Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities

Jing Li

AMSS, Chinese Academy of Sciences

Collaborator:

Zhouping Xin (IMS, CUHK)

Nonlinear PDEs and Related Topics

December 26–30 2019; National University of Singapore
Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities

Jing Li\textsuperscript{a,b}, Zhouping Xin\textsuperscript{b}\textsuperscript{*}

\begin{itemize}
\item a. Institute of Applied Mathematics, AMSS, & Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, P. R. China
\item b. The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong
\end{itemize}

Abstract

This paper concerns the existence of global weak solutions to the barotropic compressible Navier-Stokes equations with degenerate viscosity coefficients. We construct suitable approximate system which has smooth solutions satisfying the energy inequality, the BD entropy one, and the Mellet-Vasseur type estimate. Then, after adapting the compactness results due to Mellet-Vasseur [Comm. Partial Differential Equations 32 (2007)], we obtain the global existence of weak solutions to the barotropic compressible Navier-Stokes equations with degenerate viscosity coefficients in two or three dimensional periodic domains or whole space for large initial data. This, in particular, solved an open problem in [P. L. Lions. Mathematical topics in fluid mechanics. Vol. 2. Compressible models. Oxford University Press, 1998].

Keywords. compressible Navier-Stokes equations; degenerate viscosities; global weak solutions; large initial data; vacuum.

AMS subject classifications. 35Q35, 35B65, 76N10

1 Introduction and main results

The barotropic compressible Navier-Stokes equations, which are the basic models describing the evolution of a viscous compressible fluid, read as follows

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}S + \nabla P(\rho) = 0,
\end{cases}$$

\text{(1.1)}
Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities

Jing Li\textsuperscript{a,b}, Zhouping Xin\textsuperscript{b}\textsuperscript{*}
\textsuperscript{a}. Institute of Applied Mathematics, AMSS, & Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, P. R. China
\textsuperscript{b}. The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong

Abstract

This paper concerns the existence of global weak solutions to the barotropic compressible Navier-Stokes equations with degenerate viscosity coefficients. We construct suitable approximate system which has smooth solutions satisfying the energy inequality, the BD entropy one, and the Mellet-Vasseur type estimate. Then, after adapting the compactness results due to Bresch-Desjardins (2002, 2003) and Mellet-Vasseur (2007), we obtain the global existence of weak solutions to the barotropic compressible Navier-Stokes equations with degenerate viscosity coefficients in two or three dimensional periodic domains or whole space for large initial data. This, in particular, solved an open problem proposed by Lions (1998).

Keywords. compressible Navier-Stokes equations; degenerate viscosities; global weak solutions; large initial data; vacuum.

AMS subject classifications. 35Q35, 35B65, 76N10

1 Introduction and main results

The barotropic compressible Navier-Stokes equations, which are the basic models describing the evolution of a viscous compressible fluid, read as follows

\[
\begin{cases}
\rho_t + \nabla (\rho u) = 0, \\
(\rho u)_t + \nabla (\rho u \otimes u) - \nabla S + \nabla P(\rho) = 0,
\end{cases}
\]

where \( x \in \Omega \subset \mathbb{R}^N (N = 2, 3), t > 0 \), \( \rho \) is the density, \( u = (u_1, \cdots, u_N) \) is the velocity, \( S \) is the viscous stress tensor, and \( P(\rho) = a\rho^\gamma (a > 0, \gamma > 1) \) is the pressure. Without

\textsuperscript{*}This research is supported in part by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK-4041/11P, CUHK-4048/13P, a Croucher Foundation & CAS Joint Grant, NSFC/RGC Joint Research Scheme N-CUHK443/14, and a Focus Area Grant at The Chinese University of Hong Kong. The research of J. Li was partially supported by the National Center for Mathematics and Interdisciplinary Sciences, CAS, and NNSFC Grant No. 11371348. Email: aijingli@gmail.com (J. Li), zpxin@ims.cuhk.edu.hk (Z. Xin).
Contents

- Part I: Introduction
- Part II: Main Theorems
- Part III: Sketch of Proof
The barotropic compressible Navier-Stokes equations read:

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div} \mathbb{S} + \nabla P(\rho) = 0,
\end{cases}
\]  

where \( x \in \Omega \subset \mathbb{R}^N \) \((N = 2, 3), t > 0, \)
\( \rho : \text{density}, \)
\( u = (u_1, \cdots, u_N) : \text{velocity}, \)
\( P(\rho) = \rho^\gamma (\gamma > 1) : \text{pressure}, \)
\$: \text{viscous stress tensor with either}

\[ S \equiv S_1 \triangleq h \nabla u + g \text{div} u \mathbb{I}, \quad (1.2) \]

or

\[ S \equiv S_2 \triangleq h \mathcal{D} u + g \text{div} u \mathbb{I}, \quad (1.3) \]

where \( \mathcal{D} u = \frac{1}{2}(\nabla u + (\nabla u)^{\text{tr}}) \): deformation tensor,

\( \mathbb{I} \): the identical matrix,

\( h, g \) satisfy the physical restrictions

\[ h > 0, \quad h + Ng \geq 0. \quad (1.4) \]
Global existence of classical solutions away from vacuum ($h$ and $g$ are both constants):

- **1D case:**
  - Kanel (1968): isentropic case, large initial data;
  - Kazhikhov & Shelukhin (1977): full NS case, large initial data.

- **Multi-D case:**
  - Matsumura & Nishida (1980): initial data close to a non-vacuum equilibrium
Global existence of weak solutions containing vacuum states (3D, $h$ and $g$ are both constants):

- Lions (1993, 1998): large initial data, when $\gamma \geq 9/5$;
- Feireisl (2001): large initial data, when $\gamma > 3/2$;
- Jiang-Zhang (2001): $\gamma > 1$, for spherically symmetric solutions.
Theorem (Lions-Feireisl (1993,1998,2001))

If $\gamma > 3/2$ and the initial energy

$$C_0 \triangleq \frac{1}{2} \int \rho_0 |u_0|^2 \, dx + \frac{1}{\gamma - 1} \int P(\rho_0) \, dx < \infty. \quad (1.5)$$

THEN $\exists$ a global weak solution $(\rho, u)$.

