THE TODA CONJECTURE

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Consider the Gromov-Witten potential

\[ F = \sum_{g=0}^{\infty} \varepsilon^{2g} F_g \]

of \( \mathbb{CP}^1 \). Eguchi and Yang [4] have conjectured that

\[ Z = \exp(\varepsilon^{-2} F) \]

is a \( \tau \)-function of the Toda hierarchy. (Similar ideas were also proposed by Dubrovin, cf. [2].) In this paper, we will explore this conjecture using the bihamiltonian method in the theory of integrable systems.

Let \( P \) be the puncture operator, with descendents \( \tau_{k,P}, k \geq 0 \), and let \( Q \) be the operator Poincaré dual to a point, with descendents \( \tau_{k,Q} \); denote the corresponding coordinates on the large phase space by \( s_k \) and \( t_k \) respectively.

Let \( \partial \) and \( \partial Q \) be the vector fields of differentiation with respect to \( s_0 \) and \( t_0 \), let \( E = e^{\partial} \), and introduce the operators \( \nabla = \varepsilon^{-1}(E^{1/2} - E^{-1/2}) \) and \( [2] = E^{1/2} + E^{-1/2} \). Let \( u \) and \( v \) be the functions \( \nabla^2 F \) and \( \nabla \partial Q F \).

The Toda conjecture consists of the Toda equation

\[ \partial_Q^2 F = q e^u, \quad (1) \]

whose implications have been studied by Pandharipande [25], and the recursion

\[ (v \nabla + [2] \partial_Q) \langle \langle \tau_{k-1,Q} \rangle \rangle = (k + 1) \nabla \langle \langle \tau_{k,Q} \rangle \rangle. \quad (2) \]

The large phase space of \( \mathbb{CP}^1 \) may be identified with the jet-space of the space with coordinates \( u \) and \( v \), that is, it has coordinates \( \{\partial^n u, \partial^n v\}_{n \geq 0} \). The Toda conjecture implies that, in these coordinates, the flows \( \partial_{k,Q} \) are the flows of the Toda lattice hierarchy.

Eguchi and Yang also give a matrix integral representation of the Gromov-Witten potential of \( \mathbb{CP}^1 \). Studying this representation, Eguchi, Hori and

\[ ^* \text{The original paper of Eguchi and Yang [4] had the incorrect identifications } u = \partial \nabla F \text{ and } v = \partial \partial_Q F; \text{ the corrected form of the conjecture is found in Eguchi, Hori and Yang [5].} \]
Yang [5] were led to conjecture that $Z$ satisfies a sequence of constraints $z_n = 0$, $n \geq -1$, where $z_{-1} = 0$ is the string equation and $z_0 = 0$ is Hori’s equation. This conjecture is analogous to the formulation of Witten’s KdV conjecture for topological gravity in terms of an action of the Virasoro algebra, and is now called the Virasoro conjecture for $\mathbb{C}P^1$; it has recently been proved by Givental [13] (along with its generalization to higher-dimensional projective spaces).

In Section 5, we discuss the relationship between the Toda conjecture and the Virasoro conjecture for $\mathbb{C}P^1$; the main result is that, if (2) holds, then the Virasoro conjecture is equivalent of the following recursion:

$$\left( v\nabla + [2, \partial_Q] \right) \langle \langle \tau_{k-1}, P \rangle \rangle = k \nabla \langle \langle \tau_k, P \rangle \rangle + 2 \nabla \langle \langle \tau_{k-1}, Q \rangle \rangle. \quad (3)$$

We show that, when written in terms of the coordinates $\{\partial^n u, \partial^n v\}_{n \geq 0}$, the commuting flows $\partial_k P$ associated to the descendents of the puncture operator $P$ are Hamiltonian flows; in this way, we obtain a new hierarchy of Hamiltonians in involution with each other and with the flows of the Toda lattice.

In Section 6, we show that, in the presence of the Virasoro conjecture, the Toda conjecture follows once it is known to hold along the submanifold $\{s_k = 0\}_{k > 1}$ of the large phase space. Since Okounkov and Pandharipande have recently proved the Toda conjecture on this submanifold [23], the Toda conjecture is established.

Dubrovin and Zhang [3] have proved that for homogenous spaces, and in particular for $\mathbb{C}P^1$, the Virasoro conjecture determines the Gromov-Witten potential. (They actually prove the analogous result in the more general context of semisimple Frobenius manifolds.) This poses the interesting problem of understanding how the Toda conjecture might follow directly from the Virasoro conjecture.

In this paper, we work over the field $\mathbb{Q}_\varepsilon = \mathbb{Q}(\{\varepsilon\})$ of Laurent polynomials with rational coefficients. The parameter $\varepsilon$ is known in physics as the loop expansion parameter: this simply means that integrals over moduli spaces of genus $g$ are weighted by a factor of $\varepsilon^{2g}$. In the theory of the Toda lattice, $\varepsilon$ is the lattice spacing — it is the identification of the lattice spacing with the genus expansion parameter that lies at the heart of the Toda conjecture.

1 Witten’s conjecture

The Toda conjecture is the analogue for $\mathbb{C}P^1$ of a famous conjecture of Witten [27], proved by Kontsevich [18], that the Gromov-Witten potential of a point is a $\tau$-function of the KdV hierarchy. (See also Itzykson and Zuber [15] and Looijenga [14], for illuminating discussions of the proof, and Okounkov
and Pandharipande [22] for an enumerative proof.) In this section, we recall Witten’s conjecture.

Let $\mathcal{M}_{g,n}$ be the moduli space of $n$-pointed stable curves of arithmetic genus $g$, introduced by Deligne, Mumford and Knudsen; it is an orbifold of dimension $3(g - 1) + n$.

Let $\mathcal{L}_i$ be the line bundle on $\mathcal{M}_{g,n}$ whose fibre at a stable curve $(C, z_1, \ldots, z_n)$ is the cotangent line $T^*_{z_i} C$, and let $\psi_i = c_1(\mathcal{L}_i) \in H^2(\mathcal{M}_{g,n}, \mathbb{Z})$ be its first Chern class. Witten’s conjecture is a formula for the values of the intersection numbers

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}. $$

It is convenient to assemble these numbers into generating functions on the large phase space; this is a space with coordinates $t_k$, $k \geq 0$. Denote by $\partial_k$ the vector field $\partial / \partial t_k$ on the large phase space. The vector field $\partial = \partial_0$ plays a special role.

Introduce generating functions

$$F_g = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \ldots, k_n} t_{k_1} \cdots t_{k_n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g, $$

with partial derivatives $\langle \langle \tau_{k_1} \cdots \tau_{k_n} \rangle \rangle_g = \partial_{k_1} \cdots \partial_{k_n} F_g$, and let $F$ be the total potential

$$F = \sum_{g=0}^{\infty} \varepsilon^{2g} F_g,$$

with partial derivatives $\langle \langle \tau_{k_1} \cdots \tau_{k_n} \rangle \rangle = \partial_{k_1} \cdots \partial_{k_n} F$.

The total potential $F$ satisfies the string equation $\mathcal{L}_{-1} F + \frac{1}{8} t_0^2 = 0$ and the equation $\mathcal{L}_0 F + \frac{1}{8} \varepsilon^2 = 0$, where $\mathcal{L}_{-1}$ and $\mathcal{L}_0$ are the vector fields

$$\mathcal{L}_{-1} = \sum_{k=0}^{\infty} t_{k+1} \partial_k - \partial_0, \quad \mathcal{L}_0 = \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_k \partial_k - \frac{3}{2} \partial_1.$$

**Proposition 1.1.** If $f$ is a function on the large phase space such that $\partial f$ and $\mathcal{L}_{-1} f$ are constant, then $f$ is constant. If in addition, $\lambda \mathcal{L}_0 f = f$ for some constant $\lambda$, then $f = 0$.

**Proof.** We give an outline of the proof (see Section 3 of Getzler [10] for more details): if

$$(\partial + \mathcal{L}_{-1}) f = \sum_{k=0}^{\infty} t_{k+1} \partial_k f$$
is a constant, it follows that \( f \) is a constant. Hence \( \mathcal{L}_f f = 0 \); if in addition \( \lambda \mathcal{L}_f f = f \), then \( f = 0 \).

**Theorem 1.2.** Let \( u = \partial^2 F \). The functions \( \partial^k u \) (\( = \partial^{k+2} F \)), \( k \geq 0 \), form a coordinate system on the large phase space.

**Proof.** The string equation, in conjunction with the genus 0 topological recursion relation

\[
\langle \langle \tau_k \tau_\ell \tau_m \rangle \rangle_0 = \langle \langle \tau_k-1 \tau_0 \rangle \rangle_0 \langle \langle \tau_0 \tau_\ell \tau_m \rangle \rangle_0,
\]

implies that \( \partial_k (\partial^\ell u) \mid_{t=0, \epsilon=0} = \delta_{k\ell} \).

In the coordinate system \( \{ \partial^k u \}_{k \geq 0} \), the vector fields \( \mathcal{L}_{-1} \) and \( \mathcal{L}_0 \) have the formulas

\[
\mathcal{L}_{-1} = -\frac{\partial}{\partial u}, \quad \mathcal{L}_0 = -\sum_{k=0}^{\infty} (\frac{1}{2} k + 1) \partial^k u \frac{\partial}{\partial (\partial^k u)}.
\]

(1.1)

We now recall the definition of the Kortweg-deVries (KdV) hierarchy; this is a sequence of commuting vector fields on the jet-space of the affine line. Let \( \mathcal{A} \) be a differential ring (a commutative ring with differential \( \partial \)), and let \( \Psi(\mathcal{A}) \) be the algebra of pseudodifferential operators defined over \( \mathcal{A} \); this is the algebra

\[
\Psi(\mathcal{A}) = \bigcup_{N=0}^{\infty} \left\{ \sum_{i=-\infty}^{N} a_i \partial^i \bigg| \ a_i \in \mathcal{A} \right\},
\]

with product determined by the relations \( \partial^i \cdot \partial^j = \partial^{i+j} \) and

\[
\partial^i \cdot a = \sum_{j=0}^{\infty} \binom{i}{j} \partial^j a \cdot \partial^{i-j}.
\]

Let \( A \mapsto A_+ \) be the projection on the space of pseudodifferential operators

\[
\left( \sum_{i=-\infty}^{N} a_i \partial^i \right)_+ = \sum_{i=0}^{N} a_i \partial^i,
\]

and let \( A_- = A - A_+ \).

