On some regularity results for $2 - D$ Euler equations and linear elliptic b.v. problems.

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Abstract

About thirty years ago we looked for "minimal assumptions" on the data which guarantee that solutions to the $2 - D$ evolution Euler equations in a bounded domain are classical. Classical means here that all the derivatives appearing in the equations and boundary conditions are continuous up to the boundary. Following a well known device, the above problem led us to consider this same regularity problem for the Poisson equation under homogeneous Dirichlet boundary conditions. At this point, one was naturally led to consider the extension of this last problem to more general linear elliptic boundary value problems, and also to try to extend the results to more general data spaces. At that time, some side results in these directions remained unpublished. The first motivation for this note is a clear description of the route followed by us in studying these kind of problems. New results and open problems are also considered.

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1 Introduction.

In this note the central problem is the following. Looking for "minimal assumptions" on the data which guarantee that solutions to a specific stationary, or evolution, problem are classical. This problem is called here the minimal assumptions problem. We say that solutions are classical if all derivatives appearing in the equations are continuous up to the boundary on their domain of definition.

The starting point of these notes is reference [5], where the main goal was to look for minimal assumptions on the data which guarantee classical solutions to the $2 - D$ Euler equations in a bounded domain

$$\begin{align*}
\partial_t v + (v \cdot \nabla)v + \nabla \pi &= f & \text{in } Q \equiv \mathbb{R}^+ \times \Omega, \\
\text{div} \, v &= 0 & \text{in } Q; \\
v \cdot n &= 0 & \text{on } \mathbb{R} \times \Gamma; \\
v(0) &= v_0 & \text{in } \Omega,
\end{align*}$$

(1.1)
where here, and everywhere in the sequel, the initial data $v_0$ is assumed to be divergence free and tangential to the boundary. Merely for simplicity, every time we are concerned with $2-D$ Euler equations, we assume that the domain $\Omega$ is simply connected. Recall that in the $2-D$ case the curl of the velocity can be, and is here, identified with a scalar field.

We start by briefly explaining why the curl of the velocity plays a central role in equation (1.1). This follows from a quite favorable situation, fruit of the lucky combination of two distinct facts. The first one is that the second and the third equations in the well known elliptic system (3.3), see below, are still included in (1.1). Hence, at each fixed time, the solution $v(t)$ of problem (1.1) is completely determined by the scalar quantity $\text{curl} v(t)$. The second favorable feature is that, as a rule, this last quantity “is transported by the characteristics” (assume, for simplicity, that $f = 0$).

The above setup led us, in reference [5], to look for data spaces $C^*(\Omega)$, as large as possible, for which the two assumptions below hold. In the second assumption, a loss of regularity going from the curl to the gradient is deliberately allowed since in the minimal assumptions problem nothing more than continuity is required to $\nabla v$. Roughly speaking, getting more than continuity, could mean that assumptions on the data are not “minimal”.

**Assumption 1.1.** The space $C^*(\Omega)$ satisfies the following property. If $\text{curl} v_0 \in C^*(\Omega)$, and $\text{curl} f \in L^1(\mathbb{R}^+; C^*(\Omega))$, then the global solution $v$ of problem (1.1) satisfies $\text{curl} v \in C(\mathbb{R}^+; C^*(\Omega))$.

**Assumption 1.2.** The space $C^*(\Omega)$ satisfies the following property. If $\theta \in C^*(\Omega)$, then $\nabla v \in C^*(\Omega)$, and $\|\nabla v\| \leq c_0 \|\theta\|_*$, where $v$ is the solution to the problem (3.3). So, for divergence free vector fields, tangential to the boundary, the estimate $\|\nabla v\| \leq c_0 \|\text{curl} v\|_*$ holds.

It is worth noting that in this section we proceed as if we have not already found a suitable space $C^*(\Omega)$. We are just setting up the problem. We believe this helps the understanding of our approach. Following this idea, we introduce our specific functional space $C^*(\Omega)$ only in the next section.

If a functional space $C^*(\Omega)$ satisfies the two above assumptions, the following result is immediate.

**Proposition 1.1.** Assume that the functional space $C^*(\Omega)$ enjoys the above assumptions 1.1 and 1.2. Furthermore, let $\text{curl} v_0 \in C^*(\Omega)$, and $\text{curl} f \in L^1(\mathbb{R}^+; C^*(\Omega))$. Then, the global solution $v$, to problem (1.1) is classical:

(1.2) $\nabla v \in C(\mathbb{R}^+; C^*(\Omega))$.

Clearly, the above two assumptions and proposition hold for our specific space $C^*(\Omega)$. See Theorems 3.2, 3.3, and 3.4.

Assumptions 1.1 and 1.2 alone are clearly insufficient to determine good choices of spaces $C^*(\Omega)$, since arbitrarily high regularity to the solutions is not imposed. Actually, our interest in the minimal assumption problem comes from the particular situation in which data are Hölder continuous. In fact, the above two assumptions hold by setting $C^*(\Omega) = C^{0,\lambda}(\Omega)$. However a Hölder continuity assumption on the data is unnecessarily restrictive, since it adds too...
much regularity to our continuity requirement, see (3.4). On the other hand, the choice $C^*_*(\Omega) = C(\Omega)$ is too wide. In this case assumption (1.2) holds, however assumption (1.1) is false in this setting. In conclusion, a basic problem in reference [5] was to single out functional spaces $C^*_*(\Omega)$ which satisfy assumptions (1.1) and (1.2) and for which the embeddings
\begin{equation}
C^{0,\lambda}(\Omega) \subset C^*_*(\Omega) \subset C(\Omega),
\end{equation}
are strict. In conclusion, a main problem was the following.

**Problem 1.1.** Look for Banach spaces $C^*_*(\Omega)$, which strictly satisfy the inclusions (1.3), and enjoy assumptions (1.1) and (1.2).

In reference [5] we singled out a specific functional space $C^*_*(\Omega)$ which enjoys the three requirements stated in problem (1.1). We postpone the definition of our specific space $C^*_*(\Omega)$ to section 2 below. Obviously, there may exist other significant functional spaces satisfying the required properties.

