Aggregation of Foraging Swarms *

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Abstract: In this paper we consider a continuous-time anisotropic swarm model with an attraction/repulsion function and study its aggregation properties. It is shown that the swarm members will aggregate and eventually form a cohesive cluster of finite size around the swarm center. We also study the swarm cohesiveness when the motion of each agent is a combination of the inter-individual interactions and the interaction of the agent with external environment. Moreover, we extend our results to more general attraction/repulsion functions. The model in this paper is more general than isotropic swarms and our results provide further insight into the effect of the interaction pattern on individual motion in a swarm system.

Keywords: Autonomous mobile agents, biological systems, multi-agent systems, swarm intelligence, aggregation.

1 Introduction

In nature swarming can be found in many organisms ranging from simple bacteria to more advanced mammals. Examples of swarms include flocks of birds, schools of fish, herds of animals, and colonies of bacteria. Such collective behavior has certain advantages such as avoiding predators and increasing the chance of finding food. Recently, there has been a growing interest in biomimicry of foraging and swarming for using in engineering applications such as optimization, robotics, military applications and autonomous air vehicle [1]–[7]. Modeling and exploring the collective dynamics has become an important issue and many investigations have been carried out [8]–[13]. However, results on the anisotropic swarms are relatively few. The study of anisotropic swarms is very difficult

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though anisotropic swarming is a ubiquitous phenomenon, including natural phenomena and social phenomena.

Gazi and Passino [2] proposed an isotropic swarm model and studied its aggregation, cohesion and stability properties. Subsequently, Chu, Wang and Chen [4] generalized their model, considering an anisotropic swarm model, and obtained the properties of aggregation, cohesion and completely stability. The coupling matrix considered in [4] is symmetric, that is, the interaction between two individuals is reciprocal. In this paper, we will study the behavior of anisotropic swarms when the coupling matrix is completely asymmetric. The results given in this paper extend the corresponding results on isotropic swarms [2] and anisotropic swarms [4] to more general cases and further illustrate the effect of the interaction pattern on individual motion in swarm systems. Moreover, we also study the aggregation properties of the anisotropic swarm under an attractant/repellent profile.

In the next section we specify an "individual-based" continuous-time anisotropic swarm model in Euclidean space which includes the isotropic model of [2] as a special case, and we also study the agent motion when the external attractant/repellent profile is considered. In Section 3, under some assumption on the coupling matrix, we show that the swarm exhibits aggregation. In Section 4, we extend the results in Section 3 by considering a more general attraction/repulsion function. We summarize our results in Section 5.

2 Anisotropic Swarms

We consider a swarm of $N$ individuals (members) in an $n$-dimensional Euclidean space. We model the individuals as points and ignore their dimensions. The equation of motion of individual $i$ is given by

$$\dot{x}^i = \sum_{j=1}^{N} w_{ij} f(x^i - x^j), \quad i = 1, \cdots, N,$$

(1)

where $x^i \in \mathbb{R}^n$ represents the position of individual $i$; $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ with $w_{ij} \geq 0$ for all $i, j = 1, \cdots, N$ is the coupling matrix; $f(\cdot)$ represents the function of attraction and repulsion between the members. In other words, the direction and magnitude of motion of each member is determined as a weighted sum of the attraction and repulsion of all the other members on this member. The attraction/repulsion function that we consider is

$$f(y) = -y \left( a - b \exp \left( -\frac{\|y\|^2}{c} \right) \right),$$

(2)

where $a, b,$ and $c$ are positive constants such that $b > a$ and $\|y\|$ is the Euclidean norm given by $\|y\| = \sqrt{y^T y}$.

In the discussion to follow, we always assume $w_{ii} = 0, i = 1, \cdots, N$ in model (1). Moreover, we assume that there are no isolated clusters in the swarm, that is, $W + W^T$ is irreducible.
Note that the function $f(\cdot)$ is the social potential function that governs the interindividual interactions and is attractive for large distances and repulsive for small distances. By equating $f(y) = 0$, one can find that $f(\cdot)$ switches sign at the set of points defined as $\mathcal{Y} = \{y = 0 \text{ or } \| y \| = \delta = \sqrt{c \ln(b/a)}\}$. The distance $\delta$ is the distance at which the attraction and repulsion balance. Such a distance in biological swarms exists indeed [3]. Note that it is natural as well as reasonable to require that any two different swarm members could not occupy the same position at the same time.

