THE PROBABILISTIC CAUCHY PROBLEM FOR THE FOURTH ORDER SCHRÖDINGER EQUATION WITH SPECIAL DERIVATIVE NONLINEARITIES

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Abstract. Chen and Zhang [7] consider the probabilistic Cauchy problem of the fourth order Schrödinger equation
\[(i\partial_t + \varepsilon \Delta + \Delta^2)u = P_m((\partial^\alpha_x u)_{|\alpha| \leq 2}), \quad m \geq 3,\]
where \(P_m\) is a homogeneous polynomial of degree \(m\). The almost sure local well-posedness and small data global existence were obtained in \(H^s(\mathbb{R}^d)\) with the regularity threshold \(s_c - 1/2\) when \(d \geq 3\), where \(s_c := d/2 - 2/(m - 1)\) is the scaling critical regularity. For the lower regularity threshold \((d - 1)s_c/d\) with \(m = 2\) and \(s_c = \min\{1, d/4\}\) with \(m \geq 3\), we get the corresponding well-posedness of the following fourth order nonlinear Schrödinger equation
\[(i\partial_t + \varepsilon \Delta + \Delta^2)u = P_m((\partial^\alpha_x u)_{|\alpha| \leq 2}), \quad m \geq 2\]
on \(\mathbb{R}^d\) \((d \geq 2)\) with random initial data.

1. Introduction. We consider the Cauchy problem for the fourth order nonlinear Schrödinger equation with special derivative nonlinearities (4NLS) on \(\mathbb{R}^d\), \(d \geq 2\):
\[
\begin{cases}
(i\partial_t + \varepsilon \Delta + \Delta^2)u = P_m((\partial^\alpha_x u)_{|\alpha| \leq 2}), \\
u(0, x) = \phi(x) \in H^s(\mathbb{R}^d),
\end{cases}
\]
where \(\varepsilon \in \{-1, 0, 1\}\), \(u\) is a complex-valued function of \((t, x) \in \mathbb{R} \times \mathbb{R}^d\), \(\Delta = \partial_x^2_1 + \cdots + \partial_x^2_d\), \(\partial^\alpha_x = \partial_x^{\alpha_1} \cdots \partial_x^{\alpha_d}\), \(|\alpha| = \alpha_1 + \cdots + \alpha_d\) and \(P_m((\partial^\alpha_x u)_{|\alpha| \leq 2})\) is the following homogeneous polynomial of degree \(m \geq 2\)
\[
c_1 u^m + c_2 \sum_{i=1}^d (\partial_x^2_i u) u^{m-1} + c_3 \sum_{i,j=1}^d (\partial_x^i u \partial_x^j u) u^{m-2} + c_4 \sum_{i,j=1}^d (\partial_x^2_{i,j} u) u^{m-1}. \tag{1.2}
\]

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When $\varepsilon = 0$ and $c_1 = c_2 = 0$, 4NLS (1.1) is invariant under the scaling transformation

$$u_\mu(t, x) := \mu^{-2/(m-1)}u(\mu^{-4}t, \mu^{-1}x),$$

which gives rise to the scaling invariant critical regularity $s_c := d/2 - 2/(m-1)$ of (1.1) (see [7] for more details). The well-posedness of (1.1) in $H^s$ is called supercritical, critical or subcritical depending on whether $s < s_c$, $s = s_c$, or $s > s_c$, respectively.

We establish the almost sure local well-posedness and small data global existence of (1.1) for random initial data in $H^s$ with $s \in (s_d,m \vee 0, s_c]$, where

$$s_{d,m} = \begin{cases} \frac{(d-1)(s_c-1)}{d}, & m = 2; \\ s_c - \min\{1, \frac{d}{4}\}, & m \geq 3 \end{cases} \quad (1.3)$$

and $d, m$ satisfy $s_c := \frac{d}{2} - \frac{2}{m-1} > 0$.

**Remark 1.** Let $d, m \geq 2$ satisfy $s_c = d/2 - 2/(m-1) > 0$. If $m = 2$, then $d \geq 5$. If $m = 3$, then $d \geq 3$. If $m \geq 4$, then $d \geq 2$.

The 4NLS, including its special forms, arises in deep water wave dynamics, plasma physics, optical communications, cf. [11, 18, 19, 20]. The well-posedness of 4NLS with different nonlinearities was widely studied in scaling critical or supercritical regime by several authors, cf. [13, 17, 23, 26, 27]. 4NLS (1.1) is ill-posed below the scaling critical regularity in some cases, cf. [9, 24].

Recently, there are many researches, using probabilistic tools to study nonlinear dispersive equations in scaling supercritical regimes, for example [2, 3, 4, 5, 6, 7, 10, 15, 16, 21]. Bényi-Oh-Pocovnicu [2] and Lührmann-Mendelson [21] introduced randomization on $\mathbb{R}^d$, independently. Thus, it is possible to establish supercritical well-posedness of (1.1) with suitable random initial data. Hirayama and Okamoto [15] studied random data Cauchy problem for 4NLS $i\partial_t u + \Delta^2 u = \pm \partial(|u|^2 u)$ below the scaling critical regularity. Via Strichartz estimates and bilinear estimates, Chen and Zhang [7] investigated the random data Cauchy problem of 4NLS with nonlinearities containing the second order derivatives in $H^s(\mathbb{R}^d)$. The regularity threshold in [7] is $s_c - 1/2$ when $d \geq 3$.

It seems that the decay estimates (2.6) are only available for $u$. In this paper, the nonlinearities of (1.1) have special structure which can cancel out the worse interaction such that the decay estimates (2.6) are available for (1.1). Thus, we expect to lower the regularity threshold of the random data Cauchy problem of (1.1).

Applying the decay estimates (2.6), Strichartz estimates and the truncated bilinear estimates exploited by us, we establish the almost sure supercritical well-posedness of (1.1) in $H^s(\mathbb{R}^d)$ for $s \in (s_{d,m}, s_c]$, where $s_{d,m}$ is in (1.3). Obviously, $s_{d,m}$ is smaller than the regularity threshold in [7]. Because the singularity near the frequency $0$ occurs in Strichartz estimates of (1.1) when $\varepsilon = 0, 1$, we split the main proof into the low and high frequency sections. We also use symmetries method that we exploited in [7] to simplify our main proof.

### 1.1 Randomization procedure

We give the randomization of initial data based on the uniform decomposition of the frequency space. In [29], Wang, Zhao and Guo first applied the frequency uniform decomposition operators $\psi(D - n)$ to study nonlinear evolution equations with initial data in modulation spaces which contain
a class of super-critical data in Sobolev spaces, where $\psi \in S(\mathbb{R}^d)$ satisfies supp $\psi \subset [-1, 1]^d$ and
$$\sum_{n \in \mathbb{Z}^d} \psi(\xi - n) = 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$ Let $\phi \in H^s$ be a complex-valued function for some $s \in \mathbb{R}$. Wiener randomization of $\phi$ is defined as follows
$$\phi^\omega = \sum_{n \in \mathbb{Z}^d} g_n(\omega) \psi(D - n)\phi.$$ Here, $\{g_n(\omega)\}_{n \in \mathbb{Z}^d}$ is a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, F, P)$. The real and imaginary parts of $g_n(\omega)$ are independent and endowed with probability distributions $\mu_1^\omega$ and $\mu_2^\omega$, respectively. Throughout this paper, we assume that there exists $c > 0$ such that
$$\left| \int_{\mathbb{R}^d} e^{\gamma x} d\mu_j^\omega(x) \right| \leq e^{|\gamma|^2}$$ for any $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. This condition is satisfied by the standard complex-valued Gaussian random variables and the standard Bernoulli random variables.

It follows from Bernstein’s inequality that
$$\|\psi(D - n)\phi\|_{L^r_\omega} \lesssim \|\psi(D - n)\phi\|_{L^\infty_\omega}, \quad 1 \leq r \leq \tilde{r} \leq \infty,$$ where the implicit constant is independent of $n \in \mathbb{Z}^d$. We can see that there is no regularity increase for the Bernstein estimate of $\psi(D - n)\phi$ (cf. [28]). Thus, the frequency uniform decomposition can improve space integrability and then improve the Strichartz-type estimates of random data (1.4) in a way. Moreover, by [5, Lemma 3.1], there exists $C > 0$ such that
$$\left\| \sum_{n \in \mathbb{Z}^d} g_n(\omega)c_n \right\|_{L^p(\Omega)} \leq C\sqrt{p}\|c_n\|_{l^2(\mathbb{Z}^d)}, \quad p \geq 2,$$ which means that the summation of $g_n(\omega)c_n$ in $L^p$ can be controlled by the $l^2$-norm of $\{c_n\}_{n \in \mathbb{Z}^d}$ (Since $l^1 \subset l^2$, (1.6) has gained some regularity). These two points are crucial for us to consider the supercritical random data.

