Asymptotics of class numbers

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Introduction

For an order \( \mathcal{O} \) in a number field let \( h(\mathcal{O}) \) denote its class number and \( R(\mathcal{O}) \) its regulator. Proving a conjecture of C.F. Gauss, C.L. Siegel showed in [29],

\[
\sum_{d(\mathcal{O}) \leq x} h(\mathcal{O}) R(\mathcal{O}) \sim \frac{\pi^2}{36 \zeta(3)} x^{\frac{3}{2}},
\]

where the sum ranges over the set of all real quadratic orders (i.e., orders in real quadratic fields) with discriminant \( d(\mathcal{O}) \) bounded by \( x \).

For a long time it was believed to be impossible to separate the class number and the regulator. However, in 1981 P. Sarnak showed [28], using the trace formula, that

\[
\sum_{R(\mathcal{O}) \leq x} R(\mathcal{O}) \sim e^{2x} / 2x,
\]

the sum ranging over all real quadratic orders with regulator bounded by \( x \).

Sarnak established this result by identifying the regulators with lengths of closed geodesics of the modular curve \( \mathcal{H}/\text{SL}_2(\mathbb{Z}) \) (Theorem 3.1 there) and by using the prime geodesic theorem for this Riemann surface. Actually, Sarnak proved not this result but the analogue where \( h(\mathcal{O}) \) is replaced by the class number in the narrower sense and \( R(\mathcal{O}) \) by a “regulator in the narrower

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sense”. But in Sarnak’s proof the group $\text{SL}_2(\mathbb{Z})$ can be replaced by $\text{PGL}_2(\mathbb{Z})$ giving the above result. See also [11, 32].

Our goal is to generalize Sarnak’s result to number fields of higher degree. Such a generalization has resisted all efforts so far since the trace formula is not yet in a state that would make it useful in spectral geometry. For instance, as yet a proof of the absolute convergence of the spectral side of the trace formula is outstanding. However, recent partial results by W. Müller are sufficient for the case treated in this paper.

We will now formulate the main theorem. Since there are several concepts of class numbers, we have to make clear which one we use. Let $\mathcal{O}$ be an order in a number field $F$. Let $I(\mathcal{O})$ be the set of all finitely generated $\mathcal{O}$-submodules of $F$. According to the Jordan-Zassenhaus Theorem [26], the set of isomorphism classes $[I(\mathcal{O})]$ of elements of $I(\mathcal{O})$ is finite. Let $h(\mathcal{O})$ be the cardinality of the set $[I(\mathcal{O})]$, called the class number of $\mathcal{O}$.

A cubic field $F$ (i.e., a number field of degree 3 over the rationals) is either totally real or has two complex and one real embedding in which case we call it a complex cubic field. Let $O$ be the set of isomorphism classes of orders in complex cubic fields. The following is our main result.

**Theorem 0.1** As $x \to \infty$ we have

$$\sum_{\mathcal{O} \in O \atop R(\mathcal{O}) \leq x} h(\mathcal{O}) \sim \frac{e^{3x}}{3^x}.$$ 

Our method is based on a new “simple trace formula”. Such formulae have been used in the past by various authors, for instance by Deligne-Kazhdan or Kottwitz. They come about by plugging special test functions into Arthur’s trace formula. The test functions are chosen such that many terms in the trace formula vanish. The simple trace formula of this paper will be such that the geometric side only consists of orbital integrals of globally elliptic elements as opposed to locally elliptic elements which is what the previous simple trace formulae reduced to.

In Section 1 the general form of the simple trace formula is given, in which the test functions are characterized by vanishing conditions. In Section 2, test functions are constructed explicitly by twisting with virtual characters. This facilitates the computation of orbital integrals and still leaves great freedom in the choice of test functions. The convergence of the spectral side for noncompactly supported test functions is discussed in Section 3.
From this point on we restrict to the case $SL_3$. In order to separate orbital integrals of splitrank one, twisted resolvents are used. The validity of the trace formula for these non-compactly supported functions is derived via a Casimir functional calculus in Sections 4 to 7. The Prime Geodesic Theorem, which is our main result in a different guise, is given in Section 8.

In the light of Sarnak’s result and the result of the present paper, one is tempted to formulate a conjecture about the growth rate of class numbers in number fields of a given type. We must however warn that our method does not support any speculation of this kind. The reason is that we do not count orders, but rather their units which in the setting of reductive groups come about as globally elliptic elements. Only in the case when the rank of the unit group is one, it is possible to draw conclusions about class numbers from the distribution of units. Thus one is limited to real quadratic fields (Sarnak), complex cubic (present paper) or purely imaginary fields of degree 4. In the latter case a new difficulty emerges: since the degree of the field extension is not a prime, elliptic elements can no longer be identified with order-units in number fields any more, so the method gives a different asymptotic altogether.

Contents

1. A simple trace formula 4
2. Test functions 7
3. Twisting by characters 10
4. Convergence of the spectral side 13
5. Orbital integrals 17
6. Choice of the twisting character 21
7. The geometric side 23
8. The spectral side 25
9. The prime geodesic theorem 31
1 A simple trace formula

We derive a simple version of Arthur’s trace formula by inserting functions with certain restrictive properties which guarantee the vanishing of the parabolic terms on the geometric side. The trace formula for $SL(3, \mathbb{Z})$ has also been studied in [33], which unfortunately arrives at an incorrect formula due to a wrong handling of the truncation.

Let $G$ be a linear algebraic $\mathbb{Q}$-group. If $E$ is a $\mathbb{Q}$-algebra, any rational character $\chi$ of $G$ defined over $\mathbb{Q}$ defines a homomorphism $G(E) \to GL_1(E)$. If $E$ comes with an absolute value $|.|$, we define $G(E)^1$ to be the subgroup of all elements $g$ such that $|\chi(g)| = 1$ for all rational characters $\chi$ defined over $\mathbb{Q}$. We will use this notation in the cases when $E$ is $\mathbb{R}$ or the ring $\mathbb{A}$ of adeles of $\mathbb{Q}$. One should be aware that $G(\mathbb{R})^1$ could also be defined with respect to characters defined over the field $\mathbb{R}$, but this is not be the point of view in the present paper.

From now on we denote by $G$ a connected reductive linear algebraic group over $\mathbb{Q}$. If $P$ is a parabolic $\mathbb{Q}$-subgroup of $G$ with unipotent radical $N$, we have a Levi decomposition $P = LN$. Generally, we denote the group of real points of a linear algebraic $\mathbb{Q}$-group by the corresponding roman letter, so that $P = LN$. However, if $A$ is a maximal $\mathbb{Q}$-split torus of $L$, we denote by $A$ the connected component of the identity $A(\mathbb{R})^0$. One has decompositions $L(A) = L(A)^1A$, $L = MA$ (direct products) and $P^1 = MN$, where $M = L^1$.

Let $A_{\text{fin}}$ denote the subring of finite adeles, then we have direct product decompositions $A = A_{\text{fin}} \mathbb{R}$ and $G(A) = G(A_{\text{fin}})G$. Fix a maximal compact subgroup $K$ of $G$.

The geometric expansion of the trace formula

$$J_{\text{geom}}(f) = \sum_{\mathfrak{o}} J_{\mathfrak{o}}(f)$$

for $f \in C_c^\infty(G(A)^1)$ was introduced in [Π]. Here the sum runs over all classes $\mathfrak{o}$ in $G(\mathbb{Q})$ with respect to the following equivalence relation: Two elements are called equivalent if the semisimple components in their Jordan decomposition are conjugate in $G(\mathbb{Q})$. Further,

$$J_{\mathfrak{o}}(f) = \int_{G(\mathbb{Q}) \setminus G(A)^1} k_\mathfrak{o}(x) \, dx$$

for certain functions $k_\mathfrak{o}$ whose definition we will recall below. The sum and the integral converge if we replace $k_\mathfrak{o}$ by its absolute value.
The integrand in the definition of \( J_\omega(f) \) is given as

\[
\kappa_\omega(x) = \sum_P (-1)^{\dim \mathcal{A}_P/\mathcal{A}_G} \sum_{\delta \in \mathcal{P}(Q) \setminus \mathcal{G}(Q)} K_{\mathcal{P},\omega}(\delta x, \delta x) \tau_\mathcal{P}(H(\delta x) - T),
\]

where the sum runs over the standard parabolic \( Q \)-subgroup \( \mathcal{P} = \mathcal{L}\mathcal{N} \), for which we write \( \mathcal{A}_\mathcal{P} = \mathcal{A}_\mathcal{L} \) and

\[
K_{\mathcal{P},\omega}(x, y) = \sum_{\gamma \in \mathcal{L}(Q) \cap \omega} \int_{\mathcal{N}(A)} f(x^{-1} \gamma ny) \, dn.
\]

All we need to know about the factor \( \tau_\mathcal{P}(H(\delta x) - T) \) at this point is that in the case \( \mathcal{P} = \mathcal{G} \) it is identically equal to 1.

We call a function on \( \mathcal{G}(A) \) parabolically regular at the infinite place if it is supported on \( K_{\text{fin}} \times G^1 \) for some compact open subgroup \( K_{\text{fin}} \) of \( \mathcal{G}(A_{\text{fin}}) \) and vanishes on all \( G \)-conjugates of \( K_{\text{fin}} \times P^1 \) for every parabolic \( Q \)-subgroup \( \mathcal{P} \neq \mathcal{G} \).

An element \( \gamma \in \mathcal{G}(Q) \) is called \( Q \)-elliptic if it is not contained in any parabolic \( Q \)-subgroup other than \( \mathcal{G} \) itself. This notion is clearly invariant under conjugation, and we say that a class \( \omega \) is \( Q \)-elliptic if some (hence any) of its elements is so. It is known that \( Q \)-elliptic elements are semisimple, so \( Q \)-elliptic classes \( \omega \) are just conjugacy classes in \( \mathcal{G}(Q) \).

**Proposition 1.1** If \( f \in C^\infty_c(\mathcal{G}(A)^1) \) is parabolically regular at the infinite place, then \( J_\omega(f) \) vanishes unless \( \omega \) is \( Q \)-elliptic, in which case

\[
J_\omega(f) = \int_{\mathcal{G}(Q) \setminus \mathcal{G}(A)} K_{\mathcal{G},\omega}(x, x) \, dx.
\]

In light of the above remarks the proposition is a consequence of the following lemma.

**Lemma 1.2** Suppose that \( f \) satisfies the conditions of Proposition 1.1. Then \( K_{\mathcal{P},\omega}(x, x) = 0 \) for any \( x \in \mathcal{G}(A) \) unless \( \mathcal{P} = \mathcal{G} \) and the class \( \omega \) is \( Q \)-elliptic.

**Proof:** First we show for any \( \mathcal{P} \neq \mathcal{G} \) that \( f(x^{-1} q x) = 0 \) for \( q \in \mathcal{P}(A)^1 \) and \( x \in \mathcal{G}(A) \). By the assumption on the support of \( f \) we have only to consider \( q = q_{\text{fin}} q_\infty \) with \( x^{-1} q_{\text{fin}} x \in K_{\text{fin}} \), i.e., \( q_{\text{fin}} \in xK_{\text{fin}} x^{-1} \cap \mathcal{P}(A) \), a compact subgroup of \( \mathcal{P}(A) \). Any continuous quasicharacter with values in \( ]0, \infty[ \) will
be trivial on that subgroup, hence \( q_{\text{fin}} \in \mathcal{P}(\mathbb{A})^1 \). Since \( q \) was already in \( \mathcal{P}(\mathbb{A})^1 \), it follows that \( q_\infty \in \mathcal{P}(\mathbb{A})^1 \cap P = P^1 \), and so \( f(x^{-1}qx) = 0 \) due to the assumption on \( f \) applied to the parabolic \( \mathbb{Q} \)-subgroup \( \mathcal{P} \).

In particular, as \( \mathcal{L}(\mathbb{Q})\mathcal{N}(\mathbb{A}) \subset \mathcal{P}(\mathbb{A})^1 \), we see that \( f(x^{-1}\gamma nx) = 0 \) for all \( \gamma \in \mathcal{L}(\mathbb{Q}) \) and \( n \in \mathcal{N}(\mathbb{A}) \), hence \( K_{\mathcal{P},\mathfrak{o}}(x,x) = 0 \), and so \( f(x^{-1}\gamma x) = 0 \).

