The nilpotent genus of finitely generated residually nilpotent groups.

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Abstract

If $G$ and $H$ are finitely generated residually nilpotent groups, then $G$ and $H$ are in the same nilpotent genus if they have the same lower central quotients (up to isomorphism). A stronger condition is that $H$ is para-$G$ if there exists a monomorphism of $G$ into $H$ which induces isomorphisms between the corresponding quotients of their lower central series. We first consider residually nilpotent groups and find sufficient conditions on the monomorphism so that $H$ is para-$G$. We then prove that for certain polycyclic groups, if $H$ is para-$G$, then $G$ and $H$ have the same Hirsch length. We also prove that the pro-nilpotent completions of these polycyclic groups are locally polycyclic.

Keywords: Polycyclic, Residually nilpotent, Nilpotent genus, Pronilpotent completions.

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1 Introduction

Throughout this paper the groups we study will be finitely generated residually nilpotent groups. If $G$ and $H$ are two such groups, then $G$ and $H$ are in the same nilpotent genus if they have the same lower central quotients (up to isomorphism). Hanna Neumann asked whether free groups can be characterised in terms of their lower central series and as a result Baumslag [2], Bridson and Reid [5], among others, looked at parafree groups: residually nilpotent groups in the same nilpotent genus as a free group. If $H$ is a residually finite group and $\phi : G \to H$ is the inclusion of a subgroup $G$, then $(H, G)_\phi$ is called a Grothendieck Pair if the induced homomorphism $\hat{\phi} : \hat{G} \to \hat{H}$ is an isomorphism but $\phi$ is not, where $\hat{H}$ denotes the profinite completion of $H$. Bridson and Reid showed, Prop. 3.4. [5], that if $(G, H)_\phi$ is a Grothendieck pair of residually nilpotent groups, then $\phi$ induces isomorphisms on the lower central quotients. In [3],[4], Baumslag, Mikhailov and Orr studied the nilpotent genus of finitely generated residually nilpotent metabelian groups and the motivation for this work was to extend some of their results.

Definition 1.1 The nilpotent genus of $G$ is the set of all isomorphism classes of groups $H$ such that $G/\gamma_i(G) \cong H/\gamma_i(H)$, for all $i \in \mathbb{N}$.

The group $H$ is para-$G$ if there exists a monomorphism $\phi : G \to H$ which induces isomorphisms between the corresponding quotients of their lower central series:

$$\phi_i : G/\gamma_i(G) \xrightarrow{\cong} H/\gamma_i(H),$$

we say that $H$ is para-$G$ via $\phi$.

Given any set of primes $\pi$, let $\pi'$ denote its complement in the set of all primes. If $n \in \mathbb{N}$, denote by $\pi(n)$ the set of primes dividing $n$. We say that $n$ is a $\pi$-number if $\pi(n) \subseteq \pi$ and a $\pi'$-number if $\pi(n) \subseteq \pi'$. If $K \leq G$, then the $\pi$-isolator of $K$ in $G$ denoted $I_\pi(K)$ is

$$I_\pi(K) = \{ x \in G \mid x^n \in K, n \text{ is a } \pi - \text{no} \}.$$
If $K \triangleleft G$ and $G/K$ is a finitely generated nilpotent group, then
\[ I_\pi(K) = \{ x \in G \mid x^n \in K, \text{n is a } \pi - \text{no} \} \triangleleft G \]
and $I_\pi(K)/K$ is a finite $\pi$–group. Let
\[ \tau(G) = \bigcup_{i=1}^{\infty} \pi(|\text{Tor}(\gamma_i(G)/\gamma_{i+1}(G))|). \]

**Definition 1.2** Let $\tau = \tau(G)$. Suppose that $\phi : G \to H$ be a monomorphism that induces an isomorphism $G_{ab} \cong H_{ab}$ and there exists $\tau'$-numbers $n_i$ such that $\gamma_i(H)^{n_i} \leq \phi(\gamma_i(G))$, for all $i \geq 2$. Then we call $\phi$ a lower central $\tau$-monomorphism.

We shall prove that the existence of such a monomorphism is sufficient for $H$ to be para-$G$.

**Theorem 2.2.** Let $G, H$ be finitely generated residually nilpotent groups and set $\tau = \tau(G)$. Suppose that $\phi : G \to H$ is a lower central $\tau$-monomorphism. Then $H$ is para-$G$ via $\phi$ and $\tau(H) = \tau$.

In [4], Baumslag, Mikhailov and Orr found examples of groups $G$ such that the nilpotent genus of $G$ is nontrivial. We show that the nilpotent genus of $G$ is nontrivial if the nilpotent genus of a certain type of quotient is nontrivial.

**Proposition 2.6.** Let $G, H$ be finitely generated residually nilpotent groups where $\tau = \tau(G)$ is finite. Suppose that $\phi : G \to H$ is a lower central $\tau$-monomorphism. Let $N = I_\pi(\gamma_k(H))$ and $M = I_\pi(\gamma_k(G))$ where $\pi \subseteq \tau$. If $\tau(G/Z_j(M)) \subseteq \tau$, for some $j \geq 1$, then $H/Z_j(N)$ is para $G/Z_j(M)$. Furthermore if $G/Z_j(M) \not\cong H/Z_j(N)$, then $G \not\cong H$.

In §3 and §4 we will study the class $\mathcal{P}^*$ of polycyclic groups which are residually nilpotent and nilpotent (of class $c$) by abelian. Baumslag, Mikhailov and Orr [4] proved that if $G, H$ are finitely generated residually nilpotent metabelian groups, then $H$ is para-$G$, if, and only if, $G$ is para-$H$ so that this is an equivalence relation. Unfortunately we do not have as strong a result when $G$ and $H$ are polycyclic of derived length greater than 2, however we can use their work to show that, for certain polycyclic groups, if $H$ is para-$G$, then $h(G) = h(H)$, which is a necessary condition for $G$ to be also para-$H$.

**Theorem 3.4.** Let $G, H \in \mathcal{P}_c^*$ where $H$ is para-$G$ via $\phi$ and one of the following holds:

(i) $G/G''$, $H/H''$ are residually nilpotent;
(ii) $|Z_{c-1}(G') : G''|, |Z_{c-1}(H') : H''| < \infty$.

Then $h(H) = h(G)$.

