A practical method to estimate the condensate fraction of interacting and trapped Bose atoms

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We suggest a practical way to estimate the condensate fraction of an interacting dilute Bose gas confined by an external harmonic potential as a function of temperature and scattering length. It shows that an increase of the scattering length produces an exponential decrease of condensate fraction.

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Over the last decade the phenomenon of Bose-Einstein condensation of ultra-cold dilute alkali Bose gas confined in a magneto-optic trap has been a subject of fundamental interest. Even below the transition temperature some atoms leave the lowest energy state and remain in the excited state by an atom-atom interaction. This causes a depletion of the condensate number in comparison to its ground state occupation for the non-interacting case, and it depends on temperature and interaction strength both.

The study of the condensate fraction (CF), $N_0/N$, where $N$ being total number of atoms, of a dilute Bose system has a long history. The first microscopic theory was suggested by Bogoliubov. If the system is homogeneous and dilute ($\rho a^3 \ll 1$) where $\rho$ is the density, the CF at $T = 0K$ is known from the Bogoliubov theory as

$$\frac{N_0}{N} = \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1}.$$  \hspace{1cm} (1)

where $\beta = 1/k_B T$ and $\mu$ is the chemical potential. $\varepsilon_n$ is the discrete energy spectrum of the harmonic oscillator given by $\varepsilon_n = \hbar(n_x \omega_x + n_y \omega_y + n_z \omega_z) + \varepsilon_0$, where $n_x, n_y, n_z = 0, 1, 2, ...$ and $\varepsilon_0$ is the zero point energy.

We rewrite Eq. (1) as a sum of two parts: ground state and excited states as

$$N = \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1} + \sum_{n=1}^{\infty} \frac{d(n)}{e^{\beta(\varepsilon_n - \mu)} - 1}$$

$$\equiv N_0^{\text{non}} + N_e,$$ \hspace{1cm} (2)

where $d(n) = (n + 1)(n + 2)/2$ is the degeneracy of the harmonic oscillator. If we consider an isotropic trap of angular frequency $\omega_x = \omega_y = \omega_z = \omega$, the number of atoms in the condensate at temperature $T$ is well-known as

$$N_0^{\text{non}}(T) = N - \left( \frac{k_B T}{\hbar \omega} \right)^3 g_3(z) - \frac{3}{2} \left( \frac{k_B T}{\hbar \omega} \right)^2 g_2(z),$$ \hspace{1cm} (3)

where $z$ is the effective fugacity defined by $z = e^{\beta(\mu - \varepsilon_0)}$ and $g_s$ is the Bose series function defined by $g_s(z) = \sum_{n=1}^{\infty} z^n/n^n$. Eq. (3) is solved numerically to obtain $N_0^{\text{non}}(T)$. Note that the transition temperature for $N = 10^3$ trapped atoms with $\omega = 10^3\sec^{-1}$ is about $150nK$.

In the interacting system of the interaction energy in the ground state $U(N_0, a)$, the zero point energy is shifted from $\varepsilon_0$ to $\varepsilon_0 + U$. Then, it depends on number of condensate atoms $N_0$ and the $s$-wave scattering length $a$, both. In this scheme, the same ground state occupation number can be rewritten as

$$N_0(T, a) = \frac{1}{e^{\beta(\varepsilon_0 - \mu + U)} - 1}.$$ \hspace{1cm} (4)

The ground state information of a dilute Bose gas is obtained from the solution of Gross-Pitaevskii equation as

$$ih \frac{\partial \Psi_0}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \omega^2 r^2 + U_0 |\Psi_0(r)|^2 \right] \Psi_0(r, t),$$ \hspace{1cm} (5)

where $U_0(= 4\pi \hbar^2 a/m)$ is the interaction strength between atoms and $\Psi_0$ is the macroscopic wave function of the condensate which is normalized to the number of particles in the ground state, $N_0$. The macroscopic wave function $\Psi_0$ is obtained from the solution of Eq. (5) numerically and then, the ground state interaction energy $U(N_0, a)$ is given by

$$U(N_0, a) = \int \frac{U_0}{2} |\Psi_0(r)|^4 d^3 r.$$ \hspace{1cm} (6)
FIG. 1: Interaction energy per atom of harmonically trapped and interacting Bose atoms. The unit is $\hbar \omega$.

FIG. 2: Condensate number $N_0$ as a function of scattering length shown for various temperatures.

We plotted the interaction energy in Fig. 2 as a function of a dimensionless interaction strength $g(N_0, a) = 4\pi aN_0/a_{ho}$, where $a_{ho}$ is the harmonic oscillator length given by $a_{ho} = \sqrt{\hbar/m\omega}$. The $U$ is not linear but close to a logarithmic function for small $g$ [8].

A direct calculation of Eq. (4) is not reliable because Eq. (3) is too sensitive to get an accurate $z$. Instead, comparing Eq. (4) with Eq. (2), we may obtain a practical form of the CF as

$$N_0(T, a) \sim N_{0}^{\text{non}}(T) e^{-\beta U(N_0, a)}.$$  

(7)

This approximation is valid only when $|\beta U| \ll 1$. In the non-interacting limit $N_0 \rightarrow N_{0}^{\text{non}}$, and otherwise $N_0$ falls exponentially as $U$ increases.

Because $U$ itself is a function of $N_0$, the CF has to be obtained self-consistently. We obtain $N_{0}^{\text{non}}(T)$ from Eq. (3) and then put it into $g(N_0, a)$ to solve the Eq. (6) numerically. Then, substituting the solution of Eq. (6) into Eq. (7) again, we obtain the interaction energy $U$ as a function of $N_0$ and $a$. Next, substitution of $U$ into Eq. (7) again, we obtain a new $N_0$. Then, it is put back into Eq. (5) for self-consistency.

Calculations are carried out for $|\beta U| < 0.1$ with the practical variables of $N = 10^4$ and $\omega = 10^3\text{sec}^{-1}$. The results are plotted in Fig. 3. The $N_0$ shows an exponential decrease as scattering length $a$ increases. The slope is steeper for lower temperatures. As scattering length increases with other parameters remaining fixed, the depletion of the condensate is enhanced.

We suggested a practical way to estimate the condensate fraction of interacting and trapped Bose atoms in specific experimental ranges. The effect of the attractive interaction is not clear because it may increase CF a little bit for negative scattering length compared with non-interacting system. That is, there exists a possibility an attractive interaction between fermions and bosons may increase CF compared with that of bosons only.

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