BOUNDARY AND UNBounded Fredholm Modules
FOR QUANTUM PROJECTIVE SPACES
FRANCESCO D’ANDREA AND GIOVANNI LANDI

Abstract. We construct explicit generators of the K-theory and K-homology of the
coordinate algebra of ‘functions’ on quantum projective spaces. We also sketch a con-
struction of unbounded Fredholm modules, that is to say Dirac-like operators and spectral
triples of any positive real dimension.

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1. INTRODUCTION

For quantum projective spaces \( \mathbb{C}P^n_q \) we generalize some ideas of [5] and give ‘polynomial’
generators of its K-theory, that is projections whose entries are in the corresponding coor-
dinate algebra \( \mathcal{A}(\mathbb{C}P^n_q) \). Dually, we give generators of its K-homology via even Fredholm
modules \( (\mathcal{A}(\mathbb{C}P^n_q), \mathcal{H}(k), \gamma(k), F(k)) \), for \( k = 0, \ldots, n \). The ‘top’ Fredholm module – the only
one for which the representation of \( \mathcal{A}(\mathbb{C}P^n_q) \) is faithful – can be realized as the ‘conformal
class’ of a spectral triple \( (\mathcal{A}(\mathbb{C}P^n_q), \mathcal{H}(n), \gamma(n), D) \), namely we realize \( F(n) := D|D|^{-1} \) as the
‘sign’ of a Dirac-like operator. This procedure allows us to construct spectral triples of
any summability \( d \in \mathbb{R}^+ \).

In the following, without loss of generalities, the real deformation parameter is restricted
to be \( 0 < q < 1 \). Also, by \(*\)-algebra we shall mean a unital involutive associative \( \mathbb{C}\)-algebra,
and by representation of a \(*\)-algebra we always mean a unital \(*\)-representation.

The ‘ambient’ algebra for the quantum projective space is the coordinate algebra
\( \mathcal{A}(S^{2n+1}_q) \) of the unit quantum spheres. This \(*\)-algebra is generated by \( 2n + 2 \) elements

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\{ z_i, z_i^* \}_{i=0,\ldots,n} \text{ with relations } [8]:

\begin{align}
  z_i z_j &= q^{-1} z_j z_i & \forall \ 0 \leq i < j \leq n, \quad (1a) \\
  z_i^* z_j &= q z_j z_i^* & \forall \ i \neq j, \quad (1b) \\
  [z_i^*, z_j] &= (1 - q^2) \sum_{j=i+1}^{n} z_j z_j^* & \forall \ i = 0, \ldots, n-1, \quad (1c) \\
  [z_n^*, z_n] &= 0 , \quad (1d) \\
  z_0 z_n^* + z_1 z_n^* + \ldots + z_n z_n^* &= 1 . \quad (1e)
\end{align}

The *-subalgebra generated by \( p_{ij} := z_i^* z_j \) will be denoted \( \mathcal{A}(\mathbb{C}P_q^n) \), and identified with the algebra of ‘polynomial functions’ on the quantum projective space \( \mathbb{C}P_q^n \). The algebra \( \mathcal{A}(\mathbb{C}P_q^n) \) is made of (co)invariant elements for the \( U(1) \) (co)action \( z_i \to \lambda z_i \) for \( \lambda \in U(1) \). From the relations of \( \mathcal{A}(S_q^{2n+1}) \) one gets analogous relations for \( \mathcal{A}(\mathbb{C}P_q^n) \):

\begin{align}
  p_{ij} p_{kl} &= q^{\text{sign}(k-i) + \text{sign}(j-l)} p_{ki} p_{lj} & \text{if } i \neq l \text{ and } j \neq k , \\
  p_{ij} p_{jk} &= q^{\text{sign}(j-i) + \text{sign}(j-k) + 1} p_{jk} p_{ij} - (1 - q^2) \sum_{l>j} p_{jl} p_{li} & \text{if } i \neq k , \\
  p_{ij} p_{ji} &= q^{2 \text{sign}(j-i)} p_{ij} p_{ij} + (1 - q^2) \left( \sum_{l>i} q^{2 \text{sign}(j-i)} p_{jl} p_{lj} - \sum_{l>j} p_{li} p_{il} \right) & \text{if } i \neq j ,
\end{align}

