Radial prescribing scalar curvature on $RP^n$

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Abstract

The study of radial prescribing scalar curvature of X.Xu and P.C.Yang [2] in 1993 showed a nonexistence result on $S^2$. Later in 1995, W.Chen and C.Li [2] generalized the nonexistence result to higher dimensions. G.Bianchi and E.Egnell [1] suggested that there may exist some non-negative smooth radial function which cannot be scalar curvature in the standard conformal class of $RP^n$. However, in our paper, we prove that all smooth radial non-negative smooth functions which are positive on the pole could be prescribing scalar curvatures. We consider the quotient $\frac{v_1}{v_3}$ rather than the difference $v_1 - V_3$ as in [1]. This trick yields a concise argument of the existence. Consequently, their counter examples stated in [1] cannot be true on $RP^n$.

1 Introduction

First of all, we use a simple argument to show that their examples cannot be true if $K(r) = K(\frac{1}{r})$. So the counter examples constructed by G.Bianchi and E.Egnell in [1] do not exist on $RP^n$. 


We regard $\mathbb{RP}^n$ as the quotient space of the standard sphere $S^n$ under the antipodal map. $\mathbb{RP}^n$ has a canonical conformal class inheriting from the standard class of $S^n$.

As in [1], by the north pole projection which projects the equator of $S^n$ onto the unit sphere of $\mathbb{R}^n$, the prescribing scalar curvature problem on $S^n$ is reduced to the following differential equation on $\mathbb{R}^n$

\[
\Delta u + Ku^p = 0, \quad p = \frac{n + 2}{n - 2},
\]

where $K$ is a radial continuous function and $v^p = \text{sgn}(v)|v|^p$. We could write it down using polar coordinates as

\[
v''(r) + \frac{n-1}{r}v'(r) + Kv^p(r) = 0.
\]

Due to the symmetry, we should have $v'(0) = 0$. So we consider the initial value problem of the ODE:

\[
\begin{cases}
v''(r) + \frac{n-1}{r}v'(r) + Kv^p(r) = 0, \\
v(0) = \lambda, \ v'(0) = 0,
\end{cases}
\]

Noticing that this initial value problem always has a short time solution, we write it as $v_\lambda$. As we assume $K$ is the prescribing scalar curvature on $\mathbb{RP}^n$, $K(r) = K(\frac{1}{r})$ and $u$ should be invariant under the Kelvin transformation. Namely,

\[
v(r) = r^{2-n}v_\lambda(\frac{1}{r}).
\]

Thus, we need to solve the equation system

\[
\begin{cases}
v''(r) + \frac{n-1}{r}v' + Kv^p = 0, \\
v(0) = \lambda, \ v'(0) = 0,
\end{cases}
\]

(1.1)

When we take the derivative on $r = 1$, we get

\[(n-2)v(1) + 2v'(1) = 0\]

called gluing equation. So we can define the gluing function

\[G(\lambda) := (n-2)v_\lambda(1) + 2v'_\lambda(1).\]

We define the following quantities as in [1]:

\[\lambda_0 := \sup\{\alpha : v_\lambda(r) \text{ exists and is positive for all } \lambda \in (0, \alpha), \ r \in [0, 1]\}.
\]
\[ \lambda_\infty := \sup(\alpha : \nu_\lambda(r) \text{ exists and is positive for all } \lambda \in (\alpha, \infty), \ r \in [0, 1]). \]

In this paper, we always assume \( K(0) > 0, K \geq 0 \) and \( K \) is continuous.

We restate some results in [1] first.

