The Stability and Non-linear Vibration Analysis of a Circular Clamped Microplate under Electrostatic Actuation

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ABSTRACT
A study was made of the dynamic response of a circular clamped micro-plate actuated by a DC/AC voltage. The analysis considers not only the non-linear electrostatic coupling force, residual stress effect, and hydrostatic pressure acting on the upper surface, but also the squeeze-film damping effect generated by the air gap between the vibrating micro-plate and the fixed substrate. The non-linear governing equation of motion of the circular micro-plate was solved using a hybrid numerical method comprising the differential transformation and the finite difference. It was shown that the numerical results obtained for the pure DC pull-in voltage deviate by no more than 1.6% from the results presented in the literature. The effects of the actuating voltage, hydrostatic pressure, squeeze-film damping, and residual stress on the dynamic response of the clamped circular micro-plate were systematically examined. In addition, the stability of the vibrating micro-plate was investigated by reference to phase portraits.

1. Introduction
Micro-electro-mechanical systems (MEMS) devices such as micro-pumps [1], pressure sensors, and mass sensors [2,3] have found extensive use in recent years in such fields as biotechnology, image processing, as automotive components, in chemical and food processing, and so on. Such devices are essentially simple capacitors composed of two parallel micro-plates separated by a small air gap. The micro-plates typically have a square, circular or beam type of configuration, and are designed so that one of the plates (usually the upper) deforms under the application of an external electrical force, while the other (usually the lower) remains static. To optimize the performance of such devices, a detailed understanding of the basic mechanical and electrical phenomena associated with the stability and dynamic characteristics of micro-actuator systems is required.

The attractive force between the deformable electrode and the fixed electrode is generated by the application of a DC voltage, AC voltage, or a combination of the two. The electrostatic force prompts the deformable electrode to deflect toward the fixed electrode by a distance proportional to the intensity of the applied voltage. At a
certain critical value of the voltage, the attractive force will exceed the elastic restoring force, and the upper electrode collapses and makes transient contact with the lower electrode. This phenomenon is conventionally referred to as the ‘pull-in phenomenon’, with the corresponding voltage designated as the ‘pull-in voltage’ [4–6]. In practice, the pull-in voltage is sensitive to such factors as the residual stress in the upper electrode, the nonlinearity of the electrostatic coupling force, the squeeze-film damping effect, and so on. The inertia force has no effect on the deformation behavior of the upper electrode if the increase in the actuating voltage is slow; the critical voltage is thus referred to as the static pull-in voltage ($V_{pi}^{s}$). However, in most practical applications, the actuating voltage (AC, DC, or both) increases rapidly, and thus the electrode behavior depends fundamentally on the effects of inertia, damping-related forces, and so on. Under these conditions, the critical actuating voltage is referred to as the dynamic pull-in voltage ($V_{pi}^{d}$) [7]. Younis et al. [8] presented a continuous reduced-order model for predicting the static pull-in voltage and the position of electrically actuated MEMS micro-beams. Vogl and Nayfeh [9] presented an analytical continuous reduced-order model for describing the effects of geometric nonlinearities and residual stress on the dynamic behavior of a clamped, electrically actuated, circular plate. Wang et al. [10] investigated the pull-in dynamics and vibration properties of an electrostatically actuated circular micro-plate in the presence of residual stress and Casimir forces.

