FILTERED RANDOM VARIABLES, BIALGEBRAS AND CONVOLUTIONS

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Abstract

We introduce the filtered *-bialgebra which is a multivariate generalization of the unital *-bialgebra \( C\langle X, X', P \rangle \) of polynomials in noncommuting variables \( X = X^*, X'^* = X' \) and a projection \( P = P^* = P^2 \), endowed with the coproduct \( \Delta(X) = X \otimes 1 + 1 \otimes X, \Delta(X') = X' \otimes P + P \otimes X' \), with \( P \) being group-like. We study the associated convolutions, random walks and filtered random variables. The GNS representations of the limit states lead to filtered fundamental operators which are the CCR fundamental operators on the multiple symmetric Fock space \( \Gamma(H) \) over \( H = L^2(\mathbb{R}^+, G) \), where \( G \) is a separable Hilbert space, multiplied by appropriate projections. The importance of filtered random variables and fundamental operators stems from the fact that by addition and strong limits one obtains from them the main types of noncommutative random variables and fundamental operators, respectively, regardless of the type of noncommutative independence.

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1. Introduction

In this work we introduce and study basic noncommutative random variables, from which the main types of noncommutative random variables can be constructed regardless of the notion of independence.
The basic idea of introducing filtered random variables is pretty straightforward and has its origin in the definition of the convolution of measures and the associated states. Let $C[X]$ be the unital $*$-algebra of polynomials in $X^* = X$, with the coproduct

$$\Delta(X) = X \otimes 1 + 1 \otimes X. \quad (1.1)$$

If $\phi, \psi$ are states on $C[X]$, then

$$\phi \ast_c \psi = \phi \otimes \psi \circ \Delta$$

gives the convolution of states corresponding to the classical convolution of measures.

In order to define a quantum deformation of this simple model, we replace the unit in the coproduct (1.1) by a projection $P$ to get

$$\Delta(X') = X' \otimes P + P \otimes X', \quad \Delta(P) = P \otimes P \quad (1.2)$$

Then, for given state $\phi$ on $C[X']$, we define its noncommutative extension $\tilde{\phi}$ to $C[X'] \ast C[P]$ with identified units, by

$$\tilde{\phi}(P^\alpha Y^{n_1} P Y^{n_2} P \ldots Y^{n_k} P^\beta) = \phi(Y^{n_1}) \phi(Y^{n_2}) \ldots \phi(Y^{n_k})$$

where $\alpha, \beta \in \{0, 1\}$ and $n_1, \ldots, n_k \in \mathbb{N}$, called the Boolean extension [11]. The convolution

$$\tilde{\phi} \ast_B \tilde{\psi} = \tilde{\phi} \otimes \tilde{\psi} \circ \Delta$$

where $\Delta$ is given by (1.2), gives a quantum analog of the classical convolution of states, called the Boolean convolution. Note that by introducing $P$ we can deal with the usual tensor coproducts instead of the special one as in the approach of Schürmann [19]. The same holds for the $m$-free and free products of states.

This new convolution is very important since its generalization to the multivariate case, when restricted to suitable $*$-bialgebras, gives also $m$-free and free convolutions (see [4] and [11]). In the multivariate case we study the unital $*$-algebra $\hat{B}$ over $\mathbb{C}$ generated by $X_k(\sigma), P(\sigma)$ $k \in \mathbb{N}, \sigma \in P(\mathbb{N})$, where $P(\mathbb{N})$ is the power set of $\mathbb{N}$, with the involution given by $X_k(\sigma) = X_k(\sigma)^*$, $P(\sigma)^* = P(\sigma)$, and subject to the relations

$$P(\sigma) P(\tau) = P(\sigma \cap \tau), \quad P(\emptyset) = 1$$

$$P(\sigma) X_k(\tau) = X_k(\tau) P(\sigma) \quad \text{iff} \quad k \in \sigma$$

i.e. $P(\sigma)$’s are projections which “partially commute” with the variables $X_k(\sigma)$. When equipped with the coproduct and the counit

$$\hat{\Delta}(X_k(\sigma)) = X_k(\sigma) \otimes P(\sigma) + P(\sigma) \otimes X_k(\sigma)$$

$$\hat{\Delta}(P(\sigma)) = P(\sigma) \otimes P(\sigma), \quad \hat{\epsilon}(X_k(\sigma)) = 0, \quad \hat{\epsilon}(P(\sigma)) = 1,$$

the algebra $\hat{B}$ becomes a unital $*$-bialgebra called filtered $*$-bialgebra. Therefore, we are in the position to study random walks [14] and stochastic processes over $*$-bialgebras (see [1] and [18]).
We take a suitable state $\hat{\phi}$ on $\mathcal{B}$, which is obtained by lifting the tensor product state $\tilde{\phi} \otimes \infty$ on $\bigotimes_{k=1}^{\infty} C(Y_k, P_k)$ to $\mathcal{B}$ through the mapping which sends each $X_k(\sigma)$ onto $Y_k$ and $P(\sigma)$ onto the tensor product of $P_k$'s with $k \in \sigma$, where $Y_k^* = Y_k$ and $P_k$ a projection. The state $\hat{\phi}$ is our noncommutative, “filtered” analog of the classical product measure ($\sigma$’s play the role of filters due to “partial commutations”).

The corresponding convolution central limit theorem (or discrete random walk), which plays the role of a noncommutative analog of the classical multivariate central limit theorem, gives, under the usual normalization, pointwise convergence of the $N$-th convolution power

$$\hat{\phi}^{*N} = \hat{\phi} \otimes N \circ \hat{\Delta}^{N-1}$$

where $\hat{\Delta}^{N-1}$ is the $N - 1$-th iteration of the coproduct $\hat{\Delta}$. The summands produced by iterating the coproduct are called filtered random variables and can be viewed as quantum analogs of independent random vectors. It is important to note that by taking suitable linear combinations (strongly convergent series on the GNS pre-Hilbert space) of filtered random variables we obtain $m$-free (free) random variables. Thus all three basic notions of quantum independence in the axiomatic theory (tensor, free and Boolean, see [3],[20]) are covered by this scheme.

By considering random walks with continuous time, or stochastic processes over the filtered $*$-bialgebra, we obtain in the limit the vacuum expectation state in the multiple symmetric Fock space $\Gamma(\mathcal{H})$, where

$$\mathcal{H} = L^2(\mathbb{R}^+ \otimes \mathcal{G})$$

and $\mathcal{G}$ is a separable Hilbert space called the multiplicity space. The GNS representation leads to filtered fundamental operators which are the CCR fundamental operators on $\Gamma(L^2(\mathbb{R}^+) \otimes \mathcal{G})$, multiplied by projections $P^{(\sigma)}$, where $P^{(\sigma)}$ is the second quantization of the canonical projection onto subspaces of $L^2(\mathbb{R}^+) \otimes \mathcal{G}$ built from the modes (called colors) which belong to the set $\sigma$.

Fundamental operators associated with different notions of independence can be expressed in terms of the filtered ones. In particular, one can define bounded extensions to $\Gamma(L^2(\mathbb{R}^+) \otimes \mathcal{G})$ of $m$-free creation and annihilation operators introduced in [5] as strongly convergent series of filtered creation and annihilation operators, respectively. This formalism enables us not only to embed the free (or, full) Fock space over $L^2(\mathbb{R}^+)$ in $\Gamma(\mathcal{H})$, but also decompose $\Gamma(\mathcal{H})$ into an orthogonal sum of subspaces which are isomorphic to the free Fock space.

The corresponding filtered stochastic calculus is developed in [12] and it is, in fact, a generalization of the Hudson-Parthasarathy calculus [6] (see also [16]) on multiple symmetric Fock spaces [15] and includes a new version of the free calculus, originally developed for the Cuntz algebra [8], as well as gives calculi for the hierarchy of $m$-free Brownian motions introduced in [4]. In that context, see also [7] and [17].

In Section 2 we introduce the filtered $*$-bialgebra which sets the framework for a unified approach to noncommutative probability. This leads to filtered random variables, which are introduced in the more general setting of unital $*$-algebras in Section 3. Their combinatorics and the recurrence relation for the product state is given in Section 4. Convolution limit theorems are proved in Section 5. In Section 6 we introduce the
filtered fundamental operators. These are used for the GNS construction of the limit of a sequence of random walks on the filtered \*-bialgebra in Section 7. In Section 8 we determine the combinatorics of general filtered white noises. In Section 9 we study in more detail extensions of the \(m\)-free and free creation and annihilation operators to all of \(\Gamma(\mathcal{H})\). A free Fock space decomposition of \(\Gamma(\mathcal{H})\) is established.

We denote all scalar products by \(\langle . , . \rangle\) and identify operators and their ampliations if no confusion arises.

2. Filtered bialgebras and convolutions

In this section we discuss the bialgebras in our construction and the associated convolution. For general background on this, we refer the reader to [14] and [18].

For simplicity, consider first the unital \*-algebra \(C[X]\) of polynomials in the variable \(X = X^*\) endowed with the coproduct

\[
\Delta : C[X] \rightarrow C[X] \otimes C[X]
\]
given by

\[
\Delta(X) = X \otimes 1 + 1 \otimes X
\]  \quad (2.1)

and the counit \(\epsilon : C[X] \rightarrow C\) given by \(\epsilon(X) = 0\). This coproduct leads to the classical convolution of measures and thus classical convolution of states.

Namely, if \(\phi, \psi\) are states on \(C[X]\) associated with measures \(\mu, \nu\) on the real line, i.e.

\[
\phi(X^n) = \int_{\mathbb{R}} x^n d\mu(x), \quad \psi(X^n) = \int_{\mathbb{R}} x^n d\nu(x),
\]

then the convolution of states

\[
\phi \star_c \psi = \phi \otimes \psi \circ \Delta
\]
corresponds to the classical convolution of measures \(\mu \star_c \nu\) in the sense that

\[
\phi \star_c \psi(X^n) = m_n(\mu \star_c \nu),
\]

where \(m_n(\mu \star_c \nu)\) is the \(n\)-th moment of the measure \(\mu \star_c \nu\).

The coproduct is a convenient tool to produce independent random variables [14]. Namely, by applying successive iterations of \(\Delta\) to \(X\), we obtain

\[
\Delta^{N-1}(X) = \sum_{k=1}^{N} j_{k,N}(X)
\]

where \(\Delta^N := (\text{id} \otimes \Delta^{N-1}) \circ \Delta\) for \(N > 1\) with \(\Delta^1 = \Delta\), and the summands

\[
j_{l,N}(X) = 1^{\otimes (l-1)} \otimes X \otimes 1^{\otimes (N-l)}, \quad 1 \leq l \leq N,
\]
can be viewed as independent random variables with respect to the state \(\phi \otimes N\).
The so-called Boolean convolution can be obtained by considering the unital *-algebra of polynomials in two noncommuting self adjoint variables $C\langle X', P \rangle$, where $P$ is a projection, with the coproduct

$$\Delta : C\langle X', P \rangle \to C\langle X', P \rangle \otimes C\langle X', P \rangle,$$

given by

$$\Delta(X') = X' \otimes P + P \otimes X', \quad \Delta(P) = P \otimes P,$$  \hspace{0.5cm} (2.2)

and the counit $\epsilon(X') = 0$, $\epsilon(P) = 1$. It can be shown that this coproduct gives the Boolean convolution of states and thus the Boolean convolution of measures [23]. This follows from the hierarchy of freeness construction [11], but a direct proof will be presented below.

**Definition 2.1.** If $\phi$ is a state on $C[Y]$, where $Y^{*} = Y$, then its Boolean extension is the state on $C\langle Y, P \rangle$, where $P$ is a projection, given by the linear extension of

$$\tilde{\phi}(P^{\alpha}Y^{n_1}PY^{n_2}P \ldots Y^{n_k}P^{\beta}) = \phi(Y^{n_1})\phi(Y^{n_2}) \ldots \phi(Y^{n_k})$$  \hspace{0.5cm} (2.3)

where $\alpha, \beta \in \{0,1\}$ and $n_1, \ldots, n_k \in \mathbb{N}$, with $\tilde{\phi}(P) = 1$. If $\phi$ is a state on the unital *-algebra $A$, then its Boolean extension $\tilde{\phi}$ to the free product $\tilde{A} = A \ast C[P]$ (units identified) is defined in an analogous way.

The Boolean extension of a state is a state since it is obtained as the Boolean product of the state $\phi$ on $C[Y]$ and the unital *-homomorphism $h$ on $C[P]$ given by $h(P) = h(1) = 1$.

In order to have a unified model for both convolutions it is now enough to incorporate both coproducts (2.1)-(2.2) into one scheme. This is done as follows. The unital *-algebra $B = C\langle X, X', P \rangle$ where $X = X^{*}$, $X' = X'^{*}$ and $P$ is a projection, endowed with the coproduct $\Delta : B \to B \otimes B$ and counit $\epsilon : B \to C$ given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(X') = X' \otimes P + P \otimes X'$$

$$\Delta(P) = P \otimes P, \quad \epsilon(X) = \epsilon(X') = 0, \quad \epsilon(P) = 1$$

(in other words, $X'$ is $P$-primitive and $P$ is group-like), becomes a unital *-bialgebra. Both classical and Boolean convolutions are recovered from $(B, \Delta, \epsilon)$, as we show below.

**Proposition 2.2.** Let $\eta : B \to C\langle Y, P \rangle$, where $Y = Y^{*}$ and $P^2 = P = P^{*}$, be the linear and multiplicative extension of

$$\eta(X) = \eta(X') = Y, \quad \eta(P) = P, \quad \eta(1) = 1$$

and, for states $\phi, \psi$ on $C[Y]$, let $\phi_0 = \tilde{\phi} \circ \eta$, $\psi_0 = \tilde{\psi} \circ \eta$ with the convolution

$$\phi_0 \ast \psi_0 = \phi_0 \otimes \psi_0 \circ \Delta$$  \hspace{0.5cm} (2.4)

where $\Delta$ is the coproduct for $B$. Then the restrictions of $\phi_0 \ast \psi_0$ to $C[X]$ and $C[X']$, respectively, agree with $\phi \ast_{c} \psi$ and $\phi \ast_{B} \psi$, respectively.
Proof. First of all, note that $\phi_0$ and $\psi_0$ are states since $\eta$ is a unital $*$-homomorphism. That the restriction of the convolution (2.4) to $C[X]$ gives classical convolution, is obvious. In turn, the statement concerning the Boolean convolution follows from the fact that the subalgebras $C[Y] \otimes P$, $P \otimes C[Y]$ of $C(Y, P) \otimes C(Y, P)$ are Boolean independent with respect to the state $\tilde{\phi} \otimes \tilde{\psi}$. This fact can be easily seen from the following calculation:

$$\tilde{\phi} \otimes \tilde{\psi}(Y^{k_1} \otimes P)(P \otimes Y^{k_2})(Y^{k_3} \otimes P)(P \otimes Y^{n_2}) \ldots$$

$$= \tilde{\phi}(Y^{k_1} P Y^{k_2} P \ldots) \tilde{\psi}(P Y^{n_1} P Y^{n_2} \ldots)$$

$$= \phi(Y^{k_1}) \psi(Y^{n_1}) \phi(Y^{k_2}) \psi(Y^{n_2}) \ldots$$

A more general setting of the Boolean product of states was given in [11].