We are interested in the case where \( \mathcal{A} \) is the algebra of differential polynomials

\[
\mathcal{A} = \mathbb{Q}[u, \partial u, \partial^2 u, \ldots];
\]

the differential \( \partial \) acts on the generators as \( \partial (\partial^k u) = \partial^{k+1} u \).
The Lax operator is the differential operator
\[ L = \frac{1}{2} \varepsilon^2 \partial^2 + u \in \Psi(A). \]
There is a unique square root \( D \in \Psi(A) \) of \( 2 \varepsilon^{-2} L = \partial^2 + 2 \varepsilon^{-2} u \), which commutes with \( L \) and has the form
\[ D = \partial + \varepsilon^{-2} u \partial^{-1} + \ldots. \]

Let \( k \) be a natural number. The **Lax equation** is the equation
\[ \delta_k L = [(L^k D)_+, L], \]
or equivalently, \( \delta_k L = -[(L^k D)_-, L] \). From these two equations, we see that \( \delta_k L \) is an element of \( A \). Write
\[ L^k D = \sum_{i=-\infty}^{2k+1} a_i(k) \partial^i, \]
where \( a_i(k) \) is an element of \( A \). The differential polynomial \( f_k = \varepsilon^2 a_{-1}(k) \) is called the \( k \)th **Gelfand-Dickii polynomial**. Since \( \delta_k L \) is the constant term in the commutator \( -[(L^k D)_-, L] \), we see that
\[ \delta_k L = \partial f_k \in A. \]

The Lax equation determines a derivation of \( A \), defined on generators by
\[ \delta_k (\partial^n u) = \partial^n (\delta_k L) = \partial^{n+1} f_k. \]

The essential property of the Lax equation is that these flows commute: since \( \delta_k D = [(L^k D)_+, D] \), it follows that \( \delta_m (L^n D)_+ = [(L^m D)_+, L^n D]_+ \), and we see that
\[ [\delta_m, \delta_n] L = \delta_m [(L^n D)_+, L] - \delta_n [(L^m D)_+, L] \]
\[ = [\delta_m (L^n D)_+, L] + [(L^n D)_+, \delta_m L] - [\delta_n (L^m D)_+, L] - [(L^m D)_+, \delta_n L] \]
\[ = [[(L^m D)_+, L^n D]_+, L] - [[(L^n D)_+, L^m D]_+, L] - [[L^m D]_+, (L^n D)_+] = 0. \]

**Theorem 1.3.** Let \( K = \frac{1}{2} \varepsilon^2 \partial^3 + u \partial + \frac{1}{2} \partial u \). The differential polynomials \( f_k \) are characterized by two properties: the recursion \( K f_{k-1} = \partial f_k \) holds, and \( f_k \) has vanishing constant term.

**Proof.** It is clear that the recursion \( K f_{k-1} = \partial f_k \) determines \( f_k \) up to a constant, since the kernel of the linear map \( \partial : A \to A \) consists of multiples of the identity. Furthermore, \( f_k \) has vanishing constant term, since \( D|_{u=0} = \partial \), hence \( (L^k D)_-|_{u=0} \) vanishes. It remains to prove the recursion.
The vanishing of the coefficient of $\partial^i$ in the equation $[L, L^k D] = 0$ gives
\[
\partial a_{i-1}(k) = -\frac{1}{2} \partial^2 a_i(k) + \varepsilon^{-2} \sum_{j=1}^{\infty} \binom{i+j}{j} a_{i+j}(k) \partial^j u.
\]
Taking $i = -2$ and $i = -1$, we see that
\[
\partial a_{-3}(k) = -\frac{1}{2} \partial^2 a_{-2}(k) - \varepsilon^{-2} \partial u a_{-1}(k) = (\frac{1}{4} \partial^3 - \varepsilon^{-2} \partial u) a_{-1}(k).
\]
By considering the coefficient of $\partial^i$ in the equations $L^{k+1} D = (L^k D) L$, we see that
\[
a_i(k+1) = \frac{1}{2} \varepsilon^2 a_{i-2}(k) + \sum_{j=0}^{\infty} \binom{i+j}{j} \partial^j u a_{i+j}(k),
\]
and in particular, that $a_{-1}(k+1) = \frac{1}{2} \varepsilon^2 a_{-3}(k) + u a_{-1}(k)$. Taking a derivative of this equation gives
\[
\partial a_{-1}(k+1) = \frac{1}{2} \varepsilon^2 \partial a_{-3}(k) + \partial u a_{-1}(k) + u \partial a_{-1}(k),
\]
and the recursion follows.

Since $f_0 = u$, we see from Theorem 1.3 that $f_1 = \frac{1}{8} \varepsilon^2 \partial^3 u + \frac{12}{8} u^2$. In particular, $\delta_0 = \partial$, while $\delta_1 u = \frac{1}{8} (\varepsilon^2 \partial^3 u + 12 u \partial u)$ is the KdV equation.

Let
\[
\alpha_k = \langle \langle \tau_0 \tau_k \rangle \rangle - \frac{2^k}{(2k+1)!} f_k;
\]
in particular, $\alpha_0 = 0$. Witten’s conjecture has a number of equivalent formulations:

- For all $k \geq 0$, $K \langle \langle \tau_0 \tau_k \rangle \rangle = (k + \frac{1}{2}) \partial \langle \langle \tau_0 \tau_{k+1} \rangle \rangle$ (i)
- For all $k \geq 0$, $\alpha_k = 0$ (ii)
- For all $k \geq 0$, the vector field $\partial_k$ equals $\frac{2^k}{(2k+1)!} \delta_k$ (iii)

It is obvious that (ii) implies (i); let us show that (i) implies (ii). Since $\mathcal{L}_{-1} u = -1$, we see that $\mathcal{L}_{-1} (L^k D) = -(k + \frac{1}{2}) L^{k-1} D$, hence $\mathcal{L}_{-1} f_k = -(k + \frac{1}{2}) f_{k-1}$; it follows that $\mathcal{L}_{-1} \alpha_k = -\alpha_{k-1}$. Likewise, we may prove by induction, using the recursion $K f_{k-1} = \partial f_k$, that
\[
\mathcal{L}_0 f_k = -\sum_{n=0}^{\infty} \binom{n}{\frac{1}{2}} \partial^n u a_{\frac{1}{2}(\partial^n u)} f_k = -(k + 1) f_k,
\]
hence that $L_0\alpha_k = -(k+1)\alpha_k$. Together, (i) and Theorem 1.3 imply that $K\alpha_{k-1} = (k+\frac{1}{2})\partial\alpha_k$. Using Proposition 1.1, we may now argue by induction that $\alpha_k = 0$.

Likewise, it is obvious that (ii) implies (iii), since $\partial\alpha_k = (\partial_k - \frac{2^k}{(2k+1)!} \delta_k)u$. The proof of the converse is similar to the proof that (i) implies (ii).

The involutivity of the vector fields $\delta_k$ is essential to the formulation of Witten’s conjecture: it is seen to be an integrability condition for the existence of the coordinates $t_k$ on the large phase space. Clearly, this conjecture determines $\partial F$; in combination with the equations $L_{-1}F + \frac{1}{2}t_0^2 = 0$ and $L_0F + \frac{1}{8}e^2 = 0$, it follows from Proposition 1.1 that it determines $F$.

2 The Toda lattice

In this section, we introduce the Toda lattice, in the form in which it enters into the Toda conjecture: the limit in which the lattice spacing $\varepsilon$ is infinitesimal (Takasaki and Takebe [29]). We follow the approach of Kupershmidt [18].

If $(A, \partial)$ is a commutative algebra with derivation $\partial$ over $Q_{\varepsilon,q} = Q_{\varepsilon,q}$, let $E : A \to A$ be the automorphism $e^{\varepsilon\partial}$. Let $\Phi(A)$ be the algebra of twisted Laurent series with coefficients $A$; as a vector space, $\Phi(A)$ is the space of Laurent series $A((\Lambda^{-1}))$, and the product is given by the formula

$$
\sum_i a_i \Lambda^i \cdot \sum_j b_j \Lambda^j = \sum_{i,j} (E^{-j/2}a_i)(E^{j/2}b_j)\Lambda^{i+j}.
$$

Extend the automorphism $E$ to $\Phi(A)$ by letting $E$ act trivially on $\Lambda$. For example, if $A = C^\infty(\mathbb{R})$ and $\partial = d/dt$, then $\Phi(A)$ is a continuum limit of the algebra of infinite matrices $(M_{ij})_{i,j\in\mathbb{Z}}$ such that $M_{ij} = 0$ for $|i-j| \gg 0$.

Let $A \mapsto A_+$ be the projection on $\Phi(A)$ defined by the formula

$$(\sum_{i=-\infty}^N a_i \Lambda^i)_+ = \sum_{i=0}^N a_i \Lambda^i,$$

and let $A_- = A - A_+$.

The commutative algebra with derivation which we will use is

$$A = Q_{\varepsilon,q}[e^u, \partial^n u, \partial^n v | n \geq 0].$$

The derivation $\partial$ acts on the generators in the evident way:

$$\partial e^u = e^u \partial u, \quad \partial(\partial^n u) = \partial^{n+1} u, \quad \partial(\partial^n v) = \partial^{n+1} v.$$
The kernel of the operators $\partial$ and $\nabla$ on $\mathcal{A}$ equals $Q_{\varepsilon,q}$. Let $P$ be the infinite-order differential operator

$$P = \frac{\partial}{\partial v} = \sum_{g=0}^{\infty} \frac{\varepsilon^{2g} (2^{1-2g} - 1) B_{2g}}{(2g)!} \partial^{2g} = 1 - \frac{1}{24} \varepsilon^2 \partial^2 + O(\varepsilon^4).$$

Obviously, $\partial = P \circ \nabla$.

**Definition 2.1.** The Lax operator of the Toda lattice is

$$L = \Lambda + v + q e^{u} \Lambda^{-1} \in \Phi(\mathcal{A}).$$

Define elements $p_k(n) \in \mathcal{A}$, $n \geq 0$, $k \in \mathbb{Z}$, as follows:

$$L^n = \sum_{k=-\infty}^{n} p_k(n) \Lambda^k.$$

**Lemma 2.1.** $p_{-1}(n) = qe^{u} p_1(n)$

**Proof.** Taking the coefficient of $\Lambda^0$ in the equation $[L, L^n] = 0$, we see that

$$\nabla p_{-1}(n) = \nabla (qe^{u} p_1(n)).$$

Thus $p_{-1}(n) - qe^{u} p_1(n) \in Q_{\varepsilon,q}$. But when $u = v = 0$, the Lax operator equals $\Lambda + q \Lambda^{-1}$, so that $p_{-1}(n)|_{u=v=0} = q^k p_k(n)|_{u=v=0}$. \qed

The following proposition shows that the functions $p_k(n)$ have certain homogeneity properties.

**Proposition 2.2.** Let $e$ and $E$ be the vector fields

$$e = \frac{\partial}{\partial v} \quad\text{and}\quad E = \sum_{n=0}^{\infty} \partial^n v \frac{\partial}{\partial \partial^n v} + 2 \frac{\partial}{\partial u}. \quad (2.1)$$

Then $e(p_k(n)) = np_k(n) - 1$ and $E(p_k(n)) = (n - k)p_k(n)$.

