VERY ACCURATE APPROXIMATIONS FOR THE ELLIPTIC INTEGRALS OF THE SECOND KIND IN TERMS OF STOLARSKY MEANS

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Dedicated to my mother Ru-Yi Jiang

Abstract. For $a, b > 0$ with $a \neq b$, the Stolarsky means are defined by

$$S_{p,q}(a, b) = \left( \frac{p(a^p - b^p)}{q(a^q - b^q)} \right)^{1/(p-q)}$$

if $pq(p- q) \neq 0$ and $S_{p,q}(a, b)$ is defined as its limits at $p = 0$ or $q = 0$ or $p = q$ if $pq(p- q) = 0$.

The complete elliptic integrals of the second kind $E$ is defined on $(0, 1)$ by

$$E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t} dt.$$

We prove that the functions

$$F(r) = \frac{1 - (2/\pi) E(r)}{1 - S_{11/4, 7/4}(1, r')},$$

and

$$G(r) = \frac{1 - (2/\pi) E(r)}{1 - S_{5/2, 2}(1, r')}$$

are strictly decreasing and increasing on $(0, 1)$, respectively, where $r' = \sqrt{1 - r^2}$.

These yield some very accurate approximations for the complete elliptic integrals of the second kind, which greatly improve some known results.

1. Introduction

For $r \in (0, 1)$, the well-known complete elliptic integrals of the first and second kinds \cite{1, 2} are defined by

$$K(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}}, \quad K(0^+) = \frac{\pi}{2}, \quad K(1^-) = \infty$$

and

$$E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t} dt, \quad E(0^+) = \frac{\pi}{2}, \quad E(1^-) = 1,$$

respectively. These integrals can be expressed exactly in terms of Gaussian hypergeometric function

$$\binom{a}{b} \binom{c}{n} \frac{z^n}{n!} \quad |z| < 1,$$

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where \( a, b, c \in \mathbb{R} \) with \( c \neq 0, -1, -2, \ldots \), \((a)_n\) is defined by \((a)_0 = 1\) for \( a \neq 0 \) and \((a)_n = a(a+1) \cdots (a+n-1), a \neq 0\).

Indeed, we have

\[
K = \frac{\pi}{2} F \left( \frac{1}{2}, 1; 1; r^2 \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n^2}{n!} n^{2n}, \tag{1.3}
\]

\[
E = \frac{\pi}{2} F \left( -\frac{1}{2}, 1; 1; r^2 \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} n^{2n}. \tag{1.4}
\]

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory and other related fields [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13].

Let \( l(1,r) \) be the arc length of an ellipse with semiaxis 1 and \( r \in (0,1) \). Then

\[
l(1,r) = 4E'(r'), \]

where and in what follows \( r' = \sqrt{1-r^2} \). In 1883, Muir [14] presented a simple approximation for \( l(1,r) \) by \( 2\pi A_{3/2}(1,r) \), where

\[
A_p(a,b) = \left( \frac{a^p + b^p}{2} \right)^{1/p} \text{ if } p \neq 0 \text{ and } A_0(a,b) = \sqrt{ab}
\]
is the classical power mean of positive numbers \( a \) and \( b \). This mean contains some simple ones, such as \( A_{-1}(a,b) = H(a,b) \) –harmonic mean, \( A_1(a,b) = A(a,b) \) –arithmetic mean, \( A_0(a,b) = G(a,b) \) –geometric mean, and \( A_2(a,b) = S(a,b) \) –root-square mean, etc.

In 1996, Vuorinen [15] conjectured that the following inequality

\[
E(r) \geq \frac{\pi}{2} A_{3/2}(1,r') =: \frac{\pi}{2} A_1(r') \tag{1.5}
\]
holds for \( 0 \leq r \leq 1 \). This conjecture was proved in 1997 in [16] Theorem 2] by Qiu and Shen (see also [8] Theorem 1.1]). Barnard et al. [9] discovered an upper bound \((\pi/2) A_2(1,r')\) for \( E(r) \), that is,

\[
E(r) \leq \frac{\pi}{2} A_2(1,r') \quad (0 \leq r \leq 1). \tag{1.6}
\]

An improvement of (1.6) was presented by Qiu in [17] Corollary (1)] (see also [18] Theorem 22]), which states that the inequality

\[
E(r) \leq \frac{\pi}{2} A_{q_0}(1,r') \tag{1.7}
\]
is valid for all \( r \in (0,1) \) with the best constant \( q_0 = \ln 2/\ln (\pi/2) \).

Motivated by the inequalities (1.5), (1.7), some new approximations for \( E(r) \) in terms of bivariate means were presented. For example, Chu and Wang [19] Corollary 3.2] proved the inequality

\[
E(r) < \frac{\pi}{2} L_{1/4}(1,r') =: \frac{\pi}{2} A_6(r''), \tag{1.8}
\]
holds for \( r \in (0,1) \) with the best constant \( 1/4 \), where \( L_p(a,b) \) is the Lehmer mean of positive \( a \) and \( b \) defined by

\[
L_p(a,b) = \frac{a^{p+1} + b^{p+1}}{a^p + b^p}.
\]
Soon afterwards, Chu et al. [20, Theorem 3.1] gave a much better upper bound for $E(r)$. They proved the double inequality

\[
\frac{16-3\sqrt{2}}{2(2-\sqrt{2})\pi} A(1,r') - \frac{8-\sqrt{2}}{2(2-\sqrt{2})\pi} G(1,r') + \frac{8-8}{2(2-\sqrt{2})\pi} S(1,r') \leq \frac{2}{\pi} E(r) \leq \frac{9}{8} A(1,r') - \frac{5}{16} G(1,r') + \frac{3}{16} S(1,r') =: A_7(r')
\]

(1.9)

holds for $r \in (0,1)$.

Wang et al. [21, Theorem 2.4] established a sharp double inequality for $E(r)$:

\[
A_2(r') := \frac{23}{16} A(1,r') - \frac{5}{16} H(1,r') - \frac{1}{8} S(1,r') \leq \frac{2}{\pi} E(r) \leq \frac{24-5\sqrt{2}}{2\pi(3-2\sqrt{2})} A(1,r') - \frac{8-\sqrt{2}}{2\pi(3-2\sqrt{2})} H(1,r') - \frac{(16-5\pi)}{2\pi(3-2\sqrt{2})} S(1,r'),
\]

and pointed out that the lower and upper bounds in (1.10) are stronger than ones in (1.5) and (1.6), respectively. In 2013, Wang and Chu [22, Corollary 3.1] presented another improvement of (1.5) and (1.6), which states that

\[
A_3(r') := \frac{1}{128} \frac{(9r'^2 + 14r' + 9)^2}{(r' + 1)^3} \leq \frac{2}{\pi} E(r) \leq \frac{1}{\pi} \sqrt{4r'^2 + (\pi^2 - 8)r' + 4}.
\]

Very recently, Hua and Qi showed in [23, Theorem 1.3.] that the double inequality

\[
A_4(r') := \frac{1 + r' + r'^2}{2(1 + r')} + \frac{1 + r'}{8} \leq \frac{2}{\pi} E(r) \leq \left(\frac{8}{\pi} - 2\right) \frac{1 + r' + r'^2}{1 + r'} + \left(2 - \frac{6}{\pi}\right) (1 + r')
\]

(1.11)

is valid for $r \in (0,1)$.

Other various approximations for $E(r)$ can be found in [10, 21, 18, 9, 25, 26, 27, 29, 28, 30, 31, 32, 33, 34] and references therein.

For $a, b > 0$ with $a \neq b$, the Stolarsky means $S_{p,q}(a,b)$ are defined in [35] by

\[
S_{p,q}(a,b) = \begin{cases} 
\left( \frac{q(a^p - b^p)}{p(a^q - b^q)} \right)^{1/(p-q)} & \text{if } p \neq q, pq \neq 0, \\
\frac{a^p - b^p}{p(\ln a - \ln b)} & \text{if } p \neq 0, q = 0, \\
\frac{a^q - b^q}{q(\ln a - \ln b)} & \text{if } p = 0, q \neq 0, \\
\exp \left( \frac{a^p \ln a - b^p \ln b}{a^p - b^p} - \frac{p}{q} \right) & \text{if } p = q \neq 0, \\
\sqrt{ab} & \text{if } p = q = 0,
\end{cases}
\]

(1.13)

and, $S_{p,q}(a,a) = a$. This family of means contains many famous means, for example, $S_{1,0}(a,b) = L(a,b)$ the logarithmic mean, $S_{1,1}(a,b) = I(a,b)$ the identric (exponential) mean, $S_{2,1}(a,b) = A(a,b)$ the arithmetic mean, $S_{3/2,1/2}(a,b) = H_e(a,b)$ the harmonic mean, $S_{2p,1/2}(a,b) = A_{1/p}(a^p, b^p) = A_p$ the $p$-order power mean, $S_{3p,2/2}(a,b) = H_{1/p}(a^p, b^p) = H_p$ the $p$-order Heronian mean, $S_{p,0}(a,b) = L_{1/p}(a^p, b^p) = L_p$ the $p$-order logarithmic mean, $S_{p,p}(a,b) = I_{1/p}(a^p, b^p) = I_p$ the $p$-order identric (exponential) mean, etc.

Stolarsky means have many well properties, which can follow directly from the defining formula (1.13) and be found in [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. For later use, we mentioned the following properties:
(P1) For all $a, b > 0$ and $p, q \in \mathbb{R}$, $S_{p,q}(a, b)$ are increasing with both $p$ and $q$, or with both $a$ and $b$ (see [38] (2.22), (3.12)).

(P2) For fixed $c > 0$, $S_{p,2c-p}(a, b)$ is increasing in $p$ on $(-\infty, c]$ and decreasing on $[c, \infty)$ (see [50] (3.14), [44] Corollary 1.1).