1.2. Main results. The almost sure local well-posedness of (1.1) for random initial data reads as follows.

**Theorem 1.1.** Let $d, m \geq 2$, $s_c := d/2 - 2/(m - 1) > 0$ and $s \in (s_{d,m}, s_c]$, where $s_{d,m}$ is defined in (1.3). Given $\phi \in H^s$, let $\phi^\omega$ be its randomization defined in (1.4). Then for each $0 < T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:

(i) $P(\Omega_T) \leq C \exp \left[ -c(T^{-\gamma}\|\phi\|_{H^s}^2) \right]$ for some $C$, $c$, $\gamma > 0$.

(ii) For each $\omega \in \Omega_T$, there exists a unique solution $u$ to (1.1) with $u|_{t=0} = \phi^\omega$ in the class
$$S(t)\phi^\omega + C([-T, T]; H^s(\mathbb{R}^d)) \subset C([-T, T]; H^s(\mathbb{R}^d)).$$

The next theorem states almost sure global well-posedness and scattering of (1.1) for random small data.

**Theorem 1.2.** Let $d, m \geq 2$, $s_c := d/2 - 2/(m - 1) > 0$ and $s \in (s_{d,m}, s_c]$, where $s_{d,m}$ is defined in (1.3). Given $\phi \in H^s$, let $\phi^\omega$ be its randomization defined in (1.4). If $\varepsilon = -1$, there exists $C$, $c > 0$ and a set $\Omega_{\phi} \subset \Omega$ with the following properties:

(i) $P(\Omega_{\phi}) \leq C \exp \left( -c/\|\phi\|_{H^s}^2 \right) \to 0$ as $\|\phi\|_{H^s} \to 0$. 


estimates are showed in Section 3. In Section 4, we establish Theorems 1.1 and 1.2.

(ii) For each $\omega \in \Omega_\phi$, there exists a unique global solution $u$ to 4NLS (1.1) with $u_{|t=0} = \phi^\omega$ in the class

$$S(t)\phi^\omega + C(\mathbb{R}; H^{s_c}(\mathbb{R}^d)) \subset C(\mathbb{R}; H^{s}(\mathbb{R}^d)).$$

(iii) For each $\omega \in \Omega_\phi$, there exists $v^\omega_\pm \in H^{s_c}(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm \infty} \|u - S(t)(\phi^\omega + v^\omega_\pm)\|_{H^{s_c}} = 0. \quad (1.7)$$

If $\varepsilon = 1$, we have the same results (i) and (ii) on the time interval $[-T, T], T < \infty$. If $\varepsilon = 0$, the same results (i), (ii) and (iii) also hold on $\mathbb{R}$ when $d/2 - 4/(m - 1) \geq 0$.

Remark 2. (i) If $c_1 = c_2 = 0$ or $c_1 = 0$, $d/2 - 3/(m - 1) > 0$ or $T < \infty$ for the nonlinearities (1.2) of (1.1), then the condition $d/2 - 4/(m - 1) \geq 0$ is not necessary for $\varepsilon = 0$ in Theorem 1.2.

(ii) $d/2 - 4/(m - 1)$ is the scaling critical regularity of (1.1) for $\varepsilon = 0$ with no derivative nonlinearities, i.e., $c_2 = c_3 = c_4 = 0$ in (1.2).

(iii) $d/2 - 3/(m - 1)$ is the scaling critical regularity of (1.1) for $\varepsilon = 0$ with the first order derivative nonlinearities, i.e., $c_1 = c_3 = c_4 = 0$ in (1.2).

1.3. Notations. Let $\mathbb{N}$ be positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Denote $C > 0$ a constant that depends only on fixed parameters which can be different at different places. Write $a \lesssim b$ if $a \leq Cb$, and analogous for $a \gtrsim b$. We use the notation $a \sim b$ if $a \lesssim b \lesssim a$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^d)$, Sobolev Spaces $H^s := (1 - \Delta)^{-s/2}L^2$, and homogeneous Sobolev Spaces $H^s := |\nabla|^{-s}L^2$. Denote $c(\delta)$ a function tends to 0 as $\delta \to 0$ which can be different at different places. We use $\partial$ to present a first order derivative with respect to the spatial variable, for example a linear combination of $\partial_{\varepsilon_i}, i = 1, \cdots, d$ or $|\nabla|$. We put $S(t) = e^{it(\Delta^2 + \varepsilon \Delta)}$ for $\varepsilon \in \{-1, 0, 1\}$ and

$$N_m(u) := P_m((\partial^m_\varepsilon \pi)_{|\varepsilon| \leq 2}), \quad \Gamma N_m(u) := \int_0^t S(t-s)N_m(u)(s,x)ds. \quad (1.8)$$

Let $\varphi \in C_0^\infty((-2,2))$ be a smooth radial function such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $0 \leq \varphi(\xi) \leq 1$ for $1 \leq |\xi| \leq 2$. Given a dyadic number $N \in 2^{\mathbb{N}_0}$, let $\varphi_1(\xi) := \varphi(\xi)$ and $\varphi_N(\xi) := \varphi(\xi/N) - \varphi(2\xi/N)$ for $N \geq 2$. We define

$$P_N f(x) := F^{-1} \varphi_N Ff.$$ Put $P_{\leq M} := \sum_{N \leq M} P_N$, $P_{> M} := \sum_{N > M} P_N$. Set $\tilde{P}_N = P_{N/2} + P_N + P_{2N}$ (if $N_1 = 1$, we assume $P_{N_1/2} = 0$), then

$$\tilde{P}_N P_N = P_N. \quad (1.9)$$

We also define

$$Q_N f := F^{-1} \varphi_N (\tau - |\xi|^4 + \varepsilon |\xi|^2) Ff(\tau, \xi). \quad (1.10)$$

Let $Q_{\leq M} := \sum_{N \leq M} Q_N$ and $Q_{> M} := \sum_{N > M} Q_N$.

For $K \in 2^\mathbb{N}$, let $\varphi_K(\xi) := \varphi(\xi/K) - \varphi(2\xi/K)$ and $\hat{P}_K f(x) := F^{-1} \varphi_K Ff$. For $1 \leq q < \infty$, define

$$\|u(t, x)\|_{L^q_t L^2_x} := \sup_K K^\frac{q}{4} \|\hat{P}_K u\|_{L^q_t L^2_x}.$$ 1.4. Organization of the paper. In Section 2, we summarize some lemmata for the proof of the main nonlinear estimates. Consequently, the main nonlinear estimates are showed in Section 3. In Section 4, we establish Theorems 1.1 and 1.2.
2. Functional framework. We introduce the precise functional framework used in the proofs of Theorems 1.1 and 1.2. For $2 \leq q, r \leq \infty$, a pair $(q, r)$ is called biharmonic admissible and Strichartz admissible if

$$(q, r, d) \neq (2, \infty, 4), \quad \frac{4}{q} + \frac{2}{r} = \frac{d}{2} \quad \text{and} \quad (q, r, d) \neq (2, \infty, 2), \quad \frac{2}{q} + \frac{2}{r} = \frac{d}{2},$$

respectively.

We only present the improved Strichartz-type estimates for the free solution of (1.1) with random initial data. The Strichartz-type estimates are given in [7]. One can see [2, Proposition 2 and Lemma 4] and [15, Lemma 2.4] for the proof.

**Lemma 2.1** (Improved Strichartz-type estimates). Let $\varepsilon \in \{-1, 0, 1\}$, $I = [0, T]$. If $\varepsilon = 1$, suppose also $T < \infty$. Assume that $\phi \in L^2_x(\mathbb{R}^d)$ and $\phi^\omega$ is its randomization defined by (1.4). Then the following results hold:

(i) There exist $C, c > 0$ such that $P(||\phi^\omega||_{L^2_t L^r_x(I \times \mathbb{R}^d)} > \lambda) < C \exp \left( -c\lambda^2 / ||\phi||^2_{L^2_x} \right)$.

(ii) Let $(q, r)$ be biharmonic admissible with $q, r < \infty$ and $r \leq \tilde{r} < \infty$, then there exist $C, c > 0$ such that

$$P \left( ||\nabla^{2/3} S(t) \phi^\omega||_{L^q_t L^r_x(I \times \mathbb{R}^d)} > \lambda \right) < C \exp \left( -c\lambda^2 / ||\phi||^2_{L^2_x} \right).$$

(iii) Let $(q, r)$ be Strichartz admissible with $q, r < \infty$ and $r \leq \tilde{r} < \infty$, then there exist $C, c > 0$ such that

$$P \left( ||\nabla^{2/3} S(t) \phi^\omega||_{L^q_t L^r_x(I \times \mathbb{R}^d)} > \lambda \right) < C \exp \left( -c\lambda^2 / ||\phi||^2_{L^2_x} \right).$$

Specially, if $\varepsilon = -1$, we also have

$$P \left( ||\nabla^d S(t) \phi^\omega||_{L^q_t L^r_x(I \times \mathbb{R}^d)} > \lambda \right) < C \exp \left( -c\lambda^2 / ||\phi||^2_{L^2_x} \right).$$

We apply $U^p$, $V^p$ spaces to hold the solutions to (1.1). $U^p$, $V^p$ spaces serve as a development of the Bourgain spaces, and have been very effective in establishing well-posedness of various dispersive PDEs. More details can be seen in [12, 14].

