FOURIER TRANSFORMS IN EXPONENTIAL REARRANGEMENT INVARIANT SPACES

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Abstract. In this article we investigate the Fourier series and transforms for the functions defined on the \([0, 2\pi]_d\) or \(R^d\) and belonging to the exponential Orlicz and some other rearrangement invariant (r.i.) spaces.

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1 Introduction. Notations. Problem Statement.

For the (complex) measurable function \(f = f(x)\) defined on the \(X = \{x\} = T^d = [0, 2\pi]^d, \quad d = 1, 2, \ldots\) or \(X = R^d\) we denote correspondently the Fourier coefficients and transform

\[ c(n) = \int_{T^d} \exp(i(n, x)) \ f(x) \ dx, \quad F[f](t) = \int_{R^d} \exp(i(t, x)) \ f(x) \ dx, \]

where as usually

\[ F[f](t) \overset{\text{def}}{=} \lim_{M \to \infty} \int_{|x| \leq M} \exp(i(t, x)) \ f(x) \ dx, \]

\[ x = (x_1, x_2, \ldots, x_d), \quad n = (n_1, n_2, \ldots, n_d), \quad t = (t_1, t_2, \ldots, t_d), \quad dx = \prod_{j=1}^d dx_j \]

if \(x \in R^d\), and \(dx = (2\pi)^{-d} \ \prod_{j=1}^d dx_j\) in the case \(X = T^d;\)

\[ n_i = 0, \pm 1, \pm 2, \ldots, \quad (t, x) = \sum_{j=1}^d t_j x_j, \quad |n| = \max_j |n_j|, \quad |t| = \max_j |t_j|, \]
We will assume in this paper that the function \( f(\cdot) \) belongs to some Orlicz space \( L(N) = L(N; X) \) with so-called exponential \( N - \) function \( N = N(u) \) or some rearrangement invariant space \( E \) with fundamental function \( \phi_E(\delta) \), \( \delta > 0 \), and will investigate the properties of Fourier transform of \( f \), for example, [13], [11, p 193 - 197]. Our results are also some generalization of [4].

For arbitrary multiply sequence (complex, in general case) \( c \) and introduce the discrete analog of |
\[
|f|_p = \| f \|_{L_p(X, \mu)} = \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}, \quad p \geq 1.
\]

In the case \( X = R^d \) we introduce a new measure \( \nu(\cdot) \) (non-finite, in general case): for all Borel set \( A \subset R^d \)
\[
\nu(A) = \int_A \prod_{i=1}^d x_i^{-2} \, dx = \int_A \prod_{i=1}^d x_i^{-2} \cdot \prod_{i=1}^d dx_i,
\]
and will denote \( |f|_p(\nu) = \)
\[
\left[ \int_X \prod_{j=1}^d x_j^{p-2} \cdot |f(x)|^p \, \nu(dx) \right]^{1/p}.
\]

For arbitrary multiply sequence (complex, in general case) \( c(n) = c(n_1, n_2, \ldots, n_d), \)
\( n_i = 0, \pm 1, \pm 2, \ldots, n \in Z^d \) we denote as usually
\[
|c|_p = \left[ \sum_n |c(n)|^p \right]^{1/p}, \quad p \geq 1;
\]
and introduce the discrete analog of \( |f|_p(\nu) \) norm:
\[
|c|_p(\nu) = \left[ \sum_n |c(n)|^p \cdot \left( \prod_{j=1}^d n_j^{p-2} + 1 \right) \right]^{1/p}, \quad p \geq 2.
\]
Let $N = N(u)$ be some $N$–Orlicz’s function, i.e. downward convex, even, continuous differentiable for all sufficiently greatest values $u$, $u \geq u_0$, strongly increasing in the right - side axis, and such that $N(u) = 0 \iff u = 0$; $u \to \infty \Rightarrow dN(u)/du \to \infty$. We say that $N(\cdot)$ is an Exponential Orlicz Function, briefly: $N(\cdot) \in EOF$, if $N(u)$ has a view: for some continuous differentiable strongly increasing downward convex in the domain $[2, \infty]$ function $W = W(u)$ such that $u \to \infty \Rightarrow W'(u) \to \infty$

$$N(u) = N(W(u)) = \exp(W(\log |u|)), \ |u| \geq e^2.$$ For the values $u \in [-e^2, e^2]$ we define $N(W, u)$ arbitrary but so that the function $N(W, u)$ is even continuous convex strictly increasing in the right side axis and such that $N(u) = 0 \iff u = 0$. The correspondent Orlicz space on $T^d$, $R^d$ with usually Lebesgue measure with $N$– function $N(W, u)$ we will denote $L(N) = EOS(W)$; $EOS = \cup_W \{EOS(W)\}$ (Exponential Orlicz’s Space).

For example, let $m = \const > 0$, $r = \const \in R^1$,

$$N_{m,r}(u) = \exp \left[ |u|^m \left( |\log^{-m}r(C_1(r) + |u|)\right)\right] - 1,$$

$C_1(r) = e$, $r \leq 0$; $C_1(r) = \exp(r)$, $r > 0$. Then $N_{m,r}(\cdot) \in EOS$. In the case $r = 0$ we will write $N_m = N_{m,0}$.

Recall here that the Orlicz’s norm on the arbitrary measurable space $(X, A, \mu)$ $\|f\|L(N) = L(N, X, \mu)$ may be calculated by the formula (see, for example, [11], p. 73; [6], p. 66)

$$\|f\|L(N) = \inf_{v>0} \left\{ v^{-1} \left( 1 + \int_X |f(x)| \mu(dx) \right) \right\}.$$ Recall also that the notation $N_1(\cdot) << N_2(\cdot)$ for two Orlicz functions $N_1, N_2$ denotes:

$$\forall \lambda > 0 \Rightarrow \lim_{u \to \infty} N_1(\lambda u)/N_2(u) = 0.$$ We will denote for arbitrary Orlicz $L(N)$ (and other r.i.) spaces by $L^0(N)$ the closure of all bounded functions with bounded support.

