COMBINATORIAL ASPECTS OF ELLIPTIC CURVES II: RELATIONSHIP BETWEEN ELLIPTIC CURVES AND CHIP-FIRING GAMES ON GRAPHS

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CONTENTS

1. Introduction 1
2. An Enumerative Correspondence between Elliptic Curves and Wheel Graphs 3
3. Introduction to Chip-Firing Games 5
4. Critical Configurations for the $W_k(q, t)$ Graphs 7
5. The Frobenius Map and Elliptic Cyclotomic Polynomials 9
5.1. Analogues of Elliptic Cyclotomic Polynomials for Wheel Graphs 10
6. Maps between Critical Groups 11
6.1. Group Presentations 13
7. Connections to Deterministic Finite Automata 20
7.1. Another Kind of Zeta Function 21
References 23

Abstract. Let $q$ be a power of a prime and $E$ be an elliptic curve defined over $\mathbb{F}_q$. In [17], the present author examined a sequence of polynomials which express the $N_k$'s, the number of points on $E$ over the field extensions $\mathbb{F}_{q^k}$, in terms of the parameters $q$ and $N_1 = \#E(\mathbb{F}_q)$. These polynomials have integral coefficients which alternate in sign, and a combinatorial interpretation in terms of spanning trees of wheel graphs. In this sequel, we explore further ramifications of this connection. In particular, we highlight a relationship between elliptic curves and chip-firing games on graphs by comparing the group structures of both. As a coda, we construct a cyclic rational language whose zeta function is dual to that of an elliptic curve.

1. Introduction

The theory of elliptic curves is quite rich, arising in both complex analysis and number theory. In particular, they can be given a group structure using the tangent-chord method or the divisor class group of algebraic geometry [20]. This property makes them not only geometric but also algebraic objects and allows them to be used for cryptographic purposes [22].

In [17], the author started an exploration of elliptic curves from a combinatorial viewpoint. For a given elliptic curve $E$ defined over a finite field $\mathbb{F}_q$, we let $N_k = \ldots$
\#E(F_{q^k}) where $F_{q^k}$ is a $k$th degree extension of the finite field $F_q$. Because the zeta function for $E$, i.e.

$$\exp\left(\sum_{k \geq 1} \frac{N_k T^k}{k}\right) = \frac{1 - (1 + q - N_1)T + qT^2}{(1 - T)(1 - qT)},$$

only depends on $q$ and $N_1$, the sequence $\{N_k\}$ only depends on those two parameters as well. More specifically, Adriano Garsia observed that these bivariate expressions for $N_k$ are in fact polynomials with integer coefficients, which alternate in sign with respect to the power of $N_1$ [10].

This motivated the main topic of [17], which was the search for a combinatorial interpretation of these coefficients. One such interpretation discussed therein involved a sequence denoted as $W_k(q, t)$, a $(q, t)$-deformation of the number of spanning trees of a certain family of graphs known as the wheel graphs. In this sequel, we more deeply explore this combinatorial interpretation. In particular, the number of spanning trees of a graph, also known as the graph’s complexity, is an important characteristic of a graph, used to study connectivity, with applications to networks. Additionally, this quantity is known to enumerate other structures such as the order of a graph’s critical group, and as more recently observed in [15], the number of $G$-parking functions associated to graph $G$. Here we investigate the connection to critical groups for wheel graphs. We describe several properties that these critical groups share with elliptic curve groups, thus demonstrating a relationship between these structures. (See Theorem 4.)

The outline of this paper will be as follows. We start by reviewing the definitions and two theorems of [17], which are labelled as Theorems 1 and 2 below. In the present paper, we in fact provide an alternate definition of $W_k(q, t)$ and an alternative proof of Theorem 2 which did not appear in [17]. We provide this proof because it will use the same terminology that will appear elsewhere in the paper. We then switch gears, and in Section 3, discuss critical groups of graphs and the subject of chip firing games. We will include background material to try to make this paper self-contained, but most of the details of this section come from Norman Biggs [1].

In Section 4, we specialize this theory to the case of a family of graphs that we refer to as $(q, t)$-wheel graphs, and explicitly describe critical configurations for them (Theorems 5 and 6). We return to the topic of elliptic curves over finite fields in Section 5, and more closely study the Frobenius map on these varieties. This section will involve introductory material but we delay its inclusion until this place in the paper since it will not be used in earlier sections. We also describe elliptic cyclotomic polynomials, which first appeared in [17]. This will lead us to Section 6, where the main result, Theorem 8 involves wheel graph analogues of elliptic curves over the algebraic closure of $F_q$ and the Frobenius map.

We conclude Section 6 with several additional applications of this point of view. These include a characteristic equation for the wheel graph Frobenius map (Theorem 9), and explicit group presentations as expressed in Theorem 10, Corollary 2, and Theorem 12. Additionally we answer a question of Norman Biggs [4] and, in Theorem 11, generalize a result of his on the cyclicity of deformed wheel graphs [3]. In short, we will see that the critical groups of the $(q, t)$-wheel graphs decompose into at most two cyclic groups, just like elliptic curves over finite fields.
Finally in Section 7, we come full-circle and present the theory of zeta functions again. However, this time around we shall be considering zeta functions which arise in the theory of combinatorics on words. In particular, we consider particular subsets of strings arising from a given alphabet, a recognizable language in computer science terminology. Berstel and Reutenauer [6] defined a zeta function for this family of objects and it is their definition that we utilize in this section. In particular they found that in the case that a language is cyclic and recognizable by a deterministic finite automaton, then its corresponding zeta function is in fact rational. We conclude the paper by explicitly computing the zeta function for a particular family of cyclic rational languages and comparing this with the Hasse-Weil zeta function of an elliptic curve, as given in Theorem 14.

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2. An Enumerative Correspondence between Elliptic Curves and Wheel Graphs

Let $W_k$ denote the $k$th wheel graph, which consists of $(k + 1)$ vertices, $k$ of which lie in a cycle and are each adjacent to the last vertex. (We also define $W_k$ analogously in the case $k = 1$ or $k = 2$, each having a degenerate cycle of length one or two, respectively.) A spanning tree of a graph is a connected subgraph which does not contain any cycles. In the case of the wheel graphs, a spanning tree is easily defined as a collection of disconnected arcs on the rim, which each connect to the central hub along one spoke for each arc. In [17], we defined a $(q, t)$-weighting for such spanning trees $T$ by letting the exponent of $t$ be the number of spokes in $T$ and the exponent of $q$ signify the total number of edges lying clockwise with respect to the unique spoke associated to that particular arc, which we abbreviate as $\text{dist}(T)$. With this weighting in mind, the main result of [17] was the following.

**Theorem 1** (Theorem 3 of [17]). The number of points on an elliptic curve $E$ over finite field $\mathbb{F}_{q^k}$, which we denote as $N_k$, satisfies the identity

$$N_k = -W_k(q, -N_1)$$

where $N_1 = \#E(\mathbb{F}_q)$ and $W_k(q, t) = \sum_{T \text{ a spanning tree of } W_k} q^{\text{dist}(T)} t^{\#\text{spokes}(T)}$.

In this paper, we use a slightly different definition for $W_k(q, t)$, which will allow us to expand our results to other areas of combinatorics. In particular, instead of simply using the family of wheel graphs, we define a $(q, t)$-deformation of this family where the graphs are no longer simple or undirected. In other words we use a weighting scheme such that the graphs themselves change rather than the way in which we enumerate $W_k$’s spanning trees.

Define $W_k(q, t)$ to be the following directed graph (digraph) with multiple edges: We use the 0-skeleton of the wheel graph $W_k$, where we label the central vertex as
$v_0$, and the vertices on the rim as $v_1$ through $v_k$ in clockwise order. We then attach $t$ bi-directed spokes between $v_0$ and $v_i$ for all $i \in \{1, 2, \ldots, k\}$. Additionally, we attach a single counter-clockwise edge between $v_i$ and $v_{i-1}$ (working modulo $k$) for each vertex on the rim. Finally, we attach $q$ clockwise edges between $v_i$ and $v_{i+1}$ (again working modulo $k$).

**Proposition 1.** The number of directed spanning trees of $W_k(q, t)$, rooted at vertex $v_0$ equals the polynomial $\mathcal{W}_k(q, t)$.

**Proof.** By comparing this new definition with the original one from [17], we simply note that we have translated the above weighting into a scheme where we have multiple edges in $W_k(q, t)$ whenever we have a weight in $\mathcal{W}_k(q, t)$.

By transitivity we arrive at the fact that the sequence of $\{N_k\}$’s are in fact a signed version of the number of rooted spanning trees in this family of multigraphs. As an immediate application of this different characterization of $\mathcal{W}_k(q, t)$, we obtain another proof of the determinantal formula for $N_k$ which appeared in [17].

