ON THE LIFESPAN OF STRONG SOLUTIONS TO THE PERIODIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. An explicit lifespan estimate is presented for the derivative Schrödinger equations with periodic boundary condition.

1. Introduction. We consider the Cauchy problem for the following derivative nonlinear Schrödinger (DNLS) equation:

\[
\begin{cases}
    i \partial_t u + \partial_x^2 u = \lambda \partial_x(|u|^{p-1}u), & t \in [0,T), \quad x \in \mathbb{T}, \\
    u(0) = u_0, & x \in \mathbb{T}
\end{cases}
\]

on one-dimensional torus \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \), where \( p > 1 \) and \( \lambda \in \mathbb{C}\setminus\{0\} \). The aim of this paper is to study an explicit upper bound of lifespan of solutions for (1) in terms of the data \( u_0 \) in the case \( \text{Re}\lambda \neq 0 \).

The original DNLS equation is DNLS on \( \mathbb{R} \) with \( p = 3 \) and \( \lambda = -i \) with additional terms, which was derived in plasma physics for a model of Alfvén wave (see [13, 18]). By a simple computation, if \( \lambda \in i\mathbb{R} \), then we have the charge (\( L^2 \)) conservation law for solutions of DNLS with any \( p > 1 \) in the case of both torus and Euclidean space. The well-posedness for the original DNLS without additional terms has been studied, for example, in [2, 3, 7, 8, 9, 10, 12, 17, 20, 24]. Furthermore, the Cauchy problem (1) with \( p = 3 \) and \( \lambda \in i\mathbb{R} \) has also been studied, for example, in [1, 6, 11, 15, 21, 23, 24]. Especially, the global well-posedness of (1) in the frame work of \( H^s(\mathbb{T}) \) with \( s \geq 1/2 \) has been studied by Win [25] and Mosincat [14] in the case where the charge of initial data is sufficiently small. Here \( H^s(\mathbb{T}) \) denotes the standard Sobolev space defined by \( H^s(\mathbb{T}) = (1 - \Delta)^{-s/2}L^2(\mathbb{T}) \).

However, the blowup problem for DNLS is still open in a general setting, where the conservation

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law is insufficient or fails, for examples, in the case where \( p = 3, \lambda = i\mathbb{R}, \) and the charge of initial data is large. Partial results have been obtained in [22].

On the other hand, Sunagawa [19] studied the finite time blow-up of

\[
\begin{cases}
i\partial_t u + \partial_x^2 u = \lambda |u|^{p-1} \partial_x u, & t \in [0, T), \ x \in \mathbb{T}, \\
u(0) = u_0, & x \in \mathbb{T}
\end{cases}
\]

when \( p = 3 \) and \( \Re \lambda \neq 0 \), where (2) is connected with the gauge transformation of (1). Indeed let \( v \) be a solution for

\[
\begin{cases}
i\partial_t v + \partial_x^2 v = -i\partial_x(|v|^2 v), & t \in [0, T), \ x \in \mathbb{R}, \\
v(0) = v_0, & x \in \mathbb{R}.
\end{cases}
\]

Then the gauge transformed solution \( w \) defined by

\[
w(t, x) = v(t, x) \exp \left( \frac{i}{2} \int_{-\infty}^{x} |v(t, y)|^2 dy \right)
\]

satisfies

\[
i\partial_t w + \partial_x^2 w = -i|w|^2 \partial_x w, \quad t \in [0, T), \ x \in \mathbb{R}.
\]

For related subjects, we refer the reader [8, 9, 10, 11]. He showed that solutions of (2) blow up when

\[-\text{sgn}(\Re \lambda) \cdot \text{Im} \int_{\mathbb{T}} u_0(x) \partial_x u_0(x) dx > 0.\]

Namely, in the case where \( \Re \lambda \neq 0 \), solutions may blow up even when their charge is arbitrary small. We remark that the condition \( p = 3 \) plays a crucial role in his argument.

In this article, we study the finite time blowup of solutions for (1) in the case where \( \Re \lambda \neq 0 \) and \( p > 1 \) by using a simple ODE argument. We remark that in this case, the conservation law fails and will show that there exists no \( L^2 \) global solution in a certain case. For the ODE approach, we refer the reader [4, 5, 16].

