Generalised vector products and metrical trigonometry of a tetrahedron

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Abstract

We study the general rational trigonometry of a tetrahedron, based on quadrances, spreads and solid spreads, using vector products associated to an arbitrary symmetric bilinear form over a general field, not of characteristic two. This gives us algebraic analogs of many classical formulas, as well as new insights and results. In particular we derive original relations for a tri-rectangular tetrahedron.

Keywords: scalar product; vector product; symmetric bilinear form; tetrahedron; rational trigonometry; affine geometry; projective geometry

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1 Introduction

In this paper, which is a follow-up to Generalised vector products applied to affine and projective rational trigonometries in three dimensions (henceforth referred to as [20]), we will apply the framework of generalised scalar and $B$-vector products developed there to set up a framework for the rational trigonometry of a general tetrahedron in three-dimensional affine space with a general metrical structure defined over an arbitrary field, not of characteristic two.

The metrical structure of the tetrahedron is a much studied topic, since at least the time of Tartaglia, who was the first to give a formula for the volume in terms of the (squared) lengths of the sides, a formula also found by Euler and generalized by Cayley and Menger. This classical treatment of the trigonometry of the tetrahedron reached a high point with the large scale summary of Richardson [21] for the Euclidean case, involving not just lengths and angles of faces but also dihedral angles between faces and solid angles at vertices and transversal lengths between opposite edges. Since locally at a vertex a tetrahedron determines a tripod, there was also a projective aspect which connects naturally with spherical trigonometry [29]. In modern times both computational geometry in three dimensions [11] and the finite element method in space [5] have utilized tetrahedral meshes and their measurements. And of course there is naturally interest in generalizations to metrical formulas for more general polytopes, as in the rigidity results of Sabitov [22], and to explorations of analogs of
trigonometric concepts and laws to simplices in higher dimensions, for example [12]. But it is clear that our understanding of the n-dimensional story is still very much in its early days.

Along with this classical mostly Euclidean orientation, since the 19th century development of non-Euclidean geometry, the question of how trigonometry extends to hyperbolic and spherical settings has been of keen interest, and an important issue is the three-dimensional situation for hyperbolic tetrahedra, where even the volume formula originally due to Lobachevsky is much more complicated than in the Euclidean case, and much work has been done in understanding and extending this result for example the work of [18], [9] and [19].

With the advent of rational trigonometry (see [27] and [28]) the possibility emerged to reframe trigonometry in a purely algebraic fashion; where lengths and angles become secondary to purely algebraic concepts defined in terms of a symmetric bilinear form. A projective version of this theory applies also to allow a recasting of hyperbolic geometry (see [30], [31], [32] and [1]) which also extends the subject to general fields, including finite fields.

In [20] we laid out a three-dimensional version of this theory which extends classical formulas of Lagrange, Binet, Cauchy and others to this more general setting valid for arbitrary fields (not of characteristic two) and to general quadratic forms. In this follow up paper we intend to apply this technology to completely reformulating the classical trigonometry of a tetrahedron, using rational analogs also of dihedral angles and solid spreads. We hope that this will provide a bridge also to a projective version which describes the trigonometry of spherical and hyperbolic tetrahedra, and in fact some key projective formulas for that project are already contained in this present work.

Suppose that $V^3$ is the three-dimensional vector space, over a field $F$ not of characteristic 2, consisting of row vectors $v = (x, y, z)$ and that $B$ is a symmetric non-degenerate $3 \times 3$ matrix, so that $\det B \neq 0$. Then we may define a symmetric bilinear form, or $B$-scalar product, by the rule that for any two vectors $v$ and $w$

$$v \cdot_B w \equiv vBw^T.$$

This number is always an element of the field $F$.

The $B$-quadrance of a vector $v$ is

$$Q_B(v) \equiv v \cdot_B v$$

and a vector $v$ is $B$-null precisely when $Q_B(v) = 0$.

If $v$ and $w$ are non-null vectors, then the $B$-spread between them is the number

$$s_B(v, w) \equiv 1 - \frac{(v \cdot_B w)^2}{Q_B(v)Q_B(w)}.$$  

In Euclidean geometry, the quadrance and spread are typically thought of respectively as the squared distance and the squared sine of an angle; in our framework, we are using quadrances and spreads to allow for extensions of Euclidean geometry to arbitrary symmetric bilinear forms over a general field more easily.

In [20], we extended the definition of vector products in Euclidean three-dimensional vector space also to this more general three-dimensional situation. Given the usual (Euclidean) vector product
$v \times w$, for two vectors $v$ and $w$ in $\mathbb{V}^3$, we define the $B$-vector product of $v$ and $w$ to be

$$v \times_B w \equiv (v \times w) \text{adj} B$$

where $\text{adj} B = (\det B) B^{-1}$ is the adjugate of the invertible matrix $B$.

After a short review of the properties of $B$-scalar and $B$-vector products which were proven in [20], we define the fundamental trigonometric invariants in three-dimensional affine space, denoted by $A^3$, where $\mathbb{V}^3$, as given above, is its associated vector space. These include the $B$-quadrance and $B$-spread, as well as the $B$-quadrea of a triangle in $A^3$ which extends the definition of quadrea in [27].

While these three quantities featured prominently in [20], this paper introduces four new trigonometric invariants: the $B$-quadrum, the $B$-dihedral spread, the $B$-solid spread and the $B$-dual solid spread. The $B$-quadrum, which is a quadratic version of volume in our framework, has a close connection to the Cayley-Menger determinant, as seen in [11], [10] pp. 285-289] and [24] pp. 124-126]. The latter three quantities, which are analogs of angles between two planes or solid angles between three lines in our framework, have origins in projective geometry (see [28] and [29]) and thus highlight the power of using generalised vector products to explain affine rational trigonometry in three dimensions.

We can then compute these quantities for a general tetrahedron in $A^3$, for which we can then discover interesting algebraic relations. For a tetrahedron $A_0A_1A_2A_3$ with points $A_0, A_1, A_2, A_3$, we will denote, for indices $0 \leq i < j < k \leq 3$:

- by $Q_{ij}$ the $B$-quadrance between two points $A_i$ and $A_j$;
- by $A_{ijk}$ the $B$-quadrea of the triangle with points $A_i, A_j$ and $A_k$;
- by $V$ its $B$-quadrum;
- by $E_{ij}$ the $B$-dihedral spread between the planes through any three points intersecting at a line through $A_i$ and $A_j$;
- by $S_i$ the $B$-solid spread between three concurrent lines at $A_i$; and
- by $D_i$ the $B$-dual solid spread between three concurrent lines at $A_i$.

The quantity

$$\mathcal{R} \equiv \frac{16V^2}{A_{012}A_{013}A_{023}A_{123}}$$

is a key component of our study, whose geometric meaning is yet to be fully understood; we will call this the Richardson constant. Here are some examples of relations we obtain:

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \mathcal{R}$$

$$\frac{D_0}{A_{123}} = \frac{D_1}{A_{023}} = \frac{D_2}{A_{013}} = \frac{D_3}{A_{012}} = \frac{\mathcal{R}}{4}$$

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and

\[
\begin{align*}
\frac{S_0 S_1 S_2}{Q_0 Q_1 Q_2 Q_3} &= \frac{S_0 S_1 S_3}{Q_0 Q_2 Q_3 Q_1} = \frac{S_0 S_2 S_3}{Q_0 Q_1 Q_3 Q_2} = \frac{S_1 S_2 S_3}{Q_0 Q_1 Q_2 Q_3} = \frac{S_2 S_0 S_3}{Q_0 Q_1 Q_2 Q_3} = \\
&= \frac{64 Q_0^2 Q_1^2 Q_2^2 Q_3^2}{Q_1^2 Q_2^2 Q_3^2}. 
\end{align*}
\]

The first two results are rational/algebraic analogs of some results found from [21], while the last one is novel; but we will develop many more in this paper.

In a later section we specialize our formulas to the case of the tri-rectangular tetrahedron, and develop some additional key results for this important situation. We will also use this framework to derive \( B \)-quadrances between opposite lines of a tetrahedron, which are skew, as well as presenting potential directions for further research.

2 A review of three-dimensional vector algebra over a general metrical framework

We begin with the framework of the three-dimensional vector space \( \mathbb{V}^3 \) over a field \( F \) not of characteristic 2, consisting of row vectors \( v = (x, y, z) \), with the usual arithmetical structures of vector addition and subtraction, together with scalar multiplication.

