Quantum Hall effect beyond the linear response approximation

Alejandro Kunold$^1$ and Manuel Torres$^2$

$^1$Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana-Azcapotzalco, Av. San Pablo 180, México D. F. 02200, México
$^2$Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, México Distrito Federal 01000, México

(Dated: March 22, 2022)

The problem of Bloch electrons in two dimensions subject to magnetic and intense electric fields is investigated, the quantum Hall conductance is calculated beyond the linear response approximation. Magnetic translations, electric evolution and energy translation operators are used to specify the solutions of the Schrödinger equation. For rational values of the magnetic flux quanta per unit cell and commensurate orientations of the electric field relative to the original lattice, an extended superlattice is defined and a complete set of mutually commuting space-time symmetry operators are obtained. Dynamics of the system is governed by a finite difference equation that exactly includes the effects of: an arbitrary potential, an electric field orientated in a commensurable direction of the lattice, and coupling between Landau levels. A weak periodic potential broadens each Landau level in a series of minibands, separated by the corresponding minigaps; additionally the effect of the electric field in the energy spectrum is to superimpose equally spaced discrete levels, in this “magnetic Stark ladder” the energy separation is an integer multiple of $hE/aB$, with $a$ a lattice parameter. A closed expression for the Hall conductance, valid to all orders in $E$ is obtained, the leading order term reduces to the result of Thouless et al., in which $\sigma_H^{(2)}$ is quantized in units of $e^2/h$. The first order corrections exactly cancel for any miniband that is completely filled. Second order corrections for the miniband conductance are explicitly calculated as $\sigma_H^{(2)} \propto \epsilon^2 / U_0 B$, with $U_0$ the strength of the periodic potential. However the use of a sum rule shows that $\sigma_H^{(2)}$ cancels when a Landau band is fully occupied.

I. INTRODUCTION

The quantization of the Hall conductance ($\sigma_H$) in integer and fractional values is an astonishing and unexpected result, because in a real sample there are many effects that would be expected to flaw this possibility. In the integer quantum Hall effect (IQHE), the conductance is an integer multiple of $e^2/h$; simultaneously the longitudinal magnetoresistance vanishes; both results are observed within an experimental uncertainty of less than $3 \times 10^{-8}$. For the IQHE the exact quantization of the Hall conductivity was explained based on a topological reasoning by Thouless et al.\cite{4,5}, $\sigma_H$ in units of $e^2/h$ is a topological invariant, the so-called first Chern number, which can only take integer values. $\sigma_H$ is an integer multiple of $e^2/h$ if the Fermi level lies in a gap between Landau levels. The addition of a periodic potential broadens each Landau level in a series of minibands, separated by the corresponding minigaps. Contrary to the obvious suggestion that each subband carries a fraction of $e^2/h$, Thouless et al.\cite{4,5} demonstrated that each filled miniband also contributes with an integer multiple of $e^2/h$ to $\sigma_H$. The Hall conductance varies in a non-monotonous sequence as the Fermi level sweeps through contiguous minigaps. Clear experimental evidence for the internal structure of the Hofstadter butterfly spectrum has only been found very recently in the measurement of $\sigma_H$ for lateral superlattices\cite{8}. The work of Thouless et al.\cite{4,5} makes use of the Kubo linear response theory, it is of our interest to consider the structure of the Hall current when a finite electric field ($E$) is applied, in particular we shall be able to compute higher order corrections on $E$ to the Hall conductivity.

The problem of electrons moving under the simultaneous influence of a periodic potential and a magnetic field has been discussed by many authors\cite{9,10,11,12,13,14,15,16}; the spectrum displays an amazing complexity including various kinds of scaling and a Cantor set structure\cite{17}. Some of these results had been used in order to disclose the topological structure of the Hall conductance within the linear response theory. The addition of the electric field leads to the possibility of testing the exactness of the quantization beyond the linear approximation. Furthermore the problem by itself possesses a rich physical structure that makes its analysis worthwhile.

We consider the problem of an electron moving in a two-dimensional lattice in the presence of applied magnetic and electric fields. We refer to this as the electromagnetic Bloch problem (EMB). The corresponding magnetic Bloch system (MB) has a long and rich history. An early important contribution was made by Peierls\cite{2}, who suggested the substitution of the Bloch index $k$ by the operator $(p - eA)$ in the $B = 0$ dispersion relation $\epsilon(k)$, which is then treated as an effective one-band Hamiltonian. The symmetries of the MB problem were analyzed by Zak\cite{11,14,16}, who worked out the representation theory of the magnetic-translations group. The renowned Harper equation was derived assuming a tight-binding approximation in which the magnetic field acts as a perturbation that splits the Bloch bands\cite{3}. Rauh\cite{18} derived a dual Harper equation in the opposite limit of intense magnetic field\cite{19}, here the periodic potential acts
where \( \sigma = 1/\phi \) is the inverse of magnetic flux \( \phi \) in a cell in units of \( h/e \). The studies of the butterfly spectrum by Hofstader and others have since created an unceasing interest in the problem because of the beautiful self-similar structure of the butterfly spectrum. Remarkably, an experimental realization of the Hofstader butterfly was achieved considering the transmission of microwaves and acoustic waves through an array of scatterers.

The symmetries of the EMB problem were analyzed by Ashby and Miller, who constructed the group of the electric-magnetic translation operators, and worked out their irreducible representations. The properties of the electric-magnetic operators were utilized in order to derive a finite difference equation that governs the dynamics of the EMB problem when the coupling between Landau levels can be neglected. In this paper we shall derive the equation that applies under most general conditions. Magnetic translations, electric evolution and energy translation operators are used to specify the solutions of the Schrödinger equation, commensurability conditions must be implemented in order to obtain a set of mutually commuting space-time symmetry operators. In addition to the broadening of the Landau levels induced by the periodic potential, the effect of the electric field in the energy spectrum is to superimpose equally spaced discrete levels; in this “magnetic Stark ladder” the energy separation is an integer multiple of the lattice parameter. A closed expression for the Hall conductance, valid to all order in \( B \), is obtained. The leading order contribution is quantized in units of \( e^2/h \). First and second order correction are explicitly computed and shown to take the form: \( \sigma_{H}^{(1)} \left| 0 \right\rangle = 0 \) and \( \sigma_{H}^{(2)} \propto e^{3}/U^{2}B \) respectively, for any miniband that is completely filled.

The paper is organized as follows. In the next section we present the model that describes the EMB problem and construct its symmetry operators. In Section II we describe the commensurability conditions required to have simultaneously commuting operators, they are exploited in order to construct an appropriated wave function basis. We derive an effective equation in which the “evolution” is determined by a differential equation with respect to the longitudinal pseudomomentum, this equation becomes essential in order to obtain a closed expression for the Hall conductance. The derivation of the finite difference equation that governs the dynamics of the system is presented in Section II. Results for the energy spectrum are presented, and are also discussed from the perspective of the adiabatic approximation. A closed expression for the Hall conductance, valid to all order in \( B \), is obtained in II. The leading order term \( \sigma_{H}^{(0)} \) and perturbative corrections \( \sigma_{H}^{(1)} \) and \( \sigma_{H}^{(2)} \) are calculated. The last section contains a summary of our main results.

II. THE ELECTRIC-MAGNETIC BLOCH PROBLEM

A. The model

Let us consider the motion of an electron in a two-dimensional periodic potential, subject to a uniform magnetic field \( B \) perpendicular to the plane and to a constant electric field \( E \), lying on the plane according to \( E = E(\cos \theta, \sin \theta) \) with \( \theta \) the angle between \( E \) and the lattice \( x \)-axis. The dynamics of the electron is governed by a time-dependent Schrödinger equation that for convenience is written as

\[
S|\psi\rangle = \left[ \frac{1}{2m^*} (\Pi^2 + \Pi^2) + V - \Pi_0 \right]|\psi\rangle = 0, \tag{2}
\]

where \( m^* \) is the effective electron mass, \( \Pi_\mu = p_\mu + eA_\mu \), with \( p_\mu = (i\hbar \partial/\partial t, -i\nabla) \). Except for the final results for the conductivity, throughout the paper energy and lengths are measured in units of \( \hbar \omega_c = \frac{eB}{m^*} \), and \( \ell_0 = \frac{\hbar}{B} \), respectively, where \( \omega_c \) is the cyclotron frequency and \( \ell_0 \) the magnetic length. So, unless specified, we set \( \hbar = e = m^* = 1 \); although \( B \) can also be omitted from the expressions, we find convenient to explicitly display it. Covariant notation will be used to simplify the expressions, e.g., \( x_\mu = (t, x) = (t, x, y) \).

Equation (2) can be considered as an eigenvalue equation for the operator \( S \) with eigenvalue 0. The gauge potential is written in an arbitrary gauge, it includes two gauge parameters \( \alpha \) and \( \beta \), the final physical results should be of course, independent of the gauge. Hence the components of the gauge potentials are written as

\[
A_0 = \left( \beta - \frac{1}{2} \right) x \cdot E,
\]

\[
A_x = -\left( \beta + \frac{1}{2} \right) E_x t + \left( \alpha - \frac{1}{2} \right) B y,
\]

\[
A_y = -\left( \beta + \frac{1}{2} \right) E_y t + \left( \alpha + \frac{1}{2} \right) B x.
\]

The symmetrical gauge is recovered for \( \alpha = 0 \), whereas the Landau gauge corresponds to the selection \( \alpha = 1/2 \). If \( \beta = 1/2 \) all the electric field contribution appears in the vector potential, instead if \( \beta = -1/2 \) it lies in the scalar potential and the Schrödinger equation becomes time independent, however even in this case a time dependence will slip into the problem through the symmetry operators. A general two-dimensional periodic potential can be represented in terms of its Fourier decomposition

\[
V(x, y) = \sum_{r,s} v_{rs} \exp \left( \frac{2\pi i x}{a} + \frac{2\pi iy}{a} \right). \tag{4}
\]

For specific numerical results we shall use the potential

\[
V(x, y) = U_0 \left[ \cos \left( \frac{2\pi x}{a} \right) + \lambda \cos \left( \frac{2\pi y}{a} \right) \right]. \tag{5}
\]
\( \lambda \) is a parameter that can be varied in order to have an anisotropic lattice; \( \lambda = 1 \) corresponds to the isotropic limit.