In [4] Baumslag, Mikhailov and Orr proved that the pro-nilpotent completion of a $\mathcal{P}_c^*$ group is locally polycyclic, in §4 we extend their result to $\mathcal{P}^*$.

**Theorem 4.4.** If $G \in \mathcal{P}^*$, then $\hat{G}_{\text{nil}}$ is locally polycyclic.

## 2 The nilpotent genus of residually nilpotent groups

When considering the nilpotent genus, the lower central nilpotent quotients of the groups play a significant role. In later sections, we'll also consider polycyclic groups where the derived subgroup is a finitely generated nilpotent group. If $G$ is any group with operator domain $\Omega$, then $\gamma_i(G)/\gamma_{i+1}(G)$ is a right $\Omega$-module and the mapping
\[ a_1\gamma_{i+1}(G) \otimes G \to [a_g]_i\gamma_{i+2}(G). \]
is a well-defined epimorphism from $\gamma_i(G)/\gamma_{i+1}(G) \otimes \mathbb{Z}G_{ab}$ to $\gamma_{i+1}(G)/\gamma_{i+2}(G)$ so that $G_{ab}$ greatly influences the structure of $G/\gamma_i(G)$, for all $i$, (see Robinson p. 131 [8]). Some of the results that we use (which can be easily proved using induction and most likely appear elsewhere) are included in the following Lemma:

**Lemma 2.1** Let $G$ be a group and let $N \leq G$.

(i) If $NG' = G$, then the inclusion $N \hookrightarrow G$ induces epimorphisms

$$N/\gamma_i(N) \twoheadrightarrow G/\gamma_i(G);$$

(ii) If $G$ is nilpotent and $G^m \leq NG'$, for some $m$, then $\gamma_i(G)^{n_i} \leq \gamma_i(N)$ and $G^n \leq N$, for some $\pi(m)$-numbers $n_i, n$;

(iii) If $G$ is nilpotent of class $c$ and $\alpha \leq \text{Aut}(G)$ is abelian where the automorphism induced by $\alpha$ on $G_{ab}$ is the identity, then the automorphism induced by $\alpha$ on $\gamma_i(G)/\gamma_{i+1}(G)$ is the identity and $G \prec t > 0$ is nilpotent of class $c$, where $t$ acts on $G$ by $\alpha$;

(iv) If $G$ is a polycyclic group, then there exists $k$ such that $|\gamma_k(G)/\gamma_{k+j}(G)| < \infty$, for all $j \geq 1$, $\pi(|\gamma_k(G)/\gamma_{k+j}(G)|) \subseteq \pi(|\gamma_k(G)/\gamma_{k+1}(G)|)$ and

$$\tau(G) = \bigcup_{i=1}^{k} \pi(|\text{Tor}(\gamma_i(G)/\gamma_{i+1}(G))|)$$

is a finite set of primes.

**Proof.** (i) See Corollary 10.3.3. p. 155 [6].

(ii) This is true if $G$ is abelian. Let $G$ be nilpotent of class $k$ so that $\gamma_k(G) \leq Z(G)$ and $[x^l, y^l] = [x, y]^l$, for all $x \in \gamma_{k-1}(G)$, $y \in G$, $l \in \mathbb{Z}$. Assume that the result is true for nilpotent groups of class $k - 1$. Then there exists $\pi(m)$-numbers $m_i, l$ such that

$$\gamma_i(G)^{m_i} \leq \gamma_i(N)\gamma_k(G), \ G^l \leq N\gamma_k(G), \ \forall 1 \leq i \leq k - 1.$$

Therefore

$$\gamma_k(G)^{m_k} \leq \gamma_k(N)^{\gamma_k(G)}, \ G^l \leq N\gamma_k(N) \leq \gamma_k(N)$$

where $n_k = m_{k-1}, n_i = m_in_k$, for all $i < k$, and $n = nk_l$.

(iii) It is very straightforward to show this by induction.

(iv) Since $G$ is a polycyclic group, there exists $k$ such that $h(\gamma_k(G)) = h(\gamma_j(G))$, for all $j \geq k$. Let $n = |\gamma_k(G) : \gamma_{k+1}(G)|$. Then $\gamma_k(j+1)(G)/\gamma_{k+j}(G)$ has exponent dividing $n$, for all $j \geq 1$, and $\tau(G)$ is a finite set of primes. □

**Theorem 2.2** Let $G$, $H$ be finitely generated residually nilpotent groups and set $\tau = \tau(G)$. Suppose that $\phi : G \rightarrow H$ is a lower central $\tau$-monomorphism. Then $H$ is para-$G$ via $\phi$ and $\tau(H) = \tau$.

**Proof.** By the hypothesis, $\phi$ induces an isomorphism $G_{ab} \cong H_{ab}$ and therefore induces epimorphisms $\phi_i : G/\gamma_i(G) \rightarrow H/\gamma_i(H)$, by Lemma 2.1 (i). Suppose that $g \in G$ and $\phi(g) \in \gamma_i(H)$, then $g^{n_i} \in \gamma_i(G)$, where $n_i$ is a $\tau$-number, thus $g \in \gamma_i(G)$ and $\phi_i$ is an isomorphism for all $i$. Therefore $H$ is para-$G$ via $\phi$ and $\tau(H) = \tau$. □

The conditions in Theorem 2.2 can be simplified when $H'$ is nilpotent of class $2$. 

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Corollary 2.3 Let $G$, $H$ be finitely generated residually nilpotent groups where $H'$ is nilpotent of class at most 2 and set $\tau = \tau(G)$. Suppose that a monomorphism $\phi : G \to H$ induces an isomorphism $G_{ab} \cong H_{ab}$ and $\gamma_2(H)^n \leq \phi(\gamma_2(G))$, for some $n \in \tau'$. Then $H$ is para-$G$ via $\phi$.

Proof. We will replace $G$ by $\phi(G)$ so that we are assuming that $G \leq H$. If $h \in H$, then $h = gk$, for some $g \in G$, $k \in H'$, since the inclusion induces an isomorphism $G_{ab} \cong H_{ab}$.

