WEAK-STRONG UNIQUENESS FOR AN ELASTIC PLATE INTERACTING WITH THE NAVIER STOKES EQUATION

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Abstract. We show weak-strong uniqueness and stability results for the motion of a two or three dimensional fluid governed by the Navier-Stokes equation interacting with a flexible, elastic plate of Koiter type. The plate is situated at the top of the fluid and as such determines the variable part of a time changing domain (that is hence a part of the solution) containing the fluid. The uniqueness result is a consequence of a stability estimate where the difference of two solutions is estimated by the distance of the initial values and outer forces. For that we introduce a methodology that overcomes the problem that the two (variable in time) domains of the fluid velocities and pressures are not the same. The estimate holds under the assumption that one of the two weak solutions possesses some additional higher regularity. The additional regularity is exclusively requested for the velocity of one of the solutions resembling the celebrated Ladyzhenskaya-Prodi-Serrin conditions in the framework of variable domains.

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1. Introduction

The paper investigates the interaction between an elastic solid plate and a viscous incompressible fluid. For the fluid we will consider the three (or two) dimensional Navier-Stokes equations [16, 31]. For the solid we consider a shell or a plate that is modeled as a thin object of one dimension less than the fluid and which is assumed to be fixed on the top of a container (See Figure 1). For modeling on elastic plates see [10, 11] and the references therein. The fluid and the plate interact via a kinematic and a dynamic coupling condition on the moving interface.

Our main result consists in the weak-strong uniqueness of solutions for a flow in a variable 3D (or 2D) domain interacting with a 2D (or 1D) plate (see Theorem 1.2). While the regularity of the weak solutions that we use are known to be satisfied for all weak solutions we assume additional regularity of the velocity of the strong solution. Please observe, that we do not assume any additional regularity of the solid displacement. Our assumptions on the regularity index of the velocity are close to the frame of Ladyzhenskaya-Prodi-Serrin condition [37, 39, 40, 28, 26] (for more details see Remark 1.4). As a further consequence we also get a stability estimate (see Theorem 1.5).

While the existence theory for weak solutions describing flexible (thin) shells interacting with fluids has been flourishing in the past years [14, 15, 4, 19, 17, 33, 30, 34, 35, 20, 6, 32] the uniqueness and stability questions are rather untouched. The only available result for an elastic plate seems to be the work of [22]; it treats a 1D elastic beam interacting with a 2D fluid with slip-boundary conditions at the interface. Otherwise, the only weak-strong uniqueness results for fluid-structure interactions are for non-elastic solids, namely rigid objects [41, 18, 8, 5]. For fluid-structure interactions involving elastic materials there are some existence results where the uniqueness of strong solutions is inherited from the methodology of existence. These are short time uniqueness results for strong solutions [12, 13, 1, 3, 21], global uniqueness results of strong solutions for small data [9, 25] and the global uniqueness of strong solutions for a 1D visco-elastic plate interacting with a 2D fluid [24]. As a consequence of our estimates all constructed strong solutions (involving elastic plates) are unique within the class of weak solutions as well.

The applications within this framework consist in fluids interacting with various thin materials. Of particular interest are those in medicine and biology for arteries or the trachea [2, 24]. More concrete are applications of the methodology for respective a-priori estimates for numerical approximations of mathematical solutions that are developed along the concept of weak solutions [23, 42, 38].

To measure the distance between solutions it is necessary to introduce a change of variables as the domains of the velocity fields depend on the solution itself. Moreover, since the solid deformation is...
governed by a hyperbolic equation a mollification in time is unavoidable. In this paper a methodology is introduced that overcomes both obstacles with operators that conserve the property of solenoidality.

\[
\varepsilon_v = \frac{1}{2} \text{div} v
\]

\[
(\varepsilon v : \mu v - \nabla p + \rho_f f)
\]

\[\Omega_\eta(t) := \{(x,y) \in \omega \times (0,\infty) : 0 \leq y \leq \eta(t,x)\}, \ t \in [0,T].\]

Here and in the following \(x\) denotes a 2D (or 1D), \(y\) a 1D and \(z = (x,y)\) a 3D (or 2D) variable. With some misuse of notation we consider the space-time domain

\[\Omega_\eta(t) := \bigcup_{t \in [0,T]} \{t\} \times \Omega_\eta(t).\]

The motion of the fluid is described by the incompressible Navier-Stokes equations

\[
\rho_f (\partial_t v + [\nabla v]v) = \mu_f \Delta v - \nabla p + \rho_f f \quad \text{on } [0,T] \times \Omega_\eta(t),
\]

\[
\text{div} v = 0 \quad \text{on } [0,T] \times \Omega_\eta(t),
\]

where the fluid’s velocity field \(v\) and the pressure \(p\) are the unknown quantities, \(\rho_f\) is the fluid density, \(\mu_f\) the fluid viscosity and \(f\) is a given outer force (e.g. gravity). By \(\sigma(v,p) = 2\mu_f \varepsilon v - pI\) we denote the fluid stress tensor, where \(\varepsilon v := \frac{1}{2} (\nabla v + (\nabla v)^T)\) is the symmetric part of the gradient and \(I\) denotes the identity matrix in 3D, (2D). The incompressibility condition implies that the pressure is determined by the velocity field. On the non-moving parts of the container \(B_c = \omega \times \{0\} \cup \partial \omega \times [0,1]\) we assume no-slip boundary conditions

\[v = 0 \text{ on } [0,T] \times B_c.\]

The moving part of the shell satisfies a linearized plate equation of Koiter type with a source term stemming from the forces the fluid exerts on the shell

\[
\rho_s h_0 \partial_t^2 \eta + E'(\eta) = F(u,p,\eta) + \rho_s g, \text{ on } [0,T] \times \omega,
\]

with Dirichlet boundary conditions

\[\eta = 1, \ \nabla \eta = \Delta \eta = 0 \text{ on } (0,T) \times \partial \omega.\]

Here \(\eta\) is the (scalar valued) unknown, \(\rho_s\) is the solid density, \(h_0\) the thickness of the plate, \(E'(\eta)\) is the \(L^2\) gradient of the elastic energy of the plate, \(F\) are forces stemming from the fluid and \(g\) is a given outer force. Due to the troubles between hyperbolic equations and non-linearities we have to assume that \(E'(\eta)\) is a linear and elliptic operator of 4th order. For this work we assume that

\[E'(\eta) := \beta \Delta \eta_t + \alpha \Delta^2 \eta - \beta \Delta \eta + \gamma \Delta \partial_t \eta\]

with \(\alpha > 0\) and \(\beta, \gamma, \delta \geq 0\). Note that the equations for the fluid are stated in Eulerian coordinates while the equations for the solid are stated in Lagrangian coordinates.

The fluid and the shell are coupled via a kinematic and a dynamic coupling condition on the moving interface. For expressing the coupling conditions we define the variable transform from Langrangian to Eulerain coordinates

\[\psi : [0,T] \times \omega \to [0,T] \times \mathbb{R}^3, \quad (t,x) \mapsto (t,x,\eta(t,x)).\]
The dynamic coupling condition states that the total force in normal direction at the interface is zero
\begin{equation}
F(v, \eta, p) = -(0,1)^T((\nabla v - \rho \kappa) \circ \psi) n \cdot n \text{ on } [0, T] \times \omega,
\end{equation}
where \(n(t, x) = (-\nabla \eta, 1)/(1 + |\nabla \eta|^2)^{1/2}\) is the outer normal of \(\Omega_\eta(t)\) at the point \((x, \eta(x))\).

We assume a no slip kinematic boundary condition, i.e. the fluid and the structure velocity are equal at the interface
\begin{equation}
v \circ \psi = (0, \partial_t \eta)^T \text{ on } [0, T] \times \omega,
\end{equation}
To complete the equations we impose initial conditions
\begin{align}
v(0) &= v_0 \text{ on } \Omega_\eta(0), \\
\eta(0) &= \eta_0, \quad \partial_t \eta(0) = \eta^* \text{ on } \omega.
\end{align}
We will refer to (1.10)-(1.9) as FSI in the following.

By formally multiplying equation (1.1) by \(v\), (1.3) by \(\partial_t \eta\) and integrating over \(\Omega_\eta(t)\), \(\omega\) and \((0, t)\) we get (using Korn's identity Lemma 2.1 and Absorption) the energy inequality
\begin{align}
\|v(t)\|_{L^2(\Omega_\eta(t))}^2 + \|\partial_t \eta(t)\|_{L^2(\omega)}^2 + \|\nabla^2 \eta(t)\|_{L^2(\omega)}^2 + \delta \|\nabla \partial_t \eta(t)\|_{L^2(\omega)} + \int_0^t \|\nabla v(\tau)\|_{L^2(\Omega_\eta(\tau))}^2 d\tau \\
&\leq c \left(\|v_0\|_{L^2(\Omega_\eta(0))}^2 + \|\eta_0\|_{L^2(\omega)}^2 + \|\nabla^2 \eta_0\|_{L^2(\omega)}^2 + \delta \|\nabla \eta^*\|_{L^2(\omega)}^2 + \int_0^t \|f(\tau)\|_{L^2(\Omega_\eta(\tau))}^2 + \|g(\tau)\|_{L^2(\omega)} d\tau\right).
\end{align}
In the paper we use the standard notation for Lebesgue and Sobolev spaces. The weak solutions to FSI are defined in the following function spaces.
\[V_\eta(t) = \{v \in H^1(\Omega_\eta(t)) : \text{div} v = 0 \text{ in } \Omega_\eta(t), v = 0 \text{ on } B_\eta\}, \]
\[V_F = L^\infty([0, T), L^2(\Omega_\eta(t))) \cap L^2([0, T), V_\eta(t)), \]
\[V_K = W^{1,\infty}([0, T], L^2(\omega)) \cap \{\eta \in L^\infty([0, T], H^2(\omega)) : \eta = 1, \nabla \eta = 0 \text{ on } \partial \omega\} \text{ in case } \delta = 0, \]
\[V_K = W^{1,\infty}([0, T], W^{1,2}(\omega)) \cap L^\infty([0, T], H_0^2(\omega)) \text{ in case } \delta > 0, \]
\[V_S = \{(v, \eta) \in V_F \times V_K : v \circ \psi = \partial_t \eta\}, \]
\[V_T = \{(w, \xi) \in V_F \times V_K : w \circ \psi = \xi, \partial_t w \in L^2([0, T]; L^2(\Omega_\eta(t)))\} \]
For the distributional time derivative we introduce the following space
\[\widetilde{W}^{1,r} \cdot s(\Omega) := \{(f \in \tilde{W}^{1,r}(\Omega) : f = 0 \text{ on } B_\eta)\}^*. \]

1.1. Definition. Let \(f \in L^2([0, T] \times \omega \times \mathbb{R}), g \in L^2([0, T] \times \omega, \eta_0 \in H_0^2(\omega), \eta^* \in L^2(\omega) \text{ and } v_0 \in L^2(\Omega_\eta).\) Moreover, if \(\delta > 0\) let additionally \(\eta^* \in H_0^1(\omega).\) Then we call a pair \((v, \eta) \in V_S\) a weak solution to FSI if it satisfies the energy inequality (1.10), if
\begin{align}
\frac{d}{dt} &\left(\int_{\Omega_\eta(t)} v \cdot w dz\right) - \int_{\Omega_\eta(t)} v \cdot \partial_t w - 2\mu v : \varepsilon w + \rho_f(v \circ v) : \nabla \varepsilon w \\
&+ b_0 \rho_s \partial_t \left(\int_\omega \partial_t \eta_\xi dx\right) - b_0 \rho_s \int_\omega \partial_t \eta_\xi dx + (\mathcal{E}'(\eta), \xi) = \rho_f \int_{\Omega_\eta} f \cdot w dz + \rho_s \int_\omega g_\xi dx
\end{align}
for all \((w, \xi) \in V_T\) as an equation in \(D'(0, T)\) and if it attains the initial conditions in the sense of the \(L^2\) weak convergence.

1.2. Main results. Our main result is the following.

1.2. Theorem. In case that \(\omega \subset \mathbb{R}^2\) let \(r > 2\) and \(s > 3\) and in case that \(\omega \subset \mathbb{R}\) let \(r = 2\) and \(s = 2\). Assume that \((v_2, \eta_2)\) is a weak solution to FSI on \([0, T]\), such that \(\min_{[0,T] \times \eta_2} \eta_2 > 0\) and additionally that \(v_2 \in L^r(0, T; W^{1,s}(\Omega_\eta))\) and \(\partial_t v_2 \in L^2(0, T; \tilde{W}^{-1,r}(\Omega_\eta))\). Then this solution is unique in the class of weak solutions. In particular, if \((v_1, \eta_1)\) is any weak solution to FSI on \([0, T_0]\) (for any \(T_0 > 0\)) and if \(v_1(0) = v_2(0), \eta_1(0) = \eta_2(0), \partial_t \eta_1(0) = \partial_t \eta_2(0)\), then \((v_1, \eta_1) \equiv (v_2, \eta_2)\) as an equation in \(V_S\) on \([0, T_0]\).

In some situations strong solutions are known to exist. In particular, in the case of \(\omega = [0, L]\) and \(\delta > 0\) global strong solutions exist [20]. This means that our result implies the following corollary.

1.3. Corollary. In the 2D case \((\omega = [0, L])\) with \(\delta > 0\) and smooth initial values, there exists a global strong solution to FSI which is unique in the class of weak solutions.
1.4. Remark (Minimality of the regularity assumptions on \(v_2\)). Let us compare our assumptions to the case of a non-moving domain, i.e. \(\eta \equiv \eta_0\) and therefore \(\Omega_{\eta_0}\) is constant in time. And let \(v_1, v_2 \in \mathcal{V}_F\) be weak (Leray-Hopf) solutions. If additionally \(v_2\) satisfies the Ladyzhenskaya-Prodi-Serrin condition, namely \(v_2 \in L_t^r(0,T;L_x^s(\Omega))\) for \(3/s + 2/r = 1\), then from the well known regularity and uniqueness result \([37, 39, 40, 28, 26]\) on the Navier-Stokes equations it follows:

\[
\|w(t)\|^2 \leq C\|w(0)\| \exp \left( c \int_0^t \|v_2\|_{L_x^r} \, dy \right).
\]

As \(W_t^1, s(\Omega) \hookrightarrow L^\infty(\Omega)\) for all \(s > 3\) this is in particular true for \(v_2 \in L^2(0,T;W_t^1, s(\Omega))\) or \(v_2 \in L^r_t(0,T;W_t^1, 1(\Omega))\) and \(s > 3\) or \(r > 2\). Please observe that we assume that the strong solution \(v_2 \in L^r_t(0,T;W_t^1, s(\Omega_2))\) for any \(s > 3\) and any \(r > 2\) for flow and no further assumption for 2D flows.\(^1\)

The reason why we need both, \(s > 3\) and \(r > 2\) seems to be due to the fact that the deformation \(\eta_2\) is a-priori not uniformly (in time) Lipschitz (at least in case \(\delta = 0\)). Moreover, we have to assume the higher integrability is on \(\nabla v_2\) (and not on \(v_2\)) since the regularity theory for 3D fluids satisfying the Ladyzhenskaya-Prodi-Serrin condition is not yet known to be satisfied (even in some cases for 2D flows).

While the index is the same we have to assume that the negative space is smaller than generally assumed. This additional regularity seems not to be replaceable with the current state of the art; since in the case of variable geometries it might very well be unavoidable that the regularity of the pressure intervenes.

Further we prove the following stability estimate.

