MOURRE THEORY FOR TIME-PERIODIC MAGNETIC FIELDS

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Abstract

We study the Mourre theory for the Floquet Hamiltonian $\hat{H} = -i\partial_t + H(t)$ generated by the time-periodic Hamiltonian $H(t)$ with $H(t+T) = H(t)$, which describes the Schrödinger equations with general time-periodic magnetic fields.

Keywords: Mourre Theory, Time-Periodic System, Time-Periodic Magnetic Fields, Scattering Theory

MSC[81U05]

1 Introduction

Mourre theory was firstly considered by E. Mourre [12] to prove the absence of singular spectrum of Schrödinger operators, and this theory played very important role in the scattering theory, especially, in the many-body quantum scattering theory, see e.g., Derezinski and Gérard [DG]. From 1980’s to the present, this theory has been applied to the Schrödinger operators with various external fields. However, there are no results about the Mourre theory for the Schrödinger operators with time-periodic magnetic fields as far as we know and hence we prove this problem in this paper.

We study the dynamics of a charged particle, which is influenced by time-periodic magnetic fields, where we assume that the charged particle is moving in the plane $\mathbb{R}^2$ in the presence of a time-periodic magnetic field $B(t) = (0, 0, B(t))$ with $B(t+T) = B(t)$, which is always perpendicular to this plane. Then the free Hamiltonian for this system is given by

$$H_0(t) = (p - qA(t,x))^2/(2m), \quad A(t,x) = (-B(t)x_2, B(t)x_1)/2, \quad \text{on } L^2(\mathbb{R}^2)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $p = (p_1, p_2) = -i(\partial_1, \partial_2)$, $m > 0$ and $q \in \mathbb{R} \setminus \{0\}$ are the position, the momentum, the mass and the charge of the particle, respectively. The intense of the magnetic field at $t$ is denoted by $B(t) \in L^\infty(\mathbb{R})$. We let the wave function $\psi(t,x)$ is the solution to the following time-dependent Schrödinger equations:

$$i\partial_t \psi(t,x) = H_0(t)\psi(t,x), \quad \psi(0,x) = \psi_0.$$ 

Then by denoting the propagator for $H_0(t)$ by $U_0(t,0)$, we can see the wave function $\psi(t,x)$ of (2) is denoted by $\psi(t,x) = U_0(t,0)\psi_0$, where we call a family of unitary operators $\{U_0(t,s)\}_{(t,s) \in \mathbb{R}^2}$ a propagator for $H_0(t)$, if the each components $U_0(t,s)$ satisfy

$$i\partial_t U_0(t,s) = H_0(t)U_0(t,s), \quad i\partial_s U_0(t,s) = -U_0(t,s)H(s),$$

$$U_0(t,\theta)U_0(\theta,s) = U_0(t,s), \quad U_0(s,s) = \text{Id}.$$
Here we introduce the result of a paper Adachi-Kawamoto [1], which considers the scattering theory for a time-periodic pulsed magnetic field and proves the asymptotic completeness of wave operators, where we let the pulsed magnetic field is the following

$$B(t) = \begin{cases} B, & t \in [NT, NT + T_B], \\ 0, & t \in [NT + T_B, (N + 1)T], \end{cases} \quad 0 < T_B < T, \ N \in \mathbb{Z}. \quad (3)$$

In [1], the following lemma was showed and that implies the asymptotic behavior of the charged particle with respect to (3) is classified into the three types accordingly to the \(B, T_B\) and \(T\).

**Lemma 1.1.** Let \(\tilde{D} = 2\cos(qBT_B/(2m)) - qB(T - T_B)\sin(qBT_B/(2m))/2\) and \(\tilde{\lambda}_\pm := \tilde{D}/2 \pm i qB/\lambda\) in the pulsed magnetic field case, and \(\tilde{\lambda}_\pm \in \mathbb{R}\) is related to the case where \(\tilde{D} = 4\) in the pulsed magnetic field case, and \(\tilde{\lambda}_\pm \in \mathbb{R}\) holds.

Roughly speaking, the term \(\|xU_0(t, 0)\phi\|_{L^2(\mathbb{R}^3)}^2\) can be regarded like the position of the particle at \(t\), and hence [4] implies that the charged particle moves out to the origin exponentially in \(t\) by the effect of the pulsed magnetic field, where we use that either \(|\lambda_+|\) or \(|\lambda_-|\) is larger than 1 if \(\tilde{D} > 4\). We call this phenomena the particle is in exponentially scattering state in this paper.

On the other hand, Korotyaev [9] considers the same Hamiltonian \(H_0(t)\) in (1) for time-periodic magnetic fields with the condition mentioned later and proves the asymptotic completeness of wave operators by using the following representation of the free propagator:

$$U_0(t, 0) = e^{-i(y_2'(t)/y_2(t))x^2 - i y_1(t)y_2(t)p^2 - i\Omega(t)L} e^{-i\log(|y_2(t)|)A/2} S^\nu(t),$$

where, \(A = (x \cdot p + p \cdot x)/2\) and \(L = x_1p_2 - x_2p_1\) are called a generator of dilation group and the angular momentum of the charged particle, respectively. We put \(\Omega(t) = \int_0^t qB(s)/(2m)ds\) and define \(y_j(t), \ j \in \{1, 2\}\) is the solution to

$$y_j''(t) + \left(\frac{qB(t)}{2m}\right)^2 = 0, \quad \begin{cases} y_1(0) = 0, & y_1'(0) = 1, \\ y_2(0) = 1, & y_2'(0) = 0, \end{cases}$$

respectively. Moreover \(S\) is an unitary operator which satisfies \((S\phi)(x) = (-1)\phi(-x)\) and \(v(t)\) is the number of zeros of \(\zeta_1(s)\) in \(s \in [0, t]\). Korotyaev considers the scattering theory for the case where \(y_j(t)\) is described by

$$\begin{align*}
\begin{cases} y_1(t) = ty_2(t) + \chi_1(t), \\ y_2(t) = \chi_2(t), \end{cases} & \quad \text{or} \quad \begin{cases} y_1(t) = e^{\lambda t}\chi_1(t), \\ y_2(t) = e^{-\lambda t}\chi_2(t), \end{cases}
\end{align*}$$

for \(\lambda \in \mathbb{R}\) and periodic or antiperiodic functions \(\chi_1\) and \(\chi_2\). The case where \(y_j(t)\) is written by l.h.s. of (5) is related to the case where \(\tilde{D} = 4\) in the pulsed magnetic field case, and \(y_j(t)\) is written by r.h.s. of (5) is related to (4).

Now, we investigate what makes the motion of the charged particle [1]. Here we firstly introduce the following Theorem:
Theorem 1.2. Define \( \zeta_1(t), \zeta_2(t), \zeta_1'(t) \) and \( \zeta_2'(t) \) as solutions to the following equations;

\[
\zeta_j'(t) + \left( \frac{qB(t)}{2m} \right)^2 \zeta_j(t) = 0, \quad \begin{cases} \zeta_1(0) = 1, & \zeta_1'(0) = 0, \\ \zeta_2(0) = 0, & \zeta_2'(0) = 1, \end{cases}
\]

(6)

respectively, and suppose that these are continuous functions. Let \( U_0(t, 0) \) be the propagator for \( H_0(t) \), and then \( U_0(t, 0) \) is factorized as

\[
U_0(t, 0) = e^{-ia(t)x^2} e^{-ib(t)p^2} e^{i\Omega(t)L} e^{-ic(t)x^2},
\]

(7)

where \( \Omega(t) = \int_0^t qB(s)/(2m)ds, \) \( L = x_1p_2 - x_2p_1 \) and

\[
a(t) = \frac{m}{2} \left( 1 - \frac{\zeta_1'(t)}{\zeta_2(t)} \right), \quad b(t) = \frac{\zeta_2(t)}{2m}, \quad c(t) = \frac{m}{2} \left( 1 - \frac{\zeta_1(t)}{\zeta_2(t)} \right).
\]

(8)

Remark 1.3. This lemma can be proven without the periodic condition \( B(t + T) = B(t) \) (see §2).

Møller [12] studied the scattering theory for a charged particle influenced by time-periodic electric fields \( \mathbf{E}(t + T_B) = \mathbf{E}(t), \mathbf{E}(t) = (E_1(t), \ldots, E_\alpha(t)) \) with \( \int_0^{T_B} \mathbf{E}(s)ds \neq 0 \), and noticed that the propagator \( U_0^S(t, 0) \) for the Hamiltonian which describes this system was expressed by

\[
U_0^S(t, s) = \tilde{T}(t)e^{-i(t-s)H_0^S} (\tilde{T}(s))^*,
\]

(9)

where \( \tilde{T} \) was an unitary operator with \( \tilde{T}(t + T_B) = \tilde{T}(t), \tilde{T}(\tilde{T})^* = 1d \) and \( H_0^S \) was the time-independent Stark-Hamiltonian. Thus, one can obtain \( U_0^S(NT_B, 0) = \tilde{T}(0)e^{-iNT_BH_0^S} (\tilde{T}(0))^* \), and see that the asymptotic behavior of the charged particle is governed only by the propagator \( e^{-iH_0^S} \).

From this expression, many results such as Mourre theory, propagation estimate, asymptotic completeness of wave operators and so on are proven by the almost same way of time-independent case. Thus, the first aim of this paper is to find the time-independent Hamiltonian such that the propagator \( \tilde{T} \) is decomposed like \( (\tilde{T})^* \).

Here, define so-called repulsive Hamiltonian as

\[
H_R = p^2 - x^2.
\]

(10)

Then it is known that for all \( \psi \in L^2(\mathbb{R}^n) \), a particle \( e^{-i(t^2-x^2)}\psi \) is in exponentially scattering state, see Bony-Charles-Hübner-Michel [2], and hence it can be expected that, in the case where \( y_j(t) = e^{-\lambda(-1)^j} \chi_j(t) \) in \([5]\) (or \( \tilde{D} > 4 \) in pulsed magnetic case), the free propagator \( \tilde{T} \) is decomposed like

\[
U_0(t, s) \sim \tilde{\mathcal{F}}(t)e^{-i(t-s)H_R} (\tilde{\mathcal{F}}(s))^*
\]

for some unitary operator \( \tilde{\mathcal{F}} \) with \( \tilde{\mathcal{F}}(t + T) = \tilde{\mathcal{F}}(t) \) and \( \tilde{\mathcal{F}}(t)(\tilde{\mathcal{F}}(t))^* = 1d \) by the same scheme of \([11]\). By the following Lemma [14] our expectation will be proven.

Lemma 1.4. Let \( D, \sigma_D, A_D, B_D, C_D \) and \( D_D \) are the followings

\[
D = \zeta_1(T) + \zeta_2(T), \quad \sigma_D = \begin{cases} 0 & D \geq 2 \\ 1 & D \leq -2 \end{cases}, \quad D_D = \begin{cases} -\Omega(T)/B_D & D \geq 2 \\ -\Omega(T + \pi)/B_D & D \leq -2 \end{cases},
\]

\[
A_D = \begin{cases} a(T) + (a_3 - 1)^2/(8a_3b(T)) & D \geq 2 \\ a(T) - (a_3 + 1)^2/(8a_3b(T)) & D \leq -2 \end{cases},
\]

\[
B_D = \begin{cases} 2b(T)a_3(\log a_3)/(a_3^2 - 1) & D \geq 2 \\ -2b(T)b_3(\log b_3)/(b_3^2 - 1) & D \leq -2 \end{cases}, \quad C_D = \begin{cases} (a_3^2 - 1)/(4a_3b(T)) & D \geq 2 \\ (b_3^2 - 1)/(4b_3b(T)) & D \leq -2 \end{cases}
\]

(11)
where \( \Omega(t) = \int_0^t qB(s)/(2m)ds \), \( a(T), b(T), c(T) \) are equivalent to those in \([8]\). \( a_3 \) and \( b_3 \) are given as the solutions of

\[
\begin{align*}
   a_3^2 - Da_3 + 1 &= 0, \quad a_3 > 1, \\
   b_3^2 + Db_3 + 1 &= 0, \quad b_3 > 1.
\end{align*}
\]

Furthermore, denote

\[
W_0 = (B_D/T)(p^2 - C_D^2 x^2 + D_D L) + (\pi\sigma_D)/T.
\]

Then the monodromy operator \( U_0(T, 0) \) is decomposed into

\[
U_0(T, 0) = e^{-iA D x^2} e^{-iT W_0} e^{iA D x^2}, \quad U_0(NT, 0) = e^{-iA D x^2} e^{-iNT W_0} e^{iA D x^2}.
\]  \((12)\)

\( D \) is often called discriminant of Hill’s equation.

