A quantitative criteria for the coincidence problem

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The cosmic coincidence problem is a serious challenge to dark energy model. We suggest a quantitative criteria for judging the severity of the coincidence problem. Applying this criteria to three different interacting models, including the interacting quintessence, interacting phantom, and interacting Chaplygin gas models, we find that the interacting Chaplygin gas model has a better chance to solve the coincidence problem. Quantitatively, we find that the coincidence index $C$ for the interacting Chaplygin gas model is smaller than that for the interacting quintessence and phantom models by six orders of magnitude.

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I. INTRODUCTION

The type Ia supernovae observations in 1998 discovered that the expansion of the universe is accelerating [1]. This discovery of late time cosmic acceleration imposes a big challenge and provides opportunities to gravitational and particle physics. To explain the accelerated expansion, we can modify either the left hand side (modify the theory of gravity) or the right hand side (add an exotic component with negative pressure, dubbed as "dark energy") of Einstein equation. Although lots of dark energy models have been proposed in the literature, the nature of dark energy is still a mystery. For a review of dark energy models, see for example [2] and references therein.

The simplest candidate of dark energy which is consistent with current observations is the cosmological constant. Because of the many orders of magnitude discrepancy between the theoretical predication and the observation of vacuum energy, the origin of the smallness of the value of the cosmological constant is unknown. Furthermore, the cosmological constant faces the "coincidence problem", namely, why the energy density of dark energy and dark matter happens to be of the same order now? In terms of the parameter $r$,

$$r \equiv \frac{\rho_{dm}}{\rho_{de}},$$

(1)

where $\rho$ is the energy density, the subscript $de$ denotes dark energy and $dm$ labels dark matter, the coincidence problem says why $r$ becomes order of 1 now. If the current value of $r$ which is order of 1 is independent of initial conditions, then the coincidence problem is alleviated. Therefore, the resolution of the coincidence problem lies on the attractor solution. Along this line of reasoning, the coincidence problem has been extensively studied [2, 4, 5, 6].

For the quintessence model without interaction, the attractor solution with acceleration is the scaling solution with $r = 0$. Thus, the interaction between dark matter and dark energy is proposed to get nonzero $r$ attractor solution. Assuming that the interaction is turned on recently, the standard radiation and matter dominated eras can be recovered and the coincidence problem is alleviated because $r$ starts from order 1 unstable fixed point $E$. To solve the coincidence problem, $r$ should not vary too much through the whole history of the universe, in addition to having attractor solution. Thus, during most of the history of the universe, dark matter and dark energy evolve almost in the same way. In other words, during the matter domination, dark energy behaves like dark matter. The Chaplygin gas model satisfies this requirement $E$. At early times, the Chaplygin gas model behaves like dark matter; and it behaves like dark energy at late times. Furthermore, the interacting Chaplygin gas model was shown to have the attractor solution with nonzero $r$ in $E$. So it can be used to solve the coincidence problem. Here the

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Chaplygin gas model is used as a dark energy model, not as a model unifying dark matter and dark energy. Recently, a relationship between the Hubble parameter $H$ and the ratio $r$ was found in [9]. Since the evolution of the Hubble parameter $H(z)$ can be determined by observations [10, 11], the evolution of the ratio $r$ can be derived. In this letter, we discuss the evolution of $r$ for several interacting dark energy models.

The letter is organized as follows. In section II, we study the coincidence problem in the interacting quintessence (IQT) model, the interacting phantom (IPT) model and the interacting Chaplygin gas (ICG) model. We conclude the letter with some discussions in section III.

II. THREE INTERACTING MODELS

Due to the interaction between the two dark components, the equations of motion for $\rho_{de}$ and $\rho_{dm}$ become

$$\dot{\rho}_{de} + 3H\gamma_{de}\rho_{de} = -\Gamma, \quad (2)$$

and

$$\dot{\rho}_{dm} + 3H\gamma_{dm}\rho_{dm} = \Gamma, \quad (3)$$

where a dot stands for the derivative with respect to the cosmic time $t$, the barotropic index $\gamma$ is defined as

$$\gamma = 1 + \frac{p}{\rho} = 1 + w, \quad (4)$$

$w$ is the equation of state parameter, $\gamma_{dm} = 1$, and $\Gamma$ is the interaction term between dark energy and dark matter. The origin of the interaction between dark energy and dark matter is not clear and the interaction can be rather arbitrary. Here we choose the interaction term [4, 5]

$$\Gamma = 3Hc(\rho_{dm} + \rho_{de}), \quad (5)$$

where the constant $c$ is the coupling constant. A possible mechanism for this interaction from scalar-tensor theory of gravity is discussed in detail in [8, 12, 13].