To optimize the design of MEMS-based actuator devices, the squeeze-film damping effect between the upper and lower electrodes must be taken into account [4,11,12]. This damping effect is generally modeled using the Reynolds equation, which is derived from the Navier–Stokes equations and the continuity equation. In modeling the damping effect, it is assumed that the gas (usually air) in the gap can be treated as a continuum. The validity of this assumption depends on the so-called Knudsen number ($K_n$), which is defined as the ratio of the mean free path of the air particles to the film thickness. Based on the Knudsen number, air flows can be divided into four regimes [4], namely continuum flow ($K_n < 0.01$), slip flow ($0.01 < K_n < 0.1$), transitional flow ($0.1 < K_n < 10$), and free molecular flow ($K_n > 10$). Krylov and Maimon [11] investigated the transient dynamics of an electrically actuated micro-beam subject to the effects of electrostatic forces, squeeze-film damping, and the rotational inertia of a mass carried by the micro-beam. Younis [12] used a hybrid numerical analysis technique to perform multi-physics simulations of MEMs-based actuation systems. Liu et al. [13] demonstrated that the hybrid method represented a computationally efficient and precise means of obtaining the nonlinear dynamic behavior of a micro circular plate. They also utilized an analytical model based on a non-linear deflection and Reynolds equations to examine the dynamic response of an electrically actuated clamped–clamped micro-beam with piezoelectric layers and a squeeze-film damping effect [14].

In general, circular micro-plates yield a better structural flexibility than rectangular plates since they contain no corners or sharp edges to increase the residual stress following multiple depositions [15]. As a result, they are an ideal solution for such devices as MEMS-based microphones, pressure sensors, and so on. However, in predicting the performance of MEMS-based actuators, the effects of the residual stress must be properly quantified [16]. Consequently, the origins and effects of residual stress have emerged as important concerns in the microsystems development field [17,18].

However, a review of the literature revealed that most previous studies involving analysis of the dynamic behavior of MEMS-based actuator systems neglected both the effects of hydrostatic pressure and residual stress. The present study addresses this deficiency with an analytical model that describes the nonlinear dynamic response of an electrostatically actuated clamped circular micro-plate subject to squeeze-film damping, hydrostatic pressure, and residual stress effects. The governing equation of motion of the micro-plate was solved using a hybrid numerical scheme comprising the differential transformation method and the finite difference method. The validity of the proposed model was demonstrated by comparing the results obtained for the pure DC pull-in voltage of the plate with those reported in the literature. Simulations were then performed to investigate the effects of squeeze-film damping, hydrostatic pressure, and residual stress on the dynamic response of the micro-plate under various AC/DC actuation conditions.

2. The Mathematical Model

2.1. Model Description

Figure 1 presents a schematic illustration of the circular clamped micro-plate considered in this study. As shown, the device consists of a deformable upper membrane made of a linearly elastic material and a stationary lower electrode. The membrane is actuated via an electric driving force with the form $V(t) = V_{DC} + V_{AC} \sin(\omega t)$, where $V_{DC}$ is the DC polarization voltage, $V_{AC}$ is the amplitude of the applied AC voltage, and $\omega$ is the excitation frequency. Taking account of the residual stress, hydrostatic pressure, and squeeze-film damping effect, the governing equation of motion of the actuated micro-plate has the form [14]
where \( \varepsilon_0 \), \( h \), and \( g \) are the permittivity of free space, the thickness of the circular micro-plate, and the initial gap height between the upper and lower plates, respectively. In addition, \( \rho \) is the density of the circular micro-plate material, and \( w \) is the transverse deflection of the micro-plate at a distance \( r \) from the center. Notably, \( w \) is assumed to vary as a function only of the position \( r \) and time \( t \), so that \( w = w(r, t) \). In other words, the deflection of the micro-plate is independent of the polar coordinate \( \theta \). Finally, \( T_r \) is the residual stress within the plate, \( h_d \) is the hydrostatic pressure, and \( P_n \) is the net pressure. The flexural rigidity of the plate, \( D \), can be formulated as

\[
D = \frac{E h^3}{12 (1 - \nu^2)} \tag{2}
\]

where \( E \) and \( \nu \) are the Young’s modulus and Poisson ratio of the plate. Finally, \( \nabla^4 \) is the biharmonic operator and is given by \( \nabla^4 w = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \). The boundary conditions for the governing equation of motion of the circular micro-plate are defined as follows:

\[
w(r, t) = \frac{\partial w(r, t)}{\partial r} = 0 \quad \text{at} \quad r = \pm R, \tag{3}
\]

where \( R \) is the radius of the plate.