(P3) For fixed $c > 0$ and $p \in (0, 2c)$, $(1/\theta_p) S_{p,2c-p}(a, b)$ is decreasing in $p$ on $(0, c)$ and increasing on $(c, 2c)$, where $\theta_p$ is defined by

\[
\theta_p = \left(\frac{2c-p}{p}\right)^{1/(2p-2c)} \text{ if } p \neq c \text{ and } \theta_c = e^{-1/c}
\]

(see [44] Corollary 1.2).

**Remark 1.** Taking $a = b$ in (P3) yields that $(1/\theta_p) S_{p,2c-p}(a, a)$ is decreasing in $p$ on $(0, c)$ and increasing on $(c, 2c)$. Since $S_{p,2c-p}(a, a) = a$, it follows that $\theta_p$ strictly increasing in $p$ on $(0, c)$ and decreasing on $(c, 2c)$.

Now we intend to estimate for the complete elliptic integrals of the second kind $E(r)$ by the Stolarsky means of 1 and $r^2$, i.e. $S_{p,q}(1, r^2)$. Expanding in power series gives

\[
\frac{2}{\pi} E(r) - S_{p,q}(1, r^2) = -\frac{p+q-9/2}{96} r^4 - \frac{p+q-9/2}{128} r^6 + \frac{8(p+q) (2p^2+2q^2-5p-5q-550)}{45 \times 2^{14}} r^8 + O\left(r^{10}\right).
\]

In order to increase accuracy of estimate for $E(r)$, we let $p + q - 9/2 = 0$, or $q = 9/2 - p$. Then we get

\[
\frac{2}{\pi} E(r) - S_{9/2-p, p}(1, r^2) = \frac{(4p-7)(4p-11)}{5 \times 2^{14}} r^8 + O\left(r^{10}\right).
\]

Further, taking $p = 7/4$ or $11/4$ yields

\[
S_{11/4,7/4}(1, r^2) = \frac{7}{11} \frac{1-r^{11/4}}{1-r^{7/4}} := A_5(r^2),
\]

and

\[
\lim_{r \to 1^-} \left( \frac{2}{\pi} E(r) - S_{11/4,7/4}(1, r^2) \right) = \frac{2}{\pi} - \frac{7}{11} \approx 0.00025614.
\]

Letting $p = 2$ or $5/2$ yields

\[
S_{5/2,2}(1, r^2) = \left(\frac{41}{5} - r^{5/2}\right)^2 := A_8(r^2)
\]

and

\[
\lim_{r \to 1^-} \left( \frac{2}{\pi} E(r) - S_{5/2,2}(1, r^2) \right) = \frac{3}{5 \times 2^4} r^8 + O\left(r^{10}\right),
\]

These show that $S_{11/4,7/4}(1, r^2)$ and $S_{5/2,2}(1, r^2)$ may be excellent approximations for the complete elliptic integrals of the second kind. The purpose of the paper is to prove this assertion. Our main results are contained in the following theorems.
Theorem 1. The function
\[
F(r) = \frac{1 - (2/\pi) E(r)}{1 - S_{11/4,7/4}(1,r')}
\]
is strictly decreasing from \((0,1)\) onto \((11(\pi - 2)/(4\pi), 1)\). Therefore, the double inequality
\[
1 - \mu + \mu S_{11/4,7/4}(1,r') < \frac{2}{\pi} E(r) < 1 - \lambda + \lambda S_{11/4,7/4}(1,r')
\]
holds if and only if \(\mu \geq 1\) and \(\lambda \leq 11(\pi - 2)/(4\pi)\) \(\approx 0.9993\).

In particular, we have
\[
S_{11/4,7/4}(1,r') < \frac{2}{\pi} E(r) < \frac{22}{7\pi} S_{11/4,7/4}(1,r')
\]
where the coefficients 1 and 22/(7\pi) \(\approx 1.0094\) are the best constants.

Theorem 2. The function
\[
G(r) = \frac{1 - (2/\pi) E(r)}{1 - S_{5/2,2}(1,r')}
\]
is strictly increasing from \((0,1)\) onto \((1,25(\pi - 2)/(9\pi))\). Consequently, the double inequality
\[
1 - \xi + \xi S_{5/2,2}(1,r') < \frac{2}{\pi} E(r) < 1 - \eta + \eta S_{5/2,2}(1,r')
\]
holds if and only if \(\xi \geq 25(\pi - 2)/(9\pi)\) \(\approx 1.0094\) and \(\eta \leq 1\).

Particularly, it holds that
\[
\frac{22}{7\pi} S_{5/2,2}(1,r') < \frac{2}{\pi} E(r) < S_{5/2,2}(1,r')
\]
where the coefficients 25/(8\pi) \(\approx 0.99472\) and 1 are the best constants.

The paper is organized as follows. Some lemmas used to prove main results are presented in Section 2. The proof of Theorem 1 is complicated and longer, so it is independently arranged in Sections 3; while the proof of Theorem 2 is placed in Section 4. In Section 5, some interesting and applied corollaries involving the monotonicity of difference and ratio between \((2/\pi) E(r)\) and \(S_{9/2-p,p}(1,r')\) are deduced. In the last section, it is shown that our approximations \(S_{11/4,7/4}(1,r')\) and \(S_{25/2}(1,r')\) are indeed excellent by accuracy of comparing some known approximations with ours.

2. Lemmas

In order to prove our main results, we need some lemmas.

The first lemma is the "L'Hospital Monotone Rule" (see [46, 47]), which has been widely used very effectively in the study of some areas. A new and natural way to prove this class of rules can refer to [48], which is easily understood and used.

Lemma 1 ([46 Proposition 1.1], [47 Theorem 2]). For \(-\infty < a < b < \infty\), let \(f,g : [a,b] \to \mathbb{R}\) be continuous functions that are differentiable on \((a,b)\), with \(f(a) = g(a) = 0\) or \(f(b) = g(b) = 0\). Assume that \(g'(x) \neq 0\) for each \(x\) in \((a,b)\). If \(f'/g'\) is increasing (decreasing) on \((a,b)\) then so is \(f/g\).
The second lemma is a monotonicity criterion for the ratio of power series, which will be used to prove Theorem 2.

**Lemma 2 ([39]).** Let \( A(t) = \sum_{k=0}^{\infty} a_k t^k \) and \( B(t) = \sum_{k=0}^{\infty} b_k t^k \) be two real power series converging on \((-r, r)\) with \( b_k > 0 \) for all \( k \). If the sequence \( \{a_k/b_k\} \) is increasing (decreasing) for all \( k \), then the function \( t \mapsto A(t)/B(t) \) is also increasing (decreasing) on \((0, r)\).

A more general monotonicity criterion for the ratio of power series has been established in [50] recently.

For \( x > 0 \) the classical Euler’s gamma function \( \Gamma \) and psi (digamma) function \( \psi \) are defined by

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\]

respectively. In the proof of Theorem 1, we will use several inequalities for the gamma and psi functions, in which Lemma 3 is very crucial.

**Lemma 3 ([51], [52, (2.8)]).** For all \( x > 0 \) and all \( a \in (0, 1) \), it holds that

\[
\left(\frac{x}{x+a}\right)^{1-a} < \frac{\Gamma(x+a)}{x^a \Gamma(x)} < 1,
\]

or equivalently,

\[
\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}.
\]

**Lemma 4 ([53]).** For \( x \in (0, 1) \), it holds that

\[
\frac{x^2+1}{x+1} < \Gamma(x+1) < \frac{x^2+2}{x+2}, \quad x \in (0, 1),
\]

or equivalently,

\[
\frac{x^2+1}{(x+1)x} < \Gamma(x) < \frac{x^2+2}{(x+2)x}, \quad x \in (0, 1).
\]

**Lemma 5 ([54, Lemma 1.7]).** Let \( x \) be a positive real number. Then we have

\[
\psi(x+1) > \ln(x + \frac{1}{2}).
\]

**Lemma 6 ([55, Lemma 7], [48, Lemma 1]).** Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{0\} \) with \( n > m \) and let \( P_n(t) \) be an \( n \) degrees polynomial defined by

\[
P_n(t) = \sum_{i=m+1}^{n} a_i t^i - \sum_{i=0}^{m} a_i t^i,
\]

where \( a_n, a_m > 0, a_i \geq 0 \) for \( 0 \leq i \leq n - 1 \) with \( i \neq m \). Then there is a unique number \( t_{m+1} \in (0, \infty) \) to satisfy \( P_n(t_{m+1}) = 0 \) such that \( P_n(t) < 0 \) for \( t \in (0, t_{m+1}) \) and \( P_n(t) > 0 \) for \( t \in (t_{m+1}, \infty) \).
3. Proof of Theorem 1

We are in a position to prove Theorem 1.

Proof of Theorem 1. Let us consider the ratio

\[ F (r) = \frac{1 - (2/\pi) E (r)}{1 - S_{11/4,7/4} (1, r')} = \frac{11 \left( 1 - r^{7/4} \right) (1 - (2/\pi) E)}{11 \left( 1 - r^{7/4} \right) - 7 \left( 1 - r^{11/4} \right)} = \frac{f_1 (r)}{f_2 (r)}. \]

Clearly, \( f_1 (0^+) = f_2 (0^+) = 0 \).

