Mean field dynamo theory is a leading candidate to explain the large scale magnetic flux in galaxies and stars. However, controversy arises over the extent of premature quenching by the backreaction of the growing field. We distinguish between rapid dynamo action, which is required by astrophysical systems, and resistively limited dynamo action. We show how the flow of magnetic helicity is important for rapid dynamo action. Existing numerical and analytic work suggesting that mean field dynamos are prematurely quenched and resistively limited include approximations or boundary conditions which suppress the magnetic helicity flow from the outset. Thus they do not unambiguously reveal whether flux generating astrophysical dynamos are dynamically suppressed when the required helicity flow is allowed. An outflow of helicity also implies an outflow of magnetic energy and so active coronae and/or winds are a prediction of mean field dynamos. Open boundaries alone may not be sufficient and the additional physics of buoyancy and winds may be required. Some possible simulation approaches, even with periodic boxes, to test the principles are discussed. Some imitations of the “Zeldovich relation” are also addressed.
I. INTRODUCTION

A leading candidate to explain the origin of large scale magnetic flux growth in stars and galaxies has been the mean field dynamo (MFD) theory \([1,2,3,4,5,6]\). The theory appeals to a combination of helical turbulence (leading to the \(\alpha\) effect), differential rotation (the \(\Omega\) effect), and turbulent diffusion to exponentiate an initial seed mean magnetic field. Ref \([7]\) developed a formalism for describing the concept \([8]\) that helical turbulence can twist toroidal (\(\phi\)) fields into the poloidal (\(r, z\)) direction, where they can be acted upon by differential rotation to regenerate a powerful large scale toroidal magnetic field. Their formalism involved breaking the total magnetic field into a mean component \(\bar{B}\) and a fluctuating component \(b\), and similarly for the velocity field \(\bar{V}\). The mean can be a spatial mean or an ensemble average. For comparison to observations of a single astrophysical system, the ensemble average is approximately equal to the spatial average when there is a scale separation between the mean scale and the fluctuating scale. In reality, the scale separation is often weaker than the dynamo theorist desires. Nevertheless, we proceed to consider spatial averages.

Ref \([7]\) showed that \(\bar{B}\) satisfies the induction equation

\[
\frac{\partial \bar{B}}{\partial t} = -c \nabla \times \bar{E},
\]

(1)

where

\[
\bar{E} = -\left(\nabla \times \bar{B}\right)/c - \langle \mathbf{v} \times \mathbf{b} \rangle/c + \lambda \nabla \times \bar{B},
\]

(2)

where the first term describes the effect of differential rotation ("\(\Omega\)-effect"),

\[
\langle \mathbf{v} \times \mathbf{b} \rangle_i = \alpha_{ij} \bar{B}_j - \beta_{ijk} \partial_j \bar{B}_k
\]

(3)

is the "turbulent emf," and \(\lambda = \eta c^2/4\pi\) is the magnetic diffusivity defined with the resistivity \(\eta\). Here \(\alpha_{ij}\) contains Parker’s twisting ("\(\alpha\) effect") and \(\beta_{ijk}(\gg \lambda)\) contains the turbulent diffusivity. Ref \([7]\) calculated \(\bar{E}\) to first order in \(\bar{B}\) for isotropic \(\alpha_{ij}\) and \(\beta_{ijk}\) and hence the dynamo coefficients \(\alpha\) and \(\beta\) to zeroth order in \(\bar{B}\) from the statistics of the turbulence. They ignore the Navier-Stokes equation. The back-reaction on the dynamo coefficients to first order in \(\bar{B}\) and \(\bar{V}\) was calculated in Ref \([9]\), and Ref \([10]\) calculates \(\alpha\) to all orders in \(\bar{B}\) when mean field gradients are small) See also Refs \([11, 12]\) and the issues raised in \([13]\). Refs \([11,12]\) obtain a catastrophically quenched \(\alpha\) but by a different argument than we use later.
When (2) is substituted into (1), we have the mean-field dynamo equation:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\nabla \times \mathbf{B}) + \nabla \times (\alpha \mathbf{B}) - \nabla \times (\beta + \lambda) \nabla \times \mathbf{B}.
\] (4)

In the approximation that \( \mathbf{V}, \alpha \) and \( \beta \) are independent of \( \mathbf{B} \), (4) is a linear equation for \( \mathbf{B} \) which can be solved as an eigenvalue problem for the growing modes in the Sun and other bodies. Actually, a rapid growth of the fluctuating field necessarily accompanies the mean-field dynamo. Its impact upon the growth of the mean field, and the impact of the mean field itself on its own growth are controversial.

The controversy results because Lorentz forces from the growing magnetic field react back on and complicate the turbulent motions driving the field growth [14,15,16,17,18,19]. It is tricky to disentangle the back reaction of the mean field from that of the fluctuating field. Analytic studies and numerical simulations seem to disagree as to the extent to which the dynamo coefficients are suppressed by the back reaction of the mean field. But, as we will address, the disagreements may be because different problems are being solved and the results inappropriately compared.