Define the family of matrices $M_k$ by $M_1 = [-N_1]$, $M_2 = \begin{bmatrix} 1 + q - N_1 & -1 - q \\ 1 & 1 + q - N_1 \end{bmatrix}$, and for $k \geq 3$, let $M_k$ be the $k$-by-$k$ “three-line” circulant matrix

$$\begin{bmatrix}
1 + q - N_1 & -q & 0 & \ldots & 0 & -1 \\
-1 & 1 + q - N_1 & -q & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & -1 & 1 + q - N_1 & -q & 0 \\
0 & \ldots & 0 & -1 & 1 + q - N_1 & -q \\
-q & 0 & \ldots & 0 & -1 & 1 + q - N_1 \\
\end{bmatrix}.$$

**Theorem 2** (Theorem 5 in [17]). The sequence of integers $N_k = \# E(\mathbb{F}_{q^k})$ satisfies the relation

$$N_k = -\det M_k$$

for all $k \geq 1$. We obtain an analogous determinantal formula for $\mathcal{W}_k(q, t)$, in fact $\mathcal{W}_k(q, t) = \det M_k|_{N_1 = -t}$.

**Proof.** We appeal to the directed multi-graph version of the Matrix-Tree Theorem [21] to count the number of spanning trees of $W_k(q, t)$ with root given as the hub. The Laplacian $L$ of a digraph on $m$ vertices, with possibly multiple edges, is defined to be the $m$-by-$m$ matrix in which off-diagonal entries $L_{ij} = -d(i, j)$ and diagonal entries $L_{ii} = d(i)$. Here $d(i, j)$ is the number of edges from $v_i$ to $v_j$, and $d(i)$ is the outdegree of vertex $v_i$, or more simply we choose $L_{ii}$ such that each row of $L$ sums to zero. In the case of $W_k(q, t)$, the Laplacian matrix is

$$L = \begin{bmatrix}
1 + q + t & -q & 0 & \ldots & 0 & -1 & -t \\
-1 & 1 + q + t & -q & 0 & \ldots & 0 & -t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & -1 & 1 + q + t & -q & 0 & -t \\
0 & \ldots & 0 & -1 & 1 + q + t & -q & -t \\
-q & 0 & \ldots & 0 & -1 & 1 + q + t & -t \\
-t & -t & -t & \ldots & -t & -t & kt \\
\end{bmatrix}$$

where the last row and column correspond to the hub vertex. We wish to count the number of directed spanning trees rooted at the hub, and the Matrix-Tree Theorem
states that this number is given by \( \det L_0 \) where \( L_0 \) is matrix \( L \) with the last row and last column deleted. From this, we obtain the identities

\[
N_k = -W_k(q, -N_1) \\
M_k = L_0 \bigg|_{t=-N_1} \quad \text{and thus} \\
W_k(q, t) = \det L_0 \quad \text{implies} \\
-W_k(q, -N_1) = -\det L_0 \bigg|_{t=-N_1} \quad \text{so we get} \\
N_k = -\det M_k.
\]

Thus we have proven Theorem 2.

We will return to ramifications of this combinatorial identity in Section 5, after discussing another instance of the graph Laplacian.

3. Introduction to Chip-Firing Games

We step away from elliptic curves momentarily and discuss some fundamental results from the theory of chip-firing games on graphs as described by Björner, Lovász, and Shor [5]. These are also known as abelian sandpile groups as described by Dhar [7]. Gabrielov wrote one of the first papers describing the relationship between these two models [9]. The main source for the details we will use is [1], though there is an extensive literature on the subject, for example see [16] for a summary.

At first glance, this topic might appear totally unrelated to elliptic curves, but we will shortly flesh out the connection. Given a directed (loop-less) graph \( G \), we define a configuration \( C \) to be a vector of nonnegative integers, with a coordinate for each vertex of the graph, letting \( c_i \) denote the integer corresponding to vertex \( v_i \). One can think of this assignment as a collection of chips placed on each of the vertices. We say that a given vertex \( v_i \) can fire if the number of chips it holds, \( c_i \), is greater than or equal to its out-degree. If so, firing leads to a new configuration where a chip travels along each outgoing edge incident to \( v_i \). Thus we obtain a configuration \( C' \) where \( c'_j = c_j + d(v_i, v_j) \) and \( c'_i = c_i - d(v_i) \). Here \( d(v_i, v_j) \) equals the number of directed edges from \( v_i \) to \( v_j \), and \( d(v_i) \) is the out-degree of \( v_i \), which of course equals \( \sum_{j \neq i} d(v_i, v_j) \).

Many interesting problems arise from this definition. For example, it can be shown [12] that the set of configurations reachable from an initial choice of a vector forms a distributive lattice. Thus one can ask combinatorial questions such as examining the structure of this lattice as a poset. Other computations such as the minimal number or expected number of firings necessary to reach configuration \( C' \) from \( C \) are also common in dynamical systems.

A variant of the standard chip-firing game, known as the dollar game, due to Biggs [1] has the same set-up as before with three changes.

1. We designate one vertex \( v_0 \) to be the bank, and allow \( c_0 \) to be negative. All the other \( c_i \)'s still must be nonnegative.
2. To limit extraneous configurations, we presume that the sum \( \sum_{i=0}^{#V-1} c_i = 0 \). (Thus in particular, \( c_0 \) will be non-positive.)
(3) The bank, i.e., vertex $v_0$, is only allowed to fire if no other vertex can fire. Note that since we now allow $c_0$ to be negative, $v_0$ is allowed to fire even when it is smaller than its outdegree.

A configuration is stable if $v_0$ is the only vertex that can fire, and configuration $C$ is recurrent if there is a firing sequence which leads back to $C$. Note that this will necessarily require the use of $v_0$ firing. We call a configuration critical if it is both stable and recurrent.

**Proposition 2.** For any initial configuration satisfying rules (1) and (2) above, there exists a unique critical configuration that can be reached by a firing sequence, subject to rule (3).

**Proof.** See [9] for original proof, or [1] for slightly different technique. □

The critical group of graph $G$, with respect to vertex $v_0$ is the set of critical configurations, with addition given by $C_1 \oplus C_2 = C_1 + C_2$. Here $+$ signifies the usual pointwise vector addition and $C_3$ represents the unique critical configuration reachable from $C_3$. When $v_0$ is understood, we will abbreviate this group as the critical group of graph $G$, denoting it as $K(G)$.

**Theorem 3** (Gabrielov [9]). $K(G)$ is in fact an abelian (associative) group.

**Proof.** If we consider the initial configuration $C_3 = C_1 + C_2$, then by Proposition 2 there is a unique critical configuration reachable from $C_3$. Additionally, we can compute $(C_0 \oplus C_1) \oplus C_2$ or $C_0 \oplus (C_1 \oplus C_2)$ by adding together $C_0 + C_1 + C_2$ pointwise, and then reducing once at the end, rather than reducing twice. Thus associativity and commutativity follow. □

The savvy reader might have noticed that the firing of vertex $v_i$ alters the configuration vector exactly as the subtraction of the $i$th row of the Laplacian matrix. In fact, for any graph we have the following general fact.

**Proposition 3.** If $K(G)$ denotes the critical group of graph $G$, on $(k+1)$ vertices, with bank vertex $v_0$, and $L_0$ denotes the reduced Laplacian of $G$ with the row and column corresponding to $v_0$ deleted, then

$$K(G) \cong \text{coker } L_0 = \mathbb{Z}^k / \text{Im } L_0 z^k.$$ 

**Corollary 1.** $|K(G)| = \det(L_0) = \#\{\text{directed rooted spanning trees of graph } G\}$.

**Proof.** We use the algebraic fact that when a matrix $M$ is nonsingular, $|\det(M)| = |\text{coker } M|$ for the first equality. The second equality follows from the Matrix-Tree Theorem. □

Corollary 1 allows us to extend the identities of Theorems 1 and 2 to one which exhibits a reciprocity between the two families of groups described above.

**Theorem 4.** Letting $N_k(q, N_1)$ be the bivariate expression for the cardinality $|E(F_{q^k})|$ and $K(W_k(q, t))$ be the critical group on the $(k+1)$ vertex $(q, t)$-wheel graph, we have

$$|K((q, t)-W_k)| = -N_k(q, -t).$$

It is this theorem that motivates the remainder of this paper as we explore deeper properties of the $W_k(q, t)$’s and compare them to the case of elliptic curves.
4. Critical Configurations for the $W_k(q,t)$ Graphs

We begin our exploration by completely characterizing critical configurations of the $(q,t)$-wheel graphs. This new characterization of critical configurations also yields a bijection between critical configurations and spanning trees, as given in Theorem 6.

We take root and hub $v_0$ to be the bank vertex as a convention, and thus a configuration of this graph is a vector of length $k$ which encodes the number of chips on each of the rim vertices, which are labelled in clockwise order.

**Lemma 1.** A configuration $C = [c_1, c_2, \ldots , c_k]$ of the wheel graph $W_k(q,t)$ is stable if and only if $0 \leq c_i \leq q+t$ for all $1 \leq i \leq k$. Furthermore, any configuration which is not stable can be reduced to a stable one by the chip-firing rules.

**Proof.** It is clear that we disallow $c_i < 0$ as a legal configuration by our definition. If such a configuration were to come up, we could add $t$ to every value $c_i$, simulating the firing of the central vertex, until we have a nonnegative vector. If on the other hand, there exists $c_i \geq 1 + q + t$, with all other $c_i \geq 0$, then vertex $v_i$ can fire resulting in a new nonnegative configuration, with the sum of the $c_i$'s having been decreased by $t$. Thus eventually, we will arrive at a configuration with all $c_i$'s satisfying $0 \leq c_i \leq q+t$. Otherwise, if all $c_i$ are in the specified range, we have a stable configuration where no vertex except the hub can fire. □

**Lemma 2.** Let $C = [c_1, \ldots , c_k]$ be a stable configuration. Then $C$ is critical if and only if $C + [t] = [c_1 + t, \ldots , c_k + t]$ is not stable.