The definition of $U^p$, $V^p$, $U^p_{-\infty, \infty}$, $V^p_{-\infty, \infty}$ and $V^p_{-\infty, r, \infty}$ are the same as the ones presented in [7], we omit the details. We give the definition of $Z^s$, $Y^s$ spaces.

**Definition 2.2.** For $s \in \mathbb{R}$, we define $Y^s$ and $Z^s$ as the closure of $C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V^2_{-\infty, \infty}$ and $C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U^2_{-\infty, \infty}$ with respect to the norm

$$||f||_{Y^s} := \left( \sum_{N \in \mathbb{Z}^d} N^{2s} ||P_N f||^2_{L^2_x} \right)^{1/2}, \quad ||f||_{Z^s} := \left( \sum_{N \in \mathbb{Z}^d} N^{2s} ||P_N f||^2_{U^2_x} \right)^{1/2},$$

respectively. Here, $P_N$ is acting on the space variable.

There are the following embeddings: for $p > 2$,

$$\langle \nabla \rangle^{-s} U^2_{-\infty} \hookrightarrow Z^s \hookrightarrow Y^s \hookrightarrow \langle \nabla \rangle^{-s} V^2_{-\infty, \infty} \hookrightarrow \langle \nabla \rangle^{-s} U^2_{-\infty} \hookrightarrow L^\infty(\mathbb{R}; H^s).$$

Given an interval $I \subset \mathbb{R}$, we define the local-in-time versions of these spaces as restriction norms. For example, we define the $Z^s(I)$-norm by

$$||u||_{Z^s(I)} = \inf \{ ||v||_{Z^s(\mathbb{R})} : v|_I = u \}.$$

The dual property of the $Z^s$ norm is as follows. One can refer to [14] for the proof.

**Lemma 2.3.** Given $s \geq 0$, let $\varepsilon \in \{-1, 0, 1\}$, $T > 0$. Then, we have:

(i) If $\phi \in L^2$, then $||S(t)\phi||_{Z^0} \leq ||\phi||_{L^2_x}$.
Lemma 2.4. Combining with Plancherel equality and the inequality (2.18). We first show (2.6). By the definition of Fourier transform and letting $N > M$, we have

\[ \| Q > M u \|_{L^q_t L^r_x(I \times \mathbb{R}^d)} \lesssim M^{-1/q} \| u \|_{V^q_\infty(I)}, \]

(2.6)

and

\[ \| Q > M u \|_{L^q_t L^r_x(I \times \mathbb{R}^d)} + \| Q \leq M u \|_{V^q_\infty(I)} \lesssim \| u \|_{V^q_\infty(I)}. \]

(2.7)

(ii) If $2 < q < \infty$ and $(q, r)$ is biharmonic admissible, then

\[ \| u \|_{L^q_t L^r_x(I \times \mathbb{R}^d)} \lesssim \| u \|_{V^q_\infty(I)} \lesssim \| u \|_{V^q_\infty(I)}. \]

(2.8)

(iii) If $2 < q < \infty$ and $(q, r)$ is Strichartz admissible, then

\[ \| \| \nabla \|^{2/q} u \|_{L^q_t L^{2q/r}_x(I \times \mathbb{R}^d)} \lesssim \| u \|_{V^q_\infty(I)} \lesssim \| u \|_{V^q_\infty(I)} \]

Specially, for $\varepsilon = -1$, we also have

\[ \| \| \nabla \|^{2/q} u \|_{L^q_t L^{2q/r}_x(I \times \mathbb{R}^d)} \lesssim \| u \|_{V^q_\infty(I)} \lesssim \| u \|_{V^q_\infty(I)}. \]

(2.10)

Proof. We consider (i). The idea is essential due to the proof of [12, Corollary 2.18]. We first show (2.6). By the definition of Fourier transform and letting $\tau = \tau_1 = \tau_1 - |\xi|^4 + \varepsilon |\xi|^2$

\[ Q_N u = F_{r, \xi}^{-1} \varphi_N (\tau - |\xi|^4 + \varepsilon |\xi|^2) F_{L, x} u = S(t) F_{r, \xi}^{-1} \varphi_N (\tau_1) F_{L, x} S(-t) u. \]

(2.11)

Combining with Plancherel equality and the inequality $\| v \|_{B^{1, \infty}_{\infty, \infty}} \lesssim \| v \|_{V^q}$ for any $v \in V^q, 1 < q < \infty$ (see e.g. [25, Example 9], pp. 167-168.), we have

\[ \| Q_N u \|_{L^q_t L^r_x} = \| F_{r, \xi}^{-1} \varphi_N (\tau_1) F_{L, x} S(-t) u \|_{L^q_t L^r_x} \lesssim N^{-\frac{1}{q}} \| F_{L, x} S(-t) u \|_{V^q} = N^{-\frac{1}{q}} \| u \|_{V^q}. \]

Making summation on $N > M$ gives (2.6).

Second we show (2.7). It suffices to show $\| Q_1 u \|_{V^q_\infty(I)} \lesssim \| u \|_{V^q_\infty(I)}$. Recall that

\[ \| v \|_{V^q} = \sup_{\{t_k\}_{k=0}^K \in \mathcal{P}} \left( \sum_{k=1}^K \| S(-t_k) v(t_k) - S(-t_{k-1}) v(t_{k-1}) \|_{L^2_t}^q \right)^{1/q}. \]

For any $\{t_k\}_{k=0}^K \in \mathcal{P}$, the equality (2.11), Plancherel equality and Minkowski’s inequality imply

\[ \left( \sum_{k=1}^K \| Q_1 u(t_k) - Q_1 u(t_{k-1}) \|_{L^2_t}^q \right)^{\frac{1}{q}} = \left( \sum_{k=1}^K \| F_{r, \xi}^{-1} \varphi * F_{L, x} S(-t_k) u(t_k) - F_{r, \xi}^{-1} \varphi * F_{L, x} S(-t_{k-1}) u(t_{k-1}) \|_{L^2_t}^q \right)^{\frac{1}{q}} \]

\[ \lesssim \left( \sum_{k=1}^K \left( \int_{\mathbb{R}} | F_{r, \xi}^{-1} \varphi(s) \| S(s - t_k) u(t_k - s) - S(s - t_{k-1}) u(t_{k-1} - s) \|_{L^2_t} ds \right)^q \right)^{\frac{1}{q}}. \]
where

\[ R \]

It seems that (2.6) and (2.7) only hold for \( R \). Let \( \tau \) and (2.6), (2.7) hold for \( \overline{\text{Lemma 2.5.}} \)

\[ Q \]

If \( \phi \) and \( \| \phi \|_V \leq \epsilon \), then \( \| Q_{\geq 1} u \|_{V^q} \leq \| S(t) u \|_{V^q} \). We can see the singularity occurs for Strichartz estimates of \( \epsilon = 0 \), if the frequency is near 0.

Exploring the frequency interaction, we get the following truncated bilinear estimates.

**Remark 3.** It seems that (2.6) and (2.7) only hold for \( u \). In fact, assume \( \phi(\xi) : \mathbb{R}^d \to \mathbb{R} \) and \( S(t) = \mathcal{F}^{-1} e^{i t \phi(\xi)} \). Let \( Q_{N} = \mathcal{F}_{\tau, \xi}^{-1} \phi_N(\tau - \phi(\xi)) \).

\[ \overline{\text{Lemma 2.5.}} \]

Similar to (2.11),

\[ \overline{\text{Remark 3.}} \]

If \( \phi \) is even, then

\[ \overline{\text{Remark 3.}} \]

and (2.6), (2.7) hold for \( \overline{u} \).

If \( \phi \) is even e.g. \( \phi(\xi) = |\xi|^4 - |\xi|^2 \), then

\[ \overline{\text{Remark 3.}} \]

It seems that we only have

\[ \| Q_{\geq 1} u \|_{L^2_t L_x^1} \leq M^{-1/q} \| S(t) u \|_{V^q} \] and \( \| Q_{\leq 1} u \|_{V^q} \leq \| S(t) u \|_{V^q} \), where \( S(t) u \in V^q \).

**Remark 4.** (i) By (2.11), we have

\[ Q_{\geq 1} S(t) \hat{u} = S(t) \mathcal{F}_{\tau, \xi}^{-1} \hat{\phi}_{\geq 1}(\tau) \mathcal{F}_{\xi} \hat{u} = S(t) \mathcal{F}_{\tau, \xi}^{-1} \phi_{\geq 1}(\tau) \mathcal{F}_{\xi} \phi \hat{u} = 0, \quad (2.12) \]

where \( \delta \) is Dirac function.

