Abstract

In this paper, we propose a new set representation for binary vectors called logical zonotopes. A logical zonotope is constructed by XOR-ing a binary vector with a combination of binary vectors called generators. A logical zonotope can efficiently represent up to $2^r$ binary vectors using only $r$ generators. Instead of the explicit enumeration of the zonotopes’ members, logical operations over sets of binary vectors are applied directly to a zonotopes’ generators. Thus, logical zonotopes can be used to greatly reduce the computational complexity of a variety of operations over sets of binary vectors, including logical operations (e.g. XOR, NAND, AND, OR) and semi-tensor products. Additionally, we show that, similar to the role classical zonotopes play for formally verifying dynamical systems defined over real vector spaces, logical zonotopes can be used to efficiently analyze the forward reachability of dynamical systems defined over binary vector spaces (e.g. logical circuits or Boolean networks). To showcase the utility of logical zonotopes, we illustrate three use cases: (1) discovering the key of a linear-feedback shift register with a linear time complexity, (2) verifying the safety of a logical vehicle intersection crossing protocol, and (3) performing reachability analysis for a high-dimensional Boolean function.

1 Introduction

For several decades, logical systems have been used to model complex behaviors in numerous applications. By modeling a system as a collection of logical functions operating in a binary vector space, we can design models that consist of relatively simple dynamics but still capture a complex system’s behavior at a sufficient level of abstraction. For many applications, this enables researchers to analyze large-scale systems in a tractable manner. Some popular approaches to modeling logical systems are finite automatons, Petri nets, or Boolean Networks (BNs). Notably, logical systems have been used to successfully model the behavior of physical systems such as gene regulatory networks [1, 20] and robotics [17, 21]. One of the most common types of logical systems, discrete-event systems, are used for analysis in a variety of applications such as communication systems [7], manufacturing systems [18], and transportation systems [9, 11].

An important form of analysis for logical systems is reachability analysis. Reachability analysis allows us to formally verify the behavior of logical systems and provide guarantees that, for example, the system will not enter into undesired states. One of the primary challenges of reachability analysis is the need to exhaustively explore the system’s state space, which grows exponentially with the number of variables. To avoid exponential computational complexity, many reachability analysis algorithms are based on a representation called a Binary Decision Diagram (BDD). Given a proper variable ordering, BDDs can evaluate Boolean functions with linear complexity in the number of variables [12]. Due to this benefit, BDDs are widely used for verifying real-world hardware systems [3] and discrete event systems [5], in general. While BDDs play a crucial role in verification, they have well-known drawbacks, such as requiring an externally supplied variable ordering, since, in many applications, automatically finding an optimal variable ordering is an NP-complete problem [5, 6]. Due to these drawbacks, BDDs are difficult to apply to a general class of logical systems. Outside of BDDs, there are also approaches to reachability analysis for logical systems modeled as BNs, or Boolean Control Networks (BCNs) for systems with control inputs, that rely on the semi-tensor product [15]. However, these approaches are point-wise and work with an explicit representation of sets where all set members are explicitly enumerated. Furthermore, the structure matrix used in semi-tensor product-based approaches grows exponentially in size with respect to the number of states and inputs, making it challenging to apply to high-dimensional logical systems [14]. In this work, we propose a novel zonotope representation that reduces the exponential computational complexity of reachability analysis to a complexity that is up to linear in the number of the zonotope’s generators.

In real vector spaces, zonotopes already play an important role in the reachability analysis of dynamical systems [2, 10]. Classical zonotopes are constructed by taking the Minkowski sum of a real vector center and a combination of real vector generators. Through
this construction, a set of infinite real vectors can be represented by a finite number of generators. Then, by leveraging the fact that the Minkowski sum of two classical zonotopes can be computed through the Minkowski sum of their generators, researchers have formulated computationally efficient approaches to reachability analysis [2]. In this work, we take inspiration from classical zonotopes and formulate logical zonotopes. Similarly, logical zonotopes are constructed by XOR-ing a binary vector center and a combination of binary vector generators. In binary vector spaces, logical zonotopes are able to represent up to $2^r$ binary vectors using only $r$ generators, as illustrated in Figure 1 with $r = 2$. Moreover, we show that any logical operation on the generators of the logical zonotopes is either equivalent to or over-approximates the explicit application of the logical operation to each member of the represented set. Based on these results, we formulate our logical zonotope-based reachability analysis for logical systems.

Explicitly, the contributions of this work are summarized by the following:

1. we present our formulation of logical zonotopes,
2. we detail the application of logical operations and forward reachability analysis to logical zonotopes,
3. we illustrate the use of logical zonotopes in three different applications.

Our logical zonotope library is publicly available to recreate our results.