It is important to distinguish between rapid MFD action, resistively limited MFD action, and no MFD action. Rapid dynamo action describes dynamo action which proceeds at rates not greatly suppressed from the kinematic values. Resistively limited dynamo action is MFD growth that proceeds at rates strongly dependent on the magnetic Reynolds number. No MFD action implies no mean growth at all. For galaxies, rapid MFD action is necessary if the observed large scale fields are to be produced and sustained by the MFD. There the \( \alpha \) effect and the \( \Omega \) effect operate in the same volume. For the Sun, rapid dynamo action is also required but the \( \alpha \) effect operates in the convection zone, whilst the \( \Omega \) shear effect operates in the overshoot layer beneath the convection zone [20].

In section II we discuss the role of magnetic helicity in dynamo theory. We show that open boundaries, allowing magnetic helicity to escape, play an important role for rapid astrophysical dynamo action. This leads to an ambiguity in interpreting of quenching simulations. In section III we predict that astrophysical rotators with dynamos harbor steady coronae. In section IV we suggest that open boundary conditions may not be enough for a dynamo and that the magnetic helicity flow may require buoyancy or winds. We also address pitfalls of the Zeldovich relation. We conclude in section V.
II. ROLE OF MAGNETIC HELICITY CONSERVATION

Although the MFD theory predates detailed studies of MHD turbulence, the MFD may be looked upon in hindsight as a framework for studying the inverse cascade of magnetic helicity. Whether this inverse cascade is primarily local (proceeding by interactions of eddies/waves of nearby wavenumbers) or non-local (proceeding with a direct conversion of power from large to small wavenumbers) is important to understand. The simple MFD is most consistent with the latter.

From the numerical solution of approximate equations describing the spectra of energy and helicity in MHD turbulence, Ref [21] showed that the $\alpha$ effect conserves magnetic helicity ($= \int (A \cdot B) d^3x$) by pumping a positive (negative) amount to scales $> L$ (the outer scale of the turbulence) while pumping a negative (positive) amount to scales $\ll L$. Magnetic energy at the large scale was identified with the $B$ of [7]. Thus, dynamo action leading to an ever larger $B$, hence the creation of ever more large scale helicity, can proceed as long as small scale helicity of opposite sign can be removed or dissipated. Recent simulations [22] confirm this inverse cascade and the role of $H_M$ conservation. But the rate of small scale $H_M$ removal determines the rate of MFD action. Presently, simulations have invoked boundary conditions for which the growth of large scale field is resistively limited. We now discuss what this means.

A. Constraint equation for the turbulent EMF

To explore the role of boundary conditions in constraining the value of the $\alpha$ dynamo parameter, we take Ohm’s law

$$\mathbf{E} = -c^{-1} \mathbf{V} \times \mathbf{B} + \eta \mathbf{J}$$

and average the dot product with $\mathbf{B}$ to find [23]

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle = \mathbf{E} \cdot \mathbf{B} + \langle \mathbf{e} \cdot \mathbf{b} \rangle = -c^{-1} \langle \mathbf{v} \times \mathbf{b} \rangle \cdot \mathbf{B} + \eta \mathbf{J} \cdot \mathbf{B} + \langle \mathbf{e} \cdot \mathbf{b} \rangle \tag{6}$$

where $\mathbf{J}$ is the current density.

A second expression for $\langle \mathbf{E} \cdot \mathbf{B} \rangle$ also follows from Ohm’s law without first splitting into mean and fluctuating components, that is

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle = \eta \langle \mathbf{J} \cdot \mathbf{B} \rangle = \eta \mathbf{J} \cdot \mathbf{B} + \eta \langle \mathbf{j} \cdot \mathbf{b} \rangle = \eta \mathbf{J} \cdot \mathbf{B} + c^{-1} \lambda \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle. \tag{7}$$
Using (7) and (8), we have

\[-c^{-1}\langle v \times b \rangle \cdot \mathbf{B} = c^{-1}\lambda \langle b \cdot \nabla \times b \rangle - \langle e \cdot b \rangle, \tag{8}\]

which can be used to constrain \(\langle v \times b \rangle\) in the mean field theory.

**B. Necessity of magnetic helicity escape and open boundaries**

Now consider \(\mathbf{E}\) in terms of the vector and scalar potentials \(\mathbf{A}\) and \(\Phi\):

\[\mathbf{E} = -\nabla \Phi - (1/c)\partial_t \mathbf{A}. \tag{9}\]

Dotting with \(\mathbf{B} = \nabla \times \mathbf{A}\) we have

\[\mathbf{E} \cdot \mathbf{B} = -\nabla \Phi \cdot \mathbf{B} - (1/c)\mathbf{B} \cdot \partial_t \mathbf{A}. \tag{10}\]

