EXOTIC GALILEAN CONFORMAL SYMMETRY AND ITS DYNAMICAL REALISATIONS

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Abstract

The six-dimensional exotic Galilean algebra in (2+1) dimensions with two central charges $m$ and $\theta$, is extended when $m=0$, to a ten-dimensional Galilean conformal algebra with dilatation, expansion, two acceleration generators and the central charge $\theta$. A realisation of such a symmetry is provided by a model with higher derivatives recently discussed in [1]. We consider also a realisation of the Galilean conformal symmetry for the motion with a Coulomb potential and a magnetic vortex interaction. Finally, we study the restriction, as well as the modification, of the Galilean conformal algebra obtained after the introduction of the minimally coupled constant electric and magnetic fields.

1 Introduction

There are two possible ways of looking at symmetries in physics. One of them asks which symmetry is possessed by a given model, \textit{i.e.} one tries to
find a specific realisation of the symmetry generators in a model and then calculate their Lie algebra. The second method involves going the other way round, \textit{i.e.} one looks for a concrete realisation of a given symmetry algebra by constructing new models.

In 1997 we followed the second approach and presented a Lagrangian point particle model which possessed planar Galilean symmetry with two central charges \( m \) and \( \theta \). To construct such a model we had to introduce a higher-order Lagrangian \cite{2}

\[
L = \frac{m}{2} \dddot{x}_i^2 - \frac{\theta}{2} \epsilon_{ij} \dddot{x}_i \dddot{x}_j, \tag{1}
\]

where \( \theta \) corresponded to the second central charge appearing in the Poisson bracket of the two boost generators \( K_i \)

\[
\{K_i, K_j\} = \theta \epsilon_{ij}. \tag{2}
\]

Then, in \cite{1} the \( m = 0 \) limit of (1) was considered.

There are two ways of extending the Galilean symmetry:

- i) One can add the dilatation and conformal transformations preserving the Schrödinger equation. In this approach one adds to the Galilean algebra, in any spatial dimensions, two additional generators: dilatation \( D \) and expansion \( K \). The resultant algebra is called the Schrödinger algebra \cite{3-5}. However, although this symmetry was also called ‘non-relativistic conformal symmetry’ in \cite{3} it does not inherit the basic properties of relativistic conformal symmetries (vanishing of the mass parameter, the number of conformal generators being equal to the number of translations etc.)

- ii) One can perform the nonrelativistic contraction of the relativistic conformal algebra in \( D \) dimensions isomorphic to \( O(D,2) \) algebra. We supplement the Poincaré algebra generators \((P_\mu, J_{\mu\nu})\), \((\mu, \nu = 0, 1, \ldots D - 1)\) by dilatation generators \( D \) and special conformal generators \( R_\mu \). We rescale the generators in the following way: \((i = 1, \ldots D - 1)\)

\[
P_0 = \frac{H}{c}, \tag{3}
\]

\[
J_{i0} = cK_i, \tag{4}
\]

\[
R_i = c^2 F_i, \quad R_0 = cK. \tag{5}
\]

The remaining generators \( P_i, J_{ij} \) and \( D \) remain unscaled.

We see that
• i) The relation (3) implies that we should put the rest mass $m_0 = 0$ (in general we have the expansion $P_0 = m_0 c + \frac{H}{c}$).

• ii) With the rescaling (3-5) the contraction limit $c \to \infty$ does exist and describes a proper non-relativistic conformal extension of Galilean symmetries. It should be added that Negro et al introduced in [6] a family of nonrelativistic conformal algebras dependent on half-integer $l$. When $l = 1$ one obtains a nonrelativistic conformal algebra, which coincides with the one described by the $c \to \infty$ limit. We shall call this algebra the Galilean conformal algebra, with 10 generators in $D = 2+1$ and 15 generators in $D = 3+1$.

The aim of this paper is to study the symmetries of the nonrelativistic conformal models in $(2+1)$ dimensions, in the presence of the central extension $\theta$.

The paper is organised as follows. In the next section we perform the contraction of the $D = 3$ relativistic conformal algebra (isomorphic to $O(3,2)$) and show that by adding the central charge $\theta$ we derive the exotic $(2+1)$ dimensional Galilean conformal algebra. In section 3 we demonstrate that the free model introduced in [1] is a realisation of the exotic Galilean conformal symmetry. We discuss also how the Galilean conformal symmetry is modified if the Coulomb term and/or magnetic vortex interactions are added. Finally, in section 4, we show, by extending the results of [7], how the Galilean symmetry group is changed when we add to the model of [1] constant electromagnetic fields.

