Deep Convolutional Sum-Product Networks for Probabilistic Image Representations

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Abstract

Sum-Product Networks (SPNs) are hierarchical probabilistic graphical models capable of fast and exact inference. Applications of SPNs to real-world data such as large image datasets has been fairly limited in previous literature. We introduce Convolutional Sum-Product Networks (ConvSPNs) which exploit the inherent structure of images in a way similar to deep convolutional neural networks, optionally with weight sharing. ConvSPNs encode spatial relationships through local products and local sum operations. ConvSPNs obtain state-of-the-art results compared to other SPN-based approaches on several visual datasets, including color images, for both generative as well as discriminative tasks. ConvSPNs are the first pure-SPN models applied to color images that do not depend on additional techniques for feature extraction. In addition, we introduce two novel methods for regularizing SPNs trained with hard EM. Both regularizations aim at alleviating the effect of an empirically validated relation between the depth of a randomly structured SPN with random weights and an exponentially decreasing variance of the log probability that causes hard EM to perform suboptimally. ConvSPNs are the first pure-SPN models applied to color images that do not depend on additional techniques for feature extraction.

1. Introduction

Sum-Product Networks (Poon & Domingos, 2011) are probabilistic graphical models (PGM) (Koller et al., 2009) designed with computational efficiency in mind. These networks allow for exact and efficient marginal, conditional and joint inference queries and naturally deal with missing data. Other PGMs like Markov Networks and Bayesian Networks are generally unable to compute exact inferences in reasonable time for a large number of variables. SPNs can be seen as a special type of deep neural network that can be trained using the techniques commonly used in deep learning (Peharz et al., 2018), as well as other techniques used for PGMs (Zhao et al., 2016; Rashwan et al., 2018).

SPNs have been used for various visual tasks such as image completion (Poon & Domingos, 2011), classification (Poon & Domingos, 2011), classification with missing features (Jaini et al., 2018b; Peharz et al., 2018), novelty detection (Zheng et al., 2018) and more. However, SPNs have not yet excelled in domains with large amounts of high-dimensional data such as large image datasets. This work takes a first step towards enabling SPNs to successfully perform within these more challenging domains.

Computer vision has recently been dominated by convolutional neural networks (CNNs) (LeCun et al., 1990; Krizhevsky et al., 2012; Szegedy et al., 2015; He et al., 2015). CNNs share their weights across the spatial axes. Weight sharing reduces the risk of overfitting and, together with pooling operations, provides translation invariance. In addition, local connectivity ensures that features are learned locally and gradually increase in complexity when moving up the layer hierarchy. We explore similar architectures for SPNs which we refer to as Convolutional Sum-Product Networks (ConvSPNs). ConvSPNs exploit the inherent structure in image data by using local product and local sum operations, optionally with weight sharing. ConvSPNs yield generative and discriminative abilities for image data that substantially surpass SPN-based approaches from existing literature across several datasets.

Our main contributions are (i) the introduction of novel ConvSPN architectures (ii) an extensive range of experiments on image data with state-of-the-art performance among SPN models and (iii) the introduction of two new regularization methods for SPNs trained with hard EM. Both regularizations aim at alleviating the effect of an empirically validated relation between the depth of a randomly structured SPN with random weights and an exponentially decreasing variance of the log probability that causes hard EM to perform suboptimally. ConvSPNs are the first pure-SPN models applied to color images that do not depend on additional techniques for feature extraction.
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Our models leverage the efficiency of the modern computational TensorFlow framework (Abadi et al., 2015). Most of our code is part of a generic TensorFlow-based library\(^1\) which enables joint, conditional, marginal and MPE inferences as well as generative and discriminative learning for SPNs (Pronobis et al., 2017). It also supports various input distributions so that it can handle both discrete and continuous data or a mix of both.

The paper is structured as follows: Section 2 discusses the theoretical background for SPNs. Section 3 discusses related research on SPN-based approaches for computer vision. Section 4 outlines our approach to ConvSPNs. In Section 5, we propose two novel methods for regularizing SPNs that are trained with hard EM. We then compare the performance of ConvSPNs to other SPNs from related literature for several image datasets in Section 6.

