Vacuum Effective Action: Semiclassical Approach

Ilya L. Shapiro
Departamento de Física, ICE, Universidade Federal de Juiz de Fora, MG, Brazil
E-mail: shapiro@fisica.ufjf.br

Abstract. We present a brief review of quantum corrections to the action of gravity. The main attention is concentrated on the quantum theory of matter fields (QFT) on classical metric background. The list of most interesting possible applications of quantum corrections includes inflation and the Dark Energy problem. We show that both problems can be, in principle, resolved within the semiclassical theory, without invoking quantum gravity or string theory.

1. Introduction
A standard assumption in the modern gravitational physics is that the gravity action includes the Einstein-Hilbert term. The existence of the nonzero cosmological constant does not contradict any known principle and therefore the corresponding term can be also included into the Lagrangian, so that the action is

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{\tilde{g}} \left\{ R + 2\Lambda \right\} .$$

(1)

The theory based on this action passed almost all known experimental and observational tests. The main conceptual problem is the existence of singularities in the most important solutions such as cosmological and black hole ones. In the vicinity of singularity one meets energy densities and curvature tensor components of the Planck order of magnitude. One can interpret this fact such that in the corresponding regions of the space-time manifold the gravitational theory has to be modified, taking quantum corrections into account. These corrections may come from quantum gravity, supergravity or from superstring theory. In any case we have to assume their universal nature, that means gravitational action should be actually different from eq. (1) everywhere but, far from singularities the effect of extra terms should be negligible.

As far as we do not know the nature or origin of extra terms in the gravitational action, it is worthwhile to start from the most simple source of possible corrections to the gravitational action. The success of the Standard Model (SM) of particle physics shows the correctness of the quantum field theory (QFT) description of the particles interactions. One of the most important aspects of QFT is a complicated vacuum structure, with the possibility of particle creation and relevant effects of the virtual loops of matter fields. It is obvious that the vacuum quantum corrections of matter fields can modify the gravitational action. Let us emphasize that in this case we are dealing not with a kind of qualitatively new unknown physics, rather it is a relatively well known physics in a more complicated environment, that is in an external gravitational field.

1 On leave from Tomsk State Pedagogical University, Tomsk, Russia
Looking from this perspective, the most natural question is: - in which way vacuum quantum effects of matter fields do contribute to the gravitational action?

Other related questions are:
• Whether the corrections from quantum matter fields are relevant?
• How to evaluate quantum corrections?
• Should we quantize gravity?
In the present review we shall deal with the first two items, the last one will be discussed in the consequent extended version of this review.

2. Formulation of QFT in curved space: vacuum action

The introduction to QFT in curved space can be found in many books and reviews, e.g. in [1, 2, 3]. Here we just mention the most fundamental aspects of the theory.

The first step is to consistently formulate quantum theory of matter on classical curved background. The qualitatively new element is the action of vacuum, that is of an external gravitational field. The standard criteria for the action of external field are locality of the vacuum action, renormalizability and simplicity, that means one does not introduce more vacuum terms than necessary. For example, this means we can construct the vacuum actions without parameters with the inverse mass dimensions. In this way we arrive at the action of vacuum

\[ S_{vac} = S_{EH} + S_{HD}, \quad (2) \]

\[ S_{HD} = \int d^4x \sqrt{-g} \left\{ a_1 C^2 + a_2 E + a_3 \nabla^2 R + a_4 R^2 \right\}, \quad (3) \]

where \( C^2 = R_{\mu\nu\alpha\beta}^2 - 2R_{\alpha\beta}^2 + (1/3) R^2 \) is square of the Weyl tensor and \( E = R_{\mu\nu\alpha\beta}^2 - 4R_{\alpha\beta}^2 + R^2 \) is the integrand of the Gauss-Bonnet topological term. Let us remark that the presence of higher derivative terms is unavoidable in a renormalizable theory. The same concerns the cosmological term, especially in case matter fields are massive\(^2\).

The higher derivative terms in (2) are not quantum corrections, they should be introduced already at the classical level. The reason why they are not observed in the gravitational experiments is that the lower derivative Einstein-Hilbert term has coefficient \( 1/16\pi G = M_P^2 \). In order to understand the role of this coefficient, one can use the language of Feynman diagrams. For example, the Newton law is the consequence of one-graviton exchange between the two masses. Then, in order to produce a relevant impact on the Newton law, an additional degree of freedom associated with higher derivatives (see, e.g. [3]), the energy of graviton should be comparable to the Planck mass.

