INTERIOR HESSIAN ESTIMATES FOR SIGMA-2 EQUATIONS IN DIMENSION THREE

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Abstract. We prove a priori interior $C^2$ estimate for $\sigma_2 = f$ in $\mathbb{R}^3$, which generalizes Warren-Yuan’s result [19].

1. Introduction

The interior regularity for solutions of the following $\sigma_2$-Hessian equations is a longstanding problem in fully nonlinear partial differential equations,

(1.1) $\sigma_2(D^2u) = f(x, u, Du) > 0, \quad x \in B_1 \subset \mathbb{R}^n$

where $\sigma_k$ the $k$-th elementary symmetric function for $1 \leq k \leq n$.

Heinz [9] first derived this interior estimate in two dimension. For higher dimensional Monge-Ampere equations, Pogorelov constructed his famous counter-examples even for $f$ constant and convex solutions in [16]. Caffarelli-Nirenberg-Spruck studied more general fully nonlinear equations such as $\sigma_k$ equations in their seminal work [1]. And Urbas also constructed counter-examples in $k \geq 3$. Because of the counter-examples in $k \geq 3$, the best we can expect is the Pogorelov type interior $C^2$ estimates which were derived in [16, 4] and see [6, 11] for more general form. So people in this field want to know whether the interior $C^2$ estimate for $\sigma_2$ equations holds or not for $n \geq 3$. Moreover, this equation serves as a simply model for scalar curvature equations in hypersurface geometry

(1.2) $\sigma_2(\kappa_1(x), \ldots, \kappa_n(x)) = f(x, \nu(x)) > 0, \quad x \in B_1 \subset \mathbb{R}^n$

here $\kappa_1, \ldots, \kappa_n$ are the principal curvatures and $\nu$ the normal of the given hypersurface as a graph over a ball $B_1 \subset \mathbb{R}^n$. A major breakthrough was made by Warren-Yuan [19], they obtained $C^2$ interior estimate for the equation

(1.3) $\sigma_2(D^2u) = 1, \quad x \in B_1 \subset \mathbb{R}^3$

The purely interior $C^2$ estimates for solutions of equations (1.1) and (1.2) with certain convexity constraints were obtained recently by McGonagle-Song-Yuan in [14] and Guan and the author in [5].

Now we state our main result in this paper

Theorem 1. Let $u$ be a smooth solution to (1.1) on $B_{10} \subset \mathbb{R}^3$ with $\Delta u > 0$. Then we have

(1.4) $\sup_{B_{10}} |D^2u| \leq C$.

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where $C$ depends only on $\|f\|_{C^2}$, $\|{\partial f \over \partial x}\|_{L^\infty}$ and $\|u\|_{C^1}$.

In order to introduce our idea, let us briefly review the ideas for attacking this problem so far. In two dimensional case, Heinz used the Uniformization theorem to transform this interior estimate for Monge-Ampere equation into the regularity of an elliptic system and univalent of this mapping, see also [8, 13] for more details. An another interesting proof using only maximum principle was given by Chen-Han-Ou in [2, 3]. Our quantity (2.35) as in [5] can give a new proof of Heinz. The restriction for these methods is that they need some convexity conditions which are not available in higher dimension. In $\mathbb{R}^3$, a key observation made in [19] is that equation (1.3) is exactly the special Lagrangian equation which stems from the special Lagrangian geometry [7]. And an important property for the special Lagrangian equation is that the Lagrangian graph $(x, Du) \subset \mathbb{R}^n \times \mathbb{R}^n$ is a minimal surface which has mean value inequality. So Wang-Yuan [18] can also prove interior Hessian estimates for higher dimensional special Lagrangian equations. The new observation in this paper is that the graph $(x, Du)$ where $u$ satisfied equation (1.1) can be viewed as a submanifold in $(\mathbb{R}^3 \times \mathbb{R}^3, f dx^2 + dy^2)$ with bounded mean curvature. Then instead of using Michael-Simon’s mean value inequality [15] as in [19], we prove a mean value inequality for Riemannian submanifolds which is implied in Hoffman-Spruck’s paper [10] in order to remove the convexity condition in [5]. There also have Sobolev inequalities as in [15] and [10]. But it seems not easy to estimate the additional term in the Sobolev inequalities by using only Warren-Yuan’s argument. The other innovation part is that we can avoid using the Sobolev inequality by combining our new maximum principle method in [5] to solve the problem for the equations (1.1) in $\mathbb{R}^3$.

The scalar curvature equations and the higher dimensional case for $\sigma_2$ Hessian equations are still open to us.

2. Preliminary Lemmas

We call a solution of the equation (1.1) is admissible, if $u$ is smooth and $D^2u \in \Gamma_2 := \{ A : \sigma_1(A) > 0, \sigma_2(A) > 0 \}$. It follows from [1], if $D^2u \in \Gamma_2$, then $\sigma_2^k := \frac{\partial \sigma_2}{\partial u_{kl}}$ is positive definite. So the Hessian estimates can be reduced to the estimate of $\Delta u$ due to the following fact

(2.1) \hspace{1cm} \max |D^2u| \leq \Delta u.

In the rest of this article, we will denote $C$ to be constant under control (depending only on $\|f\|_{C^2}$, $\|{\partial f \over \partial x}\|_{L^\infty}$ and $\|u\|_{C^1}$), which may change line by line.

Lemma 1. Suppose $u$ satisfies the equation (1.1), we have the following equations

(2.2) \hspace{1cm} \sigma_2^{kl} u_{kli} = f_i,

and

(2.3) \hspace{1cm} \sigma_2^{kl} u_{klii} + \sum_{k \neq l} u_{kki} u_{lli} - \sum_{k \neq l} u^2_{kli} = f_{ii}.

If $f$ is a form with gradient term, there are estimates

(2.4) \hspace{1cm} |Df| \leq C(1 + \Delta u),
and

\[ (2.5) \quad -C(1 + \Delta u)^2 + \sum_p f_{u_p} u_{pij} \leq f_{ij} \leq C(1 + \Delta u)^2 + \sum_p f_{u_p} u_{pij} \]

**Proof.** The equations (2.2) and (2.3) follows from twice differential of the equation \( \sigma_2(D^2u) = f \) and the elementary fact that

\[ \frac{\partial^2 \sigma_2}{\partial u_{k} \partial u_{l}} = 1 \quad \text{for } k \neq l, \]
\[ \frac{\partial^2 \sigma_2}{\partial u_{k} \partial u_{k}} = -1 \quad \text{for } k \neq l, \]
\[ \frac{\partial^2 \sigma_2}{\partial u_{k} \partial u_{l}} = 0 \quad \text{otherwise}. \]

Moreover, by (2.1) and these direct computations

\[ (2.7) \quad f_i = f_{x_i} + f_{u_i} u_{pi}, \]
and

\[ (2.8) \quad f_{ij} = f_{x_i x_j} + f_{x_i, u_j} + f_{x_i u_j} + f_{u_i u_j} + f_{u_{p_i} u_{p_j}} + f_{u_{p_i} u_{p_j} u_{p_i}} + f_{u_{p_i} u_{p_j} u_{p_i}} + f_{u_{p_i} u_{p_j}}, \]

we get the estimates (2.4) and (2.5). \( \square \)

The second lemma is from [12].

**Lemma 2.** Suppose \( W \in \Gamma_2 \) is diagonal and \( W_{11} \geq \cdots \geq W_{nn} \), then there exist \( c_1 > 0 \) and \( c_2 > 0 \) depending only on \( n \) such that

\[ (2.10) \quad \sigma_2^{1i}(W)W_{11} \geq c_1 \sigma_2(W), \]
and for any \( j \geq 2 \)

\[ (2.11) \quad \sigma_2^{ij}(W) \geq c_2 \sigma_1(W). \]

Let us consider the quantity of \( b(x) := \log \Delta u \), we have

**Lemma 3.** If \( u \) are admissible solutions of the equations (1.1) in \( \mathbb{R}^3 \), we have

\[ (2.12) \quad \sigma_2^{ij}b_{ij} \geq \frac{1}{100} \sigma_2^{ij}b_i b_j - C \Delta u + \sum_i f_i b_i. \]

