ON COHOMOLOGY OF WITT VECTORS OF ALGEBRAIC INTEGERS AND A CONJECTURE OF HESSELHOLT

AMIT HOGADI AND SUPRIYA PISOLKAR

Abstract. Let $K$ be a complete discrete valued field of characteristic zero with residue field $k_K$ of characteristic $p > 0$. Let $L/K$ be a finite Galois extension with the Galois group $G$ and suppose that the induced extension of residue fields $k_L/k_K$ is separable. In [3], Hesselholt conjectured that $H^1(G, W_n(\mathcal{O}_L))$ is zero, where $\mathcal{O}_L$ is the ring of integers of $L$ and $W(\mathcal{O}_L)$ is the Witt ring of $\mathcal{O}_L$ w.r.t. the prime $p$. He partially proved this conjecture for a large class of extensions. In this paper, we prove Hesselholt’s conjecture for all Galois extensions.

1. Introduction

Let $p$ be a prime number and $K$ be a complete discrete valued field of characteristic zero with residue field $k_K$ of characteristic $p$. $L/K$ be a finite Galois extension of complete discrete valued fields as above with Galois group $G$. Suppose that $k_L/k_K$ is separable. In [3], Hesselholt conjectured that the proabelian group $\varprojlim H^1(G, W_n(\mathcal{O}_L))$ vanishes, where $W_n(\mathcal{O}_L)$ is the ring of Witt vectors of length $n$ in $\mathcal{O}_L$. As explained in [3], this can be viewed as an analogue of Hilbert theorem 90 for the Witt ring $W(\mathcal{O}_L)$.

In order to prove this conjecture one easily reduces to the case where $L/K$ is a totally ramified Galois extension of degree $p$ (see Lemma 4.1). For such extension, let $s = s(L/K)$ be the ramification break (see [1], III, (1.4)) in the ramification filtration of $\text{Gal}(L/K)$. Hesselholt proved the following theorem, which proves his conjecture for a large class of extensions.

Theorem 1.1 ([3]). Let $L/K$ be a totally ramified cyclic extension of order $p$ with $s > e_K/(p - 1)$. Then $\varprojlim H^1(G, W_n(\mathcal{O}_L))$ is zero.

The proabelian group $\varprojlim H^1(G, W_n(\mathcal{O}_L))$ can be identified with $H^1(G, W(\mathcal{O}_L))$ (see Corollary 2.2). The main result of this paper is,

Theorem 1.2. Let $L/K$ be a finite Galois extension of complete discrete valued fields with Galois group $G$. Then $H^1(G, W(\mathcal{O}_L))$ is zero.

Although the proof of Theorem 1.1 does not generalize (for instance due to use of [3, 2.2]), our proof, which is based on an observation on addition in Witt rings (see Lemma 3.2) relies on several ideas developed in [3]. One of these ideas which we use is the following.
Lemma 1.3. (3, 1.1) Let $L/K$ be as in Theorem 1.2. Let $m \geq 1$ be an integer and suppose that the induced map

$$H^1(G, W_{m+n}(\mathcal{O}_L)) \to H^1(G, W_n(\mathcal{O}_L))$$

is zero for $n = 1$. Then the same is true, for all $n \geq 1$. In particular

$$\lim_{\leftarrow} H^1(G, W_n(\mathcal{O}_L)) = 0$$

Thus, because of Lemmas 4.1, 1.3 and Corollary 2.2 to prove Theorem 1.2 it is enough to prove the following.

Theorem 1.4. Let $K$ be as above and $L/K$ be degree-$p$ totally ramified cyclic extension with Galois group $G$. Then there exists a positive integer $m \in \mathbb{N}$ such that the homomorphism $H^1(G, W_m(\mathcal{O}_L)) \to H^1(G, \mathcal{O}_L)$ is equal to zero.

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2. Preliminaries

In this section we show that $\lim_{\leftarrow} H^1(G, W_n(\mathcal{O}_L))$ coincides with $H^1(G, W(\mathcal{O}_L))$ (see Corollary 2.2). Note that in general group cohomology does not commute with inverse limits.

