ON SPECIAL REGULARITY PROPERTIES OF SOLUTIONS OF THE BENJAMIN-ONO-ZAKHAROV-KUZNETSOV (BO-ZK) EQUATION

A. C. Nascimento
Universidade Federal do Piauí
Campus Universitário Ministro Petrópio Portella, Ininga, 64049-550
Teresina-PI, Brazil

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Abstract. In this paper we study special properties of solutions of the initial value problem (IVP) associated to the Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation. We prove that if initial data has some prescribed regularity on the right hand side of the real line, then this regularity is propagated with infinite speed by the flow solution. In other words, the extra regularity on the data propagates in the solutions in the direction of the dispersion. The method of proof to obtain our result uses weighted energy estimates arguments combined with the smoothing properties of the solutions. Hence we need to have local well-posedness for the associated IVP via compactness method. In particular, we establish a local well-posedness in the usual $L^2(\mathbb{R}^2)$-based Sobolev spaces $H^s(\mathbb{R}^2)$ for $s > \frac{5}{4}$ which coincides with the best available result in the literature proved employing more complicated tools.

1. Introduction. In this work we are interested in the study of some special properties for solutions of the Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation. We will consider the initial-value problem (IVP) for the Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation

$$\begin{align*}
\partial_t u + \alpha \mathcal{H} \partial_x^2 u + \partial_x \partial_y^2 u + u \partial_x u &= 0, \\
u(x, y, 0) &= \phi(x, y),
\end{align*}$$

(1.1)

where $u = u(x, y, t)$ is a real-valued function, $(x, y) \in \mathbb{R}^2$, $t > 0$, $\alpha \neq 0$, and $\mathcal{H}$ stands for the Hilbert transform defined as

$$\mathcal{H} u(x, y, t) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(z, y, t)}{x - z} dz.$$  

(1.2)

The equation (1.1) is a two-dimensional generalization of the Benjamin-Ono (BO) equation

$$\partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0, \quad u = u(x, t), \quad x \in \mathbb{R}, \quad t > 0,$$

(1.3)

when the effects of long wave lateral dispersion are included. The Benjamin-Ono equation (1.3) was proposed as a model for unidirectional long internal gravity waves in deep stratified fluids (see [2] and [43]).

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The BO-ZK equation can also be considered as a non-local version of the Zakharov-Kuznetsov (ZK) equation
\[
\partial_t u + \alpha \partial_x^3 u + \partial_x \partial_y^2 u + u \partial_x u = 0, \quad u = u(x, y, t), \quad (x, y) \in \mathbb{R}^2, \quad t > 0.
\]
(1.4)

The ZK equation (1.4) was introduced by Zakharov and Kuznetsov as a higher-dimensional extension of the Korteweg-de Vries model of surface wave propagation (see [45]).

The BO-ZK equation was deduced in [21] and [31] and has applications to electromigration in thin nanoconductors on a dielectric substrate.

We observe that the following quantities are conserved
\[
\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 dx dy
\]
and
\[
\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( (\partial_y u)^2 - \alpha u \partial_x u - \frac{u^3}{3} \right) dx dy.
\]
(1.6)

Recently, Isaza, Linares and Ponce [20] discovered a propagation of regularity result for suitable solutions to the k-generalized Korteweg-de Vries equation,
\[
\begin{align*}
\partial_t u + \partial_x^3 u + u^k \partial_x u &= 0, \quad x, t \in \mathbb{R}, \quad k \in \mathbb{Z}^+, \\
u(x, 0) &= u_0(x).
\end{align*}
\]
(1.7)

They established that the unidirectional dispersion of the k-generalized Korteweg-de Vries equation (1.7) produces the following propagation of regularity phenomena [20]: if for some \( l \in \mathbb{Z}^+ \)
\[
\| \partial_x^l u_0 \|^2_{L^2((0, \infty))} < \infty,
\]
(1.8)
then for positive times the corresponding local solution \( u = u(x, t) \) satisfies
\[
\| \partial_x^j u(\cdot, t) \|^2_{L^2((a, \infty))} < \infty \quad \text{for every } a \in \mathbb{R}.
\]
(1.9)

This result tell us that the regularity (1.8) moves with infinite speed to its left as time evolves. More precisely,

**Theorem 1.1** ([20]). If \( u_0 \in H^{3/4+} (\mathbb{R}) \) and for some \( l \in \mathbb{Z}^+ \), \( l \geq 1 \) and \( x_0 \in \mathbb{R} \)
\[
\| \partial_x^l u_0 \|^2_{L^2((x_0, \infty))} = \int_{x_0}^\infty |\partial_x^l u_0(x)|^2 dx < \infty,
\]
then the solution of the IVP (1.7) provided by the local theory in [27] satisfies that for any \( v > 0 \) and \( \epsilon > 0 \)
\[
\sup_{0 \leq t \leq T} \int_{x_0 + v \epsilon - vt}^{x_0 + v \epsilon + vt} (\partial_x^j u)^2(x, t) \, dx < c,
\]
for \( j = 0, 1, \ldots, l \) with \( c = c(l; \| u_0 \|_{3/4+}, 2; \| \partial_x^l u_0 \|_{L^2((x_0, \infty))}; v; \epsilon; T) \).

In particular, for any \( t \in (0, T] \), the restriction of \( u(\cdot, t) \) to any interval \( (x_0, \infty) \) belongs to \( H^j((x_0, \infty)) \).

Moreover, for any \( v > 0, \epsilon > 0 \) and \( R > 0 \)
\[
\int_0^T \int_{x_0 + v \epsilon - vt}^{x_0 + v \epsilon + vt} (\partial_x^{l+1} u)^2(x, t) \, dx dt < c,
\]
with \( c = c(l; \| u_0 \|_{3/4+}, 2; \| \partial_x^l u_0 \|_{L^2((x_0, \infty))}; v; \epsilon; R; T) \).
Later, Kenig, Linares, Ponce and Vega [26] extended this result to the fractional exponent case with \( l > \frac{3}{4} \). The property described in Theorem 1.1 is intrinsic to suitable solutions of some nonlinear dispersive models (see for instance [35]). This result was also obtained for the Benjamin-Ono equation [18] and more recently, for the dispersion generalized Benjamin-Ono equation [37] and the fractional Korteweg-de Vries equation [38]. In the context of 2D models, analogous results for the Kadomtsev-Petviashvili II equation [19], Zakharov-Kuznetsov [36], fifth order Kadomtsev-Petviashvili II [41] and Kadomtsev-Petviashvili-Benjamin-Ono [42] equations were proved.

We recall that T. Kato in [22] showed that solution of the KdV equation satisfies

\[
\int_0^T \int_R^{-R} (\partial_x u)^2(x,t) \, dx \, dt < c(R;T;\|u_0\|_{L^2}).
\]

(1.10)
The estimate (1.10) is known as Kato’s smoothing effect. Roughly, the proof of (1.10) follows by noticing that a smooth solution to the KdV satisfies

\[
\frac{d}{dt} \int (\partial_x^l u)^2 \psi \, dx + 3 \int (\partial_x^{l+1} u)^2 \psi' \, dx
\]

\[
= \int (\partial_x^l u)^2 \psi'' \, dx + \int (\partial_x^l u)^2 \partial_x(\psi u) \, dx + \int \partial_x^l u [\partial_x^l u, \partial_x^l u] \psi \, dx
\]

(1.11)

with \( l \in \mathbb{Z}^+ \cup \{0\} \). Next, we select \( l = 0 \) and \( \psi \in C^3(\mathbb{R}) \) to be an appropriate non-negative, nondecreasing cutoff function with \( \psi' \) compactly supported and integrate in time to obtain (1.10). There exist a relation between Kato’s smoothing effect and the propagation of regularity obtained in Theorem 1.1. We will see this also in the case we are considering.

The mathematical study of the BO-ZK equation has given rise to some papers in recent years. Regarding existence and stability of solitary waves solutions for BO-ZK equation we refer [10] and [13] where the authors proved the orbital stability of ground state solutions in the energy space.

It is worth to notice that in [13] the authors established an anisotropic Gagliardo-Nirenberg type inequality whose optimal constant were later characterized by Esfahani and Pastor [11], in terms of the ground state solutions of BO-ZK equation. As an application of their results, the authors in [11] proved the uniform bound of smooth solutions in the energy space.

Regarding unique continuation properties we refer [7] and [12]. More precisely, the authors in [12] established if a sufficiently smooth solution is supported in a rectangle (for all time), then it must vanish identically. Later, in [7] the authors improved this result by showing if a sufficiently smooth local solution has, in three different times (not for all time), a suitable algebraic decay at infinity, then it must be identically zero.

Regarding well-posedness for the BO-ZK equation, we refer Cunha and Pastor [7] (see also Esfahani and Pastor [10]) which proved by using parabolic regularization method, that does not take into account the dispersive effect of the equation, the following

**Theorem 1.2.** Let \( s > 2 \). Then for any \( \phi \in H^s(\mathbb{R}^2) \), there exist \( T \in T(\|\phi\|_{H^s}) > 0 \) and a unique solution \( u \in C([0,T];H^s(\mathbb{R}^2)) \) of (1.1) with \( u(0) = \phi \) and \( u(t) \) depends on \( \phi \) continuously in the \( H^s \) norm. In addition, \( u(t) \) satisfies \( \mathcal{F}(u(t)) = \mathcal{F}(\phi), \mathcal{E}(u(t)) = \mathcal{E}(\phi) \), for all \( t \in [0,T] \).
The authors in [7] also proved local well-posedness in the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2), s_2 > 2, s_1 \geq s_2$. Esfahani and Pastor [9], following the ideas of [39] established the ill-posedness of (1.1) in the sense that it cannot be solved in the usual $L^2$-based Sobolev space by using a fixed point argument. More precisely, the map data-solution cannot be $C^2$-differentiable at the origin from $H^{s_1,s_2}(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$ for any $s_1, s_2 \in \mathbb{R}$.

Later, Cunha and Pastor [6] obtained local well-posedness for the BO-ZK equation (1.1) with $\alpha = 1$, in the Sobolev spaces $H^s(\mathbb{R}^2), s > \frac{11}{8}$. Their proof is based on the refined Strichartz estimates introduced by Koch and Tzvetkov [30] in the context of the Benjamin-Ono equation.

Recently, Ribaud and Vento [44] showed that the initial value problem associated to the dispersive generalized Benjamin-Ono-Zakharov-Kuznetsov equation

$$\partial_t u - D_x^\sigma \partial_x u + \partial_x D_y^\sigma u = u \partial_x u, \ (x, y, t) \in \mathbb{R}^3, \ 1 \leq \sigma \leq 2,$$  

(1.12)

is locally well-posed in the spaces $E^s, s > \frac{2}{\sigma} - \frac{3}{4},$ endowed with the norm $\|f\|_{E^s} = \|\|\xi\|^s + \eta^2 \xi \hat{f}\|_{L^2(\mathbb{R}^2)}$. They also proved global well-posedness in the energy space $E^2$ as long as $\sigma > \frac{5}{2}$. Observe that $E^s$ is nothing but the anisotropic Sobolev space $H^{s,2s}(\mathbb{R}^2)$. Their proof is based on the approach of the short time Bourgain spaces from Ionescu, Kenig and Tataru [17] combined with some new Strichartz estimates and modified energy.

In particular, we observe that when $\sigma = 2$ the equation (1.12) is the well-known ZK equation while for $\sigma = 1$ the equation (1.12) reduces to the BO-ZK equation. Thus, in the context of the BO-ZK, the authors in [44] obtained local well-posedness in the anisotropic Sobolev space $H^{s,2s}(\mathbb{R}^2)$ for $s > \frac{5}{2}$.

Our contribution in this direction is to improve the above results by pushing down the Sobolev regularity index of the classical $H^s(\mathbb{R}^2)$ spaces. In fact, we obtain the following

**Theorem 1.3.** Let $s > \frac{5}{7}$. The IVP (1.1) is locally well-posed in $H^s(\mathbb{R}^2)$. More precisely, there exist $T = T(||\phi||_{H^s}) > 0$ and a unique solution $u$ to (1.1), such that

(i) $u \in C([0, T]; H^s(\mathbb{R}^2)),$

(ii) $u, \ \partial_x u \in L^1([0, T]; L^\infty(\mathbb{R}^2))$ (Strichartz).

Moreover, the mapping $\phi \mapsto u \in C([0, T]; H^s(\mathbb{R}^2))$ is continuous.

The main tool of the proof of this result is a refined Strichartz inequality (see Lemma 3.6 below). This type of estimate was first introduced by Koch and Tzvetkov [30] in the context of the Benjamin-Ono equation. It was extended by Kenig and Koenig [25] for BO equation and Kenig [24] for Kadomtsev-Petviashvili (KP-I) equation. Roughly, one needs to control $||\partial_x u||_{L^1_t L^\infty_y}$.

Before to enunciate our main result for BO-ZK equation, we need to define some weights independent of the variable $y$, which are a class of real functions $\chi_{\epsilon, b} \in C^\infty(\mathbb{R})$ for $\epsilon > 0$ and $b \geq 5\epsilon$ with $\chi_{\epsilon, b}'(x) \geq 0$ and

$$\chi_{\epsilon, b}(x) = \begin{cases} 0, & x \leq \epsilon \\ 1, & x \geq b, \end{cases}$$  

(1.15)

which will be constructed in the following way. Let $\rho \in C^\infty_0(\mathbb{R})$ a function which is even, non-negative, with supp$\rho \subseteq (-1, 1)$ and $\int \rho(x)dx = 1$ and define
\[ \nu_{\epsilon,b}(x) = \begin{cases} 0, & x \leq 2\epsilon \\ \frac{1}{b - 3\epsilon}x - \frac{2\epsilon}{b - 3\epsilon}, & 2\epsilon \leq x \leq b - \epsilon \\ 1, & x \geq b - \epsilon \end{cases} \] (1.16)

with
\[ \chi_{\epsilon,b}(x) = \rho_\epsilon * \nu_{\epsilon,b}(x) \] (1.17)

where \( \rho_\epsilon(x) = \epsilon^{-1} \rho(x/\epsilon) \). It follows from definition (1.17) above that
\[ \text{supp } \chi_{\epsilon,b} \subseteq [\epsilon, \infty), \] (1.18)
\[ \text{supp } \chi'_{\epsilon,b} \subseteq [\epsilon, b]. \] (1.19)

We define \( \eta_{\epsilon,b} \) by the identities
\[ \chi'_{\epsilon,b}(x) = \eta_{\epsilon,b}^2, \text{ i.e. } \eta_{\epsilon,b} = \sqrt{\chi'_{\epsilon,b}(x)}, \] (1.20)

and observe that for any \( \epsilon > 0 \) and \( b \geq 5\epsilon \), the function \( \eta_{\epsilon,b} \in C_0^\infty(\mathbb{R}) \) with \( \text{supp } \eta_{\epsilon,b} = \text{supp } \chi'_{\epsilon,b} \).

Our main result for BO-ZK equation reads as follows

**Theorem 1.4.** Let \( s > \frac{5}{4} \). Suppose \( \phi \in H^s(\mathbb{R}^2) \) and for some \( x_0 \in \mathbb{R} \) and \( m \in \mathbb{Z}^+ \), \( m \geq 2 \) the restriction of \( \partial_x^m \phi \) to \( (x_0, \infty) \times \mathbb{R} \) belongs to \( L^2((x_0, \infty) \times \mathbb{R}) \), that is,
\[ \int_{-\infty}^\infty \int_{x_0}^\infty (\partial_x^m \phi(x,y))^2 \, dx \, dy < \infty. \] (1.21)

Then the solution \( u \) in \([0,T]\) of equation (1.1) with \( \alpha = -1 \) provided by Theorem 1.3 satisfies for any \( \nu > 0 \), \( T > 0 \), \( \epsilon > 0 \), \( b \geq 5\epsilon \)
\[ \sup_{0 \leq t \leq T} \int_{-\infty}^\infty \int_{-\infty}^\infty \left( \partial_x^l u(x,y,t) \right)^2 \chi_{\epsilon,b}(x - x_0 + \nu t) \, dx \, dy \]
\[ + \int_0^T \int_{-\infty}^\infty \int_{-\infty}^\infty \left[ D_x^{1/2} \left( \partial_x^l u(x,y,t) \eta_{\epsilon,b}(x - x_0 + \nu t) \right) \right]^2 \, dx \, dy \, dt \]
\[ \leq C, \] (1.22)

for \( l = 0, 1, 2, ..., m \) with \( C = C(T, \| \phi \|_{H^s(\mathbb{R}^2)}, \| \partial_x^m \phi \|_{L^2((x_0, \infty) \times \mathbb{R})}, \epsilon, b, \nu). \) In particular, for all times \( t \in (0,T] \) and for all \( a \in \mathbb{R} \), \( D_x^a u(t) \in L^2((a, \infty) \times \mathbb{R}) \).

Moreover, for any \( \nu > 0 \), \( \epsilon > 0 \) and \( R > 0 \)
\[ \int_0^T \int_{-\infty}^\infty \int_{x_0 + R - \nu t}^{x_0 + R} (\partial_x^m \partial_y u(x,y,t))^2 \, dy \, dx \, dt \leq C, \] (1.23)

with \( C = C(m, \nu, \epsilon, R, T, \| \phi \|_{H^s}, \| \partial_x^m \phi \|_{L^2((x_0, \infty) \times \mathbb{R})}) \). If in addition to (1.21) the exists \( x_0 \in \mathbb{R} \) such that any \( \epsilon > 0 \), \( b > 5\epsilon \)
\[ \int_{-\infty}^\infty \int_{x_0}^\infty (\partial_x^{1/2} \partial_x^m \phi(x,y))^2 \, dx \, dy < \infty, \] (1.24)

then
\[ \sup_{0 \leq t \leq T} \int_{-\infty}^\infty \int_{-\infty}^\infty \left( D_x^{1/2} \partial_x^l u(x,y,t) \right)^2 \chi_{\epsilon,b}(x - x_0 + \nu t) \, dx \, dy \]
\[ + \int_0^T \int_{-\infty}^\infty \int_{-\infty}^\infty \left[ D_x^{1/2} \left( D_x^{1/2} \partial_x^l u(x,y,t) \eta_{\epsilon,b}(x - x_0 + \nu t) \right) \right]^2 \, dx \, dy \, dt \]
\[ \leq C, \] (1.25)
for \( l = 0, 1, 2, \ldots, m \) with \( C = C(T, \| \phi \|_{H^s(\mathbb{R}^2)}, \| D_x^{1/2} \partial_x^m \phi \|_{L^2((x_0, \infty) \times \mathbb{R})}, \varepsilon, b, \nu) \). Moreover, for any \( \nu > 0, \varepsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{-\infty}^{\infty} \int_{x_0 + R - \nu t}^{x_0 + R + \nu t} (D_x^{1/2} \partial_x^m \partial_y u(x, y, t))^2 dx dy dt \leq C, \tag{1.26}
\]

with \( C = C(m, \nu, \varepsilon, R, T, \| \phi \|_{H^s}, \| D_x^{1/2} \partial_x^m \phi \|_{L^2((x_0, \infty) \times \mathbb{R})}) \).

