Aspects Of Heavy Quark Theory

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Abstract

Recent achievements in the heavy quark theory are critically reviewed. The emphasis is put on those aspects which either did not attract enough attention or cause heated debates in the current literature. Among other topics we discuss (i) basic parameters of the heavy quark theory; (ii) a class of exact QCD inequalities; (iii) new heavy quark sum rules; (iv) virial theorem; (v) applications (|V_{cb}| from the total semileptonic width and from the $B \to D^*$ transition at zero recoil). In some instances new derivations of the previously known results are given, or new aspects addressed. In particular, we dwell on the exact QCD inequalities. Furthermore, a toy model is considered that may shed light on the controversy regarding the value of the kinetic energy of heavy quarks obtained by different methods.
1 Introduction

Quark-gluon dynamics are governed by the QCD Lagrangian

\[ \mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \sum_q \bar{q} i \not{D} q + \sum_Q \bar{Q}(i \not{D} - m_Q) Q = \]

\[ \mathcal{L}_{\text{light}} + \sum_Q \bar{Q}(i \not{D} - m_Q) Q \]  

(1.1)

where \( G_{\mu\nu}^a \) is the gluon field strength tensor, the light quark fields \((u,d,s)\) are generically denoted by \( q \) and are assumed, for simplicity, to be massless, and the heavy quark fields are generically denoted by \( Q \). To qualify as a heavy quark \( Q \) the corresponding mass term \( m_Q \) must be much larger than \( \Lambda_{\text{QCD}} \). The charmed quark \( c \) can be called heavy only with some reservations and, in discussing the heavy quark theory, it is more appropriate to keep in mind \( b \) quarks. The hadrons to be considered are composed from one heavy quark \( Q \), a light antiquark \( \bar{q} \), or diquark \( qq \), and a gluon cloud which also contains light quark-antiquark pairs. The role of the cloud is, of course, to keep all these objects together, in a colorless bound state which will be generically denoted by \( H_Q \).

The light component of \( H_Q \), its light cloud\(^1\) has a complicated structure – the soft modes of the light fields are strongly coupled and strongly fluctuate. Basically, the only fact which we know for sure is that the light cloud is indeed light; typical frequencies are of order of \( \Lambda_{\text{QCD}} \). One can try to visualize the light cloud as a soft medium. The heavy quark \( Q \) is then submerged in this medium. If the hard gluon exchanges are discarded, the momentum which the heavy quark can borrow from the light cloud is of order of \( \Lambda_{\text{QCD}} \). Since it is much smaller than \( m_Q \), the heavy quark-antiquark pairs can not play a role. In other words, the field-theoretic (second-quantized) description of the heavy quark becomes redundant, and under the circumstances it is perfectly sufficient to treat the heavy quark quantum-mechanically. This is clearly infinitely simpler than any field theory. Moreover, one can systematically expand in \( 1/m_Q \). Thus, in the limit \( m_Q/\Lambda_{\text{QCD}} \to \infty \) the heavy quark component of \( H_Q \) becomes easily manageable, allowing one to use the heavy quark as a probe of the light cloud dynamics. Treating the heavy quark \( Q \) in \( H_Q \) as a non-relativistic object submerged in a soft gluon background field of the light cloud, we open the door for the use of a large variety of methods developed in Quantum Electrodynamics long ago \([1, 2]\), for instance, the Pauli expansion, the Foldy-Wouthuysen technique and so on.

The special advantages of the limit \( m_Q \to \infty \) in QCD were first emphasized by Shuryak \([3]\). The next logical step was the observation of the heavy quark symmetry \([4, 5, 6]\). The heavy quark theory in QCD was finally formalized in Refs. \([7]\) where

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\(^1\)In some papers devoted to the subject the light cloud is referred to as “brown muck”, which seems to be an unfair name for the soft components of the quark and gluon fields – perhaps we have not yet unveiled their beauty.
the systematic $1/m_Q$ expansion of the $H_Q$ dynamics was cast in the form of the effective Lagrangians. Applications of the heavy quark theory are so numerous that they comprise now a significant and, perhaps, the most active part of QCD, with a well-known record of recent successes and breakthroughs. Many exhaustive reviews are devoted to the subject [8], and we feel there is no need to repeat what has already become common knowledge in our community. Our task, as we see it, is to concentrate on several aspects of the heavy quark theory which are still controversial. These aspects, as a rule, are related to a specific structure of QCD where the gluon field has two faces. The gluon degrees of freedom play a role of a soft background medium, and, simultaneously, they are responsible for the radiative $\alpha_s$ corrections reflecting the presence of the hard component in the gluon field. These two components – soft and hard – are tightly intertwined, and in many instances it is hard to disentangle them. Without untangling the hard component one cannot build consistent $1/m_Q$ and $\alpha_s$ expansions. This is in sharp distinction with, say, the Pauli expansion in QED where this problem simply does not arise. The background electromagnetic field can be taken as soft as we want, the radiative corrections are fully calculable and small (i.e. they can be neglected in most of the problems).

The most remarkable example of the problem mentioned above is the definition of the heavy quark mass – what quantity appears in the $1/m_Q$ expansion after all? In electrodynamics the electron mass is a quantity that can be measured. Even in this case, though, one encounters some (mild) problems in theoretical description, due to the masslessness of photons. As a matter of fact, the electron never appears “alone”; it is always surrounded by a photonic cloud, manifesting itself in infrared divergences. The strategy allowing one to treat these divergences and define the mass of an “isolated” electron is well-known. The situation becomes much more complicated in QCD, where isolated quarks simply do not exist. In defining the heavy quark mass one can peel off the gluonic cloud only up to a point. Namely, we can eliminate from the definition the contribution coming from hard gluons. The contribution associated with soft gluons has to be included in the definition of the heavy quark mass simply because we do not know the precise structure of the soft-gluon component. How exactly could one separate the hard and soft-gluon components in defining the heavy quark mass? This question causes debate in the current literature, and will be one of the focal points of the present review.

Another key parameter of the heavy quark theory is the kinetic operator. Formally its expectation value can be defined as

$$\mu_\pi^2 = \frac{1}{2M_{H_Q}} \langle H_Q|\bar{Q}\vec{\pi}^2Q|H_Q \rangle, \quad \vec{\pi} = -i\vec{D}, \quad (1.2)$$

where $D$ stands for the covariant derivative (note: here and elsewhere, we always assume, unless explicitly stated to the contrary, that we are in the rest frame of the heavy hadron).

The physical meaning of $\mu_\pi^2$ is quite evident: the heavy quark inside $H_Q$ experiences a *zitterbewegung* due to its coupling to other degrees of freedom inside the
hadron. Its average spatial momentum squared is $\mu_\pi^2$. The kinetic energy operator appears in the effective Lagrangian in the next-to-leading order, and basically determines the magnitude of $1/m_Q$ corrections in a large number of expressions of practical interest – from the simplest formula for the heavy hadron masses to $B \to D^{(*)}$ formfactor at zero recoil. When one tries to proceed from formal expansions to practical analysis including perturbative $O(\alpha_s^k)$ effects, the same question – what contribution is to be included in $\mu_\pi^2$ – immediately resurfaces. One can find in the literature a spectrum of answers. The numerical value of $\mu_\pi^2$ remains rather uncertain. Various determinations of this crucial parameter contradict each other. We will present a balanced discussion of competing definitions trying to outline a procedure which seems fully consistent in QCD.

A related topic we would like to consider is a QCD analog of the virial theorem. The issue was raised recently by Neubert [9], who noted that $\mu_\pi^2$ is related to a certain transition amplitude induced by the chromoelectric field. This observation was exploited for an alternative determination of $\mu_\pi^2$, which brought a surprise. A numerical value of $\mu_\pi^2$ emerging in this way was significantly lower than the results of some other determinations. The discrepancy is a subject of ongoing discussions. We present a transparent derivation of the virial theorem in QCD, and comment on the possible causes of the discrepancy.

This review is organized as follows. First, a brief survey is given of the operator product expansion (OPE) as it is used in the heavy quark theory. This survey allows us to introduce all basic tools and formulate theoretical problems to be dealt with in the remainder of the review as they appear in their natural environment. We then proceed to relatively recent applications of the heavy quark theory in the inclusive decays of heavy hadrons. The following topics are discussed in some detail: (i) basic parameters of the heavy quark expansion; (ii) a class of exact QCD inequalities; (iii) the virial theorem and (iv) applications ($|V_{cb}|$ from the total semileptonic width and from the exclusive $B \to D^*$ transition at zero recoil). Rather than attempting a comprehensive coverage of a variety of results obtained in the last two or three years, we dwell on these selected topics which either have not attracted sufficient attention so far, or are subject to ongoing debate. A significant part of our presentation is based on the results of Ref. [10] that provides a consistent field-theoretic framework for treating various aspects of the heavy quark theory that are under intense investigation at present.

Even limiting ourselves to this rather narrow task we had to leave aside, with heavy heart, many relevant issues, e.g. duality, inclusive non-leptonic decays of heavy flavors, etc. We apologize to the authors whose results, due to space limitations, are left outside the scope of the present review. The interested readers are referred to recent talks [11].
2 Operator Product Expansion; Effective Hamiltonian

The basic theoretical tool of the heavy quark theory is the Wilson operator product expansion \[12\]. It is used in two different directions. First, one can restrict oneself to the case of a fixed number of heavy quarks \(Q\) since \(\bar{Q}Q\) fluctuations can be ignored. Most often one considers the sector with a single heavy quark \(Q\) which is treated as being submerged into a gluonic medium acting as the sole source of its interactions. At this stage we disregard any other possible interaction of the heavy quark, e.g. electromagnetic, weak and so on. The original QCD Lagrangian \([1.1]\) is formulated at very short distances, or, which is the same, at a very high normalization point \(\mu = M_0\), where \(M_0\) is the mass of an ultraviolet regulator; i.e., the normalization point is assumed to be much higher than all mass scales in the theory, in particular, \(\mu \gg m_Q\). An effective theory for describing the low-energy properties of heavy flavor hadrons is obtained by evolving the Lagrangian from the high scale \(M_0\) down to a normalization point \(\mu\) lying below the heavy quark masses \(m_Q\). This means that we integrate out, step by step, all high-frequency modes in the theory thus calculating the Lagrangian \(L(\mu)\). The latter is a full-fledged Lagrangian with respect to the soft modes with characteristic frequencies less than \(\mu\). The hard (high-frequency) modes determine the coefficient functions in \(L(\mu)\), while the contribution of the soft modes is hidden in the matrix elements of (an infinite set) of operators appearing in \(L(\mu)\). The value of this approach, outlined by Wilson long ago \([12]\), has become widely recognized and exploited in countless applications. The peculiarity of the heavy quark theory lies in the fact that the in and out states contain heavy quarks. Although we integrate out the field fluctuations with the frequencies down to \(\mu \ll m_Q\), the heavy quark fields themselves are not integrated out since we consider the sector with heavy-flavor charge \(\neq 0\). The effective Lagrangian \(L(\mu)\) acts in this sector. Since the heavy quarks are neither produced nor annihilated, any sector with the given \(Q\)-number can be treated separately from all others, as a “vacuum”.

If QCD were solved we could include all modes down to \(\mu = 0\) in our explicit evaluation of the effective Lagrangian \(\tilde{L}(\mu)\). The resulting Lagrangian would be built in terms of the fields of physical mesons and baryons rather than quarks and gluons – the latter become irrelevant degrees of freedom in the infrared limit \(\mu \to 0\). All conceivable amplitudes could be read off directly from such an effective Lagrangian and could be compared with experimental data.

This picture is of course Utopian: real QCD is not solved in closed form, and in explicit calculations of the coefficients in the effective Lagrangian one cannot put \(\mu = 0\). For decreasing values of \(\mu\) a larger and larger part of the dynamics has to be accounted for in the explicit calculation. We would like to have \(\mu\) as low as possible, definitely, \(\mu \ll m_Q\). The heavy quark can be treated as a non-relativistic object moving in a soft background field only if the latter condition is met. On the other hand, to keep computational control over the explicit calculations of the coefficient
functions we must stop at some $\mu \gg \Lambda_{\text{QCD}}$, so that $\alpha_s(\mu)/\pi$ is still a sufficiently small expansion parameter. In practice this means that the best choice (which we will always adopt) is $\mu \sim$ several units times $\Lambda_{\text{QCD}}$, i.e. 0.7 to 1 GeV. All coefficients in the effective Lagrangian obtained in this way will be functions of $\mu$.

Since $\mu$ is an auxiliary parameter, predictions for physical quantities cannot depend on $\mu$, of course. The $\mu$ dependence of the coefficients must be canceled by that coming from the physical matrix elements of the operators in $\mathcal{L}(\mu)$. However, in calculating in the hard and soft domains (i.e. above $\mu$ and below $\mu$) we make different approximations, so that the exact $\mu$ independence of the physical quantities does not take place. Since the transition from hard to soft physics is very steep, one may hope that our predictions will be insensitive to the precise choice of $\mu$ provided that $\mu \sim$ several units times $\Lambda_{\text{QCD}}$.

In descending from $M_0$ to $\mu$, the form of the Lagrangian (1.1) changes, and a series of operators of higher dimension appears. For instance, the heavy quark part of the Lagrangian takes the form

$$\mathcal{L}_{\text{heavy}} = \sum_Q \left\{ \bar{Q}(i \slashed{D} - m_Q)Q + \frac{c_G}{2m_Q} \frac{i}{2} \sigma_{\mu\nu} G_{\mu\nu} Q + \frac{d^{(\Gamma)}_{Qq}}{m_Q^2 \Gamma_q} \right\} + \mathcal{O}\left( \frac{1}{m_Q^3} \right)$$

(2.1)

where $c_G$ and $d^{(\Gamma)}_{Qq}$ are coefficient functions, $G_{\mu\nu} \equiv g G^a_{\mu\nu} t^a$ and $t^a$ is the color generator (the coupling constant is included into $G$). We often use the short-hand notation $\sigma G = \sigma_{\mu\nu} G_{\mu\nu} = \gamma_\mu \gamma_\nu G_{\mu\nu}$. The sum over the light quark flavors $q$ is shown explicitly as well as the sum over possible structures $\Gamma$ of the four-fermion operators. All masses and couplings, as well as the coefficient functions $c_G$ and $d^{(\Gamma)}_{Qq}$, depend on the normalization point.

The operators of dimension five and higher in Eq. (2.1) are due to the contribution of hard gluons, with virtual momenta from $\mu$ up to $M_0$. Here the $1/m_Q$ expansion is explicit.Implicitly, a $1/m_Q$ expansion is generated also by the first (tree-level) term in the Lagrangian (2.1),

$$\mathcal{L}_{\text{heavy}}^0 = \bar{Q}(\slashed{P} - m_Q)Q.$$  

(2.2)

Although the field $Q$ in this Lagrangian is normalized at a low point $\mu$, it carries a hidden large parameter, $m_Q$. Indeed, the interaction of the heavy quark with the light degrees of freedom enters through $\mathcal{P}_\mu = i D_\mu$. The background gluon field $A_\mu$ is weak if measured at the scale $m_Q$, which means, of course, that there is a large “mechanical” part in the $x$ dependence of $Q(x)$, known from the very beginning,

$$Q(x) = e^{-im_Qt} \tilde{Q}(x);$$

(2.3)

$\tilde{Q}(x)$ is a rescaled bispinor field which, in the leading approximation, carries no information about the heavy quark mass. It describes a residual motion of the heavy quark inside the heavy hadron [7] with typical momenta of order $\Lambda_{\text{QCD}}$. Remnants of the heavy quark mass appear in $\tilde{Q}$ only at the level of $1/m_Q$ corrections.
Equation (2.3) is written in the rest frame of \( H_Q \). In an arbitrary frame one singles out the factor \( \exp(-im_Q v_\mu x_\mu) \)

\[
Q(x) = e^{-im_Q v_\mu x_\mu} \tilde{Q}(x), \quad v_\mu = p_\mu / M_{H_Q},
\]

where \( v_\mu \) is the four-velocity of the heavy hadron.

The covariant momentum operator \( \mathcal{P}_\mu \) acting on the original field \( Q \), when applied to the rescaled field \( \tilde{Q} \), is replaced by the operator \( m_Q v_\mu + \pi_\mu \),

\[
iD_\mu Q(x) = e^{-im_Q v_\mu x_\mu} (m_Q v_\mu + iD_\mu) \tilde{Q}(x) \equiv e^{-im_Q v_\mu x_\mu} (m_Q v_\mu + \pi_\mu) \tilde{Q}(x). \quad (2.5)
\]

If not stated otherwise, we use the rescaled field \( \tilde{Q} \), omitting the tilde in all expressions where there is no risk of confusion. The rescaled field \( \tilde{Q} \) is a four-component Dirac bispinor, not a two component non-relativistic spinor which is usually introduced in the heavy quark effective theory (HQET) \[7\]. The Dirac equation \((\mathcal{P} - m_Q)Q = 0\) can be rewritten as follows in terms of \( \tilde{Q} \):

\[
\frac{1}{2} - \gamma_0 Q = \frac{\pi}{2m_Q} Q, \quad (2.6)
\]

\[
\pi_0 Q = -\frac{\pi^2 + \frac{i}{2} \sigma G}{2m_Q} Q. \quad (2.7)
\]

Armed with this knowledge one can easily expand \( \mathcal{L}_{\text{heavy}}^0 \), at the tree level, through order \( 1/m_Q^2 \):

\[
\mathcal{L}_{\text{heavy}}^0 = \tilde{Q}(i \mathcal{P} - m_Q)Q = \tilde{Q} \left[ 1 + \frac{\gamma_0}{2} \left( 1 + \frac{(\bar{\sigma} \bar{\pi})^2}{8m_Q^2} \right) \left( \pi_0 - \frac{1}{2m_Q} (\bar{\pi} \bar{\sigma})^2 \right) - \frac{1}{8m_Q^2} \left( -\bar{D} \bar{E} + \bar{\sigma} \cdot \{ \bar{E} \times \bar{\pi} - \bar{\pi} \times \bar{E} \} \right) \right] \left( 1 + \frac{(\bar{\sigma} \bar{\pi})^2}{8m_Q^2} \right) \left( 1 + \gamma_0 \right) Q + O \left( \frac{1}{m_Q^3} \right),
\]

\[
(2.8)
\]

where \( \bar{\sigma} \) denote the Pauli matrices and \((\bar{\pi} \bar{\sigma})^2 = \bar{\pi}^2 + \bar{\sigma} \bar{B} \), \( \bar{E} \) and \( \bar{B} \) denote the background chromoelectric and chromomagnetic fields, respectively, with the coupling constant \( g \) and the color matrix \( t^a \) included in the definition of these fields. There is nothing new in this Lagrangian; at the tree level it is the same as in QED.

The non-relativistic expansions in QED have been known since the thirties, see e.g. Chapter 4 of Bjorken and Drell \[1\] or Sect. 33 of the Landau and Lifshitz \[2\]. It is worth noting that

\[
\mathcal{L}_{\text{heavy}}^0 \equiv \varphi^+ (\pi_0 - \mathcal{H}_Q) \varphi, \quad (2.9)
\]

where

\[
\varphi = \left( 1 + \frac{(\bar{\sigma} \bar{\pi})^2}{8m_Q^2} \right) \left( 1 + \gamma_0 \right) Q \quad (2.10)
\]
and $\mathcal{H}_Q$ is a non-relativistic Hamiltonian,

$$
\mathcal{H}_Q = \frac{1}{2m_Q} (\vec{\pi}^2 + \vec{\sigma}\vec{B}) + \frac{1}{8m_Q^2} \left( -(\vec{D}\vec{E}) + \{\vec{E} \times \vec{\pi} - \vec{\pi} \times \vec{E}\} \right) + \mathcal{O}(1/m_Q^3) \quad (2.11)
$$

well-known (in the Abelian case) from [1, 2]. The first term in the $1/m_Q^2$ part is called the Darwin term and the second one is the convection current (spin-orbital) interaction. Equation (2.10) is the Foldy-Wouthuysen transformation which is necessary to keep the term linear in $\pi_0$ in its canonic form. For unclear reasons the Foldy-Wouthuysen transformation in the context of the heavy quark theory is sometimes referred to as “casting the Lagrangian in the NRQCD form”. For a dedicated discussion of the Foldy-Wouthuysen transformation in the heavy quark theory see Ref. [13].

At the one-loop level the coefficients in the effective Hamiltonian (2.11) get some corrections which are specific for QCD and cannot be read off the text-book QED expansion. The work on calculating the one-loop coefficients for all terms through $\mathcal{O}(1/m_Q^3)$ was recently completed [14].

The careful reader might have noticed that we consistently avoid using the term HQET. This is done deliberately. Although, in principle, HQET is one of a few convenient technical tools for the heavy quark expansion (an alternative useful approach is provided by NRQCD [15]), in many standard presentations an additional assumption is made which makes the “folklore” HQET incompatible with the OPE-based expansions. What is missing in the commonly accepted version is appreciation of the fact that all coefficients and all operators in the Lagrangian, individually, are $\mu$ dependent – as is quite obvious from the consideration above. The Lagrangian can be applied to the perturbative calculation of the Feynman graphs with heavy quarks where the characteristic virtual momenta flowing through all lines in the graphs are less than $\mu$. The contribution of all virtual momenta above $\mu$ is explicitly included in the coefficients of the effective Lagrangian.

To avoid confusion we suggest, from now on, to use distinct notations for the fundamental parameters of the Wilsonian Hamiltonian, on the one hand, and (perturbatively defined) HQET parameters, on the other. The latter correspond to tending $\mu \to 0$, after “appropriate subtraction of perturbation theory”. It is clear that the subtraction cannot be carried out consistently in all orders, so that the parameters defined in this way are to be used with caution and reservations. Since they are widely exploited in the current literature it is convenient to introduce special notations. In particular, following Ref. [16], we will denote the HQET version of the kinetic expectation value by $-\lambda_1$. Formally, $-\lambda_1$ is given by the same Eq. (1.2) as $\mu_\pi^2$; the perturbative contribution below $\mu$ is subtracted, however, from $-\lambda_1$, which results in ambiguities. A pragmatically oriented reader, who is uninterested in the discussion of the subtraction ambiguities, to be presented below, may just assume that the subtraction is done, say, at one-loop order (or at two-loop order, if calculations are carried out at the level $\mathcal{O}(\alpha_s^2)$). Even so, one should realize, that all general results obtained from the QCD equations of motion, valid for $\mu_\pi^2$ and
other parameters in the Wilsonian approach, are, generally speaking, inapplicable to the parameters of HQET. Let us parenthetically note that another key parameter appearing in the heavy quark theory, the chromomagnetic operator,

$$\mu_G^2 = \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} i \frac{1}{2} \sigma G Q | H_Q \rangle = -\frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \bar{\sigma} \bar{B} Q | H_Q \rangle + \mathcal{O}(1/m_Q), \quad (2.12)$$

depends on $\mu$ logarithmically, via its hybrid anomalous dimension \[\text{[17]}\]; the limit $\mu \to 0$ is never attempted in this case, of course. In the nomenclature of Ref. \[\text{[16]}\] the parameter $\mu_G^2$ is $3\lambda_2$.

The mass formula relating $m_Q$ to the hadronic mass is a useful and, perhaps, the simplest application of the expansion outlined above. The $1/m_Q$ corrections to the hadron mass is nothing but the expectation value of the effective Hamiltonian (2.11)

$$M_{H_Q} = m_Q + \bar{\Lambda} + \frac{1}{2m_Q} \frac{\langle H_Q | \bar{\pi}^2 + \bar{\sigma} \bar{\sigma} B | H_Q \rangle}{2M_{H_Q}} + ... = m_Q + \bar{\Lambda} + \frac{(\mu_G^2 - \mu_0^2)_{H_Q}}{2m_Q} + ... \quad (2.13)$$

If we keep only the terms up to $1/m_Q$, it does not matter whether the state $H_Q$ we average over is the asymptotic state (corresponding to $m_Q = \infty$) or the actual physical heavy-flavor state. The difference becomes noticeable only at the level $1/m_Q^2$; it is expressed in terms of a few non-local correlation functions, see Ref. \[\text{[10]}\] for further details.

The parameter $\bar{\Lambda}$ appearing in Eq. (2.13) was introduced as a HQET constant in \[\text{[18]}\]; it is associated with those terms in the effective Lagrangian $\mathcal{L}(\mu)$ (disregarded so far) which are entirely due to the light degrees of freedom. Needless to say that in the Wilsonian approach $\bar{\Lambda}$ is actually $\mu$ dependent, $\bar{\Lambda}(\mu)$. Wherever there is the menace of confusion the $\mu$ dependence of $\bar{\Lambda}$ will be indicated explicitly.

There exist a few alternative expressions for this parameter. Let us quote the one relating $\bar{\Lambda}$ to the gluon anomaly in the trace of the energy-momentum tensor,

$$\bar{\Lambda} = \frac{1}{2M_{H_Q}} \langle H_Q | \beta(\alpha_s) \frac{G^2}{4\alpha_s} | H_Q \rangle \big|_{m_Q \to \infty}. \quad (2.14)$$

This expression was obtained in Ref. \[\text{[10]}\]. Some subtleties left aside here are discussed in detail in that paper. It is an obvious counterpart of a similar relation for the nucleon mass \[\text{[19]}\].

$$M_N = \frac{1}{2M_N} \langle N | \beta(\alpha_s) \frac{G^2}{4\alpha_s} | N \rangle.$$

The renormalization properties of the operator are quite different, however, in the sector of QCD with the heavy quark. For simplicity above the light quarks are taken as massless; introduction of the light quark masses changes only technical details. If the mass term of the light quarks is set equal to zero the light quark fields do not appear explicitly in the trace of the energy-momentum tensor.
3 Basic Parameters of the Heavy Quark Expansion

3.1 The Heavy Quark Mass

This section is a brief review of the present status of determination of the heavy quark masses. An internally consistent definition of the heavy quark mass is of utmost importance for \(1/m_Q\) expansions. Numerical reliability is essential if one wants to extract accurate values of the Cabibbo-Kobayashi-Maskawa (CKM) parameters \(|V_{cb}|\) and \(|V_{ub}|\). While these remarks are obvious (in hindsight), the theoretical implications were at first not fully appreciated.