3. Quantum theory: vacuum Effective Action

Now we are in a position to deal with the main subject of our review. At the quantum level the classical action of vacuum (2) is replaced by the Effective Action (EA), \( \Gamma[g_{\mu\nu}] \), which may be defined via path integral

\[ e^{i\Gamma[g_{\mu\nu}]} = \int \mathcal{D}\phi e^{iS[\phi;g_{\mu\nu}]} . \quad (4) \]

Here \( \phi \) is the set of all matter fields and gauge ghosts, \( \mathcal{D}\phi \) is the covariant measure of functional integration. The classical action \( S[\phi; g_{\mu\nu}] \) describes matter fields, interactions between these fields, it depends on the metric (which plays the role of external parameter) and includes the action of vacuum (2).

\(^2\) On the top of that, in order to provide renormalizability in the interacting theory with scalar fields \( \varphi^k \), one has to introduce the nonminimal term \( \xi_{kl} R \varphi^k \varphi^l \), where \( \xi_{kl} \) are nonminimal parameters.
It is amusing that already at this level one can make some strong statements about possible and impossible form of quantum corrections. The EA $\Gamma[g_{\mu\nu}]$ is a well-defined diffeomorphism invariant quantity. As a consequence, $\Gamma[g_{\mu\nu}]$ can not include odd powers of the metric derivatives. Let us emphasize that this property is not related to the perturbative expansion and is valid independent on whether the EA is a local functional of the metric (indeed, it is nonlocal, as we shall discuss below). This important property of EA holds for any particular metrics, including the cosmological one. For instance, in case one detects, someday, the odd-power behaviour in the gravitational solutions, this would be an indication to a certain “new physics”, e.g. quintessence, extra dimensions, branes etc. However, it can’t be a vacuum quantum effect of known fields on purely metric background. An interesting application to cosmology is that the quantum corrections to the cosmological constant in the late universe, without scalar fields, may start from $H^2$ (here $H$ is the Hubble parameter in the late epoch Universe), but not from $H$, because this would mean an odd metric derivative. In particular, this rules out the hypothetic QCD contributions to the vacuum energy suggested in [4].

Which kind of fundamental physics may be relevant for the possible scale dependence of the vacuum energy in the present day universe? The arguments presented above lead to the relation

$$\rho_\Lambda(\text{observable}) \propto H_0^2 M_{\text{Planck}}^2,$$

where $H_0$ is the present-day Hubble parameter. As a consequence, any sort of quantum physics below the GUT scale should be irrelevant in the case of $\rho_\Lambda$. Indeed, there is no guarantee that the quadratic in $H_0$ term will be present. In this case the correction may start from higher powers of $H_0$ or even be independent on $H_0$. One can find the models of this kind in the literature [5, 6] (see also [7]).

The EA of gravity $\Gamma[g_{\mu\nu}]$ admits a loop expansion

$$\Gamma[g_{\mu\nu}] = S_{\text{vac}}[g_{\mu\nu}] + \bar{\Gamma}^{(1)} + \bar{\Gamma}^{(2)} + \bar{\Gamma}^{(3)} + \ldots,$$

where $\bar{\Gamma}^{(i)} = \frac{i}{2} \text{Tr} \ln \hat{H}$, where $\hat{H}$ is the bilinear in quantum fields part of the classical action.

The simplest and usually most important 1-loop contribution is given by the expression $\bar{\Gamma}^{(1)} = \frac{i}{2} \text{Tr} \ln \hat{H}$, where $\hat{H}$ is the bilinear in quantum fields part of the classical action.

The covariance of EA can be established using the same methods which are used for other gauge interactions. Furthermore, as we shall discuss, there are explicitly covariant calculational methods.

4. Calculational methods in curved spaces

In this section we shall discuss the existing practical methods of quantum calculations in curved spaces and their relevance for establishing the general features of renormalization and exploring the structure of finite part of EA.

- Feynman diagrams for the perturbations on flat background $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Very important early works in this area has been done using this method [8, 9]. In particular, these calculations have shown, for the first time, the necessity of higher derivative terms (3) for renormalizability and the general structure of finite quantum corrections, for both massive and massless cases. One of the advantages of using a standard usual QFT in flat space is that the locality of the counterterms is guaranteed. The shortcomings of this method are the lack of explicit covariance, difficulties in practical calculations and interpreting the results.

- One of the known covariant methods is based on the local momentum representation [10]. In general, the use of momentum representation is not possible in curved spaces, however one can use normal coordinates (see, e.g. [11] for the introduction) and perform loop calculations in the tangent space corresponding to a given point $P(x^\mu_{(0)})$. One can express all quantities of interest, such as free and interaction parts of the classical Lagrangian in form of the power series.
in $y^\mu = x^\mu - x_0^\mu$. The coefficients of this series are components of curvature tensor and its covariant derivatives in the point $P$. The local momentum representation can be used for both propagator and vertices and finally enables one to derive local quantities such as counterterms.

