Fractional Laplace transform for matrix valued functions with applications

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1. Introduction

Fractional derivative emergence dates back to the time of calculus. In 1695, L’Hospital wondered about the meaning of \( \frac{d^n}{dx^n} \) if \( n = \frac{1}{2} \), since then, researchers have been attempting to define a fractional derivative. Some of which are: Riemann-Liouville (Miller & Ross, 1993), Caputo and Fabrizio (2015) and Atangana, Baleanu, and Alsaedi (2015) definitions etc. In 2014, a new definition of fractional derivative called Conformable derivative was introduced by Khalil, Al Horani, Yousef, and Sababheh (2014).

Definition 1.1. Founded in Khalil et al. (2014).

Given a function \( y : [0, \infty) \rightarrow \mathbb{R} \), Then the “Conformable fractional derivative” of \( y \) of order \( \theta \) is defined by

\[
T_\theta(y)(s) = \lim_{\epsilon \to 0} \frac{y(s + \epsilon s^{-1-\theta}) - y(s)}{\epsilon}, \quad \text{for all } s > 0, \theta \in (0, 1).
\]

If \( y \) is \( \theta \)-differentiable in some \( (0, a) \), \( a > 0 \), and \( \lim_{\epsilon \to 0} y^{(\theta)}(s) \) exist, then define

\[
y^{(\theta)}(0) = \lim_{s \to 0} y^{(\theta)}(s),
\]

and the Conformable fractional integral is defined as

\[
The_\theta^a y(s) = The_\theta^a (s^{\theta-1} y) = \int_a^s \frac{y(x)}{x^{1-\theta}} \, dx,
\]

where the integral is the usual Riemann improper integral.

Most of the definitions give numerical solution to the problems using computer code. However, the Conformable fractional derivative is a natural definition which gives us simple and easy solutions to the problems.

For more different applications on Conformable fractional derivative we refer the reader to Tensor product technique and atomic solution of fractional Bate Man Burgers equation (Bushnaque, Al-Horani, & Khalil, 2020); Conformable fractional heat differential equation (Hammad & Khalil, 2014); Fractional Fourier series with applications (Abu Hammad & Khalil, 2014); Fractional Fourier series with separation of variables technique and it’s application on fractional differential equations (Bouchenak, Khalil, & AlHorani, 2021); Total fractional differentials with applications...
to exact fractional differential equations (ALHorani & Khalil, 2018); Fractional Cauchy Euler differential equation (Al-Horani, Khalil, & Aldaraw, 2020); Variation of parameters for local fractional non-homogenous linear differential equations (Al Horani, Hammad, & Khalil, 2016); On the nature of the conformable derivative and its applications to physics (Anderson, Camrud, & Ulness, 2018); Fractional Newton mechanics with conformable fractional derivative (Chung, 2015); Undetermined coefficients for local fractional differential equations (Khalil, Al Horani, & Anderson, 2016); On fractional vector analysis (Mhailan, Hammad, Horani, & Khalil, 2020); On conformable fractional calculus (Abdeljawad, 2015); Existence and uniqueness study of the conformable Laplace transform (Youinis, Ahmed, AlJazzazi, Al Hejaj, & Aydi, 2022); Generalization of fractional Laplace transform for higher order and its application (Ahmed, 2021) and New results on the conformable fractional Sumudu transform: theories and applications (Al-Zhour, Alrawajeh, Al-Mutairi, & Alkhasawneh, 2019).

In 2015, Abdeljawad Thabet put forward a definition of Conformable fractional Laplace transform (Abdeljawad, 2015). Now, we extend some results of the Conformable fractional Laplace transform to matrix-valued functions and we obtain certain useful theorems. Therefore, we will use the previous attained theorems to solve the following type of fractional initial value problem for matrix-valued functions

\[ Y^{(\theta)}(s) = AY(s) + g(s), \quad Y(0) = Y_0, \quad 0 < \theta \leq 1, \quad s > 0, \]

where A is a constant matrix and the components of g(s) are members of \( \mathcal{P}^\mathcal{E} \) (set of piecewise continuous functions with Conformable exponential order).

Moreover, we provide a fractional system transfer matrix and the Conformable fractional Laplace transform of the Conformable exponential matrix function which will give the solution of the following type of fractional initial value problem for matrix-valued functions

\[ \Phi^{(\theta)}(s) = A\Phi(s), \quad \Phi(0) = I, \quad 0 < \theta \leq 1, \]

where I is the \( n \times n \) identity matrix and A is a constant \( n \times n \) matrix.

