\( \mathcal{R}(p, q) \)-deformed conformal Virasoro algebra

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Abstract

This paper addresses an \( \mathcal{R}(p, q) \)-deformed conformal Virasoro algebra with an arbitrary conformal dimension \( \Delta \). Well-known deformations constructed in the literature are deduced as particular cases. Then, the special case of the conformal dimension \( \Delta = 1 \) is elucidated for its interesting properties. The \( \mathcal{R}(p, q) \)-KdV equation, associated with the deformed Virasoro algebra, is also derived and discussed. Finally, the \( (p, q) \)-deformed energy-momentum tensor, consistent with the central extension term, is computed and analyzed.

Keywords. \( \mathcal{R}(p, q) \) -- calculus; Virasoro algebra; central extension; conformal dimension; KdV equation; energy-momentum tensor.

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1 Introduction

Quantum groups and algebras, in both their construction and representation theory, with or without deformation, are exciting areas of mathematics, originated from mathematical physics of field theory and statistical mechanics. Connected with many other parts of mathematics, these topics were broadly developed over the last decades, and remain today an area of prolific research activities. This paper deals with a construction of the deformed Virasoro algebra on the line of recently introduced $R(p, q)$-deformed quantum algebras [15]. To cite a few relevant works related to the deformation of Virasoro algebra, see [3], [6] (and references therein). The Virasoro algebra with a conformal dimension $\Delta$ is related to the Korteweg-de-Vries (KdV) integrable systems, and plays an important role in physics. This motivated the series of works devoted these last years to its deformation and generalization [6], [7], [14]. In the same vein, we also realized $q$-deformed Virasoro algebras with conformal parameter $\Delta$, multiplicative and comultiplication rule for deformed generators [11], $q$-deformed central extension term [11], [3], [5], and related $q$-deformed KdV equation [7]. The relevance of $\Delta = 0$, $1/2$, $1$, and $q$-deformed energy-momentum tensor associated with the $q$-deformed central extension term [3] was highlighted. The Virasoro algebras with a conformal dimension $\Delta$ and two deformation parameters, (also called conformal $(p, q)$-Virasoro algebras), were also investigated by some authors, taking into account properties of comultiplication, deformed nonlinear $(p, q)$-KdV-equation and deformed central extension term [3]. Recently, the generalized Virasoro algebra, and related algebraic and hydrodynamics properties were studied in [14].

Motivated by the generalization of the well- known $(p, q)$-deformation in [13], we address in this paper the $R(p, q)$-deformed conformal Virasoro algebra with an arbitrary dimension $\Delta$, with a particular emphasis on properties induced by $\Delta = 1$ and $2$.

This paper is organized as follows. Main definitions and notations are briefly recalled in Section 2. In Section 3, the $R(p, q)$-deformation of the conformal Virasoro algebra with an arbitrary conformal dimension $\Delta$ is performed. The $R(p, q)$-deformed Jacobi identity is discussed, the deformation of the central extension term depending on the meromorphic function $R$ is obtained, and the $R(p, q)$-deformed conformal Virasoro algebra is constructed. Interesting results are derived for deduced known particular deformations [9], [16], [18]. Section 4 describes the case of the dimension $\Delta = 1$. Then, an $R(p, q)$-nonlinear differential equation is obtained and linked to the $R(p, q)$-deformed conformal Virasoro algebra. In Section 5, we compute the $(p, q)$-deformed energy-momentum tensor, which is consistent with the $(p, q)$-deformed central extension term. Section 6 is devoted to concluding remarks.

2 Preliminaries: basic definitions and notations

In this section, we briefly recall main definitions, notations and known results used in the sequel.

Let us consider an arbitrary conformal dimension $\Delta$ [8]. A field $\phi_{\Delta}(z)$ with the
conformal dimension $\Delta$ transforms under an infinitesimal coordinate transformation \[3\]

\[ z \mapsto z + f(z) \]

as

\[ \delta_{f(z)} \phi_{\Delta}(z) = f(z)^{1-\Delta} \partial_z (f(z)^\Delta \phi_{\Delta}(z)). \]  

(2.1)

For

\[ f(z) = z^{1+n}, \]  

(2.2)

we obtain

\[ \delta_n \phi_{\Delta}(z) := g_n(z) = (z \partial_z + \Delta(n + 1) - n) z^n \phi_{\Delta}(z), \]  

(2.3)

where the generators $g_n$ satisfy the centerless Virasoro algebra, known under the name of Witt algebra, $\mathcal{W}$,

\[ [g_n, g_m] = (m - n) g_{n+m}. \]  

(2.4)

The centrally extended Virasoro algebra $\mathcal{V}$ is spanned by the generators obeying

\[ [g_n, g_m] = (m - n) g_{n+m} + \frac{c}{12}(n^3 - n) \delta_{n+m,o}, \]  

(2.5)

with the property, for all $g_n \in \mathcal{V}$,

\[ [g_n, c] = 0. \]  

(2.6)

Let $\mathcal{R}$ be a meromorphic function defined on $\mathbb{C} \times \mathbb{C}$ by

\[ \mathcal{R}(u, v) = \sum_{s,t=-l}^{\infty} r_{st} u^s v^t, \]  

(2.7)

converging in the complex disc $\mathbb{D}_R = \{ z \in \mathbb{C} / |z| < R \}$, where $r_{st}$ are complex numbers, $l \in \mathbb{N} \setminus \{0\}$ and $R$ is the radius of convergence of the series (2.7). Let us consider the set of holomorphic functions $\mathcal{O}(\mathbb{D}_R)$ defined on $\mathbb{D}_R$.

**Definition 2.1** \[13\] Let $P$ and $Q$ be two linear operators on $\mathcal{O}(\mathbb{D}_R)$. Then, for $\Psi \in \mathcal{O}(\mathbb{D}_R)$, we have

\[ P : \Psi \mapsto P \Psi(z) = \Psi(pz), \]  

(2.8)

\[ Q : \Psi \mapsto Q \Psi(z) = \Psi(qz), \]  

(2.9)

**Definition 2.2** \[9\] The $(p, q)$–derivative and the $(p, q)$–number are defined, respectively, by:

\[ D_{p,q} : \Psi \mapsto D_{p,q} \Psi(z) = \frac{\Psi(pz) - \Psi(qz)}{z(p-q)}, \]  

(2.10)

\[ [n]_{p,q} := \frac{p^n - q^n}{p - q}, \]  

(2.11)

where $p$ and $q$, satisfying $0 < q < p \leq 1$, are two real numbers.
Definition 2.3 [13] The $\mathcal{R}(p, q)$–derivative is given by:

$$D_{\mathcal{R}(p, q)} := D_{p,q} \frac{P - Q}{P - Q} \mathcal{R}(P, Q) = \frac{p - q}{p^p - q^q} \mathcal{R}(p^p, q^q) D_{p,q},$$

(2.12)

where $P, Q$ are defined on $\mathcal{O}(\mathbb{D}_R)$, and $p, q$ verifying $0 < q < p \leq 1$, are two real numbers.

Definition 2.4 [15] The $\mathcal{R}(p, q)$–number and the $\mathcal{R}(p, q)$–factorials are defined, respectively, as follows:

1. $$[n]_{\mathcal{R}(p, q)} := \mathcal{R}(p^n, q^n) \quad \text{for} \quad n \geq 0,$$

(2.13)

2. $$[n]!_{\mathcal{R}(p, q)} := \begin{cases} 1 & \text{for} \quad n = 0 \\ \mathcal{R}(p, q) \cdots \mathcal{R}(p^n, q^n) & \text{for} \quad n \geq 1, \end{cases}$$

(2.14)

where $p, q$, satisfying $0 < q < p \leq 1$, are two real numbers.

Lemma 2.5 [15] For $\varphi(z) = z^n \in \mathcal{O}(\mathbb{D}_R)$, the $\mathcal{R}(p, q)$-derivative is given by:

$$D_{\mathcal{R}(p, q)} z^n = z^{-1} [z \partial_z]_{p,q} z^n \frac{p - q}{p^n - q^n} \mathcal{R}(p^n, q^n).$$

(2.15)

Proposition 2.6 [13] Let $P, Q$ be two linear operators and $p, q$ two real numbers, then we have the following relations

$$P = p z \partial_z \quad \text{and} \quad Q = q z \partial_z.$$

(2.16)

Definition 2.7 [13] The Jaganathan-Srinivasa $(p, q)$ factors and $(p, q)$-factorials are given by

$$[x]_{p,q} = (p - q)^{-1} (p^x - q^x),$$

and

$$[x]!_{p,q} = \begin{cases} \frac{(p,q):((p,q))}{(p-q)^x} & \text{for} \quad x \geq 1 \\ 1 & \text{for} \quad x = 0, \end{cases}$$

respectively, where $p, q$ satisfying $0 < q < p \leq 1$ are two real numbers.