In the next sections we turn back to the effective resolution of the above, and related, problems. In fact, study and resolution of the above problems opens the way to new problems. First of all, problems related to the minimal assumptions problem for more general elliptic boundary value problems. In [5], the minimal regularity problem for the elliptic system (3.3) was confined to a similar regularity problem for equation (4.2). Theorem 3.3 was sufficient for our purposes. However, at that time, as remarked in [5], we had proved an extension of this result to more general elliptic boundary value problems (the proof remained unpublished, even though we were not able to find it in the current literature). Further, in a recent paper, we extended the proof to the stationary Stokes system, see Theorem 4.2 below. Similar results hold for more general linear elliptic problems, as the reader may verify, since the proof depends only on the behavior of the associated Green’s functions.

An interesting research field is the extension of the results to larger data spaces. In the sequel we refer to two distinct possible extensions. Unfortunately, we merely obtain partial extensions. As an example of this situation, compare Theorem 4.2 with Theorem 6.1 where continuity is replaced by boundedness.

Finally, partial extensions of theorems 3.2 and 3.4 to initial data in a functional space $B^*_*(\Omega)$, which strictly contains $C^*_*(\Omega)$, are shown in section 7, see theorems 7.1 and 7.2.

**Plan of the paper:**

In section 2 we recall definition and properties of the real space $C^*_*(\Omega)$ introduced in reference [5].

In section 3 we recall some results on Euler equations and elliptic problems with $C^*_*(\Omega)$ data.

In section 4 we consider Stokes, and other elliptic problems, with data in $C^*_*(\Omega)$. In particular, we show the connection between problems (3.3) and (4.2).

In section 5 we introduce new data spaces, denoted $B^*_*(\Omega)$ and $D^*_*(\Omega)$, satisfying the inclusions
\[ C^*_*(\Omega) \subset B^*_*(\Omega) \subset D^*_*(\Omega). \]

We also introduce a new family of Banach spaces, denoted $D^{0,\alpha}(\Omega)$, a kind of weak extension of the classical family of Hölder spaces.
In section 6 the full aim would be to extend the results with data in $C^\ast(\Omega)$ to data in the new spaces $B^\ast(\Omega)$ and $D^\ast(\Omega)$. Some partial extension results are shown for solutions to the Stokes equations. Second order derivatives of solutions are bounded but, possibly, not continuous. The proofs depend essentially on the properties of the related Green’s functions.

In section 7 we try to extend the results proved in reference [5] for the Euler equations with data in $C^\ast(\Omega)$, to data in $B^\ast(\Omega)$. The extension obtained is partial, since continuity is replaced by boundedness, and external forces vanish.

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2 The real space $C^\ast(\Omega)$.

In this section, following [5], we introduce a specific space $C^\ast(\Omega)$ which enjoys the three requirements stated in problem 1.1. Furthermore, we recall some of the main properties of this space. For complete proofs see [6].

We start with some notation. In the following $\Omega$ is an open, bounded, connected set in $\mathbb{R}^n$, locally situated on one side of its boundary $\Gamma$. The boundary $\Gamma$ is of class $C^{2,\lambda}$, for some $\lambda$, $0 < \lambda \leq 1$. In considering the Euler equations it is always assumed that $n = 2$, a crucial assumption here. On the contrary, in considering the Stokes equations we assume, merely to simplify index notation, that $n = 3$. Proofs immediately apply to the $n$-dimensional case.

By $C(\Omega)$ we denote the Banach space of all real continuous functions on $\Omega$. The classical norm in this space is denoted by $\|f\|$. We also appeal to the classical spaces $C^k(\Omega)$ endowed with the standard norm $\|u\|_k$, and to the Hölder spaces $C^{0,\lambda}(\Omega)$, endowed with the standard norm $\|f\|_{0,\lambda}$. In particular $C^{0,1}(\Omega)$ is the space of Lipschitz continuous functions in $\Omega$. Further, $C^\infty(\Omega)$ denotes the set of all restrictions to $\Omega$ of indefinitely differentiable functions in $\mathbb{R}^n$.

Boldface symbols refer to vectors, vector spaces, and so on. Components of a generic vector $\mathbf{v}$ are indicated by $v_i$, with similar notation for tensors. Norms in function spaces, whose elements are vector fields, are defined in the usual way by means of the corresponding norms of the components.

We denote by $\mathbf{e}_i$, $i = 1, 2, 3$, the three Cartesian coordinate unit vectors in $\mathbb{R}^3$. When considering the $2 - D$ Euler equations, the planar motion is that generated by the couple $\mathbf{e}_1$, $\mathbf{e}_2$.

The symbols $c, c_0, c_1, \ldots$, denote positive constants depending at most on $\Omega$ and $n$. We may use the same symbol to denote different constants.

Next, we define $C^\ast(\Omega)$, and recall some properties of this functional space. Set

$$I(x; r) = \{ y : |y - x| \leq r \}, \quad \Omega(x; r) = \Omega \cap I(x; r), \quad \Omega_c(x; r) = \Omega - \Omega(x; r).$$
For $f \in C(\Omega)$ we define, for each $r > 0$,

$$
(2.1) \quad \omega_f(r) \equiv \sup_{x, y \in \Omega(x; r)} \left| f(x) - f(y) \right|.
$$

In [5] we introduced the semi-norm

$$
(2.2) \quad [f]_* = [f]_*^\delta \equiv \int_0^\rho \omega_f(r) \frac{dr}{r}.
$$

The finiteness of the above integral is known as Dini’s continuity condition, see [11], equation (4.47). In this reference, problem 4.2, it is remarked that if $f$ satisfies Dini’s condition in the whole space $\mathbb{R}^n$, then its Newtonian potential is a $C^2$ function in $\mathbb{R}^n$.

We define

$$
(2.3) \quad C_*^\delta(\Omega) \equiv \{ f \in C(\Omega) : [f]_* < \infty \}.
$$

A norm is introduced by setting

$$
\| f \|_*^\delta \equiv [f]_*^\delta + \| f \|.
$$

Since

$$
(2.4) \quad [f]_*^\rho_1 \leq [f]_*^\rho_2 \leq [f]_*^\rho_1 + 2 \left( \log \frac{\rho_2}{\rho_1} \right) \| f \|,
$$

for $0 < \rho_1 < \rho_2$, norms are essentially independent of $\rho$.