Remark 1: The anisotropic swarm model given here includes the isotropic model of [2] as a special case. Obviously, the present model [11] can better reflect the asymmetry of social, economic and psychological phenomena [14]–[19].

In the above model, the agent motion was driven solely by the interaction pattern between the swarm members, i.e., we didn’t consider the external environment’s effect on agent motion. In what follows, we will consider the external attractant/repellent profile and propose a new model.

Following [11], we consider the attractant/repellent profile $\sigma : \mathbb{R}^n \to \mathbb{R}$, which can be a profile of nutrients or some attractant/repellent substances (e.g. nutrients or toxic chemicals). We also assume that the areas that are minimum points are ”favorable” to the individuals in the swarm. For example, we can assume that $\sigma(y) > 0$ represents a noxious environment, $\sigma(y) = 0$ represents a neutral, and $\sigma(y) < 0$ represents attractant or nutrient rich environment at $y$. (Note that $\sigma(\cdot)$ can be a combination of several attractant/repellent profiles).

In the new model, the equation of motion for individual $i$ is given by

$$\dot{x}^i = -h_i \nabla_x x^i \sigma(x^i) + \sum_{j=1}^{N} w_{ij} f(x^i - x^j), \quad i = 1, \cdots, N,$$

(3)

where the attraction/repulsion function $f(\cdot)$ is same as given in (2), $h_i \in \mathbb{R}^+ = (0, \infty)$, and $w_{ij}$ is defined as before. $-h_i \nabla_x \sigma(x^i)$ represents the motion of the individuals toward regions with higher nutrient concentration and away from regions with high concentration of toxic substances. We assume that the individuals know the gradient of the profile at their positions.

In the discussion to follow, we will need the the concept of weight balance condition defined below:

**Weight Balance Condition**: consider the coupling matrix $W = [w_{ij}] \in \mathbb{R}^{N \times N}$, for all $i = 1, \cdots, N$, we assume that $\sum_{j=1}^{N} w_{ij} = \sum_{j=1}^{N} w_{ji}$.

The weight balance condition has a graphical interpretation: consider the directed graph associated with the coupling matrix, weight balance means that, for any node in this graph, the weight sum of all incoming edges equals the weight sum of all outgoing edges [5]. The weight balance condition can find physical interpretations in engineering systems such as water flow, electrical current, and traffic systems.

3
3 Swarm Aggregation

In this section, theoretic results concerning aggregation and cohesiveness of the swarms (1) and (3) will be presented. First, it is of interest to investigate collective behavior of the entire system rather than to ascertain detailed behavior of each individual. Second, due to complex interactions among the multi-agents, it is usually very difficult or even impossible to study the specific behavior of each agent.

Define the center of the swarm members as \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x^i \), and denote \( \beta_{ij} = \exp \left( -\frac{\|x^i - x^j\|^2}{c^2} \right) \). We first consider the swarm in (1), then the equation of motion of the center is

\[
\dot{x} = -\frac{a}{N} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} (x^i - x^j) \right] + \frac{b}{N} \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} w_{ij} \beta_{ij} (x^i - x^j) \right].
\]

If the coupling matrix \( W \) is symmetric, by the symmetry of \( f(\cdot) \) with respect to the origin, the center \( \bar{x} \) will be stationary for all \( t \), and the swarm described by Eqs. (1) and (2) will not be drifting on average [4]. Note, however, that the swarm members may still have relative motions with respect to the center while the center itself stays stationary. On the other hand, if the coupling matrix \( W \) is asymmetric, the center \( \bar{x} \) may not be stationary.

An interesting issue is whether the members will form a cohesive cluster and which point they will move around. We will deal with this issue in the following theorem.

**Theorem 1**: Consider the swarm in (1) with an attraction/replusion function \( f(\cdot) \) in (2). Under the weight balance condition, all agents will eventually enter into and remain in the bounded region

\[ \Omega = \left\{ x : \sum_{i=1}^{N} \|x^i - \bar{x}\|^2 \leq \rho^2 \right\}, \]

where

\[ \rho = \frac{2bM \sqrt{2c} \exp\left(-\frac{1}{2}\right)}{a\lambda_2}; \]

and \( M = \sum_{i,j=1}^{N} w_{ij} \); \( \lambda_2 \) denotes the second smallest real eigenvalue of the matrix \( L + L^T \); \( L = [l_{ij}] \) with

\[ l_{ij} = \begin{cases} -w_{ij}, & i \neq j; \\ \sum_{k=1,k \neq i}^{N} w_{ik}, & i = j; \end{cases} \]

(4)

\( \Omega \) provides a bound on the maximum ultimate swarm size.