2.1 \( B \)-scalar product

A \( 3 \times 3 \) symmetric matrix

\[
B \equiv \begin{pmatrix}
    a_1 & b_3 & b_2 \\
    b_3 & a_2 & b_1 \\
    b_2 & b_1 & a_3 
\end{pmatrix}
\]

(1)
determines a symmetric bilinear form on \( \mathbb{V}^3 \) defined by

\[
v \cdot_B w \equiv v B w^T.
\]

We will call this the \( B \)-scalar product. The associated \( B \)-quadratic form on \( \mathbb{V}^3 \) is defined by

\[
Q_B(v) \equiv v \cdot_B v
\]

and we call the number \( Q_B(v) \) the \( B \)-quadrance of \( v \). A vector \( v \) is \( B \)-null precisely when

\[
Q_B(v) = 0.
\]

The \( B \)-quadrance satisfies the obvious properties that for vectors \( v \) and \( w \) in \( \mathbb{V}^3 \) and a number \( \lambda \) in \( F \)

\[
Q_B(\lambda v) = \lambda^2 Q_B(v)
\]

as well as

\[
Q_B(v + w) = Q_B(v) + Q_B(w) + 2 (v \cdot_B w)
\]
and

\[ Q_B (v - w) = Q_B (v) + Q_B (w) - 2 (v \cdot_B w). \]

Hence the \( B \)-scalar product can be expressed in terms of the \( B \)-quadratic form by either of the two polarisation formulas

\[ v \cdot_B w = \frac{Q_B (v + w) - Q_B (v) - Q_B (w)}{2} = \frac{Q_B (v) + Q_B (w) - Q_B (v - w)}{2}. \]

The \( B \)-scalar product is non-degenerate precisely when the condition that \( v \cdot_B w = 0 \) for any vector \( v \) in \( \mathbb{V}^3 \) implies that \( w = 0 \); this will occur precisely when \( B \) is invertible. We will assume that the \( B \)-scalar product is non-degenerate throughout this paper. Finally, two vectors \( v \) and \( w \) in \( \mathbb{V}^3 \) are \( B \)-perpendicular precisely when \( v \cdot_B w = 0 \), in which case we write \( v \perp_B w \).

### 2.2 \( B \)-vector product

Define the adjugate of \( B \) from (1) to be the matrix

\[ \text{adj} \ B \equiv \begin{pmatrix} a_2 a_3 - b_1^2 & b_2 - a_3 b_3 & b_1 b_3 - a_2 b_2 \\ b_1 b_2 - a_3 b_3 & a_1 a_3 - b_2^2 & b_2 b_3 - a_1 b_1 \\ b_1 b_3 - a_2 b_2 & b_2 b_3 - a_1 b_1 & a_1 a_2 - b_3^2 \end{pmatrix}. \]

When \( B \) is invertible this is

\[ \text{adj} \ B = (\det B) B^{-1}. \]

For vectors \( v \equiv (v_1, v_2, v_3) \) and \( w \equiv (w_1, w_2, w_3) \) in \( \mathbb{V}^3 \), the usual Euclidean vector product [13, p. 65] is

\[ v \times w \equiv (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1). \]

So the \( B \)-vector product [20] of \( v \) and \( w \) is defined to be the vector

\[ v \times_B w \equiv (v \times w) \text{adj} \ B. \]

With the \( B \)-scalar and \( B \)-vector products defined, following [20] we define the following expressions involving vectors \( v_1, v_2, v_3 \) and \( v_4 \):

- \( B \)-scalar triple product:
  \[ [v_1, v_2, v_3]_B \equiv v_1 \cdot_B (v_2 \times_B v_3) \]

- \( B \)-vector triple product:
  \[ (v_1, v_2, v_3)_B \equiv v_1 \times_B (v_2 \times_B v_3) \]

- \( B \)-quadruple scalar product:
  \[ [v_1, v_2; v_3, v_4]_B \equiv (v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4) \]
\( B \)-quadruple vector product:

\[
\langle v_1, v_2; v_3, v_4 \rangle_B \equiv (v_1 \times_B v_2) \times_B (v_3 \times_B v_4)
\]

2.3 Summary of results of \( B \)-scalar and vector products

We summarize some results from [20] pertaining to \( B \)-scalar and \( B \)-vector products, \( B \)-triple products and \( B \)-quadruple products. The first result allows us to express the \( B \)-scalar triple product in terms of determinants.

**Theorem 1 (Scalar triple product theorem)** Let \( M \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \) for vectors \( v_1, v_2 \) and \( v_3 \) in \( \mathbb{V}^3 \).

Then

\[
[v_1, v_2, v_3]_B = [v_2, v_3, v_1]_B = [v_3, v_1, v_2]_B \\
= - [v_1, v_3, v_2]_B = - [v_2, v_1, v_3]_B = - [v_3, v_2, v_1]_B \\
= (\det B) [v_1, v_2, v_3] = \det (MB).
\]

The following result expresses the \( B \)-vector triple product as a linear combination of two vectors.

**Theorem 2 (Lagrange’s formula)** For vectors \( v_1, v_2 \) and \( v_3 \) in \( \mathbb{V}^3 \),

\[
\langle v_1, v_2, v_3 \rangle_B = (\det B) [(v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3].
\]

The \( B \)-scalar quadruple product can be computed as a determinantal identity involving \( B \)-scalar products, as follows.

**Theorem 3 (Binet-Cauchy identity)** For vectors \( v_1, v_2, v_3 \) and \( v_4 \) in \( \mathbb{V}^3 \),

\[
[v_1, v_2; v_3, v_4]_B = (\det B) [(v_1 \cdot_B v_3) (v_2 \cdot_B v_4) - (v_1 \cdot_B v_4) (v_2 \cdot_B v_3)].
\]

The following result immediately follows from the Binet-Cauchy identity and links the \( B \)-vector product between two vectors to their \( B \)-quadrances and their \( B \)-scalar product.

**Theorem 4 (Lagrange’s identity)** For vectors \( v_1 \) and \( v_2 \) in \( \mathbb{V}^3 \),

\[
Q_B (v_1 \times_B v_2) = (\det B) \left[ Q_B (v_1) Q_B (v_2) - (v_1 \cdot_B v_2)^2 \right].
\]

The \( B \)-vector quadruple product can be computed by using only the \( B \)-scalar triple product, as follows.

**Theorem 5 (Vector quadruple product theorem)** For vectors \( v_1, v_2, v_3 \) and \( v_4 \) in \( \mathbb{V}^3 \),

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = (\det B) ([v_1, v_2, v_4]_B v_3 - [v_1, v_2, v_3]_B v_4) \\
= (\det B) ([v_1, v_3, v_4]_B v_2 - [v_2, v_3, v_4]_B v_1).
\]
Immediately following from the $B$-vector quadruple product theorem, the following corollary will prove very useful in the study of a vector tetrahedron.

**Corollary 6** Let $M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ for vectors $v_1, v_2$ and $v_3$ in $\mathbb{V}^3$. Then

$$\langle v_1, v_2; v_1, v_3 \rangle_B = \left( (\text{det } B)^2 (\text{det } M) \right) v_1.$$

A consequence of this corollary gives us a result that will also prove useful in this paper. We present a quick proof of this result as follows.

**Theorem 7 (Scalar triple product of products)** For vectors $v_1, v_2$ and $v_3$ in $\mathbb{V}^3$,

$$[v_2 \times_B v_3, v_3 \times_B v_1, v_1 \times_B v_2]_B = (\text{det } B) ([v_1, v_2, v_3]_B)^2.$$

**Proof.** For $M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ we use Corollary 6 and the Scalar triple product theorem to obtain

$$[v_2 \times_B v_3, v_3 \times_B v_1, v_1 \times_B v_2]_B = \left( (\text{det } B)^2 (\text{det } M) \right) [v_1, v_2, v_3]_B$$

$$= (\text{det } B) ([v_1, v_2, v_3]_B)^2$$

as required. ■

### 3 Affine and vector geometry in three dimensions

For the rest of this paper, we will work over the three-dimensional affine space over a field $\mathbb{F}$ not of characteristic two, denoted by $\mathbb{A}^3$, where $\mathbb{V}^3$ is its associated vector space. While the main objects in $\mathbb{A}^3$ are points, which we denote as triples enclosed in rectangular brackets, vectors in $\mathbb{V}^3$ can be expressed as a separation between two points. In other words, for two points $X$ and $Y$ in $\mathbb{A}^3$, a vector from $X$ to $Y$ is expressed as $\overrightarrow{XY}$ and computed to be the affine difference $Y - X$.

A **line** is a pair $(A, v)$ containing a point $A$ in $\mathbb{A}^3$ and a vector $v$ in $\mathbb{V}^3$, so that a point $X$ lies on it precisely when there exists a number $\lambda$ in $\mathbb{F}$ such that

$$X - A = \lambda v.$$ 

Two lines $(A_1, v_1)$ and $(A_2, v_2)$ are equal precisely when $v_1$, $v_2$ and $\overrightarrow{A_1A_2}$ are all scalar multiples of each other. The vector $v$ is a **direction vector** for the line $(A, v)$. Given two points $X_1$ and $X_2$ both lying on a line, we can denote such a line by $X_1X_2$, so that the lines $X_1X_2$ and $Y_1Y_2$ are equal precisely when $Y_1$ and $Y_2$ both lie on the line $X_1X_2$ and vice versa.

