Perturbation Resilience and Superiorization of Iterative Algorithms

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Abstract. Iterative algorithms aimed at solving some problems are discussed. For certain problems, such as finding a common point in the intersection of a finite number of convex sets, there often exist iterative algorithms that impose very little demand on computer resources. For other problems, such as finding that point in the intersection at which the value of a given function is optimal, algorithms tend to need more computer memory and longer execution time. A methodology is presented whose aim is to produce automatically for an iterative algorithm of the first kind a “superiorized version” of it that retains its computational efficiency but nevertheless goes a long way towards solving an optimization problem. This is possible to do if the original algorithm is “perturbation resilient,” which is shown to be the case for various projection algorithms for solving the consistent convex feasibility problem. The superiorized versions of such algorithms use perturbations that drive the process in the direction of the optimizer of the given function. After presenting these intuitive ideas in a precise mathematical form, they are illustrated in image reconstruction from projections for two different projection algorithms superiorized for the function whose value is the total variation of the image.

Keywords: iterative algorithms, convex feasibility problem, superiorization, perturbation resilience, projection methods

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1. Introduction

We first motivate and describe our ideas in a not fully general context, in which superiorization is envisioned as lying in-between the methodologies of optimization and of feasibility seeking. With a feasible solution one settles for a point that just fulfills a set of constraints, whereas solving a constrained optimization problem calls for finding a feasible point that optimizes a given objective function. Generally speaking, optimization is logically and computationally a more demanding task than that of finding just any feasible point. We show that, without employing an optimization algorithm, it is possible to use certain iterative methods, designed for (the less demanding) feasibility problems, in a way that will steer the iterates toward a point that is superior, but not necessarily optimal, in a well-defined sense. The advantage of superiorization is that it allows us to solve significant problems by using powerful feasibility seeking methods, see, e.g., [10] and references therein, and reach a superior feasible point without resorting to optimization techniques. We now explain this with more details.

Many significant real-world problems are modeled by constraints that force the sought-after solution point to fulfill conditions imposed by the physical nature of the problem. Such a modeling approach often leads to a convex feasibility problem of the form

\[
\text{find } \mathbf{x}^* \in C = \bigcap_{i=1}^{I} C_i,
\]

where the sets \( C_i \subseteq \mathbb{R}^J \) are closed convex subsets of the Euclidean space \( \mathbb{R}^J \), see [2, 10] or [15, Chapter 5] for this broad topic. In many real-world problems the underlying system is very large (huge values of \( I \) and \( J \)) and often very sparse. In these circumstances projection methods have proved to be effective. They are iterative algorithms that use projections onto sets while relying on the general principle that when a family of closed and convex sets is present, then projections onto the individual sets are easier to perform than projections onto other sets, such as their intersection as in (1), that are derived from them.

Projection methods can have various algorithmic structures (some of which are particularly suitable for parallel computing) and they also possess desirable convergence properties and good initial behavior patterns [2, 15, 17, 18, 19, 27, 32]. The main advantage of projection methods, which makes them successful in real-world applications, is computational. They commonly have the ability to handle huge-size problems of dimensions beyond which more sophisticated methods cease to be efficient or even applicable due to memory requirements. (For a justification of this claim see the various examples provided in [10].) This is so because the building bricks of a projection algorithm (which are the projections onto the given individual sets) are easy to perform, and because the algorithmic structure is either sequential or simultaneous, or in-between, as in the block-iterative projection methods or in the more recently invented string-averaging projection methods. The number of sets used simultaneously in each iteration in block-iterative methods and the number and lengths of strings used in each iteration in string-averaging methods are variable, which provides great flexibility in matching the implementation of the algorithm with the parallel architecture at hand; for block-iterative methods see, e.g., [1, 3, 5, 12, 19, 23, 25, 26, 29, 30, 31] and for string-averaging methods see, e.g., [4, 6, 11, 13, 14, 22, 31, 33].
The key to superiorization is our recent discovery [6, 23, 28] that two principal prototypical algorithmic schemes of projection methods: string-averaging projections (SAP) and block-iterative projections (BIP), which include as special cases a variety of projection methods for the convex feasibility problem, are bounded perturbations resilient in the sense that the convergence of sequences generated by them continues to hold even if the iterates are perturbed in every iteration. We harness this resilience to bounded perturbations to steer the iterates to not just any feasible point but to a superior (in a well-defined sense) feasible point of (1).