After straightforward algebraic manipulation, application of Maxwell’s equations and \(\nabla \cdot \mathbf{B} = 0\), this equation implies

\[\mathbf{E} \cdot \mathbf{B} = -(1/2)\nabla \cdot \Phi \mathbf{B} + (1/2)\nabla \cdot (\mathbf{A} \times \mathbf{E})
- (1/2c)\partial_t (\mathbf{A} \cdot \mathbf{B}) = (1/2c)\partial_\mu H^\mu(\mathbf{B}) = \eta \mathbf{J} \cdot \mathbf{B}, \tag{11}\]

where

\[H^\mu(\mathbf{B}) = (H_0, H_i) = [\mathbf{A} \cdot \mathbf{B}, c\Phi \mathbf{B} - c\mathbf{A} \times \mathbf{E}] \tag{12}\]

is the magnetic helicity density 4-vector [24], and the contraction has been done with the 4 x 4 matrix \(\eta_{\mu\nu}\) where \(\eta_{\mu \nu} = 0\) for \(\mu \neq \nu\), \(\eta_{\mu \nu} = 1\) for \(\mu = \nu = 0\) and \(\eta_{\mu \nu} = -1\) for \(\mu = \nu > 0\). Taking the average of (11) gives

\[\partial_\nu H^\nu(\mathbf{B}) = -2c\langle \mathbf{E} \cdot \mathbf{B} \rangle = -2c\mathbf{E} \cdot \mathbf{B} - 2c\langle e \cdot b \rangle = -2c\eta \langle \mathbf{J} \cdot \mathbf{B} \rangle. \tag{13}\]

If, instead of starting with the total \(\mathbf{E}\) as in (9), we start with \(\mathbf{e}\) and then dot with \(\mathbf{b}\) and average, the analogous derivation replaces (13) by

\[\partial_\nu \overline{H}^\nu(\mathbf{b}) = -2c\langle \mathbf{e} \cdot \mathbf{b} \rangle, \tag{14}\]

where \(\overline{H}^\nu(\mathbf{b})\) indicates the average of \(H^\mu(\mathbf{b})\). The latter is defined like (12) but with the corresponding fluctuating quantities replacing the total quantities. Similarly, starting with \(\overline{\mathbf{E}}\) and dotting with \(\overline{\mathbf{B}}\), gives

\[\partial_\mu \overline{H}^\mu(\mathbf{B}) = -2c\overline{\mathbf{E}} \cdot \overline{\mathbf{B}}, \tag{15}\]
where $H^\mu(\mathbf{B})$ is defined as in (12) but with the corresponding mean quantities replacing the total quantities. Now consider two cases.

1. **Case 1**

In this case we consider statistically stationary turbulence in which the scale of averaging is equal to the universal scale, or equivalently for present purposes, that the overbar indicates averaging over periodic boundaries. In this case, the spatial divergence terms on the left of (13) become surface integrals and vanish. Rewriting this (14), and discarding the divergence terms gives

$$
\langle \mathbf{e} \cdot \mathbf{b} \rangle = -\frac{1}{2c} \partial_t \langle \mathbf{a} \cdot \mathbf{b} \rangle
$$

(16)

Thus, eqn (8) is resistively limited unless significant non-stationarity is allowed.

Let us apply this to the specific case of statistically stationary turbulence in which a uniform mean field is imposed over a periodic box, and for which the averaging scale is the scale of the box. Then the mean field cannot change with time, and has no gradients. This is the case of Ref [18]. Then the right side of (16) vanishes because not only are zeroth order turbulent correlations (those computed to zeroth order in the mean field) stationary, but all higher order corrections must also be stationary. The mean field does not change with time so no mean quantity varies on macroscopic time scales. Incidentally, when expanded in powers of the mean field, the lowest order contribution to (16) enters at second order [23].

For the uniform field in a periodic box, (8) then implies that

$$
-c^{-1} \langle \mathbf{v} \times \mathbf{b} \rangle \cdot \mathbf{B} = \alpha \mathbf{B}^2/c = c^{-1} \lambda \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle,
$$

(17)

where $\alpha = \alpha_{33}$ for a uniform field in the $z$ direction. Rearranging we have

$$
|\alpha| = \left| \lambda \frac{\langle b \cdot \nabla \times b \rangle}{\mathbf{B}^2} \right|.
$$

(18)

Now if we assume that $b$ and $v$ follow e.g. a $k^{-5/3}$ Kolmogorov energy spectrum, we then have

$$
|\alpha| \lesssim \left| \frac{k_0 b_0^2 \tau_0}{R_M^{3/4} \mathbf{B}^2 / v^2} \right| \sim \left| \frac{\alpha_0}{R_M^n \mathbf{B}^2 / b_0^2} \right|.
$$

(19)

