THE UPSILON INVARIANT AT 1 OF 3-BRAID KNOTS

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Abstract. We provide explicit formulas for the integer-valued smooth concordance invariant \( \upsilon(K) = \Upsilon_K(1) \) for every 3-braid knot \( K \). We determine this invariant, which was defined by Ozsváth, Stipsicz and Szabó [OSS17a], by constructing cobordisms between 3-braid knots and (connected sums of) torus knots. As an application, we show that for positive 3-braid knots \( K \) several alternating distances all equal the sum \( g(K) + \upsilon(K) \), where \( g(K) \) denotes the 3-genus of \( K \). In particular, we compute the alternation number, the dealternating number and the Turaev genus for all positive 3-braid knots. We also provide upper and lower bounds on the alternation number and dealternating number of every 3-braid knot which differ by 1.

1. Introduction

We study knots in the 3-sphere \( S^3 \), i.e. non-empty, connected, oriented, closed smooth 1-dimensional submanifolds of \( S^3 \), considered up to ambient isotopy. Two knots \( K \) and \( J \) are called concordant if there exists an annulus \( A \approx S^1 \times [0,1] \) smoothly and properly embedded in \( S^3 \times [0,1] \) such that \( \partial A = K \times \{0\} \cup J \times \{1\} \) and such that the induced orientation on the boundary of the annulus agrees with the orientation of \( K \), but is the opposite one on \( J \). Knots up to concordance form a group, the concordance group \( C \), with the group operation induced by connected sum.

In [OSS17a], Ozsváth, Stipsicz and Szabó used the Heegaard Floer knot complex to define the invariant \( \Upsilon_K \) of a knot \( K \), which induces a homomorphism from the knot concordance group to the group of real-valued piecewise linear functions on the interval \( [0,2] \). The function \( \Upsilon_K \) evaluated at \( t = 1 \), \( \upsilon(K) := \Upsilon_K(1) \), induces a homomorphism \( C \to \mathbb{Z} \). In this article, we will call \( \upsilon(K) \) upsilon of \( K \).

A 3-braid is an element of the braid group on three strands, denoted \( B_3 \). The classical presentation of \( B_3 \) with generators \( a \) and \( b \) and relation \( aba = bab \), the braid relation, was introduced by Artin [Art25]. A braid word \( \gamma \) — a word in the generators of \( B_3 \) and their inverses — defines a diagram for a (geometric) 3-braid; the generators \( a \) and \( b \) correspond to the geometric 3-braids given by braid diagrams as in Figure 1. In our figures, braid diagrams will always be oriented from bottom to top. We denote by \( \Delta \) the braid \( aba = bab \), and note that its square \( \Delta^2 = (ab)^3 \) (the positive full twist on three strands) generates the center of \( B_3 \) [Cho18 Theorem 3]. A 3-braid knot is a knot that arises as the closure \( \hat{\gamma} \) of a 3-braid \( \gamma \).

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As our main result, we determine the upsilon invariant for all 3-braid knots. More precisely, we show the following.

**Theorem 1.1.** Let $\gamma = \Delta^2 a^{-p_1}bq_1^{-1}a^{-p_2}bq_2^{-1} \cdots a^{-p_r}bq_r^{-1}$ be a braid word in the generators $a$ and $b$ of $B_3$ for some integers $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$, where $\Delta^2 = (ab)^3$. Suppose that the closure $K = \hat{\gamma}$ of $\gamma$ is a knot. Then its upsilon invariant is

$$\upsilon(K) = \frac{\sum_{i=1}^{r} (p_i - q_i)}{2} - 2\ell.$$

It follows from Murasugi’s classification of the conjugacy classes of 3-braids [Mur74, Proposition 2.1] that indeed all 3-braid knots — except for the torus knots that are closures of 3-braids — are covered by Theorem 1.1. However, for torus knots the invariant $\upsilon$ can be calculated explicitly by a combinatorial, inductive formula in terms of their Alexander polynomial [OSS17a, Theorem 1.15]; see Equation (12) below. Hence, we have indeed determined $\upsilon(K)$ for all 3-braid knots $K$.

As an application of Theorem 1.1, we show that the following invariants coincide for **positive 3-braid knots** — knots that are the closure of positive 3-braids.

**Corollary 1.2.** Let $K$ be a knot that is the closure of a positive 3-braid, i.e. an element of $B_3$ that can be written as a word in the generators $a$ and $b$ only (no inverses). Then

$$\text{alt}(K) = \text{dalt}(K) = \text{g}_T(K) = \text{A}_{s}(K) = g(K) + \upsilon(K).$$

Here, the **alternation number** $\text{alt}(K)$, **dealternating number** $\text{dalt}(K)$ and **Turaev genus** $\text{g}_T(K)$ are different ways of measuring how far the knot $K$ is from being alternating. The best known among them is certainly the first one: the alternation number $\text{alt}(K)$ of a knot $K$ was first defined by Kawauchi [Kaw10] as the minimal Gordian distance of $K$ to the set of alternating knots. In Section 5 we will review the precise definition and prove Corollary 1.2. The invariant $\text{A}_{s}(K)$ introduced by Friedl, Livingston and Zentner [FLZ17] is defined as the minimal number of double point singularities in a generically immersed concordance from a knot $K$ to an alternating knot. Lastly, $g(K)$ denotes the 3-genus of $K$, the minimal genus of a compact, connected, oriented smooth surface in $S^3$ with oriented boundary the knot $K$. 

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**Figure 1.** Generators and relation in the braid group $B_3$. 

(a) The two generators $a$ and $b$. 

(b) The braid relation $aba = bab$. 

Two other corollaries of Theorem 1.1 for positive 3-braid knots are the following.

**Corollary 1.3.** Let \( K \) be a positive 3-braid knot. Then the minimal \( r \) such that \( K \) is the closure of \( a^{p_1} b^{q_1} a^{p_2} b^{q_2} \cdots a^{p_r} b^{q_r} \) for positive integers \( p_i, q_i, i \in \{1, \ldots, r\} \), is 
\[
 r = g(K) + v(K) + 1.
\]

**Corollary 1.4.** If \( K \) and \( J \) are concordant knots that are both closures of positive 3-braids, then the minimal \( r \) from Corollary 1.3 is the same for both \( K \) and \( J \).

Proposition 3.2 in Section 3 provides a normal form for 3-braids, the Garside normal form, which is different from Murasugi’s normal form mentioned above (cf. Definition 4.1). The Garside normal form allows us to read off from a braid word whether it is conjugate to a positive braid word. In Section 6 we provide formulas for the fractional Dehn twist coefficient for all 3-braids in Garside normal form; see Corollary 6.1.

**Proof strategy for Theorem 1.1.** A crucial property of the invariant \( v \) is that it provides a lower bound on the 4-genus \( g_4(K) \) of a knot \( K \), the minimal genus of a compact, connected, oriented surface smoothly embedded in the 4-ball \( B^4 \) with oriented boundary the knot \( K \) in \( S^3 = \partial B^4 \): we have
\[
 |v(K)| \leq g_4(K) \tag{1}
\]
for any knot \( K \) [OSS17a, Theorem 1.11]. Our general strategy to find \( v(K) \) for any 3-braid knot \( K \) will be to construct a cobordism between \( K \) and another knot \( J \) for which the value of \( v \) is known. A cobordism between \( K \) and \( J \) is a smoothly and properly embedded oriented surface \( C \) in \( S^3 \times [0, 1] \) with boundary \( K \times \{0\} \cup J \times \{1\} \) such that the induced orientation on the boundary of \( C \) agrees with the orientation of \( K \) and disagrees with the orientation of \( J \). We have
\[
 |v(K) - v(J)| \leq g(C) \tag{2}
\]
for any cobordism \( C \) between \( K \) and \( J \), where \( g(C) \) denotes the genus of the cobordism; see inequality (15) in Section 4.1. This provides bounds on \( v(K) \) in terms of \( v(J) \) and \( g(C) \).

We will find such cobordisms for example by algebraic modifications of a braid word representing \( K \) and by saddle moves corresponding to the addition or deletion of generators from such braid words. We will also repeatedly make use of the trick described in Example 4.1 in Section 4.1 of looking at cobordisms of genus 1 between \( \hat{\gamma} # T_{2,2n+1} \) and \( \gamma b^{2n} \) for 3-braid words \( \gamma \) and \( n \geq 1 \).

To prove Theorem 1.1 we will first determine \( v \) for all positive 3-braid knots and then generalize our computations to all 3-braid knots. This extension was somewhat unexpected for the author since, in contrast, the same method would not work to determine slice-torus invariants [Lew14] like the invariant \( \tau \) defined by Ozsváth and Szabó [OS03] or Rasmussen’s invariant \( s \) [Ras10] for all 3-braid knots. We will elaborate on this in Section 4.4.2.

**Remark 1.5.** As we will only use properties of the upsilon invariant (see Section 2.2) and not its definition, we can similarly determine any concordance homomorphism \( C \to \mathbb{Z} \) whose absolute value bounds the 4-genus of a knot from below and which takes the same value as \( v \) on torus knots of braid index 2 and 3. An example is \( -\frac{1}{2} \) for the concordance invariant \( t \) constructed by Ballinger [Bal20] from the \( E(-1) \)
spectral sequence on Khovanov homology. The invariant $t$ defines a concordance homomorphism valued in the even integers which satisfies $|t(K)| \leq g_t(K)$ for any knot $K$. Moreover, it fulfills $t(T_{p,q}) = -2 \nu(T_{p,q})$ for the torus knots $T_{p,q}$ for any coprime positive integers $p$ and $q$. The same method we use for the proof of Theorem 1.1 shows that $t(K) = -2 \nu(K)$ for any 3-braid knot $K$.

Remark 1.6. Theorem 1.1 and a result of Erle imply that $\sigma(K) = 2\nu(K)$ for all 3-braid knots $K$ except when $K = \pm T_{3,3\ell+k}$ for odd $\ell > 0$ and $k \in \{1, 2\}$. Here $\sigma(K)$ denotes the classical signature of the knot $K$. In the exceptional cases, we have $\sigma(K) = 2\nu(K) - 2$. This observation improves a result by Feller and Krcatovich who showed that $|\nu(K) - \sigma(K)| \leq 2$ for all 3-braid knots $K$. See also Section 4.4.1.

Organization. The remainder of this article is organized as follows. In Section 2 we will provide the necessary background on (positive) braids and the upsilon invariant before providing a normal form for 3-braids (Proposition 3.2) that we call Garside normal form in Section 3. Then in Section 4, after a more detailed outline of our proof strategy (Section 4.1), we will prove Theorem 1.1 first for positive 3-braid knots (Section 4.2) and afterwards in the general 3-braid case (Section 4.3). We will prove Corollary 1.3 and Corollary 1.4 in Section 4.2. Section 4.4 will provide further context on our results. Section 5 is concerned with the proof of Corollary 1.2 (Section 5.1) and the application of our result about the upsilon invariant to alternating distances of general 3-braid knots (Section 5.2). In particular, we determine the alternation number of any 3-braid knot up to an additive error of at most 1. Finally, in Section 6 we determine the fractional Dehn twist coefficient for all 3-braids in Garside normal form.

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2. Preliminaries

We recall important concepts about knots and braids, and also the necessary properties of the upsilon invariant and the knot invariant $\tau$ coming from Heegaard Floer homology.

2.1. Knots and Braids

By a fundamental theorem of Alexander, every knot in $S^3$ can be represented as the closure of a geometric $n$-braid for some positive integer $n$. An $n$-braid is an element of the braid group on $n$ strands, denoted $B_n$, which is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2, \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

[Art25].

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1 We use the standard signature convention that the positive torus knots have negative signatures, e.g. $\sigma(T_{3,2}) = -2$. 

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We call a word in the generators of $B_n$ and their inverses a \textit{braid word}. A braid word defines a diagram for a (geometric) $n$-braid where the generators $\sigma_i$ of the braid group correspond to the geometric $n$-braids given by the braid diagrams in which the $i$-th and $(i+1)$-st strands cross once positively. In the following, we will always identify braid words with the corresponding geometric braids, and we suppress $n$ if the context is clear.

By gluing the top ends of the (oriented) strands of a geometric braid $\gamma \in B_n$ to the corresponding bottom ends, we get a knot (or link) $\hat{\gamma}$, called the \textit{closure} of $\gamma$. If $\gamma$ induces a permutation with only one cycle on the ends of its $n$ strands, then its closure $\hat{\gamma}$ is a knot and we call it an \textit{n-braid knot}. Note that conjugate braids $\gamma_0, \gamma_1 \in B_n$, denoted by $\gamma_0 \sim \gamma_1$, have isotopic closures $\hat{\gamma}_0 = \hat{\gamma}_1$. For a more detailed account on braids, we refer the reader to [BB05].

A \textit{positive braid} is an element of the braid group $B_n$ for some $n$ that can be written as a positive braid word $\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_l}$ with $s_i \in \{1, \ldots, n-1\}$. A knot is called a \textit{positive braid knot} if it can be represented as the closure of a positive braid. The set of positive braid knots contains the sets of (positive) torus knots and algebraic knots, while itself being a subset of the set of positive knots or, more generally, the frequently studied set of (strongly) quasipositive knots.