For quarks – confined degrees of freedom – there exists no natural mass definition unlike with asymptotic states. Computational convenience and theoretical consistency are the only guiding principles. Convenience suggests employing the pole mass. However, as will be explained below, the pole mass does not allow the consistent inclusion of nonperturbative (power-suppressed) corrections.

3.1.1 What, after all, is the heavy quark mass?

In quantum field theory the object we begin our work with is the Lagrangian formulated at some high scale \(M_0\). The mass \(m_0\) is a parameter in this Lagrangian; it enters on the same footing as, say, the coupling constant \(\alpha_s\) with the only difference being that it carries dimension. As with any other coupling, \(m_0\) enters in all observable quantities in a certain universal combination with the ultraviolet cutoff \(M_0\) (in renormalizable theories). Although this combination is universal, its particular form is scheme and scale-dependent.

The mass parameter \(m_0\) by itself is not observable. For calculating observable quantities at the scale \(\mu \ll M_0\) it is usually convenient to relate \(m_0\) to some mass parameter relevant to the scale \(\mu\). For instance, in quantum electrodynamics at low energies (i.e. \(E \ll m_e\)) there is an obvious “best” candidate: the actual mass of an isolated electron, \(m_e\). In the perturbative calculations it is determined as the position of the pole in the electron Green function (more exactly, the beginning of the cut). The advantages are evident: \(m_e\) is gauge-invariant and experimentally measurable.

The analogous parameter for heavy quarks in QCD is referred to as the pole quark mass, the position of the pole of the quark Green function. Like \(m_e\) it is gauge invariant. The idea of using this parameter in the quark-gluon perturbation theory apparently dates back to Ref. [20]. Unlike QED, however, the quarks do not exist as isolated objects (there are no states with the quark quantum numbers in the observable spectrum, and the quark Green function beyond a given order has neither a pole nor a cut). Hence, \(m_{\text{pole}}\) cannot be directly measured; \(m_{\text{pole}}\) exists only as a theoretical construction.

In principle, there is nothing wrong with using \(m_{\text{pole}}\) in perturbation theory where...
it naturally appears in the Feynman graphs for the quark Green functions, scattering amplitudes and so on. It may or may not be convenient, depending on concrete goals.

The pole mass in QCD is perturbatively infrared stable, order by order, like in QED. It is well-defined to every given order in perturbation theory. One cannot define it to all orders, however. Nonperturbatively, the pole mass is not infrared-stable. Intuitively this is clear: since the quarks are confined in the full theory, the best one can do is to define the would-be pole position with an intrinsic uncertainty of order $\sim \Lambda_{\text{QCD}}$ [21].

The issue of the appropriate quark mass has two aspects. At the perturbative level it is desirable to define the running mass $m_Q(\mu)$ at a scale relevant to the scale in the processes at hand, $\mu$, which would incorporate contributions of all virtual momenta (of quarks and gluons) from $M_0$ down to $\mu$. This would allow eliminating unwanted large coefficients which always emerge under unsuitable choice of expansion parameters.

If analysis is aimed at higher (power in $m_Q^{-1}$) accuracy, the use of mass parameters that can be defined to that accuracy becomes mandatory. Whatever parameter we choose, it will not coincide with the mass of any observable particle, and is bound to remain a theoretical construction. If the construction is consistent, however, and allows us to achieve the desired degree of accuracy in the predictions for the observable quantities, that is all we need. The most straightforward choice in the theory with the confined quarks is as follows. We start from the original Lagrangian at $M_0$, and evolve it down to the scale $\mu$. The original mass parameter $m_0$ evolves accordingly. The mass parameter appearing in the effective Lagrangian $\mathcal{L}(\mu)$ is $m_Q(\mu)$.

To any order $k$ the relation between the pole mass and the running short-distance one $m_Q(\mu)$ takes the form

$$m_{\text{pole}}^{(k)} = m_Q(\mu) \sum_{n=0}^{k} C_n \left( \frac{\mu}{m} \right) \left( \frac{\alpha_s(\mu)}{\pi} \right)^n, \quad C_0 = 1.$$  \hspace{1cm} (3.1)

Three questions (more exactly, three facets of one and the same question) naturally arise in connection with Eq. (3.1):

i) Does the series (3.1) converge to a reasonable number?

ii) How well does it behave numerically in low orders, say, in the second or third order in $\alpha_s$ which we usually deal with in actual calculations?

iii) What happens when we start analyzing nonperturbative corrections suppressed by powers $1/m_Q$?

The answer to the first question is negative, which entails negative consequences in the second and the third question, as we will see shortly.

Let us begin with the divergence of the perturbative series in Eq. (3.1). Although more than one reason for the factorial divergence may exist, it is easy to

\footnote{In fact, $m_{\text{pole}}^{(k)}$ depends on $\mu$ since the $\alpha_s$ series is truncated at the order $k$. This dependence shows up, however, only at the level $\mathcal{O}(\alpha_s^{k+1})$.}
Figure 1: Perturbative diagrams leading to the IR renormalon uncertainty in $m_Q^{\text{pole}}$ of order $\Lambda_{\text{QCD}}$. The number of bubble insertions in the gluon propagator can be arbitrary.

Identify a particular source, the so-called infrared renormalon singularity in the perturbative expansion of the pole mass \cite{22, 23}, see diagrams in Fig. 1. The bubble chain generates the running of the strong coupling $\alpha_s$. To leading order, it can be accounted for by inserting the running coupling constant $\alpha_s(k^2)$ in the integrand corresponding to the one-loop expression. In the non-relativistic regime, when the internal momentum $|k| \ll m_Q$, the expression is simple,

$$\delta m_Q \sim -\frac{4}{3} \int \frac{d^4k}{(2\pi)^4 i k_0} \frac{4\pi \alpha_s(-k^2)}{k^2} = \frac{4}{3} \int \frac{d^3q \alpha_s(q^2)}{4\pi^2 q^2}.$$  \hspace{1cm} (3.2)

Now, expressing the running $\alpha_s(k^2)$ in terms of $\alpha_s(\mu^2)$, $k^2 < \mu^2$,

$$\alpha_s(k^2) = \alpha_s(\mu^2) \left\{ 1 + \frac{\alpha_s(\mu^2)}{4\pi} b \ln \frac{k^2}{\mu^2} \right\}^{-1}, \quad b = \frac{11}{3} N_c - \frac{2}{3} n_f,$$  \hspace{1cm} (3.3)

and expanding $\alpha_s(k^2)$ in a power series in $\alpha_s(\mu^2)$, it is easy to find the $(n+1)$-th order contribution to $\delta m_Q$,

$$\frac{\delta m_Q^{(n+1)}}{m_Q} \sim \frac{4}{3} \frac{\alpha_s(\mu^2)}{\pi} n! \left( \frac{b \alpha_s(\mu^2)}{2\pi} \right)^n.$$  \hspace{1cm} (3.4)

Observe that the coefficients grow factorially and contribute with the same sign. Therefore, one cannot define the sum of these contributions even using the Borel transformation. The best one can do is to truncate the series judiciously. An optimal truncation leaves us with an irreducible uncertainty $\sim \mathcal{O}(\Lambda_{\text{QCD}})$ \cite{22, 23}.

Thus, the perturbative expansion per se anticipates the onset of the nonperturbative regime (the impossibility of locating the would-be quark pole to accuracy better than $\Lambda_{\text{QCD}}$). Certainly, the concrete numerical value of the uncertainty in $m_Q^{\text{pole}}$ obtained through renormalons is not trustworthy. The renormalons do not represent the dominant component of the infrared dynamics. However, they are a clear indicator of the presence of the power-suppressed nonperturbative effects, or infrared instability of $m_Q^{\text{pole}}$; the very fact that there is a correction $\mathcal{O}(\Lambda_{\text{QCD}}/m_Q)$ is beyond any doubt.

One can illustrate the emergence of this correction in the following transparent way. Consider the energy stored in the chromoelectric field in a sphere of radius $11$.
\[ R \gg 1/m_Q \] around a static color source of mass \( m_Q \),

\[ \delta \mathcal{E}_{\text{Coul}}(R) \propto \int_{1/m_Q \leq |x| < R} d^3x \vec{E}^2_{\text{Coul}} \propto \text{const} - \frac{\alpha_s(R)}{\pi} \frac{1}{R}. \] (3.5)

The definition of the pole mass amounts to setting \( R \to \infty \); i.e., in evaluating the pole mass one undertakes to integrate the energy density associated with the color source over all space assuming that it has the Coulomb form. In real life the color interaction becomes strong at \( R_0 \sim 1/\Lambda_{\text{QCD}} \); at such distances the chromoelectric field has nothing to do with the Coulomb tail. Thus, one cannot include the region beyond \( R_0 \) in a meaningful way. Its contribution which is of order \( \Lambda_{\text{QCD}} \), thus, has to be considered as an irreducible uncertainty which is power-suppressed relative to \( m_Q \),

\[ \frac{\delta_{\text{IR}} m_Q^\text{pole}}{m_Q} = \mathcal{O}\left( \frac{\Lambda_{\text{QCD}}}{m_Q} \right). \] (3.6)

It is worth noting that the pole mass was the first example where a quantity which is perturbatively infrared-stable was shown not to be stable nonperturbatively at the level \( \Lambda_{\text{QCD}} \). The observation of Refs. \[22, 23\] gave impetus to dedicated analyses of other perturbatively infrared-stable observables in numerous hard processes without OPE, in particular, in jet physics. Such nonperturbative infrared contributions, linear in \( \Lambda_{\text{QCD}}/Q \) were indeed found shortly after in thrust and many other jet characteristics (for a review and a representative list of references see e.g. \[24\]).

To demonstrate that the problem of divergence of the \( \alpha_s \) series for \( m_Q^\text{pole} \) is far from being academic, let us examine how the perturbative contribution to the \( b \) quark mass looks numerically:

\[ m_b^\text{pole} = m_b(1 \text{ GeV}) + \delta m_{\text{pert}}(\leq 1 \text{ GeV}) \simeq \]

\[ 4.55 \text{ GeV} + 0.25 \text{ GeV} + 0.22 \text{ GeV} + 0.38 \text{ GeV} + 1 \text{ GeV} + 3.3 \text{ GeV} + ..., \] (3.7)

where \( m_b(1 \text{ GeV}) \) is the running mass at \( \mu = 1 \text{ GeV} \), and \( \delta m_{\text{pert}}(\leq 1 \text{ GeV}) \) is the perturbative series taking account of the loop momenta from 1 GeV down to zero. It is quite obvious that the corrections start to blow up already in rather low orders! Expressing observable infrared-stable quantities (e.g. the inclusive semileptonic width \( B \to X_u \ell \nu \)) in terms of the pole mass will necessarily entail large coefficients in the \( \alpha_s \) corrections compensating for the explosion of the coefficients in \( m_b^\text{pole} \). We will return to this point later (Sect. 8.1.3).

Summarizing, the pole mass \textit{per se} does not appear in OPE for infrared-stable quantities. Such expansions operate with the short-distance (running) mass. Any attempt to express the OPE-based results in terms of the pole mass creates a problem making the Wilson coefficients ill-defined theoretically and poorly convergent numerically.

Concluding this section, we make a side remark concerning the \( t \)-quark mass. The peculiarity of the \( t \) quark is that it has a significant width \( \Gamma_t \sim 1 \text{ GeV} \) due to
its weak decay. The perturbative position of the pole in the propagator is, thus, shifted into the complex plane by $-\frac{i}{2}\Gamma_t$. The finite decay width of the $t$ quark introduces a physical infrared cutoff for the infrared QCD effects. In particular, the observable decay characteristics do not have ambiguity associated with the uncertainty in the pole mass discussed above. The uncertainty cancels in any physical quantity that can be measured. That is not the case, however, in the position of the pole of the $t$-quark propagator in the complex plane (more exactly, its real part). The quark Green function is not observable there, and one would encounter the very same infrared problem and the same infrared renormalon. The latter does not depend on the absolute value of the quark mass (and whether it is real or have an imaginary part). Thus, in the case of top, one would observe an intrinsic uncertainty due to the renormalon, of several hundred MeV, in relating the peak in the physical decay distributions to the position of the propagator singularity in the complex plane.

While this fact will pose no practical problem any time soon, it will do so in the future, in particular when analyzing top production at linear $e^+e^-$ colliders. The observables can be conveniently expressed in terms of a mass $m_t$ defined at the scale $\mu \approx \Gamma_t$ that is free from the renormalon ambiguities. It can be defined and measured without intrinsic limitations. It is worth trying to come to an agreement in advance what kind of top quark mass should be listed in the PDG tables, and thus avoid the problems which accompanied attempts to pinpoint the masses of $c$ and $b$ quarks.

### 3.1.2 Short distance masses

Since the pole mass is theoretically ill-defined, it is preferable to use a short-distance mass. The most popular choice is the so-called $\overline{\text{MS}}$ mass $\overline{m}_Q(\mu)$. The $\overline{\text{MS}}$ mass is not a parameter in the effective Lagrangian; rather it is a certain ad hoc combination of the parameters which is particularly convenient in the perturbative calculations using dimensional regularization. Its relation to the perturbative pole mass is known to two loops:

$$m_Q^{\text{pole}} = \overline{m}_Q(\mu) \left\{ 1 + \frac{4}{3} \frac{\alpha_s(\overline{m}_Q)}{\pi} + (1.56 b - 3.73) \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots \right\} \quad (3.8)$$

At $\mu \gtrsim m_Q$ the $\overline{\text{MS}}$ mass coincides, roughly speaking, with the running Lagrangian mass seen at the scale $\sim \mu$. However, it becomes rather meaningless at $\mu \ll m_Q$. To elucidate the point let us note that, by definition, it adds to the bare mass $m_0$ the same logarithmic contribution $\sim \alpha_s m_Q \frac{dk^2}{k^2}$ both at $k^2 > m_Q^2$, where it is indeed present, and below $m_Q$ where such a contribution is nonsensical. $\overline{m}_Q(\mu)$ logarithmically diverges when $\mu \to 0$. The actual contribution coming from domain below $m_Q$ is essentially smaller, $\sim \alpha_s \frac{dk^2}{k}$. For this reason $\overline{m}_Q(\mu)$ is not appropriate in the heavy quark theory where the possibility of evolving down to a low normalization point, $\mu \ll m_Q$, is crucial.
The genuine short-distance low-scale mass \( m_Q(\mu) \) at \( \mu \ll m_Q \) can be introduced in different ways, and it always exhibits an explicit linear \( \mu \) dependence,

\[
\frac{d m_Q(\mu)}{d \mu} = -c_m \frac{\alpha_s(\mu)}{\pi} - \ldots \tag{3.9}
\]

The coefficient \( c_m \) is definition-dependent. For the most natural choices \([27, 24, 28]\), \( c_m \) varies between \( 4/3 \) and \( 16/9 \). The difference in the definition of the running mass in the heavy quark theory can, thus, constitute \( \sim 50 \) to \( 100 \) MeV at \( \mu \sim 1 \text{ GeV} \). Once a particular definition is chosen, no ambiguity remains. A convenient physical (gauge-invariant) definition of the short-distance heavy quark mass with \( \mu \lesssim m_Q \) was outlined in \([10]\) (Sects. III.D and V.A). Technical aspects are discussed in detail in the dedicated paper \([23]\), where \( c_m = 16/9 \).

### 3.1.3 Which mass is to be used?

It is legitimate to use any short-distance mass. The normalization point \( \mu \) can be arbitrary as long as \( \mu \gg \Lambda_{\text{QCD}} \). It does not mean, however, that all masses are equally practical, since the perturbative series are necessarily truncated after a few first terms. Using an inappropriate scale makes numerical approximations bad. In particular, relying on \( \bar{m}_Q(m_Q) \) in treating the low-scale observables can be awkward. The following toy example illustrates this point.

Suppose, we would like to exploit QCD-like methods in solving the textbook problem of the positronium mass, calculating the standard Coulomb binding energy. To use the result written in terms of the \( \overline{\text{MS}} \) mass one would, first, need to evaluate \( \bar{m}_e(m_e) \). This immediately leads to technical problems: \( \bar{m}_e(m_e) \) is known only to the order \( \alpha^2 m_e \); therefore, the “theoretical uncertainty” in \( \bar{m}_e(m_e) \) would generate the error bars \( \sim \alpha^3 m_e \) in the binding energy, i.e. at least 0.01 eV. Moreover, without cumbersome two-loop calculations one would not know the binding energy even at a few eV level — although, obviously, it is known to a much better accuracy (up to \( \alpha^4 m_e \) without a dedicated field-theory analysis), and the result

\[
M_P = 2m_e(0) - \frac{\alpha^2 m_e}{4} \tag{3.10}
\]

is obtained without actual loop calculations!

Thus, working with the \( \overline{\text{MS}} \) mass we would have to deal here with the “hand-made” disaster. The reason is obvious. The relevant momentum scale in this problem is the inverse Bohr radius, \( \mu_B = r_B^{-1} \sim \alpha m_e \). Whatever contributions emerge at much shorter distances, say, at \( \mu^{-1} \sim m_e^{-1} \), their sole role is to renormalize the low-scale parameters \( \alpha \) and \( m_e \). If the binding energy is expressed in terms of these parameters, further corrections are suppressed by powers of \( \mu_B/\mu \). Exploiting the proper low-energy parameters \( \alpha(0), m_e(0) \) one automatically resums chains of corrections. In essence, this is the basic idea of the effective Lagrangian approach, which, unfortunately, is often forgotten.
 Needless to say, that in the processes at high energies it is the high-scale masses that appear directly. In particular, the inclusive width $Z \rightarrow b \bar{b}$ is sensitive to $m_b(M_Z)$; using the MS mass normalized at $\mu \sim M_Z$ is appropriate here. On the contrary, the inclusive semileptonic decays $b \rightarrow c \ell \nu$ are rather low-energy in this respect \[29\], and, to some extent, that is true even for $b \rightarrow u$.

3.1.4 The numerical values of $m_c$ and $m_b$

Accurate phenomenological determination of $m_b$ and $m_c$ at large $\mu$ (higher or comparable to the quark masses themselves) is not easy since it requires very precise data and a high degree of control over the perturbative and nonperturbative corrections. The terms $\sim (\alpha_s/\pi)^2 m_b$ by themselves constitute typically $\sim 200$ MeV, which, thus, sets the scale of accuracy one can expect.

The mass of the $c$ quark at the scale $\sim m_c \sim 1$ GeV can be obtained from the charmonium sum rules \[30\], $m_c(m_c) \sim 1.25$ GeV. The result to some extent is affected by the value of the gluon condensate. To be safe, we conservatively ascribe a rather large uncertainty,

$$m_c(m_c) = 1.25 \pm 0.10 \text{ GeV}.$$

One can argue that the precision charmonium sum rules \[30\] actually fine-tune the charmed quark mass to a much better accuracy. The argument would lead us far astray, and is irrelevant for our present purposes since the convergence of the $1/m_c$ expansion is not good anyway.

The desire to get rid of potentially large perturbative corrections in the $b$ quark mass, scaling as $m_b$, suggests examining the threshold region. The process $e^+e^- \rightarrow b\bar{b}$ in the threshold domain provides us with the opportunity of determining the low-scale running mass, along the lines suggested in Ref. \[30\]. Using dispersion relations

$$\Pi_b(q^2) = \Pi_b(0) + \frac{q^2}{2\pi^2} \int \frac{ds R_b(s)}{s(s-q^2)}$$

one evaluates the polarization operator $\Pi_b(q^2)$ (and its derivatives) induced by the vector currents $\bar{b}\gamma_\mu b$, in the complex $q^2$ plane at an adjustable distance $\Delta$ from the threshold. Such quantities are proportional to weighted integrals over the experimental cross section; the integrals are saturated in the interval $\sim \Delta$ near threshold, and are very sensitive to the mass $m_b(\Delta)$.

A dedicated analysis was carried out by Voloshin \[31\] who considered a set of relatively high derivatives of $\Pi_b$ at $q^2 = 0$. On the phenomenological side they are expressed through moments of $R_b(s)$,

$$\frac{2\pi^2}{n!} \Pi_b^{(n)}(0) = I_n = \int \frac{ds R_b(s)}{s^{n+1}} \simeq$$

$$M_{\bar{T}(1S)}^{-2(n+1)} \int ds R_b(s) \exp \left\{ -(n+1) \left( s - 4M_{\bar{T}(1S)}^2 \right) \right\}, \quad (3.12)$$
while the theoretical expressions for the very same moments are given in terms of the running quark mass and $\alpha_s$. The relevant momentum scale here is $\mu \sim m_b^2/\sqrt{n}$. Considering small-$n$ moments $I_n$, one would determine $m_b$ at the scale of the order $m_b$. The small-$n$ moments are contaminated by the contribution of $R_b$ above the open beauty threshold where experimental data are poor. Increasing $n$ shrinks the interval of saturation and, thus, lowers the effective scale. On the other hand, we can not go to too high values of $n$ where infrared effects (given, first, by the gluon condensate) explode. There is still enough room to keep the gluon condensate small and, simultaneously, suppress the domain above the open beauty threshold. In the fiducial window, $\mu$ must be large enough to ensure control over the QCD corrections. The latter requires a nontrivial summation of enhanced Coulomb terms unavoidable in the non-relativistic situation. As known from textbooks, the part of the perturbative corrections to the polarization operator, associated with the potential interaction, is governed by the parameter $\alpha/|\vec{v}|$ rather than by $\alpha$ per se. In [31] a typical scale is $\mu \sim 0.7$ to 1.5 GeV. A sophisticated resummation technique was used to access the moments as high as 16 to 20.

The crucial question is the estimate of uncertainties in determination of $m_b$. The uncertainty of the fit for $m_b$ is a meagre few MeV. This is not surprising since the theoretical expression for the moments very sharply depends on the value of $m_b$,

$$I_n \propto e^{-2n\delta m_b/m_b}.$$ (3.13)

It is clear that the uncertainty of the fit per se is overshadowed by a systematic theoretical uncertainty. The calculation of the Coulomb effects in Ref. [31] was not genuinely two-loop: the effect of running of $\alpha_s$ was accounted for only through the Brodsky-Lepage-Mackenzie (BLM) scale fixing [32]. Considering the fact that the dominant, BLM $\alpha_s^2$ effects were accounted for, it seems safe to assume that the actual accuracy of the determination of $m_b(\mu)$ with $\mu \sim 1$ GeV is not worse than $(\alpha_s(\mu)/\pi)^2 \mu \sim 30$ MeV [31]. To be on the safe side, we increase it up to 50 MeV. This number includes, additionally, possible scattering in the definitions of $m_b(\mu)$ at the level $\sim (\alpha_s/\pi)\mu^2/(2m_b)$ which can constitute 10 to 20 MeV. It is worth mentioning that the most labor-consuming analysis of the next-to-leading $\alpha_s^{n+1}/|\vec{v}|^n$ effects in [31], although absolutely necessary for determination of $\alpha_s$, led to a modest change in the numerical value of $m_b$, $\sim 15$ MeV, as compared to the ten-year old analysis of Ref. [33].

This method could be refined in several respects to yield the overall precision in the ballpark of 10 MeV. At this level the accurate definition of the renormalization procedure is mandatory. If the experimental cross section $e^+e^- \rightarrow b\bar{b}$ above the threshold were better measured in a reasonably wide interval, one could study lower

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\[3\] After this review was essentially finished it was reported [34] that the analysis of the $b\bar{b}$ sum rules had been repeated, with some rather surprising conclusions. Resummation of the full Coulomb ladder is not addressed, however, in Ref. [34]. We choose to rely on the previous results until the matter is more closely looked at.
moments $I_n$ and get an accurate determination of $m_b$ without tedious analysis of the perturbative Coulomb corrections.

Numerically, the $b$ quark mass turns out to be

$$m_b(1\text{ GeV}) = 4.64\text{ GeV} \pm 0.05\text{ GeV}.$$  \hfill (3.14)

The definition corresponds to the scheme in which $c_m = 16/9$. In other words, the “one-loop” pole mass is

$$m_b^{(1)} = 4.64\text{ GeV} + \frac{16}{9} \frac{\alpha_s}{\pi} \cdot 1\text{ GeV} \simeq 4.83\text{ GeV}$$ \hfill (3.15)

at $\alpha_s = 0.336$. The result is, of course, sensitive to the choice of $\mu$. The sensitivity to the uncertainty in $\alpha_s$ is around 10 MeV. The corresponding value of $\Lambda(1\text{ GeV}) \approx 0.6\text{ GeV}$.

