Existence Result for Non-linearly Perturbed Hardy-Schrödinger Problems: Local and Non-local cases

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain having zero in its interior $0 \in \Omega$. We fix $0 < \alpha \leq 2$ and $0 \leq s < \alpha$. We investigate a sufficient condition for the existence of a positive solution for the following perturbed problem associated with the Hardy-Schrödinger operator $L_{\gamma, \alpha} := (-\Delta)^{\alpha/2} - \frac{\gamma}{|x|^{\alpha}}$ on $\Omega$:

$$
\begin{cases}
(-\Delta)^{\alpha/2} u - \frac{\gamma u}{|x|^\alpha} - \lambda u = \frac{u^{2^*_{\alpha}(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
$$

where $2^*_{\alpha}(s) := \frac{2(n-s)}{n-\alpha}$, $\lambda \in \mathbb{R}$, $h \in C^0(\overline{\Omega})$, $h \geq 0$, $q \in (2, 2^*_{\alpha})$ with $2^*_{\alpha} := 2^*_{\alpha}(0)$, and $\gamma < \gamma_H(\alpha)$, the latter being the best constant in the Hardy inequality on $\mathbb{R}^n$.

We prove that there exists a threshold $\gamma_{crit}(\alpha)$ in $(-\infty, \gamma_H(\alpha))$ such that the existence of solutions of the above problem is guaranteed by the non-linear perturbation (i.e., $h(x)u^{q-1}$) whenever $\gamma \leq \gamma_{crit}(\alpha)$, while for $\gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha)$, it is determined by a subtle combination of the geometry of the domain and the size of the nonlinearity of the perturbations.

1 Introduction

Let $h(x)$ be a non-negative function in $C^0(\overline{\Omega})$. Given $0 < \alpha \leq 2$ and $0 \leq s < \alpha$, we consider the following perturbed problem associated with the operator $L_{\gamma, \alpha} := (-\Delta)^{\alpha/2} - \frac{\gamma}{|x|^\alpha}$ on bounded domains $\Omega \subset \mathbb{R}^n(n > \alpha)$ with $0 \in \Omega$:

$$
\begin{cases}
(-\Delta)^{\alpha/2} u - \frac{\gamma u}{|x|^\alpha} - \lambda u = \frac{u^{2^*_{\alpha}(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
$$

where $2^*_{\alpha}(s) := \frac{2(n-s)}{n-\alpha}$, $q \in (2, 2^*_{\alpha})$ with $2^*_{\alpha} := 2^*_{\alpha}(0)$, $\lambda \in \mathbb{R}$, and $\gamma < \gamma_H(\alpha) := \frac{2\alpha \Gamma_n}{\Gamma_{2\alpha} \frac{\Gamma_n}{\Gamma_{n-\alpha}}}$, the latter being the best constant in the Hardy inequality on $\mathbb{R}^n$ defined in (3). The structure of the operator $(-\Delta)^{\alpha/2}$ has broad influence on problem (1). More precisely, if $\alpha = 2$, this operator is defined as

$$
(-\Delta)^{1/2} u := -\Delta u = \sum_{i=0}^{n} \frac{\partial^2 u}{\partial x_i^2}, \quad \text{for } u \in C^2.
$$
It is well-known that the classical Laplacian $-\Delta$ is a local operator. Therefore, problem (1) involves a local equation when $\alpha = 2$. While, for $0 < \alpha < 2$, the operator $(-\Delta)^{\alpha}$ has more complicated structure. Indeed, for these values of $\alpha$, the operator is non-local and it is defined as

$(-\Delta)^{\alpha} u := F^{-1}(|2\pi \xi|^\alpha (F u)) \quad \forall \xi \in \mathbb{R}^n, \quad \text{for } u \in \mathcal{S},$

where $\mathcal{S}$ is the Schwartz class (space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^n$) and $F u$ denotes the Fourier transform of $u$. Hence, the operator $(-\Delta)^{\alpha}$ and problem (1) are non-local when $0 < \alpha < 2$.

When $(h \equiv 0)$, problem (1) has been studied in the both local and non-local cases. See [9] and [10], and the references therein. In [13], Jaber considered the local problem (i.e., $\alpha = 2$) in the Riemannian context but in the absence of the Hardy term (i.e., $\gamma = 0$). A similar problem, with the second order operator replaced by the fourth order Paneitz operator, was studied by Esposito-Robert [7] (see also Djadli-Hebey-Ledoux [6]). The author in [15] addressed questions regarding the existence and multiplicity of solutions of problem (1) in the case when $\lambda = 0$ and $1 < q < 2$ (i.e., the Concave-Convex non-linearity). In this paper, we consider the remaining cases.

By using ideas from [10] and [13], we investigate the role of the linear perturbation (i.e., $hu$), the non-linear perturbation (i.e., $hu^{q-1}$), and the geometry of the domain on the existence of a positive solution of (1). As in Jaber [13], our main tool here to investigate the existence of solutions is the following Mountain Pass Lemma of Ambrosetti-Rabinowitz [1]:

**Lemma 1.1** (Ambrosetti and Rabinowitz [1]). Let $(V, \|\|)$ be a Banach space and $\Psi : V \to \mathbb{R}$ a $C^1$—functional satisfying the following conditions:

(a) $\Psi(0) = 0$,
(b) There exist $\rho, R > 0$ such that $\Psi(u) \geq \rho$ for all $u \in V$, with $\|u\| = R$,
(c) There exists $v_0 \in V$ such that $\limsup_{t \to \infty} \Psi(tv_0) < 0$.

Let $t_0 > 0$ be such that $\|t_0 v_0\| > R$ and $\Psi(t_0 v_0) < 0$, and define

$c_{v_0}(\Psi) := \inf_{\sigma \in \Gamma} \sup_{t \in [0,1]} \Psi(\sigma(t)),$

where

$\Gamma := \{ \sigma \in C([0,1], V) : \sigma(0) = 0 \text{ and } \sigma(1) = t_0 v_0 \}.$

Then, $c_{v_0}(\Psi) \geq \rho > 0$, and there exists a Palais-Smale sequence at level $c_{v_0}(\Psi)$, that is there exists a sequence $(w_k)_{k \in \mathbb{N}} \in V$ such that

$$\lim_{k \to \infty} \Psi(w_k) = c_{v_0}(\Psi) \quad \text{and} \quad \lim_{k \to \infty} \Psi'(w_k) = 0 \quad \text{strongly in } V'.$$

Inspired by the work of Jannelli [14], it was shown in [10] that the behaviour of problem (1) is deeply influenced by the value of the parameter $\gamma$. More precisely, there exists a threshold $\gamma_{\text{crit}}(\alpha) \in (-\infty, \gamma_H(\alpha))$ such that the operator $L_{\gamma,\alpha}$ becomes critical in the following sense:

**Definition 1.2.** We say that the Hardy-Schrödinger operator $L_{\gamma,\alpha}$ is critical,

- for $0 < \alpha < 2$, if $\gamma_{\text{crit}}(\alpha) < \gamma < \gamma_H(\alpha)$ when $n \geq 2\alpha$, or $0 \leq \gamma < \gamma_H(\alpha)$ when $\alpha < n < 2\alpha$.
- for $\alpha = 2$, if $\gamma_{\text{crit}}(2) < \gamma < \gamma_H(2)$ when $n \geq 4$, or $\gamma < \gamma_H(2)$ when $n = 3$.

Otherwise, the operator $L_{\gamma,\alpha}$ is called non-critical.

Our analysis shows that the existence of a solution for problem (1) depends only on the non-linear perturbation when the operator $L_{\gamma,\alpha}$ is non-critical, while the critical case is more complicated and depends on other conditions involving both the perturbation and the global geometry of the domain. More precisely, in the non-critical case, the competition is between the linear and non-linear
perturbations, and since \( q > 2 \), the non-linear term dominates. In the critical case, this competition is more challenging as it is between the geometry of the domain (i.e., the mass) and the non-linear perturbation. In this situation, there exists a threshold \( q_{\text{crit}}(\alpha) \in (2, 2^{\ast}_n) \), where the dominant factor switches from the non-linear perturbation to the mass. The transition at \( 2^{\ast}_n \) is most interesting. We shall establish the following result.

**Theorem 1.3.** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n(n > \alpha) \) such that \( 0 \in \Omega \), and let \( 2^{\ast}_n(s) := \frac{2(n-s)}{n-\alpha}, 0 \leq s < \alpha, 0 < \alpha \leq 2, -\infty < \lambda < \lambda_1(L_{\gamma,\alpha}), \) and \( \gamma < \gamma_H(\alpha) \). We also assume that \( 2 < q < 2^{\ast}_n \), \( h \in C^0(\Omega) \) and \( h > 0 \). Then, there exists a non-negative solution \( u \in H^2_0(\Omega) \) to (1) under one of the following conditions:

1. \( L_{\gamma,\alpha} \) is not critical and \( h(0) > 0 \).

