Adiabatic regularization and Green’s function of massless scalar field in general RW spacetimes

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Abstract

We study adiabatic regularization of a coupling massless scalar field in general spatially flat Robertson-Walker (RW) spacetimes. For the conformally-coupling, the 2nd-order regularized power spectrum and 4th-order regularized stress tensor are zero, and no trace anomaly exists in general RW spacetimes. This is an extension of our previous result in de Sitter space. For the minimally-coupling, the regularized spectra are also zero in the radiation-dominant stage, the matter-dominant stage, and de Sitter space as well. The vanishing of these adiabatically regularized spectra are also confirmed by direct regularization of the Green’s functions. For a general coupling and general RW spacetimes, the regularized spectra can be negative under the conventional prescription. By going to higher order of regularization, the spectra will generally become positive, but will also acquire IR divergence which is inevitable for a massless field. To avoid the IR divergence, the inside-horizon regularization is applied. By these procedures, one will eventually achieve nonnegative, UV- and IR-convergent power spectrum and spectral energy density.

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1 Introduction

The vacuum expectation values of stress tensor and power spectrum of quantum fields in curved spacetimes have UV divergences [1–3]. To remove UV divergences, several approaches have been proposed for regularization, such as the dimensional regularization [4–7], the covariant point-splitting [8–13], and the zeta function [5, 14, 15]. These methods involve the Green’s function and are essentially equivalent [5]. The appropriate subtraction term to the Green’s function in position space is generally hard to find. Only for the massless scalar field in de Sitter space with conformal, or minimal coupling, the subtraction term has been found [16]. Different from the above approaches, the adiabatic regularization works with the $k$-modes [17–31], and by the minimal subtraction rule the power spectrum is regularized to the 2nd-order, and the stress tensor to the 4th-order, respectively. For a massive scalar field with $\omega = (k^2 + m^2)^{1/2}$ as the 0th-order frequency, this prescription is sufficient in removing all UV divergences, but sometimes removes more than necessary, and leads to negative spectra, as demonstrated in de Sitter space [16].

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it turns out, the 0th-order regularization is sufficient to achieve nonnegative, UV- and IR-convergent spectra for a massive scalar field with the conformal coupling, and, similarly, so is the 2nd-order regularization for the minimal coupling. Given the regularized power spectrum, Fourier transformation gives the regularized Green’s function which is UV and IR convergent [16].

In this paper we extend the study to general spatially flat RW spacetimes. We shall consider a massless scalar field with a coupling $\xi$, whose exact solution is available and regularization can be performed in an analytical manner. Our aim is to search for proper regularization schemes which will yield nonnegative, UV- and IR-convergent power spectrum and spectral energy density. As we shall see, the goal can be eventually achieved, but there is no universal scheme that would work for all coupling and all RW spacetimes. First, UV convergence can be easily achieved by regularization of certain order. Unlike a massive field, the 0th-order frequency of a massless field is the wavenumber $k$, so that the 2nd-order regularization for the power spectrum and, respectively, the 4th-order for the spectral stress tensor, are necessary to remove all UV divergences. In particular, for the conformal coupling $\xi = \frac{1}{6}$, the regularization of all orders are equivalent, and yield a zero power spectrum and a zero spectral stress tensor, and there is no trace anomaly. For the minimal coupling $\xi = 0$, the regularized power spectrum and spectral stress tensor are zero in several important RW spacetimes, such as the radiation-dominated (RD) expansion, the matter-dominated (MD) expansion, and the de Sitter space. For general couplings and general RW spacetimes, however, it can happen that the regularized spectra are negative as we shall see. In order to avoid the negative spectrum, we try to increase the order of regularization on the pertinent spectrum. As it turns out, this will generally yield a positive spectrum. Nevertheless, this higher-order regularized, positive spectrum generally carries new IR divergence which is characteristic of the adiabatically regularized spectra of a massless field [31]. To retain IR convergence, we shall apply the inside-horizon scheme of regularization, by which the long wavelength modes outside the horizon are fixed and only the short wavelength modes inside the horizon are regularized [31]. After these procedures, we shall finally arrive at nonnegative, UV- and IR-convergent power spectrum and spectral energy density.

The paper is organized as follows.

In Sec. 2, we derive the exact solution of a coupling massless scalar field and its Green’s function in general RW spacetimes, analyze the behaviors of the power spectrum and the spectral stress tensor, and give the prescriptions of adiabatic regularization.

In Sect. 3, for the conformal coupling $\xi = \frac{1}{6}$, we show that adiabatic regularization of various orders are equal, yielding zero power spectrum and zero stress tensor, and there is no trace anomaly in general RW spacetimes. We also give direct regularization of the Green’s function, confirming the result of adiabatic regularization.

Sect. 4 is for the minimal coupling $\xi = 0$. We show that the regularized power spectrum and stress tensor are zero in several important RW spacetimes, and the results are also confirmed by regularization of the Green’s functions. In general RW spacetimes, we show by two examples that regularized spectra can be negative, and the pertinent spectrum will become positive by going to higher-order regularization, and thereby also acquire IR divergence. The IR divergence will be avoided by the inside-horizon scheme.

Sect. 5 is for the general coupling $\xi$, the analysis is similar to Sect. 4.

Sect. 6 gives conclusion and discussions.

Appendix A lists high-$k$ expansions of the exact modes. Appendix B lists the WKB solutions and the associated subtraction terms up to 6th-order, and demonstrates the covariant conservation to each adiabatic order.
2 The massless scalar field in general RW spacetimes

In a flat Robertson-Walker spacetime

\[ ds^2 = a^2(\tau)[d\tau^2 - \delta_{ij}dx^i dx^j], \]

with the conformal time \( \tau \), a massless scalar field has the field equation \([2, 13, 24]\)

\[ (\Box + \xi R) \phi = 0, \]

where \( \Box = \frac{1}{a^4} \frac{\partial}{\partial \tau} (a^2 \frac{\partial}{\partial \tau} - \frac{1}{a^2} \nabla^2) \), \( R = 6a''/a^3 \) is the scalar curvature, and \( \xi \) is the coupling constant, and we consider a range \( 0 \leq \xi \leq \frac{1}{6} \) specifically. Write

\[ \phi(x, \tau) = \int \frac{d^3k}{(2\pi)^3/2} \left[ a_k \phi_k(\tau) e^{ikx} + a_k^{\dagger} \phi_k^*(\tau) e^{-ikx} \right] \]

where \( a_k, a_k^{\dagger} \) are the annihilation and creation operators satisfy the canonical commutation relation, and \( \phi_k(\tau) \) is the \( k \)-mode which is written as \( \phi_k(\tau) = v_k(\tau)/a(\tau) \). The equation of \( v_k \) mode is

\[ v_k'' + \left( k^2 + (\xi - \frac{1}{6})a^2 R \right) v_k = 0. \] (3)

In this paper we consider a class of power-law expanding RW spacetimes,

\[ a(\tau) = a_0 |\tau|^b, \]

where the expansion index \( b \) is a constant, \( a^2 R = 6b(b-1)\tau^{-2} \). The positive-frequency mode solution of (3) is

\[ v_k(\tau) \equiv \sqrt{\frac{\pi}{2}} \sqrt{\frac{x}{2k}} e^{\frac{\pi}{2}(\nu + \frac{1}{2})} H_{\nu}^{(1)}(x), \]

where \( x \equiv k|\tau| \), \( H_{\nu}^{(1)} \) is the Hankel functions, and

\[ \nu \equiv \sqrt{\frac{1}{4} - (6\xi - 1)b(b-1)}. \] (6)

\( \nu = \frac{1}{2} \) corresponds to three cases: the conformal coupling \( (\xi = \frac{1}{6}) \), the Minkowski spacetime \( (b = 0) \), and the RD expansion \( (b = 1) \), and the mode (5) reduces to

\[ v_k(\tau) = -i \sqrt{\frac{\pi}{2}} \sqrt{\frac{x}{2k}} H_{1/2}^{(2)}(x) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \] (7)

conformal to the mode in Minkowski spacetime. The mode (5) of a general \( \nu \) at high \( k \) also approaches to (7).

The Bunch-Davies vacuum state is defined such that

\[ a_k |0 \rangle = 0, \quad \text{for all } k. \] (8)

In a general RW spacetime the unregularized Green’s function in the vacuum is

\[ G(x^\mu, x'^\nu) = \langle 0 | \phi(r, \tau) \phi(r', \tau') | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3k \ e^{ik(r-r')} \phi_k(\tau) \phi_k^*(\tau') = \frac{1}{a(\tau)a(\tau')} \frac{|\tau|^{1/2} |\tau'|^{1/2}}{8\pi} \int_0^\infty dk k \sin(k|r-r'|) \frac{H_{\nu}^{(1)}(k\tau)H_{\nu}^{(2)}(k\tau')}{|r-r'|}. \] (9)
The integration (9) can be carried out [13, 16, 32], and the result is a hypergeometric function as the following

\[ G(\sigma) = \frac{1}{16\pi^2 a(\tau)a(\tau')} \Gamma\left(\frac{3}{2} - \nu\right) \Gamma\left(\nu + \frac{3}{2}\right) 2F1 \left[ \frac{3}{2} + \nu, \frac{3}{2} - \nu, 2, 1 + \frac{\sigma}{2} \right], \]  

(10)

where \( \sigma = \left( (\tau - \tau')^2 - (r - r')^2 \right) / (2\tau\tau') \) is the half of squared geometric distance between two points \( x^\mu \) and \( x'^\mu \). For the equal-time \( \tau = \tau' \) case the Green’s function is

\[ G(r - r') = \langle 0 | \phi(r, \tau) \phi(r', \tau) | 0 \rangle = \int_0^\infty \frac{\sin(k|r - r'|)}{|r - r'| k^2} \Delta_k^2(\tau) dk; \]  