Remark

In particular, for the whole space case, the density vanishes (in some weak sense).
Variable viscosities degenerate at vacuum

- $h = h(\rho), g = g(\rho)$
  - Liu-Xin-Yang (1998) derived the compressible Navier-Stokes equations from the Boltzmann equation by the Chapman-Enskog expansions.
  - Gerbeau-Perthame (2001), Marche (2007), Bresch-Noble (2007) (2011)
    a friction shallow-water system used in Oceanography can be written in a two-dimensional space domain $\Omega$ with $h(\rho) = g(\rho) = \rho$.
  - Geophysical flow models etc.
Open Problem (Lions (1998))

For $N = 2, 3$, the Cauchy problem is completely open with
$S = \rho \nabla u$ and $\gamma = 2$. 
Known results

- **1D with** \( h = g = \rho^\alpha (\alpha > 0) \) (free boundaries):
  - Jiang, Kanel, Makino, Okada, Qin, Xin, Yang, Yao, Zhang, Zhu, et al.

- **Weigant & Kazhikhov model** (\( h \equiv \text{const.}, g = g(\rho) \)):
  - Weigant & Kazhikhov (1995), Huang-Li (2012),
  - Jiu-Wang-Xin (2012): 2D, large initial data.
Multi-dimensional case:

- **BD entropy**

Bresch-Desjardins (2003) obtained a new a priori estimate on the spatial derivatives of the density (BD entropy) under the condition that

\[ g(\rho) = h'(\rho)\rho - h(\rho) \]

for the periodic boundary conditions and the Cauchy problem. They used the BD entropy to obtain the global existence of weak solutions to (1.1) (1.2) and (1.1) (1.3) with some additional drag terms.
Log-type energy estimate

Mellet-Vasseur (2007) obtained a new a priori Log-type energy estimate and study the stability of (1.1) (1.2) and (1.1) (1.3) without any additional drag term under the assumption of existence of smooth approximate approximation solutions satisfying energy estimate and BD entropy.
For the cases that $h, g$ are both constants (non-degenerate viscosity), the construction of the approximation solutions can be achieved by using standard Galerkin methods (Lions (1998), Feireisl et al (2001)). However, the constructions of the smooth approximation solutions remain to be challenge, which does not seem standard in the case of appearance of vacuum due to the degeneracy of viscosities.
Remark

For the cases that $h$, $g$ are both constants (non-degenerate viscosity), the construction of the approximation solutions can be achieved by using standard Galerkin methods (Lions (1998), Feireisl et al (2001)). However, the constructions of the smooth approximation solutions remain to be challenge, which does not seem standard in the case of appearance of vacuum due to the degeneracy of viscosities.
Part results for special cases are available:

- 1D: Li-Li-Xin (2008)
- Multi-D with the spherical initial data: Guo-Jiu-Xin (2008), Guo-Li-Xin (2012)

The truly multidimensional case seems much more complicated. It has been an open problem mentioned by many researchers in the area.
Open Problem

For general initial data, the problem proposed by Lions is essentially reduced to the constructions of the smooth approximation solutions satisfying the BD entropy inequality since the a priori energy estimate and Log-type one are relatively easy to be verified.
For the sake of simplicity, it is assumed that for $\alpha > 0$,

$$h(\rho) = \rho^\alpha, \quad g(\rho) = (\alpha - 1)\rho^\alpha. \quad (1.21)$$

For $\Omega = \mathbb{R}^N (N = 2, 3)$ or $\Omega = \mathbb{T}^N (N = 2, 3)$.

The initial conditions are imposed as

$$\rho(x, t = 0) = \rho_0, \quad \rho u(x, t = 0) = m_0. \quad (1.22)$$
Definition (weak solutions)

\((\rho, u)\) is said to be a weak solution to (1.1) (1.3) (1.21) (1.22) if

\[
\begin{align*}
0 \leq \rho &\in L^\infty(0, T; L^1(\Omega) \cap L^\gamma(\Omega)), \\
\nabla \rho^{(\gamma + \alpha - 1)/2} &\in L^2(0, T; (L^2(\Omega))^N), \\
\nabla \rho^{\alpha - 1/2}, \sqrt{\rho} u &\in L^\infty(0, T; (L^2(\Omega))^N), \\
h(\rho) \nabla u, h(\rho)(\nabla u)^{\text{tr}} &\in L^2(0, T; (W_{\text{loc}}^{-1,1}(\Omega))^{N \times N}), \\
g(\rho) \text{div} u &\in L^2(0, T; W_{\text{loc}}^{-1,1}(\Omega)),
\end{align*}
\]
Definition (weak solutions (Continued))

\[ \begin{aligned}
\rho_t + \text{div}(\sqrt{\rho} \sqrt{\rho u}) &= 0, \\
\rho(x, t = 0) &= \rho_0(x),
\end{aligned} \]

in \( D' \),

and if the following equality holds for all smooth test function \( \phi(x, t) \) with compact support such that \( \phi(x, T) = 0 \):

\[ \begin{aligned}
\int_\Omega m_0 \cdot \phi(x, 0) \, dx + \int_0^\infty (\sqrt{\rho}(\sqrt{\rho u})\phi_t + \sqrt{\rho} u \otimes \sqrt{\rho} u : \nabla \phi) \, dx dt \\
+ \int_0^\infty \rho^\gamma \text{div} \phi \, dx dt - \frac{1}{2} \langle h(\rho) \nabla u, \nabla \phi \rangle - \frac{1}{2} \langle h(\rho) (\nabla u)^{tr}, \nabla \phi \rangle \\
- \langle g(\rho) \text{div} u, \text{div} \phi \rangle &= 0,
\end{aligned} \]
Introduction

Definition (weak solutions (Continued))

*where*

\[
\langle h(\rho) \nabla u, \nabla \phi \rangle = - \int_{0}^{\infty} \rho^{\alpha - 1/2} \sqrt{\rho u} \cdot \Delta \phi \, dx \, dt \\
- \frac{2\alpha}{2\alpha - 1} \int_{0}^{\infty} \sqrt{\rho u} \frac{\partial}{\partial \rho} \rho^{\alpha - 1/2} \partial \phi \, dx \, dt,
\]

\[
\langle h(\rho)(\nabla u)^{tr}, \nabla \phi \rangle = - \int_{0}^{\infty} \rho^{\alpha - 1/2} \sqrt{\rho u} \cdot \nabla \text{div} \phi \, dx \, dt \\
- \frac{2\alpha}{2\alpha - 1} \int_{0}^{\infty} \sqrt{\rho u} \frac{\partial}{\partial \rho} \rho^{\alpha - 1/2} \partial \phi \, dx \, dt,
\]

\[
\langle g(\rho) \text{div} u, \text{div} \phi \rangle = - (\alpha - 1) \int_{0}^{\infty} \rho^{\alpha - 1/2} \sqrt{\rho u} \cdot \nabla \text{div} \phi \, dx \, dt \\
- \frac{2\alpha(\alpha - 1)}{2\alpha - 1} \int_{0}^{\infty} \sqrt{\rho u} \cdot \nabla \rho^{\alpha - 1/2} \text{div} \phi \, dx \, dt.
\]
Main Theorems

Condition (Conditions on the initial data)

For some $\eta_0 > 0$,

\[
\begin{cases}
0 \leq \rho_0 \in L^1(\Omega) \cap L^\gamma(\Omega), \ \nabla \rho_0^{\alpha - 1/2} \in L^2(\Omega), \\
m_0 \in L^{2\gamma/(\gamma+1)}(\Omega), \ \rho_0^{-1-\eta_0} |m_0|^{2+\eta_0} \in L^1(\Omega).
\end{cases}
\] (2.1)
Theorem 1

Let $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$. Suppose that $\alpha$ and $\gamma$ satisfy

$$\alpha > 1/2, \quad \gamma > 1, \quad \gamma \geq 2\alpha - 1. \quad (2.1)$$

Moreover, assume that the initial data $(\rho_0, m_0)$ satisfies (1.23). Then there exists a global weak solution $(\rho, u)$ to the problem (1.1) (1.3) (1.21) (1.22).