The novel idea behind the current study is to apply the Conformable fractional Laplace transform on a new type of fractional initial value problems for matrix-valued function and obtain its exact solution. The weaknesses of the current study is just the existence of the Conformable fractional Laplace transform or not (Youinis et al., 2022). However, Its strength is in obtaining an exact solution easier without the need of the computer code but the other give an approximate solution also we can use it for the nonlinear case as can be seen in Ilhem, Al Horani, and Khalil (2022).

For further details on Conformable fractional Laplace transform see Abdeljawad (2015); Ahmed (2021); Younis et al. (2022) and Al-Zhour et al. (2019).

2. Fundamentals

**Definition 2.1.** See Abdeljawad (2015) and Al-Zhour et al. (2019).

Let \( y : [0, \infty) \to \mathbb{R} \) be a real valued function and \( 0 < \theta \leq 1 \). Then the Conformable fractional Laplace transform of y is defined as

\[ \mathcal{L}_\theta[y(s)] = y_\theta(\zeta) = \int_0^\infty e^{-\frac{s}{\theta} \zeta} y(s) ds = \int_0^\infty e^{-\frac{s}{\theta} y(s)^{\theta-1}} ds. \]

provided the integral exists.

Let us have as an example for the Conformable fractional Laplace transform of the usual functions in the theorem below.

**Theorem 2.2.** See Abdeljawad (2015) and Al-Zhour et al. (2019).

Let \( a, p, c \in \mathbb{R} \) and \( 0 < \theta \leq 1 \). Then

1. \( \mathcal{L}_\theta]\xi(\zeta) = \xi, \quad \zeta > 0. \)
2. \( \mathcal{L}_\theta]\xi^p(\zeta) = \frac{p}{p + 1} \xi^p, \quad \zeta > 0, \quad \frac{p}{p + 1} > -1. \)
3. \( \mathcal{L}_\theta]\xi^p(\zeta) = \frac{1}{1 - \frac{p}{\theta} - \xi^p}, \quad \zeta > 0. \)
4. \( \mathcal{L}_\theta]\sin a^p \zeta = \frac{a}{p + 1} \sin a^p \zeta, \quad \zeta > 0. \)
5. \( \mathcal{L}_\theta]\cos a^p \zeta = \frac{a}{p + 1} \cos a^p \zeta, \quad \zeta > 0. \)
6. \( \mathcal{L}_\theta]\sin a^p \zeta = \frac{a}{p + 1} \sin a^p \zeta, \quad \zeta > 0. \)
7. \( \mathcal{L}_\theta]\cos a^p \zeta = \frac{a}{p + 1} \cos a^p \zeta, \quad \zeta > 0. \)

**Proof.** Follows by applying Definition 2.1. □

One of the nice results is the relation between the usual and the Conformable fractional Laplace transforms which is given in the theorem below.

**Theorem 2.3.** See Abdeljawad (2015) and Al-Zhour et al. (2019).

Let \( y : [0, \infty) \to \mathbb{R} \) be a function such that \( \mathcal{L}_\theta[y(s)](\zeta) = y_\theta(\zeta) \) exists. Then

\[ \mathcal{L}_\theta[y(s)](\zeta) = y_\theta(\zeta) = \mathcal{L}_\theta[y((\theta s)^{\theta-1})](\zeta), \quad 0 < \theta \leq 1, \]

where \( \mathcal{L} \) is the usual Laplace transform.

**Proof.** Back to Abdeljawad (2015) and Al-Zhour et al. (2019). □

**Theorem 2.4.** Let \( y : [0, \infty) \to \mathbb{R}, \quad g : [0, \infty) \to \mathbb{R} \) and let \( \lambda, \mu, \alpha \in \mathbb{R} \) and \( 0 < \theta \leq 1 \). Then

1. \( \mathcal{L}_\theta[y'(s) + \alpha g(s)] = \lambda y_\theta(\zeta) + \mu g_\theta(\zeta), \quad \zeta > 0. \)

\[ \mathcal{L}_\theta[y(s)](\zeta) = y_\theta(\zeta) = \mathcal{L}_\theta[y((\theta s)^{\theta-1})](\zeta), \quad 0 < \theta \leq 1, \]

where \( \mathcal{L} \) is the usual Laplace transform.
(2) \( \mathcal{L}_d \{ e^{-\beta y(s)} \} (\xi) = y_0 (\xi + a), \ \xi > |a|. \)