**Proof.** The stability of $C$ implies that the hub vertex is the only one that can fire. Configuration $C + [t]$ is either stable as well, or $C + [t]$ reduces to $C$ via the firing of vertices $v_i$ through $v_k$, each exactly once. In the case that $C + [t]$ is stable, then there exists some minimum integer $d \geq 2$ such that $C + [dt]$ is not stable, but such a configuration reduces to $C + [(d - 1)t]$, and thus $C$ does not recur. □

**Lemma 3.** Any critical configuration $[c_1, \ldots , c_k]$ will have at least one element $c_i = B$ such that $B \in \{1 + q, \ldots , q + t\}$.

**Proof.** Assume otherwise. Then $c_i \in \{0, 1, \ldots , q\}$ for all $1 \leq i \leq k$. Consequently, we may add $t$ to every $c_i$ and still obtain a stable configuration. Thus the initial configuration is not critical by Lemma 2. □

**Theorem 5.** Any configuration $C$ is critical if and only if it consists of a circular concatenation of blocks of the form

$$B, M_1, \ldots , M_j \text{ or } B, M_1, \ldots , M_j, 0 \text{ or } B, M_1, \ldots , M_j, 0, q, q, \ldots , q$$

where $B \in \{1 + q, \ldots , q + t\}$ and $M_i \in \{1, \ldots , q\}$.

We have already shown that there exists at least one $c_i = B$ with $B > q$. Thus we prove this theorem by induction on $n$, the number of such elements. Consider such a block in context, and presume it is of the form

$$\cdots , M_n^{k_n} | B_1, M_1^1, M_1^{k_1} | B_2, \cdots$$

where $M_i^p \in \{0, 1, \ldots , q\}$ and $B_p \in \{1 + q, \ldots , q + t\}$. Here $M_n^{k_n}$ and $B_2$ represent the end of the previous block and the beginning of the next block, respectively. The heart of the proof is the verification of the following proposition.
Proposition 4. A configuration in the form of \( \mathbf{1} \) cannot be recurrent unless \( M_p^{k_t} = 0 \) implies that the remaining \( M_i \)'s, i.e. \( M_p^{k_t+1} \) through \( M_p^{k_r} \), are equal to \( q \).

Proof. Without loss of generality, we will work with \( p = 1 \) and let \( j_1 = j, k_1 = k \), \( M \neq M_0 \). Assume that \( M_1 \) through \( M_i^{-1} \in \{1, 2, \ldots, q\} \). We add \( t \) to every element of \( C \), getting \( C + [t] \), and then reduce via the chip-firing rules whenever we encounter an element with value greater or equal to \( 1 + q + t \). Configuration \( C + [t] \) contains element \( B_1 + t \), with value \( \geq 1 + q + t \), but all other elements of the block are \( < 1 + q + t \). Once we replace \( B_1 + t \) with \( B_1 - 1 - q \), and its neighbors with \( M_0 + t + 1 \) and \( M_1 + q + t \), respectively, we reduce \( M_1 + q + t \) since its entry is now \( \geq 1 + q + t \). We continue inductively until we reach the end of the block or \( M_1 + q + t \) which is less than \( 1 + q + t \) since \( M_1 = 0 \) by assumption. At this point, the block looks like

\[
M_0 + t + 1 | B_1 - q, M_1^1, \ldots, M_i^{-1} - 1, q + t, M_i^1 + 1 + t, \ldots, M_i^k + t | B_2 + t.
\]

Since \( B_2 + t \geq 1 + q + t \), we can reduce this block further as

\[
M_0 + t + 1 | B_1 - q, M_1^1, \ldots, M_i^{-1} - 1, q + t, M_i^1 + 1 + t, \ldots, M_i^k + t + 1 | B_2 - 1 - q.
\]

By propagating the same reductions to the rest of the configuration, we reduce to a configuration \( C' \) which is made up of blocks of the form

\[
B_p - q, M_p^1, \ldots, M_p^{j_p-1} - 1, q + t, M_p^{j_p+1} + t, \ldots, M_p^{k_p} + t + 1
\]

in lieu of

\[
B_p, M_p^1, \ldots, M_p^{j_p-1}, 0, M_p^{j_p+1}, \ldots, M_p^{k_p}.
\]

Since \( M_i^j \leq q \), all elements of \( C' \) are less than \( 1 + q + t \) except possibly for the last elements of each block, e.g. \( M_i^k + t + 1 \). If all of the \( M_i^k \)'s are less than \( q \), then \( C' \) is stable, and thus the original configuration \( C \) is not recurrent, nor critical as assumed.

Thus, without loss of generality, assume that \( M_i^k = q \). We then can reduce block

\[
M_0 + t + 1 | B_1 - q, M_1^1, \ldots, M_i^{-1} - 1, q + t, M_i^1 + 1 + t, M_i^1 + 2 + t, \ldots, M_i^{-1} + t, q + t + 1 | B_2 - 1 - q
\]

and obtain

\[
M_0 + t + 1 | B_1 - q, M_1^1, \ldots, M_i^{-1} - 1, q + t, M_i^1 + 1 + t, M_i^1 + 2 + t, \ldots, M_i^{-1} + t + 1, 0 | B_2 - 1.
\]

By analogous logic, we must have that \( M_i^{k-1} = q \) and continuing iteratively, we reduce to

\[
M_0 + t + 1 | B_1 - q, M_1^1, \ldots, M_i^{-1} - 1, q + t + 1, 0, q, \ldots, q, q | B_2 - 1
\]

which is equivalent to

\[
M_0 + t + 1 | B_1 - q, M_1^1, \ldots, M_i^{-1}, 0, q, \ldots, q, q | B_2 - 1.
\]

Finally, \( M_0 = M_n^{k_r} \) so we indeed obtain

\[
q | B_1, M_1^1, \ldots, M_i^{-1}, 0, q, \ldots, q, q | B_2
\]

after iterating over all the blocks to the right and wrapping around. \( \square \)

From the Proposition, it is clear that any configuration built according to the hypothesis of Theorem 5 is recurrent. Stability and thus criticality follow from Lemma 3. Furthermore, since our initial format as given in (1) is in fact that of a general stable configuration, we in fact have proven both directions of Theorem 5.
We use this characterization to describe an explicit bijection between critical configurations and spanning trees.

**Theorem 6.** There exists an explicit bijection between critical configurations and spanning trees for the \((q,t)\)-wheel graphs, thereby inducing a group structure onto the set of spanning trees of \(W_k(q,t)\).

Specifically pick one of the vertices on the rim to be \(v_1\), and label \(v_2\) through \(v_k\) clockwise. Label the central hub as \(v_0\). For \(i\) between 1 and \(k\), if \(1 \leq c_i \leq q\), then fill in the arc between \(v_{i-1}\) and \(v_i\), labeling it with the number \(c_i\). (In the case of \(i = 1\) we use the arc between \(v_k\) and \(v_1\) instead.) If \(1 + q \leq c_i \leq q + t\) then fill in the spoke between \(v_0\) and \(v_i\) and label it with number \(c_i\). After filling in the edges as indicated we will get a subgraph of a spanning tree. To complete this subgraph to a tree, fill in additional arcs using the following rule: one may fill in an arc from \(v_i\) to \(v_j\) if and only if \(c_i \geq 1 + q\). For each such \(c_i\), consider the block \(c_i, M_i^1, \ldots, M_i^t\) or \(c_i, M_i^1, \ldots, M_i^t, 0, q, q, \ldots, q\) where \(1 \leq M_i^t \leq q\). Notice that \(M_i^1, \ldots, M_i^t\) corresponds to an arc extending clockwise from the associated spoke, and each edge \((v_j, v_{j+1})\) is given a label from the set \(\{1, 2, \ldots, q\}\). Finally, whenever \(c_i = 0\), there is no arc or spoke in the tree corresponding to that coordinate. However, we do fill out the graph into a tree on all vertices by choosing rim edges which lie clockwise from a coordinate of zero, but counter-clockwise from a coordinate greater than \(q\). Such edges only have the label of \(q\), and thus we recover the definition of one possible counter-clockwise edge between a given pair of consecutive rim vertices. Since this map is injective whose image has the correct cardinality, we have the desired bijection. \(\square\)

**Remark 1.** After discovering the above bijection, the author learned of the Biggs-Winkler [2] bijection via the burning algorithm for the case of undirected simple graphs. When we set \((q,t) = (1,1)\), we do indeed recover the undirected simple graphs \(W_k\) for which the above bijection and the Biggs-Winkler algorithm agree.