The remainder of the paper is organized as follows. In Section 2, we introduce the notation and preliminary definitions we will use throughout this work. In Section 3, we formulate logical zonotopes and overview the different uses of logical zonotopes. Then, in Section 4, we illustrate the application of logical zonotopes for the key discovery of a linear-feedback shift register (LFSR), verifying intersection-crossing protocols, and performing reachability analysis on a high-dimensional Boolean function. Finally, in Section 5, we conclude the work with a discussion about the potential of logical zonotopes and future work.

2 Notation and Preliminaries

In this section, we introduce details about the notation used throughout this work and preliminary definitions for logical systems, reachability, and semi-tensor products.

2.1 Notation

The set of natural and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$ respectively. We denote the binary set $\{0, 1\}$ by $\mathbb{B}$. The XOR, NOT, OR, and AND operations are denoted by $\oplus$, $\neg$, $\lor$, and $\land$, respectively. Throughout the rest of the work, with a slight abuse of notation, we omit the $\land$ from $a \land b$ and write $a \cdot b$ instead. The NAND, NOR, and XNOR are denoted by $\lor$, $\land$, and $\circ$, respectively. Later, we use the same notation for both the classical and Minkowski logical operations, as it will be clear when the operation is taken between sets or individual vectors. Like the classical AND operator, we will also omit the Minkowski AND to simplify the presentation.

Matrices are denoted by uppercase letters, e.g., $G \in \mathbb{R}^{n \times k}$, and sets by uppercase calligraphic letters, e.g., $Z \subset \mathbb{R}^n$. Vectors and scalars are denoted by lowercase letters, e.g., $a \in \mathbb{R}^n$. The identity matrix of size $n \times n$ is denoted $I_n$. We denote the Kronecker product by $\otimes$. $x \in \mathbb{B}^n$ is an $n \times 1$ binary vector. Furthermore, $\mathbb{B}^{n \times m}$ denotes the binary $n \times m$ matrix.

2.2 Preliminaries

For this work, we consider a system with a logical function $f : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}^n$:

$$x(k + 1) = f(x(k), u(k))$$

where $x(k) \in \mathbb{B}^n$ is the state and $u(k) \in \mathbb{B}^n$ is the control input. The logical function $f$ can consist of any combination of $\oplus$, $\neg$, $\lor$, $\land$, $\circ$, and $\circ$. We will represent sets of states and inputs for (1) using logical zonotopes. As will be shown, logical zonotopes are constructed using an Minkowski XOR operation, which we define as follows.

Definition 1. (Minkowski XOR) Given two sets $L_1$ and $L_2$ of binary vectors, the Minkowski XOR is defined between every two points in the two sets as

$$L_1 \oplus L_2 = \{ z_1 \oplus z_2 | z_1 \in L_1, z_2 \in L_2 \}.$$

Similarly, we define the Minkowski NOT, OR, and AND operations as follows.

$$\neg L_1 = \{ \neg z_1 | z_1 \in L_1 \}$$

$$L_1 \lor L_2 = \{ z_1 \lor z_2 | z_1 \in L_1, z_2 \in L_2 \}$$

$$L_1 \land L_2 = \{ z_1 \land z_2 | z_1 \in L_1, z_2 \in L_2 \}$$

Then, we show how logical zonotopes can be used to compute the forward reachable sets of systems defined by (1). We define the reachable sets of system (1) by the following definition.

Definition 2. (Exact Reachable Set of (1)) Given a set of initial states $X_0 \subset \mathbb{B}^n$ and a set of possible inputs $U_k \subset \mathbb{B}^n$, the exact reachable set $R_N$ of (1) after $N$ steps is

$$R_N = \{ x(N) | \forall k \in \{0, ..., N-1\} : x(k + 1) = f(x(k), u(k)), x(0) \in X_0, u(k) \in U_k \}.$$

Another commonly used operator for BCNs is the semi-tensor product [8]. Since semi-tensor products are useful in many applications, we have extended the classical definition to logical zonotopes. The classical definition for semi-tensor products is as follows.

Definition 3. (Semi-tensor product [16]) Given two matrices $M \in \mathbb{B}^{mxn}$ and $N \in \mathbb{B}^{pxq}$, the semi-tensor product, denoted by $\ast$, is computed as:

$$M \ast N = (M \otimes I_s)(N \otimes I_{s})$$

with $s$ as the least common multiple of $n$ and $p$, $s_1 = s/n$, and $s_1 = s/p$.

