Preserving torsion orders when embedding into groups with ‘small’ finite presentations

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Abstract

We give a complete survey of a construction by Boone and Collins [3] for embedding any finitely presented group into one with 8 generators and 26 relations. We show that this embedding preserves the set of orders of torsion elements, and in particular torsion-freeness. We combine this with a result of Belegradek [2] and Chiodo [6] to prove that there is an 8-generator 26-relator universal finitely presented torsion-free group (one into which all finitely presented torsion-free groups embed).

1 Introduction

One famous consequence of the Higman Embedding Theorem [11] is the fact that there is a universal finitely presented (f.p.) group; that is, a finitely presented group into which all finitely presented groups embed. Later work was done to give upper bounds for the number of generators and relations required. In [14] Valiev gave a construction for embedding any given f.p. group into a group with 14 generators and 42 relations. In [3] Boone and Collins improved this to 8 generators and 26 relations. Later in [15] Valiev further showed that 21 relations was possible. In particular we can apply any of these constructions to a universal f.p. group to get a new one with only a few generators and relations.

More recently Belegradek in [2] and Chiodo in [6] independently showed that there exists a universal f.p. torsion-free group; that is, an f.p. torsion-free group into which all f.p. torsion-free groups embed. This can be generalised even further: if $X \subseteq \mathbb{N}$ then we say a group $G$ is $X$-torsion-free if whenever $g^n = 1$ with $n \in X$ then $g = 1$; in [8, Theorem 2.11] Chiodo and McKenzie showed that there is a universal $X$-torsion-free f.p. group for any recursively enumerable set $X$.

In this paper we show that the embedding construction of Boone and Collins in [3] preserves torsion orders, the set of orders of torsion elements of a group $G$ (denoted $\text{Tord}(G)$), to show the following result.

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1
Theorem 9.1 There is a uniform algorithm for embedding a finitely presented group $A$ into another finitely presented group $H$ with 8 generators and 26 relations. Moreover, $	ext{Tord}(A) = 	ext{Tord}(H)$.

We then apply this to the main result in [8, Theorem 2.11] to show the following:

Theorem 9.2 Let $X$ be a recursively enumerable set. Then there is a universal finitely presented $X$-torsion-free group with 8 generators and 26 relations.

Taking $X = \mathbb{N}$ in the above theorem gives a universal f.p. torsion-free group with 8 generators and 26 relations. Furthermore, in Theorem 7.2 we write down an explicit presentation of a universal f.p. $X$-torsion-free group with 13 generators and 33 relations (with the exception of one extremely long relation, which we do not write down explicitly as it is complicated and depends on finding the group in [8, Theorem 2.11] explicitly).

2 Overview

This paper follows the construction given by Boone and Collins in [3] for embedding any f.p. group into one with 8 generators and 26 relations. Unfortunately, their construction in [3] is not completely self-contained. In particular, they make use of several other embedding constructions, from several other papers ([1, 4, 5, 9, 10]). The objective of this paper is thus twofold:

1. To give a complete survey of the construction by Boone and Collins in [3], and all peripheral constructions (as well as precisely where to find them), for embedding any finitely presented group into one with 8 generators and 26 relations.

2. To note the torsion preserving properties of these constructions, where necessary, to prove our main results in Theorems 9.1 and 9.2.

Thus, this paper contains a complete description of a construction for embedding an arbitrary finitely presented group $A$ into a finitely presented group $H$ with 8 generators and 26 relations following the exposition in [3], and a proof that $\text{Tord}(A) = \text{Tord}(H)$. To succeed with objective 2 we were, in a sense, forced to carry out the survey in objective 1. It was only through a careful analysis of [3], including all peripheral constructions, that we were able to complete objective 2. We now give the structure of the rest of our paper.

In Section 3 we give the necessary preliminaries and background results needed to proceed with the constructions and proofs. These are all quite standard.

In Section 4 we give an overview of the construction by Boone, Collins and Matijasevič in [4] for simulating equality in any f.p. group $A$ in a corresponding semigroup $\mathfrak{N}$ with only 2 generators and 3 relations. However, one of these relations is extremely long, which we discuss.

In Section 5 we give an overview of the construction by Collins in [10] for simulating equality in the above semigroup $\mathfrak{N}$ in a group $G$ with 7 generators and 14 relations. The purpose of $G$ is to ‘simulate’ (but not embed) our original group $A$ in a 7-generator 14-relation group $G$. We note that $G$ is built up by several HNN-extensions, starting with the free group on 2 generators, and hence is torsion-free.
In Section 6 we give an overview of the construction by Boone and Collins in 3 (which is a variant of Aanderaa’s proof of the Higman Embedding Theorem in 11) to obtain an explicit embedding of our original group $A$ into a group $K_4$ with 13 generators and 33 relations. Since this is built up entirely by HNN-extensions, starting with the group $A \ast G$, we see that this embedding preserves torsion orders.

In Section 7 we write down a (mostly) explicit presentation for the group $K_4$ constructed in Section 6 except for one particular ‘long’ relation, which we discuss. This long relation is inherited from the long relation in Section 5.