**Proof.** We may assume that \( \{u_{ij}\} \) is diagonal. The differential equation of \( b \) by using (2.3) is

\[ (2.13) \quad A := \sigma_2^{ij}b_{ij} - \epsilon \sigma_2^{ij}b_i b_j = \frac{\sigma_2^{ij}(\Delta u)_{ij}}{\sigma_1} - \frac{(1 + \epsilon)\sigma_2^{ij}(\sum_k u_{kk})^2}{\sigma_1} \]
\[ - \frac{1}{\sigma_1} \left( \sum_{k \neq p} u_{kp}^2 - \sum_{k \neq p} u_{kk} u_{pp} \right) + f \]
\[ - \frac{(1 + \epsilon)\sigma_2^{ij}(\sum_k u_{kk})^2}{\sigma_1} \]
We use (2.2) to substitute terms with $u_{iii}$ in $A$,

\[ A = \frac{6u_{1,23}^2}{\sigma_1^3} + \frac{2\sum_{k\neq p} u_{kpp}^2}{\sigma_1^3} - \frac{2\sum_{k\neq p} u_{kpp}u_{ppp}}{\sigma_1^3} - \frac{\Delta f}{\sigma_1^3} + (1 + \epsilon)\frac{\sigma_2^2}{\sigma_1^3} (\sum_{k\neq i} u_{kkk} + u_{iii})^2 \]

\[ \geq \frac{2(u_{211}^2 + u_{311}^2 + u_{122}^2 + u_{322}^2 + u_{133}^2 + u_{333}^2)}{\sigma_1} - \frac{2(u_{211} + u_{311})}{\sigma_1} (f_1 - \sigma_2^{32}u_{221} - \sigma_3^{33}u_{331}) - \frac{2(u_{112} + u_{332})}{\sigma_1} (f_2 - \sigma_2^{11}u_{112} - \sigma_3^{33}u_{332}) - \frac{2(u_{113} + u_{223})}{\sigma_1} (f_3 - \sigma_2^{11}u_{113} - \sigma_2^{22}u_{223}) - \frac{2u_{1113}u_{223} + 2u_{1123}u_{332} + 2u_{221}u_{331}}{\sigma_1} - \frac{(1 + \epsilon)\left(\sum_{k\neq i}(\sigma_2^{ii} - \sigma_2^{kk})u_{kkk} + f_1\right)^2}{\sigma_2^{ii}} - C\sigma_1 + \sum_i f_u b_i. \]

Then we can write explicitly of the second last term and use Cauchy-Schwarz inequality and Lemma 2

\[ \geq \frac{(1 + \epsilon)(\sum_{k\neq i}(\sigma_2^{ii} - \sigma_2^{kk})u_{kkk} + f_1)^2}{\sigma_2^{ii}} \]

\[ \geq \frac{(1 + 2\epsilon)(\lambda_2 - \lambda_1)u_{221} + (\lambda_3 - \lambda_1)u_{331})^2}{\sigma_2^{ii}} \]

\[ \geq \frac{(1 + 2\epsilon)(\lambda_1 - \lambda_2)u_{112} + (\lambda_3 - \lambda_2)u_{332})^2}{\sigma_2^{ii}^2} \]

\[ \geq \frac{(1 + 2\epsilon)(\lambda_2 - \lambda_3)u_{223} + (\lambda_1 - \lambda_3)u_{113})^2}{\sigma_2^{ii}^3} \]

\[ - C\epsilon/\sigma_1. \]

Due to symmetry, we only need to give the lower bound of the terms which contain $u_{221}$ and $u_{333}$. We denote these terms by $A_1$.

\[ A_1 := \frac{2u_{221}^2}{\sigma_1} + \frac{2u_{331}^2}{\sigma_1} - \frac{2(u_{221} + u_{331})f_1}{\sigma_1} - \frac{2\sigma_2^{32}u_{221}^2}{\sigma_1\sigma_2^{ii}^3} + \frac{2\sigma_2^{33}u_{331}^2}{\sigma_1\sigma_2^{ii}^3} + \frac{2(\sigma_2^{32} + \sigma_2^{33})u_{221}u_{331}}{\sigma_1\sigma_2^{ii}^3} \]

\[ - \frac{2u_{221}u_{331}}{\sigma_1} - \frac{(1 + 2\epsilon)(\lambda_2 - \lambda_1)u_{221} + (\lambda_3 - \lambda_1)u_{331})^2}{\sigma_2^{ii}^3}. \]
By Cauchy-Schwarz and Lemma 2 we have
\begin{equation}
-2(u_{221} + u_{331})f_1 \geq -2\epsilon^2 \sigma_1 (u_{221} + u_{331})^2 / \sigma_1 \sigma_2^1 - f_1^2 / 2\epsilon^2 \sigma_2^1 \sigma_1^1
\end{equation}
\begin{equation}
\geq -2\epsilon^2 \sigma_1 (u_{221} + u_{331})^2 / \sigma_1 \sigma_2^1 - C / \epsilon^2 \sigma_1.
\end{equation}

Then we get
\begin{equation}
A_1 \geq \frac{2\sigma_1^1 + 2\sigma_2^2}{\sigma_1 \sigma_2^1} u_{221}^2 + \frac{2\sigma_1^1 + 2\sigma_2^3}{\sigma_1 \sigma_2^1} u_{331}^2 + \frac{4\lambda_1}{\sigma_1 \sigma_2^1} u_{221} u_{331} - \frac{2\epsilon^2 \sigma_1 (u_{221} + u_{331})^2}{\sigma_1 \sigma_2^1}
\end{equation}
\begin{equation}
- (1 + 2\epsilon)[(\lambda_2 - \lambda_1)u_{221} + (\lambda_3 - \lambda_1)u_{331}]^2
\end{equation}
\begin{equation}
- C / \epsilon^2 \sigma_1.
\end{equation}

We will prove that $A_1 \geq -C \sigma_1$ in the following elementary but tedious two claims. Then we choose $\epsilon = \frac{1}{100}$, such that
\begin{equation}
1 + \delta \geq \frac{1 + 2\epsilon}{1 - \epsilon},
\end{equation}
where $\delta$ is small constant in the Claim 2.

In all, we will get
\begin{equation}
A \geq -C \sigma_1 + \sum_i f_i b_i.
\end{equation}

Claim 1. For fix $\epsilon \leq \frac{1}{2}$, we have
\begin{equation}
\frac{2\sigma_1^1 + 2\sigma_2^2}{\sigma_1 \sigma_2^1} u_{221}^2 + \frac{2\sigma_1^1 + 2\sigma_2^3}{\sigma_1 \sigma_2^1} u_{331}^2 + \frac{4\lambda_1}{\sigma_1 \sigma_2^1} u_{221} u_{331} \geq \frac{2\epsilon \sigma_1 (u_{221} + u_{331})^2}{\sigma_1 \sigma_2^1}.
\end{equation}

Proof. This claim follows from elementary inequality
\begin{equation}
(\sigma_2^1 + \sigma_2^2 - \epsilon \sigma_1)(\sigma_1^1 + \sigma_2^3 - \epsilon \sigma_1) - (\lambda_1 - \epsilon \sigma_1)^2
\end{equation}
\begin{equation}
= (1 - \epsilon)^2 \sigma_1^1 + (1 - \epsilon) \sigma_1 (\lambda_2 + \lambda_3) + \lambda_2 \lambda_3 - (\lambda_1 - \epsilon \sigma_1)^2
\end{equation}
\begin{equation}
\geq (1 - \epsilon)^2(\lambda_2^2 + \lambda_3^2 + 2f) + (1 + 2\epsilon)(1 - \epsilon)(f - \lambda_2 \lambda_3)
\end{equation}
\begin{equation}
+ (1 - \epsilon^2)(\lambda_2 + \lambda_3)^2 + \lambda_2 \lambda_3
\end{equation}
\begin{equation}
\geq (1 - \epsilon)^2(\lambda_2^2 + \lambda_3^2) - \epsilon(1 - 2\epsilon)\lambda_2 \lambda_3
\end{equation}
\begin{equation}
\geq 2(1 - \epsilon)^2 |\lambda_2 \lambda_3| - \epsilon(1 - 2\epsilon)\lambda_2 \lambda_3.
\end{equation}

If we assume $2(1 - \epsilon)^2 - \epsilon(1 - 2\epsilon) \geq 0$, we have above inequality nonnegative.

Claim 2. For fix $\delta \leq \frac{1}{20}$, we have
\begin{equation}
\frac{2\sigma_1^1 + 2\sigma_2^2}{\sigma_1 \sigma_2^1} u_{221}^2 + \frac{2\sigma_1^1 + 2\sigma_2^3}{\sigma_1 \sigma_2^1} u_{331}^2 + \frac{4\lambda_1}{\sigma_1 \sigma_2^1} u_{221} u_{331}
\end{equation}
\begin{equation}
\geq (1 + \delta)[(\lambda_2 - \lambda_1)u_{221} + (\lambda_3 - \lambda_1)u_{331}]^2.
\end{equation}
Proof. In order to prove this claim, we need to prove

\[
(2.51) \quad [2(\sigma_2^{11} + \sigma_2^{22})\sigma_1 - (1 + \delta)(\lambda_1 - \lambda_2)^2]
\]

\[
(2.52) \quad \times [2(\sigma_2^{11} + \sigma_2^{22})\sigma_1 - (1 + \delta)(\lambda_1 - \lambda_3)^2]
\]

\[
(2.53) \quad \geq [2\lambda_1 \sigma_1 - (1 + \delta)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)]^2.
\]

Because we have

\[
(2.54) \quad 2(\sigma_2^{11} + \sigma_2^{22})\sigma_1 - (1 + \delta)(\lambda_1 - \lambda_2)^2
\]

\[
(2.55) \quad = 2\sigma_1^2 + 2\lambda_3\sigma_1 - (1 + \delta)(\lambda_1 - \lambda_2)^2
\]

\[
(2.56) \quad = 2\lambda_1^2 + 2\lambda_2^2 + 2\lambda_3^2 + 4\sigma_1^2 + 2\lambda_3\sigma_2^{11} - (1 + \delta)(\lambda_1 - \lambda_2)^2
\]

\[
(2.57) \quad = (1 - \delta)\lambda_1^2 + (1 - \delta)\lambda_2^2 + 4\lambda_3^2 + 6\sigma + 2\delta\lambda_1\lambda_2.
\]