**Proof.** Let \( e^i = x^i - \bar{x} \). By the definition of the center \( \bar{x} \) of the swarm and the weight balance condition, we have

\[ \dot{\bar{x}} = \frac{b}{N} \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} w_{ij} \beta_{ij} (x^i - x^j) \right]. \]
Therefore, we can order the eigenvalues of $L$ in Eq. (4) and $I$ is symmetric and $W$.

To estimate $e^i$, let $V = \sum_{i=1}^{N} V_i$ be the Lyapunov function for the swarm, where $V_i = \frac{1}{2} e^{iT} e^i$. Evaluating its time derivative along the solution of system (1), we have

$$
\dot{V} = -a \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} e^{iT} (e^i - e^j) + b \sum_{i=1}^{N} e^{iT} \left( \sum_{j=1}^{N} w_{ij} \beta_{ij} (x^i - x^j) \right)
$$

$$
\leq -ae^T (L \otimes I) e + b \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \beta_{ij} \|x^i - x^j\| \|e^i\|
$$

$$
+ \frac{b}{N} \sum_{i=1}^{N} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} w_{kj} \beta_{kj} \|x^k - x^j\| \right] \|e^i\|,
$$

where $e = (e^{1T}, \cdots, e^{NT})^T$, $L \otimes I$ is the Kronecker product of $L$ and $I$ with $L$ as defined in Eq. (4) and $I$ the identity matrix of order $n$.

Note that each of the functions $\exp \left( -\frac{\|x^i - x^j\|^2}{e} \right) \|x^i - x^j\|$ is a bounded function whose maximum is achieved at $\|x^i - x^j\| = \sqrt{c/2}$ and is given by $\sqrt{c/2} \exp(-(1/2))$. Substituting this into the above inequality and using the fact that $\|e^i\| \leq \sqrt{2V}$, we obtain

$$
\dot{V} \leq -ae^T (L \otimes I) e + 2bM \sqrt{c} \exp \left( -\frac{1}{2} \right) V^{1/2}.
$$

(5)

To get further estimate of $\dot{V}$, we only need to estimate the term $e^T (L \otimes I) e$. Since

$$
e^T (L \otimes I) e = \frac{1}{2} e^T ((L + L^T) \otimes I) e,$$

we need to analyze $e^T ((L + L^T) \otimes I) e$. First, consider the matrix $L + L^T$ with $L$ defined in Eq. (4), we have $L + L^T = \tilde{l}_{ij}$, where

$$
\tilde{l}_{ij} = \left\{ \begin{array}{ll}
-w_{ij} - w_{ji}, & i \neq j, \\
2 \sum_{k=1, k \neq i}^{N} W_{ik}, & i = j.
\end{array} \right.
$$

(6)

Under the weight balance condition, we can easily see that $\lambda = 0$ is an eigenvalue of $L + L^T$ and $u = (l, \cdots, l)^T$ with $l \neq 0$ is the associated eigenvector. Moreover, since $L + L^T$ is symmetric and $W + W^T$ (hence, $L + L^T$) is irreducible, it follows from matrix theory [5] that $\lambda = 0$ is a simple eigenvalue and all the rest eigenvalues of $L + L^T$ are real and positive. Therefore, we can order the eigenvalues of $L + L^T$ as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$. 


Moreover, it is known that the identity matrix $I$ has an eigenvalue $\mu = 1$ of $n$ multiplicity and $n$ linearly independent eigenvectors

$$u^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad u^2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad u^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

By matrix theory [5], the eigenvalues of $(L + L^T) \otimes I$ are $\lambda \mu = \lambda_i$ (of $n$ multiplicity for each $i$). Next, we consider the matrix $(L + L^T) \otimes I$. $\lambda = 0$ is an eigenvalue of $n$ multiplicity and the associated eigenvectors are

$$v^1 = [u^{1T}, \ldots, u^{1T}]^T, \ldots, v^n = [u^{nT}, \ldots, u^{nT}]^T.$$

Therefore, $e^T((L + L^T) \otimes I)e = 0$ implies that $e$ must lie in the eigenspace of $(L + L^T) \otimes I$ spanned by eigenvectors $v^1, \ldots, v^n$ corresponding to the zero eigenvalue, that is, $e^1 = e^2 = \cdots = e^N$. This occurs only when $e^1 = e^2 = \cdots = e^N = 0$. However, this is impossible to happen for the swarm system under consideration, because it implies that the $N$ individuals occupy the same position at the same time. Hence, for any solution $x$ of system (1), $e$ must be in the subspace spanned by eigenvectors of $(L + L^T) \otimes I$ corresponding to the nonzero eigenvalues. Hence, $e^T((L + L^T) \otimes I)e \geq \lambda_2 \|e\|^2 = 2 \lambda_2 V$.

