A Topological Extension of General Relativity

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ABSTRACT

A set of algebraic equations for the topological properties of space-time is derived, and used to extend general relativity into the Planck domain. A unique basis set of three-dimensional prime manifolds is constructed which consists of $S^3$, $S^1 \times S^2$, and $T^3$. The action of a loop algebra on these prime manifolds yields topological invariants which constrain the dynamics of the four-dimensional space-time manifold. An extended formulation of Mach’s principle and Einstein’s equivalence of inertial and gravitational mass is proposed which leads to the correct classical limit of the theory.

It is found that the vacuum possesses four topological degrees of freedom corresponding to a lattice of three-tori. This structure for the quantum foam naturally leads to gauge groups $O(n)$ and $SU(n)$ for the fields, a boundary condition for the universe, and an initial state characterized by local thermal equilibrium. The current observational estimate of the cosmological constant is reproduced without fine-tuning and found to be proportional to the number of macroscopic black holes. The black hole entropy follows immediately from the theory and the quantum corrections to its Schwarzschild horizon are computed.

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1. Introduction

In the past decades much effort has been devoted to the incorporation of gravitation into the framework of quantum field theory which describes the strong and electroweak interactions. Most work has concentrated on a formulation in terms of the geometrical and topological properties of these interactions. Superstring theories attempt to construct a Grand Unified Theory (GUT) in which gravity emerges naturally as a spin-2 excitation of the fundamental string[1]. Perturbative and non-perturbative gravitational quantum field theory takes general relativity as a starting point. These theories try to include quantum effects either perturbatively or through a different formulation of Einstein gravity in which an object more fundamental than the metric can be identified[2]. Also, an accurate description of the early universe and the properties of black holes should derive from a quantum theory of gravitation. Many conceptual difficulties in the unification program and quantum cosmology have long been recognized: renormalizability, factor ordering, the arrow of time, and the choice of boundary conditions[2,3,4].

It is well-known that the Einstein equations describe the geometry of space-time, but do not specify its topology[5]. Heuristic arguments, based on the lack of conformal invariance in the equation of motion, suggest that on the Planck scale space-time should have a multiply-connected (“foam-like”) structure[6]. A natural approach to incorporate the effects of a non-trivial space-time topology is the path integral[7] over geometries, including topologically complicated ones[8]. To define the ground state in such a theory one should provide for a boundary condition of the universe[9,10].

In the present work the geometrical Einstein equations are taken as a starting point and are complemented by a theory for the topological properties of space-time in terms of particular prime three-manifolds. The theory will be defined on a four-dimensional topological manifold with a (3+1) split for the dynamics. The dimension is therefore fixed.

The theory requires a choice for the three-dimensional boundary of the universe at $t \approx 0$. In this paper the initial value, $B^0_i$, is imposed at $t = t_{\text{Planck}}$ and no attempt is made to define a theory at the singularity $t = 0$. This work aims at constructing a theory which constrains the topology of the universe at $t_{\text{Planck}}$ and subsequently at later times, and extends the theory of general relativity into the Planck domain. Such an extension should not compromise the continuous evolution of the Einstein equation on time scales much larger than $t_{\text{Planck}}$ and should limit to a solution of the Einstein equation for length scales much larger than $\ell_{\text{Planck}}$.

This paper is organized as follows. In Section 2 a set of algebraic equations is derived for the dynamics of three-topologies based on a loop algebra with prime manifolds as dynamical objects, and an extension of Mach’s principle into the Planck domain is suggested. Section 3 discusses the Feynman path integral and the equivalence principle, and associates the four topological degrees of freedom of the vacuum with gauge groups. In Section 4, toy models for a quantum field theory and a black hole are presented, and the importance of topological fluctuations for the cosmological constant and macroscopic black holes is discussed. Finally, Section 5 contains the conclusions and a discussion of future work.

The first half of the paper is rather mathematical, but all the topological properties which are discussed, turn out to have clear physical meanings.
2. Dynamical 3-Topologies

2.1. One-Loops and Two-Loops in Three-Space

Consider the well-known superposition paradigm of quantum physics as expressed in Feynman’s path integral. Through constructive interference of wave amplitudes along all possible paths connecting two points, a world line is defined and associated with the trajectory of a particle. In a topologically non-trivial manifold, the world line becomes a plait with individual strands. When pinched at its ends, the underlying loop structure of the plait becomes apparent. It is this structure which reflects the topological interactions. Just like fields interact in particle dynamics, loops interact in topological dynamics.

Planck scale phenomenology precludes the dynamical significance of other homotopy groups in a theory for the dynamics of three-topologies. Any one-loop in a three-manifold generates a two-surface with the topology of a cylinder in four-dimensional space-time. So time dilation preserves the one-loop structure. This conservation of loop structure is required because any embedded lower-dimensional surface has a quantum-mechanical “width” of the order of a Planck time. Conversely, for the sphere $S^2$, the resulting topology is that of a cube. So the presence of a hole in a three-dimensional cube $I^3$ inhibits the contraction of a two-loop, but $S^2 \times I$ has a cube topology and is dynamically trivial.

