Part I: Improving Computational Efficiency of Communication for Omniscience

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Abstract—Communication for omniscience (CO) refers to the problem where the users in a finite set $V$ observe a discrete multiple random source and want to exchange data over broadcast channels to reach omniscience, the state where everyone recovers the entire source. This paper studies how to improve the computational complexity for the problem of minimizing the sum-rate for attaining omniscience in $V$. While the existing algorithms rely on the submodular function minimization (SFM) techniques and complete in $O(|V|^2 \cdot \text{SFM}(|V|))$ time, we prove the strict strong map property of the nesting SFM problem. We propose a parametric (PAR) algorithm that utilizes the parametric SFM techniques and reduce the the complexity to $O(|V| \cdot \text{SFM}(|V|))$. The output of the PAR algorithm is in fact the segmented Dilworth truncation of the residual entropy for all minimum sum-rate estimates $\alpha$, which characterizes the principal sequence of partitions (PSP) and solves some related problems: It not only determines the secret capacity, a dual problem to CO, and the network strength of a graph, but also outlines the hierarchical solution to a combinatorial clustering problem.

Index Terms—communication for omniscience, Dilworth truncation, submodularity.

I. INTRODUCTION

Let there be a finite number of users indexed by the set $V$. Each user observes a distinct component of a discrete memoryless multiple random source in private. The users are allowed to exchange their observations over public noiseless broadcast channels so as to attain omniscience, the state that each user reconstructs all components in the multiple source. This process is called communication for omniscience (CO) [2], where the fundamental problem is how to attain omniscience with the minimum sum of broadcast rates. While the CO problem formulated in [2] considers the asymptotic limits as the observation length goes to infinity, a non-asymptotic model is studied in [3]–[5], in which, the number of observations is finite and the communication rates are restricted to be integral. The CO problem has a wide range of important applications, special cases, extensions, duals and interpretations.

The CO problem is dual with the secret capacity [2], which is the maximum amount of secret key that can be generated by the users in $V$ and equals to the amount of information in the entire source, $H(V)$, subtracted by the minimum sum-rate in CO. A special case of CO is called coded cooperative data exchange (CCDE) [6]–[13], where a group of users obtain parts of a packet set, say, via base-to-peer (B2P) transmissions. By broadcasting linear combinations of packets over peer-to-peer (P2P) channels, they help each other recover the entire packet set based on a suitable network coding scheme, e.g., the random linear network coding [9]. It is shown in [13]–[17] that the solutions to the secret key agreement problem, CO and CCDE rely on the submodular function minimization (SFM) techniques in combinatorial optimization [18]. In a nutshell, all solutions in [13]–[17] come down to $O(|V|^2)$ calls of solving the SFM problem. Since the polynomial order of solving the SFM is still considerably high [18, Chapter VI], it is important to study whether the order-wise complexity $|V|^2$ in the computational complexity can be further reduced. This requires a deep understanding of the structure of the CO problem and its optimal solution. It is known from previous works [15], [17] that the first critical/turning point in the principal sequence of partitions (PSP), a partition chain that is induced by the Dilworth truncation of the residual entropy function, plays a central role in solving the CO problem. This is essentially the first or coarsest partition in the PSP that is strictly finer than the partition $\{V\}$.

Another important interpretation of CO is in the extension of the Shannon’s mutual information to the multivariate case and is called the multivariate mutual information $I(V)$ [15]: $I(V)$ equals to the secret capacity. This measure was used in [19] to interpret the PSP as a hierarchical clustering result: the partitions in the PSP contain the largest user subsets $X$ with $I(X)$ strictly greater than a given similarity threshold and get coarser (from bottom to top) as this similarity threshold decreases. This coincides with a more general combinatorial clustering framework, the minimum average clustering (MAC) in [20], where both the entropy and cut functions are viewed as the inhomogeneity measure of a dataset. For the cut function, the first critical value in the PSP identifies the network strength [20], [21] and the maximum number of edge-disjoint spanning trees [22]. This, in return, well explains why the secret agreement problem in the pairwise independent network (PIN) source model, which has a graphical representation, can be solved by the tree packing algorithms in [23]–[25]. Thus, instead of only focusing on one critical point for solving the minimum sum rate problem, it is also worth studying how to improve the existing complexity $O(|V|^2 \cdot \text{SFM}(|V|))$ for determining the whole PSP.

A. Contributions

In this paper, we propose a parametric (PAR) algorithm that reduces the complexity for solving the minimum sum-rate
problem in CO and determining the PSP to $O(|V| \cdot \text{SFM}(|V|))$. The study starts with a review of the coordinate saturation capacity (CoordSatCap) algorithm in \cite{17}, Algorithm 3, which is a nesting algorithm in the modified decomposition algorithm (MDA) algorithm \cite{17}, Algorithm 1 that determines the Dilworth truncation for a given minimum sum-rate estimate $\alpha$. We prove that the SFM problem in each iteration of CoordSatCap exhibits the strong strict map property in $\alpha$, based on which, a StrMap algorithm is proposed that determines the minimizer of this SFM problem for all values of $\alpha$. The StrMap can be implemented by the existing parametric SFM (PSFM) algorithms \cite{26}–\cite{28} that complete at the same time as the SFM algorithm.

Based on the idea of CoordSatCap, we propose a PAR algorithm that iteratively calls the subroutine StrMap to update the segmented minimizer of the Dilworth truncation for all values of the minimum sum-rate estimate $\alpha$. The critical/turning points of $\alpha$ as well as the corresponding minimizers/partitions, which characterize the segmented Dilworth truncation, converge to the PSP of $V$, where the first critical value determines the minimum sum-rate for both asymptotic and non-asymptotic model. The PAR algorithm also outputs a segmented, or piecewise linear, rate vector $r_{\alpha,V} = (r_{\alpha,i} : i \in V)$ in $\alpha$ that determines an optimal rate vector for both asymptotic and non-asymptotic source models. In addition, by choosing a proper order of iterations in the PAR algorithm, the optimal rate vector also minimizes the weighted minimum sum-rate in the optimal rate vector set.

The PAR algorithm invokes $|V|$ calls of StrMap and its complexity is $O(|V| \cdot \text{SFM}(|V|))$. It also allows distributed computation and can be applied to submodular functions other than the entropy function, e.g., the cut function. The returned PSP solves the information-theoretic and MAC clustering problems in \cite{19} and \cite{20}, respectively, with the complexity reduced from the existing algorithms by a factor of $|V|$. For the cut function, the first critical point of the returned PSP determines the network strength, also the value of the secret capacity in the PIN model. The work also studies another parametric algorithm proposed in \cite{29} Fig. 3] for determining the PSP of the graph model. It is revealed that \cite{29} Fig. 3] utilizes a non-strict strong map property, based on which, we propose a StrMap subroutine specifically for \cite{29} Fig. 3] so that it also applies to any submodular function other than the cut function in a graph.

The proposed PAR algorithm also solves a successive omniscience (SO) problem, where the omniscience process takes stages: a user subset attains the local omniscience each time. In Part II \cite{30} of this paper, we derive the achievability conditions for the multi-stage SO for both asymptotic and non-asymptotic models. By using the segmented Dilworth truncation and rate vector $r_{\alpha,V}$ returned by PAR, we propose algorithms extracting the user subset and achievable rate vector for each stage of SO such that, at the final stage, the global omniscience in $V$ is attained by the minimum sum-rate.

**B. Organization**

The rest of paper is organized as follows. The system model for CO is described in Section \ref{sec:system_model} where we also introduce the notation, review the existing results for the minimum sum-rate problem, including the PSP, and derive the properties of the CoordSatCap algorithm. In Section \ref{sec:algorithm} we prove the strict strong map property and propose the PAR algorithm and its subroutine StrMap algorithm. In Section \ref{sec:analysis} we discuss how the PAR algorithm contributes to the secret agreement, network attack and combinatorial clustering problems, where the relationship between the network strength and secret capacity in the PIN model is also explained. In Section \ref{sec:implementation} we propose a distributed computation method of the PAR algorithm.

**II. SYSTEM MODEL**

Let $V$ with $|V| > 1$ be a finite set that contains all users in the system. We call $V$ the ground set. Let $Z^*_V = (Z_i : i \in V)$ be a vector of discrete random variables indexed by $V$. For each $i \in V$, user $i$ privately observes an $n$-sequence $Z^*_i$ of the random source $Z_i$ that is i.i.d. generated according to the joint distribution $P_{Z_i}$. We allow users to exchange their observed data directly to recover the source sequence $Z^*_V$. The state that each user obtains the total information in the entire multiple source is called omniscience, and the process that users communicate with each other to attain omniscience is called communication for omniscience (CO) \cite{2}.

Let $r_V = (r_i : i \in V)$ be a rate vector indexed by $V$. We call $r_V$ an achievable rate vector if the omniscience can be attained by letting users communicate at the rates designated by $r_V$. For the original CO problem formulated in \cite{2} considering the asymptotic limits as the block length $n$ goes to infinity, each dimension $r_i$ is the compression rate denoting the expected code length at which user $i$ encode his/her observations. We also study a non-asymptotic model, where $n$ is assumed to be finite. The finite linear source model \cite{34} is one of the non-asymptotic models, in which the multiple random source is represented by a vector that belongs to a finite field and each $r_i$ denotes the integer number of linear combinations of observations transmitted by user $i$. This finite linear source model is of particular interest in that it models the CCDE problem \cite{6}–\cite{8} where the users communicate over P2P channels to help each other recover a packet set. In this paper, for the omniscience problem in the non-asymptotic model, we focus on the finite linear source model. Therefore, we use the term non-asymptotic model, finite linear source model and CCDE interchangeably.

**A. Minimum Sum-rate Problem**

For a given rate vector $r_V$, let $r : 2^V \to \mathbb{R}_+$ be the sum-rate function such that

$$r(X) = \sum_{i \in X} r_i, \quad \forall X \subseteq V$$

with the convention $r(\emptyset) = 0$. The achievable rate region is characterized in \cite{2} by the set of multiterminal Slepian-Wolf constraints \cite{31}, \cite{32}:

$$\mathcal{R}_{\text{CO}}(V) = \{ r_V \in \mathbb{R}^{|V|} : r(V) \geq H(X|V \setminus X), \forall X \subseteq V \},$$

where $H(X)$ is the amount of randomness in $Z_X$ measured by the Shannon entropy \cite{33} and $H(X|Y) = H(X \cup Y) - H(Y)$.
is the conditional entropy of $Z_X$ given $Z_Y$. In a finite linear source model, the entropy function $H$ reduces to the rank of a matrix that only takes integral values.

The fundamental problem in CO is to minimize the sum-rate in the achievable rate region [2 Proposition 1]

$$R_{\text{ACO}}(V) = \min \{ r(V) : r \in R_{\text{ACO}}(V) \},$$

$$R_{\text{NCO}}(V) = \min \{ r(V) : r \in R_{\text{NCO}}(V) \},$$

for the asymptotic and non-asymptotic models, respectively. Denote by $R_{\text{ACO}}^*(V) = \{ r \in \mathbb{R}^{|V|} : r(V) = R_{\text{ACO}}(V) \}$ and $R_{\text{NCO}}^*(V) = \{ r \in \mathbb{Z}^{|V|} : r(V) = R_{\text{NCO}}(V) \}$ the optimal rate vector set for the asymptotic and non-asymptotic models, respectively. We say that the minimum sum-rate problem is solved if the value of the minimum sum-rate in (1), as well as an optimal rate vector are determined.

To efficiently solve the minimum sum-rate problem without dealing with the exponentially growing number of constraints in the linear programming, (1a) and (1b) are respectively converted to [2 Example 4] [34] [17 Corollary 6]

$$R_{\text{ACO}}(V) = \max_{P \in \Pi(V)} \sum_{C \in P} \frac{H(V) - H(C)}{|P| - 1},$$

$$R_{\text{NCO}}(V) = \max_{P \in \Pi(V)} \sum_{C \in P} \frac{H(V) - H(C)}{|P| - 1},$$

where $\Pi(V)$ denotes the set containing all partitions of $V$. It is shown in [15] that the combinatorial optimization problem in (2) can be solved based on the existing submodular function minimization (SFM) techniques in polynomial time $O(|V|^2 \cdot \text{SFM}(|V|))$.