Remark

Similar result holds for the problem (1.1) (1.2) (1.21) (1.22).
As for the three-dimensional case, it holds that

**Theorem 2**

Let $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. Suppose that $\alpha \in [3/4, 2)$ and $\gamma \in (1, 3)$ satisfy

$$\gamma \in \begin{cases} 
(1, 6\alpha - 3), & \text{for } \alpha \in [3/4, 1], \\
[2\alpha - 1, 3\alpha - 1], & \text{for } \alpha \in (1, 2). 
\end{cases} \quad (2.3)$$

Assume that the initial data $(\rho_0, m_0)$ satisfies (2.1). Then there exists a global weak solution $(\rho, u)$ to the problem (1.1) (1.2) (1.6) (1.7).

**Remark**

In particular, $\gamma \in (1, 3)$ for $\alpha = 1$. 
Let $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. Suppose that $\alpha = 1$ and $\gamma \in (1, 3)$. Assume that the initial data $(\rho_0, m_0)$ satisfies (1.23). Then there exists a global weak solution $(\rho, u)$ to the problem (1.1) (1.3) (1.21) (1.22).
Remark

If $\alpha = 1$ and $\gamma = 2$, our results give a positive answer to the open problem proposed by Lions (1998).
Recently, for a particular case: $\alpha = 1$ and $\Omega = \mathbb{T}^3$, Vasseur-Yu [Invent. math. (2016) 206:935–974] tried to give another proof of our result (Theorem 3), that is, they used the weak solutions of

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \rho^\gamma - \text{div}(\rho D(u)) &= -r_1 u - r_2 \rho |u|^2 u - \kappa \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right),
\end{align*}
\]

whose existence is shown in Vasseur-Yu [SIAM J. MATH. ANAL. 48 (2016) 1489–1511].
However, as indicated by [Lacroix-Violet & Vasseur JMPA (2018)], those a priori estimates are not sufficient to define $\nabla u$ as a function. In fact, in the proof of the key Lemma 4.2 of [Vasseur-Yu: Invent. math. (2016) 206:935-974], it seems to us that their need the assumption that $\nabla u$ is a function essentially.
Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations

Alexis F. Vasseur · Cheng Yu

Received: 2 April 2015 / Accepted: 24 April 2016 / Published online: 9 May 2016
© Springer-Verlag Berlin Heidelberg 2016

Abstract In this paper, we prove the existence of global weak solutions for 3D compressible Navier–Stokes equations with degenerate viscosity. The method is based on the Bresch and Desjardins (Commun Math Phys 238:211–223 2003) entropy conservation. The main contribution of this paper is to derive the Mellet and Vasseur (Commun Partial Differ Equ 32:431–452, 2007) type inequality for weak solutions, even if it is not verified by the first level of approximation. This provides existence of global solutions in time, for the compressible barotropic Navier–Stokes equations. The result holds for any $\gamma > 1$ in two dimensional space, and for $1 < \gamma < 3$ in three dimensional space, in both case with large initial data possibly vanishing on the vacuum. This solves an open problem proposed by Lions (Mathematical topics in fluid mechanics. Vol. 2. Compressible models, 1998).

Mathematics Subject Classification 35Q35 · 76N10

1 Introduction

The existence of global weak solutions of compressible Navier–Stokes equations with degenerate viscosity has been a long standing open problem. The objective of this current paper is to establish the existence of global weak solutions to the following 3D compressible Navier–Stokes equations:
Hence, \( \rho_\kappa^\alpha g(|v_\kappa|^2) \) converges to \( \rho^\alpha g(|u|^2) \) almost everywhere. Since \( g \) is bounded and (4.1), \( \rho_\kappa^\alpha g(|v_\kappa|^2) \) is uniformly bounded in \( L^r((0, T) \times \Omega) \) for some \( r > 1 \). Hence,

\[
\rho_\kappa^\alpha g(|v_\kappa|^2) \to \rho^\alpha g(|u|^2) \quad \text{in} \quad L^1((0, T) \times \Omega).
\]

By the uniqueness of the limit, the convergence holds for the whole sequence.

Applying this result with \( \alpha = 1 \) and \( g(|v_\kappa|^2) = \varphi_n(v_\kappa) \), we deduce (4.2). Since \( \gamma > 1 \) in 2D, we can take \( \alpha = 2\gamma - 1 < 2\gamma \); and take \( \gamma < 3 \) in 3D, we have \( 2\gamma - 1 < \frac{5\gamma}{3} \). Thus we use the above result with \( \alpha = 2\gamma - 1 \) and \( g(|v_\kappa|^2) = 1 + \tilde{\varphi}_n'(|v_\kappa|^2) \) to obtain (4.3).

With the lemma in hand, we are ready to recover the limits in (3.2) as \( \kappa \to 0 \) and \( K \to \infty \). We have the following lemma.

**Lemma 4.2** Let \( K = \kappa^{-\frac{3}{4}} \), and \( \kappa \to 0 \), for any \( \psi \geq 0 \) and \( \psi' \leq 0 \), we have

\[
- \int_0^T \int_\Omega \psi'(t) \rho_\kappa \varphi_n(u) \, dx \, dt \\
\leq 8 \| \psi \|_{L^\infty} \left( \int_\Omega \left( \rho_0|u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} + |\nabla \sqrt{\rho_0}|^2 - r_0 \log \rho_0 \right) \, dx + 2E_0 \right) \\
+ C(\| \psi \|_{L^\infty}) \int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|u|^2)) \rho_0^2 \gamma - 1 \, dx \, dt + \psi(0) \int_\Omega \rho_0 \varphi_n(u_0) \, dx
\]

(4.5)

**Proof** Here, we use \( (\rho_\kappa, u_\kappa) \) to denote the weak solutions to (2.1) verifying Proposition 1.1 with \( \kappa > 0 \).