with \( \text{sign}(0) := 0 \). The elements \( p_{ij} \) are the matrix entries of a projection \( P = (p_{ij}) \), that is \( P^2 = P = P^* \) or \( \sum_{i=0}^{n} p_{ij} p_{jk} = p_{ik} \) and \( p_{ij}^* = p_{ji} \). This projection has q-trace:

\[ \text{Tr}_q(P) := \sum_{i=0}^{n} q^{2i} p_{ii} = 1. \] (2)

The original notations of [8] are obtained by setting \( q = e^{h/2} \); the generators of [3] correspond to the replacement \( z_i \to z_{n+1-i} \), while the generators \( x_i \) used in [6] are related to ours by \( x_i = z_{n+1-i}^* \), and by the replacement \( q \to q^{-1} \).

Generators for the K-theory and K-homology of the spheres \( S_q^{2n+1} \) are in [6]; unfortunately, there is no canonical way to obtain generators of the K-theory and/or K-homology of subalgebras, unless they are dense subalgebras (for a pair \( C^* \)-algebra/pre-\( C^* \)-algebra, the \( K \)-groups coincide). That \( S_q^{2n+1} \) and \( \mathbb{C}P_q^n \) are truly different can be seen from the fact that the \( K \)-groups of the odd-dimensional spheres are \( (\mathbb{Z}, \mathbb{Z}) \), regardless of the dimension, while for \( \mathbb{C}P_q^n \) they are \( (\mathbb{Z}^{n+1}, 0) \).

That \( K_0(\mathcal{A}(\mathbb{C}P_q^n)) \cong \mathbb{Z}^{n+1} \) can be proved by viewing the corresponding \( C^* \)-algebra \( C(\mathbb{C}P_q^n) \), the universal \( C^* \)-algebra of \( \mathbb{C}P_q^n \), as the Cuntz–Krieger algebra of a graph [7]. The group \( K_0 \) is given as the cokernel of the incidence matrix canonically associated with the graph. The dual result for K-homology is obtained using the same techniques: the group \( K^0 \) is now the kernel of the transposed matrix [11]; this leads to \( K^0(\mathcal{A}(\mathbb{C}P_q^n)) = \mathbb{Z}^{n+1} \).

In [7], somewhat implicitly, there appear generators of the \( K_0 \) groups of \( C(\mathbb{C}P_q^n) \) as projections in \( C(\mathbb{C}P_q^n) \) itself. Here, we give generators of \( K_0(\mathcal{A}(\mathbb{C}P_q^n)) \) in the form of ‘polynomial functions’, so they represent elements of \( K_0(\mathcal{A}(\mathbb{C}P_q^n)) \) as well. They are also equivariant, i.e. representative of elements in \( K_0^{\mathfrak{su}(n+1)}(\mathcal{A}(\mathbb{C}P_q^n)) \). Besides, we give \( n+1 \) Fredholm modules that are generators of the homology group \( K^0(\mathcal{A}(\mathbb{C}P_q^n)) \).
2. K-homology

A useful (unital *-algebra) morphism $\mathcal{A}(S^{2n+1}) \to \mathcal{A}(S^{2n-1})$ given by the map $z_n \mapsto 0$, restricts to a morphism $\mathcal{A}(\mathbb{C}P^n_q) \to \mathcal{A}(\mathbb{C}P^{n-1}_q)$ and is heavily used in what follows. We also stress that representations of $\mathcal{A}(S^{2n+1})$, with $z_i$ in the kernel for all $i > k$, and for a fixed $0 \leq k < n$, are the pullback of representations of $\mathcal{A}(S^{2k+1})$.