### B. Electric evolution and magnetic translations

Let \((t, x) \rightarrow (t + \tau, x + R)\) be a uniform translation in space and time, where \(\tau\) is an arbitrary time and \(R\) is a lattice vector. The classical equations of motion remain invariant under these transformations; whereas the Schrödinger equation does not, the reason being the space and time dependence of the gauge potentials. Nevertheless, quantum dynamics of the system remain invariant under the combined action of space-time translations and gauge transformations. The electric and magnetic translation operators are defined as

\[
T_0(\tau) = \exp (-i \tau O_0), \quad T_j(a) = \exp (ia O_j),
\]

with \(j = x, y\) and the electric-magnetic symmetry generators are written as covariant derivatives \(O_\mu = p_\mu + \Lambda_\mu\), with the components of the dual gauge potentials \(\Lambda_\mu\) given by

\[
\begin{align*}
A_0 &= A_0 + x \cdot E, \\
A_x &= A_x + B y + E x t, \\
A_y &= A_y - B x + E_y t.
\end{align*}
\]

It is straightforward to prove that the operators in Eq. (6) are indeed symmetries of the Schrödinger equation; they commute with the operator \(S\) in Eq. (2). Similar expressions for the electric-magnetic operators were given by Ashby and Miller\cite{20}, however their definition included simultaneous space and time translations; we deemed it more convenient to separate the effect of the time evolution generated by the \(T_0\) to that of the space translations generated by \(T_j\). The following commutators can be worked out

\[
\begin{align*}
[\Pi_0, \Pi_j] &= -i E_j, \\
[O_0, O_j] &= i E_j, \\
[\Pi_\mu, O_\nu] &= 0.
\end{align*}
\]

Notice that the electric-magnetic generators \(O_\mu\) have been defined in such a form that they commute with all the velocity operators \(\Pi_\nu\). The Schrödinger equation and the symmetry operators are expressed in terms of covariant derivatives \(\Pi_\mu\) and \(O_\mu\), respectively. A dual situation in which the roles of \(\Pi_\mu\) and \(O_\mu\) are interchanged can be considered. The dual problem corresponds to a simultaneous reverse in the directions of \(B\) and \(E\). We notice that the commutators in the second line of Eq. (8) are part of the Lie algebra of the EM-Galilean two dimensional group\cite{22,23}. This group is obtained when the usual rotation and boost operators of the planar-Galilean group are replaced by their electric-magnetic generalization in which the operators are enlarged by the effect of a gauge transformation.

## III. Electric-magnetic Bloch functions

### A. Commensurability conditions

In order to construct a complete base that expands the wave function we require the symmetry operators to commute with each other. However we have

\[
T_\mu T_\nu = e^{i e(E) [O_\mu, O_\nu]}T_\nu T_\mu.
\]

A set of simultaneously commuting symmetry operator can be found if appropriated commensurate conditions are imposed, we follow a three step method to find them:

1. First we consider a frame rotated at angle \(\theta\), with axis along the longitudinal and transverse direction relative to the electric field. An orthonormal basis for this frame is given by \(e_L = (\cos \theta, \sin \theta)\), \(e_T = (-\sin \theta, \cos \theta)\) and \(e_3 = e_L \times e_T\). The electric field is parallel to \(e_L\) and the magnetic field points along \(e_3\). We assume a particular orientation of the electric field, for which the following condition holds

\[
\tan \theta = \frac{E_y}{E_x} = \frac{m_2}{m_1}
\]

where \(m_1\) and \(m_2\) are relatively prime integers. This condition insures that spatial periodicity is also found both along the transverse and the longitudinal directions. Hence, we define a rotated lattice spanned by the longitudinal \(b_L = be_L\) and transverse \(b_T = be_T\) vectors, where \(b = a \sqrt{m_1^2 + m_2^2}\). The spatial components of the symmetry generator \(O\) are projected along the longitudinal and transverse directions: \(O_L = e_L \cdot O\) and \(O_T = e_T \cdot O\). It is readily verified that \([O_0, O_T] = 0\).

2. For the rotated lattice, we regard the number of flux quanta per unit cell to be a rational number \(p/q\), that is

\[
\phi \equiv \frac{1}{\sigma} = \frac{B b^2}{2\pi} = \frac{p}{q}.
\]

We can then define a extended superlattice. A rectangle made of \(q\) adjacent lattice cells of side \(b\) contains an integer number of flux quanta. The basis vectors of the superlattice are chosen as \(q b_L\) and \(b_T\). Under these conditions the longitudinal and transverse magnetic translations \(T_L(qb) = \exp (i q b O_L)\) and \(T_T(b) = \exp (i b O_T)\) define commuting symmetries under displacements \(q b_L\) and \(b_T\). Henceforth we shall either use the subindex \((L, T)\) or \((1, 2)\) to label the longitudinal and transverse directions.

3. We observe that \(T_0\) and \(T_L(qb)\) commute with \(T_T\). Yet they fail to commute with each other:

\[
T_0(\tau) T_L(qb) = e^{-q b r E} T_L(qb) T_0(\tau).
\]
However the operators $T_0$ and $T_L(qb)$ will commute with one another by restricting time, in the evolution operator, to discrete values with period
\[ \tau = n \tau_0, \quad n \in \mathbb{Z}, \quad \tau_0 = \frac{2\pi}{qbE} = \frac{1}{\mu} \left( \frac{b}{v_D} \right), \]
where the drift velocity is $v_D = E/B$ and we utilized Eq. \[11\] to write the second equality. $b/v_D$ is the period of time it takes an electron with drift velocity $v_D$ to travel between lattice points. Eq. \[13\] can be interpreted as the condition that the ratio of two energy scales is an integer: as we shall see ($\sigma bE$) is the magnetic-Stark ladder spacing, whereas $(2\pi v_D/b)$ is the quasienergy Brillouin width; hence Eq. \[13\] represents the ratio of these two quantities.

We henceforth consider that the three conditions \[10\], \[11\], and \[13\] hold simultaneously. In this case the three EM operators: the electric evolution $T_0 \equiv T_0(\tau_0)$, and the magnetic translations $T_L \equiv T_L(qb)$, and $T_T \equiv T_T(b)$ form a set of mutually commuting symmetry operators. In addition to the symmetry operators it is convenient to define the energy translation operator
\[ T_E = \exp \left( -i \frac{2\pi}{\tau_0} t \right), \]
that produces a finite translation in energy by $2\pi/\tau_0 \equiv qbE$. $T_E$ commutes with the three symmetry operators but not with $S$. Its eigenfunctions
\[ T_E \psi = e^{ib\mu \theta} \psi, \]
define a quasitime $\theta$ modulo $\tau_0$.

### B. Wave function and generalized Bloch conditions

Having defined $T_0$, $T_L$ and $T_T$ that commute with each other and also with $S$, it is possible to seek for solutions of the Schrödinger equation labeled by the quasienergy ($\mathcal{E}$) and the longitudinal ($k_1$) and transverse ($k_2$) quasi-momentum according to
\[ T_0 |\mathcal{E}, k_1, k_2\rangle = e^{-i\tau_0 \mathcal{E}} |\mathcal{E}, k_1, k_2\rangle, \]
\[ T_L |\mathcal{E}, k_1, k_2\rangle = e^{i k_1 q b} |\mathcal{E}, k_1, k_2\rangle, \]
\[ T_T |\mathcal{E}, k_1, k_2\rangle = e^{i k_2 q b} |\mathcal{E}, k_1, k_2\rangle. \]
The magnetic Brillouin zone (MBZ) is defined by $k_1 \in [0, 2\pi/qb]$ and $k_2 \in [0, 2\pi/b]$. Similarly the quasienergy $\mathcal{E}$ is defined modulo $2\pi/\tau_0 = qbE$. If a restricted energy scheme is selected for the energy, the first energy Brillouin region is defined by the condition $\mathcal{E} \in [0, 2\pi/\tau_0]$. We shall find convenient to perform a canonical transformation to new variables according to
\[ Q_0 = t, \quad P_0 = O_0 - \frac{1}{2} E^2, \]
\[ Q_1 = \Pi_2 + E, \quad P_1 = \Pi_1, \]
\[ Q_2 = O_1 - Et, \quad P_2 = O_2 + E, \]
that satisfy the commutation rules $[Q_{\mu}, P_{\nu}] = -i \delta_{\mu \nu}$, $g_{\mu \nu} = \text{Diag}(-1, 1, 1)$. Applied to Eq. \[2\] the transformation yields for the Schrödinger equation
\[ P_0 |\psi\rangle = H |\psi\rangle, \]
\[ H = \left[ \frac{1}{2} \left( P_1^2 + Q_1^2 \right) + V(x, y) - EP_2 \right]. \]

In this equation $(x, y)$ must be expressed in terms of the new variables:
\[ \frac{x}{a} = \frac{m_1}{b} (Q_1 - P_2) - \frac{m_2}{b} (Q_2 - P_1), \]
\[ \frac{y}{a} = \frac{m_2}{b} (Q_1 - P_2) + \frac{m_1}{b} (Q_2 - P_1). \]