In Section 8 we give an overview of the construction by Collins in 9 (which is itself a refinement of an idea due to Borisov in 5) for embedding an f.p. group with many commuting generators into an f.p. group with fewer generators and relations. We note that this embedding preserves torsion orders as it is built from HNN-extensions and an amalgamated free product.

Finally, in Section 9 we apply the embedding construction from Section 8 to the finite presentation $K_4$ given in Section 7 (as is done by Boone and Collins in 3), which we know to have several commuting relations. We end up with a presentation with 8 generators and 26 relations, though we do not write it out explicitly as it is very complicated. Thus, we obtain a torsion order preserving embedding of our original group $A$ into a group with 8 generators and 26 relations.

3 Preliminaries

This paper relies heavily on studying torsion in groups, thus it is necessary to introduce some notation on this. So, let $G$ be any group. We define the set of torsion elements of $G$ to be $\text{Tors}(G) := \{ g \in G \mid \exists n > 1, g^n = 1 \}$, and the set of torsion orders of $G$ to be $\text{Tord}(G) := \{ n \in \mathbb{N} \mid n > 1, \exists g \in G, o(g) = n \}$. $G$ is said to be torsion-free if $\text{Tord}(G) = \emptyset$.

It is possible to generalise the notion of torsion, which we do as follows: given any $X \subseteq \mathbb{N}$, we define the $X$-torsion of $G$ to be $\text{Tors}^X(G) := \{ g \in G \mid \exists n \in X, n > 1, g^n = 1 \}$. For any set $X \subseteq \mathbb{N}$, we define the factor completion of $X$ to be $X_{\text{fc}} := \{ n \in \mathbb{N} \mid \exists m \geq 1, nm \in X \}$, and we say $X$ is factor complete if $X_{\text{fc}} = X$. $G$ is said to be $X$-torsion-free if $\text{Tord}(G) \cap X_{\text{fc}} = \emptyset$; equivalently, if $\text{Tors}^X(G) = \{ e \}$. We note that, in most cases, the $X$-torsion of a group behaves almost identically to the torsion of a group, when considering embedding theorems.

Crucial to our work are the standard embedding constructions of HNN extensions and amalgamated free products, which we define here for the convenience of the reader.

Let $G$ be a group and let $A_i$, $B_i$ be subgroups of $G$ for $1 \leq i \leq n$. Let $\phi_i : A_i \to B_i$ be isomorphisms for $1 \leq i \leq n$. We define the HNN-extension of $G$ over $\phi_1, \ldots, \phi_n$ to be

$$G *_{\phi_1, \ldots, \phi_n} := G * \langle t_1, \ldots, t_n \rangle / \langle \langle \{ \phi_i(g)^{-1} t_i^{-1} g t_i | g \in A_i, 1 \leq i \leq n \} \rangle \rangle^{G * \langle t_1, \ldots, t_n \rangle}$$

Let $G$ and $H$ be groups with $A$ a subgroup of $G$ and $B$ a subgroup of $H$. Let $\phi : A \to B$ be an isomorphism. We define the amalgamated free product of $G$ and $H$ over $\phi$ to be

$$G *_{\phi} H := G * H / \langle \langle \{ \phi(g)^{-1} g \mid g \in A \} \rangle \rangle^{G * H}$$

We will often be considering presentations of HNN-extensions and amalgamated free products. These can always be written by adding some generators and relations to
Then every torsion element of $K = \text{Tord}(\text{Proposition 3.2})$ is conjugate to an element of $G$. In particular, $\text{Tord}(G) = \text{Tord}(H)$.

**Proposition 3.2** [13] Theorem 11.69] Let $K := G * _{\phi} H$ be an amalgamated free product. Then every torsion element of $K$ is conjugate to an element of $G$ or $H$. In particular, $\text{Tord}(K) = \text{Tord}(G) \cup \text{Tord}(H)$.

A famous embedding theorem, originally due to Higman, Neumann and Neumann in [12], allows us to embed any countable group into a 2-generator group. Moreover, by the above, the torsion in such an embedding is well understood. We state this here, in the form needed for our later analysis:

**Theorem 3.3** Let $G$ be a finitely presented group. There is a uniform algorithm for embedding $G$ into a finitely presented group $G'$ with 2 generators, with $\text{Tord}(G) = \text{Tord}(G')$.

**Proof** This is a standard construction which can be found in [13] Theorem 11.71. The observation that this preserves torsion orders can be seen in [7] Lemma 2.16.

We will frequently need to deal with arbitrary presentations of groups and semigroups. These may have complicated relations, and are often difficult to work with. However, given any finite presentation of a semigroup (or indeed a group), there is a useful trick to re-write the presentation with only ‘short’ relators, each of length at most 3. The trade-off is that this process introduces many more generators and relators, but for our purposes it is worthwhile. We state the precise result here; note that the same process works for group presentations, but we only require it for semigroups.