2. Background

A Sum-Product Network \( S(x) \) represents a joint probability distribution over a set of random variables \( X \), where the individual variables \( X_i \) are discrete or continuous. An SPN is a rooted directed acyclic graph where the root node computes the unnormalized probability \( S(x) \) of a distribution at \( x \in X \). The leafs of an SPN correspond to individual random variables \( X_i \). These variables can be either discrete or continuous. In the discrete case, they are typically represented as Bernoulli variables, while Gaussian distributions are often used in the continuous case. In between the leaf nodes and the root, the SPN contains weighted sums and product operations. The weighted sum nodes have non-negative weights and can be interpreted as defining a probabilistic mixture of their input nodes, while products compute joint probabilities of their inputs by multiplying their values. The output of a sum node is given by \( s_j = \sum_{i \in \text{ch}(s_j)} w_{ji} C_i \) where \( C_i \) is the value of the \( i \)-th child and \( \text{ch}(s_j) \) is the set of children of \( s_j \). The output of a product node is given by \( p_j = \prod_{i \in \text{ch}(p_j)} C_i \).

Following (Poon & Domingos, 2011), we define the following concepts related to SPNs:

**Definition 1** (Scope). The scope of a node \( n \), denoted \( \text{sc}(n) \), is the set of variables that are descendants of \( n \).

In other words, the scope of a node is the union of the scopes of its children. Leaf nodes have a **singular scope** containing a single variable \( X_i \).

**Definition 2** (Validity). An SPN is valid if it correctly computes the unnormalized probability for all evidence \( e \) where \( e \subseteq X \).

A sufficient set of conditions that ensure validity consists of **completeness** and **decomposability**:

\(^1\)www.libspn.org

**Definition 3** (Completeness). An SPN is complete if all children of a sum node have identical scopes.

**Definition 4** (Decomposability). An SPN is decomposable if all children of the same product node have pairwise disjoint scopes.

In other words, the scopes of the children of product nodes should have no variables in common.

Evaluating an SPN without any evidence gives the partition function \( Z_S = \sum_{x \in X} S(x) \) which can be used as the normalization constant for computing the probability of \( x \): \( P(x) = S(x) / Z_S \). The constant \( Z_S \) is computed by setting the output of each leaf node to 1. For a Bernoulli variable \( X_i \), both indicators \( X_i \) and \( \overline{X_i} \) are set to 1. If all \( X_i \) are continuous and represented as Gaussian leaf nodes, all of the Gaussian components corresponding to \( X_i \) are set to 1.

A **normalized** SPN contains normalized sum nodes of which the weights add up to one. It has been proven that \( Z_{\overline{S}} = 1 \) and thus \( P(x) = \overline{S(x)} \) where \( \overline{S} \) is a normalized SPN (Peharz, 2015). Hence, a valid normalized SPN accurately models a probability density function or a probability mass function. Hence, for some evidence \( e \subseteq X \), the probability \( P(e) \) can be computed with a single forward pass in \( \overline{S} \).

SPNs can be trained to perform both generative tasks (e.g. using hard EM) as well as discriminative tasks (e.g. using gradient descent).

3. Related Work

In this section we cover related work on computer vision with SPNs. In addition, we briefly elaborate on related work that covers weight sharing in SPNs.

3.1. Computer Vision With Sum-Product Networks

In (Peharz et al., 2018), SPNs with random decompositions, referred to as RAT-SPNs, are shown to yield performance on par with multi-layer perceptrons (MLPs) on several image benchmarks, including the MNIST (Lecun et al., 1998) and Fashion MNIST (Xiao et al., 2017) datasets.

In (Rashwan et al., 2018), discriminative training of SPNs is accomplished through Extended Baum-Welch (EBW) instead of gradient descent methods. The SPN architecture used by (Rashwan et al., 2018) is constructed in a way similar to the image completion experiments in (Poon & Domingos, 2011), where each rectangular region is split into all possible vertical and horizontal rectangular subregions (APVAHRS). The smallest region in this APVAHRS architecture consists of a single pixel.

