Deterministic Protocols in the SINR Model without Knowledge of Coordinates

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Abstract

Much work has been developed for studying the classical broadcasting problem in the SINR (Signal-to-Interference-plus-Noise-Ratio) model for wireless device transmission. The setting typically studied is when all radio nodes transmit a signal of the same strength. This work studies the challenging problem of devising a distributed algorithm for multi-broadcasting, assuming a subset of nodes are initially awake, for the SINR model when each device only has access to knowledge about the total number of nodes in the network \(n\), the range from which each node's label is taken \([1, \ldots, N]\), and the label of the device itself. Specifically, we assume no knowledge of the physical coordinates of devices and also no knowledge of neighborhood of each node.

We present a deterministic protocol for this problem in \(O(n \log N \log n)\) rounds, assuming we have no knowledge of either the physical coordinates of devices or neighborhood of each node. There is no known polynomial time deterministic algorithm in literature for this setting, and it remains the principle open problem in this domain. A lower bound of \(\Omega(n \log N)\) rounds is known for deterministic broadcasting without local knowledge.

In addition to the above result, we present algorithms to achieve multi-broadcast in \(O(n \log N)\) rounds and create a backbone in \(O(n \log N)\) rounds, assuming that all nodes are initially awake. For a given backbone, messages can be exchanged between every pair of connected nodes in the backbone in \(O(\log N)\) rounds and between any node and its leader in the backbone in \(O(\Delta \log N)\) rounds.

Keywords: Distributed algorithms, Multi-Broadcast, Non-Spontaneous Wakeup, Backbone Creation, Signal-to-Interference-plus-Noise-Ratio model, Wireless networks, Deterministic algorithms, Strongly Selective Family based Dilution

1 Introduction

The SINR (Signal-to-Interference-plus-Noise-Ratio) model for communication for ad-hoc wireless networks has been deeply studied in recent years. It captures the effects of both the strength of devices as well as distance between devices have on a message reaching its intended target. The protocols developed in this rich area have been complex and are either randomized or use the physical coordinates of the nodes and/or the knowledge of nodes’ neighbors’ labels.

However, in the real world, it is not practical to assume that we know the physical coordinates of the nodes after they have been deployed. Similarly, the assumption that a node is aware of its neighborhood is also quite restrictive. Finally, randomized algorithms, while good, suffer from the issue that they only work with high probability. Thus, from a very pragmatic deployment perspective, the following properties are desirable for communication protocols developed for ad-hoc wireless networks: (a) They are deterministic (b) They support uncoordinated wakeup of nodes (c) They assume availability of minimal knowledge like

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*When all nodes are awake initially, it’s called a spontaneous wakeup. When a nonempty subset of all nodes is awake initially (and other nodes are passive until they receive a message), it’s called an uncoordinated wakeup.
node’s own label, maximum number of nodes \( n \), label range \( N \). (d) Have low round complexity and low communication complexity.

In this work, we develop communication protocols for this setting, assuming weak connectivity for SINR model formulated by Daum et al. \[10\] and further refined by Jurdziński and Kowalski \[22\] and weak devices, as formulated by Jurdziński et al. \[22\]. It was shown in \[27\] that the lower bound to accomplish deterministic broadcast without local knowledge is \( \Omega(n \lg N) \). We present the first polynomial time deterministic algorithm for the challenging open problem of multi-broadcast from an uncoordinated wakeup, in this setting from \[27\], that does not use the knowledge of nodes’ coordinates or the knowledge of the labels of nodes’ neighbors. The gap between upper and lower running time bounds for this problem and setting is now \( O(\lg n) \). We also develop deterministic algorithms to handle backbone creation and communication for this setting.

1.1 Our Contributions

We present deterministic protocols for the following communication problems: (a) Wakeup: Starting from a state when an arbitrary non-empty subset of nodes is awake, wake up everyone (b) Multi-broadcast: \( k > 0 \) nodes are given an initial message to be transmitted to all the rest (c) Backbone: Create a constant degree connected dominating set (CDS) with asymptotically the same diameter as the network and such that every node in the network is connected to \( \geq 1 \) node in the backbone. For devising deterministic protocols for the setting in consideration, we extensively make use of the tool known as SSF Based Dilution, introduced in \[31\]. This tool allows a node to successfully transmit a message to its neighbors, under certain conditions. It is the key to bypassing the usual requirement that nodes know their physical coordinates.

Our protocols repeatedly use the following three sub-protocols: (a) Tree-Grower, which creates a forest of trees from an arbitrary uncoordinated start state. (b) Tree-Cutter, which cuts down the trees in a forest to trees of height at most one, called stars. (c) Token-Passing-Transfer, which allows a forest of awake nodes to transmit messages to each other and surrounding nodes. We use these sub-protocols, assuming different upper bounds on the number of participating nodes, in a time-multiplexed fashion to construct Algorithm Wakeup, which runs in \( O(n \lg N \lg n) \) rounds. Our Multi-Broadcast protocol uses the above three procedures to generate constant size stars, collect messages at the leaders, and then broadcast them throughout the network. It takes \( O(n \lg N) \) rounds, assuming spontaneous wakeup. For uncoordinated wakeup, we first run Wakeup and then run Multi-Broadcast for a running time of \( O(n \lg N \lg n) \) rounds. In order to create the backbone, we develop Backbone-Creation which uses Tree-Grower and Tree-Cutter, and then runs in 4 stages to create a backbone in \( O(n \lg N) \) rounds assuming spontaneous wakeup. For uncoordinated wakeup, we run Wakeup and then Backbone-Creation in \( O(n \lg N \lg n) \) rounds. Finally, we develop Backbone-Message-Exchange for nodes within the backbone to communicate with each other in \( O(\lg N) \) rounds, and Backbone-Message-Transmit for nodes outside the backbone to communicate with their contact node within the backbone in \( O(\Delta \lg N) \) rounds. In order to achieve our results, we require message size to be \( O(\Delta^2 N) \) bits\[1\].

Our protocols extensively and repeatedly employ the tool of SSF based dilution. Coincidentally and quite interestingly, the structure of the main algorithms for the wakeup and multi-broadcasting problems have a broader structure akin to deterministic algorithms developed for broadcasting and gossiping developed in the literature for ad-hoc radio networks \[8, 10, 21, 15\]. Thus, we feel that our work serves an important role in connecting the very rich older literature on ad-hoc radio networks and the newer literature on SINR model via the use of new available tools.

1.2 Related Results

The SINR model was introduced by Gupta and Kumar \[19\], and various problems in this model have been formulated and considered in \[2, 13, 26, 28\]. Recently, a division has been introduced in the SINR model between strong devices versus weak devices \[27, 29\]. Strong devices only need to have their SINR ratio cross

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the threshold for their message to be heard by a node, but weak devices also have to satisfy a second inequality which further limits their range. Another distinction appears when nodes have knowledge of their coordinates on the Euclidean plane and when they don’t. Jurdziński et al. [23] achieve randomized broadcast with strong devices which have knowledge of their coordinates and uncoordinated wakeup in $O((D + \log(\frac{1}{\epsilon})) \log n)$ rounds with high probability, where $\alpha$ is the path loss parameter of the SINR model, $D$ is the diameter of the communication graph, and $R_s$ is the ratio between the maximum distance between two communicable nodes to the minimum distance between two communicable nodes in the communication graph. In the case of Jurdziński et al. [24, 25], when they consider spontaneous wakeup, they achieve randomized broadcast in $O(D \log n \log^2 n)$ rounds with high probability. When they consider uncoordinated wakeup, they achieve randomized broadcast in $O(D \log^2 n)$ rounds with high probability. In Jurdziński et al. [25], they achieve randomized multi-broadcast in $O(D \log^2 n + k \log n + \log^2 n)$ rounds with high probability, where $k$ messages stored at $k$ nodes need to be broadcasted.

Another division has been introduced by Daum et al. [10] and refined by Jurdziński and Kowalski [22] in the form of weak links versus strong links in the communication graph. A communication graph built on weak links has edges between nodes within range of each other. A strong link graph has edges between nodes which are within a fraction of the range of each other, where range is determined by strength of the device. Note that Daum et al. [10] define weak links assuming the presence of strong devices, whereas Jurdziński and Kowalski [22] differentiate between the strength of devices and strength of links in the communication graph. [23, 10, 24, 25] all have results for the strong links case. For the weak links case, Daum et al. [10] provide a randomized $O(n \log^2 n)$ round algorithm with uncoordinated wakeup which uses strong devices and has no knowledge of coordinates and works with high probability. Jurdziński et al. [27] use weak links and weak devices to design several deterministic algorithms. When nodes have knowledge of their coordinates and their neighbors, they present a $O(D \log^2 N)$ round algorithm to achieve broadcast from an uncoordinated wakeup. When nodes know only their coordinates, they provide two algorithms with running times $O(n \log N)$ rounds and $O(D \Delta \log^2 N)$ rounds, where $\Delta$ is the maximum degree of any node in the graph. Furthermore, they prove a lower bound of $\Omega(n \log N)$ rounds for deterministic broadcasting with uniform transmission powers and without local knowledge of immediate neighborhood. Chlebus and Vaya [21], which is a preliminary version of Chlebus et al. [6], use weak links and weak devices with no knowledge of their coordinates to achieve randomized broadcast with uncoordinated wakeup in $O(n \log^2 N)$ rounds with high probability. Reddy et al. [33] use weak devices and weak links to construct deterministic multi-broadcast algorithms with uncoordinated wakeup in the presence of different types of knowledge. When nodes know their own coordinates and the coordinates of their neighbors, they achieve multi-broadcast in $O(D \log^2 n + k \log \Delta)$ rounds. When nodes know their own coordinates but have no information of their neighbors, they achieve multi-broadcast in $O((n + k) \log n)$ rounds. When they don’t know their coordinates but know the labels of their neighbors, they achieve multi-broadcast in $O((n + k) \log n)$ rounds. We summarize relevant previous results in Table 1.

Yu et al. [35] achieve randomized multi-broadcast in an asynchronous system in the case of weak devices and weak links without knowledge of their coordinates in $O((D + k) \log n + \log^2 n)$ time-slots. Yu et al. [37] achieve randomized multi-broadcast in the case of weak devices and weak links without knowledge of their coordinates in $O(D + k + \log^2 n)$ rounds assuming spontaneous wakeup. However, they allow stations the additional power of controlling the power output.

Work has been done on dominating sets in the SINR model by Scheideler et al. [36]. They present an algorithm to find a dominating set that stabilizes in $O(\log n)$ rounds with high probability using tunable collision detection. Yu et al. [37] use a randomized algorithm and power control to create a connected dominating set in $O(\log^2 n)$ rounds with high probability. For a comprehensive survey on connected dominating sets in wireless ad hoc and sensor networks, we refer the reader to Yu et al. [40]. Creating and communicating across a backbone in the SINR model was studied by Jurdziński and Kowalski [21]. They construct a backbone in $O(\Delta \log^3 N)$ rounds. Chlebus et al. [6, 7] use randomization to create a backbone in $O(\Delta \log^{8a+3} N)$ rounds with high probability, for some positive constant $a$, assuming all nodes are initially awake. For the
setting when only some nodes are awake, they present a randomized algorithm to create a backbone in $O(n \lg^2 N + \Delta \lg^{8a+3} N)$ rounds with high probability. Reddy et al. [35] devise deterministic protocols to create a backbone when only some nodes are awake under different assumptions of knowledge. When nodes know their own coordinates and those of their neighbors, they create a backbone in $O(D \lg^2 n + k \lg \Delta)$ rounds. When nodes only know their own coordinates, backbone is created in $O(n \lg n)$ rounds. Their most interesting result is when nodes don’t know their own coordinates, but know the labels of their neighbors. Somewhat surprisingly, they present a protocol that creates a backbone in $O(n)$ rounds. Kowalski et al. [33] construct a backbone in $O(\Delta \lg^2 N)$ rounds when all nodes are initially awake and don’t know their own coordinates but know the labels and the labels of their neighbors. Jurdiński et al. [25] create a quasi-backbone structure using randomization in $O(D \lg^2 n)$ rounds with high probability, where a quasi-backbone is the assignment of probabilities to nodes that allows groups of devices within certain distance of each other to communicate.

Table 1: Comparison of running times and other features of various algorithms to solve broadcast. $D$ is diameter of graph (based on strong/weak links), $\Delta$ is max. degree of graph, $n$ is number of nodes, $N$ is the max. value of any label of node, $\zeta$ is the maximal error probability, $\alpha$ is the path loss constant, $k$ is number of nodes with messages to transmit, and $R_k$ is the maximum ratio between strong link lengths. All running times are in rounds. Running times of randomized algorithms are with high probability, except for RandUnknownBroadcast [23], in which case it’s with probability at least $1 - \zeta$.

| Algorithm                  | Randomized | With Knowledge of Coordinates | Device Type | Link Type | Achieves Multi-Broadcast | Running Time with Spontaneous Wakeup | Running Time with Uncoordinated Wakeup |
|----------------------------|------------|-------------------------------|-------------|-----------|------------------------|--------------------------------------|----------------------------------------|
| RandUnknownBroadcast [23]  | Yes        | Yes                           | Strong      | Strong    | No                     | $O(D + \lg n \lg n)$                  | $O(n \lg n)$                           |
| StrongCast [25]            | Yes        | No                            | Strong      | Strong    | No                     | $O(D \lg n \lg n)$                  | $O(D \lg n) + \lg n$                   |
| StrongCast, NoBroadcast [24]| Yes        | No                            | Strong      | Strong    | No                     | $O(D \lg n + \lg^2 n)$               | $O(D \lg n) + \lg n$                   |
| Broadcast [25]             | Yes        | No                            | Strong      | Strong    | Yes                    | $O(D \lg n + \lg^2 n)$               | $O(n \lg n)$                           |
| HarmonyCast [10]           | Yes        | No                            | Strong      | Weak      | Yes                    | $O(D + \kappa + \lg n)$              | $O(n \lg n)$                           |
| 3-Timeslot Scheme [37]     | Yes        | No                            | Weak        | Weak      | Yes                    | $O(D + \kappa + \lg n)$              | $O(n \lg N)$                           |
| Modifiedidispatch [15]     | Yes        | No                            | Weak        | Weak      | No                     | $O(n \lg N)$                         | $O(D \lg N)$                           |
| Rand-IB [26]***           | No         | Yes                           | Weak        | Weak      | No                     | $O(D \lg n)$                         | $O(n \lg N)$                           |
| SuaUB [27]                 | No         | Yes                           | Weak        | Weak      | No                     | $O(D \lg n)$                         | $O(n \lg N)$                           |
| GeneralBroadcast [27]      | No         | Yes                           | Weak        | Weak      | No                     | $O(D \lg n + \lg n)$                 | $O(\Delta \lg n)$                      |
| Local-Multicast [25]***    | No         | Yes                           | Weak        | Weak      | Yes                    | $O(D \lg n + \kappa \lg n + (g/n)\lg n)$ | $O(n \lg n)$                           |
| BTD_Traversals, BTD_Multi [34] | No         | No                            | Weak        | Weak      | Yes                    | $O(n \lg n)$                         | $O(n \lg n)$                           |

*3-Timeslot Scheme [37] requires nodes to be able to control their power. ⋆⋆⋆Local-Multicast [25] requires nodes to know coordinates of their neighbors. **BTD_Traversals, BTD_Multi [34] require nodes to know labels of their neighbors.

