The special McKay correspondence and exceptional collection

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Abstract

We show that the derived category of coherent sheaves on the quotient stack of the affine plane by a finite small subgroup of the general linear group is obtained from the derived category of coherent sheaves on the minimal resolution by adding a semiorthogonal summand with a full exceptional collection.

1 Introduction

Let $G$ be a finite small subgroup of $GL_2(\mathbb{C})$ and $Y = G\text{-Hilb}(\mathbb{C}^2)$ be the Hilbert scheme of $G$-orbits \cite{Nak}. The universal flat family over $Y$ will be denoted by $\mathcal{Z}$:

\[
\begin{array}{ccc}
\mathcal{Z} & \to & \mathbb{C}^2 \\
p & \downarrow & \\
Y & \to & \mathbb{C}^2/G.
\end{array}
\]

The Hilbert-Chow morphism $\tau$ is the minimal resolution \cite{Ish}. The integral functor

\[\Phi = q_* \circ p^* : D^b \text{coh} Y \to D^b \text{coh}[\mathbb{C}^2/G]\]

is full and faithful, and its essential image is admissible \cite{BO} Definition 2.1 since $\Phi$ has both left and right adjoints. The functor $\Phi$ is an equivalence if $G$ is a subgroup of $SL_2(\mathbb{C})$ \cite{KV, BKR}. The main result in this paper is the following:

**Theorem 1.1.** Let $G$ be a finite small subgroup of $GL_2(\mathbb{C})$ and $Y = G\text{-Hilb}(\mathbb{C}^2)$ be the Hilbert scheme of $G$-orbits in $\mathbb{C}^2$. Then there is an exceptional collection $(E_1, \ldots, E_n)$ in $D^b \text{coh}[\mathbb{C}^2/G]$ and a semiorthogonal decomposition

\[D^b \text{coh}[\mathbb{C}^2/G] = \langle E_1, \ldots, E_n, \Phi(D^b \text{coh} Y) \rangle,\]

where $n$ is the number of irreducible non-special representations of $G$.

This theorem is complementary to the works of Craw \cite{Cra} and Wemyss \cite{Wem}, which describe $D^b \text{coh} Y$ as the derived category of modules over the path algebra of a quiver with relations called the special McKay quiver. Their works in turn are based on a result of Van den Bergh \cite[VdB]{VdB} Theorem B}. 

The essential image of $\Phi$ is generated by $\{O_{C^2} \otimes \rho\}_{\rho:\text{special}}$, and its right orthogonal is generated by $\{O_0 \otimes \rho\}_{\rho:\text{non-special}}$. Hence one has

$$\langle O_0 \otimes \rho \rangle_{\rho:\text{non-special}} = \langle E_1, \ldots, E_n \rangle,$$

although $\{O_0 \otimes \rho\}_{\rho:\text{non-special}}$ do not form an exceptional collection in general.

The proof of Theorem 1.1 proceeds as follows:

1. If $G \subset GL_2(C)$ is a cyclic group, the $G$-Hilbert scheme $G$-Hilb $C^2$ is a toric variety, and special representations can be computed by continued fraction expansions $[\text{Wun87}, \text{Wun88}]$. In this case, one can explicitly construct an exceptional collection $E_1, \ldots, E_n$ in coh $[C^2/G]$ as in Theorem 2.1.

2. Let $G$ be a finite small subgroup of $GL_2(C)$ and put $G_0 = G \cap SL_2(C)$. Then $G_0$ is a normal subgroup of $G$ and $A = G/G_0$ is a cyclic group. The group $A$ acts on $Y_0 = G_0$-Hilb $C^2$ and one has

$$D^b \text{coh}[C^2/G] \cong D^b \text{coh}[Y_0/A]$$

by Theorem 3.1, which is an equivariant version of the McKay correspondence $[\text{KV00}, \text{BKR01}]$. Since $Y_0$ is a resolution of $C^2/G_0$, a resolution of $Y_0/A$ is a resolution of $C^2/G$.

3. The stack $[Y_0/A]$ may have non-trivial stabilizer groups along divisors, and the canonical stack $Y_1$ associated with the coarse moduli space $Y_0/A$ is a stack which has trivial stabilizer groups except at the singular points. There is a morphism $[Y_0/A] \rightarrow Y_1$ which can be regarded as an iteration of root constructions $[\text{AGV08}, \text{Cad07}]$. Since irreducible divisors with non-trivial stabilizer groups have coarse moduli spaces isomorphic to $\mathbb{P}^1$, this yields a semiorthogonal decomposition

$$D^b \text{coh}[Y_0/A] = \langle E_1, \ldots, E_{n_1}, D^b \text{coh} Y_1 \rangle$$

by Proposition 4.4.

4. The coarse moduli space of $Y_1$ has cyclic quotient singularities. By taking the minimal resolution of it, we obtain a resolution $Y_2$ of $C^2/G$ and a semiorthogonal decomposition

$$D^b \text{coh} Y_1 = \langle E_{n_1+1}, \ldots, E_{n_2}, D^b \text{coh} Y_2 \rangle$$

by Proposition 5.1.

5. The minimal resolution $Y$ can be obtained from $Y_2$ by contracting $(-1)$-curves. This gives the desired semiorthogonal decomposition

$$D^b \text{coh} Y_2 = \langle E_{n_2+1}, \ldots, E_n, D^b \text{coh} Y \rangle$$

by Orlov $[\text{Orl92}]$.