(ii) Strichartz estimates (2.9) give \( \| \hat{u} \|_{L_t^q L_x^\infty(\mathbb{R}^d)} \leq \| \nabla \|^{-2/4}_{L_t^q(\mathbb{R}^d)} \). We can see the singularity occurs for Strichartz estimates of \( \epsilon = 0 \), if the frequency is near 0.

Exploring the frequency interaction, we get the following truncated bilinear estimates.

**Lemma 2.5.** Let \( d \geq 2 \), \( \varepsilon \in \{ -1, 0, 1 \} \), \( I = [0, T] \). If \( \varepsilon = 0 \), \( d = 2 \) or \( \varepsilon = 1 \), we suppose additionally \( T < \infty \). Given \( N_1, N_2, N_3 \in 2^\mathbb{N}_0 \), \( N_1 \leq N_2 \) and \( N_2 > 1 \), let \( N_{\min} = \min \{ N_1, N_2, N_3 \} \), \( N_{\max} = \max \{ N_1, N_2, N_3 \} \).

(i) Given \( P_{N_1} \phi_1 \), \( P_{N_3} \phi_2 \in L^2(\mathbb{R}^d) \), then

\[ \| P_{N_1} \left( \frac{2}{\varepsilon} \right) S(t) P_{N_1} \phi_1 \|_{L^2_t L^2_x(\mathbb{R}^d)} \leq N_{\min}^{-\delta} \left( \frac{N_{\min}}{N_{\max}} \right) \frac{1}{\varepsilon^\delta} \sum_{i=1}^2 \| P_{N_i} \phi_i \|_{L^2(\mathbb{R}^d)} \] (2.13)

(ii) Given \( P_{N_1} u_1 \), \( P_{N_2} u_2 \in V^q_S(I) \), then

\[ \| P_{N_3} \left( \frac{2}{\varepsilon} \right) P_{N_3} u_3 \|_{L^2_t L^2_x(\mathbb{R}^d)} \leq N_{\min}^{-\delta} \left( \frac{N_{\min}}{N_{\max}} \right) \frac{1}{\varepsilon^\delta} \sum_{i=1}^2 \| P_{N_i} u_i \|_{V^q_S(I)}, \] (2.14)

where \( \delta > 0 \) can be arbitrarily small.
Proof of Lemma 2.5. We use I to represent
\[
\int \varphi_{N_1} \overline{g} (\xi_1 + \xi_2, |\xi_1|^4 + |\xi_2|^4 - \varepsilon (|\xi_1|^2 + |\xi_2|^2)) \left( \varphi_{N_1} \hat{\phi_1} (\xi_1) \left( \varphi_{N_2} \hat{\phi_2} (\xi_2) \right) d\xi_1 d\xi_2
\]

By the proof of [7, Lemma 2.9], we may assume \( N_3 = N_{\text{min}} \) and only need to show
\[
I \lesssim N_{\text{min}}^{d-4} \left( \frac{N_{\text{min}}}{N_{\text{max}}} \right)^2 \prod_{i=1}^2 \left\| P_{N_i} \phi_i \right\|_{L^2(\mathbb{R}^d)}.
\]

The idea is essential due to the proof of [8, Lemma 1]. We divide supp \( \varphi_{N_2} \) into disjoint cubes \( \{Q_{2,j}\}_{j \in \mathbb{Z}^+} \) of side \( N_3 \) with center \( \xi_{2,j} \), and choose a series of cubes \( Q_{1,j} \) of side \( 5N_3 \) with center \( \xi_{1,j} := -\xi_{2,j} \). Then
\[
\chi_{Q_{2,j}} = \chi_{Q_{2,j}} \cdot \chi_{Q_{1,j}} \text{ and } \sum_{j=1}^\infty \chi_{Q_{1,j}} (\xi) \leq T_d. \tag{2.15}
\]

Set \( \tilde{\varphi}_{N_1} = \varphi_{N_1} \cdot \chi_{Q_{1,j}} \) and \( \tilde{\varphi}_{N_2} = \varphi_{N_2} \cdot \chi_{Q_{2,j}} \).

For \( \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d, i = 1, 2 \), we may assume \( |\xi_j^i - \xi_1^i| = \max \{|\xi_j^i - \xi_1^i|, j = 1, \cdots, d\} \). The conditions \( N_2 \gg N_3 \) and \( |\xi_1 + \xi_2| \sim N_3 \) imply that \( |\xi_j^i - \xi_1^i| \gg |\xi_1 + \xi_2| \).

Change variable with \( \xi = \xi_1 + \xi_2, \tau = |\xi_1|^4 + |\xi_2|^4 - \varepsilon (|\xi_1|^2 + |\xi_2|^2) \). Similar calculations as that in [7] give \( d\xi d\tau = J d\xi_1 d\xi_2 \), where \( J \gtrsim N_3^3 \).

It follows from Cauchy-Schwarz inequality, canonical transform and (2.15) that
\[
I \lesssim N_{\text{min}}^{d-4} \sum_{j=1}^\infty \left\| \tilde{\varphi}_{N_1} \hat{\phi_1} \right\|_{L^2} \left\| \tilde{\varphi}_{N_2} \hat{\phi_2} \right\|_{L^2}
\]
\[
\lesssim N_{\text{min}}^{d-2} \frac{N_{\text{min}}}{N_{\text{max}}} \right\| P_{N_1} \phi_1 \right\|_{L^2(\mathbb{R}^d)} \left\| P_{N_2} \phi_2 \right\|_{L^2(\mathbb{R}^d)}.
\]

Here, Cauchy-Schwarz inequality and (2.15) lead to the last inequality. \qed

3. Probabilistic nonlinear estimates. Let \( z(t) := S(t)\phi^\omega \) and \( v(t) := u(t) - z(t) \). Then (1.1) with random initial data is equivalent to the following perturbed 4NLS
\[
\begin{aligned}
(i\partial_t + \varepsilon \Delta + \Delta^2) v &= P_m [(\partial_x^2 (\overline{v} + \overline{z})_{|\alpha| \leq 2}], \\
v(t, x) |_{t=0} &= 0.
\end{aligned}
\tag{3.1}
\]

Thus it suffices to investigate (3.1). We establish the nonlinear estimates of (3.1) in this section.

Denote \( \|Z\|_{W^s} \) the maximum of all the norms of \( Z, \partial Z \) used in the proof of Lemmas 3.1, 3.2 and 3.3 which have the following forms:
\[
\| (\nabla)^s |\nabla|^{q_1} Z \|_{L_{t}^{r_1} L_{x}^{s_1}}, \| (\nabla)^s Z \|_{L_{t}^{r_2} L_{x}^{s_2}}, \| \partial Z \|_{L_{t}^{r_3} L_{x}^{s_3}}, \| Z \|_{L_{t}^{s_4} L_{x}^{s_5}},
\]
where \( (q_1, r_1), (q_3, r_3) \) are the improved Strichartz admissible pairs; \( (q_2, r_2), (q_4, r_4) \) are the improved biharmonic admissible pairs. For \( R > 0 \), put
\[
E_{\kappa}^R := \{ \omega \in \Omega : \| \phi^\omega \|_{H^s} + \| Z \|_{W^s} \leq R \}. \tag{3.2}
\]

Lemma 2.1 implies that
\[
P(\Omega \setminus E_{\kappa}^R) \leq C \exp \left( -cR^2 / \| \phi \|_{H^s}^2 \right). \tag{3.3}
\]

Recall \( \Gamma N_m (u) \) in (1.8). We state the following global-in-time and local-in-time nonlinear estimates.
Lemma 3.1. Let \( \varepsilon \in \{-1, 0, 1\}, I = [0, T), T \leq \infty \). If \( \varepsilon = 1 \), suppose also \( T < \infty \).

Given \( d, m \geq 2 \) and \( s_c := d/2 - 2/(m-1) > 0 \), assume \( s \in (s_d, m, s_c) \) ( \( s_d \) is defined in (1.3)). Let \( \phi \in H^s(\mathbb{R}^d), \phi^\omega \) be its randomization defined in (1.4).

Then \( \omega \in E^s_R(\mathbb{R}^d) \) and \( v, v_1, v_2 \in \mathcal{Y}^{s_c}(I) \), there exists \( \delta > 0 \) sufficiently small such that

\[
\|\Gamma N_m(v + z)\|_{\mathcal{Y}^{s_c}(I)} \leq C_1(\|v\|_{\mathcal{Y}^{s_c}(I)} + R^m),
\]

\[
\|\Gamma N_m(v_1 + z) - \Gamma N_m(v_2 + z)\|_{\mathcal{Y}^{s_c}(I)} \leq C_2\left(\sum_{i=1}^{2} \|v_i\|_{\mathcal{Y}^{s_c}(I)}^{m-1} + T^dR^{m-1}\right)\|v_i - v_2\|_{\mathcal{Y}^{s_c}(I)}.
\]

For \( \varepsilon = 0 \), we need to assume \( d/2 - 4/(m-1) \geq 0 \) to get the above results on \([0, \infty)\).

