Heterogeneous hypergeometric functions with two matrix arguments and the exact distribution of the largest eigenvalue of a singular beta-Wishart matrix

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Abstract: This paper discusses certain properties of heterogeneous hypergeometric functions with two matrix arguments. These functions are newly defined but have already appeared in statistical literature and are useful when dealing with the deviation of certain distributions for the eigenvalues of singular beta-Wishart matrices. The joint density function of the eigenvalues and the distribution of the largest eigenvalue can be expressed in terms of certain heterogeneous hypergeometric functions. Exact computation of the distribution of the largest eigenvalue is conducted here for a real case.

1 Introduction

Hypergeometric functions with one or two matrix arguments appear in distribution theory in multivariate analysis. The density and distribution functions of eigenvalues of a non-singular real Wishart matrix were given by James (1964) and Constantine (1963) in terms of hypergeometric functions. The exact distributions of the largest and smallest eigenvalues were derived by Sugiyama (1967) and Khatri (1972), respectively. Ratnarajah (2004) extended these results to the complex case and applied them to the channel capacity of wireless communication systems. Recently, these classes have been generalized to a beta-Wishart ensemble or beta-Wishart matrix that includes a classical matrix of real, complex, and quaternion cases. Koev and Edelman (2014) derived exact distributions of the extreme eigenvalues for a non-singular beta-Wishart matrix. The essence of this work lies in the use of Jack polynomials instead of zonal polynomials. Garcia (2011) discussed Wishart matrices on a real finite-dimensional normed division algebra that coincides with real numbers, complex numbers, quaternions, and octonions. In situations of normed division algebra, the Stiefel manifold and Haar measure can be defined as well as in the classical real case. Garcia (2013) derived a useful integral formula for Jack polynomials over the Steifel manifold on a normed division algebra.

A singular real Wishart distribution and multivariate beta distributions were first given by Ul- ing (1994). Ratnarajah and Villancourt (2005a, 2005b) derived some results for a singular complex Wishart matrix, and Li and Xue (2010) studied a singular quaternion Wishart matrix. However, the exact distributions of the extreme eigenvalues have not been derived in singular real, complex, and quaternion cases.

In this paper, we propose the key ideas for developing the distribution theory of eigenvalues for a singular Wishart matrix. We show that the eigenvalue distributions of a singular Wishart matrix are expressed in terms of heterogeneous hypergeometric functions with two matrix arguments. In the derivation of the heterogeneous hypergeometric functions, zero eigenvalues are substituted into the parts of arguments of zonal or Jack symmetric polynomials. In Section 2, we provide some definitions and preliminaries that include classical hypergeometric functions in terms of zonal polynomials. We define the heterogeneous hypergeometric functions with two matrix arguments for them. Furthermore, we show that the integral over the Stiefel manifold can be converted into an integral over the orthogonal group in Section 3, we derive the exact distribution of the largest eigenvalue of a singular real Wishart matrix. The distribution of the largest eigenvalue is applied using the formula of Sugiyama (1967). We then conduct numerical experiments for a theoretical distribution. Numerical results are calculated using the algorithm of Hashiguchi et al. (2000) for zonal polynomials. In Section 4.
we discuss the real finite division algebra and define the heterogeneous hypergeometric functions of parameter $\beta > 0$. We restrict the parameter $\beta = 1, 2, 4$ and present some useful formulas for these values on a real finite division algebra. In Section 5 we derive distributions of a singular beta-Wishart matrix.

2 Heterogeneous hypergeometric functions $pF_q^{(m,n)}$

In this section, we consider only the case of real numbers and define heterogeneous hypergeometric functions with two matrix arguments. These often appear in the density functions of random matrices for a singular Wishart matrix. Ratnarajah and Villancourt (2005a, 2005b) used such hypergeometric functions in the derivation of the density function of a singular complex Wishart matrix. The exterior product for real matrix $X$ is written by $(dX)$ as defined in Muirhead (1982) and Gupta and Nagar (1999).

The set of all $m \times n$ matrices $H_1$ with orthonormal columns is called the Stiefel manifold, denoted by $V_{n,m}$ where $n \leq m$, $V_{n,m} = \{ H_1 \mid H_1'H_1 = I_n \}$.

We note that $V_{m,m} = O(m)$, where $O(m)$ is the orthogonal group of order $m$. If $H_1 \in V_{n,m}$, then we define

$$(H_1'dH_1) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{m} h_{ij}'dh_{ij}$$

where $H = (H_1 : H_2) \in O(m)$. The volumes of $V_{n,m}$ and $O(m)$ are given by

$$\text{Vol}(V_{n,m}) = \int_{H_1 \in V_{n,m}} (H_1'dH_1) = \frac{2^n \pi^{mn/2}}{\Gamma_n(m/2)},$$

$$\text{Vol}(O(m)) = \int_{H \in O(m)} (H'dH) = \frac{2^m \pi^{m^2/2}}{\Gamma_m(m/2)},$$

respectively, where the multivariate gamma function is

$$\Gamma_m(c) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma \left( c - \frac{i-1}{2} \right), \quad \text{Re}(c) > n - 1.$$

The differential form $(dH_1)$, defined as

$$(dH_1) = \frac{(H_1'dH_1)}{\text{Vol}(V_{n,m})} = \frac{\Gamma_n(m/2)}{2^n \pi^{mn/2}} (H_1'dH_1)$$

is normalized as

$$\int_{H_1 \in V_{n,m}} (dH_1) = 1.$$
where $X_1 = \text{diag}(x_1, \ldots, x_n)$. For integers $p, q \geq 0$ and $m \times m$ real symmetric matrices $A$ and $B$, the hypergeometric function with two matrices is defined as

$$pF_q^{(m)}(\alpha, \beta; A, B) = \sum_{k=0}^{\infty} \sum_{\kappa \in P_m^k} \frac{(\alpha_1)_\kappa \cdots (\alpha_p)_\kappa C_\kappa(A)C_\kappa(B)}{(\beta_1)_\kappa \cdots (\beta_q)_\kappa k!C_\kappa(I_m)},$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta = (\beta_1, \ldots, \beta_q)$. For one matrix argument $A$, we also define $pF_q(\alpha, \beta; A)$ as

$$pF_q(\alpha, \beta; A) = pF_q^{(m)}(\alpha, \beta; A, I_m).$$

Then, the following relationship between (3) and (4) holds,

$$pF_q^{(m)}(\alpha, \beta; A, B) = \int_{H \in O(m)} pF_q(\alpha, \beta; AHBH')(dH).$$

To discuss the density function for a singular Wishart matrix, we define the heterogeneous hypergeometric functions as follows.

