Selberg’s central limit theorem for quadratic Dirichlet L-functions over function fields

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Received: 24 May 2022 / Accepted: 4 March 2023 / Published online: 4 April 2023
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Abstract
In this article, we study the logarithm of the central value $L\left(\frac{1}{2}, \chi_D\right)$ in the symplectic family of Dirichlet $L$-functions associated with the hyperelliptic curve of genus $g$ over a fixed finite field $\mathbb{F}_q$ in the limit as $g \to \infty$. Unconditionally, we show that the distribution of $\log|L\left(\frac{1}{2}, \chi_D\right)|$ is asymptotically bounded above by the full Gaussian distribution of mean $\frac{1}{2} \log \deg(D)$ and variance $\log \deg(D)$, and also $\log|L\left(\frac{1}{2}, \chi_D\right)|$ is at least 94.27% Gaussian distributed. Assuming a mild condition on the distribution of the low-lying zeros in this family, we obtain the full Gaussian distribution.

Keywords Finite fields · Function fields · Central limit theorem · Hyperelliptic curves

Mathematics Subject Classification 11G20 · 11R29

1 Introduction
The study of the analytic properties of $L$-functions is an integral part of modern number theory due to their myriad of connections to other objects of interest. Of particular interest is trying to understand the distribution of values these $L$-functions take at various points within the critical strip, especially values along the central line $\Re(s) = \frac{1}{2}$. Perhaps the most significant unconditional result in this vein is Selberg’s central limit Theorem [25], which states that, for $T$ sufficiently large as $t$ varies in

Communicated by Tim Browning.

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the real and imaginary parts of $\log \zeta(\frac{1}{2} + it)$ are normally distributed with mean 0 and variance $\frac{1}{4} \log \log T$. Following this result, a great deal of effort has gone into generalizing this to other $L$-functions. Indeed, Selberg himself proved that for a fixed $t$, the imaginary part of $\log L(\frac{1}{2} + it, \chi)$ becomes normally distributed as $\chi$ varies among Dirichlet characters of a large prime modulus $q$. Under some widely believed conditional assumptions, Bombieri and Hejhal were able to prove similar results for a fairly general class of $L$-functions, also in the $t$-aspect. This work has been made unconditional in at least one case by Wenzhi Luo.

Recently, Radziwill and Soundararajan provided another proof of Selberg’s original result that was much shorter and easier to digest. This proof was adapted by Hsu and Wong to treat the case of $\log |L(\frac{1}{2} + it, \chi)|$ as $t$ varies in $[T, 2T]$. Within this work, they also make explicit a notion of independence between primitive Dirichlet $L$-functions predicted by Selberg. That is, they prove that for a sequence of distinct primitive Dirichlet characters and $T$ sufficiently large, as $t$ varies in $[T, 2T]$ the vector $(\log |L(\frac{1}{2} + it, \chi_1)|, \ldots, \log |L(\frac{1}{2} + it, \chi_n)|)$ becomes an $n$-variate normal distribution with mean vector $0_n$ and covariance matrix $\frac{1}{2}(\log log T)I_n$.

The above examples have something in common, which Katz and Sarnak first illuminated in their groundbreaking work. From their work we learned that the central values, $L(\frac{1}{2} + it, f)$, of an $L$-function belong in a family with symmetry type governed by the classical compact groups. In the above, the families of $L$-functions described correspond to unitary families, $U(N)$. In it was proposed that although zeros high up on the critical line follow the statistics of the unitary family that the low-lying zeros of specific families may follow the statistics of the orthogonal or symplectic matrix groups instead. This idea was demonstrated by Katz and Sarnak finding zeta-functions over function fields whose zero statistics exhibited such behaviours. Keating and Snaith made some “Selberg-type” conjectures for the orthogonal and symplectic families based on calculations from random matrix theory. These conjectures appear to be unreachable at the moment since they involve the real part of the logarithm of $L$-functions at the central point $s = \frac{1}{2}$. At present, even the best methods in analytic number theory cannot guarantee more than a positive proportion of $L$-functions in any given family is non-zero at a single point. Indeed, even if we can show that the central value is non-negative, the logarithm is still highly sensitive to zeros close to the central point (i.e. the low-lying zeros described in [13]). So what can be said about these families?

In the orthogonal case, two different approaches lead to partial results. Radziwill and Soundararajan consider $L$-functions of elliptic curves twisted by quadratic characters. They obtain a “one-sided” central limit theorem which proves one-half of the Keating and Snaith conjecture. They prove that the distribution of the values at the central point is at most Gaussian. Their method does not rely on zero density information for the family, making it flexible. Hough showed the same is true for weight $k$ cusp forms for $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ using Selberg’s approach, that is, it does rely on zero density information. Introducing a weak condition, Hough proves a conditional version of the Keating–Snaith conjecture for this same family.

In the symplectic case, we again look to Hough. Using the method of Selberg, Hough proves unconditionally that $L(\frac{1}{2}, \chi_{8d})$ for $d > 0$ fundamental discriminants...
Selberg’s central limit theorem for quadratic Dirichlet has at most a Gaussian distribution. Introducing a weak condition, Hough proves a conditional version of the Keating–Snaith conjecture for this same family.

In this paper, we prove a Selberg-type theorem in hyperelliptic ensemble over function fields, $F_q(t)$, where $q$ is a fixed odd prime power, thus, going back to Katz and Sarnak’s original idea. Let $D \in F_q[t]$ be a monic square-free polynomial. We define the primitive quadratic character associated to $D$ as the Kronecker symbol, $(\frac{D}{-})$, see Sect. 2 for more details.

There are three aspects in function fields: (i) $n$ is fixed and $q$ varies which we call “large finite field” aspect; (ii) $q$ fixed and $n$ varies which we call “large degree” aspect; (iii) both $n$ and $q$ varies which we call the double limit aspect. Our goal is to explore the “large degree” aspect. For the geometrical significance of these aspects see the Sect. 2.1.

We define hyperelliptic ensemble $\mathcal{H}_{n,q}$ or simply $\mathcal{H}_n$ as

$$\mathcal{H}_n = \{ D \in F_q[t] : D \text{ is monic, square free, and } \deg(D) = n \}.$$ 

For each $D$ in the Hyperelliptic ensemble $\mathcal{H}_n$, there is an associated hyperelliptic curve given by $C_D : y^2 = D(t)$. These curves are non-singular and of genus $g$ given by

$$2g = n - 1 - \lambda, \tag{1.1}$$

where

$$\lambda = \begin{cases} 1, & \text{if } \ n \ \text{even}, \\ 0, & \text{if } \ n \ \text{odd}. \end{cases}$$

Note that, $g \to \infty$ as $n$ does so.

In particular, we prove results for $\log |L(1/2, \chi_D)|$ in the “large degree” aspect, as such we suppress the $q$ in the notation of the set, that is $\mathcal{H}_{n,q} = \mathcal{H}_n$. For information in the remaining ranges of $\sigma \in (1/2, 1]$ for $|L(\sigma, \chi_D)|$, see work of the second author [17] and [18]. In this setting, Weil [29] has proven the Riemann hypothesis, so one might hope to provide an unconditional result of this flavour. However, as we will see, there are other obstructions towards this end.

We are able to prove the following unconditional result which shows that if we are very near the $\tfrac{1}{2}$-line then $\log |L(\tfrac{1}{2} + \sigma_0, \chi_D)|$ is normally distributed with mean $\frac{1}{2} \log d(D)$ and variance $\log d(D)$, where $d(D)$ is the degree of the polynomial $D$.

**Theorem 1.1** Let $g$ be genus of the hyperelliptic curve defined by (1.1) and $\sigma_0 = \sigma_0(g)$ be a function of $g$, tending to zero as $g \to \infty$ in such a way that $g\sigma_0 \to \infty$ but $g\sigma_0 = o\left(\sqrt{\log n}\right)$. For $D \in \mathcal{H}_n$, we consider

$$A(D) = \frac{1}{\sqrt{\log n}} \left( \log |L\left(\frac{1}{2} + \sigma_0, \chi_D\right)| - \frac{1}{2} \log n \right).$$
Then, as \( n \to \infty \)
\[
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \Delta_{A(D)} \to N(0, 1),
\]
where \( \Delta_x \) is the point mass at \( x \) and \( N(0, 1) \) is the standard normal distribution.

Theorem 1.1 leads us to the following corollary, which is an unconditional proof of “one-half” of the Keating-Snaith conjectures. That is, we show that \( \log |L(\frac{1}{2}, \chi_D)| \) has at most a Gaussian distribution.

Corollary 1.2 Let \( Z \) be a real number. As \( n \to \infty \), we have
\[
P \left( D \in \mathcal{H}_n : \frac{1}{\sqrt{\log n}} \left( \log |L\left(\frac{1}{2}, \chi_D\right)| - \frac{1}{2} \log n \right) > Z \right) \leq \frac{1}{\sqrt{2\pi}} \int_{Z}^{\infty} e^{-\frac{t^2}{2}} dt + o_Z(1).
\]

Of course, we would like to have a full result about the distribution for \( \log |L(\frac{1}{2}, \chi_D)| \) as \( D \) varies over \( \mathcal{H}_n \), we can at best provide a conditional result, and this comes from some uncertainty due to a conjecture of Chowla. Chowla’s conjecture presented in [7] asserts that all \( L \)-functions associated with quadratic characters do not vanish at the central point \( s = \frac{1}{2} \). The analogue of Chowla’s conjecture over function fields has been disproven by Li [16, Theorem 1.3]. More precisely, she finds infinitely many quadratic characters \( \chi_D \) such that \( L(\frac{1}{2}, \chi_D) = 0 \) and provides a lower bound for the number of these counterexamples. Based on the size of these lower bounds, it is still possible that Chowla’s conjecture holds for almost all quadratic characters, that is for 100% of quadratic characters. Indeed, Bui and Florea [4] proved that 94.29% of \( D \in \mathcal{H}_{2g+1} \) satisfy \( L(\frac{1}{2}, \chi_D) \neq 0 \) as \( g \to \infty \).

Theorem 1.3 Let \( Z \) be a real number. As \( n \to \infty \), we have
\[
P \left( D \in \mathcal{H}_n : \frac{1}{\sqrt{\log n}} \left( \log |L\left(\frac{1}{2}, \chi_D\right)| - \frac{1}{2} \log n \right) > Z \right) \geq \frac{19 - \cot(1/4)}{16\sqrt{2\pi}} \int_{Z}^{\infty} e^{-\frac{t^2}{2}} dt + o_Z(1).
\]

We now make a hypothesis about the distribution of low-lying zeros which is quite mild in comparison to what is expected to hold. The justification for this hypothesis is given in Sect. 2.4.