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1 Kawamata pointed out that this case also follows from his arguments $[\text{Kaw05}, \text{Kaw06}]$. 

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Steps 1, 2 and 3 are carried out in Sections 2, 3 and 4 respectively. Theorem 1.1 and its slight generalization is proved in section 5. As a corollary, we show in Section 6 that the two-dimensional Deligne-Mumford stack associated with an invertible polynomial in four variables has a full exceptional collection.

Acknowledgment: We thank Yujiro Kawamata for the remark on Step 1 above. A. I. is supported by Grant-in-Aid for Scientific Research (No.18540034). K. U. is supported by Grant-in-Aid for Young Scientists (No.20740037).

2 The case of cyclic groups

We prove the following in this section:

**Theorem 2.1.** Let $A$ be a finite small abelian subgroup of $GL_2(\mathbb{C})$ and $Y$ be the Hilbert scheme of $A$-orbits in $\mathbb{C}^2$. Then there is an exceptional collection $(E_1, \ldots, E_n)$ in $D^b coh[C^2/A]$ and a semiorthogonal decomposition

$$D^b coh[C^2/A] = \langle E_1, \ldots, E_n, D^b coh Y \rangle,$$

where $n$ is the number of indecomposable non-special representations of $G$.

To prove Theorem 2.1, we recall Wunram's description of special representations in the case of cyclic groups. For relatively prime integers $0 < q < n$, consider the cyclic small subgroup $G = \langle \frac{1}{n}(1, q) \rangle$ of $GL_2(\mathbb{C})$ generated by

$$\frac{1}{n}(1, q) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix},$$

where $\zeta$ is a primitive $n$-th root of unity. For $a \in \mathbb{Z}/n\mathbb{Z}$, let $\rho_a$ denote the irreducible representation of $G$ so that $\rho_a$ sends the above generator to $\zeta^a$.

Define integers $r, b_1, \ldots, b_r$ and $i_0, \ldots, i_{r+1}$ as follows: Put $i_0 := n, i_1 := q$ and define $i_t+2, b_t+1$ inductively by

$$i_t = b_{t+1}i_{t+1} - i_{t+2} \quad (0 < i_{t+2} < i_{t+1})$$

until we finally obtain $i_r = 1$ and $i_{r+1} = 0$. This gives a continued fraction expansion

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_r}}}$$

and $-b_t$ is the self intersection number of the $t$-th irreducible exceptional curve $C_t$.

Special representations are described as follows:

**Theorem 2.2** (Wunram [Wun87]). *Special representations are $\rho_{i_0} = \rho_{i_{r+1}}, \rho_{i_1}, \ldots, \rho_{i_r}$.***
For an integer $d$ with $0 \leq d < n$, there is a unique expression
\[ d = d_1i_1 + d_2i_2 + \cdots + d_ri_r \tag{2.1} \]
where $d_i \in \mathbb{Z}_{\geq 0}$ are non-negative integers satisfying
\[ 0 \leq \sum_{t > t_0} d_ti_t < i_{t_0} \]
for any $t_0$.

**Lemma 2.3** (Wunram [Wun87, Lemma 1]). A sequence $(d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ is obtained from an integer $d \in [0, n - 1]$ as above if and only if the following hold:

- $0 \leq d_t \leq b_t - 1$ for any $t$.
- If $d_s = b_s - 1$ and $d_t = b_t - 1$ for $s < t$, then there is $l$ with $s < l < t$ and $d_l \leq b_l - 3$.

Let $q' \in [0, n - 1]$ be the integer with $qq' \equiv 1 \mod n$. Then $\langle \frac{1}{n} (1, q) \rangle$ coincides with $\langle \frac{1}{n} (q', 1) \rangle$ as a subgroup of $GL_2(\mathbb{C})$. Introduce the dual sequence $j_0, \ldots, j_{r+1}$ by $j_0 = 0$, $j_1 = 1$ and $j_t = j_{t-1}b_{t-1} - j_{t-2}$ for $t > 1$. Then one has $j_r = q'$ and $j_{r+1} = n$.

**Lemma 2.4** (Wunram [Wun87, Lemma 2]). Let $d = d_1i_1 + \cdots + d_rj_r$ be as in (2.1) and put $f = d_1j_1 + \cdots + d_rj_r$. Then one has $0 \leq f \leq n - 1$ and $qf \equiv d \mod n$.

Let $R = \mathbb{C}[x, y]$ be the coordinate ring of $\mathbb{C}^2$ and put
\[ R_k = R/(x, y^k). \]
For an integer $d \in [0, n - 1]$ with $\rho_d$ non-special, take $t$ with $i_{t-1} > d > i_t$. Then we define
\[ E_d = R_{j_t} \otimes \rho_{d-(j_{t-1})q}. \]
Note that the socle of $E_d$ is $\mathcal{O}_0 \otimes \rho_d$ and the representations contained in $E_d$ are $\rho_{d-lq}$ for $0 \leq l < j_t$. We show that $\{E_d \mid d:\text{non-special}\}$ is a desired exceptional collection (with respect to the order of $d \in [1, n-1]$).