Note that by setting

$$
(2.5) \quad \omega_f(x; r) = \sup_{y \in \Omega(x; r)} \left| f(x) - f(y) \right|,
$$

it follows that

$$
(2.6) \quad [f]_* = \int_0^\rho \sup_{x \in \Omega} \omega_f(x; r) \frac{dr}{r}.
$$

The main properties of $C_*^\delta(\Omega)$ are the following.

**Theorem 2.1.** $C_*^\delta(\Omega)$ is a Banach space.

**Theorem 2.2.** The embedding $C_*^\delta(\Omega) \subset C(\Omega)$ is compact.

**Theorem 2.3.** The set $C^\infty(\Omega)$ is dense in $C_*^\delta(\Omega)$.

It is worth noting that the crucial property required for the space $C_*^\delta(\Omega)$ in the proofs of Theorems 1.1 and 1.2 is Theorem 2.3. This theorem is proved, see [6], by appealing to the well known mollification technique. This density result up to the boundary requires a previous, suitable extension, of the functions outside $\Omega$. The following result holds.

**Theorem 2.4.** Set $\Omega_\delta \equiv \{ x : \text{dist}(x, \Omega) < \delta \}$. There is a $\delta > 0$ such that the following statement holds. There is a linear continuous map $T$ from $C(\Omega)$ to $C(\Omega_\delta)$, such that its restriction to $C_*^\delta(\Omega)$ is continuous from $C_*^\delta(\Omega)$ to $C_*^\delta(\Omega_\delta)$, and $T f$, restricted to $\overline{\Omega}$, coincides with $f$. 

5
3 Some results on Euler equations and elliptic problems with $C^*_s(\Omega)$ data.

In this section we refer back to the presentation shown in section 1 (the reader is supposed to have it in mind). The space $C^*_s(\Omega)$ is here no more "abstract", but that defined in section 2. Motivation was discussed in section 1 in particular the justification of the single statements (below, in rigorous form).

As already recalled, in reference [5] we have considered the problem of minimal assumptions on initial data and external forces sufficient to obtain classical solutions of the $2 \times 2$ Euler equations (1.1). According to our definition, classical solution of this system means here that $\nabla v, \partial_t v, \nabla \pi \in C_0(Q)$.

By considering $C^0(\Omega)$ as the curl's data space, one has the following result, proved in [5], Theorem 1.1.

**Theorem 3.1.** Let a divergence free vector field $v_0$, tangent to the boundary, satisfy $\text{curl} v_0 \in C^0(\Omega)$, and let $\text{curl} f \in L^1(\mathbb{R}_+; C^0(\Omega))$. Then, the problem (1.1) is uniquely solvable in the large,

$$
\text{curl} v \in C(\mathbb{R}_+; C^0(\Omega)),
$$

and the estimate

$$
\| \text{curl} v(t) \| \leq \| \text{curl} v_0 \| + \int_0^t \| \text{curl} f(\tau) \| \, d\tau
$$

holds. If $\text{curl} f = 0$, then $\| \text{curl} v(t) \| = \| \text{curl} v_0 \|$.

The next step was to replace $\text{curl} v$ by $\nabla v$ in the left hand side of equation (3.2). The starting point here is that the two last equations in the elliptic system

$$
\begin{cases}
\text{curl} v = \theta & \text{in } \Omega, \\
\text{div} v = 0 & \text{in } \Omega, \\
v \cdot \mathbf{n} = 0 & \text{on } \Gamma,
\end{cases}
$$

are still included in (1.1). Furthermore, it is well known that solutions $v$ of problem (3.3) are completely determined here by the scalar quantity $\theta$ (recall the assumption $\Omega$ simply-connected). Hence, to be allowed to replace $\text{curl} v$ by $\nabla v$ in the left hand side of equation (3.2) it is sufficient to show that solutions $v$ of problem (3.3) satisfy the estimate $\| \nabla v \| \leq c \| \theta \|$. Unfortunately, this is known to be false. In other words, the data space $C^0(\Omega)$ is too wide. On the other hand, Hölder spaces are here too narrow. In fact, in this case (see [13], [10], and also [4], [12]), the above device works, since solutions of (3.3) satisfy the estimate

$$
\| \nabla v \|_{0, \lambda} \leq c \| \theta \|_{0, \lambda} \equiv c \| \text{curl} v \|_{0, \lambda}.
$$

However this estimate is unnecessarily strong in the context of our "minimal assumptions problem". So our problem was to find a functional space $C^*_s(\Omega)$, which satisfies assumptions (1.1) and (1.2) and such that the inclusions (1.3) are strict. The space $C^*_s(\Omega)$ defined in section 2 satisfies these requirements. Concerning Assumption (1.1), we have proved in [5] (see Lemma 4.4 in this reference) the following statement, may be the main result in the above paper. It is worth noting that the presence of quite general external forces leads to very substantial, additional difficulties in the proof of this result.
Theorem 3.2. Let $C_*(\Omega)$ be the Banach space defined in section 2. Assume that $\text{curl } v_0 \in C_*(\Omega)$ and $\text{curl } f \in L^1(\mathbb{R}^+; C_*(\Omega))$. Then, the curl of the global solution $v$ of problem (1.1) satisfies

$$\text{curl } v \in C(\mathbb{R}^+; C_*(\Omega)).$$

Moreover

$$(3.5) \quad \| \text{curl } v(t) \|_* \leq e^{c_1 B t} \left( 3 B + \| \text{curl } v_0 \|_* + \| \text{curl } f \|_{L^1(0,t; C_*(\Omega))} \right),$$

where

$$(3.6) \quad B = \| \text{curl } v_0 \| + \| \text{curl } f \|_{L^1(0,t; C(\Omega))}.$$
Lemma 3.5. Let \( a \in C_*(\Omega) \) and \( U \in C^{0,\delta}(\Omega;\Omega) \), \( 0 < \delta \leq 1 \). Then \( a \circ U \in C_*(\Omega) \); moreover

(3.9) \[ |a \circ U|_* \leq \frac{1}{\delta} |a|_* \]

Note that the need for the above property narrows the possible choice of the candidate spaces \( C_*(\Omega) \).

The specific role of the \( U \)'s in equation (3.9) is that of stream-lines generated by velocity fields. In an old, hand written version [3] (still conserved) the above lemma is written in terms of metric spaces. However, at that time, it seemed to us a little "out of place" to present a so simple result in an abstract form.