The problem of local broadcasting, which deals with transmitting a message to all its neighboring nodes, has been studied in [17, 20, 29, 29, 14]. A survey of approximation algorithms in the SINR model was performed by Goussevskaia et al. [18]. Other sub-models within the SINR model have also been looked at recently by Jurdiński et al. [26]. There have been various trends in the SINR model with respect to signal strength and geometric decay [1, 10, 4, 23, 6, 7]. There has also been an attempt to bridge the real world usefulness of the SINR model with the results available for the theoretically easier model of Unit Disc Graphs (UDG) in the form of Quasi-UDGs [32, 8]. Quasi-UDGs consider two nodes to be connected if the distance between them lies below a threshold $\gamma$, $0 < \gamma < 1$, disconnected if the distance is above 1 and maybe connected if the distance lies between $\gamma$ and 1. Some work has been done on converting results from one model to the other [34].

1.3 Organization of the paper

The rest of this paper is organized as follows. Section 2 introduces the SINR model as well as useful technical preliminaries. Section 3 presents the algorithm to achieve wakeup of nodes. Section 4 develops the protocol
for achieving multi-broadcast. Section 5 presents the protocols to create and utilize a backbone subnetwork. We briefly present conclusions and an open problem in Section 6.

2 Preliminaries

The SINR Model Each wireless station, \( u \), has some transmission power \( P_u \in \mathbb{R}^+ \). The Euclidean distance between two stations \( u \) and \( v \) is given by the distance function \( d(u, v) \). For a given round, let \( T \) be the set of all stations which transmit in that round. The SINR for a station \( u \)'s message at station \( v \) in that round is defined as follows:

\[
SINR(u, v, T) = \frac{P_u}{d(u, v)^\alpha} \frac{1}{N + \sum_{i \in T \setminus u} P_i d(i, v)^\alpha}.
\]

\( \alpha \geq 2 \) and \( N \geq 0 \) are fixed parameters of the model called the path loss constant and ambient noise respectively. All the results in this paper hold only for \( \alpha > 2 \). A node \( v \) receives \( u \)'s message iff the SINR ratio of \( u \)'s message at \( v \) crosses a threshold \( \beta \geq 1 \), which is also a fixed parameter of the model:

\[
\frac{P_u}{d(u, v)^\alpha} \frac{1}{N + \sum_{i \in T \setminus u} P_i d(i, v)^\alpha} \geq \beta. \tag{1}
\]

A device which only needs to satisfy Inequality 1 for its message to be heard is called a strong device. A weak device further needs to satisfy the following inequality,

\[
\frac{P_u}{d(u, v)^\alpha} \geq (1 + \epsilon)\beta N, \tag{2}
\]

where \( \epsilon > 0 \) is called the sensitivity parameter of the device. Note the relation between weak and strong devices. When \( \epsilon = 0 \), Inequality 2 reduces to Inequality 1 (in the absence of interference). In this paper, we assume that all devices have the same fixed transmission power \( P \).

We now describe what it means for devices to receive each other’s message, using definitions commonly found in the literature, cf. [27, 29]. When both Inequality 1 and 2 are satisfied in a given round, node \( v \) successfully receives node \( u \)'s transmission in that round. Node \( u \)'s transmission range is the maximum distance at which another station can be located away from it and still successfully receive a message from \( u \) when no others stations transmit. For the remainder of this paper, we use the terms ‘transmission range’ and ‘range’ interchangeably. Since all nodes have the same power, they also all have the same range, \( r \). When all nodes within range of a node \( u \) successfully receive its message in a given round, we say that \( u \) successfully transmitted in that round. Note that only if a node was a receiver in that round will it actually receive \( u \)'s message and be counted in the definition. Otherwise, the message will be discarded.

A communication graph, denoted by \( G(V, E) \), is a graph where each station is considered a node and an edge from node \( u \) to node \( v \) denotes that \( v \) is within range of \( u \). Since all nodes have same range, the graph is undirected. We assume \( G \) is connected. The weak links model [10, 22], also known as weak connectivity, refers to how edges exist iff nodes are within range of each other as opposed to within a fraction of that range (known as the strong link model). We define a star as a subset of \( G(V, E) \) forming a tree of height at most one.

Any algorithm prescribed for a station proceeds in a series of rounds, where each round corresponds to one global clock tick. In a given round, a station may act either as a receiver or a transmitter, but not both. A transmitter may transmit a message of size \( O(\Delta lg^2 N) \) bits, where \( \Delta \) is the maximum degree of any node in the network. Thus, one round of an algorithm for a given node consists of the following three steps:

1. If the node acted as a receiver in the previous round, then it receives any messages successfully sent to it in the last round.
2. The node performs some local computation, if any.

3. If the node is a transmitter in this round, then it transmits a single message.

Nodes do not have the ability to detect collisions.

Nodes which are asleep remain inactive until they receive a wakeup message from an awake node. Furthermore, these wakeup messages contain the current round number which allows all nodes to maintain synchronization.

**Grid and Pivotal Grid** Consider a 2-dimensional grid $G_x$ overlayed on the Euclidean plane such that the length of each side of a grid box is $x$. Each grid box is denoted by the coordinates of its bottom left coordinates. Therefore, a device with coordinates $(m, n)$ in the Euclidean plane will have grid coordinates $(a, b)$ on the grid $G_x$. The **pivotal grid** is the grid $G_x \cap \mathbb{Z}^2$. The significance of the pivotal grid is that any two nodes located within the same pivotal grid box are within range of each other. This has been useful in the design of various algorithms [11, 12]. The number of nodes located in a given box of the pivotal grid is not bounded. Grid boxes $x$ and $y$ are within range of each other iff there exist locations in $x$ and $y$ such that two nodes placed at these locations are within range of each other. **Box-distance** between two grid boxes with coordinates $(a_1, b_1)$ and $(a_2, b_2)$ is 0 if the two boxes intersect, else $k$ where $k = \max(min(|a_1 - a_2 - 1|, |a_2 - a_1 - 1|), min(|b_1 - b_2 - 1|, |b_2 - b_1 - 1|))$.

**Strongly Selective Family** Let $N \geq c$ and both $N$ and $c$ be positive integers. An $(N, c)$-strongly selective family, commonly shortened to (N,c)-ssf, is a family $F$ of subsets of integers from $[1, N]$ such that for any non-empty integer subset $S$ of $[1, N]$, $|S| \leq c$, for each element $x \in S$, there exists a set $R \in F$ such that $S \cap R = x$. There exist $F$’s of size $O(c^2 \lg N)^*$ which satisfy the above definition, cf. Clementi et al. [3].

**Strongly Selective Family Based Dilution** SSF based dilution is a technique that uses an $(N, c)$-ssf to allow neighbors of a node, within $\sqrt{2x}$ distance of it in a grid $G_x$, to successfully receive a message from it. Let $F$ be an $(N, c)$-ssf and let $F_1, F_2, \ldots, F_z$ be sets belonging to $F$. A node $u$ executes an $(N, c)$-ssf when $u$ is active (performs some action such as transmission) only in those rounds $i$ such that $u \in F_i$, and in other rounds $u$ just acts as a receiver. The size of $F$, $z = O(c^2 \lg N) = c_1 \lg N$. Here, $c = k^2(2d + 1)^2$, where $d$ is a constant and comes from Lemma 1 restated from [5] below.

**Lemma 1.** [Lemma 2 in [3]] For stations with same range $r$, sensitivity $\epsilon > 0$, and transmission power, for each $\alpha > 2$, there exists a constant $d$, which depends only on the parameters $\alpha, \beta$, and $\epsilon$ of the model and a constant $k$, satisfying the following property.

Let $W$ be the set of stations such that at most a constant $k$ of them want to transmit in any grid box of the grid $G_x$, $x \leq \frac{r}{\sqrt{2}}$. Let $u$ and $v$ be two stations in different grid boxes such that the distance between them, $\sqrt{2x}$, is the minimum distance between any two stations in different grid boxes in $G_x$. Let $A_u$ be the set of stations in $u$’s grid box.

If $u$ is transmitting in a round $t$ and no other station within its box or a box less than $d$ box distance away from its box is transmitting in that round, then $v$ and all stations in $A_u$ can hear the message from $u$ in round $t$.

For a given grid $G_x$, $x \leq \frac{r}{\sqrt{2}}$, $k$ is usually an upper limit on the number of nodes present in any box of $G_x$. Because of the way we use $(N, c)$-ssf’s throughout this paper, it suffices to set $k = 1000$. Due to this $c$ is a constant and by extension $c_1$ is also a constant and the number of rounds of an $(N, c)$-ssf execution is $O(\lg N)$. Furthermore, by the following theorem, restated from [3] below, we are guaranteed that executing an $(N, c)$-ssf under the required conditions allows all neighbors of a node at most $\sqrt{2x}$ away from it in $G_x$ to successfully receive its message.

\*\*An existential bound is already known in the literature. As yet unpublished recent work [5] shows that a possible explicit construction may exist in the form of an $(N, (1, c - 1))$ cover free family.
Theorem 1. [Theorem 1 in [34]] For a grid \( G_{x, x} \), \( x \leq r/\sqrt{2} \), let set of all nodes that want to transmit satisfy properties of Lemma 1. Every node in this set can successfully transmit a message to its neighbors within \( \sqrt{2}x \) distance of it in \( O(\lg N) \) rounds by executing one \((N, c)\)-ssf, where \( c = k^2(2d + 1)^2 \) (\( d \) is the constant that bounds box distance away from node within which other nodes must be silenced, taken from Lemma 4; \( k \) is the constant that bounds number of nodes from the set in any box of the grid).

**Knowledge of Stations** Nodes know the value of \( n, N \), their own label, and a common \((N, c)\)-ssf schedule. Nodes don’t know the value of their coordinates or labels of their neighbors.

**Problem Statements** Using the above tools and techniques, in the model described, we attempt to solve the following problems. Our goal is to solve each of these problems in the minimum number of rounds.

**Wakeup:** Initially, \( k \) nodes, \( 1 \leq k < n \), are awake and can transmit messages. An asleep node can be woken up if it hears a message from an awake node. Our goal is to wake up all nodes in the network.

**Multi-Broadcast:** Initially, \( k \) nodes, \( 1 \leq k \leq n \), each have a unique message. Our goal is to transmit these messages so that all nodes have all \( k \) messages.

**Backbone:** We need to create a backbone and subsequently design protocols for nodes to communicate within the backbone. A backbone is essentially an overlay network which facilitates speedy transfer of messages between nodes. More formally, according to Jurdziński and Kowalski [21], the properties that need to be satisfied for a network to be considered a backbone are:

1. The nodes of the backbone, \( H \), form a connected dominating set which induces a subgraph of the communication graph, \( G \), and have a constant degree relative to other nodes within the backbone.
2. The number of nodes in \( H \) is \( O(sc_d) \), where \( sc_d \) is the size of the smallest connected dominating set of the communication graph.
3. Each node of \( G \setminus H \) is associated with exactly one node of \( H \) which acts as its entry point into the backbone.
4. The asymptotic diameter of \( H \) is same as that of \( G \).

Our first goal is to create a backbone that satisfies the above properties. Subsequently, we need to design a protocol to allow two nodes within the backbone to quickly exchange messages with each other. Finally, we need a protocol for nodes outside the backbone to communicate with their contact node in the backbone, also called their leader.

3 Wakeup

In the wakeup problem, there are \( k \) arbitrary nodes, \( 1 \leq k < n \), which are initially awake and the task is to wake up the remaining nodes. We prove the following theorem for this problem:

**Theorem 2.** Algorithm Wakeup successfully wakes up all nodes in \( O(n \lg N \lg n) \) rounds, assuming initially at least one node is awake.

3.1 Overview of Algorithm Wakeup

Algorithm Wakeup uses three main procedures: Tree-Grower, Tree-Cutter, and Token-Passing-Transfer. We want every asleep node to receive a wake up message from an awake node. If we initially had every awake node try to transmit such a wake up message at the same time during an \((N, c)\)-ssf schedule, there would be no guarantee that the number of transmitting nodes in any given pivotal grid box would be upper bounded by a constant. So we use Tree-Grower and Tree-Cutter to reduce the number of nodes which are allowed to transmit during a given \((N, c)\)-ssf by connecting nodes together into trees and allowing at most one node from every tree to transmit during a given \((N, c)\)-ssf. Tree-Grower allows us to overlay a forest of trees.
on the network, but we can’t immediately use this forest to pass messages through the network because multiple trees might pass through the same pivotal grid box. In order to ensure that not too many nodes in the same pivotal grid try to transmit at the same time, we need to cut the size of trees down so that each tree is of height at most one using Tree-Cutter. Token-Passing-Transfer is used to actually transmit these wake up messages. Very briefly we discuss what each procedure achieves before we sketch Algorithm Wakeup.

Procedure Tree-Grower The Tree-Grower procedure, presented in Section 3.2, takes nodes which know nothing about each other and forms a forest of trees.