6
where $n = 3/4$ if the current helicity is dominated by large wavenumber and $n = 1$ if it is dominated by small wavenumber. The $v_0, b_0$ are the speed and fluctuating field magnetic energy of the dominant energy containing eddies, $k_0$ is the wavenumber for that scale, $\tau_0$ is the associated eddy turnover time and the magnetic Reynolds number $R_M = vL/\lambda$, where $L$ is the scale of the energy dominating eddies. The latter similarity in (19) follows for near equipartition between magnetic and kinetic energies on the outer scale, and we have written the kinematic $\alpha$ as $\alpha_0$. (Note that the “Pouquet correction” [21] to $\alpha$ which is $\propto \langle b \cdot \nabla \times b \rangle$ enters to the same order in the mean field as the standard $\langle v \cdot \nabla \times v \rangle$ term [10,32], so we consider either to be representative of the lowest order contribution). The value of $\alpha$ in this case is “resistively limited,” however the Reynolds number factor on the bottom does not represent a dynamical backreaction. It is an a priori implication of the imposed boundary conditions. Thus simulations which invoke periodic boundary conditions and find this level of suppression [18], may not be seeing the effect of backreaction dynamics, but of the boundary conditions.

2. Case 2

In this case, consider the system (e.g. Galaxy or Sun) volume $V <<$ universal volume. Integrating (11) over all of space, $U$, then gives

$$\int_U \mathbf{E} \cdot \mathbf{B} \, d^3x = -(1/2) \int_U \nabla \cdot \Phi \mathbf{B} \, d^3x + (1/2) \int_U \nabla \cdot (\mathbf{A} \times \mathbf{E}) \, d^3x$$

$$- (1/2c) \partial_t \int_U \mathbf{A} \cdot \mathbf{B} \, d^3x = -(1/2c) \partial_t \mathcal{H}(\mathbf{B}) = \int_U \eta \mathbf{J} \cdot \mathbf{B} d^3x,$$

where the divergence integrals vanish when converted to surface integrals at infinity. We have defined the global magnetic helicity

$$\mathcal{H}(\mathbf{B}) \equiv \int_U \mathbf{A} \cdot \mathbf{B} \, d^3x,$$

where $U$ allows for scales much larger than the mean field scales. It is straightforward to show that a parallel argument for the mean and fluctuating fields respectively leads to

$$\partial_t \mathcal{H}(\mathbf{B}) = \partial_t \int_U \mathbf{A} \cdot \mathbf{B} \, d^3x = -2c \int_U \mathbf{E} \cdot \mathbf{B} \, d^3x$$

and

$$\partial_t \mathcal{H}(\mathbf{b}) = \partial_t \int_U \langle \mathbf{a} \cdot \mathbf{b} \rangle \, d^3x = -2c \int_U \langle \mathbf{e} \cdot \mathbf{b} \rangle \, d^3x = -2c \int_U \mathbf{e} \cdot \mathbf{b} \, d^3x = \partial_t \mathcal{H}(\mathbf{b}),$$
where the penultimate equality in (23) follows from redundancy of averages.

We now split (22) and (23) into contributions from inside and outside the rotator. One must exercise caution in doing so because $\mathcal{H}$ is gauge invariant and physically meaningful only if the volume $U$ over which $\mathcal{H}$ is integrated is bounded by a magnetic surface (i.e. normal component of $\mathbf{B}$ vanishes at the surface), whereas the surface separating the outside from the inside of the rotator is not magnetic in general.

Ref [25] shows how to construct a revised gauge invariant quantity called the relative magnetic helicity. This can be written as

$$H_{R,i}(\mathbf{B}_i) = H(\mathbf{B}_i, P_o) - H(P_i, P_o) \tag{24}$$

where the two arguments represent inside and outside the body respectively, and $P$ indicates a potential field. The relative helicity of the inside region is thus the difference between the actual helicity and the helicity associated with a potential field inside that boundary. The use of $P_i$ is not arbitrary in (24), and is in fact the field configuration of lowest energy. While (24) is insensitive to the choice of external field [25], it is most convenient to take it to be a potential field as is done in (24) symbolized by $P_o$. The relative helicity of the outer region, $H_{R,o}$, is of the form (24) but with the $o$'s and $i$'s reversed. The $H_R$ is invariant even if the boundary is not a magnetic surface.

The total global helicity, in a magnetically bounded volume divided into the sum of internal and external regions, $U = U_i + U_e$, satisfies [25]

$$H(\mathbf{B}) = H_{R,o}(\mathbf{B}) + H_{R,i}(\mathbf{B}) \tag{25}$$

when the boundary surfaces are planar or spherical. This latter statement on the boundaries means the vanishing of an additional term associated with potential fields that would appear in (25). Similar equations apply for $\mathbf{B}$ and $\mathbf{b}$, so (22) and (23) can be written

$$\partial_t H(\overline{\mathbf{B}}) = \partial_t H_{R,o}(\overline{\mathbf{B}}) + \partial_t H_{R,i}(\overline{\mathbf{B}}),$$

and

$$\partial_t \mathcal{H}(\mathbf{b}) = \partial_t \mathcal{H}_{R,o}(\mathbf{b}) + \partial_t \mathcal{H}_{R,i}(\mathbf{b}) \tag{27}$$