Differentiations by the formulas

\[
\frac{dK}{dr} = \frac{E - r'^2 K}{r'^2}, \quad \frac{dE}{dr} = \frac{E - K}{r}, \quad \frac{d(KE)}{dr} = \frac{rE}{r'^2}
\]

yield

\[
\frac{f'_1 (r)}{f'_2 (r)} = \frac{1 - (2/\pi) E + 8 (r'^{1/4} - r'^{2}) (K - E) / (7\pi r^2)}{1 - r'}, \quad f_3 (0^+) = f_4 (0^+) = 0,
\]

\[
\frac{7\pi f'_1 (r)}{2 f'_1 (r)} = \frac{r'}{r} \left( 4 \frac{r'^{1/4} - r'^{2}}{r'^2} E - \frac{(8 - 7r^2) r'^{1/4} - 8r^2}{r^3 r^2} (K - E) - \frac{7 E - K}{r} \right)
\]

\[
= \frac{(8 - 3r^2) E - (8 - 7r^2) K}{r^4 r'^{3/4}} + \left( (7r^2 + 8) K - (11r^2 + 8) E \right) \frac{r'}{r'^4},
\]

\[
\frac{7\pi}{2} \left( \frac{f'_3 (r)}{f'_4 (r)} \right)' = \frac{-32K - 32E - 14r^2 K - 3r^4 K - 2r^2 E}{r^5 r'}
\]

\[ + \frac{1}{4} \frac{128K - 128E - 224r^2 K + 93r^4 K + 160r^2 E - 21r^4 E}{r^5 r'^{11/4}} \]

\[ = \frac{-f_5 (r) - r'^{-7/4} f_6 (r)}{r^5 r'}, \quad f_3 (r) = f_4 (r), \]

where

\[
f_5 (r) = 32K - 32E - 14r^2 K - 3r^4 K - 2r^2 E,
\]

\[
f_6 (r) = \frac{1}{4} (128K - 128E - 224r^2 K + 93r^4 K + 160r^2 E - 21r^4 E).
\]

If we can prove that \( f_7 (r) > 0 \) for \( r \in (0, 1) \), that is, the function \( f'_5 / f'_4 \) is decreasing on \( (0, 1) \), then so is \( f'_1 / f'_2 \) by Lemma 1 which in turn implies that \( f_1 / f_2 \) by using Lemma 1 again, that is, the function \( F \) is strictly decreasing on \( (0, 1) \). Next we prove that \( f_7 (r) > 0 \) for \( r \in (0, 1) \) stepwise.
Step 1: Expanding \( f_7 (r) \) in power series. By the expansions (1.3) and (1.4), we have

\[
f_5 (r) = 32K - 32E - 14Kr^2 - 3Kr^4 - 2r^2E = \pi \left( 32 \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^2}{n!n^2} - 32 \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n (\frac{1}{2})_{n+2n}}{n!n!} - 14 \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^2}{(n-1)!(n-1)!}r^{2n} - 3 \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^2}{(n-2)!(n-2)!}r^{2n} - 2 \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})_n (\frac{1}{2})_{n+2n}}{(n-1)!(n-1)!}r^{2n} \right)
\]

\[
= \frac{3\pi}{2} \sum_{n=3}^{\infty} \frac{n(n-6)(n-1)(n-2)(\frac{1}{2})^2}{n!n!} r^{2n} : = \frac{3\pi}{2} r^6 \sum_{n=0}^{\infty} a_n r^{2n},
\]

where

\[
a_n = \frac{(5n+9)(\frac{1}{2})_{n+1}}{n!(n+3)!} = \frac{5n+9}{n!(n+3)!} \left( \frac{\Gamma (n+3/2)}{\Gamma (1/2)} \right)^2 ;
\]

\[
f_6 (r) = \frac{1}{4} \left( 128K - 128E - 224r^2K + 93r^4K + 160r^2E - 21r^4E \right) = \pi \left( 128 \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^2}{n!n^2} - 128 \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n (\frac{1}{2})_{n+2n}}{n!n!} - 224 \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^2}{(n-1)!(n-1)!}r^{2n} - 93 \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^2}{(n-2)!(n-2)!}r^{2n} + 160 \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})_n (\frac{1}{2})_{n-2n}}{(n-1)!(n-1)!}r^{2n} - 21 \sum_{n=2}^{\infty} \frac{(-\frac{1}{2})_n (\frac{1}{2})_{n-2n}}{(n-2)!(n-2)!}r^{2n} \right)
\]

\[
= \frac{3\pi}{16} \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)(n-4)(n+15)(2n-5)(\frac{1}{2})^2}{n!n!} r^{2n} : = \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} c_n r^{2n},
\]

where

\[
c_n = \frac{(n-1)(n+18)(2n+1)(\frac{1}{2})^2}{n!(n+3)!} = \frac{(n-1)(n+18)(2n+1)}{n!(n+3)!} \left( \frac{\Gamma (n+1/2)}{\Gamma (1/2)} \right)^2 .
\]

Also, using binomial series we have

\[
r^{r-7/4} = (1 - r^2)^{-7/8} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} r^{2n} : = \sum_{n=0}^{\infty} b_n r^{2n},
\]

Then making use of the Cauchy product gives

\[
f_7 (r) = f_5 (r) - (1 - r^2)^{-7/8} f_6 (r) = \frac{3\pi}{2} r^6 \sum_{n=0}^{\infty} a_n r^{2n} + \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} c_n r^{2n} \sum_{n=0}^{\infty} b_n r^{2n}
\]

\[
= \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} \left( 8a_n + \sum_{k=0}^{n} b_{n-k} c_k \right) r^{2n} : = \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} d_n r^{2n} .
\]
**Step 2:** A simple verification yields $d_0 = d_1 = d_2 = d_3 = 0$ and $d_4$, $d_5$, $d_6$, $d_7$, $d_8$, $d_9$, $d_{10}$ are equal to

\[
\begin{align*}
35 & \quad 32768 \quad 204864 \quad 1638464 \quad 1228836288 \quad 3841471275 \quad 7897834945.
\end{align*}
\]
respectively.

**Step 3:** We prove that for $n \geq 10$,

\[
D_n := \frac{8}{7} d_{n+1} - d_n > g(n),
\]
where

\[
g(n) = \frac{4}{\pi} \frac{n(2n+1)}{(n+1)(n+2)(n+3)} - \frac{3}{7}\left(\frac{7}{8}\right)^{n/7} + \frac{128}{7\pi} \frac{n^{7/8}}{64n - 9} g_1(n),
\]
here

\[
g_1(n) = \sum_{k=2}^{n} \frac{n-k}{n-k+1} \frac{(k-1)(k+18)}{(k+1)(k+2)(k+3)} := \sum_{k=2}^{n} \alpha_{n-k} b_k.
\]

To this end, we note that for $k \geq 0$,

\[
a_{k+1} = \frac{1}{4} \frac{(2k+3)^2}{5k+9} \frac{5k+14}{k+4} a_k,
\]

\[
b_{k+1} = \frac{k+7/8}{k+1} b_k,
\]

\[
c_{k+1} = \frac{1}{4} k \frac{(2k+1)(2k+3)(k+19)}{(k-1)(k+1)(k+4)(k+18)} c_k \quad (k \geq 2).
\]

Also, it is seen that $c_0 = -3$, $c_1 = 0$, $c_k > 0$ for $k \geq 2$, and $a_k, b_k > 0$ for $k \geq 0$.

Then the sequence $D_n$ can be expressed as

\[
D_n = \frac{8}{7} d_{n+1} - d_n = \frac{8}{7} \left(8a_{n+1} + b_{n+1} c_0 + \sum_{k=2}^{n+1} b_{n+1-k} c_k\right) - \left(8a_{n} + b_{n} c_0 + \sum_{k=2}^{n} b_{n-k} c_k\right)
\]

\[
= 8 \left(\frac{8}{7} a_{n+1} - a_n\right) - 3 \left(\frac{8}{7} b_{n+1} - b_n\right) + 8 \left(\frac{8}{7} b_{n+1} - b_n\right) c_n + \sum_{k=2}^{n} \left(\frac{8}{7} b_{n+1-k} - b_{n-k}\right) c_k
\]

\[
= 8 \left(\frac{8}{7} \frac{(2n+3)^2}{7} \frac{(5n+14)}{(5n+9)(n+1)(n+4)} - 1\right) a_n - 3 \left(\frac{8}{7} \frac{n+7/8}{n+1} - 1\right) b_n
\]

\[
+ \frac{8}{7} \sum_{k=2}^{n} \frac{(2n+1)(2n+3)(n+19)}{(n-1)(n+1)(n+4)(n+18)} c_n + \sum_{k=2}^{n} \left(\frac{8}{7} \frac{n-k+7/8}{n-k+1} - 1\right) b_{n-k} c_k
\]

\[
= \frac{8}{7} \frac{(5n^2 - 6n - 29)}{(5n+9)(n+1)(n+4)} a_n - 3 \frac{n}{7n+1} b_n
\]

\[
+ \frac{2}{7} \frac{n(2n+1)(2n+3)(n+19)}{(n-1)(n+1)(n+4)(n+18)} c_n + \frac{1}{7} \sum_{k=2}^{n} \frac{n-k}{n-k+1} b_{n-k} c_k.
\]
Notice that coefficient of \( a_n \) is positive due to \( 5n^2 - 6n - 29 > 0 \) for \( n \geq 4 \), by the double inequality (2.2) in Lemma 3 it is derived that for \( k \geq 0 \),

\[
a_k > \frac{1}{2\pi} \frac{(2k + 1)(5k + 9)}{(k + 1)(k + 2)(k + 3)}
\]

\[
\frac{1}{(k + 7/8)^{1/8}} < b_k < \frac{1}{k^{1/8} \Gamma(7/8)},
\]

\[
c_k > \frac{2}{\pi} \frac{(k - 1)(k + 18)}{(k + 1)(k + 2)(k + 3)} \quad (k \geq 2).
\]

