ABSTRACT

We derive the exact frame-dragging rate inside rotating neutron stars. This formula is applied to show that the frame-dragging rate monotonically decreases from the center to the surface of the neutron star along the pole. In the case of the frame-dragging rate along the equatorial distance, it decreases initially away from the center, becomes negligibly small well before the surface of the neutron star, rises again, and finally approaches to a small value at the surface. The appearance of a local maximum and minimum in this case is the result of the dependence of frame-dragging frequency on the distance and angle. Moving from the equator to the pole, it is observed that this local maximum and minimum in the frame-dragging rate along the equator disappear after crossing a critical angle. It is also noted that the positions of the local maximum and minimum of the frame-dragging rate along the equator depend on the rotation frequency and central energy density of a particular pulsar.

Key words: dense matter – equation of state – pulsars: general – relativistic processes – stars: neutron – stars: rotation

Online-only material: color figures

1. INTRODUCTION

Compact astrophysical objects such as neutron stars and black holes are the laboratories for the study of Einstein’s general relativity in strong gravitational fields. Frame dragging is one such important general relativistic effect as demonstrated by Lense and Thirring (LT; Lense & Thirring 1918). A stationary spacetime with angular momentum shows an effect by which the locally inertial frames are dragged along the rotating spacetime. Thus any test gyroscope in such a spacetime precess with a certain frequency is called the frame-dragging frequency or the LT precession frequency \( \Omega_{\text{LT}} \). The LT precession frequency is proportional to the angular momentum as well as the compactness of the rotating astrophysical object. This effect for a test gyroscope had been calculated and was shown to fall with the inverse cube of the distance of the test gyroscope from the source and vanishes at large enough distances where the curvature effects are small. The precession frequency is thus expected to be larger near the surface of a neutron star and in its interior than at a large distance from the star.

The precise mass measurement of PSR J0348+0432 confirmed the existence of a massive neutron star \((> 2 M_\odot)\) (Antoniadis et al. 2013). It is also known that some of them are observed to possess very high angular velocities. Hence the spacetime curvature would be much higher in the surroundings of those massive neutron stars, and the frame-dragging effect also becomes very significant in the strong gravitational fields of those rotating neutron stars. It should be noted that the inertial frames are not only dragged outside the rotating neutron stars but also inside the stars.

The theoretical prescription to determine the rate of the frame-dragging precession inside the rotating neutron star was first given by Hartle (Hartle 1967). In this formalism, one can estimate the frame-dragging precession rate inside a slowly rotating \((\Omega R \ll c, \text{where } R \text{ is the radius of the pulsar, } c \text{ is the speed of the light in vacuum})\) neutron star. The final expression of frame-dragging precession rate depends solely on \(r\), the distance from the center of the star because of the slow-rotation approximation in Hartle’s formalism. It was observed that the frame-dragging frequency was higher at the center of the star than the frame-dragging frequency at the surface. The maximum frame-dragging frequency at the center \((r = 0)\) would never exceed the frequency of the rotating neutron star. The frame-dragging effect was applied to various astrophysical problems using Hartle’s formalism. Hartle studied this effect on the equilibrium structures of rotating neutron stars (Hartle 1968). The impact of frame dragging on the Kepler frequency was investigated by Glendenning and Weber (Glendenning & Weber 1994). It was also demonstrated how this effect might influence the moment of inertia of a rotating neutron star (Weber 1999). Furthermore, Morsink and Stella studied the role of frame dragging in explaining the quasi-periodic oscillations (QPOs) of accreting neutron stars (Morsink & Stella 1999). Morsink and Stella estimated the precession frequency \(v_p\) of the disk’s orbital plane about the star’s axis of symmetry as the difference between the frequency of oscillations of the particle along the longitude and latitude \((2\pi v_p = d\phi/d\tau - d\theta/d\tau)\) observed at infinity. This expression contains the total precession frequency of the disk’s orbital plane due to the LT effect as well as the star’s oblateness. Their calculation introduced the zero angular momentum observer (ZAMO), and the precession frequency \(v_p\) was observed at infinity. It was found that the LT frequency was proportional to the ZAMO frequency on the equatorial plane in the slow-rotation limit. This is similar to Hartle’s formalism (Hartle 1967), where the angular velocity \((d\phi/d\tau)\) acquired by an observer who falls freely from infinity to the point \((r, \theta)\) is taken as the rate of rotation of the inertial frame at that point relative to the distant stars.

In this manuscript we derive the exact LT precession frequency \(\Omega_{\text{LT}}\) that is measured by a Copernican observer of a gyroscope, such as the Gravity Probe B satellite in a realistic orbit (Hartle 2009). In this case, \(\Omega_{\text{LT}}\) would not only be a function of \(\omega\) but also a complicated function of other metric components even in the slow-rotation limit.

