Phase Diagram of the Spin-Orbital model on the Square Lattice

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We study the phase diagram of the spin-orbital model in both the weak and strong limits of the quartic spin-orbital exchange interaction. This allows us to study quantum phase transitions in the model and to approach from both sides the most interesting intermediate-coupling regime and in particular the $SU(4)$-symmetric point of the Hamiltonian. It was suggested earlier by Li et al [Phys. Rev. Lett. \textbf{81}, 3527 (1999)] that at this point the ground state of the system is a plaquette spin-orbital liquid. We argue that the state is more complex. There is plaquette order, but it is anisotropic: bonds in one direction are stronger than those in the perpendicular direction. This order is somewhat similar to that found recently in the frustrated $J_1 - J_2$ Heisenberg spin model.

I. INTRODUCTION

In many transition metal oxides the low-lying electron states are characterized by both spin and orbital degrees of freedom. Thus the simplest model that also takes into account the strong Coulomb repulsion of transition metal $d$-electrons is the multiband Hubbard model. For the case of one electron per site with two-fold orbital degeneracy (isospin $T = 1/2$) and the strong interaction limit Kugel and Khomskii\textsuperscript{1} derived the effective spin and isospin Hamiltonian:

$$H = \sum_{\langle i,j \rangle} (J \mathbf{S}_i \cdot \mathbf{S}_j + J \mathbf{T}_i \cdot \mathbf{T}_j + V (\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{T}_i \cdot \mathbf{T}_j) ) . \quad (1)$$

Here $S_i$ and $T_i$ are spin and isospin operators at the lattice site $i$, and the summation is over the nearest neighbor bonds $\langle i,j \rangle$. $J$ and $V$ are positive constants of approximately equal magnitude.

In general, the exchange constants in spin and isospin subsystems may differ. It is also possible that the isospin interaction $(T_i T_j)$ is not spherically symmetric. If the isospin interaction is spherically symmetric then the total symmetry of the Hamiltonian is $SU(2) \times SU(2)$. At the point $J = V/4$ the Hamiltonian has even higher $SU(4)$ symmetry. Investigation of the model at this point has attracted great interest\textsuperscript{1} in the one-dimensional case there is an exact solution for the ground state and for several types of excitations\textsuperscript{2}. For the two-dimensional case (square lattice) an exact solution is not known. A kind of mean field $SU(4)$-spin-wave theory has been developed for this case\textsuperscript{3-10} and numerical simulations are also available\textsuperscript{11}. The $SU(4)$-spin-wave theory indicates a disordered ground state at $J = V/4$. However this approach contains uncontrolled approximation, and its accuracy is uncertain. It was pointed out in Ref.\textsuperscript{1} that at $J = V/4$ the Ising part of the Hamiltonian is equivalent to the 4-state Potts model\textsuperscript{12}. This model has classical macroscopic degeneracy of the ground state, and this is another indication in favor of the disordered ground state of the spin-orbital model at the $SU(4)$ point. As a possible ground state the authors of Ref.\textsuperscript{1} suggested a liquid of plaquette $SU(4)$ singlets. This ground state is a $SU(4)$ singlet, but it has spontaneous breaking of the translational lattice symmetry. Further analytical and numerical work supports this scenario\textsuperscript{13}.

The present work has the following goals: to investigate the spin-orbital model on the square lattice, away from the $J = V/4$ point and to identify all the quantum transitions, and also to investigate the model away from the isotropic $SU(2) \times SU(2)$ line. We also present new evidence in favor of the anisotropic plaquette spin-orbital liquid in the vicinity of the $SU(4)$ point. The structure of the paper is as follows. In Sec. II we start from the limit $J \gg V/4$ and then approach smaller values of $J$, using both spin-wave and perturbation methods. In section III we consider the opposite limit, of strong quartic interaction: $J < V/4$. In this case the spin-wave theory is not valid and the only tool is a modified series expansion. Our conclusions are presented in Sec. IV.
II. THE CASE OF WEAK QUARTIC INTERACTION, $V/4 < J$