Now we consider the asymptotic behavior of a charged particle for the case of \( D^2 > 4 \). Let us define

\[
\begin{align*}
   x_w(t) &\equiv e^{itW_0} x e^{-itW_0}, \\
   \tilde{x}_\omega(t) &\equiv e^{-it(B_D D_D /T)L} x_w(t) e^{it(B_D D_D /T)L}.
\end{align*}
\]

Then, the straightforward calculation shows

\[
\tilde{x}_\omega''(t) = D^2 \tilde{x}_\omega(t), \quad D = 2B_D C_D /T.
\]

By using this, \( x_w(t) \) can be calculated explicitly, and we have

\[
x_w(t) = (\hat{R}(NB_D D_D)x) \cosh(tD) + (\hat{R}(NB_D D_D)p)(1/C_D) \sinh(tD), \quad (13)
\]

where we define

\[
\hat{R}(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}. \quad (14)
\]

By \((13)\), we notice that the charged particle is in exponentially scattering state for the case where \( D^2 > 4 \).

In this paper, we do not deal with the case where \( D^2 \leq 4 \) by the following reasons:

1. If \( D^2 < 4 \) holds, the charged particle never move out the some compact region by the influence of the magnetic field, and which implies that \( L^2(\mathbb{R}^2) = H_{pp}(U_0(T, 0)) \) holds, where \( H_{pp}(U_0(T, 0)) \) stands for the pure point spectral subspace associated with \( U_0(T, 0) \). This result can be proven by \((13)\) with \( \cosh(it\alpha) = \cos(t\alpha) \) and \( \sinh(it\alpha) = i\sin(t\alpha) \).

2. Let \( D^2 = 4 \). Noting \( C_D = 0 \) if \( D^2 = 4 \), denote \( \hat{P}_0 = -i\partial_t + p^2 + L \). In order to consider the Mourre theory for this operator, we need to obtain the selfadjoint operator \( A \) such that \( i[\hat{P}_0, A] \) is a positive valued operator on the support of the energy cut-off. However, even if the case where \( A = (x \cdot p) (p)^{-2} + (p)^{-2} (p \cdot x) \), which was introduced by Yokoyama \([15]\), to induce the positivity of commutator \( i[\hat{P}_0, A] \) is difficult since the operator \( L \) is unbounded operator. In addition to this, the scattering theory for the case \( D^2 = 4 \) with \( n = 2 \) was not proven in the both of the papers \([1]\) and \([9]\).

Be based on these reasons, we impose the following assumption on magnetic field :
**Assumption 1.5.** Suppose that $\zeta_1(t)$, $\zeta_2(t)$, $\zeta'_1(t)$ and $\zeta'_2(t)$ in (6) are continuous functions in $t$ and that the discriminant of the fundamental solutions of Hill’s equations (6) satisfy $D^2 = (\zeta_1(T) + \zeta'_2(T))^2 > 4$. Furthermore, $\zeta_2(T) \neq 0$ holds.

By the virtue of the result of Floquet, see e.g. Magnus and Winkler [10], see also Colombini and Spagnolo [4], the following Lemma holds:

**Lemma 1.6.** If $D^2 > 4$ holds, then the solution $\zeta_0(t)$ of (6) is expressed by $\zeta_0(t) = e^{\tilde{\lambda}t}\tilde{\chi}_1(t) + e^{-\tilde{\lambda}t}\tilde{\chi}_2(t)$ for $\tilde{\lambda} \neq 0$ and periodic or antiperiodic functions $\tilde{\chi}_1$ and $\tilde{\chi}_2$.

By virtue of (12), $\lambda$ and $\chi_j$ in (5) will be calculated more precisely in §3. What we emphasize here is that, we study the Mourre theory by using four quantities $\zeta_j(T)$ and $\zeta'_j(T)$ with $j \in \{1, 2\}$. Indeed, as the subconsequent of Lemma 1.4, we have the following theorem;

**Theorem 1.7.** Suppose Assumption 1.5 then for all $t \in [NT, (N + 1)T)$ the solutions of Hill’s equation $\zeta_1(t)$ and $\zeta_2(t)$ can be written as

$$
\zeta_1(t) = (\cosh(NTD) + (2AD/C_D)\sinh(NTD))\zeta_1(t - NT) + ((C_D\sinh(NTD) - (4A_D^2/C_D)\sinh(NTD))/m)\zeta_2(t - NT),
$$

$$
\zeta_2(t) = ((m/C_D)\sinh(NTD))\zeta_1(t - NT) + ((\cosh(NTD) - (2AD/C_D)\sinh(NTD)))\zeta_2(t - NT),
$$

$$
\zeta'_1(t) = (\cosh(NTD) + (2AD/C_D)\sinh(NTD))\zeta'_1(t - NT) + ((C_D\sinh(NTD) - (4A_D^2/C_D)\sinh(NTD))/m)\zeta'_2(t - NT),
$$

$$
\zeta'_2(t) = ((m/C_D)\sinh(NTD))\zeta'_1(t - NT) + (\cosh(NTD) - (2AD/C_D)\sinh(NTD))\zeta'_2(t - NT).
$$

and

$$
D = 2BD_C/D/T.
$$

Proof of this theorem can be seen in §3.3. Here we remark that the fundamental discriminant is calculated concretely, see e.g. Hodchstadt [7] and Xu [13].

**Remark 1.8.** By using $D$, $\zeta_j(T)$ and $\zeta'_j(T)$, the conditions of a magnetic field in [9] (= r.h.s. of (5)) can be written $D > 2$, $\zeta_2(T) \neq 0$ and $\zeta'_2(T) = a_3$ or $D < -2$, $\zeta_2(T) \neq 0$ and $\zeta'_2(T) = -b_3$, see §3.

Let us define $J_D(t)$ as

$$
J_D(t) = U_0(t, 0)e^{-iADx^2}e^{itW_0}.
$$

Then, by (12),

$$
J_D(t + T) = U_0(t, 0)U_0(T, 0)e^{-iADx^2}e^{iTW_0}e^{itW_0} = J_D(t)
$$

holds where we use $e^{2\pi iL} = \text{Id}$. Here we redefine $U_0(t, 0)$ as

$$
U_0(t, 0) = J_D(t)e^{-itW_0}e^{iADx^2},
$$

5
and define $\mathcal{W}(t,0)$ as a propagator for
\[ \hat{W}(t) = W_0 + \mathcal{D}(t)V \mathcal{D}(t), \tag{17} \]
where $V$ is a multiplication operator by a real-valued function $V(x)$, i.e., which satisfies $(V\phi)(x) = V(x)\phi(x)$ for all $\phi \in L^2(\mathbb{R}^2)$. Moreover, let us define $U(t,0)$ as
\[ U(t,0) = \mathcal{D}(t)\mathcal{W}(t,0)e^{iA_{D}x^2}. \]
Then, paying attention to the condition
\[ \frac{d}{dt}\mathcal{W}(t,0) = \hat{W}(t)\mathcal{W}(t,0), \]
we have
\[ \frac{d}{dt}U(t,0) = \left( \left( \frac{d}{dt}\mathcal{D}(t) \right) + \mathcal{D}(t)(W_0)\mathcal{D}(t)^* \right)U(t,0) + VU(t,0) \]
\[ \quad = (H_0(t) + V)U(t,0). \]
Thus, the operator $U(t,0)$ is a propagator for $H(t) = H_0(t) + V$, and uniqueness and the existence of the propagator $U(t,0)$ are guaranteed under the following assumption 1.9.

**Assumption 1.9.** Let $V$ satisfies $(V\phi)(x) = V(x)\phi(x)$ for all $\phi \in L^2(\mathbb{R}^2)$, and $V(x) \in C^2(\mathbb{R}^2)$ satisfies that for some $\gamma_j > 0$, $j \in \{0, 1, 2\}$, there exists $\tilde{C}_j > 0$ such that
\[ |\nabla^j V(x)| \leq \tilde{C}_j \langle x \rangle^{-\gamma_j}, \quad \langle x \rangle = (1 + x^2)^{1/2} \]
holds.

The uniqueness and the existence of $\mathcal{W}(t,0)$ are guaranteed under the assumption 1.9 too.

**Remark 1.10.** In order to consider the effect of magnetic field, we need to suppose that the charged particle moves in dimension $n = 2$ only. But the same problems for Harmonic-oscillator with time-periodic coefficients $h(t)$ case, the Hamiltonian of this system is described by
\[ H_{h,0}(t) = \frac{p^2}{(2m)} + h(t)^2x^2/(8m), \tag{18} \]
should be considered in general dimension $n$. For $H_{h,0}(t)$, by regarding $L = 0$ in $H_0(t)$, above-mentioned results Theorem 1.2, Lemma 1.4 and Proposition 1.3 and the following results Theorem 1.11 and Corollary 1.12 can be proven by the same way under the Assumption 1.5 and Assumption 1.9 with replacement $n = 2 \rightarrow n = n$ and $qB(t) \rightarrow h(t)$.

Now we define the Floquet Hamiltonian of time-periodic magnetic fields. The basic formulation $\hat{P}_0 = -i\partial_t + H_0(t)$ was considered by [9] and [1]. However, for the sake of constructing the Mourre theory, we construct it thorough [12] with a little technical approach.