On the other hand, the EM-symmetry operators and the energy translation operator take the form:
\[ T_0 = e^{-i\tau_0 P_0}, \quad T_L = e^{ib\mathcal{E} Q_2 + E P_0}, \]
\[ T_T = e^{ib P_2}, \quad T_E = e^{-i\frac{2\pi}{\tau_0} Q_0}. \]

Notice that these operators do not depend on the variables $P_1, Q_1$. Thus, it is natural to split the phase space $(Q_\mu, P_\nu)$ in the $(Q_1, P_1)$ and the $(Q_0, P_0; Q_2, P_2)$ variables. For the first set a harmonic oscillator base is used. Whereas for the subspace generated by the variables $(Q_0, P_0; Q_2, P_2)$, we observe that the operators $(T_0^T, T_T^\dagger, T_L^k, T_E^k)$ with all possible integer values of $(i, j, k, l)$ form a complete set of operators. The demonstration follows similar steps as those presented by Zak in reference\[16\]. Hence a complete set of functions, for the subspace $(Q_0, P_0; Q_2, P_2)$, is provided by the eigenfunctions of the operators $(T_0^T, T_T^\dagger, T_L^k, T_E^k)$. As starting point, we select a base of eigenvalues of the following operators
\[ A^\dagger A |\mu, \mathcal{E}, k_2\rangle = \mu |\mu, \mathcal{E}, k_2\rangle, \]
\[ P_0 |\mu, \mathcal{E}, k_2\rangle = \mathcal{E} |\mu, \mathcal{E}, k_2\rangle, \]
\[ P_2 |\mu, \mathcal{E}, k_2\rangle = k_2 |\mu, \mathcal{E}, k_2\rangle, \]
where $A$ and $A^\dagger$ are the lowering and raising operators of the $\mu$-Landau level:
\[ A = \frac{1}{\sqrt{2}} (P_1 - iQ_1), \quad A^\dagger = \frac{1}{\sqrt{2}} (P_1 + iQ_1). \]

It is straightforward to verify that these states fulfill the required eigenfunction condition \[10\] for the $T_0$ and $T_T$ operators. On the other hand $T_L$ induces a shift in the $\mathcal{E}$ and $k_2$ labels, while $T_E$ induces a shift $qbE$ in the $\mathcal{E}$ label. The previous considerations suggest that a state $|\psi, \mathcal{E}, k_2\rangle$ that is a simultaneous eigenfunction of the four operators $T_0, T_0, T_T$ and $T_T$ with the required eigenvalues (Eqs. \[16\] and \[16\]), can be constructed as a linear superposition of states of the form $[T_L^k T_T^\dagger]^{n} |\mu, \mathcal{E}, k_2\rangle$. We write down the state, and
verify their correctness:
\[
|\partial, \mathcal{E}, k_1, k_2\rangle = \sum_l \left[ T_l e^{-i\varphi k_1}\right]^l \sum_{\mu,m} C^\mu_m e^{i\sigma b Q_2 e^{i\varphi(\mathcal{E} \partial - k_1)}} |\mu, \mathcal{E}, k_2\rangle .
\]
(23)

It is easy to check that this function satisfies the three eigenvalue equations for the symmetry operators in Eq. \[10\]. Additionally it can be verified that the eigenvalue condition \[13\] for the energy translation operator is also fulfilled by imposing the periodicity condition \( C^\mu_m = C^\mu_m \). Every state in Eq. \[23\] yields a set of different eigenvalues, a condition that follows from the fact that the selected set of operators is complete, hence the base of eigenvalues in Eq. \[23\] is complete and orthonormal.

Up to this point we have constructed a base for the set of four EM translation operators, however the operator \( T_{E1} \) is not a symmetry of the problem, consequently the solution of the Schrödinger is constructed as a superposition of all the states characterized by \( \vartheta \); the state becomes
\[
|\mathcal{E}, k_1, k_2\rangle = \int d\vartheta \ C(\vartheta) |\partial, \mathcal{E}, k_1, k_2\rangle = \sum_l \left[ T_l e^{-i\varphi k_1}\right]^l \sum_{\mu,m} b^\mu_m e^{i\sigma b(Q_2 - k_1)m} |\mu, \mathcal{E}, k_2\rangle ,
\]
(24)

where
\[
b^\mu_m = \int d\vartheta \ C(\vartheta) C^\mu_m e^{i\varphi Q_2 t/m} .
\]
(25)

Equation \[24\] represents the correct state that satisfies the eigenvalue equations \[16\]. It is convenient to recast it in a compact form as
\[
|\mathcal{E}, k\rangle = \mathcal{W}(k_1) |\mathcal{E}, k_2\rangle ,
\]
(26)

where \( |\mathcal{E}, k\rangle = |\mathcal{E}, k_1, k_2\rangle \), and the operator \( \mathcal{W}(k) \) is defined as
\[
\mathcal{W}(k_1) = \sum_l \left[ T_l e^{-i\varphi k_1}\right]^l = \sum_l e^{i\varphi Q_2 t/m} ,
\]
(27)

whereas the ket \( |\mathcal{E}, k_2\rangle \) is given by
\[
|\mathcal{E}, k_2\rangle = \sum_{\mu,m} e^{i\sigma b m(Q_2 - k_1)} b^\mu_m |\mu, \mathcal{E}, k_2\rangle .
\]
(28)

The previous expression stresses the fact that the quantity \( e^{-i\sigma b m(Q_2 - k_1)} b^\mu_m \) does not depend on \( k_1 \), this will be demonstrated below Eq. \[15\]. These results will prove to be very useful to analyze the EM problem, in particular they allow us to obtain an effective Schrödinger equation where the “dynamics” is governed by the derivative with respect to the longitudinal pseudomomentum. Utilizing Eqs. \[20\] and \[24\] it is straightforward to prove the following relation
\[
P_0 |\mathcal{E}, k_1, k_2\rangle = [P_0, \mathcal{W}] |\mathcal{E}, k_2\rangle + \mathcal{W} P_0 |\mathcal{E}, k_2\rangle = (\mathcal{E} - iE \cdot \nabla_k) |\mathcal{E}, k_1, k_2\rangle .
\]
(29)

Thus the Schrödinger \[18\] can be recast as
\[
(\mathcal{E} - iE \cdot \nabla_k) |\mathcal{E}, k_1, k_2\rangle = H |\mathcal{E}, k_1, k_2\rangle .
\]
(30)

As it will be lately discussed, this form of the Schrödinger equation becomes essential in the discussion of the quantized Hall current.

The wave function takes a simply form if we adopt the \( (P_0, P_1, P_2) \) representation. In this case \( \Psi_k(p) = \langle P_0, P_1, P_2 | \mathcal{E}, k_1, k_2 \rangle \) yields
\[
\Psi_k(p) = \sum_{\mu,l,m} b^\mu_m \phi_\mu(P_1) e^{(2\pi i/b)(m + pl)k_1}
\]
\[
\delta(P_2 - E - lqbE) \delta \left( P_2 - k_2 + \frac{2\pi}{b} (m + pl) \right) ,
\]
(31)

where \( \phi_\mu(P_1) \) is the harmonic oscillator function in the \( P_1 \) representation
\[
\phi_\mu(P_1) = \langle P_1 | \mu \rangle = \frac{1}{\sqrt{\pi^{1/2} 2^\mu \mu!}} e^{-P_1^2/2} H_\mu(P_1) ,
\]
(32)

and \( H_\mu(P_1) \) is the Hermite polynomial.

On the other hand, it is sometimes useful to restore the space-time representation of the wave function. This problem provides a good example of the use of canonical transformations in quantum mechanics. Identifying the matrices that relate the \( (x_\mu, p_\mu) \) variables to \( (Q_\mu, P_\mu) \), it is possible to apply the method of reference \[26\] in order to obtain the desired transformation. The final result for the wave function in the space-time representation yields
\[
\Psi_k(t, x) = e^{i{k \cdot x - E t}} u_k(t, x) ,
\]
(33)

where the modulation function \( u(t, x) \) is given by
\[
u_k(t, x) = \frac{1}{\sqrt{2\pi}} e^{-ix_1[(\alpha - 1/2)x_2 + k_1]e^{i(\beta + 1/2)E t x_1} \times \sum_{\mu,l,m} i^\mu b^\mu_m e^{-i\varphi E t} e^{i\varphi Q_2 t/m} \phi_\mu \left( x_1 - k_2 + \frac{2\pi (m + pl)}{b} \right)} ,
\]
(34)

here \( \phi_\mu \) is the same harmonic oscillator function Eq. \[32\], but now evaluated in the space representation.

Solution \[33\] includes a superposition of Landau-type solutions originated beneath the spatial and time periodicity. The spatial periodicity is simply related to the external potential \( V \). Whereas time periodicity arises from the conditions imposed to the symmetry operators in order to produce commuting symmetries; as discussed in Eq. \[16\] the period is given by the time that takes an electron to drift between contiguous lattice points. Notice that Eq. \[33\] follows from the Bloch and Floquet theorem. However in the electric-magnetic case the modulation functions \( u(t, x) \) are not strictly periodic, instead
they satisfy the generalized Bloch conditions
\[ u(t + t_0, x_1, x_2) = e^{i\tau_0 A_0} u(t, x_1, x_2), \]
\[ u(t, x_1 + q, x_2) = e^{-iqbA_1} u(t, x_1, x_2), \]
\[ u(t, x_1, x_2 + b) = e^{-ibA_2} u(t, x_1, x_2), \]
(35)
the phases are determined by the dual-gauge potential that appears in the symmetry operators Eq. (4). It is straightforward to verify that \( u_\mu \) in Eq. (34) satisfies these conditions. The correct normalization conditions for the modulation function are obtained as follows
\[ \frac{(2\pi)^2}{qb^2} \int_{MUC} d^2x \ u^*(t, x) \ u(t, x) = 1, \]
(36)
where MUC represents the magnetic cell defined by: \( x_1 \in [0, q] \) and \( x_2 \in [0, b] \).