2 Exotic Galilean conformal group in $(2+1)$ dimensions

We define the $D = 3$ relativistic conformal algebra by adding to the $D = 3$ Poincaré algebra the following nonvanishing commutators ($\mu, \nu, \rho = 0, 1, 2; i, j = 1, 2$: $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$) (see e.g. [8])

\[
[J_{\mu\nu}, R_\rho] = \eta_{\nu\rho} R_\mu - \eta_{\mu\rho} R_\nu,
\]

\[
[R_\mu, P_\nu] = -2(\eta_{\mu\nu} D + J_{\mu\nu}),
\]

\[
[D, P_\mu] = -P_\mu, \quad [D, R_\mu] = R_\mu.
\]

We introduce $J_{\mu\nu} = (J_{ij} = \epsilon_{ij} J, J_{i0} = cK_i)$, $P_\mu = (P_i, \frac{H}{c})$, $R_\mu = (c^2 F_i, cK)$ and keep $D$ nonscaled. By performing the nonrelativistic $c \to \infty$ limit we obtain the 10-dimensional Lie-algebra, with generators $P_i$ (space translations),
$K_i$ (Galilean boosts), $J$ ($O(2)$ rotations), $H$ (time translations), $D$ (dilatations), $K$ (time expansions) and $F_i$ (accelerations). We list below only the nonvanishing commutators. First, for any vector $A_i \in (P_i, K_i, F_i)$ we have the following Lie-bracket with respect to rotations

$$[J, A_i] = \epsilon_{ij} A_j. \quad (9)$$

Furthermore

$$[H, K_i] = P_i, \quad [H, F_i] = 2K_i. \quad (10)$$

The one-dimensional conformal subalgebra (see e.g. [9]) is the following:

$$[D, H] = -H, \quad [K, H] = -2D, \quad [D, K] = K. \quad (11)$$

We have further

$$[D, P_i] = -P_i, \quad [D, K_i] = 0, \quad [D, F_i] = F_i \quad (12)$$

and finally

$$[K, P_i] = -2K_i, \quad [K, K_i] = -F_i, \quad [K, F_i] = 0. \quad (13)$$

The realisation of the Lie algebra (9-13) on the $D = (2+1)$ nonrelativistic space and time, which can be also obtained by the contraction $c \to \infty$ of the space-time differential realisation of the $D = 3$ relativistic conformal algebra (see e.g. [8]), is, after putting $x_0 = ct$, given in terms of differential operators: (see also [6]):

$$H = \partial_t, \quad P_i = -\partial_i, \quad K_i = -t\partial_i, \quad F_i = -t^2\partial_i,$$

$$J = -\epsilon_{ij} x_i \partial_j,$$

$$D = t\partial_t + x_i \partial_i, \quad K = t^2\partial_t + 2t x_i \partial_i. \quad (14)$$

The Galilean conformal algebra in $(2+1)$ dimensions can have a central extension by an ‘exotic’ parameter $\theta$. This parameter is introduced into the Lie bracket for two Galilean boosts (see [2]):

$$[K_i, K_j] = \theta \epsilon_{ij}. \quad (15)$$

As a consequence the Lie-bracket $[P_i, F_j]$ becomes also nonvanishing:

$$[P_i, F_j] = -2 \theta \epsilon_{ij}. \quad (16)$$

To prove (16) we use the relation $F_j = -[K, K_j]$ which gives, by the Jacobi identity,

$$[P_i, F_j] = [K, [K_j, P_i]] + [K_j, [P_i, K]]. \quad (17)$$
However, because \([K_j, P_i] = 0\) (as \(m = 0\)), and using \([P_i, K] = 2K_i\) as well as (15), we see that the relation (17) leads to the formula (16).

The Galilean conformal algebra with the modified relations (15), (16), in what follows, will be called the ‘exotic Galilean conformal algebra’.

We shall see that in the model introduced in [1], the exotic Galilean conformal algebra is enlarged by two further generators \(J_\pm\) extending the \(O(2)\) spatial rotations to the special linear group \(sl(2) \sim O(2, 1)\) (we put \(J_3 = \frac{i}{2}, J_\pm = J_1 \pm iJ_2\), where \(J_r (r = 1, 2, 3)\) are the standard \(O(2, 1)\) generators).

\[
[J_3, J_\pm] = \mp iJ_\pm, \quad [J_+, J_-] = 2iJ_3. \tag{18}
\]

The remaining nonvanishing commutators of \(J_\pm\) describe the \(sl(2)\) covariance relations for any two-vector \(A_i \in (P_i, K_i, F_i)\),

\[
[J_+, A_-] = -i A_+, \quad [J_-, A_+] = i A_-, \tag{19}
\]

where \(A_\pm = A_1 \pm iA_2\).