respectively. According to equation (62) of Ref [25],

$$\partial_t H_{R,i}(\mathbf{B}) = -2c\int_{U_i} \mathbf{E} \cdot \mathbf{B} d^3x + 2c\int_{D_{U_i}} (\mathbf{A_p} \times \mathbf{E}) \cdot d\mathbf{S}, \tag{28}$$
where $A_p$ is the vector potential corresponding to a potential field $P$ in $U_e$, and $DU_i$ indicates integration on the boundary surface of the rotator. Similarly, we have

$$
\partial_t \mathcal{H}_{R,i}(B) = -2c \int_{U_i} E \cdot B d^3x + 2c \int_{DU_i} (A_p \times E) \cdot dS \quad (29)
$$

and

$$
\partial_t \mathcal{H}_{R,i}(b) = -2c \int_{U_i} e \cdot b d^3x + 2c \int_{DU_i} a_p \times e \cdot dS. \quad (30)
$$

Note again that the above internal relative helicity time derivatives are both gauge invariant and independent of the field assumed in the external region. If we were considering the relative helicity of the external region, that would be independent of the actual field in the internal region.

Now if we take the averaging scale to be less than or equal to the $U_i$ scale, we can then replace (30) by

$$
\langle e \cdot b \rangle = -\frac{1}{2c} \partial_t \mathcal{H}_{R,i}(b) + \langle \nabla \cdot (a_p \times e) \rangle, \quad (31)
$$

where the brackets indicate integrating over $U_i$ or smaller. We now see that even if the first term on the right of (31) vanishes, $\langle e \cdot b \rangle$ contributes a surface term to (8) that need not vanish. The turbulent EMF is not resistively limited as in the case of the previous section. The surface term can strongly dominate the resistive contribution. Thus in a steady state for $R_M >> 1$, an outflow of magnetic helicity is likely essential to keeping the dynamo operating. For the Sun, the steady state would refer to time scales longer than the longest eddy turnover time, but shorter than the predicted 11 year Solar cycle. Note that the right of (31) is small for large magnetic Reynolds numbers, so the flux of helicity has contributions from the small and large scale field. If the surface term vanishes, helicity could instead be “injected” through the time derivative term of (13). This has known application to Tokomaks [26].

In short, steady dynamo action unrestricted by resistivity is possible in case 2 but not in case 1. This is consistent with current simulations. Case 1 is represented by Refs [21] and [22], which find that when the boundary conditions are periodic, dynamo action proceeds at resistively limited rates. In [27] action seems to proceed more rapidly, and this corresponds to a case 2 simulation, though the resolution and $R_M$ were small.

III. ENERGY FLOW TO CORONAE
Here we explore only the deposition of relative helicity to the exterior and the associated total magnetic energy without addressing how the energy converts to particles or flows. We assume that the rotator is in a steady state over the time scale of interest, so the left sides of (29) and (30) vanish. Note that in a system like the Sun where the mean field flips sign every $\sim 11$ years, the steady state is relevant for time scales less than this period, but greater than the eddy turnover time ($\sim 5 \times 10^4$ sec). Beyond the $\sim 11$ year times scales, the mean large and small scale relative helicity contributions need not separately be steady and the left hand sides need not vanish.

The helicity supply rate, represented by the volume integrals (second terms of (29) and (30)), are then equal to the integrated flux of relative magnetic helicity through the surface of the rotator. Moreover, from (13), we see that the integrated flux of the large scale relative helicity, $\equiv F_{R,i}(B)$, and the integrated flux of small scale relative helicity, $\equiv F_{R,i}(b)$, are equal and opposite. We thus have

$$F_{R,i}(B) = -F_{R,i}(b) = 2c \int_{U_i} (\alpha B^2 - \beta \nabla \times B) \, d^3x. \quad (34)$$

This shows the relation between the equal and opposite large and small scale relative helicity deposition rates and the dynamo coefficients.

Now the realizability of a helical magnetic field requires its turbulent energy spectrum, $E_k^M$, to satisfy [28]

$$E_k^M(b) \geq \frac{1}{8\pi} k |H_k(b)|, \quad (35)$$

where $H_k$ is the magnetic helicity at wavenumber $k$. The same argument also applies to the mean field energy spectrum, so that

$$E_k^M(B) \geq \frac{1}{8\pi} k |H_k(B)|. \quad (36)$$

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If we assume that the time and spatial dependences are separable in both $E^M$ and $\mathcal{H}$, then a minimum power delivered to the corona can be derived. For the contribution from the small scale field, we have

$$\dot{E}^M(b) = \int \dot{E}^M_k(b) dk \geq \frac{1}{8\pi} \int k |\mathcal{F}_{k,R,i}(b)| dk \geq \frac{k_{\text{min}}}{8\pi} |\mathcal{F}_{R,i}(b)| = \frac{k_{\text{min}}}{8\pi} |\mathcal{F}_{R,i}(B)|,$$

