New structures for colored HOMFLY-PT invariants

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Abstract In this paper, we present several new structures for the colored HOMFLY-PT (Hoste-Ocneanu-Millet-Freyd-Lickorish-Yetter-Przytycki-Traczyk) invariants of framed links. First, we prove the strong integrality property for the normalized colored HOMFLY-PT invariants by purely using the HOMFLY-PT skein theory developed by Morton and his collaborators. By this strong integrality property, we immediately obtain several symmetric properties for the full colored HOMFLY-PT invariants of links. Then we apply our results to refine the mathematical structures appearing in the Labastida-Mariño-Ooguri-Vafa (LMOV) integrality conjecture for framed links. As another application of the strong integrality, we obtain that the $q=1$ and $a=1$ specializations of the normalized colored HOMFLY-PT invariant are well-defined link polynomials. We find that a conjectural formula for the colored Alexander polynomial which is the $a=1$ specialization of the normalized colored HOMFLY-PT invariant implies that a special case of the LMOV conjecture for framed knots holds.

Keywords colored HOMFLY-PT invariant, skein theory, integrality, string duality

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1 Introduction

1.1 Colored HOMFLY-PT invariants

The colored HOMFLY-PT invariant is an invariant of framed oriented links in $S^3$ whose components are colored with partitions. It can be viewed as a generalization of the HOMFLY-PT polynomial which was first discovered by Freyd et al. [22] and Przytycki and Traczyk [76]. Based on Turaev’s work [82], the HOMFLY-PT polynomial can be derived from the quantum group invariant labeled with the fundamental representation of the quantum group $U_q(sl_N)$. More generally, the colored HOMFLY-PT invariants can be constructed by using the quantum group invariants labeled with arbitrary irreducible representations of $U_q(sl_N)$ which are labeled by partitions. We refer to [41, 46] for the detailed definitions of the colored HOMFLY-PT invariants along this way.

Historically, the quantum group approach to the colored HOMFLY-PT invariant originates from the $SU(N)$ Chern-Simons theory due to Witten [89]. The large $N$ duality between the $SU(N)$ Chern-Simons theory and topological string theory established in [24, 73, 90] makes the colored HOMFLY-PT invariant become a central object in geometry and physics. There have been spectacular developments on the colored HOMFLY-PT invariant and its related topics over the years. For example, the dependence of the
colors is governed by recurrence relations, which was proposed in [2] and proved in [23]. Moreover, these recurrence relations are conjectured to have deep relationships to the augmentation polynomials in knot contact homology [1,2]. The relationship to the topological string theory connects the colored HOMFLY-PT invariant to Gromov-Witten theory [15,47,48,56], Hilbert schemes [72], stable pair invariants [14,57], quiver varieties [39,40,53,74,79,80], knot contact homology [1,2,18,21], etc. There have also been a lot of works devoted to the computations of the colored HOMFLY-PT invariants (see [25,30,36,46,69,70,88] and the references therein).

On the other hand, it is well known that the HOMFLY-PT polynomial satisfies the fantastic skein equivalence. Therefore, in this paper, we mainly use the HOMFLY-PT skein theory as our tool to derive several new structures for colored HOMFLY-PT invariants. The equivalence of the two approaches to colored HOMFLY-PT invariants via quantum group theory and HOMFLY-PT skein theory was presented in [3,51], and we also refer the reader to Appendix B for an explanation of this equivalence.

Compared with the quantum group theory, HOMFLY-PT skein theory is more intuitional and straightforward. Therefore, in this paper, we mainly use the HOMFLY-PT skein theory as our tool to derive several new structures for colored HOMFLY-PT invariants.

1.1.1 Strong integrality
Let $L$ be a framed oriented link with $L$ components $[\alpha]$ for $\alpha = 1, \ldots, L$. Given $\bar{x} = (\lambda^1, \ldots, \lambda^L)$ and $\bar{\mu} = (\mu^1, \ldots, \mu^L)$, where each $\lambda^\alpha$ (resp. $\mu^\alpha$) for $\alpha = 1, \ldots, L$ denotes a partition of a positive integer, we define the framed full colored HOMFLY-PT invariant of $L$ colored by $[\bar{x}, \bar{\mu}]$ by

$$\mathcal{H}_{[\bar{x}, \bar{\mu}]}(L; q, a) = \mathcal{H}(L \star \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha}; q, a).$$

(1.1)

Here $\mathcal{H}(L \star \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha}; q, a)$ denotes the framed HOMFLY-PT polynomial of the link $L$ decorated by $\bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha}$, where each $Q_{\lambda^\alpha, \mu^\alpha}$ denotes the skein basis element in the full skein $\mathcal{L}$ of the annulus, and we recall their definitions in Section 2. In particular, when all $\mu^\alpha$'s are equal to the empty partition $\emptyset$, we write $Q_{\lambda^\alpha} = Q_{\lambda^\alpha, \emptyset}$ for $\alpha = 1, \ldots, L$, and define

$$\mathcal{H}_{\emptyset}(L; q, a) = \mathcal{H}(L \star \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha}; q, a).$$

Then we obtain the standard (framing independent) colored HOMFLY-PT invariant for $L$ (e.g., the definitions in [46,93]), i.e.,

$$W_{\bar{x}}(L; q, a) = q^{-\sum_{\alpha=1}^{L} \kappa_{\lambda^\alpha} w([\kappa^\alpha]) a^{-\sum_{\alpha=1}^{L} |\lambda^\alpha| w([\kappa^\alpha])} \mathcal{H}_{\emptyset}(L; q, a).$$

(1.2)

When $L$ has only one component, i.e., it is a knot which is usually denoted by $K$, we define the normalized framed full colored HOMFLY-PT invariants of $K$ as follows:

$$P_{[\lambda, \mu]}(K; q, a) = \frac{\mathcal{H}_{[\lambda, \mu]}(K; q, a)}{\mathcal{H}_{[\lambda, \mu]}(U; q, a)},$$

(1.3)

where $U$ denotes the unknot throughout this paper. Then we have the following strong integrality property for $P_{[\lambda, \mu]}(K; q, a)$.

**Theorem 1.1.** For any framed knot $K$, the normalized framed full colored HOMFLY-PT invariant satisfies

$$P_{[\lambda, \mu]}(K; q, a) \in a^2 \mathbb{Z}[q^{\pm 2}, a^{\pm 2}],$$

(1.4)

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1) In the following, when we mention a link $L$, we always assume that it is a framed oriented link with $L$ components given by $K_{\alpha}$ for $\alpha = 1, \ldots, L$. 

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where $\epsilon \in \{0, 1\}$ is determined by the following formula:

$$(|\lambda| + |\mu|)w(K) \equiv \epsilon \pmod{2},$$

(1.5)

with $w(K)$ the writhe number of $K$.

In particular, the normalized colored HOMFLY-PT invariant $P_\lambda(K; q, a)$ of $K$, defined as $P_\lambda(K; q, a) = \frac{W_\lambda(K; q, a)}{W_\lambda(U(q, a))}$, satisfies

$$P_\lambda(K; q, a) \in \mathbb{Z}[q^2, a^2].$$

(1.6)

More generally, for a link $L$ with $L$ components, one can define the $\alpha$-normalized framed full colored HOMFLY-PT invariants for $\alpha = 1, \ldots, L$ as follows. We only decorate the $\alpha$-component $K_\alpha$ by $Q_\lambda^\alpha$, keep the other components invariant, and we denote the corresponding colored HOMFLY-PT invariant of $L$ by $H_{[\lambda^\alpha, \mu^\alpha]}(L; q, a)$. Then the $\alpha$-normalized framed full colored HOMFLY-PT invariant of $L$ is defined by

$$P_{[\lambda^\alpha, \mu^\alpha]}(L; q, a) = \frac{H_{[\lambda^\alpha, \mu^\alpha]}(L; q, a)}{H_{[\lambda^\alpha, \mu^\alpha]}(U; q, a)}.$$

(1.7)

We also have the following strong integrality theorem.

**Theorem 1.2.** For any framed link $L$ with $L$ components $K_\alpha$ for $\alpha = 1, \ldots, L$, the $\alpha$-normalized framed full colored HOMFLY-PT invariant satisfies

$$P_{[\lambda^\alpha, \mu^\alpha]}(L; q, a) \in a^\epsilon \mathbb{Z}[q^{L+2}, a^{L+2}],$$

(1.8)

where $\epsilon \in \{0, 1\}$ which is determined by the following formula:

$$(|\lambda^\alpha| + |\mu^\alpha|)w(K_\alpha) + \sum_{\beta \neq \alpha} w(K_\beta) + L - 1 \equiv \epsilon \pmod{2}.$$ 

(1.9)

As a consequence of Theorem 1.2, we have the following corollary.

**Corollary 1.3.** For any framed link $L$, the framed full colored HOMFLY-PT invariants $H_{[\lambda, \mu]}(L; q, a)$ satisfy the following symmetric properties:

$$H_{[\lambda, \mu]}(L; q, -a) = (-1)^{\sum_{\alpha=1}^L (|\lambda^\alpha| + |\mu^\alpha|)} H_{[\lambda, \mu]}(L; q, a),$$

(1.10)

$$H_{[\lambda, \mu]}(L; q, -a) = (-1)^{\sum_{\alpha=1}^L (|\lambda^\alpha| + |\mu^\alpha|)w(K_\alpha)+1} H_{[\lambda, \mu]}(L; q, a),$$

(1.11)

$$H_{[\lambda, \mu]}(L; q^{-1}, a) = (-1)^{\sum_{\alpha=1}^L (|\lambda^\alpha| + |\mu^\alpha|)} H_{[\lambda, \mu]}(L; q, a).$$

(1.12)

**Remark 1.4.** We prove Theorems 1.1 and 1.2 by purely using HOMFLY-PT skein theory. It essentially uses all the crucial results developed by Morton and his collaborators. Compared with our previous work [95], the proof presented here is essentially self-contained.

**Remark 1.5.** From Corollary 1.3, one immediately obtains the following symmetries for colored HOMFLY-PT invariants $W_{[\lambda]}(L; q, a)$:

$$W_{[\lambda]}(L; -q, a) = (-1)^{\sum_{\alpha=1}^L |\lambda^\alpha|} W_{[\lambda]}(L; q, a),$$

(1.13)

$$W_{[\lambda]}(L; q, -a) = (-1)^{\sum_{\alpha=1}^L |\lambda^\alpha|} W_{[\lambda]}(L; q, a),$$

(1.14)

$$W_{[\lambda]}(L; q^{-1}, a) = (-1)^{\sum_{\alpha=1}^L |\lambda^\alpha|} W_{[\lambda]}(L; q, a),$$

(1.15)

which have appeared in previous literature (see, for example, [50, 93]).
1.1.2 Specializations

From Theorem 1.1, we obtain that the \( q = 1 \) specialization of the normalized framed full colored HOMFLY-PT invariant \( P_{[\lambda,\mu]}(K;1,a) \) belongs to the ring \( a^\mathbb{Z}[a^2] \). With a slight modification of the proof presented in [12], we obtain the formula

\[
P_{[\lambda,\mu]}(K;1,a) = P(K;1,a)^{|[\lambda]|+|[\mu]|},
\]

where \( P(K; q, a) \) is the normalized framed HOMFLY-PT polynomial.

The composite invariant studied in [12,55] is defined as follows. Given a link \( L \) with \( L \) components and \( \vec{\lambda}, \vec{\mu}, \vec{\nu} \in P^L \), let

\[
c_{[\vec{\lambda}, \vec{\mu}, \vec{\nu}] = \prod_{\alpha = 1}^{L} c_{[\lambda, \mu]}(\alpha),
\]

where each \( c_{[\lambda, \mu]}(\alpha) \) is the Littlewood-Richardson coefficient, and then the framed composite invariant of \( L \) is defined by

\[
C_{[\vec{\nu}]}(L; q, a) = \sum_{[\vec{\lambda}, \vec{\mu}, \vec{\nu}]} c_{[\vec{\lambda}, \vec{\mu}, \vec{\nu}]}(\alpha) H_{[\vec{\lambda}, \vec{\mu}, \vec{\nu}]}(L; q, a).
\]

In particular, for the knot \( K \), we prove the special framed composite invariant is the following limit which exists and satisfies the relation

\[
\lim_{q \to 1} \frac{C_{[\vec{\nu}]}(K; q, a)}{C_{[\vec{\nu}]}(U; q, a)} = P_{[\lambda, \mu]}(K; 1, a)^{|[\nu]|}.
\]

On the other hand, one can also consider the \( a = 1 \) specialization of the normalized framed full colored HOMFLY-PT invariants, and we define the full colored Alexander polynomial of \( K \) as follows:

\[
A_{[\lambda,\mu]}(K; q) = P_{[\lambda,\mu]}(K; q,1).
\]

Then Theorem 1.1 implies that

\[
A_{[\lambda,\mu]}(K; q) \in \mathbb{Z}[q^2].
\]

**Remark 1.6.** In [30], Itoyama et al. first studied the following limit of the colored HOMFLY-PT invariant:

\[
A_{\lambda}(K; q) = \lim_{a \to 1} \frac{W_{\lambda}(K; q, a)}{W_{\lambda}(U; q, a)},
\]

and they proposed that \( A_{\lambda}(K; q) = A(K; q^{[\lambda]}) \), where \( A(K; q) \) is the classical Alexander polynomial of \( K \). Later, in [93], Zhu found that this formula does not hold for general partitions. The colored Alexander polynomial in the sense of [30] can be written in our notation as

\[
A_{[\lambda,\mu]}(K; q) = q^{-\kappa_{\lambda,\mu}(K)} A_{[\lambda,\mu]}(K; q).
\]

That is also why we refer to \( A_{[\lambda,\mu]}(K; q) \) as the full colored Alexander polynomial (see also [60]).

In conclusion, we have the following standard conjecture for the colored Alexander polynomial \( A_{\lambda}(K; q) \).

**Conjecture 1.7.** For any hook partition \( \lambda \), we have

\[
A_{\lambda}(K; q) = A(K; q^{[\lambda]}).
\]
1.2 LMOV integrality structures

Based on the large $N$ duality [24, 90] and BPS counting theory in topological strings, Labastida and Mariño [41], Labastida et al. [42] and Ooguri and Vafa [73] formulated an interesting integrality conjecture for colored HOMFLY-PT invariants which was referred to as the LMOV conjecture in the later literature. Roughly speaking, the LMOV conjecture states that the LMOV function which is a plethystic transformation of the generating functions for colored HOMFLY-PT invariants inherits an amazing integrality structure. During the study of the LMOV conjecture, we find that such integrality structures appear frequently in other settings. So we think that it is worth putting these questions together to see if there is a unified theory to deal with them. In the following, we first formulate the general framework for such an integrality structure which we refer to as the LMOV integrality structure. Then we describe the subtle LMOV integrality structure for framed links in detail and present several new results as the applications of the results obtained in previous Subsection 1.1.

Let $\Lambda(\mathbf{x})$ be the ring of symmetric functions of $\mathbf{x} = (x_1, x_2, \ldots)$ over the field $\mathbb{Q}(q, a)$. For $\bar{\mathbf{x}} = (\bar{x}^1, \ldots, \bar{x}^L)$, we consider the ring $\Lambda(\bar{\mathbf{x}}) = \Lambda(\mathbf{x}) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \Lambda(\mathbf{x}^L)$ of functions separately symmetric in $\bar{x}^1, \ldots, \bar{x}^L$, where $\bar{x}^i = (x^i_1, x^i_2, \ldots)$. Suppose that $B_\lambda(\bar{\mathbf{x}})$, $\bar{\lambda} \in \mathcal{P}^L$ form a basis of $\Lambda(\bar{\mathbf{x}})$ (e.g., taking $B_\lambda(\bar{\mathbf{x}})$ as the Schur function, the Macdonald function, etc.), and consider the generating function of a series of functions $S_\lambda(\mathbf{x}, a) \in \mathbb{Q}(q, a)$, i.e.,

$$Z(\bar{\mathbf{x}}; q, a) = \sum_\lambda S_\lambda(\mathbf{x}, a) B_\lambda(\bar{\mathbf{x}}). \quad (1.24)$$

The LMOV function for $\{S_\lambda(\mathbf{x}, a) \mid \bar{\lambda} \in \mathcal{P}^L\}$ is defined as the plethystic transformation of $Z(\bar{\mathbf{x}}; q, a)$, i.e.,

$$f_\lambda(q, a) = \langle \text{Log}(Z(\bar{\mathbf{x}}; q, a)), B_\lambda(\bar{\mathbf{x}}) \rangle, \quad (1.25)$$

where $\langle \cdot, \cdot \rangle$ denotes the Hall pair in the ring of symmetric functions $\Lambda(\bar{\mathbf{x}})$.

The LMOV integrality structure states that the LMOV function will be a polynomial of $q$ and $a$ with integral coefficients. Moreover, these integral coefficients have different geometric meanings under different settings. Here, we provide two examples.

(a) Character varieties. In [29], Hausel et al. proposed a general conjecture for the mixed Hodge polynomial of the generic character varieties. In our language, they conjectured that the mixed Hodge polynomial is the LMOV function for the HLV kernel functions introduced in [29], and these integral coefficients appearing in LMOV functions can be interpreted as the mixed Hodge numbers of the generic character varieties. The integrality of the LMOV function in this case was proved by Mellit [58]. Moreover, a possible interpretation for such LMOV structures as a kind of topological quantum field theory (TQFT) was described in [59] (see [59, Subsection 1.6]). See also [13] and the references therein for several physical generalizations of the above framework.

(b) Donaldson-Thomas invariants for quivers. Given a quiver, there is an associated algebra named the chomological Hall algebra introduced by Kontsevich and Soibelman [38]. In order to study the integrality of the Donaldson-Thomas invariants, they introduced the notion of admissibility for a series of rational functions. The admissibility condition is equivalent to saying that the corresponding LMOV function carries the integrality structure in our language. Based on this observation, we started to interpret the LMOV invariant for the frame unknot in terms of quiver representation theory [53, 96]. There have been great developments in this direction (see Subsection 1.3.3 for more descriptions about it).

1.2.1 LMOV structures for framed links

Actually, the LMOV structures for framed links are more subtle. Given a framed link $\mathcal{L}$ with $L$ components $K_1, \ldots, K_L$, denote by $\bar{\tau} = (\tau^1, \ldots, \tau^L) \in \mathbb{Z}^L$ the framing of $\mathcal{L}$, i.e., $\tau^\alpha = w(K_\alpha)$ for $\alpha = 1, \ldots, L$. In the following, we use the notation $\mathcal{L}_{\bar{\tau}}$ to denote this framed link $\mathcal{L}$ if we want to emphasize its framing.
We introduce the notions of the $\vec{\tau}$-framed full colored HOMFLY-PT invariant and the $\vec{\tau}$-framed composite invariant for $\mathcal{L}_{\vec{\tau}}$ as follows:

$$H^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a) = (-1)^{\sum_{\alpha = 1}^{L}|\lambda^\alpha|} a^{-\sum_{\alpha = 1}^{L}|\lambda^\alpha|} H^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a),$$  \hfill (1.26) 
$$C^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a) = (-1)^{\sum_{\alpha = 1}^{L}|\lambda^\alpha|} \sum_{\vec{\lambda}, \vec{\mu}} a^{-\sum_{\alpha = 1}^{L}|\mu^\alpha|} c_{\vec{\lambda}, \vec{\mu}}^{\vec{\tau}} H^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a).$$  \hfill (1.27)

We remark that the above definitions of $\vec{\tau}$-framed invariants were motivated by Mariño and Vafa [56], where the term $(-1)^{\sum_{\alpha = 1}^{L}|\lambda^\alpha|}$ is essential if we consider the framing change in Chern-Simons quantum field theory. The LMOV functions for $\mathcal{L}_{\vec{\tau}}$ are defined as the character transformation of the LMOV functions $H^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a)$ and $C^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a)$ are given by

$$f^{(0)}_{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a) = \left\langle \log \sum_{\vec{\lambda}} H^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a) s_{\vec{\lambda}}(\vec{x}), s_{\vec{\lambda}}(\vec{x}) \right\rangle$$  \hfill (1.28) 
and

$$f^{(1)}_{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a) = \left\langle \log \sum_{\vec{\lambda}} C^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a) s_{\vec{\lambda}}(\vec{x}), s_{\vec{\lambda}}(\vec{x}) \right\rangle,$$  \hfill (1.29)

respectively.

The reformulated LMOV functions are defined as the character transformation of the LMOV functions as follows:

$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a) = \frac{1}{\{\vec{\mu}\}} \sum_{\vec{\lambda}} \chi^{\vec{\lambda}}(\vec{\mu}) f^{(h)}_{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a), \quad h = 0, 1. \hfill (1.30)$$

From the formula (1.7), we obtain that for $\vec{\lambda} \neq \vec{0}$, both $H^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a)$ and $C^{\vec{\lambda}}(\mathcal{L}_{\vec{\tau}}; q, a)$ contain a factor $\frac{a^{-\sum_{\alpha = 1}^{L}|\lambda^\alpha|}}{q^{-\sum_{\alpha = 1}^{L}|\lambda^\alpha|}}$. As an application of Theorem 1.2 and Corollary 1.3, we obtain the following theorem.

**Theorem 1.8.** The reformulated LMOV functions $g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a)$ for both $h = 0$ and $h = 1$ can be written in the form

$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a) = (a - a^{-1}) g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a), \hfill (1.31)$$

and $g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a)$ has the properties

$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, -a) = (-1)^{\{\vec{\mu}\}} a^{-1} g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a),$$  \hfill (1.32) 
$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; -q, a) = g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a),$$  \hfill (1.33) 
$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q^{-1}, a) = g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a).$$  \hfill (1.34)

Furthermore, the LMOV conjecture for framed colored HOMFLY-PT invariants [11, 56] and framed composite invariants [12, 55] states that for $h = 0$ and $h = 1$,

$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a) \in \mathbb{Z}[z^2, a^\pm 1]. \hfill (1.35)$$

**Remark 1.9.** We remark that the notation $z = q - q^{-1}$ will be used frequently throughout this article.

Combining Theorem 1.8, we obtain the following conjecture.

**Conjecture 1.10 (Refined LMOV conjecture for framed links).** For $h = 0$ and $h = 1$, the reformulated LMOV functions can be written as

$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a) = (a - a^{-1}) g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a), \hfill (1.36)$$

where

$$g^{(h)}_{\vec{\mu}}(\mathcal{L}_{\vec{\tau}}; q, a) \in \mathbb{Z}[z^2, a^\pm 2] \hfill (1.37)$$
and $\epsilon \in \{0, 1\}$ is determined by $|\mu| - 1 \equiv \epsilon \pmod{2}$.

In other words, there are integral invariants
\[ \tilde{N}^{(h)}_{\mu,g,Q}(L) \in \mathbb{Z} \] such that
\[ \tilde{g}^{(h)}_{\mu}(L; q, a) = \sum_{g \geq 0} \sum_{Q \in \mathbb{Z}} \tilde{N}^{(h)}_{\mu,g,Q}(L) z^{2g-2} a^{2Q+\epsilon} \in z^{-2} a^\epsilon \mathbb{Z}[z^2, a^{\pm 2}]. \] (1.39)

**Remark 1.11.** In Section 5, we also introduce the notation of special LMOV functions for framed colored HOMFLY-PT invariants and composite invariants. There is also an integrality conjecture for them (see Conjecture 5.11 in Section 5).

**Remark 1.12.** We name Conjecture 1.10 the “refined LMOV conjecture”. It should be noted that the meaning of refinement here is different from the notion “refined” used in [33], where actually the LMOV conjecture for the refined colored HOMFLY-PT invariants is proposed.

As a special case of Conjecture 1.10, we have the following conjecture.

**Conjecture 1.13.** Given a framed knot $K_{r,s}$, for any prime $p$, we have
\[ \tilde{g}^{(0)}_p(K_{r,s}; q, 1) \in z^{-2} a^\epsilon \mathbb{Z}[z^2, a^{\pm 2}], \] (1.40)

where $\epsilon = 1$ when $p = 2$ and $\epsilon = 0$ when $p$ is an odd prime.

Furthermore, we prove the following theorem in Section 7.

**Theorem 1.14.** If Conjecture 1.7 holds, then we have
\[ \tilde{g}^{(0)}_p(K_{r,s}; q, 1) \in z^{-2} \mathbb{Z}[z^2]. \] (1.41)

In particular, $\tilde{g}^{(0)}_p(T_{r,s}; q, 1) \in z^{-2} \mathbb{Z}[z^2]$ for the torus knot $T_{r,s}$.