Hypothesis 1 (Low-lying Zero Hypothesis) Let \( \theta_{j,D} \) be the eigenphases associated to (2.5). If \( y = y(g) \to \infty \) then as \( g \to \infty \) we obtain
\[
\frac{1}{|\mathcal{H}_n|} \left\{ D \in \mathcal{H}_n : \min_j |\theta_{j,D}| < \frac{1}{yg} \right\} = o(1),
\]
where \( g \) is defined by (1.1).

From here we obtain:
Theorem 1.4  Suppose that the Low-lying Zero Hypothesis holds for \( \{L(s, \chi_D)\}_{D \in \mathcal{H}_n} \). For \( D \in \mathcal{H}_n \), we consider
\[
\tilde{A}(D) = \frac{1}{\sqrt{\log n}} \left( \log |L\left(\frac{1}{2}, \chi_D\right)| - \frac{1}{2} \log n \right).
\]

Then we have
\[
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \Delta \tilde{A}(D) \to N(0, 1), \quad n \to \infty.
\]

Remark  Note that we have a uniform statement about the distribution of \( \log |L(\frac{1}{2}, \chi_D)| \) regardless of whether \( n = 2g + 1 \) or \( n = 2g + 2 \).

1.1 Plan of the paper

In Sect. 2, we will give necessary background for studying the question over function fields. In Sect. 3, we compile a list of useful lemmas to complete the proofs. In Sect. 4, we will develop an expression for \( \log L(\frac{1}{2} + \sigma_0, \chi_D) \) which makes it easier to do the necessary moment calculations. In Sect. 5, we prove Theorem 1.1. In Sect. 6, we discuss the discrepancy between \( \log |L(\frac{1}{2}, \chi_D)| \) and \( \log |L(\frac{1}{2} + \sigma_0), \chi_D)| \) and complete the proof of Theorem 1.4. In the final section we prove Corollary 1.2.

2 Background for \( L \)-functions over function fields

We begin by fixing some notation which will be used throughout the paper. We will use [22] as a general reference.

2.1 Notations and basics

Let \( q = p^e \), for \( p \) a fixed odd prime and \( e \geq 1 \) an integer. Then let \( \mathbb{F}_q \) be the finite field with \( q \) elements. The polynomial ring \( \mathbb{F}_q[t] \) has many things in common with the integers including satisfying a “prime number theorem” for its monic irreducible polynomials. For \( f \) in \( \mathbb{F}_q[t] \) we denote the degree of the polynomial as \( d(f) \) or \( \deg(f) \).

The norm of a polynomial \( f \in \mathbb{F}_q[t] \) is, for \( f \neq 0 \), define to be \( |f| = q^{d(f)} \) and for \( f = 0, |f| = 0 \).

We define
\[
\mathcal{M}_n = \{f \in \mathbb{F}_q[t] : f \text{ is monic and } d(f) = n\}, \quad \mathcal{M}_{\leq n} = \bigcup_{j \leq n} \mathcal{M}_j,
\]
\[
\mathcal{P}_n = \{f \in \mathbb{F}_q[t] : f \text{ is monic, irreducible and } d(f) = n\}, \quad \mathcal{P}_{\leq n} = \bigcup_{j \leq n} \mathcal{P}_j.
\]
and

$$\mathcal{H}_n = \{ f \in \mathbb{F}_q[t] : f \text{ is monic, square-free and } \deg(f) = n \}.$$ 

Observe that $|\mathcal{M}_n| = q^n$ and for $n \geq 1$, $|\mathcal{H}_n| = q^{n-1}(q - 1)$. Let $\Lambda(f)$ be the analogue of the Von Mangoldt function:

$$\Lambda(f) = \begin{cases} 
\deg P & \text{if } f = P^k, P \in \mathcal{P}, \\
0 & \text{otherwise}.
\end{cases}$$

The prime polynomial Theorem (see [22], Theorem 2.2) states that

$$|\mathcal{P}_n| = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right). \quad (2.1)$$

The zeta function of $A = \mathbb{F}_q[t]$, denoted by $\zeta_A(s)$, is defined by

$$\zeta_A(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_{P \in \mathcal{P}} \frac{1}{1 - |P|^{-s}}^{-1},$$

for $\Re(s) > 1$. One can show that $\zeta_A(s) = \frac{1}{1 - q^{-s}}$, and this provides an analytic continuation of zeta function to the complex plane with a simple pole at $s = 1$. Using the change of variable $u = q^{-s}$,

$$\zeta_A(s) = \sum_{f \in \mathcal{M}} u^{d(f)} = \frac{1}{1 - qu}, \quad \text{if } |u| < \frac{1}{q}.$$ 

### 2.2 Quadratic Dirichlet characters and their $L$-functions

Let $P$ be a monic irreducible polynomial, we define the quadratic character \( \left( \frac{f}{P} \right) \) by

$$\left( \frac{f}{P} \right) = \begin{cases} 
1 & \text{if } f \text{ is a square (mod } P) \text{, } P \nmid f \\
-1 & \text{if } f \text{ is not a square (mod } P) \text{, } P \nmid f \\
0 & \text{if } P \mid f.
\end{cases}$$

We extend this definition to any $D \in \mathbb{F}_q[t]$ multiplicatively and define the quadratic character $\chi_D(f)$ as $\left( \frac{D}{f} \right)$. Given any character one can define an $L$-function associated to it:

$$L(s, \chi_D) = \sum_{f \in \mathcal{M}} \frac{\chi_D(f)}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \chi_D(P) |P|^{-s}\right)^{-1}, \quad \Re(s) > 1.$$
Using the change of variable $u = q^{-s}$, we have
\[
\mathcal{L}(u, \chi_D) = \sum_{f \in \mathcal{M}} \chi_D(f) u^{d(f)} = \prod_{P \in \mathcal{P}} \left(1 - \chi_D(P) u^{d(P)}\right)^{-1}, \quad |u| < \frac{1}{q}. \tag{2.2}
\]
Observe that if $n \geq d(D)$, then
\[
\sum_{f \in \mathcal{M}_n} \chi_D(f) = 0.
\]
Thus we have $\mathcal{L}(u, \chi_D)$ is a polynomial of degree at most $d(D) - 1$. From here on, we consider $D$ as a monic, square-free polynomial. Then $\mathcal{L}(u, \chi_D)$ has a trivial zero at $u = 1$ if and only if $d(D)$ is even. Thus
\[
L(s, \chi_D) = \mathcal{L}(u, \chi_D) = (1 - u)^\lambda \mathcal{L}^*(u, \chi_D) = (1 - q^{-s})^\lambda L^*(s, \chi_D), \tag{2.3}
\]
where
\[
\lambda = \begin{cases} 
1 & \text{if } d(D) \text{ even} \\
0 & \text{if } d(D) \text{ odd},
\end{cases} \tag{2.4}
\]
and $\mathcal{L}^*(u, \chi_D)$ is a polynomial of degree
\[
2g = d(D) - 1 - \lambda,
\]
satisfying the functional equation
\[
\mathcal{L}^*(u, \chi_D) = (qu^2)^g \mathcal{L}^* \left(\frac{1}{qu}, \chi_D\right).
\]
Because $\mathcal{L}$ and $\mathcal{L}^*$ are polynomial in $u$, it is convenient to define
\[
L^*(s, \chi_D) = \mathcal{L}^*(u, \chi_D)
\]
so that the above functional equation can be rewritten as
\[
L^*(s, \chi_D) = q^{(1-2s)g} L^*(1-s, \chi_D).
\]
Additionally, since $\mathcal{L}^*$ is a polynomial, it can be written as a product of its zeros:
\[
\mathcal{L}^*(u, \chi_D) = \prod_{j=1}^{2g} \left(1 - u \sqrt{q} \alpha_j\right), \tag{2.5}
\]
where $\alpha_j = e(-\theta_j, D)$ are the reciprocals of $u_j = q^{-\frac{1}{2}} e(\theta_j, D)$, the roots of $\mathcal{L}^*$. We call $\theta_j, D$ the eigenphases of the polynomial and are described in more detail in Sect. 2.4.
The Riemann hypothesis, proved by Weil [29] is that the zeros of $L^*(u, \chi_D)$ all lie on the circle $|u| = q^{-\frac{1}{2}}$.

We define the completed $L$-function in the following way. Set $X_D(s) = |D|^{\frac{1}{2} - s} X(s)$, where

$$X(s) = \begin{cases} q^{s-\frac{1}{2}} & \text{if } d(D) \text{ odd} \\ \frac{1-q^{-s}}{1-q^{-(1-s)}} q^{-1+2s} & \text{if } d(D) \text{ even} \end{cases}$$

Let us consider

$$\Lambda(s, \chi_D) = L(s, \chi_D) X_D(s)^{-\frac{1}{2}}.$$

Then the above completed $L$-function satisfies the symmetric functional equation

$$\Lambda(s, \chi_D) = \Lambda(1 - s, \chi_D).$$

### 2.3 The logarithmic derivative

By taking logarithmic derivative of (2.2) and (2.3) we obtain

$$\frac{L'}{L}(u, \chi_D) = \sum_{n=1}^{\infty} \left( \sum_{f \in \mathcal{M}_n} \Lambda(f) \chi_D(f) \right) u^{n-1}$$

and

$$\frac{L'}{L}(u, \chi_D) = -\lambda (1-u)^{-1} - \sum_{n=1}^{\infty} \left( \sum_{j=1}^{\infty} (\sqrt{q} \alpha_j)^n \right) u^{n-1},$$

where $\lambda$ is defined by (2.4). Equating these two expressions, we obtain

$$\sum_{f \in \mathcal{M}_n} \Lambda(f) \chi_D(f) = -\lambda - \sum_{j=1}^{2g} (\sqrt{q} \alpha_j)^n.$$

We may also express the logarithmic derivative in two ways, one in terms of the zeros of $L(s, \chi_D)$ as

$$L'(s, \chi_D) = \log q \left( \frac{\lambda q^{-s}}{1-q^{-s}} + \sum_{j=1}^{2g} \frac{\alpha_j q^{\frac{1}{2} - s}}{1-\alpha_j q^{\frac{1}{2} - s}} \right)$$

(2.6)
Selberg’s central limit theorem for quadratic Dirichlet

and the other in terms of the Dirichlet series as

$$
\frac{L'(s, \chi_D)}{L(s, \chi_D)} = -\log q \sum_{f \in \mathcal{M}} \frac{\Lambda(f) \chi_D(f)}{|f|^s}.
$$

(2.7)

2.4 Spectral Interpretation

Let $C$ be a non-singular projective curve over $\mathbb{F}_q$ of genus $g$. For each extension field of degree $n$ of $\mathbb{F}_q$, denote by $N_n(C)$ the number of points of $C$ in $\mathbb{F}_{q^n}$. Then, the zeta function associated to $C$ defined as

$$
Z_C(u) = \exp \left( \sum_{n=1}^{\infty} N_n(C) \frac{u^n}{n} \right), \quad |u| < \frac{1}{\sqrt[q]{q}},
$$
is known to be a rational function of $u$ of the form

$$
Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)},
$$

Additionally, we know, $P_C(u)$ is a polynomial of degree $2g$ with integer coefficients, satisfying a functional equation

$$
P_C(u) = (qu^2)^g P_C \left( \frac{1}{qu} \right),
$$

where $g$ is defined as in (1.1).

