Pointwise strong approximation of almost periodic functions

Radosława Kranz, Włodzimierz Łenski and Bogdan Szal
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a
P O L A N D
R.Kranz@wmie.uz.zgora.pl, W.Lenski@wmie.uz.zgora.pl,
B.Szal @wmie.uz.zgora.pl

Abstract
We consider the class $GM_{2\beta}$ in pointwise estimate of the deviations in strong mean of almost periodic functions from matrix means of partial sums of their Fourier series.

Key words: Almost periodic functions; Rate of strong approximation; Summability of Fourier series

2000 Mathematics Subject Classification: 42A24

1 Introduction
Let $S^p$ ($1 < p \leq \infty$) be the class of all almost periodic functions in the sense of Stepanov with the norm
\[
\|f\|_{S^p} := \begin{cases} 
\sup_u \left\{ \frac{1}{\pi} \int_{u-\pi}^{u+\pi} |f(t)|^p \, dt \right\}^{1/p} & \text{when } 1 < p < \infty \\
\sup_u |f(u)| & \text{when } p = \infty.
\end{cases}
\]
Suppose that the Fourier series of $f \in S^p$ has the form
\[
Sf(x) = \sum_{\nu = -\infty}^{\infty} A_{\nu}(f) e^{i\lambda_{\nu} x}, \quad \text{where } A_{\nu}(f) = \lim_{L \to \infty} \frac{1}{L} \int_{0}^{L} f(t) e^{-i\lambda_{\nu} t} \, dt,
\]
with the partial sums
\[
S_{\gamma_k} f(x) = \sum_{|\lambda_{\nu}| \leq \gamma_k} A_{\nu}(f) e^{i\lambda_{\nu} x}
\]
and that $0 = \lambda_0 < \lambda_{\nu} < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3, \ldots\}$, \( \lim_{\nu \to \infty} \lambda_{\nu} = \infty \), \( \lambda_{-\nu} = -\lambda_{\nu} \), \(|A_{\nu}| + |A_{-\nu}| > 0 \). Let \( \Omega_{\alpha,p} \), with some fixed positive \( \alpha \), be the set of functions of class \( S^p \) bounded on \( U = (-\infty, \infty) \) whose Fourier exponents satisfy the condition
\[
\lambda_{\nu+1} - \lambda_{\nu} \geq \alpha \quad (\nu \in \mathbb{N}).
\]

In case $f \in \Omega_{\alpha,p}$
\[
S_{\lambda_k} f(x) = \int_0^\infty \{f(x + t) + f(x - t)\} \Psi_{\lambda_k, \lambda_k + \alpha}(t) \, dt,
\]
where
\[
\Psi_{\lambda, \eta}(t) = \frac{2 \sin \frac{(\eta - \lambda) t}{2} \sin \frac{(\eta + \lambda) t}{2}}{\pi (\eta - \lambda) t^2} \quad (0 < \lambda < \eta, \ |t| > 0).
\]

Let \( A := (a_{n,k}) \) be an infinite matrix of real nonnegative numbers such that
\[
\sum_{k=0}^{\infty} a_{n,k} = 1, \text{ where } n = 0, 1, 2, \ldots .
\]

Let us consider the strong mean
\[
H_{n,A,\gamma}^q f(x) = \left( \sum_{k=0}^{\infty} a_{n,k} |S_{\gamma_k} f(x) - f(x)|^q \right)^{1/q} \quad (q > 0).
\]

As measures of approximation by the quantity \( (2) \), we use the best approximation of \( f \) by entire functions \( g_\sigma \) of exponential type \( \sigma \) bounded on the real axis, shortly \( g_\sigma \in B_\sigma \) and the moduli of continuity of \( f \) defined by the formulas
\[
E_\sigma(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p},
\]
\[
\omega f(\delta)_X = \sup_{|t| \leq \delta} \|f(x + t) - f(x)\|_X, \quad X = C_{2\pi} \text{ or } X = S^p
\]
and
\[
w_x f(\delta)_{S^p} := \left( \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right)^{1/p},
\]
respectively.

Recently, L. Leindler \cite{4} defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by \( RBVS \), i.e.
\[
RBVS = \left\{ a := (a_n) \in U : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in U \right\}.
\]
and by $GM$ the class of general monotone coefficients defined as follows (see [10]):

$$GM = \left\{ a := (a_n) \in U : \sum_{k=m}^{2m-1} \left| a_k - a_{k+1} \right| \leq K(a) |a_m| \text{ for all } m \in U \right\}. \quad (4)$$

Then it is obvious that

$$MS \subset RBVS \subset GM.$$

In [5, 10, 11, 12] was defined the class of $\beta$–general monotone sequences as follows:

**Definition 1** Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be $\beta$–general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} \left| a_k - a_{k+1} \right| \leq K(a) \beta_m \quad (5)$$

holds for all $m$.

In the paper [12] Tikhonov considered, among others, the following examples of the sequences $\beta_n$:

1. $1\beta_n = |a_n|$, 
2. $2\beta_n = \sum_{k=\lfloor n/c \rfloor}^{\lfloor cn \rfloor} |a_k|/c$ for some $c > 1$.

It is clear that (see [12, Remark 2.1]) $GM(1\beta) = GM$ and $GM(1\beta + 2\beta) \equiv GM(2\beta)$.

Moreover, we assume that the sequence $(K(\alpha_n))_{n=0}^\infty$ is bounded, that is, that there exists a constant $K$ such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all $n$, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (3)-(5) for the sequences $\alpha_n := (a_{n,k})_{k=0}^\infty$.