Let $T$ be a torus $T = \mathbb{R}/(T\mathbb{Z})$ and the unitary groups $\mathcal{L}_0^\sigma$ and $\mathcal{L}^\sigma$, acting on $\mathcal{K} := L^2(T; L^2(\mathbb{R}^2))$, as follows:
\[ \begin{cases} 
(\mathcal{L}_0^\sigma \phi)(t) = U_0(t, t - \sigma)\phi(t - \sigma), \\
(\mathcal{L}^\sigma \phi)(t) = U(t, t - \sigma)\phi(t - \sigma), \quad \phi(t) \in \mathcal{K}.
\end{cases} \]
Thus, by Stone's theorem, we have selfadjoint operators $\hat{L}^\sigma_0$ and $\hat{L}^\sigma$ such that
\[
\begin{align*}
(\hat{L}^\sigma_0^* \hat{L}^\sigma_0 \phi)(t) &= U_0(t, t - \sigma_2)U_0(t - \sigma_2, t - \sigma_1 - \sigma_2)\phi(t - \sigma_1 - \sigma_2) = (\hat{L}^\sigma_0^* \hat{L}^\sigma_0 \phi)(t), \\
(\hat{L}^\sigma^* \hat{L}^\sigma \phi)(t) &= U(t, t - \sigma_2)U(t - \sigma_2, t - \sigma_1 - \sigma_2)\phi(t - \sigma_1 - \sigma_2) = (\hat{L}^\sigma^* \hat{L}^\sigma \phi)(t).
\end{align*}
\]
Thus, by Stone's theorem, we have selfadjoint operators $\hat{H}_0$ and $\hat{H}$ be such that
\[
\begin{align*}
(e^{-i\sigma \hat{H}_0 \phi})(t) &= (\hat{L}^\sigma_0 \phi)(t) = U_0(t, t - \sigma)\phi(t - \sigma), \\
(e^{-i\sigma \hat{H} \phi})(t) &= (\hat{L}^\sigma \phi)(t) = U(t, t - \sigma)\phi(t - \sigma).
\end{align*}
\]
Now, by (16), $\mathcal{J}_D(t + T) = \mathcal{J}_D(t)$ holds, and which implies that, for all $\phi \in \mathcal{X}$, $(\mathcal{J}_D \phi) \in \mathcal{X}$ holds, and hence we have that $\mathcal{J}_D(t)$ is an unitary operator on $\mathcal{X}$. Here, we further define $\mathcal{M}^\sigma_0$ and $\mathcal{M}^\sigma$ as
\[
\begin{align*}
(\mathcal{M}^\sigma_0 \phi)(t) &= (\mathcal{J}_D^* \mathcal{L}^\sigma_0 \mathcal{J}_D \phi)(t) = \mathcal{J}_D^*(t)e^{-i\sigma \hat{H}_0}(\mathcal{J}_D \phi)(t), \\
(\mathcal{M}^\sigma \phi)(t) &= (\mathcal{J}_D^* \mathcal{L}^\sigma \mathcal{J}_D \phi)(t) = \mathcal{J}_D^*(t)e^{-i\sigma \hat{H}}(\mathcal{J}_D \phi)(t).
\end{align*}
\]
Straightforward calculation shows
\[
\begin{align*}
(\mathcal{M}^\sigma_0 \phi)(t) &= e^{-i\sigma \mathcal{W}_0 \phi}(t - \sigma), \\
(\mathcal{M}^\sigma \phi)(t) &= \mathcal{W}(t, t - \sigma)\phi(t - \sigma).
\end{align*}
\]
These equations yield two selfadjoint operators $\mathcal{W}_0$ and $\mathcal{W}$ such that
\[
\begin{align*}
(e^{-i\sigma \mathcal{W}_0 \phi})(t) &= (\mathcal{M}^\sigma_0 \phi)(t) = \mathcal{W}_0(t, t - \sigma)\phi(t - \sigma), \\
(e^{-i\sigma \mathcal{W} \phi})(t) &= (\mathcal{M}^\sigma \phi)(t) = \mathcal{W}(t, t - \sigma)\phi(t - \sigma).
\end{align*}
\]
By using these equations, we have
\[
e^{-i\sigma \mathcal{W}_0} = \mathcal{J}_D^*e^{-i\sigma \hat{H}_0} \mathcal{J}_D, \quad e^{-i\sigma \mathcal{W}} = \mathcal{J}_D^*e^{-i\sigma \hat{H}} \mathcal{J}_D,
\]
and
\[
\mathcal{J}_D^* \hat{H}_0 \mathcal{J}_D = \hat{W}_0, \quad \mathcal{J}_D^* \hat{H} \mathcal{J}_D = \hat{W}
\]
Sum of all results, the Floquet Hamiltonian in this model can be described by
\[
\hat{W} = -i\partial_t + \mathcal{W}_0 + \mathcal{J}_D^* \mathcal{V} \mathcal{J}_D, \quad \text{on } \mathcal{X}.
\]
Throughout the followings, we denote $\hat{W}_0 = -i\partial_t + \mathcal{W}_0$ and $\hat{V} = \mathcal{J}_D^* \mathcal{V} \mathcal{J}_D$. The Mourre theory for $\mathcal{W}_0$ with respect to $L \equiv 0$ was proven by [2]. The difference between the Hamiltonian we consider and the Hamiltonian considered by [2] is whether the potential is a pseudo-differential operator or a usual multiplication operator, and hence we need to impose the more stronger assumption on $\mathcal{V}$ than [2] in order to deduce Mourre estimate.
Theorem 1.11. Suppose Assumption \[1.5\] and Assumption \[1.9\] For all \( \phi \in \mathcal{H} \), we define the unitary operators \( J \) and \( K \) as follows

\[
(J \phi)(t, x) = J_D(t) \phi(t, x), \quad (K \phi)(t, x) = e^{i(C_D/2)x^2} e^{i(1/(4C_D))p^2} \phi(t, x),
\]

and we further define \( \varphi \in C_0^\infty(\mathbb{R} \setminus \sigma_{pp}(\hat{H})) \), and a conjugate operator \( \hat{A} \) as follows

\[
\hat{A} = JK \mathcal{A}_0 K^* J^*, \quad \mathcal{A}_0 = (T/(4B_D C_D)) (\log \langle x \rangle - \log \langle p \rangle).
\]

Then, the Mourre estimate holds, i.e., there exist \( \delta > 0 \) and a compact operator \( \mathcal{K}_0 \) such that

\[
(i|\hat{H}, \hat{A}|\varphi(\hat{H})\phi, \varphi(\hat{H})\phi) \geq \delta\|\varphi(\hat{H})\phi\|^2 + (\mathcal{K}_0 \varphi(\hat{H})\phi, \varphi(\hat{H})\phi)
\]

holds.

Corollary 1.12. Under assumption \[1.5\] and Assumption \[1.9\] \( \hat{H} \) has at most countable pure point spectrum and which singular continuous spectrum is empty.

Remark 1.13. The author could not prove so-called propagation estimate by using the Mourre estimate and could not include the Columb type potential in Assumption \[1.9\] by several reasons.

2 Proof of theorem \[1.2\]

We find \( a(t), b(t), c(t) \) and \( d(t) \) be such that

\[
i \frac{d}{dt} e^{-ia(t)x^2} e^{-ib(t)p^2} e^{-id(t)L} e^{-ic(t)x^2} = H_0(t) e^{-ia(t)x^2} e^{-ib(t)p^2} e^{-id(t)L} e^{-ic(t)x^2}.
\]

Let \( \mathcal{K}(t) = e^{-ia(t)x^2} e^{-ib(t)p^2} e^{-id(t)L} e^{-ic(t)x^2} \), and then

\[
i \frac{d}{dt} \mathcal{K}(t) = \left\{ a'(t)x^2 + b'(t)(p + 2a(t)x)^2 + c'(t)e^{-ia(t)x^2}(x - 2b(t)p)^2 e^{-ib(t)p^2} + d'(t)L \right\} \mathcal{K}(t)
\]

\[
= \left\{ 2\{ 4a(t)b'(t) - 4b(t)c'(t) \} (1 - 4a(t)b(t)) \right\} A + \left\{ b'(t) + 4b(t)^2 c'(t) \right\} p^2
\]

\[
+ \left\{ a'(t) + 4a(t)^2 b'(t) + c'(t)(1 - 4a(t)b(t)^2) \right\} x^2 + d'(t)L \right\} \mathcal{K}(t)
\]

holds, where \( A = x \cdot p + p \cdot x \). This equation yields the following differential equations

\[
\begin{aligned}
\text{(eq1)} & \quad b'(t) + 4b(t)^2 c'(t) = 1/(2m) \\
\text{(eq2)} & \quad 4a(t)b'(t) - 4b(t)c'(t) + 16a(t)b(t)^2 c'(t) = 0 \\
\text{(eq3)} & \quad a'(t) + 4a(t)^2 b'(t) + c'(t) - 8a(t)b(t)c'(t) + 16a(t)^2 b(t)^2 c'(t) = q^2 B(t)^2 / (8m) \\
\text{(eq4)} & \quad d'(t) = -qB(t)/(2m).
\end{aligned}
\]

Combining (eq3) and (eq2), we have

\[
a'(t) + c'(t) - 4a(t)b(t)c'(t) = a'(t) - \left( -4b(t)c'(t) + 16a(t)b(t)^2 c'(t) \right) / (4b(t))
\]

\[
= a'(t) + a(t)b'(t)/b(t) = (qB(t))^2 / (8m).
\]
Therefore, \( a'(t)b(t) + a(t)b'(t) = (qB(t))^2b(t)/(8m) \) holds. This equation yields
\[
a(t) = \frac{1}{8mb(t)} \int_0^t (qB(s))^2b(s)ds. \tag{19}
\]
Furthermore, Combining (eq1), (eq2) and (19),
\[
c'(t) = \frac{a(t)}{2mb(t)} = \frac{1}{16mb^2(t)^2} \int_0^t (qB(s))^2b(s)ds \tag{20}
\]
holds. Using this equation and (eq1), we also have
\[
b'(t) + \frac{1}{4m^2} \int_0^t (qB(s))^2b(s)ds = \frac{1}{2m}.
\]
This equation yields the following Hill’s equation
\[
b''(t) + \left(\frac{h(t)}{2m}\right)^2 b(t) = 0, \quad b(0) = 0, \quad b'(0) = \frac{1}{2m}.
\]
Thus, noting (19), \( b(t) \) can be written as
\[
b(t) = \zeta_2(t)/(2m).
\]
By using this, we obtain
\[
a(t) = \frac{1}{8m\zeta_2(t)} \int_0^t (qB(s))^2\zeta_2(s)ds = \frac{m}{2\zeta_2(t)} \int_0^t \left(\frac{qB(s)}{2m}\right)^2\zeta_2(s)ds = \frac{-m}{2\zeta_2(t)} \int_0^t \zeta_2''(s)ds
\]
\[
= \frac{m}{2} \left(1 - \frac{\zeta_2'(t)}{\zeta_2(t)}\right).
\]
Here we note that the Wronskian of solutions of Hill’s equation (19) is constant, in particular,
\[
\zeta_1(t)\zeta_2''(t) - \zeta_1'(t)\zeta_2(t) = 1
\]
holds for all \( t \in \mathbb{R} \). Then, (20) can be denoted by
\[
c'(t) = \frac{m}{2} \left(1 - \frac{\zeta_2'(t)}{\zeta_2(t)^2}\right) = \frac{m}{2} \left(-\left(\frac{\zeta_1(t)}{\zeta_2(t)}\right)' + \left(\frac{1}{\zeta_2(t)}\right)'ight).
\]
Therefore
\[
c(t) = \frac{m}{2} \left(\frac{1 - \zeta_1(t)}{\zeta_2(t)}\right) - \frac{m}{2} \left(\frac{1 - \zeta_1(0)}{\zeta_2(0)}\right)
\]
holds. Here we put
\[
\zeta_1(t) = \rho(t)\cos(\eta(t)), \quad \zeta_2(t) = \rho(t)\sin(\eta(t)), \quad \rho(0) = 1, \quad \rho > 0, \quad \eta(0) = 0,
\]
then it can be deduced that
\[
\rho''(t) - \rho(t)^{-3} + \left(\frac{qB(t)}{2m}\right)^2 \rho(t) = 0, \quad \eta(t) = \int_0^t \rho(s)^{-2}ds.
\]
see e.g. [10]. Therefore,

\[
\frac{1 - \zeta_1(0)}{\zeta_2(0)} = \frac{\sin^2(\eta(0)/2)}{\cos(\eta(0)/2) \sin(\eta(0)/2)} = \tan(\eta(0)/2) = 0
\]

holds. This implies

\[
c(t) = \frac{m}{2} \left( \frac{1 - \zeta_1(t)}{\zeta_2(t)} \right).
\]

By proving the following lemma, we have \( K(t) \) is a propagator for \( H_0(t) \) and hence we denote \( K(t) \) by \( U_0(t, 0) \) after this Lemma.

**Lemma 2.1.** Let \( K(t) = -ia(t)x^2e^{-ib(t)p^2}e^{i\Omega(t)L}e^{-ic(t)x^2} \). Then, for every \( t \in \mathbb{R} \), \( K(t) \) satisfies the **domain invariant property**:

\[
K(t)\mathcal{D}(p^2 + x^2) \subset \mathcal{D}(p^2 + x^2).
\]

Moreover, for all \( \phi \in L^2(\mathbb{R}^2) \) and \( t \in \mathbb{R} \),

\[
\lim_{\varepsilon \to 0} \|K(t + \varepsilon) - K(t)\phi\|_{L^2(\mathbb{R}^2)} = 0
\]

holds.