2. \( L_{\gamma,\alpha} \) is critical and

\[
\begin{cases}
  h(0) > 0 & \text{if } q > q_{\text{crit}}(\alpha) \\
  c_1 h(0) + c_2 m^\gamma_{\gamma,\alpha}(\Omega) > 0 & \text{if } q = q_{\text{crit}}(\alpha) \\
  m^\gamma_{\gamma,\alpha}(\Omega) > 0 & \text{if } q < q_{\text{crit}}(\alpha).
\end{cases}
\]

Here \( c_1, c_2 \) are two positive constants that can be computed explicitly (see Subsection 2.2.3), while \( q_{\text{crit}}(\alpha) = 2^{\ast}_n - 2 \frac{\Gamma(n-\alpha-2)}{\Gamma(\alpha+2)} \in (2, 2^{\ast}_n) \). Also, \( \beta_{\ast}(\gamma,\alpha) \) (resp., \( \beta_{\ast}(\gamma,\alpha) \)) is the unique solution in \((0, \frac{\alpha}{n-\alpha})\) (resp., in \((\frac{\alpha}{n-\alpha}, n-\alpha)\)) of the equation

\[
\Psi_{\gamma,\alpha}(t) := 2^n \frac{\Gamma(n-\alpha)}{\Gamma(n-\alpha-2)} = \gamma.
\]

One can then complete the picture as follows.

| Non-linearly perturbed problem (1): with \( 0 \in \Omega : 2 < q < 2^{\ast}_n \) and \( \lambda < \lambda(L_{\gamma,\alpha}) \) |
|---|
| **Operator** | **\( L_{\gamma,\alpha} \)** | **Singularity** | **\( q \)** | **\( \lambda \)** | **Analytic. cond** | **Ext.** |
| Not Critical | \( s \geq 0 \) | \( > 2 \) | \( -\infty \) | \( h(0) > 0 \) | Yes |
| Critical | \( s \geq 0 \) | \( > q_{\text{crit}}(\alpha) \) | \( -\infty \) | \( h(0) > 0 \) | Yes |
| Critical | \( s \geq 0 \) | \( = q_{\text{crit}}(\alpha) \) | \( -\infty \) | \( c_1 h(0) + c_2 m^\gamma_{\gamma,\alpha}(\Omega) > 0 \) | Yes |
| Critical | \( s \geq 0 \) | \( < q_{\text{crit}}(\alpha) \) | \( > 0 \) | \( m^\gamma_{\gamma,\alpha}(\Omega) > 0 \) | Yes |

Theorem 1.3 suggests the following remarks:

**Remark 1.4.** The notation \( m^\gamma_{\gamma,\alpha}(\Omega) \) stands for the mass associated to the operator \( L_{\gamma,\alpha} - \lambda I \) which is defined in Theorem 1.2 in [19] for \( 0 < \alpha < 2 \), and in Proposition 3 in [19] for \( \alpha = 2 \).

**Remark 1.5.** When \( \alpha = 2 \), problem (1) becomes local, and it can be written as follows

\[
\begin{cases}
  -\Delta u - \gamma \frac{u}{|x|^2} - \lambda u = \frac{u^{2^\ast_n(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{in } \Omega \\
  u \geq 0 & \text{in } \Omega, \\
  u = 0 & \text{in } \partial \Omega,
\end{cases}
\]

where \( u \) belongs to the space \( H^2_0(\Omega) \), which is the completion of \( C^\infty_c(\Omega) \) with respect to the norm

\[
\|u\|^2_{H^2_0(\Omega)} = \int_\Omega |\nabla u|^2 dx.
\]

**Remark 1.6.** Note that the best Hardy constant \( \gamma_H(\alpha) \) and the critical threshold \( q_{\text{crit}}(\alpha) \) can be computed explicitly when \( \alpha = 2 \); see [19]. Indeed, we have

\[
\gamma_{\text{crit}}(2) := \frac{(n-2)^2}{4} - 1 \quad \text{and} \quad \gamma_H(2) := \frac{(n-2)^2}{4}.
\]
Remark 1.7. We point out that the value \( q_{\text{crit}}(\alpha) \) corresponds to the value \( q = 4 \) obtained in [3 Proposition 3]. Indeed, when \( \alpha = 2 \), \( \gamma = 0 \) and \( n = 3 \), our problem turns to the perturbed Hardy-Sobolev equation considered by Jaber [13] in the Riemannian setting. We then have that \( \beta_+(2,0) = n - \alpha = 1, \beta_-(2,0) = 0 \), and therefore
\[
q_{\text{crit}}(2) = 2\alpha - 2 \frac{\beta_+(2,0) - \beta_-(2,0)}{n - \alpha} = 6 - 2 = 4.
\]

2 The non-local case

Throughout this section, we shall assume that \( 0 < \alpha < 2 \), which means \( (-\Delta)^\frac{\alpha}{2} \) is not a local operator. We start by recalling and introducing suitable function spaces for the variational principles that will be needed in the sequel. We shall study problems on bounded domains, but will start by recalling the properties of \( (-\Delta)^\frac{\alpha}{2} \) on the whole of \( \mathbb{R}^n \), where it can be defined on the Schwartz class \( S \) (the space of rapidly decaying \( C^\infty \) functions on \( \mathbb{R}^n \)) via the Fourier transform,
\[
(-\Delta)^\frac{\alpha}{2} u = \mathcal{F}^{-1}(|2\pi \xi|^\alpha \mathcal{F}(u)).
\]
Here, \( \mathcal{F}(u) \) is the Fourier transform of \( u \), \( \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} u(x) \, dx \). See Servadei-Valdinoci [5] and references therein for the basics on the fractional Laplacian. For \( \alpha \in (0,2) \), the fractional Sobolev space \( H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \) is defined as the completion of \( C^\infty_0(\mathbb{R}^n) \) under the norm
\[
\|u\|^2_{H_0^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |2\pi \xi|^\alpha |\mathcal{F}u(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} u|^2 \, dx.
\]
By Proposition 3.6 in Di Nezza-Palatucci-Valdinoci [5] (see also Frank-Lieb-Seiringer [8]), the following relation holds: For \( u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} |2\pi \xi|^\alpha |\mathcal{F}u(\xi)|^2 \, d\xi = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} \, dx \, dy,
\]
where \( C_{n,\alpha} = \frac{2\Gamma \left( \frac{n+\alpha}{2} \right)}{\pi \Gamma \left( \frac{n-\alpha}{2} \right)} \).

The fractional Hardy inequality in \( \mathbb{R}^n \) then states that
\[
\gamma_H(\alpha) := \inf \left\{ \frac{\int_{\mathbb{R}^n} \left[ (-\Delta)^{\frac{\alpha}{2}} u \right]^2 \, dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{n+\alpha}} \, dx} : u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\} \right\} = 2^{\alpha} \frac{\Gamma^2 \left( \frac{n+\alpha}{2} \right)}{\Gamma^2 \left( \frac{n-\alpha}{2} \right)},
\]
which means that the fractional Hardy-Schrödinger operator \( L_{\gamma,\alpha} \) is positive whenever \( n > \alpha \) and \( \gamma < \gamma_H(\alpha) \). The best constant in Hardy-Sobolev type inequalities on \( \mathbb{R}^n \) defined as
\[
\mu_{\gamma,s,\alpha}(\mathbb{R}^n) := \inf_{u \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \left[ (-\Delta)^{\frac{\alpha}{2}} u \right]^2 \, dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{n+\alpha}} \, dx}{\left( \int_{\mathbb{R}^n} \frac{|u|^{2s(\gamma)}}{|x|^\gamma} \, dx \right)^{\frac{2}{s}}},
\]
We shall use the extremal of \( \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \) and its profile at zero and infinity to build appropriate test-functions for the functional under study. Note that any minimizer for (4) leads –up to a constant– to a variational solution of the following borderline problem on \( \mathbb{R}^n \),
\[
\left\{ \begin{array}{ll}
(-\Delta)^{\frac{\alpha}{2}} u - \gamma \frac{u}{|x|^{\gamma-1}} = \frac{u^{s(\gamma)}(\cdot)^{-1}}{|x|^\gamma} & \text{in } \mathbb{R}^n \\
u \geq 0; \ u \neq 0 & \text{in } \mathbb{R}^n.
\end{array} \right.
\]
Let now \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \) with 0 in its interior. We then consider the fractional Sobolev space \( H_0^{\frac{\alpha}{2}}(\Omega) \) as the closure of \( C^\infty_0(\Omega) \) with respect to the norm
\[
\|u\|^2_{H_0^{\frac{\alpha}{2}}(\Omega)} = \frac{C_{n,\alpha}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} \, dx \, dy.
\]
Note first that inequality (3) asserts that $H^\natural_0(\Omega)$ is embedded in the weighted space $L^2(\Omega, |x|^{-\alpha})$ and that this embedding is continuous. If $\gamma < \gamma_H(\alpha)$, it follows from (3) that

$$|||u||| = \left( \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} \, dx \, dy - \gamma \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}}$$

is well-defined on $H^\natural_0(\Omega)$. Since $0 \leq \gamma < \gamma_H(\alpha)$, the following inequalities then hold for any $u \in H^\natural_0(\Omega)$,

$$(1 - \frac{\gamma}{\gamma_H(\alpha)}) ||u||^2_{H^\natural_0(\Omega)} \leq ||u||^2 \leq (1 + \frac{\gamma}{\gamma_H(\alpha)}) ||u||^2_{H^\natural_0(\Omega)}.$$  \hfill (6)

Thus, $||.||$ is equivalent to the norm $||.||_{H^\natural_0(\Omega)}$.

We shall consider the following functional $\Phi : H^\natural_0(\Omega) \rightarrow \mathbb{R}$ whose critical points are solutions for \textit{I}:

For $u \in H^\natural_0(\Omega)$, let

$$\Phi(u) = \frac{1}{2} |||u|||^2 - \frac{1}{2} \int_{\Omega} u^2 \, dx - \frac{1}{2} \int_{\Omega} \frac{u^{2^*}_n(s)}{|x|^s} \, dx - \frac{1}{q} \int_{\Omega} hu^q_+ \, dx,$$

where $u_+ = \max(0, u)$ is the non-negative part of $u$. Note that any critical point of the functional $\Phi(u)$ is essentially a variational solution of \textit{I}. Indeed, we have for any $v \in H^\natural_0(\Omega)$,

$$\langle \Phi'(u), v \rangle = \frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+\alpha}} \, dx \, dy - \int_{\mathbb{R}^n} \left( \frac{\gamma u}{|x|^\alpha} + \lambda u + \frac{u^{2^*_n(s)-1}}{|x|^s} + hu^{q-1}_+ \right) v \, dx.$$

\subsection{General condition of existence}

In this section, we investigate a general condition of existence of solution for \textit{I}.