And similarly

\[
(2.58) \quad 2(\sigma_2^{11} + \sigma_2^{22})\sigma_1 - (1 + \delta)(\lambda_1 - \lambda_3)^2
\]

\[
(2.59) \quad = (1 - \delta)\lambda_1^2 + (1 - \delta)\lambda_2^2 + 4\lambda_3^2 + 6\sigma + 2\delta\lambda_1\lambda_3.
\]

We also compute right hand side of \((2.53)\)

\[
(2.60) \quad 2\lambda_1\sigma_1 - (1 + \delta)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)
\]

\[
(2.61) \quad = 2\lambda_1^2 + 2\lambda_1\sigma_2^{11} - (1 + \delta)(\lambda_1 - \lambda_3) + (1 + \delta)(\lambda_2) - (1 + \delta)(\lambda_2\lambda_3)
\]

\[
(2.62) \quad = (1 - \delta)\lambda_1^2 + 2\sigma_1 - 2\lambda_2\lambda_3 + (1 + \delta)(\sigma_1 - 2\lambda_2\lambda_3)
\]

\[
(2.63) \quad = (1 - \delta)\lambda_1^2 + (3 + \delta)f - 2(2 + \delta)\lambda_2\lambda_3.
\]

Then we have

\[
(2.64) \quad [(1 - \delta)\lambda_1^2 + (1 - \delta)\lambda_2^2 + 4\lambda_3^2 + 6\sigma + 2\delta\lambda_1\lambda_2]
\]

\[
(2.65) \quad \times [(1 - \delta)\lambda_1^2 + (1 - \delta)\lambda_2^2 + 4\lambda_3^2 + 6\sigma + 2\delta\lambda_1\lambda_3]
\]

\[
(2.66) \quad - [(1 - \delta)\lambda_1^2 + (3 + \delta)f - 2(2 + \delta)\lambda_2\lambda_3]^2
\]

\[
(2.67) \quad = [(1 - \delta)\lambda_1^2 + (3 + \delta)f][6\sigma + (5 - \delta)(\lambda_1^2 + \lambda_2^2) - 2\delta\lambda_2\lambda_3]
\]

\[
(2.68) \quad + [(3 - \delta)f + (1 - \delta)\lambda_1^2 + 4\lambda_3^2 + 2\delta\lambda_1\lambda_2]
\]

\[
(2.69) \quad \times [(3 - \delta)f + (1 - \delta)\lambda_2^2 + 4\lambda_3^2 + 2\delta\lambda_1\lambda_3]
\]

\[
(2.70) \quad + 4[(1 - \delta)f + (3 + \delta)f][2(2 + \delta)\lambda_2\lambda_3 - 4(2 + \delta)\lambda_2^2\lambda_3^2]
\]

\[
(2.71) \quad = [(1 - \delta)\lambda_1^2 + (3 + \delta)f][6\sigma + (5 - \delta)(\lambda_1^2 + \lambda_2^2) + 2(4 + \delta)\lambda_2\lambda_3]
\]

\[
(2.72) \quad + [(3 - \delta)f + (1 - \delta)\lambda_1^2 + 4\lambda_3^2][6\sigma + (5 - \delta)(\lambda_1^2 + \lambda_3^2) + 2(4 + \delta)\lambda_2\lambda_3]
\]

\[
(2.73) \quad + 2\delta\lambda_1\lambda_2[(3 - \delta)f + (1 - \delta)\lambda_2^2 + 4\lambda_3^2]
\]

\[
(2.74) \quad + 2\delta\lambda_1\lambda_3[(3 - \delta)f + (1 - \delta)\lambda_2^2 + 4\lambda_3^2]
\]

\[
(2.75) \quad + 4\delta^2\lambda_1^2\lambda_2\lambda_3 - 4(2 + \delta)^2\lambda_2^2\lambda_3^2.
\]

The point in the above computation is that term \([(1 - \delta)\lambda_1^2 + (3 + \delta)f]^2\) cancels exactly. Moreover, other terms has a lot of room to play with if you choose \(\delta\) small.

We deal with these terms and assume that \(\delta \leq \frac{1}{20}\).

\[
(2.76) \quad [(1 - \delta)\lambda_1^2 + (3 + \delta)f]
\]

\[
(2.77) \quad \times [6\sigma + (5 - \delta)(\lambda_1^2 + \lambda_2^2) + 2(4 + \delta)\lambda_2\lambda_3]
\]

\[
(2.78) \quad \geq (1 - \delta)\lambda_1^2(1 - 2\delta)(\lambda_2^2 + \lambda_3^2)
\]

\[
(2.79) \quad \geq -4\delta^2\lambda_1^2\lambda_2\lambda_3.
\]
And

\[(2.80) \quad 2\delta\lambda_1\lambda_2[(3 - \delta)f + (1 - \delta)\lambda^2_3 + 4\lambda^2_2]\]

\[(2.81) \quad + 2\delta\lambda_1\lambda_3[(3 - \delta)f + (1 - \delta)\lambda^2_2 + 4\lambda^2_3]\]

\[(2.82) \quad = 2\delta(f - \lambda_2\lambda_3)(3 - \delta)f + 2\delta(1 - \delta)(f - \lambda_2\lambda_3)(\lambda^2_2 + \lambda^2_3)\]

\[(2.83) \quad + 2\delta(3 + \delta)\lambda_1(\lambda^2_2 + \lambda^2_3)\]

\[(2.84) \quad \geq -2\delta(3 - \delta)\lambda_2\lambda_3 f - 2\delta(1 - \delta)\lambda_2\lambda_3(\lambda^2_2 + \lambda^2_3)\]

\[(2.85) \quad + 2\delta(3 + \delta)(f - \lambda_2\lambda_3)(\lambda^2_2 - \lambda_2\lambda_3 + \lambda^2_3)\]

\[(2.86) \quad \geq -2\delta(3 - \delta)\lambda_2\lambda_3 f\]

\[(2.87) \quad -2\delta\lambda_2\lambda_3(4\lambda^2_2 + 4\lambda^2_3 - (3 + \delta)\lambda_2\lambda_3).\]

We also have,

\[(2.88) \quad [(3 - \delta)f + (1 - \delta)\lambda^2_2 + 4\lambda^2_3][(3 - \delta)f + (1 - \delta)\lambda^2_3 + 4\lambda^2_2]\]

\[(2.89) \quad \geq [16 + (1 - \delta)^2]\lambda^2_2\lambda^2_3 + (3 - \delta)f(5 - \delta)(\lambda^2_2 + \lambda^2_3)\]

\[(2.90) \quad + 4(1 - \delta)(\lambda^2_2 + \lambda^2_3)\]

\[(2.91) \quad \geq 4(2 + \delta)^2\lambda^2_2\lambda^2_3 + 16f\lambda_2\lambda_3 + 6\delta(\lambda^2_2 + \lambda^2_3)^2\]

\[(2.92) \quad \geq 4(2 + \delta)^2\lambda^2_2\lambda^2_3 + 2\delta(3 - \delta)\lambda_2\lambda_3 f\]

\[(2.93) \quad + 2\delta\lambda_2\lambda_3(4\lambda^2_2 + 4\lambda^2_3 - (3 + \delta)\lambda_2\lambda_3).\]

We combine inequalities (2.79), (2.87) and (2.93) into (2.75) to verify inequality (2.83). Finally we complete the proof of this claim.

\[\Box\]

Lemma 4. Suppose \(u\) are admissible solutions of the equations (1.1) on \(B_{10} \subset \mathbb{R}^3\), then we have

\[(2.94) \quad \sup_{B_5} \sigma_1 \leq C \sup_{B_1} \sigma_1.\]

Proof. Denote \(\rho(x) = 10^2 - |x|^2\), and \(M_r := \sup_{B_r} \sigma_1\). We consider a test function in \(B_{10}\)

\[(2.95) \quad P(x) = 2\log \rho(x) + g(x \cdot Du - u) + \log \log \max\{\frac{\sigma_1}{M_1}, 20\},\]

where \(g(t) = -\frac{1}{\beta}(\log(1 - 20\max |Du| + 2\max |u| + 1))\) and \(\beta\) is a larger number to be determined later.