For instance the first order expansion for the propagator of a massive non-minimal scalar field has the form

$$G(k) = \frac{1}{k^2 + m^2} - \frac{1}{3} \frac{(1 - 3 \xi)R}{(k^2 + m^2)^2} + \frac{2}{3} \frac{R^{\mu\nu} k_\mu k_\nu}{(k^2 + m^2)^3} + O(k^{-3}).$$

Similar expansions can be obtained for spinor and vector fields. It is obvious that after replacing such propagator into the Feynman diagrams, the divergences will be covariant. One can combine this method and flat diagrams to establish the structure of renormalization is curved space.

- Schwinger-DeWitt expansion is the most useful method for deriving one-loop divergences and related quantities. The key point is the representation of the $\ln \text{Det} \hat{H}$ via the proper time integral

$$\frac{i}{2} \text{Tr} \ln \hat{H} = -\frac{i}{2} \text{Tr} \int_0^\infty \frac{ds}{s} e^{-is \hat{H}},$$

where $e^{-is \hat{H}} = \hat{U}_0(x, x'; s) \sum_{k=0}^\infty (is)^k \hat{a}_k(x, x')$ are Schwinger-DeWitt coefficients. The expression for $\hat{U}_0(x, x'; s)$ can be found in [1]. A very powerful generalization of this method has been developed in [12] (see also references therein).

For the quantum theories in four space-time dimensions the most important is the “magic” coefficient $a_2 = \text{Tr} \hat{a}_2(x, x')$, for it defines the logarithmic divergences and also related notions such as renormalization group $\beta$-functions, anomalies etc. The general structure of $a_2$ in the vacuum sector is

$$a_2 = \int d^4x \sqrt{g} \left\{ \beta_\lambda + \beta_R R + \beta_1 C^2 + \beta_2 E + \beta_3 \nabla^2 R + \beta_4 R^2 \right\}.$$

An important and relatively recent aspect of Schwinger-DeWitt technique is the practical possibility of resummation of the series for the heat kernel. One can perform such resummations in different manners (see, e.g. [13]), but the most relevant result has been achieved in [14]. The output of this approach are the terms of the lowest order in curvature tensor, but are exact in the derivatives of these components. In order to understand the importance of this result, let us remember that the second order in curvatures corresponds to the second order in $h_{\mu\nu}$ or, in other words, to the contributions to the propagator of the gravitational perturbations on flat background. Thus, the heat kernel solution opens the way to derive the quantum corrections to such important quantities as gravitational wave equation or week interaction between two matter sources.

The quantum contribution of the mentioned type coming from the single massive scalar has been derived in [15] using Feynman diagrams in the $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ approach and also, independently, from the heat kernel solution of [14]. The result has the form

$$\Gamma^{(1)}_{\text{scalar}} = \frac{1}{2(4\pi)^n} \int d^4x g^{1/2} \left\{ \frac{m^4}{2} \cdot \left( \frac{1}{\epsilon} + \frac{3}{2} \right) + \left( \xi - \frac{1}{6} \right) m^2 R \left( \frac{1}{\epsilon} + 1 \right) + \frac{1}{2} C_{\mu\nu\alpha\beta} \left( \frac{1}{160 \epsilon} + k_{W(a)} \right) C^{\mu\nu\alpha\beta} + R \left[ \frac{1}{2\epsilon} \left( \xi - \frac{1}{6} \right)^2 + k_{R(a)} \right] R \right\},$$

where $\epsilon$ is the parameter of dimensional regularization

$$\frac{1}{\epsilon} = \frac{1}{2 - \omega} + \ln \left( \frac{4\pi \mu^2}{m^2} \right) - \gamma.$$
The formfactors (nonlocal insertions produced by quantum corrections) have the form

\begin{align}
 k_W(a) &= \frac{8A}{15a^4} + \frac{2}{45a^2} + \frac{1}{150}, \\
 k_R(a) &= A \left( \xi - \frac{1}{6} \right)^2 - \frac{A}{6} \left( \xi - \frac{1}{6} \right) + \frac{2A}{3a^2} \left( \xi - \frac{1}{6} \right) + \frac{A}{9a^4} - \frac{A}{18a^2} + \frac{A}{144} + \\
 &\quad + \frac{1}{108a^2} - \frac{7}{2160} + \frac{1}{18} \left( \xi - \frac{1}{6} \right),
\end{align}

(11)

where we used notations \( A = 1 - (1/a) \ln \left[ \left( 2 + a \right) / \left( 2 - a \right) \right] \), \( a^2 = 4\nabla^2 / (\nabla^2 - 4m^2) \). Similar expressions can be obtained for massive fermion and vector fields [16]. We shall use these results when discussing the difference between quantum effects of massive and massless fields, in the next sections.

