Does the cosmological constant stay hidden?

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Abstract

We elaborate on the proposal of [Phys. Rev. Lett. 123 (2019) 13, 131302], about the hiding of the cosmological constant. We build a differential equation ruling the time evolution of the spatial average of the expansion scalar \( \langle K \rangle \). Under certain conditions for the lapse function \( N \) a solution \( \langle K \rangle = 0 \) might exist, despite the presence of a large cosmological constant \( \Lambda \). However, we show that such solution is not stable.
I. INTRODUCTION

The cosmological constant is an ingredient of General Relativity (GR) which is both theoretically satisfactory, if one thinks about Lovelock’s theorems \[1, 2\], and observationally successful, if one looks at the ΛCDM model of cosmology and its effectiveness in describing our universe on the largest scales, especially from the point of view of its accelerated expansion, discovered about two decades ago \[3–5\]. Despite these fulfillments, something with Λ seems to be problematic. We expect, on the fundamental ground of the equivalence principle, that also vacuum energy, the zero-point energy of quantum fields, should gravitate. The latter behaves then as a cosmological constant and is expected to be so much larger (although this might not be the case \[6\]) than the observed value for Λ, of order $10^{-3}$ eV, that no universe in which structures form would be possible \[7, 8\]. The first attempt to overcome this issue was put forward by Zel’dovich \[10\], but it was unsuccessful, as all the following ones until now, and the problem is known nowadays as the old cosmological constant problem. It is typically framed in the semiclassical approach to GR, and probably its solution demands a clearer comprehension of whether and how vacuum energy gravitates and a clearer understanding of how to implement quantum effects in a theory of gravitation. Moreover, observation presents us with the fact that Λ has an energy density comparable to that of matter, so a new cosmological constant problem has arisen, related with the famous coincidence problem of cosmology, which requires the explanation of how vacuum energy can be tuned in that way \[11\].

In this paper we are interested in one recent proposal, which addresses the old formulation of the cosmological constant problem, that of Ref. \[12\]. Here it is put forward the possibility that even if a huge Λ does exist at the Planck scale, it can nevertheless be averaged to vanishingly small values on macroscopic scales. This averaging to zero depends on the foliation chosen in a $3 + 1$ decomposition of spacetime, in particular on the lapse function $N$, but there are infinite possible choices of $N$ that allow to “hide” a huge Λ within the averaging process which eventually result in a vanishingly small averaged expansion scalar $\langle K \rangle$. Put in other words, a huge Λ is hidden in the foamy nature of spacetime, i.e. Wheeler’s famous “spacetime foam” \[13\].

\[1\] It seems that this had been realised already by Pauli in the 1920s, see e.g. \[9\], when he estimated the influence of the zero-point energy of the radiation field (with a cutoff at the classical electron radius) on the radius of the universe, and came to the conclusion that it “could not even reach to the moon”.

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Here, we are mostly interested in the time evolution of \(\langle K \rangle\), i.e. whether the “hiding” is preserved with time. In Ref. [12] the focus is on the Cauchy surface of the initial values on which it is argued that the averaged expansion scalar and its time-derivatives can be made vanishing for infinite choices of \(N\) and its time-derivatives. Here we follow a different path: we derive an evolution equation for \(\langle K \rangle\) and analyse the stability of its \(\langle K \rangle = 0\) solution. We find that, despite the ingenuity of the idea of the hiding, the solution \(\langle K \rangle = 0\) does not seem to be stable.

II. INITIAL VALUES FORMULATION OF GENERAL RELATIVITY AND THE AVERAGING OF SCALAR QUANTITIES

In this section we gather the relevant equations of our framework, i.e. the initial value formulation of General Relativity (GR) [12, 14–18]. Consider a 3+1 foliation of spacetime into a family of spacelike hypersurfaces \(\Sigma_t\) orthogonal to a time-like vector \(n_\mu\), normalised as \(n_\mu n^\mu = -1\). From a general metric \(g_{\mu\nu}\) one can define the projector \(h_{\mu\nu}\) as:

\[
h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu,
\]

where the plus sign comes from having chosen the positive signature for \(g_{\mu\nu}\). The metric \(g_{\mu\nu}\) can be then decomposed as follows:

\[
ds^2 = -(N^2 - N_i N^i) dt^2 + 2 N_i dx^i dt + g_{ij} dx^i dx^j = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt),
\]

where \(N(t, x)\) is the lapse function, \(N^i(t, x)\) is the shift function and \(g_{ij} = h_{ij}\). The extrinsic curvature of \(\Sigma_t\) is the symmetric part of the projection on \(\Sigma_t\) of the covariant derivative of \(n_\mu\):

\[
K_{\rho\sigma} := n_{\mu,\nu} h^\mu_{(\rho} h^\nu_{\sigma)}.
\]