By Lemma 4.1, we can handle the first term in (3.2), that is,

\[
\int_0^T \int_\Omega \psi'(t)(\rho_\kappa \varphi_n(v_\kappa)) \, dx \, dt \to \int_0^T \int_\Omega \psi'(t)(\rho \varphi_n(u)) \, dx \, dt
\]

(4.6)

and

\[
\psi(0) \int_\Omega \rho_0 \varphi'_n(v_{\kappa,0}) \, dx \to \psi(0) \int_\Omega \rho_0 \varphi'_n(u_0) \, dx
\]

(4.7)

as \( \kappa \to 0 \) and \( K = \kappa^{-\frac{3}{4}} \to \infty \).
as $\kappa \to 0$;

\[
\kappa \int_0^T \int_\Omega |\psi(t) \phi_k(\rho) \phi_k'(\rho)| \nabla \phi_k(\rho) \Delta \sqrt{\rho} \, dx \, dt \\
\leq 2C(n, \psi) \|\phi_k(\rho)\|^2 \|\Delta \sqrt{\rho}\|_{L^1(0,T;L^1(\Omega))} \\
\leq 2C(n, \psi) \kappa \xrightarrow{\kappa \to 0} 0
\]

(4.19)

as $\kappa \to 0$, where we used $|\rho \phi_k'(\rho)| \leq 1$. Finally

\[
\kappa \int_0^T \int_\Omega |\psi(t) \phi_k(\rho) \phi_k'(\rho)| \nabla \phi_k(\rho) \Delta \sqrt{\rho} \, dx \, dt \\
\leq 2C(n, \psi) \|\phi_k(\rho)\|^2 \|\Delta \sqrt{\rho}\|_{L^1(0,T;L^1(\Omega))} \\
\leq 2C(n, \psi) \kappa \xrightarrow{\kappa \to 0} 0
\]

(4.20)

For the term $S_2 = \phi_k(\rho_0) \Delta \sqrt{\rho_0} \in S_1 + S_2$, we calculate as follows

\[
\int_0^T \int_\Omega |\psi(t) \phi_k(\rho_0) \phi_k'(\rho_0)| \nabla \phi_k(\rho_0) \Delta \sqrt{\rho_0} \, dx \, dt \\
= \int_0^T \int_\Omega \psi(t) \phi_k(\rho_0) \phi_k'(\rho_0) \nabla \phi_k(\rho_0) \, dx \, dt \\
= \int_0^T \int_\Omega \psi(t) \nabla \phi_k(\rho_0) \phi_k'(\rho_0) \, dx \, dt
\]

(4.21)

For $A_1$, by part a. of Lemma 2.2, we have

\[
A_1 = \int_0^T \int_\Omega |\psi(t) \nabla \phi_k(\rho_0) \phi_k'(\rho_0)| \nabla \phi_k(\rho_0) \, dx \, dt \\
= 2 \int_0^T \int_\Omega |\psi(t) \phi_k(\rho_0) \phi_k'(\rho_0)| \rho_0 \nabla \phi_k(\rho_0) \, dx \, dt \\
+ 4 \int_0^T \int_\Omega |\psi(t) \phi_k(\rho_0) \phi_k'(\rho_0)| \nabla \phi_k(\rho_0) \, dx \, dt
\]

(4.22)

Notice that

\[
D u_k : \nabla u_k = |D u_k|^2.
\]

thus

\[
A_1 \geq 2 \int_0^T \int_\Omega |\psi(t) \phi_k(\rho_0) \phi_k'(\rho_0)| \rho_0 |D u_k|^2 \, dx \, dt \\
- 4|\psi| \|\nabla \phi_k(\rho_0)\|^2 \int_0^T \int_\Omega \rho_0 |D u_k|^2 \, dx \, dt,
\]

(4.23)

where we control $A_{12}$

\[
A_{12} \leq 4 \int_0^T \int_\Omega |\psi(t)| \left|\frac{|\phi_k(\rho_0)|^2}{1 + |\phi_k(\rho_0)|^2}\right| \rho_0 |D u_k|^2 \, dx \, dt
\]

\[
\leq 4|\psi| \|\nabla \phi_k(\rho_0)\|^2 \left(\int_\Omega \rho_0 |D u_k|^2 \, dx \right)
\]

\[
\leq 4|\psi| \|\nabla \phi_k(\rho_0)\|^2 \left(\frac{|\rho_0(0)|^2}{\gamma - 1} + |\sqrt{\rho_0}|^2 - \epsilon_0 \log \rho_0 \right) \, dx \, dt
\]

thanks to (1.16). For $A_2$, thanks to (2.8), we can control it as follows

\[
|A_2| \leq C(n, \psi) \|\sqrt{\rho_0} \nabla u_k\|_{L^1(0,T;L^1(\Omega))} \|\phi_k(\rho_0)\|_{L^1(0,T;L^1(\Omega))}
\]

\[
\times (\kappa \|\nabla \phi_k(\rho_0)\|^2 \|\phi_k(\rho_0)\|_{L^1(0,T;L^1(\Omega))})
\]

\[
\leq C \frac{\kappa^4}{\sqrt{\kappa}} = C \kappa^4 \xrightarrow{\kappa \to 0} 0
\]

(4.24)
GLOBAL WEAK SOLUTIONS TO THE COMPRESSIBLE QUANTUM NAVIER-STOKES EQUATION AND ITS SEMI-CLASSICAL LIMIT

INGRID LACROIX-VIOLET AND ALEXIS F. VASSEUR

Abstract. This paper is dedicated to the construction of global weak solutions to the quantum Navier-Stokes equation, for any initial value with bounded energy and entropy. The construction is uniform with respect to the Planck constant. This allows to perform the semi-classical limit to the associated compressible Navier-Stokes equation. One of the difficulty of the problem is to deal with the degenerate viscosity, together with the lack of integrability on the velocity. Our method is based on the construction of weak solutions that are renormalized in the velocity variable. The existence, and stability of these solutions do not need the Mellet-Vasseur inequality.

1. Introduction

Quantum models can be used to describe superfluids [12], quantum semiconductors [6], weakly interacting Bose gases [8] and quantum trajectories of Bohmian mechanics [16]. They have attracted considerable attention in the last decades due, for example, to the development of nanotechnology applications.

