Eavesdropper’s Optimal Information in Variations of Bennett-Brassard 1984 Quantum Key Distribution in the Coherent Attacks

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We calculate eavesdropper’s optimal information on raw bits in Bennett-Brassard 1984 quantum key distribution (BB84 QKD) and six-state scheme in coherent attacks, using a formula by Lo and Chau [Science 283, 2050 (1999)] with single photon assumption. We find that eavesdropper’s optimal information in QKD without public announcement of bases [Phys. Lett. A 244(1998), 489] is the same as that of a corresponding QKD with it in the coherent attack. We observe a sum-rule concerning each party’s information.

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I. INTRODUCTION

Information processing with quantum systems is an interesting field both theoretically and practically. It may innovate our fundamental conceptions on our world [1]. And it is superior to its classical counterpart in some cases: computing with quantum bits (qubits) enables factoring large numbers [4], which has remained intractable with classical computers and algorithms. In quantum key distribution (QKD) [4]- [26], it is possible for two legitimate users Alice and Bob to distribute keys with security that quantum mechanical laws afford.

Security of QKD had been (tentatively) accepted on the basis of the no-cloning theorem [27,28] and it might be the first practical quantum information processor [24]. However it is only recently that its unconditional security is proved [22]- [25]. Since the original work [5], more and more sophisticated attacks have been considered: intercept-resend strategy with orthogonal measurement in general bases, attacks with generalized (or positive operator valued) measurements, and the most general coherent (collective or joint) attacks where all qubits are coherently treated as a whole quantum system were considered in Refs. [12], [13]- [18], and [19]- [25], respectively. One of the reasons making the proof complicated is that there is inevitable residual noise in real quantum channel. And the natural noise cannot be discriminated from what Eavesdropper’s (Eve’s) tapping on the channel causes. Thus raw bits must be processed in such a way so that Eve has essentially zero information about the final corrected bits. This might be done either quantum or classical information processing. In the former case, errors are removed by quantum error correcting codes or purification protocol [29]. In the latter case, errors are corrected with certain classical error correcting codes. In particular, in case of Ref. [25] classical error correction codes associated with Calderbank-Shor-Steane [30,31] quantum error correcting codes are used. The security of the method against coherent attacks are proved in Refs. [24,25]. However, before such elegant proofs were given, Eve’s optimal information on raw bits for various attacks were estimated [23,24] for classical privacy amplification, where Eve’s information is deleted. Although it is not rigorously proven such methods are secure against coherent attacks, it seems to be so for almost practical purposes and thus the security of such methods had been widely accepted. However, the estimations have been confined to individual attacks where each qubit is separately treated [12]- [17]. Estimating Eve’s optimal information in coherent attacks was in itself an interesting and unsolved problem until formulas for it are given recently [23,24]. Thus it is worthwhile to have explicit estimation of it. In this paper, we calculate Eve’s optimal information about raw bits in BB84, six-state schemes in the coherent attack using the formula given by Lo and Chau [23], with single photon (or quantum) assumption. Then we consider multiple-basis scheme where a number of bases are adopted: we find Eve’s optimal information in multiple-basis scheme is the same as that of six-state scheme. We consider another variation of BB84 scheme, QKD without public announcement of bases [1]. We argue the formula is also valid for it. We also find Eve’s optimal information in it is the same as that of a corresponding QKD with public announcement of bases. We observe that sum of mutual information between Alice and Bob, and Eve’s information is constant in the case of BB84 scheme.

II. FORMULA FOR EAVESDROPPER’S OPTIMAL INFORMATION

Derivation of the formula for Eve’s optimal information $I_{Eve}$ in Eq. (2.8) is briefly discussed in Ref. [23]. In this section, we give a more detailed derivation of the formula in a self-consistent manner. The entanglement-based
schemes can be reduced to BB84-like scheme. Thus \( I_{Eve} \) in entanglement-based scheme which we calculate is the same as that in BB84-like scheme. First we introduce entanglement-based scheme. With the convention of Refs. \([20,23]\), the Bell basis vectors \( |\Psi^\pm\rangle = (|0\rangle \pm |1\rangle) \) and \( |\Phi^\pm\rangle = (|0\rangle \pm |1\rangle) \) are represented by two classical bits, \([24]\)

\[
\Phi^+ = 00, \quad |\Psi^+\rangle = 01, \quad |\Phi^-\rangle = 10, \quad |\Psi^-\rangle = 11. \tag{2.1}
\]

