := \inf_{Y \in \text{Surf}(X)} \kappa(Y).$$

\section{Linear functions}

Throughout this section, $X$ is a finite branched 2-complex, $\Pi$ is a suitable collection of graphs, and $C(\mathbb{R})$ is the cone provided by Theorem 4.1. The final ingredient of the main theorem is the observation that both the total function and the area are linear functions on $C(\mathbb{R})$. We start with area.

\textbf{Lemma 5.4.} There is a linear function $\text{Area}$ on $C(\mathbb{R})$ such that 

$$\text{Area} \circ \Phi(Y) = \text{Area}(Y)$$

for any $Y \in \text{Orig}_\Pi(X)$. 

33
Proof. For each face $f$ of $X$, let $l(f)$ be the number of oriented edges in $f$. (This is exactly twice the length of $f$ thought of as a cyclic graph, since each edge appears with both orientations.) Each edge $e$ of $L(\beta)$, where $\beta$ is a vertex block, is equipped with a map to $\text{Lk}_X(x_\beta)$, sending $e$ into some face $f(e)$ of $X$. Define the linear map $\text{Area}$ on the basis vector $e_\beta$ corresponding to $\beta$ by

$$\text{Area}(e_\beta) = \sum_{e \in E_{L(\beta)}} \text{Area}(f(e))/l(f(e)).$$

Now suppose that $\Phi(Y) = (t_\beta)$. Then

$$\text{Area} \circ \Phi(Y) = \sum_{\beta \in \mathcal{V}} t_\beta \sum_{e \in E_{L(\beta)}} \text{Area}(f(e))/l(f(e)).$$

By construction, the edge set of $S_Y$ consists of exactly $t_\beta$ copies of each edge of $L(\beta)$, and so this sum can be rearranged as

$$\text{Area} \circ \Phi(Y) = \sum_{e \in E_{S_Y}} \text{Area}(f(e))/l(f(e)).$$

For each face $f$ of $X$, the number of edges of $S_Y$ mapping into $f$ is exactly $\sum_{\phi(f')=f} m_\phi(f')l(f)$. Therefore, this sum becomes

$$\text{Area} \circ \Phi(Y) = \sum_{f \in F_X} \sum_{\phi(f')=f} m_\phi(f')l(f).\text{Area}(f)/l(f)$$

$$= \sum_{f \in F_X} \sum_{\phi(f')=f} m_\phi(f').\text{Area}(f)$$

$$= \sum_{f' \in F_Y} \text{Area}(f')$$

$$= \text{Area}(Y)$$

as claimed. \qed

The next lemma makes a similar observation for the Euler characteristic of the 1-skeleton $\Gamma_Y$.

**Lemma 5.5.** There is a linear function $\chi$ on $C(\mathbb{R})$ such that

$$\chi \circ \Phi(Y) = \chi(\Gamma_Y)$$

for any $Y \in \text{Orig}_{\Pi}(X)$.

**Proof.** For $\beta \in \mathcal{V}$ and $\delta_\beta$ the corresponding basis vector of $C(\mathbb{R})$, let

$$\chi(\delta_\beta) = \#\pi_0(L(\beta)) - \#V_{L(\beta)}/2.$$
Again writing \( \Phi(Y) = (t_\beta) \), the links of \( Y \) consist of \( t_\beta \) copies of the graphs \( L(\beta) \), with the path components of \( L(\beta) \) corresponding to the vertices of \( Y \) (since \( \Pi \) is suitable) and the vertices of \( L(\beta) \) corresponding to the edges of \( Y \). With the above definition of the function \( \chi \), the desired equation

\[
\chi \circ \Phi(Y) = \#V_Y - \#E_Y / 2 = \chi(\Gamma_Y)
\]

now follows immediately.

The corresponding statement for the total curvature \( \tau(Y) \) follows immediately.

**Lemma 5.6.** There is a linear function \( \tau \) on \( C(\mathbb{R}) \) such that

\[
\tau \circ \Phi(Y) = \tau(Y)
\]

for any \( Y \in \text{Orig}_\Pi(X) \).

*Proof.* Set \( \tau = \text{Area} + \chi \).