Let $\alpha$ be arbitrary number, $\alpha \geq 1$, and $N(\cdot) \in EOS(W)$ for some $W = W(\cdot)$. We denote for such a function $N = N(W, u)$ by $N^{(\alpha)}(u)$ a new $N$ – Orlicz’s function such that

$$N^{(\alpha)}(u) = C_1 |u|^\alpha, \ |u| \in [0, C_2];$$

$$N^{(\alpha)}(u) = C_3 + C_4|u|, \ |u| \in (C_2, C_5];$$

$$N^{(\alpha)}(u) = N(u), \ |u| > C_5, \ 0 < C_2 < C_5 < \infty, \ (1.1)$$

$$C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot)).$$ In the case $\alpha = m(j + 1)$, $m > 0$, $j = 0, 1, 2, \ldots$ the function $N^{(\alpha)}(u)$ is equivalent to the following Trudinger’s function:

$$N^{(\alpha)}(u) \sim N^{(\alpha)}_{[m]}(u) = \exp \left( |u|^m \right) - \sum_{l=0}^{j} u^{ml}/l!.$$
This method is described in [15], p. 42 - 47. Those Orlicz spaces are applied to the theory of non-linear partial differential equations [15].

We can define formally the spaces \( L(N^{(m)}_N) \) at \( m = +\infty \) as a projective limit at \( m \to \infty \) the spaces \( L(N^{(m)}_N) \), but it is evident that

\[
L\left(N^{(\infty)}_N\right) \sim L_\alpha + L_\infty,
\]

where the space \( L_\infty \) consists on all the a.e. bounded functions with norm

\[
|f|_\infty = \text{vraisup}_{x \in X} |f(x)|.
\]

Of course, in the case \( X = T^d \)

\[
L_\alpha + L_\infty \sim L_\infty.
\]

Here and further we will denote by \( C_k = C_k(\cdot), k = 1, 2, \ldots \) some positive finite essentially and by \( C, C_0 \) non-essentially "constructive" constants. It is very simple to prove the existence of constants \( C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot)) \) such that \( N^{(\alpha)} \) is some new exponential \( N \) Orlicz’s function. By the symbols \( K_j \) we will denote the "classical" absolute constants.

Now we will introduce some new Banach spaces. Let \( \alpha = \text{const} \geq 1 \) and \( \psi = \psi(p), p \geq \alpha \) be some continuous positive: \( \psi(\alpha) > 0 \) finite strictly increasing function such that the function \( p \to p \log \psi(p) \) is downward convex and

\[
\lim_{p \to \infty} \psi(p) = \infty.
\]

The set of all those functions we will denote \( \Psi; \Psi = \{\psi\} \). A particular case:

\[
\psi(p) = \psi(W; p) = \exp(W^*(p)/p),
\]

where

\[
W^*(p) = \sup_{z \geq \alpha} (pz - W(z))
\]

is so-called Young-Fenchel, or Legendre transform of \( W(\cdot) \). It follows from theorem of Fenchel-Moraux that in this case

\[
W(p) = [p \log \psi(W; p)]^*, \quad p \geq p_0 = \text{const} \geq 2,
\]

and consequently for all \( \psi(\cdot) \in \Psi \) we introduce the correspondent \( N \) – function by equality:

\[
N([\psi]) = N([\psi], u) = \exp \{[p \log \psi(p)]^* (\log u)\}, \quad u \geq e^2.
\]

Since \( \forall \psi(\cdot) \in \Psi, \quad d = 0, 1, \ldots \Rightarrow p^d \cdot \psi(p) \in \Psi \), we can denote

\[
\psi_d(p) = p^d \cdot \psi(p), \quad N_d([\psi]) = N_d([\psi], u) = N([\psi_d], u).
\]
For instance, if $N(u) = \exp(|u|^m)$, $u \geq 2$, where $m = \text{const} > 0$, then

$$N_d([\psi], u) \sim \exp \left( |u|^{m/(dm+1)} \right), \quad u \geq 2.$$ 

**Definition 1.** We introduce for arbitrary such a function $\psi(\cdot) \in \Psi$ the so-called $G(\alpha; \psi)$ and $G(\alpha; \psi, \nu)$ norms and correspondent Banach spaces $G(\alpha; \psi)$, $G(\alpha, \psi, \nu)$ as a set of all measurable complex functions with finite norms:

$$\|f\|G(\alpha; \psi) = \sup_{p \geq \alpha} (|f|_p / \psi(p)),$$

and analogously,

$$\|f\|G(\alpha; \psi, \nu) = \sup_{p \geq \alpha} (|f|_p(\nu) / \psi(p)).$$

For instance $\psi(p)$ may be $\psi(p) = \psi_m(p) = p^{1/m}$, $m = \text{const} > 0$; in this case we will write $G(\alpha, \psi_m) = G(\alpha, m)$ and

$$\|f\|G(\alpha, m) = \sup_{p \geq \alpha} \left( |f|_p \, p^{-1/m} \right).$$

Also formally we define

$$\|f\|G(\alpha, m) = |f|_{\alpha} + |f|_{\infty}.$$ 

**Remark 1.** It follows from Iensen inequality that in the case $X = T^d$ all the spaces $G(\alpha_1; \psi)$, $G(\alpha_2, \psi)$, $1 \leq \alpha_1 < \alpha_2 < \infty$ are isomorphic:

$$\|f\|G(\alpha; \psi) \leq \|f\|G(1; \psi) \leq \max(1, \psi(\alpha)) \|f\|G(\alpha; \psi)$$

It is false in the case $X = R^d$.