The proof of Theorem 1.4 is based on an induction argument on \( m \), combining some weighted energy estimates the properties (1.14) of the solutions to (1.1) and Gronwall’s inequality. However, the presence of a non-local operator, the Hilbert transform (1.2), lead us to use some commutator estimates with the extension of the Calderón first commutator given by Bajvsank and Coifman in [1] (see Theorem 3.5 below) in order to obtain the local smoothing effect which is crucial to carry on our argument of induction. Besides, we notice that unlike KdV equation in which the Calderón first commutator given by Bajvsank and Coifman in [1] (see Theorem 1.4 to the function \( \partial_x^m \phi \) of (1.23) and (1.26) are a generalization of the Kato local smoothing effect since we do not require \( \partial_x^m \phi \) of (1.26). It is worth notice that the proof of Theorem 1.4 can be extended to solutions of the IVP

\[
\left\{ \begin{array}{l}
\partial_t u - \mathcal{H} \partial_x^2 u + \partial_x \partial_y^2 u + u_k \partial_x u = 0, \quad (x, y, t) \in \mathbb{R}^3, k \in \mathbb{Z}, \\
u(x, y, 0) = \phi(x, y),
\end{array} \right. \tag{1.27}
\]

(iii) The solutions of the defocusing BO-ZK equation

\[
\left\{ \begin{array}{l}
\partial_t u - \mathcal{H} \partial_x^2 u + \partial_x \partial_y^2 u - u \partial_x u = 0, \quad (x, y, t) \in \mathbb{R}^3, \\
u(x, y, 0) = \phi(x, y),
\end{array} \right. \tag{1.28}
\]

still satisfies the property described in Theorem 1.4. This can be verified applying Theorem 1.4 to the function \( v(x, y, t) = u(-x, -y, t) \), where \( u(x, y, t) \) is a solution of (1.1) with \( \alpha = -1 \). In other words, Theorem 1.4 remains valid, backward in time for initial data \( \phi \) satisfying (1.21).

We present now, some immediate consequences of Theorem 1.4.

**Corollary 1.** Let \( u \) be a solution of the IVP (1.1) with \( \alpha = -1 \) corresponding to initial data \( \phi \in H^s(\mathbb{R}^2) \), \( s > \frac{5}{4} \), described in Theorem 1.3. If there exist \( k, m \in \mathbb{Z}^+ \) with \( k \geq m \) and \( a, c \in \mathbb{R} \) with \( c < a \) such that

\[
\partial_x^k \phi \in L^2((a, \infty) \times \mathbb{R}) \quad \text{but} \quad \partial_x^m \phi \notin L^2((c, \infty) \times \mathbb{R}), \tag{1.29}
\]

then for \( l = 0, 1, \ldots, k \) any \( t \in (0, T) \), \( \nu > 0 \) and \( \varepsilon > 0 \)

\[
\int_{-\infty}^{\infty} \int_{a + \varepsilon - \nu t}^{a + \varepsilon + \nu t} |\partial_x^l u(x, y, t)|^2 dx dy < \infty
\]

and for \( l = 0, 1, \ldots, m \) any \( t \in [-T, 0) \) and \( \alpha \in \mathbb{R} \)

\[
\int_{-\infty}^{\infty} \int_{a + \varepsilon - \nu t}^{a + \varepsilon + \nu t} |\partial_x^l u(x, y, t)|^2 dx dy = \infty.
\]
As aforementioned, the authors in [26] extended the result in Theorem 1.1 for the generalized KdV for initial data with restricted \( H^s((x_0, \infty)) \)-norm, \( s \) real, instead of \( H^l((x_0, \infty)) \), \( l \) integer. More precisely,

**Theorem 1.5.** Let \( u_0 \in H^{3/4+}(\mathbb{R}) \). If for some \( s \in \mathbb{R}, s > 3/4 \), and for some \( x_0 \in \mathbb{R} \)

\[
\| J^s u_0 \|_{L^2((x_0, \infty))}^2 = \int_{x_0}^{\infty} |J^s u_0(x)|^2 \, dx < \infty,
\]

then the solution \( u = u(x, t) \) of the IVP (1.7) associated to the generalized KdV equation satisfies that for any \( v > 0 \) and \( \epsilon > 0 \)

\[
\sup_{0 \leq t \leq T} \int_{x_0+\epsilon-vt}^{\infty} (J^r u)^2(x, t) \, dx < c,
\]

for \( r \in (3/4, s] \) with \( c = c(l; \|u_0\|_{3/4+}; \|J^r u_0\|_{L^2((x_0, \infty))}; v; \epsilon; T) \).

Moreover, for any \( v \geq 0 \), \( \epsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{x_0+R-vt}^{x_0+R-vt} (J^{r+1} u)^2(x, t) \, dx \, dt < c,
\]

with \( c = c(l; \|u_0\|_{3/4+}; \|J^s u_0\|_{L^2((x_0, \infty))}; v; \epsilon; R; T) \).

It is an open problem to extend this result for nonlinear dispersive models in higher dimensions.

Another, interesting problem regards decay of the solutions implying regularity. In [26] the authors proved the following result for solutions of the generalized KdV equation,

**Theorem 1.6.** If \( u_0 \in H^{3/4+}(\mathbb{R}) \) and for some \( n \in \mathbb{Z}^+, n \geq 1 \),

\[
\| x^{n/2} u_0 \|_{L^2((0, \infty))}^2 = \int_0^\infty |x^n| |u_0(x)|^2 \, dx < \infty,
\]

then the solution \( u \) of the IVP (1.7) satisfies that

\[
\sup_{0 \leq t \leq T} \int_0^\infty |x^n| |u(x, t)|^2 \, dx \leq c
\]

with \( c = c(n; \|u_0\|_{3/4+}; \|x^{n/2} u_0\|_{L^2((0, \infty))}; T) \).

Moreover, for any \( \epsilon, \delta, R > 0 \), \( v \geq 0 \), \( m, j \in \mathbb{Z}^+, \, m + j \leq n, \, m \geq 1, \)

\[
\sup_{\delta \leq t \leq T} \int_{x-vt}^{\infty} (\partial_t^m u)^2(x, t) \, dx + \int_0^T \int_{x-vt}^{x+vt} (\partial_x^{m+1} u)^2(x, t) \, dx \, dt \leq c,
\]

with \( c = c(n; \|u_0\|_{3/4+}; \|x^{n/2} u_0\|_{L^2((0, \infty))}; T; \delta; \epsilon; R; v) \).

Roughly speaking, Theorem 1.6 tell us that polynomial decay of the initial data yields to more regular solutions. On the other hand, Cunha and Pastor [7] studied persistence properties of the solutions of the BO-ZK equation and established local well-posedness of the IVP (1.1) in the weighted Sobolev class

\[
\mathcal{Z}_{s, r} = H^s(\mathbb{R}^2) \cap L^2((1 + x^2 + y^2)^r \, dx \, dy),
\]

where \( s > 2 \), \( r \geq 0 \), and \( s \geq 2r \). We would like to know whether solutions of the BO-ZK equation satisfy a property similar to the one described in Theorem 1.6.
Recently, Linares, Miyazaki and Ponce in [33] considered the following IVP associated to the generalized KdV equation with low degree of non-linearity
\[
\begin{align*}
\begin{cases}
\partial_t u + \partial_x^3 u \pm |u|^{\alpha} \partial_x u = 0, & x, t \in \mathbb{R}, \quad 0 < \alpha < 1, \\
u(x, 0) = u_0(x).
\end{cases}
\end{align*}
\]
(1.36)
They established that suitable solutions of the IVP (1.36) satisfy the propagation of regularity principle proven in Theorem 1.1 for solutions of the k-generalized KdV equation in (1.7) (see Theorem 1.6. in [33]). We would like to extend this result for the high dimensional models with non-linearity of fractional order such as generalized BO-ZK equation
\[
\begin{align*}
\begin{cases}
\partial_t u - \mathcal{H}\partial_x^2 u + |u|^{\alpha} \partial_x u + \partial_x \partial_y^2 u = 0, & (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \quad 0 < \alpha < 1, \\
u(x, y, 0) = \phi(x, y).
\end{cases}
\end{align*}
\]
(1.37)

The rest of this paper is organized in the following manner: Section 2 contains some notations that will be used in this work. Section 3 contains the proof of Theorem 1.3, our new local well-posedness result for BO-ZK equation. In Section 4 a local smoothing effect for BO-ZK equation is derived (see Proposition 3) which is an useful tool to carry on our argument of proof. Next, we perform the proof of Theorem 1.4. In Appendices we present several estimates that are crucial in our analysis.

2. Notation and the resolution space. Given any positives quantities $C, D$, by $C \lesssim D$ we mean that there exists a constant $c > 0$ such that $C \leq cD$; and, by $C \sim D$ we mean $C \lesssim D$ and $D \lesssim C$. For a real number $r$ we shall denote $r^+$ instead of $r + \epsilon$, whenever $\epsilon$ is a positive number whose value is small enough. Given two operators $A$ and $B$, we denote by $[A, B] = AB - BA$ the commutator between $A$ and $B$. By $\mathcal{F}\{u\}$ or $\hat{u}$ we will denote the Fourier transform of $u$ with respect to the space variable, while $\mathcal{F}^{-1}\{u\}$ or $\check{u}$ will denote its inverse Fourier transform. $L^p$-norms will be written as $\| \cdot \|_{L^p}$ or $\| \cdot \|_{L^{p'}}$ if no confusion is caused. For $1 \leq p, q < \infty$ and $f : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$, we define
\[
\|f\|_{L^q_t L^{p'}_y} = \left( \int_0^T \left( \int_{\mathbb{R}^2} |f(x, y, t)|^q dx dy \right)^{q/p} dt \right)^{1/q}.
\]
$\mathcal{S}(\mathbb{R}^2)$ will represent the Schwartz space. For $s \in \mathbb{R}$ (and $f \in \mathcal{S}'$) $J^s = \mathcal{F}^{-1}((1 + |\cdot|^2)^s \hat{f})$ will be the Bessel potential of order $-s$, $D^s_x f = \mathcal{F}^{-1}((|\cdot|^2)^s \hat{f})$ will be the Riesz potential of order $-s$. The space $H^s(\mathbb{R}^2)$ is the usual Sobolev space with norm $\| \cdot \|_{H^s} = \|J^s \cdot \|_{L^2}$. We define the anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$ by the norm $H^{s_1, s_2}(\mathbb{R}^2) = \{ f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{H^{s_1, s_2}} = \|\langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{f}\|_{L^2} < \infty \}$, where $\langle \cdot \rangle = 1 + |\cdot|^2$.

3. Local well posedness in $H^s(\mathbb{R}^2)$, $s > \frac{3}{2}$. In this section we will prove Theorem 1.3. We follow the argument in [24] to obtain the result. Our argument of proof uses energy estimates. The key point is to establish an estimate of the kind $\|\partial_x u\|_{L^q_t L^{p'}_y} \leq CT$, $t \in [0, T]$ where $C_T = C(T, s, \|\phi\|_{H^s})$ (see Lemma 3.8 ). To this end, we prove a refined Strichartz estimates (see Lemma 3.6 ) for solutions of the linear problem.
3.1. **Linear estimates.** Consider the IVP associated to the BO-ZK equation,

\[
\begin{aligned}
\partial_t u + \alpha \partial_x^2 u + \partial_x \partial_y^2 u &= 0, \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\
\phi(x,y) &= 0, \quad (x,y) \in \mathbb{R}^2
\end{aligned}
\]  

(3.1)

The solution of (3.1) is given by the unitary group \( \{U(t)\}_{t=-\infty}^{\infty} \) such that

\[
u(t) = U(t)\phi = \int_{\mathbb{R}^2} e^{it(\alpha |\xi|^2 + |\eta|^2) + x\xi + y\eta} \hat{\phi}(|\xi|, |\eta|) d\xi d\eta = I_t * \phi(x,y),
\]

(3.2)

where

\[
I_t(x,y) = \int_{\mathbb{R}^2} e^{it(\alpha |\xi|^2 + |\eta|^2) + x\xi + y\eta} d\xi d\eta.
\]

(3.3)

We recall the Strichartz-type estimates for solution (3.2).

**Lemma 3.1.** Let \( 0 \leq \delta < \frac{1}{2} \), \( \alpha \in \mathbb{R} \) and \( \alpha \neq 0 \). Then

\[
I_t^{\delta}(x,y) = \int_{\mathbb{R}^2} |\xi|^\delta e^{it(\alpha |\xi|^2 + |\eta|^2) + x\xi + y\eta} d\xi d\eta
\]

(3.4)

satisfies

\[
|I_t^{\delta}(x,y)| \leq C |t|^{-\frac{3+2\delta}{4}},
\]

(3.5)

where \( C > 0 \) is a constant independent of \( (x,y) \in \mathbb{R}^2 \).

**Proof.** See Lemma 2.3 in Esfahani and Pastor [10].

**Lemma 3.2.** Let \( 0 \leq \delta < \frac{1}{2} \), \( 0 \leq \theta \leq 1 \). Then

\[
\|D_x^{\delta} U(t) \phi\|_{L^p_x L^q_y} \leq C |t|^{-\frac{3+2\delta}{4}} \|\phi\|_{L^p_x L^q_y}, \quad 0 < t \leq T,
\]

(3.6)

where \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( p = \frac{2}{1-\theta} \).

**Proof.** From Stein’s interpolation theorem, inequality (3.6) follows by using (3.5) and Plancherel’s identity.

The Strichartz estimates for solutions of the IVP (3.1) are as follows

**Proposition 1.** Let \( 0 \leq \delta < \frac{1}{2} \) and \( 0 \leq \theta \leq 1 \). Then the group \( U(t) \) satisfies

\[
\|D_x^{\delta} U(t) f\|_{L^p_x L^q_y} \leq C \|f\|_{L^p_x L^q_y},
\]

(3.7)

\[
\|\int_0^T D_x^{\delta} U(t-\tau) g(\cdot, \tau) d\tau\|_{L^p_x L^q_y} \leq C \|g\|_{L^p_y L^q_y}
\]

(3.8)

\[
\|\int_0^T D_x^{\delta} U(t) g(\cdot, t) dt\|_{L^p_x L^q_y} \leq C \|g\|_{L^p_y L^q_y}
\]

(3.9)

where \( 0 < t \leq T, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \quad p = \frac{2}{1-\theta} \) and \( \frac{2}{q} = \frac{\theta(3+2\delta)}{4} \).

**Remark 2.** We observe that we have a gain of \( \frac{1}{4} \) derivatives for \( \delta \sim \frac{1}{2} \).

**Proof.** See Proposition 2.6 in [10].

**Lemma 3.3.** The following Sobolev embedding holds

\[
\|f\|_{L^p_x L^q_y} \leq C \|f\|_{L^r_x L^{r'}_y} + \|D_x^{\delta} f\|_{L^p_x L^q_y} + \|D_y^{\delta} f\|_{L^{p'}_x L^{q'}_y},
\]

(3.10)

for \( \delta > 0 \), where \( p_\delta > \frac{2}{q} \). In particular, \( p_\delta \to \infty \) as \( \delta \to 0 \).

**Proof.** See Lemma 3.4. in [29].
Theorem 3.5. Let \( \gamma < 1/4^- \) and \( \epsilon > 0 \). Then the function \( U(t) \) satisfies
\[
\|U(t)f\|_{L^p_t L^q_y} \leq C \left\{ \|D_x^{-\gamma}f\|_{L^p_y} + \|D_x^{-\gamma}D_y^\delta f\|_{L^p_y} + \|D_x^{-\gamma}D_y^\epsilon f\|_{L^q_y} \right\}. 
\] (3.11)

Proof. Hölder’s inequality in \( t \) followed by an application of Lemma 3.3 implies that
\[
\|U(t)f\|_{L^p_t L^q_y} \leq C\left\{ \|U(t)f\|_{L^p_t L^q_y} + \|D_x^\gamma U(t)f\|_{L^p_t L^q_y} + \|D_y^\epsilon U(t)f\|_{L^q_t L^p_y} \right\},
\]
for \( p_\epsilon > \frac{2}{\epsilon} \) and \( \frac{2}{q} = \frac{\theta(3+2\delta)}{4} \). We choose \( p_\epsilon = \frac{2}{1-\theta} \), and apply estimate \( (3.7) \) in Proposition 1 to obtain
\[
\|U(t)f\|_{L^p_t L^q_y} \leq C\left\{ \|D_x^{-\gamma}f\|_{L^p_y} + \|D_x^{-\gamma}D_y^\delta f\|_{L^p_y} + \|D_x^{-\gamma}D_y^\epsilon f\|_{L^q_y} \right\},
\]
where \( 0 < \gamma < \frac{\theta \delta}{2} < \frac{1}{4^-} \) provided by \( \theta \sim 1 \) and \( \delta \sim \frac{1}{2^-} \) and with the condition \( p_\epsilon = \frac{2}{1-\theta} > \frac{2}{\epsilon} \), implying that \( 1 - \epsilon < \theta \). This concludes the proof. \( \square \)

3.2. Preliminary estimates. Consider the IVP
\[
\begin{aligned}
\partial_t u + \alpha \mathcal{H}\partial_x^2 u + \partial_y \partial_y^2 u + u \partial_x u &= 0, \\
u(x, y, 0) &= \phi(x, y),
\end{aligned}
\] (3.12)

We begin this section establishing an energy estimate for solutions of the IVP (3.12). We will borrow the result of Cunha and Pastor [7] (see Theorem 1.2 above) where smooth solutions of (3.12) were obtained.

As usual we apply the operator \( J^s \) to the equation in (3.12) and multiply it by \( J^s u \). Then we use integration by parts in \( x \), combined with Kato-Ponce commutator estimates (see [23]) and Gronwall’s Lemma to obtain the following:

Lemma 3.4. If \( u \) be a solution of the Cauchy problem (3.12) with \( \phi \in H^\infty(\mathbb{R}^2) \). Then for any \( s > 0 \)
\[
\sup_{0 < t < T} \|J^s u(t)\|_{L^2_y} \leq c_s \exp\left( c_s T \int_0^T \|\partial_x u(t')\|_{L^\infty_y} dt' \right) \|J^s \phi\|_{L^2_y},
\] (3.13)
for \( 0 < t < T \).

Proof. See Lemma 3.2 in Cunha and Pastor [6]. \( \square \)

We present the following extension of the Calderón commutator Theorem [5] established by Bajvsank and Coifman in [1]. This estimate will be a crucial ingredient in the proof of Theorem 1.4.

Theorem 3.5. Let \( \mathcal{H} \) denote the Hilbert transform. For any \( p \in (1, \infty) \) and any \( l, m \in \mathbb{Z}^+ \), \( l + m \geq 1 \) there exists \( c = c(p; l; m) > 0 \) such that
\[
\|\partial_x^l [\mathcal{H} \psi] \partial_x^m f\|_{L^p} \leq c \|\partial_x^{m+l} \psi\|_{L^\infty} \|f\|_{L^p}.
\] (3.14)

Proof. See Lemma 3.1 in [8]. \( \square \)
3.3. **Refined Strichartz estimate.** The next result is fundamental in our analysis.

