THE STRUCTURE OF HIGHER RANK GRAPH C*-ALGEBRAS REVISITED

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Abstract. In this paper, we study a higher rank graph, which has a period group deduced from a natural equivalence relation on its infinite path space. We prove that the C*-algebra generated by the standard generators with equivalent pairs is a maximal abelian subalgebra of its graph C*-algebra. This is obtained as a consequence of the general theory of a pushout $P$-graph and the following structure theorem: for a class of $P$-graphs, the C*-algebras of their pullbacks are tensor products of the $P$-graph C*-algebras with commutative C*-algebras.

1. Introduction

Let $P$ be a finitely generated cancellative abelian monoid. It was first suggested to study $P$-graphs in [KP00], where higher rank graphs (or $k$-graphs) were introduced. But this has not been done until very recently. Based on some ideas in [KP00], Carlsen, Kang, Shotwell and Sims study $P$-graphs, their pullbacks and associated C*-algebras in [CKSS14]. Making use of them as a tool, they successfully describe the picture of the primitive idea space in the C*-algebra of a row-finite higher rank graph without sources, which gives an analogue of (directed) graph C*-algebras in [HS04].

In [KP00, CKSS14], $P$-graphs are mainly used as a technical tool. Interestingly, we realize that, for a large class of $P$-graphs, the C*-algebras of their pullbacks have a very nice structure. To be more precise, let $q$ be a homomorphism on $\mathbb{Z}^k$, $\Gamma$ be a $q(\mathbb{N}^k)$-graph, and $q^*\Gamma$ be the pullback of $\Gamma$ via $q$. It turns out that $C^*(q^*\Gamma)$ is the tensor product of $C^*(\Gamma)$ with the commutative C*-algebra $C^*(\ker q)$. This, unfortunately, has been overlooked in [CKSS14]. As in [CKSS14, KP00], this is in turn applied to better understand the structure of the C*-algebras for a class of higher rank graphs. This class includes all single vertex higher rank graphs as special examples, which are extensively studied in [DPY08, DPY10, DY09a, DY09b, Pow07]. To this end, we first need to generalize the notion of the period group to a $P$-graph $\Lambda$ whose vertex set is a maximal tail. Completely similar to [CKSS14], one can see that the period group $H_\Lambda$ of every such $P$-graph $\Lambda$ is a subgroup of the Grothendieck group $G(P)$ of $P$, and is determined by an equivalence relation on the infinite path space of $\Lambda$. Furthermore, $\Lambda$ has
a pushout $q(P)$-graph $\Gamma$ via the quotient map $q : \mathcal{G}(P) \to \mathcal{G}(P)/H_\Lambda$. One key property we observe is that the pushout $\Gamma$ is aperiodic. The aperiodicity of $\Gamma$ is rather natural, but to prove it needs some care. Consequently, we show that, when $\Lambda$ is a higher rank graph, the $C^*$-algebra generated by

the standard generators of $C^*(\Lambda)$ with equivalent pairs in $\Lambda$ is a maximal

abelian subalgebra (MASA) of $C^*(\Lambda)$, which turns out to be isomorphic to

the tensor product of the canonical diagonal algebra $\mathcal{D}_\Lambda$ with $C^*(H_\Lambda)$.

This note is motivated by [BNR14, CKSS14] and strongly influenced by

[CKSS14]. In Section 2, some background on $P$-graphs and the associated

$C^*$-algebras will be briefly given. The main result of Section 3 is a structure

theorem (Theorem 3.3), which roughly says that the $C^*$-algebras of a class

of pullbacks are tensor products of $P$-graph $C^*$-algebras with commutative $C^*$-algebras. Section 4 investigates the class of row-finite $P$-graph without

sources such that the vertex sets are maximal tails. One advantage of this class is that every such $P$-graph $\Lambda$ has a period group, which is a subgroup

of the Grothendieck group $\mathcal{G}(P)$ of $P$. The quotient $q : \mathcal{G}(P) \to \mathcal{G}(P)/H_\Lambda$ induces a pushout, which is a row-finite $q(P)$-graph without sources. Applying the results of Section 3 and Section 4 in Section 5 we revisit the structure of a class of higher rank graph $C^*$-algebras, obtain a distinguished MASA, and exhibit a faithful conditional expectation onto the MASA. In particular, we answer the questions posed in [BNR14] for this class.

Notation. Let $P$ be a finitely generated cancellative abelian monoid. By $\mathcal{G}(P)$, we mean the Grothendieck group of $P$.

As usual, for $m, n \in \mathbb{Z}^k$, we use $m \lor n$ and $m \land n$ to denote the coordinate-wise maximum and minimum of $m$ and $n$, respectively. For $n \in \mathbb{Z}^k$, we let $n_+ = n \lor 0$ and $n_- = -(n \land 0)$. Of course, $n = n_+ - n_-$ with $n_+ \land n_- = 0$.

2. $P$-Graphs

Let $P$ be a finitely generated cancellative abelian monoid, which is also regarded as a category with one object. $P$-graphs are a generation of higher rank graphs. They share many properties with higher rank graphs. In this section, we briefly recall some basics on $P$-graphs. Refer to [CKSS14, Section 2] for more details.

A $P$-graph is a countable small category $\Gamma$ with a functor $d : \Gamma \to P$ such that the following factorization property holds: whenever $\xi \in \Gamma$ satisfies $d(\xi) = p + q$, there are unique elements $\eta, \zeta \in \Gamma$ such that $d(\eta) = p$, $d(\zeta) = q$ and $\xi = \eta \zeta$. Clearly, any $k$-graph is an $\mathbb{N}^k$-graph. All notions on higher rank graphs, such as row-finiteness and sources, can be generalized to $P$-graphs. For $p \in P$, let $\Gamma^p = d^{-1}(p)$, and so $\Gamma^0$ is the vertex set of $\Gamma$. For $v \in \Gamma^0$, $v\Gamma = \{\xi \in \Gamma : r(\xi) = v\}$. We say that $\Gamma$ is a $row$-finite $P$-graph without sources if $0 < |v\Gamma^p| < \infty$ for all $v \in \Gamma^0$ and $p \in P$.

