Localized-endemic state transition in the susceptible-infected-susceptible model on networks

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It is a longstanding debate concerning the absence of threshold for the susceptible-infected-susceptible spreading model on networks with localized state. The key to resolve this controversy is the dynamical interaction pattern, which has not been uncovered. Here we show that the interaction driving the localized-endemic state transition is not the global interaction between a node and all the other nodes on the network, but exists at the level of super node composed of highly connected node and its neighbors. The internal interactions within a super node induce localized state with limited lifetime, while the interactions between neighboring super nodes via a path of two hops enable them to avoid trapping in the absorbing state, marking the onset of endemic state. The hybrid interactions render highly connected nodes exponentially increasing infection density, which truly account for the null threshold. These results are crucial for correctly understanding diverse recurrent contagion phenomena.

Networks [1-3] essentially capture the structure of individual interactions, contact and mobility patterns, through which information, innovation, fads, epidemics, and human behaviors spread across us [4-7]. Burst of investigations have been devoted to the susceptible-infected-susceptible (SIS) model on networks to understand, predict and control recurrent contagion phenomena, such as influenza-like diseases, computer virus, memes etc. [8, 9]. There is a threshold of the spreading rate \( \lambda \), separating the abrupt outbreak of contagion and the nonequilibrium absorbing state phase transition, where the propagation dies out. The heterogeneity of real-world networks characterized by a power law degree distribution \( P(k) \sim k^{-\gamma} \) has a nontrivial impact on the threshold. The heterogeneous mean field (HMF) theory [10], a seminal work in the thermodynamic limit, thus, \( \lambda_{c}^{HMF} = \frac{1}{\Lambda_{1}} \), where \( \Lambda_{1} \sim \max \{ \sqrt{k_{\text{max}}}, (k^{2})/k \} \) is the largest eigenvalue of the adjacency matrix. QMF readily gives the same threshold as HMF for \( 2 < \gamma < 2.5 \). While for \( \gamma > 2.5 \), the network topology subtly changes, since \( \Lambda_{1} \) shifts from \( (k^{2})/k \) to \( \sqrt{k_{\text{max}}} \). Then \( \lambda_{c}^{QMF} \) depends on the maximum degree: \( \lambda_{c}^{QMF} \approx 1/\sqrt{k_{\text{max}}} \), which is zero for any networks with divergent degree, including \( \gamma > 3 \), just at odds with HMF theory. That the hub nodes are sustainable source was deemed as the physical origin. However, Goltsev et al. showed that it is not a genuine threshold [10]. Only a finite number of nodes are active, localized around the most highly connected nodes. The genuine threshold \( \lambda_{c} \) is definitely higher than \( \lambda_{c}^{QMF} \). This research ignited a big stir. Furthermore, Lee et al. showed that when \( \lambda_{c}^{QMF} < \lambda < \lambda_{c} \), the dynamics are governed by the compelling Griffiths phase [17, 18], featuring slow dynamics enhanced by disorder in systems. Owing to the irreversible dynamical fluctuation, local active domains eventually fall into absorbing phase with exceedingly long relaxation time. They further conjectured that the phase transition is triggered by the percolation of local active domains through direct connection of hubs, which enables the hub mutual reinfection, and the threshold is finite. Grounded on the belief that a node can be infected by distant nodes, Boguñá et al. relaxed the strong requirement of direct connection of hubs for mutual reinfection [19]. The null threshold was then retrieved again.

The ongoing debate is actually a matter of the fundamental interaction pattern responsible for the localized-endemic state transition. Conventional mind merely considers interaction between directly linked nodes, whereas literature [19] made a conceptual leap by introducing global dynamical interactions. In this paper, we devise a more accurate approach to estimate the strength of global interaction, allowing convincing validation of this theory. Our analytical and numerical studies deny the function of global and long-range dynamical interactions. Instead, we show that it is the interaction between a node and its second nearest neighbors (SNN) that drives the