Proposition 2.1. Let $G$ be a finite group and $\{A_i\}_{i \in \mathbb{N}}$ be an inverse system of $G$ modules indexed by $\mathbb{N}$. For $j > i$, let $\phi_{ji}: A_j \to A_i$ denote the given maps. Then the following two statements hold.

(i) If $\phi_{ji}$ is surjective for all $j > i$ then

$$H^1(G, \lim_{\leftarrow} A_i) \to \lim_{\leftarrow} H^1(G, A_i)$$

is surjective.

(ii) If the induced maps $\phi_{ji}^G: A_j^G \to A_i^G$ are surjective for all $j > i$, then

$$H^1(G, \lim_{\leftarrow} A_i) \to \lim_{\leftarrow} H^1(G, A_i)$$

is injective.

Corollary 2.2. Let $L/K$ be a finite Galois extension of complete discrete valued fields. Then the natural map

$$\Phi : H^1(G, W(\mathcal{O}_L)) \to \lim_{\leftarrow} H^1(G, W_n(\mathcal{O}_L))$$

is an isomorphism.

Proof. By construction of Witt vectors, the projection maps

$$W_{n+1}(\mathcal{O}_L) \to W_n(\mathcal{O}_L)$$
are surjective. Thus by the above proposition, \( \Phi \) is surjective. In order to prove injectivity of \( \Phi \) we need to prove surjectivity of
\[
W_{n+1}(\mathcal{O}_L)^G \rightarrow W_n(\mathcal{O}_L)^G
\]
This follows from the fact that \( W_i(\mathcal{O}_L)^G = W_i(\mathcal{O}_K) \) for all \( i \) and from the surjectivity of the projection maps \( W_{n+1}(\mathcal{O}_K) \rightarrow W_n(\mathcal{O}_K) \). □

Proof of Proposition 2.1. (i) Suppose we are given an element \( \alpha \in \lim \leftarrow H^1(G, \mathcal{A}_i) \). This is equivalent to given \( \alpha_i \in H^1(G, \mathcal{A}_i) \) for all \( i \) such that \( \alpha_{i+1} \mapsto \alpha_i \). We now inductively construct cocycles \( a^i_g \) representing the class \( \alpha_i \) as follows. For \( i = 1 \), choose \( a^1_g \) arbitrarily. Now, suppose \( a^n_g \) has been constructed. Then construct \( a^{n+1}_g \) as follows. First start with any cocycle \( b^{n+1}_g \) which represents \( \alpha^{n+1}_n \). For an element \( b \in A_{n+1} \), let \( \overline{b} \) denote its image in \( A_n \). Thus \( \overline{b}^{n+1}_g \) is a cocycle in \( A_n \) which represents the same class as that represented by \( a^n_g \). Thus, there exists \( c \in A_n \) such that
\[
\overline{b}^{n+1}_g - a^n_g = gc - c
\]
Since by assumption, \( A_{n+1} \rightarrow A_n \) is surjective, there exists an element \( d \in A_{n+1} \) such that \( \overline{d} = c \). Now define
\[
a^{n+1}_g = \overline{b}^{n+1}_g - (gd - d)
\]
This completes the inductive construction of the cocycles \( a^i_g \). The cocycles have the property that for all \( i \) and \( g \),
\[
a^{i+1}_g \mapsto a^i_g
\]
and thus they define a cocycles with values in \( \lim \leftarrow \mathcal{A}_i \) whose class obviously maps to the element \( \alpha \) we started with.