We define it with the plus sign here, following Ref. [12], instead of the minus one, used e.g. in Ref. [17]. The antisymmetric part of the projected covariant derivative of \(n_\mu\) is the so-called twist and by construction it is absent here because the very possibility of performing a 3 + 1 foliation depends precisely on the vanishing of the twist.

The Einstein equations can be rewritten on the foliation as two constraints and two evolution equations for \(g_{ij}\) and \(K^i_j\). The Hamiltonian constraint is:

\[
R - K^i_j K_{\rho\sigma} K^{ij} + K^2 - 2 \Lambda = 16\pi G T_{\mu\nu} n^\mu n^\nu,
\]
where $T_{\mu\nu}$ is the matter energy-momentum tensor and we have also included the cosmological constant $\Lambda$. The momentum constraint is:

$$D_i(K^i_j - \delta^i_j K) = 8\pi G T_{\mu\nu} n^\mu h^\nu_i ,$$  

(5)

where $D_i$ denotes the covariant derivative with respect to the spatial metric $h_{ij} = g_{ij}$.

The evolution equation for the first fundamental form (i.e. the spatial metric) is:

$$\frac{1}{N} \dot{g}_{ij} = 2K_{ij} + \frac{1}{N}(D_j N_i + D_i N_j) ,$$

(6)

where the dot denotes derivation with respect to $t$. The evolution equation for the second fundamental form (i.e. the extrinsic curvature tensor) is:

$$\frac{1}{N} \dot{K}^i_j = -R^i_j - K K^i_j + \delta^i_j \Lambda + \frac{D^i D_j N}{N} + \frac{1}{N} (K^i_k D_j N^k - K^k_j D_k N^i + N^k D_k K^i_j)$$

$$+ 8\pi G \left[ S^i_j + \frac{1}{2} \delta^i_j (\epsilon - S^k_k) \right] , \quad S_{ij} := T_{\mu\nu} n^\mu h^\nu_j ,$$

(7)

where $R^i_j$ is the Ricci scalar of the hypersurface $\Sigma_t$.

From the two evolution equations we can infer how the determinant $g \equiv \det(g_{ij})$ evolves with time:

$$\frac{1}{N} \dot{g} = 2g \left( K + \frac{1}{N} D_k N^k \right) ,$$

(8)

and the Raychaudhuri equation for the expansion $K$, which is the trace of the extrinsic curvature:

$$\frac{1}{N} \dot{K} = -R - K^2 + 3\Lambda + \frac{D^k D_k N}{N} + \frac{1}{N} N^k D_k K + 4\pi G (3T_{\mu\nu} n^\mu n^\nu - S^k_k) .$$

(9)

A. Averaging

The main result of Ref. [12] is to show that $\langle K \rangle$ is vanishingly small despite the presence of a very large $\Lambda$. In order to do that, the argument is to prove that if $\langle K \rangle = 0$ as initial condition on a certain $\Sigma_t$, then also all the time-derivatives of $\langle K \rangle$ are zero on the same $\Sigma_t$ and thus $\langle K \rangle$ is zero at all times. This can be achieved by suitably choosing $N$ and its time-derivatives on $\Sigma_t$. This serves to prove that even if a very large cosmological constant $\Lambda$ exists, it is absent (hidden) in the macroscopic average of the expansion $\langle K \rangle$ for many observers, i.e. for many choices of $N$. Note that “macroscopic” is intended here as “much larger than the Planck size, but not as large as the size of the universe”.