The quadruple $(\mathcal{B}, \Delta, \epsilon, \phi_0)$ can be called the random walk (we follow Majid [14] in this terminology) on the pair of quantum planes. Note that on the quantum probability space level we may also study the pair $(C\langle Y, P \rangle, \tilde{\phi})$, which corresponds to (polynomial functions on) a pair of quantum real lines.

Let us consider now the multivariate generalization of the *-bialgebra $\mathcal{B}$. In classical probability, the multivariate case in an algebraic formulation would be reached if we considered the unital *-algebra $C[ X_k; k \in N]$ of polynomials in commuting variables $(X_k)_{k \in N}$, with the classical coproduct

$$\Delta(X_k) = X_k \otimes 1 + 1 \otimes X_k$$

and the counit $\epsilon(X_k) = 0$.

Let us now define a quantum analog of this multivariate *-bialgebra, of which the bialgebra $\mathcal{B}$ is the “one-dimensional” version. Thus, introduce the unital *-algebra

$$\hat{\mathcal{B}} = C\langle X_k(\sigma), P(\sigma); k \in N, \sigma \in \mathcal{P}(N) \rangle / J$$

where $X_k(\sigma) = X_k^*(\sigma)$, $P(\sigma)^* = P(\sigma)$, $\mathcal{P}(N)$ is the power set of $N$ and $J$ is the two-sided ideal generated by the relations

$$P(\sigma) P(\sigma') = P(\sigma \cap \sigma')$$

$$P(\sigma) X_k(\tau) = X_k(\tau) P(\sigma) \text{ iff } k \in \sigma$$

i.e. the projection associated with the set $\sigma$ (we call $\sigma$ a filter) “filters through” the variables $X_k(\sigma)$ if the index $k \in \sigma$.

**Proposition 2.3.** The algebra $\hat{\mathcal{B}}$, equipped with the coproduct $\hat{\Delta} : \hat{\mathcal{B}} \rightarrow \hat{\mathcal{B}} \otimes \hat{\mathcal{B}}$ and the counit $\hat{\epsilon} : \hat{\mathcal{B}} \rightarrow C$ given by

$$\hat{\Delta}(X_k(\sigma)) = X_k(\sigma) \otimes P(\sigma) + P(\sigma) \otimes X_k(\sigma)$$

$$\hat{\Delta}(P(\sigma)) = P(\sigma) \otimes P(\sigma), \quad \hat{\epsilon}(X_k(\sigma)) = 0, \quad \hat{\epsilon}(P(\sigma)) = 1$$

for all $k$ and $\sigma$, is a unital *-bialgebra called filtered *-bialgebra.
Proof. The coproduct and the counit preserve the relations (2.6)-(2.7). 

By iterating this coproduct, called filtered coproduct, we obtain the sum

$$
\hat{\Delta}^{N-1}(X_k(\sigma)) = \sum_{l=1}^{N} \hat{\gamma}_{l,N}(X_k(\sigma))
$$

of $P(\sigma)$-deformed random variables

$$
\hat{\gamma}_{l,N}(X_k(\sigma)) = P(\sigma)^{\otimes l-1} \otimes X_k(\sigma) \otimes P(\sigma)^{\otimes (N-l)},
$$

where $\sigma \in \mathcal{P}(\mathbb{N})$, $k \in \mathbb{N}$, $1 \leq l \leq N$, $N \in \mathbb{N}$.

Remark. The power set $\mathcal{P}(\mathbb{N})$ can be put in one-to-one correspondence with the Cantor set $CS$ since any infinite sequence $(q_1, q_2, q_3, \ldots)$ of 0’s and 2’s gives a unique number $q \in CS$ with the ternary expansion given by

$$
q = \sum_{n=0}^{\infty} q_n 3^n
$$

which allows us to identify the set $\sigma$ with a number $q \in CS$ according to the rule

$$
n \in \sigma \text{ iff } q_n = 0
$$

Thus, the index set which gives our discrete quantum deformation of the coproduct (2.5) plays a role similar to the interval $[0, 1]$ in the case of $q$-deformed quantum groups like $SU_q(2)$ or $U_q(su(2))$ (see [14],[10],[13]).

When we go over from quantum groups to quantum probability spaces, we identify $X_k(\sigma)$’s for all different $\sigma$ and fixed $k$, as we did by using the map $\eta$ in the “one-dimensional” case of Proposition 2.2. This is done in order to include different notions of independence in one scheme. For that purpose, we consider the mapping

$$
\hat{\eta} : \hat{\mathcal{B}} \to \bigotimes_{k=1}^{\infty} \mathbb{C}\langle Y_k, P_k \rangle
$$

given by the linear and multiplicative extension of

$$
\hat{\eta}(P(\sigma)) = P_1^{q_1} \otimes P_2^{q_2} \otimes \ldots \otimes P_k^{q_k} \otimes \ldots
$$

$$
\hat{\eta}(X_k(\sigma)) = 1_1 \otimes 1_2 \otimes \ldots \otimes 1_{k-1} \otimes Y_k \otimes 1_{k+1} \otimes \ldots
$$

where $\mathbb{C}\langle Y_k, P_k \rangle$ is the $k$-th copy of $\mathbb{C}\langle Y, P \rangle$ and the sequence $(q_1, q_2, q_3, \ldots)$ represents $\sigma$. The infinite tensor product is taken with respect to the set $\{1_k, P_k, k \in \mathbb{N}\}$ (see [5] for the formal definition).

It can be seen that $\eta$ is a unital *-homomorphism. Therefore, for given state $\phi$ on $\mathbb{C}\langle Y \rangle$, the functional

$$
\hat{\phi} = \hat{\phi}^{\otimes \infty} \circ \hat{\eta},
$$

7
is a state on $\widehat{B}$. It plays the role of a noncommutative analog of a vector state in classical probability. A generalization to vector states corresponding to products of different measures is immediate. It is enough to take

$$\hat{\phi} = \bigotimes_{k=1}^{\infty} \tilde{\phi}_k \circ \tilde{\eta},$$

(2.16)

where $\phi_k$ is a state on $C[Y_k], k \in N$.

When we take (2.15) (or, (2.16)), the quadruple $(\widehat{B}, \widehat{\Delta}, \hat{e}, \hat{\phi})$ will give our stationary filtered random walk, using the terminology of Majid [14], which carries two structures, that of the unital *-bialgebra and that of the quantum probability space. The corresponding convolution of states

$$\hat{\phi} \star \hat{\psi} = \hat{\phi} \otimes \hat{\psi} \circ \hat{\Delta}$$

(2.17)

(or, products of states) will be called the filtered convolution.

One of the main motivations to study the filtered *-bialgebras, convolutions, random walks and stochastic processes comes from the following result, which follows from our previous work [11], although it has not been stated there in terms of the filtered convolution.

**Proposition 2.4.** If $\hat{\phi}$ and $\hat{\psi}$ are of the form (2.16) and $i$ is the unital *-homomorphism

$$i : C\langle X_k, k \in N \rangle \to \widehat{B}, \quad i(X_k) = X_k(N),$$

then $\hat{\phi} \star \hat{\psi} \circ i$ agrees with the classical convolution of products of states. If $\hat{\phi}$ and $\hat{\psi}$ are of the form (2.15) and $i^{(m)}$ denotes the unital *-homomorphism

$$i^{(m)} : C[X] \to \widehat{B}, \quad i^{(m)}(X) = \sum_{k=1}^{m} (X_k(k) - X_k(k - 1))$$

where $X_k(p) = X_k(\{1, \ldots, p - 1\})$, then $\hat{\phi} \star \hat{\psi} \circ i^{(m)}$ agrees with the additive $m$-free convolution of states $\phi \star_m \psi$ on $C[X], 1 \leq m \leq \infty$.

**Proof.** The statement concerning the classical convolution of products of states is obvious since $P(N) = 1$. In turn, the second part of the proposition is non-trivial and follows from the construction of $m$-free product states and the associated *-bialgebras (see [11], Section 5, where we also refer the reader for the definition of the $m$-free convolution).

Since $C\langle Y, P \rangle$ can be viewed as a quantum pair of real lines, on the quantum probability space level we can interpret our object of interest as (polynomial functions on) the product of infinitely many quantum pairs of real lines, which is our non-commutative analog of $R^\infty$. On the bialgebra level, we have a bigger object since every variable $X_k$ admits a family of different convolutions.

3. Filtered random variables
In this section we introduce filtered random variables which are our noncommutative analogs of independent random vectors in the general setting of arbitrary unital \(*\)-algebras.

In analogy to the classical case, we obtain them by iterating the coproduct \(\hat{\Delta}\) as in (2.10). Then, we embed \(\hat{\Delta}_{l,N}(X_k(\sigma))\) into \(B^\infty\) to get
\[
P(\sigma)^{\otimes(l-1)} \otimes X_k(\sigma) \otimes P(\sigma)^{\otimes \infty}\tag{3.1}
\]
and generalize these to the arbitrary unital \(*\)-algebras. In order to do that, write (3.1) as the product
\[
(1^{\otimes(l-1)} \otimes X_k(\sigma) \otimes 1^{\otimes \infty})(P(\sigma)^{\otimes(l-1)} \otimes 1 \otimes P(\sigma)^{\otimes \infty})
\]
of an ampliation of \(X_k(\sigma)\) into \(B^{\otimes \infty}\) and a projection indexed by \(\sigma\). This shows that the definitions given below are a natural generalization of those of Section 2.

Let \((A_l)_{l \in L}\) be a family of unital \(*\)-algebras with units \(1_l\) and let \((\phi_l)_{l \in L}\) be the corresponding family of states. Consider a noncommutative probability space \((\tilde{A}_1, \tilde{\Phi}_1)\), where
\[
\tilde{\Phi}_1 = \bigotimes_{l \in L} \tilde{\Phi}_l^{\otimes \infty},
\]
and \(\tilde{\Phi}_l = A_l * C[P_l]\) is the free product with identified units, \(P_l\) being a projection, whereas \(\phi\) is the Boolean extension of \(\phi\) (Definition 2.1). The infinite tensor products are understood as in [5], with the canonical involution. This noncommutative probability space will be called the multiple probability space associated with the considered family of probability spaces since each of them appears infinitely many times in the considered tensor products. We will refer to those copies as colors. Roughly speaking, \(\tilde{A}_l^{\otimes \infty}\) and \(\tilde{\Phi}_l^{\otimes \infty}\) correspond to \(\bigotimes_{l \in L}^\infty C(Y_k, P_k)\) and \(\bigotimes_{l \in L}^\infty \Phi_l^{\otimes \infty}\) for each \(l \in L\), respectively of Section 2.

If \((H_l, \pi_l, \Omega_l)\) is the GNS triple for the pair \((A_l, \phi_l)\), then \((H_l, \tilde{\pi}_l, \Omega_l)\) is the GNS triple for \((\tilde{A}_l, \tilde{\phi}_l)\), \(l \in L\), where \(\tilde{\pi}_l\) agrees \(\pi_l\) on \(A_l\) and \(\tilde{\pi}_l(P_l)\) is the projection onto the cyclic vector \(\Omega_l\). For convenience, we can identify \(x \in A_l\) with \(\pi_l(x)\), \(P_l\) with \(P_l\), and \(\phi_l\) with the expectation state \(\langle \Omega_l, \cdot \rangle\) (see [5]).

Guided by (3.1), from projections \(P_m\) we construct projections \(P(l, \sigma)\) to be elementary tensors in \(\tilde{A}_1\) with components
\[
P(l, \sigma)_{m,k} = \begin{cases} P_m & \text{if } m \neq l \text{ and } k \notin \sigma \\ 1_m & \text{otherwise} \end{cases}
\]
where \(\sigma \in \mathcal{P}(N)\) and \(l \in L\). In the case when \(\sigma = \{1, \ldots, r - 1\}\), we will write \(P(l, r) = P(l, \sigma)\).

**Definition 3.1.** By filtered random variables we will understand elements of \(\tilde{A}_1\) which are of the form
\[
XP \tag{3.2}
\]
where \(X = X(l, k)\) is the \((l, k)\)-th ampliation of \(x \in A_l\) into \(\tilde{A}_1\) and \(P = P(l, \sigma)\), where \(l \in L\), \(k \in N\), \(\sigma \in \mathcal{P}(N)\). In particular, the unit \(1 = \bigotimes_{l \in L} 1_l^{\otimes \infty}\) is a filtered random
variable.

If $L = N$, filtered random variables can be represented as infinite matrices. Assume that $k$ numbers the rows and $l$ numbers the columns. For instance, for $x \in A_2$, $X = X(2, 3)$ and $P = P(2, 4)$ we have

$$
\begin{pmatrix}
1_1 & 1_2 & 1_3 & \ldots \\
1_1 & 1_2 & 1_3 & \ldots \\
n_1 & 1_2 & 1_3 & \ldots \\
1_1 & x & 1_3 & \ldots \\
P_1 & 1_2 & P_3 & \ldots \\
P_1 & 1_2 & 1_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots 
\end{pmatrix}
$$

Note that if $L = N$, multiplication of filtered random variables corresponds to Schur's multiplication of matrices.

**Definition 3.2.** Let $\hat{A}$ be the unital *-subalgebra of $\hat{A}_1$ generated by all filtered random variables and let $\hat{\Phi} = \Phi_1|\hat{A}$. The noncommutative probability space $(\hat{A}, \hat{\Phi})$ will be called the **filtered probability space** associated with $(A_l, \phi_l)_{l \in L}$ and the state $\hat{\Phi}$ will be called the **filtered product** of $(\phi_l)_{l \in L}$. The unital *-subalgebras of $\hat{A}$, $\hat{A}_l = \langle X \mid X = X(l, k), P = P(l, \sigma), x \in A_l, k \in N, \sigma \in \mathcal{P}(N) \rangle$, $l \in L$ will be called **filtered with respect to** $\hat{\Phi}$.