The function \( u \) satisfies conditions similar to those in Eq. (34) but in the MBZ. The MBZ is actually a torus \( T^2 \), so the edges \((k_1 = 0, k_2)\) and \((k_1 = 2\pi/q, k_2)\) must be identified as the same set of points, the wave function can differ at most by a total phase factor (similarly for the edges \((k_1, k_2 = 0)\) and \((k_1, k_2 = 2\pi/b)\)):
\[ u(k_1 + 2\pi/q, k_2) = e^{i f_1} u(k_1, k_2), \]
\[ u(k_1, k_2 + 2\pi/b) = e^{i f_2} u(k_1, k_2), \]
(37)
the functions \( f_1(k) \) and \( f_2(k) \) are related with the Hall conductance, however instead of Eq. (35), these functions are not analytically known and must be computed numerically.

IV. HARPER GENERALIZED EQUATION

A. Finite difference equation

The coefficient \( b^m_\mu \) in Eq. (24) satisfies a recurrence relation that is obtained when this base is used to calculate the matrix elements of the Schrödinger equation (18). First, let us consider the contribution arising from the periodic potential in Eq. (1). The \( x \) and \( y \) coordinates are written in terms of the new variables \((Q_1, P_1, Q_2, P_2)\) by means of Eq. (19), producing a term of the form \([(2\pi r_0 m_1)/b]_1 \) and another contribution in which \( Q_1 \) and \( P_2 \) are replaced by \( Q_2 \) and \( P_1 \). Once that \( Q_1 \) and \( P_1 \) are replaced by the raising and lowering operators in Eq. (22), one is lead to evaluate the matrix elements of the operator \( D = \exp{(z A^\dagger - z^* A)} \) that generates coherent Landau states. A calculation yields
\[ D^{\mu\nu}(z) = \langle \mu \mid \exp{(z A^\dagger - z^* A)} \mid \nu \rangle = e^{-\frac{1}{2} |z|^2} \begin{cases} (-z^*)^{\nu-\mu} \sqrt{\frac{\mu!}{\nu!}} L^{\nu}_{\mu}(2|z|^2), & \mu > \nu, \\ \mu^{\nu-\mu} \sqrt{\frac{\mu!}{\nu!}} L^{\nu}_{\mu}(2|z|^2), & \mu < \nu, \end{cases} \]
(38)
where \( L^{\mu}_{\nu} \) are the generalized Laguerre polynomial. The evaluation of the terms that include the \( P_2 \) operator is direct, because the base is a eigenvalue of this operator, on the other hand the operator \( \exp{[(2\pi r_0 m_1)/b]_1} \) acts as a translation operator that produces a shift on the index \( m \) of the \( b^m_\mu \) coefficient. Taking into account these results, it is possible to demonstrate after a lengthy calculation that the Schrödinger equation (18) becomes
\[ t=N \sum_{t=-N} A^t_m (k_1, k_2) \hat{b}_{m+t} = (\mathcal{E} + \text{Ek}_2 + \sigma b E m) \hat{b}_m . \]
(39)
Besides its dependence on the index \( m \), \( \hat{b}_m \) is a vector with \( L \) components and \( A^t_m (k_1, k_2) \) is a \( L \times L \) matrix in the Landau space, according to
\[ \hat{b}_m = (b^0_m, b^1_m, ..., b^L_m), \]
\[ (A^t_m)^{\mu\nu} = e^{-i\sigma bk_1 t} \sum_{r,s} \hat{B}^{\mu\nu}_{rs} (r, s) \delta_{l,rm_2-sm_1}, \]
(40)
In the previous expressions \( L \) is the highest Landau level included in the calculations, \( N = \text{max}\{r, s\} (m_1 + m_2) \) is related to the largest harmonic in the Fourier decomposition of the periodic potential (4), and \( B^{\mu\nu}_{rs} (r, s) \) is given by
\[ B^{\mu\nu}_{rs} (r, s) = \begin{cases} (t_{00} + \mu + \frac{1}{2}) \delta^{\mu\nu}, & r, s = 0, \\ v_{rs} D^{\mu\nu} (H_{rs}) e^{iK_{rs} e^{iM_{rs} [\sigma_b (rm_2 - sm_1) + k_2]}}, & r, s \neq 0, \end{cases} \]
(41)
the following definitions where introduced in the previous equation
\[ H_{rs} = \sqrt{\frac{2\pi}{b}} (m_2 - m_1) (r + s), \]
\[ K_{rs} = \frac{2\pi^2}{b^2} (rm_2 - sm_1) (rm_1 + sm_2), \]
\[ M_{rs} = -\frac{2\pi}{b} (rm_1 + sm_2). \]
We recall that the indexes \((r, l)\) refer to the Fourier expansion of the periodic potential Eq. (4), whereas \((m_1, m_2)\) correspond to the integers that determined the electric field orientation in Eq. (10).

Equation (39) describes the system in terms of a recurrence relation on a two dimensional \((\mu, m)\) basis. The coupling in \( m \) ranges from \( m - N \) to \( m + N \). We follow the method of reference (23) to recast the recurrence relation on \( m \) with \( N \) nearest-neighbor coupling into a tridiagonal vector recurrence relation. This is enforced by defining a \( N \) component vector
\[ \tilde{c}_m = (\tilde{b}_{N(m-1)}, \tilde{b}_{Nm}, \tilde{b}_{Nm+1}, \ldots, \tilde{b}_{N(m+1)-1}) \]
(43)
and matrices $Q^\pm_m$ and $Q_m$ with elements

$$Q_m^+(k_1,k_2) = \begin{pmatrix} A_{N,m}^- & A_{N+1,m}^- & \cdots & A_{2N,m}^- \\ A_{N,m}^- & A_{N+1,m}^- & \cdots & A_{2N,m}^- \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,m}^- & A_{N+1,m}^- & \cdots & A_{2N,m}^- \\ 0 & 0 & \cdots & A_{N(m+1)-1}^-
\end{pmatrix},$$

$$Q_m(k_1,k_2) = \begin{pmatrix} A_{N,m}^0 & A_{N+1,m}^0 & \cdots & A_{2N,m}^0 \\ A_{N,m}^0 & A_{N+1,m}^0 & \cdots & A_{2N,m}^0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,m}^0 & A_{N+1,m}^0 & \cdots & A_{2N,m}^0 \\ 0 & 0 & \cdots & A_{N(m+1)-1}^0
\end{pmatrix}$$

Using the fact that $Q^+_m = (Q_m)^\dagger$, Eq. (59) can be reorganized as a tridiagonal recurrence relation

$$Q_m^+\tilde{c}_{m-1} + Q_m\tilde{c}_m + Q_m^+\tilde{c}_{m+1} = [\mathcal{E} + Ek_2 + \sigma bEm] I_N + \sigma bED_N] \tilde{c}_m. \quad (45)$$

Here, $\tilde{c}_m$ is $N$-dimensional vector according to the definition (42), additionally each one of its components is a $L$ dimensional vector due to its dependence on the Landau index Eq. (41). $I_N$ is the unit matrix and $D_N = \text{Diag}(0,1,2,\ldots,N-1)$. Likewise, each of the components of the matrices $Q_m$, $I_N$, and $D_N$ are $L \times L$ matrices relative to the Landau contributions. In relation with Eq. (25) it was mentioned that the quantity $e^{-i\sigma bkk_1} \tilde{c}_m$ does not explicitly depend on the longitudinal pseudomomentum component $k_1$, this can be easily verified if the substitution $\tilde{c}_m' = e^{-i\sigma bkk_1} \tilde{c}_m$ is incorporated into Eq. (44) then using Eqs. (10) and (11), it is straightforward to verify that $\tilde{c}_m'$ is indeed independent of $k_1$.

Eq. (45) is one of our main results, and deserves to be emphasized. It is a generalization of the Harper equation that exactly includes the following effects: (1) an arbitrary periodic potential, (2) an electric field orientated in a commensurable direction of the lattice, and (3) the coupling between different Landau levels. So far, no approximations are involved, consequently it holds under most general conditions. In practice, only a finite number $L$ of Landau levels can be included on the calculations, but a very good convergence can be obtained with a reasonable small selection for $L$. Previous known results are recovered if some approximations are enforced: (i) if the electric field is switched off, its orientation is meaningless, so the integers $(m_1,m_2)$ can be set to $m_1 = 1$ and $m_2 = 0$, in this case Eq. (45) reduces to the model previously discussed by Petschel and Geise (22). (iii) If $E = 0$ and additionally the coupling between different Landau bands is neglected and the potential is taken as the sum of cosines given in Eq. (5), then the system reduces to a set of Harper equations, one for each Landau band.