3 Logical Zonotopes

In this section, we present logical zonotopes and overview several different aspects of their use. We start by defining the set representation of logical zonotopes. Then, we go through the application of Minkowski XOR, NOT, XNOR, AND, NAND, OR, NOR, and semi-tensor product operations on logical zonotopes. For each operation, we show whether the operation yields an exact solution or an over-approximation. Using these results, we show that when using logical zonotopes for reachability analysis on (1), we are able to compute reachable sets that over-approximate the exact
reachable sets. In addition, we present an algorithm for reducing the number of generators in a logical zonotope.

3.1 Set Representation

Inspired by the classical zonotopic set representation which is defined in real vector space [13], we propose logical zonotopes as a set representation for binary vectors. We define logical zonotopes as follows.

**Definition 4. (Logical Zonotope)** Given a point \( c_L \in \mathbb{B}^n \) and \( y_L \in \mathbb{N} \) generator vectors in a generator matrix \( G_L = [g_L^{(1)} \ldots g_L^{(y_L)}] \) \( \in \mathbb{B}^{n \times y_L} \), a logical zonotope is defined as

\[
\mathcal{L} = \left\{ x \in \mathbb{B}^n \mid x = c_L \oplus \beta_L \right\}
\]

where \( \beta_L \) is a set of binary vectors. We use the shorthand notation \( \mathcal{L} = \langle c_L, G_L \rangle \) for a logical zonotope.

**Example 1.** Consider a logical zonotope

\[
\mathcal{L} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \mid \beta_L = \{0, 1\} \right\}
\]

With two generators, it represents the following four points:

\[
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
\]

by iterating over all possible binary values of \( \beta_L \).

**Remark 1.** Logical zonotopes are defined over \( \mathbb{B}^n \) and are different from zonotopes [13], constrained zonotopes [19], and hybrid zonotopes [4] which are all defined over real vector space \( \mathbb{R}^n \).

Logical zonotopes \( \mathcal{L} \) can enclose up to \( 2^y_L \) binary vectors with \( y_L \) generators. In the following section, we will show that due to their construction, this means we can apply logical operations to a set of \( 2^y_L \) binary vectors with a reduced computational complexity.

3.2 Minkowski Logical Operations

Often, given two sets of binary vectors, we need to perform logical operations between the members of the two sets. In order to perform these logical operations efficiently, we define new logical operations that only operate on the generators of logical zonotopes, instead of the members contained within the zonotopes. We will go through each logical operation and show that when applied to logical zonotopes, they either yield exact solutions or over-approximations.

3.2.1 Minkowski XOR (\( \oplus \))

\( \oplus \) is defined as

**Lemma 1.** Given two logical zonotopes \( \mathcal{L}_1 = \langle c_L^1, G_L^1 \rangle \) and \( \mathcal{L}_2 = \langle c_L^2, G_L^2 \rangle \), the Minkowski XOR is computed exactly as

\[
\mathcal{L}_1 \oplus \mathcal{L}_2 = \langle c_L^1 \oplus c_L^2, G_L^1 \cup G_L^2 \rangle .
\]

**Proof.** Let’s denote the right hand side of (7) by \( \mathcal{L}_{12} \). We aim to prove that \( \mathcal{L}_1 \oplus \mathcal{L}_2 \subseteq \mathcal{L}_{12} \) and \( \mathcal{L}_{12} \subseteq \mathcal{L}_1 \oplus \mathcal{L}_2 \). Choose any \( z_1 \in \mathcal{L}_1 \) and \( z_2 \in \mathcal{L}_2 \).

\[
\exists \beta_L^1, z_2 = c_L^2 \oplus \beta_L^1 \beta_L^1;
\]

Let \( \beta_L^{(1)} = \beta_L^{(1,y_L^1)} \oplus \beta_L^{(1,y_L^2)} \), \( G_L^{(1)} = G_L^{(1,y_L^1)} \cup G_L^{(1,y_L^2)} \). Given that XOR is an associative and commutative gate, we have the following:

\[
\begin{align*}
&z_1 \oplus z_2 = c_L^1 \oplus (c_L^2 \oplus \beta_L^1) \oplus c_L^2 \\
&= c_L^1 \oplus c_L^2 \oplus \beta_L^1 \beta_L^1.
\end{align*}
\]

3.2.2 Minkowski NOT (\( \lnot \)), and XNOR (\( \oplus \))

**Lemma 2.** Given two logical zonotopes \( \mathcal{L}_1 = \langle c_L^1, G_L^1 \rangle \) and \( \mathcal{L}_2 = \langle c_L^2, G_L^2 \rangle \), the Minkowski NOT can be computed exactly as

\[
\mathcal{L}_1 \oplus \mathcal{L}_2 = \langle c_L^1 \oplus c_L^2, G_L^1 \cup G_L^2 \rangle .
\]

**Proof.** The proof follows directly from truth table of XOR gate and \( \mathcal{L} \cap \mathcal{L} = \langle \{z_1 \oplus \lnot a \mid z_1 \in \mathcal{L} \} \mid z_1 \in \mathcal{L} \rangle \) which results in inverting each binary vector in \( \mathcal{L} \).