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Averaging in GR is an open problem \[19\] but if we restrict ourselves to scalar quantities, such as \( K \), then a natural definition is the following \[16, 17\]:

\[
\langle X \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} X \sqrt{g} d^3 x , \quad V_{\mathcal{D}} = \int_{\mathcal{D}} \sqrt{g} d^3 x .
\]  

(10)

The average depends of course on the portion of space \( \mathcal{D} \) over which it is performed. We avoid the subscript \( \mathcal{D} \) from now on.

### III. THE EVOLUTION EQUATION FOR \( \langle K \rangle \)

Let us neglect matter with respect to the cosmological constant \( \Lambda \) (which might be thought of as the effective one, i.e. also incorporating the zero-point energy of the quantum fields, and thus it might possibly be very large). The average \( \langle K \rangle \) is, using the definition \( (10) \):

\[
\langle K \rangle = \frac{1}{V} \int K \sqrt{g} d^3 x ,
\]  

(11)

so, by taking the time derivative of this and employing Eq. \( (9) \), one gets:

\[
\langle K \rangle^* = -\frac{\dot{V}}{V} \langle K \rangle + \frac{1}{V} \int \left[ \dot{K} + \frac{\dot{g}}{2g} \right] \sqrt{g} d^3 x =
\]

\[
-\frac{\dot{V}}{V} \langle K \rangle + \frac{1}{V} \int \left[ N(-R + 3\Lambda) + D^k D_k N + N^k D_k K + K D_k N^k \right] \sqrt{g} d^3 x .
\]  

(12)

The derivative of the volume is obtained by making use of Eq. \( (8) \):\n
\[
\frac{\dot{V}}{V} = \frac{1}{V} \int \left[ \frac{\dot{g}}{2g} \right] \sqrt{g} d^3 x = \frac{1}{V} \int (NK + D_k N^k) \sqrt{g} d^3 x .
\]  

(13)

Note that the computation of the time derivative of the volume is not necessary in Ref. \[12\] because of the initial condition chosen such that \( \langle K \rangle = 0 \). Combining Eqs. \( (12) \) and \( (13) \) we then obtain the following differential equation for \( \langle K \rangle \):

\[
\langle K \rangle^* = -\left( \langle NK \rangle + B_1 \right) \langle K \rangle - \langle NR \rangle + 3 \langle N \rangle \Lambda + B_2 + \mathcal{K} ,
\]  

(14)

where \( B_{1,2} \) are boundary terms depending only on the shift and lapse functions, respectively:

\[
B_1 := \frac{1}{V} \int D_k N^k \sqrt{g} d^3 x , \quad B_2 := \frac{1}{V} \int D^k D_k N \sqrt{g} d^3 x ,
\]  

(15)

and \( \mathcal{K} \) is a boundary term involving \( K \) itself:

\[
\mathcal{K} := \frac{1}{V} \int D_k (KN^k) \sqrt{g} d^3 x .
\]  

(16)
If we denote as $v_k$ the normal vector to the boundary $\partial D$, we have that the boundary terms are surface integrals of the quantities $v_k N^k$, $v^k D_k N$ and $K v_k N^k$. Even if we fix $N$ and $N^k$, still we have the freedom of choosing an arbitrary $D$, with an arbitrary boundary $\partial D$, which amounts to the freedom of choosing an arbitrary vector field $v_k$. Therefore, at least one of the boundary terms can be looked as an arbitrary function, even if we have already fixed $N$ and $N^k$.

The term $\langle NK \rangle$ prevents us from having a closed equation for $\langle K \rangle$. However, we can overcome this hurdle by using the following lemma, proved in Refs. [16, 17]. For a generic quantity $\Psi$ and the average defined in Eq. (10) one has:

$$\langle \Psi \rangle^\bullet = \langle \dot{\Psi} \rangle + \langle NK \Psi \rangle - \langle NK \rangle \langle \Psi \rangle ,$$  \hspace{1cm} (17)$$

Choosing then $\Psi = 1/N$, one gets:

$$\langle 1/N \rangle^\bullet = - \langle \dot{N}/N^2 \rangle + \langle K \rangle - \langle NK \rangle \langle 1/N \rangle ,$$  \hspace{1cm} (18)$$

from which we obtain:

$$\langle NK \rangle = \frac{\langle K \rangle}{\langle 1/N \rangle} - \frac{\langle 1/N \rangle^\bullet + \langle \dot{N}/N^2 \rangle}{\langle 1/N \rangle} .$$  \hspace{1cm} (19)$$