The quenched mean field (QMF) theory replaces the annealed adjacent matrix in HMF by quenched one, predicting \( \lambda_{c}^{QMF} = 1/\sqrt{k_{\text{max}}} \), where \( \Lambda_{1} \sim \max \{ \sqrt{k_{\text{max}}}, (k^{2})/k \} \) is the largest eigenvalue of the adjacency matrix. QMF readily gives the same threshold as HMF for \( 2 < \gamma < 2.5 \). While for \( \gamma > 2.5 \), the network topology subtly changes, since \( \Lambda_{1} \) shifts from \( (k^{2})/k \) to \( \sqrt{k_{\text{max}}} \). Then \( \lambda_{c}^{QMF} \) depends on the maximum degree: \( \lambda_{c}^{QMF} \approx 1/\sqrt{k_{\text{max}}} \), which is zero for any networks with divergent degree, including \( \gamma > 3 \), just at odds with HMF theory. That the hub nodes are sustainable source was deemed as the physical origin. However, Goltsev et al. showed that it is not a genuine threshold [10]. Only a finite number of nodes are active, localized around the most highly connected nodes. The genuine threshold \( \lambda_{c} \) is definitely higher than \( \lambda_{c}^{QMF} \). This research ignited a big stir. Furthermore, Lee et al. showed that when \( \lambda_{c}^{QMF} < \lambda < \lambda_{c} \), the dynamics are governed by the compelling Griffiths phase [17, 18], featuring slow dynamics enhanced by disorder in systems. Owing to the irreversible dynamical fluctuation, local active domains eventually fall into absorbing phase with exceedingly long relaxation time. They further conjectured that the phase transition is triggered by the percolation of local active domains through direct connection of hubs, which enables the hub mutual reinfection, and the threshold is finite. Grounded on the belief that a node can be infected by distant nodes, Boguñá et al. relaxed the strong requirement of direct connection of hubs for mutual reinfection [19]. The null threshold was then retrieved again.

The quenched mean field (QMF) theory replaces the description of the interaction pattern responsible for the localized-endemic state transition. Conventional mind merely considers interaction between directly linked nodes, whereas literature [19] made a conceptual leap by introducing global dynamical interactions. In this paper, we devise a more accurate approach to estimate the strength of global interaction, allowing convincing validation of this theory. Our analytical and numerical studies deny the function of global and long-range dynamical interactions. Instead, we show that it is the interaction between a node and its second nearest neighbors (SNN) that drives the
emerges, however, it undoubtedly no longer holds. As the equation in this regime. It calls for new theory. Such peculiar significant discrepancy lies the manifest failure of the HMF from the benchmark get more and more remarkable, suggest-

\[
\frac{d\rho(t)}{dt} = -\bar{\delta}(k, \lambda)\rho(t) + \lambda(1 - \rho(t))\sum_{k'} a_{kk'}\rho_k(t)N\rho(k'),
\]

where \(a_{kk'} = k k' / (N(k'))\) is the annealed adjacent matrix for uncorrelated random networks. This equation suggests that \(\rho_k \propto ke^{-\alpha k}\), where \(\beta = \ln (1 + 1/\lambda) / \ln \kappa\), and \(\kappa = (k^2) / (k) - 1\) is the average branching factor. For large-size networks, the slope of the curve under double log plot \(s(k) = \beta = a(\lambda)k \gg 1\), which can explain the tremendous deviation for the highly connected nodes, yet fails to predict that \(s(k) \approx 1\) for the most small-degree nodes shown in Fig. 1.

The failure may be caused by the aforementioned rough mean field approximation. Obviously, it is considerably underestimated, especially the interaction of nearest neighbours is completely neglected. Here we circumvent such a problem by computing the strength of global dynamical interaction based on the distance. It gives us the leverage to verify Eq. (1) in a faithful way.

The long-range dynamical interaction imposed by the neighbors of a node \(i\) at distance \(\ell\) can be evaluated as

\[
\psi_i(\ell) = \lambda(\ell, \lambda) \sum_j b_{ij}\rho_j,
\]

where \(b_{ij} = 1\) if \(d_{ij} = \ell\), otherwise \(b_{ij} = 0\).

