ON THE GENERICITY OF EISENSTEIN SERIES AND THEIR RESIDUES FOR COVERS OF $GL_m$

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Abstract. Let $\tau_1^{(r)}$, $\tau_2^{(r)}$ be two genuine cuspidal automorphic representations on $r$-fold covers of the adelic points of the general linear groups $GL_{n_1}$, $GL_{n_2}$, resp., and let $E(g,s)$ be the associated Eisenstein series on an $r$-fold cover of $GL_{n_1+n_2}$. Then the value or residue at any point $s = s_0$ of $E(g,s)$ is an automorphic form, and generates an automorphic representation. In this note we show that if $n_1 \neq n_2$ these automorphic representations (when not identically zero) are generic, while if $n_1 = n_2 := n$ they are generic except for residues at $s = \frac{n \pm 1}{2n}$.

1. Introduction

The study of the Whittaker coefficients of Eisenstein series has been a major tool in the theory of automorphic forms. See Shahidi [13] and the references there. Let $G$ be a reductive group defined over a number field $F$, $P$ be a maximal parabolic subgroup of $G$ with Levi decomposition $P = MU$, and let $\tau$ be an automorphic representation of $M(A)$. Given $f_\tau$ in the space of $\tau$, one may form an Eisenstein series $E(g,s,f_\tau)$ on $G(A)$; this is an automorphic form defined by an absolutely convergent sum for $\Re(s)$ sufficiently large and by analytic continuation in general (Langlands [11]). If $\tau$ is generic, then the global Whittaker coefficient of $E$ is computed by unfolding the Eisenstein series and making use of the factorization of a global Whittaker functional into local ones, relating it to integrals involving local Whittaker functionals for $\tau$. These are computed at unramified places using the Casselman-Shalika formula. One finds that the Whittaker coefficient of $E$ may be expressed in terms of certain Langlands $L$-functions for $\tau$. Thus the question of whether or not the Eisenstein series (resp. its residue) at a given point is generic is related to the properties of the $L$-functions which appear.

The goal of this article is to study when a maximal parabolic Eisenstein series or its residue is generic for metaplectic covers of the general linear group. Recall that given a positive integer $r$, a cover of $GL_n(A)$ may be defined whenever $F$ contains a full set of $r$-th roots of unity. However, many aspects of the theory of automorphic forms change for covers. In particular, the local Whittaker model is not unique (this happens since the inverse image of the full torus in the cover is no longer abelian), and the approach outlined above does not apply. The global Whittaker coefficients may be computed as a Dirichlet series for $\Re(s)$ sufficiently large (Brubaker-Friedberg [2]), in fact one involving exponential
sums of arithmetic interest, but it is not apparent how one could use this expression to
determine non-vanishing at a particular point. Also, though there is a good understanding
of the constant terms of the metaplectic Eisenstein series (Gao \cite{5}) the residual spectrum
has been computed in only a few cases (see for example Gao \cite{6}), and in general it is not
clear how to find the Whittaker coefficients of the residues. Indeed, the determination of
such coefficients has been a long-standing open problem even for low degree covers of \(GL_2\)
(see Eckhardt-Patterson \cite{3}, Chinta-Friedberg-Hoffstein \cite{2}).

In this paper we offer a new approach to the question of genericity. This is based on
the classification of Fourier coefficients via unipotent orbits. In our main result, we give
a complete description of when such an Eisenstein series or its residue may be generic.

Theorem 1 treats the case of maximal parabolics with Levi isomorphic to \(GL_{n_1} \times GL_{n_2}\)
with \(n_1 \neq n_2\), and Theorem 2 treats the case \(n_1 = n_2\). Our method works with only minor
differences for an Eisenstein series or its residue. However, it is possible that there is no
generic residual spectrum for the general linear group induced from cuspidal datum except
at the one point we must exclude (the residue at that point is in fact not always generic).

This question is likely related to the conjectured generalized Shimura correspondence (see
Suzuki \cite{14}).

We consider only covers of the general linear group in this article. In fact for covers of
other groups we can prove a similar theorem only in special cases. The difficulty in extending
the method is due to the property that for the general linear group, if \(\pi\) is not generic then
for every unipotent orbit in the set \(O(\pi)\) (see below for the definition), the corresponding
Fourier coefficient can be written as an integration over a unipotent subgroup which contains
the constant term attached to the unipotent radical of a maximal parabolic subgroup as an
inner integration. This does not happen in other classical or exceptional groups. Though
we consider maximal parabolic subgroups, these methods can be adapted to more general
parabolic subgroups of the general linear group.

