PRINCIPAL AND DOUBLY HOMOGENEOUS QUANDLES

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Abstract

In the paper we describe the class of principal quandles and show that connected quandles can be decomposed as a disjoint union of principal quandles. We also prove that simple affine quandles are finite and they can be characterized among finite simple quandles by several different equivalent properties, as for instance being doubly homogeneous (i.e. having a doubly transitive automorphism group). A complete description of finite doubly-homogeneous quandles is provided extending the result of [Ven17] and solving [BDS17, Problem 6.7]. We also provide a classification of connected cyclic quandles with several fixed points independently from [LL18].

Introduction

Quandles are binary algebras developed in connection with knot theory [Joy82a, Mat82], Hopf Algebras [AG03] and the set theoretical solution of the Yang-Baxter equation [ESG01, ESS99]. Most of the construction and tools used in quandle theory are based on groups [ESG01, HSV16] and modules [Hou12]. In a recent paper [BS19] we developed a commutator theory for racks and quandles in the sense of [FM87], where the notion of abelianness and nilpotence are developed for general algebras. In this paper we are going to use this universal algebraic viewpoint to investigate some classes of quandles: principal, doubly homogeneous and cyclic quandles (see Section 2, 3 and 4.2 respectively).

The class of principal quandles naturally generalizes the class of affine or Alexander quandles and it has also been studied under different names as generalized Alexander quandles [CS16, CDS17]. Moreover they are provided by a right linear construction over groups (in the sense of [Sta15]). The study of principal quandles provides useful tools to understand the other classes of quandles investigated in the present paper and quandles in general.

It is known that every quandle with at least four elements and a 3-transitive automorphism group is projection [McC12, Proposition 5] (recall that a group $G$ acting on a set $Q$ is called $k$-transitive if it is transitive on the set of $k$-tuples with different entries). Quandles with transitive automorphism groups are known as homogeneous quandles and they are characterized by a well-known construction over groups [Joy82a, Theorem 7.1]. We call quandles with doubly transitive automorphism group doubly homogeneous. A characterization of such quandles has been stated as an open problem in [BDS17, Problem 6.7]. A partial answer to this problem is given in [Ven17], where the class of finite quandles with doubly transitive left multiplication group (i.e. doubly transitive quandles) is completely understood (see [Ven17, Corollary 4]). These quandles are also characterized by a precise cyclic structure of the left multiplications given by one fixed point and one non-trivial cycle. This class is contained into the class of doubly homogeneous quandles and into the class of cyclic quandles, defined and investigated in [LL18] as the class of finite quandles which cycle structure of left multiplications is given by several fixed points and one non-trivial cycle. In this paper we provide a characterization of the first class (answering to [BDS17, Problem 6.7]) and we show an alternative proof of the classification of connected cyclic quandles (the same classification is obtained in [LL18] using combinatorial techniques).

The contents of the paper are organized as follows: in Section II we collect some basic results on quandles and we introduce two new relations for quandles: the first one is related to projection
subquandles and the second one was already defined in \[BS19\] in connection with central congruences. We define the class of semiregular quandles as the class of quandles with semiregular displacement group (of which the class of principal quandles is an instance). We extend some known result for principal quandles to this class (indeed they depend just on the semiregularity of the displacement group, see Section 1.3).

In Section 2 we study congruences and automorphisms of connected principal quandles and we show that every connected quandle is a disjoint union of isomorphic copies of a principal quandle (Theorem 2.4). In Section 2.3 we deal with right linear left distributive quasigroups, studied under the name of isogroups \[Gal79\] \[Via10\]. The finite algebras of this class are the finite principal latin quandles.

The class of affine quandles is one of the most studied class of quandles and it is an example of a class of principal quandles: we show that there are no infinite simple affine quandles (Theorem 2.21).

In Section 3 we investigate a particular class of simple quandles, namely strictly simple quandles (i.e. quandles with no proper subquandle). Theorem 3.7 shows that finite simple quandles are latin if and only if they are strictly simple. The Theorem establish the converse of Theorem 3.4 of \[Via90\] for quandles, extending a result known for left-distributive quasigroup \[Gal78\]. Moreover finite simple latin quandles are affine and doubly homogeneous and their representation over elementary abelian groups is explained in Proposition 3.6 (already shown in \[AG83\] \[Joy82\] as the family of simple quandles with abelian displacement group). Moreover, strictly simple quandles are the only finite simple principal quandles (indeed a principal simple quandle \(Q\) is faithful and then latin by virtue of Proposition 1.5). As a byproduct we prove a non existing result for latin quandles of size \(2n\) with \(n\) coprime with 2, 3 and 5 (Corollary 3.9).

Using the principal decomposition for quandles we show that a finite quandle is doubly homogeneous if and only if it is a power of a strictly simple quandle (see Theorem. 3.13). In the proof we make use of Maschke theorem for representations of finite groups.

In Section 4.1 we investigate extensions of strictly simple quandles by projection quandles of prime size (in the same direction of \[Cla10\]) and we provide strong constrains on the size of such extensions in Theorem 4.8. In Section 4.2 we show an independent proof of the classification of finite connected cyclic quandles in Theorem 4.11 using the contents of the previous Section.

Notation and terminology. Given an algebra \(A\) (i.e. a set with an arbitrary subset of operations) we denote by \(\text{Aut}(A)\) the group of its automorphism. A congruence of \(A\) is an equivalence relation which respects the algebraic structure. Congruences of an algebra \(A\) form a lattice denoted by \(\text{Con}(A)\) with minimum \(0_Q = \{(a, a) : a \in Q\}\) and maximum \(1_Q = A \times A\). Note that homomorphic images and congruences are essentially the same thing since kernels of homomorphisms are congruence. If \(a \in \text{Con}(A)\) the algebra \(A/\alpha\) is called factor algebra. We denote by \(\{\gamma_n(A) : n \in \mathbb{N}\}\) and by \(\{\gamma^n(A) : n \in \mathbb{N}\}\) the elements of the lower central series and of the derived series and by \(\zeta_A\) the center of \(A\) (defined in analogy with group theory to capture the notion of solvability and centrality, see \[BS19\] \[FM87\]).

By \(H(A), S(A)\) and \(P(A)\) we denote respectively the set of isomorphism classes of factors, of subalgebras and of powers of \(A\). For further details about the universal algebraic definitions and construction we refer the reader to \[Ber12\].

We denote group operations just by juxtaposition or by + if the operation is commutative. Given an element \(g\) of a group \(G\) we denote by \(\bar{g}\) its inner automorphism, by \(N_G(S)\) (and resp. \(C_G(S)\)) the normalizer (resp. centralizer) of a subset \(S \subseteq G\). If \(\rho\) is an action of \(G\) on a set \(Q\) we denote the point-wise stabilizer of \(Q\) by \(G_\rho\) and the set-wise point-stabilizer by \(G_\xi = \{g \in G : g(S) = S\}\). The core of a subgroup \(H\) is the biggest normal subgroup of \(G\) contained in \(H\) and it is denoted by \(\text{Core}_G(H)\). The orbit of \(a\) under the action of \(G\) will be denoted by \(a^G\).

A group is called semiregular if \(G_a = 1\) for every \(a \in G\) and regular if it semiregular and transitive. Note that the pointwise-stabilizer of a transitive group is normal if and only if it is trivial (the stabilizers of a transitive group are all conjugate).
1. Preliminary results

1.1. Quandles. A quandle is an idempotent left-distributive left quasigroup, i.e. it is a binary algebra $(Q, \ast, \triangle)$ with

$$a \ast (a\backslash b) = a\backslash(a \ast b) = b,$$
$$a \ast (b \ast c) = (a \ast b) \ast (a \ast c),$$
$$a \ast a = a.$$

for every $a, b \in Q$. Equivalently a quandle is binary algebra $(Q, \ast)$ such that the left multiplication mapping $L_a : b \mapsto a \ast b$ is in the stabilizer of $a$ of the automorphism group of $Q$ for every $a \in Q$.

Example 1.1. (i) Any set $Q$ with the operation $a \ast b = b$ for every $a, b \in Q$ is a projection quandle. If $|Q| = n$ we denote such quandle by $\mathbb{P}_n$.

(ii) Let $G$ be a group and $H \subseteq G$ a subset closed under conjugation. For $a, b \in H$, let $a \ast b = aba^{-1}$. Then $\text{Conj}(H) = (H, \ast, \triangle)$ is a quandle, called conjugation quandle over $H$.

Example 1.2. Let $G$ be a group, $f \in \text{Aut}(G)$ and $H \leq \text{Fix}(f) = \{a \in G : f(a) = a\}$. Let $G/H$ be the set of left cosets of $H$ and the multiplication defined by

$$aH \ast bH = af(a^{-1})bH.$$  

The algebra $(G/H, \ast)$ is a quandle denoted by $\mathcal{Q}_{\text{Hom}}(G, H, f)$. If $H = 1$ then $Q$ is called principal (over $G$) and it will be denoted just by $\mathcal{Q}_{\text{Hom}}(G, f)$. If $G$ is abelian then $Q$ is called affine (over $G$) and in this case we use the notation $\mathcal{A}(G, f)$.

The group generated by the left multiplications mappings is denoted $\text{LMlt}(Q)$. The displacement group, defined as $\text{Dis}(Q) = \{(L_aL_b^{-1} : a, b \in Q)\}$ is a very important object in the theory of quandles as many properties of quandles are determined by its group-theoretical properties. Moreover $\text{LMlt}(Q) = \text{Dis}(Q)(L_a)$ for every $a \in Q$ and so the orbits of $\text{Dis}(Q)$ and $\text{LMlt}(Q)$ coincide and they are called connected components.

A quandle $Q$ is said to be (doubly) homogeneous if $\text{Aut}(Q)$ acts (doubly) transitively and connected if $\text{Dis}(Q)$ acts transitively. Clearly every connected quandle is homogeneous, but the orbits with respect these two groups can be very different. Let $\mathbb{P}_n$ be the conjugation quandle of size $n$, then $\text{Aut}(\mathbb{P}_n) = \text{Sym}(n)$ and $\text{Dis}(\mathbb{P}_n) = 1$. Hence, it is homogeneous but totally disconnected.

The construction given in Example 1.2 completely characterizes homogeneous quandles [Joy82a, Theorem 7.1], and so this construction is also called homogeneous representation. Connected quandles have a representation over their displacement, i.e. any connected quandle $Q$ is isomorphic to $\mathcal{Q}_{\text{hom}}(\text{Dis}(Q), \text{Dis}(Q)_a, L_a)$ [HSV16, Theorem 4.1]. Moreover, connected abelian quandles are affine [JPSZD18, Theorem 2.2].

Congruences of a quandle $Q$ induce surjective group morphisms between the displacement group of $Q$ and the displacement group of its factors. Indeed the mapping

$$\pi_\alpha : \text{Dis}(Q) \longrightarrow \text{Dis}(Q/\alpha), \quad L_xL_y^{-1} \mapsto L_{[x]} \cdots L_{[y]}^{-1}$$

can be extended to a well-defined surjective group morphism for every $\alpha \in \text{Con}(Q)$ (see [AG33, Lemma 1.8]). The kernel of $\pi_\alpha$ has the following characterization:

$$\text{Dis}^\alpha = \{h \in \text{Dis}(Q) : [h(a)] = [a], \text{ for every } a \in Q\} = \bigcap_{\alpha \in \text{Con}(Q)} \text{Dis}(Q)_{[a]},$$

where $\text{Dis}(Q)_{[a]}_{\alpha}$ is the set-wise stabilizer in $\text{Dis}(Q)$. Note that the stabilizer of a block of $\alpha$ is $\text{Dis}(Q)_{[a]}_{\alpha} = \pi_\alpha^{-1}(\text{Dis}(Q/\alpha)_{[a]}_{\alpha})$ and if $Q/\alpha$ is connected then $\text{Dis}^\alpha = \text{Cor}_{\text{Dis}(Q)}(\text{Dis}(Q)_{[a]})$ for every $a \in Q$. The set-wise stabilizer can be used to check connectedness of a quandle (we will use this criterion with no further reference through all the paper):

Lemma 1.3. [BB19, Proposition 1.3] Let $Q$ be a quandle and $\alpha \in \text{Con}(Q)$. Then $Q$ is connected if and only if $Q/\alpha$ is connected and $\text{Dis}(Q)_{[a]}_{\alpha}$ is transitive on $[a]_{\alpha}$. 