B. Existing Results

The efficiency for solving the minimum sum-rate problems in (2) relies on the submodularity of the entropy function $H$ and the induced structure in the partition lattice. It is shown in [17] that the validity of the algorithms proposed in [13] Appendix F and [14] Algorithm 3] for solving (2b) in CCDE and the MDA algorithm proposed in [17] Algorithm 1 for solving both (2a) and (2b) can be explained by the Dilworth truncation and the partition chain it forms in the estimation of $R_{\text{ACO}}(V)$ or $R_{\text{NCO}}(V)$, which is called the principal sequence of partitions (PSP). In this section, we introduce the notation and review the Dilworth truncation, PSP and the coordinate-wise saturation capacity (CoordSatCap) algorithm, an essential nesting algorithm in [13] Appendix F, [14] Algorithm 3] and [17] Algorithm 1]. The purpose is to summarize the existing results that are required to prove the strict strong map property in Section III.

1) Preliminaries: For $X \subseteq V$, let $\chi_X = (e_i : i \in V)$ be the characteristic vector of the subset $X$ such that $e_i = 1$ if $i \in X$ and $e_i = 0$ if $i \notin X$. The notation $\chi(i)$ is simplified by $\chi_i$. Let $\sqcap$ denote the disjoint union. For $X$ that contains disjoint subsets of $V$, we denote by $X = \sqcup C \in X \Rightarrow \chi$: fusion of $X$. For example, for $X = \{(3, 4), (2, 8)\}, \chi = \{2, 3, 4, 8\}$.

For partitions $P, P' \in \Pi(V)$, we denote by $P \preceq P'$ if $P$ is finer than $P'$ and $P \prec P'$ if $P$ is strictly finer than $P'$. For any $X \subseteq V$ and $P \in \Pi(V)$, $(X)_P = \{ X \cap C : C \in P \}$ denotes the decomposition of $X$ by $P$. For example, for $X = \{1, 2, 4\}$ and $P = \{\{1, 2, 3\}, \{4\}\}$, $(X)_P = \{\{1, 2\}, \{4\}\}$.

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq V$. The problem $\min \{ f(X) : X \subseteq V \}$ is a submodular function minimization (SFM) problem. It can be solved in strongly polynomial time and the set of minimizers $\arg \min \{ f(X) : X \subseteq V \}$ form a set lattice such that the smallest/minimal minimizer $\bigwedge \arg \min \{ f(X) : X \subseteq V \}$ and largest/maximal minimizer $\bigvee \arg \min \{ f(X) : X \subseteq V \}$ uniquely exist and can be determined at the same time when the SFM problem is solved [18 Chapter VI].

We call $\Phi = (\phi_1, \ldots, \phi_{|V|})$ a linear ordering/permutation of the indices in $V$ if $\phi_i < \phi_j$ for all $i, j \in \{1, \ldots, |V|\}$ such that $i \neq j$. For $i \in V$, let $\mathcal{V}_i = \{ \phi_1, \ldots, \phi_i \}$ be the set of the first $i$ users in the linear ordering $\Phi$. We call $f_{\mathcal{V}_i} : 2^{|V|} \rightarrow \mathbb{R}$ such that $f_{\mathcal{V}_i}(X) = f(X)$ for all $X \subseteq V_i$ the reduction of $f$ on $V_i$ [18 Section 3.1(a)]. For example, for $\Phi = (2, 3, 1, 4)$, $V_2 = \{2, 3\}$ and the reduction of $f$ on $V_2$ is $f_{\{2, 3\}}(X) = f(X)$ for all $X \subseteq \{2, 3\}$.

2) Dilworth Truncation: Let $\alpha \in \mathbb{R}_+$ be an estimation of the minimum sum-rate and define a set function $f_{\alpha} : 2^V \rightarrow \mathbb{R}$ such that $f_{\alpha}(X) = \alpha - H(V) + H(X), \forall X \subseteq V$ except that $f(\emptyset) = 0$. This function is the same as the residual entropy function in [15] in that it offsets/subtracts the information amount in each nonempty subset $X$ by $H(V) - \alpha$. Let $f_{\mathcal{V}_i} \equiv \alpha$ be a partition function such that $f_{\alpha}(P) = \sum_{C \in P} f_{\alpha}(C)$ for all $P \in \Pi(V)$. The Dilworth truncation of $f_{\alpha}$ is defined as

$$f_{\alpha}(V) = \min_{P \in \Pi(V)} f_{\alpha}(P).$$

The solution to (3) exhibits a strong structure in $\alpha$ that is characterized by the PSP.

3) Principal Sequence of Partitions (PSP): For a given $\alpha$, let $Q_{\alpha,V} = \bigwedge \arg \min_{P \in \Pi(V)} f_{\alpha}(P)$ be the finest minimizer of (3). The value of Dilworth truncation $f_{\alpha}(V)$ is piecewise linear and strictly increasing in $\alpha$. It is determined by $p < |V|$ critical points

$$0 \leq \alpha^{(p)} < \ldots < \alpha^{(1)} < \alpha^{(0)} = H(V),$$

with the corresponding finest minimizer $P^{(j)} = Q_{\alpha^{(j)},V} = \bigwedge \arg \min_{P \in \Pi(V)} f_{\alpha^{(j)}}(P)$ for all $j \in \{0, \ldots, p\}$ forming a partition chain

$$\{\{i\} : i \in V\} = P^{(p)} \prec \ldots \prec P^{(1)} \prec P^{(0)} = \{V\}$$

such that $Q_{\alpha,V} = P^{(p)}$ for $\alpha \in [0, \alpha(p)]$ and $Q_{\alpha,V} = P^{(j)}$ for all $\alpha \in (\alpha^{(j+1)}, \alpha^{(j)})$ and $j \in \{0, \ldots, p - 1\}$ [20]. The partition chain in (3) together with the corresponding critical values $\alpha^{(j)}$, is called the Principal Sequence of Partitions (PSP) of the ground set $V$.

The first critical point of the PSP provides the solution to the minimum sum-rate problem [17] Corollary A3]: $R_{\text{ACO}}(V) = \alpha^{(1)}$ for the asymptotic model and $R_{\text{NCO}}(V) = \alpha^{(1)}$ for the non-asymptotic model. The corresponding partition $P^{(1)}$, called the fundamental partition, equals to the finest maximizer of (2a).

The minimizers of (3) form a partition lattice such that the finest and coarsest minimizers uniquely exist [50].
4) CoordSatCap Algorithm: All of the existing algorithms in [13, 14, 17] for solving the minimum sum-rate problem in (2) run a subroutine that determines the minimum and/or the finest minimizer of the Dilworth truncation (3) for a given value of $\alpha$. This subroutine is outlined by the CoordSatCap algorithm in Algorithm 1. The idea is to keep increasing each dimension of a rate vector $r_{\alpha,V}$ in the submodular polyhedron of $f_\alpha$

$$P(f_\alpha) = \{ r_{\alpha,V} \in \mathbb{R}^{|V|}: r_{\alpha}(X) \leq f_\alpha(X), X \subseteq V \}$$

until it reaches the base polyhedron of the Dilworth truncation $f_\alpha$

$$B(f_\alpha) = \{ r_{\alpha,V} \in P(f_\alpha): r_{\alpha}(V) = f_\alpha(V) \}.$$ 

Here, $r_{\alpha,V} = (r_{\alpha,i}: i \in V)$ is a $|V|$-dimension rate vector that is parameterized by the input minimum sum-rate estimate $\alpha$ and $r_{\alpha}(X) = \sum_{i \in X} r_{\alpha,i}$, $\forall X \subseteq V$ is the sum-rate function of this rate vector. The amount of the rate increment is determined by the minimization of the set function

$$g_\alpha(\tilde{X}) = f_\alpha(\tilde{X}) - r_{\alpha}(\tilde{X}), \forall \tilde{X} \subseteq Q_{\alpha,V}. \quad (6)$$

where $Q_{\alpha,V}$ is a partition of $V$ that is iteratively updated in Algorithm 1. Here, we use the notation $Q_{\alpha,V}$ because we will show in Section III-A that $Q_{\alpha,V} = \bigwedge_{\varphi \in \Pi(V)} f_\alpha[\varphi]$, after step 6 for all $i$. The reason for considering the function $g_\alpha(\tilde{X})$ is the min-max relationship [18] Section 2.3] Lemmas 22 and 23.\footnote{The original purpose of the CoordSatCap algorithm is to determine the value of $f_\alpha(V)$ by tightening the upper bound $f_\alpha(V)$ in $P(f_\alpha)$. See [17] Appendix B]. Also note that, since $f_\alpha(X) \leq f_\alpha(X), \forall X \subseteq V, B(f_\alpha)$ and $B(f_{\alpha})$ are not equivalent in general.} \footnote{Equation (6) is the max-min theorem in [18] Section 2.3] that holds for all $i' \in V$ and $r_{\alpha,V} \in P(f_{\alpha})$. Equation (6) is proved by [1] Lemmas 22 and 23 for Algorithm 1 which states that the minimum of (6) at the $i$th iteration can be searched over the subset $V_{i'}$ or more specifically $Q_{\alpha,V_{i'}}$, a partition of $V_{i'}$.

$$\max\{ \xi: r_{\alpha,V} + \xi x' \in P(f_{\alpha}) \} = \min\{g_\alpha(\tilde{X}): \{i': i' \in X \subseteq Q_{\alpha,V}, \forall i' \in V_{i'} \} \} \quad (7a)$$

This is an important parameter in CCDE in that it is the least common multiple (LCM) of $r_{R_{\alpha}(\varphi)}(V)$ [17 Corollary 28], i.e., by letting each packet be broken into $|P(\varphi) - 1|$ chunks, the optimal rate vector $r_{R_{\alpha}(\varphi)}(V)$ is implemented based on linear codes, which saves the overall transmission rates by no more than 1 from the optimal rate vector $r_{R_{\alpha}(\varphi)}(V) \in P_N(\varphi)$.\footnote{In this case, $f_{\alpha}$ is defined as $f_{\alpha}(X) = \alpha - f(V) + f(X), \forall X \subseteq V$.}

### Algorithm 1: CoordSatCap Algorithm [17] Algorithm 3

**input**: $\alpha, f, V$ and $\Phi$

**output**: $r_{\alpha,V} \in B(f_\alpha)$ and $Q_{\alpha,V} = \bigwedge_{\varphi \in \Pi(V)} f_\alpha[\varphi]$

1. Let $r_{\alpha,V} := \alpha - H(V)$ so that $r_{\alpha,V} \in P(f_{\alpha})$.
2. Initialize $r_{\alpha,\phi_i} := f_{\alpha}(\{\phi_i\})$ and $Q_{\alpha,\phi_i} := \{\{\phi_i\}\}$;
3. for $i = 2$ to $|V|$ do
4. $Q_{\alpha,\phi_i} := Q_{\alpha,\phi_{i-1}} \cup \{\phi_i\}$;
5. $U_{\alpha,\phi_i} := \bigwedge_{\varphi \in \Pi(V)} f_\alpha[\varphi]$;
6. Update $r_{\alpha,V}$ and $Q_{\alpha,V}$:
7. $r_{\alpha,V} := r_{\alpha,V} + g_\alpha(U_{\alpha,\phi_i})$;
8. $Q_{\alpha,V} := (Q_{\alpha,V} \setminus U_{\alpha,\phi_i}) \cup \{U_{\alpha,\phi_i}\}$;
9. endfor
10. return $r_{\alpha,V}$ and $Q_{\alpha,V}$;

### III. PARAMETRIC APPROACH

While the CoordSatCap algorithm determines the Dilworth truncation $f_{\alpha}(V)$ for only one value of $\alpha$, we reveal the structural properties of the partition $Q_{\alpha,V}$, and the rate vector $r_{\alpha,V}$ in $\alpha$ and show that the objective function $g_\alpha(\tilde{X})$ in the SFM problem (6) exhibits the strict strong map property in $\alpha$. A parametric (PAR) algorithm is then proposed, which utilizes this strong map property to obtain the minimizer of (6) for all $\alpha$ so that each iteration $i$ determines $Q_{\alpha,V_i}$ and $r_{\alpha,V_i}$, in particular its reduction $r_{\alpha,V_i}$ on $V_i$, for all values of the minimum sum-rate estimate $\alpha$. Also, by choosing a proper linear ordering $\Phi$, the optimal rate vector $r_{\alpha,V}$ returned by the CoordSatCap algorithm for both asymptotic and non-asymptotic models minimizes a weighted minimum sum-rate objective function. We show in Section III-E that this PAR algorithm reduces the computational complexity for solving the minimum sum-rate problem in both asymptotic and non-asymptotic models and allows distributed computation. Note that, in this paper, when we say for all $\alpha$, we mean for all $\alpha \in [0, \inf \{H(V)\})$ since the minimum sum-rates, $R_{\alpha}(V)$ and $R_{\alpha}(V)$, must take values in $[0, \inf \{H(V)\})$.