A **plane** in $\mathbb{A}^3$ is a triple $(A, v, w)$ containing a point $A$ in $\mathbb{A}^3$ and two linearly independent vectors
v and w in \( \mathbb{V}^3 \), so that a point \( X \) lies on it if there exists numbers \( \lambda \) and \( \mu \) in \( \mathbb{F} \) such that

\[
X - A = \lambda v + \mu w.
\]

In other words, the vector \( \overrightarrow{AX} \) is a linear combination of \( v \) and \( w \), so that two planes \((A_1, v_1, w_1)\) and \((A_2, v_2, w_2)\) are equal when any of \( v_1, v_2, w_1, w_2 \) and \( \overrightarrow{A_1A_2} \) are linear combinations of any two of these. The vectors \( v \) and \( w \) are then spanning vectors for the plane \((A, v, w)\). We can associate to a plane in \( \mathbb{A} \) a \( B \)-normal vector \( n \) so that any two points \( X \) and \( Y \) in \( \mathbb{A}^3 \) lying on the plane satisfy

\[
(Y - X) \cdot B n = \overrightarrow{XY} \cdot B n = 0.
\]

A plane in \( \mathbb{A}^3 \) with three points \( X, Y \) and \( Z \) will be denoted by \( XYZ \), with two planes \( X_1Y_1Z_1 \) and \( X_2Y_2Z_2 \) being equal precisely when \( X_2, Y_2 \) and \( Z_2 \) lie on \( X_1Y_1Z_1 \) and vice versa.

A triangle in \( \mathbb{A}^3 \) is an unordered collection of three points in \( \mathbb{A}^3 \), say \( \{A_1, A_2, A_3\} \) and is denoted by \( \overrightarrow{A_1A_2} \). Such a triangle determines two vector triangles \( \{\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \overrightarrow{A_3A_1}\} \) and \( \{\overrightarrow{A_1A_3}, \overrightarrow{A_3A_2}, \overrightarrow{A_2A_1}\} \) where the vectors in the vector triangle sum to 0. By defining \( v_{ij} = \overrightarrow{A_iA_j} \) for any integer \( i \) and \( j \) between 1 and 3, we denote these vector triangles respectively by \( v_{12}v_{23}v_{31} \) and \( v_{13}v_{21}v_{32} \). A tetrahedron in \( \mathbb{A}^3 \) is an unordered collection of four points in \( \mathbb{A}^3 \), say \( \{A_0, A_1, A_2, A_3\} \), and is denoted by \( A_0A_1A_2A_3 \). An unordered collection of any two distinct points of a tetrahedron will be called an edge of the tetrahedron, and an unordered collection of any three distinct points of a tetrahedron will be called a triangle of the tetrahedron. Associated to each edge and triangle of a tetrahedron is the line and plane (respectively) that passes through the collection of points of the tetrahedron; we call these the lines and planes of the tetrahedron.

## 4 Affine rational trigonometry in three dimensions

We now define the rational trigonometric quantities that we will use to analyse a general tetrahedron over a general field and symmetric bilinear form.

The \( B \)-quadrance between two points \( A_1 \) and \( A_2 \) in \( \mathbb{A}^3 \) is the number

\[
Q_B (A_1, A_2) \equiv Q_B \left( \overrightarrow{A_1A_2} \right) = \overrightarrow{A_1A_2} \cdot B \overrightarrow{A_1A_2}.
\]

Note that \( Q_B \left( \overrightarrow{A_1A_2} \right) = Q_B \left( \overrightarrow{A_2A_1} \right) \), so that the \( B \)-quadrance between two points in \( \mathbb{A}^3 \) is independent of order.

Define Archimedes’ function \( [27] \) p. 64] as

\[
A(a, b, c) \equiv (a + b + c)^2 - 2(a^2 + b^2 + c^2)
\]
so that we also have
\[ A(a, b, c) = 4ab - (a + b - c)^2 \]
\[ = 4ac - (a + c - b)^2 \]
\[ = 4bc - (b + c - a)^2. \]

For a triangle \( \overline{A_1A_2A_3} \) with \( B \)-quadrances
\[ Q_1 \equiv Q_B(A_2, A_3) \quad Q_2 \equiv Q_B(A_1, A_3) \quad \text{and} \quad Q_3 \equiv Q_B(A_1, A_2) \]
the \( B \)-quadrea of \( \overline{A_1A_2A_3} \) is
\[ A_B(\overline{A_1A_2A_3}) \equiv A(Q_1, Q_2, Q_3). \]

By the definition of the \( B \)-quadrance, this is also equal to \( A_B(\overrightarrow{v_12}, \overrightarrow{v_31}) \) and to \( A_B(\overrightarrow{v_13}, \overrightarrow{v_21}) \). So, the \( B \)-quadrea of a triangle is simply the \( B \)-quadrea of either of its two associated vector triangles.

The following result extends the Quadrea theorem in [20] from the vector triangle setting to the affine triangle setting. As the only variant to the result is the \( B \)-quadrea of the affine triangle, we omit the proof.

**Theorem 8 (Quadrea theorem)** For a triangle \( \overline{A_1A_2A_3} \) in \( \mathbb{A}^3 \) with \( v_{ij} \equiv \overrightarrow{A_iA_j} \) for any integer \( i \) and \( j \) between 1 and 3, we have
\[
\frac{\det B}{4} A_B(\overline{A_1A_2A_3}) = Q_B(v_{12} \times_B v_{31}) = Q_B(v_{12} \times_B v_{23}) = Q_B(v_{23} \times_B v_{31})
= Q_B(v_{21} \times_B v_{13}) = Q_B(v_{21} \times_B v_{32}) = Q_B(v_{32} \times_B v_{13}).
\]

The \( B \)-quadume of a tetrahedron \( \overline{A_0A_1A_2A_3} \) is
\[ V_B(\overline{A_0A_1A_2A_3}) \equiv 4 \frac{\det B}{\det \overline{A_0A_1A_2A_3}} \left[ \overrightarrow{A_0A_1}, \overrightarrow{A_0A_2}, \overrightarrow{A_0A_3} \right]_B^2. \]

By the linearity of the \( B \)-scalar triple product, this will be unchanged if we base the vectors at another point, for example
\[
\left[ \overrightarrow{A_0A_1}, \overrightarrow{A_0A_2}, \overrightarrow{A_0A_3} \right]_B^2 = \left[ \overrightarrow{A_1A_0}, \overrightarrow{A_1A_2}, \overrightarrow{A_1A_3} \right]_B^2.
\]

The following result ensues.

**Theorem 9 (Quadreme product theorem)** If \( M \equiv \begin{pmatrix} \overrightarrow{A_0A_1} \\ \overrightarrow{A_0A_2} \\ \overrightarrow{A_0A_3} \end{pmatrix} \) then
\[ V_B(\overline{A_0A_1A_2A_3}) = 4 \det (MBM^T) = 4 \left( \det B \right) \left( \det MM^T \right) = 4 \left( \det B \right) \left( \det M \right)^2. \]

**Proof.** The first expression is immediate from the Scalar triple product theorem, and the others are just rewrites using the multiplicative property of the determinant. ■

The \( B \)-quadreme is expressed in terms of the \( B \)-quadrances as follows.
Theorem 10 (Quadrume theorem) For a tetrahedron $A_0A_1A_2A_3$ in $\mathbb{R}^3$, define $Q_{ij} \equiv Q_B(A_i, A_j)$, for integers $i$ and $j$ satisfying $0 \leq i < j \leq 3$. The $B$-quadrume of the tetrahedron $A_0A_1A_2A_3$ satisfies

$$V_B(A_0A_1A_2A_3) = \frac{1}{2} \begin{vmatrix} 2Q_{01} & Q_{01} + Q_{02} - Q_{12} & Q_{01} + Q_{03} - Q_{13} \\ Q_{01} + Q_{02} - Q_{12} & 2Q_{02} & Q_{02} + Q_{03} - Q_{23} \\ Q_{01} + Q_{03} - Q_{13} & Q_{02} + Q_{03} - Q_{23} & 2Q_{03} \end{vmatrix}.$$

Proof. From the Quadrume product theorem,

$$V_B(A_0A_1A_2A_3) = 4 \det ( MBM^T )$$

$$= 4 \begin{vmatrix} v_1 \cdot B v_1 & v_1 \cdot B v_2 & v_1 \cdot B v_3 \\ v_1 \cdot B v_2 & v_2 \cdot B v_2 & v_2 \cdot B v_3 \\ v_1 \cdot B v_3 & v_2 \cdot B v_3 & v_3 \cdot B v_3 \end{vmatrix}.$$

By the definition of the $B$-quadratic form and the polarisation formula, this becomes

$$V_B(A_0A_1A_2A_3) = \frac{1}{2} \begin{vmatrix} 2Q_{01} & Q_{01} + Q_{02} - Q_{12} & Q_{01} + Q_{03} - Q_{13} \\ Q_{01} + Q_{02} - Q_{12} & 2Q_{02} & Q_{02} + Q_{03} - Q_{23} \\ Q_{01} + Q_{03} - Q_{13} & Q_{02} + Q_{03} - Q_{23} & 2Q_{03} \end{vmatrix}$$

as required. $\blacksquare$

The determinant present in the definition is called the Cayley-Menger determinant (see [4], [10], pp. 285-289 and [24], pp. 124-126) and forms a general framework for calculating higher-dimensional trigonometric quantities of the "distance" flavour. While named after Cayley and Menger, this formula was known to Euler and dates back to work of Tartaglia.