(ii) Suppose \( \alpha \) is a class in \( H^1(G, \lim \leftarrow \mathcal{A}_i) \) which maps to zero in \( \lim \leftarrow H^1(G, \mathcal{A}_i) \), or equivalently maps to zero in \( H^1(G, \mathcal{A}_i) \) for each \( i \). Under the given assumption we will show that \( \alpha = 0 \). Choose a cocycle \( a_g \) representing \( \alpha \). By abuse of notation, we will denote the image of \( a_g \) in \( A_n \) by \( \overline{a}_g \). The \( n \) will be clear from context. For each \( n \), we will now inductively construct an element \( b_n \in A_n \) such that
\[
a_g = gb_n - b_n \quad \forall \ g \in G
\]
and for all \( n \), \( b_{n+1} \) maps to \( b_n \). For \( n = 1 \), we know that the image of \( a_g \) in \( A_1 \) is a coboundary. Thus there exists an element \( b_1 \in A_1 \) such that
\[
\overline{a}_g = gb_1 - b_1 \quad \forall \ g \in G
\]
Now suppose we have defined \( b_n \). To define \( b_{n+1} \) we first choose an element \( c_{n+1} \in A_{n+1} \) such that
\[
\overline{a}_g = gc_{n+1} - c_{n+1} \quad \forall \ g \in G
\]
However the image of \( c_{n+1} \) in \( A_n \), denoted by \( \overline{c}_{n+1} \), satisfies
\[
g \overline{c}_{n+1} - \overline{c}_{n+1} = gb_n - b_n
\]
which means, there exists a \( d \in A_n^G \) such that
\[
b_n = \overline{c}_{n+1} + d
\]
Since the map \( A_n^G \rightarrow A_n^G \) is assumed to be surjective, we can lift \( d \) to an element \( \tilde{d} \in A_n^G \). Now define
\[
b_{n+1} = c_{n+1} + \tilde{d}
\]
The \( b' \) defined above are compatible elements and hence define an element \( b \) of \( \varprojlim A_i \). Also, from the construction it is clear that
\[
a_g = gb - b \quad \forall \ g \in G
\]
holds, since it holds after taking image in \( A_i \) for all \( i \). Thus the cocycle \( a_g \) is actually a coboundary and hence the class \( \alpha \) we started with is trivial. □

3. Remarks on addition of Witt vectors

The main observation of this section is Lemma 3.2, which lies at the heart of the proof of Theorem 1.2. We first recall from ([4], II) how addition of Witt vectors is defined. For every positive integer \( n \), define polynomials \( w_n \in \mathbb{Z}[X_1, \ldots, X_n] \) by
\[
w_n(X_1, \ldots, X_n) = X_1^{p^n-1} + pX_2^{p^n-2} + p^2X_3^{p^n-3} + \cdots + p^{n-1}X_n
\]
One now defines addition of Witt vectors (thanks to Theorem 3.1) in such a way that if
\[(1) \quad (X_1, \ldots, X_n) + (Y_1, \ldots, Y_n) = (Z_1, \ldots, Z_n)\]
then
\[
w_i(X_1, \ldots, X_i) + w_i(Y_1, \ldots, Y_i) = w_i(Z_1, \ldots, Z_i) \quad \forall \ i \leq n
\]

**Theorem 3.1.** ([4], II. §6) For every positive integer \( n \), there exists a unique \( \phi_n \in \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n] \) such that
\[
w_n(X_1, \ldots, X_n) + w_n(Y_1, \ldots, Y_n) = w_n(\phi_1, \ldots, \phi_n) \quad \forall \ n \in \mathbb{N}
\]
In other words, in equation (1) above
\[
Z_i = \phi_i(X_1, \ldots, X_i, Y_1, \ldots, Y_i)
\]
Note that since \( \phi_i' \)s are polynomials with integral coefficients, the expression makes sense in all characteristics.

We now consider addition of \( p \) Witt vectors. Let
\[(x_{11}, \ldots, x_{1n}) + \ldots + (x_{p1}, \ldots, x_{pn}) = (z_1, \ldots, z_n)\]
By above discussion, for every \( i \leq n \), there exist polynomials in \( ni \) variables, \( g_i \in \mathbb{Z}[X_{11}, \ldots, X_{1i}, \ldots, X_{pi}] \) such that
\[
z_i = g_i(x_{11}, \ldots, x_{1i}, \ldots, x_{pi}, \ldots, x_{pi})
The following observation is about the nature of these polynomials.