In (Hartmann, 2014), the authors feed the features of a CNN to an SPN and train the SPN discriminatively with SGD. This is the only related work that mentions convolutions.
on images in the context of SPNs, but the convolutions themselves are not part of the SPN model.

In (Adel et al., 2015) SPNs are trained using an SVD-based structure learning algorithm which significantly outperformed other SPN learning algorithms at the time of publication. Another recent paper on structure learning for SPNs introduces the Prometheus algorithm (Jaini et al., 2018a). This paper also considers MNIST data and together with (Peharz et al., 2018) reports the best classification accuracy on the MNIST dataset thus far.

In (Poon & Domingos, 2011), image completion was performed on the Olivetti (Samaria & Harter, 1994) and Caltech (Fei-Fei et al., 2004) datasets showing superior results compared to other generative models.

Finally, (Gens & Domingos, 2012) introduce discriminative training for SPNs through gradient descent. For image classification on CIFAR-10 (Krizhevsky, 2009), they employ a patch-wise feature extraction method based on k-means and a ‘triangle’ encoding that is max-pooled and then followed by an SPN per class. At the time of publication they reported state-of-the-art accuracies on CIFAR-10.

3.2. Weight Sharing for Sum-Product Networks

Weight sharing for SPNs has been explored before in (Cheng et al., 2014) where the authors looked at language modeling. The weight sharing occurs in the very first layer of the network that is connected to one-hot word feature vectors. Hence, the first layer can be thought of as an embedding layer where the weights are constrained to be strictly positive. It is unclear to what extent the weight sharing improved the performance of the SPN based on the results discussed in (Cheng et al., 2014).

To summarize the above, previous work lacks an attempt to frame SPNs as convolutional architectures. Furthermore, the benefits of weight sharing in SPNs are not yet empirically assessed. We aim to cover both issues among others.

4. Convolutional Sum-Product Networks

We now present Convolutional SPNs (ConvSPNs), our new SPN architecture, which exploits the inherent structure of spatial data in a way similar to deep convolutional neural networks. Since our implementation is tensorized, we first explain the interpretation of each tensor dimension.

ConvSPNs consist of products or weighted sums that combine their inputs locally in a way similar to CNNs. As in most modern CNN layer implementations, ConvSPN layers are represented as 4D tensors with dimensions for (i) the samples in the batch (ii) the rows (iii) the columns and (iv) the channels. From hereon, we omit the batch dimension and discuss ConvSPNs for a single sample for simplicity.

We refer to all channels at a given location of a spatial tensor (e.g. $X[i, j, :]$) as a cell. A node corresponds to a single element of the spatial tensor, e.g. $X[i, j, k]$. Nodes are grouped in either spatial product layers or spatial sum layers.

An SPN must be valid to accurately model a joint probability distribution. As mentioned before, completeness and decomposability are sufficient properties to guarantee validity. We will now elaborate on how to guarantee both completeness in spatial sum layers and decomposability in spatial product layers. Then, we discuss how multiple convolutional layers are stacked to form a deep network.

4.1. Spatial Sum Layers

To maintain the completeness property of a sum node, it can only have child nodes with identical scopes. The input layer of a ConvSPN contains the image and consists of leaf distributions at different cells. Clearly, these cells are unique per pixel and so the first and second axis ($X[i, j, k]$ and $X[i, :, k]$ respectively) of the spatial tensor at the input can be thought of as scope axes. As a consequence, as soon as we consider different child cells ($X[i, j, :], X[u, v, :]$, $i \neq u \lor j \neq v$), we cannot connect a sum node to children from both cells at the same time. Hence, sum nodes must be connected to cell regions of exactly $1 \times 1$, so that a sum’s children correspond to all channels within a single cell. In our experiments, we consider spatial sum layers with shared weights (i.e. a $1 \times 1$ convolutional layer) as well as spatial sum layers without weight sharing.