An obvious global solution for (1) is \( u(t, x) = C \) for \( C \in \mathbb{C} \). So it is necessary to consider a set of initial data without constants in order to show the finite time blowup of (1). Here we consider the initial data and solutions with vanishing total density defined as follows:

**Definition 1.1.** For \( u_0 \in H^2(\mathbb{T}) \) satisfying \( \int_{\mathbb{T}} u_0(x) dx = 0 \), \( u \) is called a strong solution with vanishing total density of the Cauchy problem (1) if there exists \( T \in (0, \infty) \) such that \( u \in C^1([0, T); H^2(\mathbb{T})) \) satisfies (1) and \( \int_{\mathbb{T}} u(t, x) dx = 0 \) for any \( t \in [0, T) \).

**Remark 1.** Formally,

\[
\frac{d}{dt} \int_{\mathbb{T}} u(t, x) dx = (2\pi)^{1/2} \frac{d}{dt} \hat{u}(t, 0) = -i(2\pi)^{1/2} \Im \left[ -\partial_x^2 u + \lambda \partial_x(|u|^{p-1} u) \right](0) = 0.
\]

This implies that if \( \int_{\mathbb{T}} u_0(x) dx = 0 \), then \( \int_{\mathbb{T}} u(t, x) dx = 0 \) for any \( t \in [0, T) \).
In this article, for $H^2(T)$ initial data with vanishing total density, we assume the existence of strong solutions with vanishing total density. We define the lifespan $T_0$ of a strong solution $u$ to the Cauchy problem (1) by

$$T_0 = \sup\{ T > 0; \text{ u is a strong solution for (1)} \}.$$ 

Then, from the ordinary differential inequality for

$$\int_0^{2\pi} \int_0^x u(\cdot, x) u(\cdot, y) dy dx,$$

we may obtain the equivalent conditions for the finite time blowup for (1) and estimate of lifespan.

**Proposition 1.** Let $u_0 \in L^2(T)$ satisfy $\int_T u_0(x) dx = 0$. Then the following statements are equivalent:

(i) $u_0$ satisfies

$$\text{Re}\lambda \cdot \text{Im}\int_0^{2\pi} \int_0^x u_0(x) \overline{u_0(y)} dy dx > 0. \quad (3)$$

(ii) There exists $\alpha \in \mathbb{C}$ such that

$$\text{Re}\alpha \cdot \text{Re}\lambda > 0, \quad \text{Im}\left( \alpha \int_0^{2\pi} \int_0^x u_0(x) \overline{u_0(y)} dy dx \right) > 0. \quad (4)$$

If $u_0$ satisfies one of the equivalent conditions above and $u_0 \in H^2(T)$, then the corresponding strong solution with vanishing total density of the Cauchy problem (1) blows up in finite time. Moreover, the associated lifespan is estimated by

$$T_0 \leq \frac{(2\pi)^p}{(p - 1)|\text{Re}\lambda|} \left| \int_0^{2\pi} \int_0^x u_0(x) \overline{u_0(y)} dy dx \right|^{\frac{p-1}{2}}.$$

**Remark 2.** For $f \in L^2(T)$ with vanishing total density,

$$\text{Re} \int_0^{2\pi} \int_0^x f(x) \overline{f(y)} dy dx = \frac{1}{2} \int_0^{2\pi} \frac{d}{dx} \left| \int_0^x f(y) dy \right|^2 dx = \frac{1}{2} \left| \int_0^{2\pi} f(x) dx \right|^2 = 0.$$

This means

$$\int_0^{2\pi} \int_0^x f(x) \overline{f(y)} dy dx \in i\mathbb{R}.$$ 

Then, (3) and (4) can be rewritten by

$$-i\text{Re}\lambda \cdot \int_0^{2\pi} \int_0^x u_0(x) \overline{u_0(y)} dy dx > 0, \quad -i\text{Re}\alpha \cdot \int_0^{2\pi} \int_0^x u_0(x) \overline{u_0(y)} dy dx > 0,$$

respectively. This rewriting implies the equivalence between (3) and (4).

**Remark 3.** In contrast to nonlinear Schrödinger equation of nonlinearity without derivative and $\text{Re}\lambda \neq 0$, it is unknown that whether Virial type identity works or not. On the other hand, in order to show Proposition 1, we show that

$$\int_0^{2\pi} \int_0^x u(\cdot, x) u(\cdot, y) dy dx,$$

is a super solution of an ordinary differential equation and which implies that the amount blows up at a finite time.
2. Proof of Proposition 1. Let \( M_\alpha(t) = \text{Im} \left( \alpha \int_0^{2\pi} \int_0^\infty u(t, x)u(t, y) dy \, dx \right) \), where \( \alpha \) satisfies (4). Then \( M_\alpha(t) > 0 \) for sufficiently small \( t \). By a direct calculation, we have

\[
\frac{d}{dt} M_\alpha(t) = \text{Im} \left( \alpha \int_0^{2\pi} \int_0^\infty \partial_u u(t, x)u(t, y) dy \, dx \right)
+ \text{Im} \left( \alpha \int_0^{2\pi} \int_0^\infty u(t, x) \bar{\partial}_u u(t, y) dy \, dx \right)
= I_1 + I_2.
\]