Applying these inequalities to the expression of \( D_n \) yields

\[
D_n > \frac{8}{7} \frac{n(5n^2 - 6n - 29)}{(n + 1)(n + 4)} \frac{1}{2\pi} \frac{(2n + 1)(5n + 9)}{(n + 1)(n + 2)(n + 3)} - \frac{3}{7} \frac{n}{n + 1^{1/8} \Gamma(7/8)}
\]

\[
+ \frac{2}{7} \frac{n(2n + 1)(2n + 3)(n + 19)}{(n + 1)(n + 4)(n + 18)} \frac{2}{\pi} \frac{(n - 1)(n + 18)}{(n + 1)(n + 2)(n + 3)}
\]

\[
+ \frac{1}{7} \sum_{k=2}^{n} \left( \frac{n - k}{n - k + 1} \frac{2}{(n - k + 7/8)^{1/8} \pi (k + 1)(k + 2)(k + 3)} \right).
\]

(3.8)

By the known inequality

\[
(1 - x)^p < 1 - px, \quad p, x \in (0, 1)
\]

it is deduced that for \( 2 \leq k \leq n \),

\[
\frac{1}{(n - k + 7/8)^{1/8}} > \frac{1}{(n - 2 + 7/8)^{1/8}} = \frac{1}{n^{1/8} (1 - 9/(8n))^{1/8}}
\]

\[
> \frac{1}{n^{1/8} (1 - 9/(64n))} = \frac{64}{(64n - 9)^{1/8}}
\]

which is used to the last member of the right hand side in (3.8) and factoring gives

\[
D_n > \frac{4}{\pi} \frac{n(2n + 1)}{(n + 1)(n + 2)(n + 3)} - \frac{3}{7\pi} \frac{n}{(7/8)^{1/8} n + 1}
\]

\[
+ \frac{128}{7\pi} \frac{n^{7/8}}{64n - 9} \sum_{k=2}^{n} \left( \frac{n - k}{n - k + 1} \frac{2}{(k - 1)(k + 18)} \right),
\]

which proves the step.

**Step 4:** The sequence \( g_1 (n) := \sum_{k=2}^{n} \alpha_{n-k} \beta_k \) defined by (3.7) can be expressed as

\[
g_1 (n) = \frac{n^3 + 7n^2 - 12n + 18}{(n + 2)(n + 3)(n + 4)} (\psi (n + 1) + \gamma) - \frac{n(n^2 + 8n + 21)}{(n + 1)(n + 2)(n + 3)(n + 4)},
\]

where \( \psi (t) \) denotes the psi function, \( \gamma \) is the Euler’s constant.

In fact, decomposing rational function \( \alpha_{n-k} \beta_k \) into partial fractions gives

\[
\alpha_{n-k} \beta_k = \frac{n - k}{n - k + 1} \frac{(k - 1)(k + 18)}{(k + 1)(k + 2)(k + 3)}
\]

\[
= -17 \frac{n + 1}{n + 2} \frac{1}{k + 1} + 48 \frac{n + 2}{n + 3} \frac{1}{k + 2}
\]

\[
-30 \frac{n + 3}{n + 4} \frac{1}{k + 3} - \frac{n(n + 19)}{(n + 2)(n + 3)(n + 4)} \frac{1}{n - k + 1}.
\]
Hence, we get

\[
\sum_{k=2}^{n} \alpha_{n-k} \beta_k = -17 \frac{n+1}{n+2} \left( \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} - \frac{3}{2} \right) \\
+ 48 \frac{n+2}{n+3} \left( \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} + \frac{1}{n+2} - \frac{11}{6} \right) \\
- 30 \frac{n+3}{n+4} \left( \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} - \frac{25}{12} \right) \\
- \frac{n(n+19)}{(n+2)(n+3)(n+4)} \left( \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{n} \right)
\]

\[
= \frac{n^3 + 7n^2 - 12n + 24}{(n+2)(n+3)(n+4)} \sum_{k=1}^{n} \frac{1}{k} - \frac{n(11n^2 + 8n + 21)}{(n+1)(n+2)(n+3)(n+4)}
\]

which by the identity \( \sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) + \gamma \) proves the step.

**Step 5:** We show that the sequence \( g(n) \) defined by (3.6) satisfies the inequality

\[
(3.10) \quad g(n) > \frac{128}{7\pi} \frac{n^{7/8} n^3 + 7n^2 - 12n + 24}{64n - 9} \frac{n(11n^2 + 8n + 21)}{(n+1)(n+2)(n+3)(n+4)} g_2(n),
\]

for \( n \geq 4 \), where

\[
g_2(n) = \ln (n + 1/2) + \gamma - \frac{n(11n^2 + 8n + 21)}{(n+1)(n^3 + 7n^2 - 12n + 24)} \\
+ \frac{7}{32} \sqrt{n} \frac{(n+1)(2n+1)(64n-9)}{(n+1)(n^3 + 7n^2 - 12n + 24)} - \frac{3\pi}{128\Gamma(7/8)} \frac{(64n-9)(n+2)(n+3)(n+4)}{(n+1)(n^3 + 7n^2 - 12n + 24)}.
\]

The inequality (3.10) follows by the Lemma 5.

**Step 6:** We show that \( g_2(x) > 0 \) for \( x \geq 10 \).

Differentiation yields

\[
g_2'(x) = -\frac{7}{256} \frac{(896x^7 + 7346x^6 + 41013x^5 - 3438x^4 - 149813x^3 - 227064x^2 - 40824x + 864)}{x^{7/8}(x^3 + 8x^2 - 5x^2 + 12x + 24)^2} \\
+ \frac{3\pi}{128\Gamma(7/8)} \frac{(55x^6 + 3066x^5 + 17101x^4 + 216x^3 - 71514x^2 - 73824x - 33840)}{(x^4 + 8x^3 - 5x^2 + 12x + 24)^2} \\
+ \frac{2}{2x + 1} + \frac{11x^6 + 16x^5 + 182x^4 + 72x^3 - 993x^2 - 384x - 504}{(x^4 + 8x^3 - 5x^2 + 12x + 24)^2}.
\]

Application of the second inequality of (2.3) in Lemma 4 gives

\[
\Gamma\left(\frac{7}{8}\right) < \left[ \frac{x^2 + 2}{(x + 2)x} \right]_{x=7/8} = \frac{177}{101} < \frac{6}{5}
\]
and since $\pi > 3$, it is acquired that

\[ g'_2(x) > \frac{7}{256} \left( \frac{896x^7 + 7346x^6 + 41033x^5 - 3438x^4 - 149813x^3 - 227064x^2 - 40824x + 864}{x^{1/3}(x^3 + 6x^2 + 12x + 12)^2} \right) \]

\[ + \frac{3}{128} \cdot \frac{55x^6 + 3806x^5 + 17101x^4 + 216x^3 - 71514x^2 - 73824x - 33840}{x^3 + 8x^2 + 5x^2 + 12x + 12} \]

\[ + \frac{2}{2x + 1} \left( \frac{512x^6 + 15474x^7 + 153661x^6 + 638728x^5 + 482131x^4 - 2496996x^3 - 3787398x^2 - 2184000x - 341712}{(x^3 + 8x^2 + 5x^2 + 12x + 12)^2} \right) \]

\[ = \frac{1}{256(x^3 + 8x^2 + 5x^2 + 12x + 12)^2} \left( g_3(x) - \frac{7g_2(x)}{x^{1/3}} \right), \]

where

\[ g_3(x) = 896x^7 + 7346x^6 + 41033x^5 - 3438x^4 - 149813x^3 - 227064x^2 - 40824x + 864, \]

\[ g_4(x) = 512x^8 + 15474x^7 + 153661x^6 + 638728x^5 + 482131x^4 - 2496996x^3 - 3787398x^2 - 2184000x - 341712. \]

Making a change of variable $t = x - 3 \geq 7$ yields

\[ g_3(x) = 896t^7 + 26162t^6 + 342605t^5 + 2450487t^4 + 10008901t^3 \]

\[ + 22815555t^2 + 26081658t + 10797192 > 0, \]

\[ g_4(x) = 512t^8 + 27762t^7 + 607639t^6 + 7103356t^5 + 48333256t^4 \]

\[ + 194587122t^3 + 448344153t^2 + 530387220t + 235084068 > 0. \]

Thus, to prove that $g'_2(x) > 0$ for $x \geq 10$, it suffices to prove that

\[ g_5(x) := \ln \frac{g_4(x)}{2x + 1} - \ln \left( \frac{7g_2(x)}{x^{1/3}} \right) > 0. \]

Differentiation again leads to

\[ g'_5(x) = \frac{g'_4(x)}{g_4(x)} - \frac{2}{2x + 1} \frac{g'_3(x)}{g_3(x)} + \frac{7}{8} \frac{1}{x} \]

\[ = \frac{1}{8} \left( x^4 + 8x^3 + 5x^2 + 12x + 24 \right) \frac{g_6(x)}{g_3(x)g_4(x)}, \]

where

\[ g_6(x) = 6422528x^{12} + 40606280x^{11} - 29604936x^{10} - 195118044x^9 \]

\[ - 8157468886x^8 - 54727744833x^7 - 28074816632x^6 \]

\[ - 33746602635x^5 - 132036870576x^4 - 76358742474x^3 \]

\[ - 16962249960x^2 - 1692953136x - 86111424. \]

Lemma \[\text{implies that the polynomial } g_6(x) \text{ has a unique zero point } x_0 \in (0, \infty) \]

such that $g_6(x) < 0$ for $x \in (0, x_0)$ and $g_6(x) > 0$ for $x \in (x_0, \infty)$. This in combination with $g_5(7) = 56640373211408308 > 0$ indicates that $g_6(x) > 0$ for $x \geq 7$. Therefore, $g'_5(x) > 0$ for $x \geq 7$, and so

\[ g_5(x) \geq g_5(7) = \ln 4460816168 - \frac{1}{8} \ln 7 - \ln 15 - \ln 2220734176 = 0.04879 > 0, \]
Proof of Theorem 2. Utilizing (1.4) gives the coefficients $1$ and $22/7$ where the inequality holds due to the increasing property of Stolarsky means in $d_g$ which reveals that $g_2(x) > 0$ for $x \geq 10$. It follows that

$$g_2(x) \geq g_2(10) = \gamma + \ln \frac{21}{2} - \frac{516789}{252304} \ln \frac{\pi}{7} + \frac{6499299}{252304} \sqrt{10} - \frac{6905}{88922} \approx 0.037141 > 0.$$  

**Step 7:** Steps 6 and 5 show that $g(n) > 0$ for $n \geq 10$, which in conjunction with Step 3 yield

$$D_n = \frac{8}{7} d_{n+1} - d_n > g(n) > 0,$$

that is, $d_{n+1} > (7/8) d_n$ for $n \geq 10$. Taking into account Step 2, we conclude that $d_n = 0$ for $n = 0, 1, 2, 3$ and $d_n > 0$ for $n \geq 4$, and therefore, $f_7(r) = f_5(r) - r^{7/4} f_6(r) > 0$ for $r \in (0, 1)$. Thus the function $f_1/f_2$, that is, $F$, is strictly decreasing on $(0, 1)$.