1.5. Theorem. Let \((v_2, \eta_2)\) be weak solutions to FSI on \([0,T]\), such that \(\min_{[0,T] \times \omega} \eta_2 > 0\) and that additionally \(v_2 \in L^r_t(0,T;W_t^1, s(\Omega_{\eta_2}))\) and \(\dg_t v_2 \in L^2(0,T;\dot{W}^{-1, r}(\Omega_{\eta_2}))\) for any \(s > 3\) and any \(r > 2\). If \((v_1, \eta_1)\) is a weak solution to FSI on \([0,T]\), then for \(\tilde{v}_2(t,x,y) = v_2(t,x,y, \eta_2(t,x,y))\) we find that

\[
\sup_{\tau \in [0,T]}\left(\|\tilde{v}_2(t)\|_{L_x^2(\Omega_{\eta_1}(\tau))}^2 + \|\dg_t \eta_1 - \dg_t \eta_2(t)\|_{L_x^2(\omega)}^2 + \|\eta_1 - \eta_2(t)\|_{L_x^2(\omega)}^2\right)\leq C \left(\|\eta_1 - \eta_2(t)\|_{L_x^2(\omega)}^2 + \|\eta_1 - \eta_2(t)\|_{L_x^2(\omega)}^2 + \|\nabla (\eta_1 - \eta_2(t))\|_{L_x^2(\omega)}^2\right)\]

where the constant depends on \(\omega, T\), the assumed bounds on \(v_2\), the \(L^2\)-bounds of \(f_1, f_2\) and (symmetrically) on the two deformations \(\eta_1, \eta_2\) via the bounds related to the energy estimates and via Theorem 2.2\(^2\).

In particular, the constant \(C\) can be bounded a-priori in dependence of \(\omega, T\), the assumed bounds on \(v_2\) and the right hand side of the energy inequality \((1.10)\) for both solutions.

1.3. Analytical strategy & technical novelties. Usually for uniqueness (or stability estimates) one takes the difference of the two solutions or, in case of a hyperbolic evolution, its time-derivative as a test function. We wish to emphasize that due to the variable geometry depending on the solution, even uniqueness of strong solutions for longer times (provided they exist) does not follow in a straightforward manner. An additional difficulty regarding weak-strong uniqueness results is that the regularity of one solution is too low to be used as a test function. We follow the approaches developed in \([11, 7, 8]\). The idea is to resolve the difference of the systems tested by the difference of solutions into the energy inequality of the weak solution and terms containing a coupling where at least one function is sufficiently regular.

In order to make one fluid velocity a test function for the other equation we follow the methodology introduced in \([22]\) where a change of variables from one geometry to the other is introduced that conserves the solenoidality property. This surffaces to circumvent the weak regularity properties of the pressure in case of incompressible fluids.\(^3\) What can not be circumvented is the weak regularity of the time-derivative of the involved test-functions. The technical highlight is a mollification-in-time operator that conserves solenoidality in variable domains and that does not reduce the regularity (in space) significantly. The operator is introduced in Lemma 2.6. A result that might be of independent interest

\(^1\)For the notation please see the next section.

\(^2\)In unsteady incompressible problems the pressure is known to be hard to control w.r.t. the time variable even in the simplest case of Stokes equation in a fixed (smooth) geometry \([27]\).
is that this mollification can be used to show that all weak solutions do indeed have a distributional time derivative in a Bochner space involving negative Sobolev spaces (see Proposition 2.7). Finally, of further use in the future might be the estimates (especially on the convective term) which were necessary in order to stay with our assumptions that close to the Ladyzhenskaya-Prodi-Serrin conditions.

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2. Notation & preliminary results

2.1. Simplifications. In order to simplify the quite technical argument below we assume in the following that $\mathcal{E}'(\eta) \equiv \Delta^2 \eta$; as the argument can be adapted to more general $\mathcal{E}'$ in a straightforward manner. Moreover we will assume in the following that we have a fluid in 3D. In particular we assume that $\omega \subset \mathbb{R}^2$. The adaptation of the proof for $\omega \subset \mathbb{R}$ implies only simplifications and no further complications. Finally we set all constants in the equations to one (i.e. both densities, the thickness of the plate, the viscosity of the fluid).

For vector valued functions $u : \Omega \to \mathbb{R}^3$ we use $u = (u^1, u^2, u^3)^T = (u^1, u^2, u^3)^T$. The constants $c, c_1, ...$ are used as constants that are independent of $\eta$, while the constants $C, C_1, ...$ are used as constants that may depend on bounded quantities of the deformations. Both letters $c, C$ may change there actual value with every instance. Moreover, we use the notation $a \sim b$, if there are constants $c, c_1$ such that $|a| \leq c|b| \leq c_1|a|$.

2.2. Identities & Estimates. We will use Reynold’s transport theorem which for plates reads (using the fact that the third component of the outer normal times the Jacobian of the change of variables is zero boundary values on large parts of the boundary and the inequality is a straight consequence of the fundamental theorem of calculus). Korn’s identity follows by [32, Lemma 4.1].

For vector valued functions $u : \Omega \to \mathbb{R}^3$ we use $u = (u^1, u^2, u^3)^T = (u^1, u^2, u^3)^T$. The constants $c, c_1, ...$ are used as a constants that are independent of $\eta$, while the constants $C, C_1, ...$ are used as constants that may depend on bounded quantities of the deformations. Both letters $c, C$ may change their actual value with every instance. Moreover, we use the notation $a \sim b$, if there are constants $c, c_1$ such that $|a| \leq c|b| \leq c_1|a|$.

2.1. Lemma. Let $u \in H^1(\Omega_\eta)$ such that $u = 0$ on $B_c$ and $u^i(x, \eta(x)) = 0$ for all $x$, we find

$$\partial_t \left( \int_{\Omega_\eta} u(t, z) \cdot \phi(t, z) \, dz \right) = \int_{\Omega_\eta} \partial_t (u \cdot \phi) \, dz + \int_\omega u^i(t, x, \eta(x)) \phi^i(t, x, \eta(x)) \partial_t \eta(t, x) \, dx,$$

for all $\phi, \eta$ for which the above expression is well defined.

Next due to the zero boundary conditions of $u^i$ on $\partial \Omega$ we actually may use Korn’s identity which is done throughout the paper.

2.2. Theorem. For any weak solution to FSI we find that as long as $\eta > 0$ in $[0, T) \times \omega$ that $\eta \in L^2(0, T; H^{2+\sigma}(\omega))$ and $\partial_t \eta \in L^2(0, T; H^\sigma(\omega))$ for all $\sigma < \frac{1}{2}$.

An adaption of [32, Theorem 1.2] is the following corollary.

We will need the following interpolation estimate:

1. $L^a(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^2)$ for all $a \in [1, 2]$.

2.3. Lemma. For $Y \subset \mathbb{R}^2$. If $b \in L^\infty(0, T; L^2(Y))$ and $\phi \in L^2(0, T; W^{1,a}(Y))$ for all $a \in (1, 2)$, then $|b| \phi \in L^2(0, T; L^p(Y))$ for all $p \in (1, 2)$.

Proof. The result follows by Sobolev embedding and Hölder’s inequality.
2.4. Lemma. Let $(\eta, v)$ be a weak solution to FSI. Then we find that $\int_0^\eta(t,x) |v| dy \in L^2(0,T; H^1(\omega))$

$$\left\| \int_0^\eta(t,x) |v| dy \right\|_{L^2(0,T; H^1(\omega))} \leq c\|v\|_{L^2(0,T; H^1(\omega))}\|\eta\|_{L^\infty(0,T; H^2(\omega))}$$

and $\int_0^\eta(t,x) |v|^2 dy \in L^2(0,T; W^{1,1}(\omega))$

$$\left\| \int_0^\eta(t,x) |v(t)|^2 dy \right\|_{L^2(0,T; W^{1,1}(\omega))} \leq \|v_1\|_{L^2(0,T; L^2(\Omega_\delta))} + 2\|v_1\|_{L^\infty(0,T; L^2(\Omega_\delta))}\|\nabla v_1\|_{L^2([0,T] \times \omega)}$$

$$+ \|\partial_t \eta\|^2_{L^2(0,T; L^2(\omega))}\|\nabla \eta\|_{L^2(0,T; L^\infty(\omega))}.$$ 

This implies in particular that $\int_0^\eta(t,x) |v|^2 dy \in L^2([0,T] \times \omega)$.

Proof. For the first statement we calculate

$$\nabla_x \int_0^\eta(t,x) |v(x,y)| dy = \int_0^\eta(t,x) \nabla_x |v(x,y)| dy + \nabla_x \eta(t,x) \partial_t \eta,$$

which is uniformly bounded in $L^2([0,T] \times \omega)$ since $v_1 \in L^2(0,T; H^1(\Omega_\delta))$, $\partial_t \eta \in L^2(0,T; L^2(\omega))$ and $\nabla \eta \in L^\infty(0,T; L^0(\omega))$. The estimate follows using Sobolev embedding and the trace theorem [6, Lemma 6].

For the second statement we calculate

$$\nabla_x \int_0^\eta(t,x) |v|^2 dy = \int_0^\eta(t,x) 2|\nabla v| dy + |v(\eta(t,x))|^2 \nabla \eta(t,x)$$

$$= \int_0^\eta(t,x) 2|\nabla v| dy + |\partial_t \eta(t,x)|^2 \nabla \eta(t,x) =: I_1 + I_2.$$

Due to Hölder’s inequality

$$\int_\omega I_1 \leq 2\|v_1\|_{L^2(\Omega_\delta)}\|\nabla v_1\|_{L^2(\Omega_\delta)}.$$

And it is also straightforward to see

$$\int_\omega I_2 \leq \|\partial_t \eta\|^2_{L^2(\omega)}\|\nabla \eta\|_{L^\infty(\omega)}.$$ 

Thus the statement follows since $v \in L^\infty(0,T; L^2(\Omega_\delta)) \cap L^2(0,T; H^1(\Omega_\delta))$, $\partial_t \eta \in L^\infty(0,T; L^2(\omega))$ and by Theorem 2.2 $\eta \in L^2(0,T; H^{2+\sigma}(\omega)) \hookrightarrow L^2(0,T; W^{1,\infty}(\omega))$ for all $\sigma > 0$. □

2.3. Convolution. Since the regularity in space of $\partial_t \eta$ and the regularity in time for $v$ a test function is formally not sufficient to use the couple as a test function we have to introduce a mollification in time. Unfortunately, it was not possible to use the mollification introduced [32] and we have to introduce a new version. Already here the regularity of the deformation influences the regularity of the mollification sensibly due to the fact that a change of variables will be a part of the convolution kernel.

First a technical Lemma. Here we will use a mollifier with respect to time. As is the standard procedure, choose a function $j \in C_0^\infty(\mathbb{R})$ which is positive, even, has support in $(-1,1)$ and satisfies $\int_\mathbb{R} j dt = 1$, $\frac{d}{dt} j(t) \geq 0$, $\frac{d}{dt} j(t) \leq 0$ for $t \geq 0$. For $\delta > 0$ define $j_\delta(t) \equiv \delta^{-1} j(t/\delta)$. Then $j_\delta$ has support in $(-\delta, \delta)$ and otherwise the same properties as $j$.

Let $(H, (\cdot, \cdot))$ be a Hilbert space, $T > 0$. Let $u \in L^\infty(0,T; H)$ be continuous w.r.t. the weak topology on $H$ and assume that the limits $u(0) := \lim_{t \to 0} u(t)$, $u(T) := \lim_{t \to T} u(t)$ exist in the weak topology of $H$. In the following we will call the space of all such functions $C_u(0,T; H)$. Define the extension $\tilde{u} \in L^\infty(\mathbb{R}, H)$ by

$$\tilde{u}^T(t) = \begin{cases} u(t), & t \in (0,T), \\ u(0), & t \in (-\infty,0), \\ u(T), & t \in [T,\infty]. \end{cases}$$

Now for all $\delta > 0$, $t \in [0,T]$ set

$$u_\delta^T(t) = \int_\mathbb{R} j_\delta(t-s) \tilde{u}^T(s) ds.$$ 

It is well known that $u_\delta^T \in C^\infty([0,T], H)$ and $\lim_{\delta \to 0} u_\delta = u$ in $L_p(0,T; H)$ for all $1 \leq p < \infty$. Furthermore the following holds
2.5. **Lemma.** Let \( u, v \in C_w(0, T; H) \) and \( t \in (0, T] \). Then for all \( t \in [0, T] \)

\[
\lim_{\delta \to 0^+} \int_0^t \langle u, u_\delta^T \rangle - \langle u_\delta^T, v \rangle \, dt = 0
\]

and

\[
\lim_{\delta \to 0^+} \int_0^T \left( u, \frac{d}{dt} u_\delta^T \right) + \left( \frac{d}{dt} u_\delta^T, v \right) \, dt = (u(T), v(T)) - (u(0), v(0))
\]

**Proof.** In the following we omit the superscript \( T \). The first assertion holds since

\[
(u, v_\delta) - (u_\delta, v) = (u, v_\delta - v) + (v, u - u_\delta).
\]

and the weak continuity in time.

To prove the second assertion note that \( \partial_s j_\delta \) is an odd function and therefore

\[
\int_0^T \int_0^T \frac{d}{dt} j_\delta(\tau - s)(v(s), u(\tau)) \, ds \, d\tau = - \int_0^T \int_0^T \frac{d}{dt} j_\delta(\tau - s)(u(s), v(\tau)) \, d\tau \, ds.
\]

Hence

\[
\begin{align*}
&\int_0^T \left( u, \frac{d}{dt} u_\delta \right) + \left( \frac{d}{dt} u_\delta, v \right) \, dt \\
&= \int_0^T \left( u(\tau), \int_0^\tau \frac{d}{ds} j_\delta(\tau - s) \frac{\partial}{\partial s}(u(\tau)) \, ds \right) \, d\tau + \int_0^T \left( u(\tau), \int_\tau^\infty \frac{d}{ds} j_\delta(\tau - s) \frac{\partial}{\partial s}(u(\tau)) \, ds \right) \, d\tau \\
&\quad + \int_0^T \left( v(\tau), \int_0^\tau \frac{d}{ds} j_\delta(\tau - s) \frac{\partial}{\partial s}(u(\tau)) \, ds \right) \, d\tau + \int_0^T \left( v(\tau), \int_\tau^\infty \frac{d}{ds} j_\delta(\tau - s) \frac{\partial}{\partial s}(u(\tau)) \, ds \right) \, d\tau \\
&\quad := R_1(\delta) + R_2(\delta) + R_3(\delta) + R_4(\delta).
\end{align*}
\]

By symmetry it suffices to prove \( R_1(\delta) \to -\frac{1}{2}(u(0), v(0)) \) and \( R_2(\delta) \to \frac{1}{2}u(t)v(t) \). As \( \tilde{v}(s) \equiv v(0) \) for all \( s < 0 \) and \( j_\delta \) has support in \((-\delta, \delta)\) we get

\[
R_1(\delta) = \int_0^\delta (v(0), u(\tau)) \int_\tau^\delta \frac{d}{ds} j_\delta(s) \, ds \, d\tau = \int_0^\delta (v(0), u(\tau)) \int_\tau^\delta \frac{d}{ds} j_\delta(s) \, ds \, d\tau
\]

\[
= \int_0^\delta (v(0), u(\tau))(j_\delta(\delta) - j_\delta(\tau)) \, d\tau = -\frac{1}{\delta} \int_0^\delta (v(0), u(\tau))j(\frac{T}{\delta}) \, d\tau
\]

\[
= -\int_0^1 (v(0), u(\delta \tau))j(\tau) \, d\tau.
\]

By weak continuity we get

\[
\lim_{\delta \to 0^+} (v(0), u(\delta \tau))j(\tau) = (v(0), u(0))j(\tau).
\]

As \( u \in L^\infty(0, T; H) \) we get by dominated convergence

\[
\lim_{\delta \to 0^+} R_1(\delta) = -(v(0), u(0)) \int_0^1 j(\tau) \, d\tau = -\frac{1}{2}(v(0), u(0)).
\]

The convergence of \( R_2(\delta) \) is analogous. \( \square \)

Here and in the following we will always consider the extension \( \overline{\eta} \) introduced above implicitly. Meaning, that when ever necessary we extend any function to a global in (positive and negative) time object. In order to treat distributional time derivatives we will use the notation of the dual product over a variable domain by

\[
\int_0^T \langle f, \phi \rangle_{\eta(t)} \, dt := \int_0^T (f(t), \phi(t))_{\Omega_{\eta(t)}} \, dt,
\]

where \( \langle f, \phi \rangle_{\Omega_{\eta(t)}} \) is the dual product over function spaces over \( \Omega_{\eta(t)} \) which are assumed to be bilinear mappings that map into measurable functions in time.