We first show the following:

**Proposition 2.5.** The following two triangulated subcategories are equal:
\[ \langle \mathcal{O}_0 \otimes \rho \rangle_{\rho: \text{non-special}} = \langle E_d \rangle_{\rho_d: \text{non-special}}. \]

We introduce the following order $\preceq$ on $\mathbb{Z}/n\mathbb{Z}$: for $a, b \in \mathbb{Z}/n\mathbb{Z}$, we write $a \preceq b$ if $a' \leq b'$ holds for the representatives $a', b' \in \mathbb{Z} \cap [0, n-1]$ of $a, b$. We also write $x \preceq y$ for $x, y \in \mathbb{Z}$ if the inequality holds for their classes in $\mathbb{Z}/n\mathbb{Z}$.

**Lemma 2.6.** If $0 < l < j_t$, then one has $i_{t-1} \preceq lq$.

*Proof.* We can write $l = d_1j_1 + \cdots + d_{t-1}j_{t-1}$ as in (2.1) by using $\{j_i\}$ instead of $\{i_i\}$, where $(d_1, \ldots, d_{t-1}, 0, \ldots, 0)$ satisfies the condition in Lemma 2.3. Then we have $lq \equiv d_1i_1 + \cdots + d_{t-1}i_{t-1} \mod n$. Since $(d_1, \ldots, d_{t-1}, 0, \ldots, 0)$ satisfies the condition in Lemma 2.3 and is non-zero, $d_1i_1 + \cdots + d_{t-1}i_{t-1}$ is an integer in $[i_{t-1}, n-1]$. This implies the desired inequality. \qed
Note that the following hold by the definition of ≤.

**Lemma 2.7.** If \( b \neq 0 \), \( a + b \leq a \) implies \( a + b \leq b \).

**Corollary 2.8.** If \( i_{t-1} > d > i_t \), then we have \( d < d - lq \) for \( 0 < l < j_t \).

**Proof.** Since \( i_{t-1} \leq lq \) by Lemma 2.6, we apply Lemma 2.7 for \( a = lq \) and \( b = d - lq \) to obtain \( d \leq d - lq \). The equality does not hold since \((n,q) = 1\).

**Lemma 2.9.** If \( i_{t-1} > d > i_t \), then \( \rho_{d-lq} \) is non-special for \( 0 < l < j_t \).

**Proof.** Write \( d = d_i + d_{i+1} + \cdots + d_r \), put \( f = d_{i+j} + d_{i+j} + \cdots + d_r \). Then since \( \rho_d \) is non-special, we have \( f \geq 2j_t \).

Assume that \( \rho_{d-lq} \) is special. Then \( d - lq \equiv i_s \) for some \( s \) and the above corollary implies \( s < t \). Moreover, \( d \equiv i_s + lq \) yields \( f \equiv j_s + l \). On the other hand, since \( j_s \) and \( l \) are smaller than \( j_t \), we see \( j_s + l < 2j_t \). This contradicts \( n > f \geq 2j_t \).

**Proof of Proposition 2.5.** Lemma 2.9 implies that \( E_d \) belongs to \( \langle O_0 \otimes \rho \rangle_{\rho_{d} \text{non-special}} \). Moreover, note that the socle of \( E \) is \( O_0 \otimes \rho_f \). Then, for non-special \( \rho_f \), it follows from Corollary 2.8 and the reverse induction on \( f \) with respect to ≤ that \( O_0 \otimes \rho_f \) belongs to \( \langle E_d \rangle_{\rho_{d} \text{non-special}} \).

**Proposition 2.10.** \( \{ E_d \}_{\rho_{d} \text{non-special}} \) forms an exceptional collection.

**Proof.** Take \( E_d, E_{d'} \) with \( d' \leq d \) and suppose \( i_{t-1} > d > i_t \) and \( i_{t-1} > d' > i_{t'} \). To compute \( \text{Ext}^1(E_d, E_{d'}) \), consider the following projective resolution of \( E_d \):

\[
0 \rightarrow R \otimes \rho_{1+d+q} \xrightarrow{(y^j_t-x)} R \otimes \rho_{1+d+q-jq} \oplus R \otimes \rho_{d+q} \xrightarrow{(x,y^j_t)} R \otimes \rho_{d+q-jq} \rightarrow E_d \rightarrow 0.
\]

Then \( R \text{Hom}_R(E_d, E_{d'}) \) splits into the direct sum of

\[
R_{j_{t'}} \otimes \rho_{d'-d+(j_{t'}-j_t)q} \xrightarrow{\alpha} R_{j_{t'}} \otimes \rho_{d'-d-j_{t'}q}
\]

and

\[
R_{j_{t'}} \otimes \rho_{d'-d-1+(j_{t'}-j_t)q} \xrightarrow{\beta} R_{j_{t'}} \otimes \rho_{d'-d-1-j_{t'}q}
\]

where \( \alpha \) and \( \beta \) are the multiplications by \( y^j_t \). The degrees of terms of these complexes are determined so that \( \text{Hom}(E_d, E_{d'}) = (\ker \alpha)^G, \text{Ext}^1(E_d, E_{d'}) = (\coker \alpha)^G \oplus (\ker \beta)^G \) and \( \text{Ext}^2(E_d, E_{d'}) = (\coker \beta)^G \).