where the last equality follows from the first equation in (34). The last quantity is exactly the lower limit on $\dot{E}^M(B)$. Thus the sum of the lower limits on the total power delivered from large and small scales is $\frac{k_{\text{min}}}{8\pi} |\mathcal{F}_{R,i}(b)| + \frac{k_{\text{min}}}{8\pi} |\mathcal{F}_{R,i}(B)|$. Now for a mode to fit in the rotator, $k > k_{\text{min}} = 2\pi/h$, where $h$ is a characteristic scale height of the turbulent layer. Using (37), the total estimated energy delivered to the corona (=the sum of the equal small and large scale contributions) is then

$$\dot{E}^M \geq 2 \frac{k_{\text{min}}}{8\pi} |\mathcal{F}_{R,i}(b)| = 2 \frac{k_{\text{min}}}{8\pi} |\mathcal{F}_{R,i}(B)| = \frac{V}{h} \left| \alpha B^2 - \beta B \cdot \nabla \times B \right|_{\text{ave}}, \quad (37)$$

where $V$ is the volume of the turbulent rotator and $\text{ave}$ indicates volume averaged. We will assume that the two terms on the right of (37) do not cancel, and use the first term of (37) as representative.

Working in this allowed time range for the Sun ($5 \times 10^4 \text{ sec} < t < 11 \text{ yr}$) we apply (37) to each hemisphere of the Sun, and using the first term as an order of magnitude estimate, obtain

$$\dot{E}^M \gtrsim \left( \frac{2\pi R^2}{3} \right) \alpha B^2 = 10^{28} \left( \frac{R}{7 \times 10^{10} \text{ cm}} \right)^2 \left( \frac{\alpha}{40 \text{ cm/s}} \right) \left( \frac{B}{150 \text{ G}} \right)^2 \text{ erg/s} \quad (38)$$

where we have taken $\alpha \sim 40 \text{ cm s}^{-1}$, and we have presumed a field of 150G at a depth of $10^4 \text{ km}$ beneath the solar surface in the convection zone, which is in energy equipartition with turbulent kinetic motions [2].

As this energy deposition rate is available for reconnection which can generate Alfvén waves, drive winds, and energize particles, we must compare this limit with the total of downward heat conduction loss, radiative loss, and solar wind energy flux in coronal holes, which cover $\sim 1/2$ the area of the Sun. According to [29] this amounts to an approximately steady activity of $2.5 \times 10^{28} \text{ erg s}^{-1}$, about 3 times the predicted value of (38). Other supporting evidence for deposition of magnetic energy and magnetic + current helicity [30] in the Sun is discussed in Ref [31].
AGN and the Galactic ISM represent other likely sites of mean field dynamos. Ref [31] discusses the associated energy deposition to the coronae of these systems as well. For the Galaxy, \( \dot{E}^M \gtrsim (\pi R^2) \alpha B^2 \sim 10^{40}(R/12\text{kpc})^2 \times (\alpha/10^5\text{cm/s})(B/5 \times 10^{-6}\text{G})^2\text{erg/s} \) in each hemisphere. This is consistent with coronal energy input rates required by [32] and [33]. For AGN accretion disks, the deposition rate seems to be consistent with what is required from X-ray observations. Independent of the above, the most successful paradigm for X-ray luminosity in AGN is coronal dissipation of magnetic energy [34].

**IV. DISCUSSION AND OPEN QUESTIONS**

**A. Role of boundary diffusion and winds**

We have seen that a steady MFD unlimited by resistivity requires the helicity to flow through the rotator boundary. However, merely open boundary conditions may not be enough: some mechanism must sustain the diffusion of the mean field at the boundary. Near the surface, buoyancy or winds [35] and winds may have to play a role. Note however, that it is the mean field which needs to diffuse, not necessarily the total field, and not the matter [5].

For the Galaxy, turbulent diffusion of the mean magnetic field across the boundary is also required to maintain a quadrupole geometry with a net flux inside the disk. Ref [36] argues that surface diffusion of total magnetic field for the Galaxy is difficult. Thus the diffusion of the mean and/or total field out of a galaxy remains an open question. For the Sun, the solar cycle also requires diffusion of mean field through the boundary. The flow of helicity would appeal to the same dynamics needed by these constraints.

The role of boundary diffusion is essential for understanding whether the present Galactic field is primordial or produced in situ and how the solar dynamo accounts for the surface fields.

**B. Choosing simulation boundary conditions: transient vs steady phases**

Ideally, the best simulation to test the dynamo would be a fully global simulation with significant scale separation between the system size and the turbulent scale, and would properly include the boundary diffusion mechanisms with open boundary conditions. However it may be possible to invoke periodic boundary conditions to simulate rapid dynamo action in the following ways. The first represent transient phases, the last approaches represent steady action.
A first transient phase during which rapid dynamo action should incur, is the time period during which the helicity on the small scale builds up. During this period the dynamo can grow rapidly. After this period one is subject to the boundary conditions. This has been investigated numerically in Ref [37], where the dynamo $\alpha$ is shown not to be significantly quenched in this early phase.