The Riemann hypothesis, proved by Weil [29], says that the zeros of $P_C(u)$ all lie on the circle $|u| = \frac{1}{\sqrt[q]{q}}$. Thus one may give a spectral interpretation of $P_C(u)$ as the characteristic polynomial of a $2g \times 2g$ unitary matrix $\Theta_C$:

$$
P_C(u) = \det (I - u \sqrt[q]{q} \Theta_C).
$$

Thus the eigenvalues $e^{i\theta_j}$ of $\Theta_C$ correspond to the zeros, $q^{-1/2} e^{-i\theta_j}$, of $Z_C(u)$. The matrix $\Theta_C$ is called the unitarized Frobenius class of $C$.

To put this in the context of our case, note that, for a family of hyperelliptic curves $C_D : y^2 = D(x)$ of genus $g$, the numerator of the zeta function $Z_C(u)$ associated to $C_D$ is coincide with the $L$-function $\mathcal{L}^*(u, \chi_D)$, i.e., $P_C(u) = \mathcal{L}^*(u, \chi_D)$.

It is an interesting problem to study how the Frobenius classes $\Theta_C$ change as we vary the associated curve over a family of hyperelliptic curves with genus $g$. As mentioned in the introduction, there are three aspects where we can study the distribution of these Frobenius classes:

(i) Large finite field aspect. Katz and Sarnak [13] showed that that as $q \to \infty$, the Frobenius classes $\Theta_D$ become equidistributed in the unitary symplectic group.
for any continuous function $F$ on the space of conjugacy classes of $USp(2g)$. This implies that in the large finite field aspect various statistics of the eigenvalues can be computed by integrating the corresponding quantities over $USp(2g)$.

(ii) **Large degree aspect.** Since the matrices $\Theta_D$ now inhabit different spaces as $g$ grows, it is not clear how to formulate an equidistribution problem. The following analysis of Katz and Sarnak [14] illuminates one possible interpretation. We start with an even test function $f$, say, in the Schwartz space $S(\mathbb{R})$, and for any $N \geq 1$ set

$$F(\theta) := \sum_{k \in \mathbb{Z}} f\left(N \left(\frac{\theta}{2\pi} - k\right)\right).$$

$F(\theta)$ has period $2\pi$ and is localized in an interval of size $\approx 1/N \in \mathbb{R}/2\pi\mathbb{Z}$. Next, for a unitary $2g \times 2g$ matrix $U$ with eigenvalues $e^{i\theta_j}, j = 1, \ldots, 2g$, define

$$Z_f(U) := \sum_{j=1}^{2g} F(\theta_j, D).$$

Now, $Z_f(U)$ counts the number of “low-lying” eigenphases, $\theta_j, D$, in the smooth interval of length $\approx 1/N$ around the origin defined by $f$. In other words, for $j \geq 1$, the above discussion describes the distribution of the numbers

$$\frac{\theta_j, D N}{2\pi}$$

as $D$ varies over $\mathcal{H}_n$, $n \to \infty$. That is, we use this to study the distribution of the $j$-th lowest zero.

**Conjecture 2.1** (Density Conjecture) *For a fixed $q$, we have*

$$\lim_{n \to \infty} \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} Z_f(\Theta_D) = \int_{USp(2g)} Z_f(U) dU$$

*for any test function defined as above.*

It is also known that

$$\lim_{g \to \infty} \int_{USp(2g)} Z_f(U) dU \sim \int_{-\infty}^{\infty} f(x) \left(1 - \frac{\sin 2\pi x}{2\pi x}\right) dx.$$
Rudnick [23] proved the Conjecture 2.1 for \( f \in S(\mathbb{R}) \) is even, with Fourier transform \( \hat{f} \) supported in \((-2, 2)\) before Bui and Florea [4] recover the same result and find lower order terms when the support is sufficiently restricted.

As an application, the Density Conjecture 2.1 gives the distribution of zeros for a family \( L(\frac{1}{2}, \chi_D) \) near \( s = \frac{1}{2} \). As discussed above, this has immediate applications to counting how often \( L(\frac{1}{2}, \chi_D) = 0 \). More precisely, by varying the test function \( f \) in the Density Conjecture for any of the above family of hyperelliptic curves \( C_D \), one is led to (assuming the Density Conjecture) (See equation (56) of [14]):

\[
\lim_{n \to \infty} \frac{1}{|\mathcal{H}_n|} \# \{ D \in \mathcal{H}_n : L(1/2, \chi_D) \neq 0 \} = 1.
\]

Thus, it is clear that, for the above family of hyperelliptic curves, hypothesis 1 is a consequence of Conjecture 2.1.

(iii) Double limit aspect. See [13, pg. 11] where they let \( g \to \infty \) in (2.8).

3 Preliminary lemmas

This section is simply a collection of lemmas necessary for the final result.

**Lemma 3.1** Let \( k, y \) be integers such that \( 2ky \leq n \). For any sequence of complex number \( \{a(P)\}_{P \in \mathcal{P}} \), we have

\[
\left| \sum_{D \in \mathcal{H}_n} \sum_{d(P) \leq y} \frac{\chi_D(P)a(P)}{|P|^{\frac{1}{2}}} \right|^{2k} \ll q^n \left( \sum_{d(P) \leq y} \frac{|a(P)|^2}{|P|} \right)^k.
\]

**Proof** The case \( n = 2g + 1 \) is proved by Florea, [8, Lemma 8.4]. To get the result for \( n = 2g + 2 \) it is a small adaptation of Florea’s proof.

**Lemma 3.2** Let \( f \in \mathcal{M}_n \). Then

\[
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \chi_D(f^2) = \prod_{P \mid f} \left( 1 + \frac{1}{|P|} \right)^{-1} + O\left( q^{-n-1} \right).
\]

**Proof** See [4, Lemma 3.7] for \( 2g + 1 \) case. To get the result for \( n = 2g + 2 \) it is a small adaptation of their proof.

**Lemma 3.3** (Polya-Vinogradov inequality) For \( l \in \mathcal{M}_n \) not a square polynomial, let \( l = l_1^2 l_2 \) with \( l_1 \) square free. Then we get

\[
\left| \sum_{D \in \mathcal{H}_n} \chi_D(l) \right| \ll \epsilon q^\epsilon |l_1|^\epsilon.
\]
Proof See [5], Lemma 3.5 for the case $n = 2g + 1$. To obtain the result for $n = 2g + 2$ follow the same proof, there is only a minor difference to handle the extra term coming from the trivial zero $u^{-1}$.

Lemma 3.4 Let $K \geq 2$ and $\sigma_0 < 1$ be such that $K \sigma_0 < \frac{1}{2 \log q}$. Then we have

$$\sum_{d(P) \leq K} \frac{1}{|P|^{1+2\sigma_0}} = \log K + O(1).$$

Proof From the prime polynomial Theorem (2.1) and partial summation formula, we obtain

$$\sum_{d(P) \leq K} \frac{1}{|P|^{1+2\sigma_0}} = \sum_{n \leq K} \frac{1}{n q^{2\sigma_0}} + O\left( \sum_{n \leq K} \frac{1}{n^{1/2}} \right) = \log K + O(1) + O\left( q^{-\frac{K}{4}} \right).$$

Now we state a standard probabilistic lemma which tells us when the measure of a certain set is very tiny.

Lemma 3.5 (Chebyshev’s Inequality) Let $R$ be random variable defined on the subspace $\mathcal{H}_n$ of $\mathcal{M}_n$. Suppose that the second moment of $R$ is small in the sense of

$$\sum_{D \in \mathcal{H}_n} |R_D|^2 = O\left( |\mathcal{H}_n| \right).$$

Then

$$\# \{ D \in \mathcal{H}_n : R_D > T \} \ll \frac{\mathcal{H}_n}{T^2}.$$

3.1 Estimates of sums involving $\Lambda_X$

Define

$$\Lambda_X(f) = \begin{cases} 2X^2 \Lambda(f) & \text{if } d(f) \leq X \\ (X^2 - (d(f))^2 + 2d(f)X - 3X + 3d(f) - 2) \Lambda(f) & \text{if } X < d(f) \leq 2X \\ (3X - d(f) + 1)(3X - d(f) + 2) \Lambda(f) & \text{if } 2X < d(f) \leq 3X. \end{cases} \quad (3.1)$$

We have the following estimates:

Lemma 3.6 Let $k \geq 1$ be an integer and $X$ be a real number such that $X \leq \frac{n}{4k}$. Suppose that $\sigma_0 \geq 0$. Then
We first consider Selberg’s central limit theorem for quadratic Dirichlet so that In particular, taking \( k = 2 \) we obtain this expression is \( O(1) \).