(3) \( \mathcal{L}_d \{ f'(y(s)) \} (\xi) = \frac{f'(y_0(\xi))}{\xi}, \ \xi > 0. \)

(4) \( \mathcal{L}_d \{ \frac{d^\alpha}{d\xi^\alpha} y(s) \} (\xi) = (-1)^n \frac{d^n}{d\xi^n} y_0(\xi), \ \xi > 0. \)

(5) \( \mathcal{L}_d \{ y(s) + g(s) \} = y_0(\xi)g(\xi), \ \xi > 0. \)

where \( y_0 \) and \( g_0 \) are the Conformable fractional Laplace transform of the functions \( y \) and \( g \) respectively, \( y \ast g \) is the convolution product of \( y \) and \( g \) and \( l^\alpha y(s) \) is the Conformable fractional integral.

**Proof.** See Abdeljawad (2015) and Al-Zhour et al. (2019).

**Definition 2.5.** The function \( y(s), \ t \geq 0 \) is said to have Conformable exponential order \( m \) if there exists a small \( m \) and \( K > 0 \) and \( T > 0 \) such that
\[ |y(s)| \leq Ke^{ms^\theta} \text{ for all } s \geq T \text{ and } 0 < \theta \leq 1. \]

**Definition 2.6.** A function is called piecewise continuous on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval and has a finite limit at the endpoints of each subinterval.

\( \mathcal{PE} \) will be used to designate the class of all piecewise continuous functions of Conformable exponential order in the following sections. Any linear combination of functions in \( \mathcal{PE} \) is also in \( \mathcal{PE} \), according to the next theorem. The same is true for the product of two functions in \( \mathcal{PE} \).

**Theorem 2.7.** Let’s pretend that \( y(s) \) and \( g(s) \) are two \( \mathcal{PE} \) members with
\[ |y(s)| \leq M_1 e^{\alpha s^\theta}, \ s \geq C_1 \text{ and } |g(s)| \leq M_2 e^{\alpha s^\theta}, \ s \geq C_2. \]

(1) The function \( \beta y(s) + \gamma g(s) \) is also a member of \( \mathcal{PE} \) for any constants \( \beta \) and \( \gamma \). Moreover
\[ \mathcal{L}_d \{ \beta y(s) + \gamma g(s) \} = \beta \mathcal{L}_d \{ y(s) \} + \gamma \mathcal{L}_d \{ g(s) \}. \]

(2) \( \mathcal{PE} \) includes the function \( h(s) = y(s)g(s) \) as an element.

**Proof.**

(1) \( \beta y(s) + \gamma g(s) \) is a piecewise continuous function, as can be seen.

Now, let
\[ C = C_1 + C_2, \ a = \max \{ a_1, a_2 \} \text{ and } M = |\beta|M_1 + |\gamma|M_2. \]

Thus, for \( s \geq C \) we have
\[ |\beta y(s) + \gamma g(s)| \leq |\beta| |y(s)| + |\gamma| |g(s)| \leq |\beta| M_1 e^{\alpha s^\theta} + |\gamma| M_2 e^{\alpha s^\theta} \leq Me^{\alpha s^\theta}. \]

This proves that \( \beta y(s) + \gamma g(s) \) is of Conformable exponential order.

**On the other hand,**
\[ \mathcal{L}_d \{ \beta y(s) + \gamma g(s) \} = \beta \mathcal{L}_d \{ y(s) \} + \gamma \mathcal{L}_d \{ g(s) \}. \]

(2) \( h(s) = y(s)g(s) \) is a piecewise continuous function, as can be seen.

Now, letting
\[ C = C_1 + C_2, \ M = M_1 M_2, \text{ and } a = a_1 + a_2. \]

So, for \( s \geq C \) we have
\[ |h(s)| = |y(s)||g(s)| \leq M_1 M_2 e^{(a_1 + a_2)s^\theta} = Me^{s^\theta}. \]

Consequently, \( h(s) \) is of Conformable exponential order.

**3. Solution of fractional initial value problems for matrix-valued functions**

All results of Conformable fractional Laplace transform can be extended to vector-valued and matrix-valued functions. In this part, we select some results to be extended.

