Discrete \(\frac{G'}{G}\)-expansion: A method used to get exact solution of FDDE (fractional differential-difference equation) linked with NLTL (non-linear transmission line)

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Abstract—Here, we have used the discrete \(\frac{G'}{G}\)-expansion procedure with the derivative operator MR-L (modified Riemann-Liouville) and FCT (fractional complex transform) to find the exact/analytical solution of an electrical transmission line which is non-linear. Results include solutions for integer and fractional DDE. We consider two special cases of solutions: hyperbolic and trigonometric. Hyperbolic solutions indicate propagation of singular wave on the transmission line. Trigonometric solutions show propagation of complex wave.

Keywords—Differential-difference equations (DDEs), Discrete \(\frac{G'}{G}\)-expansion method, Fractional complex transform (FCT), Fractional differential-difference equation (FDDE), Fractional differential equation (FDE), Non-linear transmission line (NLTL), Ordinary differential equation (ODE), Partial fractional differential equation (PFDE).

I. INTRODUCTION

Fractional differential equations are the subset of fractional calculus in mathematics that expands regular distinction to a complicated or non-integer order [1]. This has major influence on the various subjects, such as quantum mechanics, physics, control systems, non-linear analysis in applied mathematics and other areas. In physics and engineering, many equations [2-3] have recently been expanded into non-integer orders in order to provide new models. They are given with some computational, numerical and some solution methods since solving an FDE is difficult. Similarly DDE finds applications in areas like engineering physics, applied mathematics. Different applications of DDE require specific effective method [4-7] for solutions. Some interesting DDE model can be found in literature [8-10].

Over the years, researchers are using different analysis of transmission lines [11-12] and different numerical methods to find exact/analytical solutions of DDEs linked with non-linear waves and solitons [13-15]. In [13], exact solutions for the DDEs linked with the NLTL network uses the \(\frac{G'}{G}\)-expansion process where as [14] uses fractional analysis of the same network with discrete tanh method. Similarly [15] reports a different electrical transmission line circuit employing the discrete \(\frac{G'}{G}\)-expansion process using fractional analysis.

This paper extends the application of the discrete \(\frac{G'}{G}\)-expansion procedure to find an exact solution for local FDDE (fractional differential-difference equation) of Toda network. For simplicity, it will first transform the FDE to ODE using the fractional complex transform (FCT) method and then develop the solution method.

Organisation of this paper will be the following. Section II will provide the general expansion of the above mentioned method. Section III will discuss the fractional complex transform and the procedures for transforming the fractional differential equation (FDE) into an ordinary differential equation (ODE). Section IV will explain how to derive the solution from transmission line network using the solution process. It will be followed by the evolutionary profile in section V.

II. GENERAL \(\frac{\psi'(z)}{\psi(z)}\)-EXPANSION METHOD

Fractional complex transform (FCT) is used as a gentle way of transforming FDE to ODE. FCT can be used to rewrite
the equations in form of ODE in order to get the solution of FDE.

For a function, 
\[ \xi = \xi(t, x_1, x_2, \ldots x_n). \]

Let 
\[ U(\xi) = \sum_{i=0}^{n} a_i \left[ \frac{G'(\xi)}{G(\xi)} \right]^i \]
be a solution of certain ODE, where \( G(\xi) \) satisfies
\[ G''(\xi) + \lambda \ G'(\xi) + \mu \ G(\xi) = 0. \] (1)

Here, \( G'(\xi) = \frac{dG(\xi)}{d\xi} \)
and by using homogenous balancing principal \( n \) can be determined.