By the vanishing total density, \( I_1 \) and \( I_2 \) may be computed as follows:

\[
I_1 = - \text{Re} \left( \alpha \int_0^{2\pi} i \partial_u u(t, x) \int_0^\infty u(t, y) dy \, dx \right)
= - \text{Re} \left( \alpha \int_0^{2\pi} \partial_u (\partial_u u(t, x) + \lambda (|u(t, x)|^{p-1} u(t, x))) \int_0^\infty u(t, y) dy \, dx \right)
= - \text{Re} \left( \alpha (\partial_x u(t, 2\pi) + \lambda (|u(t, 2\pi)|^{p-1} u(t, 2\pi))) \int_0^{2\pi} u(t, y) dy \right)
+ \text{Re} \left( \alpha \int_0^{2\pi} -u(t, x) \partial_x u(t, x) + \lambda |u(t, x)|^{p+1} \, dx \right)
= \text{Re} \left( \alpha \int_0^{2\pi} -u(t, x) \partial_x u(t, x) + \lambda |u(t, x)|^{p+1} \, dx \right),
\]

\[
I_2 = \text{Re} \left( \alpha \int_0^{2\pi} u(t, x) \int_0^\infty i \partial_u u(t, y) dy \, dx \right)
= \text{Re} \left( \alpha \int_0^{2\pi} u(t, x)(-\partial_x u(t, x) + \lambda (|u(t, x)|^{p-1} u(t, x))) \, dx \right)
- \text{Re} \left( \alpha (\partial_x u(t, 0) + \lambda (|u(t, 0)|^{p-1} u(t, 0))) \int_0^{2\pi} u(t, x) \, dx \right)
= \text{Re} \left( \alpha \int_0^{2\pi} u(t, x)(-\partial_x u(t, x) + \lambda (|u(t, x)|^{p-1} u(t, x))) \, dx \right).
\]

Then,

\[
\frac{d}{dt} M_\alpha(t) = - \text{Re} \alpha \cdot \int_0^{2\pi} 2 \text{Re}(u(t, x) \bar{\partial}_x u(t, x)) \, dx + 2 \text{Re} \alpha \cdot \text{Re} \lambda \|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1}
= - \text{Re} \alpha \cdot \int_0^{2\pi} \partial_x |u(t, x)|^2 \, dx + 2 \text{Re} \alpha \cdot \text{Re} \lambda \|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1}
= 2 \text{Re} \alpha \cdot \text{Re} \lambda \|u(t)\|_{L^{p+1}(\mathbb{T})}^{p+1},
\]

Since

\[
|M_\alpha(t)| \leq \|\text{Re} \alpha \|u(t)\|_{L^2(\mathbb{T})}^2 \leq (2\pi)^{\frac{2p}{p+1}} \|\text{Re} \alpha \|u(t)\|_{L^{p+1}(\mathbb{T})}^2,
\]

we have

\[
\frac{d}{dt} M_\alpha(t) \geq 2(2\pi)^{-p} \|\text{Re} \alpha\|^{-\frac{p+1}{p+1}} \alpha \cdot \text{Re} \lambda \|M_\alpha(t)\|_{L^{p+1}(\mathbb{T})}^{\frac{p+1}{p+1}}.
\]

This and \( M_\alpha(0) > 0 \) implies

\[
M_\alpha(t) \geq (M_\alpha(0))^{-\frac{p+1}{p+1}} - (p-1)(2\pi)^{-p} \|\text{Re} \alpha\|^{-\frac{p+1}{p+1}} \alpha \cdot \text{Re} \lambda t^{-\frac{p-1}{p}} > 0.
\]
Therefore
\[ T_0 \leq \inf \left\{ \frac{(2\pi)^p|\text{Re} \alpha|^{\frac{p+1}{p}}}{(p-1)|\text{Re} \lambda|} \cdot M_\alpha(0)^{-\frac{p-1}{p}} : \alpha \in \mathbb{C}, \text{Re} \alpha \cdot \text{Re} \lambda > 0 \right\} \]
\[ \leq \frac{(2\pi)^p}{(p-1)|\text{Re} \lambda|} \left| \int_0^{2\pi} \int_0^\pi u_0(x)\overline{u_0(y)}dydx \right|^{-\frac{p-1}{p}}. \]
Here the infimum may be attained with \( \alpha = \lambda \).

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