For every congruence \( \alpha \) we can define the displacement group relative to the congruence \( \alpha \) as \( \text{Dis}_\alpha = \{ (L_a L_b^{-1} : a \neq a b) \} \) which is a normal subgroup of \( \text{LMlt}(Q) \) contained in the kernel \( \text{Dis}^\alpha \). Properties as abelianess and centrality of congruences are completely determined by the properties of the relative displacement groups (see [BS19, Theorem 1.1]).

On the other hand for every subgroup in \( \text{Norm}(Q) = \{ N \triangleleft \text{LMlt}(Q) : N \leq \text{Dis}(Q) \} \) we can define the congruence \( \text{con}_{N} = \{ (a, b) \in Q \times Q : L_a L_b^{-1} \in N \} \). The pair of mappings \( \alpha \mapsto \text{Dis}_\alpha \), \( N \mapsto \text{con}_{N} \) provides a Galois correspondence between \( \text{Con}(Q) \) and \( \text{Norm}(Q) \) and its properties are described in Section 3.3 of [BS19]. This correspondence can be used to obtain information on the displacement group from the congruence lattice and vice versa. Moreover the orbit decomposition with respect to the action of \( N \in \text{Norm}(Q) \) is also a congruence and we denote it by \( \mathcal{O}_{N} \) [BS19 Lemma 2.6].

The set \( L(Q) = \text{Conj}(\{ L_a : a \in Q \}) \) is a conjugation quandle and \( L_Q : Q \to L(Q) \) is a surjective quandle morphism. We denote by \( \lambda_Q \) the kernel of such homomorphism. A quandle is called:

(i) **faithful** if \( L_Q \) is injective, i.e. \( \lambda_Q = 0Q \);

(ii) **crossed set** if \( a * b = b \) implies \( b * a = a \) for every \( a, b \in Q \);

(iii) **Latin** if all right multiplications \( R_a : b \mapsto b * a \) are bijective.

Note that any latin quandle is faithful and any faithful quandle is a crossed set. If \( Q \) is faithful the pointwise stabilizer of an element \( a \in Q \) is \( \text{Dis}(Q)_{a} = \text{Fix}(L_a) \).

Latin quandles can be also understood as left distributive (LD) quasigroup including right division \( a / b = R_b^{-1}(a) \) as a basic operation, but congruence and subalgebras of \( (Q, \ast, \setminus) \) and \( (Q, \ast, \backslash, /) \) might be different. Nevertheless if \( Q \) is finite nothing changes.

We refer the reader to [AG03, HSV16, Joy82a] for further details on the basics of quandle theory.

### 1.2. Projection subquandles

Let \( Q \) be a quandle and let \( \mathcal{P} \) be the relation defined as

\[
(a \mathcal{P} b) \quad \text{if and only if} \quad \{a, b\} \in \mathcal{P}.
\]

The relation \( \mathcal{P} \) is symmetric and reflexive, but in general is not an equivalence. Let \( [a]_{\mathcal{P}} = \{ b \in Q : a \mathcal{P} b \} \). Clearly \( [a]_{\mathcal{P}} \subseteq [a] \subseteq [a]_{\mathcal{P}} \subseteq \text{Fix}(L_a) \) and \( [a]_{\mathcal{P}} \) is a subquandle of \( Q \). Indeed if \( b, c \in [a]_{\mathcal{P}} \), then \( b, c \in \text{Fix}(L_a) \) and \( L_a \) and \( L_b \) commute. Therefore

\[
L_b^{-1}(c) * a = L_b^{-1}L_b^{-1}(a) = a,
\]

\[
a * L_b^{-1}(c) = L_aL_b^{-1}(c) = L_b^{-1}L_a(c) = L_b^{-1}(c).
\]

Hence, \( L_b^{-1}(c) \in [a]_{\mathcal{P}} \). A quandle \( Q \) is a crossed set if and only if \( [a]_{\mathcal{P}} = \text{Fix}(L_a) \) for every \( a \in Q \).

Projection subquandles of \( Q \) containing \( a \in Q \) are contained in \( [a]_{\mathcal{P}} \). Therefore, a quandle has no projection subquandles if and only if \( \mathcal{P} = 0Q \). Moreover, the action of the automorphism group respects the relation \( \mathcal{P} \). Indeed, if \( \{a, b\} \) is a projection subquandle, so it is \( \{h(a), h(b)\} \) for every \( h \in \text{Aut}(Q) \). So we can define:

\[
\text{Aut}^\mathcal{P} = \{ h \in \text{Aut}(Q) : h(a) \mathcal{P} a \text{ for every } a \in Q \},
\]

\[
\text{LMlt}^\mathcal{P} = \text{LMlt}(Q) \cap \text{Aut}^\mathcal{P},
\]

Note that both \( \text{Aut}^\mathcal{P} \) and \( \text{LMlt}^\mathcal{P} \) are normal subgroups of \( \text{Aut}(Q) \) and the orbit decomposition with respect to the action of \( \text{LMlt}^\mathcal{P} \) is a congruence of \( Q \) (see [BS19 Theorem 6.1]).

Projection subquandles are related to the cycle structure of left multiplications.

**Proposition 1.4.** Let \( Q \) be a quandle. The following are equivalent:

(i) \( \text{Fix}(L_a) = \{a\} \) for every \( a \in Q \).

(ii) \( \mathcal{P} = 0Q \) i.e. \( Q \) has no projection subquandle.

(iii) All the subquandles of \( Q \) are faithful.

**Proof.** (i) \( \Rightarrow \) (ii) If \( \{a, b\} \) is projection, then \( b \in \text{Fix}(L_a) = \{a\} \), i.e. \( a = b \)

(ii) \( \Rightarrow \) (iii) Let \( M \) be a subquandle of \( Q \). The blocks of \( \lambda_M \) are projection subquandles of \( Q \), so they are trivial and \( M \) is faithful.

(iii) \( \Rightarrow \) (i) In particular \( Q \) is faithful and so it is a crossed set. If \( b \in \text{Fix}(L_a) \), i.e. \( a * b = b \), then also \( b * a = a \), so \( M = \{a, b\} \) is a faithful projection subquandle of \( Q \). Therefore \( a = b \). \( \square \)
1.3. **Semiregular quandles.** One of the most studied classes of quandles is the class of connected affine quandles. Some of their particular properties depend just on the fact that the displacement group is semiregular. Thus, we study the family of quandles with such property. A quandle is called *semiregular* if $\text{Dis}(Q)$ is semiregular on $Q$, namely $\text{Dis}(Q)_a = 1$ for every $a \in Q$. This family is relevant since every quandle decomposes as a disjoint union of semiregular quandles (see Proposition 1.5).

Semiregularity of a quandle $Q$ is captured by the equivalence relation already defined in [4] as follows:

\[(2) \quad a\sigma_Q b \text{ if and only if } \text{Dis}(Q)_a = \text{Dis}(Q)_b.\]

Indeed $Q$ is semiregular if and only if $\sigma_Q = 1_Q$. It is easy to see that the class of semiregular quandles is closed under $\mathbf{P}$ and $\mathbf{S}$.

**Proposition 1.5.** Let $Q$ be a quandle. The classes of $\sigma_Q$ are semiregular subquandles of $Q$ and they blocks with respect to the action of $\text{Aut}(Q)$. In particular every quandle is a disjoint union of semiregular quandles.

**Proof.** Let $a \sigma_Q b$, i.e. $\text{Dis}(Q)_a = \text{Dis}(Q)_b$ and $h \in \text{Aut}(Q)$. Then $\text{Dis}(Q)_{h(a)} = h\text{Dis}(Q)_a h^{-1} = h\text{Dis}(Q)_b h^{-1} = \text{Dis}(Q)_{h(b)}$. Hence $h(a) \sigma_Q h(b)$. Moreover $\text{Dis}(Q)_{L_{\alpha_h}(b)} = L_{\alpha_h}^{\alpha_h} \text{Dis}(Q)_{b} L_{\alpha_h} = L_{\alpha_h}^{\alpha_h} \text{Dis}(Q)_{a} L_{\alpha_h} = \text{Dis}(Q)_{a}$, so $[a]_{\sigma_Q}$ is a subquandle of $Q$. The action of the displacement group of $[a]_{\sigma_Q}$ is the action of the group $D = \langle L_{\alpha} L_{\alpha}^{-1}, b \in [a]_{\sigma_Q} \rangle$ restricted to $[a]_{\sigma_Q}$. So if $h = g[a]_{\sigma_Q} \in \text{Dis}([a]_{\sigma_Q})a$ for some $g \in D$, then $g(b) = h(b) = b$ for every $b \in [a]_{\sigma_Q}$. Therefore $h = 1$ and $[a]_{\sigma_Q}$ is semiregular. \[\square\]

Let $Q$ be a quandle, $a \in Q$ and let $N_a = N_{\text{Dis}(Q)}(\text{Dis}(Q)_a)$. Then $a^{\text{Dis}(Q)} \cap [a]_{\sigma_Q} = a^{N_a}$. Indeed $h(a) \sigma_Q a$ if and only if $\text{Dis}(Q)_{h(a)} = h\text{Dis}(Q)_a h^{-1} = \text{Dis}(Q)_a$, i.e. $h \in N_a$. Moreover $\zeta_Q = \sigma_Q \cap \text{conZ}(\text{Dis}(Q)) \leq \sigma_Q$, according to [4] Proposition 5.9. The equivalence $\sigma_Q$ can be trivial and in this case $N_a = \text{Dis}(Q)_a$ and $Z(\text{Dis}(Q)) = 1$ ($Z(\text{Dis}(Q)) \leq N_a$ for every $a \in Q$).

**Proposition 1.6.** Let $Q$ be a quandle and $\alpha \in \text{Con}(Q)$.

\[(i) \quad Q/\alpha \text{ is semiregular if and only if } \text{Dis}^\alpha = \text{Dis}(Q)_{[a]_\alpha} \text{ for every } a \in Q.
\]

\[(ii) \quad \text{Let } \beta = \bigwedge_{i \in I} \alpha_i \in \text{Con}(Q). \text{ If } Q/\beta \text{ is semiregular for every } i \in I, \text{ then } Q/\beta \text{ is semiregular.} \]

**Proof.** (i) Since $\text{Dis}(Q)_{[a]_\alpha} = \sigma^{-1}_a (\text{Dis}(Q/\alpha)_{[a]_\alpha})$, then $\text{Dis}(Q/\alpha)$ is semiregular if and only if $\text{Dis}^\alpha = \text{Dis}(Q)_{[a]_\alpha}$.

(ii) Let $\beta = \bigwedge_{i \in I} \alpha_i$. Note that $\text{Dis}(Q)_{[a]_\beta} = \bigcap_{i \in I} \text{Dis}(Q)_{[a]_{\alpha_i}}$ and according to [4] Proposition 3.2(ii), we have $\text{Dis}^\beta = \bigcap_{i \in I} \text{Dis}^\alpha$. Using item (i) we get

$$\text{Dis}(Q)_{[a]_\beta} = \bigcap_{i \in I} \text{Dis}(Q/\alpha_i)_{[a]_{\alpha_i}} = \bigcap_{i \in I} \text{Dis}^\alpha = \text{Dis}^\beta.$$ 

Therefore we can conclude that $Q/\beta$ is semiregular by (i). \[\square\]

Note that if $Q/\alpha$ is semiregular then $\text{Dis}(Q)_a \leq \text{Dis}(Q)_{[a]_\alpha} = \text{Dis}^\alpha$ for every $a \in Q$.