Now we can give the conditions to be used later on. We assume that for all $n$

$$\sum_{k=m}^{2m-1} |a_{n,k} - a_{n,k+1}| \leq K \sum_{k=\lfloor m/c \rfloor}^{\lfloor cm \rfloor} \frac{a_{n,k}}{k} \quad (6)$$

holds if $\alpha_n = (a_{n,k})_{k=0}^\infty$ belongs to $GM(2\beta)$, for $n = 0, 1, 2, ...$

We have shown in [7] the following theorem:

**Theorem 2** If $f \in \Omega_{\alpha,p}$, $p \geq q$, $(a_{n,k})_{k=0}^\infty \in GM(2\beta)$ for all $n$, [7] and

$$\lim_{n \to \infty} a_{n,0} = 0 \text{ hold, then}$$

$$\left\| H_{n,A,\gamma}^q f \right\|_{S^p} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \omega^q f \left( \frac{\pi}{k+1} \right)_{S^p} \right\}^{1/q},$$

for $n = 0, 1, 2, ..., \text{ where } \gamma = (\gamma_k) \text{ is a sequence with } \gamma_k = \frac{\alpha_k}{2}$. 3
In this paper we consider the class $GM(2\beta)$ in pointwise estimate of the quantity $H_{n,A,\gamma}^q f$. Thus we present some analog of the following result of P. Pych-Taberska (see [9, Theorem 5]):

**Theorem 3** If $f \in \Omega_{\alpha,\infty}$ and $q \geq 2$, then

\[
\|H_{n,A,\gamma}^q f\|_{S^\infty} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^n \omega f \left( \frac{\pi}{k+1} \right) \right\}^{1/q} + \frac{\|f\|_{S^\infty}}{(n+1)^{1/q}},
\]

for $n = 0, 1, 2, \ldots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha_k}{2}$, $a_{n,k} = \frac{1}{n+1}$ when $k \leq n$ and $a_{n,k} = 0$ otherwise.

We shall write $I_1 \ll I_2$ if there exists a positive constant $K$, sometimes depended on some parameters, such that $I_1 \leq KI_2$.

## 2 Statement of the results

Let us consider a function $w_x$ of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties:

\[
w_x(0) = 0, \quad w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2) \text{ for any } \delta_1, \delta_2 \geq 0 \text{ with } x \text{ such that the set}
\]

\[
\Omega_{\alpha,p}(w_x) = \left\{ f \in \Omega_{\alpha,p} : \left[ \frac{1}{\delta} \int_0^\delta |\varphi_x(t) - \varphi_x(t \pm \gamma)|^p \, dt \right]^{1/p} \ll w_x(\gamma) \right\}
\]

and $w_x f(\delta) \ll w_x(\delta)$, where $\gamma, \delta > 0$.

is nonempty. It is clear that $\Omega_{\alpha,p}(w_x) \subseteq \Omega_{\alpha,p'}(w_x)$, for $p' \leq p$.

We start with proposition

**Proposition 4** If $f \in \Omega_{\alpha,p}(w_x)$ and $q > 0$, then

\[
\left\{ \frac{1}{n+1} \sum_{k=0}^{2n} \left| S_{\frac{\pi}{k+1}} f(x) - f(x) \right|^q \right\}^{1/q} \ll w_x \left( \frac{\pi}{n+1} \right) + E_{\alpha n/2}(f)_{S^p},
\]

for $n = 0, 1, 2, \ldots$

Our main results are following

**Theorem 5** If $f \in \Omega_{\alpha,p}(w_x)$, $q > 0$, $(a_{n,k})_{k=0}^\infty \in GM(2\beta)$ for all $n$, \(\{I\}\) and $\lim_{n \to \infty} a_{n,0} = 0$ hold, then

\[
H_{n,A,\gamma}^q f(x) \ll \left\{ \sum_{k=0}^\infty a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) + E_{\frac{\pi}{2k+c}}(f)_{S^p} \right] \right\}^{1/q}
\]

for some $c > 1$ and $n = 0, 1, 2, \ldots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha_k}{2}$. 

4
Theorem 6 If \( f \in \Omega_{\alpha,p}(w_x) \), \( q > 0 \), \( (a_{n,k})_{k=0}^{\infty} \in MS \) for all \( n \), (1) and \( \lim_{n \to \infty} a_{n,0} = 0 \) hold, then

\[
H_{n,A,\gamma}^q f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) + E_{\frac{\alpha k}{2}} (f)_{S_p} \right]^q \right\}^{1/q}
\]

for \( n = 0, 1, 2, \ldots \), where \( \gamma = (\gamma_k) \) is a sequence with \( \gamma_k = \frac{\alpha k}{2} \).

Remark 1 Since

\[
\left\| \frac{1}{\delta} \int_{0}^{\delta} |\varphi(t) - \varphi(t \pm \gamma)|^p \, dt \right\|_{S_p}^{1/p} \leq \omega f(\gamma)_{S_p}
\]

and

\[
\left\| w \cdot f(\delta) \right\|_{S_p} \leq \omega f(\delta)_{S_p},
\]

the analysis of the proof of Proposition 4 shows that, the estimate from Theorem 5 implies the estimate from Theorem 2 with \( p \geq q \) (without change \( q \) instead of \( q' \) in the estimate of the quantity \( \left\{ \frac{1}{n+1} \sum_{k=0}^{2n} I_3(k) \right\}^{1/q} \)). Thus, taking \( a_{n,k} = \frac{1}{n+1} \) when \( k \leq n \) and \( a_{n,k} = 0 \) otherwise, in the case \( p = \infty \), we obtain the better estimate than this one from Theorem 3 [9, Theorem 5].

3 Proofs of the results

3.1 Proof of Proposition 4

In the proof we will use the following function \( \Phi_x f(\delta, \nu) = \frac{1}{\delta} \int_{\nu}^{\nu+\delta} \varphi_x(u) \, du \), with \( \delta = \delta_n = \frac{\pi}{n+1} \) and its estimate from [6, Lemma 1, p.218]

\[
|\Phi_x f(\delta_1, \delta_2)| \leq w_x(\delta_1) + w_x(\delta_2)
\]

for \( f \in \Omega_{\alpha,p}(w_x) \) and any \( \delta_1, \delta_2 > 0 \).

Since, for \( n = 0 \) our estimate is evident we consider \( n > 0 \), only.