We proceed to work out \(\psi_k(\ell)\) for nodes with degree \(k\). With respect to random sparse graphs, the total number of neighbors at distance \(\ell\) for a node of degree \(k\) are about \(kk^{\ell-1}\). Then

\[
\psi_k(\ell) = \lambda(\ell, \lambda)k k^{\ell-1} \sum_{k'} P(k'|k, \ell)\rho_k(t),
\]

where \(P(k'|k, \ell)\) is the probability of finding a neighboring node with degree \(k'\) at distance \(\ell\) from a node of degree \(k\). For sparse uncorrelated networks, we have \(P(k'|k, \ell) = k' p(k') / \langle k'\rangle\). Substitute it into Eq. (4), we specify \(\psi_k(\ell)\) as

\[
\psi_k(\ell) = \lambda k \Theta(\mu k)^{\ell-1},
\]

where \(\Theta = \sum_{k'} k' p(k') / \langle k'\rangle\rho_k\), is the probability of a randomly selected link connected to an infected node. On the one hand, Eq. (5) suggests that only when \(\mu k > 1\) will the dynamical correlation length be divergent. On the other hand, when \(\mu k \ll 1\), the total strength of dynamical interaction imposed on nodes with degree \(k\) is \(\Psi_k = \sum_{\ell=1}^\infty \psi_k(\ell) = \lambda k \Theta / (1 - \mu k)\).

Based on the new calculation, we derive a more accurate mean-field equation compared with Eq. (2):

\[
\frac{d\rho_k(t)}{dt} = -\bar{\delta}(k, \lambda)\rho_k(t) + (1 - \rho_k(t)) \frac{1}{1 - \mu k} \lambda k \Theta.
\]

Immediately, it yields \(\rho_k \propto ke^{\alpha k}\), hence

\[
s(k) = 1 + a(\lambda)k,
\]
a coherent explanation for Fig. (1). Since $a(\lambda) \propto \lambda^2$, it only slightly deviates from $s(\lambda) = 1$ for the most small-degree nodes. Though, with increasing of $\lambda$, the deviation $a(\lambda)k$ becomes larger and larger. For the hub nodes, the deviation is so enormous that by no means can be neglected.

Although Eq. (7) can perfectly predict Fig. (1), numerical simulations demonstrate that the spreading dynamics eventually end up with the absorbing phase at the predicted threshold, suggesting a higher genuine threshold, see Fig. (3). In consequence, we have to rethink about the validity of Eq. (1), albeit there is some reasonable element.

In Eq. (3), $\beta$ plays the part of tuning the strength of dynamical interaction, which basically affects the threshold. A higher threshold implies weaker dynamical interaction. The most critical problem is then to make sense of the actual pattern of dynamical interaction, thereby, uncovering the true mechanism of the phase transition. This prompts us to study the influence of the following dynamical interactions $\Omega_k$: the first and second nearest neighbors, $\ell$ order neighbors, all the long-range neighbors, and global nodes. For the sake of clarity, let $\Omega_k = \Pi_k \Theta$, where the interaction strength coefficient

$$\Pi = \begin{cases} 
1 & \text{for } k = 1 \\
\frac{\lambda k}{(\mu k)^{\ell-1}} & \text{for } 2 \leq k < \min (n_c, n_{\max}) \\
\frac{\mu k}{1 - \mu k} & \text{for } k = \min (n_c, n_{\max}) \\
1/(1 - \mu k) & \text{for } k > \min (n_c, n_{\max})
\end{cases}$$

(8)

With these preparations, we just write down a concise equation encompassing the four cases:

$$\frac{d\rho_k(t)}{dt} = -\delta(k, \lambda)\rho_k(t) + \Pi(1 - \rho_k(t))\lambda k \Theta.$$  

(9)

By analyzing the following transcendental equation

$$1 = \lambda \Pi \sum_k k^2 P(k) \exp(a(\lambda)k),$$

(10)

we conclude that the threshold $\lambda_c \to 0$ for $\gamma > 2.5$, only if $P(k)$ decays slower than exponential and $\lambda^2 k_{\max} \to \infty$. Moreover, $\lambda_c \to 0$ at a speed much slower than $\lambda_c^{QMF}$. This is confirmed in Ref. [19] by the lifespans method. The QMF equation predicts a null threshold as well, which has nothing to do with the network heterogeneity. In contrast, here Eq. (10) highlights the important role of the heterogeneity.

Let us focus on a more relevant issue: which kind of dynamical interactions drive the onset of active phase? The answer is involved in the value of $\Pi$, which can be determined from the critical behavior

$$\rho_k \approx \Pi_k \Theta \rho_k \exp(a(\lambda)k).$$

(11)

To mitigate the fluctuation of the simulation data, we first use the accumulative function of Eq. (11) $\phi(k) = \Pi_k \Theta \int_{\max}^k k \exp(a(\lambda)k)dk$ for data fitting. Second, we exploit continuous cutoff of the maximum degree $k_{\max}$ when generating uncorrelated random networks with the configuration model [20]. After normalizing $P(k)$ using the so called structural cutoff $\int_{\max}^k P(k)dk = 1$, cut off the degree at $NP(k_{\max}) = 1$. And the number of nodes with degree $k$ is $n(k) = N P(k)$. In this way, the continuity of $\rho_k$ is guaranteed. These measures allow us to determine the value of $\Pi$ more accurately. Besides, reliable quasi-stationary simulation is also crucial.