We should note that inside the rapidly rotating stars, one should not a priori expect the similar variation of the precession rates along the equatorial and polar plane. Thus the frame-dragging frequency should depend also on the colatitude (\(\theta\)) of the position of...
the test gyroscope. This did not arise in the formalism of Hartle due to the slow-rotation limit. We also note that the LT precession must depend on both the radial distance \( r \) and the colatitude \( \theta \) (see Equation (14.34) of [Hartle 2009]) in very weak gravitational fields (far away from the surface of the rotating object).

The exact LT precession rate in strongly curved stationary spacetime had been discussed in detail by Chakraborty and Majumdar (Chakraborty & Majumdar 2014). Later, Chakraborty and Pradhan (Chakraborty & Pradhan 2013) applied this formulation in various stationary and axisymmetric spacetimes. Our main motivation for this paper is to compute the exact LT precession rate inside the rotating neutron star. In this article we avoid all types of approximations and assumptions to obtain the exact LT precession rate inside the rotating neutron stars.

The paper is organized as follows. In Section 2, we present the basic equations of the frame-dragging effect inside the rotating neutron stars. The numerical method, which has been adopted in the whole paper, is discussed in Section 3. We discuss our results in Section 4. Finally we conclude in Section 5 with a summary.

2. BASIC EQUATIONS OF FRAME-DRAGGING EFFECT INSIDE THE ROTATING NEUTRON STARS

The rotating equilibrium models considered in this paper are stationary and axisymmetric. Thus, we can write the metric inside the rotating neutron star in the following Komatsu–Eriguchi–Hachisu (KEH; Komatsu et al. 1989) form:

\[
\frac{ds}{c} = -e^{\gamma(r)} dt^2 + e^{2\sigma(r)}(dr^2 + r^2 d\theta^2) + e^{\gamma(r) - \sigma(r)} r^2 \sin^2\theta (d\phi - \omega dt)^2,
\]

where \( \gamma, \sigma, \alpha, \omega \) are the functions of \( r \) and \( \theta \) only. In the whole paper we have used the geometrized unit \((G = c = 1)\). We assume that the matter source is a perfect fluid with a stress-energy tensor given by

\[
T^{\mu\nu} = (\rho_0 + \rho_i + P) u^\mu u^\nu + P g^{\mu\nu},
\]

where \( \rho_0 \) is the rest energy density, \( \rho_i \) is the internal energy density, \( P \) is the pressure, and \( u^\mu \) is the matter four-velocity. We are further assuming that there is no meridional circulation of the matter, so the four-velocity \( u^\mu \) is simply a linear combination of time and angular Killing vectors. Now we have to calculate the frame-dragging rate based on the above metric, and this will give us the exact frame-dragging rate inside a rotating neutron star.

We know that the vector field corresponding to the LT precession covector can be expressed as

\[
\Omega_{LT} = \frac{1}{2\sqrt{-g}} \left[ g_{0i,j} \left( \partial_i - \frac{g_{0i}}{g_{00}} \partial_0 \right) - \frac{g_{0j}}{g_{00}} g_{00,j} \partial_0 \right].
\]

For the axisymmetric spacetime, the only nonvanishing component is \( g_{0i} = g_{0\phi}, \) \( i = \phi \) and \( j = r, \theta; \) substituting these in Equation (3), the LT precession frequency vector is obtained as

\[
\Omega_{LT} = \frac{1}{2\sqrt{-g}} \left[ \left( g_{0\phi,r} - \frac{g_{0\phi}}{g_{00}} g_{00,r} \right) \partial_0 - \left( g_{0\phi,\theta} - \frac{g_{0\phi}}{g_{00}} g_{00,\theta} \right) \partial_r \right].
\]

Because the above expression has been expressed in the coordinate basis, we have to convert it into the orthonormal basis. Thus, in the orthonormal basis, our choice of polar coordinates, \( \Omega_{LT} \) can be written as (Chakraborty & Majumdar 2014)

\[
\Omega_{LT} = \frac{1}{2\sqrt{-g}} \left[ \sqrt{-g} \left( g_{0\phi,\theta} - \frac{g_{0\phi}}{g_{00}} g_{00,\theta} \right) \hat{r} + \sqrt{-g} \left( g_{0\phi,r} - \frac{g_{0\phi}}{g_{00}} g_{00,r} \right) \hat{\theta} \right],
\]

where \( \hat{r} \) is the unit vector along the direction \( \theta \) and \( \hat{r} \) is the unit vector along the direction \( r \). We note that the above formulation is valid only in the time-like spacetimes, not in the light-like or space-like regions.