Let $\text{wr}(\gamma)$ denote the \textit{writhe} of a braid word $\gamma \in B_n$, i.e., the exponent sum of the word $\gamma$. If $\gamma$ is a positive $n$-braid such that $K = \hat{\gamma}$ is a knot, then, by work of Bennequin [Ben83] and Rudolph [Rud03] — the latter building on Kronheimer and Mrowka’s proof of the local Thom conjecture [KM93] — we have

\begin{equation}
\label{eq:writhe}
\text{g}_4(K) = \text{g}(K) = \frac{\text{wr}(\gamma) - n + 1}{2}.
\end{equation}

2.2. \textbf{The concordance invariants $\tau$ and $\Upsilon$}

In [OS03], Ozsváth and Szabó constructed the knot invariant $\tau$ via the knot filtration on the Heegaard Floer chain complex of $S^3$; the latter was also defined independently by Rasmussen [Ras03]. The invariant $\tau$ induces a group homomorphism $C \to \mathbb{Z}$ from the (smooth) knot concordance group $C$ to the group of integers $\mathbb{Z}$ and gives a lower bound on the 4-ball genus $g_4(K)$: we have $|\tau(K)| \leq g_4(K)$ for any knot $K$. For the torus knots $T_{p,q}$, where $p$ and $q$ are coprime positive integers, the invariant $\tau$ recovers the 3-genus [OS03, Corollary 1.7], namely we have

\begin{equation}
\label{eq:tau_torus}
\tau(T_{p,q}) = g(T_{p,q}) = \frac{(p-1)(q-1)}{2}.
\end{equation}

Moreover, it follows from [Liv04, Theorem 4 and Corollary 7] together with Equation (3) above that, for any knot $K$ that is the closure of a positive $n$-braid $\gamma$, we have

\begin{equation}
\label{eq:tau_and_g}
\tau(K) = \frac{\text{wr}(\gamma) - n + 1}{2} = g_4(K) = g(K).
\end{equation}

The invariant $\Upsilon$ was defined by Ozsváth, Stipsicz and Szabó in [OSS17a]. We will not recall the definition of $\Upsilon$ via the knot Floer complex $CFK^\infty(K)$ since the properties of $\Upsilon$ mentioned below will be enough for our later computations and we will not explicitly use the Heegaard Floer theory behind it. For an overview on the properties of $\Upsilon$, see the original article [OSS17a] or Livingston’s notes on $\Upsilon$ [Liv17]; see [Hom17] for a survey on Heegaard Floer homology and knot concordance.
For every knot $K$, the knot invariant $\Upsilon_K : [0, 1] \to \mathbb{R}$ is a continuous, piecewise linear function with the following properties [OSS17a]:

1. $\Upsilon_K(0) = 0$,
2. the slope of $\Upsilon_K(t)$ at $t = 0$ is given by $-\tau(K)$,
3. $\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t)$ for all $0 \leq t \leq 1$ and all knots $K_1$ and $K_2$,
4. $\Upsilon_{-K}(t) = -\Upsilon_K(t)$ for all $0 \leq t \leq 1$,
5. $|\Upsilon_K(t)| \leq g_3(K)t$ for all $0 \leq t \leq 1$.

Here, $-K$ is the knot obtained by mirroring $K$ and reversing its orientation. Its concordance class is the inverse of the class of $K$ in the knot concordance group $\mathcal{C}$. It follows from [89-10] that $\Upsilon$ induces a homomorphism from the concordance group to the group of real-valued piecewise linear functions on the interval $[0, 1]$.

For some classes of knots, the invariant $\Upsilon$ can be explicitly computed in terms of classical knot invariants like the signature and the Alexander polynomial.

**Proposition 2.1** [OSS17a Theorem 1.14]. We have $\Upsilon_K(t) = \frac{\sigma(K)}{2} t$ for all alternating or quasi-alternating knots $K$ and all $0 \leq t \leq 1$.

For positive torus knots, $\Upsilon_K(t)$ is completely determined by a combinatorial formula in terms of their Alexander polynomial [OSS17a, Theorem 1.15]. For torus knots of braid index 2 or 3, the following holds; see e.g. [Pel16]. For $\ell \geq 0$, we have

$$\Upsilon_{T_{2,2\ell+1}}(t) = -\tau(T_{2,2\ell+1}) \cdot t = -\ell \cdot t$$

for all $0 \leq t \leq 1$.

For $\ell \geq 0$ and $k \in \{1, 2\}$, we have

$$\Upsilon_{T_{3,3\ell+1}}(1) = \Upsilon_{T_{3,3\ell+2}}(1) + 1 = -2\ell,$$

$$\Upsilon_{T_{3,3\ell+k}}(t) = -\tau(T_{3,3\ell+k}) t = -(3\ell + k - 1) t$$

for all $0 \leq t \leq \frac{2}{3}$ and

$$\Upsilon_{T_{3,3\ell+k}}(t)$$

is linear on $[\frac{2}{3}, 1]$.

### 3. The Garside normal form for 3-braids

In this section, we provide a classification result on the conjugacy classes of 3-braids; see Proposition 3.2. This result is basically due to work of Garside [Gar69] who gave the first solution to the conjugacy problem for all braid groups $B_n$, $n \geq 3$, in 1965. Proposition 3.2 might be known to the experts, but since the explicit formulas appear to be missing from the literature, we will provide them here.

Throughout, we denote the two generators of the braid group $B_3$ by $a := \sigma_1$ and $b := \sigma_2$ which are subject to the braid relation $aba = bab$. Recall that the braid $\Delta^2 = (aba)^2 = (ab)^3$ generates the center of $B_3$.

**Remark 3.1.** Any 3-braid is conjugate to the same braid with generators $a$ and $b$ interchanged. More precisely, let $\gamma = a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ for some $r \geq 1$ and integers $p_i, q_i$, $i \in \{1, \ldots, r\}$, be a 3-braid. Then using $\Delta a = b\Delta$ and $\Delta b = a\Delta$, we have

$$\gamma = \Delta^{-1} \Delta a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} = \Delta^{-1} \Delta b^{q_1} a^{p_1} \cdots a^{p_r} b^{q_r} \Delta \sim b^{q_1} a^{p_1} \cdots b^{q_r} a^{p_r}.$$

In Proposition 3.2 we will provide a certain standard form for the conjugacy classes of 3-braids.
Proposition 3.2. Let $\gamma$ be a 3-braid. Then $\gamma$ is conjugate to one of the 3-braids
$$(A) \quad \Delta^{2\ell} a^p$$
for $\ell \in \mathbb{Z}$, $p \geq 0$,
$$(B) \quad \Delta^{2\ell} a^p b$$
for $\ell \in \mathbb{Z}$, $p \in \{1, 2, 3\}$,
$$(C) \quad \Delta^{2\ell} a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$$
for $\ell \in \mathbb{Z}$, $r \geq 1$, $p_i, q_i \geq 2$, $i \in \{1, \ldots, r\}$,
$$(D) \quad \Delta^{2\ell+1} a^{p_1} b^{q_1} \cdots a^{p_r-1} b^{q_r-1} a^{p_r}$$
for $\ell \in \mathbb{Z}$, $r \geq 1$, $p_r \geq 2$, $p_i, q_i \geq 2$,
\hspace{5cm} $i \in \{1, \ldots, r-1\}$.

If $\gamma$ is a positive 3-braid, then $\ell \geq 0$. If $\hat{\gamma}$ is a knot, then only the cases $(B)$–$(D)$ can occur and $p$ must be odd in case $(B)$, at least one of the $p_i$ and one of the $q_i$ must be odd in case $(C)$, and at least one of the $p_i$ or $q_i$ must be odd in case $(D)$.

While we will never use it in this article, we note — without proof — the following uniqueness result related to Proposition 3.2.

Remark 3.3. Up to cyclic permutation of the powers $p_1, q_1, \ldots, p_r, q_r$ in $(C)$ and $p_1, q_1, \ldots, p_{r-1}, q_{r-1}, p_r$ in $(D)$, respectively, each 3-braid is conjugate to exactly one of the 3-braids listed in Proposition 3.2. This follows from Garside’s work [Gar69]. In his notation, each of the 3-braids listed in (A)–(D) in Proposition 3.2 is the standard form of a certain element in the (so-called) summit set of $\gamma$. For 3-braids of the form $(C)$ or $(D)$, the summit set consists of those 3-braids obtained by cyclic permutation of the powers $p_1, q_1, \ldots, p_r, q_r$ in $(C)$ and $p_1, q_1, \ldots, p_{r-1}, q_{r-1}, p_r$ in $(D)$, respectively.

Definition 3.4. We call a braid word of the form in $(A)$–$(D)$ a 3-braid in Garside normal form.

Remark 3.5. The advantage of the Garside normal form over Murasugi’s normal form for 3-braids used later in Section 4.3 (see Definition 4.15) is that positive 3-braids are easier to detect in this normal form: if $\gamma$ is a positive 3-braid, then $\gamma$ is conjugate to one of the braids in (A)–(D) with $\ell \geq 0$. Since Garside’s solution to the conjugacy problem works for any $n$-braid with $n \geq 3$, one might hope to generalize an explicit standard form as in Proposition 3.2 to $n$-braids for any $n \geq 3$.

Remark 3.6. For odd $p$, case $(B)$ of Proposition 3.2 covers the torus knots of braid index 3. More precisely, if $\gamma \sim \Delta^{2\ell} a b = (a b)^{3\ell+1}$, then its closure is $\hat{\gamma} = T_{3,3\ell+1}$ for $\ell \geq 0$ and $\hat{\gamma} = -T_{3,3\ell+2}$ for $\ell < 0$, and if $\gamma \sim \Delta^{2\ell} a^3 b \sim (a b)^{3\ell+2}$, then $\hat{\gamma} = T_{3,3\ell+2}$ for $\ell \geq 0$ and $\hat{\gamma} = -T_{3,3\ell+1}$ for $\ell < 0$.

Proof of Proposition 3.2. The proof will follow from the following claim.

Claim 1. Let $\gamma$ be a positive 3-braid. Then $\gamma$ is conjugate to one of the 3-braids in (A)–(D) with $\ell \geq 0$.

We first deduce Proposition 3.2 from this claim. To that end, let $\gamma$ be any 3-braid. If $\gamma$ is a positive braid, we are done by Claim 1. If not, then $\gamma$ can be written in the form $\gamma = \Delta^m \alpha$ where $m$ is a negative integer and $\alpha$ a positive 3-braid [Gar69, Theorem 5]. In fact, inserting $\Delta^{-1} \Delta$ if $m$ is odd, we can assume $\gamma$ to be of the form $\Delta^{-2n} \alpha$ for some $n \geq 1$ and a positive 3-braid $\alpha$. The proposition then easily follows using the claim for $\alpha$. It remains to prove Claim 1.

Proof of Claim 1. A positive 3-braid $\gamma$ has the form $\gamma = a^{P_1} b^{Q_1} \cdots a^{P_R} b^{Q_R}$ for integers $R \geq 1$, $P_i, Q_i \geq 0$, $i \in \{1, \ldots, R\}$. If all the $P_i$ or all the $Q_i$ are 0, then
(possibly using Remark 3.1) γ is conjugate to $a^p$ for some $p \geq 0$ and we are in case (A) for $\ell = 0$. Possibly after conjugation and reduction of $R$, we can thus assume that all of the integers $P_i, Q_i$ are non-zero. If $P_1, Q_1 \geq 2$ applies for all $i \in \{1, \ldots, R\}$, then γ is of the form in (C) for $\ell = 0$. If $R = 1$, i.e., $\gamma = a^{P_1} b^{Q_1}$ for integers $P_1, Q_1 \geq 1$, and $P_1 = 1$ or $Q_1 = 1$, then (possibly using Remark 3.1) γ is conjugate to a braid of the form in (B).

It remains to consider the case where $R \geq 2$ and at least one of the $P_i$ or $Q_i$ is 1. In that case — if necessary after conjugation — γ contains $\Delta = aba = bab$ as a subword and is thus conjugate to $\Delta\alpha$ for some positive 3-braid α. Now, let $n \geq 1$ be maximal with the property that γ is conjugate to $\Delta^n\alpha$ for some positive 3-braid α. Then, possibly after conjugation of γ, the braid word α must be one of the following:

$$a^p \quad \text{for } p \geq 0,$$
$$a^p b \quad \text{for } p \geq 1,$$
$$a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} \quad \text{for } r \geq 1, p_i, q_i \geq 2, i \in \{1, \ldots, r\},$$
$$a^{p_1} b^{q_1} \cdots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \quad \text{for } r \geq 1, p_r \geq 2, p_i, q_i \geq 2, i \in \{1, \ldots, r-1\}.$$  

Indeed, using Remark 3.1 up to conjugation these are the only possible words such that $\Delta^n\alpha$ does not contain any additional $\Delta$ as a subword. Note that α can be the empty word, which is covered by the first case in (13) for $p = 0$. Further, note that

$$\Delta^{2f} a^{p_1} b^{q_1} \sim \Delta^{2f+1} a^{p-2}, \quad \Delta^{2f+1} a^{p_1} b^{q_1} \sim \Delta^{2f+1} a^{p-2}, \quad \Delta^{2f+1} a^{p_1} b^{q_1} \sim \Delta^{2f+1} a^{p-2},$$

$$\Delta^{2f+1} a^{p_1} b^{q_1} \sim \Delta^{2f+1} a^{p_1} b^{q_1} a^{p_2} \cdots b^{q_{r-1}} a^{p_r} \quad \text{and}$$

$$\Delta^{2f+1} a^{p_1} b^{q_1} a^{p_2} \cdots b^{q_{r-1}} a^{p_r} \sim \Delta^{2f+1} a^{p_1} b^{q_1} a^{p_2} \cdots b^{q_{r-1}} a^{p_r}$$

for any $\ell \geq 0, p \geq 1, p_i, q_i \geq 2, i \in \{1, \ldots, r\}$. It follows from a case by case analysis of the cases in (13), using (14) and taking the parity of $n$ into account, that any positive 3-braid is conjugate to one of the 3-braids in (A)–(D) with $\ell \geq 0$.

This concludes the proof of Proposition 3.2 \hfill \square

4. The upsilon invariant of 3-braid knots

In this section, we prove Theorem 1.1. Along the way, we compute the invariant $\upsilon$ for positive 3-braid knots in Garside normal form (Proposition 4.2) and prove Corollary 1.3 and Corollary 1.4.

4.1. Methodology

We first recall inequality (2) from the introduction — which will be repeatedly used in Section 4 — in more generality.

The cobordism distance $d(K, J)$ between two knots $K$ and $J$ is defined as the 4-genus $g_4(K \# J)$ of the connected sum of $K$ and the inverse of $J$. Equivalently, the cobordism distance $d(K, J)$ could be defined as the minimal genus of a smoothly and properly embedded oriented surface $C$ in $S^3 \times [0, 1]$ with boundary $K \times \{0\} \cup J \times \{1\}$ such that the induced orientation on the boundary of $C$ agrees with the orientation of $K$ and disagrees with the orientation of $J$. Suppose the genus of a cobordism $C$ between two knots $K$ and $J$ is $g(C)$. We then have $d(K, J) \leq g(C)$, so by the
properties (8)-(10) of $\Upsilon$ from Section 2.2 we get
\begin{equation}
|\Upsilon_K(t) - \Upsilon_J(t)| = |\Upsilon_{K\#-J}(t)| \leq g_4(K\#-T)t = d(K,T)t \leq g(C)t
\end{equation}
for all $0 \leq t \leq 1$. This provides bounds on $\Upsilon_K(t)$ in terms of $\Upsilon_J(t)$ and $g(C)$.

We now give an example for the cobordisms we will use later on.