**Definition 1. (Heterogeneous hypergeometric functions)**

For an $m \times m$ symmetric matrix $A$ and an $n \times n$ symmetric matrix $B$, the heterogeneous hypergeometric functions are defined as

$$pF_q^{(s,t)}(\alpha, \beta; A, B) = \sum_{k=0}^{\infty} \sum_{\kappa \in P_m^k} \frac{(\alpha_1)_\kappa \cdots (\alpha_p)_\kappa C_\kappa(A)C_\kappa(B)}{(\beta_1)_\kappa \cdots (\beta_q)_\kappa k!C_\kappa(I_s)},$$

where $s = \max(m, n)$, $t = \min(m, n)$.

We consider the case of $m \geq n$ as well as $s = m$ and $t = n$ hereafter. So we write $pF_q^{(m,n)}$ instead of (6). If $m > n$, it is clear that

$$pF_q^{(m,n)}(\alpha, \beta; I_m, B) = pF_q(\alpha, \beta; B)$$

and

$$pF_q^{(m,n)}(\alpha, \beta; A, I_n) \neq pF_q(\alpha, \beta; A).$$

For an $m \times m$ matrix $B_1 = \begin{pmatrix} B & O \\ O & O \end{pmatrix}$, we have the following relationship from (2):

$$pF_q^{(m,n)}(\alpha, \beta; A, B) = pF_q^{(m)}(\alpha, \beta; A, B_1)$$

**Lemma 1.** For an $m \times m$ positive symmetric matrix $A$ and $m \times m$ symmetric positive semi-definite matrix $B$, we have

$$\int_{H \in O(m)} C_\kappa(AHBH')(dH) = \frac{C_\kappa(A)C_\kappa(B)}{C_\kappa(I_m)}.$$ 

**Proof.** From the fundamental properties of zonal polynomials $C_\kappa(AB) = C_\kappa(A^{1/2}BA^{1/2}) = C_\kappa(BA)$ and $C_\kappa(H'AH) = C_\kappa(A)$ for any $H \in O(m)$,

$$\int_{H \in O(m)} C_\kappa(AHBH')(dH) = \int_{H \in O(m)} C_\kappa(BH'AH)(dH).$$

Let $f_\kappa(A)$ be

$$f_\kappa(A) = \int_{H \in O(m)} C_\kappa(BH'AH)(dH).$$

From the proof of Theorem 7.2.5 in Muirhead (1982), $f_\kappa(A)$ must be a multiple of the zonal polynomial $C_\kappa(A)$; that is, $f_\kappa(A) = \lambda_\kappa C_\kappa(A)$ Putting $A = I_m$ and using $f_\kappa(I_m) = C_\kappa(B)$, we have $\lambda_\kappa = \frac{C_\kappa(B)}{C_\kappa(I_m)}$. \qed
**Theorem 1.** For an \( m \times m \) positive symmetric matrix \( A \) and an \( n \times n \) symmetric matrix \( B \), we have

\[
\int_{H_1 \in V_{m,n}} C_n(AH_1BH_1')(dH_1) = \frac{C_n(A)C_n(B)}{C_n(I_m)} \tag{8}
\]

**Proof.** We refer to the proof of Lemma 9.5.3 of Muirhead (1982).

For any \( m \times (m - n) \) matrix \( G \) with orthonormal columns that are orthogonal to those of \( H_1 \), \( K \in O(m - n), H = (H_1, H_2) \in O(m) \), and \( m \times m \) matrix \( B_1 = \begin{pmatrix} B & O \\ O & O \end{pmatrix} \), and we have \( H_2 = GK \) and

\[
AHB_1H' = A(H_1, GK) \begin{pmatrix} B & O \\ O & O \end{pmatrix} \begin{pmatrix} H_1' \\ KG' \end{pmatrix} = A(H_1B, O) \begin{pmatrix} H_1' \\ KG' \end{pmatrix} = AH_1BH_1'.
\]

Because of \( dh_{n+j} = G dk_j \ (j = 1, \ldots, m - n) \), it is clear that

\[
(H'dH) = (H'_1dH_1)(K'dK).
\]

The volume \( \text{Vol}(V_{n,m}) \) is written by

\[
\text{Vol}(V_{n,m}) = \frac{\text{Vol}(O(m))}{\text{Vol}(O(m - n))}. \tag{10}
\]

From (9) and (10), we have \((dH) = (dH_1)(dK)\), which means that \( \int_{O(m-n)} (dK) = 1 \).

Therefore,

\[
\int_{H_1 \in V_{n,m}} C_n(AH_1BH_1')(dH_1) = \int_{K \in O(m-n)} (dK) \int_{H_1 \in V_{n,m}} C_n(AH_1BH_1')(dH_1) = \int_{K \in O(m-n)} \int_{H_1 \in V_{n,m}} C_n(AH_1BH_1')(dH_1)(dK) = \int_{H \in O(m)} C_n(AHB_1H')(dH) = \frac{C_n(A)C_n(B)}{C_n(I_m)}
\]

Using (2), the following relationship holds from Lemma 1 and Theorem 1.

\[
\int_{H_1 \in V_{n,m}} C_n(AH_1BH_1')(dH_1) = \int_{H \in O(m)} C_n(AHB_1H')(dH). \tag{11}
\]

The next equation (12) is immediately obtained from the above result.

\[
\int_{H_1 \in V_{n,m}} pF_q(\alpha;\beta;AH_1BH_1')(dH_1) = \int_{H \in O(m)} pF_q(\alpha;\beta;AHB_1H')(dH) \tag{12}
\]

The following Corollary 1 provides a result similar to that in (5).

**Corollary 1.** For an \( m \times m \) symmetric matrix \( A \), \( B = \text{diag}(b_1 \ldots, b_n) \) and \( m \times m \) diagonal matrix \( B_1 = \text{diag}(b_1 \ldots, b_n, 0 \ldots, 0) \) where \( m \geq n \), we have

\[
pF_q(m,n)(\alpha;\beta;A,B) = \int_{H \in O(m)} pF_q(\alpha;\beta;AHB_1H')(dH) = \int_{H_1 \in V_{n,m}} pF_q(\alpha;\beta;AH_1BH_1')(dH_1) \tag{13}
\]
Proof. From \([12]\), we have

\[
\begin{align*}
_pF_q^{(m,n)}(\alpha; \beta; A, B) &= \_pF_q^{(m)}(\alpha; \beta; A, B_1) \\
&= \int_{H \in O(m)} _pF_q(\alpha; \beta; AHB_1H')(dH) \\
&= \int_{H_1 \in V_{n,m}} _pF_q(\alpha; \beta; AHB_1H_1')(dH_1)
\end{align*}
\]

Two particular cases of \([8]\) are listed as

\[
\begin{align*}
_0F_0^{(m,n)}(A, B) &= \sum_{k=0}^{\infty} \frac{C_k(A)C_k(B)}{k!C_k(I_m)} , \quad (14) \\
_1F_0^{(m,n)}(\alpha_1; A, B) &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k C_k(A)C_k(B)}{k!C_k(I_m)} , \quad (15)
\end{align*}
\]

and from Corollary 1, the related integral formulas of \([14]\) and \([15]\) are given in Corollary 2.