In this paper, we consider the barotropic compressible quantum Navier-Stokes equations, which has been derived in [5], under some assumptions, using a Chapman-Enskog expansion in Wigner equation. In particular, we are interested in the existence of global weak solutions together with the associated semi-classical limit. The quantum Navier-Stokes equation that we are considering read as:

$$\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(pu)_t + \text{div}(\rho u \otimes u) + \nabla \rho \gamma - 2\text{div}(\sqrt{\nu \rho} S_{\nu} + \sqrt{\kappa \rho} S_{\kappa}) &= \sqrt{\rho} f + \sqrt{\kappa} \text{div}(\sqrt{\rho} M),
\end{align*}$$

(1.1)

where

$$\begin{align*}
\sqrt{\nu \rho} S_{\nu} &= \rho D u, \\
\text{div}(\sqrt{\kappa \rho} S_{\kappa}) &= \kappa \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right),
\end{align*}$$

(1.2)

and with initial data

$$\begin{align*}
\rho(0, x) &= \rho_0(x), \\
(\rho u)(0, x) &= (\rho_0 u_0)(x) \quad \text{in} \ \Omega,
\end{align*}$$

(1.3)

where $\rho$ is the density, $\gamma > 1$, $u \otimes u$ is the matrix with components $u_i u_j$, $D u = \frac{1}{2} (\nabla u + \nabla u^T)$ is the symmetric part of the velocity gradient, and $\Omega = \mathbb{T}^d$ is the $d-$dimensional torus, here $d = 2$ or 3. The vector valued function $f$, and the matrix valued function $M$ are source terms.

Date: July 25, 2016.

2010 Mathematics Subject Classification. 35Q35, 76N10.

Key words and phrases. Global weak solutions, compressible Quantum Navier-Stokes Equations, vacuum, degenerate viscosity.

Acknowledgment. A. F. Vasseur was partially supported by the NSF Grant DMS 1209420.
Note that those a priori estimates are not sufficient to define the dominated convergence Theorem. It is obtained by considering a sequence of bounded functions \( \varphi_n \), uniformly dominated by \( y \to |y| \), and converging almost everywhere to \( y \to y_i \), for a fixed direction \( i \). This provides the momentum equation for \( pu_i \) at the limit \( n \to \infty \). The key point is that, while performing this limit, the functions \( \rho, u \) are fixed. Considering, for example, the term

\[ \rho u \varphi_n(u). \]

The challenge is then to show that the renormalized solutions, are indeed, weak solutions in the general sense (see Definition 2.1). It is obtained by considering a sequence of \( \rho_u \varphi \) bounded functions \( \rho_u \varphi \) in \( L^1 \) for \( \rho_u \varphi \). Note that the boundedness of \( \rho_u \varphi \) in \( L^1 \) is enough for this procedure. Choosing the sequence of \( \varphi_n \) such that \( \| \varphi_n \|_{L^\infty} \) converges to 0, we show that the extra terms \( R_{\varphi_n} \) and \( R_{\varphi} \) converge to 0 when \( n \) converges to \( \infty \).

The main difference with [14] is that we do not need to reconstruct the energy inequality nor the control on (1.11) via the sequence of functions \( \varphi_n \). Hence, we do not need an explicit form of the terms involving second derivatives of \( \varphi \) in the definition of renormalized solutions. Those terms (for which we do not have stability) are dumped in the extra terms \( R_\varphi \) and \( R_{\varphi} \).

2. Preliminary results and main ideas

We are first working on the System (1.9) with drag forces. The definitions will be valid for all the range of parameter, \( r_0 \geq 0, r_1 \geq 0, \kappa \geq 0, \nu > 0 \). The energy and the BD entropy on solutions to (1.9) provide controls on

\[
E_\kappa(\sqrt{\rho}, \sqrt{\rho} u) = \int_\Omega \left( \rho \frac{|u|^2}{2} + (2\kappa + 4\nu^2)|\nabla \sqrt{\rho}|^2 + \rho^\gamma + r_0(\rho - \ln \rho) \right) dx,
\]

\[
D_\kappa(\sqrt{\rho}, \sqrt{\rho} u) = \int_\Omega \left( \nu |\nabla \rho^{\gamma/2}|^2 + \nu \kappa \left( |\nabla \rho^{1/4}|^4 + |\nabla^2 \sqrt{\rho}|^2 \right) + |\nu_\nu|^2 + r_0|u|^2 + r_1 \rho |u|^4 \right) dx.
\]

From these quantities, we can obtain the following a priori estimates. For the sake of completeness we show how to obtain them in the appendix.

\[
\sqrt{\rho} \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \quad \nabla \sqrt{\rho} \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \quad \nabla \rho^{\gamma/2} \in L^2(\mathbb{R}^+; L^2(\Omega))
\]

\[
\sqrt{\rho} u \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \quad \nu_\nu \in L^2(\mathbb{R}^+; L^2(\Omega)), \quad \sqrt{\kappa} \nabla^2 \sqrt{\rho} \in L^2(\mathbb{R}^+; L^2(\Omega)),
\]

\[
\kappa^{1/4} \nabla \rho^{1/4} \in L^4(\mathbb{R}^+; L^4(\Omega)), \quad r_1^{1/4} \rho^{1/4} u \in L^4(\mathbb{R}^+; L^4(\Omega)), \quad r_0^{1/2} u \in L^2(\mathbb{R}^+; L^2(\Omega)), \quad r_0 \ln \rho \in L^\infty(\mathbb{R}^+; L^1(\Omega)).
\]

Note that those a priori estimates are not sufficient to define \( \nabla u \) as a function. The statement that \( \sqrt{\rho} \nabla u \) is bounded in \( L^2 \) means that there exists a function \( \nu_\nu \in L^2(\mathbb{R}^+; L^2(\Omega)) \) such that:

\[
\sqrt{\rho} \sqrt{\rho} \nu_\nu = \text{div}(\rho u) - \sqrt{\rho} u \cdot \nabla \sqrt{\rho},
\]
Global existence of weak solutions 
to the compressible quantum Navier-Stokes 
equations with degenerate viscosity

Remark 1.1 It should be noted that the arguments in the work of Vasseur-Yu\cite{Vasseur2016} rely crucially on the assumption that the gradient of velocity field $\nabla u$ is a well-defined function, which indeed does not make sense in the presence of vacuum. In particular, in the proof of Ref. 35, Lemma 4.2, which is crucial to deduce the key Mellet-Vasseur type estimate in Ref. 35, it requires essentially that $\nabla u$ is a well-defined function.

Very recently, Lacrioe-Violet and Vasseur\cite{Lacroix-Violet2017} also studied the QNS equations and considered a new function $\mathcal{T}_v \in L^1(\Omega \times (0, T))$ satisfying

$$\sqrt{\rho} \mathcal{T}_v = \sqrt{\rho u} \nabla \sqrt{\rho}.$$ \hfill (1.16)

More precisely, they\cite{Lacroix-Violet2017} used the function $\mathcal{T}_v$, to give a new understanding of $\sqrt{\rho} \nabla u$. However, as mentioned in Ref. 35, it still does not allow one to define the gradient of velocity $\nabla u$ as a function.