Here we seek Fredholm modules (and then irreducible *-representations) for $\mathbb{C}P^n_q$ that are not the pullback of Fredholm modules (resp. irreps) for $\mathbb{C}P^{n-1}_q$. So, we look for irreps of $S^{2n+1}_q$ for which $z_n$ is not in the kernel. These are given in [6], and classified by a phase: in particular, they are inequivalent as representation of $S^{2n+1}_q$ but give the same representation of $\mathbb{C}P^n_q$ (whatever the value of the phase is). In fact, we start with representations that are not exactly irreducible, but close to it: they are the direct sum of an irreducible representation with copies of the trivial $a \mapsto 0$ representation.

**Definition 1.** We use the following multi-index notation. We let $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and, for $0 \leq i < k \leq n$, we denote by $\xi^k_i \in \{0, 1\}^n$ the array

$$\xi_i^k := (\underbrace{0, 0, \ldots, 0}_{i \text{ times}}, \underbrace{1, 1, \ldots, 1}_{k-i \text{ times}}, \underbrace{0, 0, \ldots, 0}_{n-k \text{ times}}).$$

Let $\mathcal{H}_n := \ell^2(\mathbb{N}^n)$, with orthonormal basis $|m\rangle$. For any $0 \leq k \leq n$, a representation $\pi_k^{(n)} : \mathcal{A}(S^{2n+1}_q) \to \mathcal{B}(\mathcal{H}_n)$, is defined as follows (all the representations are on the same Hilbert space). We set $\pi_k^{(n)}(z_i) = 0$ for all $i > k \geq 1$, while for the remaining generators

$$\pi_k^{(n)}(z_i) |m\rangle = q^{m_i} \sqrt{1 - q^{2(n_i+1-m_i+1)}} |m + \xi_i^k\rangle,$$

on the subspace $\mathcal{V}_k^n \subset \mathcal{H}_n$ — linear span of basis vectors $|m\rangle$ satisfying the restrictions

$$0 \leq m_1 \leq m_2 \leq \ldots \leq m_k, \quad m_{k+1} > m_{k+2} > \ldots > m_n \geq 0,$$

with $m_0 := 0$, and they are zero on the orthogonal subspace. When $k = 0$, we define $\pi_{0}^{(n)}(z_i) = 0$ if $i > 0$, $\pi_{0}^{(n)}(z_0) |m\rangle = |m\rangle$ if $m_1 > m_2 > \ldots > m_n \geq 0$, and $\pi_{0}^{(n)}(z_0) |m\rangle = 0$ in all the other cases.

As a *-algebra, $\mathcal{A}(\mathbb{C}P^n_q)$ is generated by the elements $p_{ij}$, with $i \leq j$ since $p_{ji} = p_{ij}^*$; in fact, from the tracial relation [2] one of the generators on the diagonal, say $p_{nn}$, is redundant. The computation of $\pi_k^{(n)}(p_{ij})$ for $i \leq j$ gives the following. If $m$ satisfies (3):

$$\pi_k^{(n)}(p_{ij}) |m\rangle = q^{m_i+m_j} \sqrt{1 - q^{2(m_i+1-m_j+1)}} \sqrt{1 - q^{2(m_j+1-m_i+1)}} |m - \xi_i^j\rangle$$

if $0 \leq i \leq j < k$,

$$\pi_k^{(n)}(p_{ik}) |m\rangle = q^{m_i+m_k} \sqrt{1 - q^{2(m_i+1-m_k+1)}} |m - \xi_i^k\rangle$$

if $0 \leq i < k$,

$$\pi_k^{(n)}(p_{kk}) |m\rangle = q^{2m_k} |m\rangle,$$

$$\pi_k^{(n)}(p_{ij}) |m\rangle = 0$$

if $j > k$,

and $\pi_k^{(n)}(p_{ij}) |m\rangle = 0$ if $m$ does not satisfy (3).
Each representation $\pi_k^{(n)}$ is an irreducible $*$-representation of both $\mathcal{A}(S_q^{2n+1})$ and $\mathcal{A}(\mathbb{CP}_q^n)$ when restricted to $\mathcal{V}_k^n$, and is identically zero outside $\mathcal{V}_k^n$.