Eve is supposed to prepare a state \( |\Psi^-\rangle \otimes |\Psi^-\rangle \otimes \cdots \otimes |\Psi^-\rangle \), if she does it honestly. But she may prepare other state most generally

\[
|u\rangle = \sum_{i_1, i_2, \ldots, i_N} \sum_j \alpha_{i_1, i_2, \ldots, i_N, j} |i_1, i_2, \ldots, i_N\rangle \otimes |j\rangle. \tag{2.2}
\]

where \( i_k \) denotes the state of the \( k \)-th pair, which runs from \( 00 \) to \( 11 \), \( \alpha_{i_1, i_2, \ldots, i_N, j}^s \) are some complex coefficients, and the \( |j\rangle \) values form an orthonormal basis for the ancilla. Eve gives this state to Alice and Bob, the two legitimate participants who will secretly exchange messages. On each particle, they independently and randomly performs measurements among \( S_z \) (orthogonal measurement composed of two projection operators \(|0\rangle \langle 0 | \) and \(|1\rangle \langle 1 | \)), \( S_x \) (that of \(|0\rangle \langle 0 | \) and \(|1\rangle \langle 1 | \)), and \( S_y \) (that of \(|0\rangle \langle 0 | \) and \(|1\rangle \langle 1 | \)), where \(|0\rangle = |0 \rangle + |1 \rangle, |1\rangle = |0 \rangle - |1 \rangle \) and \(|0\rangle = |0 \rangle + i|1 \rangle, |1\rangle = |0 \rangle - i|1 \rangle \). Then Alice and Bob compare their bases by public discussion and they discard their data of the case where the bases are not matched. Then Alice and Bob publicly announce some randomly chosen subsets of remaining data. They count the number \( N_{\text{para}} \) of the case where the results are the same and the number \( N_{\text{anti}} \) of the case where the results are opposite. Alice and Bob calculate Eve’s optimal information \( I_{Eve} \) about their results as a function of error rate \( D = N_{\text{para}}/(N_{\text{para}} + N_{\text{anti}}) \). When \( D \) is too high, they abort the protocol and restart it. Otherwise, they process the raw bits (the data) into final key about which Eve has essentially zero information. This completes a description on entanglement-based scheme.

Now, let us consider Eve’s optimal information. Roughly speaking, the higher the error rate \( D \) becomes, the more states other than \( |\Psi^-\rangle \otimes |\Psi^-\rangle \otimes \cdots \otimes |\Psi^-\rangle \) are contained in the state of Eq. \((2.2)\). Thus the entropy of reduced density operator of Alice and Bob’s qubits become higher and Eve can extract more information on the bits (see Eq. \((2.7)\)). More precisely, first note the following equivalences.

\[
\begin{align*}
|00\rangle \langle 00 | + |11\rangle \langle 11 | &= |\Phi^+\rangle \langle \Phi^+ | + |\Phi^-\rangle \langle \Phi^- |, \\
|00\rangle \langle 00 | + |11\rangle \langle 11 | &= |\Phi^+\rangle \langle \Phi^+ | + |\Phi^-\rangle \langle \Phi^- |, \\
|\tilde{0}\tilde{0}\rangle \langle \tilde{0}\tilde{0} | + |\tilde{1}\tilde{1}\rangle \langle \tilde{1}\tilde{1} | &= |\Phi^-\rangle \langle \Phi^- | + |\Phi^+\rangle \langle \Phi^+ |.
\end{align*}
\tag{2.3}
\]

This means that the error rate \( D \) that Alice and Bob estimate from their measurements on qubits in \( z, x \), and \( y \) basis are the same as they would have estimated using the Bell basis measurement \([23]\). So we may estimate \( D \) using the Bell basis measurement. Let us consider the state. Assume that Eve had performed Bell basis measurement on all qubits in the state and then sent them to Alice and Bob. Then Alice and Bob perform Bell basis measurement on some subsets of the qubits, according to the scheme. After Eve did the pre-measurement, the state reduces to a mixed state

\[
\rho = \sum_{i_1, i_2, \ldots, i_N} P_{i_1, i_2, \ldots, i_N} |i_1, i_2, \ldots, i_N\rangle \langle i_1, i_2, \ldots, i_N |, \tag{2.4}
\]

where

\[
P_{i_1, i_2, \ldots, i_N} = \sum_j |\alpha_{i_1, i_2, \ldots, i_N, j}|^2. \tag{2.5}
\]