#### A. Observations

Observing the values of $Q_{\alpha,V_I}$ and $r_{\alpha,V_I}$ in $\alpha$ in the CoordSatCap algorithm as the iteration index $i$ grows, we have the following result.
Proposition III.1. After step 6 in each iteration $i$ of Algorithm 2, $Q_{\alpha,V_i} = \bigwedge \arg\min_{P \in \Pi(V_i)} f_\alpha[P]$ and $r_{\alpha,V_i} \in B(f_{\alpha_i})$ for all $\alpha$.

The proof is omitted since it is a direct result that $Q_{\alpha,V_i} = \bigwedge \arg\min_{P \in \Pi(V_i)} f_\alpha[P]$ and $r_{\alpha,V_i} \in B(f_{\alpha_i})$ are returned by the call CoordSatCap($\alpha, H, V_i, \Phi$). Note that, due to the equivalence $B(f_{\alpha_i}) = \{ r_{V_i} \in \mathcal{R}_CO(V_i) : r(V_i) = \alpha \}$, we must have the reduction of the rate vector $r_{\alpha,V_i}$ on $V_i$ being $r_{\alpha,V_i} \in B(f_{\alpha_i})$ for all $\alpha$ after step 6. According to Proposition III.1, $Q_{\alpha,V_i}$ for all $\alpha$ is again characterized by the PSP of $V_i$ with the number of critical points bounded by $|V_i|$. That is, the partition $Q_{\alpha,V_i}$ and $r_{\alpha,V_i}$ are segmented in $\alpha$ and determine the solution to the minimum sum-rate problem in a subsystem $V_i$. This fact will be utilized in Section V to propose a distributed algorithm for solving the CO problem and in Part II of this paper [30] to solve the successive omniscience problem.

Example III.2. Consider a 5-user system with

$$
Z_1 = (W_b, W_c, W_d, W_h, W_i), \\
Z_2 = (W_c, W_f, W_h, W_i), \\
Z_3 = (W_b, W_c, W_e, W_j), \\
Z_4 = (W_e, W_b, W_a, W_d, W_f, W_j, W_i, W_j), \\
Z_5 = (W_a, W_b, W_c, W_d, W_f, W_i, W_j),
$$

where each $W_m$ is an independently uniformly distributed random bit.

Choose the linear ordering $\Phi = (4, 5, 2, 3, 1)$. By setting $\alpha = 3$, we call CoordSatCap($\alpha, H, V, \Phi$). We initiate $r_{3,V} = (\alpha - H(V))_X = (-7, \ldots, -7)$, update $r_{3,4} = t_3(\{4\}) = 1$ and assign $Q_{3,V_1} = \{\{4\}\}$. For $i = 2$ and $\phi_2 = 5$, consider the minimization problem $\min g_3(\hat{X}) : \{ 5 \} \in X \subseteq Q_{3,V_2}$ where $Q_{3,V_2} = \{\{4\}, \{5\}\}$. We get the minimal minimizer $U_{3,V_2} = \{\{5\}\}$ and do the updates $r_{3,2} = -7 + g_3(U_{3,V_2}) = -1$ and $Q_{3,V_2} = \{\{4\}, \{5\}\} \cup \{U_{3,V_2}\} = \{\{4\}, \{5\}\}$. In the same way, we can continue the rest of iterations. However, to show an example of Proposition III.1, we consider another value of $\alpha$ as follows.

By setting $\alpha = 6$, we call CoordSatCap($\alpha, H, V, \Phi$). We initiate $r_{6,V} = (\alpha - H(V))_X = (-4, \ldots, -4)$ and set $r_{6,4} = t_6(\{4\}) = 4$ and $Q_{6,V_1} = \{\{4\}\}$. We have $U_{6,V_2} = \bigwedge \arg\min_{g_6(X)} : \{ 5 \} \in X \subseteq Q_{6,V_2} = \{\{4\}, \{5\}\}$ and the updates $r_{6,5} = -4 + g_6(U_{6,V_2}) = 0$ and $Q_{6,V_2} = \{\{4\}, \{5\}\} \cup \{U_{6,V_2}\} = \{\{4\}, \{5\}\}$. One can verify that $\{\{4\}, \{5\}\} = \bigwedge \arg\min_{P \in \Pi(V_2)} f_6[P]$ and $\{\{4\}, \{5\}\} = \bigwedge \arg\min_{P \in \Pi(V_2)} f_6[P]$. In fact, repeating the above procedure for all $\alpha$, we have the piecewise linear $r_{\alpha, (4,5)}$ and segmented $Q_{\alpha,V_2}$ as

$$
r_{\alpha,4} = \alpha - 2, \quad \forall \alpha \in [0, 10], \\
r_{\alpha,5} = \begin{cases} \alpha - 4 & \alpha \in [0, 4], \\
0 & \alpha \in (4, 10], \end{cases} \\
Q_{\alpha,V_2} = \begin{cases} \{\{4\}, \{5\}\} & \alpha \in [0, 4], \\
\{\{4\}, \{5\}\} & \alpha \in (4, 10], \end{cases}
$$

because of the segmented

$$
\bar{U}_{\alpha,V_2} = \begin{cases} \{\{5\}\} & \alpha \in [0, 4], \\
\{\{4\}, \{5\}\} & \alpha \in (4, 10]. \end{cases}
$$

Note that the function $g_\alpha$ is segmented. For example, for $i = 3$ and $\phi_3 = 2$, the function $g_\alpha$ defined on $Q_{3,V_2} \cup \{\{2\}\}$ differs in two segments: for $\alpha \in [0, 4], g_\alpha$ takes values on $\{0, 2\}$, $\{4, 5\}$ and $\{2, 4, 5\}$ only; for $\alpha \in (4, 10], g_\alpha$ takes values on all subsets in the power set $2^\{2,4,5\}$.

Proposition III.1 suggests that we could obtain $r_{\alpha,V_i}$ and $Q_{\alpha,V_i}$ for all values of $\alpha$ in each iteration of the CoordSatCap algorithm. To do so, it is essential to discuss how to efficiently determine $U_{\alpha,i}$ for all $\alpha$. It should be noted that we automatically know $U_{\alpha,i}$ if $U_{\alpha,i}$ is obtained in that $U_{\alpha,i} = \{ C \in Q_{\alpha,i} : C \subseteq U_{\alpha,i} \} \subseteq (U_{\alpha,i})_{Q_{\alpha,i}}$. For example, for $U_{\alpha,i}$ in (9),

$$
U_{\alpha,V_2} = \begin{cases} \{\{5\}\} & \alpha \in [0, 4], \\
\{\{4\}, \{5\}\} & \alpha \in (4, 10]. \end{cases}
$$

We derive the following structural results on $Q_{\alpha,V_i}$ and $r_{\alpha,V_i}$.

Lemma III.3 (essential properties). At the end of each iteration $i$ of Algorithm 2, the rate vector $r_{\alpha,V_i} \in P(f_{\alpha})$, where $P(f_{\alpha}) = P(f_{\alpha})$, and followings hold for all $\alpha$:

(a) $r_{\alpha,V_i} = r_{\alpha}[Q_{\alpha,V_i}] = f_{\alpha}[Q_{\alpha,V_i}] = \hat{f}_{\alpha}(V_i)$;

(b) $r_{\alpha}[X] = r_{\alpha}[X] = f_{\alpha}[X]$ for all $X \subseteq Q_{\alpha,V_i}$, where $r_{\alpha}[X] = \sum_{C \in X} r_{\alpha}(C)$.

(c) For all $\alpha < \alpha'$, $Q_{\alpha,V_i} \subseteq Q_{\alpha',V_i}$ and, for all $X \subseteq Q_{\alpha,V_i}$ and $X' \subseteq Q_{\alpha',V_i}$ such that $X = X'$,

$$
r_{\alpha}[X] = \hat{f}_{\alpha}(X) < \hat{f}_{\alpha'}(X') = r_{\alpha'}[X'].
$$

The proof of Lemma III.3 is in Appendix B. We call Lemma III.3 (a) to (c) the essential properties since they hold the strict strong map property in the Theorem III.5 in Section III-B and the main theorem that ensures the validity of the PAR algorithm. In addition, in Part II [30], we show that the monotonicity of the sum-rate in Lemma III.3 (c) also guarantees the feasibility of a multi-stage SO.

B. Strong Map Property

Since $Q_{\alpha,V_i} = \bigwedge \arg\min_{P \in \Pi(V_i)} f_\alpha[P]$ after step 6 of Algorithm 1 according to Proposition III.1, $Q_{\alpha,V_i}$ satisfies the properties of the PSP in Section II-B3; i.e., the partition $Q_{\alpha,V_i}$ gets monotonically coarser as $\alpha$ increases and is segmented by a finite number of critical points. Recall that $Q_{\alpha,V_i}$ is updated by $U_{\alpha,V_i}$ in step 6. Then, we must have $U_{\alpha,V_i}$ segmented in $\alpha$ and the size of $U_{\alpha,V_i}$ increase in $\alpha$. This can be justified by the strong map property of the function $g_\alpha$, which also states that all critical points that characterize the segmented $U_{\alpha,V_i}$ can be determined by the parametric submodular function minimization (PSFM) algorithm.

This means that $U_{\alpha,V_i}$ is the decomposition of $U_{\alpha,V_i}$ by $Q_{\alpha,V_i}$. Here, we should use the value of $Q_{\alpha,V_i}$ in the minimization problem $\min g_\alpha(X) : \{ i \} \in X \subseteq Q_{\alpha,V_i}$ before the updates in step 6.
Definition III.4 (strong map [7] Section 4.1). For two distributive lattices $L_1, L_2 \subseteq 2^V$ and submodular functions $h_1 : L_1 \to \mathbb{R}$ and $h_2 : L_2 \to \mathbb{R}$, $h_1$ and $h_2$ form a strong map, denoted by $h_1 \rightarrow h_2$, if

$$h_1(Y) - h_1(X) \geq h_2(Y) - h_2(X) \quad (11)$$

for all $X, Y \in L_1 \cap L_2$ such that $X \subseteq Y$. The strong map is strict, denoted by $h_1 \rightarrow h_2$, if $h_1(Y) - h_1(X) > h_2(Y) - h_2(X)$ for all $X \subseteq Y$.

Theorem III.5. In each iteration $i$ of Algorithm 7, $g_\alpha$ forms a strict strong map in $\alpha$.

$$g_\alpha \rightarrow g_\alpha', \quad \forall \alpha, \alpha' : \alpha < \alpha'. \tag{\text{Proof}}$$

Lemma III.6. [7] Theorems 26 to 28) In each iteration $i$ of Algorithm 1 the minimal minimizer $U_{\alpha,V}$ of $\min(g_\alpha(X)) : \{ \phi_i \} \in X \subseteq Q_{\alpha,V}, i$ satisfies $U_{\alpha,V} \subseteq U_{\alpha',V}$ for all $\alpha < \alpha'$. In addition, $U_{\alpha,V}$, for all $\alpha$ is fully characterized by $q < |V_1| - 1$ critical points

$$0 \leq \alpha_q \leq \ldots \leq \alpha_3 < \alpha_0 = H(V)$$

and the corresponding minimal minimizer $S_j = U_{\alpha_j,V}$ for all $j \in \{0,\ldots,q\}$ forms a set chain

$$\{ \phi_i \} = S_q \subseteq \ldots \subseteq S_1 \subseteq S_0 = V_i$$

and $U_{\alpha,V} = S_q = \{ \phi_i \}$ for all $\alpha \in [0,\alpha_q]$ and $U_{\alpha,V} = S_j$ for all $\alpha \in (\alpha_{j+1}, \alpha_j)$ such that $j \in \{0,\ldots,q-1\}$.

Example III.7. In Example III.2, we have $U_{\alpha,V}$ in (9) characterized by the critical points $\alpha_4 = 4$ and $\alpha_0 = H(V) = 10$ with $S_3 = \{5\}$ and $S_0 = \{4,5\}$ such that $\{5\} \subseteq S_1 \subseteq S_0 = V_2$. So, $U_{\alpha,V} = S_1$ for $\alpha \in [0,\alpha_1]$ and $U_{\alpha,V} = S_2$ for $\alpha \in (\alpha_1, \alpha_0)$.