**Remark** From the definition of \( \Lambda_X(f) \), we can expand this into a sum over prime powers so that

\[
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left| \frac{1}{X^3} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1+\sigma_0}} \right|^k = O \left( \left( \frac{18k}{e} \right)^{\frac{k}{2}} + \frac{4^k}{X^k} \right).
\]

**Proof** From the prime polynomial theorem we obtain

\[
\sum_{1 \leq k \leq X} \frac{\Lambda_X(P)}{|P|^{1+\sigma_0}} \leq \frac{1}{X^3} \sum_{P \in \mathcal{M} \leq 3X} \Lambda_X(P) \chi_D(P) + \frac{1}{X^3} \sum_{P_2 \in \mathcal{M} \leq 3X} \Lambda_X(P_2) \chi_D(P_2) \leq 2k \left( \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left| \frac{1}{X^3} \sum_{P \in \mathcal{M} \leq 3X} \frac{\Lambda_X(P) \chi_D(P)}{|P|^{1+\sigma_0}} \right|^k \right).
\]

We first consider \( S_2 \): By definition, \( \chi_D(P_2) = 1 \) if \( P \nmid D \) and 0 otherwise, and the fact \( \Lambda_X(P_2) \leq 4X^2d(P) \) so we have

\[
S_2 \ll \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left| \frac{4}{X} \sum_{P_2 \in \mathcal{M} \leq 3X} \frac{d(P)}{|P|^{1+2\sigma_0}} \right|^k.
\]

From the prime polynomial theorem we obtain

\[
S_2 \ll \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left| \frac{4}{X} \sum_{2m \leq 3X} \frac{mq^m}{m^{2\sigma_0}} \right|^k \ll 6^k.
\]

For \( S_1 \) we expand the power \( k \) and then swap the order of summation giving

\[
S_1 = \frac{1}{|\mathcal{H}_n|} \frac{1}{X^{3k}} \sum_{P_1, P_2, \ldots, P_k \in \mathcal{M} \leq 3X} \frac{\Lambda_X(P_1) \Lambda_X(P_2) \cdots \Lambda_X(P_k)}{|P_1 P_2 \cdots P_k|^{1+\sigma_0}} \sum_{D \in \mathcal{H}_n} \chi_D(P_1 P_2 \cdots P_k).
\]

From Lemmas 3.2 and 3.3 we see that the values are different depending on if \( P_1 P_2 \cdots P_k \) is a square or not. Thus we consider two cases:

Case 1: \( k \) is even and \( P_1 P_2 \cdots P_k = \square \).
Applying Lemma 3.2, we must estimate
\[
\frac{1}{X^{3k}} \sum_{P_1, P_2, \ldots, P_k \in \mathcal{M}_{\leq 3X}, \ P_1 P_2 \cdots P_k = \square} \frac{\Lambda_X(P_1) \Lambda_X(P_2) \cdots \Lambda_X(P_k)}{|P_1 P_2 \cdots P_k|^{1/2 + \sigma_0}} \prod_{j=1}^k \left(1 + \frac{1}{|P_j|}\right)^{-1}.
\]

First note that \(P_1 P_2 \cdots P_k = \square\) happens precisely when one can pair up the indices so that the corresponding primes are equal thus using that \(\Lambda_X(P) \leq 4X^2 d(P)\) we have the above
\[
\ll \frac{1}{X^{3k}} \frac{k!}{(k/2)! 2^{k/2}} \left( \sum_{P \in \mathcal{M}_{\leq 3X}} \frac{(\Lambda_X(P))^2}{|P|^{1+2\sigma_0}} \left(1 + \frac{1}{|P|}\right)^{-1} \right)^{k/2}
\]
\[
\ll \frac{k! 2^{k/2}}{(k/2)! X^k} \left( \sum_{P \in \mathcal{M}_{\leq 3X}} \frac{(d(P))^2}{|P|^{1+2\sigma_0}} \left(\frac{|P|}{|P|+1}\right) \right)^{k/2}.
\]

Then, using the prime polynomial theorem and Stirling’s approximation, we obtain
\[
\ll \frac{k! 2^{k/2}}{(k/2)!} \frac{1}{X^k} \left( \sum_{P \in \mathcal{M}_{\leq 3X}} \frac{m^2 q^m}{mq^m} \right)^{k/2}
\]
\[
\ll \frac{k! 2^{k/2}}{(k/2)!} \frac{(3X(3X+1))^{k/2}}{2^{k/2} X^k} \ll \left(\frac{18k}{e}\right)^{k/2}.
\]

Case 2: \(P_1 P_2 \cdots P_k \neq \square\). In this case, using Lemma 3.3, we need to estimate:
\[
\frac{1}{\sqrt{|\mathcal{H}_n|}} \frac{1}{X^{3k}} \sum_{P_1, P_2, \ldots, P_k \in \mathcal{M}_{\leq 3X}, \ P_1 P_2 \cdots P_k \neq \square} \frac{\Lambda_X(P_1) \Lambda_X(P_2) \cdots \Lambda_X(P_k)}{|P_1 P_2 \cdots P_k|^{1/2 + \sigma_0}} |P_1 P_2 \cdots P_k|^\epsilon.
\]

Using that \(\Lambda_X(P) \leq 4X^2 d(P)\) we find
\[
\frac{1}{\sqrt{|\mathcal{H}_n|}} \frac{1}{X^{3k}} \sum_{P_1, P_2, \ldots, P_k \in \mathcal{M}_{\leq 3X}, \ P_1 P_2 \cdots P_k \neq \square} \frac{\Lambda_X(P_1) \Lambda_X(P_2) \cdots \Lambda_X(P_k)}{|P_1 P_2 \cdots P_k|^{1/2 + \sigma_0}} |P_1 P_2 \cdots P_k|^\epsilon
\]
\[
\ll \frac{1}{\sqrt{|\mathcal{H}_n|}} \frac{4^k}{X^k} \sum_{P_1, P_2, \ldots, P_k \in \mathcal{M}_{\leq 3X}, \ P_1 P_2 \cdots P_k \neq \square} \frac{d(P_1) d(P_2) \cdots d(P_k)}{|P_1 P_2 \cdots P_k|^{1/2 + \sigma_0 - \epsilon}}
\]
\[
\ll \frac{1}{\sqrt{|\mathcal{H}_n|}} \frac{4^k}{X^k} \left( \sum_{P \in \mathcal{M}_{\leq 3X}} \frac{d(P)}{|P|^{1/2 + \sigma_0 - \epsilon}} \right)^k.
\]
\[ \ll \frac{1}{\sqrt{|\mathcal{H}_n|}} \frac{4^k}{X^k} \left( \sum_{m \leq 3X} q^{m(\frac{1}{2}+\epsilon)} \right)^k \]

\[ \ll \frac{4^k q^{3kX(\frac{1}{2}+\epsilon)}}{q^{n/2}X^k} \ll (4/X)^k q^{3n/4(1/2+\epsilon)-n/2} \ll (4/X)^k. \]

To see that this is indeed \( \ll (4/X)^k \) we use the assumption \( X \leq n/4k \). Combining these estimates we have the desired result. \( \square \)

By following the proof of Lemma 3.6 we also have the following estimate.

**Lemma 3.7** Let \( X \) be a real number such that \( X \leq \frac{n}{8} \). Suppose that \( \sigma_0 \geq 0 \). Then

\[ \left| \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \frac{1}{X^2} \sum_{f \text{ monic}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2+\sigma_0}} \right|^2 = O(1). \]

### 3.2 Estimates for sums over the zeros

We may rewrite (2.6) as

\[ \frac{L'}{L}(s, \chi_D) = \log q \left( \frac{\lambda q^{-s}}{1-q^{-s}} - 2g + \sum_{j=1}^{2g} (1 - \alpha_j q^{1/2-s}) \right). \quad (3.2) \]

Notice that the non-trivial zeros appear in complex conjugate pairs and so (3.2) gives a real value for \( s \in \mathbb{R} \), as was pointed out in [1]. Thus, we may also express (3.2) as follows:

\[ \frac{L'}{L}(s, \chi_D) = \log q \left( \frac{\lambda q^{-s}}{1-q^{-s}} - g + \sum_{j=1}^{2g} \text{Re} \left( \frac{1}{1 - \alpha_j q^{1/2-s}} - \frac{1}{2} \right) \right). \]

Furthermore, if we know that \( L(\frac{1}{2}, \chi_D) \neq 0 \) then we have

\[ \frac{L'}{L}(\frac{1}{2}, \chi_D) = \log q \left( \frac{\lambda}{\sqrt{q} - 1} - g \right). \quad (3.3) \]

**Lemma 3.8** Let \( \sigma_0 > 0 \). Then for \( \sigma \geq \frac{1}{2} + \sigma_0 \), we have the inequality

\[ \left| \frac{a_j^{-1} q^{-1/2}}{(1 - \alpha_j^{-1} q^{\sigma-1/2})^2} \right| \leq (\sigma - \frac{1}{2})^{-1} \log q \left( \frac{1}{1 - \alpha_j q^{1/2-\sigma}} \right) \leq \frac{1}{\sigma_0} \log q \left( \frac{1}{1 - \alpha_j q^{-\sigma_0}} \right). \quad (3.4) \]
Furthermore, if we know that $L(\frac{1}{2}, \chi_D) \neq 0$ then there exists $c \in (0, 1)$ such that for $\sigma \geq 1/2$ we have

$$\frac{\alpha_j^{-1}q^{\sigma - 1/2}}{(1 - \alpha_j^{-1}q^{\sigma - 1/2})^2} \leq \frac{q^{\sigma - 1/2}}{1 + q^{2\sigma - 1} - 2q^{\sigma - 1/2}\cos(2\pi c)}.$$  \tag{3.5}$$

**Proof** Equation (3.4) is an verbatim generalization of Lemma 3.2 ([1]), which has been proved for the case that $q$ is a prime.

For equation (3.5), the assumption $L(\frac{1}{2}, \chi_D) \neq 0$ implies that $\alpha_j^{-1} = \exp(2\pi\theta_j D) \neq 1$ for any $j \in \{1, \ldots, 2g\}$. Therefore, there is a constant $c_D \in (0, 1)$ such that $c_D \leq \theta_j D \leq 1 - c_D$ so that

$$\frac{\alpha_j^{-1}q^{\sigma - 1/2}}{(1 - \alpha_j^{-1}q^{\sigma - 1/2})^2} \leq \frac{q^{\sigma - 1/2}}{1 + q^{2\sigma - 1} - 2q^{\sigma - 1/2}\cos(2\pi c D)}.$$  \tag{3.5}$$

$\square$

**Lemma 3.9** Let $\sigma_0 > 0$. Then for $\sigma \geq \frac{1}{2} + \sigma_0$,

$$\sum_{j=1}^{2g} \left| (\alpha_j q^{\frac{1}{2} - \sigma})X (1 - (\alpha_j q^{\frac{1}{2} - \sigma})X)^2 \right| \leq \frac{4q^{X(X - 1)(\frac{1}{2} - \sigma)}}{\sigma_0 \log q} \left( \frac{L'}{L} \left( \frac{1}{2} + \sigma_0, \chi_D \right) + 2g \log q \right).$$  \tag{3.6}$$

Furthermore, if we know $L(\frac{1}{2}, \chi_D) \neq 0$ we have for $\sigma \geq \frac{1}{2}$ there exists a $c \in (0, 1)$ such that

$$\sum_{j=1}^{2g} \left| (\alpha_j q^{\frac{1}{2} - \sigma})X (1 - (\alpha_j q^{\frac{1}{2} - \sigma})X)^2 \right| \leq 2g \frac{4q^{X(X - 1)(\frac{1}{2} - \sigma)}}{(1 + q^{2\sigma - 1} - 2q^{\sigma - 1/2}\cos(2\pi c))^3/2}.$$  \tag{3.7}$$