If \( \mathfrak{o} \) is not \( \mathbb{Q} \)-elliptic, then every \( \gamma \in \mathfrak{o} \) is contained in some parabolic \( \mathbb{Q} \)-subgroup \( \mathcal{P} \neq G \), and in view of \( \mathcal{P}(\mathbb{Q}) \subset \mathcal{P}(\mathbb{A})^1 \) we have \( f(x^{-1}\gamma x) = 0 \). Thus \( K_{\mathcal{P},\mathfrak{o}}(x,x) = 0 \). Lemma 1.2 and Proposition 1.1 follow.

We will now rewrite, in a non-adelic language, the geometric side of our simple trace formula in a special case. Let \( \mathcal{G} \) be a semisimple simply-connected linear algebraic \( \mathbb{Q} \)-group such that \( G = \mathcal{G}(\mathbb{R}) \) has no compact factors. Let \( \Gamma \) be a congruence subgroup of \( \mathcal{G}(\mathbb{A}) \), i.e., assume that there exists an open compact subgroup \( K_\Gamma \) of \( \mathcal{G}(\mathbb{A}_{\text{fin}}) \) with \( \Gamma = K_\Gamma \cap \mathcal{G}(\mathbb{Q}) \). Let \( f_\infty \in C^\infty_c(\mathcal{G}) \) be a parabolically regular function. For \( y \in G \) the orbital integral is defined as

\[
\mathcal{O}_y(f_\infty) := \int_{G_y \backslash G} f_\infty(x^{-1}yx) \, dx,
\]

where \( G_y \) denotes the centralizer of \( y \) in \( G \). We define \( f = f_{\text{fin}} \otimes f_\infty \), where \( f_{\text{fin}} \) is the characteristic function of \( K_\Gamma \) divided by the volume of that group with respect to the Haar measure of \( \mathcal{G}(\mathbb{A}_{\text{fin}}) \).

**Corollary 1.3** Under the above conditions, we have

\[
J_{\text{geom}}(f) = \sum_{[\gamma]} \text{vol}(\Gamma \backslash G_\gamma) \, \mathcal{O}_\gamma(f_\infty),
\]

where the sum on the right-hand side runs over the set of all conjugacy classes \([\gamma]\) in the group \( \Gamma \) which consist of \( \mathbb{Q} \)-elliptic elements.

**Proof:** Consider the formula for \( J_\mathfrak{o}(f) \) given in Proposition 1.1 for a \( \mathbb{Q} \)-elliptic class \( \mathfrak{o} \). The integral can be taken over \( \mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / K_\Gamma \), because the integrand is right \( K_\Gamma \)-invariant. Under our assumptions on \( \mathcal{G} \), strong approximation [18] holds, i.e., the action of \( G \) by right translation on that double quotient is transitive and hence induces an isomorphism of \( G \)-spaces

\[
\Gamma \backslash G \overset{\sim}{\longrightarrow} \mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / K_\Gamma.
\]
This isomorphism identifies suitably normalized $G$-invariant measures on these two spaces with each other, and we get

$$J_\theta(f) = \int_{\Gamma \setminus G} \sum_{\gamma \in \mathfrak{o}} f_{\text{fin}}(\gamma) f_\infty(x^{-1}\gamma x) \, dx.$$ 

The characteristic function $f_{\text{fin}}$ has the effect of restricting summation to $\mathfrak{o} \cap \Gamma$. Note that $J_\theta(|f|) < \infty$, because $|f_\infty|$ can be bounded by a nonnegative function in $C^\infty_c(G)$. Now the equality of $J_\theta(f)$ with the partial sum over $[\gamma] \in \mathfrak{o}$ in the asserted formula follows by the familiar Fubini-type argument. \qed

## 2 Test functions

In this section $G$ is a semisimple real Lie group with finite center and finitely many connected components.

We would like to use resolvent kernels as test functions in the trace formula. The convergence of the geometric side has been proved in [1] for compactly supported test functions only. Thus we are going to approximate resolvent kernels by compactly supported functions with the aid of the functional calculus of the Casimir operator.

Let $\mathfrak{g}_\mathbb{R}$ denote the Lie algebra of $G$ and $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$ its complexification. Let $B$ denote the Killing form. Let $\theta$ be the Cartan involution fixing $K$. The form $\langle X, Y \rangle = -B(\theta(X), Y)$ is positive definite on $\mathfrak{g}_\mathbb{R}$ and induces a $G$-invariant Riemannian metric on $G/K$. Let $\text{dist}(x, y)$ denote the corresponding distance function and write $d(g) = \text{dist}(gK, eK)$ for $g \in G$.

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. Every element $X$ of $U(\mathfrak{g})$ gives rise to a left-invariant differential operator written $h \mapsto X \ast h$, and a right-invariant differential operator $h \mapsto X \ast h$, $h \in C^\infty(G)$. Recall that for $p > 0$, the $L^p$-Schwartz space $C^p(G)$ is defined as the space of all $h \in C^\infty(G)$ such that, for every $n > 0$ and $X, Y \in U(\mathfrak{g})$, the seminorms

$$|h|_{p, n, X, Y} = \sup_{g \in G} |X \ast h \ast Y(g)| \Xi(g)^{-2/p}(1 + d(g))^n$$ 

are finite. Here $\Xi$ is the basic spherical function, and it suffices for our present purposes to know that there exist $r_1 > r_2 > 0$ such that $e^{-r_1 d(g)} \leq |\Xi(g)| \leq e^{-r_2 d(g)}$. If we complete the space $C^p(G)$ with respect to the seminorms involving only derivatives up to order $N$, we obtain a space $C^p_N(G)$, whose
topology can be given by a Banach norm. For each \( \tau \in \hat{K}, \) there is a subspace \( \mathcal{C}^{p}_{N}(G, \tau) \) of functions \( h \) satisfying \( \chi_{\tau} \ast h \ast \chi_{\tau} = h, \) where \( \chi_{\tau} \in C(K) \) is the idempotent associated to \( \tau. \)

We also need the space \( \mathcal{H}^{N}_{r} \) of even holomorphic functions \( \phi \) on the strip \( \{ z \in \mathbb{C} \mid | \text{Im } z | < r \} \) extending continuously to the boundary and such that the norm

\[
|\phi|_{r,N} = \sup_{|\text{Im } z| \leq r} |\phi(z)|(1 + |\text{Re } z|)^{N}
\]

is finite. Recall that a Schwartz function \( \phi \) on \( \mathbb{R} \) is called a Paley-Wiener function if its Fourier transform \( \hat{\phi}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi(y) e^{-ixy} dy \) has compact support.

For \( \pi \in \hat{G} \) and an irreducible unitary representation \( (\tau, V_{\tau}) \) of \( K, \) let \( P_{\pi,\tau} \) be the orthogonal projection defined on the space of \( \pi \) whose image is the \( \tau \)-isotypical component. Let \( C \) be the Casimir operator of \( G. \) For \( \pi \in \hat{G} \) the Casimir \( C \) acts on \( \pi \) by a scalar \( \pi(C). \)

**Proposition 2.1** Let \( 0 < p \leq 1, \) \( N \in \mathbb{N}, \) \( b \in \mathbb{R} \) and \( \tau \in \hat{K} \) be given. Then there exist \( r > 0 \) and \( N' \in \mathbb{N} \) such that for every \( \phi \in \mathcal{H}^{N}_{r} \), there is a unique function \( h_{\phi,\tau} \in \mathcal{C}^{p}_{N}(G, \tau) \) satisfying

\[
\pi(h_{\phi,\tau}) = \frac{1}{\dim \tau} \phi \left( \sqrt{-\pi(C) - b} \right) P_{\pi,\tau}
\]

for every \( \pi \in \hat{G}. \) The map \( \mathcal{H}^{N}_{r} \rightarrow \mathcal{C}^{p}_{N}(G, \tau) \) so defined is continuous. If \( \phi \) is a Paley-Wiener function, then \( h_{\phi,\tau} \) is compactly supported.

(The factor \( \frac{1}{\dim \tau} \) is put here in order to give \( \pi(h_{\phi,\tau}) \) a nice trace.)

**Proof:** The uniqueness of \( h_{\phi,\tau} \) is clear from the Plancherel theorem. In the case of an even Paley-Wiener function \( \phi, \) the existence of \( h_{\phi,\tau} \in C^{\infty}_{c}(G) \) with the required properties (except for the bounds) has been proved in \[10\], Lemmas 2.9 and 2.11. For this one considers the \( G \)-homogeneous vector bundle \( E_{\tau} = G \times V_{\tau}/K \) and identifies the space of smooth sections with \( (C^{\infty}(G) \otimes V_{\tau})^{K}. \) Then the operator \( D_{\tau} \) induced by \( -C - b \) is a generalized Laplacian in the sense of \[55\]. The operator \( \phi \left( \sqrt{D_{\tau}} \right) \) defined by functional calculus is an operator with smooth kernel \( \langle x | \phi \left( \sqrt{D_{\tau}} \right) | y \rangle, \) and by the theory of hyperbolic equations (\[31\], ch. IV) it follows that \( \phi \left( \sqrt{D_{\tau}} \right) \) has finite propagation speed (compare \[8\]). Identifying the sections of \( E_{\tau} \) with \( K \)-invariant functions as above, it follows that the \( G \)-equivariant operator \( \frac{1}{\dim \tau} \phi \left( \sqrt{D_{\tau}} \right) \) acts as a convolution operator by a function \( h_{\phi,\tau}. \)
Now observe that by the estimates in [8] there exists a constant $c > 0$ such that, for every $\phi$,
\[
\left| \langle x | \phi(\sqrt{D_\tau}) | y \rangle \right| \leq c |\hat{\phi}(\text{dist}(x, y))|.
\]
This means that there is a constant $c > 0$ such that for every $\phi$ we have
\[
|h_{\phi, \tau}(g)| \leq c |\hat{\phi}(d(g))|.
\]

Since the subspace of even Paley-Wiener functions is dense in $H_N$, we may extend the map $\phi \mapsto h_{\phi, \tau}$ by continuity provided we check that it is continuous with respect to the correct seminorms, and then the asserted formula for $\pi(h_{\phi, \tau})$ will remain valid.

By moving the contour of integration in the formula for the Fourier transform, we get $\sup |\hat{\phi}(x)| e^{r|x|} \leq c |\hat{\phi}|_{r,2}$ for some $c > 0$. Thus it follows from the above estimate that for every $p > 0$ there exist $r > 0$, $c > 0$ such that
\[
|h_{\phi, \tau}|_{p,0,1,1} \leq c |\hat{\phi}|_{r,2}.
\]
In order to estimate the derivatives, recall from [34] that for every $N$ there exist $N'$ and functions $\mu, \nu \in C_c^N(G)$ such that
\[
h = \mu * \Delta h_{\Delta} + \nu * h
\]
for every $h \in C^1(G)$, where $\Delta$ is the Laplacian on $G$ also occurring in Proposition 4.1. Together with standard properties of the functions $\Xi$ and $l$ this shows that there exists $c > 0$ such that
\[
|h|_{p,n,X,1} = |X\mu * \Delta h_{\Delta} + X\nu * h|_{p,n,1,1} \leq c(|\Delta h_{\Delta}|_{p,n,1,1} + |h|_{p,n,1,1})
\]
for $X$ of order $N$. We have a similar inequality for $|h|_{p,n,Y}$ and hence for $|h|_{p,n,X,Y}$.

Note that $\Delta$ can be chosen as $2C_K - C$, where $C_K$ is the Casimir of $K$ induced by the restriction of the Killing form of $g$. Specialising to $f \in C^p(G, \tau)$, we may replace $\Delta$ by $2d_\tau - C$, where $d_\tau = \tau(C_K)$ is a constant. Now the obvious formula
\[
(-C - b)h_{\phi, \tau} = h_{\phi, \tau}, \quad \hat{\phi}(\lambda) = \lambda^2 \phi(\lambda)
\]
allows us to deduce the required estimate. \qed
3 Twisting by characters

A virtual representation of the group $G$ is a $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional complex representation $\psi = \psi_+ \oplus \psi_-$. We define the virtual trace and determinant as

$$\text{tr} \psi(x) = \text{tr} \psi_+(x) - \text{tr} \psi_-(x), \quad \det \psi(x) = \det \psi_+(x)/\det \psi_-(x).$$

The function $\text{tr} \psi(x)$ is then called a virtual character. Note that $\text{tr}(\psi_1 \oplus \psi_2) = \text{tr} \psi_1 + \text{tr} \psi_2$ and $\text{tr}(\psi_1 \otimes \psi_2) = (\text{tr} \psi_1)(\text{tr} \psi_2)$. Thus the virtual characters form a ring, the character ring $R(G)$. If $G = G(\mathbb{R})$ for a semisimple $\mathbb{Q}$-group as in the previous section, we call a function on $G$ parabolically regular if it vanishes on $xP^xy^{-1}$ for every parabolic $\mathbb{Q}$-subgroup $P \neq G$ and every $x \in G$.