Remark 5. If \( c_1 = c_2 = 0 \) or \( c_1 = 0, d/2 - 3/(m-1) > 0 \) or \( T < \infty \) for the nonlinearities of (1.2) of 4NLS (1.1), the condition \( d/2 - 4/(m-1) \geq 0 \) is not necessary for \( \varepsilon = 0 \) in Lemma 3.1.

Lemma 3.2. Given \( d, m \geq 2 \) and \( s_c := d/2 - 2/(m-1) > 0 \), assume \( s \in (s_d, m, s_c) \) ( \( s_d \) is defined in (1.3)). Let \( \phi \in H^s(\mathbb{R}^d), \phi^\omega \) be its randomization defined in (1.4).

Then for \( I = [0, T), T \leq 1, \omega \in E^s_R(\mathbb{R}^d) \) and \( v, v_1, v_2 \in \mathcal{Y}^{s_c}(I) \), there exists \( \delta > 0 \) sufficiently small such that

\[
\|\Gamma N_m(v + z)\|_{\mathcal{Y}^{s_c}(I)} \leq \tilde{C}_1(\|v\|_{\mathcal{Y}^{s_c}(I)} + T^dR^m),
\]

\[
\|\Gamma N_m(v_1 + z) - \Gamma N_m(v_2 + z)\|_{\mathcal{Y}^{s_c}(I)} \leq \tilde{C}_2\left(\sum_{i=1}^{2} \|v_i\|_{\mathcal{Y}^{s_c}(I)}^{m-1} + T^dR^{m-1}\right)\|v_i - v_2\|_{\mathcal{Y}^{s_c}(I)}.
\]

The following Lemma gives that the dual formula (2.5) is available for the truncated nonlinearities of (1.1).

Lemma 3.3. Let \( d, m \geq 2 \) and \( s_c := d/2 - 2/(m-1) > 0 \). Given \( s > 0 \), assume \( \phi \in H^s(\mathbb{R}^d), \omega \in E^s_R(\mathbb{R}^d) \). Then for any dyadic number \( M \in 2^{\mathbb{N}_0}, I = [0, T), T \leq 1 \) and \( v \in \mathcal{Y}^{s_c}(I) \), we have

\[
P_{\leq M}N_m(v + z) \in L_t^1L_x^2(I \times \mathbb{R}^d).
\]

Proof of Lemma 3.3. Referring to the proof of Lemma in [7], we only need to consider \((\partial v)^2\) and \((\partial v)w_2\), where \( w_2 = v \) or \( z \). In fact, by Sobolev embedding, (2.1) in Lemma 2.4 and (2.8), (2.9) in Lemma 2.4, we have

\[
\|P_{\leq M}(\partial v)^2\|_{L_t^1L_x^2(I \times \mathbb{R}^d)} \lesssim \|v\|_{Y^{s_c}(I)}^2
\]

\[
\lesssim \|v\|_{Y^{s_c}(I)}^2
\]

\[
\|P_{\leq M}(\partial v \cdot w_2)\|_{L_t^1L_x^2(I \times \mathbb{R}^d)} \lesssim \|w_2\|_{Y^{s_c}(I)}^2
\]

where \( l = 1 \) or 2.

Now we show Lemma 3.1 via dual formula (2.5).

Proof of Lemma 3.1. We only prove (3.4) since (3.5) is similar. Set \( v = v \cdot \chi_{[0,T)} \) and \( z = z \cdot \chi_{[0,T)} \) with \( T < \infty \). We split the proof into the low frequency section
and the high frequency section. Referring to the proof of Lemma 3.1 in [7], we only need to treat the high frequency section and it suffices to show

$$\sum_{N_0, \ldots, N_m} N_0^{s_0} N_m^{2} \left| \int_{\mathbb{R}^{1+d}} P_{N_0} v_0 \prod_{j=1}^{l} P_{N_j} v \prod_{j=l+1}^{m} P_{N_j} z dx dt \right| \lesssim (\|v\|_{H^{s_0}}^2 + R^m), \quad (3.7)$$

where $0 \leq l \leq m$, $N_{\text{max}} = \max\{N_0, N_1, \ldots, N_m\}$ and $N_{\text{max}} \geq 10m$.

For the left side of (3.7), denote $(\tau_j, \xi_j) \in \mathbb{R}^{1+d}$ the frequency of $P_{N_j} v$ and $P_{N_j} z$. The identity $\int_{\mathbb{R}^{1+d}} f(x, t) dx dt = \int(0)$ implies that $\sum_{j=0}^{m} (\tau_j, \xi_j) = 0$. It follows from $\sum_{j=0}^{m} \xi_j = 0$ that $N_{\text{max}} \sim N_{\text{sec}}$. By $\sum_{j=0}^{m} \tau_j = 0$, we have

$$\max_{0 \leq j \leq m} |\tau_j + \varepsilon| |\xi_j|^2 - |\xi_j|^4| \geq \frac{1}{m+1} \left| \sum_{j=0}^{m} (\varepsilon |\xi_j|^2 - |\xi_j|^4) \right| \geq \frac{1}{64(m+1)} N_{\text{max}}^4. \quad (3.8)$$

Let

$$M = \max \left\{ N \in 2^{\mathbb{N}_0} | N \leq \frac{1}{128(m+1)} N_{\text{max}}^4 \right\}. \quad (3.9)$$

Denote $R_j := Q_j P_{N_j}$, with $Q_j \in \{Q_{\leq M}, Q_{> M}\}$ (see (1.10) for the definition). By (2.12), $Q_{> M}^2 = Q_{> M} S(t) P_{\infty} = 0$. In order to show (3.7), it suffices to show that for some $0 \leq k_0 \leq l$ which satisfies $R_{k_0} = Q_{\geq N_{\text{max}}} P_{N_{k_0}}$, we have

$$I := \sum_{N_0, \ldots, N_m} N_0^{s_0} N_m^{2} \left| \int_{\mathbb{R}^{1+d}} R_{k_0} v_0 \prod_{k=1}^{l} R_k v \prod_{j=l+1}^{m} P_{N_j} z dx dt \right| \lesssim (\|v\|_{H^{s_0}}^2 + R^m), \quad (3.10)$$

where $0 \leq l \leq m$.

For a set $\mathfrak{N} \subset (2^{\mathbb{N}_0})^{m+1}$ (for example, $\mathfrak{N}$ is defined by $N_0 \sim N_1 \sim \cdots \approx N_1$, $N_m \sim \cdots \sim N_{l+1}$), we use the notation $I_{\mathfrak{N}}$ as

$$I_{\mathfrak{N}} := \sum_{N_m \geq \cdots \geq N_1+1 \atop N_0 \sim \cdots \sim N_1} N_0^{s_0} N_m^{2} \left| \int_{\mathbb{R}^{1+d}} R_{k_0} v_0 \prod_{k=1}^{l} R_k v \prod_{j=l+1}^{m} P_{N_j} z dx dt \right|. \quad (3.11)$$

By the symmetries method that we used in [7], we only need to show (3.9) in the following cases:

$$N_{\text{max}} \sim N_{\text{sec}}, \quad N_1 \leq \cdots \leq N_l \leq N_0, \quad N_{l+1} \leq \cdots \leq N_m. \quad (3.12)$$

For $N_{\text{max}} \geq 10m$, we show (3.9) according to $m = 2$ and $m \geq 3$.

**Step 1:** $m = 2$. We apply the decay estimates (2.6), (2.7) in Lemma 2.4, bilinear estimates in Lemma 2.5, Strichartz estimates and biharmonic estimates to establish (3.9). We argue according to $uv$, $zz$ and $uz$ case, i.e., $l = 2, 0$ and 1.

**Step 1:** $uv$ case. By (3.11), we have $N_0 \sim N_2 \geq N_1$. If $Q_2 = Q_{\geq N_{\text{max}}}$, by $P_N P_N = P_N$ (see (1.9)), Hölder’s inequality, Cauchy-Schwarz inequality, the decay estimates (2.6), (2.7) in Lemma 2.4 and bilinear estimates (2.14), we have

$$I_{Q_2 = Q_{\geq N_{\text{max}}}} \lesssim \sum_{Q_2 = Q_{\geq N_{\text{max}}} \atop N_0 \sim N_2 \geq N_1} N_2^{s_2+2} \|Q_{\geq N_{\text{max}}} P_{N_2} v\|_{L^2_t}, \|P_{N_2} (\prod_{k=1}^{2} R_k v)\|_{L^2_t}$$

$$\lesssim \sum_{N_0 \sim N_2 \geq N_1} N_2^{s_2-\frac{d}{2}+\delta} N_1^{2+\sqrt{d}} N_1^{\frac{d}{2}-\delta} \prod_{k=0}^{2} \|P_{N_k} v\|_{V^{\frac{d}{2}}}. \quad (3.13)$$
\[ \leq \sum_{N_0 \sim N_2} N_2^{s_c} \| P_{N_0} v_0 \|_{L^2} \| P_{N_2} v \|_{L^2} \| v \|_{Y_{s_c}} \leq \| v \|^2_{Y_{s_c}}. \]

The case \( Q_0 = Q_{z_{\max}^N} \) or \( Q_1 = Q_{z_{\max}^N} \) can be treated similarly.

