Performance Analysis of LMS Filters for Estimation of Cyclostationary Signals

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Abstract

The least mean-square (LMS) filter is one of the most common adaptive linear estimation algorithms. In many practical scenarios, and particularly in digital communications systems, the signal of interest (SOI) and the input signal are jointly wide-sense cyclostationary. Previous works analyzing the performance of LMS filters for this important case assume specific probability distributions of the considered signals or specific models that relate the input signal and the SOI. In this work, we provide a general transient and steady-state performance analysis that is free of specific distributional or model assumptions. We obtain conditions for convergence and derive analytical expressions for the non-asymptotic and steady-state mean-squared error. The accuracy of our analysis is demonstrated in simulation studies that correspond to practical communications scenarios.

I. Introduction

The least mean-squares (LMS) is a widely used algorithm for adaptive linear estimation of a signal of interest (SOI) based on an input signal. The LMS algorithm is a stochastic approximation of the iterative steepest descent based implementation of the Wiener filter, when the SOI and the input signal are jointly wide-sense stationary (JWSS) [1]–[3]. This stochastic approximation involves a simple update equation which can be implemented in practical systems with low computational complexity [1, Ch. 9], [2, Ch. 10]. As the LMS is designed for JWSS signals, many works have been devoted to analyze its performance in this setup, see, e.g., [3]–[7], and also [1, Ch. 9], [2, Ch. 24]. Nonetheless, man-made signals, and specifically digitally modulated communications signals, are typically wide-sense cyclostationary (WSCS) [8, Sec. 5-7], [9, Ch. 1], [10]. Thus, in many practical communications systems, the considered signals are...
jointly WSCS (JWSCS) [8, Sec. 3.6.2] rather than JWSS. Examples include interference-limited communications [11], [12] and cognitive radio [13]. Another important example is narrowband (NB) power line communications (PLC) systems, where the channel input is a digitally modulated WSCS signal [14, Ch. 5], the channel transfer function is periodically time-varying [15], and the additive channel noise is commonly modeled as a WSCS process that is mutually independent of the channel input [14, Ch. 2], [16]. Hence, in this case, the channel input and the noisy channel output are JWSCS.

Despite the importance of the WSCS scenario, only a few works have studied the performance of adaptive algorithms in the presence of JWCS input signal and SOI. In [17] it was shown that the LMS filter coefficients are mean convergent only when the step-size approaches zero. In this case, the filter coefficients converge to the minimal time-averaged (TA) mean-squared error (MSE) filter. The performance of the LMS when applied to the adaptation of frequency shift (FRESH) filters with WSCS inputs was studied in [18]–[20]. Specifically, [18] focused on interference rejection in the presence of cyclostationary digitally modulated signals; The work [19] proposed a scheme for blind adaptation of FRESH filters using the LMS and the recursive least-squares algorithms; The work [20] studied the effect of errors in the frequency shifts on the performance of LMS-based adaptive FRESH filters with a temporally independent input signal. In [21], the LMS performance was studied for the identification of a linear system whose coefficients obey a random walk model with a WSCS Gaussian input and an additive wide-sense stationary (WSS) Gaussian noise. We note that when the random walk effect is negligible, the linear system considered in [21] becomes linear time-invariant (LTI). The work [22] analyzed the LMS performance when applied to adaptive line enhancement/cancellation for a WSCS input consisting of a Gaussian process with periodic variance plus a sine wave with random phase. Excluding [17], which studied mean convergence only, all the works mentioned above assume specific signal distributions or specific models that relate the input and SOI.

In the works [23], [24] we developed a different adaptive filter for JWSCS signals based on the TA-MSE criterion and analyzed its performance. As the adaptive filter in [23], [24] specializes the LMS only when the signals are JWSS, the performance study conducted in [23], [24] cannot be used to characterize the LMS behavior when the signals are JWSCS. In this context, it is
important to note that the empirical performance of the LMS presented in [24, Sec. V], in which some of the scenarios considered correspond to practical communications scenarios, cannot be predicted by any existing analytical LMS performance study. This is mainly due to the fact that in [24, Sec. V], non-Gaussian signals and periodically time-varying channels (as encountered, e.g., in practical NB-PLC systems) were considered, that do not satisfy the specific distributional and model assumptions made in existing LMS performance analysis tools. The lack of a reliable characterization of the behavior of LMS filters in such practical communications setups further motivates the analysis of the LMS performance for general JWSCS signals.

**Main Contributions:** In this work, we provide a new general performance analysis of the LMS algorithm in the presence of JWSCS input and SOI. We obtain a closed-form expression for the transient MSE, derive conditions for convergence, including sufficient conditions on the LMS step-size, and characterize the steady-state MSE. Unlike [18]–[22], we do not impose a specific distribution or model on the considered signals. We only make the basic assumptions that are commonly used in the analysis of adaptive algorithms. Specifically, the SOI is related to the input signal only through a general stochastic representation which arises from the linear minimum MSE (LMMSE) filter, that represents a linear periodically time-varying (LPTV) system when the SOI and the input signal are JWSCS [25, Ch. 17.5.1]. Unlike [17], we also analyze mean-square convergence and provide analytic expressions for the non-asymptotic and steady-state MSE. Furthermore, unlike [17], in the paper, convergence of the LMS filter is characterized accurately, without small step-size approximations. This is carried out by introducing a generalized definition for convergence using the theory of asymptotically periodic sequences [26].

We verify our analysis in simulation examples. In particular, we demonstrate the accuracy of our analysis in scenarios for which the LMS performance was not yet characterized, including LPTV system identification and practical NB-PLC signal recovery scenarios. Our simulation results show an excellent agreement between the theoretical and empirical performance measures. Furthermore, we demonstrate that our analysis generalizes the results of [21] when applied to LTI system identification in the presence of non-Gaussian signals.

The paper is organized as follows: Section II states the considered problem and presents the general signal model assumed, Section III details the transient and steady state performance
analysis, and Section IV presents simulation examples. Lastly, Section V provides concluding remarks. Complete proofs for the lemmas and theorems stated throughout the paper are provided in the Appendix.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Notations

We denote column vectors with lower-case boldface letters, e.g., $\mathbf{x}$. Matrices are denoted with upper-case boldface letter, e.g., $\mathbf{X}$. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and $\mathbf{0}_{nm}$ denotes the all-zero $n \times m$ matrix. Hermitian transpose, transpose, complex conjugate, and stochastic expectation are denoted by $(\cdot)^H$, $(\cdot)^T$, $(\cdot)^*$, and $\mathbb{E}\{\cdot\}$, respectively. The real part of $x$ is denoted by $\text{Re}\{x\}$, $((n))_m$ denotes the remainder of $n$ when divided by $m$, and $\otimes$ denotes the Kronecker product. The set of non-negative integers is denoted by $\mathbb{N}$. For an $n \times n$ matrix $\mathbf{X}$, $\lambda_{\text{max}}(\mathbf{X})$ denotes the largest real eigenvalue of $\mathbf{X}$, given that such exists, $\rho(\mathbf{X})$ denotes the spectral radius of $\mathbf{X}$, and $\mathbf{x} = \text{vec}(\mathbf{X})$ denotes the $n^2 \times 1$ column vector obtained by stacking the columns of $\mathbf{X}$, which is recovered from $\mathbf{x}$ via $\mathbf{X} = \text{vec}^{-1}(\mathbf{x})$. For an $n \times 1$ vector $\mathbf{y}$ and an $n^2 \times 1$ vector $\mathbf{x}$, $\|\mathbf{y}\|^2 \triangleq \mathbf{y}^H \mathbf{y}$ denotes the squared Euclidean norm and $\|\mathbf{y}\|_x^2 \triangleq \mathbf{y}^H \text{vec}^{-1}(\mathbf{x}) \mathbf{y}$ denotes its weighted version, when $\text{vec}^{-1}(\mathbf{x})$ is positive-definite. Lastly, for a set of $n \times n$ matrices $\{\mathbf{X}_k\}$ and integers $l, m$, $\prod_{k=l}^m \mathbf{X}_k$ is the product $\mathbf{X}_m \mathbf{X}_{m-1} \cdots \mathbf{X}_l$ when $m \geq l$ and $\mathbf{I}_n$ when $m < l$.