**Lemma 3.6.** Let \(0 \leq \gamma < \frac{1}{4}\), \(T \in (0, 1]\), and \(\epsilon > 0\). Suppose that \(w \in C([0,T]; H^3(\mathbb{R}^3))\) is a solution to the linear equation
\[
\partial_tw + \alpha \mathcal{H} \partial_y^2w + \partial_x \partial_y^2w = F.
\] (3.15)

Then,
\[
\|\partial_x w\|_{L^1_t L^\infty_x} \lesssim C_{T, \epsilon} \left[ \sup_{0 < t < T} \|J^\frac{3}{2} - \gamma + \epsilon \partial_x^3 w(t)\|_{L^2} + \sup_{0 < t < T} \|J^\frac{3}{2} - \gamma + \epsilon \partial_x^4 D_y^\gamma w(t)\|_{L^2} \\
+ \int_0^T (\|J^\frac{3}{2} - \gamma + \epsilon \partial_x^3 F(t)\|_{L^2} + \|J^\frac{3}{2} - \gamma + \epsilon \partial_x^4 D_y^\gamma F(t)\|_{L^2}) dt \right].
\] (3.16)

**Proof.** In order to obtain (3.16), we will use a Littlewood-Paley decomposition of \(w\) in the variable \(\xi\). Indeed, we choose \(\chi \in C_0^\infty(|\xi| < 2)\) and \(\nu \in C_0^\infty(\frac{1}{2} < |\xi| < 2)\) such that
\[
1 = \sum_{k=1}^\infty \nu(2^{-k} \xi) + \chi(\xi).
\]

Let \(\lambda = 2^k, \ k \geq 1\), we define
\[
w_0 = Q_0(w), \quad \hat{Q}_0w(\xi, \eta) = \chi(\xi)\hat{w}(\xi, \eta)
\]
\[
w_\lambda = Q_\lambda(w), \quad \hat{Q}_\lambda w(\xi, \eta) = \nu(2^{-k} \xi)\hat{w}(\xi, \eta)
\]

First, we estimate \(\|\partial_x w_0\|_{L^1_t L^\infty_x}\). In fact, noticing that \(w_0\) is solution to the integral equation
\[
\partial_x w_0(t) = U(t)\partial_x Q_0 w(0) + \int_0^t U(t-t')\partial_x Q_0 F(\cdot, t') dt',
\] (3.17)
we obtain from Hölder’s inequality in time, Corollary 2 and Bernstein’s inequalities that
\[
\|\partial_x w_0\|_{L^1_t L^\infty_x} \lesssim T^\frac{3}{2} \left( \|U(t)Q_0 w(0)\|_{L^2_t L^\infty_x} + \int_0^T \|U(t-t')Q_0 F(\cdot, t')\|_{L^2_t L^\infty_x} dt' \right)
\lesssim C_{T, \epsilon} \left( \|J^\frac{3}{2} w(0)\|_{L^2_x} + \int_0^T \|J^\frac{3}{2} F(\cdot, t)\|_{L^2_x} dt \right).
\] (3.18)

Next we estimate \(\|\partial_x w_\lambda\|_{L^1_t L^\infty_x}\) when \(\lambda = 2^k, \ k \geq 1\). To this end, we assume for simplicity that \(T = 1\) and split the interval \([0, 1] = \bigcup J_j\) in subintervals \(I_j = [a_j, b_j]\) of length \(|I_j| \sim \lambda^{-1}\), so that \(T = b_j\) for some \(j\), with \(j = 1, 2, ..., \lambda\).

From the Hölder’s inequality in \(I_j\), and the fact that \(\xi \nu(2^{-k} \xi)\) has inverse Fourier transform whose \(L^1\) norm in \(x\) is bounded by \(c\lambda\), we have
\[
\|\partial_x w_\lambda\|_{L^1_t L^\infty_x} \lesssim \sum_j \|\partial_x w_\lambda\|_{L^1_t L^\infty_x} \leq \lambda \sum_j \|w_\lambda\|_{L^1_t L^\infty_x} \leq \lambda^\frac{1}{2} \sum_j \|w_\lambda\|_{L^2_t L^\infty_x}.
\] (3.19)

Duhamel’s principle, in each \(I_j\), implies
\[
w_\lambda(t) = U(t-a_j)w_\lambda(a_j) + \int_{a_j}^t U(t-t')F_\lambda(t') dt', \quad t \in I_j
\] (3.20)
where \(F_\lambda = w_\lambda \partial_x w_\lambda\). Combining (3.19), (3.20), and Corollary 2 we have
\[
\|\partial_x w_\lambda\|_{L^1_t L^\infty_x} \lesssim \sup_{0 < t < T} \left[ \|J^\frac{3}{2} - \gamma + \epsilon \partial_x^3 w_\lambda(t)\|_{L^2_x} + \|J^\frac{3}{2} - \gamma \partial_x^4 D_y^\gamma w_\lambda(t)\|_{L^2_x} \right]
\]
Proof. Notice that we also need to control \( \|w\|_{L^\infty}\), where \( w = w_1 + w_0 \) and \( w_1 = \sum \lambda w_\lambda \). Thus, we have

\[
\|\partial_x w_1\|_{L^1_x L^\infty_y} \leq \sum \|\partial_x w_\lambda\|_{L^1_x L^\infty_y} \leq \sum_{k \geq 1} \sup_{0 < t < T} \left( \|J_x^{-\gamma + \epsilon'} F_\lambda(t')\|_{L^2_y} + \|J_x^{-\gamma + \epsilon} Q_\lambda(w)(t)\|_{L^2_y} \right)
\]

\[
+ \int_0^T \left( \|J_x^{-\gamma} D_y^t Q_\lambda(F)(t')\|_{L^2_y} + \|J_x^{-\gamma + \epsilon} Q_\lambda(F)(t')\|_{L^2_y} \right) dt' \leq \sum_{k \geq 1} 2^{-ke} \left[ \sup_{0 < t < T} \left( \|J_x^{-\gamma + 2\epsilon} Q_\lambda(w)(t)\|_{L^2_y} + \|J_x^{-\gamma + \epsilon} D_y^t Q_\lambda(w)(t)\|_{L^2_y} \right)
\]

\[
+ \int_0^T \left( \|J_x^{-\gamma + 2\epsilon} Q_\lambda(F)(t')\|_{L^2_y} + \|J_x^{-\gamma + \epsilon} D_y^t Q_\lambda(F)(t')\|_{L^2_y} \right) dt' \right].
\]

(3.22)

From (3.18) and (3.22) we obtain (3.16). This concludes the proof. \( \square \)

3.4. A priori estimates. In this section we derive an a priori estimate for the norm \( \|\partial_x u\|_{L^1_t L_x L_y^\infty} \) based on the refined Strichartz estimate deduced in Lemma 3.6. Notice that we also need to control \( \|u\|_{L^1_t L_x L_y^\infty} \) in our argument. We define \( f(T) = \|u\|_{L^1_t L_x L_y^\infty} + \|\partial_x u\|_{L^1_t L_x L_y^\infty} \) and prove the following

**Lemma 3.7.** Let \( u \) be a solution to the Cauchy problem (3.12) with \( \phi \in H^\infty(\mathbb{R}^2) \), provided by Theorem 1.2 and defined for some \( T > 0 \). Then, for any \( s > \frac{3}{4} \), there exists \( T = T(s, \|\phi\|_{H^s}) \) such that

\[
f(T) \leq C\|\phi\|_{H^s} [e^{Cf(T)} + 1],
\]

(3.23)

for a fixed, universal constant \( C \).

**Proof.** Let \( s = \frac{3}{4} + \epsilon_0 \) and choose \( \epsilon > 0 \) so that \( 2\epsilon \ll \epsilon_0 \). We first estimate \( \|\partial_x u\|_{L^1_t L_x L_y^\infty} \). Rewriting the equation (1.1) as

\[
\partial_t u + \alpha A \partial_x^2 u + \partial_x D_y^2 u = -u \partial_x u,
\]

and using Lemma 3.6, we obtain

\[
\|\partial_x u\|_{L^1_t L_x L_y^\infty} \lesssim C_{T, \epsilon} \left[ \sup_{0 < t < T} \|J_x^{-\gamma + 2\epsilon} u(t)\|_{L^2_x} + \sup_{0 < t < T} \|J_x^{-\gamma + \epsilon} D_y^t u(t)\|_{L^2_x}
\]

\[
+ \int_0^T \left( \|J_x^{-\gamma + 2\epsilon} \partial_x u(t)\|_{L^2_x} + \|J_x^{-\gamma + \epsilon} D_y^t \partial_x u(t)\|_{L^2_x} \right) dt' \right] - I + II + III + IV.
\]

We will bound each of the terms \( I, II, III \) and \( IV \) defined above. We use Lemma 3.4 to deduce that

\[
I \leq \|J^s u\|_{L^\infty_t L^2_x L^\infty_y} \lesssim c_s \|\phi\|_{H^s} \exp(c_s \int_0^T \|\partial_x u(t)\|_{L^\infty_y} dt') \lesssim c_s \|\phi\|_{H^s} \exp(c_s f(T))
\]

where \( \gamma \sim \frac{1}{4} \) and \( \epsilon > 0 \) is chosen small enough such that \( \frac{3}{2} + 2\epsilon - \gamma \leq \frac{5}{4} + \epsilon_0 = s \).

We observe that

\[
II \lesssim \sup_{0 < t < T} \left( \|D_y^t u(t)\|_{L^2_y} + \|D_x^2 \gamma + \epsilon^t D_y^t u(t)\|_{L^2_y} \right) = II_1 + II_2.
\]
Using the fact that $|\eta|^s \leq (1 + |\xi| + |\eta|)^s \leq (1 + |\xi| + |\eta|)^s$ we obtain

$$II_1 \lesssim \|u\|_{L_T^\infty H^s}.$$  

We apply Lemma A.1 estimate (A.1) to obtain

$$II_2 \lesssim \sup_{0 < t < T} \|J^{\frac{1}{2} - \gamma + 2\epsilon} u(t)\|_{L_x^2} \lesssim \|u\|_{L_T^\infty H^s},$$

where $\gamma \sim \frac{1}{4}$ and $\epsilon > 0$ is chosen so that $\frac{3}{2} - \gamma + 2\epsilon \leq \frac{5}{4} + \epsilon_0 = s$.

Therefore, from Lemma 3.4 we conclude that

$$II \lesssim \|J^s u\|_{L_T^\infty L_x^\infty} \lesssim c_s \|H^s \exp(c_s \int_0^T \|\partial_x u(\tau)\|_{L_x^\infty} d\tau \lesssim c_s \|H^s \exp(c_s f(T))\).$$

We observe that

$$III \leq \int_0^T \|(u\partial_x u)(t)\|_{L_x^2} dt + \int_0^T \|D_x^{\frac{1}{2} - \gamma + 2\epsilon}(u\partial_x u)(t)\|_{L_x^2} dt = III_1 + III_2,$$

and

$$III_1 \lesssim \int_0^T \|\partial_x u(t)\|_{L_x^\infty \partial_x u(t)} \|L_x^\infty dt \lesssim \|u\|_{L_T^\infty L_x^2} \int_0^T \|\partial_x u(t)\|_{L_x^\infty} dt.$$

Thus, an application of Lemma 3.4 leads to

$$III_1 \lesssim c_s \|H^s \exp(c_s f(T))\).$$

Once $\epsilon > 0$ is so that $0 < \frac{1}{2} - \gamma + 2\epsilon < 1$, we employ Lemma A.2 item (A.2) and Lemma 3.4, to obtain

$$III_2 \lesssim \int_0^T \|D_x^{\frac{1}{2} - \gamma + 2\epsilon} \partial_x u(t)\|_{L_x^2} \|u(t)\|_{L_x^\infty} + \|D_x^{\frac{1}{2} - \gamma + 2\epsilon} u(t)\|_{L_x^2} \|\partial_x u(t)\|_{L_x^\infty} dt$$

$$\lesssim \sup_{0 \leq t \leq T} \|D_x^{\frac{3}{2} - \gamma + 2\epsilon} u(t)\|_{L_x^2} \|u(t)\|_{L_x^1 L_x^\infty} + \sup_{0 \leq t \leq T} \|D_x^{\frac{1}{2} - \gamma + 2\epsilon} u(t)\|_{L_x^2} \|\partial_x u(t)\|_{L_x^1 L_x^\infty}$$

$$\lesssim c_s \|H^s \exp(c_s f(T))\).$$

provided by $\gamma \sim \frac{1}{4}$ with $\epsilon > 0$ chosen small enough such that $\frac{3}{2} - \gamma + 2\epsilon \leq \frac{5}{4} + \epsilon_0 = s$.

Next, we write

$$IV \leq \int_0^T \|D_y^\epsilon (u\partial_x u)(\tau)\|_{L_x^2} d\tau + \int_0^T \|D_x^{\frac{1}{2} - \gamma + 2\epsilon} D_y^\epsilon (u\partial_x u)(\tau)\|_{L_x^2} d\tau = IV_1 + IV_2.$$  

Using Lemma A.2 item (A.2) in the $y$ variable, we have

$$IV_1 \lesssim \int_0^T \left(\|D_y^\epsilon u(t)\|_{L_x^2} \|\partial_x u(t)\|_{L_x^\infty} + \|D_y^\epsilon \partial_x u(t)\|_{L_x^2} \|u(t)\|_{L_x^\infty}\right) dt = IV_{1,1} + IV_{1,2}.$$  

We argue as we did in $II_1$ to conclude that

$$IV_{1,1} \lesssim \|u\|_{L_T^\infty H^s} \int_0^T \|\partial_x u(t)\|_{L_x^\infty} dt \lesssim \|u\|_{L_T^\infty H^s f(T)}.$$  

Next, we employ Lemma A.1 to get

$$IV_{1,2} \lesssim \|J^{1+\epsilon} u\|_{L_T^\infty H^s} \int_0^T \|u(t)\|_{L_x^\infty} dt \lesssim \|u\|_{L_T^\infty H^s f(T)},$$

since $\epsilon > 0$ is chosen small enough so that $1 + \epsilon \leq \frac{5}{4} + \epsilon_0 = s$.
From Lemma 3.4, we conclude that

\[ IV_1 \lesssim \|u\|_{L^p_x H^s} \lesssim \|c_s\|_{H^s} \exp(c_s(T)). \]

Using Lemma A.2 item (A.3), with \( 0 < \frac{1}{2} - \gamma + \epsilon = \alpha, \beta = \epsilon < 1 \) we have

\[
IV_2 = \int_0^T \left| D_x^{\frac{1}{2} - \gamma + \epsilon} D_y^\gamma u(t) \right| \|D_x u(t)\|_{L_y^\infty} \, dt
\]

\[
\lesssim \int_0^T \left| D_x^{\frac{1}{2} - \gamma + \epsilon} D_y^\gamma u(t) \right| \|D_x u(t)\|_{L_y^\infty} \, dt
\]

\[
\lesssim \int_0^T \left| D_x^{\frac{1}{2} - \gamma + \epsilon} D_y^\gamma u(t) \right| \|D_x u(t)\|_{L_y^\infty} \, dt
\]

\[
\lesssim \int_0^T \left| D_x^{\frac{1}{2} - \gamma + \epsilon} D_y^\gamma u(t) \right| \|D_x u(t)\|_{L_y^\infty} \, dt
\]

\[
\lesssim \int_0^T \left| D_x^{\frac{1}{2} - \gamma + \epsilon} D_y^\gamma u(t) \right| \|D_x u(t)\|_{L_y^\infty} \, dt
\]

\[
= IV_{2,1} + IV_{2,2} + IV_{2,3} + IV_{2,4},
\]

where \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2} \).

We observe that Young’s inequality implies that

\[
|\xi|^{\frac{1}{2} - \gamma + \epsilon} |\eta|^\gamma \leq (1 + |\xi| + |\eta|)^{\frac{1}{2} - \gamma + \epsilon} (1 \leq (1 + |\xi| + |\eta|)^{\frac{1}{2} - \gamma + 2\epsilon},
\]

where \( \gamma \sim \frac{1}{4} \) and \( \epsilon > 0 \) are chosen such that \( \frac{1}{2} - \gamma + 2\epsilon \leq \frac{5}{4} + \epsilon_0 = \frac{s}{4} \).

From Placherel’s theorem and Lemma 3.4, we have

\[
IV_{2,1} \lesssim \int_0^T \left\| J^s u(t) \right\|_{L_y^\infty} \left\| \partial_x u(t) \right\|_{L_y^\infty} \, dt
\]

\[
\lesssim \sup_{0 < t < T} \left\| u(t) \right\|_{H^s} \int_0^T \left\| \partial_x u(t) \right\|_{L_y^\infty} \, dt \leq c_s \left\| \phi \right\|_{H^s} \exp(c_s(T)).
\]

We proceed similarly, using that

\[
|\xi|^{\frac{1}{2} - \gamma + \epsilon} |\eta|^\gamma \leq (1 + |\xi| + |\eta|)^{\frac{1}{2} - \gamma + \epsilon} (1 \leq (1 + |\xi| + |\eta|)^{\frac{1}{2} - \gamma + 2\epsilon},
\]

for \( \gamma \sim \frac{1}{4} \) and \( \epsilon > 0 \) such that \( \frac{3}{2} - \gamma + 2\epsilon \leq \frac{5}{4} + \epsilon_0 = \frac{s}{4} \) and using Lemma 3.4, to see that

\[
IV_{2,2} = \int_0^T \left\| D_x^{\frac{1}{2} - \gamma + \epsilon} D_y^\gamma u(t) \right\|_{L^\infty_y} \left\| u(t) \right\|_{L^\infty_y} \, dt
\]

\[
\lesssim \int_0^T \left\| J^s u(t) \right\|_{L_y^\infty} \left\| \partial_x u(t) \right\|_{L_y^\infty} \, dt \leq c_s \left\| \phi \right\|_{H^s} \exp(c_s(T)).
\]

We note that

\[
|\eta|^\gamma |\xi| \leq (1 + |\xi| + |\eta|)(1 + |\xi| + |\eta|)^\gamma \leq (1 + |\xi| + |\eta|)^{1 + \epsilon},
\]

where \( \epsilon > 0 \) is chosen such that \( 1 + \epsilon \leq \frac{5}{4} + \epsilon_0 = \frac{s}{4} \).

From Lemma 3.4, and (A.9) of the Lemma A.8, we have

\[
IV_{2,3} \leq \int_0^T \left\| D_x^{\frac{1}{2} - \gamma + \epsilon} u(t) \right\|_{L_y^\infty} \left\| D_y^\gamma \partial_x u(t) \right\|_{L_y^\infty} \, dt
\]

\[
\leq \int_0^T \left( \left\| u(t) \right\|_{L_y^\infty} + \left\| \partial_x u(t) \right\|_{L_y^\infty} \right) \left\| D_y^\gamma \partial_x u(t) \right\|_{L_y^\infty} \, dt
\]
Lemma 3.8. \[ \epsilon \in C \] for every 0 \( \leq \sup_{0 < t < T} \| D_y^k \partial_x u(t) \|_{L^2_y} \left( \int_0^T \left( \| u(t) \|_{L_y^\infty} + \| \partial_x u \|_{L_y^\infty} \right) dt \right) \]

\[ \leq \sup_{0 < t < T} \| u(t) \|_{H^s f(T)} \leq c_s \| \phi \|_{H^s \exp(c_s f(T))}. \]

We observe that,

\[ \| D_x^{1/2} u \|_{L_y^\infty L_x^2} + \| u \|_{L_y^\infty L_x^2} \leq \| J^* u \|_{L_y^\infty L_x^2} = \sup_{0 < t < T} \| u \|_{H^s}. \quad (3.24) \]

We apply Hölder’s inequality in \( t \), Remark A.8, (A.11), (A.12), Lemma 3.4, and the estimate (3.24) to obtain that

\[ IV_{2,4} = \int_0^T \| D_x^{1/2 - \gamma + \varepsilon} \partial_x u(t) \|_{L_y^2} \| D_y^2 u(t) \|_{L_y^2} dt \]

\[ \leq \| D_x^{1/2 - \gamma + \varepsilon} \partial_x u \|_{L_y^2} \| D_y^2 u(t) \|_{L_y^2} \]

\[ \leq \left( \| D_x^{1/2 - \gamma + \varepsilon + \varepsilon} \|_{L_y^2} \| D_y^2 \|_{L_y^2} \| D_y^2 u(t) \|_{L_y^2} \right) \left( \| u \|_{L_y^2} \| \partial_x u \|_{L_y^2} \right)^\varepsilon \]

\[ \leq \| \partial_x u \|_{L_y^2 L_x^\infty} \left( \sup_{0 < t < T} \| u \|_{H^s} \right) \| \partial_x u \|_{L_y^2 L_x^\infty} \]

\[ \leq c_s \| \phi \|_{H^s \exp(c_s f(T))}. \]

Combining the estimates for \( I, II, III \) and \( IV \), we obtain

\[ \| \partial_x u \|_{L_y^2 L_x^\infty} \leq c_s \| \phi \|_{H^s \exp(c_s f(T))}. \quad (3.25) \]

The estimate for \( \| u \|_{L_y^2 L_x^\infty} \), can be obtained arguing along the same lines as above to get

\[ \| u \|_{L_y^2 L_x^\infty} \leq c_s \| \phi \|_{H^s + c_s \| \phi \|_{H^s \exp(c_s f(T))}. \quad (3.26) \]

From the (3.25) and (3.26) we obtain (3.23) and hence (3.27). This finishes the proof.