**Lemma 1.** For any \( \varepsilon \in [0, 1] \),

\[
\begin{align*}
    a(\varepsilon) &= v_\lambda(\varepsilon) + \frac{\varepsilon v_\lambda'(\varepsilon)}{n-2}, \\
    b(\varepsilon) &= \frac{\varepsilon^{n-1} v_\lambda'(\varepsilon)}{n-2}, \\
    \gamma(\varepsilon) &= |a(\varepsilon)| + |b(\varepsilon)\varepsilon^{n-2}| = |v_\lambda(\varepsilon) + \frac{\varepsilon v_\lambda'(\varepsilon)}{n-2}| + |\frac{\varepsilon v_\lambda'(\varepsilon)}{n-2}|
\end{align*}
\]

Then we have the integral formulae

\[
\begin{align*}
    v_\lambda(r) &= a(\varepsilon) + b(\varepsilon)r^{2-n} - \int_\varepsilon^r s^{1-n} \int_\varepsilon^s t^{n-1} v_\lambda r Kdt ds, \\
    v_\lambda'(r) &= (2-n)b(\varepsilon)r^{1-n} - r^{1-n} \int_\varepsilon^r t^{n-1} v_\lambda r Kdt
\end{align*}
\]

and

\[
\gamma(\varepsilon) < \frac{1}{2} \left( \frac{n}{\|K\|_\infty} \right)^{\frac{1}{2n}} = |v_\lambda(r)| < 2\gamma(\varepsilon), \ \forall r \in [\varepsilon, 1].
\]

**Proof.** The integral formulae follow from the following identity

\[ (r^{n-1}v_\lambda')' = r^{n-1}(v_\lambda'' + \frac{n-1}{r} v_\lambda') = r^{n-1}(-Kv_\lambda') = -Kr^{n-1}v_\lambda'. \]

Suppose there is some value \( r \in [\varepsilon, 1] \) such that \( |v_\lambda(r)| \geq 2\gamma(\varepsilon) \), then we could pick the smallest \( r_0 \) among such values. The integral formula for \( v_\lambda(r) \) implies

\[ 2\gamma(\varepsilon) = \sup_{r \in [\varepsilon, r_0]} |v_\lambda(r)| \leq \gamma(\varepsilon) + (2\gamma(\varepsilon))^{\frac{p}{n}}\|K\|_\infty \frac{r_0^2}{2n} \leq \gamma(\varepsilon) + (2\gamma(\varepsilon))^{\frac{p}{n}}\|K\|_\infty \frac{r_0^2}{2n} < 2\gamma(\varepsilon) \]

which is a contradiction. \( \Box \)

**Corollary 1.** When \( \lambda \downarrow 0 \), we have the following uniform asymptotic formulae on \([0, 1]\)

\[
\begin{align*}
    v_\lambda(r) &= \lambda - \lambda^p \int_0^r s^{1-n} \int_0^s t^{n-1} Kdt ds + O(\lambda^{2p-1}), \\
    v_\lambda'(r) &= -\lambda^p \int_0^r t^{n-1} Kdt + O(\lambda^{2p-1})
\end{align*}
\]
Proof. Using the integral formula for $\varepsilon = 0$, we have

$$v_\lambda(r) = \lambda - \int_0^r s^{1-n} \int_0^s r^{p-1} p^p Kdtds .$$

When $\lambda \downarrow 0$, we know $v_\lambda(r) < 2\lambda$, $r \in [0, 1]$, which implies $v_\lambda(r) = \lambda + O(\lambda^p)$ uniformly on $[0, 1]$ by the integral formula. Thus $v_\lambda^p(r) = \lambda^p + O(\lambda^{2p-1})$ uniformly on $[0, 1]$. We use this in the integral formula to obtain

$$v_\lambda(r) = \lambda - \lambda^p \int_0^r s^{1-n} \int_0^s t^{p-1} Kdtds + O(\lambda^{2p-1}) , \text{ uniformly on} [0, 1].$$

The asymptotic formula for $v_\lambda'$ is deduced analogously. □

Corollary 2. $\lambda_0 > 0$.