**Proposition 3.4** Let $A = \langle a_1, \ldots, a_k \mid R_i = S_i \rangle$ be a finite semigroup presentation where $R_i, S_i$ are non-trivial for each $i$. Then we can algorithmically add new generators to $A$ to get a finite presentation of the form $A = \langle a_i \mid A_i = B_i \rangle$ where each $A_i$ is a single letter and each $B_i$ has length 2.

**Proof** Write the first relation as $x_1 \ldots x_n = y_1 \ldots y_m$. Define $a_{k+1} := x_1x_2$, $a_{k+i} := a_{k+i-1}x_{i+1}$ for $1 < i < n$, $a_{k+n+1} := y_1y_2$ and $a_{k+n+i} := a_{k+n+i-1}y_{i+1}$ for $1 < i < m$. Then we can add $a_{k+1}, \ldots, a_{k+m+n-1}$ to the generating set along with their defining relations and the relation $a_{k+n} = a_{k+n+m-1}y_n$ to the presentation and then the initial relation is redundant. Now repeat for the rest of the relations.

A set $X \subseteq \mathbb{N}$ is said to be recursively enumerable (r.e.) if there is a Turing machine, whose inputs are binary representations of integers, with halting set $X$.

Given any word $W = x_1^{a_1} \ldots x_n^{a_n}$ we will define $W^\# := x_1^{-a_1} \ldots x_n^{-a_n}$.

We finish this section by stating a crucial result, which is the motivation for this paper, and which we explicitly appeal to at the conclusion of our analysis.
Theorem 3.5 Let $X$ be a recursively enumerable set. Then there is a universal finitely presented $X$-torsion-free group $G$, i.e., $G$ is $X$-torsion-free, and for any finitely presented (even recursively presented) group $H$ is $X$-torsion-free we have an embedding $H \rightarrow G$.

Proof This was shown in [5, Theorem 2.11]. A special case where $X = \mathbb{N}$ can found in either [2] or [6].

4 Simulating an arbitrary f.p. group $A$ in a semigroup $\mathfrak{N}$

In this section we give an overview of the construction by Boone, Collins and Matijasevič in [1] for simulating equality in any f.p. group $A$ in a corresponding semigroup $\mathfrak{N}$ with only 2 generators and 3 relations. We follow the exposition given in p.1–2 of Section 1 of [3].

Let $A$ be a finitely presented group. As noted in Theorem 3.3 we can uniformly embed $A$ in a 2 generated group in a way which preserves torsion orders. So w.l.o.g. we may assume that $A = \langle a_1, a_2, R \rangle$ has 2 generators. We now will construct a semigroup $\mathfrak{N}$ and map on words $\chi : A \rightarrow \mathfrak{N}$ as in p.1–2 of [3]. We note that this construction was initially given in [1].

Define extra generators $a_3 := a_1^{-1}$ and $a_4 := a_2^{-1}$ so that each $R_i$ is now a positive word on $a_1, a_2, a_3, a_4$. Now define the semigroup

$$A_4 := \langle a_1, a_2, a_3, a_4, p \mid R_i p = p, a_j p = p a_j \quad (\forall i, j) \rangle$$

Let $a_5 := p$. Since none of the defining relations have the empty word on either side we can use Proposition 3.4 to algorithmically add extra generators so that each relation of $A_4$ has the form $a_\lambda = a_\mu a_\nu$. So we have a presentation for $A_4$ of the following form:

$$A_4 = \langle a_1, \ldots, a_r \mid A_i = B_i \rangle$$

where each $A_i$ has length 1, and each $B_i$ has length 2. By repeating relations we may assume that there are $s = 2^t$ relations for some $t \in \mathbb{N}$.

We now define the map $\psi$ on words over $\{a_1, \ldots, a_r\}$ induced by $\psi(a_i) := \beta^2 \gamma^i \beta \gamma^{r+1-i}$.

Define the semigroup

$$\Sigma := \langle \beta, \gamma \mid \psi(A_j) = \psi(B_j) \rangle$$

Then the induced map $\overline{\psi} : A_4 \rightarrow \Sigma$ is a semigroup homomorphism. Observe that $\psi(a_i)$ has length $u = r + 4$ for each $i$; so $\psi(A_j)$ has length $u$ and $\psi(B_j)$ has length $2u$. Let $x_{i,j}$ be the $i$th letter of $\psi(A_j)$ and let $y_{i,j}$ be the $i$th letter of $\psi(B_j)$. We define the ‘interlacing words’

$$M := x_{1,1} x_{1,2} \cdots x_{1,s} x_{2,1} \cdots x_{u,s}$$

$$N := y_{1,1} y_{1,2} \cdots y_{1,s} y_{2,1} \cdots y_{2u,s}$$

Now define the semigroup $\mathfrak{M}$ with presentation

$$\mathfrak{M} := \langle \beta, \gamma, \varepsilon \mid \varepsilon \beta \beta = \beta, \varepsilon \gamma \beta = \beta, \varepsilon \beta \gamma = \gamma, \varepsilon \gamma \gamma = \gamma, M = N \rangle$$

and a map $\tau$ on words over $\{\alpha, \beta, \epsilon\}$ induced by $\tau(\beta) := \sigma \alpha, \tau(\gamma) := \sigma, \tau(\epsilon) := \alpha^2$. From this, we define the semigroup

$$\mathfrak{R} := \langle \alpha, \sigma \mid \sigma \alpha^2 = \alpha^2 \sigma \alpha, \sigma \alpha = \alpha^2 \sigma^2, \tau(M) = \tau(N) \rangle$$