B. Wide-Sense Cyclostationary Stochastic Processes

A discrete-time proper-complex (PC) multivariate process $\mathbf{x}[n]$ is said to be WSCS if both its mean $\mathbb{E}\{\mathbf{x}[n]\}$ and autocorrelation function $\mathbb{E}\{\mathbf{x}[n+l]\mathbf{x}^H[n]\}$ are periodic with some period, $N_0$, w.r.t. $n$ [8, Sec. 3.5]. A pair of jointly proper-complex (JPC) processes $\mathbf{x}_1[n], \mathbf{x}_2[n]$ are said to be JWSCS with period $N_0$ if each process is WSCS with period $N_0$ and their cross-correlation function $\mathbb{E}\{\mathbf{x}_1[n+l]\mathbf{x}_2^H[n]\}$ is periodic with period $N_0$ w.r.t. $n$ [8, Sec. 3.6.2]. Note that when $\mathbf{x}_1[n], \mathbf{x}_2[n]$ are WSCS with different periods, say $N_1$ and $N_2$, and their cross-correlation function is periodic with period $N_{1,2}$, then they are also JWSCS with a period which equals the least common multiple of $N_1$, $N_2$ and $N_{1,2}$.
C. Problem Formulation

We wish to characterize the performance of the LMS filter for linear estimation of a scalar SOI \( d[n] \) based on an \( M \times 1 \) multivariate input signal \( x[n] \), where \( x[n] \) and \( d[n] \) are zero-mean, JPC, and JWSCS with period \( N_0 \). Let \( h[n] \) denote the \( M \times 1 \) random coefficients vector of the LMS filter at time instance \( n \). For a step-size \( \mu \) and initial guess \( h[0] \), the LMS update equation is given by \[ 1 \), Eq. 9.5] \n\]

\[
\begin{align*}
    h[n+1] &= h[n] + \mu \cdot x[n] \left( d[n] - \hat{d}[n] \right)^*, \quad n \geq 0, \\
\end{align*}
\]

where \( \hat{d}[n] \triangleq h_H[n]x[n] \) is the linear estimate of the SOI at time instance \( n \).

In order to characterize the performance of the LMS algorithm, we formulate the relationship between \( d[n] \) and \( x[n] \) using the LMMSE estimator. To that aim, let \( h_M[n] \) denote the deterministic coefficients vector of the LMMSE estimator of \( d[n] \) based on \( x[n] \). Clearly, \( h_M[n] \) satisfies the Wiener-Hopf equations \[ 2 \), Ch. 3.5] at each time instance \( n \). Thus, letting \( v[n] \) be the estimation error of the LMMSE filter, the SOI \( d[n] \) can be written as

\[
d[n] = h_M^H[n]x[n] + v[n].
\]

By the orthogonality principle \[ 2 \), Ch. 4.2], the estimation error of the LMMSE filter is orthogonal to the input signal, i.e., \( E\{x[n]v^*[n]\} = 0_{M \times 1} \). Note that the relationship between \( d[n] \) and \( x[n] \) in (2) is a general stochastic representation of the SOI that does not arise from a specific model.

The fact that the SOI \( d[n] \) and the input signal \( x[n] \) are zero-mean, JPC, and JWSCS, results in the following properties of the LMMSE filter coefficients vector \( h_M[n] \) and the corresponding estimation error \( v[n] \):

1) Since \( d[n] \) and \( x[n] \) are JWSCS with period \( N_0 \), it follows from \[ 25 \), Ch. 17.5.1] that \( h_M[n] \) defines an \( M \times 1 \) periodic sequence, i.e., the LMMSE filter represents an LPTV system.

2) As \( d[n] \) and \( x[n] \) are also JPC and zero-mean, it follows from (2) that \( v[n] \) is a PC, zero-mean, WSCS process, whose variance \( \sigma_v^2[n] \triangleq E\{|v[n]|^2\} \) is periodic with period \( N_0 \).

Similarly to the standard approach used for analyzing the LMS algorithm for JWSS signals, e.g., \[ 1 \)–\[ 5 \], we make the following assumptions on the signals in (2):
The estimation error of the LMMSE estimator, \( v[n_1] \), and the input signal, \( x[n_2] \), are mutually independent \( \forall n_1, n_2 \), see also [2, Ch. 15.2], [4, Sec. B.2]. This assumption is satisfied, e.g., when \( d[n] = g^H N_0[n]x[n] + z[n] \) where \( g N_0[n] \) is a deterministic LPTV filter and \( z[n] \) is a PC WSCS process independent of \( x[n] \). In this case the LMMSE filter is \( h_M[n] = g N_0[n] \) and its estimation error is \( v[n] = z[n] \). This assumption also holds when \( d[n] \) and \( x[n] \) are jointly Gaussian and temporally uncorrelated.

The random coefficients vector \( h[n] \) in (1) is independent of the instantaneous input \( x[n] \), see [1, Pg. 392], [2, Ch. 16.4]. This is satisfied when, e.g., \( x[n] \) is a temporally independent process.

The input \( x[n] \) is fourth-order cyclostationary\(^1\), i.e., its fourth-order moments are bounded and periodic with period \( N_0 \). This is satisfied when, e.g., \( x[n] \) is Gaussian and WSCS.

Assumptions AS1-AS3 are utilized in the following section to obtain explicit convergence conditions and to derive closed-form expressions for the non-asymptotic and steady-state MSE. In Section IV, we show that the analysis carried out under these assumptions provides a reliable characterization of the LMS performance in practical communications scenarios, where AS1-AS3 do not necessarily hold.

### III. LMS Performance Analysis

To facilitate the analysis, we first define several quantities that are summarized in Table I. We then derive an expression for the non-asymptotic time-evolution of the MSE, which is utilized to derive conditions for convergence. Under these conditions, we obtain an analytic expression for the steady-state MSE.

#### A. Time-Evolution of the MSE

In order to analyze the MSE performance of the LMS filter, we first define its instantaneous estimation error:

\[
e[n] \triangleq d[n] - h^H[n]x[n].
\]  

\(^1\)A similar assumption was made in the analysis of LMS with non-Gaussian WSS inputs [2, Ch. 24], where it was assumed that the fourth-order moments are time-invariant [2, Eq. (24.9)] and bounded [2, Pg. 361].
The instantaneous MSE at time index $n$ is given by $E\{|e[n]|^2\}$. To characterize the MSE time-evolution, we first obtain recursive relations for the first and (weighted) second-order statistical moments of the coefficients error vector, which, similarly to [20, Eq. (24)], is defined as:

$$\bar{h}[n] \triangleq h_{TA} - h[n],$$  \hspace{1cm} (5)

where $h_{TA}$ is the $M \times 1$ coefficients vector of the linear minimal TA-MSE (LMTA-MSE) estimator obtained from the time-averaged Wiener-Hopf equations [20, Eq. (5)]. A recursive relation for the expected coefficients error vector is given in the following lemma:

**Lemma 1** (Mean relation). *The expected coefficients error vector (5) satisfies the following recursive relation for $n \geq 0$*
\[
E\{\tilde{h}[n+1]\} = \tilde{R}_x[n]E\{\tilde{h}[n]\} - \mu \cdot C_x[n]g[n],
\]

where \(g[n], C_x[n]\), and \(\tilde{R}_x[n]\) are defined in (3b), (3c), and (3e), respectively.

[A proof is given in Appendix A]

The mean-square deviation (MSD) in filter coefficients is defined as the stochastic expectation \(E\{\|\tilde{h}[n]\|^2\}\), where \(\tilde{h}[n]\) is the coefficients error vector defined in (5). Furthermore, we define the weighted MSD as \(E\{\|\tilde{h}[n]\|^2_q\}\), where \(q\) is some \(M \times 1\) vector, such that \(Q \triangleq \text{vec}^{-1}\{q\}\) is a Hermitian positive semi-definite matrix. A recursive relation for the weighted MSD is stated in the following lemma:

**Lemma 2** (Variance relation). The weighted MSD satisfies the following recursion for \(n \geq 0\):

\[
E\{\|\tilde{h}[n+1]\|^2_q\} = E\{\|\tilde{h}[n]\|^2_{F[n]q}\} + \mu^2\|g[n]\|^2_{B[n]q} - 2\mu \cdot (g^T[n] \otimes E\{\tilde{h}[n]\}) P[n]q + \mu^2 \sigma_v^2[n] c^T_x[n]q,
\]

where \(g[n], c_x[n], F[n], B[n],\) and \(P[n]\) are defined in (3b), (3d), (3f), (3h), and (3j), respectively.