Theorem 3. Assuming that the number of nodes given as input is an upper bound on the actual number of awake nodes in the network, the execution of Procedure Tree-Grower on this set of nodes results in creating a forest in \(O(n \lg N)\) rounds with the following properties:

1. Every node is either a leader or a child and belongs to exactly one tree.
2. Every tree has exactly one leader and there is at most one leader per grid box of the pivotal grid.
3. Every node knows the labels of its parent node and children.

Procedure Tree-Cutter The Tree-Cutter procedure, presented in Section 3.3, takes the trees previously formed by Tree-Grower and creates stars, with the same guarantees as the Tree-Grower procedure.

Theorem 4. Assuming that the nodes satisfy the properties described in Theorem 3, the execution of Procedure Tree-Cutter takes \(O(n \lg n)\) rounds and results in the creation of stars that satisfy the following properties:

1. Every node is either a leader or a follower and belongs to exactly one tree.
2. Each tree is a star and has exactly one root called its leader.
3. There exists at most one leader node per grid box of the pivotal grid.

Procedure Token-Passing-Transfer Finally, Token-Passing-Transfer, presented in Section 3.4, is used to wake up all nodes which are within the communication range of the participating nodes. It assumes as input stars created by the Tree-Cutter procedure and explores them with the help of a token. In particular, when a node receives a token, it transmits a wake up message and all asleep nodes that hear this message (formally) wake up.

Theorem 5. Assuming that the participating nodes are in a configuration characterized by Theorem 4, the Token-Passing-Transfer procedure takes \(O(n \lg N)\) rounds to complete and ensures two things:

1. If wakeup or single-transmit message needs to be sent, then each participating node successfully transmits the message. Further, any incoming messages are successfully heard.
2. If number of participating nodes is bounded by 81, then every participating node successfully transmits its message and every message it may hear during execution.

Wakeup Given a set of awake nodes \(A\), after executing the 3 procedures, we are able to wake up all neighboring nodes \(B\). However, if there exist a set of asleep nodes \(C\) which are neighbors of \(B\) but not \(A\), then we will have to execute the 3 procedures once again on the set of nodes \(B\) in order to wake up these new neighbors. If the communication graph is a line, then we would have to execute these 3 procedures \(n - 1\) times for a total running time of \(O(n^2 \lg N)\). But notice that in each execution, we’re unnecessarily considering that all \(n\) nodes are participating, when only a fraction are. We use this key insight that the number of awake nodes participating in the 3 procedures may be a fraction of \(n\) in order to develop Wakeup, which runs in \(O(n \lg N \lg n)\) rounds. 
Either one round of Tree-Grower can be executed.
Or one round of Tree-Cutter can be executed.
Or one round of Token-Passing-Transfer can be executed.

Figure 1: A graphical illustration of what happens in one slot of one phase for a given node. For a given phase and slot, at most one round of Tree-Grower, Tree-Cutter, or Token-Passing-Transfer will be executed depending on what stage the particular node is in in the given slot.

Wakeup consists of executing $O(n \lg N)$ phases. Each phase consists of executing $(\lfloor \lg n \rfloor + 1)$ slots. In turn, each slot consists of 3 rounds, which are respectively meant to execute a single round of Procedure Tree-Grower, Procedure Tree-Cutter, or Procedure Token-Passing-Transfer. In a slot, a node participates in at most one of these rounds and does nothing in the other unutilized rounds. This achieves time-division-multiplexing of the executions of the three procedures. Figure 1 shows this time-division-multiplexing.

We give a top-down view of Wakeup. Algorithm Wakeup executes the three procedures for slot $i \in [1, \lfloor \lg n \rfloor + 1]$, with an estimate of $2^i$ on the upper bound of the number of participating nodes, $O(2^i)$ times, while these procedures for the largest (correct) estimate of $O(n)$ nodes are executed $O(1)$ times, because they take much longer. Thus, one phase of Algorithm Wakeup comprises of execution of a single (next) round of (one of) the three procedures for each slot $i \in \{1, \ldots, \lfloor \lg n \rfloor + 1\}$ and this is repeated till completion of the algorithm. The complete execution of the three procedures for slot $i$ is called an epoch for that slot. The $j^{th}$ epoch for slot $i$, denoted as $e_j^i$, is the sequence of phases during which the three procedures can be run for the $j + 1^{th}$ time in slot $i$. Figures 2 and 3 graphically illustrate this, with Figure 3 giving extra insight into the timing of epochs.

We now describe the execution of Wakeup from the viewpoint of a single node. A node can be in one of 6 statuses, enumerated as 1. Asleep, 2. Waiting in Slot, 3. Tree-Grower, 4. Tree-Cutter, 5. Token-Passing-Transfer, and 6. Done in Slot. If a node is asleep initially (status 1), it needs to be woken up by a wakeup message transmitted during execution of Procedure Token-Passing-Transfer; other messages are ignored by asleep nodes. A node once woken up in one slot, is awake in all slots. In addition to the wakeup message, the current phase and slot number are also transmitted for synchronization. Once a node wakes up, it waits for the starting phase of a new epoch for each slot (status 2). Note that a node may be waiting in one slot and participating in an epoch in another slot. For a given slot, once the next epoch starts, the node participates in the 3 procedures (status 3-5), as described earlier, once and subsequently does nothing in that slot till the end of Algorithm Wakeup (status 6). Every sequence of 3 rounds of the algorithm for a given node are thus referenced by phase #, slot #, and status #. This is graphically illustrated in Figure 4.

3.1.1 Proof of Theorem 2

We will argue the correctness of Wakeup as follows. First, we show new nodes will always be woken up and all nodes would eventually awaken should Wakeup be run for an unbounded number of phases. We then construct a worst case scenario (w.r.t. running time) we call a “snowball” for the sequence of nodes waking up and bound its running time. We give a lower bound to the number of nodes that would be awake after
Phase \(j\)

Phase \(j+1\)

Phase \(j+2\)

Phase \(j+3\)

Phase \(j+4\)

Phase \(j+5\)

Phase \(j+6\)

Phase \(j+7\)

Phase \(j+8\)

Figure 2: Top-down view of Wakeup.

Figure 3: A graphical illustration of the lengths of various epochs and their relationships with each other.

\(e^j_i\) is the \(j^{th}\) epoch for slot \(i\). \(t_i = 2^i \cdot 10450c_1 \lg N + 9465c_1 \lg N\)
Algorithm 1 Wakeup(No. of nodes \( n \)), run by each node \( u \)

1: type denotes the type of node: leader, follower, or neutral.
2: \( \text{my}_\text{leader} \) denotes leader of a node (for a leader, \( \text{my}_\text{leader} \) is its own label).
3: Tree \( T \) stores the children of a leader. For followers, \( T \) is empty.
4: \( \text{round}[\text{slot}] \) is used to track which round of a procedure is being executed in a particular slot.
5: \( \text{slot}, \text{status}[\text{slot}], \) and \( t_i \) are explained in Figures 2 and 3.

6: If awake set \( \text{status}[\text{slot}] \leftarrow 2 \) else \( \text{status}[\text{slot}] \leftarrow 1, \forall \text{slot} \in [1, \lfloor \lg n \rfloor + 1] \)

7: for phase \( \leftarrow 0, 4nt_1 - 1 \) do
8: for \( \text{slot} \leftarrow 1, \lfloor \lg n \rfloor + 1 \) do
9: Status 1 (Asleep): If \( u \) is not awake
10: Do nothing for 3 rounds
11: if \( u \) hears a wakeup message then
12: Wake up, synchronize \( \text{slot} \) and phase, set \( \text{status}[\text{slot}] \leftarrow 2, \forall \text{slot} \in [1, \lfloor \lg n \rfloor + 1] \)
13: Status 2 (Waiting in Slot): Else if \( \text{status}[\text{slot}] = 2 \)
14: if \( (\text{phase} \mod t_{\text{slot}}) = 0 \) then
15: \( \text{status}[\text{slot}] \leftarrow 3, \text{round}[\text{slot}] \leftarrow 1 \) and goto Status 3.
16: Do nothing for 3 rounds
17: Status 3 (Tree-Grower): Else if \( \text{status}[\text{slot}] = 3 \)
18: Execute round \( \text{round}[\text{slot}] \) of \( \text{Tree-Grower}(2^{\text{slot}}) \), increment \( \text{round}[\text{slot}] \) by one
19: if \( \text{Tree-Grower} \) is done executing then
20: Store the values returned in \( < T[\text{slot}], \text{type}[\text{slot}], \text{my}_\text{leader}[\text{slot}] > \)
21: \( \text{status}[\text{slot}] \leftarrow 4, \text{round}[\text{slot}] \leftarrow 1 \)
22: Do nothing for 2 rounds
23: Status 4 (Tree-Cutter): Else if \( \text{status}[\text{slot}] = 4 \)
24: Do nothing for 1 round
25: Execute round \( \text{round}[\text{slot}] \) of \( \text{Tree-Cutter}(T[\text{slot}], \text{type}[\text{slot}], \text{my}_\text{leader}[\text{slot}], 2^{\text{slot}}) \), increment \( \text{round}[\text{slot}] \) by one
26: if \( \text{Tree-Cutter} \) is done executing then
27: Store the values returned in \( < T[\text{slot}], \text{type}[\text{slot}], \text{my}_\text{leader}[\text{slot}] > \)
28: \( \text{status}[\text{slot}] \leftarrow 5, \text{round}[\text{slot}] \leftarrow 1 \)
29: Do nothing for 1 round
30: Status 5 (Token-Passing-Transfer): Else if \( \text{status}[\text{slot}] = 5 \)
31: Do nothing for 2 rounds
32: \( \text{msg} \leftarrow \text{Wakeup message, phase, slot} \)
33: Execute round \( \text{round}[\text{slot}] \) of \( \text{Token-Passing-Transfer}(T[\text{slot}], \text{type}[\text{slot}], \text{my}_\text{leader}[\text{slot}], 2^{\text{slot}}, \text{msg}) \), increment \( \text{round}[\text{slot}] \) by one
34: if \( \text{Token-Passing-Transfer} \) done executing then \( \text{status}[\text{slot}] \leftarrow 6 \)
35: Status 6 (Done in Slot): Else if \( \text{status}[\text{slot}] = 6 \)
36: Do nothing for 3 rounds
Figure 4: For a given phase and slot, a node will act in one of 6 ways depending on which stage it is in in that slot.

We develop a series of lemmas that show that if we run Wakeup for an unbounded number of phases, eventually all nodes will be woken up. The following fact brings out some properties of epochs.

**Fact 1.** Let $t_i = 2^i \cdot 10450c_1 \lg N + 9465c_1 \lg N$ be the size of an epoch in slot $i$. Recall that an $(N,c)$-ssf is of size $c_1 \lg N$, where $c_1$ is a constant.

1. $t_i$ is the number of phases required to run Tree-Grower, Tree-Cutter, and Token-Passing-Transfer for a given node in slot $i$.
2. $t_{i+1} = 2t_i$.
3. Any node can only start execution of Tree-Grower in slot $i$ at phase intervals of $t_i$ starting at phase 0.

We now develop some terminology. Define an **activated set in phase** $p$, **slot** $i$ to be a set of nodes such that all nodes in the set are awake in phase $p$ and have not yet completed Token-Passing-Transfer for slot $i$, i.e. a set of nodes with status $\in [2, 5]$. An **activated connected set** is an activated set of nodes such that there exists a path between any two of its nodes in the subgraph induced by the set on the communication graph. Note that a set of nodes may be an activated set or activated connected set in a slot $i$ even if a subset of the nodes completed Token-Passing-Transfer in a slot $< i$. We say that an activated connected set of nodes **successfully completes** Token-Passing-Transfer if all of its neighbors are woken up. We now talk about when a node will successfully execute Token-Passing-Transfer.

**Lemma 2.** If the number of nodes in an activated connected set $A$ is $\leq 2^i$ at the beginning of a new epoch $e^j_i$, for some $j$, then all nodes in $A$ will successfully complete execution of Token-Passing-Transfer in $e^j_i$. 

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Proof. If any new nodes are awakened after the start of epoch $e^p_i$ in slot $i$, then they must wait until $e^{j+1}_i$ before they can start executing Tree-Grower. Meanwhile, the nodes in $A$ will go on to successfully execute Token-Passing-Transfer without interference from any node not in $A$. \hfill \blacksquare

Lemma 3. For any node that is woken up at phase $p$, there exists a phase $q_i \geq p$ with corresponding slot $i$ at which the size of the activated connected set to which the node belongs in $q_i$ is $\leq 2^i$.

Proof. The maximum size of any activated connected set is $n$, which is covered by slot $\lceil \log n \rceil + 1$, so there will always exist a slot $i$ for any given phase where the size of the activated connected set will be $\leq 2^i$. \hfill \blacksquare

By Lemma 2 and Lemma 3, we have it that any node which wakes up will eventually successfully execute Token-Passing-Transfer and wake up any neighboring nodes. Since the underlying communication graph is connected and initially there is at least one awake node, there exists a path to any node from an awake node and thus after enough time, all nodes in the graph will be woken up. Thus, it is clear that if we ran Wakeup for an unbounded period of time, eventually all nodes would wake up. We now bound the time taken to wake up all nodes by constructing a worst case scenario with respect to running time.

Worst case behavior occurs when all nodes in an activated set belong to the same activated connected set. The logic is that more the nodes, larger the slot. When nodes are connected, we have to treat them as one network. If they weren’t, we could treat them as separate networks.

Lemma 4. Define the sets $S_1, S_2, \ldots, S_k$ as the sets of nodes woken up where $s_1, s_2, \ldots, s_k$ are the corresponding sizes of the sets, where nodes belong to the same set if all of them were woken up in the same round. Let $i_1, i_2, \ldots, i_k$ be the minimum values of slots such that $s_1 \leq 2^{i_1}, s_2 \leq 2^{i_2}, \ldots, s_k \leq 2^{i_k}$. The worst case scenario with respect to running time for all nodes to successfully complete Token-Passing-Transfer occurs when the following conditions hold:

1. All nodes within a set belong to the same activated connected set.
2. $i_j \geq \lceil \log_2 \left( \sum_{a=1}^{i-1} s_a \right) \rceil$, $\forall 2 \leq j \leq k$.
3. Initially only one node is awake.