Now for equation (1), the auxiliary equation (AE) can be written as,
\[ m^2 + \lambda \ m + \mu = 0 \]
which is satisfied by 
\[ m = \frac{-\lambda \pm \sqrt{\lambda^2 - 4 \mu}}{2}. \]

Then the solutions of equation (1) are
\[ G(\xi) = e^{-\frac{\lambda \xi}{2}} \left( c_1 \cosh \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) + c_2 \sinh \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) \right), \]
for \( \lambda^2 - 4 \mu > 0 \)
\[ e^{-\frac{\lambda \xi}{2}} \left( c_1 \cos \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) + c_2 \sin \left( \frac{\sqrt{\lambda^2 + 4 \mu}}{2} \xi \right) \right), \]
for \( \lambda^2 - 4 \mu < 0 \)
\[ e^{-\frac{\lambda \xi}{2}} (c_1 + c_2 \xi), \]
for \( \lambda^2 - 4 \mu = 0 \)
with arbitrary constants \( c_2 \) and \( c_1 \).

Hence \( G'(\xi) \) can be summarized as,
\[ \left\{ \begin{array}{ll}
\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4 \mu}}{2} & \left( c_2 \sin \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) + c_2 \cos \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) \right), \\
\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4 \mu}}{2} & \left( c_2 \sin \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) + c_2 \cos \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \xi \right) \right), \\
\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 + 4 \mu}}{2} & \left( c_2 \sin \left( \frac{\sqrt{\lambda^2 + 4 \mu}}{2} \xi \right) + c_2 \cos \left( \frac{\sqrt{\lambda^2 + 4 \mu}}{2} \xi \right) \right), \\
\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 + 4 \mu}}{2} & \left( c_2 \sin \left( \frac{\sqrt{\lambda^2 + 4 \mu}}{2} \xi \right) + c_2 \cos \left( \frac{\sqrt{\lambda^2 + 4 \mu}}{2} \xi \right) \right), \\
\frac{-\lambda}{2} + \frac{c_1}{c_1 + c_2 \xi}, & \text{for } \lambda^2 - 4 \mu = 0.
\end{array} \right. \]

III. FRACTIONAL COMPLEX TRANSFORM (FCT)

The key steps to establish the exact PFDE solution by using FCT are provided below. Let’s assume the PFDE
\[ P(u_1, \ldots, u_k, D^\alpha_{\xi} u_1, \ldots, D^\alpha_{\xi} u_k, D^\beta_{\xi} u_1, \ldots, D^\beta_{\xi} u_k) = 0 \] (3)
with variables \( (t, x_1, x_2, \ldots x_n) \).
Where \( P \) is the polynomial function in \( u_i \), fractional derivatives are in terms of different partial derivatives.
And \( u_i = u_i(t, x_1, x_2, \ldots x_n) \) for \( i = 1, 2, \ldots, k \).

For \( i = 1, 2, \ldots, k, j = 1, 2, \ldots, n, \) \( D^\alpha_{\xi} u_i \) as given in (3) is usually called modified Riemann-Liouville derivative of fractional \( (\alpha) \) order of \( u_i(x_j) \) w.r.t \( x_j \).

With \( \alpha \) order, Jumarie’s MR-L derivative is defined as follows:
\[ D^\alpha_{\xi} f(t) = \begin{cases} 
\frac{1}{\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t (t - \xi)^{\alpha-1} (f(t) - f(0)) dt, & 0 < \alpha < 1 \\
(f^{(m)}(t))(\xi^{\alpha-m}), & m \geq 1 
\end{cases} \] (4)
\[ \eta = \xi^\alpha \]
is the modified FCT usually utilised for some FDEs that only applicable for wave solution.

As FCT follows chain rule, applying this to the function \( u(t, x) \):
\[ \frac{\partial^\alpha u_i}{\partial t^\alpha} = \sigma_s \frac{\partial u_i}{\partial s}, \quad s = t^\alpha \]
\[ \frac{\partial^\beta u_i}{\partial x^\beta} = \sigma_x \frac{\partial u_i}{\partial x}, \quad X = x^\beta. \] (5)
Where \( \sigma_s, \sigma_x \) are the fractal indices.

To make this calculation easy, with \( 0 < \alpha \leq 1 \) the properties of MR-L derivative is chosen.
Thus the MR-L property
\[ D^\xi_{\xi} t^s = \frac{\Gamma(1+s)}{\Gamma(1+s-\alpha)} \xi^{s-\alpha} \]
(6)
can also be utilised in modified FCT, \( \eta = \xi^\alpha \).