5
Then it is easily verified that the induced map \( \tau : \mathcal{R} \to \mathcal{R} \) is a semigroup homomorphism. Finally, we let \( \chi \) be the composition of \( \psi \) and \( \tau \) (as word maps) from words over \( \{a_1, \ldots, a_r\} \) to words over \( \{\alpha, \sigma\} \). Note that this is well defined on words but is not necessarily a semigroup homomorphism. However, as noted in Theorem 1 of [3], we have the following properties.

**Theorem 4.1** [3, Theorem 1] Let \( \Phi_0 := \chi(p)\sigma\alpha^{2t} \) and let \( U \) and \( V \) be any words in \( A \).

(i) \( \chi(UV) \equiv \chi(U)\chi(V) \) as words.

(ii) \( U = V \) in \( A \) iff \( \chi(U)\Phi_0 = \chi(V)\Phi_0 \) in \( \mathcal{R} \).

(iii) Moreover for any word \( Z \) of \( \mathcal{R} \) if \( Z\Phi_0 = \chi(V)\Phi_0 \) in \( \mathcal{R} \) then there exists a word \( W \) in \( A \) such that \( W = V \) in \( A \) and \( \chi(W) \equiv Z \) as words.

**Proof** follows immediately from the definition of \( \chi \). Proof of (i) and (iii) can be found in Lemmata 2.15 and 2.16 of [3].

Thus by Theorem 4.1 (ii) we see that we can ‘simulate’ equality in \( A \) by equality of corresponding words in \( \mathcal{R} \).

5 Constructing an f.p. group \( G \) simulating the semigroup \( \mathcal{R} \)

In this section we give an overview of the construction by Collins on p.306–308 in Part I of [10], for simulating equality in the semigroup \( \mathcal{R} \) in a group \( G \) with 7 generators and 14 relations. The purpose of \( G \) is to ‘simulate’ (but not embed) our original group \( A \) in a 7-generator 14-relation group. \( G \) will be built up by several HNN-extensions, starting with the free group on 2 generators, and thus be torsion-free.

We start with \( \mathcal{R} \) from Section 4 and for convenience we identify \( \alpha \) with \( s_1 \) and \( \sigma \) with \( s_2 \). We will now define the group \( G = G(\mathcal{R}, \Phi_0) \) as in Part I of [10]. Note that, compared to the original construction in [10], \( \mathcal{R} \) will replace \( \Lambda \), and we will write \( W^\# \) instead of \( W \). First we make the following definition:

**Definition 5.1** We define the semigroup

\[
\mathcal{R}_s := \langle s_1, s_2, q \mid F_mq = qK_m \ (m = 1, \ldots, 5) \rangle
\]

where

\[
\begin{align*}
F_1 &= s_2s_1^2 \\
F_2 &= s_2s_1^2 \\
F_3 &= \tau(M) \\
F_4 &= s_1 \\
F_5 &= s_2 \\
K_1 &= s_1^2s_2s_1s_2 \\
K_2 &= s_1^2s_2^2 \\
K_3 &= \tau(N) \\
K_4 &= s_1 \\
K_5 &= s_2
\end{align*}
\]

Observe that the \( F_m \)'s (resp. \( K_m \)'s) are just the left (resp. right) sides of the relations of \( \mathcal{R} \), re-written in terms of \( s_1, s_2 \).
Definition 5.2 We define the following sequence of groups

\[ F := H_1 := \langle a, d \rangle \]
\[ H_2 := \langle H_1; s_1, s_2 \mid s_b^{-1}as_b = a, s_b^{-1}ds_b = d^6ad^6 \ (b = 1, 2) \rangle \]
\[ H_3 := \langle H_2; q \mid q^{-1}(d^ma^md^mF_m^#)q = K_md^ma^md^m \ (m = 1, \ldots, 5) \rangle \]
\[ H_4 := \langle H_3; t \mid t^{-1}at = a, t^{-1}dt = d \rangle \]
\[ G := H_5 := \langle H_4; k \mid k^{-1}ak = a, k^{-1}dk = d, k^{-1}(\Phi_0^{-1}q^{-1}tq\Phi_0)k = (\Phi_0^{-1}q^{-1}tq\Phi_0) \rangle \]

where \( \Phi_0 := \chi(p)s_2s_1^2t_1^2 \) (= \( s_2s_1s_2s_1^2s_2s_1^2s_1s_2^{-3}s_1^2 \)).

Remark In [10] \( H_2 \) is called \( G_4 \), \( H_3 \) is called \( G_2 \) and \( H_4 \) is called \( G_1 \) (with \( G_3 \) absent.)

