Group theoretical approach to the Dirac operator on $S^2$

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Abstract. In this revision we outline the group theoretical approach to formulate and solve the eigenvalue problem of the Dirac operator on the round 2-sphere conceived as the right coset $S^2 = SU(2)/U(1)$. Starting from general symmetry considerations we illustrate the formulation of the Dirac operator through left action or right action differential operators, whose properties on a right coset are quite different. The construction of the spinor space and the solution of the spectral problem using group theoretical methods is also presented.

1. Introduction

Besides their continued use in fundamental quantum field theories, Dirac operators on curved spaces have sparked renewed interest due to other reasons, notably their use in condensed matter physics where, amongst other applications, they are central to the effective field description of graphene, fullerenes and the quantum Hall effect [1, 2]. These are both examples of Dirac operators on a Riemannian manifold, wherein no timelike coordinate exists on account of the metric signature, and where the square of the Dirac operator may be interpreted as the hamiltonian of a non-relativistic fermion confined to a curved space, out of which $S^2$ stands out as one of the simplest examples. There are presently several different accounts of the Dirac operator on $S^2$ given in the literature, most remarkably [3–9], many of which rely on the standard and general approach of vielbeins. However, despite the generality of the vielbein method it is known that when we deal with a coset space $G/H$ an elegant and natural treatment that takes advantage of the symmetry is available cf. [10]; this is the so called group theoretical approach and relies mainly on representation theory. In what follows we shall review the essential aspects of the group theoretical approach to the Dirac operator’s spectral problem on $S^2 = SU(2)/U(1)$.

It has been established [11] that for a simply connected spin manifold $\mathcal{M}$ with riemannian metric tensor $g_{\mu\nu}$ and vielbeins $e^i_\mu$, such as $S^2$, the Dirac operator corresponding to the (unique) spin structure is

$$\slashed{D} = i\gamma^\mu \nabla_\mu$$

where $\nabla_\mu = \partial_\mu + \Omega_\mu$ is the covariant spinor derivative and $\Omega_\mu = -\frac{1}{4} e^j_\sigma \tilde{\nabla}_\mu e^{j\sigma}[\gamma_i, \gamma_j]$ is the spin connection corresponding to the spin structure of $\mathcal{M}$. The euclidean Dirac matrices introduced satisfy $\{\gamma_i, \gamma_j\} = 2\delta_{ij}1$, while the riemannian Dirac matrices $\gamma_\mu = e^i_\mu \gamma_i$ accordingly.
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}. \tag{2}

As mentioned before, when dealing with a coset space \( \mathcal{M} = G/H \) the Dirac operator and its eigenvalue problem take a particularly simple form in terms of representation theory. In general given a finite dimensional Lie group \( G \) and a Lie subgroup \( H \subset G \), with Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \subset \mathfrak{g} \), one can decide on constructing either a left coset space \( G\backslash H \) or a right coset space \( G/H \), being both equivalent in a sense; but once a choice has been made then the left and right group actions will be radically distinct. Essentially on a right coset \( G/H = \{ [g] = gH : g \in G \} \) the left infinitesimal actions of \( G \) give rise to Lie derivatives and the right actions to covariant derivatives. Let \( T_a \) be the generators of \( \mathfrak{g} \), we shall assume from now on that \( G \) is compact, and hence isomorphic to a group of unitary matrices, therefore we lose no generality considering that \( T_a^l = T_a \) as complex matrices, and setting the Lie algebra to be\(^1\)

\[ [T_a, T_b] = if_{abc}T_c \tag{3} \]

with real structure constants \( f_{abc} \). Given a function \( f \in L^2(G, \mathbb{C}) \) when \( G \) is compact the Peter-Weyl theorem [12] asserts that it can be expanded in the orthogonal basis formed by the elements of the Wigner D-matrices corresponding to all the unitary inequivalent irreducible representations \( D^J \) of \( G \) (i.e. the representation ring \( \hat{G} \), labelled by \( J \)).