Our motivation is the desire to create a new methodology that will significantly improve methods for the solution of inverse problems in image reconstruction from projections, intensity-modulated radiation/proton therapy (IMRT/IMPT) and in other real-world problems such as electron microscopy (EM). Our work [6, 23], as well as the examples given below, indicate that our objective is achievable and show how algorithms can incorporate perturbations in order to perform superiorization.

The superiorization methodology has in fact broader applicability than what has been discussed until now and its mathematical specification in the next section reflects this. However, all our specific examples will be chosen from the field that we used as our motivation in this introductory section.

2. Specification of the superiorization methodology

The superiorization principle relies on the bounded perturbation resilience of algorithms. Therefore we define this notion next in a general setting within \( \mathbb{R}^J \).

We introduce the notion of a problem structure \( (T, Pr) \), where \( T \) is a nonempty problem set and \( Pr \) is a function on \( T \) such that, for all \( T \in T \), \( Pr_T : \mathbb{R}^J \to \mathbb{R}_+ \), where \( \mathbb{R}_+ \) is the set of nonnegative real numbers. Intuitively we think of \( Pr_T (x) \) as a measure of how “far” \( x \) is from being a solution of \( T \). In fact, we call \( x \) a solution of \( T \) if \( Pr_T (x) = 0 \).

For example, for the convex feasibility problem (1)

\[
T = \{ \{ C_1, \ldots, C_I \} | \quad I \text{ is a positive integer and, for } 1 \leq i \leq I, \quad C_i \text{ is a closed convex subset of } \mathbb{R}^J \} \quad (2)
\]

and

\[
Pr_{\{ C_1, \ldots, C_I \}} (x) = \sqrt{ \sum_{i=1}^{I} (d(x, C_i))^2 } , \quad (3)
\]

where \( d(x, C_i) \) is the Euclidean distance of \( x \) from the set \( C_i \). Clearly, in this case \( x \) is a solution of \( \{ C_1, \ldots, C_I \} \) as defined in the previous paragraph if, and only if, \( x \in C \) as defined in (1).

**Definition 1.** An algorithm \( P \) for \( (T, Pr) \) assigns to each \( T \in T \) an algorithmic operator \( P_T : \mathbb{R}^J \to \mathbb{R}^J \). \( P \) is said to be bounded perturbations resilient if, for all \( T \in T \), the following is the case: if the sequence \( \{ (P_T)^k x \} \infty_{k=0} \) converges to a solution of \( T \) for all \( x \in \mathbb{R}^J \), then any sequence \( \{ x^k \} \infty_{k=0} \) of points in \( \mathbb{R}^J \) also converges to a solution of \( T \) provided that, for all \( k \geq 0 \),

\[
x^{k+1} = P_T (x^k + \beta_k v^k) , \quad (4)
\]
where $\beta_k v^k$ are bounded perturbations, meaning that $\beta_k$ are real nonnegative numbers such that $\sum_{k=0}^{\infty} \beta_k < \infty$ and the sequence $\{v^k\}_{k=0}^{\infty}$ is bounded.

We give next specific instances of bounded perturbations resilient algorithms for solving the convex feasibility problem as in [2] and [3], from the classes of SAP and BIP methods. We do this by defining $P_{\{C_1, \ldots, C_I\}}$ for an arbitrary but fixed element $\{C_1, \ldots, C_I\}$ of $T$ of [2] for the different algorithms $P$. For any nonempty closed convex subset $M$ of $\mathbb{R}^J$ and any $x \in \mathbb{R}^J$, the orthogonal projection of $x$ onto $M$ is the point in $M$ that is nearest (by the Euclidean distance) to $x$; it is denoted by $P_M x$.