**Lemma 1.7.** Let $Q$ be a semiregular quandle. Then $\text{LMlt}(Q) = \langle L_a \rangle \times \{L_a\}$ for every $a \in Q$.

**Proof.** The displacement group of $Q$ is semiregular, therefore $\text{Dis}(Q) \cap \{L_a\} \leq \text{Dis}(Q)_a = 1$ and so $\text{LMlt}(Q) = \langle L_a \rangle \times \{L_a\}$ since $\text{LMlt}(Q) = \text{Dis}(Q)/\langle L_a \rangle$ and $\text{Dis}(Q)$ is normal. \[\square\]

The following lemma shows that for semiregular quandles the equivalence $\mathcal{P}$ is a congruence. Note that the implication (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) holds in general

**Proposition 1.8.** Let $Q$ be a semiregular quandle. Then $Q$ is a crossed set and $\mathcal{P} = \lambda_Q$. Therefore, the following are equivalent:

\[(i) \quad Q \text{ is faithful.}\]

\[(ii) \quad Q \text{ has no projection subquandles.}\]

\[(iii) \quad \text{Right multiplications are injective.}\]
Proof. Let \( b \in \text{Fix}(L_a) \). Then \( L_b^{-1} L_a(b) = b \) i.e. \( L_b^{-1} L_a \in \text{Dis}(Q)_b = 1 \). So \( L_a = L_b \) i.e. \( a \in \lambda Q \) and \( a \in \text{Fix}(L_a) \). Therefore \( Q \) is a crossed set and \( \mathcal{P} = \lambda Q \).

(i) \( \Leftrightarrow \) (ii) Clear since \( \mathcal{P} = \lambda Q \).

(ii) \( \Rightarrow \) (iii) Let \( a * b = c * b \). Then \( L_c^{-1} L_a(b) = b \), so \( L_c = L_a \) and then \( a = c \).

(iii) \( \Rightarrow \) (i) If \( L_a = L_b \), then \( a * b = b * b \) and then \( a = b \). \( \square \)

For a semiregular quandle \( Q \) the blocks of \( \lambda Q \) are the maximal projection subquandles of \( Q \). In particular they are blocks with respect to the action of \( \text{Aut}(Q) \).

**Corollary 1.9.** Finite semiregular faithful quandles are latin.

Proof. If \( Q \) is finite and faithful, then right multiplications are injective and so bijective, i.e. \( Q \) is latin. \( \square \)

### 2. Principal quandles

#### 2.1. Principal decomposition.

Principal quandles are a particular class of homogeneous quandles. Recall that a principal quandle \( Q \) is defined as \( Q = \mathcal{Q}_{\text{Hom}}(G,f) \) and the quandle operation is

\[
 a * b = af(a^{-1}b),
\]

for every \( a, b \in G \) (if \( Q \) is affine then \( a * b = (1 - f)(a) + f(b) \)). The binary algebra defined as in \( \mathcal{Q} \) are also called right linear over the group \( G \) (for further details see \([\text{Sta15}]\)). The canonical left action of \( G \) on itself is a regular action by automorphisms of \( Q \). The action of the generators of \( \text{Dis}(Q) \) is given by

\[ L_a L_b^{-1}(a) = bf(b)^{-1}a, \]

for every \( a, b \in Q \). Then \( \text{Dis}(Q) = [G,f] = \langle \{ a f(a)^{-1} : a \in G \} \rangle \leq G \) and its action is given by the canonical left action on \( G \) which is semiregular. Therefore principal quandles are semiregular. The connected component of \( a \) is \( [G,f]a \) for every \( a \in G \). Principal quandles can be characterized as follows:

**Proposition 2.1.** \([\text{CDS17}]\) Corollary B.3] Let \( Q \) be a quandle and \( a \in Q \). The following are equivalent:

(i) \( G \leq \text{Aut}(Q) \) is a regular and \( \mathcal{L}_a \)-invariant subgroup.

(ii) \( Q \cong \mathcal{Q}_{\text{Hom}}(G,\mathcal{L}_a) \).

If \( Q \) is connected, the following are equivalent:

(i) \( \text{Dis}(Q) \) is regular.

(ii) \( Q \) is principal.

In particular if \( Q \cong \mathcal{Q}_{\text{Hom}}(G,f) \), then \( G = [G,f] \cong \text{Dis}(Q) \).

Proof. (i) \( \Rightarrow \) (ii) The group \( G \) is regular and provides a homogeneous representation given by \( Q \cong \mathcal{Q}_{\text{Hom}}(G,\mathcal{L}_a) \), since \( G_a = 1 \).

(ii) \( \Rightarrow \) (i) The left action of \( G \) is regular action by automorphisms of \( Q \).

Assume that \( Q \) is connected. If \( \text{Dis}(Q) \) is regular, then \( Q \cong \mathcal{Q}_{\text{Hom}}(\text{Dis}(Q),\mathcal{L}_a) \), i.e. \( Q \) is principal. On the other hand, if \( Q \) is connected and principal then \( \text{Dis}(Q) \) is semiregular and transitive, then regular.

Let \( Q \cong \mathcal{Q}_{\text{Hom}}(G,f) \) be a principal representation of \( G \). Then \( 1^{\text{Dis}(Q)} = [G,f] = G \) and therefore \( G \cong \text{Dis}(Q) \). \( \square \)

**Corollary 2.2.** Let \( Q \) be a latin quandle and \( a \in Q \). The following are equivalent:

(i) \( Q \) is principal.

(ii) \( \text{Dis}(Q) = \langle L_b L_a^{-1} \rangle, b \in Q \).

Proof. A set of representatives of cosets with respect to \( \text{Dis}(Q)_a \) is \( \{ L_b L_a^{-1} : b \in Q \} \). So a latin quandle \( Q \) is principal if and only if (ii) holds, by virtue of Proposition 2.1.]\( \square \)

**Corollary 2.3.** The class of principal quandles is closed under direct product.
Proof. Let \( Q_i = Q_{Hom}(G_i, f_i) : i \in I \) be a set of principal quandles. The group \( \prod_{i \in I} G_i \) is regular on \( \prod_{i \in I} Q_i \), and it is invariant under \( f = \prod_{i \in I} f_{a_i} \). So we can apply Proposition 2.1 and \( \prod_{i \in I} Q_i \cong Q_{Hom}(\prod_{i \in I} G_i, f) \).

Every connected quandle is a disjoint union of principal subquandles (the following Theorem is the counterpart of Proposition 1.5 for connected quandles).

**Theorem 2.4.** Let \( Q \) be a connected quandle. Then \( [a]_{\sigma Q} = a^N \) and it is a principal quandle over \( N_a/\mbox{Dis}(Q)_a \) for every \( a \in Q \). In particular \( Q \) is the disjoint union of isomorphic copies of a principal quandle.

Proof. Since \( Q \) is connected \( [a]_{\sigma Q} \cap a^{\mbox{Dis}(Q)} = [a]_{\sigma Q} = a^N \). Moreover \( N_a \) is transitive over \( [a]_{\sigma Q} \), it contains \( \mbox{Dis}(Q)_a \) as a normal subgroup and \( h_{[a]_{\sigma Q}}(a) = a \) if and only if \( h_{[a]_{\sigma Q}}(a) = a \). So \( G = N_a\langle [a]_{\sigma Q} \rangle = N_a/\mbox{Dis}(Q)_a \) is regular over \( [a]_{\sigma Q} \) and since it is stable under the inner automorphism of \( f = L_a\langle [a]_{\sigma Q} \rangle \) then it provides a principal representation as \( [a]_{\sigma Q} \cong Q_{\text{Hom}}(G, f) \). The quandle \( Q \) is the union of the classes with respect to \( \sigma_Q \) which are principal subquandles. Since they are blocks with respect to \( \text{Dis}(Q) \) and the action is transitive, they are all isomorphic.

**Remark 2.5.** Principal quandles are semiregular. On the other hand the connected components of semiregular quandles are principal (a principal representation is given by the displacement group). So every quandle is actually a disjoint union of principal quandles. In general they are not guaranteed to be isomorphic or to be blocks with respect of the action of the automorphism group.

Proposition 1.8 and Corollary 1.9 apply to principal quandles, so in particular they are crossed set. Principal quandles are homogeneous so left and right multiplications are all conjugate. Indeed if \( Q \) is a quandle then \( L_{b(\alpha a)} = hL_a h^{-1} \) and \( R_{b(\alpha a)} = hR_a h^{-1} \) for every \( h \in \text{Aut}(Q) \) and \( a \in Q \). Hence item (i) of Proposition 1.8 is equivalent to \( \text{Fix}(f) = \text{Fix}(L_1) = 1 \) and item (iii) of Proposition 1.8 is equivalent to injectivity of the map \( R_1 : a \mapsto af(a)^{-1} \). The following is a direct Corollary of Proposition 1.8 using that the orbits of displacement group in an affine quandle \( Q = \text{Aff}(A, f) \) are the coset with respect to \( \text{Im}(1 - f) \) and that \( R_0 = 1 - f \) (it can be found in [HSV16]).

**Corollary 2.6.** Let \( Q = \text{Aff}(A, f) \) be an affine quandle. The following are equivalent:

(i) \( Q \) is latin.

(ii) \( Q \) is connected and faithful.

If \( Q \) is finite and connected, then it is latin.

### 2.2. Subquandles and Automorphisms of principal quandles

Principal quandles are homogeneous, so every subquandle of \( Q = Q_{Hom}(G, f) \) is given by \( aS \) where \( a \in G \) and \( S \) is a subquandle containing \( 1 \). So, up to isomorphism it is enough to consider subquandles containing \( 1 \).

**Lemma 2.7.** Let \( Q = Q_{Hom}(G, f) \) be a principal quandle and \( \alpha \in \text{Con}(Q) \).

(i) Connected subquandles of \( Q \) are cosets of a \( f \)-invariant subgroup of \( G \) and they are principal.

(ii) If \( Q \) is connected then \( [a]_{\alpha} \) is principal over \( \text{Dis}(Q)_{[a]_{\alpha}} \).

**Proof.** (i) Let \( M \) be a connected subquandle of \( Q \), without loss of generality we can assume that \( 1 \in M \). Then \( M = 1^{\text{Dis}(M)} = \text{Dis}(M) = \{ (af(a)^{-1} : a \in M \} \), so \( \text{Dis}(M) \) is a \( f \)-invariant subgroup, it is regular on \( M \) and then \( M \) is principal.

(ii) If \( Q \) is connected the group \( N = \text{Dis}(Q)_{[a]_{\alpha}} \) is regular over \( [a]_{\alpha} \) and it is stable under \( f = L_a \). Therefore, \( [a]_{\alpha} \cong Q_{\text{Hom}}(N, f) \) is principal.

If \( Q = Q_{\text{Hom}}(G, f) \) is finite the size of non projection subquandles of \( Q \) and the size of \( Q \) are not coprime (indeed a subquandle \( M \) containing \( 1 \) is union of cosets with respect to \( \text{Dis}(M) \) which is an \( f \)-invariant subgroup of \( G \) and the size of connected subquandles of \( Q \) divide the size of \( Q \).

According to Lemma 1.7 LMlt\((Q_{\text{Hom}}(G, f)) \cong [G, f] \rtimes \langle f \rangle \). The following Proposition shows the structure of the automorphism group of connected principal quandles extending [AG03 Corollary 1.25] and [Hou11] Proposition 2.1.
Proposition 2.9. Let $Q$ be a connected quandle and $\alpha \in \text{Con}(Q)$.

(i) $Q/\alpha$ is principal if and only if $\text{Dis}^\alpha = \text{Dis}(Q)[a]_\alpha$ for every $a \in Q$.