**Proposition 3.1** Let $G$ be a connected semisimple group over $\mathbb{Q}$. Then there is a virtual representation $\psi$ of $G$ such that $\text{tr} \psi$ is nonzero and parabolically regular.

**Proof:** Since characters are class functions, we may restrict attention to proper parabolic $\mathbb{Q}$-subgroups $P$ of $G$. We have the Langlands decomposition $P = MAN$, where $P^1 = MN$ is defined with respect to the $\mathbb{Q}$-structure as in section 1. Since the rank of $M$ is smaller than the rank of $G$, the restriction map $R(G) \to R(M)$ is not injective. Let $\text{tr} \psi_P$ be a nonzero element of its kernel. Since $N$ is the unipotent radical of $MN$ it follows that $\text{tr} \psi_P(mn) = \text{tr} \psi_P(m) = 0$ for all $m \in M$, $n \in N$. Set

$$\psi \overset{\text{def}}{=} \bigotimes_P \psi_P,$$

where the product runs over the finite set of all conjugacy classes of proper parabolic $\mathbb{Q}$-subgroups of $G$. Then the virtual character $\text{tr} \psi$ is parabolically regular. Since $\text{tr} \psi = \prod_P \text{tr} \psi_P$ and each $\text{tr} \psi_P$ is a nonzero algebraic function on $G$, it follows that $\text{tr} \psi$ is nonzero. \hfill $\square$

Assume now that $\text{tr} \psi$ is a parabolically regular virtual character. Let $h \in C^\infty_c(G)$ and set

$$f_\infty = h \text{ tr} \psi,$$

and let $f_{\text{fin}}$ be the characteristic function of $K_F$ as in section 1.
Lemma 3.2  The geometric side of the trace formula for the test function $f = f_{\text{fin}} \otimes f_{\infty}$ above is

$$J_{\text{geom}}(f) = \sum_{[\gamma]} \text{vol}(\Gamma \backslash G_\gamma) \text{ tr } \psi(\gamma) \mathcal{O}_\gamma(h).$$

Proof:  Since $\text{ tr } \psi$ is a class function it follows that the orbital integral satisfies $\mathcal{O}_\gamma(h \text{ tr } \psi) = \text{ tr } \psi(\gamma) \mathcal{O}_\gamma(h).$  \hfill \Box

This simple form of the trace formula is quite advantageous since $\text{ tr } \psi$ is easy to compute and concerning $h \in C_\infty^\infty(G)$ we have total freedom of choice.  However, the problem remains that the spectral side of the trace formula does not simplify.  To discuss the spectral side we will consider representations $\pi \otimes \sigma$ for $\pi \in \hat{G}$ and $\sigma$ finite-dimensional.  We endow $\sigma$ with a $K$-invariant norm, so that these are admissible Hilbert representations.  They are no longer bounded, and hence $\pi \otimes \sigma(h)$ is not defined for $h \in L^1(G).$  However, the operator norm of $\sigma(g)$ and hence that of $\pi \otimes \sigma(g)$ grows at most exponentially with $d(g) = \text{ dist}(gK, eK).$  Thus there exists $p > 0$ such that the integral defining $\pi \otimes \sigma(h)$ converges for all $h \in C_0^p(G)$ and $\pi \in \hat{G}.$  Let us first prove two general facts.

Lemma 3.3  For a given operator $M$ on the space of $\pi$ we have

$$\text{ tr } (M \pi(h \text{ tr } \sigma)) = \text{ tr}(M \otimes 1)(\pi \otimes \sigma(h)).$$

Proof:  We compute

$$\text{ tr}(M \pi(h \text{ tr } \sigma)) = \text{ tr } M \int_G h(x)\pi(x) \text{ tr } \sigma(x)dx$$

$$= \text{ tr}_1 \otimes \text{ tr}_2 \left( (M \otimes 1) \int_G h(x)\pi(x) \otimes \sigma(x)dx \right),$$

where $\text{ tr}_1$ and $\text{ tr}_2$ are the traces on the first and second tensor factor.  \hfill \Box

Let $\mathcal{B}(\pi)$ denote the Banach space of all bounded linear operators on the Hilbert space of $\pi.$

Lemma 3.4  For the linear map

$$1 \otimes \text{ tr} : \mathcal{B}(\pi \otimes \sigma) \to \mathcal{B}(\pi)$$

we have

$$\|(1 \otimes \text{ tr})(T)\| \leq \dim \sigma \|T\|.$$
Proof: Let \((e_j)_j\) denote an orthonormal basis of \(\sigma\). For \(T \in B(\pi \otimes \sigma)\) and \(v \in \pi\) we have
\[
1 \otimes \text{tr}(T)(v) = \sum_j \langle T(v \otimes e_j), e_j \rangle,
\]
where the partial inner product on the right is defined as a map from \((\pi \otimes \sigma) \times \sigma\) to \(\pi\) by
\[
\langle v \otimes w, w' \rangle = \langle w, w' \rangle v.
\]
This implies \(\|\langle v \otimes w, w' \rangle\| \leq \|v \otimes w\| \|w'\|\). Hence,
\[
\|1 \otimes \text{tr}(T)(v)\| \leq \sum_j \|\langle T(v \otimes e_j), e_j \rangle\| \\
\leq \sum_j \|T(v \otimes e_j)\| \\
\leq \dim \sigma \|T\| \|v\|.
\]
The lemma follows. \(\square\)

In order to understand the representation \(\pi \otimes \sigma\) for an induced \(\pi\) the following Lemma will be needed later.

**Lemma 3.5** Let \(P = MAN\) be a parabolic subgroup of a reductive group \(G\). Let \(P^+ = M^+AN\), where \(M^+\) is a finite index subgroup of \(M\). Let \(\pi = \pi_{\xi,\nu} = \text{Ind}_{P^+}^G(\xi \otimes \nu \otimes 1)\), where \(\xi\) is an irreducible admissible representation of \(M^+\) and \(\nu \in a_C\). Let \(\sigma\) be a finite dimensional representation of \(G\). Write
\[
\sigma|_{M^+A} = \bigoplus_{j=1}^s \sigma_j \otimes \nu_j
\]
for the decomposition into irreducibles of the restriction to the reductive group \(M^+A\).

Then, after reordering the \(\sigma_j \otimes \nu_j\) if necessary, there is a \(G\)-stable filtration of \(\pi \otimes \sigma\),
\[
0 = F^0(\pi \otimes \sigma) \subset \ldots \subset F^s(\pi \otimes \sigma) = \pi \otimes \sigma
\]
with quotients
\[
F^j/F^{j-1} \cong \pi_{\xi \otimes \sigma_j, \nu + \nu_j}.
\]
Since \(G = P^+K\), the representation \(\pi_{\xi,\nu}\) has a compact model on the Hilbert space \(\text{Ind}_{K \cap M^+}^K(\xi)\) which is independent of \(\nu\). In the compact model, this filtration does not depend on \(\nu\).
Proof: Highest weight theory implies that there is a $P^+$-stable filtration

$$0 = F^0 \sigma \subset F^1 \sigma \subset \ldots \subset F^s \sigma = \sigma$$

of $\sigma$ such that $F^j \sigma / F^{j-1} \sigma$ is isomorphic with $\sigma_j \otimes \nu_j$ with $N$ acting trivially. Let $\xi_\nu$ denote the representation of $P^+$ given by $\xi_\nu(man) = a^* \xi(m)$.

The map $\Xi$ given by

$$\Xi(\varphi \otimes \nu)(x) = \varphi(x) \otimes \sigma(x)\nu$$

is a $G$-isomorphism between the representation $\pi_{\xi,\nu} \otimes \sigma$ and the induced representation $\tilde{\pi} = \text{Ind}_{P^+}^G(\xi_\nu \otimes \sigma|_{P^+})$.

Let

$$F^j(\tilde{\pi}) = \text{Ind}_{P^+}^G(\xi_\nu \otimes F^j \sigma).$$

Then the filtration $F^j(\pi \otimes \sigma) = \Xi^{-1} F^j(\tilde{\pi})$ has the desired properties. To see that the filtration does not depend on $\nu$ in the compact model, recall that the compact model lives on the space of all $f : K \to V_\xi$ such that $f(mk) = \xi(m)f(k)$ for all $m \in K \cap M^+$ and $k \in K$. Thus $\pi \otimes \sigma$ can be modelled on the space of all $f : K \to V_\xi \otimes V_\sigma$ with $f(mk) = (\xi(m) \otimes 1)f(k)$. Using the construction above it turns out that $F^j(\pi \otimes \sigma)$ coincides with the space of all such $f$ with

$$(1 \otimes \sigma(k))f(k) \in V_\xi \otimes F^j \sigma$$

for every $k \in K$. \hfill $\square$

4 Convergence of the spectral side

Arthur proved his trace formula for a smooth compactly supported test function $f$ on $G(\mathbb{A})$. We want to substitute a function of noncompact support depending on a parameter, and we need uniform convergence. We are able to prove the necessary estimates in a special case sufficient for the purpose of this paper. Before stating them, we should recall the definition of the spectral side of the trace formula.

According to [3], Theorem 8.2, one has

$$J_{\text{spec}}(f) = \sum_{\chi} J_{\chi}(f),$$
where \( \chi \) runs through conjugacy classes of pairs \((\mathcal{M}_0, \pi_0)\) consisting of a \(\mathbb{Q}\)-rational Levi subgroup \(\mathcal{M}_0\) and its cuspidal automorphic representation \(\pi_0\), the sum being absolutely convergent. The particular terms have expansions

\[
J_{\chi}(f) = \sum_{\mathcal{M}, \pi} J_{\chi, \mathcal{M}, \pi}(f),
\]

where the sum runs over all \(\mathbb{Q}\)-rational Levi subgroups \(\mathcal{M}\) of \(G\) containing a fixed minimal one (which we take to be the subgroup \(A_0\) of diagonal matrices) and, for each \(\mathcal{M}\), over all discrete automorphic representations \(\pi\) of \(\mathcal{M}(\mathbb{A})\).

Explicitly,

\[
J_{\chi, \mathcal{M}, \pi}(f) = \sum_{s \in W_{\mathcal{M}}} c_{\mathcal{M}, s} \int_{i(\mathbb{D}^2)} \sum_{\mathcal{P}} \text{tr} \left( \mathcal{M}_{\mathcal{L}}(\mathcal{P}, \nu) M(\mathcal{P}, s) \rho_{\chi, \pi}(\mathcal{P}, \nu, f) \right) d\nu.
\]

Here, for a given element \(s\) of the Weyl group of \(\mathcal{M}\) in \(G\), the Levi subgroup \(\mathcal{L}\) is determined by \(a_{\mathcal{L}} = (a_{\mathcal{M}})^s\), and \(\mathcal{P}\) runs through all parabolic subgroups of \(G\) having \(\mathcal{M}\) as a Levi component. The coefficient \(c_{\mathcal{M}, s} > 0\) is of no interest to us except in the case \(\mathcal{M} = \mathcal{G}\), where it is 1. For later use, we denote by \(J_{\chi, \mathcal{M}, \pi}^+(f)\) the same expression with the trace replaced by the trace norm.