**Proof** We first prove (3.6). Lemma 3.8 equation (3.4) gives

$$\left| (\alpha_j q^{\frac{1}{2} - \sigma})X^{+1} (1 - (\alpha_j q^{\frac{1}{2} - \sigma})X) (\alpha_j q^{\frac{1}{2} - \sigma})^{-1} \right| \leq \frac{1}{\sigma_0 \log q} \Re \left( \frac{1}{1 - \alpha_j^{-1}q^{-\sigma}} \right)$$

We may easily bound the numerator as $|((\alpha_j q^{\frac{1}{2} - \sigma})^{X+1}(1 - (\alpha_j q^{\frac{1}{2} - \sigma})^{X})^2| \leq 4q^{X(X - 1)}$. It remains to discuss the denominator:
We have $1 - \alpha_j^{-1}q^{\sigma - \frac{1}{2}} = 1 - \cos(2\pi \theta_j)q^{\sigma - \frac{1}{2}} - i \sin(2\pi \theta_j)q^{\sigma - \frac{1}{2}}$ and so

$$|1 - \alpha_j^{-1}q^{\sigma - \frac{1}{2}}| = \sqrt{(1 - \cos(2\pi \theta_j)q^{\sigma - \frac{1}{2}})^2 + (\sin(2\pi \theta_j)q^{\sigma - \frac{1}{2}})^2}$$

$$= \sqrt{1 + q^{2\sigma - 1} - 2\cos(2\pi \theta_j)q^{\sigma - \frac{1}{2}}}$$

Since $\sigma \geq \frac{1}{2} + \sigma_0$, we have

$$\frac{1}{|1 - \alpha_j^{-1}q^{\sigma - \frac{1}{2}}|} \leq \frac{1}{\sqrt{1 + q^{2\sigma_0} - 2q^{\sigma_0}}} \leq \frac{1}{\sigma_0 \log q},$$

where the last inequality follows from the Taylor expansion. Thus we have

$$\sum_{j=1}^{2g} \left| (\alpha_j q^{\frac{1}{2} - \sigma})^X (1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2 \right| \leq 4q^{X(1 - \frac{1}{2} - \sigma)} (\sigma_0 \log q)^2 \sum_{j=1}^{2g} \Re \left( \frac{1}{1 - \alpha_j q^{\sigma_0}} \right).$$

Finally, (3.2) gives us that

$$\sum_{j=1}^{2g} \Re \left( \frac{1}{1 - \alpha_j q^{\sigma_0}} \right) = \frac{1}{\log q} \frac{L'}{L} \left( \frac{1}{2} + \sigma_0 \cdot \chi_D \right) + 2g - \frac{\lambda q^{-\frac{1}{2} + \sigma_0}}{1 - q^{-\frac{1}{2} + \sigma_0}}.$$

Note that $\frac{\lambda q^{-\frac{1}{2} + \sigma_0}}{1 - q^{-\frac{1}{2} + \sigma_0}}$ is a non-negative quantity and it is too small to effect the analysis so we drop it in the upper bound. For the second statement, we begin with Lemma 3.8 equation (3.5) to say

$$\left| (\alpha_j q^{\frac{1}{2} - \sigma})^{X+1} (1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2 (\alpha_j q^{\frac{1}{2} - \sigma})^{-1} \right| \leq \left| (\alpha_j q^{\frac{1}{2} - \sigma})^{X+1} (1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2 \right| \frac{q^{\sigma - \frac{1}{2}}}{1 + q^{2\sigma - 1} - 2q^{\sigma - \frac{1}{2}} \cos(2\pi c_D)}.$$

Following the same argument as above, we get

$$\left| (\alpha_j q^{\frac{1}{2} - \sigma})^{X+1} (1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2 (\alpha_j q^{\frac{1}{2} - \sigma})^{-1} \right| \leq \frac{4q^{X(1) - \frac{1}{2} - \sigma}}{(1 + q^{2\sigma - 1} - 2q^{\sigma - \frac{1}{2}} \cos(2\pi c_D))^3/2}.$$
Summing over all \( \alpha_j \) we get
\[
\sum_{j=1}^{2g} (\alpha_j q^{1/2-\sigma} X (1 - (\alpha_j q^{1/2-\sigma} X)^2) \over (1 - \alpha_j^{-1} q^{1/2-\sigma} X)^3) \leq 2g \frac{4q(X-1)(1/2-\sigma)}{(1 + q^{2/\sigma - 1}) - 2q^{\sigma - 1/2} \cos(2\pi c D)^{3/2}}.
\]

\[
4 \text{ A formula for } \log L(\sigma, \chi_D)
\]

Throughout rest of the paper, we fix the following notations: the parameter \( g \) as defined as in (1.1), the parameter \( \sigma_0 = \frac{c}{X} \), with \( 0 < c < \frac{1}{2 \log q} \) and \( X \geq 1 \) to be chosen later appropriately. Furthermore, for \( \sigma \geq \frac{1}{2} \) we define
\[
\tilde{P}_X(\sigma, \chi_D) := \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda(f) \chi_D(f)}{d(f)^{|f|^\sigma}}.
\]

(4.1)

The purpose of this section is to prove the following proposition:

**Proposition 4.1** Let \( X \geq 1 \). Then
\[
\log L(1/2 + \sigma_0, \chi_D) = \tilde{P}_X(1/2 + \sigma_0, \chi_D) + O \left( \frac{1}{X^3} \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2 + \sigma_0}} \right)
\]
\[
+ O \left( \frac{1}{X^2} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)^{|f|^{1/2 + \sigma_0}}} \right) + O \left( \frac{g}{X} + \frac{\lambda}{X^3} \right).
\]

Furthermore, if we know that \( L(1/2, \chi_D) \neq 0 \) then
\[
\log L \left( \frac{1}{2}, \chi_D \right) = \tilde{P}_X(1/2, \chi_D) + O \left( \frac{1}{X^3} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)^{|f|^{1/2}}} \right) O \left( \frac{g}{X} + \frac{\lambda}{X^3} \right).
\]

This proposition is useful as it allows us to compute the moments of \( \log |L(1/2 + \sigma_0, \chi_D)| \) by studying the simpler expression \( \tilde{P}_X(1/2 + \sigma_0, \chi_D) \). The proposition follows immediately from the following lemma after some straightforward manipulations.

**Lemma 4.2** Let \( X \geq 1 \). Then we have
\[
\log L \left( \frac{1}{2} + \sigma_0, \chi_D \right) = \frac{1}{2X^2} \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda_X(f) \chi_D(f)}{d(f)^{|f|^{1/2} + \sigma_0}} + O \left( \frac{1}{X^3} \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2 + \sigma_0}} \right)
\]
\[
+ O \left( \frac{g}{X} + \frac{\lambda}{X^3} \right).
\]

(4.2)
Furthermore, if we know that \( L(\frac{1}{2}, \chi_D) \neq 0 \) then

\[
\log L \left( \frac{1}{2}, \chi_D \right) = \frac{1}{2X^2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\frac{1}{2}} + O \left( \frac{g}{X^3} + \frac{\lambda}{X^3} \right). \tag{4.3}
\]

We will make use of the following identity, let \( X \geq 0, \Re(s) \geq 0 \):

\[
\frac{1}{2\pi i} \int_{2}^{2+ \frac{2\pi i}{\log q}} \frac{q^{X(w-s)}}{(1-q^{-(w-s)})^3} \, dw = - \frac{(X+1)(X+2)}{2 \log q}. \tag{4.4}
\]

**Proof of Lemma 4.2** We begin by considering the following integral

\[
\mathcal{I} = \int_{2}^{2+ \frac{2\pi i}{\log q}} \frac{q^{X(w-s)}}{(1-q^{-(w-s)})^3} \, dw = \frac{2\pi i}{2\pi i} \int_{2}^{2+ \frac{2\pi i}{\log q}} \frac{q^{X(w-s)}}{(1-q^{-(w-s)})^3} \, dw.
\]

We compute this integral in two different ways. The first approach is to use (2.7), (4.4) and integrate the result term by term:

\[
\mathcal{I} = \frac{1}{2} \sum_{f \in M \leq X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\frac{1}{2}} (X - d(f) + 1)(X - d(f) + 2)
- \sum_{f \in M \leq 2X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\frac{1}{2}} (2X - d(f) + 1)(2X - d(f) + 2)
+ \frac{1}{2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\frac{1}{2}} (3X - d(f) + 1)(3X - d(f) + 2).
\]

We combine these into a single sum recognizing the weight \( \Lambda_X(f) \) defined in (3.1), so that

\[
\mathcal{I} = \frac{1}{2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\frac{1}{2}}. \tag{4.5}
\]

The second approach is to use the residue theorem, from this method, we see that there are simple poles at \( w = s \) and each zero of \( \frac{L'}{L}(w, \chi_D) \). So,

\[
\mathcal{I} = - \frac{X^2}{\log q} \frac{L'}{L}(s, \chi_D) - \frac{\lambda q^{-Xs} (1-q^{-Xs})^2}{(1-q^{-s})^3} - \sum_{j=1}^{2g} \frac{(q^{\frac{1}{2}-s} \alpha_j)^X (1-(q^{\frac{1}{2}-s} \alpha_j)^X)^2}{(1-q^{s-\frac{1}{2}} \alpha_j^{-1})^3}. \tag{4.6}
\]
Equating (4.5) and (4.6), we have

\[
\frac{1}{2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^s} = -X^2 \frac{L'(s, \chi_D)}{\log q} - \frac{\lambda q^{-Xs}(1 - q^{-Xs})^2}{(1 - q^{-s})^3} - \frac{2g}{\log q} \lambda q^{-Xs}(1 - (q^{\frac{1}{2} - s} \alpha_j)^X)^2/(1 - q^{s - \frac{1}{2} \alpha_j^{-1}})^3,
\]

which implies

\[
-L'(s, \chi_D) = \frac{\log q}{2X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^s} - \frac{\lambda q^{-Xs}(1 - q^{-Xs})^2}{(1 - q^{-s})^3} - \frac{2g}{\log q} \lambda q^{-Xs}(1 - (q^{\frac{1}{2} - s} \alpha_j)^X)^2/(1 - q^{s - \frac{1}{2} \alpha_j^{-1}})^3. \tag{4.7}
\]

We note that this implies, for \( \Im(s) = \sigma \geq \frac{1}{2} \), we have

\[
-L'(\sigma, \chi_D) = \frac{\log q}{2X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\sigma} - \frac{\lambda q^{-X\sigma}(1 - q^{-X\sigma})^2}{(1 - q^{-\sigma})^3} - \frac{2g}{\log q} \lambda q^{-X\sigma}(1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2/(1 - \alpha_j^{-1} q^{\sigma - \frac{1}{2}})^3. \tag{4.8}
\]

It remains to provide estimates for the error terms.