**Lemma 2.** The spaces $\mathcal{V}_k^n$ enjoy the properties: $\mathcal{V}_j^n \perp \mathcal{V}_k^n$ if $|j - k| > 1$, while $\mathcal{V}_k^n \cap \mathcal{V}_k^n$ is the span of vectors $|m\rangle$ with

$$0 \leq m_1 \leq m_2 \leq \ldots \leq m_k, \quad m_k > m_{k+1} > \ldots > m_n \geq 0,$$

for all $1 \leq k \leq n$. As a consequence, if $|j - k| > 1$, for all $a, b \in \mathcal{A}(S_q^{2n+1})$

$$\pi_j^{(n)}(a) \pi_k^{(n)}(b) = 0.$$

**Proof.** Let $0 \leq j < k - 2 \leq n - 2$. Due to (3), if $m_{k-1} \leq m_k$ the vector $|m\rangle$ is not in $\mathcal{V}_j^n$, and if $m_{k-1} > m_k$ the vector $|m\rangle$ is not in $\mathcal{V}_k^n$. This proves that $\mathcal{V}_j^n$ and $\mathcal{V}_k^n$ are orthogonal subspaces of $\mathcal{H}_n$. The remaining claims are straightforward. \hfill $\square$

As a corollary of previous Lemma, the maps $\pi_{\pm}^{(n)} : \mathcal{A}(S_q^{2n+1}) \rightarrow \mathcal{B}(\mathcal{H}_n)$, defined by

$$\pi_{\pm}^{(n)}(a) := \sum_{0 \leq k \leq n} \pi_{\pm}^{(n)}(a) \quad \text{and} \quad \pi_{\pm}^{(n)}(a) := \sum_{0 \leq k \leq n} \pi_{\pm}^{(n)}(a),$$

are representations of the algebra $\mathcal{A}(S_q^{2n+1})$.

**Proposition 3.** The difference $\pi_+^{(n)}(a) - \pi_-^{(n)}(a)$ is of trace class on $\mathcal{H}_n$ for all $a \in \mathcal{A}(\mathbb{CP}_q^n)$; furthermore, the trace is given by a series which – as a function of $q$ – is absolutely convergent on the open interval $0 < q < 1$.

**Proof.** It is enough to prove the claim for $a = p_{ij}$, with $0 \leq i \leq j \leq n$. The space $\mathcal{H}_n$ is the orthogonal direct sum of $\mathcal{V}_k^n \cap \mathcal{V}_k^n$, for all $1 \leq k \leq n$, plus the joint kernel of all the representations involved. By Lemma 2 on $\mathcal{V}_k^n \cap \mathcal{V}_k^n$ only the representations $\pi_{k-1}^{(n)}$ and $\pi_k^{(n)}$ are different from zero (and one contributes to $\pi_+^{(n)}$, while the other to $\pi_-^{(n)}$, according to the parity of $k$). We need to prove that $\pi_{k-1}^{(n)}(p_{ij}) - \pi_k^{(n)}(p_{ij})$ is of trace class, and that the trace is absolutely convergent for any $0 < q < 1$.