\[ f(g) = \sum_{J \in \hat{G}} \sum_{i,k \in I(J)} f^J_{ik} D^J_{ik}(g), \quad I(J) \text{ index set of } D^J, \quad f^J_{ik} \in \mathbb{C} \tag{4} \]

on the other hand, a function on the coset space \( \varphi : G/H \to \mathbb{C} \) can be naturally understood as a function over \( G \) that takes constant values on each coset class, such property is called \( H \)-equivariance, and is tantamount to \( \varphi(gh) = \varphi(g) \) for each \( h \in H \), it follows then that such functions must be expanded in the form [5]

\[ \varphi(g) = \sum_{L \in \hat{G}/H} \sum_{\ell \in I(L)} \sum_{k_0 \in L_0} \varphi^L_{ik_0} D^L_{ik_0}(g), \quad \varphi^L_{ik_0} \in \mathbb{C}. \tag{5} \]

Above \( \hat{G}/H \subset \hat{G} \) denotes the subset of irreducible representations that, when reduced under \( H \), contain the trivial representation at least once, while the subset \( L_0 \subset I(L) \) labels such trivial subrepresentation of \( H \) within \( D^L \). Typical examples of the expansion (5) include Fourier series, expansions in spherical harmonics, and harmonic expansions in general. Left and right actions can be defined on a group \( L, R : G \times G \to G \), \( (k, g) \mapsto L_kg, R_kg \), namely

\[ L_kg := kg, \quad R_kg := gk^{-1}. \tag{6} \]

Such maps are compatible with \( G \) in that \( R_g \circ R_k = R_{gk}, L_g \circ L_k = L_{gk} \). Let a differentiable function be given \( f : G \to \mathbb{C} \), define now the left action differential operators \( \mathcal{L}_\alpha \) and the right action differential operators \( \mathcal{R}_\alpha \) as the vector fields

\[ i\mathcal{L}_\alpha f(g) := \frac{df(exp(-isT_\alpha))g}{ds}\bigg|_{s=0}, \quad i\mathcal{R}_\alpha f(g) := \frac{df(exp(+isT_\alpha))}{ds}\bigg|_{s=0} \tag{7} \]

where \( exp \) is the exponential map as usual. These differential operators form a representation of \( \mathfrak{g} \) with the same commutation relations

\(^1\) Since \( G \) is compact we have written \( f_{abc} \), as the structure constants are totally antisymmetric tensors and no distinction is required between upper and lower indices.
\[ [\mathcal{L}_a, \mathcal{L}_b] = i f_{abc} \mathcal{L}_c, \quad [\mathcal{R}_a, \mathcal{R}_b] = i f_{abc} \mathcal{R}_c. \] (8)

We would like to point out that although functions on the coset \( \varphi : G/H \to \mathbb{C} \) are just a subclass of functions on \( G \), namely the \( H \)-equivariant functions, the left action is well defined on such functions \( \varphi \), but the right action is not well defined since its effect depends on the representative chosen within each coset class. In an explicit matrix representation, cf. [13], it has been found that the left and right action operators are

\[ \mathcal{L}_a = -\text{Tr} \left( T_a g \frac{\partial}{\partial g^\mathsf{T}} \right), \quad \mathcal{R}_a = \text{Tr} \left( g T_a \frac{\partial}{\partial g^\mathsf{T}} \right), \] (9)

notice that they are dual to the left and right Maurer-Cartan forms \( \theta_L, \theta_R \in \Omega^1(G) \otimes \mathfrak{g} \):

\[ \theta_L = -(dg)g^{-1}, \quad \theta_R = g^{-1}dg, \quad \langle \theta_L, \mathcal{L}_a \rangle = \langle \theta_R, \mathcal{R}_a \rangle = T_a \] (10)

2. The round sphere \( S^2 \)

It is clear that the Dirac operator depends on the metric geometry of the space, we shall consider a “round” metric on the sphere, i.e. the \( SU(2) \)-invariant metric induced by the \( G \)-invariant metric [14] on \( SU(2) \). To this end a parametrisation for \( SU(2) \) is chosen with global coordinates \( z_A, \bar{z}_B \in \mathbb{C}, \ A, B = 1, 2 \) subjected to a constraint

\[ g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ \bar{z}_1 & \bar{z}_2 \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1 \] (11)

the round sphere can be seen embedded in \( \mathbb{R}^3 \) through the cartesian coordinates \( (x_1, x_2, x_3) = x \) which relate to \( g \) through the projection of the Hopf bundle \( \pi : S^3 \to S^2, \ z \mapsto x \) given by

\[ x_a = z^\dagger \sigma_a z, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad x \cdot x = 1. \] (12)