Denote by \( S^*_k f \) the sums of the form

\[
S^*_k f(x) = \sum_{|\lambda| \leq \frac{\pi}{2}} A_{\nu}(f) e^{i\lambda \cdot x}
\]

such that the interval \( \left( \frac{\alpha k}{2}, \frac{\alpha(k+1)}{2} \right) \) does not contain any \( \lambda_{\nu} \). Applying Lemma 1.10.2 of [8] we easily verify that

\[
S^*_k f(x) - f(x) = \int_{0}^{\infty} \varphi_x(t) \Psi_k(t) \, dt,
\]
where \( \varphi_x(t) := f(x+t) + f(x-t) - 2f(x) \) and \( \Psi_k(t) = \Psi_{\frac{\alpha k + 1}{2}}(t) \), i.e.

\[
\Psi_k(t) = \frac{4 \sin \frac{\alpha t}{2} \sin \frac{\alpha (k+1)t}{4}}{\alpha \pi t^2}
\]

(see also [3, p.41]). Evidently, if the interval \( \left( \frac{\alpha}{2}, \frac{\alpha (k+1)}{2} \right) \) contains a Fourier exponent \( \lambda_\nu \), then

\[
S_{\frac{\alpha}{2}} f(x) = S_{\frac{\alpha}{2}}^* f(x) - (A_{\nu}(f) e^{i\lambda_\nu x} + A_{-\nu}(f) e^{-i\lambda_\nu x}).
\]

We can also note that if \( f \in \Omega_{\alpha,p}(w_x) \), with \( p > 1 \) and \( q > 0 \), then there exists \( q' \geq q \) such that \( q' \geq 2 \) and \( p' = \frac{q'}{q-1} \leq p \). Thus,

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha}{2}} f(x) - f(x) \right|^q \right\}^{\frac{1}{q'}} \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha}{2}} f(x) - f(x) \right|^{\frac{1}{q'}} \right\}.
\]

Since (see [1, p.78] and [2, p.7])

\[
\left\{ \sum_{\nu=-\infty}^{\infty} |A_\nu(f)|^q \right\}^{\frac{1}{q'}} \leq \|f\|_{B^{q'}} \quad \text{and} \quad \|f\|_{B^{q'}} \leq \|f\|_{S^{q'}}
\]

where \( \|\cdot\|_{B^{q'}} \), with \( (p') \geq 1 \), is the Besicovitch norm, so we have

\[
|A_{\pm\nu}(f)| = |A_{\pm\nu}(f - g_{\alpha \mu/2})| \leq \|f - g_{\alpha \mu/2}\|_{S^{q'}} \leq \|f - g_{\alpha \mu/2}\|_{S^{q'}} = E_{\alpha \mu/2}(f)s_{q'}
\]

for some \( g_{\alpha \mu/2} \in B_{\alpha \mu/2} \), with \( \alpha k/2 < \alpha \mu/2 < \lambda_\nu \). Therefore, the deviation

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha}{2}} f(x) - f(x) \right|^{\frac{1}{q'}} \right\}^{\frac{1}{q'}}
\]

can be estimated from above by

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \int_0^\infty \varphi_x(t) \Psi_{k+\kappa}(t) dt \right\}^{\frac{1}{q'}} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( E_{\alpha k/2}(f)s_{q'} \right)^{\frac{1}{q'}} \right\}
\]

\[
\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \int_0^\infty \varphi_x(t) \Psi_{k+\kappa}(t) dt \right\}^{\frac{1}{q'}} + E_{\alpha k/2}(f)s_{q'},
\]

where \( \kappa \) equals 0 or 1. Putting \( h_n = 2\pi/\alpha(n+1) \) we obtain

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \int_0^\infty \varphi_x(t) \Psi_{k+\kappa}(t) dt \right\}^{\frac{1}{q'}}
\]

\[
= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( \int_0^{h_n} + \int_{h_n}^{(n+1)h_n} + \int_{(n+1)h_n}^{\infty} \varphi_x(t) \Psi_{k+\kappa}(t) dt \right) \right\}^{\frac{1}{q'}}
\]

\[
\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^{\frac{1}{q'}} \right\}^{\frac{1}{q'}} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^{\frac{1}{q'}} \right\}^{\frac{1}{q'}} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_3(k)|^{\frac{1}{q'}} \right\}^{\frac{1}{q'}}.
\]
So, for the first term we have
\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^{q'} \right\}^{1/q'}
\]
\[
\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha \pi} \int_{0}^{h_n} \varphi_x(t) \frac{\alpha t}{t^2} \sin \frac{\alpha t}{4} (2k + 2\kappa + 1) dt \right|^q \right\}^{1/q'}
\]
\[
\leq \frac{\alpha (4n + 2\kappa + 1)}{4\pi} \int_{0}^{h_n} |\varphi_x(t)| dt \ll \left( 2 + \frac{\kappa}{n+1} \right) w_x \left( \frac{2\pi}{\alpha (n+1)} \right)^p.
\]
Next, we estimate the second term. We have
\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^{q'} \right\}^{1/q'}
\]
\[
\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha \pi} \int_{h_n}^{(n+1)h_n} \Phi_x(f(\delta_n, t) - \varphi_x(t)) \sin \frac{\alpha t}{4} (2k + 2\kappa + 1) dt \right|^q \right\}^{1/q'}
\]
\[
+ \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{4}{\alpha \pi} \int_{h_n}^{(n+1)h_n} \Phi_x(f(\delta_n, t)) \sin \frac{\alpha t}{4} (2k + 2\kappa + 1) dt \right|^q \right\}^{1/q'}
\]
\[
= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^{q'} \right\}^{1/q'} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{22}(k)|^{q'} \right\}^{1/q'}.
\]
From the Hausdorff -Young inequality \cite[Chap. XII, Th. 3.3 II]{[13]} we obtain
\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^{q'} \right\}^{1/q'}
\]
\[
\leq \frac{8}{\alpha^2} \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{\alpha}{2\pi} \int_{h_n}^{(n+1)h_n} \Phi_x(f(\delta_n, t) - \varphi_x(t)) \sin \frac{\alpha t}{4} (2\kappa + 1) \cos \frac{\alpha kt}{2} dt \right|^q \right\}^{1/q'}
\]
\[
+ \frac{8}{\alpha^2} \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \frac{\alpha}{2\pi} \int_{h_n}^{(n+1)h_n} \Phi_x(f(\delta_n, t) - \varphi_x(t)) \sin \frac{\alpha t}{4} \cos \frac{\alpha (2\kappa + 1) \sin \frac{\alpha kt}{2}}{2} dt \right|^q \right\}^{1/q'}
\]
\[
\ll \frac{1}{(n+1)^{1/q'}} \left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{\Phi f(\delta_n, t) - \varphi_x(t)}{t^2} \sin \frac{\alpha t}{4} \sin \frac{\alpha t}{4} (2\kappa + 1) \right|^{p'} dt \right\}
\]
\[
+ \left\{ \int_{h_n}^{(n+1)h_n} \left( \frac{\Phi f(\delta_n, t) - \varphi_x(t)}{t^2} \sin \frac{\alpha t}{4} \cos \frac{\alpha t}{4} (2\kappa + 1) \right|^{p'} dt \right\}
\]
\[
\ll \frac{1}{(n+1)^{1/q'}} \left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{\Phi f(\delta_n, t) - \varphi_x(t)}{t^2} \sin \frac{\alpha t}{4} \sin \frac{\alpha t}{4} (2\kappa + 1) \right|^{p'} dt \right\}
\]
and by the Minkowski inequality, for \( p' > 1 \), we have
\[
\ll \frac{1}{(n+1)^{1/q'}} \left\{ \frac{1}{n+1} \sum_{k=1}^{2n} |I_{21}(k)|^{p'} \right\}
\]
\[
\ll \frac{1}{(n+1)^{1/q'}} \frac{1}{\delta_n} \int_0^{\delta_n} \left[ \int_{h_n}^{(n+1)h_n} \frac{|\varphi_x(u + t) - \varphi_x(t)|^{p'}}{t^{p'}} dt \right]^{1/p'} du
\]
\[
\ll \frac{1}{(n+1)^{1/q')} \frac{1}{\delta_n} \int_0^{\delta_n} \left[ \int_{h_n}^{(n+1)h_n} \frac{1}{t^{p'}} \left( \int_0^t |\varphi_x(u + s) - \varphi_x(s)|^{p'} ds \right) dt \right]^{1/p'} du
\]
\[
\ll \frac{1}{(n+1)^{1/q')} \frac{1}{\delta_n} \int_0^{\delta_n} \left\{ \left( \frac{\alpha}{2\pi} \right)^p \int_0^{(n+1)h_n} |\varphi_x(u + s) - \varphi_x(s)|^{p'} ds + \right. 
\]
\[
\left. + \left( \frac{\alpha (n+1)}{2\pi} \right)^p \int_0^{(n+1)h_n} |\varphi_x(u + s) - \varphi_x(s)|^{p'} ds + \right. 
\]
\[
+ \left. \frac{p'}{1} \int_{h_n}^{(n+1)h_n} \frac{1}{t^{p'}} \left( w_x(u) \right)^{p'} dt \right\}^{1/p'} du
\]
\[ \leq \frac{1}{(n+1)^{1/q'}} \delta_n \int_0^{\delta_n} \left\{ w'_x (u) \left( \frac{\alpha}{2\pi} \right)^{p'-1} + \left( \frac{\alpha (n+1)}{2\pi} \right)^{p'-1} \right\}^{1/p'} \frac{u^{p'} - 1}{1 - p'} du \]

