IS IT POSSIBLE TO EXTEND THE DEFORMED WEYL ALGEBRA $W_q(n)$ TO A HOPF ALGEBRA?

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Abstract. We argue that the algebra $W_q(n)$, generated by $n$ pairs of deformed $q$-bosons, does not allow a Hopf algebra structure. To this end we show that it is impossible to define a comultiplication even for the usual, nondeformed case. We indicate how the comultiplication on $U_q[osp(1/2n)]$ can be used in order to construct representations of deformed (not necessarily Hopf) algebras in tensor products of Fock spaces.
In his talk in the Workshop on Harmonic Oscillators [1] L. C. Biedenharn noted that the Weyl algebra \( W_q(n) \), i.e., the infinite-dimensional unital associative algebra, generated by \( n \) pairs of \( q \)-bosons [2-4], does not allow a complete Hopf algebra structure and gave a reference in this respect to me ([1], p.73). This is indeed my point of view, which we discussed with him. I have, however, no proof of the statement. There are only indications of the impossibility to define a (nontrivial) comultiplication \( \Delta \) on \( W_q(n) \) in a sense of a bialgebra, which I would like to outline in the present letter.

To begin with we recall that for a bialgebra \( U \) the comultiplication \( \Delta \) is an (associative) algebra morphism \( U \rightarrow U \otimes U \), which is also coassociative, \( (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \). Given two representations \( \rho_1 : U \rightarrow \text{End}(V_1) \) and \( \rho_2 : U \rightarrow \text{End}(V_2) \) of \( U \) in the category of associative algebras, one can construct, using \( \Delta \), a new representation \( \rho \) of \( U \) in the following way. Let \( a \in U \) and \( \Delta(a) = \sum_i a_i \otimes b_i \). Then \( \rho(a) = \sum_i \rho_1(a_i) \otimes \rho_2(b_i) \in \text{End}(V_1) \otimes \text{End}(V_2) \) gives a representation of \( U \) in \( V_1 \otimes V_2 \). We refer to this property of \( \Delta \) as the main (representation) property of the comultiplication.

If \( L \) is a Lie algebra and \( U(L) \) - its universal enveloping algebra, then \( U(L) \) admits only one (nontrivial) comultiplication:

\[
\Delta(a) = a \otimes 1 + 1 \otimes a \quad \forall a \in L.
\]  

(1)

\[
\Delta(1) = 1 \otimes 1,
\]  

(2)

where \( 1 \) is the unity of \( U(L) \). Indeed, the most general linear map \( \Delta : L \rightarrow U(L) \otimes U(L) \) is

\[
\Delta(a) = \sum_i \varphi_i(a) \otimes x_i + \sum_j y_j \otimes \psi_j(a), \quad \forall a \in L, x_i, y_j \in U(L),
\]

where \( \varphi, \psi : L \rightarrow U(L) \) are linear maps. From the requirement that \( \Delta \) is well defined as a map, i.e.,

\[
[\Delta(a), \Delta(b)] = \Delta([a, b]) \quad \forall a, b \in L,
\]  

(3)

and using the Poincaré-Birkhoff-Witt theorem one derives that \( x_i = y_j = 1 \). Hence \( \Delta(a) = \varphi(a) \otimes 1 + 1 \otimes \psi(a) \). Moreover from (3) it follows also that \( \varphi \) and \( \psi \) are endomorphisms of the Lie algebra \( L \). The coassociativity yields \( \varphi(\varphi(a)) = \varphi(a) \), \( \psi(\psi(a)) = \psi(a) \) for all \( a \in L \). Hence \( \varphi = id \) or 0, \( \psi = id \) or 0. The comultiplications \( \Delta(a) = a \otimes 1, \Delta(a) = 1 \otimes a \) and \( \Delta(a) = 0 \) are trivial from a point of view of the main representation property of the comultiplication. Therefore the only nontrivial comultiplication is (1).

