NON-PERTURBATIVE SOLUTION OF MATRIX MODELS MODIFIED BY TRACE-SQUARED TERMS

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ABSTRACT

We present a non-perturbative solution of large $N$ matrix models modified by terms of the form $g(\text{Tr } \Phi^4)^2$, which add microscopic wormholes to the random surface geometry. For $g < g_t$ the sum over surfaces is in the same universality class as the $g = 0$ theory, and the string susceptibility exponent is reproduced by the conventional Liouville interaction $\sim e^{\alpha+\phi}$. For $g = g_t$ we find a different universality class, and the string susceptibility exponent agrees for any genus with Liouville theory where the interaction term is dressed by the other branch, $e^{\alpha-\phi}$. This allows us to define a double-scaling limit of the $g = g_t$ theory. We also consider matrix models modified by terms of the form $gO^2$, where $O$ is a scaling operator. A fine-tuning of $g$ produces a change in this operator’s gravitational dimension which is, again, in accord with the change in the branch of the Liouville dressing.
1. Introduction

Large $N$ matrix models [1] have proven to be a remarkable source of information about two-dimensional quantum gravity coupled to conformal matter with $c \leq 1$. They are the only available method for calculating sums over geometries to all orders in the genus expansion [2]. Some matrix model results have been reproduced directly in Liouville gravity [3-6], which gives us confidence that the discretized and continuum approaches describe the same theory.

The matrix models which generate conventional discretized random surfaces are formulated with only single-trace terms in the action, such as $\text{Tr} V(\Phi)$. Such models have been studied thoroughly, and their Liouville formulation is believed to be understood quite well. In gravitational dressing of any operator one encounters a two-fold ambiguity associated with the choice of branch of square root. For agreement with the conventional matrix models this ambiguity is always resolved by picking the branch which is smoothly connected with the semiclassical limit $c \to -\infty$. Therefore, many results here are qualitatively semiclassical [7].

In this paper we study a different class of matrix models whose action, in addition to the single-trace terms, also contains trace-square terms such as $g(\text{Tr} \Phi^4)^2$. Terms of this kind can glue a pair of random surfaces together at a plaquette. This contact may be thought of as a tiny neck (a wormhole), so that the network of touching surfaces may be assigned an overall genus and overall area. It is known that such microscopic wormholes are already abundant in the conventional theory with $g = 0$ [8-10]. By changing $g$ we are essentially changing the weight of some singular geometries in the path integral. It is not surprising, therefore, that a small increase in $g$ does not change the universal properties of the model. If, however, we fine tune $g$ to a finite positive value $g_t$, then the universality class of the large area phase transition changes. This phenomenon was first observed [11] in a modified one-matrix model. It was found that there exists a critical value $g_t$ such that, for $g < g_t$, the large area behavior of the sum over genus zero surfaces gives the string susceptibility exponent $\gamma = -1/2$ characteristic of pure gravity, i.e.

$$F_0(A) \sim A^{-3+\gamma} \sim A^{-7/2}.$$  

For $g > g_t$ one finds branched polymer behavior with $\gamma = 1/2$, which is not very
interesting because it corresponds to degenerate world sheets. Most interestingly, for \( g = g_t \) there exists a new type of critical behavior with string susceptibility exponent \( 1/3 \). This is the first example of a matrix model where new critical behavior occurs due to fine-tuned wormhole weights.

Since ref. [11] a number of other such modified matrix models have been studied [12-16]. In general, as the trace-squared coupling is increased to a critical value \( g_t \), the string susceptibility exponent jumps from some negative value \( \gamma \), found in a conventional matrix model, to a positive value

\[
\bar{\gamma} = \frac{\gamma}{\gamma - 1}.
\] (1.1)

Essentially equivalent results have been obtained without using matrix models, on the basis of direct combinatorial analysis [17]. For a long time the positive values of string susceptibility exponent seemed very puzzling. Recently, however, a simple continuum explanation of these critical behaviors was proposed in ref. [16].

