The coexistence of quasi-periodic and blow-up solutions in a superlinear Duffing equation\(^1\)

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Abstract

In this paper we will construct a continuous positive periodic function \(p(t)\) such that the corresponding superlinear Duffing equation

\[
x'' + a(x) x^{2n+1} + p(t) x^{2m+1} = 0, \quad n + 2 \leq 2m + 1 < 2n + 1
\]

possesses a solution which escapes to infinity in some finite time, and also has infinitely many subharmonic and quasi-periodic solutions, where the coefficient \(a(x)\) is an arbitrary positive smooth periodic function defined in the whole real axis.

Keywords: Superlinear Duffing equations; Blow up; Quasi-periodic solutions.

1. Introduction

In the early 1960’s, Littlewood \(^8\) asked whether all solutions of the second order differential equation

\[
x'' + V_x(x, t) = 0
\]  

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are bounded for all time, that is, \( \sup_{t \in \mathbb{R}}(|x(t)| + |x'(t)|) < +\infty \) holds for all solutions \( x(t) \) of Eq. (1.1).

For the Littlewood boundedness problem, during the past years, people have paid more attention to the following equation with the polynomial potentials

\[
x'' + x^{2n+1} + \sum_{k=0}^{2n} p_k(t)x^k = 0,
\]

(1.2)

where \( p_k(t+1) = p_k(t) \) \( (k = 0, 1, \cdots, 2n) \), since

\[
x'' + x^{2n+1} = 0
\]

is a very nice time-independent integrable system, of which all solutions are periodic. Thus if \( |x| \) is large enough, Eq. (1.2) can be treated as a perturbation of an integrable system, then Moser’s twist theorem could be applied to prove the boundedness of all solutions.

The first result was due to Morris [14], who proved that all solutions of the equation with the biquadratic potential

\[
x'' + 2x^3 = p(t) = p(t + 1)
\]

are bounded.

Using the famous Moser’s twist theorem [15], Diekerhoff and Zehnder [1] generalized Morris’s results to Eq. (1.2). In [1], the coefficients \( p_k(t) \) are required to be sufficiently smooth to construct a series of variable changes to transform Eq. (1.2) into a nearly integrable systems for large energies. In fact, in [1], the smoothness on \( p_k(t) \) depends on the index \( k \).

Later, Yuan [20] proved that all solutions of Eq. (1.2) are bounded if \( p_i(t) \in C^2, n + 1 \leq i \leq 2n; p_i(t) \in C^1, 0 \leq i \leq n \). Recently, we [5] obtained the same conclusion if \( p_i(t) \in C^1, n + 1 \leq i \leq 2n; p_i(t) \in C^0, 0 \leq i \leq n \).

There are other results about the boundedness problem for superlinear Duffing equations during the past years, see [2, 9, 10, 17, 18] and the references therein. As for constructing unbounded solutions for superlinear Duffing equations, there also are some results ([3, 4, 6, 7, 12, 16]). Let us recall the results in [4] and [16], Levi and You [4] proved that the equation

\[
x'' + x^{2n+1} + p(t)x^{2m+1} = 0
\]

with a special discontinuous coefficient \( p(t) = K^{[t]}_{\mod 2}, 0 < K < 1, n + 2 \leq 2m + 1 < 2n + 1 \), possesses an oscillatory unbounded solution. In
2000, Wang [16] constructed a continuous periodic function $p(t)$ such that the corresponding equation

$$x'' + x^{2n+1} + p(t)x^i = 0$$

possesses a solution which escapes to infinity in some finite time, where $n \geq 2$ and $n + 2 \leq i < 2n + 1$.

In this paper we consider the following second order differential equation

$$x'' + a(x) x^{2n+1} + p(t) x^{2m+1} = 0, \quad n + 2 \leq 2m + 1 < 2n + 1, \quad (1.3)$$

where the coefficient $a(x)$ is an arbitrary positive smooth periodic function defined in the whole real axis, will construct a continuous positive periodic function $p(t)$ and obtain the coexistence of quasi-periodic solutions and blow-up phenomena for the corresponding equation (1.3). More precisely, we will prove

**Theorem 1.1.** There exists a continuous positive periodic function $p(t)$ such that the corresponding equation (1.3) possesses a solution which escapes to infinity in some finite time, and also has infinitely many subharmonic and quasi-periodic solutions.

Firstly, we will employ the idea in [16] to construct a continuous positive periodic function $p(t)$ such that the corresponding equation (1.3) possesses a solution which escapes to infinity in some finite time. Here, we will construct the positive periodic function $p(t)$ and the blow up solution $x(t)$ simultaneously.

First of all, we observe that during the time when the curve spirals once around the origin, the action variable $I$ increases at some times and decreases at other times after the action-angle variables $(I, \theta)$ are introduced. Therefore we do not know whether the increment of $I$ is positive or negative. However we can construct a time $t_1 \ll 1$ and modify $p(t) \equiv 1$ on $[0, 1]$ so that the increment of $I$ on this time interval $[0, t_1]$ is positive and equals to

$$O \left( \frac{1}{\tau} I_0^{2m-2n+1} \right)$$

if the initial point $(I(0), \theta(0)) = (I_0, 0)$ is far enough from the origin, where the ”jump” $\frac{1}{\tau} (0 < \frac{1}{\tau} < 1)$ is critical to modify $p(t)$ and to our estimations.

Inductively, we can construct a series of times $t_1, t_2, \cdots, t_i, t_{i+1}, \cdots$ and modify $p(t)$ on $[t_i, t_{i+1}], i = 1, 2, \cdots$, so that on every such interval $[t_i, t_{i+1}]$, the increment is positive and at least

$$O \left( \frac{1}{\tau} I_0^{2m-2n+1} \right).$$
Hence, we can construct a time \( T_1 \leq \frac{1}{\tau} < 1 \), so that the curve spirals at least \( \left[ \frac{1}{\tau} \int_0^{\frac{1}{\tau}} \right] \) times around the origin on the interval \([0, T_1]\) and \( I_1 := I(T_1) > I_0 + \frac{c}{\tau'} \int_0^{\frac{2m+2}{n+2}} \) with \( c > 0 \) independent of induction steps, where \( \frac{2m+2}{n+2} > 1 \) and sufficiently large \( \tau' \) is used to ensure the blow up time not more than 1. This complete an induction step: during the interval of time \([0, T_1]\), \( I \) increases from \( I_0 \) to \( I_1 \).