**Proof of (4.2).** We handle the sum over the zeros in (4.8) by applying equation (3.6) of Lemma 3.9, which gives

\[
\log q \left| \sum_{j=1}^{2g} \left( \alpha_j q^{\frac{1}{2} - \sigma} \right)^X (1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2 \right| \leq 4q X^{\sigma - \frac{1}{2}} \left( \frac{L'}{L} \left( \frac{1}{2} + \sigma_0, \chi_D \right) + 2g \log q \right).
\]

Using the fact that \( \sigma_0 = \frac{c}{X} \), for \( c \) some positive constant, we have

\[
\log q \left| \sum_{j=1}^{2g} \left( \alpha_j q^{\frac{1}{2} - \sigma} \right)^X (1 - (\alpha_j q^{\frac{1}{2} - \sigma})^X)^2 \right| \leq 4q X^{\sigma - \frac{1}{2}} \left( \frac{L'}{c^2 \log^2 q} \left( \frac{1}{2} + \sigma_0, \chi_D \right) + 2g \log q \right).
\]
Thus, we can find some $\nu$ with $|\nu| \leq 1$ such that

$$\frac{L'}{L}(\sigma, \chi_D) = \frac{\log q}{2X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\sigma} \left(1 + \frac{4\nu q^{-1/2+\sigma}}{c^2 \log^2 q} L' \left(\frac{1}{2} + \sigma_0, \chi_D\right) + O \left(gq^{-1/2+\sigma} + \frac{\lambda q^{-X(1/2+\sigma)}}{X^2}\right)\right).$$

(4.9)

So, evaluating (4.9) at $\sigma = \frac{1}{2} + \sigma_0$, we conclude that

$$\left(1 + \frac{4\nu q^{-1/2+\sigma_0}}{c^2 \log^2 q} \right) \frac{L'}{L} \left(\frac{1}{2} + \sigma_0, \chi_D\right) = \frac{\log q}{2X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2+\sigma_0}} \left(1 + \frac{\lambda q^{-X(1/2+\sigma)}}{X^2} + O \left(gq^{-1/2+\sigma} + \lambda q^{-X(1/2+\sigma_0)}\right)\right).$$

Additionally, the choice of $c < \frac{1}{2 \log q}$ implies that $|1 + \frac{4\nu q^{-1/2+\sigma_0}}{c^2 \log^2 q}| > |1 - \frac{4}{c^2 q^r \log^2 q}| > \frac{16}{c^{1/2}} - 1 > 8.7$. Thus, we obtain

$$\frac{L'}{L} \left(\frac{1}{2} + \sigma_0, \chi_D\right) = O \left(\frac{1}{X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2+\sigma_0}}\right) + O(g).$$

(4.10)

Finally, putting together (4.10) and (4.9), we find that

$$\frac{L'}{L}(\sigma, \chi_D) = \frac{\log q}{2X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\sigma} + O \left(q^{X(1/2+\sigma)} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2+\sigma_0}}\right)$$

$$+ O \left(gq^{X(1/2+\sigma)} + \frac{\lambda q^{-X(1/2+\sigma)}}{X^2}\right).$$

Integrating with respect to $\sigma$ from $\frac{1}{2} + \sigma_0$ to $\infty$, we conclude that

$$\log L \left(\frac{1}{2} + \sigma_0, \chi_D\right) = \frac{1}{2X^2} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)|f|^{1/2+\sigma_0}}$$

$$+ O \left(\frac{1}{X^3} \sum_{f \in \mathcal{M} \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1/2+\sigma_0}}\right) + O \left(\frac{g}{X} + \frac{\lambda}{X^3}\right).$$
Proof of (4.3). We handle the sum over the zeros in (4.8) by applying equation (3.7) of Lemma 3.9, which gives

$$\log q \sum_{j=1}^{2g} \left| \frac{(\alpha_j q^{1/2} - \sigma)(1 - (\alpha_j q^{1/2} - \sigma)X)^2}{(1 - \alpha_j^{-1} q^{-1/2})^3} \right| \leq \frac{8g \log q}{X^2} \frac{q^{(X-1)(1/2 - \sigma)}}{(1 + q^{2\sigma - 1} - 2q^{1/2} \cos(2\pi cD))^{3/2}}.$$

This leads the expression (4.8) to shift into

$$-\frac{L'}{L}(\sigma, \chi_D) = \frac{\log q}{2X^2} \sum_{f \in M_{\leq 3X}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^\sigma} + O \left( \frac{\lambda q^{-X\sigma} (1 - q^{-X\sigma})^2}{X^2} \right) + O \left( \frac{g q^{(X-1)(1/2 - \sigma)}}{X^2 (1 + q^{2\sigma - 1} - 2q^{1/2} \cos(2\pi cD))^{3/2}} \right).$$

Since $\cos(2\pi cD) \neq 1$, for $\sigma \geq 1/2$, we have

$$1 + q^{2\sigma - 1} - 2q^{\sigma - \frac{1}{2}} \cos(2\pi cD) = \left( q^{\sigma - \frac{1}{2}} - \cos(2\pi cD) \right)^2 + (1 - \cos(2\pi cD))^2 \geq 2(1 - \cos(2\pi cD)).$$

Integrating with respect to $\sigma$ from $\frac{1}{2}$ to $\infty$, we conclude that

$$\log L \left( \frac{1}{2}, \chi_D \right) = \frac{1}{2X^2} \sum_{f \in M_{\leq 3X}} \frac{\Lambda_X(f) \chi_D(f)}{d(f)|f|^{1+\sigma_0}} + O \left( \frac{g}{(2(1 - \cos(2\pi cD)))^{3/2} X^3} \right) + \lambda \frac{X}{X^3}.$$

Now, we can complete the proof of Proposition 4.1:

Proof Simply apply Lemma 4.2:

$$\log L \left( \frac{1}{2} + \sigma_0, \chi_D \right) = \frac{1}{2X^2} \sum_{f \in M_{\leq 3X}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1+\sigma_0}} + O \left( \frac{1}{X^3} \sum_{f \in M_{\leq 3X}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{1+\sigma_0}} \right) + O \left( \frac{g}{X} \right) + O \left( \frac{\lambda}{X^3} \right)$$

$$= \sum_{f \in M_{\leq X}} \frac{\Lambda(f) \chi_D(f)}{d(f)|f|^{1+\sigma_0}} + \frac{1}{2X^2} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)|f|^{1+\sigma_0}} + O \left( \frac{g}{X} \right) + O \left( \frac{\lambda}{X^3} \right)$$

$$= \sum_{X < d(f) \leq 3X} \frac{\Lambda(f) \chi_D(f)}{d(f)|f|^{1+\sigma_0}} + O \left( \frac{1}{X^2} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)|f|^{1+\sigma_0}} \right) + O \left( \frac{g}{X} \right) + O \left( \frac{\lambda}{X^3} \right)$$

$$= \tilde{P}_X(1/2 + \sigma_0, \chi_D) + O \left( \frac{1}{X^2} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)|f|^{1+\sigma_0}} \right) + O \left( \frac{g}{X} \right) + O \left( \frac{\lambda}{X^3} \right)$$
\[ + O \left( \frac{1}{X^3} \sum_{f \in M \leq 3X} \frac{\Delta_f \chi_D(f) f}{|f|^{1+\sigma_0}} \right) + O \left( \frac{g}{X} + \frac{\lambda}{X^2} \right) . \]

Similarly, for the expression of \( \log L(1/2, \chi_D) \).

\[ \square \]

5 Moment estimation and Proof of the unconditional results

Recall \( \sigma_0 \) from Sect. 4. We have the following distribution result for a Dirichlet polynomial over function fields.

**Proposition 5.1** Let \( X \geq 1 \) be sufficiently large such that \( \frac{X}{n} \to 0 \) as \( n \to \infty \) but \( \log X = \log n + o(\sqrt{\log n}) \). As \( D \) varies in \( \mathcal{H}_n \), the distribution of \( \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \) is approximately normal with mean \( \sim \frac{1}{2} \log n \) and variance \( \sim \log n \).

To establish Proposition 5.1, we need to calculate the moments of \( \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log X \). We start computing moments of the following sum over irreducible polynomials:

\[ P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) = \sum_{d(P) \leq X} \frac{\chi_D(P)}{|P|^{1+\sigma_0}} . \]

**Lemma 5.2** Assume that \( 1 \leq X \leq \frac{n}{2k} \). Uniformly for all odd natural numbers \( k \leq \frac{n}{2} \) and for every \( \epsilon > 0 \),

\[ \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left( P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right)^k \ll \frac{q^{-\frac{n}{2}+\left(\frac{1}{2}+\epsilon-\sigma_0\right)Xk}}{X^k} , \]

while, uniformly for all even natural numbers \( k \ll (\log n)^\frac{1}{3} \),

\[ \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left( P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right)^k = \frac{k!}{(k/2)!(2\pi)^k} \left( \log X \right)^{\frac{k}{2}} \left( 1 + O \left( \frac{k^3}{\log X} \right) \right) . \]

**Remark** In the case of function fields, the contribution of the odd moments is negligibly small even for \( k \) large due to the fact that Lindelöf hypothesis is known. In particular, if we take \( X = \frac{n}{2k} \) then the odd moments contribute \( o(1) \).

Finally, we require an estimate of the following form:

**Lemma 5.3** Suppose that the hypothesis in Lemma 5.2 holds. Uniformly for all odd natural numbers \( k \ll \left( \frac{\log X}{(\log \log n)^2} \right)^{\frac{1}{3}} \), we have

\[ \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left( \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log X \right)^k \ll \frac{k!k^{\frac{k}{2}} \left( \log X \right)^{\frac{k}{2}} \left( \log \log n \right)^2}{(k/2)!(2\pi)^k} \left( \log X \right)^{\frac{k}{2}} , \]

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while uniformly for all even natural numbers $k \ll \left(\frac{\log X}{(\log \log n)^2}\right)^{\frac{1}{2}}$, we obtain

$$
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left( \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log X \right)^k = \frac{k!}{(k/2)!} \left( \log X \right)^{\frac{k}{2}} \left( 1 + O \left( \frac{k^3 (\log \log n)^2}{(\log X)^{\frac{1}{2}}} \right) \right).
$$

Making use of these Lemmas, we are able to complete the proofs of our main proposition and the two unconditional results:

**Proof of Proposition 5.1** From Lemma 5.3 together with the hypothesis that $\frac{X}{n} \to 0$ but $\log X = \log n + o(\sqrt{\log n})$, we conclude that the moments of $\tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log n$ asymptotically match with the moments of a Gaussian random variable with mean 0 and variance $\log n$. Since the Gaussian is determined by its moments, our proposition follows.