It follows from the decreasing property of the function $F$ on $(0, 1)$ that

$$\frac{11(\pi - 2)}{4\pi} = \lim_{r \to 1^-} F(r) < F(r) < \lim_{r \to 0^+} F(r) = 1,$$

which implies inequalities (1.18) and the first and second ones in (1.19). The third one in (1.19) is equivalent to

$$\frac{22-7\pi}{4\pi} + \frac{11(\pi - 2)}{4\pi} S_{11/4,7/4}(1, r') - \frac{22}{\pi} S_{11/4,7/4}(1, r') = -\frac{11(22-7\pi)}{8\pi} \left(S_{11/4,7/4}(1, r') - \frac{7}{11}\right) < 0,$$

where the inequality holds due to the increasing property of Stolarsky means in their variables (P1) and $r' \in (0, 1)$.

Since

$$\lim_{r \to 0^+} \frac{2\pi}{S_{11/4,7/4}(1, r')} = 1 \quad \text{and} \quad \lim_{r \to 1^-} \frac{2\pi}{S_{11/4,7/4}(1, r')} = \frac{22}{7\pi},$$

the coefficients $1$ and $22/(7\pi)$ are the best.

Thus we complete the proof. \( \square \)

**Remark 2.** From the Step 7 in previous proof, we have $d_{n+1} > (7/8) d_n$ for $n \geq 10$. This is also valid for $4 \leq n \leq 9$ by an easy verification. It is derived that $d_n > (7/8)^{n-4} d_4$ for $n \geq 5$, and so we get

$$\sum_{n=0}^{\infty} d_n r^{2n} = \sum_{n=4}^{\infty} d_n r^{2n} > d_4 \sum_{n=4}^{\infty} \left(\frac{7}{8}\right)^n r^{2n} = \frac{35}{4096} \frac{r^8}{8 - 7r^2}.$$  

Thus we obtain that

$$f_7(r) = f_5(r) - r^{7/4} f_6(r) = \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} d_n r^{2n} > \frac{105\pi}{216} \frac{r^{14}}{8 - 7r^2}.$$  

4. **Proof of Theorem 2**

Now we prove Theorems 2.

**Proof of Theorem 2.** Utilizing (1.4) gives

$$1 - \frac{2}{\pi} E(r) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n! n} \left(\frac{1}{2}\right)_n r^{2n} = \sum_{n=1}^{\infty} v_n r^{2n}.$$
Applying binomial series to (1.17) we have

\[ S_{5/2,2}(1, r') = \left( \frac{41 - r^{5/2}}{5 - 1 - r^2} \right)^2 = \frac{16}{25} \left( \frac{(1 - r^2)^{5/2} - 2 \left( 1 - r^2 \right)^{5/4} + 1}{r^4} \right) \]

\[ = \frac{16}{25} \frac{1}{r^4} \left( \sum_{n=0}^{\infty} \frac{(-\frac{5}{4})^n}{n!} r^{2n} - 2 \sum_{n=1}^{\infty} \frac{(-\frac{5}{4})^n}{n!} r^{2n} + 1 \right) \]

\[ = 1 - \frac{16}{25} \sum_{n=1}^{\infty} \frac{2 \left( -\frac{5}{4} \right) u_{n+2} - \left( -\frac{5}{4} \right) v_{n+2} r^{2n}}{(n+2)!}, \]

which implies that

\[ 1 - S_{5/2,2}(1, r') = \frac{16}{25} \sum_{n=1}^{\infty} \frac{2 \left( -\frac{5}{4} \right) u_{n+2} - \left( -\frac{5}{4} \right) v_{n+2} r^{2n}}{(n+2)!}, \]

Thus, the function \( G(r) \) can be expressed as

\[ G(r) = \frac{1 - (2/\pi) E(r)}{1 - S_{5/2,2}(1, r')} = \sum_{n=1}^{\infty} \frac{v_n}{u_n} r^{2n}, \]

where, by the formula \((a)_n = a(a+1)\cdots(n-1)\), \(v_n\) and \(u_n\) can be written as

\[ v_n = \frac{1}{2} \frac{1}{n!} \left( \frac{1}{2} \right)_{n-1} \left( \frac{1}{2} \right)_n, \]

\[ u_n = \frac{6}{5} \frac{1}{(n+2)!} \left( \frac{1}{2} \right)_{n-1} + \frac{2}{5} \frac{1}{(n+2)!} \left( \frac{3}{4} \right)_n. \]

By Lemma 2 due to \( u_{n+1} > 0 \) for \( n \geq 1 \), to prove the function \( G(r) \) is increasing on \((0, 1)\), it suffices to prove the sequence \( \{v_n/u_n\} \) is increasing for \( n \geq 1 \). In view of \( v_n > 0 \) for \( n \geq 1 \), which suffices to check that \((v_{n+1}/v_n) u_n - u_{n+1} > 0\).

A simple verification yields

\[ \frac{v_{n+1}}{v_n} = \frac{(n-1/2)(n+1/2)}{(n+1)^2}, \]

and

\[ \frac{v_{n+1}}{v_n} u_n - u_{n+1} = \frac{3}{5} \frac{(3n+1)}{(n+1)^2} \frac{1}{(n+3)!} \left( \frac{1}{2} \right)_n + \frac{1}{10} \frac{n^2 - 11n - 6}{(n+1)^2} \frac{1}{(n+3)!} \left( \frac{3}{4} \right)_n. \]

Direct computations give

\[ \frac{v_{n+1}}{v_n} u_n - u_{n+1} = 0, \quad 0, \quad 0, \quad 3, \quad 84920, \quad 8124000, \quad 81107200, \quad 84225280, \quad 671088400, \quad 2040593573, \quad 2061584302080 \]

for \( n = 1, 2, \ldots, 11 \), respectively. And, for \( n \geq 12 \), it is evident that \((v_{n+1}/v_n) u_n - u_{n+1} > 0\) due to \( n^2 - 11n - 6 > 0 \).

Consequently, we obtain

\[ 1 = \lim_{r \to 0^+} G(r) < G(r) < \lim_{r \to 1^-} G(r) = \frac{25 (\pi - 2)}{9 \pi}, \]

which implies inequalities (1.20) and the first and second ones in (1.21). The third one in (1.21) is equivalent to

\[ \frac{25(8\pi - 25)}{9 \pi} S_{2,5/2}(1, r') \cdot \frac{25}{8} \left( S_{2,5/2}(1, r') - \frac{16}{25} \right) > 0, \]
where the inequality holds due to the increasing property of Stolarsky means in their variables (P1) and \( r' \in (0, 1) \).

Since
\[
\lim_{r \to 0^+} \frac{(2/\pi) E(r)}{S_{5/2,2}(1, r')} = 1 \quad \text{and} \quad \lim_{r \to 1^-} \frac{(2/\pi) E(r)}{S_{5/2,2}(1, r')} = \frac{25}{8\pi},
\]
the coefficients 1 and 25/(8\pi) are the best.