The heavy quark masses can be measured, in principle, by studying the distributions in the semileptonic $B$ decays \cite{37,38}. Such analyses were undertaken recently \cite{37,38}. Unfortunately, the data are not good enough yet to yield a competitive determination. On the theoretical side, there are potential problems with higher-order corrections due to not too large energy release $m_b - m_c \simeq 3.5\text{ GeV}$ and/or relatively small mass of the $c$ quark. In particular, the effect of higher-order power corrections can be noticeable. The result of analysis reported in Refs. \cite{37,38} is compatible with the value (3.14), within the quoted uncertainties. This approach seems to be most natural.

In many applications one needs to know the difference between $m_b$ and $m_c$. If both masses are normalized at the same low-scale point below both masses, $m_b(\mu) - m_c(\mu)$ depends on $\mu$ weakly. This difference is well constrained in the heavy quark expansion. For example,

$$m_b - m_c = \overline{M}_B - \overline{M}_D + \mu^2 \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) + \rho^3_D - \bar{\rho}^3 \left( \frac{1}{m_c^2} - \frac{1}{m_b^2} \right) + \mathcal{O} \left( \frac{1}{m^3} \right),$$ \hfill (3.16)

where $\mu^2$ is the asymptotic expectation value of the kinetic operator, $\rho^3_D$ is the expectation value of the Darwin term given by the local four-fermion operator and $\bar{\rho}^3_D$, $\rho^3_\pi$, $\rho^3_S$ are positive non-local correlators \cite{10}. All quantities in Eq. (3.16), except the meson masses, depend on the normalization point $\mu$ which can be arbitrary. In this way we arrive at

$$m_b - m_c \simeq 3.50\text{ GeV} + 40\text{ MeV} \frac{\mu^2 - 0.5\text{ GeV}^2}{0.1\text{ GeV}^2} + \Delta M_2, \quad |\Delta M_2| \lesssim 0.015\text{ GeV}.$$ \hfill (3.17)

The estimate at $\mu^2 = 0.5\text{ GeV}^2$ appears to be in good agreement with the separate determinations of $m_b$ and $m_c$ from the sum rules in charmonia and $\Upsilon$.  

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The main uncertainty in $m_b - m_c$ is due to that in the value of $\mu_\pi^2$. The Darwin term can be reasonably estimated relying on factorization [39]: it is of order 0.1 GeV$^3$. The non-local positive matrix elements $\rho_\pi^3$ and $\rho_S^3$ are expected, generally speaking, to be of the same order. Altogether, assuming $|\rho_D^3 - \bar{\rho}_D^3| \lesssim 0.1$ GeV$^3$, one arrives at the uncertainty in $m_b - m_c$ due to the higher-order terms $\Delta M_2$ quoted above.

3.2 The chromomagnetic and kinetic operators

The non-relativistic Hamiltonian of the heavy quark (2.11) in the order $1/m_Q$ contains two operators (1.2) and (2.12). Their expectation values in the heavy meson $B$ are the key players in many applications. Let us note that the expectation values of the chromomagnetic operator in $B$ and $B^*$ are related,

$$\mu_G^2 = \frac{1}{2M_B} \langle B|\bar{b}i\sigma_{\mu\nu}G_{\mu\nu}b|B\rangle \simeq -\frac{3}{2M_B} \langle B^*|\bar{b}i\sigma_{\mu\nu}G_{\mu\nu}b|B^*\rangle. \quad (3.18)$$

The value of $\mu_G^2$ is known: since $1/2 m_Q \bar{Q}i\sigma_{\mu\nu}G_{\mu\nu}Q$ describes the interaction of the heavy quark spin with the light cloud and causes the hyperfine splitting between $B$ and $B^*$, it is easy to see that

$$\mu_G^2 \simeq \frac{3}{4} 2m_b(M_{B^*} - M_B) \simeq \frac{3}{4} (M_{B^*}^2 - M_B^2) \approx 0.36\text{ GeV}^2. \quad (3.19)$$

In the above relations the operators are normalized at the scale $\sim m_b$. Evolving them perturbatively to the normalization scale $\mu \sim 1$ GeV slightly enhances the chromomagnetic operator and the value of $\mu_G^2(\mu)$, but this effect is numerically insignificant, and we will neglect it.

In Sect. 3.1.4 we mentioned that the heavy quark mass can be extracted from the data on semileptonic $B$ decays. By the same token, the value of the kinetic operator is, in principle, directly measurable in experiment. However, it remains rather uncertain at the moment. First attempts at extracting it from the semileptonic distributions were reported [37, 38]. Although the method seems to be the most promising, so far the outcome is inconclusive. Typically, the results fall in the interval 0.2 to 0.3 GeV$^2$. We hasten to add, however, that the difference between $\mu_\pi^2$ and $-\lambda_1$ was not fully untangled. The normalization point $\mu$ should be explicitly introduced in the perturbative corrections to reveal the distinction between these two parameters. Until this is done it is unclear to which particular definition of $-\lambda_1$ the number quoted refers.\footnote{This remark applies also to other numerical determinations discussed below in this section, with the exception of Eq. (3.21).} As we will see shortly, numerically the difference between $\mu_\pi^2$ and $-\lambda_1$ is of order 0.1 GeV$^2$ ($-\lambda_1$ is smaller than $\mu_\pi^2$). Note also that
the numerical analysis of the higher-order power corrections [10] led the authors to
the conclusion that these effects are significant in the application to the semileptonic
spectra.

Historically, the first attempt to determine the average of the kinetic operator
from the QCD sum rules was Ref. [11] where a negative value $\sim -1$ GeV$^2$ was
obtained! (This result was later revised by the author.) Shortly after, a more
thoughtful application of the QCD sum rules yielded [12] $\mu^2_\pi \approx 0.6$ GeV$^2$. The
prediction was later refined by the authors [13]

$$\mu^2_\pi = (0.5 \pm 0.15) \text{ GeV}^2. \quad (3.20)$$

Meanwhile a model-independent lower bound was established [39, 45, 44, 10]

$$\mu^2_\pi > \mu^2_G \approx 0.4 \text{ GeV}^2 \quad (3.21)$$

which constrained possible values of $\mu^2_\pi$. It is worth emphasizing that the inequality
takes place for $\mu^2_\pi$ normalized at any point $\mu$, provided $\mu^2_G$ is normalized at the same
point. For large $\mu$ it becomes uninformative; so, it is in our best interests to use it
at $\mu = $ several units $\times \Lambda_{QCD} \lesssim 1$ GeV. The inequality does not necessarily hold for
$-\lambda_1$. Exact QCD inequalities of this type will be discussed in the next section.

Recently, an approach combining the QCD sum rules and the virial theorem was
exploited [46] for determination of $\mu^2_\pi$. The corresponding analysis is claimed to
produce a surprisingly precise value

$$0.1 \pm 0.05 \text{ GeV}^2, \quad (3.22)$$

noticeably below the range obtained in other analysis. We will dwell on various
aspects of this approach, in an attempt to reveal reasons explaining the discrepancy,
in the Sect. 6.

The expectation value of the kinetic operator was also estimated in the quark
models, with a spectrum of predictions: the relativistic quark model [47] gave
about 0.6 GeV$^2$ or even slightly larger; the estimates of Ref. [48] yield a close value
0.5 GeV$^2$. Attempts of extracting $\mu^2_\pi$ on the lattices are also reported [49].

The scattering of the numerical results above should not surprise the reader too
much. As was mentioned above, although the formal definition of $\mu^2_\pi$ and $-\lambda_1$ is
the same, see Eq. (1.2), these quantities are not identical, and one should expect
noticeable differences. What is surprising is the discrepancy obtained under one
and the same definition and within basically the same method. The most notable
example of this type is the sum rule rule analysis, Ball et al. [12] versus Neubert [10].
We will dwell on this point in Sect. 6.2. Now we would like to make several remarks
regarding the difference between $\mu^2_\pi$ and $-\lambda_1$. Conceptually the problem with $\lambda_1$
is similar to the problem encountered in the attempt to define the pole mass (more
exactly, $\Lambda_{HQET} = M_B - (m_b)_{pole}$).

If $\mu^2_\pi$ is defined in the context of the Wilsonian OPE, and incorporates all soft
fluctuations, including perturbative fluctuations from 0 to $\mu$, HQET assumes the
existence of a “genuinely nonperturbative” parameter

\[-\lambda_1 = \mu^2_\pi(\mu) - (\mu^2_\pi(\mu))_{\text{pert}}\]  \hspace{1cm} (3.23)

where \((\mu^2_\pi(\mu))_{\text{pert}}\) is a “perturbative part” of \(\mu^2_\pi\) coming from the domain below \(\mu\). As was emphasized long ago [50], the procedure of separating out “perturbative parts” of the condensates cannot be carried out in general, simply because the notion of perturbation theory below \(\mu\) is non-existent. The best one can do is to construct \(-\lambda_1\), starting from \(\mu^2_\pi(\mu)\), in the given (say, the first or the second) order in \(\alpha_s(\mu)\).

The Wilsonian kinetic average \(\mu^2_\pi\) is \(\mu\)-dependent. For example, to the first order in \(\alpha_s\) one has

\[\frac{d\mu^2_\pi(\mu)}{d\mu^2} = c_\pi \frac{\alpha_s}{\pi}\]  \hspace{1cm} (3.24)

where under some natural choice of the cut-off (i.e. the normalization point) \(c_\pi = 4/3\) [14].

Equation (3.24) implies that to the first order in \(\alpha_s\)

\[-\lambda^{(1)}_1 = \mu^2_\pi(\mu) - c_\pi \frac{\alpha_s}{\pi} \mu^2\]  \hspace{1cm} (3.25)

where \(\alpha_s\) is the coupling constant (which does not run in one-loop calculations) and the superscript \((1)\) indicates the one-loop nature of \(\lambda_1\) introduced in this way.

The first observation is obvious: \(-\lambda^{(1)}_1\) is smaller than \(\mu^2_\pi(\mu)\). How much smaller? The answer depends on the choice of \(\mu\); it is also affected by the value of \(\alpha_s\) in a particular one-loop calculation. If \(\mu = 0.5\) GeV the values of the kinetic operator quoted in [12] [10] must be increased by approximately 0.05 GeV\(^2\) to translate them from \(-\lambda_1\) to \(\mu^2_\pi\); at \(\mu = 0.7\) GeV the shift constitutes \(\sim 0.1\) GeV\(^2\). A similar (probably, somewhat larger) adjustment applies to empiric determinations from the semileptonic spectra [37, 38]. These shifts lie within the error bars of the corresponding analyses.

Constructing a general “ready-to-use” cut-off procedure for the Wilsonian operators normalized at \(\mu\) is a non-trivial problem going beyond the scope of this review. A brief discussion in heavy quark theory can be found in [51, 52] and on lattices in Ref. [53]; the issue will be further elaborated in a dedicated publication [54]. Here we mention a few basic points relevant to the issue of \(\mu^2_\pi\).

A consistent field-theoretic definition of the operator \(\bar{Q}(i\vec{D})^2Q\) was given using the small velocity (SV) sum rules [10, 11]. Limiting oneself to the purely perturbative level, one can readily outline other cut-off schemes allowing to calculate the perturbative \(\mu\) dependence of \(\mu^2_\pi(\mu)\). In doing so one must keep in mind that some purely perturbative cut-off schemes that have no transparent physical meaning violate the general quantum-mechanical properties of the operator in question. Therefore, one must exercise extreme caution in connecting various results regarding the condensates obtained in different approaches with each other. Moreover, perturbative cutoff procedures do not define the operators at the nonperturbative level.
Keeping in mind all these problems, one may ask whether the controversy in the current literature is to be taken seriously at all. The numerical comparison given above seems to show that the existing scattering in the value of the kinetic operator goes beyond the mismatches of different definitions. A better understanding of theoretical uncertainties in the sum rule and empiric determinations is badly needed. This task obviously must be the subject of a special investigation. In the absence of such an investigation we will focus in this review on a class of results which seems to be theoretically clean: exact QCD inequalities.

4 Exact Inequalities of the Heavy Quark Theory in the Limit $m_Q \to \infty$

Exact inequalities reflecting the most general features of QCD (e.g. the vector-like nature of the quark-gluon interaction) have been with us since the early eighties [55]. The advent of the heavy quark theory paved the way to a totally new class of inequalities among the fundamental parameters. As with the old ones, they are based on the equations of motion of QCD and certain positivity properties. All technical details of the derivation are different, however, as well as the sphere of applications.

The most well-known example is Eq. (3.21) which was subject to intense scrutiny. As a matter of fact, this expression is just one representative of a large class, including, among others, the Bjorken and Voloshin relations, and the so-called third (BGSUV) sum rule. Since the corresponding derivations are scattered in the literature, it makes sense to give here a coherent presentation of the subject.

The starting point for all inequalities is the set of the sum rules

\begin{align}
\rho^2(\mu) - \frac{1}{4} &= \sum_n |\tau_{1/2}^{(n)}|^2 + 2 \sum_m |\tau_{3/2}^{(m)}|^2, \\
\Lambda(\mu) &= 2 \left( \sum_n \epsilon_n |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m |\tau_{3/2}^{(m)}|^2 \right), \\
\frac{\mu^2_n(\mu)}{3} &= \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2, \\
\frac{\mu^2_\rho(\mu)}{3} &= -2 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2, \\
\frac{\rho^3_D(\mu)}{3} &= \sum_n \epsilon_n^3 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^3 |\tau_{3/2}^{(m)}|^2, \\
-\frac{\rho^3_L(\mu)}{3} &= -2 \sum_n \epsilon_n^3 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^3 |\tau_{3/2}^{(m)}|^2.
\end{align}
a sequence which can be readily continued further. Here $\epsilon_k$ is the excitation energy of the $k$-th intermediate state ("$P$-wave states" in the quark-model language),

$$\epsilon_k = M_{H_Q^{(k)}} - M_{P_Q} ,$$

while the functions $\tau_{1/2}^{(n)}$ and $\tau_{3/2}^{(m)}$ describe the transition amplitudes of the ground state $B$ meson to these intermediate states. We follow the notations of Ref. [56].

$$\frac{1}{2M_{H_Q}} \langle H_Q^{(3/2)} | A_\mu | P_Q \rangle = -\tau_{1/2} (v_1 - v_2)_\mu ,$$

and

$$\frac{1}{2M_{H_Q}} \langle H_Q^{(3/2)} | A_\mu | P_Q \rangle = -\frac{1}{\sqrt{2}} i \tau_{3/2} \epsilon_{\mu\alpha\beta\gamma} \epsilon^{*\alpha} (v_1 + v_2)^\beta (v_1 - v_2)^\gamma ,$$

where $1/2$ and $3/2$ mark the quantum numbers of the light cloud in the intermediate states, $j^\pi = 1/2^+$ and $3/2^+$, respectively, and $A_\mu$ is the axial current. Furthermore, the slope parameter $\rho_2^2$ of the Isgur-Wise function is defined as

$$\frac{1}{2M_{P_Q}} \langle P_Q (\vec{v}) | (Q\gamma_0 Q)_\mu | P_Q \rangle = 1 - \bar{\epsilon}^2 (\mu) \frac{\vec{v}^2}{2} + O(\vec{v}^4) .$$

All quantities on the left-hand side of Eqs. (4.1) – (4.6) depend on the normalization point $\mu$, since they are defined in the Wilsonian sense. The right-hand side also depends on this point. Thus, all sums over the excited states appearing in Eqs. (4.1) – (4.6) imply a cut-off at the excitation energy equal to $\mu$,

$$\epsilon_k \leq \mu .$$

Equation (4.1) is nothing but the Bjorken sum rule [17]. It was discussed in great detail in Ref. [57] where, among other things, the cut-off was introduced in the form (4.10) almost a decade ago! Equation (4.2) was obtained by Voloshin [58]. The expression for $\mu_\pi^2$ is the BGSUV sum rule [59]. The next one is new. The last two sum rules are obtained along the same lines; the one for the Darwin term $\rho_3^2$ was presented in [60]. A unified field-theoretic derivation of all these relations was discussed in Ref. [10] where the transition amplitude in the SV limit was considered (see also Sect. 8.2.1). Since the presentation in that paper is sufficiently pedagogical, the interested reader is referred to it directly. In this section we will give an alternative (new) derivation of Eqs. (4.3)–(4.6). A new derivation of Eq. (4.2) will be sketched in Sect. 6.1.

At first, however, we pause to make a few explanatory remarks regarding $\mu$ dependence. To avoid inconsistencies it must be done in the same way in all theoretical calculations, including perturbation theory, the definition of the heavy quark current

\footnote{Higher moments than those presented above are not very useful in practice since they are saturated at higher $\epsilon$, the $\mu$ dependence becomes too essential, etc.}
(and, hence, the Isgur-Wise function) and so on. This cut-off scheme does not coincide with the one implied by dimensional regularization, although they are related in the perturbative domain $\mu \gg \Lambda_{\text{QCD}}$. In particular, the logarithmic scale dependence of $\varrho^2(\mu)$ is the same. The bound $\varrho^2 > 1/4$ [57] holds only for this definition of the renormalized Isgur-Wise function and is not necessarily true for other definitions.

Without the cut-off at $\epsilon_k = \mu$ the sums over the excited states in the above relations diverge at high excitation energies. The high-energy tails of the sums are dual to the perturbative transition probabilities which, thus, determine the $\mu$ dependence of the operators at hand. Cutting the integrals over the excitation energies (i.e. limiting the sums by $(\epsilon_k)_{\text{max}} = \mu$) is a physical way of introducing the normalization point without violating the general properties of QCD and not endangering the quantum-mechanical aspects of the field theory. It is often convenient to use the exponential weight $e^{-\epsilon/\mu}$ in the sums. As long as the weight function is universal, all relations are the same as with a step-like cut-off.

Now, we proceed to a consideration which illustrates the physical meaning of the sum rules presented above.

Since in the heavy quark limit, $m_Q \gg \Lambda_{\text{QCD}}$, $QQ$ pairs are not produced, we will use the quantum-mechanical language with respect to $Q$ (but not the light cloud, of course). Moreover, it is convenient, at the first stage to assume $Q$ to be spinless. The $Q$ spin effects are trivially included later. Then the lowest-lying states, the $S$-wave configurations corresponding to $B$ and $B^*$, are spin-1/2 fermions, with two spin orientations of the light cloud. We shall denote them $|\Omega_0\rangle$; the spinor wavefunction of this state is $\Psi_0$. It is obvious that

$$
\langle \Omega_0|\bar{Q}(iD_j)(iD_k)Q|\Omega_0\rangle \equiv \frac{\mu^2}{3} \delta_{jk} \Psi_0^\dagger \Psi_0 - \frac{\mu_G^2}{6} \Psi_0^\dagger \sigma_{jk} \Psi_0 = \sum_n \langle \Omega_0|\pi_j|n\rangle \langle n|\pi_k|\Omega_0\rangle,
$$

(4.11)

where a complete set of intermediate states is inserted. They are spin-1/2 states (of the opposite parity with respect to $|\Omega_0\rangle$), generically denoted by $\phi^{(n)}$, and spin-3/2 states $\chi^{(n)}$. We will use the Rarita-Schwinger wavefunctions for the latter, i.e. a set of three spinors $\chi_l$ obeying the constraint $\sigma_l \chi_l = 0$. The normalization of these spinors is fixed by the sum over polarizations $\lambda$

$$
\sum_\lambda \chi_i(\lambda) \chi_j^\dagger(\lambda) = \delta_{ij} - \frac{1}{3} \sigma_i \sigma_j.
$$

(4.12)

Defining the reduced matrix elements $a_n$ and $b_m$ as

$$
\langle \phi^{(n)}|\pi_j|\Omega_0\rangle \equiv a_n \phi^{(n)} \rangle_j, \quad \langle \chi^{(m)}|\pi_j|\Omega_0\rangle \equiv b_m \chi^{(m)} \rangle_j,
$$

(4.13)

where $\phi^{(n)}$ and $\chi^{(m)}$ stand for the states as well as for their wavefunctions, we get

$$
\mu_G^2 = -6 \sum_n |a_n|^2 + 2 \sum_m |b_m|^2,
$$

(4.14)
and

\[ \mu_\pi^2 = 3 \sum_n |a_n|^2 + 2 \sum_m |b_m|^2. \quad (4.15) \]

These expressions can be immediately generalized to the actual case of the spin-1/2 quarks \( Q \). The quantities \( a_n \) and \( b_m \) are to be understood as the matrix elements of \( \bar{b_i} \Gamma D \bar{b} \) between the \( B \) meson and higher even-parity states. They are related to \( \tau_{1/2}^{(n)} \) and \( \tau_{3/2}^{(m)} \) as follows:

\[ \tau_{1/2}^{(n)} = \frac{a_n}{\epsilon_n}, \quad \tau_{3/2}^{(m)} = \frac{1}{\sqrt{3}} \frac{b_m}{\epsilon_m}; \quad (4.16) \]

and, therefore,

\[ \mu_\pi^2 = 3 \left( \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 \right), \quad \mu_\pi^2 = 3 \left( -2 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 + 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 \right). \]

Relations (4.16) are most easily obtained from the fact that the SV amplitudes of transitions to the corresponding states are given by the overlap \( \langle n(\bar{v})|B(\bar{v} = 0) \rangle \) and by exploiting Eq. (6.7) for \( |B(\bar{v})\rangle \) from Sect. 6.1.

The sum rules (4.1)–(4.6) obviously entail a set of exact QCD inequalities. The first one is nothing but the Bjorken inequality \( \rho^{2} > 1/4 \) which was already mentioned above. Others from this sequence are:

\[ \overline{\Lambda}(\mu) \geq 2 \Delta_1 \left( \rho^{2}(\mu) - \frac{1}{4} \right), \quad (4.17) \]
\[ \mu_\pi^2(\mu) \geq \frac{3}{2} \Delta_1 \overline{\Lambda}(\mu) \quad \text{and} \quad \mu_\pi^2(\mu) \geq 3 \Delta_1^2 \left( \rho^{2}(\mu) - \frac{1}{4} \right), \quad (4.18) \]
\[ \mu_\pi^2 = \mu_\pi^2 + 9 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \geq \mu_\pi^2, \quad (4.19) \]
\[ \rho_D^3(\mu) \geq \Delta_1^2 \mu_\pi^2(\mu) \quad \text{and} \quad \rho_D^3(\mu) \geq \frac{3}{2} \Delta_1^2 \overline{\Lambda}(\mu) \quad \text{and} \quad \rho_D^3(\mu) \geq 3 \Delta_1^3 \left( \rho^{2}(\mu) - \frac{1}{4} \right), \quad (4.20) \]
\[ \rho_D^3(\mu) = -\rho_D^3(\mu) + 9 \sum_n \epsilon_n^3 |\tau_{1/2}^{(n)}|^2 \geq -\rho_D^3(\mu), \quad \text{and} \quad \rho_D^3(\mu) \geq \frac{|\rho_D^3(\mu)|}{2}. \quad (4.21) \]

Here \( \Delta_1 \) is the first excitation energy. Strictly speaking, the lowest excitation of \( B \) is \( B\pi \); the pions are coupled rather weakly, however. Moreover, the \( B\pi \) intermediate state becomes irrelevant in the limit \( N_c \to \infty \). Therefore, for all practical purposes we can equate \( \Delta_1 \) to the mass difference of \( B^{**} \) and \( B \), i.e. \( \Delta_1 \approx 0.5 \text{ GeV} \).

It is important that the above inequalities are based only on general QCD relations and the heavy-quark limit: no recourse has been made to model assumptions. In particular, dynamics of the the light cloud is included in full and exactly. The inequality \( \rho_D^3 - (1/3)\rho_D^3 > 0 \) following from Eqs. (4.5) and (4.6) is useful for the zero recoil sum rule discussed in Sect. 8.2.3.
The values of \( \tau_{3/2} \) and, in particular, \( \tau_{1/2} \) are not well known at present. For the lowest states with \( \epsilon \approx 0.5 \) GeV they were evaluated in QCD sum rules. According to [61], \( \tau_{1/2}^{(1)} \approx 0.25 \); the similar estimate for \( \tau_{3/2}^{(1)} \) is \( \tau_{3/2}^{(1)} = 0.4 \pm 0.1 \) [62]. A model estimate [66] falls quite close, \( \tau_{1/2}^{(1)} = 0.25 \); the similar estimate for \( \tau_{3/2}^{(1)} \) is \( \tau_{3/2}^{(1)} \approx 0.35 \pm 0.1 \) [62]. A model estimate [56] falls quite close, \( \tau_{1/2}^{(1)} \approx \tau_{3/2}^{(1)} \approx 0.31 \). Recently, the value \( \tau_{3/2}^{(1)} \approx 0.35 \) was derived [63] from the experimental information on the yield of the excited charm states in semileptonic \( B \) decays, yet also with sizeable uncertainty. Substituting \( \tau_{1/2}^{(1)} \approx 0.25 \) in Eq. (4.19) we arrive at a stronger bound on \( \mu_2^2 \), quoted in Eq. (4.22) below.

All these inequalities are saturated if it is only the first excitation that contributes to the sums. This is explicitly so in the so-called two-doublet model of Ref. [64]. If \( \mu \) is not too large, in the ballpark 0.7 to 1 GeV, the actual situation seems to be not too far from this model. (In the two-doublet model the higher excited states are exactly dual to the perturbative corrections).