### 1.3 Related works and future directions

#### 1.3.1 The degrees of $a$ and $q$

As shown in Theorem 1.2, for a link with $L$ components, the normalized framed full colored HOMFLY-PT invariant $P_{[\lambda^\alpha, \mu^\alpha]}(L; q, a)$ lies in the ring $a^\epsilon \mathbb{Z}[q^2, a^2]$, so we can write it in the following two forms:

\[ P_{[\lambda^\alpha, \mu^\alpha]}(L; q, a) = \sum_{k=m}^M c_k(a)q^{2k}, \] (1.42)

\[ P_{[\lambda^\alpha, \mu^\alpha]}(L; q, a) = \sum_{k=n}^N d_k(q)a^{2k+\epsilon} \] (1.43)

for some integers $m, M, n$ and $N$, which depend on $L$ and the partitions $\lambda^\alpha$ and $\mu^\alpha$.

At this point, one natural question is how to estimate these integers $m, M, n$ and $N$. For the HOMFLY-PT polynomial [45], there was a classical result due to Morton [61] which states that the above integers are bounded by algebraic crossing numbers and the number of Seifert circles of the knot diagram. In [85], van der Veen generalized Morton’s result to the case of the colored HOMFLY-PT polynomial colored with the one-box partition. So it is natural to consider if there are analogous bounding formulas in our general case. Furthermore, the topological interpretation of the degrees would be another interesting question. We hope to address this question in the future work.

#### 1.3.2 Categorifications

The HOMFLY-PT polynomial can be categorified to the Khovanov-Rozansky homology [37]; after that, there have been a lot of works devoted to the categorified theory for the colored HOMFLY-PT invariants (see, e.g., [26,86,87]). Our formulas (1.4) and (1.8) suggest that the normalized framed full colored HOMFLY-PT invariants may have categorifications whose homology is finite-dimensional. It is a challenging question to construct such categorifications for them.
1.3.3 New integral link invariants

The refined LMOV Conjecture 1.10 predicts the existence of new integral link invariants $\tilde{\mathcal{N}}^{(h)}(\vec{\mu}, g, Q, \vec{\tau})$ for the framed link $\mathcal{L}_\tau$. A natural question is how to define these new integral invariants directly. In their original literature [42, 73], Labastida et al. and Ooguri and Vafa proposed that these new integral link invariants can be interpreted as the Euler characteristic number of a certain moduli space which is expected to exist. In [53], through a straightforward computation, we found these integral invariants for the framed unknot $U_\tau$ were related to the Betti number of the cohomological Hall algebra of a corresponding quiver. Then the idea of knot-quiver correspondence was further extended in [39, 40, 96] (see [17, 74, 79, 80] for more recent developments).

Obviously, these integral link invariants are fully determined by the Chern-Simons partition function of the framed link. From the point of view of topological string theory, Aganagic and Vafa [2] introduced the deformed A-polynomial for colored HOMFLY-PT invariants which can be viewed as the mirror geometry of the topological string theory. According to the large $N$ duality of Chern-Simons theory and topological string theory [24, 42, 73], these new integral link invariants are fully determined by the deformed A-polynomial. Later, the existence of the deformed A-polynomial was proved rigidly in [23].

On the other hand, there is a proposal initiated by Aganagic et al. [1] which connects the topological strings and contact homology theory. In this framework, it is conjectured that the deformed A-polynomial is equal to the augmentation polynomial from knot contact homology [71]. Therefore, there should be a geometric way to define these new integral link invariants from knot contact homology.

1.3.4 Geometry of the colored HOMFLY-PT invariants

Recall that the large $N$ duality proposes that the colored HOMFLY-PT invariant of a link in $S^3$ gives the count of all the holomorphic curves in the resolved conifold with boundary on the shifted Lagrangian conormal of this link. However, a theory of open Gromov-Witten invariants was still missing. In [19], Ekholm and Shende developed a new method to define the invariant count of holomorphic curves with Lagrangian boundary $L$. The basic idea is to use the framed HOMFLY-PT skein module $Sk(L)$ (see Section 2 for the definition of skein module) to describe the obstruction to invariance. They introduced the notion of skein-valued open Gromov-Witten invariant for a link in $S^3$ which is the generating function of colored HOMFLY-PT invariants of this link in the skein $Sk(L)$ [20, 21]. It is expected that the new structures for colored HOMFLY-PT invariants obtained in this paper would provide new insights in this counting theory of holomorphic curves.

1.3.5 Colored Kauffman invariants

It is well known that Kauffman polynomials [35] are another useful two-variable polynomial invariants of knots and links. Its colored version was established in [10] by following the work of [46].

The corresponding skein theory for the colored Kauffman invariants was established in [7, 78]. So we hope to establish the analogous structure properties for colored Kauffman invariants via Kauffman skein theory. From the point of view of Chern-Simons theory, the colored Kauffman invariants correspond to the gauge group $SO(N)/Sp(N)$. In [10], Chen and Chen proposed an analogous LMOV integrality conjecture for colored Kauffman invariants (see also [55] for a mixed version of the LMOV integrality conjecture). In [12], we have shown that the composite invariants and colored Kauffman invariants carry the same congruence skein relations. It would be interesting to consider the analogous refined LMOV integrality structure for the colored Kauffman invariants.

1.4 The structure of this paper

The rest of this paper is organized as follows. In Section 2, we survey the main results of the HOMFLY-PT skein theory and fix the notations. In Section 3, we briefly review the construction of the idempotents of Hecke algebras $H_n(q, a)$ following the work [6], and then we establish a key formula related to the idempotent which will be applied in Section 4 to prove the strong integrality for the normalized framed
full colored HOMFLY-PT invariants. In Section 5, we introduce the notion of the LMOV structure, and then we refine the LMOV integrality structure for the framed colored HOMFLY-PT invariants and composite invariants by using the results obtained in Section 4. In Sections 6 and 7, we are devoted to two specializations of the normalized framed full colored HOMFLY-PT invariants. We prove an identity for special polynomials in Section 6 and present a conjectural formula for the colored Alexander polynomial in Section 7. Finally, we prove that this conjectural formula implies a special case of the LMOV conjecture for framed knots. In Appendix A, we provide the basic notations related to partitions and symmetric functions used in this article. Then in Appendix B, we present the equivalence of the two approaches to the colored HOMFLY-PT invariants via quantum group theory and HOMFLY-PT skein theory.

2 HOMFLY-PT skein theory

Given a 3-manifold $M$ (possibly with boundary), roughly speaking, a skein module is a quotient of the free module over isotopy classes of links in $M$ by suitably chosen skein relations. Let us recall briefly the history of development of skein modules.

Motivated by the pioneering work of Jones [31], Kauffman [34] introduced the notion of the Kauffman bracket which provided a simple approach to the Jones polynomials. More precisely, the Kauffman bracket is a framed knot invariant defined by a local relation named the Kauffman bracket skein relation together with the value of the unknot. By imposing this relation and the value of the unknot, we can reduce the Kauffman bracket of any link in $\mathbb{R}^3$ to a Laurent polynomial in $q$ times the empty link. In order to extend the definition of the Jones polynomial of knots in $\mathbb{R}^3$ to a general oriented 3-manifold $M$, Turaev [83, 84] and Przytycki [75] independently introduced the notion of skein module of $M$. In particular, the Kauffman bracket skein modules of manifolds were for the first time defined in [75], which are formal linear combinations of framed unoriented links considered up to the Kauffman bracket skein relation together with the value of the unknot. Furthermore, the Kauffman bracket skein module of a surface $\Sigma$ is defined as the Kauffman bracket skein module of the cylinder $\Sigma \times [0, 1]$ over the surface $\Sigma$. In this setup, the skein module carries a natural multiplication structure and becomes an algebra which is referred to as the Kauffman bracket skein algebra. In the past decades, the Kauffman bracket skein algebras of surfaces have been proved to have connections and applications to many interesting objects such as character varieties, quantum invariants, quantum Teichmüller spaces, cluster algebras, and so on (see, e.g., [8, 9, 44, 68, 81]).

On the other hand, it is well known that there exists the two-variable generalization of the Jones polynomial, i.e., the HOMFLY-PT polynomial [22] which can be defined through two local relations named HOMFLY-PT skein relations as shown in Figure 1, where $z = q - q^{-1}$.

![Figure 1 Local relations](image-url)
Therefore, it is natural to define the HOMFLY-PT skein modules for a general oriented 3-manifold $M$ by imposing the HOMFLY-PT skein relations [62]. The HOMFLY-PT skein theory has been studied extensively by Morton and his collaborators (see [3–6, 28, 51, 52, 62] and [63, 64, 66, 67]) during the past decades. In the following context, we attempt to provide a comprehensive review of HOMFLY-PT skein theory.

### 2.1 HOMFLY-PT skein algebras

The account here largely follows those of [28, 64, 67]. Throughout this article, we work over the coefficient ring $R = \mathbb{C}[a^{\pm 1}, q^{\pm 1}, (q - q^{-1})^{-k}]$ with $k$ ranging over $\mathbb{N}$.

Suppose that $M$ is an oriented 3-manifold. A framed oriented link in $M$ is a smooth embedding of $\sqcup S^1 \times [0, 1]$ up to isotopy in $M$. Let $\mathcal{L}(M)$ be the free $R$-module spanned by framed oriented links in $M$, and $\mathcal{L}'(M) \subseteq \mathcal{L}(M)$ be the $R$-submodule generated by the skein relations in Figure 1.

**Definition 2.1.** The framed HOMFLY-PT skein module $\text{Sk}(M)$ of the manifold $M$ is defined as the quotient

$$\text{Sk}(M) := \mathcal{L}(M)/\mathcal{L}'(M).$$

By its definition, the HOMFLY-PT skein module $\text{Sk}(M)$ has several basic properties such as the following:

(i) The $\text{Sk}(M)$ is graded by the first homology group $H_1(M)$, since each skein relation involves only links in the same homology class.

(ii) An oriented embedding $f : M \to N$ induces an $R$-linear map $f_* : \text{Sk}(M) \to \text{Sk}(N)$. When $f$ is a homeomorphism, the map $f_*$ is an isomorphism.

(iii) In particular, when $F$ is an orientable surface and $M = F \times [0, 1]$, we use the notation $\text{Sk}(F)$ in place of $\text{Sk}(F \times [0, 1])$, and refer to $\text{Sk}(F)$ as the HOMFLY-PT skein module of the surface $F$. In this setup, $\text{Sk}(F)$ becomes an algebra over the coefficient ring $R$. The product is given by stacking links. The grading is additive under the product. Usually, we call $\text{Sk}(F)$ the HOMFLY-PT skein algebra of the surface $F$.

**Remark 2.2.** One can fix an $n$-point set $S \subset F$ and include $n$ arcs from $S \times \{0\}$ to $S \times \{1\}$ which are called the $n$-tangles. Similarly, all the $R$-linear combinations of $n$-tangles in $(F, S)$ up to isotopy modulo the skein relations give an algebra denoted by $\text{Sk}_n(F)$. When $n = 0$, $\text{Sk}_0(F)$ is just the skein algebra $\text{Sk}(F)$.

**Remark 2.3.** Framed links in $F \times [0, 1]$ can be represented by diagrams in $F$ with the blackboard framing from $F$. Therefore, one can also regard elements of $\text{Sk}(F)$ as diagrams in $F$ modulo Reidemeister moves II and III and skein relations as shown in Figure 1. It is easy to see that the removal of an unknot is equivalent to multiplying a scalar $s = \frac{q - 1}{q + 1}$, i.e., we have the relation shown in Figure 2.

**Remark 2.4.** For every surface $F$, we define the mirror map $- : \text{Sk}(F) \to \text{Sk}(F)$ as follows. For every tangle $T$ in $F$, we define $\overline{T}$ to be the tangle $T$ with all its crossings switched. As to the coefficient ring $R$, we define $\overline{q} = q^{-1}$ and $\overline{a} = a^{-1}$. Then these operations induce a linear automorphism with respect to the skein relation $- : \text{Sk}(F) \to \text{Sk}(F)$, which was called the mirror map on $\text{Sk}(F)$.

In the following, we present several basic examples of the HOMFLY-PT skein algebras.

![Figure 2](image-url) Removal of an unknot
2.1.1 The plane $\mathbb{R}^2$

When $F$ is the plane $\mathbb{R}^2$, it is easy to see that every element in $Sk(\mathbb{R}^2)$ can be represented as a scalar in the ring $\mathbb{R}$. For a link $L$ with a diagram $D_L$, the resulting scalar $\langle D_L \rangle \in R$ is the framed HOMFLY-PT polynomial $H(L; q, a)$ of the link $L$, i.e., $H(L; q, a) = \langle D_L \rangle$. We use the convention $\langle \cdot \rangle = 1$ for the empty diagram, so $H(U; q, a) = \frac{a - a^{-1}}{q - q^{-1}}$. The two relations shown in Figure 1 lead to

\begin{align}
H(L_+; q, a) - H(L_-; q, a) &= \lambda H(L_0; q, a), \quad (2.2) \\
H(L^{+1}; q, a) &= aH(L; q, a) \quad \text{and} \quad H(L^{-1}; q, a) = a^{-1}H(L; q, a). \quad (2.3)
\end{align}

Under our notation, the classical HOMFLY-PT polynomial of a link $L$ is given by

$$P(L; q, a) = \frac{a^{-w(L)}H(L; q, a)}{H(U; q, a)},$$

where $w(L)$ denotes the writhe number of the link $L$.

2.1.2 The rectangle

We write $H_{n,m}(q, a)$ for the skein algebra $Sk_{n,m}(F)$ consisting of $(n, m)$-tangles, where $F$ is the rectangle with $n$ inputs and $m$ outputs at the top and matching inputs and outputs at the bottom. There is a natural algebra structure on $H_{n,m}(q, a)$ if the tangles are placed one above the another. When $m = 0$, we write $H_n(q, a) = H_{n,0}(q, a)$ for brevity.

Remark 2.5. It is well known that the Hecke algebra $H_n$ of type $A_{n-1}$ is the algebra generated by $\sigma_i : i = 1, \ldots, n - 1$, subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i : |i - j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} : 1 \leq i \leq n - 1,$$
$$\sigma_i - \sigma_i^{-1} = z.$$

It is straightforward to see that the Hecke algebra $H_n$, with $z = q - q^{-1}$ and the coefficient ring extended to include $a^{\pm 1}$ and $q^{\pm 1}$, is isomorphic to the skein algebra $H_n(q, a)$ described above. In this algebra the extra variable $a$ in the coefficient ring allows us to reduce general tangles to linear combinations of braids by means of the skein relations.

The construction of the idempotent element $y_\lambda$ of $H_n(q, a)$ via skein theory was presented in [6]. This construction and the main results are briefly reviewed in the next section.

Set $T^{(n)}$ to be the element in $H_n(q, a)$ as illustrated in Figure 3, and let

$$t_\lambda = za \sum_{x \in \lambda} q^{2cn(x)} + \frac{a - a^{-1}}{z} \quad (2.5),$$

where $cn(x) = j - i$ is the content of the cell in the row $i$ and the column $j$ of the Young diagram for the partition $\lambda$. Then we have

$$T^{(n)} y_\lambda = t_\lambda y_\lambda. \quad (2.6)$$

Since a partition is uniquely determined by its contents, it is readily verified that $t_\lambda = t_\mu$ if and only if $\lambda = \mu$.

Similarly, we set $T^{(n,m)}$ to be the analogous elements in $H_{n,m}(q, a)$, and let

$$t_{\lambda, \mu} = z \left( a \sum_{x \in \lambda} q^{2cn(x)} - a^{-1} \sum_{x \in \mu} q^{-2cn(x)} \right) + \frac{a - a^{-1}}{z}. \quad (2.7)$$

It is also clear that $t_{\lambda, \mu} = t_{\lambda', \mu'}$ if and only if $\lambda = \lambda'$ and $\mu = \mu'$. 

Similarly, element $T$ of the skein direction, to give diagrams $\pi$ where $\ast$: distinguished boundary points induces a linear automorphism of the skein $C$ is represented by the closure of the braid $\sigma_{m|1} \cdots \sigma_2 \sigma_1$. The orientation of the curve around the annulus is counter-clockwise for positive $m$ and clockwise for negative $m$. The element $A_0$ is the identity element and is represented by the empty diagram. The algebra $C$ is the product of two subalgebras $C^+$ and $C^-$ generated by $\{A_m : m \in \mathbb{Z}, m \geq 0\}$ and $\{A_m : m \in \mathbb{Z}, m \leq 0\}$, respectively.

Take a diagram $X$ in the annulus and link it once with a simple meridian loop, oriented in either direction, to give diagrams $\varphi(X)$ and $\bar{\varphi}(X)$ in the annulus. This induces linear endomorphisms $\varphi$ and $\bar{\varphi}$ of the skein $C$, called the meridian maps. Each space $C^{(n,m)}$ is invariant under $\varphi$ and $\bar{\varphi}$. We set $\varphi^{(n)} = \varphi|_{C^{(n)}}$ and $\varphi^{(n,m)} = \varphi|_{C^{(n,m)}}$.

**Remark 2.6.** Rotating diagrams in the annulus $S^1 \times I$ about the horizontal axis through the distinguished boundary points induces a linear automorphism of the skein $C$, which we denote by $*: C \to C$. It is clear to see that $A^*_m = A_{-m}$, and then $(C^+)^* = C^-$ and $(C^{n,m})^* = C^{m,n}$.

**The subspace $C^{(n)} \subset C^+$.** The subspace $C^{(n)}$ is spanned by monomials in $\{A_m\}$, with $m \in \mathbb{Z}^+$, of total weight $n$, where the weight of $A_m$ is $m$. It is clear that this spanning set consists of $\pi(n)$ elements, where $\pi(n)$ denotes the number of partitions of $n$. $C^+$ is then graded as an algebra

$$C^+ = \bigoplus_{n=0}^{\infty} C^{(n)}. \tag{2.8}$$

This subspace $C^+$ can also be viewed as the image of $H_n(q,a)$ under the closure map $\hat{\cdot}$. Given the element $T^{(n)} \in H_n(q,a)$ as illustrated in Figure 3, take an element $S \in H_n(q,n)$ with $\hat{S} \in C^{(n)}$ and compose it by $T^{(n)}$. Then

$$\hat{T^{(n)}} \hat{S} = \varphi^{(n)}(\hat{S}). \tag{2.9}$$

Similarly,

$$\hat{T^{(n)}} \hat{S} = \bar{\varphi}^{(n)}(\hat{S}). \tag{2.10}$$

Then it is easy to show that the eigenvalues of $\varphi^{(n)}$ are all distinct. Indeed, set $Q_\lambda = \hat{y}_\lambda \in C^{(n)}$. Taking closures on both sides of the formula (2.6), we immediately obtain

$$\varphi^{(n)}(Q_\lambda) = t_\lambda Q_\lambda. \tag{2.11}$$

The element $Q_\lambda$ is then an eigenvector of $\varphi^{(n)}$ with the eigenvalue $t_\lambda$. There are $\pi(n)$ such eigenvectors, and the eigenvalues are all distinct. Hence all these $Q_\lambda$’s are linearly independent. Since $C^{(n)}$ is spanned

![Figure 3](image-url)
by $\pi(n)$ elements we can deduce that all the elements $Q_\lambda$ with $|\lambda| = n$ form a basis for $C^{(n)}$ and that the eigenspaces are all 1-dimensional.

It was shown in [51, 52] that $Q_\lambda$ can be expressed as an explicit integral polynomial in $\{h_m\}_{m \geq 0}$ and $C^+$ can be regarded as the ring of symmetric functions in variables $x_1, \ldots, x_N, \ldots$ with the coefficient ring $R$. In this situation, $C^{(n)}$ consists of the homogeneous functions of degree $n$. The power sum $P_n = \sum x_i^n$ is a symmetric function which can be presented in terms of the complete symmetric functions, and hence it represents a skein element which is also denoted by $P_n \in C^{(n)}$. Moreover, we have the identity

$$[n]P_n = X_n = \sum_{j=0}^{n-1} A_{n-1-j,j},$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $A_{i,j}$ is the closure of the braid

$$\sigma_{i+j}\sigma_{i+j-1}\cdots \sigma_{j+1}\sigma_j^{-1}\cdots \sigma_1^{-1}.$$

Given a partition $\mu$, we define

$$P_{\mu} = \prod_{i=1}^{l(\mu)} P_{\mu_i}.$$  

According to the Frobenius formula for the Schur function, we have

$$Q_\lambda = \sum_{\mu} \frac{\chi_\lambda(\mu)}{\delta_\mu} P_{\mu}. $$  

**The subspace $C^{(n,m)}$.** The image of $H_{n,m}(q,a)$ under the closure map is denoted by $C^{(n,m)} \subset C$. Unlike the case for $C^{(n)}$ where $C^{(n)} \cap C^{(n-1)} = \emptyset$, we have

$$C^{(n,m)} \supset C^{(n-1,m-1)} \supset \cdots \supset \begin{cases} C^{(n-m,0)}, & \min(n, m) = m, \\ C^{(0,m-n)}, & \min(n, m) = n \end{cases}$$

and

$$C^{(n,0)} = C^{(n)} (-1), \quad C^{(0,m)} = C^{(m)} (-1).$$

where the subscripts indicate the direction of the strings around the center of the annulus. We have $C^{(n_1,m_1)} \cap C^{(n_2,m_2)} = \emptyset$ if $n_1 - m_1 \neq n_2 - m_2$. Indeed, $C^{(n,m)}$ is spanned by suitably weighted monomials in

$$\{A_n, \ldots, A_1, A_0, A_{-1}, \ldots, A_{-m}\}. $$

We can then see that

$$C^{(n,m)} = (C_+^{(n)} \times C_-^{(m)} + C^{(n-1,m-1)}).$$

The spanning set of $C^{(n,m)}$ consists of $\pi(n, m)$ elements, where

$$\pi(n, m) := \sum_{j=0}^{k} \pi(n-j)\pi(m-j)$$

with $k = \min(n, m)$.

Similar to grading of $C^+$ with the $C^{(n)}$, the full skein $C$ can be written in terms of $C^{(n,m)}$ as follows:

$$C = \bigoplus_{k=-\infty}^{\infty} \left( \bigcup_{n,m \geq 0} \{C^{(n,m)} : n - m = k \} \right).$$

Now, we consider the meridian map

$$\varphi^{(n,m)} : C^{(n,m)} \to C^{(n,m)}.$$  

Morton and Hadji [65] proved the following proposition.
Proposition 2.7 (See [65, Theorem 6]). \( t_{\lambda,\mu} \)'s given by the formula (2.7) with \(|\lambda| \leq n, |\mu| \leq m \) and \(|\lambda| - |\mu| = n - m \) are eigenvalues of the meridian map \( \varphi^{(n,m)} \); moreover, they occur with multiplicity 1.

A straightforward consequence is the following corollary.

**Corollary 2.8.** There is a basis of \( C^{(n,m)} \) given by

\[
\{ Q_{\lambda,\mu} : |\lambda| \leq n, |\mu| \leq m, |\lambda| - |\mu| = n - m \}
\]

such that \( \varphi(Q_{\lambda,\mu}) = t_{\lambda,\mu}Q_{\lambda,\mu} \).

Then in [28], Hadji and Morton extended the method of [51, 52] and obtained the explicit expression for these basis elements \( \{ Q_{\lambda,\mu} \} \).

**The explicit expression for the basis elements** \( Q_{\lambda,\mu} \). Let the element \( h_m \in C^{(m)} \) be the closure of the element \( \frac{1}{\alpha_m}a_m \in H_m(q,a) \), i.e., \( h_m = \frac{1}{\alpha_m}a_m \), where \( a_m \) is the quasi-idempotent whose definition is given by the formula (3.3), and \( \alpha_m \) is the scalar given by \( \alpha_m = q^{m(m-1)/2} \prod_{k=1}^{m} \frac{q^{r_k} - q^{-r_k}}{q^{r_k} - q^{-r_k}} \in R \).

There is a readily defined involution on the skein \( C \) denoted by \( * \). It is the rotation of diagrams in the annulus \( S^1 \times [0,1] \) by \( \pi \) around the horizontal axis through the distinguished boundary points. We can see that \( (A_m)^* = A_{-m} \) so that \( (C^+)^* = C^- \). Furthermore, \( (C^{(n,m)})^* = C^{(m,n)} \).

The skein \( C^+ \) (resp. \( C^- \)) is spanned by the monomials in \( \{ h_m \}_{m \geq 0} \) (resp. \( \{ h_k^* \}_{k \geq 0} \)). Indeed, the explicit relations between \( \{ h_m \} \) and the Turaev’s basis \( \{ A_m \} \) are presented in [63, Theorem 3.6] and [66, Theorem 8]. Then the whole skein \( C \) is spanned by the monomials in \( \{ h_m, h_k^* \}_{m,k \geq 0} \).