Suppose that $H'$ is abelian and $\gamma_i(H)^n \leq \gamma_i(G)$, for some $i \geq 2$. If $x \in \gamma_i(H)$, $h \in H$, then $[x^l, h] = [x, h]^l$, for all $l$. Therefore

$$\gamma_{i+1}(H)^n \leq [\gamma_i(H)^n, H] \leq [\gamma_i(G), GH'] = [\gamma_i(G), G] = \gamma_{i+1}(G).$$

Suppose that $H'$ is nilpotent of class 2 and $\gamma_i(H)^m \leq \gamma_i(G)$, for some $i \geq 2$ and some $\pi(n)$-number $m$ where $n|m$. Set

$$a = \begin{cases} 1, & \text{if } 2 \mid n \\ 2, & \text{if } 2 \nmid n . \end{cases}$$

Since $H'$ is nilpotent of class 2,

$$[x, y]^l = [x^l, y] = [x, y^l], \quad (xy)^l = x^l y^l [y, x]^{l(l-1)/2},$$

for all $l \in \mathbb{N}$, $x, y \in H'$. This implies that if $x, y \in \gamma_{i+1}(H)$ and $x^l, y^l \in \gamma_{i+1}(G)$, then $(xy)^{am^2} \in \gamma_{i+1}(G)$ for any $\pi(n)$-number $l$, where $n \mid l$. Given any $r \in N$,

$$\gamma_{i+1}(H)^{rm^2} = [\gamma_i(H), H]^{rm^2} = [\gamma_i(H), H']^{rm^2} = [\gamma_i(H), G]^{rm^2} [\gamma_i(H), H']^{rm^2} = [\gamma_i(H), G]^{rm^2} [\gamma_i(H)^m, (H')^m] \leq [\gamma_i(H), G]^{rm^2} \gamma_{i+1}(G).$$

If $x \in \gamma_i(H)$, $g \in G$, $l \in \mathbb{N}$, then

$$[x^l, g] = [x, g] x^{l-1} \cdots x + 1 = [x, g]^l \prod_{j=1}^{l-1} [x, g, x] = [x, g]^l [x, g, x^{l-1}/2] = [x, g]^l [x, g]^{l(l-1)/2}$$

$$\implies [x, g]^{am^2} = [x^{am^2}, g][x, g]^{am} x^{-am(m^2-1)/2} \in \gamma_{i+1}(G).$$

Set $n_{i+1} = a^3 m^4$, then

$$[\gamma_i(H), G]^{n_{i+1}} \leq \gamma_{i+1}(G) \implies \gamma_{i+1}(H)^{n_{i+1}} \leq \gamma_{i+1}(G).$$

Note that $n \mid n_{i+1}$ and $n_{i+1}$ is a $\pi(n)$-number.

Therefore when $H'$ is nilpotent of class $\leq 2$ and $n$ is a $\tau'$-number the hypothesis of Theorem 2.2 is satisfied and thus $H$ is para-$G$ via the inclusion. \( \Box \)

Lemma 2.4 Suppose that $G$ is a residually-nilpotent group and $M \triangleleft G$ where $G/M$ is nilpotent. If

$$R_j = G/Z_j(M), \quad M_j = M/Z_j(M),$$

then $R_j$ is residually nilpotent, $R_j/M_j$ is nilpotent and $R_{j+1} \cong R_j/Z(M_j)$.

Proof. Given any $j \geq 1$, $M_j \triangleleft R_j$,

$$R_j/M_j = (G/Z_j(M))/(M/Z_j(M)) \cong G/M$$

is nilpotent and

$$Z(M_j) = Z(M/Z_j(M)) = Z_{j+1}(M)/Z_j(M)$$

$$\implies R_{j+1} = G/Z_{j+1}(M) \cong (G/Z_j(M))/(Z_{j+1}(M)/Z_j(M)) = R_j/Z(M_j).$$
Let $\rho : G \to R_1$ denote the projection. If $\rho(x) \in \bigcap_{i=1}^{\infty} \gamma_i(R_1)$, then
\[
x \in \bigcap_{i=1}^{\infty} Z(M) \gamma_i(G) \leq M \implies [x, y] \in \bigcap_{i=1}^{\infty} \gamma_i(G) = 1, \text{ for all } y \in M,
\]
hence $x \in Z(M)$ and $R_1$ is residually nilpotent.

Suppose that $R_j$ is residually nilpotent, for some $j \geq 1$, then, by the above, $R_{j+1} = R_j/Z(M_j)$ is also residually nilpotent. □

**Lemma 2.5** Suppose that $G$, $H$ are residually nilpotent groups and $H$ is para-$G$ via $\phi$. If $N \triangleleft H$ where $H/N$ is nilpotent and $M = \phi^{-1}(N)$, then $\phi^{-1}(Z_j(N)) = Z_j(M)$ and $\phi$ induces a monomorphism $\mu_j : G/Z_j(M) \to H/Z_j(N)$, for all $1 \leq j$.

**Proof.** Suppose that $\gamma_k(H) \leq N$, then $\gamma_k(G) \leq M$. Trivially $\phi^{-1}(Z_0(N)) = Z_0(M)$. Assume that $\phi^{-1}(Z_j(N)) = Z_j(M)$, for some $j \geq 0$. If $\phi(x) \in Z_{j+1}(N)$, then $x \in Z_{j+1}(M)$ and $\phi^{-1}(Z_{j+1}(N)) \leq Z_{j+1}(M)$. If $x \in Z_{j+1}(M)$, $y \in N$, then there exists $x_i \in M$ such that $\phi(x_i) \gamma_i(H) = y \gamma_i(H)$, for all $i \geq k$, therefore $[x, x_i] \in Z_j(M)$, $\phi([x, x_i]) \in Z_j(N)$ and
\[
[x, y] \gamma_{i=1}^{\infty}(H) = [\phi(x), \phi(x_i)] \gamma_{i=1}^{\infty}(H) \implies [\phi(x), y] \in \bigcap_{i=k}^{\infty} \gamma_{i=1}^{\infty}(H) Z_j(N) = Z_j(N),
\]
by Lemma 2.4. Therefore $\phi(x) \in Z_{j+1}(N)$ and $\phi^{-1}(Z_{j+1}(N)) = Z_{j+1}(M)$.

Since $\phi^{-1}(Z_j(N))) = Z_j(M)$, $\phi$ induces a monomorphism $\mu_j : G/Z_j(M) \to H/Z_j(N)$, for all $1 \leq j$. □

**Proposition 2.6** Let $G$, $H$ be finitely generated residually nilpotent groups where $\tau = \tau(G)$ is finite. Suppose that $\phi : G \to H$ is a lower central $\tau$-monomorphism. Let $N = I_{\tau}(\gamma_k(H))$ and $M = I_\tau(\gamma_k(G))$ where $\tau \leq \tau$. If $\tau(G/Z_j(M)) \leq \tau$, for some $j \geq 1$, then $H/Z_j(N)$ is para $G/Z_j(M)$. Furthermore if $G/Z_j(M) \not\cong H/Z_j(N)$, then $G \not\cong H$.