The constraint on \( x \) follows directly from the constraint on \( z \), that is \( z^\dagger z = 1 \), and \( \sigma_a \) are the Pauli matrices. The embedding \( U(1) \hookrightarrow SU(2) \) we shall use is the diagonal embedding, so that \( U(1) \) is generated by \( \sigma_3 \)

\[ h = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \in U(1) \subset SU(2) \] (13)

Henceforth we shall denote by \( D^J \) the spin \( J \) irreducible representation of \( SU(2) \), in particular the expansion (5) becomes the familiar spherical harmonic expansion on the sphere

\[ \varphi(g) = \sum_{L=0}^{\infty} \sum_{M=-L}^{L} \varphi_{LM}^J(g), \quad D^J_{00}(g) = Y_{LM}(x) \] (14)

where the \( D \)-Wigner matrices \( D^J_{LM}(g) \) may be found e.g. in [15], in this case we have set \( i = M \) and recognised \( S^2 = \mathbb{C} \subset SU(2) = \mathbb{C}2 \), \( L_0 = \{0\} \) and \( I(L) = \{-L, \cdots, 0, \cdots, L\} \). Consequently the representation content of smooth functions is clearly

\[ C^\infty(S^2, \mathbb{C}) \subset \bigoplus_{J=0}^{\infty} D^J. \] (15)

The Maurer-Cartan forms used in appendix Appendix A to find the Ricci curvature are
\[ \theta_L = - \left( \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2 + z_1 d\bar{z}_1 \right), \quad \theta_R = \left( \bar{z}_1 dz_1 + \bar{z}_2 d\bar{z}_2 - \bar{z}_2 d\bar{z}_2 + z_1 d\bar{z}_1 \right). \]  

(16)

The $G$-invariant metric on $G$, and the corresponding line element are related to the Maurer-Cartan forms, taking (11) into account one finds

\[ ds^2 = -\frac{1}{2} \text{Tr}[(\theta_L)^2] = -\frac{1}{2} \text{Tr}[(\theta_R)^2] = dz_1 \bar{d}z_1 + d\bar{z}_2 \bar{d}z_2. \]

(17)

We may now write the Dirac operator using either left or right action differential operators, introducing the complex basis $\mathcal{R}_\pm = \mathcal{R}_1 \pm i\mathcal{R}_2$, $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, $\mathcal{R}_0 = \mathcal{R}_3$, $\sigma_0 = \sigma_3$

\[ \mathcal{L}_+ = \bar{z}_1 \partial_2 - z_2 \partial_1, \quad \mathcal{L}_- = \bar{z}_2 \partial_1 - z_1 \partial_2, \quad \mathcal{L}_0 = \frac{1}{2}(z_2 \partial_2 - z_1 \partial_1 + \bar{z}_1 \partial_1 - \bar{z}_2 \partial_2), \]

\[ \mathcal{R}_+ = z_2 \partial_1 - \bar{z}_1 \partial_2, \quad \mathcal{R}_- = \bar{z}_2 \partial_2 - z_1 \partial_1, \quad \mathcal{R}_0 = \frac{1}{2}(z_1 \partial_1 + z_2 \partial_2 + \bar{z}_1 \partial_1 - \bar{z}_2 \partial_2), \]

(18)

(19)

in the following manner, cf. [4,5]:

\[ \mathcal{D}_L = \sigma_a \mathcal{L}_a + 1, \quad \mathcal{D}_R = \sigma_+ \mathcal{R}_+ + \sigma_- \mathcal{R}_-. \]

(20)