Proof. We want to craft a worst case scenario for our algorithm w.r.t. running time. First we argue that every node within a set $S_j$ should belong to the same activated set. Then we argue about the ordering of $i_1, i_2, \ldots, i_k$. Finally, we argue that only one node should be awake initially.

If nodes within the same set belong to 2 or more different activated connected sets, i.e. there exist two or more different unconnected clusters of sets which woke up at the same time, then the time required for all the nodes to successfully execute Token-Passing-Transfer is less than or equal to the time required if all of them belonged to the same activated connected set. This is because nodes belonging to different activated connected sets, which are not connected to each other, can successfully finish executing Token-Passing-Transfer in possibly smaller slots and hence faster than if they were all connected.

We now prove the second point of the lemma by first showing that $i_j < i_{j+1}, \forall 1 \leq j \leq k-1$ in the worst case scenario and using this to prove the claim. For any two consecutive sets $S_j$ and $S_{j+1}$, we can achieve the longest running time to successfully execute Token-Passing-Transfer for nodes of $S_{j+1}$ if $i_j < i_{j+1}$. We make use of the following claim to show this.

Claim 1. If a node $x$ finished executing Token-Passing-Transfer in slot $i$, then $x$ has already executed Token-Passing-Transfer in all slots $j, j < i$.

Proof. Consider a node which starts executing Tree-Grower in slot $i$ in phase $p$. Consider a slot $j < i$. Either the node starts executing Tree-Grower in slot $j$ in a phase $\leq p$ or a later phase. If the phase $\leq p$, since the size of an epoch in $j$ is less than in $i$, the lemma holds true.

Consider the case when the node starts executing Tree-Grower in $j$ in a phase $> p$. This is possible only if the node woke up in a slot $> j$. However, even in this case, the node would start executing Tree-Grower...
in the immediate next epoch of slot $j$. Remember that for every one epoch of slot $i$, at least 2 epochs of $j$ will complete by Fact 1. Also remember that when a new epoch in $i$ starts, so does one in $j$. Thus, the lemma holds true.

If $i_j \geq i_{j+1}$, then by Claim 1 all nodes from $S_j$ would have completed execution of $Token-Passing-Transfer$ in slot $i_{j+1}$ by the time nodes in $S_{j+1}$ run in that slot. Hence, nodes from $S_{j+1}$ can successfully finish executing $Token-Passing-Transfer$ in slot $i_{j+1}$ without having to deal with the nodes which were previously awake. If however $i_j < i_{j+1}$, then there is the possibility that nodes from $S_j$ have not started running $Tree-Grower$ in slot $i_{j+1}$ and above. In this case the nodes from $S_j$ and any sets of nodes prior to $S_j$ which haven’t run in slot $i_{j+1}$ also participate with the nodes of $S_{j+1}$. Thus, in the worst case, $i_j < i_{j+1}$, $\forall 1 \leq j \leq k - 1$.

We now use the condition $i_{j-1} < i_j$, $\forall 2 \leq j \leq k$ to prove the second point of the lemma. Consider some arbitrary $j$, such that $2 \leq j \leq k$.

$$[\lg_2 \left( \sum_{a=1}^{j-1} s_a \right)] \leq [\lg_2 \left( \sum_{a=1}^{j-1} 2^{s_a} \right)]$$

$$< [\lg_2 \left( 2^{i_{j-1} + 1} \right)] \text{ since } i_j > i_{j-1}, \forall 2 \leq j \leq k$$

$$\leq i_j$$

Hence, the second point of the lemma is proved.

We now prove the final point of the lemma. Assuming that all nodes within the same set belong to the same activated connected set, it holds that if more than one node is awake initially, they will belong to the same activated set. The larger the size of the initial set, the less the overall running time. This holds because by the logic of the previous argument of consecutive sets, only if the number of subsequently woken up nodes is larger than the current batch, will we have a worst case scenario. Since in the worst case, running times will be added up, to get the worst case running time we should start with the smallest initial set so we can add more time to the running time. The smallest set we can start with occurs when only one node is awake initially.

Now that we’ve developed a worst case scenario for nodes to wake up, we describe how to bound the running time in this scenario. Nodes wake up in a sort of “snowballing” manner, which we now describe.

If a node successfully executes $Token-Passing-Transfer$ in a slot, wakes up nodes, and then participates with those nodes in a higher slot, this forces the slot required for the new nodes to successfully execute $Token-Passing-Transfer$ to be larger. A snowball occurs when this effect starts at slot 1 and continues to the last slot, i.e. a node successfully executes $Token-Passing-Transfer$ in slot 1, wakes up new nodes, they all participate in slot 2, wake up new nodes, all of them participate in slot 3, wake up new nodes, and so on until the final slot.

In order for this effect to last to the final slot $[\lg n] + 1$, there must be at least $2^{[\lg n]}$ nodes participating. If a node participates in one such snowball, it can not be active in a subsequent snowball. Therefore, after 2 snowballs, $2 \cdot 2^{[\lg n]} \geq n$ nodes successfully execute $Token-Passing-Transfer$.

Now we show that the time required for all nodes to wake up is bounded by $4nt_1$. We do this by showing that all nodes will successfully execute $Token-Passing-Transfer$ by the end of the $4nt_1^{th}$ phase. By running Algorithm $Wakeup$ for this many phases, we guarantee that all nodes will be woken assuming initially at least one node is awake.

From Lemma 3 we get the worst case scenario with respect to running time. It occurs when nodes wake up in groups such that they belong to the same activated connected set and furthermore nodes which wake up other nodes continue to participate in subsequent epochs with newly woken up nodes. Also, only one node is awake initially. The running time of one such snowball of nodes waking each other up and participating from slot 1 to slot $[\lg n] + 1$ is upper bounded by $\sum_{j=1}^{[\lg n]+1} t_j$ phases. From our earlier argument, we see that
2 snowballs is sufficient for all nodes to successfully execute Token-Passing-Transfer. The number of phases to execute 2 snowballs is

\[
2 \cdot \sum_{j=1}^{\lfloor \lg n \rfloor + 1} t_j = 2(t_1 + 2t_1 + 2^2t_1 + \ldots + 2^{\lfloor \lg n \rfloor}t_1)
\]

\[
= 2t_1 \left( \frac{2^{\lfloor \lg n \rfloor+1} - 1}{2 - 1} \right)
\]

\[
\leq 2(2n - 1)t_1
\]

\[
\leq 4nt_1
\]

Thus, if all nodes have successfully completed execution of Token-Passing-Transfer by the end of the \(4nt_1\)th phase, it implies that all nodes are awake after that phase. Since Algorithm Wakeup runs for that many phases, it is guaranteed that all nodes are awake by the end of the algorithm.

As for the running time, Algorithm Wakeup runs for \(4nt_1 = O(n \lg N)\) phases. Each phase consists of \(3(\lfloor \lg n \rfloor + 1)\) rounds. Thus, the total running time of Algorithm Wakeup is \(O(n \lg N \lg n)\) rounds.

### 3.2 Tree-Grower

Tree-Grower is given a network with an arbitrary number of awake nodes, with no knowledge of each other. It creates a forest of directed trees as follows. Initially, each node has no idea about its neighbors. The procedure takes two nodes which are close enough and silences one of the two. The silenced node becomes the child of the other node. This process is repeated until no more transmitting nodes are within range of each other. Progress occurs because there always exist two nodes which can communicate bidirectionally with each other. In the process, a forest of trees is created.

We say that two nodes engage in bidirectional communication with each other when each node hears the message of the other. Define one phase of the Tree-Grower procedure as the execution of the 4 ssf schedules required by a node to engage in bidirectional communication with another node and possibly become its parent or child. A node uses the first 2 ssf executions to determine who it can bidirectionally communicate with. The third and fourth ssf executions are used to decide which one of them becomes child and which one becomes parent (or neither). Nodes with smaller labels become parents. Amongst two nodes who are vying to become the parent of a node, the node which is heard first becomes the parent. A node is said to be active if it has not been silenced so far.

The leader of the entire tree is the root of that tree. The procedure forces at least two nodes in each phase to engage in bidirectional communication with each other until all nodes have either become children or will become leaders. After \(n\) phases of the procedure are complete, all those nodes which are not children are anointed as leaders. We restate Theorem 3 and prove it now.

**Theorem 3.** Assuming that the number of nodes given as input is an upper bound on the actual number of awake nodes in the network, the execution of Procedure Tree-Grower on this set of nodes results in creating a forest in \(O(n \lg N)\) rounds with the following properties:

1. Every node is either a leader or a child and belongs to exactly one tree.
2. Every tree has exactly one leader and there is at most one leader per grid box of the pivotal grid.
3. Every node knows the labels of its parent node and children.

#### 3.2.1 Proof of Theorem 3

We develop a claim and two lemmas which culminate in proving that at the end of the procedure, all nodes are either leaders or children.
Algorithm 2 Tree-Grower(No. of nodes $n$), run by each node $u$

1:  $\text{type}$ denotes the type of node, either $\text{leader}$ or $\text{neutral}$.
2:  $\text{my}\_\text{leader}$ denotes the parent of a node. For a root of a tree, $\text{my}\_\text{leader}$ is its own label.
3:  Tree $T$ stores the children of the node.
4:  Set $\text{type} \leftarrow \text{leader}$ and $\text{my}\_\text{leader} \leftarrow u$ \hspace{1cm} $\triangleright$ Initially all nodes are leaders.
5:  
6:  Execute the following $n$ times
7:     if $u$ is a leader then
8:         Clear values of potential children and potential parent
9:         Execute($N,c$)-SSF: Transmit $u$’s label and store labels heard from others in $i\_\text{hear}$
10:        Execute($N,c$)-SSF: Transmit values in $i\_\text{hear}$ and store other nodes’ $i\_\text{hear}$ info
11:        if A node $v$ is in $u$’s $i\_\text{hear}$ AND $u$ is in $v$’s $i\_\text{hear}$ then
12:            Add $v$ to $\text{bidir\_comm}$
13:            Execute($N,c$)-SSF: Transmit $u$’s label and store labels of nodes, $v$, heard that are in $\text{bidir\_comm}$
14:        else if $v < u$ AND $u$ has no potential parent then
15:            Store $v$ as potential parent
16:            Add $v$ as a potential child
17:     else
18:         Set $\text{msg}$ to $u$’s label, potential parent, and potential children.
19:         Execute($N,c$)-SSF: Transmit $\text{msg}$ and store list of nodes that mention $u$ in their $\text{msg}$
20:        if $u$’s potential parent, $v$, lists $u$ as a potential child then
21:            Set $\text{my}\_\text{leader} \leftarrow v$ and $\text{type} \leftarrow \text{neutral}$
22:     for Each of $u$’s potential children, $v$ do
23:         if $v$ transmitted that $u$ is its potential parent then
24:             Add $v$ to $T$ as a child
25:     else \hspace{1cm} $\triangleright$ $u$ became a child and is silenced.
26:     Do the following 4 times: Execute($N,c$)-SSF: Do nothing

return $<T,\text{type},\text{my}\_\text{leader}>$
Claim 2. When two nodes engage in bidirectional communication with each other, at least one of them will become a child.

Proof. When two nodes $u$ and $v$ engage in bidirectional communication with each other, one of them, say $u$, will have a larger label than the other. $u$ will always become a child. Either it will become $v$’s child, or else, if it engages in bidirectional communication with another node $z$, $z < v$, as well, it will become the child of that node. \qed

Lemma 5. If there exist active nodes within range of each other at the start of the current phase of the procedure, then at least two nodes will engage in bidirectional communication.

Proof. In the given phase, there will be at least two active nodes whose distance from each other is minimum among all possible distances between active nodes. By Theorem $1$, executing an $(N, c)$-ssf for an appropriate $c$ ensures that at least those two active nodes can engage in bidirectional communication with each other. \qed

Lemma 6. The following invariant holds for every phase $i, 1 \leq i \leq n$ of the protocol: At the beginning of every phase $i$ of the Tree-Grower procedure, either at least $i-1$ nodes have become children or all nodes which are not children will stay leaders.

Proof. Prior to the first phase, none of the nodes have decided to become children to other nodes. So the invariant holds.

Assume that the invariant holds at the beginning of the $i$th phase. Now, we need to prove that it holds at the beginning of the $i+1$th phase. Now either the first part or the second part of the condition held at the beginning of the $i$th phase. Let us analyze what can happen in each case.

1. At the beginning of the $i$th phase, at least $i-1$ nodes are children.

   Until the beginning of the $i$th phase, there were still nodes which could become children. Let us look at the two cases, namely (i) when there still exist nodes which can become children and (ii) when no more nodes can become children.

   (i) By Lemma $5$, there must exist two nodes within transmission range of each other who have not bidirectionally communicated with each other till now, but can communicate in the phase $i$. By Claim $2$, at least one of them becomes a child in phase $i$, and thus the number of nodes which are children at the beginning of phase $i+1$ is at least $i$. Hence, the invariant holds in this case.

   (ii) Assume that all nodes that could become children to other nodes became children by the beginning of some phase. The number of such nodes is $i-1$. Now, at the beginning of the $i$th phase, since no new nodes could become children, the number of children is still $i-1$. After the phase is over, there will still be $i-1$ children and all other nodes will remain leaders. Hence, the invariant holds in this case as well.

2. At the beginning of the $i$th phase, all nodes which aren’t children will stay leaders.

   If this condition is true at the beginning of the $i$th phase, it will remain true at the beginning of the $i+1$th phase and thus the invariant will hold true at the beginning of that phase. The only way the invariant won’t hold is if two leaders engage in bidirectional communication now and one of them becomes a child. However, by Lemma $5$ and Claim $2$, if there still exist nodes which can become children, they should’ve become children in the previous phases. Since no new node becomes a child, the condition remains as it was previously. Hence, the invariant still holds. \qed

By Lemma $6$, at the end of the Tree-Grower procedure, every node has either become a child or a leader. Note that it is possible for some leaders to have no children, but every child must have a parent. As for the remaining properties:
1. Since a node becomes a leader only if it has no parent at the end of the Tree-Grower procedure, the only way for one tree to have more than one leader is if a child chooses two nodes as its parents. However, this is impossible because each child chooses only one parent for itself. Also, since a child chooses only one parent, it may belong to only one tree. Hence, each node belongs to exactly one tree.