Let \( \mathcal{PE} \)'s members be \( y_1(s), y_2(s), \ldots, y_n(s) \). Take a look at the following vector-valued function.

\[ Y(s) = \begin{bmatrix} y_1(s) \\ y_2(s) \\ \vdots \\ y_n(s) \end{bmatrix} \]

For \( 0 < \theta \leq 1 \), the Conformable fractional Laplace transform (C.F.L.T) of \( Y(s) \) is

\[ \mathcal{L}_d \{ Y(s) \} = \int_0^\infty Y(s)e^{-\alpha s^\theta} \, ds = \begin{bmatrix} \mathcal{L}_d \{ y_1(s) \} \\ \mathcal{L}_d \{ y_2(s) \} \\ \vdots \\ \mathcal{L}_d \{ y_n(s) \} \end{bmatrix} = \begin{bmatrix} \int_0^\infty y_1(s)e^{-\alpha s^\theta} \, ds \\ \int_0^\infty y_2(s)e^{-\alpha s^\theta} \, ds \\ \vdots \\ \int_0^\infty y_n(s)e^{-\alpha s^\theta} \, ds \end{bmatrix} = \begin{bmatrix} \mathcal{L}_d \{ y_1(s) \} \\ \mathcal{L}_d \{ y_2(s) \} \\ \vdots \\ \mathcal{L}_d \{ y_n(s) \} \end{bmatrix}. \]

Similarly, we can define the Conformable fractional Laplace transform of an \( m \times n \) matrix to be the \( m \times n \) matrix comprised of the Conformable fractional Laplace transforms of the component functions.

If each component possesses a Conformable fractional Laplace transform, we say \( Y(s) \) is Conformable fractional Laplace transformable.

**Example 3.1.** Find the Conformable fractional Laplace transform of the following vector-valued function

\[ Y(s) = \begin{bmatrix} 1 \\ s^2 \end{bmatrix} e^{\alpha s^\theta}. \]

**Solution:**

From Theorem 2.2. and Theorem 2.4 The (C.F.L.T) of \( Y(s) \) is giving as
where \(0 < \theta \leq 1\) and \(\xi > 1\).

The Conformable fractional Laplace transform’s linearity property can be utilized to get the following result.

**Theorem 3.2.** Let \(A\) be a constant \(n \times n\) matrix and \(B\) be \(n \times p\) matrix-valued function then

\[
\mathcal{L}_\theta[AB(s)] = A\mathcal{L}_\theta[B(s)].
\]

**Proof.** Let \(A = (a_{ij})\) and \(B(s) = (b_{ij}(s))\). Thus

\[AB(s) = \left( \sum_{k=1}^{n} a_{ik}b_{kp}(s) \right).
\]

Consequently

\[
\mathcal{L}_\theta[AB(s)] = \left[ \mathcal{L}_\theta \left( \sum_{k=1}^{n} a_{ik}b_{kp}(s) \right) \right] = \left( \sum_{k=1}^{n} a_{ik} \mathcal{L}_\theta [b_{kp}(s)] \right) = A\mathcal{L}_\theta[B(s)].
\]

**Theorem 3.3.** For \(0 < \theta \leq 1\) and \(t \geq 0\) the following statements are true

1. Assume that \(Y(t)\) is a continuous vector-valued function, and that the components of the fractional derivative vector \(Y^{(\theta)}(t)\) are \(\mathcal{P}\mathcal{E}\) members. Then

\[
\mathcal{L}_\theta\left[ Y^{(\theta)}(s) \right] = \xi Y_\theta(\xi) - Y(0), \quad \xi > 0.
\]

2. Allow \(Y^{(\theta)}(s)\) to be continuous, and the entries of \(Y^{(2\theta)}(s)\) to be \(\mathcal{P}\mathcal{E}\) members. Then

\[
\mathcal{L}_\theta\left[ Y^{(2\theta)}(s) \right] = \xi^2 Y_\theta(\xi) - Y^{(\theta)}(0) - \xi Y(0), \quad \xi > 0.
\]

3. Let the entries of \(Y(s)\) be members of \(\mathcal{P}\mathcal{E}\). Then

\[
\mathcal{L}_\theta\left[ Y^{(\theta)}(s) \right] = Y_\theta(\xi), \quad \xi > 0.
\]