It should be noted that while $\mathcal{L}_a : C^\infty(S^2, \mathbb{C}) \rightarrow C^\infty(S^2, \mathbb{C})$, the right operators $\mathcal{R}_\pm$ map smooth functions to (anti)holomorphic vector fields on the sphere and $\mathcal{R}_0$ obviously annihilates smooth functions, as it generates $H = U(1)$. Left operators are the Lie derivatives of the Killing directions on the sphere, and right operators are covariant derivatives, mapping between different spaces of sections of tensor bundles [16]. Notice that $\mathcal{D}_L$ has the spin connection term, +1, displayed explicitly in (20), as opposed to $\mathcal{D}_R$ where it is hidden in the definition of $\mathcal{R}$. The left version $\mathcal{D}_L$ will act on $C^\infty(S^2, \mathbb{C})$ valued spinors, while $\mathcal{D}_R$ will act on spinors valued on the ring of antiholomorphic forms, cf. section 4.

3. Equivalence of both approaches

That both expressions (20) of the Dirac operator are equivalent is not immediate but can be proven by use of the adjoint representation $Ad_G : \mathfrak{g} \rightarrow \mathfrak{g}$, defined as usual

\[ Ad(g)X = gxg^{-1}, \quad \text{for} \quad X \in \mathfrak{g}. \]

(21)

If $Ad(g)_{ab}$ are the matrix elements in the basis $\{T_a\}$ we find readily the identity

\[ Ad(g)_{ab} \mathcal{L}_b f(g) = -\mathcal{R}_a f(g), \quad \text{with} \quad Ad_{ab}(g) = \frac{1}{2} \text{Tr}(g\sigma_a g^{-1}\sigma_b) \]

notice that this is a point dependent frame rotation, indeed a local $SU(2)$ transformation. One may show that due to the embedding chosen (13) the columns of the adjoint matrix form an orthonormal frame $Ad_{ab}(g) = \xi_a^g$, being $\xi^1, \xi^2$ tangent to $S^2$ and $\xi^3 = x$ the normal vector. It follows that the induced metric on $S^2$ is the projector

\[ P_{ab} = \delta_{ab} - x_a x_b = \xi^i_a \xi^i_b, \quad i = 1, 2. \]

(23)

The left Dirac operator written in the form (1) is $\mathcal{D}_L = \sigma_a P_{ab}(\mathcal{L}_b + \frac{1}{2}\sigma_b)$, and observe that for functions $f \in C^\infty(S^2, \mathbb{C})$ the left operators are tangent $P_{ab}\mathcal{L}_b f = \mathcal{L}_a f$, a property which no longer holds for non-equivariant functions and that is central to the construction of the spinor
space; it is easy to identify $\mathcal{L}_a$ as the usual angular momentum operators, thus $\mathcal{L} + \frac{1}{2} \sigma$ is the total angular momentum of the fermion. We now proceed to prove the aforementioned equivalence

$$g^{-1} \mathcal{D}_L g = g^{-1} \sigma_a g P_{ab} \mathcal{L}_b = Ad(g)_{ca} \sigma_c P_{ab} \mathcal{L}_b = \xi^c \xi^d_a \sigma_c \xi^b_b \mathcal{L}_b = \sigma_i Ad(g)_{ib} \mathcal{L}_b = -\mathcal{D}_R.$$  

(24)

The equation above connects both versions and shows that they are unitarily equivalent through a local $SU(2)$ rotation of the spinor space, in particular it relates spinors $\Psi^{L,R}$

$$\Psi^L(g) = g \Psi^R(g)$$

(25)

4. Spinor space

Let $\mathcal{S}$ be the space of smooth spinor fields on $S^2$. The representation content of spinors is known for spin manifolds of the type $G/H$, at least in an abstract manner [10], and since the Dirac operator is a map $\mathcal{D} : \mathcal{S} \rightarrow \mathcal{S}$ it is clear from (25) that in the left and right formulations spinors will have a different structure but the same representation content, for instance

$$\mathcal{S}^L = D^{1/2} \otimes C^\infty(S^2, \mathbb{C}) \in \bigotimes_{j=0}^{\infty} D^{J+1/2} \bigotimes_{j=1}^{\infty} D^{J-1/2}, \quad \mathcal{S}^L = \mathcal{S}^L_+ \oplus \mathcal{S}^L_-$$

(26)

where each piece corresponds to a Weyl spinor. In the right action version spinors must be represented in a different manner, for definiteness let us we write

$$\Psi^{L,R} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in \mathcal{S}^{L,R}$$