However, in this case Eve’s and [Alice+Bob]’s measurements have common eigenvectors (the Bell basis), and thus Eve’s pre-measurement do not change statistics of [Alice+Bob]’s later measurement. So we may do our estimation of \( D \) for the mixed state in Eq. \((2.4)\) instead of Eq. \((2.2)\). Then \( D \) for the state is given by the following.

\[
D = \sum_{i_1, i_2, \ldots, i_N} P_{i_1, i_2, \ldots, i_N} D(i_1, i_2, \ldots, i_N), \tag{2.6}
\]

\(^1\)In this paper obvious normalization constants are omitted. It is noted that one should never think of the Bell basis vectors as direct product state since they are maximally entangled.

\(^2\)One should not regard Eve as doing the measurement in the real protocol. It is here a hypothetical one for making the estimation easier. Alice and Bob’s measurement is real and thus perturbs the state. However, it does not matter here because their measurement is local process which does not increase entanglement between them and Eve.
where \( D(i_1, i_2, \cdots, i_N) \) is error rate that the state \(|i_1, i_2, \cdots, i_N\rangle\) induces. \( D(i_1, i_2, \cdots, i_N) \) depends on the way of checking errors in schemes and will be calculated in next section. Using Eq. (2.6), we can calculate expected error rate \( D \) for a state with a certain \( \alpha_{i_1, i_2, \cdots, i_N} \)’s.

On the other hand,

\[
I_{Eve} \leq S(\rho_{AB}),
\]

where \( S \) is the von Neumann entropy and \( \rho_{AB} = \text{Tr}_{Eve}|u\rangle\langle u| \) (see Lemma 2 of the supplementary material of Ref. [23]). There are numerous sets of \( \alpha_{i_1, i_2, \cdots, i_N} \) that give rise to a certain error rate \( D \). What Eve has to do is maximizing her information for a certain error rate \( D \). Thus she has to choose one among the sets of \( \alpha_{i_1, i_2, \cdots, i_N} \) which give maximal entropy. By inspection, we can see the maximal entropy is obtained when all \(|i_1, i_2, \cdots, i_N\rangle\) giving the error rate \( D \) are prepared with equal probability \( P_{i_1, i_2, \cdots, i_N} \). Then we have

\[
I_{Eve} \leq - \sum_{i_1, i_2, \cdots, i_N} P_{i_1, i_2, \cdots, i_N} \log P_{i_1, i_2, \cdots, i_N} = \log \Omega,
\]

where \( \Omega \) is the number of distinct \(|i_1, i_2, \cdots, i_N\rangle\)’s giving an error rate \( D \). (In this paper \( \log \equiv \log_2 \).)

III. OPTIMAL INFORMATION IN BB84, SIX-STATE AND MULTIPLE-BASIS SCHEME

First we calculate \( I_{Eve} \) of BB84 scheme: let us calculate \( D(i_1, i_2, \cdots, i_N) \) for the scheme, where Alice and Bob check errors by either \(|00\rangle\langle 00| + |11\rangle\langle 11| \) or \(|00\rangle\langle 00| + |11\rangle\langle 11| \). So probability that \(|\Psi^-\rangle, |\Phi^-\rangle, |\Psi^+\rangle, \) and \(|\Phi^+\rangle\) are detected in error checking are 0, 1/2, 1/2, and 1, respectively, by Eq. (2.3). Thus,

\[
D(i_1, i_2, \cdots, i_N) = \frac{1}{N}(b \frac{a}{2} + c \frac{a}{2} + d),
\]

where \( a, b, c, \) and \( d \) are the number of elements of the set \( A = \{i_k| i_k = 1\bar{1}\}, B = \{i_k| i_k = 1\bar{0}\}, C = \{i_k| i_k = \bar{0}1\}, \) and \( D = \{i_k| i_k = 00\} \), respectively (\( k=1,2,\ldots,N \)). We note that \( \Psi^-\), \( |\Phi^-\rangle\), \( |\Psi^+\rangle\), and \( |\Phi^+\rangle\) are detected with equal probability while she has access to the qubits: if she knows which pairs of qubits will be chosen for estimation of the error rate \( D \), she can cheat by sending \( \Psi^-\) for all the chosen pairs while sending one of the four Bell states for other pairs. In order to give an error rate \( D \),

\[
D(i_1, i_2, \cdots, i_N) = \frac{1}{N}(b \frac{a}{2} + c \frac{a}{2} + d) = D.
\]