5. **The Frobenius Map and Elliptic Cyclotomic Polynomials**

One of the fundamental properties of an elliptic curve over a finite field is the existence of the Frobenius map. In particular, for a finite field \(\mathbb{F}_q\), where \(q = p^k\), \(p\) prime, the Galois group \(\text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)\) is cyclic generated by the map \(\pi: x \mapsto x^q\). (In fact \(\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)\) is also cyclic and generated by the analogous map, \(x \mapsto x^p\), but in this paper, we will always be using the map which fixes ground field \(\mathbb{F}_q\).) This map induces an associated map on varieties. Namely letting \(\overline{\mathbb{F}_q}\) denote the algebraic closure of \(\mathbb{F}_q\), by abuse of notation we also let \(\pi\) denote the map on elliptic curves.

\[
\pi : E(\overline{\mathbb{F}_q}) \to E(\overline{\mathbb{F}_q}) \\
(x,y) \mapsto (x^q, y^q)
\]

We summarize here some well-known facts about the Frobenius map on elliptic curves. An elliptic curve can be given a group structure; for example see [20] or [22]. The identity of this group is the point at infinity, which we denote as \(P_\infty\).
Lemma 4.

\[\pi(P \oplus Q) = \pi(P) \oplus \pi(Q),\]
\[\pi^k(P) = p \quad \text{if and only if} \quad P \in E(F_q^k), \quad \text{and} \]
\[E(F_q) \subset E(F_{q^{k_1}}) \subset E(F_{q^{k_2}}) \subset E(F_{q^{k_3}}) \subset \cdots \subset E(F_q)\]

whenever we have the divisibilities \(k_1|k_2, k_2|k_3, \) and so on.

Proof. See [20]. \[\Box\]

Using this Lemma, we showed in [17] that the equation \(N_k(q, N_1) = \ker(1 - \pi^k)\) can be factored, such that the left-hand-side factors into integral irreducibles simultaneously as \((1 - \pi^k)\) factors into cyclotomic polynomials with respect to \(\pi\). In particular we can get an entire sequence of such factors.

**Proposition 5** (Proposition 12 of [17]). There exists a family of bivariate irreducible integral polynomials, indexed by positive integers, which we denote as \(ECyc_d(q, N_1)\) such that

\[N_k(q, N_1) = \prod_{d|k} ECyc_d(q, N_1)\]

for all \(k \geq 1\).

We refer to these polynomials as elliptic cyclotomic polynomials, and observe the following geometric interpretation.

**Theorem 7** (Theorem 7 of [17]). For all \(d \geq 1\),

\[ECyc_d = |\ker(Cyc_d(\pi)) : E(F_q) \to E(F_q)|\]

Proof. Proposition 5 and Theorem 7 both follow from the above factorization with respect to cyclotomic polynomials. For details, see [17]. \[\Box\]

5.1. Analogues of Elliptic Cyclotomic Polynomials for Wheel Graphs.

Since

\[N_k = \prod_{d|k} ECyc_d(q, N_1)\]

and \(W_k(q, t) = -N_k\bigg|_{N_1 \to -t}\), it also makes sense to consider the decomposition

\[W_k(q, t) = \prod_{d|k} WCyc_d(q, t)\]
where $WCyc_d(q, t) = t$, and $WCyc_d(q, t) = ECyc_d|_{N_1 \rightarrow t}$ for $d \geq 2$. A few of the first several $WCyc_d(q, t)$'s are given below:

- $WCyc_1 = t$
- $WCyc_2 = t + 2(1 + q)$
- $WCyc_3 = t^2 + (3 + 3q)t + 3(1 + q + q^2)$
- $WCyc_4 = t^2 + (2 + 2q)t + 2(1 + q^2)$
- $WCyc_5 = t^4 + (5 + 5q)t^3 + (10 + 15q + 10q^2)t^2 + (10 + 15q + 15q^2 + 10q^3)t + 5(1 + q + q^2 + q^3 + q^4)$
- $WCyc_6 = t^2 + (1 + q)t + (1 - q + q^2)$
- $WCyc_7 = t^4 + (4 + 4q)t^3 + (6 + 8q + 6q^2)t^2 + (4 + 4q + 4q^2 + 4q^3)t + 2(1 + q^4)$
- $WCyc_8 = t^6 + (6 + 6q)t^5 + (15 + 24q + 15q^2)t^4 + (21 + 36q + 36q^2 + 21q^3)t^3$
  \[+ (18 + 27q + 27q^2 + 27q^3 + 18q^4)t^2 + (9 + 9q + 9q^2 + 9q^3 + 9q^4 + 9q^5)t + 3(1 + q^3 + q^6)\]
- $WCyc_9 = t^4 + (3 + 3q)t^3 + (4 + 3q + 4q^2)t^2 + (2 + q + q^2 + 2q^3)t + (1 - q + q^2 + q^3 + q^4)$
- $WCyc_{10} = t^4 + (3 + 3q)t^3 + (4 + 3q + 4q^2)t^2 + (2 + q + q^2 + 2q^3)t + (1 - q + q^2 + q^3 + q^4)$
- $WCyc_{12} = t^4 + (4 + 4q)t^3 + (5 + 8q + 5q^2)t^2 + (2 + 2q + 2q^2 + 2q^3)t + (1 - q^2 + q^4)$

**Question 1.** Is there an analogue of Theorem 7 for the family of $WCyc_d(q, t)$'s?

The answer to this question is the inspiration for the next section.

### 6. Maps between Critical Groups

Fix integers $q \geq 0$ and $t \geq 1$ for this section. Our goal is now to understand the sequence of

\[
\{ K(W_k(q, t)) \}_{k=1}^{\infty}
\]

in a way that corresponds to the chain

\[
E(\mathbb{F}_q) \subset E(\mathbb{F}_q^{t_1}) \subset E(\mathbb{F}_q^{t_2}) \subset E(\mathbb{F}_q^{t_3}) \subset \ldots \subset E(\mathbb{F}_q^{t_k})
\]

for $k_1|k_2|k_3$, etc.

**Proposition 6.** The identity map induces an injective group homomorphism between $K(W_{k_1}(q, t))$ and $K(W_{k_2}(q, t))$ whenever $k_1|k_2$. More precisely, we let $K(W_{k_1}(q, t))$ embed into $K(W_{k_2}(q, t))$ by letting $w \in K(W_{k_1}(q, t))$ map to the word $w \ldots w \in K(W_{k_2}(q, t))$ using $\frac{k_2}{k_1}$ copies of $w$.

Define $\rho$ to be the rotation map on $K(W_k(q, t))$. If we consider elements of the critical group to be configuration vectors, then we mean circular rotation of the elements to the left. On the other hand, $\rho$ acts by rotating the rim vertices of $W_k$ counter-clockwise if we view elements of $K(W_k(q, t))$ as spanning trees.

**Proposition 7.** The kernel of $(1 - \rho^{k_1})$ acting on $K(W_{k_2}(q, t))$ is subgroup $K(W_{k_1}(q, t))$ whenever $k_1|k_2$.

**Proof.** We prove both of these propositions simultaneously, by noting that chip firing is a local process. Namely, if $k_1$ divides $k_2$ and we add two configurations of $W_{k_1}(q, t)$ together pointwise to get configuration $C$, then lift $C$ to a length $k_2$ configuration $C'$ of $W_{k_2}(q, t)$ by periodically extending length $k_1$ vector $C$. Then the claim is that if $C$ reduces to unique critical configuration $C'$, then $C'$ also reduces to $C'$'s periodic extension. To see this, observe that every time vertex $v \in W_{k_1}(q, t)$ fires in the reduction algorithm, then we could simultaneously fire the set of vertices of $W_{k_2}(q, t)$ in the image of $v$ after lifting. In other words, if $v_1 \in W_{k_1}(q, t)$ fires, we fire \{v_1', v_1'+k_2/k_1, v_1'+2k_2/k_1, \ldots \} \in W_{k_2}(q, t)$ thus obtaining the lift of the configuration reached after $v$ fires. \(\square\)
Example: \([2, 4, 2] \oplus [0, 4, 1] \equiv [1, 0, 4] \text{ in } W_3(q = 3, t = 2) \) versus
\([2, 4, 2, 2, 4, 2] \oplus [0, 4, 1, 0, 4, 1] \equiv [1, 0, 4, 1, 0, 4] \text{ in } W_6(q = 3, t = 2)\)

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
2 & 2 & 0 \\
4 & 4 & 1 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
2 & 2 & 4 \\
4 & 4 & 0 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
0 & 4 & 1 \\
1 & 0 & 4 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
2 & 2 & 4 \\
4 & 4 & 0 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
2 & 2 & 4 \\
4 & 4 & 0 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
0 & 4 & 1 \\
1 & 0 & 4 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
2 & 2 & 4 \\
4 & 4 & 0 \\
\end{array}\]

\[\begin{array}{ccc}
\text{(1)} & \text{(2)} & \text{(3)} \\
0 & 4 & 1 \\
1 & 0 & 4 \\
\end{array}\]

**Figure 1.** Illustrating Propositions [1] and [2]

We therefore can define a direct limit

\[K(\mathbb{W}(q, t)) \cong \bigcup_{k=1}^{\infty} K(W_k(q, t))\]

where \(\rho\) provides the transition maps.