(27)

the distinction is that for $\mathcal{S}^R$ one has $\psi_+ \in \Omega^0(S^2, \mathbb{C}) = C^\infty(S^2, \mathbb{C})$ and $\psi_- \otimes d\bar{\zeta} \in \Omega^1(S^2, \mathbb{C})$ is an antiholomorphic 1-form in a complex local coordinate system $\{\zeta, \bar{\zeta}\}$. The components of right spinors $\Psi^R$ are naturally identified with (possibly charged) antiholomorphic differential forms [16], such identification is well known for Kähler manifolds [17], namely

$$\mathcal{S} = \bigoplus_{k=0}^{\dim \mathcal{M}} \Omega^k(\mathcal{M}, \mathbb{C}).$$

(28)

Once we know the structure of $\mathcal{S}$ we may use Lichnerowicz’s theorem [18]

$$\mathcal{D}^2 = \Delta_{\mathcal{S}} + \frac{R}{4}$$

(29)

and the chirality operator $\gamma = x \cdot \sigma$ to construct the nonzero spectrum of $\mathcal{D}$ from the nonzero spectrum of $\mathcal{D}^2$, the advantage being that $\mathcal{D}^2$ is given entirely in terms of quadratic Casimir operators and hence its spectrum becomes straightforward to compute. Let $\Psi^{(\sigma)}_{\Lambda}$ be the eigenspinor of $\mathcal{D}^2$ with eigenvalue $\Lambda$ and let $(\sigma)$ label degeneracy: $\Psi^{(\sigma)}_{\Lambda}$ has a definite chirality since $[\mathcal{D}^2, \gamma] = 0$ and we shall also verify that $\Lambda > 0$. First of all we have

$$\mathcal{D}^2 \Psi^{(\sigma)}_{\Lambda} = \Lambda \Psi^{(\sigma)}_{\Lambda},$$

(30)

2 $U(1)$ charged forms describe spinors in the spin-c structures, these correspond physically to the motion of a fermion on a topological monopole field [16].

3 Which anticommutes with either Dirac operator $\{\mathcal{D}^{L\cdot R}, \gamma\} = 0$. 

and under these conditions to each eigenspinor of $\Psi^2$ correspond exactly two eigenspinors of $D$
\[ \Psi^2(\sigma,\pm) = \Psi(\sigma) \pm \frac{D\Psi(\sigma)}{\sqrt{A}}, \quad D\Psi^2(\sigma,\pm) = \pm \sqrt{A}\Psi^2(\sigma,\pm) \]  

Lichnerowicz’s theorem implies that on a spin manifold with strictly positive curvature there cannot be zero modes, therefore on $S^2$ the full spectrum can be obtained from (31): this follows from (29) since a zero mode $\Psi_0$ would mean $D\Psi_0 = 0$, nevertheless by construction $\Delta_S$ is a positive operator, hence $(\Psi_0, \Delta_S \Psi_0) = 0$, and since $R > 0$ equation (29) would lead to the contradiction $|D\Psi_0|^2 = (\Psi_0, D^2\Psi_0) > 0$; from here it follows also $\Lambda > 0$. The spin Laplacian is $\Delta_S = -g^{\mu\nu}\nabla_\mu \nabla_\nu$, which in our case of interest can be cast into the forms
\[
\Delta^L_S = \left( L_a + \frac{P_{ab}\sigma_b}{2} \right)^2 = \left( L_a + \frac{1}{2}\sigma_a \right)^2 - \left( \frac{\sigma_0}{2} \right)^2 \\
\Delta^R_S = R^2_a = R^2_0 - R^2_0
\]

Clearly both $(\sigma)$ and $\Lambda$ are fully determined from the $SU(2), U(1)$ representation content of the spinor space, we remark that both $S^R, S^L$ have the same representation content. The full spectrum is symmetric around zero and can be found directly using $R = 2$, cf. appendix Appendix A, the degeneracy index $(\sigma)$ counts the dimension of each irreducible representation appearing in (26)
\[ \text{Spec}(D) = \{ \lambda \in \mathbb{Z}/\{0\} \}, \quad \sigma = 1, 2, \cdots, 2|\lambda| \]  