This completes the proof. \( \square \)

**Remark 3.** Denote by \( w_n = v_n - u_n \). It is easy to verify that the relation
\[
w_{n+1} - \frac{v_{n+1}}{v_n} w_n = \frac{v_{n+1}}{v_n} u_n - u_{n+1}
\]
holds for \( n \geq 0 \). From the proof of Theorem 2 we clearly see that
\[
(4.1) \quad w_{n+1} - \frac{v_{n+1}}{v_n} w_n > 0
\]
for \( n \geq 3 \), which, due to \( w_1 = w_2 = w_3 = 0 \), \( w_4 = 3 \times 2^{-14}/5 > 0 \), means that \( w_n > 0 \) for \( n \geq 4 \). This also yields
\[
S_{5/2,2}(1, r') - \frac{2}{\pi} E(r) = \sum_{n=1}^{\infty} v_n r^{2n} - \sum_{n=1}^{\infty} u_n r^{2n} = r^8 \sum_{n=4}^{\infty} w_n r^{2n-8},
\]
so we have

**Proposition 1.** The function
\[
G_1(r) = \frac{S_{5/2,2}(1, r') - \frac{2}{\pi} E(r)}{r^8}
\]
is convex and strictly increasing from \((0, 1)\) onto \(((3/5)2^{-14}, 16/25 - 2/\pi)\). Consequently, we have
\[
\frac{3}{5} \times 2^{-14} < \frac{S_{5/2,2}(1, r') - (2/\pi) E(r)}{r^8} < \frac{16}{25} - \frac{2}{\pi} \approx 0.003802.
\]

**Remark 4.** Further, the relation (4.1) also indicates that for \( n \geq 4 \)
\[
w_n > \frac{w_4}{w_4} v_n = \frac{3 \times 2^{-14}/5}{175 \times 2^{-14}} v_n = \frac{3}{875} v_n,
\]
and therefore,
\[
S_{5/2,2}(1, r') - \frac{2}{\pi} E(r) = \sum_{n=1}^{\infty} u_n r^{2n} > \frac{3}{875} \sum_{n=4}^{\infty} v_n r^{2n} = \frac{3}{875} \left( \sum_{n=1}^{\infty} v_n r^{2n} - \sum_{n=1}^{3} v_n r^{2n} \right)
\]
\[
= \frac{3}{875} \left( 1 - \frac{2}{\pi} E(r) - \left( \frac{1}{4} r^2 + \frac{3}{64} r^4 + \frac{5}{256} r^6 \right) \right),
\]
which can be stated as a proposition as follows:

**Proposition 2.** For \( r \in (0, 1) \), it holds that
\[
\frac{2}{\pi} E(r) < \frac{875}{872} S_{5/2,2}(1, r') - \frac{3}{872} \left( 1 - \frac{1}{4} r^2 - \frac{3}{64} r^4 - \frac{5}{256} r^6 \right).
\]
5. Corollaries

As direct consequences of Theorems 1 and 2 we have

**Corollary 1.** Both the functions

\[ r \mapsto \frac{2}{\pi} E(r) - S_{11/4,7/4}(1, r') \quad \text{and} \quad r \mapsto \frac{(2/\pi) E(r)}{S_{11/4,7/4}(1, r')} \]

are strictly increasing from \((0, 1)\) onto \((0, 2/\pi - 7/11)\) and \((1, 22/(7\pi))\), respectively. And therefore, we have

\begin{align*}
(5.1) & \quad 0 < \frac{2}{\pi} E(r) - S_{11/4,7/4}(1, r') < \frac{2}{\pi} - \frac{7}{11} \approx 0.00025614, \\
(5.2) & \quad 1 < \frac{(2/\pi) E(r)}{S_{11/4,7/4}(1, r')} < \frac{22}{7\pi} \approx 1.0004.
\end{align*}

**Proof.** By the increasing property of Stolarsky means in their variables and \(r' = \sqrt{1 - r^2}\), it is seen that \(S_{11/4,7/4}(1, r')\) is decreasing with respect to \(r\) on \((0, 1)\), and so both the functions

\[ r \mapsto (1 - S_{11/4,7/4}(1, r')) \quad \text{and} \quad r \mapsto \frac{1}{S_{11/4,7/4}(1, r')} \]

are positive and increasing on \((0, 1)\).

From Theorem 1 we see that \(r \mapsto (1 - F(r))\) is positive and strictly increasing on \((0, 1)\). It follows from the identity

\[ \frac{2}{\pi} E(r) - S_{11/4,7/4}(1, r') = (1 - F(r)) (1 - S_{11/4,7/4}(1, r')) \]

that the function \(r \mapsto (2/\pi) E(r) - S_{11/4,7/4}(1, r')\) is also positive and strictly increasing on \((0, 1)\). Therefore, the double inequality \((5.1)\) is valid.

Making use of the assertion proved previously, and noting that

\[ \frac{(2/\pi) E(r)}{S_{11/4,7/4}(1, r')} = 1 + \left( \frac{2}{\pi} E(r) - S_{11/4,7/4}(1, r') \right) \times \frac{1}{S_{11/4,7/4}(1, r')} \]

gives the desired assertion. Then the estimate inequalities \((5.2)\) follow. \(\square\)

Using the same technique we can prove

**Corollary 2.** Both the functions

\[ r \mapsto \frac{2}{\pi} E(r) - S_{5/2,2}(1, r') \quad \text{and} \quad r \mapsto \frac{(2/\pi) E(r)}{S_{5/2,2}(1, r')} \]

are strictly decreasing from \((0, 1)\) onto \((2/\pi - 16/25, 0)\) and \((25/(8\pi), 1)\). And therefore, we have

\begin{align*}
(5.3) & \quad -0.0033802 \approx \frac{2}{\pi} - \frac{16}{25} < \frac{2}{\pi} E(r) - S_{2.5/2}(1, r') < 0, \\
(5.4) & \quad 0.99472 \approx \frac{25}{8\pi} < \frac{(2/\pi) E(r)}{S_{2.5/2}(1, r')} < 1.
\end{align*}

Now we give a monotonicity property for the ratio \((2/\pi) E(r) / S_{9/2-p,p}(1, r')\) with respect to \(r\) for \(p \in (-\infty, 9/4]\). To this end, we need a known statement proved in [41 Theorem 5] by Losonczi (see also [56 Theorem 3.4]).
Lemma 7. For fixed $c > 0$, $0 < x < y < z$, the function
\[ p \mapsto \frac{S_{2c-p,p}(x,y)}{S_{2c-p,p}(x,z)} := R_{2c-p,p}(x,y,z) \]
is strictly decreasing on $(-\infty, c]$ and strictly increasing on $[c, \infty)$.

Corollary 3. Let $p \in (-\infty, 9/4)$. The function
\[ r \mapsto \frac{(2/\pi)E(r)}{S_{9/2-p,p}(1,r')} := R_p(r) \]
is strictly increasing from $(0,1)$ onto $(1,2/(\pi \theta_p))$ if and only if $p \in (-\infty, 7/4]$ and strictly decreasing from $(0,1)$ onto $(2/(\pi \theta_p), 1)$ if $p \in [2, 9/4]$. Consequently, we have
\begin{align*}
(5.5) \quad S_{9/2-p,p}(1,r') &< \frac{2}{\pi}E(r) < \frac{2}{\pi \theta_p} S_{9/2-p,p}(1,r') \quad \text{if } p \in (0, \frac{7}{4}], \\
(5.6) \quad \frac{2}{\pi \theta_p} S_{9/2-p,p}(1,r') &< \frac{2}{\pi}E(r) < S_{9/2-p,p}(1,r') \quad \text{if } p \in [\frac{2}{4}, \frac{9}{4}],
\end{align*}
where the coefficients $1$ and $2/(\pi \theta_p)$ are the best possible, here $\theta_p$ is defined by \([1.14]\).

Proof. (i) The necessity can be derived from
\[ \lim_{r \to 0^+} \frac{d \left( \ln R_p(r) \right)/dr}{8r^7} \geq 0. \]
From $\lim_{r \to 0^+} R_p(r) = \lim_{r \to 0^+} S_{9/2-p,p}(1,r') = 1$ and L’Hospital rule it is obtained that
\[ 1 = \lim_{r \to 0^+} \left( \frac{\ln R_p(r)}{R_p(r) - 1 S_{9/2-p,p}(1,r')} \right) \]
\[ = \lim_{r \to 0^+} \left( \frac{2/\pi E(r) - S_{9/2-p,p}(1,r')}{\ln R_p(r)} \right) = \lim_{r \to 0^+} \frac{d \left( \ln R_p(r) \right)/dr}{d \left( \frac{2/\pi E(r) - S_{9/2-p,p}(1,r')}{8r^7} \right)/dr} \]
By the expansion \([1.15]\) and L’Hospital rule we have
\[ \frac{(4p-7)(4p-11)}{5 \times 2^{14}} = \lim_{r \to 0^+} \frac{d \left( \frac{2/\pi E(r) - S_{9/2-p,p}(1,r')}{8r^7} \right)/dr}{d \left( \frac{2/\pi E(r) - S_{9/2-p,p}(1,r')}{8r^7} \right)/dr} \]
It follows that
\[ \lim_{r \to 0^+} \frac{d \left( \ln R_p(r) \right)/dr}{8r^7} = \lim_{r \to 0^+} \left( \frac{d \left( \frac{2/\pi E(r) - S_{9/2-p,p}(1,r')}{8r^7} \right)/dr}{d \left( \frac{2/\pi E(r) - S_{9/2-p,p}(1,r')}{8r^7} \right)/dr} \right) \]
\[ = \frac{(4p-7)(4p-11)}{5 \times 2^{14}}, \]
which together with $p \in (-\infty, 9/4]$ gives the necessary condition $p \in (-\infty, 7/4]$.

(ii) To prove that the condition $p \in (-\infty, 7/4]$ is necessary, we have to prove that the function
\[ r \mapsto \frac{S_{9/2-po,p}(1,r')}{S_{9/2-p,p}(1,r')} := R^*_{po,p}(r') \]
Remark 5. Increasing on $(0,1)$ if $p < (>) p_0$. Due to $r' = \sqrt{1 - r^2}$, it suffices to prove that $r' \mapsto R_{p_0,p}(r')$ is strictly decreasing (increasing) on $(0,1)$ if $p < (>) p_0$. Assume that $r_1', r_2' \in (0,1)$ with $r_1' < r_2'$. Then $1 < 1/r_2' < 1/r_1'$. By Lemma 7, we have

$$
\frac{S_{2c-p_0,p_0}(1,1/r_2')} {S_{2c-p_0,p_0}(1,1/r_1')} < (>) \frac{S_{2c-p_0,p}(1,1/r_2')} {S_{2c-p_0,p}(1,1/r_1')},
$$

which is equivalent to

$$
\frac{S_{2c-p_0,p_0}(1,1/r_2')} {S_{2c-p_0,p}(1,1/r_2')} < (>) \frac{S_{2c-p_0,p}(1,1/r_1')} {S_{2c-p_0,p}(1,1/r_1')}.
$$

From the homogeneity and symmetry of Stolarsky means with respect to their variables, this shows that $r' \mapsto R_{p_0,p}(r')$ is strictly decreasing (increasing) on $(0,1)$ if $p < (>) p_0$.

