Topological and Homological Properties of the Orbit Space of a Simple Three-Dimensional Compact Linear Lie Group

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Abstract—The question of whether an orbit space of a compact linear group is a topological manifold and a homology manifold is studied. The case of a simple three-dimensional group is considered. An upper bound is obtained for the sum of integral parts of the halved dimensions of irreducible components for a representation whose quotient is a homology manifold. This strengthens a similar result obtained previously, which gave such a bound in the case where the quotient of the representation is a smooth manifold. Most representations for which the obtained estimate holds have also been considered previously. The argument uses standard considerations of linear algebra and the theory of Lie groups and algebras and their representations.

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1. INTRODUCTION

Consider a faithful linear representation of a compact Lie group $G$ on a real vector space $V$. It is required to determine whether the topological quotient $V/G$ of this action is a topological manifold and whether it is a homology manifold. In what follows, we refer to a topological manifold simply as a manifold.

We may assume without loss of generality that $V$ is a Euclidean space, $G$ is a Lie subgroup of the group $O(V)$, and the representation $G: V$ is tautological.

The problem stated above was studied in [1] and [2] for finite groups. In the author’s papers [3]–[6], both topological and differential-geometric properties of the quotient were studied for various classes of groups, namely, for groups with commutative connected component [3] and for simple groups of classical type [4]–[6]. In the author’s papers [7]–[9], the “topological” parts of results in [3] were strengthened for groups with commutative connected component. The present paper contains similar strengthenings of results in [4] for simple three-dimensional groups.

By $G^0$ we denote the connected component of the identity element of the group $G$ and by $\mathfrak{g}$, its tangent algebra.

Suppose that $\mathfrak{g} \cong \mathfrak{su}_2$, or, equivalently, that the group $G^0$ is isomorphic to one of the groups $\text{SU}_2$ and $\text{SO}_3$.

Let $n_1, \ldots, n_L$ be the dimensions of irreducible components of a representation $\mathfrak{g}: V$ (counting multiplicities) arranged in nonincreasing order. Since the representation $G: V$ is faithful, we have $n_1 \geq \cdots \geq n_l > 1 = n_{l+1} = \cdots = n_L$, where $l \in \{1, \ldots, N\}$. Let $q(V)$ denote the number

$$\sum_{i=1}^L \left\lfloor \frac{n_i}{2} \right\rfloor = \sum_{i=1}^l \left\lfloor \frac{n_i}{2} \right\rfloor \in \mathbb{N}.$$

The main result of the paper is the following theorem.

Theorem 1. If $\mathfrak{g} \cong \mathfrak{su}_2$ and $V/G$ is a homology manifold, then $q(V) \leq 4$. 

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2. NOTATION AND AUXILIARY FACTS

This section contains auxiliary notation and statements, some of which are borrowed from the papers cited above (all new statements are provided with proofs).

We use the following shorthand notation:

- $V^G$ is the fixed point space of a representation of $G$ on $V$.
- $\text{Spec}_F A$ is the set of eigenvalues of an operator $A$ on a space over a field $F$.

Lemma 1. Let $X$ be a topological space, and let $n$ be a positive integer.

1. If $X$ is a simply connected homology $n$-sphere, then $X \cong S^n$.
2. The cone over $X$ is a homology $(n+1)$-manifold if and only if $X$ is a homology $n$-sphere.

Proof. See Theorem 2.3 and Lemma 2.6 in [2, Sec. 2].

By $\mathbb{T}$ we traditionally denote the multiplicative Lie group $\{\lambda \in \mathbb{C}; |\lambda| = 1\}$.

Given a Euclidean space $V$ and a compact group $G \subset \text{O}(V)$ with tangent algebra $\mathfrak{g} \subset \text{so}(V)$, consider any vector $v \in V$. The subspaces $\mathfrak{g}v$ and $N_v := (\mathfrak{g}v)\perp$ of $V$ are invariant under the stabilizer $G_v$ of the vector $v$. The stationary subalgebra $\mathfrak{g}_v$ of $v$ coincides with $\text{Lie} G_v$. We set $M_v := N_v \cap (N_v^G)^\perp \subset N_v$. Clearly, $N_v = N_v^G \oplus M_v \subset V$ and $G_v M_v = M_v$.

