POSITIVITY VS. SLOPE SEMISTABILITY FOR BUNDLES WITH VANISHING DISCRIMINANT

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Abstract. It is known that a strongly slope semistable bundle with vanishing discriminant is nef if and only if its determinant is nef. We give an algebraic proof of this result in all characteristics and generalize it to arbitrary proper schemes. We also address a question of S. Misra.

1. Introduction

Let $E$ be a vector bundle of rank $r$ on a smooth projective variety $X$ defined over an algebraically closed field. Inspired by the case of line bundles, one might hope that the positivity of $E$ is determined by the positivity of its characteristic classes. The bundle $O_{P^1}(n) \oplus O_{P^1}(-n)$ is an easy counterexample. To rectify this, one adds stability assumptions on $E$.

Let $H$ be an ample polarization on $X$. In characteristic zero, say that $E$ is strongly slope semistable with respect to $H$ if it is slope semistable in the usual sense. In positive characteristic, say that $E$ is strongly slope semistable with respect to $H$ if $E$ and all its iterated Frobenius pullbacks are slope semistable.

On curves the polarization is irrelevant and we just say that $E$ is strongly semistable. Here the connection between positivity and strong semistability is well known by work of Hartshorne [Har71], Barton [Bar71], and Miyaoka [Miy87]. A strongly semistable bundle $E$ on a curve is ample (or just nef) if and only if $\int_X c_1(E) > 0$ (resp. $\geq 0$). On surfaces, even on $P^2$, it is not sufficient. See Example 2.1. Furthermore, on curves $E$ is strongly semistable if and only if the twisted normalized bundle $E(-\frac{1}{r} \det E)$ is nef (equivalently $E_{nd}E$ is nef).

A link in codimension two between semistability and positivity comes from the famous Bogomolov inequality. The discriminant of $E$ is $\Delta(E) = 2rc_2(E) - (r-1)c_1^2(E)$. Assume that $E$ is strongly slope semistable with respect to some ample polarization. The classical form of the inequality states that if $X$ is a surface, then the degree of the discriminant of $E$ is non-negative. When $X$ has arbitrary dimension, the Mehta–Ramanathan theorem [MRS82] implies that $\Delta(E)$ has nonnegative degree with respect to any polarization.

Note that $\Delta(E) = 0$ on curves. This suggests a close connection between (strong) semistability and positivity for vector bundles that are extremal with respect to the Bogomolov inequality, meaning that $\Delta(E)$ is numerically trivial. We have the following known results:

Theorem 1.1. Let $X$ be a smooth projective variety of dimension $n$, and let $H$ be an ample class on $X$. Let $E$ be a reflexive sheaf of rank $r$ on $X$. The following are equivalent:

1. $E$ is strongly slope semistable with respect to $H$, and $\Delta(E) \cdot H^{n-2} = 0$.
2. $E$ is locally free and $E_{nd}E$ is nef.
3. $E$ is universally semistable (see below).

In particular, if $E$ is strongly slope semistable with respect to $H$, and $\Delta(E) \cdot H^{n-2} = 0$, then $E$ is a nef (resp. ample) vector bundle if and only if $\det E$ is nef (resp. ample).

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In the above theorem we say that $\mathcal{E}$ is \textit{universally semistable} if for every morphism $f : Y \to X$ from any projective manifold $Y$, the pullback $f^*\mathcal{E}$ is slope semistable with respect to any ample polarization on $Y$. It is sufficient to check this condition for morphisms from curves. 

When $\det \mathcal{E}$ is numerically trivial, condition (2) is simply that the vector bundle $\mathcal{E}$ is nef, equivalently $\mathcal{E}$ is \textit{numerically flat} ($\mathcal{E}$ and $\mathcal{E}^\vee$ are nef). Theorem 1.1 takes the following form:

**Theorem 1.2.** Let $X$ be a smooth projective variety of dimension $n$ over an algebraically closed field, and let $H$ be an ample polarization. Let $\mathcal{E}$ be a reflexive sheaf on $X$. The following are equivalent:

1. $\mathcal{E}$ is strongly slope semistable with respect to $H$, and $c_1(\mathcal{E}) \cdot H^{n-1} = c_2(\mathcal{E}) \cdot H^{n-2} = 0$.
2. $\mathcal{E}$ is universally semistable and all the Chern classes $c_i(\mathcal{E})$ are numerically trivial for $i \geq 1$.
3. $\mathcal{E}$ is locally free and numerically flat.

In particular, if $\mathcal{E}$ is a strongly slope semistable with respect to $H$ with $c_1(\mathcal{E})$ numerically trivial, then $\mathcal{E}$ is nef if and only if $\Delta(\mathcal{E})$ is numerically trivial.

The result fails if $\mathcal{E}$ is only assumed to be torsion free. It also fails if the condition $c_2(\mathcal{E}) \cdot H^{n-2} = 0$ in (1) is replaced by $c_2(\mathcal{E}) \cdot H^{n-2} = 0$.

The two theorems are essentially equivalent. As mentioned before, they are not new. In characteristic zero, Theorem 1.1 is proved by Nak99. It is proved by GKP16 for more general polarizations by movable curves on varieties with mild singularities. It is proved by BBB08 for principal $G$-bundles, and by BHR06 as part of their study of Higgs bundles. Note that for Higgs bundles with nontrivial Higgs field it is not known whether an analogue of (3) $\Rightarrow$ (1) holds (this is known as Bruzzo’s conjecture). See BEG19. The proofs given in the references above have important transcendental components coming from Sim92 or DPS94. The last statement of Theorem 1.1 has also been observed by MR21. In positive characteristic, a version of the result is proved algebraically by the second named author in Lan11 making crucial use of the Frobenius morphism. In characteristic zero, Lan19 Corollary 4.10 gives an algebraic proof of the implication (1) $\Rightarrow$ (3) for the more general case of Higgs bundles by reduction to positive characteristic. See also Lan13 Theorem 12. In characteristic zero, Theorem 1.2 is proved in Sim92, Theorem 2. A positive characteristic version appears in Lan11 Proposition 5.1.

A conjecture of Bloch predicts that the Chern classes of numerically flat bundles vanish in $B^*(X) \otimes \mathbb{Q}$, where $B^m(X)$ is the group of codimension $m$ cycles modulo algebraic equivalence (see Lan21 Conjecture 3.2). This would strengthen Theorem 1.2. However, this is known only if $k$ has positive characteristic (see Lan21 Corollary 3.7). In this case the result holds also for general proper schemes (see Corollary 3.10). Analogously, one can formulate a similar conjecture and a result for bundles in Theorem 1.1 (see Lan21 Proposition 3.6). If $k = \mathbb{C}$ one can prove only a weaker version of the vanishing of Chern classes in the rational cohomology $H^{2*}(X^{an}, \mathbb{Q})$, where $X^{an}$ is the complex manifold underlying variety $X$. This result would follow from Bloch’s conjecture by using the cycle map $B^*(X) \otimes \mathbb{Q} \to H^{2*}(X^{an}, \mathbb{Q})$.

1.1. **Main results.** We give algebraic proofs for Theorems 1.1 and 1.2 that are characteristic free. We avoid the application of the non-abelian Hodge theorem Sim92 Corollary 1.3 or the reduction to positive characteristic techniques of Lan11 Lan19. Moreover, we generalize both theorems to general proper schemes over an algebraically closed field (see Theorem 3.13 and Corollary 3.14).

1.2. **On a question of S. Misra.** On curves $C$ it follows from Miy87 that $\mathcal{E}$ is strongly semistable if and only if every effective divisor on $\mathbb{P}_C(\mathcal{E})$ is nef. The first named author proved in Ful11 a generalization of this for cycles of arbitrary (co)dimension in $\mathbb{P}_C(\mathcal{E})$. It is interesting to see to what extent does Miyaoka’s result carry over to a projective manifold $X$ of arbitrary dimension. The equality $\operatorname{Eff}(X) = \operatorname{Nef}(X)$ of cones of divisors is a necessary condition which held trivially on curves. Here one has the following result:
Theorem 1.3. Let $X$ be a smooth projective variety such that $\text{Eff}(X) = \text{Nef}(X)$. Let $\mathcal{E}$ be a strongly slope semistable bundle with respect to some ample polarization of $X$ and assume that $\Delta(\mathcal{E}) \equiv 0$. Then $\text{Eff}(\mathbb{P}(\mathcal{E})) = \text{Nef}(\mathbb{P}(\mathcal{E}))$.

In the characteristic zero case this result was proven by S. Misra in [Mis21, Theorem 1.2] as an application of Theorem 1.1. A similar proof works also in an arbitrary characteristic. Misra [Mis21, Question 3.11] also asks about a possible converse to this result.

Question 1.4. Let $X$ be a smooth projective variety. Let $\mathcal{E}$ be a vector bundle on $X$ such that every effective divisor on $\mathbb{P}(\mathcal{E})$ is nef. Is it true that $\mathcal{E}$ is slope semistable with respect to some (any) polarization of $X$ and $\Delta(\mathcal{E}) \equiv 0$? Or equivalently, is $\text{End} \mathcal{E}$ numerically flat?

For every $n \geq 2$, the tangent bundle $T_{\mathbb{P}^n}$ is a counterexample to the current phrasing of the question. In Example 5.7 we even give a slope unstable counterexample. However Misra also observes that under the hypothesis of Theorem 1.3 the effective and nef cones of divisors coincide on $\mathbb{P}(\text{Sym}^n \mathcal{E})$ for all $m \geq 0$. This motivates the following positive answer to a version of Question 1.4 in arbitrary characteristic.

Theorem 1.5. Let $X$ be a smooth projective variety. Let $\mathcal{E}$ be a vector bundle on $X$ such that every effective divisor on $\mathbb{P}(\text{Sym}^n \mathcal{E})$ is nef for all $m \geq 0$. Then $\mathcal{E}$ is strongly slope semistable with respect to any polarization $H$ and $\Delta(\mathcal{E}) \equiv 0$.

Our proof shows that in fact existence of only one positive even value $2m$ for which the cones $\text{Eff}(\mathbb{P}(\text{Sym}^{2m} \mathcal{E}))$ and $\text{Nef}(\mathbb{P}(\text{Sym}^{2m} \mathcal{E}))$ coincide is sufficient. The key idea is a result on plethysms which guarantees that the line bundle $(\det \mathcal{E})^{\otimes 2m}$ is a subbundle of $\text{Sym}^m \text{Sym}^{2m} \mathcal{E}$. In characteristic zero, this also follows from [BC11]. One can also see that equality $\text{Eff}(\mathbb{P}(\text{End} \mathcal{E})) = \text{Nef}(\mathbb{P}(\text{End} \mathcal{E}))$ implies that $\text{End} \mathcal{E}$ is numerically flat, which provides a satisfactory answer to the original question.

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2. Preliminaries

Let $X$ be a connected proper scheme over an algebraically closed field $k$ and $\mathcal{E}$ be a vector bundle on $X$. Note that terms vector bundle and locally free sheaf are used interchangeably (since $X$ is connected, a locally free sheaf has the same rank at every point of $X$).

2.1. Positive bundles. Let $\mathbb{P}(\mathcal{E}) = \mathbb{P}_X(\mathcal{E}) = \text{Proj}_{\mathcal{O}_X} \text{Sym}^* \mathcal{E}$ with natural bundle map $\pi : \mathbb{P}(\mathcal{E}) \to X$, and let $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. We say that $\mathcal{E}$ is ample (resp. nef) on $X$, if $\xi$ is ample (resp. nef) on $\mathbb{P}(\mathcal{E})$. The definition also makes sense for coherent sheaves.

