GENERIC NEWTON POINTS AND THE NEWTON POSET IN IWAHORI DOUBLE COSETS

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ABSTRACT. We consider the Newton stratification on Iwahori double cosets in the loop group of a reductive group. We describe a group-theoretic condition on the generic Newton point, called cordiality, under which the Newton poset (i.e. the index set for non-empty Newton strata) is saturated and Grothendieck’s conjecture on closures of the Newton strata holds. Finally, we give several large classes of Iwahori double cosets for which this condition is satisfied by studying certain paths in the associated quantum Bruhat graph.

1. INTRODUCTION

Let $F$ be a local field with ring of integers $\mathcal{O}_F$, uniformizer $t$ and residue field $\mathbb{F}_q$ of characteristic $p$. Let $\hat{F}$ denote the completion of the maximal unramified extension of $F$, and $\hat{\mathcal{O}}$ its ring of integers. Let $G$ be an unramified reductive group over $F$. Let $I$ be an Iwahori sub-group scheme of $G$ defined over $\mathcal{O}_F$. Let $B$ over $F$ be the associated Borel subgroup and fix a maximal torus $T \subset B$ of $G$, defined over $F$. Let $\hat{W}$ be the extended affine Weyl group of $G$. We then have

$$G(\hat{F}) = \prod_{x \in \hat{W}} I(\hat{\mathcal{O}}) x I(\hat{\mathcal{O}}).$$

Here for every $x \in \hat{W}$ we choose a representative in $G(\hat{F})$ which we denote again by $x$. The Frobenius $\sigma$ of $\hat{F}$ over $F$ acts on $G(\hat{F})$ and also induces an automorphism of $\hat{W}$, which we denote again by $\sigma$. To define a length function $\ell$ on $\hat{W}$ let $\pi_a$ be the affine Weyl group. Since we fixed $I$, we also obtain a length function $\ell$ and a Bruhat order $\leq$ on the infinite Coxeter group $W_a$. There is a short exact sequence

$$1 \to W_a \to \hat{W} \to \pi_1(G) \to 1,$$

where $\pi_1(G)$ is Borovoi’s fundamental group. Identifying $\pi_1(G)$ with the stabilizer of the base alcove, we obtain $\hat{W} \cong W_a \rtimes \pi_1(G)$. We extend $\ell$ and $\leq$ from $W_a$ to $\hat{W}$ by setting $\ell(\omega) = 0$ for $\omega \in \pi_1(G)$, and defining $x \leq y$ if and only if $x$ and $y$ are of the form $x' \omega$ and $y' \omega$ for some $\omega \in \pi_1(G)$ and $x' \leq y' \in W_a$.

Let $W = N_T(F)/T(\hat{F})$ be the (finite) Weyl group of $G$. Then the natural projection $\hat{W} \to W$ has kernel $X_\ast(T)$. Choosing a special vertex of the (base) alcove corresponding to $I$ we obtain a splitting $W \to \hat{W}$ which induces an isomorphism $\hat{W} \cong X_\ast(T) \rtimes W$.

We consider the decomposition of $G(\hat{F})$ into $\sigma$-conjugacy classes. For $b \in G(\hat{F})$ let $[b] = \{ g^{-1} \sigma(g) \mid g \in G(\hat{F}) \}$ be its $\sigma$-conjugacy class and let $B(G)$ be the set of $\sigma$-conjugacy classes. The elements $[b] \in B(G)$ are classified by Kottwitz in [Kot1] by two invariants: the Newton point $\nu_b \in X_\ast(T)_{\Gamma,\text{dom}}$, where $\Gamma$ denotes the absolute Galois group of $F$, and the Kottwitz point $\kappa_G(b) \in \pi_1(G)_{\Gamma}$. There is a partial ordering $\leq$ on $B(G)$ defined by $[b] \leq [b']$ if $\kappa_G(b) = \kappa_G(b')$ and $\nu_b \leq \nu_{b'}$; i.e. the difference $\nu_b - \nu_{b'}$ is a non-negative linear combination of positive coroots.

It is a notoriously difficult problem to describe for a given fixed $x \in \hat{W}$ the set $B(G)_x$ of $\sigma$-conjugacy classes $[b] \in B(G)$ such that $N_{[b],x} := [b] \cap IxI \neq \emptyset$. An obvious necessary condition for $[b] \in B(G)_x$ is that the Kottwitz points coincide; i.e. $\kappa_G(b) = \kappa_G(x)$. Recent results of Görtz, He, and Nie [GHN] give a necessary and sufficient condition determining if the unique basic element of $B(G)$ satisfying $\kappa_G(b) = \kappa_G(x)$ is indeed in $B(G)_x$. If this is the case, then it is the unique smallest
element of $B(G)_x$ with respect to the partial ordering on $B(G)$. Another element of $B(G)_x$ that is of particular interest is the unique maximal element of $B(G)_x$, which coincides with the generic $\sigma$-conjugacy class in the irreducible double coset $I x I$ denoted $[b_2]$; see the beginning of Section 2. There are descriptions of $[b_2]$ that also give finite algorithms to compute it, but they are not themselves closed formulas; compare \[Vie2\] Cor. 5.6 and \[Mil\] Cor. 3.3. A complete description of $B(G)_x$ is only known in very particular cases, such as for example if $x$ is a translation element, in which case $B(G)_x = \{[x]\}$, or if $G = \text{SL}_3$; see \[Bea\]. By \[RR\], $N_{[b],x}$ is the set of geometric points of a locally closed subscheme of $I x I$; namely, the Newton stratum associated with $[b]$. A second natural (and similarly unsolved) question is thus to describe the closure of $N_{[b],x}$ in $I x I$.

Before we explain the array of possible answers to this pair of questions that can occur for different elements $x \in \tilde{W}$, let us compare to the much better understood situation where $I$ is replaced by a hyperspecial maximal bounded open subgroup $K = G(\hat{O})$ of a split reductive group $G$ over $\mathbb{O}_F$. We remark that we expect similar behavior without the assumption that $G$ is split, but not all of this has yet been worked out in this greater generality. In this context, the set of $K$-double cosets is indexed by $X_\ast(T)_{\text{dom}}$, the set of $B$-dominant cocharacters of $T$. For a given $\mu \in X_\ast(T)_{\text{dom}}$ and $[b] \in B(G)$, the Newton stratum $[b] \cap K\mu(t)K$ for $[b]$ in $K\mu(t)K$ is non-empty if and only if $[b] \in B(G, \mu)$; i.e. whenever $\kappa_G(b) = \mu$ in $\pi_1(G)^\vee$ and $\nu_\emptyset \leq \mu$. Furthermore, $[b] \cap K\mu(t)K$ is equidimensional, and its closure is equal to the union of all Newton strata associated with $[b'] \leq [b]$; compare \[Vie2\].

Returning to the case of Iwahori-double cosets, none of these properties hold in general. In particular, the Newton strata $N_{[b],x}$ are not equidimensional, their closures are not unions of other Newton strata, and for the set $B(G)_x$, not even a general conjecture is known. For example, the set of $\sigma$-conjugacy classes associated to $x$ can be non-saturated. Here we say that the subset $B(G)_x$ of the poset $B(G)$ is saturated if for every $[b_1] \leq [b_2] \leq [b_3] \in B(G)$ such that $[b_1], [b_3] \in B(G)_x$, we also have $[b_2] \in B(G)_x$.

Recall that we denote the generic $\sigma$-conjugacy class of $I x I$ by $[b_2]$. In Definition 2.15 we define an element $x \in \tilde{W}$ to be cordial if it satisfies

$$\ell(x) - \ell(\eta(x)) = (2\rho, \nu_x) - \text{def}(b_x).$$

Here, $\rho$ is the half-sum of the positive roots, $\nu_x = \nu_{b_x}$, and for $[b] \in B(G)$, the defect $\text{def}(b)$ is defined as $r_K G - r_K J_b$ where $J_b$ is the reductive group over $F$ with

$$J_b(F) = \{g \in G(\hat{F}) \mid gb = b\sigma(g)\}.$$  

The map $\eta : \tilde{W} \to W$ is defined as follows. Write $x \in \tilde{W}$ as $x = v\lambda(t)v^{-1}w = t^\lambda w$ where $v, w \in W$, $\lambda \in X_\ast(T)$, and $t^\lambda w$ maps the base alcove to the dominant chamber. Then let $\eta(x) = \sigma^{-1}(v^{-1}w)v$.