Embed $P$ in $\mathcal{G}(P)$ and let

$$\Omega_P = \{(p, q) \in P \times P \mid q - p \in P\}.$$
Define \( d, s, r : \Omega_P \to P \) by \( d(p, q) = q - p \), \( s(p, q) = q \), and \( r(p, q) = p \). It is shown in [CKSS14, Example 2.2] that \( \Omega_P \) is a row-finite \( P \)-graph without sources.

Let \( \Lambda \) and \( \Gamma \) be two \( P \)-graphs. A \( P \)-graph morphism between \( \Lambda \) and \( \Gamma \) is a functor \( x : \Lambda \to \Gamma \) such that \( d_\Gamma(x(\lambda)) = d_\Lambda(\lambda) \) for all \( \lambda \in \Lambda \). The infinite path space of \( \Gamma \) is defined as follows:

\[
\Gamma^\infty = \{ x : \Omega_P \to \Gamma \mid x \text{ is a } P\text{-graph morphism} \}.
\]

For \( x \in \Gamma^\infty \) and \( p \in P \), there is a unique element \( \sigma^p(x) \in \Gamma^\infty \) defined by

\[
\sigma^p(x)(q, r) = x(p + q, p + r).
\]

That is, \( \sigma^p \) is a shift map on \( \Gamma^\infty \). If \( \mu \in \Gamma \) and \( x \in s(\mu)\Gamma^\infty \), then \( \mu x \) is defined to be the unique infinite path such that \( \mu x(0, p) = \mu \cdot x(0, p - d(\mu)) \) for any \( p \in P \) with \( p - d(\mu) \in P \).

**Definition 2.1.** A \( P \)-graph \( \Gamma \) is said to be aperiodic if for every \( v \in \Gamma^0 \), there is \( x \in \Gamma^\infty \) such that \( p, q \in P \) with \( p \neq q \) implies \( \sigma^p(x) \neq \sigma^q(x) \).

For a row-finite \( P \)-graph \( \Gamma \) without sources, we associate to it a universal \( \mathrm{C}^* \)-algebra \( \mathrm{C}^*(\Gamma) \) as follows.

**Definition 2.2.** Let \( \Gamma \) be a row-finite \( P \)-graph without sources. A *Cuntz-Krieger \( \Gamma \)-family* in a \( \mathrm{C}^* \)-algebra \( \mathcal{A} \) is a family \( \{ S_\lambda : \lambda \in \Gamma \} \) in \( \mathcal{A} \) such that

1. \( \{ S_v : v \in \Gamma^0 \} \) is a set of mutually orthogonal projections;
2. \( S_\mu S_\nu = S_{\mu \nu} \) whenever \( s(\mu) = r(\nu) \);
3. \( S_\lambda^* S_\lambda = S_{s(\lambda)} \) for all \( \lambda \in \Gamma \);
4. \( S_v = \sum_{\lambda \in r_0} S_\lambda S_\lambda^* \) for all \( v \in \Gamma^0 \) and \( p \in P \).

The *\( P \)-graph \( \mathrm{C}^* \)-algebra* \( \mathrm{C}^*(\Gamma) \) is the universal \( \mathrm{C}^* \)-algebra among Cuntz-Krieger \( \Gamma \)-families. We usually use \( s_\lambda \)'s to denote its generators.

It is known that

\[
\mathrm{C}^*(\Gamma) = \overline{\text{span}}\{ s_\mu s_\nu^* : \mu, \nu \in \Gamma \}.
\]

One important property is that \( \mathrm{C}^*(\Gamma) \) and the reduced \( \mathrm{C}^* \)-algebra \( \mathrm{C}_r^*(\mathcal{G}_\Gamma) \) are isomorphic, where \( \mathcal{G}_\Gamma \) is the groupoid associated to \( \Gamma \).

By the universal property of \( \mathrm{C}^*(\Gamma) \), there is a natural gauge action \( \gamma \) of the dual \( \widehat{\mathcal{G}(P)} \) of \( \mathcal{G}(P) \) on \( \mathrm{C}^*(\Gamma) \) defined by

\[
\gamma_\lambda(s_\mu) = \chi(d(\lambda)) s_{\mu} \quad (\chi \in \widehat{\mathcal{G}(P)}, \ \lambda \in \Gamma).
\]

Averaging over \( \gamma \) gives a faithful conditional expectation \( \Phi \) from \( \mathrm{C}^*(\Gamma) \) onto the fixed point algebra \( \mathrm{C}^*(\Gamma)^\gamma \). It turns out that \( \mathrm{C}^*(\Gamma)^\gamma \) is an AF algebra and

\[
\mathfrak{F}_\Gamma := \mathrm{C}^*(\Gamma)^\gamma = \overline{\text{span}}\{ s_\mu s_\nu^* : d(\mu) = d(\nu) \}.
\]

\(^1\)Note that \( \Gamma^\infty \) is denoted by \( \Gamma^\Omega \) in [CKSS14].
The (canonical) diagonal algebra $D_\Gamma$ of $C^*(\Gamma)$ is defined as
$$D_\Gamma = \text{span}\{s_\mu s_\mu^* : \mu \in \Gamma\},$$
which is the canonical MASA of $\mathcal{F}_\Gamma$.