For our case of moving boundaries we will need the following convolution result that allows to con volve with respect to the moving geometry by keeping the solenoidality.

2.6. **Lemma.** Let \( \eta \in \mathcal{V}_K \), such that \( \eta \) is bounded uniformly from below. Let \( \phi \in L^\alpha(0, T; L^q(\Omega)) \cap L^\alpha(0, T; W^{1,\alpha}(\Omega)) \) for some \( a > 1 \) and \( \alpha, \nu, q \geq 1 \). Let \( b \in L^2(0, T; L^1(\omega)) \) with \( \phi(t, x, \eta(x)) = (0, b(t, x)) \) on \([0, T] \times \omega \) (in the sense of traces).
Set \( K : [0, T] \times [0, T] \times \mathbb{R} \times \omega \to \mathbb{R}^{3 \times 3} \)

\[
K(s, t, y, x) = \begin{pmatrix}
\frac{\eta(s, x)}{\eta(t, x)} & 0 & 0 \\
0 & \frac{\eta(s, x)}{\eta(t, x)} & 0 \\
y \partial_x \left( \frac{\eta(s, x)}{\eta(t, x)} \right) & -y \partial_x \left( \frac{\eta(s, x)}{\eta(t, x)} \right) & 1
\end{pmatrix}
\]

For each \( \delta > 0 \) define \( b_\delta = b \ast j_\delta \) and

\[
\phi_\delta(t, x, y) = \int_0^T K(s, t, y, x) \phi \left( s, x, y \frac{\eta(s, x)}{\eta(t, x)} \right) j_\delta(t - s) \, ds.
\]

Then it holds for \( \nu < \infty \) that

\[
\operatorname{div} \phi_\delta = 0, \quad \phi_\delta(t, x, \eta(x)) = b_\delta(t, x)
\]

and \( \phi_\delta \to \phi \) strongly \( L^p(0, T; L^p(\Omega_\eta(t))) \) for all \( p \in [1, \nu) \).

Moreover,

1. If \( \phi \in L^2(0, T; W^{1, \alpha}((\Omega_\eta))) \) for all \( \alpha \in (1, 2) \), then \( \phi_\delta \to \phi \) converges weakly in \( L^2(0, T; W^{1, p}(\Omega_\eta(t))) \) for all \( p \in [1, 2) \).
2. If \( \phi \in L^2(0, T; W^{1, \alpha}(\Omega_\eta)) \) for all \( \alpha > 3 \), then \( \phi_\delta \to \phi \) converges weakly in \( L^2(0, T; W^{1, 2}(\Omega_\eta(t))) \).
3. If \( \phi \in W^{1, 2}(0, T; W^{-1, \psi}(\Omega_\eta)) \cap L^2(0, T; W^{1, \alpha}(\Omega_\eta)) \) for some \( \alpha > 3 \) and some \( p \in (1, 2) \), then \( \partial_t \phi_\delta \) weakly to \( \partial_t \phi \) in \( L^2(0, T; W^{-1, \psi}(\Omega_\eta)) \).

Proof. We define

\[
\phi(s, t, x, y) = K(s, t, x, y) \phi \left( s, x, y \frac{\eta(s, x)}{\eta(t, x)} \right)
\]

If we show that \( \operatorname{div} \phi(t, x, y) \equiv 0 \) then clearly also \( \operatorname{div} \phi_\delta = 0 \). We get

\[
\operatorname{div} \phi = \left( \frac{\eta(s, x)}{\eta(t, x)} \right) \phi_1 + \frac{\eta(s, x)}{\eta(t, x)} \operatorname{div} \phi \phi_1 + y \delta_y \phi_1 \nabla \left( \frac{\eta(s, x)}{\eta(t, x)} \right) \frac{\eta(s, x)}{\eta(t, x)} - \nabla \left( \frac{\eta(s, x)}{\eta(t, x)} \right) \phi_1
\]

\[
- y \nabla \left( \frac{\eta(s, x)}{\eta(t, x)} \right) \frac{\eta(s, x)}{\eta(t, x)} \delta_y \phi_1 + \eta(s, x) \delta_y \phi_2 = \eta(s, x) \delta_y \phi_2 + \delta_x \phi_1 = 0,
\]

where we used in the last line that \( \phi_1 = 0 \). Now as \( \phi(t, x, \eta(t, x)) = (0, b(t, x)) \) we get

\[
\phi(s, t, x, \eta(t, x)) = \phi(s, x, \eta(s, x)) = (0, b(s)).
\]

Thus

\[
\phi_\delta(t, x, \eta(t, x)) = \int_0^T b(s) j_\delta(t - s) \, ds = b_\delta(t, x).
\]

For the convergence result we introduce the function on the reference domain

\[
\phi_0 : [0, T] \times \omega \times [0, 1] \to \mathbb{R}^3, \quad (t, x, y) \mapsto \phi(t, x, y \eta(t, x)).
\]

Let \( p \in [1, \nu) \). First we estimate \( \phi_\delta^2 - \phi_1^2 \) in \( L^p(0, T; L^p(\Omega_\eta(t))) \). We have

\[
(\phi_\delta^2 - \phi_1^2)(t, x, y) = \int_0^T \frac{\eta(s, x)}{\eta(t, x)} \delta^2(s, x, y \eta(s, x)) - \phi_1^2(t, x, y) \eta(t, x)) j_\delta(t - s) \, ds
\]

Hence (by a change of variables) we find

\[
\int_0^T \left( \int_{\Omega_\eta(t)} |(\phi_\delta^2 - \phi_1^2)(t, x, y)|^p \, dx \, dy \right)^{\frac{1}{p}} \, dt
\]

\[
= \int_0^T \left( \int_{\omega \times [0, 1]} \left( \int_0^T (\eta(s, x) \phi_1(s, x, y \eta(s, x)) - \eta(t, x) \phi_1(t, x, y \eta(t, x))) j_\delta(t - s) \, ds \right)^p \, dz \right)^{\frac{1}{p}} \, dt
\]

\[
= \|\phi_\delta - \phi_1\|_{L^p(0, T; L^p(\omega \times [0, 1]))}
\]

for \( \varphi(t, x, y) = \eta(t, x) \phi(t, x, y) \). As \( \eta \in L^\infty(0, T; L^\infty(\omega)) \) and \( \phi \in L^p(0, T; L^q(\Omega_\eta)) \) this converges to 0 by standard convolution estimates. Next note by a similar argument that

\[
\int_0^T \left( \int_{\Omega_\eta(t)} \int_0^T \left( \phi_2^2(s, x, y \eta(s, x)) - \phi_1^2(t, x, y) \right) j_\delta(t - s) \, ds \right)^p \, dz \right)^{\frac{1}{p}} \, dt
\]

\[
\leq \|\eta\|_{L^p(0, T; L^p(\omega))} \|\phi_0^2 - \phi_1^2\|_{L^p(0, T; L^p(\omega \times [0, 1]))},
\]
which also converges to 0. Lastly

\[ \int_0^T \left( \int_{\Omega_n(t)} \left| \int_0^T y \nabla \left( \frac{\eta(s)}{\eta(t)} \right) \phi^1(s, x, y, \frac{\eta(s)}{\eta(t)}) j_\delta(t - s) ds \right|^p dz \right) \frac{dt}{p} \]

\[ = \int_0^T \left( \int_{\omega \times [0,1]} \left| \int_0^T y \eta(t) \nabla \left( \frac{\eta(s)}{\eta(t)} \right) \phi^1_0(s, x, y) j_\delta(t - s) ds \right|^p dz \right) \frac{dt}{p} \]

As \( j_\delta \) has unit integral we can compute

\[ \int_0^T \eta(t) \nabla \left( \frac{\eta(s)}{\eta(t)} \right) \phi^1_0(s) j_\delta(t - s) ds = \int_0^T \phi^1_0(j_\delta(t - s)(\nabla \eta(s) - \nabla \eta(t)) + \frac{\nabla \eta(t)}{\eta(t)}(\eta(t) - \eta(s))) \right) ds \]

\[ = \int_0^T j_\delta(t - s)(\phi_0(s) \nabla \eta(s) - \phi_0(t) \nabla \eta(t)) + j_\delta(t - s) \frac{\nabla \eta(t)}{\eta(t)}(\phi_0(s) \eta(s) - \phi_0(t) \eta(t)) \right) \]

\[ + 2 j_\delta(t - s) \nabla \eta(t)(\phi_0(t) - \phi_0(s)) \]

Thus

\[ \int_0^T \left( \int_{\Omega_n(t)} \left| \int_0^T y \nabla \left( \frac{\eta(s)}{\eta(t)} \right) \phi^1(s, x, y, \frac{\eta(s)}{\eta(t)}) j_\delta(t - s) ds \right|^p dz \right) \frac{dt}{p} \]

\[ \leq \|\nabla \eta\|_{H^1_0(\Omega_n)}^p \leq \int_0^T \left( \int_{\omega \times [0,1]} \left| \int_0^T j_\delta(t - s)(\nabla \eta(s) - \nabla \eta(t)) + \frac{\nabla \eta(t)}{\eta(t)}(\eta(t) - \eta(s))) \right) ds \right) \frac{dt}{p} \]

\[ + \int_0^T \left( \int_{\omega \times [0,1]} \left| \nabla \eta(t) \right|^p \phi^1_0 j_\delta(t) - \phi^1_0 \right) dz \]

The first term converges to 0 by standard convolution. The third term we can estimate as \( p < q \)

\[ \int_0^T \left( \int_{\omega \times [0,1]} \left| \nabla \eta(t) \right|^p \phi^1_0 j_\delta(t) - \phi^1_0 \right) dz \]

Hence this term converges to 0 as well. The third term can be estimated analogously using the assumed uniform lower bounds on \( \eta \).

As we have shown strong convergence in \( L^2(0, T; L^p(\Omega_n(t))) \) it suffices to show that \( \nabla \phi_\delta \) is bounded in \( L^2(0, T; L^p(\Omega_n(t))) \) to prove weak convergence. The estimate on the gradient is a standard exercise combining the bounds of \( \eta \) and \( \phi \) via Hölder’s inequality. We omit here most of the details, since the estimates depend on. The critical terms are for one (1), (2), (3) estimated by

\[ \|\nabla \phi\| \leq L^2(0, T; L^p(\Omega_n)) \text{ for all } p \in [1, a). \]

and for the second

For (1) \( \|\nabla^2 \eta\| \leq L^2([0, T]; L^p(\Omega_n)) \) for all \( p \in [1, 2] \) by Lemma 2;

For (2) \( \|\nabla^2 \eta\| \leq L^2([0, T]; L^2(\Omega_n)) \) as \( \phi \in L^2(L^\infty) \) by Sobolev embedding.

Next let us consider the weak time derivative. Let us take \( \psi \in \tilde{W}^{1,p}([0, T] \times \omega \times \mathbb{R}) \), such that \( \psi(t, x, y) = 0 \) for all \( x \in B_\epsilon \) and \( \|\psi\|_{\tilde{W}^{1,p}([0, T] \times \omega \times \mathbb{R})} \leq 1 \) to find that

\[ \int_0^T \left\langle \partial_t \phi_\delta, \psi \right\rangle = \int_0^T \left( \int_0^T \left( \int_{\Omega_n(t)} K(s, t, y, x) \phi^1(s, x, y, \frac{\eta(s)}{\eta(t)}) j_\delta(t - s, \psi(t, z)) ds \right) dt \right) \]

\[ + \int_0^T \left( \int_{\omega \times [0,1]} \left| \int_0^T K(s, t, y, x) \phi^1_0(s, x, y) \eta(t) j_\delta(t - s, \psi(t, z)) dz \right|^p dz \right) \frac{dt}{p} \]

\[ - \int_0^T \left( \int_{\omega \times [0,1]} \left| \int_0^T K(s, t, y, x) \phi^1_0(s, x, y) \eta(t) j_\delta(t - s, \psi(t, z)) dz \right|^p dz \right) \frac{dt}{p} \]

\[ = (I) + (II) + (III) \]
The expression \( I \) can be transferred into an integral by using partial integration in \( x_i \) and the fact that \( \phi^\prime(t, x, \eta(t), x) = 0 \) for \( i \in \{1, 2\} \) and \((t, x) \in [0, T] \times \omega:\)

\[
(I) = \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{T} \left( - \left\langle y \partial_t \partial_{x_i} \left( \frac{\eta(s, x)}{\eta(t)} \right) \phi^\prime \left( s, x, y \frac{\eta(s)}{\eta(t)} \right), \psi^3(t) \right\rangle \right) ds \, dt \\
+ \int_{\Omega_{\eta(t)}} \partial_t \left( \frac{\eta(s, x)}{\eta(t)} \right) \phi^\prime \left( s, x, y \frac{\eta(s)}{\eta(t)} \right) \cdot \psi^3(t, z) \, dz \, \int_{t-s}^{t} j_6(t-s) \, ds \, dt
\]

\[
= \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{T} \partial_{x_i} \left( \frac{\eta(s, x)}{\eta(t)} \right) \left( \partial_t \int_{0}^{\eta(t,x)} y \phi^\prime \left( s, x, y \frac{\eta(s)}{\eta(t)} \right) \cdot \psi^3(t, x, y) \, dy \right) ds \, dt \\
+ \int_{0}^{\eta(t,x)} \int_{t-s}^{t} j_6(t-s) \, ds \, dt
\]

But these expression can be estimated using that \( p^* = \frac{3p}{4p-3} \) can be assumed to be close enough to 6 such that

\[
(I) \leq C \int_{0}^{T} \| \partial_t \eta \|_{L^2(\omega)} \left( \| \phi \|_{W^{1,\gamma}(\Omega_{\eta(t)})} + \| \nabla \phi \|_{L^3(\omega)} \right) dt.
\]

This expression is bounded as \( \partial_t \eta \in L^\infty(0, T; L^2(\omega)) \), \( \| \nabla \eta \|_{L^3(\omega)} \in L^2(0, T; L^q(\Omega_{\eta(t)})) \) for all \( q \in [3, s) \). The estimate on \((III)\) is analogous (but simpler).