We now comment on the terminology of cordial elements. Fix $x \in \tilde{W}$ and $b \in G(\hat{F})$. The associated affine Deligne-Lusztig variety is defined to be the locally closed, reduced subvariety $X_\ast(b)$ of the affine flag variety with

$$X_\ast(b)(\mathbb{F}_q) = \{g \in G(\hat{F})/I(\hat{0}) \mid g^{-1}b\sigma(g) \in I(\hat{0})xI(\hat{0})\}.$$  

The notion cordial refers to the fact explained in Section 2 that this condition is equivalent to the condition that the dimension of the affine Deligne-Lusztig variety $X_\ast(b_x)$ agrees with its virtual dimension in the sense of \[Hel\]. The cordial condition is thus equivalent to the condition that this variety “has the correct dimension”. Moreover, the following theorem illustrates that the cordial condition also gives rise to especially “well-behaved” geometry for the associated Newton strata.

**Theorem 1.1.** Let $x$ be cordial. Then $B(G)_x$ is saturated, and for $[b] \in B(G)_x$ we have

(a) $N_{[b],x}$ is equidimensional, and its codimension in $I x I$ is equal to the maximal length of any chain from $[b]$ to $[b_2]$ in $B(G)_x$ (or, equivalently, in $B(G)$).

(b) $\overline{N_{[b],x}}$ is the union of all $N_{[b'],x}$ with $[b'] \in B(G)_x$ and $[b'] \leq [b]$.
Theorem 1.1 gives a condition that can be checked from the maximal element of $B(G)_x$ alone, but implies that the shape of the entire poset $B(G)_x$, as well as all dimensions and closures of the Newton strata within $IxI$, behave as nicely as the Newton strata for $K$-double cosets. The only difference that may occur is that the set $B(G)_x$ does not in general contain all elements of the form $\{[b] \in B(G) \mid [b] \leq [b_0]\}$; small elements up to a certain lower bound may be missing. In Theorem 2.19, we also prove a partial converse of this theorem showing that non-cordial elements cannot share all of these same good geometric properties.

Our next theorem explicitly identifies several families of cordial elements. For sufficiently low-rank groups, it is sometimes possible to directly calculate the Newton strata for every $x \in W$. For example, all of the questions we address here can be settled for the group $G = \text{SL}_3$ using the first author’s thesis [Bea]. For this group, all Newton strata for all $x$ are equidimensional, and part (b) of Theorem 1.1 also holds in all cases. For $G = \text{SL}_3$, an element $x$ is cordial if and only if $B(G)_x$ is saturated, and one can give a complete description of the set of cordial elements. For more details, see Example 3.1.

In general, it appears to be a fairly difficult problem to fully characterize the cordial elements in a manner which does not require specific knowledge of the generic Newton point, but we provide several interesting families of cordial elements in the following theorem.

**Theorem 1.2.** Suppose that $G$ is split, connected, and semisimple. Let $x = t^{v\lambda}w \in W$.

(a) If $x$ is in the antidominant Weyl chamber in which $v = w_0$, then $x$ is cordial. Now further suppose that for all simple roots $\alpha_i$ we have

$$\langle \alpha_i, \lambda \rangle > \begin{cases} 4\ell(w_0) & \text{if } G \text{ is classical}, \\ 12\ell(w_0) & \text{if } G \text{ is exceptional}. \end{cases}$$

(b) If any reduced expression for $\eta(x) = v^{-1}wv \in W$ uses each simple reflection at most once, then $x$ is cordial.

(c) If $x$ is in the dominant Weyl chamber in which $v = 1$, then $x$ is cordial if and only if every reduced expression for $\eta(x) = w$ avoids all non-simple reflections $s_\alpha$ such that $\ell(s_\alpha) = (2p, \alpha^+) - 1$.

The hypotheses on $G$ in Theorem 1.2 are a direct reflection of the reliance upon the first author’s formula for calculating the generic Newton point via the quantum Bruhat graph [Hil], which is stated in precisely this level of generality. The additional hypothesis on the coroot $\lambda$ which keeps $x$ sufficiently far from the walls of any Weyl chamber is referred to as superregularity. Here we formulate a stronger, but more uniform version of the superregularity hypothesis than is required for our result; compare [Hil, Theorem 3.2] to [Hil, Corollary 3.3], and see the surrounding discussion for more details. Under this superregularity hypothesis, we characterize cordiality purely in terms of calculating lengths of certain paths in the quantum Bruhat graph; see Proposition 3.2 which is key to proving parts (b) and (c) of Theorem 1.2. Those elements which use each simple reflection at most once as in part (b) are called standard parabolic Coxeter elements in Definition 3.5. We make the condition appearing in (c) precise in Definition 3.6 where we refer to those elements as small-height-avoiding. We refer the reader to Section 3.2 for further discussion of this terminology and related properties.

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2. Maximal Newton Points and Cordial Elements

The aim of this section is to define cordial elements and to prove Theorem 1.1. In the first subsection we compute the dimension of an affine Deligne-Lusztig variety $X_x(b)$ in terms of the dimension of the corresponding Newton stratum in $IxI$. In the second subsection we compare the expression obtained in this way to He’s virtual dimension of the same affine Deligne-Lusztig variety. If these two dimensions agree, we will call the element $x$ cordial.

Definition 2.1. Given an element $x \in \tilde{W}$ in the extended affine Weyl group, let $[b_x]$ be the $\sigma$-conjugacy class in the (unique) generic point of $IxI$, and thus the unique maximal element in $B(G)_x$ with respect to the partial ordering on $B(G)$. We define the maximal Newton point $\nu_x$ associated with $x$ to be the Newton point of $[b_x]$.

By definition, $\nu_x$ satisfies $\lambda \leq \nu_x$ for all Newton points $\lambda$ of elements of $B(G)_x$. The first concrete description of the maximal element of $B(G)_x$ was given by the second author [Vie2, Cor. 5.6], a weaker version of which can be expressed as $\nu_x = \max\{\nu(y) \mid y \in \tilde{W}, y \leq x\}$. Here the maximum is taken with respect to dominance order, and the elements $y$ and $x$ are related by Bruhat order. Note that this yields a finite algorithm to compute $\nu_x$, but not a closed formula. A slightly more explicit description of $\nu_x$ provided by the first author [MB Cor. 3.3] is discussed in Section 3, albeit under an additional superregularity hypothesis on $\lambda$, and for split $G$.

2.1. Comparing dimensions of Newton strata and affine Deligne-Lusztig varieties. Although we do not dispose of a closed formula for $[b_x]$ here, we can relate its Newton point $\nu_x$ to the dimension of the corresponding affine Deligne-Lusztig variety.

Lemma 2.2. Let $x \in \tilde{W}$. Then $X_x(b_x)$ is equidimensional with
\[ \dim X_x(b_x) = \ell(x) - \langle 2p, \nu_x \rangle. \]

For the proof of Lemma 2.2 we develop a more general theory for all Newton strata that will also be helpful later on. The rough idea is to express dimensions of affine Deligne-Lusztig varieties using a product structure up to finite morphism on a corresponding Newton stratum. The construction closely follows the corresponding theory for hyperspecial maximal subgroups of [VW]. We therefore replace most proofs by references to the corresponding arguments given in loc. cit. Let us first recall some well-known notions for subschemes of loop groups.

Definition 2.3. Let $\mathcal{B}$ be a subscheme of the loop group $LG$.

1. Let $x \in \tilde{W}$. Then $\mathcal{B}$ is bounded by $x$ if it is contained in the closure of $IxI$ in $LG$. It is bounded if it is contained in a finite union of double cosets $IxI$.
2. Let $I_n$ be the kernel of the projection map $I \rightarrow I(O_F/(t^n))$. Then $\mathcal{B}$ is admissible if there is an $n \in \mathbb{N}$ with $\mathcal{B}I_n = \mathcal{B}$.
3. For a bounded and admissible algebraic set with $XI_n = X$ let
\[ \dim X := \dim (X/I_n) - n \cdot \dim (G). \]

Notice that this notion of dimension is normalized in a different way than the one in [VW].