**Proof.** First, we shall prove the domain invariant property. It can be calculated that

\[
\|x^2K(t)u\|_{L^2(\mathbb{R}^2)} = \|(x + b(t)p)^2e^{-ic(t)x^2}u\|_{L^2(\mathbb{R}^2)} = \|((1 - 2b(t)c(t))x + b(t)p)^2u\|_{L^2(\mathbb{R}^2)}.
\]

Since \( \zeta_1(t) \) and \( \zeta_2(t) \) are continuous, we have that both terms

\[
b(t) = \zeta_2(t)/(2m), \quad 1 - 2b(t)c(t) = 1 - (1 - \zeta_1(t))/2
\]

are bounded for any fixed \( t \), and hence we have \( K(t)\mathcal{D}(p^2 + x^2) \subset \mathcal{D}(x^2) \). Next, we shall consider \( p^2K(t) \). By

\[
\|p^2K(t)u\|_{L^2(\mathbb{R}^2)} = \|(p - 2a(t)x)^2e^{-ib(t)p^2}e^{-ic(t)x^2}u\|_{L^2(\mathbb{R}^2)}
\]

and that

\[
1 - 4a(t)b(t)b(t) = \zeta'_2(t),
\]

\[
-2a(t) - 2c(t) + 8a(t)b(t)c(t) = \frac{m}{\zeta_2(t)} \left( \zeta_1(t)\zeta'_2(t) - 1 \right) = \frac{m\zeta'_1(t)\zeta_2(t)}{\zeta_2(t)} = m\zeta'_1(t),
\]

we have \( K(t)\mathcal{D}(p^2 + x^2) \subset \mathcal{D}(p^2) \) by using \( \zeta'_1(t) \) and \( \zeta'_2(t) \) are bounded for every fixed \( t \), where we use \( \zeta_1\zeta'_2 - \zeta'_1\zeta_2 = 1 \). Consequently, we have \( K(t)\mathcal{D}(p^2 + x^2) \subset \mathcal{D}(p^2 + x^2) \).
At last, we prove (21). For all \( \phi \in C_{0}^{\infty}(\mathbb{R}^2) \), denote \( u(t) = \|K(t)\phi\|_{L^2(\mathbb{R}^2)} \). Clearly, we can see that the \( u(t) \) is a continuous function on \( \mathbb{R} \setminus \mathcal{A}(t) \) with

\[
\mathcal{A}(t) = \{ t \in \mathbb{R}; \zeta_2(t) = 0 \}.
\]

Hence it is enough to prove \( u(t) \) is a continuous function on the neighborhood of \( t \in \mathcal{A}(t) \) in the followings. Noting that \( \zeta_1 \zeta_2' - \zeta_1' \zeta_2 = 1 \) and \( \zeta_1(t) \) is a continuous function in \( t \), \( \zeta_1(t_0) \) and \( \zeta_2'(t_0) \) are not to be 0 if \( t_0 \in \mathcal{A}(t) \). Then, we have that, for all \( \epsilon > 0 \) and \( t_0 \in \mathcal{A}(t) \), there exists \( \delta_0 > 0 \) such that

\[
\sup_{t \in [t_0 - \epsilon, t_0 + \epsilon]} \left| \zeta_1(t) \right|^{-1} < \delta_0
\]

holds. By (7), for all \( \phi \in C_{0}^{\infty}(\mathbb{R}^2) \),

\[
e^{-ia(t)x^2}e^{-ib(t)p^2}e^{i\Omega(t)L_0}e^{-ic(t)x^2}\phi(x) = e^{-ia(t)x^2}\frac{1}{4\pi ib(t)} \int e^{i(x-y)^2/(4b(t))}e^{-ic(t)y^2}\phi(\tilde{y}(t))dy
\]

holds, where we denote

\[
\tilde{y}(t) := \begin{pmatrix} \cos (\Omega(t)) & \sin (\Omega(t)) \\ -\sin (\Omega(t)) & \cos (\Omega(t)) \end{pmatrix} y = \tilde{R}(\Omega(t))y.
\]

Noting that \( 1/(4b(t)) - c(t) = m\zeta_1(t)/(2\zeta_2(t)) \),

\[
- \frac{x \cdot y}{2b(t)} + \left( \frac{1}{4b(t)} - c(t) \right)y^2 = \frac{m\zeta_1(t)}{2\zeta_2(t)} \left( y - \frac{x}{\zeta_1(t)} \right)^2 - \frac{mx^2}{2\zeta_1(t)\zeta_2(t)}
\]

holds. By the above equation and (22), we have

\[
e^{-ia(t)x^2}e^{-ib(t)p^2}e^{i\Omega(t)L_0}e^{-ic(t)x^2}\phi(x) = \frac{m}{2\pi \zeta_2(t)} e^{ig_1(t)x^2} \int e^{im(\tilde{z}-z)^2/(2\zeta_1(t)\zeta_2(t))}\phi(\tilde{z}(t)/\zeta_1(t))dz,
\]

where

\[
g_1(t) = - \left( a(t) - \frac{1}{4b(t)} + \frac{m}{2\zeta_1(t)\zeta_2(t)} \right) = \frac{-m}{2\zeta_1(t)\zeta_2(t)}(1 - \zeta_1(t)\zeta_2'(t)) = \frac{m\zeta_1'(t)}{2\zeta_1(t)}.
\]

Consequently, we have

\[
e^{-ia(t)x^2}e^{-ib(t)p^2}e^{i\Omega(t)L_0}e^{-ic(t)x^2}\phi(x) = \zeta_1(t)^{-1}e^{im\zeta_1'(t)x^2/(2\zeta_1(t))}e^{\zeta_1(t)\zeta_2(t)p^2/(2m)}\phi(\tilde{x}(t)/\zeta_1(t))
\]

and \( L^2 \)- norm of the r.h.s. of above equation is clearly continuous at the neighborhood of \( \mathcal{A}(t) \) by (22).
3 Hill’s equation

Now we further assume the intense of the magnetic fields is periodic function be such that \(B(t+T) = B(t)\). Then we can prove that, for all \(t \in [NT, NT + t_0]\), \(N \in \mathbb{N}\) and \(t_0 \in [0, T)\),

\[
U_0(t, 0) = U_0(t, NT)U_0(NT, 0) = U_0(t - NT, 0)(U_0(T, 0))^N
\]

holds. In this section, we rewrite the \(\zeta_j(t), \ j \in \{1, 2\}\) as concrete forms thorough the \(\zeta_1(T), \zeta_2(T)\) and so on. We firstly prove the following Lemma:

**Lemma 3.1.** For some \(\theta_3 \in \mathbb{R}\) and \(\theta_4 \in \mathbb{R}\), denote \(\theta_1(t) = -t\theta_3\) and \(\theta_2(t) = -\theta_3 e^{-4t\theta_3/(4\theta_3)} + \theta_4/(4\theta_3)\). Then

\[
e^{i\theta_1(t)}Ae^{-i\theta_2(t)p^2} = e^{-it(\theta_3A + \theta_4 p^2)}
\]

holds, where \(A = x \cdot p + p \cdot x\).

**Proof.** Since the operator \(\theta_3A + \theta_4 p^2\) is a selfadjoint operator, we only find \(\theta_1(t)\) and \(\theta_2(t)\) such that

\[
i\frac{d}{dt}e^{i\theta_1(t)}Ae^{-i\theta_2(t)p^2} = (\theta_3A + \theta_4 p^2) e^{i\theta_1(t)}Ae^{-i\theta_2(t)p^2} \tag{23}
\]

holds. Then we can conclude that Lemma 3.1 holds by Stone’s theorem. On the other hand, by using the equation

\[
e^{i\theta_1(t)}Ape^{-i\theta_2(t)}A = e^{-2\theta_1(t)}p,
\]

(23) can be proven easily. \(\square\)

By (7) and (8), \(U_0(T, 0)\) is denoted by

\[
U_0(T, 0) = e^{-ia(T)x^2}e^{-ib(T)p^2}e^{i\Omega(T)L}e^{ia(T)x^2}, \quad U_0(NT, 0) = e^{-ia(T)x^2}e^{-iN\theta(T)p^2}e^{iN\Omega(T)L}e^{ia(T)x^2}
\]

in the case where \(D = \zeta_1(T) + \zeta_2(T) = 2\). Hence our first aim of this section is to extend this argument for the case where \(D^2 \geq 4\).

3.1 Proof of Lemma 1.4

At first, we consider the case where \(D \geq 2\). For all \(\phi(x) \in L^2(\mathbb{R}^2_x)\),

\[
U_0(T, 0)\phi(x) = \frac{1}{4\pi ib(T)} \int e^{-ia(T)x^2}e^{i(x-y)^2/(4b(T))}e^{-ic(T)y^2}\phi(\hat{y})dy, \quad \hat{y} = R(\Omega(T))y
\]

holds. Here, denote that

\[
\hat{U}_0(T, 0) = e^{-i\alpha x^2}e^{-ia_1A}e^{-ia_2p^2}e^{i\Omega(T)L}.
\]

Then, by using the equation \(e^{-ia_1A}(\phi(x)) = e^{-2a_1}(\phi(x))\) in dimension \(n = 2\), we have

\[
\hat{U}_0(T, 0)\phi(x) = \frac{e^{-2a_1}}{4i\pi a_2} \int e^{-i\alpha x^2}e^{i(e^{-2a_1}x-y)^2/(4a_2)}e^{i\alpha y^2}\phi(\hat{y})dy.
\]
Here, if the equation $U_0(T,0)\phi = \tilde{U}_0(T,0)\phi$ holds, then we can see
\[
\frac{e^{-2a_1}}{a_2} = \frac{1}{b(T)}, \quad -a_0 + \frac{e^{-4a_1}}{4a_2} = -a(T) + \frac{1}{4b(T)}, \quad a_0 + \frac{1}{4a_2} = \frac{1}{4b(T)} - c(T).
\]
holds. Denoting $a_1 = (-1/2) \log a_3$, $a_3 > 1$, we have $a_2 = b(T)a_3$, $a_0 = a_3/(4b(T)) + a(T) - 1/(4b(T))$ and
\[
a_3^2 - 2(1 - 2(a(T) + c(T))b(T))a_3 + 1 = a_3^2 - Da_3 + 1 = 0,
\]
where we use that $2(a(T) + c(T))b(T) = (2 - \zeta_1(T) - \zeta_2(T))/2 = (1 - D)/2$ by \(S\). Here we put
\[
\Xi_1 = \frac{2b(T)a_3 \log a_3}{a_3^2 - 1}, \quad \Xi_2 = \log a_3 / 2.
\]
Then we have
\[
e^{i(\log a_3)A/2}e^{-ib(T)a_3p^2} = e^{-i(\Xi_1 p^2 - \Xi_2 A)}
\]
by using Lemma 3.1 for $t = 1$. By the simple calculation, we get
\[
\Xi_1 p^2 - \Xi_2 A = \Xi_1 \left( p - \frac{\Xi_2}{\Xi_1} x \right)^2 - \frac{\Xi_2}{\Xi_1} x^2
\]
and that yields
\[
e^{i(\log a_3)A/2}e^{-ib(T)a_3p^2} = e^{-i(2b(T)a_3(\log a_3)p^2/(a_3^2 - 1) - (\log a_3)A/2)}
\]
\[
e^{-i(2b(T)a_3(\log a_3)/(a_3^2 - 1))(p - (a_3^2 - 1)x/(4b(T)a_3)))^2 - ((a_3^2 - 1)/(4a_3 b(T)))^2 x^2}
\]
\[
e^{i((a_3^2 - 1)/(8a_3 b(T)))x^2}e^{-i(2b(T)a_3(\log a_3)/(a_3^2 - 1))(p^2 - ((a_3^2 - 1)/(4a_3 b(T)))^2 x^2)}e^{-i((a_3^2 - 1)/(8a_3 b(T)))x^2}.
\]
Therefore, Lemma 1.4 for the case where $D \geq 2$ is proven.