Furthermore, as higher rank graphs, we also have the following two important uniqueness theorems for $C^*(\Gamma)$: the gauge invariant uniqueness theorem [CKSS14, Proposition 2.7] and the Cuntz-Krieger uniqueness theorem [CKSS14, Corollary 2.8].

Applying obvious modifications, one can now easily generalize the main result of [Hop05] to $P$-graphs.

**Theorem 2.3.** ([Hop05, Theorem]) Let $\Gamma$ be a row-finite $P$-graph without sources. Then the following are equivalent.

(i) $\Gamma$ is aperiodic.

(ii) The diagonal algebra $D_\Gamma$ of $C^*(\Gamma)$ is a MASA in $C^*(\Gamma)$.

### 3. A Structure Theorem

We begin this section with the following definition.

**Definition 3.1.** Let $P$ and $Q$ be finitely generated cancellative abelian monoids, and $f : P \to Q$ be a monoid homomorphism. If $(\Gamma, d_\Gamma)$ is a $Q$-graph, the pullback of $\Gamma$ via $f$ is the $P$-graph $(f^*\Gamma, d_{f^*\Gamma})$ defined as follows: $f^*\Gamma = \{(\lambda, p) : d_\Gamma(\lambda) = f(p)\}$ with $d_{f^*\Gamma}(\lambda, p) = p$, $s(\lambda, p) = s(\lambda)$ and $r(\lambda, p) = r(\lambda)$. The composition of two paths in $f^*\Gamma$ is given by $(\mu, p)(\nu, q) = (\mu\nu, p+q)$.

The pullback $f^*\Gamma$ defined above is indeed a $P$-graph. The proof is completely similar to that of [CKSS14, Lemma 3.2] where $P = \mathbb{N}^k$. The following properties are easily derived from the very definition of pullbacks.

**Lemma 3.2.** Let $P$ and $Q$ be finitely generated cancellative abelian monoids, and $f : P \to Q$ be a monoid homomorphism. Suppose that $(\Gamma, d_\Gamma)$ is a $Q$-graph. Then we have the following.

(i) $\Gamma^0 = (f^*\Gamma)^0$.

(ii) If $\Gamma$ has no sources, then $f^*\Gamma$ has no sources. The converse is true if $f$ is surjective.

(iii) If $f^*\Gamma$ is row-finite, then so is $\Gamma$. The converse is true if $f$ is surjective.

**Proof.** (i) This can be easily seen from the following:
$$d_{f^*\Gamma}(\lambda, p) = 0 \Leftrightarrow p = 0 \Leftrightarrow d_\Gamma(\lambda) = f(0) = 0 \Leftrightarrow \lambda \in \Gamma^0.$$

(ii) This is proved by noticing $(\lambda, p) \in (f^*\Gamma)^p \Leftrightarrow \lambda \in \Gamma^{f(p)}$.

(iii) This follows from
$$v(f^*\Gamma)^p = \{(\lambda, p) \in (f^*\Gamma)^p : r(\lambda) = v\} = \{\lambda : \lambda \in v\Gamma^{f(p)}\}$$
implying $|v(f^*\Gamma)^p| = |v\Gamma^{f(p)}|$. 

\[\blacksquare\]
In the remainder of this section, we focus on monoid morphisms $f$ induced from homomorphisms on $\mathbb{Z}^k$. In this case, we can prove the following structure theorem.

**Theorem 3.3.** Let $H$ be a subgroup of $\mathbb{Z}^k$, and $q : \mathbb{Z}^k \to \mathbb{Z}^k/H$ be the quotient map. Suppose that $\Gamma$ is a row-finite $q(\mathbb{N}^k)$-graph without sources. Then

$$C^*(q^\ast \Gamma) \cong C^*(\Gamma) \otimes C(\hat{H}).$$

**Proof.** For any $h \in H$, it is shown in [CKSS14] that there is a unitary $W_h$ in the centre of $\mathcal{M}(C^*(q^\ast \Gamma))$, the multiplier algebra of $C^*(q^\ast \Gamma)$, which is given by

$$W_h = \sum_{\lambda \in \Gamma^0} \sum_{\mathbf{q}(\lambda^+)} s_{(\lambda,h^+)} s_{(\lambda,h^-)}^* = \text{s-lim}_F \sum_{\lambda \in F} \sum_{\mathbf{q}(\lambda^+)} s_{(\lambda,h^+)} s_{(\lambda,h^-)}^*.$$  \hspace{1cm} (1)

Here the limit is taking in the strict topology as $F$ increases over finite subsets of $\Gamma^0$. Furthermore, for any $\lambda \in \Gamma^q(h^+)$ one has

$$s_{(\lambda,h^+)} = W_h s_{(\lambda,h^-)} = s_{(\lambda,h^-)} W_h.$$

Assume that $H$ has rank $r$, and has generators $h_1, \ldots, h_r$ in $\mathbb{Z}^k$. If $r = 0$, let $W_r = W_0$ be the identity of the multiplier algebra of $C^*(\Gamma)$. If $r \geq 1$, for $1 \leq i \leq r$ we let $W_i := W_{h_i}$ denote the central unitaries corresponding to $h_i$ defined in [11]. By [CKSS14], $C(\hat{H}) \cong C^*(H) \cong C^*(W_1, \ldots, W_r)$, and so it is now equivalent to show

$$C^*(q^\ast \Gamma) \cong C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r).$$

Let $j$ be a section of the quotient map $q : \mathbb{Z}^k \to \mathbb{Z}^k/H$. That is, $j$ is a map from $\mathbb{Z}^k/H$ to $\mathbb{Z}^k$ satisfying $q \circ j = \text{id}_{\mathbb{Z}^k/H}$. Define an action $\Theta : \mathbb{T}^k \to \text{Aut} \left( C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r) \right)$ via

$$\Theta_t(s_{\mu} \otimes W^n) = e^{i\rho_d(\mu)} s_{\mu} \otimes e^{i\rho_{-d}} W^n \quad (t \in \mathbb{T}^k),$$

where $h = (h_1, \ldots, h_r)$, $n \cdot h = \sum_{i=1}^r n_i h_i \in H$, and $W^n = \prod_{i=1}^r W_i^{n_i}$ for $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ (cf. Remark 3.4 (1) below).