[A proof is given in Appendix B]

Notice that for a period \(N_0 = 1\), the input signal and the SOI are JWSS. In this case, the LMMSE filter coincides with the LMTA-MSE filter, introduced below (5), and therefore, by Eq. (3b), the deviation vector \(g[n]\) satisfies \(g[n] \equiv 0_{M \times 1}\). In this case, as expected, Lemma 1 coincides with the LMS mean relation for JWSS signals [2, Eq. (24.2)], and Lemma 2 coincides with the LMS variance relation for JWSS signals [2, Eq. (24.11)]. Another special case of the variance relation (7) is obtained under the following scenario: Consider the problem of identifying an LTI system with WSCS input \(x[n]\) whose output is corrupted by additive WSCS noise \(v[n]\), uncorrelated with the input. Under this scenario, one can verify that the LMMSE filter is time invariant, and therefore, the deviation vector \(g[n]\) satisfies \(g[n] \equiv 0_{M \times 1}\). In this case, if both \(x[n]\) and \(v[n]\) are Gaussian, and \(v[n]\) is WSS (recall that WSCS processes specialize WSS processes), then, by setting \(q = \text{vec}^{-1}(I_M)\), one can verify that (7) specializes the recursive characterization.
of the MSD in [21, Eq. (15)]. We note that this specialization holds only when the random walk effect in [21] is neglected, i.e., the linear system considered in [21] becomes LTI\(^2\).

In the following theorem, the recursive relations in Lemmas 1–2 are used to obtain an explicit characterization of the non-asymptotic MSE of LMS filters with JWSCS input and SOI:

**Theorem 1 (MSE time-evolution).** The instantaneous MSE of the LMS algorithm (1) satisfies

\[
E\{|e[n]|^2\} = E\left\{||\tilde{h}[n]||_{c_x[n]}^2\right\} + 2\text{Re}\left\{g^H[n]C_x[n]E\{h[n]\}\right\} + ||g[n]||_{c_x[n]}^2 + \sigma_v^2[n], \quad \forall n \geq 0, \tag{8}
\]

where the quantities \(g[n], C_x[n],\) and \(c_x[n]\) are defined in (3b)–(3d), respectively.

[A proof is given in Appendix C]

Note that the expectations \(E\{\tilde{h}[n]\}\) and \(E\left\{||\tilde{h}[n]||_{c_x[n]}^2\right\}\) in (8) can be recursively computed using (6) and (7), respectively. Also notice that when \(g[n] \equiv 0_{M \times 1}\), i.e., the LMMSE filter is LTI, the MSE time-evolution (8) specializes the corresponding result for the LMS with JWSS signals in [2, Pg. 363].

**B. Convergence Analysis and Steady-State MSE**

Here, we derive the conditions for convergence of the LMS filter with JWCS signals, and characterize its steady-state performance. We begin by stating the definition for asymptotically periodic sequence, which is equivalent to the one in [26, Def. 3.1].

**Definition 1 (Asymptotically periodic sequence).** A sequence \(p[n]\) is said to asymptotically periodic with period \(N_0\) if for every \(k \in \{0, 1, \ldots, N_0 - 1\} \equiv N_0\), the subsequence \(p_k[n] \equiv p[n \cdot N_0 + k]\) converges as \(n \to \infty\).

**Comment 1.** Note that when the \(N_0\) subsequences all converge to the same limit, the definition of asymptotically periodic sequences specializes the definition of convergent sequences.

\(^2\)While we consider the general setup of adaptive estimation of JWCS signals, for which the LMMSE filter coefficients vary periodically in time, the work [21] studied the identification of a linear system and modeled the temporal variations in the system coefficients via a non-stationary random walk process. As a result, the variance relation in [21, Eq. (15)] includes an additive variance component due to these random variations. Consequently, for LTI system identification, Eq. (7) specializes [21, Eq. (15)] when this variance term is neglected, as carried out in part of the analysis conducted in [21].
Comment 2. It follows from Def. 1 that if $p[n]$ is asymptotically periodic with period $N_0$, then $p_k[n] = p[n \cdot N_0 + k]$ converges for every finite $k \in \mathbb{N}$.

Based on Def. 1, we consider the following definitions for convergence:

**Definition 2** (Convergence in the mean). An adaptive filter with coefficients error vector $\bar{h}[n]$ is said to be **mean convergent** if $\mathbb{E}\{\bar{h}[n]\}$ is asymptotically periodic.

**Definition 3** (Mean-square stability). An adaptive filter with coefficients error vector $\bar{h}[n]$ is said to be **mean-square stable** if $\mathbb{E}\{||\bar{h}[n]||^2\}$ is asymptotically periodic.

Defs. 2–3 generalize the traditional definitions for mean convergence and mean-square stability for JWSS SOI and input signal [2, Ch. 23.2, 23.4]. These traditional definitions require the mean coefficients error $\mathbb{E}\{\bar{h}[n]\}$ to converge to $0_{M \times 1}$ and the MSD $\mathbb{E}\{||\bar{h}[n]||^2\}$ to be convergent. However, as was shown in [17, Cmt. 2], when no specific model relating the JWSCS SOI $d[n]$ and input signal $x[n]$ is assumed, then $\lim_{n \to \infty} \mathbb{E}\{\bar{h}[n]\} = 0_{M \times 1}$ only when the step-size $\mu \to 0$. Consequently, to be able to specify a non-infinitesimal step-size region which guarantees that the algorithm does not diverge, we useDefs. 2–3.

Using Lemma 1, we obtain the following necessary and sufficient condition for the LMS to be mean convergent:

**Proposition 1** (Necessary and sufficient condition for mean convergence). The LMS algorithm is mean convergent if and only if $L_{k,k}^R$, defined in (3k), satisfies

$$
\rho \left( L_{k,k}^R \right) < 1, \quad \forall k \in \mathcal{N}_0. 
$$

[Ap roof is given in Appendix D]

**Comment 3.** For a given $\mu > 0$, it is shown in Appendix D that $\lim_{n \to \infty} \mathbb{E}\{\bar{h}[n]\} = 0_{M \times 1}$, namely, the LMS satisfies the traditional mean-convergence definition used in [17], if and only if, in addition to (9), the vector $s_k$ defined in (3m) satisfies $s_k = 0_{M \times 1}, \forall k \in \mathcal{N}_0$. According to (3l) and (3m), this occurs, e.g., when the deviation vector $g[n]$ defined in (3b) satisfies $g[n] = 0_{M \times 1}$, i.e., the LMMSE filter is LTI, as in the case of JWSS signals. However, since the LMMSE filter for
JWCS signals is LPTV [25, Ch. 17.5.1], $g[n]$ is generally non-zero. Consequently, as noted in [17, Cmt. 2], the LMS filter with JWCS signals generally does not satisfy traditional definition for mean convergence for any fixed step-size $\mu > 0$.

By (3k), one can verify that (9) is satisfied if (but not only if) the spectral norm of $\tilde{R}_x[l]$ satisfies $\rho(\tilde{R}_x[l]) < 1$, $\forall l \in N_0$. Hence, we obtain the following sufficient condition on the step-size $\mu$ which guarantees mean convergence:

**Corollary 1** (Sufficient condition for mean convergence). The LMS algorithm is mean convergent if the step-size satisfies

$$0 < \mu < \frac{2}{\lambda_{\max}(C_x[k])}, \quad \forall k \in N_0. \tag{10}$$

[A proof is given in Appendix E]

Note that in the WSS case, $N_0 = 1$, Eq. (9) can be written as $\rho(\tilde{R}_x[0]) < 1$, thus (10) becomes also a necessary condition for mean convergence, and coincides with the standard mean convergence condition for WSS signals in [2, Ch. 24.2].

Next, based on the recursive relations in Lemmas 1-2, we obtain the following sufficient condition for mean-square stability:

**Theorem 2** (Mean-square stability). The LMS algorithm is both mean convergent and mean-square stable if the following conditions are satisfied:

1) Condition (10) is satisfied.