For all the conventional matrix models describing \((p, q)\) minimal models coupled to gravity, the correct scaling follows from the Liouville interaction of the form

\[
\Delta \int d^2\sigma O_{\min} e^{\alpha_+ \phi},
\]

\[
\alpha_+ = \frac{1}{2\sqrt{3}} \left( \sqrt{1 - c + 24h_{\min}} - \sqrt{25 - c} \right) = -\frac{p + q - 1}{\sqrt{2pq}}
\] (1.2)

where \(O_{\min}\) is the matter primary field of the lowest dimension,

\[
h_{\min} = \frac{1 - (p - q)^2}{4pq}.
\] (1.3)

A simple calculation reveals that the string susceptibility exponent is given by

\[
\gamma = 2 + \frac{Q}{\alpha_+},
\] (1.4)

where \(Q = \sqrt{\frac{25 - c}{3}}\). In ref. [16] it was argued that the effect of fine-tuning the touching
interaction is to replace the Liouville potential by

\[
\Delta \int d^2 \sigma O_{\text{min}} e^{\alpha - \phi},
\]

\[
\alpha_- = -\frac{1}{2\sqrt{3}}(\sqrt{1-c+24h_{\text{min}}} + \sqrt{25-c}) = -\frac{p+q+1}{\sqrt{2pq}}.
\] (1.5)

Now the string susceptibility exponent is found to be

\[
\bar{\gamma} = 2 + \frac{Q}{\alpha_-} = \frac{\gamma}{\gamma-1}.
\] (1.6)

This establishes correspondence with the matrix model results, eq. (1.1). Thus, in the Liouville description of the modified matrix models we simply have to pick the other branch of square root in gravitational dressing compared to the conventional matrix models. This proposal has a number of interesting implications that are worth studying. For instance, using the scaling arguments of ref. [6] we find that the modified sum over surfaces of genus \(G\) should obey the scaling law

\[
\frac{\partial^2 \tilde{F}_G}{\partial \Delta^2} \sim \frac{1}{\Delta^{2G+\bar{\gamma}(1-G)}}.
\] (1.7)

In the matrix models this has only been checked for \(G = 0\) and 1. If true for all \(G\), eq. (1.7) implies that it should be possible to define a double scaling limit for the modified matrix models.

1.1. Summary of Results

In this paper we carry out a non-perturbative study of various modified matrix models and confirm that eq. (1.7) is indeed valid. The simplest class of models we investigate are the modified multicritical one-matrix models [12,13],

\[
Z_k = \int \mathcal{D}\Phi e^{-N \left[ \text{Tr} V_k(\Phi) + (c_2 - \lambda)\text{Tr} \Phi^4 - \frac{1}{N} (\text{Tr} \Phi^4)^2 \right]}.
\] (1.8)

The critical potential of the \(k\)-th model with \(g = 0\) is

\[
V_k(\Phi) = \sum_{i=1}^{k} (-1)^{i+1} c_i \Phi^{2i},
\]

where \(c_i\) have been determined in ref. [2]. We choose to study the dependence of the sum over surfaces on a deformation of the potential by a term \(\sim \Phi^4\). For \(g = 0\), it is
known that $\gamma = -1/k$. The universal (leading singular) part of the sum over connected surfaces, $F = \log Z$, is a function of the scaling variable $t \sim (c_2 - \lambda)N^{2/(2-\gamma)}$. In fact, for all $g < g_t$ the sum over surfaces is in the same universality class. For the fine-tuned value $g = g_t$ the string susceptibility exponent jumps to $\bar{\gamma} = 1/(k + 1)$. At this point there exists a critical value $\lambda_c$ such that the universal part of the sum over connected surfaces, $\bar{F}$, is a function of the scaling variable $\bar{t} \sim (\lambda_c - \lambda)N^{2/(2-\bar{\gamma})}$. Our calculations reveal a remarkably simple non-perturbative relation

$$\bar{F}(\bar{t}) = \log \int_{-\infty}^{\infty} dt e^{\bar{t} + F(t)}$$

(1.9)

which connects the double scaling limit of modified matrix models with that of the conventional ones. We will show that this relation is quite general; it applies to all modified one-matrix and two-matrix models with trace-squared terms of the simplest kind, which describe $c < 1$ models coupled to gravity.

There is a way, however, to alter the relation (1.9) by changing the type of trace-squared terms added to the action. We may replace the term which is the square of the lowest dimension operator by the square of some other scaling operator $O$. Adding $gO^2$ to the action and fine-tuning $g$, we obtain a model where the gravitational dimension of the operator $O$ changes from $d$ to

$$\bar{d} = \gamma - d$$

(1.10)

(the string susceptibility exponent remains unchanged). Remarkably, this change of dimension is again reproduced in Liouville theory by a mere change of the branch of gravitational dressing. Namely, if the operator is dressed by $e^{\beta \pm \phi}$ then

$$d = 1 - \frac{\beta_+}{\alpha}, \quad \bar{d} = 1 - \frac{\beta_-}{\alpha}$$

where $e^{\alpha \phi}$ is the dressing of the Liouville potential. The relation (1.10) follows from $\beta_+ + \beta_- = -Q$. This gives us some further evidence in favor of the interpretation in ref. [16].