(ii) Let $\beta = \bigwedge_{i \in I} \alpha_i \in \text{Con}(Q)$. If $Q/\alpha_i$ is principal for every $i \in I$, then $Q/\beta$ is principal.

The class of (connected) principal quandles is not closed under $\mathbf{H}$. Indeed SmallQuandle(12,1) with $i = 1,2$ in the RIG library of GAP are principal connected quandles with a non-principal factor of size 6.

Let $Q = \mathcal{Q}_{\text{Hom}}(G,f)$. Let us denote by $\sim_N$ the equivalence provided by the left cosets partition induced by $N \leq G$ and define

$$\text{Sub}(G,f) = \{ N \leq G : f(N) = N \text{ and } [N,f] \leq \text{Core}_G(N) \}.$$ 

Note that every subgroup of $\mathit{Fix}(f)$ and every $f$-invariant normal subgroup is in $\text{Sub}(G,f)$. Moreover characteristic subgroups of $G$ are contained in $\text{Sub}(G,f)$ for every $f \in \text{Aut}(G)$. Subgroups in $\mathbf{H}$ provides congruences for every principal quandles and if a quandle is connected all its congruences arise in this way.

Theorem 2.10. Let $Q = \mathcal{Q}_{\text{Hom}}(G,f)$ be a quandle. Then

$$\{ \sim_N : N \in \text{Sub}(G,f) \} \subseteq \text{Con}(Q).$$

If $Q$ is connected then $\text{Con}(Q) = \text{Sub}(G,f)$.

Proof. Let $N \in \text{Sub}(G,f)$, $a, b \in Q$ and let $n, m \in N$. Then

$$an * bm = a \underbrace{n f(n)^{-1} f(a)^{-1} f(b) f(m)}_{\in \text{Core}_G(N)} = a f(a)^{-1} f(b) n' f(m) =$$

$$= (a * b) n' f(m) \in (a * b) N.$$ 

Then $an \sim_N a * b$. Using that $n f^{-1}(n)^{-1} = f^{-1}(n f(n)^{-1})^{-1} \in [N,f]$ we can prove similarly that $an \sim_N a \backslash b$. Then $\sim_N$ is a congruence of $Q$.

Assume that $Q$ is connected and let $\alpha \in \text{Con}(Q)$ and $N = \text{Dis}(Q)[1]$. Then $[1] = N$ is an $f$-invariant subgroup of $G$ and $\text{Dis}(Q)[a] = a\text{Dis}(Q)[1][a]^{-1}$. Therefore $b \alpha a$ if and only if $b = ag a^{-1} a = ag$ for some $g \in N$, i.e. $[a] = a N$. Moreover

$$[N,f] = \{ a f(a)^{-1} , a \in N \} = \{ L_a L_1^{-1} , a \in [1] \} \leq \text{Dis}_N \leq \text{Dis}^\alpha = \text{Core}_G(N).$$

So every congruence of $Q$ arises as the equivalence defined by left cosets of an $f$-invariant subgroup of $G$ in $\text{Sub}(G,f)$.

Corollary 2.11. Let $Q = \mathcal{Q}_{\text{Hom}}(G,f)$ be a connected quandle. The following are equivalent:

(i) $Q/\sim_N$ is principal.
(ii) $N \triangleleft G$.

If the blocks of $\sim_N$ are connected then $Q/\sim_N$ is principal and $\text{Dis}_N = \text{Dis}^\sim_N = N$.

**Proof.** We have that $N = \text{Dis}(Q)_{[1]}$ and so it contains $\text{Dis}^\sim_N$ as its core. Using Proposition 2.9

(i) we have that $Q/\sim_N$ is principal if and only if $N = \text{Dis}^\sim_N$, i.e. $N$ is normal.

If $[1]_N$ is connected, then $\text{Dis}_N$ is transitive on $[1]_N$ and so $1^\text{Dis}_N = 1^N = N$ and then $N$ is normal. Accordingly $Q/\sim_N$ is principal. \qed

**Example 2.12.** Let $Q = \mathcal{Q}_{\text{Hom}}(G, f)$ be a connected quandle and let $N = \{ a \in G : af(a)^{-1} \in Z(G) \}$.

A straightforward computation shows that $\zeta_Q = \text{con}_{Z(\text{Dis}(Q))} \sim_N$.

**2.4. Isogroups.** A Latin quandle $(Q, \ast)$ can be interpreted also as left-distributive (LD) quasigroup $(Q, \ast, \setminus, /)$ by adding right division as a basic operation. Note that if $Q$ is finite case the two structures are actually (term) equivalent since if $n$ is the order of right multiplications of $Q$ then

$$x/y = R_y^{-1}(x) = R_y^{n-1}(x) = (\ldots (x \ast y) \ast y) \ldots \ast y.$$ 

In this section we will consider principal Latin quandles as LD quasigroups and we call them **isogroups** following [Gal79, Vla10]. Isogroups are the LD quasigroups which are right linear over a group or equivalently they are isotopic to a group (see [Bel67] and [Bru58] for loop isotopy). An isogroup $Q$ will be denoted as $Q = \mathcal{Q}_{\text{Hom}}(G, f, /)$. In particular $Q$ is (polynomially) equivalent to the algebra $(G, \cdot, ^{-1}, 1, f, f^{-1}, t, t^{-1})$ where $t$ is the mapping $t : a \mapsto af(a)^{-1}$ (all the basic operations of the first algebra can be defined by using the basic operations of the second and some constant elements as $x \cdot y = x/1 \cdot y$. See [Bel12] Section 4.3 for formal definitions of term and polynomial equivalence). The following results easily follows by the polynomial equivalence and they are available in in Russian in [Gal79] and in English in [Vla10].

**Proposition 2.13.** Let $Q = \mathcal{Q}_{\text{Hom}}(G, f, /)$ be an isogroup. Then

$$\text{Con}(Q) = \text{Sub}(Q, f) = \{ \sim_N : N \triangleleft G, f(N) = N \}$$

and all its factors are isogroups.

**Proposition 2.14.** Let $Q = \mathcal{Q}_{\text{Hom}}(G, f, /)$ be an isogroup. The following are equivalent:

(i) $S \subseteq Q$ is a subalgebra of $Q$.

(ii) $S$ is a coset with respect to a subgroup invariant under $f$.

In particular, the subalgebras of $Q$ are isogroups.

An alternative proof of Proposition 2.14 is given by Lemma 2.7(ii), since every subalgebra of $Q = \mathcal{Q}_{\text{Hom}}(G, f, /)$ is also a connected subquandle of the principal quandle reduct $\mathcal{Q}_{\text{Hom}}(G, f)$ (the algebra obtained by $Q$ forgetting the $/$ operation).

If $Q = \mathcal{Q}_{\text{Hom}}(G, f, /)$ is an isogroup and $1 \in S \subseteq G$, the subquandle (resp. congruence) generated by $S$ is the smallest (resp. normal) $f$-invariant subgroup of $G$ containing $S$ i.e. $(f^j(s))^{-1}, s \in S, j \in \mathbb{Z})$ (resp. $(g^j(s)g)^{-1}, s \in S, g \in G, j \in \mathbb{Z})$. Note that if $Q$ is affine quandle, then the lattice of congruences and the lattice of subquandles containing 1 are the same (since conjugation is trivial).

All the previous results apply to finite principal latin quandles. In particular every finite connected quandle with no projection subquandles is the disjoint union of isomorphic copies of a finite principal latin quandles (combining Theorem 2.4 and Corollary 1.9).

A finite algebra has the Lagrange property if the size of every subalgebra divides the size of the algebra and it has the Sylow property if for every prime $p$ dividing its size there exists a subalgebra of size $p^n$ where $p^n$ is the maximal power of $p$ dividing the size of the algebra.

**Proposition 2.15.** [Vla10] Proposition 1.4.5] Finite principal latin quandles have the Lagrange and the Sylow property.

These properties do not extend to (connected) principal quandles. Odd order non faithful quandles have projection subquandles of size 2 and in the RIG library there exist principal quandles for which the Sylow property does not hold (SmallQuandle(36,i) with $i = 9, \ldots, 14$).
In [BS19 Section 3.4] we investigate quandles for which the pair \((\text{Dis}, \text{con})\) are mutually inverse isomorphisms of lattice, under the name quandles with CDSg property. A class of such quandles is the class of finite principal latin quandles.

**Proposition 2.16.** Finite principal latin quandles have the CDSg property.

**Proof.** Let \(Q\) be a latin quandle. All the factors of \(Q\) are latin and principal, hence faithful. Moreover, by Corollary 2.13 we have \(\text{Dis}_\alpha = \text{Dis}^\alpha\) for every congruence \(\alpha\). Then we can apply [BS19 Proposition 3.10]. \(\square\)

**Proposition 2.17.** Let \(Q\) be a finite nilpotent quandle with the CDSg property. Then \(Q\) is a principal latin quandle.

**Proof.** If a quandle \(Q\) has the CDSg property then \(Q\) is faithful, connected and \(\text{Dis}_\alpha = \text{Dis}^\alpha\) for every \(\alpha \in \text{Con}(Q)\) and this property is stable under factors. If \(Q\) is abelian then it is an affine connected and faithful quandle, hence latin. Assume that \(Q\) is nilpotent of length \(n + 1\), i.e. \(\gamma_n(Q) \leq \zeta Q\). By induction on the nilpotency length, \(Q/\gamma_n(Q)\) is principal and latin. So \(\text{Dis}(Q)_{\alpha} \leq \text{Dis}(Q)_{[\alpha]} = \text{Dis}^{\gamma_n}(Q) = \text{Dis}_{\gamma_n}(Q) \leq Z(\text{Dis}(Q))\) (the relative displacement group of a central congruence is central [BS19 Theorem 1.1]). Therefore \(\text{Dis}(Q)_{\alpha}\) is normal and then trivial, so \(Q\) is principal. The quandle \(Q\) is finite faithful and connected, \(\gamma_n(Q)\) is central and \(Q/\gamma_n\) is latin. According to [BB19 Lemma 3.4] \(Q\) is latin. \(\square\)

2.5. **Simple Affine quandles.** An affine quandle \(\text{Aff}(A, f)\) carries a natural structure of \(\mathbb{Z}[t, t^{-1}]\) modules, where the action of \(t\) is given by the automorphism \(f\). According to Proposition 2.10 congruences of affine connected quandles correspond to submodules, indeed all subgroups are normal.

**Proposition 2.18.** Let \(Q = \text{Aff}(A, f)\) be a connected affine quandle. Then
\[
\text{Con}(Q) = \{\sim_N: N \leq A, f(N) = N\},
\]
i.e. congruences of \(Q\) correspond to \(\mathbb{Z}[t, t^{-1}]\)-submodules of \(Q\).

**Proof.** Every \(f\)-invariant subgroup \(N\) satisfies \([N, f] = \text{Im}(1 - f)_{|N} \leq \text{Core}_A(N) = N\). Then we apply Theorem 2.10. \(\square\)

**Remark 2.19.** If \(Q = \text{Aff}(A, f)\) is finite and connected, then it is latin and there is a one-to-one correspondence between subquandles of \(Q\) containing 0, congruences of \(Q\) and \(f\)-invariant subgroups of \(A\) (i.e. submodules).

In order to understand simple affine quandles we can think of them as simple modules according to Proposition 2.18.

**Proposition 2.20.** Let \(Q\) be a simple affine quandle. Then \(\text{Dis}(Q) \cong \mathbb{K}^n\) where \(\mathbb{K} \in \{\mathbb{Q}, \mathbb{Z}_p\}\).

