On the best-choice prophet secretary problem

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Abstract

We study a variant of the secretary problem where candidates come from independent, not necessarily identical distributions known to us, and show that we can do at least as well as in the IID setting. This resolves a conjecture of Esfandiari et al. [4].
1 Introduction

In the classical secretary problem, an adversary chooses \( n \) numbers which are unknown to us. The numbers are then randomly ordered, and we are sequentially presented with them. We have to make an irrevocable decision about whether we would like to accept a number or not, with the goal of selecting the largest number. It is well known that this can be achieved with a probability of \( \frac{1}{e} \).

Many variations on the secretary problem have been studied. Gilbert and Mosteller [1] studied the case where the numbers are independent draws from a known distribution and found that the optimal probability with which we can win is \( \gamma \approx 0.5801 \). More recently, Esfandiari et al. [4] studied the case where the numbers come from known, independent, but not necessarily identical distributions which satisfy a “no-superstars assumption,” and showed that you can win with a probability of at least as much as in the IID setting of Gilbert and Mosteller. They also conjectured that the no-superstars assumption can be dropped. In this paper, we prove that this is indeed the case.

In fact, we show that we can win by using strategies of the following form: accept the \( i \)-th number if and only if it is the biggest seen so far, and is greater than a threshold \( \tau_i \). To find the success probability of such a strategy, we imitate the proof of Gilbert and Mosteller from the IID setting [1].

The results can be extended to the case where the distributions satisfy a negative dependence condition, and knowledge of the distributions is replaced by access to samples.

2 Statement and proof of main result

Consider the following game (“best-choice prophet secretary game”): An adversary chooses independent continuous distributions \( \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \) and generates draws \( X_1, X_2, \ldots, X_n \). The draws are then randomly ordered. We are told the distributions \( \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \) and are then sequentially presented with the draws. We have to make an irrevocable decision about whether we would like to accept a draw or not, with the goal of selecting the largest
number. What is the probability with which we can win?

**Theorem.** You can win any best-choice prophet secretary game with at least the probability with which you can win the best-choice prophet secretary game with identical distributions, i.e., with probability at least $\gamma \approx 0.5801$.

**Proof.** Fix parameters $\{d_i\}_{i=1}^n$ and let us choose $\tau_i$ such that $\Pr[\max_{k=1}^n X_k \leq \tau_i] = d_i^n$ (Correa et al. [3] call such strategies blind). Assume these parameters are monotonically decreasing in $i$. (If the $X_i$ were IID, and uniform on $[0,1]$, then $\tau_i = d_i$.)

Note the following simple lemma:

**Lemma.** Suppose that $X_1, X_2, \ldots, X_n$ are independent. If $r$ of the $X_k$ are randomly chosen, say, $X_{j_1}, X_{j_2}, \ldots, X_{j_r}$, then, $\Pr[\max_{k=1}^r X_{j_k} \leq \tau_i] \geq d_r^r$.

**Proof.** Fix a constant $T$, and suppose $\Pr[X_k \leq T] = a_k$. Then,

$$\Pr[\max_{k=1}^n X_k \leq T] = \prod_{k=1}^n a_k$$

If $r$ of the $X_k$ are randomly chosen, say, $X_{j_1}, X_{j_2}, \ldots, X_{j_r}$ then

$$\Pr[\max_{k=1}^r X_{j_k} \leq T] = \frac{\sum_{(i_1,i_2,\ldots,i_r)} \prod_{k=1}^r a_{i_k}}{\binom{n}{r}}$$

By the AM-GM inequality, this is at least

$$\left( \prod_{k=1}^n a_k^{\frac{n-1}{r}} \right)^{\frac{1}{\binom{n}{r}}} \leq \left( \prod_{k=1}^n a_k \right)^{\frac{r}{n}}$$

In particular, we can let $T = \tau_i$, and the desired result follows. \hfill $\Box$

Now consider a strategy in which we accept the $i^{th}$ number if and only if it is the biggest seen so far, and is more than its threshold $\tau_i$. We analyze the probability of winning with this strategy, following the proof of Gilbert and Mosteller [1].
Note that the probability that the $i^{th}$ number is biggest amongst the first $r$, but is less than its threshold is just

$$\Pr[\max_{k=1}^r X_{jk} \leq \tau_i] \geq \frac{d_i^r}{r}$$

since this happens if (and only if!) all the first $r$ numbers are less than $\tau_i$ and they’re ordered so that the $i^{th}$ number is biggest.

Next, note that the probability the $i^{th}$ number is less than its threshold, is biggest amongst the first $r$, but is not biggest amongst all numbers is at least

$$\frac{d_i^r}{r} - \frac{\Pr[\max_{k=1}^n X_k \leq \tau_i]}{n} = \frac{d_i^r}{r} - \frac{d_i^n}{r}$$

since the second term simply represents the probability that the $i^{th}$ number is biggest amongst the first $n$ and is less than its threshold.