Notice that, for any $(\mu, n) \in q^\ast \Gamma$, one has $n - j \circ d(\mu) \in H$ as $q(n - j \circ d(\mu)) = q(n) - d(\mu) = 0$. So there is a unique $m \in \mathbb{Z}^r$ such that $n - j \circ d(\mu) = m \cdot h$. For convenience, we also write $W^{n - j \circ d(\mu)}$ for $W^m$. Then one can verify that $\{s_{\mu} \otimes W^{n - j \circ d(\mu)} : d(\mu) = q(n)\}$ is a Cuntz-Krieger

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2 Since $C(\hat{H})$ is a commutative $C^*$-algebra, it does not matter which $C^*$-tensor product one chooses below.
$q^*\Gamma$-family. By the universal property of $C^*(q^*\Gamma)$, there is a (unique) $^*$-homomorphism $\pi$ given by

$$\pi : C^*(q^*\Gamma) \to C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r),$$

where $(\mu, n) \in q^*\Gamma$. It is easy to check that $\pi$ is equivariant between the gauge action $\gamma$ of $\mathbb{T}^k$ on $C^*(q^*\Gamma)$ and the above action $\Theta$ on $C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r)$. By the gauge invariant uniqueness theorem [KP00 Theorem 3.4], $\pi$ is injective.

It remains to show that $\pi$ is surjective. It is sufficient to show $s_\mu \otimes W^\ell \in \pi(C^*(q^*\Gamma))$ for all $\mu \in \Gamma$ and $\ell \in \mathbb{Z}^r$. To this end, first notice that

$$\pi (s_{(\lambda,h_+)s_{(\lambda,h_-)}}) = (s_\lambda \otimes W^{h_+ - jod(\lambda)}) (s_\lambda^* \otimes (W^*)^{h_- - jod(\lambda)})$$

$$= (s_\lambda s_\lambda^*) \otimes W^{h_+ + h_- - jod(\lambda)}$$

$$= s_\lambda s_\lambda^* \otimes W^{h_+}$$

$$= s_\lambda s_\lambda^* \otimes W_i.$$  \hfill (4)

Now let $(\mu, m) \in q^*\Gamma$ and $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r$. Consider $s_{(\mu,m)} W^n$, which is an element in $C^*(q^*\Gamma)$ since $W^n$ is in the multiplier algebra of $C^*(q^*\Gamma)$. We derive from (1) and (4) that

$$\pi (s_{(\mu,m)} W^n) = \pi \left( s_{(\mu,m)} \prod_{i=1}^r W_i^{n_i} \right)$$

$$= s^\ast \lim_F \sum_{v \in F} \left( s_\mu \otimes W^{m - jod(\mu)} \cdot \prod_{i} \left( \sum_{\lambda \in \mathbb{N} q(h_+)} s_\lambda s_\lambda^* \otimes W_i \right)^{n_i} \right)$$

$$= s^\ast \lim_F \sum_{v \in F} \left( s_\mu \otimes W^{m - jod(\mu)} \cdot \prod_{i} \left( \sum_{\lambda \in \mathbb{N} q(h_+)} s_\lambda s_\lambda^* \right)^{n_i} \otimes W_i^{n_i} \right)$$

$$= s^\ast \lim_F \sum_{v \in F} \left( s_\mu \prod_{i} \left( \sum_{\lambda \in \mathbb{N} q(h_+)} s_\lambda s_\lambda^* \right)^{n_i} \otimes W^{m - jod(\mu) + n} \right)$$

$$= \left\{ s^\ast \lim_F \sum_{v \in F} \sum_{\lambda \in \mathbb{N} q(h_+)} s_\lambda s_\lambda^* \right\} \otimes W^{m - jod(\mu) + n}$$

$$= s_\mu \otimes W^{m - jod(\mu) + n}. \hfill \text{In what follows, we use the usual convention that for an operator $A$, $A^k = (A^*)^{-k}$ if $k < 0.$}$$
Here the last “=” above used the fact that, for each $1 \leq i \leq r$, \[ \sum_{\lambda \in \Gamma} q(h_{i,\lambda}) s_\lambda s_\lambda^* \] is strictly convergent to the identity of the multiplier algebra of $C^*_{\Gamma}$. This fact can be proved by a standard argument (cf. the proof of Proposition 3.3 of [CKSS14]). Let $\ell'$ be the unique element in $\mathbb{Z}^r$ satisfying $m - \sum d(\mu) = \ell' \cdot h$. Replacing $n$ by $\ell - \ell'$, we deduce that $s_\mu \otimes W_{\ell}'$ is in the range of $\pi$. This proves the subjectivity of $\pi$.

Some remarks are in order.

Remark 3.4. (1) One can think the action $\Theta$ defined in (2) of the tensor product of the (canonical) gauge action of $H_{\perp}(\subset T_k)$ on the $P$-graph $C^*_{\Gamma}$ and the (canonical) gauge action of $H_{\perp}$ on $C^*(W_1, \ldots, W_r)$.

(2) From the above proof, one actually has $D_{q^* \Gamma} \cong D_{\Gamma}$, since $\pi(s(\mu, m)s(\mu, m)^*) = s_\mu s_\mu^* \otimes I$ for every $(\mu, m) \in q^* \Gamma$, i.e., the left hand side is independent of the choice of $m \in \mathbb{N}^k$.