Inductively, a series of times \( T_1, T_2, \cdots, T_i, T_{i+1}, \cdots \) are constructed such that during the interval of time \([T_k, T_{k+1}]\), \( I \) increases from \( I_k \) to \( I_{k+1} \), where \( I_{k+1} > I_k + \frac{c}{(\tau')^k} \int_0^{\frac{2m+2}{n+2}} \) with the jump \( \frac{1}{\tau'} \), where \( T_{k+1} - T_k \leq \frac{1}{\tau'} \). The reason that the jump is less and less is that we have to assure \( p(t) \) is continuous. Because the exponent \( \frac{2m+2}{n+2} > 1 \), the less and less jump cannot stop the rapid increase of \( I \). If \( \frac{1}{\tau} \) is chosen small enough, we will find that \( T_k \to T_\infty < 1 \) as \( k \to \infty \) and \( I(t) \to +\infty \) as \( t \to T_\infty \).

Once we have found the continuous positive periodic function \( p(t) \) such that the corresponding equation \((1.3)\) possesses a solution which escapes to infinity in some finite time, the remain thing is to apply the result in [11]. To this end, we first introduce this result. Consider the conservative system

\[
x'' + p(t) x^{2m+1} + e(t, x) = 0, \quad m \geq 1, \tag{1.4}
\]

where \( p(t) \) is a continuous and 1-periodic function in the time \( t \), \( e(t, x) \) is also 1-periodic in the time \( t \) and dominated by the power \( x^{2m+2} \) in a neighborhood of \( x = 0 \). Liu in [11] proved that if \( \int_0^1 p(t) dt \neq 0 \), then the trivial solution \( x = 0 \) of Eq.\((1.3)\) is stable in the Liapunov sense if and only if \( \int_0^1 p(t) dt > 0 \) by Moser’s twist theorem. Moreover, by the same argument in [1], if \( \int_0^1 p(t) dt > 0 \), then Eq.\((1.3)\) also has infinitely many subharmonic and quasi-periodic solutions with small amplitudes. Compared Eq.\((1.3)\) with Eq.\((1.4)\), since \( p(t) \) is positive, then Eq.\((1.3)\) also has infinitely many subharmonic and quasi-periodic solutions with small amplitudes.

Therefore, if we find a continuous positive periodic function \( p(t) \) such that the corresponding equation \((1.3)\) possesses a blow-up solution, then such equation also has infinitely many subharmonic and quasi-periodic solutions with small amplitudes simultaneously.

Similar to the above, if we modify \( p(t) \equiv 0 \) in \([0, 1]\), we can construct a continuous non-positive periodic function \( p(t) \) with \( \int_0^1 p(t) dt < 0 \) such that the corresponding equation \((1.3)\) possesses a solution which escapes to infinity.
in some finite time, and at the same time, by Liu's result in [11], the trivial solution \( x = 0 \) of such equation is not stable. Therefore we can obtain

**Theorem 1.2.** There exists a continuous non-positive periodic function \( p(t) \) such that the corresponding equation (1.3) possesses a solution which escapes to infinity in some finite time, and the trivial solution \( x = 0 \) is not stable.

Finally we remark that the authors in [19] also obtained the coexistence of quasi-periodic solutions and blow-up phenomena in a class of higher dimensional Duffing-type equations, and the author [13] obtained the coexistence of bounded and unbounded motions in a bouncing ball model.

The paper is organized as follows. The action-angle variables are introduced in Section 2. In Section 3 we first prove some Lemmas which will be useful later. After that, we will construct a continuous positive periodic function \( p(t) \) and a series of times \( T_k \), then obtain an unbounded solution of equation (1.3) and finish the proof of Theorem 1.1.

2. Action-angle variables

In this section we first introduce action and angle variables. Let \( y = x' \), then Eq. (1.3) is equivalent to the following Hamiltonian system

\[
x' = \frac{\partial H}{\partial y}, \quad y' = -\frac{\partial H}{\partial x},
\]

where the Hamiltonian is

\[
H(x, y, t) = \frac{1}{2} y^2 + G(x) + \frac{p(t)}{2m + 2} x^{2m+2}
\]  

(2.1)

with

\[
G(x) = \int_0^x a(s)s^{2n+1}ds.
\]  

(2.2)

In order to introduce action and angle variables, we consider the auxiliary autonomous system

\[
x' = y, \quad y' = -G'(x).
\]

Since \( a(x) > 0 \) for all \( x \in \mathbb{R} \), then \( G(x) > 0 \) for all \( x \neq 0 \), and all solutions of this system are periodic. For every \( h > 0 \), denote by \( I(h) \) the area enclosed by the closed curve

\[
\Gamma_h : \quad \frac{1}{2} y^2 + G(x) = h.
\]
That is, 
\[ I = I(h) = \int_{\Gamma_h} \sqrt{2(h - G(x))} \, dx. \]

Let \( h = h(I) \) be the inverse function of \( I = I(h) \). Define

\[ S(x, I) = \begin{cases} 
\int_{x_-}^{x} \sqrt{2(h(I) - G(s))} \, ds, & y \geq 0, \\
I - \int_{x_-}^{x} \sqrt{2(h(I) - G(s))} \, ds, & y < 0,
\end{cases} \]

where \( x_- = x_-(I) < 0 \) is determined uniquely by \( G(x_-) = h(I) \).

Now we introduce the well-known action-angle transformation

\[ y = \frac{\partial S}{\partial x}, \quad \theta = \frac{\partial S}{\partial I}. \]

Then

\[ \theta = \begin{cases} 
h'(I) \int_{x_-}^{x} \frac{ds}{\sqrt{2(h(I) - G(s))}}, & y \geq 0, \\
1 - h'(I) \int_{x_-}^{x} \frac{ds}{\sqrt{2(h(I) - G(s))}}, & y < 0.
\end{cases} \]

Denote

\[ \Psi : (\theta, I) \to (x, y), \]

then under \( \Psi \), the Hamiltonian \( H \) of (2.1) is transformed into

\[ H_1 = H \circ \Psi = h(I) + \frac{p(t)}{2m + 2} x(I, \theta)^{2m+1}, \]

and the corresponding Hamiltonian system is

\[ \begin{cases} 
\frac{d\theta}{dt} = h'(I) + p(t)x(I, \theta)^{2m+1}\partial_x x(I, \theta), \\
\frac{dI}{dt} = -p(t)x(I, \theta)^{2m+1}\partial_\theta x(I, \theta).
\end{cases} \quad (2.3) \]

In the following, we do not attempt to obtain estimates with particularly sharp constants. Indeed, we suppress all constants, and use the notations \( u \leq v \) and \( u \cdot v \) to indicate that \( u \leq cv \) and \( cu \leq v \), respectively, with some constant \( c > 0 \).