**Remark 2.** $G(\alpha; \psi)$ is a rearrangement invariant (r.i.) space. $G(\alpha, m)$ has a fundamental function $\phi(\delta; G(\alpha, m))$, $\delta > 0$, where for any rearrangement invariant space $G$

$$\phi(\delta; G) \overset{def}{=} \|I_A(\cdot)||G(\cdot), \quad \text{mes}(A) = \delta \in (0, \infty),$$

$\text{mes}(A)$ denotes usually Lebesque measure of Borel set $A$. We have:

$$\phi(\delta; G(\alpha, m)) = (e \, m \, |\log \delta|)^{-1/m}, \quad \delta > \exp(-\alpha/m),$$

$$\phi(\delta; G(\alpha, m)) = \alpha^{-1/m} \, \delta^{1/\alpha}, \quad \delta \in (0, \exp(-\alpha/m)).$$

Let us consider also another space $G(a, b, \alpha, \beta)$, $1 \leq a < b < \infty$; $\alpha, \beta \geq 0$. Here $X = R^d$ and we denote $h = \min((a+b)/2; 2a)$. We introduce the function $\zeta: (a, b) \to R^1_+$:

$$\zeta(p) = \zeta(a, b, \alpha, \beta; p) = (p - a)^{\alpha}, \quad p \in (a, h);$$

$$\zeta(p) = (b - p)^{\beta}, \quad p \in [h, b).$$
Definition 2. The space $G(a, b, \alpha, \beta)$ consists on all the complex measurable functions with finite norm:

$$||f||_{G(a, b, \alpha, \beta)} = \sup_{p \in (a,b)} ||f_p \cdot \zeta(a, b, \alpha, \beta; p)||.$$ 

The space $G(a, b, \alpha, \beta)$ is also a rearrangement invariant space. After some calculations we receive the correspondence fundamental function $\phi(\delta; G(a, b, \alpha, \beta))$. Namely, let us denote $\delta_1 = \exp(\alpha h^2/(h-a))$ and define for $\delta \geq \delta_1$

$$p_1 = p_1(\delta) = \frac{\log \delta}{2\alpha} - \left[\frac{\log^2 \delta}{4\alpha^2} - \frac{a \log \delta}{\alpha}\right]^{1/2},$$

$$\phi_1(\delta) = \delta^{1/p_1} (p_1 - a)^{\alpha};$$

and for $\delta \in (0, \delta_1) \Rightarrow$

$$\phi_1(\delta) = \delta^{1/h} (h-a)^{\alpha}.$$ 

Further, set $\delta_2 = \exp(-h^2 \beta/(b-h))$, and for $\delta \in (0, \delta_2)$ $p_2 = p_2(\delta) =$

$$-\frac{\log \delta}{2\beta} + \left[\frac{\log^2 \delta}{4\beta^2} + b \frac{\log \delta}{\beta}\right]^{1/2},$$

$$\phi_2(\delta) = \delta^{1/p_2} (b - p_2(\delta))^\beta;$$

and for $\delta \geq \delta_2$ we define

$$\phi_2(\delta) = \delta^{1/h} (b-h)^{\beta}.$$ 

We can write:

$$\phi(\delta; G(a, b, \alpha, \beta)) = \max [\phi_1(\delta), \phi_2(\delta)].$$

Note than at $\delta \to 0+$

$$\phi(\delta; G(a, b, \alpha, \beta)) \sim \max \left\{\delta^{1/h} (h-a)^{\alpha}, (\epsilon \beta b^2 \delta^{1/\beta} |\log \delta|^{-a}\right\}$$

and at $\delta \to \infty$

$$\phi(\delta; G(a, b, \alpha, \beta)) \sim \max \left\{\delta^{1/h} (b-h)^{\beta}, (2a^2 \alpha e^{-2})^{\alpha} \delta^{1/\alpha} (\log \delta)^{-a}\right\}.$$ 

For example, let us consider the function $f(x) = f(a, b; x), x \in R^1 \to R : f(x) = 0, x \leq 0$;

$$f(x) = x^{-1/b}, x \in (0,1); \ f(x) = x^{-1/a}, x \in [1,\infty);$$

then $f(a, b, \cdot) \in G(a, b, 1, 1)$ and

$$\forall \Delta \in (0, 1/2] \Rightarrow f \notin G(a, b, 1 - \Delta, 1) \cup G(a, b, 1, 1 - \Delta).$$
Analogously may be defined the "discrete" \( g(a, b, \alpha, \beta) \) spaces. Namely, let \( c = c(n_1, n_2, \ldots, n_d) \) be arbitrary multiply (complex) sequence. We say that \( c \in g(a, b, \alpha, \beta) \) if

\[
||c||g(a, b, \alpha, \beta) \overset{df}{=} \sup_{p \in (a, b)} [|c|_p (p - a)\alpha (b - p)\beta].
\]

It is evident that the non-trivial case of those spaces is only if \( \beta = 0 \); in this case we will write \( g(a, b, \alpha, 0) = g(a, \alpha) \) and

\[
||c||g(\alpha) = \sup_{p>a} |c|_p (p - a)\alpha.
\]

We denote also for \( \psi(\cdot) \in \Psi : ||c||g(\psi, \nu) = \sup_{p \geq 2} [||c|_p(\nu)/\psi(p)], \quad ||c||m(\nu) = \sup_{p \geq 2} [||c|_p(\nu) \cdot p^{-1/m}], \quad m = \text{const} > 0. \]

Our goal is investigating the boundness of Fourier operators and convergence (divergence) Fourier series and integrals in exponential Orlicz and \( G(a, b, \alpha, \beta), \ g(\psi, \nu) \) etc. spaces.

Note than our Orlicz \( N \) – functions \( N \in EOS \) does not satisfy the so-called \( \Delta_2 \) condition.