Remark 2.4. We can make several initial observations about subschemes of $LG$.

1. Let $\mathcal{B}$ be bounded. Then one easily sees that $\mathcal{B}$ is admissible if and only if there is an $n' \in \mathbb{N}$ with $I_{n'} \mathcal{B} = \mathcal{B}$. Here $n'$ can be given in terms of the bound for $\mathcal{B}$ and the integer $n$ arising in the definition of admissibility.
The dimension of a bounded and admissible subscheme of $LG$ is independent of the choice of $n$.

Similarly, one can define the codimension of a closed irreducible admissible subscheme $\mathcal{B}'$ of some bounded and admissible scheme $\mathcal{B}$. If $\mathcal{B}$ is also equidimensional, one easily sees that this codimension agrees with $\dim \mathcal{B} - \dim \mathcal{B}'$.

**Proposition 2.5.** Let $\mathcal{B}$ be a bounded subset of $LG(k)$. Then there is an integer $c \in \mathbb{N}$ such that for each $d \in \mathbb{N}$, each $g \in \mathcal{B}$ and $h \in I_d+c(k)$, there is an $l \in I_d(k)$ with $gh = l^{-1}g\sigma^*(l)$.

**Proof.** This follows from [VW], Proposition 2.3. □

**Corollary 2.6.** Let $b \in IxI$. Then the $I$-$\sigma$-conjugacy class $\mathcal{C}_b = \{ i\sigma(i)^{-1} \mid i \in I \}$ of $b$ is contained in $IxI$, admissible, and a smooth and locally closed subscheme of $LG$. Further, $\mathcal{N}[b,x]$ is admissible.

**Proof.** In both cases, admissibility follows from the previous proposition. Since $\mathcal{C}_b$ is one $I$-orbit, it is smooth and locally closed. □

The last assertion in Corollary 2.6 also follows from a corresponding assertion on Newton strata in the whole loop group by He [He, Theorem A.1].

**Definition 2.7.** Let $n \in \mathbb{N}$ and let $b \in IxI$. Here $IxI = \bigcup_{I' \subseteq I} IxI$ denotes the closure of $IxI$ in $LG$. We consider the following functor on the category $(\text{Art}/k)$ of Artinian local $k$-algebras with residue field $k$.

$$\text{Def}(b)_n : (\text{Art}/k) \to (\text{Sets}),$$

$$A \mapsto \{ \tilde{b} \in (IxI)(A) \mid \tilde{b}_k = b \} / \sim_n.$$

Here, $\tilde{b} \equiv \tilde{b}'$ if there exists a $g \in I_n(A)$ with $g_k = 1$ such that $\tilde{b}' = g^{-1}\tilde{b}\sigma(g)$. We call $\text{Def}(b)_n$ the deformation functor of level $n$ of $b$.

**Proposition 2.8.** The functor $\text{Def}(b)_n$ is pro-represented by the formal completion of $I_n \backslash IxI$ at $I_n b$, which we denote by $D_{b,n}$.

**Proof.** This is shown completely analogously to the proof of [VW] Proposition 2.9. □

For a given $n$, consider the projection morphism $D_{b,n} \to D_{b,0}$. The projection $LG \to I \backslash LG$ has sections étale locally. Thus we also have a (non-unique) section $s : D_{b,0} \to D_{b,n}$. Let $(D_{b,0} \times (I_n \backslash I))^\wedge$ denote the completion of $D_{b,0} \times (I_n \backslash I)$ at $(1,1)$.

Using that $I_n$ is normal in $I$, the section $s$ induces a (still non-canonical) morphism

$$\phi : (D_{b,0} \times (I_n \backslash I))^\wedge \to D_{b,n}$$

given by $(b,g) \mapsto g^{-1}s(b)\sigma(g)$.

**Lemma 2.9.** The morphism $\phi : (D_{b,0} \times (I_n \backslash I))^\wedge \to D_{b,n}$ is an isomorphism.

**Proof.** Compare the proof of [VW] Lemma 2.10. □

Recall from [VW] Lemma 2.11 that for any admissible $\mathbb{F}_q$-algebra $R$ with filtered index poset $\mathbb{N}_0$, the pullback by the natural morphism $\text{Spec} R \to \text{Spec} \mathbb{F}_q$ induces a bijection between the $\text{Spec} R$-valued points and the $\text{Spec} \mathbb{F}_q$-valued points of $IxI$. Thus we can associate with the formal scheme $D_{b,n}$ a scheme $D_{b,n}'$, and we have a section $D_{b,n}' \to LG$. In particular, we can study the Newton stratification on $D_{b,n}'$. For large $n$, Corollary 2.6 implies that the Newton stratification does not depend on the choice of the lift.

For $[b] \in B(G)$, let $y_b \in \widetilde{W}$ be a fundamental alcove with $[y_b] = [b]$; compare [Nic]. By [Nic] Theorem 1.3, every fundamental alcove is $P$-fundamental for some semistandard parabolic subgroup $P$. Its Levi subgroup $M$ containing $T$ centralizes the $M$-dominant Newton point $\nu$ of $y_b$. From the definition of $P$-fundamental alcoves one can then easily see that $y_b$ is also $P'$-fundamental for $P' = M'P \supset P$ where $M'$ is the centralizer of $\nu$, so $P'$ and $y_b$ are as in the theorem below. If $y$ is $P$-fundamental for some parabolic $P$, let $\overline{N}$ be the unipotent radical of the opposite
parabolic, and let \( I_N = I \cap L_N \). Then by definition of \( P \)-fundamental alcoves, we have \( y^{-1}I_{\sigma^{-1}(N)}y \subseteq I_N \).

**Theorem 2.10.** Let \( b \in IxI \). Let \( N_{\{b\}} \) be the Newton stratum of \([b]\) in \( \text{Spec} D_{b,0} \). Let \( y_b \) be a \( P \)-fundamental alcove associated with \([b]\), where \( P \) is chosen such that the Levi subgroup \( M \) of \( P \) containing \( T \) equals the centralizer of the \( M \)-dominant Newton point of \( y_b \). Then there is a finite surjective morphism

\[
((X_x(b) \times_k I_N/y_b^{-1}I_{\sigma^{-1}(N)}y_b)^\wedge)'/y_b \rightarrow N_{\{b\}}.
\]

Here again, \((\cdot)'/y_b \) denotes the scheme associated with the formal scheme obtained by completion. Furthermore, the locus in \( \text{Spec} D_{b,0} \) of elements \( I-\sigma \)-conjugate to \( b \) is smooth and equal to the image of \(((\{1\} \times_k I_N/y_b^{-1}I_{\sigma^{-1}(N)}y_b)^\wedge)'/y_b \) in \( N_{\{b\}} \).

**Proof.** This follows from essentially the same proof as \([\text{WW}, \text{Theorem 2.9}]\), which in turn was a natural generalization of the proof of Theorem 6.5 or Theorem 6.6 from \([\text{HV}]\) to unramified groups. \( \square \)

Recall that \( N_{\{b\},x} \) is the Newton stratum for \([b]\) in \( IxI \).

**Corollary 2.11.** For \( x \in \bar{W} \) and \( b \in IxI \), let \( h \in N_{\{b\},x}(k) \). Let \( g \in X_x(b)(k) \) with \( g^{-1}b_0g = h \). Denote by \( X_x(b)_h^\wedge \) and \( (N_{\{b\},x})_h^\wedge \) the completions in the two points, respectively. Assume that \(((N_{\{b\},x})_h^\wedge)'/y_b \) is irreducible. Then

\[
\dim (X_x(b)_h^\wedge)'/y_b = \dim (X_x(b)_{\{b\}}^\wedge)/y_b - (2\rho, \nu(b)).
\]

**Proof.** This follows from the previous theorem, using that \( I_N/y_b^{-1}I_{\sigma^{-1}(N)}y_b \) is irreducible and of dimension \( \dim (y_b) = (2\rho, \nu(b)) \). \( \square \)

**Corollary 2.12.** Using the notation of the previous corollary, \( \dim X_x(b) = \dim (X_x(b)_h^\wedge)/y_b - \dim (N_{\{b\},x})_h^\wedge \), where \( \dim (N_{\{b\},x})_h^\wedge \) denotes the minimal codimension of all irreducible components. Furthermore, \( X_x(b) \) is equidimensional if and only if the same holds for \( N_{\{b\},x} \).