Lemma 5.3 \( H_{i+1} \) is a HNN-extension of \( H_i \) for each \( i = 1, 2, 3, 4 \).

Proof \( H_5 \) and \( H_4 \) are HNN-extensions of \( H_4 \) and \( H_3 \) with the identity maps on the subgroups \( \langle a, d, \Phi_0^{-1}q^{-1}tq\Phi_0 \rangle \) and \( \langle a, d \rangle \) respectively. Observe that \( \langle a, d^6ad^6 \rangle \) is the free group on two generators as a word reduced on \( a \) and \( d^6ad^6 \) is reduced on \( a \) and \( d \) as well. Hence \( H_2 \) is an HNN-extension of \( H_1 \) along the isomorphism sending \( a \mapsto a \), \( d \mapsto d^6ad^6 \). Finally it can be shown (such as in Lemma 15 in Part I of [10]) that the subgroups \( \langle d^ma^md^mF_m^#, \ m = 1, \ldots, 5 \rangle \) and \( \langle K_md^ma^md^m, \ m = 1, \ldots, 5 \rangle \) are free on their given generators. Hence \( H_3 \) is a HNN-extension of \( H_2 \) along the isomorphism \( d^ma^md^mF_m^# \mapsto K_md^ma^md^m, \ m = 1, \ldots, 5 \).

Lemma 5.4 The group \( G \) is torsion-free.

Proof Using Lemma 5.3 we can apply Proposition 3.1 to get the following equality chain:

\[ \text{Tord}(G) = \text{Tord}(H_4) = \text{Tord}(H_3) = \text{Tord}(H_2) = \text{Tord}(F) = \emptyset \]

In the next section we will use this f.p. ‘simulator’ group \( G \) to simulate equality in \( \mathfrak{N} \), and hence equality in \( A \).

6 Using the simulator group \( G \) to embed \( A \) into a 13 generator 33 relation group \( K_4 \)

In this section we give an overview of the construction by Boone and Collins in p.3–4 in Section I of [3] (which is a variant of Aanderaa’s proof of the Higman Embedding Theorem in [1]) to obtain an explicit embedding of our original group \( A \) into a group \( K_4 \) with 13 generators and 33 relations. Since \( K_4 \) is built up entirely by HNN-extensions, starting with the group \( A \ast G \), we see that this embedding preserves torsion orders.

From Definition 5.2 we see that the group \( G \) has finite presentation

\[ s_1, s_2, q, t, k, a, d; \]
\[ s_j^{-1}as_j = a \quad s_j^{-1}ds_j = d^6ad^6 \]
\[ q^{-1}(d^ma^md^mF_m^#)q = K_md^ma^md^m \]
\[ ta = at \quad td = dt \]
\[ka = ak \quad kd = dk\]
\[k^{-1}(\Phi_0^{-1}q^{-1}tq\Phi_0)k = (\Phi_0^{-1}q^{-1}tq\Phi_0)\]

for all \(m = 1, \ldots, 5\), \(j = 1, 2\), and where \(\Phi_0 := \chi(p)s_2s_1^2 = s_2s_1s_1s_2s_1s_2s_1^{-3}s_2^{-2}\).

The following theorem, a combination of two results from [10], gives the key properties of \(\mathfrak{R}_*\) and \(G\) that we require.

**Theorem 6.1** [10] Lemma 0 and Technical Result (i) Let \(\Delta\) and \(\Pi\) be any words of \(\mathfrak{R}\). Then

\[\Delta\Pi = \Phi\]

\[\Delta\Pi = \Phi_0\] in \(\mathfrak{R}\) \(\iff\) \(\Delta\Pi = \Phi_0\) in \(\mathfrak{R}_*\) \(\iff\) \(k^{-1}(\Delta^#q\Pi)^{-1}t(\Delta^#q\Pi)k = (\Delta^#q\Pi)^{-1}t(\Delta^#q\Pi)\) in \(G\)

**Proof** This is proved in [10] p.307–313.

Let \(\theta(W) := \chi(W)^#\). We will now show how we simulate equality in \(A\) by equality in \(G\).

**Theorem 6.2** [3] Theorem 2] Let \(U\) be any word of \(A\). Then

\[U = 1\] in \(A\) \(\iff\) \(k^{-1}(\theta(U)q\Phi_0)^{-1}t(\theta(U)q\Phi_0)k = (\theta(U)q\Phi_0)^{-1}t(\theta(U)q\Phi_0)\) in \(G\)

**Proof** This is just Theorem 11 [11] combined with Theorem 6.1 see [3] Theorem 2).

Thus \(G\) ‘simulates’ equality in \(A\). Note that \(G\) has 7 generators and 14 relations, regardless of the original finite presentation of \(A\) that we input. Note that, while \(G\) ‘simulates’ equality in \(A\), there is no reason that \(A\) should embed into \(G\). Indeed, \(G\) is always torsion-free (Lemma 6.4), regardless of whether \(A\) has torsion or not. To embed \(A\) into an f.p. group with few generators and relators, we will now follow Aanderaa’s proof of the Higman Embedding Theorem [1], in the same way as done on p.3–5 of [3].