- Calculations on special backgrounds, especially on the de Sitter space (dS) or Anti de Sitter (AdS) spaces, are very powerful in sense one can, sometimes, calculate the whole Effective Action. Unfortunately the information obtained in this way is restricted because the curvature is constant \((\text{AdS}) \) spaces, are very powerful in sense one can, sometimes, calculate the whole Effective Action. In particular, those terms which have singular insertions such as \((m^2 - \xi R)^{-1}\) or \(\ln (m^2 - \xi R)\) are likely to be ruled out.

The disadvantages of the calculations on dS-AdS backgrounds is that one can not see the non-localities of the EA and can not distinguish higher derivative terms and the Einstein-Hilbert and cosmological terms.

5. Vacuum EA for massless and massive fields

It is important to remember that the effective action of vacuum is essentially non-local object. In some cases we can derive the non-local terms, e.g. this is the case of eq. (10), but in general our possibilities are very restricted. Let us review the status of quantum corrections starting from the simplest case.

5.1. Massless conformal fields and anomaly

Consider a set of free massless conformal fields on a background \( g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma(x)} \), where we assume that the metric \( \bar{g}_{\mu\nu} \) is fixed. We shall look only for the \( \sigma\)-dependence of EA, hence the conformal anomaly can be the starting point. In case of conformal quantum matter only conformal invariant part of the vacuum action which requires renormalization. Without loosing generality we can assume that the vacuum action has the form

\[ S_{\text{vac}} = \int d^4x \sqrt{\bar{g}} \{ a_1 C^2 + a_2 E + a_3 \nabla^2 R \}. \]

(12)

The derivation of conformal anomaly and related issues has been recently reviewed in [19], so let us just give the result for the anomaly

\[ < T_{\mu}^\mu > = - \left( \beta_1 C^2 + \beta_2 E + \beta_3 \nabla^2 R \right). \]

(13)

The \( \beta_{1,2,3} \)-functions depend on the number of the fields of different spin

\[ \left( \begin{array}{c} \beta_1 \\ -\beta_2 \\ \beta_3 \end{array} \right) = \frac{1}{360(4\pi)^2} \left( \begin{array}{c} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{array} \right) \]

(14)
The anomaly-induced EA, \( \Gamma_{\text{ind}}[\bar{g}_{\mu\nu}], \) is defined as solution of the equation

\[
- \frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta \Gamma_{\text{ind}}}{\delta \bar{g}_{\mu\nu}} = T.
\]

An explicit solution has been found in [21], with \( \beta_{1,2,3} = (w, b, c) \)

\[
\Gamma_{\text{ind}} = S_c[\bar{g}_{\mu\nu}] + \int d^4 x \sqrt{-g} \left\{ w \sigma C^2 + b \sigma (E - \frac{2}{3} \nabla^2 \bar{R}) + 2b \sigma \Delta_4 \sigma \right\} - \frac{3c - 2b}{36} \int d^4 x \sqrt{-g} R^2,
\]

where \( \Delta_4 = \nabla^2 + 2R_{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \nabla^2 + \frac{1}{3} (\nabla^\mu R) \nabla_\mu \) is a fourth derivative conformal operator, while \( S_c[\bar{g}_{\mu\nu}] \) is an arbitrary conformal functional.

The last expression is equivalent to the covariant non-local solution [21]. Furthermore, the EA can be presented in the local form by using two auxiliary scalars [22]:

\[
\Gamma_{\text{ind}} = S_c[\bar{g}_{\mu\nu}] - \frac{3c + 2b}{36(4\pi)^2} \int d^4 x \sqrt{-g(x)} R^2(x) + \int d^4 x \sqrt{-g(x)} \left\{ \frac{1}{2} \varphi \Delta_4 \varphi - \frac{1}{2} \psi \Delta_4 \psi \right\}
\]

\[
+ \varphi \left[ \frac{\sqrt{6}}{8\pi} (E - \frac{2}{3} \nabla^2 R) - \frac{a}{8\pi \sqrt{b}} C^2 \right] + \frac{a}{8\pi \sqrt{b}} \psi C^2 \right\}. \quad (16)
\]

The importance of both scalars has been explained in [19]. In our opinion (16) is the most useful form of the anomaly induced EA. The arbitrariness related to \( S_c \) is irrelevant in the FRW-like spaces, where eq. (16) is an exact one-loop result.

Some of important application of the EA (16) are following:

- Classification of the vacuum states (Boulware, Hartle-Hawking and Unruh) of a black hole using the boundary conditions for the auxiliary scalar fields \( \varphi \) and \( \psi \) [24]. This result is quite natural, because it is well known that, e.g. the Hawking effect can be obtained from the conformal anomaly [26]. However, it is remarkable that one can perform classification of all three vacuum states using the EA method. Recently, this result has been generalized for the more complicated Reissner-Nordstrom spacetime [25].