\textsuperscript{35} J. Lacrioe-Violet and A. Vasseur, "Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit," J. Math. Pures Appl. 116, 191–210 (2018).

\textsuperscript{36} A. Vasseur and C. Yu, "Global weak solutions to the compressible quantum Navier-Stokes equations with damping," SIAM J. Math. Anal. 48, 1499–1511 (2016).

\textsuperscript{37} A. Vasseur and C. Yu, "Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations," Invent. Math. 206, 935–974 (2016).
Remark

For 3D case, our Theorems 2 & 3 are valid for $\alpha = 1$ and all $\gamma \in (1, 3)$, which are in sharp contrast to the case that $h$ and $g$ are both constants, where the condition $\gamma > 3/2$ is essential in the analysis of Lions (1998) and Feireisl et al (2001). In fact, for $h$ and $g$ being both constants and $\gamma \in (1, 3/2]$, it remains completely open to obtain the global existence of weak solutions to (1.1) (1.2) for general initial data except for the spherically symmetric case (Jiang-Zhang (2001)).
**GOAL:** To construct an approximate system which has smooth solutions satisfying

1) Standard and Log-type energy estimates
2) BD entropy

- Observation: Standard and Log-type energy estimates hold for parabolized system.
- Key issue is to derive BD entropy independent of perturbation parameters
- Main technical difficulty is to obtain the positive lower bound and upper one for the density (which may depend on the parameters)
Our constructions: Consider the following approximate system

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= G, \\
\rho u_t + \rho u \cdot \nabla u - \text{div}(h_\varepsilon(\rho) \mathcal{D} u) - \nabla(g_\varepsilon(\rho) \text{div} u) + \nabla P &= H,
\end{align*}
\]  

(3.1)

where

\[
\begin{align*}
h_\varepsilon(\rho) &= \rho^\alpha + \varepsilon^{1/3}(\rho^{7/8} + \rho^{\tilde{\gamma}}), \\
g_\varepsilon(\rho) &= \rho h'_\varepsilon(\rho) - h_\varepsilon(\rho), \\
0 < \varepsilon &\leq \varepsilon_0 \overset{\Delta}{=} (2\alpha - 1)(16(\alpha + \gamma))^{-10}, \\
\tilde{\gamma} &\overset{\Delta}{=} \gamma + 1/6.
\end{align*}
\]
Sketch of Proof

- $G \triangleq \varepsilon \rho^{1/2} \text{div}(\rho^{-1/2} h'_\varepsilon(\rho) \nabla \rho)$ parabolization
  
  KEY for BD entropy and lower and upper bounds of the density

- $H \triangleq H_1 + H_2$
  
  $H_1 \triangleq \sqrt{\varepsilon}[\text{div}(h_\varepsilon(\rho) \nabla u) + \nabla(g_\varepsilon(\rho) \text{div} u)]$
  
  $\Rightarrow \int \rho |u|^{2+\varepsilon} \, dx$

  $H_2 \triangleq -e^{-\varepsilon^3} (\rho^{\varepsilon-2} + \rho^{-\varepsilon-2})u$
  
  $\Rightarrow \int (\rho^p + \rho^{-p}) \, dx$, $\forall p > 1$.

- $\int \rho |u|^{2+\varepsilon} \, dx + \int (\rho^p + \rho^{-p}) \, dx + \text{De Giorgi type method}$
  
  $\Rightarrow C^{-1}(\varepsilon) \leq \rho \leq C(\varepsilon)$. 
Step 2: BD entropy inequality

Lemma

\[ \exists C \text{ independent of } \varepsilon \text{ and } T \text{ s.t.} \]

\[ \sup_{0 \leq t \leq T} \int \left( \rho^{-1}(h'_{\varepsilon}(\rho))^2 |\nabla \rho|^2 + \varepsilon^{13/3} e^{-\varepsilon^{-3}} (\rho\varepsilon^{-2} + \bar{\gamma} - 1 + \rho^{-\varepsilon^{-2} - 1/8}) \right) dx \]

\[ + \int_0^T \int h_{\varepsilon}(\rho) |\nabla u|^2 dx dt + \int_0^T \int \rho^{\bar{\gamma}^{-3}} h_{\varepsilon}(\rho) |\nabla |^2 dx dt \leq C. \]

(4.2)
Sketch of Proof (2D Case)

Proof. Set

\[ G \triangleq \varepsilon \rho^{1/2} \text{div} (\rho^{-1/2} h_\varepsilon' (\rho) \nabla \rho), \quad \varphi_\varepsilon' (\rho) \triangleq \rho^{-1} h_\varepsilon' (\rho) \geq 0. \]

Direct computations give

\[
\frac{1}{2} \left( \int \rho |\nabla \varphi_\varepsilon (\rho)|^2 dx \right) + \int \nabla h_\varepsilon (\rho) \cdot \nabla u \cdot \nabla \varphi_\varepsilon (\rho) dx \\
+ \int \nabla h_\varepsilon (\rho) \cdot \nabla (\rho \varphi_\varepsilon' (\rho) \text{div} u) dx \\
+ \int \varphi_\varepsilon' (\rho) G \left( \Delta h_\varepsilon (\rho) - \frac{1}{2} \varphi_\varepsilon' (\rho) |\nabla \rho|^2 \right) dx = 0.
\] (4.3)
Multiplying (3.1) by $\nabla \varphi_\varepsilon(\rho)$ leads to

$$\frac{1}{1 + \sqrt{\varepsilon}} \left( \int u \cdot \nabla h_\varepsilon(\rho) \, dx \right)_t - \int h_\varepsilon(\rho) \nabla \text{div} u \cdot \nabla \varphi_\varepsilon(\rho) \, dx$$

$$- \int \nabla h_\varepsilon(\rho) \cdot \nabla u \cdot \nabla \varphi_\varepsilon(\rho) \, dx + \int g_\varepsilon(\rho) \text{div} u \Delta \varphi_\varepsilon(\rho) \, dx$$

$$+ \frac{1}{2(1 + \sqrt{\varepsilon})} \int h_\varepsilon(\rho) |\nabla u|^2 \, dx + \frac{1}{1 + \sqrt{\varepsilon}} \int P'(\rho) \varphi_\varepsilon'(\rho) |\nabla \rho|^2 \, dx$$

$$+ \frac{e^{-\varepsilon^{-3}}}{1 + \sqrt{\varepsilon}} \int (\rho^{-2} + \rho^{-\varepsilon^{-2}}) u \cdot \nabla \varphi_\varepsilon(\rho) \, dx$$