Statement 1. Any $G_0$-invariant subspace $V' \subset V$ contains a vector $v$ for which $M_v \subset (V')\perp$.

Proof. See Statement 2.2 in [3, Sec. 2].

Theorem 2. Let $v \in V$ be a vector. If $V/G$ is a homology manifold, then so are $N_v/G_v$ and $M_v/G_v$.

Proof. See Theorem 4 and Corollary 5 in [8].

Definition. A linear operator on a space over some field is called a reflection (pseudoreflection) if its fixed point space is of codimension 1 (respectively, of codimension 2).

We denote by $K$ the multiplicative Lie group $\{v \in \mathbb{H}; \|v\| = 1\}$.

Definition. The Poincaré group is the preimage of the rotation group of the dodecahedron under the covering homomorphism $K \twoheadrightarrow \text{SO}_3$.

It is well known that the Poincaré group $\Gamma_0 \subset K$ coincides with its commutator subgroup; the same can be said about the linear group $\Gamma \subset \text{GL}_R(\mathbb{H})$ obtained by restricting the action of $K$: $\mathbb{H}$ by left shifts to $\Gamma_0$.

Theorem 3. If a group $G \subset \text{O}(V)$ is finite and $V/G$ is a homology manifold, then there exist decompositions $G = G_0 \times G_1 \times \cdots \times G_k$ and $V = V_0 \oplus V_1 \oplus \cdots \oplus V_k (k \in \mathbb{Z}_{\geq 0})$ such that

- The subspaces $V_0, V_1, \ldots, V_k \subset V$ are pairwise orthogonal and $G$-invariant.
- Given any $i, j = 0, \ldots, k$, the linear group $(G_i)|_{V_j} \subset \text{O}(V_j)$ is trivial if $i \neq j$, generated by pseudoreflections if $i = j = 0$, and isomorphic to the linear group $\Gamma$ if $i = j > 0$ (in particular, $\dim V_j = 4$ for any $j = 1, \ldots, k$).

Proof. See Proposition 3.13 in [2, Sec. 3].
On the space $g$, an $\text{Ad}(G)$-invariant inner product is defined; in what follows, we identify the spaces $g$ and $g^*$ by means of this product. If $g' \subset g$ is a one-dimensional subalgebra and $V' \subset V$ is a subspace, then $g'V' \subset V$ is a subspace of dimension at most $\dim V'$.

Let us recall the definitions of $q$-stable ($q \in \mathbb{N}$) and indecomposable sets of vectors in finite-dimensional spaces over fields [3, Sec. 1], which are needed in what follows.

A decomposition of a set of vectors in a finite-dimensional linear space into components is a representation of this set as a union of subsets with linearly independent linear spans. If at least two of these linear spans are nontrivial, then the decomposition is said to be proper. We say that a set of vectors is indecomposable if it has no proper decompositions. Any set of vectors admits a unique (up to a distribution of the zero vector) decomposition into indecomposable components; moreover, each component of any decomposition is the union of some indecomposable components (again up to the zero vector).

**Definition.** We say that a finite set of vectors in a finite-dimensional space considered with taking account of the multiplicities of its elements is $q$-stable ($q \in \mathbb{N}$) if its linear span is preserved under the removal of any $q$ (or less) vectors (with multiplicities taken into account).

Given any finite set $P$ of vectors with multiplicities in a finite-dimensional space over some field, we denote the number of nonzero vectors in $P$ (counted with multiplicities) by $\|P\|$.

Suppose that the group $G^0$ is commutative, i.e., is a torus.

Any irreducible representation of the group $G^0$ is either one-dimensional or two-dimensional. Below we recall the notion of a weight of its irreducible representation introduced in [3, Sec. 1].