Another view of \(K(\mathbb{W}(q, t))\) is as the set of bi-infinite words which are (1) periodic, and (2) have fundamental subword equal to a configuration vector in \(K(W_k(q, t))\) for some \(k \geq 1\). In this interpretation, map \(\rho\) acts on \(K(\mathbb{W}(q, t))\) also. In this case, \(\rho\) is the shift map, and in particular we obtain

\[K(W_k(q, t)) \cong \text{Ker}(1 - \rho^k) : K(\mathbb{W}(q, t)) \to K(\mathbb{W}(q, t)).\]

We now can describe our variant of Theorem 7.

**Theorem 8.**

\[WCyc_d = \left| \text{Ker} \left( Cyc_d(\rho) \right) : K(\mathbb{W}(q, t)) \to K(\mathbb{W}(q, t)) \right|\]

where \(\rho\) denotes the shift map, and \(K(\mathbb{W}(q, t))\) is the direct limit of the sequence \(\{K(W_k(q, t))\}_{k=1}^{\infty}\).

**Proof.** The proof is analogous to the elliptic curve case. Since the maps \(Cyc_{d_1}(\rho)\) and \(Cyc_{d_2}(\rho)\) are group homomorphisms, we get

\[\left| \text{Ker} \left( Cyc_{d_1}(\rho) Cyc_{d_2}(\rho) \right) \right| = |\text{Ker} Cyc_{d_1}(\rho)| \cdot |\text{Ker} Cyc_{d_2}(\rho)|\]

and the rest of the proof follows as in [17].
Consequently we identify shift map $\rho$ as being the analogue of the Frobenius map $\pi$ on elliptic curves. In addition to $\rho$’s appearance in
\[ K(W_k(q, t)) \cong \text{Ker}(1 - \rho^k) : K(W(q, t)) \to K(W(q, t)) \text{ just as } \]
\[ E(F_{q^k}) = \text{Ker}(1 - \pi^k) : E(F_q) \to E(F_q). \]
and Theorem 8 another comparison with $\pi$ is highlighted below.

**Theorem 9.** As a map from $K(W(q, t))$ to itself, we get
\[ \rho^2 - (1 + q + t)\rho + q = 0. \]
Note that this quadratic is a simple analogue of the characteristic equation
\[ \pi^2 - (1 + q - N_1)\pi + q = 0 \]
of the Frobenius map $\pi$. In the case of elliptic curves, this equation is proven using an analysis of the endomorphism ring of an elliptic ring or the Tate Module. However in the critical group case, linear algebra suffices.

**Proof of Theorem 9.** Since elements of $K(W(q, t))$ are periodic extensions of some vector in $K(W_k(q, t))$ for some $k$, it suffices to prove the identity on $K(W_k(q, t))$ for all $k \geq 1$. In particular, if $\rho(C) = [c_2, c_3, \ldots, c_k, c_1]$, then we notice that $\rho^2(C) - (1 + q + t)\rho(C) + q \cdot C$ equals
\[
\begin{bmatrix}
q & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-1 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{pmatrix}
-1 + q + t \\
q \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
1 \\
q \\
\end{pmatrix}
\]
which equals $[0, 0, 0, 0, \ldots, 0, 0]$ modulo the rows of the reduced Laplacian matrix. \qed

An even more surprising connection is the subject of the next subsection.

**6.1. Group Presentations.** It is well known that an elliptic curve over a finite field has a group structure which is the product of at most two cyclic groups. This result can be seen as a manifestation of the lattice structure of an elliptic curve over the complex numbers. One way to prove this is by showing that for $gcd(N, p) = 1$, the $[N]$-torsion subgroup of $E(F_p)$ (also denoted as $E[N]$) is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/\mathbb{N}\mathbb{Z}$ and that $E[p^r]$ is either $0$ or $\mathbb{Z}/p^r\mathbb{Z}$.

Since we know that the critical group of graphs are also abelian groups, this motivates the question: what is the group decomposition of the $K(G)$’s? The case of a simple wheel graph $W_k$ was explicitly found in [11] to be
\[ \mathbb{Z}/L_k\mathbb{Z} \times \mathbb{Z}/L_k\mathbb{Z} \text{ or } \mathbb{Z}/F_{k-1}\mathbb{Z} \times \mathbb{Z}/5F_{k-1}\mathbb{Z} \]
depending on whether $k$ is odd or even, respectively. Here $L_k$ is the $k$th Lucas number and $F_k$ is the $k$th Fibonacci number.

Determining such structures of critical groups has been the subject of several papers recently, e.g. [11, 13], and a common tool is the Smith normal form of the Laplacian. We use the same method here to prove the following comparison of elliptic curves and critical groups.
**Theorem 10.** $K(W_k(q,t))$ is isomorphic to at most two cyclic groups, a property that this sequence of critical groups shares with the family of elliptic curve groups over finite fields.

**Proof.** The Smith normal form of a matrix is unchanged by

1. Multiplication of a row or a column by $-1$.
2. Addition of an integer multiple of a row or column to another.
3. Swapping of two rows or two columns.

We let $M_k$ be the $k$-by-$k$ circulant matrix $circ(1 + q - N_1, -q, 0,\ldots, 0, -1)$ as in Section 2, and let $\overline{M}_k$ denote the $k$-by-$k$ matrix $circ(1 + q + t, -q, 0,\ldots, 0, -1)$, the reduced Laplacian of the $(q,t)$-wheel graph. To begin we note after permuting rows cyclically and multiplying through all rows by $(-1)$ that we get

$$\overline{M}_k^T \equiv \begin{bmatrix}
1 & 0 & \ldots & 0 & q & -1 - q - t \\
-1 - q - t & 1 & 0 & \ldots & 0 & q \\
qu & -1 - q - t & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & q & -1 - q - t & 1 & 0
\end{bmatrix}.$$ (1)

Except for an upper-right corner of three nonzero entries, this matrix is lower-triangular with ones on the diagonal. Adding a multiple of the first row to the second and third rows, respectively, we obtain a new matrix with vector $[1, 0, 0,\ldots, 0]^T$ as the first column. Since we can add multiples of columns to one another as well, we also obtain a matrix with vector $[1, 0, 0,\ldots, 0]$ as the first row.

This new matrix will again be lower triangular with ones along the diagonal, except for nonzero entries in four spots in the last two columns of rows two and three. By the symmetry and sparseness of this matrix, we can continue this process, which will always shift the nonzero block of four in the last two columns down one row. This process will terminate with a block diagonal matrix consisting of $(k-2)$ 1-by-1 blocks of element 1 followed by a single 2-by-2 block. □

Since this proof is constructive, as an application we arrive at a presentation of the $K(W_k(q,t))$’s as the cokernels of a 2-by-2 matrix whose entries have combinatorial interpretations.

**Corollary 2.** For $k \geq 3$, the Smith normal form of $\overline{M}_k$ is equivalent to a direct sum of the identity matrix and

$$\begin{bmatrix}
q\hat{F}_{2k-4} + 1 & q\hat{F}_{2k-2} \\
\hat{F}_{2k-2} & \hat{F}_{2k} - 1
\end{bmatrix}$$

where $\hat{F}_{2k}(q,t)$ is defined as

$$\hat{F}_{2k}(q,t) = \sum_{S \subseteq \{1,2,\ldots,2k\} : \text{S contains no two consecutive elements}} q^\# \text{ even elements in } S \cdot k - \#S.$$ (2)

Notice that these are a bivariate analogue of the Fibonacci numbers.
Proposition 8. The Smith normal form of \( (-1)^k \sum_{S \subseteq \{1,2,\ldots,2k-2\}} q^\# \text{even elements in } S \left(-N_1\right)^{k-\# S} \) appeared in [17] where they had a plethystic interpretation as \( e_k[1 + q - \alpha_1 - \alpha_2] \) such that \( \alpha_1 \) and \( \alpha_2 \) are the two roots of \( qT^2 - (1 + q - N_1)T + 1 \).

Remark 2. An analogous family of polynomials, namely the \( E_\alpha \)'s, defined by \( E_k(q, N_1) = (-1)^k \sum_{S \subseteq \{1,2,\ldots,2k-2\}} q^\# \text{even elements in } S \left(-N_1\right)^{k-\# S} \) appeared in [17] where they had a plethystic interpretation as \( e_k[1 + q - \alpha_1 - \alpha_2] \) such that \( \alpha_1 \) and \( \alpha_2 \) are the two roots of \( qT^2 - (1 + q - N_1)T + 1 \).