**Example 4.1.** Among other things, we will frequently use the following trick the author first saw in [FK17, Example 4.5]. Let $\gamma$ be a 3-braid such that $K = \hat{\gamma}$ is a knot. Consider the 3-braid $\alpha := \gamma b^{2n}$ for some $n \geq 1$. Then $\hat{\alpha}$ is also a knot and there is a cobordism between $\hat{\alpha}$ and the connected sum $K\#T_{2,2n+1}$ of genus 1. This cobordism can be realized by two saddle moves (1-handle attachments) of the form shown in Figure 2(b) performed in the two circled regions of Figure 2(a). One of them is used to add a generator $b$ to the braid $\alpha$ to obtain the braid word $\gamma b^{2n+1}$ and the other is used to transform the closure of this new braid word into a connected sum of $K$ and $T_{2,2n+1}$. Recall that our braid diagrams are oriented from bottom to top.

Using $\nu(T_{2,2n+1}) = -n$ by Equation (11) and that the genus of the cobordism is 1, by (15) for $t = 1$, we have
\begin{equation}
|\nu(\hat{\alpha}) - \nu(K\#T_{2,2n+1})| \leq 1 \iff |\nu(\hat{\alpha}) - \nu(K) + n| \leq 1,
\end{equation}
which provides the lower bound $\nu(K) \geq \nu(\hat{\alpha}) + n - 1$ on $\nu(K)$. 

![Figure 2](image.png)

(a) A schematic of a cobordism between the knots $\hat{\alpha}$ and $\hat{\gamma}\#T_{2,2n+1}$ realized by two saddle moves.

(b) A saddle move.
4.2. The upsilon invariant of positive 3-braid knots

In this section, we determine the invariant \( \upsilon \) for all positive 3-braid knots.

By Proposition 3.2 and Remark 3.6, positive 3-braid knots are either the torus knots \( T_{3,3\ell+k} \) for \( \ell \geq 0 \) and \( k \in \{1, 2\} \) which have braid representatives of Garside normal form (B), or closures of positive 3-braids of Garside normal form (C) or (D) (cf. Definition 3.4). The following proposition thus proves Theorem 1.1 for all positive 3-braid knots.

**Proposition 4.2.** Let \( \gamma \) be a positive 3-braid such that \( K = \hat{\gamma} \) is a knot. Then

\[
\upsilon(K) = \begin{cases} 
-2\ell - \frac{p - 1}{2} & \text{if } \gamma \text{ is conjugate to a braid in (B)}, \\
-\frac{1}{2} \sum_{i=1}^{r} (p_i + q_i) + r - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in (C)}, \\
-\frac{1}{2} \sum_{i=1}^{r-1} (p_i + q_i) + pr + r - 2\ell - \frac{3}{2} & \text{if } \gamma \text{ is conjugate to a braid in (D)}. 
\end{cases}
\]

**Remark 4.3.** In fact, the formulas from Proposition 4.2 also give the correct upsilon invariant in terms of the Garside normal form of a 3-braid representative of a knot \( K \) if \( K \) is the closure of any 3-braid in Garside normal form (C) or (D), not necessarily a positive one. This follows from Theorem 1.1 (proved in the next section) and the observations of Section 4.4.3.

Recall that for the torus knots of braid index 3, we know the invariant \( \upsilon \) by Equation (12). In the following, we will determine the invariant \( \upsilon \) for all knots that are closures of positive 3-braids of Garside normal form (C) or (D).

We first provide an upper bound on \( \Upsilon_K(t) \) for positive 3-braid knots \( K \) and \( 0 \leq t \leq 1 \). The following inequality (17) in Lemma 4.4 could also be shown using the dealternating number and a result of Abe and Kishimoto [AK10, Lemma 2.2], whereas the main work for the upper bound on \( \upsilon \) for the knots in the second and third case in Proposition 4.2 will be to rewrite the braid words representing these knots. We use the approach below since it will also give bounds on the minimal cobordism distance between any positive 3-braid knot and an alternating knot; see Remark 4.14.

**Lemma 4.4.** Let \( \gamma = a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} \) be a positive 3-braid, where \( r \geq 1 \) and \( p_i, q_i \geq 1, i \in \{1, \ldots, r\} \), are integers such that \( K = \hat{\gamma} \) is a knot. Then

\[
\Upsilon_K(t) \leq (-g(K) + r - 1) t \quad \text{for all } 0 \leq t \leq 1.
\]

**Proof.** We claim that there is a cobordism \( C \) of genus

\[
g(C) = \frac{r - 1 + \varepsilon}{2}
\]

between \( K \) and the connected sum

\[
J_\varepsilon = T_{2, \sum_{i=1}^{r} p_i + \varepsilon} \# T_{2,q_1+\varepsilon_1} \# T_{2,q_2+\varepsilon_2} \# \cdots \# T_{2,q_r+\varepsilon_r},
\]

where \( \varepsilon_1, \ldots, \varepsilon_r, \varepsilon_p \in \{0,1\} \) are chosen such that \( J_\varepsilon \) is a connected sum of torus knots (rather than links), i.e., such that \( \sum_{i=1}^{r} p_i + \varepsilon_p, q_1 + \varepsilon_1, q_2 + \varepsilon_2, \ldots, q_r + \varepsilon_r \) are all odd, and \( \varepsilon := \varepsilon_p + \sum_{i=1}^{r} \varepsilon_i \). This cobordism \( C \) can be realized by \( r - 1 + \varepsilon \).
saddle moves as follows. Following the schematic in Figure 3 we add $\varepsilon$ generators $b$ by $\varepsilon$ saddle moves and additionally perform $r - 1$ saddle moves of the form shown in Figure 2(b) in the green circled regions of Figure 3. In Figure 3 a box on the left labeled $p_i$ or $q_i$ stands for the positive braid $a^{p_i}$ or $b^{q_i}$, respectively. The Euler characteristic of the cobordism $C$ is $\chi(C) = -r + 1 - \varepsilon$. Since $C$ is connected and — as $J_\varepsilon$ and $K$ are knots — has two boundary components, the genus of $C$ is $g(C) = \frac{-\chi(C)}{2} = \frac{r - 1 + \varepsilon}{2}$ as claimed.

Figure 3. A schematic of a cobordism between the knots $K = \hat{\gamma}$ and $J_\varepsilon = T_2, \sum_{i=1}^r p_i + \varepsilon, \#T_2, q_1 + \varepsilon, \#T_2, q_2 + \varepsilon, \# \ldots, \#T_2, q_r + \varepsilon$ realized by $r - 1 + \varepsilon$ saddle moves.

By (15), we get $|\Upsilon_K(t) - \Upsilon_{J_\varepsilon}(t)| \leq g(C)t$ for all $0 \leq t \leq 1$, hence

(19) \hspace{1cm} \Upsilon_K(t) \leq \Upsilon_{J_\varepsilon}(t) + g(C)t \hspace{1cm} \text{for all } 0 \leq t \leq 1.
By Equations (8) and (11) from Section 2.2, we have
\[ \Upsilon_{J_r}(t) = \left( -\sum_{i=1}^{r} p_i + \epsilon_p - \frac{1}{2} - q_1 + \epsilon_1 - \frac{1}{2} - q_2 + \epsilon_2 - \frac{1}{2} - \ldots - q_r + \epsilon_r - \frac{1}{2} \right) t \]
\[ = -\frac{1}{2} \left( \sum_{i=1}^{r} (p_i + q_i) - (r + 1) + \epsilon \right) t, \]
so (18) and (19) imply
\[ \Upsilon_K(t) \leq \left( -\frac{1}{2} \sum_{i=1}^{r} (p_i + q_i) - r \right) t \quad \text{for all } 0 \leq t \leq 1. \]
The claim follows, since by Equation (3), we have
\[ g(K) = \frac{\text{wr}(\gamma) - 2}{2} = \frac{\sum_{i=1}^{r} (p_i + q_i) - 2}{2}. \]

The following two lemmas improve the bound from Lemma 4.3 for knots that are closures of positive 3-braids of Garside normal form (C) or (D), respectively.

**Lemma 4.5.** Let \( \gamma = \Delta^{2\ell+1} a^p b^n \ldots a^{p_{r-1}} b^{r-1} a^p \) for some \( \ell \geq 0, r \geq 1, p_r \geq 1 \) and \( p_i, q_i \geq 1 \) for \( i \in \{1, \ldots, r-1\} \) such that \( K = \gamma \) is a knot. Then
\[ \Upsilon_K(t) \leq \left( -\sum_{i=1}^{r-1} (p_i + q_i) + p_r + r - 2\ell - \frac{3}{2} \right) t \quad \text{for all } 0 \leq t \leq 1. \]

In the proof of Lemma 4.5, we will use that in \( B_3 \), we have
\[ (ab)^{3n+1} = ab\Delta^{2n} = a^2b^3(aba^3)^{n-1}ba^n \quad \text{for all } n \geq 1, \]
where \( \Delta^2 = (ab)^2 = (ab)^3 = (ba)^3 \); see [Fel16, Proof of Prop. 22].

**Proof of Lemma 4.5.** Let \( \Sigma_\gamma = \sum_{i=1}^{r-1} (p_i + q_i) + p_r \) and note that using Equation (3), we have
\[ g(K) = \frac{3(2\ell + 1) + \Sigma_\gamma - 2}{2} = \frac{\Sigma_\gamma}{2} + 3\ell + \frac{1}{2}. \]
If \( \ell = 0 \), then \( \gamma = \Delta a^{p_1} b^{q_1} \ldots a^{p_{r-1}} b^{r-1} a^p \) is conjugate to
\[ \gamma_1 = a^{p_1+1} b^{q_1} \ldots a^{p_{r-1}} b^{r-1} a^{p_1+1}b \]
and \( \hat{\gamma}_1 = \hat{\gamma} = K \), so \( g(\hat{\gamma}_1) = \frac{\Sigma_\gamma}{2} + \frac{1}{2} \). By Lemma 4.4, we get
\[ \Upsilon_K(t) \leq (-g(\hat{\gamma}_1) + r - 1) t = \left( -\frac{\Sigma_\gamma}{2} + r - \frac{3}{2} \right) t \quad \text{for all } 0 \leq t \leq 1. \]
For \( \ell \geq 1 \), using \( \Delta^{2\ell+1} = (ab)^{3\ell} a b = (ab)^{3\ell+1} a \), we have
\[ \gamma = \Delta^{2\ell+1} a^p b^n \ldots a^{p_{r-1}} b^{r-1} a^p = (ab)^{3\ell+1} a^{p_1+1} b^{q_1} \ldots a^{p_{r-1}} b^{r-1} a^p \]
\[ \sim a^{p+2} b a^3 (aba^3)^{\ell-1} b^\ell a^{p_1+\ell+1} b^{q_1} \ldots a^{p_{r-1}} b^{r-1} a^p \]
\[ \sim a^{p+p+2} b a^3 (aba^3)^{\ell-1} b^\ell a^{p_1+\ell+1} b^{q_1} \ldots a^{p_{r-1}} b^{r-1} a^p =: \gamma_1. \]
We have \( \hat{\gamma}_1 = \hat{\gamma} = K \) and \( g(\hat{\gamma}_1) = \frac{\Sigma_\gamma}{2} + 3\ell + \frac{1}{2} \) by (21). Again, Lemma 4.4 implies 
\[ \Upsilon_K(t) \leq (-g(\hat{\gamma}_1) + r + \ell - 1)t = \left( -\frac{\Sigma_\gamma}{2} + r - 2\ell - \frac{3}{2} \right) t \quad \text{for all } 0 \leq t \leq 1, \]
which proves the claim of the lemma. \( \square \)

**Lemma 4.6.** Let \( \gamma = \Delta^{2^i}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) for some \( \ell \geq 0, r \geq 1 \) and \( p_i, q_i \geq 1 \) for \( i \in \{1, \ldots, r\} \) such that \( K = \hat{\gamma} \) is a knot. Then
\[ \Upsilon_K(t) \leq \left( -\sum_{i=1}^r (p_i + q_i) + r - 2\ell \right) t \quad \text{for all } 0 \leq t \leq 1. \]

In the proof, we will need the following statement about positive 3-braids.

**Lemma 4.7.** In B3, we have
\[ (ab)^{3n-1} = a^{2n}b(a^2b^2)^{n-1}a \quad \text{for all } n \geq 1. \]

**Proof.** Starting with the left-hand side we have
\[ (ab)^{3n-1} = (ab)(ab)^{3(n-1)}bab = a(ab)^{3(n-1)}aba, \]
which proves Equation (22) for \( n = 1 \). We now show by induction that
\[ (ab)^{3(n-1)}a = a^{2n-1}b(a^2b^2)^{n-2}a^2b \quad \text{for all } n \geq 2, \]
which implies the lemma for all \( n \geq 1 \). For \( n = 2 \), we have
\[ (ab)^3a = a(ba)^3 = a(ab)^3 = a^2baba = a^3ba^2b. \]
Assuming that (23) is true for some \( n - 1 \geq 2 \), we get
\[ (ab)^{3(n-1)}a = a(ba)(ab)^{3(n-1)}a = a(ab)^{3(n-2)}babab = a^2(ab)^{3(n-2)}aba^2b = a^2(a^{2n-3}b(a^2b^2)^{n-3}a^2b)ba^2b = a^{2n-1}b(a^2b^2)^{n-2}a^2b, \]
using the induction hypothesis in the second to last equality. \( \square \)

**Proof of Lemma 4.6.** Let \( \Sigma_\gamma = \sum_{i=1}^r (p_i + q_i) \). If \( \ell = 0 \), then by Equation (3) and Lemma 4.4 we have
\[ \Upsilon_K(t) \leq (-g(K) + r - 1)t = \left( -\frac{\Sigma_\gamma}{2} + r \right) t \quad \text{for all } 0 \leq t \leq 1. \]
For \( \ell \geq 1 \), using \( \Delta^2 = (ba)^3 \) and Lemma 4.7, we have
\[ \gamma = (ba)^{3\ell}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \sim (ab)^{3\ell-1}a^{p_1+1}b^{q_1} \cdots a^{p_r}b^{q_r+1} \sim a^{2\ell}(a^2b^2)^{\ell-1}a^{p_1+2}b^{q_1} \cdots a^{p_r}b^{q_r+1} =: \gamma_1. \]
Note that \( \hat{\gamma}_1 = \hat{\gamma} = K \) and by Equation (3), we have
\[ g(\hat{\gamma}_1) = g(K) = \frac{6\ell + \Sigma_\gamma - 2}{2} = \frac{\Sigma_\gamma}{2} + 3\ell - 1. \]
Again by Lemma 4.4 we get
\[ \Upsilon_K(t) \leq (-g(\hat{\gamma}_1) + r + \ell - 1)t = \left( -\frac{\Sigma_\gamma}{2} + r - 2\ell \right) t \quad \text{for all } 0 \leq t \leq 1. \]
We will now focus on \( v(K) = \Upsilon_K(1) \) and prove Proposition [12] by showing that the upper bounds on \( \Upsilon_K(t) \) from Lemma 4.5 and Lemma 4.6 for \( t = 1 \) are also lower bounds. We will need the following observation used in [FK17, Example 4.5] about 3-braids, which we prove here for completeness.