2.2. Numerically trivial Chern classes. In the following we write $A_*(X)$ for the Chow group of rational equivalence classes on $X$ and $A^*(X) = A^*(X \xrightarrow{id} X)$ for the operational Chow ring, i.e., the group of bivariant rational equivalence classes (see Fulton, Chapter 17). The Chern classes of $\mathcal{E}$ are operations on the Chow group (see Fulton, Chapter 3) so they are elements of $A^*(X)$. Following Fulton, Chapter 19, we say that $c_j(\mathcal{E})$ is numerically trivial if for every proper closed subscheme $Y \subseteq X$ of dimension $j$ we have $\int_X c_j(\mathcal{E}) = \int_X c_j(\mathcal{E}) \cap [Y] = 0$, where $\int_X : A_0(X) \to \mathbb{Z}$ denotes the natural degree map. Then we write $c_j(\mathcal{E}) \equiv 0$. By additivity it is sufficient to check the condition $\int_X c_j(\mathcal{E}) \cap [Y] = 0$ for all $Y$ that are irreducible and reduced. Similarly, we can define numerical triviality for any polynomial in Chern classes of $\mathcal{E}$, or even in Chern classes of finitely many bundles.
2.3. Positive polynomials. Consider \( n \geq 1 \) and grade \( \mathbb{Q}[c_1, \ldots, c_n] \) so that \( \deg c_i = i \). Let \( P(c_1, \ldots, c_n) \) be a weighted homogeneous polynomial of degree \( n \). Fulton and Lazarsfeld proved in [FLS3] that \( \int_X P(E) \geq 0 \) for every \( n \)-dimensional variety \( X \) and every nef vector bundle \( E \) on \( X \) if and only if \( P \) is a linear combination with nonnegative coefficients of Schur polynomials of degree \( n \). We call such polynomials positive. For example, the degree 1 positive polynomials are spanned over \( \mathbb{Q}_{\geq 0} \) by \( c_1 \), while the degree 2 ones are spanned by \( c_2 \) and by \( c_1^2 - c_2 \). In particular, \( c_1^2 = c_2 + c_1^2 - c_2 \) is positive, but \( c_1^2 - 2c_2 \) is not.

Let \( X \) be a smooth projective surface and let \( E \) be a nef vector bundle of rank \( r \) on it. Then \( c_1(E) \) is nef, and \( \int_X c_2(E) \) and \( \int_X (c_1^2(E) - c_2(E)) \) are nonnegative integers. If \( E \) is strongly slope semistable with respect to some polarization, Bogomolov’s inequality gives \( \int_X (2rc_2(E) - (r - 1)c_1^2(E)) \geq 0 \). However, there exist examples of strongly slope semistable vector bundles \( E \) on surfaces such that all the above positivity conditions hold for the characteristic classes of \( E \) without \( E \) being nef.

Example 2.1. Let \( E = O_{\mathbb{P}^2}(-1) \otimes \bigotimes^3(T_{\mathbb{P}^2}(-1)) \). This is strongly slope semistable since \( T_{\mathbb{P}^2} \) is strongly slope semistable. Let \( h = c_1(O_{\mathbb{P}^2}(1)) \). Then \( r = \text{rk } E = 8, c_1(E) = 4h, \) and \( c_2(E) = 16h^2 \).

We compute \( c_1^2(E) - c_2(E) = 0 \). So the characteristic classes of \( E \) suggest that \( E \) might be nef. However, the restriction of \( E \) to every line in \( \mathbb{P}^2 \) is \( O_{\mathbb{P}^1}(-1) \otimes \bigotimes^3(O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}) \) and this has \( O_{\mathbb{P}^1}(-1) \) as a summand, hence it is not nef. ∎

See also [BIHP13] Section 5 for a related example.

Convention 2.2. For the rest of this section we assume that \( X \) is a smooth projective variety of dimension \( n \).

2.4. Positive cones of cycles. Denote by \( N^c(X) \) the space of numerical classes of cycles of codimension \( c \) with real coefficients. For example \( N^1(X) \) is the real Néron–Severi space spanned by Cartier divisors modulo numerical equivalence. The space \( N^c(X) \) is finite dimensional. It contains important convex cones. For instance it contains the pseudoeffective cone \( \text{Eff}^c(X) \), the closure of the cone spanned by classes of closed subsets of codimension \( c \). It also contains the nef cone \( \text{Nef}^c(X) \), the set of classes that intersect every \( c \)-dimensional subvariety nonnegatively, the dual of the pseudoeffective cone in the complementary codimension. The nef cone and the pseudoeffective cone in codimension \( c \) each span \( N^c(X) \) and do not contain linear subspaces of \( N^c(X) \). See [FL17a].

When \( c = 1 \), we put \( \text{Eff}(X) = \text{Eff}^1(X) \) and \( \text{Nef}(X) = \text{Nef}^1(X) \). In this case, \( \text{Nef}(X) \subseteq \text{Eff}(X) \). The interior of the nef cone is the ample cone, and the interior of the pseudoeffective cone is the big cone of numerical classes with positive volume [Laz04a, Chapter 2.2].

2.5. Determinant and discriminant. Let \( F \) be a torsion free sheaf on the smooth projective \( X \). Then the singular (non-locally free) locus of \( F \) has codimension \( \geq 2 \). If \( F \) is reflexive, then the singular locus has codimension \( \geq 3 \).

One can abstractly compute Chern classes of coherent sheaves in \( A^*(X) \) by taking locally free resolutions and using the additivity of the Chern character \( \text{ch} \). The determinant line bundle \( \det F \) of a torsion free sheaf can be defined by extending from the locally free locus, or considering the reflexive hull \( (\bigwedge^{rk} F)^{\vee} \). This gives a concrete definition of \( c_1 \). The second Chern class of a reflexive sheaf can be similarly concretely defined.

Remark 2.3. Let \( f : Y \to X \) be a morphism of nonsingular projective varieties and let \( E \) be a coherent sheaf on \( X \). Then \( \text{ch}(Lf^*E) = f^* \text{ch}(E) \) in the Chow ring of \( Y \), where \( Lf^* \) is the derived pullback. In particular \( c_i(f^*E) = f^*c_i(E) \) for all \( i \) in each of the following cases:

1. \( E \) is locally free.
2. \( f \) is flat.
3. \( f \) is the inclusion of a Cartier divisor and \( E \) is torsion free.
If $i : D \hookrightarrow X$ is the inclusion of a Cartier divisor and $\mathcal{E}$ is reflexive, then $i^* \mathcal{E}$ is torsion free. If $|H|$ is a basepoint free linear system and $\mathcal{E}$ is torsion free (resp. reflexive), then $i^* \mathcal{E}$ is again torsion free (resp. reflexive) for $D$ general in $|H|$. See [HL10] Corollary 1.1.14.

The discriminant of $\mathcal{E}$ is

$$\Delta(\mathcal{E}) = 2rc_2(\mathcal{E}) - (r - 1)c_1^2(\mathcal{E}),$$

where $r = \text{rk} \mathcal{E}$. It is an element of $A^2(X)$, but we use the same notation for its image in $N^2(X)$. The class

$$\log r + \frac{c_1(\mathcal{E})}{r} - \frac{\Delta(\mathcal{E})}{2r^2} + \ldots$$

in $A^*(X)_\mathbb{R}$ is the formal logarithm of the Chern character $\text{ch}(\mathcal{E})$. It follows that $\frac{\Delta(\mathcal{E})}{2\text{rk} \mathcal{E}}$ is additive for tensor products if one of the factors is locally free, just like the slope. In particular, $\Delta(\mathcal{E} \otimes \mathcal{O}_X(D))$ for every divisor $D$ on $X$. Furthermore, note that if $c_1(\mathcal{E}) \equiv 0$, then $\Delta(\mathcal{E}) \equiv 0$ if and only if $c_2(\mathcal{E}) \equiv 0$. The vanishing $\Delta(\mathcal{E}) \equiv 0$ is equivalent to the numerical vanishing of the second Chern class $c_2$ of $\mathcal{E}(-\frac{1}{r} \text{det} \mathcal{E})$, the formal twist of $\mathcal{E}$ in the sense of [Laz04b] Chapter 6.2. If $\mathcal{E}$ is locally free, then $\Delta(\mathcal{E}) \equiv 0$ is also equivalent to the numerical vanishing of $c_2$ of $\mathcal{E}^\otimes r \otimes \text{det} \mathcal{E}^\vee$ or $\text{End} \mathcal{E}$.

2.6. Semistability. Let $H$ be an ample (or just nef) divisor on $X$. For a nonzero torsion free sheaf $\mathcal{E}$ on $X$, we define the slope by $\mu_H(\mathcal{E}) = \frac{c_1(\mathcal{E})H^{n-1}}{\text{rk} \mathcal{E}}$. We say that $\mathcal{E}$ is $\mu_H$-semistable (or slope semistable with respect to $H$) if no proper subsheaf $0 \neq \mathcal{F} \subseteq \mathcal{E}$ verifies $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$. We say that $\mathcal{E}$ is $\mu_H$-stable (or slope stable with respect to $H$) if no proper subsheaf $0 \neq \mathcal{F} \subseteq \mathcal{E}$ with $\text{rk} \mathcal{F} < \text{rk} \mathcal{E}$ has $\mu_H(\mathcal{F}) \geq \mu_H(\mathcal{E})$. When $X$ is a curve and $H$ is ample, slope semistability is independent of $H$. In this case we simply say that $\mathcal{E}$ is semistable. In positive characteristic it is useful to also consider Frobenius pullbacks.

Definition 2.4. A torsion free sheaf $\mathcal{E}$ is called strongly $\mu_H$-(semi)stable if $(F_X^m)^* \mathcal{E}$ is $\mu_H$-(semi)stable for all $m \geq 0$. Here $F_X$ denotes the Frobenius morphism in positive characteristic, and the identity morphism in characteristic zero.

Examples show that these notions are indeed stronger than $\mu_H$-(semi)stability in positive characteristic. The first example showing this is due to J.-P. Serre and it was published in [Gie71]. More precisely, the example shows that there exists a stable bundle of rank 2 and degree 1 on a genus 3 curve in characteristic 3, whose Frobenius pullback splits as direct sum of line bundles of different degrees. Nowadays there are many more examples of stable bundles that are not strongly semistable in an arbitrary positive characteristic. See, e.g., [JP15] Theorem 1.1.3 for recent examples that appear on any smooth projective curve in large characteristic.

2.7. Semistability on curves. In the sequel we will use the following well–known lemma. It goes back to R. Hartshorne [Har70] in the characteristic zero case, with a subsequent algebraic proof in any characteristic due to C. M. Barton [Bar71, Theorem 2.1]. See also [Mor98] Proposition 7.1.

Lemma 2.5. Let $C$ be a smooth projective curve defined over some algebraically closed field $k$ and let $\mathcal{E}$ be a degree 0 vector bundle on $C$. Then $\mathcal{E}$ is strongly semistable if and only if $\mathcal{E}$ is nef.

We will also use the following standard lemma (see, e.g., [LP97] Lemma 7.1.2).

Lemma 2.6. Let $\mathcal{E}$ be a semistable vector bundle on a smooth projective curve $C$ defined over some algebraically closed field $k$. Then

$$h^0(C, \mathcal{E}) \leq \text{rk} \mathcal{E} + \deg \mathcal{E}.$$
3. Numerically flat vector bundles on proper schemes

Let $X$ be a proper scheme over a field $k$.

**Definition 3.1.** A vector bundle $E$ on $X$ is called numerically flat if both $E$ and $E^\vee$ are nef.

**Remark 3.2.** A bundle $E$ is numerically flat if and only if it is nef with $c_1(E) = 0$. Indeed if $E$ and $E^\vee$ are nef, then $c_1(\text{det } E) = c_1(E)$ is numerically trivial. Conversely, if $E$ is nef with $c_1(E) = 0$ then by [Laz04b Theorem 6.2.12] any exterior power of $E$ is nef and hence $E^\vee \cong \det E^\vee \otimes \wedge^{-1} E$ is also nef.

It follows that if $E$ is numerically flat, then all tensor functors of $E$ and their duals are numerically flat, e.g., tensor powers, symmetric powers, divided powers, exterior powers, and all other Schur and co-Schur (Weyl) functors associated to $E$.

3.1. Numerical triviality of Chern classes of numerically flat bundles.

**Proposition 3.3.** Let $X$ be a proper scheme over a field $k$ and let $E$ be a numerically flat vector bundle on $X$. Then $c_j(E)$ is numerically trivial for all $j > 0$.

