LINEAR ALGEBRAIC GROUPS WITHOUT THE NORMALIZER THEOREM

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Abstract. One can develop the basic structure theory of linear algebraic groups (the root system, Bruhat decomposition, etc.) in a way that bypasses several major steps in the standard development, including the self-normalizing property of Borel subgroups.

An awkwardness of the theory of linear algebraic groups is that one must develop a lot of material about general linear algebraic groups before one can really get started. Our goal here is to show how to develop the root system, etc., using only the completeness of the flag variety and some facts about solvable groups. In particular, one can skip over the usual analysis of Cartan subgroups, the fact that $G$ is the union of its Borel subgroups, the connectedness of torus centralizers, and the normalizer theorem (a Borel subgroup is self-normalizing). The main idea is a new approach to the structure of rank 1 groups; the key step is lemma 5.

All algebraic geometry is over a fixed algebraically closed field. $G$ always denotes a connected linear algebraic group with Lie algebra $\mathfrak{g}$, $T$ a maximal torus, and $B$ a Borel subgroup containing it. We assume the structure theory for connected solvable groups, and the completeness of the flag variety $G/B$ and some of its consequences. Namely: that all Borel subgroups (resp. maximal tori) are conjugate; that $G$ is nilpotent if one of its Borel subgroups is nilpotent; that $C_G(T)_0$ lies in every Borel subgroup containing $T$; and that $N_G(B)$ contains $B$ of finite index and (therefore) is self-normalizing. We also assume known that the centralizer of a torus has the expected dimension, namely, that of the subspace of $\mathfrak{g}$ where the torus acts trivially. For these results we refer to Borel [1], Humphreys [2] and Springer [3].

In section 1 we develop a few properties of solvable groups, and in section 2 we treat the structure of rank 1 groups. The root system, etc., can then be developed in essentially the standard way, so after
the rank 1 analysis we restrict ourselves to brief comments. We are grateful to J. Humphreys, G. McNinch and T. Springer for their helpful comments.

1. Lemmas about solvable groups

First we recall from [3, §13.4] the groups that we call the positive and negative subgroups of $G$. Fix a 1-parameter group $\phi : \mathbb{G}_m \to G$. For $g \in G$ and $\lambda \in \mathbb{G}_m$, define $\operatorname{Cl}_g(\lambda) = \phi(\lambda)gg(\lambda)^{-1}$ and

$$G^+ = \left\{ g \in G \mid \lim_{\lambda \to 0} \operatorname{Cl}_g(\lambda) = 1 \right\}.$$ 

Of course, the condition $\lim_{\lambda \to 0} \operatorname{Cl}_g(\lambda) = 1$ means that $\operatorname{Cl}_g : \mathbb{G}_m \to G$ extends to a regular map $\mathbb{G}_m \cup \{0\} \to G$ sending 0 to 1. We call $G^+$ the positive part of $G$ (with respect to $\phi$). It is a group because

$$\lim_{\lambda \to 0} \phi(\lambda)gg'(\lambda)^{-1} = \lim_{\lambda \to 0} \phi(\lambda)g\phi(\lambda)^{-1} \cdot \lim_{\lambda \to 0} \phi(\lambda)g'\phi(\lambda)^{-1}$$

and

$$\lim_{\lambda \to 0} \phi(\lambda)g^{-1}\phi(\lambda)^{-1} = \left( \lim_{\lambda \to 0} \phi(\lambda)g\phi(\lambda)^{-1} \right)^{-1}$$

when the limits on the right hand sides exist. It is closed and connected because it is generated by irreducible curves containing 1. Of course, there is a corresponding subgroup $G^-$ called the negative part of $G$, got by considering limits as $\lambda$ approaches $\infty$. All of our discussion applies equally well to $G^-$. The key properties of $G^\pm$ are that they are unipotent and “large”:

**Proposition 1.** $G^+$ is unipotent, and every weight of $\mathbb{G}_m$ on the lie algebra of $G^+$ is positive. If $G$ is solvable, then the lie algebra of $G^+$ contains all the positive weight spaces for $\mathbb{G}_m$ on $\mathfrak{g}$.