Assume \(P\) attains its maximum point, say \(x_0 \in B_{10}\), and \(\frac{\sigma_1}{\sup_{B_1} \sigma_1} > 20\). Then \(P\) must attained its maximum point in the ring \(B_{10}/B_1\). Moreover, we may always assume \(\frac{\sigma_1}{\sup_{B_1} \sigma_1}\) sufficient large. We choose coordinate frame \(\{e_1, e_2, e_3\}\), such that \(D^2 u\) is diagonalized at this point. By maximum principle, at \(x_0\), we have

\[(2.96) \quad 0 = P_i = \frac{2\rho_i}{\rho} + g'x_ku_{ki} + \frac{b_i}{b - \log M_1},\]

and

\[(2.97) \quad P_{ij} = \frac{2\rho_{ij}}{\rho} - \frac{2\rho_i\rho_j}{\rho^2} g'x_ku_{ki}x_lu_{lj} + g'(u_{ij} + x_ku_{kij}) + \frac{b_{ij}}{b - \log M_1} - \frac{b_ib_j}{(b - \log M_1)^2}.\]
Contracting with $\sigma_{ij}^{2} := \frac{\partial \sigma_{ij}(P^2u)}{\partial \sigma_{ij}}$, we get
\[
\sigma_{ij}^{2} P_{ij} = -4 \sum \sigma_{ii}^{2} \frac{x_i^2}{\rho} + 8 \frac{\sigma_{ii}^{2} x_i^2}{\rho^2} + g'' \sigma_{ij}^{2} (x_i u_{ii})^2 + 2g' f + g' x_k f_k + \frac{g' x_{ji} b_i}{b - \log M_1} - \frac{\sigma_{ij}^{2} b_i^2}{(b - \log M_1)^2}.
\]
(2.100)

Using Lemma 1, we get
\[
\sigma_{ij}^{2} P_{ij} \geq -4 \sum \sigma_{ii}^{2} \frac{x_i^2}{\rho} + 8 \frac{\sigma_{ii}^{2} x_i^2}{\rho^2} + g'' \sigma_{ij}^{2} (x_i u_{ii})^2 + 2g' f + g' x_k f_k + \frac{g' b_i^2}{100(b - \log M_1)} + \frac{\sum f_{ij} b_i - C \sigma_{ij}}{b - \log M_1}.
\]
(2.101)

By (2.96),
\[
g' x_k f_k + \frac{\sum f_{ij} b_i}{b - \log M_1} \geq -C(g' + \frac{1}{\rho}).
\]
(2.104)

We may assume $\rho^2(x_0) \log \frac{\sigma_1(x_0)}{M_1} \geq C$. We also use Newton-MacLaur in inequality $\sigma_1 \sigma_2 \geq 9 \sigma_3$ to get
\[
\sigma_{ij}^{2} P_{ij} \geq -4 \sum \sigma_{ii}^{2} \frac{x_i^2}{\rho} + 8 \frac{\sigma_{ii}^{2} x_i^2}{\rho^2} + g'' \sigma_{ij}^{2} (x_i u_{ii})^2 + 2g' f + g' x_k f_k + \frac{g' b_i^2}{100(b - \log M_1)} - \frac{C \sigma_{ij}}{b - \log M_1}.
\]
(2.105)

Then we divided into two cases.

Case 1: $x_i^2 \geq \frac{1}{b}$.

From (2.96), we know
\[
0 = -4 x_1 \frac{1}{\rho} + g' x_1 u_{11} + \frac{b_1}{b - \log M_1}.
\]

We can assume
\[
g' \big| x_1 \big| u_{11} \geq \frac{12 |x_1|}{\rho}.
\]

Otherwise we get the estimate from
\[
g' (b - \log M_1) \leq g' u_{11} \leq \frac{12}{\rho}.
\]
(2.107)

Then we have
\[
\left( \frac{b_1}{b - \log M_1} \right)^2 \geq \frac{(g')^2 x_1^2 u_{11}^2}{3} \geq \frac{(g')^2 u_{11}^2}{18}.
\]
(2.108)
Inserting this inequality into (2.106), we get from Lemma 2

\[
\begin{align*}
\sigma_{ij}^2 P_{ij} & \geq -4 \sum \frac{\rho^2}{\sigma_{ii}^2} - 8 \frac{\sigma_{ij}^2 x_i^2}{\rho^2} + \frac{\sigma_{ii}^2 b_i^2}{100(b - \log M_1)} - \frac{C \sigma_1}{b - \log M_1} \\
& \geq -4 \sum \frac{\rho^2}{\sigma_{ii}^2} - 800 \sum \frac{\sigma_{ii}^2}{\rho^2} + \frac{(g')^2 \sigma_{ii}^2 u_i^2}{180} (b - \log M_1) \\
& \geq -4 \sum \frac{\rho^2}{\sigma_{ii}^2} - 800 \sum \frac{\sigma_{ii}^2}{\rho^2} + c_1 (g')^2 (b - \log M_1) \sum \sigma_{ii}^2.
\end{align*}
\]

So we obtain the estimate

\[
\rho^2(x) \log \frac{\sigma_1(x)}{M_1} \leq C \rho^2(x_0) \log \frac{\sigma_1(x_0)}{M_1} \leq C.
\]

Case 2: there exists \( j \geq 2 \), such that \( x_j^2 \geq \frac{1}{6} \).

Using (2.96), we have

\[
\beta \frac{\sigma_{jj}^2 b_j^2}{(b - \log M_1)^2} = \beta \sigma_{jj}^2 \left[ -\frac{4x_j}{\rho} + g' x_i u_{ij} \right]^2 \geq 2 \beta \sigma_{jj}^2 x_j^2 \rho^2 - 2 \beta \sigma_{jj}^2 (g')^2 (x_j u_{ij})^2.
\]

Then we get

\[
\begin{align*}
\sigma_{ij}^2 P_{ij} & \geq -4 \sum \frac{\rho^2}{\sigma_{ii}^2} + (2 \beta - 8) \frac{\sigma_{ij}^2 x_i^2}{\rho^2} - \frac{C \sigma_1}{b - \log M_1} \\
& \quad + (g'' - 2 \beta (g')^2) \sigma_{ii}^2 (x_i u_{ii})^2.
\end{align*}
\]

If we choose \( g'' - 2 \beta (g')^2 > 0 \), by Lemma 2 we have

\[
\sigma_{ij}^2 P_{ij} \geq -4 \sum \frac{\rho^2}{\sigma_{ii}^2} + (2 \beta - 8) \frac{\sigma_{ij}^2 x_i^2}{6 \rho^2} \geq \sum \frac{\sigma_{ii}^2}{\rho} (-4 + \frac{(\beta - 2) c_2}{300}).
\]

We choose constant above by the following order, first \( \beta = \frac{1200}{c_2} + 3 \), then \( g(t) = -\frac{1}{2c_2} \log(1 - \frac{\max |Du|}{2 \max |u| + 1}) \) such that \( g'' \geq 2 \beta (g')^2 \). We finally get the estimate (2.94).

Similarly, if we consider quantity \( V := \log(\sigma_1 - \sigma_3) \) we also have the following Lemma.

**Lemma 5.** Suppose \( u \) are admissible solutions of equations (1.1) on \( B_{10} \subset \mathbb{R}^3 \), then we have in \( B_4 \) that

\[
\begin{align*}
\sigma_{ij}^2 V_{ij} & \geq -C \sigma_1 + f_u V_i, \\
\sigma_1 f - \sigma_3 & \leq \frac{M_5}{\log M_5} \log(\sigma_1 f - \sigma_3).
\end{align*}
\]
Proof. For the proof of the first inequality (2.113) which is tedious but similar to Lemma 3, we give its details in the appendix. The second inequality (2.114) follows from (2.113) almost the same as the proof of Lemma 4. The proof of the second inequality will also be in the appendix. □

3. Mean value inequality.

In this section we prove a mean value type inequality

**Theorem 2.** Suppose \( u \) are admissible solutions of equations (1.1) on \( B_{10} \subset \mathbb{R}^3 \), then we have

\[
\sup_{B_1} \sigma_1 = \sigma_1(y_0) \leq C \int_{B_1(y_0)} \sigma_1(x)(\sigma_1 f - \sigma_3)dx.
\]

We prove this theorem similar as Michael-Simon [15].

**Proof.** First we know from concavity of \( \frac{\sigma_1^2}{2} \), we have

\[
\sigma_2^{ij}(\sigma_1)_{ij} \geq \Delta f - \frac{|Df|^2}{2f}, \quad \text{and} \quad \sigma_1 f - \sigma_3 \geq -c\lambda_1 \lambda_3.
\]

By direct computation, we have

\[
2rr_i = f_i |x|^2 + 2f_i x_i + 2u_k u_{kij},
\]

and

\[
2r_i r_j + 2rr_{ij} = f_{ij} |x|^2 + 2f_{ij} x_i + 2f_i x_j + 2f_{ij} x_k + 2u_k u_{kij} + 2u_k u_{kij}.
\]

Because \( \lambda_2 + \lambda_3 > 0 \), we may assume \( \lambda_3 < 0, \lambda_1 \geq \lambda_2 \geq \lambda_3 \) and \( \lambda_1 \) large,

\[
f \sigma_1 - \sigma_3 \geq f \lambda_1 + \lambda_1 \lambda_3^2 \geq -c\lambda_1 \lambda_3 \geq -c\lambda_2 \lambda_3.
\]

By equation we also have

\[
\lambda_2 \lambda_1 = f - \lambda_1 \lambda_3 - \lambda_2 \lambda_3 \leq C(f \sigma_1 - \sigma_3).
\]
We then have from (3.5), (3.6), (3.9), (3.10) and Lemma 1

\begin{align*}
\sigma^2_{ij} \psi_{ij} &= \sigma^2_{ij} (-r_i r_j \chi (\rho - r))_j \\
\sigma^2_{ij} &= -\sigma^2_{ij} r_i r_j \chi (\rho - r) - \sigma^2_{ij} r_j r_j \chi (\rho - r) + \sigma^2_{ij} r_j r_j \chi' (\rho - r) \\
\chi (\rho - r) \sigma^2_{ij} &= f \delta_{ij} + u_{ki} u_{kj} + \frac{f_{ij} |x|^2}{2} + 2 f_i x_j \\
-\chi (\rho - r) \sigma^2_{ij} u_{ki} u_{kj} &= \sigma^2_{ij} r_i r_j \chi' (\rho - r) \\
-3 (\sigma_1 f - \sigma_3) \chi + C (r^2 \chi + r \chi) (f \sigma_1 - \sigma_3) + \sigma^2_{ij} r_i r_j r_i r_j.
\end{align*}