Let us return to the case that includes the electric field. Equation (45) describes the system dynamics under most general conditions; in spite of its complicated structure the equation can be solved by the matrix continued fraction methods. However, a considerable simplification is obtained if the electric field is aligned along the $x$-lattice axis, i.e., $(m_1 = 1, m_2 = 0)$. Henceforth, we consider that the electric field is orientated along the $x$-axis $(m_1 = 1, m_2 = 0)$, so $b = a$ and additionally that the periodic potential takes the form given in (5). The dimension $N$ of the vectors $\tilde{c}_m$ and matrices in Eqs. (43) and (44) reduces to $N = \max\{(r,s)(m_1+m_2) = 1\}$, hence the dimensions of $\tilde{c}_m \equiv c_m$ and $Q$ reduce to include only the Landau indexes. Using equations (40), (41) and (44) the matrices $Q$ take the form

$$(Q_m^+)^{\mu\nu} = (Q_+)^{\mu\nu} = \frac{\lambda\pi K}{2(1+\lambda)a^2} e^{i\sigma bkk_1} D^{\mu\nu}(\sqrt{\pi\sigma}),$$

$$(Q_m)^{\mu\nu} = \frac{\pi K}{2(1+\lambda)a^2} e^{i(2\mu\sigma m + \sigma bkk_2)} D^{\mu\nu}(\sqrt{\pi\sigma}) + c.c., \quad (46)$$

here the parameter $K$ is a measure of the strength of the coupling between Landau bands

$$K = \frac{ma^2U_0(1+\lambda)}{\hbar^2\pi}. \quad (47)$$

Taking into account these simplifications and that $D_N$ cancels out, we find that Eq. (44) reduces to

$$Q^-c_{m-1} + Q_m c_m + Q^+ c_{m+1} = (\mathcal{E} + Ek_2 + \sigma bEm) c_m. \quad (48)$$

![FIG. 1: The energy spectrum inside the lowest Landau level as a function of the inverse magnetic flux $\sigma$. The energy axis is rescaled to $\epsilon = \mathcal{E}/U_0e^{-\pi\sigma/2}$. All values of $k_1 \in [0,2\pi/qb]$ are included. The parameters selected are: $m^* = 0.067m_e$, $a = 100 \text{nm}$, $U_0 = 0.5\text{meV}$, $E = 0.5\text{V/cm}$, and $k_2 = \pi/2$.](image)

**B. Numerical results**

The generalized Harper equation is given by a tridiagonal infinite recurrence relation (45). If $E$ is switched
off, the equation becomes periodic with period \( p \), in this case the equation can be recast as a finite \( pL \times pL \) matrix that can be solved by a direct diagonalization. However, the introduction of \( \mathbf{E} \) breaks the periodicity, and most of the methods used to solve the Harper equation break down. Eq. (48) was solved using an expansion of the associated Green’s function into matrix continued fractions (MCFs). The energy spectrum is determined detecting the change of sign in the Green’s functions that appear in the vicinity of a pole. The density of states can be obtained from the expression

\[
N(\mathcal{E}) = \frac{i}{\pi} \text{Tr} \tilde{G}(\mathcal{E}) = \frac{i}{\pi} \text{Im} \left[ \text{Tr} \ G^{\pm}(\mathcal{E}) \right],
\]

were the discontinuity in Green’s function is given by \( \tilde{G}(\mathcal{E}) = G^+(\mathcal{E}) - G^-(\mathcal{E}) \), and the retarded and advanced Green’s functions are defined on the side limits \( G^{\pm}(\mathcal{E}) = \lim_{\epsilon \to 0} G(\mathcal{E} \pm \epsilon) \). The numerical solution is obtained by truncating the iteration of the (MCFs) after the \( M \)-th term. The solution converges if \( M \) is large enough, in the calculations we observe that in order to obtain a convergence with a precision of one part in \( 10^7 \), the cutoff can be selected as \( M \approx 100 \).

In our calculations we have used the effective mass \( m^* = 0.067m_e \) typical for electrons in GaAs and a superlattice constant \( a = 100\text{nm} \). The rescaled energy spectrum \( \mathcal{E}'/[U_0 e^{-\pi \sigma^2/2}] \) is shown in Fig. 1 for the lowest Landau level, a weak modulation is considered \( (U_0 = 0.5\text{meV}) \) so the \( \mu - \nu \) Landau mixing is negligible. The electric field intensity is \( E = 0.5\text{V/cm} \) corresponding to a ratio of the electric to periodic energy \( \rho = eaE/U_0 = 0.02 \). In the strong magnetic region \( \sigma \in [0,1] \) the Hofstadter butterfly is clearly depicted. A distorted replica of the butterfly spectrum can be still observed in the region \( \sigma \in [1,2] \). As the magnetic field loses intensity, the effect of the electric field becomes dominant, the butterfly is replaced by discrete levels separated in regular intervals, i.e. a Stark ladder. Although, \( \rho = eaE/U_0 \) is small, there is a regime in which the electric field dominates over the periodic contribution, we can trace down the origin of this effect to Eqs. 48 and 49 showing that the periodic contribution is modulated by a factor \( e^{-\pi \sigma^2/2} \). Consequently, in addition to a small Landau mixing, the condition to preserve the butterfly spectrum can be stated as

\[
e aE \ll U_0 e^{-\pi \sigma^2/2}.
\]

This condition restricts the intensity of both the electric and magnetic fields, these are estimated as (inserting units): \( E \ll U_0/(ea) \sim 100\text{ (U_0/meV)} V/cm \) and \( B \geq \pi h/(ea^2) \sim 0.6T \). Fig. 2 shows the energy spectrum as a function of \( \sigma \) when a higher coupling strength \( (K = 6) \) produces Landau mixing. For strong magnetic field the effect of the electric field is small and the spectrum is very similar to that previously obtained by Petschel and Geisel. Three regimes can be identified in the plot: (i) For strong magnetic field (small \( \sigma \)) the Landau bands are well separated and butterfly structures are clearly identified. (ii) In the intermediate region (\( \sigma \sim 1 \)) the periodic potential induces Landau level overlapping, as shown in Fig. 3.
(iii) as $\sigma$ increase very narrow levels appear as a result of the electric field. The development of quasi discrete level induced by the electric field is displayed in Fig. 8 here the energy spectrum as a function of $E$ for the first two Landau levels is presented. For $\sigma = 1/2$ every Landau level splits in two bands, these bands evolve into a series of quasi discreet levels as the intensity of $E$ increases.

![Energy density plot](image)

**Fig. 4:** Energy density plot for the two lowest Landau levels as a function of the transversal pseudomomentum $k_2$. Three values of $\sigma = 1/2, 3/2, 5/2$ are selected, so two bands appear when $E = 0$. Three values of $E$ are considered: $\rho = e\alpha E/U_0 = 0, 0.002, 0.01$.

In Fig. 4 density plots of the density of states as a function of the $k_2$ pseudomomentum and the quasienergy $E$ are presented. The density of states was calculated from Green's function discontinuity obtained from the continued fraction expansion. Three values of $\sigma = 1/2, 3/2, 5/2$ are selected, consequently two bands appear when $E = 0$. In the superior panels it is observed that both as $E$ increases or $B$ decreases, the spectrum for a fixed value of $k_2$ evolves into a set of narrow bands. When $k_2$ is varied the spectrum becomes almost continuous, except for the small gaps that open between the bands.

Based on the previous results it follows that the EMB model preserves the band structure. For weak electric fields the bands are grouped forming the "butterfly spectrum"; whereas as the intensity of $E$ increases the bands are replaced by a series of quasi discreet levels: a "magnetic a Stark ladder".

C. Adiabatic approximation

In this section we exhibit an alternative expression for the effective Schrödinger equation that governs the dynamics of the system. Based on this formalism we shall find an approximated adiabatic solution, that throws further insight into the physical results obtained in the previous section. Let us consider the generalized Harper equation (48), taking into account that the matrix $Q_m$ is periodic in $m$ with period $p$, we find convenient to define the unitary transformation

$$d_l(\phi) = \sum_m U_{l,m}(\phi) c_m,$$

$$U_{l,m}(\phi) = \sqrt{\frac{q_\alpha}{2\pi}} e^{iqa[m/p]} \delta_{l,m \text{ mod } [p]}, \quad (51)$$

where $[m/p]$ denotes the integer part of the number, and the delta enforces the condition $l = m \text{ mod } [p]$. Notice that whereas the index $m$ in $c_m$ runs over all the integers, the corresponding label of the state $d_l(\phi)$ takes the values $l = 0, 1, 2, \ldots, p - 1$. The new vector state $d_l(\phi)$ satisfies the periodicity condition

$$d_l(\phi + \frac{2\pi}{qa}) = d_l(\phi), \quad (52)$$

and the transformation matrices fulfill the following properties

$$\sum_{m=-\infty}^{\infty} U_{l,m}(\varphi) U_{l,m'}^\dagger (\phi) = \delta_{l,l'} \delta (\phi - \varphi),$$

$$\sum_{l=0}^{p-1} \int_0^{2\pi/qa} d\phi \ U_{m,l}^\dagger (\phi) U_{l,m'} (\phi) = \delta_{m,m'}. \quad (53)$$