**Proof.** The monomorphism $\phi : G \to H$ satisfies the hypothesis of Theorem 2.2 and therefore $H$ is para-$G$ via $\phi$. Hence $\phi$ induces an isomorphism $G/\gamma_k(H) \cong H/\gamma_k(G)$ which implies that $\phi^{-1}(\gamma_k(H)) = \gamma_k(G)$, $\phi^{-1}(N) = M$ and $|N/\gamma_k(H)| = |M/\gamma_k(G)|$. Set $m = |N/\gamma_k(H)|$,
\[
R = G/Z_j(M), \quad M_j = M/Z_j(M), \quad S = H/Z_j(N), \quad N_j = N/Z_j(N).
\]
By Lemma 2.4, $R$, $S$ are residually nilpotent. By Lemma 2.5, $Z_j(M) = \phi^{-1}(Z_j(N))$ and $\phi : G \to H$ induces a monomorphism $\mu : R \to S$.

Let $\rho : G \to R$ denote the projection. If $\rho(x) \in I_{\tau}(\gamma_k(R))$, then, for some $\tau$-number $a$,
\[
x^a \in Z_j(M) \gamma_k(G) \leq M \implies x \in M
\]
and $I_{\tau}(\gamma_k(R)) = M_j$. Similarly $I_{\tau}(\gamma_k(S)) = N_j$.

By the hypothesis $(m, n_k) = 1$ and we can find $\alpha, \beta \in \mathbb{Z}$ such that $n = \alpha n_k = 1 - \beta m$. Suppose that $\phi(g) \in Z_j(N) \gamma_k(N)$, then $g \in M$, $\phi(g^n) Z_j(N) = \phi(u) Z_j(N)$, for some $u \in \gamma_k(G)$, and therefore
\[
\phi(g^n u^{-1}) \in Z_j(N) \implies g^n u^{-1} \in Z_j(M) \implies g = g^n u^{-1} u g \beta m \in Z_j(M) \gamma_k(G).
\]
Hence $\phi^{-1}(Z_j(N) \gamma_k(H)) = Z_j(M) \gamma_k(G)$ and $\mu$ induces an isomorphism
\[
R/\gamma_k(R) \cong G/Z_j(M) \gamma_k(G) \cong H/Z_j(N) \gamma_k(H) \cong S/\gamma_k(S)
\]
and hence an isomorphism $R_{ab} \cong S_{ab}$. Given any $i \geq 2$, we have the following commutative diagram

$$
\begin{array}{ccc}
\gamma_i(G) & \longrightarrow & \gamma_i(H) \\
\downarrow & & \downarrow \\
\gamma_i(R) & \longrightarrow & \gamma_i(S)
\end{array}
$$

where the rightward arrows are induced by $\phi$ and $\mu$ respectively and downward arrows are epimorphisms, therefore $\gamma_i(S)^{n_i} \leq \mu(\gamma_i(R))$ and since $\tau(R) \subseteq \tau, n_i$ is a $\tau(R)'$-number. Hence $S$ is para-$R$ via $\mu$, by Theorem 2.2.

Suppose that $\alpha : G \rightarrow H$ is an isomorphism. Then

$$
\alpha(\gamma_k(G)) = \gamma_k(H) \implies \alpha(M) = N \implies \alpha(Z_j(M)) = Z_j(N)
$$

and $\alpha$ induces an isomorphism $G/Z_j(M) \cong H/Z_j(N)$. □

3 The nilpotent genus of polycyclic groups

Every polycyclic group has a characteristic subgroup of finite index which is both residually nilpotent and nilpotent by abelian so for the rest of this paper we will focus on the class $P^*$ of polycyclic groups which are residually nilpotent and nilpotent by abelian. We will denote by $P^*_c$ the class of polycyclic groups which are residually nilpotent and nilpotent of class $c$ by abelian.

**Lemma 3.1** If $G, H$ are residually nilpotent polycyclic groups and $H$ is para-$G$ via $\phi$, then

$$
G \in P^*_c \iff H \in P^*_c.
$$

**Proof.** If $G \in P^*_c$, then $G'$ is nilpotent of class $c$. If $h_1, \ldots, h_{c+1} \in H'$, then given any $i$, there exists $g(i)_j \in G'$ such that $h_j \gamma_{i+1}(H) = \phi(g(i)_j) \gamma_{i+1}(H)$, for all $1 \leq j \leq c + 1$, and

$$
[h_1, \ldots, h_{c+1}] \gamma_{i+1}(H) = \phi([g(i)_1, \ldots, g(i)_{c+1}]) \gamma_{i+1}(H) = 1 \gamma_{i+1}(H)
$$

$$
\implies [h_1, \ldots, h_{c+1}] \in \bigcap_{i=1}^{\infty} \gamma_i(H) = 1.
$$

Therefore $H'$ is nilpotent of class $c$ and $H \in P^*_c$. Conversely suppose that $H \in P^*_c$, then $G \cong \phi(G) \in P^*_c$, for some $k \leq c$. This would imply that $H \in P^*_k$ and therefore $k = c$. □

Baumslag, Mikhailov and Orr studied the nilpotent genus of groups $G, H \in P^*_c$, and showed that if $H$ is para-$G$, then $G$ is para-$H$ so that this is an equivalence relation. They did this by constructing the telescope of a metabelian group which is a type of group localization. We will give the bare details here but further details can be found in [3]. Let $G \in P^*_c$ be a metabelian group with derived subgroup $A$, $Q = G_{ab}$, integral group ring $R = \mathbb{Z}Q$, augmentation ideal $I$ and set $S = 1 + I$. The ring $RS$ is the set of equivalence class of elements $(a, s) \in R \times S$ subject to the equivalence relation

$$(a, s) \sim (b, t) \text{ if there exists } u \in S \text{ such that } (at - bs)u = 0.$$  