$$+ \frac{1}{1 + \sqrt{\varepsilon}} \int \text{div} uh_\varepsilon'(\rho) G \, dx$$

$$\leq C \int h_\varepsilon(\rho) |D u|^2 \, dx,$$

(4.4)
(4.3) + (4.4)

\[ \varphi'_\varepsilon(\rho) = \rho^{-1} h'_\varepsilon(\rho), \quad g_\varepsilon(\rho) = \rho h'_\varepsilon(\rho) - h_\varepsilon(\rho) \Rightarrow \]

\[ \int \nabla h_\varepsilon(\rho) \cdot \nabla (\rho \varphi'_\varepsilon(\rho) \text{div} u) dx - \int h_\varepsilon(\rho) \nabla \text{div} u \cdot \nabla \varphi_\varepsilon(\rho) dx \]

\[ + \int g_\varepsilon(\rho) \text{div} u \Delta \varphi_\varepsilon(\rho) dx \]

\[ = \int (\nabla h_\varepsilon(\rho) \cdot \nabla (\rho \varphi'_\varepsilon(\rho)) - \nabla g_\varepsilon(\rho) \cdot \nabla \varphi_\varepsilon(\rho)) \text{div} u dx \]

\[ + \int (\rho \varphi'_\varepsilon(\rho) \nabla h_\varepsilon(\rho) - h_\varepsilon(\rho) \nabla \varphi_\varepsilon(\rho) - g_\varepsilon(\rho) \nabla \varphi_\varepsilon(\rho)) \cdot \nabla \text{div} u dx = 0 \]

THEN (4.3) + (4.4) \Rightarrow
\[
\frac{1}{2} \left( \int \rho |\nabla \varphi_\varepsilon(\rho)|^2 \, dx \right)_t + \frac{1}{1 + \sqrt{\varepsilon}} \left( \int \rho u \cdot \nabla \varphi_\varepsilon(\rho) \, dx \right)_t \\
+ \frac{1}{2(1 + \sqrt{\varepsilon})} \int h_\varepsilon(\rho) |\nabla u|^2 \, dx + \frac{1}{1 + \sqrt{\varepsilon}} \int P'(\rho) \varphi'_\varepsilon(\rho) |\nabla \rho|^2 \, dx \\
+ \frac{e^{-\varepsilon^{-3}}}{1 + \sqrt{\varepsilon}} \int (\rho^{\varepsilon^{-2}} + \rho^{-\varepsilon^{-2}}) u \cdot \nabla \varphi_\varepsilon(\rho) \, dx \\
+ \int \varphi'_\varepsilon(\rho) G \left( \Delta h_\varepsilon(\rho) - \frac{1}{2} \varphi'_\varepsilon(\rho) |\nabla \rho|^2 + \frac{1}{1 + \sqrt{\varepsilon}} \rho \text{div} u \right) \, dx \\
\leq C \int h_\varepsilon(\rho) |\mathcal{D} u|^2 \, dx,
\]
(4.5)
By the definition of $G$

$$G \triangleq \varepsilon \rho^{1/2} \text{div}(\rho^{-1/2} h'_\varepsilon(\rho) \nabla \rho) = \varepsilon (\Delta h'_\varepsilon(\rho) - \frac{1}{2} \varphi'_\varepsilon(\rho)|\nabla \rho|^2).$$

We have the following **KEY** observation (Since $\varphi'_\varepsilon(\rho) \geq 0$):

$$\Rightarrow \int \varphi'_\varepsilon(\rho) G \left( \Delta h'_\varepsilon(\rho) - \frac{1}{2} \varphi'_\varepsilon(\rho)|\nabla \rho|^2 + \frac{1}{1 + \sqrt{\varepsilon}} \rho \text{div} u \right) dx$$

$$\geq \frac{1}{2\varepsilon} \int \varphi'_\varepsilon(\rho) G^2 dx - \frac{\varepsilon}{2} \int \rho^2 \varphi'_\varepsilon(\rho) (\text{div} u)^2 dx$$

$$\geq \frac{1}{2\varepsilon} \int \varphi'_\varepsilon(\rho) G^2 dx - C\varepsilon \int h'_\varepsilon(\rho)|Du|^2 dx. \quad (4.6)$$
Step 3: Log-type energy estimate

Lemma

Assume that $\gamma > 1$ satisfies $\gamma \geq (1 + \alpha)/2$ in addition. Then there exists some generic constant $C$ but independent of $\varepsilon$ such that

$$\sup_{0 \leq t \leq T} \int \rho (e + |u|^2) \ln (e + |u|^2) \, dx \leq C. \quad (4.7)$$
Step 4: Lower and upper bounds of the density

**Lemma**

There exists some positive constant $C$ depending on $\varepsilon$ and $T$ such that for all $(x, t) \in \Omega \times (0, T)$

$$C^{-1} \leq \rho(x, t) \leq C. \quad (4.8)$$
Proof. 1) It is easy to show that

\[
\sup_{0 \leq t \leq T} \int \rho |u|^{2+\epsilon} \, dx + \sqrt{\epsilon} \int_0^T \int h_\epsilon(\rho) |u|^{\epsilon} |\nabla u|^2 \, dx \, dt \leq C. \quad (4.9)
\]
Sketch of Proof (2D Case)

2) $v \triangleq \sqrt{\rho}$ satisfies

$$2v_t - 2\varepsilon \text{div}(h'_\varepsilon(v^2)\nabla v) + \text{div}(uv) + u \cdot \nabla v = 0.$$  \hspace{1cm} (4.10)

For $k \geq \|v(\cdot, 0)\|_{L^\infty(\Omega)} = \|\rho_0\|_{L^\infty(\Omega)}^{1/2},$ (4.10) \times (v - k)_+ \Rightarrow

$$
\frac{d}{dt} \int (v - k)^2_+ dx + 2\alpha\varepsilon \int v^{2\alpha-2}|\nabla (v - k)_+|^2 dx \\
\leq C \int_{A_k(t)} v^{4-2\alpha}|u|^2 dx + \alpha\varepsilon \int v^{2\alpha-2}|\nabla (v - k)_+|^2 dx,
$$  \hspace{1cm} (4.11)

where $A_k(t) \triangleq \{x \in \Omega | v(x, t) > k\}.$
\[
\int_{A_k(t)} v^{4-2\alpha}|u|^2 \, dx \\
\leq C \left( \int_{A_k(t)} v^2 |u|^{2+\varepsilon} \, dx \right)^{2/(2+\varepsilon)} \left( \int_{A_k(t)} v^{(4+4\varepsilon-2(2+\varepsilon)\alpha)/\varepsilon} \, dx \right)^{\varepsilon/(2+\varepsilon)} \\
\leq C \left( \int_{A_k(t)} (\rho^{4(\alpha+1)\varepsilon^{-1}} + \rho^{-4(\alpha+1)\varepsilon^{-1}}) \, dx \right)^{\varepsilon/(2+\varepsilon)} \\
\leq C \left( \int_{A_k(t)} (\rho^{\varepsilon^{-2}} + \rho^{-\varepsilon^{-2}}) \, dx \right)^{\varepsilon(4-\varepsilon)/(6(2+\varepsilon))} |A_k(t)|^{\varepsilon/6} \\
\leq C |A_k(t)|^{\varepsilon/6},
\]
(4.12)+(4.11) \Rightarrow