**Proof.** The proof is by induction on $j$. For $j = 1$ the assertion is clear since by assumption for any proper curve $Y \subseteq X$ we have $\int_X c_1(E) \cap [Y] \geq 0$ and $\int_X c_1(E^\vee) \cap [Y] = -\int_X c_1(E) \cap [Y] \geq 0$.

Let $Y \subseteq X$ be a $j$-dimensional subvariety of $X$. Let $s_j(E^\vee) = \pi_* (\xi^{j+1})$ be the $j$-th Segre class of the dual of $E$. By [Ful98 Chapter 3.2], we have $s_j(E^\vee) = (-1)^{j+1} c_j(E) - \sum_{i=1}^{j-1} (-1)^i s_i(E^\vee) c_{j-i}(E)$. By our induction hypothesis, we deduce $\int_Y s_j(E^\vee) = (-1)^{j+1} \int_Y c_j(E)$, so is suffices to prove that $\int_Y s_j(E^\vee) = 0$. Set $Z := \mathbb{P}(E_Y)$ and $L := \mathcal{O}_{\mathbb{P}(E_Y)}(1)$. Since $E$ is nef, $E_Y$ and $L$ are also nef. By the asymptotic Riemann–Roch theorem (see [Kol96 Chapter VI, Corollary 2.14 and Theorem 2.15]) we have

$$h^0(Z, L \otimes^m) = \chi(Z, L \otimes^m) + O(m^{j+r-2}) = \int_Z c_1(L)^{j+r-1} (j+r-1)! m^{j+r-1} + O(m^{j+r-2}).$$

Since $E_Y$ is numerically flat, $\text{Sym}^m E_Y$ is also numerically flat. Hence for any ample divisor $H$ on $Y$ we have $h^0(Y, (\text{Sym}^m E_Y)(-H)) = 0$ (otherwise $\text{Sym}^m E_Y$ contains $O(H)$ contradicting the nefness of $(\text{Sym}^m E_Y)^\vee$). Again using the asymptotic Riemann–Roch theorem we get

$$h^0(Z, L \otimes^m) = h^0(Y, (\text{Sym}^m E_Y)(-H)) + h^0(H, \text{Sym}^m E_H)$$

$$= h^0(\mathbb{P}(E_H), \mathcal{O}_{\mathbb{P}(E_H)}(m)) = O(m^{j+r-2}).$$

Summing up, we have $\int_Y s_j(E^\vee) = \int_Z c_1(L)^{j+r-1} = 0$ as required. \hfill $\square$

**Remark 3.4.** The idea of proof of vanishing of highest Segre classes comes from the proof of [Ful20 Proposition 5.1], in turn inspired by the proof of the Bogomolov inequality. We avoid using this result and semistability and give a proof working in an arbitrary characteristic. The proof of an analogue of [Ful20 Proposition 5.1] would require small rewriting and the use of deep results of Ramanan and Ramanathan [RR84] on the behaviour of strong slope semistability.

**Remark 3.5.** If $X$ is a smooth complex manifold the above proposition was proven in [DPS94 Corollary 1.19] using earlier deep analytic results. If $X$ is a smooth variety and $k$ has positive characteristic the above proposition follows from [Lan11 Proposition 5.1 and Theorem 4.1]. The proof of [Lan11 Proposition 5.1] cites rather deep results from [FL83] although it uses only a much weaker and easier result of Kleiman [Kle69]. However, it also depends on [Lan04] and the proof above is much more elementary.

**Remark 3.6.** From [BG71] or [PL83], the Chern class $c_k(E)$ is nef if $E$ is a nef bundle. Thus for $E$ numerically flat the classes $c_k(E)$, $c_k(E^\vee)$, and $s_k(E^\vee)$ are nef. Using induction and the fact that Nef$^k(X)$ does not contain linear subspaces (cf. [PL17a]), this gives another argument than the one
Together with the main result of \[\text{Lan04}\] the above proposition implies the following corollary:

**Corollary 3.7.** Let \( f : X \to S \) be a flat projective morphism of noetherian schemes. Then the set of numerically flat vector bundles of fixed rank \( r \) on the fibers of \( f \) is bounded.

**Proof.** Let \( \mathcal{O}_X(1) \) be an \( f \)-very ample line bundle on \( X \) and let \( \mathcal{E} \) be a rank \( r \) numerically flat vector bundle on a geometric fiber \( X_s \) for some geometric point \( s \) of \( S \). The singular Grothendieck–Riemann–Roch theorem (see \[\text{Ful98 Corollary 18.3.1}\]) and Proposition 3.3 imply that

\[
\chi(X_s, \mathcal{E}(m)) = \int_{X_s} \text{ch} (\mathcal{E}(m)) \cap \text{Td}(X_s) = r \int_{X_s} \text{ch} (\mathcal{O}_{X_s}(m)) \cap \text{Td}(X_s) = r \chi(X_s, \mathcal{O}_{X_s}(m)).
\]

Since \( f \) is flat, for every connected component \( S_0 \) of \( S \) the Hilbert polynomial \( P_s(m) = \chi(X_s, \mathcal{O}_{X_s}(m)) \) is independent of the geometric point \( s \) of \( S_0 \). Moreover, any numerically flat vector bundle on \( X \) is slope \( \mathcal{O}_{X_s}(1) \)-semistable (the general definition of slope semistability in case of singular projective schemes can be found in \[\text{HL10 Definition and Corollary 1.6.9}\]). Therefore the required assertion follows from \[\text{Lan04 Theorem 4.4}\]. \( \square \)

**Remark 3.8.** In case \( X \) is a normal variety and \( S = \text{Spec} k \), the above corollary was proved in \[\text{Lan12 Theorem 1.1}\]. If \( S = \text{Spec} k \) and \( k \) is a finite field the above corollary was proved in \[\text{DW20 Theorem 2.4}\].

**Theorem 3.9.** Let \( X \) be a projective scheme over a perfect field \( k \) of positive characteristic. Let \( \mathcal{E} \) be a rank \( r \) vector bundle on \( X \). Then the following conditions are equivalent:

1. \( \mathcal{E} \) is numerically flat.
2. The set \( \{(F^m_X)^* \mathcal{E}\}_{m \in \mathbb{Z}_{\geq 0}} \) is bounded.
3. There exist \( m_1 > m_2 \geq 0 \) such that \((F^{m_1}_X)^* \mathcal{E} \) and \((F^{m_2}_X)^* \mathcal{E} \) are algebraically equivalent.

**Proof.** If \( \mathcal{E} \) is numerically flat then all \( \mathcal{E}_m = (F^m_X)^* \mathcal{E} \) are numerically flat, so (1) \( \Rightarrow \) (2) follows from Corollary 3.7. Assume (2). Then by definition there exists a \( k \)-scheme \( S \) of finite type and an \( S \)-flat coherent sheaf \( \mathcal{F} \) on \( X_S := X \times_k S \) such that for every \( m \in \mathbb{Z} \) there exists a geometric \( k \)-point \( s_m \) in \( S \) such that \( \mathcal{F}_{X_{s_m}} \simeq \mathcal{E}_m \). Now (3) follows by the pigeonhole principle applied to the finitely many connected components of \( S \). Assume (3). Then for all \( m \geq 0 \) the bundles \((F^{m_1+m}_X)^* \mathcal{E} \) and \((F^{m_2+m}_X)^* \mathcal{E} \) are algebraically equivalent. This implies that the family \( \{(F^{m_1+m(m_1-m_2)}_X)^* \mathcal{E}\}_{m \in \mathbb{Z}_{\geq 0}} \) is bounded. The implication (3) \( \Rightarrow \) (1) follows as in the first part of proof of \[\text{Lan11 Proposition 5.1}\]. \( \square \)

The following corollary can be thought of as a generalization of \[\text{DW20 Theorem 2.3}\] from finite fields to arbitrary perfect fields of positive characteristic. In case \( X \) is smooth the result is contained in \[\text{Lan21 Corollary 3.7}\].

**Corollary 3.10.** Let \( X \) be a projective scheme over a perfect field \( k \) of positive characteristic. Let \( \mathcal{E} \) be a numerically flat vector bundle on \( X \). Then for all \( i > 0 \) the Chern classes \( c_i(\mathcal{E}) \) are, up to torsion, algebraically equivalent to 0.

**Proof.** By the above theorem we know that there exist \( m_1 > m_2 \geq 0 \) such that \((F^{m_1}_X)^* \mathcal{E} \) and \((F^{m_2}_X)^* \mathcal{E} \) are algebraically equivalent. Since \( c_i((F^m_X)^* \mathcal{E}) = p^i c_i(\mathcal{E}) \) in \( A^*(X) \), we get

\[
0 = c_i((F^{m_1}_X)^* \mathcal{E}) - c_i((F^{m_2}_X)^* \mathcal{E}) = (p^{im_1} - p^{im_2}) c_i(\mathcal{E})
\]

in \( B^*(X) \), so \( c_i(\mathcal{E}) = 0 \) in \( B^*(X)_\mathbb{Q} \). \( \square \)
3.2. Characterizations of numerically flat bundles. Let $X$ be a proper scheme over an algebraically closed field $k$.

**Definition 3.11.** A vector bundle $E$ on $X$ is called *universally semistable* if for all $k$-morphisms $f : C \to X$ from smooth connected projective curves $C$ over $k$ the pullback $f^*E$ is semistable. We say that $E$ is *Nori semistable* if it is universally semistable and $c_1(E)$ is numerically trivial.

To better justify the terminology, note that if $E$ is universally semistable, and $f : Y \to X$ is a morphism from a projective manifold $Y$ over $k$, then for every polarization $H$ on $Y$ the pullback $f^*E$ is $\mu_H$-semistable.

If $f : Y \to X$ is a proper generically finite morphism between smooth projective varieties and $E$ is a strongly slope $H$-semistable bundle on $X$ then $f^*E$ is slope $f^*H$-semistable bundle. This motivates the following definition:

**Definition 3.12.** If $X$ is irreducible then we say that a vector bundle $E$ on $X$ is *strongly semistable* if there exist a proper generically finite $k$-morphism $f : Y \to X$ from a smooth projective $k$-variety $Y$ to $X$ and an ample divisor $H$ on $Y$ such that the bundle $f^*E$ is strongly slope $H$-semistable. In general, we say that a vector bundle $E$ on $X$ is *strongly semistable* if its restriction to every irreducible component of $X$ is strongly semistable.

A line bundle $L$ is said to be $\tau$-trivial, if $L^\otimes m$ is algebraically equivalent to $O_X$ for some $m \geq 1$. This notion is equivalent to $L$ being numerically trivial by [Kle05, Theorem 6.3]. Numerically flat bundles can be seen as a higher rank version of $\tau$-trivial bundles. In the proof we use Theorem 4.8.

**Theorem 3.13.** Let $X$ be a proper scheme over an algebraically closed field $k$. Let $E$ be a rank $r$ vector bundle on $X$. Then the following conditions are equivalent:

1. $E$ is numerically flat.
2. $E$ is Nori semistable.
3. $E$ is strongly semistable and $c_j(E)$ is numerically trivial for all $j > 0$.
4. $E$ is strongly semistable and both $c_1(E)$ and $c_2(E)$ are numerically trivial.
5. $E$ is strongly semistable and for every coherent sheaf $F$ on $X$ we have $\chi(X, E \otimes F) = r \cdot \chi(X, F)$.

**Proof.** The equivalence of (1) and (2) is well known and it follows from Lemma [25] (see, e.g., [Lan11, 1.2]). Assume that $E$ is numerically flat and let $X_0$ be some irreducible component of $X$. By [496] there exists a proper generically finite $k$-morphism $f : Y \to X_0$ from a smooth projective $k$-variety $Y$. Then $f^*E$ is numerically flat, so it is strongly $\mu_H$-semistable for every ample divisor $H$ on $X$. In particular, $E$ is strongly semistable. Now implication $(1) \Rightarrow (3)$ follows from Proposition 3.3.