* For $c = 1$ the relation is somewhat different due to the logarithmic scaling violations.
In the matrix models we, in fact, find the non-perturbative dependence on the coupling constant corresponding to $O$. If we perturb the action of the model by a term $\tau_O O$ then, for $g = 0$, the universal free energy $F$ is a function of $t$ and $t_O = \tau_O N^{2(1-d)/(2-\gamma)}$. For $g = g_t$ we obtain a new universal free energy

$$\bar{F}(t, \bar{t}_O) = \log \int_{-\infty}^{\infty} dt_O e^{t_O \bar{t}_O + F(t,t_O)}$$

(1.11)

where $\bar{t}_O \sim \tau_O N^{2(1-d)/(2-\gamma)}$. Thus, the calculation reduces to an integral over a different coupling constant from that in eq. (1.9). Now it is clear that, by simultaneously adding $n$ different types of trace-squared terms and fine tuning their coefficients we can change the gravitational dimensions of $n$ operators in the theory. The resulting partition function is a transform of the original partition function with respect to corresponding coupling constants $t_i$. The most general relation is

$$\bar{F}((\bar{t}), \{T\}) = \log \prod_{i=1}^{n} \int_{-\infty}^{\infty} dt_i e^{\sum_{j=1}^{n} t_j \bar{t}_j + F(t_i,\{T\})}$$

(1.12)

where $\{T\}$ is some set of other coupling constant that remain unintegrated.

The organization of the rest of the paper is as follows. In section 2 we derive the integration over the coupling constants in eq. (1.9) via a trick familiar from wormhole physics. In section 3 we demonstrate the change of operator gravitational dimensions and derive eqs. (1.11) and (1.12). In section 4 we study the torus free energy which is directly related to the operator content of the theory. We attempt to interpret the modified matrix model results in terms of Liouville theory. We conclude in section 5.
2. Wormholes and Integration over Coupling Constants

In this section we derive non-perturbative results for the sum over surfaces in modified matrix models.

2.1. The one-matrix models

Our method is completely general but we will first illustrate it with the simplest example, the modified $k = 2$ one-matrix model which describes pure gravity. We need to study the following integral over an $N \times N$ hermitian matrix $\Phi$,

$$
Z = \int \mathcal{D}\Phi e^{-N \left[ \text{Tr} \left( \frac{1}{2} \Phi^2 - \lambda \Phi^4 \right) - \frac{g}{2N} (\text{Tr} \Phi^4)^2 \right]}.
$$

(2.1)

A very helpful trick is to rewrite this as

$$
Z = \frac{N}{\sqrt{2\pi g}} \int_{-\infty}^{\infty} dy e^{-\frac{N^2 y^2}{2g}} \int \mathcal{D}\Phi e^{-N \text{Tr} \left( \frac{1}{2} \Phi^2 - (\lambda + y) \Phi^4 \right)}.
$$

(2.2)

This trick is generally useful when the action is perturbed by a square of an operator and has been applied extensively in wormhole physics [18]. In fact, the wormhole combinatorics leading to eq. (2.2) is the same as in four dimensions. Matrix models implement this combinatorics automatically, so that eq. (2.2) can be derived in one line. As in four dimensions we find integration over a coupling constant, which in this case is the quartic coupling. Here we are on a much firmer ground, however, because we know a great deal about the euclidean path integral, which is given by the logarithm of the matrix integral. This will allow us to obtain perfectly explicit and interesting results.

The advantage of representation (2.2) is that we know how to perform the integral over $\Phi$. The remaining integral over $y$ is one-dimensional and can be thought of as an effective theory of baby universes, which lends itself to perturbative expansion around the saddle point. Relying on the well-known solution of the one-matrix model [19, 2],
we have
\[
\log \int \mathcal{D}\Phi e^{-N \operatorname{Tr} \left( \frac{1}{2} \Phi^2 - (\lambda + y) \Phi^4 \right)} = N^2 (-a_1 x + \frac{1}{2} a_2 x^2) + F(x, N^2),
\]
\[
F(x, N^2) = N^2 \left( -\frac{2}{5} a_3 x^{5/2} + \ldots \right) + N^0 \left( -\frac{1}{24} \log x + \ldots \right) + N^{-2} (a_4 x^{-5/2} + \ldots) + \mathcal{O}(N^{-4}),
\]
x = c_2 - (\lambda + y).
\]
\]
\]
(2.3)