For \((II)\) we use \( \partial_t j_6(t-s) = \partial_t \int_{t-s}^{t} j_6(t-s) \, ds \, dt \) to find (using the 0-trace of \( j_6(t-s) \) that)

\[
(II) = \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{T} \partial_{x_i} \left( K(s, t, y, x) \phi \left( s, x, y \frac{\eta(s)}{\eta(t)} \right), j_6(t-s), \psi(t, z) \right) \right\rangle ds \, dt
\]

\[
= \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{T} \int_{\Omega_{\eta(t)}} \partial_t K(s, t, y, x) \phi \left( s, x, y \frac{\eta(s)}{\eta(t)} \right) j_6(t-s), \psi(t, z) \right\rangle ds \, dt
\]

\[
= \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{T} \int_{\Omega_{\eta(t)}} \partial_t \phi \left( s, x, y \frac{\eta(s)}{\eta(t)} \right) j_6(t-s), \psi(t, z) \right\rangle ds \, dt
\]

First observe that \( I_1 = 0 \). The estimates on \( I_2, I_3 \) are similar to the estimate of \( I \) above. Now, finally \( I_4 \) is estimated using the assumption on \( \partial_t \phi \). We define \( K^T(s, t, y, x) \) in such a way that

\[
I_4 = \int_{0}^{T} \int_{0}^{T} \int_{\Omega_{\eta(t)}} \partial_t \phi \left( s, x, y \frac{\eta(s)}{\eta(t)} \right), K^T(t-s, y, x) \psi(t, z) \right\rangle \right\rangle ds \, dt
\]

\[
= \int_{0}^{T} \int_{0}^{T} \int_{\Omega_{\eta(t)}} \partial_t \phi \left( s, x, y \frac{\eta(s)}{\eta(t)} \right), K^T(t-s, y, x) \psi \left( s, x, y \frac{\eta(t)}{\eta(s)} \right) \right\rangle \right\rangle ds \, dt
\]

This implies that

\[
I_4 \leq \int_{0}^{T} \int_{\Omega_{\eta(t)}} \| \partial_t \phi(s) \|_{W^{1,\gamma}(\Omega_{\eta(t)})} \| K^T(t-s, y, x) \psi \left( s, x, y \frac{\eta(t)}{\eta(s)} \right) \|_{W^{1,\gamma}(\Omega_{\eta(t)})} j_6(t-s) \, ds \, dt,
\]

which is uniformly bounded using \( \| \nabla \psi \|_{L^2(\Omega_{\eta(t)})} \) and \( \| \psi \|_{L^2(\omega)} \) for all \( p \in [1, 2] \).

2.4. The distributional time derivatives. En pasent we include here a result that is independent of our main result but might be important for further use. Here a meaning is given to the distributional time derivative of solutions.
2.7. Proposition. Let \((v,p,\eta)\) be a weak solution satisfying [1.11], then if \(v \in L^2(0,T; W^{1,q}(\Omega))\) for \(s \geq 2\), then
\[
\partial_t v + [\nabla v] v \in L^2(0,T; (W^{1,q}_{0,\text{div}}(\Omega))^*),
\]
for any \(q \in (2,\infty)\) if \(s = 2\) and \(q = 2\) if \(s > 2\).

This means that for \(\phi \in L^2(0,T; W^{1,q}_{0,\text{div}}(\Omega))\) we find that
\[
(\partial_t v + [\nabla v] v, \phi) dt = -\int_0^T \int_{\Omega_n} \nabla v \cdot \nabla \phi dx dt.
\]

Moreover, \((\partial_t v + [\nabla v] v, \partial^2_t \eta) \in L^2(0,T; W^*)\) for
\[
W = \{(\phi, b) \in W^{1,q}_{0,\text{div}}(\Omega) \times W^{2,2}(\omega) : \phi(t, x, \eta(x)) = b(t, x)\}
\]
for any \(q \in (2,\infty)\) if \(s = 2\) and \(q = 2\) if \(s > 2\).

In particular, for all \((\phi, b) \in W\) we find that
\[
\int_0^T (\partial_t v + [\nabla v] v, \phi) + (\partial^2_t \eta, b) dt = -\int_0^T \int_{\Omega_n} \nabla v \cdot \nabla \phi dx dt + \int_0^T \int_{\omega} \nabla^2 \eta \cdot \nabla^2 b dx dt.
\]

Proof. Let \(\phi \in L^2(0,T; W^{1,q}_{0,\text{div}}(\Omega))\). First observe, that if (additionally) \(\partial_t \phi \in L^2(0,T \times \Omega)\) and \(\nabla \phi \in L^\infty(0,T; L^2(\Omega))\), then (as \(|v|^2 \in L^1(1)^2\)) we find
\[
\int_0^T (\partial_t v + (v \cdot \nabla) v, \phi) \eta \omega = \int_0^T \int_{\Omega_n} (v(T) \cdot \phi(T)) dz = \int_0^T \int_{\Omega_n} \langle v \cdot \phi, v \rangle dz dt = -\int_0^T \int_{\Omega_n} \nabla v \cdot \nabla \phi \eta \omega dz dt.
\]

Hence, by taking the mollification introduced in Lemma 2.6 (here \(b \equiv 0\)), we find that
\[
\int_0^T (\partial_t v + (v \cdot \nabla) v, \phi) \eta \omega = -\int_0^T \int_{\Omega_n} \nabla v \cdot \nabla \phi \eta \omega dz dt,
\]
which implies the result by passing with \(\delta \rightarrow 0\) by the convergence result of Lemma 2.6. This allows to give the left hand side a well defined meaning: hence the domain of the left hand side can accordingly be extended. The proof of the second identity is analogous.

\[\square\]

3. Proof of the main result

3.1. The set-up. Throughout this section let \((v_1, \eta_1), (v_2, \eta_2)\) be weak solutions to FSI for initial conditions \(v_1(0) = v_{1,0}, v_2(0) = v_{2,0}, \eta_1(0) = \eta_{1,0}, \eta_2(0) = \eta_{2,0}\) and \(\partial_t \eta_1(0) = \eta_{1,0}, \partial_t \eta_2(0) = \eta_{2,0}\). Let \(v_2\) satisfy the additional regularity assumption \(v_2 \in L^r(0,T; H^{1,s}(\Omega_2))\), \(\partial v_2 \in L^2(0,T; H^{-1,s}(\Omega_2))\) for some \(s > 3, r > 2\). Note that as \(\partial_t \eta_1 = tr_{\eta_1}(v_1)\) and \(\partial_t \eta_2 = tr_{\eta_2}(v_2)\) we have by the trace theorem for moving boundaries (see [6] Lemma 6))
\[
\partial_t \eta_1 \in L^2(0,T; H^1(\omega)), \quad \partial_t \eta_2 \in L^r(0,T; W^{2,3-s}(\omega))
\]
for all \(l \in (0,1/2)\). By Theorem 2.2 we find additionally that
\[
\eta_1 \in L^2(0,T; H^{2+l}(\omega)), \quad \eta_2 \in L^r(0,T; H^{2+l}(\omega)), \quad l \in (0,1/2).
\]

We define the variable in time domains
\[
\Omega_1 := \Omega_{t_1} \quad \text{and} \quad \Omega_2 := \Omega_{t_2}.
\]

Since most of the computations will be given on the domain of the weak solution \(\Omega_1\) we introduce for \(u : [0,T] \times \Omega_1 \rightarrow \mathbb{R}\) the notation
\[
\|u(t)\|_{k,p} := \|u(t)\|_{W^{k,p}(\Omega_{t_1}(\omega))}, \quad \|u(t)\| := \|u(t)\|_{L^2(\Omega_{t_1}(\omega)), \quad (u(t), w(t)) := (u(t), w(t)|_{\eta_1},
\]
whenever well defined. Recall also, that in case a function \(b : [0,T] \times \omega \rightarrow \mathbb{R}\) we will extend it constantly to \(b : [0,T] \times \omega \times \mathbb{R} \rightarrow \mathbb{R}\) without further notice. For such function we use
\[
\|b(t)\|_{k,p} := \|b(t)\|_{W^{k,p}(\omega)}, \quad \|b(t)\| := \|b(t)\|_{L^p(\omega)}, \quad (u(t), w(t)) := (u(t), w(t)|_{\omega}.
\]

\[3\]The expression (2.3) seems to be the appropriate definition of a weak time derivative in the setting of fluid-structure interaction.
The first step of the proof is to introduce a Diffeomorphism \( \psi : \Omega_1 \rightarrow \Omega_2 \) to compare the velocity fields on the same domain. We define such a \( \psi \) explicitly by

\[
\gamma : \omega \rightarrow (0, \infty), \quad x \mapsto \frac{\eta_2(x)}{\eta_1(x)}.
\]

\[
\psi : [0, T] \times \omega \times \mathbb{R} \rightarrow [0, T] \times \omega \times \mathbb{R} \quad (t, x, y) \mapsto (t, x, \gamma(t, x) y).
\]

Then \( \psi(t \times \Omega_1) = (t) \times \Omega_2 \) for all \( t \in [0, T] \). Note however that this transformation does not conserve the property of vanishing divergence. For that we follow the approach in [22]. Define the \( 3 \times 3 \) matrix

\[
J(t, x, y) = D_2 \psi(t, x, y) = \begin{pmatrix}
I_2 & 0 \\
y \nabla \gamma(t, x) & \gamma(t, x)
\end{pmatrix},
\]

\[
\tilde{J} = J \circ \psi^{-1} = \begin{pmatrix}
I_2 & 0 \\
-y \gamma^{-1} \nabla \gamma & \gamma^{-1}(t, x)
\end{pmatrix}.
\]

Now for \( w : [0, T] \times \Omega_2 \rightarrow \mathbb{R}^3 \) set \( \tilde{w} = \gamma J^{-1}(w \circ \psi) \) and for \( u : [0, T] \times \Omega_1 \rightarrow \mathbb{R}^3 \) set \( \tilde{u} = \gamma^{-1} \tilde{J} u \circ \psi^{-1} \).

The next lemma shows that \((\tilde{w}, \tilde{\xi})\) is an admissible and solenoidal testfunction for \((v_1, \eta_1)\) if an admissible and solenoidal \((w, \xi)\) is for \((v_2, \eta_2)\) and \((\tilde{u}, \tilde{\xi})\) is an an admissible and solenoidal testfunction for \((u_1, \eta_1)\) if \((u, \xi)\) is an admissible and solenoidal for \((v_2, \eta_2)\).

3.1. Lemma. Let \( w \in L^1(0, T; W^{1,q}(\Omega_2; \mathbb{R}^3)) \), \( u : [0, T] \rightarrow \Omega_1 \) (sufficiently smooth). The following holds

1. If \( \text{div} \, w = \text{div} \, u = 0 \) then \( \text{div} \, \tilde{w} = \text{div} \, \tilde{u} = 0 \).
2. \( u^3(t, x, \eta_2(x)) = \tilde{u}^3(t, x, \eta_1(x)) \), \( u^3(t, x, \eta_1(x)) = \tilde{u}^3(t, x, \eta_2(x)) \).
3. \( (u - \tilde{w}) \circ \psi^{-1} = \gamma J^{-1}(u - w) \) and \( (\tilde{u} - u) \circ \psi = \gamma^{-1} J(u - \tilde{w}) \).

Proof. We calculate

\[
\gamma J^{-1} = \begin{pmatrix}
\frac{\gamma I_2}{y} & 0 \\
y \gamma^{-1} \nabla \gamma & 1
\end{pmatrix}, \quad \gamma^{-1} \tilde{J} = \begin{pmatrix}
\frac{\gamma^{-1} I_2}{y} & 0 \\
y \gamma^{-1} \nabla \gamma & 1
\end{pmatrix}.
\]

Thus it is sufficient to prove (1) and (2) for \( \tilde{w} \) as for \( \tilde{u} \) we just have to replace \( \gamma \) by \( \gamma^{-1} \) everywhere. We get

\[
\tilde{w} = (\gamma w' \circ \psi, -y \nabla \gamma \cdot w' \circ \psi + w^2 \circ \psi),
\]

As \( \psi(x, \eta_1) = (x, \eta_2) \) this directly yields the second assertion. For the divergence we find

\[
\text{div}_x \tilde{w}' = \nabla \gamma \cdot w' \circ \psi + \gamma \text{div}_x (w' \circ \psi) = \nabla \gamma \cdot w' \circ \psi + \gamma ((\text{div}_x w') \circ \psi + (\partial_y w') \circ \psi) \cdot y \nabla \gamma)
\]

and using \( \partial_y (w \circ \psi) = \gamma (\partial_y w) \circ \psi \)

\[
\partial_y w^2 = -\nabla \gamma \cdot w' \circ \psi + \gamma (-y \nabla \gamma \cdot (\partial_y w') \circ \psi + (\partial_y w^2) \circ \psi).
\]

Thus \( \text{div} \, w_1 = 0 \) gives \( \text{div} \, \tilde{w} = \gamma (\text{div}_x w) \circ \psi = 0 \). For (3) note first that

\[
J^{-1} \circ \psi^{-1} = \begin{pmatrix}
I_2 & 0 \\
-y \gamma^{-2} \nabla \gamma & \gamma^{-1}
\end{pmatrix} = \tilde{J}^{-1}
\]

This gives

\[
(u - \tilde{w}) \circ \psi^{-1} = u \circ \psi^{-1} - \gamma (J^{-1} \circ \psi^{-1}) w = \gamma \tilde{J}^{-1}(\gamma^{-1} \tilde{J} u \circ \psi^{-1} - w) = \gamma \tilde{J}^{-1}(\tilde{u} - w).
\]

Lastly

\[
(u - \tilde{w}) \circ \psi = \gamma^{-1} J u - w \circ \psi = \gamma^{-1} J((u - \tilde{w})).
\]

\( \square \)

For notational purposes set

\[
\eta_1 - \eta_2 = \eta, \quad w_1 = v_1 - \tilde{v}_2, \quad w_2 = \tilde{v}_1 - v_2,
\]

\[
v_2 \circ \psi = \tilde{v}_2, \quad v_1 \circ \psi^{-1} = \tilde{v}_1, \quad w_2 \circ \psi = \tilde{w}_2, \quad w_1 \circ \psi^{-1} = \tilde{w}_1, \quad f_2 = f_2 \circ \psi.
\]

Note that by Lemma 3.1

\[
\tilde{w}_2 = \gamma^{-1} J w_1, \quad \tilde{w}_1 = \gamma \tilde{J}^{-1} w_2,
\]

and with a slight misuse of notation

\[
\tilde{v}_{1,\delta} = \gamma^{-1} \tilde{J} v_{1,\delta} \circ \psi^{-1}, \quad \tilde{v}_{2,\delta} = \gamma J^{-1} v_{2,\delta} \circ \psi, \quad w_{2,\delta} = \tilde{v}_{1,\delta} - v_{2,\delta}, \quad w_{1,\delta} = v_{1,\delta} - \tilde{v}_{2,\delta}.
\]

\footnote{4 Here and in the following we use \((I_2, 0)\) for \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\).}
3.2. **A-priori estimates.** Before we turn to the main argument we collect some results that show our test-functions are admissible and that the error terms due to the geometric convolution in time are converging to 0.

3.2. **Remark.** The following estimates we will use frequently in the following. They are consequences of Hölder’s inequality and the embeddings $H^1(\omega) \hookrightarrow L^p(\omega)$ ($p \in [1, \infty]$) and in case $q < 3$, that $W^{1,q}(\Omega_i) \hookrightarrow L^r(\Omega_i)$ for all $r < 3q/(3 - q)$ $(i = 1, 2$ here and in the following). See [30] for a reference.