**Proof.**

1. By using Definition 2.1. and integration by parts, we have

\[
\mathcal{L}_\theta[Y(s)] = \int_0^\infty Y(s)e^{-\xi s}ds = \left[ \int_0^\infty e^{-\xi^\frac{\theta}{\theta-1} s}ds \right] e^{-\xi \frac{\theta}{\theta-1} s} = \frac{1}{\xi} \Gamma\left(1 + \frac{\theta}{\theta-1}\right) = \frac{1}{\xi - \frac{1}{\theta - 1}}
\]

2. Similarly, by applying Definition 2.1. and integration by parts, we get a result.

3. By using result (1) in this theorem, we have

\[
\mathcal{L}_\theta[D^{(\theta)}(Y(t))] = \xi \mathcal{L}_\theta\left[ Y^{(\theta)}(t) \right] - Y(0).
\]

By using definition of Conformable fractional integral we get \(Y^{(\theta)}(0) = 0\), then we obtain

\[
Y_\theta(\xi) = \frac{\mathcal{L}_\theta\left[ Y^{(\theta)}(t) \right]}{\xi}.
\]

**Example 3.4.** Consider the following fractional initial value problem

\[
Y^{(\theta)}(s) = AY(s) + g(s), \quad Y(0) = Y_0, \quad 0 < \theta \leq 1, \quad s > 0(1)
\]

where \(A\) is a constant matrix and the components of \(g(s)\) are members of \(\mathcal{P}\mathcal{E}\).

We can write using the above theorems

\[
\xi Y_\theta(\xi) - Y_0 = AY_\theta(\xi) + g_\theta(\xi).
\]

Thus

\[
(\xi I - A)Y_\theta(\xi) = Y_0 + g_\theta(\xi),
\]

where \(I\) is the identity matrix, \(\mathcal{L}_\theta[g(s)] = g_\theta(\xi)\) and \(\mathcal{L}_\theta[Y(s)] = Y_\theta(\xi)\).

If \(\xi\) is not an eigenvalue of matrix \(A\) so matrix \((\xi I - A)\) is invertible and in this case we get

\[
Y_\theta(\xi) = (\xi I - A)^{-1}(Y_0 + g_\theta(\xi)).
\]
From Theorem 2.2. and Theorem 2.4. we obtain

\[ Y(0)(s) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} Y(s) + \begin{bmatrix} e^{2s/\theta} \\ \frac{s}{\theta} \end{bmatrix}, \quad Y(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \]

Thus

\[ Y_0(\xi) = (\xi I - A)^{-1}(Y_0 + g_0(\xi)) \]

The fractional system transfer matrix is given by \((\xi I - A)^{-1}\) according to (2). This matrix represents the Conformable fractional Laplace transform of the Conformable exponential matrix function \(e^{\xi \theta}A\), as we will demonstrate.

**Theorem 4.1.** The Conformable exponential matrix function \((e^{\xi \theta}A)\) is the solution of the following fractional initial value problem for matrix-valued functions

\[ \Phi^p(s) = A\Phi(t), \quad \Phi(0) = I, \quad 0 < \theta \leq 1, \]

where \(I\) is the \(n \times n\) identity matrix and \(A\) is a constant \(n \times n\) matrix.

**Proof.** Taking the Conformable fractional Laplace transform of both sides yields

\[ \xi \Phi_0(\xi) - \Phi(0) = A\Phi_0(\xi). \]

\[ \xi \Phi_0(\xi) = l = A\Phi_0(\xi). \]

If \(\xi\) is not an eigenvalue of matrix \(A\) so matrix \((\xi I - A)\) is invertible and in this instance, we conclude that

\[ \Phi_0(\xi) = L_0(\Phi(s)) = (\xi I - A)^{-1} = L_0(e^{\xi \theta}A). \]

Hence, a result as required. \(\square\)

As an application, we will solve the following fractional initial value problems for matrix-valued functions in the below examples.

**Example 4.2.**

\[ Y^p(s) = AY(s), \quad Y(0) = I, \quad 0 < \theta \leq 1, \]

where \(A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\) and \(I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\).

By Theorem 4.1, we have

\[ Y_0(\xi) = (\xi I - A)^{-1} = \begin{bmatrix} 1/\xi & 0 \\ 0 & 1/\xi \end{bmatrix}. \]

By Theorem 2.2, we obtain the solution

\[ Y(s) = L_0^{-1}[Y_0(\xi)] = \begin{bmatrix} e^{\xi \theta} & 0 \\ 0 & e^{\xi \theta} \end{bmatrix}. \]

Hence a result as required.