**Example 1.** Let $\ast_{l \in L} A_l$ denote the free product of $(A_l)_{l \in L}$ with non-identified units. Fix $k \in N$, $\sigma \in \mathcal{P}(N)$ and define a *-homomorphism $j^{(k, \sigma)} : \ast_{l \in L} A_l \to \hat{A}$ as the linear extension of

$$
j^{(k, \sigma)}(x_1 \ldots x_n) = X_1(l_1, k)P(l_1, \sigma) \ldots X_n(l_n, k)P(l_n, \sigma)
$$

for $x_i \in A_{l_i}$, $l_1 \neq l_2 \neq \ldots \neq l_n$. Let us define the mapping $i : \ast_{l \in L} A_l \to \bigotimes_{l \in L} A_l$ as the linear extension of

$$
i(x_1 \ldots x_n) = i_{l_1}(x_1) \ldots i_{l_n}(x_n)
$$

where $i_l$ are canonical *-homomorphic embeddings of $A_l$ into $\bigotimes_{l \in L} A_l$. Then,

$$
\hat{\Phi} \circ j^{(k, \sigma)} = \begin{cases} 
\bigotimes_{l \in L} \phi_l \circ i & \text{if } k \in \sigma \\
\ast_{l \in L} \phi_l & \text{if } k \notin \sigma 
\end{cases}
$$
where $*_B \phi_l$ denotes the Boolean (or, 1-free) product of states $(\phi_l)_{l \in L}$ on $*_L \mathcal{A}_l$. In other words, for fixed $k, \sigma$, the set $\{X(l, k)P(l, \sigma) : l \in L\}$ is a family tensor independent r.v. if $k \in \sigma$ and Boolean independent r.v. if $k \notin \sigma$ (see [7]).

**Example 2.** As we showed in [11], the Boolean product is just the first-order approximation of the free product of states in free probability [24]. Higher order approximations given by the modified hierarchy of $m$-free products [5] (the case with non-identified units of the usual hierarchy of freeness of [11]) can also be obtained from the filtered product. Namely, let $m \in \mathbb{N}$, and define

$$\tilde{j}^{(m)} : *_{l \in L} \mathcal{A}_l \to \hat{\mathcal{A}}$$

as the linear extension of

$$\tilde{j}^{(m)}(x_1 \ldots x_n) = \tilde{j}_{l_1}^{(m)}(x_1) \ldots \tilde{j}_{l_n}^{(m)}(x_n)$$

where $x_i \in \mathcal{A}_{l_i}$, $l_1 \neq l_2 \neq \ldots \neq l_n$, and

$$\tilde{j}_{l_i}^{(m)}(x) = \sum_{k=1}^{m} X(l, k)(P(l, k) - P(l, k - 1))$$

for $x \in \mathcal{A}_l$. Then

$$\hat{\Phi} \circ \tilde{j}^{(m)} = *_{l \in L} \phi_l$$

where $*_{l \in L} \phi_l$ denotes the modified $m$-free product of states. If $m = \infty$, the series given by the representation of (3.3) converges strongly on the GNS pre-Hilbert space (see [5]). Moreover,

$$\tilde{j}_{l_i}^{(\infty)}(1_l) = 1, \ l \in L$$

and thus $\hat{\Phi} \circ \tilde{j}^{(\infty)}$ is well-defined on the free product of $\mathcal{A}_l$, $l \in L$, with identified units and agrees on it with the free product of states (for details, see [5]). Thus, the variables $\tilde{j}_{l_i}^{(m)}(x)$, $x \in \mathcal{A}_l$, $l \in L$, are $m$-free random variables for $m \in \mathbb{N}$ and free random variables if $m = \infty$.

**4. Combinatorics**

Let us now introduce a new class of partitions which is crucial to the combinatorics of filtered random variables.

**Definition 4.1.** Let $\vec{k} = (k_1, \ldots, k_n)$ and $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be color and filter tuples of natural numbers and sets of natural numbers, respectively. A partition $R = \{R_1, \ldots, R_q\}$ of the set $\{1, \ldots, n\}$ will be called $(\vec{k}, \vec{\sigma})$-adapted if and only if it satisfies the conditions

$$(A1). \ \forall \ 1 \leq q \leq n \ \forall \ i, j \in R_q \ \ k_i = k_j$$
(A2). If \( i < m < j \), where \( i, j \in R_q \) and \( m \notin R_q \), then \( k_i = k_j \in \sigma_m \).

The collection of all \((\vec{k}, \vec{\sigma})\)-adapted partitions (pair partitions) will be denoted by \( \mathcal{P}_n(\vec{k}, \vec{\sigma}) \) \((\mathcal{P}_n^{\text{pair}}(\vec{k}, \vec{\sigma}))\). The partitions of \( \{1, \ldots, n\} \) which are not \((\vec{k}, \vec{\sigma})\)-adapted will be called \((\vec{k}, \vec{\sigma})\)-non-adapted.

In other words, \( \mathcal{P}_n(\vec{k}, \vec{\sigma}) \) is the subset of all partitions \( \mathcal{P}_n \) of \( \{1, \ldots, n\} \) which are adapted to the tuples \( \vec{k} \) and \( \vec{\sigma} \) in the following sense: (A1) colors corresponding to the elements of the same block have to match, (A2) between the elements of a given block there are no filters associated with other blocks which separate them. In particular, if \( k_i = k, \sigma_i = \sigma \) for all \( 1 \leq i \leq n \), then the two extreme cases are given by

\[
\mathcal{P}_n(\vec{k}, \vec{\sigma}) = \begin{cases} 
\mathcal{P}_n & \text{if } k \in \sigma \\
\mathcal{P}_n^{\text{int}} & \text{if } k \notin \sigma
\end{cases}
\]

where \( \mathcal{P}_n^{\text{int}} \) denotes the interval partitions of \( \{1, \ldots, n\} \). In turn, if \( \sigma_i = N \) for all \( i = 1, \ldots, n \), then

\[
\mathcal{P}_n(\vec{k}, \vec{\sigma}) = \mathcal{P}_n(\vec{k})
\]

where \( \mathcal{P}_n(\vec{k}) \) denotes all partitions \( R \) of the set \( \{1, \ldots, n\} \) such that \( k_i = k_j \iff i, j \) belong to the same block of \( R \) (this corresponds to the classical multivariate case).

**Definition 4.2.** If \( R \) is \((\vec{k}, \vec{\sigma})\)-non-adapted, then the unique coarsest subpartition of \( R \) which is \((\vec{k}, \vec{\sigma})\)-adapted will be denoted by \( R(\vec{k}, \vec{\sigma}) \).

**Examples.** Consider the partition

\[
R = \{\{1, 3, 5\}, \{2, 4\}\}
\]

of \( \{1, \ldots, 5\} \) and let the color tuple be given by \( \vec{k} = (1, 1, 2, 1, 1) \). Then \( R \) is not \((\vec{k}, \vec{\sigma})\)-adapted for any \( \vec{\sigma} \) since it does not satisfy (A1). If we take now the filter tuple \( \vec{\sigma} = (\sigma_1, \ldots, \sigma_5) \) given by \( \sigma_i = \{1, \ldots, r_i - 1\} \), with \( r_1 = r_3 = r_5 = 1, r_2 = r_4 = 2 \), then

\[
R(\vec{k}, \vec{\sigma}) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}.
\]

In turn, if we take \( \vec{\tau} = (\tau_1, \ldots, \tau_5) \), where \( \tau_i = \{1, \ldots, s_i - 1\} \) and \( s_1 = s_5 = 1 \) and \( s_2 = s_3 = s_4 = 2 \), then

\[
R(\vec{k}, \vec{\tau}) = \{\{1, 5\}, \{2, 4\}, \{3\}\}.
\]

We will see that partitions which are not \((\vec{k}, \vec{\sigma})\)-adapted are less important since they do not survive in the limit theorems. Therefore, there is an analogy with the crossing and non-crossing partitions in free probability, the \((\vec{k}, \vec{\sigma})\)-adapted playing a similar role to non-crossing partitions, whereas the non- \((\vec{k}, \vec{\sigma})\)-adapted behave like crossing partitions.

Let us give a recurrence formula for moments of filtered random variables, or “filtered moments”. It is convenient to introduce the following notions.
DEFINITION 4.3. Given a tuple of pairs \(((l_1,k_1),\ldots,(l_n,k_n))\), we will say that \((l_j,k_j)\) is a singleton if \((l_j,k_j) \neq (l_i,k_i)\) for all \(i \neq j\). If \((l_i,k_i) = (l_j,k_j)\) for \(i < j\) such that there is no \(i < r < j\) for which \((l_r,k_r) = (l_i,k_i)\) and there exists \(i < m < j\) such that \(l_m \neq l_i\) and \(k_i \notin \sigma_m\), then we will say that the filter \(\sigma_m\) separates \((l_i,k_i)\) and \((l_j,k_j)\).

PROPOSITION 4.4. Let \(X_i := X(l_i,k_i), P_i := P(l_i,\sigma_i), 1 \leq i \leq n\), where \(x_i \in \mathcal{A}_i, l_i \in L, k_i \in \mathbb{N}, \sigma_i \in \mathcal{P}(\mathbb{N}), \) with \(n \in \mathbb{N}\), and \((l_1,k_1) \neq (l_2,k_2) \neq \ldots \neq (l_n,k_n)\). Then

\[
\hat{\Phi}(X_1 P_1 X_2 P_2 \ldots X_n P_n) = \hat{\Phi}(X_1 P_1) \hat{\Phi}(X_2 P_2 \ldots X_n P_n)
\]

if \((l_1,k_1)\) is a singleton, or if there exists a filter \(\sigma_m\) which separates \((l_1,k_1)\) and \((l_r,k_r)\), where \(r\) is the first index for which \((l_1,k_1) = (l_r,k_r)\), and otherwise

\[
\hat{\Phi}(X_1 P_1 X_2 P_2 \ldots X_n P_n) = \hat{\Phi}(X_2 P_2 \ldots X_1 X, P_r \ldots X_n P_n)
\]

Proof. These formulas follow from the definition of filtered random variables. \(\square\)

PROPOSITION 4.5. Under the assumptions of Proposition 4.4,

\[
\hat{\Phi}(X_1 P_1 \ldots X_n P_n) = \hat{\Phi}(X_{B_1}) \ldots \hat{\Phi}(X_{B_r}) = \phi_{l(B_1)}(x_{B_1}) \ldots \phi_{l(B_r)}(x_{B_r})
\]

where \(R\) is the partition associated with the tuple \((l_1,\ldots,l_n)\), \(B_1,\ldots,B_r\) are the blocks of \(R(k,\sigma)\), \(X_B = \prod_{j \in B} X_j\) and \(x_B = \prod_{j \in B} x_j\) are products taken in the natural order, and \(l(B)\) is the index \(l \in L\) associated with block \(B\).

Proof. This is a straightforward consequence of Proposition 4.4 and the fact that if \(j_1,\ldots,j_r\) are elements of the same block \(B \in R(k,\sigma)\), then

\[
\hat{\Phi}(X_{j_1} P_{j_1} \ldots X_{j_r} P_{j_r}) = \hat{\Phi}(X_{j_1} \ldots X_{j_r}) = \phi_{l(B)}(x_{j_1} \ldots x_{j_r})
\]

(Definition 4.2 is crucial here). \(\square\)

Example. Let \(L = \mathbb{N}\) and take \(x_1, x_3 \in \mathcal{A}_1\) and \(x_2, x_4 \in \mathcal{A}_2\). Then the partition \(R = \{\{1,3\},\{2,4\}\}\) is associated with the tuple \((l_1,l_2,l_3,l_4) = (1,2,1,2)\). Let us consider two cases of color and filter tuples: (i) \(\vec{k} = (1,1,1,1), \vec{\sigma} = (1,2,2,1)\) and (ii) \(\vec{m} = (1,1,1,1), \vec{\tau} = (1,2,1,1)\). Then the corresponding “filtered moments” can be obtained by refinement of \(R\) and represented in terms of diagrams. By \(p^{k \cdot r}\) we understand the number \(p\) associated with color \(k\) and filter \(\{1,\ldots,r-1\}\).
The corresponding moments are given by

\[(i) \quad \phi_1(x_1x_3)\phi_2(x_2x_4), \quad (ii) \quad \phi_1(x_1x_3)\phi_2(x_2)\phi_2(x_4),\]

respectively. We can see how the filters make certain connections in the partition \( R \) disappear.

5. Convolution limit theorems

In this section we will prove the central limit theorem and Poisson's limit theorem for filtered convolutions of states on the bialgebra \( \hat{\mathcal{B}} \). We choose the convolution formulation for clarity of exposition, but the general case, based on the filtered product of states, is done in an analogous fashion.

The combinatorics of filtered convolution powers

\[
\hat{\phi}^\otimes N = \hat{\phi}^\otimes N \circ \hat{\Delta}^N^{-1}
\]

where \( N \in \mathbb{N} \), is given by Lemma 5.1. To a large extent we follow our approach for the convolution powers of \( q \)-deformed states on \( U_q(\text{su}(2)) \) given in [10].

**Lemma 5.1.** Let \( \vec{k} = (k_1, \ldots, k_n), \vec{\sigma} = (\sigma_1, \ldots, \sigma_n) \), where \( k_i \in \mathbb{N}, \sigma_i \in \mathcal{P}(\mathbb{N}), 1 \leq i \leq n \), and let \( N \in \mathbb{N} \). Then

\[
\hat{\phi}^\otimes N(X_{k_1}(\sigma_1) \ldots X_{k_n}(\sigma_n)) = \sum_{p=1}^{n} (N)_p \sum_{R = (R_1, \ldots, R_p) \in \mathcal{P}_n \times R(\vec{k}, \vec{\sigma})} \prod_{B \in R} \hat{\phi}(X_B)
\]  

where \( (N)_p = N(N-1) \ldots (N-p+1) \) and \( X_B = \prod_{i \in B} X_{k_i}(\sigma_i) \) for the block \( B \) of the partition \( R(\vec{k}, \vec{\sigma}) \), with the product taken in the natural order.

**Proof.** Denote \( X_1 = X_{k_1}(\sigma_1), \ldots, X_n = X_{k_n}(\sigma_n) \). Using the notation of (2.11), we have

\[
\hat{\phi}^\otimes N(X_1 \ldots X_n) = \sum_{l_1, \ldots, l_n = 1}^{N} \hat{\phi}^\otimes N(\tilde{j}_{l_1,N}(X_1) \ldots \tilde{j}_{l_n,N}(X_n))
\]
and
\[ \hat{j}_{l_1,N}(X_1) \ldots \hat{j}_{l_n,N}(X_n) = \prod_{m=1}^{n} P(\sigma_m)^{l_m-1} \otimes X_m \otimes P(\sigma_m)^{N-l_m}. \]

The tuple \((l_1, \ldots, l_n)\) defines a partition \(R\) of the set \(\{1, \ldots, n\}\) in the usual way. Namely, if \(\{l_1, \ldots, l_n\} = \{k_1, \ldots, k_r\}\), where \(k_j's\) are all different, then \(R_j = \{i : l_i = k_j\}\). Thus, from (2.15) we get
\[ \hat{\phi}^N(\hat{j}_{l_1,N}(X_1) \ldots \hat{j}_{l_n,N}(X_n)) = \prod_{i=1}^{r} \hat{\phi}(\xi^R(X_1 \ldots X_n)) \]

where \(\xi_i^R\) is a multiplicative extension of the mapping
\[ \xi_i^R(X_p) = \begin{cases} P(\sigma_p) & \text{if } p \notin R_i, \\ X_p & \text{if } p \in R_i. \end{cases} \]

Now, if \(1 \leq r \leq n\), then for each partition \(R\) consisting of \(r\) blocks, there are \((N)_r\) tuples \((l_1, \ldots, l_n)\) which give the same contribution \(\prod_i \hat{\phi}(\xi_i^R(X_1 \ldots X_n))\) (the same combinatorial argument is presented in [10] in more detail). Thus
\[ \hat{\phi}^N(X_1 \ldots X_n) = \sum_{r=1}^{n} (N)_p \sum_{R = \{R_1, \ldots, R_p\}}^{r} \prod_{i=1}^{r} \hat{\phi}(\xi_{i=1}^R(X_1 \ldots X_n)). \]

Finally, note that
\[ \prod_{j=1}^{r} \hat{\phi}(\xi_i^R(X_1 \ldots X_n)) = \hat{\phi}(X_{B_1}) \ldots \hat{\phi}(X_{B_r}) \]

where \(B_1, \ldots, B_r\) are blocks of the partition \(R(\vec{k}, \vec{\sigma})\) since every block \(R_j\) of \(R\) splits up into subblocks for which all \(k_i's\) are the same and are not separated by any filters due to the way \(\tilde{\phi}\) separates words. It is also worth noting that \(\hat{\phi}(X_B) = \phi(X^\#)\) where \# stands for the number of elements.