From (5), we have

$$\dot{V} \leq -a \lambda_2 V + 2bM \sqrt{c} \exp(-\frac{1}{2}) V^{1/2}$$

$$= -\left[a \lambda_2 V^{1/2} - 2bM \sqrt{c} \exp(-\frac{1}{2})\right] V^{1/2}$$

$$< 0$$

whenever

$$V > \left(\frac{2bM \sqrt{c} \exp(-1/2)}{a \lambda_2}\right)^2.$$

Therefore, any solution of system (1) will eventually enter into and remain in $\Omega$. \hfill \Box

Remark 2: The discussions above explicitly show the effect of the coupling matrix $W$ on aggregation and cohesion of the swarm.

Remark 3: The weight balance condition is more general than the case when the coupling matrix $W$ is a symmetric matrix [2, 4].

Remark 4: Theorem 1 shows that the swarm members will aggregate and form a bounded cluster around the swarm center.

Remark 5: From Theorem 1, we see that, under the weight balance condition, the motion of the swarm center only depends on the repulsion between the swarm members. From the above discussions, we know that if we ignore the influence on agent motion from external environment, under the weight balance condition, the motion of the swarm center only depends on the repulsion between the swarm members, and all agents...
will eventually enter into and remain in a bounded region around the swarm center. In what follows, we will study the aggregation properties of the swarm system when the attractant/repellent profile is taken into account.

The equation of the motion of the swarm center now becomes

\[
\dot{x} = -\frac{1}{N} \sum_{i=1}^{N} h_i \nabla_x \sigma(x_i) - \frac{a}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} w_{ij} - \sum_{j=1}^{N} w_{ji} \right) x_i + \frac{b}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} w_{ij} \beta_{ij} (x_i - x_j) \right).
\]

Before we discuss cohesiveness of the swarm, we first make an assumption.

**Assumption 1:** There exists a constant \( \sigma > 0 \) such that

\[
\| \nabla \sigma(y) \| \leq \sigma, \quad \text{for all } y.
\]

Assumption 1 implies that the gradient of the profile is bounded. This assumption is reasonable since almost all profiles we encounter such as plane and Gaussian profiles are with bounded gradient.

The following theorem shows that the swarm system still exhibits aggregation behavior when the external profile is taken into account.

**Theorem 2:** Consider the swarm in (3) with an attraction/replusion function \( f(\cdot) \) in (2). Under the weight balance condition and Assumption 1, all agents will eventually enter into and remain in the bounded region

\[
\Omega = \left\{ x : \sum_{i=1}^{N} \| x_i - \overline{x} \| ^2 \leq \rho^2 \right\},
\]

where

\[
\rho = \frac{2bM \sqrt{2c} \exp(-\frac{1}{2}) + 4\sigma (\sum_{i=1}^{N} h_i)}{a \lambda_2};
\]

and \( M \) and \( \lambda_2 \) are defined as in Theorem 1. \( \Omega \) provides a bound on the maximum ultimate swarm size.

**Proof.** Let \( e^i = x^i - \overline{x} \). By the definition of the center \( \overline{x} \) of the swarm and the weight balance condition, we have

\[
\dot{\overline{x}} = -\frac{1}{N} \sum_{i=1}^{N} h_i \nabla_x \sigma(x_i) + \frac{b}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} w_{ij} \beta_{ij} (x_i - x_j) \right).
\]

Define the Lyapunov function as \( V = \sum_{i=1}^{N} V_i \), where \( V_i = \frac{1}{2} e^{i T} e^i \). Evaluating its time
derivative along solution of the system (3), we have

\[ \dot{V} = -a \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} e^{iT}(e^i - e^j) + b \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \beta_{ij} e^{iT}(e^i - e^j) \]

\[ \quad - \frac{b}{N} \sum_{i=1}^{N} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} w_{kj} \beta_{kj} e^{iT}(x^k - x^j) \right] \]

\[ \quad - \sum_{i=1}^{N} e^{iT} \left[ h_i \nabla_{x^i} \sigma(x^i) - \frac{1}{N} \sum_{i=1}^{N} h_i \nabla_{x^i} \sigma(x^i) \right]. \]