2. If there exist two nodes within a grid box of the pivotal grid, it means that they are within range of each other. If there are two leaders in one grid box of a pivotal grid after the algorithm is over, it means that two active nodes within range of each other did not participate in bidirectional communication with each other. By Lemma 5 and Lemma 6, and the fact that there are $n$ phases, there exists a phase of the procedure in which they will bidirectionally communicate with each other and one of them will become a child.

3. Any node becomes a child of another node only when it communicates with that node and both agree to the relationship. Therefore, no node can enter into a parent-child relationship with another node unless both nodes are aware of the existence of such a relationship.

The running time of Tree-Grower can easily be calculated. We run a for loop $n$ times. Within this for loop, we run 4 ssf schedules, each of which takes $O(\lg N)$ rounds. Therefore, the total running time is $O(n \lg N)$.

3.3 Tree-Cutter

Tree-Cutter takes the forest of trees created by Tree-Grower and reduces the height of every tree to at most one, i.e., forms stars, while creating new stars if necessary.

For a given leader, each of its immediate children is called its follower. A node can be a follower of only one leader. A node which is not a leader is said to align itself to a leader when it hears from a leader and decides to make that leader its parent. It then becomes that node’s follower. During the procedure, a node can have one of the following three statuses: leader, follower, or neutral. A neutral node is a node which has not aligned itself to a leader (or anointed as a leader yet).

Define one phase of Tree-Cutter as 5 executions of an ssf schedule. For a group of nodes, passing a token amongst themselves, the token is just a message sent or received by the nodes. In a given phase, a potential leader is a neutral node which has a token.

The Tree-Cutter procedure works as follows:

1. The first 3 executions are uniquely used by potential leaders to elect at most one leader among themselves per grid box of the pivotal grid, using a sub-procedure Potential-Leader-Election.

2. The next ssf execution is used to announce whether the node is a leader or someone’s follower.

3. The final ssf execution is used to pass the token, if any, to another node as follows. If the node is a neutral node or has no children, then it passes the token back to the node it received the token from. If the node has any children, then it passes the token to one of them in Depth First Search (DFS) manner. Note that, if the node became a leader during the execution of Potential-Leader-Election, then it generates and keeps a new token with itself.

The Potential-Leader-Election sub-procedure takes place amongst neutral nodes who receive a token. They initially are potential leaders and transmit that during an ssf execution. By the property of a strongly selective family, and since at most a constant number of tokens are present in each grid box, there exists a round for each token holder in which only it will transmit, successfully, to all others within its range. The nodes then execute a second ssf schedule and transmit the information of which nodes they heard in each round of the ssf schedule. Each token holder uses this information (and ssf schedule) to figure out which round of the ssf schedule was the round in which only it successfully transmitted. Then, one final ssf schedule is executed where the node transmits at most once during the entire execution. In this execution, if the number of the current round is equal to the round number in which the given node is the sole potential leader, then it becomes a leader.
leader within its range to transmit, then it transmits and becomes a leader. If before its round comes up, it hears another potential leader transmit, then it becomes a follower of that node. Once it becomes either a leader or a follower, it remains inactive for the remaining rounds. After all rounds are over, all nodes are either leaders or followers. Furthermore, if a node has become a leader, then every potential leader, within its range, who was not already a follower must become its follower. Since all nodes within a grid box are within range of each other, there can exist at most one leader per grid box.

Now, we restate Theorem 4 and prove it.

**Theorem 4.** Assuming that the nodes satisfy the properties described in Theorem 3, the execution of Procedure Tree-Cutter takes $O(n \lg N)$ rounds and results in the creation of stars that satisfy the following properties:

1. Every node is either a leader or a follower and belongs to exactly one tree.
2. Each tree is a star and has exactly one root called its leader.
3. There exists at most one leader node per grid box of the pivotal grid.

**Algorithm 3** Potential-Leader-Election(terminology type), run by each node $u$

1. **type** denotes the type of a node, either **leader**, **follower**, or neutral.
2. **my_leader** denotes the leader of a node. For a leader, **my_leader** is its own label.
3. If $u$ is a neutral node and has a token then
4.   Execute($N, c$)-SSF: Transmit that $u$ is a potential leader and store list of each node, $v$, who is a potential leader and what round, $\text{round\_heard}$, it transmitted in
5.   Execute($N, c$)-SSF: Transmit all pairs of $<v, \text{round\_heard}>$ heard
6. Determine in which round, $R$, $u$ alone was heard by every potential leader within range
7. Execute ($N, c$)-SSF: In each round of the execution, do the following:
8.   If $u$ is a neutral node then
9.     if current round = $R$ then $\triangleright$ $u$’s turn to transmit that $u$ is a leader.
10.     Transmit that $u$ is the leader
11.     $\text{type} \leftarrow \text{leader}, \text{my\_leader} \leftarrow u$
12.   else
13.     $\triangleright$ if $u$ hears a node $v$ say it’s the leader, then set $\text{type} \leftarrow \text{follower}, \text{my\_leader} \leftarrow v$
14.     else $\triangleright$ $u$ became either a leader or a follower.
15.     Do nothing
16.   else $\triangleright$ $u$ doesn’t participate in leader election.
17. Do the following 3 times: Execute($N, c$)-SSF: Do nothing
18. return $<\text{type}, \text{my\_leader}>$

### 3.3.1 Proof of Theorem 4

We first prove that at end of the procedure, there exists at most one leader per grid box of the pivotal grid. We then prove that at the end of the procedure, all nodes will be either leaders or followers. Subsequently, we prove that all trees formed and created are stars at the end of the procedure. Finally, we bound the running time of Tree-Cutter.

We now prove that there exists at most one leader per grid box of the pivotal grid. We show this using an intertwined induction proof between two invariants, where we use each to prove the other for the next step of the induction.

**Inv1:** At the beginning of any phase of Tree-Cutter, at most one leader can exist in a grid box of the pivotal grid.
Algorithm 4 Tree-Cutter(Tree $T$, Type $type$, Node my_old_parent, No. of nodes $n$), run by each node $u$

1: $type$ denotes the type of node, either leader, follower, or neutral.
2: my_old_parent denotes the parent of a node at the start of Tree-Cutter. For a leader, my_old_parent is its own label.
3: my_leader denotes the leader of a node. For a leader, my_leader is its own label.
4: Tree $T$ stores the children of the node. After Tree-Cutter is over, only leaders maintain $T$.
5: 
6: if $u$ is a leader then
7: Set my_leader ← $u$ and create a token
8: Execute the following $2 \cdot 947 \cdot (n + 1) - 1$ times
9: Case 1: If $u$ is a leader or follower
10: Potential-Leader-Election($type$)
11: if $u$ has a token then
12: Execute($N,c$)-SSF: Transmit $u$'s label, $type$, and my_leader. If leader, update $T$ by adding any nodes that have made $u$ their leader and removing those that made someone else their leader
13: if $u$ is a leader then
14: target ← one of the children of $T$, chosen in DFS manner.
15: else if $u$ had received the token from $u$'s leader then
16: target ← one of the as yet untraversed children of $u$'s tree $T$
17: else $u$ had received the token from my_old_parent.
18: target ← my_old_parent
19: Execute($N,c$)-SSF: Pass token to target
20: else
21: Execute($N,c$)-SSF: If leader, add any nodes that made $u$ their leader as children in $T$
22: Execute($N,c$)-SSF: Listen to see if $u$ receives a token
23: Case 2: Else if $u$ is a neutral node
24: if $u$ has a token then
25: $<type,my\_leader>$ ← Potential-Leader-Election($type$)
26: if $u$ is a leader then
27: Execute($N,c$)-SSF: Transmit that $u$ is a leader
28: Execute($N,c$)-SSF: Pass token back to whomever sent it to $u$
29: Generate and keep a new token for $u$
30: else $\triangleright$ Potential leaders who became followers instead.
31: Execute($N,c$)-SSF: Transmit that $u$ is a follower of $u$'s leader
32: Execute($N,c$)-SSF: Pass token back to whomever sent it to $u$
33: else
34: Potential-Leader-Election($type$)
35: Execute($N,c$)-SSF:
36: if $u$ hears a node $v$ say it is a leader AND $u$ is still a neutral node then
37: my_leader ← $v$, type ← follower
38: Execute($N,c$)-SSF: Listen to see if $u$ receives a token
39: if $u$ is a follower then Clear $T$
40: return $<T,type,my\_leader>$
Inv$_2$: At the beginning of any phase of Tree-Cutter, at most a constant number of tokens are present in any grid box of the pivotal grid.

We intertwine the proofs by assuming Inv$_2$ is true when proving Inv$_1$ holds for a given phase and by using Inv$_2$ to inductively prove Inv$_2$.

We now start with Inv$_1$ that at the beginning of any phase of the procedure, at most one leader can exist in a grid box of the pivotal grid. Assuming Inv$_2$ holds true, we prove the equivalent of Inv$_1$, captured by Lemma 8 that for any grid box of the pivotal grid, at most one node can become a leader. Every leader is either already present at the beginning of Tree-Cutter or created using Potential-Leader-Election. We first show in Lemma 7 that assuming a constant number of token holders (and by extension potential leaders) exist in any grid box of the pivotal grid at the beginning of a phase of Tree-Cutter, Potential-Leader-Election elects at most one leader per grid box of the pivotal grid containing potential leaders.

**Lemma 7.** If at most a constant number of potential leaders exist in any grid box of the pivotal grid at the beginning of a phase of Tree-Cutter, then at the end of the Potential-Leader-Election procedure in that phase, all those potential leaders become either leaders or followers. Furthermore, at most one leader will be created in those grid boxes that previously contained potential leaders.

**Proof.** For each potential leader, in the first ssf schedule, call the round in which it was heard by all other potential leaders within range its **main round**. Define **participating nodes** of a given phase as all potential leaders and those potential leaders which have become leaders or followers. For a given phase of Tree-Cutter, we show that the following invariant holds for every participating node at the beginning of each round of the execution of the third ssf schedule in Potential-Leader-Election: **At most one participating node will become a leader per grid box of the pivotal grid.** All participating nodes within range of a created leader are followers, though not necessarily of that leader.

Prior to the first round, since no potential leader has become a leader, the invariant holds true.

Assume that the invariant holds true up to the current round. If no node transmits in the current round, then the invariant continues to hold.

For a given grid box of the pivotal grid, if in the current round, a node within that grid box transmits, then all other potential leaders in that box become followers of that node since there are a constant number of potential leaders in any grid box and by Theorem 1 they hear the transmitter. If there were followers of other nodes within that grid box, they remain of the same type. So the invariant holds at the beginning of the next round.

Thus, after all rounds of the ssf schedule are executed, at most one of the potential leaders becomes a leader per grid box of the pivotal grid and all participating nodes within its range are followers, though not necessarily of it. Thus, all potential leaders in any given grid box have either become a leader or followers at the end of the Potential-Leader-Election procedure.

Now, we are ready to prove Inv$_1$ holds true in given phase assuming that Inv$_2$ holds true up to and including the current phase.

**Lemma 8.** If for all phases $\leq i$ of Tree-Cutter, there are a constant number of token holders in any grid box of the pivotal grid, then for each grid box of the pivotal grid, at most one node can become a leader in phase $i$. Furthermore, once a leader is elected in a grid box, all other nodes within its range and within its grid box must become its followers or another leader.

**Proof.** There are only two ways in which leaders may arise in a grid box:

1. Originally a leader existed in that grid box.

   At the start of Tree-Cutter, every grid box of the pivotal grid has at most one leader by Theorem 1. This leader will transmit that it is a leader and all nodes within range (other nodes in the grid box) will become followers of it if they weren’t already someone else’s followers. This is because other nodes will hear the transmission of the leader due to Theorem 1.
2. A leader was elected by the Potential-Leader-Election procedure. If a grid box of the pivotal grid has a leader, then it will announce its status when it was created and all surrounding nodes in the grid box will not be able to become potential leaders. If a grid box does not have a leader yet and tokens make their way to neutral nodes within that grid box, then Potential-Leader-Election will be run on those nodes. By Lemma \[\text{we are guaranteed that at most one leader will be elected per grid box and the remaining potential leaders within range will become followers. Then, during the rest of Tree-Cutter in that phase, the newly elected leader will transmit its status; since there are a constant number of token holders, and by extension transmitters in any grid box of the pivotal grid, Theorem \[\text{guarantees that other nodes within range will successfully hear its message and become followers of it if they weren’t already followers of another leader.}\]

We now prove Inv\(_2\), captured by Lemma \[\text{that at the beginning of each phase of Tree-Cutter, the number of tokens present in a grid box of the pivotal grid is at most some constant number. We do this by showing that the number of tokens that can travel to any grid box is a constant and the number of tokens that can be created in a given grid box is a constant. Thus, at any time, there are only a constant number of tokens in any grid box of the pivotal grid. To show that only a constant number of tokens may travel to any grid box of the pivotal grid, we argue that the farthest distance a token can travel is at most 2\(r\) away from its leader. This is assuming a constant number of tokens are present in any grid box of the pivotal grid during the phases that the token moves.}\]

**Lemma 9.** Consider a leader with a token in phase \(i\). That token can travel a distance of at most 2\(r\) away from its leader assuming that each grid box of the pivotal grid has a constant number of tokens in phases \(i - 1\), \(i\), and \(i + 1\).

**Proof.** We prove this by first constraining the type of nodes a leader can pass its token to and then constraining the distance the token can travel from its leader. The types of nodes that can exist in the network are leaders, neutral nodes, and followers. We show that a leader can only pass its token to a follower.