Now, if $p_0 = 7/4$ and $p \in (-\infty, 7/4)$, then $r' \mapsto R_{p_0,p}(r')$ is positive and strictly increasing on $(0,1)$. While Corollary 1 tells us that $r \mapsto R_{p_0}(r)$ is also strictly increasing on $(0,1)$.

Corollary 2 reveals that $r \mapsto R_{p}(r)$ is also strictly increasing on $(0,1)$.

The proof is finished.

Remark 5. Let $p = 9/8, 3/2, 7/4; 2, 9/4$ in Corollary 3. Then by the monotonicity of $S_{2c-p_0,p}$ and $(1/\theta_p) S_{2c-p_0,p}$ in $p$ on $(0,c)$ given in (P2) and (P3) we have

$$
He_{9/4}(1,r') < A_{3/2}(1,r') < S_{11,4/7,4}(1,r') < \frac{2}{\pi} E(r)
$$

$$
< \frac{22}{7\pi} S_{11,4/7,4}(1,r') < \frac{25/3}{\pi} A_{3/2}(1,r') < \frac{2 \times 34/9}{\pi} He_{9/4}(1,r'),
$$

$$
\frac{2e^{4/9}}{\pi} I_{9/4}(1,r') < \frac{25}{8\pi} S_{5/2,2}(1,r') < \frac{2}{\pi} E(r) < S_{5/2,2}(1,r') < I_{9/4}(1,r'),
$$

where $He_p(a,b) = He(a^p,b^p)^{1/p}$ and $I_p(a,b) = I(a^p,b^p)^{1/p}$ are the $p$-order Hermite mean and identric (exponential) mean of positive numbers $a$ and $b$, respectively.

Using expansion (1.15) and Corollaries 1, 2 together with the property of Stolarsky means (P2), we obtain immediately

Corollary 4. For $p \in (-\infty, 9/4]$, the inequality

$$
\frac{2}{\pi} E(r) > S_{9/2-p_0,p}(1,r')
$$

holds for all $r \in (0,1)$ if and only if $p \in (-\infty, 7/4]$. The inequality (5.7) reverses for $p \in [2, 9/4]$. 

Remark 6. Taking \( p = -9/2, -9/4, 0, 9/8, 3/2, 7/4; 2, 9/4 \) in Corollary 3, we get immediately
\[
A_{3/2} (1, r') \quad < \quad \sqrt{G (1, r') H \epsilon_{9/2} (1, r')} \quad < \quad L_{9/2} (1, r') \quad < \quad H \epsilon_{9/4} (1, r')< S_{11/4,7/4} (1, r') < \frac{2}{\pi} E (r') < S_{5/2,2} (1, r') < I_9 (1, r')
\]
(5.8)
holds for \( r \in (0, 1) \), where \( L_\alpha (a, b) = L (a^\alpha, b^\alpha)^{1/p} \) is the \( p \)-order logarithmic mean of positive numbers \( a \) and \( b \).

Remark 7. For \( a, b > 0 \) with \( a \neq b \), the Toader mean \( T (a, b) \) is defined in [57] by
\[
T (a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} \, dt.
\]
An easy transformation yields
\[
T (a, b) = \begin{cases} 
\frac{2}{\pi} a E \left( \sqrt{1 - (b/a)^2} \right) & \text{if } a > b, \\
\frac{2}{\pi} b E \left( \sqrt{1 - (a/b)^2} \right) & \text{if } a < b.
\end{cases}
\]
Thus, all our results can be rewritten in the form of Toader mean, for example, inequalities [1.19, 1.24] and [5.5] are equivalent to
\[
S_{11/4,7/4} (a, b) < T (a, b) < \frac{22}{4\pi} \max (a, b) + \frac{11(\pi - 2)}{4\pi} S_{11/4,7/4} (a, b) < \frac{22}{7\pi} S_{11/4,7/4} (a, b),
\]
\[
\frac{25}{8\pi} S_{2,5/2} (a, b) < \frac{-2(8\pi - 25)}{9\pi} \max (a, b) + \frac{25(\pi - 2)}{9\pi} S_{2,5/2} (a, b) < T (a, b) < S_{2,5/2} (a, b),
\]
\[
A_{3/2} (a, b) \quad < \quad \sqrt{G(a, b) H \epsilon_{9/2} (a, b)} \quad < \quad L_{9/2} (a, b) \quad < \quad H \epsilon_{9/4} (a, b) \quad < \quad S_{11/4,7/4} (a, b) \quad < \quad T (a, b) \quad < \quad S_{5/2,2} (a, b) \quad < \quad I_9 (a, b).
\]

6. Comparisons with some known approximations

Let \( \mathcal{A} (r) \) be the given approximation for \((2/\pi) E (r)\) and let \( \Delta (r) = \mathcal{A} (r) - (2/\pi) E (r) \) denote the error. In general, the most important criterion to measure of accuracy of the given approximation \( \mathcal{A} (r) \) for \((2/\pi) E (r)\) should be the maximum absolute error \( \max_{r \in (0, 1)} \left| \Delta (r) \right| \). Due to \( r \in (0, 1) \), however, similar to Barnard et al.’s opinion in [9], if \( \Delta (r) = \varepsilon \eta \varepsilon_{2n0} + O (\varepsilon_{2n0}^{-2}) \) with \( \varepsilon \eta_0 \neq 0 \) by expanding in Maclaurin series, then the leading item \( \delta_0 := \varepsilon \eta_0 \varepsilon_{2n0} \) can be viewed as a measure of accuracy of the given approximation \( \mathcal{A} (r) \) for \((2/\pi) E (r)\). For this reason, we call \( \mathcal{A} (r) \) an \( n_0 \)-order approximation for \((2/\pi) E (r)\). And, \( \mathcal{A} (r) \) is called an \( n_0 \)-order lower (upper) approximation for \((2/\pi) E (r)\) if \( \Delta (r) \leq (\geq) 0 \) for all \( r \in (0, 1) \).

In most cases, the greater the order \( n_0 \) is, the higher the accuracy of approximation \( \mathcal{A} (r) \) for \((2/\pi) E (r)\) is.

Of course, a desirable approximation \( \mathcal{A} (r) \) also has the simplicity of the expression. Unfortunately, there exists frequently, certain negative correlation between the accuracy and simplicity.

Now we choose those approximations which have higher accuracy mentioned in Introduction to compare with our ones.
6.1. For some lower approximations. In Tables 1, the values in the third column are derived by expanding in power series, while \( \max_{r \in (0, 1)} |\Delta_i(r)| \) for \( i = 1, 2, 5 \) in the fourth column are from [8, Theorem 1.1], [21, Theorem 2.5], (5.1), respectively.

| \( A_i(r') \) | Expressions | \( \delta_0^{(i)} := \varepsilon_{n_0} 2^{n_0} \) | \( \max_{r \in (0, 1)} |\Delta_i(r)| \) |
|---|---|---|---|
| \( A_1(r') \) | \( A_{23/2}(1, r') \) | \( \frac{1}{2^{12}} r^{12} \) | \( 2 - \frac{23 - 2 \sqrt{2}}{\pi} \approx 0.0062581 \) |
| \( A_2(r') \) | \( \frac{23 H(1, r') - 5 H(1, x') - 25(1, r')}{16} \) | \( \frac{1}{2^{12}} r^{12} \) | \( 2 - \frac{23 - 2 \sqrt{2}}{\pi} \approx 0.0062581 \) |
| \( A_3(r') \) | \( \frac{1}{128} \left( \frac{(9x^4 + 14x^4 + 9)^2}{(r+1)^4} \right) \) | \( \frac{1}{2^{12}} r^{12} \) | \( 2 - \frac{23 - 2 \sqrt{2}}{\pi} \approx 0.0062581 \) |
| \( A_4(r') \) | \( \frac{1}{2^{21} r^6 + 1 + r^6}{8} \) | \( \frac{2048}{2^{21} r^6} \) | \( 2 - \frac{23 - 2 \sqrt{2}}{\pi} \approx 0.0062581 \) |
| \( A_5(r') \) | \( S_{11/4, 7/4}(1, r') \) | \( \frac{1}{72} 2^{12} r^{12} \) | \( 2 - \frac{23 - 2 \sqrt{2}}{\pi} \approx 0.0062581 \) |

Moreover, we can prove

**Lemma 8.** Let \( A_i(x) (i = 1, 2, 3, 4, 5) \) be defined on \((0, \infty)\) by (6.5), (6.10), (6.11), (6.12) and (1.10), respectively (see also Table 1). Then the inequalities

\[
A_5(x) > A_3(x) > A_2(x) > A_1(x) > A_4(x)
\]

hold for \( x > 0 \) with \( x \neq 1 \).