**Proof.** Apply the previous corollary to all elements contained in just one irreducible component of \( N_{\{b\},x} \). \( \square \)

We are now able to prove Lemma 2.2 as an immediate consequence.

**Proof of Lemma 2.2.** Apply Corollary 2.12 to \([b_x]\), and use that in this case the Newton stratum is irreducible and of codimension 0 in \( IxI \). \( \square \)

### 2.2. Virtual dimension and cordality

We recall from \([\text{He}1, \S 10.1]\) the notion of virtual dimension. For \( x \in \bar{W} \) and \([b]\) in \( B(G) \) with \( \kappa_G(b) = \kappa_G(x) \), define

\[
d_x(b) = \frac{1}{2}(\ell(x) + \ell(\eta(x)) - \text{def}(b)) - (2\rho, \nu(b))
\]

to be the virtual dimension of the pair \((x, [b])\).

**Lemma 2.13 (He).** Let \( x \in \bar{W} \) and \([b]\) in \( B(G) \) with \( \kappa_G(x) = \kappa_G(b) \). Then,

\[
\dim X_x(b) \leq d_x(b).
\]

**Proof.** For groups \( G \) such that the action of \( \sigma \) on \( \bar{W} \) is trivial, this is \([\text{He}1, \text{Corollary 10.4}]\). The generalization to the present more general context follows from the same proof, using that the dimension formula for affine Deligne-Lusztig varieties in affine Grassmannians has been proven in the meantime in \([\text{Ham1}]\). \( \square \)

We now combine this lemma with the formula for \( \dim X_x(b) \) from the preceding subsection.

**Lemma 2.14.** Let \( x \in \bar{W} \), and let \([b_x]\) in \( B(G) \) be the generic \( \sigma \)-conjugacy class in \( IxI \). Then

\[
\ell(x) - \ell(\eta(x)) \leq (2\rho, \nu_x) - \text{def}(b_x).
\]
Proof. By Lemma 2.13, we have $\dim X_x(b) \leq d_x(b)$ for all $[b]$ with $\kappa_G(b) = \kappa_G(x)$, and in particular for $[b_x]$. Together with Lemma 2.2, we have

$$\ell(x) - \langle 2\rho, \nu_x \rangle \leq \frac{1}{2} (\ell(x) + \ell(\eta(x)) - \operatorname{def}(b) - \langle 2\rho, \nu_x \rangle),$$

which is equivalent to the above inequality. \hfill \Box

**Definition 2.15.** Let $x \in \tilde{W}$. Let $[b_x] \in B(G)$ be the generic $\sigma$-conjugacy class in $I_xI$. Then $x$ is called **cordial** if

$$\ell(x) - \ell(\eta(x)) = \langle 2\rho, \nu_x \rangle - \operatorname{def}(b_x).$$

In other words, $x$ is cordial if and only if $\dim X_x(b_x) = d_x(b_x)$. 

**Example 2.16.** Suppose that $x = \ell w_0 w \in \tilde{W}$ so that $x$ is in the antidominant Weyl chamber. Then by [Vie2] Cor. 5.6, $\nu_x = \lambda$. Thus $\operatorname{def}(b_x) = 0$, and $\langle 2\rho, \nu_x \rangle = \ell(\lambda) = \ell(x) - \ell(w_0 w_0) = \ell(x) - \ell(\eta(x))$. Hence all $x$ in the antidominant Weyl chamber are cordial, which proves the first assertion (a) of Theorem 1.2.

We are now ready to prove our first main theorem. The idea (first used in [Vie1]) is to combine a strong version of purity of the Newton stratification with upper bounds on the dimension of the Newton strata obtained via Corollary 2.12; see [Vie3, §5] for an overview.

**Proof of Theorem 1.1.** Let $[b] \in B(G)_x$ and $g \in X_x(b)$. Let $b' = g^{-1}b\sigma(g)$. By Lemma 2.13, the dimension of the completion of $X_x(b)$ in $g$ can be estimated as

$$(1) \quad \dim X_x(b)_g \leq \dim X_x(b) \leq d_x(b) = \frac{1}{2} (\ell(x) + \ell(\eta(x)) - \operatorname{def}(b) - \langle 2\rho, \nu(b) \rangle).$$

Together with Corollary 2.11, we obtain

$$\operatorname{codim}(N_{[b], x})_g \geq \frac{1}{2} (\ell(x) - \ell(\eta(x)) + \operatorname{def}(b) - \langle 2\rho, \nu(b) \rangle))$$

$$(2) \quad \frac{1}{2} (\operatorname{def}(b) - \operatorname{def}(b_x) - \langle 2\rho, \nu(b) - \nu_x \rangle),$$

where the last equality holds because $x$ is cordial. By (a slight correction of) [Chai] Theorem 7.4, together with the main result of [Kot2] and [Ham2] Prop. 3.8, the right hand side of this estimate is equal to the length of every maximal chain of elements in $B(G)$ from $[b]$ to $[b_x]$. For a detailed discussion of this result and of what needs to be modified in Chai’s theorem compare [Vie3, Theorem 3.4] and its proof.

Thus the conditions of [Vie3, Lemma 5.12] are satisfied. This implies that the Newton stratification on the scheme associated with the completion of $I_xI$ in $b'$ satisfies the analogue of the assertions of the theorem. As all of the above holds for every $g \in X_x(b)$, and in particular for all elements contained in exactly one irreducible component, the theorem follows. \hfill \Box

**Remark 2.17.** Theorem 5.3 of [He4] shows that if $x$ is in the shrunken Weyl chamber, and the basic locus is non-empty, then $\dim X_x(b) = d_x(b)$ for the basic $[b] \in B(G)_x$. A necessary and sufficient criterion for non-emptiness of the basic locus is given in [GHN]. In this case, our theorem shows that if $x$ is cordial, then $B(G)_x = \{ [b] : [b] \leq [b_x] \}$.

**Corollary 2.18.** Let $x$ be cordial. Then for every $[b] \in B(G)_x$ we have that $X_x(b)$ is equidimensional of dimension $\dim X_x(b) = d_x(b)$.

**Proof.** The proof of the preceding theorem also shows that for every $b'$ contained in only one irreducible component of the Newton stratum, the codimension $\operatorname{codim}(N_{[b], x})_{[b']} \geq \frac{1}{2} (\operatorname{def}(b) - \operatorname{def}(b_x) - \langle 2\rho, \nu(b) - \nu_x \rangle)$; i.e. to the right hand side of (2). Thus all inequalities in (1) have to be equalities. As this holds for every $g$, the corollary follows. \hfill \Box

We now present a partial converse to Theorem 1.1.

**Theorem 2.19.** Suppose that $x \in \tilde{W}$ is not cordial. Assume that there is a $[b] \in B(G)_x$ such that $\dim X_x(b) = d_x(b)$. Then there is a $[b'] \in B(G)$ such that
finite Weyl group $S$ fixed base alcove. We also identify $S$ the set of positive roots. Let $G$

on $\text{semisimple}$. Thus for the remainder of the paper we make these additional assump-

imal Newton point $x$ of $G$.

Proof. We have $\dim X_x(b_x) < d_x(b_x)$, and hence

\[
\dim N_{[b_x]} - \dim N_{[b]} = \dim X_x(b_x) - \dim X_x(b) + (\langle 2\rho, \nu(b_x) - \nu(b) \rangle)
\]

\[
= \frac{1}{2} \left( \text{def}(b) - \text{def}(b_x) - (\langle 2\rho, \nu(b) - \nu_v) \rangle. \right.
\]

Therefore, by the argument following (2), the difference in dimensions is less than

the length of every maximal chain in $B(G)$ between $[b]$ and $[b_x]$, and the theorem

follows.

Along these same lines, one could also formulate more precise statements relating

d_x(b_x) - \dim X_x(b_x) to the number of $[b']$ as in Theorem 2.19

3. Families of cordial elements

Characterizing the cordial elements in $\tilde{W}$ requires a good description of the max-

imal Newton point $\nu_x$. The most useful known description of $\nu_x$ uses paths in the

quantum Bruhat graph and is available for groups $G$ which are split, connected, and

semisimple. Thus for the remainder of the paper we make these additional assumptions on $G$.