A leader cannot pass its token to another leader or neutral node. If the leader already existed prior to the current phase, then it would have already transmitted its status; and surrounding nodes would have aligned themselves to it, if they were not already aligned (and, thus could not become leaders). If the leader was created through Potential-Leader-Election in the previous phase of Tree-Cutter, then any neutral nodes in the previous phase with a token would have participated and can’t become leaders by Lemma \[\text{Thus, a leader, \(x\), can only pass its token to followers. There are two types of followers. Those who were previously \(x\)’s children in its tree \(T\), but have realigned themselves to another leader. These nodes will announce their realignment and send the token back to \(x\). The other type of node is a follower of \(x\). A follower will pass its token to a child within its sub-tree, if it has a sub-tree. The child will either be of type follower or neutral. It won’t be of type leader, since newly created leaders announce their new status immediately and cease to belong to their old sub-tree. If it is of type follower, then it will transmit its realignment and return the token. If it is of type neutral, after the potential leader election is done, it will either be a follower of another leader or a leader itself. In either case, it will return the token back. Thus, any token can move at most two hops away from its leader. Since the maximum distance of each hop is at most \(r\), any token can travel a distance of at most 2\(r\) away from its leader. Note that both times the token is passed to reach a distance 2\(r\) away from the leader, since there are a constant number of tokens and by extension transmitting nodes in any grid box of the pivotal grid, by Theorem \[\text{the passing of the token (message transfer) is successful.}\]

We are now ready to prove Inv\(_2\). We use a strongly inductive proof where, assuming Inv\(_2\) holds true for all phases up to and including the given phase, we use it and Inv\(_1\) (which we proved holds true for a given phase when the Inv\(_2\) holds true for all phases up to and including that phase) to prove that Inv\(_2\) holds true for the next phase of Tree-Cutter.
Lemma 10. The following invariant holds at the beginning of any phase of the Tree-Cutter procedure: the number of tokens present in a grid box of the pivotal grid is at most some constant number.

Proof. Prior to the start of the first phase, there is at most one token per grid box of the pivotal grid (since at most one leader per grid box exists by Theorem [3]).

Assume that at the beginning of all phases $\leq i^{th}$ phase, we began with a constant number of tokens per grid box. Now, we need to show that at the beginning of the $i+1^{th}$ phase, we also begin with a constant number of tokens.

By Lemma [8], the maximum distance a token can travel is at most $2r$ away from its leader. For a given grid box $b$, there are 44 surrounding grid boxes from which a token may make its way into $b$ by traveling a distance of at most $2r$. Therefore, only a constant number of tokens can enter any grid box by the beginning of phase $i$. By Lemma [8], a grid box can have at most only one new leader (and by extension token) created within its boundaries. Therefore, the total number of tokens that can be present within a grid box at the beginning of the next phase is some constant number.

Lemma [8] guarantees that so long as we start any phase with a constant number of token holders per grid box, the number of leaders elected per grid box is at most one. By Lemma [11], the number of the token holders in any grid box at the beginning of any phase of the Tree-Cutter procedure is at most a constant number. Thus, after the execution of the Tree-Cutter procedure, there exists at most one leader per grid box of the pivotal grid.

We now prove that after the execution of Tree-Cutter, all nodes become either leaders or followers. Let $K = 947$. Consider the following invariant, which holds for $1 \leq i \leq n+1$.

$Inv_3$: At the beginning of phase $2Ki$ of the Tree-Cutter procedure, at least $i-1$ nodes have either become a leader and transmitted that they are a leader or become a follower and transmitted that they are a follower of another node.

Prior to the $2K^{th}$ phase, $Inv_3$ holds trivially.

Assume that $Inv_3$ holds up to the beginning of the $2Ki^{th}$ phase. We show that it holds at the beginning of the $2K(i+1)^{th}$ phase. At the beginning of the $2K(i+1)^{th}$ phase, at least $i-1$ nodes have either become a leader and transmitted or become a follower and transmitted.

Now, we want to find out the amount time it takes for a token to reach a node which can be a potential leader or a follower which has not transmitted yet, assuming that the tree to which that token belongs has not been fully explored already. All nodes are either leaders, followers or neutral nodes. We want to show that there exists a token which will definitely reach a node which has not transmitted yet within $2K$ phases of the Tree-Cutter procedure or else all nodes have transmitted. Consider a token at the start of the $2K^{th}$ phase. This token can be either present in a leader, a follower, or a neutral node. Let us consider these cases:

1. **In a leader or follower:** If the leader has not yet transmitted, then it will transmit at the beginning of this phase and $Inv_3$ will hold at the beginning of the $2K(i+1)^{th}$ phase. If the follower has not yet transmitted, it originally belonged to its leader’s tree and it will transmit at the beginning of this phase and the invariant will hold at the beginning of the $2K(i+1)^{th}$ phase. If the follower has already transmitted, i.e. it originally belonged to another leader’s tree and has realigned itself to its current leader, then we must find the maximum amount of time it takes before we find a node which has not transmitted. Note that the nodes which don’t increase the value of $i$ of our invariant are those nodes which have not been explored in the current tree, but have already transmitted. Such nodes are those that were previously aligned to other leaders and subsequently transmitted their realignment to the current leader. Let us call these nodes **disloyal nodes** for the duration of this proof. We upper bound the number of disloyal nodes we encounter before we encounter a node which has not transmitted or till we finish exploring the tree. First, consider how disloyal nodes come to be. They previously belonged to other leaders, but were within range of the leader to which the current token belongs. Let us call this leader $x$. Now, at most 21 grid boxes are within range of $x$. Furthermore, by Lemma [9] and [10], any grid box can receive tokens from at most 44 unique leaders. So for each of these 21 grid boxes, it is possible that at most 45 nodes (44 entering tokens plus one originating token) realigned to $x$. Therefore, we can
encounter at most 945 disloyal nodes in one tree. If the token is at the leader, in the worst case it has
to traverse all disloyal nodes before it gets to a node which hasn’t yet transmitted. This takes at most
$2 \cdot 945 + 1 < 2 \cdot K$ phases. If the token is at a follower which has already transmitted, then the number
of phases required to reach a node which hasn’t yet transmitted is even less. If no non-disloyal node
is found after all tokens have traversed this distance, it means that all trees have been traversed and
that all nodes have transmitted that they are leaders or followers. This is because, if no non-disloyal
node is found, then all nodes that were traversed were disloyal nodes. Since we have traversed all the
disloyal nodes prior to the start of the next phase, it is implied that there are no new nodes to traverse
at the beginning of the $2K(i + 1)^{th}$ phase. Hence, the invariant would hold at the beginning of the
$2K(i + 1)^{th}$ phase.

2. In a neutral node: In this case, the node is a potential leader. Then, by the Potential-Leader-Election
procedure, at the end of this phase the node will become either a leader or a follower and transmit.
So by the beginning of the $2K(i + 1)^{th}$ phase, at least $i$ nodes have become followers or leaders and
transmitted.

At the end of the procedure, we would be in the $2K(n + 1)^{th}$ phase of the Tree-Cutter procedure and by
Inv3, $n$ nodes would have declared themselves leaders or followers. Thus, all nodes would have been covered.
Furthermore, since nodes can be followers of only one leader, each node belongs to exactly one tree.

We now prove that every tree is a star with exactly one root node. Only leaders can be a root and
each tree has only one leader, thus every tree has one root. We now prove that the trees are of height at
most one. Every node becomes either a leader or a follower by the end of Tree-Cutter. Because of this, all
trees contain either leaders or followers. Since any tree can have at most one leader, all its children must be
followers. Furthermore, a follower cannot have any children after the execution of the Tree-Cutter procedure,
because any of its children will either become leaders, in which case they will have their own trees rooted at
themselves or else followers of some leader, in which case they will be children of that leader. Thus, at the
end of the Tree-Cutter procedure, every tree is a star.

Regarding the running time, there are $O(n)$ phases, since $K$ is a constant. Each phase of the proce-
dure consists of 5 executions of an ssf schedule. Each execution of an ssf schedule takes $O(\lg N)$ rounds
to complete. Thus, the entire procedure takes $O(n \lg N)$ rounds to complete. This concludes the proof of
Theorem 4. □

We prove one more lemma about the Tree-Cutter procedure which will be used in a subsequent section on
the backbone algorithm.

**Lemma 11.** At the end of Tree-Cutter, assuming that there exist at least two leaders, there exists at least
one other leader within 3 hops of any leader.

**Proof.** Consider a leader $x$ at the end of Tree-Cutter procedure. All children of that leader would be made
followers of either that leader or another leader. If they were made children of another leader, that means
that in 2 hops from $x$ there exists a leader. Let us assume that this is not the case. Now consider the
followers of $x$. It must occur that there exists one node within range of one of the followers of $x$ or else
there are no more nodes. If this is not the case, it means that there exist nodes which have no means of
communicating with $x$ and this implies that the communication graph is disconnected. But according to our
model, that is not the case. Therefore, let us consider that there exists at least one node within range of one
of the followers assuming there exist nodes which are not followers of $x$. The node can only be a leader or
a follower by Theorem 4. Then within 3 hops of $x$, there exists a leader. Therefore, for every leader, there
exists at least one other leader within 3 hops of that leader. □

3.4 Token-Passing-Transfer

The goal of Token-Passing-Transfer is to ensure that every participating node’s message is successfully
transmitted. It is used in two different ways depending on if the application is wakep or multi-broadcast.
When used for wakeup, the message that needs to be transmitted is a special *wakeup* message (and some additional info) which allows neighboring nodes to wakeup. Here, all participating nodes are transmitting the same message and so all that is required is that all participating node successfully transmit this wakeup message once. When used for multi-broadcast, it is again used in two ways. The first way requires that a specific message be transmitted by each participating node and other messages heard are stored. We call such messages that need to be sent *single-transmit* messages. The second way it is used requires that each participating node successfully transmits its message, if any, and successfully transmits any message heard during the execution of the procedure. We restate Theorem 5 and prove it now.

**Theorem 5.** Assuming that the participating nodes are in a configuration characterized by Theorem 4, the Token-Passing-Transfer procedure takes $O(n \log N)$ rounds to complete and ensures two things:

1. If wakeup or single-transmit message needs to be sent, then each participating node successfully transmits the message. Further, any incoming messages are successfully heard.

2. If number of participating nodes is bounded by 121, then every participating node successfully transmits its message and every message it may hear during execution.

**Algorithm 5** Token-Passing-Transfer(Tree $T$, Type $type$, Node $my\_leader$, No. of nodes $n$, Message $msg$), run by each node $u$

1: $type$ denotes the type of node, either *leader* or *follower*.
2: $my\_leader$ denotes the leader of a node. For a leader, $my\_leader$ is its own label.
3: Tree $T$ stores the children of a leader. For followers, $T$ is empty.
4: $msg\_to\_transmit$ stores all messages that need to be transmitted

5: $msg\_to\_transmit \leftarrow msg$
7: if $u$ is a leader then Create token
8: Execute the following 488$n$ times
9: if $u$ has a token then
10: Execute($N,c\_SSF$): If $msg$ contains wakeup or single-transmit message, then transmit $msg$. Else, transmit message from $msg\_to\_transmit$ with lowest associated label. Remove this message from $msg\_to\_transmit$ after execution of SSF. Add any messages heard to $msg\_to\_transmit$ if not heard before
11: else
12: Execute($N,c\_SSF$): Add any new messages heard to $msg\_to\_transmit$ if not heard before
14: Execute($N,c\_SSF$): Listen to see if $u$ receives a token

3.4.1 Proof of Theorem 5

We first prove that if a wakeup message or single-transmit message needs to be transmitted, all participating nodes will transmit the message by the end of procedure. We subsequently prove that if the number of participating nodes is bounded by 121, then each participating node will successfully transmit its message as well any message it could hear over the course of the procedure.

We now prove if each participating node needs to transmit a wakeup or single-transmit message, it will successfully do so by the end of the procedure. Further, we also show that any participating node will hear any incoming messages. Since tokens can be passed to nodes at most one hop away from a leader, the maximum number of nodes with tokens that can transmit from a particular grid box is bounded by a constant. Furthermore, since there are only a constant number of grid boxes within range of a grid box
containing a node, the number of nodes within range of a given node which can transmit at any given time is bounded by a constant. By Theorem 1, by choosing a sufficiently large constant $c$, using an $(N, c)$-ssf, we can guarantee that a participating node will successfully transmit its wakeup message and hear any incoming messages.

We now show that if the number of participating nodes is bounded by 121, then each node will successfully transmit all messages it has and hears during the course of the procedure. Since there are at most $n$ nodes totally in the network, there can be at most $n$ messages that are heard and need to be transmitted over the course of the procedure. Each message, once it is heard by one of the participating nodes, needs to be transmitted to the remaining participating nodes and then transmitted by each participating node. So the token must be passed around all the participating nodes twice. Since the number of participating nodes is bounded by 121, it will take at most $484n (\leq 488n)$ executions of the two $(N, c)$-SSFs to achieve this.

As to the running time, every node executes two $(N, c)$-ssf schedules of length $O(\lg N)$ rounds $484n$ times, for a total running time of $O(n \lg N)$ rounds.

4 Multi-Broadcast

The problem of multi-broadcast is to transmit the information held by $k$ nodes, $1 \leq k \leq n$, to all nodes in the network. Our result is captured by the following theorem.

**Theorem 6.** Assuming all nodes are initially awake, the Algorithm Multi-Broadcast achieves multi-broadcast in $O(n \lg N)$ rounds.

This algorithm takes the trees formed by Tree-Cutter and uses them to transmit information through the network. Each leader engages in four token passing routines with its followers. In the first routine, every node transmits the label of its leader to surrounding nodes. The second routine is used by followers to transmit all pairs of leader labels it heard in the previous step and which node it heard it from to its leader. Each leader then chooses the minimum set of its children such that for each leader label heard, there exists a child that is a neighbor of a node originally having that leader label. Each leader generates a new tree $T'$ with this subset of its children. The third routine is used by each leader to collect all messages that need to be transmitted by all its children. In the final routine, the new trees generated previously are used to transmit all messages throughout the network.