Given two partitions \( \lambda \) and \( \mu \) with \( l \) and \( r \) parts, we first construct an \( (l + r) \times (l + r) \)-matrix \( M_{\lambda,\mu} \) with entries in \( \{ h_m, h_k^* \}_{m,k \in \mathbb{Z}} \) as follows. We let \( h_m = 0 \) if \( m < 0 \) and \( h_k^* = 0 \) if \( k < 0 \), and then

\[
M_{\lambda,\mu} = \begin{pmatrix}
  h_{\mu_1}^* & h_{\mu_2}^* & \cdots & h_{\mu_r}^* \\
  h_{\mu_1}^* & h_{\mu_2}^* & \cdots & h_{\mu_r-l}^* \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{\mu_1-(r-1)}^* & h_{\mu_1-(r-2)}^* & \cdots & h_{\mu_1-l}^* \\
  h_{\lambda_1-r} & h_{\lambda_1-(r-1)} & \cdots & h_{\lambda_1+l-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{\lambda_1-l} & h_{\lambda_1-l+r} & \cdots & h_{\lambda_l}
\end{pmatrix}
\]

(2.23)

It is easy to see that the subscripts of the diagonal entries in the \( h \)-rows are the parts \( \lambda_1, \lambda_2, \ldots, \lambda_l \) of \( \lambda \) in order, while the subscripts of the diagonal entries in the \( h^* \)-rows are the parts \( \mu_1, \mu_2, \ldots, \mu_r \) of \( \mu \) in reverse order.

Then \( Q_{\lambda,\mu} \) is defined as the determinant of the matrix \( M_{\lambda,\mu} \), i.e.,

\[
Q_{\lambda,\mu} = \det M_{\lambda,\mu}.
\]

(2.24)

**Example 2.9.** For two partitions \( \lambda = (4,2,2) \) and \( \mu = (3,2) \), we have

\[
Q_{\lambda,\mu} = \det \begin{pmatrix}
  h_2^* & h_1^* & 1 & 0 & 0 \\
  h_4^* & h_5^* & h_2^* & 1 \\
  h_6 & h_3 & h_4 & h_5 & h_6 \\
  0 & 1 & h_1 & h_2 & h_3 \\
  0 & 0 & 1 & h_1 & h_2
\end{pmatrix}.
\]

(2.25)

Furthermore, we have

\[
Q_{\lambda,\mu} = \sum_{\sigma,\rho,\nu} (-1)^{|\sigma|} c_{\sigma,\rho}^\lambda c_{\mu,\nu}^\rho \varphi(\rho,\theta)Q_{\rho,\theta}Q_{\theta,\nu}.
\]

(2.26)
where $c_{\sigma,\rho}^\lambda$ is the Littlewood-Richardson coefficient which is determined by the product formula for Schur functions

$$s_\sigma(x)s_\rho(x) = \sum_\lambda c_{\sigma,\rho}^\lambda s_\lambda(x). \quad (2.27)$$

Moreover, the basis elements $Q_{\lambda,\mu}$ of $C$ have the property that the product of any two is a non-negative integral linear combination of basis elements.

### 2.2 Colored HOMFLY-PT invariants

Let $L$ be a framed link with $L$ components with a fixed numbering. For diagrams $Q_1, \ldots, Q_L$ in the skein model of the annulus with the positive oriented core $C^+$, we define the decoration of $L$ with $Q_1, \ldots, Q_L$ as the link

$$L \star \bigotimes_{i=1}^L Q_i, \quad (2.28)$$

which is derived from $L$ by replacing every annulus $L$ by the annulus with the diagram $Q_1$ such that the orientations of the cores match. Each $Q_i$ has a small blackboard neighborhood in the annulus which makes the decorated link $L \bigotimes_{i=1}^L Q_i$ into a framed link (see Figure 4 for a framed trefoil $K$ decorated by the skein element $Q$).

The framed colored HOMFLY-PT polynomial of $L$ is defined to be the framed HOMFLY-PT polynomial of the decorated link $L \star \bigotimes_{i=1}^L Q_i$, which is given by

$$\mathcal{H} \left( L \star \bigotimes_{i=1}^L Q_i; q, a \right) = \left\langle L \star \bigotimes_{i=1}^L Q_i \right\rangle. \quad (2.29)$$

In particular, if we choose $L$ basis elements $Q_{\lambda,\emptyset}$ with $\alpha = 1, \ldots, L$ in the full skein $C$, then we have the following definition.

**Definition 2.10.** The **framed full colored HOMFLY-PT invariant** of the framed link $L$ denoted by $\mathcal{H}_{[\vec{\lambda},\vec{\mu}]}(L; q, a)$, is defined as the framed HOMFLY-PT polynomial of the decorated link

$$L \star \bigotimes_{\alpha=1}^L Q_{\lambda,\mu^\alpha},$$

i.e.,

$$\mathcal{H}_{[\vec{\lambda},\vec{\mu}]}(L; q, a) := \mathcal{H} \left( L \star \bigotimes_{\alpha=1}^L Q_{\lambda,\mu^\alpha}; q, a \right). \quad (2.30)$$

In particular, when $\mu^\alpha = \emptyset$, $Q_{\lambda,\emptyset} = Q_{\lambda}$. In this case, the framing factor for $Q_{\lambda}$ is given by $q^{-\kappa_\lambda} a^{-|\lambda|}$. So we can add a framing factor to eliminate the framing dependency. It makes the framed colored HOMFLY-PT invariant $\mathcal{H}_{[\vec{\lambda}]}(L; q, a)$ into a framing independent invariant.

![Figure 4](Color online) $K$ decorated by $Q$
Definition 2.11. The (framing independent) colored HOMFLY-PT invariant for the link $L$ is given by
\[ W_{\vec{\lambda}}(L; q, a) = q^{-\sum_{\alpha=1}^{L} |\lambda^\alpha| w(K^\alpha)} \Big( \sum_{\alpha=1}^{L} \kappa_{\lambda^\alpha} |w(K^\alpha)| \Big) H_{\vec{\lambda}}(L; q, a), \] (2.31)
where $\vec{\lambda} = (\lambda^1, \ldots, \lambda^L) \in \mathcal{P}_L$.

Example 2.12. For the unknot $U$,
\[ W_{\mu}(U; q, a) = H_{\mu}(U; q, a) = \prod_{x \in \mu} \left( aq^{cn(x)} - a^{-1}q^{-cn(x)} \right). \] (2.32)

By the formula (2.26), we have
\[ H_{[\lambda, \mu]}(U; q, a) = \sum_{\sigma, \rho, \nu} (-1)^{|\sigma|} c_{\lambda, \mu}^{\sigma} W_{\rho}(U; q, a) W_{\nu}(U; q, a). \] (2.33)

Definition 2.13. Given a framed knot $K$, the normalized framed full HOMFLY-PT invariant of $K$ is defined as follows:
\[ P_{[\lambda, \mu]}(K; q, a) = \frac{H_{[\lambda, \mu]}(K; q, a)}{H_{[\lambda, \mu]}(U; q, a)}. \] (2.34)

In other words,
\[ H_{[\lambda, \mu]}(K; q, a) = P_{[\lambda, \mu]}(K; q, a) H_{[\lambda, \mu]}(U; q, a). \] (2.35)

If we draw the framed knot $K$ in the annulus as the closure of a 1-tangle, then decorating it by $Q \in \mathcal{C}$ gives a diagram of $K \star Q$ in the annulus (see Figure 5 for the case where $K$ is the trefoil knot).

This construction induces a linear map $T_K : \mathcal{C} \rightarrow \mathcal{C}$. Suppose that $Q$ is the eigenvector of $T_K$ with the eigenvalue $t_K(q, a)$. Then
\[ K \star Q = T_K(Q) = t(K; q, a)Q = t(K; q, a)U \star Q. \] (2.36)

In particular, when $Q = Q_{\lambda, \mu}$, taking the HOMFLY-PT polynomial, we obtain
\[ t(K; q, a) = \frac{H_{[\lambda, \mu]}(K; q, a)}{H_{[\lambda, \mu]}(U; q, a)}. \] (2.37)

Therefore, $P_{[\lambda, \mu]}(K; q, a)$ is also referred to as the framed HOMFLY-PT 1-tangle invariant in [64].

For a link $L$ with $L$ components, let $\vec{\nu} = (\nu^1, \ldots, \nu^L)$, $\vec{\lambda} = (\lambda^1, \ldots, \lambda^L)$ and $\vec{\mu} = (\mu^1, \ldots, \mu^L) \in \mathcal{P}_L$, and set $c_{\lambda, \mu}^{\nu} = \prod_{\alpha=1}^{L} c_{\lambda^\alpha, \mu^\alpha}^{\nu^\alpha}$, where $c_{\lambda^\alpha, \mu^\alpha}$ is the Littlewood-Richardson coefficient.

Definition 2.14. We define the framed composite invariants for $L$ as follows:
\[ C_{\vec{\nu}}(L; q, a) = \sum_{\vec{\lambda}, \vec{\mu}} c_{\vec{\lambda}, \vec{\mu}}^{\vec{\nu}} H_{[\vec{\lambda}, \vec{\mu}]}(L; q, a). \] (2.38)

Figure 5 (Color online) The element $K \star Q$ in the annulus.
3 Idempotents in Hecke algebras

In this section, we first briefly review the construction of the idempotent of the Hecke algebra $H_n(q,a)$ via skein theory (see [6]). Then we establish an important formula which will be used in the next section.

3.1 Young diagrams

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of $n$. Then $\lambda$ can be represented by a Young diagram which is a collection of $n$ cells arranged in rows, with $\lambda_1$ cells in the first row and $\lambda_2$ cells in the second row up to $\lambda_l$. Usually, we denote both the partition and its Young diagram by $\lambda$. The conjugate of $\lambda'$ of $\lambda$ is the Young diagram whose rows form the columns of $\lambda$. For the cell in the $i$-th row and the $j$-th column of $\lambda$ we write $(i,j) \in \lambda$, and refer to $(i,j)$ as the coordinates of the cell.

A standard tableau $T' = T(\lambda)$ is an assignment of the numbers $1$ to $n$ to the cells of $\lambda$ such that the numbers increase from the left to the right along the rows and from the top to the bottom down the columns. In particular, $T(\lambda)$ will denote the tableau where the cells of the Young diagram are numbered from $1$ to $n$ along the rows. Note that the transposition of rows and columns does not take $T(\lambda)$ to $T(\lambda')$. We define the permutation $\pi_\lambda$ by $\pi_\lambda(i) = j$, where the transposition of $\lambda$ carries the cell $i$ in $T(\lambda)$ to the cell $j$ in $T(\lambda')$.

Let $\lambda$ and $\mu$ be Young diagrams with $|\lambda| = |\mu| = n$. We say that a permutation $\pi \in S_n$ separates $\lambda$ from $\mu$ if no pair of numbers in the same row of $T(\lambda)$ is mapped by $\pi$ to the same row of $T(\mu)$. For example, the permutation $\pi_\lambda$ separates $\lambda$ from its conjugate $\lambda'$.

Denote by $R(\lambda) \subseteq S_n$ the subgroup of the permutations which preserve the rows of $T(\lambda)$. It is easy to see that if $\pi$ separates $\lambda$ from $\mu$ then so does $\rho \pi \sigma$ for any $\rho \in R(\lambda)$ and $\sigma \in R(\mu)$, and conversely, if $\pi$ separates $\lambda$ from $\lambda'$ then $\pi = \rho \pi_\lambda \sigma$ with $\rho \in R(\lambda)$ and $\sigma \in R(\lambda')$. We say that $\lambda$ and $\mu$ are inseparable if no permutation $\pi \in S_n$ separates $\lambda$ from $\mu$.

For every permutation $\pi \in S_n$, there exists a unique braid $\omega_\pi$ (called a positive permutation braid) which is uniquely determined by the following properties:

(i) all the strings are oriented from the top to the bottom;
(ii) for $i = 1, \ldots, n$, the $i$-th string joins the point numbered $i$ at the top of the braid to the point numbered $\pi(i)$ at the bottom of the braid;
(iii) all the crossings occur with positive signs and each pair of strings crosses at most once.

**Lemma 3.1** (See [6]). Let $\pi \in S_n$ be a permutation which separates $\lambda$ from $\lambda'$. Then there are $\rho \in R(\lambda)$ and $\sigma \in R(\lambda')$ such that

$$\omega_\pi = \omega_\rho \omega_\pi \omega_\sigma. \tag{3.1}$$

**Proof.** Since $\pi$ separates $\lambda$ from $\lambda'$, there are $\rho \in R(\lambda)$ and $\sigma \in R(\lambda')$ such that $\pi = \rho \pi_\lambda \sigma$. By the definition of $\omega_\pi$, the pairs of strings which start in the same row of $T(\lambda)$ or end in the same row of $T(\lambda')$ would never cross. The only pairs which cross in $\omega_\rho$ or $\omega_\sigma$ are in the same row of $T(\lambda)$ or $T(\lambda')$, respectively. It follows that $\omega_\rho \omega_\pi \omega_\sigma$ is a positive permutation braid. Hence $\omega_\pi = \omega_\rho \omega_\pi \omega_\sigma$. □

3.2 Constructions and properties

Let $F = R_n^a$ be a rectangle with $n$ inputs at the top and $n$ outputs at the bottom. Let $H_n(q,a)$ be the skein $S_{kn}(R_n^a)$ of $n$-tangles (see Figure 6 for an element in $H_n(q,a)$).

Composing $n$-tangles by placing one above another induces a product which makes $H_n(q,a)$ into the Hecke algebra of type $A_{n-1}$ with the coefficient ring $R$. $H_n(q,a)$ has a presentation generated by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2, \tag{3.2}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad |i-j| = 1,$$

$$(\sigma_i - q)(\sigma_i + q^{-1}) = 0, \quad i = 1, 2, \ldots, n-1.$$
The Hecke algebra $H_n$ can be viewed as the quantum deformation of the group algebra $\mathbb{C}[S_n]$ of the symmetric group $S_n$, whose idempotents are described by the classical Young symmetrisers. For a Young diagram $\lambda$, its Young symmetriser is the product of the sum of permutations which preserve the rows of $T(\lambda)$ and the alternating sum of permutations which preserve columns. In order to construct the idempotents in the Hecke algebra, the idea is to make the suitable quantum deformation of the classical Young symmetrisers. The two simplest idempotents in $H_n$, corresponding to the single row and the single column Young diagrams, are given algebraically by Jones [32]. Their skein version is described by Morton [62] in terms of positive permutation braids $\omega_{\pi}$ ($\pi \in S_n$). Then Gyoja [27] constructed the idempotents for general $\lambda$. Finally, Aiston and Morton [6] provided the construction of the idempotents in the HOMFLY-PT skein $H_n(q,a)$ via skein theory.

We first define Jones’s row and column elements $a_j$ and $b_j$, following the account in [62]. Write $E_n(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) = \sum_{\pi \in \mathcal{S}_n} \omega_{\pi}$ for the sum of positive permutation braids. We define

$$a_n = E_n(q\sigma_1, q\sigma_2, \ldots, q\sigma_n) = \sum_{\pi \in \mathcal{S}_n} q^{l(\pi)} \omega_{\pi},$$

$$b_n = E_n(-q^{-1}\sigma_1, -q^{-1}\sigma_2, \ldots, -q^{-1}\sigma_n) = \sum_{\pi \in \mathcal{S}_n} (-q)^{-l(\pi)} \omega_{\pi}. \quad (3.3)$$

**Lemma 3.2.** The element $a_n$ (resp. $b_n$) can be factorised in $H_n(q,a)$ with $\sigma_i + q^{-1}$ (resp. $\sigma_i - q$) as a left or a right factor for any $i \in \{1, 2, \ldots, n-1\}$. If we define two linear homomorphisms $\varphi$ and $\psi$ from the Hecke algebra $H_n(q,a)$ to the ring of scalars $R$ defined by $\varphi(\sigma_i) = -q^{-1}$ and $\psi(\sigma_i) = q$, respectively, for $i = 1, \ldots, n-1$, then for all $T \in H_n(q,a)$, we have

$$a_n T = T a_n = \psi(T) a_n, \quad b_n T = T b_n = \varphi(T) b_n. \quad (3.4)$$

**Proof.** This lemma was first proved in [62]. For the sake of completeness, we provide the proof here. Given $i \in \{1, 2, \ldots, n-1\}$, we can pair the permutations as follows. For each permutation $\pi$, consider its composite $\pi' = \pi \circ (i, i+1)$ with transposition $(i, i+1)$, and then $\pi'(i) = \pi(i+1)$ and $\pi'(i+1) = \pi(i)$. Hence exactly one of them preserves the order of $i$ and $i+1$, and suppose that it is $\pi$ such that $\pi(i) < \pi(i+1)$. Consider the positive braid $\omega_{\pi}$, and $\sigma_i = \omega_{(i,i+1)}$. It is clear that $\omega_{\pi} \sigma_i$ is also a positive braid, and then $\omega_{\pi'} = \omega_{\pi} \sigma_i$.

Therefore,

$$a_n = \sum_{\pi(i) < \pi(i+1)} q^{l(\pi)} \omega_{\pi} + \sum_{\pi(i) > \pi(i+1)} q^{l(\pi')} \omega_{\pi'}$$

$$= \sum_{\pi(i) < \pi(i+1)} q^{l(\pi)} \omega_{\pi} + \sum_{\pi(i) < \pi(i+1)} q^{l(\pi)+1} \omega_{\pi} \sigma_i$$

$$= a_n^{(i)} (1 + q \sigma_i), \quad (3.5)$$

where $a_n^{(i)} = \sum_{\pi(i) < \pi(i+1)} q^{l(\pi)} \omega_{\pi}$.

By using the quadratic relations of the Hecke algebra $(\sigma_i + q^{-1})(\sigma_i - q) = 0$ for $1 \leq i \leq n-1$, we have

$$a_n \sigma_i = a_n^{(i)} (1 + q \sigma_i) \sigma_i = qa_n^{(i)} (1 + q \sigma_i) = qa_n. \quad (3.6)$$
On the other hand, given $i \in \{1, \ldots, n - 1\}$, one can also consider the composition of permutations $\pi' = (i, i + 1) \pi$. Then
\[
a_n = \sum_{\pi^{-1}(i) < \pi^{-1}(i+1)} q^{l(\pi)} \omega_\pi + \sum_{\pi^{-1}(i) > \pi^{-1}(i+1)} q^{l(\pi')} \omega_{\pi'},
\]
\[
= \sum_{\pi^{-1}(i) < \pi^{-1}(i+1)} q^{l(\pi)} \omega_\pi + \sum_{\pi^{-1}(i) < \pi^{-1}(i+1)} q^{l(\pi)+1} \sigma_\pi \omega_\pi
\]
\[
= (1 + qa_i) a_n^{(i)}, \tag{3.7}
\]
where $a_n^{(i)} = \sum_{\pi^{-1}(i) < \pi^{-1}(i+1)} q^{l(\pi)} \omega_\pi$. By this formula we also obtain $\sigma_i a_n = qa_n$. Hence we prove the first part of the formula (3.4), and the proof of the second part of (3.4) is similar.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$, we define
\[
E_\lambda = a_{\lambda_1} \otimes a_{\lambda_2} \otimes \cdots \otimes a_{\lambda_l}, \tag{3.8}
\]
\[
F_\lambda = b_{\lambda_1} \otimes b_{\lambda_2} \otimes \cdots \otimes b_{\lambda_l}. \tag{3.9}
\]
Let $H_\lambda$ be the subalgebra of $H_n(q,a)$ generated by $\{\omega_\mu, \rho \in R(\lambda)\}$. Then $E_\lambda, F_\lambda \in H_\lambda$ and
\[
E_\lambda T = TE_\lambda = \psi(T) E_\lambda,
\]
\[
F_\lambda T = TF_\lambda = \varphi(T) F_\lambda \tag{3.10}
\]
for any $T \in H_\lambda$ by Lemma 3.2.

**Lemma 3.3.** Let $\lambda$ and $\mu$ be two Young diagrams of $n$ cells, and $\pi \in S_n$ be a permutation which does not separate $\lambda$ from $\mu$. Then
\[
E_\lambda \omega_\pi F_\mu = 0 = F_\lambda \omega_\pi E_\mu. \tag{3.11}
\]

**Proof.** This proof essentially follows [6, Lemma 4.4]. Since $\pi$ does not separate $\lambda$ from $\mu$, there are two cells in the same row of $\lambda$ which are sent to two cells in the same row of $\mu$ by $\pi$. Then one can find $\rho \in R(\lambda)$ and $\sigma \in R(\mu)$ such that $\pi' = \rho \pi \sigma$ sends the two adjacent cells $i$ and $i + 1$ in the same row of $\lambda$ to the two adjacent cells $j$ and $j + 1$ in the same row of $\mu$. Then we have $\omega_\pi = T^{\omega_\pi} T'$ for some $T \in H_\lambda$ and $T' \in H_\mu$. Therefore,
\[
E_\lambda \omega_\pi F_\mu = \psi(T) \varphi(T') E_\lambda \omega_\pi F_\mu. \tag{3.12}
\]
So we only need to show $E_\lambda \omega_\pi F_\mu = 0$. As to the permutation $\pi'$, we have $\pi'(i+1) = \pi(i) + 1 = j + 1$, where $(i, i + 1) \in R(\lambda)$ and $(j, j + 1) \in R(\mu)$. By the adjacency, we obtain $\sigma_i \omega_{\pi'} = \omega_{\pi'} \sigma_j$. Hence,
\[
(q^{-1} + \sigma_i) \omega_{\pi'} = \omega_{\pi'} (q^{-1} + \sigma_j). \tag{3.13}
\]
By the formulas (3.5) and (3.7), we have $E_\lambda = T(q^{-1} + \sigma_i)$ and $F_\mu = (q - \sigma_j) T'$ for some $T \in H_\lambda$ and $T' \in H_\mu$, and then
\[
E_\lambda \omega_{\pi'} F_\mu = T(q^{-1} + \sigma_i) \omega_\pi' (q - \sigma_j) T' = T \omega_{\pi'} (q^{-1} + \sigma_j) (q - \sigma_j) T' = 0. \tag{3.14}
\]
The proof of the second identity in (3.11) is similar.

**Corollary 3.4.** Given a Young diagram $\lambda$,
\[
E_\lambda \omega_\pi F_\mu = \begin{cases} (-1)^{l(\sigma)} q^{l(\rho)-l(\sigma)} E_\lambda \omega_{\pi', \nu} F_{\lambda^\nu}, & \text{if } \omega_\pi = \omega_\rho \omega_{\pi', \nu} \omega_\sigma, \\ 0, & \text{otherwise} \end{cases} \tag{3.15}
\]
and
\[
F_\lambda \omega_\pi E_\mu = \begin{cases} (-1)^{l(\sigma)} q^{l(\rho)-l(\sigma)} F_\lambda \omega_{\pi', \nu} E_{\lambda^\nu}, & \text{if } \omega_\pi = \omega_\nu \omega_{\pi', \nu} \omega_\rho, \\ 0, & \text{otherwise} \end{cases} \tag{3.16}
\]
for some $\rho \in R(\lambda)$ and $\sigma \in R(\lambda^\nu)$.
Proof. By Lemma 3.3, if \( \pi \) does not separate \( \lambda \) from \( \lambda' \), then \( E_\lambda \omega_\pi F_{\lambda'} = 0 \). Otherwise, there are some \( \rho \in R(\lambda) \) and \( \sigma \in R(\lambda') \) such that \( \omega_\rho = \omega_\rho \omega_\pi \omega_\rho \). Then
\[
E_\lambda \omega_\pi F_{\lambda'} = E_\lambda \omega_\rho \omega_\pi \omega_\rho \omega_\pi F_{\lambda'} = \phi(\omega_\rho) E_\lambda \omega_\pi F_{\lambda'} = (-1)^{(\sigma)} q^{(\rho)} E_\lambda \omega_\pi F_{\lambda'} \tag{17.17}
\]
Similarly, if \( \pi \) does not separate \( \lambda' \) from \( \lambda \), by Lemma 3.3, \( F_{\lambda'} \omega_\pi E_\lambda = 0 \). Otherwise, there are some \( \rho \in R(\lambda) \) and \( \sigma \in R(\lambda') \) such that \( \omega_\pi = \omega_\rho \omega_\pi \omega_\rho \), since \( \omega_\pi \omega_\rho = \omega_\rho \) separates \( \lambda' \) from \( \lambda \). Then we obtain the second formula.