Similarly, we can perform the XNOR exactly as follows:

\[
\mathcal{L}_1 \oplus \mathcal{L}_2 = \langle c_L^1 \oplus c_L^2, G_L^1 \cup G_L^2 \rangle .
\]

3.2.3 Minkowski AND (\( \land \))

**Lemma 3.** Minkowski AND between two logical zonotopes as follows:

**Lemma 4.** Given two logical zonotopes \( \mathcal{L}_1 = \langle c_L^1, G_L^1 \rangle \) and \( \mathcal{L}_2 = \langle c_L^2, G_L^2 \rangle \), the Minkowski AND can be over-approximated by \( \mathcal{L}_\wedge = \langle c_L^1, G_L^1 \rangle \) where

\[
\mathcal{L}_1 \wedge \mathcal{L}_2 \subseteq \left\langle c_L^1 \cap c_L^2, G_L^1 \cup G_L^2 \right\rangle.
\]

**Proof.** Choose \( z_1 \in \mathcal{L}_1 \) and \( z_2 \in \mathcal{L}_2 \). Then, we have

\[
\exists \beta_L^1, z_1 = c_L^1 \oplus \beta_L^1, \quad \exists \beta_L^2, z_2 = c_L^2 \oplus \beta_L^2.
\]

\[
\beta_L^1 \oplus \beta_L^2.
\]
where

\[ \mathcal{L} = \{ x \in \mathbb{R}^{m \times n} \mid x = C_L \bigoplus_{i=1}^{\gamma_{C_L}} p_{i,2}^{(i)}(z), p_{i,2}^{(i)} = \{ 0, 1 \} \}. \]

We again use the shorthand notation \( \mathcal{L} = \langle C_L, G_L \rangle \) for a logical matrix zonotope.

We define the Minkowski semi-tensor product with a slight abuse of the notation.

\[ L_1 \kappa L_2 = \{ z_1 \vee z_2 \mid z_1 \in L_1, z_2 \in L_2 \} \]

(18)

We define the Minkowski semi-tensor product between two logical matrix zonotopes as follows.

**Lemma 3.** Given two logical matrix zonotopes \( L_1 = \langle C_{L_1}, G_{L_1} \rangle \in \mathbb{R}^{m \times n} \) and \( L_2 = \langle C_{L_2}, G_{L_2} \rangle \in \mathbb{R}^{p \times q} \), the Minkowski semi-tensor product can be over-approximated by \( L_{C_{L_1}, G_{L_2}} \).

\[ L_1 \kappa L_2 \subseteq L_{C_{L_1}, G_{L_2}}. \]

(19)

where

\[ C_{L_1} \kappa C_{L_2} = \left( C_{L_1} \bigoplus_{i=1}^{\gamma_{C_{L_1}}} g_{L_1}^{(i)}(z), C_{L_2} \bigoplus_{i=1}^{\gamma_{C_{L_2}}} g_{L_2}^{(i)}(z) \right) \]

\[ g_{L_1}^{(i)}(z), g_{L_2}^{(i)}(z) \]

(20)

Combining the factors in

\[ \hat{\beta}_{C_{L_1}, C_{L_2}} = \left\{ \delta^{(i)}(1; \gamma_{C_{L_1}}), \delta^{(i)}(2; \gamma_{C_{L_2}}), \delta^{(i)}(1; \gamma_{C_{L_1}}), \delta^{(i)}(2; \gamma_{C_{L_2}}) \right\} \]

results in having \( z_1, z_2 \in L_\gamma \), and thus \( L_1 \cap L_2 \subseteq L_\gamma \). □

3.2.4 Minkowski NAND (\( \& \))

Given that we are able to do Minkowski XOR and NOT operations, we will be able to do the Minkowski NAND as follows.

**Corollary 2.** Given two logical zonotopes \( L_1 = \langle c_{L_1}, G_{L_1} \rangle \) and \( L_2 = \langle c_{L_2}, G_{L_2} \rangle \), the Minkowski NAND can be over-approximated by

\[ L_1 \lor L_2 \subseteq (L_1 \land L_2) \land (L_1 \land L_2). \]

(16)

\[ L_1 \lor L_2 \subseteq (L_1 \land L_2) \land (L_1 \land L_2). \]

(17)

3.2.5 Minkowski OR (\( \lor \)) and NOR (\( \lor \))

Given that we are able to NAND two sets which is a universal gate operation, we will be able to over-approximate the following logical Minkowski operations as shown next:

\[ L_1 \lor L_2 \subseteq (L_1 \land L_2) \land (L_1 \land L_2). \]