The initial condition is defined as

\[
w(r, 0) = \frac{\partial w(r, 0)}{\partial t} = 0. \tag{4}
\]

The squeeze-film pressure acting on the circular plate is assumed to be symmetrically distributed and is modeled using the following linearized gas-film Reynolds equation [13]:

\[
\frac{\partial^2 P_n}{\partial r^2} + \frac{1}{r} \frac{\partial P_n}{\partial r} = \frac{12 \mu_{\text{eff}}}{P_g g} \left( g \frac{\partial P_n}{\partial t} + P_n \frac{\partial H}{\partial t} \right), \tag{5}
\]

where \( H \) and \( \mu \) represent the variable distance between the two electrodes \( (H = g - w) \) and the effective viscosity \( (\mu_{\text{eff}}) \) of the air in the air gap, respectively. The effective viscosity, \( \mu_{\text{eff}} \), is defined as \( \mu_{\text{eff}} = \mu_0 / (1 + 6 K_n) \), where \( \mu_0 \) is the absolute air viscosity and \( K_n \) is the Knudsen number [5]. The Knudsen number is expressed as \( K_n = \lambda / H \), where \( \lambda \) is the molecular mean free path of air (in this case) and has a value of \( \lambda = 0.064 \, \mu\text{m} \). For the device shown in Figure 1, the variable distance between the two plates is approximately 1 \( \mu\text{m} \), and hence \( K_n \) is very small. Consequently, the effective viscosity of the air can be approximated as the absolute air viscosity. In Equation (5), \( P \) denotes the absolute pressure in the air gap while \( P_a \) is the ambient pressure. The net pressure in the gap is therefore given by \( P_n = P - P_a \). The pressure boundary conditions for the plate are given as follows:

\[
P_n(R, t) = 0, \quad \frac{\partial P_n(0, t)}{\partial r} = 0. \tag{6}
\]

### 2.2. Hybrid Method Model

For analytical convenience in solving the governing equation of motion of the clamped micro-plate, let the transverse displacement \( w \) be normalized with respect to the initial gap between the electrodes, the radial distance \( r \) be normalized with respect to the plate radius, and the time \( t \) be normalized with respect to a constant \( \tilde{T} \) with the form \( \tilde{T} = \sqrt{\rho h R^2 / D} \). In other words,

\[
\tilde{w} = w / g, \quad \tilde{r} = r / R, \quad \tilde{t} = t / \tilde{T}. \tag{7}
\]
In addition, let the excitation frequency \( \omega \) be normalized by taking the product of \( \omega \) and the time constant \( \bar{T} \) (i.e., \( \bar{\omega} = \omega \bar{T} \)). Finally, let the following non-dimensional parameters also be defined:

\[
P = \frac{p}{P_c}, \quad B = \frac{h_f}{P}, \quad Q = \frac{R_s}{2D}, \quad T = \frac{T_r}{b/D}, \quad \bar{h}_j = h_j/b_D, \quad I = \frac{P_c}{b_D}.
\]  

(8)

Substituting Equations (7) and (8) into Equations (1), (3) and (4), the dimensionless governing equation of motion for the circular micro-plate is obtained as

\[
\frac{\partial^4 \bar{w}}{\partial \bar{r}^4} + \frac{2 \partial^3 \bar{w}}{\partial \bar{r}^3} + \frac{2 \partial^2 \bar{w}}{\partial \bar{r}^2} - \frac{\partial \bar{w}}{\bar{r}} - T_r \frac{\partial^2 \bar{w}}{\partial \bar{r}^2} - T_c \frac{\partial \bar{w}}{\bar{r}} = \frac{\bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2}{(1 - \bar{\omega})^2} + \bar{h}_d - \bar{P}L.
\]

(9)