4.2. Spatial Product Layers

To satisfy decomposability of product nodes, the scopes of a product’s children need to be pairwise disjoint (so $\text{sc}(C_i) \cap \text{sc}(C_j) = \emptyset$, $\forall i, j \in \text{ch}(p)$ s.t. $i \neq j$ where $\text{ch}(p)$ is the set of children of the product node). A spatial product layer combines the nodes of the preceding layer through local products of $k \times k$ patches. The scopes of these products form the union of the scopes under the current patch. In the non-log domain, local products cannot be framed as convolutions. For numerical stability, SPNs are implemented to propagate log probabilities. In the log-domain products become sums. Hence, local products can be implemented as convolutions in the log domain with unit weights. We refer to this interpretation as a convolutional log-product (CLP). The kernels of CLPs can be thought of as ‘one-hot’ in the sense that they have zeros in all but one channel per cell.

It is often infeasible to generate all possible permutations of product children in a CLP layer. For example, if there are 16 input channels and a patch size of $2 \times 2$, then there are $16^4$ possible combinations of product children per patch. To limit the number of products, one can take a random subset of combinations. A special case of a limited subset can
be obtained by computing depth-wise convolutions, so that
the number of output channels equals the number of input
channels (i.e. there is exactly one product per channel).

4.3. Stacking Layers

Stacking layers in a ConvSPN requires a careful design to
preserve SPN validity. Spatial sums do not change the spa-
tial layout of the scopes. However, CLPs combine patches
of multiple cells, which consequently changes the spatial
layout of the scopes. We propose two ways of building
multi-layered ConvSPNs and elaborate on how to choose
the strides, patch size and dilation rates for CLPs so that we
ensure decomposability.

4.3.1. Non-overlapping ConvSPNs

The most straightforward ConvSPN architecture uses CLPs
with strides as large as their kernel size. Let the input
of a CLP be a tensor \( I \in \mathbb{R}^{h_i \times w_i \times d_i} \) (height \( h_i \), width
\( w_i \) and depth \( d_i \)). The output of the layer is a tensor
\( O \in \mathbb{R}^{(h_i/h_r) \times (w_i/w_r) \times d_o} \) and its filter \( F \in \mathbb{R}^{h_r \times w_r \times d_i \times d_o} \).
By using strides of \( h_r \times w_r \), the neighboring patches do not
overlap. As a result, the scopes in \( O \) at different cells are all
disjoint and thus the ConvSPN is still decomposable at \( O \).
Equally sized strides and filters can be applied recursively.
Whenever the spatial dimensions of the input to a CLP are
not a multiple of the kernel size, we can pad accordingly
by nodes with ‘no evidence’ and empty scopes, meaning
that they always have probability one. This conveniently
corresponds to zero-padding in the log domain.

4.3.2. Wicker ConvSPN

A more general architecture, which we call wicker Con-
vpSPN, relies on a more complex spatial pattern of products.
As opposed to non-overlapping ConvSPNs, wicker Con-
vSPNs allow for unit strides without the loss of decompos-
ability. Figure 1 shows an example of a ConvSPN with a
wicker pattern in 1D, where scopes are enumerated from 0
through 7. Although the kernel size remains 2 throughout
all product layers in the architecture, we ensure decompos-
ability by doubling the dilation rate (Yu & Koltun, 2016) at
each CLP, starting at a dilation rate of 1. Note that (Yu &
Koltun, 2016) also advocate the use of exponentially increas-
ing dilations, albeit for convolutional neural networks, to
circumvent the loss of feature resolution or coverage which
would otherwise occur with strides larger than 1 or pooling
layers. The maximum scope cardinality of each cell starts
at 1 at the input and then increases to 2, 4 and 8 after ap-
plying the first, second and third product respectively. Due
to padding, the actual scope cardinality might be smaller
for cells that are near the spatial boundaries. Figure 1 also
reflects the sparsity in the kernels of the log products as each
product node only has one child per input channel.