**Step 1-2:** \( zz \) case. Note that \( Q_0 = Q_{z_{\max}^N} \) in this case. We need to consider: \( N_0 \sim N_2 \geq N_1 \) and \( N_2 \sim N_1 \geq N_0 \) by (3.11).

**Case 1:** \( N_0 \sim N_2 \geq N_1 \). If \( N_1 \leq N_2^{1/(d-1)} \), by Hölder’s inequality, (2.6), (2.7) in Lemma 2.4 and bilinear estimates (2.13), we have

\[ I_{N_0 \sim N_2 \geq N_1} \leq \sum_{N_0 \sim N_2 \geq N_1} N_2^{s_c+2} \| P_{N_0} v_0 \|_{L^2} \| P_{N_1} z \|_{L^2} \| P_{N_2} z \|_{L^2} \sum_{N_0 \sim N_2 \geq N_1} N_1^{d-\delta - s} N_2^{s_c - \frac{d}{2} + \delta - s} \prod_{j=1}^2 \| (\nabla)^s P_{N_j} \xi \|_{L^2} \leq \sum_{N_2} N_2^{s_c - 1 - \frac{d}{2} + c(\delta)} R^2 \leq R^2 \]

for \( \omega \in E_R^s \), \( \frac{d-1}{d} (s_c - 1) < s \leq s_c \) and \( \delta > 0 \) is sufficiently small.

If \( N_1 \geq N_2^{1/(d-1)} \), it follows from Hölder’s inequality, inequalities (2.1), (2.2) in Lemma 2.1 and (2.6), (2.7) in Lemma 2.4 that

\[ I_{N_1 \geq N_2^{1/(d-1)}} \leq \sum_{N_0 \sim N_2 \geq N_1} N_2^{s_c+2} \| Q_{z_{\max}^N} P_{N_0} v_0 \|_{L^2} \| P_{N_1} z \|_{L^2} \| P_{N_2} z \|_{L^2} \sum_{N_0 \sim N_2 \geq N_1} \frac{(2+\delta)}{d} \| (\nabla)^s P_{N_1} z \|_{L^2} \| (\nabla)^s P_{N_2} z \|_{L^2} \leq \sum_{N_2} N_2^{s_c - 1 + c(\delta)} - \frac{d}{2} + s R^2 \leq R^2 \]

for \( \omega \in E_R^s \), \( s > \max \{ 0, \frac{d-1}{d} (s_c - 1) \} \). Here, we apply biharmonic estimate to \( P_{N_1} z \) due to the singularity occurring in Strichartz estimates for \( \epsilon = 0, 1 \).

**Case 2:** \( N_2 \sim N_1 \geq N_0 \). Referring to the above case: \( N_1 \geq N_2^{1/(d-1)} \), it follows from Hölder’s inequality, inequalities (2.1), (2.2) in Lemma 2.1 and (2.6), (2.7) in Lemma 2.4 that

\[ I_{N_2 \sim N_1 \geq N_0} \leq \sum_{N_2 \sim N_1 \geq N_0} N_2^{s_c+2} \| Q_{z_{\max}^N} P_{N_0} v_0 \|_{L^2} \| P_{N_1} z \|_{L^2} \| P_{N_2} z \|_{L^2} \sum_{N_2 \sim N_1 \geq N_0} \frac{(2+\delta)}{d} \| (\nabla)^s P_{N_2} z \|_{L^2} \leq \sum_{N_2} N_2^{s_c - 1 + c(\delta)} - 2 \times R^2 \leq R^2 \]

as long as \( \omega \in E_R^s \) and \( s > \max \{ 0, (s_c - 1)/2 \} \).
Step 1-3: vz case. We need to consider: $N_0 \sim N_2 \gtrsim N_1$ and $N_0 \sim N_1 \gtrsim N_2$ by (3.11).

**Case 1:** $N_0 \sim N_2 \geq N_1$. First, we consider $N_1 \leq N_0^{1/(d-1)}$. If $Q_1 = Q_{N_1}$, by Hölder’s inequality, (2.6), (2.7) in Lemma 2.4 and bilinear estimates (2.14), we have

$$I_{Q_1 \sim Q_{N_1} \sim N_0} \lesssim \sum_{N_0 \sim N_1 \gtrsim N_1} N_0^{s_0 + 2} \|R_1 v\|_{L^2_{t,x}} \left\| \bar{P}_N (R_0 v_0 P_{N_2} z) \right\|_{L^2_{t,x}}$$

$$\lesssim \sum_{N_0 \gtrsim N_1} N_0^{s_0 - 3/2 + \delta - s} N_1^{3/2 - \delta + s} \|P_N v\|_{V^2_s} \|P_{N_2} (\nabla)^s \phi^w\|_{L^2}$$

$$\lesssim \sum_{N_0} N_0^{s_0 - \frac{3}{2} + \frac{3}{2(d-1)} + c(\delta - s)} \|v\|_{Y^{s_c} R} \lesssim \|v\|_{Y^{s_c} R}$$

as long as $\omega \in E_R^s$ and $s > \max\{0, s_c - \frac{3}{2} + \frac{3}{2(d-1)}\} = \max\{0, (1 - \frac{1}{d-1})s_c - 1\}$. If $Q_0 = Q_{N_2}$, we can argue in the same way.

Second, we consider $N_1 \sim N_0^{1/(d-1)}$. If $Q_1 = Q_{N_0}$, it follows from Hölder’s inequality, (2.6), (2.7), (2.9) in Lemma 2.4 and (2.2) in Lemma 2.1 that

$$I_{Q_1 \sim Q_{N_0} \sim N_1} \lesssim \sum_{Q_1 = Q_{N_0} \sim N_1 \sim N_0} N_0^{s_0 + 2} \|R_1 v\|_{L^2_{t,x}} \left\| \bar{P}_N (R_0 v_0 P_{N_2} z) \right\|_{L^2_{t,x}}$$

$$\lesssim \sum_{N_0 \sim N_1} N_0^{s_0 - \frac{2}{d} - s} N_1^{s_c - s} \|P_N v\|_{V^2_s} \|P_{N_2} (\nabla)^s \phi^w\|_{L^2}$$

$$\lesssim \sum_{N_0} N_0^{s_0 - \frac{2}{d} - 1 + c(\delta - s)} \|v\|_{Y^{s_c} R} \lesssim \|v\|_{Y^{s_c} R}$$

provided that $\omega \in E_R^s$ and $s > \max\{0, (1 - \frac{1}{d-1})s_c - 1\}$. If $Q_0 = Q_{N_0}$, we can substitute $R_0 v_0$ and $R_1 v$ with $R_1 v_1$ and $R_0 v_0$ in the above inequalities, respectively.

**Case 2:** $N_0 \sim N_1 \geq N_2$. We may assume $Q_1 = Q_{N_1}$ by $N_0 \sim N_1$. By Hölder’s inequality, (2.6), (2.7), (2.9) in Lemma 2.4 and (2.1) in Lemma 2.1, we have

$$I_{Q_1 \sim Q_{N_1} \sim N_2} \lesssim \sum_{Q_1 \sim Q_{N_1} \sim N_2} N_1^{s_0 + 2} \|R_1 v\|_{L^2_{t,x}} \left\| \bar{P}_N (R_0 v_0 P_{N_2} z) \right\|_{L^2_{t,x}}$$

$$\lesssim \sum_{N_1 \sim N_2} N_1^{s_0 - \frac{2}{d} - s} N_2^{s_c - s} \|P_N v\|_{V^2_s} \|P_{N_2} v_0\|_{V^2_s} \|P_{N_2} (\nabla)^s \phi^w\|_{L^2}$$

$$\lesssim \sum_{N_1} N_1^{s_0 - s_c - 1 + c(\delta - s)} \|v\|_{Y^{s_c} R} \lesssim \|v\|_{Y^{s_c} R}$$

as long as $\omega \in E_R^s$ and $s > 0$. 

Considering all the above cases, the inequality (3.9) holds for \( m = 2 \) in the conditions:

\[
s > \max \left\{ 0, \frac{d-1}{d} (s_c - 1), \left( 1 - \frac{1}{d} \right) s_c - 1 \right\} = \max \left\{ 0, \frac{d-1}{d} (s_c - 1) \right\}.
\]

**Step 2:** \( m \geq 3 \). We apply the decay estimates (2.6), Strichartz estimates (2.2), (2.9) and biharmonic estimates (2.1), (2.8) to establish (3.9).