Algorithm 2: Parametric (PAR) Algorithm

input : $f, V$ and $\Phi$
output: segmented variables $r_{\alpha,V} \in B(f_{\alpha})$ and $Q_{\alpha,V} = \bigwedge \arg\min_{P \in L(V)} f_{\alpha}[P]$ for all $\alpha$

1. $r_{\alpha,V} := (\alpha - H(V))V_{\alpha}$ for all $\alpha$;
2. $r_{\alpha,\phi_i} := f_{\alpha}(\{ \phi_i \})$ and $Q_{\alpha,V} = \{ \{ \phi_i \} \}$ for all $\alpha$;
3. for $i = 2$ to $|V|$ do
4. $Q_{\alpha,V} := Q_{\alpha,V-1} \cup \{ \{ \phi_i \} \}$ for all $\alpha$;
5. Obtain the critical points $\{ \alpha_j : j \in \{0,\ldots,q\} \}$ and $\{ (S_j : j \in \{0,\ldots,q\}) \}$ that determine the minimal minimizer $U_{\alpha,V}$ of $\min(g_\alpha(X)) : \{ \phi_i \} \in X \subseteq Q_{\alpha,V}$ for all $\alpha$ by the StrMap algorithm in Algorithm 3;
6. Let $\Gamma_j := (\alpha_{j+1}, \alpha_j)$ for all $j \in \{0,\ldots,q-1\}$ and $\Gamma_0 := [0,\alpha_0]$.
7. for each $j \in \{0,\ldots,q\}$, update $r_{\alpha,V}$ and $Q_{\alpha,V}$ by $r_{\alpha,V} := r_{\alpha,V} + g_\alpha(S_j)\chi_{\alpha,V}$;
8. endfor
9. return $r_{\alpha,V}$ and $Q_{\alpha,V}$ for all $\alpha$.

We continue the procedure in Example III.2 for $i = 3$ and $\phi_3 = 2$ by considering the problem $\min(g_\alpha(X)) : \{ \{2\} \} \subseteq Q_{\alpha,V} \subseteq \{ \{2\} \}$ where $Q_{\alpha,V}$ and $r_{\alpha,V}$ are in (8). We have

$$U_{\alpha,V} = \{ \{2\} \} \subseteq \{ \{4,5\} \} \subseteq \{ \{4,5\} \} \subseteq \{ \{2,4,5\} \}$$

that is determined by the critical points $\alpha_1 = 8$ and $\alpha_0 = H(V) = 10$ with $S_1 = \{2\}$ and $S_2 = \{4,5\}$ such that $\{2\} \subseteq S_1 \subseteq S_0 = V_3 = \{2,4,5\}$. After the updates in step 6 we have

$$r_{\alpha,V} = \begin{cases} (\alpha - 10, \ldots, \alpha - 6, \alpha - 2, \alpha - 4) & \alpha \in [0,4], \\ (\alpha - 10, \ldots, \alpha - 6, \alpha - 2, \alpha - 2) & \alpha \in [4,8], \\ (\alpha - 10, \alpha - 10, \alpha - 10, \alpha - 2, 0) & \alpha \in (8,10], \\ \end{cases}$$

$$Q_{\alpha,V} = \begin{cases} \{ \{2\}, \{4\}, \{5\} \} & \alpha \in [0,4], \\ \{ \{4,5\}, \{2\} \} & \alpha \in [4,8], \\ \{ \{2,4,5\} \} & \alpha \in (8,10]. \\ \end{cases}$$

C. Parametric Algorithm

Lemma III.6 directly leads to the PAR algorithm in Algorithm 2 where the values of $Q_{\alpha,V}$ and $r_{\alpha,V}$ are determined for all $\alpha$ in each iteration $i$. We call Algorithm 2 a parametric algorithm since the variables $U_{\alpha,V}, Q_{\alpha,V}$ and $r_{\alpha,V}$ are parameterized by the minimum sum-rate estimate $\alpha$. For the minimum sum-rate problems in (2), the input linear ordering $\Phi$ can be arbitrarily chosen. In Section III-D we show how to choose $\Phi$ to minimize a weighted sum-rate problem.

Example III.8. We apply the PAR algorithm to the system in Example III.2. First, initiate $r_{\alpha,i} = \alpha - H(V) = \alpha - 10$ for all $\alpha \in V$ and $\alpha$. For $i = 1$, we get $Q_{\alpha,V} = \{\{4\}\}$ and $r_{\alpha,4} = f_\alpha(\{4\}) = \alpha - 2$ for all $\alpha$. See Fig. 11. As shown in Example III.7 for $i = 2$, we get $U_{\alpha,V} = \{\{4\}\}$, so that the updated $r_{\alpha,V}$ and $Q_{\alpha,V}$ are in (8) for $i = 3$, we get $U_{\alpha,V}$...
in (12) so that the updated $r_{\alpha,V_3}$ and $Q_{\alpha,V_3}$ are in (13). See Fig. (b) and (c), respectively.

For $i = 4$ and $\phi_4 = 3$, consider the problem

$$\min\{g_0(\bar{X}) : \{3\} \in \mathcal{X} \subseteq Q_{\alpha,V_4}\}$$

where $Q_{\alpha,V_4} = Q_{\alpha,V_3} \cup \{\{3\}\}$. We have the critical points $\alpha_3 = 7$ and $\alpha_0 = H(V) = 10$ with $\tilde{S}_1 = \{3\}$ and $\tilde{S}_0 = \{2,3,4,5\}$ such that $\{3\} = \tilde{S}_1 \subseteq \tilde{S}_0 = V_4$ and

$$U_{\alpha,V_4} = \begin{cases} 
\{3\} & \alpha \in [0,7], \\
\{2,3,4,5\} & \alpha \in (7,10]. 
\end{cases}$$

We use $U_{\alpha,V_4}$ to update $r_{\alpha,V}$ and $Q_{\alpha,V_4}$ for all $\alpha$ as in step 6 and get

$$r_{\alpha,V} = \begin{cases} 
(\alpha - 10, -6 - 6, -6, -2, -6 - 4) & \alpha \in [0,4], \\
(\alpha - 10, -6, -6, -6, -2, -2) & \alpha \in [4,7], \\
(\alpha - 10, -6, -6, -6, -6 - 2, -2) & \alpha \in (7,8], \\
(\alpha - 10, 2, 0, -6, -2, 2) & \alpha \in (8,10],
\end{cases}$$

$$Q_{\alpha,V_4} = \begin{cases} 
\{\{2\}, \{3\}, \{4\}, \{5\}\} & \alpha \in [0,4], \\
\{\{4,5\}, \{2\}, \{3\}\} & \alpha \in (4,7], \\
\{\{2,3,4,5\}\} & \alpha \in (7,10].
\end{cases}$$

See Fig. (d).

For $i = 5$ and $\phi_5 = 1$, we have the critical points for the problem

$$\min\{g_0(\bar{X}) : \{1\} \in \mathcal{X} \subseteq Q_{\alpha,V_5}\}$$

being $\alpha_2 = 6$, $\alpha_1 = 6.5$ and $\alpha_0 = H(V) = 10$ with $\tilde{S}_2 = \{1\}$, $\tilde{S}_1 = \{1,4,5\}$ and $\tilde{S}_0 = \{1,\ldots,5\}$ such that

$$U_{\alpha,V} = \begin{cases} 
\{1\} & \alpha \in [0,6], \\
\{1,4,5\} & \alpha \in (6,6.5], \\
\{1,\ldots,5\} & \alpha \in (6.5,10].
\end{cases}$$

After the updates in step 6

$$r_{\alpha,V} = \begin{cases} 
(\alpha - 5, -6 - 6, -6, -6, 0 - 4) & \alpha \in [0,4], \\
(\alpha - 5, -6 - 6, -6, -6, -2, 0) & \alpha \in (4,6], \\
(1, -6 - 6, -6, -6, -6 - 2, 0) & \alpha \in (6,6.5], \\
(1 - 2, -6 - 6, -6, -6, -6 - 2, 0) & \alpha \in (6.5,7], \\
(0, -6 - 6, -6, -6, -6, -6) & \alpha \in (7,8], \\
(0, 2, 0, -6, -6, -6) & \alpha \in (8,10],
\end{cases}$$

$$Q_{\alpha,V} = \begin{cases} 
\{\{1\}, \ldots, \{5\}\} & \alpha \in [0,4], \\
\{\{4,5\}, \{1\}, \{2\}, \{3\}\} & \alpha \in (4,6], \\
\{\{1,4,5\}, \{2\}, \{3\}\} & \alpha \in (6,6.5], \\
\{\{1,\ldots,5\}\} & \alpha \in (6.5,10].
\end{cases}$$

(16)

See Fig. (e). For the final segmented partition $Q_{\alpha,V}$, the corresponding PSP has the critical points $\alpha^{(3)} = 4$, $\alpha^{(2)} = 6$ and $\alpha^{(1)} = 6.5$ and $\alpha^{(0)} = H(V) = 10$ with $\mathcal{P}^{(3)} = \{\{1\}, \ldots, \{5\}\}$, $\mathcal{P}^{(2)} = \{\{4,5\}, \{1\}, \{2\}, \{3\}\}$, $\mathcal{P}^{(1)} = \{\{1,4,5\}, \{2\}, \{3\}\}$ and $\mathcal{P}^{(0)} = \{\{1,\ldots,5\}\}$ so that we know $\mathcal{R}_{\alpha,V}(V) = \alpha^{(1)} = 6.5$ is the minimum sum-rate for the asymptotic model and $\mathcal{P}^{(1)} = \{\{4,5\}, \{1\}, \{2\}, \{3\}\}$ is fundamental partition. We also know an optimal achievable rate vector $r_{6.5,V} = (1,0.5,0.5,4.5,0) \in \mathcal{R}_{\alpha,V}(V)$, which has the LCM $|\mathcal{P}^{(1)}| - 1 = 2$ so that it is implementable by network coding schemes with 2-packet-splitting in CCDE. The results also provide the solution to the non-asymptotic model: the minimum sum-rate solutions to $\mathcal{R}_{\alpha,V}(V) = \mathcal{R}_{\alpha,V}(V) = [\mathcal{R}_{\alpha,V}(V)]_{\{\alpha\}}$ is an optimal achievable rate vector $\tilde{r}_{\alpha,V}$.

The remaining problem is how to obtain $\alpha_{j,S}$ and $\tilde{S}_j$ in Lemma III.6. Recall that, $\alpha_{j,S}$ and $\tilde{S}_j$ are used to update $Q_{\alpha,V_i}$ for all $\alpha$ in step 6 of Algorithm 2 and the resulting $Q_{\alpha,V_i}$ determines the PSP of $V_i$. We can still use Lemma A.1 in Appendix A to adapt the value of $\alpha$ to search for $\tilde{S}_j$. This results in the StRmap algorithm in Algorithm 3. All $\tilde{S}_j$s are determined by the call StRmap($\{m\} : m \in V_i$, $\{V_i\}$).

---

\footnote{We used the same source model in Example III.2 as in [17] so that the results can be compared and verified: the minimum sum-rate solutions to both asymptotic and non-asymptotic model are consistent with [17].}
Algorithm 3: Strong Map (StrMap) Algorithm

input : $\mathcal{P}_d, P_a \in \Pi(V_i)$ such that $\mathcal{P}_d \subset P_a$ (We assume the $Q_{\alpha,V_i}$ and $g_{\alpha}$ for all $\alpha$ are the global variables.)
output: A subset of $\{\tilde{S}_i: j \in \{0, \ldots, q\}\}$ for the problem min\{$g_{\alpha}(\tilde{X}): \{\phi_i\} \in \tilde{X} \subseteq Q_{\alpha,V_i}\}$ in step 3 of Algorithm 2

1. $\alpha := H(V) = \max_{\mathcal{P}_d \in \Pi(V_i)} \frac{H[P_d]}{H[P_a]}$
2. $\mathcal{S} := \{\arg\max_{\mathcal{P}_d \in \Pi(V_i)} \{\phi_i\} \in \tilde{X} \subseteq Q_{\alpha,V_i}\}$;
3. $\mathcal{P} := (Q_{\alpha,V_i} \setminus \mathcal{S}) \cup \{\tilde{S}\}$;
4. if $\mathcal{P} = \mathcal{P}_d$ then return $\{\tilde{S}\}$;
5. else return StrMap($\mathcal{P}_d, \mathcal{P}$) $\cup$ StrMap($\mathcal{P}, \mathcal{P}_a$) ;

The corresponding critical values $\alpha_{j}$ can be obtained by the property of the strict strong map below.