2. Notation and Preliminaries

Let \(n_1, n_2 \geq 1\) and let \(P_{n_1,n_2}\) denote the maximal parabolic subgroup of \(GL_{n_1+n_2}\) whose
Levi part \(M\) is isomorphic to \(GL_{n_1} \times GL_{n_2}\) embedded in \(GL_{n_1+n_2}\) by \((g_1, g_2) \mapsto \text{diag}(g_1, g_2)\).
We will assume without loss that \(n_1 \geq n_2\). Let \(r \geq 1\), and let \(F\) be a number field containing
a full set of \(r\)-th roots of unity \(\mu_r\). For each \(m \geq 1\) we let \(GL_m^{(r)}(\mathbb{A})\) denote an \(r\)-fold cover of
\(GL_m(\mathbb{A})\). This is a central extension of \(GL_m(\mathbb{A})\) by \(\mu_r\); that is, it consists of ordered pairs
\((g, \zeta)\) with \(g \in GL_m(\mathbb{A})\) and \(\zeta \in \mu_r\), with multiplication \((g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 \sigma(g_1, g_2))\)
where \(\sigma : GL_m(\mathbb{A})^2 \to \mu_r\) is a two-cocycle. See Kazhdan-Patterson \cite{10}. There are cocycles
giving rise to non-isomorphic extensions but in fact they all agree on the subgroup \(GL_m(\mathbb{A})_0^{(r)}\)
consisting of \((g, \zeta)\) with \(\det(g) \in (\mathbb{A}^\times)^r F^\times\). Because of this, we fix one such extension but it
does not matter which for the sequel. If \(H\) is any subgroup of \(GL_m(\mathbb{A})\) we let \(\tilde{H}\) denote its
inverse image in \(GL_m^{(r)}(\mathbb{A})\).

Let \(\tau_i^{(r)}, i = 1, 2\), denote genuine irreducible cuspidal automorphic representations of the
groups \(GL_{n_i}^{(r)}(\mathbb{A})\). Then one can construct an Eisenstein series \(E_\tau^{(r)}(g, s)\) on \(GL_{n_1+n_2}^{(r)}(\mathbb{A})\).
This depends on a choice of test vectors, but we shall suppress this from the notation. This
construction is given in detail in Brubaker and Friedberg \cite{1}. Alternative constructions are
due to Suzuki \cite{14} and to Takeda \cite{15}. In all cases, before inducing one must restrict to the
subgroup \( S := GL_{n_1}(\mathbb{A})_0^{(r)} \times_{\mu_r} GL_{n_2}(\mathbb{A})_0^{(r)} \) of \( \tilde{M}(\mathbb{A}) \) to take into account that the full inverse images of the two \( GL_{n_1}(\mathbb{A}) \) in \( \tilde{M}(\mathbb{A}) \) do not commute. Since our computations all take place inside the subgroup of \( GL_{n_1+n_2}(\mathbb{A}) \) generated by \( S \) and by unipotents, the minor differences in these constructions does not affect the results. The complex parameter \( s \) is normalized so that \( E_{\tau}^{(r)}(g, s) \) has functional equation under \( s \mapsto 1 - s \).

If \( U \) is any unipotent subgroup of \( GL_m \), \( U(\mathbb{A}) \) splits in the \( r \)-fold cover by means of the trivial section \( u \mapsto (u, 1) \). We write \( U(\mathbb{A}) \) for its isomorphic image in the cover \( GL_m^{(r)}(\mathbb{A}) \) under this section. If \( \varphi \) is any automorphic form on \( GL_m^{(r)}(\mathbb{A}) \), let \( \varphi^U \) denote the constant term of \( \varphi \) along \( U \)

\[
\varphi^U(g) = \int_{U(F) \backslash U(\mathbb{A})} \varphi(ug) \, du.
\]

Then the following lemma is standard and follows from the cuspidality of the representations \( \pi_i^{(r)} \) (see for example Moeglin-Waldspurger [12], II.1.7).