The $(p, q)-$deformation of quantum algebras gives rise to a realization of a $(p, q)$-oscillator [9]. The main properties of this deformation can be found in [9] and [13].
**Definition 2.8** The Chakrabarty and Jagannathan \((p^{-1}, q)\) factors and \((p^{-1}, q)\)-factorials are given by:
\[
[x]_{p^{-1},q} = (p^{-1} - q)^{-1}(p^{-x} - q^x),
\]
and
\[
[x]!_{p^{-1},q} = \begin{cases} 
1 & \text{for } x = 0 \\
\frac{(p^{-1},q):((p^{-1},q)^x)}{(p^{-1}-q)^x} & \text{for } x \geq 1,
\end{cases}
\]
where \(p, q\) satisfying \(0 < q < p \leq 1\) are two real numbers.

**Definition 2.9** \([13]\) The \((p,q)\)-Quesne factors and factorials are given by:
\[
[n]_{p,q}^Q = (-p^{-1} + q)^{-1}(p^n - q^{-n}),
\]
and
\[
[n]_{p,q}^Q! = \begin{cases} 
1 & \text{for } n = 0 \\
\frac{((p,q^{-1}):(p,q^{-1})^n)}{(-p^{-1}+q)^n} & \text{for } n \geq 1.
\end{cases}
\]
The above mentioned deformations are recoverable in the work \([17]\) as particular cases. Finally, let \(p, q, \nu, \mu\) be four real numbers such that \(0 < pq < 1\), \(p^{-\mu} < q^{-\nu-1}\), \(p > 1\), and \(g\) a real function of two-parameter deformation \(p\) and \(q\) verifying \(g(p, q) \to 1\) as \((p, q) \to (1,1)\). The \((p, q, \mu, \nu, g)\)-factorial and the \((p, q, \mu, \nu, g)\)-number are given by \([17]\):
\[
[n]_{p,q,g}^{\mu,\nu} ≡ \begin{cases} 
[n]_{p,q,g}^{\mu,\nu} [n-1]_{p,q,g}^{\mu,\nu} \cdots [2]_{p,q,g}^{\mu,\nu} [1]_{p,q,g}^{\mu,\nu} & \text{if } n = 1, 2, \ldots \\
1 & \text{if } n = 0,
\end{cases}
\]
\[
[1]_{p,q,g}^{\mu,\nu} = g(p, q) \frac{p^n - q^{-n}}{p^{\mu} q - q^{-1}},
\]
respectively.

### 3 \(R(p, q)\)-deformed conformal algebra

In this section, we derive an \(R(p, q)\)–extension of a conformal algebra with an arbitrary conformal dimension \(\Delta\) \([8]\). The \(R(p, q)\)-generators are computed using the analogue of the \(R(p, q)\)–Leibniz rule. The \(R(p, q)\)-deformed Witt and Virasoro algebras are built and discussed. A result deduced by Chaichian et al in \([3]\) is recovered in a specific case.
3.1 General construction

Let us consider an infinitesimal $R(p, q)$-transformation for a field $\phi_\Delta(z)$ as:

$$\delta^{R(p,q)}_{f(z)} \phi_\Delta(z) := f(z)^{1-\Delta} D_{R(p,q)}(z) \Delta \phi_\Delta(z).$$  \hspace{1cm} (3.1)

For $f(z) := z^{1+n}$, we define the $R(p, q)$-Virasoro generators $L_n^\Delta$ by:

$$L_n^{(\Delta)} \phi_\Delta(z) := z^{(1+n)(1-\Delta)} D_{R(p,q)}(z) \Delta \phi_\Delta(z)).$$  \hspace{1cm} (3.2)

According to the $R(p, q)$-derivative (2.12), we use the following lemma.

**Lemma 3.1** The $R(p, q)$-Leibniz rule is given by:

$$D_{R(p,q)}(\varepsilon(z)^\Delta \phi_\Delta(z)) = K h(P,Q) z^{\Delta} \{ \varepsilon(pz)^\Delta \phi_\Delta(pz) - \varepsilon(qz)^\Delta \phi_\Delta(qz) \},$$  \hspace{1cm} (3.3)

where $K = (p-q)^{-1}$,

$$h(P,Q) = \frac{p-q}{p^P - q^Q} R(p^P, q^Q).$$  \hspace{1cm} (3.4)

$\phi_\Delta$ is an arbitrary primary field with the conformal dimension $\Delta$, $\varepsilon \in O(D_R)$. $P, Q$ are two linear operators on $O(D_R)$, while $p, q$ are two real numbers satisfying $0 < q < p \leq 1$.

**Proof:** Let us consider two functions $f(z)$ and $g(z)$ belonging to $O(D_R)$. Then using the $R(p, q)$-derivative, we get

$$D_{R(p,q)}(f(z)g(z)) = \frac{p-q}{p^P - q^Q} R(p^P, q^Q) D_{R(p,q)}((f(z)g(z)).$$  \hspace{1cm} (3.5)

Setting $f(z) = \varepsilon(z)^\Delta$, $g(z) = \phi_\Delta(z)$ and using the $(p, q)$-Leibniz rule [13], the result is immediately obtained. \hfill \Box

**Proposition 3.2** The $R(p, q)$-generators of the $R(p, q)$-deformed Virasoro algebra are given by:

$$L_n^{(\Delta)} \phi_\Delta(z) = [z \partial_z + \Delta(n+1) - n] z^n \omega_n^{\Delta}(p, q) \phi_\Delta(z),$$  \hspace{1cm} (3.6)

where

$$\omega_n^{\Delta}(p, q) = (p-q)(p^{-\Delta(n+1)} - q^{-\Delta(n+1)})^{-1} R(p^\Delta(n+1), q^\Delta(n+1)).$$  \hspace{1cm} (3.7)

**Proof:** The proof uses (3.2), and Lemmas 2.5 and 3.1. \hfill \Box

**Remark 3.3**

(i) For the choice of the meromorphic function $R(x, y) = (p-q)^{-1}(\frac{1}{x} - \frac{1}{y})$, we get the $(p, q)$-Virasoro generators given by Chakrabarti et al [8].
(ii) The $\mathcal{R}(p, q)$-Virasoro generators $L_n^{(\Delta)}$ can be expressed in terms of the $(p, q)$-Virasoro generators $e_n^\Delta$, i.e.

$$L_n^{(\Delta)} = w_n^{(\Delta)}(p, q)e_n^\Delta$$

(3.8)

**Proposition 3.4** The $\mathcal{R}(p, q)$-deformed generators $L_n^{(\Delta)}$ satisfy the following commutation relation:

$$\left[L_n^{(\Delta)}, L_m^{(\Delta)}\right] = \tilde{X}_n L_n^{(\Delta)} L_m^{(\Delta)} - \tilde{Y}_n L_m^{(\Delta)} L_n^{(\Delta)}$$

$$= K \left\{ p^{N_\Delta} (\tilde{X}_n p^{-n} - \tilde{Y}_n p^{-m}) - q^{N_\Delta} (\tilde{X}_n q^{-n} - \tilde{Y}_n q^{-m}) \right\}$$

$$\times L_{n+m}^{(\Delta)},$$

(3.9)

where

$$\begin{align*}
\tilde{X}_n &= (pq)^n \frac{[n(\Delta-1)][\Delta m]}{[n][m]} \frac{w_{n+m}^{(\Delta)}(p, q)}{w_n^{(\Delta)}(p, q)} \\
\tilde{Y}_n &= (pq)^m \frac{[m(\Delta-1)][\Delta n]}{[n][m]} \frac{w_{n+m}^{(\Delta)}(p, q)}{w_m^{(\Delta)}(p, q)} \\
N_\Delta &= z\partial_z + \Delta \\
K &= (p - q)^{-1}.
\end{align*}$$

(3.10)

**Proof:** It is obtained by a straightforward computation. \qed

It is worth mentioning the following relevant particular deformed quantum algebras together with their conformal characterisation and properties, which are derived from the above developed formalism:

1. For $\mathcal{R}(x, y) = (p - q)^{-1} (\frac{1}{x} - \frac{1}{y})$, and the limit $(p, q \to 1)$ in (3.9), we get the results given by Chakrabarti et al. [8] and Chaichian et al. [3], respectively.