The equivalence class of $(a, s)$ is denoted by $a/s$. Likewise $A_S$ is the set of equivalence class of elements $(a, s) \in A \times S$ subject to the equivalence relation

$$(a, s) \sim (b, t) \text{ if there exists } u \in S \text{ such that } (at - bs)u = 0.$$  

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Therefore, \( \phi \) is straightforward to see that

\[
\text{Proof. We will first show in both cases that}
\]

Then, \( \text{Theorem 3.4 implies that} \)

\[
\mu \text{ is residually nilpotent, } S \text{ does not contain any zero divisors of } A \text{ so that } \tau_a \text{ is monic and also the mapping } \tau : A \to A_S \text{ is monic where } a \to a/1. \text{ If } \alpha \in \text{Hom}_R(A, B) \text{ where } B \text{ is also an } R\text{-module, then we can extend it to } \alpha_S \in \text{Hom}_R(A_S, B_S). \text{ We can form the semi-direct product } P = A_S \rtimes G \text{ and setting } K = \{(a/1, a^{-1}) \mid a \in A\}, \text{ the telescope of } G \text{ is the factor group } G_S = P/K. \text{ Viewing } G \text{ and } A_S \text{ as subgroups of } G_S, \text{ set } G_S = GA_S. \text{ If } S = \{s_1, s_2, \ldots\}, \text{ } t_n = s_1 s_2 \ldots s_n, \text{ } G_0 = G \text{ and } G_i = G_{ti}, \text{ then, by Lemma 5.6 [4], } G \cong G_i, \text{ for all } i \text{ and } G_S = \bigcup_{i=1}^{\infty} G_i.
\]

**Theorem 3.2** Let \( G, H \in \mathcal{P}_1^* \) and \( \phi : G \to H \) a homomorphism that induces an isomorphism

\[
\phi_2 : G_{ab} \to H_{ab}.
\]

Then \( h(H) \leq h(G). \)

**Proof.** By Lemma 2.1 (i) \( \phi \) induces epimorphisms \( \phi_i : G/\gamma_i(G) \to H/\gamma_i(H) \). As in Theorem 5.8 [4], \( \phi \) induces an epimorphism \( \phi_S : G_S \to H_S \). By Lemma 5.6 [4], \( G_S \) is a union of subgroups \( G_i \) with \( G \cong G_i \). Since \( H \) is finitely generated, we can find a \( k \) such that \( H \leq \phi_S(G_k) \) and \( h(H) \leq h(G_k) = h(G) \). \( \square \)

**Corollary 3.3** Let \( G, H \in \mathcal{P}^* \) and \( \phi : G \to H \) a homomorphism that induces an isomorphism \( \phi_2 : G_{ab} \to H_{ab} \). Suppose that \( N \triangleleft H \) where \( N \leq H' \) and \( \tilde{H} = H/N \in \mathcal{P}_1^* \). Set \( M = \phi^{-1}(N) \) and \( \tilde{G} = G/M \), so that \( \phi \) induces a monomorphism \( \mu : G \to \tilde{H} \).

\[
\tilde{G} \cong G \cong H_{ab} \cong \tilde{H}_{ab}.
\]

Both \( \tilde{G} \) and \( \tilde{H} \) are metabelian and residually nilpotent so that \( \mu \) satisfies the hypothesis of Theorem 3.2 and hence \( h(\tilde{H}) \leq h(\tilde{G}) \). Since \( \mu \) is a monomorphism this implies that \( h(\tilde{H}) = h(\tilde{G}) \). \( \square \)

**Theorem 3.4** Let \( G, H \in \mathcal{P}_c^* \) where \( H \) is para-\( G \) via \( \phi \) and one of the following holds:

(i) \( G/G'' \), \( H/H'' \) are residually nilpotent;

(ii) \( |Z_{c-1}(G') : G''|, |Z_{c-1}(H') : H''| < \infty \).

Then \( h(H) = h(G) \).

**Proof.** We will first show in both cases that \( h(G/G'') = h(H/H'') \).

(i) It is straightforward to see that \( \phi(G'') \leq H'' \). Suppose that \( \phi(x) \in H'' \), then

\[
\phi(x)\gamma_i(H) \leq (H/\gamma_i(H))''.
\]

for all \( i \), and, as \( \phi_i \) is an isomorphism, this implies that

\[
x\gamma_i(G) \in G''\gamma_i(G)/\gamma_i(G) \implies x \in \bigcap_{i=1}^{\infty} G''\gamma_i(G) = G''.
\]

Therefore \( \phi^{-1}(H'') = G'' \) and, by Corollary 3.3, \( h(G/G'') = h(H/H'') \).
(ii) By Lemma 2.4, $H/Z_{c-1}(H') \in \mathcal{P}_c$, by Lemma 2.5, $\phi^{-1}(Z_{c-1}(H')) = Z_{c-1}(G')$ and so, by Corollary 3.3, $h(G/Z_{c-1}(G')) = h(H/Z_{c-1}(H'))$. By the hypothesis
\[ h(G/G') = h(G/Z_{c-1}(G')) = h(H/Z_{c-1}(H')) = h(H/H'). \]

In both cases $h(\phi(G)H'/H') = h(G/G') = h(H/H')$. Since $\phi_2 : G_{ab} \to H_{ab}$ is an isomorphism,
\[ h(G_{ab}) = h(H_{ab}) \implies h(G'/G'') = h(G/G') - h(G_{ab}) = h(H/H') - h(H_{ab}) = h(H'/H'). \]
Therefore $\phi(G')\gamma_2(H')/\gamma_2(H')$ has finite index in $H'/\gamma_2(H')$. As $H'$ is a finitely generated nilpotent group, $\phi(G')$ has finite index in $H'$, by Lemma 2.1 (ii), so that $h(G') = h(H')$ and
\[ h(G) = h(G') + h(G_{ab}) = h(H') + h(H_{ab}) = h(H). \]
\[ \square \]

When $h(G) = h(H)$ we have the following partial converse to Theorem 2.2:

**Lemma 3.5** Let $G, H \in \mathcal{P}_c$ where $H$ is para-$G$ via $\phi$ and $h(G) = h(H)$. Then $\phi$ is a lower central $\pi$-monomorphism, for some set of primes $\pi$ such that $\pi'$ is a finite set.