**Lemma 1.** Let \( E_{\tau}^{(r)}(g) \) be either a residue of \( E_{\tau}^{(r)}(g, s) \) at a specific point \( s_0 \), or the value of \( E_{\tau}^{(r)}(g, s) \) at \( s = s_0 \). If \( U \) is the unipotent radical of a parabolic subgroup of \( GL_{n_1+n_2} \), then the constant term \( E_{\tau}^{(r),U}(g) \) is zero for all choices of data unless \( U \) is a conjugate of the unipotent radical of \( P_{n_1,n_2} \) or of \( P_{n_2,n_1} \).

Given an automorphic representation \( \pi \) of a reductive group \( G \), one may attach a set of integrals over unipotent groups to each unipotent orbit \( O \) of \( G(\mathbb{C}) \); we call these (generalized) Fourier coefficients. See Ginzburg [8]. The set \( O(\pi) \) consists of the unipotent orbits \( O \) with two properties: first, \( \pi \) has a non-zero Fourier coefficient with respect to \( O \), and second, all Fourier coefficients with respect to unipotent orbits \( O' \) that are greater than or not comparable with \( O \) are identically zero. The definitions extend to metaplectic covers without change since all unipotent subgroups split in covers. For \( G = GL_{n_1+n_2} \), unipotent orbits correspond to partitions of \( n_1 + n_2 \), by the Jordan decomposition.

Let \( \mathcal{E}_{\tau}^{(r)} \) be the automorphic representation generated by \( E_{\tau}^{(r)}(g) \) in any one of the cases treated in Lemma 1. We have

**Proposition 1.** The representation \( \mathcal{E}_{\tau}^{(r)} \) is either generic, or \( O(\mathcal{E}_{\tau}^{(r)}) = (n_1n_2) \).

**Proof.** We will assume that \( \mathcal{E}_{\tau}^{(r)} \) is not generic, and prove that \( O(\mathcal{E}_{\tau}^{(r)}) = (n_1n_2) \). We fix some notations.

For \( 1 \leq k \leq n_1 + n_2 - 1 \), let \( U_k \) denote the unipotent radical of the standard parabolic subgroup of \( GL_{n_1+n_2} \) whose Levi part is \( GL_k \times GL_{n_1+n_2-k} \), embedded in \( GL_{n_1+n_2} \) by \( (a_1, \ldots, a_k, h) \mapsto \text{diag}(a_1, \ldots, a_k, h) \). Let \( \psi : F\backslash A \to \mathbb{C} \) be a fixed nontrivial additive character, and let \( \psi_{U_k} \) be the character of \( U_k(F) \backslash U_k(\mathbb{A}) \) given by

\[
\psi_{U_k}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{k,k+1}).
\]

For \( 1 \leq m \leq n_1 + n_2 \), let \( V_m \) denote the unipotent subgroup of \( GL_{n_1+n_2} \) generated by all matrices of the form \( I_{n_1+n_2} + r_{m+1}e_{m,m+1} + \cdots + r_{n_1+n_2}e_{n_1+n_2} \), where \( e_{i,j} \) is the matrix whose \((i,j)\) entry is one and whose other entries are zero. Notice that we can identify the group \( V_{k+1} \) with the quotient \( U_{k+1}/U_k \).
For $1 \leq k \leq n_1 + n_2 - 1$ let $I_k$ denote the integral

$$I_k = \int_{U_k(F) \backslash U_k(\mathbb{A})} E^{(r)}(ug) \psi U_k(u) \, du,$$

where $E^{(r)}(g)$ is a vector in the space of $E^{(r)}$. It follows from Ginzburg \[8\] that this Fourier coefficient is attached to the unipotent orbit $(k + 1)^{n_1 + n_2 - k - 1}$. Notice also that if $k = n_1 + n_2 - 1$, then the Fourier coefficient given by \[1\] is the Whittaker coefficient. By our assumption it is zero for all choices of data.

We will study the vanishing of $I_k$ as we vary over vectors in $E^{(r)}$. To begin, expand $E^{(r)}(g)$ along the unipotent group $U_1$. Since the Weyl group together with any one-parameter unipotent subgroup $x_{\alpha}(r)$ corresponding to any root $\alpha$ generates the adelic special linear group, it follows that a nontrivial automorphic form cannot equal its constant term along the unipotent radical of a maximal parabolic. We deduce that for $k = 1$ the integral $I_k$ is not zero for some choice of data.