Now we give some estimates on \( h(I) \) and \( x(I, \theta) \). For this purpose, we first give some properties of the potential function \( G \).
Lemma 2.1. For all \( x \neq 0 \), we have

\[
x^{2n+2} \leq G(x) \leq x^{2n+2}, \quad |G'(x)| \leq |x|^{2n+1},
\]

\[
|G''(x)| \leq (|x|^{2n+1} + x^{2n}), \quad \left| \frac{G(x)}{G'(x)} \right| \leq |x|, \quad \left| \frac{G(x)G''(x)}{G'(x)^2} \right| \leq (|x| + 1).
\]

Proof. Since the periodic function \( a(x) \) is positive, then there exist two positive constants \( m, M \) such that \( 0 < m \leq a(x) \leq M \) for all \( x \in \mathbb{R} \) and from the expression (2.2) of \( G(x) \), we know that for \( x > 0 \),

\[
\frac{m}{2n+2}x^{2n+2} \leq G(x) = \int_0^x a(s)s^{2n+1}ds \leq \frac{M}{2n+2}x^{2n+2},
\]

and if \( x < 0 \),

\[
\frac{m}{2n+2}x^{2n+2} \leq G(x) = \int_0^{-x} a(-s)s^{2n+1}ds \leq \frac{M}{2n+2}x^{2n+2},
\]

which yields the first inequality of this lemma.

If \( x > 0 \), then \( mx^{2n+1} \leq G'(x) = a(x)x^{2n+1} \leq Mx^{2n+1} \); if \( x < 0 \), then \( Mx^{2n+1} \leq G'(x) = a(x)x^{2n+1} \leq mx^{2n+1} \). Combining this two inequalities, one can obtain the second inequality.

Since \( G''(x) = a'(x)x^{2n+1} + (2n+1)a(x)x^{2n} \), if we let \( m_1 \leq a'(x) \leq M_1 \) for all \( x \in \mathbb{R} \), where \( m_1 < 0, M_1 > 0 \) are two constants, then \( m_1 x^{2n+1} + (2n+1)mx^{2n} \leq G''(x) \leq M_1 x^{2n+1} + (2n+1)Mx^{2n} \) for \( x > 0 \), and \( M_1 x^{2n+1} + (2n+1)mx^{2n} \leq G''(x) \leq m_1 x^{2n+1} + (2n+1)Mx^{2n} \) for \( x < 0 \), therefore \( |G''(x)| \leq \max\{|m_1|, M_1\}|x|^{2n+1} + (2n+1)Mx^{2n} \leq (|x|^{2n+1} + x^{2n}) \), which is the third inequality.

For \( x > 0 \), \( \frac{m}{M}x \leq \frac{G(x)}{G'(x)} \leq \frac{M}{m}x \), and if \( x < 0 \), \( \frac{M}{m}x \leq \frac{G(x)}{G'(x)} \leq \frac{m}{M}x \), then the fourth inequality holds.

If \( x > 0 \), \( \frac{m m_1}{(2n+2)m^2}x + \frac{(2n+1)m^2}{(2n+2)m^2} \leq G(x)G''(x) \leq \frac{M M_1}{(2n+2)m^2}x + \frac{(2n+1)M^2}{(2n+2)m^2} \), and if \( x < 0 \), \( \frac{m M_1}{(2n+2)m^2}x + \frac{(2n+1)m^2}{(2n+2)m^2} \leq G(x)G''(x) \leq \frac{M m_1}{(2n+2)m^2}x + \frac{(2n+1)M^2}{(2n+2)m^2} \), hence

\[
\left| \frac{G(x)G''(x)}{G'(x)^2} \right| \leq (|x| + 1).
\]

Up to now, we have finished the proof of the lemma. \( \square \)
Lemma 2.2. For sufficiently large $h > 0$, we have
\[
h^\frac{1}{2+\frac{1}{2n+2}} \cdot \leq I(h) \leq \cdot h^\frac{1}{2+\frac{1}{2n+2}},
\]
\[
h^{-\frac{1}{2}+\frac{1}{2n+2}} \cdot \leq I'(h) \leq \cdot h^{-\frac{1}{2}+\frac{1}{2n+2}},
\]
\[
|I''(h)| \leq h^{-\frac{3}{2}+\frac{1}{2n+2}}.
\]

Proof. Let $x_- < 0 < x_+$ defined by $G(x_-) = G(x_+) = h$ for any $h > 0$, then
\[
I(h) = 2\sqrt{2} \int_{x_-}^{x_+} \sqrt{h - G(x)}\,dx.
\]
By Lemma 2.1, we know that
\[
h^\frac{1}{2n+2} \cdot \leq x_+(h), \left|x_-(h)\right| \leq \cdot h^\frac{1}{2n+2},
\]
which implies that
\[
I(h) \leq \cdot h^\frac{1}{2+\frac{1}{2n+2}}.
\]
On the other hand, if we let $\bar{x} > 0$ determined by $G(\bar{x}) = \frac{h}{2}$, then
\[
h^\frac{1}{2n+2} \cdot \leq \bar{x}(h) \leq \cdot h^\frac{1}{2n+2},
\]
and for $0 \leq s \leq \bar{x}$,
\[
\sqrt{h - G(s)} \geq \sqrt{h - G(\bar{x})} \geq h^\frac{1}{2},
\]
thus
\[
I(h) \geq 2\sqrt{2} \int_0^\bar{x} \sqrt{h - G(s)}\,ds \geq \cdot h^\frac{1}{2+\frac{1}{2n+2}}.
\]
Now we prove the second inequality. From the expression of $I(h)$, we know that
\[
I'(h) = \sqrt{2} \int_{x_-}^{x_+} \frac{dx}{\sqrt{h - G(x)}} = \sqrt{2} \int_{x_-}^0 \frac{dx}{\sqrt{h - G(x)}} + \sqrt{2} \int_0^{x_+} \frac{dx}{\sqrt{h - G(x)}}.
\]
The second term can be rewritten as follows
\[
\int_0^{x_+} \frac{dx}{\sqrt{h - G(x)}} = \int_0^\bar{x} \frac{dx}{\sqrt{h - G(x)}} + \int_\bar{x}^{x_+} \frac{dx}{\sqrt{h - G(x)}}.
\]
For $0 \leq x \leq \bar{x}$, we have