Assigning to the estimate of \( \tau_{1/2}^{(1)} \) above an uncertainty \( \pm 30\% \), one obtains from Eq. (4.19)

\[
\mu_2^2(0.7 \text{ GeV}) = (0.6 \pm 0.15) \text{ GeV}^2. 
\]

(4.22)

Moreover, assuming that \( \bar{\rho}^2(\mu) = 0.8 \pm 0.2 \) and \( \Delta_1 = 0.5 \) GeV we get

\[
\sqrt{\Lambda(\mu)} = 0.35 \text{ to } 0.75 \text{ (0.5 to 1) GeV,} 
\]

(4.23)

\[
\mu_2^2(\mu) = 0.25 \text{ to } 0.56 \text{ (0.38 to 0.8) GeV}^2, 
\]

(4.24)

\[
\rho_3^2(\mu) = 0.13 \text{ to } 0.28 \text{ (0.2 to 0.4) GeV}^3. 
\]

(4.25)

The first interval assumes complete saturation by the first excitation, while the interval indicated in parenthesis assumes 70% saturation.

The exact QCD inequalities presented above take place at any \( \mu \), provided that all parameters refer to the Wilsonian formulation. At large \( \mu \) they become non-informative since all dimensionful parameters (except \( \mu_2^2, \rho_3^2 \)) are dominated by the large perturbative pieces scaling like the corresponding power of \( \mu \).

Concluding this section, let us stress again that the information encoded in the exact inequalities is model-independent. It makes sense to impose the corresponding constraints in any analysis aimed at extracting the set of the fundamental parameters, say, from the semileptonic spectra. If a particular determination leads to results incompatible with these inequalities, one may be sure that something is wrong either with data or with the analysis.

In the practical applications discussed in the later sections we will use, as a central value, \( \mu_2^2 = 0.5 \) GeV\(^2 \) normalized at 0.5 to 0.7 GeV.

---

\(^6\)The experimentally measured slope is centered around 0.9, with the errors \( \sim \pm 0.2 \). To get \( \bar{\rho}^2(\mu) \) from this slope one must eliminate \( 1/m_c \) and hard perturbative corrections. According to estimates in [63, 64], this amounts to subtracting \( \sim 0.1 \) from the experimental number.
5 More on $\mu_\pi^2 > \mu_G^2$

In view of importance of this particular inequality a few additional comments are in order. Unlike quantum-mechanical potential problems, in QCD the heavy quark kinetic operator is expressed in terms of the covariant derivatives $\pi_j = -iD_j = -i\partial_j - A_j$ which contain the vector potential $A_j$. The presence of $A_j$ leads to non-commutativity of different spatial components of the momentum operator in the presence of the chromomagnetic field $\vec{B}$,

$$\left[\pi_j, \pi_k\right] = iG_{jk} = -i\epsilon_{jkl}B_l.$$  \hfill (5.1)

This non-commutativity immediately leads to the lower bound on the expectation value of $\vec{\pi}^2$, in full analogy with the uncertainty principle in quantum mechanics. The simplest quantum-mechanical formulation was suggested in Ref. [45],

$$\langle B | (\vec{\sigma} \vec{\pi})^2 | B \rangle = \langle B | \vec{\pi}^2 + \vec{\sigma} \vec{B} | B \rangle = \mu_\pi^2 - \mu_G^2 > 0.$$ \hfill (5.2)

This inequality has a transparent physical interpretation. We deal here with the Landau precession of a colored, i.e. “charged” particle in the (chromo)magnetic field. Hence, one has $\langle p^2 \rangle \geq |\vec{B}|$. Literally speaking, in the $B^*$ meson the quantum-mechanical expectation value of the chromomagnetic field is suppressed, $\langle B_\perp \rangle = -\mu_G^2/3$. It completely vanishes in the $B$ meson. However, the essentially non-classical nature of $\vec{B}$ (e.g. $\langle \vec{B}^2 \rangle \geq 3 \langle \vec{B} \rangle^2$), in turn, enhances the bound which then takes the same form as in the external classical field.

The SV sum rules [44, 10] translate the quantum-mechanical derivation of the above inequality to the field-theoretical language; the difference $\mu_\pi^2(\mu) - \mu_G^2(\mu)$ is represented as the imaginary part of the transition operator which is given by a sum of certain transition probabilities.

6 Virial Theorem

In this section we continue the discussion of the kinetic operator from a different perspective. As was mentioned, the expectation value of $\vec{\pi}^2$ in the $B$ meson was estimated in the QCD sum rules directly [12] and [46] \textit{via} the virial theorem. A strong discrepancy between the two determinations, which do not overlap within their respective error bars, prompts us to revisit the issue. Two questions arise. A practical one is what result is more trustworthy and why. A more theoretical problem is what went wrong with the evaluation of the uncertainties in the competing calculations?

Clearly, only the authors themselves could answer these questions in full. We would like to suggest, however, some qualitative insights which, hopefully, may shed light on the discrepancy and be helpful in future refined treatments of the issue. We must start, however, with the brief description of the virial relation itself.
6.1 Alternative derivations

The kinetic and chromomagnetic matrix elements are defined via the expectation values of the corresponding operators in $B$ meson. In Ref. [9] a certain relation for the matrix elements of $\bar{Q}\vec{\pi}\pi Q$ was obtained involving a new, chromoelectric operator $\bar{Q}g\vec{E}Q$. The transition matrix element of $\bar{Q}g\vec{E}Q$ between the states with different velocities was considered, and it was found that

$$\frac{1}{2M_{HQ}}\langle H_Q(v')|\bar{Q}iG^\mu\nu Q|H_Q(v)\rangle = \frac{1}{3} (v^\mu v'^\nu - v^\nu v'^\mu) \left\{ \frac{1}{2M_{HQ}}\langle H_Q(v)|\bar{Q}\vec{\pi}\pi Q|H_Q(v)\rangle + O(vv' - 1) \right\}.$$  

(6.1)

This equation is valid for the spinless $H_Q$; for non-vanishing spins it is valid after averaging over polarizations of $H_Q$. In the rest frame of the initial hadron ($\vec{v} = 0$) the relation is nontrivial for $\{\mu,\nu\} = \{0,k\}$; then $G_{0k} = -E_k t^a$ where $\vec{E}$ is the chromoelectric field. Then Eq. (6.1) takes the form

$$\frac{1}{2M_{HQ}}\langle H_Q(v')|\bar{Q}(iE^i)Q|H_Q(0)\rangle = \frac{v^n}{3} \mu^2 + ...$$  

(6.2)

where the ellipses denote $O(\vec{v}^2)$ terms which will be systematically omitted.

Below several complementary pedagogical derivations will be presented. First, however, a few words as to why Eq. (6.1) is sometimes referred to as a virial theorem [9]. Assume that, instead of QCD with its messy dynamics of infinite number of degrees of freedom residing in the light cloud, we consider a non-relativistic bound-state problem: a heavy quark interacting with a lighter one via a potential. Using the formalism of non-relativistic quantum mechanics it is easy to rewrite the left-hand side of Eq. (6.2) as follows:

$$\langle e^{-im_0\vec{v}\times g\vec{E}(\vec{x})} \rangle \rightarrow m_Q v^i \langle gx^i E^i(\vec{x}) \rangle = m_Q v^i \langle gx^i \nabla_i V(\vec{x}) \rangle,$$

(6.3)

where the angle brackets denote averaging over the given bound state (in the rest frame), $V$ is the binding potential and, we restore the coupling constant contained in $G_{\mu\nu}$. Equation (6.3) now takes the text-book form of the virial theorem,

$$\langle \pi^2_k \rangle = m \langle x_k \partial_k V(x) \rangle.$$  

(6.4)

No sum over $k$ is implied here. (Note that in quantum mechanics $\langle m x_k \partial_k V(x) \rangle$ remains the same being calculated for the heavy “quark” $Q$ or the lighter one $q$.)

Equation (6.1) might be helpful in various applications. It was exploited, in particular, in studying the properties of the kinetic energy operator under mixing [67, 68].

The most elementary derivation of Eq. (6.1) is as follows. If $\vec{v} = 0$ and $|\vec{v}'| \ll 1$ then

$$\langle H_Q(v')|\bar{Q}(\pi_0\vec{v}_i - \pi_i\pi_0)Q|H_Q(v)\rangle = \langle H_Q(v')|\bar{Q}(\pi_0 - \pi\vec{v}' + \pi\vec{v}')\pi_i Q|H_Q(v)\rangle,$$
where all π’s act to the right, and we have used the equation of motion neglecting the terms suppressed by 1/mQ. Moreover, to the first order in $\vec{v}'$ the operators π on the right-hand side can be considered as acting to the left (the difference is the full derivative which is obviously $O(|\vec{v}'|^2)$). When $\pi_0 - \vec{π} \vec{v}'$ acts to the left it again produces zero due to the equation of motion and we are left with

$$\langle H_Q(v')|\bar{Q}(\vec{π} \vec{v}'\pi_i)Q|H_Q(v)\rangle$$

(6.5)

which is equivalent to Eq. (6.1).

It is instructive to give a different derivation, more directly related to the quantum-mechanical aspect of the problem. Consider

$$\langle H_Q(0)|\pi_j\pi_k|H_Q(0)\rangle = \langle H_Q(0)|\pi_j\pi_0^{-1}\pi_0\pi_k|H_Q(0)\rangle =$$

$$\langle H_Q(0)|\pi_j\pi_0^{-1}(\pi_0\pi_k - \pi_k\pi_0)|H_Q(0)\rangle.$$  

(6.6)

The operator $\pi_0^{-1}(\vec{v}\vec{π})$ acting on $|H_Q\rangle$ is nothing but the generator of the boost along direction $\vec{v}$:

$$|H_Q(\vec{v})\rangle = |H_Q(0)\rangle + \pi_0^{-1}\vec{v}\vec{π}|H_Q(0)\rangle + O(\vec{v}^2).$$

(6.7)

This is a useful relation nicely elucidating the meaning of the small velocity sum rules. It represents the first-order perturbation theory in $\delta H = \vec{v}\vec{π}$; the unperturbed Hamiltonian is $\mathcal{H}_0 = \pi_0$ (further details can be found in [10], Eq. (178) and Sect. VI).

In the second-quantized notations the very same relation takes the form

$$|H_Q(\vec{v})\rangle = |H_Q(0)\rangle + \int d^3\vec{x} \bar{Q}\pi_0^{-1}\vec{v}\vec{π}Q(x)|H_Q(0)\rangle + O(\vec{v}^2).$$

(6.8)

Keeping in mind that $\langle H_Q(0)|\pi_i|H_Q(0)\rangle$ vanishes and invoking the definition

$$[\pi^0, \pi^k] = iG^{0k} = -iE^k,$$

(6.9)

we immediately rewrite Eq. (6.6) in the form

$$\langle H_Q(0)|\bar{Q}\vec{π}\pi_kQ|H_Q(0)\rangle = \langle H_Q(\vec{v})|\bar{Q}iG_{0k}Q|H_Q(0)\rangle$$

(6.10)

which is again the virial relation.

The simple derivations above are formulated in terms of the on-mass-shell states. In some applications (e.g. the QCD sum rules), rather than dealing with the on-mass-shell states, one works with the Green functions and amplitudes induced by interpolating currents. For future applications we find it useful to formulate yet another derivation of the virial theorem based on the technique of the Green functions. In this approach we consider the three-point functions induced by appropriately chosen currents, and then use the reduction formula. The relevant correlators are treated in the background field method (for a review see [89]), although actually we use very little of the background field formalism. The advantage of this approach is its manifest field-theoretic nature. One operates with full QCD and takes the limit $m_Q \to \infty$, as in Ref. [3].
Consider the three-point function depicted in Fig. 2. The central vertex in this triangle is the operator $G^{0i}$. Two other vertices are generated by the interpolating currents

$$J = \bar{Q}i\gamma_5 q \quad \text{and} \quad J^I = \bar{q}i\gamma_5 Q,$$

(6.11)

where the pseudoscalar current is chosen merely for definiteness. The sides of the triangle are Green’s functions of the quarks in the background gluon field. The reduction theorem tells us that to get the transition amplitude $\langle H_Q | \bar{Q}G^{0i}Q | H_Q \rangle$ from this three-point function we amputate it: multiply by $(p^2 - M_{HQ}^2)$ and $(p'^2 - M_{HQ}^2)$ and tend $p^2$ and $p'^2$ to $M_{HQ}^2$. This singles out the double pole whose residue is proportional to the meson-to-meson transition on $G^{0i}$.

The expression for the three-point function of Fig. 2 has the form

$$i \text{Tr} \left\{ i\gamma_5 \frac{1}{p' + \gamma - m_Q} G^{0i} \frac{1}{p + \gamma - m_Q} i\gamma_5 \frac{1}{\gamma - m_q} \right\} ,$$

(6.12)

where $\pi$ is the momentum operator, $\text{Tr}$ implies taking trace in the color, Lorentz and (abstract) momentum spaces, and $p$ and $p'$ are external momenta flowing through the currents. In the non-relativistic limit, keeping only terms linear in $\vec{v}'$, one can readily rewrite Eq. (6.12) as follows:

$$i \text{Tr} \left\{ \frac{1}{(\varepsilon + \pi_0) - \vec{p}' \vec{v}'} G^{0i} \frac{1}{(\varepsilon + \pi_0)} \right\} i\gamma_5 \frac{\gamma_0 + 1 - \vec{v}' \vec{\gamma}}{2} \frac{1}{\gamma - m_q} \frac{\gamma_0 + 1}{2} i\gamma_5 \right\} ,$$

(6.13)

where $\varepsilon + m_Q$ is the time component of the external momentum $p$ (see Fig. 2), and we set $\varepsilon' = \varepsilon$, since only the double pole of the type

$$\frac{1}{(\varepsilon - \Lambda)^2}$$

is of interest for singling out the ground state $H_Q$ from the tower of states created by the interpolating currents $J$.

Now, let us use the fact that

$$G^{0i} = -i[\pi^0, \pi^i] = -i \left\{ (\varepsilon + \pi_0)\pi^i - \pi^i(\varepsilon + \pi_0) \right\} .$$

(6.15)
Substituting Eq. (6.15) in Eq. (6.13) and expanding the denominator in \( \vec{v}' \) we observe that the square bracket reduces to

\[
\left[ \frac{1}{(\varepsilon + \pi_0) - \pi' \vec{v}'} \left( (\varepsilon + \pi_0) \pi^i - \pi'(\varepsilon + \pi_0) \right) \right] \rightarrow \text{terms with single pole} + \\
\frac{1}{(\varepsilon + \pi_0) \pi'} \pi^i \frac{1}{(\varepsilon + \pi_0)} + \mathcal{O}(\vec{v}'^2).
\]

(6.16)

In the limit \( \varepsilon \to \bar{\Lambda} \), the residue of the double pole (in the second line in Eq. (6.16)) is just

\[
\frac{1}{3} v'^n \frac{1}{2M_{HQ}} \langle H_Q | \bar{Q} \pi^2 Q | H_Q \rangle,
\]

which proves Eq. (6.2). Note that it is crucial to single out the double-pole contribution.

By the same token, a similar consideration would lead us to Eq. (94) from Ref. [10], and, eventually, to the sum rule (4.2). Consider, instead of the three-point function (6.12), the two-point function

\[
i \text{Tr} \left\{ i \gamma_5 \frac{1}{p + \not{\!\! p} - m_Q} i \gamma_5 \frac{1}{\not{\!\! p} - m_q} \right\},
\]

(6.17)

with the external momentum \( p = m_Q v_\mu + \epsilon_\mu \). It is assumed that \( v_\mu = \{ v_0, \vec{v} \} \) where \( |\vec{v}| \neq 0 \) but \( |\vec{v}| \ll 1 \) and \( \epsilon_\mu = \{ \epsilon, 0 \} \). We then take the limit \( m_Q \to \infty \), expand in \( \vec{v} \) keeping the terms quadratic in this parameter, isolate the double pole (6.14) and compare the result with the phenomenological representation for the same two-point function. Equation (94) from Ref. [10] is immediately reproduced.

### 6.2 Virial theorem and QCD sum rules

In this section we dwell on the problem of determining the kinetic energy from different QCD sum rules.

The technical reason of the existing discrepancy is quite evident: in the sum rule analyzed by Ball et al. [42] the free-quark diagram gives a contribution in the theoretical side of the sum rule. This contribution is then “corrected” by \( \mathcal{O}(\alpha_s) \) terms and nonperturbative condensates, a standard situation in the QCD sum rules technology. On the other hand, Neubert suggests [46] considering the virial partner of \( \bar{Q} \pi^2 \), and analyzing the QCD sum rule for \( \bar{Q} \bar{E}Q \) rather than for \( \bar{Q} \pi^2 Q \). In this case the free-quark contribution on the right-hand side is absent. The theoretical side of the sum rules is solely due to interaction terms which must be kept relatively small, by necessity. Although such a regime is much less studied than the standard one, some experience is still available.

To get an idea of how the QCD sum rules work in such problems it is instructive to consider a toy model which was used for this purpose previously more than once
We mean illustrating the sum-rule approach in the three-dimensional harmonic oscillator.

Assume that we have an infinitely heavy “quark” \( Q \) and a lighter but still non-relativistic “quark” \( q \) with mass \( m \ll m_Q \), interacting via the harmonic oscillator potential,

\[
V(\vec{r}) = \frac{m\omega^2\vec{r}^2}{2} \tag{6.18}
\]

where \( \vec{r} \) is the distance between \( Q \) and \( q \).

Needless to say, this model is exactly solvable. We will pretend, however, that we do not know it, and will calculate the matrix element of interest using the sum-rule approach. Parallelizing QCD we will consider the expectation value of the kinetic energy and its virial partner \( \frac{1}{2}x_i\partial_i V(x) \). Since the potential is quadratic, the virial partner is the potential energy operator itself,

\[
E_{\text{pot}} = \frac{m\omega^2\vec{r}^2}{2}. \tag{6.19}
\]

The kinetic energy operator in the model at hand has the form

\[
E_{\text{kin}} = \frac{\vec{p}^2}{2m} = -\frac{1}{2m}\Delta. \tag{6.20}
\]

As is well-known, the expectation values of the \( E_{\text{pot}} \) and \( E_{\text{kin}} \) are indeed always equal to each other in the harmonic oscillator; for the ground state

\[
\langle E_{\text{kin}} \rangle_0 = \langle E_{\text{pot}} \rangle_0 = \frac{3\omega}{4}. \tag{6.21}
\]

Note that

\[
E_{\text{kin}} + E_{\text{pot}} = H \tag{6.22}
\]

where \( H \) is the Hamiltonian of the system.

The important element is that, like in QCD, the potential energy vanishes for free particles and appears only when the interaction is included. Thus, its calculation is technically similar to the QCD sum rule for \( \bar{Q}\vec{E}Q \). The kinetic energy, clearly, is nonzero even for free – but moving – quarks and, in this respect, is the counterpart of \( \bar{Q}\vec{\pi}^2 Q \).

Now, in the spirit of the sum-rule approach, we pretend that the only quantities we are able to calculate reliably are the correlation functions at short (Euclidean) times. Namely, we consider the following amplitudes: at the initial moment of time the quarks are at zero separation, at final (Euclidean) time \( T = \tau_1 + \tau_2 \) they are at zero separation again; at \( \tau_1 \) the insertion of the operator \( E_{\text{kin}} \) or \( E_{\text{pot}} \) is made. These amplitudes are perfect analogues of the Borel-transformed two- and three-point functions in the QCD sum rules.

All relevant formulae are collected in Ref. \[61\] where a toy-model calculation in a related problem is carried out. The two-point function is

\[
S(T) = K(0, T|0, 0) = \left(\frac{m\omega}{2\pi}\right)^{3/2} \frac{1}{(\sinh\omega T)^{3/2}}, \tag{6.23}
\]
where \( K(\vec{r}, \tau | \vec{0}, 0) \) is the amplitude (time-dependent Green function) of the propagation from the point \((\vec{0}, 0)\) to the point \((\vec{r}, \tau)\) in the Euclidean time,

\[
K(\vec{r}, \tau | \vec{0}, 0) = \sum_n \psi_n^*(\vec{0}) \psi_n(\vec{r}) e^{-E_n \tau} = 
\left( \frac{m \omega}{2 \pi} \right)^{3/2} \exp \left( -\frac{m \omega}{2 \sinh(\omega \tau)} \vec{r}^2 \cosh(\omega \tau) \right). 
\]

(6.24)

The three-point functions with the insertion of \( E_{\text{pot}} \) and \( E_{\text{kin}} \) are

\[
S_{\text{kin}} = \int d^3 \vec{r} K(\vec{0}, \tau_1 + \tau_2 | \vec{r}, \tau_1) E_{\text{kin}} K(\vec{r}, \tau_1 | \vec{0}, 0), 
\]

(6.25)

plus the same expression with the insertion of \( E_{\text{pot}} \). Following the standard routine of the QCD sum-rule practitioners we will analyze the sum rules at the symmetric point, \( \tau_1 = \tau_2 = T/2 \). Using Eq. (6.24) above it is easy to find that

\[
S_{\text{kin}} = \frac{3}{4} \omega S \frac{\cosh(\omega T) + 1}{\sinh(\omega T)}, \quad S_{\text{pot}} = \frac{3}{4} \omega S \frac{\cosh(\omega T) - 1}{\sinh(\omega T)}, 
\]

(6.26)

where \( S \) is defined in Eq. (6.23).

Let us review how the sum rule technology works. As a classic example, we determine the ground state energy of the three-dimensional oscillator. To this end we will analyze two sum rules

\[
S = \sum_{n=0}^\infty \psi_n(0) \psi_n^*(0) e^{-E_n T} = \left( \frac{m T^{-1}}{2 \pi} \right)^{3/2} \left( 1 - \frac{\omega^2 T^2}{4} + \frac{19 \omega^4 T^4}{480} + \ldots \right), 
\]

(6.27)

and

\[
S_{\text{kin}} + S_{\text{pot}} = \sum_{n=0}^\infty \psi_n(0) \psi_n^*(0) E_n e^{-E_n T} = 
\frac{3 T^{-1}}{2} \left( \frac{m T^{-1}}{2 \pi} \right)^{3/2} \left( 1 + \frac{\omega^2 T^2}{12} - \frac{19 \omega^4 T^4}{288} + \ldots \right). 
\]

(6.28)

Equation (6.27) is a two-point function sum rule, Eq. (6.28) is a three-point function sum rule with the insertion of the Hamiltonian. The right-hand sides in both expressions are given in the small-\( T \) expansion since we pretend not to know the exact solutions. Note, that the leading terms in these expansions are \( \omega \)-free; they correspond to the propagation of free “quarks” and are analogous to the free-quark diangle and triangle diagrams in QCD. The subsequent terms are proportional to powers of \( \omega \) and are analogous to the condensate corrections in the sum-rule approach, and, indeed, can be easily calculated without the exact solution at hand.

To single out the ground state contribution it is desirable to go to as high values of \( T \) as possible since all higher states are exponentially suppressed at large \( T \). We cannot go to the asymptotically large values of \( T \), however, since the power...
corrections blow up. Upon inspecting Eqs. (6.27) and (6.28) it becomes clear that if we want to keep the power corrections under control (i.e. the last correction kept to be less than $\sim 30\%$) we can not go higher than $T \sim 1.5 \omega^{-1}$. At such values of $T$ the suppression of the higher states in the sum rules, although quite visible, is still not good enough. Thus, the relative contribution of the first excited state in Eq. (6.27) is $(3/2)e^{-2\omega T}$, while in Eq. (6.28) its relative weight is $(7/2)e^{-2\omega T}$. We can see it in a different way: taking the ratio of Eqs. (6.28) to (6.27) would produce $3\omega/2$ if all higher excitations were fully suppressed. Instead, we get $\sim 0.75(3\omega/2)$ at $T \sim 1.5 \omega^{-1}$.

If we aim at better accuracy we have to have at least a rough idea of the contribution coming from the excited states. The standard strategy in the sum rule analyses is representing all states higher than the ground state by the free-quark approximation, starting from some effective continuum threshold $\epsilon_c$. The particular form of the corresponding spectral densities is determined by the free-quark term in the power expansion. In the sum rules (6.27) and (6.28) we have, respectively,

$$\sum_{n=1}^{\infty} \psi_n(0)\psi_n^*(0)e^{-E_n T} \approx \left(\frac{m}{2\pi}\right)^{3/2} \frac{2}{\sqrt{\pi}} \int_{\epsilon_c}^{\infty} d\epsilon \epsilon^{1/2} e^{-\epsilon T}$$

and

$$\sum_{n=1}^{\infty} \psi_n(0)\psi_n^*(0)E_n e^{-E_n T} \approx \frac{3}{2} \left(\frac{m}{2\pi}\right)^{3/2} \frac{4}{3\sqrt{\pi}} \int_{\epsilon_c}^{\infty} d\epsilon \epsilon^{3/2} e^{-\epsilon T}. \quad (6.29)$$

Transferring the “continuum” contribution to the right-hand side of the sum rules we arrive at

$$\psi_0(0)\psi_0^*(0)e^{-E_0 T} = \left(\frac{m}{2\pi}\right)^{3/2} \left(\frac{2}{\sqrt{\pi}} \int_{\epsilon_c}^{\epsilon_0} d\epsilon \epsilon^{1/2} e^{-\epsilon T} - \frac{\omega^2 T^2}{4} + \frac{19\omega^4 T^4}{480} + \ldots\right)$$

and

$$\psi_0(0)\psi_0^*(0)E_0 e^{-E_0 T} = \frac{3}{2} \left(\frac{m}{2\pi}\right)^{3/2} \left(\frac{4}{3\sqrt{\pi}} \int_{0}^{\epsilon_c} d\epsilon \epsilon^{3/2} e^{-\epsilon T} + \frac{\omega^2 T^2}{12} - \frac{19\omega^4 T^4}{288} + \ldots\right). \quad (6.30)$$

Assume that our idea of the excited state contributions is roughly correct. If $T$ can be chosen large enough so that the higher states are sufficiently suppressed, on the one hand, and small enough to keep the power expansion in the right-hand side under control, on the other, the ratio of the two expressions must be close to $E_0 = (3/2\omega)$. The interval of $T$ satisfying the above conditions is called window (fiducial domain). Of course, the constancy ($T$ independence) of the ratio is valid only to the extent one can neglect the higher-order power corrections and the error of the continuum approximation. Thus, we expect the ratio to be approximately constant. The solid curve in Fig. 3 shows the ratio, as a function of $T$ in the fiducial domain (the plot corresponds to $\epsilon_c = 2.5\omega$; the value of the “continuum threshold” can be fitted in its turn). We see a clear-cut plateau, and the height of the plateau differs from the exact ground state energy ($E_0 = 3\omega/2$) by at most $\pm 5\%$. 