Any two-dimensional irreducible representation of the group $G^0$ is a two-dimensional complex representation of the group $G^0$. To this end, we naturally assign a weight, that is, a Lie group homomorphism $\lambda: G^0 \to \mathbb{T}$, to this representation and identify it with a vector $\lambda \in g^*$, which we treat as the differential. The one-dimensional representation of the group $G^0$ is assigned the weight $\lambda := 0 \in g^*$.

The classes of isomorphic irreducible representations of the group $G^0$ are characterized by weights $\lambda \in g^* = g$ determined up to multiplication by $(-1)$.

Let $P \subset g$ be the set of weights $\lambda \in g$ corresponding to the decomposition of the representation $G^0: V$ into a direct sum of irreducible representations (with multiplicities taken into account). The set $P \subset g$ is independent of the choice of the decomposition (up to the multiplication of weights by $(-1)$).

Since the representation $G: V$ is faithful, we have $\langle P \rangle = g$.

**Theorem 4** (see [9, Theorem 4]). Suppose that $V/G$ is a homology manifold and $P \subset g$ is a $2$-stable set. Then the representation $G: V$ is the direct product of representations $G_l: V_l (l = 0, \ldots, p)$ such that

1. For any $l = 0, \ldots, p$, the quotient $V_l/G_l$ is a homology manifold.
2. $|G_0| < \infty$.
3. For any $l = 1, \ldots, p$, the group $G_l$ is infinite and the set of weights of the representation $G_l: V_l$ is indecomposable and $2$-stable and contains no zeros.

**Theorem 5.** Suppose that $\dim G = 1$ and $P \subset g$ is a $2$-stable set. If $V/G$ is a homology manifold, then $\|P\| = 3$.

**Proof.** We may assume without loss of generality that $0 \notin P$. Indeed, by Theorem 4, the representation $G: V$ is the direct product of representations $G_l: V_l (l = 0, 1)$ whose quotients are homology manifolds, $|G_0| < \infty$, $|G_1| = \infty$, and the set of weights of the representation $G_1: V_1$ is indecomposable and $2$-stable and contains $\|P\|$ nonzero weights and no zero weights.

Since $0 \notin P$, it follows that the space $V$ carries a $G^0$-invariant complex structure. If the group $G \subset \mathbb{O}(V)$ contains no complex reflections, then the desired assertion follows from Theorem 6 in [9].

The general case can be reduced (see [3, Secs. 3, 7]) to the case of a representation of a one-dimensional group without complex reflections whose set of weights is obtained from $P$ by multiplying all weights by nonzero scalars.
Corollary 1. If \( \dim G = 1 \) and \( V/G \) is a homology manifold, then \( \|P\| \leq 3 \) (or, equivalently, \( \dim(\mathfrak{g}V) \leq 6 \)).

Proof. Suppose that \( \|P\| > 3 \). Then the set \( P \subset \mathfrak{g} \) is \( 2 \)-stable. According to Theorem 5, \( \|P\| = 3 \). We have obtained a contradiction.

Let us recall basic facts on representations of the complex Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) and its compact real form \( \mathfrak{su}_2 \).

For any \( m \in \mathbb{N} \), there exists a unique (up to isomorphism) \( m \)-dimensional irreducible representation \( \rho_m \) of the algebra \( \mathfrak{sl}_2(\mathbb{C}) \), namely, \( (m-1) \)-th symmetric power of its (obviously symplectic) tautological representation. For even (odd) \( m \), the representation \( \rho_m \) is symplectic (respectively, orthogonal). Therefore,

- For any \( m \in \mathbb{N} \), there exists a unique (up to isomorphism) \( m \)-dimensional irreducible complex representation of \( \mathfrak{su}_2 \), namely, \( \tilde{\rho}_m := (\rho_m)|_{\mathfrak{su}_2} \).
- For even \( m \), the representation \( \tilde{\rho}_m \) is irreducible over \( \mathbb{R} \), and for odd \( m \), it has an invariant real form.
- All pairwise nonisomorphic irreducible real representations of \( \mathfrak{su}_2 \) are precisely the realizations of the representations \( \tilde{\rho}_m (m:2) \) and the real forms of the representations \( \tilde{\rho}_m (m \neq 2) \); their dimensions are pairwise distinct and range over all positive integers not congruent to \( 2 \mod 4 \).