The number \( \Omega \) of \( i_1, i_2, \cdots, i_N \)’s that satisfies Eq. (3.2) is given by

\[
\Omega = \sum_{b+c+d=D} \frac{N!}{a! b! c! d!}.
\]

Among many summed terms, \( \Omega \) is dominated by maximal (typical) one. Thus we obtain

\[
\log \Omega = \text{Max} \{-a \log \frac{a}{N} + b \log \frac{b}{N} + c \log \frac{c}{N} + d \log \frac{d}{N}\}. \tag{3.4}
\]

By inspection, we can see that the maximum is obtained when \( b = c \). Then with Eq. (3.2),

\[
\log \Omega = \text{Max} \{(N - 2ND + d) \log \frac{N - 2ND + d}{N} + 2(ND - d) \log \frac{ND - d}{N} + d \log \frac{d}{N}\}. \tag{3.5}
\]

The maximum is obtained when the term’s differential is zero or \( d = ND^2 \).

\[
\log \Omega = -N \{(1 - 2D + D^2) \log(1 - 2D + D^2) + 2(D - D^2) \log(D - D^2) + D^2 \log D^2\}. \tag{3.6}
\]

Before comparing it with \( I_{Eve} \) for incoherent attacks (Eq. (65) of Ref. [17]), our \( I_{Eve} \) should be divided by \( 2N \) since it is the information about \( N \) pairs of particles. Then,

\[
I_{Eve} \leq -\frac{1}{2} \{(1 - 2D + D^2) \log(1 - 2D + D^2) + 2(D - D^2) \log(D - D^2) + D^2 \log D^2\},
\]

\[
= -[D \log D + (1 - D) \log(1 - D)].
\]
Eq. (3.7) is plotted in Fig. 1 among others.

Next we calculate $I_{Eve}$ of the six-state scheme in the same way: in the scheme one of the three measurements in Eq. (2.3) is performed with equal probabilities. So we obtain

$$D(i_1, i_2, \cdots, i_N) = \frac{1}{N} \left( \frac{2}{3} b + \frac{2}{3} c + \frac{2}{3} d \right). \tag{3.8}$$

In order to give an error rate $D$,

$$\frac{1}{N} \left( \frac{2}{3} b + \frac{2}{3} c + \frac{2}{3} d \right) = D. \tag{3.9}$$

The maximum is obtained when $b = c = d = ND/2$. Then,

$$I_{Eve} \leq -\frac{1}{2} \left\{ (1 - \frac{3}{2} D) \log(1 - \frac{3}{2} D) + \frac{3}{2} D \log \frac{D}{2} \right\}. \tag{3.10}$$

As we see in Fig. 1, $I_{Eve}$ of Eq. (3.10) is lower than that of Eq. (3.8), which means that the six-state scheme is more advantageous than the BB84 scheme in the case of coherent attacks, too.

Now we address the multiple-basis scheme. In the scheme many bases are adopted while two and three bases are adopted in the BB84 and six-state scheme, respectively. We assume the bases are uniformly distributed on the Bloch sphere. (Schemes with non-uniform distributions do not seem to be more advantageous than the uniform one.) It is shown in Ref. 10 that the multiple basis scheme does not give more security than the six-state scheme within the individual attack. So we can expect that this is the case in the coherent attack. Here we show that the multiple-basis scheme is indeed no more advantageous than the six-state scheme in the coherent attack: let us compute the average probability $p$ that a Bell state induces parallel result when they are measured in one of the many bases uniformly distributed on the Bloch sphere. We can easily see that $p(|\Psi^\mp \rangle) = 0$ since $|\Psi^\mp \rangle$ induces only anti-parallel results for any basis. We can also see

$$p(|\Phi^\mp \rangle) = \int_0^{\pi} p(|\Phi^\mp \rangle, \theta) d\Omega = \int_0^{\pi} \sin^2 \frac{\theta}{2} e^{-\frac{\pi}{3}} d\theta = \frac{2}{3}, \tag{3.11}$$