Let $k_0$ be maximal such that the integral $I_k$ is not zero for some choice of data. We claim first that $k_0 \leq n_1 - 1$. Indeed, suppose $k_0 \geq n_1$. Since $k_0 \leq n_1 + n_2 - 2$, we can expand \[1\] along the group $V_{k_0+1}$. The group $\text{GL}_{n_1+n_2-k_0-1}(F)$ acts on the characters appearing in this expansion with two orbits. First, the nontrivial orbit will contribute a sum such that each of its summands is the integral $I_{k_0+1}$ (at varying $g$). By the maximality of $k_0$ this sum is zero. The second contribution to the expansion is from the constant term. However, since $k_0 \geq n_1$, it follows from Lemma \[1\] that this term is also zero for all choices of data. This is a contradiction. Thus $k_0 \leq n_1 - 1$.

We next show that $k_0 = n_1 - 1$ or $k_0 = n_2 - 1$. Indeed, suppose that $n_2 \leq k_0 \leq n_1 - 1$. If $k_0 \neq n_1 - 1$, then as in the previous case, we expand along the group $V_{k_0+1}$. We get a contradiction, either to the maximality of $k_0$ or to Lemma \[1\]. Similarly, suppose that $1 \leq k_0 \leq n_2 - 1$. Then arguing as above we deduce that $k_0 = n_2 - 1$.

Consider the case when $k_0 = n_1 - 1$. Expand the integral \[1\] along the group $V_{n_1}$. By the maximality of $k_0$, the contribution to the expansion from the nontrivial orbit is zero. Thus only the constant term contributes, and we deduce that $I_{n_1-1}$ is equal to

$$I_{n_1-1}^0 = \int_{U_{n_1}(F) \backslash U_{n_1}(\mathbb{A})} E^{(r)}(ug) \psi(u_{1,2} + u_{2,3} + \cdots + u_{n_1-1,n_1}) \, du.$$

Next expand the integral \[2\] along the group $V_{n_1+1}$. By Lemma \[1\] the contribution from the constant term is zero. From this we deduce that the integral

$$I_{n_1+1}^0 = \int_{U_{n_1+1}(F) \backslash U_{n_1+1}(\mathbb{A})} E^{(r)}(ug) \psi_{U_{n_1+1}}^0(u) \, du$$

is not zero for some choice of data, where

$$\psi_{U_{n_1+1}}^0(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n_1-1,n_1} + u_{n_1+1,n_1+2}).$$

Continuing this process, we deduce that the integral

$$I_{n_1+n_2-1}^0 = \int_{U_{n_1+n_2-1}(F) \backslash U_{n_1+n_2-1}(\mathbb{A})} E^{(r)}(ug) \psi_{U_{n_1+n_2-1}}^0(u) \, du$$

is not zero for all choices of data.
is not zero for some choice of data, with
\[ \psi^0_{U_{n_1+n_2+1}}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n_1-1,n_1} + u_{n_1+1,n_1+2} + \cdots + u_{n_1+n_2-1,n_1+n_2}). \]

A similar result may be obtained in the case when \( k_0 = n_2 - 1 \). In this case expanding along \( V_{n_2} \), only the constant term contributes a nonzero term to the expansion. After this step, in expanding in larger \( V_m \), only the non-constant terms contribute. We deduce that the integral
\[ I_{n_1+n_2-1}' = \int_{U_{n_1+n_2-1}(F) \backslash U_{n_1+n_2-1}(A)} E_\tau^{(r)}(ug) \psi^0_{U_{n_1+n_2-1}}(u) \, du \]
is not zero for some choice of data, where
\[ \psi^0_{U_{n_1+n_2+1}}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n_2-1,n_2} + u_{n_2+1,n_2+2} + \cdots + u_{n_1+n_2-1,n_1+n_2}). \]

Since the Fourier coefficients given by the integrals (3) and (4) are associated with the unipotent orbit \((n_1n_2)\), the Proposition follows. In particular in both cases the integrals are not zero for some choice of data.

\[ \square \]

3. Main Theorems

3.1. The case when \( n_1 > n_2 \). Our first main theorem treats the case that the two factors of the Levi have different sizes.

**Theorem 1.** Suppose that \( n_1 > n_2 \), and that \( E_\tau^{(r)} \) is the automorphic representation generated by the value of \( E_\tau^{(r)}(g,s) \) at a specific point \( s = s_0 \) or by the residue of \( E_\tau^{(r)}(g,s) \) at a specific point \( s = s_0 \), and is not identically zero. Then \( E_\tau^{(r)} \) is generic.