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Figure 3: Sum rules for the three-dimensional harmonic oscillator. The solid curve presents $\langle H \rangle$ (in units of $\frac{3}{2}\omega$) vs. $T$. The dashed curve gives $\langle E_{\text{kin}} \rangle$ in units of $\frac{3}{4}\omega$, the dotted curve $\langle E_{\text{pot}} \rangle$ in units of $\frac{3}{4}\omega$. The horizontal axis is $T$ in units of $\omega^{-1}$. The exact results for the ground state energy, kinetic energy and potential energy are $\frac{3}{2}\omega$, $\frac{3}{4}\omega$, $\frac{3}{4}\omega$, respectively.

So far the application of the sum rule technique is successful. What happens, if instead of the sum of the kinetic and potential energy operators we would analyze them separately?

From the exact solution of the problem we know that the most drastic change is the occurrence of the off-diagonal transitions in the three-point sum rules for $S_{\text{kin}}$ and $S_{\text{pot}}$. These off-diagonal contributions are absent, by construction, in $S$ and are obviously absent in the sum rule for $\langle H \rangle$, too, since the off-diagonal matrix elements of the Hamiltonian vanish. On the contrary, the off-diagonal matrix elements of the operators $E_{\text{kin}}$ and $E_{\text{pot}}$ do not vanish and are numerically relatively large compared to the diagonal matrix elements. For instance, $\langle \vec{r}^2 \rangle_{01} = -2 \langle \vec{r}^2 \rangle_{00}$. Moreover, if the first excited state appearing in $S$ or $\langle H \rangle$ has the excitation energy $2\omega$ (above the ground state), the first excited state in $\langle E_{\text{kin}} \rangle$ and $\langle E_{\text{pot}} \rangle$ is separated from the ground state by only $\omega$, as explained in detail in Ref. [61]. The fact that the excited states lie closer to the ground state and their relative weight is relatively large makes it harder to suppress them sufficiently/and or approximate them by the free-quark continuum. The fiducial domain is bound to shrink.

A closer inspection shows that the situation is especially bad in the sum rule for $\langle E_{\text{pot}} \rangle$, since in this case the continuum simply vanishes in the free quark approximation. This is obvious from the definition of the operator, and is clearly seen from the small-$T$ expansion of the second expression in Eq. (6.26). Following the standard strategy we would be forced merely to neglect all higher states. The exact solution shows [61] that the excited states contribution consists of two components – the diagonal transitions and off-diagonal ones – and each component separately does not vanish in the free quark approximation and is large, but they have opposite relative signs. For highly excited states the sign-alternating terms, after summation, are smeared to zero, so this approximation of “no continuum” is not bad. The first excitation, however, is too conspicuous and is not annihilated by the second one. For values of $T$ up to $\sim 1.5\omega^{-1}$ the first excitation plays an important role in the sum
rule, and effectively screens the ground state contribution. Accepting the "standard" continuum model (i.e. no continuum in the case at hand) we significantly underestimate the expectation value of $E_{\text{pot}}$ in the ground state. Figure 3 illustrates the trend. Assuming that the fiducial domain is the same as in the sum rule for $S$ or $\langle H \rangle$ we get around 0.5 of the exact value. As a matter of fact this means that the fiducial domain shrinks to zero in this case.

Analysis of the kinetic energy operator is better, but not much better. The diagonal and off-diagonal transitions are the same in the absolute value as above (this is obvious from Eq. (6.26)), but the relative sign is now positive. The free quark approximation gives a non-vanishing continuum contribution. This is always good news. The bad news is that the lowest lying excited state (the off-diagonal transition), even after smearing, is not well-described by the free-quark curve. On average, the free-quark curve falls below the actual value of the (smeared) spectral density. To make a relatively precise prediction we would have to go to such high values of $T$ where the power expansion, truncated at a couple of the first terms, fails. If we stay inside the former fiducial domain, we with necessity overestimate the value of the kinetic energy in the ground state. The outcome of the sum-rule calculation includes a positive and rather significant contamination from the first excitation. The extent of this contamination is clear from Fig. 3. Instead of $3\omega/4$ we get roughly $1.5 \cdot 3\omega/4$. The sum is very close to the exact result, $3\omega/2$, but since the divergences go in the opposite directions, the difference, instead of being zero in accord with the virial theorem, is quite large.

What lessons can be inferred for the QCD calculations? Although the toy model discussed above is far too simple to closely follow the QCD pattern, some features are still rather close. The virial partner of $\vec{\pi}^2$ in QCD is $\vec{E}$. It is quite obvious that the sum rule for $\vec{E}$ does not have the free-quark graph. In other words, just like in the case of $\langle E_{\text{pot}} \rangle$ the free-quark continuum vanishes \[10\]. This was believed to be a positive feature of the analysis. We see now that it is not. We expect a large physical spectral density in close proximity to the ground state contribution (caused by off-diagonal transitions), and of the opposite sign. To suppress it one would need to go to such high values of the Borel parameter where – alas – we cannot go without losing control over the power expansion. The screening of the ground state contribution by the off-diagonal transitions results in an underestimated value of $\langle g\vec{E} \rangle$, which through the virial theorem leads to an underestimated value of $\vec{\pi}^2$.

At the same time, and for the same reason, the calculation of Ref. [42], which follows the standard pattern, is expected to give an overestimated prediction for $\vec{\pi}^2$. Whether the genuine value of $\mu_\pi^2$ is closer to the one edge of the interval or to the other – we do not know. It seems clear, however, that the error bars in Eqs. (3.20) and (3.22) reflect only uncertainties in various parameters entering the sum sum-rule calculations and do not reflect the fact that the continuum saturation in the problem at hand is likely to deviate from the standard pattern.\[7\]

\[7\] The authors of Ref. [2] estimated the uncertainty of the continuum model reflected in Eq. (3.24) at the 30% level, i.e. 0.15 GeV$^2$. 
The actual value of $\vec{\pi}^2$ lies, most probably, somewhere in between the two estimates. Unfortunately, the potential interaction is strictly speaking non-existent in QCD, let alone the quadratic potential of the oscillator model. Therefore, mechanically averaging the estimates (3.20) and (3.22) is unlikely to suppress the off-diagonal transitions and yield a more reliable approximation to the genuine value of $\mu_\pi^2$. In no way do we suggest such an averaging. It is clear that more sophisticated models for the off-diagonal transitions are needed in both calculations. Once we realize that the standard continuum approximation is inadequate, developing a better one does not seem to be a hopeless problem.

In summary, it seems safe to say, that the result obtained in Ref. [46] significantly underestimates the kinetic operator. The sum rule technology for $\vec{\pi}^2$ seems to operate in a somewhat better environment than that in the case of $\vec{E}$. This conclusion is indirectly supported by the fact that the alternative analysis relying on the exact QCD inequalities (plus the data on the slope of the Isgur-Wise function and the observed spectrum of the excited states) favors larger values of $\mu_\pi^2$.

### 7 OPE for Inclusive Weak Decays

The number of applications of the heavy quark expansion dramatically grows when one includes external (non-QCD) interactions of the heavy quark, e.g. electromagnetic, weak and so on. A variety of problems can be formulated in the language of the OPE – inclusive heavy quark decays are the best-studied example of this kind. In this case, the operator product expansion is built in a slightly different way. The expansion parameter is not necessarily $1/m_Q$, but rather it is regulated by the energy release in the problem at hand, or by other external parameters. The general consideration runs parallel to the treatment of $\sigma(e^+e^- \to \text{hadrons})$. One describes the decay rate into an inclusive final state $f$ in terms of the imaginary part of a forward scattering operator (the so-called transition operator) evaluated to second order in the weak interactions [71, 72]

$$
\text{Im} \hat{T}(Q \to f \to Q) = \text{Im} \int d^4x \, i \, T \left( \mathcal{L}_W(x) \mathcal{L}_W^\dagger(0) \right) \tag{7.1}
$$

where $T$ denotes the time ordered product and $\mathcal{L}_W$ is the relevant weak Lagrangian at the normalization point higher or about $m_Q$. The space-time separation $x$ in Eq. (7.1) is fixed by the inverse energy release. If the latter is sufficiently large in the decay, one can express the non-local operator product in Eq. (7.1) as an infinite sum of local operators $O_i$ of increasing dimensions. The width for $H_Q \to f$ is then obtained by averaging $\text{Im} \hat{T}$ over the heavy-flavor hadron $H_Q$,

$$
\frac{\langle H_Q | \text{Im} \hat{T}(Q \to f \to Q) | H_Q \rangle}{2M_{H_Q}} \propto \Gamma(H_Q \to f) = 
$$

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with $V_{\text{CKM}}$ denoting the appropriate combination of the CKM parameters. A few comments are in order to elucidate the content of Eq. (7.2).

(i) The parameter $\mu$ in Eq. (7.2) is the normalization point, indicating that we explicitly evolved from $m_Q$ down to $\mu$. The effects of momenta below $\mu$ are lumped into the matrix elements of the operators $O_i$.

(ii) The coefficients $\tilde{c}_i^{(f)}(\mu)$ are dimensionful, they contain powers of $1/m_Q$ that go up with the dimension of the operator $O_i$. Sometimes it is convenient to introduce dimensionless coefficients $c_i^{(f)}(\mu) = m_d^{-3} \tilde{c}_i^{(f)}(\mu)$ where $d_i$ denotes the dimension of the operator $O_i$. The dimensionless coefficients $c_i^{(f)}$ depend on the ratio of final-to-initial-state quark masses. Using the normalization introduced in Eq. (7.2), one obtains on dimensional grounds

\[
\tilde{c}_i^{(f)}(\mu) \frac{1}{2M_{H_Q}} \langle H_Q | O_i | H_Q \rangle (\mu) \sim \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}^{d_i-3}}{m_Q^{d_i-3}} \cdot \frac{\alpha_s \mu^{d_i-3}}{m_Q^{d_i-3}} \right)
\]

with $d_i$ denoting the dimension of operator $O_i$. The contribution from the lowest-dimensional operator obviously dominates in the limit $m_Q \to \infty$.

(iii) It seems natural then that the expansion of total rates can be given in powers of $1/m_Q$. The master formula (7.2) holds for a host of different integrated heavy-flavor decays: semileptonic, nonleptonic and radiative transitions, CKM-favored or suppressed, etc. For semileptonic and nonleptonic decays, treated through order $1/m_Q^3$, it takes the following form:

\[
\Gamma(H_Q \to f) = \frac{G_F^2 m_Q^5}{192 \pi^3} |V_{\text{CKM}}|^2 \times \left[ c_3^{(f)}(\mu) \frac{\langle H_Q | Q \bar{Q} | H_Q \rangle (\mu)}{2M_{H_Q}} + c_5^{(f)}(\mu) m_Q^{-2} \frac{\langle H_Q | Q \bar{Q} | H_Q \rangle (\mu)}{2M_{H_Q}} + \sum_i c_{6,i}^{(f)}(\mu) m_Q^{-3} \frac{\langle H_Q | (Q \Gamma q)(\bar{Q} \Gamma \bar{q}) | H_Q \rangle (\mu)}{2M_{H_Q}} + \mathcal{O}(1/m_Q^4) \right].
\] (7.3)

We pause here to make a few explanatory remarks on this particular expression. First, the main statement of OPE is that there is no correction of order $1/m_Q$ [73]. This is particularly noteworthy because the hadron masses, which control the phase space, do contain such a correction: $M_{H_Q} = m_Q \left( 1 + \Lambda/m_Q + \mathcal{O}(1/m_Q^2) \right)$; the parameter $\Lambda$, different for different hadrons, does not enter the width! The reason for the absence of the $1/m_Q$ correction in the total widths is two-fold: the corrections to the expectation value of the leading QCD operator $\bar{Q}Q(\mu)$ is only $\sim \mu^2/m_Q^2$, and there is no independent QCD operator of dimension 4 for forward matrix elements. Since the coefficients functions are purely short-distance, infrared effects neither can
penetrate into them. The absence of the dimension-4 operator in HQET was noted in [72, 74].

A physically more illuminating way to realize the absence of corrections of order $1/m_Q$ is to realize that the bound-state effects in the initial state (mass shifts, etc.) do generate corrections of order $1/m_Q$ to the total width – as does hadronization in the final state. Yet local color symmetry demands that they cancel against each other, as can explicitly be demonstrated in simple models. It is worth realizing that this is a peculiar feature of QCD interactions – other dynamical realizations of strong confining forces would, generally, destroy the exact cancelation.

Second, the leading nonperturbative corrections are $\sim O(1/m_Q^2)$, i.e. small in the total decay rates for beauty hadrons: $(\mu/m_b)^2 \sim$ few % if $\mu \lesssim 1$ GeV. The first calculation of the leading nonperturbative corrections in the decays of heavy flavors was done in [73, 75, 76, 35].

Third, the four-quark operators $(\bar{Q}\Gamma_i q)(\bar{q}\Gamma_i Q)$ depend explicitly on the light-quark flavors denoted by $q$. They, therefore, generate differences in the weak transition rates for the different hadrons of a given heavy flavor. Their effects were calculated already in mid-eighties [77].

Fourth, the short-distance coefficients $c_i^{(f)}(\mu)$ in practice are calculated in perturbation theory. However it is quite conceivable that certain nonperturbative effects arise also in the short-distance regime. They are believed to be rather small in beauty decays [78].

Fifth, a new matrix element appearing in OPE, not discussed so far, is the scalar heavy quark density. Its expansion originally established in Ref. [73] takes the form

$$\langle H_Q | \bar{Q} Q | H_Q \rangle = \langle H_Q | \bar{Q} \gamma_0 Q | H_Q \rangle + \frac{\langle H_Q | \bar{Q} \left( \pi^2 + \frac{i}{2} \sigma G \right) Q | H_Q \rangle}{2m_Q^2} + O(1/m_Q^4). \quad (7.4)$$

Since $\langle H_Q | \bar{Q} \gamma_0 Q | H_Q \rangle = 2M_{H_Q}$, the spectator ansatz indeed emerges as the asymptotic scenario universal for all types of hadrons, and holds up to $1/m_Q^2$ corrections. In addition to $\bar{Q}\pi^2 Q$, the second dimension-five operator is the chromomagnetic operator $\bar{Q}i\sigma GQ$. Since $\bar{Q}\tilde{D}^2 Q$ is not a Lorentz scalar, it does not appear independently in Eq. (7.3).

Equation (7.4) is readily obtained in the heavy quark expansion if one uses proper non-relativistic heavy-quark spinors incorporating the Foldy-Wouthuysen transformation. In the context of HQET this was suggested in Ref. [13]. When the Foldy-Wouthuysen transformation is ignored, the straightforward evaluation of the scalar density in HQET leads to incorrect $1/m_Q^2$ terms. Recovering the additional terms required a certain revision of the standard HQET strategy [73, 80]. At present, the full-QCD derivation and that based on HQET are in perfect agreement. We thus

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8Expanding $\langle H_Q | \bar{Q} i\sigma GQ | H_Q \rangle / m_Q^2$ also yields contributions of order $1/m_Q^3$; those are, however, practically insensitive to the light quark flavors.
Figure 4: a) Spectator effect leading to $1/m_b$ correction in the decay width $b \to s + \gamma$. b) A different cut of the same diagram leads to an electromagnetic correction in the hadronic decay width. Terms $1/m_b$ and $1/m_b^2$ cancel out in the sum of two decay widths. The solid dot in the vertex denotes the penguin-induced $b \to s\bar{q}q$ interaction.

have, for example, for the pseudoscalar mesons,

$$
\frac{1}{2M_{PQ}} \langle P_Q | \bar{Q}Q | P_Q \rangle = 1 - \frac{\mu^2}{2m_Q^2} + \frac{3}{8} \frac{M_{\bar{Q}Q}^2 - M_{PQ}^2}{m_Q^2} + \mathcal{O}(1/m_Q^3) \quad (7.5)
$$

The reason for the kinetic operator term to appear is quite transparent. The first two quantities on the right-hand side of the equation represent the mean value of the factor $\sqrt{1 - \vec{v}^2}$ reflecting the time dilation which slows down the decay of the quark $Q$ moving inside $H_Q$ [35, 81, 82].

Finally, Eqs. (7.3)–(7.4) show that the two dimension-five operators do produce differences in $B$ versus $\Lambda_B/\Xi_B$ versus $\Omega_B$ decays of order $1/m_Q^2$. To a small extent they can also differentiate $B$ and $B_s$ via the $SU(3)$ breaking in their expectation values. Differences in the transition rates inside the meson family are generated at order $1/m_Q^3$ by dimension-six four-quark operators. They are usually estimated in the vacuum saturation approximation which – although cannot be exact – represents a reasonable starting approximation. There is an intriguing way to check factorization experimentally [81]: similar four-fermion operators enter semileptonic $b \to u$ transition rates. Moreover, in the heavy quark limit the four-fermion operators populate mainly the transitions into the hadronic states with low energy, and thus show up, for example, in the end-point domain of the lepton spectrum where their relative effect is enhanced. Considering the difference of the decay characteristics of the charged and neutral $B$’s in this domain, one can measure these matrix elements and even feel their scale dependence. Further details regarding the “flavor-dependent” preasymptotic effects can be found in [83].

It is important to keep in mind that the OPE approach discussed in this section implies that all decay channels induced by a given term in the short-distance Lagrangian in Eq. (7.1) are included. It is not enough to consider the states that appear at the free-quark level. The final-state interactions can annihilate, for example, the $\bar{c}c$ quark pair into light hadrons; electromagnetic interaction, if considered, can do the same. Disregarding the channels that can emerge due to such final state interactions can violate the general theorems: even $1/m_Q$ terms can appear in such
incomplete “inclusive” widths. For example, the correction to the $b \rightarrow s + \gamma$ width does have nonperturbative corrections scaling like $1/m_b$ due to the effect of weak annihilation (in mesons, or weak scattering in baryons) of the light quarks with emission of a hard photon in the penguin-induced weak decay $b \rightarrow s\bar{q}q$, see Fig. 4a. This effect would only cancel against the (virtual) electromagnetic correction to the hadronic penguin-induced width, Fig. 4b. [84, 81].

8 Applications of Heavy Quark Expansion

If in the first part of the review we were more concerned with the theoretical foundations, here we pass to applied problems. The number of applications is quite large, and continues to grow. We will limit ourselves to those applications where the heavy quark expansion is exploited as a tool for extracting such fundamental parameter as $|V_{cb}|$, the element of the CKM matrix. Below we dwell on two methods often briefly called inclusive and exclusive approaches.

8.1 $|V_{cb}|$ from the total semileptonic $B$ width

The idea of extracting $|V_{cb}|$ from the inclusive semileptonic transition rate $b \rightarrow c\ell\nu$ is quite evident. If $m_b \rightarrow \infty$ the parton formula becomes exact. Comparing the decay rate in this limit to the theoretical expression one could determine $|V_{cb}|$. In the real world, where $m_b$ is finite, analysis of various pre-asymptotic corrections, which became possible after the advent of the heavy quark expansion, is needed.

Dimensional arguments tell us that the total decay width scales as the fifth power of mass, but not which mass it is. A priori possible choices would be the heavy flavor quark or hadron masses which agree to leading order, $(M_{HQ} - m_Q)/m_Q \sim O(\Lambda/m_Q)$. However, in the case of beauty $M_B^5 \approx 1.5m_b^5$. This difference, which would represent a power-suppressed nonperturbative correction, has to be brought under theoretical control if one wants to extract reliable numerical values for the CKM parameters from the measurement of the total semileptonic $B$ width. In particular, the theoretical uncertainties intrinsic to the pole masses must be exterminated.

The heavy quark expansion yields an unambiguous answer. Not surprisingly, everything is expressed in terms of the quark masses \cite{73, 76, 35}:

$$\Gamma_{sl}(B) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192 \pi^3} \left\{ z_0 \left( 1 - \frac{\mu^2}{2m_b^2} \right) - 2 \left( 1 - \frac{m_c^2}{m_b^2} \right) \frac{\mu_G^2}{m_b^2} \frac{2\alpha_s z_0^{(1)} 3\pi}{m_b^2} + \ldots \right\}$$

(8.1)

where ellipses stand for higher order perturbative and/or power corrections; $z_0, z_0^{(1)}$ are known parton phase space factors depending on $m_c^2/m_b^2$ (see, e.g. \cite{85}). The $1/m_b^2$ power corrections to $\Gamma_{sl}$ are rather small, about $-5\%$, and thus the direct impact of the higher-order power corrections is negligible. One obviously has to be careful in dealing with the quark mass in Eq. (8.1) since it enters in a high power. Our
strategy is to first present a complete theoretical formula which explicitly exhibits theoretical uncertainties. Then we inspect the extracted value for $|V_{cb}|$. Finally, we detail the criticism that sometimes is raised against such an analysis.

The first-order perturbative correction term $z^{(1)}_0 \left( m_c^2 / m_b^2 \right)$ is essentially the same as for $\mu \rightarrow e\bar{\nu}\nu$ known since the mid-fifties [86]. Recently the QCD calculation was improved by computing the second-order [87] and all-order [85] corrections arising due to the running of $\alpha_s$ in the first-order diagrams (i.e. the BLM part of $\alpha_s^k$ corrections). These are presumably dominant terms. They have a small numerical impact on the width, though. For completeness, the second-order BLM terms are included below.

Evaluating Eq. (8.1) we find

$$|V_{cb}| = 0.0419 \left( \frac{\text{BR}(B \rightarrow X_c \ell \nu)}{0.105} \right)^{\frac{1}{2}} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \left( 1 - 0.012 \left( \frac{\mu^2 - 0.5 \text{ GeV}^2}{0.1 \text{ GeV}^2} \right) \right) \times \left( 1 - 0.01 \frac{\delta m_b(\mu)}{50 \text{ MeV}} \right) \left( 1 + 0.006 \frac{\alpha_s^{\text{MS}}(1 \text{ GeV}) - 0.336}{0.02} \right) \left( 1 + 0.007 \frac{\tilde{\rho}^3}{0.1 \text{ GeV}^3} \right). \quad (8.2)$$

Here we have relied on the mass $m_b(\mu)$, see Eqs. (3.14) – (3.15), and on the value of the low-energy coupling $\alpha_s^{\text{MS}}(1 \text{ GeV}) \simeq \alpha_s^V(2.3 \text{ GeV}) = 0.336$ obtained in the dedicated analysis of [31]. The charmed quark mass $m_c(\mu)$ is then obtained using the relation (3.16) which introduces also the parameter $\tilde{\rho}$. Furthermore, $\tilde{\rho}^3$ reflects the dependence on the $1/m_Q^2$ terms. Electroweak corrections are still neglected. It is worth noting that all expressions for the perturbative coefficients depend on the concrete definition of mass and relevant operators. This dependence propagates into numerical expressions, which, however, give the same final result for observables for commensurate input. The numbers shown above correspond to the definitions discussed at length in the preceding sections.

The proper evaluation of the theoretical uncertainties is a real challenge which we now discuss. The last four terms in parentheses in Eq. (8.2) exhibit some of the uncertainties; others will be considered shortly.

(i) As was mentioned, the perturbative corrections are known exactly to first order in $\alpha_s$. Their impact on $|V_{cb}| \sim \left( \Gamma_{\text{sl}}^{-1/2} \right)_{\text{theor}}$ is only about 5%. The (presumably dominant) BLM part of higher-order corrections is now known to all orders; already in the second order the impact is less than 1%. The only remaining uncertainty, thus, is the genuine, non-BLM second-order corrections. They are not known completely, but there are good reasons to believe that they are indeed small.

There exist some enhanced higher-order corrections not related to the running of $\alpha_s$, that are specific to the inclusive widths [29]. They appear if one uses the masses normalized at a high scale $\mu \gtrsim m_b$. They have been accounted for in the analyses [14,28], but were not included in Refs. [88, 89].