(21)

Combining the factors in \( \beta_{C_{L_1}, C_{L_2}} = \left\{ \hat{\beta}_{C_{L_1}, C_{L_2}}, \beta_{C_{L_1}, C_{L_2}} \right\} \) results in having \( z \in L_\gamma \), and thus \( L_1 \lor L_2 \subseteq L_\gamma \). □

3.3 Reachability Analysis

We aim to over-approximate the exact reachable region of (1) which is defined in Definition 2 as follows:

**Theorem 1.** Given a logical function \( f: \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}^n \) in (1) and starting from initial set \( \bar{R}_0 \) where \( x(0) \in \bar{R}_0 \), then the reachable region computed as

\[ \bar{R}_{k+1} = f(\bar{R}_k, \mathcal{U}_k) \]

(22)

using logical zonotopes over-approximates the exact reachable set, i.e., \( \bar{R}_{k+1} \supseteq \bar{R}_{k+1} \).

**Proof.** The logical function consists in general of XOR and NOT operations and any logical operations constructed from the NAND. ∀\( x(0) \in \bar{R}_0 \), and ∀\( u(k) \in \mathcal{U}_k \), we are able to compute Minkowski XOR and NOT exactly using Lemma 1 and Corollary 1 and over-approximate Minkowski NAND using Lemma 2 and Corollary 2. Thus, \( \bar{R}_{k+1} \supseteq \bar{R}_k \). □

3.4 Generators Reduction

In certain scenarios, we might need to find a logical zonotope that contains at least the given binary vectors. One way to do that is as follows:

**Lemma 4.** Given a list \( S_p \) of \( p \) binary vectors in \( \mathbb{B}^n \), the logical zonotope \( L_p = \langle c_{L_p}, G_{L_p} \rangle \) with \( S_p^{(i)} \in L_p \), ∀\( i = 1, \ldots, p \) is given by

\[ c_{L_p} = S_p^{(1)}, \]

(23)

\[ g_{L_p}^{(i-1)} = S_p^{(i)} \ominus c_{L_p}, \quad \forall i = 2, \ldots, p. \]

(24)

**Proof.** By considering the truth table of all values of \( \beta_{C_p} \), we can find that the evaluation of \( L_p \) results in \( c_{L_p} = S_p^{(1)} \) at one point and \( g_{L_p}^{(i-1)} \ominus c_{L_p} = S_p^{(i)} \ominus c_{L_p} \ominus c_{L_p} = S_p^{(i)}, \quad \forall i = 2, \ldots, p \) at other points. □

After finding a logical zonotope containing the given binary vectors, reducing the number of generators without sacrificing any
Algorithm 1: Function to reduce to reduce the number of generators of a logical zonotope.

**Input:** A logical zonotope $L = (c_L, G_L)$ with large number of generators.

**Output:** A logical zonotope $L_r = (c_{L_r}, G_{L_r})$ with $|G_{L_r}| \leq |G_{L_r}|$ generators.

1. $B_L \leftarrow \text{evaluate}(L)$ // Compute a list $B_L$ of all binary vectors contained in $L$.
2. for $i = 1 : |G_L|$ do
   3. $B_L = \text{evaluate}(L \setminus g_L^{(i)})$ // Compute a list $B_L$ of all binary vectors contained in $L$ without the generator $g_L^{(i)}$.
   4. if $\text{isequal}(B_L, B_{L_r})$ then
   5. $g_L = \text{removeGenerator}(g_L^{(i)})$
   6. $L_r = (c_{L_r}, G_{L_r})$

unique binary vector would be helpful. Thus, we propose Algorithm 1 to reduce the number of generators while maintaining the same contained individual vectors. We first compute all the different binary vectors contained in the input logical zonotope $L$ in Line 1. Then, the algorithm checks the effect of removing each generator by computing the binary vectors contained in the logical zonotope without the removed generator in Line 3. The chosen generator will be deleted if its removal does not remove a unique binary vector in Lines 17 and 5.

4 Case Studies

To illustrate the use of logical zonotopes, we present three different use cases. We first show how logical zonotopes can drastically improve the complexity of exhaustively searching for the key of a LFSR. Then, we formulate an intersection crossing problem, where we compute the computational complexity of BDDs, BCN-based semi-tensor products, and logical zonotopes when verifying the safety of two vehicle’s intersection crossing protocol. Finally, we demonstrate the use of logical zonotopes for conducting reachability analysis on a high-dimensional Boolean function and show the advantages of logical zonotopes when high-dimensional logical systems. The code to recreate each use case is publicly available.