We have chosen to separate the free energy into its singular part, $F$, and the leading non-singular parts of order $N^2$ which play an important role in our discussion. From the leading order solution of ref. [19] we know the coefficients
\[
a_1 = 4, \quad a_2 = 576, \quad a_3 = 6144 \sqrt{3}, \quad c_2 = \frac{1}{48}.
\]

In the double scaling limit, $t = x N^{4/5} a_3^{2/5}$ is held fixed so that the subleading parts of $F$ at each order in $N$ become negligible. Thus, in this limit
\[
F(x, N^2) = F(t) = -\frac{2}{5} t^{5/2} - \frac{1}{24} \log t + \frac{7}{2160} t^{-5/2} + \mathcal{O}(t^{-10})
\]
(2.4)

where we neglected $x$-independent additive terms. The expansion of $F(t)$ follows from the fact that $\chi(t) = \frac{df}{dt}$ satisfies the Painlevé equation [2]
\[
\frac{d^2 \chi}{dt^2} = 3(t - \chi^2).
\]

From eqs. (2.2) and (2.3), we have
\[
Z = \frac{N}{\sqrt{2\pi g}} \int_{-\infty}^{\infty} dx e^{f(x)},
\]
(2.5)
\[
f(x) = N^2 \left[ (c_2 - \lambda)^2 + 2x(c_2 - a_1 g - \lambda) - x^2 (1 - a_2 g) \right] + F(x, N^2).
\]

This may be thought of as an effective theory of baby universes. The interaction vertex of $n$ baby universes arises from surfaces with $n$ punctures (each puncture is generated by insertion of operator $\operatorname{Tr} \Phi^4$). The explicit quadratic term in $x$ gives a kind of mass
term (inverse propagator) for the baby universe. Thus, the mass-squared is given by
\[ m^2 = \frac{1}{g} - a_2 \]
with the first contribution coming from the wormhole term in the matrix model, and the second from the degenerate sphere consisting of two plaquettes. We will analyze the three cases where the mass-squared is positive, negative and zero separately. The essential observation is that, since \( f(x) \) is of order \( N^2 \), we may develop a large \( N \) expansion by integrating around the saddle point \( x_s \) given by \( f'(x_s) = 0 \).

Let us now show that, in the massive case \( g < \frac{1}{a_2^2} \) the sum over surfaces is in the same universality class as the \( g = 0 \) theory. Defining
\[ \Delta = \lambda_c - \lambda, \quad \lambda_c = c_2 - a_1g \]
we find that the location of the saddle point is
\[ x_s = \frac{\Delta}{1 - a_2g} + O(\Delta^{3/2}) \]
Here the best way to analyze eq. (2.5) is by shifting the integration variable, \( z = x - \frac{\Delta}{1 - a_2g} \). Discarding some non-singular terms in \( \Delta \), we have
\[
\log Z(\Delta, N^2) = \log \int_{-\infty}^{\infty} N dz \exp \left[ -\frac{N^2(1 - a_2g)}{2g} z^2 + F \left( z + \frac{\Delta}{1 - a_2g}, N^2 \right) \right].
\]
After rescaling the variables, \( t = \frac{\Delta}{1 - a_2g} N^{4/5} a_3^{2/5} \), \( \tilde{z} = z N^{4/5} a_3^{2/5} \), we arrive at
\[
\log Z(\Delta, N^2) = \log \int_{-\infty}^{\infty} N^{1/5} d\tilde{z} \exp \left[ -\frac{N^{2/5} a_3^{-4/5}(1 - a_2g)}{2g} \tilde{z}^2 + F(t + \tilde{z}) \right].
\]
In the double-scaling limit, the gaussian term in the integrand becomes a delta-function. Therefore, the singular part of the sum over surfaces satisfies
\[
\log Z(\Delta, N^2) = F(t),
\]
where \( F(t) \) is given in eq. (2.4). This explicitly establishes the universality for \( g < 1/a_2 \). Thus, in their massive phase, the baby universes are irrelevant.\(^*\)