**Proof.** The vector fields $e$ and $E$ both commute with $\partial$, hence with the operator $E$. It follows that they induce derivations of the algebra $\Phi(\mathcal{A})$. Since $e(L) = 1$, we see that $e(L^n) = nL^{n-1}$, and expanding in powers of $\Lambda$, that $e(p_k(n)) = np_k(n) - 1$.

Likewise, since $E(L) = v + 2qe^{u} \Lambda^{-1} = (1 - \Lambda \partial_\Lambda)L$ and $\Lambda \partial_\Lambda$ is a derivation of $\Phi(\mathcal{A})$, we see that

$$E(L^n) = (n - \Lambda \partial_\Lambda)L^n;$$

expanding in powers of $\Lambda$, it follows that $E(p_k(n)) = (n - k)p_k(n)$. \qed
The $n$th Toda flow is determined by the Lax equation

$$\delta_n L = \varepsilon^{-1}[L^n_+, L] = -\varepsilon^{-1}[L^n_-, L].$$

Since $L^n_+$ involves only positive powers of $\Lambda$ and $L^n_-$ involves only negative powers of $\Lambda$, the coefficient of $\Lambda^i$ in $\delta_n L$ vanishes unless $i$ equals 0 or $-1$, and

$$\delta_n L = \nabla p_{n-1}(u) + q e^n \nabla p_0(n) \Lambda^{-1}.$$

There is a unique derivation $\delta_n$ of the algebra $\mathcal{A}$, which commutes with $\partial$ and is characterized by the formulas $\delta_n u = \nabla p_0(n)$ and $\delta_n v = \nabla p_{n-1}(u)$. In particular, the derivation $\delta_1$ is the original Toda flow:

$$\delta_1 u = \nabla v, \quad \delta_1 v = q \nabla e u.$$

Eliminating $v$, we obtain the Toda equation $\delta_2^2 u = q \nabla^2 e u$.

Being defined by Lax equations, the Toda flows commute: by the formula

$$\delta_m L^n_+ + [L^n_+, \delta_n L] = -\delta_n L^n_+ + [L^n_-, \delta_m L] = -\delta_n L^n_+ + [L^n_-, \delta_m L],$$

$$\delta_m L^n_+ + [L^n_+, \delta_n L] = [[L^n_+, L^n_+]_+, L] = 0.$$

Let $\Omega_{\mathcal{A}}$ be the algebra of differential forms

$$\Omega^*_{\mathcal{A}} = \mathcal{A}[d(\partial^n u), d(\partial^n v) \mid n \geq 0],$$

generated over $\mathcal{A}$ by Grassmann variables $\{d(\partial^n u), d(\partial^n v)\}_{n \geq 0}$ of degree 1. There is a unique derivation $\partial$ on $\Omega_{\mathcal{A}}$ which agrees with the derivation $\partial$ on $\mathcal{A}$ and commutes with the exterior differential $d$.

The space of functional differential forms is the cokernel of $\partial$:

$$\mathcal{R}^* = \Omega^*_{\mathcal{A}}/\partial \Omega^*_{\mathcal{A}}.$$

The image of an element $\alpha \in \Omega^*_{\mathcal{A}}$ in $\mathcal{R}^*$ is denoted $\int \alpha \, dt$; this notation is intended to indicate that integration by parts is permitted under the integral sign:

$$\int \partial \alpha \wedge \beta \, dt = -\int \alpha \wedge \partial \beta \, dt.$$

The exterior differential $d$ on $\Omega^*_{\mathcal{A}}$ induces a differential on $\mathcal{R}^*$. Elements of $\mathcal{R}^0$ are called functionals.

There is a natural identification between $\mathcal{R}^1$ and $\mathcal{A} du \oplus \mathcal{A} dv$, since

$$\int f_k d(\partial^k u) \, dt = \int (-\partial)^k f_k \, du \, dt$$

and

$$\int g_k d(\partial^k v) \, dt = \int (-\partial)^k g_k \, dv \, dt.$$
Under this identification, the exterior differential \( d : R^0 \to R^1 \) may be written
\[
d = \delta_u du + \delta_v dv,
\]
where \( \delta_u \) and \( \delta_v : R^0 \to A \) are the variational derivatives (also known as Euler-Lagrange operators)
\[
\delta_u = \sum_{k=0}^{\infty} (-\partial)^k \frac{\partial}{\partial (\partial^k u)}, \quad \delta_v = \sum_{k=0}^{\infty} (-\partial)^k \frac{\partial}{\partial (\partial^k v)}.
\]
(2.2)

**Lemma 2.3.** The residue \( \text{Res} : \Phi(\Omega A) \to R \), defined by the formula
\[
\text{Res}(a_n \Lambda^n) = \begin{cases} \int a_0 dt, & n = 0, \\ 0, & \text{otherwise}, \end{cases}
\]
vanishes on graded commutators and satisfies \( d\text{Res}(f) = \text{Res}(df) \).

**Proof.** It is clear that \( d\text{Res}(f) = \text{Res}(df) \), and that \( \text{Res}[a\Lambda^k, b\Lambda^\ell] \) vanishes unless \( k + \ell = 0 \), while \( \text{Res}[a\Lambda^k, b\Lambda^{-k}] = (E^{k/2} - E^{-k/2})(ab) \in \partial \Omega_A \).

The functionals \( h_n = \frac{1}{n+1} \text{Res}(L^{n+1}) = \frac{1}{n+1} \int p_0(n + 1) \, dt \in R^0 \) are the Hamiltonians of the Toda lattice hierarchy.

**Proposition 2.4.** \( \delta_u h_n = p_{-1}(n) \) and \( \delta_v h_n = p_0(n) \)

**Proof.** It follows from Lemma 2.3 that \( dh_n = \text{Res}(L^n dL) \). Since
\[
dL = dv + qe^u du \Lambda^{-1},
\]
we see that \( \text{Res}(L^n dL) = \int (p_0(n) dv + qe^u p_1(n) du) dt \).

In the dispersionless limit \( \varepsilon \to 0 \), the algebra \( \Phi(A) \) becomes the commutative algebra of Laurent series \( A((\Lambda^{-1})) \), and it is straightforward to calculate a generating function for these Hamiltonians (Fairlie and Strachan [7]).

**Proposition 2.5.**
\[
\lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} t^n p_0(n) = \left(1 - 2tv + t^2(v^2 - 4qe^u)\right)^{-1/2}
\]

**Proof.** Let \( w = v - t^{-1} \). Let \( \gamma \) be a small circular contour around the origin of the complex plane. Using the residue formula, we may write the generating
function which we wish to calculate as

\[
\lim_{\varepsilon \to \infty} \sum_{n=0}^{\infty} t^n p_0(n) = \frac{1}{2\pi i} \int \frac{1}{1 - t(\Lambda + v + qe^u\Lambda^{-1})} \frac{d\Lambda}{\Lambda}.
\]

\[
= -\frac{1}{2\pi i} \int_{\gamma} \frac{1}{(\Lambda + \frac{1}{2}w + \frac{1}{4}(w^2 - 4qe^u)^{1/2})(\Lambda + \frac{1}{2}w - \frac{1}{4}(w^2 - 4qe^u)^{1/2})} \frac{d\Lambda}{\Lambda}.
\]

The two poles of the integrand are at \( \frac{1}{2}w + \frac{1}{2}(w^2 - 4qe^u)^{1/2} = O(t) \) and \( \frac{1}{2}w - \frac{1}{2}(w^2 - 4qe^u)^{1/2} = -t^{-1} + O(1) \) respectively; thus, for sufficiently small values of \( t \), only the first contributes, with residue 1, and the generating function equals \( t^{-1}(w^2 - 4qe^u)^{-1/2} = (1 - 2tv + t^2(v^2 - 4qe^u))^{-1/2} \).

**Corollary 2.6.** Let \( P_n(x) \) be the \( n \)th Legendre polynomial. Then

\[
p_0(n) = (v^2 - 4qe^u)^{n/2} P_n(v/(v^2 - 4qe^u)^{1/2}).
\]

**Proof.** The Legendre polynomials have generating function

\[
\sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-1/2}.
\]

Setting \( x = v/(v^2 - 4qe^u)^{1/2} \) and \( t = t(v^2 - 4qe^u)^{1/2} \), the result follows. \( \square \)

The following theorem of Kupershmidt [18] gives a pair of formulas for the derivations \( \delta_n \), in terms of \( h_n \), and \( h_{n-1} \) respectively. (He also proves a third formula for \( \delta_n \), in terms of \( h_{n-2} \).)

**Theorem 2.7.** Let \( C_u = e^{-1}q( E^{1/2} \cdot e^u \cdot E^{1/2} - E^{-1/2} \cdot e^u \cdot E^{-1/2}) \). Then

\[
\delta_n \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 0 & \nabla \\ \nabla & 0 \end{bmatrix} \begin{bmatrix} \delta_n h_n \\ \delta_u h_n \end{bmatrix} = \begin{bmatrix} C_u & v \nabla \\ \nabla v & e^{-1}(E - E^{-1}) \end{bmatrix} \begin{bmatrix} \delta_n h_{n-1} \\ \delta_u h_{n-1} \end{bmatrix}.
\]

**Proof.** We have already proved the first of these formulas. To prove the second formula, observe that by the equations \( L^n = L^{n-1}L = LL^{n-1} \), we have

\[
p_k(n) = E^{-1/2}p_{k-1}(n-1) + (E^{k/2}v)p_k(n-1) + q(E^{(k+1)/2}e^u)E^{1/2}p_{k+1}(n-1)
\]

\[
= E^{1/2}p_{k-1}(n-1) + (E^{-k/2}v)p_k(n-1) + q(E^{-(k+1)/2}e^u)E^{-1/2}p_{k+1}(n-1).
\]

In particular, the equality of these two formulas for \( p_0(n) \) shows that

\[
\nabla p_{-1}(n) = q\nabla(e^u p_1(n)). \tag{2.3}
\]
Further, taking $E^{1/2}$ times the first formula minus $E^{-1/2}$ times the second, with $k$ respectively $-1$ and 0, gives

$$\nabla p_{-1}(n) = v \nabla p_{-1}(n - 1) + C_a p_0(n - 1)$$

$$\nabla p_0(n) = \nabla (vp_0(n - 1)) + \epsilon^{-1} q(E - E^{-1}) (e^u p_1(n - 1)).$$

Since $q(E - E^{-1})(e^u p_1(n - 1)) = (E - E^{-1}) p_{-1}(n - 1)$ by Lemma 2.3, the result follows.

**Corollary 2.8.** $p_0(n) = vp_0(n - 1) + [2] p_{-1}(n - 1)$

**Proof.** Since $\nabla p_0(n) = \nabla (vp_0(n - 1) + [2] p_{-1}(n - 1))$, we see that

$$p_0(n) - vp_0(n - 1) - [2] p_{-1}(n - 1) \in \mathbb{Q}_{\varepsilon,q}.$$  

To show that this vanishes, we evaluate it at the point $u = v = 0$: since $L|_{u=v=0} = \Lambda + qA$, we see that

$$p_0(n)|_{u=v=0} = (qp_1(n - 1) + p_{-1}(n - 1))|_{u=v=0} = 2p_{-1}(n - 1)|_{u=v=0},$$

as required.