Lemma III.9 ([37] Theorem 31). For all $\alpha, s$ and $\mathcal{S}$ that characterize $U_{\alpha,V_i}$ of the minimal minimizer of $\min\{g_{\alpha}(\tilde{X}): \{\phi_i\} \in \tilde{X} \subseteq Q_{\alpha,V_i}\}$ in Lemma II.6

$$r_{\alpha_{j}}(\tilde{S}_{j-1} \setminus \mathcal{S}_{j}) = H(\tilde{S}_{j-1}) - H(\mathcal{S}_{j}), \forall j \in \{1, \ldots, q\}.$$ 

D. Minimum Weighted Sum-rate Problem

Let $w_{V} = (w_{i}: i \in V) \in \mathbb{R}_{+}^{\left|V\right|}$ be a positive weight vector and $w_{V}^{r_{V}} = \sum_{i \in V} w_{i} \cdot r_{i}$ be the weighted sum-rate of $r_{V}$. The minimum weighted sum-rate problem is to search a rate vector that minimizes the $w_{V}^{r_{V}}$ in the optimal rate region:

$$\min\{w_{V}^{r_{V}}: r_{V} \in A_{VCO}(V)\}, \quad (17a)$$

$$\min\{w_{V}^{r_{V}}: r_{V} \in A_{NCO}(V)\}, \quad (17b)$$

for the asymptotic and non-asymptotic models, respectively. It is shown in [13], [14] that, by choosing a proper linear ordering $\Phi$, the solution is returned by the CoordSatCap algorithm. This method is described in [17] Theorem 35 as follows. For a given weight vector $w_{V}$, we call $\Phi = (\phi_{1}, \ldots, \phi_{|V|})$ a linear ordering w.r.t. $w_{V}$ if $w_{\phi_{1}} \leq \cdots \leq w_{\phi_{|V|}}$ and the calls CoordSatCap$(R_{ACO}(V), H, V, \Phi)$ and CoordSatCap$\{R_{NCO}(V), H, V, \Phi\}$ return the minimizers of (17a) and (17b), respectively. By knowing that that $r_{\alpha_{j}}$ returned by the PAR algorithm is exactly the same as the one returned by the call CoordSatCap$(\alpha, H, V, \Phi)$ for all $\alpha$ and the relations $B(f_{R_{ACO}(V)}) = A_{VCO}(V)$ and $B(f_{R_{NCO}(V)}) = A_{NCO}(V)$, the following properties hold straightforwardly.

Corollary III.10. For a given weight vector $w_{V}$, choose the linear ordering $\Phi$ w.r.t. $w_{V}$. The rate vector $r_{\alpha_{j}}$ returned by the call PAR$(H, \Phi, V)$ provides solutions to the minimum weighted sum-rate problems [17] at $\alpha = R_{ACO}(V)$ and $\alpha = R_{NCO}(V)$, i.e., $r_{\alpha_{j}}(V) \in \{\arg\min_{w_{V}^{r_{V}}: r_{V} \in A_{VCO}(V)}\}$ and $r_{\alpha_{j}}(V) \in \{\arg\min_{w_{V}^{r_{V}}: r_{V} \in A_{NCO}(V)}\}$, respectively.

E. Complexity

The PSP invokes $\left|V\right|$ calls of the StrMap algorithm. As explained in Appendix B the StrMap algorithm can be implemented by the PSFM algorithms in [26–28] that have the same asymptotic complexity as the SFM algorithm. Therefore, the minimum sum-rate problem in (2), as well as the minimum weighted sum-rate problem in (17), for both asymptotic and non-asymptotic models can be solved by the PAR algorithm in $O(|V| \cdot SFM(|V|))$ time. As compared to the existing computation time $O(|V|^2 \cdot SFM(|V|))$ of the MDA algorithm in [17] and the algorithms in [13, 14] for the finite linear source model, the complexity is reduced by a factor of $|V|$. In addition, the PAR algorithm allows distributed computation. See Section V.

IV. RELATED PROBLEMS

CO was first formulated in [2] based on the secret key agreement problem for the purpose of determining the secret capacity $C_{S}(A)$, the largest rate that the secret key can be generated by the active users in $A \subseteq V$ with the rest users in $V \setminus A$ being the helpers. The secret capacity is shown in [2] Example 4] to be upper bounded by a multivariate mutual dependence, which is proved to be tight when $A = V$ in [34]. The relationship with the PSP became clearer in the further studies on the case $A = V$ of the secret key agreement and the CO problems in [15], [17]. In this section, we show the contribution of the PAR algorithm to the existing related problems.

A. Secret Capacity

The secret capacity in the case when $A = V$ is [2] Example 4]

$$C_{S}(V) = I(V), \quad (18)$$

where

$$I(V) = \min_{P \in \Pi(V); |P| > 1} \left(\frac{D(P_{ZV})}{\prod_{P \in P} P_{ZC}} - 1\right) \quad (19)$$

The term $I(V)$ is called the shared information in [38] and multivariate mutual information in [15], that measures the mutual dependence in $Z_{V}$. Based on [15], we have the duality relationship between $C_{S}(V)$ and $R_{ACO}(V)$ [2]: Theorem 1] [5].

$$R_{ACO}(V) = H(V) - C_{S}(V) \quad (20)$$

which states that the omniscience is attained by the minimum sum-rate $R_{ACO}(V)$ if the users in $V$ only exchange over broadcast channels the amount of information that is not known to all. It is shown in [15] that $I(V) = H(V) - \alpha(1^{12})$ and

$^{12}$Here, ‘asymptotic’ refers to the asymptotic limits of the complexity notation $O(\cdot)$: for the actual running time $\alpha(\cdot)$, the asymptotic complexity is $O(b(|V|))$ if $\lim_{|V| \rightarrow \infty} \frac{\alpha(|V|)}{b(|V|)} = c$ for some constant $c$.

$^{13}$It is shown $C_{S}(V) \leq I(V)$ in [2] Example 4, which is proved to be tight in [34].
the fundamental partition $\mathcal{P}^{(1)}$ is the finest/minimal minimizer of $\mathcal{H}$. Thus, the solutions to both the CO and the secret agreement problems are provided by the PSP. The PAR algorithm reduces the existing complexity $[13]-[17]$ for determining the secret capacity $C_S(V)$ from $O(|V|^2 \cdot \text{SFM}(|V|))$ to $O(|V| \cdot \text{SFM}(|V|))$.

### B. Clustering

For the data points in $V$, let $f$ be the normalized submodular set function such that $f(X)$ measures the inhomogeneity of the data points in subset $X \subseteq V$. The entropy function $H$ and (graph) cut function $\kappa$ are two typical examples of the inhomogeneity measures. For a (non-overlapping) clustering result $\mathcal{P}$ such that $|\mathcal{P}| > \beta$ for some $\beta \in (0, |V|)$, the clustering cost $f(\mathcal{P}) = \sum_{C \in \mathcal{P}} f(C)$ is normalized by the increment number of clusters $|\mathcal{P}| - \beta$ and the problem $\min_{\mathcal{P} \in \Pi(V)} f(\mathcal{P})$ is called the \(\beta\)-minimum average cost (\(\beta\)-MAC) clustering $[20]$, to which the solution for all $\beta$ is fully determined by the minimizers of the Dilworth truncation $f_\lambda(V) = \min_{\mathcal{P} \in \Pi(V)} f_\lambda(\mathcal{P}) = \min_{\mathcal{P} \in \Pi(V)} \sum_{C \in \mathcal{P}} f_\lambda(C)$, where $f_\lambda = f(X) - \lambda$ for all $X \subseteq V$. As stated in [20] Lemma 3], the problem of \(\beta\)-MAC clustering is equivalent to determining the PSP of $V$.

By letting $\lambda = f(V) - \alpha$, $f_\lambda$ is equivalent to $f_\alpha$ defined in Section II-B and all results derived in this paper also hold for all $\lambda$. The minimal minimizer $Q_{\alpha,V}$ is $\Lambda \arg \min_{\mathcal{P} \in \Pi(V)} f_\lambda(\mathcal{P})$ of the critical points $\lambda(V) = H(V) - \alpha(\lambda)$ for all $\lambda \in (0, \ldots, \alpha)$ determining the PSP of $V$. See also Lemma A.3 in Appendix A. The \(\beta\)-MAC clustering (when $\beta = 1$) is of particular interest in that it generalizes the network strength and can be extended to an information-theoretic clustering framework.

1) **Network Strength and Pairwise Independent Network (PIN) Model:** For $f$ being the cut function $\kappa$ of a graph, the \(\beta\)-MAC clustering determines the network strength $\sigma(V)$ $[21]$:

$$\sigma(V) = \frac{1}{2} \min_{P \in \Pi(V) : |P| > 1} \frac{\kappa(P)}{|P| - 1},$$

(21)

where each $P \in \Pi(V)$ denotes a multi-way cut. The cost this multi-way cut incurs is $\kappa(P) = \sum_{C \in P} \kappa(C)$. The network strength is a measure of connectivity in the view of the optimal network attack problem $[21][13]$ which is determined by the first critical point in the PSP $\sigma(V) = \frac{\alpha(1)}{2}(H(V) - \alpha(1))$. It was shown in [21] that $\sigma(V)$ can be obtained in $O(|V|^2 \cdot \text{MaxFlow}(|V|))$ time, where MaxFlow($|V|$) denotes the complexity of the max-flow/min-cut algorithm [40] applied to a graph with $|V|$ nodes. This complexity is reduced to $O(|V| \cdot \text{MaxFlow}(|V|))$ in [41]. The network strength also denotes the secret capacity in the pairwise independent network (PIN) source model $[23]-[25]$, which has a graphical representation.

In a PIN model, we have each terminal being $Z_i = (W_{ii'} : i' \in V \setminus \{i\})$, where the pairs of r.v.s in $\{(W_{ii'}, W_{ij}) : i, j \in V, i \neq j\}$ are independent of each other. The entropy function is $H(V) = \sum_{i, i' \in V, i \neq i'} H(W_{ii'}, W_{ii'})$ and $H(X) = \sum_{i, i' \in X, i \neq i'} H(W_{ii'}, W_{ii'}) + \sum_{i, i' \notin X} H(W_{ii'})$ for all $X \subseteq V$. Thus, the secret capacity $C_S(V)$ of the PIN model is $\min_{\mathcal{P} \in \Pi(V)} \sum_{C \in \mathcal{P}} f_\lambda(C)$ determined by the \(\beta\)-MAC clustering $[20]$. The following notation $\lambda(V)$ is $\frac{\alpha(1)}{2}(H(V) - \alpha(1))$. It was shown in [21] that $\sigma(V)$ reduces to $\frac{\alpha(1)}{2}(H(V) - \alpha(1))$ in [24] Theorem 3.4]

$$C_S(V) = \sigma(V) = \frac{1}{2} \min_{P \in \Pi(V) : |P| > 1} \frac{\kappa(P)}{|P| - 1},$$

(22)

where $\kappa$ is the cut function of the undirected graph $G = (V, E)$ with the weight of each edge $(i, i')$ being $I(W_{ii'}; W_{ii'})$.

See Example V.1. The idea of solving the secret agreement problem in the PIN model via the tree packing algorithms, e.g., [42], is based on the relationship [22] Section 5.1 [43], [44]: the maximum number of edge-disjoint spanning trees is $<0(V)>$. More directly, any algorithm determining the network strength $\sigma(V)$, e.g., [21], [41], can also be applied to the secret agreement problem in the PIN model.

2) **Information-theoretic Clustering:** The authors in [19] extended the 1-MAC ($\beta = 1$) clustering problem based on the measure $I(V)$ in [19]. It is shown in [19] Theorem 3] that $C = \cup\{X \subseteq V : I(X) > \lambda, |X| > 1\}$ for all $C \in Q_{\lambda,V}$ such that $|C| > 1$. The interpretation is that any non-singleton $C \in Q_{\lambda,V}$ is the maximal subset with similarity $I(C)$ strictly greater than a given threshold $\lambda$. In this sense, all critical points $\lambda(\lambda)$ and partitions $\mathcal{P}(\lambda)$ in the PIN form a hierarchical clustering result. For example, replacing $\alpha$ by $H(V) - \lambda$ for $Q_{\alpha,V}$ in $[14]$, we have

$$Q_{\lambda,V} = \begin{cases} \{2, 3, 4, 5\} & \lambda \in [0, 3), \\ \{4, 5\}, \{2\}, \{3\} & \lambda \in [3, 6), \\ \{2\}, \{3\}, \{4\}, \{5\} & \lambda \in [6, +\infty), \end{cases}$$

(23)

corresponding to the dendrogram in Fig. 2(c). Here, $I(\{4, 5\}) = 6$ and $I(\{4, 5\} \cup X) \leq \lambda$ for all $X \subseteq \{2, 3\}$ and any similarity threshold $\lambda \in [3, 6]$, i.e., $\{4, 5\}$ is the maximal subset with a shared information $I(\{4, 5\})$ strictly greater than $\lambda$. Therefore, $\{4, 5\} \in Q_{\lambda,V}$ for all $\lambda \in [3, 6]$. In the region $\lambda \in [0, 3]$, $I(\{4, 5\}) > \lambda$ means that users 4 and 5 should be clustered, i.e., $\{4, 5\}$ must be contained in some cluster/subset in $Q_{\lambda,V}$. But, in this case, $\{2, 3, 4, 5\}$ is the maximal subset with $I(\{2, 3, 4, 5\}) > \lambda$. In Part II of this paper [40], we show that the dendrogram indicates a bottom-up successive omniscience asymptotic for the approximative model.