Let $\Phi$ be the set of relative roots of $G$ over $\tilde{F}$ with respect to $T$, and let $\Phi^+$ be

the set of positive roots. Let $S$ be the basis of $\Phi$ of simple roots corresponding to the

fixed base alcove. We also identify $S$ with the set of simple reflections in $W$. The

finite Weyl group $W$ acts on $\mathbb{R}^r$ as a finite reflection group, where $r$ is the rank of

$G$. The set of reflections in $W$ is defined as $R = \{ws\alpha^{-1} | s \in S, w \in W\}$. There

is a bijection between $\Phi^+$ and $R$. More precisely, let $\alpha \in \Phi^+$ and write $\alpha = w(\alpha_i)$

for some simple root $\alpha_i$ and $w \in W$. Then $\alpha$ corresponds to the well-defined reflection

$s_\alpha := ws_\alpha w^{-1} \in W$. Throughout the paper we denote simple reflections by $s_i$ (the

index being a roman letter), and reflections associated with a positive root (which may

or may not be simple) by $s_\alpha$ (the index being a greek letter).

3.1. Cordial elements and the quantum Bruhat graph. The primary tool in

the proof of Theorem 1.2 (b) and (c) is a labeled directed graph associated with the

group $G$ called the quantum Bruhat graph. We now review some key properties of

this graph and its relation to maximal Newton points.

Definition 3.1 ([FGP]). We construct the quantum Bruhat graph $\Gamma_G$ as follows.

(1) The vertices of the graph are the elements $w \in W$. Here $W$ is the finite Weyl

group of $G$.

(2) Draw a directed edge $w \to ws_\alpha$ for any $\alpha \in \Phi^+$ if either of the following is

satisfied:

\[
w \to ws_\alpha \text{ if } \ell(ws_\alpha) = \ell(w) + 1, \ \text{or} \n\]

\[
w \to ws_\alpha \text{ if } \ell(ws_\alpha) = \ell(w) - (2\rho, \alpha^\vee) + 1. \n\]

(3) Label the edge $w \to ws_\alpha$ by the corresponding root $\alpha$.

Figure 1 shows the quantum Bruhat graph for $G = SL_3$. As in Figure 1 we can

can always draw $\Gamma_G$ such that vertices are ranked by length increasing upward, in which

case the first type of edge (colored blue) always points upward and the second type

(colored red) downward; this will be our convention throughout the paper. Note that

the upward edges correspond precisely to the covering relations in Bruhat order, and

so we can also view the vertices in $\Gamma_G$ as a ranked partially ordered set. We write

$v \overset{w}{\prec} w$ if $v \leq w$ in Bruhat order and $\ell(v) = \ell(w) - 1$ to denote such a covering relation.

Define the weight of an edge in the quantum Bruhat graph $\Gamma_G$ as follows.

(1) An upward edge $w \to ws_\alpha$ carries no weight.

(2) A downward edge $w \to ws_\alpha$ carries a weight of $\alpha^\vee$. 

The weight of a path in $\Gamma_G$ is the sum of the weights over all of the edges in the path. For example, in $\Gamma_{S_3}$ from Figure 1, the weight of each of the three shortest paths from $s_1s_2 = s_12$ to $s_2$ equals $\alpha_1^\vee + \alpha_2^\vee$. In general, given any $u, v \in W$, by [Pos, Lemma 1] there always exists a path in $\Gamma_G$ from $u$ to $v$, and all paths of minimal length between $u$ and $v$ have the same weight.

Since $G$ is split, connected, and semisimple, then under the superregularity hypothesis guaranteeing that $x = t^{v\lambda}w \in \overline{W}$ is sufficiently far from the walls of any Weyl chamber, the maximal Newton point $\nu_x$ can be computed from the weight of certain paths in the quantum Bruhat graph $\Gamma_G$. More specifically, [Mil, Cor. 3.3] says that the maximal Newton point $\nu_x$ can be expressed as

$$\nu_x = \ell - \alpha_2^\vee,$$

where $\alpha_2^\vee$ denotes the weight of any path of minimal length from $w^{-1}v$ to $v$ in $\Gamma_G$.

Denote by $d_G(u, v)$ the minimum length among all paths in $\Gamma_G$ from $u$ to $v$; the choice of notation represents the fact that $d_G(u, v)$ equals the distance between these two elements in the graph $\Gamma_G$. As an important special case, denote the minimum length of any path in $\Gamma_G$ from $w$ to the identity which uses exclusively downward edges by $d_1(w)$. We remark that such a path always exists, since by definition of $\Gamma_G$ any reduced expression for $w$ determines an all downward path from $w$ to the identity having $\ell(w)$ edges. We say that any path in $\Gamma_G$ from $u$ to $v$ which uses exactly $d_G(u, v)$ edges realizes $d_G(u, v)$. Similarly, any downward path in $\Gamma_G$ from $w$ to 1 consisting of exactly $d_1(w)$ edges realizes $d_1(w)$.

We are now able to characterize the cordial elements under our additional superregularity hypothesis in a purely combinatorial manner which does not require any explicit knowledge of the maximal Newton point.

**Proposition 3.2.** Let $x = t^{v\lambda}w \in \overline{W}$, and assume that $\lambda$ is superregular in the sense of Theorem 1.2. Then $x$ is cordial if and only if $d_G(w^{-1}v, w) = \ell(w^{-1}v) = \ell(\eta(x))$.

**Proof.** First note by (3) that $\nu_x$ is integral under our superregularity hypothesis on $\lambda$. Therefore, $\text{def}(b_x) = 0$ in this case, and so $x$ is cordial if and only if $\ell(x) - \ell(\eta(x)) = \langle 2\rho, \nu_x \rangle$. Now recall a length formula for $x$ from [LS, Lemma 3.4], which applies since $\lambda$ is both regular and dominant:

$$\ell(x) = \ell(t^{v\lambda}) - \ell(w^{-1}v) + \ell(v) = \langle 2\rho, \lambda \rangle - \ell(w^{-1}v) + \ell(v).$$

Combine Equations (3) and (4) to write

$$\ell(x) - \langle 2\rho, \nu_x \rangle = \left(\langle 2\rho, \lambda \rangle - \ell(w^{-1}v) + \ell(v)\right) - \langle 2\rho, \lambda - \alpha_2^\vee \rangle$$

$$= \langle 2\rho, \alpha_2^\vee \rangle - \ell(w^{-1}v) + \ell(v),$$

where $\alpha_2^\vee$ is the weight of any minimal length path $p$ in $\Gamma_G$ from $w^{-1}v$ to $v$. Therefore, $x$ is cordial if and only if $\langle 2\rho, \alpha_2^\vee \rangle - \ell(w^{-1}v) + \ell(v) = \ell(\eta(x))$. It thus suffices to show

*Figure 1. The quantum Bruhat graph $\Gamma_G$ for $W = S_3.*
that
\[ (5) \quad \langle 2\rho, \alpha'_x \rangle - \ell(w^{-1}v) + \ell(v) = d\Gamma(w^{-1}v, v). \]

Note that the quantity \(-\ell(w^{-1}v) + \ell(v)\) equals the difference in rank in the poset \(\Gamma_G\) from the beginning to the end of the path \(p\), where the quantity is positive, negative, or zero according to whether the rank of the final vertex of \(p\) is higher, lower, or the same as the rank of its initial vertex. For ease of reference, denote this quantity by \(\Delta \text{rk}(p) = -\ell(w^{-1}v) + \ell(v)\). Recall that we draw an edge \(w \rightarrow ws_\alpha\) in \(\Gamma_G\) if and only if
\[ \ell(ws_\alpha) = \begin{cases} \ell(w) + 1, & \text{or} \\ \ell(w) - \langle 2\rho, \alpha'_x \rangle + 1, \end{cases} \]
where the edges of the first type are directed upward and the second type are directed downward. Therefore, each upward edge in \(p\) contributes +1 to \(\Delta \text{rk}(p)\), and each downward edge in \(p\) labeled by \(\alpha\) contributes \(-\langle 2\rho, \alpha'_x \rangle + 1\) to \(\Delta \text{rk}(p)\). Denote the roots labeling the downward edges by \(\alpha_{d_i}\) for \(i = 1, \ldots, d\) where \(d\) equals the number of downward edges in \(p\). Denote the number of upward edges in \(p\) by \(u\). We can thus write
\[ \Delta \text{rk}(p) = u + \sum_{i=1}^{d} (-\langle 2\rho, \alpha'_{d_i} \rangle + 1) \]
\[ = (u + d) - \sum_{i=1}^{d} \langle 2\rho, \alpha'_{d_i} \rangle \]
\[ = d\Gamma(w^{-1}v, v) - \sum_{i=1}^{d} \langle 2\rho, \alpha'_{d_i} \rangle. \]