To define $P_{\{C_1, \ldots, C_I\}}$ for the SAP instances, we make use of index vectors, which are not ordered ordered sets $t = (t_1, \ldots, t_I)$, where $N$ is an arbitrary positive integer, whose elements $t_n$ are in the set $\{1, \ldots, I\}$. For an index vector $t$ we define the composite operator

$$P[t] = P_{C_{t_N}} \cdots P_{C_{t_1}}. \tag{5}$$

A finite set $\Omega$ of index vectors is called fit if, for each $i \in \{1, \ldots, I\}$, there exists $t = (t_1, \ldots, t_I) \in \Omega$ such that $t_n = i$ for some $n \in \{1, \ldots, N\}$. If $\Omega$ is a fit set of index vectors, then a function $\omega: \Omega \to \mathbb{R}_{++} = (0, \infty)$ is called a fit weight function if $\sum_{t \in \Omega} \omega(t) = 1$. A pair $(\Omega, \omega)$ consisting of a fit set of index vectors and a fit weight function defined on it was called an amalgamator in [4]. For each amalgamator $(\Omega, \omega)$, we define the algorithmic operator $P_{\{C_1, \ldots, C_I\}} : \mathbb{R}^J \to \mathbb{R}^J$ by

$$P_{\{C_1, \ldots, C_I\}} x = \sum_{t \in \Omega} \omega(t) P[t] x. \tag{6}$$

For this algorithmic operator we have the following bounded perturbations resilience theorem.

**Theorem 1.** [6 Section II] If $C$ of [4] is nonempty, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of nonnegative real numbers such that $\sum_{k=0}^{\infty} \beta_k < \infty$ and $\{v^k\}_{k=0}^{\infty}$ is a bounded sequence of points in $\mathbb{R}^J$, then for any amalgamator $(\Omega, \omega)$ and any $x^0 \in \mathbb{R}^J$, the sequence $\{x^k\}_{k=0}^{\infty}$ generated by

$$x^{k+1} = P_{\{C_1, \ldots, C_I\}} \left( x^k + \beta_k v^k \right), \quad \forall k \geq 0, \tag{7}$$

converges, and its limit is in $C$. (The statement of this theorem in [6] is for positive $\beta_k$s, but the proof given there applies to nonnegative $\beta_k$s.)

**Corollary 1.** For any amalgamator $(\Omega, \omega)$, the algorithm $P$ defined by the algorithmic operator $P_{\{C_1, \ldots, C_I\}}$ is bounded perturbations resilient.

**Proof.** Assume that for $T = \{C_1, \ldots, C_I\}$ the sequence $\{(P_T)^k x\}_{k=0}^{\infty}$ converges to a solution of $T$ for all $x \in \mathbb{R}^J$. This implies, in particular, that $C$ of [4] is nonempty. By Definition 1, we need to show that any sequence $\{x^k\}_{k=0}^{\infty}$ of points in $\mathbb{R}^J$ also converges to a solution of $T$ provided that, for all $k \geq 0$, [4] is satisfied when the $\beta_k v^k$s are bounded perturbations. Under our assumptions, this follows from Theorem 1. \qed

Next we look at a member of the family of BIP methods. Considering the convex feasibility problem [4], for $1 \leq u \leq U$, let $B_u$ be an ordered set $\{b_{u,1}, \ldots, b_{u,|B_u|}\}$ of elements of $\{1, \ldots, I\}$ ($|B_u|$ denotes the cardinality of $B_u$). We call such a $B_u$ a block and define the (composite) algorithmic operator $Q_{\{C_1, \ldots, C_I\}} : \mathbb{R}^J \to \mathbb{R}^J$ by

$$Q_{\{C_1, \ldots, C_I\}} = Q_U \cdots Q_1. \tag{8}$$
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where, for \( x \in \mathbb{R}^J \) and \( 1 \leq u \leq U \),
\[
Q_u x = \frac{1}{R} \sum_{i \in B_u} P_{i} x + \frac{R - |B_u|}{R} x,
\]
and
\[
R = \max \{|B_u| \mid 1 \leq u \leq U\}.
\]

The iterative procedure \( x^{k+1} = Q_{\{C_1, \ldots, C_1\}} x^k \) is a member of the family of BIP methods. For this algorithmic operator we have the following bounded perturbations resilience theorem.