Lemma 3.5. Given a Young diagram \( \lambda \), we have the following identities:
\[
F_{\lambda'} \omega_{\pi}^{-1} E_\lambda = F_{\lambda'} \omega_{\pi}^{-1} E_\lambda = F_{\lambda'} \omega_{\pi}^{-1} E_\lambda \tag{18.18}
\]
Proof. Taking any one of the crossings of \( \omega_{\pi}^{-1} \), we see that the skein relation gives
\[
\omega_{\pi}^{-1} = \omega_{\pi}^{-1} - (q - q^{-1}) L_0, \tag{19.19}
\]
where \( L_0 \) arises from smoothing this crossing. Clearly, the corresponding permutation determined by \( L_0 \) does not separate \( \lambda' \) from \( \lambda \). With a slight modification of the proof of Lemma 3.3, we have \( F_{\lambda'} L_0 E_\lambda = 0 \). Hence
\[
F_{\lambda'} \omega_{\pi}^{-1} E_\lambda = F_{\lambda'} \omega_{\pi}^{-1} E_\lambda \tag{20.20}
\]
Then switch the other crossings in \( \omega_{\pi}^{-1} \). Using the above process inductively, we obtain the formula (18.18). 

We now define the idempotent elements \( y_\lambda \in H_n(q,a) \) for each partition \( \lambda \) as follows. Let \( e_\lambda = E_\lambda \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1} \). Then by using Lemma 3.3 and Corollary 3.4, we straightforwardly compute \( e_\lambda e_\mu = 0 \) for \( \lambda \neq \mu \), and
\[
e_\lambda = \alpha_\lambda e_\lambda \tag{21.21}
\]
for some scalar \( \alpha_\lambda \in R \). With the help of a formula established in [91] via \( SU(N) \) skein theory, Aiston [3] computed that
\[
\alpha_\lambda = \prod_{(i,j) \in \lambda} q^{1-i} [\lambda_i + \lambda_j - i - j + 1]. \tag{22.22}
\]
Then we define the idempotent \( y_\lambda \) as follows:
\[
y_\lambda = \frac{1}{\alpha_\lambda} e_\lambda. \tag{23.23}
\]

Finally, we prove the following proposition which will be used in the next section.

Proposition 3.6. For any \( \pi \in S_n \), we have
\[
y_{\lambda \pi} \omega_\pi y_\lambda = \begin{cases} (-1)^{(\sigma)} q^{(\pi)} y_\lambda, & \text{if } \omega_\pi = \omega_{\pi} \omega_{\rho}^{-1} \omega_\rho \text{ for some } \rho \in R(\lambda) \text{ and } \sigma \in R(\lambda'), \\ 0, & \text{otherwise}. \end{cases} \tag{24.24}
\]
Proof. We only need to compute \( e_{\lambda} \omega_{\pi} e_\lambda \). By the second formulas in Corollary 3.4 and Lemma 3.5, when \( \omega_\pi = \omega_{\pi} \omega_{\rho}^{-1} \omega_\rho \), we obtain
\[
e_{\lambda} \omega_{\pi} e_\lambda = E_{\lambda} \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1} \omega_{\pi} E_{\lambda} \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1} = E_{\lambda} \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1} \omega_{\pi} E_{\lambda} \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1} = (-1)^{(\sigma)} q^{(\rho)} E_{\lambda} \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1} E_{\lambda} \omega_{\pi} F_{\lambda'} \omega_{\pi}^{-1}.
\]
This completes the proof.

4 The strong integrality theorem

4.1 The refined coefficient formula

In this subsection, we study the resolutions in the skein $H_{n,p}(q,a)$ carefully, and we obtain a refined formula for the coefficients appearing in the resolutions. We follow the notations in [64]. The diagram in a rectangle with $n$ outputs and $m$ inputs at the top, matched at the bottom is called an $(n,m)$-tangle. The resulting skein of the rectangle, denoted by $H_{n,m}(q,a)$, is an algebra of dimension $(n+m)!$ over the ring $R$, where the product is induced by placing one tangle above another.

A framed knot $K$ can be represented as a 1-tangle $T(K)$ by a single knotted arc as in Figure 7 for $K$ being the trefoil knot.

The $(n,m)$-parallel of $T(K)$, denoted by $T_{n,m}(K)$ in the skein $H_{n,m}(q,a)$, is constructed by drawing $n+m$ parallel oriented strands to the arc $T(K)$, with $n$ oriented in one sense and $m$ in the other. Clearly, $T_{n,m}(K) \in H_{n,m}(q,a)$.

Applying the standard resolution procedures to the tangle $T_{n,m}(K) \in H_{n,m}(q,a)$, we can write $T_{n,m}(K)$ as an $R$-linear combination of $(n+m)!$ totally descending tangles without closed components. We denote the set of all such tangles by

$$S = \{ T(i) : i = 1, \ldots, (n+m)! \}.$$ 

In particular, for the case $m = 0$, such tangles are the positive permutation braids, denoted by $\{ \omega_\pi : \pi \in S_n \}$, with strings oriented from the top to the bottom, while for $n = 0$, they are the positive permutation braids, denoted by $\{ \omega_\rho^* : \rho \in S_m \}$, with string orientations from the bottom to the top.

For each tangle, we can count the number $k$ of its arcs which connect input and output points at the top. Then $0 \leq k \leq \min(n,m)$. We can divide the set $S$ into two parts

$$S = S^{(0)} \cup S^{(1)},$$

where $S^{(0)}$ consists of tangles with $k = 0$ in $S$, while $S^{(1)}$ consists of tangles with $k \geq 1$ in $S$. In particular, the tangles with $k = 0$ in $S^{(0)}$ have the form $\omega_\pi \otimes \omega_\rho^*$ for some $\pi \in S_n$ and $\rho \in S_m$, where $\otimes$ denotes juxtaposition of tangles side by side.

Figure 7 (Color online) The 1-tangle
We can write $T_{n,m}(K)$ in the skein $H_{n,m}(q,a)$ as
\[ T_{n,m}(K) = T_{n,m}^{(0)}(K) + T_{n,m}^{(1)}(K), \]
where $T_{n,m}^{(0)}(K)$ is an $R$-linear combination of tangles in $S^{(0)}$ and $T_{n,m}^{(1)}(K)$ is an $R$-linear combination of tangles in $S^{(1)}$. Then
\[ T_{n,m}^{(0)}(K) = \sum_{\pi \in S_n, \rho \in S_m} c_{\pi,\rho}(K)(\omega_{\pi} \otimes \omega_{\rho}^*), \]
and we have the following lemma.

**Lemma 4.1.** $c_{\pi,\rho}(K) \in \mathbb{Z}[a^\pm, z]$ and the powers of $z$ appearing in the coefficients $c_{\pi,\rho}(K)$ are simultaneously even (resp. odd) if $l(\pi) + l(\rho)$ is even (resp. odd).

**Proof.** Given an $(n,m)$-tangle $T$, let $L(T)$ denote the number of the closed components of $T$. Denote by $T^*$ the tangle after removing all the closed components of $T$, and by $l(T)$ the number of the components after closing the tangle $T^*$. It is clear that $l(\omega_{\pi}) = l(c(\pi))$, the length of the partition $c(\pi)$ which denotes the cycle type of the permutation $\pi$.

If we switch or smooth a crossing of sign $\pm 1$ in a tangle $T \in H_{n,m}(q,a)$, we obtain tangles $T_{\pm}$ and $T_0$ which satisfy the skein relation
\[ T = T_{\pm} \pm zT_0. \]
It is obvious that we have $l(T_{\pm}) + l(T_0) = l(T) + L(T) + L(T_0) \pm 1$. Then
\[ z^{l(T)}T = z^{l(T_{\pm})+L(T_{\pm})}T_{\pm} \pm z^zT_{\pm}z^{L(T_0)+L(T_0)}T_0, \]
where $\epsilon$ takes the value 0 or 2.

By induction and using Reidemeister moves, we obtain
\[ z^{l(T)}T = \sum_{\bar{T}(i) \in S} c_{(i)}z^{l(\bar{T}(i))}T_{(i)}, \]
where $c_{(i)} \in \mathbb{Z}[a^\pm, z^2]$, and $\bar{T}(i)$ is the tangle $T_{(i)}$ plus $L(\bar{T}(i))$ null-homotopic closed curves without crossings. Then we deduce $z^{l(\bar{T}(i))}\bar{T}(i) = (a - a^{-1})^{L(\bar{T}(i))}T_{(i)}$ by removing these null-homotopic closed curves.

In conclusion, we have the following expansion for any $T \in H_{n,m}(q,a)$:
\[ z^{l(T)}T = \sum_{T_{(i)} \in S} c_{(i)}z^{l(T_{(i))}T_{(i)}}, \]
with $c_{(i)} \in \mathbb{Z}[a^\pm, z^2]$.

In particular, for $T_{n,m}(K) \in H_{n,m}(q,a)$, it is easy to see that $L(T_{n,m}(K)) = 0$ and $l(T_{n,m}(K)) = n + m$, and we have
\[ T_{n,m}(K) = \sum_{T_{(i)} \in S} c_{(i)}z^{l(T_{(i))}-(n+m)}T_{(i)}. \]
Since $S = S^{(0)} \cup S^{(1)}$ and $S^{(0)} = \{\omega_{\pi} \otimes \omega_{\rho}^*: \pi \in S_n, \rho \in S_m\}$, we obtain
\[ T_{n,m}(K) = \sum_{\pi \in S_n, \rho \in S_m} c_{\pi,\rho}z^{l(\omega_{\pi} \otimes \omega_{\rho}^*)-(n+m)}\omega_{\pi} \otimes \omega_{\rho}^* + \sum_{T_{(i)} \in S^{(1)}} c_{(i)}z^{l(T_{(i))}-(n+m)}T_{(i)} \]
with coefficients $c_{\pi,\rho}$, $c_{(i)} \in \mathbb{Z}[a^\pm, z^2]$.

Therefore, the coefficient $c_{\pi,\rho}(K)$ in the formula (4.3) is given by
\[ c_{\pi,\rho}(K) = c_{\pi,\rho}z^{l(\omega_{\pi} \otimes \omega_{\rho}^*)-(n+m)}. \]
Clearly,
\[ l(\omega_\pi \otimes \omega_\rho) = l(\omega_\pi) + l(\omega_\rho) = l(c(\pi)) + l(c(\rho)). \]

By the relation (4.12) which is proved in the following Lemma 4.2, we obtain
\[ c_{\pi,\rho}(K) = c_{\pi,\rho} \cdot l(c(\pi)) + l(c(\rho)) = c_{\pi,\rho} \cdot l(\pi) + l(\pi + 2k) \]
for some \( k \in \mathbb{Z} \). Combining \( c_{\pi,\rho} \in \mathbb{Z}[a^{\pm 1}, z^2] \), we complete the proof.

**Lemma 4.2.** Given a permutation \( \pi \in S_n \), we have the following identity:
\[ l(\pi) + l(c(\pi)) \equiv n \pmod{2}. \]  

**Proof.** For every permutation \( \pi \in S_n \), its length \( l(\pi) \) can be obtained by calculating the minimal number of the crossings in the positive braid \( \omega_\pi \). When \( n = 2 \), we have \( S_2 = \{ \pi_1 = (1)(2), \pi_2 = (12) \} \). Hence \( c(\pi_1) = (11) \) and \( c(\pi_2) = (2) \). It is clear that
\[ l(\pi_1) + l(c(\pi_1)) = l(\pi_2) + l(c(\pi_2)) = 2. \]  

So Lemma 4.2 holds when \( n = 2 \).

Now we assume that Lemma 4.2 holds for \( n \leq d - 1 \). As to \( n = d \), we consider a permutation \( \pi \in S_d \).

We first study the special case where \( \pi \) has the cycle form \( \pi = \pi'(d) \), with \( \pi' \) a permutation in \( S_{d-1} \). It is easy to see that \( l(\pi) = l(\pi') \) and \( l(c(\pi)) = l(c(\pi')) + 1 \). By the induction hypothesis, we have
\[ l(\pi') + l(c(\pi')) \equiv d - 1 \pmod{2}. \]  

Thus we obtain
\[ l(\pi) + l(c(\pi)) \equiv d \pmod{2}. \]  

Thus Lemma 4.2 holds for \( \pi \in S_d \) with the cycle form \( \pi'(d) \) (\( \pi' \in S_{d-1} \)).

For the general case, we can assume that \( \pi \) has the cycle form \( \pi = \sigma \tau \), where \( \tau \) is the cycle containing the element \( d \) as the form \( (i_1 \cdots i_d) \) for \( \{i_1, \ldots, i_d\} \subset \{1, \ldots, d-1\} \), \( 1 \leq j \leq d - 1 \), and \( \sigma \) is a cycle in \( S_{d-j-1} \). Hence,
\[ l(c(\pi)) = l(c(\sigma)) + l(c(\tau)). \]  

By the property of permutations, the number of the crossings between \( \omega_\sigma \) and \( \omega_\tau \) must be an even number, and thus
\[ l(\pi) \equiv l(\sigma) + l(\tau) \pmod{2}. \]  

Combining (4.16), (4.17) and the induction hypothesis, we have
\[ l(\pi) + l(c(\pi)) \equiv d \pmod{2}. \]  

So we finish the proof of Lemma 4.2.

**Lemma 4.3.** The powers of \( a \) in the coefficients \( c_{\pi,\rho}(K) \) are simultaneously even (resp. odd) if \( (n + m)w(K) \) is even (resp. odd).

**Proof.** Applying the Reidemeister moves RI and RIII and the HOMFLY-PT skein relation to the diagram \( T_{n,m}(K) \) in \( H_{n,m}(q,a) \), we eventually obtain the final diagrams in the following forms: \( T^\alpha_{(i)} \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma} \) for \( \alpha, \beta, \gamma \in \mathbb{N} \), where \( T^\alpha_{(i)} \) denotes the tangle \( T_{(i)} \) with \( \alpha \) positive kinks, \( U \) denotes the unknot (i.e., the trivial knot), and \( P \) denotes the unknot with one positive kink.

Let \( S(K) \) be the set of all such final diagrams. Then we have
\[ T_{n,m}(K) = \sum_{T^\alpha_{(i)} \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma} \in S(K)} c_{\alpha,\beta,\gamma}(K) T^\alpha_{(i)} \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma} \]  

(4.19)
and the coefficients $c_{(i),\alpha,\beta,\gamma}(K) \in \mathbb{Z}[z]$.

For any tangle $T \in H_{n,m}(q,a)$, if we switch or smooth a crossing of sign $\pm 1$ in $T$, we obtain tangles $T_{\pm 1}$ and $T_0$. The key observation is that we have

$$w(T_0) + L(T_0) + l(T_0) \equiv 0 \pmod{2}.$$  \hfill (4.20)

Therefore,

$$w(T_0) + L(T_0) + l(T_0) \equiv w(T_{\pm}) + L(T_{\pm}) + l(T_{\pm}) \pmod{2}$$

\hfill (4.21)

In particular, when $T_{(i)} \in S^{(i)}$, we know that $T_{(i)} = \omega_\tau \otimes \omega_\rho^*$ for some $\pi \in S_n$ and $\rho \in S_m$. It is easy to obtain that

$$w((\omega_\tau \otimes \omega_\rho^*)^{\gamma} \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma}) = l(\pi) + l(\rho) + \alpha + \gamma,$$

$$L((\omega_\tau \otimes \omega_\rho^*)^{\gamma} \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma}) = \beta + \gamma,$$

$$l((\omega_\tau \otimes \omega_\rho^*)^{\gamma} \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma}) = l(c(\pi)) + l(c(\rho)).$$

According to Lemma 4.2, we have

$$l(\pi) + l(c(\pi)) \equiv n \pmod{2},$$

$$l(\rho) + l(c(\rho)) \equiv p \pmod{2}.$$  \hfill (4.23)

Combining the formulas (4.21) and (4.22), we obtain

$$\alpha + \beta \equiv (n + m)^2 w(K) \pmod{2}.$$  \hfill (4.24)

Finally, applying the Reidemeister move RI to the tangle $(\omega_\tau \otimes \omega_\rho^*)^\alpha \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma}$, we obtain

$$(\omega_\tau \otimes \omega_\rho^*)^\alpha \otimes U^{\otimes \beta} \otimes P^{\otimes \gamma} = a^{\alpha + \gamma}(\omega_\tau \otimes \omega_\rho^*) \otimes U^{\otimes \beta} \otimes U^{\otimes \gamma}$$

$$= a^{\alpha + \gamma} (a - a^{-1})^{\beta + \gamma} \omega_\tau \otimes \omega_\rho^*.$$  \hfill (4.25)

It is obvious that the degrees of $a$ in the term $a^{\alpha + \gamma} (a - a^{-1})^{\beta + \gamma}$ have the same parity with $\alpha + \beta$ which is equal to $(n + m)^2 w(K) \pmod{2}$. Note that the coefficient $c_{\pi,\rho}(K)$ is contributed by the terms

$$c_{\pi,\rho}(K) = c_{(i),\alpha,\beta,\gamma}(K) \frac{a^{\alpha + \gamma} (a - a^{-1})^{\beta + \gamma}}{z^{\beta + \gamma}}.$$  \hfill (4.26)

Therefore, the degrees of $a$ appearing in the coefficient $c_{\pi,\rho}(K)$ are equal to $(n + m)^2 w(K) \pmod{2}$. \hfill \Box

In conclusion, combining Lemmas 4.1 and 4.3 together, we obtain the following refined coefficient theorem.

**Theorem 4.4.** The coefficient $c_{\pi,\rho}(K)$ given by the formula (4.3) satisfies the following strong integrality:

$$c_{\pi,\rho}(K) \in q^{\epsilon_1} a^{\epsilon_2} \mathbb{Z}[q^{\pm 2}, a^{\pm 2}],$$

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$ are determined by

$$l(\pi) + l(\rho) \equiv \epsilon_1 \pmod{2},$$

$$(n + m)w(K) \equiv \epsilon_2 \pmod{2}.$$  \hfill (4.27)
4.2 Proof of the strong integrality theorem

In this subsection, we apply the technique used in [64] together with Proposition 3.6 and Theorem 4.4 to prove the following strong integrality theorem.

**Theorem 4.5.** For any framed knot $K$, the normalized framed full colored HOMFLY-PT invariant

$$P_{|\lambda,\mu|}(K; q, a) \in a^2\mathbb{Z}[q^{\pm 2}, a^{\pm 2}],$$

where $\epsilon \in \{0, 1\}$ is determined by

$$(|\lambda| + |\mu|)w(K) \equiv \epsilon \pmod{2}.$$  

**Proof.** First, it is easy to see that the meridian map $\varphi$ commutes with the map $T_K$. The eigenvalues of $\varphi$ are distinct and nonzero, and then any eigenvector $Q_{\lambda,\mu}$ of $\varphi$ is also an eigenvector of $T_K$. Suppose that $t(\lambda, \mu)$ is the eigenvalue of $T_K$ for its eigenvector $Q_{\lambda,\mu}$. By the formula (2.36), we only need to prove the strong integrality for $t(\lambda, \mu)$.

For $Q_\lambda \in C_{|\lambda|,0}$ and $Q_\mu^* \in C_{0,|\mu|}$, their products $Q_\lambda Q_\mu^* \in C_{|\lambda|,|\mu|}$. It is shown in [28] that

$$Q_{\lambda,\mu} = Q_\lambda Q_\mu^* + W,$$

where $W \in C_{|\lambda|-1,|\mu|-1}$. Since $T_K(Q_{\lambda,\mu}) = t(\lambda, \mu)Q_{\lambda,\mu}$, we have

$$T_K(Q_\lambda Q_\mu^*) = t(\lambda, \mu)Q_\lambda Q_\mu^* + V,$$

where $V \in C_{|\lambda|-1,|\mu|-1}$.

One can express $T_K(Q_\lambda Q_\mu^*)$ as the closure of the element $(y_\lambda \otimes y_\mu^*)_{T,n,m}(K)$ in $H_{n,m}(q, a)$. By the formula (4.2),

$$(y_\lambda \otimes y_\mu^*)_{T,n,m}(K) = (y_\lambda \otimes y_\mu^*)_{T,n,m}(K) + (y_\lambda \otimes y_\mu^*)_{T,n,m}(K).$$

Clearly, the closure of $(y_\lambda \otimes y_\mu^*)_{T,n,m}(K)$ denoted by $V'$ belongs to $C_{n-1,m-1}$, and

$$(y_\lambda \otimes y_\mu^*)_{T,n,m}(K) = \sum_{\pi \in S_u, \rho \in S_m} c_{\pi,\rho}(K)(y_\lambda \omega_\pi \otimes y_\mu^* \omega_\rho).$$

It is easy to see that the closures of $y_\lambda \omega_\pi$ and $y_\mu^* \omega_\rho$ are equal to $c(\pi, \lambda)Q_\lambda$ and $c(\rho, \mu)Q_\mu^*$, respectively. By Proposition 3.6, we have

$$c(\pi, \lambda) \in q^{(\tau(\pi))}\mathbb{Z}[q^2],$$

$$c(\rho, \mu) \in q^{(\tau(\rho))}\mathbb{Z}[q^2].$$

Therefore, the closure of $(y_\lambda \otimes y_\mu^*)_{T,n,m}(K)$ is equal to $C(\lambda, \mu)Q_\lambda Q_\mu^*$ with

$$C(\lambda, \mu) = \sum_{\pi \in S_u, \rho \in S_m} c_{\pi,\rho}(K)c(\pi, \lambda)c(\rho, \mu).$$

Hence we have

$$T_K(Q_\lambda Q_\mu^*) = C(\lambda, \mu)Q_\lambda Q_\mu^* + V',$$

where $V' \in C_{n-1,m-1}$.

Therefore, compared with the formula (4.31), $C(\lambda, \mu) = t(\lambda, \mu)$ is the normalized framed full colored HOMFLY-PT invariant. Applying Theorem 4.4 to the coefficients $c_{\pi,\rho}(K)$ and the formulas (4.34) and (4.35), we obtain

$$C(\lambda, \mu) \in a^2\mathbb{Z}[q^{\pm 2}, a^{\pm 2}],$$

where $\epsilon \in \{0, 1\}$ is determined by

$$(|\lambda| + |\mu|)w(K) \equiv \epsilon \pmod{2}.$$  

This completes the proof.
Remark 4.6. In [43], Le proved the strong integrality for the normalized quantum group invariant of a knot \( K \) colored by an irreducible module over any simple Lie algebra \( \mathfrak{g} \) via quantum group theory. It is easy to see that Theorem 4.5 implies Le’s strong integrality when \( \mathfrak{g} \) is the Lie algebra \( \mathfrak{sl}_N \mathbb{C} \).

**Corollary 4.7.** For any framed knot \( K \), we have

\[
\mathcal{H}(\lambda, \mu)(K; -q, a) = (-1)^{(|\lambda|+|\mu|)}\mathcal{H}(\lambda, \mu)(K; q, a), \\
\mathcal{H}(\lambda, \mu)(K; q, -a) = (-1)^{(|\lambda|+|\mu|)(w(K)+1)}\mathcal{H}(\lambda, \mu)(K; q, a).
\]

**Proof.** Using the expression for the colored HOMFLY-PT invariant of the unknot (2.32), it is straightforward to obtain

\[
W_\lambda(U; q, -a) = (-1)^{|\lambda|} W_\lambda(U; q, a), \\
W_\lambda(U; -q, a) = (-1)^{|\lambda|} W_\lambda(U; q, a),
\]

where in the second identity, we need to use the following identity for the partition \( \lambda \):

\[
(-1)^{|\lambda|} = (-1)^{\sum_{x \in \lambda} (h(x)+con(x))}.
\]

Then, by the formula (2.33),

\[
\mathcal{H}(\lambda, \mu)(U; q, -a) = (-1)^{|\lambda|+|\mu|}\mathcal{H}(\lambda, \mu)(U; q, a), \\
\mathcal{H}(\lambda, \mu)(U; -q, a) = (-1)^{|\lambda|+|\mu|}\mathcal{H}(\lambda, \mu)(U; q, a).
\]

Finally, combining Theorem 4.5 and the formula (2.35) together, we complete the proof.

**Corollary 4.8.** For any framed knot \( K \), we have

\[
\mathcal{H}(\lambda, \mu)(K; q^{-1}, a) = (-1)^{(|\lambda|+|\mu|)}\mathcal{H}(\lambda, \mu)(K; q, a).
\]

**Proof.** By using the formula (2.24) for \( Q_{\lambda, \mu} \), we have

\[
Q_{\lambda, \mu} |_{q \rightarrow -q^{-1}} = Q_{\lambda, \mu}^{-\varepsilon}.
\]

It follows that

\[
\mathcal{H}(\lambda, \mu)(K; -q^{-1}, a) = \mathcal{H}(\lambda, \mu)(K; q, a).
\]

Then combining the formula (4.40), we complete the proof.