2.2. Loop Algebra

To facilitate the continuous evolution of the Einstein equation on large scales and for long times, and still include quantum effects, time is regarded as a discrete coordinate with a time step equal to the Planck time. If one assumes that the topology of space-time on the Planck scale is non-trivial, then knowledge of its precise structure requires some formal measurement through matter degrees of freedom which cannot measure beyond a Planck time or Planck scale without influencing the observed system itself. Conversely, the effects of any change in topology, although instantaneous in terms of topological invariants like Betti numbers (kinematics), cannot be distinguished dynamically through interactions within a Planck time. It is assumed that in the absence of matter degrees of freedom, this behavior holds as well. This is a kind of semi-classical approximation in which the four-manifold with Planck scale structure (the wave function) is separated into the Einstein solution (a slowly varying amplitude) and the quantum foam (a rapidly varying phase).

In analogy with the first order (in time) nature of the Schrödinger equation, any equation for the discrete evolution of three-topologies is assumed to require only one initial value. Consequently, for any time $t_1$ the solution one Planck time later can only depend on the topology at time $t_1$.

In the present work the following equation will be used to determine the topology of the universe at times $t = (k + 1)t_{\text{Planck}}$ in the absence of matter:

$$\sum_j [T_i T_j^{k+1}, T_j^{k+1}] / P_j = (TT^\dagger + T^\dagger T) B^k_i, \quad k = 0, 1, 2, ...$$ (1)
with \([T \mathcal{T}_1, T^\dagger \mathcal{T}_2] \equiv T(\mathcal{T}_1 \times T^\dagger \mathcal{T}_2) - T^\dagger(\mathcal{T}_2 \times T \mathcal{T}_1)\), and \(B_i^0 = \alpha_i^0 \mathcal{P}_i\) the three-boundary of the universe at \(t = t_{\text{Planck}}\). The basis set \(\mathcal{B} \equiv \{\mathcal{P}_i\} = \{\mathcal{T}_i/\alpha_i\}\) consists of prime manifolds \(\mathcal{P}_i\) with multiplicities \(\alpha_i\). A three-manifold is called prime iff it is not the connected sum of two three-manifolds none of which is diffeomorphic to the three-sphere.

The word three-manifold is assumed to imply closed (compact), connected and oriented. In three dimensions any manifold \(M\) is diffeomorphic to the (chiral) unique, finite, and linear decomposition

\[
M = \bigoplus_i \mathcal{T}_i = \bigoplus_i (\alpha_i \mathcal{P}_i).
\]

This connected sum is an associative and commutative operation in the category of oriented three-manifolds and orientation preserving homeomorphisms. The connected sum of any manifold \(M\) and \(S^3\) is homeomorphic to \(M\)

\[
M \oplus S^3 \sim M.
\]

The operators \(T\) (annihilation) and \(T^\dagger\) (creation) satisfy

\[
T(\mathcal{P}_1 \times \mathcal{P}_2) = T(\mathcal{P}_1) \times \mathcal{P}_2 + \mathcal{P}_1 \times T(\mathcal{P}_2),
\]

\[
T(\mathcal{P}_1 \oplus \mathcal{P}_2) = T(\mathcal{P}_1) + T(\mathcal{P}_2),
\]

\[
TT^\dagger(\mathcal{P}) = k\mathcal{P},
\]

\[
T^\dagger T(\mathcal{P}) = l\mathcal{P},
\]

with \(k, l \in \mathbb{Z}\). The \(+\) rather than \(\oplus\) on the right-hand side of Equations (4a) and (4b) signifies the fact that a direct sum should not be taken at this stage, because \(\mathcal{P}_1\) and \(\mathcal{P}_2\) do not have the same dimension as \(T(\mathcal{P}_1)\) and \(T(\mathcal{P}_2)\). The action of \(T\) and \(T^\dagger\) on a prime manifold \(\mathcal{P}\) is

\[
T^\dagger \mathcal{P} = S^1 \times \mathcal{P}
\]

\[
T \mathcal{P} = \sum_i Q_i,
\]

with \(T S^3 = 0\), i.e. the trivial element. The sum in Equation (5b) extends over all homotopically inequivalent loops and \(Q\) denotes \(\mathcal{P}\) with a loop shrunk to a point. The summation reflects the fact that there is only one type of loop in the theory. If the primes in question are chiral, i.e. do not admit orientation reversing homeomorphisms, then both right- and left-handed orientations, \((\mathcal{P})\) and \((-\mathcal{P})\) should be included in the decomposition (2).

### 2.3. Derivation of the Equation of Motion

The combination of three-dimensional prime manifolds and their topological invariants associated with the discrete time loop algebra of the \(T\) and \(T^\dagger\) operators renders Equation (1) unique, as follows.
A basis set of prime manifolds is adopted in Equation (1) because these objects cannot be decomposed in two or more topologically non-trivial submanifolds. Also, any theory based on the concept of a topological manifold in dimension $n$, ultimately needs to incorporate the topological invariants of the primes in that dimension. This is analogous to the use of a differentiable manifold in the description of Einstein gravity. In four dimensions the identities associated with the Riemann tensor impose their own constraints on the dynamics of the theory.