Now we consider the case $D \leq 2$. Denote
\[
\tilde{U}_0(T,0) = (-1)e^{-ib_0 x^2}e^{-ib_1 A}e^{-ib_2 p^2}e^{ib_3 x^2}e^{i(\Omega(T) + \pi)L},
\]
where $(-1)$ stands for the antiperiodic part. By the same calculation of \([25]\) and \([26]\), we have $e^{i\pi L}\phi(x) = \phi(-x)$. Paying attention to this, we also have
\[
\tilde{U}_0(T,0)\phi(x) = \frac{-e^{-2b_1}}{4i\pi b_2} \int e^{-ib_0 x^2}e^{i(e^{-2b_1}x+y)^2/(4b_2)}e^{ib_3 y^2} \phi(y) dy.
\]
By the same way in the case where $D \geq 2$,
\[
-\frac{e^{-2b_1}}{b_2} = \frac{1}{b(T)}, \quad -b_0 + \frac{e^{-4b_1}}{4b_2} = -a(T) + \frac{1}{4b(T)}, \quad b_0 + \frac{1}{4b_2} = \frac{1}{4b(T)} - c(T)
\]
hold. By denoting $b_1 = (-1/2) \log b_3$, $b_3 > 0$, and using the above equations, we have $b_2 = -b(T)b_3$, $b_0 = -b_3/(4b(T)) + a(T) - 1/(4b(T))$ and
\[
b_3^2 + Db_3 + 1 = 0, \quad b_3 > 1
\]
hold. By using the same way for the case where $D \geq 2$, we obtain the Lemma 1.4.
3.2 Hill’s equation

Denoting

\[ W_0 = (B_D/T)(p^2 - C_D^2x^2 + D_DL), \]

\[ U_0(T,0) \] can be written as

\[ U_0(T,0) = (-1)^{\sigma_D}e^{-iA_Dx^2}e^{-iTW_0}e^{iA_Dx^2}, \]

\[ U_0(NT,0) = (-1)^{\sigma_D}N^2e^{-iA_Dx^2}e^{-iT^2W_0}e^{iA_Dx^2}, \]

where \( A_D, B_D, C_D \) and \( D_D \) are defined in (11). Now we consider the asymptotic behavior of the charged particle \( e^{-it\tilde{W}_0}\phi_0, \phi_0 \in L^2(\mathbb{R}^2) \) as \( t \to \infty \). Let us define the position and the momentum of this particle at the time \( t \) as

\[
\begin{pmatrix} x_w(t) \\ p_w(t) \end{pmatrix} = e^{it\tilde{W}_0} \begin{pmatrix} x \\ p \end{pmatrix} e^{-it\tilde{W}_0}, \quad \begin{pmatrix} \tilde{x}_w(t) \\ \tilde{p}_w(t) \end{pmatrix} = e^{it(B_DD_D/T)L} \begin{pmatrix} x_w(t) \\ p_w(t) \end{pmatrix} e^{-it(B_DD_D/T)L}.
\]

Then straightforward calculation shows

\[
\begin{pmatrix} \tilde{x}_w''(t) \\ \tilde{x}_w'(t) \end{pmatrix} = (2B_DC_D/T)^2 \begin{pmatrix} \tilde{x}_w(t) \\ \tilde{p}_w(t) \end{pmatrix}
\]

and

\[
\begin{aligned}
\quad & \left\{ \begin{array}{l}
x_{L,1}(t) - (B_DD_D/T)x_{L,2}(t) = 0, \\
x_{L,2}(t) + (B_DD_D/T)x_{L,1}(t) = 0,
\end{array} \right. \\
\quad & \left\{ \begin{array}{l}
p_{L,1}(t) - (B_DD_D/T)p_{L,2}(t) = 0, \\
p_{L,2}(t) + (B_DD_D/T)p_{L,1}(t) = 0,
\end{array} \right. \\
\quad & \left\{ \begin{array}{l}
x_{L,1}(t) = \hat{R}(B_DD_D/T)x, \\
p_{L}(t) = \hat{R}(B_DD_D/T)p,
\end{array} \right. \\
\quad & \hat{R}(t) := \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.
\end{aligned}
\]

Here for \( x_L(t) = T(x_{L,1}(t),x_{L,2}(t)) \) and \( p_L(t) = T(p_{L,1}(t),p_{L,2}(t)) \) (25) yields

\[
\left\{ \begin{array}{l}
x_L(t) = \hat{R}(B_DD_D/T)x, \\
p_L(t) = \hat{R}(B_DD_D/T)p,
\end{array} \right. \\
\hat{R}(t) := \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.
\]

Define

\[ D = 2B_DC_D/T, \]

and then \( x_w(t) \) and \( p_w(t) \) can be calculated explicitly by using (24) and (26);

\[
\begin{aligned}
\quad & x_w(t) = x_L(t) \cosh(tD) + (1/C_D)p_L(t) \sinh(tD), \\
\quad & p_w(t) = p_L(t) \cosh(tD) + C Dx_L(t) \sinh(tD).
\end{aligned}
\]

Here, the condition \( D^2 > 4 \) appears naturally by the argument without using \( W_0 \); Let \( \tilde{x}(t) = \tilde{U}_0(t,0)^*xU_0(t,0) \) and \( \tilde{p}(t) = \tilde{U}_0(t,0)^*pU_0(t,0) \) with \( \tilde{U}(t,0) = e^{-it\tilde{W}_0}U_0(t,0), \) then, \( \tilde{x}'(t) = \tilde{p}(t)/m \) and \( \tilde{p}'(t)/m = -(qB(t)/(2m))^2\tilde{x}(t) \) hold and hence we have Hill’s equation

\[
\tilde{x}''(t) + \left( \frac{qB(t)}{2m} \right)^2 \tilde{x}(t) = 0, \quad \tilde{x}(0) = x, \quad \tilde{x}'(0) = p/m.
\]
and differential equation $\ddot{p}(t) = m\ddot{x}(t)$. Using fundamental solutions $\zeta_1$ and $\zeta_2$, we have

$$
\begin{pmatrix}
\dot{x}(t) \\
\dot{p}(t)
\end{pmatrix}
= 
\begin{pmatrix}
\zeta_1(t) & \zeta_2(t)/m \\
m\zeta_1'(t) & \zeta_2'(t)
\end{pmatrix}
\begin{pmatrix}
x(t) \\
p(t)
\end{pmatrix}.
$$

(29)

Thus, by putting

$$
\mathcal{A} = 
\begin{pmatrix}
\zeta_1(T) & \zeta_2(T)/m \\
m\zeta_1'(T) & \zeta_2'(T)
\end{pmatrix},
$$

we have

$$
U_0(NT,0)^{*} \begin{pmatrix} x \\ p \end{pmatrix} U_0(NT,0) = \mathcal{A}^N \begin{pmatrix} \hat{R}(\Omega(NT))x \\ \hat{R}(\Omega(NT))p \end{pmatrix}.
$$

and hence we can see that the asymptotic behavior of the particle $U(NT,0)\phi$ is determined by the absolute value of eigenvalues of $\mathcal{A}$. Saying concretely, let $\lambda_0 \in \mathbb{R}$ is the solution of

$$
det(\mathcal{A} - \lambda_0) = \lambda_0^2 - (\zeta_1(T) + \zeta_2'(T))\lambda_0 + 1 = 0.
$$

Here we use $\zeta_1\zeta_2' - \zeta_1'\zeta_2 = 1$. Then, the charged particle has only bound state if and only if $(\zeta_1(T) + \zeta_2'(T))^2 < 4$, the charged particle act like linearly uniform motion if and only if $(\zeta_1(T) + \zeta_2'(T))^2 = 4$ and the charged particle is in exponentially scattering state if and only if $(\zeta_1(T) + \zeta_2'(T))^2 > 4$.

### 3.3 Solutions of Hill’s equation

We calculate the explicit forms of fundamental solutions by using $a(T)$, $b(T)$, $c(T)$, $\zeta_j(t - NT)$ and $\zeta_j'(t - nT)$ in this subsection. Let us define

$$
\begin{pmatrix}
x(t) \\
p(t)
\end{pmatrix}
= U_0(t,0)^{*} \begin{pmatrix} x \\ p \end{pmatrix} U_0(t,0)
$$

(30)

with $t \in [NT, (N + 1)T]$. Here, we see that $U_0(t,0) = U_0(t - NT,0)U_0(NT)$ and

$$
\begin{pmatrix}
x(t - NT) \\
p(t - NT)
\end{pmatrix}
= 
\begin{pmatrix}
\zeta_1(t - NT) & \zeta_2(t - NT)/m \\
m\zeta_1'(t - NT) & \zeta_2'(t - NT)
\end{pmatrix}
\begin{pmatrix}
\hat{R}(\Omega(t - NT))x \\
\hat{R}(\Omega(t - NT))p
\end{pmatrix}.
$$

(31)

Now we calculate

$$
\begin{pmatrix}
x_N \\
p_N
\end{pmatrix}
= U_0(NT,0)^{*} \begin{pmatrix} x \\ p \end{pmatrix} U_0(NT,0).
$$

explicitly. We know that $U_0(NT,0) = e^{-iADx^2}e^{-iNTW_0}e^{iADx^2}$. Thus, by putting $\cosh(NTD) = S_N$ and $\sinh(NTD) = S_N$,

$$
\begin{pmatrix}
x_N \\
p_N
\end{pmatrix}
= (-1)^{\sigma DN} e^{-iADx^2}e^{iNTW_0} \begin{pmatrix}
\hat{x} \\
\hat{p} - 2AD\hat{x}
\end{pmatrix} e^{iNTW_0}e^{iADx^2}
$$

$$
= (-1)^{\sigma DN} e^{-iADx^2} \begin{pmatrix}
\hat{x}C_N + (S_N/C_D)\hat{p} \\
(C_DS_N - 2AD\hat{x}) + (C_N - 2ADSN/C_D)\hat{p}
\end{pmatrix} e^{iADx^2}
$$

$$
\equiv (-1)^{\sigma DN} \begin{pmatrix}
A_{1,N} & A_{2,N} \\
A_{3,N} & A_{4,N}
\end{pmatrix} \begin{pmatrix}
\hat{x} \\
\hat{p}
\end{pmatrix}
$$
hold by \((27)\) and \((28)\), where \(\hat{\xi} = \hat{R}(BD_DN)x\), \(\hat{p} = \hat{R}(BD_DN)p\) and

\[
\begin{align*}
A_{1,N} &= \cosh(NTD) + (2A_D/C_D) \sinh(NTD) \\
A_{2,N} &= (1/C_D) \sinh(NTD) \\
A_{3,N} &= C_D \sinh(NTD) - (4A_D^2/C_D) \sinh(NTD) \\
A_{4,N} &= \cosh(NTD) - (2A_D/C_D) \sinh(NTD).
\end{align*}
\]