**Theorem 2.** \([23]\) If \( C \) of (11) is nonempty, \( \{1, \ldots, I\} = \bigcup_{u=1}^{U} B_u \), \( \{\beta_k\}_{k=0}^{\infty} \) is a sequence of nonnegative real numbers such that \( \sum_{k=0}^{\infty} \beta_k < \infty \), \( \{v^k\}_{k=0}^{\infty} \) be a bounded sequence of points in \( \mathbb{R}^J \), then for any \( x^0 \in \mathbb{R}^J \), the sequence \( \{x^k\}_{k=0}^{\infty} \) generated by
\[
x^{k+1} = Q_{\{C_1, \ldots, C_1\}} (x^k + \beta_k v^k), \quad \forall k \geq 0,
\]
converges, and its limit is in \( C \). (This is a special case of Theorem 2 in \([23]\) given here without a relaxation parameter. Also, that theorem is stated for positive \( \beta_k \), but the proof given there applies to nonnegative \( \beta_k \).)

**Corollary 2.** The algorithm \( Q \) defined by the algorithmic operator \( Q_{\{C_1, \ldots, C_1\}} \) is bounded perturbations resilient.

**Proof.** Replace in the proof of Corollary 1 \( P \) by \( Q \) and Theorem 1 by Theorem 2. \( \square \)

Further bounded perturbations resilience theorems are available in a Banach space setting, see \([7, 8]\). Thus the theory of bounded perturbations resilient algorithms already contains some solid mathematical results. As opposed to this, the superiorization theory that we present next is at the stage of being a collection of heuristic ideas, a full mathematical theory still needs to be developed. However, there are practical demonstrations of its potential usefulness; see \([0, 23, 28]\) and the illustrations in Section 3 below.

For a problem structure \( (T, \mathcal{P}_T) \), \( T \in T, \varepsilon \in \mathbb{R}^J_+ \) and a sequence \( S = \{x^k\}_{k=0}^{\infty} \) of points in \( \mathbb{R}^J \), we use \( O(T, \varepsilon, S) \) to denote the \( x \in \mathbb{R}^J \) that has the following properties: \( \mathcal{P}_T(x) \leq \varepsilon \) and there is a nonnegative integer \( K \) such that \( x^K = x \) and, for all nonnegative integers \( \ell < K \), \( \mathcal{P}_T(x^\ell) > \varepsilon \). Clearly, if there is such an \( x \), then it is unique. If there is no such \( x \), then we say that \( O(T, \varepsilon, S) \) is undefined.

The intuition behind this definition is the following: if we think of \( \mathcal{S} \) as the (infinite) sequence of points that is produced by an algorithm (intended for the problem \( T \)) without a termination criterion, then \( O(T, \varepsilon, S) \) is the output produced by that algorithm when we add to it instructions that makes it terminate as soon as it reaches a point at which the value of \( \mathcal{P}_T \) is not greater than \( \varepsilon \). The following result is obvious.

**Lemma 1.** If \( \mathcal{P}_T \) is continuous and the sequence \( S \) converges to a solution of \( T \), then \( O(T, \varepsilon, S) \) is defined and \( \mathcal{P}_T (O(T, \varepsilon, S)) \leq \varepsilon \).

Given an algorithm \( P \) for a problem structure \( (T, \mathcal{P}_T) \), a \( T \in T \) and an \( \bar{x} \in \mathbb{R}^J \), let \( R(T, \bar{x}) = \{ (P_T)^k \bar{x} \}_{k=0}^{\infty} \). For a function \( \phi : \mathbb{R}^J \to \mathbb{R} \), the superiorization methodology should provide us with an algorithm that produces a sequence \( S(T, \bar{x}, \phi) = \{x^k\}_{k=0}^{\infty} \), such that for any \( \varepsilon \in \mathbb{R}^J_+ \) and \( x \in \mathbb{R}^J \) for which
\( \mathcal{P}_{RT}(\bar{x}) > \varepsilon \) and \( O(T, \varepsilon, R(T, \bar{x})) \) is defined, \( O(T, \varepsilon, S(T, \bar{x}, \phi)) \) is also defined and \( \phi(O(T, \varepsilon, S(T, \bar{x}, \phi))) < \phi(O(T, \varepsilon, R(T, \bar{x}))) \). This is of course too ambitious in its full generality and so here we analyze only a special case, but one that is still quite general. We now list our assumptions for the special case for which we discuss details of the superiorization methodology.