(3) If $r = 0$, namely, $\Gamma$ and $q^* \Gamma$ are isomorphic monoids, then $C^*_\Gamma \cong C^*(q^* \Gamma)$, as expected.

Theorem 3.3 really describes the structure of $C^*$-algebras for higher rank graphs being pullbacks. Some important and interesting consequences will be exhibited in Section 5. But we need to discuss a general theory of $P$-graphs first. This is given in the next section.

4. The Period Group and a Pushout

The notion of a maximal tail plays an important role in [CKSS14, KPT1], and is also essential to this note. In order to adapt it to $P$-graphs, we replace [CKSS14, Definition 4.1(b)] by (ii) below; but they are easily seen to be equivalent for higher rank graphs.

Definition 4.1. Let $\Lambda$ be a row-finite $P$-graph without sources. A nonempty subset $T$ of $\Lambda^0$ is called a maximal tail if

(i) for every $v_1, v_2 \in T$ there is $w \in T$ such that $v_1\Lambda w \neq \emptyset$ and $v_2\Lambda w \neq \emptyset$.

(ii) for every $v \in T$ and $p \in P$ there is $\lambda \in v\Lambda^0$ such that $s(\lambda) \in T$.

(iii) for $w \in T$ and $v \in \Lambda^0$ with $v\Lambda w \neq \emptyset$, we have $v \in T$.

Let $\Lambda$ be a $P$-graph such that $\Lambda^0$ is a maximal tail. Define an equivalence relation $\sim$ on $\Lambda$ as follows:

\[ \xi \sim \eta \iff s(\xi) = s(\eta) \text{ and } \xi x = \eta x \text{ for all } x \in s(\xi)\Lambda^\infty. \]

If $\xi \sim \eta$, obviously one also has $r(\xi) = r(\eta)$ automatically.

This equivalence relation respects sources and ranges, and so $\Lambda/\sim$ is a category:

\[ r([\xi]) = [r(\xi)], \quad s([\xi]) = [s(\xi)], \quad [\xi][\eta] = [\xi\eta]. \]
Define
\[ H_\Lambda = \{ d(\xi) - d(\eta) : \xi, \eta \in \Lambda, \xi \sim \eta \}, \]
and
\[ \Lambda_{\text{Per}}^0 = \left\{ v \in \Lambda^0 \mid \text{any } \xi \in v\Lambda, p \in P \text{ with } d(\xi) - p \in H_\Lambda \implies \text{there is } \eta \in v\Lambda^p \text{ such that } \xi \sim \eta \right\}. \]

The following simple uniqueness result is essentially from [CKSS14, Theorem 4.2 (3)]. We single it out below to highlight that \( r(\lambda) \) is not required to be in \( \Lambda_{\text{Per}}^0 \). This is what we really need later.

**Lemma 4.2.** Let \( \Lambda \) be a \( P \)-graph such that \( \Lambda^0 \) is a maximal tail. Let \( \lambda \sim \mu \) in \( \Lambda \) be such that \( \lambda \sim \mu \) with the same degree. Then \( \lambda = \mu \).

**Proof.** Fix \( x \in s(\lambda)\Lambda^\infty \). Since \( \lambda \sim \mu \), we have \( \lambda x = \mu x \). This implies \( \lambda = (\lambda x)(0,d(\lambda)) = (\lambda x)(0,d(\mu)) = (\mu x)(0,d(\mu)) = \mu \).

The following theorem is an analogue of [CKSS14, Theorem 4.2].

**Theorem 4.3.** Let \( \Lambda \) be a row-finite \( P \)-graph without sources such that \( \Lambda^0 \) is a maximal tail. Then we have the following.

(i) \( H_\Lambda \) is a subgroup of \( G(P) \).

(ii) \( \Lambda_{\text{Per}}^0 \) is a nonempty hereditary subset of \( \Lambda^0 \), such that for all \( p, q \in P \) with \( p - q \in H_\Lambda \) and for all \( x \in \Lambda_{\text{Per}}^0 \Lambda^\infty \) one has \( \sigma^p(x) = \sigma^q(x) \).

(iii) If \( r(\xi) \in \Lambda_{\text{Per}}^0 \) and \( d(\xi) - p \in H_\Lambda \), there is a unique \( \eta \in r(\xi)\Lambda^p \) such that \( \xi \sim \eta \).

(iv) Let \( q : G(P) \to G(P)/H_\Lambda \) be the quotient map. Then \( \Gamma = \Lambda_{\text{Per}}^0 \Lambda/\sim \) is a \( q(P) \)-graph with degree map \( \tilde{d} := q \circ d \).

(v) \( \Lambda_{\text{Per}}^0 \Lambda \) is isomorphic to the pullback \( q^*\Gamma \) via \( \lambda \mapsto ([\lambda],d(\lambda)) \), and so it is row-finite and has no sources.

Theorem 4.3 can be proved completely similar to [CKSS14, Theorem 4.2] with obvious modifications. However, two points should be mentioned here. First of all, we do not have a generalization of the conclusion of [CKSS14, Proposition 4.4] that “The set \( \Sigma^\min_\Lambda \) of minimal elements of \( \Sigma_\Lambda \setminus \{0\} \) is finite and generates \( \Sigma_\Lambda \) as a monoid” to \( P \)-graphs. But that conclusion is not used anywhere. Secondly, in the proof of [CKSS14, Lemma 4.5], one can replace \( n = n_+ - n_- \) by the fact that, for any \( g \in G(P) \), one has \( g = g_1 - g_2 \) for some \( g_1, g_2 \in P \).

One useful fact ([CKSS14, Lemma 4.5]) obtained during the course of the proof of [CKSS14, Theorem 4.2] is the following that will be used several times later.