2) The covariance matrix $C_x[k]$ is non-singular and its entries are bounded $\forall k \in N_0$.

3) The step-size $\mu$ satisfies\(^3\) for all $k \in N_0$:

$$\mu < \min\left\{ \frac{1}{\lambda_{\max}(A^{-1}[k]B[k])}, \frac{1}{\lambda_{\max}(H[k])} \right\}, \tag{11}$$

where $A[k]$, $B[k]$, and $H[k]$ are defined in (3g)–(3i), respectively.

[A proof is given in Appendix F]

\(^3\)If $H[k]$ does not have any real positive eigenvalues, then condition (11) is replaced with $\mu < \frac{1}{\lambda_{\max}(A^{-1}[k]B[k])}$. 
Note that since the covariance matrix $C_x[k]$ is required to be non-singular, then $A[k]$ defined (3g) is non-singular, thus (11) is well-defined. We also note that when the signals are JWSS, i.e., $N_0 = 1$, then Thm. 2 specializes the sufficient condition for stability of LMS with JWSS non-Gaussian signals in [2, Eq. (24.24)].

Lastly, as stated in the following theorem, when the conditions in Thm. 2 are satisfied, and the number of iterations $n$ approaches infinity, the transient MSE (8) converges to a periodic sequence that characterizes the steady-state MSE and takes the following form:

**Theorem 3 (Steady-state MSE).** If the conditions stated in Thm. 2 are satisfied, then the transient MSE (8) is an asymptotically periodic sequence such that $\xi[k] \overset{\Delta}{=} \lim_{n \to \infty} E \{|e[n \cdot N_0 + k]|^2\}$ takes the form:

$$
\xi[k] = \mu^2 \sum_{l=1}^{N_0+k-1} z_{1(l))N_0}^T L_{l,1,k}^F \left(I_{M^2} - L_{k,k}^F\right)^{-1} c_x[k]
- 2\mu \text{Re}\{g^H[k]C_x[k]s_k\} + \|g[k]\|^2_{c_x[k]} + \sigma_v^2[k],
$$

(12)

where $g[n]$, $C_x[n]$, $c_x[n]$, $s_k$, $L_{k,l}^F$, and $z_k$ are defined in (3b)–(3d) and (3m)–(3o), respectively. [A proof is given in Appendix G]

Notice that for $N_0 = 1$, i.e., $x[n]$ and $d[n]$ are JWSS, then, as noted in the discussion following Lemma 2, $g[n] \equiv 0_{M \times 1}$. In this case, the second and third summands of (12) vanish, and the first summand reduces to the excess steady-state MSE of the LMS with JWSS signals in [2, Thm. 24.1]. Thus, Thm. 3 specializes the steady-state MSE characterization in [2, Thm. 24.1] when the SOI and input are JWSS.

**C. Discussion**

First, we note that, as discussed in the previous subsection, when the signals are JWSS, the generalized performance analysis presented in this paper coincides with the standard performance analysis of the LMS with JWSS signals in [2, Ch. 24]. We further note that when the signals are not JWSS, then the vector $g[n]$ (3b), which represents the deviation of the LMMSE filter from an LTI system, and essentially, the periodic dynamic of the signals (and is thus zero for the JWSS setup), has a dominant effect on the LMS behavior. This is observed, e.g., in the temporal
statistical moments of the coefficients error characterized by the recursive relations in (6) and (7), and also in the instantaneous MSE in (8). Consequently, the presented performance analysis quantifies the effect of these periodic dynamic on the performance of the LMS filter, compared to the JWSS setup. Finally, note that the deviation vector $g[n]$ does not appear in [21]. This is due to the fact that for the specific system model and the additive noise considered in [21], it can be shown that the LMMSE estimator of the SOI is LTI, and thus $g[n]$ is the zero vector.

IV. NUMERICAL EXAMPLES

Here, we demonstrate the theoretical analysis in a simulation study. We focus on two scenarios:

1) A scenario which satisfies AS1-AS3, whose purpose is to verify the theoretical analysis and to numerically compare our analysis with the state-of-the-art.

2) An NB-PLC signal recovery scenario, which demonstrates the accuracy of the analysis in a practical scenario when AS1-AS3 are not necessarily satisfied.

All empirical performance measures were obtained via 10000 Monte Carlo simulations.

A. Scenario 1: Theoretical Analysis Verification

We begin with an example in which AS1-AS3 are satisfied. To that aim, we construct JWSSC input signal $x[n]$ and SOI $d[n]$, under the following settings: Let $N_u$, $N_x$, $N_h$, and $N_v$ be positive integers, and let $u[n]$ be an $M \times 1$ i.i.d. random process with $M = 8$ i.i.d. entries that are uniformly distributed over the unit circle. The input signal $x[n]$ is a zero-mean WSCS given by

$$x[n] \triangleq \frac{1}{\sqrt{N_u}} \left( 1 + 0.5 \cos \left( \frac{2\pi n}{N_x} \right) \right) \sum_{k=0}^{N_u-1} u[N_u \cdot n + k].$$

We set the $M \times 1$ LMMSE coefficients vector $h_M[n]$ in (2) such that

$$(h_M[n])_k \triangleq \left( 1 + \frac{1}{5 \cdot N_h} \cdot ((n))_{N_h} \right) e^{-0.5 |k|}.$$

Furthermore, we set the estimation error process $v[n]$ in (2) to be a zero-mean temporally uncorrelated Gaussian process independent of $x[n]$, with variance

$$\sigma_v^2[n] \triangleq 10^{-6} \cdot \left( 1 + \frac{0.1}{((n))_{N_u} + 1} \right)^2.$$
The SOI $d[n]$ is related to $x[n]$ via (2). Note that $d[n]$ and $x[n]$ are JWSCS with period $N_0$, which is the least common multiple of $N_x$, $N_h$, and $N_v$, and that this setup satisfies $AS1$-$AS3$, for any selection of $N_u$, $N_x$, $N_h$, and $N_v$.

In order to verify the theoretical analysis in Thms. 1-3, we first fix $N_u = 8$, $N_x = 40$, $N_h = 10$, and $N_v = 5$. Note that under this setting, the input signal $x[n]$ is non-Gaussian, the LMMSE filter $h_M[n]$ is an LPTV system, and the estimation error $v[n]$ is a WSCS process. Fig. 1 depicts the theoretical MSE for this setting, computed via (8), as compared to the empirical MSE, for step sizes $\mu = \{0.01, 0.04\}$. Fig. 2 depicts the theoretical steady-state TA-MSE for this setting (recall that the MSE converges to a periodic sequence) computed by time-averaging (12) over the period $N_0$, the stability threshold, computed via (11), and the empirical steady-state TA-MSE. It is illustrated in Fig. 1 that, when $AS1$-$AS3$ are satisfied, the time-evolution of the MSE is accurately characterized by the theoretical analysis in Thm. 1. An excellent agreement between the empirical and theoretical performance measures is also observed in Fig. 2. One sees that, indeed, the conditions in Thm. 2 guarantee mean-square stability when the step-size is not larger than the threshold. Furthermore, one can notice that (12) accurately characterizes the steady-state performance of the LMS filter after convergence is obtained.

Next, we compare our analysis to the one provided in [21], for the problem of LTI system...
identification. We note that in [21], the system input is assumed to be a Gaussian WSCS process and the additive noise in the system output is assumed to be a WSS Gaussian process. Note that for \(N_h = N_v = 1\), the SOI \(d[n]\) and the input signal \(x[n]\) defined above are related via an LTI filter whose output is corrupted by additive WSS Gaussian noise. Also note that by the multivariate central limit theorem [27, Pg. 912], as \(N_u\) increases, the input vector \(x[n]\) converges in distribution to a multivariate Gaussian random vector. Thus, for sufficiently large \(N_u\), the LMS performance for this setting can be characterized by [21, Eq. (15)]\(^4\). As [21] focused on characterizing the MSD in the LMS coefficients, \(E\left\{\|\hat{h}[n]\|_2^2\right\}\), Figs. 3–4 depict the empirical filter coefficients MSD, compared to the theoretical MSD, evaluated using Lemma 2 with \(q = \text{vec}^{-1}(I_M)\) and using [21, Eq. (15)], for step-size \(\mu = 0.1\), \(N_x = 40\), and \(N_u = \{64, 1\}\). In Fig. 3 we observe that for \(N_u = 64\), i.e., when the distribution of the input signal is approximately Gaussian, then, as expected, both our analysis and [21, Eq. (15)] accurately characterize the filter coefficients MSD. However, in Fig. 4 we observe that for \(N_u = 1\), [21, Eq. (15)] does not provide an accurate characterization of the empirical performance, as the distribution of the input signal is not Gaussian. On the other hand, an excellent agreement between the empirical MSD and

\(^4\)As noted in footnote 2, [21] modeled temporal variations in the coefficients of a linear system using as a random walk process. Thus, [21, Eq. (15)] includes an additive variance component due to these variations. Consequently, [21, Eq. (15)] characterizes the LMS performance for LTI system identification when this variance term is neglected, as performed in part of the analysis in [21].
the theoretical MSD computed using Lemma 2 is observed, as unlike [21], our analysis is not limited to a specific distribution of the input signal.