The first indication that there exists no comultiplication on \( W_q(n) \) is based on the observation that it is impossible to define it in the form (1) even for the undeformed Weyl algebra \( W(n) \). In
order to show this, denote by $H_n$ the (abstract) Heisenberg Lie algebra, namely the algebra with $2n + 1$ generators $c, b_1^+, \ldots, b_n^+$, which satisfy the commutation relations

$$[b_i^-, b_j^+] = \delta_{ij}c, \quad [b_i^+, b_j^-] = [c, b_j^+] = [c, b_i^-] = 0, \quad i, j = 1, \ldots, n. \quad (4)$$

Let $U(H_n)$ be its universal enveloping algebra (with unity $1$). Then eqs.(1,2) define a comultiplication on $U(H_n)$ and in particular

$$\Delta(c) = c \otimes 1 + 1 \otimes c. \quad (5)$$

The Weyl algebra $W(n)$ is a factor algebra of $U(H_n)$ with respect to the ideal generated from the element $c - 1$. In other words, $W(n)$ is the associative algebra with free generators $c, b_1^+, \ldots, b_n^+$, the relations (4), and one more relation,

$$c = 1. \quad (6)$$

In view of (6) one can write $W(n)$ in its usual form:

$$[b_i^-, b_j^+] = \delta_{ij}1, \quad [b_i^+, b_j^-] = [b_i^-, b_j^-] = 0, \quad i, j = 1, \ldots, n. \quad (7)$$

Now it is evident that the comultiplication (1,2) is incompatible with the extra relation (6). Indeed, from (6) one has $\Delta(c) = \Delta(1)$, i.e., $c \otimes 1 + 1 \otimes c = 1 \otimes 1$ and hence $2(1 \otimes 1) = 1 \otimes 1$, which is impossible. Certainly, we would have arrived to the same conclusion working directly with $W(n)$, i.e., avoiding the Heisenberg algebra $H_n$.

The second indication of a lack of a comultiplication is based on the observation that the undeformed Weyl algebra $W(n)$ has only one (nontrivial irreducible) unitary representation $\rho_{\text{Fock}}$, namely the one, realized in the well known Fock module $V_{\text{Fock}}$. By unitary representation we understand, as usually, that the representation of the corresponding Weyl group is unitary (see, for instance [5]). Assume that $\Delta$ exists on $W(n)$. Then from the main property of the comultiplication it follows that $V_{\text{Fock}} \otimes V_{\text{Fock}}$ is also a $W(n)$ module. On the other hand, the tensor product of unitary modules is also a unitary module. Hence $V_{\text{Fock}} \otimes V_{\text{Fock}}$ has also to be an unitary module, i.e. to carry a unitary representation of $W(n)$. Therefore $V_{\text{Fock}} \otimes V_{\text{Fock}}$ could be only a direct sum of different copies of Fock modules and a module $V_0$, which is a direct sum of one-dimensional trivial modules,

$$V_{\text{Fock}} \otimes V_{\text{Fock}} = V_0 \oplus \sum_i \oplus(V_{\text{Fock}})_i. \quad (8)$$

In order to show that the decomposition (8) is impossible, we use the circumstance that $W(n)$ can be viewed as a factor algebra of another algebraic structure, namely of the universal
enveloping algebra $U[osp(1/2n)]$ of the orthosymplectic Lie superalgebra $osp(1/2n)$ [6]. The algebra $U[osp(1/2n)]$ is a free algebra of $2n$ generators $B_{1}^{±}, \ldots, B_{n}^{±}$, which satisfy the relations $(\xi, \eta, \epsilon = \pm 1, i, j, k = 1, 2, \ldots, n; [x, y] = xy - yx, \{x, y\} = xy + yx)$

$$[[B_{i}^{x}, B_{j}^{y}], B_{k}^{z}] = (\epsilon - \xi)\delta_{ik}B_{j}^{y} + (\epsilon - \eta)\delta_{jk}B_{i}^{x}. \quad (9)$$

The operators $B_{1}^{±}, \ldots, B_{n}^{±}$ are called para-Bose (pB) operators [7] and they are odd generators of $U[osp(1/2n)]$. The Bose operators satisfy the relations (9); a morphism of $U[osp(1/2n)]$ onto $W(n)$ is given simply by a replacement $B_{i}^{±} \rightarrow b_{i}^{±}$. Therefore any $W(n)$ module and in particular $V_{\text{Fock}}$, $V_{\text{Fock}} \otimes V_{\text{Fock}}$ and $V_{0}$ should be also a $U[osp(1/2n)]$ module. Now it is not difficult to run into a contradiction with the decomposition (8) and hence with the assumption that a comultiplication on $W(n)$ exists. We do not go into the details of the representation theory of the para-Bose operators. We mention only that $V_{\text{Fock}} \otimes V_{\text{Fock}}$ is well defined as a $U[osp(1/2n)]$ module through the (graded) comultiplication

$$\Delta(B_{i}^{±}) = B_{i}^{±} \otimes 1 + 1 \otimes B_{i}^{±} \quad \forall i = 1, \ldots, n. \quad (10)$$

The decomposition of $V_{\text{Fock}} \otimes V_{\text{Fock}}$ into irreducible $U[osp(1/2n)]$ modules is known [8] and it contains representations corresponding to the statistics order of 2. Such representations are not present in the right hand side of the decomposition (8).