4.1 Exhaustive Search for the Key of a LFSR

LFSRs are used intensively in many stream ciphers in order to generate pseudo random randoms longer keys from the input key. For simplicity we consider 60-bits LFSR $A$ initialized with the input key $K_A$ with length $I_k$. The operations on bit level are shown in Figure 2 where

$A[1] = A[60] \oplus A[59] \oplus A[58] \oplus A[14],$

output $= A[60] \oplus A[59].$

Each bit of the output of the LFSR is XOR-ed with the message $m_A[i]$ to obtain one bit of the ciphertext $c_A[i]$.

Now consider we aim to obtain the input key $K_A$ using exhaustive search by trying out $2^{I_k}$ key values that can generate the cipher $c_A$ from $m_A$ with worst case complexity $O(2^{I_k})$ where $I_k = 60$ is the key length. Instead we propose to use logical zonotopes in Algorithm 2 to decrease the complexity of the search algorithm. We start by defining a logical zonotope $L_B$ which contains 0 and 1 in line 1. Initially, we assign a logical zonotope to each bit of LFSR $A$ in line 4 except the first two bits. Then, we set the first two bits of LFSR $A$ to one of the $2^2$ options of comb list in line 6. Then, we call the LFSR with the assigned key bits to get a list of logical zonotopes $G_A$ with a misuse of notations. The pseudo random output of logical zonotopes $G_A$ is XOR-ed with the message $m_A$ to get a list of ciphertext logical zonotopes $C_A$. If any cipher of the list $c_A$ is not included in the corresponding logical zonotope $C_A$, then the assigned two digits in line 6 are wrong and we do not need to continue finding values for the remaining bits of LFSR $A$. After finding the correct two bits with $c_A \in C_A$, we continue by assigning a zero to bit by bit in line 12. Then we generate the pseudo random numbers $G_A$ and XOR-ed it with the $m_A$ to get the list of cipher logical zonotopes $C_A$. The cipher logical zonotopes $C_A$ are checked to contain the list of ciphers $c_A$ and assign $K_A$ in line 16, accordingly. The worst case complexity of Algorithm 2 is $O(I_k)$. We measured the execution time of Algorithm 2 with different key sizes in comparison to the worst case execution time of traditional search. To compute the execution time of the traditional search, we multiply the number of iterations by the average execution time of a single iteration.

4.2 Safety Verification of an Intersection Crossing Protocol

In this example, we consider an intersection where two vehicles need to pass through the intersection, while avoiding collision. We encode their respective crossing protocols as logical functions and verify the safety of their protocols through reachability analysis using BDDs, a BCN semi-tensor product-based approach, and logical zonotopes. We compare the computational times of each approach for different time horizons of the reachability analysis.

We denote whether vehicle $i$ is passing the intersection or not at time $k$ by $p_i(k)$. Then, we denote whether vehicle $i$ came first or not at time $k$ by $c_i(k)$. We use control inputs $u_i^r(k)$ and $u_i^l(k)$ to denote the decision of vehicle $i$ to pass or to come first at time $k$, respectively. For each vehicle $i = 1, 2$, the intersection passing

| Key Size | Algorithm 2 | Traditional Search |
|----------|-------------|---------------------|
| 30       | 1.97        | $1.18 \times 10^6$  |
| 60       | 4.76        | $1.26 \times 10^{15}$ |
| 120      | 7.95        | $1.46 \times 10^{33}$ |
Algorithm 2: Exhaustive search for LFSR key using logical zonotopes

**Input:** A message $m_A$ and its ciphertext $c_A$ with length $l_m$

**Output:** The used Key $K_A$ with length $l_k$ in encrypting $m_A$

1. $B_B$ = enclosePoints([0 1]) // enclose the points 0 and 1 by a logical zonotope
2. comb = {00, 01, 10, 11}
3. for $i = 3 : l_k$ do
   4. $K_A[i] = L_B$ // assign the key bits as a logical zonotope
5. for $i = 1 : 4$ do
   6. $K_A[1:2] = \text{comb}[i]$
   7. $G_A = \text{LFSR}(K_A)$ // generate pseudo random numbers from the key $K_A$
   8. $C_A = G_A \oplus m_A$
   9. if $\neg \text{contains}(C_A,c_A)$ then
      10. continue; // continue if $c_A \notin C_A$
   11. for $j = 3 : l_k$ do
      12. $K_A[j] = 0$.
      13. $G_A = \text{LFSR}(K_A)$ // generate pseudo random numbers from the key $K_A$
      14. $C_A = G_A \oplus m_A$
      15. if $\neg \text{contains}(C_A,c_A)$ then
         16. $K_A[i] = 1$ // assign if $c_A \notin C_A$
      17. if $\text{isequal}(K_A \oplus m_A,c_A)$ then
         18. return $K_A$