In the SV limit, when

$$\xi \equiv (m_b - m_c)/(m_b + m_c) \ll 1,$$
the perturbative corrections to the widths can be related \[89\] to the zero-recoil renormalization of the axial current, which was recently calculated by Czarnecki \[90\] to two loops. In this way one obtains for the second-order non-BLM coefficient

\[
a_2^0(\xi) = \frac{85}{36} - \left(\frac{5}{2} - \ln 2\right) \frac{\pi^2}{9} - \frac{\zeta(3)}{6} + \mathcal{O}\left(\frac{\mu^2}{m_cm_b}, \xi^2\right) \approx 0.179 + \mathcal{O}\left(\frac{\mu^2}{m_cm_b}, \xi^2\right)
\]

\[(8.3)\]

where \(a_i\) are defined in Eq. (8.9) below. If \(\xi = 0\) it is clear that \(a_2^0(\xi)\) is rather small numerically. At the actual value of \(\xi \approx 0.5\) the coefficient \(a_2^0\) is different, of course, but hardly can be large if the appropriate value of \(\mu\) is used. For example, the second-order non-BLM term in \(d\Gamma_{sl}/dq^2\) at \(q^2 = q^2_{\text{max}} = (m_b - m_c)^2\) constitutes approximately \(0.1(\alpha_s/\pi)^2\) for the actual quark masses, according to Ref. \[90\]. The same correction to \(d\Gamma_{sl}/dq^2\) at \(q^2 = 0\) was recently evaluated \(\approx 1.25(\alpha_s/\pi)^2\) in Ref. \[91\], thus demonstrating that the non-BLM effects are moderate. Altogether, we assess a \(\pm 2\%\) uncertainty in \(\Gamma_{sl}\) due to higher-order perturbative corrections, as a conservative estimate. There is an additional uncertainty associated with the experimental error bars in \(\alpha_s\), but it is minor.

(ii) The power corrections in \(\Gamma_{sl}\) scaling like \(\Lambda^2_{\text{QCD}}/(m_b - m_c)^3\), are at the 1% level, i.e. tiny. They emerge from several sources. First, there are \(1/m_b\) corrections in the expectation values of the kinetic and chromomagnetic operators over the actual \(B\) meson state. They are small and, in any case, covered by the existing uncertainties in the values of the condensates. The “genuine” corrections are given by the local Darwin term \(\rho_3^D\) and the “spin-orbital” expectation value \(\rho_3^{LS}\) \[10\] appearing in Eq. (2.11). The latter, in turn, appears also as a \(1/m_b\) correction in the expectation value of \(\langle B|\bar{b}\gamma^\mu\gamma^5Gb|B\rangle\) which enters the \(1/m_b^2\) term in the widths. Thus, this effect is also negligible. It is worth noting that \(\rho_3^{LS}\) is expected to be suppressed in the ground-state mesons \[10\].

The largest direct effect is due to the Darwin term; relying on the approximation \(m_c^2/m_b^2 \ll 1\) and using the calculations of Refs. \[92, 40\] one arrives at the estimate \[
\Gamma_{sl}(b \rightarrow c) \propto \left(z_0 - 0.01 \frac{\rho_3^D}{0.1 \text{ GeV}^3}\right) \approx z_0 \left(1 - 0.02 \frac{\rho_3^D}{0.1 \text{ GeV}^3}\right).
\]

\[(8.4)\]

Finally, if \(m_c\) is related to \(m_b\) via the mass formula \(\text{(3.16)}\) – so far, the most accurate method – one has an indirect dependence on \(\rho_3^D\) and \(\bar{\rho}^3\) through \(m_b - m_c\). Even though the value of \(\rho_3^D\) is reasonably well estimated, the overall dependence on it practically cancels out in \(\Gamma_{sl}\) in this method. As a result, the extracted value of \(|V_{cb}|\) is sensitive only to non-local correlators \(\rho_3^{\pi\pi}\) and \(\rho_3^S\) as far as the \(1/m_Q^3\) terms are concerned. Both \(\rho_3^{\pi\pi}\) and \(\rho_3^S\) are positive, see \[10\]. The corresponding shift in \(\Gamma_{sl}\) is thus, probably, negative but below 2%.

\[\text{There is a minor disagreement in the relevant expressions in the two works. Numerically it is negligible.}\]
(iii) The calculation of the width could be affected by violations of duality, which are conceptually related to the asymptotic nature of the OPE expansions. This is the least understood ingredient of the theoretical analysis. Some model considerations are given in Ref. [78]. With the energy release in the semileptonic widths \( \gtrsim 3.5 \text{ GeV} \), the duality violation is expected to be negligible, below 1% level, and not to exceed the effect of \( 1/m_Q^3 \) corrections.

Assembling all pieces together and assuming, conservatively, that at present \( |\delta m_b| < 50 \text{ MeV} \) and \( |\bar{\rho}| < 0.1 \text{ GeV}^3 \), we arrive at the model-independent estimate

\[
|V_{cb}| = 0.0419 \left( \frac{\text{BR}(B \to X_c \ell \nu)}{0.105} \right)^{\frac{1}{2}} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \times \\
\left( 1 - 0.012 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2} \right) \cdot (1 \pm 0.015_{\text{pert}} \pm 0.01 m_b \pm 0.012),
\]

where the last error reflects \( m_Q^{-3} \) and higher power corrections, including possible deviations from duality.

8.1.1 \( \Gamma(B \to X_u \ell \nu) \)

Similar to the treatment of \( \Gamma(B \to X_c \ell \nu) \) it is straightforward to relate the value of \( |V_{ub}| \) to the total semileptonic width \( \Gamma(B \to X_u \ell \nu) \) [28]:

\[
|V_{ub}| = 0.00465 \left( \frac{\text{BR}(B^0 \to X_u \ell \nu)}{0.002} \right)^{\frac{1}{2}} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \cdot (1 \pm 0.025_{\text{pert}} \pm 0.03 m_b) .
\]

The dependence on \( \mu_\pi^2 \) is practically absent here.

Accurate measurement of the inclusive \( b \to u \ell \nu \) width is difficult and for a long time seemed unfeasible. However, recently ALEPH announced the first direct measurement [93]:

\[
\text{BR}(B \to X_u \ell \nu) = 0.0016 \pm 0.0004 .
\]

It is not clear how reliable and model-independent are the quoted error bars in this complicated analysis. It certainly will be clarified in the future. Taking these numbers at their face value, one would arrive at the model-independent result

\[
|V_{ub}|/|V_{cb}| = 0.098 \pm 0.013 .
\]

The theoretical uncertainty in translating \( \Gamma(B \to X_u \ell \nu) \) into \( |V_{ub}| \) [28] is a few times smaller than the above experimental one.

8.1.2 Caveat

Deviation from duality is the most vulnerable element of the theoretical prediction. Reliable information on this aspect of QCD is scarce. The only model specifically
designed to address the issue is presented in Ref. [78]. If it can be relied upon, at least for orientation, we can safely neglect deviations from duality in the semileptonic decays of $B$ mesons, but not in those of $D$ mesons. And indeed, a parallel analysis of $\Gamma(D \to X_s \ell \nu)$ [82, 94, 92], along the lines discussed above, shows that the expression for $\Gamma(D \to X_s \ell \nu)$, similar to Eq. (8.1), falls short of the experimental number roughly by a factor of two, provided that the modern values of the fundamental parameters ($m_c, \mu^2_\pi$ and so on) are used.

8.1.3 Why has $\Gamma_{sl}(B)$ been sometimes discarded in the quest for $|V_{cb}|$?

The status of the radiative corrections to the widths had been the subject of some controversy recently, but finally the issue seems to be settled. The reason for apparently contradicting opinions was related, once again, to the problem of the heavy quark mass in the context of the heavy quark expansion. We will briefly sketch the problem and explain the answers.

The inclusive width can be expressed in terms of any well-defined mass parameter. Its perturbative component through a certain order can conveniently be given in terms of the pole masses. Yet they are not defined with the necessary accuracy so as to include power-suppressed corrections. As discussed before, the pole mass contains long-distance contributions of order $\Lambda_{QCD}$. With the width being a short-distance quantity we can then infer that the indefiniteness $\sim \Lambda_{QCD}$ in the quark pole mass has its counterpart in an irreducible uncertainty $\sim \Lambda_{QCD}/m_Q$ in the perturbative corrections if those are evaluated with pole masses as input. This conclusion has been proven in the most general terms by virtue of OPE where the emergence of the $1/m_Q$ infrared renormalon was noted [22]. It was also substantiated by the perturbative calculation. A concrete analysis in the framework of the BLM resummation was performed later in [85]. Consider the perturbative expansion of $\Gamma_{sl}$ in terms of the pole masses:

$$\Gamma_{sl} = \frac{G_F^2 m_b^5}{192 \pi^3} |V_{cb}|^2 \cdot z_0 \left( \frac{m_c^2}{m_b^2} \right) \cdot c_3^{(sl)}, \quad c_3^{(sl)} = \left( 1 + a_1^{(p)} \frac{\alpha_s}{\pi} + a_2^{(p)} \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots \right)$$ (8.8)

One finds that $a_1^{(p)} \simeq -1.7$ is followed by a huge coefficient [87, 85] $a_2^{(p)} \simeq -10$ to $-20$ for $b \to c$ (the exact value depends on the choice of the expansion parameter $\alpha_s$) or even $a_2^{(p)} \simeq -30$ for $b \to u$. If this were the end of the story, this fact would mean that the $\alpha_s^2$ corrections by themselves reduce the width significantly, namely, by about 10%, suggesting, at first sight, that the unknown third (and higher) order corrections are sizeable. In that case one could argue that the theoretical uncertainties in extracting $|V_{cb}|$ from $\Gamma_{sl}(B)$ are considerably larger than stated above. It would be natural to be concerned [87] whether such an extraction can be trusted at all!

The key point that eliminates the concern, is as follows. When one calculates the radiative correction factor $c_3^{(sl)}$ through order $k$, one has to evaluate the pole
mass likewise to that order – since the pole mass is only a formal perturbative construction. Then one finds that $m^{(k)}_{\text{pole}}$ receives sizeable increase as well: $m_{\text{pole}}$ breaths as $k$ changes. The shifts are correlated so that the width remains almost the same numerically.

In doing so one must be careful always to use the very same perturbative approximation both in masses $m^{(k)}_{\text{pole}}$ and in the coefficient $c_3$. Speaking theoretically, the perturbative approximations to the width computed in such a strictly correlated way contain no uncertainty $\sim O(\Lambda_{\text{QCD}}/m_Q)$. Uncertainties $\sim O(\Lambda_{\text{QCD}}^2/m_Q^2)$ still remain due to spurious terms $\sim (\delta m_{\text{pole}}/m_Q)^2$ having no OPE interpretation. On the practical side, the latter problem emerges at the one-percent level only – unless one attempts to continue to too high orders.

If one, however, exploits the short-distance masses, say the masses normalized at the scale $\sim 1$ GeV, all these problems are completely avoided and one does not have to rely on what seems to be a numerical miracle. Indeed, one can use a fixed number for $m_{b,c}$ which, unlike $m_{\text{pole}}$, does not breath. Moreover, one finds then that

$$\Gamma_{\text{sl}} = \frac{G_F^2 m_b^2(\mu)}{192\pi^3} |V_{cb}|^2 \cdot z_0 \left( \frac{m_c^2(\mu)}{m_b^2(\mu)} \right) \cdot \left( 1 + a_1(\mu) \left( \alpha/s/\pi \right) + a_2(\mu) \left( \alpha/s/\pi \right)^2 + ... \right) \quad (8.9)$$

with $a_1(\mu) \approx -1$, $a_2(\mu) \approx 1$, etc. The exact values of $a_k(\mu)$ depend on the scheme used. The same is true for $m_{b,c}(\mu)$. The scheme dependence affects the overall result for $\Gamma_{\text{sl}}$ only at a percent level. Such variations are unavoidable since the perturbative series are truncated.

To reiterate, the suspicion of a large and uncontrollable impact of the perturbative corrections on the absolute values of the semileptonic widths, quite popular in the literature in the past, was mainly due to theoretical subtleties with the pole mass. In particular, it was tacitly assumed that the quark pole mass has an unambiguous value, exact up to a hundred MeV, or so. Then it was observed that:

(a) It is difficult to accurately extract $m_{b}^{\text{pole}}$ from experiment. In any particular calculation one can identify effects left out, which can change its value by $\sim 200$ MeV. This uncertainty leads to a theoretical error $\delta_1 \approx 10\%$ in $\Gamma_{\sl}(B)$.

(b) When routinely calculating $\Gamma_{\sl}(B)$ in terms of the pole masses, there are significant higher order corrections $\delta_2 \approx 10\%$.

Thus the conclusion was made: $\Gamma_{\sl}(B)$ cannot be calculated with accuracy better than $\sim 20\%$, and, correspondingly, at best $\delta |V_{cb}|/|V_{cb}| \sim 10\%$.

Both observations (a) and (b) above are correct, beyond any doubt. If one more step is taken, however, the conclusion is invalidated – one should take into account the fact that the origin of these two uncertainties is actually the same, and therefore they practically cancel each other.

Let us mention that the problems associated with the pole mass sometimes surface in a superficially different form, leading (in the past) to the same feeling, that $\delta |V_{cb}|/|V_{cb}|$ is as large as $\sim 10\%$. Quite often, inflated error bars in $m_b$ are quoted, see e.g. [96], where the uncertainty in $m_b$ is set 200 to 300 MeV. One may
notice that this variation actually reflects the change in the values of $m_b$ emerging in different perturbative definitions of $m_b$. Such a procedure is like comparing a quantity renormalized in the MS and $\overline{\text{MS}}$ schemes at the same scale without including the non-trivial translation between those two definitions. The large perturbative uncertainty $\sim 20\%$ in the value of $\Gamma_{\text{sl}}$ in [96] was deduced not from the analysis of the absolute prediction for the semileptonic widths. Instead, it was based on a comparison of the perturbatively-improved calculation of $\Gamma_{\text{sl}}$ to an ad-hoc quantity dubbed $\Gamma_{\text{tree}}$; the latter was supposed to represent a numerical estimate of $\Gamma_{\text{sl}}$ before any radiative corrections are included. However, $\Gamma_{\text{tree}}$ involves the fifth power of the mass which was presumed to be $m_{\text{pole}}$ in [96]. Then everything depends on what is the value of $m_{\text{pole}}$ in the tree-level calculations. The large uncertainty stated in [96] can be traced back to the assumption that this tree-level mass is given by the actual all-order sum of the perturbative series emerging when one attempts to calculate the would-be pole mass of the $b$ quark in terms of $\bar{m}_b(m_b)$. This procedure relies on summation of the series similar to Eq. (3.7), which is particularly unnatural in the context of the tree-level evaluation. A finite value for this tree-level mass, $\simeq 5.05 \text{ GeV}$, used in [96], was due to a trick with a non-summable series, a principal-value prescription for the Borel integral, and so was the ratio $\Gamma/\Gamma_{\text{tree}} = 0.77 \pm 0.05$. Values other than 0.77 could have been obtained as well, invoking other prescriptions. In any case, it is clear that such definition of $\Gamma_{\text{tree}}$ is not appropriate for evaluating the impact of the perturbative corrections.

The epilogue of the long story with the determination of $|V_{cb}|$ from the inclusive width can be summarized as follows. After all refinements and thorough analysis of the corrections, we are basically back to square one. The expression for $|V_{cb}|$ given in Ref. [44] has changed by less than one percent (if one inputs the same values of the experimental parameters as in [44]). What became clearer in the last two years is the acceptable range of the underlying key QCD parameters entering the heavy quark expansion, and the size of theoretical corrections.

8.2 $|V_{cb}|$ from $\Gamma(B \to D^*\ell\nu)$ at Zero Recoil

Introduction of the universal Isgur-Wise function [6] was a crucial step in the evolution of heavy quark theory. One aspect is of a special significance in applications: the fact that it is normalized to unity at zero recoil [4, 5, 6]. As was emphasized later (see e.g. [77]) one can exploit this feature for determinations of $|V_{cb}|$ from the exclusive $B \to D^*$ semileptonic transition extrapolated to zero recoil. To this end one measures the differential rate, extrapolates to the point of zero recoil and gets the quantity $|V_{cb}F_{D^*}(0)|$, where $F_{D^*}$ is the axial $B \to \ell\nu D^*$ formfactor. To extract $|V_{cb}|$ it is necessary to know $|F_{D^*}(0)|$, which, although close to unity, still deviates from unity due to various perturbative and nonperturbative corrections. In the real world

$$F_{D^*}(0) = 1 + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right) + \delta_1/m^2 + \delta_1/m^3 + ...$$
1 + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right) + \mathcal{O}\left(\frac{1}{m_c^2}\right) + \mathcal{O}\left(\frac{1}{m_s m_b}\right) + \mathcal{O}\left(\frac{1}{m_b^2}\right) + \ldots \quad (8.10)

The absence of $1/m_Q$ corrections in $|F_{D^*(0)}|$ was noted in passing in Ref. [5]. This fact was cast in the form of a theorem by Luke [74]. This is nothing but the heavy-quark analog of the Ademollo-Gatto theorem for the $SU(3)_f$ breaking effects which is routinely exploited in determinations of $|V_{us}|$ from $K \to \pi\ell\nu$ and semileptonic hyperon decays.

The task of precision determination of $|V_{cb}|$ from the exclusive transition requires a detailed dynamical analysis of various pre-asymptotic corrections in Eq. (8.10). The perturbative part, albeit technically complicated, is at least conceptually transparent. The theory of $1/m_Q^2$ corrections is more challenging.

The urgent need in evaluation of the $1/m_Q^2$ corrections in Eq. (8.10) for practical purposes was realized quite early [16]. In these days the theory of the power corrections in heavy quarks was immature, our knowledge was scarce, so that it was hard to decide even the sign of $\delta_{1/m^2}$. The general impression was that this deviation is quite small, $\lesssim 2\%$ [15]. At the present stage we learned much more about the nonperturbative $1/m_Q^2$ and perturbative corrections. We discuss them in turn, and then proceed to numerical analysis and survey of the literature. At first, a brief excursion into the theory is undertaken to sketch the main ingredients of the direction where most of the advances have been achieved – the sum rules for heavy flavor transitions [44, 10]. Although related to those presented in Sect. 4, these sum rules are not identical (with a few exceptions, see Sect. 8.2.2) since we account for $1/m_Q$ effects. They express the moments of the inclusive probabilities in terms of the quark masses and the expectation values of certain local operators. Since the transition probabilities are non-negative, the sum rules lead to constraints on the exclusive form factors.

### 8.2.1 Heavy flavor sum rules: generalities

The sum rules are derived in QCD using the standard methods of the short-distance expansion [10]. One starts the analysis from the forward transition amplitude

$$T^{(12)}(q_0; \vec{q}) = \frac{1}{2M_B} \int d^3x \, dx_0 e^{i\vec{q}_0 \cdot x_0} \langle B|c\Gamma^{(1)} b(x), \bar{b}\Gamma^{(2)} c(0)]|B\rangle \quad (8.11)$$

where $\Gamma^{(1,2)}$ are some spin structures or, more generally, local operators. The transition amplitude contains a lot of information about the decay probabilities. As usual in QCD, one cannot calculate it completely in the physical domain of $q$. The amplitude (8.11) has several cuts corresponding to different physical processes [72].

The discontinuity at the physical cut $q_0 < M_B - \sqrt{M_B^2 + \vec{q}^2}$ describes the inclusive decay probabilities at a given energy released into the final hadronic system. The cut continues further than the domain accessible in the actual decays, see Fig. 5.
Figure 5: Cuts of the transition amplitude in the complex $q_0$ plane. The physical cut for the weak decay starts at $q_0 = M_B - (M^2_D + \vec{q}^2)^{1/2}$ and continues towards $q_0 = -\infty$. Other physical processes generate cuts starting near $q_0 = \pm(M^2_{Bc} + \vec{q}^2)^{1/2}$ (one pair); another pair of cuts originates at close values of $q_0$. An additional channel opens at $q_0 \gtrsim 2m_b + m_c$.

In QCD we calculate the amplitude (8.11) away from all its cuts. Essentially, it can be expanded in the inverse powers of the distance from the physical cut, $\epsilon$:

$$\epsilon = M_B - \sqrt{M^2_D + \vec{q}^2} - q_0 = m_b - \sqrt{m^2_c + \vec{q}^2} + \delta(q^2) - q_0,$$

where

$$\delta(q^2) = (M_B - m_b) - (\sqrt{M^2_D + \vec{q}^2} - \sqrt{m^2_c + \vec{q}^2}) \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{m_Q}\right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_Q}\right).$$

The “off-shellness” $\epsilon$ must be chosen in such a way that that $|\epsilon|$ and $|\epsilon \cdot \arg \epsilon| \gg \Lambda_{\text{QCD}}$, but, simultaneously, $|\epsilon| \ll \sqrt{m^2_c + \vec{q}^2}$, $m_b$. The second requirement allows us to “resolve” the contributions of the separate cuts.

How does the deep Euclidean expansion of $T^{(12)}$ help constrain the amplitude in the physical domain of real $\epsilon$ lying just on the cut? The dispersion relation immediately tells us that the coefficients in the expansion of the amplitude in powers of $1/\epsilon$ are given by the corresponding moments of the spectral density in $T^{(12)}$,

$$T^{(12)}(\epsilon; \vec{q}) = \frac{1}{\pi} \int \frac{\text{Im}T^{(12)}(\epsilon'; \vec{q})}{\epsilon' - \epsilon} d\epsilon' = -\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{1}{\epsilon^{k+1}} \int \text{Im}T^{(12)}(\epsilon'; \vec{q}) \epsilon^k d\epsilon'.$$

Various subtleties going far beyond the scope of the present review, are discussed in Refs. [10] and [78].

On the other hand, we can build the large-$\epsilon$ expansion of the transition amplitude per se, treating it as the propagation of the virtual heavy quark submerged into a soft medium. The expansion takes the general form

$$T^{(12)}(q_0; \vec{q}) = \frac{1}{2M_B} \int d^3x d\sigma \bar{b}(x) \Gamma^{(1)} \left( m_c + iD - \vec{q} \right) \frac{1}{m^2_c - (iD - q)^2 - \frac{1}{2} \sigma G} \Gamma^{(2)} b(0) |B\rangle =$$

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\begin{align*}
\langle \Gamma^{(1)}(m_c + m_b \gamma_0 - q + \not{q}) \rangle & = \\
\langle \Gamma^{(1)}(m_c + (E_c + \epsilon - \delta(q^2)) \gamma_0 + \not{q} \gamma + \not{q}) \rangle \times \\
\sum_{n=0}^{\infty} \left[ \frac{2(E_c + \epsilon - \delta(q^2)) \pi_0 + 2 \not{q} \pi + \pi^2 + \frac{i}{2} \sigma G}{-(\epsilon - \delta(q^2))(2E_c + \epsilon - \delta(q^2))^{(n+1)}} \right]^n \Gamma^{(2)} \rangle \). \tag{8.14}
\end{align*}

Here \( E_c = \sqrt{m_c^2 + \vec{q}^2} \), and we use the short-hand notation

\[
(... = \frac{1}{2M_B} \int d^3x \, dx_0 \, e^{ix_0 \vec{q}} \langle B| \bar{b}(x) ... b(0)|B \rangle \). 
\]

In the first equation we used the full QCD fields and then passed to the low-energy fields according to Eqs. (2.3)–(2.5). Picking up the corresponding term \( 1/\epsilon^{k+1} \) in the expansion over \( 1/\epsilon \) at \( \epsilon \gg \Lambda_{\text{QCD}} \) and evaluating the expectation value of the resulting local operators (e.g., \( \bar{b}(0) \pi \mu \pi b(x) \equiv \delta^4(x) \bar{b} \pi \mu b(0) \)), one gets the sum rules sought for. Taking \( k = 0 \) yields the sum rule for the equal-time commutator of the currents \( \bar{c} \Gamma^{(1)} b \) and \( \bar{b} \Gamma^{(2)} c \); \( k = 1 \) selects the commutator for the time derivative, etc. After this brief general overview we now outline a few concrete applications \[44, 10\].

### 8.2.2 Sum rules for \( \mu_{\pi}^2 \) and \( \mu_G^2 \)

Considering the semileptonic transitions driven by the pseudoscalar weak current \( J_5 = \int d^4x \, (\bar{c} i \gamma_5 b)(x) \) (i.e., at zero recoil, \( \vec{q} = 0 \)) one obtains the sum rule \[10\] for the structure function \( w^{(5)} \)

\[
\frac{1}{2\pi} \int_0^\mu w^{(5)}(\epsilon) \, d\epsilon = \left( 1/2m_c - 1/2m_b \right)^2 \left( \mu_{\pi}^2(\mu) - \mu_G^2(\mu) \right) \). \tag{8.15}
\]

The normalization point \( \mu \) of \( \mu_{\pi}^2 \) and \( \mu_G^2 \) is introduced through the cut-off of the integral on the left-hand side. Since the structure functions are non-negative, one arrives at the conclusion that \( \mu_{\pi}^2(\mu) \geq \mu_G^2(\mu) \), which is the field-theoretic analog of the difference of the sum rules \[1.15\] and \[1.14\].