On the other hand, recall that the weight of the path \(p\) is defined to be \(\sum \alpha'_d \) summing over all the downward edges, so that by linearity we can rewrite (6) as
\[ \Delta \text{rk}(p) = d\Gamma(w^{-1}v, v) - \left(2\rho, \sum_{i=1}^{d} \alpha'_{d_i} \right) = d\Gamma(w^{-1}v, v) - \langle 2\rho, \alpha'_x \rangle. \]

Therefore,
\[ d\Gamma(w^{-1}v, v) = \langle 2\rho, \alpha'_x \rangle + \Delta \text{rk}(p) = \langle 2\rho, \alpha'_x \rangle - \ell(w^{-1}v) + \ell(v), \]
confirming (5) and concluding the proof. \(\Box\)

**Remark 3.3.** In general, we always have \(d\Gamma(w^{-1}v, v) \leq \ell(w^{-1}wv)\). More precisely, by taking any reduced expression for \(w^{-1}wv = s_{i_1} \cdots s_{i_k}\) and following the edges labeled by the simple roots \(\alpha_{i_1}, \ldots, \alpha_{i_k}\) in order, we obtain a path from \(w^{-1}v\) to \(v\) which has exactly \(\ell(w^{-1}wv)\) edges. Therefore, under the superregularity hypothesis, cordial elements are precisely those for which no shorter path exists from \(w^{-1}v\) to \(v\).

We now provide an example which illustrates how to use Proposition 3.2 to identify families of cordial elements. Recall that we already considered this case (in greater generality) in Example 2.16.

**Example 3.4.** Suppose that \(x = t^\lambda w \in \bar{W}\) so that \(x\) is in the antidominant Weyl chamber. If \(\lambda\) is superregular in the sense of Theorem 1.2, we want to show also with this new method that \(x\) is cordial. By Proposition 3.2, it suffices to prove that \(d\Gamma(w^{-1}w_0, w_0) = \ell(\eta(x))\). Since the end vertex of the path in \(\Gamma_G\) is the longest element \(w_0\), and since every upward edge only increases the length by one, any path of minimal length ending at \(w_0\) is necessarily a path containing only upward edges. Comparing rank, any minimal path from \(w^{-1}w_0\) to \(w_0\) then has exactly \(\ell(w_0) - \ell(w^{-1}w_0) = \ell(w_0) - \ell(w_0w)\) edges. Recall from [BB Corollary 2.3.3] that \(\ell(w_0w) = \ell(w_0) - \ell(w)\) and \(\ell(w_0w_0w) = \ell(w)\) for all \(w \in \bar{W}\). Therefore, for these elements, we have
\[ \ell(\eta(x)) = \ell(w_0w_0w) = \ell(w) = \ell(w_0) - \ell(w^{-1}w_0) = d\Gamma(w^{-1}w_0, w_0). \]
By Proposition 3.2, \(x\) is cordial. Compare Theorem 1.2 (a), which we recall was proved in Example 2.16 without any superregularity hypothesis.
3.2. Standard parabolic Coxeter and small-height-avoiding elements. In this section, we develop the necessary background to study the latter two families of cordial elements identified in Theorem 1.3.

The reflection length of $w \in W$ is the minimal number of reflections required to express $w$ as a product of elements in $R$; namely,

$$\ell_R(w) = \min \{ r \in \mathbb{N} \mid w = s_{\beta_1} \cdots s_{\beta_r} \text{ for } s_{\beta_i} \in R \}.$$ 

By definition, $\ell_R(w) \leq \ell(w)$. We now recall a characterization of those elements such that $\ell_R(w) = \ell(w)$.

Definition 3.5. The element $w \in W$ is a standard parabolic Coxeter element if each simple reflection is used at most once in any (equivalently every) reduced expression for $w$.

As the terminology suggests, standard parabolic Coxeter elements are those which are Coxeter elements in some standard parabolic subgroup of $W$. (We remark that standard parabolic Coxeter elements have also appeared by other names in the literature; for example, they are called boolean in \cite{RT}.) By \cite{BDSW} Lemma 2.1, the element $w$ is standard parabolic Coxeter if and only if $\ell_R(w) = \ell(w)$, a property which will be critical in the proof of Theorem 1.2 (b).

Next we define a slightly more general family of elements in $W$, which properly contains the standard parabolic Coxeter elements.

Definition 3.6. We say $w \in W$ contains the element $v \in W$ if there exist $u, u' \in W$ such that $w = uvu'$ and $\ell(w) = \ell(u) + \ell(v) + \ell(u')$. An element $w \in W$ is called small-height-containing if $w$ contains a non-simple reflection $s_{s_0}$ such that $\ell(s_{s_0}) = \langle 2\rho, \alpha \rangle - 1$. Otherwise, we say that $w$ is small-height-avoiding.

Note that all simple reflections $\alpha$ satisfy $\ell(s_{s_0}) = \langle 2\rho, \alpha \rangle - 1$, so we intentionally exclude these. Also note that the small-height-avoiding condition cannot be verified by looking at only one reduced expression, as the example $s_{1213} = s_{1231}$ in type $A_3$ illustrates.

This terminology is inspired by the related notion of short-braid-avoiding elements, which are those elements of $W$ which do not contain a subexpression of the form $s_is_js_i$ in any reduced expression; see \cite{Fan}. If $G$ is simply-laced, then for any $\alpha \in \Phi^+$ we have $\ell(s_{s_0}) = \langle 2\rho, \alpha \rangle - 1$ by \cite{BFP} Lemma 4.3, and so the notions of small-height-avoiding and short-braid-avoiding coincide in this case. More generally, for any $\alpha \in \Phi^+$ we always have $\ell(s_{s_0}) \leq \langle 2\rho, \alpha \rangle - 1$, and the inequality may be strict. Rewriting this expression, we see that $\ell(s_{s_0}) \geq \ell(s_{s_0}) + 1$, and so those reflections which we avoid in Definition 3.6 are precisely those whose height is as "small" as it could possibly be. There is also a relationship between the small-height-avoiding and fully commutative elements defined in \cite{Sto}, which are those for which any reduced expression can be obtained from any other by means of only commuting relations. In the simply-laced case, it follows from \cite{Sto} Proposition 2.1 that all of these notions coincide.

Example 3.7. As an example which illustrates the relations among these families, we identify the standard parabolic Coxeter, small-height-avoiding, short-braid-avoiding, and fully commutative elements for $G$ of type $C_2$. In this case, $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^4 = 1 \rangle$ so that the four reflections are $s_1, s_2, s_{121}$, and $s_{212}$, and the other nontrivial elements (all of which are rotations in $\mathbb{R}^2$) are $s_{12}, s_{21},$ and $w_0 = s_{1212}$. The standard parabolic Coxeter elements are thus $\{1, s_1, s_2, s_{12}, s_{21}\}$, which coincides here with the set of short-braid-avoiding elements. All of the elements besides $w_0$ are fully commutative. To determine the small-height-avoiding elements, we must further identify the coroots which correspond to each non-simple reflection. We follow the convention that $\alpha_1$ is the short simple root and $\alpha_2$ the long one. Then

$$s_{121} \leftrightarrow \alpha_1 + \alpha_2 \leftrightarrow \alpha_1 + \alpha_2^\vee,$$

$$s_{212} \leftrightarrow 2\alpha_1 + \alpha_2 \leftrightarrow 2\alpha_1 + 2\alpha_2^\vee.$$ 

We thus see that $\ell(s_{121}) = \langle 2\rho, \alpha_1^\vee + \alpha_2^\vee \rangle - 1$, so that small-height-avoiding elements cannot contain $s_{121}$. By contrast, $3 = \ell(s_{212}) \neq \langle 2\rho, \alpha_1^\vee + 2\alpha_2^\vee \rangle - 1 = 5$, so $s_{212}$ does
not need to be avoided. Therefore, the set of small-height-avoiding elements in $C_2$ is $\{1, s_1, s_2, s_{12}, s_{21}, s_{212}\}$, which sits properly between the sets of standard parabolic Coxeter (or equivalently, short-braid-avoiding) and fully commutative elements.