A. The evolution of $r$ in three interacting models

Combining Equations (2), (3) and (5), one obtains the variation of $r$ with respect to the cosmic time,

$$\dot{r} = 3Hr \left[ \gamma - 1 + \frac{c(1 + r)^2}{r} \right]. \quad (6)$$

The evolution of $r$ for the IQT and IPT models with constant $\gamma$ has been discussed in [3, 6, 9]. The attractor solution is [3, 6]

$$r_s = \frac{w + \sqrt{w^2 + 4cw}}{w - \sqrt{w^2 + 4cw}} \quad (7)$$

with the condition $0 < c < |w|/4$. For the ICG model, $p_{ch} = -A/\rho_{ch}$, so

$$\gamma_{ch} = 1 - \frac{H_0^4}{H^4(1 + r)^2}A', \quad (8)$$

where $A' = A(3H_0^2)^{-2}\kappa^4$ and $\kappa^2 = 8\pi G$. The interacting Chaplygin gas model was studied in detail in [8]. The attractor solution is $r_s = c/(1 - c)$ with the condition $0 < c < 1$.

With the help of Friedmann equation, the differential equation of $r$ with respect to $H$ reads [5],

$$\frac{dr}{dH} = \frac{-2r(1 + r)}{H(\gamma + r)} \left[ \gamma - 1 + \frac{\kappa^2\Gamma(1 + r)^2}{9H^3r} \right]. \quad (9)$$

In terms of the dimensionless variable $u$, $v$ which are defined as

$$u = (3H_0^2)^{-1}\kappa^2\rho_{dm}, \quad (10)$$
\( r = u/v \) as a function of \( \ln(1 + z) \) in ΛCDM, IQT, IPT and ICG models. The initial conditions and the parameter are taken as: \( u_{|z=0} + \Omega_b = 0.3, v_{|z=0} = 0.7, \) and \( \Omega_b = 0.0389. \)

\[
v = (3H_0^2)^{-1} \kappa^2 \rho_{de},
\]

(11)

Equations (2) and (3) become

\[
\frac{du}{dx} = -3u + 3c(u + v),
\]

(12)

\[
\frac{dv}{dx} = -3v(1 + w_{de}) - 3c(u + v),
\]

(13)

where \( x = \ln a = -\ln(1 + z) \). The Friedmann equation becomes

\[
\frac{H^2}{H_0^2} = u + v + \Omega_b(1 + z)^3,
\]

(14)

where \( \Omega_b \) is the fraction of the baryon energy density. The above equations describe the evolution history of the universe. We solve the above equations numerically for the ΛCDM, IQT, IPT, and ICG models, and the results are shown in Figure 1. The Hubble constant \( H_0 = 70 \) kms\(^{-1}\)Mpc\(^{-1}\) [11]. The baryon energy density \( \Omega_b = 0.0389 \) [14]. The equation of state parameter \( w_q = -0.9 \) for the IQT model and \( w_p = -1.1 \) for the IPT model. The coupling constant between dark energy and dark matter \( c_q = 1.0 \times 10^{-4} \) for the IQT model, \( c_p = 1.0 \times 10^{-4} \) for the IPT model, and \( c_{Cp} = 0.04 \) for the ICG model. Note that for the IQT model, observations require \( c < 2.3 \times 10^{-3} \) [6], so we take \( c = 1.0 \times 10^{-4} \) [9].

Note that there is no interaction between the baryon and dark energy because the standard model of particle physics is well constrained and the coupling between dark energy and ordinary matter is strongly constrained by the solar system experiments [15]. From Figure 1 it is clear that the ICG model solves the coincidence problem because the ratio \( r \) does not change much during the evolution of the universe and the current ratio \( r_s \) is the attractor. In the low redshift region, the three models almost behave in the same way. But in the high redshift region, the ratio \( r \) in the ICG model is effectively a constant, but it continually increases to a very high value for the IQT and IPT models.

Currently there is no data available for the evolution of the ratio \( r(z) \) yet. However, by solving equation (9), we can express \( H \) as a function of \( r, H = H(r) \). For some special cases, especially for the quintessence and phantom models with constant equation of state parameter \( w \), the explicit function forms were obtained in [8]. Recently, the Hubble data \( H(z) \) become available by using the age of the oldest galaxies [10, 11], and it has been used to constrain cosmological parameters [16].

In Figure 2, we show the evolution of \( H \) with respect to \( r \). Form Figure 2, one sees that in the ICG model, the ratio \( r \) is almost a constant for \( H/H_0 \geq 2 \), while in the IQT and IPT models, \( r \) increases with respect to \( H \).

In figure 3, we show the evolution of \( H \) with respect to the redshift \( z \).
From Figures 1, 2, and 3, we see that the ICG model is a much better model for overcoming the coincidence problem. But in what degree? We need a quantitative criteria for the selection. We shall discuss this point in the next subsection.