2.3.1. Left-Hand Side

The left-hand side of Equation (1) contains the dynamics of the theory and reflects the interaction between all the prime manifolds: The total of variations in $T_i$ with respect to all the primes $\{P_j\}$. The commutator on the left-hand side reduces to the, topologically trivial, identity $[T, T^\dagger]T_i = T_i$ for $T_j = S^3$.

It is easy to show that the expression for the commutator in Equation (1) is the only multi-linear combination of operators and primes which is anti-symmetric under the simultaneous interchange of operators and primes, and limits to the identity for the three-sphere. The anti-symmetry property is necessary to assure that: 1) The net effect of an interaction and its reverse is zero. 2) The sum over both $i$ and $j$ in Equation (1) leads to a scalar expression which is invariant under the interchange of any two “vectors” $T_i$ and $T_j$ for time $k + 1$ as required by the decomposition (2). 3) This scalar is anti-symmetric under the interchange of the non-commuting creation and annihilation operators $T$ and $T^\dagger$. If the latter property were absent than the physical system which would result from the complete interchange of creation and annihilation processes would be described by the identical equation of motion. Anti-symmetry in $T$ and $T^\dagger$ yields a theory which has an arrow of time.

2.3.2. Right-Hand Side

The right-hand side of Equation (1) contains a scalar operator which assures the conservation of the topological properties of the primes. One seeks an operator $O$ linear in $T$ and $T^\dagger$, as these are the only objects acting on the primes, which assigns a unique number to every prime. Also, the topological invariant should not depend on the dynamics. That is, the order in which the creation and annihilation operators act on the prime through $O$ does not matter.

It is easy to show that the right-hand side of Equation (1) is the only invariant, linear and symmetric in $T$ and $T^\dagger$ and unique to each prime, which yields unity for the three-sphere. The latter requirement implements the identity (3) which shows that in the decomposition (2), the three-sphere has an effective multiplicity of unity.
The primes are the fundamental building blocks of the topological manifold and interactions among the primes cannot change their intrinsic properties. The right-hand side of Equation (2) then contains the number representative of the “prime quantum”. As such, it provides a topological constraint on the evolution of the multiplicities of the primes.

2.4. Some Relevant Properties of Prime Manifolds

Most of the results quoted below can be found in[11]. A manifold is called chiral iff it does not allow for an orientation reversing self-diffeomorphism. A three-manifold is non-chiral iff no prime in its prime decomposition is chiral. The three-sphere is non-chiral, but isometries (point identifications) of $S^3$ like SO(3) or its cover SU(2) are chiral.

A three-manifold $M$ is called irreducible, if every embedded two-sphere in $M$ bounds a three-ball. Clearly, an irreducible three-manifold is prime and also has a trivial second homotopy group. Interesting enough, the converse is also true with the exception of the handle manifold $S^1 \times S^2$, which is the only non-irreducible three-prime. This is directly related to the fact that black holes can increase their size and mass by accretion. That is, they have a “throat”.

An additional class of primes are those with an infinite fundamental group which in addition are sufficiently large. The fundamental group of such a sufficiently large three-manifold contains as a subgroup the fundamental group of a Riemannian surface. The latter property means physically that a non-contractible loop on an embedded surface is also not contractible within the ambient three-manifold. A sufficient condition for an irreducible manifold to be sufficiently large is that the first homology group is infinite. These irreducible primes fall into the class of $K(\pi, 1)$ spaces (Eilenberg-MacLane spaces), whose only non-vanishing homotopy group is the first. No connected sum containing at least one $K(\pi, 1)$ admits a Riemannian metric of everywhere positive scalar curvature. Moreover, if it admits a nowhere negative scalar curvature metric then it must be flat. That is, the manifold must be the three-torus or one of its quotients[12].

A three-manifold is called nuclear iff it is the space-like boundary of a Lorentz four-manifold with SL(2;C) spin structure, which is isomorphic with SO(3,1;R). It can be shown that a three-manifold is nuclear iff the number of nuclear primes in its prime decomposition is odd. Both $S^1 \times S^2$ and $T^3$ are nuclear unlike the primes $S^1 \times R_g$, where $R_g$ is a Riemannian surface of genus $g$.

The three-torus is the only known perfect group or Steinberg group St(3;Z). This group is a central $Z_2$ extension of SL(3;Z). A perfect group is its own commutator subgroup, where the commutator subgroup contains elements like $(xyx^{-1}y^{-1})$ to assure that $[x, y] = 0$. It is not likely that some of the less well studied three-primes which are not sufficiently large change the uniqueness of this result.

2.5. Interpretation
The physical interpretation of Equation (1) is as follows. In the absence of matter degrees of freedom and on the Planck scale, space acquires a multiply-connected topology as a function of the discrete time coordinate \( t = (k + 1)t_{\text{Planck}} \). This foam-like appearance of space-time is a direct consequence of the absence of conformal invariance in the Einstein equation[6].

Although Equation (1) possesses an arrow of time, its direction is not specified. If one sees the creation of the universe as “something from nothing” then the direction of time as given in Equation (1) is to be preferred since the evolution of the universe turns out not to depend strongly on the initial state (see §2.7). A reversed arrow of time leads to an exponentially decaying solution in Equation (7) below. This would also suggest that the initial quantum fluctuations required in the “something from nothing” scenario cannot grow and would have to be very fine-tuned.