**Lemma 4.8.** In \( B_3 \), we have
\[
(24) \quad a^{2n+1}b \left( a^2b^2 \right)^n = (ab)^{3n+1} \quad \text{and} \quad b^{2n+1}a \left( b^2a^2 \right)^n = (ba)^{3n+1} \quad \text{for all } n \geq 0.
\]

**Proof.** We prove the first statement by induction. For \( n = 0 \), the equality is clearly true. For \( n = 1 \), using \( \Delta a = b \Delta \) and \( \Delta b = a \Delta \), we have
\[
a^3ba^2b^2 = a^2\Delta ab^2 = a^2ba\Delta b = a\Delta^2b = \Delta^2ab = (ab)^4.
\]

We now assume that (24) is true for some \( n-1 \geq 0 \). Using the induction hypothesis and the equality for \( n = 1 \), we get
\[
a^{2n+1}b \left( a^2b^2 \right)^n = a^2 (ab)^{(3n-1)+1} a^2b^2 = a^3b\Delta^{2(n-1)}a^2b^2
\]
\[
= \Delta^{2(n-1)}a^3ba^2b^2 = (ab)^{(3n-1)}(ab)^4 = (ab)^{3n+1}.
\]

\( \square \)

**Lemma 4.9.** Let \( \gamma = \Delta^{2\ell+1}a^{p_1}b^{q_1} \cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \) for some \( \ell \geq 0 \), \( r \geq 1 \), \( p_r \geq 3 \) and \( p_i, q_i \geq 2 \) for \( i \in \{1, \ldots, r-1\} \) such that \( K = \gamma \) is a knot. Then
\[
v(K) = -\frac{\sum_{i=1}^{r-1} (p_i + q_i) + p_r}{2} + r - 2\ell - \frac{3}{2}.
\]

**Proof of Lemma 4.9** Let \( \Sigma_\gamma = \sum_{i=1}^{r} (p_i + q_i) + p_r \). From Lemma 4.5, it follows directly that \( v(K) = \Upsilon_K(1) \leq -\frac{\Sigma_\gamma}{2} + r - 2\ell - \frac{3}{2} \), so we are left to show that \( v(K) \geq -\frac{\Sigma_\gamma}{2} + r - 2\ell - \frac{3}{2} \). To that end, consider
\[
\gamma = \Delta^{2\ell+1}a^{p_1}b^{q_1} \cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \sim \Delta^{2\ell}a\Delta a^{p_1}b^{q_1} \cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1}
\]
\[
= \Delta^{2\ell}bab^2a^{p_1}b^{q_1} \cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1} =: \gamma_1,
\]
where we used \( a\Delta = abab = bab^2 \). Note that \( \gamma_1 = \gamma = K \). Now, define
\[
\alpha := b^{2\ell+1}a^{p_1}b^{q_1} \cdots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1}
\]
and note that \( \alpha \) is a knot. By assumption, we have \( p_r - 1 \geq 2 \). There is a cobordism between \( \alpha \) and the connected sum \( T_{2,2r+1} \# \gamma_1 = T_{2,2r+1} \# K \) of genus 1 by using two saddle moves similar to the two saddle moves illustrated in Figure 2 from Example 4.1. Similarly as in (16) from Example 4.1 we have \( v(K) \geq v(\alpha) + r - 1 \). In order to find a lower bound for \( v(\alpha) \), note that there is a cobordism \( C \) between \( \alpha \) and the torus knot \( T = T_{3,3(\ell+r)+1} \) of genus \( g(C) = \frac{\Sigma_\gamma}{2} - 2\ell + \frac{1}{2} \). Here we think of \( T \) as the closure of the braid word \( \beta = \Delta^{2\ell}b^{2\ell+1}a(b^2a^2)^r \), which is equal to \( \Delta^{2r}(ba)^{3r+1} = (ba)^{3(\ell+r)+1} \) as 3-braids by Lemma 4.8. The cobordism \( C \) between \( \alpha \) and \( T = \beta \) can thus be realized by
\[
p_1 - 2 + q_1 - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 3 = \Sigma_\gamma - 4r + 1
\]
saddle moves corresponding to the deletion of the same number of generators \( a \) and \( b \) from the braid word \( \alpha \) to obtain \( \beta \). Hence the Euler characteristic of the cobordism \( C \) is \( \chi(C) = -\Sigma_\gamma + 4r - 1 \). Since \( C \) is connected and has two boundary
components (as \( \tilde{\alpha} \) and \( T = \tilde{\beta} \) are knots), the genus of \( C \) is indeed \( g(C) = \frac{\Sigma_2}{2} - 2r + \frac{1}{2} \).

Now, by (15) and Equation (12), we have
\[
v(\tilde{\alpha}) \geq v(T) - g(C) = -2(\ell + r) - \left( \frac{\Sigma_2}{2} - 2r + \frac{1}{2} \right) = -\frac{\Sigma_2}{2} - 2\ell - \frac{1}{2}
\]

It follows that
\[
v(K) \geq v(\tilde{\alpha}) + r - 1 \geq -\frac{\Sigma_2}{2} + r - 2\ell - \frac{3}{2}
\]
as claimed, hence the statement of the lemma. \( \square \)

**Lemma 4.10.** Let \( \gamma = \Delta^{2\ell}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) for some \( \ell \geq 0, r \geq 1, p_r, q_r \geq 3 \) and \( p_i, q_i \geq 2 \) for \( i \in \{1, \ldots, r-1\} \) such that \( K = \hat{\gamma} \) is a knot. Then
\[
v(K) = -\frac{1}{2} \sum_{i=1}^{r} (p_i + q_i) + r - 2\ell.
\]

**Proof of Lemma 4.10.** The proof uses similar ideas as the proof of Lemma 4.9. Let \( \Sigma_{\gamma} = \sum_{i=1}^{r} (p_i + q_i) \). By Lemma 4.6 we have \( v(K) \leq -\frac{\Sigma_{\gamma}}{2} + r - 2\ell \), so it remains to show that \( v(K) \geq -\frac{\Sigma_{\gamma}}{2} + r - 2\ell \). To that end, we consider
\[
\gamma = \Delta^{2\ell}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \sim \Delta^{2\ell}ba^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r-1} =: \gamma_1.
\]

Note that \( \hat{\gamma}_1 = \hat{\gamma} = K \). We define
\[
\alpha := a^{2r}\gamma_1 = a^{2r}\Delta^{2\ell}ba^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r-1} \sim \Delta^{2\ell}ba^{2r}b^{q_1} \cdots a^{p_r}b^{q_r-2} =: \alpha_1.
\]

Then \( \alpha_1 = \hat{\alpha} \) is a knot and by assumption we have \( q_r - 2 \geq 1 \). There is a cobordism between \( \alpha \) and \( T_{2,2r+1} # \gamma_1 = T_{2,2r+1} # K \) of genus 1 by using two saddle moves similar to the cobordism considered in Example 4.1 and in the proof of Lemma 4.9 hence \( v(K) \geq v(\alpha_1) + r - 1 \). To find a lower bound for \( v(\alpha_1) \), we observe that there is a cobordism \( C \) between the knot \( \alpha_1 \) and the knot \( \beta \), where
\[
\beta = \Delta^{2\ell}ba^{2r}b(a^2b^2)^{r-1}a^{3b}.
\]

Using Equation (123) from Lemma 1.8 for \( n - 1 \), in \( B_3 \), we have
\[
ba^{2n}b(a^2b^2)^{n-1}a^2 = ba(ab)^{3(n-1)+1}a^2 = ba\Delta^{2(n-1)}aba^2 = \Delta^{2n} \quad \text{for all } n \geq 1.
\]

We thus have \( \beta = \Delta^{2\ell}ba^{3r}ab = (ab)^{3(\ell+r)+1} \), so the closure of \( \beta \) is the torus knot \( T = T_{3,3(\ell+r)+1} \) with \( v(T) = -2(\ell + r) \) by Equation (12). The cobordism \( C \) between \( \alpha_1 \) and \( T = \beta \) can be realized by
\[
p_1 - 2 + q_1 - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 3 + q_r - 3 = \Sigma_\gamma - 4r - 2
\]
saddle moves corresponding to the deletion of the same number of generators \( a \) and \( b \) from the braid word \( \alpha_1 \) to obtain \( \beta \). By a similar Euler characteristic argument as in the proofs of Lemma 4.4 and Lemma 4.9 the genus of this cobordism is \( g(C) = \frac{\Sigma_{\gamma}}{2} - 2r - 1 \). Note that here we used \( p_r \geq 3 \) and \( q_r \geq 3 \). Now, by (15), we have
\[
v(\alpha_1) \geq v(T) - g(C) = -\frac{\Sigma_\gamma}{2} - 2\ell + 1, \quad \text{hence}
\]
\[
v(K) \geq v(\alpha_1) + r - 1 \geq -\frac{\Sigma_\gamma}{2} + r - 2\ell.
\]
\( \square \)
Lemma 4.11. Let $\gamma = \Delta^{2\ell}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r}$ for some $\ell \geq 0$, $r \geq 2$, $p_i, q_i \geq 2$ for $i \in \{1, \ldots, r\}$. Suppose that $q_r \geq 3$ and $p_k \geq 3$ for some $1 \leq k < r$ and that $K = \hat{\gamma}$ is a knot. Then

$$v(K) = -\frac{\sum_{i=1}^{r} (p_i + q_i)}{2} + r - 2\ell.$$ 

Proof. We proceed similar as in the proof of Lemma 4.10, but here we will look at a different cobordism to obtain a lower bound for $v(\hat{\alpha_1})$. The steps of the proof are exactly the same until then, so we consider

$$\gamma = \Delta^{2\ell}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \sim \Delta^{2\ell}ba^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r-1} =: \gamma_1$$

and define

$$\alpha := a^{2r}\gamma_1 \sim \Delta^{2\ell}ba^{2r}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r-2} =: \alpha_1.$$ 

Again, we have $v(K) \geq v(\hat{\alpha_1}) + r - 1$. Now, in order to find a lower bound for $v(\hat{\alpha_1})$, we observe that there is a cobordism $C$ between $\hat{\alpha_1}$ and the knot $\hat{\beta}$, where

$$\beta = \Delta^{2\ell}ba^{2r}b(a^{2}\ell b)\gamma_1^{-1}a^{3}b^{2}(a^{2}2b) r^{-k-1}a^{2}b.$$ 

We find the cobordism $C$ by the deletion of generators from the braid word $\beta$ to obtain $\alpha_1$, where we use the assumptions $q_r \geq 3$ and $p_k \geq 3$. In fact, the cobordism can be realized by

$$p_1 - 2 + q_1 - 2 + \cdots + p_{k-1} - 2 + q_{k-1} - 2 + p_{k} - 3 + q_{k} - 2$$

$$+ p_{k+1} - 2 + q_{k+1} - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 + p_{r} - 2 + q_{r} - 3$$

$$= \Sigma_{\gamma} - 4r - 2$$

saddle moves, so its genus is $g(C) = \frac{\Sigma_{\gamma}}{2} - 2r - 1$. Using $a^{2k-1}b(a^{2}2b)k^{-1} = (ab)^{3k-2}$ by Lemma 4.8, we have

$$\beta = \Delta^{2\ell}ba^{2r}b(a^{2}2b)\gamma_1^{-1}a^{3}b^{2}(a^{2}2b) r^{-k-1}a^{2}b$$

$$= \Delta^{2\ell}ba^{2r}b\Delta^{2(k-1)}a^{3}b^{2}(a^{2}2b) r^{-k-1}a^{2}b$$

$$\sim \Delta^{2(\ell+k-1)}a^{3}b^{2}(a^{2}2b) r^{-k-1}a^{2}b a^{2r-2k+1}$$

$$= \Delta^{2(\ell+k-1)+1}(a^{2}b) r^{-k+1}a^{2r-2k+1} =: \beta_1.$$ 

Note that by our assumptions on $\ell, r$ and $k$, we have $\ell + k - 1 \geq 0$, $r - k + 1 \geq 2$ and $2r - 2k + 1 \geq 3$, so $\beta_1$ has the form of the braid words considered in Lemma 4.9. We thus have

$$v(\beta) = v(\beta_1) = -\frac{4(r - k + 1) + 2r - 2k + 1}{2} + (r - k + 2) - 2(\ell + k - 1) - \frac{3}{2}$$

$$= -2(\ell + r).$$

By (13), we have

$$v(\hat{\alpha_1}) \geq v(\beta) - g(C) = -\frac{\Sigma_{\gamma}}{2} - 2\ell + 1,$$ 

hence

$$v(K) \geq v(\hat{\alpha_1}) + r - 1 \geq -\frac{\Sigma_{\gamma}}{2} + r - 2\ell.$$ 

□
Proof of Proposition 4.2. The first case of Proposition 4.2 follows from Remark 3.6 and Equation (12). Lemma 4.10 and 4.11 together prove the second case, Lemma 4.9 proves the third case. Note that up to conjugation, by Remark 3.1 and the remarks in Proposition 3.2 it is no restriction to assume that \( p_r \geq 3 \) in Lemma 4.9 and that \( q_r \geq 3 \) and either \( p_r \geq 3 \) or \( p_k \geq 3 \) for some \( 1 \leq k < r \) in Lemma 4.10 and 4.11 respectively. □

Before we proceed with the general case where the knot \( K \) is given as the closure of any 3-braid, let us prove the following corollaries of our results in this section.

Corollary 4.12 (Corollary 4.3). Let \( K \) be a knot that is the closure of a positive 3-braid. Then

\[
r = g(K) + v(K) + 1
\]

is minimal among all integers \( r \geq 1 \) such that \( K \) is the closure of a positive 3-braid \( a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) for integers \( p_i, q_i \geq 1, i \in \{1, \ldots, r\} \).

Proof. By Lemma 4.4 we have

\[
v(K) \leq -g(K) + r - 1 \quad \iff \quad g(K) + v(K) + 1 \leq r
\]

whenever \( K \) is the closure of a positive 3-braid \( a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) for integers \( r \geq 1, p_i, q_i \geq 1, i \in \{1, \ldots, r\} \). It remains to show that we can always find a positive braid representative for \( K \) of the form \( a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) with \( r = g(K) + v(K) + 1 \).