- Another interesting application is the inflationary model based on the action

\[
S_{\text{total}} = -\frac{M_P^2}{16\pi} \int d^4 x \sqrt{-g} (R + 2\Lambda) + S_{\text{matter}} + S_{\text{vac}} + \Gamma_{\text{ind}}.
\]

Let us consider this model in some details. The equation of motion for \( a(t) = \exp(\sigma(t)), \)\n
\[
dt^2 = a(\eta) d\eta \] has the form

\[
\frac{\ddot{a}}{a} + 3\dot{a}^2 + \frac{\dot{a}^2}{a^2} - \left( 5 + \frac{4b}{c} \right) \frac{\dot{a}^2}{a^2} - \frac{M_P^2}{8\pi c} \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{2\Lambda}{3} \right) = 0,
\]

where we took \( k = 0 \) case only. The particular solution of Starobinsky [27] is

\[
a(t) = a_0 \exp(Ht),
\]

where Hubble parameter \( H = \dot{a}/a \) is

\[
H = \frac{M_P}{\sqrt{-32\pi b}} \left( 1 \pm \sqrt{1 + \frac{64\pi b \Lambda}{3M_P^2}} \right)^{1/2}.
\]

\[3\] An alternative but qualitatively similar representation has been suggested in [23].
For $0 < \Lambda \ll M_P^2$ one can distinguish two solutions

\begin{align}
\text{a) } H \approx \sqrt{\Lambda/3}; \quad \text{and} \quad \text{b) } H \approx \sqrt{-\frac{M_P^2}{16\pi b} - \frac{\Lambda}{3}} \approx \frac{M_P}{\sqrt{-16\pi b}}.
\end{align}

(21)

The first one corresponds to the late universe with dominating cosmological constant and is not affected by quantum corrections. The second solution is a very fast inflation driven by quantum correction. No inflaton is needed, its role is played by the massive high derivative mode of the field $\sigma(x)$. This solution is the cornerstone of the renowned Starobinsky model [27, 28, 29].

In order to better understand the status of this model, consider the perturbations of the conformal factor $\sigma(t) \rightarrow \sigma(t) + y(t)$. The criterion for a stable inflation is [27, 30]

\begin{align}
c > 0 \iff N_1 < \frac{1}{3} N_{1/2} + \frac{1}{18} N_0,
\end{align}

(22)

The original Starobinsky model deals with the unstable $c < 0$ case only. An alternative version of the Starobinsky model has been suggested in [30, 31]. Suppose we have a supersymmetric particle physics model in the UV limit ($H \gg M_F$). For instance, in case of Minimal Supersymmetric Standard Model the particle content is $(N_{1,1/2}, N_0) = (12, 32, 104)$. According to (22) this provides a stable inflation. Then inflation is independent on initial data.

On the other hand, at the lower energies we know SUSY is broken and, according to all available data, there is a Minimal Standard Model with $(N_{1,1/2}, N_0) = (12, 24, 4)$. In this case the inflation is unstable. The natural interpretation is that, in the course of inflation, the typical energy of the gravitational quanta decreases and at some point sparticles do decouple from gravity. The point of decoupling is $H = M_*$, where $M_* \sim M_{SUSY}$ means the scale where sufficient amount of sparticles decouple. One of the relevant questions is whether the decoupling really take place in the gravitational sector?

5.2. Decoupling theorem for gravity

The useful quantity to observe decoupling is the physical $\beta$-function. Let us start from the higher derivative parameters. Using the momentum-subtraction scheme and eq. (10), in case of scalar field and the coefficient of the Weyl term [15, 16]

\begin{align}
\beta_1 = -\frac{1}{(4\pi)^2} \left( \frac{1}{18a^2} - \frac{1}{180} - \frac{a^2 - 4}{6a^4} A \right).
\end{align}

(23)

The UV ($m \to 0$) and IR ($m \to \infty$) limits correspond to $a \to 2$ and $a \to 0$. Then we meet the limits

\begin{align}
\beta_1^{UV} = -\frac{1}{120(4\pi)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right) \quad \text{and} \quad \beta_1^{IR} = -\frac{1}{1680(4\pi)^2} \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right).
\end{align}

(24)

The last formula is nothing but the Appelquist & Carazzone decoupling theorem [32] for gravity. Our calculations [15] have shown it holds in all higher derivative sector. The change of sign of the coefficient $\beta_3 = c$ for the transition between supersymmetric and non-supersymmetric cases is smooth [16] and this strongly supports the scheme of anomaly-induced inflation.