A second time scale over which rapid dynamo action could be seen in a periodic box, even after the small scale helicity saturates, would be facilitated by significant scale separation between the box size and the scale of the input turbulent scale. One could inject kinetic helicity only into a sub-volume of the box and observe the inverse cascade of magnetic helicity and magnetic energy in this sub-volume before the material diffuses all the way across the box. During this transient period, even if the small scale helicity saturates, the surface terms can play an important role. The length of this period of course depends on the extent of scale separation.

Finally, there may be a way to use a periodic box to actually get steady rapid dynamo action. One can inject helicity inside the sub-volume described above, and invoke a one-way diffusive valve out of this region. The helicity of one sign should grow inside the forced region, while the opposite sign would be shed to the exterior region. One could then surround the box with a resistive layer so that this external helicity dissipates rather than feeds back into the system.

C. Previous analytical and numerical suppression results

To date, analytic and numerical studies that suggest resistively limited dynamo action either (1) invoke periodic boundary conditions (as described above), and/or (2) are 2-D, or (3) do not distinguish between zeroth order isotropic components of the turbulence and the higher order anisotropic perturbations for a weak mean field [31]. This existence of an alternative explanation for the catastrophic suppression makes the computed suppression ambiguous. This does not mean that the physical concepts found in the strong suppression results are invalid, but just that they may be valid only for the restricted cases considered.

The limitations can be specifically pinpointed. For example, [38] derives the now famous “Zeldovich relation” $\langle b^2 \rangle / B^2 \sim R_M$. But this relation is for $2 - D$, ignores boundary terms, and is non-trivial only when the magnetic energy is dominated on small scales (unlike 3-D simulation results).
The Zeldovich relation arises from first deriving, in the absence of boundary terms, the evolution equation for the average of the square of the vector potential from an initially uniform magnetic field in 2-D. We choose the vector potential to be in the \( z \) direction, and thus we have \( A = A_z \). The resulting evolution equation is

\[
\partial_t \langle A_z^2 \rangle = -2\lambda \langle (\nabla A_z)^2 \rangle. \tag{39}
\]

Note that there is only a single term on the right hand side because of the boundary conditions. We now appeal to the positive definite nature of the last term to rewrite the equation

\[
\partial_t \langle A_z^2 \rangle = -2\lambda \langle A_z^2 \rangle / \delta^2(t), \tag{40}
\]

where \( \delta(t) \) represents the dominant scale of the turbulent field. In 2-D, when surface terms are ignored, the mean field cannot change. However the small scale field can grow in response to the turbulent stretching. In analytic studies [eg. 16] and numerical simulations [e.g. 37] (which happens to be 3-D) this type of behavior can be described by a decrease of \( \delta(t) \), that is \( d\delta(t)/dt < 0 \), as the turbulent eddies cascade to smaller and smaller scales. Equipartition between magnetic and kinetic energy is first reached on the dissipative scale as there the eddy turnover time is the shortest. When the field peaks at this scale, the dissipation rate in (40) is maximized because \( \delta(t) \) reaches its minimum. The end of kinematic regime occurs when the field saturates at the resistive scale. In fact the maximum dissipation rate occurs if \( R_M \sim b^2 / B^2 \). (Note however, that if \( R_M >> b^2 / B^2 \), saturation at \( b^2 = v^2 \) would occur before \( R_M \) enters the relation.)

Soon after the maximum dissipation is achieved, \( d\delta/dt > 0 \). That is, the scale \( \delta(t) \) increases as the dynamic regime sets in and equipartition is reached on successively larger and larger scales. The dissipation rate then decreases in the non-linear dynamic regime. This is the “suppression” of dissipation that is directly implied by (40). We refer to this as “dissipation” rather than “diffusion” because the only term determining the time evolution of \( \langle A_z^2 \rangle \) has an explicit \( \lambda \) as a coefficient.

In fact, the original Zeldovich result was meant to be a limitation on the growth of \( \langle b^2 \rangle \) from a fixed \( \overline{B} \), the latter of which was restricted not to change due to 2-D calculation and the choice of boundary conditions. Nothing was said about the turbulent diffusion of field lines or the ability of the mean field to grow. Zeldovich recognized that the 2-D case did not necessarily imply
much about the 3-D case, the case when boundary terms are included, or the case when $\mathcal{B}$ is allowed to grow.

That being said, there are some important related issues to understand in future work. The observation that at least the 2-D Lagrangian chaos properties of the flow seem to change in the presence of a weak mean field for turbulence in a periodic box [39] needs to be understood in relation to the imposed boundary conditions and the shape of the magnetic energy spectrum. The connection between this observed restricted 2-D diffusion and the Zel’dovich relation is actually subtle, and also requires further investigation. One point to consider in this context is that when the diffusion time across the box is comparable to the Alfvén crossing time across the box, numerical effects may be playing a strong role in determining the results and must be kept in mind.