Applying this transformation to the generalized Harper equation (48) yields

$$H_M (k_1 + \phi, k_2) d (k_1 + \phi, k_2) =$$

$$\left( E + E k_2 - iE \frac{\partial}{\partial \phi} \right) d (k_1 + \phi, k_2), \quad (54)$$

where the Hamiltonian is reduced to a $p \times p$ block form

$$H_M (k_1 + \phi, k_2) = \left( \begin{array}{ccccc} Q_0 & Q^+ & 0 & 0 & \ldots & Q^- \\ Q^- & Q_1 & Q^+ & 0 & \ldots & 0 \\ 0 & Q^- & Q_2 & Q^+ & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Q^+ & 0 & 0 & 0 & \ldots & Q_{p-1} \end{array} \right) \quad (55)$$

here $Q^\pm \equiv Q^-(k_1 + \phi, k_2), Q_m \equiv Q_m(k_1 + \phi, k_2) \text{ and } Q^\pm \equiv Q^+(k_1 + \phi, k_2)$. Taking into account that each block in (55) is a $L \times L$ matrix determined by the Landau indexes, we have obtained that the transformation (51) reduces the infinite dimensional representation of the Schrödinger equation to a $pL \times pL$ representation. The price is that a non-local derivative term respect to parameter $\phi$ has been added. In the absence of electric field, Eq. (53) represents a finite dimensional eigenvalue problem, that can be solved by direct diagonalization. We observe from Eqs. (54) and (55) that $\phi$ appears to
be directly related the longitudinal quasimomentum $k_1$, in fact if we redefine $k_1 + \phi \rightarrow k_1$ in Eq. (54) yields

$$H_M (k_1, k_2) d (k_1, k_2) = \left( \mathcal{E} + E k_2 - i E \frac{\partial}{\partial k_1} \right) d (k_1, k_2).$$

(56)

This expression confirms the form of the Schrödinger equation (50) previously discussed, in which the “dynamics” is determined by a differential equation with respect to the longitudinal quasimomentum. However we now have an explicit finite-dimensional matrix representation for the Hamiltonian. Let the “instantaneous” eigenstates of $H_M$ be $h^{(\alpha)}$ with energies $\Delta^{(\alpha)}$, i.e.

$$H_M h^{(\alpha)} (k_1, k_2) = \Delta^{(\alpha)} h^{(\alpha)} (k_1, k_2),$$

(57)

where $\alpha$ labels the band state of the non-perturbed problem. A solution of Eq. (56) can readily be obtained in the adiabatic approximation as follows

$$d^{(\alpha)} (k_1, k_2) = h^{(\alpha)} (k_1, k_2) \times$$

$$\exp \left[ -\frac{\mathcal{E}}{E} k_1 + i \frac{1}{E} \int_0^{k_1} d\phi \Delta^{(\alpha)} (\phi, k_2) + i \gamma^{(\alpha)} (k_1, k_2) \right],$$

(58)

where the Berry phase $\gamma^{(\alpha)}$ is determined from the substitution of the previous expression in Eq. (56)

$$\gamma^{(\alpha)} (k_1, k_2) = i \int_0^{k_1} d\phi h^{(\alpha)*} (\phi, k_2) \frac{\partial}{\partial \phi} h^{(\alpha)} (\phi, k_2).$$

(59)

The energy eigenvalue is then determined by the periodicity condition (52), hence the change of the phase of the wave function (58) must be an integral multiple of $2\pi$, consequently in the spectrum in the adiabatic approximation is given as

$$\mathcal{E}^{(\alpha)} (k_2) = nEa\sigma + \frac{qa}{2\pi} \int_0^{2\pi/qa} dk_1 \Delta^{(\alpha)} (k_1, k_2)$$

$$+ \frac{qaE}{2\pi} \gamma^{(\alpha)}(2\pi/qa, k_2), \quad n = 0, \pm 1, \ldots$$

(60)

This result deserves some comments. The energy spectrum in the presence of electric and magnetic field contains a series of discrete levels separated by

$$\Delta \mathcal{E} = E a\sigma = \frac{hE}{aB}.$$

(61)

where we have used Eq. (11) and restored units. These levels are similar to the Wannier levels that appear when an electric field is applied to an electron in a periodic potential, the energy separation being proportional to $aE$. In the present case, the band structure parallel to the electric field is replaced by a set of discrete steps, this “magnetic-Stark” ladder are characterized by a separation proportional to the electric field intensity, but inversely proportional to both the lattice separation and the magnetic field. The existence of these levels can be explained by the following argument. In the presence of simultaneous $E$ and $B$ fields, the electron travels between lattice points in a time $\tau = aB/E$. As long as the electron does not tunnel into another band, the motion appears as periodic, with frequency $\omega = 2\pi/\tau$, corresponding to a series of energy levels whose separation $\Delta \mathcal{E} = h\omega$ coincides with the result in Eq. (61). This magnetic-Stark ladder is combined with the Hofstadter spectrum represented by the second term in Eq. (50) and the contribution of the Berry phase, the competition between these three factors was discussed in the numerical analysis of the previous section. We notice that the Berry phase is written as the integral of the longitudinal component of the Berry connection $A_\alpha (k)$ discussed by Kohmoto (see Eq. (66)).

V. HALL CONDUCTIVITY

In order to calculate the electric current for a system of independent electrons let us consider the fermionic field $\psi$

$$\psi (t, x) = \sum_\alpha \int \frac{d^2k}{(2\pi)^2} b_\alpha (k) \varphi^{*}_\alpha (k, t, x),$$

(62)

expanded in terms of the EMB states $\varphi_\alpha$, that according to the Floquet Eq. (33) are related with the solution of the Schrödinger as $\Psi_\alpha = e^{-itE\frac{\partial}{\partial k}} \varphi_\alpha$. The creation and annihilation operators $b_\alpha$ and $b_\alpha^*$ fulfill the usual fermion anti-commutation relations $\{b_\alpha (k), b_\alpha' (k')\} = \delta_{\alpha,\alpha'} \delta (k' - k)$, and zero for other anticommutators. The current density operator $\psi \mathcal{J} \psi$ takes the form

$$\mathcal{J} = \sum_{\alpha, \alpha'} \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2k'}{(2\pi)^2} b_\alpha^* (k') b_\alpha (k) \langle \alpha', k' | \Pi | \alpha, k \rangle,$$

(63)

where

$$\langle \alpha', k' | \Pi | \alpha, k \rangle = \int d^2x \varphi^{*}_\alpha (x, k) \Pi \varphi_\alpha (x, k).$$

(64)

In order to evaluate the previous expression, we first consider the matrix element of $\varphi$. Using the replacement $\varphi (k; t, x) = e^{ikx} u (k; t, x)$, that follows from the Bloch theorem Eq. (33), it can be readily demonstrated that

$$\langle \alpha', k' | x | \alpha, k \rangle = -i \nabla_k \delta (k' - k) \delta_{\alpha,\alpha'}$$

$$+ \delta (k' - k) \mathcal{A}_{\alpha'}^{\alpha} (k),$$

(65)

with the definition

$$\mathcal{A}_{\alpha'}^{\alpha} (k) = \frac{(2\pi)^2}{qa^2} \int_{\text{MUC}} d^2x u^{*}_\alpha (x, k) \nabla_k u_{\alpha'} (x, k).$$

(66)
The subscript means that the spatial integration is restricted to a magnetic unit cell. We observe that the diagonal element $A_{\alpha}(k) = \mathcal{A}^{\alpha,\alpha}(k)$ is the induced Berry connection\cite{5,6,7}. The derivation of Eq. (56) for Bloch states is well known (see for example\cite{8}), the demonstration is easily extended for the MB states\cite{9} and also for the EMB states\cite{10}. Using the relation $\mathbf{\Pi} = i [\mathbf{H}, \mathbf{x}]$, the effective Schrödinger equation \cite{11} and the result in (66) one finds

$$\langle \alpha', k' | \mathbf{\Pi} | \alpha, k \rangle = [\mathbf{\Pi}_{\alpha}(k) \delta_{\alpha', \alpha} + \mathbf{\Pi}_{\alpha', \alpha}(k) (1 - \delta_{\alpha', \alpha})] \delta(k' - k),$$

where

$$\mathbf{\Pi}_{\alpha}(k) = \nabla_k \mathcal{E}_{\alpha}(k) - E \cdot \nabla_k \mathcal{A}_{\alpha}(k),$$

$$\mathbf{\Pi}_{\alpha', \alpha}(k) = i [\mathcal{E}_{\alpha'}(k) - \mathcal{E}_{\alpha}(k) + i E \cdot \nabla_k] \mathcal{A}_{\alpha', \alpha}^*(k).$$

Substituting (67) into (63) the electric current becomes

$$\mathbf{J} = \sum_{\alpha} \int \frac{d^2 k}{(2\pi)^2} b^*_{\alpha}(k) b_{\alpha}(k) [\nabla_k \mathcal{E}_{\alpha}(k) - E \cdot \nabla_k \mathcal{A}_{\alpha}(k)]$$

$$+ \sum_{\alpha, \alpha'} \int \frac{d^2 k}{(2\pi)^2} b^*_{\alpha}(k) b_{\alpha'}(k) [\mathcal{E}_{\alpha'}(k) - \mathcal{E}_{\alpha}(k)] \mathcal{A}_{\alpha', \alpha}^*(k).$$

At very low temperature the electrons occupy all the levels that are below the Fermi energy. So the state of the system in the independent particle approximation can be written in the form

$$|\Psi\rangle = \prod_{\alpha \leq \nu_F} \prod_k b_{\alpha}(k) |0\rangle,$$

where $|0\rangle$ is the vacuum, $\nu_F$ is the number of filled bands below the Fermi level and the state is normalized in such a way that the number of charge carriers is given as $N = \nu_F S/(\hbar^2)$, where $S$ is the area of the sample. For the state (70) the electric current becomes

$$\mathbf{J} = \sum_{\alpha \leq \nu_F} \int \frac{d^2 k}{(2\pi)^2} [\nabla_k \mathcal{E}_{\alpha}(k) - E \cdot \nabla_k \mathcal{A}_{\alpha}(k)].$$

The first term on the right hand side represents the usual velocity group contribution. Additionally there is a novel contribution arising from gradient of the Berry connection along the longitudinal electric field direction.