We remark that the solvability hypothesis in the last part is unnecessary, but we will need only the solvable case. The general case requires the structure theory which depends on theorem 3.

**Proof.** See [3, theorem 13.4.2] for the first claim; the idea is to embed $G$ into $\text{GL}_n$ and diagonalize the $\mathbb{G}_m$ subgroup. We will prove the second claim, since the proof in [3] includes the nonsolvable case and therefore relies on the Bruhat decomposition. We will use induction on $\dim G_u$; the 0-dimensional case is trivial.

So suppose $\dim G_u > 0$ and choose a connected subgroup $N$ of $G_u$, normal in $G$, with $G_u/N \cong \mathbb{G}_a$. We write $\pi$ for the associated map $G_u \to \mathbb{A}^1$. By the action of $\mathbb{G}_m$ on $G_u/N$, we have $\pi \circ \operatorname{Cl}_g(\lambda) = \pi(g) \cdot \lambda^n$ for some $n \in \mathbb{Z}$. If $n \leq 0$ then $\pi \circ \operatorname{Cl}_g$ does not extend to a regular map
\[ G_\mathbb{m} \cup \{0\} \to \mathbb{A}^1 \text{ sending } 0 \text{ to } 0 \text{ unless } \pi(g) = 0, \text{ so a group element outside } N \text{ cannot lie in } G^+. \text{ So } G^+ = N^+ \text{ and we use induction.} \\

So suppose \( n > 0 \), i.e., the action is by a positive character. Choose a linear representation \( V \) of \( G_\mathbb{m} \) and an embedding (as a variety) of \( G_u \) into \( PV \) which is equivariant with respect to the conjugation action of \( G_\mathbb{m} \). Choose a \( G_\mathbb{m} \)-invariant linear subspace of \( PV \) containing \( 1 \in G_u \), whose tangent space there is complementary to that of \( N \). By dimension considerations, its intersection with \( G_u \) contains an invariant curve \( C \) that passes through \( 1 \) and does not lie in \( N \). By passing to a component we may assume that \( C \) is irreducible, so it is the closure of the orbit of some \( g \in C \). The map \( \text{Cl}_g : G_\mathbb{m} \to G_u \) extends to a regular map from \( P^1 \) to \( PV \). Because \( 1 \) lies in the closure of the orbit, \( \text{Cl}_g \) sends at least one of \( 0 \) or \( \infty \) to \( 1 \). It cannot send \( \infty \) to \( 1 \), because \( \pi \circ \text{Cl}_g(\lambda) = (\text{nonzero constant}) \cdot \lambda^n \) for some \( n > 0 \), which admits no regular extension \( G_\mathbb{m} \cup \{\infty\} \to \mathbb{A}^1 \). Therefore \( \lim_{\lambda \to 0} \text{Cl}_g(\lambda) = 1 \), so \( g \in G^+ \). This shows that \( G^+ \) projects onto \( G_u/N \). It also contains \( N^+ \), to which the inductive hypothesis applies. The proposition follows. \( \square \)

An immediate consequence is that a connected solvable group \( G \) is generated by its subgroups \( G^+, G^- \) and \( C_G(\phi(G_\mathbb{m}))_0 \), since together their Lie algebras span \( \mathfrak{g} \). Next, we need a theorem on orbits of solvable groups. The proof does not use the structure theorem for solvable groups, and indeed can be used to prove it. It can be simplified slightly if one assumes the structure theorem.

**Theorem 2.** If \( G \) is solvable and acts on a variety, then no orbit contains any complete subvariety of dimension \( > 0 \).

**Proof.** We assume the result known for solvable groups of smaller dimension than \( G \). (The 0-dimensional case is trivial.) If \( X \) is the variety, \( x \in X \), \( G_x \) its stabilizer and \( Y \) its orbit, then the natural map \( G/G_x \to Y \) is generically finite (since \( G/G_x \) and \( Y \) have the same dimension), hence finite (by homogeneity). If \( Y \) contained a complete subvariety of positive dimension then the preimage in \( G/G_x \) would also be complete. So it suffices to treat the case \( X = G/G_x \).