Furthermore, by assumption, we have

\[ \dot{V} \leq -a \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} e^{iT}(e^i - e^j) + b \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \beta_{ij} \|x^i - x^j\| \|e^i\| \]

\[ \quad + \frac{b}{N} \sum_{i=1}^{N} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} w_{kj} \beta_{kj} \|x^k - x^j\| \|e^i\| \right] \]

\[ \quad + \sum_{i=1}^{N} \|h_i \nabla_{x^i} \sigma(x^i) - \frac{1}{N} \sum_{i=1}^{N} h_i \nabla_{x^i} \sigma(x^i)\| \|e^i\| \]

\[ \leq -a \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} e^{iT}(e^i - e^j) \]

\[ \quad + 2bM \sqrt{c} \exp(-\frac{1}{2}) V^{1/2} + 2 \sqrt{2\sigma} \left( \sum_{i=1}^{N} h_i \right) V^{1/2}. \]

By analogous discussions as in the proof of Theorem 1, we have

\[ \dot{V} \leq -a \lambda_2 V + 2bM \sqrt{c} \exp(-\frac{1}{2}) V^{1/2} + 2 \sqrt{2\sigma} \left( \sum_{i=1}^{N} h_i \right) V^{1/2} \]

\[ = - \left[ a \lambda_2 V^{1/2} - 2bM \sqrt{c} \exp(-\frac{1}{2}) - 2 \sqrt{2\sigma} \left( \sum_{i=1}^{N} h_i \right) \right] V^{1/2} \]

\[ < 0 \]

whenever

\[ V > \left( \frac{2bM \sqrt{c} \exp(-1/2) + 2 \sqrt{2\sigma} \left( \sum_{i=1}^{N} h_i \right)}{a \lambda_2} \right)^2. \]

Therefore, any solution of system (3) will eventually enter into and remain in \( \overline{\Omega} \).

**Remark 6:** Theorem 2 shows that, with bounded attractant/repellent profile, the swarm members will aggregate and form a bounded cluster around the swarm center. The motion of the swarm center depends on the repulsion between the swarm members and the weighted average of the gradient of the profile evaluated at the current positions of the individuals.

Of course, not all the profiles are bounded. In the case of unbounded profile, in order to ensure the swarm to be ultimately bounded, the gradient of the profile at \( x^i \) should
have a "sufficiently large" component along \( e^i \) so that the influence of the profile does not affect swarm cohesion. The following theorem addresses this issue.

**Theorem 3**: Consider the swarm in (3) with an attraction/replusion function \( f(\cdot) \) in (2). Assume that there exist constants \( A^i_\sigma, i = 1, \cdots, N \), with \( A_\sigma = \min A^i_\sigma > -\frac{a\lambda_2}{2} \) such that
\[
e^{iT} \left[ h^i_i \nabla_{x^i} \sigma(x^i) - \frac{1}{N} \sum_{k=1}^{N} h^k_k \nabla_{x^k} \sigma(x^k) \right] \geq A^i_\sigma \|e^i\|^2
\]
for all \( x^i \) and \( x^k \). Then, under the weight balance condition, all agents will eventually enter into and remain in the bounded region
\[
\Omega = \left\{ x : \sum_{i=1}^{N} \|x^i - \pi\|^2 \leq \rho^2 \right\},
\]
where
\[
\rho = \frac{2bM\sqrt{c}\exp(-\frac{1}{2})}{a\lambda_2 + 2A_\sigma};
\]
and \( M \) and \( \lambda_2 \) are defined as in Theorem 1. \( \Omega \) provides a bound on the maximum ultimate swarm size.

**Proof.** Following the proof of Theorem 2, from (7), we have
\[
\dot{V} \leq -a \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} e^{iT}(e^i - e^j) + b \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \beta_{ij} \|x^i - x^j\| \|e^i\| \\
+ \frac{b}{N} \sum_{i=1}^{N} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} w_{kj} \beta_{kj} \|x^k - x^j\| \|e^i\| - A_\sigma \|e^i\|^2 \right] \leq -a \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} e^{iT}(e^i - e^j) \\
+ 2bM\sqrt{c}\exp(-\frac{1}{2})V^{1/2} - 2A_\sigma V.
\]
By analogous discussions as in the proof of Theorem 1, we have
\[
\dot{V} \leq -(a\lambda_2 + 2A_\sigma)V + 2bM\sqrt{c}\exp(-\frac{1}{2})V^{1/2} \\
= -\left[ (a\lambda_2 + 2A_\sigma)V^{1/2} - 2bM\sqrt{c}\exp(-\frac{1}{2}) \right] V^{1/2} < 0
\]
whenever
\[
V(x) > \left( \frac{2bM\sqrt{c}\exp(-1/2)}{a\lambda_2 + 2A_\sigma} \right)^2.
\]
Therefore, any solution of system (3) will eventually enter into and remain in \( \Omega \). \( \square \)
4 Further Extensions