By Lemma 3.1 of [1] or our Corollary 1, we have when $0 < \lambda << 1$,

$$(n - 2)v_\lambda(1) + 2v_\lambda'(1) = (n - 2)\lambda + o(\lambda) > 0 .$$

If $K \in C^1$, $K(r) = K(0) + K_\rho r^\rho + o(r^\rho), K_\rho \neq 0, \rho \in (\frac{n(n-2)}{n+2}, n]$, by Lemma 3.4 in [1], we have $\lambda_\infty = \infty$ or

$$(n - 2)v_\lambda(1) + 2v_\lambda'(1) = \frac{n(n-2)}{K(0)} \lambda^{\frac{2}{n-2}} (2-n)\lambda^{-1} + o(\lambda^{-1}) < 0, \text{ when } \lambda \uparrow \infty .$$

Lemma 2. If $\exists \lambda > 0$ s.t. $(n - 2)v_\lambda(1) + 2v_\lambda'(1) = 0$, then the equation has a positive solution on $[0, \infty)$.

Proof. If $v_\lambda > 0$ in $[0, 1]$, one could use the Kelvin transformation

$$v_\lambda(r) = r^{2-n} v_\lambda(\frac{1}{r})$$

to expand the solution from $[0, 1]$ to $[0, \infty)$.

Now we could assume the smallest $\lambda$ such that the gluing equation holds is $\lambda_1$. If $\lambda_0 = \infty$, then $v_\lambda > 0$ in $[0, 1]$ always holds. If $\lambda_0 < \infty$, from the definition, we know

$$v_{\lambda_0}(r) \geq 0, \forall r \in [0, 1] .$$

Thus, by the integral formula

$$v_\lambda'(r) = -r^{1-n} \int_0^r t^{p-1} v_\lambda^p Kdtds$$

we know $v_\lambda'(1) < 0$. Thus

$$(n - 2)v_{\lambda_0}(1) + 2v_{\lambda_0}'(1) < 0 .$$
Meanwhile, we know
\[(n - 2)v(1) + 2v'(1) = (n - 2)\lambda + o(\lambda) > 0, \text{ when } \lambda \downarrow 0 .\]

Hence \(0 < \lambda_1 < \lambda_0\) and \(v_{\lambda_1} > 0\) in \([0,1]\) by the definition of \(\lambda_1\).

\[\square\]

**Corollary 3.** If \(\lambda_0 < 0\), then (1.1) has a positive solution on \([0,1]\).

**Proof.** Since \(v_{\lambda_0}\) is decreasing and non-negative from the continuous reliability on parameters, we know \(v_{\lambda_0}\) exists on \([0,1]\). As in the proof of Lemma 2, \(G(\lambda_0) < 0\) and \(G(\lambda) < 0\) for small \(\lambda\). Thus \(\exists \lambda_1 \in (0,\lambda_0)\) such that \(G(\lambda_1) = 0\) and \(v_{\lambda_1}\) is the desired solution.

\[\square\]

**Theorem 1.** If \(K \in C^1(0,1), K(r) = K(0) + K_\rho r^\rho + o(r^\rho), K_\rho \neq 0, \rho \in (\frac{n(n-2)}{n^2}, n]\), then a positive solution of (1.1) on \([0,\infty)\) exists.

**Proof.** From Lemma 3.4 in [1],
\[(n - 2)v(1) + 2v'(1) = \left(\frac{n(n-2)}{K(0)}\right) \frac{2}{n-2} (2 - n)\lambda^{-1} + o(\lambda^{-1}) < 0, \ \lambda \uparrow \infty .\]

If \(\lambda \uparrow \infty\), then \(\lambda_0 < 0\). So the result follows from Corollary 3.