Figure 1 illustrates an example of a fully convolutional SPN
architecture. In practice, we can construct several arbitrary
valid decompositions after any number of layers. We can
further generalize the wicker ConvSPN architecture. First,
we can use the same connectivity pattern with kernel sizes
larger than two. In addition, striding can be used to reduce
the number of decompositions. In the extreme case, the
strides are as large as the kernel size so that there is exactly
one decomposition (see Section 4.3.1). Figure 1 depicts
the other extreme without any strides, resulting in eight
decompositions. Finally, the wicker pattern also extends to
a number of spatial dimensions larger than 2.

5. Regularization For Generative SPNs

Several regularization techniques have been proposed for
SPNs. In (Poon & Domingos, 2011), \( L_0 \) regularization
was used for training SPNs with hard EM. In (Peharz et al.,
2018), dropout and input evidence masking was introduced
for discriminative SPNs trained with gradient descent.

To further improve the performance of ConvSPNs, we intro-
duce two regularization methods for SPNs trained with hard
EM. Hard EM has proven to be an effective algorithm to
train generative SPNs as it overcomes the vanishing gradient
problem. Hard EM is accomplished by backpropagating the
‘hard’ gradients. Starting at the root node, we determine
the path \( W_x \), which is obtained by taking the maximum
weighted child of a sum node, or by splitting the path for
each product node to all of its children. For each sum node,
the weights are determined by normalizing the path counts
\( \ell_{ji} \) (the number of times the weight was part of a path):
\( w_{ji} = \ell_{ji} / \sum_k \ell_{jk} \). Although hard EM would formally
require MPE inference in the forward pass, we follow the
suggested modification in (Poon & Domingos, 2011) to use
marginal inference in the forward pass instead.

Figure 2 displays how the variance of \( \log(c_i) \) (where \( c_i \)
is the probability of a sum child) decreases exponentially
as we move from the leaf nodes to the root in a randomly
structured SPN with random weights.\(^2\) The variances attain
extremely small values, suggesting that the (log) probability
of sum inputs in the network converge to the same value
as we move from leaf nodes to the root. Consequently, the
value of the weights are the only terms that matter to select
the maximum weighted sum input. In our preliminary exper-
iments, we indeed observed that when randomly structured
SPNs are trained with ‘vanilla’ online hard EM, the distri-
bution of the weight counts for a single sum node tends to
concentrate on a single weight with the other weights never
being selected as the winner. As a result, the largest weight
of the sum thereafter dominates the path selection.

\(^2\)The figure was generated using binarized MNIST data and an
SPN with 2 children per product and 4 mixtures for each group of
products with the same scope.
To alleviate the strong and immediate convergence to single weights in a sum node, we propose to either use unweighted sum inputs in path selection or path selection by sampling.

5.1. Unweighted Sum Inputs

To eliminate the effect of the sum weights on the path selection, we use the unweighted sum inputs to select the path. As a consequence, the weights of a sum node will never enforce a single winning child on their own. We assess this regularization method in Section 6. Path selection by unweighted sum inputs has also been explored in another parameter learning algorithm in (Kalra et al., 2018). However, the experiments in (Kalra et al., 2018) do not assess the influence of this decision. In our experiments, we provide a comparison of unweighted vs. weighted sum children in path selection. We argue that the improved performance
is attributable to alleviating the problem of weights dominating path selection as a consequence of log probabilities converging to zero variance. Because of this convergence, it is important to break ties when selecting the winning child, which we accomplished by randomly sampling from the children with the maximum log probability.

5.2. MPE Path Sampling

Another way of alleviating the deterministic behavior of vanilla hard EM path selection is to sample the winning sum child with the probability given by normalizing the weighted probability of the sum child: \( P(w_{ji} \in W_x) = S_i(x)w_{ji}/(\sum_k S_k(x)w_{jk}) \). In practice, we found that merely sampling does not necessarily yield optimal results. Therefore, we perform the sampling only with some probability \( p_{\text{PS}} \) and take the actual max child otherwise. Our experiments show the positive impact of this approach.