**Lemma 3.2.** Let $R$ be a ring, $p$ be a prime, $n \in \mathbb{N}$ and $W_n(R)$ be the ring of Witt vectors of length $n$. Let $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in W_n(R)$ for $1 \leq i \leq p$. Let

$$(z_1, z_2, \cdots, z_n) := \sum_{i=1}^{p} (x_{i1}, x_{i2}, \cdots, x_{in})$$

(1) For all $1 \leq \ell \leq n$ there exists a polynomial expression $f_\ell \in \mathbb{Z}[\{x_{ij}\}]$ where $1 \leq i \leq p, 1 \leq j \leq \ell - 1$ such that

$$z_\ell = \sum_{i=1}^{p} x_{i\ell} + f_\ell$$

where each monomial of $f_\ell$ has degree $\geq p$.

(2) There exists a polynomial expression $h_{\ell-2} \in \mathbb{Z}[\{x_{ij}\}]$ where $1 \leq i \leq p, 1 \leq j \leq \ell - 2$ such that

$$f_\ell = \frac{\sum_{i=1}^{p} x_{i,\ell-1}^p - (\sum_{i=1}^{p} x_{i,\ell-1})^p}{p} + \frac{1}{p} \sum_{j=1}^{p-1} \binom{p}{j} \sum_{i=1}^{p} x_{i,\ell-1} f_{j-1}^p + h_{\ell-2}$$

and each monomial appearing in $h_{\ell-2}$ has degree $\geq p^2$.

**Proof.** (1) By definition of addition of Witt vectors in Witt ring we have

$$\sum_{i=1}^{p} w_\ell(x_{i1}, \ldots, x_{ip}) = w_\ell(z_1, \ldots, z_\ell)$$

Using the expression for the polynomials $w_\ell$ and rearranging, we get

$$z_\ell = \sum_{i=1}^{p} x_{i\ell} + f_\ell$$

where

$$f_\ell = \frac{1}{p^{\ell-1}} \left( \sum_{i=1}^{p} x_{i,1}^{p-1} - z_1^{p-1} \right) + \cdots + \frac{1}{p} \left( \sum_{i=1}^{p} x_{i,(\ell-1)}^p - z_{\ell-1}^p \right)$$

The claim that $f_\ell$ has integral coefficients follows from Theorem 3.1. Note that each $z_t, t \leq \ell$ in the above expression is again a polynomial expression in the variables $x'_{ij}s, j \leq t$. It is straightforward to observe from the expression of $f_\ell$ that every monomial appearing in the expression has degree $\geq p$.

(2) Substitute $z_{\ell-1} = \sum_{j=1}^{p} x_{j,(\ell-1)} + f_{\ell-1}$ in the expression of $f_\ell$ and rewrite $f_\ell$ as

$$f_\ell = \left( \frac{\sum_{i=1}^{p} x_{i,\ell-1}^p}{p} - (\sum_{i=1}^{p} x_{i,\ell-1})^p \right) - \frac{1}{p} \sum_{j=2}^{p-1} \binom{p}{j} \left( \sum_{i=1}^{p} x_{i,(\ell-1)}^p \right)^{p-j} \cdot f_{j-1}^p + h_{\ell-2}$$

where
\[ h_{\ell-2} = -\frac{1}{p} f_{\ell-1}^p + \frac{1}{p} \left( x_{1(\ell-2)}^p + x_{2(\ell-2)}^p + \cdots + x_{p(\ell-2)}^p - z_{\ell-2}^p \right) + \cdots \]

Note that since \( p \) is a prime number, every binomial coefficient \( \binom{p}{j} \) with \( 1 \leq j < p \), is divisible by \( p \). Thus the first two terms in the above expressions of \( f \) have integral coefficients. Since we know that \( f \) has integral coefficients, it follows that \( h_{\ell-2} \) has integral coefficients too. Moreover, since all monomials appearing in \( f_{\ell-1} \) have degree \( \geq p \), all monomials appearing in \( f_{\ell-1}^p \) have degree \( \geq p^2 \). It is also clear that for \( 1 \leq i \leq \ell - 2 \), all monomials in

\[ \frac{1}{p^{e_i}} \left( x_{1i}^{p^{e_i}-i} + x_{2i}^{p^{e_i}-i} + \cdots + x_{pi}^{p^{e_i}-i} - z_i^{p^{e_i}-i} \right) \]

have degree \( \geq p^2 \). This shows that all monomials appearing in the expression of \( h_{\ell-2} \) have degree \( \geq p^2 \).