Given two lines $l_1$ and $l_2$ in $\mathbb{R}^3$ with respective direction vectors $v_1$ and $v_2$, we define the $B$-spread between them as

$$s_B(l_1, l_2) \equiv 1 - \frac{(v_1 \cdot B v_2)^2}{Q_B(v_1)Q_B(v_2)}.$$

Lagrange’s identity allows us to rewrite this as

$$s_B(l_1, l_2) = \frac{Q_B(v_1 \times_B v_2)}{( \det B ) Q_B(v_1)Q_B(v_2)}.$$

The following result, originally from [27], p. 82] and proven with $B$-vector products in [20], computes the $B$-quadra of a triangle in terms of its $B$-quadrances and $B$-spreads. We state it without proof here.

Theorem 11 (Quadrea spread theorem) For a triangle $A_1A_2A_3$ with $B$-quadrances

$$Q_1 \equiv Q_B(A_2, A_3) \quad Q_2 \equiv Q_B(A_1, A_3) \quad \text{and} \quad Q_3 \equiv Q_B(A_1, A_2)$$

as well as $B$-spreads

$$s_1 \equiv s_B(A_1A_2, A_1A_3) \quad s_2 \equiv s_B(A_1A_2, A_2A_3) \quad \text{and} \quad s_3 \equiv s_B(A_1A_3, A_2A_3)$$

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and $B$-quadrea $A \equiv A_B (A_1 A_2 A_3)$, we have that

$$A = 4Q_1 Q_2 s_3 = 4Q_1 Q_3 s_2 = 4Q_2 Q_3 s_1.$$ 

Given two planes $\Pi_1$ and $\Pi_2$ in $\mathbb{A}^3$ with $B$-normal vectors $n_1$ and $n_2$ respectively, we define the $B$-dihedral spread between them to be

$$E_B (\Pi_1, \Pi_2) \equiv 1 - \left( \frac{(n_1 \cdot B n_2)^2}{Q_B (n_1) Q_B (n_2)} \right).$$

This is clearly independent of the rescaling of normal vectors. Note the similarities between the definition of the $B$-spread and the $B$-dihedral spread; this is a central theme in projective rational trigonometry (see [7] and [28]). The $B$-dihedral spread can also be rewritten using Lagrange’s identity as

$$E_B (\Pi_1, \Pi_2) = \frac{Q_B (n_1 \times B n_2)}{(\det B) Q_B (n_1) Q_B (n_2)}. $$

The $B$-dihedral spread satisfies the following property.

**Theorem 12 (Dihedral spread theorem)** Let $\Pi_1$ be a plane with spanning vectors $v$ and $w_1$, and $\Pi_2$ be a plane with spanning vectors $v$ and $w_2$, so that these two planes meet at a line with direction vector $v$. Then,

$$E_B (\Pi_1, \Pi_2) = \frac{(\det B) [v, w_1, w_2]^2 Q_B (v)}{Q_B (v \times B w_1) Q_B (v \times B w_2)}. $$

**Proof.** We use the rearrangement of the definition of the $B$-dihedral spread using Lagrange’s identity to write

$$E_B (\Pi_1, \Pi_2) = \frac{Q_B ((v \times B w_1) \times B (v \times B w_2))}{(\det B) Q_B (v \times B w_1) Q_B (v \times B w_2)}. $$

By Corollary 6,

$$E_B (\Pi_1, \Pi_2) = \frac{Q_B \left( \left[ (\det B)^2 (\det M) \right] v \right)}{(\det B) Q_B (v \times B w_1) Q_B (v \times B w_2)} = \frac{(\det B)^3 (\det M)^2 Q_B (v)}{Q_B (v \times B w_1) Q_B (v \times B w_2)},$$

where $M \equiv \begin{pmatrix} v \\ w_1 \\ w_2 \end{pmatrix}$. By the Quadrume product theorem,

$$E_B (\Pi_1, \Pi_2) = \frac{(\det B)^3 (\det M)^2 Q_B (v)}{Q_B (v \times B w_1) Q_B (v \times B w_2)} = \frac{(\det B) [v, w_1, w_2]^2 Q_B (v)}{Q_B (v \times B w_1) Q_B (v \times B w_2)}$$

as required. ■

Take three concurrent lines $l_1$, $l_2$ and $l_3$ in $\mathbb{A}^3$ with respective direction vectors $v_1$, $v_2$ and $v_3$. We
define the $B$-solid spread between them as

$$s_B (l_1, l_2, l_3) = \frac{([v_1, v_2, v_3]_B)^2}{(\det B) Q_B (v_1) Q_B (v_2) Q_B (v_3)}.$$

The $B$-solid spread satisfies the following identity.

**Theorem 13 (Solid spread theorem)** Suppose three lines $l_1$, $l_2$ and $l_3$ in $A^3$ meet at a single point $O$ with respective direction vectors $v_1$, $v_2$ and $v_3$. Furthermore, define the planes

$$\Pi_{12} \equiv (O, v_1, v_2), \quad \Pi_{13} \equiv (O, v_1, v_3) \quad \text{and} \quad \Pi_{23} \equiv (O, v_2, v_3).$$

Then,

$$s_B (l_1, l_2, l_3) = E_B (\Pi_{12}, \Pi_{13}) s_B (l_1, l_2) s_B (l_1, l_3) = E_B (\Pi_{12}, \Pi_{23}) s_B (l_1, l_2) s_B (l_2, l_3) = E_B (\Pi_{13}, \Pi_{23}) s_B (l_1, l_3) s_B (l_2, l_3).$$

**Proof.** By rewriting the definition of the $B$-spread using Lagrange’s identity, we have

$$s_B (l_1, l_2) = \frac{Q_B (v_1 \times_B v_2)}{(\det B) Q_B (v_1) Q_B (v_2)} \quad \text{and} \quad s_B (l_1, l_3) = \frac{Q_B (v_1 \times_B v_3)}{(\det B) Q_B (v_1) Q_B (v_3)}.$$

Given that

$$E_B (\Pi_{12}, \Pi_{13}) = \frac{(\det B) [v_1, v_2, v_3]_B Q_B (v_1)}{Q_B (v_1 \times_B v_2) Q_B (v_1 \times_B v_3)},$$

compute the product of the above three quantities to get our desired result. The other results follow by symmetry. $\blacksquare$

Given three concurrent lines $l_1$, $l_2$ and $l_3$ in $A^3$, we construct three lines $k_{12}$, $k_{13}$ and $k_{23}$ with respective direction vectors

$$n_{12} \equiv v_1 \times_B v_2, \quad n_{13} \equiv v_1 \times_B v_3 \quad \text{and} \quad n_{23} \equiv v_2 \times_B v_3$$

so that all six lines are concurrent and $k_{12}$ is $B$-perpendicular to $l_1$ and $l_2$, $k_{13}$ is $B$-perpendicular to $l_1$ and $l_3$, and $k_{23}$ is $B$-perpendicular to $l_2$ and $l_3$. We then define the $B$-dual solid spread between lines $l_1$, $l_2$ and $l_3$ to be

$$D_B (l_1, l_2, l_3) \equiv S_B (k_{12}, k_{13}, k_{23}).$$

We now present an analog to the Solid spread theorem for $B$-dual solid spreads.

**Theorem 14 (Dual solid spread theorem)** Suppose three lines $l_1$, $l_2$ and $l_3$ in $A^3$ meet at a single point $O$ with respective direction vectors $v_1$, $v_2$ and $v_3$. Furthermore, define the planes

$$\Pi_{12} \equiv (O, v_1, v_2), \quad \Pi_{13} \equiv (O, v_1, v_3) \quad \text{and} \quad \Pi_{23} \equiv (O, v_2, v_3).$$
Then,
\[
D_B(l_1, l_2, l_3) = s_B(l_1, l_2) E_B(\Pi_{12}, \Pi_{13}) E_B(\Pi_{12}, \Pi_{23}) \\
= s_B(l_1, l_3) E_B(\Pi_{12}, \Pi_{13}) E_B(\Pi_{13}, \Pi_{23}) \\
= s_B(l_2, l_3) E_B(\Pi_{12}, \Pi_{23}) E_B(\Pi_{13}, \Pi_{23}).
\]

**Proof.** First we construct three lines \(k_{12}, k_{13}\) and \(k_{23}\) with respective direction vectors

\[ n_{12} \equiv v_1 \times_B v_2, \quad n_{13} \equiv v_1 \times_B v_3 \quad \text{and} \quad n_{23} \equiv v_2 \times_B v_3 \]

so that these three lines are concurrent to \(l_1, l_2\) and \(l_3\). By the Scalar triple product of products theorem, we know that

\[ [n_{12}, n_{13}, n_{23}]_B = (\det B) ([v_1, v_2, v_3]_B)^2 \]

so that

\[
D_B(l_1, l_2, l_3) = \frac{([n_{12}, n_{13}, n_{23}]_B)^2}{(\det B) Q_B(n_{12}) Q_B(n_{13}) Q_B(n_{23})} \\
= \frac{(\det B) [v_1, v_2, v_3]_B^4}{Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3) Q_B(v_2 \times_B v_3)}.
\]