All the above formulas can be generalized to the case of links.

**Theorem 4.9.** For any framed link \( \mathcal{L} \), we have

\[
\mathcal{H}(\lambda, \mu)(\mathcal{L}; -q, a) = (-1)^{\sum_{n=1}^{L} (|\lambda^n|+|\mu^n|)}\mathcal{H}(\lambda, \mu)(\mathcal{L}; q, a), \\
\mathcal{H}(\lambda, \mu)(\mathcal{L}; q, -a) = (-1)^{\sum_{n=1}^{L} (|\lambda^n|+|\mu^n|)(w(K_n)+1)}\mathcal{H}(\lambda, \mu)(\mathcal{L}; q, a), \\
\mathcal{H}(\lambda, \mu)(\mathcal{L}; q^{-1}, a) = (-1)^{\sum_{n=1}^{L} (|\lambda^n|+|\mu^n|)}\mathcal{H}(\lambda, \mu)(\mathcal{L}; q, a).
\]

Suppose that \( \mathcal{L} \) is a link with \( L \) components \( K_1, \ldots, K_L \), and we denote it by \( \mathcal{L} = K_1 \vee K_2 \vee \cdots \vee K_L \). We cut the component \( K_n \) open and obtain a 1-tangle, and we can draw \( \mathcal{L} \) in the annulus as the closure of this 1-tangle. Decorating \( K_n \) with a diagram \( Q_\alpha \) gives a diagram \( K_1 \vee \cdots \vee K_n \ast Q_\alpha \vee \cdots \vee K_L \) in \( \mathcal{C} \), and it induces an \( R \)-linear map \( T_{\mathcal{L}}^{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \).

Similarly, the eigenvectors of \( T_{K_n}^{\mathcal{C}} \) are given by \( Q_{\lambda, \mu} \), and we denote the eigenvalue of \( T_{K_n}^{\mathcal{C}} \) corresponding to the eigenvector \( Q_{\lambda, \mu} \) by \( t_{K_n}^{\mathcal{C}}(\lambda, \mu) \).

Then the identity

\[
K_1 \vee \cdots \vee K_n \ast Q_{\lambda, \mu} \vee \cdots \vee K_L = T_{K_n}^{\mathcal{C}}(Q_{\lambda, \mu}) = t_{K_n}^{\mathcal{C}}(\lambda, \mu)Q_{\lambda, \mu} = t_{K_n}^{\mathcal{C}}(\lambda, \mu)U \ast Q_{\lambda, \mu}
\]
Proposition 4.10. Under the above settings, we have
\[ t^L_{K_0}(\lambda, \mu) = \frac{H(K_1 \otimes \cdots \otimes K_\alpha \ast Q_{\lambda, \mu} \otimes \cdots \otimes K_L; q, a)}{H(U \ast Q_{\lambda, \mu}; q, a)}. \]  
(4.54)

With a slight modification of the proof of Theorem 4.5, one can show the following proposition.

**Proposition 4.10.** Under the above settings, we have
\[ t^L_{K_0}(\lambda, \mu) \in a^*\mathbb{Z}[q^{\pm 1}, a^{\pm 2}], \]  
(4.55)

where \( \epsilon \in \{0, 1\} \) is determined by
\[ (|\lambda| + |\mu|)w(K_\alpha) + \sum_{\beta \neq \alpha} w(K_\beta) + (L - 1) \equiv \epsilon \pmod{2}. \]  
(4.56)

By the formulas (2.26) and (2.14), we obtain
\[ Q_{\lambda, \mu} = \sum_{\sigma, \rho, \nu} (-1)^{|\sigma|} c^\lambda_{\sigma, \rho, \nu} c^\mu_{\sigma, \rho, \nu} Q_\rho Q_\nu. \]

We define the link \( \tau \) and \( \hat{\tau} \) such that \( \tau \) is determined by
\[ L \ast \bigotimes_{\alpha = 1}^L Q_{\lambda^\alpha, \mu^\alpha} = K_1 \ast Q_{\lambda^1, \mu^1} \otimes \cdots \otimes K_\alpha \ast Q_{\lambda^\alpha, \mu^\alpha} \otimes \cdots \otimes K_L \ast Q_{\lambda^L, \mu^L}, \]

Therefore,
\[ L \ast \bigotimes_{\alpha = 1}^L Q_{\lambda^\alpha, \mu^\alpha} = K_1 \ast Q_{\lambda^1, \mu^1} \otimes \cdots \otimes K_\alpha \ast Q_{\lambda^\alpha, \mu^\alpha} \otimes \cdots \otimes K_L \ast Q_{\lambda^L, \mu^L}, \]

and \( \tau^\alpha \) and \( \delta^\alpha \) denote the indexes \( \tau^\alpha \) and \( \delta^\alpha \), respectively, which do not appear in the summation.

We define the link
\[ L_{\tau^1, \delta^1, \cdots, \tau^\alpha, \delta^\alpha, \cdots, \tau^L} = K_1 \ast X_{\tau^1} X^*_{\delta^1} \otimes \cdots \otimes K_\alpha \ast X_{\tau^\alpha} X^*_{\delta^\alpha} \otimes \cdots \otimes K_L \ast X_{\tau^L} X^*_{\delta^L}, \]

By the formula (4.55) and a careful computation of the writhe numbers for this link, we obtain
\[ t^L_{K_0}(\lambda^\alpha, \mu^\alpha) \in a^*\mathbb{Z}[q^{\pm 1}, a^{\pm 2}], \]  
(4.60)

where \( \epsilon \in \{0, 1\} \) is determined by the following formula:
\[ (|\lambda^\alpha| + |\mu^\alpha|)w(K_\alpha) + \sum_{\beta \neq \alpha} (|\lambda^\beta| + |\mu^\beta|)w(K_\beta) + 1 \equiv \epsilon \pmod{2}. \]  
(4.61)

Therefore,
\[ H\left( L \ast \bigotimes_{\alpha = 1}^L Q_{\lambda^\alpha, \mu^\alpha}; q, a \right) = \sum_{\tau^\beta, \delta^\beta, \beta \neq \alpha} C_{\tau^1, \delta^1, \cdots, \tau^\alpha, \delta^\alpha, \cdots, \tau^L} \prod_{\beta = 1, \beta \neq \alpha} \frac{1}{\{\tau^\beta\}\{\delta^\beta\}} \times t^L_{K_0}(\lambda^\alpha, \mu^\alpha)H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; q, a). \]  
(4.62)

Combining the following formulas:
\[ H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; q, -a) = (-1)^{|\lambda^\alpha| + |\mu^\alpha|}H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; q, a), \]  
(4.63)
\[ H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; -q, a) = (-1)^{|\lambda^\alpha| + |\mu^\alpha|}H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; q, a), \]  
(4.64)
\[ H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; q, -a) = (-1)^{|\lambda^\alpha| + |\mu^\alpha|}H(U \ast Q_{\lambda^\alpha, \mu^\alpha}; q, a), \]  
(4.65)

we finish the proof of Theorem 4.9.
5 The refined LMOV integrality structure

5.1 Symmetric functions and plethysms

Recall that the power sum symmetric function of infinite variables $\mathbf{x} = (x_1, \ldots, x_N, \ldots)$ is defined by $p_n(\mathbf{x}) = \sum_i x_i^n$. We refer to Appendix A for more detailed definitions about the symmetric functions. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, we define

$$p_\lambda(\mathbf{x}) = \prod_{j=1}^l p_{\lambda_j}(\mathbf{x}).$$

The Schur function $s_\lambda(\mathbf{x})$ is determined by the Frobenius formula

$$s_\lambda(\mathbf{x}) = \sum_{\mu} \chi_{\lambda/\mu} p_\mu(\mathbf{x}).$$

(5.1)

Let $\Lambda(\mathbf{x})$ be the ring of symmetric functions of $\mathbf{x} = (x_1, x_2, \ldots)$ over the ring $\mathbb{Q}(q, a)$, and $\langle \cdot, \cdot \rangle$ be the Hall pair on $\Lambda(\mathbf{x})$ determined by

$$\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda, \mu}.$$  

(5.2)

For $\mathbf{x} = (x^1, \ldots, x^L)$, denote by $\Lambda(\mathbf{x}) := \Lambda(x^1) \otimes \cdots \otimes \Lambda(x^L)$ the ring of functions separately symmetric in $x^1, \ldots, x^L$, where $x^i = (x^i_1, x^i_2, \ldots)$. We study functions in the ring $\Lambda(\mathbf{x})$. For $\tilde{\mu} = (\mu^1, \ldots, \mu^L) \in \mathcal{P}^L$, we let

$$a^{\tilde{\mu}}(\mathbf{x}) = a_{\mu^1}(x^1) \cdots a_{\mu^L}(x^L) \in \Lambda(\mathbf{x})$$

be homogeneous with degree $(|\mu^1|, \ldots, |\mu^L|)$. Moreover, the Hall pair on $\Lambda(\mathbf{x})$ is given by

$$\langle a_1(x^1) \cdots a_L(x^L), b_1(x^1) \cdots b_L(x^L) \rangle = \langle a_1(x^1), b_1(x^1) \rangle \cdots \langle a_L(x^L), b_L(x^L) \rangle$$

(5.3)

for $a_1(x^1) \cdots a_L(x^L), b_1(x^1) \cdots b_L(x^L) \in \Lambda(\mathbf{x})$. For $d \in \mathbb{Z}_+$, we define the $d$-th Adams operator $\Psi_d$ as the $\mathbb{Q}$-algebra map on $\Lambda(\mathbf{x})$, i.e.,

$$\Psi_d(f(\mathbf{x}; q, a)) = f(\mathbf{x}^d; q^d, a^d).$$

(5.4)

Denote by $\Lambda(\mathbf{x})^+$ the set of symmetric functions with degree greater than or equal to 1. The plethystic exponential $\Exp$ and logarithmic $\Log$ are inverse maps

$$\Exp : \Lambda(\mathbf{x})^+ \rightarrow 1 + \Lambda(\mathbf{x})^+, \quad \Log : 1 + \Lambda(\mathbf{x})^+ \rightarrow \Lambda(\mathbf{x})^+,$$

(5.5)

respectively, defined by (see [29])

$$\Exp(f) = \exp \left( \sum_{d \geq 1} \frac{\Psi_d(f)}{d} \right), \quad \Log(f) = \sum_{d \geq 1} \frac{\mu(d)}{d} \Psi_d(\log(f)),$$

(5.6)

where $\mu$ is the Möbius function. It is clear that

$$\Exp(f + g) = \Exp(f)\Exp(g), \quad \Log(fg) = \Log(f) + \Log(g)$$

(5.7)

and $\Exp(x) = \frac{1}{1 - x}$, if we use the expansion

$$\log(1 - x) = -\sum_{d \geq 1} \frac{x^d}{d}. $$
5.2 LMOV functions

Consider a series of functions \(S_{\tilde{\lambda}}(q, a) \in \mathbb{Q}(q, a)\), where \(\tilde{\lambda} \in \mathcal{P}^{L}\). We introduce the partition function for \(\{S_{\tilde{\lambda}}(q, a) \mid \tilde{\lambda} \in \mathcal{P}^{L}\}\), which is the following generating function:

\[
Z(\bar{x}; q, a) = \sum_{\tilde{\lambda} \in \mathcal{P}^{L}} S_{\tilde{\lambda}}(q, a)s_{\tilde{\lambda}}(\bar{x}). \quad (5.8)
\]

**Definition 5.1.** The *LMOV function* for the series \(\{S_{\tilde{\lambda}}(q, a) \mid \tilde{\lambda} \in \mathcal{P}^{L}\}\) is given by

\[
f_{\tilde{\lambda}}(q, a) = \langle \text{Log}(Z(\bar{x}; q, a)), s_{\tilde{\lambda}}(\bar{x}) \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the Hall pair in the ring of symmetric functions \(\Lambda(\bar{x})\).

In particular, if we take \(x^\alpha = (x_\alpha, 0, 0, \ldots)\) for all \(\alpha = 1, \ldots, L\), i.e.,

\[
\bar{x} = ((x_1, 0, \ldots), (x_2, 0, \ldots), \ldots, (x_L, 0, \ldots)),
\]

then we obtain the following special partition function for \(\{S_{r_1, \ldots, r_L}(q, a)\}\) which is given by

\[
Z(x_1, x_2, \ldots, x_L; q, a) = \sum_{r_1, \ldots, r_L \geq 0} S_{r_1, \ldots, r_L}(q, a)x_1^{r_1} \cdots x_L^{r_L}. \quad (5.10)
\]

**Definition 5.2.** The *special LMOV function* for the series \(\{S_{r_1, \ldots, r_L}(q, a)\}\) is

\[
f_{r_1, \ldots, r_L}(q, a) = [x_1^{n_1} \cdots x_L^{n_L}]\text{Log}Z(x_1, x_2, \ldots, x_L; q, a), \quad (5.11)
\]

where the notation \([x_1^{n_1} \cdots x_L^{n_L}]f(x_1, \ldots, x_L)\) denotes the coefficients of \(x_1^{n_1} \cdots x_L^{n_L}\) in the function \(f(x_1, \ldots, x_L)\).

We introduce the notion of the \(F\)-invariant \(F_{\hat{\mu}}(q, a)\) (resp. \(\mathcal{F}_{r_1, \ldots, r_L}(q, a)\)) for \(\{S_{\tilde{\lambda}}(q, a)\}\) (resp. \(\{S_{r_1, \ldots, r_L}(q, a)\}\)) which is determined by the formula

\[
\log Z(\bar{x}; q, a) = \sum_{\hat{\mu}} F_{\hat{\mu}}(q, a)p_{\hat{\mu}}(\bar{x}) \quad (5.12)
\]

(resp. \(\log Z(x_1, x_2, \ldots, x_L; q, a) = \sum_{r_1, \ldots, r_L \geq 0} \mathcal{F}_{r_1, \ldots, r_L}(q, a)x_1^{r_1} \cdots x_L^{r_L}\)).

Then by the formula (A.10) in Appendix A, we obtain

\[
F_{\hat{\mu}}(q, a) = \sum_{\Lambda \in \mathcal{P}(P^L), |\Lambda| = \hat{\mu} \in \mathcal{P}^L} \Theta_{\Lambda}Z_{\Lambda}(q, a) \quad (5.13)
\]

and

\[
\mathcal{F}_{r_1, \ldots, r_L}(q, a) = \sum_{|\Lambda| = (r_1, \ldots, r_L)} \Theta_{\Lambda}S_{\Lambda}(q, a). \quad (5.14)
\]

By a straightforward computation, we obtain

\[
f_{\tilde{\lambda}}(q, a) = \langle \text{Log}(Z), s_{\tilde{\lambda}}(\bar{x}) \rangle
\]

\[
= \left\langle \sum_{d \geq 1} \frac{\mu(d)}{d} \circ \Psi_d \left( \sum_{\hat{\mu}} F_{\hat{\mu}}(q, a)p_{\hat{\mu}}(\bar{x}) \right), s_{\tilde{\lambda}}(\bar{x}) \right\rangle
\]

\[
= \left\langle \sum_{d \geq 1} \frac{\mu(d)}{d} \sum_{\hat{\mu}} F_{\hat{\mu}}(q^d, a^d)p_{d\hat{\mu}}(\bar{x}), s_{\tilde{\lambda}}(\bar{x}) \right\rangle
\]

\[
= \left\langle \sum_{d \geq 1} \frac{\mu(d)}{d} \sum_{\hat{\mu}} F_{\hat{\mu}}(q^d, a^d)\sum_{\hat{\nu}} \chi_d(d \cdot \hat{\nu})s_{\hat{\nu}}(\bar{x}), s_{\tilde{\lambda}}(\bar{x}) \right\rangle
\]
\[ \sum_{\mu} \chi_{\tilde{\mu}}(\bar{\mu}) \sum_{d|\bar{\mu}} \frac{\mu(d)}{d} F_{\bar{\mu}/d}(q^d, a^d) \] (5.15)

and

\[ f_{n_1,\ldots,n_L}(q, \bar{a}) = \sum_{d \geq 1} \frac{\mu(d)}{d} F_{\bar{\mu}/d}(q^d, a^d). \] (5.16)

**Remark 5.3.** Here, we mention that when \( \tilde{\lambda} = (\lambda^1, \ldots, \lambda^L) = ((n_1), \ldots, (n_L)) \), according to their definitions, \( f_{n_1,\ldots,n_L}(q, a) \neq f_{n_1,\ldots,n_L}(q, a) \). That is why we use the different symbols here.

**Definition 5.4.** We define the reformulated LMOV function for \( \{S_{\tilde{\lambda}}(q, a)\} \) as a character transformation of the LMOV functions, i.e.,

\[ g_{\tilde{\mu}}(q, a) = \frac{1}{|\tilde{\mu}|} \sum_{\tilde{\lambda}} f_{\tilde{\lambda}}(q, a) \chi_{\tilde{\lambda}}(\bar{\mu}). \] (5.17)

Plugging the formula (5.15) into (5.17), we obtain

\[ g_{\tilde{\mu}}(q, \bar{a}) = \frac{\hbar_{\tilde{\mu}}}{|\tilde{\mu}|} \sum_{d|\tilde{\mu}} \frac{\mu(d)}{d} F_{\bar{\mu}/d}(q^d, a^d). \] (5.18)

**Remark 5.5.** The original definition of the reformulated LMOV function [41,42,49] is slightly different from here, which is

\[ T_{\lambda,\mu}(x) = \sum_{\nu} \chi_{\nu}(\mu) \chi_{\nu}(\nu) \bar{p}_\nu(x). \]

In particular,

\[ T_{\lambda,\mu}(q^\nu) = \sum_{\mu} \chi_{\nu}(\mu) \chi_{\nu}(\nu) \prod_{i=1}^{\nu} \frac{1}{\nu_i} \]

if one lets \( q^\nu = (q^{-1}, q^{-3}, q^{-5}, \ldots) \). In [41,42,49], the reformulated LMOV function for \( \{S_{\tilde{\lambda}}(q, a)\} \) is defined as

\[ \hat{f}_{\tilde{\mu}}(q, a) = \sum_{\tilde{\lambda}} f_{\tilde{\lambda}}(q, a) T_{\tilde{\lambda},\tilde{\mu}}(q^\nu). \]

Therefore, we have

\[ \hat{f}_{\tilde{\lambda}}(q, a) = \chi_{\tilde{\lambda}}(\bar{\mu}) \sum_{\tilde{\mu}} \frac{1}{|\tilde{\mu}|} \sum_{d|\tilde{\mu}} \frac{\mu(d)}{d} F_{\bar{\mu}/d}(q^d, a^d). \] (5.19)

It is obvious that the two definitions by the formulas (5.17) and (5.19) are related by a character transformation

\[ g_{\tilde{\mu}}(q, a) = \sum_{\tilde{\lambda}} \chi_{\tilde{\lambda}}(\bar{\mu}) \hat{f}_{\tilde{\lambda}}(q, a). \] (5.20)

**5.3 The refined LMOV conjecture**

Given a framed link \( \mathcal{L} \) with \( L \) components \( \mathcal{K}_1, \ldots, \mathcal{K}_L \), denote by \( \bar{\tau} = (\tau^1, \ldots, \tau^L) \in \mathbb{Z}^L \) the framing of \( \mathcal{L} \), i.e., \( \tau^\alpha = w(\mathcal{K}_\alpha) \) for \( \alpha = 1, \ldots, L \). In the following, we use the notation \( \mathcal{L}_{\bar{\tau}} \) to denote this framed link \( \mathcal{L} \) if we want to emphasize its framing.

For \( \tilde{\lambda} = (\lambda^1, \ldots, \lambda^L), \bar{\mu} = (\mu^1, \ldots, \mu^L), \bar{\nu} = (\nu^1, \ldots, \nu^L) \in \mathcal{P}^L \), we set \( e_{\lambda,\mu}^{\nu} = \prod_{\alpha=1}^{L} c_{\lambda^\alpha,\mu^\alpha}^{\nu^\alpha} \), where \( c_{\lambda^\alpha,\mu^\alpha}^{\nu^\alpha} \) is the Littlewood-Richardson coefficient.

**Definition 5.6.** The \( \bar{\tau} \)-framed full colored HOMFLY-PT invariant for the framed link \( \mathcal{L}_{\bar{\tau}} \) is given by

\[ H_{\tilde{\lambda}}(\mathcal{L}_{\bar{\tau}}; q, a) = (-1)^{\sum_{\alpha=1}^{L} |\lambda^\alpha| |\tau^\alpha| - \sum_{\alpha=1}^{L} |\lambda^\alpha| |\nu^\alpha|} H_{\tilde{\lambda}}(\mathcal{L}_{\bar{\tau}}; q, a). \] (5.21)
Definition 5.7. The $\tilde{\tau}$-framed composite invariant for $\mathcal{L}_\tilde{\tau}$ is given as follows:

$$C_{\tilde{\tau}}(\mathcal{L}_\tilde{\tau}; q, a) = (-1)^{\sum_{\alpha=-1}^{1} |\alpha^\tau| |\alpha^\rho|} \sum_{\tilde{\lambda}, \tilde{\mu}} a^{-\sum_{\alpha=-1}^{1} |\alpha^\tau| |\alpha^\rho|} e_{\tilde{\lambda}, \tilde{\mu}}^\tilde{\tau} \mathcal{H}_{\tilde{\lambda}, \tilde{\mu}}(\mathcal{L}_\tilde{\tau}; q, a).$$  

(5.22)

By using Theorem 4.9, we obtain

$$H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, -a) = (-1)^{|\tilde{\lambda}|} H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; -q, a) = (-1)^{|\tilde{\lambda}|} H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q^{-1}, a) = (-1)^{|\tilde{\lambda}|} H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, a),$$

(5.23)

which hold similarly for the framed composite invariant $C_{\tilde{\tau}}(\mathcal{L}_\tilde{\tau}; q, a)$.

We denote the reformulated LMOV functions for $\tilde{\tau}$-framed colored HOMFLY-PT invariants \( \mathcal{H}_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, a) \) and $\tilde{\tau}$-framed composite invariants \( C_{\tilde{\tau}}(\mathcal{L}_\tilde{\tau}; q, a) \) by \( g^{(0)}_{\tilde{\mu}}(\mathcal{L}_\tilde{\tau}; q, a) \) and \( g^{(1)}_{\tilde{\mu}}(\mathcal{L}_\tilde{\tau}; q, a) \), respectively. Similarly, we write \( f^{(0)}_{\tilde{\mu}}(\mathcal{L}_\tilde{\tau}; q, a) \) and \( f^{(1)}_{\tilde{\mu}}(\mathcal{L}_\tilde{\tau}; q, a) \) for the corresponding special LMOV functions for the $\tilde{\tau}$-framed colored HOMFLY-PT invariants \( \mathcal{H}_{\tilde{\tau}}(\mathcal{L}_\tilde{\tau}; q, a) \) and $\tilde{\tau}$-framed composite invariants \( C_{\tilde{\tau}}(\mathcal{L}_\tilde{\tau}; q, a) \), respectively.

By the formula (5.23), we obtain that the corresponding $F$-invariants satisfy

$$F_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, -a) = (-1)^{|\tilde{\mu}|} F_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$F_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; -q, a) = (-1)^{|\tilde{\mu}|} F_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$F_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q^{-1}, a) = (-1)^{|\tilde{\mu}|} F_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a).$$

(5.24)

It follows that

$$g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, -a) = (-1)^{|\tilde{\mu}|} g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; -q, a) = g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q^{-1}, a) = g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a)$$

(5.25)

and

$$f_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, -a) = (-1)^{|\tilde{\mu}|} f_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$f_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; -q, a) = (-1)^{|\tilde{\mu}|} f_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

$$f_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q^{-1}, a) = (-1)^{|\tilde{\mu}|} f_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a)$$

(5.26)

for both $h = 0$ and $h = 1$.

Remark 5.8. We know that $g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a)$ belongs to the ring $\mathbb{Q}[q^\pm, a^\pm]$ with $(q^r - q^{-r})$ as its denominators, and the formula (5.25) implies that there exists some $d_0 \in \mathbb{N}$ such that

$$g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a) \in z^{-2d_0} a^\epsilon \mathbb{Q}[[z^2]][a^\pm 2],$$

(5.27)

where $\epsilon \in \{0, 1\}$ which is determined by $|\tilde{\mu}| \equiv \epsilon \pmod{2}$.

From the formula (4.62), we know that for $\tilde{\lambda} \neq 0$, both $H_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, a)$ and $C_{\tilde{\lambda}}(\mathcal{L}_\tilde{\tau}; q, a)$ contain a factor $\frac{q-a^{-1}}{q-q^{-1}}$. We obtain the following theorem.