In the limit $J \gg V$ the Hamiltonian (1) describes two almost independent Heisenberg subsystems. Both of them are ordered antiferromagnetically, and so the $SU(2) \times SU(2)$ symmetry is spontaneously broken. Therefore there are two Goldstone excitations, spin-wave ($s$-wave) and isospin-wave ($t$-wave), that determine the low energy physics of the model. The ground state is a direct product of the Néel-ordered spin subsystem and of the Néel-ordered isospin subsystem, with two sublattices in each subsystem. In each sublattice we introduce a Dyson-Maleev transformation of spin and isospin operators. For the A sublattice,

$$
S^z_i = S - a^+_i a_i, \quad T^z_i = T - c^+_i c_i, \\
S^-_i = a^+_i, \quad T^-_i = c^+_i, \\
S^+_i = (2S - a^+_i a_i) a_i, \quad T^+_i = (2T - c^+_i c_i) c_i.
$$

(2)

and for the B sublattice,

$$
S^z_j = -S + b^+_j b_j, \quad T^z_j = -T + d^+_j d_j, \\
S^-_j = -b^+_j, \quad T^-_j = -d^+_j, \\
S^+_j = -b^+_j (2S - b^+_j b_j), \quad T^+_j = -d^+_j (2T - d^+_j d_j).
$$

(3)

We are interested in $S = T = 1/2$, but for now we keep them as parameters. After the transformation the Hamiltonian (1) takes the following form

$$
H = \frac{1}{2} (J - V T^2) S^2 z N + (J - V T^2) \sum_{(i,j)} \left( S(a^+_i a_i + b^+_j b_j - a_i b_j - a^+_i b^+_j) + \frac{1}{2} a^+_i (b^+_j - a^+_i) b^+_j \right)
$$

$$
- \frac{1}{2} (J - V S^2) T^2 z N + (J - V S^2) \sum_{(i,j)} \left( T(c^+_i c_i + d^+_j d_j - c_i d_j - c^+_i d^+_j) + \frac{1}{2} c^+_i (d^+_j - c^+_i) d^+_j \right)
$$

$$
- \frac{1}{2} V S^2 T^2 z N + V S T \sum_{(i,j)} (a^+_i a_i + b^+_j b_j - a_i b_j - a^+_i b^+_j)(c^+_i c_i + d^+_j d_j - c_i d_j - c^+_i d^+_j).
$$

(4)

Here $N$ is the number of lattice sites and $z = 4$ is the number of nearest neighbors. In eq. (4) the part containing only the operators $a$ and $b$ describes the spin subsystem with a renormalized exchange constant, $J \rightarrow J - VT^2$, while the part containing only the $c$ and $d$ operators describes the isospin subsystem with similar renormalization, $J \rightarrow J - VS^2$. The other terms describe the interaction between the subsystems.

In the linear spin wave approximation we neglect all quartic terms in the Hamiltonian (1). After this the standard Bogoliubov transformation diagonalization gives the dispersion relation for $s$- and $t$-waves: $\omega_s(k) = (J - VT^2)S z \sqrt{1 - \gamma^2(k)}$, $\omega_t(k) = (J - VS^2)T z \sqrt{1 - \gamma^2(k)}$, where $\gamma(k) = \frac{1}{2} (\cos k_x + \cos k_y)$. The staggered magnetization in this approximation is independent of $J/V$, and is

$$
\langle S_z \rangle = S - 0.1966, \\
\langle T_z \rangle = T - 0.1966.
$$

(5)

One can also easily take into account single loop corrections to the linear spin wave approximation. To do this, following the usual procedure, we make all possible decouplings in the quartic terms in the Hamiltonian (1). This gives the following effective Hamiltonian

$$
H = -\frac{1}{2} (J - V T^2)(S^2 - \kappa^2) z N - \frac{1}{2} (J - V S^2)(T^2 - \kappa^2) z N - \frac{1}{2} V (S^2 T^2 + 4 \kappa^4) z N
$$

$$
+ J_s \sum_{(i,j)} (a^+_i a_i + b^+_j b_j - a_i b_j - a^+_i b^+_j) + J_T \sum_{(i,j)} (c^+_i c_i + d^+_j d_j - c_i d_j - c^+_i d^+_j).
$$