Now, we can apply the above Equation (5) to determine the exact frame-dragging rate inside the rotating neutron star of which the metric could be determined from the line-element (1). The various metric components can be read from the metric. Likewise,

\[
\sqrt{-g} = r^2 e^{2\gamma(r)} \sin \theta.
\]

On an orthonormal coordinate basis, the exact LT precession rate inside the rotating neutron star is

\[
\Omega_{LT} = \frac{e^{-(\omega + \sigma)}}{2(\omega^2 r^2 \sin^2 \theta - e^{2\gamma})} \cdot \left[ \sin^2 \theta [r^3 \omega^2 \omega_r \sin^2 \theta + e^{2\omega}(2\omega + r \omega_r - 2 \omega \sigma_r)]\hat{r} + [r^2 \omega^2 \omega_r \sin^3 \theta + e^{2\omega}(2\omega \cos \theta + \omega_r \sin \theta - 2 \omega \sigma_r \sin \theta)]\hat{\theta} \right],
\]

and the modulus of the above LT precession rate is

\[
\Omega_{LT} = |\Omega_{LT}(r, \theta)| = \frac{e^{-(\omega + \sigma)}}{2(\omega^2 r^2 \sin^2 \theta - e^{2\gamma})} \cdot \left[ \sin^2 \theta [r^3 \omega^2 \omega_r \sin^2 \theta + e^{2\omega}(2\omega + r \omega_r - 2 \omega \sigma_r)]^2 + [r^2 \omega^2 \omega_r \sin^3 \theta + e^{2\omega}(2\omega \cos \theta + \omega_r \sin \theta - 2 \omega \sigma_r \sin \theta)]^2 \right]^{1/2}.
\]
As a vector quantity the expression of $\Omega_{LT}$ (Equation (7)) depends on the coordinate frame (i.e., polar coordinates $(r, \theta)$, which is used here), but the modulus of $\Omega_{LT}$ or $|\Omega_{LT}|$ (Equation (8)) must be coordinate frame independent. We use only the modulus of $\Omega_{LT}$ in the rest of our manuscript.

To calculate the frame-dragging precession frequency at the center of the neutron star, we substitute $r = 0$ in Equation (8), and we obtain

$$\Omega_{LT}|_{r=0} = \left[\frac{e^{-(\sigma+\omega)}}{2} - (4\omega^2 + (\omega, \omega - 2\omega \sigma, \omega)^2 \sin^2 \theta + 4\omega \cos \theta \sin \theta(\omega, \omega - 2\omega \sigma, \omega)) \right]_{r=0}. \quad (9)$$

Solving numerically $\omega, \theta$ and $\sigma, \theta$ at the center, we get the value zero for both of them. Thus, we obtain the frame-dragging precession rate

$$\Omega_{LT}|_{r=0} = \omega e^{-(\sigma+\omega)}|_{r=0} \quad (10)$$

at the center ($r = 0$) of a rotating neutron star.

Following KEH we can write the general relativistic field equations determining $\sigma$, $\gamma$, and $\omega$ as

$$\Delta [\sigma e^{\frac{r}{2}}] = S_\sigma(r, \mu) \quad (11)$$

$$\left(\Delta + \frac{1}{r} \partial_r - \frac{\mu}{r^2} \partial_\mu\right) [\gamma e^{\frac{r}{2}}] = S_\gamma(r, \mu) \quad (12)$$

$$\left(\Delta + \frac{2}{r} \partial_r - \frac{2\mu}{r^2} \partial_\mu\right) [\omega e^{\frac{r}{2}}] = S_\omega(r, \mu), \quad (13)$$

where

$$\Delta \equiv \frac{\partial^2}{r^2} + \frac{2}{r} \partial_r + \frac{1}{r^2} \left[ 1 - \frac{\mu^2}{r^2} \right] \partial_\mu + \frac{1}{r^2 (1 - \mu^2)} \partial^2_\phi$$

(14)

is the flat-space spherical coordinate Laplacian, and $\mu = \cos \theta$ and $S_\sigma$, $S_\gamma$, $S_\omega$ are the effective source terms that include the nonlinear and coupling terms. The effective source terms are given by

$$S_\sigma(r, \mu) = e^{r/2} \left\{ 8\pi (\rho_0 + \rho_i + P)e^{2\alpha} \left[ \frac{1 + v^2}{1 - v^2} + r^2 (1 - \mu^2) e^{-2\sigma} \right] - \omega_\sigma r + \frac{1}{r^2} \omega_\omega \mu + \frac{1}{r^2} \omega_\omega \mu \right\}, \quad (15)$$

$$S_\gamma(r, \mu) = e^{r/2} \left[ 16\pi e^{2\alpha} P + \frac{\gamma}{2} \left( 16\pi e^{2\alpha} P - \frac{1}{2} \gamma_\sigma - \frac{1}{2} \gamma_\mu \right) \right], \quad (16)$$

$$S_\omega(r, \mu) = e^{r/2-\sigma} \left\{ -16\pi \frac{(\rho_0 + \rho_i + P)(\Omega - \omega)}{1 - v^2} e^{2\alpha} - \omega \left[ -8\pi \frac{(\rho_0 + \rho_i)(1 + v^2) + 2P v^2}{1 - v^2} e^{2\alpha} - \frac{1}{r^2} \left( \frac{1}{2} \gamma_\sigma + 2\sigma_\mu \right) \right] \right\} + \frac{\mu}{r^2} \left( \frac{1}{2} \gamma_\sigma + 2\sigma_\mu \right) + \sigma_\sigma - \frac{1}{4} \gamma_\sigma + \frac{1}{4} \gamma_\mu \right\}, \quad (17)$$