As a representation of \( G \), \( \ker \alpha \) is the direct sum of \( \rho_{d'-d-jq} \) for \( 0 \leq l < j_t \). Assume that \( \rho_{d'-d-jq} \) is trivial, i.e., \( d - d' \equiv lq \). If \( j = 0 \), then Lemma 2.6 implies \( i_{t-1} \leq lq \), which contradicts \( 0 \leq d' \leq d < i_{t-1} \) and \( d - d' \equiv lq \). Therefore, we obtain \( l = 0 \) and \( d = d' \). Thus \( (\ker \alpha)^G = 0 \) if \( d = d' \) and it is one-dimensional if \( d = d' \). \( \coker \alpha \) is the direct sum of \( \rho_{d'-d-(j_{t'-1})q} \) for \( 0 \leq l < j_{t'} \). Assume \( \rho_{d'-d-(j_{t'-1})q} \) is trivial. Then we see \( d - d' + i_{t'} \equiv lq \), which again contradicts Lemma 2.6. Hence we obtain \( (\coker \alpha)^G = 0 \). In a similar way, we can show \( (\ker \beta)^G = (\coker \beta)^G = 0 \) and we are done.

Since \( \langle O_0 \otimes \rho \rangle_{\rho_{d} \text{non-special}} \) is the right orthogonal complement of the essential image of \( \Phi \), Propositions 2.5 and 2.10 imply Theorem 2.1.
3 Equivariant McKay correspondence

Let $G$ be any finite subgroup of $GL_2(\mathbb{C})$ and put $G_0 = G \cap SL(2, \mathbb{C})$. Then $G_0$ is a normal subgroup of $G$ and $A = G/G_0$ is a cyclic group. The group $A$ acts on $Y_0 = G_0$-Hilb $\mathbb{C}^2$.

**Theorem 3.1.** There is an equivalence

$$D^b \text{coh}[\mathbb{C}^2/G] \cong D^b \text{coh}[Y_0/A]$$

**Proof.** We regard each side as the derived category of $G$-equivariant coherent sheaves on $\mathbb{C}^2$ and that of $A$-equivariant coherent sheaves on $Y_0$ respectively. Let $Z \subset Y_0 \times \mathbb{C}^2$ be the universal subscheme. Then we can define an integral functor $\Phi : D^b \text{coh}[Y_0/A] \to D^b \text{coh}[\mathbb{C}^2/G]$ by

$$\Phi(\alpha) = \pi_{\mathbb{C}^2*}(O_Z \otimes \pi_{Y_0*}(\epsilon(-)))$$

where $\pi_{\mathbb{C}^2}$ and $\pi_{Y_0}$ are projections from $Y_0 \times \mathbb{C}^2$ and $\epsilon : D^b \text{coh}[Y_0/A] \to D^b \text{coh}[Y_0/G]$ is the pull-back functor, which can be regarded as lifting $A$-actions to $G$-actions by the surjection $G \to A$. The adjoint functor $\Psi$ is defined by

$$\Psi(\alpha) = (\pi_{Y_0*}(O_Z[2] \otimes \det\rho_{\text{Nat}} \otimes \pi_{\mathbb{C}^2*}(-)))^{G_0}$$

where $\rho_{\text{Nat}}$ is the representation of $G$ given by $G \subset GL_2(\mathbb{C})$. This is both left and right adjoint since $Y_0$ is crepant. Note that if we restrict $G$-actions to $G_0$ actions and forget $A$-actions, we can also define functors $\Phi' : D^b \text{coh}Y_0 \to D^b \text{coh}[\mathbb{C}^2/G_0]$ and its adjoint $\Psi'$ in the same way as above, which are equivalences by [BKR01]. Let $\alpha$ be any object of $D^b \text{coh}[Y_0/A]$ and consider the adjunction morphism $\nu : \alpha \to \Psi\Phi(\alpha)$. If we restrict $G$-action to $G_0$-action, then the morphism $\nu$ becomes an isomorphism since $\Phi$ and $\Psi$ are equivalences. As a result, the morphism $\nu$ itself must be an isomorphism. We can also show that $\Phi\Psi(\beta) \to \beta$ is an isomorphism for any object $\beta$ of $D^b \text{coh}[\mathbb{C}^2/G]$ in the same way, and hence $\Phi$ and $\Psi$ are equivalences. \qed

4 Root constructions and semiorthogonal decompositions

In this section, we show that a smooth Deligne-Mumford stack containing divisors with non-trivial stabilizers can be replaced with another stack without such divisors by removing a semiorthogonal summand from its derived category of coherent sheaves. This operation is inverse to the root construction with respect to a line bundle with a section, defined independently in [AGV08] and [Cad07].

4.1 The root stack of a line bundle

Let $L$ be a line bundle on a Deligne-Mumford stack $X$ and $L$ be the principal $\mathbb{G}_m$-bundle on $X$ associated with $L$. For a positive integer $r$, the stack $\sqrt{rL/X}$ of the $r$-the roots of $L$ is defined as follows: An object of $\sqrt{rL/X}$ over a scheme $T$ is a triple $(\varphi, M, \phi)$ consisting of a morphism $\varphi : T \to X$, a line bundle $M$ on $T$ and an isomorphism $\phi : M^{\otimes r} \iso \varphi^*L$. Morphisms of triples are defined in the obvious way. The stack $\sqrt{rL/X}$
is an essentially trivial gerb over $X$ banded by $\mu_r$, and the natural projection is denoted by $\pi_X: \sqrt{L/X} \to X$. Let $(\mathcal{M}, \Phi)$ be the universal object on $\sqrt{L/X}$, so that $\mathcal{M}$ is a line bundle on $\sqrt{L/D}$ and $\Phi: \mathcal{M}^{\otimes r} \to \pi^* L$ is an isomorphism of line bundles.