\textbf{Theorem 2.1.} Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n (n > \alpha)$ such that $0 \in \Omega$, and let $2^*_\alpha(s) := \frac{2(n-s)}{n-\alpha}$, $0 \leq s < \alpha$, $-\infty < \lambda < \lambda_1(L, \omega)$, and $0 \leq \gamma < \gamma_H(\alpha)$. We consider $2 < q < 2^*_\alpha$, $h \in C^0(\Omega)$ and $h \geq 0$. We also assume that there exists $w \in H^\natural_0(\Omega)$, $w \neq 0$ and $w \geq 0$ such that

$$\sup_{t \geq 0} \Phi(tw) < \frac{\alpha - s}{2(n-s)} \mu_{\gamma, s, \alpha}^{\frac{n}{n-s}}.$$  \hfill (7)

Then, problem \textit{I} has a non-negative solution in $H^\natural_0(\Omega)$.

We split the proof in three parts:

\subsubsection{The Palais-Smale condition below a critical threshold}

In this subsection, we prove the following

\textbf{Proposition 2.2.} If $c < \frac{\alpha - s}{2(n-s)} \mu_{\gamma, s, \alpha}^{\frac{n}{n-s}}$, then every Palais-Smale sequence $(u_k)_{k \in \mathbb{N}}$ for $\Phi$ at level $c$ has a convergent subsequence in $H^\natural_0(\Omega)$.

\textbf{Proof of Proposition 2.2.} Assume $c < \frac{\alpha - s}{2(n-s)} \mu_{\gamma, s, \alpha}^{\frac{n}{n-s}}$ and let $(u_k)_{k \in \mathbb{N}} \in H^\natural_0(\Omega)$ be a Palais-Smale sequence for $\Phi$ at level $c$, that is $\Phi(u_k) \rightarrow c$ and $\Phi'(u_k) \rightarrow 0 \quad \text{in} \quad (H^\natural_0(\Omega))^\prime$, \hfill (8)
where \((H^\Phi_0(\Omega))'\) denotes the dual of \(H^\Phi_0(\Omega)\).

We first prove that \((u_k)_{k \in \mathbb{N}}\) is bounded in \(H^\Phi_0(\Omega)\). One can use \(u_k \in H^\Phi_0(\Omega)\) as a test function in \(\mathbf{S}\) to get that

\[
\|u_k\|^2 - \lambda \int_\Omega u_k^2 \, dx = \int_\Omega (u_k)^2 \frac{2^*_x(s)}{|x|^s} \, dx + \int_\Omega h(u_k)^q \, dx + o(\|u_k\|) \quad \text{as} \quad k \to \infty. \tag{9}
\]

On the other hand, from the definition of \(\Phi\), we deduce that

\[
\|u_k\|^2 - \lambda \int_\Omega u_k^2 \, dx = 2\Phi(u_k) + \frac{2}{2^*_\alpha(s)} \int_\Omega (u_k)^{2^*_x(s)} \frac{2^*_x(s)}{|x|^s} \, dx + \frac{2}{q} \int_\Omega h(u_k)^q \, dx. \tag{10}
\]

It follows from the last two identities that as \(k \to \infty\),

\[
2\Phi_q(u_k) = \left(1 - \frac{2}{2^*_\alpha(s)}\right) \int_\Omega (u_k)^{2^*_x(s)} \frac{2^*_x(s)}{|x|^s} \, dx + \left(1 - \frac{2}{q}\right) \int_\Omega h(u_k)^q \, dx + o(\|u_k\|). \tag{11}
\]

This coupled with the Palais-Smale condition \(\Phi(u_k) \to c\), and the fact that \(h \geq 0\) yield

\[
2c = \left(1 - \frac{2}{2^*_\alpha(s)}\right) \int_\Omega (u_k)^{2^*_x(s)} \frac{2^*_x(s)}{|x|^s} \, dx + \left(1 - \frac{2}{q}\right) \int_\Omega h(u_k)^q \, dx + o(1)
\]

\[
\geq \left(1 - \frac{2}{2^*_\alpha(s)}\right) \int_\Omega (u_k)^{2^*_x(s)} \frac{2^*_x(s)}{|x|^s} \, dx + o(1) \quad \text{as} \quad k \to \infty.
\]

Thus,

\[
\left(1 - \frac{2}{q}\right) \int_\Omega h(u_k)^q \, dx = O(1) \quad \text{as} \quad k \to \infty.
\]

We finally obtain

\[
\|u_k\|^2 - \lambda \int_\Omega u_k^2 \, dx \leq O(1) + o(\|u_k\|) \quad \text{as} \quad k \to \infty.
\]

Using that \(\lambda < \lambda_1(L_{\gamma,\alpha})\) and \(\gamma < \gamma_H(\alpha)\), we get that

\[
0 < \left(1 - \frac{\gamma}{\gamma_H(\alpha)}\right) \left(1 - \frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right) \|u_k\|^2_{H^\Phi_0(\Omega)} \leq \left(1 - \frac{\lambda}{\lambda_1(L_{\gamma,\alpha})}\right) \|u_k\|^2 \leq \|u_k\|^2 - \lambda \int_\Omega u_k^2 \, dx \leq O(1) + o(\|u_k\|) \quad \text{as} \quad k \to \infty.
\]

We then deduce that \((u_k)_{k \in \mathbb{N}}\) is bounded in \(H^\Phi_0(\Omega)\), which implies that there exists \(u \in H^\Phi_0(\Omega)\) such that, up to a subsequence,

\[
\begin{align*}
(1) & \quad u_k \to u \text{ weakly in } H^\Phi_0(\Omega), \\
(2) & \quad u_k \to u \text{ strongly in } L^{p_1}(\Omega) \text{ for all } p_1 \in [2, 2^*_\alpha). \\
(3) & \quad u_k \to u \text{ strongly in } L^{p_2}(\Omega, |x|^{-s} \, dx) \text{ for all } p_2 \in [2, 2^*_\alpha(s)).
\end{align*} \tag{12}
\]

We now claim that, up to a subsequence, we have

\[
\|u_k - u\|^2 = \int_\Omega \frac{(u_k - u)^{2^*_x(s)}}{|x|^s} \, dx + o(1) \quad \text{as} \quad k \to \infty, \tag{13}
\]
We therefore have
\[ \frac{\alpha - s}{2(n - s)} \| u_k - u \|^2 \leq c + o(1) \quad \text{as } k \to \infty. \] (14)

Indeed, straightforward computations yield
\[ o(1) = \langle \Phi'(u_k) - \Phi'(u), u_k - u \rangle \]
\[ = \| u_k - u \|^2 - \lambda \int_{\Omega} (u_k - u)^2 \]
\[ - \int_{\Omega} (u_k - u) \frac{(u_k^+ - u^+_k)^{(s-1)} - u^+_{k+} - u_{k+})^{(s-1)}}{|x|^s} \, dx \]
\[ + \int_{\Omega} h(u_k - u) \left[ (u_k^+)^{(s-1)} - (u^+_k)^{(s-1)} \right] \, dx \quad \text{as } k \to \infty. \]

We first write
\[ \int_{\Omega} (u_k - u) \frac{(u_k^+ - u^+_k)^{(s-1)} - u^+_{k+} - u_{k+})^{(s-1)}}{|x|^s} \, dx = \int_{\Omega} \frac{(u_k^+)^{(s-1)}}{|x|^s} \, dx - \int_{\Omega} \frac{u^+_k}{|x|^s} \, dx \]
\[ - \int_{\Omega} \frac{u^+_{k+} - u_{k+}}{|x|^s} \, dx + \int_{\Omega} \frac{u^+_{k+} - u_{k+}}{|x|^s} \, dx. \]

It now follows from integral theory that
\[ \lim_{k \to \infty} \int_{\Omega} \frac{u^+_{k+} - u_{k+}}{|x|^s} \, dx = \int_{\Omega} \frac{u^+_{k+} - u_{k+}}{|x|^s} \, dx = \lim_{k \to \infty} \int_{\Omega} \frac{(u_k^+)^{(s-1)}}{|x|^s} \, dx. \]

So, we get that
\[ \int_{\Omega} (u_k - u) \frac{(u_k^+ - u^+_k)^{(s-1)} - u^+_{k+} - u_{k+})^{(s-1)}}{|x|^s} \, dx = \int_{\Omega} \frac{(u_k^+)^{(s-1)}}{|x|^s} \, dx - \int_{\Omega} \frac{u^+_k}{|x|^s} \, dx. \]

In order to deal with the right hand side of the last identity, we use the following basic inequality:
\[ \left| (u_k^+ - u^+_k - (u_k - u)^+_{k+}) \right| \leq c \left( u^+_{k+} - u_k - u + (u_k - u)^+_{k+} \right), \]
for some constant \( c > 0 \). We multiply both sides of the above inequality by \( |x|^{-s} \) and take integral over \( \Omega \), and then use \( (12) \) to get that
\[ \lim_{k \to \infty} \int_{\Omega} \frac{(u_k^+ - u^+_k - (u_k - u)^+_{k+})}{|x|^s} \, dx = \int_{\Omega} \frac{u^+_{k+} - u_{k+}}{|x|^s} \, dx. \]

We therefore have
\[ \int_{\Omega} (u_k - u) \frac{(u_k^+ - u^+_k - (u_k - u)^+_{k+})}{|x|^s} \, dx = \int_{\Omega} \frac{(u_k - u)^+_{k+}}{|x|^s} \, dx + o(1) \quad \text{as } k \to \infty. \]

In addition, the embeddings \( (12) \) yield that
\[ \int_{\Omega} (u_k - u)^2 = o(1) \quad \text{as } k \to \infty, \]
and
\[ \int_{\Omega} h(u_k - u) \left[ (u_k)^{-1}_+ - u^{-1}_+ \right] dx = \int_{\Omega} (u_k - u)^2 dx + o(1) \text{ as } k \to \infty. \]

Plugging back the last three estimates into (15) gives (13). On the other hand, since \( u \) is a weak solution of (1), then \( \Phi(u) \geq 0 \), and since \( \Phi(u_k) \to c \) as \( k \to \infty \), it follows that \( \frac{\alpha - s}{2(n - s)} \|u_k - u\|^2 \leq c + o(1) \). This proves the claim.