Let us comment on the items in the integrand. Let \(\rho(\mathcal{P}, \nu)\) be the representation of \(\mathcal{G}(\mathbb{A})\) which is induced from the representation of \(\mathcal{P}(\mathbb{A})\) in \(L^2(\mathcal{M}(\mathbb{Q}) \backslash \mathcal{M}(\mathbb{A})) \cong L^2(\mathcal{N}(\mathbb{A}) \mathcal{P}(\mathbb{Q}) \backslash \mathcal{P}(\mathbb{A}))\) twisted by \(\nu\). If one starts the induction with the subspace of the \(\pi\)-isotypical component spanned by certain residues of Eisenstein series coming from \(\chi\), one gets a subrepresentation which is denoted by \(\rho_{\chi, \pi}(\mathcal{P}, \nu)\). We let \(\rho_{\chi, \pi}(\mathcal{P}, \nu, f)\) act in the space of \(\rho(\mathcal{P}, \nu)\) by composing it with the appropriate projector. Further, there is a meromorphic family of standard intertwining operators \(M_{\mathcal{Q}|\mathcal{P}}(\nu)\) between dense subspaces of \(\rho(\mathcal{P}, \nu)\) and \(\rho(\mathcal{Q}, \nu)\) defined by an integral for \(\text{Re} \nu\) in a certain chamber. The operator \(M(\mathcal{P}, s)\) is \(M_{s|\mathcal{P}}(0)\) followed by translation with a representative of \(s\) in \(G(\mathbb{Q})\). And finally, \(\mathcal{M}_{\mathcal{L}}(\mathcal{P}, \nu)\) is obtained from such intertwining operators by a limiting process.

The decomposition in terms of \(\chi\) is only there for technical reasons. In general, it is unknown whether the sum over \(\mathcal{M}\) and \(\pi\) can be taken outside the sum over \(\chi\) in order to obtain an expansion in terms of the distributions

\[
J_{\mathcal{M}, \pi}(f) = \sum_{\chi} J_{\chi, \mathcal{M}, \pi}(f),
\]
which would be given by expressions that are analogous to $J_{\chi,\mathcal{M},\pi}(f)$ but with $\rho_{\chi,\pi}(\mathcal{P}, \nu)$ replaced by

$$\rho_{\pi}(\mathcal{P}, \nu) = \bigoplus_{\chi} \rho_{\chi,\pi}(\mathcal{P}, \nu).$$

This is a problem of absolute convergence, hence of the finiteness of

$$J^+_{\text{spec}}(f) = \sum_{\chi, \mathcal{M}, \pi} J^+_{\chi,\mathcal{M},\pi}(f).$$

In [22], this problem was reduced to certain conditions on local intertwining operators, which are known to be satisfied in some cases. We will check below that those conditions are satisfied in the situation of interest to us.

Thus, we specialize to the linear algebraic group $G = \text{SL}_3$. Let $G = G(\mathbb{R})$ be the group of real points. We fix maximal compact subgroups $K = \text{SO}_3 \subset G$ and $K_p = \text{SL}_3(\mathbb{Z}_p) \subset \text{SL}_3(\mathbb{Q}_p)$ for all primes $p$, and we set $K_{\text{fin}} = \prod_p K_p$.

**Theorem 4.1** Let $f = f_{\text{fin}} \otimes f_{\infty}$, where $f_{\text{fin}}$ is the characteristic function of $K_{\text{fin}}$ and $f_{\infty}$ is a $K$-finite function in Harish-Chandra’s $L^1$-Schwartz space $C^1(G)$. Then $J^+_{\text{spec}}(f) < \infty$. Moreover, there exists $N > 0$ with the following property. For any subset $\Pi$ of pairs $(\mathcal{M}, \pi)$ as above, there exists $c > 0$ such that

$$\sum_{(\mathcal{M}, \pi) \in \Pi} J^+_{\chi,\mathcal{M},\pi}(f) \leq c \sup_{(\mathcal{M}, \pi) \in \Pi} \sup_{\nu \in \mathfrak{i}_{\text{ad}}^* \mathcal{M}} \|\rho_{\pi}(\mathcal{P}, \nu, (1 + \Delta)^N f)\|.$$

Here we have fixed a $K$-invariant norm on the dual of the real Lie algebra of $G$ and denoted by $\Delta$ the corresponding element of the universal enveloping algebra. The superscript $+$ indicates that the trace has been replaced by the trace norm.

**Proof:** Our first assertion, which concerns absolute convergence, would follow from Theorem 0.2 of [22] if we could verify conditions 1) and 2') of that theorem. Once this is done, our second assertion will be a byproduct of the proof. Indeed, in the course of proving Lemma 6.2 of [22], equation (6.15) was used to estimate the operator norm of $\rho_{\chi,\pi}(\mathcal{P}, \nu, (1 + \Delta)^N f)$ in terms of the $L^1$-norm of $(1 + \Delta)^N f$. If we omit that step and consider only the terms with $(\mathcal{M}, \pi) \in \Pi$, the asserted bound will follow.
The aforementioned condition 1) is a uniform bound on the derivatives of the local intertwining operators $R_{Q|P}(\pi_p, \nu)_{K_p}$ for all automorphic representations $\pi = \pi_\infty \otimes \bigotimes_p \pi_p$ of $\mathcal{M}(\mathbb{A})^1$, all primes $p$ and all open compact subgroups $K_p$ of $\mathcal{G}(\mathbb{Q}_p)$, where the subscript $K_p$ indicates restriction to the subspace of $K_p$-fixed vectors of the representation induced from $\pi_p$. In contrast to Theorem 0.2 of [22], our claim concerns only a fixed test function $f_{\text{fin}}$ which is biinvariant under a particular maximal compact subgroup $K_p$, and hence we need only verify condition 1) for that group. However, $K_p$ as chosen above is hyperspecial, and $R_{Q|P}(\pi_p, \nu)_{K_p}$ is the identity, so that the condition is automatically satisfied.

Condition 2′) is a uniform bound on the derivatives of the local intertwining operators $R_{Q|P}(\pi_\infty, \nu)_\tau$ for all $\pi$ as above and all $K$-types $\tau$, where the subscript $\tau$ indicates restriction to that $K$-type in the representation induced from $\pi_\infty$. Since we consider a fixed $K$-finite function $f_\infty$, we need only verify the condition for finitely many $K$-types, and the uniformity of the required bound in $\tau$ is no issue. However, the bound does have to be uniform in $\pi_\infty$, which still allows us to split the set of those representations $\pi_\infty$ into a finite number of subsets and check the condition for each of them. For the subset of tempered representations, condition 2′) follows from results of Arthur (cf. [22], Proposition 6.4).

For our group $\mathcal{G} = \text{SL}_3$, we have either $\mathcal{M} = \mathcal{A}_0$ or $\mathcal{M} = \mathcal{G}$ or $\mathcal{M} \cong \text{S(GL}_2 \times \text{GL}_1)$. In the first case, $\pi_\infty$ is just a character of $\mathcal{A}_0(\mathbb{R})$, hence tempered. In the second case $\mathcal{M} = \mathcal{G}$, the induced representation $\rho_\pi(\mathcal{G})$ coincides with $\pi$, and the intertwining operator is the identity, so that condition 2′) is trivially satisfied.

This leaves us with the case of the intermediate Levi subgroups. The map $g \mapsto (g, \det g^{-1})$ is a $\mathbb{Q}$-rational isomorphism from $\text{GL}_2$ to $\text{S(GL}_2 \times \text{GL}_1)$. Thereby we may identify $\mathcal{M}$ with $\text{GL}_2$, $K \cap M$ with $\text{O}_2$ and $K_p \cap M(\mathbb{Q}_p)$ with $\text{GL}_2(\mathbb{Z}_p)$. Thus, our only remaining concern are the automorphic representations $\pi$ of $\mathcal{M}(\mathbb{A})^1 \cong \text{GL}_2(\mathbb{A})^1$ occurring in $L^2(\mathcal{M}(\mathbb{Q}) \backslash \mathcal{M}(\mathbb{A})^1)$ for which $\pi_\infty$ is non-tempered and which have a $K_{\text{fin}} \cap M(\mathbb{A})$-fixed vector. For such a representation, $\pi_\infty$ must occur in the space of $L^2$-functions on $\mathcal{M}(\mathbb{Q}) \backslash \mathcal{M}(\mathbb{A})^1 / K_{\text{fin}} \cap M(\mathbb{A}) \cong \text{GL}_2(\mathbb{Z}) \backslash \text{GL}_2(\mathbb{R})^1$,

where the last isomorphism of right $\text{GL}_2(\mathbb{R})^1$-spaces is due to the fact that $\mathbb{Q}$ has class number one. The superscript 1 refers to the subgroup of elements with determinant of absolute value 1. Since $\text{GL}_2(\mathbb{Z})$ contains elements with
determinant $-1$, this quotient is isomorphic to $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$ as an $\text{SL}_2(\mathbb{R})$-space. As $\pi_\infty$ is non-tempered, its Casimir eigenvalue does not exceed $1/4$ in the usual normalization. It follows from Roelcke’s eigenvalue estimate \cite{27} (cf. \cite{16}, ch. 11, Prop. 2.1) that $\pi_\infty$ lies in the space of constants, so it must be the trivial representation.

For a single representation $\pi_\infty$ and $K$-type $\tau$, the norm of the derivative of $R_{Q|P}(\pi_\infty, \nu)_\tau$ is certainly bounded by a polynomial in $\|\nu\|$ for $\nu$ outside a sufficiently large compact subset $\Omega$ of the line $i\mathfrak{a}^*_M$, because the operator is a rational function of $\nu$. However, the function in question is smooth and therefore bounded on $\Omega$, so condition 2') is satisfied. □

5 Orbital integrals

We will continue to focus on the group $G = \text{SL}_3(\mathbb{R})$. For later use we will fix some notation. Let $P_0 = M_0 A_0 N_0$ be the minimal parabolic subgroup of all upper triangular matrices in $G$. We fix $A_0$ to be the group of all diagonal matrices with positive entries and determinant one. Let $P_1 = M_1 A_1 N_1$ be the parabolic subgroup of all matrices in $G$ with last row of the form $(0, 0, \ast)$. We fix $A_1$ to be the group of all diagonal matrices of the form $\text{diag}(a, a, a^{-2})$, $a > 0$. The group $M_1$ is isomorphic to the group $\text{GL}_2(\mathbb{R})$ consisting of all real two by two matrices with determinant equal to $\pm 1$.

For $j = 0, 1$ let $\rho_j \in \mathfrak{a}^*_j$ be the modular shift of $P_j$. So by definition for $a \in A_0$ we have $\det(a|n_j) = a^{2\rho_j}$, where $n_j$ is the Lie algebra of $N_j$. Further let $\rho_{M_1} \in \mathfrak{a}^*_1$ be the modular shift of the parabolic $P_0 \cap M_1$. Then by definition $\det(a|n_0 \cap m_1) = a^{2\rho_{M_1}}$. Note that $\rho = \rho_1 + \rho_{M_1}$.

The Killing form $B$ on the real Lie algebra $\mathfrak{g}_\mathbb{R} = \text{sl}_3(\mathbb{R})$ is given by

$$B(X, Y) = \text{tr} \text{ad}(X) \text{ad}(Y) = 6 \text{tr} XY$$

for $X, Y \in \mathfrak{g}_\mathbb{R}$. We will use the same letter for its complexification as a symmetric bilinear form on $\mathfrak{g}$ as well as for the corresponding quadratic form $B(X) = B(X, X)$. Let $\theta$ be the Cartan involution fixing $K$. Then $\theta(x) = t^{-1}x^{-1}$ for $x \in G$ and $\theta(X) = -tX$ for $X \in \mathfrak{g}_\mathbb{R}$. The Killing form gives a natural identification between $\mathfrak{a}$ and its dual $\mathfrak{a}^*$. Thus it also gives an invariant form on the latter space. Note that the sum $\rho = \rho_1 + \rho_{M_1}$ of the last paragraph is orthogonal with respect to $B$. Therefore, $B(\rho) = B(\rho_1) + B(\rho_{M_1})$.

We are going to apply Proposition \ref{2.1} with a special choice of the number $b$ and the representation $\tau$ of $K \cong \text{SO}_3$. Recall that for each $k = 0, 1, 2, \ldots$
there is an irreducible representation $\delta_{2k}$ of dimension $2k + 1$ and that this exhausts the set $\hat{K}$ of irreducible representations of $K$ up to equivalence. For a virtual $K$-representation $\tau = \tau_+ - \tau_-$ we define $h_{\phi,\tau} = h_{\phi,\tau_+} - h_{\phi,\tau_-}$. We choose the virtual representation 

$$\tau_0 = \delta_4 - \delta_2 - 2\delta_0$$

of $K$, fix $b = B(\rho_1) = 1/4$ and set $h_{\phi} = h_{\phi,\tau_0}$. The reason for this choice will become transparent in the next lemma.