Before giving the proof of Corollary 2, we show the following more general result. Define \( \widetilde{M}_k \) as the following \( k \)-by-\( k \) matrix:

\[
\widetilde{M}_k = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & A & B \\
-\Delta & 1 & 0 & \ldots & 0 & 0 & C & D \\
q & -\Delta & 1 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -\Delta & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & q & -\Delta & W & X \\
0 & 0 & 0 & \ldots & 0 & q & Y & Z \\
\end{bmatrix}.
\]

Proposition 8. The Smith normal form of \( \widetilde{M}_k \) is equivalent to

\[
\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & a & b \\
0 & 0 & \ldots & 0 & c & d \\
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
a & b \\
c & d \end{bmatrix} = \begin{bmatrix}
\Delta & 1 \\
-q & 0 \end{bmatrix}^{k-2} \begin{bmatrix}
A & B \\
C & D \end{bmatrix} + \begin{bmatrix}
W & X \\
Y & Z \end{bmatrix}.
\]

Proof. We represent the last two columns of \( \widetilde{M}_k \) as

\[
\begin{bmatrix}
a''_m & b''_m \\
a'_{m+1} & b'_{m+1} \\
\vdots & \vdots \\
a''_k & b''_k \\
\end{bmatrix},
\]

and letting

\[
\begin{bmatrix}
a''_m & b''_m \\
a'_{m+1} & b'_{m+1} \\
\vdots & \vdots \\
a''_k & b''_k \\
\end{bmatrix} = \begin{bmatrix}
a''_m & b''_m \\
a'_{m+1} & b'_{m+1} \\
\vdots & \vdots \\
a''_k & b''_k \\
\end{bmatrix},
\]

signify the last two columns after completing the steps outlined above, i.e. subtracting a multiple of the first row from the second and third row, and then using the first column to cancel out the entries \( a''_1 \) and \( b''_1 \).

Continuing inductively, we get the relations

\[
\begin{align*}
a''_m &= \Delta a''_{m-1} + a'_m \\
b''_m &= \Delta b''_{m-1} + b'_m \\
a'_{m+1} &= qa''_{m-1} + a_{m+1} \\
b'_{m+1} &= qb''_{m-1} + b_{m+1},
\end{align*}
\]

which we encode as the matrix equation

\[
\begin{bmatrix}
a''_m & b''_m \\
a'_{m+1} & b'_{m+1} \\
\end{bmatrix} = \begin{bmatrix}
\Delta & 1 \\
-q & 0 \end{bmatrix} \begin{bmatrix}
a''_{m-1} & b''_{m-1} \\
a'_{m} & b'_{m} \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
a_{m+1} & b_{m+1} \end{bmatrix}.
\]
Letting $a_3, b_3, \ldots, a_{k-2}, b_{k-2} = 0$ and using 
\[
\begin{bmatrix}
\Delta & 1 \\
-q & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
W & X \\
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 \\
Y & Z \\
\end{bmatrix}
= 
\begin{bmatrix}
W & X \\
Y & Z \\
\end{bmatrix}
\]
we obtain the desired result.

We now wish to consider the special case $\Delta = 1 + q + t$, 
\[
\begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix}
= 
\begin{bmatrix}
q & -\Delta \\
0 & q \\
\end{bmatrix},
\]
and 
\[
\begin{bmatrix}
W & X \\
Y & Z \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
-\Delta & 1 \\
\end{bmatrix}.
\]
To simplify our expression further, we utilize the following formula for a specific sequence of matrix powers.

**Lemma 5.** For all $m \geq 2$,
\[
\begin{bmatrix}
1 + q + t & 1 \\
-q & 0 \\
\end{bmatrix}^m = 
\begin{bmatrix}
\hat{F}_{2m} & \hat{F}_{2m-2} \\
-q\hat{F}_{2m-2} & -q\hat{F}_{2m-4} \\
\end{bmatrix}.
\]

**Proof.** We verify the result for $m = 2$ using the fact that
\[
\begin{align*}
\hat{F}_0 &= 1 \\
\hat{F}_2 &= t + (1 + q) \\
\hat{F}_4 &= t^2 + (2 + 2q)t + (1 + q + q^2) = (1 + q + t)^2 - q.
\end{align*}
\]

The product 
\[
\begin{bmatrix}
1 + q + t & 1 \\
-q & 0 \\
\end{bmatrix}
\begin{bmatrix}
\hat{F}_{2m} & \hat{F}_{2m-2} \\
-q\hat{F}_{2m-2} & -q\hat{F}_{2m-4} \\
\end{bmatrix}
\]
equals
\[
\begin{bmatrix}
(1 + q + t)\hat{F}_{2m} - q\hat{F}_{2m-2} & (1 + q + t)\hat{F}_{2m-2} - q\hat{F}_{2m-4} \\
-q\hat{F}_{2m} & -q\hat{F}_{2m-2} \\
\end{bmatrix}.
\]
Thus it suffices to demonstrate 
\[
\hat{F}_{2m+4} = (1 + q + t)\hat{F}_{2m+2} - q\hat{F}_{2m}
\]
by recursion. This recurrence was proven in [17]; the proof is a generalization of the well-known recurrence $F_{2m+4} = 3F_{2m+2} - F_{2m}$ for the Fibonacci numbers.

Namely, the polynomial $\hat{F}_{2m+4}$ is a $(q, t)$-enumeration of the number of chains of $2m + 4$ beads, with each bead either black or white, and no two consecutive beads both black. Similarly $(1 + q + t)\hat{F}_{2m+2}$ enumerates the concatenation of such a chain of length $2m + 2$ with a chain of length 2. One can recover a legal chain of length $2m + 4$ this way except in the case where the (2m + 2)nd and (2m + 3)rd beads are both black. Since this forces the (2m + 1)st and (2m + 4)th beads to be white, such cases are enumerated by $q\hat{F}_{2m}$ and this completes the proof. With this recurrence, Lemma 5 is proved.

**Proof of Corollary** Here we give the explicit derivation of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in terms of the $\hat{F}_k(q, t)$’s. By Proposition and Lemma we let $m = k - 2$ and we obtain 
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}
\begin{bmatrix}
\hat{F}_{2k-4} & \hat{F}_{2k-6} \\
-q\hat{F}_{2k-6} & -q\hat{F}_{2k-8} \\
\end{bmatrix}
\begin{bmatrix}
q & -1 - q - t \\
0 & q \\
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 0 \\
-1 - q - t & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
q\hat{F}_{2k-4} + 1 & -(1 + q + t)\hat{F}_{2k-4} + q\hat{F}_{2k-6} \\
-q^2\hat{F}_{2k-6} - 1 - q - t & (1 + q + t)q\hat{F}_{2k-6} - q^2\hat{F}_{2k-8} + 1 \\
\end{bmatrix}
\]
when reducing $\overline{M}_k^T$ to a 2-by-2 matrix with an equivalent Smith normal form.
We apply the recursion $\hat{F}_{2m+4} = (1 + q + t)\hat{F}_{2m+2} - q\hat{F}_{2m}$ followed by adding a multiple of $(1 + q + t)$ times the first row to the second row, and then use the recursion again to get $\begin{bmatrix} q\hat{F}_{2k-4} + 1 \\ q\hat{F}_{2k-2} \end{bmatrix} - \hat{F}_{2k-2}$. Finally we multiply the second row column through by $(-1)$ and take the transpose, thereby obtaining the desired result.

If one plugs in specific integers for $q$ and $t$ ($q \geq 0, t \geq 1$), then one can reduce the Smith normal form further. In general, the Smith normal form of a 2-by-2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as $\text{diag}(d_1, d_2)$ where $d_1 = \gcd(a, b, c, d)$. The group $K(W_k(q, t))$ is cyclic if and only if $d_1 = 1$ in this case.

**Question 2.** How can one predict what choices of $(k, q, t)$ lead to a cyclic critical group, and can we more precisely describe the group structure otherwise?

As Biggs discusses in [3], being able to find families of graphs with cyclic critical groups might have cryptographic applications just as it is important to find prime powers $q$ and elliptic curves $E$ such that the groups $E(F_q)$ are cyclic.

Answering this question for $W_k(q, t)$'s is difficult since the Smith normal form of even a 2-by-2 matrix can vary wildly as the four entries change, altering the greatest common divisor along with them. However, we give a partial answer to this question below, after taking a segway into a related family of graphs.

**Remark 3.** In [3], Biggs shows that a family of deformed wheel graphs (with an odd number of vertices) have cyclic critical groups. We are able to obtain a generalization of this result here by using Proposition 8. The author thanks Norman Biggs [4] for bringing this family of graphs to the author’s attention.

Biggs defined $\tilde{W}_k$ by taking the simple wheel graph $W_k$ with $k$ rim vertices and adding an extra vertex on one of the rim vertices. Equivalently, $\tilde{W}_k$ can be constructed from $W_{k+1}$ by removing one spoke. We construct a $(q, t)$-deformation of this family by defining $\tilde{W}_k(q, t)$ as the graph $W_{k+1}(q, t)$ where all edges, i.e. spokes, connecting vertex $v_0$ and $v_1$ are removed.

With such a deformation, it is no longer true that this entire family of graphs have cyclic critical groups, but the next theorem gives a precise criterion for cyclicity and further gives an explicit formula for the smaller of the two invariant factors otherwise.

**Theorem 11.** For all $k \geq 1$, let $Q_k = 1 + q + q^2 + \cdots + q^k$. If $\gcd(t, Q_k) = 1$ then the critical group of $\tilde{W}_k(q, t)$ is cyclic. Otherwise, if we let $d_1 = \gcd(t, Q_k)$, then $\tilde{W}_k(q, t) \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_1 d_2\mathbb{Z}$.