2 Formulations of main results.

**Theorem 1.** Let \( X = [0, 2\pi]^d \) and \( \psi \in \Psi. \) Then the Fourier operators \( s_M[\cdot] \) are uniformly bounded in the space \( L(N[\psi]) \) into the other Orlicz’s space \( L(N_d[\psi]) \):

\[
\sup_{M \geq 1} ||s_M[f]||L(N_d[\psi]) \leq C_6(d, \psi) ||f||L(N[\psi]). \tag{2.1}
\]

**Theorem 2.** Let now \( X = R^d, \ \psi \in \Psi \) and \( \alpha = \text{const} > 1. \) The Fourier operators \( S_M[\cdot] \) are uniformly bounded in the space \( L(N^{(\alpha)}[\psi]) \) into the space \( L(N^{(\alpha)}_d[\psi]) \):

\[
\sup_{M \geq 1} ||S_M[f]||L(N^{(\alpha)}_d[\psi]) \leq C_7(\alpha, d, \psi) ||f||L(N^{(\alpha)}[\psi]). \tag{2.2}
\]

Since the function \( N[\psi] \) does not satisfies the \( \Delta_2 \) condition, the assertions (2.1) and (2.2) does not mean that in general case when \( f \in L(N^{(\alpha)}_d[\psi]) \)

\[
\lim_{M \to \infty} ||s_M[f] - f||L(N_d[\psi]) = 0, \tag{2.3}
\]

\[
\lim_{M \to \infty} ||S_M[f] - f||L(N^{(\alpha)}_d[\psi]) = 0; \tag{2.4}
\]
see examples further. But it is evident that propositions (2.3) and (2.4) are true if correspondently
\[ f \in L^0(N_d[\psi]), \quad f \in L^0(N_d^{(\alpha)}[\psi]). \]
Also it is obvious that if \( f \in L(N_d[\psi]), X = [0, 2\pi]^d \) or, in the case \( X = R^d \), \( f \in L(N_d^{(\alpha)}[\psi]), \) then for all EOF \( \Phi(\cdot) \) such that \( \Phi \ll N_d[\psi] \) or \( \Phi \ll N_d^{(\alpha)}([\psi]) \) the following implications hold:
\[ \forall f \in L(N_d[\psi]) \Rightarrow \lim_{M \to \infty} ||s_M[f] - f||L(\Phi) = 0, \quad X = [0, 2\pi]^d; \quad (2.5) \]
\[ \forall f \in L(N_d[\psi]) \Rightarrow \lim_{M \to \infty} ||S_M[f] - f||L(\Phi) = 0, \quad X = R^d. \quad (2.6) \]

**Theorem 3.** Let \( \Phi(\cdot) \) be an EOF and let \( N(\cdot) = L^{-1}(u) \in EOF \), where \( L(y), \, y \geq \exp(2) \) is a positive slowly varying at \( u \to \infty \) strongly increasing continuous differentiable in the domain \([\exp(2), \infty)\) function such that the function
\[ W(x) = W_L(x) = \log L^{-1}(\exp x), \quad x \in [2, \infty) \]
is again strong increasing to infty together with the derivative \( dW/dx \). In order to the implication (2.5) or, correspondently, (2.6) holds, it is necessary and sufficient that \( \Phi \ll L(N_d[\psi]) \), or, correspondently \( \Phi \ll L(N_d^{(\alpha)}[\psi]) \).

For instance, the conditions of theorem 3 are satisfied for the functions \( N = N_{m,r}(u) \).

**Theorem 4.** Let \( f(\cdot) \in G(1, b, \alpha, 0), \alpha > 0 \). Then \( F[f] \in L(N_{1/\alpha}^{(2)}) \) and
\[ \sup_{M \geq 1} ||S_M[f]||L(N_{1/\alpha}^{(2)}) \leq C_8(\alpha, N) ||f||G(1, b, \alpha, 0). \quad (2.7) \]

**Theorem 5.** Let \( \{\phi_k(x), k = 1, 2, \ldots\} \) be an orthonormal uniform bounded:
\[ \sup_{k,x} |\phi_k(x)| < \infty \]
sequence of functions on some non-trivial measurable space \((X, A, \mu)\) and (in the \( L_2(X, \mu) \) sense)
\[ f(x) = \sum_{k=1}^{\infty} c(k) \phi_k(x). \]

**A.** If \( c \in g(\psi, \nu), \) then
\[ ||f||L(N_1[\psi], X, \mu) \leq C_9 \cdot (\max(1, \sup_{k,x} |\phi_k(x)|)) \cdot ||c||g(\psi, \nu), \quad (2.8) \]

**B.** Let \( c \in g(\alpha) \) for some \( \alpha \in (0, 1] \). We assert that
\[ ||f||L(N_{1/\alpha}[\psi], X, \mu) \leq C_{10}(\alpha) \cdot (\max(1, \sup_{k,x} |\phi_k(x)|)) \cdot ||c(\cdot)||g(\alpha). \]

**Theorem 6.** If \( f \in G(\alpha; \psi, \nu), \) where \( \alpha \geq 2 \), then
\[ \sup_{M \geq 1} ||S_M[f]||L(N_d^{(\alpha)}[\psi]) \leq C_{11}(\alpha, \psi, N, \nu) ||f||G(\alpha; \psi, \nu). \quad (2.9) \]
3 Auxiliary results

Theorem 7. Let \( N(u) = N(W, u) = \exp(W(\log u)) \), \( u > e^2 \), \( \psi(p) = \exp(W^*(p)/p) \), \( p \geq 2 \), and \( X = T^d \). We propose that the Orlicz's norm \( \| \cdot \| L(N) \) and the norm \( \| \cdot \| G(\psi) \) are equivalent. Moreover, in this case \( f \neq 0 \), \( f \in G(\psi) \) (or \( f \in L(N(W(\cdot), u)) \)) if and only if \( \exists C_{12}, C_{13}, C_{14} \in (0, \infty) \Rightarrow \forall u > C_{14} \)

\[
T(|f|, u) \leq C_{12} \exp(-W(\log (u/C_{13}))),
\]

where for each measurable function \( f : X \to \mathbb{R} \)

\[
T(|f|, u) = \text{mes}\{x : |f(x)| > u\}.
\]

Proof of theorem 7. A). Assume at first that \( f \in L(N), f \neq 0 \). Without loss of generality we suppose that \( \|f\|_{L(N)} = 1/2 \).

\[
\int_X N(W, |f(x)|) \, dx \leq 1 < \infty.
\]

The proposition (3.1) follows from Chebyshev’s inequality such that in (3.1) \( C_{12} = 1, C_{13} = C_{14} = 1/\|f\|_{L(N)}, f \neq 0 \).