We want to describe the action of our kernel in representations $\pi_{\xi,\nu}$ of the principal series of $G$. Here $P = MAN$ is a parabolic subgroup, $\xi$ a representation in the discrete series of $M$ and $\nu \in \mathfrak{a}^*$. Following [19], we write $\xi = \text{Ind}_{M_0}^M \omega$, where $M_0$ is a certain subgroup of finite index in $M$ and $\omega$ belongs to the discrete series of $M_0$. In our situation, $M_1^+ = M_1^0 \cong \text{SL}_2(\mathbb{R})$ and $M_0^+ = M_0$.

A $K$-finite function $f \in L^1(G)$ is called a pseudo-cusp form if $\text{tr} \pi(f) = 0$ for every $\pi \in \hat{G}$ which is induced from the minimal parabolic $P_0$.

Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{p}$ is the orthocomplement of $\mathfrak{t} = \text{Lie}_C(K)$ with respect to the Killing form.

**Lemma 5.1** Let $\pi = \pi_{\xi,\nu}$ with $\xi = \text{Ind}_{M_0}^M \omega$ as above.

(i) If $P = P_1$ and we denote $\mathfrak{p}_{M_1} = \mathfrak{p} \cap \mathfrak{m}_1$, then

$$\text{tr} \pi(h_{\phi}) = \phi(|\nu|) \dim \text{Hom}_{K \cap M_1^0}(\omega, \bigwedge^{\text{odd}} \mathfrak{p}_{M_1} - \bigwedge^{\text{even}} \mathfrak{p}_{M_1}).$$

Note in addition that

$$\bigwedge^{\text{odd}} \mathfrak{p}_{M_1} - \bigwedge^{\text{even}} \mathfrak{p}_{M_1} \cong (S^+ - S^-) \otimes (S^- - S^+)$$

as a representation of the spin group $\text{Spin}(B|_{\mathfrak{p}_{M_1}})$, where $S^\pm$ are the half-spin representations.

(ii) If $P = P_0$, then $\text{tr} \pi(h_{\phi}) = 0$, so $h_{\phi}$ is a pseudo-cusp form.

**Proof:** Since $\pi$ can be considered as induced from $M^+AN$, Frobenius reciprocity implies that

$$\text{tr} \pi(h_{\phi}) = \phi\left(\sqrt{-\pi}(C) - B(\rho_1)\right) \dim \text{Hom}_K(\tau_0, \pi)$$

$$= \phi\left(\sqrt{-\pi}(C) - B(\rho_1)\right) \dim \text{Hom}_{K \cap M^+}(\omega, \tau_0)$$
in terms of the virtual dimension of a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space. In case (i), we have \( K \cap M_1^0 \cong \text{SO}_2 \), and it is straightforward to check that

\[
\tau_0|_{K \cap M_1^0} \cong \bigwedge^{\text{odd}} p_{M_1} - \bigwedge^{\text{even}} p_{M_1}.
\]

If \( \Lambda \in \mathfrak{b}^* \) is the infinitesimal character of \( \xi \), then \( \Lambda + \nu \) is the infinitesimal character of \( \pi \), hence

\[
\pi(C) = B(\Lambda + \nu) - B(\rho) = B(\Lambda) - |\nu|^2 - B(\rho_1) - B(\rho_{M_1}),
\]

as \( \nu \) is imaginary and orthogonal to \( \Lambda \). Lemma 2.4 of \([20]\) says that \( \text{tr} \, \pi(h_\phi) \) vanishes unless \( B(\Lambda) = B(\rho_{M_1}) \), so we get the asserted formula.

For the proof of (ii), observe that

\[
M_0 = \{ \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i \in \{\pm 1\}, \varepsilon_1\varepsilon_2\varepsilon_3 = 1 \},
\]

a group whose characters are \( \xi_i(\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)) = \varepsilon_i, \ i = 1, 2, 3 \), together with the trivial character \( \xi_0 \). It is clear that

\[
\delta_0|_{M_0} = \xi_0, \quad \delta_2|_{M_0} = \xi_1 + \xi_2 + \xi_3, \quad \delta_4|_{M_0} = 2\xi_0 + \xi_1 + \xi_2 + \xi_3
\]

(e.g., by dimensional reasons, using the fact that the \( \xi_i \) are conjugate under the normalizer of \( M_0 \) in \( K \)). This implies that \( \dim \text{Hom}_{M_0}(\xi_i, \tau_0) = 0 \).

Let \( g \in G = \text{SL}_3(\mathbb{R}) \) be regular. Then the centralizer \( G_g \) is a maximal torus of \( G \), so either it is conjugate to the torus of all diagonal elements of \( G \) or to \( H = A_1 B \), where \( B \cong \text{SO}_2 \) is the compact maximal torus of \( M_1 \).

We say that \( g \) is of split rank 2 in the former case and of split rank 1 in the latter. If \( g \) is of split rank 1, then there is \( a_g b_g \in A_1 B \) and \( x \in G \) such that \( g = x a_g b_g x^{-1} \). Here \( a_g \) is uniquely determined, and so is \( x a_g x^{-1} \), the split part of \( g \). For \( g \in G \) of split rank one with split part \( \exp X \), we define its length by \( l(g) = \sqrt{B(X)} \). Since \( \theta \) acts trivially on \( A_1 \), we have \( l(g) = d(a_g) \).

We will follow the conventions of \([15]\) about the normalization of Haar-measures on \( G \) and its Lie subgroups. For \( g \in G \), we use the notation

\[
D(g) = \det(\text{Id} - \text{Ad}(g^{-1}))|_{\mathfrak{g}/\mathfrak{g}_g}.
\]

**Proposition 5.2** Let \( g \in G \) be regular, and let \( \phi \in \mathcal{H}_N \), where \( r \) and \( N \) are such that \( h_\phi \in C^1_\mathbb{R}(G) \). If the split rank of \( g \) is 2 then \( O_g(h_\phi) = 0 \). If the split rank of \( g \) is 1, then

\[
O_g(h_\phi) = |D(g)|^{-1/2} \hat{\phi}(l(g)).
\]
Proof: If $\phi$ is a Paley-Wiener function, then $h_\phi \in C^\infty_c(G)$ and in particular, $h_\phi \in C^2(G)$. Using Lemma 5.1 the proposition follows from Lemma 4.3 of [19]. For the general case one approximates $\phi$ by Paley-Wiener functions.

Proposition 2.1 can be applied to the function

$$\phi_N^\lambda(x) = (N - 1)! (x^2 + \lambda)^{-N},$$

in which case the operator $\phi_N^\lambda(\sqrt{D_\tau})$ equals $(N - 1)!$ times the resolvent $(D_\tau + \lambda)^{-N}$. The corresponding convolution kernel $h_{\phi_N^\lambda}$ will be denoted by $R_{\lambda,\tau}^N$. If $|\text{Im} \sqrt{-\lambda}| > r$, then $\phi_N^\lambda \in \mathcal{H}_{2N}^r$. Thus, if $|\text{Im} \sqrt{-\lambda}|$ and $N$ are large enough (e.g., if $N$ and $\lambda > 0$ are large enough), then $R_{\lambda,\tau}^N \in C^1_0(G)$ and, for all for $\pi \in \hat{G}$,

$$\pi(R_{\lambda,\tau}^N) = (N - 1)! \frac{(-\pi(C) - B(\rho_1) + \lambda)^{-N}}{\dim \tau} P_{\pi,\tau}.$$ 

Again, for a virtual representation $\tau = \tau_+ - \tau_-$ we set $R_{\lambda,\tau}^N = R_{\lambda,\tau_+}^N - R_{\lambda,\tau_-}^N$, in particular $R_{\lambda}^N = R_{\lambda,\tau_0}^N$.

**Proposition 5.3** Let $\lambda > 0$ and $N$ be large as above. If the splitrank of $g$ is 2 then $O_g(R_{\lambda}^N) = 0$. If the splitrank of $g$ is 1, then

$$O_g(R_{\lambda}^N) = \frac{1}{|D(g)|^{1/2}} \left( -\frac{\partial}{\partial \lambda} \right)^{N-1} e^{-l(g)\sqrt{\lambda}} \frac{e^{-|x|\sqrt{\lambda}}}{2\sqrt{\lambda}}.$$ 

These orbital integrals are real and positive.

**Proof:** The convergence follows from the fact that $R_{\lambda}^N \in C^2_0(G)$ for large $\lambda$ and $N$. The formula from Prop. 5.2 can be specialised using

$$\hat{\phi}_{\lambda}^N(x) = \left( -\frac{\partial}{\partial \lambda} \right)^{N-1} e^{-|x|\sqrt{\lambda}} \frac{1}{2\sqrt{\lambda}}.$$ 

If $\lambda$ is real, then $\hat{\phi}_{\lambda}^N$ is positive, because one can see by induction that

$$\hat{\phi}_{\lambda}^N(x) = |x|^{2N-1} p_N \left( \frac{1}{|x| \sqrt{\lambda}} \right) e^{-|x|\sqrt{\lambda}}$$

for some polynomial $p_N$ of degree $2N - 1$ with nonnegative coefficients. □
6 Choice of the twisting character

For the group $\text{SL}_3(\mathbb{R})$ we will now give an explicit example of a virtual character which is parabolically regular. For this let $\text{st} : G \to \text{GL}_3(\mathbb{C})$ denote the standard representation of $G$, and let $\eta = S^2(\text{st})$ be its symmetric square. Then the dimension of $\eta$ is 6. Consider the virtual representation

$$\psi = \sum_{j=0}^{6} (-1)^j \wedge^j \eta.$$  

For $x \in G$ we have $\text{tr} \psi(x) = \det(1 - \eta(x))$.

Lemma 6.1 Let $P = MAN$ be a proper parabolic subgroup of $G$. Then $\text{tr} \psi(mn) = 0$ for every $mn \in MN$. So $\text{tr} \psi(x)$ is parabolically regular.

Proof: For $x \in G$ we have $\text{tr} \psi(x) = 0$ if $\eta(x)$ has 1 as an eigenvalue. Since $\eta(x)$ is the symmetric square of $x$ it has 1 as an eigenvalue if $x$ has $\pm 1$ as an eigenvalue. So we have to show this for $x = mn$. There are three conjugacy classes of proper parabolic subgroups of $G$. The first is given by the group $P_0 = M_0A_0N_0$. The group $M_0N_0$ is the subgroup of all upper triangular matrices with $\pm 1$ on the diagonal. The claim follows. The next parabolic is $P_1 = M_1A_1N_1$. The group $M_1A_1$ is the centralizer of $A_1$, so $M_1$ is isomorphic to the group of real two by two matrices of determinant $\pm 1$. Again the claim follows. The third parabolic $P_2$ is obtained from $P_1$ by reflection along the second diagonal. The lemma is now clear.

We now consider the Cartan subgroup $H_0$ of diagonal matrices in $G$. Then its Lie algebra $a_0$, which equals the Lie algebra of the connected component $A_0$, is the Lie algebra of diagonal matrices in $g$, i.e.

$$a_0 = \{ \text{diag}(a, b, c) \mid a, b, c \in \mathbb{C}, \ a + b + c = 0 \}.$$  

The weight lattice is generated by

$$\lambda_1(\text{diag}(a,b,c)) = a, \quad \lambda_2(\text{diag}(a,b,c)) = b.$$  

The weights of the standard representation $\text{st}$ are $\lambda_1, \lambda_2, -\lambda_1 - \lambda_2$. The dominant weights are those of the form $a\lambda_1 + b\lambda_2$ with $a, b \in \mathbb{Z}$, $a \geq b \geq 0$. For instance, the modular shift of the minimal parabolic $P_0$ is $\rho = 2\lambda_1 + \lambda_2$. 
and the modular shift $\rho_1 \in \mathfrak{p}_1^*$ of $P_1$, restricted to $a_0$, is $\rho_1 = \frac{3}{2}(\lambda_1 + \lambda_2)$. The modular shift of the minimal parabolic $P_0 \cap M_1$ of $M_1$, extended trivially to $a_1$, is $\rho_{M_1} = \frac{1}{2}(\lambda_1 - \lambda_2)$.