In proof of the implication $(3) \Rightarrow (5)$ we use singular Riemann–Roch [Ful98, Chapter 18]. Let $f : X \to Y = \text{Spec}k$ be the structural morphism. We denote by $[E]$ the class of $E$ in $K^0(X)$ and by $[F]$ the class of $F$ in $K_0(X)$. By [Ful98, Theorem 18.3] we have canonical maps $\tau_X : K_0(X) \to A_*(X)_Q$ and $\tau_Y : K_0(Y) \to A_*(Y)_Q = Q$, which satisfy the following equalities:

$$\chi(X, E \otimes F) = \tau_Y f_*( [E] \otimes [F]) = f_* \tau_X ( [E] \otimes [F]) = f_* (\text{ch}(E)) \cap \tau_X ([F]) = r \tau_Y f_*( [F]) = r \chi(X, F).$$

To prove that $(5)$ implies $(1)$ we can assume that $X$ is irreducible. Then by assumption there exist a proper generically finite $k$-morphism $f : Y \to X$ from a smooth projective $k$-scheme $Y$ to $X$ and an ample divisor $H$ on $Y$ such that the bundle $f^*E$ is strongly slope $H$-semistable. By the Leray spectral sequence and the projection formula we have

$$\chi(Y, f^*E(mH)) = \sum_i (-1)^i \chi(X, R^if_* (f^*E(mH))) = \sum_i (-1)^i \chi(X, E \otimes R^if_* O_Y(mH))$$

$$= r \sum_i (-1)^i \chi(X, R^if_* O_Y(mH)) = r \chi(Y, O_Y(mH)).$$
So by the implication (4) ⇒ (2) of Theorem 4.8 we see that \( f^*E \) is numerically flat. Since \( f \) is surjective, this implies that \( E \) is also numerically flat.

The implication (3) ⇒ (4) is obvious, so it is sufficient to prove that (4) ⇒ (1). Without loss of generality we can assume that \( X \) is irreducible and there exist a proper generically finite \( f : Y \to X \) from a smooth projective \( k \)-variety \( Y \) and an ample divisor \( H \) on \( Y \) such that the bundle \( f^*E \) is strongly \( \mu_H \)-semistable. Then \( c_1(f^*E) \) and \( c_2(f^*E) \) are numerically trivial. So by the implication (1) ⇒ (2) of Theorem 4.8 we see that \( f^*E \) is numerically flat. As before this implies that \( E \) is also numerically flat.

**Corollary 3.14.** Let \( E \) be a vector bundle on \( X \). Then the following conditions are equivalent:

1. \( E \) is strongly semistable and \( \Delta(E) \) is numerically trivial.
2. \( \text{End}_E \) is nef.
3. \( E \) is universally semistable.

**Proof.** If \( E \) is strongly semistable and \( \Delta(E) \equiv 0 \) then \( \text{End}_E \) is strongly semistable and both \( c_1(\text{End}_E) \) and \( c_2(\text{End}_E) \) are numerically trivial. So \( \text{End}_E \) is numerically flat, which proves (1) ⇒ (2). If \( \text{End}_E \) is nef then it is also numerically flat (as it isomorphic to its dual) and hence it is Nori semistable. This implies (3). To prove that (3) ⇒ (1) it is sufficient to prove that \( \Delta(E) \equiv 0 \). But \( \text{End}_E \) is universally semistable with trivial determinant, so it is Nori semistable. Hence it is numerically flat and \( \Delta(E) = c_2(\text{End}_E) \equiv 0 \). □

4. **Algebraic proofs of Theorems 1.1 and 1.2**

4.1. **Restriction theorems.** In the sequel we frequently use the following strengthening of the Mehta–Ramanathan theorem for sheaves with vanishing discriminant. The result follows from [Lan04, Theorem 5.2] with a different proof from the Mehta–Ramanathan theorem.

**Lemma 4.1.** Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) and let \( H \) be an ample divisor class on \( X \). Let \( E \) be a torsion free sheaf with \( \Delta(E) \equiv 0 \). Then there exists \( m_0 = m_0(X, H, \text{rk } E) \geq 1 \) such that for all \( m \geq m_0 \)

1. If \( E \) is strongly \( \mu_H \)-stable and \( E_D \) is torsion free for some normal divisor \( D \in |mH| \), then \( E_D \) is strongly \( \mu_{HD} \)-stable.
2. If \( E \) is strongly \( \mu_H \)-semistable, then for general \( D \in |mH| \) the restriction \( E_D \) is strongly \( \mu_{HD} \)-semistable.

**Proof.** In characteristic zero pick \( m_0 \) such that \( |mH| \) is basepoint free for all \( m \geq m_0 \). In positive characteristic we also need to exceed a constant \( \beta_\varepsilon \) depending on \( X, H \), and \( \text{rk } E \). See the inequality in [Lan04, Theorem 5.2]. (1) follows immediately from [Lan04, Theorem 5.2]. We obtain (2) as a consequence of (1) as in [Lan04, Corollary 5.4]. The factors (successive quotients) in any Jordan–Hölder filtration of \( E \) are \( \mu_H \)-stable. As in [Lan04, Theorem 5.4] we observe that they also have numerically trivial discriminant. Their restriction to a general \( D \) in a basepointfree \( |mH| \) is again torsion free. In characteristic zero then (2) follows from (1). In positive characteristic strong \( \mu_H \)-semistability also takes into account the countably many Frobenius pullbacks. We remark that there exists some \( s_0 \) such that the factors in a Jordan–Hölder filtration of \( (F_X^s)^*E \) are strongly \( \mu_H \)-stable. Then for a general divisor \( D \in |mH| \) the restrictions of the factors in a Jordan–Hölder filtration of the sheaves \( \{(F_X^s)^*E\}_{s \leq s_0} \) to \( D \) are torsion free. The restrictions of the factors in a Jordan–Hölder filtration of \( (F_X^s)^*E \) for all \( s \geq s_0 \) to such a divisor are also torsion free since they are pullbacks of those of \( (F_X^s)^*E \). □

4.2. **The surface case.**

**Proposition 4.2.** Let \( (X, H) \) be an amply polarized smooth projective surface defined over an algebraically closed field \( k \). Let \( E \) be a strongly \( \mu_H \)-semistable locally free sheaf of rank \( r \) with \( c_i(E) \equiv 0 \) for \( i = 1, 2 \). Then
(1) For any line bundle $L$ on $X$ and any $i \in \{0, 1, 2\}$ we have $h^i(X, L \otimes \text{Sym}^m E) = O(m^{-i})$.

(2) $E$ is numerically flat.

Proof. Without loss of generality we can assume that $H$ is very ample. Since $\dim \mathbb{P}(E) = r + 1$, the claimed growth rate in (1) is two degrees lower than expected. The proof is similar to the proof of the Bogomolov inequality in [HL10, Theorem 7.3.1].

By [RR84, Theorem 3.23 and the remark at the end of Section 4] the bundle $\text{Sym}^m E$ is strongly $\mu_H$-semistable. Since $\Delta(\text{Sym}^m E) \equiv 0$ Lemma [4.1] implies that if $D \in |H|$ is a general divisor then $(\text{Sym}^m E)_D$ is strongly semistable of degree 0. By Lemma [2.6] we have

$$h^0(D, (L \otimes \text{Sym}^m E)_D) \leq (1 + \deg D) \cdot \text{rk} \text{Sym}^m E = O(m^{-1})$$

From the short exact sequence

$$0 \to (L \otimes \text{Sym}^m E)(-H) \to L \otimes \text{Sym}^m E \to (L \otimes \text{Sym}^m E)_D \to 0$$

we have

$$h^0(X, L \otimes \text{Sym}^m E) \leq h^0(D, (L \otimes \text{Sym}^m E)_D) + h^0(X, (L \otimes \text{Sym}^m E)(-H)).$$

Changing $H$ by its multiple (which does not depend on $m$) if necessary, we can assume that $L(-H)$ has negative degree with respect to $H$ (e.g., we can assume that $L^\vee(H)$ is effective). Then the bundle $(L \otimes \text{Sym}^m E)(-H)$ is $\mu_H$-semistable with negative slope so it does not have any nonzero sections. Therefore $h^0(X, L \otimes \text{Sym}^m E) = O(m^{-1})$.

By Serre’s duality, $h^2(X, L \otimes \text{Sym}^m E) = h^0(X, (\text{Sym}^m E)^\vee \otimes \omega_X \otimes L^\vee)$. The bundle $(\text{Sym}^m E)^\vee$ is also strongly semistable with numerically trivial Chern classes. An analogous proof to the case $i = 0$ gives $h^2(X, L \otimes \text{Sym}^m E) = O(m^{-1})$. Note that in positive characteristic this equality does not follow formally from the previous case applied to $E^\vee$.

Finally, to prove that $h^1(X, L \otimes \text{Sym}^m E)$ grows at most like $O(m^{-1})$, it is sufficient to prove that $\chi(X, L \otimes \text{Sym}^m E) = O(m^{-1})$. Let $\pi : \mathbb{P}(E) \to X$ be the bundle map and let us set $\xi := c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. Since the Chern classes $c_1(E)$ and $c_2(E)$ are both numerically trivial, we have $\xi^r = 0$. By the Riemann–Roch theorem

$$\chi(X, L \otimes \text{Sym}^m E) = \chi(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* L) = \int_{\mathbb{P}(E)} \exp(m \xi + \pi^* c_1(L)) \cap \text{Td} \mathbb{P}(E),$$

so the coefficients of $m^{r+1}$ and $m^r$ in the expression above are 0. The claim is proved.

(2) Let $f : C \to X$ be a morphism from a smooth projective curve and let $C' \subset X$ be the (possibly singular) image of $f$. Consider the restriction sequence

$$0 \to (\text{Sym}^m E)(-C') \to \text{Sym}^m E \to \text{Sym}^m E_{C'} \to 0.$$

Let $\xi_C = c_1(\mathcal{O}_{\mathbb{P}(f^* E)}(1))$. If $f^* E$ is not strongly semistable, then since it has degree 0, it follows from Lemma [2.5] that $\xi_C$ is not nef. It is however big since some Frobenius pullback of $f^* E$ has a strongly semistable subbundle of positive degree, so an ample subbundle. Bigness is invariant under birational pullback (cf. [Laz04a, Chapter 2.2]) and even under dominant generically finite pullback, hence $h^0(C', \text{Sym}^m E_{C'}) = h^0(\mathbb{P}(E_{C'}), \mathcal{O}_{\mathbb{P}(E_{C'})}(m))$ grows like $O(m^r)$. However, we have

$$h^0(C', \text{Sym}^m E_{C'}) \leq h^0(X, \text{Sym}^m E) + h^1(X, (\text{Sym}^m E)(-C')).$$