Theorem 5.9. The reformulated LMOV functions $g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a)$ for both $h = 0$ and $h = 1$ can be written in the following form:

$$g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a) = (a - a^{-1}) g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

(5.28)

and $g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a)$ has the properties

$$g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, -a) = (-1)^{|\tilde{\mu}| - 1} g_{\tilde{\mu}}^{(h)}(\mathcal{L}_\tilde{\tau}; q, a),$$

(5.29)
The LMOV conjecture for framed colored HOMFLY-PT invariants \[11,56\] which is an extension form of the original LMOV conjecture for (unframed) colored HOMFY-PT invariants \[41,42,49,73\] states that
\[
g_{\mu}^{(0)}(\mathcal{L}; q, a) \in \mathbb{Z}[\mathbb{Z}, a]^{\pm 1}
\]
for
\[
f_{\mu}^{(0)}(\mathcal{L}; q, a) \in (q - q^{-1})^{-1}\mathbb{Z}[q^{\pm 1}, a^{\pm 1}].
\]

Later, the LMOV conjecture for framed colored HOMFLY-PT invariants was generalized to the case of the framed composite invariants \[12,55\] which states that \(g_{\mu}^{(1)}(\mathcal{L}; q, a)\) and \(f_{\mu}^{(1)}(\mathcal{L}; q, a)\) satisfy the formulas (5.32) and (5.33), respectively.

By Theorem 5.9, we obtain the following conjecture.

**Conjecture 5.10** (Refined LMOV conjecture for framed links). For \(h = 0\) and \(h = 1\), the reformulated LMOV functions can be written as
\[
g_{\mu}^{(h)}(\mathcal{L}; q, a) = (a - a^{-1})g_{\mu}^{(h)}(\mathcal{L}; q, a),
\]
where
\[
g_{\mu}^{(h)}(\mathcal{L}; q, a) \in z^{-2}a^2\mathbb{Z}[z, a^{\pm 2}],
\]
and \(\epsilon \in \{0, 1\}\) is determined by \(|\mu| - 1 \equiv \epsilon \pmod{2}\).

In other words, there are integral invariants
\[
\tilde{N}_{\mu,g,q}^{(h)}(\mathcal{L}; q, a) \in \mathbb{Z}
\]
such that
\[
\tilde{g}_{\mu}^{(h)}(\mathcal{L}; q, a) = \sum_{q > 0} \sum_{Q \in \mathbb{Z}} \tilde{N}_{\mu,g,q}^{(h)}(\mathcal{L}; q, a) z^{2Q+2}a^{2Q+\epsilon} \in z^{-2}a^2\mathbb{Z}[z^2, a^\pm 2].
\]

Similarly, as to the special LMOV functions, we also have the following integrality conjecture.

**Conjecture 5.11.** For \(h = 0\) or \(h = 1\), the special LMOV functions can be written as
\[
f_{\mu}^{(h)}(\mathcal{L}; q, a) = \frac{(a - a^{-1})}{(q - q^{-1})}f_{\mu}^{(h)}(\mathcal{L}; q, a),
\]
where
\[
f_{\mu}^{(h)}(\mathcal{L}; q, a) \in a^2\mathbb{Z}[a, a^\pm 2],
\]
and \(\epsilon \in \{0, 1\}\) is determined by \(|\mu| \equiv \epsilon \pmod{2}\). In other words, there are integral invariants
\[
\tilde{N}_{\mu,i,j}^{(h)}(\mathcal{L}; q, a) \in \mathbb{Z}
\]
such that
\[
(q - q^{-1})f_{\mu}^{(h)}(\mathcal{L}; q, a) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \tilde{N}_{\mu,i,j}^{(h)}(\mathcal{L}; q, a) q^i a^{2j+\epsilon} \in a^2\mathbb{Z}[q^{\pm 1}, a^{\pm 2}].
\]

**Remark 5.12.** Liu and Peng \[49\] first studied the mathematical structures of the LMOV conjecture for general links without framing contribution, which is equivalent to the LMOV conjecture for colored HOMFY-PT invariants \(W_{\mathcal{X}}(\mathcal{L}; q, a)\). They provided a proof for this case by using cut-and-join analysis and the cabling technique \[49\]. Motivated by the work \[56\], Chen et al. \[11\] applied the HOMFLY-PT skein theory to study the mathematical structures of the LMOV conjecture for \(\tau\)-framed colored HOMFLY-PT invariants \(H_{\mathcal{X}}(\mathcal{L}; q, a)\).
5.4 A special case of the refined LMOV conjecture

From the formula (5.18), we have

\[
\tilde{g}^{(b)}_{\mu} (\mathcal{L}; q, a) = \frac{1}{a - a^{-1}} \tilde{g}^{(b)}_{\mu} (\mathcal{L}; q, a) = \frac{1}{a - a^{-1}} \frac{\mu(d)}{\mu(\hat{d})} \sum_{a | \hat{d}} F^{(h)} (\mathcal{L}; q^{\alpha}, a^{\alpha}).
\]  

(5.42)

In particular, for a framed knot \( \mathcal{K}_\tau \) (i.e., a link with only one component) with \( \tau = w(\mathcal{K}) \) and \( \mu = (p) \), by the formula (5.13), we obtain

\[
F^{(0)}_{(p)} (\mathcal{K}_\tau; q, a) = \frac{1}{p} (-1)^{p\tau} \mathcal{H} (\mathcal{K}_\tau \ast P_p; q, a).
\]  

(5.43)

Then the refined LMOV Conjecture 5.10 implies the following result.

**Conjecture 5.13.** For any prime \( p \), we have

\[
\tilde{g}^{(0)}_{\mu} (\mathcal{K}_\tau; q, a) = \frac{1}{a - a^{-1}} \frac{1}{(p)} \mathcal{H} (\mathcal{K}_\tau \ast P_p; q, a) - (-1)^{(p-1)\tau} \Psi_p (\mathcal{H} (\mathcal{K}_\tau; q, a))
\]  

\[\in z^{-2} a^{\tau} \mathbb{Z}[a^2, a^{2\tau}],\]

(5.44)

where \( \epsilon = 1 \) when \( p = 2 \) and \( \epsilon = 0 \) when \( p \) is an odd prime.

**Remark 5.14.** In other words, the refined LMOV Conjecture 5.10 implies that for any prime number \( p \),

\[
\{p\} \mathcal{H}_p (K; q, a) - (-1)^{(p-1)w(K)} \Psi_p (\{1\} \mathcal{H} (K; q, a)) \in [p]^{2} a^{\tau} \mathbb{Z}[a^{2}, a^{2\tau}],
\]

(5.45)

where \( \epsilon = 0 \) when \( p = 2 \) and \( \epsilon = 1 \) when \( p \) is an odd prime. Formula (5.45) is referred to as Hecke lifting in [11]. In order to prove it, in [11], we introduced the notion of congruence skein relations for colored HOMFLY-PT invariants which may have independent interest elsewhere.

5.5 New integral link invariants

The refined LMOV Conjecture 5.10 predicts the new integral link invariants \( \tilde{N}^{(b)}_{\tilde{\mu}, g, \mathcal{Q}} (\mathcal{L}) \) and \( \tilde{N}^{(b)}_{\tilde{\mu}, i, j} (\mathcal{L}) \) for the framed link \( \mathcal{L} \). A central question in this direction is how to define these new integral invariants directly by the geometric/algebraic/combinatoric method and find the geometric meaning for them.

As to the integral invariants \( \tilde{N}^{(0)}_{\tilde{\mu}, g, \mathcal{Q}} (\mathcal{L}) \), there is an interpretation from the physics literature [42,73]. It was conjectured in [73] that for the link \( \mathcal{L} \), one can construct a Lagrangian submanifold \( C_{\mathcal{L}} \) of the resolved conifold with \( b_1 (C_{\mathcal{L}}) = L \) which is the number of the components of \( C_{\mathcal{L}} \). Let \( \gamma_\alpha, \alpha = 1, \ldots, L \) be the one-cycles representing a basis of \( H_1 (C_{\mathcal{L}}; \mathbb{Z}) \). Let \( \mathcal{M}_{g, h, \mathcal{Q}} \) be the conjectural moduli space of Riemann surfaces of genus \( g \) with \( h \) holes embedded into the resolved conifold such that there are \( h_\alpha \) holes ending on the cycles \( \gamma_\alpha \) for \( \alpha = 1, \ldots, L \). The action of the symmetric group \( \Sigma_{h_1} \times \cdots \times \Sigma_{h_L} \) on the Riemann surfaces is given by exchanging the \( h_\alpha \) holes that end on \( \gamma_\alpha \). Then the integral invariants \( \tilde{N}^{(0)}_{\tilde{\mu}, g, \mathcal{Q}} (\mathcal{L}) \) can be interpreted as the Euler number \( \chi (S^{(h)}_{\mu} (\mathcal{M}_{g, h, \mathcal{Q}})) \), where

\[
S_{\tilde{\mu}} = S_{\mu} \otimes \cdots \otimes S_{\mu^L},
\]

and \( S_{\mu^0} \) is referred to as the Schur functor.

For the integral invariants \( \tilde{N}^{(0)}_{\tilde{\mu}, i, j} (U_\tau) \) for the framed unknot \( U_\tau \), we related them to the Betti numbers of the cohomological Hall algebra or the quiver variety of a corresponding quiver in [53,95]. Then the idea of the knot-quiver correspondence was further extended in [39,40] (see [17,74,79,80] for more recent developments).

Obviously, these integral link invariants are fully determined by the Chern-Simons partition function of the link. From the point of view of topological string theory, Aganagic and Vafa [2] introduced the \( \alpha \)-deformed A-polynomial which encodes the mirror geometry of the topological string theory. According
to the large $N$ duality of Chern-Simons theory and topological string theory, these new integral link invariants are fully determined by the $a$-deformed A-polynomial.

For a series of colored HOMFLY-PT invariants $\{\mathcal{H}_n(K; q, a)\}_{n \geq 0}$ of the symmetric representation of a knot $K$, we introduce two operators $M$ and $L$ acting on $\{\mathcal{H}_n(K; q, a)\}_{n \geq 0}$ as follows:

$$MH_n = q^n \mathcal{H}_n, \quad LH_n = \mathcal{H}_{n+1},$$

and then $LM = qML$.

**Definition 5.15.** The noncommutative $a$-deformed A-polynomial for series $\{\mathcal{H}_n(q, a)\}_{n \geq 0}$ is a polynomial $\hat{A}(M, L; q,a)$ of operators $M$ and $L$ such that

$$\hat{A}(M, L; q, a)\mathcal{H}_n(q, a) = 0 \quad \text{for } n \geq 0,$$

and $A(M, L; a) = \lim_{q \to 1} \hat{A}(M, L; q, a)$ is called the $a$-deformed A-polynomial.

The existence of the $a$-deformed A-polynomial was proved rigorously in [23].

**Example 5.16.** As to the framed unknot $U_\tau$, the noncommutative $a$-deformed A-polynomial for $U_\tau$ is as follows:

$$\hat{A}_{U_\tau}(M, L; q, a) = (-1)\tau(qM - 1)L - M^\tau(a^{\frac{1}{2}} q^{\frac{1}{2}} M - a^{-\frac{1}{2}} q^\frac{1}{2}),$$

and the $a$-deformed A-polynomial is

$$A_{U_\tau}(M, L; a) = \lim_{q \to 1} \hat{A}(M, L; q, a) = (-1)\tau(M - 1)L - M^\tau(a^{\frac{1}{2}} M - a^{-\frac{1}{2}}).$$

From this formula, one can compute the explicit formula for the integral invariant $\tilde{N}_{m,0,\emptyset}(U_\tau)$ (see [94]).

On the other hand, there is a proposal initiated by Aganagic et al. [1] which connects the topological strings and contact homology theory. In this framework, it is conjectured that the $a$-deformed A-polynomial is equal to the augmentation polynomial of knot contact homology after variable transformations. Therefore, there should be a geometric way to define these new integral link invariants from knot contact homology.

### 6 Special polynomials

From this section, we begin to study two specializations of the normalized framed full colored HOMFLY-PT invariants $\mathcal{P}_{\lambda,\mu}(K; q, a)$.

By Theorem 4.5, we know that when $q = 1$, $\mathcal{P}_{\lambda,\mu}(K; q = 1, a)$ is a well-defined polynomial of $a$ lying in the ring $\mathbb{Z}[a]$.

For a knot $K$ and a partition $\lambda \in \mathcal{P}$, Dunin-Barkowski et al. [16] introduced the following special polynomial for the colored HOMFLY-PT invariant of the knot $K$:

$$H^K_\lambda(a) = \lim_{q \to 1} \frac{W_\lambda(K; q, a)}{W_\lambda(U; q, a)}, \quad \text{(6.1)}$$

After testing many examples [16,30], they proposed the following conjectural formula:

$$H^K_\lambda(a) = H^{(1)}_{\lambda,\emptyset}(a)^{|\lambda|}. \quad \text{(6.2)}$$

A rigorous mathematical proof of the formula (6.2) was provided in [49] and [93] with different methods. According to the formula (6.1), indeed, we have

$$H^K_\lambda(a) = \lim_{q \to 1} a^{-|\lambda|w(K)} \mathcal{P}_{\lambda,\emptyset}(K; q, a) = a^{-|\lambda|w(K)} \mathcal{P}_{\lambda,\emptyset}(K; 1, a). \quad \text{(6.3)}$$
Note that when \( \lambda = (1) \),
\[
\mathcal{P}_{(1),0}(K; q, a) = \frac{\mathcal{H}(K; q, a)}{\mathcal{H}(U; q, a)},
\]
which will be denoted by \( \mathcal{P}(K; q, a) \) for brevity in the following.

Therefore, the formula (6.2) can be rewritten as
\[
\mathcal{P}_{(1),0}(K; 1, a) = \mathcal{P}_{(1),0}(K; 1, a)^{|\lambda|} = \mathcal{P}(K; 1, a)^{|\lambda|}.
\]

Motivated by the above results, it is natural to introduce the following definition.

**Definition 6.1.** Given a knot \( K \) and partitions \( \lambda, \mu \in \mathcal{P}_+ \), the special normalized framed full colored HOMFLY-PT invariant of \( K \) is defined by \( \mathcal{P}_{(\lambda, \mu)}(K; 1, a) \).

A direct consequence of Theorem 4.5 gives the following corollary.

**Corollary 6.2.** For any knot \( K \), we have
\[
\mathcal{P}_{(\lambda, \mu)}(K; 1, a) \in a^\epsilon \mathbb{Z}[a^2],
\]
where \( \epsilon \in \{0, 1\} \) is determined by \( (|\lambda| + |\mu|)w(K) \equiv \epsilon \pmod{2} \).

As a generalization of the formula (6.2), we proved the following result in [12, Theorem 5.2].

**Theorem 6.3.** For any knot \( K \) and partitions \( \lambda, \mu \in \mathcal{P}_+ \),
\[
\mathcal{P}_{(\lambda, \mu)}(K; 1, a) = \mathcal{P}(K; 1, a)^{|\lambda|+|\mu|}.
\]

Since the special polynomial plays an important role in studying the genus expansion of the quantum invariant, in this article, we also introduce the special polynomial for the composite invariants.

**Definition 6.4.** Given a knot \( K \), the special composite invariant is defined as the following limit:
\[
\mathcal{D}_\lambda(K; a) = \lim_{q \rightarrow 1} \frac{C_\lambda(K; q, a)}{C_\lambda(U; q, a)}.
\]

**Theorem 6.5.** For any knot \( K \), we have
\[
\mathcal{D}_\lambda(K; a) = \mathcal{P}(K; 1, a)^{|\lambda|}.
\]

We need the following lemma for the algebraic structure of the HOMFLY-PT polynomial.

**Lemma 6.6** (See [45, 92]). For a link \( \mathcal{L} \) with \( L \) components \( K_\alpha, \alpha = 1, \ldots, L \), the HOMFLY-PT polynomial \( P(\mathcal{L}; q, a) \) of \( \mathcal{L} \) can be written in the following form:
\[
P(\mathcal{L}; q, a) = \sum_{g \geq 0} p_{2g+1-L}(a)(q - q^{-1})^{2g+1-L}.
\]

Moreover,
\[
p_{1-L}(a) = a^{-\text{tr}(\mathcal{L})}(a - a^{-1})^{L-1} \prod_{\alpha=1}^{L} p_0^{K_\alpha}(a),
\]
where \( p_0^{K_\alpha}(a) \) is the HOMFLY-PT polynomial of the \( \alpha \)-th component \( K_\alpha \) of the link \( \mathcal{L} \) with \( q = 1 \), i.e.,
\[
p_0^{K_\alpha}(a) = P(K_\alpha; 1, a) = a^{-w(K_\alpha)} P(K_\alpha; 1, a).
\]

By the definition of the HOMFLY-PT polynomial, we have
\[
\mathcal{H}(\mathcal{L}; q, a) = \sum_{g \geq 0} \hat{p}_{2g+1-L}(a)(q - q^{-1})^{2g-L},
\]
where \( \hat{p}_{2g+1-L}(a) = a^w(\mathcal{L}) p_{2g+1-L}(a)(a - a^{-1}) \). Hence,
\[
p_{1-L}(a) = a^\sum_{\alpha=1}^{L} w(K_\alpha)(a - a^{-1}) \prod_{\alpha=1}^{L} p_0^{K_\alpha}(a) = (a - a^{-1})^L \prod_{\alpha=1}^{L} P(K_\alpha; 1, a).
\]

We now prove Theorem 6.5.
Proof of Theorem 6.5. We consider the skein element

\[ Q^\lambda = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} Q_{\mu,\nu}, \]  

(6.14)

where

\[ c^\lambda_{\mu,\nu} = \sum_{A,B} \frac{\chi_\mu(A)\chi_\nu(B)}{\delta A \delta B} \chi_\lambda(A \cup B) \]  

(6.15)

and

\[ Q_{\mu,\nu} = q_{\mu} Q^*_{\mu} + \sum_{\sigma \neq 0} (-1)^{|\sigma|} c^\mu_{\sigma,\rho} c^\nu_{\sigma',\tau} Q_{\rho} Q^*_\tau. \]  

(6.16)

By using the Frobenius formula (2.14), we obtain

\[ Q^\lambda = \sum_{\mu,\nu} \sum_{A,B} \frac{\chi_\mu(A)\chi_\nu(B)}{\delta A \delta B} \chi_\lambda(A \cup B)Q^*_\mu Q^*_\nu \\
+ \sum_{\mu,\nu} c^\lambda_{\mu,\nu} \sum_{\sigma \neq 0} (-1)^{|\sigma|} c^\mu_{\sigma,\rho} c^\nu_{\sigma',\tau} Q_{\rho} Q^*_\tau \\
= \sum_{A,B} \frac{\chi_\lambda(A \cup B)}{\delta A \delta B} P_A P_B^* + \sum_{\mu,\nu} c^\lambda_{\mu,\nu} \sum_{\sigma \neq 0} (-1)^{|\sigma|} c^\mu_{\sigma,\rho} c^\nu_{\sigma',\tau} Q_{\rho} Q^*_\tau \\
= \sum_{|A|+|B|=|\lambda|} \frac{\chi_\lambda(1|\lambda|)}{\delta(1|A|) \delta(1|B|)} P_{(1|A|)}^* P_{(1|B|)}^* + \sum_s LT_s. \]  

(6.17)

The main observation is that every term in the first summation

\[ \sum_{|A|+|B|=|\lambda|} \frac{\chi_\lambda(1|\lambda|)}{\delta(1|A|) \delta(1|B|)} P_{(1|A|)}^* P_{(1|B|)}^* \]  

(6.18)

contains $|A|+|B|=|\lambda|$ components in the skein $C$, while the remaining terms $LT_s$ in the second summation have the components less than $|\lambda|$.

By definition, we have

\[ C_\lambda(K; q, a) = \sum_{|A|+|B|=|\lambda|} \frac{\chi_\lambda(1|\lambda|)}{\delta(1|A|) \delta(1|B|)} H(K \ast P_{(1|A|)}^* P_{(1|B|)}^*; q, a) \\
+ \sum_s H(K \ast LT_s; q, a) \]  

(6.19)

and

\[ C_\lambda(U; q, a) = \sum_{|A|+|B|=|\lambda|} \frac{\chi_\lambda(1|\lambda|)}{\delta(1|A|) \delta(1|B|)} \left( \frac{a - a^{-1}}{q - q^{-1}} \right)^{|\lambda|} \\
+ \sum_s H(U \ast LT_s; q, a). \]  

(6.20)

Since $K \ast P_{(1|A|)}^* P_{(1|B|)}^*$ is a link with $|A|+|B|=|\lambda|$ components, according to the expansion formula (6.12), we have

\[ H(K \ast P_{(1|A|)}^* P_{(1|B|)}^*; q, a) = \sum_{g \geq 0} K_{\gamma_{2g+1-(|\lambda|)}} P_{(1|A|)}^* P_{(1|B|)}^*(a) (q - q^{-1})^{2g-|\lambda|}. \]  

(6.21)
For the link $\mathcal{K} \ast LT_s$ with the number of components $L(\mathcal{K} \ast LT_s) \leq |\lambda| - 1$, we also have

$$
\mathcal{H}(\mathcal{K} \ast LT_s; q, a) = \sum_{g \geq 0} \hat{p}_{2g+1-L(\mathcal{K} \ast LT_s)}^{\mathcal{K} \ast LT_s}(a)(q - q^{-1})^{2g-L(\mathcal{K} \ast LT_s)}.
$$

(6.22)

Since

$$
\frac{\chi_\mathcal{L}(1)}{\delta(1)} \neq 0
$$

when $|A| + |B| = |\lambda|$, by direct calculation, we obtain

$$
\lim_{q \to 1} \frac{\mathcal{C}_\mathcal{L}(\mathcal{K}; q, a)}{\mathcal{C}_\mathcal{L}(U; q, a)} = \frac{\sum_{|A|+|B| = |\lambda|} \frac{1}{\delta_1(1)} \hat{p}_{1-|\lambda|}^{\mathcal{K} \ast P_{1|\lambda|}(P_{1|\lambda|}^*) \mathcal{P}_{1|\lambda|}}(a)}{(a - a^{-1})^{|\lambda|} \sum_{|A|+|B| = |\lambda|} \frac{1}{\delta_1(1)} \hat{p}_{1-|\lambda|}^{\mathcal{K} \ast P_{1|\lambda|}(P_{1|\lambda|}^*) \mathcal{P}_{1|\lambda|}}(a)}.
$$

(6.23)

Moreover, the formula (6.13) implies

$$
\frac{\mathcal{K} \ast P_{1|\lambda|} \mathcal{P}_{1|\lambda|}^* - \mathcal{P}_{1|\lambda|}}{(a - a^{-1})^{|\lambda|} \mathcal{P}_{1|\lambda|}}(a) = \mathcal{P}_{1|\lambda|} \mathcal{P}_{1|\lambda|}^* - \mathcal{P}_{1|\lambda|}^* \mathcal{P}_{1|\lambda|}.
$$

(6.24)

since the number of components $L(\mathcal{K} \ast P_{1|\lambda|} \mathcal{P}_{1|\lambda|}^*) = |A| + |B| = |\lambda|$. Therefore, we have

$$
\lim_{q \to 1} \frac{\mathcal{C}_\mathcal{L}(\mathcal{K}; q, a)}{\mathcal{C}_\mathcal{L}(U; q, a)} = \mathcal{P}_{1|\lambda|}.
$$

(6.25)

This completes the proof.

\[ \square \]

### 7 Colored Alexander polynomials

In [30], Itoyama et al. also considered another limit for colored HOMFLY-PT invariants

$$
A_\lambda(\mathcal{K}; q) = \lim_{a \to 1} \mathcal{W}_\lambda(\mathcal{K}; q, a).
$$

(7.1)

When $\lambda = (1)$, by its definition, we obtain

$$
A_{(1)}(\mathcal{K}; q) := A(\mathcal{K}; q)
$$

which is the Alexander polynomial for the knot $\mathcal{K}$, so we call $A_\lambda(\mathcal{K}; q)$ the colored Alexander polynomial of $\mathcal{K}$. It motivates us to consider the limit of the full framed colored HOMFLY-PT invariants.

**Definition 7.1.** The full (framed) colored Alexander polynomial of $\mathcal{K}$ is defined by

$$
A_{|\lambda|}(\mathcal{K}; q) := \lim_{a \to 1} \mathcal{H}_{|\lambda|}(\mathcal{K}; q, a).
$$

(7.2)

In particular, when $|\mu| = \emptyset$, we have the relationship

$$
A_{|\lambda,\emptyset|}(\mathcal{K}; q) = q^{c_{\lambda,\emptyset}} A_\lambda(\mathcal{K}; q)
$$

(7.3)

according to their definitions.