For a dominant weight $\lambda$ let $W_\lambda$ denote the irreducible representation of $G$ with highest weight $\lambda$. Then $W_0$ is the trivial representation and $st = W_{\lambda_1}$.

**Lemma 6.2** The virtual representation $\psi$ decomposes as

$$\psi = 2W_0 - W_{2\lambda_1} - W_{2\lambda_1 + 2\lambda_2} + W_{3\lambda_1 + \lambda_2} + W_{3\lambda_1 + 2\lambda_2} - W_{3\lambda_1} - W_{3\lambda_1 + 3\lambda_2}.$$ 

Note that $W_{3\lambda_1 + \lambda_2}$ and $W_{3\lambda_1 + 2\lambda_2}$ are dual to each other as are $W_{3\lambda_1}$ and $W_{3\lambda_1 + 3\lambda_2}$.

**Proof:** The representation $\eta$ contains the highest weight $2\lambda_1$. Weyl's dimension formula shows that $\dim W_{2\lambda_1} = 6 = \dim \eta$, hence

$$\eta = W_{2\lambda_1}.$$ 

Similarly we get $\wedge^2 \eta = W_{3\lambda_1 + \lambda_2}$ and $\wedge^3 \eta = W_{3\lambda_1} + W_{3\lambda_1 + 3\lambda_2}$. Since $\wedge^6 \eta$ is the trivial representation, the wedge product induces dualities $\eta^* = \wedge^5 \eta$ and $\wedge^2 \eta^* = \wedge^4 \eta$. The automorphism $g \mapsto \{g\}^{-1}$ interchanges each of the representations $st$ and $\wedge^j \eta$ with its contragredient. Its composition with a suitable Weyl group element preserves dominant weights and interchanges $\lambda_1$ with $\lambda_1 + \lambda_2$.

**Lemma 6.3** For elements $g \in G$ of splitrank one we have $\det(1 - \eta(g)) < 0$ and

$$\det(1 - \eta(g)) \sim -e^{l(g)} \quad \text{as } l(g) \to \infty.$$ 

**Proof:** For $a, b, c \in \mathbb{C}^\times$ we have

$$\det(1 - \eta(\text{diag}(a, b, c))) = (1 - a^2)(1 - b^2)(1 - c^2)(1 - ab)(1 - bc)(1 - ca).$$

An element $g \in G$ of splitrank one is conjugate in $\text{SL}_2(\mathbb{C})$ to the element $\text{diag}(e^{r+i\theta}, e^{-r-i\theta}, e^{-2r})$, for which $l(g) = 6|r|$. The claim follows by inspection.
7 The geometric side

Let \( \psi \) be the virtual representation of \( G \) given in Section 6. Let \( \phi \) be a Paley-Wiener function and define

\[
f_\infty^\phi(x) = h_\phi(x) \, \text{tr} \, \psi(x).
\]

Let \( f_\phi = f_\infty^\phi \otimes f_\text{fin} \), where \( f_\text{fin} \) is the characteristic function of \( K_\text{fin} = \prod_p \text{SL}_3(\mathbb{Z}_p) \). Let \( \mathcal{E}(\Gamma) \) denote the set of all conjugacy classes \([\gamma]\) in \( \Gamma \) which are of split rank one. Note that, according to our definition, such \( \gamma \) are regular.

**Proposition 7.1** The geometric side of the trace formula for \( f_\phi \) is

\[
\sum_{[\gamma] \in \mathcal{E}(\Gamma)} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \, \mathcal{O}_\gamma(h_\phi) \, \det(1 - \eta(\gamma)).
\]

**Proof:** It follows from Lemma 6.1 that \( f_\infty^\phi \) is parabolically regular, so by Corollary 1.3 the geometric side of the trace formula takes the form

\[
\sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \, \mathcal{O}_\gamma(f_\infty^\phi).
\]

The orbital integral of \( f_\infty^\phi \) can be computed as

\[
\mathcal{O}_\gamma(f_\infty^\phi) = \mathcal{O}_\gamma(h_\phi \, \text{tr} \, \psi) = \mathcal{O}_\gamma(h_\phi) \, \text{tr} \, \psi(\gamma),
\]

since \( \text{tr} \, \psi \) is invariant under conjugation. It remains to show that the sum can be reduced to the regular classes of split rank one. For this let \( \gamma \in \Gamma \) with \( \text{tr} \, \psi(\gamma) \neq 0 \). Then, by the proof of Lemma 6.1, \( \gamma \) does not have \( \pm 1 \) as an eigenvalue.

**Lemma 7.2** Let \( \gamma \in \text{SL}_3(\mathbb{Z}) \). Suppose \( \gamma \) does not have \( \pm 1 \) as an eigenvalue. Then \( \gamma \) is regular and the \( \mathbb{Q} \)-subalgebra \( \mathbb{Q}(\gamma) \) generated by \( \gamma \) is a cubic field equal to the centralizer of \( \gamma \) in \( \text{Mat}_3(\mathbb{Q}) \). This field is complex iff \( \gamma \) has split rank 1.

**Proof:** Suppose that \( \gamma \) has a rational eigenvalue \( \nu \). Since the characteristic polynomial is monic and has integer coefficients, \( \nu \) is an algebraic integer. Being rational, \( \nu \) must be an integer, and since \( \gamma^{-1} \) also has integer coefficients, \( \nu = \pm 1 \).
If we exclude this case, then the characteristic polynomial of \( \gamma \) is irreducible, hence its roots are distinct and \( \mathbb{Q}(\gamma) \) is a cubic field. Complexification shows that the centralizer \( F \) of \( \gamma \) in \( \text{Mat}_3(\mathbb{Q}) \) is three-dimensional and commutative. By comparison of degree we see that \( \mathbb{Q}(\gamma) = F \).

If \( \nu \) is an eigenvalue of \( \gamma \), then the map which assigns to any element of \( F \) its eigenvalue in the \( \nu \)-eigenspace of \( \gamma \) is an isomorphism of \( F \) onto \( \mathbb{Q}(\nu) \). Since \( \gamma \) has split rank one iff it has only one real eigenvalue, the Lemma follows.

**Proposition 7.3** Let \( \phi = \phi_N^\lambda \), so that \( h_{\phi} = R_N^\lambda \). For \( N, \lambda \gg 0 \) the trace formula is valid for the resulting test function \( f_N^\lambda \), and Proposition 7.1 remains valid.

**Proof:** Given \( N' \) and \( r > 0 \), choose \( N \) and \( \lambda \) sufficiently large so that \( \phi_N^\lambda \in \mathcal{H}_N^r \) according to Proposition 5.3. We want to approximate \( \phi_N^\lambda \) in this space by a sequence of Paley-Wiener functions \( \phi_n \) such that \( \hat{\phi}_n(x) \) does not change sign and tends to \( \hat{\phi}_N^\lambda(x) \) monotonically. Thus, let \( \chi \in C^\infty_c(\mathbb{R}) \) be even, monotonically decreasing on \( \mathbb{R}^+ \) and such that \( \chi(x) = 1 \) for \( |x| \leq 1 \). It is easy to check that the function \( \phi_n \) whose Fourier transform is \( \chi(x/n) \hat{\phi}_N^\lambda(x) \) does the job. Proposition 2.1 now implies that, given \( p > 0 \) and \( N'' \), we may choose \( N \) and \( \lambda \) such that \( h_{\phi_n} \) converges to \( R_N^\lambda \) in \( C^1_{N''}(G) \). Since \( \text{tr} \psi(g) \) grows only exponentially with \( d(g) \), we see that, if \( p \) was small enough, the sequence \( h_{\phi_n} \) converges to \( R_N^\lambda \) in \( C^1_{N''}(G) \).

It follows from Theorem 4.1 that \( f_{\infty} \mapsto J_{\text{spec}}(f_{\infty} \otimes f_{\text{fin}}) \) is a continuous linear functional on \( C^1(G) \). It is clear that any continuous linear functional on \( C^1(G) \) extends to \( C^1_{N''}(G) \) for sufficiently large \( N'' \). Thus, if we set \( f_n = f_{\phi_n} \), then \( J_{\text{spec}}(f_n) \to J_{\text{spec}}(f_N^\lambda) \) as \( n \to \infty \). The trace formula implies that \( J_{\text{geom}}(f_n) \to J_{\text{spec}}(f_N^\lambda) \) as \( n \to \infty \) as well.

By Proposition 5.3 and Lemma 6.3 all the terms on the geometric side of the trace formula for \( f_n \) have the same sign, and each term tends to the corresponding term for \( f_N^\lambda \) monotonically. Thus, we may pass to the limit \( n \to \infty \) on the geometric side by monotone convergence. □

An element \( \gamma \) of \( \Gamma = \text{SL}_3(\mathbb{Z}) \) is called *primitive* if it is not of the form \( \delta^n \) for any \( \delta \in \Gamma \) and any natural number \( n \neq 1 \). For every regular \( \gamma \in \Gamma \) there is a primitive \( \gamma_0 \in \Gamma \) such that \( \gamma = \gamma_0^\mu \) for some \( \mu \in \mathbb{N} \). If \( \gamma \) is of splitrank two then \( \gamma_0 \) is uniquely determined. If \( \gamma \) is of splitrank one then the split part of \( \gamma_0 \) is uniquely determined. The normalization of Haar
measures chosen \[15\] implies that for each regular \(\gamma \in \Gamma\) of splitrank 1 we have \(\text{vol}(\Gamma_\gamma \backslash G_\gamma) = l(\gamma_0)\).

**Corollary 7.4** The geometric side of the trace formula for \(f_N^\lambda\) equals

\[
\sum_{[\gamma] \in \mathcal{E}(\Gamma)} l(\gamma_0) \frac{\det(1 - \eta(\gamma))}{|D(\gamma)|^{1/2}} \left( -\frac{\partial}{\partial \lambda}\right)^{N-1} e^{-l(\gamma)\sqrt{\lambda}} \frac{1}{2\sqrt{\lambda}}.
\]

**8 The spectral side**

We have proved in Proposition 7.3 that the spectral side of the trace formula with the test function \(f_N^\lambda\) converges for sufficiently large positive \(\lambda\) and \(N\). Now we want to show that it extends meromorphically as a function of \(\lambda\) to a sufficiently large subset of the complex plane.

In the notation of section 4, \(J_{\text{spec}}(f_N^\lambda)\) is a sum of terms \(J_{\mathcal{M},\pi}(f_N^\lambda)\), where \(\mathcal{M}\) is a Levi \(\mathbb{Q}\)-subgroup of \(\mathcal{G}\) and \(\pi\) is a square-integrable automorphic representation of \(\mathcal{M}(\mathbb{A})\). The contributions with \(\mathcal{M} = \mathcal{G}\) are the easiest ones:

\[
J_{\mathcal{G},\pi}(f_N^\lambda) = \text{tr} \pi(f_N^\lambda) = \text{tr} \pi_\infty(R_N^\lambda \text{tr} \psi) \prod_p \text{tr} \pi_p(f_p).
\]

For any prime \(p\), we have \(\pi_{\text{triv},p}(f_p) = 1\) due to our choice of \(f_{\text{fin}}\), while the factor at the infinite place can be computed using Lemma 3.3 giving

\[
J_{\mathcal{G},\pi}(f_N^\lambda) = \text{tr} \pi_\infty \otimes \psi(R_N^\lambda), \quad \pi = \pi_{\text{triv}}.
\]

We need the explicit result only for the trivial representation \(\pi_{\text{triv}}\) of \(\mathcal{G}(\mathbb{A})\).