**Proof of Theorem 1.1** Take $\sigma_0 = \frac{X}{n}$. We begin with applying the first half of Proposition 4.1 to write $\log L \left( \frac{1}{2} + \sigma_0, \chi_D \right)$ in terms of $\tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right)$ and then apply Proposition 5.1. Viewing $\log L \left( \frac{1}{2} + \sigma_0, \chi_D \right)$ as sum of two random variable say $Y_D + Z_D$, where $Y_D = \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right)$ and $Z_D$ is its error term. The hypothesis $g \sigma_0 \to \infty$ and $g \sigma_0 = o(\sqrt{\log n})$ implies that $\log X = \log n + O(\log n)$, which satisfies the hypothesis of Proposition 5.1. So, Proposition 5.1 tells us $\tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log n$ is approximately normally distributed with mean 0 and variance $\log n$. Therefore it is enough to show that (see Lemma 2.9 of [10])

$$
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} |Z_D|^2 = o (\log n).
$$

Alternatively one can also use Lemma 3.5 to show that the random variable $Z_D$ contributes negligibly small. Note that

$$
Z_D \ll \frac{1}{X^3} \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda_X(f) \chi_D(f)}{|f|^1 + \sigma_0} + \frac{1}{X^2} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{d(f)|f|^1 + \sigma_0} + \frac{g}{X} + \frac{\lambda}{X^3}.
$$

Finally, using Lemma 3.6 with $k = 2$ and Lemma 3.7, we conclude the above claim together with the hypothesis that $\frac{X}{n} = o(\sqrt{\log n})$.

For the proof of Theorem 1.3 we need the following result of Bui and Florea on the non-vanishing of $L(1/2, \chi_D)$.

**Theorem 5.4** ([4], Corollary 1) We have

$$
\mathbb{P} \left( D \in \mathcal{H}_n : L(1/2, \chi_D) \neq 0 \right) \geq \frac{19 - \cot(1/4)}{16} = 0.9427\ldots
$$

as $n \to \infty$. 

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Proof of Theorem 1.3  Let us consider the following events:

We call the event $\mathcal{A}$ as $D \in \mathcal{H}_n$ such that $\frac{1}{\sqrt{\log n}} (\log |L(\frac{1}{2}, \chi_D)| - \frac{1}{2} \log n) > Z$. We call the event $\mathcal{B}$ as $D \in \mathcal{H}_n$ such that $L(1/2, \chi_D) \neq 0$ and $\mathcal{B}^c$ such that $L(1/2, \chi_D) = 0$. We denote $\mathbb{P}(\mathcal{A} \setminus \mathcal{B})$ as the conditional probability. Observe that

$$
\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B}) + \mathbb{P}(\mathcal{A} \cap \mathcal{B}^c) = \mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A} \setminus \mathcal{B}) \mathbb{P}(\mathcal{B}),
$$

since $\mathbb{P}(\mathcal{A} \cap \mathcal{B}^c) = 0$.

Assume that the event $\mathcal{A}$ happens. The proof will go through in a similar fashion as of Theorem 1.1. We first use Proposition 4.1 to write $\log L(\frac{1}{2}, \chi_D)$ in terms of $\tilde{P}_X(\frac{1}{2}, \chi_D)$, and then apply Proposition 5.1 with $X = X_0$. Finally, we use Lemma 3.7 to show the error term coming from Proposition 4.1 is negligibly small. This gives us that

$$
\mathbb{P}(\mathcal{A} \setminus \mathcal{B}) = \frac{1}{\sqrt{2\pi}} \int_Z^{\infty} e^{-t^2/2} dt.
$$

Note that, Theorem 5.4 gives us $\mathbb{P}(\mathcal{B}) \geq \frac{19 - \cot(1/4)}{16} = 0.9427 \ldots$. Hence, we conclude that

$$
\mathbb{P}(\mathcal{A}) \geq \frac{19 - \cot(1/4)}{16\sqrt{2\pi}} \int_Z^{\infty} e^{-t^2/2} dt,
$$

which concludes the proof.

5.1 Proofs of Lemma 5.3 and Lemma 5.2

Proof of Lemma 5.2 Expanding $k$-th power and interchanging summations, we have

$$
\sum_{d(P_i) \leq X} \frac{1}{P_1 \cdots P_k \prod_{i=1}^k \tau(P_i^\sigma_0)} \sum_{D \in \mathcal{H}_n} \chi_D(P_1 \cdots P_k).
$$

Case 1 Let $k$ be even and $P_1 \cdots P_k = \Box$.

Subcase 1.1 Suppose there are exactly $\frac{k}{2}$ distinct primes say $Q_1, \ldots, Q_{k/2}$ with

$$
d(Q_1) \leq d(Q_2) \leq \ldots \leq d(Q_{k/2}).
$$

Using Lemma 3.2, we obtain

$$
\sum_{d(Q_i) \leq X} \frac{1}{Q_1 \cdots Q_{k/2} \prod_{i=1}^{k/2} \tau(Q_i^\sigma_0)} \sum_{D \in \mathcal{H}_n} \chi_D((Q_1 \cdots Q_{k/2})^2)\
$$

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\[
\begin{align*}
= |\mathcal{H}_n| \times \frac{k!}{(k/2)!2^{k/2}} \sum_{d(Q_i) \leq X} \prod_{i=1}^{k/2} \frac{1}{(|Q_i| + 1)|Q_i|^{2\sigma_0}} + O \left( \frac{k!}{(k/2)!2^{k/2}} \frac{1}{|Q_1 \cdots Q_{k/2}|} \right) \\
= |\mathcal{H}_n| \times \frac{k!}{(k/2)!2^{k/2}} \sum_{d(Q_i) \leq X} \prod_{i=1}^{k/2} \frac{1}{(|Q_i| + 1)|Q_i|^{2\sigma_0}} + O \left( \frac{k!}{(k/2)!2^{k/2}} \frac{1}{(\log X)^{k/2}} \right)
\end{align*}
\]

Now, if we fix \(Q_1, \ldots, Q_{k/2-1}\) and vary over \(Q_{k/2}\) in the main term of the above expression. Then, the sum becomes

\[
\sum_{d(Q_{k/2}) \leq X} \frac{1}{(|Q_{k/2}| + 1)|Q_{k/2}|^{2\sigma_0}} + O \left( \frac{k}{2} \right).
\]

Applying Lemma 3.4 with \(c < \frac{1}{2 \log q}\), we find that

\[
s_n = |\mathcal{H}_n| \times \frac{k!}{(k/2)!2^{k/2}} (\log X + O(k))^{k/2}
\]

\[
= |\mathcal{H}_n| \times \frac{k!}{(k/2)!2^{k/2}} (\log X)^{k/2} \left( 1 + O \left( \frac{k}{\log X} \right) \right)^{k/2}.
\]

**Subcase 1.2** Let there are \(Q_1, \ldots, Q_r\) distinct primes with

\[
d(Q_1) \leq \cdots \leq d(Q_k)
\]

where \(r < \frac{k}{2}\). Therefore, Lemma 3.4 gives us

\[
s_n \leq |\mathcal{H}_n| \times \sum_{r < \frac{k}{2}} \frac{k!}{r!2^r} \binom{k-r}{r} \left( \sum_{d(P) \leq X} \frac{1}{(|P| + 1)|P|^{2\sigma_0}} \right)^r
\]

\[
+ O \left( \sum_{r < k/2} \frac{k!}{r!2^r} \binom{k-r}{r} (\log X)^r \right)
\]

\[
\leq |\mathcal{H}_n| \times \sum_{r < \frac{k}{2}} \frac{k!}{r!2^r} \binom{k-r}{r} (\log X + O(1))^r + O \left( \sum_{r < k/2} \frac{k!}{r!2^r} \binom{k-r}{r} (\log X)^r \right)
\]

\[
\ll |\mathcal{H}_n| \times \frac{k^3k!}{(k/2)!2^{k/2} \log X} (\log X)^{k/2} + O \left( \frac{k^3k!}{(k/2)!2^{k/2} \log X} (\log X)^{k/2} \right).
\]
Case 2 Let \( k \) be even and \( P_1 \ldots P_k \neq \Box \). Then we use Lemma 3.3 to get

\[
s_n \ll \sqrt{|H_n|} \sum_{d(P_i) \leq X} \frac{1}{P_1 \ldots P_k |P_i|^{1+\sigma_0-\epsilon}} \ll \frac{q^{\frac{n}{2} + \left(\frac{1}{2} - \sigma_0 + \epsilon\right)X_k}}{X^k}.
\]

Case 3 Let \( k \) be odd. In this case \( P_1 \ldots P_k \neq \Box \). In a similar manner in Case 2, we come across the following

\[
s_n \ll \frac{q^{\frac{n}{2} + \left(\frac{1}{2} - \sigma_0 + \epsilon\right)X_k}}{X^k}.
\]

Proof ofLemma 5.3 We write

\[
\tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) = \sum_{d(P) \leq X} \frac{\Lambda(P)\chi_D(P)}{|P|^{1+\sigma_0}} + \frac{1}{2} \sum_{d(P) \leq X} \frac{\Lambda(P^2)\chi_D(P^2)}{|P|^2|P|^{1+2\sigma_0}} + O \left( \frac{\sum_{d(P) \leq \frac{X}{2}} 1}{|P|^{2+3\sigma_0}} \right).
\]

It is clear that the third sum is \( O(1) \). So the contribution of \( P^k, k \geq 3 \) in \( \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \) is \( O(1) \). Therefore, Lemma 3.4 yields

\[
\tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) = \frac{1}{2} \sum_{d(P) \leq X} \frac{\chi_D(P)}{|P|^{1+\sigma_0}} + \frac{1}{2} \sum_{d(P) \leq \frac{X}{2}} \frac{1}{|P|^{1+2\sigma_0}} + O(1)
\]

\[
= \sum_{d(P) \leq X} \frac{\chi_D(P)}{|P|^{1+\sigma_0}} + \frac{1}{2} \log X + O(\log \log n),
\]

where the error term \( O(\log \log n) \) comes from the sum over \( P \) such that \( P \mid D \). Therefore,

\[
\tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log X = P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) + O(\log \log n).
\]

This implies that for some positive constant \( c \),

\[
\sum_{D \in H_n} \left( \tilde{P}_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) - \frac{1}{2} \log X \right)^k = \sum_{D \in H_n} \left( P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right)^k
\]

\[
+ O \left( \sum_{D \in H_n} \sum_{r=1}^{k-1} \binom{k}{r} (c \log \log n)^{k-r} \left| P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right|^r \right).
\]

(5.1)
The first sum will be handled by Lemma 5.2. To handle reminder term we need to estimate the sum

\[ \sum_{D \in \mathcal{H}_n} \left| P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right|^r, \quad r \leq k - 1. \]

If \( r \) is odd then Cauchy-Schwarz inequality and Lemma 5.2 with even moments gives us

\[ \sum_{D \in \mathcal{H}_n} \left| P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right|^r \leq \left( \sum_{D \in \mathcal{H}_n} \left( P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right)^{r-1} \right)^{1/2} \times \left( \sum_{D \in \mathcal{H}_n} \left( P_X \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right)^{r+1} \right)^{1/2} \]

\[ \ll |\mathcal{H}_n| \left( \frac{(r-1)! (r+1)!}{(r-1)! (r+1)! 2^r} \right)^{1/2} (\log X)^r \]

\[ = |\mathcal{H}_n| \left( \frac{(r-1)! r^{1/2}}{(r-1)! 2^{r-1}} \right) (\log X)^r. \]

On the other hand, if \( r \) is even then directly use the asymptotic formula for even moments. Finally, plugging these estimates with the Lemma 5.2 to the sum (5.1), we complete the proof.