\[ \leq \frac{1}{(n+1)^{1/q'}} w_x (\delta_n) \left\{ 1 + (n+1)^{p'-1} + p' \left[ \left( \frac{1}{1 - p'} \right) \frac{1}{h_n} \right]^{1/p'} \right\} \]

\[ \leq w_x (\delta_n) (n+1)^{-1/q'} (n+1)^{1-1/p'} = w_x \left( \frac{\pi}{n+1} \right). \]

And

\[ \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |J_{22}(k)|^{q'} \right\}^{1/q'} \]

\[ = \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \frac{4}{\alpha \pi} \int_{h_n}^{(n+1)h_n} \Phi_x f (\delta_n, t) \frac{\sin \alpha t}{t^2} \left( \frac{\alpha}{4} (2k + 2\kappa + 1) dt \right)^{q'} \right\}^{1/q'} \]

\[ = \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \frac{4}{\alpha \pi} \int_{h_n}^{(n+1)h_n} \Phi_x f (\delta_n, t) \frac{\sin \alpha t}{t^2} \frac{d}{dt} \left( \frac{-\cos \frac{\alpha t}{4} (2k + 2\kappa + 1)\right)\left(2\kappa + 2\kappa + 1\right) dt \right)^{q'} \right\}^{1/q'} \]

\[ \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \frac{4}{\alpha \pi} \Phi_x f \left( \delta_n, \frac{2\pi}{\alpha(n+1)} \right) \frac{\cos \frac{\pi}{2(n+1)} (2k + 2\kappa + 1)}{\frac{\alpha}{4} (2k + 2\kappa + 1)} \left\{ \frac{\alpha}{4} (2k + 2\kappa + 1) \right\}^{q'} \right\}^{1/q'} \]

\[ + \frac{4}{\alpha \pi} \Phi_x f \left( \delta_n, \frac{2\pi}{\alpha(n+1)} \right) \sin \frac{\pi}{2(n+1)} \left( \frac{\alpha}{4} (2k + 2\kappa + 1) \right)^2 \left\{ \frac{\alpha}{4} (2k + 2\kappa + 1) \right\}^{q'} \]

\[ + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \frac{4}{\alpha \pi} \int_{h_n}^{(n+1)h_n} \frac{d}{dt} \left( \Phi_x f (\delta_n, t) \frac{\sin \alpha t}{t^2} \right) \frac{\cos \frac{\alpha t}{4} (2k + 2\kappa + 1)\right)\left(2\kappa + 2\kappa + 1\right) dt \right\}^{q'} \]

9
\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{221}(k)|^q \right\}^{1/q'} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{222}(k)|^q \right\}^{1/q'}.
\]

For the first term, using (7), we obtain
\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{221}(k)|^q \right\}^{1/q'} \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( \frac{4}{(k+1)\pi^3} \Phi_x f \left( \delta_n, \frac{2\pi}{\alpha (k+1)} \right) \right)^q \right\}^{1/q'}
\]
\[
+ \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( \frac{2(n+1)}{(k+1)\pi^2} \left( w_x (\delta_n) + w_x \left( \frac{2\pi}{\alpha (n+1)} \right) \right) \right)^q \right\}^{1/q'}
\]
\[
\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( \frac{4}{(k+1)\pi^3} \left( w_x (\delta_n) + w_x \left( \frac{2\pi}{\alpha (n+1)} \right) \right) \right)^q \right\}^{1/q'}
\]
\[
\leq w_x (\delta_n) + w_x \left( \frac{2\pi}{\alpha (n+1)} \right) + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( \frac{1}{k+1} w_x \left( \frac{2\pi}{\alpha (n+1)} \right) \right)^q \right\}^{1/q'}
\]
\[
\leq w_x \left( \frac{\pi}{n+1} \right) + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} w_{xq}^q \left( \frac{\pi}{k+1} \right) \right\}^{1/q'} \leq 2 w_x \left( \frac{\pi}{n+1} \right).
\]