B). Inversely, assume that \( f, f \neq 0 \) is a measurable function, \( f : X \to \mathbb{R}^1 \) such that

\[
T(|f|, u) \leq \exp(-W(\log u)), u \geq e^2.
\]

We have by virtue of properties of the function \( W \):

\[
\int_X N(|f(x)|/e^2) \, dx = \int_{\{x : f(x) \leq e^2\}} + \int_{\{x : |f(x)| > e^2\}} = I_1 + I_2;
\]

\[
I_1 \leq \int_X N(1) \, dx = N(1),
\]

\[
I_2 \leq \sum_{k=2}^{\infty} \int_{e^k < |f| \leq e^{k+1}} \exp(W(|f(x)|/e^2)) \, dx \leq \sum_{k=2}^{\infty} \exp((W(k-1)) \, T(|f|, k) \leq \sum_{k=2}^{\infty} \exp(W(k-1) - W(k)) < \infty.
\]

Thus, \( f \in L(N(W)) \) and

\[
\int_X N(|f(x)|/e^2) \, dx \leq N(1) + \sum_{k=2}^{\infty} \exp(W(k-1) - W(k)) < \infty.
\]

C). Let now \( f \in G(\psi) \); without loss of generality we can assume that \( \|f\|_{G(\psi)} = 1 \). We deduce for \( p \geq 2 \) :

\[
\int_X |f(x)|^p \, dx \leq \psi^p(p).
\]
We obtain using again the Chebyshev’s inequality:

\[ T(|f|, u) \leq u^{-p} \psi^p(p) = \exp \left[ -p \log u + p \log \psi(p) \right], \]

and after the minimization over \( p \): \( u \geq \exp(2) \) \( \Rightarrow \)

\[ T(|f|, u) \leq \exp \left( - \sup_{p \geq 2} (p \log u - p \log \psi(p)) \right) = \exp \left( (p \log \psi(p))^*(\log u) \right) = \exp(-W(\log u)). \]

D). Suppose now that \( T(|f|, x) \leq \exp(-W(\log x)), x \geq \exp(2) \). We conclude:

\[ \int_X |f(x)|^p \, dx = p \int_0^\infty x^{p-1} T(|f|, x) \, dx = p \int_0^{\exp(2)} x^{p-1} T(|f|, x) \, dx + \]

\[ p \int_{\exp(2)}^\infty x^{p-1} T(|f|, x) \, dx \leq p \int_0^{\exp(2)} x^{p-1} \, dx + p \int_{\exp(2)}^\infty x^{p-1} T(|f|, x) \, dx \leq \]

\[ e^{2p} + \int_{\exp(2)}^\infty px^{p-1} \exp(-W(\log x)) \, dx = \]

\[ e^{2p} + p \int_2^\infty \exp(py - W(y)) \, dy, \quad p \geq 2. \]

We obtain using Laplace’s method and theorem of Fenchel - Moraux ([8], p. 23 - 25):

\[ \int_X |f(x)|^p \, dx \leq e^{2p} + C^p \exp \left( \sup_{y \geq 2} (py - W(y)) \right) = e^{2p} + \]

\[ C^p \exp(W^*(p)) = e^{2p} + C^p \exp(p \log \psi(p)) \leq C^p \psi^p(p). \]

Finally, \( ||f||G(\psi) < \infty \).

For example, if \( m > 0, \ r \in \mathbb{R}, \) then

\[ f \in L \left( N_{m,r} \right) \iff \sup_{p \geq 2} \left[ |f|_p \, p^{-1/m} \log^{-r} p \right] < \infty \iff \]

\[ T(|f|, u) \leq C_0(m, r) \exp \left( -C(m, r) u^m \left( \log^{-mr} u \right) \right), \quad u \geq 2. \]

**Remark 3.** If conversely

\[ T(|f|, x) \geq \exp(-W(\log x)), \quad x \geq e^2, \]

then for sufficiently large values of \( p; \ p \geq p_0 = p_0(W) \geq 2 \)

\[ |f|_p \geq C_0(W) \psi(p), \quad C_0(W) \in (0, \infty). \quad (3.2) \]

**Remark 4.** In this proof we used only the condition \( 0 < mes(X) < \infty \). Therefore, our conclusions in theorem 7 are true in this more general case.
Theorem 8. Let $\psi \in \Psi$. We assert that $f \in L^0(N[\psi])$, or, equally, $f \in G^0(\psi)$ if and only if

$$\lim_{p \to \infty} |f|_p/\psi(p) = 0.$$  \hfill (3.3)

Proof. It is sufficient by virtue of theorem 7 to consider only the case of $G(\psi)$ spaces.

1. Denote $G^{00}(\psi) = \{ f : \lim_{p \to \infty} |f|/\psi(p) = 0 \}$. Let $f \in G^0(\psi)$, $f \neq 0$. Then for arbitrary $\delta = const > 0$ there exists a constant $B = B(\delta, f(\cdot)) \in (0, \infty)$ such that

$$\|f - fI(|f| \leq B)\|G(\psi) \leq \delta/2.$$  

Since $|f|I(|f| \leq B)| \leq B$, we deduce

$$|fI(|f| \leq B)|_p/\psi(p) \leq B/\psi(p).$$

We obtain using triangular inequality for sufficiently large values $p : p \geq p_0(\delta) = p_0(\delta, B) \Rightarrow |f|_p/\psi(p) \leq \delta/2 + B/\psi(p) \leq \delta,$

as long as $\psi(p) \to \infty$ at $p \to \infty$. Therefore $G^0(\psi) \subset G^{00}(\psi)$.

(The set $G^{00}(\psi)$ is a closed subspace of $G(\psi)$ with respect to the $G(\psi)$ norm and contains all bounded functions.)