3. PROOF OF THE MAIN RESULTS

This section deals with the proof of Theorem 1.

In the remaining part of the paper, we will assume that \( \mathfrak{g} \cong \mathfrak{su}_2 \), i.e., the group \( G^0 \) is isomorphic to \( \text{SU}_2 \) or \( \text{SO}_3 \). Let \( V_0 := V^{G^0} \subset V \). In the notation and under the conventions of Sec. 1, we have \( L-l = \dim V_0 \) and \( V_0 \neq V \), the numbers \( n_1, \ldots, n_l \) are the dimensions of the irreducible components of the representation \( \mathfrak{g} : V_0^l \) (with multiplicities taken into account), and each of them is either a multiple of \( 4 \) or odd. If \( \mathfrak{g}' \subset \mathfrak{g} \) is a proper subalgebra, then \( \dim \mathfrak{g}' = 1 \) and the subspace \( \mathfrak{g}'V \subset V \) has dimension \( 2q(V) \). Therefore, \( 2q(V) \leq \dim V \).

It suffices to prove the theorem in the case of \( V_0 = 0 \) (where the representation \( G^0 : V \) has no one-dimensional irreducible components). Indeed, there exists a vector \( v \in V_0 \) such that \( M_v \subset V_0^l \) (see Statement 1). We have \( \mathfrak{g}v = 0, N_v = V, G_v \supset G^0, (G_v)^0 = G^0, \) and \( \mathfrak{g}_e = \mathfrak{g} \). We also have \( M_v = (V^{G^0})^l \supset (V^{G^0})^l = V_0^l \supset M_v, \) whence \( M_v = V_0^l \). According to Theorem 2, if \( V/G \) is a homology manifold, then so is \( M_v/G_v \). As to decompositions into irreducible components of the representations of the group \( G^0 = (G_v)^0 \) on the spaces \( V \) and \( M_v \), the latter is obtained from the former by deleting all one-dimensional components. Therefore, \( q(V) = q(M_v) \).

In what follows, we assume that \( V/G \) is a homology manifold and \( V_0 = 0 \). We must prove that \( q(V) \leq 4 \).

Suppose that there exists a vector \( v \in V \) for which \( \dim G_v = 1 \). Then \( \dim \mathfrak{g}_v = 1 \) and \( \dim(\mathfrak{g}v) = 2 \). Moreover, by Theorem 2, \( N_v/G_v \) is a homology manifold. According to Corollary 1, we have \( \dim(\mathfrak{g}_v N_v) \leq 6, 2q(V) = \dim(\mathfrak{g}_v V) \leq \dim(\mathfrak{g}_v N_v) + \dim(\mathfrak{g}v) \leq 8, \) and \( q(V) \leq 4 \).

In what follows, we assume that the space \( V \) contains no vectors with one-dimensional stabilizer and that \( q(V) > 4 \). In this case,

- \( \dim V \geq 2q(V) > 8 \).
- \( G^0 \cong \text{SU}_2 \).
- The dimension of each irreducible component of the representation \( G^0 : V \), as well as of any of its subrepresentations, is a multiple of \( 4 \).
(Ker Ad) ∩ G^0 = Z(G^0) = {±E} ⊂ O(V).

Any operators \( g \in G \) and \( \xi \in g^{Ad(g)} \) on the space \( V \) commute. Therefore, for any \( g \in G \), the subspaces \( V^g \), \((E - g)V \subset V\) are \( (g^{Ad(g)}) \)-invariant; if \( Ad(g) = E \), then they are \( g \)-invariant, and therefore, \( rk(E - g) : 4 \).