We consider three cases. First, if \( G_x \) contains \([G, G]\) then it is normal, so \( G/G_x \) is an affine variety and cannot contain a complete subvariety of positive dimension. Second, if \( G_x \) surjects to \( G/[G, G] \), then \([G, G]\) acts transitively on \( G/G_x \), and by the inductive hypothesis applied to \([G, G]\), \( G/G_x \) cannot contain a complete subvariety of positive dimension. Finally, suppose \( G_x \) neither contains \([G, G]\) nor surjects to \( G/[G, G] \). Set \( H \) equal to the group generated by \( G_x \) and \([G, G]\). We will use the fact that \( G/G_x \) maps to \( G/H \) with fibers that are copies
of $H/G_x$. As in the first case, $G/H$ is an affine variety, so any complete subvariety of $G/G_x$ lies in the union of finitely many copies of $H/G_x$. But the inductive hypothesis applied to $H$ shows that every complete subvariety of $H/G_x$ is a finite set of points. Therefore the same conclusion applies to $G/G_x$. □

2. Rank One Groups

In this section, $G$ is connected and non-solvable of rank 1. To goal is:

**Theorem 3.** $G$ modulo its unipotent radical admits an isogeny to $PGL_2$.

There is a standard argument that reduces this to proving that $T$ lies in exactly two Borel subgroups. We must modify this slightly because we are not assuming the normalizer theorem. Since $N := N_G(B)$ is self-normalizing, it fixes only one point of $G/N$, so the stabilizers of distinct points of $G/N$ are the normalizers of distinct Borel subgroups. The fixed points of $T$ in $G/N$ correspond to Borel subgroups that $T$ normalizes, hence lies in. Now we use the theorem that a torus acting on a $d$-dimensional projective variety has at least $d + 1$ fixed points. Since $G$ is not solvable, $G/B$ has dimension $> 0$, so $G/N$ does too. Therefore $T$ lies in at least two Borel subgroups. And if we prove that it lies in exactly two, then we can also deduce $\dim G/N = 1$. Then it is easy to see that $G/N \cong P^1$ and prove theorem 3. So our aim is to prove that $T$ lies in exactly two Borel subgroups.

Using the positive and negative subgroups, we will construct two Borel subgroups containing $T$, and then show that there are no more. Suppose $\phi : \mathbb{G}_m \to T$ is a parametrization of $T$ (meaning $\phi$ is an isomorphism) and $B$ a Borel subgroup containing $T$. Call $B$ positive (with respect to $\phi$) if it contains $G^+$ and negative if it contains $G^-$. Obviously, $B$ is positive with respect to one parametrization of $T$ if and only if it is negative with respect to the other. Here are the basic properties of positive and negative Borel subgroups.

**Lemma 4.** Suppose $\phi : \mathbb{G}_m \to T$ a parametrization of the maximal torus $T$. Then

1. $T$ lies in a positive Borel subgroup and in a negative Borel subgroup;
2. if $B$ (resp. $B'$) is a positive (resp. negative) Borel subgroup containing $T$, then every Borel subgroup containing $T$ lies in $\langle B, B' \rangle$;
3. every Borel subgroup containing $T$ is either positive or negative, but not both; and
(4) $N_G(T)$ contains an element acting on $T$ by inversion.

Proof. (1) $G^+$ is connected, unipotent and normalized by $T$. Therefore $T^G$ lies in some Borel subgroup, which is then positive. And similarly for $G^-$.

(2) Suppose $B''$ is a Borel subgroup containing $T$. Then $B''^+$ lies in $B$ since $B$ is positive, $B''^-$ lies in $B'$ since $B'$ is negative, and $C_{B''}(T)_0$ lies in both $B$ and $B'$ because $C_{G}(T)_0$ lies in every Borel subgroup containing $T$. Since $B''$ is generated by $B''^+, B''^-$ and $C_{B''}(T)_0$, it lies in $\langle B, B' \rangle$.