The corresponding dimensionless boundary conditions are given as

\[
\frac{\partial \bar{w}}{\partial \bar{r}}(\bar{r}, \bar{t}) = 0 \quad \text{at} \quad \bar{r} = \pm 1.
\]

(10)

Finally, the initial condition is given by

\[
\bar{w}(\bar{r}, 0) = \frac{\partial \bar{w}}{\partial \bar{t}}(\bar{r}, 0) = 0.
\]

(11)

Substituting Equation (7) into Equations (5) and (6), the dimensionless linearized one-dimensional Reynolds equation is obtained as

\[
\frac{\partial^2 \bar{P}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{P}}{\partial \bar{r}} = \sigma \left( \frac{\partial \bar{U}}{\partial \bar{t}} - \frac{\partial \bar{w}}{\partial \bar{t}} \right),
\]

(12)

where \( \sigma = 12\nu_c R^2 / P_c a^2 \bar{T} \) is the squeeze number and is dimensionless. In the present case, the elastic damping force generated by the compression of the gas in the gap between the two electrodes is negligible compared to the viscous damping force generated by the gas [19]. Consequently, the gas can be considered as incompressible, and thus adequately described by the expression given in Equation (12).

The electrostatic force term \( \bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2 / (1 - \bar{\omega})^2 \) in Equation (9) can be approximated by the following Taylor series expansion:

\[
\frac{\bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2}{(1 - \bar{\omega})^2} = \bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2 + \frac{1}{2} \bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2 + \frac{1}{3!} \bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2 + \ldots.
\]

(13)

Neglecting terms higher than the fifth order and substituting Equation (13) into Equation (9), the nonlinear governing equation of the clamped micro-beam is finally obtained as

\[
\bar{Q}(1 + 3\bar{\omega}^2 + 4\bar{\omega}^2 + 5\bar{\omega}^2 )^2 \times \bar{Q}[V_{DC} + V_{AC} \sin(\bar{\omega}t)]^2 \left[ V_{DC} + V_{AC} \sin(\bar{\omega}t) \right] + \frac{V_{DC}^2}{2} + \frac{V_{AC}^2}{2} \sin(2\bar{\omega}t) + \bar{h}_d - \bar{P}L.
\]

(14)

In this study, Equation (14) was solved using the hybrid differential transformation and finite difference method described in [6,14]. The solution procedure commences by discretizing the equation of motion with respect to the time domain \( t \) using the differential transformation method, i.e.

\[
T \left[ \frac{\partial^4 \bar{w}}{\partial \bar{r}^4} \right] = \frac{d^4 W(\bar{r}, k)}{d\bar{r}^4},
\]

\[
T \left[ \frac{\partial^3 \bar{w}}{\partial \bar{r}^3} \right] = \frac{(k + 1)(k + 2)}{H^2} W(\bar{r}, k + 2),
\]

\[
T \left[ \frac{\partial^2 \bar{w}}{\partial \bar{r}^2} \right] = \frac{2 \bar{d}^3 W(\bar{r}, k)}{\bar{r} \bar{d}^3},
\]

\[
T \left[ \frac{\partial \bar{w}}{\partial \bar{r}} \right] = \frac{1}{\bar{r}} \frac{d^2 W(\bar{r}, k)}{d\bar{r}^2},
\]

\[
T \left[ \bar{h}_d \right] = \bar{h}_d \delta(k),
\]

\[
T \left[ 4\bar{d}^3 V_{DC}^2 \right] = 4V_{DC}^2 \sum_{\ell=1}^{k} (3 + 1\ell - k) W(\bar{r}, \ell) W(\bar{r}, k - \ell),
\]

\[
T \left[ \bar{P}L \right] = \bar{L} \times P(\bar{r}, k),
\]

\[
T \left[ \frac{\partial \bar{P}}{\partial \bar{t}} \right] = \frac{(k + 1)}{H} P(\bar{r}, k + 1).
\]

(15)