(6)

Here we use the notations:
\[ \tilde{J}_s = (J - VT^2)(1 - \kappa) + 2VT^2 \kappa, \]
\[ \tilde{J}_t = (J - VS^2)(1 - \kappa) + 2VS^2 \kappa, \]
\[ \kappa = \frac{1}{S} \left( \langle a_i^d a_i^\dagger \rangle - \langle a_i^a a_i^\dagger \rangle \right) = \frac{1}{T} \left( \langle c_i^d c_i^\dagger \rangle - \langle c_i^d d_i^\dagger \rangle \right) = \frac{1}{T} \left( \langle d_i^d d_i^\dagger \rangle - \langle c_i^d c_i^\dagger \rangle \right). \] (8)

The value of \( \kappa \) can be calculated in the linear spin-wave approximation or it can be found self-consistently. Both values are very close. For further analysis we take the self-consistent value: \( \kappa = -0.07897/S \). The excitation spectra for \( s \) - and \( t \)-waves are then

\[ \omega_s(k) = \tilde{J}_s Sz \sqrt{1 - \gamma^2(k)}, \]
\[ \omega_t(k) = \tilde{J}_t Tz \sqrt{1 - \gamma^2(k)}. \] (9)

The difference from the linear spin-wave approximation is only in the renormalized values of \( \tilde{J} \). The staggered magnetization is exactly the same as that in the linear spin wave approximation, see eq. (5). The values of the effective exchange constants \( \tilde{J}_s \) and \( \tilde{J}_t \) vanish at some critical value of the bare exchange constant \( J \). One finds from eqs. (5) that at \( S = T = 1/2 \) the critical value is

\[ J_c = VS^2 S - 3\kappa \frac{S - \kappa}{S - \kappa} = 0.3182V. \] (10)

The staggered magnetization given by eqs. (5) remains constant on approaching this point, and therefore the quantum phase transition at \( J/V \approx 0.32 \) is of first-order.

In the above analysis we have taken into account explicitly only the \( s \) - and \( t \)-waves. However along with these two excitations there are also the so called spin-isospin excitations (\( st \)-wave) carrying simultaneous spin and isospin flips. In the spin-wave approach the \( st \)-excitations are not treated explicitly. They appear in the calculation via the quartic interaction term in the Hamiltonian (8) that we treat in the one-loop approximation. However as one approaches the transition point this interaction becomes more important and our way of treating it is questionable. Moreover, it is known that in the 1D model at \( J = V/4 \) for \( S = T = 1/2 \) the \( s \)- \( t \)- and \( st \)-excitations are almost degenerate at \( k \to 0 \), see Ref. (4). This raises further questions about the validity of our implicit treatment of the \( st \)-excitation. In the remainder of this Section we treat the \( st \) excitations explicitly.

The spin-wave approach is not convenient for the explicit consideration of the \( st \)-excitation. To address this problem we use a kind of series expansion method. One takes the Ising part of the Hamiltonian (8) as a zeroth-order approximation and the transverse terms are treated as perturbations proportional to the introduced small parameters \( x \) and \( y \). So we represent the Hamiltonian (8) in the following form

\[ H = H_0 + H_s + H_t + H_{st}, \]
\[ H_0 = \sum_{\langle i,j \rangle} \left( JS_i^z S_j^z + JT_i^z T_j^z + VS_i^z S_j^z + T_i^z T_j^z \right), \]
\[ H_s = \frac{x}{2} \sum_{\langle i,j \rangle} (J + VT_i^z T_j^z) (S_i^z S_j^z + S_i^z S_j^z), \]
\[ H_t = \frac{y}{2} \sum_{\langle i,j \rangle} (J + VS_i^z S_j^z) (T_i^z T_j^z + T_i^z T_j^z), \]
\[ H_{st} = \frac{xy}{4} V \sum_{\langle i,j \rangle} (S_i^z S_j^z + S_i^z S_j^z) (T_i^z T_j^z + T_i^z T_j^z). \]

The properties of the initial spherically symmetric Hamiltonian (8) are recovered in the limit \( x = y \to 1 \).