It is easy to see that the decoupling theorem derived from the EA (10) is incomplete. The origin of the physical $\beta$-function (23) is the nontrivial momentum dependence of the corresponding formfactor in (10). Therefore, the momentum-independent formfactors will give zero $\beta$-functions for the cosmological (CC) and Newton $G$ constants. Why this happens? One possible answer is as follows. An expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ works well for higher
derivative terms, but not for $\Lambda$ and $G$. E.g. in the UV limit, running means the presence of a $f(\nabla^2) = \ln (\nabla^2/\mu^2)$-like formfactor. In QED, in the UV limit we meet the term

$$-\frac{e^2}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^4}{3(4\pi)^2} F_{\mu\nu} \ln \left( -\frac{\nabla^2}{\mu^2} \right) F^{\mu\nu}.$$ 

Similarly in gravity it is possible to insert $C_{\mu\nu\alpha\beta} f(\nabla^2) C_{\mu\nu\alpha\beta}$ or $R f(\nabla^2) R$ in the higher derivative sector. However, no insertion is possible for $\Lambda$ and $1/G$, since $\nabla^2 \Lambda = 0$ and $\nabla^2 R$ is a total derivative. In order to see $\beta_\Lambda$ and $\beta_{1/G}$, the calculations on a flat background are not appropriate. One has to perform calculations of some relevant observable at least on de Sitter or, better, on some dynamical background.

5.3. Light massive fields
The decoupling regime corresponds to the situation when the mass of the quantum field is much greater than the metric derivatives. Another interesting possibility is to evaluate the EA of massive but very light fields, when the mass can be seen as a kind of perturbation compared to the massless theory. The last can be chosen to possess a local conformal symmetry.

The conformal invariance of scalar and fermion actions is violated by the masses. Also the Einstein-Hilbert action is not conformal. Compared to the method presented in the section 5.1, one can not use the conformal anomaly to derive quantum corrections. But, this can be changed if we apply the conformal description of the massive theory in a way close to the cosmon model [33, 34].

Let us replace the dimensional parameters in both matter and gravitational sectors by the powers of a new auxiliary scalar field $\chi$ according to

$$m_s^2 \to \frac{m_s^2}{M^2} \chi^2 \quad \text{(scalar mass)}, \quad m_f \to \frac{m_f}{M} \chi \quad \text{(fermion mass)},$$

$$\frac{1}{16\pi G} R \to \frac{M_p^2}{16\pi M^2} \left(R \chi^2 + 6 (\partial \chi)^2 \right), \quad \Lambda \to \frac{\Lambda}{M^2} \chi^2. \quad (25)$$

Here $M$ is a new dimensional parameter. In order to provide the local conformal invariance we postulate that the auxiliary field $\chi$ transforms as $\chi \to e^{-\sigma} \chi$, $\sigma = \sigma(x)$. Under the procedure (25), the massive terms for the matter fields are replaced by Yukawa and quartic scalar interactions between physical fields and the new auxiliary scalar $\chi$. The new actions in both matter and gravitational sectors become conformal invariant. The classical Noether identity for the new vacuum action has the form

$$T = \left( -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{-g}} \chi \frac{\delta}{\delta \chi} \right) S_{\text{vac}}[g_{\mu\nu}, \chi] = 0. \quad (26)$$

The conformal invariance is violated by a conformal anomaly and one can derive the effective action of the background fields $g_{\mu\nu}, \chi$ using the anomaly-induced effective action scheme. The divergences of the theory in the conformal representation have the form

$$\Gamma^{(1)}_{\text{div}} = -\frac{\mu^{n-4}}{(n-4)} \int d^n x \sqrt{-g} \left\{ wc^2 + hE + c\nabla^2 R + \frac{f}{M^2} [R \chi^2 + 6 (\partial \chi)^2] + \frac{g}{M^2} \chi^4 \right\}, \quad (27)$$

where the coefficients $w, b$ and $c$ are given by Eq. (14) and

$$f = \frac{1}{3(4\pi)^2} \sum_f N_f m_f^2 \quad \text{and} \quad g = \frac{1}{2(4\pi)^2} \sum_s N_s m_s^2 - \frac{2}{(4\pi)^2} \sum_f N_f m_f^4. \quad (28)$$
Here the sums are taken over massive fermion $f$ and scalar $s$ fields with the masses $m_f$ and $m_s$ correspondingly, $N_f$ and $N_s$ being multiplicities.