Note that \( \delta(k) \) in the third item is defined as follows:

\[
\delta(k) = \begin{cases} 
1 & \text{for } k = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

(16)
Note also that $W(\bar{r}, k)$ is the spectrum of $\tilde{w}(\bar{r}, k)$, $P(\bar{r}, k)$ is the spectrum of $P(\bar{r}, k)$, $k$ and $\ell$ are transformation parameters, and $H$ is the time interval. Having applied the differential transformation method to the coupled governing equation and associated boundary conditions and initial conditions, the transverse displacement of the plate is discretized spatially in the radial direction using the finite difference approximation method based on fourth-order and second-order accurate central difference formulae.

3. Numerical Results and Discussion

Table 1 shows the material properties and dimensions of the clamped micro-plate. Table 2 compares the results obtained by the hybrid numerical method for the pull-in voltage under a pure DC voltage with those obtained by the CoSolve and macromodel methods presented in [9]. As shown, in the absence of residual stress, the hybrid numerical method predicts a pull-in voltage of 320 V. By contrast, the CoSolve and macromodel predict pull-in voltages of 319 and 315 V, respectively. In other words, the pull-in voltage computed using the hybrid method deviates by no more than 1.5% from the values presented in the literature. Similarly, for a residual stress of 500 MPa, the pull-in voltage obtained by the hybrid method is within 1.6% of the values obtained using the CoSolve and macromodel methods. In other words, the accuracy of the proposed method has been confirmed.

![Figure 2. Variation of dimensionless center-point displacement of a circular micro-plate over time with and without squeeze-film damping effect.](image)

Figure 2 shows the variation over time of the center-point displacement of the circular clamped micro-plate with and without the squeeze-film damping effect. Note that the plate radius is assumed to be 200 μm. In addition, the DC voltage is assigned a constant value of 4.5 V and the residual stress and AC voltage are set as $T_r = 0$ MPa and $V_{AC} = 1$V, respectively. As expected, the amplitude of the circular clamped micro-plate deflection drops when the squeeze-film damping effect is taken into account (see inset). Figure 3(a) and (b) show the phase portraits of the actuator system without and with the squeeze-film damping effect, respectively. In both cases, the orbit forms a closed structure, indicating that the system is stable.

Figure 4 shows the variation of the dimensionless center deflection of the circular clamped micro-plate with the hydrostatic pressure with different DC actuation voltage and residual stress. It can be seen that for a DC voltage of 40 V and no residual stress (green solid-line), the pull-in hydrostatic pressure is equal to 96.4 kPa. For a DC voltage of 80 V and no residual stress (blue solid-line), the pull-in hydrostatic pressure is 24.5 kPa. In other words, the pull-in hydrostatic pressure drops with an increase in DC voltage. For a DC voltage of 40 V and a residual stress of 20 MPa (dashed green line), the pull-in hydrostatic pressure drops to 44 kPa. Thus, a higher DC voltage once again results in a lower pull-in hydrostatic pressure. However, for a given actuation voltage, the pull-in hydrostatic voltage increases when the effects of residual stress are taken into account.

### Table 1. Material and geometry parameters of circular micro-plate.

| Symbol   | Parameters                  | Value/unit |
|----------|-----------------------------|------------|
| $E$      | Young's modulus             | 130 GPa    |
| $\nu$    | Poisson's ratio             | 0.23       |
| $\rho$   | Density                     | $2.33 \times 10^3$ kg/m$^3$ |
| $\varepsilon_0$ | Permittivity of free space | $8.8542 \times 10^{-12}$ F/m |
| $h$      | Thickness of circular micro-plate | 1 µm |
| $g$      | Initial gap                 | 1 µm       |
| $R$      | Radius of plate             | 45 µm      |
| $P_a$    | Ambient pressure            | 1 bar      |

### Table 2. Comparison of present analytical results with those from the literature for the pull-in voltage of clamped circular micro-plate.