Figure 2. Average variance of \( \log(c_i) \) (where \( c_i \) is the probability of the \( i \)-th child of a sum) for different layers in a sum-product network with random structure and random initial weights for MNIST data. The SPN used products with 2 children and 4 sums per scope. There appears to be an exponentially decreasing variance as the line is almost perfectly straight when viewed using log-scale, except for the values at a distance greater than 14. The latter is presumably caused by floating point precision errors.

6. Experiments

In this section we explore the discriminative and generative capabilities of ConvSPNs on visual tasks. As a generative task, we use image completion. We used image classification to evaluate our ConvSPNs on discriminative problems.

6.1. Learning and Hyperparameters

To train the generative SPNs, we use online hard EM (Poon & Domingos, 2011) with the regularization methods as proposed in Section 5. In addition, we use Gaussian leaf nodes of which the parameters are learned as proposed in (Kalra et al., 2018). We perform image completion by computing the marginal posterior probability at the Gaussian leaf distributions through partial derivatives (Darwiche, 2003). The partial derivatives are used as coefficients to linearly combine the modes of the leaf distributions, similar to (Poon & Domingos, 2011). The linear combinations form the actual predicted completion values. For discriminative SPNs, we use the AMSGrad optimizer (Reddi et al., 2018) to perform gradient descent as it proved to be more stable than the Adam optimizer (Kingma & Ba, 2014) in preliminary experiments across several datasets and learning rates. For discriminative SPNs, we only train the location parameters of either Gaussian or Cauchy leaf nodes. Color pixels are represented as a product of three univariate leaves corresponding to the red, green and blue channels. For all experiments, we performed sample-wise normalization by subtracting the mean and dividing by the standard deviation. We chose to use depth-wise convolutions as soon as the number of input channels exceeds 4. We generate all possible combinations of product children otherwise.

Several other SPN approaches, including those in (Gens & Domingos, 2012) or (Hartmann, 2014), require sub-SPNs per class after which the class-specific SPNs are combined by a single sum node at the root of the SPN. Consequently, these SPN architectures scale poorly when the number of classes is large. In contrast, our ConvSPNs achieve state-of-the-art results with only a single stack of product and sum layers which is shared for all classes followed by a layer of \( K \) sums (where \( K \) is the number of classes) and a root node. Hence, our ConvSPNs are scalable to datasets with potentially thousands of classes.
Whenever we apply dropout (Peharz et al., 2018), we do that throughout the entire network, except for the final sum node with the latent class indicator variable to avoid zero probability outputs.

Table 1 displays the default hyperparameter settings for our experiments. Since we employ a kernel size of $2 \times 2$ for all experiments, we generally require $\lceil \log_2(n) \rceil + 1$ CLPs for $n \times n$ images to join all scopes in the center as well as near the image boundaries (see also Figure 1), which corresponds to $2(\lceil \log_2(n) \rceil + 1)$ layers in total when including sum layers. Since our datasets vary in image sizes, we display the hyperparameters for different dimensions in Table 1.

### 6.2. Datasets

The MNIST dataset (Lecun et al., 1998) contains 60,000 $28 \times 28$ gray-scale images of handwritten digits of which 55,000 are used for training and 5,000 are used for testing with 10 different classes. The Fashion MNIST (Xiao et al., 2017) dataset has the same number of images and image dimensions but contains images of pieces of cloth. The Olivetti dataset (Samaria & Harter, 1994) contains face images of $64 \times 64$ pixels. For the Caltech dataset (Fei-Fei et al., 2004) we use the same 100 $\times 100$ crops as used in (Poon & Domingos, 2011). For Olivetti and Caltech we used the same train and test splits as used in (Poon & Domingos, 2011) to ensure a fair comparison. Finally, we used the CIFAR-10 dataset (Krizhevsky, 2009) with 55,000 train images and 5,000 test images of size $32 \times 32$ in color, again with 10 different classes.