**Lemma 4.1.** The abelian category $\text{coh} \sqrt{L/X}$ is equivalent to the direct sum of $r$ copies of $\text{coh} X$.

**Proof.** For any coherent sheaf $\mathcal{F}$ on $\sqrt{L/D}$ and any integer $i$, one has adjunction morphisms $\pi^* \pi_* \mathcal{F} \otimes M^{\otimes i} \to \mathcal{F} \otimes M^i$, whose direct sum gives the morphism

$$
\bigoplus_{i=0}^{r-1} \pi^*(\pi_*(\mathcal{F} \otimes M^{\otimes (-i)})) \otimes M^{\otimes i} \to \mathcal{F}.
$$

One can show that this morphism is an isomorphism by working on a local chart where the line bundle $L$ is trivial and the root stack is the quotient of $X$ by the trivial action of $\mu_r$. The same local consideration also shows that $\pi^* \text{coh} X \otimes M^{\otimes i}$ for $i = 0, \ldots, r - 1$ are mutually orthogonal. \qed

### 4.2 The root stack of a line bundle with a section

Let $(L, \sigma)$ be a pair of a line bundle $L \to X$ and a section $\sigma: X \to L$. The stack $\sqrt{(L, \sigma)/X}$ of the $r$-th roots of $(L, \sigma)$ is defined as follows. An object of the stack $\sqrt{(L, \sigma)/X}$ over $\varphi: T \to X$ is a triple $(M, \phi, \tau)$, where $(M, \phi)$ is an object of $\sqrt{L/X}$ over $\varphi$ and $\tau$ is a section of $M$ such that $\phi(\tau^m) = \sigma$. If $Y$ is the zero locus of $\sigma$, then the restriction of $\sqrt{(L, \sigma)/X}$ to $X \setminus Y$ is isomorphic to $X \setminus Y$, and the restriction of $\sqrt{(L, \sigma)/X}$ to $Y$ is the $r$-th infinitesimal neighborhood of $\sqrt{L/Y}$ in its universal line bundle.

Assume that $X$ is a smooth Deligne-Mumford stack and $j: D \to X$ is a closed embedding of a smooth divisor. There is a closed embedding $\sqrt{\mathcal{O}_D(D)/D} \to \sqrt{(\mathcal{O}(D), 1)/X}_D$ sending an $r$-th root $M$ of $\mathcal{O}_D(D)$ to the same $M$ together with the zero section. The composition of this morphism with the embedding $\sqrt{\mathcal{O}(D), 1/X}_D \to \sqrt{(\mathcal{O}(D), 1)/X}$ will be denoted by $j$, which fits into the commutative diagram

$$
\begin{array}{ccc}
\sqrt{\mathcal{O}_D(D)/D} & \xrightarrow{j} & \sqrt{(\mathcal{O}(D), 1)/X} \\
\pi_D \downarrow & & \downarrow \pi_X \\
D & \xrightarrow{j} & X.
\end{array}
$$

The universal line bundle on $\sqrt{(\mathcal{O}(D), 1)/X}$ will be denoted by $\mathcal{M}$.

**Lemma 4.2.** (i) The functor $j_* \pi_D^*: D^b(\text{coh} D) \to D^b(\text{coh} \sqrt{(\mathcal{O}(D), 1)/X})$ is fully faithful.

(ii) One has a semiorthogonal decomposition

$$
D^b(\text{coh} \sqrt{(\mathcal{O}(D), 1)/X}) = \langle j_* \pi_D^* D^b(\text{coh} D) \otimes \mathcal{M}^{\otimes r-1}, \ldots, j_* \pi_D^* D^b(\text{coh} D) \otimes \mathcal{M}, \pi_X^* D^b(\text{coh} X) \rangle.
$$
Proof. (i) For any objects $\alpha$ and $\beta$ of $D^b(\text{coh } D)$ and any $q \in \mathbb{Z}$, we show that the natural morphism

$$\text{Hom}^q(\alpha, \beta) \to \text{Hom}^q(j_*\pi_D^*\alpha, j_*\pi_D^*\beta) \cong \text{Hom}^q(j^*j_*\pi_D^*\alpha, \pi_D^*\beta)$$

(4.1)
is an isomorphism. We may assume that $\alpha$ and $\beta$ are sheaves. Then we have

$$H^i(j^*j_*\pi_D^*\alpha) \cong \begin{cases} \pi_D^*\alpha & i = 0, \\ \pi_D^*\alpha \otimes \mathcal{M}^{-1} & i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

(4.2)

Lemma \ref{lem:adjunction} shows that the group $\text{Hom}^q(\pi_D^*\alpha \otimes \mathcal{M}^{-1}, \pi_D^*\beta)$ vanishes and $\text{Hom}^q(\pi_D^*\alpha, \pi_D^*\beta) \cong \text{Hom}^q(\alpha, \beta)$ for any $q$, so that (4.1) is an isomorphism.

(ii) The subcategory $\pi_X^*(D^b(\text{coh } X))$ is admissible since the functor $\pi_X^*$ has both right and left adjoints. The subcategories $j_*\pi_D^*D^b(\text{coh } D) \otimes \mathcal{M}^{\otimes i}$ are also admissible since the functor $j_*\pi_D^*$ has both left and right adjoints and the functor $\bullet \otimes \mathcal{M}^{\otimes i}$ is an equivalence.