**Corollary 2.** For an \(m \times m\) symmetric matrix \(A, B = \text{diag}(b_1, \ldots, b_n)\) and a nonnegative integer \(r\), we have

\[
\begin{align*}
_0F_0^{(m,n)}(A, B) &= \int_{H_1 \in V_{n,m}} \text{etr}(AH_1BH_1') (dH_1) \\
_1F_0^{(m,n)}(\alpha_1; A, B) &= \int_{H_1 \in V_{n,m}} |I_m - AH_1BH_1'|^{-\alpha_1} (dH_1)
\end{align*}
\]

where \(\text{etr}(\cdot) = \exp(\text{tr}(\cdot))\) and \(|A|\) is the determinant of the matrix \(A\).

Proof. For an \(m \times m\) diagonal matrix, \(B_1 = \text{diag}(b_1, \ldots, b_n, 0, \ldots, 0)\) and \((H_1, H_2) \in H\), where \(H_1 \in V_{n,m}\). The special cases of \([4]\) are represented as \(_0F_0^{(m,n)}(A) = \text{etr}(A)\) and \(_1F_0^{(m,n)}(\alpha_1; A) = |I_m - A|^{-\alpha_1}\). Then

\[
\begin{align*}
_0F_0^{(m,n)}(A, B_1) &= \int_{H \in O(m)} _0F_0^{(m)}(AHB_1H') (dH) \\
&= \int_{H \in O(m)} \text{etr}(AHB_1H') (dH) \\
&= \int_{H_1 \in V_{n,m}} \text{etr}(AH_1BH_1') (dH_1).
\end{align*}
\]

The identity \([17]\) can also be derived in the same way as above.

**Corollary 3.** For an \(m \times m\) symmetric matrix \(A\) and \(B = \text{diag}(b_1, \ldots, b_n)\), we have

\[
\begin{align*}
_0F_0^{(m,n)}(A + I_m, B) &= \text{etr}(A) \_0F_0^{(m,n)}(A, B)
\end{align*}
\]

Proof. For an \(m \times m\) diagonal matrix \(B_1 = \text{diag}(b_1, \ldots, b_n, 0, \ldots, 0)\) and \((H_1, H_2) \in H\) where \(H_1 \in V_{n,m}\),

\[
\begin{align*}
_0F_0^{(m,n)}(I_m + A, B) &= \int_{H_1 \in V_{n,m}} _0F_0^{(m,n)}(I_m + A)H_1BH_1' (dH_1) \\
&= \int_{H \in O(m)} _0F_0^{(m)}((I_m + A)HB_1H') (dH) \\
&= \text{etr}(B) \int_{H \in O(m)} \text{etr}(AHB_1H') (dH) \\
&= \text{etr}(B) \_0F_0^{(m,n)}(A, B_1) \\
&= \text{etr}(B) \_0F_0^{(m,n)}(A, B)
\end{align*}
\]

\(\square\)
3 Exact distribution of the largest eigenvalue of a singular Wishart matrix

Suppose that an \( m \times n \) real Gaussian random matrix \( X \) is distributed as \( X \sim N_{m,n}(O, \Sigma \otimes I_n) \), where \( O \) is the \( m \times n \) zero matrix, \( \Sigma > 0 \), and \( \otimes \) is the Kronecker product. This means that the column vectors of \( X \) are an i.i.d. sample of size \( n \) from \( N_m(0, \Sigma) \), where \( 0 \) is the \( m \)-dimensional zero vector. The non-zero eigenvalues of \( \Sigma \) are denoted by \( \lambda_1, \lambda_2, \ldots, \lambda_m \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0 \). Then the random matrix \( W = XX' \) is called a non-singular real Wishart matrix. The eigenvalues of \( W \) are denoted by \( \ell_1, \ldots, \ell_m \), with \( \ell_1 > \ell_2 \cdots > \ell_m > 0 \). If \( n < m \), then \( W \) is said to be a singular real Wishart matrix. The first \( n \) eigenvalues are not zero and the remaining \( m - n \) eigenvalues \( \ell_{n+1}, \ldots, \ell_m \) are all zero. Uhlig (1994) derived that the density function of \( W \) for a singular case is given as

\[
f(W) = \frac{\pi^{-(mn+n^2)/2} \etr(-\Sigma^{-1}W/2)(\det L_1)^{(n-m-1)/2}}{2^{mn/2} \Gamma_n(n/2)(\det \Sigma)^{n/2}}
\]

where \( L_1 = \text{diag}(\ell_1, \ldots, \ell_n) \). Srivastava (2003) showed that the joint density function of \( \ell_1, \ldots, \ell_n \) is given as

\[
C \left( \prod_{i=1}^{n} \ell_i^{(m-n-1)/2} \right) \left( \prod_{i<j} \left( \ell_i - \ell_j \right) \right) \int_{H_1 \in V_{nm}} \etr \left( -\frac{1}{2} \Sigma^{-1} H_1 L_1 H_1' \right) \] (18)

where \( C = \frac{2^{-n/2} \pi^{n/2} |\Sigma|^{-n/2}}{\Gamma_n(\frac{n}{2}) \Gamma_n(\frac{m}{2})} \). From (14), the equation (18) is also expressed as

\[
C \left( \prod_{i=1}^{n} \ell_i^{(m-n-1)/2} \right) \left( \prod_{i<j} \left( \ell_i - \ell_j \right) \right) \text{qF}_0^{(m,n)} \left( -\frac{1}{2} \Sigma^{-1}, L_1 \right) \] (19)

The following two lemmas are required in order to integrate (19) with respect to \( \ell_2, \ldots, \ell_n \).

**Lemma 2.** Let \( L = \text{diag}(\ell_1, \ldots, \ell_n) \) and the length of \( \kappa \) be equal to or less than \( n \). Then the following equation holds.

\[
\int_{1>\ell_1>\ell_2>\cdots>\ell_n>0} |L|^{-(n+1)/2} |I_n - L|^{u-(n+1)/2} C_n(L) \prod_{i<j} (\ell_i - \ell_j) \prod_{i=1}^{n} d\ell_i = \frac{\Gamma_n(n/2) \Gamma_n(t, \kappa) \Gamma_n(u, \kappa) C_n(I_n)}{\pi^{n/2} \Gamma_n(t + u, \kappa)},
\]

where \( \text{Re}(t) > (n - 1)/2, \text{Re}(u) > (n - 1)/2, \Gamma_n(\alpha, \kappa) = (\alpha, \kappa) \Gamma(\alpha) \).