We abandon the discrete-time approach, which synchronously updates the states of all the nodes with a fixed time interval such as $\Delta t = 1$, in that it will lead to incorrect result, especially when the recovery rate is unit [21]. Instead, the continuous-time approach is employed in this paper. Once the system visits the absorbing state where all the nodes recover,
we randomly choose a node or select the last active node as the seed to continue the evolution. Such updating scheme can effectively avoid the disadvantage of other schemes such as using the hub or randomly selecting a evolutionary configuration from the memory, which result in a susceptibility curve with multiple ambiguous peaks. Despite such distinction, our tests show that all of these schemes produce similar results near the threshold.

The values of $\Pi$ and $\alpha(\lambda_c)$ are obtained from the fitting curve of $\phi(k)$. In the examples portrayed by Fig. 2, for $2 < \gamma < 2.5$, the empirical value of the interaction strength coefficient $\Pi \approx 1$ ($\alpha = 0$), which is consistent with HMF theory. For $\gamma > 2.5$, however, $\Pi \ll \Pi^G$, namely, the theoretical expectation of the global interaction. More specifically, it is striking that

$$\Pi \approx \lambda \kappa.$$  \hspace{1cm} (12)

That is, it is the SNN that makes an impact.

Above arguments are further confirmed by analyzing the threshold. The corresponding threshold predicted by different interactions can be numerically computed by plugging the empirical value of $a(\lambda_c)$ into Eq. 10. As shown in Fig. 3, the actual threshold is much bigger than the prediction of global interaction. Yet, it is fairly close to the prediction of the SNN. Combine Eq. 12 with Eq. 10, the transcendental equation permits $\lambda_c < \lambda_{HMF}$ for $2.5 < \gamma < 3$, which is demonstrated in Ref. [4] and Fig. 3.

When $2.5 < \gamma < 3$, as shown in Fig. 2 and Fig. 3, the error between the theoretical and numerical results for $\Pi$ and $\lambda_c$ is notable. Because the networks in this regime are still rather dense, and there may be multiple paths between a node and its distant neighbors, which produce stronger interaction than that of tree-like structure. For $\gamma > 3$, the networks become sparse, accordingly, inducing very tiny error. Our calculation provides a lower bound for estimating $\Pi$.

Our comprehensive results of theoretical and numerical studies explicitly convey that it is safe to rule out the long-range interaction highlighted in Eq. 1. And only the SNN interaction should appear in Eq. 9. Therefore, we think the correct mean-field equation for $\gamma > 2.5$ near the critical point is

$$\frac{d\rho_k(t)}{dt} = -\delta(k, \lambda)\rho_k(t) + (1 - \rho_k(t)) \lambda^2 k \kappa \Theta. \hspace{1cm} (13)$$

One may cannot help asking why the SNN interaction? Eq. 13 is intended for super nodes, suggesting a markedly different physical picture for the transition. Leaf nodes centered around a node of degree $k$ group into a super node, which on average has $k \kappa$ neighboring super nodes. See Fig. 4 for the sketch. Since neighboring super nodes are connected by chains of 2 hops, through which they interact with each other, the corresponding infection rate is $\lambda^2$. We call it SNN interaction. A super node will inevitably visit the absorbing state, on account of the dynamical fluctuation. If the SNN interaction is too weak, it cannot be reactivated by neighboring ones, or rather, it is trapped in the absorbing state. Thanks to the broad distribution of the lifespan, those active domains localized around highly connected nodes, e.g., $k \gtrsim 1/\lambda^2$, will annihilate slowly. The system is in the so called Griffiths phase.