Now,

\[ E_B(\Pi_{12}, \Pi_{13}) = \frac{(\det B) [v_1, v_2, v_3]_B^2 Q_B(v_1)}{Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3)}, \quad E_B(\Pi_{12}, \Pi_{23}) = \frac{(\det B) [v_1, v_2, v_3]_B^2 Q_B(v_2)}{Q_B(v_1 \times_B v_2) Q_B(v_2 \times_B v_3)} \]

and, by Lagrange’s identity,

\[ s_B(l_1, l_2) = \frac{Q_B(v_1 \times_B v_2)}{(\det B) Q_B(v_1) Q_B(v_2)}. \]

Compute the product of the above three quantities to get our desired result. The other results follow by symmetry. ■

### 5 Rational trigonometry of a general tetrahedron

In what follows, we consider a tetrahedron \(\Delta_0 A_1 A_2 A_3\) in \(\mathbb{A}^3\). For integers \(i\) and \(j\) satisfying \(0 \leq i < j \leq 3\), the \(B\)-quadrances between any two points \(A_i\) and \(A_j\) of \(\Delta_0 A_1 A_2 A_3\) will be denoted by \(Q_{ij}\); the \(B\)-quadrea associated to the triangle \(\Delta_0 A_i A_j\) of \(\Delta_0 A_1 A_2 A_3\) will be denoted by \(A_{ijk}\), for integers \(i, j\) and \(k\) satisfying \(0 \leq i < j < k \leq 3\); and its \(B\)-quadrum will be denoted by \(V\).

The \(B\)-spreads between two lines \(A_i A_j\) and \(A_i A_k\) of \(\Delta_0 A_1 A_2 A_3\) will be denoted by \(s_{i,j,k}\), for an integer \(i\) satisfying \(0 \leq i \leq 3\) and integers \(j\) and \(k\) distinct from \(i\) satisfying \(0 \leq j, k \leq 3\), and the \(B\)-dihedral spreads between two planes \(A_i A_j A_k\) and \(A_i A_j A_l\), for integers \(i\) and \(j\) satisfying \(0 \leq i < j \leq 3\) and distinct integers \(k\) and \(l\) between \(0\) and \(3\) which are also distinct from \(i\) and \(j\), will be denoted by \(E_{ij}\).

Finally, the \(B\)-solid spreads and \(B\)-dual solid spreads between the three lines \(A_i A_j, A_i A_k\) and \(A_i A_l\)
will be denoted respectively by $S_i$ and $D_i$, for an integer $i$ satisfying $0 \leq i \leq 3$, and distinct integers $j$, $k$ and $l$ between 0 and 3 which are also distinct from $i$.

The Quadrea theorem and Quadrume theorem gives us expressions for $A_{012}$, $A_{013}$, $A_{023}$ and $V$ in terms of the six $B$-quadrances of $A_0A_1A_2A_3$. In terms of the $B$-quadrances and $B$-spreads of $A_0A_1A_2A_3$, we use the Quadrea spread theorem to express the $B$-quadreas as

\[
A_{012} = 4Q_{01}Q_{02}s_{0;12} = 4Q_{01}Q_{12}s_{1;02} = 4Q_{02}Q_{12}s_{2;01}
\]

\[
A_{013} = 4Q_{01}Q_{03}s_{0;13} = 4Q_{01}Q_{13}s_{1;03} = 4Q_{03}Q_{13}s_{3;01}
\]

\[
A_{023} = 4Q_{02}Q_{03}s_{0;23} = 4Q_{02}Q_{23}s_{2;03} = 4Q_{03}Q_{23}s_{3;02}
\]

and

\[
A_{123} = 4Q_{12}Q_{13}s_{1;23} = 4Q_{12}Q_{23}s_{2;13} = 4Q_{13}Q_{23}s_{3;12}.
\]

5.1 The Alternating spreads theorem

The following gives relations between face spreads of a tetrahedron.

**Theorem 15 (Alternating spreads theorem)** For a tetrahedron $A_0A_1A_2A_3$ with $B$-spreads $s_{i;jk}$, for $i$, $j$ and $k$ distinct integers with $0 \leq j < k \leq 3$, we have

\[
s_{1;02}s_{2;03}s_{3;01} = s_{1;03}s_{2;01}s_{3;02}.
\]

**Proof.** From the Quadrea spread theorem, we know that

\[
A_{012} = 4Q_{01}Q_{12}s_{1;02} = 4Q_{02}Q_{12}s_{2;01}
\]

so that

\[
\frac{s_{1;02}}{s_{2;01}} = \frac{Q_{02}}{Q_{01}}.
\]

This is also a direct consequence of the Spread law in the triangle $A_0A_1A_2$. Similarly we have the relations

\[
\frac{s_{2;03}}{s_{3;02}} = \frac{Q_{03}}{Q_{02}} \quad \text{and} \quad \frac{s_{3;01}}{s_{1;03}} = \frac{Q_{01}}{Q_{03}}.
\]

The required result follows by taking the product of these three relations and cancelling all of the quadrances.

Note that all the $B$-spreads in the formula involve the index 0 on the right hand side; so, including this relation, three other relations hold which will correspond to the other points of the tetrahedron.

5.2 Results for $B$-dihedral spreads

The following result establishes a formula for the $B$-dihedral spread of a tetrahedron in terms of its $B$-quadrances, $B$-quadreas and $B$-quadreme.

**Theorem 16 (Tetrahedron dihedral spread formula)** For a tetrahedron $A_0A_1A_2A_3$ with $B$-quadrances $Q_{ij}$ for $0 \leq i < j \leq 3$, $B$-quadreas $A_{012}$, $A_{013}$, $A_{023}$ and $A_{123}$, and $B$-quadreme $V$, the $B$-dihedral
spread $E_{01}$ can be expressed as
\[ E_{01} = \frac{4Q_{01}V}{A_{012}A_{013}}. \]

**Proof.** Let
\[ v_1 \equiv A_0A_1, \quad v_2 \equiv A_0A_2 \quad \text{and} \quad v_3 \equiv A_0A_3. \]

By the Dihedral spread theorem
\[ E_{01} = \frac{(\det B) [v_1, v_2, v_3]^2_B Q_B (v_1)}{Q_B (v_1 \times_B v_2) Q_B (v_1 \times_B v_3)}. \]

By the Quadrea theorem
\[ Q_B (v_1 \times_B v_2) = \frac{\det B}{4} A_B (A_0A_1A_2) = \frac{\det B}{4} A_{012} \]
and
\[ Q_B (v_1 \times_B v_3) = \frac{\det B}{4} A_B (A_0A_1A_3) = \frac{\det B}{4} A_{013}. \]

Given that
\[ V = \frac{4}{\det B} [v_1, v_2, v_3]^2_B \]
we combine the above results to get
\[ E_{01} = \frac{4Q_{01}V}{A_{012}A_{013}} \]
as required. ■

Similarly we have that
\[ E_{02} = \frac{4Q_{02}V}{A_{012}A_{023}}, \quad E_{03} = \frac{4Q_{03}V}{A_{013}A_{023}}, \quad E_{12} = \frac{4Q_{12}V}{A_{012}A_{123}}, \quad E_{13} = \frac{4Q_{13}V}{A_{013}A_{123}} \quad \text{and} \quad E_{23} = \frac{4Q_{23}V}{A_{023}A_{123}}. \]

The following result, a rational version of a result from [21], allows us to form a relationship between the products of opposite $B$-dihedral spreads and the products of opposite $B$-quadreces. For somewhat mysterious reasons, the quantity
\[ R \equiv \frac{16V^2}{A_{012}A_{013}A_{023}A_{123}} \]
is of significance in the study of the rational trigonometry of a tetrahedron. Unfortunately we do not currently have a good geometric interpretation of this quantity, although the two-dimensional analog is the quadratic curvature of the circumcircle of a triangle, and the following theorem does provide a partial answer. The quantity $R$ is the rational equivalent of a quantity denoted by $h^2$ in [21], and hence we will call it the Richardson constant.

**Theorem 17 (Dihedral spread ratio theorem)** For a tetrahedron $A_0A_1A_2A_3$ with $B$-quadreces $Q_{ij}$ for $0 \leq i < j \leq 3$, $B$-quadreces $A_{012}$, $A_{013}$, $A_{023}$ and $A_{123}$, $B$-quadreces $V$ and $B$-dihedral spreads $E_{ij}$ for $0 \leq i < j \leq 3$, we have
\[ \frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = R. \]
Proof. From the equivalent formulation of the Dihedral spread theorem for $A_0A_1A_2A_3$, we have

\[ E_{01}E_{23} = \frac{4Q_{01}V}{A_{012}A_{013} A_{023}A_{123}} = RQ_{01}Q_{23} \]

and similarly

\[ E_{02}E_{13} = RQ_{02}Q_{13} \quad \text{and} \quad E_{03}E_{12} = RQ_{03}Q_{12} \]

Divide each result through by $Q_{01}Q_{23}$, $Q_{02}Q_{13}$ and $Q_{03}Q_{12}$ respectively to obtain our desired result.