In order to state the central limit theorem, let us introduce the gradation on \(\hat{B}\) given by \(d(X_k(\sigma)) = 1\) and \(d(P(\sigma)) = 0\) for all \(k\) and \(\sigma\). Then, for \(N \in \mathbb{N}\), define
\[ D_{1/\sqrt{N}}(W) = \frac{1}{N^{d(W)}/2} W \]

where \(W\) is a word in \(\hat{B}\) and \(d(W)\) is its degree.

**Corollary 5.2.** Consider a family of states \(\phi_N\) on \(C[Y]\), where \(N \in \mathbb{N}\) and suppose that the limits
\[ \lim_{N \to \infty} \phi_N(Y^k) = Q(k) \]
exist and are finite for all \(k \in \mathbb{N}\). Then
\[ \lim_{N \to \infty} \hat{\phi}^N(X_{k_1}(\sigma_1) \ldots X_{k_n}(\sigma_n)) = \sum_{R \in P_n(\vec{k}, \vec{\sigma})} \prod_{B \in R} Q(\#B) \quad (5.2) \]
where \( \#B \) is the number of elements in the block \( B \).

**Proof.** It is an immediate consequence of Lemma 5.1 since if \( R \in \mathcal{P}_n \setminus \mathcal{P}_n(\bar{k}, \bar{\sigma}) \), then the number of blocks in \( R(\bar{k}, \bar{\sigma}) \) is strictly greater than the number of blocks in \( R \) which makes the contribution from \( R \) disappear as \( N \to \infty \).

\[ \square \]

**Theorem 5.3. (Central limit theorem)** Let \( k_i \in \mathbb{N}, \sigma_i \in \mathcal{P}(\mathbb{N}), i = 1, \ldots, n. \)

Suppose that \( \hat{\phi}(X_{k_i}(\sigma_i)) = 0 \) and \( \hat{\phi}(X_{k_i}^2(\sigma_i)) = 1 \) for \( i = 1, \ldots, n. \) If \( n \) is even, then

\[
\lim_{N \to \infty} \hat{\phi}^\ast_N \circ D_{1\sqrt{N}}(X_{k_1}(\sigma_1) \ldots X_{k_n}(\sigma_n)) = |\mathcal{P}^\text{pair}(\bar{k}, \bar{\sigma})| \tag{5.3}
\]

and, if \( n \) is odd, the limit vanishes.

**Proof.** It is enough to use Lemma 5.1 and notice that if there is a singleton in \( R \), then there is no contribution from such a partition to the right hand side of (5.1). In turn, if there are no singletons, then \( (N)^p/N^{n/2} \to 0 \) unless \( 2p = n \). That means that in the limit only pair-partitions may give a non-zero contribution. However, note that those pair partitions which are not \((\bar{k}, \bar{\sigma})\)-adapted give zero since in that case the number of blocks of \( R(\bar{k}, \bar{\sigma}) \) is strictly greater than the number of blocks of \( R \) and

\[
\prod_{B \in R(\bar{k}, \bar{\sigma})} \hat{\phi}(X_B) = 0 \text{ by the mean zero assumption.} \tag{5.3}
\]

\[ \square \]

**Example 1.** Note that if \( \sigma_i = \mathbb{N} \) for all \( 1 \leq i \leq n \) and all \( n \), we obtain \( |\mathcal{P}^\text{pair}(\bar{k})| \) on the RHS of (5.3) which gives the moments of the classical multivariate Gaussian law.

**Example 2.** Here we give some one-dimensional examples. If \( k_i = k \) and \( \sigma_i = \sigma \) for \( 1 \leq i \leq n \) and all \( n \), then we obtain the Gaussian law if \( k \in \sigma \) and the 1-free (or, Boolean) central limit law corresponding to the discrete measure \( \mu^{(1)} = 1/2(\delta_{-1} + \delta_1) \) if \( k \notin \sigma \). In turn, if we take

\[
\Delta^{(m)} = \Delta \circ \bar{t}^{(m)} \tag{5.4}
\]

where \( \bar{t}^{(m)} \) is given by (3.3), we obtain the \( m \)-free coproduct defined in [11], for which the convolution powers tend to the \( m \)-free central limit laws and approximate pointwise the Wigner semi-circle law for \( m = \infty \). For details, see [4].

**Theorem 5.4. (Poisson’s limit theorem)** Under the assumptions of Corollary 5.2, suppose that \( Q(k) = \lambda \) for all \( k \in \mathbb{N} \), where \( \lambda > 0 \). Then

\[
\lim_{N \to \infty} \hat{\phi}^\ast_N(X_{k_1}(\sigma_1) \ldots X_{k_n}(\sigma_n)) = \sum_{R \in \mathcal{P}_n(\bar{k}, \bar{\sigma})} \lambda^{b(R)} \tag{5.5}
\]

where \( b(R) \) is the number of blocks of \( R \).

**Proof.** It is an immediate consequence of Corollary 5.2.

\[ \square \]

**Example 1.** Let us first give some one-dimensional examples. Again, if \( k_i = k \) and \( \sigma_i = \sigma \) for \( 1 \leq i \leq n < \infty \), then we obtain the classical Poisson law for \( k \in \sigma \) and the 1-free (or Boolean) Poisson law for \( k \notin \sigma \) corresponding to the discrete measure \( \mu^{(1)}_\lambda = 1/(1 + \lambda)(\delta_0 + \lambda \delta_{1+\lambda}) \). Considering linear combinations of sample sums as in the
preceeding example, we obtain the \( m \)-free Poisson laws for \( m \in \mathbb{N} \) and the free Poisson law \([21]\) for \( m = \infty \) (see \([4]\)).

\[\text{Example 2.}\] If we take \( \hat{\phi} \) given by (2.16), i.e. corresponding to the product of measures, then we can generalize Lemma 5.1 and Corollary 5.2 to the effect that if
\[
\lim_{N \to \infty} \phi_{s,N}(Y^k) = \lambda_s \quad \text{where} \quad \lambda_s > 0, \ s \in \mathbb{N},
\]
then the RHS of (5.5) takes the form
\[
\sum_{R \in \mathcal{P}_n(\vec{k}, \vec{\sigma})} \lambda_{s_1} \lambda_{s_2} \ldots \lambda_{s_p}
\]
where \( s_1, \ldots, s_p \) correspond to the blocks \( B_1, \ldots, B_p \) of the partition \( R(\vec{k}, \vec{\sigma}) \) and denote their colors (which are the same within one block by (A1) of Definition 4.1). These moments are the moments of the multivariate classical Poisson law.

6. Filtered Fundamental Operators

In this section we recall basic facts concerning multiple symmetric Fock spaces over \( \mathcal{K} \equiv L^2(\mathbb{R}^+) \), which will be the underlying space for the filtered fundamental processes.

Let \( \mathcal{G} \) be a separable Hilbert space with a countably infinite fixed orthonormal basis \((e_n)_{n \in \mathbb{N}}\). It is called the \textit{multiplicity space}. By a \textit{multiple symmetric Fock space} over \( \mathcal{K} \) we understand the symmetric Fock space over \( \mathcal{H} = L^2(\mathbb{R}^+, \mathcal{G}) \cong L^2(\mathbb{R}^+) \otimes \mathcal{G} \equiv \mathcal{K} \otimes \mathcal{G} \), namely
\[
\Gamma(\mathcal{H}) = \mathcal{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}
\]
where \( \mathcal{H}^{\otimes n} \) denotes the \( n \)-th symmetric tensor power of \( \mathcal{H} \) and \( \Omega \) is the vacuum vector, with the scalar product given by \( \langle \Omega, \Omega \rangle = 1, \ \langle \Omega, u \rangle = 0 \) and
\[
\langle u_1 \circ \ldots \circ u_n, v_1 \circ \ldots \circ v_m \rangle = \delta_{n,m} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \langle u_1, v_{\sigma(1)} \rangle \ldots \langle u_n, v_{\sigma(n)} \rangle
\]
where
\[
u_1 \circ \ldots \circ u_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(n)}
\]
and \( \mathcal{S}_n \) denotes the symmetric group of order \( n \).

Denote by \( \mathcal{H}^{(\sigma)} \) the linear subspace of \( \mathcal{H} \) spanned by all \( u \in \mathcal{H} \) of the form
\[
u = \sum_{k \in \sigma} u^{(k)} \otimes e_k
\]
where \( \sigma \in \mathcal{P}(\mathbb{N}) \). In particular, we put \( \mathcal{H}^{(0)} = \{0\} \). The set \( \sigma \) will be called a \textit{filter} and the associated canonical projection will be denoted \( \Pi^{(\sigma)} : \mathcal{H} \to \mathcal{H}^{(\sigma)} \) with \( v^{(\sigma)} = \Pi^{(\sigma)} v \) for any \( v \in \mathcal{H} \). Then let \( P^{(\sigma)} : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}^{(\sigma)}) \) be the second quantization of \( \Pi^{(\sigma)} \).

Thus, if \( \varepsilon(v) \) is an exponential vector in \( \Gamma(\mathcal{H}) \), i.e.
\[
\varepsilon(v) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^{\otimes n}
\]
with \( v^{0} = \Omega \), we have \( P^{(\sigma)} \varepsilon(u) = \varepsilon(u^{(\sigma)}) \).

Of special importance will be subspaces \( \mathcal{H}^{(r)} \) of \( \mathcal{H} \) spanned by all \( u \in \mathcal{H} \) of the form

\[
u = \sum_{k=1}^{r-1} u^{(k)} \otimes e_{k},\]

where \( r > 1 \), i.e. here \( \sigma = \{1, \ldots, r-1\} \); we set \( \mathcal{H}^{(1)} = \{0\} \). In \( \Gamma(\mathcal{H}) \), we will use the finite particle domain \( \Gamma_{0}(\mathcal{H}) \), i.e. the linear space generated by vectors of the form

\[v_1 \circ v_2 \circ \ldots \circ v_n\]

where \( v_1, \ldots, v_n \in \mathcal{H}, n \in \mathbb{N} \).

Since \( \mathcal{H} \) can be viewed as a direct sum of infinitely many copies of \( \mathcal{K} \) and we need some convenient terminology concerning the numbering of those copies, we will refer to them as colors. Thus, in the direct sum decomposition

\[\mathcal{H} = \bigoplus_{k \in \mathbb{N}} \mathcal{K} \otimes e_{k}\]

the \( k \)-th summand will be associated with the \( k \)-th color and we will say that non-zero vectors from that summand are of \( k \)-th color. In addition, to the zero vector we assign the 0-th color.

By filtered creation and annihilation operators we will understand operators given by

\[
a^{(\sigma)\ast}(f \otimes e_{k}) = a^{\ast}(f \otimes e_{k})P^{(\sigma)} \quad (6.1) \\
a^{(\sigma)}(f \otimes e_{k}) = P^{(\sigma)}a(f \otimes e_{k}), \quad (6.2)
\]

respectively, where \( a^{\ast}(f \otimes e_{k}) \) and \( a(f \otimes e_{k}) \) are the usual boson creation and annihilation operators (see [P]). Thus, filtered creation operators first “filter out particles of colors which are not in \( \sigma \) and then create a particle of given color”, whereas the filtered annihilation operators “first annihilate a particle of a given color and then filter out particles of colors which are not in \( \sigma \)”.

In addition, we define

\[
a^{(k,\sigma)\circ}(v_1 \circ v_2 \circ \ldots \circ v_n) = a^{(k)\circ}P^{(\sigma \cup \{k\})} \quad (6.3)
\]

and call filtered number operators. In an analogous fashion one can define exchange operators.

**Proposition 6.1.** The finite particle domain \( \Gamma_{0}(\mathcal{H}) \) is contained in the domains of filtered fundamental operators. Furthermore, the following relations hold:

\[
a^{(\sigma)}(f \otimes e_{k})(v_1 \circ v_2 \circ \ldots \circ v_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (\langle f, v_j^{(k)} \rangle \nu_1^{(\sigma)} \circ \ldots \circ \nu_j \circ \ldots \circ \nu_n^{(\sigma)}),
\]

\[
a^{(\sigma)\ast}(f \otimes e_{k})(v_1 \circ v_2 \circ \ldots \circ v_n) = \sqrt{n+1}(f \otimes e_{k}) \circ v_1^{(\sigma)} \circ \ldots \circ v_n^{(\sigma)},
\]

\[
a^{(k,\sigma)\circ}(v_1 \circ v_2 \circ \ldots \circ v_n) = \sum_{j=1}^{n} v_1^{(\sigma)} \circ \ldots \circ (v_j^{(k)} \otimes e_{k}) \circ \ldots \circ v_n^{(\sigma)}
\]
with \( a^{(\sigma)}(f \otimes e_k) \Omega = 0 \), \( a^{(\sigma)\ast}(f \otimes e_k) \Omega = f \otimes e_k \) and \( a^{(\kappa,\sigma)\ast} \Omega = 0 \), where \( v_1, \ldots, v_n \in H \), \( k \in \mathbb{N} \), \( \sigma \in \mathcal{P}(\mathbb{N}) \), \( n \in \mathbb{N} \).

**Proof.** The first statement follows from the definitions (6.1)-(6.3) and an analogous property of the canonical (CCR) operators and the fact that the projections \( P^{(\sigma)} \) leave the finite particle domain invariant. Similarly, the relations follow immediately from the analogous formulas for the canonical (CCR) operators (we use Hudson-Parthasarathy’s normalization).