Recalling the definition of the normalized framed full colored HOMFLY-PT invariant, actually, we have

$$
A_{|\lambda,\mu|}(\mathcal{K}; q) = \mathcal{P}_{|\lambda,\mu|}(\mathcal{K}; q, a = 1).
$$

(7.4)

By Theorem 4.5, we obtain that $\mathcal{P}_{|\lambda,\mu|}(\mathcal{K}; q, 1)$ is a well-defined polynomial of $q$ lying in the ring $\mathbb{Z}[q^2]$. 


Corollary 7.2.  The full colored Alexander polynomial $A_{|\lambda|,\mu}(\mathcal{K};q)$ is well defined and
\begin{equation}
A_{|\lambda|,\mu}(\mathcal{K};q) \in \mathbb{Z}[q^2].
\end{equation}

In [16], Dunin-Barkowski et al. studied the properties for $A_{\lambda}(\mathcal{K};q)$, and they proposed the conjectural formula
\begin{equation}
A_{\lambda}(\mathcal{K};q) = A(\mathcal{K};q^{|\lambda|}).
\end{equation}
However, in [93], we found that the formula (7.6) does not hold for the non-hook partition such as $\lambda = (2, 2)$, and we modified the above formula in the following form.

Conjecture 7.3.  When $\lambda$ is a hook partition,
\begin{equation}
A_{\lambda}(\mathcal{K};q) = A(\mathcal{K};q^{|\lambda|}).
\end{equation}

We proved the following results in [93].

Theorem 7.4.  Conjecture 7.3 holds for the torus knot $T_{m,r}$.

Remark 7.5.  Based on some concrete computations, we found that the analogous conjecture does not hold for the full colored Alexander polynomial (7.4).

In the following, we show that Conjecture 7.3 can be reduced to an identity for the characters of the Hecke algebra.

First, we recall that every hook partition of weight $d$ can be presented as the form $(m + 1, 1, \ldots, 1)$ with length $n + 1$ for some $m, n \in \mathbb{Z}_{\geq 0}$, denoted by $(m|n)$, with $m + n + 1 = d$. It is clear that $\kappa_{(m|n)} = (m - n)d$.

By the property of character theory of the symmetric group, one has
\begin{equation}
\chi_{\lambda}((d)) = \begin{cases} (-1)^n, & \text{if } \lambda \text{ is a hook partition } (m|n), \\ 0, & \text{otherwise}. \end{cases}
\end{equation}

Given a knot $\mathcal{K}$, let $b$ be a braid presentation of $\mathcal{K}$. Suppose that $b$ has $r$ strands. Given a partition $\lambda$ of weight $|\lambda| = d$, we construct a $d$-cabling braid $b^{(d,\ldots,d)}$, which means that $b^{(d,\ldots,d)}$ is the new braid constructed by replacing every strand in $b$ with $d$ parallel strands. Recall the idempotent $y_{\lambda}$ of the Hecke algebra introduced in Section 3, and let
\begin{equation}
X_{\lambda}(b) = b^{(d,\ldots,d)} \cdot (y_{\lambda} \otimes \cdots \otimes y_{\lambda})
\end{equation}
be the element in the Hecke algebra $H_{dr}(q, a)$. Given a partition $\Lambda$ of weight $|\Lambda| = dr$, denote by $\zeta^\Lambda$ a character of the Hecke algebra $H_{dr}(q, a)$.

Then we have the following formula due to Lin and Zheng [46]:
\begin{equation}
W_{\lambda}(\mathcal{K};q,a) = q^{-\kappa_{\lambda}(\mathcal{K})}a^{-|\lambda|w(\mathcal{K})} \sum_{|\Lambda|=dr} \zeta^{\Lambda}(X_{\lambda}(\mathcal{K})) s_{\lambda}^*(q,a),
\end{equation}
where $s_{\lambda}^*(q,a) = \prod_{x \in \lambda} \frac{q^{\kappa(\lambda)} - q^{-\kappa(\lambda)}}{q^{\kappa(\lambda)} - q^{-\kappa(\lambda)}}$ is the colored HOMFLY-PT invariant of the unknot.

Hence,
\begin{align*}
A_{(m|n)}(\mathcal{K};q,a) &= \lim_{a \to 1} \frac{W_{(m|n)}(\mathcal{K};q,a)}{s_{(m|n)}^*(q,a)} \\
&= \lim_{a \to 1} q^{-d(m-n)w(\mathcal{K})}a^{-dw(\mathcal{K})} \sum_{|\Lambda|=rd} \zeta^{\Lambda}(X_{(m|n)}(\mathcal{K})) \sum_{q(\Phi)=rd} \frac{\chi_{\lambda}(\Phi) \prod_{j=1}^{\mu(\Phi)} \frac{[\kappa(\Phi)]_{\mu_j}}{[\kappa(\Phi)]}}{[\kappa(\Phi)]} \\
&= \lim_{a \to 1} \frac{q^{-d(m-n)w(\mathcal{K})} \sum_{|\Lambda|=rd} \zeta^{\Lambda}(X_{(m|n)}(\mathcal{K})) \frac{\chi_{(m|n)}(\mu)}{[\kappa(\mu)]}}{d} + \sum_{|\Lambda|=dr} \frac{\mu(\Phi)}{[\kappa(\Phi)]} \prod_{j=1}^{\mu(\Phi)} \frac{[\kappa(\Phi)]_{\mu_j}}{[\kappa(\Phi)]} \\
&= \lim_{a \to 1} \frac{(-1)^n \frac{d}{[d]} + \sum_{|\Lambda|=rd} \frac{\kappa(\mu)}{[\kappa(\mu)]} \prod_{j=1}^{\mu(\Phi)} \frac{[\kappa(\Phi)]_{\mu_j}}{[\kappa(\Phi)]}}{d}.
\end{align*}
\[ q^{-d(m-n)w(K)}(-1)^n \sum_{k+l+1=rd} \zeta^{(k|l)}(X(m|n)(K))(-1)^l. \]  

(7.11)

Therefore, Conjecture 7.3 is reduced to the following identity for the characters of the Hecke algebra:

\[ q^{-d(m-n)w(K)}(-1)^n \sum_{k+l+1=dr} (-1)^l \zeta^{(k|l)}(X(m|n)(K)) = \sum_{k+l+1=r} (-1)^l \zeta^{(k|l')}(X(1)) \bigr|_{q^{-q^d}}. \]  

(7.12)

For the torus knot \( T_{r,s} \) which is defined to be the closure of \((\sigma_1 \cdots \sigma_{r-1})^s\) with \( r \) and \( s \) relatively prime, Lin and Zheng [46] computed that

\[ \zeta^\Lambda(X_\Lambda(T_{r,s})) = c_{\lambda,r}^\Lambda q^{-s\kappa_\Lambda + s\kappa_\Lambda/r}, \]  

(7.13)

where

\[ c_{\lambda,r}^\Lambda = \sum_\mu \frac{\lambda(n)(\mu)}{\delta_{\mu}} \chi(\lambda)(r\mu). \]  

(7.14)

In particular, when \( \Lambda = (m|n) \) with \( m+n+1 = d \) and \( \Lambda = (k|l) \) with \( k+l+1 = rd \), then \( q^{c(m|n)} = q^{(m-n)d} \) and \( q^{c(k|l)} = q^{(k-l)rd} \). Hence,

\[ \zeta^{(k|l)}(X(m|n)(T_{r,s})) = \sum_{|\mu| = d} \frac{\lambda(m|n)(\mu)}{\delta_{\mu}} \chi(\lambda)(r\mu)q^{-ds(m-n)}q^{ds(k-l)}. \]  

(7.15)

Therefore,

\[
\begin{align*}
&= q^{-ds(m-n)} \sum_{|\mu| = d} \frac{\lambda(m|n)(\mu)}{\delta_{\mu}} \sum_{k+l+1=rd} \chi(\lambda)(r\mu)(-1)^l q^{ds(k-l)} \\
&= q^{-ds(m-n)} \sum_{|\mu| = d} \frac{\lambda(m|n)(\mu)}{\delta_{\mu}} \prod_{j=1}^{|\mu|} \{drs\} \sum_{|\mu| = d} \frac{\lambda(m|n)(\mu)}{\delta_{\mu}} \prod_{j=1}^{|\mu|} \{drs\} \\
&= q^{-ds(m-n)} \{drs\} \sum_{|\mu| = d} \frac{\lambda(m|n)(\mu)}{\delta_{\mu}} \prod_{j=1}^{|\mu|} \{drs\} \\
&= q^{-ds(m-n)} \{drs\} \sum_{|\mu| = d} \frac{\lambda(m|n')(\mu)}{\delta_{\mu}} \sum_{m'+n'+1=rd} \chi(m'|n')(\mu)(-1)^n' q^{rsd(m'-n')} \\
&= q^{-ds(m-n)} \{drs\} \sum_{|\mu| = d} \frac{\lambda(m|n')(\mu)}{\delta_{\mu}} (-1)^n q^{drs(m-n)},
\end{align*}
\]  

(7.16)

where we have used the formula (7.18) in Lemma 7.6 twice.

Finally, we obtain

\[ A_{(m|n)}(T_{r,s}; q) = \frac{\{d\} \{drs\} \{ds\}}{\{dr\} \{ds\}}. \]  

(7.17)

Compared with the Alexander polynomial for the torus knot, Theorem 7.4 is proved.

**Lemma 7.6 (See [93, Lemma 6.5]).** Given a partition \( B \), we have the following identity:

\[ \sum_{a+b+1=|B|} \chi(a|b)(B)(-1)^b u^{a-b} = \prod_{j=1}^{|B|} \left( u^{B_j} - u^{-B_j} \right) \frac{u^{B_j}}{u - u^{-1}}. \]  

(7.18)
Proof. According to [54, Problem 14, p.49], taking \( t = u \), we have
\[
\prod_i \frac{1 - u^{-1}x_i}{1 - ux_i} = E(-u^{-1})H(u) = 1 + (u - u^{-1})s(a | b)(x)(-1)^b a^{a-b}.
\]
(7.19)

since
\[
s(a | b)(x) = \sum_{\lambda} \chi(a | b)(\lambda) \frac{z_{\lambda}}{z_{\lambda}} p_{\lambda}(x)
\]
and
\[
E(-u^{-1})H(u) = \frac{H(u)}{H(u^{-1})} = \exp \left( \sum_{r \geq 1} \frac{p_r(x)}{r} (u^r - u^{-r}) \right)
\]
\[= \prod_{r \geq 1} \exp \left( \frac{p_r(x)}{r} (u^r - u^{-r}) \right)
\]
\[= \prod_{r \geq 1} \sum_{m_r \geq 0} \frac{p_r(x)^{m_r}}{r^{m_r} m_r!} (u^r - u^{-r})^{m_r}
\]
\[= \sum_{\lambda} \frac{p_{\lambda}(x)}{\lambda} \prod_{j=1}^{(\lambda)} (a^{\lambda_j} - u^{-\lambda_j}).
\]
(7.20)

Compared with the coefficients of \( p_B(x) \) in (7.20), the formula (7.18) is obtained.

In the rest of this section, we show that Conjecture 7.3 is closely related to Conjecture 5.13 which is implied by the refined LMOV integrality conjecture for framed links.

Recall the definition of the colored Alexander polynomial
\[
A(m | n)(\mathcal{K}; q) = \lim_{a \to 1} q^{-m-n} d_{\mathcal{K}}(a^{-d_{\mathcal{K}}} \mathcal{K} \ast Q(m | n); q, a)
\]
\[= q^{-(m-n)d_{\mathcal{K}}} (-1)^n d(q^d - q^{-d}) \lim_{a \to 1} \frac{\mathcal{H}(\mathcal{K} \ast Q(m | n); q, a)}{(a^d - a^{-d})}.
\]
(7.21)

By Conjecture 7.3, we obtain
\[
\lim_{a \to 1} \frac{\mathcal{H}(\mathcal{K} \ast Q(m | n); q, a)}{(a^d - a^{-d})} = (-1)^n q^{m-n} d(q^d - q^{-d}) A(m | n)(\mathcal{K}; q)
\]
\[= (-1)^n q^{m-n} d(q^d - q^{-d}) A(\mathcal{K}; q^d).
\]
(7.22)

Then by using the Frobenius formula (2.14), we obtain
\[
\lim_{a \to 1} \frac{\mathcal{H}(\mathcal{K} \ast P_d; q, a)}{a^d - a^{-d}} = \lim_{a \to 1} \sum_{m+n+1=d} (-1)^n \frac{\mathcal{H}(\mathcal{K} \ast Q(m | n); q, a)}{a^d - a^{-d}}
\]
\[= A(\mathcal{K}; q^d) \sum_{m+n+1=d} q^{m-n} d_{\mathcal{K}}(\mathcal{K}).
\]
(7.23)

On the other hand, for the definition of the Alexander polynomial, we have
\[
A(\mathcal{K}; q) = (q - q^{-1}) \lim_{a \to 1} \frac{\mathcal{H}(\mathcal{K}; q, a)}{a - a^{-1}}.
\]
(7.24)

Then
\[
\lim_{a \to 1} \Psi_d \left( \frac{\mathcal{H}(\mathcal{K}; q, a)}{a - a^{-1}} \right) = \frac{A(\mathcal{K}; q^d)}{(q^d - q^{-d})}.
\]
(7.25)
Therefore, we obtain
\[
\lim_{a \to 1} \left( \frac{\mathcal{H}(\mathcal{K} \ast P; q, a)}{a^{d} - a^{d-1}} - (-1)^{(d-1)w(\mathcal{K})} \Psi_d \left( \frac{\mathcal{H}(\mathcal{K}; q, a)}{a - a^{-1}} \right) \right) 
\]
\[
= \frac{A(\mathcal{K}; q^d)}{d(q^d - q^{-d})} \left( \sum_{m+n+1=d} q^{(m-n)d_1(\mathcal{K})} - (-1)^{(d-1)w(\mathcal{K})} \right). 
\] (7.26)

For any prime \( p \), let
\[
G_p(\mathcal{K}; q, a) = \frac{\mathcal{H}(\mathcal{K} \ast P; q, a)}{a - a^{-1}} - (-1)^{(p-1)w(\mathcal{K})} \Psi_p(\mathcal{H}(\mathcal{K}; q, a)) 
\]
\[
= p \lim_{a \to 1} \left( \frac{\mathcal{H}(\mathcal{K} \ast P; q, a)}{a^p - a^{-p}} - (-1)^{(p-1)w(\mathcal{K})} \Psi_p(\mathcal{H}(\mathcal{K}; q, a)) \right) 
\]
\[
= \frac{A(\mathcal{K}; q^p)}{(q^p - q^{-p})} \left( \sum_{m+n+1=p} q^{(m-n)d_1(\mathcal{K})} - (-1)^{(p-1)w(\mathcal{K})} \right). 
\] (7.27)

Conjecture 5.13 states that
\[
\tilde{g}_p^{(0)}(\mathcal{K}; q, a) = \frac{G_p(\mathcal{K}; q, a)}{\{p\}} \in z^{-2}Z[z^2]. 
\] (7.28)

We have the following theorem which supports the conjectural formula (7.28).

**Theorem 7.7.** If Conjecture 7.3 holds, then we have
\[
\tilde{g}_p^{(0)}(\mathcal{K}; q, 1) \in z^{-2}Z[z^2]. 
\] (7.29)

**Proof.** By the formula (7.26), we obtain
\[
G_p(\mathcal{K}; q, 1) = \lim_{a \to 1} \left( \frac{\mathcal{H}(\mathcal{K} \ast P; q, a)}{a - a^{-1}} - (-1)^{(p-1)w(\mathcal{K})} \Psi_p(\mathcal{H}(\mathcal{K}; q, a)) \right) 
\]
\[
= p \lim_{a \to 1} \left( \frac{\mathcal{H}(\mathcal{K} \ast P; q, a)}{a^p - a^{-p}} - (-1)^{(p-1)w(\mathcal{K})} \Psi_p(\mathcal{H}(\mathcal{K}; q, a)) \right) 
\]
\[
= \frac{A(\mathcal{K}; q^p)}{(q^p - q^{-p})} \left( \sum_{m+n+1=p} q^{(m-n)d_1(\mathcal{K})} - (-1)^{(p-1)w(\mathcal{K})} \right). 
\] (7.30)

By Lemma 7.8, there is a function
\[
\alpha_p^w(\mathcal{K})(q) \in Z[z^2] 
\] (7.31)

such that
\[
\sum_{m+n+1=p} q^{(m-n)d_1(\mathcal{K})} - (-1)^{(p-1)w(\mathcal{K})} = \frac{\{p\}^2}{z^2} \alpha_p^w(\mathcal{K})(q). 
\] (7.32)

Therefore,
\[
\tilde{g}_p^{(0)}(\mathcal{K}; q, 1) = \frac{G_p(\mathcal{K}; q, a)}{\{p\}} = z^{-2}A(\mathcal{K}; q^p)\alpha_p^w(\mathcal{K})(q). 
\] (7.33)

Moreover, by using the properties for the Alexander polynomial in our notation, we have
\[
A(\mathcal{K}; -q^p) = A(\mathcal{K}; q^p), \quad A(\mathcal{K}; q^{-p}) = A(\mathcal{K}; q^p), 
\] (7.34)

which imply that
\[
A(\mathcal{K}; q^p) \in Z[z^2]. 
\] (7.35)

We finish the proof of the formula (7.29).

**Lemma 7.8.** For any prime number \( p \) and integer \( \tau \in \mathbb{Z} \), there is a function \( \alpha_p^\tau(z) \in Z[z^2] \) such that
\[
(q^p \tau)^{p-1} + (q^p \tau)^{p-3} + \ldots + (q^p \tau)^{-1} - p(1)(p-1)^\tau = [p]^2 \alpha_p^\tau(z). 
\] (7.36)
Proof. First, we prove the case \( p = 2 \). In this case, we only need to show that there is a function \( \alpha_2^\tau(z) \) in the ring \( \mathbb{Z}[z^2] \) such that

\[
q^{2\tau} + q^{-2\tau} - 2(-1)^\tau = (q^2 + q^{-2} + 2)\alpha_2^\tau(z)
\]

for any \( \tau \in \mathbb{N} \). It clearly holds when \( \tau = 0 \) and \( \tau = 1 \). For \( \tau \geq 2 \), we prove it by induction. Suppose that it holds when \( s \leq \tau \). Now for \( s = \tau + 1 \), indeed, we have the identity

\[
q^{2(\tau+1)} + q^{-2(\tau+1)} - 2(-1)^{\tau+1}
= (q^2 + q^{-2} + 2)(\alpha_2^\tau(z)(z^2 + 2) - \alpha_2^{\tau-1}(z) + 2(-1)\tau),
\]

(7.37)

Therefore, by induction, we have

\[
q^{2(\tau+1)} + q^{-2(\tau+1)} - 2(-1)^{\tau+1}
= (q^2 + q^{-2} + 2)(\alpha_2^\tau(z)(z^2 + 2) - \alpha_2^{\tau-1}(z) + 2(-1)^\tau),
\]

(7.38)
i.e., we obtain

\[
\alpha_2^{\tau+1}(z) = \alpha_2^\tau(z)(z^2 + 2) - \alpha_2^{\tau-1}(z) + 2(-1)^\tau.
\]

(7.39)

When \( p \) is an odd prime, given \( n \in \mathbb{N} \), we introduce the function

\[
Q_n(x) = \frac{x^n - x^{-n}}{x - x^{-1}} = x^{n-1} + x^{n-3} + \cdots + x^{-(n-3)} + x^{-(n-1)}.
\]

(7.40)

Then we consider the following function:

\[
f_p^\tau(x) = \frac{(x^{\tau}p^{-1} + (x^{\tau}p)p^{-3} + \cdots + (x^{\tau}p)^{-3} + (x^{\tau}p)^{-3} + (x^{\tau}p)^{-3} - p}{Q_p(x)^2}.
\]

(7.41)

Let

\[
g_p^\tau(x) = (x^{\tau}p)p^{-1} + (x^{\tau}p)p^{-3} + \cdots + (x^{\tau}p)^{-3} + (x^{\tau}p)^{-3} + (x^{\tau}p)^{-3} - p.
\]

(7.42)

Since all the roots of \( Q_p(x) \) are given by \( \exp\left(\frac{2\pi i j}{p}\right) \) for \( j = 1, 2, \ldots, p - 1, \) for any root \( x_0 \) of \( Q_p(x) \), by direct computations, it is easy to obtain \( g_p^\tau(x_0) = 0 \) and \( (g_p^\tau)'(x_0) = 0 \). Therefore, \( x_0 \) is a double root of the polynomial \( g_p(x) \), and then it is easy to see that \( f_p^\tau(x) \) is a Laurent polynomial with integral coefficients which can be written as

\[
f_p^\tau(x) = c_{-\ell}x^{-\ell} + \cdots + c_0 + \cdots + c_\ell x^\ell.
\]

(7.43)

Moreover, by the definition of \( f_p^\tau(x) \), it is obvious that

\[
f_p^\tau(x) = f_p^\tau(x^{-1}),
\]

(7.44)
\[
f_p^\tau(-x) = f_p^\tau(x).
\]

(7.45)

So we must have \( i = j, c_k = c_{-k} \) and \( c_k = 0 \) for odd \( k \). Finally, it is easy to obtain that \( f_p^\tau(x) \in \mathbb{Z}[(x - x^{-1})^2] \).

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we define in the obviously, $\lambda$ is the transposed Young diagram of $\lambda$. $m$ equals the number of boxes in the $\lambda$. In this appendix, we fix the notations related to partitions and symmetric functions used in this paper.

Appendix A Partitions and symmetric functions

In this appendix, we fix the notations related to partitions and symmetric functions used in this paper.

A partition $\lambda$ is a finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$. The length of $\lambda$ is the total number of parts in $\lambda$ and denoted by $l(\lambda)$. The weight of $\lambda$ is defined by $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$. If $|\lambda| = d$, we say that $\lambda$ is a partition of $d$ and denoted by $\lambda \vdash d$. The automorphism group of $\lambda$, denoted by $\text{Aut}(\lambda)$, contains all the permutations that permute parts of $\lambda$ by keeping it as a partition. Obviously, $\text{Aut}(\lambda)$ has the order

$$|\text{Aut}(\lambda)| = \prod_{i=1}^{l(\lambda)} m_i(\lambda)!,$$

where $m_i(\lambda)$ denotes the times that $i$ occurs in $\lambda$.

Every partition is identified with a Young diagram. The Young diagram of $\lambda$ is a graph with $\lambda_i$ boxes in the $i$-th row for $i = 1, 2, \ldots, l(\lambda)$, where we have enumerated the rows from the top to the bottom and the columns from the left to the right. Given a partition $\lambda$, we define the conjugate partition $\lambda^\vee$ whose Young diagram is the transposed Young diagram of $\lambda$: the number of boxes in the $j$-th column of $\lambda^\vee$ equals the number of boxes in the $j$-th row of $\lambda$ for $1 \leq j \leq l(\lambda)$. For the box in the $i$-th row and the $j$-th column of $\lambda$, we write $(i, j) \in \lambda$, and refer to $(i, j)$ as the coordinates of the box. For $x = (i, j) \in \lambda$, we define $c_n(x) = j - i$ and $h_l(x) = \lambda_i + \lambda_j' - i - j + 1$.

The following numbers associated with a given partition $\lambda$ are used frequently in this article:

$$\mathcal{A}_\lambda = \prod_{j=1}^{l(\lambda)} j^{m_j(\lambda)} m_j(\lambda)! \quad \text{and} \quad k_\lambda = \sum_{j=1}^{l(\lambda)} \lambda_j(\lambda_j - 2j + 1).$$

Obviously, $k_\lambda$ is an even number and $k_\lambda = -k_{\lambda^\vee}$.

In the following, we use the notation $\mathcal{P}_+$ to denote the set of all the partitions of positive integers. Let $0$ be the partition of $0$, i.e., the empty partition. Define $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$, and $\mathcal{P}^L$ to be the $L$ tuple of $\mathcal{P}$.