We will use Proposition 3.2. In fact, if \( K \) is the closure of a positive braid \( \gamma \) of the form in (C) with \( \ell \geq 0 \), then \( g(K) + v(K) + 1 = r + \ell \) by Equation (3) applied to \( \gamma \), Lemma 4.10 and Lemma 4.11. Moreover, we have

\[
\gamma = a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \quad \text{if} \quad \ell = 0 \quad \text{and}
\]

\[
\gamma \sim a^{2\ell b(a^2b^2)^{\ell-1}}a^{p_1+2b^{q_1}} \cdots a^{p_r}b^{q_r+1} \quad \text{if} \quad \ell \geq 1
\]

by the proof of Lemma 4.6. These give the desired braid representatives for \( K \). Furthermore, if \( K \) is represented by a positive braid \( \gamma \) of the form in (D) with \( \ell \geq 0 \), then \( g(K) + v(K) + 1 = r + \ell \) by Equation (3) and Lemma 4.8, and we have

\[
\gamma \sim a^{p_1+1}b^{q_1} \cdots a^{p_r-1}b^{q_r-1}a^{p_r+1}b \quad \text{if} \quad \ell = 0 \quad \text{and}
\]

\[
\gamma \sim a^{p_r+2ba^3(ab)^{\ell-1}}a^{p_1+\ell+1}b^{q_1} \cdots a^{p_r-1}b^{q_r-1} \quad \text{if} \quad \ell \geq 1
\]

by the proof of Lemma 4.5. Finally, if \( K = T_{3,3\ell+k} \) for \( \ell \geq 0 \) and \( k \in \{1, 2\} \), then by Equation (4) and Equation (12), we have \( g(K) + v(K) + 1 = \ell + 1 \) and \( T_{3,3\ell+1} \) and \( T_{3,3\ell+2} \) are represented by the positive 3-braids \( (ab)^{3\ell+1} = a^{2\ell+1}b(a^2b^2)^{\ell} \) and \( (ab)^{3\ell+2} = a^{2\ell+3}b(a^2b^2)^{\ell} \), respectively, by Lemma 4.8 and Lemma 4.7. □

Corollary 4.13 (Corollary 4.4). If \( K \) and \( J \) are concordant knots that are both closures of positive 3-braids, then the minimal \( r \) from Corollary 4.12 is the same for both \( K \) and \( J \).

Proof. If \( K \) and \( J \) are concordant, then their 4-genus and their upsilon invariants are equal. So by Equation (3) from Section 2.1 and by Corollary 4.12 positive 3-braids with closures \( K \) and \( J \), respectively, will have the same minimal \( r \). □

Remark 4.14. Let \( A_g(K) \) denote the minimal genus of a cobordism between a knot \( K \) and an alternating knot, i.e. the cobordism distance \( d(K, \{\text{alternating knots}\}) \).
By [FLZ17, Theorem 8], we have \( |\tau(K) + \nu(K)| \leq A_g(K) \) for any knot \( K \). It thus follows from our results in this section that
\[
\frac{r + \ell - 1}{2} \leq A_g(K) \leq \frac{r + \ell - 1 + \varepsilon}{2}
\]
for any knot \( K \) that is the closure of a positive 3-braid in Garside normal form (C) or (D), where \( \varepsilon \geq 0 \) is an integer depending on \( K \). The lower bound uses Proposition 4.2 and Equation (5) from Section 2.2; see also the proof of Corollary 4.12. The upper bound follows from the proofs of Lemma 4.5 and Lemma 4.6, see also the proof of Lemma 4.4. Note that for most positive 3-braid knots, we have \( \varepsilon > 0 \), so we do not get an equality.

A shorter proof of Lemma 4.4 without cobordisms follows from a result of Abe and Kishimoto on the dealternating number of positive 3-braid knots. Indeed, we have
\[
|\Upsilon_K(t) + g(K)t| \overset{(5)}{=} |\Upsilon_K(t) + \tau(K)t| \leq \text{alt}(K)t \leq dalt(K)t \leq (r - 1)t \quad \text{for all } 0 \leq t \leq 1.
\]

The definitions of the dealternating number \( dalt(K) \) and the alternation number \( \text{alt}(K) \) of a knot \( K \) and more details on the inequalities used here will be provided in Section 5.

4.3. PROOF OF THEOREM 1.1

It remains to show Theorem 1.1 when \( K \) is the closure of a not necessarily positive 3-braid. We first recall a result of Murasugi, which implies that indeed all 3-braid knots except for the torus knots of braid index 3 are covered by Theorem 1.1.

Let \( \gamma \) be a 3-braid. Then, by [Mur74, Proposition 2.1], \( \gamma \) is conjugate to one and only one of the 3-braids
\[
\begin{align*}
(a) & \quad \Delta^{2\ell} a^p \quad \text{or} \quad \Delta^{2\ell + 1} \quad \text{for } \ell \in \mathbb{Z}, \ p \in \mathbb{Z}, \\
(b) & \quad \Delta^{2\ell} ab \quad \text{or} \quad \Delta^{2\ell} (ab)^2 \quad \text{for } \ell \in \mathbb{Z}, \\
(c) & \quad \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r} \quad \text{for } \ell \in \mathbb{Z}, \ r \geq 1, \ p_i, q_i \geq 1, \ i \in \{1, \ldots, r\}.
\end{align*}
\]

Definition 4.15. We call a braid word of the form in (a)–(c) a 3-braid in Murasugi normal form.

Remark 4.16. The closures of the 3-braids in Murasugi normal form (C) are links of two (if \( p \) is odd) or three components and the closures of the 3-braids in Murasugi normal form (D) are the torus knots of braid index 3 (cf. Remark 3.6).

If \( \ell = 0 \) in case (c), the braid word \( \gamma = a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r} \) for integers \( r \geq 1 \) and \( p_i, q_i \geq 1, \ i \in \{1, \ldots, r\} \), gives rise to an alternating braid diagram. If \( K = \hat{\gamma} \) is a knot, by Proposition 2.1 we thus have \( \nu(K) = \frac{\sigma(K)}{2} \) in that case and the statement of Theorem 1.1 follows directly from a result by Erle on the signature of 3-braid knots.

Proposition 4.17 ([Erl99 Theorem 2.6]). Let \( \gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r} \) for integers \( \ell \in \mathbb{Z}, \ r \geq 1 \) and \( p_i, q_i \geq 1 \) for \( i \in \{1, \ldots, r\} \) such that \( K = \hat{\gamma} \) is a knot. Then
\[
\sigma(K) = \sum_{i=1}^{r} (p_i - q_i) - 4\ell.
\]
We still need to show Theorem 1.1 when $K$ is the closure of a 3-braid in Murasugi normal form (c) with $\ell \neq 0$. The proof will follow from the following two lemmas.

**Lemma 4.18.** Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$ for some $\ell \geq 1$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then

$$\Upsilon_K(t) \leq \left( \sum_{i=1}^{r} (p_i - q_i) \right) t - 2\ell t \quad \text{for all } 0 \leq t \leq 1.$$  

**Lemma 4.19.** Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$ for some $\ell \geq 0$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then

$$v(K) \geq \frac{\sum_{i=1}^{r} (p_i - q_i)}{2} - 2\ell.$$  

**Proof of Theorem 1.1.** For $\ell \geq 1$, the statement of the theorem follows directly from Lemma 4.18 and Lemma 4.19. If $\ell < 0$, the knot $-K$ is represented by the braid word $\Delta^{-2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$, with $-\ell \geq 1$ and accordingly we have

$$v(-K) = v(K) - 2\ell.$$  

Using that $v(-K) = -v(K)$, we have

$$v(K) \geq \frac{\sum_{i=1}^{r} (p_i - q_i)}{2} - 2\ell.$$  

The remainder of this section is devoted to prove the above lemmas.

**Proof of Lemma 4.18.** We first consider the case where $p_1 \geq 2$ and $\ell \geq 2$. Using $\Delta a^{-1} = ab$ and $(ab)^{3n+2} = b^{n+1}a(b^3ab)^{n-1}b^3ab^3$ for all $n \geq 1$ (Fel16 Proof of Prop. 22), we have

$$\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r} = \Delta^{2(\ell - 1)+1} ab a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$$

$$= (ab)^{3(\ell - 1)+2} b^{q_1} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r} \sim (ab)^{3(\ell - 1)+2} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r+1}$$

$$\sim a(b^3ab)^{\ell - 2} b^{3} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r+\ell + 1} =: \gamma_1.$$  

Now, we claim that there is a cobordism $C$ of genus $g(C) = \ell + r - 2 + \varepsilon$ between the closure $K$ of $\gamma_1$ and the connected sum

$$J_z = -T_{2, p_1 - 1 - \varepsilon_1} \# - T_{2, p_2 - \varepsilon_2} \# \cdots \# - T_{2, \varepsilon_r} \# T_{2, 5\ell - 1 + \varepsilon_q},$$

where we choose $\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that $J_z$ is a connected sum of torus knots, i.e. such that $\sum_{i=1}^{r} q_i + 5\ell - 1 + \varepsilon_q, p_1 - 1 - \varepsilon_1, p_2 - \varepsilon_2, \ldots, p_r - \varepsilon_r$ are all odd; and $\varepsilon = \varepsilon_q + \sum_{i=1}^{r} \varepsilon_i$. This cobordism $C$ can be realized using $\ell + r - 1 + \varepsilon$ saddle moves as follows. On the one hand, we add $\sum_{i=1}^{r} \varepsilon_i$ generators $a$ and $\varepsilon_q$ generators $b$ to the braid word $\gamma_1$, on the other hand, we perform $\ell + r - 1 + \varepsilon$ saddle moves of the form as the $r - 1$ saddle moves used in the proof of Lemma 4.4 to get a connected sum of torus knots. The Euler characteristic of $C$ is $\chi(C) = -\ell - r - 1 - \varepsilon$. Since $C$ is connected and has two boundary components (as $K$ and $J_z$ are knots),
the genus of \( C \) is \( g(C) = -\frac{\chi(C)}{2} = \frac{\ell + r - 1 + \varepsilon}{2} \) as claimed. By Equations 8 and 11, we have

\[
\Upsilon_J(t) = \left( \frac{\sum_{i=1}^{r} (p_i - q_i) - \varepsilon - r - 5\ell + 1}{2} \right) t \quad \text{for all } 0 \leq t \leq 1
\]

and by 15, we get

\[
\Upsilon_K(t) \leq \Upsilon_J(t) + g(C)t = \left( \frac{\sum_{i=1}^{r} (p_i - q_i)}{2} - 2\ell \right) t \quad \text{for all } 0 \leq t \leq 1.
\]

If \( p_1 \geq 2 \) and \( \ell = 1 \), then

\[
\gamma \sim (ab)^2a^{-p_1+1}b^{q_1} \cdots a^{-p_r}b^{q_r+1} \sim ab^2a^{-p_1+1}b^{q_1} \cdots a^{-p_r}b^{q_r+2} =: \gamma_1,
\]

and similarly as above, there is a cobordism \( C \) of genus \( g(C) = \frac{\ell + \varepsilon}{2} \) between the closure \( K \) of \( \gamma_1 \) and the connected sum

\[
J_\varepsilon = -T_{2p_1-\varepsilon_1} \# - T_{2p_2-\varepsilon_2} \# \cdots \# - T_{2p_r-\varepsilon_r} \# T_{2\sum q_i + 4 + \varepsilon_q},
\]

where we choose \( \varepsilon_1, \ldots, \varepsilon_r, \varepsilon_q \in \{0, 1\} \) such that \( J_\varepsilon \) is a connected sum of torus knots and \( \varepsilon = \varepsilon_q + \sum_{i=1}^{r} \varepsilon_i \). The claim follows also in this case from Equations 8 and 11, and the inequality in 15.

It remains to show the claim when \( p_1 = 1 \). In that case, using \( \Delta a^{-1} = ab \), we have

\[
\gamma = \Delta^{2\ell}a^{-1}b^{q_1} \cdots a^{-p_r}b^{q_r} = \Delta^{2\ell-1}ab^{q_1+1} \cdots a^{-p_r}b^{q_r} \sim \Delta^{2\ell-1}b^{q_1+1} \cdots a^{-p_r}b^{q_r+1}.
\]

If \( \ell = 1 \), then \( \gamma \) is conjugate to \( \gamma_1 = ab^{q_1+2}a^{-p_2}b^{q_2} \cdots a^{-p_r}b^{q_r+2} \) and if \( \ell \geq 2 \), then using Equation 20 from Section 4.2 we have

\[
\gamma \sim \Delta^{2(\ell-1)+1}b^{q_1+1}a^{-p_2}b^{q_2} \cdots a^{-p_r}b^{q_r+1} = (ba)^{3(\ell-1)+1}b^{q_1+2}a^{-p_2}b^{q_2} \cdots a^{-p_r}b^{q_r+1}
\]

\[
\sim ab^3(ba)^{\ell-2}b^{q_1+1}a^{-p_2}b^{q_2} \cdots a^{-p_r}b^{q_r+3} =: \gamma_1.
\]

In both cases, there is a cobordism \( C \) of genus \( g(C) = \frac{\ell + r - 2 + \varepsilon}{2} \) between the closure \( K \) of \( \gamma_1 \) and the connected sum

\[
J_\varepsilon = -T_{2p_2-\varepsilon_2} \# \cdots \# - T_{2p_r-\varepsilon_r} \# T_{2\sum q_i + 5\ell-1 + \varepsilon_q},
\]

where we choose \( \varepsilon_1, \ldots, \varepsilon_r, \varepsilon_q \in \{0, 1\} \) such that \( J_\varepsilon \) is a connected sum of torus knots and \( \varepsilon = \varepsilon_q + \sum_{i=1}^{r} \varepsilon_i \). Using 8, 11, and 15 again, the claim follows. \( \square \)

We will need the following two technical lemmas for the proof of Lemma 4.19.

**Lemma 4.20.** Let \( \gamma = \Delta^{2\ell}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) for some \( \ell \geq 0, r \geq 1 \) and integers \( p_i, q_i \) such that \( p_i < 0 \) or \( p_i \geq 2 \), and \( q_i < 0 \) or \( q_i \geq 2 \), for any \( i \in \{1, \ldots, r\} \). Moreover, assume that \( K = \hat{\gamma} \) is a knot. Then

\[
\nu(K) \geq \frac{\sum_{i=1}^{r} (p_i + q_i)}{2} + r - 2\ell - \# \{ i \mid p_i < 0 \} - \# \{ i \mid q_i < 0 \},
\]

where \( \# A \) denotes the cardinality of the set \( A \).
Lemma 4.21. Let \( \gamma = \Delta^{2\ell+1}a^{p_1}b^{q_1} \cdots a^{p_r}b^{q_r} \) for some \( \ell \geq 0 \), \( r \geq 1 \) and integers \( p_i, q_i \) such that \( p_i < 0 \) or \( p_i \geq 2 \) for any \( i \in \{1, \ldots, r\} \) and \( q_i < 0 \) or \( q_i \geq 2 \) for any \( i \in \{1, \ldots, r-1\} \). Moreover, assume that \( K = \tilde{\gamma} \) is a knot. Then

\[
v(K) \geq \frac{1}{2} \sum_{i=1}^{r} (p_i + q_i) + p_r + r - 2\ell - \frac{3}{2} - \# \{ i \mid p_i < 0 \} - \# \{ i \mid q_i < 0 \}.
\]

For the proof of Lemma 4.20 and Lemma 4.21, we refer the reader to the very end of this section; we will first prove Lemma 4.19 using these lemmas.