1. For all $s \in (1, \infty)$, $p \in [1, s)$ and $f \in L^s(\Omega_i)$, $g \in H^1(\omega)$
   \[ \|fg\|_{L^p(\Omega_i)} \leq C \|f\|_{L^s(\Omega_i)} \|g\|_{H^1(\omega)}. \]

2. For all $p \in (1, 2)$, $q \in (6p/(6 - p), 3)$, $f \in H^{1,q}(\Omega_i)$ and $g \in L^2(\Omega_i)$
   \[ \|fg\|_{L^p(\Omega_i)} \leq C \|f\|_{H^{1,q}(\Omega_i)} \|g\|_{L^2(\Omega_i)}. \]

3. If $p, q, f$ are as above and $g \in H^2(\omega)$ 1. and 2. give in particular
   \[ \|fg\|_{H^{1,p}(\Omega_i)} \leq C \|f\|_{H^{1,q}(\Omega_i)} \|g\|_{H^2(\omega)}. \]

3.3. **Lemma.** Let $(v_1, \eta_1), (v_2, \eta_2) \in V_S$ weak solutions of FSI, $(v_2, \eta_2)$ satisfying the additional regularity assumptions. Then

1. $\gamma$ satisfies the following estimates for a.e. $t \in [0, T]$.
   \[ \|\gamma(t) - 1\|_{H^2(\omega)} \leq C \||\eta(t)\|_{H^2(\omega)} \leq C \|\partial_t \gamma(t)\|_{L^2(\omega)} \leq C \||\partial_t \eta(t)\|_{L^2(\omega)} + C \||\eta(t)\|_{L^2(\omega)}. \]

   The same estimates hold for $\gamma^{-1}$.

2. $\nabla \gamma \in L^\infty(0, T; L^q(\omega))$ for all $q \in [1, \infty)$
   \[ \|\nabla \gamma(t)\|_{L^q(\omega)} \leq C \||\eta(t)\|_{H^2(\omega)} \]
   and the same holds for $\gamma^{-1}$.

3. $\hat{v}_1 \in L^\infty(0, T; L^p(\Omega_2)) \cap L^2(0, T; W^{1,p}(\Omega_2))$ for all $p \in (1, 2)$ and $\|\hat{v}_1\|_{W^{1,p}(\Omega_2)} \leq C \|v_1\|_{1,2}$ for all $p \in [1, 2)$.

4. $\partial_t \hat{v} \in L^2(0, T; W^{-1,p'}(\Omega_1))$ for all $p' \in [1, r)$.

**Proof.** (1) and (2)

It holds
\[
\gamma - 1 = \frac{\eta_2 - \eta_1}{\eta_1} \leq C|\eta|
\]
\[
\gamma = \frac{\partial_t \eta_2}{\eta_1} - \frac{\eta_2 \partial_t \eta_1}{\eta_1^2} \leq C(\partial_t \eta_2||\eta|| + |\partial_t \eta|),
\]
\[
\nabla \gamma = \frac{\nabla \eta_2}{\eta_1} - \frac{\eta_2 \nabla \eta_1}{\eta_1^2} \leq C(\|\nabla \eta_2||| + |\nabla \eta|),
\]
\[
\partial_{x, x}^2 \gamma = \eta_1^{-2}(\partial_{x, x}^2 \eta_2 \eta_1 + \partial_{x_1} \eta_2 \partial_{x_2} \eta_1 - \partial_{x_1} \eta_2 \partial_{x_2} \eta_1 - \eta_2 \eta_1 \partial_{x, x} \eta_1) - 2\partial_{x_1} \eta_1 \partial_{x_1} \gamma
\]
\[\leq C(\|\nabla^2 \eta_2||| + |\nabla \eta_2||| + |\nabla^2 \eta| + |\nabla \eta| + |\nabla^2 \eta_2| + |\nabla \eta_2||| + |\nabla \eta|)
\]

(1) and (2) now follow from the embeddings $H^2(\omega) \hookrightarrow W^{1,q}(\omega) \hookrightarrow L^\infty(\omega)$ for all $q \in [1, \infty)$. The results for $\gamma^{-1}$ follow by replacing the roles of $\eta_1$ and $\eta_2$. 

(3.4)

We calculate

\[ \partial_\gamma (\gamma^{-1} J) = \begin{pmatrix} 1 & 0 \\ -y \partial_\gamma \nabla (\gamma^{-1}) & 0 \end{pmatrix}, \quad \partial_\eta (\gamma^{-1} J) = \begin{pmatrix} 0 & 0 \\ -\nabla (\gamma^{-1}) & 0 \end{pmatrix} \]

Hence

\[ |\partial_\gamma (\gamma^{-1} J)| + |\partial_\eta (\gamma^{-1} J)| \leq C (|\nabla (\gamma^{-1})| + |\nabla (\gamma^{-1})|^2 + |y \nabla (\gamma^{-1})|) \]

Observe further, that by Lemma 2.4 \[ \int_0^t (t, x) |\nabla (\gamma^{-1})| \text{ for all } q \in [1, \infty), \] which implies (using also (2)) that

\[ |\nabla^2 \gamma||v_1||_1 |, |\nabla^2 \gamma||\tilde{v}_1||_1 | \in L^2(0, T; L^p(\Omega_1)) \text{ and } |\nabla^2 \gamma||v_2||_1 |, |\nabla^2 \gamma||\tilde{v}_2||_1 | \in L^2(0, T; L^p(\Omega_2)) \text{ for all } p \in [1, 2] \]

Now by (3.2)

\[ |\partial_\gamma (\gamma^{-1} J \tilde{v}_1)| \leq C (|\nabla (\gamma^{-1})| + |\nabla (\gamma^{-1})| + |\nabla (\gamma^{-1})||\tilde{v}_1| + |\nabla (\gamma^{-1})||\nabla (v_1) \circ \psi^{-1}|) \]

Thus the assertion for \( \psi_1 \) follows using also (1), (2) and Remark 3.2.

(4)

This estimate is analogous to (3) in Lemma 2.6. Let us take \( \psi \in \tilde{W}^{1,p'}(\omega \times \mathbb{R}) \), such that \( \psi(t, x, y) = 0 \) for all \( x \in B_0 \) and \( ||\psi||_{\tilde{W}^{1,p'}(\omega \times \mathbb{R})} \leq 1 \) to find that

\[ \int_0^T (\partial_t \tilde{v}_2, \psi) dt = \int_0^T (\partial_t (\gamma J^{-1}) \tilde{v}_2, \psi) dt + \int_0^T (J^{-1} \partial_t v_2, \psi)_{\Omega_2} dt + \int_0^T \int_{\Omega_1} J^{-1} \partial_\gamma v_2 \partial_\gamma \cdot \psi dz dt \]

The estimates on the first and the third term are now straightforward using the assumptions on \( v_2 \).

In the first term it is important to observe that the terms involving \( \partial_\gamma \gamma \) are always coupled to \( v_2^* \). Using the fact that \( v_2^*(t, x, \eta_2(t, x)) = 0 \) for all \( (t, x) \in [0, T] \times \omega \), we may use integration by parts in \( x \) direction and find

\[ (\partial_t (\gamma J^{-1}) \tilde{v}_2, \psi) \leq C \int_{\Omega_1} |\partial_\gamma \gamma| (|\nabla \gamma| |\nabla \tilde{v}_2| |\psi| + |v_2|_{L^\infty(\Omega_2)} |\tilde{v}_2| |\nabla \psi|) \]

But these expression can be estimated using that \( p^* = \frac{3p}{3p-2} \) can be assumed to be close enough to 6 such that

\[ \int_0^T (\partial_t (\gamma J^{-1}) \tilde{v}_2, \psi) dt \leq C \int_0^T \|\partial_\gamma \gamma\| (\||\nabla \tilde{v}_2| |\nabla \gamma|\|_{L^2+3-q/2}) |\psi|_{p^*} + |v_2|_{\tilde{W}^{1,p'}(\Omega_2)} |\psi|_{1,p} dt \]

This expression is bounded since \( \partial_\eta \in L^\infty(\Omega) \) and \( |\nabla \gamma| |\nabla \tilde{v}_2| \in L^2(0, T; L^q(\Omega_1)) \) for all \( q \in [3, s] \).

At this point we choose \( t \in [0, T] \) such that all involved quantities do have a Lebesgue point at this time instance. Without any further notice we extend all quantities via \( \psi(t) \) constant on \((-\infty, 0]\) and \([t, \infty)\).

Next we take the convolution introduced in Lemma 2.6 on \( w_2 \) and \( \tilde{v}_2 \). We will need the following convergences:

3.4. Lemma. The following expressions are all well defined and convergence to zero as \( \delta \to 0 \):

(3.4)

\[ \int_0^t \langle \partial_t v_2, w_2 - w_{2,\delta} \rangle_{\eta_2} + \langle |\nabla v_2| v_2, w_2 - w_{2,\delta} \rangle_{\eta_2} + \langle \varepsilon \varepsilon w_2, \varepsilon w_2 - \varepsilon w_{2,\delta} \rangle_{\eta_2} dt \]

(3.5)

\[ \int_0^t \langle v_1 \otimes v_1, \nabla \tilde{v}_2 - \nabla \tilde{v}_{2,\delta} \rangle dt \]

(3.6)

\[ \langle v_1(t), \tilde{v}_2(t) - \tilde{v}_{2,\delta}(t) \rangle - \int_0^t \langle v_1, \partial_t \tilde{v}_2 - \partial_t \tilde{v}_{2,\delta} \rangle + \langle \varepsilon v_1, \varepsilon \tilde{v}_2 - \varepsilon \tilde{v}_{2,\delta} \rangle dt \]

Moreover, \( (\partial_\eta \psi, \tilde{v}_{2,\delta}) \) is a valid testfunction for the weak formulation of \( (\eta_1, v_1) \) and the terms

\[ (\partial_\eta \psi_2, v_2, \eta_2), (\varepsilon \varepsilon w_2, \varepsilon w_{2,\delta}), (|\nabla v_2| v_2, |\nabla \tilde{v}_2| \tilde{v}_2, \tilde{v}_{2,\delta})_{\eta_2} \in L^1(0, T) \text{ uniformly in } \delta \]

Proof. For (3.4) we know that \( w_2 \in L^2(0, T; W^{1,p}(\Omega_2)) \) for all \( p \in [1, 2] \) by Lemma 5.3. Hence by Lemma 2.6 \( w_2 - w_{2,\delta} \to 0 \) weakly in \( L^2(0, T; W^{1,p}(\Omega_2)) \) for all \( p \in [1, 2] \). Since it is a valid argument for \( \partial_t v_2 \in L^2(0, T; W^{1,p'}(\Omega_2)) \) and since \( \nabla v_2 \in L^2(0, T; W^{1,s}(\Omega_2)) \) for \( s > 3 \) it yields the convergence of the first and third term. Moreover, it was shown in Lemma 5.3 (6) that \( |\nabla v_2| v_2 \in L^2(0, T; L^q(\Omega_2)) \) for some \( q > (6/5) \). Since we may assume \( p \in [1, 2] \) such that \( W^{1,p}(\Omega_2) \to L^p \) the convergence of the second term follows again from the weak convergence of \( w_{2,\delta} \) in \( L^2(0, T; W^{1,p}(\Omega_2)) \).
In (3.5), we will show that all involved terms are uniformly bounded. The uniform bounds imply that all weakly converging sub-sequences converge to 0, by the uniqueness of the weak limits. The critical term here is \( \int_0^T \int_{\Omega_2} |v_1 \otimes v_1 \cdot \nabla(\partial_x \gamma \tilde{v}_2,\delta)| \, dz \, dt \). All other terms can be estimated in a straightforward manner and we skip the details. Using the uniform bounds on \( \eta_1, \eta_2, \frac{1}{\eta_1}, \frac{1}{\eta_2} \) we find

\[
\int_{\Omega_1} |v_1 \otimes v_1 \cdot \nabla(\partial_x \gamma \tilde{v}_2)| \, dz \, dt \\
\leq C \int_0^T \int_{\Omega_1} |v_1|^2 |\tilde{v}_2| \, dy \left((|\nabla \eta_1| + |\nabla \eta_2|)(1 + |\nabla \eta_2|) + |\nabla \eta_2||\nabla^2 \eta_1|\right) \, dx \\
+ C \int_{\Omega_1} |v_1|^2 |\nabla \tilde{v}_2| \, dy \left(1 + |\nabla \eta_1|^2 + |\nabla \eta_2|^2\right) \, dz =: I_1 + I_2.
\]

Using Lemma 2.6 and Hölder’s inequality in space we can estimate

\[
I_1 \leq C \|v_2\|_{L^\infty(\Omega_2)} \int_0^T \int_{\Omega_1} |v_1|^2 \, dy (|\nabla \eta_1| + |\nabla \eta_2|) \left(1 + |\nabla \eta_2| + |\nabla^2 \eta_2|\right) \, dx \\
\leq C \|v_2\|_{L^\infty(\Omega_2)} \left(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty}\right) (\|\nabla \eta_1\|_{2,2} + \|\nabla \eta_2\|_{2,2} + 1) \int_0^T |v_1|^2 \, dy, \\
\leq C \|v_2\|_{L^\infty(\Omega_2)} \left(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty}\right) (\|v_1\|^2 + \|v_1\| \|\nabla v_1\| + \|\partial_t \eta_1\| |\nabla \eta_1|) \\
\leq C \|v_2\|_{W^{1,p}(\Omega_2)} + 1^2 \left(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty}\right) (\|v_1\|^2 + \|\partial_t \eta_1\|^2 + C\|v_1\|_{2,2}^2.
\]

Since \( v_2 \in L^r(0, T; W^{1,p}(\Omega_2)) \) for some \( r > 2 \) and \( \eta_1, \eta_2 \in L^q(0, T; W^{1,\infty}(\omega)) \) for all \( q < \infty \) (Theorem 2.2) \( \|v_2\|_{L^\infty(\Omega_2)} \left(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty}\right) \in L^q(0, T) \). As additionally \( v_1 \in L^\infty(0, T; L^2(\Omega_1)) \cap L^2(0, T; H^1(\Omega_1)) \) and \( \partial_t \eta_1 \in L^\infty(0, T; L^2(\omega)) \) the last term is bounded in time.