where $\Omega$ is the angular velocity of the matter as measured at infinity, and $v$ is the proper velocity of the matter with respect to a ZAMO. The proper velocity of the matter is given by

$$v = (\Omega - \omega)r e^{-\sigma} \sin \theta \quad (18)$$

and the coordinate components of the four-velocity of the matter can be written as

$$u^\mu = \frac{e^{-(\sigma+\gamma)/2}}{\sqrt{1 - v^2}} [1, 0, 0, \Omega]. \quad (19)$$
Following Cook, Shapiro, and Teukolsky (Cook et al. 1992), we can write another field equation that determines $\alpha$ and is given by

$$
\alpha_{\mu} = \frac{1}{2} (\gamma_{\mu} + \sigma_{\mu}) - \{(1 - \mu^2)(1 + r \gamma_{r})^2 + \mu - (1 - \mu^2)\gamma_{\mu}\}^{-1}
$$

where

$$
\gamma_{\mu} = \frac{1}{2} \left[ r^2 (\gamma_{rr} + \gamma_{r}^2) - (1 - \mu^2)(\gamma_{\mu}^2 + \gamma_{\mu}) \right] [-\mu + (1 - \mu^2)\gamma_{\mu}] + \frac{3}{2} \mu \gamma_{\mu} [-\mu + (1 - \mu^2)\gamma_{\mu}]
$$

and

$$
\sigma_{\mu} = \frac{1}{4} [(1 - \mu^2)^2] r [2r^2 \omega_{r}\omega_{\mu} - \mu r^2 \omega_{\mu} + 2r^4 \gamma_{r}\omega_{r}\omega_{\mu}] - r^2 (1 - \mu^2)\gamma_{\mu} [r^2 \omega_{r}^2 - (1 - \mu^2)\omega_{\mu}^2].
$$

(20)

3. NUMERICAL METHOD

Here we adopt the rotating neutron star (rns) code based on the KEH (Komatsu et al. 1989) method and written by Stergioulas (Stergioulas 1995) to obtain the frame-dragging rate inside the rotating neutron stars. The equations for the gravitational and matter fields were solved on a discrete grid using a combination of integral and finite difference techniques. The computational domain of the problem is $0 \leq r \leq \infty$ and $0 \leq \mu \leq 1$. It is easy to deal with finite radius rather than the infinite domain via a coordinate transformation to a new radial coordinate $s$, which covers the infinite radial span in a finite coordinate interval $0 \leq s \leq 1$. This new radial coordinate $s$ is defined by

$$
r = r_e \frac{s}{1-s}.
$$

(21)

Thus, $s = (1/2)$ represents the radius of the equator ($r_e$) of the pulsar, and $s = 1$ represents the infinity.

The three elliptical field Equations (11)–(13) were solved by an integral Green’s function approach following the KEH. Taking into account the equatorial and axial symmetry in the configurations, we can find the three metric coefficients $\sigma$, $\gamma$, and $\omega$, which can be written as

$$
\sigma(s, \mu) = -e^{-s} \sum_{n=0}^{\infty} P_{2n}(\mu) \left[ \left( \frac{1-s}{s} \right)^{2n+1} \int_0^{\pi} \frac{d\mu'}{(1-s')^{2n+2}} \int_0^{1} d\mu' P_{2n}(\mu') \tilde{S}_\sigma(s', \mu') \right]
$$

$$
+ \left( \frac{s}{1-s} \right)^{2n-1} \int_0^{1} d\mu' P_{2n}(\mu') \tilde{S}_\sigma(s', \mu') \right].
$$

(22)

$$
\gamma(s, \mu) = -\frac{2e^{-s}}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(2n - 1)\theta]}{(2n - 1) \sin \theta} \left[ \left( \frac{1-s}{s} \right)^{2n} \int_0^{\pi} \frac{d\mu'}{(1-s')^{2n+2}} \int_0^{1} d\mu' \sin[(2n - 1)\theta'] \tilde{S}_\gamma(s', \mu') \right]
$$

$$
+ \left( \frac{s}{1-s} \right)^{2n-2} \int_0^{1} d\mu' \sin[(2n - 1)\theta'] \tilde{S}_\gamma(s', \mu') \right].
$$

(23)

$$
\omega(s, \mu) \equiv r_e \omega(s, \mu) = -e^{-(s-\frac{1}{2})} \sum_{n=0}^{\infty} \frac{P_{2n-1}(\mu)}{2n(2n - 1) \sin \theta} \left[ \left( \frac{1-s}{s} \right)^{2n+1} \int_0^{\pi} \frac{d\mu'}{(1-s')^{2n+2}} \int_0^{1} d\mu' \sin \theta' P_{2n-1}(\mu') \tilde{S}_\omega(s', \mu') \right]
$$

$$
+ \left( \frac{s}{1-s} \right)^{2n-2} \int_0^{1} d\mu' \sin \theta' P_{2n-1}(\mu') \tilde{S}_\omega(s', \mu') \right].
$$