**LEMMA 6.2.** Filtered creation and annihilation operators satisfy the following relations on the finite particle domain:

\[
a^{(\sigma)}(f \otimes e_k)a^{(\tau)\ast}(g \otimes e_l) - a^{(\tau)\ast}(g \otimes e_l)a^{(\sigma)}(f \otimes e_k)P^{(\tau)} = \delta_{k,l}(f, g)P^{(\sigma \cap \tau)}
\]

for any \( k, l \in \mathbb{N} \), \( \sigma, \tau \in \mathcal{P}(\mathbb{N}) \), \( f, g \in K \).

**Proof.** In the proof given below we understand that the equations hold on the finite particle domain, but it remains valid on the whole intersection of the domains of the considered filtered operators. Using canonical commutation relations (CCR) of the form

\[
a(f \otimes e_k)a^\ast(g \otimes e_l) - a^\ast(g \otimes e_l)a(f \otimes e_k) = \delta_{k,l}(f, g).
\]

we obtain

\[
a^{(\sigma)}(f \otimes e_k)a^{(\tau)\ast}(g \otimes e_l) = P^{(\sigma)}a(f \otimes e_k)a^\ast(g \otimes e_l)P^{(\tau)}
\]

\[
= P^{(\sigma)}a^\ast(g \otimes e_l)a(f \otimes e_k)P^{(\tau)} + \delta_{k,l}(f, g)P^{(\sigma \cap \tau)}.
\]

Now, note that if \( l \notin \sigma \), then

\[
P^{(\sigma)}a^\ast(g \otimes e_l)a(f \otimes e_k)P^{(\tau)} = 0
\]

and we obtain

\[
a^{(\sigma)}(f \otimes e_k)a^{(\tau)\ast}(g \otimes e_l) = \delta_{k,l}(f, g)P^{(\sigma \cap \tau)}.
\]

Consider now the case \( l \in \sigma \). Then

\[
P^{(\sigma)}a^\ast(g \otimes e_l)a(f \otimes e_k)P^{(\tau)} - a^\ast(g \otimes e_l)P^{(\sigma \cap \tau)}a(f \otimes e_k)P^{(\tau)}
\]

\[
= a^\ast(g \otimes e_l)(P^{(\sigma)} - P^{(\sigma \cap \tau)})a(f \otimes e_k)P^{(\tau)}
\]

since \( P^{(\sigma)} \) commutes with \( a^\ast(g \otimes e_l) \) for \( l \in \sigma \). Note that if \( \sigma \subseteq \tau \), then \( P^{(\sigma)} = P^{(\sigma \cap \tau)} \) and thus the above expression vanishes. In turn, if \( \tau \subseteq \sigma \), then \( P^{(\sigma)} - P^{(\sigma \cap \tau)} = P^{(\sigma)} - P^{(\tau)} \). However, \( P^{(\tau)} : \Gamma(H) \rightarrow \Gamma(H^{(\tau)}) \) and \( a(f \otimes e_k) \) leaves \( \Gamma(H^{(\tau)}) \) invariant, hence when we apply \( P^{(\sigma)} - P^{(\tau)} \), we can see that the above expression also vanishes. Therefore, if \( l \in \sigma \), we obtain

\[
a^{(\sigma)}(f \otimes e_k)a^{(\tau)\ast}(g \otimes e_l) - a^{(\tau)\ast}(g \otimes e_l)P^{(\sigma \cap \tau)}a(f \otimes e_k)P^{(\tau)}
\]

\[
= a^{(\sigma)}(f \otimes e_k)a^{(\tau)\ast}(g \otimes e_l) - a^{(\tau)\ast}(g \otimes e_l)a^{(\sigma)}(f \otimes e_k)P^{(\tau)}
\]

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\( = \langle f, g \rangle \delta_{k,l} P^{(\sigma \cap \tau)} \).

Combining the two cases \( l \in \sigma \) and \( l \notin \sigma \) ends the proof. \( \square \)

Let us finally define the fundamental processes associated with the filtered fundamental operators introduced in this section. They will appear in Sections 5-7 when finding GNS realizations of limit states. They will also serve as integrators in the filtered calculus developed in [8]. Thus, in connection with (6.1)-(6.3), let

\[
A_{t}^{(k,\sigma)*} = a^{(\sigma)*}(\chi_{[0,t]} \otimes e_{k}), \quad (6.4)
\]

\[
A_{t}^{(k,\sigma)} = a^{(\sigma)}(\chi_{[0,t]} \otimes e_{k}), \quad (6.5)
\]

\[
A_{t}^{(k,\sigma)\circ} = \lambda(I_{[0,t]} \otimes |e_{k}\rangle\langle e_{k}|)P^{(\sigma \cup \{k\})}, \quad (6.6)
\]

\[
A_{t}^{(0,\sigma)} = tP^{(\sigma)}, \quad (6.7)
\]

where \( t \geq 0, \; k \in \mathbb{N}, \; \sigma \in \mathcal{P}(\mathbb{N}), \; I_{[0,t]} \) denotes the operator of multiplication by the characteristic function \( \chi_{[0,t]} \) on \( L^{2}(\mathbb{R}^{+}) \), and \( \lambda(H) \) denotes the differential second quantization of \( H \in \mathcal{B}(\mathcal{H}) \). The families of processes given by (6.4)-(6.7) will be called filtered creation, annihilation, number and time processes, respectively. When speaking of all of them, we will call them filtered fundamental processes. By filtered Brownian motion we will understand the unital \(*\)-algebra generated by filtered creation and annihilation operators indexed by time intervals.

### 7. Random walk on the filtered bialgebra

In this section we show that a limit of continuous-time random walks on the filtered \(*\)-bialgebra gives the filtered Brownian motion. This gives a multivariate Brownian motion on the multiple symmetric Fock space which satisfies the properties required by the axioms for white noise on \(*\)-bialgebras given in [1] and [18] and includes quantum Brownian motions for different types of independence [2]. For the first quantum version of the Wiener process, see [3]. We follow the notation used in [9] for the random walk on \( U_{q}(su(2)) \).

Instead of \( \hat{\mathcal{B}} \), we choose to work with a slightly more general unital \(*\)-bialgebra \( \hat{\mathcal{C}} \), also called filtered \(*\)-bialgebra, which is defined to be the unital \(*\)-algebra over \( \mathbb{C} \) generated by \( X_{k}(\sigma), X_{k}^{*}(\sigma) \) and \( P(\sigma) \), where \( k \in \mathbb{N}, \; \sigma \in \mathcal{P}(\mathbb{N}) \) subject to relations (2.6)-(2.7), where \( P(\sigma) \) is a projection for each \( \sigma \in \mathcal{P}(\mathbb{N}) \) (this of course also means that \( P(\sigma) \) commutes with \( X_{k}(\tau) \) for \( k \in \sigma \), with the coproduct in which \( X_{k}(\sigma) \) and \( X_{k}^{*}(\sigma) \) are both \( P(\sigma) \)-primitive and \( P(\sigma) \) is group-like (cf. (2.8)-(2.9)).

Let \( k \in \mathbb{N}, \; \sigma \in \mathcal{P}(\mathbb{N}), \; N \in \mathbb{N} \), and consider a sequence of continuous-time random walks on \( \hat{\mathcal{C}} \) given by

\[
\hat{\Delta}_{s,t}^{N}(X_{k}^{*}(\sigma)) = \sum_{l=N_{s}+1}^{N_{t}} P(\sigma)^{\otimes(l-1)} \otimes X_{k}^{*}(\sigma) \otimes P(\sigma)^{\otimes\infty}, \quad (7.1)
\]

\[
\hat{\Delta}_{s,t}^{N}(P(\sigma)) = 1^{\otimes N_{s}} \otimes P(\sigma)^{\otimes(N_{t}-N_{s})} \otimes 1^{\otimes\infty}, \quad (7.2)
\]
where \(0 \leq s \leq t < \infty\), \(N_t = E[tN]\), with \(X_k^i(\sigma) \in \{X_k(\sigma), X_k^i(\sigma)\}\).

It is easy to check that for each pair \((s, t)\) and natural number \(N\), the mapping

\[
\hat{\Delta}^N_{s,t} : \hat{C} \to \hat{C}^{\otimes \infty}
\]

given by the linear and multiplicative extension of \((7.1)-(7.2)\) is a unital \(*\)-homomorphism and the triple \((\hat{C}^{\otimes \infty}, (\Delta^N_{s,t})_{0 \leq s \leq t}, \hat{\phi}^{\otimes \infty})\) satisfies for each \(N \in \mathbb{N}\) the properties required from a stochastic process over the bialgebra \(\hat{C}\) given in [1]. In particular,

\[
\Delta^N_{s,t} \ast \Delta^N_{t,r} = \Delta^N_{s,r}
\]

for all \(0 \leq s \leq t \leq r\), where \(\Delta^N_{s,t} \ast \Delta^N_{t,r} = M \circ (\Delta^N_{s,t} \otimes \Delta^N_{t,r}) \circ \Delta\) with \(M(a \otimes b) = ab\). For details on stochastic processes over \(*\)-bialgebras see [1] and [18].

In this \(*\)-bialgebra formulation, further preparations are similar to those which lead to the central limit theorem. Namely, for a given state \(\phi\) on \(C(Y, Y^*)\) we denote by \(\tilde{\phi}\) its Boolean extension to \(C(Y, Y^* , P)\), where \(P\) is a projection and we set \(\hat{\phi} = \tilde{\phi}^{\otimes \infty} \circ \eta\), where

\[
\eta : \hat{C} \to \bigotimes_{k=1}^{\infty} C(Y_k, Y^*_k, P_k)
\]

is defined by the \(*\)-multiplicative extension of formulas \((2.13)-(2.14)\).

Below we will study the limit of distributions of the mixed moments of \((7.1)\) as \(N \to \infty\) and find the GNS representation of the limit state. Let us remark that more general sample sums indexed by \(f \in L^2_\varepsilon(\mathbb{R}^+)\) can also be given and the proofs of this section will still hold.

**Theorem 7.1.** Let \(Z_i = X_k(\sigma_i), Z^*_i = X^*_k(\sigma_i)\), where \(k_i \in \mathbb{N}, \sigma_i \in \mathcal{P}(\mathbb{N})\), \(i = 1, \ldots, n\). Suppose that \(\hat{\Phi} = \hat{\phi}^{\otimes \infty}\) with \(\phi(Y) = 0\) and the only non-vanishing second-order moment of \(\phi\) is given by \(\phi(YY^*) = 1\). Then

\[
\lim_{N \to \infty} N^{-n/2} \hat{\Phi}(\hat{\Delta}^N_{s,t,1}(Z_1^1) \ldots \hat{\Delta}^N_{s,r,n}(Z_n^n)) = \varphi(a^{\sigma_1}(v_1) \ldots a^{\sigma_n}(v_n))
\]

where \(Z_i^i \in \{Z_i, Z^*_i\}\), \(v_i = \chi_{[s_i, t_i]} \otimes e_{k_i}, i = 1, \ldots, n\), and \(\varphi(.) = \langle \Omega, \Omega \rangle\) is the vacuum expectation in \(\Gamma(H)\).

**Proof.** From the general invariance principle [22] and the combinatorics of the filtered central limit theorem it follows that for even \(n = 2p\) we have

\[
LHS = \sum_{R \in \mathcal{P}^{\text{pair}}(k, \sigma)} \delta(R) \prod_{m=1}^{p} \langle v_{\alpha(m)}, v_{\beta(m)} \rangle
\]

where the blocks of \(R\) consist of two-element sets \(\{\alpha(i), \beta(i)\}\), with \(\alpha(i) < \beta(i), i = 1, \ldots, p\) and \(\delta(R) = 1\) if for the given partition \(R\) we have \(Z^2_{\alpha(i)} = Z_{\alpha(i)}\) and \(Z^2_{\beta(i)} = Z^*_{\beta(i)}\) and otherwise \(\delta(R) = 0\). It is clear that if \(n\) is odd, then \(LHS = 0\).

It is clear that \(RHS = 0\) if \(n\) is odd, too – it is enough to use the properties of creation and annihilation operators following from Proposition 6.1. Therefore assume
that \( n = 2p \). Next, notice that in order that \( LHS = RHS \) it is enough to show the following claim:

\[
\varphi(a_1^\ast \ldots a_n^\ast) = \sum_{R \in \mathcal{P}_{2p}^{\text{pair}}(\overset{\circ}{k},\overset{\circ}{\sigma})} \delta(R) \prod_{m=1}^{p} \langle v_{a(m)}, v_{b(m)} \rangle
\]

where, for simplicity, we denote \( a_j^\ast = a^\ast j^2(v_j) \).

We begin with the simplest case, i.e.

\[
\varphi(a_1 a_2^\ast \ldots a_{2p-1} a_2^\ast) = \langle v_1, v_2 \rangle \langle v_3, v_4 \rangle \ldots \langle v_{2p-1}, v_{2p} \rangle
\]

the second expression being formally written as a sum since we have at most one partition contributing to it. This is the beginning of an induction procedure. Namely, it is enough to show that from the claim being true for all expectations of orders \( \leq 2p-2 \) and for \( \varphi(a_1^\ast \ldots a_i^\ast a_{i+1} \ldots a_{2p}^\ast) \) it follows that it also holds for the expectation of the form \( \varphi(a_1^\ast \ldots a_{i+1} a_i^\ast \ldots a_{2p}^\ast) \).

Suppose that \( k_i \notin \sigma_{i+1} \). Then

\[
\varphi(a_1^\ast \ldots a_{i+1} a_i^\ast \ldots a_{2p}^\ast) = \langle v_{i+1}, v_{i+1} \rangle \varphi(a_1^\ast \ldots P(\sigma_i \cap \sigma_{i+1}) \ldots a_{2p}^\ast)
\]

\[
= \sum_{R \in \mathcal{P}_{2p}^{\text{pair}}(\overset{\circ}{k},\overset{\circ}{\sigma} \cup \{i, i+1\})} \delta(R) \prod_{m=1}^{p} \langle v_{a(m)}, v_{b(m)} \rangle
\]

where \( \mathcal{P}_{n}^{\text{pair}}(\overset{\circ}{k},\overset{\circ}{\sigma} \cup \{i, i+1\}) \) denotes all \( (\overset{\circ}{k}, \overset{\circ}{\sigma}) \)-adapted pair partitions with

\( \overset{\circ}{k} = (k_1, \ldots, k_{i-1}, k_{i+2}, \ldots, k_n), \overset{\circ}{\sigma} = (\sigma_1, \ldots, \sigma_{i+1}, \sigma_i, \ldots, \sigma_n) \)

in which \( (i, i + 1) \) forms a pairing. The first equality follows from filtered relations of Lemma 6.2, whereas the second – from the inductive assumption and the fact that if \( k_i \notin \sigma_{i+1} \), then

\( \forall R \in \mathcal{P}_{2q}^{\text{pair}}(\overset{\circ}{k}, \overset{\circ}{\sigma}) \exists R' \in \mathcal{P}_{2q-2}^{\text{pair}}(\overset{\circ}{k}_i, \overset{\circ}{\sigma}) : R = R' \cup \{i, i + 1\} \),

where \( \overset{\circ}{k}_i = (k_1, \ldots, k_{i-1}, k_{i+2}, \ldots, k_n) \), \( \overset{\circ}{\sigma} = (\sigma_1, \ldots, \sigma_{i-1}, \zeta(\sigma_{i+2}), \ldots, \zeta(\sigma_n)) \), and

\[
\zeta(\sigma_l) = \begin{cases} 
\sigma_l \cap \sigma_i \cap \sigma_{i+1} & \text{if } (s, l) \text{ is a pairing for } s < i \\
\sigma_l & \text{otherwise}
\end{cases}
\]

where \( i + 2 \leq l \leq n \).