Lemma 3.8. Let \( s > \frac{5}{4} \). There exist \( T = T(s, \| \phi \|_{H^s}) \) and a constant \( C_T = C_T(s, \| \phi \|_{H^s}) \) such that

\[ \| u \|_{L_y^\infty H^s} \leq c_s \| \phi \|_{H^s} \text{ and } f(T) \leq C_T, \quad t \in [0, T]. \]

Proof. From Lemma 3.7 estimate (3.23), we define

\[ F(X, \varepsilon) = X - C \varepsilon \exp(CX) - C \varepsilon \]

and observe that \( F(0, 0) = 0, \frac{\partial F}{\partial X}(0, 0) = 1 \) and \( \frac{\partial F}{\partial \varepsilon}(0, 0) = -2C \). By the Implicit Function Theorem, there exists \( \varepsilon_0 > 0 \) and a smooth function \( A(\varepsilon) \), which is increasing in \( \varepsilon \), so that

\[ F(A(\varepsilon), \varepsilon) = 0, \]

for every \( 0 < \varepsilon < \varepsilon_0 \).

Note that if \( X \leq 0 \), we have \( F(X, \varepsilon) = X - C \varepsilon \exp(CX) - C \varepsilon < 0 \), which implies that \( A(\varepsilon) > 0 \) for \( \varepsilon > 0 \). Moreover, since \( \frac{\partial F}{\partial \varepsilon}(0, 0) = 1 \), \( F(\cdot, \varepsilon) \) is increasing near \( A(\varepsilon) \) provided that \( 0 < \varepsilon < \varepsilon_0 \) is chosen small enough. Assume \( \| \phi \|_{H^s} \leq \varepsilon_0 \) and set \( M = A(\| \phi \|_{H^s}) \).

Then

\[ f(0) \leq C \| \phi \|_{H^s} \leq C \| \phi \|_{H^s} + C \| \phi \|_{H^s} \exp(CA(\| \phi \|_{H^s}) = A(\| \phi \|_{H^s}) = M. \]
Suppose \( f(T) > M \) for some \( T \in (0, 1) \) and define \( T_0 = \inf \{ T \in (0, 1); f(T) > M \} \). So, \( T_0 > 0 \), \( f(T_0) = M \) and there exists a decreasing sequence \((T_n)\), converging to \( T_0 \) such that \( f(T_n) > M \) for \( n \) sufficiently large.

From (3.23) we have \( F(f(T)), \|\phi\|_{H^s} \leq 0 \) for all \( T \in [0, 1] \). On the other hand, since \( F(\cdot, \|\phi\|_{H^s}) \) is increasing near \( M \), we have

\[
0 \geq F(f(T_n)), \|\phi\|_{H^s} > F(f(T_0)), \|\phi\|_{H^s} = F(M, \|\phi\|_{H^s}) = F(A(\|\phi\|_{H^s}), \|\phi\|_{H^s}) = 0,
\]

for \( n \) sufficiently large and \( \|\phi\|_{H^s} \leq \varepsilon_0 \), which is an absurd. So, we conclude that \( f(T) \leq M \) for all \( T \in (0, 1) \).

Next, note that \( u(x,y,t) \) is a solution of (3.12), with initial data \( \phi \), if and only if \( u_\lambda(x,y,t) = \lambda^2 u(\lambda^2 x, \lambda^2 y, \lambda^4 t) \) is a solution of (3.12) with initial data \( \phi_\lambda(x,y,t) = \lambda^2 \phi(\lambda^2 x, \lambda^2 y) \). Since \( \|\phi_\lambda\|_{L^2} = \lambda^2 \|\phi\|_{L^2} \), \( \|D_x^2 \phi_\lambda\|_{L^2} = \lambda^2 \|D_x^2 \phi\|_{L^2} \), \( \|D_{xy}^2 \phi_\lambda\|_{L^2} = \lambda^4 \|D_{xy}^2 \phi\|_{L^2} \), we see that all the exponents of \( \lambda \) are different of zero, so we can first choose \( \lambda = \lambda(\|\phi\|_{H^s}) \) such that \( \|\phi_\lambda\|_{H^s} \leq \varepsilon_0 \), and apply the above conclusion to \( u_\lambda \) to obtain (3.27). Finally, an application of Lemma 3.4 concludes the proof of (3.27).

### 3.5. Local well posedness result.
We follow a similar argument to that employed in [16] (see [7] for more details). We will use essentially Lemmas 3.4 and 3.7 to obtain our result. Let \( \phi \in H^s, s > \frac{5}{4} \) and \( \varepsilon > 0 \), we consider the Cauchy problem

\[
\begin{aligned}
\partial_t u + \alpha \Delta u + \alpha \beta \partial_t \partial^2 u + u \partial_x u &= 0, \\
u(x, y, 0) &= \rho_\varepsilon * \phi = \phi_\varepsilon,
\end{aligned}
\]

(3.29)

where \( \rho \in C_0^\infty(\mathbb{R}^2) \) with
1. \( \int_{\mathbb{R}^2} \rho(x) dx = 1 \),
2. \( \int_{\mathbb{R}^2} x^\alpha \rho(x) dx = 0 \), for any \( \alpha = (\alpha_1, \alpha_2) \in (Z^+)^2 \) with \( |\alpha| = \alpha_1 + \alpha_2 > 0 \), and \( \rho_\varepsilon(x) = \varepsilon^{-1} \phi(x/\varepsilon) \), we find \( \phi_\varepsilon \in H^s \cap H^\infty(\mathbb{R}^2) \), such that \( \|\phi - \phi_\varepsilon\|_{H^s} \to 0, \varepsilon \to 0 \) and \( \|\phi_\varepsilon\|_{H^s} \leq 2 \|\phi\|_{H^s} \). From Theorem 1.2 and its proof, it follows, the problem (3.29) has a unique solution \( u_\varepsilon \) such that

\[
u_\varepsilon \in C((0, T]; H^\infty(\mathbb{R}^2)),
\]

with \( T \) independent of \( \varepsilon > 0 \). From Lemma 3.7, there exists \( T = T(\|\phi\|_{H^s}) \) so that

\[
\|\partial_x u_\varepsilon\|_{L^1_t L^\infty_x} + \|u_\varepsilon\|_{L^1_t L^\infty_x} \leq C_T.
\]

(3.30)

From Lemma 3.4 we have

\[
\sup_{0 < t < T} \|u_\varepsilon(t)\|_{H^s} \leq C_T,
\]

(3.31)

with \( C_T = C_T(s, \|\phi\|_{H^s}) \). From (3.31) we can conclude there exists \( u \in H^s(\mathbb{R}^2) \) with \( 0 < t < T \) such that

\[
u_\varepsilon(t) \rightarrow u(t)
\]

in \( H^s(\mathbb{R}^2) \) uniformly in \( \varepsilon \), when \( \varepsilon \to 0 \). Moreover, we have

\[
\|u(t)\|_{H^s}^2 = \lim_{\varepsilon \to 0} \|u_\varepsilon(t)\|_{H^s} \leq \limsup_{\varepsilon \to 0} \|u_\varepsilon(t)\|_{H^s} \leq C \|u(t)\|_{H^s}.
\]

(3.32)
We multiply (3.32) by \( \omega \) and integrate by parts to obtain

\[
\frac{d}{dt} \| \omega(t) \|_{L^2} \leq C(\| \partial_x u_\epsilon \|_{L^\infty} + \| \partial_x u_\epsilon \|_{L^\infty}) \| \omega(t) \|_{L^2}.
\]  
(3.33)

From (3.30) and Gronwall’s inequality we have

\[
\sup_{0 < t < T} \| (u_\epsilon - u_\epsilon)(t) \|_{L^2} \to 0,
\]  
(3.34)

as \( \epsilon, \epsilon' \to 0 \). Thus, we conclude that \( \{ u_\epsilon \} \) is a Cauchy sequence in \( L^\infty([0, T] : L^2(\mathbb{R}^2)) \) and \( \partial_x u_\epsilon^2 \) converges to \( \partial_x u^2 \) in the distributional sense. Then, \( u \) satisfies (3.12) in the distributional sense. Therefore, we have obtained

\[
\begin{align*}
    u_\epsilon & \to u \text{ in } L^2 \text{ uniformly in } \epsilon \\
    u_\epsilon & \to u \text{ in } H^s \text{ uniformly in } \epsilon.
\end{align*}
\]

Thus, \( u \in L^\infty([0, T]; H^s) \cap C([0, T]; L^2) \). The uniqueness of \( u \) follows from the same use of Gronwall’s inequality as in (3.34). In fact, Suppose that \( u_1 \) and \( u_2 \) are two solutions of IVP (3.12). We set \( w = u_1 - u_2 \) and observe that \( w \) satisfies

\[
\partial_t w + \alpha \mathcal{H} \partial_y^3 w + \partial_x \partial_y^2 w + \frac{1}{2} \partial_x ((u_1 + u_2)w) = 0,
\]  
(3.37)

with \( w(\cdot, 0) = \phi_1 - \phi_2 \). We multiply (3.37) by \( w \), integrate by parts to obtain that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} w^2 dx dy = -\frac{1}{2} \int_{\mathbb{R}^2} \partial_x ((u_1 + u_2)w) dx dy - \frac{1}{4} \int_{\mathbb{R}^2} \partial_x (u_1 + u_2)w^2 dx dy.
\]

Using Hölder’s inequality we obtain that

\[
\frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 \lesssim \left( \| \partial_x u_1 \|_{L^1_y L^\infty_x} + \| \partial_x u_2 \|_{L^1_y L^\infty_x} \right) \| w \|_{L^2}^2,
\]

which implies from Gronwall’s inequality that

\[
\sup_{t \in [0, T]} \| w(\cdot, t) \|_{L^2_y} \leq C K \| \phi_1 - \phi_2 \|_{L^2_y},
\]  
(3.38)

where \( K = \max \{ \| \partial_x u_1 \|_{L^1_y L^\infty_x}, \| \partial_x u_2 \|_{L^1_y L^\infty_x} \} \). Therefore, estimate (3.38) implies that \( u_1 = u_2 \) since \( \phi_1 = \phi_2 = \phi \). This concludes the proof of uniqueness.

It remains to show that \( u \in C([0, T]; H^s) \). In fact, let \( \varphi \in H^s, \| \varphi \|_{H^s} = 1 \) we have

\[
\| \varphi \|_{H^s} = \sup_{\| \varphi \|_{H^s} = 1} | (\varphi, \varphi)_{H^s} | \leq \lim \inf_{t \searrow 0} \| u(t) \|_{H^s} \leq \lim \sup_{t \searrow 0} \| u(t) \|_{H^s} \leq \| \varphi \|_{H^s},
\]

which implies

\[
\lim \sup_{t \searrow 0} \| u(t) \|_{H^s} = \| \varphi \|_{H^s}.
\]

Thus, continuity at zero is a consequence of weak continuity. If \( \tau \in (0, T] \) we obtain continuity from continuity at zero and uniqueness. Since the equation in (3.12) is invariant under the transformations \( (x, y, t) \to (-x, -y, \tau - t) \), we obtain continuity to left. This completes the proof of existence and uniqueness of a solution \( u \in C([0, T]; H^s) \) for the problem (3.12). Continuous dependence is a consequence of the Bona-Smith argument [4].
4. Propagation of regularity of solutions of the BO-ZK equation. In this section we establish Theorem 1.4 the main result of this work. Roughly, we will show that solutions of the IVP
\[
\begin{cases}
\partial_t u - \mathcal{H} \partial_y^2 u + \partial_x \partial_y^2 u + u \partial_x u = 0, \\
u(x, y, 0) = \phi(x, y),
\end{cases}
\] (4.1)
founded previously satisfies some special properties. We start our task by proving several needed estimates in our analysis.

Before stating our result we define the class of solutions to the IVP (4.1) to which it applies. We shall consider solutions provided by Theorem 1.3, satisfying the properties (1.14) namely, for \(s > \frac{5}{4}\), the solution \(u\) provided by Theorem 1.3 satisfies, in addition to belong to \(C([0, T]; H^s(\mathbb{R}^2))\)
\[
u, \partial_x u \in L^1([0, T]; L^\infty(\mathbb{R}^2)).
\] (4.2)
We also deduce a Kato’s smoothing effect (see Proposition 3) that will be an important ingredient of our analysis.

4.1. Preliminares. We present some useful properties of the weights functions \(\chi_{e, b}, \phi_{e, b}, \phi_e, \psi_e \in C^\infty(\mathbb{R})\) for \(e > 0\) and \(b \geq 5e\) with \(\chi_{e, b}'(x) \geq 0\) and
\[
\chi_{e, b}(x) = \begin{cases} 
1, & x \geq b, \\
0, & x \leq \epsilon 
\end{cases}
\] originally constructed in [20] and [26] to prove our main result.

**Proposition 2.** Given \(e > 0\) and \(b \geq 5e\), it holds
(1) \(\chi_{e/5, e}(x) = 1\), on \(\text{supp} \chi_{e, b}\),
(2) \(\chi_{e, b}(x) = \chi_{e/5, e}(x) \chi_{e, b}(x)\),
(3) \(\chi_{e, b}'(x) \leq c\chi_{e, 3b+e}'(x) \chi_{e, 3b+e}(x)\),
(4) \(\chi_{e, b}'(x) \leq c\chi_{e, 5e}'(x)\),
(5) \(|\chi_{e, b}(x)| \leq c_j \chi_{e/3, b+\epsilon}(x) \forall j \geq 1\),
(6) \(\text{supp}(\phi_{e, b}), \text{supp}(\phi_e) \subset [\epsilon/4, b]\),
(7) \(\phi_{e, b}(x) = \phi_{e, b} = 1, \ x \in [\epsilon/2, \epsilon]\),
(8) \(\text{supp}(\psi_e) \subseteq (-\infty, \epsilon/2]\),
(9) for \(x \in \mathbb{R}, \chi_{e, b}(x) + \phi_{e, b}(x) + \psi_e(x) = 1\) and \(\chi_{e, b}^2(x) + \phi_{e, b}^2(x) + \psi_e(x) = 1\).

**Proof.** See [18, 19, 20, 26].

**Remark 3.** From Proposition 2 and the fact that
\[
\chi_{e, b}'(x) = \eta_{e, b}'^2, \ \text{i.e.} \ \eta_{e, b} = \sqrt{\chi_{e, b}'(x)},
\] provided by definition (1.20) we notice that the function \(\eta_{e, b}\) satisfies
\[
(\eta_{e, b}'(x))^2 \leq c_{e, b} \chi_{e, 3b+\epsilon}'(x).
\] (4.4)

We introduce now a local smoothing properties of the BO-ZK equation.

**Proposition 3.** Let \(s > \frac{5}{4}\) and \(\phi \in H^s(\mathbb{R}^2)\). Then the corresponding solution \(u\) of (4.1) satisfies, for any \(R > 0\) and \(T > 0\),
\[
\int_0^T \int_{-\infty}^\infty \int_{-R}^R [(D_x^2 D_y^2 u(x, y, t))^2 + (J_x^2 \partial_y u(x, y, t))^2] \, dx \, dy \, dt \leq C(R, T, \|\phi\|_{H^s}).
\] (4.5)
Proof. We consider \( \psi = \psi(x) \in C^\infty(\mathbb{R}) \) an increasing function such that \( \psi' \in C_0^\infty(\mathbb{R}) \). Formally, we apply \( J_x^s \) to equation (4.1) multiply by \( J_x^s u \), and integrate the result in the space variable to obtain,

\[
\frac{1}{2} \frac{d}{dt} \int (J_x^s u)^2 \psi \, dx \, dy + \frac{1}{2} \int (J_x^s \partial_\eta u)^2 \psi' \, dx \, dy - \int J_x^s \mathcal{H} \partial_\eta^2 u J_x^s u \psi \, dx \, dy + \int J_x^s(u \partial_\eta u) J_x^s u \psi \, dx \, dy = 0. \tag{4.6}
\]

We obtain from integration by parts that

\[
- \int J_x^s \mathcal{H} \partial_\eta^2 u J_x^s u \psi \, dx \, dy
= \int \partial_\eta J_x^s u \partial_\eta J_x^s u \psi \, dx \, dy + \int \partial_\eta \mathcal{H} J_x^s u J_x^s u \psi' \, dx \, dy = A + B. \tag{4.7}
\]

Using the fact that Hilbert transform is skew symmetric, from integration by parts we have

\[
A = - \int J_x^s \partial_\eta u \mathcal{H}(J_x^s \partial_\eta u \psi) \, dx \, dy
= - \int J_x^s \partial_\eta u \mathcal{H} J_x^s \partial_\eta u \psi \, dx \, dy - \int J_x^s \partial_\eta u \mathcal{H}, \psi \, J_x^s \partial_\eta u \, dx \, dy
= -A - \int J_x^s \partial_\eta u \mathcal{H}, \psi \, J_x^s \partial_\eta u \, dx \, dy.
\]

This implies that

\[
A = -\frac{1}{2} \int J_x^s \partial_\eta u \mathcal{H}, \psi \, J_x^s \partial_\eta u \, dx \, dy = \frac{1}{2} \int J_x^s \partial_\eta u \mathcal{H}, \psi \, J_x^s \partial_\eta u \, dx \, dy. \tag{4.8}
\]

Using the commutator estimate (3.14) of the Theorem 3.5, we obtain

\[
|A| \leq C \| J_x^s u \|_{L_x^2} \| \partial_\eta \mathcal{H}, \psi \| J_x^s \partial_\eta u \|_{L_x^2} \leq C \| J_x^s u(t) \|_{L_x^2}^2. \tag{4.9}
\]

Next, we use the notation \( \eta^2 = \psi' \) to rewrite \( B \) as follows

\[
B = \int \mathcal{H} J_x^s \partial_\eta u \eta J_x^s u \, \eta \, dx \, dy
= \int J_x^s u J_x^s u \eta \mathcal{H}, \psi \, J_x^s \partial_\eta u \, dx \, dy
= \int J_x^s u J_x^s \mathcal{H} \partial_\eta J_x^s \partial_\eta u \eta \, dx \, dy - \int J_x^s u \mathcal{H}(J_x^s \eta \eta') \, dx \, dy - \int J_x^s u \mathcal{H}, \eta \, J_x^s \partial_\eta u \, dx \, dy
= \int [D_x^{1/2}(J_x^s u \eta)]^2 \, dx \, dy - \int J_x^s u \mathcal{H}(J_x^s \eta \eta') \, dx \, dy - \int J_x^s u \mathcal{H}, \eta \, J_x^s \partial_\eta u \, dx \, dy
= B_1 + B_2 + B_3. \tag{4.10}
\]

We notice that \( B_1 \) in (4.10) is positive and represents the smoothing effect.