5.3 Results for $B$-solid spreads

We now present a formula for calculating the $B$-solid spreads of a tetrahedron in terms of its $B$-quadrances and $B$-quadrume.

Theorem 18 (Tetrahedron solid spread formula) For a tetrahedron $A_0A_1A_2A_3$ with $B$-quadrances $Q_{ij}$ for $0 \leq i < j \leq 3$ and $B$-quadrume $V$, the $B$-solid spread $S_0$ can be expressed as

\[ S_0 = \frac{V}{4Q_{01}Q_{02}Q_{03}}. \]

Proof. Defining $v_1 \equiv \overrightarrow{A_0A_1}$, $v_2 \equiv \overrightarrow{A_0A_2}$ and $v_3 \equiv \overrightarrow{A_0A_3}$, the definition of the $B$-solid spread

\[ S_0 = \frac{([v_1, v_2, v_3]_B)^2}{(\det B)Q_B (v_1) Q_B (v_2) Q_B (v_3)}, \]

can be rewritten using the formula for the quadrume $V$ as

\[ S_0 = \frac{V}{4Q_{01}Q_{02}Q_{03}} \]

which is our desired result.

Similarly, we have

\[ S_1 = \frac{V}{4Q_{01}Q_{12}Q_{13}}, \quad S_2 = \frac{V}{4Q_{02}Q_{12}Q_{23}} \quad \text{and} \quad S_3 = \frac{V}{4Q_{03}Q_{13}Q_{23}}. \]

We present an interesting result regarding the ratio of $B$-solid spreads.

Theorem 19 (First solid spread ratio theorem) For a tetrahedron $A_0A_1A_2A_3$ with $B$-quadrances $Q_{ij}$ for $0 \leq i < j \leq 3$, $B$-quadrume $V$, and $B$-solid spreads $S_k$ for $0 \leq k \leq 3$, we have

\[ \frac{S_0}{S_1} = \frac{Q_{12}Q_{13}}{Q_{02}Q_{03}}. \]

Proof. This is an immediate consequence of the Tetrahedron solid spread formula, as

\[ \frac{S_0}{S_1} = \left( \frac{V}{4Q_{01}Q_{02}Q_{03}} \right) \div \left( \frac{V}{4Q_{01}Q_{12}Q_{13}} \right) = \frac{Q_{12}Q_{13}}{Q_{02}Q_{03}}. \]
Similar formulas hold also for other ratios, for example

\[
\frac{S_0}{S_2} = \frac{Q_{12}Q_{23}}{Q_{01}Q_{03}} \quad \text{and} \quad \frac{S_1}{S_3} = \frac{Q_{03}Q_{23}}{Q_{01}Q_{12}} \quad \text{etc.}
\]

**Theorem 20 (Second solid spread ratio theorem)** For a tetrahedron \( \overline{A_0A_1A_2A_3} \) with \( B \)-quadrances \( Q_{ij} \) for \( 0 \leq i < j \leq 3 \), \( B \)-quadrume \( \mathcal{V} \), and \( B \)-solid spreads \( S_k \) for \( 0 \leq k \leq 3 \) we have

\[
\frac{S_0S_1}{S_2S_3} = \frac{Q_{23}^2}{Q_{01}^2}.
\]

**Proof.** This is an immediate consequence of the First solid spread ratio theorem, as

\[
\frac{S_0S_1}{S_2S_3} = \left( \frac{S_0}{S_2} \right) \left( \frac{S_1}{S_3} \right) = \left( \frac{Q_{12}Q_{23}}{Q_{01}Q_{03}} \right) \left( \frac{Q_{03}Q_{23}}{Q_{01}Q_{12}} \right) = \frac{Q_{23}^2}{Q_{01}^2}.
\]

Similarly, we will have

\[
\frac{S_0S_2}{S_1S_3} = \frac{Q_{14}^2}{Q_{02}^2} \quad \text{and} \quad \frac{S_0S_3}{S_1S_2} = \frac{Q_{12}^2}{Q_{03}^2}.
\]

We may also derive a result pertaining to the ratio of the product of three \( B \)-solid spreads to the product of three \( B \)-quadrances; this is a new result that is unique to this paper, which can only be understood by using the framework of rational trigonometry.

**Theorem 21 (Third solid spread ratio theorem)** For a tetrahedron \( \overline{A_0A_1A_2A_3} \) with \( B \)-quadrances \( A_{012}, A_{013}, A_{023} \) and \( A_{123} \), \( B \)-quadrume \( \mathcal{V} \), \( B \)-solid spreads \( S_0, S_1, S_2 \) and \( S_3 \), we have

\[
\frac{S_0S_1S_2}{Q_{03}Q_{13}Q_{23}} = \frac{S_0S_1S_3}{Q_{02}Q_{12}Q_{23}} = \frac{S_0S_2S_3}{Q_{01}Q_{12}Q_{13}} = \frac{S_1S_2S_3}{Q_{01}Q_{02}Q_{03}} = \frac{\mathcal{V}^3}{64Q_{01}^2Q_{02}^2Q_{03}^2Q_{12}^2Q_{13}^2Q_{23}^2}.
\]

**Proof.** By the equivalent formulation of the Solid spread theorem for \( \overline{A_0A_1A_2A_3} \), we have

\[
S_0S_1S_2 = \frac{\mathcal{V}}{4Q_{01}Q_{02}Q_{03}} \quad \text{and} \quad S_0S_1S_3 = \frac{\mathcal{V}}{4Q_{02}Q_{12}Q_{13}} \quad \text{and} \quad S_0S_2S_3 = \frac{\mathcal{V}}{4Q_{01}Q_{03}Q_{12}} \quad \text{and} \quad \mathcal{V}^3 = 64Q_{01}^2Q_{02}^2Q_{03}^2Q_{12}^2Q_{13}^2Q_{23}^2.
\]

Divide both sides by \( Q_{03}Q_{13}Q_{23} \) to get our desired result. The other results follow by symmetry.

**5.4 Results for \( B \)-dual solid spreads**

We now present a formula for the \( B \)-dual solid spread of a tetrahedron in terms of its \( B \)-quadreas and \( B \)-quadrume.

**Theorem 22 (Tetrahedron dual solid spread formula)** For a tetrahedron \( \overline{A_0A_1A_2A_3} \) with \( B \)-quadreas \( A_{012}, A_{013}, A_{023} \) and \( A_{123} \), \( B \)-quadrume \( \mathcal{V} \), the \( B \)-dual solid spread \( D_0 \) can be expressed as

\[
D_0 = \frac{4\mathcal{V}^2}{A_{012}A_{013}A_{023}}.
\]
**Proof.** Using the vectors \( \mathbf{v}_1 \equiv \overrightarrow{A_0 A_1}, \mathbf{v}_2 \equiv \overrightarrow{A_0 A_2} \) and \( \mathbf{v}_3 \equiv \overrightarrow{A_0 A_3} \), as we saw in the proof of the Dual solid spread theorem,

\[
D_0 = \frac{(\det B) [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]_B^4}{Q_B(\mathbf{v}_1 \times_B \mathbf{v}_2) Q_B(\mathbf{v}_1 \times_B \mathbf{v}_3) Q_B(\mathbf{v}_2 \times_B \mathbf{v}_3)}.
\]

But we know that

\[
\nu = \nu_B(\overrightarrow{A_0 A_1 A_2 A_3}) \equiv \frac{4}{\det B} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]_B^2.
\]

and that

\[
Q_B(\mathbf{v}_1 \times_B \mathbf{v}_2) = \frac{\det B}{4} A_{012},
\]

\[
Q_B(\mathbf{v}_1 \times_B \mathbf{v}_3) = \frac{\det B}{4} A_{013},
\]

\[
Q_B(\mathbf{v}_2 \times_B \mathbf{v}_3) = \frac{\det B}{4} A_{023}
\]

so that substituting we get

\[
D_0 = \frac{(\det B) (\frac{\det B}{4})^2 \nu^2}{4 \nu^2 A_{012} A_{013} A_{023}} = \frac{4 \nu^2}{A_{012} A_{013} A_{023}}.
\]

Similarly we have

\[
D_1 = \frac{4 \nu^2}{A_{012} A_{013} A_{123}}, \quad D_2 = \frac{4 \nu^2}{A_{012} A_{023} A_{123}}, \quad D_3 = \frac{4 \nu^2}{A_{013} A_{023} A_{123}}.
\]

The following result outlines the ratio of \( B \)-dual solid spreads to \( B \)-quadreas, one that is a rational analog of another classical result from [21]: it acts similarly to the Sine law in classical trigonometry, albeit in a very different context. The Richardson constant \( R \) will be present here as well.