**Definition 6.3** Let \(C = \langle c_1, c_2 \rangle\) be an isomorphic copy of \(A\). (Recall from the beginning of Section 4 that w.l.o.g. \(A\) is 2-generated.) Define \(k_0 := (q\Phi_0)k(q\Phi_0)^{-1}\) and the following groups

\[K_1 := C \ast G\]

\[K_2 := \langle K_1; b_1, b_2 \mid b_1^{-1}s_jb_i = s_j, b_1^{-1}c_jb_i = c_j, b_1^{-1}k_0b_i = k_0c_1^{-1} \quad (i, j = 1, 2) \rangle\]

\[K_3 := \langle K_2; f \mid f^{-1}\theta(ai)^{ef} = \theta(ai)^{e}, f^{-1}k_0f = k_0 \quad (i = 1, 2, \epsilon = \pm1) \rangle\]

\[K_4 := \langle K_3; h \mid h^{-1}th = tf, h^{-1}k_0h = k_0, h^{-1}s_jh = s_j \quad (j = 1, 2) \rangle\]

**Lemma 6.4** [3] Theorem 3 (i) \(K_{i+1}\) is a HNN-extension of \(K_i\) for \(i = 1, 2, 3\).

**Proof** This is shown on p.5–6 of [3].

Given how clear the construction of \(K_4\) is, we can deduce its torsion orders.

**Lemma 6.5** \(\text{Tord}(K_4) = \text{Tord}(A)\).

**Proof** Combining Lemmata 6.3 and 6.4 with Propositions 6.1 and 6.2 gives that

\[\text{Tord}(K_4) = \text{Tord}(K_3) = \text{Tord}(K_2) = \text{Tord}(K_1) = \text{Tord}(C) \cup \text{Tord}(G) = \text{Tord}(A) \cup \emptyset\]
The reason for constructing $K_4$ was to embed $A$ into an f.p. group with few generators and relations. We now see that $K_4$ can indeed be given by such a ‘small’ finite presentation.

**Lemma 6.6** [3, Theorem 3 (ii)] *The defining relations of $C$ are redundant in $K_4$.*

**Proof** We will follow the proof given on p.4 of [3]. Let $W_a$ be a word of $A$ with $W_a = 1$ in $A$. Let $W_b$ and $W_c$ be copies of $W_a$ with the $a_i$’s replaced with the corresponding $b_i$’s and $c_i$’s. To prove the result it is enough to show that $W_c = 1$ in $K_4$ without using any of the relations of $C$. Theorem 6.2 tells us that

$$k^{-1}((\theta(W_a)q\Phi_0)^{-1}t(\theta(W_a)q\Phi_0))k = (\theta(W_a)q\Phi_0)^{-1}t(\theta(W_a)q\Phi_0)$$

$$\Rightarrow k_0^{-1}\theta(W_a)^{-1}t\theta(W_a)k_0 = \theta(W_a)^{-1}t\theta(W_a)$$

(1)

Now conjugate this by $h$ and using the relations introduced in $K_4$ we obtain

$$h^{-1}k_0^{-1}\theta(W_a)^{-1}t\theta(W_a)k_0h = h^{-1}\theta(W_a)^{-1}t\theta(W_a)h$$

$$\Rightarrow k_0^{-1}h^{-1}\theta(W_a)^{-1}t\theta(W_a)hk_0 = h^{-1}\theta(W_a)^{-1}t\theta(W_a)h$$

$$\Rightarrow k_0^{-1}\theta(W_a)^{-1}h^{-1}t\theta(W_a)k_0 = \theta(W_a)^{-1}h^{-1}t\theta(W_a)$$

$$\Rightarrow k_0^{-1}\theta(W_a)^{-1}tf\theta(W_a)k_0 = \theta(W_a)^{-1}tf\theta(W_a)$$

(2)

Now multiply the inverse of (1) by (2) and then use the relations introduced in $K_3$ to get

$$k_0^{-1}\theta(W_a)^{-1}f\theta(W_a)k_0 = \theta(W_a)^{-1}f\theta(W_a)$$

$$\Rightarrow k_0^{-1}\theta(W_a)^{-1}f\theta(W_a)W_bk_0f = \theta(W_a)^{-1}f\theta(W_a)W_bf$$

$$\Rightarrow k_0^{-1}W_bk_0 = W_b$$

Now use the relations introduced in $K_2$ to get

$$W_bW_c = W_b$$

$$\Rightarrow W_c = 1$$

which is exactly what we wanted to show.

So $K_4$ has a presentation with the 2 generators of $C$ (we discarded the relations of $C$), the generators (7) and relations (14) of $G$, and the stable letters (4) and HNN-relations (19) of the chain of three HNN-extensions from $C * G$ to $K_4$. Counting these up, we see that we have embedded $A$ into an f.p. group $K_4$ with 13 generators and 33 relations, regardless of the input presentation for $A$.