**Proof.** Assume that \( \Re(s) \) is sufficiently large that the Eisenstein series \( E_\tau^{(r)}(g,s) \) is given as a convergent series. In this case, let \( I_{n_1-1}(g,s) \) denote the integral (1) with \( k = n_1 - 1 \). We shall compute this integral in two ways. Assuming that this Eisenstein series is not generic, we shall derive a contradiction.

First, from the proof of Proposition (1) we have \( I_{n_1-1}(g,s) = I_{n_1}^0(g,s) \). Next we unfold the Eisenstein series. This leads us to consider representatives of the space of double cosets \( P_{n_1,n_2}(F) \backslash GL_{n_1+n_2}(F) / P_{n_1,n_2}(F) \). Using the cuspidality of \( \tau_1^{(r)} \) we see that only the identity contributes a non-zero term. So we find that
\[ I_{n_1-1}(g,s) = I_{n_1}^0(g,s) = f_{W_1^{(r)},\tau_2^{(r)}}(g,s), \]
where \( W_1^{(r)} \) denotes the Whittaker coefficient of the representation \( \tau_1^{(r)} \) and \( f \) is the associated function in the induced space.

To compute \( I_{n_1-1}(g,s) \) in a second way, let \( w = \begin{pmatrix} I_{n_1} & I_{n_2} \\ I_{n_1} & I_{n_2} \end{pmatrix} \). Using the left-invariance properties of \( E_\tau^{(r)}(g,s) \), we obtain
\[ I_{n_1-1}(g,s) = \int_{L(F) \backslash L(A)} \int_{N_{n_1}(F) \backslash N_{n_1}(A)} E_\tau^{(r)} \left( \begin{pmatrix} I_{n_2} & u_1 \\ I_{n_1} & I_{n_1} \end{pmatrix} \right) wg, s \, \psi_{N_{n_1}}(u_1) \, du_1 \, dl. \]
Here $L \cong \text{Mat}_{(n_1-1)\times n_2}$ is the subgroup of $\text{Mat}_{n_1\times n_2}$ consisting of matrices with bottom row zero, the group $N_{n_1}$ is the maximal upper triangular unipotent subgroup of $GL_{n_1}$, and $\psi_{N_{n_1}}$ is the Whittaker character of $N_{n_1}$.

Next we perform a certain root exchange. The notion of root exchange was defined in Ginzburg-Rallis-Soudry [9], Section 7.1. Let $Y_0 \cong \text{Mat}_{n_2\times(n_1-1)}$ be the subgroup of $\text{Mat}_{n_2\times n_1}$ consisting of all matrices with first column zero. Then using [9], Lemma 7.1, we deduce that (6) is equal to

$$
\int_{L(\mathbb{A}) \backslash Y_0(\mathbb{A})} \int_{N_{n_1}(\mathbb{A})\backslash N_{n_1}(F)} \int_{N_{n_1}(\mathbb{A})} E^{(r)}_\tau \left( \begin{pmatrix} I_{n_2} & y \\ I_{n_1} \end{pmatrix} \begin{pmatrix} I_{n_2} & u_1 \\ l & I_{n_1} \end{pmatrix} w g, s \right) \psi_{N_{n_1}}(u_1) dy du_1 dl.
$$

Expand the integral along the unipotent subgroup $V_{n_2+1}$ which consists of all matrices of the form $I_{n_1+n_2} + r_1 e_{1,n_2+1} + r_2 e_{2,n_2+1} + \cdots + r_{n_1} e_{n_1,n_2+1}$. The group $GL_{n_2}(F)$ acts on the characters appearing in this expansion. It follows from Proposition 11 and the assumption that $E^{(r)}_\tau(g,s)$ is not generic that $O(E^{(r)}_\tau(\cdot,s)) = (n_1 n_2)$. In particular this means that the contribution to the expansion from the nontrivial orbit is zero. Hence only the constant term of the expansion contributes. We see that $I_{n_1-1}(g,s)$ is equal to

$$
\int_{L(\mathbb{A}) \backslash N_{n_1}(F) \backslash N_{n_1}(\mathbb{A})} \int_{N_{n_1}(\mathbb{A}) \backslash N_{n_1}(\mathbb{A})} \int_{Y(\mathbb{A})} \int_{Y(\mathbb{A})} E^{(r)}_\tau \left( \begin{pmatrix} I_{n_2} & y \\ I_{n_1} \end{pmatrix} \begin{pmatrix} I_{n_2} & u_1 \\ l & I_{n_1} \end{pmatrix} w g, s \right) \psi_{N_{n_1}}(u_1) dy du_1 dl.
$$

Here $Y = \text{Mat}_{n_2\times n_1}$.