We claim that

\begin{align*}
\sigma^2_{ij} r_i r_j &\leq (f \sigma_1 - \sigma_3) (1 + Cr).
\end{align*}

In fact,

\begin{align*}
\frac{\sigma^2_{ij} (f_i |x|^2 + 2 f x_i + 2 u_{ki} u_{kj}) (f_j |x|^2 + 2 f x_j + 2 u_{ki} u_{kj})}{4 r^2} &\leq C (f \sigma_1 - \sigma_3) r + \frac{\sigma^2_{ij} (f x_i + u_i u_i)^2}{r^2}.
\end{align*}

Moreover, we have following elementary properties

\begin{align*}
(f \sigma_1 - \sigma_3) \delta_{ij} - f \sigma^2_{ij} &= \sigma^2_{kl} u_{ki} u_{lj}.
\end{align*}

and

\begin{align*}
f \sigma^2_{ij} u_{ii} x_i^2 - 2 f \sigma^2_{ij} x_i u_i u_i + f \sigma^2_{ii} u_i^2 &\geq f \sigma^2_{ij} (u_i x_i - u_i)^2.
\end{align*}

Then we have

\begin{align*}
\frac{\sigma^2_{ij} (f x_i + u_i u_i)^2}{r^2} &\leq (f \sigma_1 - \sigma_3).
\end{align*}

We obtain from (3.10) and (3.15) that

\begin{align*}
\sigma^2_{ij} \psi_{ij} &\leq (\sigma_1 f - \sigma_3) [-3 \chi + C (r^2 \chi + r \chi) + (1 + r) r \chi'].
\end{align*}

Then multiply both side by \(\sigma_1\) and integral on the domain \(\mathcal{B}_{10}\),

\begin{align*}
\int_{\mathcal{B}_{10}} \sigma_1 \sigma^2_{ij} \psi_{ij} \, dx &\leq \rho^4 \frac{d}{d \rho} \left( \int_{\mathcal{B}_{10}} \sigma_1 \chi (\rho - r) (\sigma_1 f - \sigma_3) \, dx \right) + C \int_{\mathcal{B}_{10}} \sigma_1 r^2 \chi (\sigma_1 f - \sigma_3) \, dx \\
&+ C \int_{\mathcal{B}_{10}} r \sigma_1 \chi (\rho - r) (\sigma_1 f - \sigma_3) \, dx.
\end{align*}

By (2.12), we have

\begin{align*}
-C \int_{\mathcal{B}_{10}} \sigma^2_{ij} \psi \, dx + \int_{\mathcal{B}_{10}} f_{ui} (\sigma_1) \psi \leq \int_{\mathcal{B}_{10}} \sigma_1 \sigma^2_{ij} \psi_{ij} \, dx.
\end{align*}
Inserting (3.26) into (3.29), we get

\[
\frac{d}{d\rho} \int_{\mathcal{B}_\rho} \sigma_1 \chi (\rho - r) \rho^3 (\sigma_1 f - \sigma_3) dx \leq \frac{C \int_{\mathcal{B}_\rho} r \sigma_1 \chi (\rho - r) (\sigma_1 f - \sigma_3) dx}{\rho^4} + \frac{C \int_{\mathcal{B}_\rho} \sigma_1^2 \chi' (\sigma_1 f - \sigma_3) dx}{\rho^4} + \frac{C \int_{\mathcal{B}_\rho} \psi dx - \int_{\mathcal{B}_\rho} f_{u_i} (\sigma_1)_i \psi dx}{\rho^4}.
\]

(3.28)

Because \( \chi, \chi' \) and \( \psi \) are all supported in \( \mathcal{B}_\rho \), we deal with right hand side of above inequality term by term. For the first term, we have

\[
\frac{C \int_{\mathcal{B}_\rho} r \sigma_1 \chi (\rho - r) (\sigma_1 f - \sigma_3) dx}{\rho^4} \leq \frac{C \int_{\mathcal{B}_\rho} \sigma_1 \chi (\rho - r) (\sigma_1 f - \sigma_3) dx}{\rho^4}.
\]

(3.30)

Then for the second term, we integral from \( \delta \) to \( R \),

\[
\int_{\delta}^{R} \int_{\mathcal{B}_\rho} \sigma_1^2 \chi' (\sigma_1 f - \sigma_3) dx d\rho \leq \int_{\delta}^{R} \int_{\mathcal{B}_\rho} \sigma_1 \chi (\sigma_1 f - \sigma_3) dx d\rho \leq \int_{\mathcal{B}_\rho} \sigma_1 \chi (\sigma_1 f - \sigma_3) dx|_{\delta}^{R} + \int_{\delta}^{R} 2 \int_{\mathcal{B}_\rho} \sigma_1 \chi (\sigma_1 f - \sigma_3) dx d\rho.
\]

(3.31)

(3.32)

(3.33)

(3.34)

For the last term, we use (3.33) and definition of \( \psi \) to estimate

\[
\frac{C \int_{\mathcal{B}_\rho} \sigma_1^2 \psi dx - \int_{\mathcal{B}_\rho} f_{u_i} (\sigma_1)_i \psi dx}{\rho^4} \leq \frac{C \int_{\mathcal{B}_\rho} \sigma_1^2 \psi dx - \int_{\mathcal{B}_\rho} f_{u_i} (\sigma_1) r_i \rho \chi dx}{\rho^4} \leq \frac{C \int_{\mathcal{B}_\rho} \sigma_1^2 \psi dx + \int_{\mathcal{B}_\rho} \sigma_1^2 r \chi dx}{\rho^4} \leq \frac{C \int_{\mathcal{B}_\rho} \sigma_1 \chi (\rho - r) (\sigma_1 f - \sigma_3) dx}{\rho^4}.
\]

(3.35)

(3.36)

(3.37)

Then we combine (3.30), (3.34) and (3.37) with integrating from \( 0 \leq \delta \) to \( R \leq 10 \),

\[
\int_{\mathcal{B}_\rho} \frac{\sigma_1 (\sigma_1 f - \sigma_3) \chi (\delta - r)}{\delta^3} dx \leq \int_{\mathcal{B}_\rho} \frac{\sigma_1 (\sigma_1 f - \sigma_3) \chi (R - r)}{\delta^3} dx.
\]

(3.38)

Letting \( \chi \) approximate the characteristic function of the interval \(( -\infty, 0 )\), in an appropriate fashion, we obtain,

\[
\int_{\mathcal{B}_\rho} \frac{\sigma_1 (\sigma_1 f - \sigma_3)}{\delta^3} \leq C \int_{\mathcal{B}_\rho} \frac{\sigma_1 (\sigma_1 f - \sigma_3)}{R^3}.
\]

(3.39)

Because the graph \(( x, Du )\) where \( u \) satisfied equation (1.1) can be viewed as a three dimension smooth submanifold in \( (\mathbb{R}^3 \times \mathbb{R}^3, f dx^2 + dy^2) \) with volume form exactly \( \sigma_1 f - \sigma_3 \). We note that the boundedness of the mean curvature is encoded
in the above proof. Moreover, the geodesic ball with radius \( \delta \) of this submanifold is comparable with \( \mathcal{B}_\delta \). Then let \( \delta \to 0 \), we finally get
\[
\sigma_1(y_0) \leq C\frac{\int_{\mathcal{B}_\delta(y_0)} \sigma_1(\sigma_1 f - \sigma_3)dx}{R^3} \leq C\frac{\int_{\mathcal{B}_\delta(y_0)} \sigma_1(\sigma_1 f - \sigma_3)dx}{R^3}.
\]

\[\square\]

4. Proof of the theorem

**Proof.** From Theorem 2, Lemma 5 and Lemma (4), we have at maximum point \( x_0 \) of \( \bar{B}_1(0) \)
\begin{align}
M_1 & \leq \int_{B_1(x_0)} \sigma_1(\sigma_1 f - \sigma_3)dx \\
& \leq \frac{M_5}{\log^2 M_5} \int_{B_1(x_0)} \log(\sigma_1 f - \sigma_3) \sigma_1 dx \\
& \leq \frac{C M_1}{\log^2 M_1} \int_{B_1(x_0)} \sigma_1 \log \sigma_1 dx.
\end{align}

Recall that
\[
\sigma_{ij}^2 b_{ij} \geq \frac{1}{100} \sigma_{ij}^2 b_{ij} - C \sigma_1 + \sum_i f_u b_i,
\]
We have integral version of this inequality for any \( r < 5 \),
\[
\int_{B_r} -\sigma_{ij}^2 \phi_i b_i \geq c_0 \int_{B_r} \phi \sigma_{ij}^2 b_{ij} b_j - C \int_{B_r} \sum_i f_u b_i \phi.
\]
for all non-negative \( \phi \in C_0^\infty \). We choose different cutoff functions all are denoted \( 0 \leq \phi \leq 1 \), which support in larger ball \( B_{r+1}(x_0) \) and equals to 1 in smaller ball \( B_r(x_0) \) and \( |\nabla \phi| + |\nabla^2 \phi| \leq C \).
\begin{align}
\int_{B_1(x_0)} \phi b_{\sigma_1} & \leq \int_{B_2(x_0)} \phi b_{\sigma_1} \\
& \leq C(\int_{B_2(x_0)} b + \int_{B_2(x_0)} |\nabla b|) \\
& \leq C(1 + \int_{B_2(x_0)} |\nabla b|).
\end{align}