### 3.3. Two additional families of cordial elements.

The goal of this section is to prove parts (b) and (c) of Theorem 1.2. For part (c), we first require two more technical lemmas as stepping stones to Proposition 3.10, which allows us to focus exclusively on paths in $G$ with all downward edges.

**Lemma 3.8.** Let $s_\beta \in R$ be a non-simple reflection such that $\ell(s_\beta) = (2\rho, \beta^\vee) - 1$ for some $\beta \in \Phi^+$, and suppose that $s_\beta s_\alpha < s_\beta$ for some $\alpha \in \Phi^+$. Then $s_\beta s_\alpha = s_{\gamma_1} s_{\gamma_2}$, where $\ell(s_\beta s_\alpha) = \ell(s_{\gamma_1}) + \ell(s_{\gamma_2})$ and $\ell(s_{\gamma_1}) = (2\rho, \gamma_1^\vee) - 1$.

**Proof.** For any reduced expression $s_\beta = s_{i_1} \cdots s_{i_m}$, the condition $s_\beta s_\alpha < s_\beta$ together with the Strong Exchange Property implies that there is a reduced expression $s_\beta s_\alpha = s_{i_1} \cdots s_{i_l} \cdots s_{i_m}$ for some $1 \leq l \leq m$. Moreover, since $s_\beta$ is a reflection, $\ell(s_\beta)$ is odd, and we may choose the reduced expression for $s_\beta$ to be palindromic by [BFP, Lemma 4.1]. For $l = (m + 1)/2$, the resulting expression for $s_\beta s_\alpha$ is trivial, and the hypothesis $s_\beta s_\alpha < s_\beta$ is not satisfied. Thus for symmetry reasons, it is enough to consider the cases where $l > (m + 1)/2$. In this situation, we have $s_\alpha = s_{i_m} \cdots s_{i_l} \cdots s_{i_1}$, and $\ell(s_\alpha) < \ell(s_\beta) - 1 = \ell(s_\beta s_\alpha) = \ell(s_\alpha s_{\beta_0})$. By [BB Prop. 4.4.6], this inequality implies that $\ell(s_\alpha) > 0$. By the same proposition, $\ell(s_\beta s_\alpha) < \ell(s_\beta)$ implies that $s_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta < 0$. Since $\alpha$ and $\beta$ are positive, $(\alpha, \beta^\vee)$ also has to be positive. Therefore, (7) $$(s_\alpha(\beta))^\vee = s_\alpha^\vee(\beta^\vee) = \beta^\vee - c^\vee$$ for some integral $c > 0$.

Now, recalling that $l > (m + 1)/2$, we will prove that we may choose $\gamma_1 = \alpha$ and $\gamma_2 = s_\alpha(\beta)$ to satisfy the conclusion of the lemma. Certainly, $s_\beta s_\alpha = s_\alpha(s_\alpha s_\beta s_\alpha) = s_{\gamma_1} s_{\gamma_2} s_\alpha$. We next show that this product is length-additive. Since $\ell(s_\beta) - 1 = \ell(s_\beta s_\alpha) = \ell(s_\alpha s_\beta) + \ell(s_\alpha)$, length-additivity is implied by the following claim.

**Claim.** $\ell(s_{s_\alpha(\beta)}) + \ell(s_\alpha) \leq \ell(s_\beta) - 1$.

For every positive root $\gamma$ we have $\ell(s_\gamma) \leq (2\rho, \gamma^\vee) - 1$ by [BFP, Lemma 4.3], and we assumed equality for $\gamma = \beta$. Then we have (8) $$\ell(s_{s_\alpha(\beta)}) \leq (2\rho, (s_\alpha(\beta))^\vee) - 1 \leq (2\rho, (s_\beta)^\vee - c^\vee) - 1 \leq (2\rho, \beta^\vee - c^\vee) - 1 \leq \ell(s_\beta) - \ell(s_\alpha) - 1,$$ which proves the claim.

Furthermore, since both sides of the inequality in the claim are equal, each of the inequalities in (8) also has to be an equality. From the first line of (8), we see that $\ell(s_{s_\alpha(\beta)}) = (2\rho, (s_\alpha(\beta))^\vee) - 1$. Finally, since $\ell(s_\beta) = (2\rho, \beta^\vee) - 1$ by hypothesis, the last equality in (8) yields $\ell(s_\alpha) = (2\rho, \alpha^\vee) - 1$, which completes the proof. □

**Lemma 3.9.** Let $w \in W$, and suppose that $w < ws_\alpha$ for some $\alpha \in \Phi^+$. Then $$d_i(w) \leq d_i(ws_\alpha) + 1.$$

**Proof.** Let $w \in W$, and suppose that $w < ws_\alpha$ for some $\alpha \in \Phi^+$. Consider any path in $G$ realizing $d_i(ws_\alpha)$, which then corresponds to a length-additive expression as a product of reflections of the form $ws_\alpha = s_{\beta_1} \cdots s_{\beta_k}$, where each of the reflections satisfies $\ell(s_{\beta_i}) = (2\rho, \beta_i^\vee) - 1$.

On the other hand, since $w < ws_\alpha$ is a cocover, then for any reduced expression $ws_\alpha = s_{i_1} \cdots s_{i_l}$, we have $w = s_{i_1} \cdots s_{i_\ell} \cdots s_{i_k}$ for some $1 \leq \ell \leq k$ by the Strong Exchange Property. Further, since $\ell(w) = \ell(ws_\alpha) - 1$, then the expression $w =$
\[ s_1 \cdots s_i \cdots s_k \] is still reduced. Therefore, \( w \) has a reduced expression of the form \( w = s_{i_1} \cdots (s_{i_j} \cdots s_{i_m}) \cdots s_{i_k} \), where \( s_i = s_{i_j} \) is the single factor removed from the reflection \( s_{i_j} = s_{i_1} \cdots s_{i_m} \). Since the entire expression for \( w \) remains reduced when removing \( s_{i_j} \) then the expression \( s_{i_1} \cdots s_{i_m} \) is also reduced. Defining \( s_\gamma = s_{i_1} \cdots s_{i_m} \), we then see that \( s_\beta, s_\gamma < s_{i_j} \), and the hypotheses of Lemma 3.8 are satisfied. Therefore, we may write \( s_\beta s_\gamma = s_{\gamma_1} s_{\gamma_2} \), where \( \ell(s_\beta, s_\gamma) = \ell(s_{\gamma_1}) + \ell(s_{\gamma_2}) \) and \( \ell(s_{\gamma_2}) = (2p, \gamma^*) = 1 \).

Altogether, we have thus shown that we have a length-additive expression for \( w \) as a product of reflections of the form \( w = s_{\beta_1} \cdots s_{\beta_{j-1}} s_{\gamma_1} s_{\gamma_2} s_{\beta_{j+1}} \cdots s_{\beta_n} \), where each of the reflections in the product satisfies the criterion for drawing a downward edge in \( G \). Therefore, this expression corresponds to a downward path of length \( r + 1 = d_1(ws_\alpha) + 1 \) from \( w \) to \( 1 \) in \( G \), and so \( d_1(w) \leq d_1(ws_\alpha) + 1 \).

Lemma 3.9 provides the foundation for the proof of Proposition 3.10 which allows us to trade paths from \( w \) to the identity containing upward edges for a path of the same length that uses exclusively downward edges.