5. The role of the constraint

Finally we should clarify what is the meaning of the differential operators we have introduced, for instance $\partial_1 = \frac{\partial}{\partial z_1}$ would usually imply that $z_1$ can be varied in the argument of a function $f(z_1, z_2, \bar{z}_1, \bar{z}_2)$ leaving all other arguments constant, but this is clearly impossible as we have the constraint (11). In a sense the constraint plays a trivial role since the left and right differential operators we are interested in are all tangent to $S^3$. A way to show that we may formally operate on $z_A, \bar{z}_B$ as if they were all independent is to consider a function $f$ on $S^3$ as the restriction of a smooth extension $\tilde{f}$ to $\mathbb{C}^2$, in the spirit of [4]
\[
\tilde{f}(z_A, \bar{z}_B) = f(z_A, \bar{z}_B)\Delta(z^1 z - 1) \in C^\infty(\mathbb{C}^2, \mathbb{C}), \quad \tilde{f} = f|_{S^3}
\]
where $\Delta(y)$ is a smooth function that goes rapidly to zero for $y \neq 0$ while $\Delta(0) = 1$ and $f$ is formally extended considering $z_A, \bar{z}_B$ as independent variables. Now we may verify that the central property required is that the constraint $z^1 z - 1$ must be annihilated by the differential operators (understood as vector fields on $\mathbb{C}^2$), $\mathcal{R}_a(z^1 z - 1) = \mathcal{L}_a(z^1 z - 1) = 0$; it is very easy to check that this is indeed the case and we have for instance
\[
\mathcal{R}_a[f|_{S^3}] = \mathcal{L}_a[f(z_A, \bar{z}_B)\Delta(z^1 z - 1)]|_{S^3} = (z^1 z - 1)\mathcal{R}_a[f(z_A, \bar{z}_B)]|_{S^3} = \mathcal{R}_a[f].
\]

Thus we may operate as if $z_A, \bar{z}_B$ were all independent variables and obtain the same result after restriction to $S^3$. As pointed out in [4] this is a consequence of the fact that $\mathcal{L}_a, \mathcal{R}_b$ may be understood as flows on $S^3$, and consider functions on $S^3$ as functions of its polar coordinates whereupon such differential operators depending only of these polar coordinates will act. This justifies the use of the computation rules $\frac{\partial\mathcal{L}_a}{\partial z_B} = \frac{\partial\mathcal{R}_a}{\partial \bar{z}_B} = \delta^B_A$ and $\frac{\partial\mathcal{L}_a}{\partial \bar{z}_B} = \frac{\partial\mathcal{R}_a}{\partial z_B} = 0$ throughout the calculations, notwithstanding the constraint.

$^4$ It is also possible to take the limit when $\Delta$ becomes singular, one such example is $k \to +\infty$ and $\Delta(y) = e^{-ky^2}$, the result (36) is independent of the $\Delta$ chosen.
Remarks
When a space has a high degree of symmetry and can be realised as a coset space there is a particularly simple treatment of field theory problems using representation theory. If the space admits a spin structure then it is feasible to construct the spinor space and spin bundle using representation theory and to solve the spectral problem for the Dirac operator essentially through quadratic Casimir operators of the involved groups, rendering the problem very simple. We have reviewed the details corresponding to \( S^2 \) and shown both approaches, based on left and right action differential operators, which are naturally induced from the group structure. One makes use of Lie derivatives and the other of covariant derivatives, in the first case the spin connection is displayed explicitly, and both are ultimately equivalent. Although we have discussed here only a simple example the general approach works for a wide variety of spaces \( G/H \), and constitutes a significant computational advantage over the vielbein approach, in addition to the advantages provided by the natural use of the symmetry involved.

Acknowledgments
I. Huet acknowledges support under the SNI funding system, S. Gutiérrez is thankful to the UAM for continued financial support.