**Assumptions**

(i) \( \langle T, \mathcal{P}_r \rangle \) is a problem structure such that \( \mathcal{P}_{rT} \) is continuous for all \( T \in T \).

(ii) \( \mathcal{P} \) is a bounded perturbation resilient algorithm for \( \langle T, \mathcal{P}_r \rangle \) such that, for all \( T \in T \), \( \mathcal{P}_{rT} \) is continuous and, if \( x \) is not a solution of \( T \), then \( \mathcal{P}_{rT}(\mathcal{P}_{T}x) < \mathcal{P}_{rT}(x) \).

(iii) \( \phi \) is a convex function.

We now describe, under these assumptions, the algorithm to produce the sequence \( S(T, \bar{x}, \phi) = \{x^k\}_{k=0}^{\infty} \).

The algorithm assumes that we have available a summable sequence \( \{\gamma_\ell\}_{\ell=0}^{\infty} \) of positive real numbers. It is easy to generate such sequences; e.g., we can use \( \gamma_\ell = a^\ell \), where \( 0 < a < 1 \). The algorithm generates, simultaneously with the sequence \( \{x^k\}_{k=0}^{\infty} \), sequences \( \{v^k\}_{k=0}^{\infty} \) and \( \{\beta_k\}_{k=0}^{\infty} \). The latter will be generated as a subsequence of \( \{\gamma_\ell\}_{\ell=0}^{\infty} \). Clearly, the resulting sequence \( \{\beta_k\}_{k=0}^{\infty} \) of positive real numbers will be summable. We first specify the algorithm and then discuss it. The algorithm depends on the specified \( \bar{x}, \phi, \{\gamma_\ell\}_{\ell=0}^{\infty}, \mathcal{P}_{rT} \) and \( \mathcal{P}_T \). It makes use of a logical variable called \( \text{continue} \) and also of the concept of a subgradient of the convex function \( \phi \).

**Superiorized Version of Algorithm P**

(i) set \( k = 0 \)

(ii) set \( x^k = \bar{x} \)

(iii) set \( \ell = 0 \)

(iv) repeat

(v) set \( g \) to a subgradient of \( \phi \) at \( x^k \)

(vi) if \( ||g|| > 0 \)

(vii) then set \( v^k = -g / ||g|| \)

(viii) else set \( v^k = g \)

(ix) set \( \text{continue} = \text{true} \)

(x) while \( \text{continue} \)

(xi) set \( \beta_k = \gamma_\ell \)

(xii) set \( y = x^k + \beta_k v^k \)

(xiii) if \( \phi(y) \leq \phi(x^k) \) and \( \mathcal{P}_{rT}(\mathcal{P}_T y) < \mathcal{P}_{rT}(x^k) \) then

(xiv) set \( x^{k+1} = \mathcal{P}_T y \)

(xv) set \( \text{continue} = \text{false} \)

(xvi) set \( \ell = \ell + 1 \)

(xvii) set \( k = k + 1 \)
Sometimes it is useful to emphasize the function $\phi$ for which we are superimizing, in which case we refer to the algorithm above as the $\phi$-superiorized version of algorithm $P$. It is important to bear in mind that the sequence $S$ produced by the algorithm depends also on the initial point $\bar{x}$, the selection of the subgradient in Line (v) of the algorithm, the summable sequence $\{\gamma^\ell\}_{\ell=0}^\infty$, and the problem $T$. In addition, the output $O(T, \varepsilon, S)$ of the algorithm depends on the stopping criterion $\varepsilon$.

**Theorem 3.** Under the Assumptions listed above, the Superiorized Version of Algorithm $P$ will produce a sequence $S(T, \bar{x}, \phi)$ of points in $\mathbb{R}^J$ that either contains a solution of $T$ or is infinite. In the latter case, if the sequence $\{(P_T^k x)\}_{k=0}^\infty$ converges to a solution of $T$ for all $x \in \mathbb{R}^J$, then, for any $\varepsilon \in \mathbb{R}_{++}$, $O(T, \varepsilon, S(T, \bar{x}, \phi))$ is defined and $\phi(O(T, \varepsilon, S(T, \bar{x}, \phi))) \leq \varepsilon$.

