1 Problem

Estimate the period $\tau$ of a “simple” harmonic oscillator consisting of a zero-rest-length massless spring of constant $k$ that is connected to a rest mass $m_0$ (with the other end of the spring fixed to the origin), taking into account the relativistic mass.

2 Solution

2.1 Quick Estimates

Ignoring relativistic effects, the angular frequency $\omega_0$ and the period $\tau_0$ of the oscillator are,

$$\omega_0 = \sqrt{\frac{k}{m_0}}, \quad \tau_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m_0}{k}}. \tag{1}$$

In this approximation, the oscillating mass has position and velocity,

$$x = A \cos \omega_0 t, \quad v = -A \omega_0 \sin \omega_0 t. \tag{2}$$

In general, the oscillating mass has (time-dependent) relativistic mass,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \approx m_0 \left(1 + \frac{v^2}{2c^2}\right), \tag{3}$$

where $c$ is the speed of light in vacuum. We expect that the period $\tau$ of oscillation of the relativistic mass can be approximated as,

$$\tau \approx \frac{2\pi}{\omega} = 2\pi \sqrt{\langle m \rangle / k} > \tau_0, \tag{4}$$

where $\langle m \rangle > m_0$ is an appropriate average of the relativistic mass. This might be the time average,

$$\langle m \rangle_t = \frac{1}{\tau} \int_0^\tau m(t) \, dt \approx \frac{m_0}{\tau} \int_0^\tau \left(1 + \frac{v^2}{2c^2}\right) \, dt \approx m_0 \left(1 + \frac{1}{2\tau_0 c^2} \int_0^{\tau_0} A^2 \omega_0^2 \cos^2 \omega_0 t \, dt\right)$$

$$= m_0 \left(1 + \frac{A^2 \omega_0^2}{4c^2}\right) = m_0 \left(1 + \frac{k A^2}{4m_0 c^2}\right), \tag{5}$$

in which case,

$$\tau \approx \tau_0 \sqrt{1 + \frac{k A^2}{4m_0 c^2}} \approx \tau_0 \left(1 + \frac{k A^2}{8m_0 c^2}\right), \quad \langle m \rangle = \langle m \rangle_t. \tag{6}$$
However, it could be that the spatial average is more appropriate,

\[
\langle m \rangle_x = \frac{1}{A} \int_{0}^{A} m(x) \, dx \approx \frac{m_0}{A} \int_{0}^{A} \left(1 + \frac{v^2}{2c^2}\right) \, dx \approx m_0 \left[1 + \frac{1}{2Ac^2} \int_{0}^{A} A^2 \omega_0^2 \left(1 - \frac{x^2}{A^2}\right) \, dx \right]
\]

\[
= m_0 \left(1 + \frac{A^2 \omega_0^2}{4c^2}\right) = m_0 \left(1 + \frac{kA^2}{3m_0c^2}\right),
\]

noting that \(\sin \omega t = \sqrt{1 - \cos^2 \omega t} \approx \sqrt{1 - x^2/A^2}\), in the approximation that oscillating mass has \(x\)-coordinate \(x = A \cos \omega t\). In this case,

\[
\tau \approx \tau_0 \sqrt{1 + \frac{kA^2}{3m_0c^2}} \approx \tau_0 \left(1 + \frac{kA^2}{6m_0c^2}\right), \quad \langle m \rangle = \langle m \rangle_x.
\]

As many other averages of the relativistic mass can be imagined, we seek a method that clarifies what type of approximation is best.

### 2.2 A Better Estimate

A different approach is to note that the motion is periodic with spatial amplitude \(A\), and so the period \(\tau\) can be computed as,

\[
\tau = 4 \int_{0}^{A} \frac{dt}{dx} \, dx = 4 \int_{0}^{A} \frac{dx}{v}.
\]

Total energy \(E\) is conserved in this example,

\[
E = mc^2 + \frac{kx^2}{2} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} + \frac{kx^2}{2} = m_0c^2 + \frac{kA^2}{2},
\]

where the potential energy of the system is \(kx^2/2\), such that,

\[
\frac{1}{v} = \frac{\tau_0}{2\pi} \frac{1 + k(A^2 - x^2)/2m_0c^2}{\sqrt{A^2 - x^2} \sqrt{1 + k(A^2 - x^2)/4m_0c^2}} \approx \frac{\tau_0}{2\pi} \left(\frac{1}{\sqrt{A^2 - x^2}} + \frac{3k \sqrt{A^2 - x^2}}{8m_0c^2}\right).
\]

Hence,

\[
\tau \approx \frac{2\tau_0}{\pi} \left(\int_{0}^{A} \frac{dx}{\sqrt{A^2 - x^2}} + \frac{3k}{8m_0c^2} \int_{0}^{A} \sqrt{A^2 - x^2} \, dx\right) = \tau_0 \left(1 + \frac{3kA^2}{16m_0c^2}\right).
\]

The correction term in this result is 2\% larger than that in the estimate (8) based on the spatial average of the relativistic mass.

The “exact” period of a relativistic harmonic oscillator can be given as an elliptic integral. A series approximation to this integral is given in [2].

\^There is a sign error in the correction term of eq. (7-150), p. 325 of [1], which corresponds to eq. (11) of the present note. Thanks to Bill Jones for pointing this out.
References

[1] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, 1980),
http://kirkmcd.princeton.edu/examples/mechanics/goldstein_3ed.pdf

[2] L.A. MacColl, *Theory of the Relativistic Harmonic Oscillator*, Am. J. Phys. 25, 535 (1957), http://kirkmcd.princeton.edu/examples/mechanics/maccoll_ajp_25_535_57.pdf