We only need to estimate \( \int_{B_2(x_0)} |\nabla b| \). We use
\[
\sigma_1 \sigma_{ij}^2 \geq c \delta_{ij},
\]
to get
\[
\int_{B_2(x_0)} |\nabla b| \leq \int_{B_2(x_0)} \sqrt{\sigma_{ij}^2 b_{ij} b_j \sqrt{\sigma_1}}.
\]

By Holder inequality, we use (4.4),
\begin{align}
\int_{B_2(x_0)} |\nabla b| & \leq \left( \int_{B_2(x_0)} \sigma_{ij}^2 b_{ij} b_j \right)^{\frac{1}{2}} \left( \int_{B_2(x_0)} \sigma_1 \right)^{\frac{1}{2}} \\
& \leq C \int_{B_3(x_0)} \phi^2 \sigma_{ij}^2 b_{ij} b_j.
\end{align}
Then using (4.5), we get

\[
\int_{B_3(x_0)} \phi^2 \sigma_{ij}^3 b_i b_j \leq C \left( \int_{B_3(x_0)} \phi^2 \sigma_{ij}^3 b_i b_j + 1 \right)
\]

By Cauchy-Schwarz inequality

\[
\int_{B_3(x_0)} \phi^2 \sigma_{ij}^3 b_i b_j \leq C \epsilon \int_{B_3(x_0)} \phi^2 \sigma_{ij}^3 b_i b_j + \frac{1}{\epsilon} \sigma_{ij}^3 \phi_i \phi_i + \frac{C}{\epsilon}
\]

We Choose \( \epsilon \) small such that \( C \epsilon \leq \frac{1}{2} \),

\[
\int_{B_3(x_0)} \phi^2 \sigma_{ij}^3 b_i b_j \leq C.
\]

Finally, combine (4.3), (4.8), (4.12) and (4.18), we get the estimate

\[
\log M_1 \leq C.
\]

5. Appendix

We prove a differential Equation of \( V = \log(f \sigma_1 - \sigma_3) \) in Lemma [5]

**Proof.** As in Lemma [3] we assume that \( \{u_{ij}\} \) is diagonal. The differential equations of \( V \) are

\[
V_i = \frac{(\sigma_1)_i f + \sigma_1 f_i - (\sigma_3)_i}{\sigma_1 f - \sigma_3},
\]

and

\[
V_{ij} = \frac{(\sigma_1)_{ij} f + (\sigma_1)_i f_j + (\sigma_1)_j f_i + \sigma_1 f_{ij} - (\sigma_3)_{ij}}{\sigma_1 f - \sigma_3} - \frac{[(\sigma_1)_i f + \sigma_1 f_i - (\sigma_3)_i][(\sigma_1)_j f + \sigma_1 f_j - (\sigma_3)_j]}{(\sigma_1 f - \sigma_3)^2}.
\]
We contract with $\sigma_{2}^{ij}$ and use (2.3),

\begin{align*}
\sigma_{2}^{ij}V_{ij} &= \frac{f\sigma_{2}^{ij}u_{kkij} + 2\sigma_{2}^{ij}(\sigma_{1})_{fi} + \sigma_{3}^{ij}f_{ij} - \sigma_{2}^{ij}(\sigma_{3})_{ij}}{\sigma_{1}f - \sigma_{3}} \\
&\quad - \frac{\sigma_{2}^{ij}(\sigma_{1})_{f} + \sigma_{1}f_{i} - (\sigma_{3})_{i}}{(\sigma_{1}f - \sigma_{3})^{2}} \\
(5.5) &= -f \sum_{i, k \neq p} u_{kki}u_{p pi} + f \sum_{i, k \neq p} u_{kpi}^{2} + 2\sigma_{2}^{ij}(\sigma_{1})_{i}f_{i} \\
(5.6) &= \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}} \sum_{k \neq p} \sigma_{2}^{ij}(\sigma_{1})(\lambda|k)p u_{kki}u_{p pi} + \sum_{k \neq p} \sigma_{2}^{ij}(\sigma_{1})(\lambda|k)p u_{kki}^{2} \\
(5.7) &+ \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}} \sum_{k \neq p} \sigma_{2}^{ij}(\sigma_{1})(\sigma_{3})_{i}^{2} f_{ij} \\
(5.8) &= \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}} + \frac{f\Delta f - \sigma_{3}^{ij}f_{ij}}{\sigma_{1}f - \sigma_{3}}
\end{align*}

We divided above equation into following four parts,

\begin{align*}
I : &= \sum_{i, k \neq p} (-f - \sigma_{2}^{ij}(\sigma_{1})(\lambda|k)p + \sigma_{3}^{ij})u_{kki}u_{p pi} \\
(5.10) &= \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}} \\
(5.11) &= -\sum_{i, k \neq p} \sigma_{2}^{ij}(\lambda_{i} + (\sigma_{1})(\lambda|k)p)u_{kki}u_{p pi} \\
(5.12) &= \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}}.
\end{align*}

\begin{align*}
II : &= \sum_{i, k \neq p} (f + \sigma_{2}^{ij}(\sigma_{1})(\lambda|k)p - \sigma_{3}^{ij})u_{kpi}^{2} \\
(5.13) &= \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}} \\
(5.14) &= \sum_{i, k \neq p} \sigma_{2}^{ij}(\lambda_{i} + (\sigma_{1})(\lambda|k)p)u_{kpi}^{2} \\
(5.15) &= \frac{\sigma_{1}f - \sigma_{3}}{\sigma_{1}f - \sigma_{3}}.
\end{align*}

\begin{align*}
III : &= -\frac{\sigma_{2}^{ij}(\sigma_{1})(\sigma_{1})(\sigma_{3})_{i}^{2}}{(\sigma_{1}f - \sigma_{3})^{2}} \\
(5.16) &= -\frac{\sigma_{3}^{ij}(\sum_{k} \lambda_{k}u_{kkki}u_{p pi} + \sigma_{1}f_{i})^{2}}{(\sigma_{1}f - \sigma_{3})^{2}}.
\end{align*}

\begin{align*}
IV : &= \frac{2\sigma_{2}^{ij}(\sigma_{1})_{fi}}{\sigma_{1}f - \sigma_{3}} + \frac{\sigma_{3}^{ij}f_{ij}}{\sigma_{1}f - \sigma_{3}} + \frac{f\Delta f - \sigma_{3}^{ij}f_{ij}}{\sigma_{1}f - \sigma_{3}}.
\end{align*}

Then we use following equation to replace the $u_{kkki}$ term above,

\begin{align*}
\sum_{i \neq k} \sigma_{2}^{ij}u_{iiik} + \sigma_{2}^{kk}u_{kkki} = f_{i}.
\end{align*}
We have

\begin{align}
I &= -2 \sum_{k \neq p, i \neq p, i \neq k} \sigma_{ii}^2 \lambda_i u_{kki} u_{ppi} \\
&= \frac{-2 \sum_{p} \sum_{k \neq p} \sigma_{kk}^2 u_{kki} \sigma_{pp}^2 u_{ppp}}{\sigma_1 f - \sigma_3} \\
&- 2 \sum_{p} \sum_{k \neq p} \sigma_{kk}^2 u_{kki} \sigma_{pp}^2 u_{ppp} \\
&= \frac{-2 \sum_{k \neq p, i \neq p, i \neq k} \sigma_{ii}^2 \lambda_i u_{kki} u_{ppi}}{\sigma_1 f - \sigma_3} \\
&\quad + 2 \sum_{p} \sum_{k \neq p} \sigma_{kk}^2 u_{kki} \sigma_{pp}^2 u_{ppp} \\
&\quad + 2 \sum_{k \neq p, i \neq p, i \neq k} \sigma_{ii}^2 \lambda_i u_{kki} u_{ppi} \\
&= \frac{-2 \sum_{k \neq p} \sigma_{kk}^2 u_{kki} \sigma_{pp}^2 u_{ppp}}{\sigma_1 f - \sigma_3} + 2 \sum_{p} \sum_{k \neq p} \frac{(\sigma_{kk}^2)^2 (u_{kki})^2}{\sigma_1 f - \sigma_3}. \\
\end{align}