Consider any vector \( v \in V \). If the subalgebra \( g_v \subset g \) is proper, then \( dim g_v = 1 \), which contradicts the assumption. Therefore, we have either \( g_v = g \) or \( g_v = 0 \). In the former case, \( G_v \supset G^0 \) for \( v \in V^{G^0} = V_0 = 0 \). Therefore, if \( v \neq 0 \), then \( g_v = 0, |G_v| < \infty \), the map \( g \mapsto (gv), \xi \mapsto (\xi v) \), is a linear isomorphism, and \( g(\xi v) = (Ad(g)\xi)v \) for any \( g \in G_v \) and \( \xi \in g \), whence \( (\xi v \in V^g) \Leftrightarrow (\xi \in g^{Ad(g)}) \).

Thus, if \( v \neq 0 \) and \( g \in G_v \), then \( (gv)^g = (g^{Ad(g)})v \).

For any \( g \in G \) and \( \xi \in g \), let \( \varphi_{g,\xi} \) denote the linear map of the space \( V \) into the outer direct sum of two copies of \((E - g)V\) defined by \( v \mapsto ((E - g)v, (E - g)\xi v) \).

**Lemma 2.** For any \( g \in G \) and \( \xi \in g \setminus (g^{Ad(g)}) \), one has \( Ker \varphi_{g,\xi} = 0 \).

**Proof.** If \( v \neq 0 \) and \( v \in Ker \varphi_{g,\xi} \), then \((E - g)v = (E - g)\xi v = 0\), i.e., \( g \in G_v \) and \( \xi v \in V^g \), whence \( \xi \in g^{Ad(g)} \), which contradicts the assumption.

**Corollary 2.** If \( g \in G \) and \( Ad(g) \neq E \), then \( dim V \leq 2 \cdot rk(E - g) \).

Our immediate goal is to prove the following theorem.

**Theorem 6.** For any \( v \in V \setminus \{0\} \), one has \( [G_v, G_v] = G_v \subset Ker Ad \).

**Proof of Theorem 6.** Take any vector \( v \in V \setminus \{0\} \).

Let \( \pi \) denote the homomorphism \( G_v \rightarrow O(N_v) \), \( g \mapsto g|_{N_v} \), and let \( H_v \) denote the subgroup \( \pi(G_v) \subset O(N_v) \). By Theorem 2, \( N_v / G_v \) is a homology manifold. Moreover, \( |G_v| < \infty \). According to Theorem 3, there exist decompositions \( H_v = H_0 \times H_1 \times \cdots \times H_k \) and \( N_v = W_0 \oplus W_1 \oplus \cdots \oplus W_k \) (\( k \in \mathbb{Z}_{\geq 0} \)) such that

- The subspaces \( W_0, W_1, \ldots, W_k \subset N_v \) are pairwise orthogonal and \( G_v \)-invariant.
- Given any \( i, j = 0, \ldots, k \), the linear group \((H_i)|_{W_j} \subset O(W_j)\) is trivial if \( i \neq j \), generated by pseudoreflections if \( i = j = 0 \), and isomorphic to the linear group \( \Gamma \) if \( i = j > 0 \) (in particular, \( dim W_j = 4 \) for any \( j = 1, \ldots, k \)).

Recall that \( \Gamma = [\Gamma, \Gamma] \). Therefore, \( H_i = [H_i, H_i] \) (\( i = 1, \ldots, k \)).

If \( g \in G_v \), then \( rk(E - g) - dim((E - g)N_v) = dim((E - g)(gv)) = rk(E - Ad(g)) \leq 2 \); in the case of \( Ad(g) = E \), we have \( rk(E - g) = dim((E - g)N_v) \).

**Lemma 3.** If \( g \in G_v \) and \( dim((E - g)N_v) \leq 2 \), then \( g = E \).

**Proof.** By assumption, we have \( rk(E - g) \leq 4 \). If \( Ad(g) \neq E \), then \( dim V \leq 2 \cdot rk(E - g) \leq 8 \), while \( dim V > 8 \). Therefore, \( Ad(g) = E \). It follows that first, \( rk(E - g) : 4 \), and second,

\[
rk(E - g) = dim((E - g)N_v) \leq 2.
\]

Thus, \( rk(E - g) = 0 \), and \( g = E \).