**7 The explicit presentation of the group $K_4$ with 13 generators and 33 relations**

We now give a (mostly) explicit presentation of $K_4$, and state its properties.
Theorem 7.1 There is a uniform algorithm for embedding a finitely presented group $A$ into another finitely presented group $K_4$ with 13 generators and 33 relations. Moreover, $\text{Tord}(A) = \text{Tord}(K_4)$, and $K_4$ is given by the presentation

$$s_1, s_2, q, t, k, a, d, c_1, c_2, b_1, b_2, f, h;$$

$$s_j^{-1}s_j = a \quad s_j^{-1}ds_j = d^6ad^6$$

$$q^{-1}(dads_1)^2s_1^{-2})q = s_2^2s_1s_2dad$$

$$q^{-1}(d^2ad^2s_1^{-1}s_1^{-2})q = s_1^2s_2^2d^2ad^2$$

$$q^{-1}(d^3ad^3\tau(M)^#)q = \tau(N)d^3ad^3$$

$$q^{-1}(d^3jad^3js_j^{-1})q = s_jd^3jad^3j$$

$$ta = at \quad td = dt$$

$$ka = ak \quad kd = dk$$

$$k^{-1}(\Phi_0^{-1}q^{-1}tq\Phi_0)k = (\Phi_0^{-1}q^{-1}tq\Phi_0)$$

$$b_i^{-1}s_jb_i = s_j \quad b_i^{-1}c_jb_i = c_j \quad b_i^{-1}k_0b_i = k_0c_i^{-1}$$

$$f^{-1}\theta(a_i)^{\epsilon}b_i^{\epsilon}f = \theta(a_i)^{\epsilon} \quad f^{-1}k_0f = k_0$$

$$h^{-1}th = tf \quad h^{-1}k_0h = k_0 \quad h^{-1}s_jh = s_j$$

for all $i = 1, 2, j = 1, 2, \epsilon = \pm 1$ and with the shorthand

$$\Phi_0 := \chi(p)s_2^2s_1^2 \quad (= s_2s_1s_2s_1s_2^2s_1s_2^2)$$

$$k_0 := (q\Phi_0)k(q\Phi_0)^{-1}$$

where $r$ and $t$ are the integers determined in Section 3.

Proof By Lemma 6.1 we have $C$ (which is an isomorphic copy of $A$) embedding into $K_4$. By Lemma 6.3 we have $\text{Tord}(A) = \text{Tord}(K_4)$. Finally by Lemma 6.6 the defining relations of $C$ in $K_4$ are redundant and so $K_4$ has the given presentation.

We can apply this construction to certain universal groups to get new ones with few generators and relations.

Theorem 7.2 Let $X$ be a recursively enumerable set. Then there is a universal finitely presented $X$-torsion-free group with 13 generators and 33 relations with a presentation as given in Theorem 7.1.

Proof Apply Theorem 7.1 to any universal finitely presented $X$-torsion-free group, for example the one from Theorem 3.3.

Remark We note that although most of the relations of $K_4$ are determined explicitly, the relation $q^{-1}(d^3ad^3\tau(M)^#)q = \tau(N)d^3ad^3$ is extremely long and depends entirely on our initial choice of finite presentation for $A$. As such we cannot give a completely explicit presentation for these universal groups using this method. It would be interesting to see a construction of a completely explicit presentation for a universal f.p. torsion-free group. We have not been able to do this, as the Turing machine needed in Theorem 3.3 (coming from [8, Theorem 2.11]) appears to be very difficult to construct. Theoretically this should be possible by the Church-Turing Thesis.
8 Embedding an f.p. group with several commuting generators into one with fewer generators and fewer relations

In this section we give an overview of the construction by Collins in [9] (which is itself a refinement of an idea due to Borisov in [5]) for embedding an f.p. group with many commuting generators into an f.p. group with fewer generators and relations. Borisov’s original construction dealt with the case where one generator $c$ commutes with a set of generators $\{u_j\}$. Collins’ refinement deals with the more general case, where one set of generators $\{c_j\}$ simultaneously commute with another set $\{u_k\}$ (that is, $c_j u_k = u_k c_j$ for all $j, k$).

We note that this embedding preserves torsion orders, as it is built from HNN-extensions and an amalgamated free product. Due to the length of many of the relations, an explicit form will not be given.