To estimate \( I_2 \) note that \( v_1 \in L^2(0, T; H^1(\Omega_1)) \) \( \leftrightarrow \) \( L^2(0, T; L^2(\Omega_1)) \) for all \( \alpha \in [1, 6] \) we find for all \( \alpha < 3/2 \) (i.e. \( \frac{3}{2} \)) \( \|v_1\|_{\alpha} \leq \|v_1\|_{2} \|v_1\|_{\frac{3}{2}} \leq \|v_1\|_{\alpha} \|v_1\|_{1,2} \). Now choose \( p > 1, q > 3 \) such that \( qp < 2 \) and \( pq' < 3/2 \).

\[
I_2 \leq C \left(\|\eta_1\|_{2,2} + \|\eta_2\|_{2,2}\right) \|v_2\|^2 \|v_1\|^2 \leq C \|v_2\|_{W^{1,p}(\Omega_2)} \|v_1\|_{1,2}
\]

which is bounded in time due to the regularities on \( v_2 \) and \( v_1 \). We continue with (3.5). We write

\[
\int_0^t (v_1, \partial_t \tilde{v}_2 - \partial_t \tilde{v}_{2,\delta}) \, dt = \int_0^t \left(\gamma J^{-T}v_1, \partial_t \tilde{v}_2 - \partial_t \tilde{v}_{2,\delta}\right) \, dt + \int_0^t (v_1, \partial_t (\gamma J^{-1})(\tilde{v}_2 - \tilde{v}_{2,\delta})) \, dt \\
= \int_0^t \left(\gamma J^{-T}v_1, \partial_t \tilde{v}_2 - \partial_t \tilde{v}_{2,\delta}\right) \, dt + \frac{2}{\alpha} \int_0^t (v_1, \partial_t (\gamma \tilde{v}_2 - \tilde{v}_{2,\delta})) \, dt \\
- \frac{2}{\alpha} \int_0^t \left(y(\partial_x \gamma, v_1(\tilde{v}_2^2 - \tilde{v}_{2,\delta}^2)) \right) \, dt =: (i) + (ii) + (iii)
\]

The term (i) converges to 0 by Lemma 2.6 using that by an analogous estimate to Lemma 3.3 (3) we find that \( \gamma J^{-T} \tilde{v}_1 \in L^2(W^{1,p}(\Omega_2)) \) for all \( p \in (1, 2) \). The term (ii) converges directly by Lemma 2.6 and Lemma 3.3. On the term (iii) we integrate by parts to find that

\[
|iii| \leq \int_0^t \int_{\Omega_1} |\partial_t \gamma| |\nabla (v_1(\tilde{v}_2^2 - \tilde{v}_{2,\delta}^2))| \, dz \, dt
\]

which can be bounded uniformly (using Lemma 2.6 and Lemma 3.3 again) and therefore converges to 0. The estimate on the part involving symmetric gradients is straightforward using the bounds in Lemma 2.6 and Lemma 3.3. It remains to show that the first term in (3.3) converges. For that we simply use the fact that we chose \( t \) to be a Lebesgue point of all involved quantities. Hence by the very definition of \( \tilde{v}_{2,\delta} \), we find that

\[
\lim_{\delta \to 0} (v_1(t), \tilde{v}_{2,\delta}(t)) = (v_1(t), \tilde{v}_2(t)).
\]

For the last statement observe that for all \( p \in [1, 2] \) by the calculations in Lemma 3.3 that \( w_2 = v_2 - \tilde{v}_1 \in L^2(0, T; W^{1,p}(\Omega_2)) \) and therefore by Lemma 2.6 \( w_2, \tilde{w} \in L^2(0, T; W^{1,p}(\Omega_2)) \). This holds in particular for
\( p = r' \) which yields that the first two terms are in \( L^1(0,T) \). Further, since \( \nabla v_2 \in L^2(0,T; L^s(\Omega_2)) \) for \( s > 3 \) Hölder’s inequality implies for some \( q > \frac{6}{5} \)

\[
\|\nabla v_2\|_{L^q(\Omega_2)} \leq \|v_2\|_{L^2(\Omega_2)} \|\nabla v_2\|_{L^{2s}(\Omega_2)}.
\]

Choosing \( q > 6/5 \) such that \( (2/q') < s \) bounds the right hand side in \( L^2([0,T]) \). As by embedding \( w_{2,\delta} \in L^2(0,T; L^s(\Omega_2)) \) for all \( a \in [1,6] \) we find that \( \|\nabla v_2\|_{L^2} \cdot w_2 \in L^1(0,T; L^1(\Omega_2)) \).

\[
\square
\]

### 3.3. The stability estimate (Proof of Theorem 1.5)

We have collected all the necessary notations and estimates to start the stability estimate. The estimate is derived by testing first the equation of \( (v_2, \eta_2) \) by \( (w_{2,\delta}, \partial_t \eta_2) \), second the energy inequality for \( (v_1, \eta_1) \) and finally testing \( (v_1, \eta_1) \) with \( (\tilde{v}_{2,\delta}, \partial_t \eta_{2,\delta}) \).

Testing the equation of \( (v_2, \eta_2) \) by \( (w_{2,\delta}, \partial_t \eta_2) \), integration by parts and Reynolds’ transport theorem give

\[
\int_0^t \langle \partial_t v_2 + [\nabla v_2] v_2, w_{2,\delta} \rangle_{\eta_2} + \langle \varepsilon v_2, \varepsilon w_{2,\delta} \rangle_{\eta_2} - \langle f_2, w_{2,\delta} \rangle_{\eta_2} dt
\]

\[
+ (\partial_t \eta_2, \partial_t \eta_2) - (\partial_t \eta_{2,0}, \partial_t \eta_0) - \int_0^t (\partial_t \eta_2, \partial_t^2 \eta_2) - (\Delta \eta_2, \Delta \eta_2) - (\rho_2, \partial_t \eta_2) dt = 0.
\]

We can write this

\[
\int_0^t \langle \partial_t v_2 + [\nabla v_2] v_2, w_{2,\delta} \rangle_{\eta_2} + \langle \varepsilon v_2, \varepsilon (w_2 - w_{2,\delta}) \rangle_{\eta_2} - \langle f_2, w_2 - w_{2,\delta} \rangle_{\eta_2} dt
\]

\[
+ (\partial_t \eta_2, \partial_t \eta_2) - (\partial_t \eta_{2,0}, \partial_t \eta_0) - \int_0^t (\partial_t \eta_2, \partial_t^2 \eta_2) - (\Delta \eta_2, \Delta \eta_2) - (\rho_2, \partial_t \eta_2) dt = K_{1,\delta}
\]

where

\[
K_{1,\delta} := \int_0^t \langle \partial_t v_2 + [\nabla v_2] v_2, w_2 - w_{2,\delta} \rangle_{\eta_2} + \langle \varepsilon v_2, \varepsilon (w_2 - w_{2,\delta}) \rangle_{\eta_2} - \langle f_2, w_2 - w_{2,\delta} \rangle_{\eta_2}
\]

Then \( K_{1,\delta} \to 0 \) for \( \delta \to 0 \) by Lemma 3.3.

The next step is to transform the equation for \( v_2, \eta_2 \) to the domain \( \Omega_1 \). In particular we want to prove an estimate for

\[
\int_0^t \langle \partial_t \tilde{v}_2 + \nabla \tilde{v}_2 \tilde{v}_2 + \nabla w_1, (\tilde{v}_2, w_1) \rangle dt
\]

First compute

\[
\langle \partial_t \tilde{v}_2, w_1 \rangle = \langle \gamma J^{-1} \partial_t \tilde{v}_2 + \partial_t (\gamma J^{-1}) \tilde{v}_2, w_1 \rangle
\]

\[
= \langle \tilde{J}^{-1} ((\partial_t \tilde{v}_2) \circ \psi^{-1}), \tilde{w}_1 \rangle_{\eta_2} + \langle \partial_t (\gamma J^{-1}) \tilde{v}_2, w_1 \rangle.
\]

By chain rule we get

\[
(\partial_t \tilde{v}_2) \circ \psi^{-1} = \partial_t v_2 + y \gamma^{-1} \partial_t \gamma \partial_y v_2,
\]

Also using \( w_2 = \gamma^{-1} \tilde{J} \tilde{w}_1 \) (cf. (3.11)) this gives

\[
\tilde{J}^{-1} (\partial_t \tilde{v}_2) \circ \psi^{-1} \cdot \tilde{w}_1 = \partial_t v_2 \cdot w_2 + \partial_t v_2 \cdot (\tilde{J}^{-t} \tilde{w}_1 - w_2) + y \gamma^{-1} \partial_t \gamma \tilde{J}^{-t} \partial_y v_2 \cdot \tilde{w}_1
\]

\[
= \partial_t v_2 \cdot w_2 + \partial_t v_2 \cdot (\tilde{J}^{-t} - \gamma^{-1} \tilde{J}) \tilde{w}_1 + y \gamma^{-1} \partial_t \gamma \tilde{J}^{-t} \partial_y v_2 \cdot \tilde{w}_1,
\]

which yields

\[
\begin{align*}
\langle \partial_t v_2, w_2 \rangle_{\eta_2} &= \langle \partial_t \tilde{v}_2, w_1 \rangle - \langle \partial_t (\gamma J^{-1}) \tilde{v}_2, w_1 \rangle - \langle \partial_t v_2, (\tilde{J}^{-t} - \gamma^{-1} \tilde{J}) \tilde{w}_1 \rangle_{\eta_2} \\
+ \langle y \gamma^{-1} \partial_t \gamma \tilde{J}^{-t} \partial_y v_2, \tilde{w}_1 \rangle_{\eta_2} &= : (\partial_t \tilde{v}_2, w_1) + R_1.
\end{align*}
\]

**Estimate of \( R_1 \).** With similar estimates as in the proof of Lemma 5.3 (1) we get

\[
|\tilde{J}^{-t} - \gamma^{-1} \tilde{J}| \leq C(1 - \gamma + |\nabla \gamma|), \quad |\nabla (\tilde{J}^{-t} - \gamma^{-1} \tilde{J})| \leq C(|\nabla \gamma| + |\nabla^2 \gamma|)
\]

Hence as in the proof of Lemma 5.3 (1) we have (using also Lemma 5.3 (1))

\[
\|((\tilde{J}^{-t} - \gamma^{-1} \tilde{J}) \tilde{w}_1)\|_{W^{1,2}(\Omega_2)} \leq C\||\gamma||_{2,2}\|w_1\|_{1,2}
\]
for all $q \in [1, 2)$. This yields for $p' \in (2, r)$
\[ \left\langle \partial_t v_2, (\tilde{J}^{-1} - \gamma^{-1} \tilde{J}) \tilde{w}_1 \right\rangle_{q_2} \leq \left\| \partial_t v_2 \right\|_{W^{1, p'}(\Omega_2)} \left\| (\tilde{J}^{-1} - \gamma^{-1} \tilde{J}) \tilde{w}_1 \right\|_{W^{1, r}(\Omega_2)} \]
\[ \leq C_{r} \left\| \partial_t v_2 \right\|_{W^{1, p'}(\Omega_2)} \left\| \tilde{w}_1 \right\|_{W^{1, r}(\Omega_2)} \left\| \tilde{J}^{-1} - \gamma^{-1} \tilde{J} \right\|_{L^p(\Omega_2)} \]
By Remark 5.2 we have for $p \in (1, 3/2)$, $q \in (p, 3/2)$ and $a \in (6q/(6-q), 2)$
\[ \left\| \partial_t \gamma \right\|_{L^p(\Omega_2)} \leq \left\| \nabla \gamma \right\|_{L^2(\Omega_2)} \left\| \partial_t \gamma \right\|_{L^2(\Omega_2)} \leq \left\| \nabla \gamma \right\|_{L^2(\Omega_2)} \left\| \partial_t \gamma \right\|_{W^{1, r}(\Omega_2)} \]
Thus by (3.10) and Lemma 5.3 we get for $p = s' \in (1, 3/2)$
\[ \left\langle y \gamma^{-1} \partial_t \gamma^{-1} \partial_y v_2, \tilde{w}_1 \right\rangle_{q_2} \leq C \left\| \partial_t \gamma \right\|_{L^p(\Omega_2)} \left\| \nabla \gamma \right\|_{L^2(\Omega_2)} \left\| \partial_t \gamma \right\|_{W^{1, r}(\Omega_2)} \left\| \tilde{w}_1 \right\|_{W^{1, r}(\Omega_2)} \]
Next compute
\[ \partial_t (\gamma J^{-1}) = \left( \begin{array}{cc} \partial_t \gamma & 0 \\ -\gamma \partial_t \nabla \gamma & 0 \end{array} \right) \]
By H"older’s inequality we get for all $p \in (3, s)$ and $q = 2(p/2)' < 6$
\[ \left\| \nabla \tilde{v}_2 \right\|_{W^{1, p}(\Omega_2)} \leq \left\| \nabla \tilde{v}_2 \right\|_{p, \Omega_2} \left\| v_1 \right\|_q \leq \left\| v_2 \right\|_{W^{1, r}(\Omega_2)} \left\| w_1 \right\|_{1, 2}, \]
also
\[ \left\| \tilde{v}_2 \right\|_{W^{1, p}(\Omega_2)} \leq \left\| \tilde{v}_2 \right\|_{p, \Omega_2} \left\| w_1 \right\|_{1, 2} \leq \left\| v_2 \right\|_{W^{1, r}(\Omega_2)} \left\| w_1 \right\|_{1, 2} \]
This yields
\[ \left( \partial_t (\gamma J^{-1}), \tilde{v}_2, w_1 \right) \leq C \left\| \partial_t \gamma \right\|_{L^p(\Omega_2)} \left\| \nabla \gamma \right\|_{L^2(\Omega_2)} \left\| \partial_t \gamma \right\|_{W^{1, r}(\Omega_2)} \left\| \tilde{v}_2 \right\|_{W^{1, r}(\Omega_2)} \left\| w_1 \right\|_{1, 2} \]
In conclusion
\[ (3.11) \quad |R_1| \leq C_{r} \left( \left\| \tilde{v}_2 \right\|_{W^{1, p}(\Omega_2)} \left\| \tilde{v}_2 \right\|_{W^{1, r}(\Omega_2)} \right) \left\| \tilde{v}_2 \right\|_{W^{1, r}(\Omega_2)} \left\| w_1 \right\|_{1, 2} \]
To simplify Notation in the next step, for a Matrix $A \in \mathbb{R}^{3 \times 3}$ we denote the symmetric part of it as $A^s = \frac{1}{2} (A + A^t)$. We get by transformation and chain rule
\[ \langle \varepsilon v_2, \varepsilon w_2 \rangle_{q_2} = (\gamma (\nabla \tilde{v}_2 J^{-1})^s, (\nabla \tilde{v}_2 J^{-1})^s) \]
By (5.1)
\[ \gamma \nabla \tilde{w}_2 J^{-1} = \gamma \nabla (\gamma J w_1) J^{-1} = J \nabla w_1 J^{-1} + \gamma \nabla (\gamma J) w_1 J^{-1} \]
\[ = \nabla w_1 + \nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1} + \gamma \nabla (\gamma J) w_1 J^{-1}. \]
and using $\hat{\epsilon}_2 = \gamma J^{-1} v_2$
\[ \nabla \hat{\epsilon}_2 J^{-1} = \nabla \hat{\epsilon}_2 + \nabla \hat{\epsilon}_2 (J^{-1} - I) = \nabla \hat{\epsilon}_2 + \nabla ((I - \gamma J^{-1}) \hat{\epsilon}_2) + \nabla \hat{\epsilon}_2 (J^{-1} - I) \]
\[ = \nabla \hat{\epsilon}_2 + (I - \gamma J^{-1}) \nabla \hat{\epsilon}_2 - \nabla (\gamma J^{-1}) \hat{\epsilon}_2 + \nabla \hat{\epsilon}_2 (J^{-1} - I) \]
Hence
\[ (\gamma (\nabla \tilde{v}_2 J^{-1})^s) : (\nabla \tilde{w}_2 J^{-1})^s \]
\[ = \left( (\nabla \tilde{v}_2 J^{-1})^s, \varepsilon \varepsilon w_1 + \nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1} + \gamma \nabla (\gamma J w_1 J^{-1})^s \right) \]
\[ = (\varepsilon \varepsilon \varepsilon w_1) + \left( (\nabla \tilde{v}_2 J^{-1})^s, \left[ \nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1} + \gamma \nabla (\gamma J w_1 J^{-1})^s \right] \right) \]
\[ + \left( \left[ (I - \gamma J^{-1}) \nabla \hat{\epsilon}_2 - \nabla (\gamma J^{-1}) \hat{\epsilon}_2 + \nabla \hat{\epsilon}_2 (J^{-1} - I) \right]^s, \varepsilon \varepsilon w_1 \right) \]
\[ =: (\varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon w_1) + R_2. \]
Estimate of $R_2$. By the definition of $J$ it is straightforward to see that
\[ |J^{-1}| \leq C (1 + |\nabla \gamma|), \]
\[ |J^{-1} - I| + |J - I| + |\gamma J^{-1} - I| \leq C (|\gamma - 1| + |\nabla \gamma|). \]
By Hölder’s inequality we get \( \|\nabla \tilde{v}_2 \|_{L^{6/5}} \leq \|\nabla \tilde{v}_2 \|_3 \|\nabla w_1 \|_2 \) and thus for \( p = 6/5 \)
\[
\left( (\nabla \tilde{v}_2 J^{-1})^s, [\nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1}]^s \right) + \left( [I - \gamma J^{-1}] \nabla \tilde{v}_2 + \nabla \tilde{v}_2 (J^{-1} - I) \right)^s, \varepsilon \omega_1 \right)
\leq C \|\nabla \gamma \|_{3p'} \|\nabla \tilde{v}_2 \|_p \|\nabla w_1 \|_{p'} \leq C \|\nabla \omega_2 \|_s \|\nabla w_1 \|_2 \leq C' \|\nabla w_1 \|_{2,1}^2 + \|\nabla w_1 \|_2^2
\]