(24)

where $P_n(\mu)$ are the Legendre polynomials, $P_n^m(\mu)$ are the associated Legendre polynomials, and $\sin(n\theta)$ is a function of $\mu$ through $\theta = \cos^{-1} \mu$. The effective sources could be defined as

$$
\tilde{S}_\sigma(s, \mu) = r^2 S_\sigma(s, \mu)
$$

(25)

$$
\tilde{S}_\gamma(s, \mu) = r^2 S_\gamma(s, \mu)
$$

(26)

$$
\tilde{S}_\omega(s, \mu) = r_e r^2 S_\omega(s, \mu).
$$

(27)

The advantages of this Green’s function approach for solving the elliptical field equations is that the asymptotic conditions on $\sigma$, $\gamma$, and $\omega$ are imposed automatically. The numerical integration of Equations (22)–(24) is straightforward. These integrations
are solved using the rna code, and we obtain the value of frame-dragging precession rate inside the rotating neutron star using Equation (8).

3.1. Equation of State (EoS) of Dense Matter

Recent observations of PSR J0348+0432 have reported the measurement of a \(2.01 \pm 0.04 \, M_\odot\) neutron star (Antoniadis et al. 2013). This is the most accurately measured highest neutron star mass so far. The accurately measured neutron star mass is a direct probe of dense matter in its interior. This measured mass puts a strong constraint on the equation of state (EoS).

EoS of dense matter are used as inputs in the calculation of frame dragging in a neutron star interior. We adopt three EoS in this calculation. We are considering EoS of \(\beta\)-equilibrated hadronic matter (Banik et al. 2004). The chiral EoS is based on the QCD motivated chiral SU(3)\(_L\) \(\times\) SU(3)\(_R\) model (Hanaszke et al. 2000) and includes hyperons. We exploit the density-dependent (DD) relativistic mean model to construct the DD2 EoS (Typel et al. 2010). Here the nucleon–nucleon interaction is mediated by the exchange of mesons, and the DD nucleon–meson couplings are obtained by fitting properties of finite nuclei. The other EoS is the Akmal, Pandharipande, and Ravenhall (APR) EoS calculated in the variational chain summation method using Argonne V18 nucleon–nucleon interaction and a fitted three-nucleon interaction along with relativistic boost corrections (Akmal et al. 1998). We calculate the static mass limits of neutron stars using those three EoS. Maximum masses and the corresponding radii of rotating neutron stars at the mass shedding limits are also shown in the tables. These results show that maximum masses in all three cases are above \(2 \, M_\odot\) and are compatible with the benchmark measurement mentioned above.

4. RESULTS AND DISCUSSION

We divide our results into two parts: in the first part we show the frame-dragging effect in some pulsars that rotate with fixed values of Kepler frequencies \(\Omega_K\) and central densities \(\epsilon_c\). Next we consider pulsars whose masses and rotational periods are known from observations. Rotational frequencies of observed pulsars are generally much lower than their Kepler frequencies (\(\Omega < \Omega_K\)).

4.1. Pulsars Rotate with Their Kepler Frequencies \(\Omega = \Omega_K\)

Figure 1 displays the frame-dragging frequency (or LT precession frequency) as a function of radial distance for APR EoS. Panel (a) of the figure represents the results along the equator, whereas panel (b) represents those along the pole. For both panels of Figure 1, we consider rotating neutron stars with central energy densities of \(5.2435 \times 10^{14}\), \(6.404 \times 10^{14}\), and \(7.534 \times 10^{14} \, g/cm^3\), and their corresponding Kepler frequencies are \(4000\) (online version: red), \(5000\) (online version: green), and \(6000\) (online version: blue) s\(^{-1}\), respectively, whereas masses of the rotating compact stars in three cases range from \(\sim 0.6\) to \(\sim 1.7 \, M_\odot\). The Kepler periods \(P_K\) = 1.57 ms, 1.26 ms, and 1.05 ms correspond to the above Kepler frequencies. Here and throughout the paper, \(\Omega_{LT}\) is measured in a Copernican frame. For the cases in panel (a), frame-dragging frequencies decrease initially with increasing distance from the center of the neutron star, respectively. The surface of the neutron star along the pole located around \(0.6 \, r_c\), but the plot is still valid beyond the surface of the pole because our formalism is applicable to regions outside the pulsar.

(A color version of this figure is available in the online journal.)