If
\[(n - 2)v(1) + 2v'(1) = \left(\frac{n(n-2)}{K(0)}\right) \frac{2}{n-2} (2 - n)\lambda^{-1} + o(\lambda^{-1}) < 0, \ \lambda \uparrow \infty ,\]

from
\[(n - 2)v(1) + 2v'(1) = (n - 2)\lambda + o(\lambda) > 0, \text{ when } \lambda \downarrow 0 \]

we know that there exists one \(\lambda > 0\) such that \((n - 2)v(1) + 2v'(1) = 0\). Thus the result follows from Lemma 1.

\[\square\]

**2 Main result**

One could check that the solution of
\[
\begin{cases}
v''(r) + \frac{2}{n-2} v' + K(0)v^\rho = 0, \\
v(0) = \lambda, \ v'(0) = 0,
\end{cases}
\]
is
\[V_\lambda(r) = \frac{\lambda}{[1 + \frac{\rho^\beta K(0)}{\rho -2}]^{\frac{1}{\beta}}}, \ \beta = \frac{p-1}{2} = \frac{2}{n-2} .\]

**Lemma 3.** If \(\exists \epsilon > 0\) such that \(K \in C^\infty[0,\epsilon]\) and \(v_{\lambda}\) exists on \([0,1]\) for all \(\lambda > 0\), then \(G(\lambda) > 0\) for \(\lambda\) large enough.
Proof. Consider $T_A := \frac{v_A}{T_A}$. Since $v_A = V_AT_A$, we know that $T_A$ satisfies the equation

$$V''_AT_A + (2V'_A + \frac{n-1}{r}V_A)T'_A + (V''_A' + \frac{n-1}{r}V_A')T_A + KV_A^p T_A^P = 0$$

$$\iff V''_AT_A + (2V'_A + \frac{n-1}{r}V_A) - KV_A^p T_A + KV_A^p T_A^P = 0$$

$$\iff r^5 T''_A + [n-1 - \frac{2K(0)}{n} \frac{A^2r^2}{1 + \frac{k(0)}{n} r^2 A^2}]r^4 T'_A + K \frac{A^2r^2}{[1 + \frac{k(0)}{n} r^2 A^2]^2} r(-T_A + T_A^P) = 0.$$

Thus, when $A \to +\infty$, the ODEs uniformly converge to

$$r^5 T''_\infty + (3-n)r^4 T'_\infty = 0.$$  

This equation has all solutions of the form

$$T_\infty = Cr^{n-2} + D.$$  

Now we need the assumption $K \in C^\infty([0,\varepsilon))$. Since the family of ODEs are degenerate at $r = 0$, we should consider the initial value data $P = \{T(0) = 1, T^{(k)}(0) = 0, k = 1, 2, \ldots \}$. Being subject to the initial data $P$, all the ODEs have only one solution. As $v_A(0) = V_A(0), v'_A(0) = V'_A(0)$, we know from the equation that $v_A^{(k)}(0) = V_A^{(k)}(0)$ and hence $T_A^{(k)}(0) = 0, \forall k = 1, 2, \ldots$ for log $T_A = \log v_A - \log V_A$. Thus the unique solution is $T_A$ and for the limit equation, the unique solution is $T_\infty = 1$. Thus, we know that

$$T_A \to 1 \text{ and } T'_A \to 0,$$

$$\frac{v'_A}{v_A} - \frac{V'_A}{V_A} = \frac{T'_A}{T_A} \to 0.$$  

Since

$$\lim_{A \to \infty} \frac{V_A'(1)}{V_A(1)} = 2 - n,$$

we have

$$\lim_{A \to \infty} \frac{v'_A(1)}{v_A(1)} = 2 - n.$$  

Consequently, for $A$ large enough, we have $G(\lambda) = (n-2)v_A(1) + 2v'_A(1) < 0$. □

**Theorem 2.** If $K(0) > 0, K \geq 0, K$ is continuous and $\exists \varepsilon > 0$ s.t. $K \in C^\infty([0,\varepsilon))$, then the equation (1.1) has a positive solution on $[0,\infty)$.

**Proof.** If $A_0 < \infty$, we already know the existence from Corollary 3. So we only need to treat the case $A_0 = \infty$ which means $v_A$ exists and is positive on $[0,1]$ for every $A > 0$. In this case, Lemma 2 and Lemma 3 deduce the result. □

6
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