**Proof.** The displacement group of \(Q\) has no characteristic subgroup. Therefore, either \(n\text{Dis}(Q) = 0\) for some \(n\) and so it has finite exponent or \(n\text{Dis}(Q) = \text{Dis}(Q)\) for every \(n \in \mathbb{N}\) and so it is divisible. In the first case the exponent is \(p\) and so \(Q\) is a simple \(\mathbb{Z}_p[t, t^{-1}]\) module. In the second case \(\text{Dis}(Q)\) is a direct product of a power of \(Q\) and its torsion part. Since \(\text{Dis}(Q)\) has no characteristic subgroup, then \(\text{Dis}(Q)\) is torsion free and so it is a power of \(Q\) and hence it is a simple \(\mathbb{Q}[t, t^{-1}]\) module. If \(K\) is a field the ring \(\mathbb{K}[t, t^{-1}]\) is a principal ideal domain and its simple modules are finite dimensional \(K\)-vector spaces (as \(\mathbb{K}[t, t^{-1}]\) is a localization of the polynomial ring \(K[t]\)). \(\square\)

**Theorem 2.21.** Let \(Q\) be an affine quandle and \(|Q| > 2\). Then \(Q\) is simple if and only if \(Q \cong \text{Aff}(\mathbb{Z}_p^n; f)\) and \(f\) acts irreducibly.

**Proof.** Let \(A = \mathbb{Q}^n\) and let \(Q = \text{Aff}(A, f)\). Then \(f\) is a \(\mathbb{Q}\)-linear map and let \(F\) be the matrix with coefficients in \(Q\) with respect to a basis \(e_1, \ldots, e_n\). Let \(k\) be the l.c.m. of the denominators of \(\{F_{i,j}, F_{-i,j}^{-1} : 1 \leq i, j \leq n\}\) Then \(F\) and \(F^{-1}\) are elements of the subring of the matrices with coefficients in \(\mathbb{Z}[k^{-1}]\). The subgroup \(K = \langle f^j(e_1), j \in \mathbb{Z} \rangle\) is a \(\mathbb{Z}[t, t^{-1}]\)-submodule of \(A\) and it is contained in \(\mathbb{Z}[k^{-1}]^n\) which is a proper subring of \(\mathbb{Q}^n\). Therefore \(Q\) is not simple. Hence necessarily \(\text{Dis}(Q) \cong \mathbb{Z}_p^n\) and \(Q \cong \text{Aff}(\mathbb{Z}_p^n; f)\) is simple if and only if \(f\) acts irreducibly. \(\square\)
3. Doubly Homogeneous Quandles

3.1. Strictly simple quandles. In this section we investigate strictly simple quandles and we prove one of the main results in Theorem 3.7. A proper subquandle of Q is a subquandle for which the underlying set is a proper subset Q with more than one element.

Definition 3.1. A quandle is said to be strictly simple if it has no proper subquandles.

Minimal proper subquandles (with respect to inclusion) of finite quandles are strictly simple quandles. Strictly simple quandles are simple, since the blocks of congruences are subquandles and so they are connected and faithful [Jy82a Lemma 1].

Lemma 3.2. A quandle Q is strictly simple if and only if it is generated by any pair of its elements. Moreover \( \text{Aut}(Q) \) is a Frobenius group.

Proof. Let \( a, b \in Q \), then \( Sq(a, b) \) has more than one element and so it is equal to \( Q \). If \( h \in \text{Aut}(Q)_a \cap \text{Aut}(Q)_b \), then \( h = 1 \) and so \( \text{Aut}(Q) \) is a Frobenius group (it is transitive and it contains \( L_a \in \text{Aut}(Q)_a \)).

Theorem 3.3. Let \( Q \) be a finite quandle and \(|Q| > 2\). Then \( Q \) is strictly simple if and only if \( Q \cong \text{Aff}(\mathbb{Z}_p^n, f) \) and \( f \) acts irreducibly.

Proof. (\( \Rightarrow \)) The group \( \text{LMlt}(Q) \) is a Frobenius group (Lemma 3.2). By Theorem 1 of [Th059], it has a normal regular nilpotent subgroup \( N \). The quandle \( Q \) is connected, so \( \text{Dis}(Q) = N \) (Proposition 2.1) and therefore \( Q \) is a simple nilpotent quandle [BS19 Theorem 1.2]. Simple nilpotent algebras are abelian, and so \( Q \) is affine and then we can conclude by Theorem 2.21.

(\( \Leftarrow \)) Assume that \( Q \cong \text{Aff}(\mathbb{Z}_p^n, f) \) and \( f \) has no proper invariant subgroups. By Corollary 2.14, \( Q \) has no proper subquandles. □

According to Proposition 3.3, strictly simple quandles correspond to irreducible representations of cyclic groups. Irreducible cyclic subgroups lie in irreducible cyclic subgroups of maximal order, called Singer cycle, and it is known that the order of a Singer cycle in \( \text{GL}_n(p) \) is \( p^n - 1 \). An example of a Singer cycle is the group of linear transformation given by \( \{ \lambda : \alpha \mapsto \lambda \alpha : \lambda \in \mathbb{F}_p^n \} \) where \( \mathbb{F}_p \) is the finite field on \( p \) elements. According to the isomorphism theorem in [Nel03], \( \text{Aff}(\mathbb{F}_p^n, \lambda) \cong \text{Aff}(\mathbb{F}_p^n, \lambda) \) if and only if \( f \) and \( g \) are conjugate in \( \text{Aut}(A) \). All Singer cycles are conjugate [Sho92 Theorem 2.3.3], so strictly simple quandles are isomorphic to \( \text{Aff}(\mathbb{F}_p^n, \lambda) \) for some \( \lambda \in \mathbb{F}_p^n \).

Recall that if \( G \) is a group acting on a set \( Q \), \( G \) is doubly transitive on a set \( Q \) if and only if \( G_a \) is transitive on \( Q \setminus \{a\} \) for every \( a \in Q \).

Proposition 3.4. Let \( Q \) be a strictly simple quandle of size \( p^n \). Then \( \text{Aut}(Q) \cong Z_p^n \rtimes Z_{p^n-1} \) and \( Q \) is doubly-homogeneous.

Proof. Let \( Q = \text{Aff}(Z_p^n, f) \) be a strictly simple quandle. Since \( f \) acts irreducibly on \( Z_p^n \) then \( C = \text{CGL}_n(\mathbb{F}_p)(f) \) is a Singer cycle [Sho92 Theorem 2.3.5] and then \( C \cong Z_{p^n-1} \). In particular, according to Corollary 2.8, \( \text{Aut}(Q)_0 = C \), so it has size \( p^n - 1 \). The quandle \( Q \) is generated by 0 and every \( a \in Q \), so \( \text{Aut}(Q)_{0,a} = 1 \) and therefore \( \text{Aut}(Q)_0 \) is transitive over \( Q \setminus \{0\} \). □

The connected components of conjugation quandle over the automorphism group of a strictly simple quandle are affine connected quandles.

Proposition 3.5. Let \( Q = \text{Aff}(Z_p^n, f) \) be a strictly simple quandle and let \( C = \text{Conj}(\text{Aut}(Q)) \). Then the connected component of \( (a, g) \) in \( C \) is isomorphic to the connected quandle \( \text{Aff}(Z_p^n, g) \) for every \( g \neq 1 \).

Proof. The automorphism group of \( Q \) is \( \text{Aut}(Q) \cong Z_p^n \rtimes Z_{p^n-1} \). Therefore \( (b, h)(a, g)(b, h)^{-1} = (b + h(a), hg)(h^{-1}(-b), h^{-1}) = ((1 - g)(b) + h(a), g) \).

If \( g \neq 1 \) then \( 1 - g \) is a bijection, since \( \text{Aut}(Q)_0 \cap \text{Aut}(Q)_b = 1 \) for every \( b \in Q \). Then the orbit of \( (a, g) \) is \( Z_p^n \times \{g\} \), therefore \( (a, g)^{\text{LMlt}(C)} \cong \text{Aff}(Z_p^n, g) \) which is connected since \( \text{Fix}(g) = 0 \). □

In particular, if \( Q \) is a strictly simple quandle of size \( p^n \), then \( C \) contains every strictly simple quandle of size \( p^n \) (indeed they are isomorphic to \( \text{Aff}(Z_p^n, g) \) where \( g \in \text{CGL}_n(\mathbb{F}_p)(f) \) acts irreducibly).
3.2. Finite simple latin quandles. In this section we give a characterization of finite simple latin quandles in terms of several equivalent properties. First we show that finite doubly-homogeneous quandles are either projection or latin.

**Lemma 3.6.** Let \( Q \) be a finite doubly-homogeneous quandle. Then \( Q \) is either projection or latin.

**Proof.** Let \( Q \) be not a projection quandle, then for every \( a \in Q \) there exists \( b \in Q \) such that \( b \circ a \neq a \). Let \( c \in Q \), then there exists \( f \in \text{Aut}(Q)_a \) such that \( c = f(b \circ a) \), since \( \text{Aut}(Q) \) is doubly transitive. Then \( c = f(b \circ a) = f(b) \circ R_a(f(b)) \). So \( R_a \) is surjective and then bijective for every \( a \in Q \). □

Using Corollary 3.4 and Lemma 3.6 we can finally prove one of the main theorems. Note that Theorem 1.4 of [Ste01] uses the classification of finite simple groups.

**Theorem 3.7.** Let \( Q \) be a finite simple quandle and \( |Q| > 2 \). The following are equivalent:

(i) \( Q \) is abelian.
(ii) \( Q \) is strictly simple.
(iii) \( Q \) is doubly-homogeneous.
(iv) \( Q \) is latin.

**Proof.** (i) ⇒ (ii) By Theorem 3.4 of [Va90], every finite simple abelian algebra has no proper subalgebras.

(ii) ⇒ (iii) It follows by Proposition 3.3.

(iii) ⇒ (iv) Since \( |Q| > 2 \) and it is simple, \( Q \) is not a projection quandle. According to Lemma 3.3, \( Q \) is latin.

(iv) ⇒ (i) Using Theorem 1.4 of [Ste01], \( \text{Dis}(Q) \) is solvable, so \( Q \) is solvable ([BS19, Theorem 1.2]) and simple, then abelian. □

Doubly-transitive quandles described in [Ven17] are the doubly-homogeneous quandles for which the left multiplication group coincides with the automorphism group (they are isomorphic to \( \text{Aff}(\mathbb{F}_q, \lambda) \) where \( \lambda \) is a generator of \( \mathbb{F}_q \)).

As a consequence of the Sylow and Lagrange properties for principal latin quandles and the characterization of simple latin quandles (Theorem 3.7) we have the following corollary.

**Corollary 3.8.** Let \( Q \) be a finite principal latin quandle and let \( p \) be a prime dividing \( |Q| \). Then there exists a strictly simple subquandle of \( Q \) of size a power of \( p \).

**Proof.** Let \( p \) be a prime dividing \( |Q| \). Then there exists a \( p \)-Sylow subquandle of \( S_p \) and it is principal and latin (2.13). Then every minimal subquandle of \( S_p \) (with respect to inclusion) is a strictly simple subquandle of size a power of \( p \), since \( S_p \) has the Lagrange property. □

As a corollary of Theorem 3.7 we obtain that there are no latin quandles of size \( 2n \) with \( n \) coprime with 30, based on non-existing known result for connected quandles of size 2p for \( p > 5 \).

**Corollary 3.9.** There is no latin quandle of size \( 2n \) with \( n \) coprime with 2, 3 and 5.

**Proof.** Let \( Q \) be a counterexample of minimal size. The quandle \( Q \) is not simple because its size is not a power of a prime. There are no connected quandles of size \( 2p \) for \( p > 5 \) [McC12], so \( n \) is not prime. For every \( \alpha \in \text{Con}(Q) \) either \([a]_\alpha\) or \( Q/\alpha \) has size \( 2m \) with \( m < n \) coprime with 30, violating the minimality of the size of \( Q \). □

3.3. Classification of Doubly-homogeneous quandles. In this section we show that all finite doubly homogeneous quandles are powers of a strictly simple quandles. Automorphism group of doubly transitive quandles acts transitively on the set of 2-generated subquandles, so they are all isomorphic and this property holds also in all factors. Recall that finite doubly homogeneous quandles are either projection or latin (see Lemma 3.6) and that latin quandles are solvable. In the following we denote by \( Sg(X) \) the subquandle generated by a subset \( X \).