We get a contradiction from part (1). \qed

4.3. Local freeness via the vanishing of the discriminant.

**Lemma 4.3.** Let $(X, H)$ and $(Y, A)$ be amply polarized smooth projective varieties defined over an algebraically closed field $k$. Let $E$ be a $\mu_H$-semistable ($\mu_H$-stable, strongly $\mu_H$-semistable or strongly $\mu_H$-stable) torsion free sheaf on $X$. Then $pr^*_X E$ is $\mu_L$-semistable (respectively $\mu_L$-stable, strongly $\mu_L$-semistable or strongly $\mu_L$-stable) for $L = pr^*_X H + pr^*_Y A$. 
Proof. Let \( r = \text{rk} \mathcal{E} \) and denote \( n = \text{dim} X \) and \( m = \text{dim} Y \). Let \( \mathcal{F} \subseteq \mathcal{pr}^*_X \mathcal{E} \) be a subsheaf of rank less than \( r \). For \( x \in X \) and \( y \in Y \), let \( \mathcal{F}_x \coloneqq \mathcal{F}|_{\{x\} \times Y} \) and \( \mathcal{F}_y \coloneqq \mathcal{F}_{X \times \{y\}} \). For general points \( x \in X(k) \) and \( y \in Y(k) \) we have \( \mathcal{F}_x \subseteq \mathcal{O}^\mathcal{pr}_Y \) and \( \mathcal{F}_y \subseteq \mathcal{E} \). Since \( \mathcal{E} \) is \( \mu_H \)-semistable and \( \mathcal{O}^\mathcal{pr}_Y \) is \( \mu_A \)-semistable, we deduce that
\[
\frac{c_1(\mathcal{F}) \cdot \mathcal{pr}^*_X H^{n-1} \mathcal{pr}^*_Y A^m}{(A^m) \cdot \text{rk} \mathcal{F}} = \frac{c_1(\mathcal{F}_x) \cdot H^{n-1}}{\text{rk} \mathcal{F}} \leq \mu_H(\mathcal{E}) \quad \text{and} \quad \frac{c_1(\mathcal{F}) \cdot \mathcal{pr}^*_X H^n \mathcal{pr}^*_Y A^{m-1}}{(H^n) \cdot \text{rk} \mathcal{F}} = \frac{c_1(\mathcal{F}_y) \cdot A^{m-1}}{\text{rk} \mathcal{F}} \leq 0.
\]
Then
\[
\mu_L(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot \mathcal{F}^{m+1}}{\text{rk} \mathcal{F}} \leq \frac{c_1(\mathcal{F}) \cdot \left( \binom{n+m-1}{n-1} \right) \mathcal{pr}^*_X H^{n-1} \mathcal{pr}^*_Y A^m + \binom{n+m-1}{n-1} \mathcal{pr}^*_X H^n \mathcal{pr}^*_Y A^{m-1}}{\text{rk} \mathcal{F}} \leq \mu_H(\mathcal{E} \cdot \mathcal{X}^*) + \mu_H(\mathcal{E} \cdot \mathcal{X}^*) = \mu_L(\mathcal{pr}^*_X \mathcal{E}).
\]
If \( \mathcal{E} \) is \( \mu_H \)-stable then \( \frac{c_1(\mathcal{F}) \cdot \mathcal{pr}^*_X H^{n-1} \mathcal{pr}^*_Y A^m}{(A^m) \cdot \text{rk} \mathcal{F}} = \frac{c_1(\mathcal{F}_y) \cdot H^{n-1}}{\text{rk} \mathcal{F}} < \mu_H(\mathcal{E}) \) and we get \( \mu_L(\mathcal{F}) < \mu_L(\mathcal{pr}^*_X \mathcal{E}) \) as required.

Applying the above assertions for slope semistability and slope stability to all Frobenius pullbacks gives immediately the assertions for strong slope semistability and strong slope stability. \( \square \)

**Proposition 4.4.** Let \( X \) be a smooth projective variety of dimension \( n \) defined over an algebraically closed field \( k \) and let \( H \) be an ample polarization on \( X \). Let \( \mathcal{E} \) be a strongly \( \mu_H \)-stable torsion-free sheaf on \( X \) with \( c_1(\mathcal{E}) = 0 \). Then the following conditions are equivalent:

1. \( \mathcal{E} \) is reflexive and \( c_2(\mathcal{E}) \cdot H^{n-2} = 0 \).
2. \( \mathcal{E} \) is locally free and numerically flat.
3. \( c_j(\mathcal{E}) = 0 \) for all \( j \geq 1 \).
4. The normalized Hilbert polynomial \( \frac{1}{\text{rk} \mathcal{E}} \cdot \chi(X, \mathcal{E}(mH)) \) of \( \mathcal{E} \) is equal to the Hilbert polynomial of \( \mathcal{O}_X \).

**Proof.** We argue by induction on \( n \). If \( n = 1 \), the equivalence of all four conditions is tautological, with (2) being implied by Lemma [2.5]. Let us assume that \( n = 2 \). Since every reflexive sheaf on a smooth surface is locally free, the implication (1) \( \Rightarrow \) (2) follows from Proposition [1.2].

The implication (2) \( \Rightarrow \) (3) follows from Proposition [3.3] and (3) \( \Rightarrow \) (4) follows from the Hirzebruch–Riemann–Roch theorem. To prove (4) \( \Rightarrow \) (1), consider the exact sequence \( 0 \to \mathcal{E} \to \mathcal{E}^{\mathcal{pr}} \to Q \to 0 \), where \( Q \) has a finite support. Since \( c_1(\mathcal{E}^{\mathcal{pr}}) = c_1(\mathcal{E}) = 0 \) and \( \mathcal{E}^{\mathcal{pr}} \) is strongly \( \mu_H \)-stable, the Bogomolov type inequality (see [Lan04, Theorem 3.2]) gives \( \int_X \Delta(\mathcal{E}^{\mathcal{pr}}) \geq 0 \). Our assumption on the Hilbert polynomial of \( \mathcal{E} \) and the Hirzebruch–Riemann–Roch theorem imply that \( \int_X c_2(\mathcal{E}) = 0 \).

Therefore with \( r = \text{rk} \mathcal{E} \)
\[
\int_X \Delta(\mathcal{E}^{\mathcal{pr}}) = 2r \int_X c_2(\mathcal{E}^{\mathcal{pr}}) = 2r \int_X (c_2(\mathcal{E}) + c_2(Q)) = 2r \int_X c_2(Q) \leq 0,
\]
which gives \( \int_X c_2(Q) = 0 \). But then \( Q = 0 \) and \( \mathcal{E} \) is reflexive.

Let us now assume that the result holds for varieties of dimension \( < n \), where \( n \geq 3 \).

(1) \( \Rightarrow \) (2)

Let \( \mathcal{E} \) be reflexive, strongly \( \mu_H \)-stable, with \( c_1(\mathcal{E}) \) numerically trivial and \( c_2(\mathcal{E}) \cdot H^{n-2} = 0 \). Let \( \iota : D \to X \) be the inclusion of a general member of \( |mH| \) for sufficiently large \( m \). Then \( D \) is smooth of dimension \( n-1 \). The restriction \( \iota^* \mathcal{E} \) is still reflexive by the general choice of \( D \) and it is strongly \( \mu_{H^\iota} \)-stable by Lemma [3.4]. We also have \( c_2(\iota^* \mathcal{E}) \cdot \iota^* H^{n-3} = c_2(\mathcal{E}) \cdot H^{n-2} \) by Remark [2.3].

Using the implication (1) \( \Rightarrow \) (4) on \( D \) we see that
\[
\chi(X, \mathcal{E}(mH)) - \chi(\mathcal{E}((m-1)H)) = \chi(D, \iota^* \mathcal{E}(mH)) = r \chi(D, \mathcal{O}_D(mH)) = r(\chi(X, \mathcal{O}_X(mH)) - \chi(\mathcal{O}_X((m-1)H))).
\]

Now assume that \( \mathcal{E} \) is not locally free. For sufficiently large \( m \), let \( \iota : D \to X \) be an embedding of a member of \( |mH| \) that is general among those that pass through one of the points where \( \mathcal{E} \) is
not locally free. By [DH91] we know that $D$ is smooth. We also know that the restriction $i^*\mathcal E$ is torsion-free. So Lemma 4.1 implies that $i^*\mathcal E$ is strongly $\mu_*H$-stable. By the above, we also know that
\[
\chi(D, i^*\mathcal E(mH)) = \chi(X, \mathcal E(mH)) - \chi(\mathcal E((m - 1)H)) = r^\chi(X, \mathcal O_X(mH)) - \chi(\mathcal O_X((m - 1)H)) = r^\chi(D, \mathcal O_D(mH)).
\]
Then the induction assumption implies that $i^*\mathcal E$ is locally free. By [Lan19, Lemma 1.14] we deduce that $\mathcal E$ is locally free around $D$, a contradiction. Thus $\mathcal E$ is locally free and we need to prove that it is numerically flat. Let $f : C \to X$ be a morphism from a smooth projective curve. Let $A$ be any ample polarization on $C$ and let $\mathcal C \subset X \times C$ be the embedding of the graph of $C$. Denote by $pr_X : X \times C \to X$ and $pr_C : X \times C \to C$ the projections onto the two factors. Let us also set $L := pr_X^*H + pr_C^*A$ and $\tilde{\mathcal E} := pr_X^*\mathcal E$. Then $\tilde{C} \simeq C$ and $f^*\mathcal E \simeq \tilde{\mathcal E}$, so it is sufficient to check that $\tilde{\mathcal E}_{\tilde{C}}$ is semistable of degree 0. We have $c_2(\tilde{\mathcal E}) \cdot L^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} pr_X^*(c_2(\tilde{\mathcal E}) \cdot H^{n-1-j}) \cdot pr_C^*A^j$. For dimension reasons, all terms except possibly $j = 1$ vanish. The term $j = 1$ also vanishes from the assumption $c_2(\mathcal E) \cdot H^{n-2} = 0$. Therefore we have $c_1(\mathcal E) \equiv 0$ and $c_2(\mathcal E) \cdot L^{n-1} = 0$. By Lemma 4.3 we also know that $\mathcal E$ is strongly $\mu_L$-stable. By [DH91, Theorem 3.1] for large $m$ there exists a chain of smooth varieties $S = X_2 \subset X_3 \subset \ldots \subset X_n = X \times C$ containing $\tilde{C}$ such that all $X_i \in [mL_{X_i+1}]$ are smooth. Then Lemma 4.1 implies that $\tilde{\mathcal E}_S$ is strongly $\mu_{L_S}$-stable with $c_1(\tilde{\mathcal E}_S) \equiv 0$ and $\int_S c_2(\tilde{\mathcal E}_S) = 0$. So the required assertion follows from Proposition 1.2.

(2) $\Rightarrow$ (3) follows from Proposition 4.3.

(3) $\Rightarrow$ (4) follows from the Hirzebruch–Riemann–Roch theorem.

(4) $\Rightarrow$ (1) Let $\mathcal E$ be a rank $r$ torsion free, strongly $\mu_H$-stable sheaf with $c_1(\mathcal E) \equiv 0$ and $\chi(X, \mathcal E(mH)) = r^\chi(X, \mathcal O_X(mH))$ for all $m \in \mathbb{Z}$. Comparing coefficients of these polynomials at $m^{n-2}$ we see that $c_2(\mathcal E) \cdot H^{n-2} = 0$. Consider the exact sequence
\[
0 \to \mathcal E \to \mathcal E^{\vee\vee} \to Q \to 0,
\]
where $Q$ is a torsion sheaf on $X$. Then $\mathcal E^{\vee\vee}$ is reflexive, strongly $\mu_H$-stable sheaf with $c_1(\mathcal E^{\vee\vee}) \equiv 0$. As in the surface case, the Bogomolov type inequality (see [Lan04, Theorem 3.2]) gives
\[
\Delta(\mathcal E^{\vee\vee}) \cdot H^{n-2} = 2rc_2(Q) \cdot H^{n-2} \geq 0,
\]
which implies $\Delta(\mathcal E^{\vee\vee}) \cdot H^{n-2} = c_2(Q) \cdot H^{n-2} = 0$. Using already proven implication (1) $\Rightarrow$ (4) we see that Hilbert polynomials of $\mathcal E$ and $\mathcal E^{\vee\vee}$ coincide. Therefore the Hilbert polynomial of $Q$ vanishes. This implies that $Q = 0$ and hence $\mathcal E$ is reflexive with $c_2(\mathcal E) \cdot H^{n-2} = 0$.

The following known example shows that reflexivity assumption is necessary in condition (1) of Proposition 4.4.

Example 4.5. Let $X$ be a smooth projective variety of dimension $n \geq 3$. Let $\mathcal I \subset \mathcal O_X$ be the ideal sheaf of a nonempty closed subset of codimension $j \geq 3$. Then $\mathcal I$ is torsion-free, strongly slope semistable with respect to any polarization, and $c_1(\mathcal I)$ and $c_2(\mathcal I)$ are numerically trivial. However $c_j(\mathcal I)$ is not numerically trivial and of course $\mathcal I$ is not locally free.

Corollary 4.6. Let $\mathcal E$ be a reflexive strongly $\mu_H$-semistable sheaf with $c_1(\mathcal E) \equiv 0$ and $c_2(\mathcal E) \cdot H^{n-2} = 0$. Then $\mathcal E$ is locally free and numerically flat. Moreover, every factor in a Jordan–Hölder filtration of $\mathcal E$ is also locally free and numerically flat.

This proof is analogous to [Nak99, Proposition 2.5] and [Lan19, Section 2.2].