(11)

the auto-correlation function is

\[ G(0) = \langle 0 | \phi(r, \tau) \phi(r, \tau) | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3 k |\phi_k(\tau)|^2 = \int_0^\infty \Delta_k^2(\tau) \frac{dk}{k}, \]  

(12)

and the power spectrum is

\[ \Delta_k^2(\tau) = \frac{k^3}{2\pi^2 a^2(\tau)} |v_k(\tau)|^2 = \frac{x^3}{8\pi a^2(\tau)\tau^2} |H_\nu^{(1)}(x)|^2. \]  

(13)

The power spectrum is shown in Fig.1 for \( b = 2 \) and \( \xi = \frac{1}{8} \), which is UV divergent. The asymptotic behaviors of power spectrum can be analyzed by the series expansion. At high-\( k \), the power spectrum for general \( \xi \) and \( b \) is

\[ \Delta_k^2(\tau) = \frac{1}{4\pi^2 a^2 \tau^2} \left[ x^2 - \frac{(6\xi - 1)(b - 1)b}{2} - \frac{3(6\xi - 1)(b - 1)b [(1 - 6\xi)(b^2 - b) - 2]}{8x^2} - \frac{5(6\xi - 1)(b - 1)b [(1 - 6\xi)(b^2 - b) - 2]}{16x^4} \right] + \ldots \]  

(14)

The first two terms are quadratic and logarithmic UV divergent, leading to an infinite auto-correlation \( G(0) \). Our task is to perform adiabatic regularization and arrive at a power spectrum which should be: 1) UV convergent, 2) IR convergent, 3) nonnegative.
By the minimal subtraction rule [28], the first two terms of (14) are subtracted off under the 2nd-order regularization,

$$\Delta^2_{k \text{ reg}} = \frac{k^3}{2\pi^2 a^2} \left( |v_k(\tau)|^2 - |v^{(2)}_k(\tau)|^2 \right), \quad (15)$$

where $|v^{(2)}_k(\tau)|^2$ is the subtraction term given by (B.13), formed from the 2nd-order WKB approximate solution. The 2nd-order regularized power spectrum (15) is UV convergent, however, it can be negative for certain values of $\xi$ and $b$, as can be checked by the dominant, third term of Eq.(14),

$$-\frac{3(6\xi - 1)(b - 1)b [(1 - 6\xi)(b^2 - b) - 2]}{8x^2} \quad (16)$$

at high $k$. When this term is negative, in order to obtain a positive power spectrum, we shall try the 4th-order regularized power spectrum,

$$\Delta^2_{k \text{ reg}} = \frac{k^3}{2\pi^2 a^2} \left( |v_k(\tau)|^2 - |v^{(4)}_k(\tau)|^2 \right), \quad (17)$$

where $|v^{(4)}_k(\tau)|^2$ is given by (B.21), constructed from the 4th-order WKB approximate solution, and removes all the first three terms of Eq.(14). This usually yields a positive, UV-convergent power spectrum, which is dominated at high $k$ by the fourth term of Eq.(14)

$$-\frac{5(6\xi - 1)(b - 1)b [(1 - 6\xi)(b^2 - b) - 2] [(1 - 6\xi)(b^2 - b) - 6]}{16x^4} \quad (18)$$

We have checked that (18) is positive when (16) is negative. We shall demonstrate this procedure by examples in later sections.

We examine the low $k$ behavior of $\Delta^2_k$. For $\xi = \frac{1}{6}$, or $b = 1$, it has only one term,

$$\Delta^2_k = \frac{k^3}{2\pi^2 a^2} \frac{1}{2k}, \quad (19)$$

which holds also for all $k$. For a general $\xi$ by (A.6), it is

$$\Delta^2_k(\tau) \simeq \frac{2^{2\nu}}{8\pi^3 a^2 \tau^2} \Gamma(\nu)^2 x^{3-2\nu} \propto k^{3-2\nu}. \quad (20)$$

In regard to inflationary cosmology, if the scalar field $\phi$ is used to model the perturbed inflaton scalar field during inflation, the power spectrum $\Delta^2_k$ at low $k$ gives the primordial spectrum of scalar field perturbations, which is often written in a form of

$$\Delta^2_k \propto k^{n_s - 1}.$$  

Thus, one reads off the scalar spectral index

$$n_s = 4 - 2\nu = 4 - 2\sqrt{\frac{1}{4} - (6\xi - 1)b(b - 1)}. \quad (21)$$

The currently observed value is $n_s \simeq 0.96$, which for $\xi = 0$ corresponds to an expansion index $b \simeq -1.02$ during inflation [30, 31]. In general RW spacetimes, the power spectrum (20) with $\xi = 0$ is IR convergent for $-1 < b < 2$, and is IR divergent for $b \leq -1$ or $b \geq 2$. In this paper we do not discuss the issue of IR divergence in the unregularized power spectrum, which can be avoided either by certain initial condition or by some precedent
For each $k$ the pressure is
\begin{equation}
\rho = \langle T^0_0 \rangle = \int_0^\infty \rho_k \frac{dk}{k},
\end{equation}
where the spectral energy density is
\begin{equation}
\rho_k = \frac{k^3}{4\pi^2a^4} \left[ |v_k'|^2 + k^2|v_k|^2 + (6\xi - 1) \left( \frac{a'}{a}(v_k'v_k^* + v_kv_k'^*) - \left( \frac{a'}{a} \right)^2|v_k|^2 \right) \right],
\end{equation}
the trace of stress tensor is
\begin{equation}
\langle T^\mu_\mu \rangle = \frac{1}{2\pi^2a^4} \int k^2dk (6\xi - 1) \left[ |v_k'|^2 - \frac{a'}{a}(v_k'v_k^* + v_kv_k'^*) - k^2|v_k|^2 \right.
- \left( \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 \right)|v_k|^2 + (1 - 6\xi) \frac{a''}{a} |v_k|^2 \right],
\end{equation}
the pressure is
\begin{equation}
p = -\frac{1}{3} \langle T^i_i \rangle = \int_0^\infty p_k \frac{dk}{k},
\end{equation}
where the spectral pressure is
\begin{align}
p_k &= \frac{k^3}{4\pi^2a^4} \left[ \frac{1}{3} |v_k'|^2 + \frac{1}{3} k^2 |v_k|^2 + 2(\xi - \frac{1}{6}) \left( \frac{a'}{a}(v_k'v_k^* + v_kv_k'^*) - \left( \frac{a'}{a} \right)^2|v_k|^2 \right) - 4(\xi - \frac{1}{6}) \left( |v_k'|^2 - \frac{a'}{a}(v_k'v_k^* + v_kv_k'^*) - k^2|v_k|^2 - \left( \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 \right)|v_k|^2 \right.
\right.
- \left. 6(\xi - \frac{1}{6}) \frac{a''}{a} |v_k|^2 \right].
\end{align}
For each $k$-mode, the unregularized spectral stress tensor satisfies the covariant conservation $\rho'_k + 3\frac{a''}{a}(\rho_k + p_k) = 0$, as can be checked by the field equation (3). The spectral...
energy density and pressure have the following high-$k$ expansions

\[
\rho_k = \frac{1}{4\pi^2a^4\tau^4}\left[ x^4 - \frac{(6\xi - 1)b^2x^2}{2} + \frac{3(6\xi - 1)^2(b - 1)b^2(b + 1)}{8} \right.
\]

\[
+ \frac{5(6\xi - 1)^2(b - 1)b^2(b + 2)}{16x^2} \left[ (1 - 6\xi)(b^2 - b) - 2 \right]
\]

\[
+ \frac{35(6\xi - 1)^2b^2(b - 1)(b + 3)[b(b - 1)(6\xi - 1) + 2][b - 1)(b(6\xi - 1) + 6]}{128x^4}
\]

\[
+ ... \right], \tag{29}
\]

\[
p_k = \frac{1}{4\pi^2a^4\tau^4}\left[ x^4 - \frac{(6\xi - 1)b(b + 2)x^2}{3} + \frac{(6\xi - 1)^2(b - 1)b(b + 1)(b + 4)}{6} \right.
\]

\[
+ \frac{5(6\xi - 1)^2(b - 1)b(b + 2)(b + 6)}{48x^2} \left[ (1 - 6\xi)(b^2 - b) - 2 \right]
\]

\[
+ \frac{35(6\xi - 1)^2(b - 1)b(b + 3)(b + 8)[(b - 1)(b(6\xi - 1) + 2)[(b - 1)(b(6\xi - 1) + 6]}{384x^4}
\]

\[
+ ... \right], \tag{30}
\]

containing quartic, quadratic, and logarithmic UV divergences. We search for a regularized stress tensor that should satisfy the following criteria: 1) UV convergent, 2) IR convergent, 3) the spectral energy density is nonnegative. A negative pressure is allowed, so we focus on the spectral energy density in this paper. By the minimal subtraction rule [17], the first three divergent terms of $\rho_k$ in (29) are subtracted off under the 4th-order regularization,

\[
\rho_{k,\text{reg}} = \rho_k - \rho_{k,4}\tag{31}
\]

where $\rho_{k,4}$ is the 4th-order subtraction term given by (B.37). The resulting $\rho_{k,\text{reg}}$ is always UV convergent, and if it is also positive, our goal is achieved. Like the power spectrum, however, the 4th-order regularized $\rho_k$ can be negative for certain values of $\xi$ and $b$, as is determined the dominant fourth term of Eq.(29) at high $k$,

\[
\frac{5(6\xi - 1)^2(b - 1)b^2(b + 2)(1 - 6\xi)(b^2 - b) - 2}{16x^2}. \tag{32}
\]

When this happens, in order to get a positive spectral energy density, we shall try the 6th-order regularization

\[
\rho_{k,\text{reg}} = \rho_k - \rho_{k,6}\tag{33}
\]

where the subtraction term $\rho_{k,6}$ is given by Eq.(B.39). Under this subtraction, the first four terms of (29) will be removed, and the fifth term

\[
\frac{35(6\xi - 1)^2b^2(b - 1)(b + 3)b(b - 1)(6\xi - 1) + 2}{128x^4}
\]

remains and dominates the 6th-order regularized spectral energy density at high $k$. As we shall see, for RW spacetimes used in cosmology, this often gives a positive, UV-convergent regularized spectral energy density. For RW spacetimes rarely used in cosmology, say $-3 < b < -2$, and for a coupling

\[
\frac{b^2 - b - 2}{6b^2 - 6b} < \xi < \frac{1}{6}, \tag{35}
\]
both Eq.(32) and Eq.(34) will be negative for high $k$. Then one has to seek the 8th-order regularized $\rho_k$ which is dominated by the following term

$$\frac{1}{4\pi^2 a^4 \tau^4} \left[ -\frac{1}{256x^6} 63(b-1)(b+4) b^2 (1-6\xi)^2 \left( (b-1)b(6\xi-1)+2 \right) (b-1)b(6\xi-1) + 6 \right] \left( (b-1)b(6\xi-1)+12 \right).$$

This term is positive with a coupling in the range of (35). Thus, a positive UV-convergent spectral energy density can be achieved by this procedure.