**Proof.** By definition $\phi$ induces an isomorphism $G/\gamma_i(G) \cong H/\gamma_i(H)$, for any $i \geq 2$, and so $h(\gamma_i(G)) = h(\gamma_i(H))$. Since $\gamma_i(H)$ and $\gamma_i(G)$ are finitely generated nilpotent groups, this implies that $\gamma_i(H)^{n_i} \leq \phi(\gamma_i(G))$, for some $n_i$. By Lemma 2.1 (iv), there exists $k$ such that $|\gamma_k(G)/\gamma_{k+j}(G)| < \infty$, for all $j \geq 1$, and $\pi(m_j) \leq \pi(m_1)$ where $m_j = \exp(\gamma_k(G)/\gamma_{k+j}(G))$. If $h \in \gamma_k(H)$, then $h^{n_{k+j}} \in \phi(\gamma_{k+j}(G))$. Set
\[ \rho = \bigcup_{i=1}^k \pi(n_i) \cup \pi(m_1) \text{ and } \pi = \rho', \text{ then } \phi \text{ is a lower central } \pi\text{-monomorphism}. \]
\[ \square \]

Note we cannot stipulate that $\tau(G) \leq \pi$. Let $G_1, H_1 \in \mathcal{P}_c$ where $h(G_1) = h(H_1)$, $\tau_1 = \tau(G_1)$ and $\mu : G_1 \to H_1$ is a lower central $\tau_1$-monomorphism. Then $H_1$ is para-$G_1$ via $\mu$, by Theorem 2.2. Suppose that $p | n_2$ where $\gamma_2(H_1)^{n_2} \leq \mu(\gamma_2(G_1))$ and $n_2$ is a $\tau_1$-number. Set $G = C_p \times G_1$, $H = C_p \times H_1$ and define $\phi : G \to H$ by
\[ (x, y) \mapsto (x, \mu(y)) \]

Then $H$ is para-$G$ via $\phi$, $h(G) = h(H)$ and $\gamma_2(H)^{n_2} \leq \phi(\gamma_2(G))$ but $\tau = \tau(G) = \tau_1 \cup \{p\}$ so that $\phi$ is not a lower central $\tau$-monomorphism.

In [4], Baumslag, Mikhailov and Orr found a finitely generated abelian by cyclic group $G$ such that the nilpotent genus of $G$ is nontrivial. The following results have been proven with an aim to extend their example.

**Lemma 3.6** Let $N$ be a finitely generated torsion-free nilpotent group of class $c$ and suppose that $\langle \alpha_i \mid 1 \leq i \leq r \rangle \leq \text{Aut}(N)$ is abelian and there exists $a_i \in \mathbb{N}$ such that
\[ \prod_{i=0}^{a_i-1} \alpha_i^i(x) \in N', \]
for all $x \in N$, $1 \leq i \leq r$. Let $A$ be the free abelian group on $\{t_1, \ldots, t_r\}$ and set $x^{t_i} = \alpha_i(x)$, for all $x \in N$, and $H = N \rtimes A$ using this action. If $a = \gcd(a_i)$ and $b = \text{lcm}(a_i)$, then $H$ is a residually finite $b$-group, $N = I_{\pi(a)}(H')$ and $\tau(H) = \pi(a)$. In particular if $a = b = p$ is a prime, then $H \in \mathcal{P}^p$.

**Proof.** If $x \in N$, then
\[ 1N' = x^{t_i^{a_i-1} + \cdots + t_i + 1}N' = x[x, t_i^{a_i-1}] \cdots x[x, t_i]xN' \implies x^{a_i} \in H' \implies x^a \in H' \]
and $N = I_{\pi(a)}(H^1)$. If $x, y \in N$, $u, v \in A$, then

$$[xu, yv]^{\alpha_3}(H) = (x^a[y][x, y][u, y])^{\alpha_3}(H) = [x, y]^a[x, y]^a[u, y]^a \gamma_3(H)$$

so that $\gamma_2(H)/\gamma_3(H)$ is a finite group of exponent dividing $a$ and $\tau(H) = \pi(a)$.

Given any $x \in N$, $1 \leq i \leq r$,

$$x^{t_i - 1} = x^{(t_i^{r_1} + \ldots + t_i^{r_1+1})^{(i-1)}} \in N' \implies x^b N' = xN',$$

for all $t \in A$, and $K = N \times A^b$ is a finitely generated torsion-free nilpotent group of class $c$, by Lemma 2.1 (iii) and hence is a residually finite $b$-group. Since $K \triangleleft H$ and $H/K$ is a finite $b$-group, $H$ is also a residually finite $b$-group. If $b = p$, then the finite $p$-quotients are nilpotent so that $H$ is a residually nilpotent group. It is straightforward to see that $H$ is polycyclic and nilpotent by abelian so $H \in \mathcal{P}^*$. □

From now on we will be considering the case where $N$ is a finitely generated torsion-free nilpotent group and $\alpha \in \text{Aut}(N)$. We will denote by $T$ the infinite cyclic group on $t$ and set $H = N \rtimes T$ where $t$ acts on $N$ by $\alpha$. We also consider a subgroup $M \leq N$ where $M \triangleleft H$ and we set $G = M \rtimes T$. We will also assume that $H$ is not nilpotent.

**Theorem 3.7** Let $N$ be a finitely generated torsion-free nilpotent group of class at most $2$, $\alpha \in \text{Aut}(N)$ such that

$$\Pi_{i=0}^{p-1} \alpha^i(y) \in N',$$

for all $y \in N$ and some prime $p$, and $N_{ab} = < \overline{\alpha}(a) | 0 \leq i \leq p - 2 >$, for some $a \in N_{ab}$, where $\overline{\alpha}$ is the induced automorphism in $\text{Aut}(N_{ab})$. If $M \triangleleft H$, $\Pi_{i=0}^{p-1} \alpha^i(y) \in M'$, for all $y \in M$ and $N_{ab} \leq M \leq N$, for some $n$ prime to $p$, then $H$ is para-$G$ via inclusion.