The PAR algorithm solves the $\beta$-MAC and information-theoretic clustering problems in $O(|V| \cdot \text{SFM}(|V|))$ time, which is faster than the existing computation time $O(|V|^2 \cdot \text{SFM}(|V|))$ [20] Algorithm SPLIT] [19] Algorithm 3. In [19], replacing the entropy function $H$ by the cut function $\kappa$ with $\kappa(V) = 0$ of a graph, we have the network strength $\sigma(V) = \frac{1}{2}I(V)$. The call PAR($\kappa, V, \Phi$) for any linear ordering $\Phi$ returns the PSP of the graph, of which the first critical value determines to the network strength in $[21]$ and the secret capacity $C_S(V)$ of the PIN model in $C_S(V) = \sigma(V) = \frac{1}{2}I(V)$. In this call, the solution to the SFM problem $[70]$ is the min-cut and the StrMap algorithm (Algorithm 3) can be implemented by
Critical value, the PAR algorithm returns all critical values of the PSP. The X is applied to the 17 run the Q rate problem can be solved when the size of the ground set in V

the parametric max-flow algorithm [46]. The parametric max-flow algorithm [46] and max-flow algorithm [40] complete in the same time. The complexity of PAR in this case is O(|V| · MaxFlow(|V|)) 16

Independently, Kolmogorov proposed another parametric algorithm in [45] Fig. 3 for determining the PSP specifically for the graph model, which also completes in O(|V| · MaxFlow(|V|)) time. In Appendix B we show that Kolmogorov’s algorithm [45] Fig. 3 is also based on a non-strict strong map property and propose a StrMapKolmogorov algorithm to allow it to be applicable to general submodular functions f.

V. DISTRIBUTED COMPUTATION

An observation about Fig. 1 is that the minimum sum-rate problem can be solved when the size of the ground set V_i is gradually increasing in the order of i = 1, 2, . . . , |V|. Replacing α by H(V) − λ in the PAR algorithm, we have Q_{λ,V_i} = \arg\min_{P \in \Pi(V_i)} f_\lambda[P] for all λ at the end of each iteration i. For λ interpreted as the estimate of the shared/multivariate mutual information, or the secret capacity, based on the dual relationship 20, f_\lambda(X) = H(X) − λ, \forall X \subseteq V is the called residual entropy function and the critical points of Q_{λ,V_i} are \lambda^{(j)} = H(V) − \alpha^{(j)} for all j \in \{0, . . . , p\}. To run the i\_th iteration, only the knowledge of the first i users in V_i = \{\phi_1, . . . , \phi_i\} is required: the value of H(X) for all X \subseteq V_i and the rate vector r_{\lambda,V_{i−1}} has that has been updated in previous iterations. This suggests a distributed and adaptive computation of the PSP of V by the PAR algorithm as in Algorithm 4. For example, when the DistPAR algorithm is applied to the 5-user system in Example 11.2 we get the Dilworth truncation \tilde{f}_\lambda(V_i) being the same as in Fig. 1 for \lambda = H(V) − α. We show another example below.

For the graph model, while the algorithm in 41 only searches the first critical value, the PAR algorithm returns all critical values of the PSP. The PSP of the graph model also provides the solution to the optimal network attack problem 21 for all λ, where λ is interpreted as the per-subgraph payoff.

In Algorithm 4 for all \lambda’ means for all nonnegative values of λ since the minimizer of \min_{P \in \Pi(V)} f_\lambda[P] is always \{V\} for all λ < 0.

Algorithm 4: Distributed computation of PSP by PAR algorithm (DistPAR)

input : f, V and \Phi
output: segmented variables r_V \in B(\tilde{f}_\lambda) and Q_{λ,V} = \arg\min_{P \in \Pi(V)} f_\lambda[P] for all λ
1 Let user \phi_1 initiate r_{\lambda,\phi_1} := −λ and Q_{λ,V_1} := \{\phi_1\} for all λ and pass to user \phi_2;
2 for i = 2 to |V|, let user \phi_i do
3 r_{\lambda,\phi_i} := −λ for all λ;
4 Q_{λ,V_i} := Q_{λ,V_{i−1}} \cup \{\phi_i\} for all λ;
5 For function g_\lambda(X) = f_\lambda(X) − r_\lambda(X), \forall X \subseteq Q_{λ,V_i}, obtain the critical points \{\lambda_j : j \in \{0, . . . , q\}\} and \{\tilde{S}_j : j \in \{0, . . . , q\}\} that determine the minimal minimizer \tilde{U}_{\lambda,V_i} of \min \{g_\lambda(X) : \{\phi_i\} \neq \emptyset, \lambda \in \lambda_j\} for all λ by the StrMapDistPAR algorithm (Algorithm 3);
6 for j = 0 to q do
7 r_{\lambda,V} := r_{\lambda,V} + g_\lambda(\tilde{S}_j)\phi_i;
8 Q_{λ,V_i} := (Q_{λ,V_i} \setminus \tilde{S}_j) \cup \{\tilde{S}_j\};
9 for all λ \in [\lambda_j, \lambda_{j+1});
10 Pass the results r_{\lambda,V_i} and Q_{λ,V_i} as well as function f(X) for all X \subseteq V_i to user \phi_{i+1};
11 endfor
12 return r_V and Q_{λ,V} for all λ;

Example V.1. Consider a 4-user system with 
Z_1 = (W_1, W_2), Z_2 = (W_2, W_3), Z_3 = (W_2, W_3), Z_4 = (W_2, W_3),
where each W_m is an independent uniformly distributed random bit. Here, Z_{1,2,3} forms a PIN model. The corresponding undirected graph G = (V_3, E) is shown in Fig. 2(e), where the weight of edge \{i, i’\} is I(Z_i, Z_{i'}).

We run the DistPAR algorithm in Algorithm 4 for f = H and the linear ordering \Phi = (1, 2, 3, 4) as follows. First, user 1 initiates r_{\lambda,\phi_1} = r_{\lambda,\phi_1} = −λ and Q_{\lambda,V_1} = \{\{1\}\} for all λ \in [0, +∞) and passes them to user 2. User 2 initiates r_{\lambda,2} = −λ for all λ. For Q_{λ,V_2} = Q_{λ,V_1} \cup \{\{2\}\} = \{\{1\}, \{2\}\}, the minimal minimizer of \min_{\lambda} (g_\lambda(X) : 2 \in X \subseteq Q_{λ,V_2}) is \tilde{U}_{λ,V_2} = \{\{1\}, \{2\}\} for \lambda \in [0, 1) and \tilde{U}_{λ,V_2} = \{\{2\}\} for \lambda \in [1, +∞). After step 6 he/she obtains Q_{λ,V_2} as in Fig. 2(b) and the rate vector

\begin{align*}
r_{\lambda,V_2} &= \begin{cases} (2 - \lambda, 1) & \lambda \in [0, 1), \\ (2 - \lambda, 2 - \lambda) & \lambda \in [1, +∞). \end{cases}
\end{align*}

User 3 obtains r_{\lambda,V_2} and Q_{λ,V_2} from user 2 and gets

\begin{align*}
\tilde{U}_{λ,V_3} &= \begin{cases} \{\{1\}, \{2\}, \{3\}\} & \lambda \in [0, 1), \\ \{\{1\}\} & \lambda \in [1, 1.5), \\ \{\{3\}\} & \lambda \in [1.5, +∞). \end{cases}
\end{align*}

so that after step 6 he/she obtains Q_{λ,V_3} in Fig. 2(c) and

\begin{align*}
r_{\lambda,V_3} &= \begin{cases} (2 - \lambda, 1, 0) & \lambda \in [0, 1), \\ (2 - \lambda, 2 - \lambda, \lambda - 1) & \lambda \in [1, 1.5), \\ (2 - \lambda, 2 - \lambda, 2 - \lambda) & \lambda \in [1.5, +∞), \end{cases}
\end{align*}

Note, the first critical point \lambda^{(1)} = 1.5 equals the secret capacity C_5(|1, 2, 3|) of the first three users. Alternatively, for
\[ \hat{f}_\lambda(V_i) \text{ and } \mathcal{Q}_{\lambda,V_i} \]

\[ \hat{f}_\lambda(V_2) \text{ and } \mathcal{Q}_{\lambda,V_2} \]

\[ \hat{f}_\lambda(V_3) \text{ and } \mathcal{Q}_{\lambda,V_3} \]

\[ \hat{f}_\lambda(V) \text{ and } \mathcal{Q}_{\lambda,V} \]

\[ \hat{f}_\lambda(V) \text{ and } \mathcal{Q}_{\lambda,V} \]

\[ \hat{f}_\lambda(V_2) \text{ and } \mathcal{Q}_{\lambda,V_2} \]

\[ \hat{f}_\lambda(V_3) \text{ and } \mathcal{Q}_{\lambda,V_3} \]

\[ \text{PIN model } Z_{V_i} \]

Fig. 3. The piecewise linear decreasing Dilworth truncation \( \hat{f}_\lambda(V_i) \) in \( \lambda \) and the segmented partition \( \mathcal{Q}_{\lambda,V_i} \) obtained at the end of each iteration of the distributed PAR (DistrPAR) algorithm (Algorithm 4) when it is applied to the 4-user system in Example V.1. Here, \( \mathcal{Q}_{\lambda,V} \) in (a) to (d) characterizes the PIN model, for which, the secret agreement problem can be represented by the undirected graph in (e) with the weight of each edge \((i,i')\) being \( f(Z_i;\bar{Z}_{i'}) \). The strength of this graph is \( \sigma(V_3) = 1.5 \), which equals the secret capacity \( C_S(V_3) \) and determines the minimum sum-rate \( R_{\text{ACO}}(V_3) \) based on the dual relationship 20: \( R_{\text{ACO}}(V_3) = H(V_3) - C_S(V_3) = 1.5 \).

\( \kappa \) being the cut function of the undirected graph in Fig. 3(e), one can show that \( \frac{1}{2} \min_{\mathcal{P} \in \mathcal{P}(V)} |\mathcal{P}| \frac{\kappa(\mathcal{P})}{|\mathcal{P}|} = 1.5 = \lambda^{(1)} = C_S(\{1,2,3\}) \), which equals the network strength \( \sigma(\{1,2,3\}) \).

User 3 passes \( \mathcal{Q}_{\lambda,V_3} \) and \( r_{\lambda,V_3} \) to user 4, where \( \mathcal{Q}_{\lambda,V} \) in Fig. 3(d) and

\[ r_{\lambda,V} = \begin{cases} (2 - \lambda, 1, 0, 1) & \lambda \in [0, 1), \\ (2 - \lambda, 2 - \lambda, \lambda - 1, 2 - \lambda) & \lambda \in [1, 1.5), \\ (2 - \lambda, 2 - \lambda, 2 - \lambda, 2 - \lambda) & \lambda \in [1.5, +\infty) \end{cases} \]

are obtained.

For obtaining the minimal minimizer \( \mathcal{U}_{\lambda,V} \) in each iteration \( i \), one can still apply the StrMap algorithm (Algorithm 5) to get \( \{\hat{S}_j : j \in \{0, \ldots, q\}\} \). Based on Lemma [III.9] the critical points are \( \lambda_j = H(V) - \alpha_j \).

Independently, we propose a StrMapDistPAR algorithm in Appendix C that determines the segmented \( \mathcal{U}_{\lambda,V} \).

In the DistrPAR algorithm, the complexity incurred at each user is \( O(SFM(|V|)) \). At the end of each iteration, \( \mathcal{Q}_{\lambda,V} \) determines all the critical points \( \lambda_j \) and partitions \( P_j \) in the PSP of \( V_i \). The first critical point \( \lambda^{(1)} \) equals the secret capacity \( C_S(V_i) \) and shared/multivariate mutual information \( I(V_i) \).