With these facts in hand, we can prove the most general form of the rationality theorem.

**Theorem 5.7.** Let \( X \) be a finite branched 2-complex and let \( \Pi \) be any suitable set of graphs. If \( \text{Ess}_\Pi(X) \) is non-empty then:

(i) the supremum and infimum of \( \kappa(Y) \) over all \( Y \in \text{Ess}_\Pi(X) \) are the extrema of an explicit linear-programming problem; furthermore,

(ii) the supremum is attained by some \( Y_{\text{max}} \in \text{Ess}_\Pi(X) \) and the infimum is attained by some \( Y_{\text{min}} \in \text{Ess}_\Pi(X) \).

*Proof.* We give the proof for the supremum, since the proof for the infimum is identical.

Since \( \kappa \) does not depend on any choice of origami, the supremum is also the supremum of \( \kappa(Y) \) over all \( Y \in \text{Orig}_\Pi(X) \), by Proposition 3.19. By Lemmas 5.4 and 5.6, the projective function

\[
\kappa = \tau / \text{Area}
\]

on \( C(\mathbb{R}) \setminus 0 \) is such that \( \kappa(Y) = \kappa \circ \Phi(Y) \) for each \( Y \in \text{Orig}_\Pi(X) \). On the compact, rational polytope

\[
P = C(\mathbb{R}) \cap \{\text{Area}(t) = 1\}
\]

35
the simplex algorithm implies that the supremum of $\kappa$ is achieved at a vertex of $P$, equal to a rational vector $t_{\text{max}}$. Thus, by construction, $\kappa(t_{\text{max}})$ is the desired supremum, which proves item (i). Since $t_{\text{max}}$ is rational and $\kappa$ is projective, we may replace $t_{\text{max}}$ by a multiple so that $t_{\text{max}} \in C(\mathbb{Z})$. By the surjectivity guaranteed by Theorem 4.1, there is $Y_{\text{max}} \in \text{Orig}_\Pi(X)$ such that $\Phi(Y_{\text{max}}) = t_{\text{max}}$, so the supremum is attained, proving item (ii) for the supremum.

Since $\text{Irred}(X)$ and $\text{Surf}(X)$ are instances of $\text{Ess}_\Pi(X)$, Theorems A and C follow immediately.

**References**

[1] Danny Calegari. Stable commutator length is rational in free groups. *Jour. Amer. Math. Soc.*, 22(4):941–961, 2009.

[2] Joseph Helfer and Daniel T. Wise. Counting cycles in labeled graphs: the nonpositive immersion property for one-relator groups. *Int. Math. Res. Not. IMRN*, 9:2813–2827, 2016.

[3] Larsen Louder and Henry Wilton. Stackings and the $W$-cycles conjecture. *Canad. Math. Bull.*, 60(3):604–612, 2017.

[4] Larsen Louder and Henry Wilton. One-relator groups with torsion are coherent. *Math. Res. Lett.*, 27(5):1499–1512, 2020.

[5] Larsen Louder and Henry Wilton. Uniform negative immersions and the coherence of one-relator groups. *arXiv:2107.08911*, 2021.

[6] Larsen Louder and Henry Wilton. Negative immersions for one-relator groups. *Duke Math. J.*, 171(3):547–594, 2022.

[7] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[8] Jean-Pierre Serre. *Arbres, amalgames, $SL_2$*, volume 46 of *Astérisque*. Société Mathématique de France, Paris, 1977.

[9] John R. Stallings. Topology of finite graphs. *Inventiones Mathematicae*, 71(3):551–565, 1983.
[10] Henry Wilton. One-ended subgroups of graphs of free groups. *Geom. Topol.*, 16:665–683, 2012.

[11] Henry Wilton. Essential surfaces in graph pairs. *J. Amer. Math. Soc.*, 31(4):893–919, 2018.

[12] Henry Wilton. Rational curvature invariants for 2-complexes. *arXiv:2210.09853*, 2022.

[13] Daniel T. Wise. Coherence, local indicability and nonpositive immersions. *J. Inst. Math. Jussieu*, 21(2):659–674, 2022.

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK
E-mail address: h.wilton@maths.cam.ac.uk