**Lemma 3.10.** Let \( Q \) be a finite doubly-homogeneous quandle. Then there exists a strictly simple quandle \( M \) such that \( M \cong Sg([a]_\alpha, [b]_\alpha) \leq Q/\alpha \) for every \( \alpha \in \text{Con}(Q) \) and every \([a]_\alpha \neq [b]_\alpha \in Q/\alpha \).
Proof. If \( Q \) is projection, then \( Sg([a]_\alpha, [b]_\alpha) \cong \mathcal{P}_2 \) for every \( \alpha \in \text{Con}(Q) \) and every \( [a]_\alpha, [b]_\alpha \in Q/\alpha \).

Let \( Q \) be latin and let \( M \) be a strictly simple quandle containing \( a, b \in Q \), then \( M = Sg(a, b) \) and \( h(Sg(a, b)) = Sg(h(a), h(b)) \cong M \) for every \( h \in \text{Aut}(Q) \). Since \( \text{Aut}(Q) \) is doubly transitive, every pair of elements of \( Q \) generates a subquandle isomorphic to \( M \). Let \( \alpha \in \text{Con}(Q) \), then \( Sg([a]_\alpha, [b]_\alpha) \) is the image of \( M \) with respect to the canonical map \( a \mapsto [a]_\alpha \). The quandle \( M \) is simple, so whenever \( [a]_\alpha \neq [b]_\alpha \) then \( Sg([a]_\alpha, [b]_\alpha) \cong M \). \( \square \)

Let \( Q \) be a finite latin doubly homogeneous quandle. We define \( M_Q \) as the unique strictly simple quandle in \( HS(Q) \) up to isomorphism. We call \( M_Q \) the \textit{minimal quandle} of \( Q \).

**Proposition 3.11.** Let \( Q \) be a finite latin doubly homogeneous quandle. Then \( Q \) is a nilpotent quandle of prime power size.

**Proof.** Assume that \( |M_Q| \) is a power of \( p \). We prove by induction on the solvability length of \( Q \) that if all 2-generated subquandles are strictly simple and isomorphic then \( |Q| \) is a power of \( p \). If \( Q \) is abelian, then in particular it is principal and latin. If \( p, q \) are primes dividing \( |Q| \) then \( Q \) has a strictly simple subquandle of size a power of \( p \) and of size a power of \( q \) by virtue of Corollary 3.8. Then \( p = q \) since all 2 generated subquandles are isomorphic and so the size of \( Q \) is a power of a \( p \). Let \( Q \) be solvable of length \( n \), i.e. \( \gamma^{n-1}(Q) \) is abelian. The blocks of \( \gamma^{n-1}(Q) \) are abelian latin subquandles and then affine. Using again Corollary 3.8 the blocks have size a power of \( p \). By induction, \( Q/\gamma^{n-1}(Q) \) has size a power of \( p \), so \( |Q|/\gamma^{n-1}(Q)||[a]| \) is a power of \( p \). Finally, by [BS19, Theorem 1.4], \( Q \) is nilpotent. \( \square \)

Let \( M = \text{Aff}(\mathbb{F}_p^n, f) \) be a strictly simple quandle. Then its powers are given by

\[
M^n = \text{Aff}(\mathbb{F}_p^m, f \times \ldots \times f).
\]

The subquandles of \( M^n \) are subspaces invariant under \( f^{*n} = f \times \ldots \times f \) and so they correspond to subrepresentations of the cyclic group generated by \( f^{*n} \). We can Maschke theorem for group representations to \( M^n \), since the order of \( f^{*n} \) is coprime with \( p \) and its subrepresentation are direct product of irreducible representations, which are all isomorphic to \( M \). Therefore every subquandle of \( M^n \) is isomorphic to \( M^k \) for some \( k \leq n \).

On the other hand \( M \) can be understood as \( M = \text{Aff}(\mathbb{F}_p^m, \lambda) \) and \( M^n \) as \( \text{Aff}(\mathbb{F}_p^{mn}, \lambda^n) \), where \( \lambda \) denote the scalar multiplication by \( \lambda \).

**Remark 3.12.** In the next theorem we are using a well-know concept in universal algebra: every algebra embeds into a direct product of some of its subdirectly irreducible (SI) factors (an algebra is called subdirectly irreducible if the intersection of all the proper congruences of \( A \) is non-trivial). In particular if \( Q = \text{Aff}(A, f) \) and \( M, N \) are strictly simple subquandles of \( Q \) containing 0, then \( M \cap N = \{0\} \) and so \( \sim_M \wedge \sim_N = 0_Q \). So, if \( Q \) is finite and SI then there exists a unique strictly simple subquandle containing 0.

**Theorem 3.13.** Let \( Q \) be a finite latin quandle. The following are equivalent:

(i) \( Q \) is doubly-homogeneous quandle;

(ii) \( Q \cong M_Q \) where \( M_Q \) is the minimal quandle of \( Q \).

**Proof.** (i) \( \Rightarrow \) (ii) First we show that \( Q \) is principal. According to Proposition 3.11 \( Q \) is nilpotent and so \( 0_Q \leq \sigma_Q \leq \sigma_{Q} \) and the classes of \( \sigma_Q \) are blocks with respect to the action of \( \text{Aut}(Q) \). The automorphism group of \( Q \) is doubly-transitive and then primitive, so \( \sigma_Q = 1_Q \), i.e. \( Q \) is principal and latin.

Let \( Q_{\text{Hom}}(G, f) \) be a principal representation of \( Q \). Since \( \text{Aut}(Q)_{1} = C_{\text{Aut}(Q)}(f) \) is transitive on \( G \setminus \{1\} \), \( G \) is elementary abelian and so \( Q \) and all its factors are affine.

Let \( Q/\alpha \) be a subdirectly irreducible factor of \( Q \). According to Lemma 3.10 every pair of elements of \( Q/\alpha \) generates a strictly simple subquandle isomorphic to \( M_Q \). By Remark 3.12 every pair of elements 0, \( a \in Q/\alpha \) generates the same strictly simple subquandles. Therefore \( Q/\alpha \cong M_Q \).

The quandle \( Q \) embeds into a product of some of its subdirectly irreducible factors, thus \( Q \) embeds into \( M_Q^n \) for some \( n \in \mathbb{N} \) and therefore \( Q \cong M_Q^n \) for some \( k \leq n \).
(ii) ⇒ (i) Let \( q = p^n \). Up to isomorphism, there exists \( \lambda \in \mathbb{F}_q^* \) such that \( M_Q \cong \text{Aff}(\mathbb{F}_q, \lambda) \) and \( Q \cong \text{Aff}(\mathbb{F}_q, \lambda) \). Then \( \text{GL}_n(\mathbb{F}_q) \leq \text{GGL}_n(p)(\lambda) = \text{Aut}(Q)_0 \). Therefore, \( \text{Aut}(Q)_0 \) is transitive over \( Q \setminus \{0\} \).

\[ \square \]

4. Cyclic Quandles

4.1. Extensions of strictly simple quandles by projection quandles. Let \( Q \) be a (connected) quandle and \( \alpha \in \text{Con}(Q) \). We called \( Q \) a (connected) extension of \( Q/\alpha \) by \([a]\). We study some particular extensions of quandles in the same direction of \([\text{Cla}10]\), in which extensions of affine quandles by projection quandles of size 2 have been investigated. We will focus on extensions of strictly simple quandles by projection quandles of prime size. We first shows the properties of the congruence lattice of such extensions using the Galois connection between the congruence lattice of a quandle and the lattice of normal subgroups of the left multiplication group contained in the displacement group (denoted by \( \text{Norm}(Q) \)). Recall that for faithful quandles a congruence \( \alpha \) is abelian (resp. central) if and only if \( \text{Dis}_\alpha \) is abelian (resp. central), \([\text{BS}19]\) Corollary 5.4. Moreover if \( \alpha \) is a minimal congruence then \( \text{Dis}_\alpha \) is a minimal elements of \( \text{Norm}(Q) \): indeed if \( N < \text{Dis}_\alpha \) then \( \mathcal{O}_N < \alpha \) and so \( \mathcal{O}_N = Q_0 \) and accordingly \( N = 1 \).

Lemma 4.1. Let \( p \) be a prime and let \( Q \) be a finite faithful connected extension of a strictly simple quandle \( Q/\alpha \) by a projection quandle of prime power size. If \( \alpha \) is a minimal congruence then \( \alpha \) is abelian.

Proof. To prove that \( \alpha \) is abelian we just need to show that \( \text{Dis}_\alpha \) is Abelian, since \( Q \) is faithful. The subgroup \( \text{Dis}_\alpha \) is generated by the family of non-trivial subgroups \( K_{[a]} = \{ L_bL_c^{-1} : b, c \in [a] \} \) for \( [a] \in Q/\alpha \), which are all normal in \( \text{Dis}^a \). The length of the orbits of \( K_{[a]} \) divides the length of the orbits of \( \text{Dis}^a \) which is \( |[a]| \). The subgroup \( K_{[a]} \) acts trivially on \([a]\) and since \( Q = S_G([a], b) \) whenever \( b \in [a] \), then it acts semiregularly on the block \([b]\) and then \( |K_{[a]}| \) divides the size of the blocks and so it is a \( p \)-group. Since \( K_{[a]} \cap K_{[b]} = 1 \), hence \( |K_{[a]}|, |K_{[b]}| = 1 \). So \( \text{Dis}_\alpha \) is generated by a family of commuting \( p \)-subgroups and so it is nilpotent. Let \( D \) be the derived subgroup of \( \text{Dis}_\alpha \). Then \( D \in \text{NNorm}(Q) \) and it is a proper subgroup of \( \text{Dis}_\alpha \), therefore \( D = 1 \).

Lemma 4.2. Let \( Q \) be a finite connected extension of a strictly simple quandle \( Q/\alpha \) by a prime size projection quandle. Then \( \alpha = \gamma_Q \) and it is the unique proper congruence of \( Q \). If \( Q \) is faithful \( \zeta_Q = 0_Q \).

Proof. The congruence \( \alpha \) is a minimal congruence of \( Q \) since its blocks have prime size and it is maximal since the factor \( Q/\alpha \) is simple. The quandle \( Q \) is not affine since it contains projection subquandles. The factor \( Q/\alpha \) is abelian so \( \gamma_Q \leq \alpha \) and then \( \alpha = \gamma_Q \). Let \( \beta \neq \alpha \) be a congruence of \( Q \). Then \( \alpha \cap \beta = 0_Q \) and \( Q \) embeds into \( Q/\alpha \times Q/\beta \). The factor \( Q/\alpha \) has no proper subquandle, therefore \( [a]_\alpha \) projects onto \( Q/\alpha \). Therefore \( |Q/\beta|[a]_\beta \geq |Q/\beta||Q/\alpha| \) and accordingly \( Q \cong Q/\alpha \times Q/\beta \) and \( Q/\beta \cong [a]_\alpha \) is not connected, contradiction. Hence \( \alpha \) is the unique proper congruence of \( Q \). If \( Q \) is faithful, the blocks of central congruences are connected according to \([\text{BB}13]\) Corollary 3.2. Then \( \alpha \) is not central and so \( \zeta_Q = 0_Q \).

The properties of the congruence lattice influence the structure of the displacement group and its subgroups.

Corollary 4.3. Let \( Q \) be a finite faithful connected extension of a strictly simple quandle \( Q/\gamma_Q \) by a projection quandle of prime size \( p \). Then \( |Q/\gamma_Q| \) and \( p \) are coprime and \( Z(\text{Dis}(Q)) = 1 \).