Furthermore as in the proof of Lemma 3.3, we get for \( p \in (3, s) \) (i.e. \( p' \in (s', 3/2) \))
\[
\left( (\nabla \tilde{v}_2 J^{-1})^s, (\nabla (\gamma^{-1} J) w_1 J^{-1})^s \right) \leq C \|1 + |\nabla \gamma| + |\nabla \gamma|^2 + |\nabla \gamma|^3 \|\nabla \tilde{v}_2 \|_p \|\nabla \gamma^{-1} \| \|\nabla \omega_1 \|_p \|\nabla \omega_1 \|_p' \leq C \|\nabla \omega_2 \|_s \|\nabla w_1 \|_{1,s}^2 \leq C \|\nabla w_1 \|_{2,1}^2 + C \|\nabla \omega_1 \|_2^2 + \|\nabla w_1 \|_{1,s}^2
\]

In conclusion
\[
(3.13) \quad \| R_2 \| \leq C \|\nabla \omega_2 \|_{2,1}^2 \|\nabla \tilde{v}_2 \|_{2,1}^2 + \|\nabla w_1 \|_{1,2}^2
\]

Next by chain rule and (3.11) we get
\[
(3.14) \quad \langle [\nabla v_2] v_2, w_2 \rangle_{\eta_2} = ([\nabla \tilde{v}_2] \gamma^{-1} J^{-1} \tilde{v}_2, \gamma^{-1} J w_1) = ([\nabla \tilde{v}_2] \gamma^{-1} J^{-1} \tilde{v}_2, \gamma^{-1} J w_1)
\]

\[
= ([\nabla \tilde{v}_2] \tilde{v}_2, w_1) + ([\nabla \tilde{v}_2] \tilde{v}_2, (\gamma^{-1} J - I) w_1)
\]

\[
= ([\nabla \tilde{v}_2] \tilde{v}_2, w_1) + ([\nabla ((I - \gamma^{-1} J) \tilde{v}_2)] \tilde{v}_2, w_1) + ([\nabla \tilde{v}_2] \tilde{v}_2, (\gamma^{-1} J - I) w_1)
\]

\[
: = ([\nabla \tilde{v}_2] \tilde{v}_2, w_1) + R_3
\]

**Estimate on R_3.** With similar estimates as above we can conclude
\[
\langle (\nabla (\gamma^{-1} J) \tilde{v}_2) \tilde{v}_2, w_2 \rangle \leq C \|\nabla \tilde{v}_2 \|_{\infty} \|\gamma^{-1} - J \|\nabla \tilde{v}_2 \| w_1 \| \leq C \|\nabla \omega_2 \|_{2,1}^2 \|\nabla \tilde{v}_2 \|_{2,1}^2 \|\nabla \omega_2 \|_{2,1}^2 + \| w_1 \|_{1,2}^2
\]

Additionally
\[
\langle (\nabla ((I - \gamma^{-1} J) \tilde{v}_2)] \tilde{v}_2, w_2 \rangle \leq C \|\nabla \tilde{v}_2 \|_{\infty} \|\nabla \tilde{v}_2 \|_{\infty} \|\nabla (\gamma^{-1} J) \| \|\nabla \tilde{v}_2 \|_{\infty} \|\nabla (\gamma^{-1} J) \| \|\nabla \tilde{v}_2 \|_{\infty} \|\nabla \omega_2 \|_{2,1}^2 + \| w_1 \|_{1,2}^2
\]

Thus
\[
(3.15) \quad \| R_3 \| \leq C \|\nabla \omega_2 \|_{2,1}^2 \|\nabla \tilde{v}_2 \|_{2,1}^2 + \| w_1 \|_{1,2}^2
\]

Lastly by transformation rule and (3.11)
\[
(3.16) \quad \langle f_2, w_2 \rangle_{\eta_2} = (\tilde{f}_2, \gamma \tilde{w}_2) = (\tilde{f}_2, w_1) - (\tilde{f}_2, (I - J) w_1) = (\tilde{f}_2, w_1) - R_4.
\]

We find for all \( p \in (1, \infty) \)
\[
(3.17) \quad R_4 \leq C \|((1 - \gamma)|\|p\|') \|\tilde{f}_2 \|_{\infty} \| w_1 \|_{p} \leq C \| f_2 \|_{L^p(\Omega_2)} \|\nabla \omega_2 \|_{2,1}^2 + \| w_1 \|_{1,2}^2
\]

Adding (3.10), (3.12), (3.14), (3.16) and integrating over \((0, t)\) we get
\[
\int_0^t \langle \partial_t \tilde{v}_2 + [\nabla \tilde{v}_2] \tilde{v}_2, w_1 \rangle + \langle \varepsilon \tilde{v}_2, \varepsilon w_1 \rangle - (\tilde{f}_2, w_1) \ dt
\]

\[
= \int_0^t \langle \partial_t v_2 + [\nabla v_2] v_2, w_2 \rangle_{\eta_2} + \langle \varepsilon v_2, \varepsilon w_2 \rangle_{\eta_2} - (f_2, w_2) \ dt + R,
\]

where \( R = \int_0^t R_1 + R_2 + R_3 + R_4 \ dt \). By (3.11), (3.13), (3.15), (3.17) we get
\[
(3.19) \quad \| R \| \leq \int_0^t h_1(t) (\|\nabla \omega_2 \|_{2,1}^2 + \|\partial_t \omega_2 \|_{2,1}^2 + \| w_1 \|_{2,1}^2) \ dt,
\]

\[
h_1(t) = C \| v_2 \|_{L^1(\Omega_2)} + \|\partial_t v_2 \|_{L^1(\Omega_2)} + \| f_2 \|_{L^1(\Omega_2)} \in L^1([0, T]).
\]
We can now estimate the differences of the solutions, namely we estimate

\[
I := \frac{1}{2} \|w_1\|^2 + \int_0^t \|\varepsilon w_1\|^2 dt + \frac{1}{2} (\|\partial_t \eta\|^2 + \|\Delta \eta\|^2) dt
\]

\[
= \frac{1}{2} \|v_1\|^2 + \int_0^t \|\varepsilon v_1\|^2 dt + \frac{1}{2} (\|\partial_t \eta_1\|^2 + \|\Delta \eta_1\|^2) dt
\]

\[
- (v_1(t), \hat{v}_2(t)) - \int_0^t (\varepsilon v_1, \varepsilon \hat{v}_2) dt - (\partial_t \eta_1, \partial_t \eta_2) + (\Delta \eta_1, \Delta \eta_2)
\]

\[
+ \frac{1}{2} (\|\varepsilon \hat{v}_2\|^2 + \|\partial_t \eta_2\|^2 + \|\Delta \eta_2\|^2) - \int_0^t (\varepsilon \hat{v}_2, \varepsilon w_1) dt.
\]

The energy inequality for \((v_1, \eta_1)\) gives

\[
I \leq \frac{1}{2} (\|v_{1,0}\|^2 + \|\eta_{1,0}\|^2 + \|\Delta \eta_{1,0}\|^2) + \int_0^t (f_1, v_1) dt + (g_1, \partial_t \eta_1) dt
\]

\[
- (v_1(t), \hat{v}_2(t)) - \int_0^t (\varepsilon v_1, \varepsilon \hat{v}_2) dt - (\partial_t \eta_1, \partial_t \eta_2) - (\Delta \eta_1, \Delta \eta_2)
\]

\[
+ \frac{1}{2} (\|\varepsilon \hat{v}_2\|^2 + \|\partial_t \eta_2\|^2 + \|\Delta \eta_2\|^2) - \int_0^t (\varepsilon \hat{v}_2, \varepsilon w_1) dt
\]

By (3.8) and (3.13) we get

\[
- \int_0^t (\varepsilon \hat{v}_2, \varepsilon w_1) dt = \int_0^t (\partial_t \hat{v}_2 + (\nabla \hat{v}_2) \hat{w}_1, v_1) dt + (\partial_t \eta_2, \partial_t \eta_3) - (\eta_{2,0}^\ast, \eta_0^\ast)
\]

\[
- \int_0^t \int_0^t \partial_t \eta_2 \partial_t^2 \eta_3 - (\Delta \eta_2, \Delta \partial_t \eta_3) + (g_2, \partial_t \eta_3) dt + K_3^1 + R.
\]

Reynold’s transport theorem and \(\hat{v}_2(x, \eta_1(x)) = \partial_t \eta_2(x)\) gives

\[
\int_0^t (\partial_t \hat{v}_2, v_1 - \hat{v}_2) dt = - \frac{1}{2} (\|\varepsilon \hat{v}_2\|^2 - \|\varepsilon \hat{v}_2, 0\|^2 - (\partial_t \eta_1, (\partial_t \eta_2)^2)) + \int_0^t (\partial_t \hat{v}_2, v_1) dt
\]

Inserting this calculation in (3.20) yields

\[
I \leq \frac{1}{2} (\|v_{1,0}\|^2 + \|\varepsilon \hat{v}_2\|^2) - (v_1(t), \hat{v}_2(t)) + \int_0^t (v_1, \partial_t \hat{v}_2) - (\varepsilon v_1, \varepsilon \hat{v}_2) + (f_1, v_1) - (f_2, w_1) dt
\]

\[
+ \frac{1}{2} (\|\eta_{1,0}\|^2 + \|\Delta \eta_{1,0}\|^2 + \|\partial_t \eta_2(t)\|^2 + \|\Delta \eta_2(t)\|^2) - (\partial_t \eta_1(t), \partial_t \eta_2(t)) - (\Delta \eta_1(t), \Delta \eta_2(t))
\]

\[
+ (\partial_t \eta_2(t), \partial_t \eta_3(t)) - (\eta_{2,0}^\ast, \eta_0^\ast) - \int_0^t (\partial_t \eta_2, \partial_t^2 \eta_3) - (\Delta \eta_2, \Delta \partial_t \eta_3) - (g_1, \partial_t \eta_1) + (g_2, \partial_t \eta_3) dt
\]

\[
+ \int_0^t (\nabla \hat{v}_2, \hat{v}_1) + \frac{1}{2} (\partial_t \eta_1, (\partial_t \eta_2)^2) + K_3^1 + R
\]

We denote the first line of the right hand side as \(I_1\) the second and third line as \(I_2\) and the fourth line as \(I_3\). We calculate that

\[
\frac{1}{2} (\|v_{1,0}\|^2 + \|\varepsilon \hat{v}_2\|^2) + \int_0^t (f_1, v_1) dt - (f_2, w_1) dt = (v_{1,0}, \hat{v}_2, 0) + \frac{1}{2} \|v_{1,0} - \hat{v}_2, 0\|^2 + \int_0^t (f_1, \hat{v}_2) + (f_1 - \hat{f}_2, w_1) dt.
\]

Thus

\[
I_1 = (v_{1,0}, \hat{v}_2, 0) - (v_1(t), \hat{v}_2(t)) + \int_0^t (v_1, \partial_t \hat{v}_2) - (\nabla v_1, \nabla \hat{v}_2) + (f_1, \hat{v}_2) dt
\]

\[
+ \frac{1}{2} \|v_{1,0} - \hat{v}_2, 0\|^2 + \int_0^t (f_1 - \hat{f}_2, w_1) dt
\]

We write the first line as

\[
(v_{1,0}, \hat{v}_2, 0) - (v_1(t), \hat{v}_2(t)) + \int_0^t (v_1, \partial_t \hat{v}_2, 0) - (\nabla v_1, \nabla \hat{v}_2) + (f_1, \hat{v}_2) dt
\]

\[
= (v_{1,0}, \hat{v}_2, 0) - (v_1(t), \hat{v}_{2,\delta}(t)) + \int_0^t (v_1, \partial_t \hat{v}_{2,\delta}) - (\nabla v_1, \nabla \hat{v}_{2,\delta}) + (f_1, \hat{v}_{2,\delta}) dt + K_{2,\delta},
\]
Now we use the equation \( \text{vor} \) into the parts that depend solely on \( \eta_2 \) and the rest:

\[
I_2 = \frac{1}{2}(\|\Delta_{\eta_2}\|^2 - \|\partial_t\eta_2\|^2) + \int_0^t (\partial_t\eta_2, \partial_t\Delta_{\eta_2}) dt + \int_0^t (\partial_t\eta_2, \partial_t\eta_2) - (\Delta_{\eta_2}, \partial_t\Delta_{\eta_2}) dt
\]

We denote the first line by \( I_{21} \) and find that

\[
I_{21} = \frac{1}{2}(\|\eta_2\|^2 + \|\Delta_{\eta_2}\|^2)
\]

where \( K_{3,\delta} \to 0 \) for \( \delta \to 0 \) by Lemma \[\text{2.5}\].