2. The Jagannathan-Srinivasa generators can be obtained by taking $\mathcal{R}(s, t) = (p - q)^{-1} (s - t)$ leading to generators $L_n^{(\Delta)}$ acting on the conformal field $\phi_\Delta$ as:

$$L_n^{(\Delta)} \phi_\Delta(z) = -(pq)^{\Delta(n+1)} [z\partial_z + \Delta(1 + n) - n] z^n \phi_\Delta(z),$$

(3.11)

and satisfying the algebraic structure (3.9) with

$$\tilde{X}_n = -(pq)^{n-\Delta} \frac{[n(\Delta-1)][\Delta m]}{[n][m]}$$

(3.12)

and

$$\tilde{Y}_n = -(pq)^{m-\Delta} \frac{[m(\Delta-1)][\Delta n]}{[n][m]}.$$ 

(3.13)

3. The deformed Chakrabarti-Jagannathan algebra [9] corresponds to the choice $\mathcal{R}(s, t) = (p^{-1} - q)^{-1} s^{-1} (1 - st)$ yielding the generators $\tilde{L}_n$ acting as:

$$\tilde{L}_n \phi_\Delta(z) = -(pq)^{\Delta(n+1)} [z\partial_z + \Delta(1 + n) - n] z^n \phi_\Delta(z),$$

(3.14)
and obeying the algebraic structure \([3,9]\) with

\[
\tilde{X}_{\Delta} = -(pq)^{n-\Delta} \frac{[n(\Delta - 1)][\Delta m][\Delta(n + m + 1)]^p_{\mu,\nu}}{[\Delta(n + 1)][\Delta(m + 1)][\Delta(n + m + 1)]}.
\]

and

\[
\tilde{Y}_{\Delta} = -(pq)^{m-\Delta} \frac{[m(\Delta - 1)][\Delta n][\Delta(n + m + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)][\Delta(m + 1)][\Delta(n + m + 1)]}.
\]

4. Putting \(R(s, t) = ((-p^{-1} + q)t)^{-1}(st - 1)\), we obtain the generalized Quesne deformed algebra \([17]\) with the generator \(\tilde{L}_n\) acting as:

\[
\tilde{L}_n \phi_{\Delta}(z) = -(pq)^{\Delta(n + 1)} \frac{[\Delta(n + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)]^q_{p,\mu}} [z\partial_z + \Delta(1 + n) - n] z^n \phi_{\Delta}(z), \quad (3.15)
\]

and satisfying the relation \([3,9]\) with

\[
\tilde{X}_{\Delta} = -(pq)^{n-\Delta} \frac{[n(\Delta - 1)][\Delta m][\Delta(n + m + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)][\Delta(m + 1)][\Delta(n + m + 1)]}.
\]

and

\[
\tilde{Y}_{\Delta} = -(pq)^{m-\Delta} \frac{[m(\Delta - 1)][\Delta n][\Delta(n + m + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)][\Delta(m + 1)][\Delta(n + m + 1)]}.
\]

5. Taking \(R(x, y) = g(p, q) w_{xy}^{\mu,\nu} \frac{x_{q-p} - y}{q-p - 1} y\), (see properties in \([13]\)), we obtain the deformed Hounkonnou-Ngompe generalized algebra induced by the generators \(\tilde{L}_n\) such that:

\[
\tilde{L}_n \phi_{\Delta}(z) = -(pq)^{\Delta(n + 1)} \frac{[\Delta(n + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)]^q_{p,\mu}} [z\partial_z + \Delta(1 + n) - n] z^n \phi_{\Delta}(z), \quad (3.16)
\]

and the commutation relation \([3,9]\) with

\[
\tilde{X}_{\Delta} = -(pq)^{n-\Delta} \frac{[n(\Delta - 1)][\Delta m][\Delta(n + m + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)][\Delta(m + 1)][\Delta(n + m + 1)]}.
\]

and

\[
\tilde{Y}_{\Delta} = -(pq)^{m-\Delta} \frac{[m(\Delta - 1)][\Delta n][\Delta(n + m + 1)]^q_{\mu,\nu}}{[\Delta(n + 1)][\Delta(m + 1)][\Delta(n + m + 1)]}.
\]
6. The generalized algebra derived by Hounkonnou-Bukweli in [15] with

\[ [n]^{\mu,\nu}_{p,q,g} = g(p,q) \frac{q^n}{p^m} [\eta]^{Q}_{p,q,g}, \]

(3.17)
yields the deformed conformal generators \( \tilde{L}_n \)

\[ \tilde{L}_n = -g(p,q) \left( (pq) \frac{q^n}{p^m} \right)^{\Delta(n+1)} \frac{[\Delta(n+1)]^Q_{p,q,g}}{[\Delta(n+1)]_{p,q}^Q} [z \partial_z + \Delta(1+n) - n] z^n, \]

(3.18)

which satisfy the relation (3.9) with

\[ \dot{X}_\Delta = -(pq)^{n-\Delta} g(p,q)^{-1} \left( \frac{q^n}{p^m} \right)^{n\Delta} \frac{[n(\Delta - 1)][\Delta m]}{[n][m]} \]

\[ \times \frac{[\Delta(n + m + 1)]^Q_{p,q,g}}{[\Delta(n + m + 1)]^Q_{p,q,g}} \frac{[\Delta(n + 1)][\Delta(m + 1)]}{[\Delta(n + 1)][\Delta(m + 1)]^Q_{p,q,g}}, \]

(3.19)

and

\[ \dot{Y}_\Delta = -(pq)^{m-\Delta} g(p,q)^{-1} \left( \frac{q^n}{p^m} \right)^{m\Delta} \frac{[m(\Delta - 1)][\Delta n]}{[m][n]} \]

\[ \times \frac{[\Delta(n + m + 1)]^Q_{p,q,g}}{[\Delta(n + m + 1)]^Q_{p,q,g}} \frac{[\Delta(n + 1)][\Delta(m + 1)]}{[\Delta(n + 1)][\Delta(m + 1)]^Q_{p,q,g}}. \]

(3.20)

7. The deformation of the Chaichian et al [3] Virasoro algebra is performed by considering two parameters, \( p > 0 \) and \( q > 0 \),

\[ \lambda := \sqrt{pq} \]

and \( \theta := \sqrt{q} \),

(3.21)

with the numbers

\[ [\theta^x] = \lambda^{1-x} [\theta^x], \]

(3.22)

and

\[ [\theta^y] = (\theta - \theta^{-1})^{-1}(\theta^x - \theta^{-x}). \]

(3.23)

The associated \( \mathcal{R}(p,q) \)-deformed generators \( \tilde{L}_n \) given by:

\[ \tilde{L}_n \phi_\Delta(z) = T_n^\Delta(\lambda) [z \partial_z + \Delta(1+n) - n] \theta^n w_\Delta^\lambda(\theta, \theta^{-1}) \phi_\Delta(z), \]

(3.24)

where

\[ T_n^\Delta(\lambda) = \lambda^{n-1-\Delta(n+1)} \mathcal{R}(\lambda^{-\Delta(n+1)}, \lambda^{-\Delta(n+1)}), \]

(3.25)

satisfy the commutation relation:

\[ [\tilde{L}_n, \tilde{L}_m]_{x,y} = K_\theta \left\{ \theta^{N\Delta}(x\theta^n - y\theta^m) - \theta^{-N\Delta}(x\theta^n - y\theta^m) \right\} \tilde{L}_{n+m}, \]

(3.26)
where

\[
\begin{align*}
  x &= \lambda^{1-\Delta}(n+m) \left[ n(\Delta - 1) \right] \theta [\Delta n] \theta \frac{w_\Delta^m(\theta, \theta^{-1})}{[n]\theta[m] \theta w_\Delta^m(\theta, \theta^{-1})} \\
y &= \lambda^{1-\Delta}(n+m) \left[ m(\Delta - 1) \right] \theta [\Delta n] \theta \frac{w_\Delta^m(\theta, \theta^{-1})}{[n]\theta[m] \theta w_\Delta^m(\theta, \theta^{-1})} \\
K_\theta &= (\theta - \theta^{-1})^{-1}.
\end{align*}
\]

Let us now construct the $\mathcal{R}(p, q)$-deformed conformal Witt algebra by using the $\mathcal{R}(p, q)$-deformed conformal algebra \eqref{3.9}.