Thus, we obtain

\[
\begin{pmatrix}
x(t) \\
p(t)
\end{pmatrix} = U_0(NT,0)^* \begin{pmatrix}
\zeta_1(t-NT) & \zeta_2(t-NT)/m \\
m\zeta_1'(t-NT) & \zeta_2'(t-NT)
\end{pmatrix} \begin{pmatrix}
\hat{R}(\Omega(t-NT))x \\
\hat{R}(\Omega(t-NT))p
\end{pmatrix} U_0(NT,0)
= (-1)^{\sigma_D N} \begin{pmatrix}
\zeta_1(t-NT) & \zeta_2(t-NT)/m \\
m\zeta_1'(t-NT) & \zeta_2'(t-NT)
\end{pmatrix} \begin{pmatrix}
A_{1,N} & A_{2,N} \\
A_{3,N} & A_{4,N}
\end{pmatrix} \begin{pmatrix}
\hat{R}(\Omega(t-NT))\hat{x} \\
\hat{R}(\Omega(t-NT))\hat{p}
\end{pmatrix}
= (-1)^{\sigma_D N} \begin{pmatrix}
B_{1,N}(t) & B_{2,N}(t) \\
B_{3,N}(t) & B_{4,N}(t)
\end{pmatrix} \begin{pmatrix}
\hat{R}(\Omega(t)x) \\
\hat{R}(\Omega(t)p)
\end{pmatrix},
\]

where

\[
\begin{align*}
B_{1,N}(t) &= A_{1,N}\zeta_1(t-NT) + (A_{3,N}/m)\zeta_2(t-NT) \\
B_{2,N}(t) &= A_{2,N}\zeta_1(t-NT) + (A_{4,N}/m)\zeta_2(t-NT) \\
B_{3,N}(t) &= mA_{1,N}\zeta_1'(t-NT) + A_{3,N}\zeta_2'(t-NT) \\
B_{4,N}(t) &= mA_{2,N}\zeta_1'(t-NT) + A_{4,N}\zeta_2'(t-NT).
\end{align*}
\]

On the other hand, we see that

\[
\begin{pmatrix}
x(t) \\
p(t)
\end{pmatrix} = \begin{pmatrix}
\zeta_1(t) & \zeta_2(t)/m \\
m\zeta_1'(t) & \zeta_2'(t)
\end{pmatrix} \begin{pmatrix}
\hat{R}(\Omega(t)x) \\
\hat{R}(\Omega(t)p)
\end{pmatrix},
\]

hold and which yields

\[
\begin{align*}
\zeta_1(t) &= B_{1,N}(t), \\
\zeta_1'(t) &= B_{3,N}(t)/m, \\
\zeta_2(t) &= mB_{2,N}(t), \\
\zeta_2'(t) &= B_{4,N}(t),
\end{align*}
\]

for all \(t \in [NT, (N + 1)T]\). Indeed, \(\zeta_1(t)\zeta_2'(t) - \zeta_1'(t)\zeta_2(t) = 1\) holds by

\[
A_{1,N}A_{4,N} - A_{2,N}A_{3,N} = \cosh^2(NTD) - \sinh^2(NTD) = 1.
\]

We now rewrite the condition of the magnetic fields of \((9)\) by using \(D\), \(\zeta_j(T)\) and \(\zeta_j'(T)\). If \(\zeta_1(t) = e^{-\lambda t}\chi_1(t), \lambda > 0\) holds, it must be proven that \(A_{1,N} = A_{3,N} = O(e^{-\lambda N})\). Suppose that \(\zeta_2(T) \neq 0\), which yields \((1/C_D) \neq \infty\). Here paying attention to

\[
NTD = 2NC_DBD = \begin{cases}
N \log (a_3) > 0, & D > 2, \\
-N \log (b_3) < 0, & D < -2,
\end{cases}
\]

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we have

\[ A_{1,N} = \begin{cases} 
  a_3^N (1 + 2A_D/C_D)/2 + O(a_3^{-N}), & D > 2, \\
  b_3^N (1 - 2A_D/C_D)/2 + O(b_3^{-N}), & D < -2, 
\end{cases} \]

Thus, we have \( A_{1,N} < Ce^{-\lambda N} \) with \( \lambda > 0 \) if \( \zeta'_2(T) = a_3 \) or \( \zeta'_2(T) = -b_3 \) holds.

**Remark 3.2.** In the case where \( D = 2 \),

\[(1/C_D) \sinh(NTD) = (2a_3b(T)(a_3^N + 1)/(a_3^N (a_3 + 1)))(a_3^{N-1} + a_3^{N-2} + ... + a_3 + 1) \]

\[\rightarrow 2Nb(T) \quad \text{as} \quad a_3 \to 1\]

holds. The case where \( D = -2 \) can be calculated by the same way.

## 4 Relative compactness of resolvent of Floquet Hamiltonian

Now we prove so-called relative compactness of weighted resolvent. Let us define \( X_f(z) = f(\tilde{H}_0 - z)f \) with \( (f\phi)(t, x) = f(|x|)\phi(t, x), \phi \in \mathcal{H} \) and \( z \in \mathcal{C}\setminus\mathcal{R} \) where \( f \in C_0^\infty(\mathcal{R}) \) satisfies \( f(s) = 1 \) if \( s \leq R_0 \) and \( f(s) = 0 \) if \( s > R_0 + 1 \) for some \( R_0 > 0 \). Here we prove that \( X_f(z) \) is a compact operator on \( \mathcal{H} \) for all \( \text{Im} \, z \in \mathcal{R}\setminus\{0\} \) and \( \text{Re} \, z \in \mathcal{R} \). By the same argument of Yajima [14],

\[ (X_f(z)\phi)(t, x) = if(|x|) \sum_{N=1}^\infty \int_0^T e^{i(t+NT-s)x}U_0(t + NT, s)(f\phi)(s)ds \]

\[+ if(|x|) \int_0^t e^{i(t-s)x}U_0(t, s)(f\phi)(s)ds \]  \hspace{1cm} (35)

holds. Here we supposed that \( \text{Im} \, z > 0 \). For the case of \( \text{Im} \, z < 0 \) can be calculated by the same way. By Theorem [12] we see that

\[ fU_0(\tau, s)(f\phi)(s, x) \]

\[= e^{-ia(\tau)x^2}e^{i(\Omega(\tau) - \Omega(s))L}f(|x|)e^{-ib(\tau)p^2}e^{-i(c(\tau) - c(s))x^2}(e^{ib(s)p^2}f(|x|)e^{-ia(s)x^2})\phi(s, x) \]

\[= e^{i\tilde{\Omega}L}e^{-ia(\tau)x^2}f(|x|) \lim_{\epsilon \to 0} \frac{1}{(4\pi i)^2b(\tau)b(s)} \int e^{i|x-z|^2/(4\epsilon)}e^{-ez^2}e^{-i\tilde{c}(\tau,s)z^2} \int e^{-i(z-y)^2/(4b(s))}g(s, y)dydz \]

where \( \tilde{\Omega} = \Omega(\tau) - \Omega(s), \tilde{c}(\tau, s) = c(\tau) - c(s) \) and \( g(s, y) = e^{-ia(s)y^2}f\phi(s, y) \). After the simple calculation, we also see that

\[ (fU_0(\tau, s)f\phi)(\tau, x) = e^{i\tilde{\Omega}L}e^{-ia(\tau)x^2}f(|x|) \lim_{\epsilon \to 0} \frac{1}{(4\pi i)^2b(\tau)b(s)} \int g(s, y)dy \int e^{g_2(x, y, z)}dz \]

with

\[g_2(x, y, z) \]

\[= -\left( \epsilon + i \frac{1}{4} \left( \frac{1}{b(s)} - \frac{1}{b(\tau)} \right) + i\tilde{c}(\tau, s) \right) z^2 + i \frac{1}{2} \left( \frac{x}{b(\tau)} - \frac{y}{b(s)} \right) \cdot z + i \frac{1}{4} \left( \frac{x^2}{b(\tau)} - \frac{y^2}{b(s)} \right). \]
Define $\Gamma_1(\tau, s) = \zeta_1(s)/\zeta_2(s) - \zeta_1(\tau)/\zeta_2(\tau)$. Then by the definitions of $b(\tau)$ and $c(\tau)$, $g_2(x, y, z)$ can be calculated that

$$
g_2(x, y, z) = -\left(\varepsilon + \frac{im\Gamma_1(\tau, s)}{2}\right) \left(z + \frac{i}{2(\varepsilon + im\Gamma_1(\tau, s)/2)} \left(\frac{mx}{\zeta_2(\tau)} - \frac{my}{\zeta_2(s)}\right)^2\right)
- \frac{1}{4} \left(\varepsilon + \frac{im\Gamma_1(\tau, s)}{2}\right)^{-1} \left(\frac{mx}{\zeta_2(\tau)} - \frac{my}{\zeta_2(s)}\right)^2 + \frac{mi}{2} \left(\frac{x^2}{\zeta_2(\tau)} - \frac{y^2}{\zeta_2(s)}\right).
$$

Here, we put

$$
\Gamma_2(\tau, s, x, y) = \frac{i}{2m\Gamma_1(\tau, s)} \left(\frac{mx}{\zeta_2(\tau)} - \frac{my}{\zeta_2(s)}\right)^2 + \frac{mi}{2} \left(\frac{x^2}{\zeta_2(\tau)} - \frac{y^2}{\zeta_2(s)}\right).
$$

Then, we have

$$
\lim_{\varepsilon \to 0} \int e^{g_2(x,y,z)} dz = \frac{2\pi}{m|\Gamma_1(\tau, s)|} e^{\Gamma_2(\tau, s, x, y)},
$$

and we also have

$$
(fU_0(\tau, s)f\phi)(\tau, x) = \left(\frac{2\pi}{m|\Gamma_1(\tau, s)|}\right) \frac{m^2}{4(\pi i)^2\zeta_2(\tau)\zeta_2(s)} e^{i(\Omega(t)-\Omega(s))_t} e^{-i\alpha(t)x^2} f(|x|) \int e^{\Gamma_2(\tau, s, x, y)} (g(s, y)dy).
\tag{36}
$$

**Theorem 4.1.** Suppose $D^2 \geq 4$. Let $(f\phi)(t, x) = f(|x|)\phi(t, x)$, $\phi \in \mathcal{K}$ and define $X_f$ as

$$
X_f = (f(\hat{H}_0 - z)^{-1})
$$

for all $z \in C\setminus R$. Then $X_f$ is a compact operator on $\mathcal{K}$.

**Proof.** Define

$$
\Gamma_3(\tau, s) = \zeta_2(\tau)\zeta_2(s)|\Gamma_1(\tau, s)|, \quad S^N_1(t, s) = \{(t, s) \in [0, T)^2; |\Gamma_3(t + NT, s)| < \varepsilon\},
$$

$$
S^N_2(t, s) = [0, T)^2 \setminus S^N_1
$$

for sufficiently small $\varepsilon > 0$. Then

$$
\sum_{N=0}^{\infty} \left\|f(|x|) \int_{S^N_1(t,s)} e^{i(t+NT-s)z} U_0(t + NT, s)(f\phi)(s, x) ds \right\|_{\mathcal{K}}
\leq C \sum_{N=0}^{\infty} e^{-|\text{Im } z|NT} |S^N_1(t, s)|_{\mathcal{K}}, \quad |S^N_1(t, s)|_{\mathcal{K}} = \left|\int_{S_1(t,s)} dsdt\right|.
$$

holds. On the other hand, define $\mathcal{I}$ as

$$
(\mathcal{I}\phi)(t, x) \equiv i \sum_{N=0}^{\infty} \int_{S^N_2(t,s)} f(|x|) e^{i(t+NT-s)z} U_0(t + NT, s)(f\phi)(s, x) ds,
$$
and \(I(t, x)\) as

\[
I(t, x) := C_M \sum_{N=0}^{\infty} \int \mathcal{S}_N^N(t, s) e^{i(t+NT-s)x} (\Gamma_3(t + NT, s))^{-1} e^{-ia(t+NT)x^2} f(|x|) \times \int e^{V_2(t+NT,s,y)} e^{ia(s)y^2} f(|y|) ds dy
\]

with \(C_M = -m/(2\pi)\). Noting that \(\Gamma_3(t, s) = |\Gamma_1(t, s)|\zeta_2(t)\zeta_2(s)\) holds, we easily see that

\[
\int_0^T \int I(t, x) dx dt < \infty
\]

by the definition of \(\mathcal{S}_N^N(t, s)\), and it implies \(I\) is a compact operator. By these results, we have that

\[
(X_f(z)\phi)(t, x) = I(t, x)\phi(t, x) + \varepsilon(t, x)\phi(t, x),
\]

where \(\varepsilon(t, x)\) is a bounded operator satisfies \(\|\varepsilon(t, x)\| \leq C \sum_{N=0}^{\infty} e^{-|\text{Im}z|NT} |\mathcal{S}_1^N(t, s)|_\mathscr{L}\). By using the following Lemma 4.2 we have the Theorem 4.1.