**Lemma 3.** Let \( X_1 = \text{diag}(1, x_2, \ldots, x_n) \) and \( X_2 = \text{diag}(x_2, \ldots, x_n) \) with \( x_2 > \cdots > x_n > 0 \). Then the following equation holds:

\[
\int_{1>x_2>\cdots>x_n>0} |X_2|^{t-(n+1)/2} C_n(X_1) \prod_{i=2}^{n} (1 - x_i) \prod_{i<j} (x_i - x_j) \prod_{i=1}^{n} dx_i = \frac{(nt + k)(\Gamma_n(n/2)/\pi^{n/2}) \Gamma_n(t, \kappa) \Gamma_n((n + 1)/2) C_n(I_n)}{\Gamma_n(t + (n + 1)/2, \kappa)}.
\]

From Lemma 2, Sugiyama (1967) derived Lemma 3 for the derivation of the exact distribution of \( \ell_1 \) for a non-singular real Wishart matrix. Shionozaki et al. (2018) gave the distribution of the largest eigenvalue under an elliptical population using Lemmas 2 and 3. In the case of a singular Wishart matrix, the exact distribution of \( \ell_1 \) of \( W \) is given in Theorem 2.

**Theorem 2.** Let \( W \sim W_m(n, \Sigma) \), where \( m > n \). Then the distribution function of the largest eigenvalue \( \ell_1 \) of \( W \) is given as

\[
\Pr(\ell_1 < x) = \frac{\Gamma_n((n + 1)/2)(\frac{x}{2})^{nm/2}}{\Gamma_n((n + m + 1)/2)|\Sigma|^{n/2}} \text{1F}_1^{(m,n)} \left( \frac{m}{2}, \frac{n + m + 1}{2}; -\frac{1}{2} x \Sigma^{-1}, I_n \right), \] (20)
Proof. The joint density of $\ell_1, \ell_2, \ldots, \ell_n$ is given in [19] as

\[
f(\ell_1, \ldots, \ell_n) = C(\det L_1)^{(m-n-1)/2} \prod_{i<j}^{n}(\ell_i - \ell_j) \prod_{i=1}^{n} (1 - x_i) \prod_{2 \leq l < j}^{m} (x_i - x_j) C_{\kappa}(X_1)
\]

where $X_1 = \text{diag}(1, x_2, \ldots, x_n)$ and $X_2 = \text{diag}(x_2, \ldots, x_n), x_2 > \cdots > x_n > 0$. Moreover, integrating $f(\ell_1)$ with respect to $\ell_1$, we obtain the distribution function of $\ell_1$ as

\[
\Pr(\ell_1 < x) = \frac{\Gamma_n((n+1)/2)}{\Gamma_n((n+m+1)/2)} \sum_{k=0}^{\infty} \sum_{\kappa \in P_k} \frac{C_{\kappa}(-\frac{1}{2} x - \frac{1}{2} \Sigma^{-1}) C_{\kappa}(I_n)}{k! C_k(I_m)}
\]

The zonal polynomials $C_{\kappa}(I_m)$ are expressed, for the length of partition $p > 0$, as

\[
C_{\kappa}(I_m) = \frac{2^{2k} k! (m/2)_k \prod_{i=1}^{p} (2\kappa_i - 2k_j - i + j)}{\prod_{i=1}^{p} (2\kappa_i + p - i)!}.
\]

Using the heterogeneous hypergeometric function, we obtain the distribution function of $\ell_1$ as

\[
\Pr(\ell_1 < x) = \frac{\Gamma_n((n+1)/2)}{\Gamma_n((n+m+1)/2)} \sum_{k=0}^{\infty} \sum_{\kappa \in P_k} \frac{(n/2)_k C_{\kappa}(-\frac{1}{2} x - \frac{1}{2} \Sigma^{-1})}{\prod_{i=1}^{p} (2\kappa_i + p - i)!}
\]

where we note that $(m/2)_k/C_k(I_m) = (n/2)_k/C_k(I_n)$ from [21].

**Corollary 4.** Let $W \sim W_m(n, I_m)$, with $m > n$. Then the distribution function of the largest eigenvalue $\ell_1$ of $W$ is given as

\[
\Pr(\ell_1 < x) = \frac{\Gamma_n((n+1)/2)}{\Gamma_n((n+m+1)/2)} \exp \left( - \frac{nx}{2} \right) I_{1/2}(n+1, n+m+1; x/2)
\]

**Proof.** From Theorem 2, we have

\[
\Pr(\ell_1 < x) = \frac{\Gamma_n((n+1)/2)}{\Gamma_n((n+m+1)/2)} \exp \left( - \frac{nx}{2} \right) I_{1/2}(n+1, n+m+1; x/2)
\]

The distribution function is translated to the series of positive terms using the Kummer relation as

\[
I_{1/2}(a, c; x) = \text{etr}(x) I_{1/2}(c-a, c; x)
\]

\[\square\]
The distribution function \( (22) \) is an infinite series. The truncated distribution function \( (22) \) up to the \( K \)th degree is denoted by

\[
F_K(x) = \frac{\Gamma_n \left\{ \left( \frac{n+1}{2} \right) \frac{nm}{2} \right\}}{\Gamma_n \left\{ \left( n+m+1 \right) \right\}} \exp \left( -\frac{nx}{2} \right) \sum_{k=0}^{K} \sum_{\kappa \in P_k} \left\{ \left( \frac{n+1}{2} \right) \right\}_{\kappa} \left\{ \left( n+m+1 \right) \right\}_{\kappa} \frac{C_{\kappa} \left( \frac{n}{2} I_n \right)}{k!} \tag{24}
\]

The empirical distributions based on \( 10^6 \)-trial Monte Carlo simulations are denoted by \( F_{\text{sim}}(x) \). Figure 1 shows the comparison of \( F_{\text{sim}}(x) \) and \( F_K(x) \) for \( K = 10, 30, 60 \). If \( K = 60 \), then the truncated series \( F_K(x) \) reaches that near \( x = 40 \). Table 1 shows the comparison of percentile points between \( F_{\text{sim}}^{-1}, F_K^{-1} \), and \( F^{-1} \), where the exact \( F^{-1} \) is calculated by the method of Chiani (2014). All percentile points have the same precision. In the case of \( n = 3 \), we need about 60 and 90 terms in the hypergeometric series for \( m = 10 \) and 50, respectively. However, more terms in the hypergeometric series are needed and much longer calculation time is required compared with the case for \( n = 2 \). We observe that the hypergeometric series in (24) converges slowly when the dimension \( n \) increases.

| Table 1: Percentile points of \( \ell_1 \) of \( W \sim W_m(3, I_m) \) |
|------------------------|------------------------|------------------------|------------------------|
| \( \alpha \) | \( F_{\text{sim}}^{-1} \) | \( F_{60}^{-1} \) | \( F_{90}^{-1} \) | \( \alpha \) | \( F_{\text{sim}}^{-1} \) | \( F_{90}^{-1} \) |
| 0.01 | 7.75 | 7.75 | 7.75 | 0.01 | 46.2 | 46.2 |
| 0.05 | 9.74 | 9.74 | 9.74 | 0.05 | 50.9 | 50.9 |
| 0.50 | 16.2 | 16.2 | 16.2 | 0.50 | 64.4 | 64.4 |
| 0.95 | 25.9 | 25.9 | 25.9 | 0.95 | 81.3 | 81.4 |
| 0.99 | 31.1 | 31.1 | 31.1 | 0.99 | 89.6 | 89.7 |