**Proposition 3.5** Let $\mathcal{L}^{(\Delta)}_n$ be the $\mathcal{R}(p, q)$-deformed generators, $m$ and $n$ two numbers belonging to the set of natural numbers. For two real numbers $p, q$ satisfying $0 < q < p \leq 1$, and all field $\phi_\Delta(z)$ of conformal dimension $\Delta$, the following commutation relation holds:

\[
[\mathcal{L}^{(\Delta)}_n, \mathcal{L}^{(\Delta)}_m]_{\tilde{X}, \tilde{Y}} \phi_\Delta(z) = \mathcal{R}(p^{n-m}, q^{n-m}) \mathcal{L}^{(\Delta)}_{n+m} \phi_\Delta(z),
\]

with

\[
\begin{align*}
  \tilde{X} &= (p - q) \mathcal{R}(p^{n-m}, q^{n-m}) \chi_{nm}(p, q) \\
  \tilde{Y} &= (p - q) \mathcal{R}(p^{n-m}, q^{n-m}) \chi_{mn}(q, p).
\end{align*}
\]

and

\[
\chi_{nm}(p, q) = \left\{ p^{N\Delta} (p^n - \frac{[m(\Delta - 1)][\Delta n]}{n(\Delta - 1)[\Delta m]} q^{m-n} p^n) - q^{N\Delta} (q^n - \frac{[m(\Delta - 1)][\Delta n]}{n(\Delta - 1)[\Delta m]} q^{n-m}) \right\}^{-1}.
\]

**Proof:** It uses the closed algebraic structure \eqref{3.9}. By setting

\[
\chi_{nm}(p, q) = \left\{ p^{N\Delta} (p^n - \frac{\tilde{X}_\Delta}{X_\Delta} p^n - \frac{\tilde{Y}_\Delta}{X_\Delta} q^n) - q^{N\Delta} (q^n - \frac{\tilde{Y}_\Delta}{X_\Delta} q^n) \right\}^{-1},
\]

we obtain:

\[
[\mathcal{L}^{(\Delta)}_n, \mathcal{L}^{(\Delta)}_m]_{\tilde{X}_\Delta, \tilde{Y}_\Delta} = K \left\{ p^{N\Delta} (\tilde{X}_\Delta p^n - \tilde{Y}_\Delta p^m) - q^{N\Delta} (\tilde{X}_\Delta q^n - \tilde{Y}_\Delta q^m) \right\} \mathcal{L}^{(\Delta)}_{n+m}
\]

\[
= K \tilde{X}_\Delta \left\{ p^{N\Delta} (p^n - \frac{\tilde{Y}_\Delta}{X_\Delta} p^m) - q^{N\Delta} (q^n - \frac{\tilde{Y}_\Delta}{X_\Delta} q^m) \right\} \mathcal{L}^{(\Delta)}_{n+m}.
\]
Replacing $\tilde{X}_\Delta$ and $\tilde{Y}_\Delta$ by their respective expressions yields:

$$
\chi_{nm}(p, q) = \left\{ p^{N_\Delta}(p^{-n} - \frac{m(\Delta - 1)}{n(\Delta - 1)} [\Delta n] \frac{q^{m-n}}{p^n}) - q^{N_\Delta}(q^{-n} - \frac{m(\Delta - 1)}{n(\Delta - 1)} [\Delta m] \frac{p^{m-n}}{q^n}) \right\}^{-1},
$$

(3.32)

and

$$
\hat{X} = (p - q) L(p^{n-m}, q^{n-m}) \chi_{nm}(p, q).
$$

We explicitly compute $\hat{Y}$ in a similar way, and the required result naturally comes by collecting different quantities in (3.31).

□

Remark 3.6 For a meromorphic function $R(x, y) = (p - q)^{-1}(x^{-1} - y^{-1})$ and the use of the transformation $q \leftarrow p, p^{-1} \leftarrow q$, the result given by Chakrabarti et al [8] is recovered.

The $R(p, q)$-deformed Jacobi identity is derived in the following Lemma.

**Lemma 3.7** Let $L_n$ be the $R(p, q)$-deformed generators of the deformed conformal algebra, and $\hat{X}, \hat{Y}$ be the coefficients of the commutation relation (3.28). For all $n, m,$ and $k$ belonging to $\mathbb{N}$, the $R(p, q)$-deformed Jacobi identity is given by:

$$
\sum_{(u, v, l) \in C(n, m, k)} (pq)^{-1}[p^n + q^u] [L_u, [L_v, L_l]] \hat{X}, \hat{Y} = 0,
$$

(3.33)

where $C(n, m, k)$ denotes the cyclic permutation of $(n, m, k)$.

**Proof:**

$$
[L_n, [L_m, L_k]] = \alpha_{nmk}^\Delta [e_n, [e_m, e_k]],
$$

(3.34)

where

$$
\alpha_{nmk}^\Delta(p, q) = \omega_n^\Delta(p, q) \omega_m^\Delta(p, q) \omega_k^\Delta(p, q),
$$

(3.35)

and $e_n$ are the $(p, q)$-deformed generators. By mimicking step by step [8], we obtain:

$$
(pq)^{-k} \frac{[2n]}{\alpha_{nmk}^\Delta[n]} [L_n, [L_m, L_k]] \hat{X}, \hat{Y} = (pq)^{-k} \frac{[2n]}{[n]} [e_n^\Delta, [e_m^\Delta, e_k^\Delta]] R_{mk, Sm} R_{n(m+k), S(m+k)n},
$$

(3.36)

and, finally, the result follows. □

A central extension of the $R(p, q)$-deformed Witt algebra (3.28) is governed by the commutation relations:

$$
[\tilde{L}_n, \tilde{L}_m] \hat{X}, \hat{Y} = R(p^{n-m}, q^{n-m}) \tilde{L}_{n+m} + \delta_{n+m, 0} \tilde{C}_n^R(p, q),
$$

(3.37)

and

$$
[\tilde{L}_k, \tilde{C}_n^R(p, q)] \hat{X}, \hat{Y} = 0,
$$

(3.38)
where

\[
\begin{align*}
\hat{X}_k &= (p - q) \mathcal{R}(p^k, q^k) \chi_{k0}(p, q) \\
\hat{Y}_k &= (p - q) \mathcal{R}(p^k, q^k) \chi_{0k}(q, p),
\end{align*}
\]

with the central charge given by the factorization

\[
\tilde{C}_n^R(p, q) = \tilde{\Gamma}(N \Delta) C_n^R(p, q),
\]

as shown in the sequel.

**Lemma 3.8** \(\tilde{\Gamma}(N \Delta)\) satisfies the following identities:

1. For all \(\Delta\)

\[
\hat{X}_k \tilde{L}_k^\Delta \tilde{\Gamma}(N \Delta) - \hat{Y}_k \tilde{\Gamma}(N \Delta) \tilde{L}_k^\Delta = 0.
\]

2. For \(\Delta = 1/2\)

\[
(pq)^{k/2} \tilde{L}_k^\Delta \tilde{\Gamma}(N \Delta) + \tilde{\Gamma}(N \Delta) \tilde{L}_k^\Delta = 0.
\]

3. For \(\Delta = 2\)

\[
\tilde{L}_k^\Delta \tilde{\Gamma}(N \Delta) - (p^{-k} + q^{-k}) \tilde{\Gamma}(N \Delta) \tilde{L}_k^\Delta = 0.
\]

**Proof:**

1. From the relations (3.38) and (3.40), we obtain (3.41).

2. For \(\Delta = 1/2\), we obtain

\[
\chi_{k0}(p, q) = \left\{ p^{N \Delta} \left( p^{-k} - \frac{[k/2]}{[-k/2]} q^{-k} \right) - q^{N \Delta} \left( q^{-k} - \frac{[k/2]}{[-k/2]} p^{-k} \right) \right\}^{-1},
\]

and

\[
[k/2] = -(pq)^{k/2}[-k/2]
\]

giving

\[
\chi_{k0}(p, q) = \left( p^{-k/2} + q^{-k/2} \right)^{-1} \left( p^{N \Delta-k/2} - q^{N \Delta-k/2} \right)^{-1}.
\]

Hence,

\[
\hat{X}_k = (p - q) \mathcal{R}(p^{-k}, q^{-k}) \left( p^{-k/2} + q^{-k/2} \right)^{-1} \left( p^{N \Delta-k/2} - q^{N \Delta-k/2} \right)^{-1}.
\]

By analogy, we obtain

\[
\hat{Y}_k = -(p - q) \mathcal{R}(p^{-k}, q^{-k}) \left( p^{k/2} + q^{k/2} \right)^{-1} \left( p^{N \Delta-k/2} - q^{N \Delta-k/2} \right)^{-1}.
\]

Substituting (3.47) and (3.48) in (3.41) provides (3.42).
Proposition 3.9 The $\mathcal{R}(p, q)$-deformed central charge $\tilde{C}_n^R(p, q)$ is provided by:

$$\tilde{C}_n^R(p, q) = C(p, q)(p^n + q^n)^{-1}(pq)^{\Delta_n} \mathcal{R}(p^{n-1}, q^{n-1})\mathcal{R}(p^n, q^n)\mathcal{R}(p^{n+1}, q^{n+1}),$$  

(3.54)

where $p, q$ are two real numbers verifying $0 < q < p \leq 1$, $C(p, q)$ is a function of $(p, q)$, and $\alpha_{n \Delta m k}^R(p, q)$ is given by (3.35).