In summary, the agreement between the empirical and theoretical performance measures observed in this numerical study illustrate the validity of the theoretical study when ASI-AS3 are satisfied. Furthermore, the comparison with [21] for LTI system identification demonstrates the benefit of our general analysis compared to previous works.

B. Scenario 2: NB-PLC Signal Recovery

Next, we evaluate the theoretical analysis in Thms. 1-3 in a practical scenario which corresponds to signal recovery in NB-PLC channels. The SOI $d[n]$ is the input signal to an NB-PLC channel. Here, $d[n]$ is an orthogonal frequency division multiplexing signal with 36 subcarriers, each modulated via a QPSK constellation, with 12 cyclic prefix samples. It follows from [10] that $d[n]$ is WSCS with period $N_0 = 48$. The channel output, $r[n]$, is given by

$$r[n] = \sum_{l=0}^{\infty} g[n, l] d[n - l] + w[n]$$

[28, Sec. III], where $g[n, l]$ is an LPTV filter with period $N_0$, generated as in [28] following the IEEE P1901.2 standard [29]. The additive channel noise $w[n]$ (note that this is not the estimation error $v[n]$ in (2)) is a WSCS Gaussian process with period\(^5\) $N_0$, generated using the model [30], with a set of parameters taken from [30, Tbl. 2]. This scenario is illustrated in Fig 5. The input to the LMS filter, $x[n]$, is obtained via

---

\(^5\)In NB-PLC channels, the periods of the WSCS information signal, LPTV channel transfer function, and WSCS noise, are not necessarily the same [15]. However, as these periods are typically commensurate [29], the statistical moments of the considered signals and the LPTV channel can be treated as if they have the same period, which equals the least common multiple of the different periods.
multivariate preprocessing of \( r[n] \) that produces \( (x[n])_k = r[n-k], \ k \in \{0, 1, \ldots, M - 1\} \), with \( M = 8 \). The LMMSE filter \( h_M[n] \) in (2) is obtained from the orthogonality principle, i.e., \( E\{x[n]d^*[n]\} = E\{x[n]x^H[n]\}h_M[n] \), and \( v[n] \) is obtained as the estimation error for \( h_M[n] \). We note that the entire set of assumptions \textit{AS1-AS3} is not satisfied in this scenario. The signal-to-noise ratio, defined as \( \frac{\sum_{k=0}^{N-1} E\{|r[n-k]-w[n-k]|^2\}}{\sum_{k=0}^{N-1} E\{|w[n-k]|^2\}} \), is set to 12 dB.

We note that the analysis in [21] is not applicable under this scenario due to the following reasons: 1) The work [21] is designated for system identification, while the considered NB-PLC scenario corresponds to signal recovery; 2) NB-PLC channels are modeled as LPTV systems with additive WSCS noise, while in [21] a linear non-periodically time-varying system is assumed, where the temporal variations in the system coefficients obey a random walk process. Furthermore, in [21] the additive noise is a WSS process.

Fig. 6 depicts the empirical transient MSE and its theoretical values, computed via (8), for step sizes \( \mu = \{0.01, 0.04\} \). The empirical steady-state TA-MSE is depicted in Fig. 7, compared to the stability threshold, computed via (11), and to the theoretical steady-state TA-MSE, computed by time-averaging (12). The results in Fig. 6 demonstrate that even in practical scenarios where \textit{AS1-AS3} are not necessarily satisfied, there is a very good agreement between the theoretical and the empirical performance. Additionally, it is observed in Fig. 6 that due to the dominant periodic dynamics in the NB-PLC scenario, the LMS exhibits significant periodic variations in the MSE. In Fig. 7 we observe that Thm. 2 provides a reliable prediction of the stability threshold of the LMS filter. Furthermore, one can notice that Thm. 3 accurately characterizes the empirical steady-state performance, and that there is only a small gap between the theoretical and empirical measures, which arises from the fact \textit{AS1-AS3} are not satisfied here. To conclude,
these results illustrate the accuracy of the proposed analysis in practical scenarios in which assumptions \textit{AS1-AS3} are not satisfied.

V. CONCLUSIONS

This paper provides a general performance analysis of the LMS algorithm for estimation of WSCS signals. A complete characterization of the time-evolution of the MSE and the first and second-order statistical moments of the coefficients error vector was provided. Sufficient
conditions for convergence and stability were obtained, and the steady-state MSE was derived. The simulation results demonstrate the accuracy of the theoretical performance measures and the stability thresholds.

APPENDIX

The following properties are repeatedly used in the sequel:

1) For any matrix triplet $A_1, A_2, A_3$ of compatible dimensions, it holds that (see [32, Ch. 9.2]),

$$\text{vec} (A_1 A_2 A_3) = (A_3^T \otimes A_1) \text{vec} (A_2).$$

(13)

2) For any pair of square matrices $A_1, A_2$ of identical dimensions, it holds that (see [32, Ch. 9.2]),

$$\text{Tr} \{A_1^T A_2\} = \text{vec} (A_1)^T \text{vec} (A_2).$$

(14)

3) Since $x[n]$ is WSCS, if follows from $\text{AS3}$ that $C_x[n], c_x[n], \bar{R}_x[n], F[n], A[n], B[n], H[n],$ and $P[n]$, defined in (3c)-(3j), are all periodic with period $N_0$.

A. Proof of Lemma 1

From (2) it follows that the instantaneous estimation error $e[n]$ (4) can be written as

$$e[n] = \bar{h}^H[n] x[n] + g^H[n] x[n] + v[n].$$

(A.1)

From the definition of the coefficients error vector (5) it follows that $\forall n \in \mathbb{N}$

$$\bar{h}[n+1] = h_{TA} - h[n+1]$$

$$(a) \quad \bar{h}[n] - h[n] - \mu \cdot x[n] e^*[n]$$

$$(b) \quad \bar{h}[n] - \mu \cdot x[n] \left( \bar{h}^H[n] x[n] + g^H[n] x[n] + v[n] \right)^*$$

$$(c) \quad R_x[n] \bar{h}[n] - \mu \cdot x[n] x^H[n] g[n] - \mu \cdot x[n] v^*[n],$$

(A.2)

where $(a)$ follows from (1); $(b)$ follows from (A.1); and $(c)$ follows from the definition of $R_x[n]$ in (3a). Applying the stochastic expectation to (A.2) yields
\[
E\{\bar{h}[n+1]\} = E\{R_x[n]\bar{h}[n]\} - \mu \cdot E\{x[n]x^H]\} g[n] - \mu \cdot E\{x[n]v^*[n]\} \\
\overset{(a)}{=} R_x[n]E\{\bar{h}[n]\} - \mu \cdot C_x[n]g[n],
\]
where \((a)\) follows since \(\bar{h}[n]\) and \(R_x[n]\) are mutually independent by AS2, and since \(x[n]\) and \(v[n]\) are zero-mean and mutually independent by AS1.  