Because \( t_{\text{Planck}} > 0 \), every three-manifold \( M_k \) at discrete time \( k \) generates a “small” four-manifold \( M_k \times t_{\text{Planck}} \). The solutions to Equation (1) are envisaged to describe the topology of the universe as it evolves with time. Therefore, the expansion of the universe needs to be provided for as well.

Black holes, associated with the \( S^1 \times S^2 \) handles or wormholes, increase their mass through merging or mass accretion. Conversely, three-tori are irreducible and remain restricted to the Planck scale. Their intrinsic size is also assumed equal \( \ell_{\text{Planck}} \). So during the evolution of the universe the number of these primes per Planckian volume cannot exceed unity, and complete merging on sub-Planck scales is assumed. On scales smaller than \( \ell_{\text{Planck}} \), the concept of a prime manifold should perhaps be replaced by an even more primitive structure like a Borel ring.

Equation (1) is purely topological (non-spatial) in nature. When one computes that the multiplicity of, say, the handle manifold is \( N \) at time \( n \) then these mini black holes are created randomly throughout the entire universe. The assumption here is that the statistical properties of space-time on the Planck scale are homogeneous.

### 2.6. Basis Set and Formal Solutions

Currently, no complete list of three-primes is available. Especially the not sufficiently large primes are poorly studied. A basis set will be chosen here which incorporates nuclearity and commutativity.

The basis set in question consists of the non-chiral primes \( S^1 \times S^2 (\alpha) \), \( S^3 (\beta) \) and \( T^3 (\gamma) \). 1) The non-irreducible and nuclear handle manifold appears as part of the large scale solution to the Einstein equation, i.e. the Schwarzschild solution. Since the solution holds for any super-Planck scale, the three-prime should be included. 2) A unit element is required for the large scale vacuum limit. 3) All other primes are confined to the Planck scale and are required to be nuclear for Lorentz invariance. Also, the indistinguishability of group elements (individual loops) under the loop algebra and the superposition principle will be shown to demand a perfect group structure for the three-primes with the matter fields defined on them, hence \( T^3 \) (see §3.1).
The above choice yields the following set of equations at time $k$

\[\begin{align*}
2\alpha_k^2 + \alpha_k \beta_k + 4\alpha_k \gamma_k &= 3\alpha_{k-1}, \\
\beta_k^2 + 2\alpha_k \beta_k + 4\beta_k \gamma_k &= \beta_{k-1}, \\
4\gamma_k^2 + 2\alpha_k \gamma_k + \beta_k \gamma_k &= 7\gamma_{k-1},
\end{align*}\]  

with general solution for $k > 0$ in the absence of matter degrees of freedom:

\[\begin{align*}
\alpha_k &= 3^k \frac{\alpha_0}{\beta_0} \beta_k, \\
\gamma_k &= 7^k \frac{\gamma_0}{\beta_0} \beta_k.
\end{align*}\]  

Note that $\alpha_k/\beta_k$ and $\gamma_k/\beta_k$ are invariant under changes ($\Delta \beta_k$) for $k > 0$. In §3.2 it will be shown that these are the quantities relevant for the computation of the Feynman propagator. Despite the trivial appearance of the basis set and the solution, there are some interesting consequences (see §3 and §4).

### 2.7. Initial Values and Possible Histories

The initial values $B_i^0$ determine the evolutionary path in the topological state space and Equation (3) demands that the initial values are invariant under the transformation

\[B_i^0 \rightarrow B_i^0 \oplus S^3,\]

with $S^3$ the homotopy 3-sphere. Therefore, only the integer ratios $\alpha_0/\beta_0$ and $\gamma_0/\beta_0$ are initial value data. For $k \geq 0$ the identity (8) is then satisfied manifestly, but the ratios $x \equiv \alpha_0/\beta_0$ and $y \equiv \gamma_0/\beta_0$ still need to be specified. Different evolutions are associated with different choices of these ratios.

For $\ell \sim \ell_{\text{Planck}}$ merging and coalescence suggest values for $x$ and $y$ close to unity. Values $x = y = 1$ seem natural if one considers that quantum fluctuations in the metric scale like $\ell_{\text{Planck}}/\ell$. Still, $x = 0$ is even more natural since the matter degrees of freedom will create black holes. Also, if the number of nuclear primes in a three-manifold is even, it does not bound a Lorentz manifold. The choice of $(x, y) = (0, 1)$ then corresponds merely to the requirement of Lorentz, or rather $SL(2;C)$, invariance and the superposition principle on a topological manifold. In this sense one can also argue that $x > 1$ is likely to give rise to a closed universe which will recollapse in a Planck time.
2.8. Matter Degrees of Freedom

If black holes are formed, the topology of space-time, and hence its evolution is modified. If the black hole is described by a Schwarzschild metric with the topology of a wormhole then its formation increases the multiplicity of the handle manifold, $S^1 \times S^2$, by one. Likewise, the merging of two black holes, or complete evaporation, has the opposite effect. In general the number of handles $h$, associated with the matter degrees of freedom, will satisfy

$$\frac{dh}{dt} = -R_e - R_m + R_f,$$

(9)

where $R_e$, $R_m$, and $R_f$ denote the rates in s$^{-1}$ for complete evaporation, merging, and formation, respectively. The fate of the singularity upon evaporation is unclear, but if it is naked, $R_e$ should be zero. Here it is assumed that no remnant remains. For the massive black holes in the current epoch of the universe $R_e/H \ll 1$ will hold, with $H$ the Hubble constant. The formation, complete evaporation or merging of black holes thus resets the right-hand side of Equation (7) at some time $k_1$ and the matter-free evolution resumes until the next event.