The effect of the \( H_s \) perturbation \( (y = 0) \) alone leads to the the same reduction of the staggered magnetization as in the pure Heisenberg model. The average spin is reduced from \( 1/2 \) to

\[ \langle S_i^z \rangle = 1/2 - x^2/9 - x^4/225 - ... \to 0.307. \] (12)

The numerical value corresponds to \( x = 1 \), see Ref. (4). The antiferomagnetically ordered isospin subsystem background results only in a renormalization of exchange constant \( J \to J - V/4 \). However it does not effect the magnetization as it does not depend on \( J \). The same effect occurs with \( H_t \) acting alone on the isospin subsystem.

An additional reduction of the staggered magnetization appears due to the simultaneous action of \( H_s \) and \( H_t \) \((s + t\) channel). For example first \( H_s \) flips the spins at nearest sites and then \( H_t \) flips the isospins at the same sites. So a
virtual fluctuation with $st$-flips at neighboring sites is created. After that the fluctuation can collapse back to the Ising ground state using the same chain of spin and isospin flips. Thus the effect of the staggered magnetization reduction arises in 4th order of perturbation theory and hence it is proportional to $\sim (J - V/4)^4/J^4$. Near the critical value $J \approx J_c \approx 0.32V$, this reduction is so small that there is no point to calculate it more accurately. A more important effect arises from the $H_{st}$ term ($st$-channel) in the Hamiltonian (11). This perturbation creates, in first order, a pair of $st$-flips at the neighboring sites. The matrix element of the perturbation is $xyV/4$, and the energy of the virtual excitation is $\Delta E = 6J$. Hence the reduction of the staggered magnetization due to this mechanism is

$$\delta \langle S_z \rangle = -4 \frac{(xyV/4)^2}{(\Delta E)^2} = -\frac{1}{9} \left( \frac{xyV/4}{J} \right)^2.$$ (13)

At the critical value $J \approx J_c \approx 0.32V$ and at $x = y = 1$ this gives $\delta \langle S_z \rangle \approx -0.07$. This is the additional reduction of the value presented in eq. (12). We see that the effect of $st$-excitations is not completely negligible, but it is not qualitatively important: the staggered magnetization remains finite at the critical point. This supports our conclusion that the phase transition from the Néel×Néel state at $J > J_c \approx 0.32V$ to some other state at $J < J_c$ is of first order. Unfortunately because of the first-order phase transition this analysis does not give any insight into the structure of the ground state at $J < J_c$. A different approach is needed for consideration of the strong quartic interaction limit.

### III. THE STRONG QUARTIC INTERACTION CASE, $0 < J < V/4$

We will see that in the limit $J < V/4$ the $st$-excitations are crucially important. This makes this regime qualitatively different from that considered in the previous section, and this is why the spin-wave approach is not helpful at $J < V/4$. We base our analysis on series expansions using a representation of the Hamiltonian (1) in the form (11). There are two immediately obvious possibilities for the ground state of the system at small $J$: a) The Néel×Ferromagnetic state ($N\times F$), where one subsystem has Néel ordering and the other is ordered ferromagnetically, b) The Stripe×Stripe state ($S\times S$), where one subsystem has collinear magnetic order along one crystal axis and the other subsystem has collinear magnetic order in the perpendicular direction, see Fig.1. The $N\times F$ ground state spontaneously violates the $SU(2) \times SU(2)$ symmetry of the Hamiltonian (1). The $S\times S$ state in addition violates the $C_{4v}$ group of the square lattice, so the total spontaneously broken symmetry in this case is $SU(2) \times SU(2) \times C_{4v}$.

Now let us look at the Hamiltonian (11). One can easily show that in the Ising limit (i.e. $x = y = 0$) the $N\times F$ and $S\times S$ states have the same energy. Moreover there is an infinite set of other degenerate states because a simultaneous spin-isospin flip on any lattice site ($st$- wave) does not cost any energy (we consider the $S = T = 1/2$ case). So in the Ising approximation there is an infinite macroscopic degeneracy of the ground state. What we want to demonstrate first is that the quantum fluctuations stabilize the $S\times S$ state.