**Proof.** Notice, that the reduced Laplacian matrix for $\tilde{W}_k(q, t)$ agrees with matrix $M_{k+1}$ except in the first entry corresponding to the outdegree of $v_1$. After taking the transpose, cyclically permuting the rows, and multiplication by $(-1)$, we obtain a matrix adhering to the hypothesis of Proposition 8 with $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} q & -1 - q \\ 0 & q \end{bmatrix}$
and \[
\begin{bmatrix}
W & X \\
Y & Z
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1 - q + N_1 & 1
\end{bmatrix}.
\]
Thus, the 2-by-2 matrix for this case equals
\[
\begin{bmatrix}
\hat{F}_{2k-2} & \hat{F}_{2k-4} \\
-q\hat{F}_{2k-4} & -q\hat{F}_{2k-6}
\end{bmatrix}
\begin{bmatrix}
q & -1 - q \\
0 & q
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 \\
-1 - q - t & 1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
q\hat{F}_{2k-2} + 1 & -(1 + q)\hat{F}_{2k-2} + q\hat{F}_{2k-4} \\
-q\hat{F}_{2k-4} - 1 - q - t & (1 + q)q\hat{F}_{2k-4} - q^2\hat{F}_{2k-6} + 1
\end{bmatrix}.
\]

As in Corollary \[2\] we add \((1 + q + t)\) times the first row to the second row, and then multiply the second column by \((-1)\) to arrive at
\[
\begin{bmatrix}
a' & b' \\
c' & d'
\end{bmatrix} = \begin{bmatrix}
q\hat{F}_{2k-2} + 1 & (1 + q)\hat{F}_{2k-2} - q\hat{F}_{2k-4} \\
q\hat{F}_{2k} & (1 + q)\hat{F}_{2k} - q^2\hat{F}_{2k-2} + 1
\end{bmatrix} = \begin{bmatrix}
q\hat{F}_{2k-2} + 1 & \hat{F}_{2k} - t\hat{F}_{2k-2} \\
q\hat{F}_{2k} & \hat{F}_{2k+2} - t\hat{F}_{2k-1}
\end{bmatrix}.
\]

We can reduce this by plugging in specific values for \(q\) and \(t\) and checking whether or not \(Q_k = 1 + q + \cdots + q^k\) and \(t\) share a common factor. We know there exist a unique \(d_1\) and \(d_2\) such that \(\begin{bmatrix}a' & b' \\
c' & d'\end{bmatrix} = \begin{bmatrix}d_1 & 0 \\
0 & d_1d_2\end{bmatrix}\). We begin by showing that \(d_1\) must divide \(t\). Suppose otherwise; then looking at the off-diagonal entries \(b'\) and \(c'\), we see \(d_1\) divides \(q\hat{F}_{2k}\) and \(\hat{F}_{2k} - t\hat{F}_{2k-2}\) but not \(t\), and so \(d_1\) must divide either \(q\) or \(\hat{F}_{2k-2}\). However, \(d_1\) must also divide the top left entry, which is \(q\hat{F}_{2k-2} + 1\). Thus we get a contradiction, and conclude that \(d_1|t\).

This greatly limits the possibilities for \(d_1\). Furthermore, if we work modulo \(t\), the equivalence class of \((a', b', c', d')\) (modulo \(d_1\)) does not change.

Letting \(t = 0\) in \(\hat{F}_{2k}\) is equivalent to counting subsets of \(\{1, 2, \ldots, 2k\}\) of size \(k\) with no two elements consecutive. We can choose the subsets of all odds numbers, which will have weight 1. If we then pick element \(2k\) instead of \(2k - 1\), we get a subset of weight \(q\), and inductively, we get a weighted sum of \(1 + q + q^2 + \cdots + q^k\) where the last weight corresponds to the subset of all even numbers. Thus the desired 2-by-2 matrix reduces to
\[
\begin{bmatrix}
q(1 + q + q^2 + \cdots + q^{k-1}) + 1 & 1 + q + q^2 + \cdots + q^k \\
q(1 + q + q^2 + \cdots + q^k) & q^2 + \cdots + q^{k+1}
\end{bmatrix},
\]
modulo \(t\), the gcd of the entries is the quantity \(Q_k\).

Thus we conclude that \(d_1|Q_k\). Combining this fact with \(d_1|t\), we conclude \(d_1|\gcd(Q_k, t)\). However, since we know that \(d_1 \equiv Q_k \pmod{t}\), we have integers \(m_1, m_2\) such that \(t = m_1d_1\), \(Q_k = (m_1m_2 + 1)d_1\), and so \(\gcd(Q_k, t) = d_1 \cdot m_3\) where \(m_3 = \gcd(m_1, m_1m_2 + 1) = 1\). Thus we obtain the equality \(d_1 = \gcd(Q_k, t)\) as desired.

\[\Box\]

**Remark 4.** Notice, that if \(q = 1\) and \(t = 1\) we have \(d_1 = 1\), hence cyclicity in this case. This result was proven by Biggs for case of odd \(k\), and the above proof of Theorem \[1\] specializes to give an alternate proof of this result.

**Remark 5.** The above proof can also be adapted to demonstrate values of \((k, q, t)\) for which the original critical groups, \(K(W_k(q, t))\), are not cyclic. Namely, by pushing through the same argument, we get \(K(W_k(q, t))\) is not cyclic whenever \(K(W_k(q, t))\) is not cyclic. Unfortunately, we do not get an if and only if criterion nor a precise formula for the smaller invariant factor in this case. This is due to the fact that there are cases where \(d_1\) does not divide \(t\) for the undeformed wheel graphs. In particular, this allows the simple wheel graphs to have non-cyclic critical groups.
In addition to a presentation for $K(W_k(q, N_1))$, we also get a more explicit presentation of elliptic curves $E(F_q)$ in certain cases.

**Theorem 12.** If $E(F_q) \cong \mathbb{Z}/N_1\mathbb{Z}$, as opposed to the product of two cyclic groups, and $End(E) \cong \mathbb{Z}[\pi]$, then

$$E(F_q) \cong \mathbb{Z}^k/M_k\mathbb{Z}^k$$

for all $k \geq 1$. That is, $E(F_q)$ is the cokernel of the image of $M_k$. Furthermore, there exists a point $P \in E(F_q)$ with property $\pi^m(P) \neq P$ for all $1 < m < k$ such that we can take $\mathbb{Z}^k$ as being generated by $\{P, \pi(P), \ldots, \pi^{k-1}(P)\}$ under this presentation.

**Proof.** A theorem of Lenstra [13] says that an ordinary elliptic curve over $F_q$ has a group structure in terms of its endomorphism ring, namely,

$$E(F_q) \cong End(E) / (\pi^k - 1).$$

Wittman [23] gives an explicit description of the possibilities for $End(E)$, given $q$ and $E(F_q)$. It is well known, e.g. [20], that the endomorphism ring in the ordinary case is an order in an imaginary quadratic field. This means that

$$End(E) \cong O_g = \mathbb{Z} \oplus g\delta\mathbb{Z}$$

for some $g \in \mathbb{Z}_{\geq 0}$ and $\delta = \sqrt{D}$ or $\frac{1 + \sqrt{D}}{2}$ according to $D$’s residue modulo 4. Wittman shows that for a curve $E$ with conductor $f$, the possible $g$’s that occur satisfy $g/f$ as well as

$$n_1 = \gcd(a - 1, g/f).$$

The conductor $f$ and constant $a$ are computed by rewriting the Frobenius map as $\pi = a + f\delta$, and $n_1$ is the unique positive integer such that $E(F_q) \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ ($n_1|n_2$).

We focus here on the case when $g = f$ and $End(E) \cong \mathbb{Z}[\pi]$. In particular, $n_1$ must be equal to one in this case, and so the condition that $End(E) = \mathbb{Z}[\pi]$ is actually a sufficient hypothesis. Since $E(F_q) \cong \mathbb{Z}[\pi]/(1 - \pi^k)$ in this case, we get

$$E(F_q) \cong \mathbb{Z}[x]/(x^2 - (1 + q - N_1)x + q, x^k - 1)$$

with $x$ transcendent over $Q$. Thus

$$E(F_q) \cong \mathbb{Z}\{1, x, x^2, \ldots, x^{k-1}\} /$$

\[\left(x^2 - (1 + q - N_1)x + q, x^3 - (1 + q - N_1)x^2 + qx, \ldots, x^{k-1} - (1 + q - N_1)x^{k-2} + qx^{k-3}, 1 - (1 + q - N_1)x^{k-1} + qx^{k-2}, x - (1 + q - N_1) + qx^{k-1}\right)\]

and using matrix $M_k$, as defined above, we obtain the desired presentation for $E(F_q)$ in this case. \hfill $\square$

**Question 3.** What can we say in the case of another endomorphism ring, or the case when $E(F_q)$ is not cyclic?

**Question 4.** Even more generally, are there other families of varieties $F$ and other families of graphs $G$ such that the Jacobian groups of $F$ correspond to the critical groups of $G$?
7. Connections to Deterministic Finite Automata

We conclude this paper with connection to yet a third field of mathematics. A deterministic finite automaton (DFA) is a finite state machine $M$ built to recognize a given language $L$, i.e., a set of words in a specific alphabet. To test whether a given word $\omega$ is in language $L$ we write down $\omega$ on a strip of tape and feed it into $M$ one letter at a time. Depending on which state the machine is in, it will either accept or reject the character. If the character is accepted, then the machine’s next state is determined by the previous state and the relevant character on the strip. As the machine changes states accordingly, and the entire word is fed into the machine, if all letters of $\omega$ are accepted, then $\omega$ is an element of language $L$.