From the explicit expressions given above, we see that $\pi_{k-1}^{(n)}(p_{ij})$ and $\pi_k^{(n)}(p_{ij})$ are both zero, if $j > k$. For $j = k$, $\pi_{k-1}^{(n)}(p_{ik})$ is zero and $\pi_k^{(n)}(p_{ik})$ has matrix coefficients bounded by $q^{m_k}$. For $j = k - 1$, one uses the inequality $|1 - \sqrt{1 - x^2}| \leq x$ — valid for $0 \leq x \leq 1$ — to prove that $\pi_{k-1}^{(n)}(p_{ij}) - \pi_k^{(n)}(p_{ij})$ still has matrix coefficients bounded by $q^{m_k}$. For $0 \leq i < j < k - 2$, the operators $\pi_{k-1}^{(n)}(p_{ij})$ and $\pi_k^{(n)}(p_{ij})$ coincide on $\mathcal{V}_k^n \cap \mathcal{V}_k^n$. The observation that the series

$$\sum_{\text{m satisfying (4)}} q^{m_k} = \sum_{m_k = n-k}^{\infty} \binom{m_k + k - 1}{k - 1} \binom{m_k}{n - k} q^{m_k}$$

is absolutely convergent for $0 < q < 1$ concludes the proof. \hfill $\square$

Assembling things together, a 1-summable even Fredholm module for $\mathcal{A}(\mathbb{CP}_q^n)$ is obtained by using the representation $\pi^{(n)} := \pi_+^{(n)} \oplus \pi_-^{(n)}$ on $\mathcal{H}_n := \mathcal{H}_n \oplus \mathcal{H}_n$, the obvious
grading operator $\gamma(n)$ and, as usual,

$$F(n) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(6)

All this is valid for any $n \geq 1$. Additional $n - 1$ Fredholm modules are obtained by the same construction for $\mathcal{A}(\mathbb{C}P^n_j)$, $1 \leq j \leq n - 1$, and by pulling them back to $\mathcal{A}(\mathbb{C}P^n_q)$. The last Fredholm module is the pullback of the canonical non-trivial Fredholm module on $\mathbb{C}$, given on $\mathbb{C} \oplus \mathbb{C}$ by the representation $c \mapsto c \oplus 0$ and by usual grading and operator $F$.

For $0 \leq k \leq n$ and $N \in \mathbb{N}$, we denote $[\mu_k]$ the class of the Fredholm module $$(\mathcal{A}(\mathbb{C}P^n_q), \mathcal{H}(k), \pi^{(k)}, \gamma(k), F(k)).$$

3. K-theory

We need some notation. The $q$-analogue of an integer number $n$ is given by

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}.$$ 

This is defined for $q \neq 1$ and equals $n$ in the limit $q \to 1$. For any $n \geq 1$, define

$$[n]! := [n][n-1] \ldots [1],$$

to be the $q$-factorial, with $[0]! := 1$, and the $q$-multinomial coefficients as

$$[j_0, \ldots, j_n]! := \frac{[j_0 + \ldots + j_n]!}{[j_0]! \ldots [j_n]!}.$$ 

With $N \geq 0$, let $\Psi_N = (\psi^N_{j_0, \ldots, j_n})$ be the vector-valued function on $S^{2n+1}_q$ with components

$$\psi^N_{j_0, \ldots, j_n} := [j_0, \ldots, j_n]!^{1/2} q^{-\frac{1}{2} \sum r<s j_r j_s (z_{j_n} \ldots z_{j_0})^*}, \quad \forall j_0 + \ldots + j_n = N;$$

there are $\binom{N+n}{n}$ of them. Then $\Psi_N^\dagger \Psi_N = 1$ and $P_N := \Psi_N \Psi_N^\dagger$ is a projection; the proof is in [3], and is a generalization of the case $n = 2$ in [5]. In particular $P_1 = P$ is the ‘defining’ projection in Sec. 1 of the algebra $\mathcal{A}(\mathbb{C}P^n_q)$.