For the CO problem in \( V_i \), the value \( \alpha^{(1)} = H(V_i) - \lambda^{(1)} \) equals the minimum sum-rate \( R_{\text{ACO}}(V_i) \) and \( r_{\lambda^{(1)},V_i} \) is an optimal rate vector in the asymptotic model; \( [\alpha^{(1)}] = H(V_i) - [\lambda^{(1)}] = R_{\text{ACO}}(V_i) \) and \( r_{[\lambda^{(1)}],V_i} \) is an optimal rate vector in the non-asymptotic model. For example, consider the CO problem in the PIN model formed by the first 3 users in Example V.1. The minimum sum-rate is \( R_{\text{ACO}}(V_3) = \alpha^{(1)} = H(V_3) - \lambda^{(1)} = 1.5 \) and \( r_{1.5,V} = (0.5, 0.5, 0.5) \) is an optimal rate vector in the asymptotic model; The minimum sum-rate \( R_{\text{ACO}}(V_3) = [\alpha^{(1)}] = H(V_3) - [\lambda^{(1)}] = 2 \) and \( r_{1,V} = (1, 1, 0) \) is an optimal rate vector in the non-asymptotic model. It means that the local omniscience problem in \( V_i \) is solved before the global omniscience. This fact will be utilized in Part II [30] for solving the SO problem, where it is also shown that an optimal local omniscience in \( X \subseteq V_i \) can be directly determined from \( r_{\lambda,V_i} \).

DistrPAR is also an adaptive approach where \( \mathcal{Q}_{\lambda,V} \) is adapted from \( \mathcal{Q}_{\lambda,V_i} \) based on the minimal minimizer \( \mathcal{U}_{\lambda,V_i} \) of \( \min \{g_{\lambda}(\hat{X}) : \{\phi_i \} \subseteq X \subseteq \mathcal{Q}_{\lambda,V_i-1} \} \) for all \( \lambda \). The value of \( \mathcal{Q}_{\lambda,V} \) converges to \( \mathcal{Q}_{\lambda,V} \) at the last user \( V_i \). This is particularly useful when the users complete recording their observations at different times. In this case, \( \phi_i \) in the linear ordering \( \Phi \) denotes that user \( \phi_i \) is the \( i \)th user that finishes observing \( Z_{\phi_i} \).

Thus, instead of waiting for all users having the data ready, the PAR algorithm can be implemented in the first-come-first-serve manner. The forwarding of the value of the entropy function \( H(X) \) in step 7 is also not difficult in CCDE: the source \( Z_{V_i} \) in a finite linear source model can be represented by a \( |V_i| \)-column matrix, which determines the value of \( H(X) \) for all \( X \subseteq V_i \).

VI. CONCLUSION

This paper proposed a PAR algorithm that reduces the complexity of solving the minimum sum-rate problem and determining the PSP in communication for omniscience and other related problems by a factor of \(|V|\). We observed the existing CoordSatCap algorithm that determines the Dilworth truncation \( f_{\alpha}(V) \) in the minimum sum-rate estimate \( \alpha \), which is segmented by the critical values and also characterizes the PSP \( \{P_j : j \in \{1, \ldots, p\}\} \). We proved that the objective function in a SFM problem in CoordSatCap exhibits strict strong map property so that the minimizer for all \( \alpha \) is found by \( O(1) \) calls of the PSFM algorithm that completes in \( O(SFM(|V|)) \) time. Based on this fact, we proposed a PAR algorithm that obtains the PSP in \( O(|V| \cdot SFM(|V|)) \) time. We showed the distributed implementation of PAR by proposing the DistrPAR algorithm, which iteratively adapts the Dilworth truncation \( f_{\alpha}(V) \) of the subsystem \( V_i \) for all \( \alpha \) as \( i \) increases. It converges to \( \hat{f}_{\alpha}(V) \) finally where the first critical point \( \alpha^{(1)} \) and partition \( P^{(1)} \) provide the solutions to the minimum sum-rate problem for both asymptotic and non-asymptotic models.

The PSP returned by the PAR or DistrPAR algorithm also provides the solutions to the secret key agreement, optimal network attack, information-theoretic and \( \beta \)-MAC clustering problems.
Algorithm 5: Decomposition Algorithm (DA) [20 Algorithm SPLIT] [36 Algorithm II]

\[\begin{align*}
\text{input: } & P^{(j)}, P^{(j')} \text{ in the PSP of } V \text{ such that } P^{(j')} < P^{(j)}, \\
\text{output: } & \{P^{(j)}, P^{(j+1)}, \ldots, P^{(j')}\}. \\
1. & \alpha := H(V) - \frac{f(\phi_{P^{(j')}}(V)) - f(\phi_{P^{(j)}}(V))}{\mu_{P^{(j')}} - \mu_{P^{(j)}}}, \\
2. & (r_{a,V}, Q_{a,V}) := \text{CoordSatCap}(\alpha, f, V, \Phi) \text{ where } \Phi \text{ is an arbitrarily chosen linear ordering of } V; \\
3. & \text{if } Q_{a,V} = P^{(j')} \text{ then return } \{P^{(j)}, P^{(j')}\}; \\
4. & \text{else return } \text{DA}(P^{(j)}, Q_{a,V}) \cup \text{DA}(Q_{a,V}, P^{(j')}). 
\end{align*}\]

In addition to the brief discussion on the related problems in Section IV, it is worth studying how the PAR algorithm contributes to the recent developments in secret key agreement problem in [47], [48] and the agglomerative approach for the information-theoretic clustering problem in [49]. The study also highlighted the importance of the PSFM algorithm. Given the fact that the PSFM algorithm is adapted from an existing SFM algorithm, it is worth discussing whether the minimum norm algorithm [50], which is the most practically efficient SFM algorithm, can also be adapted to a parametric one.

**APPENDIX A**

**PROPERTIES OF PSP IN \(\alpha\) AND DECOMPOSITION ALGORITHM**

For \(f\) being a submodular function, e.g., the entropy \(H\) or the cut \(\kappa\) function, the solution to the minimization \(\min_{P \in \Pi(V)} f_{\lambda}(P)\), where \(f_{\lambda}(P) = \sum_{C \in P} f(C)\) and \(f_{\lambda}(C) = f(C) - \lambda\), is segmented in \(\alpha\) by critical points \(\alpha(j)\), or \(\alpha(j) = H(V) - \lambda(j)\), and \(\{P^{(j)}: \{0, \ldots, p\}\}\) as described in Section I-B3. The \(\alpha(j)\) and \(\alpha(j')\) satisfy the following lemma.

**Lemma A.1** (20 Sections 2.2 and 31 [36 Definition 3.8]). For any two \(P^{(j)}\) and \(P^{(j')}\) such that \(j < j'\) (or \(j' < j\)) and \(\alpha(j) < \alpha(j')\),

\[\lambda = \frac{f(P^{(j')}) - f(P^{(j)})}{P^{(j')}} - P^{(j)},\]

and \(\alpha = f(V) - \lambda.\) The followings hold.

(a) If \(j + 1 = j'\), \(\lambda = \lambda(j')\) and \(\alpha = \alpha(j')\);

(b) If \(j + 1 < j'\), \(\lambda(j) < \lambda(j')\) and \(\alpha(j) < \alpha(j')\).

Based on Lemma A.1, the call DA(\{i\} : i \in V\} \{V\}) of the decomposition algorithm (DA) in Algorithm 5 returns all partitions in \(\{P^{(j)}: j \in \{0, \ldots, p\}\}\) of the PSP. The corresponding critical points \(\alpha(j)\) or \(\lambda(j)\) can be determined by Lemma A.1(a). The MDA algorithm in [17 Algorithm 1] is a revised version of the DA algorithm for the purpose of determining only the first partition \(P^{(1)}\), which determines the solution to the minimum sum-rate problem in CO. Lemma A.1 also ensures the validity of StrMap algorithm in Algorithm 3.

**APPENDIX B**

**PROOF OF LEMMA III.3**

The fact \(r_{a,V} \in P(f_a)\) holds throughout Algorithm 1 is shown in [20 Section 4.2] [17 Lemma 19] and the equality of two polyhedra \(P(f_a) = P(f_a)\) is proved in [18 Theorem 25]. (a) is the result in [20 Theorem 8 and Lemma 9].

We prove (b) and (c) as follows. All \(C \in Q_{a,V}\) are tight sets [20 Section 4.2], i.e., \(r_a(C) = f_a(C), \forall C \in Q_{a,V}\). In addition, for each \(C \in Q_{a,V}\), \(r_a(C) = f_a(C) \leq f_a(C)\) since \(r_{a,V} \in P(f_a) = P(f_a)\). But, \(f_a(C) \leq f_a(C)\), too, based on the definition of Dilworth truncation [3]. So, \(r_a(C) = f_a(C) = f_a(C)\) for all \(C \in Q_{a,V}\). Therefore, we also have \(X = \bigcup \text{argmin}_{P \in \Pi(V)} f(P)\), \(\forall X \subseteq Q_{a,V}\). because, otherwise, either \(Q_{a,V} \notin \text{argmin}_{P \in \Pi(V)} f(P)\) or \(Q_{a,V}\) is not the finest minimizer. Therefore, (b) holds. (c) also holds because of the properties of the PSP in Section I-B3.

**APPENDIX C**

**ALTERNATIVE TO STRMAP FOR DISTRPAR ALGORITHM**

Similar to the StrMap algorithm, we derive the properties in Lemma C.1 below and prove the StrMapDistPar algorithm in Algorithm 4 for determining the minimal minimizer \(U_{\lambda,V}\), of \(\min\{g_{\lambda}(X): \phi_i \in X \subseteq Q_{\lambda,Y}\}\) in each iteration \(i\) of the DistrPAR algorithm (Algorithm 4).

**Lemma C.1.** Consider the critical points \(\{\lambda_i: j \in \{0, \ldots, q\}\}\) and \(\{S_i: j \in \{0, \ldots, q\}\}\) that determine the minimal minimizer \(U_{\lambda,V}\) of \(\min\{g_{\lambda}(X): \phi_i \in X \subseteq Q_{\lambda,Y}\}\) in Algorithm 4. The followings hold:

(a) \(g_{\lambda}(S_{j-1}) = g_{\lambda}(S_j)\) for all \(j \in \{1, \ldots, q\}\);

(b) For any \(j, j' \in \{0, \ldots, q\}\) such that \(j < j'\), let

\[\lambda = \frac{f(S_j \setminus S_{j'}) + f(S_j) - f(S_j)}{|S_j \setminus S_{j'}|},\]

where \(\lambda_d \leq Q_{\lambda,Y,V}\).

(i) If \(P_d \notin Q_{\lambda,Y,V}\), let \(P^{(i)}\) in the PSP of \(V_{i-1}\) such that \(P_d = P^{(i)} \cup \{\phi_i\}\). Then, the corresponding critical value \(\lambda^{(i)} > \lambda_d\) and \(\lambda_{d+1} < \lambda < \lambda^{(i)}\);

(ii) If \(P_d = Q_{\lambda,Y,V}\), then \(\lambda_{d+1} < \lambda < \lambda_{d+1} = 1\) for \(j = 1\) and \(\lambda = \lambda_{d+1} = 1 + 1 = 2\).

**Proof:** (a) is a result in [37] Theorem 31 of the strict strong map: for \(S_j = \bigcup \text{argmin}_{P \in \Pi(V)} f(P)\), \(\forall \phi_i \in X \subseteq Q_{\lambda,Y,V-1} = \{\phi_i\}\) and \(S_{j-1} = \bigcup \text{argmin}_{P \in \Pi(V)} g_{\lambda}(X)\), \(\forall \phi_i \in Y \subseteq Q_{\lambda,Y-1} = \{\phi_i\}\) for all \(j \in \{1, \ldots, q\}\).

By converting \(g_{\lambda}(S_{j-1}) = g_{\lambda}(S_j)\) to \(r_{\lambda_d}(S_{j-1} \setminus S_j) = f(S_{j-1}) - f(S_j)\) and (23) to \(f(S_j) - f(S_{j'}) = f(S_j \setminus S_{j'})\) for all \(j < j'\), we prove (b) by contradiction. If \(P_d \notin Q_{\lambda,Y,V}\) and let \(P^{(i)}\) be one of the partitions in the PSP of \(V_{i-1}\), which characterizes the segmented \(Q_{\lambda,Y,V}\) for all \(\lambda\), such that \(P^{(i)} \cup \{\phi_i\}\). If \(\lambda < \lambda_{d+1}\), we have \(P^{(i)} = \bigcup \text{argmin}_{P \in \Pi(V)} f(P)\) for all \(X \subseteq Q_{\lambda,Y,V}\) and \(\lambda = H(V) - \alpha\) in Lemma 3.3. If \(\lambda \geq \lambda^{(i)}\), we have \(P^{(i)} = \bigcup \text{argmin}_{P \in \Pi(V)} f(P)\) for all \(X \subseteq Q_{\lambda,Y,V}\). So, we must have \(\lambda_{d+1} \leq \lambda < \lambda^{(i)}\) and (b)-(i) holds.