The differential realisation of the generators \(J_\pm\) is given by

\[
J_\pm = \pm i x_\pm \partial_{x_\pm}, \tag{20}
\]

where \(x_\pm = x_1 \pm ix_2\) and \(\partial_{x_\pm} = \frac{\partial}{\partial x_\pm} = \frac{1}{2}(\partial x_1 \mp i\partial x_2)\).

It is an open question whether the extension by the generators \(J_\pm\) always exists if we deal with Lagrangian models in \(D = (2 + 1)\) which are invariant under the exotic Galilean conformal symmetry.

### 3  Galilean conformal symmetry \((D = (2 + 1))\)
in dynamical models

The planar model introduced in [1], in the noninteracting case, is defined by the first order Lagrangian\(^1\)

\[
L_0 = P_i(\dot{x}_i - y_i) - \frac{\theta}{2} \epsilon_{ij} y_i \dot{y}_j, \tag{21}
\]

This expression is the \(m \rightarrow 0\) limit of the model introduced in [2].

The model described by (21) possesses the full exotic Galilean conformal symmetry. Indeed, the corresponding Lie algebra in \(D = (2 + 1)\) is realised as the Poisson-bracket (PB) algebra expressed in terms of phase space variables \(x_i, y_i\) and \(P_i\) as follows:

\[
H = P_i y_i, \tag{22}
\]

\(^1\)cp. to [1] we have changed the sign of \(\theta\) in accordance with (15).
Let us observe that the second term in formula (1) is invariant under the $\text{Sp}(2) \sim O(2,1) \sim \text{SL}(2)$ transformations

\begin{equation}
  x_i = A_{ij} x_j, \quad A^T \epsilon A = \epsilon
\end{equation}

i.e. the $O(2)$ generator $J$ should be supplemented by the generators $J_\pm = J_1 \pm i J_2$ satisfying the $O(2,1)$ algebra (13). In order to obtain the extended Galilean conformal invariance of the first order Lagrangian (21) we should add to the transformation (29) the relations:

\begin{equation}
  y'_i = A_{ij} y_j, \quad P'_i = (A^{-1})_{ij} P_j.
\end{equation}

Moreover, the generators $J_\pm$ have in the phase space of our model the following realisation:

\begin{equation}
  J_\pm = -\frac{\theta}{4} y^2 \pm \frac{i}{2} x P_\pm.
\end{equation}

To prove that all the generators ((22-28) and (31)) are conserved we use the equations of motion (EOM) derived from the Lagrangian $L_0$

\begin{equation}
  \dot{x}_i = y_i, \quad \dot{P}_i = 0,
\end{equation}

\begin{equation}
  \dot{y}_i = \frac{1}{\theta} \epsilon_{ik} P_k.
\end{equation}

These equations can be written as Hamilton’s equations

\begin{equation}
  \dot{Y} = \{Y, H\}, \quad Y \in (x_i, y_i, P_i)
\end{equation}

by using the following nonvanishing fundamental PBs

\begin{equation}
  \{x_i, P_k\} = \delta_{ik}, \quad \{y_i, y_k\} = \frac{\epsilon_{ik}}{\theta}
\end{equation}

obtained from (21) by the canonical procedure due to Faddeev and Jackiw [10].
By means of these PBs it is straightforward to show that the generators (22-28) satisfy the exotic Galilean conformal algebra extended by the generators \( J_\pm \) (see section 2).

Let us consider now the model (21) as describing the motion with the Coulomb and magnetic vortex interactions \( r = (x_1^2 + x_2^2)^{\frac{1}{2}} \)

\[
L_{\text{int}} = \frac{\lambda}{r} - \frac{g}{r^2} \epsilon_{ij} y_i x_j. \tag{35}
\]

In this case the variables \( x_i \) and \( y_i \) can be treated as coordinate differences, i.e. as invariant under the translations, boosts and constant accelerations. Therefore, we should consider the representation of the Galilean conformal algebra with six vanishing generators

\[
P_i = K_i = F_i = 0. \tag{36}
\]

The four remaining generators \((H, D, K \text{ and } J)\) form an \( O(2,1) \oplus O(2) \) algebra described by the algebra (11) supplemented by the Abelian \( O(2) \) rotation generator. Because the second term in (35) is, like the free model, invariant under \( Sp(2) \sim O(2,1) \) transformations (29), when \( \lambda = 0 \) and \( g \neq 0 \) the model possesses the symmetry \( O(2,1) \otimes O(2,1) \), described by the algebras (11) and (18).