Proof: According to the relations (3.37) and (3.38), we obtain

$$[\tilde{L}_m, [\tilde{L}_n, \tilde{L}_k]] = \mathcal{R}(p^{m-k}, q^{m-k})\mathcal{R}(p^{m-k+n}, q^{m-k+n})\tilde{L}_{n+m+k} + \mathcal{R}(p^{m-k}, q^{m-k})\delta_{n+k+m,0}\tilde{\Gamma}(N\Delta)C_n^R(p, q),$$  

(3.55)

and by analogy

$$[\tilde{L}_m, [\tilde{L}_k, \tilde{L}_n]] = \mathcal{R}(p^{n+k}, q^{n+k})\mathcal{R}(p^{n-k+m}, q^{n-k+m})\tilde{L}_{n+m+k} + \mathcal{R}(p^{n+k}, q^{n+k})\delta_{n+k+m,0}\tilde{\Gamma}(N\Delta)C_m^R(p, q),$$  

(3.56)

and

$$[\tilde{L}_k, [\tilde{L}_n, \tilde{L}_m]] = \mathcal{R}(p^{n-m}, q^{n-m})\mathcal{R}(p^{n-m+k}, q^{n-m+k})\tilde{L}_{n+m+k} + \mathcal{R}(p^{n-m}, q^{n-m})\delta_{n+k+m,0}\tilde{\Gamma}(N\Delta)C_k^R(p, q).$$  

(3.57)

Using the $\mathcal{R}(p, q)$-deformed Jacobi identity, we obtain

$$\sum_{(u,v,l) \in \mathbb{C}(n,m,k)} \frac{(pq)^{-l}(p^u + q^u)}{\alpha_{u,v,l}^\Delta(p, q)} \mathcal{R}(p^{l-v}, q^{l-v})\delta_{u+v+l,0}C_u^R(p, q) = 0,$$  

(3.58)
where \( C(n, m, k) \) denotes the cyclic permutation of \( (n, m, k) \), leading to the following form of \( C_n^R(p, q) \):

\[
C_n^R(p, q) = C(p, q)(p^n + q^n)^{-1}(pq)^nR(p^{n-1}, q^{n-1})R(p^n, q^n)R(p^{n+1}, q^{n+1}),
\]

with the solution of (3.42) given by:

\[
\tilde{\Gamma}(N_{\Delta}) = (pq)^{N_{\Delta}/2}.
\]

Then, using the relations (3.40), (3.59), and (3.60), we obtain the required result.

**Theorem 3.10** Let \( \tilde{L}_n, n \in \mathbb{Z} \), be the deformed generators and \( \Delta \) a conformal dimension. Then, the \( R(p, q) \)-deformed Virasoro algebra is driven by the following commutation relations

\[
[\tilde{L}_n, \tilde{L}_m]_{\tilde{X}, \tilde{Y}} = [n - m]\tilde{L}_{n+m} + \delta_{n+m,0}\tilde{C}_n^R(p, q),
\]

where \( \tilde{C}_n^R(p, q) \) given by (3.34) is the central charge commuting with all \( \tilde{L}_n \), i.e.

\[
[\tilde{L}_k, \tilde{C}_n^R(p, q)]_{\tilde{X}_k, \tilde{Y}_k} = 0,
\]

with \( \tilde{X}_k, \tilde{Y}_k \) furnished by (3.29).

Note that the result obtained by Chakrabarti et al [8] can be retrieved here by taking \( R(s, t) = (p - q)^{-1}(s^{-1} - t^{-1}) \).

**Remark 3.11** The next particular cases are also worthy of attention. In each case, we give the commutation relation and the central element.

1. The \( R(p, q) \)-deformed Jagannathan-Srinivasa [18] Virasoro algebra:

\[
[\tilde{L}_n, \tilde{L}_m]_{\tilde{X}, \tilde{Y}} = [n - m]\tilde{L}_{n+m} + \delta_{n+m,0}\tilde{C}_n^R(p, q),
\]

with

\[
\begin{aligned}
\tilde{X} &= (p^{n-m} - q^{n-m})\chi_{nm}(p, q) \\
\tilde{Y}_{\Delta} &= (p^{n-m} - q^{n-m})\chi_{nm}(p, q).
\end{aligned}
\]

\[
\tilde{C}_n^R(p, q) = C(p, q)(pq)^{N_{\Delta}+n}(p^n + q^n)^{-1}[n - 1][n][n + 1].
\]

2. The \( R(p, q) \)-deformed Chakrabarti-Jagannathan [9] Virasoro algebra:

\[
[\tilde{L}_n, \tilde{L}_m]_{\tilde{X}, \tilde{Y}} = [n - m]_{p^{-1}, q}\tilde{L}_{n+m} + \delta_{n+m,0}\tilde{C}_n^R(p, q),
\]

where

\[
\begin{aligned}
\tilde{X} &= (p - q)[n - m]_{p^{-1}, q}\chi_{nm}(p, q) \\
\tilde{Y}_{\Delta} &= (p - q)[n - m]_{p^{-1}, q}\chi_{nm}(p, q) \\
\tilde{C}_n^R(p, q) &= C(p, q)(pq)^{N_{\Delta}+n}(p^n + q^n)^{-1}[n - 1]_{p^{-1}, q}[n][n + 1]_{p^{-1}, q}.
\end{aligned}
\]

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3. The $\mathcal{R}(p, q)$-deformed Generalized $q$-Quesne $^{[17]}$ Virasoro algebra:

$$[\tilde{L}_n, \tilde{L}_m]_{\tilde{X}, \tilde{Y}} = [n - m]_{\mu, \nu}^{Q} [\tilde{L}_n + \delta_{n+m,0} \tilde{C}_n^R(p, q)],$$  \hspace{1cm} (3.68)

where

$$\begin{align*}
\tilde{X} &= (p - q)[n - m]^{\mu, \nu}_{\mu, \nu} \chi_{nm}(p, q) \\
\tilde{Y}_\Delta &= (p - q)[n - m]^{\mu, \nu}_{\mu, \nu} \chi_{nm}(p, q) \\
\tilde{C}_n^R(p, q) &= C(p, q)(pq)^{\frac{N}{N + n}(p^n + q^n)} - 1 [n - 1]^{Q}_{\mu, \nu} [n + 1]^{Q}_{\mu, \nu}.
\end{align*}$$  \hspace{1cm} (3.69)

4. The $\mathcal{R}(p, q)$-deformed Hounkonnou-Ngompe generalized $^{[17]}$ Virasoro algebra:

$$[\tilde{L}_n, \tilde{L}_m]_{\tilde{X}, \tilde{Y}} = [n - m]^{\mu, \nu}_{\mu, \nu} [\tilde{L}_n + \delta_{n+m,0} \tilde{C}_n^R(p, q)],$$  \hspace{1cm} (3.70)

with

$$\begin{align*}
\tilde{X} &= (p - q)[n - m]^{\mu, \nu}_{\mu, \nu} \chi_{nm}(p, q) \\
\tilde{Y}_\Delta &= (p - q)[n - m]^{\mu, \nu}_{\mu, \nu} \chi_{nm}(p, q) \\
\tilde{C}_n^R(p, q) &= C(p, q)(pq)^{\frac{N}{N + n}(p^n + q^n)} - 1 [n - 1]^{Q}_{\mu, \nu} [n + 1]^{Q}_{\mu, \nu}.
\end{align*}$$  \hspace{1cm} (3.71)

5. The $\mathcal{R}(p, q)$-deformed Hounkonnou-Bukweli $^{[15]}$ Virasoro algebra:

$$[\tilde{L}_n, \tilde{L}_m]_{\tilde{X}, \tilde{Y}} = g(p, q) \left(\frac{q^n}{p^n}\right)^{n - m} [n - m]^{Q}_{\mu, \nu} \chi_{nm}(p, q)$$  \hspace{1cm} (3.72)

where

$$\begin{align*}
\tilde{X} &= g(p, q)(p - q) \left(\frac{q^n}{p^n}\right)^{n - m} [n - m]^{Q}_{\mu, \nu} \chi_{nm}(p, q) \\
\tilde{Y}_\Delta &= g(p, q)(p - q) \left(\frac{q^n}{p^n}\right)^{n - m} [n - m]^{Q}_{\mu, \nu} \chi_{nm}(p, q) \\
\tilde{C}_n^R(p, q) &= C(p, q)g(p, q)^{\frac{N}{N + n}(p^n + q^n)} \left(\frac{q^n}{p^n}\right)^{3n} - 1 [n + 1]^{Q}_{\mu, \nu} [n + 1]^{Q}_{\mu, \nu}.
\end{align*}$$  \hspace{1cm} (3.73)