\(^*\) In section 2.2 we show that this is true is general. The massive baby universes are irrelevant due to the general relation \( d > \gamma/2 \), where \( d \) is the gravitational dimension of the operator inserted by the baby universe, and \( \gamma \) is the string susceptibility. In the case just studied, \( d = 0 \) and \( \gamma = -1/2 \).
For the tachyonic case $g > 1/a_2$, the saddle point near $x = 0$ becomes unstable, but a stable saddle point appears at

$$x_s = \tilde{x} + \alpha \Delta^{1/2} + O(\Delta), \quad \alpha > 0,$$

where $\tilde{x} > 0$ is determined by the equation $f''(\tilde{x}) = 0$. We also deduce that $f'(\tilde{x}) = N^2 \Delta/g$. Expanding $f(x)$ in powers of $\Delta$, we find that the leading singularity in the planar limit is

$$\log Z(\Delta, N^2) \sim N^2 \Delta^{3/2}$$

which is indicative of the branched polymer phase. This behavior is quite generic because it does not depend on the precise form of $F(x, N^2)$.

The massless case $g = 1/a_2$ is critical. Here the position of the saddle point acquires a new scaling,

$$x_s = \left( \frac{\Delta a_2}{a_3} \right)^{2/3} + O(\Delta).$$

Integrating around the saddle point in eq. (2.5), we find

$$\log Z = \left[ f - \frac{1}{2} \log \left( -\frac{f''}{N^2} \right) + \frac{f^{(4)}}{8(f'')^2} - \frac{5}{24} \frac{(f^{(3)})^2}{(f'')^3} \right]_{x=x_s} + O(1/N^4). \quad (2.6)$$

We find that all the terms in this expansion are important. After some calculation, we arrive at

$$\log Z = N^2 \left( \frac{3}{5} \tilde{\Delta}^{5/3} + \ldots \right) + N^0 \left( -\frac{7}{36} \log \tilde{\Delta} + \ldots \right) + N^{-2} \left( \frac{77}{960} \tilde{\Delta}^{-5/3} + \ldots \right) + O(N^{-4}), \quad \tilde{\Delta} = \Delta a_2 a_3^{-2/5}.$$

Thus, the singularity for any genus occurs as $\Delta \to 0$. The structure of the leading singular terms suggests that we may now define double scaling limit by keeping the variable $\bar{t} \sim \Delta N^{6/5}$ fixed. In fact, this follows directly from eq. (2.5). For $g = 1/a_2$ the
singular part of the sum over surfaces, given by $\bar{F} = \log Z$, becomes

$$\bar{F}(\Delta, N^2) = \log \int_{-\infty}^{\infty} dx N e^{N^2 a_2 \Delta x + F(x, N^2)}.$$  \hspace{1cm} (2.7)

Introducing scaling variables

$$t = x N^{4/5} a_3^{2/5}, \quad \bar{t} = \Delta N^{6/5} a_2 a_3^{-2/5},$$

we find that

$$\bar{F}(\Delta, N^2) = \bar{F}(\bar{t}) = \log \int_{-\infty}^{\infty} dt e^{\bar{t} + F(t)}.$$  \hspace{1cm} (2.8)

This is our main result, which establishes a simple relation between the double-scaling limits in the modified and conventional matrix models. It is uncertain whether eq. (2.8) has a truly non-perturbative meaning: there are well-known problems in defining $F(t)$ non-perturbatively. Even if they are overcome, it is not clear if the integral over $t$ will converge. What is certain is that eq. (2.8) determines the sum over modified surfaces of any genus, i.e. it works to all orders of perturbation theory. Using saddle-point techniques, summarized in eqs. (2.5) and (2.6), we may generate the large $\bar{t}$ expansion of $\bar{F}$ directly from the integral representation (2.8) and the large $t$ expansion of $F$.

Eq. (2.8) applies equally well to matrix models with non-symmetric potentials, which are more basic because they do not double the degrees of freedom. For such models we divide $F(t)$ from eq. (2.4) by 2 to obtain the sum over surfaces. After a redefinition of $t$, we find

$$F(t) = -\frac{2}{5} t^{5/2} - \frac{1}{48} \log t + \frac{7}{8640} t^{-5/2} + \mathcal{O}(t^{-10}).$$