Mass accretion and partial evaporation of black holes does not increase the multiplicity of topological fluctuations associated with the handle manifold. It does however increase or decrease the amplitude of the fluctuation and thereby the effective scale on which matter and topology can interact.

Equation (9) modifies the values of the prime multiplicities at some time $k_1$, and subsequently the evolutionary path through state space. Its solution derives from the motion of matter (see Equation (10) below) in analogy with the coupled system formed by matter and metric in general relativity: Mach’s principle as interpreted by Einstein. Now one has, changes in topology and the local quantum motion of matter are determined by the shape and global topology of space-time, and vice versa.

3. The Feynman Path Integral on a Multiply-Connected Space

3.1. The Quantum Foam

The motion of matter (a world line) derives from a superposition of wave amplitudes which are determined by the actions associated with the followed paths. Imagine then an $S^3$ like space with matter degrees of freedom. Suppose this space, with the quantum fields defined on it, is contorted topologically into something which resembles an ensemble of three-tori. The three-tori now add three more homotopic classes of paths. Furthermore, the fields in quantum mechanics reflect the concept of probability amplitudes spread over space. Rather than merely identifying the edges of a region to create a closed three-torus topology, the extended nature of the wave amplitudes should be preserved and the various
three-tory connected through three-ball surgery, i.e. a connected sum. This results in four homotopically inequivalent paths. In effect, a quadruplet of quantum fields is constructed which will be called “multiplication”.

As one performs the path integral in the quantum foam, the various classes with fields are indistinguishable, beyond their non-homotopy, from one Planckian volume to the other. As such, it should not matter in which order the different group elements, i.e. non-homotopic loops, are chosen at each “point” along a path. This condition can only be satisfied if the group is perfect, i.e. $T^3$. Therefore, the construction above is the only one possible for a single-type loop algebra. It is thus found that at the Planck scale the vacuum has $e_M = 4$ topological degrees of freedom corresponding to a lattice of three-tori.

3.2. The Propagator

The inclusion of the quantum foam topology should yield a modified Feynman propagator which has the correct asymptotic form on scales much larger than $\ell_{\text{Planck}}$. The following Ansatz is adopted for the amplitude between space-time points $a$ and $b$

$$G(a, b) = \kappa \sum_j x_j \int_{\text{paths}} e^{iS_{a\nu}^{\nu}[m_j](x)} Dx^\mu,$$

where the sum over $j$ includes all topologically non-trivial primes and $x_j$ is the multiplicity of prime $P_j$ per Planckian volume. Furthermore, $m_j$ denotes the number of homotopically inequivalent paths between $a$ and $b$ for a given prime $P_j$

$$m_j P_j \equiv (T^1 T + 1) P_j. \tag{11}$$

This includes the path resulting from three-ball surgery. Only the invariant ratios of the multiplicities of the non-trivial primes $S^1 \times S^2$ and $T^3$ enter in the amplitude of Equation (10). The value of $\kappa$ is determined from

$$G(a, b) = \sum_c G(a, c) G(c, b). \tag{12}$$

Given the number of homotopically inequivalent paths associated with a prime, it is further assumed here that the wormhole associated with an $S^1 \times S^2$ handle is not traversible by matter. That is, a particle path through the wormhole has a divergent action if the particle’s de Broglie wavelength is smaller then the Schwarzschild radius. This issue of wormhole traversibility is not settled, but it appears that the divergent Riemann curvature associated with a black hole should be sufficient to completely destroy the identity of the traversing matter.
3.2.1. Particles

The action in Planck units for a free relativistic particle of rest mass \( m \) is proportional to the arclength

\[
S^{ab}_{p} = -\hbar^{-1}m \int_{p} (g_{\mu\nu} dx^{\mu} dx^{\nu})^{1/2},
\]

(13)

where all the symbols have their usual meaning. Even when no interactions between the fields occur on the intersections where the three-tori are connected, the various paths of the homotopic classes will merge. At these points, it appears that the physical system for which the lagrangian needs to be written down consists of \( e_{M} \) identical particles of mass \( m \). Therefore, the action will increase by a factor of \( e_{M} = m_{j} \), which leads to \( S^{ab}_{j}[m_{j}] = m_{j}S^{ab}_{p} \). The number \( m_{j} \) is then a multiplier in the exponent for the wave amplitude. This is a classical rewording of the multiplication phenomenon and is useful for semi-classical approximations.