We consider the non-null case of \( W \sim W_1(2, \Sigma) \) where \( \Sigma \neq I_2 \). The density function of eigenvalue \( \ell_1 \) of \( W \) is given as

\[
f(\ell_1) = \frac{1}{2 \sqrt{\text{det} \Sigma}} \exp \left( -\frac{1}{2} \frac{\ell_1}{\lambda_2} \right) \frac{1}{\Gamma\left( \frac{1}{2} \right)} \frac{1}{\Gamma\left( 1 ; \frac{1}{2} \right)} \left( \frac{1}{\lambda_2} \right) \frac{1}{\Gamma\left( 1 ; \frac{1}{2} \right)} \frac{1}{\Gamma\left( 1 ; \frac{1}{2} \right)} \left( \frac{1}{\lambda_1} \right) \left( \frac{1}{\lambda_2} \right) \tag{25}
\]

where \( \Sigma = \text{diag}(\lambda_1, \lambda_2) \) and \( a = -\frac{1}{2} \ell_1 (1/\lambda_2 - 1/\lambda_1) \). The derivation of (25) is presented in the appendix. Figure 2 (a) shows the comparison of (25) and \( 10^6 \) Monte Carlo simulation results under \( \Sigma = \text{diag}(5, 2) \). Figure 2 (b) shows the line for \( F_{100}(x) \) and the dot plots for numerical integration of (25). They have almost the same precision.
4 Stiefel manifold over a real finite-dimensional division algebra

Let $\mathbb{F}_\beta$ denote a real finite-dimensional division algebra such that $\mathbb{F}_1 = \mathbb{R}$, $\mathbb{F}_2 = \mathbb{C}$, and $\mathbb{F}_4 = \mathbb{H}$ for $\beta = 1, 2, 4$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively, and $\mathbb{H}$ is the quaternion division algebra over $\mathbb{R}$. We restrict the parameter $\beta$ to values of $\beta = 1, 2, 4$, and denote $\mathbb{F}_\beta^{m \times n}$ by the set of all $m \times n$ matrices over $\mathbb{F}_\beta$, where $m \geq n$. The conjugate transpose of $X \in \mathbb{F}_\beta^{m \times n}$ is written by $X^* = \overline{X}^t$ and we say that $X$ is Hermitian if $X^* = X$. The set of all Hermitian matrices is denoted by $S^\beta(m) = \{X \in \mathbb{F}_\beta | X^* = X\}$. The eigenvalues of a Hermitian matrix are all real. If the eigenvalues of $X \in S^\beta(m)$ are all positive, then we say that it is positive definite and write $X > 0$. The exterior product $(dX)$ for $X \in \mathbb{F}_\beta^{m \times n}$ was defined in Mathi (1997) and García (2011). In a similar manner $\int \Gamma_n(dH)$, we define the Stiefel manifold and the unitary group over $\mathbb{F}_\beta$ as

$$V_{n,m}^\beta = \{H_1 \in \mathbb{F}_\beta^{m \times n} | H_1^* H_1 = I_n\}, \quad U_m^\beta = V_{n,m}^\beta = \{H \in \mathbb{F}_\beta^{m \times m} | H^* H = H H^* = I_m\}$$

, respectively. If $\beta = 1, 2, 4$, then $U_m^\beta$ are the real orthogonal group, unitary group, and symplectic group, respectively. The $\beta$-multivariate gamma function for $c \in \mathbb{F}_\beta$, $\Gamma_n^\beta(c)$, is defined by

$$\Gamma_n^\beta(c) = \int_{X > 0} |X|^{-(m-1)\beta/2} \text{etr}(-X)(dX) = \pi^{\frac{m(m-1)\beta}{4}} \prod_{i=1}^{m} \Gamma \left\{c - \frac{(i-1)\beta}{2}\right\}$$

where $\Re(c) > \frac{(m-1)\beta}{2}$. We define $(H_1^* dH_1)$ and $\text{Vol}(V_{n,m}^\beta)$ by

$$\langle H_1^* dH_1 \rangle = \int_{H_1 \in V_{n,m}^\beta} \langle H_1^* dH_1 \rangle = \int_{H_1 \in V_{n,m}^\beta} \text{Vol}(V_{n,m}^\beta) = \int_{H_1 \in V_{n,m}^\beta} (H_1^* dH_1) = \frac{2^n \pi^m n^{n\beta/2}}{\Gamma_n^\beta(m\beta/2)}$$

, respectively, where $H_1 \in V_{n,m}^\beta$ and $H = (H_1 | H_2) = (h_1, \ldots, h_n | h_{n+1}, \ldots, h_m) \in U_m^\beta$. Another differential form $(dH_1)$ defined by

$$\langle dH_1 \rangle = \frac{(H_1^* dH_1)}{\text{Vol}(V_{n,m}^\beta)} = \frac{\Gamma_n^\beta(m\beta/2)}{2^n \pi^m n^{n\beta/2}} (H_1^* dH_1)$$

is normalized such as $\int_{H_1 \in V_{n,m}^\beta} (dH_1) = 1$. For a partition $\kappa$, the $\beta$-generalized Pochhammer symbol of parameter $a > 0$ is defined as

$$(a)_\kappa^{\beta} = \prod_{i=1}^{m} \left(a - \frac{i-1}{2} \beta\right).$$
The Jack polynomial $C_{\kappa}^{\beta}(X)$ is a symmetric polynomial in $x_1, \ldots, x_m$; these are eigenvalues of $X$. See Stanley (1989) and Koev and Edelman (2006) for the relevant detailed properties. If $\beta = 1, 2$, then Jack polynomials are referred to as zonal polynomials and Shur polynomials, respectively Li and Xue (2009) proposed zonal polynomials and hypergeometric functions of quaternion matrix arguments for $\beta = 4$.

**Definition 2.** For $A \in S^{\beta}(m)$ and $B \in S^{\beta}(n)$, the heterogeneous hypergeometric functions of parameter $\beta$ are defined as

$$pF_q^{(\beta; m, n)}(\alpha, \beta; A, B) = \sum_{k=0}^{\infty} \sum_{\kappa \in \mathcal{P}_n} \frac{(\alpha_1)^{\beta_1} \cdots (\alpha_p)^{\beta_p} C_{\kappa}^{\beta}(A)C_{\kappa}^{\beta}(B)}{k! C_{\kappa}^{\beta}(I_m)},$$

(26)

where $m \geq n$.

Gross and Richards (1987) provided some properties of the integral formula for hypergeometric functions on a division algebra. The following two equations are needed for the derivation of Theorems 3 and 4 respectively. We utilize the complexification $\mathbb{C}S^{\beta}(m) = S^{\beta}(m) + iS^{\beta}(m)$. That is, $\mathbb{C}S^{\beta}(m)$ consists of all matrices $Z$ of the form $Z = X + iY$, where $X,Y \in S^{\beta}(m)$, and $i^2 = -1$. We also write $\text{Re}(Z) = X$ and $\text{Im}(Z) = Y$ for $Z = X + iY$.