In order to estimate the remaining terms in (4.10) we use the fact that \( \mathcal{H} \) is an isometry in \( L^2 \) and the commutator estimate (3.14) of the Theorem 3.5, to obtain

\[
|B_2 + B_3| \leq \| J_x^s u \eta \|_{L_x^2} \| J_x^s u \|_{L_x^2} + \| J_x^s u \eta \|_{L_x^2} \| \mathcal{H}, \eta \| \| J_x^s \partial_\eta u \|_{L_x^2} \leq c \| J_x^s u \|_{L_x^2}^2 \| J_x^s u \|_{L_x^2} \tag{4.11}
\]

Integration by parts and Kato-Ponce’s commutator estimate [23] implies that

\[
\left| \int J_x^s(u \partial_\eta u) J_x^s u \psi \, dx \, dy \right|
\]
Combining the estimates (4.7) – (4.12), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \int (J_x^s u)^2 \, dx \, dy + \frac{1}{2} \int \left( J_x^s \partial_y u \right)^2 \, dx \, dy + \int \left[ D_x^{1/2}(J_x^s u \eta) \right]^2 \, dx \, dy \\
\leq c \| \partial_x u \|_{L^\infty_x} \left\| J_x^s u \right\|_{L^2_x}^2 + c \left( \| \partial_x u \|_{L^\infty_x} \left\| J_x^s \partial_x u \right\|_{L^2_x} \right) + \left\| J_x^s u \right\|_{L^\infty_x} \left\| \partial_x u \right\|_{L^2_x} \left\| J_x^s u \right\|_{L^2_x}.
\]  
(4.12)

Using the a priori estimate (3.13) the fact that the solution \( u \) provided by the Theorem 1 satisfies (4.2), i.e., \( \| \partial_x u \|_{L^1_t L^\infty_x} < \infty \) and integrating in time we conclude that
\[
\int_0^T \left( D_x^{1/2}(J_x^s u \eta) \right)^2 \, dx \, dy + \int_0^T \left( J_x^s \partial_y u \right)^2 \, dx \, dy \leq C(T, \| \phi \|_{H^s}, \psi).
\]  
(4.14)

Therefore, from Lemma A.4, for \( \psi' \in C_0^\infty(\mathbb{R}) \) we obtain
\[
D_x^{1/2}(J_x^s u \eta) + J_x^s(\partial_y u \psi') \in L^2(\mathbb{R}^2 \times [0, T]).
\]

This finishes the proof. \( \square \)

4.2. Proof of Theorem 1.4. By translation if necessary, we may assume that \( x_0 = 0 \). We apply \( \partial_x^l \) to equation (4.1), multiply by \( \partial_x^l u \chi_{\epsilon,b}(x + \nu t) \) and integrate in \( \mathbb{R}^2 \). If we have enough regularity to apply integration by parts we obtain that

\[
\frac{1}{2} \frac{d}{dt} \int (\partial_x^l u)^2 \, \chi \, dx \, dy - \frac{\nu}{2} \int (\partial_x^l u)^2 \, \chi' \, dx \, dy + \frac{1}{2} \int (\partial_x^l \partial_y u)^2 \, \chi' \, dx \, dy \\
- \int \partial_x^l \mathcal{H} \partial_x^l u \partial_x^l u \chi \, dx \, dy + \int \partial_x^l (u \partial_x^l u) \partial_x^l u \chi \, dx \, dy = 0.
\]  
(4.15)

We recall that \( g > 0 \) is a function with \( \int_0^T g(t) \, dt \leq C \). Notice that, eventually, we will mix several cases together to perform our analysis and then we will carry on denoting by \( g \) a generic nonnegative integrable function on \([0, T]\).

Case \( l = 1 \) in (1.21). We observe that for \( l = 0, 1 \) the result follows directly from the well-posedness of the the IVP (4.1) with \( \phi \equiv u(0) \in H^{\frac{1}{2} +} (\mathbb{R}^2) \) provided by Theorem 1.3.

Since given \( T > 0, \epsilon > 0, b > 5\epsilon, \nu > 0 \), there exist \( c > 0 \) and \( R > 0 \) such that
\[
\chi_{\epsilon,b}(x + \nu t) \leq c 1_{[-R,R]}(x) \text{ for all } (x,t) \in \mathbb{R} \times [0,T].
\]  
(4.16)

Thus, we obtain that
\[
\int_0^T |A_1(t)| \, dt \leq c \nu \int_0^T (\partial_x^l u)^2 \, \chi' \, dx \, dy \, dt \leq CT \| u \|_{L^\infty_x H^1} \lesssim \| u \|_{L^\infty_x H^{\frac{1}{2} +}}.
\]  
(4.17)

We observe that the term \( A_2 \) is positive and provides a smoothing effect.

To estimate the terms \( A_3 \) and \( A_4 \) we consider the two cases \( l = 0 \) and \( l = 1 \) separately. Regarding the first case \( l = 0 \) we use integration by parts to rewrite the term \( A_3 \) in (4.15) as follows
\[ A_3 = \int \mathcal{H} \partial_x u \partial_x u \, \chi' \, dx \, dy + \int \mathcal{H} \partial_x uu' \, dx \, dy = A_{31} + A_{32} \]  

(4.18)

Using the fact that Hilbert transform is skew symmetric, from integration by parts we have

\[ A_{31} = -A_3 - \int \partial_x u[\mathcal{H}, \chi] \partial_x u \, dx \, dy. \]

Thus, we have

\[ A_{31} = -\frac{1}{2} \int \partial_x u[\mathcal{H}, \chi] \partial_x u \, dx \, dy = \frac{1}{2} \int u \partial_x[\mathcal{H}, \chi] \partial_x u \, dx \, dy. \]  

(4.19)

Using the commutator estimate (3.14), we obtain

\[ \sup_{0 \leq t \leq T} |A_{31}(t)| \leq C \|u\|_{L^2_t} \|\partial_x[\mathcal{H}, \chi] \partial_x u\|_{L^2_y} \leq C \|u(t)\|^2_{L^2_y} = C \|\phi\|^2_{L^2_y}. \]  

(4.20)

Now, we use that \( \eta^2 = \chi' \) to rewrite \( A_{32} \) as follows

\[ A_{32} = \int \mathcal{H} \partial_x u \eta \eta \, dx \, dy \]

\[ = \int u \eta \mathcal{H}(\partial_x u \eta) \, dx \, dy - \int u \eta[\mathcal{H}, \eta] \partial_x u \, dx \, dy \]

\[ = \int u \eta \mathcal{H}(u \eta') \, dx \, dy - \int u \eta[\mathcal{H}, \eta] \partial_x u \, dx \, dy \]

\[ = \int \left[ D_x^{1/2}(u \eta) \right]^2 \, dx \, dy - \int u \eta \mathcal{H}(u \eta') \, dx \, dy - \int u \eta[\mathcal{H}, \eta] \partial_x u \, dx \, dy \]

\[ = A_{321} + A_{322} + A_{323}. \]

(4.21)

First of all, we observe that \( A_{321}(t) \) in (4.21) is positive and represents the smoothing effect.

To estimate \( A_{322} \) we use the fact that \( \mathcal{H} \) is an isometry in \( L^2 \) to write

\[ \int_0^T |A_{322}(t)| \, dt \leq C \int_0^T \|u \eta\|_{L^2_y} \|\eta\|_{L^2_y} \|\eta'\|_{L^2_y} \, dt \leq CT \|u\|^2_{L^2_T L^2(\mathbb{R}^2)} \lesssim \|\phi\|^2_{L^2_y}. \]  

(4.22)

From the commutator estimate (3.14) in Theorem 3.5, and integration in time interval \([0, T]\) we see that

\[ \int_0^T |A_{323}(t)| \, dt \leq C \int_0^T \|u \eta\|_{L^2_y} \|\mathcal{H}, \eta\|_{L^2_y} \|\partial_x u\|_{L^2_y} \, dt \]

\[ \leq C \int_0^T \|u \eta\|_{L^2_y} \|\eta\|_{L^2_y} \|\partial_x u\|_{L^2_y} \, dt \leq CT \|u\|^2_{L^2_T L^2(\mathbb{R}^2)} \lesssim \|\phi\|^2_{L^2_y}. \]  

(4.23)

To estimate the term \( A_4 \) in (4.15) we use Sobolev embedding, the conservation of \( L^2 \) norm of the solutions (1.5) and (4.16), to obtain that

\[ A_4 = \int u \partial_x uu' \chi' \, dx \, dy = -\frac{1}{3} \int u^2 \chi' \, dx \, dy \leq C \|u\|_{L^2_T H^{\frac{5}{2}}} \int u^2 \chi' \, dx \, dy \lesssim \|u\|_{L^2_T H^{\frac{5}{2}}}. \]

Next, we consider the case \( l = 1 \). In order to control the contribution of \( A_3 \) in (4.15) we perform integration by parts and write

\[ A_3 = \int \partial_x \mathcal{H} \partial_x u \partial_x^2 u \, \chi \, dx \, dy + \int \partial_x \mathcal{H} \partial_x u \partial_x u \, \chi' \, dx \, dy = A_{31} + A_{32} \]  

(4.24)
An argument similar to the one used in (4.19) leads to
\[ A_{31} = -\frac{1}{2} \int \partial_z^2 u [\mathcal{H}, \chi] \partial_z^2 u dx dy = \frac{1}{2} \int u \partial_z^2 [\mathcal{H}, \chi] \partial_z^2 u dx dy. \] (4.25)

Using the commutator estimate (3.14) we obtain
\[ \sup_{0 \leq t \leq T} |A_{31}(t)| \leq C \|u\|_{L^2_y} \|\partial_z^2 [\mathcal{H}, \chi] \partial_z^2 u\|_{L^2_y} \leq C \|u(t)\|^2_{L^2_y} = C \|\phi\|^2_{L^2_y}. \] (4.26)

Similar to the argument in (4.21)
\[ A_{32} = \int [D_x^{1/2} (\partial_x u \eta)]^2 dx dy - \int \partial_x u \eta \mathcal{H}(\partial_x u \eta') dx dy - \int \partial_x u \eta \mathcal{H}, \eta \partial_z^2 u dx dy = A_{321} + A_{322} + A_{323} \] (4.27)

Firstly, we observe that \( A_{321}(t) \) in (4.27) is positive and represents the smoothing effect.

To estimate \( A_{322} \) we use the fact that \( \mathcal{H} \) is an isometry in \( L^2 \) to obtain
\[ \int_0^T |A_{322}(t)| dt \leq C \int_0^T \|\partial_x u \eta\|_{L^2_y} \|\partial_x u \eta'\|_{L^2_y} dt \]
\[ \leq C T \|u\|_{L^2_y}^2 H^1(\mathbb{R}^2) \lesssim \|u\|_{L^2_y} H^{\frac{5}{4} +}(\mathbb{R}^2). \] (4.28)

The commutator estimate (3.14), after integration in time interval \([0, T]\), Sobolev embedding and the conservation of \( L^2 \) norm of the solutions (1.5) leads to
\[ \int_0^T |A_{323}(t)| dt \leq C \int_0^T \|\partial_x u \eta\|_{L^2_y} \|\mathcal{H}, \eta \partial_z^2 u\|_{L^2_y} dt \]
\[ \leq C \int_0^T \|\partial_x u \eta\|_{L^2_y} \|u\|_{L^2_y} dt \lesssim \|u\|_{L^2_y} H^{\frac{5}{4} +}(\mathbb{R}^2). \] (4.29)

To estimate the term \( A_4 \) in (4.15) we use integration by parts to observe that
\[ A_4 = \frac{1}{2} \int \partial_x u (\partial_x u)^2 \chi dx dy - \frac{1}{2} \int u (\partial_x u)^2 \chi' dx dy = A_{41} + A_{42} \]
one has
\[ |A_{41}| \leq \|\partial_x u\|_{L^\infty_y} \int (\partial_x u)^2 \chi dx dy \] (4.30)
and
\[ |A_{42}| \leq \|u\|_{L^\infty_y} \int (\partial_x u)^2 \chi' dx dy. \] (4.31)

Hence by Sobolev embedding and (4.16) after integration in the time interval \([0, T]\) one gets
\[ \int_0^T |A_{42}(t)| dt \leq \|u\|_{L^\infty_T H^{\frac{5}{4} +}} \int_0^T \int (\partial_x u)^2 \chi' dx dy dt \lesssim \|u\|^3_{L^\infty_T H^{\frac{5}{4} +}}. \] (4.32)

Inserting the above information in (4.15), Gronwall’s inequality and (4.2) yield the estimate
\[ \sup_{0 \leq t \leq T} \int (\partial_x^2 u)^2 \chi_{x,b}(x + \nu t) dx dy + \int_0^T \int [D_x^{1/2} (\partial_x^2 u \eta_{x,b}(x + \nu t))]}^2 dx dy dt \]
\[ + \int_0^T \int (\partial_x^2 \partial_y u)^2 \chi' (x + \nu t) dx dy dt \leq C. \] (4.33)

This gives the result for all indices \( l = 0, 1 \).
Case \( l = 1 \) in (1.24). After apply \( D_x^{1/2} \partial_x \) to (4.1), multiply the resulting by \( D_x^{1/2} \partial_x u \chi \) and integrate in the space variable we get the identity

\[
\frac{1}{2} \frac{d}{dt} \int (D_x^{1/2} \partial_x u)^2 \chi dxdy - \nu \int (D_x^{1/2} \partial_x u)^2 \chi' dxdy + \frac{1}{3} \int (D_x^{1/2} \partial_x \psi u)^2 \chi' dxdy
\]

\[
- \int D_x^{1/2} \partial_x \mathcal{H} \partial_x^2 u D_x^{1/2} \partial_x u \chi dxdy + \int D_x^{1/2} \partial_x (u \partial_x u) D_x^{1/2} \partial_x u \chi dxdy = 0. \tag{4.34}
\]

We use that \( \eta^2 = \chi' \) to write

\[
|A_1| = c \| D_x^{1/2} \partial_x u \|_{L_y^2} \leq \| D_x^{1/2} \partial_x u \|_{L_y^2}, \tag{4.35}
\]

We notice that the first term on the RHS of (4.35) is bounded, after integration in time, by the former case \( l = 1 \) in (1.22) estimate (4.33). To estimate the term \( A_2 \) on the RHS of (4.35) we use the fact that \( \chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi(x) = 1 \forall x \in \mathbb{R} \) to write

\[
A_{12} = \| [D_x^{1/2}, \chi_{\epsilon,b}] \partial_x (u \chi_{\epsilon,b}) \|_{L_y^2} = A_{121} + A_{122} + A_{123}. \tag{4.36}
\]

We employ Lemma A.5, to obtain

\[
|A_{121} + A_{122}| \leq c \| \partial_x (u \chi_{\epsilon,b}) \|_{L_y^2} + c \| \partial_x (u \phi_{\epsilon,b}) \|_{L_y^2}, \tag{4.37}
\]

which is bounded, after integration in time, by the former case of order \( l = 1 \) and the estimate (4.17) since \( \phi_{\epsilon,b} \) is compactly supported.

Since

\[
\text{dist}(\text{supp}(\eta_{\epsilon,b}), \text{supp}(\psi)) \geq \frac{\epsilon}{2},
\]

we employ Lemma A.7, to obtain

\[
|A_{123}| = \| [D_x^{1/2}, \chi_{\epsilon,b}] \partial_x (u \psi) \|_{L_y^2} \leq C. \tag{4.38}
\]

Thus, we obtain

\[
\int_0^T |A_1(t)| dt \leq c \int_0^T (D_x^{1/2} \partial_x u)^2 \chi' dxdy dt \leq C. \tag{4.39}
\]

Regarding the term \( A_3 \), using integration by parts we have

\[
A_3 = \int \partial_x^2 \mathcal{H} D_x^{1/2} u \partial_x^2 D_x^{1/2} u \chi dxdy + \int \mathcal{H} \partial_x^2 D_x^{1/2} u \partial_x D_x^{1/2} u \chi' dxdy = A_{31} + A_{32}. \tag{4.40}
\]

A similar argument to that one used in (4.19) – (4.76) yields

\[
\sup_{0 \leq t \leq T} |A_{31}(t)| \leq C \| D_x^{1/2} u \|_{L_y^2} \| \partial_x^2 \mathcal{H} \|_{H^{1/2}} \| \partial_x^2 D_x^{1/2} u \|_{L_y^2}
\]

\[
\leq C \| D_x^{1/2} u(t) \|_{L_y^2}^2 \| u \|_{L_y^2}^{1/2} \| \mathcal{H} \|_{H^{1/2}}. \tag{4.41}
\]

Now, we use the fact that \( \eta^2 = \chi' \) and a familiar argument analogous to that one used in (4.21) to write

\[
A_{32} = \int [D_x^{1/2}(D_x^{1/2} \partial_x u \eta)]^2 dxdy - \int D_x^{1/2} \partial_x u \eta \mathcal{H}(D_x^{1/2} \partial_x u \eta') dxdy
\]

\[
- \int D_x^{1/2} \partial_x u \eta \mathcal{H}(D_x^{1/2} \partial_x u \eta') dxdy
\]
We recall that $A_{321}(t)$ in (4.42) is positive and represents the smoothing effect. Now, an argument similar to that one employed in (4.22) – (4.23) provide us

$$\int_0^T (|A_{322}(t)| + |A_{323}(t)|)dt$$

$$\leq \int_0^T (D_x^{1/2} \partial_x u)^2 \chi'(x)dxdydt + \int_0^T (D_x^{1/2} \partial_x u)^2 (\eta')^2 dxdydt + cT \|u\|^2_{L_y^\infty H_x^4}. \tag{4.43}$$

The two term on RHS of (4.43) are similar, and the first one is simply a multiply of the term $A_1$ above, so it is treated exactly as we did before in (4.39).