Substituting this into (2.8), and generating the saddle-point expansion, we find the modified sum over surfaces for non-symmetric models,

$$F(\bar{t}) = \frac{3}{5} \bar{t}^{5/3} - \frac{13}{72} \log \bar{t} + \frac{257}{3840} \bar{t}^{-5/3} + \mathcal{O}(\bar{t}^{-10/3})$$

where we have explicitly calculated the contributions up to genus 2.
Extension of the methods presented above to the $k$-th multicritical one-matrix model is quite straightforward. First we rewrite eq. (1.8) as

$$Z_k = \frac{N}{\sqrt{2\pi g}} \int_{-\infty}^{\infty} dy e^{-\frac{N^2 y^2}{2g}} \int \mathcal{D}\Phi e^{-N \text{Tr} (V_k(\Phi) + (c_2 - \lambda - y)\Phi^4)}$$  \hspace{1cm} (2.9)$$

Using the variable $x = c_2 - \lambda - y$, we find

$$Z_k = \frac{N}{\sqrt{2\pi g}} \int_{-\infty}^{\infty} dx e^{f_k(x)} ,$$

$$f_k(x) = \frac{N^2}{2g} \left[(c_2 - \lambda)^2 + 2x(c_2 - a_1 g) - x^2(1 - a_2 g)\right] + F_k(x, N^2) ,$$

$$F_k(x, N^2) = -\frac{k}{2k+1} a_3 N^2 x^{(2k+1)/k} + \ldots$$  \hspace{1cm} (2.10)$$

In the double scaling limit of the $g = 0$ theory, $t \sim x N^{2k/(2k+1)}$ is held fixed, and the sum over surfaces is given by $F_k(t)$. Analysis of the saddle-point expansion shows that the theory is in the same universality class for any $g < 1/a_2$.

For $g = 1/a_2$, we instead have

$$F_k(\Delta, N^2) = \log \int_{-\infty}^{\infty} dx N e^{N^2 a_2 \Delta x + F_k(x, N^2)} ,$$

$$\Delta = c_2 - \frac{a_1}{a_2} - \lambda .$$

Introducing scaling variables

$$t = x N^{2k/(2k+1)} a^k_{3/(2k+1)} , \quad \bar{t} = \Delta N^{(2k+2)/(2k+1)} a_2 a_3^{-k/(2k+1)}$$

we arrive at the modified sum over surfaces in the double scaling limit,

$$F_k(\bar{t}) = \log \int_{-\infty}^{\infty} dt e^{\bar{t} + F_k(t)} .$$  \hspace{1cm} (2.11)$$

From the fact that the modified sum over surfaces is a function of $\bar{t}$, and that the original expansion was in powers of $1/N^2$, it follows that the genus $G$ contribution scales.
as $t^{(2k+1)(1-G)/(k+1)}$. Using the fact that, for non-symmetric matrix potentials,

$$F_k(t) = -\frac{k}{2k+1} t^{(2k+1)/k} - \frac{k-1}{24k} \log t + \sum_{j=1}^{\infty} \alpha_j(k) t^{-j(2k+1)/k}$$  \hspace{1cm} (2.12)

we generate the genus expansion of $\bar{F}_k$ with the saddle-point methods,

$$\bar{F}_k(\bar{t}) = \frac{k+1}{2k+1} \bar{t}^{2-\frac{1}{k+1}} - \frac{1}{k+1} \left( \frac{k-1}{24} + \frac{1}{2} \right) \log \bar{t} + O(\bar{t}^{-2+\frac{1}{k+1}}).$$

This confirms the known result [12, 13] that on a sphere the string susceptibility exponent is $\bar{\gamma} = \frac{1}{k+1}$. Thus, the order of the phase transition for planar surfaces has changed from the third to the second. We have also established that the susceptibility exponent at genus $G$ is $\bar{\gamma} + G(2 - \bar{\gamma})$, in agreement with eq. (1.7). Since this genus dependence has such a natural explanation in Liouville theory, it provides a solid argument in favor of the Liouville interpretation of the modified matrix models [16].

### 2.2. The two-matrix models

In this section we consider a more general class of matrix models and show that, with the simplest kind of trace-squared terms, the non-perturbative relation (1.9) applies to them as well.