3.2.2. Fields

More importantly, in a quantum field theory one wants to integrate over all fields between two fixed field configurations on space-like surfaces and include an interaction potential in the Lagrangian density. In this case the action in Planck units for a single \( n \)-vector multiplet of scalar fields \( \phi_{i} \) and potential \( V(\phi) \) would be of the form

\[
S[\phi] = \hbar^{-1} \int d^{4}x \frac{1}{2} \partial_{\mu}\phi\partial^{\mu}\phi - V(\phi),
\]

(14)

where all the symbols have their usual meaning. For the three-torus lattice one has \( e_{M} = 4 \) topological degrees of freedom. Hence, the interaction potential is a fourth order polynomial of the scalar fields

\[
V(\phi) = \mu^{2}\phi_{i}\phi_{i} + \lambda(\phi_{i}\phi_{i})^{2},
\]

(15)

where \( \lambda \) and \( \mu^{2} \) are real. It is well-known that these types of potentials are invariant under the action of the general rotation groups \( O(n) \) and the unitary groups \( SU(n) \). Obviously, these symmetries are local and generate gauge theories, although it is not apparent that the \( \phi_{i} \) are fundamental enough objects for a GUT (see also \( \S \)5).

3.3. Thermal Equilibrium of the Initial State

The multiplicated matter degrees of freedom, can be absorbed by black holes. It is thus found that in the Planckian universe and in the presence of sinks, i.e. non-irreducible primes...
$S^1 \times S^2$, a fraction of the irreducible fluctuation energy is continuously being stored in mini black holes. Simultaneous black hole evaporation will then lead to local thermalization at roughly the Planck temperature, as has been suggested in[10].

Furthermore, for a large number of fluctuations and a $T^3$ lattice, it follows from the central limit theorem that the deviations from the Planck temperature are driven to values much smaller than $M_{\text{Planck}}$ under the action of multiplication. The distribution function of the average sum of $n$ independent identically distributed Gaussian random variables of dispersion $s = 1$ in Planck units, has a total dispersion of $s/n^{1/2}$. The probability for an $x = 2s$ fluctuation is then approximately

$$p = e^{1/2} e^{-1/2s^2} / (2\pi s^2)^{1/2} \approx 3 \times 10^{-4},$$

more than two orders of magnitude smaller than in a simply-connected space-time.

This process, being a direct consequence of the quantum foam, suppresses fluctuations on scales larger than the particle horizon and is scale free by nature. This explains the success of the Harrison-Zeldovich power spectrum in reproducing the large scale structure of the universe.

### 3.4. The Equivalence Principle

The motion of matter as determined by Equation (10) depends on both general relativistic and quantum interference effects. The accelerations associated with the shape of the world line are now in part caused by the non-trivial topological space-time. Still, the different classes of paths can only be distinguished homotopically. Therefore, the Gedanken experiment usually referred to as “Einstein’s lift” still holds in small, but larger than $\ell_{\text{Planck}}$, regions of space. It is proposed then that the equivalence of inertial and gravitational mass holds irrespective of the underlying $T^3$ topology.

Furthermore, no local measurement of the gravitational field can distinguish one homotopic equivalence class from the other and Einstein gravity remains valid. In a stronger sense, no local measurement of any sort can make this distinction and the strong Principle of Equivalence should hold. One may wonder how the required GL(4) symmetry of general relativity can be facilitated. The $T^3$ primes have an intrinsic size of $\ell_{\text{Planck}}$, so general relativity in the large scale and long time limit can be envisaged as the way the lattice of three-tori bends and twists.

### 3.5. Evaluation of the Propagator

In the (3+1) Arnowitt-Deser-Misner formulation of geometric dynamics, one uses lapse and shift functions to integrate forward in time. The Einstein equation is independent of
topology and the action, say the arclength or the integrated lagrangian density, in Equation (10) is covariant. The (3+1) procedure then assures that no matter what lapse and shift functions are chosen, the four-geometry, action, and the topology are evolved consistently from one space-like slice to the next. When calculating the prime multiplicity (an invariant) per Planckian volume, one should use the integral of the covariant three-volume element \((\text{det}(g_{ij}))^{1/2}dx^i\). With the three-metric \(g_{ij}\) from the (3+1) decomposition of \(g_{\mu \nu}\).

In practice, the path \(a - b\) above should be much longer than the de Broglie wavelength of some particle. So for de Broglie wavelengths larger than the scale of a topological fluctuation, the propagator in Equation (10) will limit to its classical value associated with the three-sphere. For scales larger than \(\ell_{\text{Planck}}\) the three-tori are compactified and only traversible wormholes could contribute to the interference effects. If wormholes are traversible then no time difference is allowed between their “throats” because as three-space is evolved, paths from future directions will contribute to the wave amplitudes which is an obvious violation of causality. Of course, non-traversibility does not change the homotopical nature of the wormholes.

### 3.6. Correspondence Principle

It is desirable that the Poincaré conjecture is true. Namely, that a compact, simply-connected three-manifold is homeomorphic to the three-sphere. If a fake three-cell were to exist, the structure of the algebraic equations or the form of the propagator (the motion of matter) would not change. Also, the large scale limit provided by the Einstein equation does not depend on the existence or non-existence of a fake three-cell. Still, the theory would yield two or more non-homeomorphic, but loop homotopic, classical limits. One would then need to speculate about the physics, not represented by the loop algebra, to which this phenomenon corresponds. Conversely, the fact that non-homeomorphic, but still homotopic, primes exist in four dimensions and higher (e.g. \(S^4 \leftrightarrow S^2 \times S^2\)), suggests a topological origin for the spatial dimension of the universe.