**Proof.** Assume that the sequence $S(T, \bar{x}, \phi)$ produced by the Superiorized Version of Algorithm $P$ does not contain a solution of $T$. We first show that in this case the algorithm generates an infinite sequence $\{x^k\}_{k=0}^\infty$. This is equivalent to saying that, for any $x^k$ that has been generated already, the condition in Line (xiii) of the algorithm will be satisfied sooner or later (and hence $x^{k+1}$ will be generated). This needs to happen, because as long as the condition is not satisfied we keep resetting (in Line (xii)) the value of $\beta_k$ to $\gamma^\ell$, with ever increasing values of $\ell$. However, $\{\gamma^\ell\}_{\ell=0}^\infty$ is a summable sequence of positive real numbers, and so $\gamma^\ell$ is guaranteed to be arbitrarily small if $\ell$ is sufficiently large. Since $v^k$ is either a unit vector in the direction of the negative subgradient of the convex function $\phi$ at $x^k$ or is the zero vector (see Lines (v)–(viii)), $\phi(x^k + \beta_k v^k) \leq \phi(x^k)$ must be satisfied if the positive number $\beta_k$ is small enough. Also, since $\mathcal{P}_T(P_T x^k) < \mathcal{P}_T(x^k)$ and $P_T$ and $\mathcal{P}_T$ are continuous (Assumptions (ii) and (i), respectively), we also have that $\mathcal{P}_T(P_T(x^k + \beta_k v^k)) < \mathcal{P}_T(x^k)$ if $\beta_k$ is small enough. This completes the proof that the condition in Line (xiii) of the algorithm will be satisfied and so the algorithm will generate an infinite sequence $S(T, \bar{x}, \phi)$. Observing that we have already demonstrated that the $\beta_k v^k$ are bounded perturbations, and comparing (11) with Lines (xii) and (xiv), we see that (by the bounded perturbation resilience of $P$) the assumption that the sequence $\{(P_T^k x)\}_{k=0}^\infty$ converges to a solution of $T$ for all $x \in \mathbb{R}^J$ implies that $S(T, \bar{x}, \phi)$ also converges to a solution of $T$. Thus, applying Lemma 1 we obtain the final claim of the theorem. $\square$

Unfortunately, this theorem does not go far enough. To demonstrate that a methodology leads to superimizing we should be proving (under some assumptions) a result like $\phi(O(T, \varepsilon, S(T, \bar{x}, \phi))) < \phi(O(T, \varepsilon, R(T, \bar{x})))$ in place of the weaker result at the end of the statement of the theorem. Currently we do not have any such proofs and so we are restricted to providing practical demonstrations that our methodology leads to superimizing in the desired sense. In the next section we provide such demonstrations for the Superiorized Version of Algorithm $P$, for two different $P$s.

### 3. Illustrations of the superiorization methodology

We illustrate the superiorization methodology on a problem of reconstructing a head cross-section (based on Figure 4.6(a) of [27]) from its projections using both an SAP and a BIP algorithm. (All the computational work reported in this section was done
Figure 1(a) shows a...choose P in Figure 1 of the initial point (U of Q(b) and (d), respectively. More

and their TV-tomography would be considered a variant of the algebraic reconstruction techniques J(5)-(6) with

Ω = 82

by all the hyperplanes, we have here an instance of the convex feasibility problem with a

collection of the data are collected.

For our illustration, we chose the SAP algorithm P_{C_1,...,C_1} as determined by [5, [5] with \Omega = \{(1,...,I)\} and \omega (1,...,I) = 1. This is a classical method that in
tomography would be considered a variant of the algebraic reconstruction techniques (ART) [27]. Chapter 11]. For the BIP algorithm we chose Q_{C_1,...,C_1} as determined by [5-10] with U = 82 and each block corresponding to one of the 82 sets of parallel

lines along which the data are collected.