Concerning the 2 – D Euler equations we also refer the reader to [14], where the treatment of this particular problem is not very dissimilar to that followed in reference [5].

4 Stokes, and other elliptic problems, in \( C_*(\Omega) \).

We start by showing how to confine the minimal regularity problem for the elliptic system (3.3), treated in Theorem 3.3, to a similar, simpler, regularity problem for equation (4.2). A classical argument shows that the solution \( v \) of the linear elliptic system (3.3) can be obtained by setting

(4.1) \[ v = \text{Rot} \psi, \]

where the scalar field \( \psi \) solves the problem

(4.2) \[ \begin{cases} -\Delta \psi = \theta & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases} \]

We appeal here to a typical approach in studying planar motions. The scalar \( \psi \) is a simpler representation of \( \psi e_3 \), a vector field normal to the plane of motion. Furthermore, Rot \( \psi \) is defined by setting Rot \( \psi \equiv \text{curl} (\psi e_3) \), a vector field lying in the plane of motion. It may be obtained by a \( \frac{\pi}{2} \) counter-clockwise rotation of \( \nabla \psi \).

In reference [5] we have stated the following result.

Theorem 4.1. Let \( \theta \in C_*(\Omega) \) and let \( \psi \) be the solution to problem (4.2). Then \( \psi \in C^2(\Omega) \), moreover, \( \| \psi \|_2 \leq c_0 \| \theta \|_* \).

Theorem 3.3 follows immediately from this result, by appealing to the explicit expression (4.1) of the solution \( v \) of problem (3.3).

Theorem 4.1 was stated in [5] as Theorem 4.5. Actually, in this last reference the result was claimed for more general linear elliptic boundary value problems. However the proof remained unpublished. Recently, in reference [6], we followed the same lines to obtain a corresponding result for the Stokes system, see Theorem 4.2 below (in section 6 we show a partial extension of this theorem to larger functional spaces).

Consider the Stokes system (see, for instance, [10], [15], [21])

(4.3) \[ \begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \]
If \( f \in C(\Omega) \), this problem has a unique solution \((u, p) \in C^1(\Omega) \times C(\Omega)\), where \( p \) is defined up to a constant. The solution is given by

\[
\begin{align*}
    u_i(x) &= \int_{\Omega} G_{ij}(x, y) f_j(y) \, dy, \\
    p(x) &= \int_{\Omega} g_j(x, y) f_j(y) \, dy,
\end{align*}
\]

where \( G \) and \( g \) are respectively the Green’s tensor and vector associated with the above boundary value problem. Furthermore, the following estimates hold.

\[
\begin{align*}
    |G_{ij}(x, y)| &\leq \frac{C}{|x-y|}, \\
    |g_j(x, y)| &\leq \frac{C}{|x-y|^2}, \\
    \left|\frac{\partial G_{ij}(x, y)}{\partial x_k}\right| &\leq \frac{C}{|x-y|^2}, \\
    \left|\frac{\partial g_j(x, y)}{\partial x_k}\right| &\leq \frac{C}{|x-y|^3},
\end{align*}
\]

The positive constant \( C \) depends only on \( \Omega \). For an overview on the classical theory of hydrodynamical potentials, and the construction of the Green functions \( G \) and \( g \), we refer to chapter 3 of the classical treatise [15]. The estimates (4.5) are contained in equations (46) and (47) in this last reference. They may also be found in [17]; see also [9] and [22]. The estimates (4.5) are a particular case of a set of much more general results, due to many authors. See, for instance, [1], [17], [18], [19] and [20].

It is well known, see [9], that for every \( f \in C^0(\Omega) \) the solution \((v, p)\) to the Stokes system (4.3) belongs to \( C^2(\Omega) \times C^1(\Omega) \). Hence, as above, Hölder spaces look too strong as data spaces for getting classical solutions. On the other hand, as above, it is well known that \( f \in C(\Omega) \) does not guarantee classical solutions. In reference [4], by following [3], we proved the following result.

**Theorem 4.2.** For every \( f \in C^*(\Omega) \) the solution \((u, p)\) to the Stokes system (4.3) belongs to \( C^2(\Omega) \times C^1(\Omega) \). Moreover, there is a constant \( c_0 \), depending only on \( \Omega \), such that the estimate

\[
\|u\|_2 + \|\nabla p\| \leq c_0 \|f\|_* , \quad \forall f \in C^*(\Omega),
\]

holds.

A partial generalization of the above theorem to data in a larger space is presented in section 6; see Theorem 6.1.

## 5 Spaces \( B_*(\Omega) \), and \( D_*(\Omega) \).

The results obtained in the framework of \( C_*(\Omega) \) spaces, described in sections 3 and 4 immediately lead us to consider their possible extension to larger functional spaces of continuous functions. Below we consider functional spaces \( B_*(\Omega) \) and \( D_*(\Omega) \), such that, in particular

\[
C_*(\Omega) \subset B_*(\Omega) \subset D_*(\Omega).
\]

In [3] we have introduced the space \( B_*(\Omega) \), as follows. For each \( f \in C(\Omega) \), we define the semi-norm

\[
(f)_* = \sup_{x \in \Omega} \int_0^\rho \frac{\omega_f(x; r)}{r} \, dr,
\]
and the functional space

\[ B_\ast(\Omega) \equiv \{ f \in C(\Omega) : \langle f \rangle \ast < +\infty \} \]

dowered with the norm

\[ \| f \| \ast \equiv \langle f \rangle \ast + \| f \| . \]

The reader should compare (5.1) with (2.6). Obviously, \( \langle f \rangle \ast \leq [f] \ast \). Actually, the \( B_\ast(\Omega) \) norm is "much weaker". In [3] we have shown that the inclusion \( C_\ast(\Omega) \subset B_\ast(\Omega) \) is proper, by constructing oscillating functions which belong to \( B_\ast(\Omega) \) but not to \( C_\ast(\Omega) \). This construction was recently published in reference [7]. Further, we may show that \( B_\ast(\Omega) \) is compactly embedded in \( C(\Omega) \), and that (2.4) still holds for the \( B_\ast(\Omega) \) semi-norm.