We prove (b) and (c) as follows. All \(C \in Q_{a,V}\) are tight sets [20 Section 4.2], i.e., \(r_a(C) = f_a(C)\), \(\forall C \in Q_{a,V}\). In addition, for each \(C \in Q_{a,V}\), \(r_a(C) = f_a(C) \leq f_a(C)\) since \(r_{a,V} \in P(f_a) = P(f_a)\). But, \(f_a(C) \leq f_a(C)\), too, based on the definition of Dilworth truncation [3]. So, \(r_a(C) = f_a(C) = f_a(C)\) for all \(C \in Q_{a,V}\). Therefore, we also have \(X = \bigcup \text{argmin}_{P \in \Pi(V)} f(P)\), \(\forall X \subseteq Q_{a,V}\). because, otherwise, either \(Q_{a,V} \notin \text{argmin}_{P \in \Pi(V)} f(P)\) or \(Q_{a,V}\) is not the finest minimizer. Therefore, (b) holds. (c) also holds because of the properties of the PSP in Section I-B3.
Therefore, we must have each iteration \( r \in \alpha \{ \lambda < \lambda \} \) as the push-relabel MaxFlow algorithm. The same technique can be determined in the same asymptotic time as the push-relabel MaxFlow algorithm. Therefore, we must have \( \lambda < \lambda \) in [26]–[28].

**APPENDIX E**

**KOLMOGOROV’S ALGORITHM**

For determining the PSP of a graph, Kolmogorov proposed Algorithm 7 in [29] Fig. 3 with the parametric MaxFlow being the subroutine, the contribution of which is similar to the PAR algorithm: it reduces the previous complexity \( O(|V|^2 \cdot \text{MaxFlow}(|V|)) \) for determining the network strength and the maximum number of edge-disjoint spanning trees [21], [22] to \( O(|V| \cdot \text{MaxFlow}(|V|)) \). Algorithm 7 is based on the SFM problem \( \min \{ h_\lambda(X) : \phi_i \in X \subseteq V \} \) for

\[
h_\lambda(X) := f_\lambda(X) - r_\lambda(X), \quad \forall X \subseteq V,
\]

where the rate vector \( r_\lambda \) is updated in steps 4 and 5 in each iteration to maintain the monotonicity: \( r_{\lambda,i} \leq r_{\lambda',i} \) for all \( i \in V \) and \( \lambda < \lambda' \). It is show in [29] Lemmas 4 and 5 that the minimizer of \( \min \{ h_\lambda(X) : \phi_i \in X \subseteq V \} \) forms a ‘nesting’ set sequence in \( \lambda \), which, for \( f \) being the cut function, can be determined by only one call of the parametric MaxFlow algorithm in [46]. We show in Theorem E.1 and Lemma E.2 below that this ‘nesting’ property is also due to the strong map property, which is conditioned on the monotonicity of \( r_\lambda \).

**Theorem E.1.** In each iteration \( i \) of Algorithm 7 \( h_i \) forms a non-strict strong map sequence in \( \lambda \), i.e., \( h_i \prec h_{\lambda,i} \) for all \( \lambda \) and \( \lambda' \) such that \( \lambda < \lambda' \).

**Proof:** For any \( X, Y \subseteq V \) such that \( X \subseteq Y \) and \( \phi_i \notin Y \setminus X \), we have \( h_\lambda(X) - h_\lambda(X) - h_{\lambda,i}(Y) + h_{\lambda,i}(X) = r_\lambda(Y \setminus X) \leq 0 \). But, this inequality does not hold strictly for all \( X \subseteq Y \) such that \( \phi_i \notin Y \setminus X \) since \( r_\lambda \) is only nonincreasing, instead of strictly increasing, in \( \lambda \). Based on Definition E.1 the theorem holds.

**Lemma E.2.** Theorem 26 to 28] In each iteration \( i \) of the Algorithm 7 the minimal minimizer \( U_{\lambda,V} = \argmin \{ h_\lambda(X) : \phi_i \in X \subseteq V \} \) satisfies \( U_{\lambda,V} \supseteq U_{\lambda',V} \) for all \( \lambda < \lambda' \). \( U_{\lambda,V} \) for all \( \lambda \) is fully characterized by \( q' < |V| - 1 \) critical points

\[
0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{q'} < \lambda_{q'+1} = +\infty
\]

and the corresponding minimal minimizer \( S_j = U_{\lambda_j,V} \) forms a set chain

\[
V = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_{q'} = \{ \phi_i \}
\]

such that \( U_{\lambda_j,V} = S_j \) for all \( \lambda \in [\lambda_j, \lambda_{j+1}) \) and \( j \in \{0, \ldots, q'\} \).

We derive the properties based on the strong map property in Theorem E.1 below and show that Algorithm 7 can determine the PSP for the general submodular function \( f \).

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**APPENDIX D**

**STRMAP BY PSFM**

In [46], the push-relabel MaxFlow algorithm in [40] was extended to a parameterized one based on the fact: if the capacities of edges from the source node and to the sink node are monotonically changing with a real-valued parameter \( \alpha \), the max-flows/min-cuts for a finite number of monotonic values of \( \alpha \) can be determined in the same asymptotic time as the push-relabel MaxFlow algorithm. The same technique was further applied to extend the SFM algorithms to the PSFM ones in [26]–[28]. But, all PSFs in [26]–[28] requires a finite number of monotonic values of \( \alpha \) as the inputs. For solving the problem \( \min \{ g_\alpha(x) : \{i \in X \subseteq Q_\alpha \} \} \) where the critical values of \( \alpha \) are not known in advance, the StrMap algorithm can be implemented in the same way as the Slicing algorithm in [26] Section 4.2.
**Algorithm 7: Komolgorov’s Algorithm** [28] Fig. 3

**input**: \( f, V \) and \( \Phi \).

**output**: \( \mathcal{Q}_\lambda V = \arg \min_{P \in \Pi(V)} f_\lambda(P) \) and \( \mathbf{r}_\lambda \in B(f_\lambda) \) for all \( \lambda \).

1. Initiate \( \mathbf{r}_\lambda V := (-\lambda, \ldots, -\lambda) \) and \( \mathcal{Q}_\lambda V := \{\{i\} : i \in V\} \) for all \( \lambda \);
2. for \( i = 1 \) to \( |V| \) do
3. For function \( h_\lambda(X) := f_\lambda(X) - r_\lambda(X) \), obtain the critical points \( \{\lambda_j : j \in \{0, \ldots, q'\}\} \) and \( \{S_j : j \in \{0, \ldots, q'\}\} \) that determine the minimal minimizer \( \min_{U_\lambda V} \{h_\lambda(X) : \phi \in X \subseteq V\} \);
4. Update \( r_{\lambda, \phi} := f_\lambda(U(i)) \) for all \( \lambda \in [\lambda_j, \lambda_j+1) \);
5. foreach \( m \in V \) such that \( m \neq \phi \) do find \( j^* \in \{0, \ldots, q'\} \) such that \( m \in U_{j^* - 1} \setminus U_{j^*} \), and update \( r_{\lambda, m} := \min \{r_{\lambda, m}, r_{\lambda, m, *}\} \) for all \( \lambda \);
6. for \( j = 0 \) to \( q' \) do
7. endfor
8. return \( \mathcal{Q}_\lambda V \) and \( \mathbf{r}_\lambda \) for all \( \lambda \);

**Lemma E.3.** In each iteration \( i \) of Algorithm 7 consider the critical points \( \{\lambda_j : j \in \{1, \ldots, q'\}\} \) and \( \{S_j : j \in \{1, \ldots, q'\}\} \) that characterize the minimal minimizer \( \min_{U_\lambda V} \{h_\lambda(X) : \phi \in X \subseteq V\} \). For any two \( j, j^* \in \{0, \ldots, q'\} \) such that \( j < j^* \), let the value of \( \lambda \) holds

\[
\lambda(S_j, S_{j^*}) = f(S_j) - f(S_{j^*}).
\] (27)

**Proof.**

(a) If \( j + 1 = j^* \), \( \lambda = \lambda_j \);

(b) If \( j + 1 < j^* \), \( \lambda_{j+1} \leq \lambda \leq \lambda_j \).

For the monotonicity \( r_\lambda V \geq r_\lambda V \) for all \( \lambda < \lambda' \), we have \( \lambda < \lambda' \) if \( r_\lambda V > r_{\lambda'} \) and \( \lambda' \leq \lambda \) if \( r_\lambda V \geq r_{\lambda'} \).

Based on Lemma E.2 for \( j < j^* \) and a sufficiently small \( \epsilon > 0 \), we have \( \lambda_{j+1} - \epsilon \leq \lambda_j \leq \lambda_{j+1} \), and \( r_{\lambda_j} S_j = r_{\lambda_{j+1}} S_j \).

Therefore, we have

\[
r_{\lambda_j} S_j \leq r_{\lambda_j} S_j S_{j^*} < r_{\lambda_{j+1}} S_j S_{j^*}.
\]

Thus,

\[
\lambda_{j+1} - \epsilon \leq \lambda \leq \lambda_j \text{ for any sufficiently small } \epsilon, \text{ which, due to the continuity of } f_\lambda(V) \text{ in } \lambda, \text{ is equivalent to } \lambda_{j+1} \leq \lambda \leq \lambda_j \text{ and reduces to } \lambda = \lambda_j \text{ in the case when } j + 1 = j^*.
\]

Based on Lemma E.3 the call StrMapKomolgorov(V, \{\phi_i\}) of Algorithm 8 returns all \( S_j \)s that segments the minimal minimizer \( \min_{U_\lambda V} \{h_\lambda(X) : \phi \in X \subseteq V\} \) in each iteration.

**Algorithm 8: StrMapKomolgorov(S_j, S_{j'})** Find \( \{S_j : j \in \{1, \ldots, q'\}\} \) in step 3 of the Komolgorov’s algorithm (Algorithm 7)

**input**: \( S_j, S_{j' }, \) such that \( S_j \supseteq S_{j'} \).

**output**: \( \{S_j, S_{j+1}, \ldots, S_{j'}\} \).

1. Determine \( \lambda \) such that \( r_\lambda S_j = f(S_j) - f(S_j') \);
2. \( U_\lambda V := \arg \min_{U_\lambda V} \{h_\lambda(X) : \phi \in X \subseteq V\} \);
3. if \( U_\lambda V = S_j \) then
   4. Obtain \( U_{\lambda - \epsilon} V := \arg \min_{U_{\lambda - \epsilon} V} \{h_\lambda(X) : \phi \in X \subseteq V\} \) for a small \( \epsilon > 0 \);
   5. if \( U_{\lambda - \epsilon} V = S_j \) then return \( \{U(i), U(i')\} \);
   6. else return StrMapKomolgorov(S_j, U_{\lambda - \epsilon} V) \cup StrMapKomolgorov(U_{\lambda - \epsilon} V, S_{j'})

The corresponding critical points \( \lambda_j \)s can be obtained by Lemma E.3a. Again, the StrMapKomolgorov algorithm can be implemented by the PSFM algorithms [26–28]. It should be noted that in Algorithm 7 we need to check whether \( U_{\lambda - \epsilon} \), the minimal minimizer of the problem \( \min \{h_\lambda(X) : \phi \in X \subseteq V\} \), equals to \( S_j \) for a small \( \epsilon > 0 \) before terminating the recursion. The reason is to avoid missing the subsets \( S_j \)s, such that \( S_j \supseteq S_{j'} \) with the critical point \( \lambda_j \in (\lambda_{j'}, \lambda_j) \). Therefore, the value of \( \epsilon \) should be chosen sufficiently small for the validity of Algorithm 7. This is because the strong map property of \( h_\lambda \) is non-strict.

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19We denote \( r_{\lambda, i} \geq r_{\lambda, i} \) for all \( i \in V \) and \( r_{\lambda, V} > r_{\lambda, V} \) if \( r_{\lambda, i} \geq r_{\lambda, i} \) for all \( i \in V \) and at least one inequality holds strictly.

20In the case when \( j + 1 < j^* \), we could have \( \lambda = \lambda_j \) as in Lemma E.3b with \( U_\lambda V = S_{j'} \), where, if the recursion is terminated, the critical values \( \lambda_{j'} \) will not be searched.