For the second term, using (7) and the Hausdorff-Young inequality [13, Chap. XII, Th. 3.3 II] we have
\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{222}(k)|^q \right\}^{1/q'} \leq \left\{ \frac{1}{(n+1)^{1/q'+1}} \int_{h_n}^{(n+1)h_n} \left| \frac{d}{dt} \left( \Phi_x f (\delta_n, t) \sin \frac{\alpha t}{t^2} \right) \right| \cos \frac{\alpha t}{4} (2k + 2\kappa + 1) dt \right\}^{1/q'}
\]
\[
\leq \left\{ \frac{1}{(n+1)^{1/q'+1}} \int_{h_n}^{(n+1)h_n} \left| \frac{d}{dt} \left( \Phi_x f (\delta_n, t) \sin \frac{\alpha t}{t^2} \right) \right|^{q'} dt \right\}^{1/p'}
\]
\[
\leq \left\{ \int_{h_n}^{(n+1)h_n} \left| \frac{d}{dt} \left( \Phi_x f (\delta_n, t) \sin \frac{\alpha t}{t^2} \right) \right|^{q'} dt \right\}^{1/p'}
\]
\[
\leq \frac{1}{(n+1)^{1/q'+1}} \left\{ \int_{h_n}^{(n+1)h_n} \left( \frac{1}{\delta_n} \left| \varphi_x (\delta_n + t) - \varphi_x (t) \right| \sin \frac{\pi t}{t^2} \right)^{p'} dt \right\}^{1/p'} + |\Phi_x f (\delta, t)| \left( \frac{\pi t^2 \cos \frac{\pi t}{t^4} - 2t \sin \frac{\pi t}{t^4}}{t^4} \right)^{p'} dt \right\}^{1/p'}
\]

\[
\ll \frac{1}{(n+1)^{1/q'+1}} \left[ \int_{h_n}^{(n+1)h_n} \left( \frac{1}{\delta_n} \left| \varphi_x (\delta_n + t) - \varphi_x (t) \right| \right)^{p'} dt \right]^{1/p'} + \left\{ \int_{h_n}^{(n+1)h_n} \left( w_x (\delta_n) \right)^{p'} dt \right\}^{1/p'}
\]

\[
\leq \frac{1}{(n+1)^{1/q'+1}} \left[ \frac{1}{\delta_n} \left\{ \frac{\alpha}{2\pi} \int_{0}^{(n+1)h_n} \left| \varphi_x (\delta_n + u) - \varphi_x (u) \right|^{p'} du \right\}^{(n+1)h_n} + \frac{\alpha(n+1)}{2\pi} \int_{0}^{h_n} \left| \varphi_x (\delta_n + u) - \varphi_x (u) \right|^{p'} du \right\}^{1/p'}
\]

\[
+ \frac{1}{\delta_n} \left\{ \int_{h_n}^{(n+1)h_n} t^{-p'} (w_x (\delta_n))^{p'} dt \right\}^{1/p'}
\]
\[ \ll \frac{1}{(n+1)^{1/q'+1}} \left[ \frac{1}{\delta_n} w_x (\delta_n) + \frac{1}{\delta_n} w_x (\delta_n) (n+1)^{1-1/p'} \right. \\
+ w_x (\delta_n) (n+1)^{2-1/p'} + (n+1) w_x \left( \frac{\pi}{n+1} \right) (n+1)^{1-1/p'} \right] \\
\ll w_x \left( \frac{\pi}{n+1} \right). \]

For the third term we obtain

\[ \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{31}(k)|^{q'} \right\}^{1/q'} \]

\[ \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \sum_{\mu=1}^{\infty} \int_{(n+1)h_{n,\mu}}^{(n+1)h_{n,\mu+1}} |\varphi_x (t) - \Phi_x f (\delta_k, t)| \Psi_{k+\kappa} (t) \, dt \right\}^{q'/q'} \]

\[ + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \sum_{\mu=1}^{\infty} \int_{(n+1)h_{n,\mu}}^{(n+1)h_{n,\mu+1}} \Phi_x f (\delta_k, t) \Psi_{k+\kappa} (t) \, dt \right\}^{q'/q'} \]

\[ = \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{31}(k)|^{q'} \right\}^{1/q'} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{32}(k)|^{q'} \right\}^{1/q'} \]

and

\[ |I_{31}(k)| \leq \frac{4}{\alpha \pi} \sum_{\mu=1}^{\infty} \int_{(n+1)h_{n,\mu}}^{(n+1)h_{n,\mu+1}} |\varphi_x (t) - \Phi_x f (\delta_k, t)| t^{-2} \, dt \]

\[ \leq \frac{4}{\alpha \pi} \sum_{\mu=1}^{\infty} \int_{(n+1)h_{n,\mu}}^{(n+1)h_{n,\mu+1}} \left[ \frac{1}{\delta_k t^2} \int_0^{\delta_k} |\varphi_x (t) - \varphi_x (t+u)| \, du \right] \, dt \]

\[ = \frac{4}{\alpha \pi} \frac{1}{\delta_k} \int_0^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \int_{(n+1)h_{n,\mu}}^{(n+1)h_{n,\mu+1}} \frac{1}{t^2} |\varphi_x (t) - \varphi_x (t+u)| \, dt \right\} \, du \]

\[ = \frac{4}{\alpha \pi} \frac{1}{\delta_k} \int_0^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \int_0^t \left[ \int_0^{t-s} |\varphi_x (s) - \varphi_x (s+u)| \, ds \right] \right\} \, du \]