In reference [3] we considered the problem (4.2) with data in \( B_\ast(\Omega) \), and have proved that the the first order derivatives of \( \psi \) are Lipschitz continuous in \( \Omega \). Hence, second order derivatives are bounded. However we were (and are) not able to prove the continuity of these derivatives (the continuity result would hold if the density theorem [2.3] were to hold with \( C_\ast(\Omega) \) replaced by \( B_\ast(\Omega) \); an interesting open problem). This led us, at that time, to replace in the published work [5] the space \( B_\ast(\Omega) \) by the more handy space \( C_\ast(\Omega) \). Actually, in [3], the result was proved for linear elliptic boundary value problem whose solutions are given by

\[ u(x) = \int_{\Omega} G(x, y) f(y) \, dy, \]

where the scalar Green function \( G(x, y) \) satisfies the estimates stated in (4.5).

Recently, in reference [7], we have published the proof of this result in a more general form, since \( B_\ast(\Omega) \) was replaced by the larger space \( D_\ast(\Omega) \), defined below. In section 6 we appeal to the same ideas to extend the result proved in [7] to the Stokes problem (4.3), see Theorem 6.1. Obviously, all the results proved for data in \( D_\ast(\Omega) \) hold for data in \( B_\ast(\Omega) \).

The space \( D_\ast(\Omega) \) is defined as follows. Set

\[ S(x; r) = \{ y \in \Omega : |y - x| = r \} \]

and define, for \( f \in C(\Omega) \), \( x_0 \in \Omega \), and \( r > 0 \), the quantity

\[ \omega_f(x_0; r) \equiv \sup_{y \in S(x_0; r)} |f(y) - f(x_0)|. \]

Further, we define the semi-norm

\[ (f) \ast \equiv \sup_{x_0 \in \Omega} \int_0^R \omega_f(x_0; r) \frac{dr}{r}, \]

and the related functional space

\[ D_\ast(\Omega) \equiv \{ f \in C(\Omega) : (f) \ast < \infty \}. \]

A norm in \( D_\ast(\Omega) \) is introduced by setting \( \| f \| \ast = (f) \ast + \| f \| \). We remark that (2.4) still holds for the \( D_\ast(\Omega) \) semi-norm.
Note that, compared to \( B_\ast(\Omega) \), the space \( D_\ast(\Omega) \) is defined by replacing in (5.1) the expression of \( \omega_f(x; r) \) shown in (2.5) by the expression given by (5.4). Sets \( \Omega(x; r) \) are replaced by sets \( S(x; r) \). Also note that \( S(x; r) \) is a proper subset of \( \partial \Omega(x; r) \) if the distance of \( x \) to the boundary \( \partial \Omega \) is less than \( r \).

It is worth noting that the above substitution in the definition of \( C_\ast(\Omega) \) is irrelevant. It leaves this space invariant.

The results obtained in the framework of \( C_\ast(\Omega) \) spaces also lead us to consider the problem of their restriction to smaller functional spaces, instead of extension to larger spaces. The main motivation, within the realm of solutions to second order linear elliptic boundary value problems, can be illustrated as follows. If \( f \in C_0, \lambda(\Omega) \), the second order derivatives of the solution satisfy \( \nabla^2 u \in C_0, \lambda(\Omega) \). Let's say, for brevity, that they fully "remember" their origin. On the other hand, if the data \( f \) is in \( C_\ast(\Omega) \), then the second order derivatives of the solution are merely continuous. Roughly speaking, they completely "forget" that \( f \) produces a finite the integral on the right hand side of (2.2). This situation leads us to look for data spaces, between Hölder and \( C_\ast(\Omega) \) spaces, for which solutions "remember", at least partially, their origin. The following is a significant example of a functional space of "intermediate type". Define, for each \( \alpha > 0 \), the semi-norm

\[
[f]_{0,\alpha} \equiv \sup_{x, y \in \Omega} \frac{|f(x) - f(y)|}{\log \frac{1}{|x - y|} - \alpha},
\]

and the related norm \( \|f\|_{0,\alpha} \equiv [f]_{0,\alpha} + \|f\| \). Next, define functional spaces \( D_{0,\alpha}(\Omega) \) in the obvious way. Roughly speaking, we have replaced in the definition of Hölder spaces the quantity \( \frac{1}{|x - y|} \) by \( \log \frac{1}{|x - y|} \).

This similitude leads us to call these spaces Hölderlog (Hölder-logarithmic) spaces. The family of Hölderlog spaces enjoys some typical, significant property. For instance, \( D_{0,\alpha}(\Omega) \) is a Banach space, and \( C^\infty(\Omega) \) is a dense subspace. Furthermore, for \( 0 < \beta < 1 < \alpha \), and \( 0 < \lambda \leq 1 \), the following strict embeddings

\[
C^{0, \lambda}(\Omega) \subset D^{0, \alpha}(\Omega) \subset C^\ast(\Omega) \subset D^{0, \beta}(\Omega) \subset C(\Omega)
\]

hold. Note that \( D^{0, 1}(\Omega) \subset C^\ast(\Omega) \) is false. The embeddings \( D^{0, \alpha}(\Omega) \subset D^{0, \beta}(\Omega) \subset C(\Omega) \), for \( \alpha > \beta > 0 \), and the embeddings \( D^{0, \alpha}(\Omega) \subset C^\ast(\Omega) \), for \( \alpha > 1 \), are compact.

In forthcoming papers we consider boundary value problems with data in \( D^{0, \alpha}(\Omega) \). For a second order linear elliptic problem we show, in reference [8], that if \( f \in D^{0, \alpha}(\Omega) \), for some \( \alpha > 1 \), then \( \nabla^2 u \in D^{0, (\alpha - 1)}(\Omega) \). There is just a "partial loss of regularity". Full application to the Euler problem (1.1), will be also shown.
6 The Stokes equations with data in \( D_*(\overline{\Omega}) \). Uniform boundedness of \( \nabla^2 u \) and \( \nabla p \).

In this section we consider the Stokes system and show that the first order derivatives of the velocity \( u \), and the pressure \( p \), are Lipschitz continuous in \( \overline{\Omega} \) for given external forces in \( D_*(\overline{\Omega}) \) (so, in particular, in \( B_*(\overline{\Omega}) \)). We prove the following result.