Table 2: Execution Time (seconds) for verifying an intersection crossing protocol.

| Steps | Logical Zonotopes | BDD | BCN |
|-------|-------------------|-----|-----|
| 10    | 0.06              | 0.14| 0.16|
| 50    | 0.22              | 0.42| 0.79|
| 100   | 0.39              | 1.30| 2.44|
| 1000  | 2.60              | 8.36| 11.11|

The protocol is represented by the following:

$$p_1(k+1) = u_1^0(k) \neg p_1(k) \neg c_1(k). \quad (25)$$

Then, the logic behind coming first for each vehicle $i = 1, 2$ is written as the following:

$$c_i(k+1) = \neg p_1(k+1)(u_1^0(k) \lor (\neg p_1(k)p_1(k+1))). \quad (26)$$

To perform reachability analysis, we initialize the crossing problem with the following conditions:

$$p_1(0) = 1, p_2(0) = 0, c_1(0) = 1, c_2(0) = 0. \quad (27)$$

To verify the passing protocol is always safe, under any decision made by each vehicle, we perform reachability analysis under the following uncertain control inputs:

$$u_1^0(k) \in \{0, 1\}, u_2^0(k) \in \{0, 1\}, k = 0, \ldots, N, \quad (28)$$

$$u_1^0(k) \in \{0, 1\}, u_2^0(k) \in \{0, 1\}, k = 0, \ldots, N. \quad (29)$$

Then, we compare performing reachability analysis using BDDs, BCNs, and logical zonotopes. We construct BDDs for each formula and execute the reduced form of the BDDs with uncertainty. We illustrate a reduced BDD of (25) in Figure 3. For the semi-tensor product-based approach with BCNs, we write the state $x(k) = (\kappa_{i=1}^2 \rho_i(k)) \odot (\kappa_{i=1}^2 c_i(k))$. We write input $u(k) = (\kappa_{i=1}^2 u_i^0(k)) \odot (\kappa_{i=1}^2 u_i^0(k))$. The structure matrix $L$, which encodes (25)-(26), is a $2^4 \times 2^8$ matrix where 4 is the number of the states and 8 is the number of states and inputs. We perform reachability analysis for the BCN using $x(k+1) = L \circ u(k) \circ x(k)$ for all possible combinations. For reachability analysis with logical zonotopes, we represent each variable in (25)-(26) with a logical zonotope. We first compute the initial zonotope $R_0$ using Lemma 4 which contains the initial and certain states (27). Then, using Theorem 1, we compute the next reachable sets as logical zonotopes.

The execution time of the three approaches is presented in seconds in Table 2. We note that reachability analysis using logical zonotopes provides better run-times when compared with reachability analysis with BDDs and semi-tensor products. Moreover, as the time horizon of the reachability analysis increases, the run-time of the reachability analysis with logical zonotopes increases slower than the other two methods.

4.3 Reachability Analysis on a High-Dimensional Boolean Function

We consider the following Boolean function with $B_i \in \mathbb{B}_{20}$ and $U_i \in \mathbb{B}_{20}$, $i = 1, 2, 3$:

$$B_1(k+1) = U_1(k) \lor (B_2(k) \odot B_1(k)), \quad (30)$$

$$B_2(k+1) = B_2(k) \odot (B_1(k) \wedge U_2(k)), \quad (31)$$

$$B_3(k+1) = B_3(k) \wedge (U_2(k) \odot U_3(k)). \quad (32)$$

For our reachability analysis, we initially assign sets of 10 possible values to $B_1(0), B_2(0)$ and $B_3(0)$. Then, we compare the execution time of reachability analysis starting from this initial condition using BDDs and logical zonotopes. We do not compare with the semi-tensor product-based approach in this example since the size structure matrix is intractable for high-dimensional systems. For the supplied variable ordering, the reachability analysis using BDDs could not be completed in a reasonable amount of time, so we...
Table 3: Execution Time (seconds) for reachability analysis of a high-dimensional Boolean function.

| Steps N | Logical Zonotopes | BDD |
|---------|-------------------|-----|
| 3       | 11.33             | 3 \times 10^4 |
| 5       | 55.90             | 5 \times 10^4 |
| 10      | 134.02            | 10 \times 10^6 |

instead used the average execution time for one iteration and multiplied that time to get the total time for the reachability analysis. The results are shown in Table 3.

5 Conclusion

In this work, we propose a novel set representation for binary vectors called logical zonotope. Logical zonotopes can represent 2^N binary vectors using only y generators. We prove that applying different Minkowski logical operations to logical zonotopes always yields either exact solutions or over-approximations. Moreover, we demonstrate that since the different Minkowski operations only work with the generators of the logical zonotopes, we can greatly reduce the computational complexity of search and reachability algorithms. In general, logical zonotopes allow for a variety of computationally-efficient analyses for logical systems. In future work, we are investigating the potential of logical zonotopes for analyzing hybrid systems and are continuing to explore the practical application of logical zonotopes in new use cases.