In turn, if \( k_i \in \sigma_{i+1} \), then using Lemma 6.2 again, we obtain

\[
\varphi(a_1^\ast \ldots a_{i+1} a_i^\ast \ldots a_{2p}^\ast)
= \langle v_{i+1}, v_i \rangle \varphi(a_1^\ast \ldots P(\sigma_i \cap \sigma_{i+1}) \ldots a_{2p}^\ast) + \varphi(a_1^\ast \ldots a_i^\ast a_{i+1} P(\sigma_i) \ldots a_{2p}^\ast).
\]
By the inductive assumption and similar arguments as above, the first term gives
\[
\sum_{R \in \mathcal{P}^\text{pair}_{2p}(\hat{k},\hat{\sigma}|i,i+1)} \delta(R) \prod_{m=1}^{p} \langle v_{\alpha(m)}, v_{\beta(m)} \rangle,
\]
whereas the second, a sum over all the remaining partitions from \(\mathcal{P}^\text{pair}_{2p}(\hat{k},\hat{\sigma})\) (it is disjoint from the first since \((i,i+1)\) cannot form a pairing as in the associated creation-annihilation pair the annihilation operator follows the creation operator), namely
\[
\sum_{R \in \mathcal{P}^\text{pair}_{2p}(\hat{k},\hat{\sigma}) \setminus \mathcal{P}^\text{pair}_{2p}(\hat{k},\hat{\sigma}|i,i+1)} \delta(R) \prod_{m=1}^{p} \langle v_{\alpha(m)}, v_{\beta(m)} \rangle
\]
by the inductive assumption. Note that the projection \(P^{(\sigma_i)}\), which follows the annihilation operator in the second term, ensures that the annihilation operator \(a^{(\sigma_i+1)}(v_{i+1})\) in the original expression cannot be paired off with any creation operator (standing to the right of this annihilation operator) of color \(k \not\in \sigma_i\).

Adding now those two expressions, we can see that the claim holds for
\[
\varphi(a_1^* \ldots a_{i+1}^* a_i^* \ldots a_{2p}^*),
\]
which finishes the proof. \(\square\)

**Example.** If \(k_i = k, \sigma_i = \sigma\) for \(1 \leq i \leq n\) and arbitrary \(n\), then we obtain the CCR Brownian motion if \(k \in \sigma\) and Boolean Brownian motion if \(k \not\in \sigma\). By taking linear combinations of sample sums corresponding to \(m\)-free (free) independent random variables, we obtain the \(m\)-free (free) Brownian motion \([4]\).

### 8. Filtered White Noise

In this section we define the general notion of filtered white noise, determine its combinatorics and study the example of filtered Poisson white noises. Our approach largely parallels that used by Speicher for free white noise \([21]\).

**Definition 8.1.** Let \(\text{Int}(\mathbb{R}^+)\) denote the intervals in \(\mathbb{R}^+\). An \(s\)-dimensional filtered white noise consists of a unital \(*\)-algebra \(\mathcal{C}\), a state \(\rho\) on \(\mathcal{C}\) and a family of finitely additive mappings \(\text{Int}(\mathbb{R}^+) \to \mathcal{C}\),
\[
I \to (c_I(k,\sigma;1), \ldots, c_I(k,\sigma;s)), \quad k \in \mathbb{N}, \quad \sigma \in \mathcal{P}(\mathbb{N})
\]
such that
1. for any pairwise disjoint intervals \(I(1), \ldots, I(n)\),
\[
\rho(c_{I(l_1)}(k_1,\sigma_1;q_1) \ldots c_{I(l_n)}(k_n,\sigma_n;q_n)) = \rho(c_{B_1}) \ldots \rho(c_{B_r}) \quad (8.1)
\]
where \(B_1, \ldots, B_r\) are the blocks of \(R(\tilde{k},\tilde{\sigma})\), with \(R\) being the partition associated with \((l_1, \ldots, l_n)\) and \(c_B\) denotes the product, taken in the natural order, of \(c_{I(l_i)}(k_i,\sigma_i;q_i)\)’s
for \( i \in B \),

(ii) the distribution \( \rho_I = \rho|_{C_I} \) depends only on the Lebesgue measure of the interval \( I \), where \( C_I \) denotes the unital *-algebra generated by \( c_I(k, \sigma; q), k \in \mathbb{N}, \sigma \in \mathcal{P}(\mathbb{N}) \) and \( 1 \leq q \leq s \).

**Lemma 8.2** Let \((C, \rho, (c_I(k, \sigma; 1), \ldots, c_I(k, \sigma; n)))_{I \in \text{Int}(\mathbb{R}^+), k \in \mathbb{N}, \sigma \in \mathcal{P}(\mathbb{N})}\) be an \( s \)-dimensional filtered white noise and let \( c_I(k, \sigma) = c_{(k,t)}(k, \sigma) \) for \( k \in \mathbb{N}, \sigma \in \mathcal{P}(\mathbb{N}) \). Then

\[
\rho(c_I(k_1, \sigma_1; q_1) \ldots c_I(k_n, \sigma_n; q_n)) = \sum_{R \in \mathcal{P}_n(k, \sigma)} \prod_{B \in R} Q_I(B)
\]

where

\[
Q_I(B) = Q_I(k, (\sigma_i)_{i \in B}, (q_i)_{i \in B}) = \lim_{N \to \infty} \rho(c_{k/N}(k, \sigma_{i(1)}; q_{i(1)}) \ldots c_{l/N}(k, \sigma_{l(r)}; q_{i(s)}))
\]

and \( B = \{i(1), \ldots, i(m)\} \) with \( i(1) < \ldots < i(m) \) with \( k = k(B) = k_{i(j)} \) for all \( i(j) \in B \).

**Proof.** By additivity of the filtered white noise, we can split up each \( c^{(k, \sigma; q)}_I \) into a sum of \( N \) summands:

\[
c_I(k, \sigma; q) = \sum_{I(l)} c_{I(l)}(k, \sigma; q)
\]

where \( I(l) = [(l - 1)t/N, lt/N] \). Moreover, the summands have the same distributions.

Now, note that from (8.1) it follows that we can use the same combinatorial argument as in Corollary 5.2 providing the limits

\[
Q_I(B) = \lim_{N \to \infty} N \rho(c_{I(l)}(k, \sigma_1; q_{i(1)}) \ldots c_{I(l)}(k, \sigma_n; q_{i(r)}))
\]

\[
= \lim_{N \to \infty} N \rho(c_{I(l)}(k, \sigma_1; q_{i(1)}) \ldots c_{I(l)}(k, \sigma_n; q_{i(r)}))
\]

exist for all \( l, k \in \mathbb{N}, i_1, \ldots, i_r, 1 \leq r \leq n \), where \( k = k(B) \) and the dependence of the limit on \( k, \sigma_{i(1)}, \ldots, \sigma_{i(r)} \) and \( q_{i(1)}, \ldots, q_{i(r)} \) is suppressed. Existence of such limits follows from an induction procedure which is analogous to that in the free case [21]. □

**Example.** A 2-dimensional filtered Gaussian noise is obtained from \( c_I(k, \sigma; 1) = A_I^{(k, \sigma)} \), \( c_I(k, \sigma; 2) = A_I^{(k, \sigma)} \), given by (6.4)-(6.5), for any \( k \in \mathbb{N}, \sigma \in \mathcal{P}(\mathbb{N}), t \geq 0 \) with \( \rho = \varphi \), the vacuum expectation in \( \Gamma(\mathcal{H}) \). Then

\[
\varphi(c_I(k_1, \sigma_1; q_1) \ldots c_I(k_n, \sigma_n; q_n)) \begin{cases} 
\sum_{R \in \mathcal{P}_n(\overline{k, \sigma})} \prod_{B \in R} Q_I(B) & \text{n even} \\
0 & \text{n odd}
\end{cases}
\]

where the generator \( Q_I \) does not depend of \( k(B) \) and \( \sigma_i \)’s and is given by

\[
Q_I(B) = Q_I(i(1), i(2)) = \begin{cases} 
t & \text{if } q_{i(1)} = 1, q_{i(2)} = 2 \\
0 & \text{otherwise}
\end{cases}
\]

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Note that the filtered Gaussian noise was obtained before as the GNS representation of the limit state of the invariance principle (in a slightly more general version).

Below we will find the expectations of the filtered (multivariate) Poisson noise constructed from filtered fundamental processes given by (6.4)-(6.7), which is also a way to justify the correctness of our definition of filtered number operators (6.3).

**THEOREM 8.3.** For any \( k \in \mathbb{N} \), \( \sigma \in \mathcal{P}(\mathbb{N}) \) and \( t \geq 0 \), let

\[
\Lambda_t^{(k,\sigma)} = A_t^{(k,\sigma)} + A_t^{(k,\sigma)^*} + A_t^{(k,\sigma)} + A_t^{(0,\sigma)}
\]

and let \( \varphi \) be the vacuum expectation state in \( \Gamma(\mathcal{H}) \). Then

\[
\varphi(\Lambda_t^{(k_1,\sigma_1)} \cdots \Lambda_t^{(k_n,\sigma_n)}) = \sum_{R \in \mathcal{P}_n(k,\sigma)} b(R),
\]

where \( k_1, \ldots, k_n \in \mathbb{N}, \sigma_1, \ldots, \sigma_n \in \mathcal{P}(\mathbb{N}) \) and \( b(R) \) is the number of blocks of \( R \).

**Proof.** First of all, notice that if \( I(1), \ldots, I(r) \) are disjoint intervals in \( \mathbb{R}^+ \), then

\[
\varphi(\Lambda_{I(1)}^{(k_1,\sigma_1)} \cdots \Lambda_{I(r)}^{(k_n,\sigma_n)}) = \varphi(\Lambda_{B_1}) \cdots \varphi(\Lambda_{B_r})
\]

where \( \Lambda_{[s,t]} = \Lambda_t - \Lambda_s \) and \( B_1, \ldots, B_r \) are the blocks of \( R(k,\sigma) \), with \( R \) being the partition associated with the tuple \( (l_1, \ldots, l_n) \). This fact follows from the continuous tensor product decomposition of \( \Gamma(\mathcal{H}) \) with respect to time and the fact that all summands of \( \Lambda_t^{(k,\sigma)} \) have the form \( a(I(l))p(I(l),\sigma) \), where \( p(I(l),\sigma) \) plays the role of \( \mathcal{P}(l,\sigma) \) in (3.1), whereas \( a(I(l)) \) is an elementary tensor which has units at all sites associated with \( I(m) \)'s for \( m \neq l \). Note that they are not filtered random variables in the sense of Definition 3.1, however (8.1) still holds. Moreover, the distribution \( \phi_t = \varphi|\mathcal{C}_t \), where \( \mathcal{C}_t \) is the unital \(^*\)-algebra generated by \( \Lambda_t^{(k,\sigma)} \), \( k \in \mathbb{N}, \sigma \in \mathcal{P}(\mathbb{N}) \), depends only on the Lebesgue measure \( \lambda(I) \) of every expectation is in fact a polynomial in the length of \( I \). This can be seen by using Proposition 6.1.

In view of Lemma 8.2, it suffices to show that

\[
\lim_{N \to \infty} \varphi(\Lambda_t^{(k,\sigma_1)} \cdots \Lambda_t^{(k,\sigma_n)}) = t
\]

for any \( k \in \mathbb{N} \) and \( \sigma_1, \ldots, \sigma_n \in \mathcal{P}(\mathbb{N}) \).

Looking at the action of the fundamental filtered operators on the finite particle domain (Proposition 6.1), we can see that in the considered expectation each creation-annihilation pair as well as each time operator produce \( t \), whereas each number operator produces an integer. Therefore, we obtain

\[
\varphi(\Lambda_t^{(k,\sigma)}) = \varphi(A_t^{(0,\sigma)}) = t,
\]

\[
\varphi(\Lambda_t^{(k,\sigma_1)} \Lambda_t^{(k,\sigma_2)}) = \varphi(A_t^{(k,\sigma_1)} A_t^{(k,\sigma_2)^*}) + o(t)
\]

\[
\varphi(\Lambda_t^{(k,\sigma_1)} \cdots \Lambda_t^{(k,\sigma_n)}) = \varphi(A_t^{(k,\sigma_1)} A_t^{(k,\sigma_2)^*} \cdots A_t^{(k,\sigma_{n-1})^*} A_t^{(k,\sigma_n)^*}) + o(t)
\]
\[ t + o(t) \]

for \( n > 2 \) and any \( k \in \mathbb{N}, \sigma_1, \ldots, \sigma_n \in \mathcal{P}(\mathbb{N}) \). Hence

\[
\lim_{N \to \infty} N\varphi(\Lambda^{(k,\sigma_1)}_{t/N} \cdots \Lambda^{(k,\sigma_n)}_{t/N}) = t
\]

which enables us to use Lemma 8.4 and obtain the desired form of the expectation. \( \square \)

The above theorem gives the combinatorics of the filtered Poisson noise, defined by (8.1). As it contains infinitely many colors and filters, this combinatorics involves multivariate expectations (when speaking of a 1-dimensional noise we mean one “type” of operator, although it has infinitely many “copies”). It can be noted that if \( k \in \sigma \), then \( \Lambda^{(k,\sigma)}_{t/N} \) for fixed \( k \) and \( \sigma \) gives classical Poisson white noise and if \( k \notin \sigma \), then \( \Lambda^{(k,\sigma)}_{t/N} \) gives Boolean (or, 1-free) Poisson white noise (cf. Theorem 5.4). Moreover, \( m \)-free and free Poisson white noises are obtained from linear combinations of the same type as in (3.3) and (5.4). In general, also on the level of white noise, filtered Gaussian and Poisson’s white noises are also the building blocks of other Gaussian and Poisson’s white noises since the latter can be obtained from the former by addition or strong limits.