In [11] we discussed the solution of the effective Schrödinger equation\cite{12} in terms of the adiabatic approximation. However, if we consider the matrix element of Eq. (56) taken between the exact EMB states that solve Eq. (30) we find that the energy eigenvalue fulfills the relation

$$\mathcal{E}_{\alpha}(k) = \Delta_{\alpha}(k) + E \cdot \mathcal{A}_{\alpha}(k_1, k_2) - E k_2,$$

where

$$\Delta_{\alpha}(k) = \langle u_{\alpha, k} | H_M | u_{\alpha, k} \rangle.$$

The eigenvalues in (72) and the previous results in (60) look similar; however the adiabatic approximation makes use of the approximate instantaneous solution\cite{13} in order to evaluate the quantities that appear in Eq. (60). Instead the evaluation of (72) requires the exact solution of the Schrödinger equation\cite{14}. Substituting Eq. (72) into the expression (71) for the current, this reduces to the form

$$\mathbf{J} = \frac{\nu_F}{2\pi p} e_T - \frac{1}{2\pi p} E \sum_{\alpha \leq \nu_F} \int \frac{d^2 k}{2\pi} \Omega_{\alpha}(k),$$

the Berry curvature $\Omega_{\alpha}(k) = [\nabla_k \times \mathcal{A}_{\alpha}(k)]_3$. Eq. (74) shows that the longitudinal magnetoresistance exactly cancels, regardless of the value of $E$. On the other hand, the Hall conductance can be read from Eq. (74) and written (restoring units) in a compact form as

$$\sigma_H = \frac{e^2}{h} \sum_{\alpha \leq \nu_F} \sigma_{\alpha}, \quad \sigma_{\alpha} = \frac{1}{p} (1 - q \eta_{\alpha}),$$

where $\eta_{\alpha}$ is the $\alpha$ subband contribution to the conductance given as

$$\eta_{\alpha} = \frac{1}{q} \int \frac{d^2 k}{2\pi} \Omega_{\alpha}(k)_3 = \frac{1}{q} \int_{\text{CMBZ}} \frac{d k}{2\pi} \mathcal{A}_{\alpha}(k),$$

here CMBZ denotes the contour of the MBZ and the Stoke’s theorem was used to write the second equality. We find it convenient to multiply and divide by $q$ the contribution of $\eta_{\alpha}$ that appears in Eq. (76), the reason is that the Brillouin zone $k_1 \in [0, 2\pi/qa], k_2 \in [0, 2\pi/a]$ can be divided in $q$ restricted Brillouin zones, RMBZ: $k_1 \in [0, 2\pi/qa], k_2 \in [2\pi j/qa, 2\pi j + 1)/qa]$ with $j = 0, \ldots, q - 1$. The wave function satisfies periodicity conditions that connect the solutions in different RMBZ and thus the integrals in Eq. (77) can be evaluated in any of the RMBZ giving the same result.\cite{15}

An expression similar to Eq. (74) was originally discussed by Thouless et al\cite{16} and Kohmoto\cite{17}. However, their work is based on the Kubo linear response theory, consequently only terms linear in $E$ appear in that case. Our result reduces to that of Thouless and Kohmoto if we drop the electric field in Eq. (56), in other words if we evaluate the conductivity in Eq. (77) using the zero order solution Eq. (57). As far as the gap condition is satisfied the argument that shows that the expression in Eq. (77) is related to the first Chern number can be applied to prove that $\eta_{\alpha}$ is quantized. However as the electric field is increased the high density of levels makes it practically impossible to resolve the conductance contributions for each miniband, as far as the original Hofstadter miniband
is concerned this fact can be evaluated as a nonlinear contribution to \( \sigma_\alpha \) that will lead to a break down of its quantization. Higher order corrections to the Hall conductivity can be calculated by a perturbative evaluation of Eq. \( \text{(56)} \). We observe that the term \( \bar{E} \) of Eq. \( \text{(56)} \) can be considered as a perturbative potential; hence a series expansion in \( E \) can be worked out. The Hall conductance for the \( \alpha \)-band is expanded as

\[
\sigma_\alpha = \sigma_\alpha^{(0)} + \sigma_\alpha^{(1)} E + \sigma_\alpha^{(2)} E^2 + \ldots .
\]  

Starting with the zero-order solution of Eq. \( \text{(79)} \), \( u_\alpha^{(0)}(t, x) = \langle t, x | \alpha^{(0)} \rangle \), the leading contribution to the Berry connection is simply given by

\[
\mathcal{A}_\alpha^{(0)} = i \left\langle \alpha^{(0)} \right| \nabla_k \left\langle \alpha^{(0)} \right| .
\]  

Then based on definition \( \text{(77)} \) and the condition \( \text{(37)} \), fulfilled by the wave function on the boundary of the MBZ, it follows that \( \eta_\alpha^{(0)} \) is the change of the wavefunction’s phase around the integration loop. The wave function has zeros in the interior of the MBZ thus \( \eta_\alpha^{(0)} \) must be an integer. \( \eta_\alpha^{(0)} \) is evaluated numerically. The contour integration around the MBZ in Eq. \( \text{(77)} \) is numerically unstable; thus it is more convenient to evaluate the surface integral of the Berry curvature.

From Eqs. \( \text{(73)} \) and \( \text{(74)} \), \( \Omega_\alpha^{(0)} \) is written as \( \Omega_\alpha^{(0)} = i \left[ \left( \nabla_{k_i} \alpha^{(0)} \right) \left( \nabla_{k_j} \alpha^{(0)} \right) - \left( \nabla_{k_j} \alpha^{(0)} \right) \left( \nabla_{k_i} \alpha^{(0)} \right) \right] \). Inserting a complete state \( \sum_{\beta} \left\langle \beta^{(0)} \right| \beta^{(0)} \rangle \) inside the dot product and using the identity

\[
\left( \epsilon_\beta^{(0)} - \epsilon_\alpha^{(0)} \right) \left\langle \alpha^{(0)} \left| \frac{\partial H}{\partial k_i} \right| \beta^{(0)} \right\rangle = \left\langle \alpha^{(0)} \left| \frac{\partial H}{\partial k_i} \right| \beta^{(0)} \right\rangle ,
\]  

that can be deduced from the partial derivative of Eq. \( \text{(81)} \), the Berry curvature can be written in the form

\[
\left[ \Omega_\alpha^{(0)}(k) \right] = \sum_{\beta \neq \alpha} \left[ \frac{\left\langle \alpha^{(0)} \left| \frac{\partial H}{\partial k_\gamma} \right| \beta^{(0)} \right\rangle \left( \frac{\partial H}{\partial k_\gamma} \right) \left\langle \beta^{(0)} \left| \frac{\partial H}{\partial k_\gamma} \right| \alpha^{(0)} \right\rangle}{\left( \omega_{\beta \alpha} \right)^2} - \text{C.c.} \right] ,
\]  

where \( \omega_{\beta \alpha} = \epsilon_\beta^{(0)} - \epsilon_\alpha^{(0)} \). Eq. \( \text{(81)} \) is well suited for numerical evaluations. Fig. \( \text{5} \) shows the distribution of the Berry curvature plotted in the magnetic Brillouin zone when the inverse magnetic flux is selected as \( \sigma = 1/3 \) and the coupling strength \( K = 0.2 \). In this case, each Landau level \( \mu \) is splitted in three subbands, results are presented for the first three Landau levels. It is verified that \( \eta_\alpha^{(0)} \) always yields an integer value; here \( \alpha = (\mu, n) \) labels the Landau level \( \mu \) and the \( n \) internal miniband. In this example we find: \( \eta_{\mu,1}^{(0)} = 1 \), \( \eta_{\mu,2}^{(0)} = -2 \), and \( \eta_{\mu,3}^{(0)} = 1 \), regardless of \( \mu \). The numbers \( q \) \( \nu \) \( p \) are relatively prime, so there must be integer numbers \( u \) \( v \) \( w \) such that \( up + vq = 1 \). We can identify \( v = \eta_\alpha^{(0)} \), hence \( (1 - q \eta_\alpha^{(0)})/p \) is also an integer, but according to Eq. \( \text{(70)} \) this number is the Hall conductance of the subband \( \alpha \); thus \( \sigma_\alpha^{(0)} \) is quantized in units of \( e^2/h \). As the coupling strength \( K \) increases some of the bands crosses (see Fig. \( \text{5} \) and the values of \( \eta_\alpha^{(0)} \) are exchanged. Fig. \( \text{6} \) shows the plot of \( \Omega_\alpha^{(0)} \) for \( \sigma = 1/3 \) and \( K = 0.28 \). The Berry curvatures for the first Landau level are modified, now: \( \eta_{1,1}^{(0)} = 1 \), \( \eta_{1,2}^{(0)} = 1 \), and \( \eta_{1,3}^{(0)} = -2 \). The profiles of \( \Omega_\alpha^{(0)} \) for small values of \( K \) coincide with those that appear in the semiclassical dynamics approach introduced by Chang and Niu,\( \text{20} \), our results include the effects produced by Landau mixing.

Table \( \text{V} \) shows the values of \( \eta_\alpha^{(0)} \) for various selections of \( \sigma \) when \( K \) is small. If the coupling between Landau levels is small, the following sum rules are satisfied

\[
\sum_n \eta_{\mu,n}^{(0)} = 0, \quad \sum_n \sigma_{\mu,n}^{(0)} = 1.
\]  

Here the sum includes all the subbands within a given Landau level. The last result guarantees that the Hall conductivity of a completely filled Landau level takes the value \( e^2/h \). However as the Fermi energy sweeps through a Landau level, the partial contributions of the subband within each Landau level, although integer multiples of \( e^2/h \), do not follow a monotonous behavior. If the Fermi energy lies in the \( n \)-minigap, the accumulated conductance \( \zeta_\alpha \) is defined as \( \zeta_{\mu,n} = \sum_{j=1}^{\infty} \sigma_{\mu,j} \). The \( n \)-minigap conductance satisfies the Diophantine equation \( n = \lambda q + p \zeta_{\mu,n} \), \( (|\lambda| \leq p/2) \) in agreement with the results obtained by Thouless et al.\( \text{37} \).