The conformal anomaly is proportional to the integrand of (27)

$$<T> = - \left\{ wC^2 + bE + c\nabla^2R + \frac{f}{M^2} [R\chi^2 + 6(\partial\chi)^2] + \frac{g}{M^4} \chi^4 \right\}.$$  

(29)

In the standard way, described in section 5.1, one can derive the anomaly-induced effective action of the background fields $g_{\mu\nu}$ and $\chi$

$$\Gamma_{ind} = S_{c}[g_{\mu\nu}, \chi] + \int d^4x \sqrt{-g} \left\{ w\bar{C}^2\sigma + b(\bar{E} - \frac{2}{3}\bar{\nabla}^2\bar{R})\sigma + 2b\sigma\Delta\sigma + \frac{f}{M^2} [\bar{R}\chi^2 + 6(\partial\chi)^2]\sigma + \frac{g}{M^4} \chi^4 \sigma \right\} - \frac{3c + 2b}{36} \int d^4x \sqrt{-\bar{g}} R^2.$$  

(30)

The last step is to fix the conformal unitary gauge $\chi = \bar{\chi} e^{-\sigma} = M$, such that the classical Einstein-Hilbert and cosmological terms acquire their standard form

$$\Gamma = S_{HD} + \bar{\Gamma}_{ind}[\bar{g}_{\mu\nu}, \sigma] - \int d^4x \sqrt{-\bar{g}} e^{2\sigma}[\bar{R} + 6(\nabla\sigma)^2] \cdot \left( \frac{1}{16\pi G} - f \cdot \sigma \right) - \int d^4x \sqrt{-\bar{g}} e^{4\sigma} \cdot \left( \frac{\Lambda}{8\pi G} - g \cdot \sigma \right),$$  

(31)

where $\bar{\Gamma}_{ind}[\bar{g}_{\mu\nu}, \sigma]$ has been defined in (16).

The higher-derivative part of the Eq. (31) is identical to that for the massless fields. Furthermore, there is a strong link between the anomaly-induced effective action Eq. (31) and the quantum corrections coming from the renormalization group [31]. Indeed, (31) can be considered as a local generalization of the renormalization group improved classical action. One can regard Eq. (31) as a leading-log approximation for the effective action for the massive fields. This approximation picks up the logarithmic quantum corrections and is reliable in the high energy region where masses of the fields are much smaller than the Hubble constant $H$.

6. Application of quantum corrections: massive fields

Let us consider two interesting applications for the EA of massive fields.

6.1. Modified Starobinsky model

Let us now consider some particular examples of the quantum corrections for massive fields presented in the previous section. In the subsection 5.1 we have discussed the inflation produced by the quantum effects of supersymmetric quantum matter. The main advantage is the independence of initial data and the main problem is the graceful exit to the FRW stage. In particular, the second inflationary solution in (21) have time-independent energy scale and it is unclear why the sparticle should decouple in the course of inflation.

Now, let us remember that some of the particles are massive and replace the quantum term $\bar{\Gamma}_{ind}$ in the EA (17) by the eq. (31). The equation for $\sigma(t)$ gets modified (see [31] for the details) and the numerical solution can be well approximated by

$$\sigma(t) = H_0 t - \frac{H_0^2}{4} \bar{f} t^2,$$  

(32)

where we used $\Lambda = \bar{g} = 0$ for simplicity. We observe that the inflation is becoming slower and at some moment the scale characteristic $H$ is approaching the point of decoupling $H = M_*$. Thus, everything fits and we may have a consistent natural inflation free of fine-tuning and without need for an inflaton.
There is, however, a very subtle point. The total amount of inflationary e-folds \( N_e \) can be easily calculated from (32), it depends only on the ratio \( M_P/M_* \). For example, for the TeV-scale SUSY breaking we have \( N_e = 10^{32} \). However, only last 60 – 70 of these e-folds are physically relevant. In order to extract the physical information one needs, therefore, an EA in the region when \( H \propto M_* \). However, this is the most difficult problem, because we have no model or approximation for this case. The expressions like (10) corresponds to the small curvature approximation, exactly as more simple expressions for the Schwinger-DeWitt coefficients. On the other end of the energy scale we have a unique approach leading to (31), which may be a good approximation, but only for the limit of very light fields. No model exists for the intermediate situation. Until such model will be achieved, the only way to study inflation would be through the inflaton paradigm. At the same time, the very fact that the inflation can be, in principle, explained by quantum effects of matter fields and without, kind of a qualitatively new physics, looks rather interesting.

6.2. Running cosmological constant

As we have seen in the previous section, at the moment there is no available method for calculating the physical \( \beta \)-function for the cosmological constant. However, since this is a very interesting issue, one can approach it from the phenomenological side. Let us assume that the Appelquist & Carazzone-like quadratic decoupling holds for a CC and consider the consequences for the present-day Universe. One has to associate the scale \( \mu \equiv H \) with the Hubble parameter [35]. Let us notice that this identification of the scale has certain advantages over other possible choices [35, 36].