We can deduce that $j_*\pi_D^*D^b(\text{coh } D) \otimes \mathcal{M}^{\otimes i}$ are right orthogonal to $\pi_X^*(D^b(\text{coh } X))$ for $1 \leq i \leq r - 1$ from

$$\text{Hom}(\pi_X^*\alpha, j_*(\pi_D^*\beta \otimes \mathcal{M}^{\otimes i})) \cong \text{Hom}(j^*\pi_X^*\alpha, \pi_D^*\beta \otimes \mathcal{M}^{\otimes i})$$

$$\cong \text{Hom}(\pi_D^*\alpha, \pi_D^*\beta \otimes \mathcal{M}^{\otimes i}) = 0,$$

where $\bar{j} : D \to X$ is the closed immersion. Similarly, (4.2) implies

$$\text{Hom}(j_*\pi_D^*\alpha \otimes \mathcal{M}^{\otimes k}, j_*\pi_D^*\beta \otimes \mathcal{M}^{\otimes i}) = 0$$

for $1 \leq k < l \leq r - 1$.

It remains to show that any object $\mathcal{E}$ of $D^b\text{coh } \sqrt{\mathcal{O}(D)}/\mathcal{X}$ is obtained from objects of $j_*\pi_D^*(D^b\text{coh } D) \otimes \mathcal{M}^{\otimes i}$ for $1 \leq i \leq r - 1$ and $\pi_X^*D^b\text{coh } \mathcal{X}$ by taking shifts and cones. Since $\pi_X^*$ is an isomorphism outside $D$, the mapping cone $\text{Cone}(\pi_X^*\mathcal{X}, \mathcal{E} \to \mathcal{E})$ of the adjunction morphism is supported on $\sqrt{\mathcal{O}_D}/\mathcal{D}$. It follows that $\mathcal{E}$ can be obtained from $\pi_X^*\mathcal{X}, \mathcal{E}$ and an object supported on $\sqrt{\mathcal{O}_D}/\mathcal{D}$ by taking cones. An object supported on $\sqrt{\mathcal{O}_D}/\mathcal{D}$ is obtained from objects of $j_*D^b\text{coh } \sqrt{\mathcal{O}_D}/\mathcal{D}$ by taking cones, which in turn can be obtained from objects of $j_*\pi_D^*D^b(\text{coh } D) \otimes \mathcal{M}^{\otimes i}$ for $0 \leq i \leq r - 1$ by Lemma \ref{lem:adjunction}. Finally, we have to show that an object of $j_*\pi_D^*D^b(\text{coh } D)$ is obtained from objects of $\pi_X^*D^b\text{coh } \mathcal{X}$ and $j_*\pi_D^*D^b(\text{coh } D) \otimes \mathcal{M}^{\otimes i}$ for $1 \leq i \leq r - 1$. If $\alpha$ is a sheaf in $D^b(\text{coh } D)$, then $\pi_X^*j_*\alpha$ has a filtration whose factors are $j_*\pi_D^*\alpha \otimes \mathcal{M}^{\otimes i}$ for $0 \leq i \leq r - 1$. Thus $j_*\pi_D^*\alpha$ is obtained from $\pi_X^*j_*\alpha$ and $j_*\pi_D^*\alpha \otimes \mathcal{M}^{\otimes i}$ for $1 \leq i \leq r - 1$ by taking shifts and cones. This concludes the proof of Lemma \ref{lem:adjunction}.

Corollary 4.3. If both $\mathcal{X}$ and $\mathcal{D}$ have full exceptional collections, then so does the root stack $\sqrt{\mathcal{O}(D), 1}/\mathcal{X}$.

4.3 Iteration of root constructions

Let $X$ be a variety with at worst quotient singularity and $X^{\text{can}}$ be its canonical stack \cite[Section 4]{FMN07}. For prime divisors $D_1, \ldots, D_s$ on $X^{\text{can}}$ and positive integers $r_1, \ldots, r_s$,
the fiber product
\[ X = \sqrt[\text{can}]{(\mathcal{O}(D_1), 1)} \times_{X_{\text{can}}} \cdots \times_{X_{\text{can}}} \sqrt[\text{can}]{(\mathcal{O}(D_s), 1)} \]
is obtained by iterated root constructions as in [Cad07]. If \( \sum D_i \) is a simple normal crossing divisor, then \( X \) is also a smooth Deligne-Mumford stack. The stack \( X \) is characterized by the properties that it has the same coarse moduli space as \( X_{\text{can}} \), it is isomorphic to \( X_{\text{can}} \) outside \( \cup D_i \) and the pull back of \( D_i \) is \( r_i \) times a prime divisor for each \( i \).

Let \( D_1, \ldots, D_s \) be the prime divisors on \( X \) corresponding to \( D_1, \ldots, D_s \).