## 3 Hamiltonian operators and the Toda lattice

In this section, we introduce the variational Schouten Lie algebra; this is an infinite dimensional analogue of the usual Schouten Lie algebra, developed by Dorfman and Gelfand [9]. We explain how it may be used to give an alternative approach to the Toda lattice. For more details, together with applications to other hierarchies, see Dorfman [1], Getzler [11], Manin [20] and Olver [24].

We start by introducing the free graded commutative algebra $\Lambda_{\infty}$ over the algebra $A$, with generators $\{\partial^n \theta_v, \partial^n \theta_u\}_{n \geq 0}$ of degree 1. The derivation $\partial$ is extended to $\Lambda_{\infty}$ by the formulas

$$\partial(\partial^n \theta_v) = \partial^{n+1} \theta_v,$$

$$\partial(\partial^n \theta_u) = \partial^{n+1} \theta_u.$$  

The kernel of $\partial$ is spanned by $1 \in \Lambda_{\infty}$, and the cokernel of $\partial$ is denoted $\mathcal{L}$. Denote the image of an element $F \in \Lambda_{\infty}$ in $\mathcal{L}$ by $\int F \, dt$.

The variational derivatives $\delta_u$ and $\delta_v$ on $\Lambda_{\infty}$ are defined by the same formulas (2.3) as on $A$, while the associated Grassmann variational derivatives, which we denote by $\delta^u$ and $\delta^v$, are defined by the formulas

$$\delta^u = \sum_{n=0}^{\infty} (-\partial)^n \frac{\partial}{\partial(\partial^n \theta_u)},$$

$$\delta^v = \sum_{n=0}^{\infty} (-\partial)^n \frac{\partial}{\partial(\partial^n \theta_v)}.$$
Since all of these operators vanish on the image of $\partial$, they descend to linear operators from $\mathcal{L}$ to $\Lambda_\infty$. The **Schouten bracket** is the bilinear operation on $\mathcal{L}$ defined by the formula

$$[\int F \, dt, \int G \, dt] = \int (\delta^n F \delta_u G + \delta^n F \delta_v G + (-1)^{|F|}(\delta_u F \delta^n G + \delta_v F \delta^n G)) \, dt.$$ 

With this bracket, the graded vector space $\mathcal{L}$ becomes a graded Lie algebra: $[f, g] = (-1)^{|f||g|}[f, g]$ and $[f, [g, h]] = [[f, g], h] + (-1)^{|f|+1}(|g|+1)[g, [f, h]]$. (Proofs of these formulas may be found in [1].)

The Lie subalgebra $\mathcal{L}^1$ of $\mathcal{L}^\bullet$ is isomorphic to the Lie algebra of derivations of $\mathcal{A}$ commuting with $\partial$, under the map

$$X = \int (f \theta_u + g \theta_v) \, dt \in \mathcal{L}^1 \mapsto \sum_{n=0}^\infty \left( \partial^n f \frac{\partial}{\partial (\partial^n u)} + \partial^n g \frac{\partial}{\partial (\partial^n v)} \right) \in \text{Der}(\mathcal{A}).$$

For example, the vector fields $e$ and $E$ of (2.1) correspond to $\int \theta_v \, dt$ and $\int (v \theta_v + 2 \theta_u) \, dt$; we will denote these elements of $\mathcal{L}^1$ by $e$ and $E$ as well.

An element $\mathcal{H}$ of $\mathcal{L}^2$ defines a graded derivation $\delta_{\mathcal{H}}$ of degree 1 on the graded Lie algebra $\mathcal{L}$, by the formula $\delta_{\mathcal{H}} = [\mathcal{H}, -]$.

If $[\mathcal{H}, \mathcal{H}] = 0$, $\mathcal{H}$ is called a **Hamiltonian operator**.

**Proposition 3.1.** If $\mathcal{H}$ is a Hamiltonian operator, $\delta_{\mathcal{H}}$ is a differential, the bracket $\{f, g\}_{\mathcal{H}}$ is a Lie bracket, and $[\delta_{\mathcal{H}} f, \delta_{\mathcal{H}} g] = \delta_{\mathcal{H}} \{f, g\}_{\mathcal{H}}$.

**Proof.** By the graded Jacobi rule,

$$\delta_{\mathcal{H}}(\delta_{\mathcal{H}} f) = [\mathcal{H}, [\mathcal{H}, f]] = \frac{1}{2}[[\mathcal{H}, \mathcal{H}], f] = 0,$$

showing that $\delta_{\mathcal{H}}$ is a differential.

We have

$$\{f, g\}_{\mathcal{H}} + \{g, f\}_{\mathcal{H}} = [\delta_{\mathcal{H}} f, g] + [\delta_{\mathcal{H}} g, f] = \delta_{\mathcal{H}} \{f, g\}_{\mathcal{H}}.$$

Since $[f, g]$ vanishes, we see that $\{f, g\}_{\mathcal{H}}$ is antisymmetric.

By the graded Jacobi rule, we have

$$\{\{f, g\}_{\mathcal{H}}, h\}_{\mathcal{H}} = [\delta_{\mathcal{H}} \{\delta_{\mathcal{H}} f, g\}_{\mathcal{H}}, h] = [[\delta_{\mathcal{H}} f, \delta_{\mathcal{H}} g], h]$$

$$= [\delta_{\mathcal{H}} f, \delta_{\mathcal{H}} g] - [\delta_{\mathcal{H}} g, \delta_{\mathcal{H}} f] = \{f, \{g, h\}_{\mathcal{H}}\}_{\mathcal{H}} - \{g, \{f, h\}_{\mathcal{H}}\}_{\mathcal{H}}.$$

Finally, we have $[\delta_{\mathcal{H}} f, \delta_{\mathcal{H}} g] = \delta_{\mathcal{H}} \{\delta_{\mathcal{H}} f, g\}_{\mathcal{H}}$. 

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A bihamiltonian structure is a pair of Hamiltonian operators \((\mathcal{H}, \mathcal{H}_0)\) such that \(\mathcal{H} + \lambda \mathcal{H}_0\) is a Hamiltonian operator for all \(\lambda\), or equivalently, such that \([\mathcal{H}, \mathcal{H}_0] = 0\). We now reformulate Theorem 2.7 in terms of a bihamiltonian structure.

**Theorem 3.2.** The operators

\[
\mathcal{H} = \frac{1}{2} \int \left[ \theta_v, \theta_u \right] \left[ \nabla_v \varepsilon^{-1}(E - E^{-1}) \right] \left[ \theta_u \right] dt
\]

\[
= \int \left( \varepsilon^{-1} \theta_u \mathcal{E}_\theta v + \varepsilon^{-1} q \varepsilon^u(E^{-1/2} \theta_v)(E^{1/2} \theta_v) \right) dt, \quad \text{and}
\]

\[
\mathcal{H}_0 = \frac{1}{2} \int \left[ \theta_v, \theta_u \right] \left[ \begin{array}{c} 0 \\ \nabla \end{array} \right] \left[ \begin{array}{c} \theta_v \\ \theta_u \end{array} \right] dt = \int \theta_v \nabla \theta_u dt
\]

give a bihamiltonian structure.

**Proof.** It is clear that \(\delta_v \mathcal{H} = \theta_v(\nabla \theta_u)\). Using the formula

\[
\delta^u = \sum_{i=-\infty}^{\infty} E^{-i/2} \frac{\partial}{\partial (E^{i/2} \theta_v)},
\]

we see that \(\delta^u \mathcal{H} = \theta_v(\nabla \theta_u) + C_u \theta_v\). It follows that

\[
\delta^u \mathcal{H} \delta_v \mathcal{H} = C_u \theta_v \theta_v(\nabla \theta_u)
\]

\[
= \varepsilon^{-1} q \varepsilon^u(E^{1/2} \theta_v)(E^{1/2} \theta_v) \varepsilon^{-1} q \varepsilon^u(E^{-1/2} \theta_v)(E^{-1/2} \theta_v) \theta_v(\nabla \theta_u).
\]

Likewise, since \(\delta_u \mathcal{H} = \varepsilon^{-1} q \varepsilon^u(E^{-1/2} \theta_v)(E^{1/2} \theta_v)\) and \(\delta^u \mathcal{H} = \nabla(\theta_v) + [2] \nabla \theta_u\), it follows that

\[
\delta^u \mathcal{H} \delta_u \mathcal{H} = \varepsilon^{-1} q \varepsilon^u [2](\nabla \theta_u)(E^{-1/2} \theta_v)(E^{1/2} \theta_v).
\]

We conclude that

\[
\delta^u \mathcal{H} \delta_v \mathcal{H} + \delta^u \mathcal{H} \delta_u \mathcal{H} = \varepsilon^{-1} q((E - E^{-1/2})(E^{1/2} \theta_v)(E^{1/2} \theta_v)\theta_u)
\]

\[
+ (1 - E^{-1/2})(E^{1/2} \theta_v)(E^{1/2} \theta_v)\theta_v(E^{1/2} \theta_v)
\]

\[
+ (E^{1/2} - 1)(E^{1/2} \theta_v)(E^{1/2} \theta_v)(E^{-1/2} \theta_v)(E^{-1/2} \theta_v)),
\]

and hence that \([\mathcal{H}, \mathcal{H}] = 0\). The proof that \([\mathcal{H} + \lambda \mathcal{H}_0, \mathcal{H} + \lambda \mathcal{H}_0] = 0\) is a formal consequence of this formula, since \(\mathcal{H} + \lambda \mathcal{H}_0\) is obtained from \(\mathcal{H}\) by translating \(v\) by \(\lambda\).

The Hamiltonian operators \(\mathcal{H}_0\) and \(\mathcal{H}\) have nice commutation relations with the vector fields \(e\) and \(\mathcal{E}\): it is easily checked that \([e, \mathcal{H}] = \mathcal{H}_0\) and \([e, \mathcal{H}_0] = 0\), and that \([\mathcal{E}, \mathcal{H}] = 0\) and \([\mathcal{E}, \mathcal{H}_0] = -\mathcal{H}_0\).
Denote the Poisson bracket associated to \( \mathcal{H} \) by \( \{ f, g \}_\mathcal{H} \), and the Poisson bracket associated to \( \mathcal{H}_0 \) by \( \{ f, g \}_{0} \); then Theorem 2.7 may be reformulated as the identity
\[
[\mathcal{H}_0, h_n] = [\mathcal{H}, h_{n-1}],
\]
or equivalently, as the pair of equations
\[
\nabla(\delta_v h_n) = \nabla(v \delta_v h_{n-1}) + \varepsilon^{-1}(E - E^{-1})\delta_u h_{n-1} \tag{3.2}
\]
\[
\nabla(\delta_u h_n) = C_u \delta_v h_{n-1} + v \nabla \delta_u h_{n-1} \tag{3.3}
\]
Using (3.1), we obtain another proof that the flows associated to the Hamiltonians \( h_n \) commute; this proof has the advantage that it does not depend on the Lax equation, and so applies to prove the involutivity of more general hierarchies.