**Lemma 8.1**

\[
\frac{1}{(N-1)!} J_{\mathcal{G},\pi_{\text{triv}}}(f_N^\lambda) = -2 \left(\lambda - \frac{9}{4}\right)^{-N} - 2 \left(\lambda - \frac{49}{36}\right)^{-N} - 4 \left(\lambda - \frac{1}{4}\right)^{-N}
\]

**Proof:** The value in question is

\[
\frac{1}{(N-1)!} \text{tr} \psi(R_N^\lambda) = \sum_\sigma [\psi : \sigma] \sum_{\tau \in K} (-\sigma(C) - B(\rho_1) + \lambda)^{-N}[\sigma|_K : \tau][\tau_0 : \tau],
\]

where \(\sigma\) runs through the irreducible representations of \(G\) occurring in \(\psi\). The coefficients \([\psi : \sigma]\) have been calculated in Lemma 3.2 so it remains to determine \([\sigma|_K : \tau]\) and \(\sigma(C)\). All of these numbers are unchanged if we replace \(\sigma\) by \(\sigma^*\).
For each dominant weight \( \lambda \) occurring we determine the decomposition of \( W_\lambda|_K \) from its weights and compute

\[
W_\lambda(C) = B(\lambda + \rho) - B(\rho)
\]

using \( B(a\lambda_1 + b\lambda_2) = \frac{1}{9} (a^2 + b^2 - ab) \). The result is

\[
\begin{align*}
W_{2\lambda_1}|_K &= \delta_0 + \delta_4, & W_{2\lambda_1}(C) &= \frac{10}{9}, \\
W_{3\lambda_1 + \lambda_2}|_K &= \delta_2 + \delta_4 + \delta_6, & W_{3\lambda_1 + \lambda_2}(C) &= \frac{16}{9}, \\
W_{3\lambda_1}|_K &= \delta_2 + \delta_6, & W_{3\lambda_1}(C) &= 2.
\end{align*}
\]

Since \( B(\rho_{M_1}) = \frac{1}{12} \), the lemma follows from this. \( \square \)

Let \( \Pi_\infty(\tau_0, \psi) \) be the set of all admissible irreducible representations \( \eta \) of \( G \) which are subquotients of \( \pi \otimes \psi \) for some nontrivial \( \pi \in \hat{G} \) and such that \( \eta \) contains a \( K \)-type in \( \tau_0 \). Let

\[
S(\tau_0, \psi) = \{ \eta(C) + B(\rho_1) \mid \eta \in \Pi_\infty(\tau_0, \psi) \}.
\]

This is a closed subset of \( \mathbb{C} \). Note that \( B(\rho_1) = \frac{1}{4} \) in our normalization. Let \( \Omega(\tau_0, \psi) = \mathbb{C} \setminus S(\tau_0, \psi) \).

**Proposition 8.2** The expression \( J_{\text{spec}}(f_\lambda^N) - J_{G,\pi_{\text{triv}}}(f_\lambda^N) \) extends to a holomorphic function of \( \lambda \in \Omega(\tau_0, \psi) \).

**Proof:** Let us write

\[
J_{\text{spec}}(f_\lambda^N) - J_{G,\pi_{\text{triv}}}(f_\lambda^N) = \sum_{(M, \pi) \in \Pi} J_{M, \pi}(f_\lambda^N),
\]

where \( \Pi \) consists of all pairs different from \( (G, \pi_{\text{triv}}) \). We want to apply Theorem 4.1 to show that the integral-series on the right-hand side converges normally for \( \lambda \in \Omega(\tau_0, \psi) \) and hence represents a holomorphic function. Thus, we have to find a uniform bound on the operator norms of \( \rho_{\pi}(P, \nu, f) \) for \((\pi, M) \neq (G, \pi_{\text{triv}}), \nu \in i\mathfrak{a}_M, \) parabolics \( P \) with Levi component \( M \) and \( f = (\Delta + 1)^N f_\lambda^N \), where \( \lambda \) runs through a compact subset of \( \Omega(\tau_0, \psi) \). These operators are direct sums of copies of

\[
\text{Ind}^G_{P(\mathfrak{a})}(\pi, \nu, f) = \text{Ind}^G_{P}(\pi_\infty, \nu, f_\infty) \otimes \text{Ind}^G_{P(\mathfrak{a}_{\text{fin}})}(\pi_{\text{fin}}, \nu, f_{\text{fin}}),
\]

for
hence have the same operator norm as the latter. By our choice of $f_{\text{fin}}$, the second factor is the projection onto the subspace of $K_{\text{fin}}$-fixed vectors, hence of norm one. This leaves us with the norm of the factor at the infinite place. Thus, we focus attention on an irreducible component of $\text{Ind}^G_G(\pi_\infty, \nu)$, which is a nontrivial unitary representation of $G$. Our assertion will be a consequence of the following result, where we use the symbol $\pi$ in a different sense for simplicity of notation.

**Lemma 8.3** There is a uniform bound on the operator norms of

$$\pi((\Delta + 1)^N(R_{\lambda,\tau}^N \text{tr } \sigma))$$

for all nontrivial $\pi \in \hat{G}$, all constituents $\sigma$ of $\psi$, all constituents $\tau$ of $\tau_0$ and $\lambda$ in a compact subset of $\Omega(\tau_0, \psi)$.

**Proof:** Recalling that $\Delta = 2C_K - C$, we have

$$\pi((\Delta + 1)^N(R_{\lambda,\tau}^N \text{tr } \sigma)) = \pi(R_{\lambda,\tau}^N \text{tr } \sigma)\pi(2C_K - C + 1)^N.$$

Since $R_{\lambda}^N$ is $K$-finite, here the operator $C_K$ can be estimated by a constant, so the second factor behaves like polynomial of degree $N$ in $\pi(C)$. Applying Lemma 3.4 to $T = \pi \otimes \sigma(R_{\lambda,\tau}^N)$, we get

$$\|\pi(R_{\lambda,\tau}^N \text{tr } \sigma)\| \leq \dim \sigma \|\pi \otimes \sigma(R_{\lambda,\tau}^N)\|$$

$$= (N - 1)! \dim \sigma \|\pi \otimes \sigma(-C - B(\rho_1) + \lambda)^{-N}P_\tau\|.$$

According to [30] every nontrivial $\pi \in \hat{G}$ is a quotient of a representation which is parabolically induced from a unitary representation of a proper parabolic subgroup $P = MAN$, i.e., $\pi = \pi_{\xi,\nu}$, where $\xi \in \hat{M}$ and $\nu \in i\mathfrak{a}^*$. Of course, it suffices to consider the standard parabolics $P_0$, $P_1$ and $P_2$. In the case of the maximal parabolics $P_1$ and $P_2$, if $\xi$ itself is parabolically induced from a unitary representation of a proper parabolic subgroup, we use induction in stages to regard $\pi$ as induced from the minimal parabolic $P_0$. In this way, the complementary series corresponds to nonunitary parameters $\nu \in \mathfrak{a}^*$ with $\text{Re}(\nu) = t\rho \in \mathfrak{a}_0^*$, where $0 < t < 1/2$.

Thus, let $\pi = \pi_{\xi,\nu}$ be an induced representation. There is a natural $K$-stable grading $Gr^j$ of $\pi \otimes \sigma$ underlying the filtration from Lemma 3.5. The space $Gr^j$ is defined to be the set of all $f : V \to V_{\xi} \otimes V_{\sigma}$ such that

$$(1 \otimes \sigma(k))f(k) \in V_{\xi} \otimes (\sigma_j \otimes \nu_j).$$
Let $P_j$ be the projection to $Gr^j$. Then
$$P_j f(k) = (1 \otimes \sigma(k^{-1}))(1 \otimes Pr_{\sigma_j \otimes \nu_j})(1 \otimes \sigma(k)) f(k).$$

With respect to this grading the operator
$$\pi \otimes \sigma(-C - B(\rho_1) + \lambda)^N P_\tau$$
is a triangular matrix whose entries are polynomials in $\nu$ of degree $\leq 2N$. The diagonal entries have the form
$$(-\eta(C) - B(\rho_1) + \lambda)^N$$
with $\eta$ being a subquotient of $\pi \otimes \sigma$, and their leading term in $\nu$ is $B(\nu)^N$. Hence the inverse matrix is triangular and its entries are rational functions in $\nu$ which tend to zero as fast as $B(\nu)^{-N}$ as $\nu \to \infty$. Moreover, these functions have no poles at points $\nu$ parametrising $\eta \in \Pi_\infty(\tau_0, \psi)$ if $\lambda \in \Omega(\tau_0, \psi)$, as follows from the definition of the latter set. This implies that the norm $\|\pi \otimes \sigma(R_{\lambda, \tau}^N)\|$ times $(1 + |\pi(C)|)^N$ is bounded.

The bound will depend on $\xi$, but for $P$ being the minimal parabolic, the group $M$ is finite and there are only finitely many $\xi$. For the maximal parabolics we may assume that $\xi$ is not induced itself, so it is one-dimensional or a (limit of) discrete series representation. As there are only finitely many such $\xi$ for which some $\xi \otimes \sigma_j$ has a $K \cap M$-type in $\tau|_{K \cap M}$, we get a uniform bound on the operator norm in question, and the lemma follows. □

As noted above, Proposition 8.2 is thereby proved, too. Next we show that the set $\Omega(\tau_0, \psi)$, to which we have meromorphically continued the spectral side, is large enough for our goals, in particular, that it contains the main pole at $\lambda = \frac{9}{4}$.

**Proposition 8.4** There exists $\alpha < \frac{3}{2}$ such that $\Re(\pm \sqrt{\lambda}) \leq \alpha$ for every $\lambda \in S(\tau_0, \psi)$.

For the proof we will need two prerequisites. First, let $S_\alpha$ be the set of all $z \in \mathbb{C}$ with $|\Re(z)| \leq \alpha$, and let $S_\alpha^2 = \{z^2 \mid z \in S_\alpha\}$. A computation shows that
$$S_\alpha^2 = \left\{x + iy \mid x + \frac{y^2}{4\alpha^2} \leq \alpha^2 \right\}.$$
Lemma 8.5 Let $V_\mathbb{R}$ be a finite dimensional real vector space with a positive definite symmetric bilinear form $B_\mathbb{R}$. Let $V, B$ be their complexifications. Then every $v \in V$ can be written as $v = \text{Re} v + i \text{Im} v$ for $\text{Re} v, \text{Im} v \in V_\mathbb{R}$.
For every $v \in V$ and all $\beta > 0, c \geq 0$ we have
\[ B(\text{Re} v) - c \in S^2_{\beta} \Rightarrow B(v) - c \in S^2_\alpha, \]
where $\alpha = \sqrt{\beta^2 + c}$.

Proof: We first show the case $c = 0$, so $\alpha = \beta$. For $v \in V$ write $B(v) = x + iy$. Then $x = B(\text{Re} v) - B(\text{Im} v)$ and $y = 2B(\text{Re} v, \text{Im} v)$. The Cauchy-Schwartz inequality implies $y^2 \leq 4B(\text{Re} v)B(\text{Im} v)$ and thus
\[ x + \frac{y^2}{4\alpha^2} \leq B(\text{Re} v) + B(\text{Im} v) \left( \frac{B(\text{Re} v)}{\alpha^2} - 1 \right). \]
Now $B(\text{Re} v) \in S^2_\alpha$ implies $B(\text{Re} v) \leq \alpha^2$ and hence $x + \frac{y^2}{4\alpha^2} \leq B(\text{Re} v) \leq \alpha^2$, whence the claim.

For the general case simply observe that $S^2_\beta + c \subset S^2_\alpha \subset S^2_\alpha + c$. \hfill \Box

We will also need some elementary facts about representations of $\text{SL}_2(\mathbb{R})$. For $n \in \mathbb{Z}$, consider the character of the subgroup of upper triangular matrices which takes value $\text{sgn}(a)a^n$ on a matrix with upper left entry $a$, and let $\pi_n$ be the normalised induced representation of $\text{SL}_2(\mathbb{R})$. Then $\pi_n$ has a unique subrepresentation $\xi^+_n$ (resp. $\xi^-_n$) whose $\text{SO}_2$-types are bounded only from below (resp. only from above). In fact,
\[ \xi^+_n|_{\text{SO}_2} \cong \bigoplus_{m=0}^{\infty} \varepsilon_{\pm(n+2m+1)}, \]
where $\varepsilon_r, r \in \mathbb{Z}$, are the characters of $\text{SO}_2$. For $n > 0$ (resp. $n = 0$), the representations $\xi^+_n$ are irreducible and constitute the discrete series (resp. its limits). In the notation of [17], $\xi^+_n = D^\pm_{n+1}$.

Lemma 8.6 If $\zeta$ is a $k+1$-dimensional irreducible representation of $\text{SL}_2(\mathbb{R})$, then $\xi^+_n \otimes \zeta$ has a filtration whose subquotients are isomorphic to $\xi^\pm_{n+m}$ with $|m| \leq k$ and $m \equiv k \pmod{2}$. 