The sum rule \[1.14\] for \( \mu_G^2 \) can be also easily obtained in this way. For example, one can consider the antisymmetric part (with respect to \( i \) and \( j \)) of the correlator of the vector currents \( (\Gamma^{(1)} = \gamma_i, \, \Gamma^{(2)} = \gamma_j) \). Another possibility is to turn directly to the sum rule for the correlator of the non-relativistic currents \( \bar{c} \pi_j b \) and \( \bar{b} \sigma \pi c \) in the leading order in \( 1/m_Q \). The OPE guarantees that all such relations are equivalent.

\[10\] Corrections in the overall normalization of the current are ignored as irrelevant. The structure functions are defined, e.g. in \[69\].
8.2.3 $F_{D^*}$ at zero recoil

The axial current $\bar{c}\gamma_\mu\gamma_5 b$ produces $D^*$, $D\pi$ and higher excitations in semileptonic $B$ decays at zero recoil. A straightforward derivation yields the following sum rule for this current,

$$|F_{D^*}|^2 + \frac{1}{2\pi} \int_{\epsilon>0} w^A(\epsilon) \, d\epsilon = \xi_A(\mu) - \Delta^A_{1/m^2} - \Delta^A_{1/m^3} + \mathcal{O}\left(\frac{1}{m^4}\right), \quad (8.16)$$

where

$$\Delta^A_{1/m^2} = \frac{\mu_G^2}{3m_c^2} + \frac{\mu_G^2 - \mu_Z^2}{4} \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b}\right), \quad (8.17)$$

and $\langle D^*(\bar{q} = 0)|\bar{c}\gamma_\mu\gamma_5 b|B\rangle = \sqrt{2M_B M_{D^*}} F_{D^*}\epsilon_i$. Furthermore, $w^A$ denotes the corresponding structure function of the heavy hadron excited by the axial current. Its integral describes the contributions from excited charm states with mass $M_i = M_{D^*} + \epsilon_i$, up to $\epsilon_i \leq \mu$:

$$\frac{1}{2\pi} \int_{\epsilon>0} w^A(\epsilon) \, d\epsilon = \sum_{\epsilon_i < \mu} |F_i|^2 \quad (8.18)$$

Contributions from excitations with $\epsilon$ higher than $\mu$ are dual to perturbative contributions and get lumped into the coefficient $\xi_A(\mu)$ of the unit operator, the first term on the right-hand side of Eq. (8.16).

The role of $\mu$ is thus two-fold: in the left-hand side it acts as an ultraviolet cutoff in the effective low-energy theory, and by the same token determines the normalization point for the local operators; simultaneously, it defines the infrared cutoff in the Wilson coefficients.

Equation (8.16) immediately implies

$$|F_{D^*}|^2 = \xi_A(\mu) - \Delta^A_{1/m^2} - \sum_{\epsilon_i < \mu} |F_i|^2 > \xi_A(\mu) - \Delta^A_{1/m^2} \quad (8.19)$$

due to the positivity of $|F_i|^2$. (For a moment we forget about the cubic corrections, $\Delta^A_{1/m^3}$.) The perturbative coefficient $\xi_A(\mu)$ is obtained considering the sum rules (8.16) and (8.15) in perturbation theory. To order $\alpha_s$ [10]

$$\xi_A(\mu) = 1 + 2 \frac{\alpha_s}{\pi} \left[\frac{m_b + m_c}{m_b - m_c} \ln \frac{m_b}{m_c} - \frac{8}{3} + \frac{1}{3} \left(\frac{\mu_G^2}{m_c^2} + \frac{\mu_Z^2}{m_b^2} + \frac{2\mu^2}{3m_c m_b}\right)\right]. \quad (8.20)$$

The most complicated part of the genuine second-order correction in $\xi_A(\mu)$ was calculated in Refs. [10]. It turned out to be small. The two-loop calculation of $\xi_A(\mu)$ was recently completed [100]. The BLM-resummation of the one-loop result was carried out in [51] and placed the numerical value of $\xi_A(\mu)$ somewhere in between the tree-level and one-loop estimates, see Fig. 6. This analysis suggests that at $\mu \approx 0.5\text{ GeV}$, the value of $\eta_A(\mu) \equiv (\xi_A(\mu))^{1/2}$ is $0.99 \pm 0.01$ with the uncertainty coming from the higher-order corrections and terms $\sim (\mu/m_c)^3$. 

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The nonperturbative $1/m_Q^2$ power corrections $\delta_{1/m^2}$ to $F_{D^*}$, Eq. (8.10), are negative. They consist of two terms: the first, $-\frac{1}{2} \Delta_{1/m^2}^A$, is known explicitly. The second, the sum over the excited states, also scales as $1/m_Q^2$, it is negative, but is unknown otherwise. It cannot be calculated without additional dynamical assumptions. $-\delta_{1/m^2}$ exceeds $\mu^2/(6m_c^2) \simeq 0.035$ in magnitude. The sum over the excited states replaces the wavefunction overlap deficit of the simple quantum-mechanical analysis of the transition probabilities. We will return to this, the most difficult part of the analysis, after briefly discussing $1/m_Q^3$ terms in the heavy quark expansion.

It is not difficult to calculate the $1/m_Q^3$ term in the sum rule. Its explicit form depends on the convention for the $D = 5$ operators in Eq. (8.17). Assuming that $\mu^2_\pi$ and $\mu^2_G$ there are understood as the expectation values of the operators $\bar{b}(i\tilde{D})^2b$ and $\bar{b}\frac{i}{2}\sigma G b$ over the actual $B$ mesons rather than the asymptotic states, we get

$$\Delta_{1/m^3}^A = \frac{\rho^3_D - \frac{1}{3} \rho^3_{LS}}{4m_c^3} + \frac{1}{6m_c m_b} \left( \frac{1}{m_c} - \frac{1}{2m_b} \right) (\rho^3_D + \rho^3_{LS}).$$

(8.21)

If the asymptotic values of the $D = 5$ matrix elements are used, the additional $1/m_b^3$ parts (see [10]) come from the expectation values in Eq. (8.17). The $1/m_Q^3$ correction to the inclusive sum of the decay probabilities turns out to be moderate, and we will not dwell on this issue further.

The excited states in the sum rule are higher excitations of $D^*$ and non-resonant $D\pi,...$ states. Their contribution in the right-hand side of Eq. (8.19) is negative. What can be said about its absolute value? Unfortunately, no model-independent answer to this question exists at present. The best we can do today is to assume that the sum over the excited states (cut-off at $\mu$) is a fraction $\chi$ of the local term given by $\mu^2_\pi$ and $\mu^2_G$,

$$\frac{1}{2\pi} \int_{\epsilon>0}^{\mu} w^A(\epsilon) \, d\epsilon = \sum_{\epsilon<\mu} |F_i|^2 = \chi \Delta_{1/m^2}^A,$$

(8.22)
where on general grounds $\chi \sim 1$. Some idea of the value of $\chi$ can be obtained from the $D\pi$ contribution which is readily calculable by virtue of the soft-pion technique. The result exhibits the proper scaling behavior in all relevant parameters \cite{12}. Its absolute value depends on the pion coupling to the heavy mesons. The estimate of this constant was improved \cite{101}. With this improvement, the $D\pi$ contribution to $\chi$ falls in the interval 0.15 to 0.35. It is natural to think that the resonant contribution is several times larger. To be conservative for numerical estimates we accept $\chi = 0.5 \pm 0.5$. Rather arbitrarily we limit $\chi$ by unity on the upper side. Note that the larger the value of $\chi$ the stronger the deviation of $F_{D^*}(0)$ from unity. In this way we arrive at

$$F_{D^*} \simeq \eta_A(\mu) - (1 + \chi) \left[ \frac{\mu^2_G}{6m^2_c} + \frac{\mu^2_\pi - \mu^2_G}{8} \left( \frac{1}{m^2_c} + \frac{1}{m^2_b} + \frac{2}{3m_c m_b} \right) \right] - \delta_{1/m^3}. \quad (8.23)$$

Assembling all pieces together and assuming our standard input for $\mu^2_\pi$ and the low-scale quark masses ($m_c = 1.25 \text{ GeV}$, $m_b = 4.75 \text{ GeV}$) we get \cite{7}

$$F_{D^*} \simeq 0.91 - 0.013 \frac{\mu^2_\pi - 0.5 \text{ GeV}^2}{0.1 \text{ GeV}^2} \pm 0.02_{\text{excit}} \pm 0.01_{\text{pert}} \pm 0.025_{1/m^3}. \quad (8.24)$$

Estimates of the uncertainties in the $O(1/m^2_c)$ corrections and the contributions from the higher excitations are not very firm and reflect a reasonably optimistic viewpoint of the convergence of the heavy quark expansion. Altogether we get

$$F_{D^*} \simeq 0.91 \pm 0.06, \quad (8.25)$$

where the optimistic uncertainty $\pm 0.1 \text{ GeV}^2$ is ascribed to $\mu^2_\pi$.

The uncertainty can presumably be reduced in the future by one percentage point by accurately measuring $\mu^2_\pi$. An additional 0.01 can be removed via a dedicated experimental study of the transitions to the excited $P$-wave states. Using current theoretical technologies, it seems impossible to overcome the 5% barrier of the model-independent theoretical accuracy in this exclusive decay.

In summary, the QCD-based analysis favors a significantly larger deviation of $F_{D^*}(0)$ from unity than those \textit{en vogue} three years ago. The $1/m^2_c$ shift by $-0.035$ is definitely model-independent. Values of $|\delta_{1/m^2}|$ close to the minimal possible scenario 0.035 are achievable only if there are practically no zero-recoil transitions $B \to D\pi, D^{**}, \ldots$ up to the mass range 2.5 to 2.6 GeV, and, additionally, the overall yield of the lowest $P$-wave states in the sum rule for $\mu^2_\pi$ is totally suppressed. Needless to say, that this is hardly possible. In any case such a distinct pattern can also be checked experimentally. Although the present data are not conclusive yet, there is no evidence for such a feature.

\footnote{In perturbation theory $\chi = 1$ in the first order, but this relation changes in higher orders.}

\footnote{The central numerical value here is larger by 0.015 than that cited in [14] for the same $\mu^2_\pi$. 0.01 is due to a shift in the perturbative factor $\eta_A(\mu)$ (0.98 was used in [14]). The remaining 0.005 reflects a different definition of $\mu^2_\pi$ at the level of perturbative corrections.}

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8.2.4 Quantum-mechanical interpretation

The sum rules (8.16) have a transparent interpretation in the framework of conventional quantum mechanics \[10\]. From the gluon point of view the semileptonic decay of the \(b\) quark is an instantaneous replacement of \(b\) by \(c\) quark. The overall probability of the produced state to hadronize to some final state is exactly unity, which corresponds to the first term on the right-hand side of Eq. (8.16). The nonperturbative corrections in the sum rule appear since the normalization of the weak current \(\bar{c}\gamma_\mu \gamma_5 b\) is not unity and depends, in particular, on the external gluon field. Expressing the current in terms of the non-relativistic fields used in quantum mechanics one has, for example:

\[
\bar{c}\gamma_\mu \gamma_5 b \leftrightarrow \sigma_k \left( \frac{(\bar{\sigma} i \vec{D})^2 \sigma_k}{8m_c^2} + \frac{\sigma_k (\bar{\sigma} i \vec{D})^2}{8m_b^2} - \frac{(\bar{\sigma} i \vec{D}) \sigma_k (\bar{\sigma} i \vec{D})}{4m_cm_b} \right) + \mathcal{O}\left( \frac{1}{m^3} \right). \tag{8.26}
\]

The second term immediately yields the correction to the normalization, which is present on the right-hand side of the sum rule. It is curious to note that the first HQET analysis of \(1/m^2\) corrections (see, e.g., \[16, 65\]) does not include the first two terms in the brackets which contain the dominant effect \(\sim 1/m_c^2\). They appear, however, in the approach of the Mainz group \[13\].

8.2.5 Digression in the literature

In view of the practical importance of the value of \(F_{D^*}(0)\) a more detailed comment on the literature seems in order. We will dwell on a combined analysis \[102\] often cited in this context as the analysis with the best accuracy and reliability available today on the theoretical market. Our task is to warn the potential reader of hidden assumptions and inconsistencies.

The basic idea of Ref. \[102\] was supplementing the sum rules of \[44, 10\] by a certain symmetry relation obtained previously \[16\]. In this way one gets rid of the sum over the excited states for the price of expressing the deviation of the formfactor \(F_{D^*}\) from unity in terms of two other unknown zero-recoil formfactors for the vector current in pseudoscalar and vector mesons \(\ell_{P,V}\), and an unknown hadronic correlator \(\lambda^2_G\):

\[
\delta_{1/m^2} = -\left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \left( \frac{\ell_V}{2m_c} - \frac{\ell_P}{2m_b} \right) + \frac{1}{4m_cm_b} \left( -\frac{4}{3}\mu^2_\pi + 6\mu^2_G - \lambda^2_G \right). \tag{8.27}
\]

Here \(\ell_P\) and \(\ell_V\) are defined via the forward matrix elements

\[
\frac{1}{2\sqrt{M_B M_D}}(D|\bar{c}\gamma_0 b|B) = \eta_V \left( 1 - \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 \ell_P \right),
\]

\[
\frac{1}{2\sqrt{M_B^* M_{D^*}}}(D^*(i)|\bar{c}\gamma_0 b|B^*(j)) = \eta_V \delta_{ij} \left( 1 - \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 \ell_V \right), \tag{8.28}
\]

\[53\]
\( \ell_p \), theoretically, observable in the decays \( B \to \tau \ell \nu \). The same sum rules derived in \([14, 11]\) for \( F_{D(*)} \) apply to \( \ell_{PV} \) as well. The parameter \( \lambda_G^2 \) is a sum of three other unknown correlators introduced in \([14, 11]\). The strategy of Ref. \([102]\) was to obtain constraints on the introduced unknown parameters following from the sum rules \((8.16)-(8.15)\), and, then, in turn, to find the allowed interval for \( \delta_{1/m^2} \).

Expressing the phenomenologically relevant formfactor \( F_D^\ast \) in terms of two other unknown formfactors and one unknown correlator is clearly of no help unless a new dynamical input is provided. It was provided in a form of an ad hoc prescription for \( \lambda_G^2 \) which was set to be very small. This step was crucial since without information on \( \lambda_G^2 \) the relation \((8.27)\) provides no constraint.

Another crucial element of the analysis was exploiting the quark model to estimate \( \ell_P \) and \( \ell_V \). The quantities

\[
-\left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 \ell_{PV}
\]

were identified with the deviations from unity of the wavefunction overlap between the charm and beauty states, associated with the \( c, b \) quark mass splitting. The complete relation has, however, the form \([11]\)

\[
\langle \Psi_D | \Psi_B \rangle = 1 - \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 \ell_P + \frac{\mu_\pi^2 - \mu_G^2}{2} \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2
\]

\[
\langle \Psi_{D(*)}^j | \Psi_{B(*)}^i \rangle = \delta_{ij} \left[ 1 - \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 \ell_V + \frac{\mu_\pi^2 + \frac{1}{3} \mu_G^2}{2} \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right)^2 \right], \quad (8.29)
\]

where \( \Psi_{B(*)}, D(*) \) are the normalized wavefunctions for charm and beauty. The terms with \( \mu_\pi^2 \) and \( \mu_G^2 \) were missed in Ref. \([102]\). These terms emerge due to the Foldy-Wouthuysen transformation. Thus, even accepting the quark model for guidance, we observe inconsistency in evaluating the dominant \( 1/m^2_Q \) part of the correction. This led to the surprising statement on the insensitivity of the estimates for \( \delta_{1/m^2} \) to the value of kinetic operator. This insensitivity, together with discarding \( 1/m^3_Q \) corrections, was the reason behind a smaller overall uncertainty in \( F_{D^*} \) than indicated in Eq. \((8.25)\).

Another potential problem of this work refers to the treatment of radiative \( (\alpha_s^k) \) corrections. The problem arises once one addresses the issue of higher-order summation ignoring the basic aspect of OPE. Postulating that the perturbative factor \( \eta_A(\mu) \) in Eq. \((8.23)\) and its vector counterpart \( \eta_V(\mu) \) coincide with the sum of purely perturbative corrections involving all virtual momenta (including those below \( \Lambda_{QCD} \)) we are bound to run into troubles. Physically, this is double-counting since the domain below \( \mu \) was already included in the matrix elements. It is explicit in this approach: the sum rules of \([14, 10]\) are valid only if the perturbative coefficients are

\[13\quad \text{As a matter of fact, its meaning is seen from relation \((8.16)\) in the quantum-mechanical interpretation at } m_b = m_c. \text{ It represents a sum of the transition probabilities in the spin-flip processes.} \]
understood in the Wilsonian sense; they definitely do not hold in the interpretation used in [102]. Formally the resummed series itself tells us of an inconsistency: the resummed values of the perturbative factors $\eta_V$, $\eta_A$ cannot be defined at the level $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_c^2)$; they develop unphysical imaginary parts and so on. This irreducible ambiguity in thus defined $\eta_A$ was estimated to be 2.4% [103]. It is impossible to get rid of it in the analysis [102], since the hadronic parameters $\ell_P$, $\ell_V$ etc. are treated as fixed positive numbers. This fact was pointed out in [103] and admitted in [102].

A practical manifestation of these theoretical problems was later noted in Ref. [104]. It was found that already in the second order the incorporation of the perturbation theory in the approach of [102] suffers from large perturbative corrections $\sim 3-5\%$. Combining these observations it is logical to conclude that the combined analysis [102] loses information rather than adds it and worsens the accuracy. It seems unlikely that following this line of reasoning in the future we could narrow the possible interval for $F_{D^*}$. Evaluating deviations from the symmetry limit for $\ell_P$ and $\ell_V$ is neither easier nor more difficult than evaluating $F_{D^*}$ itself. At the moment there seems to be no ground for more optimistic attitude.

8.2.6 Analyticity and unitarity constraints on the formfactor

Extracting $|V_{cb}|$ from the exclusive decays implies extrapolating data to zero recoil. Near zero recoil statistics in the decays $B \to D^*\ell\nu$ is very limited, and the result for the differential decay rate at this point is sensitive to the way one extrapolates the experimental data to $q^2 = 0$. Most simply this is done through linear extrapolation. Noticeable curvature of the formfactor would change the experimental value for $|V_{cb}F_{D^*}(0)|$. Under the circumstances it is natural to try to get independent theoretical information on the $q^2$ behavior of the formfactor near zero recoil.

Some time ago it was emphasized [105] that additional constraints on the $q^2$ behavior follow from analytic properties of the $B \to D^*$ formfactor considered as a function of the momentum transfer $q^2$, combined with certain unitarity bounds. In particular, in the dispersion integral

$$F(q^2) = \frac{1}{\pi} \int ds \frac{\text{Im} F(s)}{s-q^2}$$

(8.30)

the contribution of the physical $s$-channel domain $q^2 > (M_B + M_{D^*})^2$ is bound since $|F|^2$ describes the exclusive production of $B \bar{D}$ and cannot exceed the total $b\bar{c}$ cross section. The integral (8.30) receives important contribution from the domain below the open $b\bar{c}$ threshold, where $F(q^2)$ has narrow pole-like singularities corresponding to a few lowest $b\bar{c}$ bound states. Introducing a set of unknown residues and making plausible estimates of the positions of these bound states, on the one hand, and adding a small non-resonant subthreshold contribution, on the other hand, Refs. [106] suggested an ansatz for the formfactor in the whole decay domain. Since the residues are unknown and the momentum transfer in the actual decays $B \to D^{(*)}\ell\nu$ varies in a rather narrow range, the advantages of this parameterization over the standard polynomial fit are not clear at the moment.
Surprisingly, a much more stringent (compared to what follows from Ref. \[106\]) relation between the slope and the curvature of the formfactor (i.e., \(F'(q^2)\) and \(F''(q^2)\) at zero recoil) was found in \[107\]. This relation was hastily incorporated in some experimental analyses. Below we argue that one should be extremely cautious in relying on the relation \[107\] due to a hidden assumption which makes the results very vulnerable and unstable.

The suggestion of Ref. \[107\] was two-fold: consider the scalar formfactor for the \(B\to D\) transition, and discard the resonant subthreshold contribution. In the heavy quark limit the scalar formfactor is related to usual axial-vector one \(F(q^2)\). If one neglects the contribution coming from the resonances below the open \(b\bar{c}\) threshold (in this channel they are scalar \(b\bar{c}\) relatives of the \(\chi_\text{bc}\) and \(\chi_\text{c}\) states) and retains only the the non-resonant continuum similar to the one used in \[106\], stringent bounds do indeed follow from the dispersion representation. The non-resonant continuum, however, can be safely disregarded numerically. It is 50 to 100 times smaller than what can be expected from the resonant subthreshold contribution \[108\]. To give an idea why it happens it is sufficient to recall that in the good old pole-dominance models, the subthreshold poles due to \(\chi_\text{bc}\) saturate the formfactor completely. Reinstating the \(\chi_\text{bc}\) subthreshold resonances at appreciable level suppresses the predictive power of the dispersion approach, as is explained in detail e.g. in Refs. \[109\]; no advantage compared to a more careful analysis of \[106\] emerges.

An additional problem one immediately encounters is the necessity of including \(1/m_c\) corrections translating the results for the scalar current into those referring to the axial formfactor. Such corrections to the derivatives over the velocity transfer were evaluated \[110\], and turned out to be very significant even for more stable inclusive transitions. For further comments see Ref. \[108\].

The lesson we would like to draw is as follows: it is not advisable to impose the model-dependent relation \[107\] in experimental extrapolations to zero recoil, where model dependence is undesirable.

9 Challenges in Nonleptonic Beauty Decays

Concluding the review it is impossible to avoid mentioning some unsolved mysteries of the heavy quark theory. Today’s question marks carry the seeds of tomorrow’s advances. Basically there are two problems where our theoretical understanding is lagging behind. Both are related to non-leptonic decays. The most placid solution (which we would prefer, of course) is experiment evolving towards theory, to meet its expectations. If this does not happen, some major adjustments in our theoretical ideas seem to be inevitable.
9.1 Semileptonic branching ratio of the $B$ meson and $\Gamma(B \to c\bar{c}s\bar{q})$

The theoretical attitude to this problem oscillates with time. Twenty years ago it was believed that the parton model would give a sufficiently accurate prediction, $\text{BR}(b \to c\ell\nu) \sim 15\%$. Then it was realized that the $\alpha_s$ radiative corrections were important \[11\]. The present situation is given in Ref. \[112\]. The general tendency associated with the $\alpha_s$ corrections is lowering the value of $\text{BR}(b \to c\ell\nu)$ down to $\sim 11.5$ to $13.5\%$. This is still noticeably higher than the experimental number \[113\]:

$$\text{BR}(B \to X\ell\nu) = 10.43 \pm 0.24\%.$$  \hspace{1cm} (8.31)

Nonperturbative corrections lower the semileptonic branching ratio further; yet being of order $1/m_b^2$ they are numerically quite small, $\Delta_{\text{nonpert}}\text{BR}_{sl}(B) \simeq 0.5\%$, which seems to be insufficient to close the gap between the expectation and the data \[114\].

The issue of the semileptonic branching ratio must be considered in conjunction with the charm yield $n_c$, the number of charm states emerging from $B$ decays. To measure $n_c$ one assigns charm multiplicity one to $D$, $D_s$, $\Lambda_c$ and $\Xi_c$ and two to charmonia. Zero is assigned to the charmless hadronic final state. It is obvious that

$$n_c \simeq 1 + \text{BR}(\bar{B} \to c\bar{c}s\bar{q}) - \text{BR}(\bar{B} \to \text{no charm}).$$  \hspace{1cm} (8.32)

The experimental situation with $n_c$ is as follows. The CLEO group finds $n_c = 1.134 \pm 0.043$ whereas the ALEPH collaboration reports $n_c = 1.23 \pm 0.07$. While both experimental numbers \[113\] are consistent with each other and consistent with a “canonical” value 1.15, the CLEO number clearly favors lower values ($\sim 1.15$) while ALEPH does not rule out $n_c$ as large as 1.3. The statistical average is $n_c = 1.16 \pm 0.04$.

Further reduction of the theoretical prediction for the branching ratio can be provided either by higher-order perturbative corrections (which are amenable to analysis, at least, in principle) or by largely uncontrollable duality violations (usual scape-goat). In the later case it is natural to suspect $\Gamma(\bar{B} \to c\bar{c}s\bar{q})$ since the energy release in $b \to c\bar{c}s$ is not very large.

With the advent of the heavy quark theory the question of the compatibility of the existing theoretical ideas with the data on $\text{BR}(B \to X_c\ell\nu)$ and $n_c$ acquired a solid footing \[114\]. While $n_c \approx 1.15$ came out naturally, the excess of $\text{BR}(B \to X_c\ell\nu)$ was an obvious challenge. Shortly after, the attitude changed. A natural desire to have all problems peacefully settled prevailed, see e.g. the summary talk \[93\] where the general conclusion leans towards the absence of any problem. A combination of two factors was crucial in this respect. Large values of $\alpha_s$ fashionable two or three years ago (corresponding to $\alpha_s(M_Z) = 0.125$ or even higher) enhance the nonleptonic width and, hence, suppress $\text{BR}(B \to X_c\ell\nu)$ down to 11 or even 10.5%. Simultaneously $n_c$ jumps up to 0.125, but since the ALEPH data were newer, their significance was overemphasized, and it was tempting to close one’s eyes on the CLEO data.
The present perception of the issue seems more balanced; it acknowledges the existence of the problem, see e.g. [116, 117]. The only theoretical ingredient one needs to reveal the problem is the statement that the $b \to c\bar{c}s$ channel is to blame for the discrepancy in the semileptonic branching ratio. It is conceivable that $\text{BR}(\bar{B} \to c\bar{c}s\bar{q})$ is actually larger than it is usually inferred from the explicit quark-gluon calculation, either due to higher order $\alpha_s$ corrections or due to deviations from duality. Say, if $\text{BR}(\bar{B} \to c\bar{c}s\bar{q}) \sim 0.25$, rather than 0.15, (which is in line with the fresh CLEO data) and the charmless modes are negligible, this would bring the predicted semileptonic branching ratio pretty close to the observed one, but $n_c$ becomes at least two standard deviations higher than the current average. A possible way out is to assume the charmless modes (say, $b \to s + \text{gluon}$) at the level of 10%. According to Eq. (8.32), this will bring $n_c$ down to the acceptable value. Reference [116] suggests that such high yields of $b \to s + \text{gluon}$ are attainable in QCD, which is extremely unlikely, to put it mildly. According to Ref. [117] new physics is supposed to be responsible.