**Proposition 3.3.** Let \( f_n \) and \( g_n \) be sequences of Hamiltonians such that
\[
[\mathcal{H}_0, f_n] = [\mathcal{H}, f_{n-1}] \quad \text{and} \quad [\mathcal{H}_0, g_n] = [\mathcal{H}, g_{n-1}]
\]
for \( n > 0 \), and \( \{ f_0, f \} = 0 \) for all \( f \). Then \( \{ f_m, g_n \}_0 = 0 \) for all \( m, n \geq 0 \). In particular, the Hamiltonian vector fields associated to \( f_m \) and \( g_n \) commute.

**Proof.** If \( m > 0 \), we have
\[
\{ f_m, g_n \}_0 = [[\mathcal{H}_0, f_m], g_n] = [[\mathcal{H}, f_{m-1}], g_n] = -[[\mathcal{H}, g_n], f_{m-1}]
\]
\[
= -[[\mathcal{H}_0, g_{n+1}], f_{m-1}] = \{ f_{m-1}, g_{n+1} \}_0.
\]
Thus, it suffices to prove the proposition for \( m = 0 \), for which it is clear. \( \square \)

It is an immediate consequence of Proposition 2.2 that
\[
[e, h_k] = kh_{k-1} \quad \text{and} \quad [E, h_k] = (k + 1)h_k. \tag{3.4}
\]
If \( a \in \mathbb{Z} \), let \( \mathcal{L}^i(a) \) be the generalized eigenspace
\[
\mathcal{L}^i(a) = \bigcup_{n > 0} \ker((ad(E) + i - a - 1)^n);
\]
then \( \mathcal{L}(a), \mathcal{L}(b) = \mathcal{L}(a + b) \). Since \( [E, \mathcal{H}_0] = -\mathcal{H}_0 \), we see that \( \mathcal{H}_0 \in \mathcal{L}(0) \), hence the differential \( \delta_0 \) preserves the graded subspace \( \mathcal{L}(a) \). Also, we see that \( \mathcal{H} \in \mathcal{L}(1) \).

**Theorem 3.4.** The complex \( \mathcal{L}(a) \) has vanishing cohomology unless \( a = -1 \) or 0, while the nonzero cohomology groups of \( \mathcal{L}(-1) \) and \( \mathcal{L}(0) \) are as follows:
\[
H^0(\mathcal{L}(-1), \delta_0) = \langle \int 1 \, dt, \int u \, dt \rangle \quad H^0(\mathcal{L}(0), \delta_0) = \langle \int v \, dt \rangle
\]
\[
H^1(\mathcal{L}(-1), \delta_0) = \langle \int \theta_v \, dt \rangle \quad H^1(\mathcal{L}(0), \delta_0) = \langle \int \theta_u \, dt, \int (u\theta_u - v\theta_v) \, dt \rangle
\]
\[
H^2(\mathcal{L}(0), \delta_0) = \langle \int \theta_u \theta_v \, dt \rangle
\]
Proof. Write elements of the cone $M$ of the linear map $\nabla : \Lambda_\infty / \mathbb{Q}_{\varepsilon,q} \to \Lambda_\infty$ as $(F,G)$, where $F \in \Lambda_\infty$ and $G \in \Lambda_\infty / \mathbb{Q}_{\varepsilon,q}$. If $D$ is a linear operator from $\Lambda_\infty / \mathbb{Q}_{\varepsilon,q}$ to $\Lambda_\infty$, let $iD : M \to M$ be the operator

$$iD(F,G) = (DG, 0).$$

The differential of $M$ equals $i\nabla$.

The differential $\delta_0$ on $L$ lifts to a differential

$$\delta = \sum_{n=0}^{\infty} ((\nabla \partial^n \theta_u) \partial \theta^n_v + (\nabla \partial^n \theta_v) \partial \theta^n_u)$$

on $M$, and the map $\tau : M \to L$ which sends $(F,G)$ to $\int F \, dt$ is a chain homotopy equivalence between the complexes $(M, \delta + i\nabla)$ and $(L, \delta_0)$.

Let $S$ be the chain homotopy

$$S = \sum_{n=0}^{\infty} ((P \partial^n v) \partial \theta^{n+1} u + (P \partial^n u) \partial \theta^{n+1} v),$$

let $U = i(v \theta_u + u \theta_v)$, and let $P$ be the semisimple operator

$$P = \sum_{n=0}^{\infty} ((\partial^n u) \partial \theta^n u + (\partial^n v) \partial \theta^n v) + \sum_{n=1}^{\infty} ((\partial^n \theta_v) \partial \theta^n \theta_u + (\partial^n \theta_u) \partial \theta^n \theta_v).$$

We have $[\delta + i\nabla, S] = e^{-U} Pe^U = P + U$. Thus, the cohomology of the complex $(M, d + i\nabla)$ equals the kernel

$$\langle (1, 0), (\theta_u, 0), (\theta_v, 0), (\theta_u \theta_v, 0), (u, -\theta_v), (v, -\theta_u), (u \theta_u - v \theta_v, \theta_u \theta_v) \rangle \subset M$$
of $[\delta + i\nabla, S]$. Applying $\tau$, the theorem follows. \hfill $\Box$

4 The Toda conjecture

Having introduced the Toda lattice hierarchy in the last two sections, we can now formulate the Toda conjecture. Recall the definition of the Gromov-Witten invariants of $\mathbb{C}P^1$. (A good review of the subject is Manin [21].) Let $\mathcal{M}_{g,n,d}$ be the moduli stack of stable maps of genus $g$ and degree $d$, with $n$ marked points, to $\mathbb{C}P^1$. Let $ev_i : \mathcal{M}_{g,n,d} \to \mathbb{C}P^1$, $1 \leq i \leq n$, be the map

$$ev_i(f : C \to \mathbb{C}P^1, z_1, \ldots, z_n) = f(z_i).$$

defined by evaluating a stable map $f : C \to \mathbb{C}P^1$ at the $i$th marked point $z_i \in C$. Let

$$[\mathcal{M}_{g,n,d}]^{\text{virt}} \in H_{2(2g-2+2d+n)}(\mathcal{M}_{g,n,d}, \mathbb{Q})$$
be the virtual fundamental class.

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Definition 4.1. Let $L_i$ be the line bundle on $\overline{M}_{g,n,d}$ whose fibre at the stable map $(f : C \to \mathbb{CP}^1, z_1, \ldots, z_n)$ is the line $T^*_z C$, and let $\psi_i = c_1(L_i)$ be its first Chern class.

Let $\gamma_p \in H^0(\mathbb{CP}^1, \mathbb{Z})$ and $\gamma_q \in H^2(\mathbb{CP}^1, \mathbb{Z})$ be the cohomology classes Poincaré dual to the fundamental class and to a point. Given $k_i \in \mathbb{N}$ and $a_i \in \{ P, Q \}$, define

$$\langle \tau_{k_1, a_1} \cdots \tau_{k_n, a_n} \rangle_g = \sum_{d=0}^{\infty} q^d \int_{[\overline{M}_{g,n,d}]^{\text{virt}}} \ev_1^* \gamma_{a_1} \cdots \ev_n^* \gamma_{a_n} \cup \psi_1^{k_1} \cdots \psi_n^{k_n} \in \mathbb{Q}[q].$$

We write $P$ and $Q$ instead of $\tau_0, p$ and $\tau_0, q$.

The large phase space $M$ is the formal manifold with coordinates $\{ s_k, t_k \}_{k \geq 0}$. Define

$$t_k^a = \begin{cases} s_k, & a = P, \\ t_k, & a = Q. \end{cases}$$

The genus $g$ Gromov-Witten potential $F_g$ of $\mathbb{CP}^1$ is the generating function on the large phase space given by the formula

$$F_g = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \ldots, k_n} \prod_{i=1}^{n} t_{k_i}^a \int_{[\overline{M}_{g,n,a}]^{\text{virt}}} \langle \tau_{k_1, a_1} \cdots \tau_{k_n, a_n} \rangle_g,$$

and $F = \sum_{g=0}^{\infty} e^{2g} F_g$ is the total Gromov-Witten potential.

Denote the constant vector fields $\partial / \partial t_k^a$ on the large phase space by $\partial_{k,a}$; in particular, write $\partial$ and $\partial_Q$ for $\partial_{0,P}$ and $\partial_{0,Q}$. Just as in the theory of the Toda lattice, denote the operator $e^{\epsilon \partial}$ by $E$, and the operators $\epsilon^{-1}(E^{1/2} - E^{-1/2})$ and $E^{1/2} + E^{-1/2}$ by $\nabla$ and [2]. The partial derivatives of $F$ are denoted $\partial_{k_1, a_1} \cdots \partial_{k_n, a_n} F$.

The potential $F$ satisfies the string equation $L_{-1} F + s_0 t_0 = 0$ (Witten [27]) and Hori’s equation $L_0 F + s_0^2 = 0$ (Hori [14]), where $L_{-1}$ and $L_0$ are the vector fields

$$L_{-1} = \sum_{k=0}^{\infty} \left( t_{k+1} \partial_{k,Q} + s_{k+1} \partial_{k,P} \right) - \partial$$

and

$$L_0 = \sum_{k=0}^{\infty} \left( (k+1) t_k \partial_{k,Q} + k s_k \partial_{k,P} + 2 s_{k+1} \partial_{k,Q} \right) - \partial_{1,P} - 2 \partial_Q.$$ (4.2)

The analogue of Proposition [14] holds, with essentially the same proof.

Proposition 4.1. If $f$ is a function on the large phase space such that $\nabla f$ and $L_{-1} f$ are constant, then $f$ is constant. If in addition, $\lambda L_0 f = f$ for some constant $\lambda$, then $f = 0$. 

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There is also an analogue for \( \mathbb{CP}^1 \) of Theorem 1.2. 

**Theorem 4.2.** Let \( u = \nabla^2 F \) and \( v = \nabla \partial_Q F \). The functions \( \{ \partial^n u, \partial^n v \}_{n \geq 0} \) form a coordinate system on the large phase space. The origin \( s_k = t_k = 0 \) of the large phase space has coordinates

\[
\partial^n v = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise}, \end{cases} \quad \partial^n u = 0.
\]

**Proof.** The string equation, in conjunction with the genus 0 topological recursion relation

\[
\langle \langle \tau_{k-1,a} P \rangle \rangle_0 = \langle \langle \tau_{k-1,a} Q \rangle \rangle_0 \langle \langle \tau_{k-1,b} \tau_{m,c} \rangle \rangle_0 + \langle \langle \tau_{k-1,a} \rangle \rangle_0 \langle \langle \partial_Q P \rangle \rangle_0 \langle \langle \partial_Q Q \rangle \rangle_0 \langle \langle \tau_{k-1,b} \tau_{m,c} \rangle \rangle_0,
\]

implies that \( \partial_k P(\partial^n u)|_{s^*_k = t^*_k = 0} = \partial_k Q(\partial^n v)|_{s^*_k = t^*_k = 0} = \delta_{kn} + O(\varepsilon) \).