**Lemma 4.2.** For all sufficiently small \(\varepsilon > 0\), there exists sufficiently small \(\delta > 0\) such that

\[
|\mathcal{S}_1^N(t, s)|_\mathscr{L} \leq \delta.
\]

holds where \(\mathcal{S}_1^N(t, s)\) and \(|\mathcal{S}_1^N(t, s)|_\mathscr{L}\) are equivalent to those in the proof of theorem 4.1.

**Proof.** We see that

\[
\Gamma_3(t + NT, s) = \zeta_2(t + NT)\zeta_1(s) - \zeta_1(t + NT)\zeta_2(s),
\]

\[
= (mA_{2,N}\zeta_1(t) + A_{4,N}\zeta_2(t))\zeta_1(s) - (A_{1,N}\zeta_1(t) + A_{3,N}/m)\zeta_2(s).
\]

Hence we define that

\[
\mathcal{M}_1(t) = \zeta_2(t + NT)\zeta_1(s) - \zeta_1(t + NT)\zeta_2(s) = \Gamma_3(t + NT, s),
\]

\[
\mathcal{M}_2(t) = \zeta_1(t + NT)\zeta_1(s) + \zeta_2(t + NT)\zeta_2(s),
\]

\[
\mathcal{N}_1(s) = \zeta_2(t + NT)\zeta_1(s) - \zeta_1(t + NT)\zeta_2(s) = \Gamma_3(t + NT, s),
\]

\[
\mathcal{N}_2(s) = \zeta_1(t + NT)\zeta_1(s) + \zeta_2(t + NT)\zeta_2(s),
\]

then simple calculation shows that \(\zeta_1'(t + NT)\zeta_2(t + NT) - \zeta_1(t + NT)\zeta_2'(t + NT) = (A_{1,N}A_{4,N} - A_{2,N}A_{3,N}) = 1\) and

\[
\mathcal{M}_1'(t)\mathcal{M}_2(t) - \mathcal{M}_1(t)\mathcal{M}_2'(t) = (\zeta_1(s)^2 + \zeta_2(s)^2)
\]

\[
\mathcal{N}_1'(s)\mathcal{N}_2(s) - \mathcal{N}_1(s)\mathcal{N}_2'(s) = -(\zeta_1(t + NT)^2 + \zeta_2(t + NT)^2)
\]

holds. By the virtue of the above equations, we have that if \(\mathcal{M}_1(t) = 0\), then, \(\mathcal{M}_1'(t) \neq 0\) holds. That means the zero points of \(\Gamma_3(t + NT, s)\) in \(t\) has at most single multiplicity. On the same arguments for \(\mathcal{N}\) show that the zero point of \(\Gamma(t + NT, s)\) in \(s\) have at most single multiplicity. Thus (38) holds.
5 Mourre theory

In this section, we consider the Mourre estimate for the case where $D^2 > 4$. This discussion is based on [2]. Let us define

$$\hat{W} = -i\partial_t + \Sigma_1 p^2 + \Sigma_2 x^2 + \Sigma_3 L + \Sigma_4 + J^* V J$$

with $\Sigma_1 \Sigma_2 < 0$. Here $L = x_1 p_2 - x_2 p_1$ and $J$ is a unitary operator on $\mathcal{K}$ and satisfies that for some bounded functions $\theta_j(t)$, $j \in \{1,2,3,4\}$,

$$J \left( \begin{array}{c} x \\ p \end{array} \right) J^* = \left( \begin{array}{cc} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{array} \right) \left( \begin{array}{c} x \\ p \end{array} \right).$$

In this section, we suppose that $V$ satisfies Assumption [19]. For time-periodic magnetic fields, we choose $J = \mathcal{J}_D$ and

$$D > 2 \Rightarrow \Sigma_1 = B_D/T, \quad \Sigma_2 = -(B_D C_D^2)/T, \quad \Sigma_3 = -\Omega(T)/T, \quad \Sigma_4 = 0,$$

$$D < -2 \Rightarrow \Sigma_1 = B_D/T, \quad \Sigma_2 = -(B_D C_D^2)/T, \quad \Sigma_3 = -(\Omega(T) + \pi)/T, \quad \Sigma_4 = \pi/T.$$ Denote that $\hat{H} = \hat{H}^{} \hat{W} J^*$ and

$$X_0 = \left( \sqrt{-\Sigma_2/\Sigma_1} \right)/2, \quad X_1 = -\left( \sqrt{-\Sigma_1/\Sigma_2} \right)/4.$$ Let us define $K$ as follows:

$$\hat{K} = e^{i X_1 p^2} e^{-i X_0 x^2} \hat{W} e^{i X_0 x^2} e^{-i X_1 p^2} = -i\partial_t + 2X_0 \Sigma_1 A + \Sigma_3 L + e^{i X_1 p^2} e^{-i X_0 x^2} \hat{J}^* V J e^{i X_0 x^2} e^{-i X_1 p^2}.$$ Here, by [2], one can expect the candidate of conjugate operator $\mathcal{A}_0$ for $\hat{K}$ is

$$\mathcal{A}_0 = (\log \langle x \rangle - \log \langle p \rangle)/(8X_0 \Sigma_1).$$

Let us write $\hat{K} = \hat{K}_0 + \hat{V}$, $\hat{V} = e^{i X_1 p^2} e^{-i X_0 x^2} \hat{J}^* V J e^{i X_0 x^2} e^{-i X_1 p^2}$. Then, paying attention to $[-i\partial_t, \mathcal{A}_0] = [L, \mathcal{A}_0] = 0$, $i[\hat{K}_0, \mathcal{A}_0] = i(x \cdot p + p \cdot x, \log \langle x \rangle - \log \langle p \rangle)/4 = ((p)^2 (p)^{-2} + x^2 (x)^{-2})/2 = 1 - ((p)^{-2} + (x)^{-2})/2$ holds. For all $R \gg 1$, $F(|p^2 + x^2| \leq R)$ is a compact operator on $\mathcal{K}$. Thus, there exists a compact operator $\mathcal{K}_0$ such that

$$i[\hat{K}_0, \mathcal{A}_0] = 1 - F(|p^2 + x^2| \geq R) \langle p \rangle^{-2} + \langle x \rangle^{-2} F(|p^2 + x^2| \geq R) + \mathcal{K}_0$$

holds. By the Gårding inequality, for sufficiently large $R \gg 1$,

$$F(|p^2 + x^2| \geq R) \langle p \rangle^{-2} + \langle x \rangle^{-2} F(|p^2 + x^2| \geq R) < 3/4$$

holds. This equation yields following Mourre estimate; Let us define $\varphi$ as in Theorem [11] and

$$\mathcal{A} = e^{i X_0 x^2} e^{-i X_1 p^2} \mathcal{A}_0 e^{i X_1 p^2} e^{-i X_0 x^2}$$

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Then, we have
\[
(i[\hat{W}_0, \mathcal{A}]\varphi(\hat{H})\phi, \varphi(\hat{H})\phi) = (i[\hat{K}_0, \mathcal{A}]e^{iX_1 p^2} e^{-iX_0 x^2} \varphi(\hat{H})\phi, e^{iX_1 p^2} e^{-iX_0 x^2} \varphi(\hat{H})\phi)
\geq 1/4 + (K_0 e^{iX_1 p^2} e^{-iX_0 x^2} \varphi(\hat{H})\phi, e^{iX_1 p^2} e^{-iX_0 x^2} \varphi(\hat{H})\phi).
\]

Hence, one can see the conjugate operator of \(\hat{H}\) is
\[
\hat{A} = J\mathcal{A} J^* \equiv (\log((\theta_1 x + \theta_2 p)) - \log((\theta_3 x + \theta_4 p)))/\{8X_0 \Sigma_1\},
\]
where we suppose that \(\theta_j\) is defined as bounded function. (in the case of time-periodic magnetic fields, we calculate it in §4.1 and proves that these are bounded function). Indeed, putting \(\psi_0 = \varphi(\hat{H})\phi\), we can obtain
\[
(i[\hat{H}, \hat{A}] \psi_0, \psi_0) = (Ji[\hat{W}_0, \mathcal{A}] J^* \psi_0, \psi_0)
+ (i[V, \log((\theta_1 x + \theta_2 p)) - \log((\theta_3 x + \theta_4 p))]/\{8X_0 \Sigma_1\}) \psi_0, \psi_0
\geq 1/4 + (K_0 e^{iX_1 p^2} e^{-iX_0 x^2} \psi_0, e^{iX_1 p^2} e^{-iX_0 x^2} \psi_0)
+ (i[V, \log((\theta_1 x + \theta_2 p)) - \log((\theta_3 x + \theta_4 p))]/\{8X_0 \Sigma_1\}) \psi_0, \psi_0
\]
holds. Here we need to prove that \(\varphi(\hat{H}) i[V, \log((\theta_1 x + \theta_2 p)) \varphi(\hat{H})\) and \(\varphi(\hat{H}) i[V, \log((\theta_3 x + \theta_4 p))\varphi(\hat{H})\) are compact operators. We only calculate about \(i[V, \log((\theta_1 x + \theta_2 p))\). Denote \(h_1 \in C_0^\infty(\mathbb{R})\) and \(h_2, h_2^1 \in L^\infty(\mathbb{R})\) as follows
\[
h_1(t) = \begin{cases} 1, & |t| \leq 1/2, \\ 0, & |t| \geq 1, \end{cases} \quad h_2(t) = \begin{cases} 1, & t \leq \sqrt{2}, \\ 0, & t \geq \sqrt{2}, \end{cases} \quad h_2^1(t) = 1 - h_2(t).
\]
Noting that
\[
\varphi(\hat{H}) i[V, \log((\theta_1 x + \theta_2 p))] \varphi(\hat{H}) = (-1/2) \varphi(\hat{H}) i[\log(1 + (\theta_1 x + \theta_2 p)^2)], V] \varphi(\hat{H}),
\]
we define \(L_{R_0}(t) = h_1(t/R_0) \log(1 + t)\) for some \(R_0 > 0\), and apply the Helffer-Sjöstrand formula (Helffer-Sjöstrand [3], see also [2]) to \(L_{R_0}((\theta_1 x + \theta_2 p)^2)\). Then we have
\[
L_{R_0}((\theta_1 x + \theta_2 p)^2) = (2\pi i)^{-1} \int \partial_z l_{R_0}(z)(z - (\theta_1 x + \theta_2 p)^2)^{-1} dz d\bar{z},
\]
where \(l_{R_0}\) stands for an almost analytic extension of \(L_{R_0}\), which is defined as
\[
l_{R_0}(z) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left( \frac{d^n}{d\tau^n} (h_1(\tau/R_0) \log(1 + \tau)) \right) \kappa^n h_1(\kappa/(2 \langle \tau \rangle)), \quad z = \tau + i\kappa.
\]
Simple calculation shows that there exist sufficiently small constant \(0 < \varepsilon_2 \ll 1\) and a constant \(\varepsilon_1 > 0\), which is independent of \(R_0\), such that
\[
|\partial_z l_{R_0}(z)| \leq c_1 \langle z \rangle^{-1-\varepsilon_2} |\text{Im} z |^M, \quad M \in \mathbb{N}
\]
holds. Then
\[
i[L_{R_0}((\theta_1 x + \theta_2 p)^2/R_0), V] = \frac{\theta_2}{2\pi i} \int \partial_z l_{R_0}(z)(z - (\theta_1 x + \theta_2 p)^2)^{-1} \left( (\theta_1 x + \theta_2 p) \cdot \nabla V + \nabla V \cdot (\theta_1 x + \theta_2 p) \right)(z - (\theta_1 x + \theta_2 p)^2)^{-1} dz d\bar{z}
\]