Proof. The statement is clear if $n = 1$ so we assume that $n \geq 2$. We perform induction on the rank $r$ of $\mathcal E$. The case $r = 1$ or more generally $\mathcal E$ is strongly $\mu_H$-stable is Proposition 4.4. Thus we may assume that $\mathcal E$ is not strongly $\mu_H$-stable. First, let us assume that there exists an exact sequence
\[
0 \to \mathcal E_1 \to \mathcal E \to \mathcal E_2 \to 0
\]
where $\mathcal{E}_1$ is $\mu_H$-stable, strongly $\mu_H$-semistable and reflexive and $\mathcal{E}_2$ is strongly $\mu_H$-semistable and torsion free, both $\mathcal{E}_1$ and $\mathcal{E}_2$ are nonzero sheaves and $\mu_H(\mathcal{E}_1) = \mu_H(\mathcal{E}_2) = 0$. By the Hodge index theorem we have
\[
0 = \frac{\Delta(\mathcal{E}) \cdot H^{n-2}}{r} = \frac{\Delta(\mathcal{E}_1) \cdot H^{n-2}}{r_1} + \frac{\Delta(\mathcal{E}_2) \cdot H^{n-2}}{r_2} - \frac{r_1 r_2}{r} \left( \frac{c_1(\mathcal{E}_1)}{r_1} - \frac{c_1(\mathcal{E}_2)}{r_2} \right)^2 \cdot H^{n-2}
\]
\[
\geq \frac{\Delta(\mathcal{E}_1) \cdot H^{n-2}}{r_1} + \frac{\Delta(\mathcal{E}_2) \cdot H^{n-2}}{r_2}.
\]
Using the Bogomolov type inequality \cite[Theorem 3.2]{Lan04} for $\mathcal{E}_1$ and $\mathcal{E}_2$, we see that $\Delta(\mathcal{E}_1) \cdot H^{n-2} = \Delta(\mathcal{E}_2) \cdot H^{n-2} = 0$. Equality in the Hodge index inequality implies also that $c_1(\mathcal{E}_1)$ and $c_1(\mathcal{E}_2)$ are numerically trivial. The induction hypothesis directly applies only to the reflexive sheaf $\mathcal{E}_1$ (not the torsion free $\mathcal{E}_2$) and we deduce that it is locally free and numerically flat. But we also have an exact sequence
\[
0 \to \mathcal{E}_2 \to \mathcal{E}_2^\vee \to Q \to 0,
\]
where $Q$ is supported in codimension at least 2. Since $\mathcal{E}_2^\vee$ is also strongly $\mu_H$-semistable, $\Delta(\mathcal{E}_2^\vee) \cdot H^{n-2} \geq 0$. But $\Delta(\mathcal{E}_2^\vee) \cdot H^{n-2} = 2r_2(Q) \cdot H^{n-2} \leq 0$, so $\Delta(\mathcal{E}_2^\vee) \cdot H^{n-2} = 0$ and $Q$ is supported in codimension at least 3. By the induction hypothesis, $\mathcal{E}_2^\vee$ is locally free and every factor in a Jordan–Hölder filtration of $\mathcal{E}_2^\vee$ is also locally free and numerically flat. An Ext computation using that $Q$ has codimension at least 3 shows that we have a commutative diagram
\[
\begin{array}{c}
\begin{array}{c}
0 \\
0
\end{array} \to \\
\begin{array}{c}
\mathcal{E}_1 \\
\mathcal{E}
\end{array} \to \\
\begin{array}{c}
\mathcal{E}_2 \\
\mathcal{E}_2^\vee
\end{array} \to \\
\begin{array}{c}
0 \\
0
\end{array}
\end{array}
\]
for some sheaf $\mathcal{E}'$ that is then necessarily locally free. See \cite[Proposition 2.5]{Nak99} or \cite[Lemma 1.12]{Lan19} for details. The middle vertical arrow is an isomorphism since $\mathcal{E}$ and $\mathcal{E}'$ are both reflexive, and isomorphic on the locally free locus of $\mathcal{E}_2$. This implies that $\mathcal{E}$ is locally free and $\mathcal{E}_2$ is reflexive, so we can apply the induction assumption also to $\mathcal{E}_2$.

To finish the proof one needs to deal with the case when $\mathcal{E}$ is $\mu_H$-stable but not strongly $\mu_H$-stable. Then we can apply the above arguments for some Frobenius pull-back $(F^m_X)^* \mathcal{E}$. Since local freeness and numerical flatness of $(F^m_X)^* \mathcal{E}$ implies local freeness and numerical flatness of $\mathcal{E}$, we get the required assertion.

\begin{corollary}
Let $\mathcal{E}$ be a reflexive strongly $\mu_H$-semistable sheaf of rank $r$ with $\Delta(\mathcal{E}) \cdot H^{n-2} = 0$. Then $\mathcal{E}$ is locally free and $\End \mathcal{E}$ is numerically flat. Moreover, every factor in a Jordan–Hölder filtration of $\mathcal{E}$ is also locally free and its endomorphism bundle is numerically flat.
\end{corollary}

\begin{proof}
We use a finite cover to extract an $r$-th root of $det \mathcal{E}$ and reduce to the case $c_1(\mathcal{E}) \equiv 0$ where Corollary \ref{corollary} applies. Let $\varphi : Y \to X$ be a Bloch–Gieseker cover (see \cite[Lemma 2.1]{BG71}), i.e., a finite surjective map from a smooth projective variety $Y$ such that $\varphi^* det \mathcal{E} = \mathcal{L}^\otimes r$ for some line bundle $\mathcal{L}$ on $Y$. The map $\varphi$ is flat so $\varphi^* \mathcal{E}$ is reflexive. By Remark \ref{remark} we also have $c_i(\varphi^* \mathcal{E}) = \varphi^* c_i(\mathcal{E})$ for all $i$. In particular, we have $det(\varphi^* \mathcal{E} \otimes \mathcal{L}^\vee) = \mathcal{O}_Y$ and $\Delta(\varphi^* \mathcal{E}) \cdot \varphi^* H^{n-2} = 0$. Furthermore, $\varphi^* H$ is ample, and $\varphi^* \mathcal{E}$ is strongly $\mu_{\varphi^* H}$-semistable. Then $\varphi^* \mathcal{E} \otimes \mathcal{L}^{-1}$ is reflexive, strongly $\mu_{\varphi^* H}$-semistable, with trivial determinant, and $\Delta(\varphi^* \mathcal{E} \otimes \mathcal{L}^{-1}) \cdot \varphi^* H^{n-2} = \Delta(\varphi^* \mathcal{E}) \cdot \varphi^* H^{n-2} = 0$. From the results above we deduce that $\varphi^* \mathcal{E} \otimes \mathcal{L}^{-1}$ is locally free and numerically flat. Hence $\varphi^* \End \mathcal{E} = \End(\varphi^* \mathcal{E} \otimes \mathcal{L}^{-1})$ is also numerically flat. This implies that $\mathcal{E}$ is locally free and $\End \mathcal{E}$ is numerically flat. The last part follows analogously. The only difference is that the pull-back of $\mu_H$-stable sheaf need not be $\mu_{\varphi^* H}$-stable and it is only $\mu_{\varphi^* H}$-semistable. So one needs to take a refinement of the pull-back of a Jordan–Hölder filtration of $\mathcal{E}$ to a Jordan–Hölder filtration of $\varphi^* \mathcal{E}$ and then use Corollary \ref{corollary}.
\end{proof}
4.4. Main theorems in the smooth case.

**Theorem 4.8.** Let $X$ be a smooth projective variety of dimension $n$ defined over an algebraically closed field $k$ and let $H$ be an ample polarization on $X$. Let $E$ be a torsion free sheaf on $X$. Then the following conditions are equivalent:

1. $E$ is reflexive, strongly $\mu_H$-semistable and $\text{ch}_1(E) \cdot H^{n-1} = \text{ch}_2(E) \cdot H^{n-2} = 0$.
2. $E$ is locally free and numerically flat.
3. $E$ is strongly $\mu_H$-semistable and $c_j(E) \equiv 0$ for all $j \geq 1$.
4. $E$ is strongly $\mu_H$-semistable and the normalized Hilbert polynomial of $E$ equals to the Hilbert polynomial of $\mathcal{O}_X$.

In particular, if $E$ is a strongly slope semistable vector bundle on $X$ with $c_1(E) \equiv 0$, then $E$ is nef if and only if $\Delta(E) \equiv 0$.

**Proof.** The conditions $\text{ch}_1(E) \cdot H^{n-1} = 0$ and $\text{ch}_2(E) \cdot H^{n-2} = 0$ imply that $c_1(E) \cdot H^{n-2}$ is numerically trivial and $\Delta(E) \cdot H^{n-2} = 0$ by the Bogomolov inequality and by the Hodge index theorem on surfaces (see [Lan11, Lemma 4.2]). The condition $c_1(E) \cdot H^{n-2} = 0$ implies that $c_1(E)$ is numerically trivial by [Ful98, Example 19.3.3]. Therefore (1) $\Rightarrow$ (2) follows from Corollary 4.6. The implication (2) $\Rightarrow$ (3) follows from Proposition 3.3. The proofs of implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are analogous to the proofs of the corresponding implications in Proposition 4.4.

If $E$ is a nef vector bundle with $c_1(E) \equiv 0$, then $E$ is numerically flat and in particular $\Delta(E) \equiv 0$. Conversely, if $E$ is a strongly semistable vector bundle with $c_1(E) \equiv 0$, and $\Delta(E) \equiv 0$, then $\text{ch}_2(E) \equiv 0$ and so $E$ is numerically flat.

Even in the locally free case, one cannot replace condition (3) in Theorem 4.8 with $c_j(E) \cdot H^{n-j} = 0$ for all $j \geq 1$, or the condition $\text{ch}_2(E) \cdot H^{n-2} = 0$ in (1) with $c_2(E) \cdot H^{n-2} = 0$.

**Example 4.9.** Let $X$ be a smooth projective surface of Picard rank at least 3. Let $H, L, L'$ be divisors on $X$ with $H$ ample such that the intersection pairing on span$(H, L, L') \subseteq N^1(X)$ has diagonal matrix with respect to the basis $(H, L, L')$. Let $E = \mathcal{O}_X(L) \oplus \mathcal{O}_X(L')$. It is strongly $\mu_H$-semistable. Furthermore, we have $c_1(E) \cdot H = 0$ and $\int_X c_2(E) = 0$. However, $E$ is not numerically flat since $c_1(E) = L + L'$ is not numerically trivial. Note that in this case $\int_X \text{ch}_2(E) < 0$.

**Theorem 4.10.** Let $X$ be a smooth projective variety of dimension $n$ defined over an algebraically closed field $k$ and let $H$ be an ample polarization on $X$. Let $E$ be a reflexive sheaf of rank $r$ on $X$. Then the following conditions are equivalent:

1. $E$ is strongly $\mu_H$-semistable and $\Delta(E)$ is numerically trivial.
2. $E$ is strongly $\mu_H$-semistable and $\Delta(E) \cdot H^{n-2} = 0$.
3. $E$ is locally free and the twisted normalized bundle $E(-\frac{1}{r} \det E)$ is nef.
4. $E$ is locally free and $\text{End} E$ is nef.
5. For every morphism $f : C \to X$ from a smooth projective curve, $f^*E$ is semistable.
6. For every morphism $f : Y \to X$, where $Y$ is a smooth projective variety, $f^*E$ is strongly slope semistable with respect to any ample polarization on $Y$.

In particular, if $E$ is strongly $\mu_H$-semistable and $\Delta(E) \cdot H^{n-2} = 0$, then $E$ is locally free. Furthermore, it is nef (resp. ample) if and only if $\det E$ is nef (resp. ample).

**Proof.** (1) $\Rightarrow$ (2) is trivial. For locally free $E$, the nefness of $\varphi^*E \otimes L'$, where $\varphi : Y \to X$ is a finite cover as in Corollary 4.7. Since this bundle has trivial determinant, it is equivalent to the nefness (equivalently numerical flatness) of $\text{End} \varphi^*E$ and then to that of $\text{End} E$. We get the implications (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) by Corollary 4.7. We also have (4) $\Rightarrow$ (1) by Proposition 3.3.