At the origin of the large phase space, we have

\[
\partial^n v = \sum_{k=1}^{\infty} \sum_{g=0}^{\infty} \frac{\varepsilon^{2g+2k-1}}{(2k-1)!} \langle \langle P^{2k-1+n} Q \rangle \rangle_g.
\]

The moduli space \( \mathcal{M}_{g,2k+n} \) only contributes to this sum if it has virtual dimension 1, that is, if \( 2(g - 1 + d + k) + n = 1 \). The only solution of this equation is \( \mathcal{M}_{0,3} \cong \mathbb{CP}^1 \), which contributes the coefficient 1 to \( \partial v \). The argument for \( \partial^n u \) is similar: at the origin in the large phase space,

\[
\partial^n u = \sum_{k=1}^{\infty} \sum_{g=0}^{\infty} \frac{\varepsilon^{2g+2k-2}}{2k-1!(2k)!} \langle \langle P^{2k+n} \rangle \rangle_g.
\]

The only contributions to this sum come from moduli spaces \( \mathcal{M}_{g,2k+n} \) such that \( 2(g - 1 + d + k) + n = 0 \); this equation has no solutions. 

The Toda conjecture describes the vector fields \( \partial_k Q \) in the coordinate system \( \{ \partial^n u, \partial^n v \}_{n \geq 0} \); it may be formulated as any one of the equivalent conditions in the following theorem.

The vector fields \( \mathcal{L}_{-1} \) and \( \mathcal{L}_0 \) commute with \( \partial \); written in the coordinate system \( \{ \partial^n u, \partial^n v \}_{n \geq 0} \), they may be identified with \( -e \) and \( -\mathcal{E} \) (see (2.1)).

Introduce the constraints

\[
\alpha^v_{k,Q} = \nabla(\langle \langle \tau_{k-1,Q} \rangle \rangle) - \frac{1}{k!}\delta_k h_k, \quad \alpha^u_{k,Q} = \partial_Q(\langle \langle \tau_{k-1,Q} \rangle \rangle) - \frac{1}{k!}\delta_u h_k.
\]

Observe that \( \alpha^v_{1,Q} \) and \( \alpha^u_{0,Q} \) vanish. By Proposition 2.2, we have

\[
\mathcal{L}_{-1} \alpha^v_{k,Q} = -\alpha^v_{k-1,Q}, \quad \mathcal{L}_0 \alpha^v_{k,Q} = -k \alpha^v_{k,Q}, \quad \mathcal{L}_0 \alpha^u_{k,Q} = -(k+1) \alpha^u_{k,Q}. \quad (4.3)
\]

The Toda conjecture describes the vector fields \( \partial_k Q \) in the coordinate system \( \{ \partial^n u, \partial^n v \}_{n \geq 0} \). It may be formulated as any one of the equivalent conditions in the following theorem.
Theorem 4.3. Let $D$ be the differential operator
\[ D = v \nabla + [2] \partial_Q. \]
The following are equivalent:

- For all $k > 0$, $D(\langle \tau_{k-1}, Q \rangle) = (k + 1) \nabla(\langle \tau_k, Q \rangle)$, and $\partial_Q^2 F = qe^u$. (i)
- For all $k > 0$, $k! \nabla(\langle \tau_{k-1}, Q \rangle) = \delta_k h_k$ and $k! \partial_Q(\langle \tau_{k-1}, Q \rangle) = \delta_u h_k$ (ii)
- For all $k > 0$, the vector field $k! \partial_{k-1,Q}$ equals $\delta_k$ (iii)

Proof. Introduce the constraint
\[ y_k = D(\langle \tau_k, Q \rangle) - (k + 2)\nabla(\langle \tau_{k+1}, Q \rangle). \]
We may reformulate the conditions of the theorem in terms of the constraints $y_k$, $\alpha_{k,Q}^u$ and $\alpha_{k,Q}^v$ as follows:

- For all $k \geq 0$, $y_k = 0$, and $\alpha_{1,Q}^u = 0$ (i)
- For all $k > 0$, $\alpha_{k,Q}^u = 0$ (ii)
- For all $k > 0$, $\nabla \alpha_{k,Q}^v = 0$ (iii)

The third of these reformulations follows from the formulas $k! \nabla \alpha_{k,Q}^v = (k! \partial_{k-1,Q} - \delta_k)u$ and $k! \nabla \alpha_{k,Q}^u = (k! \partial_{k-1,Q} - \delta_k)v$.

Lemma 4.4. We have $y_{k-1} = va_{k,Q}^v + [2] \alpha_{k,Q}^u - (k + 1)\alpha_{k+1,Q}^u$ and
\[ \partial_Q y_{k-1} = C_{u} \alpha_{k,Q}^v + v \nabla \alpha_{k,Q}^v - (k + 1) \nabla \alpha_{k+1,Q}^v. \]

Proof. The first formula follows from the calculation
\[ (k + 1) \alpha_{k+1,Q}^v = (k + 1) \nabla(\langle \tau_k, Q \rangle) - \frac{1}{k+1} \delta_v h_{k+1} = D(\langle \tau_{k-1}, Q \rangle) - \frac{1}{k+1} (v \delta_v h_k + [2] \delta_u h_k) - y_{k-1}. \]
To prove the second formula, observe that
\[ (k + 1) \nabla \alpha_{k+1,Q}^u = (k + 1) \nabla \partial_Q(\langle \tau_k, Q \rangle) - \frac{1}{k+1} \delta_v h_{k+1} = \partial_Q D(\langle \tau_{k-1}, Q \rangle) - \frac{1}{k+1} C_u \delta_u h_k - \frac{1}{k+1} v \nabla \delta_v h_k - \partial_Q y_{k-1}. \]
Since $\partial_Q^2(\langle \tau_{k-1}, Q \rangle) = \partial_{k-1,Q} e^u = e^u \nabla^2(\langle \tau_{k-1}, Q \rangle)$ and $\partial_Q v = \nabla e^u$, we see that
\[ \partial_Q D(\langle \tau_{k-1}, Q \rangle) = (\partial_Q v) \nabla(\langle \tau_{k-1}, Q \rangle) + v \partial_Q \nabla(\langle \tau_{k-1}, Q \rangle) + [2] \partial_Q^2(\langle \tau_{k-1}, Q \rangle) \]
\[ = (\nabla e^u) \nabla(\langle \tau_{k-1}, Q \rangle) + v \partial_Q \nabla(\langle \tau_{k-1}, Q \rangle) + [2] (e^u \nabla^2(\langle \tau_{k-1}, Q \rangle)). \]
A short calculation shows that \((\nabla e^u)f + [2](e^u \nabla f) = C_u f\), proving the second formula.

By (4.3) and Proposition 4.1, we see that if \(\nabla \alpha_i^* = 0\) for \(i \leq k\), then \(\alpha_{k+1}^* = 0\); in particular, (iii) implies (ii). By Lemma 4.4, it is clear that (ii) implies (i) and that (i) implies (iii).

Condition (iii) of Theorem 4.3 has recently been proved by Okounkov and Pandharipande [23] on the submanifold \(\{s_k = 0\}_{k>1}\) of the large phase space; as we will see in Section 6, in conjunction with the Virasoro conjecture, this establishes the Toda conjecture.

5 The Toda conjecture and the Virasoro conjecture

The Virasoro conjecture for \(\mathbb{C}P^1\) says that the functions \(z_k, k \geq -1\), vanish, where \(z_k\) is given by the formula

\[
z_k = \sum_{m=0}^{\infty} \left( c_k^m + 1 \right) t_m \langle \langle \tau_m + k, Q \rangle \rangle + \epsilon_k^m s_m \langle \langle \tau_m + k, P \rangle \rangle + 2d_k^m s_m \langle \langle \tau_m + k - 1, Q \rangle \rangle \\
+ \sum_{i+j=k} i! j! \left( e^2 \langle \langle \tau_{i-1, Q} \tau_{j-1, Q} \rangle \rangle + \langle \langle \tau_{i-1, Q} \rangle \rangle \langle \langle \tau_{j-1, Q} \rangle \rangle \right) + \delta_{k,-1} s_0 t_0 + \delta_{k,0} s_0^2.
\]

Here, \(c_k^m = m(m+1)\ldots(m+k)\), and \(d_k^m = e_k(m,\ldots,m+k)\), where \(e_k(x_0,\ldots,x_k)\) is the \(k\)th elementary symmetric function, and \(s_m = s_m - \delta_{m,1}\). This conjecture was made by Eguchi, Hori and Yang [5], motivated by their matrix integral representation of the Gromov-Witten potential of \(\mathbb{C}P^1\). The string equation and Hori’s equation are the special cases with \(k = -1\) and \(k = 0\) respectively. The formulas

\[
L_{-1} z_k = -(k+1) z_{k-1}, \quad L_0 z_k = -k z_k,
\]

may be proved by direct calculation.

The Virasoro conjecture for \(\mathbb{C}P^1\) has been proved by Givental [13]; his proof uses Kontsevich’s localization theorem for Gromov-Witten invariants of toric varieties [17], together with results from Dubrovin’s theory of semisimple Frobenius manifolds [3].