Remember that, in the \( \overline{\text{MS}} \) scheme, \( \beta_\Lambda \sim m^4 \), \( m \) being mass of a quantum matter field [3]. Then the quadratically suppressed expression is [37]

\[
H \frac{d\Lambda}{dH} = \beta_\Lambda = \sum_i c_i \frac{H^2}{m_i^2} \times m_i^4 = \frac{\sigma}{(4\pi)^2} M^2 H^2 ,
\]

(33)

where \( M \) is an unknown mass parameter and \( \sigma = \pm 1 \). The expression \( \sigma M^2 \) is the algebraic sum of the contributions of all virtual fields, the ones of the heaviest particles being the most relevant. The sign \( \sigma \) depends on whether fermions or bosons dominate in the particle spectrum. One can notice that (33) is nothing else but the equivalent of the simplest possible expression (5).

Let us assume, for a moment, that \( M^2 = M_P^2 \). Then we find

\[
|\beta_\Lambda| \sim 10^{-47} \text{GeV}^4,
\]

(34)

that is close to the existing supernovae and CMB data for the vacuum energy density. Therefore, the renormalization group may, in principle, explain the variation of the vacuum energy with the change of the scale (which, in presence of matter, means also variation of time) without introducing special entities like quintessence. It proves useful to characterize the intensity of the running by a new parameter

\[
\nu = \frac{\sigma}{12 \pi} \frac{M^2}{M_P^2}.
\]

(35)

\( \nu = 0 \) means there is no running and we come to the usual \( \Lambda \)CDM model. The value of \( \nu \) depends on the particle content of the gauge model, especially on the high energy part of the mass spectrum. The single particle of the mass \( M_P \), for instance, gives \( \nu \approx 0.026 \).

It is clear that the renormalization group can not solve the famous CC problem [39], neither the coincidence problem. The fine-tuning of the vacuum CC is performed at the instant when we define the initial point \( \Lambda_0 \) of the renormalization group flow for \( \Lambda \) [35]. However, the running
of the CC may perform in the range comparable to the observed CC, making the coincidence problem less severe.

Two examples of a cosmological model with running CC has been developed in [37, 40]. Along with the renormalization group equation (33), one also needs the Friedmann equation

\[ H^2 = \frac{8\pi G}{3} \left( \rho + \Lambda \right), \tag{36} \]

and the conservation law which is in fact manifestation of the general covariance of an unknown EA of vacuum and matter. The last can be chosen in many possible ways and the problem is closely related to our inability to calculate the EA (or physical \( \beta \)-function) for the massive fields. The scale dependence of the CC implies its time dependence. Then, in order to satisfy the conservation law we need some additional input.

For instance, we can admit the possibility of energy exchange between vacuum and matter sectors of the theory. The resulting model is exactly soluble and can be compared to the SNIa observations [37]. The density perturbations, which have been explored using both analog models [41, 42] and direct calculus [43] have shown a very strong dependence on the parameter \( \nu \). In particular, the models with \( |\nu| > 10^{-4} \) are ruled out. At the same time the values \( |\nu| \leq 10^{-6} \) are in a perfect agreement with the observational data (see \[43\] and references therein). Let us notice that \( |\nu| \sim 10^{-6} \) corresponds to the particle theory with a desert in the mass spectrum between the GUT and Planck scales.

An alternative model [40] satisfies the conservation law via the scale dependence of the Newton constant \( G \). The main advantages of this approach are that: \( i) \) it implies an independent covariance of EA in the vacuum and matter sectors; \( ii) \) avoids an obvious problem with overproduction of massless or extremely light (dark) matter particles. It is interesting that the scale dependence of the Newton constant \( G \) should manifest also at the astrophysical level. It turns out that the model with \( \nu \approx 10^{-6} \) leads to the approximately “correct” shape of rotation curves of the starts in the galaxies, essentially alleviating the Dark Matter problem.

As we can see, it would be extremely interesting to have a consistent QFT model for the EA behind the possible CC running. For this end one needs to develop some method for deriving the remnant non-local structures in the EA of a massive fields. So far, we have no such model. One can prove that the EA responsible for the CC running cannot be of a finite polynomial order in curvatures [37], but this does not constitute the proof of no-running.

7. Conclusions

The effective approach to QFT in curved space-time may tell us a lot about gravitational physics, especially in cosmology. Perhaps the most interesting problem is evaluation of vacuum effective action for massive quantum fields. Working in this direction one may prove or disprove the possibility of a time-dependent cosmological constant. The same calculation is vital for further development of the anomaly-induced inflation model.

Within the existing formalism of QFT in curved space and known calculational techniques we can learn something relevant about the possible form of quantum corrections, e.g. rule out the \( \mathcal{O}(H) \)-type corrections to the CC. Furthermore we can see that some problems of modern cosmology, such as a possible time variation of CC and inflation, can be solved within the semiclassical effective approach, without invoking the fundamental concepts like quantum gravity or string theory. Moreover, until the semiclassical contributions are not properly explored, there is no chance to get safe information concerning these fundamental theories. From this perspective the most important problems are indeed the investigation of quantum corrections, especially those coming from the massive fields.

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