Table 1

Maximum Gravitational Masses \((M_G/M_\odot)\), Equatorial Radii \((R)\), and Their Corresponding Central Energy Densities \((\epsilon_c)\) for Static \((\Omega = 0)\) and Keplerian Limit \((P = P_K = 2\pi/\Omega_K)\) with Different EoS, where \(P_K\) Is the Kepler Period in Milliseconds

| EoS    | \(P\) (ms) | \(\epsilon_c\) \(\times 10^{15} \, g/cm^3\) | \(M_G/M_\odot\) | \(R\) (km) |
|--------|------------|---------------------------------|----------------|-----------|
| APR    | Static     | 2.78                            | 2.190           | 9.93      |
|        | 0.6291     | 1.50                            | 2.397           | 14.53     |
| DD2    | static     | 1.94                            | 2.417           | 11.90     |
|        | 0.7836     | 1.00                            | 2.677           | 17.53     |
| Chiral | static     | 1.99                            | 2.050           | 12.14     |
|        | 0.8778     | 1.11                            | 2.353           | 18.17     |
and encounter a local minimum at a distance \( r_{\min} \sim 0.5 r_e \), which is well below the surface. It is interesting to note here that the frame-dragging frequencies in all three cases rise again and attain a local maximum at the distance \( r_{\max} \sim 0.7 r_e \) and finally drop to smaller values at the surface. On the other hand, the frame-dragging frequencies along the pole smoothly vary from large values at the center to smaller values at the surface, as evident from panel (b) of Figure 1.

Figure 2 shows the frame-dragging frequency along the equator (panel (a)) and along the pole (panel (b)) for the DD2 EoS, whereas Figure 3 exhibits the frame-dragging frequency along the equatorial distance (panel (a)) and polar distance (panel (b)) for the chiral EoS. In both figures, results are shown for Keplerian frequencies \( \Omega_K = 4000, 5000, \) and 6000 \( \text{s}^{-1} \). However, the central energy densities corresponding to the Keplerian frequencies mentioned above are different for the three EoSs. The behavior of frame-dragging frequencies along the equator and pole in Figures 2 and 3 are qualitatively similar to the results of Figure 1. In Figures 1–3, as the rotation frequency \( \Omega_e \) and the central energy density \( \varepsilon_c \) increase, the frame-dragging frequencies increase, and also the local maxima and minima shift toward the surface of the neutron star along the equator for all three EoSs. It reveals an important conclusion that the ratio of the positions of the local maxima and minima to the radius of the neutron star must depend on \( \Omega \) and \( \varepsilon_c \) for a particular pulsar.

It can be easily seen from Equation (8) that the LT frequency inside a neutron star is a function of both the radial distance \( r \) and colatitude \( \theta \). The colatitude plays a major role in determining the exact frame-dragging frequency at a particular point inside the rotating neutron star, as evident from Figures 1–3. We obtain the LT frequency at the pole by just plugging \( \theta = 0 \) into Equation (8), and it is given by \( \Omega_{\text{LT}} = e^{-(\sigma + \omega)} \Omega \). It should be noted here that the LT frequency is connected to \( \omega \), which appears as the nonvanishing metric component in the metric of the rotating star. According to the theorem by Hartle, the dragging of inertial frames as represented by \( \omega \) with respect to a distant observer decreases smoothly as a function of \( r \) from a large value at the center of the star to a smaller value at the surface (Weber 1999) for both equatorial and polar cases. In this formalism the frame-dragging frequency depends solely on \( r \). For a fixed value of \( r \), one gets the same frequency from the equator to the pole inside the rotating neutron star. We find the behavior of the LT frequency along the pole in panel (b) of Figures 1–3 to be similar to that obtained in Hartle’s formalism. However, our results along the equator are quite different from what was obtained using Hartle’s formalism (Weber 1999). It is evident from Figures 1–3 that the plots are smooth along the pole but not along the equator. For the calculation of the LT frequency along the equator, we find that the second term of Equation (8) does not contribute. Further investigation of the first term involving metric components \( \sigma, \omega \), and their derivatives reveals that this term is responsible for the local maxima and minima along the equator as reported above. The appearance of local maxima and minima in the LT frequencies along the equator may be attributed to the dependence of \( \Omega_{\text{LT}} \) on \( r \) and \( \theta \). As a consistency check, we obtain two solutions for the local maximum and minimum after extremizing Equation (8) with respect to \( r \). Details are given in the Appendix.

The normalized frame-dragging values at the center (\( \bar{\Omega}_e = \Omega_{\text{LT,center}}^c / \Omega \)) and the surface (\( \bar{\Omega}_e = \Omega_{\text{LT,center}}^s / \Omega \)) of the star models with three EoSs are recorded in Table 2. It is noted that the normalized frame-dragging value at the star’s center is at a maximum and
However, for a particular EoS, the normalized frame-dragging value at the star’s center and surface is higher for a fast-rotating star. The inertial frame-dragging effect is higher at the surface of the neutron star along the pole than at the surface of the neutron star along the equator. As the latitude (θ) increases from the equator to the pole, the height between the maximum and minimum of ΩLT diminishes, and after a certain critical angle (θcr) both extrema disappear and the plot is smooth like LT along the pole. The critical angle can be seen from the three-dimensional plot in Figure 4.