We now show that
\[ \lim_{k \to \infty} u_k = u \text{ in } H^2_0(\Omega). \] (16)

Indeed, test the inequality (13) on \( u_k - u \), and use (12) and (20) to obtain that
\[ \int_{\Omega} \frac{(u_k - u)^2}{|x|^s} dx \leq \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \frac{2^*}{2} \|u_k - u\|^2 (s - \gamma) + o(1). \] (17)

Combining this with (13), we get
\[ \|u_k - u\|^2 \left( 1 - \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \frac{2^*}{2} \|u_k - u\|^{2^* - 2} + o(1) \right) \leq o(1). \]

It then follows from the last inequality and (13) that
\[ \left( 1 - \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \frac{2^*}{2} \left( \frac{2(n - s)}{\alpha - s} \right) c^* \right) \|u_k - u\|^2 \leq o(1). \]

Note that the assumption \( c < \frac{\alpha - s}{2(n - s)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \frac{2^*}{2} \) implies that
\[ \left( \frac{2(n - s)}{\alpha - s} \right) c^* < \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \frac{2^*}{2}, \]
and therefore
\[ \left( 1 - \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \frac{2^*}{2} \left( \frac{2(n - s)}{\alpha - s} \right) c^* \right) > 0. \]

Thus, \( \|u_k - u\| \to 0 \) as \( k \to \infty \), and this proves (16).

Finally, we have that \( \Phi(u) = c \), since the functional is continuous on \( H^2_0(\Omega) \).

\[ \square \]

2.1.2 Mountain pass geometry and existence of a Palais-Smale sequence

In this subsection, we prove the following

**Proposition 2.3.** For every \( w \in H^2_0(\Omega) \setminus \{0\} \) with \( w \geq 0 \), there exists an energy level \( c \), with
\[ 0 < c \leq \sup_{t \geq 0} \Phi(tw), \] (18)
and a Palais-Smale sequence \( (u_k) \) for \( \Phi \) at level \( c \), that is
\[ \Phi(u_k) \to c \text{ and } \Phi'(u_k) \to 0 \text{ in } (H^2_0(\Omega))^*. \]

**Proof.** We show that the functional \( \Phi \) satisfies the hypotheses of the mountain pass lemma (11). It is standard to show that \( \Phi \in C^1(H^2_0(\Omega)) \) and clearly \( \Phi(0) = 0 \), so that (a) of Lemma (11) is satisfied.
For (b), we show that 0 is a strict local minimum. For that, we first need to recall the definition of 
\(\lambda_1(L_{\gamma,\alpha})\), which is the first eigenvalue of the operator \(L_{\gamma,\alpha}\) with Dirichlet boundary condition, and the best constant in the fractional Hardy-Sobolev inequality on domain \(\Omega\), that is

\[
\mu_{\gamma,s,\alpha}(\Omega) := \inf_{u \in H^s_0(\Omega) \setminus \{0\}} \frac{c_{\gamma,s,\alpha}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} dxdy - \gamma \int_{\Omega} \frac{|u|^2}{|x|^\alpha} dx.
\]

(19)

As in the local case, one can show by translating, scaling and cutting off the extremals of \(\mu_{\gamma,s,\alpha}(\mathbb{R}^n)\) that

\[
\mu_{\gamma,s,\alpha}(\Omega) = \mu_{\gamma,s,\alpha}(\mathbb{R}^n).
\]

See Proposition 6.1 in [10] for more detail. Therefore, we have that

\[
\lambda_1(L_{\gamma,\alpha}) \int_{\Omega} |w|^2 dx \leq ||w||^2 \quad \text{and} \quad \mu_{\gamma,s,\alpha}(\mathbb{R}^n)(\int_{\Omega} \frac{|w|^{2(*)}}{|x|^s} dx)^{\frac{2}{2(*)}} \leq ||w||^2.
\]

In addition, it follows from (12) that there exists a positive constant \(S > 0\) such that

\[
S(\int_{\Omega} h|w|^q dx)^{\frac{2}{q}} \leq ||w||^2.
\]

Hence,

\[
\Phi(w) \geq \frac{1}{2} ||w||^2 - \frac{1}{2} \frac{\lambda}{\lambda_1(L_{\gamma,\alpha})} ||w||^2 - \frac{1}{q} S^{-\frac{2}{q}} ||w||^q
\]

\[
- \frac{1}{2^*_a(s) \mu_{\gamma,s,\alpha}(\mathbb{R}^n)} \frac{2^*_a(s)}{2} ||w||^{2^*_a(s)}
\]

\[
= ||w||^2 \left( \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1(L_{\gamma,\alpha})} \right) - \frac{1}{q} S^{-\frac{2}{q}} ||w||^{q-2}
\]

\[
- \frac{1}{2^*_a(s) \mu_{\gamma,s,\alpha}(\mathbb{R}^n)} \frac{2^*_a(s)}{2} ||w||^{2^*_a(s)-2} \right).
\]

Since \(\lambda < \lambda_1(L_{\gamma,\alpha})\), \(q \in (2, 2^*_a)\) and \(s \in [0, \alpha)\), we have that \(1 - \frac{\lambda}{\lambda_1(L_{\gamma,\alpha})} > 0\), \(q - 2 > 0\) and \(2^*_a(s) - 2 > 0\), respectively. Thus, we can find \(R > 0\) such that \(\Phi(w) \geq \rho\) for all \(w \in H^s_0(\Omega)\) with \( ||w||_{H^s_0(\Omega)} = R\).

Regarding (c), we have

\[
\Phi(tw) = \frac{t^2}{2} ||w||^2 - \frac{t^2 \lambda}{2} \int_{\Omega} w^2 dx - \frac{t^{2^*_a(s)}}{2^*_a(s)} \int_{\Omega} \frac{w^{2^*_a(s)}}{|x|^s} dx - \frac{t^q}{q} \int_{\Omega} hw^q dx,
\]

hence \(\lim_{t \to \infty} \Phi(tw) = -\infty\) for any \(w \in H^s_0(\Omega) \setminus \{0\}\) with \(w_+ \neq 0\), which means that there exists \(t_w > 0\) such that \(||tw||_{H^s_0(\Omega)} > R\) and \(\Phi(tw) < 0\), for \(t \geq t_w\). In other words,

\[
0 < \rho \leq \inf\{\Phi(w) : ||w||_{H^s_0(\Omega)} = R\} \leq \sup_{\gamma \in \mathcal{F}} \Phi(\gamma(1)) \leq \sup_{t \geq 0} \Phi(tw),
\]

where \(\mathcal{F}\) is the class of all path \(\gamma \in C([0,1]; H^s_0(\Omega))\) with \(\gamma(0) = 0\) and \(\gamma(1) = t_w w\).

The rest follows from the Ambrosetti-Rabinowitz lemma [11].
2.1.3 End of Proof of Theorem 2.1

We assume that there exists there exists \( w \in H^s_0(\Omega), w \neq 0, w \geq 0 \) such that

\[
\sup_{t \geq 0} \Phi(tw) < \frac{\alpha - s}{2(n - \alpha)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-\alpha}{2}}.
\]

Existence: It follows from Proposition 2.3 that there exist \( c_w > 0 \) and a Palais-Smale sequence \((u_k)_{k \in \mathbb{N}}\) for \( \Phi \) at level \( c_w \) such that

\[
c_w \leq \sup_{t \geq 0} \Phi(tw).
\]

It is obvious now that \( c_w < \frac{\gamma}{2(n-\alpha)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-\alpha}{2}}. \) So, we can apply Proposition 2.2 to get that there exists \( u \in H^s_0(\Omega) \) such that, up to a subsequence, \( u_k \to u \) strongly in \( H^s_0(\Omega) \) as \( k \to \infty \), and \( \Phi'(u) = 0 \). This means that \( u \) is a solution for (1).

Positivity: We test (1) against \( u_- \in H^s_0(\Omega) \). Then, arguing as in the proof of Lemma 5.2 in [10], we get that \( \|u_-\|_{H^s_0(\mathbb{R}^n)} = 0 \), which implies that \( u_- \equiv 0 \), and therefore \( u \geq 0 \) on \( \Omega \).