**Example 4.3.**

\[ Y^p(s) = AY(s), \quad Y(0) = I, \quad 0 < \theta \leq 1, \]

where \(A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) and \(I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\).
By Theorem 4.1, we have
\[
Y_0(\xi) = (\xi I - A)^{-1} = \begin{pmatrix} \xi & -1 \\ -1 & \xi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\xi - 1} & \frac{1}{\xi^2 - 1} \\ \frac{1}{\xi - 1} & \frac{1}{\xi^2 - 1} \end{pmatrix}.
\]

By Theorem 2.2, we conclude that the solution is
\[
Y(s) = L^{-1}_0[Y_0(\xi)] = \begin{pmatrix} \cosh \frac{s^0}{\theta} & \sinh \frac{s^0}{\theta} \\ \sinh \frac{s^0}{\theta} & \cosh \frac{s^0}{\theta} \end{pmatrix}.
\]

Indeed:
We are going to prove that
\[
Y(s) = L^{-1}_0[Y_0(\xi)] = \begin{pmatrix} \cosh \frac{s^0}{\theta} & \sinh \frac{s^0}{\theta} \\ \sinh \frac{s^0}{\theta} & \cosh \frac{s^0}{\theta} \end{pmatrix} = e^{s^0A}.
\]

It is clear that
\[
A = P^{-1}DP \quad \text{and} \quad e^{s^0A} = P^{-1}e^{s^0D}P,
\]
where \( P = (v_1, v_2) \) is the passage matrix and \( v_1, v_2 \) are the eigenvectors corresponding to the eigenvalues \( \lambda_1, \lambda_2 \) respectively and \( D = \text{diag}(\lambda_1, \lambda_2) \).

In order to determine the eigenvectors of matrix \( A \) we must first determine the eigenvalues \( \lambda_1, \lambda_2 \) by solving the equation \( (A - \lambda I)X = 0 \), where \( I \) is the identity matrix and for some nonzero vector \( X \).

Hence, we get
\[
\lambda_1 = 1, \quad \lambda_2 = -1.
\]

Also, we find
\[
P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Then, we calculate the inverse matrix of \( P \) to obtain
\[
P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Finally
\[
e^{s^0A} = P^{-1}e^{s^0D}P
\]
\[
= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{s^0} & 0 \\ 0 & e^{-s^0} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]
\[
= \begin{pmatrix} \frac{e^{s^0} + e^{-s^0}}{2} & \frac{e^{s^0} - e^{-s^0}}{2} \\ \frac{e^{s^0} - e^{-s^0}}{2} & \frac{e^{s^0} + e^{-s^0}}{2} \end{pmatrix} = \begin{pmatrix} \cosh \frac{s^0}{\theta} & \sinh \frac{s^0}{\theta} \\ \sinh \frac{s^0}{\theta} & \cosh \frac{s^0}{\theta} \end{pmatrix}.
\]

Hence a result as required.

5. Conclusion
It is quite complicated to find the exact solution for Riemann–Liouville and Caputo fractional differential equations and initial value problems even in the linear scalar case. More details and information on methods solving fractional initial value problems for Caputo and Riemann–Liouville sense can be founded in Bushnaq et al. (2022); Vinh An, Vu, and Van Hoa (2017) and Hristova, Agarwal, and O’Regan (2020). Since, the formulas for the exact solutions are important tools in fractional models. In this paper, we introduce new exact solution to the fractional initial value problems for matrix-valued functions called Conformable fractional Laplace transform method. Our method was illustrated on two types of fractional initial problems for vector-valued functions and matrix-valued functions as mentioned previously. We conclude that the Conformable fractional Laplace transform is an easy and simple method which gives us exact solution to this kind of problems. It is a known fact that Laplace transform is a famous mathematical tool for linear operators, but it is extremely difficult to deal with nonlinear operators. Our interest future work is to develop our method for solving fractional initial and boundary values problems specially the nonlinear case as can be seen in Ilhem et al. (2022). Moreover, the latest publications on fractal theory can be founded in Ain, Anjum, and He (2021); Ain et al. (2022); Anjum, Ain, and Li (2021); Anjum, He, and He (2021).

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