Table 2. Unsupervised image completion. Regularization methods include $L_0$, path sampling (PS), unweighted inputs (UI). The uppermost results for Olivetti and Caltech are taken from (Poon & Domingos, 2011). $L_2$ B corresponds to $L_2$ distance for the case when the bottom half of the image is masked and $L_2$ L is the corresponding distance where the left half is masked.

| DATASET  | REG  | ARCHITECTURE | $L_2$ B | $L_2$ L |
|----------|------|--------------|---------|---------|
| OLIVETTI | $L_0$ | APVAHRS      | 918     | 942     |
| OLIVETTI | –    | WICKERConvSPN| 917     | 879     |
| OLIVETTI | UI   | WICKERConvSPN| 859     | 875     |
| OLIVETTI | PS   | WICKERConvSPN| 799     | 756     |
| OLIVETTI | PS, UI | WICKERConvSPN | 666 | 721 |
| CALTECH  | $L_0$ | APVAHRS      | 3270    | 3551    |
| CALTECH  | –    | WICKERConvSPN| 2971    | 2979    |
| CALTECH  | PS   | WICKERConvSPN| 2851    | 2971    |
| CALTECH  | UI   | WICKERConvSPN| 2570    | 2529    |
| CALTECH  | PS, UI | WICKERConvSPN | 2504 | 2441 |
| MNIST    | PS, UI | WICKERConvSPN | 3058 | 2541 |
| FMNIST   | PS, UI | WICKERConvSPN | 1689 | 2215 |

### 6.3. Results: Generative Learning

Table 2 shows the results for generative unsupervised learning. We task the model with a difficult problem of recreating half of an image based on its other half. We mask either the left or the bottom half of images in each dataset. The performance is assessed based on $L_2$ distance between the original data and the completions. We provide the average $L_2$ distance over the pixels scaled to the range of $[0, 255]$ on test data. Our ConvSPN models consistently outperform the SPNs in (Poon & Domingos, 2011). In addition, the table shows a clear gain in performance for our regularization approaches (MPE path sampling and using unweighted sum inputs) for both the Olivetti and Caltech datasets. Best performance is obtained for both types of regularization combined. This supports the suspicion that the decreasing variances of log probabilities are detrimental to the generative performance of SPNs trained with hard EM.

### 6.4. Results: Discriminative Learning

Table 3 provides an overview of our ConvSPN results together with the results from related work discussed in Section 3. For both MNIST and Fashion MNIST, we present results for a variety of configurations. In a subset of the experiments here, we augmented the data by taking random translations, rotations and crops.

![Figure 3. A random selection of completions for left-occluded test images from the Olivetti dataset. We alternate rows of original images (with masked part highlighted in red) and images with reconstructions done using wicker ConvSPNs. The average $L_2$ distance for this run was 713 (left) and 672 (right) which substantially outperforms (Poon & Domingos, 2011).](image)
Table 3. Results of discriminative experiments on multiple datasets for different regularizations: product dropout (PD), input dropout (ID), data augmentation (DA) and path sampling (PS) as well as weight sharing (WS). Models marked with * are pure-SPN approaches (the remaining approaches rely on an additional model for feature extraction).