The effective recovery rate $\delta(k, \lambda)$ was initially introduced in Eq. 1. The reason was presented as follows: “on long time scales node $i$ is considered as susceptible only when the node and all of its nearest neighbors in the original graph are susceptible”. On this ground, there is no pronounced limitation to the applicable regime of Eq. 1. It seems to be a more accurate counterpart for the QMF equation. This is of course not the case for $2 < \gamma < 2.5$. Here we relate $\delta(k, \lambda)$ to localized state. A super node in localized state alone has exponential growth lifespan $\tau \sim e^{\alpha(\lambda)k}$. Once it recovers, it can be reinfected by neighboring ones and return to localized state, provided that the SNN interaction is strong enough to overcome the irreversible dynamical fluctuation. In this circumstance, we have to replace the spontaneous recovery rate by the effective one $\delta(k, \lambda) = \tau^{-1}$ in the vicinity of critical point. The reactivation triggers the localized-endemic state transition, namely, there are nonvanishing fraction of active nodes in the end.

To summarize, we have proposed a new mean field equation for SIS model, which uncovers the underlying interaction pattern and the physical mechanism for localized state to endemic state transition. For $2 < \gamma < 2.5$, the outbreak of spreading largely corresponds to a branching process. For $\gamma > 2.5$, however, localized state is allowed and things get drastically different. Highly connected nodes and its leaf nodes constitute a super node. Instead of the global interaction, the SNN interaction enables super nodes to get over the irreversible dynamical fluctuation, giving rise to the Griffiths phase to active phase transition.

We theoretically demonstrate the absence of threshold in the thermodynamic limit for heterogeneous networks with $P(k)$ that decays slower than exponential. The threshold converges to zero at a speed much slower than $\lambda_{HMF}^G$, even though $(k^3)$ is...
finite. The null threshold stems from the joint impacts of the internal interaction of a super node maintaining the localized state, and the SNN interaction reactivating neighboring super nodes. The hybrid interaction pattern makes highly connected nodes far more active than the prediction of HMF theory as demonstrated in Fig. (1).

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[1] M. E. J. Newman, Networks: An Introduction. Oxford University Press, Inc. 2010.
[2] A.-L. Barabási, Network science. Cambridge University Press, 2016.
[3] S. Boccaletti, et al. Phys. Rep. 424, 175 (2006).
[4] V. Belik, T. Geisel, D. Brockmann, Phys. Rev. X 1, 011001 (2011).
[5] D. Centola, Science, 329, 1194 (2010).
[6] I. Hanski, Nature 396, 41 (1998).
[7] D. Brockmann, D. Helbing, Science 342, 1337 (2013).
[8] R. Pastor-Satorras, C. Castellano, P. V. Mieghem, and A. Vespignani, Rev. Mod. Phys. 87, 925 (2015).
[9] W. Wang, M. Tang, H. E. Stanley, L. A. Braunstein, Rep. Prog. Phys. 80, 036603 (2017).
[10] R. Pastor-Satorras, A. Vespignani, Phys. Rev. Lett. 86, 3200 (2001).
[11] M. Shrestha, S. V. Scarpino, and C. Moore, Phys. Rev. E 92, 022821 (2015).
[12] C.-R. Cai, Z.-X. Wu, M. Z. Q. Chen, P. Holme, J.-Y. Guan, Phys. Rev. Lett. 116, 258301 (2016).
[13] R. Parshani, S. Carmi, and S. Havlin, Phys. Rev. Lett. 104, 258701 (2010).
[14] S. C. Ferreira, C. Castellano, and R. Pastor-Satorras, Phys. Rev. E 86, 041125 (2012).
[15] C. Castellano, R. Pastor-Satorras, Phys. Rev. Lett. 105, 218701 (2010).
[16] A. V. Goltsev, S. N. Dorogovtsev, J. G. Oliveira, and J. F. F. Mendes, Phys. Rev. Lett. 109, 128702 (2012).
[17] H. K. Lee, P.-S. Shim, and J. D. Noh, Phys. Rev. E 87, 062812 (2013).
[18] M. A. Muñoz, R. Juhász, C. Castellano, and G. Ódor, Phys. Rev. Lett. 105, 128701 (2010).
[19] M. Boguñá, C. Castellano, R. Pastor-Satorras, Phys. Rev. Lett. 111, 068701 (2013).
[20] M. Catanzaro, M. Boguñá, R. Pastor-Satorras, Phys. Rev. E 71, 027103 (2005).
[21] P. G. Fennell, S. Melnik, and J. P. Gleeson, Phys. Rev. E 94, 052125 (2016).
[22] R. S. Sander, G. S. Costa, and S. C. Ferreira, Phys. Rev. E 94, 042308 (2016).