And

\begin{align}
II &= 2 \sum_{k \neq p, i \neq p, i \neq k} \sigma_{ii}^2 \lambda_i u_{kpi}^2 \\
&= \frac{2 \sum_{k \neq p} \sigma_{pp}^2 \sigma_{kk}^2 u_{kpp}^2}{\sigma_1 f - \sigma_3} + \frac{2 \sum_{k \neq p} \sigma_{pp}^2 \sigma_{kk}^2 u_{kpp}^2}{\sigma_1 f - \sigma_3} \\
&= \frac{4 f u_{23}^2}{\sigma_1 f - \sigma_3} + \frac{2 \sum_{k \neq p} \sigma_{pp}^2 \sigma_{kk}^2 u_{kpp}^2}{\sigma_1 f - \sigma_3}. \\
\end{align}

\begin{align}
III &= -\sigma_{ii}^2 \left[ \sum_{k \neq i} \lambda_k \sigma_{kk}^2 u_{kki} + \lambda_1 (f_1 - \sigma_{kk}^2 u_{kki}) + \sigma_1 f_i \right] \\
&= -\frac{(\sigma_{kk}^2)^2}{(\sigma_1 f - \sigma_3)^2} \left[ \sum_{k \neq i} (\lambda_k - \lambda_1) \sigma_{kk}^2 u_{kki} + \lambda_1 f_i + \sigma_1 f_i \right] \\
&= -\frac{(\sigma_{kk}^2)^2}{(\sigma_1 f - \sigma_3)^2} \left[ \sum_{k \neq i} (\lambda_k - \lambda_1) \sigma_{kk}^2 u_{kki} + \lambda_1 f_i + \sigma_1 f_i \right]. \\
\end{align}

There is an identity

\begin{align}
\sigma_1 f - \sigma_3 &= \sigma_{11}^2 \sigma_{22}^2 \sigma_{33}^2.
\end{align}
Then we get

\[ III = -\frac{(\lambda_3 - \lambda_1)^2 \sigma_3^2 u_{331}^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} - \frac{2(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)u_{221}u_{331}}{(\sigma_1 f - \sigma_3)} \]

\[ -\frac{(\lambda_2 - \lambda_1)^2 \sigma_2^2 u_{221}^2}{(\sigma_1 f - \sigma_3) \sigma_3^2} + \frac{2(\lambda_1 + \sigma_1)f_1(\lambda_1 - \lambda_3)u_{331}}{(\sigma_1 f - \sigma_3) \sigma_2^2} \]

\[ + 2(\lambda_1 + \sigma_1)f_1(\lambda_1 - \lambda_2)u_{221} - \frac{\sigma_2^1(\lambda_1 + \sigma_1)^2 f_1^2}{(\sigma_1 f - \sigma_3)^2} \]

\[ \frac{-(\lambda_3 - \lambda_2)^2 \sigma_3^2 u_{332}^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} - \frac{2(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)u_{112}u_{332}}{(\sigma_1 f - \sigma_3)} \]

\[ -\frac{(\lambda_1 - \lambda_2)\sigma_2^1 u_{112}^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} + \frac{2(\lambda_2 + \sigma_1)f_2(\lambda_2 - \lambda_3)u_{332}}{(\sigma_1 f - \sigma_3) \sigma_2^2} \]

\[ + \frac{2\lambda_2 - \lambda_1)u_{112}f_2(\lambda_1 + \lambda_2)}{(\sigma_1 f - \sigma_3) \sigma_2^2} - \frac{\sigma_2^2(\lambda_2 + \sigma_1)^2 f_2^2}{(\sigma_1 f - \sigma_3)^2} \]

\[ \frac{-(\lambda_3 - \lambda_3)^2 \sigma_2^2 u_{113}^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} - \frac{(\lambda_2 - \lambda_3)^2 \sigma_2^2 u_{223}^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} \]

\[ -\frac{2(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)u_{113}u_{233}}{(\sigma_1 f - \sigma_3) \sigma_2^2} - \frac{\sigma_2^3(\lambda_3 + \sigma_1)^2 f_3^2}{(\sigma_1 f - \sigma_3)^2} \]

\[ + \frac{2(\lambda_3 - \lambda_1)u_{113}(\lambda_3 + \sigma_1) f_3}{(\sigma_1 f - \sigma_3) \sigma_2^2} + \frac{2(\lambda_3 - \lambda_2)u_{223}(\lambda_3 + \sigma_1) f_3}{(\sigma_1 f - \sigma_3) \sigma_2^2} \]

We can also compute \( I \) and \( II \) explicitly,

\[ I = \frac{2 \sum \sigma^k_{kk} u_{kk1} f_1}{\sigma_1 f - \sigma_3} - \frac{2 \sum \sigma^k_{kk} u_{kk2} f_2}{\sigma_1 f - \sigma_3} - \frac{2 \sum \sigma^k_{kk} u_{kk3} f_3}{\sigma_1 f - \sigma_3} \]

\[ + \frac{2(\sigma^2_{22})^2(u_{221})^2}{\sigma_1 f - \sigma_3} + \frac{2(\sigma^2_{33})^2(u_{331})^2}{\sigma_1 f - \sigma_3} + \frac{2(\sigma^2_{11})^2(u_{112})^2}{\sigma_1 f - \sigma_3} \]

\[ + \frac{2(\sigma^3_{33})^2(u_{332})^2}{\sigma_1 f - \sigma_3} + \frac{2(\sigma^3_{11})^2(u_{113})^2}{\sigma_1 f - \sigma_3} + \frac{2(\sigma^2_{22})^2(u_{223})^2}{\sigma_1 f - \sigma_3} \]

\[ + \frac{4(\sigma^2_{11})^2(\lambda_1 + \sigma_2^2 \sigma^3_{33}) u_{221} u_{331}}{\sigma_1 f - \sigma_3} + \frac{4(\sigma^2_{22})^2(\lambda_2 + \sigma_2^1 \sigma^3_{33}) u_{112} u_{332}}{\sigma_1 f - \sigma_3} \]

\[ + \frac{4(\sigma^3_{33})^2(\lambda_3 + \sigma_2^2 \sigma^3_{11}) u_{113} u_{223}}{\sigma_1 f - \sigma_3} \]

And

\[ II = \frac{4f u_{223}^2}{\sigma_1 f - \sigma_3} + \frac{2 \sum \sigma^2_{pp} \sigma^k_{2k} u_{pp}^2}{\sigma_1 f - \sigma_3} \]

\[ = \frac{4f u_{223}^2}{\sigma_1 f - \sigma_3} + \frac{2(\sigma^2_{11})^2(\lambda_1 + \sigma_2^2 \sigma^3_{33}) u_{221} u_{331}}{\sigma_1 f - \sigma_3} + \frac{2(\sigma^2_{22})^2(\lambda_2 + \sigma_2^1 \sigma^3_{33}) u_{221} u_{331}}{\sigma_1 f - \sigma_3} \]

\[ + \frac{2\sigma^2_{22}\sigma^1_{11} u_{122}^2}{\sigma_1 f - \sigma_3} + \frac{2\sigma^2_{33}\sigma^1_{33} u_{332}^2}{\sigma_1 f - \sigma_3} \]

\[ + \frac{2\sigma^3_{33}\sigma^1_{11} u_{133}^2}{\sigma_1 f - \sigma_3} + \frac{2\sigma^3_{33}\sigma^2_{22} u_{233}^2}{\sigma_1 f - \sigma_3} \]
We also have from Lemma 11

\begin{align}
(5.51) \quad IV & \geq \frac{2\sigma^1_2(\sigma_1)_1 f_1}{\sigma_1 f - \sigma_3} + \frac{2\sigma^2_2(\sigma_1)_2 f_2}{\sigma_1 f - \sigma_3} + \frac{2\sigma^3_2(\sigma_1)_3 f_3}{\sigma_1 f - \sigma_3} \\
(5.52) & - C\sigma_1 + f_u[\sigma_1 f_1 + f(\sigma_1)_i - (\sigma_3)_i] \\
(5.53) & = \frac{2\sigma^1_2(\sigma_2^{ij}u_{221} - \sigma^j_2u_{331})}{\sigma_1 f - \sigma_3} + u_{221} + u_{331}f_1 \\
(5.54) & + \frac{2\sigma^2_2(\sigma_1^j u_{112} - \sigma^j_2u_{332})}{\sigma_1 f - \sigma_3} + u_{112} + u_{332}f_2 \\
(5.55) & + \frac{2\sigma^3_2(\sigma_1^j u_{113} - \sigma^j_2u_{223})}{\sigma_1 f - \sigma_3} + u_{223} + u_{113}f_3 \\
(5.56) & - C\sigma_1 + f_u V_i \\
(5.57) & = \frac{2((\lambda_2 - \lambda_1)u_{221} + (\lambda_3 - \lambda_1)u_{331})f_1}{\sigma_1 f - \sigma_3} \\
(5.58) & + \frac{2((\lambda_2 - \lambda_1)u_{112} + (\lambda_3 - \lambda_2)u_{332})f_2}{\sigma_1 f - \sigma_3} \\
(5.59) & + \frac{2((\lambda_2 - \lambda_1)u_{223} + (\lambda_1 - \lambda_3)u_{113})f_3}{\sigma_1 f - \sigma_3} \\
(5.60) & - C\sigma_1 + f_u V_i.
\end{align}