4 Deformation of conformal algebra with $\Delta = 1$

In this section, we consider the algebra given by the relation (3.9) for $\Delta = 1$, and derive the relation between the $\mathcal{R}(p, q)$-deformed Virasoro algebra and the $\mathcal{R}(p, q)$-KdV equation. Relevant particular cases of the $(p, q)$-deformed KdV equation are also deduced.
**Proposition 4.1** Let $\phi(z)$ be the fields of the conformal dimension $\Delta = 1$, and $p$ and $q$ two real numbers verifying $0 < q < p \leq 1$. Then the $\mathcal{R}(p, q)$-deformed generators $\mathcal{L}^1_n$ such that

$$L^1_n \phi(z) = [z\partial_z + 1]z^n w^1_n(p, q)\phi(z)$$  \hspace{1cm} (4.1)

define an algebra obeying the following commutation relation

$$[\mathcal{L}^1_n, \mathcal{L}^1_m]_{x,y} \phi(z) = \mathcal{R}(p^{n-m}, q^{n-m})q^{N_1-m}\mathcal{L}^1_{n+m} \phi(z),$$  \hspace{1cm} (4.2)

with

$$\begin{cases} 
\hat{x} = \hat{x}_{nm}(p, q) \\
\hat{y} = p^{-n+m}\hat{x}_{nm}(p, q),
\end{cases}$$  \hspace{1cm} (4.3)

and

$$\hat{x}_{nm}(p, q) = \frac{\mathcal{R}(p^{n-m}, q^{n-m})}{[m-n]} \frac{w^1_{n+m}(p, q)}{w^1_n(p, q) w^1_m(p, q)}. \hspace{1cm} (4.4)$$

**Proof:** It follows from the definition of the commutator:

$$[\mathcal{L}^1_n, \mathcal{L}^1_m]_{x,y} \phi(z) = \hat{x}\mathcal{L}^1_n\mathcal{L}^1_m \phi(z) - \hat{y}\mathcal{L}^1_m\mathcal{L}^1_n \phi(z). \hspace{1cm} (4.5)$$

Let us now rewrite the $\mathcal{R}(p, q)$-deformed generators as follows:

$$\tilde{L}^1_n \phi(z) = q^{-N_1} \mathcal{L}^1_n \phi(z),$$  \hspace{1cm} (4.6)

or, equivalently, by using Proposition 2.6:

$$\tilde{L}^1_n \phi(z) = (p - q)^{-1} \left( \frac{p}{q} \phi(p/qz) - \phi(z) \right) z^n w^1_n(p, q). \hspace{1cm} (4.7)$$

**Lemma 4.2** The $\mathcal{R}(p, q)$-generators $\tilde{L}^1_n$ defined by (4.6) satisfy the following relations:

$$[\tilde{L}^1_n, \tilde{L}^1_m]_{x,y} \phi(z) = \mathcal{R}(p^{n-m}, q^{n-m})\tilde{L}^1_{n+m} \phi(z) \hspace{1cm} (4.8)$$

where

$$\begin{cases} 
x = q^{-n+m}\hat{x}_{nm}(p, q) \\
y = p^{-n+m}\hat{x}_{nm}(p, q)
\end{cases}$$  \hspace{1cm} (4.9)

or, equivalently,

$$[\tilde{L}^1_n, \tilde{L}^1_m] \phi(z) = [m - n]p^{N_1-m}q^{-N_1+n}K^1_{nm}(p, q)\tilde{L}^1_{n+m} \phi(z). \hspace{1cm} (4.10)$$

with

$$K^1_{nm}(p, q) = \frac{w^1_n(p, q) w^1_m(p, q)}{w^1_{n+m}(p, q)}. \hspace{1cm} (4.11)$$
Remark 4.3  Note that:

1. Using the symmetry $p \rightarrow q$ and $q \rightarrow p^{-1}$, we obtain the result given in [8] from the relations (4.9) and (4.10).

2. The $\mathcal{R}(p, q)$—deformed $su(1, 1)$ subalgebra is generated by the commutation relations:

   - $\left[\tilde{L}^1_{0}, \tilde{L}^1_{1}\right]_{x,y}\phi(z) = \mathcal{R}(p^{-1}, q^{-1})\tilde{L}^1_{1}\phi(z)$ (4.12)

     where

     \[ x = q\chi_{01}(p, q) \quad \text{and} \quad y = p\hat{\chi}_{01}(p, q) \] (4.13)

   - $\left[\tilde{L}^1_{-1}, \tilde{L}^1_{0}\right]_{x,y}\phi(z) = \mathcal{R}(p^{-1}, q^{-1})\tilde{L}^1_{-1}\phi(z)$ (4.14)

     where

     \[ x = q\hat{\chi}_{-10}(p, q) \quad \text{and} \quad y = p\hat{\chi}_{-10}(p, q) \] (4.15)

   - $\left[\tilde{L}^1_{-1}, \tilde{L}^1_{1}\right]\phi(z) = [2]p^{N_1-1}q^{-N_1-1}K_{-10}(p, q)\tilde{L}^1_{0}\phi(z)$. (4.16)

3. The commutation relation (4.16) defines an $\mathcal{R}(p, q)$—deformation of the Witten algebra [21] in the case of the vertex models.

4. The $\mathcal{R}(p, q)$ generators defined by

   \[ \tilde{\mathcal{L}}_n\phi(z) := \lambda^{-1}\tilde{L}^1_{n}\phi(z). \] (4.17)

   satisfy the commutation relation

   \[ \left[\tilde{\mathcal{L}}_n, \tilde{\mathcal{L}}_m\right]_{x,y}\phi(z) = \mathcal{R}(\theta^{m-n}, \theta^n)\tilde{\mathcal{L}}_{n+m}\phi(z) \] (4.18)

   where

   \[ \begin{aligned} 
   x &= \lambda^{m-n}\theta^{n-m}\bar{\chi}_{nm}(\theta, \theta^{-1}) \\
   y &= \lambda^{m-n}\theta^{m-n}\bar{\chi}_{nm}(\theta, \theta^{-1}). 
   \end{aligned} \] (4.19)
4.1 $\mathcal{R}(p, q)$-deformed nonlinear equation

Chaichian et al \[7\] used the well known connection between the Virasoro algebra and the KdV equation to derive a $q-$ deformed KdV equation corresponding to a $q-$ Virasoro algebra. Chakrabarti et al \[8\] used a similar formalism to study the correlation between a $(p, q)-$Virasoro algebra and a $(p, q)-$KdV equation. The method used in these works is based on the construction of a current defining a bi-Hamiltonian structure which satisfies a nonlinear evolution equation. We follow the same procedure to derive a $\mathcal{R}(p, q)$-deformed nonlinear differential equation corresponding to the algebra (4.23).

The central extension of the algebra (4.8) is generated by the commutation relation

$$[\tilde{L}_n, \tilde{L}_m]_{x,y} = \mathcal{R}(p^{n-m}, q^{n-m}) \tilde{L}_{n+m} + \delta_{n+m,0} \tilde{C}_n^R(p, q),$$

where $x$ and $y$ are given by (4.9), and, for all $\tilde{L}_k$,

$$[\tilde{L}_k, \tilde{C}_n^R(p, q)]_{x_k, y_k} = 0,$$

with $x_k = q^{-k} \chi_{ko}$ and $y_k = p^{-k} \chi_{ok}$.