We mention that the four-divergence of the subtraction terms to the stress tensor is zero under regularization of each order, so that the covariant conservation is respected by the regularized stress tensor respects (See (B.45)–(B.50) in Appendix B.)

We examine the low-$k$ behavior of the stress tensor. For $\xi = \frac{1}{6}$, or $b = 1$, Eq.(6) gives $\nu = \frac{1}{2}$, and (25) (28) reduce to

$$\rho_k = \frac{k^3}{4\pi^2 a^4} \left[ |v_k'|^2 + k^2 |v_k|^2 \right] = \frac{k^3}{4\pi^2 a^4} k, \quad p_k = \frac{1}{3} \rho_k,$$

which are UV divergent and IR convergent. For $\xi = 0$, (25) (28) reduce to

$$\rho_k = \frac{k^3}{4\pi^2 a^2} \left[ \left( \frac{v_k}{a} \right)'^2 + k^2 \left( \frac{v_k}{a} \right)^2 \right],$$

$$p_k = \frac{k^3}{4\pi^2 a^2} \left[ \left( \frac{v_k}{a} \right)'^2 - \frac{1}{3} k^2 \left( \frac{v_k}{a} \right)^2 \right],$$

shown in Fig. 2, and at low $k$ they reduce to

$$\frac{(\rho_k_{unreg}, p_k_{unreg})}{(16\pi a^4 \tau^4)^{-1}} \text{ with } \xi = 0, \ b = 1$$

![Figure 2: Red Dash: unregularized $\rho_k$, Blue Solid: unregularized $p_k$. For $\xi = 0$ and $b = 1.$](image)

$$\rho_k \simeq \begin{cases} \frac{2^{2+2\nu}}{16\pi^3 a^4 \tau^4} \Gamma(\nu+1)^2 x^{3-2\nu}, & b \geq \frac{1}{2}, \\ \frac{2^{2-2\nu}}{16\pi^3 a^4 \tau^4} x^{2\nu+3}, & 0 < b < \frac{1}{2}, \\ \frac{\pi^4}{4\pi^2 a^4 \tau^4} \Gamma(\nu)^2 x^4, & b = 0, \\ \frac{2^{2\nu}}{16\pi^3 a^4 \tau^4}, & b < 0, \end{cases}$$

(40)
where \( \nu = b - \frac{1}{2} \) if \( b \geq \frac{1}{2} \), and \( \nu = \frac{1}{2} - b \) if \( b < \frac{1}{2} \). Both \( \rho_k \) and \( p_k \) are IR convergent when \(-2 < b < 2\), as shown in Fig. 2, which includes the RD stage \((b = 1)\) and the de Sitter inflation \((b = -1)\). But when \( b = \pm 2 \), or \( b > 2 \), or \( b < -2 \), \( \rho_k \) and \( p_k \) are IR divergent. The issue of IR divergence of the unregularized spectra has been analyzed in Ref. [33,34], and in this paper we shall not discuss it further. Regularization may take IR convergent \( \rho_k \) and \( p_k \) into IR divergent, just as with the power spectrum, and the scheme of inside-horizon regularization can be used to retain IR convergence [16].

3 Regularization for \( \xi = \frac{1}{6} \)

Now we implement adiabatic regularization of the conformally-coupling massless scalar field. the mode with \( \xi = \frac{1}{6} \) is given by Eq. (7),

\[
v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}},
\]

valid for an arbitrary expansion index \( b \). The power spectrum (13) has only one term,

\[
\Delta^2_k(\tau) = \frac{k^3}{2\pi^2a^2} \frac{2k}{2k},
\]

so that the 0th-order regularization is sufficient, yielding a vanishing regularized spectrum

\[
\Delta^2_{k\text{reg}} = \frac{k^3}{2\pi^2a^2} \left( |v_k(\tau)|^2 - |v_k(0)|^2 \right) = \frac{k^3}{2\pi^2a^2} \left( \frac{1}{2k} - \frac{1}{2k} \right) = 0.
\]

The spectral energy density and pressure in Eq. (37) have only one term, and the trace \( \langle T_{\mu}^{\mu} \rangle_k = 0 \). The 0th-order subtraction terms are given by (B.33) (B.34), and are just equal to the unregularized stress tensor. Hence, the regularized stress tensor is zero,

\[
\rho_{k\text{reg}} = \rho_k - \rho_{kA0} = \frac{k^4}{4\pi^2a^4} - \frac{k^4}{4\pi^2a^4} = 0,
\]

\[
p_{k\text{reg}} = p_k - p_{kA0} = \frac{k^4}{12\pi^2a^4} - \frac{k^4}{12\pi^2a^4} = 0,
\]

and the regularized trace is also zero,

\[
\langle T_{\beta}^3 \rangle_{k\text{reg}} = \rho_{k\text{reg}} - 3p_{k\text{reg}} = 0.
\]

The above calculations show two important features of the conformally-coupling massless scalar field. First, the vanishing spectra (44)–(47) hold for a general scale factor \( a(\tau) \). Thus, in any flat RW spacetime, the power spectrum and the stress tensor are all regularized to zero and there is no trace anomaly of the conformally-coupling massless scalar
field. This is a generalization of the result in de Sitter space [16] to general RW spacetimes. Second, for \( \xi = \frac{1}{6} \), the vanishing regularized spectra (44)–(47) hold for any order of adiabatic regularization, because the subtraction terms of any order are equal to those of the 0th-order. (See (B.41)–(B.44) in Appendix B). The above results of adiabatic regularization also follow from a direct regularization of Green’s function. For \( \nu = \frac{1}{2} \), the unregularized Green’s function (10) reduces to

\[
G(\sigma) = -\frac{1}{16\pi^2a(\tau)a(\tau')\tau\tau'}^2.
\]  

which has one term, and is UV divergent at \( \sigma = 0 \). To remove this UV divergence, the natural choice for the subtraction term is

\[
G(\sigma)_{\text{sub}} = -\frac{1}{16\pi^2a(\tau)a(\tau')\tau\tau'}^2.
\]

and the regularized Green’s function is simply given by

\[
G(\sigma)_{\text{reg}} = G(\sigma) - G(\sigma)_{\text{sub}} = 0.
\]

This vanishing Green’s function confirms the vanishing power spectrum of (44), as they are the Fourier transformation to each other. Consequently the regularized stress tensor is also vanishing as it is constructed from the Green’s function.

### 4 Regularization for \( \xi = 0 \)

Next we do regularization for the minimally-coupling \( \xi = 0 \). As listed in Appendix B, the subtraction terms of various orders depend on the expansion index \( b \) through \( a(\tau) \). We shall consider some specific values of \( b \). (The case \( b = -1 \) of de Sitter space was already worked out in Ref. [16].)

We first consider the RD expansion stage. With \( b = 1 \), the definition (6) gives \( \nu = \frac{1}{2} \) true for any \( \xi \). The mode \( v_k \) becomes the same as (42) and the power spectrum \( \Delta_k^2 \) becomes the same as (43). We use the 2nd-order regularization, and obtain a zero regularized power spectrum

\[
\Delta_k^2_{\text{reg}} = \frac{k^3}{2\pi^2a^2} \left( |v_k(\tau)|^2 - |v_k(2)(\tau)|^2 \right) = 0.
\]

Also the regularization of the corresponding Green’s function is the same as those given by Eqs.(48) (49) (50). (Notice that for \( a \propto \tau \), one has \( a'' = a''' = ... = 0 \), so that the 0th-, 2nd- and higher order substraction terms are equal, \( |v_k(0)(\tau)|^2 = |v_k(2)(\tau)|^2 = |v_k(4)(\tau)|^2 = ... = \frac{1}{2k} \). See (B.5) (B.13) (B.20) (B.28) in Appendix B.) The spectral energy density (38) and spectral pressure (39) become

\[
\rho_k = \frac{1}{4\pi^2a^4\tau^4} \left( x^4 + \frac{x^2}{2} \right),
\]

\[
p_k = \frac{1}{4\pi^2a^4\tau^4} \left( \frac{x^4}{3} + \frac{x^2}{2} \right),
\]

which contain quartic and quadratic UV divergences. The 2nd-order substraction terms (B.35) (B.36) are

\[
\rho_k A_2 = \frac{k^4}{4\pi^2a^4} \left( \frac{1}{2x^2} + 1 \right),
\]
\[ p_{kA2} = \frac{k^4}{12\pi^2a^4} \left( \frac{3}{2x^2} + 1 \right), \]  

so the regularized stress tensor are vanishing

\[ \rho_{k\text{reg}} = \rho_k - \rho_{kA2} = 0, \]  

\[ p_{k\text{reg}} = p_k - p_{kA2} = 0. \]  

(By the way the 4th-order subtraction terms (B.37) (B.38) happen to be equal to those of the 2nd-order since \( a'' = a''' = 0 \).) Thus, in the RD stage, the 2nd-order regularization is sufficient to remove all the divergences of the power spectrum and stress tensor of a minimally-coupling massless field. This is similar to what happens in de Sitter space, in which the 2nd-order regularization also brings to zero the power spectrum and stress tensor of a minimally-coupling massless scalar field [16]. The result can be also derived in terms of Green’s function. For \( b = 1 \), the unregularized Green’s function (10) is the same as (48), and so is the subtraction term is chosen as (49), so the regularized Green’s function is \( G(\sigma)_{\text{reg}} = G(\sigma) - G(\sigma)_{\text{sub}} = 0 \), thus the regularized power spectrum is zero and the regularized stress tensor is also zero.