4. PROOF OF THE MAIN THEOREM

We will prove Theorem 1.2 in this section.

Lemma 4.1. Let \( p \) be a prime number and \( L/K \) be a finite Galois extension of complete discrete fields with \( G = \text{Gal}(L/K) \). Suppose that \( k_L/k_K \) is separable. Then the following two statements are equivalent.

(i) \( H^1(G, W(O_L)) = 0 \) for all extensions \( L/K \) as above.
(ii) \( H^1(G, W(O_L)) = 0 \) for all \( L/K \) as above which are ramified and of degree \( p \).

Proof. \( (i) \implies (ii) \) is obvious. Now we prove \( (i) \) assuming \( (ii) \).

Let \( L/K \) be any Galois extension of complete discrete valued fields. Let \( L' \) be the maximal subfield of \( L \) which is tamely ramified over \( K \). The extension \( L'/K \) is Galois and let \( H = \text{Gal}(L/L') \). Since \( L'/K \) is tame, \( O_{L'} \) is a projective \( O_K[G/H] \) module (see [2], I. Theorem(3)) which can be used to show the vanishing of \( H^1(G/H, W(O_{L'})) \). Moreover, because of the following inflation-restriction exact sequence

\[ 0 \to H^1(G/H, W(O_{L'})) \xrightarrow{\text{inf}} H^1(G, W(O_L)) \xrightarrow{\text{res}} H^1(H, W(O_L)) \]

vanishing of \( H^1(G, W(O_L)) \) is implied by that of \( H^1(H, W(O_L)) \). Thus without loss of generality, we may replace \( K \) by \( L' \) and assume that our extension \( L/K \) is totally wildly ramified Galois extension. Thus \( G \) is a \( p \)-group. Since any \( p \)-group has a normal subgroup of index \( p \), again by induction and inflation-restriction exact sequence, we reduce ourselves to the case when \( L/K \) is of degree \( p \). But in this case the vanishing of \( H^1(G, W(O_L)) \) is guaranteed by \( (ii) \). This proves the lemma.

\[ \square \]
Let $G$ be any finite cyclic group with a generator $\sigma$. Let $M$ a $G$-module. Then the cohomology group $H^i(G, M)$ is isomorphic to the $i^{th}$ cohomology group of the complex

$$M \xrightarrow{1-\sigma} M \xrightarrow{\text{tr}} M \xrightarrow{1-\sigma} M \xrightarrow{\text{tr}} M \to \cdots$$

where for $a \in M$, $\text{tr}(a) = \sum_{g \in G} ga$. Thus in the case at hand, where $L/K$ is a cyclic Galois extension, we have a canonical isomorphism

$$H^1(G, W_m(O_L)) \cong W_m(O_L)^{\text{tr}=0}/(\sigma - 1)W_m(O_L)$$

Henceforth, for $K$ as before, we assume $L/K$ is a totally ramified cyclic extension of degree $p$. For such an extension we will denote by $s$ the ramification break. To prove the theorem 1.4 we need following lemmas and results from [3].

**Lemma 4.2.** ([3], 2.4) Let $L/K$ be as above. Suppose that $x \in O_L^{\text{tr}=0}$ represents a non-zero class in $H^1(G, O_L)$. Then $v_L(x_1) \leq s - 1$.