We consider now the generators defined as follows:

$$t_n := \left( \frac{q}{p} \right)^{N_1} \tilde{L}_n,$$

(4.22)

Then, using the algebra (4.10), the generators (4.22) satisfy the following deformed Virasoro algebra

$$[t_n, t_m] = [m - n] p^{N_1 - m} q^{-N_1 + n} K_{nm}^1(p, q) t_{n+m} + \delta_{n+m,0} \tilde{C}_n^R(p, q),$$

(4.23)

where the deformed central charge commuting with all generators $t_i, i \in \mathbb{Z}$, is expressed as:

$$\tilde{C}_n^R(p, q) = C(p, q) p^{N_1} q^{N_1} (p^n + q^n)^{-1} \mathcal{R}(p^{n-1}, q^{n-1}) \mathcal{R}(p^n, q^n) \mathcal{R}(p^{n+1}, q^{n+1}).$$

(4.24)

Definition 4.4 Let $t_n, n \in \mathbb{Z}$, be the generators considered in (4.22), and $x \in \mathbb{R}$. Then the $\mathcal{R}(p, q)-$deformed current is given by:

$$v(x) := \sum_{n \in \mathbb{Z}} t_n e^{-inx}.$$  

(4.25)

This definition is consistent with the $(p, q)-$deformed current given in [8]:

$$u(x) := \sum_{n \in \mathbb{Z}} e_n e^{-inx}$$

(4.26)
After computation, we obtain
\[ v(x) := \sum_{n \in \mathbb{Z}} w_n^1(p, q)e_n e^{-inx}. \] (4.27)

After computation, we obtain
\[ v(x) = \alpha u(x) - \gamma_{nm}(x) \] (4.28)
where \( \alpha = \sum_{n \in \mathbb{Z}} w_n^1(p, q)\), \( \gamma_{nm}(x) = \sum_{n \neq m} w_n^1(p, q)e_m e^{-inx} \) and \( u(x) \) is given by (4.26).

Then, we arrive at the following commutation relation
\[ \left[ v(x), v(y) \right] = \alpha^2 \left[ u(x), u(y) \right]. \] (4.29)

The latter yields, after some algebra,
\[
\frac{1}{2\pi i} \left[ v(x), v(y) \right] = \alpha^2 \frac{\theta}{2 \sin \epsilon} \left( e^{-2\epsilon \partial_x} v(x) - v(x)e^{2\epsilon \partial_x} \right) \lambda^{-2N_1} \delta(x - y) \\
- \frac{\theta^3}{2 \sinh 2\epsilon \partial_x} \sinh \epsilon \partial_x \sinh \epsilon(\partial_x + i) \sinh \epsilon \partial_x \sinh \epsilon(\partial_x - i) \sinh^3 \epsilon \\
\times \lambda^{-2N_1} \delta(x - y),
\] (4.30)
where \( \lambda \) and \( \theta = e^{-i\epsilon}, \epsilon \in \mathbb{R}^* \) are given by the relation (3.21). This generates the \( \mathcal{R}(p, q) \)-deformed KdV equation as follows:
\[
\frac{dv}{dx} = \frac{\alpha^2 \theta}{4 \sin \epsilon} \left( e^{-2\epsilon \partial_x} v(x) - v(x)e^{2\epsilon \partial_x} \right) \left( \lambda^{-2N_1} v(x) + v(x)\lambda^{-2N_1} \right) \\
- \frac{\alpha^2 \theta^3}{2 \sinh 2\epsilon \partial_x} \sinh \epsilon \partial_x \sinh \epsilon(\partial_x + i) \sinh \epsilon \partial_x \sinh \epsilon(\partial_x - i) \sinh^3 \epsilon \\
\times \left( \lambda^{-2N_1} v(x) + v(x)\lambda^{-2N_1} \right), \] (4.31)
which reduces to the Chakrabarti et al [8] KdV equation in the particular case of \( \mathcal{R}(s, t) = (p - q)^{-1}(s^{-1} - t^{-1}) \).

**Remark 4.5** We can easily deduce the nonlinear differential equations associated to particular algebras described in the previous sections. Without loss of generality, for the computation of \( \alpha \), we consider only the \( \mathbb{Z}_+ \) part, and find the following results:

1. **The Jagannathan-Srinivasa [18] deformed KdV equation:**
\[
\frac{dv}{dx} = \frac{\alpha^2 \theta}{4 \sin \epsilon} \left( e^{-2\epsilon \partial_x} v(x) - v(x)e^{2\epsilon \partial_x} \right) \left( \lambda^{-2N_1} v(x) + v(x)\lambda^{-2N_1} \right) \\
- \frac{\alpha^2 \theta^3}{2 \sinh 2\epsilon \partial_x} \sinh \epsilon \partial_x \sinh \epsilon(\partial_x + i) \sinh \epsilon \partial_x \sinh \epsilon(\partial_x - i) \sinh^3 \epsilon \\
\times \left( \lambda^{-2N_1} v(x) + v(x)\lambda^{-2N_1} \right),
\] (4.32)
where \( \alpha = \frac{pq}{pq - 1} \), with \( |pq| < 1 \).
2. The deformed Chakrabarty et al \[9\] KdV equation:

\[
\frac{dv}{dx} = \frac{\alpha^2 \theta}{4 \sin \epsilon} \left( e^{-2\epsilon \partial_x} v(x) - v(x) e^{2\epsilon \partial_x} \right) \left( \lambda^{-2N_1} v(x) + v(x) \lambda^{-2N_1} \right) \\
- \frac{\alpha^2 \theta^3}{2 \sinh 2\epsilon \partial_x} \frac{\sinh \epsilon (\partial_x + i) \sinh \epsilon (\partial_x - i)}{\sin^3 \epsilon} \\
\times \left( \lambda^{-2N_1} v(x) + v(x) \lambda^{-2N_1} \right),
\]

where

\[
\alpha = \frac{p - q}{p^{-1} - q} \left( -\frac{q}{q - p} - \frac{pq}{1 - pq} - \frac{p^2}{1 - p^2} \right)
\]

\[|\frac{q}{p}| < 1 \text{ and } |pq| < 1.\]

3. The generalized Quesne \[17\] KdV equation is given by:

\[
\frac{dv}{dx} = \frac{\alpha^2 \theta}{4 \sin \epsilon} \left( e^{-2\epsilon \partial_x} v(x) - v(x) e^{2\epsilon \partial_x} \right) \left( \lambda^{-2N_1} v(x) + v(x) \lambda^{-2N_1} \right) \\
- \frac{\alpha^2 \theta^3}{2 \sinh 2\epsilon \partial_x} \frac{\sinh \epsilon (\partial_x + i) \sinh \epsilon (\partial_x - i)}{\sin^3 \epsilon} \\
\times \left( \lambda^{-2N_1} v(x) + v(x) \lambda^{-2N_1} \right),
\]

with

\[
\alpha = \frac{p - q}{q - p^{-1}} \left( -\frac{q^2}{1 - q^2} + \frac{p}{p - q} - \frac{pq}{1 - pq} \right)
\]

\[|\frac{q}{p}| < 1 \text{ and } |pq| < 1.\]

4. The deformed Hounkonnou-Ngompe generalized \[17\] KdV equation:

\[
\frac{dv}{dx} = \frac{\alpha^2 \theta}{4 \sin \epsilon} \left( e^{-2\epsilon \partial_x} v(x) - v(x) e^{2\epsilon \partial_x} \right) \left( \lambda^{-2N_1} v(x) + v(x) \lambda^{-2N_1} \right) \\
- \frac{\alpha^2 \theta^3}{2 \sinh 2\epsilon \partial_x} \frac{\sinh \epsilon (\partial_x + i) \sinh \epsilon (\partial_x - i)}{\sin^3 \epsilon} \\
\times \left( \lambda^{-2N_1} v(x) + v(x) \lambda^{-2N_1} \right),
\]

where

\[
\alpha = g(p, q) \frac{p - q}{q - p^{-1}} \left( \frac{q^\nu}{p^\mu - q^\nu} + \frac{q^{\nu+1}}{p^\mu+1 - q^{\nu+1}} - \frac{q^{\nu+2}}{p^\mu-1 - q^{\nu+2}} + \frac{q^{\nu+2}}{p^\mu - q^{\nu+2}} \right)
\]

with \[|\frac{q}{p}| < 1\]

5 \(\mathcal{R}(p, q)\)—deformed energy-momentum tensor for the case \(\Delta = 2\)

In this section, we compute the \(\mathcal{R}(p, q)\)—deformed energy-momentum tensor for the conformal dimension \(\Delta = 2\).
5.1 \((p, q)\) – deformed energy-momentum tensor

Let us start with the simplest case of the \((p, q)\)-deformed algebra given in [8] for \(\Delta = 2\). The corresponding generators are given as follows:

\[
L_n^2 \phi_2(z) = z^{-(n+1)} D_{p,q} (z^{2(n+1)} \phi_2(z)),
\]

satisfying the commutation relation

\[
[L_n^2, L_m^2]_{x_2,y_2} \phi(z) = K \left\{ p^{N_2} (x_2 p^n - y_2 p^{-m}) - q^{N_2} (x_2 q^n - y_2 q^{-m}) \right\} L_{n+m}^2 \phi(z)
\]

\[
= (pq)^n [m - n] (p^{N_2} + q^{N_2}) L_{n+m}^2 \phi(z),
\]

with

\[
\begin{cases}
  y_2 = (pq)^n (p^m + q^m) \\
y_2 = (pq)^m (p^n + q^n) \\
K = (p - q)^{-1}.
\end{cases}
\]