We next consider the matter-dominated (MD) stage, \( b = 2 \). The mode (5) with \( \nu = \frac{3}{2} \) becomes

\[ v_k(\tau) = \frac{e^{ix}}{\sqrt{2k}}(1 + \frac{i}{x}), \]  

and the power spectrum (13) becomes

\[ \Delta_k^2(\tau) = \frac{k^3}{2\pi^2a^2} \left( \frac{1}{2k} + \frac{1}{2k^3\tau^2} \right). \]  

By the 2nd-order regularization, using \( |v_k^{(2)}|^2 \) from (B.13), the regularized spectrum is zero

\[ \Delta_{k\text{reg}}^2 = \frac{k^3}{2\pi^2a^2} \left( |v_k(\tau)|^2 - |v_k^{(2)}(\tau)|^2 \right) = 0. \]  

The spectral stress tensor (38) (39) for \( b = 2 \) become

\[ \rho_k = \frac{1}{4\pi^2a^4\tau^4} \left( x^4 + 2x^2 + \frac{9}{2} \right), \]  

\[ p_k = \frac{1}{4\pi^2a^4\tau^4} \left( \frac{x^4}{3} + \frac{4x^2}{3} + \frac{9}{2} \right), \]  

which contain quartic, quadratic and log UV divergences. To remove these divergences, we apply the 4th-order regularization, and the subtraction terms (B.37) and (B.38) are

\[ \rho_{kA4} = \frac{k^4}{4\pi^2a^4} \left( 1 + \frac{2}{x^2} + \frac{9}{2x^4} \right), \]  

\[ p_{kA4} = \frac{k^4}{12\pi^2a^4} \left( 1 + \frac{4}{x^2} + \frac{27}{2x^4} \right). \]  

The regularized spectral energy density and pressure are zero

\[ \rho_{k\text{reg}} = \rho_k - \rho_{kA4} = 0, \]  

\[ p_{k\text{reg}} = p_k - p_{kA4} = 0. \]
These results can be also derived in terms of Green’s function. For $\nu = \frac{3}{2}$ we can directly integrate Eq.(9) to obtain the Green’s function

$$G(\sigma) = \frac{1}{(2\pi)^3 a(\tau)a(\tau')} \int \frac{1}{k^3} e^{i k \cdot (r-r')-i k (\tau-\tau')} \left( \frac{1}{k} + i \left( \frac{1}{\tau} - \frac{1}{\tau'} \right) \frac{1}{k} + \frac{1}{k^2 \tau \tau'} \right)$$

$$= \frac{1}{8\pi^2 a(\tau)a(\tau')|\tau\tau'|} \left( -\frac{1}{\sigma} - \ln \sigma \right),$$

in which both terms are UV convergent and should be removed. So the subtraction term is taken to be

$$G(\sigma)_{sub} = \frac{1}{8\pi^2 a(\tau)a(\tau')|\tau\tau'|} \left( -\frac{1}{\sigma} - \ln \sigma \right),$$

resulting in

$$G(\sigma)_{reg} = G(\sigma) - G(\sigma)_{sub} = 0,$$

which agrees with the vanishing regularized spectra (60) (65) (66).

From the spectra (14) (29) (30) with $\xi = 0$, we see that, after subtracting their respective divergent terms, all the remaining convergent terms are proportional a common factor

$$b(b-2)(b-1)(b+1),$$

which is vanishing for $b = 0, \pm 1, 2$. Hence, for the minimally-coupling massless scalar field, the regularized power spectrum and stress tensor are zero in the Minkowski spacetime, the RD stage, the MD stage, and the de Sitter space, the latter case was shown in Ref. [16].

What about a general index $b$? In the following we consider two quasi de Sitter inflation models with $b \simeq -1$. For the model $b = -1.02$, as shown in Fig. 3, the 2nd-order regularized $\Delta^2_{k,reg}$ is positive, UV convergent and IR divergent. But the 4th-order regularized $\rho_{k,reg}$ is negative as shown in Fig. 4. This is implied by the dominant, fourth term of (29) of $\rho_k$,

$$\frac{5(b-2)(b-1)b^2(b+1)(b+2)}{16x^2},$$

which is negative for $b = -1.02$. This is the phenomenon of negative spectra that we have mentioned early around Eqs.(32) (33). In order to get a positive spectral energy density, we proceed to compute the 6th-order regularized $\rho_{k,reg}$ in Eq.(33), which is dominated by the fifth term of (29). As it turns out, the 6th-order regularized $\rho_{k,reg}$ is positive and UV
convergent, as shown in Fig.4. As for the IR divergences in the regularized spectra, we adopt the inside-horizon scheme [31] as the following. The UV divergences come from the high $k$ modes, whereas the low $k$ modes do not cause UV divergence. So we need only regularize the short wavelength modes inside the horizon during the expansion ($k \gtrsim 1/|\tau_1|$ where $\tau_1$ is a fixed time during the expansion), and the long wavelength modes outside the horizon are kept unchanged. Under this scheme, UV divergences are removed and IR divergences are avoided. Thus, for the case $b = -1.02$ under consideration, the spectral energy density is regularized by

$$\rho_k(\tau)^\text{reg} = \begin{cases} \rho_k - \rho_k A_6, & \text{for } k \geq \frac{1}{|\tau_1|}, \\ \rho_k, & \text{for } k < \frac{1}{|\tau_1|}, \end{cases}$$

(71)

The regularization is performed instantaneously at a fixed $\tau_1$. The result is plotted in Fig. 5. Thus, the IR divergence is avoided, and a positive, UV- and IR-convergent spectral energy density is achieved, and the regularized energy density is $\rho_{\text{reg}} = \int_0^\infty \rho_{k\text{reg}} \frac{dk}{k} = 0.622 \left(\frac{b/a_0}{16\pi}\right)^4$ at $|\tau| = 1$.

For the model $b = -0.98$, the 2nd-order regularized power spectrum is dominated by the third term of (14) and is negative, as shown in Fig. 6. So we proceed to calculate the 4th-order regularized power spectrum according to Eq.(17), which is positive, UV-
Figure 6: For $\xi = 0, b = -0.98$: the 2nd-order $\Delta^2_{k, reg}$ is negative; the 4th-order $\Delta^2_{k, reg}$ is UV convergent and IR divergent.

Figure 7: For $\xi = 0, b = -0.98$: the inside-horizon regularization for $\Delta^2_{k, reg}$ according to Eq.(72).

The resulting power spectrum is plotted in Fig. 7. Thus, the IR divergence is avoided, and a positive, UV- and IR-convergent power spectrum is achieved. By the Fourier transformation of (72) according to the formula (11), we obtain the corresponding regularized Green’s function $G(r - r')_{reg}$, which is UV finite and IR convergent, as shown in Fig. 8. The 4th-order regularized spectral energy density is positive and UV convergent, as plotted in Fig. 9, and the regularized energy density is $\rho_{reg} = \int_0^\infty \rho_{k, reg} \frac{dk}{k} = 0.228(b/a_0)^4/16\pi$ at $|\tau| = 1$. The above two examples show that the inside-horizon scheme is effective in avoiding IR divergences.

5 Regularization for general $\xi$

Now we explore adiabatic regularization for a general coupling $\xi$, search for proper regularization schemes that would yield nonnegative, UV- and IR-convergent spectra $\Delta^2_k$ and
Figure 8: For $\xi = 0$, $b = -0.98$: the regularized Green’s function $G(|r - r'|)_{\text{reg}}$ is Fourier transform of the regularized power spectrum in Fig. 7.

Figure 9: For $\xi = 0, b = -0.98$: the 4th-order $\rho_{k_{\text{reg}}}$ is positive, UV convergent and IR log divergent.
\[ \Delta^2_{k_{\text{reg}}}/(8\pi a^2\tau^2)^{-1} \text{ with } \xi=1/8, \; b=2 \]

Figure 10: For \( b = 2 \) and \( \xi = 1/8 \), the 2nd-order regularized \( \Delta^2_k \) is negative, and the 4th-order regularized \( \Delta^2_k \) is positive, and UV convergent.

\[ \rho_{k_{\text{reg}}}/(16\pi a^4\tau^4)^{-1} \text{ with } \xi=1/8, \; b=2 \]

Figure 11: For \( b = 2 \) and \( \xi = 1/8 \), Red Dashed: the 4th-order regularized \( \rho_{k_{\text{reg}}} \) is negative; Blue Solid: the 6th-order regularized \( \rho_{k_{\text{reg}}} \) is positive, and \( k^{-4} \) UV convergent at high \( k \).

\( \rho_k \). We shall consider several values of \( b \) in several interesting cosmological models. By the minimal subtraction rule, the 2nd-order regularization for \( \Delta^2_k \) and the 4th-order regularization for \( \rho_k \) are default, and we shall go to higher order regularization when a negative spectrum appears.