I_k'(t) + \alpha \varepsilon \int \rho^{\alpha-1}|\nabla(v - k)_+|^2 dx \leq C\nu_k^{\varepsilon/6}, \quad (4.13)

where

I_k(t) \triangleq \int (v - k)_+^2(x, t)dx, \quad \nu_k \triangleq \sup_{0 \leq t \leq T} |A_k(t)|.
Let

\[ I_k(\sigma) = \sup_{0 \leq t \leq T} I_k(t). \]

(4.13) \( \Rightarrow \) \[ I_k(\sigma) + \int \rho^{\alpha^{-1}} |\nabla (v - k)_+|^2 (x, \sigma) dx \leq C \nu_k^{\varepsilon/6}, \]

\( \Rightarrow \) \[ I_k(\sigma) + \|\nabla (v - k)_+ (\cdot, \sigma)\|_{L^{24/(12+\varepsilon)}(\Omega)}^2 \]
\[ \leq I_k(\sigma) + \int \rho^{\alpha^{-1}} |\nabla (v - k)_+|^2 (x, \sigma) dx \left( \int \rho^{12(1-\alpha)/\varepsilon} (x, \sigma) dx \right)^{\varepsilon/12} \]
\[ \leq C \nu_k^{\varepsilon/6}. \]
Then, for any $h > k \geq \|v(\cdot, 0)\|_{L^\infty(\Omega)}$, 

\[
|A_h(t)|(h - k)^2 \\
\leq \|(v - k)_+ (\cdot, t)\|^2_{L^2(\Omega)} \\
\leq \|(v - k)_+ (\cdot, \sigma)\|^2_{L^2(\Omega)} \\
\leq C\|(v - k)_+ (\cdot, \sigma)\|^2_{L^{24/\varepsilon}(\Omega)} |A_k(\sigma)|^{1-\varepsilon/12} \\
\leq C \left( \|(v - k)_+ (\cdot, \sigma)\|^2_{L^2(\Omega)} + \|\nabla (v - k)_+ (\cdot, \sigma)\|^2_{L^{24/(12+\varepsilon)}(\Omega)} \right) \nu_k^{1-\varepsilon/12} \\
\leq C \nu_k^{1+\varepsilon/12},
\]
⇒ \nu_h \leq C(h - k)^{-2}\nu_k^{1+\varepsilon/12},

De Giorgi-type lemma ⇒ \sup_{0 \leq t \leq T} \|\rho\|_{L^\infty(\Omega)} \leq \tilde{C}. \quad (4.14)

3) Applying similar arguments to the equation of \rho^{-1} shows

\sup_{(x,t) \in \Omega \times (0,T)} \rho^{-1}(x, t) \leq C,

for some positive constant \( C \geq \tilde{C}. \)
Step 5: Higher order estimates

**Lemma**

For any $p > 2$, there exists some constant $C$ depending on $\varepsilon, p,$ and $T$ such that

$$
\int_0^T \left( \| (\rho, u)_t \|_{L^p(\Omega)}^p + \| (\rho, u) \|_{W^{2,p}(\Omega)}^p \right) dt \leq C. \quad (4.15)
$$
For constants $p_0$ and $\varepsilon$ satisfying

$$p_0 = 50, \quad 0 < \varepsilon \leq \varepsilon_1 \triangleq \min\{10^{-10}, \eta_0\},$$

with $\eta_0$ as in (1.23), we consider the following approximate system

$$\begin{aligned}
\rho_t + \text{div}(\rho u) &= \varepsilon v \Delta v + \varepsilon v \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0}, \\
\rho u_t + \rho u \cdot \nabla u - \text{div}(\rho D u) + \nabla P &= \sqrt{\varepsilon} \text{div}(\rho \nabla u) + \varepsilon |\nabla v|^2 \nabla v \cdot \nabla u - \varepsilon \rho^{-p_0} u - \varepsilon \rho |u|^3 u,
\end{aligned}$$

(5.1)

where $v \triangleq \rho^{1/2}$. 
Sketch of Proof (3D Case)

Lemma

There exists some generic constant $C$ independent of $\varepsilon$ such that

$$
\sup_{0 \leq t \leq T} \int (\rho |u|^2 + \rho + \rho^\gamma + \varepsilon \rho^{-p_0}) dx + \int_0^T \int \rho |D u|^2 dx dt \\
+ \varepsilon \int_0^T \int (|\nabla v|^4 + |\nabla v|^2 |u|^2 + |\nabla v|^4 |u|^2 + \rho^{-p_0} |u|^2 + \rho |u|^5) dx dt \\
+ \varepsilon^2 \int_0^T \int \rho^{-2p_0-1} dx dt \leq C.
$$

(5.2)
Lemma (BD entropy)

\[ \exists \ C \ \text{independent of} \ \varepsilon \ \text{such that} \]

\[
\sup_{0 \leq t \leq T} \int (|\nabla v|^2 + \varepsilon |\nabla v|^4) \, dx + \int_0^T \int (\rho |\nabla u|^2 + \rho^{-2} |\nabla \rho|^2) \, dx \, dt \\
+ \varepsilon \int_0^T \int ((\Delta v)^2 + |\nabla v|^2 |\nabla^2 v|^2) \, dx \, dt + \varepsilon^2 \int_0^T \int |\nabla v|^4 |\nabla^2 v|^2 \, dx \, dt \\
\leq C. \tag{5.3}
\]
Observation:

\[
\int \text{div}(|\nabla v|^2 \nabla v) \Delta v \, dx
\]

\[= - \int |\nabla v|^2 \nabla v \cdot \nabla \Delta v \, dx \tag{5.4}\]

\[= \int |\nabla v|^2 |\nabla^2 v|^2 \, dx + \frac{1}{2} \int |\nabla |\nabla v|^2|^2 \, dx.\]
Open Problems:

- Shallow water models;
- Full compressible Navier-Stokes system with viscosity and heat conduction depending on temperature;
- ...
Thank You!