9. A free Fock space decomposition of \( \Gamma(\mathcal{H}) \)

In this section we embed the free and \( m \)-free Fock spaces over \( \mathcal{K} \), denoted by \( \mathcal{F}(\mathcal{K}) \), \( \mathcal{F}^{(m)}(\mathcal{K}) \), \( m \in \mathbb{N} \), respectively, in the multiple symmetric Fock space \( \Gamma(\mathcal{H}) \), where \( \mathcal{H} = \mathcal{K} \otimes \mathcal{G} \), and extend the \( m \)-free and free creation and annihilation operators to bounded operators on \( \Gamma(\mathcal{H}) \). We assume that \( \mathcal{K} = L^2(\mathbb{R}_+) \).

Let us introduce the following linear combinations of filtered creation and annihilation operators, respectively:

\[
\begin{align*}
\ell^{(m)}(f) &= \sum_{k=1}^m (a^{(k)}(f \otimes e_k) - a^{(k-1)}(f \otimes e_k)) \\
\ell^{(m)}(f) &= \sum_{k=1}^m (a^{(k)}(f \otimes e_k) - a^{(k-1)}(f \otimes e_k))
\end{align*}
\]

where \( m \in \mathbb{N} \). We will call \( \ell^{(m)}(f) \), \( \ell^{(m)}(f) \), the extended \( m \)-free creation and annihilation operators, respectively. In order to compare them with the \( m \)-free creation and annihilation operators \( a^{(m)^*}(f), a^{(m)}(f) \) introduced in [4], let us recall the definition of the latter.

First, the \( m \)-free Fock space over \( \mathcal{K} \) is the truncation of order \( m \) of the free Fock space, namely

\[
\mathcal{F}^{(m)}(\mathcal{K}) = C\omega_m \oplus \bigoplus_{k=1}^m \mathcal{K}^\otimes k
\]

where \( \omega_m \) is the vacuum unit vector, with the canonical scalar product. The \( m \)-free creation operators are then given by

\[
a^{(m)^*}(f) : \mathcal{F}^{(m)}(\mathcal{K}) \to \mathcal{F}^{(m)}(\mathcal{K})
\]

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A^{(m)}(f) f_1 \otimes \ldots \otimes f_n = \begin{cases} f \otimes f_1 \otimes \ldots \otimes f_n & \text{if } 1 \leq n < m \\ 0 & \text{if } n = m \end{cases}

with A^{(m)}(f)\omega_m = f and the m-free annihilation operators

A^{(m)}(f) : \mathcal{F}^{(m)}(\mathcal{K}) \rightarrow \mathcal{F}^{(m)}(\mathcal{K})

A^{(m)}(f) f_1 \otimes \ldots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \ldots \otimes f_n

if 1 \leq n \leq m and A^{(m)}(f)\omega_m = 0. Note that A^{(m)*}(f), A^{(m)}(f) are bounded on \mathcal{F}^{(m)}(\mathcal{K}) since they are truncations of order m of free creation and annihilation operators A^{*}(f), A(f) on the free Fock space \mathcal{F}(\mathcal{K}), respectively. We will see below that l^{(m)*}(f), l^{(m)}(f) are bounded extensions of A^{(m)*}(f), A^{(m)}(f), f \in \mathcal{K}, respectively, to \Gamma(\mathcal{H}).

If we set m = \infty in the formulas for extended m-free creation and annihilation operators, we obtain operators which we denote l^{*}(f) and l(f), respectively which will be called extended free creation and annihilation operators. They, too, are bounded extensions of free creation and annihilation operators A^{*}(f), A(f), f \in \mathcal{K}, to all of \Gamma(\mathcal{H}), respectively.

Thus, we identify two notations: l^{(\infty)*}(f) \equiv l^{*}(f), l^{(\infty)}(f) \equiv l(f), f \in \mathcal{K}. In that context we will understand that P^{(\infty)} \equiv I. In general, in this section we will often assume for convenience that m \in \mathbb{N}^{*} = \mathbb{N} \cup \{\infty\}. However, certain results will be stated for m = \infty separately in order to single out the free case.

Remark. On the finite particle domain \Gamma_0(\mathcal{H}), spanned by \Omega and vectors of the form

(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})

where f_1, \ldots, f_n \in \mathcal{K}, k_1 \leq k_2 \leq \ldots \leq k_n, n \in \mathbb{N}, the series given by (9.1)-(9.2) for m = \infty are strongly convergent since only a finite number of terms do not vanish when acting on vectors of finite "color support" and thus give well-defined operators with domains dense in \Gamma(\mathcal{H}). A similar feature was exhibited by the series representation of free random variables obtained from the construction of the hierarchy of freeness ([11], [5]). We will see below that they have bounded extensions to \Gamma(\mathcal{H}).

Let us first determine the action of m-free creation and annihilation operators on \Gamma_0(\mathcal{H}).

**Proposition 9.1.** Let f, f_1, \ldots, f_n \in \mathcal{K} and k_1 \leq k_2 \leq \ldots \leq k_n, m \in \mathbb{N}^{*}. Then

l^{(m)*}(f)\Omega = f \otimes e_1,

l^{(m)}(f)\Omega = 0,

l^{(m)*}(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) = \mathbb{I}_{\{m \geq k_n+1\}}\sqrt{(n+1)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \circ (f \otimes e_{k_n+1})},

l^{(m)}(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) = \mathbb{I}_{\{m \geq k_n\}}\frac{1}{\sqrt{n}}(f, f_n)\delta_{k_n,k_{n-1}}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_{n-1} \otimes e_{k_{n-1}})
where it is understood that \( k_0 = 0 \).

**Proof.** Note that

\[
\begin{align*}
    a^{(k)}(f \otimes e_k) - a^{(k-1)}(f \otimes e_k) &= a^*(f \otimes e_k)P^{[k-1]} \\
    a^{(k)}(f \otimes e_k) - a^{(k-1)}(f \otimes e_k) &= P^{[k-1]}a(f \otimes e_k)
\end{align*}
\]

where

\[
P^{[k-1]}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) = \delta_{k-1,k_n}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})
\]

since \( P^{[k-1]} = P^{(k)} = P^{(k-1)} \) and \( k_j \leq k_n \) for all \( j = 1, \ldots, n \). Thus,

\[
\begin{align*}
l^{(m)}(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) &= \sum_{k=1}^{m} a^*(f \otimes e_k)P^{[k-1]}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \\
&= \sum_{k=1}^{m} \delta_{k-1,k_n}a^*(f \otimes e_k)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \\
&= \mathbb{I}_{\{m \geq k_n+1\}}a^*(f \otimes e_{k_n+1})(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \\
&= \mathbb{I}_{\{m \geq k_n+1\}}\sqrt{n+1}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \circ (f \otimes e_{k_n+1}).
\end{align*}
\]

Next, if \( n > 1 \), then

\[
\begin{align*}
l^{(m)}(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) &= \sum_{k=1}^{m} P^{[k-1]}a(f \otimes e_k)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{m} \sum_{j=1}^{n-1} \langle f \otimes e_k, f_j \otimes e_{k_j} \rangle P^{[k-1]}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_j \otimes e_{k_j}) \circ \ldots \circ (f_n \otimes e_{k_n}) \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{m} \sum_{j=1}^{n-1} \langle f, f_j \rangle \delta_{k,k_j} \delta_{k-1,k_n}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_j \otimes e_{k_j}) \circ \ldots \circ (f_n \otimes e_{k_n}) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{k=1}^{m} \langle f, f \rangle \delta_{k,k_n} \delta_{k-1,k_n-1}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_{n-1} \otimes e_{k_{n-1}}) + \ldots \\
&= \mathbb{I}_{\{m \geq k_n\}} \frac{1}{\sqrt{n}} \langle f, f \rangle \delta_{k_n,k_n+1}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_{n-1} \otimes e_{k_{n-1}})
\end{align*}
\]

where the last equality follows again from the fact that \( k_j \leq k_n \) for \( j \leq n \), which makes the first sum vanish. If \( n = 1 \), then

\[
l^{(m)}(f)f_1 \otimes e_{k_1} = \sum_{k=1}^{m} \delta_{k,k_1} \langle f, f_1 \rangle P^{[k-1]}\Omega = \delta_{k_1,1} \langle f, f_1 \rangle \Omega \equiv \delta_{k_1,1} \langle f, f_1 \rangle
\]

The action on the vacuum vector is immediate. \( \square \)
Corollary 9.2. In particular, if \( m = \infty \), we obtain
\[
l^*(f) \Omega = f \otimes e_1,
\]
\[
l^*(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})
\]
\[
= \sqrt{(n+1)}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \circ (f \otimes e_{k_{n+1}}),
\]
\[
l(f) \Omega = 0,
\]
\[
l(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})
\]
\[
= \frac{1}{\sqrt{n}} \langle f, f_1 \rangle \delta_{k_1,k_{n+1}}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_{n-1} \otimes e_{k_{n-1}}).
\]

Theorem 9.3. For any \( m \in \mathbb{N}^* \) and \( f, g \in K \), the operators \( l^{(m)*}(f) \) and \( l^{(m)}(f) \) have unique bounded extensions to \( \Gamma(\mathcal{H}) \), are adjoints of each other, and satisfy the following relation:
\[
l^{(m)}(g)l^{(m)*}(f) = \langle g, f \rangle P^{(m)}.
\]

Proof. By Proposition 9.1 we have
\[
l^{(m)}(f)l^{(m)*}(g)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})
\]
\[
= l^{(m)}(f)\mathbb{I}_{\{m \geq k_{n+1}\}} \sqrt{n+1}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \circ (g \otimes e_{k_{n+1}})
\]
\[
= \mathbb{I}_{\{m \geq k_{n+1}\}} \langle f, g \rangle \langle f_1 \otimes e_{k_1} \rangle \circ \ldots \circ (f_n \otimes e_{k_n})
\]
where \( k_1 \leq k_2 \leq \ldots \leq k_n \), which proves that the relation holds on \( \Gamma_0(\mathcal{H}) \). The proof of adjointness goes as follows.
\[
\langle (f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}), l^{(m)*}(f)(g_1 \otimes e_{l_1}) \circ \ldots \circ (g_p \otimes e_{l_p}) \rangle
\]
\[
= \mathbb{I}_{\{m \geq k_{p+1}\}} \sqrt{p+1}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) \circ (g_1 \otimes e_{l_1}) \circ \ldots \circ (g_p \otimes e_{l_p}) \circ (f \otimes e_{k_{p+1}})
\]
\[
= \mathbb{I}_{\{m \geq k_{p+1}\}} \sqrt{\frac{n}{n!}} \delta_{n,p} \delta_{l_1,1} \ldots \delta_{l_{p-1},1} \delta_{l_p,k_{n+1}} \langle f_1, g_1 \rangle \ldots \langle f_{n-1}, g_{n-1} \rangle \langle f_n, f \rangle.
\]
On the other hand,
\[
\langle l^{(m)}(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}), (g_1 \otimes e_{l_1}) \circ \ldots \circ (g_p \otimes e_{l_p}) \rangle
\]
\[
= \mathbb{I}_{\{m \geq k_n\}} \frac{1}{\sqrt{n(n-1)!}} \delta_{n-1,p} \delta_{l_1,1} \ldots \delta_{l_{p-1},1} \delta_{l_p,k_{n+1}} \langle f_1, g_1 \rangle \ldots \langle f_{n-1}, g_{n-1} \rangle \langle f_n, f \rangle
\]
where the following expression for the scalar product
\[
\langle (f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}), (g_1 \otimes e_{l_1}) \circ \ldots \circ (g_p \otimes e_{l_p}) \rangle
\]
\[
= \frac{1}{n!} \delta_{n,p} \delta_{l_1,1} \ldots \delta_{l_{p-1},1} \delta_{l_p,k_n} \langle f_1, g_1 \rangle \ldots \langle f_n, g_n \rangle.
\]
is obtained from the canonical scalar product on \( \Gamma(\mathcal{H}) \). Therefore, we have
\[
\langle l^{(m)}(f)x, y \rangle = \langle x, l^{(m)*}(f)y \rangle
\]
for \( x, y \in \Gamma_0(\mathcal{H}) \). Now, note that
\[
\|l^{(m)}(f)(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})\|^2 = \mathbb{I}_{\{m \geq k_n+1\}}\|f\|^2\|f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n})\|^2
\]
hence \( l^{(m)}(f) \) has a unique bounded extension to \( \Gamma(\mathcal{H}) \) of norm \( \|l^{(m)}(f)\| = \|f\| \) and thus the annihilation operator \( l^{(m)}(f) \) has also a unique bounded extension to \( \Gamma(\mathcal{H}) \) of norm \( \|l^{(m)}(f)\| = \|f\| \).

\[\square\]

Acting with the \( m \)-free creation and annihilation operators on \( \Omega, m \in \mathbb{N} \), and taking the closure, we recover a subspace isomorphic to the \( m \)-free Fock space \( \mathcal{F}^{(m)}(\mathcal{K}) \). Thus, denote by \( \bar{\mathcal{F}}^{(m)}(\mathcal{K}) \) the closure of the space \( \bar{\mathcal{F}}_0^{(m)}(\mathcal{K}) \) spanned by \( \Omega \) and vectors of the form
\[
(f_n \otimes e_1) \circ \ldots \circ (f_1 \otimes e_n)
\]
where \( f_1, \ldots, f_n \in \mathcal{K}, 1 \leq n \leq m \) if \( m \) is finite. Similarly, denote by \( \bar{\mathcal{F}}(\mathcal{K}) \) the closure of \( \bar{\mathcal{F}}_0(\mathcal{K}) \) spanned by vectors of the above form with arbitrary \( n \in \mathbb{N} \). We obtain
\[
\langle (f_n \otimes e_1) \circ \ldots \circ (f_1 \otimes e_n) \rangle, (g_m \otimes e_1) \circ \ldots \circ (g_1 \otimes e_m) \rangle = \delta_{n,m} \frac{1}{n!}\langle f_1, g_1 \rangle \ldots \langle f_n, g_n \rangle
\]
by the orthogonality of \( e_1, \ldots, e_n \).

**Corollary 9.4.** The \( m \)-free Fock space \( \mathcal{F}^{(m)}(\mathcal{K}) \) is isomorphic to \( \bar{\mathcal{F}}^{(m)}(\mathcal{K}) \). The free Fock space \( \mathcal{F}(\mathcal{K}) \) is isomorphic to \( \bar{\mathcal{F}}(\mathcal{K}) \).

**Proof.** The unitary isomorphism from \( \mathcal{F}_0(\mathcal{K}) \) to \( \bar{\mathcal{F}}_0(\mathcal{K}) \) is given by
\[
f_1 \otimes \ldots \otimes f_n \rightarrow \sqrt{n!}(f_n \otimes e_1) \circ \ldots \circ (f_1 \otimes e_n)
\]
and thus extends uniquely to \( \mathcal{F}(\mathcal{K}) \) (its restrictions give the result for \( \mathcal{F}^{(m)}(\mathcal{K}) \)). \( \square \)

Thus, for each \( m \in \mathbb{N} \), we obtain the filtration
\[
\bar{\mathcal{F}}^{(1)}(\mathcal{K}) < \ldots < \bar{\mathcal{F}}^{(m)}(\mathcal{K}) < \ldots < \bar{\mathcal{F}}(\mathcal{K})
\]
in which \( \bar{\mathcal{F}}^{(m)}(\mathcal{K}) \) is an invariant subspace for the \( C^* \)-algebra
\[
C^{(m)} = C^*(1, l^{(m)}(f))| f \in \mathcal{K}\]
and \( \bar{\mathcal{F}}(\mathcal{K}) \) is an invariant subspace for the \( C^* \)-algebra
\[
C = C^*(1, l^*(f)), f \in \mathcal{K}\).