First we consider \( b = 1 \) for the RD stage with a general \( \xi \). The analysis between (51)–(57) is also valid for a general \( \xi \), and the results are \( \Delta^2_{k_{\text{reg}}} = \rho_{k_{\text{reg}}} = p_{k_{\text{reg}}} = 0 \).

Next we consider \( b = 2 \) for the MD stage with a general \( \xi \). For illustration, \( \xi = 1/8 \) is taken in the following, other values of \( \xi \) can be analyzed in the same fashion. As shown in Fig. 10, the 2nd-order regularized \( \Delta^2_{k_{\text{reg}}} \) is negative, dominated by the third term of Eq.(14), and the 4th-order regularized \( \Delta^2_{k_{\text{reg}}} \) is positive and UV convergent, but IR divergent. As shown in Fig.11, the 4th-order regularized \( \rho_{k_{\text{reg}}} \) is negative, dominated by the fourth term in Eq.(29), and the 6th-order regularized \( \rho_{k_{\text{reg}}} \) is positive, dominated by the fifth term of (29).

Then, we consider \( b = -1 \), de Sitter space. We plot the regularized power spectra for \( \xi = 1/8 \) in Fig.12, the 2nd-order regularized \( \Delta^2_{k_{\text{reg}}} \) is negative, the 4th-order regularized \( \Delta^2_{k_{\text{reg}}} \) is positive and UV convergent, but IR divergent. As shown in Fig.13, the 4th-order regularized \( \rho_k \) is positive and UV and IR convergent. (Here \( \frac{2\ddot{a}a'}{a'} - \frac{a''^2}{a'} - \frac{4\dddot{a}a'}{a'} = 0 \) for de Sitter space, so that the 2nd-order subtraction term (B.35) is equal to the 4th-order...
Finally, we consider the quasi de Sitter inflation model with \( b = -1.098 \) and \( \xi = \frac{1}{8} \). As shown in Fig. 14, the 2nd-order regularized \( \Delta^2_k \) is negative, the 4th-order regularized \( \Delta^2_k \) is positive and UV convergent but IR divergent. Fig.15 shows that the 4th-order regularized \( \rho_k \) is positive, IR finite and UV convergent. For the model with \( b = -1.02 \) and \( \xi = \frac{1}{8} \), as shown in Fig. 16, the 2nd-order regularized \( \Delta^2_k \) is negative; the 4th-order regularized \( \Delta^2_k \) is positive and UV convergent, but IR divergent. Fig.17 shows that the 4th-order regularized \( \rho_k \) is positive and UV convergent, but IR log divergent.

In the above when IR divergence appears, the inside-horizon scheme can apply as in Sect. 4, and we do not repeat the details to save room. Hence, for general \( \xi \) and \( b \), one can achieve positive, UV- and IR-convergent power spectrum and spectral energy density.

### 6 Conclusion and Discussions

We have studied adiabatic regularization of a massless scalar field in general RW space-times with a coupling \( \xi \) (in a range \( 0 \leq \xi \leq \frac{1}{8} \) for specific in this paper), and this extends our previous study in de Sitter space [16]. The analytical expressions of the power spectrum, the corresponding Green’s function, and the spectral stress tensor are given, and all contain UV divergences. Our goal is to find the appropriate schemes of regularization that will achieve nonnegative, UV- and IR-convergent power spectrum and spectral en-
Figure 14: For $b = -0.98$ and $\xi = 1/8$, the 2nd-order regularized $\Delta^2_k$ is negative, the 4th-order regularized $\Delta^2_k$ is positive and UV convergent.

Figure 15: For $b = -0.98$ and $\xi = 1/8$, the 4th-order regularized $\rho_k$ is positive and UV convergent.

Figure 16: For $b = -1.02$ and $\xi = 1/8$, The 2nd-order regularized $\Delta^2_k$ is negative, and the 4th-order regularized $\Delta^2_k$ is positive and UV convergent.
energy density. Adiabatic regularization respects the covariant conservation of stress tensor to each order. For the massless field, the UV divergences are generally removed when the power spectrum is regularized to the 2nd-order, and respectively the stress tensor to the 4th order. The nonnegativeness, however, is not always ensured by the conventional prescription, and this is our main concern in this paper. Through several examples, we have found that there is no regularization scheme of fixed-order which would work for all coupling and all RW spacetimes. An adequate scheme depends upon the coupling ξ and the expansion index b.

Several interesting cases are very simple. For the conformally-coupling ξ = 1/6 in Sect 3, we have found that the regularized power spectrum and stress tensor are zero, no trace anomaly exists in general RW spacetimes. This new result is a generalization of that in de Sitter space [16]. In literature on the trace anomaly one strangely started with a massive field to discuss the stress tensor of a massless field. If one starts straightforwardly with a massless scalar field, one simply obtains a zero stress tensor and never comes up with a trace anomaly, as demonstrated in this paper. The trace anomaly claimed in literature is an artifact of inadequate regularization. For the RD expansion in Sect 4, the regularized spectra are also zero for any coupling ξ. This is because in this case the wave equation (3) is conformal to those in the Minkowski spacetime, and the regularization amounts to the normal-ordering. Another simple case is the minimally-coupling ξ = 0 in Sect 4, the regularized spectra are zero for b = 0, ±1, 2, corresponding to the Minkowski space, the de Sitter space, the radiation-dominated stage, and the matter-dominated stage. In the above simple cases, we have also carried out direct regularization of Green’s functions in position space, and found that the regularized Green’s functions are zero too, confirming the zero spectra by adiabatic regularization.

For the cases of general ξ and b in Sect 5, as we have found, regularized spectra of the massless scalar field can be negative under the conventional regularization. To avoid the negative spectra, we have performed higher order regularization to the pertinent spectrum. Specifically, if the power spectrum is negative under the 2nd-order regularization, we calculate its 4th-order regularization, and similarly if the spectral energy density is negative under the 4th-order regularization, we calculate its 6th-order regularization. As it turns out, the resulting higher-order regularization spectrum will usually become positive for the RW spacetimes commonly used in cosmology. In some rarely-used RW spacetimes, the 6th-order regularized spectral energy density may be still negative, then we go to the 8th-order which will eventually yield a positive spectral energy density.

A massless field may carry IR divergence in the unregularized spectra as summarized in

![Figure 17: For b = −1.02 and ξ = 1/8, the 4th-order regularized ρk_reg is positive at high k.](image-url)
(20) (40) (41), and how to avoid was analyzed in Ref. [33,34]. In this paper we have focused on the IR divergences that are caused by regularization. In particular, IR divergence occurs when going to higher order regularization for general $\xi$ and $b$. These new IR divergences can be avoided by the inside-horizon scheme of regularization, as demonstrated by the examples in Fig.5 and Fig.7. The details of inside-horizon scheme was discussed in Ref. [31], and we just mention the following two related points. First, the scalar field considered in this paper is linear, and its $k$-modes are independent of each other, unlike the nonlinear fields [35]. Under the inside-horizon scheme, each regularized short wavelength mode respects the covariant conservation to pertinent order, the unregularized long wavelength modes also respect the covariant conservation. Thus the total stress tensor respects the covariant conservation by the inside-horizon scheme. Second, the long wavelength modes outside the horizon correspond to the wave band of the observed CMB temperature anisotropies and polarization [30, 31]. When the scalar field is used to model the cosmological perturbations, the inside-horizon scheme reserves the spectra perturbations of the long wavelength band, so that the observed primordial spectrum will not affected by regularization.

Through these detailed investigations, we come to the conclusion: for a coupling massless scalar field in general RW spacetimes, the nonnegative, UV- and IR-convergent power spectrum and spectral energy density can be achieved by adiabatic regularization, with the help of higher order scheme and the inside-horizon scheme when necessary.

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A  High \( k \) expansions of exact modes

We list some asymptotic expressions following from the analytical solution for a general RW spacetime which are used in the context.

At high \( k \), the mode \( v_k \) of (5) approaches to

\[
v_k \simeq \frac{1}{\sqrt{2k}} e^{ix} \left( 1 + i \frac{(4\nu^2 - 1)}{8x} - \frac{(16\nu^4 - 40\nu^2 + 9)}{128x^2} - \frac{i(64\nu^6 - 560\nu^4 + 1036\nu^2 - 225)}{3072x^3} \right. \\
+ \frac{(256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025)}{98304x^4} \\
+ \left. \frac{i(-1024\nu^{10} + 42240\nu^8 - 561792\nu^6 + 2764960\nu^4 - 4228884\nu^2 + 893025)}{3932160x^5} \right) \ .
\]

where the first term corresponds to the positive-frequency mode in Minkowski spacetime, and other terms are due to expansion effects. The squared mode at high \( k \) is

\[
|v_k|^2 = \frac{\pi x}{4k} |H^{(2)}_\nu(x)|^2 = \frac{1}{2k} \left( 1 + \frac{4\nu^2 - 1}{8x^2} + \frac{3(16\nu^4 - 40\nu^2 + 9)}{128x^4} \right. \\
+ \frac{5(2\nu - 5)(2\nu - 3)(2\nu - 1)(2\nu + 1)(2\nu + 3)(2\nu + 5)}{1024x^6} \\
+ \left. \cdots \right) \ .
\]

The time derivatives to the 4th adiabatic order are given by

\[
|v_k'|^2 = k \left( \frac{1}{2} - \frac{(4\nu^2 - 1)}{16x^2} - \frac{(16\nu^4 - 104\nu^2 + 25)}{256x^4} - \frac{(64\nu^6 - 2096\nu^4 + 4876\nu^2 - 1089)}{2048x^6} + \cdots \right) \ .
\]

\[
|\frac{v_k'}{a}|^2 = b^{-2} H^2 \left( \frac{x^2}{2k} + \frac{8b^2 + 1 - 4\nu^2}{16k} - \frac{16\nu^4 - (64b^2 + 128b + 104)\nu^2 + (16b^2 + 32b + 25)}{256kx^2} \right. \\
+ \frac{(216b^2 + 864b + 1089) - (960b^2 + 3840b + 4876)\nu^2 + (384b^2 + 1536b + 2096)\nu^4 - 64\nu^6}{2048kx^4} \\
+ \cdots \right) \ .
\]

From these, one obtains the high-\( k \) expansions of \( \rho_k \) and \( p_k \) in Eq.(29) and (30) in the context.