**B. Proof of Lemma 2**

Set \(Q = \text{vec}^{-1}\{q\}\). The weighted squared Euclidean norm can be formulated as

\[
\|\bar{h}[n+1]\|_q^2 = \bar{h}^H[n+1]Q\bar{h}[n+1] \\
\overset{(a)}{=} \left( R_x[n] \bar{h}[n] - \mu \cdot x[n] (x^H[n] g[n] + v^*[n]) \right)^H Q \\
\times \left( R_x[n] \bar{h}[n] - \mu \cdot x[n] (x^H[n] g[n] + v^*[n]) \right) \\
= \bar{h}^H[n] R_x[n] Q R_x[n] \bar{h}[n] \\
+ \mu^2 \cdot (g^H[n] x[n] + v[n]) x^H[n] Q x[n] (x^H[n] g[n] + v^*[n]) \\
- 2\mu \cdot \text{Re} \left\{ \bar{h}^H[n] R_x[n] Q x[n] (x^H[n] g[n] + v^*[n]) \right\},  
\]
where \((a)\) follows from (A.2). Applying the stochastic expectation to (B.1), combined with AS2 yields

\[
E\left\{ \|\bar{h}[n+1]\|_q^2 \right\} = E\left\{ \bar{h}^H[n] E\{R_x[n] Q R_x[n]\} \bar{h}[n] \right\} \\
- 2\mu \cdot \text{Re} \left\{ E\left\{ \bar{h}^H[n] \right\} E\{R_x[n] Q x[n] x^H[n] \} g[n] \right\} \\
+ \mu^2 \cdot g^H[n] E\{x[n] x^H[n] Q x[n] x^H[n]\} g[n] \\
+ \mu^2 \cdot E\{ |v[n]|^2 x^H[n] Q x[n] \}.
\]

In the following we simplify the expressions for each of the summands in (B.2). First we note that combining (13) with definitions (3f), (3h), and (3j) yields

\[
E\{R_x[n] Q R_x[n]\} = \text{vec}^{-1}\{F[n] q\},  
\]
\[
E\{R_x[n] Q x[n] x^H[n]\} = \text{vec}^{-1}\{B[n] q\},  
\]
\[
E\{R_x[n] Q x[n] x^H[n]\} = \text{vec}^{-1}\{B[n] q\},  
\]

\[\text{vec}^{-1}\{F[n] q\},  \]

\[\text{vec}^{-1}\{B[n] q\},  \]
and
\[
E\{x[n]x^H[n]Qx[n]x^H[n]\} = \text{vec}^{-1}\{P[n]q\}, \tag{B.3c}
\]
respectively. Thus, combining (13) with (B.3a) yields
\[
E\{\bar{h}^H[n]E\{R_x[n]QR_x[n]\}\bar{h}[n]\} = E\{||\bar{h}[n]||^2_{F[n]q}\}. \tag{B.4}
\]
Furthermore, combining (13) with (B.3c) yields
\[
E\{\bar{h}^H[n]\} E\{R_x[n]Qx[n]x^H[n]\} g[n] = \left(g^T[n] \otimes E\{\bar{h}^H[n]\}\right) P[n] q. \tag{B.5}
\]
and (13) combined with (B.3b) yields
\[
g^H[n] E\{x[n]x^H[n]Qx[n]x^H[n]\} g[n] = ||g[n]||^2_{B[n]q}. \tag{B.6}
\]
Lastly, from AS1 it follows that
\[
E\{|v[n]|^2x^H[n]Qx[n]\} = \sigma_v^2[n] E\{x^H[n]Qx[n]\}
\]
\[
\quad \overset{(a)}{=} \sigma_v^2[n] \text{Tr}\{C_x[n]Q\}
\]
\[
\quad \overset{(b)}{=} \sigma_v^2[n] c_x^T[n] q. \tag{B.7}
\]
where (a) follows from the definition of $C_x[n]$ (3c), and (b) follows from (14). Plugging (B.4), (B.5), (B.6), and (B.7) into (B.2) yields (7).

\[\square\]

C. Proof of Theorem 1

From (A.1) it follows that
\[
E\{|e[n]|^2\} = E\left\{\left|\bar{h}^H[n]x[n] + g^H[n]x[n] + v[n]\right|^2\right\}
\]
\[
\quad \overset{(a)}{=} E\left\{\left|\bar{h}[n] + g[n]\right|^H x[n]\right\} + E\{|v[n]|^2\}
\]
\[
\quad \overset{(b)}{=} E\left\{\left|\bar{h}^H[n]x[n]\right|^2\right\} + 2\Re\{g^H[n] E\{x[n]x^H[n]\} E\{\bar{h}[n]\}\}
\]
\[
+ g^H[n] E\{x[n]x^H[n]\} g[n] + \sigma_v^2[n], \tag{C.1}
\]
where (a) follows from AS1, and (b) follows from AS2. From AS2 it also follows that
\[
E\left\{ \left| \mathbf{h}^H[n] \mathbf{x}[n] \right|^2 \right\} = E\left\{ \mathbf{h}^H[n] E\left\{ \mathbf{x}[n] \mathbf{x}^H[n] \right\} \mathbf{h}[n] \right\}
\]
\[
\overset{(a)}{=} E\left\| \mathbf{h}[n] \right\|_c^2[n],
\]
where (a) follows from the definition of \( c_x[n] \) (3d). Hence, plugging (C.2) into (C.1) yields (8).

\[ \square \]

D. Proof of Proposition 1

First, as in [24, Appendix F], we define the decimated components decomposition [25, Sec. 17.2] of the coefficients error vector as follows:
\[
\bar{\mathbf{h}}_k[n] \triangleq \bar{\mathbf{h}}[n \cdot N_0 + k], \quad k \in \mathbb{N}_0, n \in \mathbb{N}.
\]
(D.1)

It thus follows that \( \forall n > N_0 \)
\[
E\{\bar{\mathbf{h}}_k[n+1]\} = E\{\bar{\mathbf{h}}[n \cdot N_0 + N_0 + k]\}
\]
\[
\overset{(a)}{=} \bar{\mathbf{R}}_x[((k-1)N_0)] E\{\bar{\mathbf{h}}[n \cdot N_0 + N_0 + k - 1]\} - \mu \cdot C_x[((k-1)N_0)] \mathbf{g}[((k-1)N_0]
\]
\[
\overset{(b)}{=} \mathbf{L}^\mathbf{R}_{k,k} E\{\bar{\mathbf{h}}_k[n]\} - \mu \cdot \mathbf{f}_k,
\]
where (a) follows from (6) and since \( \bar{\mathbf{R}}_x[n] \), \( C_x[n] \), and \( \mathbf{g}[n] \), defined in (3e), (3c), and (3b), are all periodic with period \( N_0 \); (b) follows from repeating the recursion \( N_0 \) times and plugging the expressions for \( \mathbf{L}^\mathbf{R}_{k,k} \) and \( \mathbf{f}_k \) defined in (3k) and (3l). Repeating (D.2) \( n \) times yields
\[
E\{\bar{\mathbf{h}}_k[n+1]\} = \left( \mathbf{L}^\mathbf{R}_{k,k} \right)^{n+1} E\{\bar{\mathbf{h}}_k[0]\} - \mu \sum_{m=0}^{n} \left( \mathbf{L}^\mathbf{R}_{k,k} \right)^m \mathbf{f}_k.
\]
(D.3)

Therefore, \( E\{\bar{\mathbf{h}}_k[n]\} \) converges regardless of the initial value \( E\{\bar{\mathbf{h}}_k[0]\} \) if and only if
\[
\lim_{n \to \infty} \left( \mathbf{L}^\mathbf{R}_{k,k} \right)^n = 0_{M \times M},
\]
which is satisfied if and only if \( \rho\left( I_{k,k}^\mathbf{R} \right) < 1 \) [31, Ch. 7.10]. This proves condition (9).

Additionally, we note that when (9) is satisfied, then
\[
\lim_{n \to \infty} \sum_{m=0}^{n} \left( \mathbf{L}^\mathbf{R}_{k,k} \right)^m = \left( I_{M} - \mathbf{L}^\mathbf{R}_{k,k} \right)^{-1} \]
Ch. 7.10], thus
\[
\lim_{n \to \infty} E\{ \bar{h}_k[n] \} = -\mu \cdot \left( I_M - \mathbf{L}_{k,k}^{\mathbf{R}} \right)^{-1} f_k = -\mu \cdot s_k, 
\]
which is defined in (3m). It follows from (D.4) that \( \lim_{n \to \infty} E\{ \bar{h}_k[n] \} = 0_{M \times 1} \) if and only if \( f_k = 0_{M \times 1} \), which proves the statement in Comment 3.