So we combine above inequalities

\begin{align}
(5.61) \quad \sigma^ij_{ij} & \geq \frac{4\lambda_2u_{221}f_1\sigma^2_2}{(\sigma_1 f - \sigma_3)\sigma^2_n} - \frac{4\lambda_3u_{331}f_1\sigma^3_2}{(\sigma_1 f - \sigma_3)\sigma^2_n} \\
(5.62) & - \frac{4\lambda_1u_{112}f_2\sigma^1_2}{(\sigma_1 f - \sigma_3)\sigma^2_n} + \frac{4\lambda_3u_{332}f_2\sigma^3_2}{(\sigma_1 f - \sigma_3)\sigma^2_n} \\
(5.63) & - \frac{4\lambda_1u_{113}f_3\sigma^1_2}{(\sigma_1 f - \sigma_3)\sigma^2_n} + \frac{4\lambda_2u_{223}f_3\sigma^3_2}{(\sigma_1 f - \sigma_3)\sigma^2_n} \\
(5.64) & + \frac{2\sigma^2_2\sigma^3_2u_{221}u_{331}}{\sigma_1 f - \sigma_3} + \frac{2\sigma^1_2\sigma^3_2u_{112}u_{332}}{\sigma_1 f - \sigma_3} \\
(5.65) & + \frac{2\sigma^1_2\sigma^2_2u_{113}u_{223}}{\sigma_1 f - \sigma_3} + \frac{4fu_{123}^2}{\sigma_1 f - \sigma_3} \\
(5.66) & + \frac{\sigma^3_2[4f + (\sigma^2_2)^2]u_{331}}{(\sigma_1 f - \sigma_3)\sigma^2_n} + \frac{\sigma^2_2[4f + (\sigma^3_2)^2]u_{221}}{(\sigma_1 f - \sigma_3)\sigma^2_n} \\
(5.67) & + \frac{\sigma^3_2[4f + (\sigma^3_2)^2]u_{332}}{(\sigma_1 f - \sigma_3)\sigma^2_n} + \frac{\sigma^1_2[4f + (\sigma^3_2)^2]u_{112}}{(\sigma_1 f - \sigma_3)\sigma^2_n} \\
(5.68) & + \frac{\sigma^1_2[4f + (\sigma^1_2)^2]u_{113}}{(\sigma_1 f - \sigma_3)\sigma^2_n} + \frac{\sigma^2_2[4f + (\sigma^3_2)^2]u_{223}}{(\sigma_1 f - \sigma_3)\sigma^2_n} \\
(5.69) & - C\sigma_1 + f_u V_i.
\end{align}
Square these terms,
\begin{align}
(5.70) \quad \sigma^2_{ij} V_{ij} & \geq \frac{\sigma^2_{22} \sigma_2^3(u_{331} + u_{221})^2}{\sigma_1 f - \sigma_3} + \frac{\sigma^2_{22} \sigma_2^3(u_{332} + u_{112})^2}{\sigma_1 f - \sigma_3} \\
(5.71) & + \frac{\sigma^2_{11} \sigma_2^2(u_{113} + u_{223})^2}{\sigma_1 f - \sigma_3} + \frac{4f u_{123}^2}{\sigma_1 f - \sigma_3} \\
(5.72) & + \frac{4f \sigma_2^3(u_{331} - \frac{f_1}{7} \lambda_1)^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} + \frac{4f \sigma_2^3(u_{221} - \frac{f_1}{7} \lambda_1)^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} \\
(5.73) & + \frac{4f \sigma_2^3(u_{332} - \frac{f_1}{7} \lambda_2)^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} + \frac{4f \sigma_2^3(u_{112} - \frac{f_2}{7} \lambda_2)^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} \\
(5.74) & + \frac{4f \sigma_2^3(u_{113} - \frac{f_2}{7} \lambda_1)^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} + \frac{4f \sigma_2^3(u_{223} - \frac{f_2}{7} \lambda_2)^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} \\
(5.75) & - C \sigma_1 + f u_i V_i.
\end{align}

In above inequality we have used
\begin{align}
(5.76) \quad \frac{\sigma^2_{22} \sigma_2^3 \lambda_1^2 \lambda_2^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} & \leq \frac{C \sigma^2_{22} \lambda_1^2 \lambda_2^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} \\
(5.77) & = \frac{C \sigma^2_{22} (f - \lambda_1 \sigma_2^1) \lambda_2^2}{(\sigma_1 f - \sigma_3) \sigma_2^2} \\
(5.78) & \leq C \sigma_1 + \frac{C \sigma^2_{22} \sigma_2^1 \lambda_1^2}{\sigma_1 f - \sigma_3} \\
(5.79) & \leq C \sigma_1 + \frac{C \sigma^2_{22} (f \lambda_1 - \sigma_3)}{\sigma_1 f - \sigma_3} \\
(5.80) & \leq C \sigma_1.
\end{align}

So we have proved this lemma. \(\square\)

5.1. **Proof of (2.114).** Denote \(\rho(x) = 5^2 - |x|^2\), and \(M_r := \sup_{B_r} \sigma_1\). We consider test function
\[(5.81) \quad \varphi(x) = 4 \log \rho(x) + h \left( \frac{|Du|^2}{2} \right) + \log \log \left( \frac{(\sigma_1 f - \sigma_3) \log \frac{\rho}{M_5}}{M_5 \log(\sigma_1 f - \sigma_3)} \right) + 20\]
in \(B_5\), where \(h(t) = -\frac{1}{2} \log(1 - \frac{t}{2\max|Du|})\).

Assume \(\varphi\) attains its maximum point \(z_0 \in \Omega\), and \(\frac{(\sigma_1 f - \sigma_3) \log \frac{\rho}{M_5}}{M_5 \log(\sigma_1 f - \sigma_3)} > 20\). Then \(\varphi\) must attained its maximum point in the ring \(B_5\). Moreover, we may always assume \(\frac{(\sigma_1 f - \sigma_3) \log \frac{\rho}{M_5}}{M_5 \log(\sigma_1 f - \sigma_3)}\) sufficient large. We choose coordinate frame \(\{e_1, e_2, e_3\}\), such that \(D^2 u\) is diagonalized at this point. We denote \(M = \log \frac{M_5}{\log \frac{\rho}{M_5}}\).

By maximum principle, at \(z_0\), we have
\[(5.82) \quad 0 = \varphi_i = \frac{4 \rho_i}{\rho} + h' u_k u_{ki} + \frac{V_i - \frac{22}{35} V}{V - M - \log V}.
\]

and
So we get
\begin{align}
(5.83) \quad \varphi_{ij} &= \frac{4\rho_{ij}}{\rho} - \frac{4\rho_i\rho_j}{\rho^2} + h''u_ik_iu_iu_j + h'(u_kj_1u_k + u_uk_{ij}) \\
&\quad + \frac{V_{ij} - V_i + V_j}{V - M - \log V} - \frac{V_iV_j}{(V - M - \log V)^2}(1 - \frac{1}{V})^2.
\end{align}

Contracting with \(\sigma_{ij}^{ii} := \frac{\partial \sigma_i(D^2\varphi)}{\partial u_{ij}}\), we get
\begin{align*}
\sigma_{ij}^{ii}\varphi_{ij} &\geq -8\frac{\sigma_{ij}^{ii}}{\rho} - 16\frac{\sigma_{ij}^{ii}x_i^2}{\rho^2} + h''\sigma_{ij}^{ii}u_i^2u_j^2 + h'(\sigma_1f - 3\sigma_3) \\
&\quad + h'u_if_i + \frac{\sigma_{ij}^{ii}V_{ii}(1 - \frac{1}{V})}{V - M - \log V} - \frac{\sigma_{ij}^{ii}V_i^2}{(V - M - \log V)^2}(1 - \frac{1}{V})^2.
\end{align*}

By (2.113),
\begin{align}
(5.85) \quad \sigma_{ij}^{ii}V_{ii} &\geq -C\sigma_1 + f_iV_i.
\end{align}

And by (5.82),
\begin{align}
(5.86) \quad f_iV_i - \frac{V_i - V}{V - M - \log V} + h'u_if_i &\geq -C\left(\frac{1}{\rho} + h'\right)
\end{align}

Then combine this inequality and (5.82), we have
\begin{align}
(5.87) \quad \sigma_{ij}^{ii}\varphi_{ij} &\geq -8\frac{\sigma_{ij}^{ii}}{\rho} - 48\frac{\sigma_{ij}^{ii}x_i^2}{\rho^2} + (h'' - 2(h')^2)\sigma_{ij}^{ii}u_i^2u_j^2 \\
&\quad + \frac{h'}{2}(\sigma_1f - 3\sigma_3).
\end{align}

If we choose \(h(t) = -\frac{t}{2}\log(1 - \frac{t}{2\max|Du|^2})\) such that \(h'' - 2(h')^2 \geq 0\), we have
\begin{align}
(5.89) \quad \sigma_{ij}^{ii}\varphi_{ij} &\geq -4\frac{\sigma_{ij}^{ii}}{\rho} - 1200\frac{\sigma_{ij}^{ii}x_i^2}{\rho^2} + h\frac{M_5\log(\sigma_1f - \sigma_3)}{\log^\frac{1}{2}M_5} \\
&\quad \geq -1300\frac{\sigma_{ij}^{ii}}{\rho^2} + h\frac{M_5\log^\frac{1}{2}M_5}{2}.
\end{align}

So we get
\begin{align}
(5.91) \quad \rho^4(z_0)\log\frac{(\sigma_1f - \sigma_3)(z_0)\log^\frac{1}{2}M_5}{M_5\log(\sigma_1f - \sigma_3)(z_0)} &\leq 3\rho^4(z_0)\log M_5 \leq C.
\end{align}

Finally, we obtain in \(B_4\)
\begin{align}
(5.92) \quad (\sigma_1f - \sigma_3)(x) &\leq \frac{M_5\log(\sigma_1f - \sigma_3)}{\log^\frac{1}{2}M_5}.
\end{align}

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