We summarize. We have shown that it is impossible to define a comultiplication (and hence to introduce a Hopf algebra structure) neither on $W(n)$, considered as an associative algebra, nor in $W(n)$, considered as an associative superalgebra. It is difficult to imagine that at $q \neq 1$ there might exists $\Delta$, which is undefined for $q = 1$. Certainly, the arguments we have given above are only indications, but not a proof of the conjecture that the deformed Weyl algebra $W_{q}(n)$ cannot be turned into a Hopf (super)algebra.

At present it is known how to deform $U[osp(1/2n)]$ to a Hopf algebra $U_{q}[osp(1/2n)]$ [9,10]. It has been shown that the latter is freely generated by deformed pB operators $B_{1}^{±}, \ldots, B_{n}^{±}$ [11,12]. Moreover, it has been proved that $W_{q}(n)$ is a factor algebra of $U_{q}[osp(1/2n)]$ [13], i.e., that there exists a morphism

$$\pi : U_{q}[osp(1/2n)] \rightarrow W_{q}(n). \quad (11)$$

If $W_{q}(n)$ has only one (nontrivial irreducible) representation, namely the (deformed) Fock representation, then there will be no difficulty to prove the conjecture. To our knowledge, however, a proof of the uniqueness of the Fock representation does not exists.

In conclusion we mention that the comultiplication $\Delta$ of $U_{q}[osp(1/2n)]$ together with the morphism $\pi$ (see(11)) can replace in several respects the lack of a comultiplication on $W_{q}(n)$ as far as
representations of deformed algebras (not necessarily Hopf algebras) are concerned. We have in mind all subalgebras of $U_q[osp(1/2n)]$, among them $U_q[gl(m)]$, $m = 1, \ldots, n$ (which are Hopf subalgebras), $U_q[sp(2m)]$ $m = 1, \ldots, n$ (which are not Hopf subalgebras of $U_q[osp(1/2n)]$; see, for example, [14] for the case $n = 1$) and several others. In order to be more concrete, define by induction a morphism

$$\Delta^{(k)} = [(\otimes_{i=1}^{k-2} id) \otimes \Delta] \circ \Delta^{(k-1)}, \quad \Delta^{(2)} = \Delta, \quad \Delta^1 = id.$$

(12)

of $U_q[osp(1/2n)]$ into $\otimes_{i=1}^k U_q[osp(1/2n)]$. The map

$$\pi^{(k)} = (\otimes_{i=1}^k \pi) \circ \Delta^{(k)} : U_q[osp(1/2n)] \to \otimes_{i=1}^k W_q(n)$$

(13)

is a morphism of $U_q[osp(1/2n)]$ into $\otimes_{i=1}^k W_q(n)$ and therefore it is a good substitute for a comultiplication in a sense that $\pi^{(k)}$ helps to construct representations in tensor products of Fock spaces. Indeed if $A$ is any subalgebra of $U_q[osp(1/2n)]$ and $\rho_{Fock}$ - the representation of the Weyl algebra $W_q(n)$ in the deformed Fock space $V_{Fock}$, then the map

$$\rho^{(k)} = (\otimes_{i=1}^k \rho_{Fock}) \circ \pi^{(k)}$$

(14)

defines a representation of $A$ in $\otimes_{i=1}^k V_{Fock}$. In this way one can use the Fock representations of $A$ in order to construct new representations of the same algebra. Thus, despite of the lack of a comultiplication on $W_q(n)$, one can construct representations of various deformed algebras in any tensor power of Fock spaces.

I am thankful to Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for the kind hospitality at the International Center for Theoretical Physics.

The research was supported through contract Φ-215 of the Committee of Science of Bulgaria.
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