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holds. Here we can obtain that for sufficiently small $0 < \gamma_2 \ll 1$
\begin{equation}
|\nabla V|(z - (\theta_1 x + \theta_2 p)^2)^{-1} \langle x \rangle^{\gamma_2} \leq C(|\text{Im } z|^{-1} + |\text{Im } z|^{-1 - \gamma_2/2}(1 + \langle z \rangle^{\gamma_2/2})) \tag{40}
\end{equation}
holds. Indeed, by using
\begin{align*}
(z - (\theta_1 x + \theta_2 p)^2)^{-1} x^2 &= -2\theta_2^2(z - (\theta_1 x + \theta_2 p)^2)^{-2} + x^2(z - (\theta_1 x + \theta_2 p)^2)^{-1} \\
-4i\theta_2 x \cdot (\theta_1 x + \theta_2 p)(z - (\theta_1 x + \theta_2 p)^2)^{-2} - 8\theta_2^2(\theta_1 x + \theta_2 p)^2(z - (\theta_1 x + \theta_2 p)^2)^{-3}
\end{align*}
and
\begin{equation}
|\langle z - t^2 \rangle^{-1} \langle t \rangle| = \left| \langle z - t^2 \rangle^{-1}(h_2(t \langle z \rangle^{-1/2}) + h_2(t \langle z \rangle^{-1/2})) \langle t \rangle \right| \\
\leq \sqrt{2}(\langle z \rangle^{1/2} |\text{Im } z|^{-1} + \langle z \rangle^{-1/2}), \tag{41}
\end{equation}
we can get
\begin{align*}
\left\| \nabla V |^{2/\gamma_2}(z - (\theta_1 x + \theta_2 p)^2)^{-1} \langle x \rangle^{2} \right\|_{\mathcal{X}} &\leq C \left( |\text{Im } z|^{-1} + |\text{Im } z|^{-2}(1 + \langle z \rangle) \right) \tag{42} \\
\left\| \nabla V|^{0}(z - (\theta_1 x + \theta_2 p)^2)^{-1} \langle x \rangle^{0} \right\|_{\mathcal{X}} &\leq |\text{Im } z|^{-1}. \tag{43}
\end{align*}
By interpolating (42) and (43), we have (40). By (39), (40) and (41), one can obtain that
\begin{equation}
\left\| \partial_z R_0(z) \right\| \langle (\theta_1 x + \theta_2 p) \rangle |(z - (\theta_1 x + \theta_2 p)^2)^{-1}| \left\| \nabla V|^{2/\gamma_2}(z - (\theta_1 x + \theta_2 p)^2)^{-1} \langle x \rangle^{\gamma_2} \right\|_{\mathcal{X}} \\
\leq C \langle z \rangle^{\varepsilon_2 - 7/2 + \gamma_2/2} |\text{Im } z|^{-1 - 2/\gamma_2} \leq C \langle z \rangle^{-5/2 + \varepsilon_2} \tag{44}
\end{equation}
for some constant $C > 0$, which is independent on $R_0$. Noting that $\langle z \rangle \geq \tau \geq s/2$ on the support of $h_1^s(\tau/s)$,
\begin{equation}
|h_1(\tau/R_1) - h_1(\tau/R_2)| \leq C \int_{R_2}^{R_1} |s^{-1 - \varepsilon_3} h_1^s(\tau/s)| d\langle z \rangle^{\varepsilon_3}
\end{equation}
holds for some small $\varepsilon_3 > 0$. The above equation and (44) yield
\begin{align*}
(i[L_{R_1},((\theta_1 x + \theta_2 p)^2),V] - i[L_{R_2},((\theta_1 x + \theta_2 p)^2),V]) \langle x \rangle^{\gamma_2} \cdot \langle x \rangle^{-\gamma_2} \psi_0, \psi_0, \rightarrow 0, \quad \text{as } R_1, R_2 \rightarrow \infty,
\end{align*}
by taking $\varepsilon_2 + \varepsilon_3 \leq 1/4$, and hence we have that $i\varphi(H)[V, \log ((\theta_1 x + \theta_2 p))]\varphi(H)$ is a compact operator since $\langle x \rangle^{-\gamma_2} \varphi(H)$ is a compact operator by Theorem 4.1. By (44), one can also obtain that $i[[H,A],A]$ is a bounded operator under Assumption 1.9. Other conditions, which is necessary for to prove the Mourre theory, see e.g. [3], can be proven easily since $i[H,A]$ is a bounded operator.

### 5.1 Calculation of $\theta_j$

Now, we calculate $\theta_j$ defined above, where we suppose $D > 2$. We only need to calculate
\begin{equation}
\left( \begin{array}{c}
x_\theta \\
p_\theta 
\end{array} \right) = J_D e^{i X_0 x^2} e^{-i X_1 p^2} \left( \begin{array}{c}
x \\
p 
\end{array} \right) e^{i X_1 p^2} e^{-i X_0 x^2} J_D^*.
\end{equation}
Denote that
\[ Q_1(t) = \begin{pmatrix} 1 + 4X_0X_1 & -2X_1 \\ -2X_0 & 1 \end{pmatrix}, \]
\[ Q_2(t) = \begin{pmatrix} \cosh(tD) + (2A_D/C_D) \sinh(tD) & (1/C_D) \sinh(tD) \\ 2A_D \cosh(tD) + C_D \sinh(tD) & \cosh(tD) \end{pmatrix}, \]
\[ Q_3(t) = \begin{pmatrix} \zeta'_2(t) & -\zeta_2(t)/m \\ -m\zeta'_1(t) & \zeta_1(t) \end{pmatrix}, \]
\[ \hat{R}_1(t) = \hat{R}(\Omega_1(t)), \quad \Omega_1(t) = \Omega(T)t/T - \Omega(t). \]

Then,
\[ \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix} = Q_3(t)Q_2(t)Q_1(t)\hat{R}_1(t) \] (45)

holds. Indeed, straightforward calculation shows that
\[ e^{-ia(T)x^2}e^{itW_0} \begin{pmatrix} x \\ p \end{pmatrix} e^{-itW_0}e^{ia(T)x^2} = Q_2(t)\hat{R}(\Omega(T)t/T) \begin{pmatrix} x \\ p \end{pmatrix} \] (46)

holds by (27) and (28). Moreover, if the matrix \( Q_4(t) \) and \( \hat{R}_0(t) \) satisfy that
\[ U_0(t,0) \begin{pmatrix} x \\ p \end{pmatrix} U_0(t,0)^* = Q_4(t)\hat{R}_0(t) \begin{pmatrix} x \\ p \end{pmatrix}, \]
then,
\[ Q_4(t)\hat{R}_0(t)U_0(t,0)^* \begin{pmatrix} x \\ p \end{pmatrix} U_0(t,0) = Q_4\hat{R}_0(t)Q_3(t)\hat{R}(\Omega(t)) \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} x \\ p \end{pmatrix} \] (47)

holds. By using (29) and \( \zeta_1\zeta'_2 - \zeta'_1\zeta_2 = 1 \), we have \( Q_4(t)^{-1} = Q_3(t), \hat{R}_0(t) = \hat{R}(\Omega(t))^{-1}. \) By (46) and (47), we obtain (45). For the case where \( D < -2 \) can be calculated by the same way.

A Model

Now we consider the model of the magnetic fields which satisfies Assumption 1.5. For the case where the magnetic fields can be written by \( B(t) = \lambda_0\theta(t) \) with arbitrarily time periodic function \( \theta(t) \) are considered by [4] and it is proven that there exists an interval \( \Lambda \subset (-\infty, \infty) \) such that for all \( \lambda_0 \in \Lambda, \ D^2 > 4 \) holds. However, it is difficult to calculate \( \Lambda \) precisely and to calculate an additional condition \( \zeta_2(T) \neq 0. \)

To avoid this, we consider the case where \( B(t) \) is defined as the following three pulsed magnetic field:
\[
B(t) = \begin{cases} 
B_1 & 0 \leq t \leq T_1 \\
B_2 & T_1 \leq t \leq T_2, \quad 0 < T_1 \leq T_2 < T, \quad B(t + T) = B(t).
\end{cases}
\]
Then fundamental solutions of Hill’s equation (6) are
\[ \zeta_1(t) = \begin{cases} 
\cos(\omega_1 t/2), \\
\cos(\omega_1 T_1/2) \cos(\omega_2(t - T_1)/2) - (\omega_1/\omega_2) \sin(\omega_1 T_1/2) \sin(\omega_2(t - T_1)/2), \\
\alpha_1 \cos(\omega_3(t - T_2)/2) + (2\alpha_2/\omega_3) \sin(\omega_3(t - T_2)/2), \\
(2/\omega_1) \sin(\omega_1 t/2), \\
(2/\omega_1) \sin(\omega_1 T_1/2) \cos(\omega_2(t - T_1)/2) + (2/\omega_2) \cos(\omega_1 T_1/2) \sin(\omega_2(t - T_1)/2), \\
\beta_1 \cos(\omega_3(t - T_2)/2) + (2\beta_2/\omega_3) \sin(\omega_3(t - T_2)/2) 
\end{cases} \]
with
\[ t \in \begin{cases} 
[0, T_1], \\
[T_1, T_2], \\
[T_2, T], 
\end{cases} \]
\[ \omega_1 = qB_1/m, \quad \omega_2 = qB_2/m, \quad \omega_3 = qB_3/m, \quad \alpha_1 = \zeta_1(T_2), \quad \beta_1 = \zeta_2(T_2), \quad \alpha_2 = \zeta_1'(T_2), \quad \beta_2 = \zeta_2'(T_2). \]
Then
\[ \zeta_1(T) + \zeta_2'(T) = (\alpha_1 + \beta_2) \cos(\omega_3(T - T_2)/2) + (2\alpha_1/\omega_3 - \omega_3\beta_1/2) \sin(\omega_3(T - T_2)/2) \]
holds. By denoting
\[ c_1 = \cos(\omega_1 T_1/2), \quad c_2 = \cos(\omega_2(T_2 - T_1)/2), \quad c_3 = \cos(\omega_3(T - T_2)/2), \]
\[ s_1 = \sin(\omega_1 T_1/2), \quad s_2 = \sin(\omega_2(T_2 - T_1)/2), \quad s_3 = \sin(\omega_3(T - T_2)/2), \]
we have
\[ D = \zeta_1(T) + \zeta_2'(T) = \left(2c_1c_2 - \frac{\omega_1^2 + \omega_2^2}{\omega_1\omega_2} s_1 s_2\right) c_3 - \left(\frac{\omega_2^2 + \omega_3^2}{\omega_2\omega_3} c_1 s_2 + \frac{\omega_1^2 + \omega_3^2}{\omega_1\omega_3} s_1 c_2\right) s_3. \]
It is difficult to deduce a general condition such that Assumption 1.3 holds. Thus, we see an example. Set
\[ c_1 = c_2 = c_3 = s_1 = s_2 = s_3 = 1/\sqrt{2}, \]
and then
\[ D \leq 2c_1 c_2 c_3 - 2s_1 s_2 c_3 - 2c_1 s_2 s_3 - 2s_1 c_2 s_3 = -2\sqrt{2} < -2 \]
holds if \( \omega_1\omega_2 > 0, \omega_3\omega_2 > 0 \) and \( \omega_1\omega_3 > 0 \) hold. Furthermore
\[ \zeta_2(T) = (1/\sqrt{2})(1/\omega_1 + 1/\omega_2 + 1/\omega_3 - \omega_2/(\omega_1\omega_3)) \]
holds. Then, for example, by putting \( \omega_1 = \omega_2 = \omega_3, \) we have \( \zeta_2(T) \neq 0. \)

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