**Proof.** (i) The first inequality in (6.1) is equivalent to

\[
A_5(x^4) - A_3(x^4) = \frac{7}{11} \frac{(x^{11} - 1)}{(x^7 - 1)} - \frac{(9x^8 + 14x^4 + 9)^2}{128 (x^4 + 1)^3} > 0
\]

Factoring yields

\[
A_5(x^4) - A_3(x^4) = \frac{1}{1408} \frac{(x - 1)^5}{(x^7 - 1)(x^4 + 1)^3} h_1(x) > 0,
\]

where

\[
h_1(x) = 5x^{16} + 35x^{15} + 140x^{14} + 420x^{13} + 966x^{12} + 1722x^{11} + 2268x^{10} + 2415x^9 + 2362x^8 + 2415x^7 + 2268x^6 + 1722x^5 + 966x^4 + 420x^3 + 140x^2 + 35x + 5.
\]

(ii) A direct computation leads to the second inequality in (6.1) is equivalent to

\[
A_3(x) - A_2(x) = \frac{1}{128} \frac{(9x^2 + 14x + 9)^2}{(x + 1)^3} - \frac{23 \frac{1}{2} + 2 \sqrt{1 + x^2}}{16} \frac{2}{1 + 2x} > 0,
\]

where the inequality holds due to

\[
\left( \frac{\sqrt{2(1 + x^2)}}{16} \right)^2 - \left( \frac{1}{128} \frac{36x + 34x^2 + 36x^3 + 11x^4 + 11}{(x + 1)^3} \right)^2 > 0.
\]

(iii) It has been shown in [21, Lemma 3.2] that the third inequality in (6.1) holds for \( x > 0 \) with \( x \neq 1 \).
Expressions

\[ \begin{align*}
A_1 (x^2)^3 - A_4 (x^2)^3 &= \left( \frac{1 + x^3}{2} \right)^2 - \left( \frac{1 + x^2 + x^4}{2 (1 + x^2)} + \frac{1 + x^2}{8} \right)^3 \\
&= \frac{(x - 1)^6 (3x^6 + 18x^5 - 3x^4 + 28x^3 - 3x^2 + 18x + 3)}{(x^2 + 1)^3} > 0.
\end{align*} \]

This completes the proof. \(\square\)

**Remark 8.** In Table 1, both \(A_1 (r')\) and \(A_4 (r')\) are 4-order lower approximations. And for the accuracy, since \(\varepsilon_4^{(1)} < \varepsilon_4^{(5)}\) and \(\max_{r \in (0, 1)} |\Delta_1 (r)| < \max_{r \in (0, 1)} |\Delta_4 (r)|\), so the former is better than the latter. And, the last inequality in (6.7) also proves this assertion.

While \(A_2 (r')\), \(A_3 (r')\) and \(A_5 (r') (= S_{11/4, 7/4} (1, r'))\) are 6-order lower approximations. In view of \(\varepsilon_6^{(3)} < \varepsilon_6^{(5)}\) and

\[ \max_{r \in (0, 1)} |\Delta_5 (r)| \ll \min \left( \max_{r \in (0, 1)} |\Delta_2 (r)|, \max_{r \in (0, 1)} |\Delta_3 (r)| \right), \]

we claim that the accuracy of \(A_5 (r') (= S_{11/4, 7/4} (1, r'))\) is far superior to \(A_2 (r')\) and \(A_3 (r')\). And, the first and second inequalities in (6.7) confirm similarly prove this claim.

To sum up, our lower approximation \(A_5 (r') (= S_{11/4, 7/4} (1, r'))\) for \((2/\pi) E (r)\) is the best of all five ones listed in Table 1.

6.2. For some upper approximations. In Tables 2, the values in the third column are derived by expanding in power series, while \(\max_{r \in (0, 1)} |\Delta_i (r)|\) for \(i = 7, 8\) in the fourth column are from [20] Theorem 3.2, [8, 9], respectively.

| \(A_i (r')\) | Expressions | \(\varepsilon_i^{(i)} := \varepsilon_i^{(1)} x^{2n_0}\) | max_{r \in (0, 1)} |\Delta (r)| |
| --- | --- | --- | --- |
| \(A_6 (r')\) | \(L_{1/4} (1, r')\) | \(- \frac{1}{\pi} \frac{\pi}{2}\) | \(\geq 1 - \frac{7}{10} \approx 0.36338\) |
| \(A_7 (r')\) | \(18A (1, r') - 5G (1, r') + 35 (1, r')\) | \(\frac{7}{28} r^{12}\) | \(\frac{18 + 3\sqrt{2} - \sqrt{2}}{10} \approx 0.058463\) |
| \(A_8 (r')\) | \(S_{5/2, 2} (1, r')\) | \(\frac{1}{28} x^8\) | \(\frac{3}{28} \approx 0.0033802\) |

Further we have

**Lemma 9.** Let \(A_i (x) (i = 6, 7, 8)\) be defined on \((0, \infty)\) by (1.8), (1.9), (1.17), respectively (see also Table 2). Then (i) the inequality

\[ \max (A_7 (x), A_8 (x)) < A_6 (x) \]

hold for \(x > 0\) with \(x \neq 1\); (ii) there is a \(x_0 \in (0, 1)\) such that \(A_8 (x) > A_7 (x)\) for \(x \in (0, x_0)\) and \(A_8 (x) < A_7 (x)\) for \(x \in (x_0, 1)\).

**Proof.** (i) It has been proven in [20] Lemma 5.1 that \(A_7 (x) < A_6 (x)\). To prove \(A_8 (x) < A_6 (x)\), it suffices to prove that \(A_8 (x^4) - A_6 (x^4) < 0\). Straightforward calculation gives

\[ A_8 (x^4) - A_6 (x^4) = \frac{16 (x^{10} - 1)^2}{25 (x^8 - 1)^2} - \frac{x^5 + 1}{x + 1} = - \frac{(x - 1)^6 (x + 1)}{25 (x^8 - 1)^2} \times (9x^8 + 20x^7 + 44x^6 + 60x^5 + 74x^4 + 60x^3 + 44x^2 + 20x + 9), \]

which is obviously negative for \(x > 0\) with \(x \neq 1\).
(ii) We have
\[
A_8(x^2) - A_7(x^2) = \frac{16 (x^5 - 1)^2}{25 (x^4 - 1)^2} - \left( \frac{91 + 2x^2}{8} \right) - \frac{5}{16x^2} + \frac{3}{16} \sqrt{\frac{1 + x^4}{2}}.
\]
(6.3) \\
\frac{h_2(x) - h_3(x)}{400} = \frac{h_2(x) - h_3(x)}{400 h_2(x) + h_3(x)},
where
\[
h_2(x) = \frac{31x^8 + 187x^7 + 118x^6 + 49x^5 + 430x^4 + 49x^3 + 118x^2 + 187x + 31}{(x^2 + 1)^2 (x + 1)^2},
\]
\[
h_3(x) = 75 \sqrt{\frac{1 + x^4}{2}}.
\]
Making a change of variable \(u = x + 1/x\) yields
\[
h_2(x) = x \frac{31u^4 + 187u^3 - 6u^2 - 512u + 256}{u^2 (u + 2)} \quad \text{and} \quad h_3(x) = 75x \sqrt{\frac{u^2 - 2}{2}},
\]
and therefore,
\[
h_2^2(x) - h_3^2(x) = \frac{x^2 (u - 2)^2}{2 u^4 (u + 2)^2} h_4(u),
\]
where
\[
h_4(u) = 3703u^6 + 14124u^5 - 16260u^4 - 98560u^3 - 23040u^2 + 98304u - 32768.
\]
Since \(u = x + 1/x \geq 2\), replacing \(u \) by \(v + 2\) leads us to
\[
h_4(v + 2) = 3703v^6 + 58560v^5 + 347160v^4 + 928800v^3 + 1014000v^2 + 1440000v - 288000,
\]
where \(v > 0\).

It follows from Lemma 6 that the polynomial \(h_4(v + 2)\) has a unique zero point \(v_1 \in (0, \infty)\) such that \(h_4'(v + 2) < 0\) for \(v \in (0, v_1)\) and \(h_4'(v + 2) > 0\) for \(v \in (v_1, \infty)\). Numeric computation gives \(v_1 \in (0.399475162, 0.399475163)\). This together with (6.4) and (6.3) reveals that
\[
A_8(x^2) - A_7(x^2) \begin{cases} > 0 & \text{for } x \in (0, x_1), \\ < 0 & \text{for } x \in (x_1, 1), \end{cases}
\]
where \(x_1 = 2 + v_1\), that is, \(x_1 = \left(v_1 + 2 - \sqrt{v_1(v_1 + 4)}\right)/2 \approx 0.53689\), which proves the second assertion, where \(x_0 = x_1^2 \approx 0.28825\).

This completes the proof. \(\square\)

Remark 9. From Table 2, as upper approximations, \(A_6(r') (= L_{1/4}(1,r'))\) and \(A_8(r') (= S_{5/2,3}(1,r'))\) have 8-order accuracy, but the facts \(|\varepsilon_4^{(6)}| < |\varepsilon_4^{(6)}|\) and \(\max_{r \in (0,1)} |\Delta_8(r')| < \max_{r \in (0,1)} |\Delta_8(r')|\) together with the inequality (6.3) show that the accuracy of \(A_8(r')\) is higher than \(A_6(r')\).

The upper approximation \(A_7(r')\) has 6-order accuracy, but its maximum absolute error is greater than our approximation \(A_8(r')\)’s. By the second assertion of Lemma 6, it is seen that there is a unique \(r_0 \in (0, 1)\) such that \((2/\pi) E(r) < A_7(r') < A_8(r')\) for \(r \in (0, r_0)\) and \((2/\pi) E(r) < A_8(r') < A_7(r')\) for \(r \in (r_0, 1)\), where \(r_0 = \sqrt{1 - x_0^2} \approx 0.95756\).
Conjecture 1. Let $\theta_p = 2/\pi$, that is,

$$\left(\frac{9/2-p}{p}\right)^{1/(2p-9/2)} = \frac{2}{\pi}$$

on $(0, 9/4]$. Then there is a $r_0 \in (0, 1)$ such that the function

$$H(r) = 1 - \frac{(2/\pi) E(r)}{1 - S_{9/2-p,p}(1, r')},$$

is strictly increasing on $(0, r_0)$ and strictly decreasing on $(r_0, 1)$. Consequently, it holds that

$$\frac{2}{\pi} E(r) < S_{9/2-p,r_0}(1, r')$$

for $r \in (0, 1)$ with the best constant $p_0$.

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