\[ + 2 \int_{(n+1)h_{n,\mu}}^{(n+1)h_{n,\mu+1}} \left[ \frac{1}{t^2} \int_0^t |\varphi_x (s) - \varphi_x (s+u)| \, ds \right] \, dt \, du \]
Next, we will estimate the term $f_\delta$ and therefore $\zeta$. Since \[
\lim_{\zeta \to \infty} \frac{1}{\zeta^2} \int_0^\zeta |\varphi_x (s) - \varphi_x (s + u)| \, ds \leq \lim_{\zeta \to \infty} \frac{1}{\zeta^2} w_x (u) \leq \lim_{\zeta \to \infty} \frac{1}{\zeta} w_x (\delta_k) \leq \lim_{\zeta \to \infty} \frac{1}{\zeta} w_x (\pi) = 0,
\]
and therefore
\[
|I_{31}(k)| \leq \frac{1}{\delta_k} \int_0^{\delta_k} \frac{\alpha}{2\pi} \left[ \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} |\varphi_x (s) - \varphi_x (s + u)| \, ds \right] du \quad \text{and}
+ \frac{1}{\delta_k} \int_0^{\delta_k} w_x (u) du \sum_{\mu=1}^\infty \left[ \int_0^{\infty} \left( \int_0^{t} |\varphi_x (s) - \varphi_x (s + u)| \, ds \right) dt \right] du
\]
\[
\ll \frac{1}{\delta_k} \int_0^{\delta_k} w_x (u) du + w_x (\delta_k) \sum_{\mu=1}^\infty \frac{1}{(n+1)h_n \mu^2}
\]
\[
\ll w_x (\delta_k).
\]

Next, we will estimate the term $|I_{32}(k)|$. So,
\[
I_{32}(k) = \frac{2}{\alpha \pi} \sum_{\mu=1}^\infty \int_0^{(n+1)h_n \mu} \Phi_x f (\delta_k, t) \frac{d}{dt} \left( \frac{\cos \frac{\alpha t (k+\kappa)}{2}}{\frac{\alpha (k+\kappa)}{2}} + \frac{\cos \frac{\alpha t (k+\kappa+1)}{2}}{\frac{\alpha (k+\kappa+1)}{2}} \right) dt
\]
\[
= \frac{2}{\alpha \pi} \sum_{\mu=1}^\infty \int_0^{(n+1)h_n \mu} \Phi_x f (\delta_k, t) \frac{d}{dt} \left( \frac{\cos \frac{\alpha t (k+\kappa)}{2}}{\frac{\alpha (k+\kappa)}{2}} + \frac{\cos \frac{\alpha t (k+\kappa+1)}{2}}{\frac{\alpha (k+\kappa+1)}{2}} \right) dt
\]
\[
+ \frac{2}{\alpha \pi} \sum_{\mu=1}^\infty \int_0^{(n+1)h_n \mu} \frac{d}{dt} \left( \frac{\Phi_x f (\delta_k, t)}{t^2} \right) \left( \frac{\cos \frac{\alpha t (k+\kappa)}{2}}{\frac{\alpha (k+\kappa)}{2}} - \frac{\cos \frac{\alpha t (k+\kappa+1)}{2}}{\frac{\alpha (k+\kappa+1)}{2}} \right) dt
\]
\[
= I_{321}(k) + I_{322}(k)
\]

Since $f \in \Omega_{\alpha,p}$, thus for $x$ (using (7))
\[
\lim_{\zeta \to \infty} \left| \frac{\Phi_x f (\delta_k, \frac{2\pi}{\alpha} \zeta)}{2\pi \alpha \zeta^2} \right| \leq \lim_{\zeta \to \infty} \frac{w_x (\delta_k) + w_x (\frac{2\pi}{\alpha} \zeta)}{\zeta^2} \ll \lim_{\zeta \to \infty} \frac{w_x (\delta_k) + \zeta w_x (\frac{2\pi}{\alpha})}{\zeta^2 k} \ll \frac{w_x (\pi)}{\zeta^2} \lim_{\zeta \to \infty} \frac{1 + \zeta}{\zeta^2} = 0,
\]
\[
13
\]
and therefore

\[
I_{321} (k) = \frac{2}{\alpha \pi} \sum_{\mu = 1}^{\infty} \left[ \frac{\Phi_x f \left( \delta_k, \frac{2 \pi}{\alpha} \right)}{\left[ \frac{2 \pi}{\alpha} \right]^2} \left( - \cos \left[ \frac{\pi (\mu + 1) (k + \kappa)}{\alpha (k + \kappa + 1)} \right] \right. \right.
\]

\[
+ \cos \left[ \frac{\pi (\mu + 1) (k + \kappa + 1)}{\alpha (k + \kappa + 1)} \right) \right]
\]

\[
- \frac{\Phi_x f \left( \delta_k, \frac{2 \pi}{\alpha} \right)}{\left[ \frac{2 \pi}{\alpha} \right]^2} \left( - \cos \left[ \frac{\pi \mu (k + \kappa)}{\alpha (k + \kappa + 1)} \right] + \cos \left[ \frac{\pi \mu (k + \kappa + 1)}{\alpha (k + \kappa + 1)} \right] \right) \right] \]

\[
= - \frac{2}{\alpha \pi} \frac{\Phi_x f \left( \delta_k, \frac{2 \pi}{\alpha} \right)}{\left[ 2 \pi / \alpha \right]^2} \left( - \frac{(-1)^{(k+\kappa)}}{\alpha (k+\kappa+1)} + \frac{(-1)^{(k+\kappa+1)}}{\alpha (k+\kappa+1)} \right) \]

\[
= - \frac{1}{\pi^3} \Phi_x f \left( \delta_k, \frac{2 \pi}{\alpha} \right) (-1)^{(k+\kappa+1)} \left( \frac{1}{k+\kappa+1} + \frac{1}{k+\kappa} \right). \]

Using (7), we get

\[
|I_{321} (k)| \ll \frac{1}{\pi^3 \left( k + 1 \right)} |\Phi_x f \left( \delta_k, 2 \pi / \alpha \right)| \leq \frac{2}{\pi^3 \left( k + 1 \right)} (w_x (\delta_k) + w_x (2 \pi / \alpha)). \]