The power symetric function of infinite variables $x = (x_1, \ldots, x_N, \ldots)$ is defined by

$$p_n(x) = \sum_i x_i^n.$$
The Schur function \( S_{\lambda}(x) \) is determined by the Frobenius formula
\[
s_{\lambda}(x) = \sum_{\mu} \frac{\chi_{\lambda}(\mu)}{\delta_{\mu}} p_{\mu}(x),
\]
where \( \chi_{\lambda} \) is the character of the irreducible representation of the symmetric group \( S_{|\lambda|} \) corresponding to \( \lambda \). We have \( \chi_{\lambda}(\mu) = 0 \) if \(|\mu| \neq |\lambda|\). The orthogonality of the character formula gives
\[
\sum_{\lambda} \chi_{\lambda}(\mu) \chi_{\lambda}(\nu) = \delta_{\mu,\nu},
\]
and
\[
\sum_{\lambda} \chi_{\lambda}(\mu) \chi_{\lambda}(\nu) = \delta_{\mu,\nu}.
\]

We introduce the multiple-index variable \( \vec{x} = (x^1, \ldots, x^L) \), where \( x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots) \) for \( \alpha = 1, \ldots, L \).

Given \( \vec{X} = (\lambda^1, \ldots, \lambda^L) \) and \( \vec{\mu} = (\mu^1, \ldots, \mu^L) \in P^L \), we introduce the following notations:
\[
[\vec{X}, \vec{\mu}] = ([\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]),
\]
\[
\vec{\delta} = \prod_{\alpha=1}^L \delta_{\mu^\alpha},
\]
\[
\vec{\mu} = (\mu^1, \ldots, \mu^L),
\]
\[
\chi_{\vec{X}}(\vec{\mu}) = \prod_{\alpha=1}^L \chi_{\lambda^\alpha}(\mu^\alpha),
\]
\[
s_{\vec{X}}(\vec{\mu}) = \prod_{\alpha=1}^L s_{\lambda^\alpha}(x^\alpha),
\]
\[
p_{\vec{X}}(\vec{\mu}) = \prod_{\alpha=1}^L p_{\mu^\alpha}(x^\alpha).
\]

Now, we consider the set \( P^L \), and one can define the order of \( P^L \) as follows. For any \( \vec{X}, \vec{\mu} \in P^L \), \( \vec{X} \sim \vec{\mu} \) if and only if
\[
\sum_{\alpha=1}^L |\lambda^\alpha| > \sum_{\alpha=1}^L |\mu^\alpha|
\]
or
\[
\sum_{\alpha=1}^L |\lambda^\alpha| = \sum_{\alpha=1}^L |\mu^\alpha|,
\]
and there is a \( \beta \) such that \( \lambda^\alpha = \mu^\alpha \) for \( \alpha < \beta \) and \( \lambda^\beta > \mu^\beta \). Define
\[
\vec{X} \cup \vec{\mu} = (\lambda^1 \cup \mu^1, \ldots, \lambda^L \cup \mu^L),
\]
and \((\emptyset, \ldots, \emptyset)\) is the empty element. Then \( P^L \) is a partitionable set (see [49]). For a partitionable set \( S \), one can define the partition with respect to \( S \), a finite sequence of nonincreasing non-minimum elements in \( S \). We call it an \( S \)-partition.

Let \( P(P^L) \) be the set of all the \( P^L \)-partitions. For a \( P^L \)-partition \( \Lambda \), denote by \( l(\Lambda) \) the length of \( \Lambda \). One can also define the automorphism group of \( \Lambda \) as follows. Given \( \Omega \in \Lambda \), denote by \( m_{\Omega}(\Lambda) \) the times that \( \Omega \) occurs in the parts of \( \Lambda \). Then we have
\[
|\text{Aut}(\Lambda)| = \prod_{\Omega \in \Lambda} m_{\Omega}(\Lambda)!
\]
We define the following quantity associated with \( \Lambda \):
\[
\Theta_{\Lambda} = (-1)^{l(\Lambda) - |l(\Lambda) - 1|} \frac{|\text{Aut}(\Lambda)|}{|\text{Aut}(\Lambda)|}.
\]
We have
\[
\log \left( \sum_{\vec{X} \in P^L} Z_{\vec{X}} p_{\vec{X}}(\vec{\mu}) \right) = \sum_{\Lambda \in P(P^L)} Z_{\Lambda} \Theta_{\Lambda} p_{\vec{X}}(\vec{\mu}).
\]
Appendix B  Quantum group invariants and colored HOMFLY-PT invariants

In this appendix, we briefly review the definition of the quantum group invariant of the link (see [41,46]).

Appendix B.1 Quantum group invariants

Let $\mathfrak{g}$ be a complex simple Lie algebra and $q$ be a nonzero complex number which is not a root of unity. Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra of $\mathfrak{g}$. The ribbon category structure of finite-dimensional $U_q(\mathfrak{g})$-modules provides the following objects.

(1) For each pair of $U_q(\mathfrak{g})$-modules $V$ and $W$, there is a natural isomorphism $\hat{R}_{V,W}: V \otimes W \rightarrow W \otimes V$ such that

$$\hat{R}_{U \otimes V,W} = (\hat{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \hat{R}_{V,W}),$$  \hspace{1cm} (B.1)

hold for all the $U_q(\mathfrak{g})$-modules $U$, $V$, and $W$. The naturality means

$$(y \otimes x)\hat{R}_{V,W} = \hat{R}_{V',W'}(x \otimes y)$$  \hspace{1cm} (B.2)

for $x \in \text{Hom}_{U_q(\mathfrak{g})}(V,V')$ and $y \in \text{Hom}_{U_q(\mathfrak{g})}(W,W')$. These equalities imply the braiding relation

$$(\hat{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \hat{R}_{U,V})(\hat{R}_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes \hat{R}_{U,V})(\hat{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \hat{R}_{V,W}).$$  \hspace{1cm} (B.3)

(2) There exists an element $K_{2\rho} \in U_q(\mathfrak{g})$, where $\rho$ denotes the half-sum of all the positive roots of $\mathfrak{g}$ such that

$$K_{2\rho}(v \otimes w) = K_{2\rho}(v) \otimes K_{2\rho}(w)$$  \hspace{1cm} (B.4)

for $v \in W$ and $w \in W$. Furthermore, for any $z \in \text{End}_{U_q(\mathfrak{g})}(V \otimes W)$ with $z = \sum_i x_i \otimes y_i$, $x_i \in \text{End}(V)$ and $y_i \in \text{End}(W)$, one has the partial quantum trace

$$\text{tr}_W(z) = \sum_i \text{tr}(y_i K_{2\rho}) : x_i \in \text{End}_{U_q(\mathfrak{g})}(V).$$  \hspace{1cm} (B.5)

(3) For every $U_q(\mathfrak{g})$-module $V$, there is a natural isomorphism $\theta_V: V \rightarrow V$ satisfying

$$\theta_V^{\pm1} = \text{tr}_V \hat{R}^{\pm1}_{V,V}.$$  \hspace{1cm} (B.6)

The naturality means that $x \cdot \theta_V = \theta_{V'} \cdot x$ for $x \in \text{Hom}_{U_q(\mathfrak{g})}(V,V')$.

In the following, we only consider the finite-dimensional irreducible representations of $U_q(\mathfrak{g})$. These representations are labeled by the highest weights $\Lambda$, and the corresponding modules are denoted by $V_{\Lambda}$. In this case,

$$\theta_{V_{\Lambda}} = q^{(\Lambda,\Lambda + 2\rho)} \mathrm{id}_{V_{\Lambda}}.$$  \hspace{1cm} (B.7)

Let $B_n$ be the braid group of $n$ strands that is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with two defining relations: (i) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$; (ii) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if $|i-j| = 1$. Every link can be presented by the closure of some element, i.e., a braid in $B_n$ for some $n$. This kind of braid representation of a link is not unique. In the following, we first fix a braid representation of a link, and then we construct the quantum group invariant of the link by this braid, and finally it turns out that such construction is independent of the choice of the braid representation.

Let $\mathcal{L}$ be an oriented link with $L$ components $\mathcal{K}_1, \ldots, \mathcal{K}_L$ labeled by $L$ irreducible $U_q(\mathfrak{g})$-modules $V_{\lambda_1}, \ldots, V_{\lambda_L}$, respectively. Choose a closed braid representation $b$ of $\mathcal{L}$ with $b \in B_m$ for some positive integer $m$. Then for the $j$-th strand in the braid $b$ we associate an irreducible module $V_{\alpha_j}$ if the $j$-th strand belongs to the component $\mathcal{K}_{\alpha_j}$ with $\alpha_j \in \{1, 2, \ldots, L\}$. In this way, the braid $b$ is associated with a total module

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_m}.$$
Due to the fundamental works \[77,83\] of Reshetikhin and Turaev, the quantum trace in an irreducible module choice of the sufficiently large integer \(N\). The above definition of the colored HOMFLY-PT invariant does not depend on the choice of \(N\). In other words, given a link \(L\), there is an integer \(N_0\) (which depends on \(L\)) such that for any \(N \geq N_0\), substituting \(q^N\) with \(a\) in the formula (B.14) will yield the same result. We refer to [82] (see [82, Lemma 4.2.4]) for the statement about how to define the HOMFLY-PT polynomial via the quantum group \(U_q(\mathfrak{sl}_N)\).
Appendix B.3  HOMFLY-PT polynomials

In the following, we first introduce the variant HOMFLY-PT skein relations

\[ x^{-1}L_+ - x^{-1}L_- = (q - q^{-1})L_0, \]  
\[ L(+1) = (xa)L \]  

to define the variant skein algebra \( Sk^<(v)(F) \) of a surface \( F \), where \( L(+) \) denotes the link which is constructed by adding a positive kink to \( L \).

Given a framed link \( L \), the variant skein algebra \( Sk^<(v)(\mathbb{R}^2) \) of the plane \( \mathbb{R}^2 \) will lead to the variant framed HOMFLY-PT polynomial \( H^<(v)(L) \) of \( L \) which is the polynomial of \( x, q \) and \( a \). Let \( w(L) \) be the writhe number of \( L \). A key observation is that the framing independent HOMFLY-PT polynomial \((xa)^{-w(L)}H^<(v)(L)\) does not involve the variable \( x \), i.e.,

\[ (xa)^{-w(L)}H^<(v)(L) = a^{-w(L)}H(L). \]

We also have the variant Hecke algebra \( H^<(v)(x, q, a) \) which is the variant skein algebra \( Sk^<(v)(R_n^a) \) of the rectangle \( R_n^a \). Correspondingly, the idempotent \( y^{(v)}_\lambda \) in \( H^<(v)(x, q, a) \) is given by

\[ y^{(v)}_\lambda = \frac{1}{\alpha_\lambda} E^{(v)}_\lambda(a) \omega_{\pi} E^{(v)}_\lambda(b) \omega_{\pi}^{-1}, \]

where

\[ E^{(v)}_\lambda(a) = a^{(v)}_{\lambda_1} \otimes a^{(v)}_{\lambda_2} \otimes \cdots \otimes a^{(v)}_{\lambda_n} \]

and

\[ E^{(v)}_\mu(b) = b^{(v)}_{\mu_1} \otimes b^{(v)}_{\mu_2} \otimes \cdots \otimes b^{(v)}_{\mu_m} \]

for any partitions \( \lambda \) and \( \mu \) with

\[ a^{(v)}_\lambda = \sum_{\pi \in S_n} (x^{-1}q)^{(\pi)} \omega_{\pi} \quad \text{and} \quad b^{(v)}_\lambda = \sum_{\pi \in S_n} (-x^{-1}q)^{(\pi)} \omega_{\pi} \in H^<(v)(x, q, a). \]

Let \( V \) be the module of the fundamental representation of \( U_q(sl_N) \), and \( \{ K^{\pm}_i, E_i, F_i \mid 1 \leq i \leq N - 1 \} \) be the set of the generators of \( U_q(sl_N) \). With a suitable basis \( \{ v_1, \ldots, v_N \} \) of \( V \), the fundamental representation is given by the matrices

\[ K_i = qE_{i,i} + q^{-1}E_{i+1,i+1} + \sum_{j \neq i} E_{j,j}, \]

\[ E_i = E_{i,i+1}, \]

\[ F_i = E_{i+1,i}, \]

where \( E_{i,j} \) is the \( N \) by \( N \) matrix with 1 in the \((i,j)\)-position and 0 elsewhere. The action of the universal \( R \)-matrix on \( V \otimes V \) is given by

\[ \hat{R}_{V,V} = q^{-\frac{1}{2}} \left( q \sum_{1 \leq i \leq N} E_{i,i} \otimes E_{i,i} + \sum_{1 \leq i \neq j \leq N} E_{j,j} \otimes E_{i,i} + (q - q^{-1}) \sum_{1 \leq i < j \leq N} E_{j,j} \otimes E_{i,i} \right). \]

It gives

\[ q^{\frac{1}{2}} \hat{R}_{V,V}(v_i \otimes v_j) = \begin{cases} q(v_i \otimes v_j), & i = j, \\ v_j \otimes v_i, & i < j, \\ v_j \otimes v_i + (q - q^{-1})v_i \otimes v_j, & i > j. \end{cases} \]

Then it is straightforward to verify the following identity:

\[ q^{\frac{1}{2}} \hat{R}_{V,V} - q^{-\frac{1}{2}} \hat{R}_{V,V}^{-1} = (q - q^{-1})\text{id}_{V \otimes V}. \]
Moreover, we have
\[
\theta_{V}^{\pm 1} = q^{\pm (N - \frac{1}{2})} \text{id}_{V}.
\] (B.24)

Given a framed link \( L \), we color all of its components by the fundamental representation \( V \). We choose a closed braid representation \( \hat{b} \) of \( L \) with \( b \in B_{M} \) for some positive integer \( M \). Then every strand of \( b \) is labeled by the fundamental representation \( V \). Since
\[
\text{tr}_{V \otimes M} \Phi(b)
\] (B.25)
is a framed link invariant of \( L \), and it satisfies the relations (B.23) and (B.24), comparing the variant skein relations (B.15) and (B.16), we obtain the following proposition.

**Proposition B.4.** The framed link invariant \( \text{tr}_{V \otimes M} \Phi(b) \) is equal to the variant HOMFLY-PT polynomial \( \mathcal{H}^{(v)}(L) \) of \( L \) after the substitutions of \( x \) by \( q^{-\frac{1}{2}} \) and \( a \) by \( q^{N} \), i.e.,
\[
\text{tr}_{V \otimes M} \Phi(b) = \mathcal{H}^{(v)}(L) \bigg|_{x=q^{-\frac{1}{2}}, a=q^{N}}.
\] (B.26)

Let \( \hat{H}_{n}^{N}(q) \) be the specialization of \( H^{(v)}(x, q, a) \) by letting \( x = q^{-\frac{1}{2}} \) and \( a = q^{N} \). Then by the relation (B.23), the homomorphism
\[
\Phi : \mathbb{C}B_{n} \to \text{End}_{U_q(sl_N)}(V \otimes^n)
\]actually gives a homomorphism \( \varphi \) from \( \hat{H}_{n}^{N}(q) \) to \( \text{End}_{U_q(sl_N)}(V \otimes^n) \) via
\[
q^{\frac{1}{2}} \sigma \rightarrow q^{\frac{1}{2}} \Phi(\sigma).
\]Finally, we let \( \hat{y}^{(v)}_{\alpha} \in \hat{H}_{n}^{N}(q) \) be the specialization of the idempotent \( y^{(v)}_{\alpha} \) in \( H^{(v)}_{n}(x, q, a) \).

Given \( L \) partitions \( \lambda^{1}, \ldots, \lambda^{L} \), let \( L \) be a framed link with \( L \) components \( K_{1}, \ldots, K_{L} \). Suppose that \( b \in B_{M} \) is a braid representation of \( L \), and the \( j \)-th strand of \( b \) belongs to the component \( K_{\alpha_j} \) for \( \alpha_j \in \{1, 2, \ldots, L\} \). Let
\[
b^{(|\lambda^{\alpha_1}|,\ldots,|\lambda^{\alpha_m}|)} \in B_{M}
\]be the braid obtained by cabling the \( j \)-th strand of \( b \) with \( |\lambda^{\alpha_j}| \) parallel ones, where \( M = |\lambda^{\alpha_1}| + \cdots + |\lambda^{\alpha_m}| \). By this construction, the closure of the braid
\[
b^{(|\lambda^{\alpha_1}|,\ldots,|\lambda^{\alpha_m}|)}(\hat{y}^{(v)}_{\lambda^{\alpha_1}} \otimes \cdots \otimes \hat{y}^{(v)}_{\lambda^{\alpha_m}})
\]is just the decorated link \( L \ast \bigotimes_{\alpha=1}^{L} Q_{\alpha^{\lambda}}^{(v)} \) since \( (\hat{y}^{(v)}_{\lambda^{\alpha_1}})^2 = \hat{y}^{(v)}_{\lambda^{\alpha_1}} \).

By Proposition B.4, we have
\[
\text{tr}_{V \otimes M} \varphi(b^{(|\lambda^{\alpha_1}|,\ldots,|\lambda^{\alpha_m}|)}(\hat{y}^{(v)}_{\lambda^{\alpha_1}} \otimes \cdots \otimes \hat{y}^{(v)}_{\lambda^{\alpha_m}})) = \mathcal{H}^{(v)}(L \ast \bigotimes_{\alpha=1}^{L} Q_{\alpha^{\lambda}}^{(v)}) \bigg|_{x=q^{-\frac{1}{2}}, a=q^{N}}.
\] (B.27)

By the construction of the idempotent \( y^{(v)}_{\lambda} \) (see the formula (B.18)) and the formula (B.17), we conclude that
\[
(xa)^{-w(b^{(|\lambda^{\alpha_1}|,\ldots,|\lambda^{\alpha_m}|)})} \mathcal{H}^{(v)}(L \ast \bigotimes_{\alpha=1}^{L} Q_{\alpha^{\lambda}}^{(v)}) = a^{-w(b^{(|\lambda^{\alpha_1}|,\ldots,|\lambda^{\alpha_m}|)})} \mathcal{H}(L \ast \bigotimes_{\alpha=1}^{L} Q_{\alpha^{\lambda}}^{(v)}).
\] (B.28)

Thus we have
\[
\mathcal{H}^{(v)}(L \ast \bigotimes_{\alpha=1}^{L} Q_{\alpha^{\lambda}}^{(v)}) = x^{w(b^{(|\lambda^{\alpha_1}|,\ldots,|\lambda^{\alpha_m}|)})} \mathcal{H}(L \ast \bigotimes_{\alpha=1}^{L} Q_{\alpha^{\lambda}}^{(v)}).
\] (B.29)
Then the formula (B.27) leads to
\[
\text{tr}_{V^\otimes m} \varphi(b^{(|\lambda|,\ldots,|\lambda^m|)}) \cdot (\tilde{y}_\lambda^{(v)} \otimes \cdots \otimes \tilde{y}_{\lambda^m}^{(v)}) = q^{-\sum_{\alpha=1}^m (|\lambda|,\ldots,|\lambda^m|)} H \left( \bigotimes_{\alpha=1}^L Q^{\lambda_\alpha} \right) \bigg|_{q=q^N}. \quad (B.30)
\]

On the other hand, since \( \tilde{y}_\lambda^{(v)} \) is an idempotent in the algebra \( \hat{H}_N \), it follows that
\[
\varphi(\tilde{y}_\lambda^{(v)}) \in \text{End}_{U_q(sl_N)}(V^\otimes |\lambda|)
\]
is an idempotent in \( \text{End}_{U_q(sl_N)}(V^\otimes |\lambda|) \). Moreover, we have
\[
\varphi(\tilde{y}_\lambda^{(v)}) V^\otimes |\lambda| = V_\lambda. \quad (B.32)
\]

Finally, if we color all the components \( K_1,\ldots,K_L \) of the framed link \( L \) by irreducible \( U_q(sl_N) \)-modules \( V_{\lambda_1},\ldots,V_{\lambda_L} \), respectively, then we have the following cabling formula.

**Proposition B.5.** Under the above settings,
\[
\text{tr}_{V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}} (\varphi(b)) = \text{tr}_{V^\otimes m} \varphi(b^{(|\lambda|^1,\ldots,|\lambda|^m)}) \cdot (\tilde{y}_{\lambda_1}^{(v)} \otimes \cdots \otimes \tilde{y}_{\lambda_m}^{(v)}) \quad (B.33)
\]

**Proof.** We refer to [46, Lemma 3.2] for the basic idea of this proof. Let \( \chi_{n,n'} \) be the following \((n,n')\)-crossing braid:
\[
\chi_{n,n'} = \prod_{i=1}^{n'} (\sigma_{i+n-1} \sigma_{i+n-2} \cdots \sigma_1) \in B_{n+n'} \quad (B.34)
\]
Applying the identities in (B.1) inductively, we obtain
\[
\varphi(\chi_{n,n'}) = \tilde{R}_{V^\otimes n',V^\otimes n} \quad (B.35)
\]
Let \( \lambda \) and \( \lambda' \) be two partitions with \(|\lambda| = n\) and \(|\lambda'| = n'\). Then
\[
\varphi(\tilde{y}_\lambda^{(v)}) V^\otimes n = V_\lambda
\]
and
\[
\varphi(\tilde{y}_{\lambda'}^{(v)}) V^\otimes n = V_{\lambda'}. \quad (B.36)
\]
We obtain
\[
\varphi(\tilde{y}_\lambda^{(v)} \otimes \tilde{y}_{\lambda'}^{(v)}) \cdot \varphi(\chi_{n,n'}) = \varphi(\tilde{y}_\lambda^{(v)}) \otimes \varphi(\tilde{y}_{\lambda'}^{(v)}) \cdot \tilde{R}_{V^\otimes n',V^\otimes n} = \tilde{R}_{V^\otimes n',V^\otimes n} \cdot \varphi(\tilde{y}_\lambda^{(v)}) \otimes \varphi(\tilde{y}_{\lambda'}^{(v)}). \quad (B.36)
\]
Therefore, by applying the formula (B.36) iteratively, we have
\[
\text{tr}_{V^\otimes n} \left( \varphi(b^{(|\lambda|^1,\ldots,|\lambda|^m)}) \cdot (\tilde{y}_{\lambda_1}^{(v)} \otimes \cdots \otimes \tilde{y}_{\lambda_m}^{(v)}) \right)
\]
\[
= \text{tr}_{V^\otimes |\lambda|^1 \otimes \cdots \otimes V^\otimes |\lambda|^m} \left( \varphi(b) \cdot \varphi(\tilde{y}_{\lambda_1}^{(v)} \otimes \cdots \otimes \tilde{y}_{\lambda_m}^{(v)}) \right)
\]
\[
= \text{tr}_{V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}} (\varphi(b)). \quad (B.37)
\]
This completes the proof. \( \square \)

**Theorem B.6.** The two definitions (B.14) and (2.31) for the colored HOMFLY-PT invariant \( W_{\lambda_1,\ldots,\lambda_L}(L;q,a) \) are the same.

**Proof.** According to the definition (B.14),
\[
W_{\lambda_1,\ldots,\lambda_L}(L;q,a)
\]
\[
= q^{-\sum_{\alpha=1}^L \alpha H(K_\alpha)}(\kappa^\alpha + |\lambda| - \frac{|\lambda\cap\beta|}{2} + \frac{d}{2} \sum_{\alpha<\beta} \eta(K_\alpha,K_\beta) |\lambda|^2 |\lambda^\beta|) \text{tr}_{V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}} (\Phi(b)) \bigg|_{q=q^N} \quad (B.38)
\]
By the formulas (B.30) and (B.33) and the following identity:

$$
\sum_{\alpha=1}^{L} w(K_{\alpha}) |\lambda(\alpha)|^2 + 2 \sum_{1 \leq \alpha < \beta \leq L} lk(K_{\alpha}, K_{\beta}) |\lambda(\alpha)| \cdot |\lambda(\beta)| = w(\beta(n_{1}, \ldots, n_{m})) \tag{B.39}
$$

we obtain

$$
W_{\lambda_1, \ldots, \lambda_L}(L; q, a) = q^{-\sum_{\alpha=1}^{L} w(K_{\alpha}) a} - \sum_{\alpha=1}^{L} |\lambda(\alpha)| w(K_{\alpha}) \mathcal{H} \left( L \star \bigotimes_{\alpha=1}^{L} Q_{\lambda(\alpha)}; q, a \right). \tag{B.40}
$$

This completes the proof. \hfill \Box

**Remark B.7.** Comparing the character formula (7.10) for the colored HOMFLY-PT invariants obtained in [46], we have

$$
\mathcal{H} \left( L \star \bigotimes_{\alpha=1}^{L} Q_{\lambda(\alpha)}; q, a \right) = \sum_{|\lambda|=M} \zeta(\varphi((b^{n_{1}, \ldots, n_{m}} y_{\lambda(1)} \otimes \cdots \otimes y_{\lambda(m)}))) s_{\lambda}^{*}(q, a). \tag{B.41}
$$

This formula would be useful when we would like to apply the character theory of the Hecke algebra to study the HOMFLY-PT skein theory.