Proof. Connected quandles of prime power size are nilpotent \(([\text{BS}19]\) Proposition 6.5) and the orbits of \( Z(\text{Dis}(Q)) \) are contained in the block of \( \zeta_Q \) \(([\text{BS}19]\) Lemma 5.12). Since \( \zeta_Q = 0_Q \) then \( Z(\text{Dis}(Q)) = 1 \) and \( |Q| \) is not a power of a prime, so \( p \) is coprime with \( |Q/\gamma_Q| \).

Lemma 4.4. Let \( Q \) be a faithful connected quandle and \( \alpha \in \text{Con}(Q) \). If \( \text{Dis}_\alpha \) is cyclic, then \( \alpha \) is central.
Proof. Let \( \psi : \text{LMlt}(Q) \longrightarrow \text{Aut}(\text{Dis}_\alpha) \), be the automorphism that defines the conjugation action of \( \text{LMlt}(Q) \) on \( \text{Dis}_\alpha \). Since \( \text{Aut}(\text{Dis}_\alpha) \) is abelian, then \( \text{Dis}(Q) = \gamma_1(\text{LMlt}(Q)) \leq \ker(\psi) \). So \( \text{Dis}_\alpha \) is central in \( \text{Dis}(Q) \) and \( \alpha \) is central. \( \square \)

Recall that if \( \alpha \) is abelian, then \( (\text{Dis}_\alpha)_a = (\text{Dis}_\alpha)_b \) whenever \( b \alpha a \in \text{BS19, Theorem 1.2} \).

**Proposition 4.5.** Let \( Q \) be a finite faithful connected extension of a strictly simple quandle \( Q/\gamma_\alpha \) by a projection quandle of prime size \( p \). Then \( \text{Dis}_{\gamma_\alpha} \cong \mathbb{Z}_p^2 \) if \( \text{Dis}_{\gamma_\alpha} = \gamma_{\gamma_\alpha} \text{Dis}(Q)_a \) and \( N(\text{Dis}(Q)_a) \leq \text{Dis}_{\gamma_\alpha} \) for every \( a \in Q \).

**Proof.** Let \( \alpha = \gamma_\alpha \). The subgroup \( \text{Dis}_{\gamma_\alpha} \) is not trivial, so \( \mathcal{O}_{\text{Dis}_{\gamma_\alpha}} = \alpha \), i.e. \( \text{Dis}_{\gamma_\alpha} \) is transitive on each block of \( \alpha \) and so \( \text{Dis}_\alpha = \text{Dis}_{\gamma_\alpha} \text{Dis}(Q)_a \) and \( |\text{Dis}_\alpha| = p(\text{Dis}_\alpha)_a \). The quandle \( Q \) is generated by \([a], b \) whenever \( b \not\in [a] \), and since \((\text{Dis}_\alpha)_a \) fixes the block \([a] \) then \((\text{Dis}_\alpha)_a,b = 1 \) whenever \( b \not\in [a] \). So \( |(\text{Dis}_\alpha)_a| = |b(\text{Dis}_\alpha)_a| \) and since stabilizer \( (\text{Dis}_\alpha)_a \) is normal in \( \text{Dis}_\alpha \) then \(|(\text{Dis}_\alpha)_a| \) divides \( p \). If \( \text{Dis}_\alpha \) is cyclic then \( \alpha \) is central by Lemma 4.4. According to Lemma 4.3, the center of \( \text{Dis}_\alpha \) is trivial and so \( \text{Dis}_\alpha \cong \mathbb{Z}_p^2 \). If \( h \in N(\text{Dis}(Q)_a) \) then \( h \in N((\text{Dis}_\alpha)_a) \), i.e. \((\text{Dis}_\alpha)_b(\alpha) = (\text{Dis}_\alpha)_a \) and therefore \( h \in \text{Dis}_\alpha \). \( \square \)

We are going to discuss the properties of the extensions in this section according to the equivalence \( \sigma_\alpha \).

**Lemma 4.6.** Let \( Q \) be a finite faithful connected extension of a strictly simple quandle \( Q/\gamma_\alpha \) by a projection quandle of prime size. Then either \( \sigma_\alpha = \gamma_\alpha \) or \( \sigma_\alpha = 0_\alpha \). In particular, the following are equivalent:

(i) \( \text{Dis}_{\gamma_\alpha} = \text{Dis}_{\gamma \alpha} \).

(ii) \( \gamma_\alpha = \sigma_\alpha \).

(iii) \( \text{Dis}(Q)_{a,b} = 1 \) whenever \( b \not\in [a] \).

**Proof.** Let \( \alpha = \gamma_\alpha \). The block \([a]_{\gamma_\alpha} \) is the orbit of \( a \) under the normalizer of \( \text{Dis}(Q)_a \) which is contained in \( \text{Dis}_\alpha \) by Proposition 4.4. Therefore \( \sigma_\alpha \leq \alpha \). The size of the \( \sigma_\alpha \) blocks divides the size of \( Q \) then either \( \sigma_\alpha = \alpha \) or \( \sigma_\alpha = 0_\alpha \).

(i) \( \Rightarrow \) (ii) Let \( b \in [a] \). Then \( \text{Dis}(Q)_a = (\text{Dis}_\alpha)_a = (\text{Dis}_\alpha)_b = \text{Dis}(Q)_b \), so \( \alpha \leq \sigma_\alpha \).

(ii) \( \Rightarrow \) (iii) If \( h \in \text{Dis}(Q)_{a,b} \), then \( h \) fixes \([a] \) and \( b \) and accordingly \( h = 1 \) (\( Q \) is generated by \([a] \) and \( b \)).

(iii) \( \Rightarrow \) (i) The subgroup \(|\text{Dis}(Q)_a| = |b(\text{Dis}(Q)_a)| \leq p \). Therefore \(|\text{Dis}_\alpha| = p|\text{Dis}(Q)_a| \leq p^2 \). So \( \text{Dis}_{\gamma_\alpha} = \text{Dis}_\alpha \) since \(|\text{Dis}_\alpha| = p^2 \). Prop\( 4.5 \).

By virtue of Proposition 4.5, \( \sigma_\alpha = \gamma_\alpha \) if and only if \( N(\text{Dis}(Q)_a) = \text{Dis}_{\gamma_\alpha} \) and \( \sigma_\alpha = 0_\alpha \) if and only if \( N(\text{Dis}(Q)_a) = \text{Dis}(Q)_a \).

If \( Q \) is a faithful connected extension of a strictly simple quandle \( Q/\gamma_\alpha \) by a projection quandle of prime size then \( \text{Dis}_{\gamma_\alpha} = \gamma_1(\text{Dis}(Q)) \) is a maximal characteristic subgroup and \( Q/\gamma_\alpha \) is affine over \( G/\gamma_\alpha(\text{G}) \) by [BHR9, Proposition 2.6]. The subgroup \( \text{Dis}_{\gamma_\alpha} \) is a minimal normal subgroup of \( \text{LMlt}(Q) \) and since \( \gamma_2(\text{G}) \leq \text{Dis}_{\gamma_\alpha} \) [BS19, Proposition 3.3] then \( \gamma_2(\text{G}) = \text{Dis}_\alpha \) (because \( G \) is nilpotent). Moreover \( \gamma_2(\text{Dis}(Q)) \) and \( \text{Dis}(Q)/\gamma_1(\text{Dis}(Q)) \) are elementary abelian group with coprime size (see Corollary 4.3 and Proposition 4.5).

The next Proposition describes the structure of a group with such properties.

**Proposition 4.7.** Let \( p, q \) be different primes, \( m, n \in \mathbb{N} \) and let \( G \) be a group such that \( \gamma_2(\text{G}) \cong \mathbb{Z}^m_p \times \mathbb{Z}^n_q \) and \( \gamma_1(\text{G}) \) is a maximal characteristic subgroup. If \( \gamma_1(\text{G}) = \gamma_2(\text{G}) \) then \( G \cong \mathbb{Z}^m_p \times \mathbb{Z}^n_q \), otherwise \( G \cong \mathbb{Z}^m_p \times K \) where \( K \) is a special \( q \)-group with \( \text{Z}(K) \cong \gamma_1(\text{G})/\gamma_2(\text{G}) \).

**Proof.** If \( \gamma_1(\text{G}) = \gamma_2(\text{G}) \) then the short exact sequence

\[ 1 \longrightarrow \gamma_1(\text{G}) \longrightarrow G \longrightarrow G/\gamma_1(\text{G}) \longrightarrow 1 \]

splits as a semidirect product, since \( \gamma_1(\text{G}) \) is abelian and the order of \( G/\gamma_1(\text{G}) \) and \( \gamma_1(\text{G}) \) are coprime.

Otherwise, the factor group \( K = G/\gamma_2(\text{G}) \) is a 2-step nilpotent group and so \( \gamma_1(\text{K}) \leq \text{Z}(K) \). Let \( \pi : G \longrightarrow G/\gamma_2(\text{G}) \) be the canonical projection, then the characteristic subgroup \( L = \pi(\text{Z}(\text{K})) \)
contains $\gamma_1(G)$. The subgroup $L$ is a proper subgroup of $G$ since $K$ is not abelian. Then either $L = \gamma_1(G)$ or $L = G$. If $L = G$, then $G$ is 2-step nilpotent, so $G = K$ and $\gamma_1(K) \leq Z(K)$ so $\gamma_1(K) = Z(K)$. Otherwise $Z(K) = \gamma_1(G)/\gamma_2(G)$. The same argument shows that the Frattini subgroup of $K$ equals the center.

Since $K$ is nilpotent of length 2, the commutator mappings $[\cdot, \cdot]: K \to Z(K), \ x \mapsto [x, a]$ are homomorphisms and they factorize through the canonical projection onto $K/Z(K) \cong G/\gamma_1(G)$ which is an elementary $q$-abelian group. Moreover if $\{a_i : 1 \leq i \leq n\}$ is a basis of $G/\gamma_1(G)$ then $Z(K) \cong \gamma_1(G)/\gamma_2(G)$ is generated by $\{[a_i, a_j]z_i : 1 \leq i, j \leq n\}$ [Sim94, Proposition 9.2.5]. Since the generators are images of the factorized commutator mappings they have order $q$. Then $Z(K)$ is an elementary $q$-abelian group and then $K$ is a special $q$-group. Thus, the short exact sequence

$$1 \to \gamma_2(G) \to G \to G/\gamma_2(G) \to 1$$

splits as a semidirect product. □

We can apply Proposition 4.1 to the displacement group of extensions of strictly simple quandles by a prime size projection quandles. Note that the two cases in Proposition 4.1 correspond exactly to the two cases in Lemma 4.6

**Theorem 4.8.** Let $Q$ be a finite faithful connected extension of a strictly simple quandle $Q/\gamma_Q$ of size $q^n$ by a projection quandle of prime size $p$. Then

(i) if $\sigma_Q = 0$ then $|Q| = 6$.

(ii) If $\sigma_Q$ is faithful then $q = 2$, $n = 2k$, $p = 2^k + 1$ and

$$\text{Dis}(Q) \cong \mathbb{Z}_{2^{k+1}} \rtimes_p H$$

where $H$ is an extraspecial 2-group and the action $\rho$ is faithful.

**Proof.** Let $\alpha = \gamma_Q$, $G = \text{Dis}(Q)$, $f = T_a \in \text{Aut}(G)$ and $H = \text{Fix}(f) = \text{Dis}(Q)_\alpha$. The block $[a]_\alpha$ is projective and it is isomorphic to $Q/(\text{Dis}_\alpha, (\text{Dis}_\alpha)_\alpha, f|_{\text{Dis}_\alpha})$. Fixed a suitable basis $e_1, e_2$ with $e_1 \in (\text{Dis}_\alpha)_\alpha$, the restriction of $f$ to $\text{Dis}_\alpha$ is

$$f = f|_{\text{Dis}_\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

We can apply Proposition 4.1 to $G$ by virtue of the above remark.