4.1 Proof of Theorem 6

We argue the correctness of Multi-Broadcast as follows. We consider the case of a single message that needs to be broadcast using the original stars of each leader and bound the time taken until all nodes have received the message by $O(n \lg N)$ rounds. Since each modified star $T'$ is at most as large as its original star $T$ while still guaranteeing network connectivity, the previous bound acts as an upper bound to single message broadcast on these modified stars. We then show that the size of these modified stars $T'$ constructed by each leader consist of at most 120 children each and use that fact to show that the message can only be delayed by an additional $O(n \lg N)$ rounds by other messages that need to be broadcast. Thus, regardless of how many messages need to be broadcast, by running Token-Passing-Transfer using the modified stars $T'$, we can achieve multi-broadcast.

Now consider a single message that needs to be broadcast throughout the entire network during the final call to Token-Passing-Transfer. We show that this message will be reach all nodes in at most $8nc_1 \lg N$ rounds. Since the stars used in the final call to Token-Passing-Transfer are subgraphs of the stars produced at the end of Tree-Cutter, we can consider the running time bound on the original stars as an upper bound on the time taken on the modified stars. We consider all stars created by Tree-Cutter and arguing the following properties after Token-Passing-Transfer has finished. We show that if a message reaches one node of a star, it will spread to and be transmitted by all nodes of that star in a given amount of time. We then argue that so long as all nodes that have heard the message so far transmit it, the message will be propagated to nodes in stars where no node has heard the message until then. Finally, we bound the time taken to propagate this
message to all nodes in the network and thus by running Token-Passing-Transfer for an appropriate amount of time, we ensure the success of broadcast.

Consider all the stars that exist within the grid after Tree-Cutter is executed. Consider a single message that needs to be broadcast in the network. Order the stars in nondecreasing order of the time at which the first node of that star will eventually hear the message. Order the stars \( T_1, T_2, \ldots, T_k \) and let \( s_1, s_2, \ldots, s_k \) be the number of nodes contained in the respective stars. Note that \( n = \sum_{i=1}^{k} s_i. \)

We first upper bound the time taken by all nodes within a given star to transmit a message recently heard by one node of that star.

**Lemma 12.** From the time one node in a star \( T_i \) hears a message, it takes at most \( 8s_i c_1 \lg N \) time for all nodes in the star to transmit that message, where \( c_1 \) is a constant.

**Proof.** Assume that node \( x \) in the star receives the message. If it is not the root of the star, then it will take at most \( 4s_i c_1 \lg N \) rounds until \( x \) receives the token. This is because there are \( s_i \) nodes totally and the token can reside in each of the nodes for \( 2c_1 \lg N \) rounds at a time, assuming eachssf schedule takes \( c_1 \lg N \) rounds to complete. Furthermore, for every time a child gets the token, the root gets it back, so we can treat it as the token having to pass through at most \( 2s_i \) nodes before \( x \) receives the token. Therefore, it would take at most \( 4s_i c_1 \lg N \) rounds for \( x \) to receive the token.

Once \( x \) receives the token, it will transmit. Once it transmits, its leader will know the message. \( x \) will then pass the token back to its leader, who will then transmit. Once the leader transmits, all nodes in the star will know the message. Now, the leader will pass the token to all its children, who have not transmitted the message, who will transmit once they get the token. Since it takes at most \( 4s_i c_1 \lg N \) rounds to pass the token to all nodes, the total number of rounds taken for all nodes in a star to transmit a message once one of them hears the message is at most \( 8s_i c_1 \lg N. \)

We now show that a message originating at one star will eventually reach every star.

**Lemma 13.** Consider a message being transmitted in the network. The following invariant holds true for \( 1 \leq i \leq k. \) Let \( T_1 \) to \( T_i \) each have at least one node who has heard the message at a particular time \( t. \) Now after all nodes in the stars \( T_1 \) to \( T_i \) have transmitted the message, there exists at least one node in at least star \( T_{i+1} \) which has heard the message or else all nodes in the network have transmitted the message.
Proof. Initially one node of $T_1$ knows the message. Once all nodes within $T_1$ transmit the message, at least one new node outside the star $T_1$ would’ve heard the message or else all nodes have transmitted. If all nodes in the network belong to $T_1$, then after all nodes transmit, the invariant holds. Assuming that there exists a node outside $T_1$, then at least one node from at least $T_2$ must hear the message. This is because if this does not occur, it implies that all other nodes are at a distance $> r$ from all nodes within star $T_1$. This is because all nodes within range of a node should hear that node given that we’re running $(N,c)$-ssfs and there are only a constant number of token holders transmitting per grid box. However, this implies that the communication graph of all nodes is disconnected and that is a contradiction to the assumptions of our model. Hence, it must occur that at least one node not belonging to $T_1$, and thus belonging to $T_2$, is within range of one of the nodes of $T_1$ and thus hears the message. Therefore, initially, the invariant holds.

Assuming that the invariant holds true for the first $i$ stars, then it must be the case that at least one node from star $T_{i+1}$ hears the message after all nodes from $T_1$ to $T_i$ transmit or all nodes in the network have transmitted. If all nodes have transmitted, then there won’t be a star $T_{i+1}$ and the invariant holds. Assume that this is not the case. Then one node from $T_{i+1}$ heard the message. Now we should show that it is either the case that once all nodes in $T_{i+1}$ transmit the message, either all nodes in the network have transmitted or else at least one node from $T_{i+2}$ heard the message. If all nodes in the network belong to the first $i + 1$ stars, then once all nodes from those stars transmit, all nodes in the network would have transmitted. In case there exists a node outside the first $i + 1$ stars, then due to the earlier argument of the connectedness of the underlying communication graph and the use of $(N,c)$-ssfs, it is the case that at least one node from at least $T_{i+2}$ hears the message. Therefore, the invariant holds.

Now we are ready to show that a single message originating at some node will reach all nodes after running Multi-Broadcast.

Lemma 14. After the execution of Tree-Grower and Tree-Cutter, the final call to Token-Passing-Transfer achieves broadcast of a single message starting at any node in $8sc_1 \log N$ rounds.

Proof. Lemma 12 guarantees that once one node within a star hears the message, eventually all nodes within that star will hear the message and Lemma 13 guarantees that so long as all nodes within a star transmit, at least one node from a new star will hear the message. Initially, at least one node of one star knows the message. Therefore, by continuous application of Lemma 12 followed by Lemma 13 it is clear that eventually all nodes of all stars will hear the message. We want to calculate the worst case time that it might take for this to happen. This occurs when all nodes in star $T_i$ must transmit before any node in $T_{i+1}$ hears the message and this occurs for all $1 \leq i \leq k - 1$. This implies that the total time for all nodes to transmit the message (and by extension hear it)

$$= \sum_{i=1}^{k} 8s_ic_1 \log N$$

$$= 8c_1 \log N \left( \sum_{i=1}^{k} s_i \right)$$

$$= 8c_1 \log N(n).$$

Since each node runs Token-Passing-Transfer for this long, a given message originating at some node will be broadcast to all nodes.

We now argue that the additional delay imposed on any message to travel through the network is not too much.

Lemma 15. Any message being broadcast via the final call to Token-Passing-Transfer may be delayed by other messages by at most $968c_1 \log N$ rounds during its transit.

Proof. We now show that each star $T'$ generated by a leader in the procedure consists of at most 120 children. For a given leader $x$ in a grid box, there are at most 120 surrounding grid boxes within distance $3r$ from it. Thus, there are at most 120 other leaders within 3 hops from any given leader and at most 120 children of
Each message, once it is heard by one of the participating nodes, needs to be transmitted to the remaining participating nodes and then transmitted by each participating node. So the token must be passed around all the participating nodes at most twice. We say a message $m_1$ delays another message $m_2$ if for some star there is a node that has both $m_1$ and $m_2$ and needs to transmit $m_1$ before transmitting $m_2$. Once a message $m_1$ delays a message $m_2$ from being transmitted, $m_1$ never subsequently delays $m_2$ again. The only reason $m_2$ might have to wait at another node for $m_1$ to be transmitted is because there existed another message $m_3$ with a lower associated label than $m_1$ or $m_2$. In this case, the delay to $m_2$ is attributed to $m_3$. For any given message, there are at most $n-1$ other messages that can delay it. Thus an additional delay of at most $968nc_1 \lg N$ rounds may be imposed on a message as it travels through the network.

Thus by Lemmas 14 and 15 any and every message in the network is successfully broadcast in $976nc_1 \lg N$ rounds. Thus algorithm Multi-Broadcast achieves multi-broadcast. The running time of Multi-Broadcast is $O(n \lg N)$ rounds as each of the 3 procedures used takes $O(n \lg N)$ rounds.

5 Backbone Subnetwork

There are three algorithms required to turn a network into a functioning backbone. Algorithm Backbone-Creation creates the backbone. Algorithm Backbone-Message-Exchange is used to communicate between nodes within the backbone. Algorithm Backbone-Message-Transmit is used to communicate between nodes within the backbone and nodes outside of it. Algorithm Backbone-Message-Exchange and Algorithm Backbone-Message-Transmit are constantly run in a time division multiplexing fashion.

5.1 Backbone-Creation

Algorithm Backbone-Creation uses the stars produced from the Tree-Cutter algorithm and turns them into a backbone subnetwork.

Theorem 7. Assuming all nodes are initially awake, the Algorithm Backbone-Creation creates a backbone subnetwork in running time $O(n \lg N)$ rounds.

Define a connector node $v$ for a given leader $u$ as a follower of $u$ which, in the backbone, is a node in the path between $v$ and another leader at most 3 hops away from it. Algorithm Backbone-Creation sets up all leaders as members of the backbone and then for each leader, chooses a constant size subset of its followers to be connectors.

Once Tree-Grower and Tree-Cutter are run, Algorithm Backbone-Creation proceeds in 4 stages. Let us assume each stage consists of a leader passing the token to all its followers and allowing them as well as itself to transmit. The first stage consists of all followers announcing that they are followers of a given leader. The second stage consists of followers announcing their leaders and $< \text{leader}, \text{follower}>$ pairs they hear in the previous stage. Leaders designate some their followers to act as connectors to other leaders. The third stage consists of leaders and their followers transmitting who is a connector and who isn’t. The final stage consists of all nodes belonging to the backbone transmitting that they belong. Thus, nodes develop a local view of the backbone.

5.1.1 Proof of Theorem 7

The proof of the theorem proceeds as follows. First we prove message transmission during the algorithm will be successful, i.e. the conditions to use SSF based dilution are satisfied in the algorithm. We then go on the prove that the algorithm creates a network that satisfies the properties of a backbone. Finally, we bound the running time of the algorithm.
Algorithm 7 Backbone-Creation(No. of nodes $n$), run by each node $u$

1: $\textit{type}$ denotes the type of node, either $\textit{leader}$ or $\textit{follower}$.
2: $\textit{my\_leader}$ denotes the leader of a node. For a leader, $\textit{my\_leader}$ is its own label.
3: Tree $T$ stores the children of a leader. For followers, $T$ is empty.
4: Tree $H \leftarrow \bot$ $\triangleright H$ is the local view of the backbone network from $u$’s perspective. If $u$ is part of the backbone, then $H$ is rooted at $u$ and its children are all nodes in the backbone network within range of $u$. If not then $H = \bot$.
5: $\langle T, \textit{type}, \textit{parent} \rangle \leftarrow \text{Tree-Grower}(n)$
6: $\langle T, \textit{type}, \textit{my\_leader} \rangle \leftarrow \text{Tree-Cutter}(T, \textit{type}, \textit{parent}, n)$
7: Execute($N, c$)-SSF: If $u$ is a leader, transmit the number of $u$’s children. If $u$ is a child, listen for this value from $u$’s leader.
8: Set $\textit{num\_phases} \leftarrow 2 \ast$ (the no. of children of $u$’s leader)
9: **Stage 1:**
10: Case 1: $u$ is a leader
11: 
12: for Every child in $T$ do
13: Execute($N, c$)-SSF: Store a list of all labels of other leaders’ followers and labels of their leaders that $u$ hears. Add these followers to $u$’s list of connectors
14: Execute($N, c$)-SSF: Pass token to the child
15: Do the following 2 times: Execute($N, c$)-SSF: Do nothing
16: Case 2: $u$ is a follower
17: Execute the following $\textit{num\_phases}$ times
18: if $u$ has a token then
19: Execute($N, c$)-SSF: Transmit that $u$ is a follower of $u$’s leader and store a list of all labels of other leaders’ followers and labels of their leaders that $u$ hears
20: Execute($N, c$)-SSF: Pass token back to $u$’s leader
21: else
22: Execute($N, c$)-SSF: Store a list of all labels of other leaders’ followers and labels of their leaders that $u$ hears
23: Execute($N, c$)-SSF: Listen to see if $u$ receives a token
24: **Stage 2:**
25: Case 1: $u$ is a leader
26: for Every child in $T$ do
27: Execute($N, c$)-SSF: Do nothing
28: Execute($N, c$)-SSF: Pass token to the child
29: Execute($N, c$)-SSF: Store list of all triplets $<u$’s follower, other leader’s follower, other leader> that $u$ hears from $u$’s children
30: Execute($N, c$)-SSF: Do nothing
31: Case 2: $u$ is a follower
32: Execute the following $\textit{num\_phases}$ times
33: if $u$ has a token then
34: Execute($N, c$)-SSF: Transmit the list of all other leader’s followers and their leaders that $u$ heard in Stage 1
35: Execute($N, c$)-SSF: Pass token back to $u$’s leader
36: else
37: Execute($N, c$)-SSF: Do nothing
38: Execute($N, c$)-SSF: Listen to see if $u$ receives a token
Stage 3:

Case 1: \( u \) is a leader

Calculate the shortest routes to every other leader \( u \) has heard about. Add intermediary nodes to list of connectors

for Every child in \( T \) do

Execute\((N, c)\)-SSF: Transmit the list of connectors

Execute\((N, c)\)-SSF: Pass token to the child

Execute\((N, c)\)-SSF: If \( u \) hears of a node within range of it being a connector, add it to \( u \)'s list of connectors

Execute\((N, c)\)-SSF: Do nothing

Case 2: \( u \) is a follower

Execute the following \textit{num\_phases} times

if \( u \) has a token then

Execute\((N, c)\)-SSF: Transmit the list of \( u \)'s leader's connectors and listen to see if \( u \) is designated a connector by another leader

Execute\((N, c)\)-SSF: Pass token back to \( u \)'s leader

else

Execute\((N, c)\)-SSF: Store the list of connectors \( u \)'s leader transmits and listen to see if another leader designated \( u \) as a connector

Execute\((N, c)\)-SSF: Listen to see if \( u \) receives a token

Stage 4:

Case 1: \( u \) is a leader

Set \( u \) as root of \( H \)

for Every child in \( T \) do

Execute\((N, c)\)-SSF: Transmit that \( u \) belongs to the backbone

Execute\((N, c)\)-SSF: Pass token to the child

Execute\((N, c)\)-SSF: Add any nodes \( u \) hears from as children of \( H \) and to \( u \)'s list of connectors if they are not already there

Execute\((N, c)\)-SSF: Do nothing

Case 2: \( u \) is a follower

Execute the following \textit{num\_phases} times

Case a: \( u \) is a connector and has a token

Execute\((N, c)\)-SSF: Transmit that \( u \) is a connector. Add any nodes \( u \) hears from as children of \( H \)

Execute\((N, c)\)-SSF: Pass token back to \( u \)'s leader

Case b: \( u \) is a connector and doesn’t have a token

Set \( u \) as root of \( H \)

Execute\((N, c)\)-SSF: Add any nodes \( u \) hears from as children of \( H \)

Execute\((N, c)\)-SSF: Listen to see if \( u \) receives a token

Case c: \( u \) is not a connector

Execute\((N, c)\)-SSF: Do nothing

Execute\((N, c)\)-SSF: If \( u \) has a token, pass it back to \( u \)'s leader. If \( u \) doesn’t have a token, listen to see if \( u \) receives one

return \(< H >\)
The algorithm initially calls *Tree-Grower* and *Tree-Cutter*. As proved earlier, those procedures work as intended. We show that subsequently in the algorithm, when a node transmits a message, nodes within range of it can hear that message.