E. Proof of Corollary 1

It follows from (3k) that \( \forall k \in \mathbb{N}_0 \),
\[
\rho \left( \mathbf{L}_{k,k}^{\mathbf{R}} \right) = \rho \left( \prod_{l=k}^{N_0-1+k} \mathbf{R}_x \left[ ((l))_{N_0} \right] \right) \leq \prod_{l=k}^{N_0-1+k} \rho \left( \mathbf{R}_x \left[ ((l))_{N_0} \right] \right) \prod_{l=0}^{N_0-1} \rho \left( \mathbf{R}_x \left[ l \right] \right), \tag{E.1}
\]
where (a) follows from [31, Ch. 5.2]. From (E.1) it follows that condition (9) is satisfied if (but not only if) \( \rho \left( \mathbf{R}_x \left[ l \right] \right) < 1 \) \( \forall l \in \mathbb{N}_0 \). Since \( \mathbf{R}_x \left[ l \right] = I_M - \mu \cdot C_x[l] \), where \( C_x[l] \) is Hermitian positive semi-definite, it follows that \( \rho \left( \mathbf{R}_x \left[ l \right] \right) < 1 \) if and only if (10) is satisfied.

F. Proof of Theorem 2

From the proof of Prop. 1 it follows that if (9) is satisfied, then \( E\{ \bar{h}_k[n] \} \) converges to \( -\mu \cdot s_k \), where \( \bar{h}_k[n] \) is defined in (D.1). For \( k \in \mathbb{N}_0, n \in \mathbb{N} \), define
\[
\bar{b}[nN_0 + k] \triangleq E\{ \bar{h}_k[n] \} + \mu \cdot s_k, \tag{F.1}
\]
and
\[
\bar{p}[n] \triangleq \mathbf{P}^T[n] \left( \mathbf{g}[n] \otimes \bar{b}^*[n] \right). \tag{F.2}
\]
As the entries of \( C_x[l] \) are bounded \( \forall l \in \mathbb{N}_0 \), it follows from (D.3) combined with AS3 that, when (10) is satisfied, then, the entries of \( \bar{p}[n] \) are bounded \( \forall n \in \mathbb{N} \) and \( \lim_{n \to \infty} \bar{p}[n] = 0_{M^2 \times 1} \).
From Lemma 2 it follows that for any $M^2 \times 1$ vector $\mathbf{q}$ such that $\mathbf{Q} = \text{vec}^{-1}\{\mathbf{q}\}$ is Hermitian positive semi-definite, and $\forall n > N_0$, we have that

\[
E\left\{ \left\| \tilde{h}_k [n+1] \right\|_{\mathbf{q}}^2 \right\} = E\left\{ \left\| \tilde{h}_k [nN_0 + N_0 + k] \right\|_{\mathbf{q}}^2 \right\} \\
= E\left\{ \left\| \tilde{h}_k [nN_0 + N_0 + k - 1] \right\|_{\mathbf{F}[((k-1))N_0]}^2 \right\} \\
\quad + 2\mu^2 \cdot \left( \mathbf{g}^T [((k-1))N_0] \otimes \mathbf{s}^H_{((k-1))N_0} \right) \mathbf{P} [((k-1))N_0] \mathbf{q} \\
\quad + \mu^2 \cdot \sigma^2_{\mathbf{s}} [((k-1))N_0] \mathbf{c}_{\mathbf{x}}^T [((k-1))N_0] \mathbf{q} + \mu^2 \| \mathbf{g} [((k-1))N_0] \|_{\mathbf{B}[((k-1))N_0]}^2 \mathbf{q} \\
\quad - 2\mu \cdot \left( \mathbf{g}^T [((k-1))N_0] \otimes \tilde{b}^H [nN_0 + N_0 + k - 1] \right) \mathbf{P} [((k-1))N_0] \mathbf{q} \\
\quad \overset{(a)}{=} E\left\{ \left\| \tilde{h}_k [nN_0 + N_0 + k - 1] \right\|_{\mathbf{F}[((k-1))N_0]}^2 \right\} \\
\quad + \mu^2 \cdot \mathbf{z}_{((k-1))N_0}^T \mathbf{q} - 2\mu \cdot \tilde{\mathbf{p}}^T [nN_0 + N_0 + k - 1] \mathbf{q} ,
\]

where $(a)$ follows by plugging (30) and (F.2), recalling that

\[
\| \mathbf{g} [n] \|_{\mathbf{B}[b]|\mathbf{q}}^2 = (\mathbf{g}^T [n] \otimes \mathbf{g}^H [n]) \mathbf{B} [n] \mathbf{q}.
\]

Using (F.3), we next show that if (11) is satisfied, then $E\left\{ \left\| \tilde{h}_k [n] \right\|_{\mathbf{q}}^2 \right\}$ converges to a fixed and finite value for $n \to \infty$, $\forall k \in \mathcal{N}_0$. To that aim, define:

\[
\mathbf{a}_k \triangleq \left( \sum_{l=k}^{N_0-1+k} \mathbf{z}_{((l))N_0}^T \mathbf{L} \mathbf{F}_{l+1,k} \right)^T ,
\]

and

\[
\mathbf{b}_k [n] \triangleq 2 \left( \sum_{l=k}^{N_0-1+k} \tilde{\mathbf{p}}^T [nN_0 + l] \mathbf{L} \mathbf{F}_{l+1,k} \right)^T .
\]

Again, since the entries of $\mathbf{C}_x[k]$ are bounded and from AS3 it follows that $\mathbf{a}_k$ and $\mathbf{b}_k [n]$ are bounded $\forall k \in \mathcal{N}_0$. Using these definitions, repeating (F.3) $N_0$ times results in

\[
E \left\{ \left\| \tilde{h}_k [n+1] \right\|_{\mathbf{q}}^2 \right\} = E \left\{ \left\| \tilde{h}_k [n] \right\|_{\mathbf{L}_{k,k}}^2 \right\} + \mu^2 \cdot \mathbf{a}_k^T \mathbf{q} - 2\mu \cdot \mathbf{b}_k^T [n] \mathbf{q} .
\]

Following [2, Ch. 24.2] and [24, Appendix F], we use (F.5) to formulate $M^2$ state-space
Recursions for each \( k \in N_0 \) as follows:

\[
E \left\{ \left\| \mathbf{f}_k[n+1] \right\|^2_{(L_{k,k}^F)^T \mathbf{q}} \right\} = E \left\{ \left\| \mathbf{f}_k[n] \right\|^2_{(L_{k,k}^F)^T \mathbf{q}} \right\} + \mu^2 \mathbf{a}_k^T (L_{k,k}^F)^T \mathbf{q} - 2\mu \mathbf{b}_k^T [n] (L_{k,k}^F)^T \mathbf{q}, \quad (F.6)
\]

\( l \in \{0, 1, \ldots, M^2 - 1 \} \). Let \( \{\alpha_l\}_{l=0}^{M^2-1} \) be the coefficients of the characteristic polynomial of \( L_{k,k}^F \) \cite[Eq. 23.31]{pg. 648}. It follows from the Cayley-Hamilton theorem \cite[Eq. 532]{pg. 648} and from the linearity of the weighted Euclidean norm \cite[Eq. (23.31)]{pg. 492} that

\[
E \left\{ \left\| \mathbf{f}_k[n] \right\|^2_{(L_{k,k}^F)^T \mathbf{q}} \right\} = -\sum_{l=0}^{M^2-1} \alpha_l E \left\{ \left\| \mathbf{f}_k[n] \right\|^2_{(L_{k,k}^F)^T \mathbf{q}} \right\}.
\]

Therefore, by defining the \( M^2 \times 1 \) vectors \( \mathbf{f}_k[n] \), \( \mathbf{a}_k \), and \( \mathbf{b}_k[n] \), such that

\[
(\mathbf{f}_k[n])_l = E \left\{ \left\| \mathbf{f}_k[n] \right\|^2_{(L_{k,k}^F)^T \mathbf{q}} \right\},
\]

\[
(a_k)_{l} = \mathbf{a}_k^T (L_{k,k}^F)^T \mathbf{q},
\]

and

\[
(b_k[n])_l = \mathbf{b}_k^T [n] (L_{k,k}^F)^T \mathbf{q}.
\]