Figure 4. 3D plots of ΩLT of the pulsar that is rotating with Ωc = 5000 s⁻¹ as a function of θ and cos θ for (a) DD2 EoS and (b) APR EoS. (A color version of this figure is available in the online journal.)

### Table 2
Normalized Angular Velocities of the Local Inertial Frame Dragging at the Surface Ωc at and the Center Ωc of the Neutron Stars that are Rotating at Their Respective Kepler Periods (PK = 2π/Ωc) as Measured by a Distant Observer

| PK (ms) | Along the Equator | Along the Pole |
|---------|-------------------|----------------|
|         | APR | DD2 | Chiral | APR | DD2 | Chiral |
| Ωc      | 1.57 | 0.008 | 0.013 | 0.019 | 0.046 | 0.069 | 0.099 |
|         | 1.26 | 0.016 | 0.029 | 0.040 | 0.087 | 0.139 | 0.184 |
|         | 1.05 | 0.031 | 0.059 | 0.070 | 0.151 | 0.252 | 0.287 |
| Ωc      | 1.57 | 0.242 | 0.286 | 0.356 | 0.242 | 0.286 | 0.356 |
|         | 1.26 | 0.354 | 0.457 | 0.580 | 0.354 | 0.457 | 0.580 |
|         | 1.05 | 0.515 | 0.723 | 0.884 | 0.515 | 0.723 | 0.884 |

### Table 3
Normalized Angular Velocities of the Local Inertial Frame Dragging at the Surface Ωc at and the Center Ωc of Some Known Rotating Neutron Stars

| Name of the Pulsar | PK (ms) | Along the Equator | Along the Pole |
|--------------------|---------|-------------------|----------------|
|                    | APR | DD2 | Chiral | APR | DD2 | Chiral |
| J1807−2500B        | 4.19 | 0.099 | 0.075 | 0.064 | 0.156 | 0.120 | 0.105 |
| Ωc                 | 22.70 | 0.095 | 0.073 | 0.062 | 0.154 | 0.122 | 0.106 |
| B1257+12           | 6.22 | 0.122 | 0.091 | 0.077 | 0.188 | 0.145 | 0.126 |
| J1807−2500B        | 4.19 | 0.707 | 0.548 | 0.516 | 0.707 | 0.548 | 0.516 |
| Ωc                 | 22.70 | 0.685 | 0.538 | 0.502 | 0.685 | 0.538 | 0.502 |
| B1257+12           | 6.22 | 0.825 | 0.632 | 0.601 | 0.825 | 0.632 | 0.601 |

falls off on the surfaces of the equator and pole for the three EoSs irrespective of whether the compact star is rotating slowly or fast. However, for a particular EoS, the normalized frame-dragging value at the star’s center and surface is higher for a fast-rotating star with PK = 1.05 ms than for a slowly rotating star with PK = 1.57 ms for both cases along the equator and pole. One can see another interesting thing from Table 2: Ωc is always higher at the pole than at the equator for a particular pulsar. This is due to the effect of rotation frequency Ω (of the star) for which pole is nearer to the center than the surface, as is evident from Table 2. Thus, the inertial frame-dragging effect is higher at the surface of the neutron star along the pole than at the surface of the neutron star along the equator.

Now we investigate the dependence of local maxima and minima in ΩLT along the equator on the angle θ. In Figure 4, ΩLT is shown as a function of θ defined by Equation (21) and cos θ for DD2 (panel(a)) and chiral (panel (b)) EoSs. The local maxima and minima in ΩLT along the equator are clearly visible in both panels of Figure 4. It has been already noted that ΩLT along the pole decreases smoothly from the center to the surface. As the latitude (θ = π/2 − θ) increases from the equator to the pole, the height between the maximum and minimum of ΩLT diminishes, and after a certain critical angle (θcr) both extrema disappear and the plot is smooth like the plot along the pole. The critical angle can be seen from the three-dimensional plot in Figure 4. This value of the critical angle is μ ≈ 0.5 or θcr = 30° where the local maximum (rmax) and minimum (rmin) disappear. So, if we plot ΩLT vs. θ at the point rmax for a specific Kepler frequency, (namely, Ωc = 5000 s⁻¹), we find that the frame-dragging frequency increases from the equator to the pole for the specific rmax, as exhibited by Figure 4.

4.2. Pulsars Rotate with Their Frequencies Ω < ΩK

Now we apply our exact formula of ΩLT to three known pulsars. The three pulsars chosen for this purpose are J1807−2500B, J0737−3039A, and B1257+12. The periods of those pulsars are given in Table 3. The masses of those pulsars are also known and range from 1.337 to 1.5 M⊙. Furthermore, we adopt the same EoSs in this calculation as considered in the previous subsection.
Though the periods of these pulsars are larger than the Keplerian periods, the calculation of $\Omega_{LT}$ inside these real pulsars are equally important like the cases with Kepler frequencies already demonstrated.