At low \( k \), the mode \( v_k \) of (5) and the related squared modes are given by

\[
v_k \simeq \left( \frac{x}{2} \right)^{-\nu-\frac{1}{2}} \frac{\Gamma(\nu)}{\sqrt{2\pi k}} e^{i\frac{\pi}{4}(\nu-\frac{1}{2})},
\]

\[
|v_k|^2 \simeq x^{-2\nu} |\tau|^{2\nu-2} \frac{\Gamma(\nu)^2}{\pi},
\]

\[
|\frac{v_k}{a}|^2 \simeq a_0^{-2} x^{-2\nu} |\tau|^{1-2b} \frac{2^{2\nu-2} \Gamma(\nu)^2}{\pi}, \]

and the time derivatives are

\[
v_k' \simeq \frac{|\tau|}{\tau} \left( \frac{x}{2} \right)^{-\nu-\frac{1}{2}} k^{1/2} \frac{\Gamma(\nu) - 2\Gamma(\nu + 1)}{4\sqrt{2\pi}} e^{i\frac{\pi}{4}(\nu-\frac{1}{2})},
\]
\[ |v_k'|^2 \simeq \frac{\left(\frac{x}{2}\right)^{-2\nu-1} k \left(\Gamma(\nu) - 2\Gamma(\nu + 1)\right)^2}{32\pi}, \quad (A.9) \]

\[ \left(\frac{v_k}{a}\right)' = \frac{|\tau| a_0^{-1} \left(\frac{x}{2}\right)^{-\frac{1}{2} - \nu} k^{\frac{1}{2} + b} (1 - 2b) \Gamma(\nu) - 2\Gamma(\nu + 1) \right) \frac{e^{i\pi}}{2\sqrt{2\pi}} }, \quad (A.10) \]

\[ |\left(\frac{v_k}{a}\right)'|^2 = a_0^2 x^{-2\nu} |\tau|^{-1-2b} \frac{2^{2\nu} \left[ (1 - 2b) \Gamma(\nu) - 2\Gamma(\nu + 1) \right]^2}{16\pi}, \quad (A.11) \]

From these follow the low-\( k \) expansions of \( \rho_k \) and \( p_k \) in Eq.(40) and Eq.(41).

**B The 0th-, 2nd-, and 4th-order adiabatic subtraction terms**

The WKB approximate solution [17,18,24–26,36] of the massless scalar field equation (3) is written as the following

\[ v_k^{(n)}(\tau) = (2W_k(\tau))^{-1/2} \exp \left[ -i \int_{\tau'}^\tau W_k(\tau') d\tau' \right], \quad (B.1) \]

where the effective frequency is

\[ W_k(\tau) = \left[ k^2 + (\xi - \frac{1}{6}) a^2 R - \frac{1}{2} \left( \frac{W''_k}{W_k} - \frac{3}{2} \left( \frac{W'_k}{W_k} \right)^2 \right) \right]^{1/2}. \quad (B.2) \]

The WKB solution of \( W_k \) is obtained by iteratively solving (B.2) to a desired adiabatic order. Take the 0th-order [24],

\[ W_k^{(0)} = k, \quad (B.3) \]

and the 0th-order adiabatic mode

\[ v_k^{(0)}(\tau) = (2k)^{-1/2} e^{-i \int_{\tau'}^\tau k d\tau'}. \quad (B.4) \]

The 0th-order quantities that appear in the 0th-order subtraction terms are

\[ |v_k^{(0)}|^2 = \frac{1}{2W_k^{(0)}} = \frac{1}{2k}, \quad (B.5) \]

\[ |v_k^{(0)}'|^2 = \frac{1}{2} \left( \frac{(W_k^{(0)')^2}}{4(W_k^{(0)})^2} + W_k^{(0)} \right) = \frac{k}{2}, \quad (B.6) \]

\[ v_k^{(0)'}v_k^{(0)*} + v_k^{(0)}v_k^{(0)*'} = 0, \quad (B.7) \]

\[ |\left(\frac{v_k^{(0)}}{a}\right)'|^2 = \frac{1}{a^2} \left( |v_k^{(0)}'|^2 - \frac{a'}{a} (v_k^{(0)'}v_k^{(2)*'} + v_k^{(2)}v_k^{(0)*'}) + \frac{a'}{a} \right)^2 |v_k^{(0)}|^2 \]

\[ = \frac{1}{a^2} \left( \frac{k}{2} + \frac{a'}{a} \right)^2 \frac{1}{2k}. \quad (B.8) \]

These 0th-order subtraction terms are independent of \( \xi \). The 2nd-order adiabatic mode

\[ v_k^{(2)}(\tau) = (2W_k^{(2)}(\tau))^{-1/2} e^{-i \int_{\tau'}^\tau W_k^{(2)}(\tau') d\tau'}. \quad (B.9) \]
The 2nd-order effective frequency is given by first iteration of (B.2)

$$W_k^{(2)} = \left[k^2 + (\xi - \frac{1}{6})a^2R - \frac{1}{2} \left(\frac{W_k^{(0)}}{W_k^{(2)}} - \frac{3}{2} \left(\frac{W_k^{(2)}}{W_k^{(0)}}\right)^2\right)\right]^\frac{1}{2}.$$ \text{(B.10)}

Keeping only two time derivatives of \(a(\tau)\) gives

$$W_k^{(2)} = \sqrt{k^2 + 6(\xi - \frac{1}{6})\frac{a''}{a}} \approx k + 3(\xi - \frac{1}{6}) \frac{1}{k} a' \frac{a''}{a},$$ \text{(B.11)}

$$(W_k^{(2)})^{-1} \approx \frac{1}{k} - 3(\xi - \frac{1}{6}) \frac{a''/a}{k^3}.$$ \text{(B.12)}

The 2nd-order subtraction term for the power spectrum is

$$|v_k^{(2)}|^2 = \frac{1}{2W_k^{(2)}} = \frac{1}{2k} - \frac{3}{2}(\xi - \frac{1}{6}) \frac{a''/a}{k^3}.$$ \text{(B.13)}

The 2nd-order subtraction term for \(\rho_k\) and \(p_k\) also involves the following terms

$$|v_k^{(2)}|^2 = \frac{1}{2} \left(\frac{(W_k^{(2)})^2}{4(W_k^{(2)})^3} + W_k^{(2)}\right) = \frac{k}{2} + (\xi - \frac{1}{6}) \frac{3a''}{2ka},$$ \text{(B.14)}

$$v_k^{(2)} v_k^{(2)*} = - \frac{W_k^{(2)}}{2(W_k^{(2)})^2} \approx 0,$$ \text{(B.15)}

$$\left|\frac{v_k^{(2)}}{a}\right|^2 = \frac{1}{a^2} \left(|v_k^{(2)}|^2 - \frac{a'}{a} (v_k^{(2)} v_k^{(2)*} + v_k^{(2)} v_k^{(2)*}) + \left(a'\right)^2 |v_k^{(2)}|^2 \right)$$

$$= \frac{k}{2a^2} + \frac{1}{2ka^2} \left(a'\right)^2 + (\xi - \frac{1}{6}) \frac{3a''}{2ka^3}.$$ \text{(B.16)}

These 2nd-order subtraction terms dependent on \(\xi\). A very important property of a massless scalar field is that \(W_k^{(2)} = W_k^{(0)} = k\) for the conformal coupling \(\xi = \frac{1}{6}\). Actually, \(W_k^{(n)} = W_k^{(0)} = k\) holds for an arbitrary \(n\)-th-order, as is implied by the iteration formula (B.2).

The 4th-order adiabatic mode is defined by

$$v_k^{(4)}(\tau) = (2W_k^{(4)}(\tau))^{-1/2}e^{-i\int W_k^{(4)}(\tau)d\tau},$$ \text{(B.17)}

the 4th-order effective frequency is given by iteration

$$W_k^{(4)} = \left[k^2 + (\xi - \frac{1}{6})a^2R - \frac{1}{2} \left(\frac{W_k^{(2)}}{W_k^{(4)}} - \frac{3}{2} \left(\frac{W_k^{(4)}}{W_k^{(2)}}\right)^2\right)\right]^\frac{1}{2}.$$ \text{(B.18)}

Keeping up to four time derivatives, one obtains

$$W_k^{(4)} = k + \frac{3(\xi - \frac{1}{6}) a''}{k} a' - \frac{9}{2}(\xi - \frac{1}{6})^2 a''^2$$

$$- \left(\xi - \frac{1}{6}\right) \frac{3}{4k^3} \left(a''^2/a - a'^2/a^2 - \frac{2a''a'^2}{a^2} + \frac{2a'^2a''}{a^3}\right).$$ \text{(B.19)}
The 6th-order effective frequency is derived by iteration putting together yields

\[
(W_k^{(6)})^{-1} = \frac{1}{k} - 3(\xi - \frac{1}{6}\frac{1}{k^3}\frac{a''}{a}) + (\xi - \frac{1}{6})\frac{3}{4k^3}\left(\frac{a'''}{a} - \frac{a''^2}{a^2} + 2\frac{a'''a''}{a^3} - 2\frac{a'''a'}{a^2}\right) + (\xi - \frac{1}{6})^2\frac{27}{2k^5}\frac{a''^2}{a^2}.
\]