### 3.7. Topological Amplification and GUT Inflation

The standard model has a singularity which is conventionally taken at time \(t = 0\). As one approaches the singularity, the temperature and density diverge. Thus, no initial value problem can be defined at the singularity. To avoid unknown quantum gravitational effects, the hot big bang scenario is defined at a temperature comfortably below the Planck mass. This construction requires extreme fine-tuning of the density and assumes thermal equilibrium across \(\sim 10^{80}\) causally disconnected regions.

Inflation theory provides a natural explanation for this flatness and horizon problem\[14\]. Still, one is left with the question how to get to a universe in approximate
thermal equilibrium at a temperature of $10^{19}$ GeV with sufficient energy stored in various degrees of freedom which are subsequently inflated to yield a post-GUT universe. The multiplication mechanism with black hole sinks of §3.2 provides a natural explanation for this initial condition.

It is clear from Equations (7) that the multiplicity of the fluctuations $\alpha_k/\beta_k$ and $\gamma_k/\beta_k$ grows exponentially with time in the absence of matter degrees of freedom. For the three-tori this implies that even when the universe expands exponentially with an e-folding time of $2\ell_{\text{Planck}}$, their multiplicity per Planckian volume will be of the order of unity for all time. Nevertheless, the solution is dynamical and one might speculate about the effects of time reversal, where the lattice is broken up in smaller isolated regions[15].

As the universe expands and cools away from its multiplicatated initial state, the multiplicity of the handle manifold will increase because black holes cannot evaporate efficiently at very high initial temperatures and primordial black holes will form through singular collapse. Since the universe is not expanding exponentially with time, merging events between black holes in the early universe will suppress the handle multiplicity. As the temperature drops, evaporation becomes more and more efficient. In fact, for the choice $x = 0$, the formation of primordial black holes will coincide with the latter epoch[15].

The GUT symmetry should be broken when the average energy density of the universe has dropped below threshold. Under the assumption that the exponential increase in the scale factor of the universe is a robust consequence of inflation, the handle multiplicity will become frozen in and the multiplicity per Planckian volume will decrease to a value much smaller than unity. Therefore, at late times the value of $\alpha_n/\beta_n$ is determined by the formation of macroscopic black holes. This kind of result follows from the fact that black holes can exist on all scales and therefore “defy” compactification when allowed to grow in mass through accretion and merging.

4. Toy Models

4.1. Handles and Quantum Field Theory

The initial values $\mathcal{X} = (x, y) = (0, 1)$ yield a 1-level solution $\mathcal{T} = (\alpha_1/\beta_1, \gamma_1/\beta_1) = (1, 7)$ if a primordial black hole is formed. Let $v\ell_{\text{Planck}}^3$ denote the volume of the universe. In the semi-classical approximation the propagator is proportional to

$$G \sim \frac{7}{v} e^{4iS_1^\gamma} + \frac{1}{v} e^{iS_2^\gamma}. \quad (17)$$

For non-traversable wormholes $S_2^\gamma \to \infty$, the additional interference term in the propagator then has a rapidly varying phase which prevents constructive interference. This corresponds to an effective optical depth through the quantum foam. This optical depth is roughly given by the integral of $\frac{\alpha_n}{\beta_n}/v$ multiplied by the effective radius of a black hole. For values of the optical depth close to unity, thermalization as described above will be effective.
In a full quantum field theory the following phenomenology then emerges. In analogy with the above model, handles can connect onto the $T^3$ lattice and generate additional interaction terms for the fundamental symmetry group on the three-torus. These correspond to the extended form of Mach’s principle in which the global topology causes local topology change. On the Planck scale, wormhole traversibility seems less of a burden on causality and interference effects between the two primes can occur.

The meaning of the solution (7) is now also apparent. The $S^1 \times S^2$ multiplicity per Planckian volume corresponds to the weights of the above diagrams at various times. From the discussion above, $\alpha_n/\beta_n << 1$ and handles should provide a rather small term to the $T^3$ quantum field theory. The lattice of three-tori in turn provides a number of symmetry groups for the interacting fields. The SL(2;C), O(n) and SU(n) groups have already been identified and §5 discusses the deeper ties with superstring theory. In a companion paper[15], a more general model will be constructed which goes beyond phenomenology.

4.2. Black Hole Properties

As a star collapses to a black hole, notwithstanding the possibility of a stable equation of state for nuclear matter above neutronium densities, it must reach a point where quantum effects become important. Topological multiplication then distributes the vacuum over various degrees of freedom. Although a black hole has no (or very little) hair, one would expect this process to have a clear macroscopic characteristic. It has been established above that $T^3$ is the only nuclear prime which is also a perfect group. Since in the late time limit three-tori dominate the quantum foam, one has $e_M = 4$.