First we address the Ising model coupled to gravity, which is well known to be described by a two-matrix model [20]. Its simplest modified version was introduced in ref. [16],

$$Z_{\text{Ising}} = \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-N [S(\Phi_1) + S(\Phi_2) + k \text{Tr} \Phi_1 \Phi_2]},$$

$$S(\Phi) = \text{Tr} \left( \Phi^2 - \lambda \Phi^4 \right) - \frac{g}{N} \left( \text{Tr} \Phi^4 \right)^2.$$  \hspace{1cm} (2.13)

We have added trace-squared terms of the form

$$g[\left( \text{Tr} \Phi_1^4 \right)^2 + \left( \text{Tr} \Phi_2^4 \right)^2]$$  \hspace{1cm} (2.14)

which implies that the value of the Ising spin at the two ends of a wormhole is required
to be the same. Rewriting eq. (2.14) as

$$\frac{g}{2} (\text{Tr} (\Phi_1^4 + \Phi_2^4))^2 + \frac{g}{2} (\text{Tr} (\Phi_1^4 - \Phi_2^4))^2$$  \hspace{1cm} (2.15)$$

and applying our trick to each of the two terms, we arrive at

$$Z_{\text{Ising}} = \frac{N^2}{2\pi g} \int_{-\infty}^{\infty} dy dv e^{-\frac{N^2(y^2 + v^2)}{2g}} \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-NS},$$  \hspace{1cm} (2.16)$$

$$S = \text{Tr} [\Phi_1^2 + \Phi_2^2 + k\text{Tr} \Phi_1 \Phi_2 - (\lambda + y)(\Phi_1^4 + \Phi_2^4) - v(\Phi_1^4 - \Phi_2^4)] .$$

We perform the matrix integral first, and save the integrals over $v$ and $y$ until the end. The matrix integral describes the Ising model in magnetic field $v$ [20]. If we tune $k$ to its critical value and define $x = c_2 - \lambda - y$, where $c_2$ is the critical quartic coupling, then

$$\log \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-NS} = N^2(-a_1x + \frac{1}{2}a_2x^2) + F_0(x, N^2) + F_1(x, v, N^2) ,$$

$$F_0(x, N^2) = N^2(-\frac{3}{7}a_3x^{7/3} + \ldots) + N^0(-\frac{1}{12}\log x + \ldots) + \mathcal{O}(N^{-2}) ,$$

$$F_1(x, v, N^2) = \sum_{n=1}^{\infty} \frac{v^{2n}}{(2n)!} \langle (\text{Tr} (\Phi_1^4 - \Phi_2^4))^{2n} \rangle .$$  \hspace{1cm} (2.17)$$

The $v^{2n}$ vertex in the diagrammatic expansion for $v$ is given by the connected correlation function of $2n$ operators $\text{Tr} (\Phi_1^4 - \Phi_2^4)$. This operator is the gravitationally dressed spin field, which is known to have gravitational dimension $1/6$. Therefore,

$$\langle (\text{Tr} (\Phi_1^4 - \Phi_2^4))^2 \rangle = N^2(b + b'x^{2/3} + \ldots) + N^0x^{-5/3} + \ldots$$

The inverse propagator (mass-squared) for $v$ is then given by $\left(\frac{1}{g} - b\right)$. By an explicit calculation, following the results of ref. [20], we determine

$$a_2 = \frac{84024}{625}, \quad b = \frac{41256}{625} .$$

It is important that $a_2 > b$. Let us imagine dialing $g$ up. For $g < 1/a_2$ both the $x$ and $v$ baby universes are massive and contribute only subleading terms to the free energy.

* In the next section we will relax this condition and allow a spin flip when a wormhole is traversed. We will see that more general theories can be constructed this way.
Here we find the same universality class as the unmodified Ising model. For \( g = 1/a_2 \), \( x \) becomes massless and changes the critical behavior. \( v \) is still massive, however, so that the integral over \( v \) has the form

\[
\int_{-\infty}^{\infty} N dv \exp \left[ -\frac{1}{2} N^2 v^2 (a_2 - b) + F_{\text{universal}}(x, v, N^2) \right].
\]

In terms of the rescaled variables \( \tilde{v} = v N^{2(1-d)/(2-\gamma)} \) and \( t \sim x N^{2(2-\gamma)} \), this becomes

\[
\int_{-\infty}^{\infty} N^{\frac{2d-\gamma}{2-\gamma}} d\tilde{v} \exp \left[ -\frac{1}{2} N^{\frac{4d-2\gamma}{2-\gamma}} \tilde{v}^2 (a_2 - b) + F_1(t, \tilde{v}) \right].
\]

If \( 2d - \gamma > 0 \), then in the double scaling limit the gaussian factor becomes a delta function, so that \( \tilde{v} \) is frozen at zero. Since \( F_1(t, 0) = 0 \), the \( \tilde{v} \) integral does not contribute to the effective action for \( x \). Thus, all the dominant terms in \( Z_{\text{Ising}} \) can be calculated from

\[
Z_{\text{Ising}} = \frac{N}{\sqrt{2\pi g}} \int_{-\infty}^{\infty} dx e^{f(x)},
\]