Projections $P_{-N}$ are then obtained from $P_N$ with simple substitutions. We notice that the elements $z'_i = q^i z^*_i$ satisfy the defining relations of $\mathcal{A}(S^2_{q^{-1}})$; the non-trivial relation to check is (1c). Multiplying (1c) by $q^{2i}$ and summing for $i \geq k$ it becomes

$$\sum_{i \geq k} q^{2i} z^*_i z_i = q^{2k} \sum_{i \geq k} z_i z^*_i,$$

now subtracting $q^{-2}$ times the same equation evaluated for $k + 1$, we get

$$q^{2k} z^*_k z_k + (1 - q^{-2}) \sum_{i > k} q^{2i} z^*_i z_i = q^{2k} z^*_k z_k,$$

that means

$$[z'^*_k, z'_k] = (1 - q^{-2}) \sum_{j=i+1}^n z'_j z'^*_j.$$
Vectors $\Psi_{-N} = (\psi_{j_0, \ldots, j_n}^{-N})$ are obtained out of $\Psi_N$ by first replacing $q$ with $q^{-1}$, and then replacing $z_i$ with $z_i^*$. Since the $q$-analogue is invariant under $q \to q^{-1}$, we obtain:

$$
\psi_{j_0, \ldots, j_n}^{-N} := [j_0, \ldots, j_n]_1^2 q^{i \sum_{r<s} j_r j_s (z_{r0}^*)^*} \ldots (z_{n0}^*)^* \\
= [j_0, \ldots, j_n]_1^2 q^{i \sum_{r<s} j_r j_s + \sum_{r=0}^n r j_r z_{r0}^*} \ldots z_{n0}^*, \quad \forall j_0 + \ldots + j_n = N.
$$

Then $\Psi_{-N}^\dagger \Psi_{-N} = 1$ and $P_{-N} := \Psi_{-N} \Psi_{-N}^\dagger$ is a projection.

For $N \in \mathbb{N}$ we denote by $[P_{-N}]$ be the class of the projection $P_{-N}$. In the previous section we gave Fredholm module classes $[\mu_k]$ for each $0 \leq k \leq n$.

**Proposition 4.** For all $N \in \mathbb{N}$ and for all $0 \leq k \leq n$ it holds that

$$
\langle [\mu_k], [P_{-N}] \rangle := \text{Tr}_{H_k}(\pi^{(k)}_+ - \pi^{(k)}_-)(\text{Tr} P_{-N}) = \binom{N}{k},
$$

with $\binom{N}{k} := 0$ when $k > N$.

**Proof.** We have

$$
\langle [\mu_k], [P_{-N}] \rangle = \sum_{m \in \mathbb{N}^k, m_k = 0}^k \sum_{j=0}^k (-1)^j \langle m | \pi_j \text{Tr} P_{-N} | m \rangle.
$$

The integer $\langle [\mu_0], [P_{-N}] \rangle$ is the trace of the projection $P_{-N}$ evaluated at the classical point (the unique 1-dimensional representation): one finds $\langle [\mu_0], [P_{-N}] \rangle = 1$. The integer $\langle [\mu_k], [P_{-N}] \rangle$ being given, for $1 \leq k \leq n$, by a series which is absolutely convergent – as a function of $q$ – in the open interval $]0, 1[$, it can be computed in the $q \to 0^+$ limit (cf. Sec. 5.3 in [2] or [4]). The case $N = 0$ is trivial; we focus on $N \geq 1$. We notice that $q^{\frac{1}{2}n(n-1)[n]} = 1 + O(q)$ for all $n \geq 0$. This implies

$$
q^{\sum_{r<s} j_r j_s [j_0, \ldots, j_n]} = 1 + O(q)
$$

for all $j_0, \ldots, j_n \geq 0$. Thus,

$$
(\psi_{j_0, \ldots, j_n}^{-N})^* = \{1 + O(q)\} q^{\sum_{r=0}^n r j_r} (z_{r0}^*)^* \ldots (z_{n0}^*)^*.
$$

Since matrix elements of $z_i, z_i^*$ (in the representations) are $O(1)$, we can conclude that $(\psi_{j_0, \ldots, j_n}^{-N})^* = O(q)$ unless $\sum_{r=0}^n r j_r = 0$, which is equivalent to $j_0 = N$. Hence, for the pairing with the $k$-th Fredholm module, when $1 \leq k \leq n$, we get, for any $0 \leq j \leq k$,

$$
\langle m | \pi_j^{(k)}(\text{Tr}(P_{-N})) | m \rangle = \langle m | \pi_j^{(k)}(z_0^N(z_0^N)^*) | m \rangle + O(q).
$$