Collecting the above we arrive at

\[
I \leq (v_{1,0}, \dot{v}_{2,0}) - (v_1(t), \dot{v}_{2,t}(t)) + \int_0^t (v_1, \partial_t\dot{v}_{2,t}) - (\varepsilon v_1, \varepsilon \dot{v}_{2,t}) + (f_1, \dot{v}_{2,t}) dt
\]

Now we use the equation vor \((v_1, \eta_1)\) and test it with \( \dot{v}_{2,t} \):

\[
(v_{1,0}, \dot{v}_{2,0}) - (v_1(t), \dot{v}_{2,t}(t)) + \int_0^t (v_1, \partial_t\dot{v}_{2,t}) - (\varepsilon v_1, \varepsilon \dot{v}_{2,t}) + (f_1, \dot{v}_{2,t}) dt
\]

\[
= -\int_0^t (v_1 \otimes v_1, \nabla \dot{v}_{2,t}) dt + (\partial_t\eta_1(t), \partial_t\eta_2(t)) - (\eta_{1,t}, \eta_{2,t})
\]

\[
-\int_0^t (\partial_t\eta_1, \partial_t\eta_2) - (\Delta_{\eta_1}, \Delta_{\eta_2}) + (g_1, \partial_t\eta_2) dt
\]

Note that

\[
\frac{1}{2}(\|\eta_{1,t}\|^2 + \|\eta_{2,t}\|^2 + \|\Delta_{\eta_1}\|^2 + \|\Delta_{\eta_2}\|^2)
\]

\[
= (\eta_{1,0}, \eta_{2,0}) + (\Delta_{\eta_1,0}, \Delta_{\eta_2,0}) + \frac{1}{2}(\|\eta_{1,0} - \eta_{2,0}\|^2 + \|\Delta_{\eta_1,0} - \Delta_{\eta_2,0}\|^2)
\]

and

\[
(g_1, \partial_t\eta_1) - (g_2, \partial_t\eta_2) = (g_1, \partial_t\eta_1) - (g_2, \partial_t\eta_1)
\]

\[
+ (g_1, \partial_t\eta_2) + (g_1 - g_2, \partial_t\eta) + (g_2, \partial_t\eta - \partial_t\eta_3).
\]

This gives

\[
I \leq -\int_0^t ([\nabla \dot{v}_{2,t}] v_1) dt + \frac{1}{2}(\|v_{1,0} - \dot{v}_{2,0}\|^2 + \|\eta_{1,0} - \eta_{2,0}\|^2 + \|\Delta_{\eta_1,0} - \Delta_{\eta_2,0}\|^2)
\]

\[
+ \int_0^t (f_1 - \tilde{f}_2, w_1) dt + (g_1 - g_2, \partial_t\eta) dt + K_{2,\delta} + K_{3,\delta}+ K_{4,\delta} + I_3
\]
where
\[ K_{4,\delta} = (\Delta \eta_{1,0}, \Delta \eta_{2,0}) - (\Delta \eta_{1}(t), \Delta \eta_{2}(t)) + \int_0^t (\Delta \eta_{2}, \partial_t \Delta \eta_{1,\delta}) + (\Delta \eta_{1}, \partial_t \Delta \eta_{2,\delta}) \, dt + (\partial_t \eta_{1,\delta} - \partial_t \eta_{1}, \partial_t \eta_{2}) \]
\[ + \int_0^t (g_1, \partial_t \eta_{2} - \partial_t \eta_{2,\delta} + (g_2, \partial_t \eta - \partial_t \eta_{\delta}) \, dt \]

**Proof that \( K_{4,\delta} \to 0 \).** The first and third line of \( K_{4,\delta} \) converge to 0 again by Lemma 2.5. We write the second line as
\[ \left( \partial_t \eta_{1}(t), \partial_t \eta_{2,\delta}(t) - (\eta_{1,0}', \eta_{2,0}') - \int_0^t (\partial_t \eta_{2}, \partial_t^2 \eta_{1,\delta}) + (\partial_t \eta_{1}, \partial_t^2 \eta_{2,\delta}) \, dt \right) \]
\[ = (\partial_t \eta_{1}(t), \partial_t \eta_{2,\delta}(t) - \partial_t \eta_{2}(t)) + (\partial_t \eta_{1}(t), \partial_t \eta_{2}(t)) - (\eta_{1,0}', \eta_{2,0}') - \int_0^t (\partial_t \eta_{2}, \partial_t^2 \eta_{1,\delta}) + (\partial_t \eta_{1}, \partial_t^2 \eta_{2,\delta}) \, dt, \]
which also converges to 0 for \( \delta \to 0 \) by Lemma 2.5. Thus \( K_{4,\delta} \to 0 \) for \( \delta \to 0 \).

We continue by writing
\[ - \int_0^t (v_1 \otimes v_1, \nabla \hat{v}_{2,\delta}) \, dt = - \int_0^t (v_1 \otimes v_1, \nabla \hat{v}_2) \, dt + \int_0^t (v_1 \otimes v_1, \nabla \hat{v}_2 - \nabla \hat{v}_{2,\delta}) \, dt \]
\[ = - \int_0^t (v_1 \otimes v_1, \nabla \hat{v}_2) \, dt + K_{5,\delta}, \]
where \( K_{5,\delta} \to 0 \) by Lemma 3.4. Inserting this and the definition of \( I_3 \) in (3.23) finally yields
\[ I \leq \int_0^t -(v_1 \otimes v_1, \nabla \hat{v}_2) + (\|\nabla \hat{v}_2\|_2, v_1) + \frac{1}{2}(\partial_t \eta_{1,0}, (\partial_t \eta_2)^2) \, dt \]
\[ + \frac{1}{2}(\|v_1, \hat{v}_2\|^2 + \|\eta_{1,0}' - \eta_{2,0}'\|^2 + \|\eta_{1,0} - \eta_{2,0}\|^2) + \int_0^t (f_1 - f_2, w_1) + (g_1 - g_2, \partial_t \eta) \, dt \]
\[ + R + K_{1,\delta} + K_{2,\delta} + K_{3,\delta} + K_{4,\delta} + K_{5,\delta} \]

The first line can be estimated as follows. As \( \div v_1 = 0 \) we get by Gauß-integral formula
\[ (\|\nabla \hat{v}_2\|_1, v_1) = \frac{1}{2}(\partial_t \eta_2)^2, \partial_t \eta_1 \]
Hence
\[ (v_1 \otimes v_1, \nabla \hat{v}_2) = (\|\nabla \hat{v}_2\|_1, v_1 - \hat{v}_2) + (\|\nabla \hat{v}_2\|_1, \hat{v}_2) = (\|\hat{v}_2\|_1, v_1) + \frac{1}{2}(\partial_t \eta_2)^2, \partial_t \eta_1 \]
Thus we get
\[ - \int_0^t (v_1 \otimes v_1, \nabla \hat{v}_2) + (\|\nabla \hat{v}_2\|_1, v_1) + \frac{1}{2}(\partial_t \eta_{1,0}, (\partial_t \eta_2)^2) \, dt = - \int_0^t (\|\nabla \hat{v}_2\|_1, v_1) \, dt \]
We can estimate this term the same way as (3.5) in Lemma 3.4 by replacing \( v_1 \) by \( v_1 \) and \( \partial_t \eta_1 \) by \( \partial_t \eta \). We find
\[ (\|\nabla \hat{v}_2\|_1, v_1) \leq C \epsilon (\|v_2\|_{W^{1,\infty}(\Omega_2)} + 1)^2 (\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty} + 1)^2 (\|w_1\|^2 + \|\partial_t \eta\|^2 + \epsilon \|w_1\|_{1,2}^2) \]
Thus
\[ \int_0^t (\|\nabla \hat{v}_2\|_1, v_1) \, dt \leq \int_0^t h_2(t)(\|\eta_1\|^2 + \|w_1\|^2 + \epsilon \|\nabla w_1\|_2^2) \, dt, \]
\[ h_2(t) = (\|v_2\|_{W^{1,\infty}(\Omega_2)} + 1)^2 (\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty} + 1)^2. \]
As \( v_2 \in L^r(0, T; W^{1,\infty}(\Omega_2)) \) \( (r > 2) \) and \( \eta_1, \eta_2 \in L^p(0, T; W^{1,\infty}(\omega)) \) for all \( p \in [1, \infty) \) (by Theorem 2.2 and interpolation) we get \( h_2 \in L^1([0, T]) \).

Thus recalling the estimate on \( R \) (3.19) we get
\[ \int_0^t (\|\nabla \hat{v}_2\|_1, v_1) \, dt + R \leq \int_0^t h(t)(\|\eta_1\|^2 + \|w_1\|^2 + \epsilon \|\nabla w_1\|_2^2) \, dt, \]
\[ h = h_1 + h_2 \in L^1([0, T]). \]
Since $K_{i,\delta} \to 0$ for $\delta \to 0$ ($i=1,\ldots,5$) the last estimate leads to
\[
\frac{1}{2}(\|w_1\|^2 + \|\partial_t \eta\|^2 + \|\Delta \eta\|^2) + \int_0^t \|\varepsilon w_1\|^2 \, dt \\
\leq \frac{1}{2}(\|v_{i,0} - \tilde{v}_{i,0}\|^2 + \|\eta_{i,0}^* - \eta_{2,0}^*\|^2 + \|\Delta \eta_{i,0} - \Delta \eta_{2,0}\|^2) + \int_0^t \|f_1 - \tilde{f}_2\|_2^2 + \|g_1 - g_2\|_2^2 \, dt \\
+ \int_0^t h(t)(\|\eta_{i,2}\|^2 + \|\partial_t \eta\|^2 + \|\tilde{w}_1\|^2) + \epsilon \|w_1\|^2 \, dt \\
\] 
As $\eta$ is 0 on the boundary $\|\eta\|_{2,\infty} \sim \|\Delta \eta\|$. Korn’s inequality and the 0 trace of $w_1$ on $B_\varepsilon$ implies that $\|w_1\|_{1,2} \sim \|\varepsilon w_1\|_2$. Hence choosing $\epsilon < 1$ small enough we can apply Gronwall’s Lemma. This implies a stability estimate in terms of $w_1$. In order to change to $v_1 - \tilde{v}_2$ one uses $\|w_1\|_2 \leq \|v_1 - \tilde{v}_2\|_2 + \|\tilde{v}_2 - \tilde{v}_2\| \leq \|v_1 - \tilde{v}_2\|_2 + C\|\eta\|_{1,2};$ the estimate on the gradients is analogous. This finishes the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $(v_1, \pi_1, \eta_1)$ be a weak solution (with $\eta > 0$) on $[0, T]$ for any $T > 0$. Then the stability estimate implies that $\|w_1\| = \|\partial_t \eta\| = \|\eta\|_{2,\infty} = 0$ a.e. in $[0, T]$. As $\eta = \eta_1 - \eta_2 = 0$ we have $\Omega_1 = \Omega_2$ and in particular the transformation $\psi$ is the identity, $\gamma = 1, J = 1$. Thus $\tilde{v}_2 = v_2$ and $w_1$ = 0 gives $v_1 = v_2$. This proves Theorem 1.2. □

**References**

[1] H. Beirão da Veiga. On the existence of strong solutions to a coupled fluid-structure evolution problem. *J. Math. Fluid Mech.*, 6:21–52, 2004.

[2] Tomas Bodnar, Giovanni P. Galdi, and Šárka Nečasová, editors. *Fluid-Structure Interaction and Biomedical Applications*. Birkhäuser/Springer, Basel, 2014.

[3] M. Boulaikia and S. Guerrero. Regular solutions of a problem coupling a compressible fluid and an elastic structure. *J. Math. Pures Appl. (9)*, 94(4):341–365, 2010.

[4] Muriel Boulakia. Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid. *J. Math. Fluid Mech.*, 9(2):262–294, 2007.

[5] Marco Brahim. Energy equality and uniqueness of weak solutions of a viscous incompressible fluid + rigid body system with Navier slip-with-friction conditions in a 2d bounded domain. *Journal of Mathematical Fluid Mechanics*, 21(2)-23, 2019.

[6] Dominic Breit and Sebastian Schwarzacher. Compressible fluids interacting with a linear-elastic shell. *Archive for Rational Mechanics and Analysis*, 228:495–562, 2018.

[7] Yann Brenier, Camillo De Lellis, and László Székelyhidi Jr. Weak-strong uniqueness for measure-valued solutions. *Communications in mathematical physics*, 305(2):351–361, 2011.

[8] Nikolai V. Chemetov, Boris Muha, and Šárka Nečasová. Weak-strong uniqueness for fluid-rigid body interaction problem with slip boundary condition. *arXiv preprint arXiv:1710.01383*, 2017.

[9] Igor Chueshov, Irina Lasiecka, and Justin T. Webster. Evolution semigroups in supersonic flow-plate interactions. *J. Differential Equations*, 254(4):1741–1773, 2013.

[10] Philippe G. Clariet. *Mathematical elasticity. Vol. II*, volume 27 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1997. Theory of plates.

[11] Philippe G. Clariet. *Mathematical elasticity. Vol. III*, volume 39 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 2000. Theory of shells.

[12] Daniel Coutand and Steve Shkoller. Motion of an elastic solid inside an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, 176:25–102, 2005.

[13] Daniel Coutand and Steve Shkoller. The interaction between quasilinear elastodynamics and the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 179:303–352, 2006.

[14] B. Desjardins and M. J. Esteban. On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations*, 25(7-8):1399–1413, 2000.

[15] B. Desjardins, M. J. Esteban, C. Grandmont, and P. Le Tallec. Weak solutions for a fluid-elastic structure interaction model. *Rev. Mat. Complut.*, 14(2):523–538, 2001.

[16] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Nonlinear steady problems.

[17] Giovanni P. Galdi. Mathematical problems in classical and non-Newtonian fluid mechanics. In *Hemodynamical flows*, volume 37 of *Oberwolfach Semi.*, pages 121–273. Birkhäuser, Basel, 2008.

[18] Olivier Glass and Franck Sueur. Uniqueness results for weak solutions of two-dimensional fluid-solid systems. *Arch. Ration. Mech. Anal.*, 218(2):907–944, 2015.

[19] Céline Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *SIAM J. Math. Anal.*, 40:716–737, 2008.

[20] Céline Grandmont and Matthieu Hillairet. Existence of global strong solutions to a beam-fluid interaction system. *Arch. Ration. Mech. Anal.*, 220:1283–1333, 2016.

[21] Céline Grandmont, Matthieu Hillairet, and Julien Lequeurre. Existence of local strong solutions to fluid-beam and fluid-rod interaction systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(4):1105–1149, 2019.

[22] G. Guidoboni, Marcello Guidorzi, and Mariarosaria Padula. Continuous dependence on initial data in fluid-structure motions. *Journal of Mathematical Fluid Mechanics*, 14:1–32, 01 2010.
[23] M. Heil, A. Hazel, and J. Boyle. Solvers for large-displacement fluid–structure interaction problems: segregated versus monolithic approaches. *Computational Mechanics*, 43(1):91–101, 2008.

[24] AE Hosoi and L Mahadevan. Peeling, healing, and bursting in a lubricated elastic sheet. *Physical review letters*, 93:137802, 2004.

[25] Mihaela Ignatova, Igor Kukavica, Irena Lasiecka, and Amjad Tuffaha. Small data global existence for a fluid-structure model. *Nonlinearity*, 30(2):848–898, 2017.

[26] L. Iskauriaza, G. A. Serégin, and V. Šverák. $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk*, 58(2(350)):3–44, 2003.

[27] Herbert Koch and Vsevolod A Solonnikov. $L^q$-estimates of the first-order derivatives of solutions to the nonstationary stokes problem. In *Nonlinear Problems in Mathematical Physics and Related Topics I*, pages 203–218. Springer, 2002.

[28] O. A. Ladyženskaja. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967.

[29] Daniel Lengeler. Weak solutions for an incompressible, generalized Newtonian fluid interacting with a linearly elastic Koiter type shell. *SIAM Journal on Mathematical Analysis*, 46(4):2614–2649, 2014.

[30] Daniel Lengeler and Michael Ružička. Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell. *Archive for Rational Mechanics and Analysis*, 211(1):205–255, 2014.

[31] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63:193–248, 1934.

[32] B. Muha and S. Schwarzacher. Existence and regularity for weak solutions for a fluid interacting with a non-linear shell in 3d. *arXiv preprint arXiv:1906.01962*, 2019.

[33] Boris Muha and Sunčica Čanić. Existence of a Weak Solution to a Nonlinear Fluid–Structure Interaction Problem Modeling the Flow of an Incompressible, Viscous Fluid in a Cylinder with Deformable Walls. *Arch. Ration. Mech. Anal.*, 207(3):919–968, 2013.

[34] Boris Muha and Sunčica Čanić. Existence of a solution to a fluid–multi-layered-structure interaction problem. *J. Differential Equations*, 256:658–706, 2014.

[35] Boris Muha and Sunčica Čanić. Existence of a weak solution to a fluid–elastodynamic structure interaction problem with the navier slip boundary condition. *Journal of Differential Equations*, 260:8550–8589, 2016.

[36] Boris Muha and Sunčica Čanić. Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy. *Interfaces Free Bound.*, 17(4):465–495, 2015.

[37] Giovanni Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.

[38] Thomas Richter. *Fluid-structure interactions: models, analysis and finite elements*, volume 118. Springer, 2017.

[39] James Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

[40] James Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pages 69–98. Univ. of Wisconsin Press, Madison, Wis., 1963.

[41] V. N. Starovoitov. Nonuniqueness of a solution to the problem on motion of a rigid body in a viscous incompressible fluid. *Journal of Mathematical Sciences*, 130(4):4893–4898, Oct 2005.

[42] Helena Švihlová. Flow of biological fluids pabel specific geometries. *PhD thesis, Faculty of Mathematics and Physics, Charles University*, 2017.