6 The difference between \( L \)-functions and Proof of Theorem 1.4

We start with the following proposition which shows that the difference between logarithm of \( L \)-functions around critical point is small enough.

**Proposition 6.1** Suppose that the Low-lying Zero Hypothesis holds for \( \{L(s, \chi_D)\}_{D \in \mathcal{H}_n} \). Also assume that \( g \sigma_0 \rightarrow \infty \) and \( g \sigma_0 = o(\sqrt{\log n}) \) as \( g \rightarrow \infty \). Then there exist a subset \( \mathcal{H}_n' \subset \mathcal{H}_n \) of measure \( 1 - o(1) \) for which

\[ \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n'} \left| \log \left| L \left( \frac{1}{2}, \chi_D \right) \right| - \log \left| L \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right| \right| = o(\sqrt{\log n}). \]

**Proof** We choose parameters \( \sigma_0, X := X(n) \) and \( y = y(n) \) such that

\[ \sigma_0 = \frac{c}{X}, \quad \frac{yg}{X} = o(\sqrt{\log n}). \]

Therefore, the hypothesis means that

\[ \frac{g}{X} \rightarrow \infty \quad \text{and} \quad \frac{g}{X} = o(\sqrt{\log n}), \quad \text{as} \quad g \rightarrow \infty. \]
Also, the Low-lying Zero Hypothesis implies there is a subset $\mathcal{H}'_n \subset \mathcal{H}_n$ of measure $1 - o(1)$ so that for all $D \in \mathcal{H}'_n$ and for all zeros of $L(q^{-1/2} e(\theta_j, D), \chi_D)$, the condition

$$\min_j |\theta_j, D| > \frac{1}{yX}$$

holds. Recall that

$$\frac{L'(s, \chi_D)}{L(s, \chi_D)} = \log q \left( \frac{\lambda q^{-s}}{1 - q^{-s}} - g + \sum_{j=1}^{2g} \Im \left( \frac{1}{1 - \alpha_j q^{-s} - \frac{1}{2}} \right) \right). \quad (6.1)$$

Since $D \in \mathcal{H}'_n$, we have $L(\frac{1}{2}, \chi_D) \neq 0$. Then using the fact that $(6.1)$ is real for $s \in \mathbb{R}$, we obtain

$$\log \left| L \left( \frac{1}{2} + \sigma, \chi_D \right) \right| - \log \left| L \left( \frac{1}{2}, \chi_D \right) \right| = \int_{0}^{\sigma_0} \frac{L'(s, \chi_D)}{L(s, \chi_D)} \, ds$$

$$= -g\sigma_0 \log q + \frac{\log q}{2} \sum_{j=1}^{2g} \Im \int_{0}^{\sigma_0} \frac{1 + \alpha_j q^{-\sigma}}{1 - \alpha_j q^{-\sigma}} \, ds + \lambda \log \left( \frac{1 - q^{-1/2 - \sigma_0}}{1 - q^{-1/2}} \right)$$

$$= -g\sigma_0 \log q + \frac{\log q}{2} \sum_{j=1}^{2g} \log \left( \frac{q^{\sigma_0} + q^{-\sigma_0} - 2 \cos(2\pi \theta_j, D)}{2 - 2 \cos(2\pi \theta_j, D)} \right) + \lambda \log \left( \frac{1 - q^{-1/2 - \sigma_0}}{1 - q^{-1/2}} \right). \quad (6.2)$$

We observe that

$$\Im \left( \frac{1}{1 - \alpha_j q^{-\sigma_0} - \frac{1}{2}} \right) = \frac{1 - q^{-2\sigma_0}}{2 \left( 1 - q^{-\sigma_0} \cos(2\pi \theta_j, D) + q^{-2\sigma_0} \right)}. \quad (6.3)$$

Now we use the inequality $\log(1 + x) < x$, combined with the fact that $D \in \mathcal{H}'_n$ and $(6.3)$ to get

$$\left| \log \frac{q^{\sigma_0} + q^{-\sigma_0} - 2 \cos(2\pi \theta_j, D)}{2 - 2 \cos(2\pi \theta_j, D)} \right| \leq \frac{y(q^{\sigma_0} + q^{-\sigma_0} - 2)}{2 - 2 \cos(2\pi \theta_j, D)} \ll \frac{y\sigma_0(1 - q^{-2\sigma_0})}{2 - 2 \cos(2\pi \theta_j, D)}$$

$$\ll \frac{y\sigma_0(1 - q^{-2\sigma_0})}{1 - 2q^{-\sigma_0} \cos(2\pi \theta_j, D) + q^{-2\sigma_0}}$$

$$\ll y\sigma_0 \Im \left( \frac{1}{1 - \alpha_j q^{-\sigma_0} - \frac{1}{2}} \right),$$

where the first inequality comes from $|\theta_j, D| X > \frac{1}{y}$ for each $j$. 
On the real line, from (6.1), we see that

\[
\frac{L'}{L} \left( \frac{1}{2} + \sigma_0, \chi_D \right) = \log q \left( \frac{\lambda q^{-\frac{1}{2} - \sigma_0}}{1 - q^{-\frac{1}{2} - \sigma_0}} - g + \sum_{j=1}^{2g} \Re \left( \frac{1}{1 - \alpha_j q^{-\sigma_0} - \frac{1}{2}} \right) \right).
\]

From this expression of the logarithmic derivative of \( L(s, \chi_D) \) on the real line, we have

\[
\left| \log \frac{q^{\sigma_0} + q^{-\sigma_0} - 2 \cos(2\pi \theta_j, D)}{2 - 2 \cos(2\pi \theta_j, D)} \right| \ll y\sigma_0 \frac{L'}{L} \left( \frac{1}{2} + \sigma_0, \chi_D \right) + yg\sigma_0. \tag{6.4}
\]

Using (4.10), we deduce that

\[
\frac{L'}{L} \left( \frac{1}{2} + \sigma_0, \chi_D \right) = O \left( \frac{1}{X^2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{\frac{1}{2} + \sigma_0}} \right) + O(g). \tag{6.5}
\]

After inserting the bound (6.5) into (6.4), we obtain

\[
\log \left| L \left( \frac{1}{2}, \chi_D \right) \right| - \log \left| L \left( \frac{1}{2} + \sigma_0, \chi_D \right) \right| \ll g\sigma_0 + yg\sigma_0 + \frac{y\sigma_0}{X^2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{\frac{1}{2} + \sigma_0}}.
\]

Therefore, it is enough to prove that

\[
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n'} \left| \frac{y\sigma_0}{X^2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{\frac{1}{2} + \sigma_0}} \right|^2 = O(y^2) = o \left( \log n \right).
\]

Using \( \Lambda_X(P) \leq 4X^2d(P) \) and prime polynomial theorem, we have

\[
\frac{1}{X^2} \sum_{d(f) \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{\frac{1}{2} + \sigma_0}} \ll \sum_{d(P) \leq 3X} d(P)q^{-\frac{1}{2} - \sigma_0}d(f) \ll \sum_{n \leq 3X} q^{\frac{n}{2} - \sigma_0} \ll q^{\frac{3X}{2}}.
\]

Since \( |\mathcal{H}_n \setminus \mathcal{H}_n'| = o(1) \), so Lemma 3.6 and the fact that \( \sigma_0 = c/X \) yields

\[
\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n'} \left| \frac{y\sigma_0}{X^2} \sum_{f \in M \leq 3X} \frac{\Lambda_X(f) \chi_D(f)}{|f|^{\frac{1}{2} + \sigma_0}} \right|^2 = O(y^2) + o \left( (y/X)^2 q^{3X-n} \right).
\]

The above choice of \( X \) gives us \( O(y^2) \). This completes the proof.

**Proof of Theorem 1.4** The proof follows from the Theorem 1.1 and Proposition 6.1.
7 Proof of Corollary 1.2

Following the computation in (6.2) we obtain

\[
\log |L\left(\frac{1}{2}, \chi_D\right)| - \log |L\left(\frac{1}{2} + \sigma_0, \chi_D\right)| = g \sigma_0 \log q - \frac{1}{2} \sum_{j=1}^{2g} \log \left(1 + \frac{q^{\sigma_0} + q^{-\sigma_0} - 2}{2 - 2 \cos(2\pi \theta_j, D)}\right) - \lambda \log \left(1 - q^{-\frac{1}{2} - \sigma_0}\right).
\]

Note that the summands in the second term are all non-negative which leads the second term of the above expression to be a negative value. Therefore, we deduce that

\[
\log |L\left(\frac{1}{2}, \chi_D\right)| \leq \log |L\left(\frac{1}{2} + \sigma_0, \chi_D\right)| + g \sigma_0 \log q - \lambda \log \left(1 - q^{-\frac{1}{2} - \sigma_0}\right).
\]

From the hypothesis that \(g \sigma_0 = o\left(\sqrt{\log n}\right)\), we obtain

\[
\frac{\log |L\left(\frac{1}{2}, \chi_D\right)|}{\sqrt{\log n}} \leq \frac{\log |L\left(\frac{1}{2} + \sigma_0, \chi_D\right)|}{\sqrt{\log n}} + o(1).
\]

Hence, the theorem follows from Theorem 1.1.

Acknowledgements This work was carried out during the tenure of a NBHM Fellowship (funded by DAE) for the first author at ISI Kolkata, India. This work was carried out during the tenure of an NSERC PDF (funded by the government of Canada) for the second author at the Centre de Recherches Mathématiques, Montréal, QC, Canada. The authors also thank the anonymous referees for their valuable comments and insightful suggestions that have improved the quality of the manuscript.

Data Availability We confirm that all the data are included in the article.

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