Acknowledgement

This paper has received funding from the European Union’s Horizon 2020 research and innovation programme under grant agreement No. 830927.

References

[1] Tatsuya Akutsu, Satoru Miyano, and Satoru Kuhara. 1999. Identification of genetic networks from a small number of gene expression patterns under the Boolean network model. In Bioinformatics’99. World Scientific, 17–28.
[2] Matthias Althoff. 2010. Reachability analysis and its application to the safety assessment of autonomous cars. Ph.D. Dissertation. Technische Universität München.
[3] Jesse Bingham. 2015. Universal boolean functional vectors. In Formal Methods in Computer-Aided Design. IEEE, 25–32.
[4] Trevor J Bird, Herschel C Pangborn, Neera Jain, and Justin P Koeln. 2021. Hybrid zonotopes: A new set representation for reachability analysis of mixed logical dynamical systems. arXiv preprint arXiv:2106.14831 (2021).
[5] M Byrodt, Bengt Lennartsson, Arash Vahidi, and Knut Akesson. 2006. Efficient reachability analysis on modular discrete-event systems using binary decision diagrams. In 2006 8th International Workshop on Discrete Event Systems. IEEE, 288–293.
[6] Gianpiero Cabodi, Paolo Camurati, Luciano Lavagno, and Stefano Quer. 1997. Disjunctive partitioning and partial iterative squaring: An effective approach for symbolic traversal of large circuits. In Proceedings of the 34th annual Design Automation Conference. 728–733.
[7] Christos G Cassandras and Stéphane Lafortune. 2008. Introduction to discrete event systems. Springer.
[8] Daizhan Cheng, Hongsheng Qi, and Ancheng Xue. 2007. A survey on semi-tensor product of matrices. Journal of Systems Science and Complexity 20, 2 (2007), 304–322.
[9] Eric Dallal, Alessandro Colombo, Domitilla Del Vecchio, and Stéphane Lafortune. 2017. Supervisory control for collision avoidance in vehicular networks using discrete event abstractions. Discrete Event Dynamic Systems 27, 1 (2017), 1–44.
[10] Antoine Girard. 2005. Reachability of Uncertain Linear Systems Using Zonotopes. In Hybrid Systems: Computation and Control, Manfred Morari and Lothar Thiele (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 291–305.
[11] Alessandro Giua and Carla Seatzu. 2008. Modeling and supervisory control of railway networks using Petri nets. Transactions on automation science and engineering 5, 3 (2008), 431–445.
[12] Alan John Hu. 1996. Techniques for efficient formal verification using binary decision diagrams. stanford university.
[13] Wolfgang Kühn. 1998. Rigorously computed orbits of dynamical systems without the wrapping effect. Computing 61, 1 (1998), 47–67.
[14] Thomas Leifeld, Zhihua Zhang, and Ping Zhang. 2019. Overview and comparison of approaches towards an algebraic description of discrete event systems. Annual Reviews in Control 48 (2019), 80–88.
[15] Fangfei Li and Yang Tang. 2017. Robust Reachability of Boolean Control Networks. IEEE/ACM Transactions on Computational Biology and Bioinformatics 14, 3 (2017), 740–745.
[16] Hongsheng Qi and Daizhan Cheng. 2009. Analysis and control of Boolean networks: A semi-tensor product approach. In 7th Asian Control Conference. IEEE, 1352–1356.
[17] Andrea Roli, Mattia Manfroni, Carlo Pincioli, and Mauro Birattari. 2011. On the design of Boolean network robots. In European Conference on the Applications of Evolutionary Computation. 43–52.
[18] M Schuh, M Zgorzelski, and J Lunze. 2015. Experimental evaluation of an active fault–tolerant control method. Control Engineering Practice 43 (2015), 1–11.
[19] Joseph K Scott, Davide M Raimondo, Giuseppe Roberto Marseglia, and Richard D Braatz. 2016. Constrained zonotopes: A new tool for set-based estimation and fault detection. Automatica 69 (2016), 126–136.
[20] Ilya Shmulevich, Edward R Dougherty, Seungchun Kim, and Wei Zhang. 2002. Probabilistic Boolean networks: a rule-based uncertainty model for gene regulatory networks. Bioinformatics 18, 2 (2002), 261–274.
[21] Johan Thunberg, Petter Ögren, and Xiaoming Hu. 2011. A boolean control network approach to pursuit evasion problems in polygonal environments. In International Conference on Robotics and Automation. IEEE, 4506–4511.