Let us consider the simplest quantum fluctuations: spin or isospin flips at nearest sites, $s$- and $t$-waves. These fluctuations are generated by $H_s$ and $H_t$ correspondingly, see eq. (11). A straightforward second-order perturbation theory calculation gives

$$E_{N\times F} = -\left( \frac{V}{8} + \frac{x^2 V + 4J}{24} \right) N,$$

$$E_{S\times S} = -\left( \frac{V}{8} + \left( x^2 + y^2 \right) \frac{(V + 4J)^2}{4(3V - 4J)} \right) N.$$ (14)

Thus the $S \times S$ state has lower energy than the $N \times F$ state for $x = y$. So, in the following, we consider only the $S \times S$ state. However eq. (14) does not answer the question posed above: what is the excitation energy of the $st$-wave above the $S \times S$ background? If it is still zero then the background is unstable. Analysis of the second-order perturbation theory result shows that the $st$-excitation blocks some $s$- and $t$-fluctuations, and hence the $st$-excitation energy is nonzero. A calculation gives the following value of the $st$-gap

$$\Delta_{st} = E_{st} - E_{S\times S} = 2(x^2 + y^2) \left( J + \frac{V}{4} \right)^2 \left( J + \frac{V}{4} \right) \left( \frac{4}{3V - 4J} - \frac{1}{V} - \frac{1}{3V - 8J} \right),$$ (15)

where $E_{st}$ is the energy of the $S \times S$-background with one $st$-excitation. In this order of perturbation theory the $st$-excitation is dispersionless. The gap $\Delta_{st}$ vanishes at $J = 0$ and $J = V/4$. However the gap is positive in the interval $0 < J < V/4$ and hence the quantum fluctuations do, in fact, stabilize the $S \times S$-state.

One can also calculate the dispersion relations of $s$- and $t$-waves. To first-order in $H_s$ and $H_t$ the results are
\begin{align}
\omega_s(k) &= V/2 - x(V/4 - J) \cos(k_x a) , \\
\omega_t(k) &= V/2 - y(V/4 - J) \cos(k_y a) .
\end{align}

We see that these energies are substantially higher than \(\Delta_{st}\). Therefore below we concentrate on the \(st\)-waves that drive all the critical dynamics in the system.

The excitation energy of two remote \(st\)-waves is \(2\Delta_{st}\). However if two \(st\)-waves are localized at the nearest diagonal sites of the lattice then suppression of quantum fluctuations is reduced and a direct calculation shows that the energy of such a configuration is just \(\Delta_{st}\). This means that two \(st\)-waves attract each other and the binding energy is \(\epsilon_b = 2\Delta_{st} - \Delta_{st} \approx \Delta_{st}\). We would like to stress that this is not an effect of a simple potential attraction, this is the effect of a quantum bag: suppression of quantum fluctuations.

We have found that there is a quantum bag attraction between the \(st\)-waves. In this situation it is quite natural to put more \(st\)-waves in the bag. Three excitations do not fit naturally into the square lattice, and therefore we consider four \(st\)-excitations combined into a plaquette bag as shown in Fig. 2. This is the plaquette excitation. The corresponding excitation energy, the plaquette gap \(\Delta_P\), is equal to

\[
\Delta_P = 2(x^2 + y^2) \left( J + \frac{V}{4} \right)^2 \left( \frac{7}{3V - 4J} - \frac{4}{3V} - \frac{1}{3V + 4J} - \frac{2}{3V - 8J} \right) .
\]