**Proof.** Let $x \in N$ where $xN' = a$. Then, by Lemma 2.1 (i),

$$N = < x^t | 0 \leq i \leq p - 2 > .$$

By Lemma 3.6, $G$ and $H$ are residually nilpotent groups, $\tau(G) = \tau(H) = p$, $M = I_p(G')$, $N = I_p(H')$ and $H_{ab} \cong C_p \times < t >$. In order to apply Corollary 2.3, we will show that the inclusion of $G$ into $H$ induces an isomorphism $G_{ab} \cong H_{ab}$ and $\gamma_2(H)^n \leq \gamma_2(G)$. Since $M^p \leq G'$ and, by Lemma 2.1 (ii), $(N')^m \leq M' \leq G'$, for some $m$ prime to $p$, it follows that $M \cap N' \leq G'$. We can find $\alpha$ such that $\alpha \equiv 1 \mod p$. If $y \in M$, then $y = x^{k_1 t_1} \ldots x^{k_r t_r}$, for some $r, k_j, l_j \in Z$ and

$$y^{\alpha n}x^{-\alpha n k_1 t_1} \ldots x^{-\alpha n k_r t_r} \in M \cap N' \leq G'$$

$$\implies yG' = y^{\alpha n}G' = x^{\alpha n k_1 t_1} \ldots x^{\alpha n k_r t_r} G' = x^{\alpha n}G'$$

where $k = \sum_{j=1}^{r} k_j$. Therefore $G_{ab} \cong C_p \times < t >$ and the inclusion of $G$ in $H$ induces an isomorphism $G_{ab} \cong H_{ab}$ where $x^{\alpha n} \in G'$. If $h \in H'$, then $h^n \in M$ and, since $G_{ab} \cong H_{ab}$, $h^n \in G'$. Therefore, by Corollary 2.3, $H$ is para-$G$ via the inclusion. □

**Theorem 3.8** Let $N$ be a finitely generated free nilpotent group of class $2$. Suppose that $\mu \in \text{Aut}(N_{ab})$ has order $p$, for some prime $p$, is fixed-point free and $N_{ab} = < \mu^i(a) | 0 \leq i \leq p - 2 >$, for some $a \in N_{ab}$, then $\mu$ lifts to an automorphism $\alpha \in \text{Aut}(N)$.

Let $\bar{H}$ be the semi-direct product of $N_{ab}$ by $T$ with $t$ acting on $N_{ab}$ by $\mu$. Suppose that $A \leq N_{ab}$ where $A \triangleleft H$ and $N_{ab}^m \leq A$, for some $m$ prime to $p$, then $G = A \rtimes T$. If $M \leq N$ where $M \triangleleft H$, $M \cap N' = M'$ and $MN'/N' = A$, then $H$ is para-$G$ where $G = M \rtimes T$. Furthermore if $G \not\cong H$, then $G \not\cong H$. 9
Proof. By Theorem 2.1, $\mu$ can be lifted to $\alpha \in \text{Aut}(N)$. Since $\mu$ has order $p$ and is fixed-point free, $\Pi p^{-1}(1)σ^{-1}(b) = 1$, for all $b \in N_{ab}$ and therefore $\Pi p^{-1}α^{-1}(g) ∈ N_1$, for all $g ∈ N$. If $g ∈ M$, then $\Pi p^{-1}α^{-1}(g) ∈ M ∩ N_1 = M′$. Both $M$ and $N$ satisfy the hypothesis of Lemma 3.6, therefore $M = I_p(G′)$, $N, N′ = I_p(H′)$. From the hypothesis, $N_{ab}^m ≤ A$ which implies that $N_n^m ≤ MN_1$ and therefore $N_n^m ≤ M$, for some $π(m)$-number $n$, by Lemma 2.1 (ii). Hence $\tilde{H}$ is para-$\tilde{G}$ and $\hat{H}$ is para-$\hat{G}$, by Theorem 3.7. Since $M′ = M ∩ N′ = M ∩ Z(N)$, by Lemma 2.5, $M′ = Z(M)$ and

$$\tilde{G} = A × T = MN′/N′ × T = M/M′ × T = M/Z(M) × T = G/Z(M).$$

Hence, by Proposition 2.6, if $\tilde{G} \neq \hat{H}$, then $G \neq H$. □

4 Pronilpotent Completions

The aim in this section is to extend the Baumslag, Mikhailov and Orr result, Theorem 8.1, that the pro-nilpotent completion of a residually nilpotent metabelian polycyclic group is locally polycyclic to all groups in $P^n$. Given any group $G$, the pro-nilpotent completion of $G$, $\hat{G}_{n\ell}$, is the inverse limit of the lower central quotient groups of $G$:

$$\hat{G}_{n\ell} = \lim_{\leftarrow} G/γ_i(G).$$

Some obvious facts are:

(i) if $G$ is a residually nilpotent group, then $G$ embeds in $\hat{G}_{n\ell}$;

(ii) if $G$ is soluble of derived length $d$, then $\hat{G}_{n\ell}$ is also soluble of derived length $d$;

(iii) if $H$ is para-$G$, then $\hat{G}_{n\ell} ≅ \hat{H}_{n\ell}$.

Recall that a group $G$ is said to be HNN-free if it has no subgroups that are nontrivial HNN-extensions

Lemma 4.1 Let $G$ be a finitely generated soluble group then the following properties are equivalent:

(a) $G$ is polycyclic;

(b) $<x>^{<g>}$ is finitely generated for all $x, g ∈ G$;

(c) $G$ is HNN-free.

Proof. (a) ⇔ (b) by 4.6.4.

(b) ⇔ (c) Lemma 1 □

Lemma 4.2 Let $G$ be a group, let $N \triangleleft G$ and set $Q = G/N$. If $Q$ is HNN-free, then the following properties are equivalent:

(a) $G$ is HNN-free;

(b) if $g ∈ G$ and $H^g ≤ H ≤ N$, then $H = H^g$;

(c) $<x>^{<g>}$ is finitely generated for all $x ∈ N, g ∈ G$.

Proof. (a) ⇒ (b) and (c), by Lemma 1.

(b) ⇒ (a) By Lemma 1, $G$ is HNN-free if, and only if, $K^g ≤ K ≤ G$ implies that $K^g = K$. Since $Q$ is HNN-free, $K^g = K$ if, and only if, $(K ∩ N)^g = K ∩ N$.