2.2 Test functions estimates for the non-local case

This section is devoted to prove an important result (Proposition 2.6) which plays a crucial role in the proof of Theorem 1.3. Since \( \Phi \) satisfies the Palais-Smale condition only up to level \( \frac{\gamma}{2(n-\alpha)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-\alpha}{2}} \), we need to check which conditions on \( \gamma, q, h \) and the mass of the domain \( \Omega \) guarantee that there exists a \( w \in H^s_0(\Omega) \) such that

\[
\sup_{t \geq 0} \Phi(tw) < \frac{\alpha - s}{2(n - \alpha)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n-\alpha}{2}}. \tag{21}
\]

We shall use the test functions constructed in [10] to obtain the general condition of existence (21). More precisely, we use the extremal of \( \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \) and its profile at zero and infinity to introduce appropriate test functions. We need such test functions to estimate the general existence condition (21). Unlike the case of the Laplacian \( (\alpha = 2) \), no explicit formula is known for the best constant \( \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \) nor for the extremals where it is achieved. The existence of such extremals was proved in [11] under certain condition on \( \gamma \) and \( s \). We therefore need to describe their asymptotic profile whenever they exist. This was recently considered in Ghoussoub-Robert-Shakerian-Zhao [10], where the following is proved:

**Theorem 2.4.** Assume \( 0 \leq s < \alpha < 2, n > \alpha \) and \( 0 \leq \gamma < \gamma_H(\alpha) \). Then, any positive extremal \( u \in H^s_0(\mathbb{R}^n) \) for \( \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \) satisfies \( u \in C^1(\mathbb{R}^n \setminus \{0\}) \) and

\[
\lim_{x \to 0} |x|^{-\beta_+(\gamma,\alpha)} u(x) = \lambda_0 \quad \text{and} \quad \lim_{|x| \to \infty} |x|^{-\beta_-(\gamma,\alpha)} u(x) = \lambda_\infty,
\]

where \( \lambda_0, \lambda_\infty > 0 \) and \( \beta_-(\gamma,\alpha) \) (resp., \( \beta_+(\gamma,\alpha) \)) is the unique solution in \( (0, \frac{n-\alpha}{2}) \) (resp., in \( (\frac{n-\alpha}{2}, n-\alpha) \)) of the equation

\[
\Psi_{n,\alpha}(t) := 2^n \frac{\Gamma\left(\frac{n-\alpha}{2}\right) \Gamma\left(\frac{\alpha + t}{2}\right)}{\Gamma\left(\frac{n-\alpha+t}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)} = \gamma.
\]

**Remark 2.5.** We point out that the functions \( u_1(x) = |x|^{-\beta_-(\gamma,\alpha)} \) and \( u_2(x) = |x|^{-\beta_+(\gamma,\alpha)} \) are the fundamental solutions for the fractional Hardy-Schrödinger operator \( L_{\gamma,\alpha} := (-\Delta)^{\frac{\gamma}{2}} - \frac{\gamma}{|x|^\alpha} \) on \( \mathbb{R}^n \).

Indeed, a straightforward computation yields (see Section 2 in [10])

\[
L_{\gamma,\alpha} |x|^{-\beta} = (\Psi_{n,\alpha}(\beta) - \gamma) |x|^{-\beta} = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{for} \quad \beta \in \{\beta_-(\gamma,\alpha), \beta_+(\gamma,\alpha)\},
\]

which implies that \( \beta_+(\gamma,\alpha) \) and \( \beta_-(\gamma,\alpha) \) satisfy \( \Psi_{n,\alpha}(\beta) = \gamma \).
We then establish the following.

**Proposition 2.6.** There exists \( \tau_l > 0 \) for \( l = 1, \ldots, 5 \) such that

1) If \( 0 \leq \gamma \leq \gamma_{crit}(\alpha) \), then

\[
\sup_{t \geq 0} \Phi(tv) = \frac{\alpha - s}{2(n-s)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n}{n-s}} - \tau_1 h(0)\epsilon^{n-q_{crit}} + o(\epsilon^{n-q_{crit}}); \tag{23}
\]

2) If \( \gamma_{crit}(\alpha) < \gamma < \gamma_{H}(\alpha) \), then

\[
\sup_{t \geq 0} \Phi(tv) = \frac{\alpha - s}{2(n-s)} \mu_{\gamma,s,\alpha}(\mathbb{R}^n)^{\frac{n}{n-s}} + \begin{cases} 
-\tau_2 h(0)\epsilon^{n-q_{crit}} + o(\epsilon^{n-q_{crit}}) & \text{if } q > q_{crit} \\
-\tau_3 h(0) + \tau_4 m_{\alpha}^{\gamma,\alpha}\epsilon^{\beta_+(\gamma,\alpha) - \beta_-(\gamma,\alpha)} + o(\epsilon^{\beta_+(\gamma,\alpha) - \beta_-(\gamma,\alpha)}) & \text{if } q = q_{crit} \\
-\tau_5 m_{\gamma,\alpha}^{\gamma,\alpha}\epsilon^{\beta_+(\gamma,\alpha) - \beta_-(\gamma,\alpha)} + o(\epsilon^{\beta_+(\gamma,\alpha) - \beta_-(\gamma,\alpha)}) & \text{if } q < q_{crit},
\end{cases} \tag{24}
\]

where \( q_{crit} := 2^* - \frac{2\beta_+(\alpha) - \beta_-(\alpha)}{n-\alpha} \in (2, 2^*_\alpha). \)

**Remark 2.7.** Throughout this section, we may use the following notations:

\[ \beta_+(\alpha) := \beta_+(\gamma, \alpha) \quad \text{and} \quad \beta_-(\alpha) := \beta_-(\gamma, \alpha) \]

### 2.2.1 Test function for non-critical case:

We fix cut-off function \( \eta \in C^\infty_c(\Omega) \) such that

\( \eta \equiv 1 \) in \( B_{\delta}(0) \) and \( \eta \equiv 0 \) in \( \mathbb{R}^n \setminus B_{2\delta}(0) \) with \( B_{4\delta}(0) \subset \Omega. \) \( \tag{25} \)

Let \( U_\alpha \in H^\frac{n}{2}(\mathbb{R}^n) \) be an extremal for \( \mu_{\gamma,s,\alpha}(\mathbb{R}^n) \). It follows from Theorem \[ \text{[2,4]} \] that, up to multiplying by a nonzero constant, \( U_\alpha \) satisfies for some \( \kappa > 0, \)

\[
(-\Delta)^{\frac{n}{2}} U_\alpha - \frac{\gamma}{|x|^\alpha} U_\alpha = \kappa \frac{U_\alpha^{2^*_\alpha}(\cdot)^{(1-2^*_\alpha)}}{|x|^n} \text{ weakly in } H^\frac{n}{2}(\mathbb{R}^n). \tag{26}
\]

Moreover, \( U_\alpha \in C^1(\mathbb{R}^n \setminus \{0\}) \), \( U_\alpha > 0 \) and

\[
\lim_{|x| \to \infty} |x|^\beta_+(\alpha) U_\alpha(x) = 1. \tag{27}
\]

Consider

\[ u_\epsilon(x) := \epsilon^{-\frac{n-\alpha}{2}} U(\epsilon^{-1} x) \text{ for } x \in \mathbb{R}^n \setminus \{0\}. \]

It follows from Proposition 3.1 in \[ \text{[10]} \] that \( U_{\epsilon,\alpha} := \eta u_\epsilon, \alpha \in H^\frac{n}{2}(\Omega). \)

### 2.2.2 Test function for the critical case:

The test function in the critical case is more delicate and has more complicated structure compared to the non-critical case. In order to build a suitable test function, we use the fractional Hardy singular interior mass of a domain associated to the operator \( L_{\gamma,\alpha} \) which was introduced by Ghoussoub-Robert-Shakerian-Zhao \[ \text{[10]} \].

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We first summarize the results which will be needed in this section: Assume that \( \gamma > \gamma_{crit}(\alpha) \). It follows from Theorem 1.2 in [10] that there exists \( H : \Omega \setminus \{0\} \to \mathbb{R} \) such that

\[
\begin{align*}
H &\in C^1(\Omega \setminus \{0\}), \quad \xi H \in H^\frac{2}{n}(\Omega) \quad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n \setminus \{0\}), \\
(-\Delta)^\frac{\alpha}{2} H - \left( \frac{2}{|x|^n} + a \right) H &= 0 \quad \text{weakly in } \Omega \setminus \{0\} \\
H &> 0 \quad \text{in } \Omega \setminus \{0\} \\
H &= 0 \quad \text{in } \partial \Omega \\
\text{and } \lim_{x \to 0} |x|^{\beta_+(\alpha)} H(x) &= 1.
\end{align*}
\]

Note that the second identity means that for any \( \varphi \in C_c^\infty(\Omega \setminus \{0\}) \), we have that

\[
\frac{C_{n,\alpha}}{2} \int_{(\mathbb{R}^n)^2} \frac{(H(x) - H(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+\alpha}} \, dx \, dy - \int_{\mathbb{R}^n} \left( \frac{\gamma}{|x|^\alpha} + a \right) H \varphi \, dx = 0. \tag{28}
\]

Let now \( \eta \) be as in (25). Following the construction of the singular function \( H \) in the proof of Theorem 1.2 in [10], we get that there exists \( g \in H^\frac{2}{n}(\Omega) \) such that

\[
H(x) := \frac{\eta(x)}{|x|^{\beta_+(\alpha)}} + g(x) \quad \text{for } x \in \Omega \setminus \{0\},
\]

where \( g \) satisfies

\[
(-\Delta)^\frac{\alpha}{2} g - \left( \frac{\gamma}{|x|^\alpha} + a \right) g = f, \tag{29}
\]

\[
g(x) = \frac{m_{\gamma,\alpha}^\alpha(\Omega)}{|x|^{\beta_-(\alpha)}} + a \left( \frac{1}{|x|^{\beta_-(\alpha)}} \right) \quad \text{as } x \to 0, \quad \text{and } |g(x)| \leq C|x|^{-\beta_-(\alpha)} \text{ for all } x \in \Omega. \tag{30}
\]

Here \( m_{\gamma,\alpha}^\alpha(\Omega) \) is the fractional Hardy singular interior mass of a domain associated to the operator \( L_{\gamma,\alpha} \). We refer the readers to Section 5 of [10] for the definition and properties of the Mass in detail. Define the test function as

\[
T_{\epsilon,\alpha}(x) = \eta u_{\epsilon}(x) + \epsilon^{\beta_+(\gamma,\alpha) - \beta_-(\gamma,\alpha)} g(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{0\},
\]

where

\[
\eta u_{\epsilon}(x) := e^{-\frac{\alpha}{2} \epsilon} U_\alpha(\epsilon^{-1} x) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},
\]

and \( U \in H^\frac{2}{n}(\mathbb{R}^n) \) is such that \( U > 0, U \in C^1(\mathbb{R}^n \setminus \{0\}) \) and satisfies (20) above for some \( \kappa > 0 \) and also (21). It is easy to see that \( T_{\epsilon,\alpha} \in H^\frac{2}{n}(\Omega) \) for all \( \epsilon > 0 \).