Proof of Lemma 4.19. Let \( k \) be the number of exponents \( q_j \) of \( \gamma \) with \( q_j = 1 \) and let \( J = \{ \gamma_1, \ldots, \gamma_k \} \) for \( 0 \leq k \leq r \) be the set of indices such that \( q_j = 1 \) if and only if \( j \in J \). For all \( j \in J \), we rewrite the subword \( a^{-1}b^{q_j} \) of \( \gamma \) using \( \Delta a^{-1}ab = a^{-1} \Delta a^{-1} = \Delta b^{-1}a^{-1} \).

Note that if \( j, j+1 \in J \), then \( a^{-1}b^{q_j}a^{-1}b^{q_{j+1}} = \Delta^2 a^{-1}b^{-1}a^{-1} = \Delta^2 b^{-1}a^{-1} \). After rewriting \( a^{-1}b^{q_j} \) for all \( j \in J \), the braid \( \gamma \) is conjugate to \( \gamma_1 = \Delta^{2k+1}a^{-1} \) for some 3-braid \( \alpha \) which is of the form

\[
\alpha = \begin{cases} 
\bar{\alpha}b^{q_1} \cdots a^{p_r}b^{q_r} & \text{for } n = r - \frac{k}{2} \\
b^{q_1}a^{p_1} \cdots b^{q_r}a^{p_r}b^{q_r} & \text{for } n = r - \frac{k}{2} 
\end{cases}
\]

where \( \sum_{i=1}^{n} (\bar{p_i} + \bar{q_i}) = \sum_{i=1}^{r} (-p_i + q_i) - 3k \) and where \( \bar{p_i} \) and \( \bar{q_i} \) fulfill the assumptions of Lemma 4.20 and Lemma 4.21, respectively, i.e. where \( \bar{p_i} < 0 \) or \( \bar{p_i} \geq 2 \) and \( \bar{q_i} < 0 \) or \( \bar{q_i} \geq 2 \) for any \( i \). The number of negative exponents in \( \alpha \) equals the number of negative exponents in \( \gamma \), so

\[
\# \{ i \mid \bar{p_i} < 0 \} + \# \{ i \mid \bar{q_i} < 0 \} = r.
\]

If \( k \) is even, by Lemma 4.20 we get

\[
v (\gamma) \geq \frac{1}{2} \sum_{i=1}^{n} (\bar{p_i} + \bar{q_i}) + n - (2\ell + k) - \# \{ i \mid \bar{p_i} < 0 \} - \# \{ i \mid \bar{q_i} < 0 \}
\]

\[
= \frac{1}{2} \sum_{i=1}^{r} (-p_i + q_i) - 3k
\]

\[
+ r - \frac{k}{2} - (2\ell + k) - r = \frac{1}{2} \sum_{i=1}^{r} (p_i - q_i) - 2\ell.
\]

Similarly, if \( k \) is odd, the claim follows from Lemma 4.21. \( \square \)

It remains to prove Lemma 4.20 and Lemma 4.21.

Proof of Lemma 4.20. We will modify the braid word \( \gamma \) in \( 2r \) steps, where each step corresponds to one of the \( 2r \) exponents \( p_i, q_i, i \in \{1, \ldots, r\} \), of \( \gamma \). In every step, we will either just conjugate \( \gamma \) (if the corresponding exponent is positive) or perform a cobordism of genus 1 between the closure of \( a^{2n+1} \gamma \) or \( b^{2n+1} \gamma \) and the connected sum \( T_{2,2n+1} \# \tilde{\gamma} \) for some \( n \geq 0 \) — similarly as the cobordism described in Example 4.1 and used in the proofs of Lemma 4.9, Lemma 4.10 and Lemma 4.11.
We now describe these steps in more detail. First, let $\gamma_{i,q}' = \gamma$ and define

$$
\gamma_{i,q} = \begin{cases} 
\Delta^2 b^{\epsilon_{i,p}} a^{p_{i+1}} \cdot \cdots \cdot a^{p_r} \cdot b^\nu \cdot a^{q_i} & \text{if } p_i < 0 \\
\gamma_{i,1,p} & \text{if } p_i > 0,
\end{cases}
$$

so that $\gamma_{i,1,p}' = \Delta^2 b^{\epsilon_{i,p}} a^{p_{i+1}} \cdot \cdots \cdot a^{p_r} b^\nu a^{q_i}$ for some $\tilde{p}_1 \geq 2$ (note that we assumed $p_1 < 0$ or $p_1 \geq 2$). Here, if $p_1 < 0$, we choose $\epsilon_{i,p} \in \{0,1\}$ such that $-p_1 + 2 + \epsilon_{i,p}$ is even and $\gamma_{1,p}'$ is a knot. Second, let $\epsilon_{i,q} \in \{0,1\}$ such that $-q_1 + 2 + \epsilon_{i,q}$ is even if $q_1 < 0$, and define

$$
\gamma_{1,q} = \begin{cases} 
b^{-q_1+2+\epsilon_{i,q}} \gamma_{1,p}' & \text{if } q_1 < 0 \\
\gamma_{1,1,q}' & \text{if } q_1 > 0,
\end{cases}
$$

so that $\gamma_{1,q}' = \Delta^2 a^{p_2} b^{q_2} \cdot \cdots \cdot a^{p_r} b^\nu a^{q_1}$ for some $\tilde{p}_1, \tilde{q}_1 \geq 2$. Inductively, for any $1 \leq i \leq r$, we let

$$
\gamma_{i-1,q} = \begin{cases} 
\Delta^2 a^{p_{i+1}} \cdot \cdots \cdot a^{p_r} b^\nu a^{q_i} & \text{if } q_i < 0 \\
\gamma_{i-1,1,q}' & \text{if } q_i > 0,
\end{cases}
$$

and define $\gamma_{i,q}'$ similarly as $\gamma_{i,q}'$. Inductively, after $2r$ steps, we get the positive 3-braid

$$
\gamma_{r,q}' = \Delta^2 a^{q_1} b^\nu \cdot \cdots \cdot a^{q_r} b^\nu
$$

with

$$
\tilde{p}_i = \begin{cases} 
2 + \epsilon_{i,p} & \text{if } p_i < 0 \\
p_i & \text{if } p_i > 0,
\end{cases} \quad \text{and} \quad \tilde{q}_i = \begin{cases} 
2 + \epsilon_{i,q} & \text{if } q_i < 0 \\
q_i & \text{if } q_i > 0,
\end{cases}
$$

for all $1 \leq i \leq r$; so that $\tilde{p}_1, \tilde{q}_1, \ldots, \tilde{p}_r, \tilde{q}_r \geq 2$. By Proposition 4.2, we have

$$
v(\gamma_{r,q}') = -\sum_{p_i > 0} p_i + \sum_{q_i > 0} q_i + \sum_{p_i < 0} (2 + \epsilon_{i,p}) + \sum_{q_i < 0} (2 + \epsilon_{i,q}) + r - 2\ell.
$$

Now, note that if $p_1 < 0$ for some $1 \leq i \leq r$, then there is a cobordism of genus 1 between $\gamma_{i,q}'$ and $T_{2,2m+1} \# \gamma_{i-1,q}$ by using two saddle moves, where $m = \frac{-p_1 + 2 + \epsilon_{i,p}}{2}$, so similarly as in [10] from Example 4.1 we have

$$
v(\gamma_{i-1,q}') \geq v(\gamma_{i,p}') + m - 1 = v(\gamma_{i,p}') + \frac{-p_1 + \epsilon_{i,p}}{2}.
$$
Similarly, if \( q_i < 0 \) for some \( 1 \leq i \leq r \), then \( v(\gamma'_{r,p}) \geq v(\gamma'_{i,q}) + \frac{q_i - \varepsilon_{i,p}}{2} \). In addition, if \( p_i > 0 \), then we have \( v(\gamma'_{r,p}) = v(\gamma'_{i,q}) \), and if \( q_i > 0 \), then \( v(\gamma'_{r,p}) = v(\gamma'_{i,q}) \).

We conclude

\[
v(\gamma) = v\left(\gamma_{0,q}'\right) \geq v\left(\gamma_{r,q}'\right) + \sum_{i=1}^{r} \frac{-p_i + \varepsilon_{i,p}}{2} + \sum_{i=1}^{r} \frac{-q_i + \varepsilon_{i,q}}{2} \]

\[
= -\sum_{i=1}^{r} \frac{(p_i + q_i)}{2} + r - 2\ell \]

Proof of Lemma 4.21. The strategy of the proof is the same as in the proof of Lemma 4.20. Here, we need \( 2r - 1 \) steps corresponding to the \( 2r - 1 \) exponents \( p_1, q_1, \ldots, p_{r-1}, q_{r-1}, p_r \) of \( \gamma \). The steps are similar as in the proof of Lemma 4.20 the only change is that we multiply \( \gamma'_{i-1,q} \) by a power of \( b \) if \( p_i < 0 \), and \( \gamma'_i \) by a power of \( a \) if \( q_i < 0 \) (since \( a\Delta^{2r+1} = \Delta^{2r+1}b \) and \( b\Delta^{2r+1} = \Delta^{2r+1}a \)). Thus, starting with \( \gamma'_{0,q} = \gamma \), after \( 2r - 1 \) steps we obtain the positive 3-braid

\[
\gamma_{r,p} = \Delta^{2r+1}a^\bar{p}_1 b^\bar{q}_1 \ldots a^\bar{p}_{r-1} b^\bar{q}_{r-1} a^\bar{p}_r
\]

with

\[
\bar{p}_i = \begin{cases} 
2 + \varepsilon_{i,p} & \text{if } p_i < 0, \\
p_i & \text{if } p_i > 0,
\end{cases} \quad \text{and} \quad \bar{q}_i = \begin{cases} 
2 + \varepsilon_{i,q} & \text{if } q_i < 0, \\
q_i & \text{if } q_i > 0.
\end{cases}
\]

By Lemma 4.20 we have

\[
v(\gamma_{r,p}) = -\sum_{i=1}^{r} \frac{p_i + \sum_{i=1}^{r-1} q_i + \sum_{i=1}^{r} (2 + \varepsilon_{i,p}) + \sum_{i=1}^{r-1} (2 + \varepsilon_{i,q})}{2} + r - 2\ell - \frac{3}{2}.
\]

Since the steps we performed have similar effects on \( v(\gamma) \) as the ones in the proof of Lemma 4.20 we get

\[
v(\gamma) = v\left(\gamma_{0,q}'\right) \geq v\left(\gamma_{r,p}'\right) + \sum_{i=1}^{r} \frac{-p_i + \varepsilon_{i,p}}{2} + \sum_{i=1}^{r-1} \frac{-q_i + \varepsilon_{i,q}}{2} \]

\[
= -\sum_{i=1}^{r} \frac{(p_i + q_i)}{2} + r - 2\ell \]

\[\square\]

4.4. Further Discussion of Theorem 1.1

In this section, we provide some further context on our main result. In particular, in Section 4.4.2 we will discuss why it might be surprising that our proof strategy works for all 3-braid knots.
4.4.1. Comparison of upsilon and the classical signature. By Theorem 1.1 and Proposition 4.17, we have

$$\sigma(K) = 2\upsilon(K)$$

for any knot $K$ that is the closure of a 3-braid $\gamma = \Delta^{2\ell}a^{-p_1}b_{q_1}\cdots a^{-p_r}b_{q_r}$ for integers $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$. Computations of the signature for torus knots (and links) of braid index 3, first done by Hirzebruch, Murasugi and Shinoda [Mur74, Proposition 9.1, pp. 34-35], together with Equation (12) from Section 2.2 imply that the equality in (25) is in fact true for all 3-braid knots $K$ except for the cases that $K = \pm T_{3,3\ell+1}$ for odd $\ell > 0$ or $K = \pm T_{3,3\ell+2}$ for odd $\ell > 0$. In the exceptional cases, we have $\sigma(K) = 2\upsilon(K) - 2$. As mentioned in the introduction, this improves the inequality $|\upsilon(K) - \frac{\sigma(K)}{2}| \leq 2$ for all 3-braid knots $K$ in [FK17 Proposition 4.4].

It was shown in [OSS17b Theorem 1.2] that $|\upsilon(K) - \frac{\sigma(K)}{2}|$ gives a lower bound on the nonorientable smooth 4-genus of a knot $K$, denoted $\gamma_4(K)$, the minimal first Betti number of a nonorientable surface in $B^4$ that meets the boundary $S^3$ along $K$. The similarity of the invariant $\upsilon$ and the classical signature $\sigma$ on 3-braid knots $K$ described above clearly does not lead to a good lower bound on $\gamma_4(K)$.

However, the equality $\sigma(K) = 2\upsilon(K)$ for most 3-braid knots is actually no great surprise when noting that in fact $|\upsilon(K) - \frac{\sigma(K)}{2}| \leq 1$ must be true for all 3-braid knots $K$ for the following reason. It is not hard to see that for every 3-braid knot $K$, there is a nonorientable band move to a 2-bridge knot $J$, which is alternating [Goo72]. This implies that the nonorientable cobordism distance $d_\gamma(K, J) = \gamma_4(K\# - J)$ between $K$ and $J$ is bounded from above by 1. On the other hand, using that $\upsilon$ and $\sigma$ induce homomorphisms $\mathbb{C} \rightarrow \mathbb{Z}$ (see Section 2.2 and [Mur65]), the inequality $|\upsilon(K) - \frac{\sigma(K)}{2}| \leq \gamma_4(K)$ implies that

$$\left| \upsilon(K) - \frac{\sigma(K)}{2} \right| = \left| \upsilon(K\# - J) - \frac{\sigma(K\# - J)}{2} \right| \leq d_\gamma(K, J) \leq 1,$$

where we used $\upsilon(J) = \frac{\sigma(J)}{2}$ by Proposition 2.1.

Note that a similar argument shows that $|\upsilon(K) - \frac{\sigma(K)}{2}| \leq 2$ for all 4-braid knots $K$, using two nonorientable band moves to transform $K$ into a 2-bridge link, which is also alternating.