**Theorem 6.1.** Let \( f \in D_*(\overline{\Omega}) \), and let \((u, p)\) be the solution to problem \([43, 13] \). There is a constant \( C \), which depends only on \( \Omega \), such that

\[
\| u \|_{1,1} + \| p \|_{0,1} \leq C \| f \|_*. 
\]

So \( \nabla^2 u, \nabla p \in L^\infty(\Omega) \).

**Proof.** In the following we merely consider the velocity, since the pressure is treated similarly (see also \([6]\)). Let \( e_i(x) \), \( i = 1, 2, 3 \), denote three constant vector fields in \( \mathbb{R}^3 \), everywhere equal to the corresponding cartesian coordinate unit vector \( e_i \). Define the auxiliary systems

\[
\begin{align*}
-\Delta v_i(x) + \nabla q_i(x) &= e_i(x) \quad \text{in } \Omega, \\
\nabla \cdot v_i &= 0 \quad \text{in } \Omega, \\
v_i &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

Clearly, \( v_i \) and \( q_i \) are smooth. Fix a constant \( K(\Omega) \) such that

\[
\| v_i \|_{1,1} + \| q_i \|_{0,1} \leq K(\Omega),
\]

for \( i = 1, 2, 3 \). Next, in correspondence to each point \( x_0 \in \Omega \), define the auxiliary system (a kind of “tangent problem” at point \( x_0 \))

\[
\begin{align*}
-\Delta v(x_0, x) + \nabla q(x_0, x) &= f(x_0, x) \quad \text{in } \Omega, \\
\nabla \cdot v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

where \( f(x_0, x) \equiv f(x_0), \forall x \in \Omega \), is a constant vector in \( \Omega \). Since

\[
f(x_0, x) = \sum_i f_i(x_0) e_i(x),
\]

the functions \( v(x_0, x) \) and \( q(x_0, x) \) are smooth for each fixed \( x_0 \). Moreover,

\[
\| v(x_0, \cdot) \|_{1,1} + \| q(x_0, \cdot) \|_{0,1} \leq K |f(x_0)| \leq K \| f \|.
\]

Recall that \( K \) is independent of \( x_0 \). For convenience set \( v(x) = v(x_0, x) \), and so on. By setting \( w(x) \equiv u(x) - v(x) \), one has

\[
w_i(x) = \int_\Omega G_{ij}(x, y) \left( f_j(y) - f_j(x_0) \right) dy.
\]

Furthermore,

\[
\partial_k w_i(x) - \partial_k w_i(x_0) = \int_\Omega \left( \partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y) \right) \left( f_j(y) - f_j(x_0) \right) dy,
\]

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where \( \partial_k \) stands for differentiation with respect to \( x_k \), and \( \partial_k w_i(x_0) \) means the value of \( \partial_k w_i(x) \) at the particular point \( x = x_0 \).

Clearly

\[
|\partial_k w_i(x) - \partial_k w_i(x_0)| \leq \int_{\Omega} |\partial_k G_{i,j}(x, y) - \partial_k G_{i,j}(x_0, y)| |f_j(y) - f_j(x_0)| \, dy.
\]

By setting \( \rho = |x - x_0| \) one gets

\[
|\partial_k w_i(x) - \partial_k w_i(x_0)| \leq \int_{\Omega(x_0; 2\rho)} |\partial_k G_{i,j}(x, y) - \partial_k G_{i,j}(x_0, y)| |f_j(y) - f_j(x_0)| \, dy + \int_{\Omega(x_0; 2\rho)} |\partial_k G_{i,j}(x, y) - \partial_k G_{i,j}(x_0, y)| |f_j(y) - f_j(x_0)| \, dy
\]

\begin{equation}
(6.6)
\end{equation}

\[
\equiv I_1(x_0, x, \rho) + I_2(x_0, x, \rho).
\]

By appealing to (1.6), we show that

\[
I_1(x_0, x, \rho) \leq C \left( \int_{\Omega(x_0; 2\rho)} \frac{C}{|x_0 - y|^2} |f_j(y) - f_j(x_0)| \, dy \right) + J_1(x_0, x, \rho) + J_2(x_0, x, \rho).
\]

(6.7)

By setting \( r = |x_0 - y| \), and by appealing to polar-spherical coordinates centered in \( x_0 \), one easily shows that \( J_1(x_0, x, \rho) \leq C \rho \omega(x_0; 2 \rho) \), where \( C \) depends only on \( \Omega \). Similarly, \( J_2(x_0, x, \rho) \leq C \rho \omega(x; 3 \rho) \). It follows that (recall definition (5.5))

\[
I_1(x_0, x, \rho) \leq C \rho \omega(3 \rho),
\]

where \( C \) does not depend on the particular points \( x_0, x \in \Omega \), and \( \rho = |x - x_0| \).

On the other hand, by appealing to the mean-value theorem and to (1.4), we get

\[
|\partial_k G_{i,j}(x, y) - \partial_k G_{i,j}(x_0, y)| \leq C \rho |x' - y|^{-3} \leq C \rho 2^3 |x_0 - y|^{-3},
\]

for each \( y \in \Omega_{c}(x_0; 2 \rho) \), where the point \( x' \) belongs to the straight segment joining \( x_0 \) to \( x \). Consequently,

\[
I_2(x_0, x, \rho) \leq c \rho \int_{\Omega(x_0; 2\rho)} |f_j(y) - f_j(x_0)| \frac{dy}{|x_0 - y|^3} \leq c \rho \int_{2\rho}^{R} \omega_f(r) \frac{dr}{r}.
\]

Hence,

\[
I_2(x_0, x, \rho) \leq c \rho \int_{0}^{R} \omega_f(r) \frac{dr}{r} = c \rho (f)_{\ast}.
\]

Next, by appealing to equation (6.6), and to the estimates proved above for \( I_1 \) and \( I_2 \), we show that

\[
|\nabla w(x) - \nabla w(x_0)| \leq c \rho ((f)_{\ast} + \omega_f(3 \rho)).
\]
Consequently,
\[
| \nabla u(x) - \nabla u(x_0) | \leq | \nabla w(x) - \nabla w(x_0) | + | \nabla v(x) - \nabla v(x_0) |
\]
\[
\leq c \rho \left( (f)_* + \omega f(3\rho) + K \| f \| \right).
\]
So,
(6.8) \[
\frac{| \nabla u(x) - \nabla u(x_0) |}{|x - x_0|} \leq C \| f \|_*, \quad \forall x, x_0 \in \Omega, x \neq x_0.
\]
This proves (6.1) for the velocity \( u \). Similar calculations lead to the corresponding result for the pressure.