The integration over $Y$ is the constant term of the Eisenstein series along the unipotent radical of parabolic subgroup $P_{n_2,n_1}$. Unfolding the Eisenstein series, we obtain

$$
I_{n_1-1}(g,s) = \int_{L(\mathbb{A}) \backslash Y(\mathbb{A})} f_{W_1^{(r)},n_1}^{(r)} \left( w^{-1} \begin{pmatrix} I_{n_2} & y \\ I_{n_1} \end{pmatrix} \begin{pmatrix} I_{n_2} & u_1 \\ l & I_{n_1} \end{pmatrix} w g, s \right) dy dl.
$$

Now let $g = \begin{pmatrix} a I_{n_1} \\ I_{n_2} \end{pmatrix}$, where $a$ is an $r$-th power. Compare the expressions (5) and (7).

Write $I_{n_1-1}(a,s)$ for $I_{n_1-1}(g,s)$ for the above matrix $g$. From (5) we obtain

$$
I_{n_1-1}(a,s) = f_{W_1^{(r)},n_1}^{(r)}(g,s) = |a|^{n_1 n_2} f_{W_1^{(r)},r}^{(r)}(e,s) = |a|^{n_1 n_2} I_{n_1-1}(e,s).
$$

However, substituting the matrix $g$ into integral (7) and conjugating it to the left, we obtain

$$
I_{n_1-1}(a,s) = |a|^{n_1 n_2 - n_2} I_{n_1-1}(e,s).
$$

Here we obtain a factor of $|a|^{n_1(n_1-1)n_2}$ from the change of variables in $L$, and a factor of $|a|^{-n_1 n_2}$ from the change of variables in $Y$.

Expressions (8) and (9) produce a contradiction. We conclude that the Eisenstein series $E^{(r)}_\tau(g,s)$ is generic when $\Re(s)$ is large. Also, from the meromorphic continuation of the Eisenstein series, we deduce that $I_{n_1-1}(g,s)$ is a meromorphic function of $s$. If we consider the case when $E^{(r)}_\tau$ is a value of the Eisenstein series at a specific point $s_0$, it follows from (5) that $I_{n_1-1}(g,s_0)$ is not zero for some choice of data, and hence once again expressions (8)
and (9) produce a contradiction. Finally, if $E^{(r)}_\tau$ is a nonzero residue of the Eisenstein series at a point $s_0$, then it follows from the equality (5) that the residue at $s_0$ of $I_{n_1-1}(g, s)$ is zero for all choices of data. However, this residue is an inner integration in (3) in the case that $E^{(r)}_\tau(g) = \text{Res}_{s=s_0} E^{(r)}_\tau(g, s)$. This gives a contradiction to Proposition 1.

This completes the proof of Theorem 1. □

3.2. The case when $n_1 = n_2$. In this section we treat the case $n_1 = n_2$. This case is fundamentally different, for there are situations in which the representation $E^{(r)}_\tau$ is not generic. Indeed, let $n_1 = n_2 = 2$. Take a cuspidal theta representation $\Theta^{(2)}$ defined on the group $GL_2^2(\mathbb{A})$ where $\chi$ is a character of $F^\times \backslash \mathbb{A}^\times$ which is not the square of a character. These representations were constructed in Gelbart and Piatetski-Shapiro [7]. Let $\tau_i^{(2)} = \Theta^{(2)}_\chi$ for $i = 1, 2$. Then it is not hard to check that the corresponding Eisenstein series has a simple pole at $s = 3/4$, and the residue representation is not generic. Indeed, the unramified constituent of the residue at a finite place $\nu$ is the theta representation of the group $GL_4^2$ with $\chi_\nu$ as its central character. It is well known that this representation is not generic. Thus the residue is not generic.

We shall show:

**Theorem 2.** Suppose that $n_1 = n_2 = n$, and that $E^{(r)}_\tau$ is the automorphic representation generated by the value of $E^{(r)}_\tau(g, s)$ at a specific point $s = s_0$, and is not identically zero. Then $E^{(r)}_\tau$ is generic. Suppose instead that $E^{(r)}_\tau$ is the automorphic representation generated by the residue of $E^{(r)}_\tau(g, s)$ at a specific point $s = s_0$, and is not identically zero. Then it is generic provided $s_0 \neq \frac{n + 1}{2n}$.  