**Remark 2.8.** One can summarize the properties of \( \beta_+(\gamma,\alpha) \) and \( \beta_-(\gamma,\alpha) \) which will be used freely in this section:

- \( \beta_-(\alpha) + \beta_+(\alpha) = n - \alpha \)
- \( \beta_+(\alpha) - \beta_-(\alpha) > \alpha \) when \( \gamma < \gamma_{crit}(\alpha) \)
- \( \beta_+(\alpha) - \beta_-(\alpha) = \alpha \) when \( \gamma = \gamma_{crit}(\alpha) \).
- \( \beta_+(\alpha) - \beta_-(\alpha) < \alpha \) when \( \gamma > \gamma_{crit}(\alpha) \)
2.2.3 Proof of Proposition 2.6

We shall use the test functions $U_\epsilon$ (resp., $T_\epsilon$) constructed in Subsection 2.2.1 resp., Subsection 2.2.2 for the case when the operator $L_{\gamma,\alpha}$ is non-critical (resp., critical) to obtain the general condition of existence. We define the test-functions then as follows:

$$v_{\epsilon,\alpha} = \begin{cases} U_{\epsilon,\alpha} = \eta u_{\epsilon,\alpha} & \text{if } 0 \leq \gamma \leq \gamma_{\text{crit}}(\alpha) \\ T_{\epsilon,\alpha} := U_\epsilon + \epsilon \frac{\beta_+(\alpha)-\beta_-(\alpha)}{2} g(x) & \text{if } \gamma_{\text{crit}}(\alpha) < \gamma < \gamma_H(\alpha). \end{cases} \quad (31)$$

We expand $\Phi(tv_\epsilon)$ in the following way:

$$\Phi(tv_\epsilon) = \frac{t^2}{2} I_\epsilon - \frac{t^2}{2} \int_0^1 J_\epsilon - \frac{t^q}{q} K_\epsilon \quad \text{as } \epsilon \to 0,$$

where

$$I_\epsilon := \|v_\epsilon\|^2 - \lambda \|v_\epsilon\|_{L^2(\Omega)}^2, \quad J_\epsilon := \int_\Omega \frac{|v_\epsilon|^{2^*(s)}(\cdot)}{|x|^s} dx \quad \text{and} \quad K_\epsilon := \int_\Omega h|v_\epsilon|^q dx.$$

We start by recalling the important estimates obtained in [10]. Indeed, it was proved there that there exist positive constants $c_1, c_2, c_3$ such that

$$I_\epsilon = \begin{cases} \kappa \int_{\mathbb{R}^n} \frac{U_\epsilon^{2^*(s)}(\cdot)}{|x|^s} dx - c_1 \lambda \epsilon^\alpha + o(\epsilon^\alpha) & \text{if } 0 \leq \gamma < \gamma_{\text{crit}}(\alpha) \\ \kappa \int_{\mathbb{R}^n} \frac{|U_\epsilon^{2^*(s)}(\cdot)|^\alpha}{|x|^s} dx - c_2 \epsilon^\alpha \ln \epsilon^{-1} + o(\epsilon^\alpha \ln \epsilon^{-1}) & \text{if } \gamma = \gamma_{\text{crit}}(\alpha) \\ \kappa \int_{\mathbb{R}^n} \frac{|U_\epsilon^{2^*(s)}(\cdot)|^\alpha}{|x|^s} dx - c_3 m_\alpha^\alpha \eta^{-\alpha} + 2\kappa \epsilon^{\beta_+(\alpha)-\beta_-(\alpha)} \theta_\epsilon + o(\epsilon^{\beta_+(\alpha)-\beta_-(\alpha)}) & \text{if } \gamma_{\text{crit}}(\alpha) < \gamma < \gamma_H(\alpha), \end{cases} \quad (32)$$

as $\epsilon \to 0$, and

$$J_\epsilon = \begin{cases} \int_{\mathbb{R}^n} \frac{U_\epsilon^{2^*(s)}(\cdot)}{|x|^s} dx + o(\epsilon^{\beta_+(\alpha)-\beta_-(\alpha)}) & \text{if } 0 \leq \gamma \leq \gamma_{\text{crit}}(\alpha) \\ \int_{\mathbb{R}^n} \frac{|U_\epsilon^{2^*(s)}(\cdot)|^\alpha}{|x|^s} dx + 2\kappa \epsilon^{\beta_+(\alpha)-\beta_-(\alpha)} \theta_\epsilon + o(\epsilon^{\beta_+(\alpha)-\beta_-(\alpha)}) & \text{if } \gamma_{\text{crit}}(\alpha) < \gamma < \gamma_H(\alpha), \end{cases} \quad (33)$$

as $\epsilon \to 0$. Here, $\theta_\epsilon := \int_{\mathbb{R}^n} \frac{2^*(s)-1}{|x|^s} h \eta dx$ and we have $\lim_{\epsilon \to 0} \theta_\epsilon = 0$; see (57) in [10]. We are then left with estimating $K_\epsilon$.

**Estimate for $K_\epsilon$**: We will consider two following cases.

**Case 1**: $0 \leq \gamma \leq \gamma_{\text{crit}}(\alpha)$. We split $K_\epsilon$ into two integrals as follows

$$K_\epsilon = \int_{\Omega} h|U_\epsilon|^q dx = \int_{B_\delta} h|U_\epsilon|^q dx + \int_{\Omega \setminus B_\delta} h|U_\epsilon|^q dx.$$

We start by estimating the first term:

$$\int_{B_\delta} h|U_\epsilon|^q dx = e^{-\beta_+(\alpha)} \int_{B_\delta} h(x)|U_\epsilon(x)|^q dx = e^{-\beta_+(\alpha)} \int_{B_{\delta/2}} h(x)|U_\epsilon(x)|^q dx \quad \text{and} \quad e^{-\beta_+(\alpha)} \int_{\mathbb{R}^n \setminus B_{\delta/2}} h(x)|U_\epsilon(x)|^q dx.$$
Note that we used the change of variable $x = \epsilon X$. From the asymptotic (27) and the fact that $q > 2$, it then follows that
\[
\int_{B_{\delta}} h|U_{\epsilon}|^q d\xi = \epsilon^{n-q\frac{n-\alpha}{2}}h(0) \int_{\mathbb{R}^{n}} |U(X)|^q dX + O(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)})
\]
\[
= \epsilon^{n-q\frac{n-\alpha}{2}}h(0) \int_{\mathbb{R}^{n}} |U(X)|^q dX + o(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}).
\]

Following the same argument that we treat the second integral in the last term yields that
\[
\int_{\Omega \setminus B_{\delta}} h|U_{\epsilon}|^q d\xi = O(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}) = o(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}).
\]

Therefore,
\[
\int_{\Omega} h|U_{\epsilon}|^q d\xi = \epsilon^{n-q\frac{n-\alpha}{2}}h(0) \left[ \int_{\mathbb{R}^{n}} |U(X)|^q dX \right] + o(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}).
\]

It now follows from Remark 2.8 that
\[
\beta_{+}(\alpha) - \beta_{-}(\alpha) \geq \alpha \quad \text{if} \quad 0 \leq \gamma \leq \gamma_{crit}(\alpha).
\]

On the other hand, the condition $2 < q < 2^{*}$ implies that
\[
0 < n - q \frac{n-\alpha}{2} < \alpha.
\]

Combining the last two inequalities, we then get that
\[
n - q \frac{n-\alpha}{2} < \beta_{+}(\alpha) - \beta_{-}(\alpha),
\]
and therefore
\[
K_{\epsilon} = \epsilon^{n-q\frac{n-\alpha}{2}}h(0) \left[ \int_{\mathbb{R}^{n}} |U(X)|^q dX \right] + o(\epsilon^{n-q\frac{n-\alpha}{2}}),
\]
when $0 \leq \gamma \leq \gamma_{crit}(\alpha)$.

Case 2: $\gamma_{crit}(\alpha) < \gamma < \gamma_{H}(\alpha)$. In order to estimate $K_{\epsilon}$ in the critical case, we need the following inequality: For $q > 2$, there exists $C = C(q) > 0$ such that
\[
||X + Y||^q - |X|^q - qXY|X|^{q-2} \leq C(|X|^{q-2}Y^2 + |Y|^q)
\]
for all $X, Y \in \mathbb{R}$.