**Lemma 16.** For Algorithm *Backbone-Creation* post the call to *Tree-Cutter*, using an \((N,c)\)-ssf schedule for a sufficiently large constant \(c\), where only nodes with tokens can transmit, allows all nodes within range of a token holder to hear its message.

**Proof.** We show this by showing that the number of tokens that can be active in a grid box of the pivotal grid in any round of Algorithm *Backbone-Creation* post the call to *Tree-Cutter* is a constant. In Algorithm *Backbone-Creation*, post the call to *Tree-Cutter*, every leader has one token associated with it and can only send this token to nodes one hop away from it. Since there exists at most one leader per grid box of the pivotal grid at the end of *Tree-Cutter*, the number of tokens coming into a node can be at most 20 plus one token already belonging to a leader of the box. Thus, at most 21 tokens, a constant number, can be active in any grid box at any given time. By Theorem 1, we know that using an \((N,c)\)-ssf schedule for a sufficiently large \(c\) allows all nodes within range of a token holder to hear its message.

We now prove that *Backbone-Creation* indeed creates a backbone. According to Jurdziński and Kowalski [21], the properties that need to be satisfied for a network to be considered a backbone are:

1. The nodes of the backbone, \(H\), form a connected dominating set which induces a subgraph of the communication graph, \(G\), and have a constant degree relative to other nodes within the backbone.
2. The number of nodes in \(H\) is \(O(s_{c,d})\), where \(s_{c,d}\) is the size of the smallest connected dominating set of the communication graph.
3. Each node of \(G\setminus H\) is associated with exactly one node of \(H\) which acts as its entry point into the backbone.
4. The asymptotic diameter of \(H\) is same as that of \(G\).

We show that each of these properties are satisfied.

1. We first show that \(H\) is a connected dominating set in (a). In (b) we prove that the internal degree of nodes in \(H\) is constant. In (c), we argue that \(H\) induces a subgraph of \(G\).

(a) By Lemma 11, if there exists more than one leader, then for every leader, there exists at least one other leader within 3 hops of it. Since \(G\) is connected, it must be the case that there exists a path between any two leaders at most 3 hops apart. Therefore, if we show that the algorithm connects every leader to all leaders within 3 hops of it, then \(H\) is connected. Since all followers are within range of at least one leader, \(H\) would be a connected dominating set.

We now prove that *Backbone-Creation* connects every leader to all leaders within 3 hops of it. In Algorithm *Backbone-Creation*, once leaders gain knowledge of leaders in their vicinity, they will create connections to all such leaders. We will now show that leaders will have knowledge of all leaders within 3 hops of themselves. Note that leaders cannot exist within range of each other, i.e. within 1 hop of each other at the end of *Tree-Cutter*.

Now, every leader \(x\) can hear from nodes within range of it. These nodes are either its followers or other leaders’ followers. If a node is a follower of another leader \(y\), then a connection can be made to \(y\) through one of \(y\)’s followers. If a node is \(x\)’s follower, then if there exists another leader \(y\) within range of that node, then \(y\) can be connected to through one of \(x\)’s followers. So all leaders within 2 hops of a leader will be connected to it.

Now consider the case where two leaders are 3 hops apart. Since two leaders cannot be in range of each other, this implies that the nodes comprising the 2 intermediate hops are followers. If there exists a follower within range of each leader and these followers are within range of each other, then in the first stage of *Backbone-Creation*, these two followers will become aware of each other.
In the second stage, the leaders will become aware of each other and the possible connection between them. Since leaders choose connections so that all known leaders are connected to, these leaders will become connected to each other. Thus, if two leaders exist within 3 hops of each other, they will be connected to each other. Thus, $H$ is a connected dominating set.

(b) To prove that the degree of any node in $H$ relative to other nodes in $H$ is a constant, let us consider a leader $x$. It will be surrounded by at most some $X$ leaders who are at most 3 hops from it. For each such leader $y$, $x$ specifies one of its followers as a connector to $y$ and $y$ may specify another of $x$’s followers as a connector to $x$. Thus, there can be at most $2X$ connectors amongst $x$’s followers. Similarly, each of these $X$ leaders can also have $2X$ followers. Since leaders cannot be within range of each other at the end of Tree-Cutter, the maximum number of nodes $x$ could be connected to is $2X + 2X^2 = O(X^2)$. Similarly, any connector in $H$ will be surrounded by at most $2X + 2X^2 + X = O(X^2)$ nodes, where the extra $X$ comes from the surrounding $X$ leaders. Since nodes in $H$ are either leaders or connectors, if we show that $X$ is a constant, then the degree of any node in $H$ will be a constant.

There is at most one leader per grid box of the pivotal grid. There are a constant number of grid boxes within distance 3r (maximum distance between two nodes 3 hops away from each other) from a given grid box in the pivotal grid. Thus, $X$ is a constant and the internal degree of nodes in $H$ is a constant.

(c) In the final stage of the algorithm, only nodes who are leaders or connectors transmit their status using a token passing system. By Lemma 16, any node in $H$ will learn about other nodes in $H$ within range of it. Thus, the set of nodes in $H$ induces a subgraph of $G$.

2. We now need to prove that the number of nodes of $H$ is at most a constant multiple of the minimum sized connected dominating set of $G$. Let us first get an upper bound on the number of nodes in $H$. We already derived an upper bound on the number of neighbors of each leader in $H$ as $2X + 2X^2$. Assuming that there are $L$ leaders in $H$, we can arrive at an upper bound for the number of nodes in $H$ as $L(2X + 2X^2) = O(LX^2) = O(L)$ since $X$ is a constant. We need to show that $L = O(s_{c,d})$, since then the number of nodes in $H$ will be $O(L) = O(s_{c,d})$, which is the required property.

Consider all grid boxes containing nodes of $G$. Since the minimum connected dominating set must span the entire graph, there must exist at least one node in the set per 20 grid boxes, since the range of a node in one grid box allows it to reach at most 20 other grid boxes. Hence, if altogether $g$ grid boxes are occupied by nodes in $G$, then at least $\frac{g}{20}$ nodes will be present in the minimum connected dominating set, so $s_{c,d} = \Omega(g)$. Now there exists at most one leader per grid box so $L = O(g) = O(s_{c,d})$. Therefore, the number of nodes in $H$ is $O(s_{c,d})$.

3. We need to show that every node in $G\setminus H$ is associated with exactly one entry point in the backbone. In our case, we choose the leaders to act as the unique entry points. All leaders are nodes of $H$ according to Algorithm Backbone-Creation. Therefore, the only nodes remaining in $G\setminus H$ are followers. Every follower identifies with exactly one leader and that leader is a part of $H$. Hence, the property is satisfied.

4. Finally, we must prove that the asymptotic diameters of $G$ and $H$ are the same. Every node of $G$ is contained within the backbone or within one hop of a node of the backbone. One property of our model is that any two nodes within range of each other have an edge between them in the communication graph $G$. Since $H$ is an induced subgraph of $G$, the asymptotic diameter of the two graphs must be the same.

Thus, a backbone is created after using Backbone-Creation. Since each of the 4 stages consists of token passing by each leader to some subset of its neighbors using $(N, c)$-ssf schedules, the additional running time is $O(\Delta \lg N)$ rounds. This is because each $(N, c)$-ssf schedule takes $O(\lg N)$ rounds and there will be at most $O(\Delta)$ such schedules run since $\Delta$ is the maximum number of nodes surrounding a given node. Adding in the running times of Tree-Grower and Tree-Cutter, the total running time of Algorithm Backbone-Creation is $O(n \lg N)$ rounds. □
5.2 Communication in Backbone Subnetwork

Algorithm *Backbone-Message-Exchange* is used to transmit messages between two nodes in the backbone and Algorithm *Backbone-Message-Transmit* is used by a node outside the backbone to transmit a message to its leader.

**Theorem 8.** Algorithm *Backbone-Message-Exchange* guarantees the exchange of a message between every pair of connected nodes in the backbone in time $O(lgN)$ rounds. Algorithm *Backbone-Message-Transmit* guarantees the transmission of a message from a node outside the backbone to its leader in the backbone in time $O(\Delta lgN)$ rounds.

Algorithm *Backbone-Message-Exchange* involves having every leader continuously take part in a token passing routine with all its connectors. Algorithm *Backbone-Message-Transmit* involves every leader taking part in a continuous token passing routine with all of its followers. When a node gets the token, it can transmit a message if it has one.

**Algorithm 8** Backbone-Message-Exchange(Tree $T$, Tree $H$, Type $type$, Node $my\_leader$), run by each node $u$

1: $type$ denotes the type of node, either leader or follower.
2: $my\_leader$ denotes the leader of a node. For a leader, $my\_leader$ is its own label.
3: $T$ stores the children of a leader. For followers, $T$ is empty.
4: $num\_phases$ is no. of phases to run algorithm. If leader, $num\_phases$ is two times no. of children in $T$, else it’s 2.
5: For a node in backbone, tree $H$ stores other nodes in backbone within range. For a node not in the backbone, $H$ is empty.
6:  
7: if $u$ is a leader then
8: Add every node that is $u$’s child in both $T$ and $H$ to list of connectors
9: for Every connector, $v$, in $u$’s list of connectors do
10: Execute($N,c$)-SSF: Transmit $u$ message, if any, and $num\_phases$
11: Execute($N,c$)-SSF: Pass token to $v$
12: Execute($N,c$)-SSF: Store any new messages $u$ hears
13: Execute($N,c$)-SSF: Do nothing
14: else ▷ $u$ is a connector.
15: Execute the following $num\_phases$ times
16: if $u$ has a token then
17: Execute($N,c$)-SSF: Transmit $u$’s message, if any. Store any new messages $u$ hears
18: Execute($N,c$)-SSF: Pass token back to $u$’s leader
19: else
20: Execute($N,c$)-SSF: Store any new messages $u$ hears and update $num\_phases$ if needed
21: Execute($N,c$)-SSF: Listen to see if $u$ receives a token

5.2.1 Proof of Theorem 8

We first prove that message transmissions are successful in both algorithms and then go on to bound their running times. Every leader has one token that it passes only to its followers or connectors, i.e. nodes within range $r$. Leaders are the same ones created after using Tree-Cutter and by Theorem 4 there is at most one leader per grid box of the pivotal grid. There are 21 grid boxes within range of a node (including the node’s own grid box). Thus, at most 21 tokens can be present in a grid box at any given time and by Theorem 1 using an $(N,c)$-ssf with a sufficiently large $c$ allows all transmitting nodes (those with tokens) to successfully transmit their messages.
Algorithm 9 Backbone-Message-Transmit($T$, Type $type$, Node $my\_leader$), run by each node $u$

1: $type$ denotes the type of a node, either leader or follower.
2: $my\_leader$ denotes the leader of a node. For a leader, $my\_leader$ is its own label.
3: Tree $T$ stores the children of a leader. For followers, $T$ is empty.
4: $num\_phases$ is no. of phases to run algorithm. If leader, $num\_phases$ is two times no. of children in $T$,
   else it’s 2.
5: if $u$ is a leader then
6: for Every child, $v$, in $T$ do
7: Execute($N,c$)-SSF: Transmit $num\_phases$
8: Execute($N,c$)-SSF: Pass token to $v$
9: Execute($N,c$)-SSF: Store any new messages $u$ hears
10: Execute($N,c$)-SSF: Do nothing
11: else $\triangleright$ Node is a follower.
12: Execute the following $num\_phases$ times
13: if $u$ has a token then
14: Execute($N,c$)-SSF: Transmit $u$’s message, if any
15: Execute($N,c$)-SSF: Pass token back to $u$’s leader
16: else
17: Execute($N,c$)-SSF: Listen for new value of $num\_phases$ from leader and update if necessary
18: Execute($N,c$)-SSF: Listen to see if $u$ receives a token

6 Conclusions

In this work, we show several applications of the technique known as SSF Based Dilution. We use it to
provide the first deterministic algorithm for multi-broadcast from uncoordinated wakeup when knowledge
of nodes’ physical coordinates and neighborhoods is not known. Additionally, we use SSF Based Dilution
to construct algorithms to deterministically solve multi-broadcast and backbone creation for spontaneous
wakeup. We present an open problem we feel is of interest.

Open Problem: $Tree\_Grower$ requires message size of $O(\Delta \lg N)$ bits and $Tree\_Cutter$ requires message
size of $O(\Delta \lg^2 N)$ bits. Alternate algorithms with lesser requirements would be advantageous.

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