**Lemma 4.3.** ([3], 2.1) Let $L/K$ be as above. For all $a \in O_L$,

$$v_K(\text{tr}(a)) \geq (v_L(a) + s(p - 1))/p$$

**Lemma 4.4.** ([3], 2.2.) Let $L/K$ be as above. For all $a \in O_L$,

$$v_K(\text{tr}(a^p) - \text{tr}(a)^p) = e_K + v_L(a)$$

**Lemma 4.5.** Let $x = (x_1, x_2, \cdots, x_n) \in W_n(O_L)^{\text{tr}=0}$ then for all $1 \leq \ell \leq n$

$$-\text{tr}(x_\ell) = \frac{\text{tr}(x_\ell^p) - \text{tr}(x_{\ell-1})^p}{p} - C.\text{tr}(x_{\ell-1})^p + h_{\ell-2}$$

where $C$ is the integer defined by

$$C = \frac{1}{p} \sum_{j=1}^{p-1} (-1)^j \binom{p}{j}$$

and $h_{\ell-2}$ is a polynomial expression in $(x_1, \cdots, x_{\ell-2})$ and it’s all $p - 1$ conjugates. Further each monomial of $h_{\ell-2}$ is of degree $\geq p^2$.

**Proof.** Since $x \in W_n(O_L)^{\text{tr}=0}$ we have

$$\sum_{i=1}^{p}(\sigma^{i-1}x_1, ..., \sigma^{i-1}x_n) = (0, \ldots, 0)$$

Thus the above claim follows directly from Lemma 3.2(2) by making the substitutions

$$x_{ij} = \sigma^{i-1}x_j \quad 1 \leq i \leq n, \quad 1 \leq j \leq n$$

and

$$z_i = 0 \quad \forall \ 1 \leq i \leq n$$

□
Lemma 4.6. For \( \ell \geq 2 \), \( h_{\ell-2} \in \mathcal{O}_K \). Further

\[
v_K(h_{\ell-2}) \geq p \cdot \min\{v_L(x_i) \mid 1 \leq i \leq \ell - 2\}
\]

Proof. The polynomial expression for \( h_{\ell-2} \) in \( x_1, ..., x_n \) and its conjugates can be seen to be invariant under the Galois action. Hence it belongs to \( \mathcal{O}_K \). Further since \( h_{\ell-2} \) is a sum of monomials in \( x_i^s, i \leq \ell - 2 \) and its conjugates, each of degree \( \geq p^2 \), we have

\[
v_L(h_{\ell-2}) \geq p^2 \cdot \min\{v_L(x_i) \mid 1 \leq i \leq \ell - 2\}
\]

The lemma now follows from the fact that \( v_L(h_{\ell-2}) = p \cdot v_K(h_{\ell-2}) \).

Proof of Theorem 1.4. By the Lemma 4.2, to prove the Theorem 1.4 it is sufficient to find \( M \in \mathbb{N} \) such that, for all \( x = (x_1, ..., x_M) \in W_M(\mathcal{O}_L)^{tr=0} \), \( v_L(x_1) \geq s \).

Step(1): Let \( n \) be a positive integer and \((x_1, ..., x_n) \in W_n(\mathcal{O}_L)^{tr=0} \). We will prove by induction on \( \ell \) that \( v_L(x_\ell) \geq \frac{s(p-1)}{p} \) for \( 1 \leq \ell \leq n - 1 \).

By Lemma 4.3 and using the fact that \( h_0 = 0 \), \( tr(x_1) = 0 \) we have

\[
-tr(x_2) = \frac{1}{p}(tr(x_1^p) - tr(x_1)^p)
\]

But by Lemma 4.3 \( v_K(tr(x_2)) \geq \frac{s(p-1)}{p} \). Thus

\[
v_K(tr(x_1^p) - tr(x_1)^p) - e_K = v_K(tr(x_2)) \geq \frac{s(p-1)}{p}
\]

By Lemma 4.4 \( v_K(tr(x_1^p) - tr(x_1)^p) = v_L(x_1) + e_K \). Therefore \( v_L(x_1) \geq \frac{s(p-1)}{p} \). This proves the claim for \( \ell = 1 \).