It is worth noting that our proofs depend only on having suitable estimates for the Green’s functions. For instance, the argument applied in the above proof to study the system (4.3) with data in \( D_s^*(\Omega) \) can be applied to the system (4.2) with data in \( D_s(\Omega) \), since the scalar Green’s function \( G(x, y) \) related to this last problem satisfies exactly the estimates claimed in equation (4.5) for the components \( G_{ij}(x, y) \). It follows that Theorem 4.1 still holds in the above “weak form”, for data in \( D_s(\Omega) \). Further, in correspondence with Theorem 3.3, we get the following statement.

**Theorem 6.2.** Let \( \theta \in D_s(\Omega) \), and let \( v \) be the solution of problem (3.3). Then \( \nabla v \in L^\infty(\Omega) \), and \( \| \nabla v \|_{L^\infty(\Omega)} \leq c_0 \| \theta \|_* \). So, for divergence free vector fields, tangent to the boundary, the estimate
(6.9) \[
\| \nabla v \|_{L^\infty(\Omega)} \leq c_0 \| \text{curl } u \|_*
\]
holds.

This specific case will be useful in considering the Euler equations with data in \( B_s(\Omega) \). For more results and comments on the above subject we refer to [7].

### 7 The space \( B_s(\Omega) \) and the Euler equations.

Concerning possible extensions of the results obtained for the 2 - D evolution Euler equations, from \( C_s(\Omega) \) to \( B_s(\Omega) \), we show here a partial result in this direction. We clearly pay the price of the loss of regularity for solutions to the auxiliary elliptic system (3.3). This leads us to replace continuity in time by boundedness in time.

Furthermore, we simplify our task, in a quite substantial way, by assuming that external forces vanish, instead of assuming the very stringent condition \( \text{curl } f \in L^1(\mathbb{R}^+; B_s(\Omega)) \).

Below, we prove the following weak extension of theorem 3.2.

**Theorem 7.1.** Let \( v \) be the solution to the Euler equations (1.1), where the initial data \( v_0 \) is divergence free, tangent to the boundary, and satisfies \( \text{curl } v_0 \in B_s(\Omega) \). Furthermore, suppose \( f = 0 \). Then \( \text{curl } v \in L^\infty(0,T; B_s(\Omega)) \), and there is a constant \( C_T \) (an explicit expression can be easily obtained) such that
(7.1) \[
\| \text{curl } v(t) \|_* \leq C_T \| \text{curl } v_0 \|_*,
\]
for a.a. \( t \in (0, T) \).
A weak extension of theorem 3.4 follows immediately from theorem 7.1 together with Theorem 6.2. One has the following result.

**Theorem 7.2.** Under the assumptions of theorem 7.1 the estimate

\[ \| \nabla v \|_{L^\infty(Q_T)} \leq C_T \| \text{curl} \, v_0 \|, \]

holds almost everywhere in \( Q_T \).

To prove Theorem 7.1, we appeal to some estimates previously obtained in a more general form in reference [5]. For clarity, instead of stating these estimates in the weakest form, strictly necessary to prove the theorem 7.1 below, we rather prefer to show some more general formulations of the estimates. This allows us to present a short overview on the structure of the proof of theorem 3.8, suitable for readers interested in a deeper examination of reference [5]. In order to make an easier link with this last reference, we appeal here to the notation used in [5] (compare, for instance, (7.3) and (7.5) below with (3.3) and (4.2), respectively).

As already remarked, the velocity \( v(t) \), at each time \( t \), can be obtained from the vorticity \( \zeta(t) \equiv \text{curl} v(t) \), by setting, for each fixed \( t \), \( \theta = \zeta(t) \) in the elliptic system

\[
\begin{cases}
\text{curl} v = \theta & \text{in } \Omega, \\
\text{div} v = 0 & \text{in } \Omega, \\
\text{curl} v \cdot n = 0 & \text{on } \Gamma.
\end{cases}
\]

On the other hand, the solution to this system is given by

\[ v = \text{Rot} \psi, \]

where \( \psi \) is the solution of the elliptic problem

\[
\begin{cases}
-\Delta \psi = \theta & \text{in } \Omega, \\
\psi = 0 & \text{on } \Gamma.
\end{cases}
\]

So, at least in principle, we may obtain the velocity from the vorticity. However, since the vorticity is not a priori known, we start from a "fictitious vorticity" \( \theta(x, t) \), and look for a fixed point \( \theta = \zeta \). In the sequel we replace "fictitious vorticity" simply by "vorticity", and so on for other quantities. From each suitable "vorticity" we obtain a "velocity", by appealing to \( \text{Rot} \) and \( \text{curl} \). From this "velocity" we construct streamlines \( U(s, t, x) \), by appealing to Lagrangian coordinates. Finally, a well know technique (here dimension 2 is crucial) gives a correspondent fictitious "vorticity" \( \zeta \). So, a map \( \theta \to \zeta \) is, formally, well defined. A rigorous fixed point was obtained in reference [5] in the framework of \( C(\overline{\Omega}) \) spaces, as follows:

Fix an arbitrary positive time \( T \), an initial data \( v_0 \), and an external force \( f \). Set \( \zeta_0 \equiv \text{curl} v_0 \), \( \phi \equiv \text{curl} f \), and define (see (3.6))

\[ B = \| \zeta_0 \| + \int_0^T \| \phi(t) \| \, dt. \]

Further, define the convex, bounded, closed subset of \( C(\overline{Q_T}) \),

\[ K = \{ \theta \in C(\overline{Q_T}) : \| \theta \|_T \leq B \}. \]
From now on, the symbol \( \theta = \theta(x, t) \) denotes an arbitrary element of \( \mathbb{K} \). As already explained, the idea is to prove the existence and uniqueness of a fixed point in \( \mathbb{K} \), for a suitable map \( \Phi \), such that to this fixed point there corresponds a solution of the Euler equation (1.1) with the above given data. The map \( \Phi[\theta] = \zeta \) is defined as the following composition of single maps:

\[
(7.8) \quad \Phi: \quad \theta \rightarrow \psi \rightarrow v \rightarrow U \rightarrow \zeta.
\]

Given \( \theta = \theta(x, t) \in \mathbb{K} \) we get \( \psi = \psi(x, t) \) by solving the elliptic system (7.5), where \( t \) is treated as a parameter. The crucial estimates for \( \psi(x) \) follow from

\[
\psi(x) = \int_{\Omega} g(x, y) \, dy,
\]

where \( g \) is the Green function associated to problem (7.5). Knowing \( \psi \), the velocity \( v \) is obtained by setting \( v(x, t) = \text{Rot}\psi(x, t) \).