We calculate the normalized angular velocity at the center and surface of these pulsars with APR, DD2, and chiral EoSs. It is noted from Table 3 that the behavior of the normalized angular velocity from the center to the surface or along the pole and equator is qualitatively the same, as shown in Table 2.

We also plot the frame-dragging frequency ($\Omega_{LT}$) as a function of radial distance along the equator (panel (a)) and pole (panel (b)) for three EoSs in Figures 5–7. The frame-dragging frequency behaves smoothly along the pole from the center to the surface as shown in panel (b) of these figures. The results in panel (a) of the figures show features of local maxima and minima along the equator similar to that found in Figures 1–3. We note that all of the local minima of $\Omega_{LT}$ are located around $r_{\text{min}} \sim 0.7r_e$, and the local maxima are located around $r_{\text{max}} \sim 0.9r_e$ in Figures 5–7.

We also plot the $\Omega_{LT}$ of pulsar J0737–3039A as a function of $s$ and $\cos \theta$ for DD2 (panel (a)) and APR (panel (b)) EoSs in Figure 8. It is noted from Figure 8 that the value of $\theta_{cr}$ is around 30° for the pulsar J0737–3039A for DD2 and APR EoSs.
We have derived the exact frame-dragging frequency inside the rotating neutron star without making any assumption on the metric components and energy-momentum tensor. We show that the frequency must depend both on $r$ and $\theta$. It may be recalled that the frame-dragging frequency depends only on $r$ in Hartle’s formalism because of the slow-rotation approximation. We predict the exact frame-dragging frequencies for some known pulsars as well as neutron stars rotating at their Keplerian frequencies. We have also estimated LT precession frequencies at the centers of these pulsars without imposing any boundary conditions on them. We have found local maxima and minima along the equator due to the dependence of $\Omega_{LT}$ on the colatitude ($\theta$) inside pulsars. The positions of local maximum and minimum depend on the frequency $\Omega$ and the central density $\varepsilon_c$ of the particular pulsar. Furthermore, it is observed that the local maximum and minimum in $\Omega_{LT}$ along the equator disappear at a critical angle $\theta_{cr}$.

QPOs in magnetars were studied by various groups. These studies in several cases were carried out by considering spherical and nonrotating relativistic stars having a dipolar magnetic field configuration (Sotani et al. 2007). It would be worth investigating this problem for rotating relativistic stars. In particular, we are studying the effect of our exact frame-dragging formulation on the magnetic field distribution in the star and its implications on QPOs. This will be published in the future.

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APPENDIX

**CONSISTENCY CHECK OF LOCAL MAXIMUM AND MINIMUM IN $\Omega_{LT}$**

To find the local maximum and minimum in $\Omega_{LT}$ we differentiate Equation (8) with respect to $r$ and obtain

$$\frac{d\Omega_{LT}}{dr} = \Omega_{LT} \left[ - (\alpha_r + \sigma_r) - 2 \frac{r \omega \sin^2 \theta (r \omega_r + \omega) - \sigma r e^{2\sigma}}{\omega r^2 \sin^2 \theta - e^{2\sigma}} \right] + \frac{1}{\Omega_{LT}} \frac{e^{-2(\alpha + \sigma)}}{4(\omega r^2 \sin^2 \theta - e^{2\sigma})^2} \cdot \left\{ A \sin^2 \theta \left[ \omega r^2 \sin^2 \theta (3 \omega \omega_r + 2r \omega_r^2 + r \omega \omega_{rr}) + 2 \sigma r e^{2\sigma} (2 \omega + r \omega_r - 2 \omega \sigma_r) \right] + e^{2\sigma} (3 \omega_r + r \omega_{rr} - 2r \omega \sigma_r - 2 \omega \sigma_r - 2r \sigma_r) \right\} \left[ \frac{\omega r}{\omega \sin^3 \theta (2 \omega \omega_r + 2r \omega_r \omega_r + r \omega \omega_{rr})} + B \left[ \omega r \sin^3 \theta (2 \omega \omega_r + 2r \omega_r \omega_r + r \omega \omega_{rr}) \right] + 2 \sigma_r e^{2\sigma} (2 \omega \cos \theta + \omega_r \sin \theta - 2 \omega \sigma_r \sin \theta - 2 \omega \sigma_r \sin \theta + 2 \omega(\omega \sigma_r + \omega_r \sigma_r)) \right\},$$

(A1)

where

$$A = r^3 \omega^2 \omega_r \sin^2 \theta + e^{2\sigma} (2 \omega + r \omega_r - 2 \omega \sigma_r),$$

(A2)

$$B = r^2 \omega^2 \omega_r \sin^3 \theta + e^{2\sigma} (2 \omega \cos \theta + \omega_r \sin \theta - 2 \omega \sigma_r \sin \theta)$$

(A3)

and

$$\omega_{r\theta} = \frac{\partial^2 \omega}{\partial \theta \partial r}.$$
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