Procedures to adopt for FDE to ODE transformation is presented below in different steps.

**Step 1:** It is assumed
\[ u_i(t, x_1, x_2, \ldots x_n) = U_i(\xi) \]
with
\[ \xi = e^{c \frac{t^n}{\alpha}} + k_1 \frac{x_{1}^a}{\alpha} + k_2 \frac{x_{2}^a}{\alpha} + \cdots + k_0 \]
(7)
where \( c, k_1, k_2 \) are unevaluated constants.

Using this above mentioned transform along with equation (5), equation (3) can be transformed into ODE as below,
\[ \tilde{P} (U_1, \ldots, U_k, c D^\xi_{\xi} U_1, \ldots, c D^\xi_{\xi} U_k, k_1 D^\xi_{\xi} U_1, \ldots, k_1 D^\xi_{\xi} U_k, \ldots, k_2 D^\xi_{\xi} U_1, \ldots, k_2 D^\xi_{\xi} U_k, \ldots, k_n D^\xi_{\xi} U_1, \ldots, k_n D^\xi_{\xi} U_k, \ldots) = 0 \] (8)
(2)
The general representation of the transformation for \( t = 1,2,\ldots,k \) and \( r = 1,2,\ldots,N \) can be written as,
\[
D_t^{ra} U_i (t, x, x_1, x_2, \ldots, x_n) = c^r D_x^{r_i} U_i (\xi),
\]
\[
D_x^{ra} U_i (t, x, x_1, x_2, \ldots, x_n) = k^r_i D_x^{r_i} U_i (\xi).
\]
Step 2: To express the solution of equation (8) in terms of \( \frac{c'(\xi)}{G(\xi)} \), the polynomial function can be assumed as:
\[
U(\xi) = \sum_{i=0}^{m_j} a_{ij} \left[ \frac{G'(\xi)}{G(\xi)} \right]^i, \quad j = 1,2,\ldots,k
\]
where \( G(\xi) \) is the root of equation (1).
And \( a_{ij} (i = 0,1,\ldots,m_j \text{ and } j = 1,2,\ldots,k) \) is a constant which can be evaluated by homogenous balance between non-linear terms and highest-order derivatives appearing in equation (8) with the condition \( a_{ij} \neq 0 \).

Step 3: After substituting equation (9) into equation (8), and using equation (1), the coefficients of the terms containing different powers of \( \frac{c'(\xi)}{G(\xi)} \) equating to zero a set of algebraic equations in terms of \( \lambda, \mu, \delta \alpha \) are obtained.

Step 4: Substituting the solutions of the set of algebraic equations obtained in step 3 in equation (2), an interesting type of exact solution of equation (3) is expected to be established. Equation (3) should be turned into ODE for \( \alpha = 1 \) value.

IV. FRACTIONAL NON-LINEAR LC LADDER TRANSMISSION LINE EQUATION

![Figure-1. Non-linear TL network](image)

The above figure is a non-linear transmission line with \( n \) identical unit cells. Each cell contains a linear inductors and non-linear capacitors which vary with voltage \( V_n \) applied to the \( n \)th cell. So it can be written as,
\[
C(V_n) \approx C_0 (1 - \alpha V_n + b V_n^2)
\]
Applying KCL in the above network, below DDE is obtained,
\[
L C_0 \frac{d^2 V_n}{dt^2} - a L C_0 \frac{d^2 V_n^2}{dt^2} + b L C_0 \frac{d^2 V_n^3}{dt^2} = V_{n+1} + V_{n-1} - 2 V_n
\]
The constants $a_0, a_1, c, k$ are obtained by solving the equations (17).