Numerically flat bundles are universally slope semistable, in fact universally strongly slope semistable. We obtain (4) $\Rightarrow$ (6) $\Rightarrow$ (5). Then non-torsion semistable sheaves on smooth projective curves are torsion free, in particular locally free. General complete intersection curves of
high degree passing through a given point \( x \in X \) are smooth by [DH91]. Assuming (5), we obtain that \( \mathcal{E} \) is locally free. By precomposing \( f : C \to X \) with iterates of the Frobenius \( F_C \), we see that (5) is equivalent to the analogous statement for strong semistability. On \( C \), the strong semistability of \( f^* \mathcal{E} \) is equivalent to the nefness of \( f^* \mathcal{E} \). We deduce that (5) \( \Rightarrow \) (4).

For the last statements, if \( \mathcal{E} \) is strongly \( \mu_H \)-semistable with \( \Delta(\mathcal{E}) \cdot H^{n-2} = 0 \), then \( \mathcal{E} \) is locally free by Corollary 4.7. Clearly if \( \mathcal{E} \) is nef (resp. ample), then \( \det \mathcal{E} \) is nef (resp. ample). The implication (2) \( \Rightarrow \) (3) and the identity \( \mathcal{E} = \left( \mathcal{E}(\det \mathcal{E}) \right) \left( \frac{1}{r} \det \mathcal{E} \right) \) give the converse. \( \square \)

**Remark 4.11.** In the above theorem one can also give another condition analogous to (3) of Theorem 4.8. See [Lan19, Theorem 2.2] for the precise formulation. We leave an easy proof of this result along the above lines to the interested reader.

**Remark 4.12.** If \( X \) is a complex projective manifold then homological equivalence over \( \mathbb{Q} \) implies numerical equivalence. Classically they are known to agree for divisors. Lieberman [Lie68] also proved it for codimension 2 cycles using the hard Lefschetz theorem. So in this case proving that \( \Delta(\mathcal{E}) \) is 0 in \( H^4(X, \mathbb{Q}) \) is equivalent to proving that it is numerically trivial.

5. ON MISRA’S QUESTION

**Definition 5.1.** Let \( X \) be a projective variety defined over an algebraically closed field \( k \). We say that \( X \) is 1-homogeneous if \( \text{Nef}(X) = \overline{\text{Eff}}(X) \).

Curves, or more generally varieties of Picard rank 1, and homogeneous spaces are 1-homogeneous. By [Miy87] Theorem 3.1 and remark on p. 464] if \( \mathcal{E} \) is a vector bundle on a smooth projective curve then \( \mathbb{P}(\mathcal{E}) \) is 1-homogeneous if and only if \( \mathcal{E} \) is strongly semistable.

**Remark 5.2.**

(i) If \( X \) is a projective variety with Picard number 2 and \( A \) and \( B \) are globally generated line bundles, but not big, and not proportional in \( N^1(X) \), then \( X \) is 1-homogeneous and \( A \) and \( B \) span the boundary rays of \( \text{Nef}(X) = \overline{\text{Eff}}(X) \).

(ii) If \( X \) is projective with Picard number 1 and dimension \( n \), and \( \mathcal{E} \) is a vector bundle of rank \( r \) on \( X \) such that \( \mathcal{E} \) can be generated by fewer than \( n + r \) global sections, then \( \mathbb{P}(\mathcal{E}) \) is 1-homogeneous. (We get an induced morphism \( f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N \) for some \( N < \dim \mathbb{P}(\mathcal{E}) \). In particular \( r \geq 2 \) and \( \mathbb{P}(\mathcal{E}) \) has Picard rank 2. The fibers of \( \pi : \mathbb{P}(\mathcal{E}) \to X \) are embedded by \( f \). Thus, if \( H \) is a very ample line bundle on \( X \), then \( \pi^*H \) and \( f^*\mathcal{O}_{\mathbb{P}^N}(1) \) satisfy the requirements of (i).)

5.1. **Misra’s theorem in an arbitrary characteristic.** The following theorem generalizes [Mis21, Theorem 1.2] to an arbitrary characteristic:

**Theorem 5.3.** Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) and let \( \mathcal{E} \) be a strongly slope semistable bundle with respect to some ample polarization of \( X \). Let us also assume that \( \Delta(\mathcal{E}) \equiv 0 \). Then the following conditions are equivalent:

1. \( X \) is 1-homogeneous,
2. \( c_1(\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(D)) \) is nef for every effective divisor \( D \) on \( \mathbb{P}(\mathcal{E}) \),
3. \( \mathbb{P}(\mathcal{E}) \) is 1-homogeneous.

**Proof.** We specify the adjustments needed to port the proof of [Mis21, Theorem 1.2] to positive characteristic. Once can use Theorem 4.10 instead of [Mis21, Theorem 2.1]. Apart from that the implication (1) \( \Rightarrow \) (2) uses the fact that symmetric powers of \( \mathcal{E} \) are semistable. This follows either from the Ramanan–Ramanathan theorem (see the proof of Proposition 4.2) or one can use Theorem 4.10 and the fact that symmetric powers of nef bundles are nef.

Finally, the proof of (1) \( \Rightarrow \) (2) uses the duality between the cone of strongly movable curves and pseudo-effective divisors. This fact also holds in positive characteristic by [FL17b, Theorem 2.22] (see also [Das20, Theorem 1.4]). The rest of the proof is the same as in [Mis21]. \( \square \)
5.2. Syzygy bundle counterexamples to Question 1.4

**Definition 5.4.** Let $X$ be a projective variety. Let $V$ be a globally generated vector bundle on $X$. The associated **syzygy bundle** $M_V$ is the kernel of the natural evaluation morphism $H^0(X, V) \otimes O_X \xrightarrow{ev} V$. Put $E_V = M_V'$. This is a globally generated vector bundle.

For example the Euler sequence on $\mathbb{P}^n$ gives $M_{O_{\mathbb{P}^n}(1)} = O_{\mathbb{P}^n}(1)$. If $L$ is a globally generated line bundle, and $\varphi : X \rightarrow \mathbb{P}^N$ is the induced morphism with $\varphi^*O_{\mathbb{P}^N}(1) = L$, then $M_L = \varphi^*\Omega_{\mathbb{P}^N}(1)$ and $E_L = \varphi^*T_{\mathbb{P}^N}(-1)$.

**Remark 5.5.** If $V$ is a globally generated vector bundle on $X$ with $\dim X > \rk V$, then $r = \rk E_V = h^0(X, V) - \rk V$ and $E_V$ is generated by $h^0(X, V) = r + \rk V < \dim X + r$ global sections. In particular, if $X$ has Picard rank 1 then $\mathcal{P}(E_V)$ is 1-homogeneous by Remark 5.2.

**Proposition 5.6.** For $n \geq 2$ we have that

(i) $\mathcal{P}(T_{\mathbb{P}^n})$ is 1-homogeneous.

(ii) $T_{\mathbb{P}^n}$ is strongly slope semistable with respect to the hyperplane class.

(iii) $\Delta(T_{\mathbb{P}^n}) \neq 0$.

(iv) The restriction of $T_{\mathbb{P}^n}$ to every line in $\mathbb{P}^n$ is unstable.

**Proof.** (i). Apply Remark 5.5 to $V = O_{\mathbb{P}^n}(1)$. (ii) is classical and it follows, e.g., from the Bott vanishing. (iii). By direct computation, $\Delta(T_{\mathbb{P}^n}) = \Delta(T_{\mathbb{P}^n}(-1)) = n + 1$. (iv). $T_{\mathbb{P}^n}$ restricts as $O(2) \oplus O(1)^{2n-1}$ on every line. \(\square\)

The easy counterexample above is slope semistable. We also give a slope unstable example inspired in part by suggestions of S. Misra and D. S. Nagaraj.

**Example 5.7.** On $\mathbb{P}^3$ consider the globally generated bundle $V = O_{\mathbb{P}^3}(1) \oplus O_{\mathbb{P}^3}(2)$. Consider the associated syzygy bundle $M_V$ and let $E = E_V = M_V'$. Then $E$ is slope unstable, has positive discriminant, and $\mathcal{P}(E)$ is 1-homogeneous. The bundle $E$ has rank 12, $c_1(E) = 3$ and $c_2(E) = 7$. It is an immediate computation that $\Delta(E) > 0$. We have that $M_V = M_{O_{\mathbb{P}^3}(1)} \oplus M_{O_{\mathbb{P}^3}(2)}$. The summands have slopes $-1/3$ and respectively $-2/9$. Thus $M_V$ and its dual $E$ are unstable. Since $\rk V = 2 < \dim \mathbb{P}^3$, we get that $\mathcal{P}(E)$ is 1-homogeneous by Remark 5.5. \(\square\)

We list related problems asking if our counterexamples are the simplest/smallest possible.

**Question 5.8.**

(1) Does there exist a complex projective manifold $X$ of Picard rank 1 and dimension at least 2 supporting an ample and globally generated line bundle $L$ such that the syzygy bundle $M_L$ is $\mu_L$-unstable, but $\mathcal{P}(M_L')$ is 1-homogeneous? \(^1\)

(2) Does there exist a complex projective surface $X$ supporting a slope unstable $E$ such that $\mathcal{P}(E)$ is 1-homogeneous?

(3) Are there any $\mu_H$-unstable bundles $E$ with $\Delta(E) \cdot H^{n-2} = 0$ such that $\mathcal{P}(E)$ is 1-homogeneous?

5.3. A positive result.

**Lemma 5.9.** Let $V$ be a free module of rank $r$ over a commutative ring $k$. Then for any $a, b \geq 1$ there exist

(1) a surjection of $GL(V)$-modules

$$\text{Sym}^a(\text{Sym}^b V) \twoheadrightarrow \text{Sym}^{ab} V,$$

\(^1\)Sch05 constructs an example on curves. The semistability of syzygy bundles is an active topic of research. We refer to [EL92, ELM13, BPMGNO19] and the references therein for a history of the problem.
(2) an inclusion of GL(\ V)-modules
\[ (\bigwedge^r V)^{\otimes 2a} \hookrightarrow \text{Sym}^r(\text{Sym}^{2a} V). \]

Moreover, the composition
\[ (\bigwedge^r V)^{\otimes 2a} \hookrightarrow \text{Sym}^r(\text{Sym}^{2a} V) \to \text{Sym}^{2ra} V \]

is zero.

Over \( \mathbb{C} \), assertion (2) is a particular case of the main result of [BCI+], which also applies to other even partitions than \( \lambda = ((2a)^r) \).

Proof. We have a canonical surjection \( \bigotimes_{i=1}^{2a} \text{Sym}^b V \to \text{Sym}^{ab} V \) coming from the symmetric multiplication. By definition we also have a canonical surjection \( \bigotimes_{i=1}^{2a} \text{Sym}^b V \to \text{Sym}^a(\text{Sym}^b V) \). Using the universal property of the symmetric product we get an induced map \( \text{Sym}^a(\text{Sym}^b V) \to \text{Sym}^{ab} V \), which is also surjective. This gives the first assertion.

To prove the second assertion we reduce to the case \( k = \mathbb{Z} \). For \( k = \mathbb{Z} \) we construct an explicit non-zero map of GL(V)-modules that has an associated matrix with an entry equal to 1. The map is then a split inclusion as a morphism of \( \mathbb{Z} \)-modules, hence base changing to any commutative ring \( k \) is still injective.