We write
\[
K_{\epsilon} = \int_{\Omega} h|T_{\epsilon}|^q d\xi = \int_{B_{\delta}} h|U_{\epsilon} + \epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)} g(x)|^q d\xi + O(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}),
\]
where the last term came from the fact that
\[
\int_{\Omega \setminus B_{\delta}} h|T_{\epsilon}|^q d\xi = O(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}) = o(\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)}).
\]

Let now $X = U_{\epsilon}$ and $Y = \epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)} g(x)$ in the above inequality. Taking integral from both sides then leads us to
\[
\int_{B_{\delta}} h|U_{\epsilon} + \epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)} g(x)|^q d\xi = \int_{B_{\delta}} h|U_{\epsilon}|^q d\xi + q\epsilon^{\beta_{+}(\alpha)-\beta_{-}(\alpha)} \int_{B_{\delta}} hg|U_{\epsilon}|^{q-1} d\xi + R_{\epsilon},
\]
where
\[ R_{\varepsilon} = O\left(\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)}\right) \int_{B_{\delta}} h|U_{\varepsilon}|^{-q} y^2 dx + \varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)} \int_{B_{\delta}} h g^q dx \]
\[ = O(\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)}). \tag{36} \]

Regarding the second term, we have
\[ qe^{\beta_+(\alpha) - \beta_-(\alpha)} \int_{B_{\delta}} h g |U_{\varepsilon}|^{-q} \alpha^2 dx = O(\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)} + (n - q^{2/\alpha})) + O(\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)}) \]
\[ = o(\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)}) \text{ for all } q \in (2, 2^n). \tag{37} \]

Combining (34), (35) and (37), we get that there exist a constant $C > 0$ such that
\[ K_{\varepsilon} = C h(0) e^{n - q^{2/\alpha}} + o(e^{n - q^{2/\alpha}}) + o(\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)}) \text{ if } \gamma_{\text{crit}}(\alpha) < \gamma < \gamma_H(\alpha). \]

We point out that the situation in the critical case is more delicate, and unlike the non-critical case, we have that both
\[ \beta_+(\alpha) - \beta_-(\alpha) \text{ and } n - q^{n - \alpha} \]
are in the interval $(0, \alpha)$. Therefore, there is a competition between the terms $\varepsilon^{\beta_+(\alpha) - \beta_-(\alpha)}$ and $e^{n - q^{2/\alpha}}$. In order to find the threshold, namely $\gamma_{\text{crit}}$, we equate the exponents of the $\varepsilon$ terms, and solve the equation for $q$ to get that
\[ q_{\text{crit}} = 2^n - 2 \frac{\beta_+(\alpha) - \beta_-(\alpha)}{n - \alpha}. \]

One should note that $q_{\text{crit}} \in (2, 2^n)$, since $\alpha > \beta_+(\alpha) - \beta_-(\alpha) > 0$ in the critical case. This implies that
\[ \left\{ \begin{array}{ll}
o(e^{n - q^{2/\alpha}}) + o(e^{\beta_+(\alpha) - \beta_-(\alpha)}) = o(e^{n - q^{2/\alpha}}) & \text{if } q > q_{\text{crit}} 
o(e^{n - q^{2/\alpha}}) + o(e^{\beta_+(\alpha) - \beta_-(\alpha)}) = o(e^{\beta_+(\alpha) - \beta_-(\alpha)}) & \text{if } q = q_{\text{crit}} 
o(e^{n - q^{2/\alpha}}) + o(e^{\beta_+(\alpha) - \beta_-(\alpha)}) = o(e^{\beta_+(\alpha) - \beta_-(\alpha)}) & \text{if } q < q_{\text{crit}}. \end{array} \right. \]

We finally obtain
\[ K_{\varepsilon} = \left\{ \begin{array}{ll}
c_4 h(0) e^{n - q^{2/\alpha}} + o(e^{n - q^{2/\alpha}}) & \text{if } q > q_{\text{crit}} 
c_5 h(0) e^{\beta_+(\alpha) - \beta_-(\alpha)} + o(e^{\beta_+(\alpha) - \beta_-(\alpha)}) & \text{if } q = q_{\text{crit}} 
c_6 h(0) e^{\beta_+(\alpha) - \beta_-(\alpha)} & \text{if } q < q_{\text{crit}}. \end{array} \right. \tag{38} \]
for some $c_4, c_5 > 0$, and as long as $\gamma_{\text{crit}}(\alpha) < \gamma < \gamma_H(\alpha)$.

We now define
\[ I_0 := \lim_{\varepsilon \to 0} I_{\varepsilon} = \kappa \int_{\mathbb{R}^n} \frac{U_{\varepsilon}(s)}{|x|} dx \quad \text{and} \quad J_0 := \lim_{\varepsilon \to 0} J_{\varepsilon} = \int_{\mathbb{R}^n} \frac{U_{\varepsilon}(s)}{|x|} dx, \]
and it is easy to check that
\[ \lim_{\varepsilon \to 0} K_{\varepsilon} = 0 \quad \text{for all cases.} \]

In the next step, we claim that, up to a subsequence of $(u_{\varepsilon})_{\varepsilon > 0}$, there exists $T_0 := T_0(n, s, \alpha) > 0$ such that
\[ \sup_{t \geq 0} \Phi(t u_{\varepsilon}) = \phi(t \Psi(v_{\varepsilon})) \xrightarrow{s/2} T_{\varepsilon} \frac{T_{\varepsilon}^q}{q} K_{\varepsilon} + o(K_{\varepsilon}), \tag{39} \]
where
\[ \Psi(v_\epsilon) = \frac{\|v_\epsilon\|^2 - \lambda \|v_\epsilon\|^2_{L^2(\Omega)}}{(\int_{\Omega} \frac{|v_\epsilon|^2_{L^2(\Omega)}}{|x|^s} dx)^{\frac{n-s}{n}}} = \frac{I_\epsilon}{J_{\epsilon}^{\frac{n-s}{n}}}. \]

The proof of this claim goes exactly as Step II in [13, Proposition 3]. We omit it here.

Let us now compute \( \Psi(v_\epsilon) \). It follows from [32] and [33] that there exist positive constants \( c_6, c_7, c_8 \) such that
\[
\Psi(v_\epsilon) = \mu_{\gamma, s, \alpha}(\mathbb{R}^n) \left( 1 + \begin{cases} 
-c_6 \lambda \epsilon^n + o(\epsilon^n) & \text{if } 0 < \gamma < \gamma_{crit}(\alpha) \\
-c_7 \lambda \epsilon^n \ln \epsilon^{-1} + o(\epsilon^n \ln \epsilon^{-1}) & \text{if } \gamma = \gamma_{crit}(\alpha) \\
-c_8 \alpha \epsilon^{\beta_+(\alpha) - \beta_-(\alpha)} + o(\epsilon^{\beta_+(\alpha) - \beta_-(\alpha)}) & \text{if } \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha) 
\end{cases} \right).
\]

We are now going to estimate \( \sup_{t \geq 0} \Phi(v_\epsilon) \). This will be done again by considering two cases:

**Case 1:** \( 0 \leq \gamma \leq \gamma_{crit}(\alpha) \). In this case, plugging (35) and (40) into (39) implies that there exist positive constants \( c_9, c_{10}, c_{11}, c_{12} > 0 \) such that
\[
\sup_{t \geq 0} \Phi(v_\epsilon) = \frac{\alpha - s}{2(n-s)} \mu_{\gamma, s, \alpha}(\mathbb{R}^n) \frac{n-s}{n} - c_9 \lambda \epsilon^n - c_{10} h(0) \epsilon^{n-q\frac{n-a}{2}} + o(\epsilon^n) + o(\epsilon^{n-q\frac{n-a}{2}}),
\]
when \( 0 \leq \gamma < \gamma_{crit}(\alpha) \), and
\[
\sup_{t \geq 0} \Phi(v_\epsilon) = \frac{\alpha - s}{2(n-s)} \mu_{\gamma, s, \alpha}(\mathbb{R}^n) \frac{n-s}{n} - c_{11} \lambda \epsilon^n \ln(\epsilon^{-1}) - c_{12} h(0) \epsilon^{n-q\frac{n-a}{2}} + o(\epsilon^n \ln(\epsilon^{-1})) + o(\epsilon^{n-q\frac{n-a}{2}}),
\]
when \( \gamma = \gamma_{crit}(\alpha) \).

Recall that \( \alpha > n - q\frac{n-a}{2} \), since \( q > 2 \). This implies that
\[ o(\epsilon^n) + o(\epsilon^{n-q\frac{n-a}{2}}) = o(\epsilon^{n-q\frac{n-a}{2}}). \]

Thus, there exist a positive constant \( \tau_1 \) such that, for every \( 0 \leq \gamma \leq \gamma_{crit}(\alpha) \), we have
\[
\sup_{t \geq 0} \Phi(v_\epsilon) = \frac{\alpha - s}{2(n-s)} \mu_{\gamma, s, \alpha}(\mathbb{R}^n) \frac{n-s}{n} - \tau_1 h(0) \epsilon^{n-q\frac{n-a}{2}} + o(\epsilon^{n-q\frac{n-a}{2}}),
\]

**Case 2:** \( \gamma_{crit}(\alpha) < \gamma < \gamma_H(\alpha) \). The critical case needs a careful analysis as a new phenomena happens in this situation. We shall show that there is a competition between the geometry of the domain, the mass (i.e., \( \epsilon^{\beta_+(\alpha) - \beta_-(\alpha)m_{\gamma, \lambda}^\alpha} \)), and the non-linear perturbation (i.e., \( \epsilon^{n-q\frac{n-a}{2}} h(0) \)).