Then

\[
|v_k^{(4)}|^2 = (2W_k^{(4)})^{-1},
\]

\[
|v_k^{(4)}|^2 = \frac{k}{2} + (\xi - \frac{1}{6})\frac{3a''}{2ka} - (\xi - \frac{1}{6})^2\frac{9}{4k^3}\frac{a'''}{a^2} - (\xi - \frac{1}{6})\frac{3}{8k^3}\left(\frac{a''}{a} - \frac{a'''}{a^2} + 2\frac{a'''a'}{a^3} - \frac{2a''^2}{a^3}\right),
\]

\[
(v_k^{(4)},v_k^{(4)*},v_k^{(4)},v_k^{(4)*}) = -\frac{W_k^{(4)}}{2(W_k^{(4)})^2} = -\frac{3}{2}(\xi - \frac{1}{6})(\frac{a'''}{k^3a} - \frac{a'''a''}{k^3a^2}),
\]

putting together yields

\[
|v_k^{(4)}| = \frac{1}{a^2}\left[|v_k^{(4)}|^2 + \frac{a'}{a}(v_k^{(4)}v_k^{(4)*} + v_k^{(4)}v_k^{(4)*'}) + \left(\frac{a'}{a}\right)^2|v_k^{(4)}|^2\right] = \frac{1}{a^2}\left[\frac{k}{2} + \frac{a'}{2a} + (\xi - \frac{1}{6})\frac{3a''}{2ka} - (\xi - \frac{1}{6})^2\frac{9}{4k^3}\frac{a'''}{a^2} - \frac{3}{8k^3}\left(\frac{a''}{a} - \frac{a'''}{a^2} - \frac{6a'a'''}{a^2} + \frac{10a''^2a''}{a^3}\right)\right].
\]

These 4th-order subtraction terms also dependent on ξ. The portions up to two time derivatives of the above reduce to the 2nd-order results.

Similarly, the 6th-order adiabatic mode

\[
v_k^{(6)}(\tau) = (2W_k^{(4)}(\tau))^{-1/2}e^{-i\int W_k^{(4)}(\tau')d\tau'}.
\]

The 6th-order effective frequency is derived by iteration

\[
W_k^{(6)} = k^2 + (\xi - \frac{1}{6})a^2R - \frac{1}{2}\left(\frac{W_k^{(4)'}}{W_k^{(4)}} - \frac{3}{2}\left(\frac{W_k^{(4)}}{W_k^{(4)}}\right)^2\right)^{1/2}.
\]

Keeping up to six time derivatives, one obtains

\[
W_k^{(6)} = k + \frac{3(\xi - \frac{1}{6})a''}{a} - \frac{9}{2}(\xi - \frac{1}{6})^2\frac{a'''}{a^2k^5} - (\xi - \frac{1}{6})\frac{3}{4k^3}\left(\frac{a'''}{a} - \frac{a''^2}{a^2} + 2\frac{a'''a''}{a^3} + 2\frac{a''^2}{a^3}\right) + \frac{27}{2}(\xi - \frac{1}{6})^3\frac{a'''^2}{a^3k^5} + \frac{(\xi - \frac{1}{6})^2}{k^5}\left(\frac{45a'''}{a^2} - \frac{27a'''^2}{a^2} + \frac{27a''^2a''}{a^2} + \frac{153a''^2a''}{8a^4} - \frac{99a''a'''a''}{4a^3}\right) + \frac{(\xi - \frac{1}{6})}{k^5}\left(\frac{3a'''}{4a^2} + \frac{3a''^2}{8a^5} - \frac{3a''^2a'}/a^2 - \frac{21a''^2a''}{16a^3} + \frac{9a''^2a'^2}{4a^3} - \frac{9a''a''^2}{2a^4} + \frac{9a''a''^2}{4a^4} + \frac{6a''a'a''}{a^3}\right).
\]
and

\[
(W_k^{(6)})^{-1} = \frac{1}{k} - \frac{3(\xi - \frac{1}{6})a''}{k^3} + \frac{3(\xi - \frac{1}{6})}{4k^5} \left( \frac{a''' - a'''}{a''} + \frac{2a'^2a'' - 2a'a'''}{a^2} \right) + \frac{27(\xi - \frac{1}{6})^2a''}{2k^4} \]

\[
- \frac{135(\xi - \frac{1}{6})^3a'^3}{2k^7a^3} + \frac{(\xi - \frac{1}{6})^2}{k^7} \left( - \frac{45a'''}{8a^2} + \frac{45a'^3}{4a^3} - \frac{45a'''a'''}{4a^2} - \frac{225a^2a''}{8a^4} \right) + \frac{135a'''a''a'''}{4a^3} \]

\[
- \frac{9a'''}{2a^4} + \frac{9a''a'}{2a^5} + \frac{27a^2a'''}{4a^4} - \frac{6a'a''}{a^3}. \tag{B.28}
\]

Then

\[
|v_k^{(6)}|^2 = (2W_k^{(6)})^{-1}, \tag{B.29}
\]

\[
|v_k^{(6)}|^2 = \frac{1}{2} \left( \frac{(W_k^{(6)})'}{4W_k^{(6)}} + W_k^{(6)} \right)
\]

\[
= \frac{k}{2} + \frac{3(\xi - \frac{1}{6})a''}{2ka} - \frac{9(\xi - \frac{1}{6})^2a'''}{4k^3a^2} - \frac{3(\xi - \frac{1}{6})}{8k^3} \left( \frac{a''' - a'''}{a''} - \frac{2a'^2a'' - 2a'a'''}{a^2} \right) \]

\[
+ \frac{27(\xi - \frac{1}{6})^2a'^3}{4k^5a^3} + \frac{(\xi - \frac{1}{6})^2}{k^5} \left( \frac{63a'''}{16a^2} - \frac{27a'^3}{8a^3} + \frac{27a'''a'''}{8a^2} - \frac{171a^2a'''}{16a^4} - \frac{117a'''a'a'''}{8a^2} \right) \]

\[
+ \frac{(\xi - \frac{1}{6})}{k^5} \left( \frac{3a''' - 3a'''}{8a^2} - \frac{9a'^3}{16a^3} - \frac{3a'''a'}{8a^2} - \frac{21a'''a''}{32a^2} + \frac{9a'''a'''}{8a^3} - \frac{9a'''}{4a^4} \right) \tag{B.30}
\]

\[
(v_k^{(6)})^* v_k^{(6)} = \frac{k}{2} - \frac{3(\xi - \frac{1}{6})a''}{2ka} \left( \frac{a'''}{a''} - \frac{a'a'''}{a^2} \right) + \frac{(\xi - \frac{1}{6})}{k^5} \left( \frac{27a'''}{2a^2} - \frac{27a'''}{2a^2} \right) \]

\[
+ \frac{(\xi - \frac{1}{6})}{8a^2} \left( \frac{3a''' - 9a'''}{2a^2} + \frac{9a'''}{4a^3} - \frac{9a'''}{4a^2} + \frac{9a'''}{4a^3} \right). \tag{B.31}
\]

Putting together yields

\[
\left| \frac{v_k^{(6)}}{a} \right|^2 = \frac{1}{a^2} \left[ |v_k^{(6)}|^2 - \frac{a'}{a} (v_k^{(6)} v_k^{(6)*} + v_k^{(6)} v_k^{(6)*}) + (\frac{a'}{a})^2 |v_k^{(6)}|^2 \right]
\]

\[
= \frac{1}{a^2} \left[ \frac{k}{2} + \frac{a'^2}{2ka^2} + \frac{3(\xi - \frac{1}{6})a''}{2ka} - \frac{9(\xi - \frac{1}{6})^2a'''}{4k^3a^2} - \frac{3(\xi - \frac{1}{6})}{8k^3} \left( \frac{a''' - a'''}{a''} - \frac{2a'^2a'' - 2a'a'''}{a^2} - \frac{6a'a''}{a^2} \right) \right]
\]

\[
+ \frac{10a'^2a'''}{a^3} + \frac{27(\xi - \frac{1}{6})^3a'^3}{4k^5a^3} + \frac{(\xi - \frac{1}{6})^2}{k^5} \left( \frac{63a'''}{16a^2} - \frac{27a'^3}{8a^3} + \frac{27a'''a'''}{8a^2} + \frac{495a^2a'''}{16a^4} \right) \]

\[
- \frac{225a'''a'a'''}{8a^3} + \frac{(\xi - \frac{1}{6})}{k^5} \left( \frac{3a''' - 3a'''}{32a^2} - \frac{9a'^3}{16a^3} - \frac{3a'''a'}{8a^2} + \frac{21a'''a''}{32a^2} + \frac{21a'''a'''}{8a^3} \right) \tag{B.32}
\]
The portions up to four time derivatives of the above reduce to the 4th-order results.

The 0th-order subtraction term for spectral energy density and pressure

$$\rho_{kA0} = \frac{k^3}{4\pi^2a^4} \left[ |v_k^{(0)}|^2 + k^2 |v_k^{(0)}|^2 \right] + (6\xi - 1) \left( \frac{a'}{a} v_k^{(0)} v_k^{(0)*} + \frac{a'}{a} v_k^{(0)} v_k^{(0)*} \right) - \frac{a'^2}{a^2} |v_k^{(0)}|^2 \right] = \frac{k^4}{4\pi^2a^4}, \quad (B.33)$$

$$p_{kA0} = \frac{k^3}{4\pi^2a^4} \left[ \frac{1}{3} |v_k^{(0)}|^2 + \frac{1}{3} k^2 |v_k^{(0)}|^2 + 2(\xi - \frac{1}{6}) \left( -2|v_k^{(0)}|^2 + 3 \frac{a'}{a} v_k^{(0)} v_k^{(0)*} + v_k^{(0)} v_k^{(0)*} \right) \right] - \frac{3(\frac{a'}{a})^2 |v_k^{(0)}|^2 + 2k^2 |v_k^{(0)}|^2 + 12\xi \frac{a''}{a} |v_k^{(0)}|^2 \right] = \frac{k^4}{12\pi^2a^4}, \quad (B.34)$$

which are independent of $\xi$.