For our purposes we consider an automaton $M_G$ with three states, which we label as $A$, $B$, and $C$. In state $A$ we either accept a character in $\{1 + q, \ldots, q + t\}$ and return to state $A$, accept a character in $\{1, \ldots, q\}$ and move to state $B$, or accept the character 0 and move to state $C$.

On the other hand, in state $B$ we either accept a character in $\{1 + q, \ldots, q + t\}$ and move to state $A$, accept a character in $\{1, \ldots, q\}$ and return to state $B$, or accept character 0 and move to state $C$.

Finally, in state $C$ we either accept a character in $\{1 + q, \ldots, q + t\}$ and move to state $A$, or accept character $q$ and return to state $C$. A character in $\{1, \ldots, q\}$ is not accepted while in state $C$. This DFA is illustrated here, with its transition matrix also given.

![Figure 2. Deterministic finite automaton $M_G$.](image)

We consider the set of words $L(q, t)$ which are accepted by $M_G$ with the properties (1) the initial state of $M_G$ is the same as its final state, and (2) $M_G$ is in state $A$ at some point while verifying $\omega$. Comparing definitions, we observe that the set of such words is in fact the set of critical configurations, as described in Section 4. We can in fact characterize this set even more concretely.

**Proposition 9.** The set $L(q, t)$ is a regular language, i.e., a set of words which can be described by a DFA $D_L$. In particular, word $\omega$ is in $L(q, t)$ if and only if $\omega$ is admissible by $D_L$. 

Proof. Regular languages can be built by taking complements, the Kleene star, unions, intersections, images under homomorphisms, and concatenations. Thus we can prove \( L(q, t) \) is regular by decomposing it as the union over all cyclic shifts, a homomorphism, of concatenation of the blocks of form \( B, M_1, M_2, \ldots, M_k \). \( \Box \)

More explicitly, we can also use \( M_G \) to build a DFA recognizing \( L(q, t) \), thus giving a second proof. First, machine \( M_G \) as described is not technically a DFA since we are not specifying which of the three states is the initial state and what state the DFA moves to from state \( C \) when it encounters a character in \( \{0, 1, 2, \ldots, q - 1\} \). We also have the added restrictions that a word is only admissible if the DFA goes through state \( A \) along its path, and that words admitted by closed paths in this DFA.

However, this can be easily rectified. First, we add four additional states: an initial state \( I \), two states \( \tilde{B} \tilde{C} \), and a dead state \( D \). Start state \( I \) connects to states \( A, \tilde{B} \) and \( \tilde{C} \), moving to \( A \) if the first letter is \( \geq 1 + q \), moving to \( \tilde{C} \) if the first letter is 0, and moving to \( \tilde{B} \) otherwise. Additionally, state \( \tilde{B} \) connects to \( A, \tilde{B} \), and \( \tilde{C} \) just as \( B \) connects to \( A, B \), and \( C \); similarly, \( \tilde{C} \) connects to \( A \) and \( \tilde{C} \) just as \( C \) connects to \( A \) and \( C \). When the machine is in state \( C \) or \( \tilde{C} \), and a character from \( \{0, 1, 2, \ldots, q - 1\} \) is read, the machine moves to the dead state \( D \) which always loops back to itself. Letting states \( A, B, \) and \( C \) be the only final/terminal states of this DFA, we now have the property that a word is only admissible if the DFA goes through state \( A \) at some point along its path.

We now have to deal with the restriction that a word is admissible only if the word induces a cycle of states in the DFA. To this end, we expand the DFA even further essentially copying it three times and making sure the terminal states correspond to the first state reached, i.e. immediately following the start state.

7.1. Another Kind of Zeta Function. Returning to the original formulation, critical configurations correspond to closed paths in DFA \( M_G \) which go through state \( A \). Since a cycle involving both states \( B \) and \( C \) but not state \( A \) is impossible, the only cycles we need to disallow are those containing only state \( B \) and those cycles containing only state \( C \). Such words, i.e. the set \( \mathcal{L}(q, t) \) is a cyclic language since the set is closed under circular shift (more precisely \( uv \in \mathcal{L}(q, t) \) if and only if \( vu \in \mathcal{L}(q, t) \) for all \( u, v \)).

Regular cyclic languages such as \( \mathcal{L}(q, t) \) were studied in [6], and we can even define a zeta function for them. The zeta function of a cyclic language \( L \) is defined as

\[
\zeta(L, T) = \exp \left( \sum_{k=1}^{\infty} \frac{W_k}{k} T^k \right)
\]

where \( W_k \) is the number of words of length \( k \). Alternatively, this can be written as

\[
\zeta(L, T) = \exp \left( \sum_{\text{allowed closed paths } P} (\# \text{ words admissible by path } P) T^k \right).
\]

**Theorem 13** (Berstel and Reutenauer). The zeta function of a cyclic and regular language is rational.

**Proof.** See [6] or [19]. \( \Box \)

This observation motivated Berstel and Reutenauer to ask the following question.
Question 5. For a given algebraic variety $V$ with zeta function $Z_V(T)$ (also rational by Dwork [8]), can we find a cyclic regular language $L$ such that $\zeta(L, T) = Z_V(T)$?

We will come back to this question momentarily.

The trace of an automaton $A$ is the language of words generated by closed paths in $A$. Such a language is always cyclic and regular by construction, and in fact has a zeta function with an explicit formula.

**Proposition 10.**

$$\zeta(\text{trace}(A)) = \frac{1}{\det(I - MT)}$$

where $M$ encodes the number of directed edges between state $i$ and state $j$ in $A$.

This matrix is in fact the transition matrix provided above with the example of automaton $M_G$.

**Proof.** We omit this proof, again referring the reader to [6]. However, we also take this opportunity to mention that the proof is an application of MacMahon’s Master Theorem [14] which relates the generating function of traces to a determinantal formula, or more precisely the characteristic polynomial of a matrix. Moreover, analogies between the zeta function of a language and the zeta function of a variety are even clearer since the proof of the Weil conjectures via étale cohomology also involve such determinantal expressions. □

**Theorem 14.** For $\mathcal{L}(q, t)$ as given in Proposition 9, the zeta function $\zeta(\mathcal{L}(q, t))$ equals

$$\frac{(1 - qT)(1 - T)}{1 - (1 + q + W_1)T + qT^2}.$$  

**Proof.** Using the above terminology, we can describe the set of critical configurations of $W_k(q, t)$ as the language obtained by taking the trace of $M_G$ minus the trace of cycles only containing state $B$ minus the trace of cycles only containing state $C$. We again note that all other circuits with the same initial and final state necessarily need to contain state $A$ since there are no cycles containing both state $B$ and $C$ but not $A$. There is no way to go from state $C$ to state $B$ without going through state $A$ first, given the definition of $M_G$.

Thus the zeta function of this cyclic language is given as

$$\frac{\det([1 - qT])\det([1 - T])}{\det(I - MT)}$$

where the factor of $\det([1 - qT])$ correspond to the trace of cycles containing state $B$ alone, and $\det([1 - T])$ corresponds to the trace of cycles containing state $C$ alone. On the other hand, matrix $M$ is the 3-by-3 matrix encoded by the number of directed edges between the various states.

$$
\begin{bmatrix}
t & q & 1 \\
t & q & 1 \\
t & 0 & 1 \\
\end{bmatrix}
$$

Thus we arrive at the desired expression for $\zeta(\mathcal{L}(q, t))$, namely

$$\exp \left( \sum_{k=1}^{\infty} \frac{W_k}{k} T^k \right) = \frac{(1 - qT)(1 - T)}{1 - (1 + q + W_1)T + qT^2}$$
where $W_k$ equals the number of primitive cycles in $M_G$, which contain state $A$ but starting at any of the three states.

At this point, we have yet another proof of the Theorem which states $N_k = -W_k(q, -N_1)$. The reasoning being

$$
\exp \left( \sum_{k \geq 1} \frac{W_k}{k} T^k \right) = \frac{(1 - qT)(1 - T)}{1 - (1 + q + t)T + qT^2}
$$

$$
= \left( \frac{1 - (1 + q + t)T + qT^2}{1 - qT}(1 - T) \right)^{-1}
$$

$$
= \left( Z(E, T) \big|_{N_1 = -t} \right)^{-1}
$$

$$
= \exp \left( -\sum_{k \geq 1} \frac{N_k}{k} T^k \right) \bigg|_{N_1 = -t}.
$$

Additionally, we have answered Berstel and Reutenauer’s question for elliptic curves, up to an issue of sign. Perhaps other cyclic languages, constructed from the critical groups of graphs or otherwise, correspond to other algebraic varieties analogously.

In this paper, we have continued the study of [17] which explored the theory of elliptic curves over finite fields with an eye towards combinatorial results. The relationship between elliptic curves and spanning trees appears even more pronounced than one would have guessed from the motivation of Theorem. Not only do we have formal identities relating the number of spanning trees of wheel graphs and number of points on elliptic curves, but we also have connections between the corresponding group structures of these two families of objects. Characterizations of critical groups in terms of combinatorics on words also appears fruitful. The connections described here inspire further exploration for connections between these three topics.

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