where $p(|\Phi^\mp \rangle, \theta)$ is the probability density that $|\Phi^\mp \rangle$ induces parallel results for a measurement along a basis that makes an angle $\theta$ with $z$ axis and $\Omega$ is the solid angle. In a similar way,

$$p(|\Psi^\pm \rangle) = p(|\Phi^\pm \rangle) = \frac{2}{3}. \tag{3.12}$$

Thus for the multiple-basis scheme we have the same equation as Eq. (3.9). Accordingly, $I_{Eve}$ of this scheme is the same as that of six-state scheme. We can also consider a multiple-basis scheme where the bases are uniformly distributed in $z - x$ plane. We can also show in a similar way that this multiple basis scheme in the plane is no more advantageous than the BB84 scheme:

$$p(|\Psi^\pm \rangle) = \frac{1}{2}, \quad p(|\Phi^\pm \rangle) = \frac{2}{3}. \tag{3.13}$$

IV. OPTIMAL INFORMATION IN QKD WITHOUT PUBLIC ANNOUNCEMENT OF BASES

Here we show that $I_{Eve}$ of QKD without public announcement of bases is the same as that of a corresponding one with public announcement of bases. Let us consider a scheme corresponding to BB84. In the scheme, Eve knows which and which qubits are encoded in the same basis while she does not know which basis between $z$ and $x$ they are. In this case the probability that Eve will make a right guess of the encoding bases is still $1/2$, which is the same as that in the case of BB84 scheme. Thus Eq. (3.11) is also valid and later procedures for calculation of $I_{Eve}$ are the same as that of BB84 scheme. So $I_{Eve}$ of QKD without public announcement of bases is the same as that of BB84 scheme. The idea of QKD without public announcement of bases is straightforwardly applied to the six-state scheme. And we can see that $I_{Eve}$ of the six-state QKD without public announcement of bases is the same as that of the six-state scheme with it. However, if $I_{Eve}$ of both schemes are the same, we can say that QKD without public announcement of bases is more advantageous than either BB84 scheme or six-state scheme: while in either BB84 scheme or six-state scheme full information about the encoding bases are given to Eve after the qubits have arrived at Bob, in QKD without public announcement of bases only partial information (which and which are the same basis) about the encoding bases are given to Eve. To summarize this section, these facts suggest that QKD without public announcement of bases is at least as secure as either BB84 or six-state scheme even in coherent attacks.
V. DISCUSSION AND CONCLUSION

It is interesting that the sum of $I_{Eve}$ of BB84 scheme in Eq. (3.7) and

$$I_{AB} = 1 + D \log D + (1 - D) \log(1 - D)$$  \hspace{1cm} (5.1)

is constant. That is,

$$I_{Eve} + I_{AB} = 1.$$  \hspace{1cm} (5.2)

This indicates something is conserved. Roughly speaking, QKD could be interpreted by the quantum information conservation since the total quantum information that Alice have sent is conserved, the more quantum information Eve gets, the less quantum information given to Bob. It should be noted that the results for $I_{Eve}$ in this paper are asymptotically valid in the limit when the number of employed qubits become large.

In conclusion, we have calculated eavesdropper’s optimal information $I_{Eve}$ on raw bits in BB84 and six-state scheme in coherent attacks, using the formula (Eq. (2.8)) by Lo and Chau, assuming single quanums are used. We have shown that $I_{Eve}$ in multiple-basis scheme is the same as that of six-state scheme. We have considered QKD without public announcement of bases: we found that $I_{Eve}$ in it is the same as that of a corresponding QKD with public announcement of bases in the coherent attacks. This fact suggests that QKD without public announcement of bases is as secure as either BB84 or six-state scheme in coherent attacks, too. We observed that $I_{Eve} + I_{AB} = 1$ in the case of BB84 scheme.

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\(^3\) Attempts for establishment of rigorous quantum information conservation theorems can be found in Refs. 32,33.
FIGURE CAPTION:
the solid line (the upper one): $I_{Eve}$ in BB84 scheme, Eq. (3.7)
the dotted line (the middle one): $I_{Eve}$ in six-state scheme, Eq. (3.10)
the dot-dashed line (the lower one): Eq. (65) of Ref. [17]
the dashed line: $I_{AB}$, Eq. (5.1)