**Step 2-1:** We consider \( \omega_k = v \) for all \( k \in \{1, \cdots, m\} \), i.e., \( l = m \). By (3.11), we only need to consider the case \( N_0 \sim N_m \geq N_{m-1} \). If \( Q_l = Q_{\geq N_l} \), by Hölder’s inequality, Sobolev embedding \( H^{s_c+\frac{m}{2}} \hookrightarrow L^{(m-1)d} \), the embedding \( V_{-\infty,c,S} \hookrightarrow L^\infty(\mathbb{R}; L^2) \) and (2.6), (2.7), (2.8), (2.9) in Lemma 2.4, we get

\[
\sum_{Q_l = Q_{\geq N_l}} \sum_{N_0 \sim N_m \geq \cdots \geq N_1} N_{s_c+2}^m \left| \int R_0 v_0 \prod_{k=1}^m R_k v dx dt \right| \\
\leq \sum_{Q_l = Q_{\geq N_l}} \sum_{N_0 \sim N_m \geq \cdots \geq N_1} N_{s_c+2}^m \left\| R_0 v_0 \right\| \left\| R_m v \right\| \left\| R_1 v \right\| \prod_{k=2}^{m-1} \left\| R_k v \right\|_{L^\infty_k L_x^{(m-1)d}} \\
\leq \sum_{N_m \geq \cdots \geq N_1} N_{s_c+1}^{m-1} \left\| P_{N_m} v_0 \right\|_{V^S_\delta} \left\| P_{N_m} v \right\|_{V^S_\delta} \prod_{k=1}^{m-1} N_k^{s_c+\frac{1}{2}} \left\| P_{N_k} v \right\|_{V^S_\delta} \\
\leq \sum_{N_m \geq \cdots \geq N_1} N_{s_c+1}^{m-1} \left\| P_{N_m} v_0 \right\|_{V^S_\delta} \left\| P_{N_m} v \right\|_{V^S_\delta} \delta v \leq \| v \|_{Y^{s_c+}}^m.
\]

The case \( Q_k = Q_{\geq N_k} \) (\( k = 2, \cdots, m-1 \)) can be similarly treated.

If \( Q_0 \) or \( Q_m = Q_{\geq N_m} \), we assume \( Q_m = Q_{\geq N_m} \) by symmetry. It follows from Hölder’s inequality, Sobolev embedding and (2.6), (2.7), (2.8), (2.9) in Lemma 2.4 that

\[
I \sum_{Q_m = Q_{\geq N_m}} \sum_{N_0 \sim N_m \geq \cdots \geq N_1} N_{s_c+2}^m \left| \int R_0 v_0 \right\| \left\| R_m v \right\| \left\| R_1 v \right\| \prod_{k=1}^{m-1} \left\| R_k v \right\|_{L_x^{2(2^\ast+1)(m-1)}} \\
\leq \sum_{Q_m = Q_{\geq N_m}} \sum_{N_0 \sim N_m \geq \cdots \geq N_1} N_{s_c+2}^m \left\| P_{N_m} v_0 \right\|_{V^S_\delta} \left\| P_{N_m} v \right\|_{V^S_\delta} \prod_{k=1}^{m-1} N_k^{s_c+\frac{1}{2}} \left\| P_{N_k} v \right\|_{V^S_\delta} \\
\leq \sum_{N_m \geq \cdots \geq N_1} N_{s_c+1}^{m-1} \left\| P_{N_m} v_0 \right\|_{V^S_\delta} \left\| P_{N_m} v \right\|_{V^S_\delta} \delta v \leq \| v \|_{Y^{s_c+}}^m.
\]
\[ \lesssim \sum_{N_m} N_{m}^{s_c} \| P_{N_m} v \|_{\Delta_x^2} \| P_{N_m} v_0 \|_{\Delta_x^2} \| v \|_{Y^{s_c}}^{m-1} \lesssim \| v \|_{Y^{s_c}}^m. \]

We use Cauchy-Schwarz inequality to sum over \( N_m \) to get the last inequality.

**Step 2-2:** We consider \( \omega_j = z \) for all \( j \in \{1, \cdots, m\} \), i.e., \( l = 0 \). In this case, \( Q_0 = Q_{N_0}^4 \) holds. By (3.11), it suffices to consider: \( N_0 \sim N_m \geq N_{m-1} \) and \( N_m \sim N_{m-1} \geq N_0 \).

**Case 1:** \( N_0 \sim N_m \geq N_{m-1} \). It follows from Hölder’s inequality, (2.6), (2.7) in Lemma 2.4 and (2.1), (2.2) in Lemma 2.1 that

\[
I_{Q_0 = Q_{N_0}^4, N_0 \sim N_m \sim \cdots \sim N_{m-1}} \lesssim \sum_{Q_0 = Q_{N_0}^4} \sum_{N_0 \sim N_m \sim \cdots \sim N_{m-1}} N_{N_0}^{s_c+2} \| R_0 v_0 \|_{L_t^q} \prod_{j=1}^{m-1} \| P_{N_j} z \|_{L_t^q} \left\| \frac{(2+\delta)(m-1)d}{L_x} P_{N_j} \right\|_{L_t^{2} L_x^{\frac{2(2+\delta)(m-1)d}{(2+\delta)(m-1)d}}} \\
\times \| \frac{(2+\delta)(m-1)d}{L_x} P_{N_m} \|_{L_t^{2} L_x^{\frac{2(2+\delta)(m-1)d}{(2+\delta)(m-1)d}}} \| v \|_{\Delta_x^d}^{m-1} \lesssim \sum_{N_m} N_{m}^{s_c-1-s+\epsilon(\delta)} R^m \leq R^m
\]

provided that \( \omega \in E^s_R \) and \( s > \max\{0, s_c - 1\} \).

**Case 2:** \( N_m \sim N_{m-1} \geq N_0 \). We can argue via the same inequality as that in Case 1 for \( \omega \in E^s_R \) and \( s > \max\{0, s_c - 1\} \).

**Step 2-3:** We consider \( \omega_k = v, 1 \leq k \leq l; \omega_j = z, l + 1 \leq j \leq m \), i.e., \( 1 \leq l \leq m - 1 \). We need to consider the following three cases: \( N_0 \sim N_l \geq N_m, N_0 \sim N_m \geq N_l \) and \( N_m \sim N_{m-1} \geq N_0, N_l \) by (3.11).

**Case 1:** \( N_0 \sim N_l \geq N_m \). If \( Q_0 \) or \( Q_l = Q_{N_l}^4 \), we may assume \( Q_0 = Q_{N_0}^4 \) by symmetry. Through Hölder’s inequality, Sobolev embedding, (2.1) in Lemma 2.1 and (2.6), (2.7), (2.9) in Lemma 2.4, we have

\[
I_{Q_0 = Q_{N_0}^4, N_m \sim \cdots \sim N_{m-1} \sim N_{m+1}} \lesssim \sum_{Q_0 = Q_{N_0}^4} \sum_{N_m \sim \cdots \sim N_{m-1} \sim N_{m+1}} N_{N_0}^{s_c+2} \| R_0 v_0 \|_{L_t^q} \prod_{k=1}^{l-1} \| R_k v \|_{L_t^q L_x^{\frac{(2+\delta)(m-1)d}{(2+\delta)(m-1)d}}} \| R_l v \|_{L_t^q L_x^{\frac{(2+\delta)(m-1)d}{(2+\delta)(m-1)d}}} \|
\times \prod_{j=l+1}^{m} \| P_{N_j} z \|_{L_t^q} \left\| \frac{(2+\delta)(m-1)d}{L_x} P_{N_j} \right\|_{L_t^{2} L_x^{\frac{2(2+\delta)(m-1)d}{(2+\delta)(m-1)d}}} \\
\| R_k v \|_{L_t^q L_x^{\frac{(2+\delta)(m-1)d}{(2+\delta)(m-1)d}}} \| P_{N_l} v \|_{V^2_{\Delta_x^d}} \| v \|_{\Delta_x^d}^{m-1} \lesssim \sum_{N_{m+1} \sim \cdots \sim N_l \sim N_0} N_{N_0}^{s_c-1-s+\epsilon(\delta)} R^m \leq R^m
\]
as long as \( \omega \in E^R_R \) and \( s > 0 \). Here, we use the fact that \( l \leq m - 1 \).