It only remains to handle the term $A_3$ in (4.34). Since $\chi_\epsilon = \chi_{\epsilon/5, \epsilon} \equiv 1$ on $\text{supp} \chi$, we write

$$D_x^{1/2} \partial_x(u \partial_x u) \chi_\epsilon = -\frac{1}{2} [D_x^{1/2} \partial_x, \chi_\epsilon] \partial_x(u)^2 + [D_x^{1/2} \partial_x, u \chi_\epsilon] \partial_x u + u \chi_\epsilon D_x^{1/2} \partial_x^2 u$$

$$= -\frac{1}{2} [D_x^{1/2} \partial_x, \chi_\epsilon] \partial_x((u \chi_{\epsilon,b})^2) + (u \phi_{\epsilon,b})^2 + (u^2 \psi_\epsilon)$$

$$+ [D_x^{1/2} \partial_x, u \chi_\epsilon] \partial_x(u \chi_{\epsilon,b} + u \phi_{\epsilon,b} + u \psi_\epsilon) + u \chi_\epsilon D_x^{1/2} \partial_x^2 u$$

$$= A_{41}(t) + A_{42}(t) + A_{43}(t) + A_{44}(t) + A_{45}(t) + A_{46}(t) + A_{47}(t). \tag{4.44}$$

Firstly, we rewrite $A_{41}$ as follows

$$A_{41}(t) = -\frac{1}{2} \mathcal{H}[D_x^{1+1/2}, \chi_\epsilon](u \chi_{\epsilon,b})^2 - \frac{1}{2} \mathcal{H}(\chi_\epsilon) D_x^{1+1/2} \partial_x((u \chi_{\epsilon,b})^2). \tag{4.45}$$

Combining (A.2), Lemma A.5 and Lemma A.6 in the variable $x$ and then Hölder’s inequality in the variable $y$, we have

$$\|A_{41}(t)\|_{L_y^2} = \left\| [D_x^{1+1/2}, \chi_\epsilon] \partial_x((u \chi_{\epsilon,b})^2) \right\|_{L_y^2} \lesssim \left\| D_x^{1+1/2} (u \chi_{\epsilon,b}) \right\|_{L_y^2} \|u\|_{L_y^\infty} + \|u\|_{L_y^2} \|u\|_{L_y^\infty} \tag{4.46}$$

and similarly

$$\|A_{42}(t)\|_{L_y^2} = \left\| [D_x^{1+1/2}, \chi_\epsilon] \partial_x((u \phi_{\epsilon,b})^2) \right\|_{L_y^2} \lesssim \left\| D_x^{1+1/2} (u \phi_{\epsilon,b}) \right\|_{L_y^2} \|u\|_{L_y^\infty} + \|u\|_{L_y^2} \|u\|_{L_y^\infty}. \tag{4.47}$$

We recall that by construction

$$\text{dist}(\text{supp}(\chi_{\epsilon,b}), \text{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2},$$

so an application of Lemma A.7 in the variable $x$ and then Hölder’s inequality in the variable $y$, yields

$$\|A_{43}\|_{L_y^2} = \left\| [D_x^{1/2} \partial_x, \chi_{\epsilon,b}] \partial_x(u^2) \right\|_{L_y^2} \lesssim \|u\|_{L_y^2} \|u\|_{L_y^\infty}, \tag{4.48}$$

where the first identity is obtained taking $(\epsilon, b)$ instead of $(\epsilon/5, \epsilon)$. 

$$= A_{321} + A_{322} + A_{323}. \tag{4.42}$$
Analogously to (4.53) we write
\[ A_{44}(t) = -\frac{1}{2} \left[ D_x^{1+1/2}, u \psi \right] \partial_x (u \psi_{x,b}) - \frac{1}{2} \left[ \mathcal{H}, u \chi \right] D_x^{1+1/2} \partial_x (u \psi_{x,b}). \] (4.49)

Now, we apply the commutator estimates (3.14) and Lemma A.5 in the variable \( x \) and then Hölder’s inequality in the variable \( y \), to obtain
\[ \| A_{44}(t) \|_{L^2_y} \lesssim \| \partial_x (u \psi_{x,b}) \|_{L^\infty_y} \| D_x^{1+1/2} (u \psi_{x,b}) \|_{L^2_y}, \] (4.50)

and
\[ \| A_{45}(t) \|_{L^2_y} \lesssim \| \partial_x (u \phi_{x,b}) \|_{L^\infty_y} \| D_x^{1+1/2} (u \phi_{x,b}) \|_{L^2_y} + \| \partial_x (u \psi_{x,b}) \|_{L^\infty_y} \| D_x^{1+1/2} (u \psi_{x,b}) \|_{L^2_y}. \] (4.51)

where we have chosen \((\epsilon, b)\) instead of \((\epsilon/5, \epsilon)\). Since the supports of \( \chi_{x,b} \) and \( \psi_\epsilon \) are separated, a familiar argument similar to that one performed in (4.48) yields
\[ \| A_{46} \|_{L^2_y} = \left\| u \chi_{x,b} \partial_x D_x^{1+1/2} (u \psi_\epsilon) \right\|_{L^2_y} \lesssim \| u \|_{L^2_y} \| u \|_{L^\infty_y}, \] (4.52)

where the first identity is obtained taking \((\epsilon, b)\) instead of \((\epsilon/5, \epsilon)\). Next, we write
\[ D_x^{1+1/2} (u \chi_{x,b}) = D_x^{1+1/2} u \chi_{x,b} + [D_x^{1+1/2}, \chi_{x,b}] (u \chi_{x,b} + u \phi_{x,b} + u \psi_\epsilon). \] (4.53)

We notice that the \( L^2 \) norm of the first term on the RHS of (4.53) is the very quantity to be estimated. The control of the \( L^2 \) of the second one can be performed by employing a familiar argument, similar to that one used in the analysis of the terms \( A_{41}, A_{42} \) and \( A_{43} \) above.

Hölder’s inequality, Proposition 2 and Theorem A.3 in the variable \( x \) provide us
\[ \| D_x^{1+1/2} (u \phi_{x,b}) \|_{L^2_x} \lesssim \| u \|_{L^4_x} \| D_x^{1+1/2} \phi_{x,b} \|_{L^4_x} + \left\| \sum_{\beta \leq 1} \frac{1}{\beta!} \partial_\beta \phi_{x,b} D_x^{\beta-1} u \right\|_{L^2_x} \]
\[ \lesssim \| D_x^{1/2} u \|_{L^2_x} \| u \|_{L^2_x} + \| \phi_{x,b} D_x^{1+1/2} u \|_{L^2_x} + \| \partial_\beta \phi_{x,b} \mathcal{H} D_x^{1/2} u \|_{L^2_x} \]
\[ \lesssim \| D_x^{1/2} u \|_{L^2_x} + \| u \|_{L^2_x} + \| \beta \leq 1 \| \partial_\beta \phi_{x,b} \mathcal{H} D_x^{1/2} u \|_{L^2_x}. \] (4.54)

Next, employ the \( L^2 \) norm in the variable \( y \) to obtain
\[ \| D_x^{1+1/2} (u \phi_{x,b}) \|_{L^2_y} \lesssim \| D_x^{1/2} u \|_{L^2_y} + \| u \|_{L^2_y} + \| \beta \leq 1 \| \partial_\beta \phi_{x,b} \mathcal{H} D_x^{1/2} u \|_{L^2_y}. \] (4.55)

The second term on the RHS of (4.55) can be controlled, by a familiar argument similar to that one performed in the analysis of \( A_1 \) in (4.39) while the third one is bounded by Sobolev’s embedding.

Lastly, we apply integration by parts to obtain
\[ A_{47}(t) = -\frac{1}{2} \int \partial_x u \chi_{x} (D_x^{1/2} \partial_x u)^2 \chi dx dy - \frac{1}{2} \int u (\chi_{x} \chi)^\prime (D_x^{1/2} \partial_x u)^2 dx dy \]
\[ = A_{471} + A_{472}. \] (4.56)
A familiar argument yields

$$|A_{411}(t)| \lesssim \|\partial_x u\|_{L^\infty_{xy}} \int (D_x^{1/2} \partial_x u)^2 \chi dxdy.$$  

The last integral term is the quantity to be estimated, and the other term will be controlled after integration in time by (4.2). We apply Sobolev’s embedding to obtain

$$|A_{4112}(t)| \lesssim \|u\|_{L^p_{x,y} H^{3/2}} \int (D_x^{1/2} \partial_x u)^2 (\chi_1 \chi') dxdy,$$

where the last term is bounded, after integration in time, by using a familiar argument analogous to that one used in the analysis of the term $A_1$ in (4.39) above.

We insert the above information contained in (4.39) – (4.56) in (4.34), apply Gronwall’s inequality and (4.2) to obtain

$$\sup_{0 \leq t \leq T} \int (D_x^{1/2} \partial_x u)^2 \chi_{1,b}(x + \nu t) dxdy + \frac{1}{2} \int_0^T \int (D_x^{1/2} \partial_x \rho u)^2 \chi' dxdydt$$
$$+ \int_0^T \int \left[ D_x^{1/2}(D_x^{1/2} \partial_x u\eta)\right]^2 dxdydt \leq C. \quad (4.57)$$

This gives the desired estimate (1.25) with $l = 1$. Next, we consider the case $l = 2$.

**Case** $l = 2$ in (1.21). So, (4.15) can be written as follows

$$\frac{1}{2} \frac{dt}{dt} \left( \begin{array}{l} \int (\partial_x^2 u)^2 \chi dxdy - \frac{\nu}{2} \int (\partial_x^2 u)^2 \chi dxdy + \frac{1}{2} \int (\partial_x^2 \rho u)^2 \chi dxdy \\ A_1 \end{array} \right)$$
$$- \frac{1}{2} \int \partial_x^2 \mathcal{H} \partial_x^2 u \partial_x^2 u \chi dxdy + \frac{1}{2} \int \partial_x^2 (u \partial_x u) \partial_x^2 u \chi dxdy = 0. \quad (4.58)$$

We write

$$\mathcal{H}(\partial_x^2 u \eta_{1,b}) = [\mathcal{H}, \eta_{1,b}] \partial_x^2 u + D_x^{1/2}(D_x^{1/2} \partial_x \eta_{1,b}) - [D_x^{1/2}, \eta_{1,b}] D_x^{1/2} \partial_x u$$
$$= I_1 + I_2 + I_3. \quad (4.59)$$

We notice that $\|I_2\|_{L_{xy}}$ is bounded, after integration in time, by the former case (4.57). The term $\|I_1\|_{L_{xy}}$ is controlled by using commutator estimate (3.14) while $\|I_3\|_{L_{xy}}$ is bounded, after integration in time, by employing a similar argument to that one used in the analysis (4.46) – (4.48) above, combined with the former case (4.57). Since $\eta_{1,b} = \chi_{1,b}$ we obtain

$$\int_0^T |A_1(t)| dt = \int_0^T \int (\partial_x^2 u)^2 \chi dxdydt \lesssim \int_0^T \|\mathcal{H}(\partial_x^2 u \eta_{1,b})\|_{L_{xy}} dt \leq C. \quad (4.60)$$

To control the contribution of $A_3$ in (4.58) we perform integration by parts and write

$$A_3 = \int \partial_x \mathcal{H} \partial_x^2 u \partial_x^2 u \chi dxdy + \int \partial_x \mathcal{H} \partial_x^2 u \partial_x^2 u \chi dxdy = A_{31} + A_{32}. \quad (4.61)$$

Using the same argument performed to obtain (4.19) yields

$$A_{31} = -\frac{1}{2} \int \partial_x^3 u \mathcal{H} \partial_x^2 u dxdy = \frac{1}{2} \int u \partial_x^3 [\mathcal{H}, \chi] \partial_x^2 u dxdy. \quad (4.62)$$
From the commutator estimate (3.14) we obtain
\[
\sup_{0 \leq t \leq T} |A_{31}(t)| \leq C \|u\|_{L^2_y} \|\partial_x^3 [H, \chi]\|_{L^2_y} \leq C \|u(t)\|_{L^2_y}^2 = C \|\phi\|_{L^2_y}^2.
\] (4.63)

Now, we use that \(\eta^2 = \chi'\) to rewrite \(A_{32}\) and perform the same argument given in (4.21) to get
\[
A_{32} = \int [D_x^{1/2}(\partial_x^2 u \eta)]^2 dx dy - \int \partial_x^2 u \eta H(\partial_x^2 u \eta') dx dy - \int \partial_x^2 u \eta [H, \eta] \partial_x^2 u dx dy
= A_{321} + A_{322} + A_{323}.
\] (4.64)

First, we observe that \(A_{321}(t)\) in (4.77) is positive and represents the smoothing effect.

To estimate \(A_{322}\) we use the fact that \(H\) is an isometry in \(L^2\), Proposition 2, (4.4) and the estimate (4.60) to obtain
\[
\int_0^T |A_{322}(t)| dt \leq c \int_0^T \|\partial_x^2 u \eta\|_{L^2_y} \|\partial_x^2 \eta\|_{L^2_y} dt
\leq \int_0^T \int (-\partial_x^2 u)^2 \chi' dx dy dt + \int_0^T \int (-\partial_x^2 u)^2 (\eta')^2 dx dy dt
\leq \int_0^T \int_{-\infty}^{R} \int_{-R}^{R} (-\partial_x^2 u)^2 dx dy dt \leq C.
\] (4.65)

Using the commutator estimate (3.14), after integration in time and the estimate (4.60) one finds that
\[
\int_0^T |A_{323}(t)| dt \leq \int_0^T \|\partial_x^2 u \eta\|_{L^2_y} \|[H, \eta]\|_{L^2_y} dt
\leq cT \|u\|_{L^2_y}^2 + \int_0^T \int_{-\infty}^{R} \int_{-R}^{R} (-\partial_x^2 u)^2 dx dy dt \leq C.
\] (4.66)

Finally, we consider \(A_4\) in (4.58). Using integration by parts, we see that
\[
|A_4| = \frac{5}{2} \int_{R^2} \partial_x u (\partial_x^2 u)^2 \chi dx dy - \frac{1}{2} \int_{R^2} u (\partial_x^2 u)^2 \chi' dx dy
\leq c \|\partial_x u\|_{L^\infty_{x,y}} \int_{R^2} (\partial_x^2 u)^2 \chi dx dy + c \|u\|_{L^\infty_y} \int_{R^2} (\partial_x^2 u)^2 \chi' dx dy.
\] (4.67)

The first term on the RHS of (4.67) is the very quantity to be estimated while the last term can be handled as we did in the former case \(A_1\) estimate (4.60).

Inserting the above information (4.60) – (4.67) in (4.58) applying Gronwall’s inequality and (4.2) we obtain that
\[
\sup_{0 \leq t \leq T} \int (\partial_x^2 u)^2 \chi_{e,b}(x + vt) dx dy + \int_0^T \int \left[ D_x^{1/2}(\partial_x^2 u \eta_{e,b}(x + vt))\right]^2 dx dy dt
+ \int_0^T \int (\partial_x^2 \partial_y u)^2 \chi'(x + vt) dx dy dt \leq C.
\] (4.68)

This gives the result for the case \(l = 2\).

**Case** \(l = 2\) in (1.24). We assume that (1.21) holds for \(l = 2\) and that (1.22) holds for any \(\epsilon > 0\) and \(b \geq 5\epsilon\) with \(x_0 = 0\) and \(l = 2\) we shall prove (1.25). We
apply $D_x^{1/2} \partial_x^2$ to (4.1), multiply the result by $D_x^{1/2} \partial_x^2 u \chi$ and integrate in the space variable to get the identity

$$\frac{1}{2} \frac{d}{dt} \int (D_x^{1/2} \partial_x^2 u)^2 \chi dxdy - \nu \int (D_x^{1/2} \partial_x^2 u)^2 \chi' dxdy + \frac{1}{2} \int (D_x^{1/2} \partial_x^2 \partial_y u)^2 \chi dxdy$$

$$- \int A_1 + A_2 + A_3 + A_4 \frac{d}{dt} \int (D_x^{1/2} \partial_x^2 \partial_y u)(D_x^{1/2} \partial_x^2 u \chi) dxdy = 0. \quad (4.69)$$

Firstly, using that $\eta^2 = \chi'$ we write

$$|A_1| = c\|D_x^{1/2} \partial_x^2 u \eta_{x,b}\|_{L^2_y}$$

$$\lesssim \|D_x^{1/2} (\partial_x^2 u \eta_{x,b})\|_{L^2_y} + \|[D_x^{1/2}, \eta_{x,b}] \partial_x^2 u\|_{L^2_y} = A_{11} + A_{12}. \quad (4.70)$$

The first term on the RHS of (4.70) is bounded, after integration in time, by the former case $l = 2$ in (1.22) estimate (4.68). Regarding the term $A_{12}$ on the RHS of (4.70) we use the fact that $\chi_{x,b}(x) + \phi_{x,b}(x) + \psi_{x}(x) = 1 \forall x \in \mathbb{R}$ to write

$$A_{12} = \|[D_x^{1/2}, \eta_{x,b}] \partial_x^2 (u\chi_{x,b} + u\phi_{x,b} + u\psi_x)\|_{L^2_y} = A_{121} + A_{122} + A_{123}. \quad (4.71)$$

We employ Lemma A.5, to obtain

$$|A_{121} + A_{122}| \lesssim c\|\partial_x^2 (u\chi_{x,b})\|_{L^2_y} + c\|\partial_x^2 (u\phi_{x,b})\|_{L^2_y}, \quad (4.72)$$

which is bounded, after an application of Leibniz rule from calculus and integration in time, by the former cases of order $l = 1, 2$ and the smoothing effect (4.60). Notice that we have used the fact that $\phi_{x,b}$ is compactly supported, so the familiar argument in (4.60) can be employed.

Next, we recall that by construction

$$\text{dist}(\text{supp}(\eta_{x,b}), \text{supp}(\psi_x)) \geq \frac{\epsilon}{2},$$

so applying Lemma A.7, we have

$$|A_{123}| = \|[D_x^{1/2}, \eta_{x,b}] \partial_x^2 (u\psi_x)\|_{L^2_y} \lesssim c\|u\|_{L^2_y} \leq C. \quad (4.73)$$

Concerning the term $A_3$, we use integration by parts to write

$$A_3 = \int \partial_x^2 \mathcal{H} D_x^{1/2} u \partial_x^2 D_x^{1/2} u \chi dxdy + \int \mathcal{H} \partial_x^2 D_x^{1/2} u \partial_x^2 D_x^{1/2} u \chi' dxdy = A_{31} + A_{32}. \quad (4.74)$$

We perform a similar argument to that one used in (4.19) to obtain

$$A_{31} = -\frac{1}{2} \int \partial_x^2 D_x^{1/2} u [\mathcal{H}, \chi] \partial_x^2 D_x^{1/2} u dxdy = \frac{1}{2} \int D_x^{1/2} u \partial_x^2 [\mathcal{H}, \chi] \partial_x^2 D_x^{1/2} u dxdy. \quad (4.75)$$

From the commutator estimate (3.14) we obtain

$$\sup_{0 \leq t \leq T} |A_{31}(t)| \leq C\|D_x^{1/2} u\|_{L^2_y} \|\partial_x^3 [\mathcal{H}, \chi] \partial_x^2 D_x^{1/2} u\|_{L^2_y}$$

$$\leq C\|D_x^{1/2} u(t)\|_{H^{\frac{1}{2}}}^2 \lesssim \|u\|_{L^T_x H^{\frac{1}{2}}}^2. \quad (4.76)$$

Now, we use the fact that $\eta^2 = \chi'$ and the same argument given in (4.21) to write

$$A_{32} = \int [D_x^{1/2} (D_x^{1/2} \partial_x^2 u \eta)]^2 dxdy - \int D_x^{1/2} \partial_x^2 u \eta [D_x^{1/2} \partial_x^2 u \eta'] dxdy$$

$$- \int D_x^{1/2} \partial_x^2 u \eta [\mathcal{H}, \eta] D_x^{1/2} \partial_x^2 u dxdy$$
\[ A_{321} + A_{322} + A_{323}. \] (4.77)

Firstly, we observe that \( A_{321}(t) \) in (4.77) is positive and represents the smoothing effect. Now, an argument similar to that one in (4.65) yields

\[ \int_0^T |A_{322}(t)| dt \leq \int_0^T \left( (D_x^{1/2} \partial_x^2 u)^2 \chi dx dy \right) dt + \int_0^T \left( (D_x^{1/2} \partial_x^2 u)^2 (\eta')^2 dx dy dt. \] (4.78)

Firstly, we remark that the two terms on the RHS of (4.78) are similar, thus we restrict ourselves the first one. In fact, the first one is nothing but a multiply of the term \( A_1 \) above, so it is treated exactly as we did before in (4.70). Finally, a familiar argument yields

\[ \int_0^T |A_{323}(t)| dt \leq \int_0^T \left( ||D_x^{1/2} \partial_x^2 u||_{L_x^2} \left[ ||\mathcal{H}, \eta||D_x^{1/2} \partial_x^3 u||_{L_x^2} \right] dt \right. \]

\[ \leq \int_0^T \left( (D_x^{1/2} \partial_x^2 u)^2 \chi dx dy \right) dt + cT \|u\|^2_{L_t^\infty H_x^3}. \] (4.79)

Once again, the first term on the RHS of (4.79) is treated exactly as we did exactly as we did before for \( A_1 \) in (4.70). Finally, we consider the \( A_4 \) term in (4.69). Since \( \chi_{\epsilon} = \chi_{\epsilon/5, \epsilon} \equiv 1 \) on \( \text{supp} \chi \), we rewrite \( A_4 \) as follows

\[ A_4 = \int D_x^{1/2} \partial_x^2 (u \partial_x u) \chi \partial_x D_x^{1/2} \partial_x^2 u \chi dx dy \]

\[ = \frac{1}{2} \int \left[ D_x^{1/2} \partial_x^2 \chi \partial_x ((u \partial_x \epsilon, b)^2 + (u \phi_{\epsilon, b})^2 + (u^2 \phi_{\epsilon, b})) \right] D_x^{1/2} \partial_x^2 u \chi dx dy \]

\[ + \int u \chi \partial_x D_x^{1/2} \partial_x^2 u \chi dx dy + \int u \chi \partial_x D_x^{1/2} \partial_x^2 u \chi dx dy. \] (4.80)

We employ Proposition 2 in the first two terms on the RHS of (4.80) to obtain

\[ A_4 = \frac{1}{2} \int \left[ D_x^{1/2} \partial_x^2 \chi \partial_x ((u \partial_x \epsilon, b)^2 + (u \phi_{\epsilon, b})^2 + (u^2 \phi_{\epsilon, b})) \right] D_x^{1/2} \partial_x^2 u \chi dx dy \]

\[ + \int u \chi \partial_x D_x^{1/2} \partial_x^2 u \chi dx dy + \int u \chi \partial_x D_x^{1/2} \partial_x^2 u \chi dx dy \]

\[ = \sum_{1 \leq k \leq 6} \int A_{4k}(t) D_x^{1/2} \partial_x^2 u \chi dx dy + \int u \chi D_x^{1/2} \partial_x^2 u \partial_x^{1/2} \partial_x^2 u \chi dx dy. \] (4.81)

Hölder’s inequality yields

\[ |A_4| \leq ||D_x^{1/2} \partial_x^2 u||_{L_x^2} \sum_{1 \leq k \leq 6} \|A_{4k}(t)\|_{L_x^2} + \tilde{A}_{47}(t). \] (4.82)

In order to estimate \( \tilde{A}_{47}(t) \) on the RHS of (4.82) we apply integration by parts to obtain

\[ \tilde{A}_{47}(t) = -\frac{1}{2} \int \partial_x u \chi \left( (D_x^{1/2} \partial_x^2 u)^2 \right) \chi dx dy - \frac{1}{2} \int u (\chi \partial_x \epsilon) (D_x^{1/2} \partial_x^2 u)^2 dx dy \]

\[ = A_{471} + A_{472}. \] (4.83)

Firstly,

\[ |A_{471}(t)| \leq ||\partial_x u||_{L_x^\infty} \int (D_x^{1/2} \partial_x^2 u)^2 \chi dx dy, \]
where the last integral is the very quantity to be estimated, and the other term will be controlled after integration in time by (4.2). Sobolev’s embedding yields

\[ |A_{472}(t)| \lesssim \|u\|_{L^\infty_t H^1_x} \int (D_x^{1/2} \partial_x^2 u)^2 (\chi \epsilon \chi)' \, dx \, dy, \]

where the last term can be controlled using a familiar argument analogous to that one performed in the analysis of the term \( A_1 \) above.