Substituting this result into Eq. (14) we obtain a closed differential equation for $\langle K \rangle$:

$$\langle K \rangle^\bullet = - f_2 \langle K \rangle^2 + f_1 \langle K \rangle + f_0 ,$$  \hspace{1cm} (20)$$

i.e. a Riccati-type equation, with:

$$f_2 := \frac{1}{\langle 1/N \rangle} , \quad f_1 := \frac{\langle 1/N \rangle^\bullet + \langle \dot{N}/N^2 \rangle}{\langle 1/N \rangle} - B_1 , \quad f_0 = - \langle NR \rangle + 3 \langle N \rangle \Lambda + B_2 + \mathcal{K} .$$  \hspace{1cm} (21)$$

Since:

$$f_1 = - \frac{\dot{f}_2}{f_2} + f_2 \langle \dot{N}/N^2 \rangle - B_1 ,$$  \hspace{1cm} (22)$$

we can cast Eq. (20) as follows:

$$(f_2 \langle K \rangle)^\bullet = - (f_2 \langle K \rangle)^2 + (f_2 \langle \dot{N}/N^2 \rangle - B_1)(f_2 \langle K \rangle) + f_2 f_0 ,$$  \hspace{1cm} (23)$$

using $f_2 \langle K \rangle$ as the unknown function.

Since $N > 0$, i.e. the lapse function is strictly positive because an arrow of time is established, then $f_2 > 0$, provided $\sqrt{g} > 0$. If singularities develop, for example due to the gluing technique which allows us to choose a vanishing initial $\langle K \rangle$, see e.g. [20], $\sqrt{g}$ might diverge somewhere in the averaging region badly enough to make, despite the integration, $\langle 1/N \rangle$ diverging and thus $f_2$ to vanish. We do not consider this possibility here and simply assume $f_2 > 0$ from now on.
A. Neglecting the boundary terms

Let us now focus on the simplest case, in which we neglect the boundary terms $B_{1,2}$ and $\mathcal{K}$. The latter makes Eq. (23) especially tricky since it contains $K$ itself, so we assume the boundary terms, being surface terms, to be negligible with respect to the “bulk” terms. Incidentally, this assumption is equivalent to the one in which we neglect the shift, explicitly used in Ref. [12], and $B_2$, also assumed in Ref. [12], but less explicitly.

Thanks to our assumptions we can cast Eq. (23) as follows:

\[(f_2\langle K \rangle)^* = -(f_2\langle K \rangle)^2 + f_2\langle \dot{N}/N^2 \rangle (f_2\langle K \rangle) + f_2(-\langle NR \rangle + 3\langle N \rangle \Lambda ) . \tag{24}\]

The solution $\langle K \rangle = 0$ exists if:

\[\langle NR \rangle = 3\langle N \rangle \Lambda . \tag{25}\]

The same condition is required in Ref. [12], after their Eq. (7), but only on the initial values hypersurface. Here we need it to hold true throughout the whole time evolution. This seems already somehow problematic because the time dependence of the right hand side of Eq. (25) comes only from $N$ and $g$, whereas on the left hand side we have also the time dependence of $R$. Let us assume anyway, that such a choice is possible and $\langle K \rangle = 0$ is a solution. The question is now, is this solution stable?

It is not difficult to see from Eq. (24), with $\langle NR \rangle = 3\langle N \rangle \Lambda$, that $\langle K \rangle = 0$ is a stable solution if:

\[f_2\langle \dot{N}/N^2 \rangle < 0 . \tag{26}\]

This can be achieved only if $\dot{N} < 0$, provided again that no singularities for which $g = 0$ develop. But since $N > 0$, $N$ cannot arbitrarily decrease. At a certain time we must expect $\dot{N}$ to vanish and possibly to change sign. When this happens, $\langle K \rangle$ would start to grow away from $\langle K \rangle = 0$.