In turn,

$$
\langle [\mu_k], [P_{-N}] \rangle = \\
= \lim_{q \to 0^+} \sum_{j=1}^k (-1)^{j-1} \text{Tr}_{V_{j-1} \cap V_j} \left\{ \langle m | \pi_j^{(k)}(z_0^N(z_0^N)^*) | m \rangle - \langle m | \pi_j^{(k)}(z_0^N(z_0^N)^*) | m \rangle \right\}.
$$
But \( \pi_j^{(k)}(z_0^N(z_0^N)^*) \) is the identity operator on \( \mathcal{V}_0^k \), while for \( j \geq 1 \) we have
\[
\pi_j^{(k)}(z_0^N(z_0^N)^*) |m\rangle = \begin{cases} 
(1 - q^{2m_1})(1 - q^{2(m_1-1)}) \ldots (1 - q^{2(m_1-N+1)}) |m\rangle & \text{if } m_1 \geq N, \\
0 & \text{otherwise}.
\end{cases}
\]

Then, in the difference \( \pi_j^{(k)}(z_0^N(z_0^N)^*) - \pi_j^{(k)}(z_0^N(z_0^N)^*) \), the unique non-zero contribution comes from the term with \( j = 1 \). Since
\[
\langle m | \pi_1^{(k)}(z_0^N(z_0^N)^*) |m\rangle = \begin{cases} 
1 + O(q) & \text{if } m_1 \geq N, \\
0 & \text{otherwise},
\end{cases}
\]
we get
\[
\langle [\mu_k], [P_{-N}] \rangle = \lim_{q \to 0^+} \text{Tr}_{\mathcal{V}_0^k \cap \mathcal{V}_1^k} \left\{ 1 - \langle m | \pi_1^{(k)}(z_0^N(z_0^N)^*) |m\rangle \right\} = \sum_{N>m_1>m_2>\ldots>m_k \geq 0} 1 = \binom{N}{k},
\]
with the notation \( \binom{N}{k} = 0 \) if \( k > N \). This concludes the proof. \( \square \)

**Proposition 5.** The elements \([\mu_0], \ldots, [\mu_n]\) are generators of \( K^0(\mathcal{A}(\mathbb{CP}_q^n)) \), and the elements \([P_0], \ldots, [P_{-n}]\) are generators of \( K_0(\mathcal{A}(\mathbb{CP}_q^n)) \).

**Proof.** Let \( M \in \text{Mat}_{n+1}(\mathbb{Z}) \) be the matrix having entries \( M_{ij} := \langle [\mu_i], [P_{-j}] \rangle \), for \( i, j = 0, 1, \ldots, n \). Since
\[
\sum_{k=j}^i (-1)^{i+k} \binom{i}{k} \binom{j}{k} = \begin{cases} 
1 & \text{if } i = j, \\
\sum_{r=i-k=0}^{i-j} (-1)^{r+1} \binom{i}{r} \binom{j}{i-j} = (1-1)^{i-j}(i) = 0 & \text{if } i > j,
\end{cases}
\]
the matrix \( M \) has inverse \( M^{-1} \in GL(n+1, \mathbb{Z}) \) with matrix entries \( (M^{-1})_{ij} = (-1)^{i+j} \binom{i}{j} \). This proves that the above-mentioned elements are a basis of \( \mathbb{Z}^{n+1} \) as a \( \mathbb{Z} \)-module, which is equivalent to say that they generate \( \mathbb{Z}^{n+1} \) as abelian group. \( \square \)