4 Enlargement of the exotic Galilean symmetry in the presence of constant electromagnetic fields

In this section we couple minimally the model of [1] to the constant electromagnetic fields \( B \) and \( E_i \) and show how the exotic Galilean conformal algebra in \( D = (2 + 1) \) is modified into an enlarged Galilean symmetry. In such a case we consider \( E_i \) and the corresponding canonical conjugate momenta \( \pi_i \) as additional phase space variables (cp [7]).

The minimal coupling principle gives

\[
H_0 \rightarrow H = H_0|_{P_i \rightarrow P_i - A_i} - A_0, \tag{37}
\]

which, for the constant fields \( B \) and \( E_i \), modifies \( L_0 \) (21) in the following way

\[
L_0 \rightarrow L = P_i \dot{x}_i - (P_i + \frac{B}{2} \epsilon_{ij} x_j)y_i - \frac{\theta}{2} \epsilon_{ij} y_i \dot{y}_j + E_i x_i + \dot{E}_i \pi_i. \tag{38}
\]

Note that (38) leaves the PBs (34) unchanged and now they have to supplemented by

\[
\{E_i, \pi_j\} = \delta_{ij}. \tag{39}
\]
The EOM which follow from the Lagrangian (38) are given by
\[ \dot{x}_i = y_i, \quad \dot{\pi}_i = x_i \quad (40) \]
\[ \dot{P}_i = \frac{B}{2} \epsilon_{ik} y_k + E_i, \quad (41) \]
\[ \dot{y}_i = \frac{1}{\theta} (\epsilon_{ik} P_k - \frac{B}{2} x_i) \quad (42) \]
\[ \dot{E}_i = 0. \quad (43) \]

The EOM for \( x_i \) follows from (40-43):
\[ \ddot{x}_i = -\frac{B}{2} \dot{x}_i + \frac{\epsilon_{ik}}{\theta} E_k. \quad (44) \]

From (44) we can read off the symmetries of the model. They are given by:

- **Space translations:** \( \delta x_i = a_i \) with \( \delta B = \delta E_k = 0 \)
- **Rotations:** \( \delta x_i = -\varphi \epsilon_{ik} x_k \) with \( \delta B = 0 \) and \( \delta E_i = -\varphi \epsilon_{ik} E_k \).
- **Boosts:** \( \delta x_i = b_i t \) with \( \delta B = 0 \) and \( \delta E_i = -B \epsilon_{ij} b_j \)

corresponding to the physically required transformation properties of the electromagnetic fields in the nonrelativistic limit.

In addition we have also the additional \( O(2,1) \) invariance with the generators \( (J_\pm, J_3) \) which satisfy (18). To see this we rewrite (44) in terms of the complex variables \( (A_\pm = A_1 \pm iA_2 \text{ for any vector } A_i) \) getting
\[ \ddot{x}_+ = -\frac{B}{\theta} \dot{x}_+ - \frac{i}{\theta} E_+ \quad (45) \]
\[ \ddot{x}_- = -\frac{B}{\theta} \dot{x}_- + \frac{i}{\theta} E_- \quad (46) \]
and observe that \( \delta_+ (46) \rightarrow (45) \) and \( \delta_- (45) \rightarrow (46) \) where \( \delta_\pm \) act on coordinates and electric fields as
\[ \delta_+ x_+ = -i \epsilon x_+, \quad \delta_+ E_+ = i \epsilon E_+ \quad (47) \]
\[ \delta_- x_+ = i \epsilon x_-, \quad \delta_- E_+ = -i \epsilon E_- \quad (48) \]
with all other variations being zero.