Case-1:

For $\lambda^2 - 4 \mu > 0$, $\xi = c \frac{\xi}{\mu} + k_1 \frac{\xi}{\mu} + \xi_0$ and $0 < \xi \leq 1$

$$a_0 = \frac{a(\lambda + \sqrt{\lambda^2 - 4\mu})}{3b\sqrt{\lambda^2 - 4\mu}}, \quad a_1 = \frac{2a}{3b\lambda^2 - 4\mu}$$

$$k = \pm \frac{2a\sqrt{\xi}}{\sqrt{-(2a^2 - 9b)(\lambda^2 - 4\mu)}}, \quad c = \frac{6a\sqrt{\xi}}{(2a^2 - 9b)\sqrt{\lambda^2 - 4\mu}}$$

For which the solution will be,

$$V_n(t, x) = U = a_0 + a_1 \left( \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[ c_1 \sinh \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) + c_2 \cosh \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) \right] \right)$$

Case-2:

For $\lambda^2 - 4 \mu < 0$, $\xi = c \frac{\xi}{\mu} + k_1 \frac{\xi}{\mu} + \xi_0$ and $0 < \xi \leq 1$

$$a_0 = \frac{-a(\lambda + \sqrt{\lambda^2 - 4\mu})}{3b\sqrt{\lambda^2 - 4\mu}}, \quad a_1 = \frac{-2a}{3b\lambda^2 - 4\mu}$$

$$k = \pm \frac{2a\sqrt{\xi}}{\sqrt{-(2a^2 - 9b)(\lambda^2 + 4\mu)}}, \quad c = \frac{6a\sqrt{\xi}}{(2a^2 - 9b)\sqrt{\lambda^2 + 4\mu}}$$

For which the solution will be,

$$V_n(t, x) = U = a_0 + a_1 \left( \frac{-\lambda}{2} + \frac{\sqrt{-\lambda^2 + 4\mu}}{2} \left[ -c_1 \sin \left( \frac{-\lambda^2 + 4\mu}{2} \xi \right) + c_2 \cos \left( \frac{-\lambda^2 + 4\mu}{2} \xi \right) \right] \right)$$

Case-3: As the coefficients $a_0$ and $a_1$ are undefined, $\lambda^2 - 4 \mu = 0$ does not have any solutions.

V. RESULTS AND DISCUSSION

To understand the implications of the equations derived above we consider hyperbolic and trigonometric solutions only. For the third rational case, $a_0$ and $a_1$ become undetermined because their denominator contains ($\lambda^2 - 4 \mu$), which is zero. This makes the solution undetermined and hence we exclude this. For both these (hyperbolic and trigonometric) cases we use $C_1 = 0, C_2 = 1, \alpha = 0.21, b = 0.197, \frac{1}{LC_0} = 3, \delta = 1, \mu = 3$. But we take $\lambda$ as 4 in hyperbolic and 1 in trigonometric case.

(a)

(b)
Figure-2. Evolution of voltage with time $V_n(t, x)$ for case-1 (hyperbolic) at different locations on TL for $\alpha = 1$

Figure-3. Evolution of voltage with time $V_n(t, x)$ for case-1 (hyperbolic) at different locations on TL for $\alpha = 0.5$
Figure 4. Evolution of voltage with time $V_0(t, x)$ for case-2 (trigonometric) at different locations on TL for $\alpha = 1$.
For the trigonometric case with $\alpha = 1$ the evolution of wave is shown in figure 4(a), (b), (c), (d) at same locations as for the hyperbolic case. Unlike in the hyperbolic case this indicates the evolution of a complex wave with time which oscillates aperiodically between positive and negative voltages. The figure 5(a), (b), (c), (d) repeats it for $\alpha = 0.5$. In figure 4 the voltage generation is random in nature. In figure 5 the voltage generation is more ordered and accelerate as one moves along the transmission line.

VI. CONCLUSION

Using appropriate parameters, we were able to obtain three exact solutions as we can see in case 1,2,3 are hyperbolic, trigonometric, and rational respectively, in order to achieve our goal of establishing exact solution of FDDE of the given network. Finally we conclude by stating that, for problem modelling and the resolution of un-differentiable numerical in fractional order time-space, the fractional calculus is an useful tool. Using this tool is has been changed how the voltage evolves with time along a non-linear transmission line. This will find application in non-linear discrete systems like all electrical switches and periodic non-linear structures. In future this model can be expanded to consider coupled non-linear transmission line and planar non-linear TL.

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