(3) The condition that a Borel subgroup containing $T$ contains at least one of $G^\pm$ is independent under conjugation by $N(T)$. Since $N(T)$ acts transitively on the Borel subgroups containing $T$, and we have exhibited positive and negative Borel subgroups, every Borel subgroup containing $T$ contains at least one of $G^\pm$ and hence is either positive or negative. If a Borel subgroup $B$ containing $T$ were both positive and negative, then (2) would imply that it is the only Borel subgroup containing $T$, contradicting the fact that $T$ lies in at least 2 Borel subgroups.

(4) By (1), positive and negative Borel subgroups exist, and by (3) they are distinct. The result follows from the fact that $N_G(T)$ acts transitively on the Borel subgroups containing $T$. □

Now we can give the key step in our approach to the structure theorem for rank 1 groups.

Lemma 5. Every maximal torus of $G$ lies in exactly two Borel subgroups, one positive and one negative.

Proof. By induction on the dimension of a Borel subgroup. If it has dimension 1 then it is abelian, so $G = B$. Therefore the base case is dimension 2. Choose a maximal torus $T$ and a parametrization of it. We already know that $T$ lies in a positive and a negative Borel subgroup. The key point is that any two positive Borel subgroups coincide. For otherwise their unipotent radicals would be distinct subgroups of $G^+$, hence generate a unipotent group of dimension $> 1$. This is impossible because $\dim B = 1$. Similarly, there is only one negative Borel subgroup and the base case is proven.

Now we prove the inductive step; suppose $B$ has dimension at least 3, and suppose without loss that it is positive. Consider the action of $B$ on $G/N$; there is a unique fixed point because the only Borel subgroup that $B$ normalizes is itself. Consider an orbit of minimal positive dimension. By theorem (2), it contains no complete subvarieties of dimension $> 0$. On the other hand, its closure is complete and is got
by adjoining lower-dimensional orbits. By minimality, this means that its closure is got by adjoining a single point, so the orbit is a curve. Therefore there exists a Borel subgroup $B'$ for which $B \cap N(B')$ has codimension 1 in $B$. That is, $I := (B \cap B')_0$ has codimension 1 in each of $B$ and $B'$. There are two possibilities: $I = B_u = B_u'$, or $I$ contains a torus. In the first case, $\langle B, B' \rangle$ normalizes $I$, and a Borel subgroup in $\langle B, B' \rangle/I$ has no unipotent part. This forces $\langle B, B' \rangle$ to be solvable, which is impossible.

Therefore $I$ contains a torus $T$, and $I_u$ has codimension 1 in each of $B_u$ and $B'_u$. Now, $T$ normalizes $B_u$ and $B'_u$, hence their intersection, hence $I_u$. Also, $B_u$ normalizes $I_u$ because it is only one dimension larger and is nilpotent. Similarly for $B'_u$. Therefore $\langle B, B' \rangle$ normalizes $I_u$, which has dimension $> 0$ since $\dim B > 2$. We apply induction to $\langle B, B' \rangle/I_u$, and then pull back to conclude the following. $B$ and $B'$ are the only Borel subgroups of $\langle B, B' \rangle$ containing $T$, and they are exchanged by an element of $N_{\langle B, B' \rangle}(T)$ that inverts $T$. This implies that $B'$ is a negative Borel subgroup of $G$. Finally, lemma 4 implies that any Borel subgroup of $G$ containing $T$ lies in $\langle B, B' \rangle$, hence equals $B$ or $B'$.

This lemma implies theorem 3, and from then on one can follow the standard development. We make only the following remarks.

**Bruhat decomposition:** in the absence of the normalizer theorem, one should define the Weyl group $W$ as the subgroup of $N_G(T)/C_G(T)$ generated by the reflections coming from roots. Then one can prove $G = BWB$ as in [1, §14], [2, §28] or [3, §8.3].

**Normalizers:** The theorem $N_G(B) = B$ follows immediately from the Bruhat decomposition, and implies that $W$, as defined here, is all of $N_G(T)/C_G(T)$, so that our definition agrees with the usual one.

**Connectedness of torus centralizers:** this can be deduced from the Bruhat decomposition and a standard fact about reflection groups: the pointwise stabilizer of a linear subspace is generated by the reflections that stabilize it.

**References**

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