**Lemma 4.4.** Suppose that \( \Lambda \) is a row-finite \( P \)-graph without sources such that \( \Lambda^0 \) is a maximal tail. For \( v \in \Lambda^0 \), let
\[ \Sigma_v = \{(p,q) \in P \times P : \sigma^p(x) = \sigma^q(x) \text{ for all } x \in v\Lambda^\infty \}\]

\(^4\)In [CKSS14], \( H_\Lambda \) and \( \Lambda_{\text{Per}}^0 \) below are denoted by \( \text{Per}(\Lambda) \) and \( H_{\text{Per}} \), respectively.
and $\Sigma_\Lambda = \bigcup_{v \in \Lambda^0} \Sigma_v$. Then
\[(p, q) \in \Sigma_\Lambda \iff p - q \in H_\Lambda.\]

The following is now straightforward. We record it for future reference.

**Corollary 4.5.** Let $\Lambda$ be a row-finite $P$-graph without sources such that $\Lambda^0$ is a maximal tail. Then $\Lambda$ is aperiodic if and only if $H_\Lambda = \{0\}$.

## 5. A distinguished MASA

In this section, we exhibit some interesting and important applications of our results in Sections 3 and 4. But because of some technical reasoning, we have to restrict ourselves to the following class of higher rank graphs $\Lambda$: $\Lambda$ is a row-finite higher rank graph without sources such that $\Lambda^0$ is a maximal tail and $\Lambda^0_{\text{per}} = \Lambda^0$. Clearly, this class exhausts all single vertex higher rank graphs studied in [DPY08, DPY10, DY09a, DY09b].

### 5.1. The aperiodicity of $\Lambda/\sim$.

Let $\sim$, $H_\Lambda$, and $\Gamma = \Lambda/\sim$ be the same as in Section 4. The main result of this subsection is that $\Gamma$ is aperiodic. Intuitively, this is very natural and simple: $\Gamma$ is obtained by removing all periods of $\Lambda$, and so $\Gamma$ should have only the trivial period. However, to prove this needs some care.

For convenience, we summarize some properties of $\Gamma$, which are inherited from $\Lambda$, as follows, so that we can apply some results of Section 4 to $\Gamma$.

**Lemma 5.1.** Let $\Gamma = \Lambda/\sim$. Then $\Gamma$ is a row-finite $q(P)$-graph without sources such that $\Gamma^0$ is a maximal tail.

The following result is in the same vein with [KP00, Proposition 2.9].

**Proposition 5.2.** The quotient map $q : \mathcal{G}(P) \to \mathcal{G}(P)/H_\Lambda$ defines a homeomorphism $q_* : \Gamma^\infty \to \Gamma^\infty$ by $q_* x = \hat{x}$, where
\[\hat{x}(q(m), q(n)) = [x(m, n)].\]

**Proof.** First observe that
\[\Omega_{q(P)} = \{(q(m), q(n)) : m, n \in P \text{ such that } n - m \in q(P)\}.\]

For this, let $m', n' \in P$ be such that $q(n') - q(m') \in q(P)$. Then $n' - m' = p_0 + \ell_0$ for some $p_0 \in P$ and $\ell_0 \in H_\Lambda$. Let $m := m'$ and $n := n' - \ell_0 = m + p_0$. Clearly, one has $m, n, n - m \in P$ and $q(n') - q(m') = q(n) - q(m)$.

In the sequel, we prove that $q_*$ is well-defined. We need to show the following: If $m, n, m', n' \in P$ satisfy $q(m) = q(m')$ and $q(n) = q(n')$, then
\[x(m, n) \sim x(m', n').\] \hfill (5)

It is easy to see $s(x(m, n)) = s(x(m', n'))$. In fact, notice that $s(x(m, n)) = r(\sigma^n(x))$ and $s(x(m', n')) = r(\sigma'^n(x))$. But it follows from Lemma 4.4 that $\sigma^n(x) = \sigma'^n(x)$ as $n - n' \in H_\Lambda$. This of course implies $r(\sigma^n(x)) = r(\sigma'^n(x))$, and so $s(x(m, n)) = s(x(m', n'))$. 

Now arbitrarily choose \( y \in s(x(m, n))\Gamma^\infty \). To obtain \([5]\), one needs to show \( x(m, n)y = x(m', n')y \). Since \( \Lambda_0^0 = \Lambda^0 \) and \((n - m) - (n' - m') \in H_{\text{Per}}\), it follows from Theorem \([4.3]\) (iii) that there is a (unique) \( \mu \) in \( \Lambda \) satisfying
\[
d(\mu) = n' - m' \quad \text{and} \quad \mu \sim x(m, n).
\]
Applying Lemma \([4.4]\) again yields \( \sigma^m x = \sigma^{m'} x \). Then
\[
\mu \sigma^n x = x(m, n)\sigma^n x = \sigma^m x = \sigma^{m'} x.
\]
So
\[
\mu = (\mu \sigma^n x)(0, n' - m') = \sigma^m x(0, n' - m') = x(m', n'),
\]
which proves \([5]\).