Proposition 4.4. If \( D_i \cong \mathbb{P}^1 \) for each \( i \) and \( \sum D_i \) is a simple normal crossing divisor, then there exist an exceptional collection \((E_1, \ldots, E_\ell)\) and a semiorthogonal decomposition
\[ D^b \text{coh} X = \langle E_1, \ldots, E_\ell, \pi^* D^b \text{coh} X_{\text{can}} \rangle. \] (4.3)

Proof. Put
\[ X_1 = \sqrt[\text{can}]{(\mathcal{O}(D_2), 1)} \times_{X_{\text{can}}} \cdots \times_{X_{\text{can}}} \sqrt[\text{can}]{(\mathcal{O}(D_s), 1)} \]
and let \( D \subset X_1 \) be the prime divisor corresponding to \( D_1 \). Then \( X \) is isomorphic to \( X_{\text{can}} \) and one has a semiorthogonal decomposition
\[ D^b \text{coh} X = \langle j_* \pi_1^* D^b (\text{coh} D) \otimes \mathcal{M}^\text{can}, \ldots, j_* \pi_1^* D^b (\text{coh} D) \otimes \mathcal{M}, \pi_1^* D^b \text{coh} X_1 \rangle \]
by Lemma 4.2. Since \( D \) is a smooth divisor whose coarse moduli space is isomorphic to \( D_1 \cong \mathbb{P}^1 \), the derived category \( D^b \text{coh} D \) and hence the right orthogonal to \( \pi^* D^b \text{coh} X_1 \) in \( D^b \text{coh} X \) has a full exceptional collection [GL87]. Now the assertion follows from induction on \( s \).

\[ \square \]

5 Semiorthogonal decomposition for the canonical stack

Let \( X \) be a surface with at worst quotient singularities and consider the diagram
\[ \begin{array}{ccc}
Z & \overset{q}{\longrightarrow} & X \\
\downarrow p & & \downarrow \pi \\
Y & \overset{\tau}{\longrightarrow} & X
\end{array} \]
where \( X \) is the canonical stack associated with \( X \), \( \tau : Y \to X \) is the minimal resolution, and \( Z \) is the reduced part of the fiber product \( Y \times_X X \). We consider the integral functor
\[ \Phi := q_* \circ p^* : D^b (\text{coh} Y) \to D^b (\text{coh} X), \]
whose right adjoint will be denoted by \( \Psi \).

Proposition 5.1. Assume \( X \) has only cyclic quotient singularities. Then \( \Phi \) is fully faithful and there is a semiorthogonal decomposition
\[ D^b \text{coh} X = \langle E_1, \ldots, E_\ell, \Phi (D^b \text{coh} Y) \rangle \]
where \( E_1, \ldots, E_\ell \) is an exceptional collection.
Proof. If \( X \) is the global quotient \( \mathbb{C}^2/G \) for a finite small subgroup \( G \) of \( GL_2(\mathbb{C}) \), the proof of [Ish02, Theorem 3.1] shows that \( Z \) is the quotient stack of the universal subscheme in \( Y \times \mathbb{C}^2 \) by the action of \( G \) under the identification of \( Y \) with \( G\text{-Hilb}(\mathbb{C}^2) \). Hence the assertion in this case follows from Theorem 2.1.

In the general case, the composition \( \Psi \circ \Phi \) is an integral functor with respect to some kernel object \( \mathcal{P} \) on \( Y \times Y \). By the local case above, \( \mathcal{P} \) is etale locally the structure sheaf of the diagonal. Hence the kernel of \( \Psi \circ \Phi \) is a line bundle on the diagonal, which implies that \( \Phi \) is fully faithful. Since the singularities of \( X \) are isolated, the semiorthogonal decomposition comes from local contributions around each singular points, where the assertion holds by the global quotient case above. \( \square \)

This yields Step 4 in Introduction, completing the proof of Theorem 1.1. By replacing Theorem 2.1 with Theorem 1.1 in the proof of Proposition 5.1, one obtains the following:

**Theorem 5.2.** Let \( \mathcal{X} \) be the canonical stack associated with a surface \( X \) with at worst quotient singularities, and \( Y \) be the minimal resolution of \( X \). Then there is a full and faithful functor
\[
\Phi : D^b \text{coh} Y \to D^b \text{coh} \mathcal{X}
\]
and a semiorthogonal decomposition
\[
D^b \text{coh} \mathcal{X} = \langle E_1, \ldots, E_\ell, \Phi(D^b \text{coh} Y) \rangle
\]
where \( E_1, \ldots, E_\ell \) is an exceptional collection.

This is a global analogue of Theorem 1.1.

### 6 Invertible polynomials

Let \( n \) be a positive integer. An integer \( n \times n \)-matrix \( A = (a_{ij})_{i,j=1}^n \) with non-zero determinant gives a polynomial \( W \in \mathbb{C}[x_1, \ldots, x_n] \) by
\[
W = \sum_{i=1}^n x_1^{a_{i1}} \cdots x_n^{a_{in}}.
\]
Non-zero coefficients of \( W \) can be absorbed by rescaling \( x_i \). A polynomial obtained in this way is called an invertible polynomial if it has an isolated critical point at the origin. The quotient ring \( R = \mathbb{C}[x_1, \ldots, x_n]/(W) \) is naturally graded by the abelian group \( L \) generated by \( n+1 \) elements \( \vec{x}_i \) and \( \vec{c} \) with relations
\[
a_{i1} \vec{x}_1 + \cdots + a_{in} \vec{x}_n = \vec{c}, \quad i = 1, \ldots, n.
\]
The abelian group \( L \) is the group of characters of \( K \) defined by
\[
K = \{ (\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}} = \cdots = \alpha_1^{a_{nn}} \cdots \alpha_n^{a_{nn}} \}.
\]
The group \( G_{\text{max}} \) of maximal diagonal symmetries is defined as the kernel of the map
\[
\begin{align*}
K \cup & \to \mathbb{C}^\times \\
(\alpha_1, \ldots, \alpha_n) & \mapsto \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}}.
\end{align*}
\]
so that there is an exact sequence

\[ 1 \to G_{\text{max}} \to K \to \mathbb{C}^\times \to 1 \]

of abelian groups. Let

\[ \mathcal{X} = \left[ (W^{-1}(0) \setminus \{0\})/K \right] \]

be the quotient stack of \( W^{-1}(0) \setminus \{0\} \) by the natural action of \( K \). It is a smooth Deligne-Mumford stack since \( W \) has an isolated critical point at the origin and the action of \( K \) at any point in \( W^{-1}(0) \setminus \{0\} \) has a finite isotropy group.