We now turn our attention to the higher order corrections to the Hall conductance. From the first order corrections to the wave function, the corresponding con-
As already mentioned the relevant part of the Hamiltonian is numerically evaluated in a similar fashion as those terms in Eq. (81). The evaluation of this phase is unstable, it was previously explained how to determine its integral around the MBZ, some number are quoted in Table V. We can now exploit a particular gauge invariance of the problem. Suppose that \( u_{\alpha,k} \) satisfies the Schrödinger (56), then so does \( u_{\alpha,k} = e^{ig(k)} u_{\alpha,k} \), where \( g(k) \) is a smooth function of \( k_1 \) and \( k_2 \). A valid gauge transformation, must not be singular, in such a way that its topological signature \( \eta^{(0)} \) remains untouched. Then, along a contour \( \Gamma \), the phase of \( u_{\alpha,k} \) can be choose in such a way that the Berry connection takes the form \( \mathcal{A}^{(0)} = 2\pi \eta^{(0)} h(k) + i \nabla_k \rho(k) \), where \( h(k) \) is a selected function of \( k \) which integrates around the contour of the MBZ to 1. The term \( \nabla_k \rho(k) \) does not modify the value of \( \eta^{(0)} \), in particular it can be adjusted to have the same value in opposite sides of the MBZ. The gauge transformation cannot be implemented in all the magnetic Brillouin zone, it will be valid only if \( \Gamma \) encloses all the phase wave function singularities (zeros of \( u_{\alpha} \)); we are interested in the case in which \( \Gamma \) coincides with CMBZ. With all these considerations the second order contribution to the Hall conductance can be recast as

\[
\tilde{\eta}^{(2)} = \sum_{\beta \neq \alpha} \int_{\text{CMBZ}} dk \cdot \left( \frac{\partial H}{\partial k_1} \frac{\partial H}{\partial k_2} \right) \alpha^{(0)} \frac{\partial \alpha^{(0)}}{\partial k_1} \beta^{(0)} \frac{\partial \beta^{(0)}}{\partial k_2} + (1 \leftrightarrow i) \right] .
\] (85)

A suitable function \( h(k) \) can be selected as the gauge potential of a unit flux tube in \( k \)-space: \( h(k) = \frac{1}{2\pi}(k_1 - k_2)/(k_1^2 + k_2^2) \). In Table V we quote some of the values obtained for \( \tilde{\eta}^{(2)} \) for the selected Landau subbands. It is observed that as the internal structure of
The coupling between Landau levels is small, so the results are the same, regardless of the \( \mu \)-Landau level. The sum rules \( \Sigma_k = \eta_0^{(0)} \) and \( \Sigma_k^{(2)} \) are easily verified. The coefficients are symmetric respect to \( \sigma = 1/2 \), the values for \( \sigma \rightarrow 1 - \sigma \) are obtained with the replacement \( \eta_0^{(0)} \rightarrow -\eta_0^{(0)} \) and \( \eta_0^{(2)} \rightarrow \tilde{\eta}_0^{(2)} \) respectively; however \( \sigma_n \) does not share this symmetry.

### Table I: Zero and Second Order Coefficients \( \eta_0^{(0)} \) and \( \eta_0^{(2)} \) for \( \sigma = 1/3, 1/5 \) and \( 1/7 \)

| \( n \) | \( \eta_0^{(0)} \) | \( \eta_0^{(2)} \) | \( \eta_0^{(0)} \) | \( \eta_0^{(2)} \) | \( \eta_0^{(0)} \) |
|-------|---------------|---------------|---------------|---------------|---------------|
| 1     | 0.416         | -0.018        | 0.0229        |               |               |
| 3     | 1             | -4.152        | 110.2766      |               |               |
| 2     | -2            | 17.836        | 117.3056      |               |               |
| 1     | 0.006         | 0.0006        | 0.0003        |               |               |

Consequently, even if each miniband presents \( \sigma_H^{(2)} \) \( \alpha \)-dependent and \( \eta_0^{(2)} \) \( \alpha \) dependence. From the sum rule \( \Sigma_n \) it follows that \( \sigma_H^{(2)} \alpha \) cancels for a completely filled Landau band. Hence the nonlinear correction to the conductance is expected to be of order \( \mathcal{O}(E^2) \) for a filled miniband, whereas for a complete Landau level the correction is expected to be of order \( \mathcal{O}(E^3) \). We find that in order to preserve the self-similarity structure of the Butterfly spectrum, the electric field is restricted according to Eq. \( \text{[89]} \) and additional restriction arises from the condition that the nonlinear correction to the Hall conductance remains small as given by Eq. \( \text{[79]} \).

### VI. SUMMARY

In conclusion, we address the electric-magnetic problem, and the calculation of the Hall conductance beyond the linear response approximation. We presented a thoroughly discussion of the symmetries of the EMB problems and of the construction of the wave function and effective secular equation. The dynamics of the system is governed by a finite difference equation that exactly includes the effects of: an arbitrary periodic potential, an electric field oriented in a commensurable direction of the lattice, and coupling between Landau levels. In addition to the broadening of the Landau levels induced by the periodic potential, the effect of the electric field in the energy spectrum is to superimpose equally spaced discrete levels; in this “magnetic Stark ladder” the energy separation is an integer multiple of \( hE/aB \). A closed expression for the Hall conductance, valid to all orders in \( E \) is obtained. The leading order contribution is quantized in units of \( e^2/h \). The first order correction exactly vanishes, whereas the second order correction shows a \( \sigma_H^{(2)} \alpha \) dependence. From the sum rule \( \Sigma_n \) it follows that \( \sigma_H^{(2)} \alpha \) cancels for a completely filled Landau band. Hence the nonlinear correction to the conductance is expected to be of order \( \mathcal{O}(E^2) \) for a filled miniband, whereas for a complete Landau level the correction is expected to be of order \( \mathcal{O}(E^3) \). We find that in order to preserve the self-similarity structure of the Butterfly spectrum, the electric field is restricted according to Eq. \( \text{[89]} \) and additional restriction arises from the condition that the nonlinear correction to the Hall conductance remains small as given by Eq. \( \text{[79]} \).

### Acknowledgments

We acknowledge the partial financial support endowed by CONACyT through grants No. G32736-E, U42046 and 42026-F.

*Email: akb@correo.auc.unam.mx
† Email: torres@fisica.unam.mx

1. K. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).
2. D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
3. D. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).
4. Q. Niu, D. Thouless, and Y.-S. Wu, Phys. Rev. B 31, 3372 (1985).
5. M. Kohmoto, Annals of Phys. 160, 343 (1985).
6. J. E. Avron and R. Seiler, Phys. Rev. Lett. 54, 259 (1985).
7. D. Simon, Phys. Rev. Lett. 51, 2167 (1983).
8. C. Albrecht, J. H. Smet, K. von Klitzing, D. Weiss,
V. Umansky, and H. Schweizer, Phys. Rev. Lett. 86, 147 (2001).
9 R. Peierls, Z. Phys. 80, 763 (1933).
10 P. G. Harper, Proc. Phys. Soc. A 68, 874 (1955).
11 J. Zak, Phys. Rev. 134, A1602 (1964).
12 M. Y. Azbel’, Sov. Phys. JETP 19, 634 (1964).
13 A. Rau, Phys. Status Solidi B 69, K9 (1975).
14 I. Dana and J. Zak, Phys. Rev. B 28, B811 (1982).
15 P. G. Harper, J. Phys.: Condens. Matter 3, 3047 (1991).
16 J. Zak, Phys. Rev. Lett. 79, 533 (1997).
17 D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
18 U. Kuhl and H. J. Stöckmann, Phys. Rev. Lett. 80, 3232 (1998).
19 O. Richoux and V. Pagneux, Europhys. Lett. 59, 34 (2002).
20 N. Ashby and S. Miller, Phys. Rev. B 139, A428 (1965).
21 A. Kunold and M. Torres, Phys. Rev. B 61, 9879 (2000).
22 F. T. Hadjioannou and N. V. Sarlis, Phys. Rev. B 54, 5334 (1996).
23 F. T. Hadjioannou and N. V. Sarlis, Phys. Rev. B 56, 9406 (1997).
24 J. Zak, Phys. Rev. Lett. 71, 2623 (1993).
25 J. Zak, Phys. Rev. Lett. 19, 1385 (1967).
26 M. Moshinsky and C. Quesne, Journal of Mathematical Physics 12, 1772 (1971).
27 H. Risken, *The Fokker-Planck Equation-Methods of Solution and Applications* (Springer-Verlag, Berlin, 1984), vol. 18 of *Springer Series in Synergetics*, chap. 9.
28 G. Petschel and T. Geisel, Phys. Rev. Lett. 71, 239 (1993).
29 H. Haken, *Quantum Field Theory of Solids* (North-Holland, London, 1976), chap. 14.
30 J. Callaway, *Quantum Theory of the Solid State* (Academic Press, Inc., 1974), chap. 6.
31 M.-C. Chang and Q. Niu, Phys. Rev. B 53, 7010 (1996).
32 A. Kunold, Ph.D. thesis, Universidad Nacional Autónoma de México, Instituto de Física, Apartado Postal 20-364, México D.F. 01000, México (2003).