For $Y \in \mathbb{C}S^{\beta}(n)$, $t = (n-1)/2$, $\text{Re}(a) > (n-1)/2$ and $\text{Re}(b) > (n-1)/2$, we have

$$\int_{0 < U < I_n} |U|^{a-t-1}I_n - U|^{b-t-1}C_{\kappa}^{\beta}(UY)(dU) = \frac{\Gamma_n^{\beta}(a, \kappa)}{\Gamma_n^{\beta}(a + b, \kappa)}\Gamma_n^{\beta}(b)C_{\kappa}^{\beta}(Y).$$

(27)

For $Y \in S^{\beta}(m)$ and $Z \in \mathbb{C}S^{\beta}(n)$ with $\text{Re}(Z) > 0$, we have

$$\int_{U > 0} \text{etr}(-UZ)|U|^{a-t-1}C_{\kappa}^{\beta}(UY)(dU) = (a)^{\beta} \Gamma_n^{\beta}(a)|Z|^{-a}C_{\kappa}^{\beta}(YZ^{-1}).$$

(28)

The integral representation for the functions $1F_1^{(\beta; m, n)}$ and $2F_1^{(\beta; m, n)}$ in (26) are given in Theorem 3

**Theorem 3.** For $X \in S^{\beta}(m)$ and $Y \in \mathbb{C}S^{\beta}(n)$, the function $1F_1^{(\beta; m, n)}$ is represented by the following integral representation:

$$1F_1^{(\beta; m, n)}(a; c; X, Y) = \frac{\Gamma_n^{\beta}(c)}{\Gamma_n^{\beta}(a)\Gamma_n^{\beta}(c-a)} \int_{0 < U < I_n} 0F_0^{(\beta; m, n)}(X, UY)|U|^{a-t-1}I_n - U|^{c-a-t-1}(dU),$$

where $c > a + (n-1)/2 > (n-1)/\beta$.

For arbitrary $a_1$, the function $2F_1^{(\beta; m, n)}(a_1, a; c; X, Y)$ also has the integral representation

$$2F_1^{(\beta; m, n)}(a_1, a; c; X, Y) = \frac{\Gamma_n^{\beta}(c)}{\Gamma_n^{\beta}(a)\Gamma_n^{\beta}(c-a)} \int_{0 < U < I_n} 1F_0^{(\beta; m, n)}(a_1; X, UY)|U|^{a-t-1}I_n - U|^{c-a-t-1}(dU),$$

where $c > a + (n-1)/2 > (n-1)/\beta$ and $||Y|| < 1$, where $||Y||$ is the maximum of the absolute values of the eigenvalues of $Y$.

Proof. The desired result is obtained by expanding $0F_0^{(\beta; m, n)}$ and $1F_0^{(\beta; m, n)}$ in integrand and integrating term by term using identities (27), respectively.

**Theorem 4.** For $X \in S^{\beta}(m)$, $Y \in S^{\beta}(n)$, and $Z \in \mathbb{U}^{\beta}(n)$ with $\text{Re}(Z) > 0$, we have

$$\int_{U > 0} \text{etr}(-UZ)|U|^{a-t-1}pF_q^{(\beta; m, n)}(\alpha; \beta; X, UY)(dU) = \Gamma_n^{\beta}(a)|Z|^{-a}pF_{q+1}^{(\beta; m, n)}(\alpha, a; \beta; X, YZ^{-1}),$$

where $p < q$, $\text{Re}(a) > (n-1)/2$; or $p = q$, $\text{Re}(a) > (n-1)/2$, and $||Z|| < 1$. 


Proof. The result is immediately obtained by expanding $pF_q(\beta;m,n)$ in the integrand and integrating term by term using identities [28].

The integral formula over the Stiefel manifold in Section 4 can be extended to the general case of division algebra. Concerning the equations [8] and [12], these formulas can be extended easily to the following equations. For $A \in S^\beta(m)$ and $B \in S^\beta(n)$, we have

$$\int_{H_1 \in V^\beta_{n,m}} C^\beta_\kappa(AH_1BH_1^*)(dH_1) = \frac{C^\beta_\kappa(A)C^\beta_\kappa(B)}{C^\beta_\kappa(I_m)}$$

and

$$\int_{H_1 \in V^\beta_{n,m}} C^\beta_\kappa(H_1BH_1')(dH_1) = \int_{H \in O(m)} C^\beta_\kappa(HBH')(dH)$$

where $B_1 = \begin{pmatrix} B & O \\ O & O \end{pmatrix}$. García (2013) showed that the denominator on the right side of (29) was evaluated as $C^\beta_\kappa(I_r)$ instead of $C^\beta_\kappa(I_m)$, where $r = \text{rank}(B)$. If $A = I_m$ in (29), then (29) and (30) imply the following well-known property of Jack polynomials:

$$C^\beta_\kappa(B_1) = \int_{H \in O(m)} C^\beta_\kappa(HBH')(dH).$$

**Theorem 5.** For $A \in S^\beta(m)$, $B \in S^\beta(n)$, and $H_1 \in V^\beta_{n,m}$, where $H = (H_1, H_2) \in U^\beta(m)$ and $\alpha_1$ is a non-negative integer, then we have

$$pF_q(\beta;m,n)(\alpha;\beta; A, B) = \int_{H_1 \in V^\beta_{n,m}} pF_q(\beta;m,n)(\alpha;\beta; AH_1BH_1^*)(dH_1)$$

$$aF_0(\beta;m,n)(A, B) = \int_{H_1 \in V^\beta_{n,m}} \text{etr}(AH_1BH_1^*) (dH_1)$$

$$1F_0(\beta;m,n)(\alpha_1; A, B) = \int_{H_1 \in V^\beta_{n,m}} |I_m-AH_1BH_1^*|^{-\alpha_1}(dH_1)$$

$$aF_0(\beta;m,n)(I_m + A, B) = \text{etr}(B) aF_0(\beta;m,n)(A, B)$$

Proof. The derivation is the same as that for the real case in Section 2.

5 **Singular beta-Wishart matrix**

In this section, we define the singular beta-Wishart matrix on a real finite division algebra. This matrix covers the singular real, complex, and quaternion Wishart matrices. We derive the density function of the singular beta-Wishart distributions and some distributions of eigenvalues. The singular beta-Wishart distributions are denoted by $W^\beta_m(n, \Sigma)$.