Now, note that the probability that the $i^{th}$ number is less than its threshold, is biggest amongst the first $r$, but the $r+1^{th}$ number is biggest amongst all numbers is at least

$$\frac{d_i^r}{r} - \frac{d_i^n}{n}$$

since it’s equally likely that each of the last $n - r$ numbers turns out to be the biggest amongst them.

Furthermore, note that the above is also actually the probability no number before the $r + 1^{th}$ is chosen, the $i^{th}$ number is biggest amongst the first $r$, but the $r + 1^{th}$ number is biggest amongst all numbers. This is because if the biggest number amongst the first $r$ is less than its threshold, it cannot be chosen; certainly, no number after it (amongst the first $r$) can be chosen; no number before it can be chosen either since it wouldn’t have been able to meet its threshold (since the $\tau_i$ are decreasing).

Thus we conclude that the the probability no number before the $r + 1^{th}$ is chosen, and the $r + 1^{th}$ number is biggest amongst all numbers is at least
The probability that the $r + 1^{\text{th}}$ number is biggest amongst all numbers but is less than its threshold is just

\[
\Pr[\max_{k=1}^{n} X_k \leq \tau_{r+1}] = \frac{d_{r+1}^n}{n}
\]

We conclude, that the probability of winning the game with the $r + 1^{\text{th}}$ number is at least

\[
\left( \sum_{i=1}^{r} \frac{d_i^r}{r} - \frac{d_i^n}{n} \right) - \frac{d_{r+1}^n}{n}
\]

The probability of winning with the first number is

\[
\frac{1 - d_1^n}{n}
\]

Since it is known that a threshold algorithm of this sort is optimal for IID distributions [1, 2], we conclude that we can always do at least as well as we do in the IID setting (where \( \Pr[\max_{k=1}^{r} X_{jk} \leq \tau_i] = d_i^r \)).

This also immediately yields that the optimal win probability in the IID setting is decreasing in \( n \), since if \( Z \) is a deterministic 0 random variable and \( X_1, X_2, \ldots, X_{n+1} \) are IID uniform \([0, 1]\), then

\[
\Pr[\text{winning with draws } X_1, X_2, \ldots, X_n] = \Pr[\text{winning with draws } X_1, X_2, \ldots, X_n, Z] \\
\geq \Pr[\text{winning with draws } X_1, X_2, \ldots, X_{n+1}]
\]

In particular, this means that the worst case for the IID setting arises when we take the limit as \( n \to \infty \).

The optimal value of \( d_{n-i} \) does not depend on \( n \). Explicit expressions for the optimal values of \( d_{n-i} \) can be found (\( d_{n-i} = 1 - \frac{x}{i} + O\left(\frac{1}{i^2}\right) \), \( \int_0^\infty \frac{e^{-x}}{x}dx = 1 \)).
We can thus evaluate the success probability in the limit as \( n \to \infty \) by summing the expression for the probability of winning with the \( r+1 \)th number over all values of \( r \). The success probability turns out to equal \((e^c - c - 1) \int_c^{\infty} \frac{e^{-x}}{x} dx + e^{-c} \approx 0.5801\) (see [2] for details of these calculations).

Remark 1. We can achieve the same result when the distributions are possibly discrete. To do this, we employ the same trick used in [5]. Replace \( D_i \) with a bivariate distribution, where the first coordinate is from \( D_i \) and the second coordinate is a value drawn independently and uniformly from \([0,1]\). Impose the lexicographic order on \( \mathbb{R}^2 \), and then, there is no difficulty in determining the value of \( \tau_i \) from \( d_i \). While implementing the strategy, we can generate a random value from \([0,1]\) for every draw presented to us to decide whether it is large enough to accept, and the guarantee on the win probability is the same.

Remark 2. We can also achieve the same result under a negative dependence condition on the \( X_i \). For example, if we write \( Y_i = 1_{X_i \leq T} \), and the \( Y_i \) are conditionally negative associated, then, we know that after conditioning on \( \prod_{i \in A \cap B} Y_i = 1 \), we have

\[
E \left[ \prod_{i \in A \setminus B} Y_i \right] \cdot E \left[ \prod_{i \in B \setminus A} Y_i \right] \geq E \left[ \prod_{i \in A \setminus B \cup B \setminus A} Y_i \right]
\]

which implies (with no conditioning) that

\[
E \left[ \prod_{i \in A} Y_i \right] \cdot E \left[ \prod_{i \in B} Y_i \right] \geq E \left[ \prod_{i \in A \cup B} Y_i \right] \cdot E \left[ \prod_{i \in A \cap B} Y_i \right]
\]

Now, if we define \( g(A) = \log E \left[ \prod_{i \in A} Y_i \right] \), then \( g \) is a submodular set function. The inequality of the lemma can then be proved by AM-GM, followed by Han’s inequality for submodular set functions (see pages 14–15 in [6] for a statement and proof of Han’s inequality). The rest of the proof, after the lemma, works the same.