In Sections 2 and 3 we considered a specific attraction/repulsion function $f(y)$ as defined in (2). In this section, we will consider a more general attraction/repulsion function $f(y)$. Here $f(y)$ is still the social potential function that governs the interindividual interactions and is assumed to have a long range attraction and short range repulsion nature. Following [10], we make the following assumptions on the social potential function:

**Assumption 2.** The attraction/repulsion function $f(\cdot)$ is of the form

$$f(y) = -y[f_a(\|y\|) - f_r(\|y\|)], y \in \mathbb{R}^n, \quad (8)$$

where $f_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ represents (the magnitude of) attraction term and has a long range, whereas $f_r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ represents (the magnitude of) repulsion term and has a short range, and $\mathbb{R}_+$ stands for the set of nonnegative real numbers, $\|y\| = \sqrt{y^T y}$ is the Euclidean norm.

**Assumption 3.** There are positive constants $a, b$ such that for any $y \in \mathbb{R}^n$,

$$f_a(\|y\|) = a, \quad f_r(\|y\|) \leq \frac{b}{\|y\|}. \quad (9)$$

That is, we assume a fixed linear attraction function and a bounded repulsion function.

Analogous to Theorems 1–3, in this case, we can also obtain the following three theorems.

**Theorem 4:** Consider the swarm in (5) with an attraction/repulsion function $f(\cdot)$ in (8) and (9). Under the weight balance condition, all agents will eventually enter into and remain in the bounded region

$$\Omega^* = \left\{ x : \sum_{i=1}^{N} \|x^i - \bar{x}\|^2 \leq \rho^2 \right\},$$

where $\rho = \frac{4hM}{a\lambda_2}$; and $\lambda_2$ and $M$ are defined as in Theorem 1; $\Omega^*$ provides a bound on the maximum ultimate swarm size.

**Theorem 5:** Consider the swarm in (3) with an attraction/repulsion function $f(\cdot)$ in (8) and (9). Under the weight balance condition and Assumption 1, all agents will eventually enter into and remain in the bounded region

$$\overline{\Omega}^* = \left\{ x : \sum_{i=1}^{N} \|x^i - \bar{x}\|^2 \leq \rho^2 \right\},$$

where

$$\rho = \frac{4hM + 4\sigma(\sum_{i=1}^{N} h_i)}{a\lambda_2};$$

and $M$ and $\lambda_2$ are defined as in Theorem 1. $\overline{\Omega}^*$ provides a bound on the maximum ultimate swarm size.
Theorem 6: Consider the swarm in (3) with an attraction/replusion function \( f(\cdot) \) in (8) and (9). Assume that there exist constants \( A_i \), \( i = 1, \ldots, N \), with \( A_\sigma = \min_i A_i > -\frac{a\lambda_2}{2} \) such that
\[
e^{-t} \left[ \frac{1}{N} \sum_{k=1}^{N} h_k \nabla_x \sigma(x^k) \right] \geq A_\sigma \| e^i \|^2
\]
for all \( x^i \) and \( x^k \). Then, under the weight balance condition, all agents will eventually enter into and remain in the bounded region
\[
\Omega^* = \left\{ x : \sum_{i=1}^{N} \| x^i - \bar{x} \|^2 \leq \rho^2 \right\},
\]
where
\[
\rho = \frac{4bM}{a\lambda_2 + 2A_\sigma};
\]
and \( M \) and \( \lambda_2 \) are defined as in Theorem 1. \( \Omega^* \) provides a bound on the maximum ultimate swarm size.

Following the proof of Theorems 1–3, we can prove Theorems 4–6 analogously.

5 Conclusions

In this paper, we have considered an anisotropic swarm model and analyzed its aggregation. Under the weight balance condition, we show that the swarm members will aggregate and eventually form a cohesive cluster of finite size around the swarm center. The model given here is a generalization of the models in [2], [4], and [11], and can better reflect the asymmetry of social, economic and psychological phenomena [14]–[19].

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