Identifying \( \Lambda \) with \( q^* \Gamma \) via Theorem \([4.3]\) (v), we can also easily check that the inverse of \( q^* \) is given by
\[
q^* : \Gamma^\infty \to \Lambda^\infty, \quad x \mapsto q^* x : (m, n) \mapsto (x(q(m), q(n)), n - m).
\]

The rest is proved similar to \([KP00, \text{Proposition 2.9}]\). \(\blacksquare\)

**Theorem 5.3.** Let \( \Gamma = \Lambda/\sim \). Then \( \Gamma \) is an aperiodic \( q(P) \)-graph.

**Proof.** By Theorem \([4.3]\) and Corollary \([4.5]\) it is equivalent to show that \( \text{Per}(\Gamma) = \{0\} \).

For convenience, let \( q_2 \) be the quotient map from \( \Omega_P \) onto \( \Omega_Q \) defined by \( q_2(m, n) = (m + H_\Lambda, n + H_\Lambda) \). Let \( x \in \Lambda^\infty \) be an infinite path of \( \Lambda \). Define \( p_1 : \Lambda \to \Gamma \) by
\[
p_1(\lambda) = [\lambda] \quad (\lambda \in \Lambda).
\]

By Proposition \([5.2]\) the following diagram is commuting:
\[
\begin{array}{ccc}
\Omega_P & \xrightarrow{x} & \Lambda \\
\downarrow{q_2} & & \downarrow{p_1} \\
\Omega_{q(P)} & \xrightarrow{x} & \Gamma
\end{array}
\]

We now suppose that \( \mu, \nu \) are two paths in \( \Lambda \) such that \([\mu]\) and \([\nu]\) are equivalent in \( \Gamma \):
\[
[\mu] \sim_\Gamma [\nu].
\]
In what follows, we show that \( \mu \) and \( \nu \) are actually equivalent in \( \Lambda \):
\[
\mu \sim_\Lambda \nu.
\]
Once this is done, we have that \([\mu] = [\nu]\), which proves that \( \text{Per}(\Gamma) = \{0\} \).

To this end, arbitrarily choose \( x \in s(\mu)\Lambda^\infty \). Note that \( s(\mu) = s(\nu) \) as \( s([\mu]) = s([\nu]) \). By Proposition \([5.2]\) \( \dot{x} \in s([\mu])\Gamma^\infty \). Thus one obtains the
following consecutive implications:
\[ \mu \sim_{\Gamma} \nu \]
\[ \Rightarrow [\mu] \hat{x} = [\nu] \hat{x} \]
\[ \Rightarrow ([\mu] \hat{x})(q_2(0, n)) = ([\nu] \hat{x})(q_2(0, n)) \]
\[ \Rightarrow [\mu](0, q(n) - d([\mu])) = [\nu](0, q(n) - d([\nu])) \]
\[ \text{(for all } q(n) \text{ such that } q(n) - d([\mu]) \in P) \]
\[ \Rightarrow [\mu](0, d([\mu])) = [\nu](0, d([\nu])) \]
\[ \Rightarrow [x(0, d([\mu]))] = [x(0, d([\nu]))] \text{ (by the definition of } \hat{x}). \]

Thus we have \( d(\mu) - d(\nu) \in H_{\Lambda} \) by the very definition of \( H_{\Lambda} \). So there is a unique \( \nu' \in \Lambda \) such that
\[ \mu \sim_{\nu'} \text{ and } d(\nu') = d(\nu) \]
by Theorem 4.2 (iii). Hence \( \mu x = \nu' \hat{x} \), and so \([\mu] \hat{x} = [\nu'] \hat{x}\). Combining this with \([\mu] \hat{x} = [\nu] \hat{x}\) (see the first implication above) gives
\[ [\nu] \hat{x} = [\nu'] \hat{x}. \]

But we have \( d([\nu]) = d([\nu']) \) from \( d(\nu) = d(\nu') \). By Proposition 5.2 and the uniqueness of Lemma 4.2 (for the \( q(P) \)-graph \( \Gamma \)), we get \([\nu] = [\nu']\). Applying Lemma 5.1 (to the \( P \)-graph \( \Lambda \)) again, we have \( \nu = \nu' \) as \( d(\nu) = d(\nu') \). This proves \( \mu \sim_{\nu} \) from (6), and so \( \Gamma \) is aperiodic.

Identifying \( \Lambda \) with \( q^* \Gamma \), one has \( x(m, n) = ([x(m, n)], n - m) \). Then \( p_1 \) in the above proof is nothing but the projection onto the first position.

5.2. A distinguished MASA. The main aim of this subsection is to answer the questions asked in [BNR14] for row-finite higher rank graph \( \Lambda \) without sources, such that \( \Lambda^0 \) is a maximal tail and \( \Lambda_{\Per}^0 = \Lambda^0 \). We show that the \( C^* \)-algebra \( C^*(s_\mu s_\nu^*: \mu \sim \nu) \) is a MASA in \( C^*(\Lambda) \), and that there is a faithful conditional expectation from \( C^*(\Lambda) \) onto it.

The following corollary is an immediate consequence of Lemmas 5.1 and Theorems 5.2.

**Corollary 5.4.** Let \( \Lambda \) be a row-finite higher rank graph without sources, such that \( \Lambda^0 \) is a maximal tail and \( \Lambda_{\Per}^0 = \Lambda^0 \). Then
\[ C^*(\Lambda) \cong C^*(\Gamma) \otimes C(\hat{H}_{\Lambda}), \]
where the pushout \( \Gamma = \Lambda / \sim \) is an aperiodic row-finite \( q(\mathbb{N}^k) \)-graph without sources such that \( \Gamma^0 \) is a maximal tail.

Before giving the main result of this section, a simple lemma first:

**Lemma 5.5.** Let \( \Lambda \) be a row-finite higher rank graph without sources, such that \( \Lambda^0 \) is a maximal tail, and \( \mu, \nu \in \Lambda \). Then \( \mu \sim_{\nu} \) if and only if \((\mu, \nu) = (w\mu', w\nu') \) for some \( w, \mu', \nu' \in \Lambda \) such that \( d(w) = d(\mu) \wedge d(\nu) \) and \( \mu' \sim_{\nu'} \).
So the canonical diagonal algebra by Theorem 3.3 and Corollary 5.4. Also, \( \Gamma \) is aperiodic by Corollary 5.4.