**Proof.** As in the previous Section we shall assume that the representation $E^{(r)}_\tau$ is not generic and determine the cases in which we can derive a contradiction. Though the argument is similar to the one above, there is an important technical difference. When $n_1 = n_2$, there is only one unipotent radical of a parabolic (up to conjugation) that may have a nonzero constant term, rather than two distinct ones. See Lemma 1.

Let $E^{(r)}_\tau$ be as above, and write $n = n_1 = n_2$. Assume that $\Re(s)$ is large. It follows from Proposition 1 that $\mathcal{O}(E^{(r)}_\tau) = (n^2)$. The corresponding Fourier coefficient is described in a more general way in Friedberg-Ginzburg [4], equations (6) and (7). We recall its definition.

Consider the subgroup $V_{2, n}$ of $GL_{2n}$ consisting of all matrices of the form

$$
\begin{pmatrix}
I_2 & X_{1,2} & * & * & \cdots & * \\
I_2 & X_{2,3} & * & \cdots & * \\
I_2 & X_{3,4} & \cdots & * \\
I_2 & \cdots & * \\
I_2 & \ddots & * \\
I_2 
\end{pmatrix}
$$

with $I_2$ appearing $n$ times and each $X_{i,j}$ a matrix of size 2. Define a character $\psi_{V_{2,n}}$ on $V_{2, n}(F) \backslash V_{2, n}(\mathbb{A})$ by

$$
\psi_{V_{2, n}}(v) = \psi(\text{tr}(X_{1,2} + X_{2,3} + \cdots + X_{n-1,n})�)
$$
The corresponding Fourier coefficient is given by

\[
E^{(r)}_\tau(vg, s) \psi_{V_{2,n}}(v) \, dv.
\]

(10)

Consider the diagonal embedding of $GL_2$ into $GL_{2n}$, $h \mapsto \iota(h) = \text{diag}(h, h, \ldots, h)$. If $h \in GL_2(F)$, then it is immediate that the integral \[[10]\]
is left-invariant under $\iota(h)$. However, if $r$ does not divide $n$, then $\iota$ does not extend to an embedding of $GL_2(\mathbb{A})$ into the $r$-fold cover. This means that restricting to $g = \iota(h)$, the integral \[[10]\] defines a $GL_2(F)$-invariant function on a non-trivial cover of $GL_2(\mathbb{A})$, and which can thus not be the trivial function. Arguing as in Friedberg-Ginzburg \[4\], Proposition 3 ((9) and following), we deduce that the representation $E^{(r)}_\tau$ has a nontrivial Fourier coefficient corresponding to the unipotent orbit \[((n + 1)(n - 1))\]. This a contradiction.

Hence we may assume that $r$ divides $n$, and that the integral \[[10]\] is left-invariant under all matrices $\iota(h)$ with $h \in GL_2(\mathbb{A})$. Let $h = \begin{pmatrix} a & 0 \\ a^{-1} & 1 \end{pmatrix}$, and let $I(a, s)$ denote the integral \[[10]\] with $g = \iota(h)$. It follows from the left invariance properties of the above Fourier coefficient that $I(a, s) = I(e, s)$.

Next we compute the integral \[[10]\] in a different way. Let $w$ denote the Weyl element of $GL_{2n}$ with $w_{1, 2i - 1} = w_{n + i, 2i} = 1$ for all $1 \leq i \leq n$ and all other entries zero. Conjugating by this element from left to right we obtain

\[
I(a, s) = \int_{L(F) \backslash L(\mathbb{A})} \int_{N_n(F) \backslash N_n(\mathbb{A})} \int_{N_n(F) \backslash N_n(\mathbb{A})} \int_{Y(F) \backslash Y(\mathbb{A})} E^{(r)}_\tau \left( \begin{pmatrix} u_1 & y \\ u_2 & 1 \end{pmatrix} \begin{pmatrix} I_n & I_n \\ l & 1 \end{pmatrix} \psi_{N_n}(u_1) \psi_{N_n}(u_2) \, dy \, du_1 \, du_2 \, dl.
\]

Here we change the notation from prior Sections: the groups $L$ and $Y$ are each the group consisting of all matrices in $x \in \text{Mat}_{n \times n}$ such that $x_{i,j} = 0$ for all $j < i$.