In this section, we study the constraints

\[
x_k = D \langle \langle \tau_k, P \rangle \rangle - (k+1) \nabla \langle \langle \tau_{k+1}, P \rangle \rangle - 2 \nabla \langle \langle \tau_k, Q \rangle \rangle.
\]

Our main result is that if the Toda and Virasoro conjectures hold, then the constraints \(x_k\) vanish.
Theorem 5.1.

\[ Dz_k - \nabla z_{k+1} = \sum_{m=0}^{\infty} (c_k^{m+1} t_m y_{m+k} + c_k^m \partial_m x_{m+k} + 2 d_k^m \partial_m y_{m+k-1}) \]

\[ + \sum_{i+j=k} i! j! (\varepsilon^2 \partial_{i-1} + [2] \langle \tau_{i-1} \rangle) y_{j-1} \]

Proof. We make use of the following formulas: \([\partial_{i,Q}, D] = \nabla \langle \tau_{i,Q} \rangle \nabla,\]

\[ [D, t_k] = \delta_{k,0} [2], \quad [D, s_k] = \frac{1}{2} \delta_{k,0} (\varepsilon^2 \partial_0 + v [2]), \]

\[ \nabla (fg) = \frac{1}{2} [2] f \nabla g + \frac{1}{2} \nabla f [2] g, \]

\[ D(fg) = \frac{1}{2} [2] f Dg + \frac{1}{2} Df [2] g + \frac{1}{2} \varepsilon^2 \partial_0 (\nabla f \nabla g). \]

It follows that

\[ D \sum_{m=0}^{\infty} (c_k^{m+1} t_m \langle \tau_{m+k,Q} \rangle + c_k^m \partial_m \langle \tau_{m+k,P} \rangle + 2 d_k^m \partial_m \langle \tau_{m+k-1,Q} \rangle) \]

\[ = \sum_{m=0}^{\infty} (c_k^{m+1} t_m D \langle \tau_{m+k,Q} \rangle + c_k^m \partial_m D \langle \tau_{m+k,P} \rangle + 2 d_k^m \partial_m D \langle \tau_{m+k-1,Q} \rangle) \]

\[ + c_k^1 [2] \langle \tau_{k,Q} \rangle + d_k^0 (\varepsilon^2 \partial_0 + v [2]) \langle \tau_{k-1,Q} \rangle \]

\[ = \sum_{m=0}^{\infty} (c_{k+1}^{m+1} t_m \nabla \langle \tau_{m+k+1,Q} \rangle + c_k^m \partial_m \nabla \langle \tau_{m+k+1,P} \rangle + 2 d_k^m \partial_m \nabla \langle \tau_{m+k,Q} \rangle) \]

\[ + \sum_{m=0}^{\infty} (c_k^{m+1} t_m y_{m+k} + c_k^m \partial_m x_{m+k} + 2 d_k^m \partial_m y_{m+k-1}) \]

\[ + c_k^1 [2] \langle \tau_{k,Q} \rangle + d_k^0 (\varepsilon^2 \partial_0 + v [2]) \langle \tau_{k-1,Q} \rangle \]

\[ = \nabla \sum_{m=0}^{\infty} (c_{k+1}^{m+1} t_m \langle \tau_{m+k+1,Q} \rangle + c_k^m \partial_m \langle \tau_{m+k+1,P} \rangle + 2 d_k^m \partial_m \langle \tau_{m+k,Q} \rangle) \]

\[ + \sum_{m=0}^{\infty} (c_k^{m+1} t_m y_{m+k} + c_k^m \partial_m x_{m+k} + 2 d_k^m \partial_m y_{m+k-1}) \]

\[ + d_k^0 (\varepsilon^2 \partial_0 + v [2]) \langle \tau_{k-1,Q} \rangle \]
\[ \mathcal{D} \sum_{i+j=k} i! j! (e^2 \langle \tau_{i-1,0} \tau_{j-1,0} \rangle + \langle \tau_{i-1,0} \rangle \langle \tau_{j-1,0} \rangle) \]

\[ = \sum_{i+j=k} i! j! (e^2 \partial_{i-1,0} \mathcal{D} \langle \tau_{j-1,0} \rangle - e^2 [\partial_{i-1,0}, \mathcal{D}] \langle \tau_{j-1,0} \rangle) \]

\[ + [2] \langle \tau_{i-1,0} \rangle \mathcal{D} \langle \tau_{j-1,0} \rangle + \frac{1}{2} e^2 \partial_{q} (\nabla \langle \tau_{i-1,0} \rangle \nabla \langle \tau_{j-1,0} \rangle) \]

\[ = \sum_{i+j=k} i! j! (e^2 \nabla \langle \tau_{i-1,0} \tau_{j-1,0} \rangle + [2] \langle \tau_{i-1,0} \rangle \nabla \langle \tau_{j-1,0} \rangle) \]

\[ + \sum_{i+j=k} i! j! (e^2 \partial_{i-1,0} + [2] \langle \tau_{i-1,0} \rangle) y_{j-1} \]

\[ = \nabla \sum_{i+j=k+1} i! j! (e^2 \langle \tau_{i-1,0} \tau_{j-1,0} \rangle + \langle \tau_{i-1,0} \rangle \langle \tau_{j-1,0} \rangle) \]

\[ - k! (e^2 \nabla \partial_{Q} + v [2] \langle \tau_{k-1,0} \rangle) + \sum_{i+j=k} i! j! (e^2 \partial_{i-1,0} + [2] \langle \tau_{i-1,0} \rangle) y_{j-1}. \]

The theorem follows for \( k > 0 \) on adding the results of these two calculations; the cases \( k = -1 \) and \( k = 0 \) are similar, and we leave them to the reader.

**Corollary 5.2.** If \( y_{k} = 0 \) for all \( k \geq 0 \) and \( z_{k} = 0 \) for all \( k \geq -1 \), then \( x_{k} = 0 \) for all \( k \geq 0 \).

**Proof.** If \( y_{k} \) and \( z_{k} \) vanish for all \( k \), then the formula of Theorem 5.1 becomes, for \( k \geq -1 \),

\[ x_{k+1} = \sum_{m=1}^{\infty} \binom{k+m}{m-1} s_{m} x_{k+m}. \]

The result follows by induction on the order of vanishing of the constraints \( x_{k} \) at the origin of the large phase space.

Assuming the Toda and Virasoro conjectures, we will now show that there are Hamiltonians \( g_{k} \in \mathbb{R}^{0} \) such that \( \partial_{k,0} u = \frac{1}{2} \nabla \delta u g_{k} \) and \( \partial_{k,0} v = \frac{1}{2} \nabla \delta u g_{k} \). We may construct the Hamiltonian \( g_{0} \) explicitly: the equations \( \partial u = \nabla \delta u g_{0} \) and \( \partial v = \nabla \delta u g_{0} \) have the solution

\[ g_{0} = \int u v \, dt = \sum_{g=0}^{\infty} \frac{e^{2g} (2^{1-2g} - 1) B_{2g}}{(2g)!} \int u \partial^{2g} v \, dt. \quad (5.2) \]

Note that \( \delta_{0} g_{0} \) lies in the centre of \( \mathcal{L} \), since

\[ [\delta_{0} g_{0}, \int F \, dt] = [\int (\partial u \theta_{u} + \partial v \theta_{v}) \, dt, \int F \, dt] = \int \partial F \, dt = 0. \quad (5.3) \]

We have not been able to find an explicit formula for the Hamiltonians \( g_{k} \); instead, we construct them using a method due to Gelfand and Dorfman.

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Theorem 5.3. There is a unique sequence of Hamiltonians $g_k \in \mathcal{L}^0(k)$ starting with $g_0 = \int uPv \ dt$ such that $[\mathcal{H}_0, g_k] = [\mathcal{H}, g_{k-1} - \frac{2}{k} h_{k-1}]$.

Proof. Uniqueness is clear: if $k > 0$, the recursion $[\mathcal{H}_0, g_k] = [\mathcal{H}, g_{k-1} - \frac{2}{k} h_{k-1}]$ determines $g_k$ up to an element of $H^0(\mathcal{L}(k), \delta_0)$, and this cohomology group vanishes by Theorem 3.4.

We will construct $g_k$ by induction.

Lemma 5.4. The vector field $X_k = [\mathcal{H}, g_{k-1} - \frac{2}{k} h_{k-1}] \in \mathcal{L}^1(k)$ satisfies $\delta_0 X_k = 0$.

Proof. We have

$$\delta_0 X_k = [\mathcal{H}_0, [\mathcal{H}, g_{k-1} - \frac{2}{k} h_{k-1}]] = -[\mathcal{H}, [\mathcal{H}_0, g_{k-1} - \frac{2}{k} h_{k-1}]].$$

If $k > 1$, we have

$$[\mathcal{H}, [\mathcal{H}_0, g_{k-1} - \frac{2}{k} h_{k-1}]] = [\mathcal{H}, [\mathcal{H}, g_{k-2} - 2(\frac{1}{k} + \frac{1}{k-1}) h_{k-2}]] = 0.$$

If $k = 1$, we see that $\delta_0 X_1 = -[\mathcal{H}, [\mathcal{H}_0, g_0 - 2h_0]] = [\delta_0 g_0, \mathcal{H}]$, which vanishes by (5.3). 

It follows from this lemma that the cohomology class of $X_k$ is an element of $H^1(\mathcal{L}(k), \delta_0)$. For $k > 0$, this cohomology group vanishes by Theorem 3.4, hence there is an element $g_k$ of $\mathcal{L}^0(k)$ such that $\delta_0 g_k = X_k$. 

Corollary 5.5. Denote the Hamiltonian vector field $\delta_0 g_k$ associated to the Hamiltonian $g_k$ by $\tilde{\delta}_k$:

$$\tilde{\delta}_k v = \nabla \delta_0 g_k, \quad \tilde{\delta}_ku = \nabla \delta_0 g_k.$$

The flows $\tilde{\delta}_k$ commute with each other, and with the flows $\delta_k$.

Proof. Apply Theorem 3.3 to the sequences of Hamiltonians $h_k$ and $g_k$. 

It is not hard to find an explicit formula for $g_1$:

$$g_1 = \int \left(\frac{1}{2} uP(v^2 + [2] e^u) + \frac{1}{2} v([2] P - 2) v - 2 q e^u\right) \ dt. \quad (5.4)$$

The equation

$$[\mathcal{H}_0, g_1] = [\mathcal{H}, g_0 - 2h_0] = [\mathcal{H}, g_0] - 2[\mathcal{H}_0, h_1]$$
Lemma 5.7. For \( u \) By the definitions of amounts by (3.2) and (3.3) to the pair of equations
\[
\nabla (\delta_v g_1) = \nabla (v \delta_v g_0 + [2] \delta_u g_0 - 2 \delta_h h_1) = \nabla (v P u + [2] (P - 2) v)
\]
\[
\nabla (\delta_u g_1) = C_u \delta_v g_1 + v \nabla \delta_v g_1 - 2 \nabla \delta_h h_1
\]
\[
= \frac{1}{2} q \nabla (e^u [2] P u) + \frac{1}{2} q [2] (e^u \partial u) + v \nabla P v - 2 q \nabla e^u
\]
\[
= q \nabla (\frac{1}{2} e^u [2] P u + \frac{1}{2} P (v^2 + [2] e^u) - 2 e^u),
\]
and it is easily seen that these are satisfied.

Proposition 5.6. For \( k > 0 \), \([e, g_k] = k g_{k-1}\). For \( k \geq 0 \),
\[
[E, g_k] = (k + 1) g_k + 2 h_k.
\]

Proof. We prove these formulas by induction: it is easily checked, using the explicit formula (3.4) for \( g_1 \), that \([e, g_1] = g_0\), and, using the explicit formula (5.2) for \( g_0 \), that \([E, g_0] = g_0 + 2 h_0\).