In view of Lemma 3, we have \( Ker \pi = \{E\} \subset G_v \); i.e., \( \pi \) is an isomorphism \( G_v \rightarrow H_v \). Setting \( G_i := \pi^{-1}(H_i) \subset G_v (i = 0, \ldots, k) \), we see that

- \( G_v = G_0 \times G_1 \times \cdots \times G_k \).
The group $G_0$ is generated by the elements $g \in G_v$ for which $\dim((E - g)N_v) \leq 2$ (and hence it is trivial by Lemma 3).

- Each of the groups $G_i, i = 1, \ldots, k$, coincides with the commutator subgroup of itself.
- If $i \in \{1, \ldots, k\}$ and $g \in G_i \setminus \{E\}$, then $N_{g}^g = N_{v} \cap W_i^\perp$ and $(E - g)N_v = W_i$, whence $\dim((E - g)N_v) = 4$ and $\text{rk}(E - g) \leq 6$.

Summarizing, we conclude that $G_v = G_1 \times \cdots \times G_k = [G_v, G_v]$.

**Lemma 4.** Each of the groups $\text{Ad}(G_i), i = 1, \ldots, k$, is commutative.

**Proof.** Suppose that there exists a number $i \in \{1, \ldots, k\}$ and elements $g, h \in G_i$ such that the operators $\text{Ad}(g)$ and $\text{Ad}(h)$ do not commute.

We have $\text{Ad}(g), \text{Ad}(h) \not\in E$; moreover, $\mathfrak{g}^{\text{Ad}(g)}, \mathfrak{g}^{\text{Ad}(h)} \subset \mathfrak{g}$ are distinct one-dimensional subspaces. Therefore, $\mathfrak{g}^{\text{Ad}(h)} = \mathbb{R}\xi (\xi \in (\mathfrak{g}^{\text{Ad}(h)}) \setminus (\mathfrak{g}^{\text{Ad}(g)}))$, whence

$$\xi V^h \subset V^h \quad \text{and} \quad (g^v)^h = (g^{\text{Ad}(h)})v = \mathbb{R}(\xi v).$$

Since $g, h \in G_i \setminus \{E\}$, it follows that first, $\text{rk}(E - h) \leq 6$, and second, $N_{g}^g = N_{v}^h = N_{v} \cap W_i^\perp$, $V^h = N_{g}^g \oplus (\mathbb{R}(\xi v)), (E - g)\xi V^h \subset (E - g)V^h = \mathbb{R}((E - g)\xi v)$, and $\dim(\varphi_{g,\xi}V^h) \leq 2$. By Lemma 2, we have $\text{Ker} \varphi_{g,\xi} = 0$, which implies that $\dim V^h = \dim(\varphi_{g,\xi}V^h) \leq 2$, and therefore,

$$\dim V = \text{rk}(E - h) + \dim V^h \leq 8,$$

while $\dim V > 8$. This contradiction completes the proof of the lemma. \hfill \Box

For any $i = 1, \ldots, k$, we have $\text{Ad}(G_i) = \text{Ad}([G_i, G_i]) = [\text{Ad}(G_i), \text{Ad}(G_i)] = \{E\}$, which is equivalent to $G_i \subset \text{Ker} \text{Ad}$. Therefore, $G_v = G_1 \times \cdots \times G_k \subset \text{Ker} \text{Ad}$.

This completes the proof of Theorem 6. \hfill \Box

**Corollary 3.** For any $v \in V \setminus \{0\}$, one has $G_v \cap G^0 = \{E\}$.

**Proof.** In view of Theorem 6, $G_v \subset \text{Ker} \text{Ad}$ and $G_v \cap G^0 \subset (\text{Ker} \text{Ad}) \cap G^0 = \{\pm E\} \subset \mathbf{O}(V)$. \hfill \Box