(c) ⇒ (b) Suppose $H^g ≤ H ≤ N$, for some $g ∈ G$. If $x ∈ H$, then $<x>^{<g>}$ is finitely generated and hence there exists $n > 0$ such that

$$<x>^{<g>} = \langle x, x^{g^{-1}}, \ldots, x^{g^{-n}} \rangle.$$

Therefore $x^{g^{-n-1}} ∈ H^{g^{-n}}$ and $x ∈ H^g$. □
Lemma 4.3 Let \( G \) be polycyclic, \( A \triangleleft G \) be abelian and \( Q = G/A \), we view as \( A \) as a \( \mathbb{Z}Q \)-module. Then for each \( t = xA \in Q \), there exists polynomials \( \alpha, \beta \)

\[
\alpha(t) = c_0 + c_1 t + \cdots + c_{m-1} t^{m-1} \pm t^m,
\]

\[
\beta(t) = d_0 + d_1 t + \cdots + d_{n-1} t^{n-1} \pm t^n
\]

such that

\[
\alpha(t) = a\beta(t^{-1}) = 0, \quad \text{for all } a \in A.
\]

Furthermore \( \langle a \rangle^{<x>} \) is finitely generated by \( m + n - 1 \) conjugates of \( a \) by powers of \( x \).

**Proof.** As in Lemma 8.3 [4], given any \( a \in A \) we can find \( \alpha_a, \beta_a \), such that \( a\alpha_a(t) = a\beta_a(t^{-1}) = 0 \). Therefore

\[
a(\alpha_a(t)t^j) = (a\beta_a(t^{-1}))t^{-j} = a(\beta_a(t^{-1})t^{-j}), \quad \forall j \geq 0.
\]

As \( G \) is polycyclic, \( A \) is finitely generated as an abelian group, say \( A = \langle a_1, \ldots, a_n \rangle \). Let

\[
\alpha = \alpha_{a_1} \cdots \alpha_{a_n}, \quad \beta = \beta_{a_1} \cdots \beta_{a_n}.
\]

Then \( a_i\alpha(t) = a_i\beta(t^{-1}) = 0 \), for all \( 1 \leq i \leq n \), and hence \( a\alpha(t) = a\beta(t^{-1}) = 0 \), for all \( a \in A \). Note we can rewrite \( a\alpha(t) = 0 = a\beta(t^{-1}) \) using group notation as

\[
1 = a^{\alpha(x)} = a^{c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} \pm x^m} = a^{c_0} a^{c_1 x} \cdots a^{c_{m-1} x^{m-1}} a^{\pm x^m}
\]

\[
1 = a^{\beta(x)} = a^{d_0 + d_1 x + \cdots + d_{n-1} x^{n-1} \pm x^n} = a^{d_0} a^{d_1 x} \cdots a^{d_{n-1} x^{n-1}} a^{\pm x^n}
\]

and so

\[
\langle a \rangle^{<x>} = \langle a^{x^{-n}}, \ldots, a^{x^{-1}}, a, a^x, \ldots, a^{x^n} \rangle.
\]

\( \square \)

**Theorem 4.4** If \( G \in \mathcal{P}^* \), then \( \hat{G}_{nil} \) is locally polycyclic.

**Proof.** Let \( A = Z(G') \) and \( Q = G/A \). Then, by Lemma 2.4, \( Q \) is residually nilpotent and hence \( Q \in \mathcal{P}^* \). Note if \( h(A) = 0 \), then \( A \) and hence \( G' \) are finite. As \( G \) is residually nilpotent, this would in fact imply that \( G \) is nilpotent. The result would follow trivially so we shall assume that \( h(A) \neq 0 \) and hence \( h(Q) < h(G) \). Set

\[
A_i = A\gamma_i(G)/\gamma_i(G), \quad G_i = G/\gamma_i(G), \quad Q = Q/\gamma_i(Q),
\]

\[
\hat{A} = \lim_{\leftarrow} A_i, \quad \hat{G}_{nil} = \lim_{\leftarrow} G_i, \quad \hat{Q}_{nil} = \lim_{\leftarrow} Q_i,
\]

then \( \hat{A} \) embeds into \( \hat{G}_{nil} \), we will now view it as a subgroup, and \( \hat{G}_{nil}/\hat{A} \) embeds into \( \hat{Q}_{nil} \). We will use induction on the Hirsch length, \( h(G) \), of \( G \). By the remarks above we are assuming that \( h(Q) < h(G) \). If \( h(G) = 1 \), then \( Q \) is finite and so is nilpotent. Therefore \( \hat{Q}_{nil} = Q \) and \( \hat{G}_{nil} \) is an abelian-by-finite group and hence locally polycyclic. Suppose \( h(G) > 1 \), as \( Q \in \mathcal{P}^* \), \( Q \) satisfies the hypothesis and \( h(Q) < h(G) \), we can assume that \( \hat{Q}_{nil} \) is locally polycyclic. By Lemma 4.1 and Lemma 4.2, \( \hat{G}_{nil} \) is locally polycyclic, if, and only if, \( \langle a \rangle^{<x>} \) is finitely generated, for all \( a \in \hat{A}, \ x \in \hat{G}_{nil} \). We follow the notation of Lemma 8.5, [4]. Given \( a \in \hat{A}, \ x \in \hat{G}_{nil} \), let

\[
a(n) = a_1 a_2 \cdots a_n \gamma_{n+1}(G), \quad x(n) = x_1 x_2 \cdots x_n \gamma_{n+1}(G)
\]

where \( a_j \in A \cap \gamma_j(G) \) and \( x_j \in \gamma_j(G) \). Then

\[
(a^x)(n) = a(n)^x(n) = (a_1 \cdots a_n)^{x_1} \cdots x_n \gamma_{n+1}(G) = (a_1 \cdots a_n)^{x_1} \gamma_{n+1}(G).
\]
Set \( t = x_1A \in Q \), then, by Lemma 4.3, there exists \( \alpha \) and \( \beta \) such that

\[
ba(t) = b\beta(t^{-1}) = 0, \text{ for all } b \in A.
\]

Therefore

\[
(a^{\alpha}(x))(n) = a(n)^{\alpha(x(n))} = (a_1 \ldots a_n)^{\alpha(x_1)} \gamma_{n+1}(G) = 1 \gamma_{n+1}(G),
\]

\[
(a^{\beta(x^{-1})})(n) = a(n)^{\beta(x(n)^{-1})} = (a_1 \ldots a_n)^{\beta(x_1^{-1})} \gamma_{n+1}(G) = 1 \gamma_{n+1}(G), \forall n \in \mathbb{N}
\]

\[
\Rightarrow a^{\alpha(x)} = a^{\beta(x^{-1})} = 1.
\]

Hence \( < a >^{<x>} \) is finitely generated and \( \hat{G}_{nil} \) is locally polycyclic. □

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