| DATA  | ALGORITHM | ARCHITECTURE | LEAF DIST. | REG/SHARING | AUTHORS/NOTES | ACC.  |
|-------|-----------|--------------|------------|-------------|---------------|-------|
| MNIST | EBW       | APVAHRS      | BERNOLLI   | NONE        | (RASHWAN ET AL., 2018)* | 95.07% |
| MNIST | DSPN-SVD  | LEARNED      | N(µ, σ²)  | NONE        | (ADEL ET AL., 2015)* | 97.34% |
| MNIST | PROMETHEUS | LEARNED      | N(µ, σ²)  | NONE        | (JAINI ET AL., 2018A)* | 98.37% |
| MNIST | SGD       | CNN + SPN    | N(µ, σ²)  | NONE        | (HARTMANN, 2014) | 98.34% |
| MNIST | ADAM      | RAT-SPN      | N(µ, σ²σ²I) | PD, ID      | (PEHRARZ ET AL., 2018)* | 98.19% |
| MNIST | AMSGRAD   | CONVSPN      | N(µ, σ²)  | PD, ID      | Ours*         | 98.43% |
| MNIST | AMSGRAD   | CONVSPN      | CAUCHY(x₀, γ) | PD, ID | Ours*         | 98.72% |
| MNIST | AMSGRAD   | CONVSPN      | CAUCHY(x₀, γ) | PD, DA, WS | Ours*         | 98.99% |
| MNIST | AMSGRAD   | CONVSPN      | CAUCHY(x₀, γ) | PD, DA | Ours*         | 99.19% |
| FMNIST| ADAM      | RAT-SPN      | N(µ, σ²²I) | PD, ID      | (PEHRARZ ET AL., 2018)* | 89.52% |
| FMNIST| AMSGRAD   | CONVSPN      | CAUCHY(x₀, γ) | PD, ID | Ours*         | 90.56% |
| FMNIST| AMSGRAD   | CONVSPN      | CAUCHY(x₀, γ) | PD, DA, WS | Ours*         | 90.58% |
| FMNIST| AMSGRAD   | CONVSPN      | CAUCHY(x₀, γ) | PD, DA | Ours*         | 91.63% |
| CIFAR-10| SGD      | CNN + SPN    | N(µ, σ²)  | NONE        | (HARTMANN, 2014) | 53.29% |
| CIFAR-10| AMSGRAD  | CONVSPN      | CAUCHY(x₀, γ) | PD, DA | Ours*         | 68.21% |
| CIFAR-10| SGD      | K-MEANS + SPN| N(µ, σ²²) | NONE        | (GENS & DOMINGOS, 2012) | 83.76% |

First of all, we see that all realizations of ConvSPNs outperform other SPN based approaches found in the literature for all but one dataset. Second, we can see that using Cauchy leaves leads to a considerable improvement over Gaussian leaves. The best results are obtained when combining dropout with data augmentation. Table 3 also shows results for using weight sharing (WS) in the first sum layer of the ConvSPN. Although the performance is slightly inferior compared to the ConvSPN architectures without weight sharing, it is still superior to all alternatives, while reducing the number of trainable parameters by 25 percent compared to architectures without weight sharing. Weight sharing is likely to result in improved accuracy for datasets with even more complex local features and a clear benefit of translation invariance. Lastly, we provided results for CIFAR-10 showing that ConvSPNs clearly outperform the combination of CNNs and SPNs from (Hartmann, 2014), while being inferior to (Gens & Domingos, 2012). However, our ConvSPNs are the only pure-SPN models for color images to date.

7. Discussion

This paper introduced Convolutional Sum-Product Networks (ConvSPNs) that bring SPNs closer to some of the best performing representations for complex visual data. We introduced two approaches to stack layers in a ConvSPN while preserving its validity. Wicker ConvSPNs are the most general form of valid ConvSPNs that allow for overlapping patches through exponentially increasing dilation rates. ConvSPNs outperform other SPN-based approaches for generative and discriminative tasks for image completion and classification. Moreover, ConvSPNs are the first pure-SPN models used for color images without additional feature extraction methods. As opposed to our ConvSPNs, alternative methods with additional feature extraction cannot be used for generative tasks such as image completion.

We introduced two regularization methods for hard EM which yield further improvements for generative tasks across several datasets. Further analysis on the phenomenon motivating our approaches is likely to provide a deeper understanding of generative learning techniques for SPNs.

As opposed to CNNs, ConvSPNs are probabilistic models which naturally deal with missing inputs and can be used for conditional, joint or marginal queries and can deal with both generative and discriminative tasks. However, ConvSPNs still show inferior classification performance compared to CNNs. We intend to further draw from the pool of ideas applied to CNNs and derive theoretical frameworks unifying the different approaches. Additionally, we continue investigating other types of leaf distributions (beyond Gaussian and Cauchy) for representing features in image data.

The proposed architecture is applicable to a wide range of spatial data beyond pure images and 2 dimensions. The ability to model such data probabilistically is extremely beneficial in certain domains such as robotics.

We hope to encourage further development and application of spatial SPN architectures by releasing the code as part of a general SPN learning library.³

³www.libspn.org
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