$$h^{-\frac{1}{2}} \cdot \leq \frac{1}{\sqrt{h - G(x)}} \leq \cdot h^{-\frac{1}{2}},$$

which together with (2.1) implies that

$$h^{-\frac{1}{2} + \frac{1}{2n+2}} \cdot \leq \int_{0}^{\bar{x}} \frac{dx}{\sqrt{h - G(x)}} \leq \cdot h^{-\frac{1}{2} + \frac{1}{2n+2}}.$$

If $\bar{x} \leq x \leq x_+$, then

$$h - G(x) = G(x_+) - G(x) = G'(\xi)(x_+ - x), \quad \xi \in (x, x_+) \subset (\bar{x}, x_+)$$

and

$$h^{1 - \frac{1}{2n+4}} \cdot \leq \bar{x}^{2n+1} \cdot \leq G'(\xi) = a(\xi)\xi^{2n+1} \leq \cdot x_+^{2n+1} \leq \cdot h^{1 - \frac{1}{2n+2}},$$

which implies that

$$\frac{h^{\frac{1}{2} + \frac{1}{4n+4}}}{\sqrt{x_+ - x}} \cdot \leq \frac{1}{\sqrt{h - G(x)}} \leq \cdot \frac{h^{-\frac{1}{2} + \frac{1}{4n+4}}}{\sqrt{x_+ - x}}.$$

Combining the last equation with the fact that

$$h^{\frac{1}{2n+4}} \cdot \leq \int_{\bar{x}}^{x_+} \frac{dx}{\sqrt{x_+ - x}} \leq \cdot h^{\frac{1}{2n+4}},$$

we obtain

$$h^{-\frac{1}{2} + \frac{1}{2n+2}} \cdot \leq \int_{\bar{x}}^{x_+} \frac{dx}{\sqrt{h - G(x)}} \leq \cdot h^{-\frac{1}{2} + \frac{1}{2n+2}},$$

and thus

$$h^{-\frac{1}{2} + \frac{1}{2n+2}} \cdot \leq \int_{0}^{x_+} \frac{dx}{\sqrt{h - G(x)}} \leq \cdot h^{-\frac{1}{2} + \frac{1}{2n+2}}.$$

Similarly, one can prove that

$$h^{-\frac{1}{2} + \frac{1}{2n+2}} \cdot \leq \int_{x_-}^{0} \frac{dx}{\sqrt{h - G(x)}} \leq \cdot h^{-\frac{1}{2} + \frac{1}{2n+2}},$$

which completes the proof of the second inequality.
Finally we prove the estimate on $I''(h)$. From [2], we know that

$$I''(h) = \frac{\sqrt{2}}{h} \int_{x_-}^{x_+} \left( \frac{1}{2} - \frac{G(x)G''(x)}{G'(x)^2} \right) \frac{dx}{\sqrt{h - G(x)}}.$$  

By Lemma [2.1] for sufficiently large $h > 0$, we have

$$\left| \frac{G(x)G''(x)}{G'(x)^2} \right| \leq \cdot \cdot (|x| + 1) \leq \cdot \cdot h^{1/(2n+2)},$$

and thus

$$|I''(h)| \leq \cdot \cdot h^{n/(2n+2)} - 1 I'(h) \leq \cdot \cdot h^{-\frac{3}{2} + \frac{1}{n+1}},$$

which finishes the proof of this lemma. \hfill \Box

Since $h = h(I)$ is the inverse function of $I = I(h)$, we immediately obtain

**Lemma 2.3.** For sufficiently large $I > 0$, we have

$$I^{2n+2} \cdot \leq h(I) \leq \cdot \cdot I^{2n+2}, \quad I^{n+2} \cdot \leq h'(I) \leq \cdot \cdot I^{n+2}, \quad |h''(I)| \leq \cdot \cdot I^{-\frac{1}{n+2}}.$$

Now we give some estimates on $x(I, \theta)$.

**Lemma 2.4.** For sufficiently large $I > 0$, we have

$$\partial_{\theta} x > 0, \text{ when } y > 0, \quad \partial_{\theta} x < 0, \text{ when } y < 0;$$

$$|x| \leq \cdot \cdot I^{\frac{1}{n+2}}, \quad |\partial_{\theta} x| \leq \cdot \cdot I^{\frac{1}{n+2}}, \quad |\partial_I x| \leq \cdot \cdot I^{-\frac{n}{n+2}}.$$

**Proof.** The inequality $|x| \leq \cdot \cdot I^{\frac{1}{n+2}}$ is obvious. According to the definition of $\theta$, when $y > 0$ (or $0 > \theta > \frac{1}{2}$), we have

$$\theta = h'(I) \int_{x_-}^{x} \frac{ds}{\sqrt{2(h(I) - G(s))}}.$$  

Differentiating the above equality with respect to $\theta$ yields that

$$1 = \frac{h'(I)}{\sqrt{2(h(I) - G(x))}} \partial_{\theta} x,$$

which implies that

$$\partial_{\theta} x = \frac{\sqrt{2(h(I) - G(x))}}{h'(I)}.$$  

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Therefore, $\partial_x y > 0$ for $0 < \theta < \frac{1}{2}$, and by Lemma 2.3, $\partial_x y \leq \cdot I^{\frac{1}{n+2}}$ holds.

Similarly, when $y < 0$ (or $0 < \theta < 1$), we have

$$\partial_x y = -\frac{\sqrt{2(h(I) - G(x))}}{h'(I)},$$

hence $\partial_x y < 0$ and $|\partial_x y| \leq \cdot I^{\frac{1}{n+2}}$.