4.4.2. On the proof technique. As mentioned in the introduction, it came as a surprise to the author that our proof strategy works not only for positive 3-braid knots, but for all 3-braid knots. Let us make this more precise.

The proofs in Section 4.2 and Section 4.3 imply, for any 3-braid knot $K$, the existence of cobordisms $C_1$ and $C_2$ of genus $g(C_1)$ and $g(C_2)$ between $K$ and (connected sums of) torus knots $T_1$ and $T_2$, respectively, such that

$$g(C_1) + g(C_2) = |\upsilon(T_2) - \upsilon(T_1)|$$

and

$$\upsilon(K) = \upsilon(T_1) + g(C_1) = \upsilon(T_2) - g(C_2).$$

For example, for knots $K$ that are closures of positive 3-braids of Garside normal form [1D], the proof of Lemma 4.5 shows the existence of such a cobordism $C_1$ for $T_1 = J_{\ell}$ as in the proof of Lemma 4.4 and the existence of such a cobordism $C_2$ between $K$ and $T_2 = T_{3,3\ell+1}\# - T_{2,2r+1}$ follows from the proof of Lemma 4.9.
The same strategy would work to determine the concordance invariants \( s \) and \( \tau \) for all positive 3-braid knots \( K \). Indeed, every positive 3-braid knot can be realized as the slice of a cobordism \( C \) between the unknot \( U \) and a torus knot \( T \) of braid index 3 such that \( g(C) = |\tau(U) - \tau(T)| = |s(U) - s(T)| \) \[\text{\cite{PLL22} Proposition 4.1.}\] However, in contrast, there are 3-braid knots where this strategy provably fails to determine \( s \) and \( \tau \). A concrete example is the 3-braid knot \( 10_{125} \) — the closure of \( a^{-5}b^3c \) \[\text{\cite{LM21} — which is not squeezed \cite{PLL22} Example 3.1.}\] This means that every cobordism \( C \) between two connected sums of torus knots \( T_1 \) and \( T_2 \) that has \( 10_{125} \) as a slice satisfies \( g(C) > |\tau(T_2) - \tau(T_1)| = |s(T_2) - s(T_1)| \).

### 4.4.3. Comparison of the normal forms for 3-braids

An algorithm described in \cite{BM93} Section 7 as Schreier’s solution to the conjugacy problem \cite{Sch24} can be used to convert 3-braids in Garside normal form (cf. Definition 4.15) to 3-braids in Murasugi normal form (cf. Definition 4.15): If \( \gamma \) is a 3-braid of Garside normal form \( \text{(C)} \), then

\[
\gamma \sim \Delta^{2(\ell+r)}a^{-1}b^r-2a^{-1}b^r-2 \cdots a^{-1}b^r-2a^{-1}b^r-2,
\]

and if \( \gamma \) is of Garside normal form \( \text{(D)} \), then

\[
\gamma \sim \Delta^{2(\ell+r)}a^{-1}b^r-2a^{-1}b^r-2 \cdots a^{-1}b^r-2a^{-1}b^r-2a^{-1}b^r-2,
\]

In addition, it is easy to see how 3-braids of Garside normal form \( \text{(A)} \) or \( \text{(B)} \) are conjugate to braids of Murasugi normal form \( \text{(C)} \) or \( \text{(D)} \).

### 5. On alternating distances of 3-braid knots

In this section, we prove Corollary \[\text{\textbf{5.2}}\] from the introduction and provide lower and upper bounds on the alternation number and dealternating number of any 3-braid knot which differ by 1.

#### 5.1. Alternating distances of positive 3-braid knots

We will prove the following proposition.

**Proposition 5.1.** Let \( K \) be a knot that is the closure of a positive 3-braid. Then

\[
\text{alt}(K) = \text{dalt}(K) = \tau(K) + \nu(K)
\]

\[
= \begin{cases} 
\ell & \text{if } K \text{ is the torus knot } T_{3,3\ell+k} \text{ for } \ell \geq 0, k \in \{1, 2\}, \\
\ell + \ell - 1 & \text{if } K \text{ is the closure of a braid of the form in (C) or (D),}
\end{cases}
\]

where \( \text{(C)} \) and \( \text{(D)} \) refer to the Garside normal forms from Proposition \[\text{\textbf{5.2}}\].

**Remark 5.2.** Some of the cases in Proposition \[\text{\textbf{5.1}}\] have already been proved by other authors. Indeed, Feller, Pohlmann and Zentner used the observation \[\text{\textbf{20}}\] below to show that \( \text{alt}(T_{3,3\ell+k}) = \ell \) for all \( \ell \geq 0, k \in \{1, 2\} \) \[\text{\cite{PPZ18} Theorem 1.1.}\] The upper bound they used was provided by \cite{Kan10} Theorem 8]; in fact, the equality had already been shown by Kanenobu in half of the cases, namely when \( \ell \) is even. Moreover, Abe and Kishimoto \cite{AK10} Theorem 3.1 showed that \( \text{alt}(K) = \tau(K) + \nu(K) \) if \( K \) is a knot that is the closure of a positive 3-braid of the form in \( \text{(C)} \). However, to the best of this author’s knowledge, it is new that \( \text{alt}(K) = \text{alt}(K) = \nu(K) \) for all positive 3-braid knots \( K \). Recall that \( \tau(K) = \text{g}(K) \) for all positive 3-braid knots \( K \) by Equation \[\text{\textbf{5}}\] from Section \[\text{\textbf{2.1}}\].
Before we prove Proposition 5.1, let us provide the necessary definitions and background. The Gordian distance $d_G(K, J)$ between two knots $K$ and $J$ is the minimal number of crossing changes needed to transform a diagram of $K$ into a diagram of $J$, where the minimum is taken over all diagrams of $K$ [Mur85]. The \textit{alternation number} $\text{alt}(K)$ of a knot $K$ is defined as the minimal Gordian distance of the knot $K$ to the set of alternating knots [Kaw10], i.e.

$$\text{alt}(K) = \min \{d_G(K, J) \mid J \text{ is an alternating knot} \}.$$ 

The \textit{dealternating number} $\text{dalt}(K)$ of a knot $K$ is defined via a more diagrammatic approach [ABB+92]: it is the minimal number $n$ such that $K$ has a diagram that can be turned into an alternating diagram by $n$ crossing changes. It follows from the definitions that

$$\text{alt}(K) \leq \text{dalt}(K) \quad (26)$$

for any knot $K$ and $\text{alt}(K) = \text{dalt}(K) = 0$ if and only if $K$ is alternating. Note that there are families of knots for which the difference between the alternation number and the dealternating number becomes arbitrarily large [Low15, Theorem 1.1].

In the proof of Proposition 5.1, we will use that

$$|\tau(K) + \upsilon(K)| \leq \text{alt}(K) \quad (27)$$

for any knot $K$. In fact, for all alternating knots $K$, we have

$$\tau(K) = \frac{s(K)}{2} = -\upsilon(K) = -\frac{\Upsilon_K(t)}{t} = -\frac{\sigma(K)}{2} \quad (28)$$

for any $t \in (0, 1]$ (see [OS03, Theorem 1.4], [Ras10, Theorem 3] and [OSS17a, Theorem 1.14]), where $s$ denotes Rasmussen’s concordance invariant from Khovanov homology [Ras10]. It follows from [Abe09, Theorem 2.1] — which builds on ideas of Livingston [Liv04, Corollary 3] — that the absolute value of the difference of any two of the invariants in (28) is a lower bound on $\text{alt}(K)$. It was first observed in [FPZ18] that the upsilon invariant fits very well in this context (see also [FLZ17, Lemma 8]).

Another main ingredient of our proof of Proposition 5.1 is the inequality

$$\text{dalt}(\gamma) \leq r - 1 \quad (29)$$

for any positive 3-braid $\gamma = a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r}$ with integers $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$ [AK10, Lemma 2.2].

\textbf{Proof of Proposition 5.1.} Let $K$ be a knot that is the closure of a positive 3-braid $\gamma$ of the form in (C) or (D) from Proposition 3.2 with $\ell \geq 0$. We claim that then

$$r + \ell - 1 = \tau(K) + \upsilon(K) = |\tau(K) + \upsilon(K)| \leq \text{alt}(K) \leq \text{dalt}(K) \leq r + \ell - 1, \quad (30)$$

which implies the statement of the proposition for these knots. The two equalities in (30) directly follow from our computations of $\upsilon(K)$ in Proposition 4.2 and Equation (5) applied to $\gamma$. The first two inequalities are direct consequences of the inequalities (27) and (26). Finally, the last inequality follows from inequality (29) applied to the particular braid representatives of $K$ considered in the proof of Corollary 4.12.

For torus knots of braid index 3, the statement follows analogously. More precisely, if $K = T_{3,3t+k}$ for $t \geq 0$ and $k \in \{1, 2\}$, then by Equations (4) and (12), we have $|\tau(K) + \upsilon(K)| = \ell$. In addition, the inequality in (29) applied to the particular
braid representatives of $K$ considered in the proof of Corollary 4.12 implies that $\text{dalt}(T_{3,3\ell+k}) \leq \ell$. □

From Proposition 5.1 it is easy to deduce that the alternating positive 3-braid knots are precisely the unknot and the connected sums $T_{2,2p+1}\#T_{2,2q+1}$ of two torus knots of braid index 2 for $p, q \geq 0$. This was already known; in fact, the stronger statement is true that the only prime alternating positive braid knots are the torus knots of braid index 2 [Baa13, Corollary 3]. Note that by [Mor79] (see also [BM93, Corollary 7.2]), the only composite 3-braid knots are the connected sums $T_{2,2p+1}\#T_{2,2q+1}$ for $p, q \in \mathbb{Z}$.

By [Abe09, Theorem 1.1], the only torus knots with alternation number 1 are the torus knots $T_{3,4}$ and $T_{3,5}$. A knot with dealternating number 1 is called almost alternating.

**Corollary 5.3.** A positive 3-braid knot is almost alternating if and only if it is one of the torus knots $T_{3,4}$ and $T_{3,5}$ or it is represented by a braid of the form

$$g^1b^1a^2b^2, \quad \Delta g^1b^1a^2, \quad \Delta^2g^1b^1 \quad \text{or} \quad \Delta^3g^1$$

for some integers $p_1, p_2, q_1, q_2 \geq 2$.

**Proof.** This follows directly from Proposition 5.1. □

**Remark 5.4.** In particular, the seven positive 3-braid knots with crossing number 12 (cf. [LM21]) are all almost alternating.

**Remark 5.5.** Our results imply that the Turaev genus equals the alternation number for all positive 3-braid knots. Indeed, let $K$ be a knot that is the closure of a positive braid of the form in (C) or (D) with $\ell \geq 0$. Then we have

$$(31) \quad g_T(K) = \text{alt}(K) = \text{dalt}(K) = r + \ell - 1,$$

where $g_T(K)$ denotes the Turaev genus of the knot $K$. The Turaev genus $g_T(K)$ of a knot $K$ is another alternating distance [Low15], which was first defined in [DFK+08] as the minimal genus of a Turaev surface $F(D)$, where the minimum is taken over all diagrams $D$ of $K$. The Turaev surface $F(D)$ is a closed orientable surface embedded in $S^3$ associated to the diagram $D$. It is formed by building the natural cobordism between the circles in the two extreme Kauffman states (the all-A-state and the all-B-state) of the diagram $D$ via adding saddles for each crossing of $D$, and then capping off the boundary components with disks. More details on the definition can be found e.g. in a survey by Champanerkar and Kofman [CK14].

The equality $g_T(K) = \text{dalt}(K)$ in (31) easily follows from Proposition 5.1, the inequalities $|g(T(K)) + \frac{\sigma(K)}{2}| \leq g_T(K)$ [DL11, Theorem 1.1] and $g_T(K) \leq \text{dalt}(K)$ [AK10, Cor. 5.4], and the fact that $\sigma(K) = 2\nu(K)$ for all knots that are closures of positive braids of Garside normal form (C) or (D) (see Section 4.1).

It is not known whether the alternation number and the Turaev genus of a knot are in general comparable: it is not known whether $\text{alt}(K) \leq g_T(K)$ for all knots $K$ (see [Low15, Question 3]). However, it was shown by Abe and Kishimoto that $g_T(T_{3,3\ell+k}) = \text{dalt}(T_{3,3\ell+k}) = \ell$ for all $\ell \geq 0$ and $k \in \{1, 2\}$ [AK10, Theorem 5.9], so $g_T(K) = \text{alt}(K) = \text{dalt}(K)$ is true for all positive 3-braid knots.

**Remark 5.6.** In [FLZ17], Friedl, Livingston and Zentner introduce the invariant $\mathcal{A}_\ell(K)$, the minimal number of double point singularities in a generically immersed concordance from a knot $K$ to an alternating knot. In the case that the altern
knot is the unknot, this is the well studied invariant $c_4(K)$ called the 4-dimensional clasp number \cite{Shi74}. A sequence of crossing changes in a diagram of a knot $K$ leading to a diagram of an alternating knot $J$ realizes an immersed concordance from $K$ to $J$ where any crossing change gives rise to a double point singularity in the concordance. We thus have $A_s(K) \leq \text{alt}(K)$ for any knot $K$, which resembles the inequality $c_4(K) \leq u(K)$ between the 4-dimensional clasp number and the unknotting number $u(K)$ of $K$. Moreover, we have $|\nu(K) + \tau(K)| \leq A_s(K)$ for any knot $K$ \cite{FLZ17} Theorem 18, so Proposition 5.1 implies $A_s(K) = \text{alt}(K)$ for all positive 3-braid knots $K$.

We are now ready to prove Corollary 1.2 from the introduction.

**Proof of Corollary 1.2.** The corollary follows directly from Proposition 5.1, Remark 5.6 and Remark 5.6. \hfill $\square$

### 5.2. Bounds on the alternation number of general 3-braid knots

In the following, we turn our attention to 3-braid knots in general, which are not necessarily the closure of positive 3-braids. We will use that

$$\left| s(K) \right| \leq \text{alt}(K)$$

for any knot $K$, which follows from \cite{Abe09} Theorem 2.1, see also equation (28) from Section 5.1. Rasmussen’s invariant $s$ was computed for all 3-braid knots in Murasugi normal form (cf. Definition 4.15) by Greene.