If \( Q_1 = Q_{\geq N_0^4} \), we assume \( l \geq 2 \) otherwise this reduces to \( Q_1 = Q_{\geq N_1^4} \). Through Hölder's inequality, Soblev embedding and the embedding \( V^2_{-r_x,S} \rightarrow L^\infty(\mathbb{R}; L^2) \), (2.1) in Lemma 2.1 and (2.6), (2.7), (2.8), (2.9) in Lemma 2.4, we have

\[
I \quad Q_1 = Q_{\geq N_0^4} \\
N_m \geq \cdots \geq N_{l+1} \\
N_0 \sim \cdots \sim N_l \geq \cdots \geq N_1
\]

\[
\leq \sum_{Q_1 = Q_{\geq N_0^4}} \sum_{N_m \geq \cdots \geq N_{l+1}} \sum_{N_0 \sim \cdots \sim N_l \geq \cdots \geq N_1} N_{l+1}^{s+2} \| R_1 v \|_{L^{2+m} L^\infty} \| R_0 v_0 \|_{L^2 L^\infty} \sum_{j = l+1}^m \| P_{N_j} z \|_{L^{2+m} L^\infty} \sum_{j = l+1}^m \| P_{N_j} v \|_{L^2 L^\infty} \| P_{N_j} v \|_{L^\infty L^\infty} \leq \sum_{N_l} \sum_{N_m} \sum_{N_0 \sim \cdots \sim N_l \geq \cdots \geq N_1}
\]

\[
\leq \sum_{Q_1 = Q_{\geq N_0^4}} \sum_{N_m \geq \cdots \geq N_{l+1}} \sum_{N_0 \sim \cdots \sim N_l \geq \cdots \geq N_1} N_{l+1}^s \| R_1 v \|_{L^{2+m} L^\infty} \| R_0 v_0 \|_{L^2 L^\infty} \sum_{j = l+1}^m \| P_{N_j} z \|_{L^{2+m} L^\infty} \sum_{j = l+1}^m \| P_{N_j} v \|_{L^2 L^\infty} \| P_{N_j} v \|_{L^\infty L^\infty} \leq \sum_{N_l} \sum_{N_m} \sum_{N_0 \sim \cdots \sim N_l \geq \cdots \geq N_1}
\]

for \( \omega \in E^R_R \) and \( s > 0 \). The case \( Q_k = Q_{\geq N_0^4} \) \((k = 2, \cdots, l - 1)\) can be treated similarly.

**Case 2:** \( N_0 \sim N_m \geq N_l \). If \( Q_k = Q_{\geq N_0^4} \), \( k = 1, \cdots, l \), we may assume \( Q_1 = Q_{\geq N_0^4} \). Thanks to Hölder’s inequality, Soblev embedding, the embedding \( V^2_{-r_x,S} \rightarrow L^\infty(\mathbb{R}; L^2) \), (2.1), (2.2) in Lemma 2.1 and (2.6), (2.7), (2.8), (2.9) in Lemma 2.4, we have

\[
I \quad Q_1 = Q_{\geq N_0^4} \\
N_m \geq \cdots \geq N_{l+1} \\
N_0 \sim \cdots \sim N_l \geq \cdots \geq N_1
\]
\[
\lesssim \sum_{N_m} N_m^{s_c - 1 + c(\delta) - 8} \prod_{k=1}^l N_k^{s_c + \frac{2\delta}{(m-3)(d+\delta)}} \|P_{N_k} v\|_{L_\infty^2 R^{m-l}}
\]
\[
\lesssim \sum_{N_m} N_m^{s_c - 1 + c(\delta)} \|v\|_{Y^{s_c} R^{m-l}} \lesssim \|v\|_{Y^{s_c} R^{m-l}}
\]
for \(\omega \in E_R^s\) and \(s > \max\{0, s_c - 1\}\).

If \(Q_0 = Q_{\geq N_0}^l\), we argue according to: \(2 \leq d < 4\) and \(d \geq 4\). First, assume \(d \geq 4\). Through Hölder’s inequality, Sobolev embedding \(H^{s_c + 2\delta/(d+\delta)} \hookrightarrow L_{x}^{(m-1)(d+\delta)/2}\), (2.1), (2.2) in Lemma 2.1 and (2.6), (2.7) in Lemma 2.4, we have

\[
\lesssim \sum_{Q_0=Q_{\geq N_0}^l} N_m^{s_c - \frac{2\delta}{(m-3)(d+\delta)}} \prod_{k=1}^l N_k^{s_c + \frac{2\delta}{(m-3)(d+\delta)}} \|P_{N_k} v\|_{Y^{s_c} R^{m-l}} \lesssim \|v\|_{Y^{s_c} R^{m-l}}
\]

for \(\omega \in E_R^s\) and \(s > \max\{0, s_c - 1\}\). If \(l = m - 1\), for \(P_{N_m} z\), we substitute \(L_\infty^{2+\delta}\) with \(L_\infty^{2}\).

Second, assume \(2 \leq d < 4\). Referring to the above case: \(d \geq 4\), we have

\[
\lesssim \sum_{Q_0=Q_{\geq N_0}^l} N_m^{s_c - \frac{2\delta}{(m-3)(d+\delta)} - s} \prod_{k=1}^l N_k^{s_c + \frac{2\delta}{(m-3)(d+\delta)}} \|P_{N_k} v\|_{Y^{s_c} R^{m-l}} \lesssim \|v\|_{Y^{s_c} R^{m-l}}
\]

for \(\omega \in E_R^s\) and \(s > \max\{0, s_c - 1\}\).

**Case 3:** \(N_m \sim N_{m-1} \geq N_0, N_l\). We assume \(l \leq m - 2\) otherwise this is reduced to the case \(N_m \sim N_l \geq N_0\).
If $Q_0 = Q_{\geq N_m^1}$, we can argue via the same inequalities as that in the case $Q_0 = Q_{\geq N_m^1}$ of Case 2: $N_m \sim N_0 \gtrsim N_1$. For $Q_k = Q_{\geq N_m^k}$ ($k = 1, \cdots, l$), we assume $Q_1 = Q_{\geq N_m^1}$ and argue in the following two cases.

(i) $d = 3, 3 \leq m \leq 4$ or $d \geq 4$. Note that $d(m - 1) - 4(m - 2) > 0$. Via Hölder’s inequality, Soblev embedding $H^{s_c} \hookrightarrow L_x^{[(m-1)d]/d}$, (2.6), (2.7) in Lemma 2.4 and (2.1), (2.2) in Lemma 2.1, we obtain

$$I \quad Q_1 = Q_{\geq N_m^l}$$
$$N_m \sim N_0 \gtrsim N_1$$
$$N_m \gtrsim \cdots \gtrsim N_{l+1}$$

$$\leq \sum_{Q_1 = Q_{\geq N_m^l}} N_m^{s_c+2} \| R_1 v \|_{L_x^{2+4} L_t^{(m-1)d}/s} \| R_0 v_0 \|_{L_x^\infty L_t^{(m-1)d}/s} \prod_{k=2}^l \| R_k v \|_{L_x^{2+4} L_t^{(m-1)d}/s}$$

and argued in the following two cases.

(ii) $d = 3, m \geq 5$ or $d = 2$. Referring to the above case, by $H^{2-d-\frac{2-d}{d}} \hookrightarrow L_x^{[(m-1)d]/2}$, we have

$$I \quad Q_1 = Q_{\geq N_m^l}$$
$$N_m \sim N_0 \gtrsim N_1$$
$$N_m \gtrsim \cdots \gtrsim N_{l+1}$$

$$\leq \sum_{Q_1 = Q_{\geq N_m^l}} N_m^{s_c+2} \| R_1 v \|_{L_x^{2+4} L_t^{(m-1)d}/s} \| R_0 v_0 \|_{L_x^\infty L_t^{(m-1)d}/s} \prod_{k=2}^l \| R_k v \|_{L_x^{2+4} L_t^{(m-1)d}/s}$$
Considering all the above cases, for $m \geq 3$, the inequality (3.9) holds provided that $s > 0$ and $s > s_c - \min\{1, d/4\}$.

**Proof of Lemma 3.2.** The Proof of Lemma 3.2 is almost the same as that of [7, Lemma 3.2], we omit the details.

4. **The proof of well-posedness result.** The proofs of Theorems 1.1 and 1.2 are the same as that of [7]. We only give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For $\bar{C}_1, \bar{C}_2$ in Lemma 3.2, let $\eta$ satisfy $\bar{C}_1 \eta^{m-1} \leq 1/2$, $\bar{C}_2 \eta^{m-1} \leq 1/8$. Given $R > 0$, set $T = \min\{\eta/2\bar{C}_1 R^m, 1/4\bar{C}_2 R^{m-1}\}^{1/\delta}$. Define the ball $B_\eta$ by
\[
B_\eta = \{v \in Z^{s_c}([0,T]) : C([0,T]; H^{s_c}) : \|v\|_{Z^{s_c}([0,T])} \leq \eta\}.
\]
Put $\Omega_T = E_R^\eta$ (see (3.2)). By the inequality (3.3), we have
\[
P(\Omega_T^c) \leq C \exp(-c/T^\gamma \|\phi\|^2_{H^4}),
\]
where $\gamma = 2\delta/m$. Lemma 3.2 implies that for any $\omega \in \Omega_T$, the mapping $v \to \Gamma[M_n(v + Z)]$ is a contraction on $B_\eta$, which leads the almost sure local-in-time well-posedness.

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