Remark 3. There is a function \( f \) so that with just \( f(\epsilon) \) (independent of \( n \)) samples from each distribution \( D_i \) (but without knowledge of the distributions themselves), we can win with a probability of \( \gamma - \epsilon \). We include a proof...
sketch: $f_1(\epsilon), f_2(\epsilon)$, and $f_3(\epsilon)$ are functions which tend to 0 as $\epsilon \to 0$ which we can optimize (but we can think for the proof $f_1(\epsilon) \approx f_2(\epsilon) \approx \frac{\epsilon}{10}, f_3(\epsilon) \approx -\frac{\epsilon}{100 \log \epsilon}$):

Note that we can safely ignore the last $f_1(\epsilon)$ fraction of draws, and never choose them, and this will effect our success probability by at most $f_1(\epsilon)$.

Furthermore, as long as $i \leq (1 - f_1(\epsilon))n$, for $d_i = 1 - \frac{\epsilon}{n-1} + O\left(\frac{1}{(n-1)^2}\right)$ (the optimal value), $d_i$ is at least $\approx e^{-\frac{\epsilon}{10}}$ and at most $\approx e^{-\frac{\epsilon}{100 \log \epsilon}}$.

These values of $d_i$ are bounded away from 0 and 1, so with sufficiently many samples, we can find decreasing thresholds $T_i$ amongst numbers in the samples (which estimate the “real” thresholds we would’ve used with full knowledge of the distribution) so that with probability $1 - f_2(\epsilon)$, we have that $\Pr[\max_{k=1}^n X_k \leq T_i]$ is within a factor of $1 + f_3(\epsilon)$ of $d_i$ for every $i$ simultaneously (where $X_i$ are the draws we must choose from). We can then use these estimated thresholds to win with probability close to $\gamma$.

Let us indicate how this works (this is similar to Lemma 8 in [5]):

Suppose we have a distribution $\mathcal{D}$, and independent of $\mathcal{D}$, probabilities $p_i$, with $p_i > \delta$. Fix a parameter $\varepsilon$. Let $X$ be a random variable with distribution $\mathcal{D}$. (We are thinking $\mathcal{D}$ is the distribution of $\max_{k=1}^n X_k$, $p_i = d_i, \delta \approx e^{-\frac{\epsilon}{10}},$ and $\varepsilon \approx f_3(\epsilon)$.)

Suppose we draw $m$ samples independently from $\mathcal{D}$, and we define for $k \in \mathbb{N}, m_k = \frac{m}{(1+\varepsilon)^k}, M_k$ is the $m_k$ smallest sample, and $A_k$ is such that $\Pr[X \leq A_k] = \frac{1}{(1+\varepsilon)^k}$. We know that the expected number of samples $\leq A_k$ is $m_k$, and the number of samples $\leq A_k$ is a sum of independent Bernoulli random variables, so by the multiplicative Chernoff bound, with a very high probability, the number of samples is actually between $m_k - 1$ and $m_k + 1$, and by the union bound, the same is true for every $k$ such that $\frac{1}{(1+\varepsilon)^k} > \frac{\delta}{1+\varepsilon}$.

It follows that $M_{k-1} \geq A_k \geq M_{k+1}$ with high probability, or in other words, we have $A_{k+1} \leq M_k \leq A_{k-1}$ with high probability. So if $p_i \in \left[\frac{1}{(1+\varepsilon)^k+1}, \frac{1}{(1+\varepsilon)^k}\right)$, then $\Pr[X \leq M_k]$ is a good estimate for $p_i$ (up to factors of $(1+\varepsilon)$) for every $i$ simultaneously, with high probability, as required.

Now, if $\Pr[\max_{k=1}^n X_k \leq T_i]$ is within a factor of $1 + f_3(\epsilon)$ of $d_i$, then we must have that $\Pr[\max_{k=1}^n X_j \leq T_i]$ is within a factor of $1 + f_3(\epsilon)$ of $d_i$ as well.
where $\Pr[\max_{k=1}^r X_j \leq T_i]$ is the probability that $r$ randomly chosen draws are all less than $T_i$.

From the expression for the success probability (remembering that we only need to consider this for $r \leq (1 - f_1(\epsilon)n)$), we see that using the estimated thresholds (which are good with a probability of $(1 - f_2(\epsilon))$), we win with probability $\gamma - \epsilon$, as long as $f_1(\epsilon) f_2(\epsilon)$, and $f_3(\epsilon)$ are sufficiently small (which can all be achieved with sufficiently many samples).

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