Now we prove the estimate on $\partial_I x$. From [2], when $y \geq 0$ (or $0 \leq \theta \leq \frac{1}{2}$), we have

$$\partial_I x = \sqrt{2(h(I) - G(x))} \int_{x^2}^{x} \frac{L(I, s)ds}{\sqrt{2(h(I) - G(s))}} + \frac{h'(I) G(x)}{h(I) G'(x)},$$

where

$$L(I, x) = \frac{-h''(I)}{h'(I)} - \frac{h'(I)}{2h(I)} \left(1 - \frac{2G(x)G''(x)}{G'(x)^2}\right).$$

According to Lemmas 2.1, 2.3, $|L| \leq \cdot I^{\frac{n+1}{n+2}}$ and $\left|\frac{h'(I) G(x)}{h(I) G'(x)}\right| \leq \cdot I^{-\frac{n}{n+2}}$ hold.

Also, since for $x_+ \leq s \leq x$, $\frac{\sqrt{2(h(I) - G(x))}}{\sqrt{2(h(I) - G(s))}} \leq 1$, then $|\partial_I x| \leq \cdot I^{-\frac{n}{n+2}}$. One can obtain the same estimate for $y < 0$. Thus, we have finished the proof of this lemma.

If we define $x_1, x_2, x_3$ by

$$x(I, \theta) = I^{\frac{1}{n+2}} x_1(I, \theta), \quad \partial_I x(I, \theta) = I^{-\frac{n}{n+2}} x_2(I, \theta), \quad \partial_x x(I, \theta) = I^{\frac{1}{n+2}} x_3(I, \theta),$$

then they are bounded functions for sufficiently large $I$ and all $\theta \in \mathbb{R}$, that is, there are three positive constants $B_1, B_2, B_3$ such that

$$|x_i(I, \theta)| \leq B_i, \quad i = 1, 2, 3. \quad (2.5)$$

Furthermore, there exist two positive constants $C_1, C_2$ such that

$$-x_1(I, \theta) \geq C_1, \quad x_3(I, \theta) \geq C_2 \quad (2.6)$$

for sufficiently large $I$ and $\theta \in [\frac{1}{16}, \frac{3}{16}]$. Moreover, we can rewrite system (2.3) into

$$\begin{aligned}
\frac{d\theta}{dt} &= h'(I) + p(t) I^{\frac{2m+1-n}{n+2}} x_1(I, \theta) x_2(I, \theta), \\
\frac{dI}{dt} &= -p(t) I^{\frac{2m+1}{n+2}} x_1(I, \theta) x_3(I, \theta). \quad (2.7)
\end{aligned}$$

By Lemma 2.3 we know that $I^{\frac{n}{n+2}} \cdot \leq h'(I) \leq \cdot I^{\frac{n}{n+2}}$. Also by our assumption, $2m + 1 \leq 2(n - 1) + 1$, thus $2m + 1 - n \leq n - 1$ and the right of the first equation in (2.7) is dominated by $h'(I)$ for sufficiently large $I$. 

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3. The proof of Theorem 1.1

Now we define \( p(t) \) in \([0, 1]\). We will construct a time \( t_1 < 1 \) and modify \( p^0(t) \equiv 1 \) on \([0, 1]\) so that the action of one solution of (2.7) increases in \([0, t_1]\). We divide the construction into two steps: first, we construct a piecewise continuous function \( p^1(t) \) so that the action of one solution of (2.7) obtains a positive increment in \([0, t_1]\) as we expect. Then we modify this function \( p^1(t) \) into a continuous one in such a way that the modification does not influence the estimate we had obtained before.

Without loss of generality, we assume that the function \( a(x) \) is even. Denote the corresponding Hamiltonian system (2.7) with the coefficient function \( p(t) \) by \( X_p \). Suppose the solution \((I(t), \theta(t))\) of \( X_p^0 \) with \((I(0), \theta(0)) = (I_0, 0)\) at \( t = 0 \) arrives at \((I_{\frac{1}{4}}, \frac{1}{4})\) at \( t = t_{\frac{1}{4}} \ll 1 \), where \( I_0 \) is a sufficiently large constant which will be determined later. Define \( p^1_4(t) \) be a piecewise continuous function as follows

\[
p^1_4(t) = \begin{cases} 
1, & t \in [0, t_{\frac{1}{4}}], \\
1 - \sigma, & t \in (t_{\frac{1}{4}}, 1],
\end{cases}
\]

where \( 0 < \sigma < 1 \) is the jump, which is used to control the increment of \( I \).

Suppose the solution \((I(t), \theta(t))\) of \( X_{p^1_4} \) with \((I(0), \theta(0)) = (I_0, 0)\) at \( t = 0 \) arrives at \((I_{\frac{3}{4}}, \frac{3}{4})\) at \( t = t_{\frac{3}{4}} \ll 1 \). Define \( p^2_4(t) \) be a piecewise continuous function as follows

\[
p^2_4(t) = \begin{cases} 
1, & t \in [0, t_{\frac{1}{4}}], \\
1 - \sigma, & t \in (t_{\frac{1}{4}}, t_{\frac{3}{4}}], \\
1, & t \in (t_{\frac{3}{4}}, 1].
\end{cases}
\]

Suppose the solution \((I(t), \theta(t))\) of \( X_{p^2_4} \) with \((I(0), \theta(0)) = (I_0, 0)\) at \( t = 0 \) arrives at \((I_{\frac{1}{4}}, \frac{3}{4})\) at \( t = t_{\frac{1}{4}} \ll 1 \). Define \( p^3_4(t) \) be a piecewise continuous function as follows

\[
p^3_4(t) = \begin{cases} 
1, & t \in [0, t_{\frac{1}{4}}], \\
1 - \sigma, & t \in (t_{\frac{1}{4}}, t_{\frac{3}{4}}], \\
1, & t \in (t_{\frac{3}{4}}, 1].
\end{cases}
\]
Suppose the solution \((I(t), \theta(t))\) of \(X_{\mathbf{p}_1}\) with \((I(0), \theta(0)) = (I_0, 0)\) at \(t = 0\) arrives at \((I_1, 1)\) at \(t = t_1 \ll 1\). Define \(p_1(t)\) be a piecewise continuous function as follows
\[
p_1(t) = \begin{cases} 
1, & t \in [0, t_4], \\
1 - \sigma, & t \in (t_4, t_4], \\
1, & t \in (t_4, t_1], \\
1 - \sigma, & t \in (t_1, 1], \\
1, & t \in (1, 1].
\end{cases}
\tag{3.1}
\]
That is to say, the solution \((I(t), \theta(t))\) of \(X_{p_1}\) with \((I(0), \theta(0)) = (I_0, 0)\) at \(t = 0\) arrives at \((I_1, 1)\) at \(t = t_1 \ll 1\), arrives at \((I_1, 1)\) at \(t = t_1 \ll 1\), arrives at \((I_1, 1)\) at \(t = t_1 \ll 1\), which finishes one cycle of the construction of \(p(t)\).