Let the $m \times n$ beta-Gaussian random matrix $X$ be distributed as $X \sim N^\beta_{m,n}(M, \Sigma \otimes \Theta)$, where $M$ is an $m \times m$ mean matrix and $\Sigma$ and $\Theta$ are $m \times m$ and $n \times n$ positive definite matrices, respectively. $N^\beta_{m,n}$ denotes a beta-normal distribution. The density functions of $X$ are given as

$$\frac{1}{(2\pi)^{n/2} \Gamma(n/2)^{\beta/2} \exp \left( -\frac{1}{2} \text{tr} \Sigma^{-1}(X - M)^{-1}(X - M)^* \right) \text{etr} \left( -\frac{\beta}{2} \Sigma^{-1}W \right) \left( \det L_1 \right)^{\beta(n-m+1)/2 - 1}}$$

Let $X \sim N^\beta_{m,n}(O, \Sigma \otimes I_n)$; that is, $M = O$ and $\Theta = I_n$. Then the $m \times m$ singular beta-Wishart matrix is defined as $W = XX^*$, where $m > n$. Real($\beta = 1$) and complex($\beta = 2$) singular Wishart distributions were obtained by Uling (1994), Sirivastava (2003), and Ratnarajah and Villancourt (2005b). García (2013) derived some useful Jacobians of the transformation for singular matrices. Let $W \sim W^\beta_n(n, \Sigma)$. From the Jacobian of the transformation of Corollary 1 in García (2013), the density function of $W$ is given as

$$f(W) = \frac{\pi^{n\beta(n-m)/2} (\det \Sigma)^{-\beta n/2}}{(2\pi^{-1})^{\beta mn/2} \Gamma_n^\beta(n/2)} \text{etr} \left( -\frac{\beta}{2} \Sigma^{-1}W \right) \left( \det L_1 \right)^{\beta(n-m+1)/2 - 1}$$

(31)
Theorem 6. The joint density of eigenvalues (19) can be expanded on a division algebra. Sirivastava (2003) gave the joint density as

\[ f(\ell_1, \ldots, \ell_n) \propto \int_{H_1 \in V_{n,m}^\beta} \text{etr} \left( -\frac{1}{2} \Sigma^{-1} H_1 L_1 H_1^* \right) (dH_1). \tag{32} \]

However, the right hand side of (32) is not represented in terms of heterogeneous hypergeometric functions. On the other hand, Ratnarajah and Villancourt (2005b) gave the joint density of eigenvalues of a singular complex Wishart matrix as

\[ f(\ell_1, \ldots, \ell_n) \propto \int_{H_1 \in V_{n,m}^\beta} \text{etr} \left( -\Sigma H_1 L_1 H_1^* \right) (dH_1) \]

\[ = \binom{2}{m,n} (0, \Sigma, L_1). \tag{33} \]

However, they did not prove the equation given above. From (29), the equation (33) is easily proved. Theorem 6 presents the joint density of eigenvalues of a singular beta-Wishart matrix as follows. The joint density of eigenvalues [19] can be expanded on a division algebra.

**Theorem 6.** Let \( W \sim W_m^\beta(n, \Sigma) \); then the joint density of eigenvalues \( \ell_1, \ldots, \ell_n \) of \( W \) is given as

\[ f(\ell_1, \ldots, \ell_n) = \frac{(2\beta-1)^{-\beta nm/2} \pi^{\beta n^2/2+r}}{\left| \Sigma \right|^{\beta n/2} \Gamma_n^{\beta /2} \left( m, \frac{\beta}{2} \right) \Gamma_n^{-\beta /2} \left( m, \frac{\beta}{2} \right)} (\det L_1)^{\beta(m-n+1)/2-1} \]

\[ \times \prod_{i<j}(\ell_i - \ell_j)^{\beta} \binom{2}{m,n} \left( -\beta \Sigma^{-1}, L_1 \right). \tag{34} \]

where

\[ r = \begin{cases} 
0 & \beta = 1 \\
-\beta n/2 & \beta = 2, 4. 
\end{cases} \]

**Proof.** The Jacobian of the transformation \( W = H_1 L_1 H_1^* \) given in García (2013) is

\[ (dW) = 2^{-n/2} \prod_{i=1}^n \ell_i^{\beta(m-n)} \prod_{i<j}(\ell_i - \ell_j)^{\beta} (dL) \wedge (H_1^* dH_1). \tag{35} \]

Using identities [35] for the density function of a singular beta-Wishart matrix \( f(W) \) and integrating with respect to \( H_1 \) over the Stiefel manifold \( V_{n,m}^\beta \), we have

\[ f(\ell_1, \ldots, \ell_n) = \frac{(2\beta-1)^{-\beta nm/2} \pi^{\beta n^2/2+r}}{\left| \Sigma \right|^{\beta n/2} \Gamma_n^{\beta /2} \left( m, \frac{\beta}{2} \right) \Gamma_n^{-\beta /2} \left( m, \frac{\beta}{2} \right)} (\det L_1)^{\beta(m-n+1)/2-1} \]

\[ \times \prod_{i<j}(\ell_i - \ell_j)^{\beta} \int_{H_1 \in V_{n,m}^\beta} \text{etr} \left( -\frac{\beta}{2} \Sigma^{-1} H_1 L_1 H_1^* \right) (dH_1) \]

From Theorem 5, we have the desired result. \( \square \)

To derive the exact distributions of the largest eigenvalue of a singular beta-Wishart matrix, we extend Lemma 2 to the case of division algebra.

**Lemma 4.** Let \( L = \text{diag}(\ell_1, \ldots, \ell_n) \) and let the length of \( \kappa \) be equal to or less than \( n \); then the following equation holds.

\[ \int_{1 > \ell_1 > \ell_2 > \cdots > \ell_n > 0} \left| L \right|^{a-p-1} \left| I_n - L \right|^{b-p-1} C_n^\beta(L) \prod_{i < j}(\ell_i - \ell_j)^{\beta} \prod_{i=1}^n d\ell_i \]

\[ = \frac{\Gamma_n^{\beta}(n\beta/2)\Gamma_n^{\beta}(a, \kappa)\Gamma_n^{\beta}(b)C_n^\beta(I_n)}{\pi^{n^2}\beta/2+r\Gamma_n^{\beta}(a+b, \kappa)} \tag{36} \]
Proof. Let $Y = I_n$ and $U = HLH^*$ in [27]. From Proposition 3 in García (2011), the different form $(dU)$ is represented as

$$(dU) = 2^{-n} n^r \prod_{i<j}(\lambda_i - \lambda_j)^\beta (dL)(H^* dH),$$

Using the above different form $(dU)$ and integrating $(H^* dH)$ with respect to $H$ over $U_n^\beta$, we have the desired result. \qed

**Theorem 7.** Let $X_1 = \text{diag}(1, x_2, \ldots, x_n)$ and $X_2 = \text{diag}(x_2, \ldots, x_n)$ with $x_2 > \cdots > x_n > 0$; then the following equation holds

$$\int_{1 > x_2 > \cdots > x_n > 0} |X_2|^{a-t-1} C^\beta_\kappa(X_1) \prod_{i=2}^n (1 - x_i)^\beta \prod_{i<j} (x_i - x_j)^\beta \prod_{i=2}^n dx_i = (na + k)(\Gamma_n^\beta(\beta a/2)/\pi^{\beta^2/2+r}) \Gamma_n^\beta(a, \kappa) \Gamma_n^\beta(t + 1)n^\beta(I_n)$$