2. Inversely, assume that $f \in G^{00}(\psi)$. We deduce denoting $f_B = f_B(x) = f(x)I(|f| > B)$ for some $B = const \in (0, \infty)$:

$$\forall Q \geq 2 \Rightarrow \lim_{B \to \infty} |f_B|_Q = 0.$$  

Further,

$$\|f_B\|G(\psi) = \sup_{p \geq 2} |f_B|_p/\psi(p) \leq \max_{p \leq Q} |f_B|_p/\psi(p) + \sup_{p > Q} |f_B|_p/\psi(p) \overset{def}{=} \sigma_1 + \sigma_2;$$

$$\sigma_2 = \sup_{p > Q} |f_B|_p/\psi(p) \leq \sup_{p \geq Q} (|f|_p/\psi(p)) \leq \delta/2$$

for sufficiently large $Q$ as long as $f \in G^{00}(\psi)$. Let us now estimate the value $\sigma_1$:

$$\sigma_1 \leq \max_{p \leq Q} |f_B|_p/\psi(2) \leq \delta/2$$

for sufficiently large $B = B(Q)$. Therefore,

$$\lim_{B \to \infty} \|f_B\|G(\psi) = 0, \ f \in G^0(\psi).$$
Theorem 9. Let \( \psi(\cdot) = \psi_N(\cdot), \theta(\cdot) = \theta_\Phi(\cdot) \) be a two functions on the classes \( \Psi \) with correspondent \( N - \) Orlicz’s functions \( N(\cdot), \Phi(\cdot) : \)

\[
N(u) = \exp \left\{ [p \log \psi(p)]^*(\log u) \right\},
\]

\[
\Phi(u) = \exp \left\{ [p \log \theta(p)]^*(\log u) \right\}, \quad u \geq \exp(2).
\]

We assert that \( \lim_{p \to \infty} \psi(p)/\theta(p) = 0 \) if and only if \( N(\cdot) \gg \Phi(\cdot) \).

**Proof** of theorem 9. A). Assume at first that \( \lim_{p \to \infty} \psi(p)/\theta(p) = 0 \).

Denote \( \epsilon(p) = \psi(p)/\theta(p) \), then \( \epsilon(p) \to 0, p \to \infty \).

Let \( \{ f_\zeta, \zeta \in Z \} \) be arbitrary bounded in the \( G(\psi) \) sense set of functions:

\[
\sup_{\zeta \in Z} ||f_\zeta||_{G(\psi)} = \sup_{\zeta \in Z} \sup_{p \geq 2} |f_\zeta|_p/\psi(p) = C < \infty,
\]

then

\[
\sup_{\zeta \in Z} |f_\zeta|_p/\theta(p) \leq C \epsilon(p) \to 0, p \to \infty.
\]

It follows from previous theorem that \( \forall \zeta \in Z f_\zeta \in G^0(\theta) \) and that the family \( \{ f_\zeta, \zeta \in Z \} \) has uniform absolute continuous norm. Our assertion follows from lemma 13.3 in the book [6].

B). Inverse, let \( \Phi(\cdot) << N(\cdot) \). Let us introduce the measurable function \( f : X \to R \) such that \( \forall x \geq \exp(2) \)

\[
\exp \left( -2 [p \log \psi(p)]^*(\log x) \right) \leq T(|f|, x) \leq \exp \left( - [p \log \psi(p)]^*(\log x) \right).
\]

Then (see theorem 7)

\[
f(\cdot) \in G(\psi), \quad C_{15}(\psi) \psi(p) \leq |f|_p \leq C_{14}(\psi) \psi(p), \quad p \geq 2.
\]

Since \( f \in G(\psi) \), \( \Phi \ll N \), we deduce that \( f \in G^0(\theta) \), and, following,

\[
\lim_{p \to \infty} |f|_p/\theta(p) = 0.
\]

Therefore, \( \lim_{p \to \infty} \psi(p)/\theta(p) = 0 \).

**Theorem 10.** Let now \( X = R^d \) and \( \psi \in \Psi \). We assert that the norms

\[
|| \cdot ||_{L(N^{(\alpha)}(\psi), [\psi])} \quad \text{and} \quad || \cdot ||_{G(\alpha, \psi)}, \quad \alpha \geq 1 \quad \text{are equivalent.}
\]

**Proof.** 1. Let \( \forall p \geq \alpha \Rightarrow |f|_p \leq \psi(p), \quad f \neq 0 \). From Chebyshev inequality follows that

\[
\lim_{v \to \infty} T(|f|, v) = 0.
\]

We can choose the values \( v, C_{17}, C_{18} \), such that \( 0 < C_{17} < C_{18} < \infty, \quad v \in (0, \infty) \) and

\[
C_{17} \leq T(|f|, v) \leq C_{18} < \infty.
\]
Let us consider for some sufficiently small value \( \epsilon \in (0, \epsilon_0) \), \( \epsilon_0 \in (0, 1) \) the following integral:

\[
I_{\alpha,N}(f) = \int_X N^{(\alpha)}(\epsilon|f(x)|) \, dx = I_1 + I_2,
\]

where

\[
I_1 = \int_{\{x: |f(x)| \leq v\}} N^{(\alpha)}(\epsilon|f(x)|) \, dx, \quad I_2 = \int_{\{x: |f(x)| > v\}} N^{(\alpha)}(\epsilon|f(x)|) \, dx.
\]

Since for \( z \geq v \)

\[
N^{(\alpha)}(z) \leq C_{19}(\alpha, N(\cdot)) \cdot N(z),
\]

we have for the set \( X(v) = \{x, |f(x)| > v\} \), using the result of theorem 7 for the space with finite measure:

\[
I_2 = \int_{X(v)} N^{(\alpha)}(\epsilon|f(x)|) \, dx \leq C_{20}(\alpha, N, \epsilon) \|f\|L(N^{(\alpha)}, X(v)) \leq C_{21}(\alpha, \epsilon, \psi) \sup_{p \geq \alpha} \|f\|_{L^p}(X(v))/\psi(p) < \infty.
\]

Further, since for \( z \in (0, v) \) \(\Rightarrow\)

\[
N^{(\alpha)}(\epsilon z) \leq C_{22}(v, \alpha, \epsilon) |z|^\alpha,
\]

we have:

\[
I_1 \leq C_{22}(\cdot) \int_X |f(x)|^\alpha \, dx < \infty.
\]

Thus, \( f \in L(N^{(\alpha)}[\psi]) \), \( \|f\|L(N^{(\alpha)}[\psi]) < \infty \).