Combining (A.2), Lemma A.5 and Lemma A.6 in the variable \( x \) and then Hölder’s inequality in the variable \( y \), one gets

\[
\|A_{41}(t)\|_{L^2_y} = \left\| \left[ D_x^{2+1/2}, \chi \epsilon \right] \partial_x ((u \chi_{\epsilon, b})^2) \right\|_{L^2_y} \\
\lesssim \left\| D_x^{2+1/2} (u \chi_{\epsilon, b}) \right\|_{L^2_y} \|u\|_{L^\infty_y} + \|u\|_{L^2_y} \|u\|_{L^\infty_y} \quad (4.84)
\]

and

\[
\|A_{42}(t)\|_{L^2_y} = \left\| \left[ D_x^{2+1/2}, \chi \epsilon \right] \partial_x ((u \phi_{\epsilon, b})^2) \right\|_{L^2_y} \\
\lesssim \left\| D_x^{2+1/2} (u \phi_{\epsilon, b}) \right\|_{L^2_y} \|u\|_{L^\infty_y} + \|u\|_{L^2_y} \|u\|_{L^\infty_y}. \quad (4.85)
\]

Regarding the last quadratic term we apply Lemma A.7 in the variable \( x \) and then Hölder’s inequality in the variable \( y \), to obtain

\[
\|A_{43}(t)\|_{L^2_y} = \left\| \left[ D_x^{2+1/2}, \chi \epsilon \right] \partial_x ((u^2 \psi)') \right\|_{L^2_y} \lesssim \|u\|_{L^2_y} \|u\|_{L^\infty_y}.
\]

Applying Lemma A.5 in the variable \( x \) and then Hölder’s inequality in the variable \( y \), we have

\[
\|A_{44}(t)\|_{L^2_y} = \left\| \left[ D_x^{2+1/2}, u \chi \epsilon \right] \partial_x (u \chi_{\epsilon, b}) \right\|_{L^2_y} \\
\lesssim \|\partial_x (u \chi_{\epsilon, b})\|_{L^\infty_y} \left\| D_x^{2+1/2} (u \chi \epsilon) \right\|_{L^2_y} + \|\partial_x (u \chi \epsilon)\|_{L^\infty_y} \left\| D_x^{2+1/2} (u \chi_{\epsilon, b}) \right\|_{L^2_y} \\
\lesssim \|\partial_x (u \chi_{\epsilon, b})\|_{L^\infty_y} \left\| D_x^{2+1/2} (u \chi_{\epsilon, b}) \right\|_{L^2_y}, \quad (4.86)
\]

where we have chosen \((\epsilon, b)\) instead of \((\epsilon/5, \epsilon)\). Similarly, we obtain

\[
\|A_{45}(t)\|_{L^2_y} = \left\| \left[ D_x^{2+1/2}, u \chi \epsilon \right] \partial_x (u \phi_{\epsilon, b}) \right\|_{L^2_y} \quad (4.87) \\
\lesssim \|\partial_x (u \phi_{\epsilon, b})\|_{L^\infty_y} \left\| D_x^{2+1/2} (u \chi_{\epsilon, b}) \right\|_{L^2_y} + \|\partial_x (u \chi_{\epsilon, b})\|_{L^\infty_y} \left\| D_x^{2+1/2} (u \phi_{\epsilon, b}) \right\|_{L^2_y}.
\]

We recall that by construction

\[
\text{dist}(\text{supp}(\chi_{\epsilon, b}), \text{supp}(\psi)) \geq \frac{\epsilon}{2},
\]

so a familiar application of Lemma A.7 provide us

\[
\|A_{46}\|_{L^2_y} = \left\| u \chi_{\epsilon, b} \partial_x D_x^{2+1/2} (u \psi')_\epsilon \right\|_{L^2_y} \lesssim \|u\|_{L^2_y} \|u\|_{L^\infty_y}, \quad (4.88)
\]

where the first identity is obtained taking \((\epsilon, b)\) instead of \((\epsilon/5, \epsilon)\). In order to finish the estimates in (4.84) – (4.87) only remains to bound \(\|D_x^{2+1/2} (u \chi_{\epsilon, b})\|_{L^2_y}, \|D_x^{2+1/2} (u \phi_{\epsilon, b})\|_{L^2_y}\) and \(\|D_x^{2+1/2} (u \psi')_\epsilon\|_{L^2_y}\). The second and third term above are
analogous, so we will restrict ourselves to the analysis of the last one. We perform the analysis of the first term by observing that
\[ D^{2+1/2}_x(uX_{\epsilon,b}) = D^{2+1/2}_x u_{X_{\epsilon,b}} + [D^{2+1/2}_x, X_{\epsilon,b}] (u_{X_{\epsilon,b}} + u\phi_{p,b} + u\psi_x). \] (4.89)
We notice that the $L^2$ norm of the first term on the RHS of (4.89) is the very quantity to be estimated while the control of the $L^2$ of the second one can be performed by using a familiar argument, similar to that one employed in (4.71) above.

An application of Hölder’s inequality, Proposition 2 and Theorem A.3 in the variable $x$ yields
\[
\left\| D^{2+1/2}_x(u\phi_{p,b}) \right\|_{L^2_y} \lesssim \|u\|_{L^1_y} \left\| D^{2+1/2}_x \phi_{p,b} \right\|_{L^2_y} + \frac{1}{2} \sum_{\beta \leq 2} \left\| \frac{1}{\partial_x^\beta} \phi_{p,b} \right\|_{L^2_y} \left\| \partial_x^{\beta+1} u \right\|_{L^2_y}.
\]
\[
\lesssim \left\| D^{1/2}_x u \right\|_{L^2_y} \left\| u \right\|_{L^2_y}^{1/2} + \left\| \chi_{r/8, b+\epsilon/4} D^{2+1/2}_x u \right\|_{L^2_y}
\]
\[
+ \left\| \chi_{r/8, b+\epsilon/4} \partial_x^{1/2} u \right\|_{L^2_y} + \left\| D^{1/2}_x u \right\|_{L^2_y}
\]
\[
\lesssim \left\| D^{1/2}_x u \right\|_{L^2_y} + \left\| u \right\|_{L^2_y} + \left\| \eta_{r/24, b+7\epsilon/24} D^{2+1/2}_x u \right\|_{L^2_y}
\]
\[
+ \left\| \eta_{r/24, b+7\epsilon/24} \partial_x^{1/2} u \right\|_{L^2_y}.
\]
(4.90)

Next, an argument similar to that one employed in (4.55) provide us
\[
\left\| D^{2+1/2}_x(u\phi_{p,b}) \right\|_{L^2_y} \lesssim \left\| D^{1/2}_x u \right\|_{L^2_y} + \left\| u \right\|_{L^2_y} + \left\| \eta_{r/24, b+7\epsilon/24} D^{2+1/2}_x u \right\|_{L^2_y}
\]
\[
\lesssim \left\| D^{1/2}_x u \right\|_{L^2_y} + \left\| \eta_{r/24, b+7\epsilon/24} \partial_x^{1/2} u \right\|_{L^2_y}.
\]
(4.91)

The second and third term on the RHS of (4.91) can be controlled, after integration in time, as we did before for $A_1$ in (4.70) and (4.39), respectively, taking $(\epsilon, b)$ instead of $(\epsilon/24, b + 7\epsilon/24)$.

We insert the above information (4.70) – (4.90) in (4.69), apply Gronwall’s inequality and (4.2) to obtain
\[
\sup_{0 \leq t \leq T} \int (D^{1/2}_x \partial_x^2 u)^2 \chi_{\epsilon,b}(x + \nu t) dxdy + \frac{1}{2} \int_0^T \int (D^{1/2}_x \partial_x^2 \partial_y u)^2 \chi' dxdydt
\]
\[
+ \int_0^T \int [D^{1/2}_x (D^{1/2}_x \partial_x^2 u \eta)]^2 dxdydt \leq C.
\]
(4.92)

This gives the desired estimate (1.25) with $l = 2$.

We follows an induction argument by assuming that (1.22) holds for $l \leq m \in \mathbb{Z}^+$, $m \geq 3$. More precisely, we assume
\[
\sup_{0 \leq t \leq T} \int (\partial_x^l u)^2 \chi_{\epsilon,b}(x + \nu t) dxdy + \int_0^T \left[ D^{l/2}_x (\partial_x^l \eta \cdot (x + \nu t)) \right]^2 dxdydt
\]
\[
+ \int_0^T \int (\partial_x^l \partial_y u)^2 \chi' (x + \nu t) dxdydt \leq C,
\]
(4.93)
for $l = 1, 2, ..., m$, $m \geq 3$, for any $\epsilon > 0, b > 5\epsilon, \nu > 0$, with $\partial_x^{m+1} \phi \in L^2((0, \infty) \times \mathbb{R})$ and prove that if (1.21) with $x_0 = 0$ and $l = m + 1$ holds, then
a) (1.24) and (1.25) hold with $x_0 = 0$ and $l = m$, and
b) (1.22) holds with $x_0 = 0$ for $l = m + 1$.

**Part a:** Firstly, we notice that from the hypothesis (1.21) with $l = m + 1$ and since $\phi \in H^{\frac{3}{2}}(\mathbb{R}^2)$, an interpolation argument yields (1.24) with $l = m$.

An argument similar to that performed in (4.69) provides the identity

\[
\frac{1}{2} \frac{d}{dt} \int (D_x^{1/2} \partial_x^m u)^2 \chi dx dy - \frac{\nu}{2} \int (D_x^{1/2} \partial_x^m u)^2 \chi' dx dy + \frac{1}{2} \int (D_x^{1/2} \partial_x^m \partial_y u)^2 \chi dx dy
\]

\[- \int D_x^{1/2} \partial_x^{m+1} \phi u D_x^{1/2} \partial_x^m u \chi dx dy + \int D_x^{1/2} \partial_x^m (u \partial_x u) D_x^{1/2} \partial_x^m u \chi dx dy = 0. \tag{4.94}\]

An analogous argument to that one used in (4.70) yields

\[|A_1| = c\|D_x^{1/2} \partial_x^m u \partial_{x,b}\|_{L^2_y} \leq \|D_x^{1/2} \partial_x^m u \partial_{x,b}\|_{L^2_y} + \|D_x^{1/2} \partial_x^m u\|_{L^2_y} = A_{11} + A_{12}. \tag{4.95}\]

The term $A_{11}$ is bounded, after integration in time, by induction hypothesis (4.93). In order to control term $A_{12}$ we perform a similar argument to that one used in the analysis (4.71)–(4.73) above, applying the smoothing effect as (4.60) combined with (4.93) for $l = 1, 2, \ldots, m$.

Concerning the term $A_3$ in (4.94) a familiar argument provides

\[A_3 = \int \partial_x^{m+1} H D_x^{1/2} u \partial_x^{m+1} D_x^{1/2} u \chi dx dy + \int H \partial_x^{m+1} D_x^{1/2} u \partial_x^m D_x^{1/2} u \chi dx dy = A_{31} + A_{32}. \tag{4.96}\]

A similar argument to that one used in (4.75) yields

\[A_{31} = -\frac{1}{2} \int \partial_x^{m+1} D_x^{1/2} u [H, \chi] \partial_x^{m+1} D_x^{1/2} u dx dy = \frac{(-1)^m}{2} \int D_x^{1/2} u \partial_x^{m+1} [H, \chi] \partial_x^{m+1} D_x^{1/2} u dx dy. \tag{4.97}\]

From the commutator estimate (3.14) we obtain

\[\sup_{0 \leq t \leq T} |A_{31}(t)| \leq C\|D_x^{1/2} u\|_{L^2_y} \|\partial_x^{m+1} [H, \chi] \partial_x^{m+1} D_x^{1/2} u\|_{L^2_y} \leq C\|D_x^{1/2} u(t)\|_{L^2_y}^2 \lesssim \|u\|_{L_T H^{\frac{3}{2},+}}^2. \tag{4.98}\]

Using the fact that $\eta^2 = \chi'$ and the same argument employed in (4.21) we obtain

\[A_{32} = \int \left[ D_x^{1/2} (D_x^{1/2} \partial_x^m u \eta) \right]^2 dx dy - \int D_x^{1/2} \partial_x^m u \eta H(D_x^{1/2} \partial_x^m u \eta') dx dy
\]

\[- \int D_x^{1/2} \partial_x^m u \eta [H, \eta] D_x^{1/2} \partial_x^{m+1} u dx dy = A_{321} + A_{322} + A_{323}. \tag{4.99}\]

Firstly, we observe that $A_{321}(t)$ in (4.99) is positive and represents the smoothing effect. Next, an argument similar to that one performed in (4.78)–(4.79) yields

\[\int_0^T (|A_{322}(t)| + |A_{323}(t)|) dt \leq \int_0^T (D_x^{1/2} \partial_x^m u)^2 \chi' dx dy dt + \int_0^T (D_x^{1/2} \partial_x^m u)^2 (\eta')^2 dx dy dt + cT\|u\|_{L_T H^{\frac{3}{2},+}}^2. \tag{4.100}\]
The two terms on the RHS of (4.100) are similar and can be treated in the same manner as we did before for $A_1$ in (4.70).

Only remains to estimate $A_4$ to finish this part of the proof. We will assume $l$ an even integer. The case where $l$ is odd follows by an argument similar to the case $l = 1$. A familiar argument similar to that one used in (4.81) yields

$$A_4 = -\frac{1}{2} \int [D_x^{1/2} \partial_x^m, \chi_x] \partial_x ((u \chi_{e,b})^2 + (u \phi_{e,b})^2 + (u^2 \psi_e)) D_x^{1/2} \partial_x^m u \chi dxdy$$

$$+ \int [D_x^{1/2} \partial_x^m, u \chi_x] \partial_x (u \chi_{e,b} + u \phi_{e,b} + u \psi_e) D_x^{1/2} \partial_x^m u \chi dxdy$$

$$+ \int u \chi_x D_x^{1/2} \partial_x^{m+1} u D_x^{1/2} \partial_x^m u \chi dxdy$$

$$= \sum_{1 \leq k \leq 6} \int A_{4k}(t) D_x^{1/2} \partial_x^m u \chi dxdy + \int u \chi_x D_x^{1/2} \partial_x^{m+1} u D_x^{1/2} \partial_x^m u \chi dxdy. \quad (4.101)$$

An application of Hölder’s inequity provide us

$$|A_4| \leq \|D_x^{1/2} \partial_x^m u \chi\|_{L^2_y} \sum_{1 \leq k \leq 6} \|A_{4k}(t)\|_{L^2_y} + \widetilde{A}_{47}(t). \quad (4.102)$$

A similar argument to that one used in (4.83) implies

$$\widetilde{A}_{47}(t) = -\frac{1}{2} \int \partial_x u \chi_x (D_x^{1/2} \partial_x^m u)^2 \chi dxdy - \frac{1}{2} \int u (\chi_x \chi)' (D_x^{1/2} \partial_x^m u)^2 dxdy$$

$$= A_{471} + A_{472}. \quad (4.103)$$

The terms $A_{471}$ and $A_{472}$ can now be controlled, exactly as we did before in the analysis of (4.83). More precisely, we obtain

$$|A_{471}(t)| \lesssim \|\partial_x u\|_{L^\infty_y} \int (D_x^{1/2} \partial_x^m u)^2 \chi dxdy,$$

and

$$|A_{472}(t)| \lesssim \|u\|_{L^\infty_t H^{1/2}_x} \int (D_x^{1/2} \partial_x^m u)^2 (\chi_x \chi)' dxdy,$$

where the last integral term above can be controlled using an argument similar to that one performed in the analysis of the term $A_1$ in (4.95).