**Lemma 6.1.** The coarse moduli space of \( \mathcal{X} \) is a rational variety. Moreover, each codimension one irreducible component of the locus where \( \mathcal{X} \) has non-trivial stabilizers is also rational and these components form a simple normal crossing divisor.

**Proof.** Since the \( K \)-action on \((\mathbb{C}^\times)^n\) is free, the open dense substack

\[ \mathcal{U} = \left[ (W^{-1}(0) \cap (\mathbb{C}^\times)^n)/K \right] \]

of \( \mathcal{X} \) is a scheme, which is an affine linear subspace of

\[ \left[ (\mathbb{C}^\times)^n/K \right] \cong (\mathbb{C}^\times)^{n-1} \]

considered as an open subscheme of \( \mathbb{C}^{n-1} \). This shows that \( \mathcal{X} \) is rational. A divisor with a non-trivial generic stabilizer is the closure of either

\[ W^{-1}(0) \cap \{x_i = 0\} \cap \{x_k \neq 0 \text{ for } k \neq i\} \]

for some \( i \) or

\[ \{x_i = x_j = 0\} \cap \{x_k \neq 0 \text{ for } k \neq i, j\} \]

for some \( i \neq j \). (If \( \{x_i = x_j = 0\} \) is not contained in \( W^{-1}(0) \), then \( W^{-1}(0) \cap \{x_i = x_j = 0\} \) has codimension greater than one.) The quotient of the former also contains an affine subspace of a tours, and the quotient of the latter is a toric stack. Hence they are rational.

Since the stabilizer group of any point on \( \mathcal{X} \) are abelian and therefore locally diagonalizable, the union of such divisors has normal crossings. Moreover, at each point on the union, different local components have different stabilizer subgroups in \( K \). Hence the union has simple normal crossings. \( \square \)

Now assume \( n = 4 \) so that \( \dim \mathcal{X} = 2 \) and let \( Y \) be the minimal resolution of the coarse moduli space of \( \mathcal{X} \). Since \( Y \) is a rational surface, one has a full exceptional collection on \( Y \) by Orlov [Orl92]. Let \( \mathcal{X}^{\text{can}} \) be the canonical stack associated with the coarse moduli space of \( \mathcal{X} \). Then Theorem 5.2 gives a full exceptional collection on \( \mathcal{X}^{\text{can}} \). Since \( \mathcal{X} \) can be obtained by successive root constructions from \( \mathcal{X}^{\text{can}} \), Proposition 4.4 and Lemma 6.1 give the following:

**Corollary 6.2.** The two-dimensional Deligne-Mumford stack associated with an invertible polynomial in four variables has a full exceptional collection.
References

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398. MR 2450211 (2009k:14108)

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554 (electronic). MR MR1824990 (2002f:14023)

[BO] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, arXiv:alg-geom/9506012.

[Cad07] Charles Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427. MR 2306040 (2008g:14016)

[Cra] Alastair Craw, *The special mckay correspondence as an equivalence of derived categories*, arXiv:0704.3627.

[FMN07] Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric DM stacks*, arXiv:0708.1254, 2007.

[GL87] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297. MR MR915180 (89b:14049)

[Ish02] Akira Ishii, *On the McKay correspondence for a finite small subgroup of GL(2, C)*, J. Reine Angew. Math. **549** (2002), 221–233. MR MR1916656 (2003d:14021)

[Kaw05] Yujiro Kawamata, *Log crepant birational maps and derived categories*, J. Math. Sci. Univ. Tokyo **12** (2005), no. 2, 211–231. MR MR2150737 (2006a:14021)

[Kaw06] , *Derived categories of toric varieties*, Michigan Math. J. **54** (2006), no. 3, 517–535. MR MR2280493 (2008d:14079)

[KV00] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. **316** (2000), no. 3, 565–576. MR MR1752785 (2001h:14012)

[Nak01] Iku Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic Geom. **10** (2001), no. 4, 757–779. MR MR1838978 (2002d:14006)

[Ori92] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 4, 852–862. MR MR1208153 (94e:14024)

[VdB04] Michel Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. **122** (2004), no. 3, 423–455. MR MR2057015 (2005e:14023)
[Wem] Michael Wemyss, *The GL(2) McKay correspondence*, arXiv:0809.1973.

[Wun87] J. Wunram, *Reflexive modules on cyclic quotient surface singularities*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 221–231. MR MR915177 (88m:14023)

[Wun88] Jürgen Wunram, *Reflexive modules on quotient surface singularities*, Math. Ann. 279 (1988), no. 4, 583–598. MR MR926422 (89g:14029)

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