**Proposition 3.10.** Let \( w \in W \). Then \( \det(w, 1) = d_1(w) \).

**Proof.** Define \( m \) to be the minimal number of upward edges contained in any path in \( G \) realizing \( \det(w, 1) \). We have to prove that \( m = 0 \). Assume that \( m \geq 1 \), and let \( p \) be such a path. Denote the upward edges in \( p \) by \( u_i \rightarrow u_\beta \), encountered in the order \( i = 1, \ldots, m \) as we travel along the path. Consider the subpath of \( p \) which starts at \( u_m \). Since the edge \( u_m \rightarrow u_m s_{\beta_m} \) is upward, then the length only increases by one. Define \( m \rightarrow u_m s_{\beta_m} \). Lemma 3.9 then says that \( \det(u_m) \leq d_1(u_m s_{\beta_m}) + 1 \). Therefore, the subpath of \( p \) beginning at \( u_m \), which continues upward to \( u_m s_{\beta_m} \), contains at least as many edges as any path realizing \( \det(u_m) \). Define a new path \( p_m \) in \( G \) from \( w \) to \( 1 \) by following the original path \( p \) until the vertex \( u_m \), after which we follow any path down to \( 1 \) realizing \( d_1(u_m) \). By Lemma 3.9 and the fact that \( p \) realizes \( \det(w, 1) \), the length of the path \( p_m \) also equals \( \det(w, 1) \). However, the path \( p_m \) has \( m - 1 \) upward edges, contradicting the minimality of \( m \) and proving that indeed \( m = 0 \).

We are now prepared to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Recall that part (a) was already proved in Example 2.16 and so it remains only to prove parts (b) and (c). Let \( x = v^{13}w \in W \), and suppose that \( \lambda \) is superregular in the sense of Theorem 1.2 and Proposition 3.2, then says that \( x \) is cordial if and only if \( \det(w^{-1}v, v) = \ell(w^{-1}v) \).

(b) We first prove that if \( \eta(x) = v^{-1}wa \) is a standard parabolic Coxeter element, then \( x \) is cordial. Consider any path which realizes \( \det(w^{-1}v, v) = m \), say \( w^{-1}v \rightarrow w^{-1}vs_{\beta_1} \rightarrow w^{-1}vs_{\beta_1}s_{\beta_2} \rightarrow \cdots \rightarrow w^{-1}vs_{\beta_1} \cdots s_{\beta_m} = v \).

Note that \( v^{-1}w = s_{\beta_1} \cdots s_{\beta_m} \) so this path corresponds to an expression for \( \eta(x) \) as a product of \( m \) reflections. By definition, \( \ell_R(\eta(x)) \leq m \), but since \( \eta(x) \) is standard parabolic Coxeter, by BDSW Lemma 2.1 we have \( \ell(\eta(x)) = \ell_R(\eta(x)) \leq m = \det(w^{-1}v, v) \).

The opposite inequality follows from Remark 3.3. Therefore, if \( \eta(x) \) is a standard parabolic Coxeter element, we see that \( \det(w^{-1}v, v) = \ell(\eta(x)) = \ell(v^{-1}w) \), and so \( x \) is cordial by Proposition 3.2.

(c) We now prove that if \( x \) is in the dominant Weyl chamber, then \( x \) is cordial if and only if \( \eta(x) = w \) is small-height-avoiding. Since \( v = 1 \) when \( x \) is dominant, by Proposition 3.2 we aim to prove that \( \det(w^{-1}, 1) = \ell(w) \) if and only if \( w \) is small-height-avoiding. Note that \( w \) is small-height-avoiding if and only if \( w^{-1} \) is small-height-avoiding, and recall that \( \ell(w) = \ell(w^{-1}) \). Therefore, in fact it suffices to prove that \( \det(w, 1) = \ell(w) \) if and only if \( w \) is small-height-avoiding.

First suppose that \( w \) is small-height-containing. By definition, there exists an expression for \( w \) of the form \( w = ws_{\beta}v \), where \( \ell(w) = \ell(u) + \ell(s_{\beta}) + \ell(v) \), for some \( u, v \in W \) and \( s_{\beta} \) some non-simple reflection such that \( \ell(s_{\beta}) = (2p, \beta^*) = 1 \). Taking
any reduced expressions for \( u \) and \( v \), say \( u = s_{i_1} \cdots s_{i_k} \) and \( v = s_{j_1} \cdots s_{j_l} \), we can construct the following path in \( \Gamma_G \)
\[
\begin{align*}
w & \xrightarrow{\alpha_{i_k}} ws_{i_k} \xrightarrow{\alpha_{i_{k-1}}} \cdots \xrightarrow{\alpha_{i_1}} ws_{j_l} \cdots s_{j_1} s_{j_l} \xrightarrow{\beta} \cdots s_{j_1} \xrightarrow{\alpha_{j_1}} \cdots \xrightarrow{\alpha_{j_1}} 1.
\end{align*}
\]

Each edge exists because length is additive in the expression \( w = us_{i_1}v \), which means that at each step in this path the length drops by precisely \( \ell(s_{i_m}) = \ell(s_{j_n}) \) or \( \ell(s_{j_3}) = (2p, \beta') - 1 \) as required for a downward edge in \( \Gamma_G \). Since \( s_{j_3} \) is non-simple, then \( \ell(s_{j_3}) \geq 3 \), which means that the length of this particular path is at most \( \ell(w) - 2 \). Therefore, \( d_\Gamma(w,1) \leq \ell(w) - 2 < \ell(w) \) in this case, and so \( x \) is not cordial by Proposition \( 3.2 \).

Conversely, assume that \( w \) is small-height-avoiding. We aim to show that \( d_\Gamma(w,1) = \ell(w) \). Recall Proposition \( 3.10 \) which says that \( d_\Gamma(w,1) = d_1(w) \), and so there exists a path \( p \) consisting of all downward edges which also minimizes length among all paths from \( w \) to \( 1 \). By the definition of the downward edges in \( \Gamma_G \), this path corresponds to an expression \( w = s_{\beta_1} \cdots s_{\beta_r} \) such that the length decreases by exactly \( (2p, \beta') - 1 \) for all \( 1 \leq i \leq r \) when right multiplying \( w \) by \( s_{\beta_i} \). Note, however, that length cannot decrease by more than \( \ell(s_{\beta_i}) \) when right multiplying by \( s_{\beta_i} \). On the other hand, we always have \( \ell(s_{\beta_i}) \leq (2p, \beta') - 1 \), and so in fact \( \ell(s_{\beta_i}) = (2p, \beta') - 1 \) for all \( 1 \leq i \leq r \). Therefore, the expression \( w = s_{\beta_1} \cdots s_{\beta_r} \) is also length-additive. By definition of small-height-avoiding, \( w \) cannot contain any non-simple reflection \( s_{\beta_i} \) such that \( \ell(s_{\beta_i}) = (2p, \beta') - 1 \). This means that each reflection in the expression \( w = s_{\beta_1} \cdots s_{\beta_r} \) must in fact be simple, and so \( \ell(w) = d_1(w) = d_\Gamma(w,1) \). The element \( x \) is thus cordial by Proposition \( 3.2 \).

Example 3.11. For \( G = SL_3 \), the Newton stratification of each double coset \( IxI \) has been computed in [Bea]. Note, however, that our description below corrects an error in the tables at the end of loc. cit. In \( SL_3 \), all Newton strata are equidimensional, and the closure of any Newton stratum \( [b] \cap IxI \neq \emptyset \) in \( IxI \) is equal to the union of all \( [b'] \cap IxI \) such that \( [b'] \in B(G)_x \) and \( [b'] \subset [b] \). Write \( x = v^\alpha w \), and first assume that \( v = 1 \), i.e. \( x \) is in the dominant Weyl chamber, and that \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) with \( |\lambda_i - \lambda_{i+1}| \neq 1 \). Then \( x \) is non-cordial if and only if \( w = w_0 \). Thus in this case, we obtain exactly the condition of Theorem \( 1.2 \) (b) or equivalently (c), but under a much weaker regularity assumption on \( \lambda \). Furthermore, all non-cordial elements (even without any regularity assumption) are of the form \( xw \) for some non-cordial \( x \) in the dominant Weyl chamber and \( \omega \) normalizing \( I \). For \( x \) outside the dominant Weyl chamber with \( v \in \{ s_1, s_2, w_0 \} \), there exist cordial elements which are not covered by Theorem \( 1.2 \) applied directly to \( x \) or to \( xw \) for any \( \omega \) normalizing \( I \).

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