Now we estimate the differences \(I_1 - I_0\) and \(t_1 - t_0\).

**Lemma 3.1.** If \(I_0\) is sufficiently large, then
\[
I_0^{-\frac{n}{n+2}} \leq t_4 \leq I_0^{-\frac{n}{n+2}},
\]
\[
I_0^{-\frac{2m+2-n}{n+2}} \leq I_1 - I_0 \leq I_0^{-\frac{2m+2-n}{n+2}}.
\]

**Proof.** Because \(x_1 < 0, x_3 > 0\) for \(\theta \in (0, \frac{1}{4})\), then \(\frac{dI}{dt} > 0\) for \(t \in (0, t_4)\) and thus \(I(t)\) is an increasing function in this interval. Integrating the first equation of (2.7) from \(t = 0\) to \(t = t_4\) yields that
\[
t_4 \leq \frac{1}{4} \left( h'(I_0) - B_1^{2m+1} B_2 I_0^{\frac{2m+1-n}{n+2}} \right)^{-1} \leq \cdot I_0^{-\frac{n}{n+2}} \tag{3.2}
\]
for sufficiently large \(I_0 > 0\), here we use the estimate on \(h'(I)\) in Lemma 2.3 and the bound \(B_i\) of \(x_i\) in (2.3).

From the second equation of (2.7), we have
\[
\frac{dI}{I^{\frac{2m+2-n}{n+2}}} = -x_1(I, \theta)^{2m+1} x_3(I, \theta)dt. \tag{3.3}
\]
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Since $x_1 < 0, x_3 > 0$ for $\theta \in (0, \frac{1}{4})$, and also integrating the above equation (3.3) from $t = 0$ to $t = t_{\frac{1}{4}}$, one can obtain

$$\frac{n + 2}{2m - n} \left( I_{\frac{1}{4}}^{\frac{2m-n}{n+2}} - I_0^{\frac{2m-n}{n+2}} \right) = \int_0^{t_{\frac{1}{4}}} x_1(I, \theta)^{2m+1} x_3(I, \theta) dt, \quad (3.4)$$

and by (3.2), we get

$$\frac{n + 2}{2m - n} \left( I_0^{\frac{2m-n}{n+2}} - I_{\frac{1}{4}}^{\frac{2m-n}{n+2}} \right) \leq cB_1^{2m+1} B_2 I_0^{-\frac{n}{n+2}},$$

where the constant $c > 0$ is given by (3.2), which implies that

$$I_{\frac{1}{4}}^{\frac{2m-n}{n+2}} \geq I_0^{\frac{2m-n}{n+2}} - \bar{c} I_0^{-\frac{n}{n+2}} = I_0^{\frac{2m-n}{n+2}} \left( 1 - \bar{c} I_0^{-\frac{2m-2n}{n+2}} \right),$$

here the constant $\bar{c} = c^{2m-n} B_1^{2m+1} B_2 > 0$. Hence we obtain

$$I_{\frac{1}{4}} \leq I_0 \left( 1 - \bar{c} I_0^{-\frac{2m-2n}{2m-n}} \right)^{-\frac{n+2}{2m-n}},$$

which leads to

$$I_{\frac{1}{4}} - I_0 \leq \cdot I_0^{-\frac{2m-2n}{2m-n}}.$$

On the other hand, from the first equation of (2.7), we have

$$t_{\frac{1}{4}} \geq \frac{1}{4} \left( h'(I_{\frac{1}{4}}) + B_1^{2m+1} B_2 I_0^{-\frac{2m-n}{n+2}} \right)^{-1} \geq \cdot \left( I_{\frac{1}{4}}^{-\frac{n}{n+2}} + I_{\frac{1}{4}}^{\frac{2m+1-n}{n+2}} \right)^{-1} \geq \cdot I_{\frac{1}{4}}^{-\frac{n}{n+2}} \geq \cdot I_0^{-\frac{n}{n+2}}.$$

Finally, it follows from (2.6) and (3.4) that

$$\frac{n + 2}{2m - n} \left( I_0^{\frac{2m-n}{n+2}} - I_{\frac{1}{4}}^{\frac{2m-n}{n+2}} \right) \geq cC_1^{2m+1} C_2 \left( t_{\frac{1}{4}} - t_{\frac{n}{n+2}} \right).$$
Similarly, the following estimate
\[ t_{\frac{3}{10}} - t_{\frac{1}{10}} \geq \cdot I_{0}^{-\frac{n}{n+2}} \]
holds. Combining the two inequalities above yields that
\[ I_{0}^{2m-n \over n+2} - I_{\frac{1}{4}}^{2m-n \over n+2} \geq \cdot I_{0}^{-\frac{n}{n+2}}, \]
and
\[ I_{\frac{1}{4}} - I_{0} \geq \cdot I_{0}^{2m+2-n \over n+2}. \]

**Lemma 3.2.** If \( I_{0} \) is sufficiently large, then
\[ I_{0}^{-\frac{n}{n+2}} \cdot t_{\frac{3}{10}} - t_{\frac{1}{10}} \leq \cdot I_{0}^{-\frac{n}{n+2}}, \]
\[ (1 - \sigma)I_{0}^{2m+2-n \over n+2} \cdot \leq I_{\frac{1}{4}} - I_{\frac{1}{2}} \leq \cdot (1 - \sigma)I_{0}^{2m+2-n \over n+2}. \]

**Proof.** Because \( x_{1} > 0, x_{3} > 0 \) for \( \theta \in (\frac{1}{4}, \frac{1}{2}) \), then \( {\text{df}} \over {\text{dt}} < 0 \) for \( t \in (t_{1}, t_{1}) \) and thus \( I(t) \) is an decreasing function in this interval. Integrating the first equation of (2.7) from \( t = t_{\frac{1}{4}} \) to \( t = t_{\frac{1}{2}} \) yields that
\[ I_{\frac{1}{4}}^{-\frac{n}{n+2}} \cdot t_{\frac{3}{10}} - t_{\frac{1}{10}} \leq \cdot I_{\frac{1}{4}}^{-\frac{n}{n+2}}. \]  
(3.5)