Appendix A. Curvature on Coset Spaces
In what follows we found it convenient to use the following notation: \( \mu \) labels local coordinates on \( M = G/H \), \( a \) labels the basis of \( \mathfrak{g} \) which splits into \( \alpha, \beta \) that label \( \mathfrak{h} \) and \( i, j, k, l, m \) that label the remaining elements of the basis (corresponding to tangent directions on \( G/H \)), notice that this choice implies \( f_{\alpha\beta} = 0 \); we follow closely the discussion found in [18,19]. Quite generally we may set up a local coordinate system \( x^\mu \) on \( G/H \) and choose a representative on each coset class \( g(x) \in G \) corresponding to the point labelled by \( x^\mu \), the associated right Maurer-Cartan form will be

\[
e^a_\mu dx^\mu \otimes T_a = g^{-1}(x)dg(x) \in \Omega^1(G/H) \otimes \mathfrak{g}
\]

such form decomposes accordingly into \( e^a_\mu dx^\mu \otimes T_a = e^i_\mu dx^\mu \otimes T_i + e^\alpha_\mu dx^\mu \otimes T_\alpha = e^i \otimes T_i + e^\alpha \otimes T_\alpha \).

The vielbein is the tangent part of the Maurer-Cartan form, that is \( e^i = \theta^i_k \) is the vielbein for \( G/H \), and taken together with the connection 1-form \( \omega^i_k \) determine the curvature and torsion of \( M \) through Cartan’s structure equations

\[
de\omega^i_k + \omega^i_j \wedge \omega^j_k = \Omega^i_k \tag{A.1}
\]

\[
d\omega^i_k + \omega^j_k \wedge \omega^i_j = 0 \tag{A.2}
\]

in these equations \( \omega^i_k = \Gamma^i_{jk}e^j_\mu dx^\mu \) with \( \Gamma^i_{jk} \) the Christoffel symbols in the vielbein basis. The Levi-Civita connection is compatible with the \( G \)-invariant metric and has vanishing torsion \( \Omega^i = 0 \). Equation (A.1) can be compared against the Maurer-Cartan equation for \( G \)

\[
d\theta_L + \theta_L \wedge \theta_L = 0 \tag{A.3}
\]

leading, after using Jacobi’s identity, to the connection and curvature forms

\[
\omega^i_k = \frac{1}{2} f^i_{jk}e^j + f^i_{\alpha\beta}e^\alpha \tag{A.4}
\]

\[
\Omega^i_k = \frac{1}{4} (2 f^i_{\alpha\beta}f^\alpha_{km} + f^i_{kj}f^j_{lm} - f^i_{kj}f^j_{km}) e^l \wedge e^m. \tag{A.5}
\]
If we denote by $Ad_{G/H}$ the restriction of the adjoint representation $Ad_G$ to $H$ within the tangent space of $G/H$ [5], and the normalisation for the quadratic Casimir is fixed by $C_2 := T_a T_a$, then we find the Ricci scalar curvature to be

$$R = \frac{1}{2} \left( \mathrm{Tr} C_2(Ad_G) - \mathrm{Tr} C_2(Ad_H) - \mathrm{Tr} C_2(Ad_{G/H}) \right). \quad (A.6)$$

In the particular case of symmetric spaces (where $f_{ijk} = 0$, meaning that the torsion of the canonically induced connection vanishes), such as $S^n$ one has

$$R = f^i_{\alpha \kappa} F^\alpha_{ik} = \mathrm{Tr} C_2(Ad_{G/H}) = \frac{1}{2} \left( \mathrm{Tr} C_2(Ad_G) - \mathrm{Tr} C_2(Ad_H) \right). \quad (A.7)$$

For $S^2$ we may use polar coordinates and choose for instance $G(H, \phi)$ be determined by

$$(z_1, z_2) = \left( \cos(\theta/2) e^{-i \varphi/2}, \sin(\theta/2) e^{i \varphi/2} \right)$$

leading to $e^1 = d\theta$, $e^2 = \sin \theta d\phi$, $e^3 = -\cos \theta d\phi$ and $\omega^1 = \omega^2 = 0$, $\omega^1 = -\omega^2 = \cos \theta d\phi$. Either method produces $R = 2$ for the Ricci scalar curvature of the unit sphere, a standard result found e.g. in [13] and that was obtained also by use of group theory in (A.7).

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