Indeed, it follows from plugging (38) and (41) into (39) that there exist constants \( c_{12}, c_{13}, c_{14} > 0 \) such that
\[
\sup_{t \geq 0} \Phi(t w_\epsilon) = \Upsilon - c_{12} m_{\gamma, \lambda}^\alpha \epsilon^{\beta_+(\alpha) - \beta_-(\alpha)} + o(\epsilon^{\beta_+(\alpha) - \beta_-(\alpha)})
\]
\[
+ \begin{cases} 
-c_{13} h(0) \epsilon^{n-q\frac{n-a}{2}} + o(\epsilon^{n-q\frac{n-a}{2}}) & \text{if } q > q_{crit} \\
-c_{14} h(0) \epsilon^{\beta_+(\alpha) - \beta_-(\alpha)} + o(\epsilon^{\beta_+(\alpha) - \beta_-(\alpha)}) & \text{if } q = q_{crit} \\
o(\epsilon^{\beta_+(\alpha) - \beta_-(\alpha)}) & \text{if } q < q_{crit}.
\end{cases}
\]

Following our analysis in the critical case of estimating \( K_\epsilon \), one can then summarize the competition results as follows.

| Competitive Terms | \( q > q_{crit} \) | \( q = q_{crit} \) | \( q < q_{crit} \) |
|-------------------|---------------------|---------------------|---------------------|
| \( \epsilon^{n-q\frac{n-a}{2}} h(0) \) | Dominate            | Equally Dominate    | ×                    |
| \( \epsilon^{\beta_+(\alpha) - \beta_-(\alpha)} m_{\gamma, \lambda}^\alpha \) | \( \times \) | Equally Dominate    | Dominate            |
Therefore, we finally deduce that there exists $\tau_l > 0$ for $l = 2, \ldots, 5$ such that

$$
\sup_{t \geq 0} \Phi(t\omega_c) = \alpha - s \frac{\mu_{\gamma, s, \alpha}(\mathbb{R}^n)_{\text{max}}}{2(n-s)\mu_{\gamma, s, \alpha}(\mathbb{R}^n)_{\text{med}}}
$$

$$
+ \begin{cases}
-\tau_2 h(0)e^{n-q \frac{2}{n-s}} + o(e^{n-q \frac{2}{n-s}}) & \text{if } q > q_{\text{crit}} \\
-(\tau_3 h(0) + \tau_4 m_{\gamma, \lambda}^\alpha)e^{\beta_+(-\alpha) - \beta_-(\alpha)} + o(e^{\beta_+(-\alpha) - \beta_-(\alpha)}) & \text{if } q = q_{\text{crit}} \\
-\tau_5 m_{\gamma, \lambda}^\alpha e^{\beta_+(-\alpha) - \beta_-(\alpha)} + o(e^{\beta_+(-\alpha) - \beta_-(\alpha)}) & \text{if } q < q_{\text{crit}}.
\end{cases}
$$

This complete the proof of Proposition 2.6.

2.3 Proof of Theorem 1.3: The non-local case

Assume that $0 < \alpha < 2$. Theorem 1.3 is now a direct consequence of Theorem 2.1 and Proposition 2.6.

3 The local case

In this section, our aim is to prove the existence of positive solutions for (2), and show that Theorem 1.3 still holds true when $\alpha = 2$. To this end, we use the results by Ghoussoub-Robert [19] in the local setting (i.e., $\alpha = 2$), and follow the proofs of Theorem 2.1 and Proposition 2.6 to obtain the general existence condition (i.e., condition (21) when $\alpha = 2$) and the desired estimates corresponding to (43) and (44).

We consider the case when $\alpha = 2$, that is when the operator $(-\Delta)\frac{\alpha}{2}$ boils down to the well-known Laplacian operator $-\Delta$. Problem (1) can therefore be written as

$$
\begin{cases}
-\Delta u - \frac{\gamma}{|x|^2} u + \frac{u^{2^*_s(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{in } \partial \Omega,
\end{cases}
$$

(41)

where $s \in [0, 2)$, $2^*_s(s) := \frac{2(n-s)}{n-s}$, $\gamma < \gamma_H(2) = (\frac{n-2}{4})^\frac{4}{n-2}$, and $u$ belongs to the space $H^1_0(\Omega)$ which is the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$
\|u\|_{H^1_0(\Omega)}^2 = \int_\Omega |\nabla u|^2 dx.
$$

Moreover, the norm

$$
\|u\|_2 = \left[ \int_\Omega \left( |\nabla u|^2 dx - \gamma \frac{u^2}{|x|^s} \right) dx \right]^\frac{1}{2}
$$

is well-defined on $H^1_0(\Omega)$, and is equivalent to $\|\cdot\|_{H^1_0(\Omega)}$. The corresponding functional to problem (2) is

$$
\Phi_2(u) = \frac{1}{2} \|u\|_2^2 - \lambda \int_\Omega u^2 dx - \frac{1}{2}\int_\Omega \frac{u^{2^*_s(s)}}{|x|^s} dx - \frac{1}{q} \int_\Omega h u^q dx.
$$

We point out that the results proved in Sections 2.3 and 4 in [19] provide us the desired tools to show that Theorem 2.1 and Proposition 2.6 still hold in the local case (i.e., when $\alpha = 2$). One can indeed use these results, and follow the proofs of Theorem 2.1 and Proposition 2.6 to establish the following.
Theorem 3.1. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n (n > 2)$ such that $0 \in \Omega$, and let $2_{\gamma}^*(s) := \frac{2(n-s)}{n-2}$, $0 \leq s < 2$, $-\infty < \lambda < \lambda_1(\Omega,s)$, and $\gamma < \gamma_\mu(2) = \frac{(n-2)^2}{4}$. We consider $2 < q < 2^*$, $h \in C^0(\bar{\Omega})$ and $h \geq 0$. We also assume that there exists $w \in H^1_0(\Omega)$, $w \not\equiv 0$ and $w \geq 0$ such that

$$\sup_{t \geq 0} \Phi(tw) < \frac{2-s}{2(n-s)}\mu_{\gamma,s,2}(\mathbb{R}^n)^{\frac{n-s}{2}}.$$  \hspace{1cm} (42)

Then, problem (2) has a non-negative solution in $H^1_0(\Omega)$.

Remark 3.2. Recall that the best constant in the Hardy-Sobolev inequalities is defined as

$$\mu_{\gamma,s,2}(\mathbb{R}^n) = \inf_{u \in H^1_0(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} (|\nabla u|^2 dx - \gamma |u|^{2_q}) dx}{(\int_{\mathbb{R}^n} |u|^{2_q} dx)^{\frac{2}{2_q}}},$$

where $H^1_0(\mathbb{R}^n)$ is the completion of $C_\infty^0(\mathbb{R}^n)$ with respect to the norm $\|u\|_{H^1_0(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx$.

Proposition 3.3. Let $\Upsilon_2 := \frac{2-s}{2(n-s)}\mu_{\gamma,s,2}(\mathbb{R}^n)^{\frac{n-s}{2}}$. Then, there exists $\tau_{l,2} > 0$ for $l = 1, \ldots, 5$ such that

1) If $L_{\gamma,2}$ is not critical, then

$$\sup_{t \geq 0} \Phi(tv_{\epsilon,2}) = \Upsilon_2 - \tau_{1,2}h(0)\epsilon^{n-\frac{2}{n-2}} + o(\epsilon^{n-\frac{2}{n-2}});$$

2) If $L_{\gamma,2}$ is critical, then

$$\sup_{t \geq 0} \Phi(tv_{\epsilon,2}) = \Upsilon_2 + \begin{cases} -\tau_{2,2}h(0)\epsilon^{n-\frac{2}{n-2}} + o(\epsilon^{n-\frac{2}{n-2}}) & \text{if } q > q_{\text{crit}}(2) \\ -\tau_{4,2}m_{\gamma,2}(\Omega)\epsilon^{\beta_+(2) - \beta_-(2)} + o(\epsilon^{\beta_+(2) - \beta_-(2)}) & \text{if } q = q_{\text{crit}}(2) \\ -\tau_{5,2}m_{\gamma,2}(\Omega)\epsilon^{\beta_+(2) - \beta_-(2)} + o(\epsilon^{\beta_+(2) - \beta_-(2)}) & \text{if } q < q_{\text{crit}}(2), \end{cases}$$

where $q_{\text{crit}}(2) := 2^* - 2\frac{\beta_+(2) - \beta_-(2)}{n-2} \in (2, 2^*)$.

3.1 Proof of Theorem 1.3: The local case

Assume that $\alpha = 2$. Theorem 1.3 is a direct consequence of Theorem 3.1 and Proposition 3.3.

References

[1] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973), 349 - 381.

[2] T. Aubin, Le probleme de Yamabe concernant la courbure scalaire, Differential Geometry Pescola 1985. Springer Berlin Heidelberg, 1986. 66-72.

[3] G. M. Bisci, V. D. Radulescu and R. Servadei, Variational methods for nonlocal fractional problems, Vol. 162. Cambridge University Press, 2016.

[4] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical-Sobolev exponents, Comm. Pure Appl. Math., 36, (1983), 437-477.

[5] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), no. 5, 521-573.
[6] Z. Djadli, E. Hebey, and M. Ledoux, *Paneitz-type operators and applications*, Duke Mathematical Journal 104.1 (2000): 129-170.

[7] P. Esposito, and F. Robert, *Mountain pass critical points for Paneitz-Branson operators*, Calculus of Variations and Partial Differential Equations 15.4 (2002): 493-517.

[8] R. L. Frank, E. H. Lieb, and R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. 21 (4) (2008) 925-950.

[9] N. Ghoussoub and F. Robert, *The Hardy-Schrödinger operator with interior singularity: The remaining cases*, Submitted (2016). arXiv:1612.08355 Hal Id : hal-01422509.

[10] N. Ghoussoub, F. Robert, S. Shakerian and M. Zhao, *Mass and asymptotics associated to fractional Hardy-Schrödinger operators in critical regimes*, April 2017, https://arxiv.org/pdf/1704.08658.pdf.

[11] N. Ghoussoub and S. Shakerian, *Borderline variational problems involving fractional Laplacians and critical singularities*, Advanced Nonlinear Studies, 15 (2015) 527-555.

[12] I. W. Herbst, *Spectral theory of the operator $(p^2 + m^2)^{1/2} - \frac{Ze^2}{r}$*, Comm. Math. Phys. 53 (1977), no. 3, 285-294.

[13] H. Jaber, *Mountain pass solutions for perturbed Hardy-Sobolev equations on compact manifolds*, Analysis (Berlin) 36 (2016), no. 4, 287296.

[14] E. Jannelli, *The role played by space dimension in elliptic critical problems*, Journal of Differential Equations 156.2 (1999): 407-426.

[15] S. Shakerian, *Multiple Positive Solutions for Nonlocal Elliptic Problems Involving the Hardy Potential and Concave-Convex Nonlinearities*, arXiv preprint arXiv:1708.01369 (2017).