From the second equation of (2.7), we have
\[ (1 - \sigma)I_{\frac{1}{4}}^{-\frac{n}{n+2}} \cdot \leq I_{\frac{1}{4}}^{2m-n \over n+2} - I_{\frac{1}{4}}^{2m-n \over n+2} \leq \cdot (1 - \sigma)I_{\frac{1}{4}}^{-\frac{n}{n+2}}, \]
and thus
\[ (1 - \sigma)I_{\frac{1}{4}}^{2m+2-n \over n+2} \cdot \leq I_{\frac{1}{4}} - I_{\frac{1}{2}} \leq \cdot (1 - \sigma)I_{\frac{1}{4}}^{2m+2-n \over n+2}. \]  
(3.6)

By Lemma 3.1, we have
\[ I_{\frac{1}{4}} \leq I_{0} \left( 1 + cl_{0}^{2m-2n \over n+2} \right) \]
with some constant \( c > 0 \), which implies that
\[ I_{\frac{1}{4}}^{-\frac{n}{n+2}} \geq \cdot I_{0}^{-\frac{n}{n+2}} \]  
(3.7)
holds for sufficiently large $I_0 > 0$. Meanwhile it follows from (3.6) that

$$I_\frac{1}{2} \geq \cdot I_\frac{1}{4},$$

which together with $I_\frac{1}{4} > I_0$ implies that

$$I_\frac{1}{2} \geq \cdot I_0, \quad (3.8)$$

and

$$I_{\frac{n}{n+2}} \leq \cdot I_0^{\frac{n}{n+2}}. \quad (3.9)$$

Combining (3.5), (3.7) with (3.9), we obtain

$$I_{\frac{n}{n+2}} \cdot \leq t_\frac{1}{4} - t_\frac{1}{2} \leq \cdot I_0^{\frac{n}{n+2}}.$$

Also, according to (3.6), (3.8) and $I_\frac{1}{4} \geq I_0$, one can obtain the second inequality in this lemma. \square

Using the same method, one can prove the following result.

**Lemma 3.3.** If $I_0$ is sufficiently large, then

$$I_{\frac{n}{n+2}} \cdot \leq t_\frac{1}{4} - t_\frac{1}{2} \leq \cdot I_0^{\frac{n}{n+2}},$$

$$I_{\frac{n}{n+2}} \cdot \leq t_1 - t_\frac{1}{4} \leq \cdot I_0^{\frac{n}{n+2}},$$

$$I_{\frac{2m+2-n}{n+2}} \cdot \leq I_{\frac{1}{4}} - I_\frac{1}{2} \leq \cdot I_0^{\frac{2m+2-n}{n+2}},$$

$$(1 - \sigma)I_0^{\frac{n}{n+2}} \cdot \leq I_{\frac{1}{4}} - I_1 \leq \cdot (1 - \sigma)I_0^{\frac{2m+2-n}{n+2}}.$$ 

Combining Lemmas 3.1, 3.2 and 3.3, we can obtain immediately the estimates on the time $t_1$ when the curve spirals once around the origin and the increment of the action variable $I_1 - I_0$.

**Lemma 3.4.** If $I_0$ is sufficiently large, then

$$I_{\frac{n}{n+2}} \cdot \leq t_1 \leq \cdot I_0^{\frac{n}{n+2}},$$

$$\sigma I_{\frac{2m+2-n}{n+2}} \cdot \leq I_1 - I_0 \leq \cdot \sigma I_0^{\frac{2m+2-n}{n+2}}.$$
Now we modify the piecewise continuous function $p^1(t)$ of (3.1) into a continuous one. Being short of signs, we keep the notations unchanged in the process of modification. For example, $p^1(t)$ denotes the continuous function modified from the original piecewise continuous function $p^1(t)$.

First we modify $p^1(t)$ on the interval $[t_\frac{1}{4}, t_\frac{1}{4} + I_0^{-\eta}]$ ($\eta > \frac{n}{n+2}$) to be $\sigma(t_\frac{1}{4} - t) I_0^n + 1$. It is easy to see that $\{(t, p^1(t)) : t \in [t_\frac{1}{4}, t_\frac{1}{4} + I_0^{-\eta}]\}$ is the line segment connecting $(t_\frac{1}{4}, 1)$ and $(t_\frac{1}{4} + I_0^{-\eta}, 1-\sigma)$.

In view of the mean value theorem, there must exist a unique new time $t_\frac{3}{4}$ such that $\theta(t_\frac{3}{4}) = \frac{1}{2}$ if we let

$$p^1_\frac{3}{4}(t) = \begin{cases} 
1, & t \in [0, t_\frac{1}{4}], \\
\sigma(t_\frac{1}{4} - t) I_0^n + 1, & t \in (t_\frac{1}{4}, t_\frac{1}{4} + I_0^{-\eta}], \\
1 - \sigma, & t \in (t_\frac{1}{4} + I_0^{-\eta}, t_\frac{3}{4} - I_0^{-\eta}], \\
\sigma(t - t_\frac{3}{4}) I_0^n + 1, & t \in (t_\frac{3}{4} - I_0^{-\eta}, t_\frac{3}{4}], \\
1, & t \in (t_\frac{3}{4}, 1].
\end{cases}$$

Similarly, there exist the unique $t_\frac{3}{4}$ and $t_1$ such that $\theta(t_\frac{3}{4}) = \frac{3}{4}$ and $\theta(t_1) = 1$ for $X_{p^1}$ with

$$p^1(t) = \begin{cases} 
1, & t \in [0, t_\frac{1}{4}], \\
\sigma(t_\frac{1}{4} - t) I_0^n + 1, & t \in (t_\frac{1}{4}, t_\frac{1}{4} + I_0^{-\eta}], \\
1 - \sigma, & t \in (t_\frac{1}{4} + I_0^{-\eta}, t_\frac{3}{4} - I_0^{-\eta}], \\
\sigma(t - t_\frac{3}{4}) I_0^n + 1, & t \in (t_\frac{3}{4} - I_0^{-\eta}, t_\frac{1}{4}], \\
1, & t \in (t_\frac{3}{4}, t_\frac{1}{4}], \\
\sigma(t_\frac{1}{4} - t) I_0^n + 1, & t \in (t_\frac{1}{4}, t_\frac{1}{4} + I_0^{-\eta}], \\
1 - \sigma, & t \in (t_\frac{1}{4} + I_0^{-\eta}, t_1 - I_0^{-\eta}], \\
\sigma(t - t_1) I_0^n + 1, & t \in (t_1 - I_0^{-\eta}, t_1], \\
1, & t \in (t_1, 1].
\end{cases}$$

Now the newest coefficient is already a continuous function. It is easy to check that Lemmas (3.1–3.4) still hold with different constants after this modification in view of $I_0^{-\eta} \ll I_0^{\frac{n}{n+2}}$. 