The value of \(\Delta_P\) versus \(4J/V\) is plotted in Fig. 3, as the solid line. For comparison in the same figure we plot the value of \(\Delta_{st}\) given by eq. (13). The gap \(\Delta_P\) vanishes at \(J = V/4\) and in the interval \(0.239V < J < 0.25V\) it is negative. This means that in this interval the \(S \times S\)-state is unstable with respect to condensation of plaquette excitations. In the phase diagram shown in Fig. 4 this region is separated from that with smaller values of \(J\) by the vertical dashed line \(J \approx 0.24V\). Naively one would say that a crossing of this line corresponds to a second-order phase transition from the \(S \times S\)-state to a state with the plaquette order considered in Ref. 3. However, one needs to examine this more carefully. As we mentioned above, the \(S \times S\)-state violates the \(SU(2) \times SU(2) \times C_{4v}\) symmetry of the Hamiltonian. On the other hand the plaquette state at \(J = V/4\) considered in Ref. 12 violates only \(Z_2 \times Z_2\)-symmetry (plaquette state has 2-fold degeneracy in \(x\)-direction and 2-fold degeneracy in \(y\)-direction). As we have demonstrated above, the \(Z_2 \times Z_2\) order parameter appears in the second order phase transition at \(J \approx 0.24V\). However the question arises: how do the \(SU(2) \times SU(2) \times C_{4v}\) order parameters disappear when going from small \(J\) to \(J = V/4\)? To confirm this scenario we have to find two additional quantum phase transitions: 1) disappearance of \(SU(2) \times SU(2)\) order parameters, 2) restoration of \(C_{4v}\) order.

The \(SU(2) \times SU(2)\) order parameter is the usual staggered magnetization. So let us calculate reduction of the staggered magnetization. There are four types of relevant quantum fluctuations that reduce the magnetization A) spin flips at nearest sites, i.e. a virtual creation of two \(s\)-waves, B) isospin flips at nearest sites, i.e. a virtual creation of two \(t\)-waves, C) simultaneous spin and isospin flips at nearest sites, i.e. a virtual creation of two \(st\)-waves, D) simultaneous spin and isospin flips at four plaquette sites, i.e. a virtual creation of a plaquette excitation. We cannot use a conventional perturbation expansion approach because, in the Ising approximation, the excitation energies \(\Delta_{st}\) and \(\Delta_P\) vanish, see eqs. (13) and (18). Instead we use the Tamm-Dancoff method\(^2\) that accounts for all low energy physical excitations. The average value of \(S_z\) is equal to

\[
\langle S_z \rangle = \frac{1}{2} - 2 \left( \frac{t_s}{\Delta_s} \right)^2 - 4 \left( \frac{t_{st}}{\Delta_{st}} \right)^2 - 4 \left( \frac{t_P}{\Delta_P} \right)^2 .
\]

Here \(\Delta_s = 3V/4 - J\) and \(\Delta_{st}, \Delta_P\) given by eqs. (13), (18) are the excitation energies corresponding to the fluctuations A, C, and D. The matrix elements \(t_s, t_{st}\), and \(t_P\) are the amplitudes of creation of these fluctuations from the Ising ground state. The fluctuation B (isospin flips at nearest sites) does not contribute to the reduction of \(\langle S_z \rangle\). The coefficients 2, 4, and 4 in eq. (19) give the number of possibilities for a given virtual excitation to contribute to the reduction of \(\langle S_z \rangle\). The amplitude \(t_s\) arises in the first order of perturbation theory in \(H_s\), the amplitudes \(t_{sp}\) and \(t_P\) arise in the second order in \(H_{st}\) or in the combined fourth order of \(H_s\) and \(H_t\). Direct calculation give the following results

\[
t_s = \frac{x}{2} \left( \frac{V}{4} + J \right) ,
\]

\[
t_{sp} = \frac{x^2 y^2}{2} \left( \frac{J + \frac{V}{2}}{4V - J} \right)^2 \left( J - \frac{V}{4} \right)^2 \left( \frac{1}{V} + \frac{2}{V - 2J} \right) + \frac{x^2 y^2}{4} \left( \frac{J + \frac{V}{2}}{4V - J} \right)^2 V ,
\]

\[
t_P = \frac{x^3 y^2}{2} \left( \frac{J + \frac{V}{2}}{4V - J} \right)^2 \left( \frac{(J + \frac{V}{4})^2}{2V} + \frac{(J - \frac{V}{4})^2}{V - 2J} \right) .
\]
Substitution of these amplitudes into eq. (19) allows us to find the reduction of the staggered magnetization. In the final analysis we set $x = y$. At any given value of $J$ the magnetization vanishes at some particular $x$. This is the location of the second-order phase transition from the magnetically ordered phase to the spin-orbital liquid. In the phase diagram shown in Fig. 4 the solid line separates these two states.