\( l \in \{0, 1, \ldots, M^2 - 1 \} \), and the \( M^2 \times M^2 \) matrix \( \hat{F}_k^T \) such that \( \hat{F}_k^T \) is the companion matrix of the characteristic polynomial of \( L_{k,k}^F \) \cite[Eq. 648]{pg. 648}, the state-space recursions (F.6) can be written as a set of \( N_0 \) multivariate difference equations

\[
\hat{H}_k[n+1] = \hat{F}_k \hat{H}_k[n] + \mu^2 \mathbf{a}_k - 2\mu \mathbf{b}_k[n], \quad (F.7)
\]

\( k \in N_0, n \geq 0 \). Note that \( \forall k \in N_0 \), Eq. (F.7) represents an \( M^2 \times M^2 \) multivariate LTI system with input signal \( \mu^2 \mathbf{a}_k - 2\mu \mathbf{b}_k[n] \) and output signal \( \hat{H}_k[n] \). Since the entries of \( \hat{p}[n] \) are bounded \( \forall n \in \mathbb{N} \) and \( \lim_{n \to \infty} \hat{p}[n] = 0_{M^2 \times 1} \), it follows that the entries of \( \mathbf{b}_k[n] \) are also bounded \( \forall n \in \mathbb{N} \) and that \( \lim_{n \to \infty} \mathbf{b}_k[n] = 0_{M^2 \times 1} \). It therefore follows from \cite[Ch. 23.4]{pg. 532} that if \( \rho(\hat{F}_k) < 1 \) then \( \hat{H}_k[n] \) is bounded and tends to a steady-state value for \( n \to \infty \), thus \( E \left\{ \left\| \hat{H}_k[n] \right\|^2_{\mathbf{q}} \right\} \) is convergent.

So far we have shown that if condition (9) is satisfied, the entries of \( C_x[k] \) are bounded \( \forall k \in N_0 \), and \( \rho(\hat{F}_k) < 1 \) for all \( k \in N_0 \), then \( \hat{H}[n] \) is mean-square stable. We now show that the latter condition is satisfied when the constraints on \( A[k]^{-1}B[k] \) and \( H[k] \) in (11). Note that
it follows from [2, Pg. 346] that the eigenvalues of $\mathbf{F}_k$ are the eigenvalues of $\mathbf{L}_{k,k}^F$. Similarly to (E.1), it can be shown that $\forall k \in \mathcal{N}_0$, $\rho(\mathbf{L}_{k,k}^F) \leq \prod_{l=0}^{N_0-1} \rho(\mathbf{F}[l])$. Therefore, if $\rho(\mathbf{F}[k]) < 1$ for all $k \in \mathcal{N}_0$, then mean-square stability is obtained. Note that $\mathbf{F}[k]$ is Hermitian, thus its eigenvalues are real, and $\rho(\mathbf{F}[k]) < 1$ if and only if all the eigenvalues of $\mathbf{F}[k]$ are in the $(-1, 1)$ interval. Also note that $\mathbf{F}[k] = \mathbf{I}_{M^2} - \mu \mathbf{A}[k] + \mu^2 \mathbf{B}[k]$, where $\mathbf{A}$ and $\mathbf{B}$ are positive semi-definite, and $\mu > 0$. Note that all the eigenvalues of $\mathbf{F}[k]$ are smaller than 1 if and only if $\mathbf{I}_{M^2} - \mathbf{F}[k]$ is positive definite, or equivalently, $\mathbf{A}[k] - \mu \mathbf{B}[k]$ is positive definite. This is obtained when $\mathbf{A}[k]$ is positive definite, which is obtained when $\mathbf{C}_x[k]$ is positive definite, and the step-size satisfies $\mu < \frac{1}{\max(\mathbf{A}^{-1}[k]\mathbf{B}[k])}$ [7, Appendix A]. Next, note that all the eigenvalues of $\mathbf{F}[k]$ are larger than $-1$ if and only if $2\mathbf{I}_{M^2} - \mu \mathbf{A}[k] + \mu^2 \mathbf{B}[k]$ is positive definite. It follows from [7, Appendix A] that this is satisfied for all $\mu > 0$ when $\mathbf{H}[k]$ has no real positive eigenvalues, and for $\mu < \frac{1}{\max(\mathbf{H}[k])}$ when $\mathbf{H}[k]$ has at least one real positive eigenvalue. Combining the conditions used in the proof yields the theorem.

\[ \square \]

G. Proof of Theorem 3

For $k \in \mathcal{N}_0$, $n \in \mathbb{N}$, using definition (D.1) and the periodicity of $\mathbf{g}[n]$, $\mathbf{C}_x[n]$, and $\sigma_v^2[n]$, the instantaneous MSE (8) can be written as

$$
\mathbb{E}\left\{ |e[n \cdot N_0 + k]|^2 \right\} = \mathbb{E}\left\{ \left\| \hat{\mathbf{h}}_k[n] \right\|_{\mathbf{C}_x[k]}^2 \right\} + 2\text{Re}\left\{ \mathbf{g}^H[k] \mathbf{C}_x[k] \mathbb{E}\{\hat{\mathbf{h}}_k[n]\} \right\} + \left\| \mathbf{g}[k] \right\|_{\mathbf{C}_x[k]}^2 + \sigma_v^2[k].
$$

(G.1)

When Thm. 2 is satisfied, then the adaptive filter is mean convergent and mean-square stable as in Defs. 2–3, respectively. Thus, letting $n \to \infty$ in (G.1) yields

$$
\lim_{n \to \infty} \mathbb{E}\left\{ |e[n \cdot N_0 + k]|^2 \right\} = \lim_{n \to \infty} \mathbb{E}\left\{ \left\| \hat{\mathbf{h}}_k[n] \right\|_{\mathbf{C}_x[k]}^2 \right\} + 2\text{Re}\left\{ \mathbf{g}^H[k] \mathbf{C}_x[k] \lim_{n \to \infty} \mathbb{E}\{\hat{\mathbf{h}}_k[n]\} \right\} + \left\| \mathbf{g}[k] \right\|_{\mathbf{C}_x[k]}^2 + \sigma_v^2[k].
$$

(G.2)

Next, recalling the definitions of $\mathbf{a}_k$ and $\mathbf{b}_k[n]$ stated in (F.4), it follows from (F.5) that
∀ \in N_0, for \n \to \infty it holds that

\begin{align*}
\lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_q^2 \right\} &= \lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_{L_{k,F}^2}^2 \right\} + \mu^2 \cdot a^T_k q - 2 \mu \cdot \lim_{n \to \infty} b^T_k [n] q \\
&= \lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_{L_{k,F}^2}^2 \right\} + \mu^2 \cdot a^T_k q,
\end{align*}

where (a) follows since \lim_{n \to \infty} b_k [n] = 0_{M \times 1}. Thm. 2 guarantees that \lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_q^2 \right\} and \lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_{L_{k,F}^2}^2 \right\} both exist and are finite. Thus, from the linearity of the weighted Euclidean norm [2, Eq. (23.31)] we have

\begin{align*}
\lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_{(I_{M^2} - L_{k,F}^2)}^2 \right\} &= \mu^2 a^T_k q,
\end{align*}

\begin{equation}
(G.3)
\end{equation}

∀ \in N_0. Setting \mathbf{q} = (I_{M^2} - L_{k,F}^2)^{-1} \mathbf{c}_x[k] in (G.3) yields

\begin{align*}
\lim_{n \to \infty} E \left\{ \| \hat{\mathbf{h}}_k [n] \|_{(I_{M^2} - L_{k,F}^2)}^2 \right\} &= \mu^2 a^T_k \left( I_{M^2} - L_{k,F}^2 \right)^{-1} \mathbf{c}_x[k] \\
&= \mu^2 \sum_{l=k}^{N_0+k-1} \mathbf{z}_l^T \mathbf{I}_{M^2} \left( I_{M^2} - L_{k,F}^2 \right)^{-1} \mathbf{c}_x[k],
\end{align*}

\begin{equation}
(G.4)
\end{equation}

where (a) follows from (F.4a). Plugging (G.4) and (D.5) into (G.2) yields (12).
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