A black hole is a thermodynamic object which, from Bekenstein’s analogy, emits black body radiation of a temperature

$$T_{BH} = \frac{\kappa}{8\pi c},$$

where $\kappa = 1/4M$ denotes the surface gravity and the constant $c$ is still to be determined. One also has that the black hole entropy is given by

$$S_{BH} = cA,$$

with $A$ the surface area of the black hole which is uniquely determined by its mass, charge and angular momentum. In fact, up to a numerical factor, the latter quantities comprise all the information about the black hole. That is, the entropy represents the total amount of information about the black hole interior not accessible to observers. From the equivalence of the $T^3$ group elements beyond non-homotopicity it then follows that $c = 1/e_M$.

This result is the same as Hawking’s derivation of the black hole temperature [16]. The latter author did not make any assumptions about the quantum foam topology, but showed that pair production could sustain an energy flow out to infinity corresponding to a non-zero effective temperature of the black hole. Only in a relativistic field theory can pair production occur. For this, one has to require that the dominating Planck scale prime is also the boundary of a Lorentz four-manifold SO(3,1;R), i.e. is nuclear. Conversely, the
perfect group property alone makes $T^3$ unique, which derives purely from loop homotopy, i.e. superposition. Once more one is confronted with the fact that black holes can exist on all scales and thereby form an inescapable link between macroscopic and microscopic physics.

### 4.3. Observational Diagnostics of the Quantum Foam

As mentioned above, the current epoch is characterized by a $T^3$ lattice. Furthermore, it is a prediction of the theory that the handle multiplicity at late times is determined by the number of macroscopic black holes.

One can estimate that the current total number of black holes in the universe is less than $10^{16}$. But if primordial black holes with masses of roughly $10^{15}$ g were produced copiously in the early universe, then this number may be more than $p = 20$ orders of magnitude larger and still be consistent with observational constraints[17].

In any case, at the current epoch where the GUT fields have undergone their phase transitions, the number of mini black holes spontaneously created in the quantum foam divided by the total number of Planckian volumes is a direct measure of the cosmological constant. \( \Lambda < 10^{-168 + p} \, m_{\text{Planck}}^2 \) consistent with the observational limit of $10^{-120} \, m_{\text{Planck}}^2$ without any fine-tuning. Observations of both the Hubble constant $H = \dot{R}/R$ and the deceleration parameter $q_0 = -\ddot{R}/R H^2$, with $R$ the scale factor of the universe, yield the cosmological constant. Since an upper bound of $q_0 < 5$ yields $\Lambda/m_{\text{Planck}}^2 < 10^{-120}$, accurate knowledge of supernova rates as a function of time and the nature of the remnant, neutron star or black hole, is extremely valuable.

One interesting thought is the following. The formation of mini black holes is assumed to occur homogeneously over space. Within this assumption, a number of events will occur within the Schwarzschild radii of black holes and contribute to their mass $M$. The magnitude of this effect should be proportional to $M^3$. Therefore, the Schwarzschild radius of an isolated, non-accreting, black hole of $10^9$ solar masses, should grow by 1 part in $10^{19}$ over a period of one year, or by 1 part in $10^3$ over a period of one year if the number of primordial black holes corresponds to $p = 16$. That is, cold black holes get colder.

### 5. Discussion and Outlook

In this work an algebraic equation has been derived and a basis set of prime manifolds obeying a loop algebra constructed, in order to determine the topological dynamics of the universe. Einstein gravity has been extended into the quantum domain assuming that the equivalence principle holds independent of topology. It has been shown that the vacuum possesses four topological degrees of freedom corresponding to a $T^3$ lattice. A physical
boundary condition for the universe was found. Natural solutions have been obtained for
the thermal equilibrium of the initial state, O(n) and SU(n) gauge groups for the fields,
black hole entropy, and the cosmological constant.

A problem to pursue further is the shape and amplitude of the frozen-in quantum
fluctuations after the epoch of inflation. These may have observable consequences for
the large scale structure of the universe at various cosmological redshifts. Given the large and
growing data sets available on the history of structure formation in the universe, tight
constraints can be derived[15].

There appear to be a number of deep topological and group theoretical connections.
Adopting the superposition principle in a topological manifold leads one to the loop
homotopy which yields a unique Planck scale structure through the three-torus. The $T^3$
topology assures Lorentz covariance or more general SL(2;C) invariance on the bounded
four-manifold. The latter group is the complexification of SL(2;R), also known as the
projection group. Recall that $T^3$ is a $Z_2$ extension of the discrete modular group SL(2;Z).
Modular invariance in string theory is crucial for supersymmetry and the finiteness of the
multi-loop amplitude. In fact, modular invariance forces self-duality of the string lattice
which restricts one to the $E_8 \times E_8$ and Spin(32)/$Z_2$ groups necessary for the anomaly
cancellation in the 26-dimensional heterotic string. These relations deserve further scrutiny.

A posteriori it appears that despite the obvious simplicity of the solutions (7), they
have a profound meaning. The very fact that the expressions only involve three-tori and
handles gives rise to a natural frame work in which to do Planck scale quantum field theory.
The solutions quoted above are an indication of this. In §4.1 a phenomenological model was
presented to develop a full quantum field theory with a special role for the handle manifold.
The link with superstring theory suggested above seems very interesting.

Obviously, many questions remain to be answered. The present work has been
cconcerned with the construction of an exploratory model to facilitate further research, and
a more complete theory will be presented in a companion paper.

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