| Residual Stress (MPa) | Pull-in voltage (V) | Error |
|-----------------------|---------------------|-------|
|                       | Hybrid Method (H.M) | CoSolve [9] | macromodel [9] | $e_1(\%)$ | $e_2(\%)$ |
| 0                     | 320                 | 319    | 315          | 0.3     | 1.5      |
| 500                   | 363                 | 369    | 364          | 1.6     | 0.3      |

Notes:

\[
e_1(\%) = \frac{|\text{CoSolve} - \text{H.M}|}{\text{CoSolve}} \times 100\%
\]

\[
e_2(\%) = \frac{|\text{macromodel} - \text{H.M}|}{\text{macromodel}} \times 100\%
\]
In general, the results show that when the hydrostatic pressure increases, the center-point deflection of the micro-plate increases nonlinearly. In addition, it is seen that for dimensionless times of less than 800, the applied DC voltage is unstable and less peak deflection occurs. Finally, it is seen that for a hydrostatic pressure of 1.1 kPa, the micro-plate oscillates in a stable manner about the maximum deflection point. However, for a higher hydrostatic pressure of 1.2 kPa, the pull-in phenomenon occurs, and the clamped micro-plate collapses onto the lower electrode.

Figure 5 shows the dynamic variation of the center-point deflection of the circular clamped micro-plate with different AC voltages and hydrostatic pressures applied. (Note that the DC voltage is assumed to be 94 V, and \( T_r = 0 \) and \( \bar{\omega} = 1 \)). In general, the results show that when the hydrostatic pressure increases, the center-point deflection of the micro-plate increases nonlinearly. In addition, it is seen that for dimensionless times of less than 800, the applied DC voltage is unstable and less peak deflection occurs. Finally, it is seen that for a hydrostatic pressure of 1.1 kPa, the micro-plate oscillates in a stable manner about the maximum deflection point. However, for a higher hydrostatic pressure of 1.2 kPa, the pull-in phenomenon occurs, and the clamped micro-plate collapses onto the lower electrode.
Figure 6 shows the phase portraits of the clamped micro-plate given various values of the hydrostatic pressure. For hydrostatic pressures of 0–1.1 kPa, the phase portraits have the form of stable periodic orbits. In other words, the system remains in a stable condition. However, the size of the orbit increases with an increasing hydrostatic pressure, and the system becomes unstable at a hydrostatic pressure of 1.2 kPa.

Figure 7 shows the dynamic deflection of the micro-plate under an applied DC voltage of 40 V and an AC voltage of 15 V. The results indicate that the micro-plate transits from an initial transient state to a final steady state. Figure 8(a) and (b) show the phase portrait and Poincaré map for the actuation conditions considered in Figure 7. The phase portrait has the form of a closed curve, while the Poincaré map is a solitary point. In other words, the motion of the clamped micro-plate is stable and periodic.

4. Conclusions

In this study, a hybrid numerical method comprising differential transformation finite difference was used to investigate the nonlinear dynamic behavior of a circular clamped micro-plate subject to residual stress, hydrostatic pressure, and squeeze-film damping effects. The results have shown that the hybrid method successfully captures the effects of the applied AC voltage, residual stress, and hydrostatic pressure in determining the pull-in event within the system. It has been shown that the peak displacement of the circular clamped micro-plate drops when the squeeze-film damping effect is taken into account. In addition, the micro-plate deflection increases with an increasing voltage or hydrostatic pressure. The pull-in voltage drops with an increase in hydrostatic pressure, but increases with an increasing residual stress. Given appropriate AC actuation voltages, the phase portraits of the clamped micro-plate show closed orbits, while the Poincaré maps are single point. In other words, the operation of the circular clamped micro-plate can be confined to the stable region through an appropriate specification of the actuation conditions. In general, the results presented in this study confirm the validity of the hybrid numerical method as a tool for the design and analysis of circular clamped micro-plates and similar MEMs-based actuation devices.