B. A quantitative criteria for coincidence

The interacting models are often invoked to solve the coincidence problem. As we have mentioned before, the attractor behavior in the interacting model is a necessary ingredient to overcome the coincidence problem. For the attractor solution, the ratio $r$ is independent of the initial conditions, and therefore greatly softens the coincidence problem. For such models, the ratio only depends on the parameters of the model, but is independent of the initial
conditions. Therefore there is no problem of fine-tuning the initial conditions. However, we still need to choose appropriate model parameters to get the right value of \( r_s \). In this sense, there exists the problem of fine-tuning the parameters. If the ratio \( r \) does not change much during the whole history of the universe for most of the parameters, then the coincidence problem can be solved. Based on these considerations, we suggest a quantitative criteria for judging the severity of the coincidence problem. We divide the coincidence problem into two smaller problems. The first is the coincidence between the value of \( r_e \) at early time and that of \( r_0 \) at present. The second is the coincidence between \( r_0 \) and the attractor value (if it exists) \( r_f \). To describe the coincidence problem quantitatively, we define the indices of early coincidence and late coincidence, respectively,

\[
C_e = \frac{r_e}{r_0},
\]

\[
C_f = \frac{r_f}{r_0}.
\]

The closer to 1 \( C_e \) or \( C_f \) is, the better the coincidence problem is overcome. However we can not define the index of the coincidence as \( C_e C_f \), since \( C_e \) and \( C_f \) maybe varies in the opposite direction, which makes the coincidence problem less severe. For example, if \( C_e = 10^{10} \) and \( C_f = 10^{-10} \), the \( C_e C_f = 1 \). Therefore, we introduce a function,

\[
F(x) = \begin{cases} 
  x, & x > 1 \\
  1/x, & 0 \leq x \leq 1 
\end{cases}
\]

(17)

By using the function \( F(x) \), we can define a proper index of coincidence \( C \) in the whole history of the universe as follows,

\[
C = F(C_e)F(C_f).
\]

(18)

which evades the above mentioned problem.

We must take a standard epoch in the early universe to determine \( C_e \). A most natural choice is \( z \to \infty \). However, we should not expect a phenomenological model can describe the whole evolution history of the universe, especially the physics of the very early universe. In this letter, we take \( z = 100 \) as “early universe”.

From equation (6), the fixed point of \( r \) is

\[
w + c \frac{(r_s + 1)^2}{r_s} = 0,
\]

which yields,

\[
r_s = -\left(1 + \frac{w}{2c}\right) \pm \sqrt{\frac{w}{c} + \frac{w^2}{4c^2}}.
\]

(19)

Since \( r \) should be real, so

\[
\frac{w}{2c} \geq 0,
\]

(21)

or

\[
\frac{w}{2c} \leq -2.
\]

(22)

The condition (21) leads to negative \( r_s \), which is not physical. Hence, for the existence of the fixed point, the parameters \( w \) and \( c \) are required to satisfy the condition (22). If \( w \) is a function of cosmic time, like the case in the Chaplygin gas model, then in (22) we take the the value of \( w \) as \( \lim_{z \to -1} w \). For the ICG model, we take \( c = 0.06, A' = 0.4 \), which is consistent with observations [1]. For the IQT model, we take \( c = 1.0 \times 10^{-4} \).

We apply this criteria to the three interacting models. The result is shown in table I. From table I, it is clear that the ICG model evades the coincidence problem. Quantitatively, the coincidence index \( C \) for the interacting Chaplygin gas model is smaller than that for the interacting quintessence and phantom models by six orders of magnitude.
III. CONCLUSION AND DISCUSSION

In order to study the coincidence problem, we propose a quantitative criteria for the determination of the severity of the coincidence problem.

First, we study the evolution of the ratio $r$ of dark matter to dark energy in the IQT, IPT, and ICG models. Though presently we do not have the information about $r(z)$, we can express $r$ with the Hubble parameter $H$. We found the evolutions of $r$ with respect to the redshift $z$ and the Hubble parameter $H$ for the three interacting models. We also found the evolution of the Hubble parameter $H(z)$ and compared it with the observational data. We find that the ICG model solves the coincidence problem.

Through a detailed analysis of the coincidence problem, we suggest a quantitative criteria for the determination of the severity of the coincidence problem. Applying this criteria to the IQT, IPT, and ICG models, we find that the ICG model solves the coincidence problem and the coincidence index $C$ for the interacting Chaplygin gas model is smaller than that for the interacting quintessence and phantom models by six orders of magnitude.

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|        | IQT | IPT | ICG |
|--------|-----|-----|-----|
| $C_e$  | $10^4$ | $10^7$ | 0.149 |
| $C_f$  | $10^{-4}$ | $10^{-4}$ | 0.149 |
| $C$    | $10^8$ | $10^9$ | 45   |

TABLE I: Coincidence indices for the IQT, IPT and ICG models.