Let \( \lambda = ((2a)^r) = (2a, ..., 2a) \) be a partition of \( 2ar \), i.e., we have a rectangle of size \( (r \times 2a) \). Let \( \Sigma \) be the set of all tableaux \( T \) of shape \( \lambda \) with the entries in \( [1, r] = \{1, \ldots, r\} \) so that in each column we have a permutation of the set \( [1, r] \) and the first column corresponds to an even permutation. We set \( \text{sgn} \ T = \prod_{i=1}^{2a} \text{sgn} \ \sigma_i \), where \( \sigma_i \) is the permutation of \( [1, r] \) corresponding to the \( i \)-th column of \( T \) and \( \text{sgn} \ \sigma \) is the sign of permutation \( \sigma \). Now we define the map
\[ \varphi : \prod_{i=1}^{2a} \left( \prod_{j=1}^{r} V \right) \to \text{Sym}^r(\text{Sym}^{2a} V) \]

by setting
\[ \varphi((v_{11}, \ldots, v_{r1}), \ldots, (v_{12a}, \ldots, v_{r2a})) = \sum_{T \in \Sigma} \text{sgn} \ T \prod_{j=1}^{r} \left( \prod_{i=1}^{2a} v_{T(j,i),i} \right). \]

Since this map is multilinear it factors to the map
\[ \varphi : (V^\otimes r)^{\otimes 2a} = \bigotimes_{i=1}^{2a} \left( \bigotimes_{j=1}^{r} V \right) \to \text{Sym}^r(\text{Sym}^{2a} V) \]

Note that this last map is alternating in each set of variables \( (v_{1m}, \ldots, v_{rm}) \), where \( m \in [1, 2a] \). Since we work over \( \mathbb{Z} \), it is sufficient to check that the corresponding multilinear form is antisymmetric. This is clear for \( m > 1 \) as exchanging \( v_{im} \) with \( v_{jm} \) defines a bijection on the set \( \Sigma \) that replaces the tableau \( T \) with another tableau with exchanged entries between \( (i, m) \) and \( (j, m) \) places. For \( m = 1 \) it follows from the fact that exchanging \( v_{i1} \) and \( v_{j1} \) defines a bijection on the set \( \Sigma \) that replaces the tableau \( T \) with another tableau with the same first column but exchanged \( i \)-th and \( j \)-th entries on all of the remaining \( 2a - 1 \) columns. This changes the sign with which the corresponding product is taken. Therefore \( \varphi \) is antisymmetric also in the variables \( (v_{11}, \ldots, v_{r1}) \). This implies that the formula
\[ \bigotimes_{i=1}^{2a} (v_{i1} \wedge \ldots \wedge v_{ri}) \to \sum_{T \in \Sigma} \text{sgn} \ T \prod_{j=1}^{r} \left( \prod_{i=1}^{2a} v_{T(j,i),i} \right) \]
defines a map of $\text{GL}(V)$-modules
\[
(\bigwedge^r V)^{\otimes 2a} \to \text{Sym}^r(\text{Sym}^{2a} V).
\]
If $(e_1, \ldots, e_r)$ is a basis of $V$, the element $\bigotimes_{i=1}^{2a}(e_1 \wedge \ldots \wedge e_r)$ is mapped to
\[
W = \sum_{T \in \Sigma} \text{sgn } T \prod_{j=1}^r \left( \prod_{i=1}^{2a} e_{T(j,i)} \right).
\]
Note that $\text{Sym}^r(\text{Sym}^{2a} V)$ has a standard basis corresponding to
\[
\prod_{j=1}^r \left( \prod_{i=1}^{2a} e_{n_{ij}} \right),
\]
where $\sum_i n_{ij} = 2a$ for $j = 1, \ldots, r$. If we write $W$ in this basis, the coefficient at the element $\prod_{j=1}^r e_j^{2a}$ is equal to 1, so the corresponding map is non-zero.

To see the last part of the lemma, it is sufficient to remark that we have
\[
\sum_{T \in \Sigma} \text{sgn } T \prod_{j=1}^r \prod_{i=1}^{2a} e_{T(j,i)} = \sum_{T \in \Sigma} \text{sgn } T \prod_{i=1}^{2a} \prod_{j=1}^r e_j = 0
\]
in $\text{Sym}^{2ra} V$.

**Remark 5.10.** [Wei90, Example 1.9] shows that the plethysm $\text{Sym}^5(\text{Sym}^3 \mathbb{C}^5)$ does not contain $(\bigwedge^5 \mathbb{C}^5)^{\otimes 3}$ as a $\text{GL}(\mathbb{C}, 5)$-submodule. Thus the parity condition in the above lemma is necessary.

**Corollary 5.11.** Let $E$ be a rank $r$ vector bundle on some scheme $X$ defined over some commutative ring $k$. Then for any $a, b \geq 1$ we have

1. a canonical surjection
\[
\text{Sym}^a(\text{Sym}^b E) \twoheadrightarrow \text{Sym}^{ab} E,
\]
2. a canonical inclusion
\[
(\text{det } E)^{\otimes 2a} \hookrightarrow \text{Sym}^r(\text{Sym}^{2a} E)
\]
on onto a subbundle.

Moreover, the composition
\[
(\text{det } E)^{\otimes 2a} \hookrightarrow \text{Sym}^r(\text{Sym}^{2a} E) \rightarrow \text{Sym}^{2ra} E
\]
is zero.

**Proof.** The corollary follows immediately from the previous lemma. For the convenience of the reader we recall the idea of proof. Let $V$ be a free $k$-module of rank $r$ and let $P \to X$ be the principal $\text{GL}(V)$-bundle associated to $E$. Then for any $\text{GL}(V)$-module $W$ we have the associated vector bundle $P(W)$ and maps of $\text{GL}(V)$-modules induce the corresponding maps of vector bundles. Applying this construction to maps from Lemma 5.9 we get the corresponding maps from the corollary.

When $E$ is a $\mu_H$-semistable bundle on $(X, H)$ such that $\Delta(E) \equiv 0$, [Mis21] Lemma 2.3 observes that $\text{Sym}^m E$ is also $\mu_H$-semistable and $\Delta(\text{Sym}^m E) \equiv 0$ for all $m \geq 0$. If furthermore $X$ is 1-homogeneous, then it follows from [Mis21, Theorem 1.2] that $\mathbb{P}(\text{Sym}^m E)$ is 1-homogeneous for all $m \geq 0$. Question 1.3 should also consider $m \geq 1$. 

\[\square\]
Example 5.12. On \( \mathbb{P}^2 \) consider \( \mathcal{E} = \text{Sym}^2(T_{\mathbb{P}^2}(-1)) \). Then \( \mathbb{P}(\mathcal{E}) \simeq \text{Hilb}^2\mathbb{P}^2 \). The divisor \( E \) of nonreduced length 2 subschemes of \( \mathbb{P}^2 \) is contracted by the birational Hilbert–Chow morphism. In particular, it is effective, but not a nef divisor. If \( L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) and \( H \) is the pullback of the class of a line in \( \mathbb{P}^2 \) then \( E \) is linearly equivalent to \( 2(L - H) \). See [FL17b, Section 7.2] for details.

Another perspective at this example is as follows. Since \( \mathcal{E} \) has rank 2, Corollary 5.11 and comparison of ranks imply that we have a short exact sequence of vector bundles

\[
0 \to (\det \mathcal{E})^{\otimes 2} \to \text{Sym}^2(\text{Sym}^2 \mathcal{E}) \to \text{Sym}^4 \mathcal{E} \to 0.
\]

This gives a short exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2} \to (\text{Sym}^2(\text{Sym}^2 T_{\mathbb{P}^2}(-1)))(-2) \to (\text{Sym}^4(T_{\mathbb{P}^2}(-1)))(-2) \to 0.
\]

In particular, we see that \( (\text{Sym}^2(\text{Sym}^2 T_{\mathbb{P}^2}(-1)))(-2) \) is effective. However, it is not nef, e.g., because its quotient \( (\text{Sym}^4(T_{\mathbb{P}^2}(-1)))(-2) \) restricts to \( \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(2) \) on every line \( L \) in \( \mathbb{P}^2 \).

Remark 5.13. Let \( \mathcal{E} \) be a vector bundle on \( X \). Then one can easily see that the following conditions are equivalent:

1. \( \mathbb{P}(\mathcal{E}) \) is 1-homogeneous,
2. If \( D \) is a divisor on \( X \) such that \( (\text{Sym}^m \mathcal{E})(D) \) is effective for some \( m \geq 0 \) then \( (\text{Sym}^m \mathcal{E})(D) \) is nef.

Theorem 5.14. Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \). Let \( \mathcal{E} \) be a vector bundle of rank \( r \) on \( X \). Then \( \mathcal{E} \) is strongly slope semistable with respect to any ample polarization and \( \Delta(\mathcal{E}) \equiv 0 \) if any of the following conditions hold:

1. For every \( m \geq 0 \), we have that \( \mathbb{P}(\text{Sym}^m \mathcal{E}) \) is 1-homogeneous.
2. For every divisor \( D \) on \( X \) and every \( m, l \geq 0 \), we have that \( (\text{Sym}^l(\text{Sym}^m \mathcal{E}))(D) \) is effective if and only if it is nef.
3. There exists \( m \geq 1 \) such that \( \text{Sym}^r(\text{Sym}^{2m} \mathcal{E}) \otimes (\det \mathcal{E}^\vee)^{\otimes 2m} \) is nef.
4. \( \mathbb{P}(\mathcal{E}^\otimes r) \) is 1-homogeneous.
5. \( \mathbb{P}(\text{End} \mathcal{E}) \) is 1-homogeneous.

Proof. The equivalence of (1) and (2) follows from Remark 5.13. We focus on (2). By Corollary 5.11 for all \( m \geq 1 \), the bundle \( \text{Sym}^r(\text{Sym}^{2m} \mathcal{E}) \) contains \( (\det \mathcal{E})^{\otimes 2m} \). Thus \( \text{Sym}^r(\text{Sym}^{2m} \mathcal{E}) \otimes (\det \mathcal{E}^\vee)^{\otimes 2m} \) is effective and hence it is also nef. Since Corollary 5.11 implies that \( \text{Sym}^{2mr} \mathcal{E} \otimes (\det \mathcal{E}^\vee)^{\otimes 2m} \) is a quotient of \( \text{Sym}^r(\text{Sym}^{2m} \mathcal{E}) \otimes (\det \mathcal{E}^\vee)^{\otimes 2m} \), it is also nef. Since nefness for (twisted) vector bundles is homogeneous (cf. Lazarsfeld, Theorem 6.2.12), or [FM21, Lemma 3.24 and Remark 3.10]), we deduce that \( \mathcal{E}(-\frac{1}{r} \det \mathcal{E}^\vee) \) is nef. Conclude by Theorem 1.11. This argument also handled (3).

(4) We have a natural inclusion \( \det \mathcal{E} = \Lambda^r \mathcal{E} \hookrightarrow \mathcal{E}^\otimes r \). It is obtained by dualizing the natural surjection \( (\mathcal{E}^\vee)^{\otimes r} \to \Lambda^r(\mathcal{E}^\vee) \). It shows that \( \mathcal{E}^\otimes r \otimes \det \mathcal{E}^\vee \) is effective. By the assumption on the positive cones, it is then also nef. Hence so is its quotient \( \text{Sym}^r \mathcal{E} \otimes \det \mathcal{E}^\vee \). Argue as above.

(5) We have a natural inclusion \( \mathcal{O}_X \hookrightarrow \text{End} \mathcal{E} \) induced by sending \( 1 \in \mathcal{O}_X(U) \) to \( \text{id}_{\mathcal{E}(U)} \). Therefore \( \text{End} \mathcal{E} \) is effective and hence our assumption implies that it is nef. Now Theorem 4.11 implies the required assertion.

Together with Theorem 5.3, this implies the following result:

Corollary 5.15. Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) and let \( \mathcal{E} \) be a vector bundle of rank \( r \) on \( X \). Then the following conditions are equivalent:

1. \( \mathbb{P}(\text{Sym}^m \mathcal{E}) \) is 1-homogeneous for every \( m \geq 0 \).
2. \( \mathbb{P}(\text{Sym}^{2m} \mathcal{E}) \) is 1-homogeneous for some \( m \geq 1 \).
3. \( \mathbb{P}(\mathcal{E}^\otimes r) \) is 1-homogeneous.
4. \( \mathbb{P}(\text{End} \mathcal{E}) \) is 1-homogeneous.
5. The bundle \( \text{End} \mathcal{E} \) is nef and \( X \) is 1-homogeneous.
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