9.2 Lifetimes of Heavy-Flavor Hadrons

As stated before, differences between meson and baryon decay widths arise already in order $1/m_Q^2$. The lifetimes of the various mesons get differentiated effectively first in order $1/m_Q^3$. A detailed review can be found in [83]; here we will comment only briefly on the issue.

Because the charm quark mass is not much larger than typical hadronic scales one can expect to make only semi-quantitative predictions on the charm lifetimes, in particular for the charm baryons. The agreement of the predictions with the data is surprisingly good. (It is quite possible, though, that future more precise measurements of the $\Xi_c$ and $\Omega_c$ lifetimes might reveal serious deficiencies.)

As far as the beauty lifetimes are concerned there is much less “plausible deniability” when predictions fail. Table [1] contains the world averages of published data together with the predictions. The latter were actually made before data (or data of comparable sensitivity) became available.

Data and predictions on the meson lifetimes are completely and non-trivially consistent. Yet even so, a comment is in order for proper orientation. The numerical prediction is based on the assumption of factorization at a typical hadronic scale which is commonly taken as the one where $\alpha_s(\mu_{\text{had}}) \simeq 1$. The related uncertainty has been emphasized in [119], however the recent QCD sum rule estimates (though carried out in a simplified manner) did not support the conjecture of [119] about the significant impact of nonfactorizable contributions. While there is no justification for factorization at $\mu \sim m_b$, there exists ample circumstantial evidence in favor of approximate factorization at a typical hadronic scale – from the QCD sum rule calculations, to lattice evaluations, to $1/N_c$ arguments. More to the point,

\footnote{It is unlikely that significant new data will appear before the turn of the millennium.}
Table 1: QCD Predictions for Beauty Lifetimes

| Observable | QCD Expectations (1/m_b expansion) | Ref. | Data from [83] |
|------------|-----------------------------------|------|----------------|
| \(\tau(B^-)/\tau(B_d)\) | \(1 + 0.05 (f_B/200 \text{ MeV})^2\) | [84] | 1.04 ± 0.04 |
| \(\bar{\tau}(B_s)/\tau(B_d)\) | \(1 \pm O(0.01)\) | [118] | 0.97 ± 0.05 |
| \(\tau(\Lambda_b)/\tau(B_d)\) | \(\gtrsim 0.9\) | [118] | 0.77 ± 0.05 |

the validity of factorization can be probed in semileptonic decays of \(B\) mesons in an independent way, as was pointed out in [81].

The prediction on \(\tau(\Lambda_b)\) versus \(\tau(B_d)\) seems to be in conflict with the data. However, the experimental situation has not been fully settled yet. The difference between \(\langle \tau(\Lambda_b)/\tau(B_d) \rangle_{\text{exp.}} \simeq 0.77\) and \(\tau(\Lambda_b)/\tau(B_d)\)_{\text{theor.}} \simeq 0.9\) represents a large discrepancy. A failure of that proportion cannot be rectified unless one adopts a new paradigm in evaluating baryonic expectation values of the four-fermion operators. Two recent papers [121, 119] have re-analyzed the relevant quark model calculations and found:

\[
\tau(\Lambda_b)/\tau(B_d) \equiv 1 - \Delta(\Lambda_bB), \quad \Delta(\Lambda_bB) \sim 0.03 \text{ to } 0.12. \tag{8.33}
\]

There are large theoretical uncertainties in \(\Delta(\Lambda_bB)\) since the baryon lifetimes reflect the interplay of several contributions of different signs. Yet one cannot boost the size of \(\Delta(\Lambda_bB)\) much beyond the 10% level: to achieve the latter one had to go beyond a usual description of baryons when light quarks are “soft”. A similar conclusion has been reached by the authors of Ref. [122] who analyzed the relevant baryonic matrix elements through QCD sum rules.

10 Conclusions and Outlook

Heavy quark theory is now a mature branch of QCD. Many practically important applied problems that defied theoretical understanding for years, are now tractable. At the same time, all natural limitations of QCD take place also in the heavy quark theory. The infrared part of dynamics is parameterized rather than solved. Therefore, every new success based on the general properties of the quark-gluon interactions, is a precious asset. The most important stages of the success story are heavy quark symmetry itself, introduction of the universal Isgur-Wise function and combining Wilson’s approach with the heavy quark expansion. The exact inequalities of the heavy quark theory is another link of the same chain.

The most clear-cut recent manifestation of the power of the heavy quark theory is the framework it provides the determination of |\(V_{cb}\)|. It is remarkable that the values of |\(V_{cb}\)| that emerged from exploiting two theoretically complementary approaches are very close. The progress was not for free: it became possible only due to essential refinements of the theoretical tools in the last several years, which prompted us,
in particular, that the zero-recoil $B \to D^*$ formfactor $F_{D^*}$ is probably close to 0.9, significantly lower than previous expectations. The decrease in $F_{D^*}$ and more accurate experimental data which became available shortly after, reduced the gap between the exclusive and inclusive determinations of $|V_{cb}|$. There is a hint in the experimental data that some discrepancy may still persist: the central value of $|V_{cb}|$ from $B \to D^*$ decay seems to be somewhat lower than that from $\Gamma_{\text{sl}}(B)$. Both theoretical values, however, depend to a certain extent on the precise magnitude of $\mu_\pi^2$, as is seen from Eqs. (8.5) and (8.24). In the exclusive and inclusive formulae the dependence is rather similar in magnitude but opposite in sign. It is tempting to think that the actual value of $\mu_\pi^2$ is somewhat larger than the “canonical” 0.5 GeV$^2$. Increasing it by about 0.2 GeV$^2$ makes the two results much closer. We hasten to add, though, that the existing experimental error bars are such that any speculations on $\mu_\pi^2$ are premature. Moreover, theoretical uncertainties in the exclusive formfactor also preclude us from the above adjustment of $\mu_\pi^2$. ($F_{D^*}$ can well be, say, 0.87 even at the canonical value $\mu_\pi^2 = 0.5$ GeV$^2$). Future accurate measurements will, hopefully, allow one to directly measure – through comparison with $\Gamma_{\text{sl}}(B)$ – the exclusive formfactor with accuracy better than that achieved by today’s theory. Thus, we will get a new source of information on intricacies of the strong dynamics in a so far rather poorly known regime.

A large number of applications of the heavy quark theory are based on duality. Although this notion becomes exact at asymptotically high energies, at finite energies (momentum transfers) certain deviations must be present. How fast duality sets in and how large are these deviations are important questions. These and similar questions are among most difficult, with virtually no or very little progress. Determinations of $|V_{cb}|$ we discussed rely on the assumption that approximate duality between the actual hadronic amplitudes and the quark-gluon ones sets in already at the excitation energies $\sim 0.7$ to 1 GeV. While there are no experimental indications so far that this is not the case (at least, in the semileptonic physics), the proof is not known either. If that is not true, and duality starts only above 1 GeV, most probably one would have to abandon the idea of accurate determination of $|V_{cb}|$ from the exclusive $B \to D(\ast)$ transitions. The only option still open will be the inclusive semileptonic decays where the energy release is large, $\sim 3.5$ GeV. Of course, in such a pessimistic scenario (which, we believe, is unlikely) the theoretical precision in $|V_{cb}|$ will hardly exceed 5%.

What lies ahead? There are problems (e.g. duality violations) which we simply do not know how to attack. Solution of other problems seem to be possible in the near future. A practically important problem of this type is perturbation theory in the context of Wilson’s approach, where the soft parts of all diagrams have to be removed from the $\alpha_s$ series. This will lead us to more accurate estimates obtained in the heavy quark expansion. Another topical problem is constructing a reference model of the semileptonic decays where the transition amplitudes will be saturated by a minimal set of resonances and satisfy all constraints following from the heavy quark theory. This list can be continued. We are looking forward to new exciting
developments in the near future.

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**References**

[1] J. D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics* (Mc Graw-Hill, 1964).

[2] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, *Quantum Electrodynamics*, 2nd Edition (Pergamon Press, Oxford, 1982).

[3] E. Shuryak, *Nucl. Phys.* B198 (1982) 83.

[4] S. Nussinov and W. Wetzel, *Phys. Rev.* D36 (1987) 130.

[5] M. Voloshin and M. Shifman, *Yad. Fiz.* 47 (1988) 801 [Sov. J. Nucl. Phys. 47 (1988) 511].

[6] N. Isgur and M. Wise, *Phys. Lett.* B232 (1989) 113; *Phys. Lett.* B237 (1990) 527.

[7] E. Eichten and B. Hill, *Phys. Lett.* B234 (1990) 511; H. Georgi, *Phys. Lett.* B240 (1990) 447.

[8] N. Isgur and M.B. Wise, in *B Decays*, Ed. B. Stone, 2nd edition (World Scientific, Singapore, 1994), p. 231; H. Georgi, in *Perspectives in the Standard Model*, Proceedings of the Theoretical Advanced Study Institute, Boulder, Colorado, 1991, Ed. R.K. Ellis, C.T. Hill, and J. D. Lykken (World Scientific, Singapore, 1992); T. Mannel, *Heavy Quark Mass Expansions in QCD*, in Proc. of the Workshop *QCD – 20 Years Later*, Eds. P. M. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1993), vol. 2, p. 634; B. Grinstein, *Ann. Rev. Nucl. Part. Sci.* 42 (1992) 101; I. Bigi, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, in *B Decays*, Ed.
S. Stone, 2nd edition (World Scientific, Singapore 1994), p. 132;
M. Neubert, *Phys. Reports* **245** (1994) 259;
M. Voloshin, *Surv. High En. Physics*, **8** (1995) 27;
B. Grinstein, *An Introduction to Heavy Mesons*, in Proc. 6th Mexican School of
Particles and Fields, Eds. J. C. D’Olivo, M. Moreno, and M. A. Perez,
(World Scientific, Singapore, 1995) [hep-ph/9508227];
M. Shifman, *Lectures on Heavy Quarks in Quantum Chromodynamics*, in *QCD
and Beyond*, Proc. Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95), Ed. D.E. Soper (World Scientific, Singapore, 1996),
p. 409 [hep-ph/9510377].

[9] M. Neubert, *Phys. Lett.* **B322** (1994) 419.

[10] I.I. Bigi, M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Rev.* **D52** (1995)
196.

[11] A. Falk, [hep-ph/9609380](hep-ph/9609380) and [hep-ph/9610363](hep-ph/9610363); M. Neubert, [hep-ph/9702310](hep-ph/9702310);
M. Wise, [hep-ph/9703269](hep-ph/9703269).

[12] K. Wilson, *Phys. Rev.* **179** (1969) 1499.

[13] S. Balk, J.G. Körner and D. Pirjol, *Nucl. Phys.* **B428** (1994) 499.

[14] B. Blok, J.G. Körner, D. Pirjol and J.C. Rojas, *Nucl. Phys.* **B496** (1997) 358;
A. Manohar, *Phys. Rev.* **D56** (1997) 230, and references therein.

[15] W.E. Caswell and G.P. Lepage, *Phys. Lett. B* **167** (1986) 437;
G.P. Lepage, L. Magnea, C. Nakleh, U. Magnea, and K. Hornbostel, *Phys.
Rev. D* **46** (1992) 4052.

[16] A. Falk and M. Neubert, *Phys. Rev.* **D47** (1993) 2965.

[17] A. Falk, B. Grinstein and M. Luke, *Nucl. Phys.* **B357** (1991) 185.

[18] M. Luke, *Phys. Lett. B* **252** (1990) 447.

[19] M. Shifman, A. Vainshtein and V. Zakharov, *Phys. Lett. B* **78B** (1978) 443
[Reprinted in *The Standard Model Higgs Boson*, Ed. M. Einhorn (North-
Holland, Amsterdam, 1991), p. 84].

[20] V. Novikov, L.B. Okun, M. Shifman, A. Vainshtein, M. Voloshin, and V.
Zakharov, *Phys. Rep.* **41** (1978) 1.

[21] I.I. Bigi, N.G. Uraltsev, *Phys. Lett. B* **321** (1994) 412.

[22] I. Bigi, M. Shifman, N. Uraltsev, A. Vainshtein, *Phys. Rev. D* **50** (1994) 2234.

[23] M. Beneke, V. Braun, *Nucl. Phys. B* **426** (1994) 301.
[24] V. Braun, hep-ph/9610212.

[25] V.A. Khoze, in Future Physics and Accelerators, Proc. First Arctic Workshop on Future Physics and Accelerators, Saariselka, Finland, August 1994, Eds. M. Chaichian, K. Huitu, and R. Orava (World Scientific, Singapore, 1995), p. 458 [hep-ph/9412239].

[26] N. Gray, D.J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C 48 (1990) 673.

[27] M. Voloshin, Phys. Rev. D46 (1992) 3062.

[28] N. Uraltsev, Int. J. Mod. Phys. A11 (1996) 515.

[29] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, hep-ph/9704242, to appear in Phys. Rev. D.

[30] M. Shifman, A. Vainshtein, M. Voloshin and V. Zakharov Phys. Lett. B77 (1978) 80.

[31] M.B. Voloshin, Int. J. Mod. Phys. A10 (1995) 2865.

[32] S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, Phys. Rev. D28 (1983) 228; G.P. Lepage and P.B. Mackenzie, Phys. Rev. D48 (1993) 2250.

[33] M. Voloshin and Yu. Zaitsev, Usp. Fiz. Nauk 152 (1987) 361 [Sov. Phys. - Uspekhi 30 (1987) 553].

[34] M. Jamin and A. Pich, hep-ph/9702276.

[35] I. Bigi, M. Shifman, N. Uraltsev, A. Vainshtein, Phys. Rev. Lett. 71 (1993) 496.

[36] M. Voloshin, Phys. Rev. D51 (1995) 4934.

[37] V. Chernyak, Phys. Lett. B387 (1996) 173.

[38] M. Gremm, A. Kapustin, Z. Ligeti, and M. Wise Phys. Rev. Lett. 77 (1996) 20.

[39] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, Int. Journ. Mod. Phys. A9 (1994) 2467.

[40] M. Gremm and A. Kapustin, Phys. Rev. D55 (1997) 6924.

[41] M. Neubert, Phys. Rev. D46 (1992) 1076.

[42] P. Ball and V. Braun, Phys. Rev. D49 (1994) 2472.
[43] E. Bagan, P. Ball, V. Braun and P. Gosdzinsky, *Phys. Lett.* B342 (1995) 362.

[44] M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Rev.* D51 (1995) 2217.

[45] M. Voloshin, *Surv. High En. Phys.* 8 (1995) 27.

[46] M. Neubert, *Phys. Lett.* B389 (1996) 727.

[47] F. De Fazio, *Mod. Phys. Lett.* A11 (1996) 2693.

[48] D.S. Hwang, C.S. Kim and W. Namgung, [hep-ph/9608392](http://arxiv.org/abs/hep-ph/9608392) to appear in *Phys. Lett. B*.

[49] V. Gimenez, G. Martinelli and C.T. Sachrajda, *Nucl. Phys.* B486 (1997) 227.

[50] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Nucl. Phys.* B249 (1985) 445.

[51] N.G. Uraltsev, *Nucl. Phys.* B491 (1997) 303.

[52] X. Ji, [hep-ph/9507322](http://arxiv.org/abs/hep-ph/9507322) (unpublished).

[53] X. Ji, [hep-ph/9506216](http://arxiv.org/abs/hep-ph/9506216) (unpublished).

[54] I. Bigi et al., paper in preparation.

[55] D. Weingarten, *Phys. Rev. Lett.* 51 (1983) 1830; S. Nussinov, *Phys. Rev. Lett.* 51 (1983) 1081; 52 (1983) 1966; C. Vafa and E. Witten, *Nucl. Phys.* B234 (1984) 173; E. Witten, *Phys. Rev. Lett.* 51 (1983) 2351.

[56] N. Isgur and M. Wise, *Phys. Rev.* D43 (1991) 819.

[57] J.D. Bjorken, in *Proceedings of the 4th Rencontres de Physique de la Vallée d’Aoste*, La Thuille, Italy, 1990, Ed. M. Greco (Editions Frontières, Gif-Sur-Yvette, France, 1990), p. 583.

[58] M. Voloshin, *Phys. Rev.* D46 (1992) 3062.

[59] I. Bigi, A. Grozin, M. Shifman, N. Uraltsev, A. Vainshtein, *Phys. Lett.* B339 (1994) 160.

[60] C.-K. Chow and D. Pirjol, *Phys. Rev.* D53 (1996) 3998.

[61] B. Blok and M. Shifman, *Phys. Rev.* D47 (1993) 2949; this paper contains a large number of misprints, though.

[62] P. Colangelo, reported at 3d BaBar Physics Workshop, Paris, June 1997 (paper in preparation); for the first calculation, see P. Colangelo, G. Nardulli and N. Paver, *Phys. Lett.* B293 (1992) 207.
[63] A. Leibovich, Z. Ligeti, I. Stewart and M. Wise, *Phys. Rev. Lett.* **78** (1997) 3995, and hep-ph/9705467.

[64] L. Koyrakh, hep-ph/9607443.

[65] M. Neubert, *Phys. Rep.* **245** (1994) 259.

[66] H. Davoudiasl and A. Leibovich, hep-ph/9702341.

[67] M. Neubert, *Phys. Lett.* **B393** (1997) 110.

[68] G. Amoros and M. Neubert, *Phys. Lett.* **B394** (1997) 377; A. Grozin and M. Neubert, *Nucl. Phys.* **B495** (1997) 81.

[69] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Fortsch. Phys.* **32** (1984) 585.

[70] M. Shifman, *Annu. Rev. Nucl. Part. Sci.* **33** (1983) 199; and references therein.

[71] M. Voloshin, M. Shifman, *Yad. Fiz.* **41** (1985) 187 [Sov. J. Nucl. Phys. **41** (1985) 120].

[72] J. Chay, H. Georgi and B. Grinstein, *Phys. Lett.* **B 247** (1990) 399.

[73] I. Bigi, N. Uraltsev and A. Vainshtein, *Phys. Lett.* **B 293** (1992) 430.

[74] M. Luke, *Phys. Lett.* **B252** (1990) 447.

[75] B. Blok and M. Shifman, *Nucl. Phys.* **B399** (1993) 441 and 459.

[76] I. Bigi, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, *The Fermilab Meeting*, Proc. of the 1992 DPF meeting of APS, C.H. Albright et al. (World Scientific, Singapore 1993), vol. 1, p. 610.

[77] M. Voloshin and M. Shifman, ZhETF **91** (1986) 1180 [JETP **64** (1986) 698].

[78] B. Chibisov, R. Dikeman, M. Shifman and N. Uraltsev, *Int. J. Mod. Phys.* **A12** (1997) 2075.

[79] A. Falk, M. Luke and M. Savage, *Phys. Rev.* **D49** (1994) 3367.

[80] A. Manohar and M. Wise, *Phys. Rev.* **D49** (1994) 1310.

[81] I. Bigi and N. Uraltsev, *Nucl. Phys.* **B423** (1994) 33.

[82] I. Bigi and N. Uraltsev, *Z. Phys.* **C62** (1994) 623.

[83] G. Bellini, I.I. Bigi and P. Dornan, preprint UND-HEP-96-BIG02, to appear in *Phys. Rep.*
[84] I. Bigi and N. Uraltsev, *Phys. Lett.* B 280 (1992) 271.

[85] P. Ball, M. Beneke and V. Braun, *Phys. Rev.* D 52 (1995) 3929.

[86] R.E. Behrends et al., *Phys. Rev.* 101 (1956) 866.

[87] M. Luke, M. Savage and M. Wise, *Phys. Lett.* B 343 (1995) 329; B 345 (1995) 301.

[88] P. Ball and U. Nierste, *Phys. Rev.* D 50 (1994) 5841.

[89] M. Shifman and N.G. Uraltsev, *Int. J. Mod. Phys.* A 10 (1995) 4705.

[90] A. Czarnecki, *Phys. Rev. Lett.* 76 (1996) 4124;
A. Czarnecki and K. Melnikov, [hep-ph/9703277](http://arxiv.org/abs/hep-ph/9703277), to appear in *Nucl. Phys. B*.

[91] A. Czarnecki and K. Melnikov, *Phys. Rev. Lett.* 78 (1997) 3630.

[92] B. Blok, R. Dikeman and M. Shifman, *Phys. Rev.* D 51 (1995) 6167.

[93] ALEPH collaboration, PA05-059 “Inclusive Measurement of Charmless Semileptonic Branching Ratio of B-Hadrons”, contribution to the XXVIII Int. Conf. on High Energy Physics, Warsaw, Poland 25-31 July 1996.

[94] M. Luke and M. Savage, *Phys. Lett.* B 321 (1994) 88;
Z. Ligeti and Y. Nir, *Phys. Rev.* D 49 (1994) 4331;
V. Chernyak, *Nucl. Phys.* B 457 (1995) 96.

[95] M. Beneke, V. Braun and V. Zakharov, *Phys. Rev. Lett.* 73 (1994) 3058.

[96] M. Neubert, in Proc. XVII International Symposium on Lepton–Photon Interactions, 10-15 Aug. 1995, Beijing, Ed. Zhi-Peng Zheng and He-Sheng Chen (World Scientific, Singapore, 1996), p. 298 [hep-ph/9511409](http://arxiv.org/abs/hep-ph/9511409).

[97] M. Neubert, *Phys. Lett.* B 264 (1991) 455.

[98] For a review see M. Neubert, in *The Building Blocks of Creation - From Micro-fermions to Megaparsecs*, 1993 TASI Proceedings, Eds. S. Raby and T. Walker (World Scientific, Singapore, 1994) p. 125 [hep-ph/9404296](http://arxiv.org/abs/hep-ph/9404296).

[99] B. Blok, L. Koyrakh, M. Shifman and A. Vainshtein, *Phys. Rev.* D 49 (1994) 3356.

[100] A. Czarnecki, K. Melnikov and N. Uraltsev, [hep-ph/9706311](http://arxiv.org/abs/hep-ph/9706311).

[101] V. Belyaev, V. Braun, A. Khodjamirian, and R. Rückl, *Phys. Rev.* D 51 (1995) 6177.

[102] M. Neubert, *Phys. Lett.* B 338 (1994) 84.
[103] M. Neubert and C.T. Sachrajda, *Nucl. Phys.* **B438** (1995) 235.

[104] A. Kapustin, Z. Ligeti, M.B. Wise and B. Grinstein, *Phys. Lett.* **B375** (1996) 327.

[105] E. de Rafael and J. Taron, *Phys. Lett.* **B282** (1992) 215.

[106] C. Boyd, B. Grinstein, R. Lebed, *Phys. Lett.* **B353** (1995) 306; *Nucl. Phys.* **B461** (1996) 461.

[107] I. Caprini and M. Neubert, *Phys. Lett.* **B380** (1996) 376.

[108] N. Uraltsev, *Acta Phys. Polon.* **B28** (1997) 755 (Proceedings of the Third International Symposium on Radiative Corrections in Cracow, Poland, 1-5 August 1996); [hep-ph/9612349](http://arxiv.org/abs/hep-ph/9612349).

[109] C. Dominguez, Körner and D. Pirjol, *Phys. Lett.* **B301** (1993) 257; Ref. [100](http://www.arxiv.org/abs/hep-ph/9512419) and references therein.

[110] A. Vainshtein, *Proc. Int. Europhys. Conf. on High Energy Physics*, Brussels, July 1995, Eds. J. Lemonne et al. (World Scientific, Singapore 1996), p. 470 [hep-ph/9512419](http://arxiv.org/abs/hep-ph/9512419).

[111] For a review, see G. Altarelli, S. Petrarca, *Phys. Lett.* **B261** (1991) 303.

[112] E. Bagan, P. Ball, V. Braun and P. Gosdzinsky, *Phys. Lett.* **B342** (1995) 362; (E) **B374** (1996) 363; E. Bagan, P. Ball, B. Fiol and P. Gosdzinsky, *Phys. Lett.* **B351** (1995) 546.

[113] R.M. Barnett *et al.* (PDG), *Phys. Rev.* **D54** (1996) 1.

[114] I.Big, B.Blok, M.Shifman and A Vainshtein, *Phys. Lett.* **B323** (1994) 408.

[115] T. Browder, *Acta Phys. Polon.* **B28** (1997) 805 (Proceedings of the Third International Symposium on Radiative Corrections in Cracow, Poland, 1-5 August 1996); [hep-ph/9612491](http://arxiv.org/abs/hep-ph/9612491).

[116] I. Dunietz, J. Incandela, F. Snider, and H. Yamamoto, [hep-ph/9612421](http://arxiv.org/abs/hep-ph/9612421).

[117] A. Kagan and J. Rathsman, [hep-ph/9701300](http://arxiv.org/abs/hep-ph/9701300).

[118] I. Big, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, in *B Decays*, Ed. S. Stone, 2nd edition (World Scientific, Singapore 1994), p. 132.

[119] M. Neubert and C.T. Sachrajda, *Nucl. Phys.* **B483** (1997) 339.

[120] M. Baek, J. Lee, C. Liu and H. Song, Preprint SNUTP 97-064.

[121] N. Uraltsev, *Phys. Lett.* **B376** (1996) 303.

[122] P. Colangelo and F. De Fazio *Phys.Lett.* **B387** (1996) 37.