We now perform a series of root exchanges together with certain Fourier expansions. Using the fact that $\mathcal{O}(E^{(r)}_\tau) = (n^2)$, we obtain

\[
I(a, s) = \int_{L(\mathbb{A})} \int_{N_n(F) \backslash N_n(\mathbb{A})} \int_{N_n(F) \backslash N_n(\mathbb{A})} \int_{Y'(F) \backslash Y'(\mathbb{A})} E^{(r)}_\tau \left( \begin{pmatrix} u_1 & y' \\ u_2 & 1 \end{pmatrix} \begin{pmatrix} I_n & I_n \\ l & 1 \end{pmatrix} \psi_{N_n}(u_1) \psi_{N_n}(u_2) \, dy' \, du_1 \, du_2 \, dl.
\]

Here $Y' = \text{Mat}_{n \times n}$. Notice that the integral over $Y'$ is the constant term along the unipotent radical of the parabolic subgroup $P_{n,n}$. Hence, since $\Re(s)$ is large, unfolding the Eisenstein series, we deduce that $I(a, s)$ is equal to

\[
\int_{L(\mathbb{A})} f_{W_1, W_2} \left( \begin{pmatrix} I_n \\ l \\ I_n \end{pmatrix} \psi_{N_n}(u_1) \psi_{N_n}(u_2) \, dy' \, dl + \int_{L(\mathbb{A})} f_{W_1, W_2} \left( w_1 \begin{pmatrix} I_n & y' \\ l & 1 \end{pmatrix} \begin{pmatrix} I_n & I_n \\ l & 1 \end{pmatrix} \psi_{N_n}(u_1) \psi_{N_n}(u_2) \, dy' \, dl.
\]

Here \( w_1 = \begin{pmatrix} I_n \\ I_n \end{pmatrix} \), and $W_i$ denotes the Whittaker coefficient of the representation $\tau^{(r)}_i$.

Suppose that $a$ is an $r$-th power. Conjugating the matrix $g$ to the left we obtain

\[
I(a, s) = |a|^{2n^2 s - n^2 + n} I_1(s) + |a|^{-2n^2 s + n^2 + n} I_2(s).
\]

(11)
Here

\[ I_1(s) = \int_{L(\lambda)} f_{W_1,W_2} \left( I_n \begin{pmatrix} l & w \\ I_n & s \end{pmatrix} \right) dl \]

and

\[ I_2(s) = \int_{L(\lambda) Y'(\lambda)} \int f_{W_1,W_2} \left( w_1 \begin{pmatrix} I_n & y' \\ l & I_n \end{pmatrix} \right) \left( w_1 \begin{pmatrix} I_n & y' \\ l & I_n \end{pmatrix} \right) s' dl. \]

Since \( I(a,s) = I(e,s) \) (and so is independent of \( a \) and also non-zero), and since the numbers \( 2n^2s - n^2 + n \) and \( -2n^2s + n^2 + n \) cannot vanish at the same value of \( s \), \( \text{[11]} \) will produce a contradiction unless either \( I_1(s) \) is zero for all choices of data and \(-2n^2s + n^2 + n = 0\), or \( I_2(s) = 0 \) for all choices of data and \( 2n^2s - n^2 + n = 0 \). From this we deduce that for \( \Re(s) \) large, the representation \( E^{(r)} \) must be generic. This is also true if \( s \) is not equal to \( \frac{n-1}{2n} \) or to \( \frac{n+1}{2n} \).

Suppose that \( s = \frac{n+1}{2n} \). Arguing similarly to Friedberg-Ginzburg \( \text{[4]} \), it is not hard to check that the integral \( I_1(s) \) is zero for all choices of data if and only if \( f_{W_1,W_2}(e,s) \) is zero for all choices of data. Hence, unless \( E^{(r)} \) is the residue at \( s = \frac{n+1}{2n} \), then it must be generic. By the functional equation of the Eisenstein series (Moeglin-Waldspurger \( \text{[12]} \)), the same holds at \( s = \frac{n-1}{2n} \).

This completes the proof of the Theorem. \( \square \)

In a similar way one may show that an Eisenstein series associated to a non-maximal parabolic formed from irreducible cuspidal representations is once again generic as long as the complex tuple \( s \) is in general position.

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