Similarly

\[
I_{322} (k) = \frac{2}{\alpha \pi} \sum_{\mu = 1}^{\infty} \int_{(n+1)h_n \mu}^{(n+1)h_n \mu+1} \left( \frac{4 \Phi_x f \left( \delta_k, t \right)}{t^2} - \frac{2 \Phi_x f \left( \delta_k, t \right)}{t^3} \right)
\]

\[
\cdot \left( \cos \frac{\alpha (k+\kappa)}{2} + \cos \frac{\alpha (k+\kappa+1)}{2} \right) dt \]

and

\[
|I_{322} (k)| \ll \frac{8}{\alpha^2 (k + 1) \pi} \sum_{\mu = 1}^{\infty} \left[ \int_{(n+1)h_n \mu}^{(n+1)h_n \mu+1} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{\delta_k t^2} dt \right]
\]

\[
+ 2 \int_{(n+1)h_n \mu}^{(n+1)h_n \mu+1} \frac{|\Phi_x f \left( \delta_k, t \right)|}{t^3} dt \]

\[
\leq \frac{8}{\alpha^2 (k + 1) \pi \delta_k} \sum_{\mu = 1}^{\infty} \int_{(n+1)h_n \mu}^{(n+1)h_n \mu+1} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{t^2} dt
\]

\[
+ \frac{16}{\alpha^2 (k + 1) \pi} \sum_{\mu = 1}^{\infty} \int_{(n+1)h_n \mu}^{(n+1)h_n \mu+1} \frac{w_x (\delta_k) + w_x (t)}{t^3} dt \]

14
\[ \ll \frac{1}{(k+1)\delta_k} w_x(\delta_k) + \frac{1}{k+1} \sum_{\mu=1}^{\infty} \left( w_x(\delta_k) + w_x \left( \frac{2\pi (\mu+1)}{\alpha} \right) \right) \frac{\alpha^2}{4\pi^2 \mu^3} \]

\[ \ll w_x(\delta_k) + \frac{1}{k+1} \left[ w_x(\delta_k) \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} + \sum_{\mu=1}^{\infty} w_x \left( \frac{2\pi (\mu+1)}{\alpha} \right) \right] \]

\[ \ll w_x(\delta_k) + \frac{1}{k+1} \left( w_x(\delta_k) + w_x \left( \frac{4\pi}{\alpha} \right) \sum_{\mu=1}^{\infty} \frac{\mu+1}{\mu^3} \right) \]

\[ \ll w_x(\delta_k) + \frac{1}{k+1} \left( w_x(\delta_k) + w_x \left( \frac{4\pi}{\alpha} \right) \right) . \]

Therefore

\[ |I_3(k)| \ll w_x(\delta_k) + \frac{1}{k+1} \left( w_x(\delta_k) + w_x \left( \frac{2\pi}{\alpha} \right) + w_x \left( \frac{4\pi}{\alpha} \right) \right) \]

and thus

\[ \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_3(k)|^q \right\}^{1/q} \ll \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( w_x \left( \frac{\pi}{k+1} \right) + \frac{1}{k+1} w_x \left( \frac{\pi}{\alpha} \right) \right)^q \right\}^{1/q} \]

\[ \ll \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( w_x \left( \frac{\pi}{k+1} \right) \right)^q \right\}^{1/q} \leq w_x \left( \frac{\pi}{n+1} \right) . \]

and the desired result follows. □

### 3.2 Proof of Theorem 5

For some \( c > 1 \)

\[ H_{n, A, \gamma}^q f(x) = \left\{ \sum_{k=0}^{2^{|c|}-1} a_{n,k} \left| S_{\frac{k}{2^c}} f(x) - f(x) \right|^q \right\}^{1/q} + \sum_{c=|c|}^{\infty} \left\{ \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \left| S_{\frac{k}{2^c}} f(x) - f(x) \right|^q \right\}^{1/q} \]

\[ \ll \left\{ \sum_{k=0}^{2^{|c|}-1} a_{n,k} \left| S_{\frac{k}{2^c}} f(x) - f(x) \right|^q \right\}^{1/q} + \left\{ \sum_{m=|c|}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \left| S_{\frac{k}{2^c}} f(x) - f(x) \right|^q \right\}^{1/q} \]

\[ = I_1(x) + I_2(x) . \]
Using Proposition 4 and denoting the left hand side of the inequality from its
by \( F_n \), i.e. \( F_n = w_x \left( \frac{x}{n+1} \right) + E_{an/2} (f) \), we get

\[
I_1 (x) \leq \left\{ \sum_{k=0}^{2^{|c|}-1} a_{n,k} \frac{k/2 + 1}{k/2 + 1} \sum_{l=k/2}^{k} \left| S_{\frac{l}{k/2}} f (x) - f (x) \right| \right\}^{1/q}
\]

\[
= \left\{ \sum_{k=0}^{2^{|c|}-1} a_{n,k} \frac{1}{k/2 + 1} \sum_{l=k/2}^{k} \left| S_{\frac{l}{k/2}} f (x) - f (x) \right| \right\}^{1/q}
\]

\[
\leq \left\{ \sum_{k=0}^{2^{|c|}-1} a_{n,k} F_{k/2} \right\}^{1/q} .
\]

By partial summation, our Proposition 4 gives

\[
I_2 (x) = \sum_{m=0}^{\infty} \left[ \sum_{k=2^m}^{2^{m+1}-2} (a_{n,k} - a_{n,k+1}) \sum_{l=2^m}^{k} \left| S_{\frac{l}{k/2}} f (x) - f (x) \right|^q \right]
\]

\[
+ a_{n,2^{m+1}-1} \sum_{l=2^m}^{2^{m+1}-1} \left| S_{\frac{l}{k/2}} f (x) - f (x) \right|^q
\]

\[
\leq \sum_{m=0}^{\infty} \left[ \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| F_{k/2} \alpha_{2^m/2} \right]
\]

\[
+ 2^m a_{n,2^{m+1}-1} F_{k/2} \alpha_{2^m/2}
\]

\[
= \sum_{m=0}^{\infty} 2^m F_{\alpha_{2^m/2}} \left[ \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| + a_{n,2^{m+1}-1} \right] .
\]

Since (6) holds, we have

\[
|a_{n,s+1} - a_{n,r}|
\]

\[
\leq |a_{n,r} - a_{n,s+1}| \leq \sum_{k=r}^{s} |a_{n,k} - a_{n,k+1}|
\]

\[
\leq \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| \ll \sum_{k=2^m}^{2^{m+1}-2} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2) ,
\]

whence

\[
a_{n,s+1} \ll a_{n,r} + \sum_{k=2^m}^{2^{m+1}-2} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2)
\]
and
\[ 2^m a_{n,2^{m+1}-1} = \frac{2^m}{2^{m-1}} \sum_{r=2^m}^{2^{m+1}-2} a_{n,2^{m+1}-1} \]
\[ \ll \sum_{r=2^m}^{2^{m+1}-2} \left( a_{n,r} + \sum_{k=\left\lceil 2^m/c \right\rceil}^{\left\lceil 2^m \right\rceil} \frac{a_{n,k}}{k} \right) \]
\[ \ll \sum_{r=2^m}^{2^{m+1}-1} a_{n,r} + 2^m \sum_{k=\left\lceil 2^m/c \right\rceil}^{\left\lceil 2^m \right\rceil} \frac{a_{n,k}}{k}. \]