### 4. On Dirac operators

For any \( a \in \mathcal{A}(\mathbb{CP}_q^n) \), the matrix coefficients of \( \pi_+^{(n)}(a) - \pi_-^{(n)}(a) \), where \( \pi_\pm \) are the representations in \([5]\), go to zero exponentially. This fact suggests using these representations to construct even spectral triples for \( \mathcal{A}(\mathbb{CP}_q^n) \) (as was done for the standard Podleś sphere in Prop. 5.1 in \([3]\)) with Dirac operator having sign given by \([6]\). One obtains spectral triples of any summability \( d \in \mathbb{R}^+ \). For example, an \( n \)-summable spectral triple is obtained by defining the operator \( D = |D|F_{(n)} \) on \( \mathcal{H}_n = \mathcal{H}_n \oplus \mathcal{H}_n \) as
\[
|D| |m\rangle := (m_1 + \ldots + m_n) |m\rangle.
\]
The multiplicity of the eigenvalue \( \pm \lambda, \lambda \in \mathbb{N} \), is \( \binom{n+1}{n-1} \). This is a polynomial in \( \lambda \) of order \( n-1 \), and so the metric dimension is \( n \), as claimed. The bounded commutators condition follows from the identity \([D, a] = |D|[F_{(n)}, a] + ||D|, a|F_{(n)}\) and the observation that:
i) the first term is bounded, since $|D|$ has a spectrum that is polynomially divergent and $[F_{(n)}, a]$ has matrix coefficients that go to zero exponentially;

ii) the second term is bounded as well, since one sees that the explicit expression of $\pi_{\pm}(p_{ij})$ are sums of bounded shift operators, which are eigenvectors of $[|D|, \cdot]$.

With some extra work, by setting $|D|^{\langle m \rangle} := (m_1 + \ldots + m_n)^{n/d} |m\rangle$ for any $d \in \mathbb{R}^+$, one gets a $d$-summable spectral triple. We expect all these $d$-summable triples to be regular.

On the other hand, $0^+$-summable (non-regular) spectral triples for quantum projective spaces have been recently constructed in [3].

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REFERENCES

[1] J. Cuntz, “On the homotopy groups for the space of endomorphisms of a C*-algebra”, in: Operator Algebras and Group Representations, Pitman, London, 1984, pp. 124–137.

[2] F. D’Andrea and L. Dąbrowski, Local Index Formula on the Equatorial Podleś Sphere, Lett. Math. Phys. 75 (2006), no. 3, 235–254; [arxiv:math/0507337].

[3] ———, Dirac operators on Quantum Projective Spaces, preprint [arxiv:0901.4735], 2009.

[4] F. D’Andrea, L. Dąbrowski, G. Landi, and E. Wagner, Dirac operators on all Podleś spheres, J. Noncomm. Geom. 1 (2007), no. 2, 213–239; [arxiv:math/0606480].

[5] F. D’Andrea and G. Landi, Antiself-dual Connections on the Quantum Projective Plane: Monopoles, [arxiv:0903.3551], 2009.

[6] E. Hawkins and G. Landi, Fredholm Modules for Quantum Euclidean Spheres, J. Geom. Phys. 49 (2004), no. 3–4, 272–293; [arxiv:math/0210139].

[7] J.H. Hong and W. Szymański, Quantum Spheres and Projective Spaces as Graph Algebras, Commun. Math. Phys. 232 (2002) 157–188.

[8] L. Vaksman and Ya. Soibelman, The algebra of functions on the quantum group $SU(n + 1)$ and odd-dimensional quantum spheres, Leningrad Math. J. 2 (1991), 1023–1042.

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, CHEMIN DU CYCLOTRON 2, B-1348, LOUVAIN-LA-NEUVE, BELGIUM
E-mail address: francesco.dandrea@uclouvain.be

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI TRIESTE, VIA A. VALERIO 12/1, I-34127 TRIESTE, ITALY, AND INFN, SEZIONE DI TRIESTE, TRIESTE, ITALY
E-mail address: landi@univ.trieste.it