**Definition 8.1** Suppose we have a finite presentation of a group
\[
\Gamma = \langle x_i, c, u_j \mid D, cu_j = u_j c \rangle
\]
with $D$ a set of relations. We construct the following groups:

\[
\Gamma_1 := \langle \Gamma; a, d \mid a^{-1}ca = c, d^{-1}cd = c \rangle
\]

\[
C' := \langle c' \rangle \quad \text{(an isomorphic copy of } C = \langle c \rangle)\]

\[
K := \langle C'; x, y \mid x^{-1}c'x = c', y^{-1}c'y = c' \rangle
\]

\[
\Gamma_2 := \langle \Gamma_1 \ast K \mid c = c', d = y, u_j a^{-1}d a^j = x^{-j}y a^j \rangle
\]

\[
\Gamma_3 := \langle \Gamma_2; b \mid b^{-1}cb = c, b^{-1}ab = d, b^{-1}db = x \rangle
\]

**Lemma 8.2** We have the following extensions

(i) $\Gamma_1$ is an HNN-extension of $\Gamma$

(ii) $K$ is an HNN-extension of $C'$

(iii) $\Gamma_2$ is an amalgamated free product of $\Gamma_1$ and $K$

(iv) $\Gamma_3$ is an HNN-extension of $\Gamma_2$

**Proof** These are shown in the proof of [9, Theorem].

The clear construction of $\Gamma_3$ from $\Gamma$ allows us to understand the torsion orders of $\Gamma_3$.

**Lemma 8.3** $\text{Tord}(\Gamma_3) = \text{Tord}(\Gamma)$.

**Proof** As $C \leq \Gamma$ we have $\text{Tord}(C) \subseteq \text{Tord}(\Gamma)$. So Lemma 8.2 now implies that $\text{Tord}(\Gamma_3) = \text{Tord}(\Gamma) \cup \text{Tord}(C) = \text{Tord}(\Gamma)$, via Propositions 3.1 and 3.2.

Borisov [5] used the above construction to embed $\Gamma$ into an f.p. group with fewer generators and relations; we follow the exposition of this as given by Collins in [9].
Theorem 8.4 [9, Theorem 1](Borisov’s Theorem) Define the map on words induced by \( \lambda(c) := c \), \( \lambda(x_i) = x_i \) and \( \lambda(u_j) := b^{-2}a^{-j}bab^{-1}a^ib^2a^{-j}b^{-1}a^{-1}b^j \).

Define the group \( \Gamma' := \langle x_i, a, b, c \mid \lambda(D), ac = ca, bc = cb \rangle \)

Then the induced map \( \overline{\lambda} : \Gamma \to \Gamma' \) is an embedding of groups with \( \text{Tord}(\Gamma') = \text{Tord}(\Gamma) \).

Proof We can remove excess generators and relations to see that \( \Gamma_3 \) is the same group as \( \Gamma' \) and that we get the given embedding \( \overline{\lambda} \).

Collins [9] was able to refine Borisov’s construction, to deal with the case where there are two sets of commuting generators.

Theorem 8.5 [9, Theorem 2] Suppose we have a finite group presentation \( \Gamma = \langle x_i, c_j, u_k \mid D, c_j u_k = u_k c_j \rangle \)

Then there is a map \( \lambda \) (which can be found algorithmically) such that \( \Gamma \) embeds into \( \Gamma' := \langle x_i, a, b, p, q \mid \lambda(D), ap = pa, bp = pb, aq = qa, bq = qb \rangle \)

with \( \text{Tord}(\Gamma') = \text{Tord}(\Gamma) \).

Moreover, by setting \( J = |\{c_j\}| \) and \( K = |\{u_k\}| \), we see that \( \Gamma' \) has \( J + K - 4 \) fewer generators, and \( JK - 4 \) fewer relations, than \( \Gamma \) (assuming \( J + K \geq 4 \) and \( JK \geq 4 \)).

Proof This is similar to the above, see [9, Theorem 2] for details of the changes required. The construction is still formed entirely of HNN-extensions and amalgamated free products, so we still have \( \text{Tord}(\Gamma') = \text{Tord}(\Gamma) \).

Although Theorems 8.4 and 8.5 are embedding theorems, they should be seen as a sort of clever rewriting process.

9 Embedding \( K_4 \) into a group with 8 generators and 26 relations

We can now use Borisov’s Theorem, and its variant by Collins, to prove our main results.

Theorem 9.1 There is a uniform algorithm for embedding a finitely presented group \( A \) into another finitely presented group \( H \) with 8 generators and 26 relations. Moreover, \( \text{Tord}(A) = \text{Tord}(H) \).

Proof In \( K_4 \) replace the generators \( a, d, k \) with \( a_0 := q\Phi_0 a\Phi_0^{-1}q^{-1}, d_0 := q\Phi_0 d\Phi_0^{-1}q^{-1} \) and \( k_0 := q\Phi_0 k\Phi_0^{-1}q^{-1} \). We now have relations for \( k_0 \) commuting with \( a_0, d_0, f, h \) and \( t \). So applying Theorem 8.4 we can reduce the number of generators by 3 and and the number of relations by 3. Also we have relations for \( b_1 \) and \( b_2 \) commuting with \( s_1, s_2, c_1 \) and \( c_2 \). So we can apply Theorem 8.5 to reduce the number of generators by 2 and the number of relations by 4.

Theorem 9.2 Let \( X \) be a recursively enumerable set. Then there is a universal finitely presented \( X \)-torsion-free group with 8 generators and 26 relations.

Proof Apply Theorem 9.1 to any universal finitely presented \( X \)-torsion-free group, for example the one from Theorem 3.5.
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