Thus
\[ I_{2}^{q} (x) \ll \sum_{m=\left\lceil c \right\rceil}^{\infty} \left\{ \frac{2^m F_{\alpha 2^m/2} q}{q^{\alpha 2^m/2}} \sum_{k=\left\lceil 2^m/c \right\rceil}^{\left\lceil 2^m \right\rceil} \frac{a_{n,k}}{k} + F_{\alpha 2^m/2} q^{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\}. \]

Finally, by elementary calculations we get
\[ I_{2}^{q} (x) \ll \sum_{m=\left\lceil c \right\rceil}^{\infty} \left\{ \frac{2^m F_{\alpha 2^m/2} q}{q^{\alpha 2^m/2}} \sum_{k=\left\lceil 2^m/c \right\rceil}^{\left\lceil 2^m \right\rceil} \frac{a_{n,k}}{k} + F_{\alpha 2^m/2} q^{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\} \]
\[ \ll \sum_{m=\left\lceil c \right\rceil}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \frac{F_{\alpha k/2}}{2^{m+1}} + \sum_{m=\left\lceil c \right\rceil}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \frac{F_{\alpha k/2}}{2^{m+1}} \]
\[ = \sum_{m=\left\lceil c \right\rceil}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \frac{F_{\alpha k/2}}{2^{m+1}} + \sum_{m=\left\lceil c \right\rceil}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \frac{F_{\alpha k/2}}{2^{m+1}} \]
\[ + \sum_{m=\left\lceil c \right\rceil}^{\infty} \sum_{r=1}^{\left\lceil c \right\rceil} \sum_{k=2^{m-r}}^{2^{m-1}} a_{n,k} \frac{F_{\alpha k/2}}{2^{m+1}} + \sum_{m=\left\lceil c \right\rceil}^{\infty} \sum_{r=0}^{\left\lceil c \right\rceil-1} \sum_{k=2^{m+r}}^{2^{m+1}} a_{n,k} \frac{F_{\alpha k/2}}{2^{m+1}} \]
\[ \leq \sum_{r=1}^{[c]} \sum_{k=2^{[r]}}^\infty a_{n,k}F_{\alpha k/2}^q + \sum_{r=0}^{[c]-1} \sum_{k=2^{[r]+[r]}}^\infty a_{n,k}\frac{F_{\alpha k}^q}{2^{[r]+[r]}} + \sum_{k=2^{[c]}}^\infty a_{n,k}\frac{F_{\alpha k}^q}{2^{[r]+[r]}} \]

\[ \ll \sum_{k=0}^\infty a_{n,k}\frac{F_{\alpha k}^q}{2^{[r]+[r]}}. \]

Thus we obtain the desired result. \(\square\)

### 3.3 Proof of Theorem 6

If \((a_{n,k})_{k=0}^\infty \in MS\) then \((a_{n,k})_{k=0}^\infty \in GM (\gamma \beta)\) and using Theorem 5 we obtain

\[ H_{n,A,\gamma}^q f(x) \leq \left\{ \sum_{k=0}^\infty a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} \]

\[ + \left\{ \sum_{k=0}^{(k+1)2^{[c]}-1} \sum_{m=k2^{[c]}}^\infty a_{n,m} \left[ E_{\alpha m} (f)_{S_p} \right]^q \right\}^{1/q} \]

\[ \leq \left\{ \sum_{k=0}^\infty a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} \]

\[ + \left\{ \sum_{k=0}^{(k+1)2^{[c]}-1} \sum_{m=k2^{[c]}}^\infty a_{n,m} \left[ E_{\alpha m} (f)_{S_p} \right]^q \right\}^{1/q} \]

\[ \ll \left\{ \sum_{k=0}^\infty a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) + E_{\alpha k} (f)_{S_p} \right]^q \right\}^{1/q} \]

This ends our proof. \(\square\)

### References

[1] A. Avantaggiati, G. Bruno and B. Iannacci, The Hausdorff-Young theorem for almost periodic functions and some applications, Nonlinear analysis, Theory, Methods and Applications, Vol. 25, No. 1, (1995), pp. 61-87.
[2] A. D. Bailey, Almost Everywhere Convergence of Dyadic Partial Sums of Fourier Series for Almost Periodic Functions, Master of Philosophy, A thesis submitted to School of Mathematics of The University of Birmingham for the degree of Master of Philosophy, September 2008.

[3] A. S. Besicovitch, Almost periodic functions, Cambridge, 1932.

[4] L Leindler, On the uniform convergence and boundedness of a certain class of sine series, Analysis Math., 27 (2001), 279-285.

[5] L. Leindler, A new extension of monotone sequence and its application, J. Ineq. Pure and Appl. Math., 7(1) (2006), Art. 39, 7 pp.

[6] W. Lenski, Pointwise strong and very strong approximation of Fourier series, Acta Math. Hung., 115(3), 207, p.215-233.

[7] W. Lenski and B. Szal, Strong approximation of almost periodic functions, submitted.

[8] B. L. Levitan, Almost periodic functions, Gos. Izdat. Tekh-Teoret. Liter., Moscov 1953 (in Russian).

[9] P. Pych-Taberska, Approximation properties of the partial sums of Fourier series of almost periodic functions, Studia Math. XCVI (1990), 91-103.

[10] S. Tikhonov, Trigonometric series with general monotone coefficients, J. Math. Anal. Appl., 326(1) (2007), 721-735.

[11] S. Tikhonov, On uniform convergence of trigonometric series. Mat. Zametki, 81(2) (2007), 304-310, translation in Math. Notes, 81(2) (2007), 268-274.

[12] S. Tikhonov, Best approximation and moduli of smoothness: Computation and equivalence theorems, J. Approx. Theory, 153 (2008), 19-39.

[13] A. Zygmund, Trigonometric series, Cambridge, 2002.e, 2002.