Thus we have found the transition from the magnetically ordered $S \times S$-state to the spin-orbital liquid. However we do not see any mechanism for restoration of the $C_{4v}$-symmetry violated in the $S \times S$-state. These considerations lead us to the following phase diagram, see Fig. 4. The phase (I) below the solid line is the Stripe-$\times$Stripe-symmetry which has a nonzero staggered magnetization and also spontaneously violates the $C_{4v}$-symmetry of the Hamiltonian. The phase (II) is the spin-orbital liquid. It has no magnetization, but still must violate the $C_{4v}$-symmetry. So the symmetry of this state is exactly the same as the symmetry of the columnar dimer spin liquid in the frustrated $J_1 - J_2$ Heisenberg model at $0.38 < J_2/J_1 < 0.5$, see Refs. 14, 15. Certainly we cannot claim that the state (II) is a kind of dimer quantum liquid, but its symmetry is the same. Finally, the state (III) on the right hand side of the dashed line is a plaquette spin-orbital liquid because of the condensation of the plaquette excitations at $J \approx 0.24V$. However it is not an isotropic plaquette liquid considered in Refs. 16. According to our symmetry considerations the $C_{4v}$ symmetry must be violated, so this is a plaquette liquid with bonds in one direction different from that in the perpendicular direction. One can say that it is a Stripe-Plaquette-Correlated quantum spin liquid. It violates the $Z_2 \times Z_2 \times C_{4v}$ symmetry of the Hamiltonian. This state is similar to that found recently in the frustrated $J_1 - J_2$ Heisenberg model at $0.5 < J_2/J_1 < 0.6$, see Refs. 18, 19.

Our analysis is based on the Tamm-Dancoff expansion at small $x$. One certainly cannot guarantee that there is not an additional phase transition line somewhere at $x \sim 1$. In our opinion such a qualitative difference is unlikely, but it is possible. This is why the problem certainly requires further numerical analysis.

**IV. CONCLUSIONS**

We have analyzed the properties of the spin-orbital model on the square lattice in the weak and strong coupling limits. We show that in the case of weak quartic interaction the usual spin-wave approach is valid. There are the st-quasiparticles that are not adequately described by this approach, but in this limit these quasiparticles just give small corrections. In the strong quartic interaction limit the spin-wave approach can not be applied because all the critical dynamics is driven by the composite st-quasiparticles. The usual series expansion method also cannot be applied because of the infinite classical degeneracy. To analyze the situation we apply the Tamm-Dancoff method that proved to be efficient in this limit.

At $J/V > 0.32$ the ground state of the model is a direct product of the Néel ordered spin subsystem and the Néel ordered isospin subsystem. There is a first order phase transition to the spin-orbital quantum liquid at the point $J/V \approx 0.32$. We argue that this quantum liquid also has a structure. At $0 < J/V < (J/V)_c$ it is a stripe liquid with spontaneously violated $C_{4v}$-symmetry. At the critical point $(J/V)_c$ there is a second-order phase transition to the stripe-plaquette-correlated liquid that spontaneously breaks $Z_2 \times Z_2 \times C_{4v}$ symmetry. The Tamm-Dancoff method gives the following estimate for the critical point: $(J/V)_c \approx 0.24$. Thus we argue that the stripe-plaquette-correlated quantum spin-orbital liquid exists in a narrow interval $0.24 < J/V < 0.32$ around the $SU(4)$ symmetric point $J/V = 0.25$. The structure of this state is more complex than has been believed previously: it violates an additional $C_{4v}$-symmetry of the Hamiltonian.

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FIG. 1. The structure of the Stripe $\times$ Stripe state.

FIG. 2. Plaquette fluctuation in $S \times S$ background

FIG. 3. The plaquette energy gap $\Delta_P$ is shown by solid line. The energy gap $\Delta_{st}$ of the state with one $st$-flip is shown by the dashed line.

FIG. 4. Phase diagram $x - J/V$ of the spin-orbital model on the square lattice. (I) is the Stripe$\times$Stripe phase with non-zero average spin and isospin; (II) spin-orbital liquid; (III) plaquette spin-orbital liquid; (IV) Néel$\times$Néel state.