(37)

Proof. Let $b = t + 1$. With the translation of $x_i = \ell_i/\ell_1$ for $i = 2, \ldots, n$, the left side of (36) is given as

$$\int_0^1 \ell_1 n a + k - 1 \int_{1 > x_2 > \cdots > x_n > 0} |X_2|^{a-t-1} C^\beta_\kappa(X_1) \prod_{i=2}^n (1 - x_i)^\beta \prod_{i<j} (x_i - x_j)^\beta \prod_{i=2}^n dx_i$$

we note that $\int_0^1 \ell_1 n a + k - 1 = 1/na + k$. \qed

**Theorem 8.** Let $W \sim W_m^\beta(n, \Sigma)$; then the largest eigenvalue $\ell_1$ of $W$ is given as

$$\Pr(\ell_1 < x) = \frac{(2^{\beta - 1}) - \beta mn/2 \Gamma_n^\beta\{(n - 1)\beta/2 + 1\}x^{mn/2}}{\Gamma_n\{(n + m - 1)\beta/2 + 1\}x^{n\beta/2}} 1_{F_1(\beta, m, n)} \left(\frac{m\beta}{2}, \frac{(n + m - 1)\beta}{2} + 1; -\frac{\beta}{2}x^{\Sigma^{-1}}; I_n\right)$$

(38)

Proof. This proof is presented in the same way as that for Theorem 2 in Section 3. We consider the joint density of eigenvalues [34], the translation of $x_i = \ell_i/\ell_1$ for $i = 2, \ldots, n$, and [37] in order to integrate $x_2, \ldots, x_n$ in [34]. Moreover, integrating the density functions $f(\ell_1)$ with respect to $\ell_1$, the density functions of $\ell_1$ are obtained by

$$\Pr(\ell_1 < x) = \frac{(2^{\beta - 1}) - \beta mn/2 \Gamma_n^\beta\{(n - 1)\beta/2 + 1\}x^{mn/2}}{\Gamma_n\{(n + m - 1)\beta/2 + 1\}x^{n\beta/2}} \sum_{k=0}^\infty \sum_{\kappa \in P_k^\beta} (m\beta/2)_k C^\beta_{\kappa}\left(-\frac{2}{\beta}x^{\Sigma^{-1}}\right) C^\beta_{\kappa}(I_n)$$

(39)

The Jack polynomials $C^\beta_{\kappa}(I_m)$ are expressed as

$$C^\beta_{\kappa}(I_m) = \frac{(2^{\beta - 1})^{2k}k!}{j_\kappa} \left(\frac{m}{2\beta}\right)$$

(40)

where

$$j_\kappa = \prod_{(i,j) \in \kappa} h^{\kappa}_{\kappa}(i,j) h^{\kappa}_{\kappa}(i,j)$$

and $h^{\kappa}_{\kappa}(i,j) \equiv \kappa_j - i + 2\beta(\kappa_i - j + 1)$ and $h^{\kappa}_{\kappa}(i,j) \equiv \kappa_j' - i + 1 + 2\beta(\kappa_i - j)$ are the upper and lower hook lengths at $(i,j) \in \kappa$, respectively. See Koev and Edelman (2006) for details. Using the identities [10], we obtain [38]. \qed
6 Conclusion

In this study, we considered the exact distributions of the largest eigenvalue of a singular beta-Wishart matrix. To derive the distributions of a singular beta-Wishart matrix, we defined heterogeneous hypergeometric functions with two matrix arguments. We provided a formula (29) that differs from Theorem 3 in Garcia (2013). Numerical computations were performed with a small sample of size \( n \). Approximate distributions for distribution function (38) are still required. Furthermore, numerical computations for large sample sizes are planned as part of future work. Finally, the holonomic gradient method (HGM) proposed by Hashiguchi et al. (2013, 2018) may also be applicable to such computations.

7 Appendix. Derivation of the \( f(\ell_1) \) for \( m = 2 \)

Let \( W \sim W_2(1, \Sigma) \); then the density function of \( W \) is given as

\[
f(W) = \frac{1}{2\pi \sqrt{|\Sigma|}} \ell_1^{-1} \text{etr}
\left(-\frac{1}{2} \Sigma^{-1} W\right)
\]

Because \( m = 2 \), we consider the case of \( n = 1 \) and just one eigenvalue \( \ell_1 \). The spectrum decomposition of \( W \) is given as

\[
W = H L H' = \ell_1 h_1 h_1'
= \begin{pmatrix}
\ell_1 \cos^2 \theta & \ell_1 \cos \theta \sin \theta \\
\ell_1 \sin \theta \cos \theta & \ell_1 \sin^2 \theta
\end{pmatrix}
\]

where \( H = (h_1, h_2) \), \( h_1 = (\cos \theta, \sin \theta)' \), \( h_2 = (-\sin \theta, \cos \theta)' \) and \( L = \text{diag}(\ell_1, 0) \). We note that \( h_2 \) vanishes because of \( \ell_2 = 0 \). Thus, the density function \( f(\ell_1, \theta) \) is given as

\[
f(\ell_1, \theta) = \frac{1}{2\pi} \frac{1}{\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2\lambda_1}\right) \times \exp\left\{-\frac{1}{2} \ell_1 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \sin^2 \theta\right)\right\}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( \Sigma \). We integrate the density function of \( f(\ell_1, \theta) \) as

\[
\int_0^{2\pi} f(\ell_1, \theta) d\theta = \frac{1}{4\pi} \frac{1}{\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2\lambda_1}\right) \exp(a \sin^2 \theta) d\theta
\]

\[
= \frac{1}{4\pi} \frac{1}{\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2\lambda_1}\right) \sum_{k=0}^{\infty} \int_0^{2\pi} \frac{a^k}{k!} (\sin^2 \theta)^k d\theta
\]

where \( a = -\frac{1}{2} \ell_1 (\frac{1}{\lambda_2} - \frac{1}{\lambda_1}) \). Hence we have

\[
f(\ell_1) = \frac{1}{4\pi} \frac{1}{\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2\lambda_1} \ell_1\right) \sum_{k=0}^{\infty} \frac{2\pi a^k (2k - 1)!!}{k! 2k!!}
\]

\[
= \frac{1}{2\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2\lambda_1} \ell_1\right) \frac{1}{F_1\left(\frac{1}{2}; 1; a\right)}
\]

Moreover, we obtain the mean and variance of \( \ell_1 \) as

\[
\int_0^{\infty} \ell_1 f(\ell_1) d\ell_1 = \lambda_1 + \lambda_2
\]

\[
\int_0^{\infty} \ell_1^2 f(\ell_1) d\ell_1 = \lambda_1 \lambda_2,
\]

respectively.
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