At present there is a dearth of simulations on 3-D turbulent diffusion of the mean field. However molecular cloud simulations [40] which do not explicitly consider an application to dynamo theory, can actually be interpreted to imply the absence of catastrophic quenching of mean field diffusion.

Finally, we note that Zel’dovich et al. produced a second approach to this relation between the mean field strength and $R_M$ which was thought to be applicable to 3-D [4]. In 3-D, a logarithmic dependence on $R_M$ arises, while the method also seems to reproduce the 2-D result in the 2-D limit. However, this approach may be flawed as it does not distinguish between isotropic and anisotropic components of the turbulence. This will be discussed further elsewhere.

**D. Coronal Activity**

The estimated energy deposition rates are consistent with the coronal + wind power from the Sun, Galaxy and Seyfert Is [31]. The helical properties also seem to agree well in the solar case where they can be observed [30]. The steady flow of magnetic energy into coronae thus provides an interesting connection between mean field dynamos and coronal dissipation paradigms in a range of sources. A reasonably steady (over time scales long compared to turbulent turnover time scales), active corona with multi-scale helical structures, provides a self-consistency check for a dynamo production of magnetic field in which there is exponential field growth inside the body on small and large scales. If the growth rate were only linear, the corona and wind output might be more episodic with fluctuations not by many orders of magni-
tude but by an order of magnitude or less. The latter seems to be consistent with observations of the Sun and coronae of Seyfert AGN [41], suggesting exponential field growth inside the rotator.

**E. Role of magnetic Prandtl number**

Much of the numerical and analytic work to date with respect to the mean field dynamo problem has focused on unit Prandtl number $Pr \equiv \lambda/\eta$ (the ratio of the magnetic to kinetic viscosities.) In nature, this number is rarely near unity. In the Sun $Pr << 1$, whereas in the Galaxy $Pr >> 1$. There are currently several numerical [19,42,43] and analytic studies [43] addressing the role of the magnetic Prandtl number. For unit Prandtl number, the magnetic and kinetic energies approach equipartition from the the input scale down to the dissipation scale. For $Pr >> 1$, Ref [41] finds that the magnetic energy dominates the turbulent energy at scales below the viscous cutoff scale, and that the magnetic energy initially peaks on these tiny scales. With time, the peak seems to migrate back toward the input scale, although runs have not been done long enough to see what the saturated state is like. These simulations do not have a large enough inertial range, and so are far from being able to assess the power build up at scales even larger than the forcing scales, as required to test the MFD. nor do they consider helical forcing.

Ref [21] finds that the non-local inverse cascade is not much affected by the large $Pr$ when compared to the small $Pr$, but initially sees a similar development of the small scale magnetic energy as in Ref [42]. The dynamic range is also limited. Refs [43] appears consistent with this. The work also suggests that the contribution to $\alpha$ from the current helicity might exceed that from the kinetic helicity for large Prandtl number. More work is needed to understand the effect of large and small Prandtl numbers.

**V. CONCLUSIONS**

We have emphasized that the MFD represents a framework for understanding an inverse cascade of magnetic helicity: Kinetic helicity imposed at small scales pumps magnetic helicity from small to large scales through an inverse cascade [21,22,44]. This process seems to be non-local [22] in that helicity “jumps” directly from small to large scales. Accompanying the magnetic helicity is a growth of magnetic energy.

We suggest that the MFD can in principle proceed much more rapidly for real astrophysical rotators, when compared to simulations in periodic boxes.
in which the turbulence is homogeneous. For such simulations, surface terms are ignored and so the magnetic helicity is nearly conserved for large magnetic Reynolds numbers. Generating magnetic helicity and magnetic energy on the large scales from an MFD then requires generating a compensating magnetic helicity at or below the input scale. In a steady state, the small scale helicity would drain only through resistive dissipation. In this case the growth rate of the large scale field is resistively limited.

But in real astrophysical systems, the boundary terms relax the helicity conservation constraint. The growth rate of the large scale helicity (and thus large scale field) is then limited by the rate at which the compensating helicity flows out the boundary. This rate can in principle be much faster than the resistively limited rate. The message is that the boundary terms are likely important for astrophysical MFD’s and so backreaction studies which show suppression but do not allow boundary terms leave the ambiguity as to whether the suppression is actually due to the dynamic backreaction or is simply due to the boundary conditions.

Note that the role of shear for the generation of magnetic

A steady flow of magnetic helicity into the corona is expected for an astrophysical rotator harboring a vigorous $\alpha - \Omega$ mean field dynamo. The helicity escape rate leads to a lower limit on the total magnetic energy deposition into the corona. When the corona itself is turbulent, there should also be an inverse cascade of magnetic helicity in any wind driven outward, thus the dominant magnetic helicity scale would appear to increase on increasingly large distances from the source.

In addition to the conclusions above, we discussed a number of open issues: the choice of boundary conditions for simulations, the need to understand the boundary diffusion in real systems, some implications of previous analytic work, the predicted variability for a corona, and the role of the magnetic Prandtl number.

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