We have
\[
\delta_0[e, g_k] = [e, \delta_0 g_k] = [e, [H, g_{k-1} - \frac{2}{k} h_{k-1}]]
\]
\[
= [[e, H], g_{k-1} - \frac{2}{k} h_{k-1}] + [H, [e, g_{k-1} - \frac{2}{k} h_{k-1}]]
\]
\[
= \delta_0(g_{k-1} - \frac{2}{k} h_{k-1}) + [H, (k-1) g_{k-2} - \frac{2(k-1)}{k} h_{k-2}]
\]
\[
= \delta_0(g_{k-1} - \frac{2}{k} h_{k-1}) + \delta_0((k-1) g_{k-2} + \frac{2}{k} h_{k-1}) = \delta_0(k g_{k-1}).
\]
Since the cohomology group \( H^0(L(k-1), \delta_0) \) vanishes for \( k > 1 \), the formula for \([e, g_k] \) follows.

The argument for \( E \) is similar: we have
\[
\delta_0[E, g_k] = [\delta_0 E, g_k] + [E, \delta_0 g_k] = \delta_0 g_k + [E, [H, g_{k-1} - \frac{2}{k} h_{k-1}]]
\]
\[
= \delta_0 g_k + [H, [E, g_{k-1} - \frac{2}{k} h_{k-1}]]
\]
\[
= \delta_0 g_k + [H, k g_{k-1}] = \delta_0((k + 1) g_k + 2 h_k).
\]
Since the cohomology group \( H^0(L(k), \delta_0) \) vanishes for \( k > 0 \), the formula for \([E, g_k] \) follows. \( \square \)

Introduce the constraints
\[
\alpha^u_{k,P} = \nabla \langle \tau_k, P \rangle - \frac{1}{M} \delta_v g_k, \quad \alpha^v_{k,P} = \partial_\tau \langle \tau_k, P \rangle - \frac{1}{M} \delta_u g_k.
\]

By the definitions of \( u \) and \( v \), we see that \( \alpha^u_{k,P} \) and \( \alpha^v_{k,P} \) vanish.

Lemma 5.7. For \( k > 0 \), \( L_{-1} \alpha^u_{k,P} = - \alpha^v_{k-1,P} \) and
\[
L_0 \alpha^u_{k,P} = - k \alpha^v_{k,P} - 2 \alpha^v_{k,Q}, \quad L_0 \alpha^v_{k,P} = -(k+1) \alpha^u_{k,P} - 2 \alpha^u_{k,Q}.
\]
Proof. By the string equation, \( \mathcal{L}_{-1} \langle \langle \tau_{k,P} \rangle \rangle = \langle \langle \tau_{k-1,P} \rangle \rangle \). Since \([e, \delta_v] = [e, \delta_u] = 0\), we see that \( \mathcal{L}_{-1} \delta_v g_k = -k \delta_v g_{k-1} \) and \( \mathcal{L}_{-1} \delta_u g_k = -k \delta_u g_{k-1} \). This shows that \( \mathcal{L}_{-1} \alpha_{k,P}^* = -\alpha_{k,P}^* \).

By Hori’s equation \( z_0 = 0 \), we see that \( \mathcal{L}_0 \nabla \langle \langle \tau_{k,P} \rangle \rangle = k \nabla \langle \langle \tau_{k,P} \rangle \rangle + 2 \langle \langle \tau_{k-1,Q} \rangle \rangle \),

\[
\mathcal{L}_0 \partial_Q \langle \langle \tau_{k,P} \rangle \rangle = -(k+1) \partial_Q \langle \langle \tau_{k,P} \rangle \rangle - 2 \partial_Q \langle \langle \tau_{k-1,Q} \rangle \rangle.
\]

Since \([E, \delta_v] = -\delta_v\) and \([E, \delta_u] = 0\), we see that \( \mathcal{L}_{-1} \delta_v g_k = -k \delta_v g_k - 2 \delta_v h_k \) and \( \mathcal{L}_0 \delta_u g_k = -(k+1) \delta_u g_k - 2 \delta_u h_k \). This yields the formulas for \( \mathcal{L}_0 \alpha_{k,P}^* \).

Theorem 5.8. Assume that the Toda conjecture holds. Each of the following conditions are equivalent to the Virasoro conjecture:

(i) for all \( k > 0 \), \( \mathcal{D} \langle \langle \tau_{k-1,P} \rangle \rangle = k \nabla \langle \langle \tau_{k,P} \rangle \rangle + 2 \langle \langle \tau_{k-1,Q} \rangle \rangle \)

(ii) for all \( k > 0 \), \( k! \nabla \langle \langle \tau_{k,P} \rangle \rangle = \delta_v g_k \) and \( k! \partial_Q \langle \langle \tau_{k,P} \rangle \rangle = \delta_u g_k \)

(iii) for all \( k \geq 0 \), the vector field \( k! \partial_{k,P} \) equals \( \tilde{\delta}_k \)

Proof. We may reformulate the conditions of the theorem in terms of the constraints \( x_k \), \( \alpha_{k,P}^u \) and \( \alpha_{k,P}^v \) as follows:

(i) for all \( k \geq 0 \), \( x_k = 0 \)

(ii) for all \( k \geq 0 \), \( \alpha_{k,P}^u = \alpha_{k,P}^v = 0 \)

(iii) for all \( k \geq 0 \), \( \nabla \alpha_{k,P}^u = \nabla \alpha_{k,P}^v = 0 \)

We have already shown that the Virasoro conjecture implies (i); let us prove the converse. We argue by induction that \( z_k \) vanishes, starting with Hori’s equation \( z_0 = 0 \). If \( z_{k-1} = 0 \), then Theorem 5.1 together with (i) implies that \( \nabla z_k = 0 \). We conclude by (5.1) and Proposition 4.1 that \( z_k = 0 \).

The proof of the following lemma is analogous to that of Lemma 4.4.

Lemma 5.9. We have \( x_{k-1} = v \alpha_{k-1,P}^u + [2] \alpha_{k-1,P}^u - k \alpha_{k,P}^v - 2 \alpha_{k,Q}^v \) and \( \partial_Q x_{k-1} = C_u \alpha_{k-1,P}^u + v \nabla \alpha_{k-1,P}^u - k \nabla \alpha_{k,P}^u - 2 \nabla \alpha_{k,Q}^u \).

By Lemma 5.7 and Proposition 4.1, we see that if \( \nabla \alpha_{i,P}^* = 0 \) for \( i \leq k \), then \( \alpha_{k,P}^* = 0 \); in particular, (iii) implies (ii). Lemma 5.9 shows that (ii) implies (i) and that (i) implies (iii).

\[ \square \]

Corollary 5.10. The Toda and Virasoro conjectures determine the Gromov-Witten potential \( F \) of \( \mathbb{CP}^1 \) up to a constant (that is, an element of \( \mathbb{Q}_{c,q} \)).
Proof. The Toda and Virasoro conjectures determine the vector fields $\partial_{k,Q}$ and $\partial_{k,P}$ in the coordinate system $\{\partial^n u, \partial^n v\}_{n \geq 0}$ on the large phase space. It follows that the coordinates $\{s_k, t_k\}_{k \geq 0}$ are determined up to constants of integration; but these constants are fixed by Theorem 4.2. By inversion, we see that $u$ is determined as a function of $\{s_k, t_k\}_{k \geq 0}$. Integrating twice, using Lemma 4.1 and the string equation, we see that $F$ is determined up to a constant.

In order to determine the constant term of $F$, we may use the divisor equation

$$ q \frac{\partial F}{\partial q} + \sum_{k=0}^{\infty} s_{k+1} \frac{\partial F}{\partial t_k} = \partial_{Q} F. $$

This fixes the constant up to an element of $Q_{\epsilon}$; however, such constant terms correspond to Gromov-Witten invariants of degree 0, and no moduli space $\overline{M}_{g,n,d}$ has vanishing virtual dimension if $d = 0$. Thus, this constant vanishes.

6 Propagating the Toda conjecture

Consider the submanifold $L \subset M$ of the large phase space on which $s_k = 0$, $k > 1$. Okounkov and Pandharipande [23] have proved the Toda conjecture on this submanifold; that is, they prove that $k! \partial_{k-1,Q} \partial_{Q} F$ equals $\delta_k$ along $L$, for all $k > 0$. Our results allow us to prove that this, in conjunction with the Virasoro conjecture, implies the full Toda conjecture.

Suppose that the constraints $\nabla_{\alpha} v^u_{k,Q}$ and $\nabla_{\alpha} u_{k,Q}$ vanish to order $N$ along $L$; the theorem of Okounkov and Pandharipande is the case $N = 1$. We now argue by induction. The proof of Theorem 4.3 shows that the constraints $y_k$ vanish to order $N$ along $L$. (Here, we use the fact that the vector fields $L_{-1}$ and $L_0$ are tangential to $L$, as may be seen by inspection of the explicit formulas (4.1) and (4.2).)

Applying Theorem 5.1, we see that the constraints $x_k$ vanish to order $N$ along $L$. The proof of Theorem 5.8 shows that the constraints $\nabla_{\alpha} v^u_{k,P}$ and $\nabla_{\alpha} u_{k,P}$ vanish to order $N$ along $L$, in other words, that the vector fields $k! \partial_{k,P}$ and $\delta_k$ are equal to order $N$ along $L$.

To prove the induction step, we must show that the vector field $k! \partial_{k-1,Q} - \delta_k$ vanishes to order $N+1$ along $L$, in other words, that $[\ell! \partial_{\ell,P}, k! \partial_{k-1,Q} - \delta_k]$ vanishes to order $N$ along $L$ for all $\ell > 1$. We have

$$ [\ell! \partial_{\ell,P}, k! \partial_{k-1,Q} - \delta_k] = [\ell! \partial_{\ell,P}, k! \partial_{k-1,Q}] - [\delta_{\ell}, \delta_k] $$

$$ - [\ell! \partial_{\ell,P} - \delta_{\ell}, k! \partial_{k-1,Q}] + [\ell! \partial_{\ell,P} - \delta_{\ell}, k! \partial_{k-1,Q} - \delta_k]. $$

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Obviously, \([\ell! \partial_{\ell,P}, k! \partial_{k-1,Q}]\) vanishes; the commutator \([\delta_{\ell}, \delta_{k}]\) vanishes by Corollary 5.5; the vector field \([\ell! \partial_{\ell,P} - \delta_{\ell}, \delta_{k}]\) vanishes to order \(N\) along \(L\), while the vector field \([\ell! \partial_{\ell,P} - \delta_{\ell}, k! \partial_{k-1,Q} - \delta_{k}]\) vanishes to order \(2N - 1 \geq N\) along \(L\).

Acknowledgments

I thank B. Dubrovin, T. Eguchi, B. Feigin, A. Orlov, R. Pandharipande, T. Shiota, C.-S. Xiong, Y. Zhang and the referee for stimulating my interest in this subject and for their helpful suggestions.

I wish to thank Kyoji Saito and Masa-Hiko Saito, and all of the other organizers and participants in the memorable year 1999–2000 at RIMS, Kyoto University devoted to “Geometry of String Theory.”

The research of the author is supported in part by NSF grants DMS-9704320 and DMS-0072508.

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