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We will modify $p^0$ inductively and denote the function obtained and the corresponding solution with $(I_0, 0)$ as the initial point by $p^i$ and $(I^i(t), \theta^i(t))$ with $(I^i(t_i), \theta^i(t_i)) = (I_i, i)$, respectively.

Suppose that we have obtained $p^0, p^1, \ldots, p^i$. The function $p^{i+1}$ defined on $[0, 1]$ is constructed by modifying $p^i$ on the interval $[t_i, t_{i+1}]$, where $t_{i+1}$ satisfies $\theta^{i+1}(t_{i+1}) = i + 1$ in the same way as above if we regard $I_i, t_i$ as $I_0, t_0$.

All the lemmas are true after the modification.

In the process of constructing $p^i$, we keep the jump $\sigma = \frac{1}{\tau}$ ($\tau \geq 2$) unchanged until $i = j_1$. Then we let $\sigma = \frac{1}{\tau}$ and keep it unchanged until $i = j_2$. Inductively, we choose $\sigma = \frac{1}{\tau}$ when $\theta \in [j_k-1, j_k]$, where $j_0 = 0, j_1, j_2, \ldots$ are defined as below.

Let $j_1 = \left[ \frac{1}{\tau} I_0^{n+2} \right]$, where $[x]$ denotes the integer part of $x$ and $\tau' > 0$ is used to control time and will be determined later. It follows that

$$T_1 := t_{j_1} \leq j_1 \cdot t_1 \cdot \leq \frac{1}{\tau'} I_0^{n+2} \cdot I_0^{-n} \cdot \leq \frac{1}{\tau'}.$$

On the interval $[0, T_1]$, since $I_{k+1} - I_k \geq \cdot \frac{1}{\tau'} I_0^{2m+2-n}$ for $k = 0, 1, \ldots, j_1 - 1$, we have

$$I_{j_1} := I^{j_1}(T_1) \geq \cdot \left[ \frac{1}{\tau'} I_0^{n+2} \right] \cdot \frac{1}{\tau'} I_0^{2m+2-n} \geq \cdot \frac{1}{\tau'} I_0^{2m+2}.$$

Suppose we have defined $T_0 = 0, T_1, \ldots, T_i$ according to the above method and the following are tenable for $k = 1, 2, \ldots, i$:

$$j_k - j_{k-1} = \left[ \frac{1}{\tau'} I_0^{n+2} \right], \quad T_k - T_{k-1} \cdot \leq \frac{1}{\tau'} k,$$

$$\sigma = \tau_k = \frac{1}{\tau'} k, \quad I_{j_k} := I^{j_k}(T_k) \geq \cdot \frac{1}{(\tau')^k} I_0^{2m+2}.$$

Set

$$j_{i+1} - j_i = \left[ \frac{1}{\tau'} I_0^{n+2} \right], \quad \sigma = \tau_{i+1} = \frac{1}{\tau'} i+1, \quad T_{i+1} := t_{j_{i+1}},$$

similar to the above discussion, we have

$$I_{j_{i+1}} := I^{j_{i+1}}(T_{i+1}) \geq \cdot \frac{1}{(\tau')^i+1} I_0^{2m+2}, \quad T_{i+1} - T_i \cdot \leq \frac{1}{\tau'} i+1.$$
Consequently,
\[ T_{i+1} \cdot \leq \sum_{k=1}^{i+1} \frac{1}{\tau' k} \cdot \leq \frac{1}{\tau'} < 1 \]
if \( \tau' > 0 \) is sufficiently large.

Let
\[ \lim_{i \to \infty} T_i = T_\infty, \quad \lim_{i \to \infty} p^i(t) = p(t), \]
since
\[ \max_{t_1, t_2 \in [T_k, T_\infty]} |p(t_1) - p(t_2)| \leq \frac{1}{\tau' k}, \quad \lim_{t \to T_\infty} p(t) = 1, \]
then \( p(t) \) can be extended to a continuous positive 1-periodic function.

**Lemma 3.5.** If \( I_0 \) is sufficiently large, then
\[ I_{j_k} \geq I_0^k, \]
where the constant \( l > 1 \).

**Proof.** First, by the assumption \( 2m + 1 \geq n + 2 \), if we let \( l = \frac{2m+1}{n+2} + \frac{1}{2(n+2)}, \)
then \( l > 1 \), and for sufficiently large \( I_0 \) we have
\[ I_{j_1} \geq \frac{1}{\tau' \tau} I_0^\frac{2m+2}{n+2} \geq \frac{1}{\tau' \tau} I_0^\frac{2}{n+2} I_0^i \geq I_0^i. \]
If
\[ I_{j_k} \geq I_0^k, \]
then
\[ I_{j_k+1} \geq \frac{1}{(\tau' \tau)^{k+1}} I_{j_k}^{\frac{2m+2}{n+2}} \geq \frac{1}{(\tau' \tau)^{k+1}} I_0^\frac{2m+2}{n+2} \]
\[ \geq \frac{1}{(\tau' \tau)^{k+1}} I_0^i \]
\[ \geq \frac{I_0^i}{(\tau' \tau)^{k+1}} I_0^{k+1} \]
\[ \geq I_0^{k+1}. \]

**Proof of Theorem 1.** By Lemma 3.5, one has
\[ \min_{t \in [T_i, T_{i+1}]} I(t) \geq I_{j_i} \geq I_0^i. \]

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Since \( l > 1 \), then \( I(t) \to +\infty \) as \( t \to t_{\infty} \). Therefore, Eq. (1.3) in Theorem 1.1 possesses an unbounded solution defined in the interval \([0, T_{\infty})\), and Theorem 1.1 is proved.

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