The 2nd-order subtraction term for the spectral energy density and pressure

$$\rho_{kA2} = \frac{k^4}{4\pi^2a^4} \left[ 1 - (\xi - \frac{1}{6}) \frac{3 a'^2}{k^2 a^2} \right] = \frac{1}{4\pi^2a^4} \left[ x^4 - \frac{(6\xi - 1)b^2 x^2}{2} \right], \quad (B.35)$$

$$p_{kA2} = \frac{k^4}{12\pi^2a^4} \left[ 1 + (\xi - \frac{1}{6}) \frac{1}{k^2} \left( 6 \frac{a''}{a} - 9 \frac{a'^2}{a^2} \right) \right] = \frac{1}{4\pi^2a^4} \left[ x^4 - \frac{(6\xi - 1)b(b + 2)x^2}{6} \right], \quad (B.36)$$

which depend on $\xi$. It is important that the derivatives are kept only up to second order in these 2nd-order subtraction terms, and this will ensure the covariant conservation to the 2nd adiabatic order.

The 4th-order subtraction term for the spectral energy density and pressure

$$\rho_{kA4} = \frac{k^4}{4\pi^2a^4} \left[ 1 - (\xi - \frac{1}{6}) \frac{3 a'^2}{k^2 a^2} - (\xi - \frac{1}{6})^2 \frac{9}{2k^4} \left( 2 \frac{a'' a'}{a^2} - \frac{a'^2}{a^2} - 4 \frac{a'' a'^2}{a^3} \right) \right] = \frac{1}{4\pi^2a^4} \left[ x^4 - \frac{(6\xi - 1)b^2 x^2}{2} + \frac{3(6\xi - 1)^2(b - 1)b^2(b + 1)}{8} \right], \quad (B.37)$$

$$p_{kA4} = \frac{k^4}{12\pi^2a^4} \left[ 1 + (\xi - \frac{1}{6}) \frac{1}{k^2} \left( 6 \frac{a''}{a} - 9 \frac{a'^2}{a^2} \right) \right. \left. + (\xi - \frac{1}{6})^2 \frac{9}{2k^4} \left( 2 \frac{a'' a'}{a^2} - \frac{10 a'' a'}{a^2} - \frac{5 a'^2}{a^2} + \frac{16 a'' a'^2}{a^3} \right) \right] = \frac{1}{4\pi^2a^4} \left[ x^4 - \frac{(6\xi - 1)b(b + 2)x^2}{6} + \frac{(6\xi - 1)^2(b - 1)b(b + 1)(b + 4)}{8} \right]. \quad (B.38)$$
The 6th-order subtraction term for spectral energy density and pressure

\[
\rho_{k,6} = \frac{k^4}{4\pi^2a^4} \left[ 1 - \frac{3(\zeta - \frac{1}{6})a'^2}{k^2a^2} - \frac{9(\zeta - \frac{1}{6})^2}{2k^4} \left( \frac{2a''a'}{a^2} - \frac{a''^2}{a^2} - \frac{4a'^2}{3a^2} \right) \right. \\
\left. + \frac{(\xi - \frac{1}{6})^3}{k^6} \left( - \frac{27a'^3}{a^3} - \frac{243a'^2a'^2}{2a^4} + \frac{81a''a'a''}{3a^3} \right) + \frac{(\xi - \frac{1}{6})^2}{k^6} \left( \frac{9a''^2}{8a^2} + \frac{9a'^2}{4a^3} + \frac{9a''a'}{4a^2} \right) - \frac{9a''a''}{4a^2} \right. \\
\left. - \frac{9a''a''}{a^2} + \frac{18a''a'^2}{a^4} - \frac{18a'^4a''}{a^5} + \frac{99a'^2a'^2}{8a^4} - \frac{27a''a'a''}{4a^2} \right] \\
= \frac{1}{4\pi^2a^4r^4} \left[ x^4 - \frac{(6\xi - 1)b^2x^2}{2} + \frac{3(6\xi - 1)^2(b - 1)b^2(b + 1)}{8} \\
\left. + \frac{5(6\xi - 1)^2(b - 1)b^2(b + 2)[(1 - 6\xi)(b^2 - b) - 2]}{16x^2} \right] \tag{B.39}
\]

which is the first four terms of \( \rho_k \) in (29).

\[
\rho_{k,6} = \frac{k^4}{12\pi^2a^4} \left[ 1 + \frac{(\xi - \frac{1}{6})}{k^2} \left( \frac{6a''}{a} - \frac{9a'^2}{a^2} \right) \right. \\
\left. + \frac{9(\xi - \frac{1}{6})}{2k^4} \left( \frac{2a''a'}{a^2} - \frac{10a''a'}{a^2} - \frac{5a'^2}{a^3} \right) \right. \\
\left. + \frac{(\xi - \frac{1}{6})^3}{k^6} \left( - \frac{81a'^2}{a^2} + \frac{135a'^2}{a^3} - \frac{81a''a''}{a^2} - \frac{1215a'^2a'^2}{2a^4} + \frac{567a''a'a''}{a^3} \right) \\
\left. + \frac{(\xi - \frac{1}{6})^2}{k^6} \left( - \frac{9a''a''}{4a} + \frac{81a'^2}{8a^2} - \frac{63a'^2}{8a^3} + \frac{63a''a''}{4a^2} + \frac{18a''a''}{a^2} - \frac{54a''a'^2}{a^3} \right) \\
\left. + \frac{108a'^2a'^3}{a^4} - \frac{108a'^2a'^3}{a^5} + \frac{1071a'^2a'^2}{8a^4} - \frac{423a''a'a''}{4a^3} \right] \\
= \frac{1}{4\pi^2a^4r^4} \left[ \frac{x^4}{3} - \frac{(6\xi - 1)b(b + 2)x^2}{6} + \frac{(6\xi - 1)^2(b - 1)b(b + 1)(b + 4)}{8} \\
\left. + \frac{5(6\xi - 1)^2(b - 1)b(b + 2)(b + 6)[(1 - 6\xi)(b^2 - b) - 2]}{48x^2} \right] \tag{B.40}
\]

which is the first four terms of \( p_k \) in (30).

As we have mentioned earlier, for the conformally-coupling \( \xi = \frac{1}{6} \), one has

\[
(W_k^{(0)})^{-1} = (W_k^{(2)})^{-1} = (W_k^{(4)})^{-1} = (W_k^{(6)})^{-1} = \frac{1}{k}, \tag{B.41}
\]

\[
|v_k^{(0)}|^2 = |v_k^{(2)}|^2 = |v_k^{(4)}|^2 = |v_k^{(6)}|^2 = \frac{1}{2k^2}, \tag{B.42}
\]

\[
\rho_{k,A0} = \rho_{k,A2} = \rho_{k,A4} = \rho_{k,A6}; \tag{B.43}
\]

\[
p_{k,A0} = p_{k,A2} = p_{k,A4} = p_{k,A6}; \tag{B.44}
\]

so that the 0th-, 2nd-, 4th- and 6th-order subtraction terms are all equal. Inspection of iteration (B.2) tells that, for \( \xi = \frac{1}{6} \), the subtraction terms of any order are the same as (B.41)–(B.44).

Now we show that the four-divergence of the subtraction terms of the stress tensor is zero at each adiabatic order

\[
\langle T_{\mu\nu\rho\sigma} \rangle_{A_n} = 0. \tag{B.45}
\]
Time derivative of Eq. (B.39) gives
\[
\rho_{kA6}' = \frac{1}{16\pi^2 k^2 a_10} a' \left[ 648(\xi - \frac{1}{6})^2 a'^4 a'' - 3(\xi - \frac{1}{6}) a^4 \left( -3(\xi - \frac{1}{6}) a^{'''} + 12k^2(\xi - \frac{1}{6}) a^{''''} + 8k^4 a'' \right) \\
+ 36(\xi - \frac{1}{6})^2 a a'^2 \left( 108(\xi - \frac{1}{6}) - 19 \right) a''^2 - 18a''' a' \\
- 36(\xi - \frac{1}{6})^2 a^2 \left( 6(\xi - \frac{1}{6}) - 1 \right) a'^3 + a^2 \left( 14k^2 a'' - 9a''' \right) + 2(45(\xi - \frac{1}{6}) - 7) a''' a' a'' \\
+ 9(\xi - \frac{1}{6}) a^3 \left( 8k^4 a'^2 + 2(\xi - \frac{1}{6}) (16k^2 a''' - 5a''') a' \\
+ (\xi - \frac{1}{6}) \left( 6(6(\xi - \frac{1}{6}) - 1) a''^2 + 4k^2 a''^2 + (36(\xi - \frac{1}{6}) - 5) a''' a'' \right) \right) - 16k^6 a^5 \right],
\]
(B.46)
and combining (B.39) and (B.40) gives
\[
3\frac{a'}{a} (\rho_{kA6} + p_{kA6}) = -\rho_{kA6}', \quad (B.47)
\]
so that the four-divergence of the 6th-order subtraction terms is zero
\[
\rho_{kA6}' + 3\frac{a'}{a} (\rho_{kA6} + p_{kA6}) = 0. \quad (B.48)
\]
Similarly, it is checked that
\[
\rho_{kA n}' + 3\frac{a'}{a} (\rho_{kA n} + p_{kA n}) = 0 \quad (B.49)
\]
is valid for \( n = 0, 2, 4 \). Hence, the covariant conservation
\[
\langle T^{\mu\nu(n)} \rangle_{\text{reg}} = \langle T^{\mu\nu} \rangle - \langle T^{\mu\nu} \rangle_{A n} = 0 \quad (B.50)
\]
is respected by the regularized stress tensor at each order we need in this paper.