The next step is to get \( \zeta \), from \( v \). We introduce the streamlines \( U \) associated with the "velocity" \( v(x, t) \) obtained in the previous step. The streamlines \( U(s, t, x) \) are the solution to the system of ordinary differential equations

\[
(7.9) \quad \begin{cases}
\frac{d}{ds} U(s, t, x) = v(s, U(s, t, x)), & \text{for } s \in [0, T], \\
U(t, t, x) = x.
\end{cases}
\]

\( U(s, t, x) \) denotes the position at time \( s \) of the physical particle which occupies the position \( x \) at time \( t \). A main tool is here the following estimate (see equation (2.6) in [5]).

\[
|U(s, t, x) - U(s_1, t_1, x_1)| \leq c_1 B |s - s_1| + c_2 (1 + c_1 B) \left( |x - x_1|^{\delta} + |t - t_1|^{\delta} \right),
\]

where \( c_1 \) depends only on \( \Omega \), \( \delta \equiv e^{-c_1 B T} \), and \( c_2 = \max\{1, e R\} \), where \( R \) denotes the diameter of \( \Omega \). Knowing \( U \), we set (5, equation (2.8))

\[
(7.11) \quad \zeta(t, x) = \zeta_0(U(0, t, x)) + \int_0^t \phi(s, U(s, t, x)) \, ds
\equiv \zeta_1(t, x) + \zeta_2(t, x),
\]

where, as already remarked, \( \zeta_0 \equiv \text{curl} v_0 \), and \( \phi \equiv \text{curl} f \). The curl of the solution is here expressed separately in terms of the curls of the initial data and of the external forces. The main estimates for these two terms were proved in [5], respectively in lemmas 4.3 and 4.2. The reader may verify that the control of the external forces term is much more involved than that of the initial data term.

The composition map \( \Phi[\theta] = \zeta \) turns out to be well defined over \( \mathbb{K} \), by appealing to (7.8). In the proof of theorem 3.1 in [5], we close the above scheme by showing that \( \Phi(\mathbb{K}) \subseteq \mathbb{K} \), and that there is a (unique) fixed point in \( \mathbb{K} \). Finally, it was proved that this fixed point is the curl of the solution to the Euler equations (1.1). The velocity follows from the curl by appealing to (7.3).

After this flying visit to the proof of theorem 3.1 we prove the theorem 7.1.
Proof of theorem 7.1 A main tool in proving the regularity theorem 3.2 for data in $C^\infty(\Omega)$ is the lemma 3.5 (see the Lemma 4.1 in [3]). The absence of external forces $f$ lead us to revive below, directly, the simple idea used in the proof of this lemma, without appealing to the original statement itself.

We deal with solutions whose existence is already guaranteed by theorem 3.1. We merely want to show the additional regularity claimed in theorem 7.1. Since in this theorem the external forces vanish, the following very simplified form of (7.10) holds.

\[(7.12) \quad |U(0, t, x) - U(0, t, y)| \leq K|x - y|^\delta,\]

where here $\delta = e^{-c_1 BT}$, and $B = \|\zeta_0\|$. Following (7.11), and taking into account that $\zeta_2$ vanishes, one has $\zeta = \zeta_1$. So the curl of the solution $v$ to the Euler equation (1.1) is simply given by

$\zeta(t, x) = \zeta_0(U(0, t, x)).$

It follows that

\[(7.13) \quad \omega_\zeta(t; x; r) = \sup_{y \in \overline{\Pi}(x; r)} |\zeta(t, x) - \zeta_1(t, y)| = \sup_{y \in \overline{\Pi}(x; r)} |\zeta_0(U(0, t, x)) - \zeta_0(U(0, t, y))|.
\]

Further, by appealing to (7.12), one gets

\[(7.14) \quad \omega_\zeta(t; x; r) \leq \omega_{\zeta_0}(U(0, t, x); K r^\delta).
\]

So, by recalling definition (5.1), one has

\[(7.15) \quad \langle \zeta(t) \rangle_* \equiv \sup_{x \in \overline{\Pi}} \int_0^\rho \omega_\zeta(t; x; r) \frac{dr}{r} \leq \sup_{x \in \overline{\Pi}} \int_0^\rho \omega_{\zeta_0}(U(0, t, x); K r^\delta) \frac{dr}{r}.
\]

Since

\[
\{U(0, t, x) : x \in \overline{\Pi}\} = \overline{\Pi},
\]

it follows, by appealing to the change of variables $\tau = K r^\delta$, that

\[(7.16) \quad \langle \zeta(t) \rangle_* \leq \sup_{x \in \overline{\Pi}} \int_0^\rho \omega_{\zeta_0}(\overline{\Pi}; K r^\delta) \frac{dr}{r} = \frac{1}{\delta} \sup_{x \in \overline{\Pi}} \int_0^K \rho^{\delta} \omega_{\zeta_0}(x; \tau) \frac{d\tau}{\tau} \equiv \frac{1}{\delta} \langle \zeta_0 \rangle_* K \rho^{\delta},
\]

with obvious notation. On the other hand, $\|\zeta(t)\| = \|\zeta_0\|$, for all $t$. Since (2.4) also applies for $B_*$ semi-norms, one shows that

\[
\|\zeta(t)\|_* \leq C_T \|\zeta_0\|_*,
\]

for all $t \in [0, T]$. Theorem 7.1 is proved. Theorem 7.2 follows by appealing to Theorem 6.2.

It would be interesting to prove theorem 7.1 in the presence of external forces, even in a simplified version, for instance, $\text{curl} f \in C(\mathbb{R}^+; B_*(\overline{\Pi}))$. We believe that a (possibly modified) version of this result holds by appealing to the measure preserving properties of the streamlines, together with the control of the linear dimensions of figures in finite time.
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