The function \phi for which we superiorized is defined so that, for any \bar{x} \in \mathbb{R}^J, \phi (\bar{x}) is the total variation (TV) of the corresponding 243 \times 243 image. If the pixel values of this image are g_{g,h}, then the value of the TV is defined to be

\[
\sum_{g=1}^{242} \sum_{h=1}^{242} \sqrt{(q_{g+1,h} - q_{g,h})^2 + (q_{g,h+1} - q_{g,h})^2}.
\]

(12)

For the TV-Superiorized Versions of the Algorithms P_{C_1,...,C_1} and Q_{C_1,...,C_1} of the previous paragraph we selected \bar{x} to be the origin (the vector of all zeros) and \gamma_\ell = 0.999^\ell. Also, we set \varepsilon = 0.01 for the stopping criterion, which is small compared to the \Pr_T of the initial point (Pr_T (\bar{x}) = 330.208).

For each of the four algorithms (P_{C_1,...,C_1}, Q_{C_1,...,C_1} and their TV-
superiorized versions), the sequence S that is produced by it is such that the output

O(T, \varepsilon, S) is defined; see Figures 1(b)-(e) for the images that correspond to these outputs. Clearly, the superiorized reconstructions in Figures 1(c) and (e) are visually superior to their not superiorized versions in Figures 1(b) and (d), respectively. More

importantly from the point of view of our theory, consider Table 1. As stated in the last paragraph of the previous section, we would like to have that \phi (O(T, \varepsilon, S(T, \bar{x}, \phi))) < \phi (O(T, \varepsilon, R(T, \bar{x}))). While we are not able to prove that this is the case in general, Table 1 clearly shows it to be the case for the two algorithms discussed in this section.

Table 1: Values of TV for the outputs of the various algorithms. The second column is for the superiorized versions and the third column is for the original versions.

| Algorithm | \phi (O(T, \varepsilon, S(T, \bar{x}, \phi))) | \phi (O(T, \varepsilon, R(T, \bar{x}))) |
|-----------|---------------------------------|---------------------------------|
| Variant of ART | 441.50 | 1,296.44 |
| Variant of BIP | 444.15 | 1,286.44 |

using SNARK09 [24]; the phantom, the data, the reconstructions and displays were all generated within this same framework. Figure 1(a) shows a 243 \times 243 digitization of the head phantom with J = 59,049 pixels. An \bar{x} \in \mathbb{R}^J is interpreted as a vector of pixel values, whose components represent the average X-ray linear attenuation coefficients (measured per centimeter) within the 59,049 pixels. Each pixel is of size 0.0752 \times 0.0752 (measured in centimeters). The pixel values range from 0 to 0.5639. For display purposes, any value below 0.204 is shown as black (gray value 0) and any value above 0.21675 is shown as white (gray value 255), with a linear mapping of the pixel values into gray values in between (the same convention is used in displaying reconstructed images in Figures 1(b)-(e)).

Data were collected by calculating line integrals across the digitized image for 82 sets of equally spaced parallel lines, with I = 25,452 lines in total. Each data item determines a hyperplane in \mathbb{R}^J. Since the digitized phantom lies in the intersection of all the hyperplanes, we have here an instance of the convex feasibility problem with a nonempty C, satisfying the first condition of the statements of Theorem 1 and 2.

For our illustration, we chose the SAP algorithm P_{C_1,...,C_1} as determined by [5-10] with \Omega = \{(1,...,I)\} and \omega (1,...,I) = 1. This is a classical method that in

tomography would be considered a variant of the algebraic reconstruction techniques (ART) [27]. Chapter 11]. For the BIP algorithm we chose Q_{C_1,...,C_1} as determined by [5-10] with U = 82 and each block corresponding to one of the 82 sets of parallel

lines along which the data are collected.

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\[
\sum_{g=1}^{242} \sum_{h=1}^{242} \sqrt{(q_{g+1,h} - q_{g,h})^2 + (q_{g,h+1} - q_{g,h})^2}.
\]

(12)

For the TV-Superiorized Versions of the Algorithms P_{C_1,...,C_1} and Q_{C_1,...,C_1} of the previous paragraph we selected \bar{x} to be the origin (the vector of all zeros) and \gamma_\ell = 0.999^\ell. Also, we set \varepsilon = 0.01 for the stopping criterion, which is small compared to the \Pr_T of the initial point (Pr_T (\bar{x}) = 330.208).