Moreover, each \( \bar{\mathcal{F}}(\mathcal{K}) \) is only one copy of the free Fock space in \( \Gamma(\mathcal{K}) \) and it turns out that one can decompose \( \Gamma(\mathcal{H}) \) into a countable direct sum of subspaces isomorphic to the free Fock space and invariant under \( C \). In the sequel we will concentrate on this decomposition, in other words on what is “between” \( \bar{\mathcal{F}}(\mathcal{K}) \) and \( \Gamma(\mathcal{H}) \).

In order to determine this, we need to take a closer look at the kernel of the annihilation operators. Let \( \{d_n\}_{n=1}^{\infty} \) be an orthonormal basis in \( \mathcal{K} \). Note that the set consisting of \( \Omega \) and vectors of the form
\[
(d_{i_1} \otimes e_{k_1}) \circ \ldots \circ (d_{i_n} \otimes e_{k_n})
\]
where $k_1 \leq k_2 \leq \ldots \leq k_n$ and $i_r \leq i_{r+1}$ whenever $k_{i_r} = k_{i_{r+1}}$, is an orthogonal basis in $\Gamma(\mathcal{H})$ (the ordering of indices is used for convenience, which is possible due to the fact that the product is symmetrized). Denote by $\hat{\mathcal{D}}$ the subset of this basis consisting of $\Omega$ and vectors of the above form for which $k_n \neq k_{n-1} + 1$, i.e. the last two vectors are of identical colors or their colors differ by more than 1 if $n > 1$, and the last color is not equal to 1 if $n = 1$. By normalizing the vectors from $\hat{\mathcal{D}}$ we get $\mathcal{D} = \{ x/\|x\| \mid x \in \hat{\mathcal{D}} \}$ which is an orthonormal set. Let $\mathcal{D}^{(m)} = \mathcal{D} \cap \Gamma^{m+1}$, where $\Gamma^{m+1} = \Gamma(\mathcal{H}^{(m+1)})$. We understand that $\mathcal{D}^{(\infty)} = \mathcal{D}$.

**Proposition 9.5.** $\mathcal{D}^{(m)} \subseteq \ker l^{(m)}(f)$ for any $m \in \mathbb{N}^*$ and $f \in \mathcal{K}$.

*Proof.* This follows from Proposition 9.1 due to the presence of $\delta_{k_n,k_{n-1}+1}$ on the right-hand side of the formula for the annihilation operators. $\square$

**Proposition 9.6.** Let $m \in \mathbb{N}^*$. Then

$$\sum_{s=1}^{\infty} l^{(m)}(d_s^*) l^{(m)}(d_s) = I - P_{[\mathcal{D}^{(m)}] \oplus \Gamma^{(m)}}$$

where $P_{[\mathcal{D}^{(m)}] \oplus \Gamma^{(m)}}$ is the projection onto $[\mathcal{D}^{(m)}] \oplus \Gamma^{(m)}$ and $\Gamma^{(m)} = \Gamma(\mathcal{H} \ominus \mathcal{H}^{(m+1)})$. In particular,

$$\sum_{s=1}^{\infty} l^*(d_s) l(d_s) = I - P_{\mathcal{D}}$$

thus $\mathcal{C} \cong \mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is the Cuntz algebra.

*Proof.* It can be seen from Theorem 9.3 that

$$l^{(m)}(d_s^*) l^{(m)}(d_s) = Q_s,$$

where $Q_s$ is the projection onto the subspace spanned by vectors of the form

$$(d_{s_1} \otimes e_{k_1}) \circ \ldots \circ (d_{s_{n-1}} \otimes e_{k_{n-1}}) \circ (d_s \otimes e_{k_n})$$

where $k_1 \leq \ldots \leq k_{n-1} = k_n - 1 < k_n \leq m$ (cf. [25]). These subspaces are pairwise orthogonal and span the orthogonal complement of $[\mathcal{D}^{(m)}] \oplus \Gamma^{(m)}$, which proves the first formula. The second formula is just a special case when $m = \infty$ and, together with Theorem 9.3, it implies that the $C^*$-algebra generated by $l^*(f)$, $f \in \mathcal{K}$, is isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ since $\mathcal{K}$ is countably separable. $\square$

Let us introduce the following notation on $\Gamma_0(\mathcal{H})$:

$$u \odot w = u_1 \circ \ldots \circ u_r \circ w_1 \circ \ldots \circ w_n$$

where $u = u_1 \circ \ldots \circ u_r$, $w = w_1 \circ \ldots \circ w_n$. 

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Proposition 9.7. If \( x = x_1 \circ \ldots \circ x_r, z = z_1 \circ \ldots \circ z_r, u = u_1 \circ \ldots \circ u_n, v = v_1 \circ \ldots \circ v_n, \)
where \( x_i, z_i \in \mathcal{H}_1, 1 \leq i \leq r \) and \( z_j, v_j \in \mathcal{H}_2, 1 \leq j \leq n, \) and \( \mathcal{H}_1, \mathcal{H}_2 \) are two orthogonal subspaces of \( \mathcal{H}, \) then
\[
\langle x \circ u, z \circ v \rangle = \frac{r!n!}{(r+n)!} \langle x, z \rangle \langle u, v \rangle.
\]

Proof. Using the orthogonality of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) and the formula for the scalar product in \( \Gamma(\mathcal{H}), \) we obtain
\[
\langle x \circ u, z \circ v \rangle = \frac{1}{(r+n)!} \sum_{\sigma \in S_r} \sum_{\tau \in S_n} \langle x_1, z_{\sigma(1)} \rangle \ldots \langle x_r, z_{\sigma(r)} \rangle \langle u_1, v_{\tau(1)} \rangle \ldots \langle u_n, v_{\tau(n)} \rangle
= \frac{r!n!}{(r+n)!} \langle x, z \rangle \langle u, v \rangle.
\]
\( \square \)

The \( C^* \)-algebra \( \mathcal{C} \) is a \( C^* \)-subalgebra of \( \mathcal{B}(\Gamma(\mathcal{H})). \) Denote the faithful representation of \( \mathcal{C} \) on \( \Gamma(\mathcal{H}) \) by \( \pi. \) Since \([\mathcal{C}x] \) is for each \( x \in \mathcal{D} \) a closed subspace of \( \Gamma(\mathcal{H}), \) which is invariant under each operator \( A \) in \( \mathcal{C}, \) the mapping \( A \to A[\mathcal{C}x] \) is a cyclic representation of \( \mathcal{C} \) on \([\mathcal{C}x] \) with cyclic vector \( x. \) Denote this representation by \( \pi_x. \) We will show below that \( \pi \) is a direct sum of cyclic representations \( \pi_x, x \in \mathcal{D}. \)

Theorem 9.8. The multiple symmetric Fock space has the direct sum decomposition
\[
\Gamma(\mathcal{H}) = \bigoplus_{x \in \mathcal{D}} [\mathcal{C}x]
\]
where \([\mathcal{C}x] \cong \mathcal{F}(\mathcal{K}), \) according to which
\[
\pi = \bigoplus_{x \in \mathcal{D}} \pi_x
\]
where \( \pi_x \cong \rho, \) and \( \rho \) is the free Fock space representation of \( \mathcal{C}. \)

Proof. If \( x = \hat{x}/\|x\| \in \mathcal{D}, \) where \( \hat{x} \) is of the form
\[
\hat{x} = (d_1 \otimes e_{k_1}) \circ \ldots \circ (d_i \otimes e_{k_i})
\]
with \( k_r = l, \) then \([\mathcal{C}x] \) is the closed subspace of \( \Gamma(\mathcal{H}) \) spanned by vectors of the form
\[
x \circ (f_n \otimes e_{l+1}) \circ \ldots \circ (f_1 \otimes e_{l+n})
\]
where \( f_1, \ldots, f_n \in \mathcal{K}. \) Clearly, \([\mathcal{C}x] \) is invariant under \( \mathcal{C}. \) Let us show that for each \( x \in \mathcal{D}, [\mathcal{C}x] \cong \mathcal{F}(\mathcal{K}). \)

For that purpose, define the linear mapping
\[
U_x : \mathcal{F}_0(\mathcal{K}) \to [\mathcal{C}x]
\]
by
\[
U_x(\omega) = x
\]
\[ U_x(f_1 \otimes \ldots \otimes f_n) = \sqrt{(r+n)! \over r!} x \circ (f_n \otimes e_{l+1}) \circ \ldots \circ (f_1 \otimes e_{l+n}). \]

This mapping is scalar-product preserving since
\[
\langle U_x f_1 \otimes \ldots \otimes f_n, U_x g_1 \otimes \ldots \otimes g_m \rangle = \delta_{n,m} \frac{(r+n)!}{r!} \langle x \circ (f_n \otimes e_{l+1}) \circ \ldots \circ (f_1 \otimes e_{l+n}), x \circ (g_n \otimes e_{l+1}) \circ \ldots \circ (g_1 \otimes e_{l+n}) \rangle
\]
\[
= n! \langle (f_n \otimes e_{l+1}) \circ \ldots \circ (f_1 \otimes e_{l+n}), (g_n \otimes e_{l+1}) \circ \ldots \circ (g_1 \otimes e_{l+n}) \rangle
\]
\[
= \langle f_1, g_1 \rangle \ldots \langle f_n, g_n \rangle,
\]
and therefore has a unique extension to \( F(K) \). It is not hard to see that \([Cx] \perp [Cx']\) for \( x \neq x'\) and that \( \Gamma(\mathcal{H}) \) is a direct sum of \([Cx]\) for all \( x \in D\).

It remains to be shown that \( U_x \) intertwines between \( \pi_x \) and the free Fock space representation \( \rho \) of \( C \) on \( F(K) \). We have
\[
\pi_x(l^x(f)) U_x(\omega) = \sqrt{r+1} x \otimes (f \otimes e_{r+1}) = U_x(f) = U_x \rho(l^x(f)) \omega
\]
and
\[
\pi_x(l^x(f)) U_x(f_1 \otimes \ldots \otimes f_n)
\]
\[
= \sqrt{(r+n+1)! \over r!} x \circ (f_n \otimes e_{l+1}) \circ \ldots \circ (f_1 \otimes e_{l+n}) (f \otimes e_{l+n+1})
\]
\[
= U_x(f \otimes f_1 \otimes \ldots \otimes f_n)
\]
\[
= U_x \rho(l^x(f))(f_1 \otimes \ldots \otimes f_n)
\]
for any \( f_1, \ldots, f_n, f \in K, n \geq 1 \). Similarly, \( \pi_x(l(f)) U_x \Omega = l(f)x = 0 = U_x \rho(l(f)) \Omega \) and
\[
\pi_x(l(f)) U_x(f_1 \otimes \ldots \otimes f_n)
\]
\[
= \pi_x(l(f)) \sqrt{(r+n)! \over r!} x \circ (f_n \otimes e_{l+1}) \circ \ldots \circ (f_1 \otimes e_{l+n})
\]
\[
= \sqrt{(r+n-1)! \over r!} (f, f_1) x \circ (f_n \otimes e_{l+1}) \circ \ldots \circ (f_2 \otimes e_{l+n-1})
\]
\[
= \langle f, f_1 \rangle U_x(f_2 \otimes \ldots \otimes f_n)
\]
\[
= U_x \rho(l(f))(f_1 \otimes \ldots \otimes f_n).
\]
Therefore \( \pi_x(a) U_x = U_x \rho(a) \) also for any \( a \in C \). This finishes the proof.

Let us finally define extended \( m \)-free number operators. Guided by the definitions of extended creation and annihilation operators, we set
\[
l^{(m)\circ} = \sum_{k=1}^{m} (a^{(k,k)} - a^{(k,k-1)\circ})
\]
where \( m \in \mathbb{N}^* \) and \( a^{(k,r)\circ} = a^{(k,\sigma)\circ} \) for \( \sigma = \{1, \ldots, r-1\} \) and \( a^{(k,\sigma)\circ} \) is given by (6.3).

Let us determine the action of extended number operators on the finite particle domain.

**Proposition 9.9.** Let \( f_1, \ldots, f_n \in \mathcal{K}, \, k_1 \leq \ldots \leq k_n, \, m \in \mathbb{N}^* \). The finite particle domain \( \Gamma_0(\mathcal{H}) \) is contained in the domains of extended \( m \)-free number operators and

\[
l^{(m)\circ}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) = \begin{cases} N_{k_n}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) & \text{if } k_j + 1 = k_n \leq m \text{ for } j < n \\ 0 & \text{otherwise} \end{cases}
\]

where \( N_k = \#\{i | k_i = k\} \).

**Proof.** We have

\[
l^{(m)\circ}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) = \sum_{k=1}^{m} a^{(k)\circ}(P(\{1, \ldots, k\}) - P(\{1, \ldots, k-2, k\}))(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) = \begin{cases} a^{(k_n)\circ}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) & \text{if } k_j + 1 = k_n \leq m \text{ for } j < n \\ 0 & \text{otherwise} \end{cases}
\]

\[
= \begin{cases} N_{k_n}(f_1 \otimes e_{k_1}) \circ \ldots \circ (f_n \otimes e_{k_n}) & \text{if } k_j + 1 = k_n \leq m \text{ for } j < n \\ 0 & \text{otherwise} \end{cases}
\]

This ends the proof. \( \square \)

In other words, \( l^{(m)\circ} \) “counts” particles of the highest color \( k_n \) if that one is smaller or equal to \( m \) and the second highest color is equal to \( k_n - 1 \). Otherwise, the extended free number operator gives zero. In particular, on \( \tilde{\mathcal{F}}^{(m)}_0(\mathcal{K}) \) we obtain

\[
l^{(m)\circ}\Omega = 0
\]

\[
l^{(m)\circ}(f_1 \otimes e_{n}) \circ \ldots \circ (f_n \otimes e_{1}) = (f_1 \otimes e_{n}) \circ \ldots \circ (f_n \otimes e_{1})
\]

for \( 1 \leq n \leq m \).

It can be seen that, contrary to the case of extended \( m \)-free creation and annihilation operators, the operators \( l^{(m)\circ} \) are not bounded on \( \Gamma(\mathcal{H}) \). Clearly, they are bounded on \( \tilde{\mathcal{F}}^{(m)}(\mathcal{H}) \) and, in fact, it can be shown that they are bounded on \( [\mathcal{C}x] \) for each \( x \in \mathcal{D} \).

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