2). We prove now the inverse inclusion. Let \( f \in L\left(N^{(\alpha)}[\psi]\right) \) and

\[
\|f\|L\left(N^{(\alpha)}[\psi]\right) = 1.
\]

Hence for some \( \epsilon > 0 \)

\[
\int_X N^{(\alpha)}(\epsilon|f(x)|) \, dx < \infty.
\]

It follows from the proof of theorem 7 and the consideration of two cases: \( |z| \leq v; \ |z| > v \) the following elementary inequality: at \( p \geq \alpha \) and for all \( z > 0 \) \(\Rightarrow\)

\[
|z|^p \leq C_{23}(\alpha, \epsilon, N) N^{(\alpha)}(\epsilon|z|) \cdot \psi^p(p).
\]

We obtain for all values \( p, \ p \geq \alpha \):

\[
\int_{R^n} |f(x)|^p \, dx \leq C_{24}(\alpha, \epsilon, \psi) \psi^p(p), \quad \|f\|G(\alpha; \psi) < \infty.
\]

Note that theorems 7 - 10 are some generalizations of [1, p. 201 - 233], [15], p.29 and [8], section 1. The case of Hilbert transform of generalized functions \( \{f\} \) is consider in [2].
4 Proofs of main results.

At first we consider the case Orlicz spaces, i.e. if the function $f$ belongs to some exponential Orlicz space.

Proof of theorems 1, 2. Let $X = [0, 2\pi]^d$ and $f \in L(N[\psi])$ for some $\psi \in \Psi$. Without loss of generality we can assume that $\|f\|L(N[\psi]) = 1$. From theorem 7 follows that

$$\forall p \geq 2 \Rightarrow |f|_p \leq C_{25}(\psi) \psi(p).$$

From the classical theorem of M. Riesz (see [3], p.100 - 103; [9]) follows the inequality:

$$\sup_{M \geq 1} |s_M[f]|_p \leq K_1^d p^d \psi(p), \quad K_1 = 2\pi.$$

It follows again from theorem 7 that

$$\sup_{M \geq 1} \| s_M[f] - f \|L(N_d[\psi]) \leq K_1^d + 1 < \infty.$$

For example, if $N(u) = N_m(u) = \exp(|u|^m) - 1$ for some $m = \text{const} \geq 1$, then

$$\sup_{M \geq 1} \| s_M[f] \| - f \|L(N_{m/(dm+1)}) \leq C_{26}(d, m) \| f \|L(N_m). \quad (4.1)$$

The "continual" analog of M.Riesz's inequality, namely, the case $X = \mathbb{R}$, $L(N) = L_p(\mathbb{R})$, $p \geq 2$:

$$\sup_{M \geq 2} |S_M[f]|_p \leq K_2^d p^d |f|_p, \quad K_2 = 1$$

is proved, for example, in [10], p.187 - 188. [9]. This fact permit us to prove also theorem 2.

Lemma 1. We assert that the "constant" $m/(dm+1)$ in the estimation (3.5) is exact. In detail, for all $m \geq 1$ there exists $g = g_m(\cdot) \in L(N_m)$ such that $\forall \Delta \in (0, m)$

$$\sup_{M \geq 1} \| s_M[g] \|L\left(N_{(m-\Delta)/(dm+1)}\right) = \infty.$$

Proof of lemma 1. It is enough to prove that

$$\exists g \in L(N_m), \quad \| H[g] \|L(N_{(m-\Delta)/(dm+1)}) = \infty,$$

where $H[g]$ denotes the Hilbert transform on the $[0, 2\pi]^d$, see [12], p. 193 - 197; [13]. Also it is enough to consider the case $d = 1$.

Let us introduce the function

$$g(x) = g_m(x) = |\log(x/(2\pi))|^{1/m}.$$

Since for $u > 0$

$$\text{mes}\{x : g_m(x) > u\} = \exp(-u^m),$$
we conclude \( g_m(\cdot) \in L(N_m) \setminus L^0(N_m) \) (theorem 7). Further, it is very simple to verify using the formula for Hilbert transform ([3], p.192) that
\[
C_{28}(m) \left( |\log(x/(2\pi))|^{(m+1)/m} + 1 \right) \leq |H[g_m](x)| \leq C_{29}(m) \left( |\log(x/(2\pi))|^{(m+1)/m} + 1 \right).
\]
Hence \( \forall u \geq 2 \)
\[
\exp \left( -C_{29}(m) u^{m/(m+1)} \right) \leq \text{mes}\{x, |H[g_m](x)| > u\} \leq \exp \left( -C_{30}(m) u^{m/(m+1)} \right).
\]
It follows again from theorem 7 that
\[
H[g_m] \in L(N_{m/(m+1)}) \setminus L(N_{m/(m+1)}).
\]
Hence \( \forall \Delta \in (0, m) \Rightarrow H[g_m] \notin L(N_{(m-\Delta)/(m+1)}) \).

**Proof** of theorem 3. Let us consider the following function:
\[
z(x) = z_L(x) = \sum_{n=8}^{\infty} n^{-1} L(n) \sin(nx).
\] (4.2)

It is known from the properties of slowly varying functions ([14], p. 98 - 101) that the series (4.2) converge a.e. and at \( x \in (0, 2\pi] \)
\[
C_0 L(1/x) \leq z(x) \leq CL(1/x).
\]
Therefore, at \( u \in [\exp(2), \infty) \)
\[
L^{-1}(Cu) \leq T(|z|, u) \leq L^{-1}(C_0 u).
\]
It follows from theorem 7 and (3.2) that
\[
z(\cdot) \in L(N) \setminus L^0(N), \ N(u) = L^{-1}(u), \ u \geq \exp(2).
\]
From theorem 8 follows that
\[
0 < C_0 \leq |z|_p/\psi(p) \leq C < \infty, \ p \geq 2, \ \psi(p) = \exp(W^*(p)/p).
\] (4.3)

Note as a consequence that the series (4.2) does not converge in the \( L(N) \) norm, as long as the system of functions \{\sin(nx)\} is bounded and hence in the case when the series (4.2) converge in the \( L(N) \) norm \( \Rightarrow z(\cdot) \in L^0(N) \).