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References

[1] Bertarelli E, Corigliano A, Greiner A, Korvink JG. Design of high stroke electrostatic micropumps: a charge control approach with ring electrodes. Microsyst Technol. 2011;17:165–173.

[2] Melvas P, Kalvesten E, Stemme G. A surface-micromachined resonant-beam pressure-sensing structure. J Micromech Syst. 2001;10:498–502.

[3] Batra RC, Porfiri M, Spinello D. Vibrations of narrow microbeams predeformed by an electric field. J Sound Vib. 2008a;309:600–612.

[4] Liu CC, Wang CC. Numerical investigation into nonlinear dynamic behavior of electrically-actuated clamped-clamped micro-beam with squeeze-film damping effect. Appl Math Model. 2014;38:3269–3280.

[5] Krylov S. Lyapunov exponents as a criterion for the dynamic pull-in instability of electrostatically actuated microstructures. Int J Nonlinear Mech. 2007;42:626–642.

[6] Chen CK, Lai HY, Liu CC. Nonlinear dynamic behavior analysis of micro electrostatic actuator based on a continuous model under electrostatic loading. ASME J Appl Mech. 2011;78:031003–1–031003–9.

[7] Alsaleem FM, Younis MI, Ouakad HM. On the nonlinear resonances and dynamic pull-in of electrostatically actuated resonators. J Micromech Microeng. 2009;19(4):045 013-14.

[8] Younis MI, Abdel-Rahman EM, Nayfeh A. A reduced-order model for electrically actuated microbeam-based MEMS. J Microelectromech Syst. 2003;12:672–680.

[9] Vogl GW, Nayfeh AH. A reduced-order model for electrically actuated clamped circular plates. J Micromech Microeng. 2005;15:684–690.

[10] Wang Y-G, Lin W-H, Li X-M, Feng Z-J. Bending and vibration of an electrostatically actuated circular micro-plate in presence of Casimir force. Appl Math Model. 2011;35:2348–2357.

[11] Krylov S, Maimon R. 'Pull-in dynamics of an elastic beam actuated by continuously distributed electrostatic force'. ASME J Vib Acoust. 2004;126(3):332–342.

[12] Younis MI. Modeling and simulation of microelectromechanical systems in multi-physics fields [PhD dissertation]. Blacksburg (VA): Virginia Polytechnic Institute and State University; 2004.

[13] Liu CC, Li JH, Hsieh MC. Numerical study of nonlinear micro circular plate analysis using hybrid method (H.M.). Appl Math Inf Sci. 2015;9(1L):245–250.

[14] Liu CC, Liu CH. Analysis of nonlinear dynamic behavior of electrically-actuated micro-beam with piezoelectric layers and squeeze-film damping effect. Nonlinear Dyn. 2014;77(4):1349–1361.

[15] Liao LD, Chao CP, Huang CW, Chiu CW. dc dynamic and static pull-in predictions and analysis for electrostatically actuated clamped circular micro-plate based on a continuous model. J Micromech Microeng. 2010;20:025013.

[16] Talebian S, Rezazadeh G, Fathalilou M, Toosi B. Effect of temperature on pull-in voltage and natural frequency of an electrostatically actuated microplate. Mechatronics. 2010;20(6):666–673.

[17] Rezazadeh G, Tahmasebi A, Zuhbstov M. Application of piezoelectric layers in electrostatic MEM actuators: controlling of pull-in voltage. Microsyst Technol. 2006;12(12):1163–1170.

[18] Nabian A, Rezazadeh G, Almassi M, Borgheei AM. On the stability of a functionally graded rectangular micro-plate subjected to hydrostatic and nonlinear electrostatic pressures. Acta Mechanica Solida Sinica. 2013;26(2):205–220.

[19] Starr JB. Squeeze film damping in solid state accelerometer. Tech. Digest, IEEE Solid State Sensor and Actuator Workshop, USA; 1990 June. p. 44–47.