Proof: By Lemma 3.5, $\pi_n \otimes \zeta$ has a filtration with subquotients isomorphic to $\pi_{n+m}$ with $|m| \leq k$ and $m \equiv k \pmod{2}$. This induces a filtration on $\xi_n^+ \otimes \zeta$ of length at most $k + 1$ whose subquotients are subrepresentations of the subquotients of the previous filtration. Since

$$\zeta|_{SO_2} \cong \varepsilon_{-k} \oplus \varepsilon_{-k+2} \oplus \cdots \oplus \varepsilon_k,$$

the $SO_2$-types $\varepsilon_r$ of $\xi_n^+ \otimes \zeta$ are bounded from below and have multiplicity $k + 1$ for large $r$. This identifies those subrepresentations of $\pi_{n+m}$ uniquely. The case of $\xi_n^- \otimes \zeta$ is analogous.

Proof of Proposition 8.4: We are going to show that $\alpha = \sqrt{19/12}$ has the required property, i.e., that $\eta(C) + B(\rho_1) \in S_2^\alpha$ for all $\eta \in \Pi_\infty(\tau_0, \psi)$. Note that for this $\alpha$ we have $\alpha = \sqrt{\beta^2 + c}$ for $\beta = \sqrt{3/2}$ and $c = 1/12$. Like in the proof of Proposition 8.2, we will use the classification of $\hat{G}$ from [30].

Consider the representation $\pi = \pi_{\xi,\nu}$ induced from the parabolic subgroup $P = MAN$ of $G$, where $\xi \in \hat{M}$ and $\nu \in a^*$. If $\eta$ is a subquotient of $\pi \otimes \sigma$, then by Lemma 3.5 there is a $j$ such that $\eta$ is a subquotient of $\pi_{\xi \otimes \sigma_j, \nu + \nu_j}$.

Let us start with the case that $\pi$ is induced from the minimal parabolic $P_0 = M_0 A_0 N_0$ of all upper triangular matrices, where $M_0$ is the group of all diagonal matrices with entries $\pm 1$ and $N_0$ is the group of all upper triangular matrices with ones on the diagonal. We have

$$\eta(C) + B(\rho_1) = B(\nu + \nu_j) - B(\rho_{M_1}) = B(\nu + \nu_j) - \frac{1}{12}.$$ 

In this case $\nu_j$ runs through the weights of $\sigma$. For the principal series, we have $\nu \in a_0^*$ purely imaginary. Since we discuss the complementary series as induced from $P_0$, we also have to consider $\nu \in a_0^*$ with $Re(\nu) = t \rho$, $0 < t \leq \frac{1}{2}$. In the same manner, we can at once handle the representations induced from the one-dimensional representations of $P_1$ or $P_2$ by embedding them into principal series for the parameter $\frac{1}{2} \rho$.

Refering to Lemma 3.5, we see that it suffices to show $B(t \rho + \nu_j) - \frac{1}{12} \in S_2^\beta$ for $0 \leq t \leq \frac{1}{2}$. On a Weyl orbit of weights $\nu_j \in a_0^*$, this function obtains its largest value for dominant weights. Those occurring in $\psi$ are of the form $\nu_j = a \lambda_1 + b \lambda_2$ for $0 \leq b \leq a \leq 3$. The maximum is obtained for $a = b = 3$ and $t = \frac{1}{2}$, where we get $B(3 \lambda_1 + 3 \lambda_2 + \rho/2) - \frac{1}{12} = \frac{\beta}{2} = \beta^2$.

It remains to consider the case when $\pi = \pi_{\xi,\nu}$ is induced from a maximal parabolic, $\xi$ belongs to the (limit of) discrete series and $\nu$ is purely imaginary.
Since all maximal parabolics are conjugate under the automorphism group of $G$, it suffices to consider $P_1 = M_1 A_1 N_1$. Then $M_1 \cong \text{GL}_2(\mathbb{R})^1$ has two connected components, and $\xi = \text{Ind}_{M_1}^G(\xi^+)$, where $\xi^+$ is in the (limit of) discrete series of $M_1^0 \cong \text{SL}_2(\mathbb{R})$. Since conjugation by the nontrivial element of $M_1 / M_1^0$ switches holomorphic with antiholomorphic discrete series, we may assume that $\xi^+ \cong \xi^+_n$ for some $n \geq 0$ in the notation of Lemma 8.6. By induction in stages, we consider $\pi$ as being induced from $P_1^0 = M_1^0 A_1 N_1$, so $\eta$ is a subquotient of $\pi_{\xi_{n+m}^+} \otimes \eta^{\nu_j}$ for some irreducible constituent $\zeta$ of $\sigma_j$, now denoting a representation of $M_1^0$.

The highest weight of $\zeta$ is of the form $k_1 \rho_{M_1} \in \mathfrak{a}_{M_1}^*$, where $k$ is a non-negative integer because $\rho_{M_1} = \frac{1}{2}(\lambda_1 - \lambda_2)$ happens to be the generator of the weight lattice of $A_{M_1}$. Now $\dim \zeta = k + 1$, and from Lemma 8.6 we conclude that $\eta$ is a subquotient of $\pi_{\xi_{n+m}} \otimes \eta^{\nu_j}$ for some $|m| \leq k$. The condition $\eta \in \Pi_\infty(\tau_0, \psi)$ means that $\eta$ has a $K$-type in common with $\tau_0$, and by Frobenius reciprocity this implies that $\xi_{n+m}^+$ has a $K_{M_1}$-type in common with $\tau_0|_{K_{M_1}}$. Since the only constituents of $\tau_0$ are $\delta_{2k}$ with $|k| \leq 2$ and

$$\delta_{2k}|_{K_{M_1}} \cong \varepsilon_{-k} \oplus \varepsilon_{-k+1} \oplus \cdots \oplus \varepsilon_k,$$

this imposes the restriction $n + m \leq 1$. Remembering that $n \geq 0$, we see that the infinitesimal character of $\xi_{n+m}^+$, which is $(n + m)\rho_{M_1}$, lies in the segment connecting $\rho_{M_1}$ with the lowest weight $\omega_j$ of $\sigma_j$, hence is of the form $u \omega_j + t \rho_{M_1}$ with $|u| \leq 1$ and $0 \leq t \leq 1$. But $\pm \omega_j + \nu_j$ is an $\mathfrak{a}_0$-weight occurring in $\psi$, and so the infinitesimal character of $\eta$, which is $(n + m)\rho_{M_1} + \nu + \nu_j$, can be written as $\omega + \nu + t \rho_{M_1}$, where $\omega$ is in the convex hull of the weights occurring in $\psi$ and $0 \leq t \leq 1$. Thus

$$\eta(C) + B(\rho_1) = B(\omega + \nu + t \rho_{M_1}) - B(\rho_{M_1}).$$

If we replace $\rho_{M_1}$ by its Weyl-conjugate $\frac{1}{2} \rho$, it follows from the calculation done in the case of $P_0$ that $\eta(C) + B(\rho_1) \leq \frac{3}{2} = \beta^2$. $\Box$

9 The prime geodesic theorem

Note that if $\gamma \in \Gamma$ is of splitrank one, then $\gamma$ is not conjugate to its inverse $\gamma^{-1}$. So let $\mathcal{E}(\Gamma)$ be the set $\mathcal{E}(\Gamma)$ of conjugacy classes of split rank one modulo the equivalence relation $[\gamma] \sim [\gamma^{-1}]$. Let $\mathcal{E}_0^+(\Gamma)$ be the subset of primitive classes.
For \([\gamma] \in \mathcal{E}_0(\Gamma)\) let \(N(\gamma) = e^{l(\gamma)}\) and define for \(x > 0\),
\[
\pi(x) = \#\{[\gamma] \in \mathcal{E}_0^\pm(\Gamma) \mid N(\gamma) \leq x\}.
\]

**Theorem 9.1 (Prime Geodesic Theorem)**

> For \(x \rightarrow \infty\) we have the asymptotic formula
\[
\pi(x) \sim \frac{x}{\log x}.
\]

Our main result Theorem 0.1 can be deduced from Theorem 9.1 as follows. If \([\gamma] \in \mathcal{E}_0(\Gamma)\), then by Lemma 7.2 the centralizer \(F_\gamma\) of \(\gamma\) in \(\text{Mat}_3(\mathbb{Q})\) is a complex cubic field and the set \(O_\gamma = F_\gamma \cap \text{Mat}_3(\mathbb{Z})\) is an order in \(F_\gamma\) whose unit group is generated by \(\gamma\). We claim that every order \(O\) occurs \(h(O)\) times in this way. The corresponding claim for a division algebra instead of \(\text{Mat}_3(\mathbb{Q})\) is shown in [9], section 2. In that section the restriction to a division algebra was made to secure that the centralizer \(F_\gamma\) would be a field. In the present paper this information is obtained from Lemma 7.2. Thus Theorem 0.1 follows from the Prime Geodesic Theorem.

It remains to prove Theorem 9.1. For \(\gamma \in \mathcal{E}(\Gamma)\) let
\[
c_\gamma = -\frac{\det(1 - \eta(\gamma))}{|D(\gamma)|^{1/2}} e^{-l(\gamma)/2}.
\]

Since \(|\rho_1| = \frac{1}{2}\), we have \(|D(\gamma)| \sim e^{l(\gamma)}\) and hence \(c_\gamma \rightarrow 1\) as \(l(\gamma) \rightarrow \infty\) by Lemma 6.3.

**Proposition 9.2** Let
\[
L(s) = \sum_{[\gamma] \in \mathcal{E}^\pm(\Gamma)} l(\gamma_0) c_\gamma e^{-l(\gamma)s},
\]
where \(\gamma_0\) is a primitive element whose power is \(\gamma\). Then \(L(s)\) converges for \(\text{Re}(s) > 1\) and extends to a meromorphic function on \(\{\text{Re}(s) > 1 - \varepsilon\}\) for some \(\varepsilon > 0\). It has a simple pole of residue 1 at \(s = 1\) and is otherwise analytic on \(\{\text{Re}(s) \geq 1\}\).
Proof: For $\lambda \gg 0$ let $M_N(\lambda)$ be $-\frac{1}{2}$ times the geometric side of the trace formula for $f_N^\lambda$. By Corollary 7.4,

$$M_N(\lambda) = -\sum_{[\gamma] \in \mathcal{E}^\pm(\Gamma)} l(\gamma_0)c_\gamma \left(-\frac{\partial}{\partial \lambda}\right)^{N-1} \frac{e^{-l(\gamma)(\sqrt{\lambda}-\frac{1}{2})}}{2\sqrt{\lambda}}.$$ 

For $\text{Re}(s) \gg 0$ we compute formally at first,

$$M_N(s(s + 1) + \frac{1}{4}) = (-1)^N \sum_{[\gamma]} l(\gamma_0)c_\gamma \left(\frac{\partial}{\partial (s(s + 1))}\right)^{N-1} \frac{e^{-sl(\gamma)}}{2s + 1}$$

$$= (-1)^N \sum_{[\gamma]} l(\gamma_0)c_\gamma \left(\frac{1}{2s + 1} \frac{\partial}{\partial (s)}\right)^{N-1} \frac{e^{-sl(\gamma)}}{2s + 1}$$

$$= (-1)^N \frac{1}{2s + 1} D^{N-1} L(s),$$

where $D$ is the differential operator $D = \frac{\partial}{\partial s} \frac{1}{2s+1}$.

We deduce from the proof of Proposition 5.3 that there exists $C > 0$ such that, for $\lambda > 0$,

$$\left(-\frac{\partial}{\partial \lambda}\right)^{N-1} \frac{e^{-|x|\sqrt{\lambda}}}{2\sqrt{\lambda}} \geq C \frac{e^{-|x|\sqrt{\lambda}}}{\lambda^{N-1/2}},$$

which shows that $M_1(\lambda)$ is convergent for $\lambda \gg 0$. From this and Propositions 8.2 and 8.4 we infer the claim on the analytic continuation. Since $L(s)$ is a Dirichlet series with positive coefficients by Lemma 6.3 and Proposition 5.3, its analytic continuation also implies convergence. Proposition 9.2 is proven. The Prime Geodesic Theorem follows from the Proposition by the Wiener-Ikehara Theorem as in [7]. □
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