For each of the four algorithms (P_{C_1,...,C_1}, Q_{C_1,...,C_1} and their TV-
superiorized versions), the sequence S that is produced by it is such that the output

O(T, \varepsilon, S) is defined; see Figures 1(b)-(e) for the images that correspond to these outputs. Clearly, the superiorized reconstructions in Figures 1(c) and (e) are visually superior to their not superiorized versions in Figure 1(b) and (d), respectively. More

importantly from the point of view of our theory, consider Table 1. As stated in the last paragraph of the previous section, we would like to have that \phi (O(T, \varepsilon, S(T, \bar{x}, \phi))) < \phi (O(T, \varepsilon, R(T, \bar{x}))). While we are not able to prove that this is the case in general, Table 1 clearly shows it to be the case for the two algorithms discussed in this section.

Table 1: Values of TV for the outputs of the various algorithms. The second column is for the superiorized versions and the third column is for the original versions.

| Algorithm | \phi (O(T, \varepsilon, S(T, \bar{x}, \phi))) | \phi (O(T, \varepsilon, R(T, \bar{x}))) |
|-----------|---------------------------------|---------------------------------|
| Variant of ART | 441.50 | 1,296.44 |
| Variant of BIP | 444.15 | 1,286.44 |
Figure 1: A head phantom (a) and its reconstructions from underdetermined consistent data obtained for 82 views using: (b) a variant of ART, (c) TV-superiorized version of the same variant of ART, (d) a block-iterative projection method, and (e) TV-superiorized version of the same block-iterative projection method. The same initial point and stopping criterion were used in all cases; see the text for details.
A final important point that is illustrated by the experiments in this section is that, from the practical point of view, TV-superiorization is as useful as TV-optimization. This is because a realistic phantom, such as the one in Figure 1(a), is unlikely to be TV-minimizing subject to the constraints provided by the measurements. In fact, the TV value of our phantom is 450.53, which is larger than that for either of the TV-superiorized reconstructions in the second column of Table 1. While an optimization method should be able to find an image with a lower TV value, there is no practical point for doing that. The underlying aim of what we are doing is to estimate the phantom from the data and producing an image whose TV value is further from the TV value of the phantom than that of our superiorized reconstructions is unlikely to be helpful towards achieving this aim.

4. Discussion and conclusions

Stability of algorithms under perturbations is generally studied in numerical analysis with the aim of proving that an algorithm is stable so that it can “endure” all kinds of imperfections in the data or in the computational performance. Here we have taken a proactive approach designed to extract specific benefits from the kind of stability that we term perturbation resilience. We have been able to do this in a context that includes, but is much more general than, feasibility-optimization for intersections of convex sets.

Our premise has been that (1) there is available a bounded perturbations resilient iterative algorithm that solves efficiently certain type of problems and (2) we desire to make use of perturbations to find for these problems solutions that, according to some criterion, are superior to the ones to which we would get without employing perturbations. To accomplish this one must have a way of introducing perturbations that take into account the criterion according to which we wish to “superiorize” the solutions of the problems.

We have set forth the fundamental principle, have given some mathematical formulations and results, and have shown potential benefits (in the field of image reconstruction from projections). However, the superiorization methodology needs to be studied further from the mathematical, algorithmic and computational points of view in order to unveil its general applicability to inverse problems. As algorithms are developed and tested a dialog on algorithmic developments must be accompanied by mathematical validation and applications to simulated and real data from various relevant fields of applications.

Validating the concept means proving precise statements about the behavior of iterates \( \{x^k\}_{k=0}^{\infty} \) generated by the superiorized versions of algorithms. Under what conditions do they converge? Can their limit points be characterized? How would different choices of the perturbation coefficients \( \beta_k \) and the perturbation vectors \( v_k \) affect the superiorization process? Can different schemes for generating the \( \beta_k \)'s be developed, implemented, investigated? Enlarging the arsenal of bounded perturbation resilience algorithms means generalizing existing proofs of bounded perturbations resiliency of algorithms and developing new theories that will bring more algorithms into the family of bounded perturbations resilient algorithms. Further developments should include the problem of finding a common fixed point of a family of operators (a direct generalization of the convex feasibility problem) and studying the behavior of superiorization algorithms in inconsistent situations when the underlying solution set is empty. Thus we view the material in this paper as only an initial step in a
promising new field of endeavor for solving inverse problems.

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