There exists an embedding $\mathbb{T} \hookrightarrow G^0$; hence we can identify the group $\mathbb{T}$ with its image under this embedding and regard it as a subgroup of $G^0$. By Corollary 3, any irreducible subrepresentation of the representation $\mathbb{T}: V$ is faithful and hence isomorphic to the representation $\mathbb{T}: \mathbb{C}$ by multiplications. Thus, the space $V$ carries a complex structure, in which the action $\mathbb{T}: V$ is implemented by multiplication by scalars. All operators of the group $\text{Ker} \text{Ad}$ commute with all operators of the group $G^0 \supset \mathbb{T}$, and $(\text{Ker} \text{Ad}) \cap G^0 = \{\pm 1\} \subset \mathbb{T}$. Therefore, $\text{Ker} \text{Ad}$ is a finite subgroup of $\mathbf{GL}_\mathbb{C}(V)$, and each operator $g$ of this group is semisimple over the field $\mathbb{C}$ and satisfies the relation $(\text{Spec}_\mathbb{C} g) \subset \mathbb{T} \subset G^0$.

Let $H$ denote the subgroup of $G$ generated by the union all subgroups $G_v, v \in V \setminus \{0\}$. According to Theorem 6, we have $H \subset \text{Ker} \text{Ad}$.

**Proposition 1.** For any $g \in \text{Ker} \text{Ad}$, one has $(\text{Spec}_\mathbb{C} g) \subset (gH) \cap \mathbb{T}$.

**Proof.** Consider any element $\lambda \in (\text{Spec}_\mathbb{C} g)$. There exists a vector $v \in V \setminus \{0\}$ for which $gv = \lambda v$. We have $\lambda \in gG_v \subset gH$. \hfill \Box

**Lemma 5.** For any $g \in \text{Ker} \text{Ad}$, one has $g^2 = E$.

**Proof.** As was mentioned, $H \subset \text{Ker} \text{Ad}$. This implies

$$gH \subset g(\text{Ker} \text{Ad}) = \text{Ker} \text{Ad}.$$ 

By Proposition 1, we have $(\text{Spec}_\mathbb{C} g) \subset (gH) \cap \mathbb{T} \subset (\text{Ker} \text{Ad}) \cap \mathbb{T} = \{\pm 1\} \subset \mathbb{T}$. \hfill \Box
Corollary 4. The group $\text{Ker Ad}$ is commutative.

Corollary 5. For any $v \in V \setminus \{0\}$, one has $G_v = \{E\}$.

Proof. The statement follows from Theorem 6 and Corollary 4. □

Corollary 6. The subgroup $H \subset G$ is trivial.

Lemma 6. The inclusion $\text{Ker Ad} \subset G^0$ holds.

Proof. Let $g \in \text{Ker Ad}$ be any element. Proposition 1 and Corollary 6 imply $(\text{Spec}_C g) \subset \{g\} \cap T$. At the same time, we have $(\text{Spec}_C g) \neq \emptyset$, whence $g \in T \subset G^0$. □

Since $\text{Aut}(g) = \text{In}(g)$, we have $\text{Ad}(G) = \text{Ad}(G^0)$. This, together with Lemma 6, implies

$$G = G^0(\text{Ker Ad}) = G^0 \cong \text{SU}_2.$$

Therefore, $\pi_3(G) \cong \pi_3(\text{SU}_2) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Let $S \subset V$ be the unit sphere, and let $M$ be the quotient $S/G$.

We have $\dim V > 8$ and $\dim S > 7$; hence $S$ and $M$ are connected topological spaces and $\pi_k(S) = \{e\}$ ($k = 1, \ldots, 7$). According to Corollary 5, the action $G$: $S$ is free. The factorization map $S \to M$ is a locally trivial bundle with fiber $G$. The corresponding exact homotopy sequence gives the relations $\pi_k(M) \cong \pi_{k-1}(G)$ ($k = 2, \ldots, 7$) and $\pi_1(M) \cong G/G^0 = \{e\}$. By Lemma 1, we have $M \cong S^m$ ($m := \dim S - 3 > 4$); on the other hand, $\pi_4(M) \cong \pi_3(G) \cong \mathbb{Z}$. We have obtained a contradiction, which completes the proof of Theorem 1.

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