We can also write down an explicit, though rather formal, solution of Eq. (24) by exploiting a very useful property of Riccati equations, which allows us, if we know a particular solution say $\mathcal{F}$, to write down a general solution as follows:

\[f_2\langle K \rangle = \mathcal{F} + \frac{\Phi(t)}{C + \int^t dt' \Phi(t')} , \quad \Phi = \exp \left\{ \int^t dt' \left[ -2\mathcal{F}(t') + f_2\langle \dot{N}/N^2 \rangle \right] \right\} , \tag{27}\]

where $C$ is some integration constant. For the particular solution $\langle K \rangle = 0$ whose existence
we have assumed, the general solution becomes then:

\[ \langle K \rangle = \frac{1}{f_2 C + \int^t dt' \Phi(t')} \cdot \Phi = \exp \left( \int^t dt' f_2 \langle \dot{N}/N^2 \rangle \right) . \]  

(28)

A reflection of the instability above mentioned is seen in the denominator of Eq. (28). Indeed, 

\[ 1/C \] is the initial value of 

\[ f_2 \langle K \rangle . \] So, if 

\[ f_2 \langle K \rangle \] is vanishingly small but not with fixed sign, 

then \( C \) is large and positive or negative. When \( C < 0 \), the denominator:

\[ C + \int^t dt' \Phi(t') , \]  

(29)

might diverge, because \( \int^t dt' \Phi(t') \) is always positive. In particular, we expect this to happen if \( f_2 \langle \dot{N}/N^2 \rangle > 0 \), because in this case \( \Phi \) is a growing function. If \( C > 0 \) there is no such divergence, but still \( \langle K \rangle \) should in principle grow away from \( \langle K \rangle = 0 \) if \( f_2 \langle \dot{N}/N^2 \rangle > 0 \), at least until the \( -(f_2 \langle K \rangle)^2 \) term in the Riccati equation (24) dominates on the \( f_2 \langle \dot{N}/N^2 \rangle (f_2 \langle K \rangle) \) one, in which case \( \langle K \rangle \) starts again to decrease.

In conclusion, the proposal of Ref. [12] seems to be problematic because even if the solution \( \langle K \rangle = 0 \) does exist, it is not stable.

As a final remark, note that according to Ref. [16] the ratio:

\[ \frac{\dot{V}}{V} = \langle NK \rangle , \]  

(30)

where we have neglected the shift, cf. Eq. [13], can be interpreted as an averaged Hubble factor and therefore one might argue that \( \langle NK \rangle \) is the quantity we should focus on, rather than \( \langle K \rangle \). It is not difficult to find, neglecting boundary terms:

\[ \langle NK \rangle^\bullet = -\langle NK \rangle^2 + \langle \dot{N}/N \rangle \langle NK \rangle \]

\[ + \left( \langle \dot{N}/N^2 \rangle^\bullet + \langle (\dot{N}/N^2)^\bullet \rangle + 3\langle N^2 \rangle \Lambda - \langle N^2 R \rangle + \langle ND_k D_k N \rangle \right) , \]  

(31)

as the evolution equation for \( \langle NK \rangle \), whose structure is similar to that of the evolution equation for \( \langle K \rangle \) and to which a similar analysis applies. An advantage of Eq. (31) over Eq. (24) is that we do not have to worry about \( f_2 \) being strictly positive, but just on having:

\[ \langle \dot{N}/N^2 \rangle^\bullet + \langle (\dot{N}/N^2)^\bullet \rangle + 3\langle N^2 \rangle \Lambda - \langle N^2 R \rangle + \langle ND_k D_k N \rangle = 0 , \]  

(32)

in order to guarantee the \( \langle NK \rangle = 0 \) solution, although this would be unstable anyway.
IV. DISCUSSION AND CONCLUSIONS

In this paper we have investigated the idea proposed in Ref. [12], about hiding a possibly huge cosmological constant into the foamy nature of spacetime at the Planck scale. Through a simple definition of average, we built an evolution equation describing the time evolution of \( \langle K \rangle \) and analyse the stability of its \( \langle K \rangle = 0 \) solution, provided that this exists. Unfortunately, it seems that such solution is not stable, because a necessary condition for stability is \( \dot{N} < 0 \), which is not admissible throughout the whole evolution since \( N > 0 \). Therefore, we conclude that, at least for the very simple case considered, the hiding of \( \Lambda \) is not preserved in time.

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