All symmetries extending conformally our exotic (2+1) dimensional Galilean algebra, \( i.e. \) accelerations \( F_i \), dilatations \( D \) and expansions \( K \) do not preserve our EOM (44).
The generators of the symmetries that remain can be easily constructed. The Hamiltonian \( H \) can be read off from (38)

\[
H = (P_i + \frac{B}{2} \epsilon_{ij} x_j) x_i - E_i x_i. \tag{49}
\]

The space translation generators \( P_i \) are obtained by integrating (41)

\[
P_i = P_i - \frac{B}{2} \epsilon_{ij} x_j - E_i t. \tag{50}
\]

Angular momentum generator \( J \) is obtained from (26) by adding the term generating rotations of the \( E_i \) fields

\[
J = \epsilon_{ij} x_i P_j - \frac{\theta}{2} y_i^2 + \epsilon_{ij} E_i \pi_j. \tag{51}
\]

The boost generators \( K_i \) are constructed in close analogy with the corresponding expression in [7]

\[
K_i = (P_i + \frac{E_i}{2} t) t + B \epsilon_{ij} \pi_j + \theta \epsilon_{ij} y_j. \tag{52}
\]

The operators \( J_{\pm} \) are obtained from (31) by adding to it the term generating the required transformations (47-48) of the electric fields

\[
J_{\pm} = -\frac{\theta}{4} y_{\pm}^2 \mp i \frac{1}{2} (x_{\pm} P_{\pm} - E_{\pm} \pi_{\pm}). \tag{53}
\]

Using the EOM (40-43) one sees immediately that all the generators (49-53) are conserved. Furthermore, using the PBs (34) and (39) it can be shown that they generate the desired transformations.

The enlarged Galilean symmetry algebra follows by the use of the fundamental PBs. We find the following changes of the (2+1) dimensional exotic Galilean algebra:

\[
\{ P_i, H \} = 0 \quad \rightarrow \quad \{ P_i, H \} = E_i \tag{54}
\]

\[
\{ P_i, P_j \} = 0 \quad \rightarrow \quad \{ P_i, P_j \} = -B \epsilon_{ij}. \tag{55}
\]

The PBs (54) imply additional relations closing the algebra with respect to \( E_i \)

\[
\{ J, E_i \} = \epsilon_{ik} E_k, \quad \{ E_i, K_j \} = -B \epsilon_{ij}, \tag{56}
\]

\[
\{ J_+, E_- \} = -i E_+, \quad \{ J_-, E_+ \} = i E_- \tag{57}
\]

We note that \( B \) plays the role of a new central element in the exotic Galilean symmetry algebra, enlarged by the generators \( J_{\pm} \) and \( E_i \). This 10-dimensional extended (2+1) dimensional Galilean algebra with two central charges (\(\theta, B\)) describes the symmetries of the model (38).
5 Conclusions

Scale and conformal transformations in nonrelativistic models were first introduced in a way which preserves the Schrödinger equation [3-5]. Such a procedure permits to add, in any dimensions, only two generators $D$ and $K$ forming together with the Hamiltonian $H$ an $O(2, 1)$ algebra of one dimensional conformal transformations of the time variable (see also [9]).

In this paper we have considered other extensions of the Galilean nonrelativistic transformations by adding to them conformal translations in the target space $x_i$ which describe the constant accelerations. This has given us a nonrelativistic conformal symmetry which can be shown to be derivable as the nonrelativistic, $i.e. c \to \infty$ limit of the relativistic conformal symmetry. We obtain in the relativistic and nonrelativistic cases the same number of generators and the requirement that the mass parameter vanishes.

Following our earlier considerations [1,2] we have considered in this context the nonrelativistic dynamics in (2+1) dimensions, with the Galilean algebra endowed with two central charges $m$ and $\theta$. We have found that our ‘genuine’ conformal extension of the Galilean symmetry requires $m = 0$ and exhibits the symmetries of the model recently considered in [11]. It seems that the second central extension parameter $\theta \neq 0$ is necessarily required if we wish to construct a nonrelativistic conformally invariant free model. We see that for the Galilean conformal symmetry the parameter $\theta$ plays the analogous role to the mass $m$ of standard Galilean symmetries: it permits the explicit dynamical realisation of the symmetry algebra. As the exotic central extension is possible only in $D = (2 + 1)$ it is interesting to ask whether only in this dimension the Galilean conformal transformations may represent an invariance group of a dynamical model.

The Galilean conformal symmetries can be also supersymmetrised by a suitable contraction of the relativistic superconformal algebras. In particular, in $D = (2 + 1)$, such a structure is obtained by the contraction of the well known $OSp(N; 4)$ superalgebra. The supersymmetric extension of the Galilean conformal algebra is different from the ‘so-called’ Schrödinger superalgebra (see e.g. [11]) and is currently under consideration.

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