(i) If $\sigma_Q = 0$ then $\gamma_1(G) = \gamma_2(G) = N(\text{Fix}(f))$. So $G \cong \mathbb{Z}_{p^m} \rtimes_p \mathbb{Z}_{q^n}$ and $\rho$ is faithful since $Z(G) = 1$. If $h \in \mathbb{Z}_{q^n}$ then

$$\rho_h = \begin{bmatrix} x & y \\ y & t \end{bmatrix}$$

with $y \neq 0$ since $h$ does not normalize $\text{Fix}(f)$. Using that $\rho_{f(h)} = f|_{\text{Fix}_h}(f)$ and that $\rho_h \rho_f(h) = \rho_{f(h)} \rho_h$, we have $2y = 0$ and so $p = 2$. Hence $q^n$ divides $|GL_2(2)| = 6$. Therefore $q^n = 3$ and $|Q| = 6$.

(ii) If $\sigma_Q = 0$ then $\gamma_1(G) = \gamma_2(G)$ and $N(\text{Fix}(f)) = \text{Fix}(f)$. Then $G \cong \mathbb{Z}_{p^m} \rtimes_p \mathbb{Z}_{q^n}$ and $K$ is a special $q$-group. According to Proposition 4.1, $\gamma_1(G) = \gamma_2(G) \text{Fix}(f)$ and then $\text{Fix}(f) \cong (\text{Dis}_\alpha)_\alpha \rtimes_p Z(K)$. If $\rho_h = 1$ then $h \in N(\text{Fix}(f)) \cap K = \text{Fix}(f) \cap K = Z(K)$ and so $h \in Z(G) = 1$. So the action $\rho$ is faithful. Every $z \in Z(K)$ normalizes $(\text{Dis}_\alpha)_\alpha$ and using the equation of the orbits for the action of $z$ over the 1-dimensional subspaces of $\gamma_2(G)$ we have that

$$p + 1 = e + nq$$

where $e \geq 1$ is the number of the eigenspaces of $z$. Then $e = 2$ and $z$ is diagonalizable. Moreover

$$F(zvz^{-1}) = Fp_z(v) = f(z)F(v)f(z)^{-1} = \rho_{f(z)}F(v) = \rho_zF(v).$$

for every $v \in \gamma_2(G)$. Equation (9) implies that $\rho_z$ is a scalar matrix, so $Z(K)$ embeds into $\mathbb{Z}_{p-1}$ and therefore it is cyclic. Then $K$ is an extraspecial $q$-group, $q$ divides $p - 1$ and $n = 2k$.

Let $q > 2$. Since $q$ divides $p - 1$, by virtue of (8) the action of a non central element $h \in K$ is
diagonalizable with different eigenvalues since $h$ does not normalize $(\text{Dis}_a)_a$. Let $z$ be a generator of $\mathcal{Z}(K)$ and $g \in K$ which does not centralize $h$, then

$$(10) \quad [\rho_h, \rho_g] = \rho_z^s$$

for some $1 \leq s \leq p - 2$. From (10) follows that $\rho_z^s = 1$ and therefore $g = 2$, contradiction. Let $q = 2$ and let $s$ be the order of left multiplications of $Q/\alpha$. Then $s$ divides $|Q/\alpha| = 2^{2k} - 1 = (2^k - 1)(2^k + 1)$ since $Q/\alpha$ is strictly simple. Then $f^s(h) = htd$ for some $t \in Z(H)$ and $d \in \text{Dis}_\alpha$ for every non-central element $h \in H$. Then

$$(11) \quad \rho_{f^s(h)} = f^s \rho_h f^{-s} = \rho_h \rho_t,$$

and $\rho_h$ is like in (7) with $y \neq 0$ since $h$ does not normalize $(\text{Dis}_a)_a$. If (11) holds then $sy = 0 \pmod{p}$ i.e. $p$ divides $s$, so

either $p$ divides either $2^k + 1$ or $2^k - 1$. (*)

The centralizer of $\rho_h$ have size $2^{2k}$ for every non-central element $h \in K$. According to (8) the number of eigenspaces of $h$ is either $0$ or $2$. If the action of $\rho_h$ is irreducible, the centralizer of $\rho_h$ in the image of $\rho$ is cyclic [Sho92, Theorem 2.3.5] and then it has order $4$ and $k = 1$. Since $p$ divides $4 - 1 = 3$ then $p = 3 = 2^1 + 1$.

If $\rho_h$ is diagonalizable with different eigenvalues (otherwise $h$ normalizes $(\text{Dis}_a)_a$) then the centralizer of $\rho_h$ is given by the diagonal matrices so $2^{2k}$ divides $(p - 1)^2$. Hence $2^k$ divides $p - 1$, and in particular $p \geq 2^k + 1$. From (*), it follows $p = 2^k + 1$.

In the RIG library the unique example of quandles as in Theorem (ii) is SmallQuandle(12,10).

4.2. Classification of Connected Cyclic Quandles. In this section we investigate a classification of quandles defined by a particular cycle structure of the left multiplications. This class has been already studied in [LL18] using combinatorial tools: we present an alternative approach making use of the results in the previous section.

Definition 4.9. A finite quandle $Q$ is called cyclic with $f$ fixed points if for every $a \in Q$, the cycle structure of $L_a$ is given by $f$ fixed points and one cycle of size $|Q| - f$.

We complete the classification of cyclic quandles in two steps: first we show that if the number of fixed points is prime then $|Q| = 6$. Then we show that any cyclic quandle with $f$ fixed points has a cyclic factor with a prime number of fixed points and from that we deduce that $|Q| = 6$ as well.

Let us recall some results from [LL18].

Lemma 4.10. [LL18] Proposition 2.4, Theorem 2.1, Theorem 2.2] Let $Q$ be a cyclic quandle with $f$ fixed points. Then $Q$ is connected if and only if $|Q| > 2f$ and $\mathcal{P}$ is an equivalence relation with $[a]_\mathcal{P} = \text{Fix}(L_a)$.

In the following we are making use of the following subgroup

$$(12) \quad (L_a)^\mathcal{P} = (L_a) \cap \text{Aut}^\mathcal{P}.$$  

Proposition 4.11. Let $Q$ be a connected cyclic quandle with $f$ fixed points. Then $\mathcal{P}$ is the unique maximal congruence of $Q$ and $Q/\mathcal{P}$ is a doubly transitive quandle.

Proof. Since $(L_a)$ is regular on $Q \setminus [a]_\mathcal{P}$ then the group $(L_a)/(L_a)^\mathcal{P}$ acts regularly on the set of $\mathcal{P}$-blocks $Q/\mathcal{P} \setminus \{[a]\}$. So we have that

$$(13) \quad |Q/\mathcal{P}| - 1 = \frac{|L_a|}{|(L_a)^\mathcal{P}|} = \frac{(|Q/\mathcal{P}| - 1)f}{|(L_a)^\mathcal{P}|}.$$  

Therefore the subgroup $\mathcal{P}$ has order $f$ and so it is regular on each block of $\mathcal{P}$ different from $[a]$. The subgroup $\text{LMlt}^\mathcal{P}$ contains $(L_a)^\mathcal{P}$ for each $a \in Q$, therefore $\text{LMlt}^\mathcal{P}$ is transitive on each block. Then $\mathcal{P} = \text{O}_{\text{LMlt}}^\mathcal{P}$ and according to [BS19] Lemma 2.6] $\mathcal{P}$ is a congruence. Let $a, b \in Q$ in different classes with respect to $\mathcal{P}$. Then $a^{L_a} \cap b^{L_b} \subseteq \text{Sg}(a, b)$. By virtue of Lemma 4.10 $a^{L_a} = Q \setminus [b]_\mathcal{P}$ and $b^{L_b} = Q \setminus [a]_\mathcal{P}$, therefore $Q$ is generated by $a$ and $b$. Hence, the blocks of $\mathcal{P}$ are
the unique maximal subquandles of $Q$, and accordingly $\mathcal{P}$ is the unique maximal congruence of $Q$. The order of left multiplication in the factor $Q/\mathcal{P}$ is $|Q/\mathcal{P}| - 1$ according to [13]. Then the factor $Q/\mathcal{P}$ is doubly transitive [13, Corollary 4].

**Lemma 4.12.** Let $Q$ be a connected cyclic quandle. Then $Q$ is faithful.

**Proof.** Assume that $L_a = L_b$. Then necessarily $a \mathcal{P} b$ and there exists $c \notin [a] \mathcal{P}$, and $h \in (L_a)^{\mathcal{P}}$ such that $b = h(a)$. So $L_b = L_h(a) = hL_a h^{-1} = L_a$. Then $h(a \ast c) = a \ast h(c) = a \ast c$ and $[a \ast c] \neq [c]$ ($Q/\mathcal{P}$ is affine and connected, so it has no projection subquandle). Since $(L_a)^{\mathcal{P}}$ acts regularly on each block different of $[c] \mathcal{P}$, then $h = 1$, i.e. $a = b$. \hfill $\square$

We can apply Theorem 1.8 to cyclic quandles with a prime number of fixed points. Indeed if $Q$ is such a quandle, $Q$ is faithful, $Q/\mathcal{P}$ is strictly simple and $\text{Dis}(Q)_{a \mathcal{P} b} = 1$ whenever $b \notin [a]$ ($Q$ is generated by $a$ and $b$).

**Corollary 4.13.** Let $Q$ be a connected cyclic quandle with $p$ fixed points where $p$ is a prime. Then $p = 2$ and $|Q| = 6$.

**Theorem 4.14.** Let $Q$ be a connected cyclic quandle with $f$ fixed points. Then $f = 2$ and $|Q| = 6$.

**Proof.** The quandle $Q$ is generated by any pair of elements such that $[a] \mathcal{P} \neq [b] \mathcal{P}$. Then $(L_a)^{\mathcal{P}} \cap (L_b)^{\mathcal{P}} = \text{LMlt}(Q)_a \cap \text{LMlt}(Q)_b = 1$ and so $Z_f^2 \cong (L_a)^{\mathcal{P}} \times (L_b)^{\mathcal{P}} \leq \text{ LMlt}^{\mathcal{P}}$. Since $\text{LMlt}(Q)_a \leq \text{LMlt}^{\mathcal{P}}$ is regular on $[b]$, $|\text{LMlt}^{\mathcal{P}}| = f |\text{LMlt}(Q)_a| \leq f^2$. Hence $\text{LMlt}^{\mathcal{P}} \cong Z_f^2$.

Let $q$ be a prime dividing $f$. Then $K = q \text{LMlt}^{\mathcal{P}} \cong qZ_f \times qZ_f$ is a characteristic subgroup of $\text{LMlt}^{\mathcal{P}}$ and so it is normal in $\text{LMlt}(Q)$. Then the congruence $\alpha = \mathcal{O}_K$ is contained in $\mathcal{P}$ and it has blocks of size $\frac{q}{p}$. The factor $Q/\alpha$ is a cyclic quandle, since $Q/\alpha = \text{ Fix}(L_{[a]}) \cup \{ L_{[a]}(b) : k \in \mathbb{N} \}$ of size $|Q/\alpha| = q/|\mathcal{P}|$. By virtue of Corollary 1.13, $q = 2$ and $|Q/\mathcal{P}| = 3$. Hence $f = 2^k$ and the subgroups $2^4 \text{LMlt}^{\mathcal{P}}$ for every $0 \leq l \leq k$ provide a chain of congruences with cyclic factor of size $3 \cdot 2^l$. An exhaustive computer search (for instance on the RIG database on GAP) shows that there are no such quandles for $k = 2$. Hence $k = 1$ and $|Q| = 6$. \hfill $\square$

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