**Corollary 5.7.** Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \cdots a^{-p_r} b^{q_r}$ for some $\ell \in \mathbb{Z}, r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then

$$|\ell| - 1 \leq \text{alt}(K) \leq \text{dalt}(K) \leq |\ell| \quad \text{if } \ell \neq 0.$$

**Proof of Corollary 5.7.** The lower bound on the alternation number follows from the inequality \cite{Abe09} Theorem 2.1 and the values of the invariant $s$ for $K = \hat{\gamma}$ \cite{Gre14} Proposition 2.4, namely

$$s(K) = \begin{cases} -\sum_{i=1}^{r} (p_i - q_i) + 6\ell - 2 & \text{if } \ell > 0, \\ -\sum_{i=1}^{r} (p_i - q_i) + 6\ell + 2 & \text{if } \ell < 0. \end{cases}$$

Moreover, it follows from \cite{AK10} Theorem 2.5 that $\text{dalt}(\hat{\gamma}) \leq |\ell|$.

**Remark 5.8.** An alternative way to prove the upper bound on $\text{dalt}(K)$ in Corollary 5.7 for $\ell \geq 1$ follows from our observations in the proof of Lemma 4.18. In fact, the braid diagrams given by the braid representatives $\gamma_1$ of $K = \hat{\gamma}$ considered in that proof can easily be transformed into alternating diagrams by $\ell$ crossing changes: it is enough to change the positive crossings corresponding to the single generators $a$ in $\gamma_1$ to negative crossings; we obtain generators $a^{-1}$ in the corresponding braid words which then correspond to alternating braid diagrams.

---

These computations were generalized to all links that are closures of 3-braids in \cite{Mar19}.
Remark 5.9. If $K$ is represented by a 3-braid of Garside normal form (C) or (D) (see Definition 3.4), then using the observations in Section 4.4.3, Corollary 5.7 implies

\[ |r + \ell| - 1 \leq \text{alt}(K) \leq \text{dalt}(K) \leq |r + \ell| \quad \text{if } |r + \ell| > 0 \quad \text{and} \]

\[ \text{alt}(K) = \text{dalt}(K) = 0 \quad \text{if } |r + \ell| = 0. \]

By Proposition 5.1, the lower bound in (33) is sharp whenever $K$ is the closure of a positive 3-braid of Garside normal form (C) or (D). However, there are examples where the upper bound in (33) is sharp. The two easiest such examples in terms of crossing number are the non-alternating knots $8_{20}$ and $8_{21}$, which are represented by the 3-braids (cf. [LM21])

\[ a^3b^{-1}a^{-3}b^{-1} \sim \Delta^{-3}a^7 \quad \text{and} \]

\[ a^3ba^{-2}b^{-2} \sim \Delta^{-2}a^3b^2a^2b^3, \]

respectively. The lower bound on the alternation number from (33) is $|r + \ell| - 1 = 0$ in both cases. Indeed, by [Bal08, Theorem 8.6] both knots are quasialternating, so all the invariants from equation (28) are equal [Bal08, Proposition 1.4], [MO08], [OSS17a].

Remark 5.10. In a similar fashion as Corollary 5.7, the Turaev genus of all 3-braid knots was determined up to an additive error of at most 1 by Lowrance in [Low11, Proposition 4.15] using his computation of the Khovanov width for these knots. More precisely, we have

\[ |\ell| - 1 \leq g_T(K) \leq |\ell| \quad \text{if } \ell \neq 0 \]

for any knot $K$ that is represented by $\gamma = \Delta^2a^{-p_1}b^{q_1} \cdots a^{-p_r}b^{q_r}$ for some $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$.

6. The fractional Dehn twist coefficient of 3-braids in Garside normal form

In this section, we compute the fractional Dehn twist coefficient of any 3-braid in Garside normal form (cf. Definition 3.4).

The fractional Dehn twist coefficient is a homogeneous quasimorphism on the braid group $B_n$ that assigns to any $n$-braid $\gamma$ a rational number $\omega(\gamma)$. Here, a quasimorphism on a group $G$ is any map $\varphi: G \to \mathbb{R}$ such that

\[ \sup_{(a,b) \in G \times G} |\varphi(ab) - \varphi(a) - \varphi(b)| =: D_\varphi < \infty, \]

where $D_\varphi$ is called the defect of $\varphi$. A quasimorphism $\varphi: G \to \mathbb{R}$ is called homogeneous if $\varphi(a^k) = k\varphi(a)$ for all $k \in \mathbb{Z}$ and $a \in G$. Any homogeneous quasimorphism is invariant under conjugation, so $\omega(\gamma)$ is invariant under the conjugacy class of $\gamma$.

The fractional Dehn twist coefficient first appeared in [GO89] in a different language. It can be defined for mapping classes of general surfaces with boundary, where we here view braids as mapping classes of the $n$ times punctured closed disk. Malyutin defined the fractional Dehn twist coefficient $\omega: B_n \to \mathbb{R}$, $n \geq 2$, for all braid groups and showed that its defect is 1 if $n \geq 3$ and 0 if $n = 2$ [Mal04, Theorem 6.3]. We refer the reader to [Mal04] for a more detailed account.
Corollary 6.1. Let \( \gamma \) be a 3-braid. Then its fractional Dehn twist coefficient is

\[
\omega(\gamma) = \begin{cases} 
\ell & \text{if } \gamma \text{ is conjugate to a braid in } [A], \\
\frac{p+1}{6} + \ell & \text{if } \gamma \text{ is conjugate to a braid in } [B], \\
r + \ell & \text{if } \gamma \text{ is conjugate to a braid in } [C] \text{ or } [D],
\end{cases}
\]

where \([A] - [D]\) refer to the Garside normal forms from Proposition 3.2.

Remark 6.2. The fractional Dehn twist coefficient was computed for 3-braids in Murasugi normal form (cf. Definition 4.15) in [HKK+21, Proposition 6.6].

In the proof of Corollary 6.1, we will use that the fractional Dehn twist coefficient of any 3-braid \( \gamma \) is completely determined by the writhe \( \text{wr}(\gamma) \) and the homogenized upsilon invariant \( \tilde{\upsilon} \) of \( \gamma \): we have

\[
\omega(\gamma) = \tilde{\upsilon}(\gamma) + \frac{\text{wr}(\gamma)}{2},
\]

for any 3-braid \( \gamma \). The invariant \( \tilde{\upsilon} \) is another real-valued homogeneous quasimorphism on the braid group \( B_3 \) which can be defined as

\[
\tilde{\upsilon}: B_3 \to \mathbb{R}, \quad \gamma \mapsto \tilde{\upsilon}(\gamma) = \lim_{k \to \infty} \frac{\upsilon(\gamma_{6k}ab)}{6k}.
\]

More generally, Brandenbursky [Bra11, Theorem 2.6] showed that a homogeneous quasimorphism \( B_n \to \mathbb{R} \) can be assigned to any concordance homomorphism \( C \to \mathbb{R} \) that is bounded above by a constant multiple of the 4-genus. We refer the reader to [Bra11] or [FH19, Appendix A] for more details on homogenized concordance invariants.

Proposition 6.3. Let \( \gamma \) be a 3-braid. Then

\[
\tilde{\upsilon}(\gamma) = \begin{cases} 
- \frac{p}{3} - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in } [A], \\
- \frac{p+1}{3} - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in } [B], \\
- \frac{1}{2} \sum_{i=1}^{n} (p_i + q_i) + r - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in } [C], \\
- \frac{1}{2} \sum_{i=1}^{n} (p_i + q_i) + r - 2\ell - \frac{3}{2} & \text{if } \gamma \text{ is conjugate to a braid in } [D].
\end{cases}
\]

Proof of Proposition 6.3. We will use that \( \tilde{\upsilon}(\alpha\beta) = \tilde{\upsilon}(\alpha) + \tilde{\upsilon}(\beta) \) if \( \alpha \) and \( \beta \) commute [FH19, Lemma A.1]. In particular, for any 3-braid \( \gamma \) and any \( \ell \in \mathbb{Z} \), we have

\[
\tilde{\upsilon}(\Delta^{2\ell} \gamma) = \tilde{\upsilon}(\Delta^{2\ell}) + \tilde{\upsilon}(\gamma).
\]

Moreover, by the definition of \( \tilde{\upsilon} \), Equation (12) and the homogeneity of \( \tilde{\upsilon} \), we have

\[
\tilde{\upsilon}(\Delta^{2\ell}) = -2\ell \quad \text{for all } \ell \in \mathbb{Z}.
\]

We will now compute \( \tilde{\upsilon}(\gamma) \) for the positive 3-braids \( \gamma \) of the form \([A] - [D]\), i.e. assuming \( \ell \geq 0 \) in \([A] - [D]\). The statement of Proposition 6.3 will then follow from (35) and (36).

First, let \( \gamma = \Delta^{2\ell} \alpha^p \) for some \( \ell \geq 0, p \geq 0 \). If \( p = 0 \), we have \( \tilde{\upsilon}(\gamma) = -2\ell \) by (36). If \( p \geq 1 \), we have

\[
\gamma_{6k}ab = \Delta^{12tk\ell} a^{6pk} ab \sim \Delta^{12tk+1} a^{6pk-1},
\]
so by Lemma 4.9 for $k \geq 1$, we get
\[
v(\gamma^{6k}ab) = -\frac{6pk - 1}{2} + 1 - 12\ell k - \frac{3}{2} = -3pk - 12\ell k, \quad \text{hence}
\]
\[
\bar{v}(\gamma) = \lim_{k \to \infty} v(\gamma^{6k}ab) = \lim_{k \to \infty} -\frac{3pk - 12\ell k}{6k} = -\frac{p}{2} - 2\ell.
\]

Second, let $\gamma = \Delta^{2\ell}a^p b$ for some $\ell \geq 0$, $p \in \{1, 2, 3\}$. We have
\[
\gamma^{6k}ab = \Delta^{12\ell k} (ab)^{6k} ab = \Delta^{12\ell k+4k} ab \quad \text{if } p = 1,
\]
\[
\gamma^{6k}ab = \Delta^{12\ell k} (a^2b^2ab)^{3k} ab = \Delta^{12\ell k+6k} ab \quad \text{if } p = 2, \text{ and}
\]
\[
\gamma^{6k}ab = \Delta^{12\ell k} (a^3b^3babababab)^{2k} ab = \Delta^{12\ell k+8k} ab \quad \text{if } p = 3.
\]

By Equation (12), we get
\[
\bar{v}(\gamma) = \lim_{k \to \infty} -\frac{12\ell k - (2p + 2)k}{6k} = -2\ell - \frac{p + 1}{3}.
\]

Third, let $\gamma = \Delta^{2\ell}a^{p_1}b^{q_1} \ldots a^{p_r}b^{q_r}$ for some $\ell \geq 0$, $p_i, q_i \geq 2$, $i \in \{1, \ldots, r\}$. Then
\[
\gamma^{6k}ab = \Delta^{12\ell k} (a^{p_1}b^{q_1} \ldots a^{p_r}b^{q_r})^{6k} ab
\]
\[
\sim \Delta^{12\ell k+1} a^{p_1-1}b^{q_1} \ldots a^{p_r}b^{q_r} (a^{p_1}b^{q_1} \ldots a^{p_r}b^{q_r})^{6k-1}
\]
\[
\sim \Delta^{12\ell k+1} (b^{p_1}a^{p_2}b^{q_2} \ldots a^{p_r}b^{q_r} a^{p_1})^{6k-1} b^{q_1} a^{p_2}b^{q_2} \ldots a^{p_r} b^{p_1+q_r-1},
\]
where $p_1 + q_r - 1 \geq 3$. By Lemma 4.9 we have
\[
v(\gamma^{6k}ab) = -3k \sum_{i=1}^{r} (p_i + q_i) + 6kr - 12\ell k - 1, \quad \text{hence}
\]
\[
\bar{v}(\gamma) = -\frac{1}{2} \sum_{i=1}^{r} (p_i + q_i) + r - 2\ell.
\]

Finally, let $\gamma = \Delta^{2\ell+1}a^{p_1}b^{q_1} \ldots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}$ for some $\ell \geq 0$, $r \geq 1$, $p_r \geq 2$, $p_i, q_i \geq 2$, $i \in \{1, \ldots, r-1\}$. Then
\[
\gamma^{6k}ab = \Delta^{12\ell k} (\Delta a^{p_1}b^{q_1} \ldots a^{p_{r-1}}b^{q_{r-1}}a^{p_r})^{6k} ab
\]
\[
= \Delta^{12\ell k} (\Delta^2 b^{p_1}a^{q_1} \ldots b^{p_{r-1}}a^{q_{r-1}}b^{p_r}a^{q_1}a^{p_1}a^{p_2}a^{p_3} \ldots a^{p_{r-1}}b^{q_{r-1}}a^{p_r})^{3k} ab
\]
\[
= \Delta^{12\ell k+6k} (b^{p_1} \ldots b^{p_r} a^{p_1} \ldots a^{p_r})^{3k} ab
\]
\[
\sim \Delta^{12\ell k+6k+1} a^{p_1}b^{q_1}a^{p_2} \ldots b^{p_r} a^{p_1} \ldots a^{p_r} (b^{p_1} \ldots b^{p_r} a^{p_1} \ldots a^{p_r})^{3k-2}
\]
\[
b^{p_1} \ldots b^{p_r} a^{p_1} \ldots a^{p_r+1} b^{p_1+1},
\]
where $p_r + 1, p_1 + 1 \geq 3$. By Lemma 4.10 we have
\[
v(\gamma^{6k}ab) = -3k \left( \sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + 6kr - 12\ell k - 9k - 1, \quad \text{hence}
\]
\[
\bar{v}(\gamma) = -\frac{1}{2} \left( \sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + r - 2\ell - \frac{3}{2}.
\]
Proof of Corollary 6.4. This follows directly from Proposition 6.3, Equation (34), and a straightforward calculation of the writhe of the braids in (A)–(D).

Remark 6.4. If \( \gamma \) is a 3-braid conjugate to a braid of the form in (C) or (D) such that \( \hat{\gamma} \) is a knot, then Proposition 6.3 and Theorem 1.1 imply \( \tilde{\upsilon}(\gamma) = \upsilon(\hat{\gamma}) \). If \( \gamma \) additionally is a positive 3-braid, then \( \omega(\gamma) = r + \ell = g(\hat{\gamma}) + \upsilon(\hat{\gamma}) + 1 \) is the minimal number from Corollary 6.3

Remark 6.5. Our computation of \( \omega(\gamma) \) in Corollary 6.1 together with [FH19, Theorem 1.3] completely determines \( \tilde{\Upsilon}(t)(\gamma) \) for all \( 0 \leq t \leq 1 \) for any 3-braid \( \gamma \), where \( \tilde{\Upsilon}(t)(\gamma) \) is the homogenization of the invariant \( \Upsilon(t) : C \to \mathbb{R} \), defined similarly as the homogenization \( \tilde{\upsilon} \) of \( \upsilon \).

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