Let us suppose now that for some EOF \( \Phi(\cdot) \) with correspondence function \( \theta(p) \) the series (4.2) convergence in the \( L(\Phi) \) norm. Assume converse to the assertion of theorem 3, or equally that
\[
\lim_{p \to \infty} \theta(p)/\psi(p) > 0.
\] (4.4)
Since the system of functions \( \{\sin(nx)\} \) is bounded, \( z(\cdot) \in L^0(\Phi) \). By virtue of theorem 8 we conclude that
\[
\lim_{p \to \infty} |z|_p/\theta(p) = 0.
\]
Thus, we obtain from (4.3)
\[
\lim_{p \to \infty} \psi(p)/\theta(p) = 0,
\]
in contradiction with (4.4). The cases \( X = [0,2\pi]^d, X = \mathbb{R}^d \) are considered as well as the case \( X = [0,2\pi] \).

Now we consider the case when \( f \in G(a,b,\alpha,\beta) \).

**Proof** of theorem 4. Let \( f \in G(1,b,\alpha,0) \), \( ||f||_G(1,b,\alpha,0) = 1 \). Then \( \forall q \in (1,2] \)
\[
|f|_q \leq (q-1)^{-\alpha}.
\]
We denote \( p = q/(q-1) \), then \( p \in [2,\infty) \). We will use the classical result of Hardy - Littlewood, Hausdorff - Young ([3], p.193; [16], p. 93; [17], p. 26):
\[
|F[f]|_p \leq (2\pi)^{1/2} |f|_q.
\]
We have in our case:
\[
|F[f]|_p \leq (2\pi)^{1/2} (q-1)^{-\alpha} = (2\pi)^{1/2}(p/(p-1) - 1)^{-\alpha} < (2\pi)^{1/2} p^a.
\]
Our proposition it follows from theorem 10.

**Proof** of theorem 5. **A.** We will use the classical result of Paley and F.Riesz ([19], p.120). Let \( \{\phi_k(x), k = 1,2,\ldots\} \) be some orthonormal bounded sequence of functions. Then \( p \geq 2 \Rightarrow 
\]
\[
|f|_p \leq K_3 \cdot p \cdot \left( 1 + \sup_{k,x} |\phi_k(x)| \right) \cdot \left( \sum_k |c(k)|^p \left( |k|^{p-2} + 1 \right) \right)^{1/p},
\]
where \( K_3 \) is an absolute constant, \( f(x) = \sum_k c(k) \phi_k(x) \).

Let \( c(\cdot) \in g(\psi,\nu) \) and \( ||c||g(\psi,\nu) = 1 \). By definition of the \( g(\psi,\nu) \) norm
\[
\sum_k |c(k)|^p \left( k^{p-2} + 1 \right) \leq ||c||^p g(\psi,\nu) \cdot \psi^p(p).
\]
Therefore
\[
|f|_p \leq K_3 C_{31} p \cdot \psi(p),
\]
and by virtue of theorem 7 \( f(\cdot) \in L(N_1[\psi]) \).

**B.** Here we use the ”discrete” inequality of Hausdorff - Young, Hardy - Littlewood (see [16], p.101; [17], p.26:)
\[
|f|_p \leq K_4 |c|_q, \ p \geq 2, \ q = p/(p-1), \ K_4 = 2\pi.
\]
If $||c||g(\alpha) = 1$, then

$$|c|_q \leq (q - 1)^\alpha, |f|_p \leq K_4 p^\alpha, \ p \geq 2.$$ 

Again from theorem 7 follows that $f \in L \left( N_{1/\alpha} \right)$.

**Proof** of theorem 6. The analog of inequality (4.5) in the case

$$F[f](t) = \int_R \exp(itx)f(x)dx, \ d = 1,$$

namely:

$$|F[f]|_p \leq K_5 p |f|_p(\nu),$$

when $f(\cdot) \in L_p(\nu) \subset G(\nu, \alpha, \psi)$, see, for example, in ([16], p. 108). Hence for all $p \geq \alpha$

$$||F[f]||L \left( N^{(\alpha)}_d[\psi] \right) \leq K_5 ||f||G(\alpha; \psi, \nu).$$

(The generalization on the case $d \geq 2$ is evident).

Note that the moment estimations for the wavelet transforms and Haar series are described for example in the books [7], p. 21; [10], p.297 etc. It is easy to generalize our results on the cases Haar’s or wavelet series and transforms.

In detail, it is true in this cases the moment estimation for the partial sums (wavelet’s or Haar’s)

$$|P_M[f]|_p \leq K_6 |f|_p, \ X = [0, 1], \ p \geq 1,$$

where $K_6 = 13$ for Haar series on the interval $X = [0, 1]$ and $K_6 = 1$ for the classical wavelet series, in both the cases $X = [0, 1]$ and $X = R$. Hence $\forall \psi \in \Psi, f(\cdot) \in L(N[\psi])$

$$\sup_{M \geq 1} ||P_M[f] - f||L(N[\psi]) \leq (K_6 + 1) ||f||L(N[\psi]). \quad (4.6)$$

But (4.6) does not mean in general case the convergence

$$\lim_{M \to \infty} ||P_M[f] - f||L(N[\psi]) = 0, \quad (4.7)$$

as long as if (4.7) is true, then $f(\cdot) \in L^0(N[\psi])$ and conversely if $f \in L^0(N[\psi])$, then (4.7) holds.

For the different generalizations of wavelet series the estimation (4.6) with constants $K_6$ not depending on $p, \ p \geq 2$ see, for example, in the books ([5], p. 221; [18], p. 69).

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