If \( \Lambda \) is aperiodic, then clearly \( C_{\Gamma}(\Lambda) =: \mathcal{D} \) is isomorphic to its pullback \( \pi^* \Gamma \). Keeping the same notation as in the proof of Theorem 3.3 we have that

\[
C^*(\Lambda) \cong C^*(\pi^* \Gamma) \cong C^*(\Gamma) \otimes C(\hat{H}) \cong C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r)
\]

by Theorem 3.3 and Corollary 5.4. Also, \( \Gamma \) is aperiodic by Corollary 5.4.

So the canonical diagonal algebra \( \mathcal{D} \) of \( C^*(\Gamma) \) is a MASA by Theorem 2.3. Hence \( \mathcal{D}_\Gamma \otimes C(\hat{H}) = \mathcal{D} \) is isomorphic to a MASA in \( C^*(\Lambda) \) [Was76, Was08]. But \( \mathcal{D}_\Gamma \cong \mathcal{D}_\Lambda \) by Remark 3.4 and Theorem 4.3, so \( \mathcal{D}_\Lambda \cong \mathcal{D}_\Lambda \otimes C^*(W_1, \ldots, W_r) \) is a MASA in \( C^*(\Lambda) \).

In the sequel, slightly abusing the notation, for \( 1 \leq i \leq r \), we also use \( W_i \) to denote the unitary in \( \mathcal{Z}(\mathcal{M}(C^*(\Lambda))) \) corresponding to the one in \( \mathcal{Z}(\mathcal{M}(C^*(q^* \Gamma))) \) mentioned above. We next prove

\[
\mathcal{D}_\Lambda \otimes C^*(W_1, \ldots, W_r) \cong \mathcal{D}_\Lambda C^*(W_1, \ldots, W_r).
\]

Let \( \tilde{\pi} \) be the natural homomorphism from \( \mathcal{D}_\Gamma \otimes C^*(W_1, \ldots, W_r) \cong \mathcal{D}_\Lambda \otimes C^*(W_1, \ldots, W_r) \) onto \( \mathcal{D}_\Lambda C^*(W_1, \ldots, W_r) \) (cf. Remark 3.4), let \( \iota_1 \) be the inclusion of \( \mathcal{D}_\Lambda C^*(W_1, \ldots, W_r) \) to \( C^*(\Lambda) \), and \( \iota_2 \) be the inclusion of \( \mathcal{D}_\Gamma \otimes C^*(W_1, \ldots, W_r) \) to \( C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r) \). Then one can verify that \( \iota_1 \tilde{\pi} = \pi \iota_2 \), where \( \pi \) is the isomorphism defined in (3). That is, the following diagram

\[
\begin{array}{ccc}
\mathcal{D}_\Gamma \otimes C^*(W_1, \ldots, W_r) & \xrightarrow{\tilde{\pi}} & C^*(\Gamma) \otimes C^*(W_1, \ldots, W_r) \\
\downarrow \iota_2 & & \downarrow \pi \\
\mathcal{D}_\Lambda C^*(W_1, \ldots, W_r) & \xrightarrow{\iota_1} & C^*(\Lambda)
\end{array}
\]
commutes. Then one can easily check that \( \tilde{\pi} \) is injective, and so it is a \(*\)-isomorphism.

We now show that 
\[
\mathcal{D}_\Lambda C^*(W_1, \ldots, W_r) = C^*(s_\mu s_\nu^* : \mu \sim \nu).
\]

Obviously, the left hand side is contained in the right hand side. To show the other inclusion, we make use of Lemma 5.5. Let \( \mu \sim \nu \). Then \( (\mu, \nu) = (w\mu', w\nu') \) with \( d(w') = d(\mu) \wedge d(\nu) \) and \( \mu' \sim \nu' \). Note \( d(\mu') \wedge d(\nu') = 0 \). Let \( h = d(\mu') - d(\nu') \). By [CKSS14, Proposition 3.3], the central unitary multiplier \( W_h \) defined by 
\[
W_h = \sum_{[\lambda]} s_{[\lambda]}(d(\mu')) s_{[\lambda]}^*(d(\nu'))
\]
satisfies 
\[
s_{[\lambda]}(d(\mu')) = W_h s_{[\lambda]}(d(\nu')).
\]
In particular, this implies \( s_{\mu'} = W_h s_{\nu'} \). Hence 
\[
s_\mu s_\nu^* = W_h s_{\mu'} s_\nu^* s_{\mu'}^* = W_h s_{\nu'} s_{\nu'}^* \in \mathcal{D}_\Lambda C^*(W_1, \ldots, W_r).
\]
This ends our proof.

The following corollary is straightforward by Corollary 5.4 and Theorem 5.6, as \( \mathcal{D}_\Gamma \) is the canonical MASA of the AF-algebra \( \mathfrak{F}_\Gamma \), the fixed point algebra of the gauge action of \( C^*(\Gamma) \).

**Corollary 5.7.** There is a faithful conditional expectation from \( C^*(\Lambda) \) onto the MASA \( C^*(s_\mu s_\nu^* : \mu \sim \nu) \).

Let us end by remarking the simplicity of \( C^*(\Gamma) \) and the centre of \( C^*(\Lambda) \).

**Corollary 5.8.** Suppose that \( \Lambda \) is cofinal. Then \( C^*(\Gamma) \) is simple and the centre of \( C^*(\Lambda) \) is \( C^*(W_1, \ldots, W_r) \cong \mathcal{C}(\hat{H}_\Lambda) \).

**Proof.** By Proposition 5.2, it is easy to see that \( \Gamma \) is also cofinal. By Theorem 5.3, \( \Gamma \) is aperiodic. As [KP00, Proposition 4.8], one can see that \( C^*(\Gamma) \) is simple. The rest of the corollary can be easily seen from Corollary 5.4. 

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