Now assume that for all \( i \leq \ell - 1 \), \( v_L(x_i) \geq \frac{s(p-1)}{p} \). We will prove \( v_L(x_\ell) \geq \frac{s(p-1)}{p} \).

By Lemma 4.3 we have

\[
-tr(x_{\ell+1}) = \frac{tr(x_\ell^p) - tr(x_\ell)^p}{p} - C \cdot tr(x_\ell)^p + h_{\ell-1}
\]

Thus, using Lemma 4.3 we get

\[
v_L(x_\ell) = v_K(\frac{tr(x_\ell^p) - tr(x_\ell)^p}{p}) \geq \inf\{v_K(tr(x_{\ell+1})), v_K(C \cdot tr(x_\ell)^p), v_K(h_{\ell-1})\}
\]

Using Lemma 4.3 we have

\[
v_K(tr(x_{\ell+1})) \geq \frac{s(p-1)}{p} \quad \text{and} \quad v_K(C \cdot tr(x_\ell)^p) \geq s(p-1)
\]
By Lemma 4.6 and by induction hypothesis \( v_K(h_{\ell-1}) \geq s(p-1) \). Combining the above, we get

\[
v_L(x_\ell) \geq \frac{s(p-1)}{p}
\]

Step(2): Now we will show existance of \( M \in \mathbb{N} \) such that for all \( x_1 \in W_M(O_L) \), \( v_L(x_1) \geq s \). For any positive integer \( n \) and \( (x_1, \ldots, x_n) \in W_n(O_L)^{tr=0} \), by Step(1) we have

\[
v_L(x_i) \geq \frac{s(p-1)}{p}, \quad \forall \ 1 \leq i \leq n - 1.
\]

For a fixed \( n \), and \( 1 \leq i \leq n - 1 \), we claim that

\[
v_L(x_{n-i}) \geq \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i-1}}\right)
\]

We prove this by induction on \( i \). For \( i = 1 \), this is the claim that

\[
v_L(x_{n-1}) \geq \frac{s(p-1)}{p}
\]

which follows from Step(1). Now assuming the claim for a general \( 1 \leq i \leq n - 2 \), we will prove it for \( i + 1 \). By induction hypothesis

\[
v_L(x_{n-i}) \geq \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i-1}}\right)
\]

Therefore by using Lemma 4.3 we get

\[
v_K(tr(x_{n-i})) \geq \frac{v_L(x_{n-i}) + s(p-1)}{p} \geq \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i}}\right)
\]

By Lemma 4.5

\[-tr(x_{n-i}) = \frac{tr(x_{n-i+1}) - tr(x_{n-(i+1)})^p - C \cdot tr(x_{n-(i+1)})^p + h_{n-(i+2)}}{p}
\]

By Lemma 4.3 \( v_K(C \cdot tr(x_{n-(i+1)})^p) \geq s(p-1) \). By Step(1) and Lemma 4.6 \( v_K(h_{n-(i+2)}) \geq s(p-1) \). Thus, using Lemma 4.4

\[
v_L(x_{n-(i+1)}) = v_K \left(\frac{tr(x_{n-(i+1)})^p - tr(x_{n-(i+1)})^p}{p}\right)
\]

\[
\geq \min \{ v_K(tr(x_{n-i})), v_K(C \cdot tr(x_{n-(i+1)})^p), v_K(h_{n-(i+2)}) \}
\]

\[
\geq \min \left\{ \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p}^{i-1}\right), s(p-1), s(p-1) \right\}
\]

\[
= \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i-1}}\right)
\]

This proves the claim. Hence

\[
v_L(x_1) \geq \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{n-2}}\right)
\]
We know that as \( n \to \infty \), 
\[
\frac{s(p-1)}{p} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{n-2}} \right) \to s.
\]
There exists an integer \( M \), such that
\[
\frac{s(p-1)}{p} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{M-2}} \right) > s - 1
\]
Since \( v_L \) is a discrete valuation, for such \( M \) and for any \( (x_1, ..., x_M) \in \Omega_L^{tr=0} \), we have shown that
\[
v_L(x_1) \geq s
\]
\[\square\]

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