According to [2] and [22], the energy-momentum \(\mathcal{T}(z)\) has the conformal dimension two but does not transform as a primary field [3]. In the undeformed case,

\[
\mathcal{T}(z) \longrightarrow (\partial_z \phi(z))^2 + \phi(z) + c \left( \frac{\partial_z^{(3)} \phi(z)}{\partial_z} - 3/2 \left( \frac{\partial_z^{(2)} \phi(z)}{\partial_z} \right)^2 \right),
\]

where \(c\) is the central charge and \(\partial_z^{(n)}(z)\) is the \(n\)-th order derivative. The infinitesimal form is given by:

\[
\delta_\epsilon \mathcal{T}(z) = \left( \epsilon(z) \partial_z + 2 \partial \epsilon(z) \right) \mathcal{T}(z) + c \partial_z^{(3)} \epsilon(z).
\]

Putting \(\epsilon(z) = z^{n+1}\), we have

\[
\delta_n \mathcal{T}(z) = l_n \mathcal{T}(z) + cg(n)z^{n-2},
\]

where

\[
l_n \phi(z) = [z \partial_z + n + 2] z^n \phi(z)
\]

and \(g(n) = (n - 1)n(n + 1)\).

If the equation (5.7) satisfies

\[
[\delta_m, \delta_n] \mathcal{T}(z) = (m - n) \delta_{n+m} \mathcal{T}(z),
\]

then the central term can be obtained.

The \((p, q)\) – analogue of equation (5.6) is written as:

\[
\delta_{p,q} \mathcal{T}_{p,q}(z) = L_n^2 \mathcal{T}_{p,q}(z) + c(p, q)z^{n-2},
\]
where $\mathcal{T}_{p,q}(z)$ is the $(p, q)$—deformed energy-momentum tensor and $L_n^2$ is defined by (5.1). We consider now $c_0(p, q)$ as the central extension term for the $(p, q)$—deformed conformal algebra. Using (5.2), we obtain

$$[\delta_m^{p,q}, \delta_n^{p,q}]_{x_2, y_2}\mathcal{T}_{p,q}(z) = (pq)^n[m - n](p^{N_2} + q^{N_2})\delta_{n+m}^{p,q}\mathcal{T}_{p,q}(z), \quad (5.10)$$

where $x_2$ and $y_2$ are given by equation (5.3).

According to the relations (5.9), and (5.10), and using (5.2), we get

$$\nu_{nm}(p, q)c_m(p, q) - \mu_{nm}(p, q)c_n(p, q) = \alpha_{nm}(p, q)c_{n+m}(p, q), \quad (5.11)$$

with $\nu_{nm}(p, q) = (pq)^n(p^m + q^m)[2n + m]$, $\mu_{nm}(p, q) = (pq)^{m-n}(p^n + q^n)[2m + n]$ and $\alpha_{nm}(p, q) = (m - n)(p^{m+n} + q^{m+n})$. Putting $m = 1$ and $n = 0$ in the equation (5.11), we have $c_0(p, q) = 0$. But if $m = -n$, we get $c_n(p, q) = c_{-n}(p, q)$. Thus, we have $c_1(p, q) \neq 0$.

Now defining the tensor such that $c_1(p, q) = 0$ imposes to shift $\mathcal{T}_{p,q}(z)$ as follows:

$$\hat{\mathcal{T}}_{p,q}(z) = \mathcal{T}_{p,q}(z) + \frac{\beta(p, q)}{[2]} z^{-2}, \quad (5.12)$$

where $\beta$ is a constant depending on the parameters $p$ and $q$. Thus the function $c_n(p, q)$ takes the following form

$$\hat{c}_n(p, q) = c_n(p, q) + \beta(p, q)\frac{[2n]}{[2]} \quad (5.13)$$

Hence, if we choose $\beta = -c_1(p, q)$, then $\hat{c}_1(p, q) = 0$. Finally, we get from (5.11) the following equation

$$(p^m + q^m)[m - 2]\hat{c}_m(p, q) = (pq)^{-2}(p^{m-1} + q^{m-1})[m + 1]\hat{c}_{m-1}(p, q). \quad (5.14)$$

Its solution is the $(p, q)$—deformed central extension given in [8].

### 5.2 $\mathcal{R}(p, q)$—deformed energy-momentum tensor

The corresponding $\mathcal{R}(p, q)$ generators for $\Delta = 2$ are given as follows:

$$\mathcal{L}^{(2)}_n(z) = \frac{\mathcal{D}_{R(p,q)}}{(2(1+n))}(z^{2(1+n)}\phi(z)), \quad (5.15)$$

and satisfy the following relation

$$[\mathcal{L}^2, \mathcal{L}^2_{m}]_{X_2, Y_2}\phi(z) = K \left\{ p^{N_2}(\bar{X}_2p^{-m} - \bar{Y}_2p^{-m}) - q^{N_2}(\bar{X}_2q^{-m} - \bar{Y}_2q^{-m}) \right\}$$

$$\times \mathcal{L}^2_{n+m}\phi(z) = (pq)^nK_{nm}(p, q)[m - n](p^{N_2} + q^{N_2})L^2_{n+m}\phi(z), \quad (5.16)$$

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where

\[
\begin{align*}
\tilde{X}_2 &= (pq)^n(p^n + q^m)K_{nm}(p,q) \\
\tilde{Y}_2 &= (pq)^m(p^n + q^n)K_{nm}(p,q) \\
N_2 &= z\partial_z + 2 \\
K_{nm}(p,q) &= \frac{w_{n+m}^2(p,q)}{w_n^2(p,q)w_m^2(p,q)} \\
K &= (p - q)^{-1}.
\end{align*}
\]

According to the equation (5.6), the $\mathcal{R}(p, q)$—deformed infinitesimal form can be given by:

\[
\delta_n\mathcal{T}_{\mathcal{R}(p, q)}(z) = L^2_n\mathcal{T}_{\mathcal{R}(p, q)}(z) + \tilde{C}^R_n(p, q)z^{-2},
\]

where $L^2_n$ are the deformed generators given by (5.16), $\tilde{C}^R_n(p, q)$ is the central charge of the $\mathcal{R}(p, q)$—deformed conformal algebra, and $\mathcal{T}_{\mathcal{R}(p, q)}(z)$ is the $\mathcal{R}(p, q)$—deformed energy-momentum tensor. Therefore, the $\mathcal{R}(p, q)$—deformed infinitesimal form satisfies the following commutation relation

\[
[\delta_m, \delta_n]\tilde{X}_2, \tilde{Y}_2 \mathcal{T}_{\mathcal{R}(p, q)}(z) = (pq)^nK_{nm}(p, q)[m - n](p^{N_2} + q^{N_2})\delta_{n+m}\mathcal{T}_{\mathcal{R}(p, q)}(z),
\]

where $\tilde{X}_2, \tilde{Y}_2, \text{and } K_{nm}(p, q)$ are given by (5.17), which can be deduced by a straightforward computation. Furthermore, using the relations (5.16), (5.18), and (5.19), we obtain

\[
\nu^R_{nm}(p, q)C^R_m(p, q) - \mu^R_{nm}(p, q)C^R_n(p, q) = \alpha^R_{nm}(p, q)C^R_{n+m}(p, q),
\]

where $\nu^R_{nm}(p, q) = (pq)^n(p^m + q^m)[2n + m]w_n^2(p, q)$, $\mu^R_{nm}(p, q) = (pq)^m(p^n + q^n)[2m + n]w_m^2(p, q)$, and $\alpha^R_{nm}(p, q) = (pq)^n[m - n](p^{n+m} + q^{m+n})$.

The $\mathcal{R}(p, q)$—deformed tensor is then given by:

\[
\tilde{T}_{\mathcal{R}(p, q)}(z) = \mathcal{T}_{\mathcal{R}(p, q)}(z) + \frac{\gamma(p, q)}{2}z^{-2}.
\]

6 Concluding remarks

In this paper, we have constructed an $\mathcal{R}(p, q)$—deformed conformal Virasoro algebra with an arbitrary conformal dimension $\Delta$. Wellknown deformed algebras, investigated in the literature, have been deduced as particular cases. A special attention has been paid to the specific case of the conformal dimension $\Delta = 1$ for its interesting properties. The $\mathcal{R}(p, q)$—nonlinear differential equation has been derived, and its link to the $\mathcal{R}(p, q)$—deformed Virasoro algebra has been established. The $\mathcal{R}(p, q)$—deformed energy-momentum tensor, consistent with the obtained deformed central extension term, has been computed.
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