**Theorem 23 (Dual solid spread and quadrea ratio theorem)** For a tetrahedron \( \overrightarrow{A_0 A_1 A_2 A_3} \) with \( B \)-quadreas \( A_{012}, A_{013}, A_{023} \) and \( A_{123} \), \( B \)-quadreme \( \nu \), \( B \)-dual solid spreads \( D_0, D_1, D_2 \) and \( D_3 \), and Richardson constant \( R \), we have

\[
\frac{D_0}{A_{123}} = \frac{D_1}{A_{023}} = \frac{D_2}{A_{013}} = \frac{D_3}{A_{012}} = \frac{R}{4}.
\]

**Proof.** By the equivalent formulation of the Dual solid spread theorem for \( \overrightarrow{A_0 A_1 A_2 A_3} \), we have

\[
D_0 = \frac{4 \nu^2}{A_{012} A_{013} A_{023}}.
\]

Divide through by \( A_{123} \) to get

\[
\frac{D_0}{A_{123}} = \frac{4 \nu^2}{A_{012} A_{013} A_{023} A_{123}} = \frac{R}{4}.
\]
The other results follow by symmetry. ■

5.5 Tetrahedron skew quadrance formula

The familiar formula for the projection of one vector onto another ([3, p. 206] and [25, p. 174]) holds also for more general bilinear forms; we define the B-projection of a vector $v$ in the direction of the vector $u$ as the vector

$$(\text{proj}_u v)_B \equiv \frac{u \cdot B v}{Q_B (u)} u.$$

For a tetrahedron $\overrightarrow{A_0A_1A_2A_3}$ with all the above quantities defined, Hilbert and Cohn-Vossen [16 pp. 13-17] established in the Euclidean case that the pairs of lines $(A_0A_1, A_2A_3)$, $(A_0A_2, A_1A_3)$ and $(A_0A_3, A_1A_2)$ of $\overrightarrow{A_0A_1A_2A_3}$ are skew, i.e. their meets do not exist. We now define

$$n_{01;23} \equiv \overrightarrow{A_0A_1} \times_B \overrightarrow{A_2A_3}, \quad n_{02;13} \equiv \overrightarrow{A_0A_2} \times_B \overrightarrow{A_1A_3} \quad \text{and} \quad n_{03;12} \equiv \overrightarrow{A_0A_3} \times_B \overrightarrow{A_1A_2}$$

so that we define

$$R_{01;23} \equiv Q_B \left( \text{proj}_{n_{01;23}} \overrightarrow{P_0P_{23}} \right), \quad R_{02;13} \equiv Q_B \left( \text{proj}_{n_{02;13}} \overrightarrow{P_0P_{13}} \right) \quad \text{and} \quad R_{03;12} \equiv Q_B \left( \text{proj}_{n_{03;12}} \overrightarrow{P_0P_{12}} \right)$$

to be the skew B-quadrances of $\overrightarrow{A_0A_1A_2A_3}$ associated to the respective pairs of opposing lines $(A_0A_1, A_2A_3)$, $(A_0A_2, A_1A_3)$ and $(A_0A_3, A_1A_2)$, where $P_{ij}$ is an arbitrary point on the line $A_iA_j$ for integers $i$ and $j$ satisfying $0 \leq i < j \leq 3$. This quantity is independent of the selection of the $P_{ij}$’s, since if the two lines don’t meet, then the points on the line will lie on separate planes which are parallel.

We establish a formula for the skew-B-quadrances of a tetrahedron based on its B-quadrances and B-quadrume. We use [23] as inspiration to prove this result in our framework.

**Theorem 24 (Tetrahedron skew quadrance formula)**: For a tetrahedron $\overrightarrow{A_0A_1A_2A_3}$ with B-quadrances $Q_{ij}$, B-quadrume $\mathcal{V}$, and skew B-quadrances $R_{01;23}$, $R_{02;13}$ and $R_{03;12}$, we have

$$R_{01;23} = \frac{\mathcal{V}}{4Q_{01}Q_{23} - (Q_{02} + Q_{13} - Q_{03} - Q_{12})^2}$$

$$R_{02;13} = \frac{\mathcal{V}}{4Q_{02}Q_{13} - (Q_{01} + Q_{23} - Q_{03} - Q_{12})^2}$$

and

$$R_{03;12} = \frac{\mathcal{V}}{4Q_{02}Q_{13} - (Q_{01} + Q_{23} - Q_{03} - Q_{12})^2}.$$

**Proof.** For integers $i$ satisfying $1 \leq i \leq 3$, define vectors $v_i \equiv \overrightarrow{A_0A_i}$ so that we may define $n_{01;23} \equiv v_1 \times_B (v_3 - v_2)$. By the definition of skew B-quadrances, we have

$$R_{01;23} = Q_B \left( \text{proj}_{n_{01;23}} \overrightarrow{A_0A_2} \right) = Q_B \left( \frac{n_{01;23} \cdot_B v_2}{Q_B (n_{01;23})} n_{01;23} \right)$$

$$= \frac{\left[ (v_1 \times_B (v_3 - v_2)) \cdot_B v_2 \right]^2}{Q_B (v_1 \times_B (v_3 - v_2))} = \frac{\left[ v_1, v_3 - v_2, v_2 \right]_B^2}{Q_B (v_1 \times_B (v_3 - v_2))}. $$

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Define \( M \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \) so we may use the bilinearity properties of the \( B \)-scalar and \( B \)-vector products, as well as the Scalar triple product theorem, to rewrite the numerator as

\[
[v_1, v_3 - v_2, v_2]_B^2 = ( [v_1, v_3, v_2]_B - [v_1, v_2, v_2]_B )^2 = [v_1, v_2, v_3]_B = (\det M \det B)^2.
\]

Furthermore, the denominator becomes

\[
Q_B (v_1 \times_B (v_3 - v_2)) = \det B \left( Q_01 Q_23 - [(v_1 \cdot_B v_3) - (v_1 \cdot_B v_2)]^2 \right)
\]

by Lagrange’s identity. Use the polarisation formula to obtain

\[
Q_B (v_1 \times_B (v_3 - v_2)) = \det B \left( Q_{01} Q_{23} - \left( \frac{Q_{02} + Q_{13} - Q_{03} - Q_{12}}{4} \right)^2 \right)
\]

Combine the results for the numerator and denominator with the Quadrume product theorem to get

\[
R_{01;23} = \frac{4 (\det M \det B)^2}{(\det B) \left( 4Q_{01} Q_{23} - (Q_{02} + Q_{13} - Q_{03} - Q_{12})^2 \right)}
\]

as required. The other results follow by symmetry. □

It is curious to note that the denominator of the Tetrahedron skew quadrance formula is a rational form of Bretschneider’s formula \([6]\) for the quadrea of a general quadrangle (a collection of four coplanar points) in terms of the six quadrances between any two of its points (see \([7, 11] \) and \([17, pp. 204-205]\)).

6 Tri-rectangular tetrahedron

To finish we apply the framework devised in this paper to study a particularly fundamental type of tetrahedron, which is the analog of a right triangle in the three-dimensional setting. Just as many problems in metrical planar geometry can be resolved into right triangles, and in spherical or elliptic geometry Napier’s rules highlight the importance of right spherical or elliptic triangles, so the tri-rectangular tetrahedron plays a special role in three-dimensional geometry.
We set \( \overrightarrow{A_0A_1A_2A_3} \) to be a tetrahedron in \( \mathbb{A}^3 \) with all its trigonometric invariants denoted as above. Introducing 
\[
\mathbf{v}_1 \equiv \overrightarrow{A_0A_1}, \quad \mathbf{v}_2 \equiv \overrightarrow{A_0A_2} \quad \text{and} \quad \mathbf{v}_3 \equiv \overrightarrow{A_0A_3}
\]
we define \( \overrightarrow{A_0A_1A_2A_3} \) to be \( \mathbb{B} \)-tri-rectangular \( [2, \text{pp. } 91-94] \) at the point \( A_0 \) precisely when \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \) are all mutually \( \mathbb{B} \)-perpendicular, that is when
\[
\mathbf{v}_1 \cdot \mathbb{B} \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbb{B} \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbb{B} \mathbf{v}_3 = 0.
\]

While we can also similarly define a \( \mathbb{B} \)-tri-rectangular tetrahedron at another point of \( \overrightarrow{A_0A_1A_2A_3} \), we may suppose for the purposes of this study, and without loss of generality, that the tetrahedron \( \overrightarrow{A_0A_1A_2A_3} \) is \( \mathbb{B} \)-tri-rectangular at \( A_0 \).

Then by the definition of the \( \mathbb{B} \)-spread we have
\[
s_{0;12} = s_{0;13} = s_{0;23} = 1.
\]
Furthermore, since \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \) are all mutually \( \mathbb{B} \)-perpendicular we deduce that
\[
E_{01} = E_{02} = E_{03} = 1.
\]
By the Solid spread projective theorem, we then obtain \( S_0 = 1 \).