A familiar argument similar to that one performed in (4.84) – (4.85) yields

$$\|A_{41}(t)\|_{L^2_y} = \left\| [D_x^{m+1/2}, \chi_x] \partial_x ((u \chi_{e,b})^2) \right\|_{L^2_y}$$

$$\lesssim \left\| D_x^{m+1/2} (u \chi_{e,b}) \right\|_{L^2_y} \|u\|_{L^\infty_y} + \|u\|_{L^2_y} \|u\|_{L^2_y}, \quad (4.104)$$

$$\|A_{42}(t)\|_{L^2_y} = \left\| [D_x^{m+1/2}, \chi_x] \partial_x ((u \phi_{e,b})^2) \right\|_{L^2_y}$$

$$\lesssim \left\| D_x^{m+1/2} (u \phi_{e,b}) \right\|_{L^2_y} \|u\|_{L^\infty_y} + \|u\|_{L^2_y} \|u\|_{L^2_y}, \quad (4.105)$$

Similarly, we have

$$\|A_{43}(t)\|_{L^2_y} = \left\| [D_x^{m+1/2}, \chi_x] \partial_x ((u^2 \psi_e)) \right\|_{L^2_y} \lesssim \|u\|_{L^2_y} \|u\|_{L^\infty_y},$$
Once again, we employ the same procedure as in the previous analysis in (4.86) – (4.87) to obtain
\[
\| A_{44}(t) \|_{L_x^2} = \left\| \left[ D_x^{m+1/2}, u \chi_{\epsilon} \right] \partial_x (u \chi_{\epsilon}) \right\|_{L_x^2} \\
\lesssim \| \partial_x (u \chi_{\epsilon}) \|_{L_x^2} \left\| D_x^{m+1/2}(u \chi_{\epsilon}) \right\|_{L_x^2},
\]
and
\[
\| A_{45}(t) \|_{L_x^2} = \left\| \left[ D_x^{m+1/2}, u \chi_{\epsilon} \right] \partial_x (u \phi_{\epsilon,b}) \right\|_{L_x^2} \\
\lesssim \| \partial_x (u \phi_{\epsilon,b}) \|_{L_x^2} \left\| D_x^{m+1/2}(u \chi_{\epsilon}) \right\|_{L_x^2} \\
+ \| \partial_x (u \chi_{\epsilon}) \|_{L_x^2} \left\| D_x^{m+1/2}(u \phi_{\epsilon,b}) \right\|_{L_x^2}.
\]
Analogously to the analysis in (4.88) we obtain
\[
\| A_{46} \|_{L_x^2} = \left\| u \chi_{\epsilon,b} \partial_x D_x^{m+1/2}(u \psi_{\epsilon}) \right\|_{L_x^2} \lesssim \| u \|_{L_x^2} \| u \|_{L_x^2}.
\]
To conclude the estimates in (4.104) – (4.107), we observe that \( \chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_{\epsilon}(x) = 1 \) \( \forall x \in \mathbb{R} \) and write
\[
D_x^{m+1/2}(u \chi_{\epsilon,b}) = D_x^{m+1/2}u \chi_{\epsilon,b} + \left[ D_x^{m+1/2}, \chi_{\epsilon,b} \right] (u \chi_{\epsilon,b} + u \phi_{\epsilon,b} + u \psi_{\epsilon}).
\]
Similarly to the analysis in (4.89), we notice that the \( L^2 \) norm of the first term on the RHS of (4.109) is the very quantity to be estimated. The second term can be controlled by using a familiar argument, similar to that one employed in (4.71) above, combining local theory, smoothing effect, estimate (4.93) and the former case for \( l = 1, 2, ..., m \).

A familiar argument similar to that one used in (4.90) – (4.91) provide us
\[
\left\| D_x^{m+1/2}(u \phi_{\epsilon,b}) \right\|_{L_x^2} \lesssim \| D_x^{1/2}u \|_{L_x^2} + \| u \|_{L_x^2} + \sum_{\beta \in O_1(m)} \frac{1}{|\beta|} \left\| \partial_{\phi_{\epsilon,b}} D_x^{m-\beta+1/2} u \right\|_{L_x^2} \\
+ \sum_{\beta \in O_2(m)} \frac{1}{|\beta|} \left\| \partial_{\phi_{\epsilon,b}} \mathcal{H} D_x^{m-\beta+1/2} u \right\|_{L_x^2},
\]
with \( O_1(m), O_2(m) \) denoting the odd integers and even integers in \( \{1, 2, ..., m\} \) respectively.

We employ Proposition 2 to obtain
\[
\sum_{\beta \in O_1(m)} \frac{1}{|\beta|} \left\| \partial_{\phi_{\epsilon,b}} D_x^{m-\beta+1/2} u \right\|_{L_x^2} \\
\lesssim \sum_{\beta \in O_1(m)} \frac{1}{|\beta|} \left\| \chi_{\epsilon/8, b+\epsilon/4} D_x^{m-\beta+1/2} u \right\|_{L_x^2} \\
\lesssim \sum_{\beta \in O_1(m)} \frac{1}{|\beta|} \left\| \eta_{\epsilon/24, b+7\epsilon/24} D_x^{m-\beta+1/2} u \right\|_{L_x^2}.
\]
The term on the RHS of (4.111) can be controlled, after integration in time, by applying the induction hypothesis (4.93) for \( l = 1, 2, ..., m \) combined with the same argument performed in the analysis of \( A_1 \) in (4.70) with \((\epsilon, b)\) instead of \((\xi/24, b + 7\epsilon/24)\).
Similarly, one gets
\[
\sum_{\beta \in O_2(m)} 1 \beta! \left\| \partial^\beta \phi \mathcal{H} D^{-\beta + 1/2}_x u \right\|_{L^2_y} \\ \leq \sum_{\beta \in O_2(m)} 1 \beta! \left\| \partial^\beta \mathcal{H} D^{-\beta + 1/2}_x u \right\|_{L^2_y} \\ \leq \sum_{\beta \in O_2(m), \beta \neq m} 1 \beta! \left\| \partial^\beta \eta / 24 \partial^2 \mathcal{H} D^{-\beta + 1/2}_x u \right\|_{L^2_y} + \|u\|_{L^\infty \mathcal{H}^{5/4}}. 
\] (4.112)

Finally, employing an argument similar to that used in (4.109), we write
\[
\eta \mathcal{H} D^{m+1/2}_x u = \mathcal{H}(D^{m+1/2}_x u \eta) + \mathcal{H} \eta D^{m+1/2}_x(u \partial_x + u \partial_y + 2 u \partial_x + 2 u \partial_y) 
\] (4.113)
in order to control the term on the RHS of (4.112), by combining local theory, smoothing effect and the induction hypothesis (4.93) for \(l = 1, 2, \ldots, m\).

\textbf{Part b:} Next, we assume that (1.22) holds with \(x_0 = 0\) for \(l = m + 1\), our induction hypothesis, i.e., for \(l = 1, 2, \ldots, m\).

A familiar argument yields
\[
\frac{1}{2} \frac{d}{dt} \int (\partial^{m+1}_x u)^2 \chi dx \, dy - \frac{\nu}{2} \int (\partial^{m+1}_x u)^2 \chi' dx \, dy + \frac{1}{2} \int (\partial^{m+1}_y u)^2 \chi' dx \, dy - \int \mathcal{H} \partial^{m+1}_x \partial^2_x u \partial^{m+1}_x u \chi dx \, dy + \int \partial^{m+1}_x (u \partial_x u) \partial^{m+1}_x u \chi dx \, dy = 0. 
\] (4.115)

For the first term we write
\[
\mathcal{H}(\partial^{m+1}_x u \eta) = [\mathcal{H}, \eta] \partial^{m+1}_x u + D^{1/2}_x (D^{1/2}_x \partial^m \eta) - [D^{1/2}_x, \eta] \partial^{1/2}_x \partial^m u \\
= I_1 + I_2 + I_3. 
\] (4.116)

We notice that \(\|I_2\|_{L^2_y}^2\) is bounded, after integration in time, by the induction hypothesis (1.25) with \(l = m\). The term \(\|I_1\|_{L^2_y}^2\) is controlled by using commutator estimate (3.14) while \(\|I_3\|_{L^2_y}^2\) is bounded, after integration in time, by employing a similar argument to that one used in the analysis (4.71) – (4.73) above, combined with induction hypothesis (1.25) for \(l = m\).

In order to control the contribution of \(A_3\) in (4.115) we write
\[
A_3 = \int \partial^{m+1}_x \mathcal{H} \partial_x u \partial^{m+2}_x u \chi dx \, dy + \int \partial^{m+1}_x \mathcal{H} \partial_x u \partial^{m+1}_x u \chi' dx \, dy = A_{31} + A_{32}. 
\] (4.117)

A familiar argument yields
\[
\sup_{0 \leq t \leq T} \|A_{31}(t)\| \leq C \|u\|_{L^2_y}^2 \|\partial^{m+2}_x [\mathcal{H}, \chi] \partial^{m+2}_x u\|_{L^2_y} \leq C \|u(t)\|_{L^2_y}^2 = C \|\phi\|_{L^2_y}^2. 
\] (4.118)
Similar to the argument in (4.21)

\[ A_{32} = \int \left[ D_x^{1/2}(\partial_x^{m+1} u \eta) \right]^2 dx dy - \int \partial_x^{m+1} u \eta \mathcal{H}(\partial_x^{m+1} u \eta') dx dy \]

\[ - \int \partial_x^{m+1} u \eta \mathcal{H}(\eta, \eta') \partial_x^{m+2} u dx dy = A_{321} + A_{322} + A_{323} \quad (4.119) \]

Once again, we observe that \( A_{321}(t) \) in (4.119) is positive and represents the smoothing effect. To take care of the remaining terms on the RHS of (4.119) we employ a familiar argument similar to that one used in the analysis (4.65) – (4.66), combined with the former case \( A_1 \) above, treated in (4.116).

Finally, one just needs to handle the contribution of the nonlinear term \( A_4 \) in (4.115). From integration by parts and Leibniz rule, we have

\[
A_4 = \int \partial_x^{m+1} (u \partial_x u) \partial_x^{m+1} u \chi dx dy
\]
\[
= c_0 \int u (\partial_x^{m+1} u)^2 \chi' dx dy + c_1 \int \partial_x u (\partial_x^{m+1} u)^2 \chi dx dy
\]
\[
+ c_2 \int \partial_x^2 u \partial_x^{m+1} u \partial_x^{m+1} u \chi dx dy + \sum_{l=1}^{m-1} c_l \int \partial_x^l u \partial_x^{m+2-l} u \partial_x^{m+1} u \chi dx dy
\]
\[
= A_{40} + A_{41} + A_{42} + \sum_{l=3}^{m-1} A_{4l}. \quad (4.120)
\]

We notice that \( A_{40} \) is bounded, after integration in time, by the former case \( A_1 \) above, treated in (4.116). Next, we have

\[
|A_{41}(t)| \leq c \|\partial_x u\|_{L_\infty} \int (\partial_x^{m+1} u)^2 \chi dx dy. \quad (4.121)
\]

We use (4.2) to bound the first term on the right hand side of (4.121) when we latter apply Gronwall’s Lemma. The last integral term is the quantity to be estimated.

We now consider the term \( A_{42} \) in (4.120). Firstly, we denote \( \bar{\chi} = \chi_{\epsilon/5,c} \). Using the identity \( \chi_{\epsilon,b} = \chi_{\epsilon/5,c} \), Hölder’s inequality, Sobolev embedding, Young’s inequality, we obtain that

\[
|A_{42}(t)| \leq c \int (\partial_x^2 u \chi_{\epsilon/5,c} \partial_x^m u \chi_{\epsilon/5,c})^2 dx dy \quad (4.122)
\]

We begin observing that the last term on the right hand side of (4.122) is the very quantity to be estimated. The remaining terms are bounded, after integration in time interval, by the former cases \( l = 2, 3 \) combined with (4.93).
To estimate $\sum_{i=3}^{m-1} A_i(t)$ in (4.120) we employ an argument analogous to that in (4.122) to get

$$|A_i(t)| \lesssim \left| \frac{\partial^i_x u(x, y, t)}{\rho^i} \right|_{L^2_y}^2 + \left| \frac{\partial^{i+1}_x u(x, y, t)}{\rho^{i+1}} \right|_{L^2_y}^2 + \left| \frac{\partial^{i+1}_x u(x, y, t)}{\rho^{i+1}} \right|_{L^2_y}^2$$

where the last term integral on the right hand side of (4.123) is the quantity to be estimated. To deal with the remaining terms on the RHS of (4.123) we observe they are all of order $\leq m$ so bounded, after integration in time interval, by the former cases $t = 1, 2, 3, ... , m$ combined with (4.93). This completes the induction argument.

4.3. Limiting argument. Next, we use a limiting standard argument to justify the previous computations for arbitrary $\phi \in H^s(\mathbb{R}^2)$ with $s > T$. Fix $\rho \in C_0^\infty(\mathbb{R}^2)$ with supp $\rho \subset B_1(0) = \{z \in \mathbb{R}^2 : |z| < 1\}$, $\rho \geq 0$, $\int \rho(z) \, dz = 1$ and

$$\rho_\mu(z) = \frac{1}{\mu^2} \rho\left(\frac{z}{\mu}\right), \quad \mu > 0.$$ 

For $\mu > 0$ the solutions $u^\mu$ of the initial value problem (4.1) corresponding to smoothed data $\phi^\mu = \rho_\mu * \phi$, satisfies

$$u^\mu \in C([0, T] : H^\infty).$$

Hence we may conclude

$$\sup_{0 \leq t \leq T} \int_0^\infty \int_0^\infty \left( \frac{\partial^i_x u^\mu(x, y, t)}{\rho^i} \right)^2 \chi(x, y) \, dx \, dy$$

$$+ \int_0^T \int_0^\infty \int_0^\infty \left( D_x^2 \left( \frac{\partial^i_x u^\mu(x, y, t)}{\rho^i} \right) \eta(x, y) \right) \, dx \, dy \, dt$$

$$\leq C = C(\nu, \epsilon, T; ||\phi^\mu||_{H^s}; ||\partial^m_x \phi^\mu||_{L^2((0, \infty) \times \mathbb{R}^2)}).$$

Too see that this bound is independent of $\mu > 0$, first we note

$$||\phi^\mu||_{H^s} \leq ||\rho_\mu||_{L^\infty} ||\phi||_{H^s} \leq ||\phi||_{H^s}.$$ 

Since $\chi \equiv 0$ for $x < \epsilon$, restricting $0 < \mu < \epsilon$ it follows

$$(\partial^i_x \phi^\mu)^2 \chi(x, y) = (\rho_\mu * \partial^i_x \phi \mid_{[0, \infty)})^2 \chi(x, y).$$

Thus by Young’s inequality

$$\int_0^\infty (\partial^i_x \phi^\mu)^2 \, dx \, dy = \int_0^\infty (\rho_\mu \ast \partial^i_x \phi \mid_{[0, \infty)})^2 \, dx \, dy$$

$$\leq ||\rho_\mu||_{L^\infty} \int_0^\infty (\partial^i_x \phi)^2 \, dx \, dy \leq ||\partial^i_x \phi||_{L^2((0, \infty) \times \mathbb{R}^2)}.$$

From the a priori estimate of the local well-posedness result, $s > 5/4$, 

$$||u^\mu||_{L^\infty H^s} \leq c(||\phi^\mu||_{H^s}) \leq c(||\phi||_{H^s})$$

and so we may replace the bound $C = C(\mu)$ with $\tilde{C}$ as in (1.22). Using the continuous dependence of the solution upon the initial data we have

$$\sup_{0 \leq t \leq T} ||u^\mu(t) - u(t)||_{H^s} \downarrow 0 \quad \text{as} \quad \mu \downarrow 0 \quad \text{for} \quad s > 5/4.$$ 

Combining this fact with the $\mu$–uniform bound $\tilde{C}$, weak compactness and Fatou’s lemma, one gets (1.22). This completes the proof of Theorem 1.4.
Appendix A. Nonlinear estimates.

Lemma A.1. For $\alpha, \beta > 0$ we have
\[ \|D_x^\alpha D_y^\beta u\|_{L^2(\mathbb{R}^2)} \leq C \|J^{\alpha+\beta} u\|_{L^2(\mathbb{R})}. \]  

Proof. It follows directly from Young’s inequality and Plancherel’s theorem. \qed

Lemma A.2 (Leibniz rule for fractional differentiation). (i) For $0 < \alpha < 1$, we have
\[ \|D_x^\alpha (fg)\|_{L^2(\mathbb{R})} \leq C \left\{ \|D_x^\alpha f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{q_1}(\mathbb{R})} + \|D_x^\alpha g\|_{L^{p_2}(\mathbb{R})} \|f\|_{L^{q_1}(\mathbb{R})} \right\} \]
for $1 < p_1, q_1, r_1, s_1 \leq \infty$, such that $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} + \frac{1}{s_1}$.

(ii) For $0 < \alpha, \beta < 1$, we have
\[ \|D_x^\alpha D_y^\beta (fg)\|_{L^2(\mathbb{R})} \leq C \left\{ \|D_x^\alpha f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{q_1}(\mathbb{R})} + \|D_x^\alpha g\|_{L^{p_2}(\mathbb{R})} \|f\|_{L^{q_2}(\mathbb{R})} \right\} \]
for $1 < p_i \leq \infty$, $1 < q_i \leq \infty$, such that $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{q_1}$.

Proof. The proof of (A.2) were established by Kenig, Ponce and Vega [28]. The proof of (A.3) is given in [40]. \qed

Recently, D. Li [32] extended fractional Leibniz rule for the nonlocal operator $D^s$, $s > 0$ and related ones, including various end-point situations.

Theorem A.3. Let $s > 0$ and $1 < p < \infty$. Then for any $s_1, s_2 \geq 0$ with $s = s_1 + s_2$, and any $f, g \in S(\mathbb{R})$, the following hold
\[ \left\| D^s(fg) - \sum_{\alpha \leq s_1} \frac{1}{\alpha!} \partial_x^\alpha f D_x^{s_1-\alpha} g - \sum_{\beta \leq s_2} \frac{1}{\beta!} \partial_x^\beta g D_x^{s_2-\beta} f \right\|_{L^p} \lesssim \|D^{s_1}f\|_{L^p_c} \|D^{s_2}g\|_{L^p_2}, \]  
where $1 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and the operator $D_x^{s_1}$ is defined via Fourier transform as
\[ \hat{D}_x^{s_1} = i^{-s_1} \partial_\zeta^s (\zeta^s) \hat{g}(\zeta). \]

Proof. See [32]. \qed

Lemma A.4. For each $s \in \mathbb{R}$ and $\sigma > \frac{d}{2}$ there exists a constant $C = C_{s,\sigma} > 0$ such that for all $\varphi \in S(\mathbb{R}^d)$ and $f \in H^{s-\frac{1}{2}}(\mathbb{R}^d)
\[ \|D^s \varphi f\|_{L^2} \leq C \|\varphi\|_{L^\infty} \|f\|_{H^{s-\frac{1}{2}}} \]
where $l = |s - 1| + 1 + \sigma$.

Proof. See [14] Lemma 6.16 p. 202. \qed

D. Li [32] also obtained a family of refined Kato-Ponce type inequalities for the nonlocal operator $D^s$. Among these, he proved that

Lemma A.5. Let $1 < p < \infty$. Let $1 < p_1, p_2, p_3, p_4 \leq \infty$ satisfy
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]

Therefore

(a) If $0 < s \leq 1$, then
\[ \|D^s(fg) - fD^s g\|_{L^p} \lesssim \|D^{s-1} \partial_x f\|_{L^{p_1}} \|g\|_{L^{p_2}}. \]
Lemma A.8. Let \( 0 < \epsilon < \epsilon_1 \) small enough so that 
\[
\|D_x^{\frac{1}{2} - \gamma + \epsilon} u\|_{L_\infty^p} \lesssim \|u\|_{L_\infty^q} + \|\partial_x u\|_{L_\infty^q}.
\] (A.9)

b) If \( \epsilon_0 > 0 \) is a constant chosen small enough, then the following holds true. There exist
\[
\begin{cases}
2 < p_1, p_2 < \infty & \text{with } \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}, \frac{1}{q_1} + \frac{1}{q_2} = 1, \\
1 < q_1, q_2 < \infty
\end{cases}
\] (A.10)

0 \( \theta < 1 \) and 0 \( \epsilon < \epsilon_1 = \delta_1(\epsilon_0, \theta) \ll 1 \) such that
\[
\|D_x^{\frac{1}{2} - \gamma + \epsilon} \partial_x u\|_{L_p^p L_{L_\infty}^q} \lesssim \|\partial_x u\|_{L_p^q} \|J_x^{\frac{1}{2} - \gamma + \epsilon} u\|_{L_p^q}^{1 - \theta}.
\] (A.11)

\[
\|D_\gamma u\|_{L_p^p L_{L_\infty}^q} \lesssim \left(\|u\|_{L_p^q} \right)^{1 - \theta} \left(\|D_\gamma u\|_{L_p^q} + \|u\|_{L_p^q}\right)^\theta,
\] (A.12)

for all 0 \( \epsilon < \epsilon_1 \).

Proof. The argument of proof is analogous to the one given by Kenig in [24] (see for instance Lemma 4.6 in [34]). \( \square \)

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E-mail address: ailtoncn@impa.br