Because of this, it is natural to parametrize a \( \mathbb{B} \)-tri-rectangular tetrahedron \( \overrightarrow{A_0A_1A_2A_3} \) by the quadrances
\[
Q_{01} \equiv K_1, \quad Q_{02} \equiv K_2 \quad \text{and} \quad Q_{03} \equiv K_3.
\]
These quantities represent the \( \mathbb{B} \)-quadrances of \( \overrightarrow{A_0A_1A_2A_3} \) emanating from the point \( A_0 \). Doing this, we use Pythagoras' theorem (see \([15] \) and \([20] \)) to obtain
\[
Q_{12} = K_1 + K_2, \quad Q_{13} = K_1 + K_3 \quad \text{and} \quad Q_{23} = K_2 + K_3.
\]
By the Quadrume theorem, the \( \mathbb{B} \)-quadrume of \( \overrightarrow{A_0A_1A_2A_3} \) is
\[
\gamma = \frac{1}{2} \begin{vmatrix} 2K_1 & 0 & 0 \\ 0 & 2K_2 & 0 \\ 0 & 0 & 2K_3 \end{vmatrix} = 4K_1K_2K_3.
\]
Then by the Quadrea spread theorem
\[
A_{012} = 4Q_{01}Q_{02} = 4K_1K_2
\]
\[
A_{013} = 4Q_{01}Q_{03} = 4K_1K_3
\]
and
\[
A_{023} = 4Q_{02}Q_{03} = 4K_2K_3
\]
are three of its \( \mathbb{B} \)-quadreas. To obtain \( A_{123} \), we rely on the following generalization of a classical
result from [8], which provides a parallel to Pythagoras’ theorem for $B$-quadras of a $B$-tri-rectangular tetrahedron.

**Theorem 25 (de Gua’s theorem)** For a $B$-tri-rectangular tetrahedron $A_0A_1A_2A_3$ at $A_0$, we have that

$$A_{123} = A_{012} + A_{013} + A_{023}.$$  

**Proof.** By the definition of the $B$-quadra,

$$A_{123} = A(Q_{12}, Q_{13}, Q_{23})$$

$$= (2(K_1 + K_2 + K_3))^2 - 2 \left( (K_1 + K_2)^2 + (K_1 + K_3)^2 + (K_2 + K_3)^2 \right)$$

$$= 4(K_1^2 + K_2^2 + K_3^2 + 2K_1K_2 + 2K_1K_3 + 2K_2K_3) - 4(K_1^2 + K_2^2 + K_3^2 + K_1K_2 + K_1K_3 + K_2K_3)$$

$$= 4K_1K_2 + 4K_1K_3 + 4K_2K_3$$

as required. ■

The result of the Quadrea spread theorem can be rearranged to obtain the remaining $B$-spreads, which are

$$s_{1;02} = \frac{S_3K_1K_2}{K_1(K_1 + K_2)}, \quad s_{1;03} = \frac{K_1K_3}{K_1(K_1 + K_3)}, \quad s_{1;23} = \frac{K_1K_2 + K_1K_3 + K_2K_3}{(K_1 + K_2)(K_1 + K_3)},$$

$$s_{2;01} = \frac{K_2K_3}{K_2(K_1 + K_2)}, \quad s_{2;03} = \frac{K_2K_3}{K_2(K_2 + K_3)}, \quad s_{2;13} = \frac{K_1K_2 + K_1K_3 + K_2K_3}{(K_1 + K_2)(K_2 + K_3)},$$

$$s_{3;01} = \frac{K_1K_3}{K_3(K_1 + K_3)}, \quad s_{3;02} = \frac{K_3K_3}{K_3(K_2 + K_3)} \text{ and } s_{3;12} = \frac{K_1K_2 + K_1K_3 + K_2K_3}{(K_1 + K_3)(K_2 + K_3)}.$$  

By the Tetrahedron dihedral spread formula, the remaining $B$-dihedral spreads are

$$E_{12} = \frac{K_3(K_1 + K_2)}{K_1K_2 + K_1K_3 + K_2K_3}, \quad E_{13} = \frac{K_2(K_1 + K_3)}{K_1K_2 + K_1K_3 + K_2K_3} \text{ and } E_{23} = \frac{K_1(K_2 + K_3)}{K_1K_2 + K_1K_3 + K_2K_3}.$$  

The following elegant relation between $B$-dihedral spreads then becomes visible.

**Theorem 26 (Tri-rectangular dihedral spread theorem)** For a $B$-tri-rectangular tetrahedron $A_0A_1A_2A_3$ at $A_0$, we have that

$$E_{12} + E_{13} + E_{23} = 2.$$  

**Proof.** Use the above quantities to immediately obtain our result, as follows:

$$E_{12} + E_{13} + E_{23} = \frac{K_3(K_1 + K_2)}{K_1K_2 + K_1K_3 + K_2K_3} + \frac{K_2(K_1 + K_3)}{K_1K_2 + K_1K_3 + K_2K_3} + \frac{K_1(K_2 + K_3)}{K_1K_2 + K_1K_3 + K_2K_3} = 2.$$  

■

By the Tetrahedron solid spread formula, the remaining $B$-solid spreads are

$$S_1 = \frac{K_2K_3}{(K_1 + K_2)(K_1 + K_3)}, \quad S_2 = \frac{K_1K_3}{(K_1 + K_2)(K_2 + K_3)} \text{ and } S_3 = \frac{K_1K_2}{(K_1 + K_3)(K_2 + K_3)}.$$
Recall that another version of Pythagoras’ theorem in the plane is that if \( \overline{A_0A_1A_2} \) has a right angle at \( A_0 \) then
\[
s_B(\overline{A_0A_1A_2}) = s_B(\overline{A_0A_2A_1}) + s_B(\overline{A_0A_2A_1}) = 1.
\]

Here is a three-dimensional extension of this involving solid spreads.

**Theorem 27 (Tri-rectangular solid spread theorem)** For a \( B \)-tri-rectangular tetrahedron \( \overline{A_0A_1A_2A_3} \) at \( A_0 \), we have that
\[
(1 - S_1 - S_2 - S_3)^2 = 4S_1S_2S_3.
\]

**Proof.** With the values of \( S_1, S_2 \) and \( S_3 \) above, we have
\[
1 - S_1 + S_2 + S_3 = 1 - \frac{K_2K_3}{(K_1 + K_2)(K_1 + K_3)} - \frac{K_1K_3}{(K_1 + K_2)(K_2 + K_3)} - \frac{K_1K_2}{(K_1 + K_3)(K_2 + K_3)}
\]
so that
\[
(1 - S_1 - S_2 - S_3)^2 = \frac{4K_1^2K_2^2K_3^2}{(K_1 + K_2)^2(K_1 + K_3)^2(K_2 + K_3)^2} = 4S_1S_2S_3
\]
as required. ■

It appears interesting to ask if this result extends in some fashion to more general tetrahedra. As for the dual solid spreads, the value at \( A_0 \) is
\[
D_0 = 1
\]
because the normals to the lines meeting there are the lines themselves. Note that this is consistent with
\[
D_0 = \frac{4V^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}} = \frac{4(4K_1K_2K_3)^2}{(4K_1K_2)(4K_1K_3)(4K_2K_3)} = 1
\]
which uses the Tetrahedron dual solid spread formula. Using this same formula, we also get
\[
D_1 = \frac{4V^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{123}} = \frac{4(4K_1K_2K_3)^2}{K_2K_3(K_1K_2 + 4K_1K_3 + 4K_2K_3)}
\]
and similarly
\[
D_2 = \frac{K_1K_3}{K_1K_2 + K_1K_3 + K_2K_3} \quad \text{and} \quad D_3 = \frac{K_1K_2}{K_1K_2 + K_1K_3 + K_2K_3}.
\]

The following result pertaining to \( D_1, D_2 \) and \( D_3 \) then follows.

**Theorem 28 (Tri-rectangular dual solid spread)** For a \( B \)-tri-rectangular tetrahedron \( \overline{A_0A_1A_2A_3} \) at \( A_0 \), we have that
\[
D_1 + D_2 + D_3 = 1.
\]
Proof. With the values of $D_1$, $D_2$ and $D_3$ above, we compute that

$$D_1 + D_2 + D_3 = \frac{K_2K_3}{K_1K_2 + K_1K_3 + K_2K_3} + \frac{K_1K_3}{K_1K_2 + K_1K_3 + K_2K_3} + \frac{K_1K_2}{K_1K_2 + K_1K_3 + K_2K_3} = 1.$$ 


7 Further directions

There is clearly a big step in going from the two-dimensional to the three-dimensional situation in trigonometry. One of the reasons is simply that the number of objects can increased considerably; instead of just three points, three lines and a triangle, we have four points, six lines, four faces, and a tetrahedron, and so the range of metrical notions must also expand to include the various configurations that are possible when we combine these in various ways.

So when we contemplate higher dimensional trigonometry, the situation will become much more involved even when we restrict to the case of the simplex, and will also require the addition of higher dimensional invariants. We can expect algebraic relations from the very simple, as in the previous case of the tri-rectangular tetrahedron, to the enormously complicated and intricate. We are really only at the beginning of a comprehensive understanding of the geometry of space.

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