(2.18)

\[
f(x) = \frac{N^2}{2g} \left[ (c_2 - \lambda)^2 + 2x(c_2 - a_1 g - \lambda) - x^2 (1 - a_2 g) \right] + F_0(x, N^2) .
\]

From here on the calculation is analogous to those encountered in the one-matrix model. Introducing scaling variables

\[
\tilde{t} \sim (c_2 - \frac{a_1}{a_2} - \lambda) N^{8/7}, \quad t \sim x N^{6/7}
\]

we arrive at the relation (1.9).

\* This holds here because the gravitational dimension of the spin operator is \( d = 1/6 \) and \( \gamma = -1/3 \). In fact, this holds in general because in all conventional matrix models \( d \geq 0 \) and \( \gamma < 0 \). We conclude that all massive baby universes are irrelevant.
The calculation presented above for the Ising model can be carried over almost verbatim to any modified two-matrix model of the form

\[
Z = \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-N [S_p(\Phi_1) + S_p(\Phi_2) + k \text{Tr} \Phi_1 \Phi_2]},
\]

\[
S_p(\Phi) = \text{Tr} (\Phi^2 - \lambda \Phi^4 + \ldots + \tau \Phi^{2p-2}) - \frac{g}{N} (\text{Tr} \Phi^4)^2.
\]

The parameters of the potential \(S_p(\Phi)\) can be tuned [26] in such a way that this describes an arbitrary minimal model coupled to gravity. For any such model, the modified sum over surfaces \(\bar{F}(\bar{t})\) is related to the conventional sum \(F(t)\) by eq. (1.9). The general validity of this relation implies the generality of eq. (1.1). To show this, note that

\[
\bar{t} \sim \Delta N^2/(2-\gamma), \quad t \sim x N^2/(2-\gamma).
\]

Since \(\bar{t} t \sim \Delta x N^2\), we have

\[
\frac{1}{2-\bar{\gamma}} + \frac{1}{2-\gamma} = 1
\]

from which eq. (1.1) follows. Thus, if the asymptotic expansion of \(F(t)\) is in powers of \(t^{2-\gamma}\), then the asymptotic expansion of \(\bar{F}(\bar{t})\) is in powers of \(\bar{t}^{2-\bar{\gamma}}\).

2.3. \(c = 1\)

One interesting theory that remains to be discussed is the \(c = 1\) model coupled to gravity. We will consider compact target space of radius \(R\), which is described by matrix quantum mechanics at finite temperature [21]. The path integral that generates the sum over touching surfaces is [14,15]

\[
Z = \int \mathcal{D}\Phi(t) e^{-N \int_0^{2\pi R} dt \left[ \text{Tr} \left( \frac{1}{2} \Phi^2 + \frac{1}{2} \Phi^2 - \lambda \Phi^3 \right) - \frac{g}{N} (\text{Tr} \Phi^3)^2 \right]},
\]
with \( \Phi(2\pi R) = \Phi(0) \). Let us introduce the normal mode operators

\[
P = \int_0^{2\pi R} dt \text{Tr} \Phi^3(t),
\]

\[
C_n = \frac{1}{\sqrt{2}} \int_0^{2\pi R} dt \cos \frac{nt}{R} \text{Tr} \Phi^3(t),
\]

\[
S_n = \frac{1}{\sqrt{2}} \int_0^{2\pi R} dt \sin \frac{nt}{R} \text{Tr} \Phi^3(t),
\]

and write the trace-squared term as a sum of squares

\[
\int_0^{2\pi R} dt \left( \text{Tr} \Phi^3 \right)^2 = \frac{1}{2\pi R} \left( P^2 + \sum_{n=1}^{\infty} (C_n^2 + S_n^2) \right)
\]

The operators \( C_n \) and \( S_n \) are known to have gravitational dimension \( d = n/2R \) [22]. Using the by now familiar trick, we introduce a “baby universe variable” for each squared operator in the action to derive

\[
Z \sim \int dy_0 \prod_{n=1}^{\infty} dy_n dz_n e^{-\frac{R N^2}{6}(y_0^2 + y_n^2 + z_n^2)}
\]

\[
\int \mathcal{D}\Phi(t) e^{-N \int_0^{2\pi R} dt \left[ \text{Tr} \left( \frac{1}{2} \dot{\Phi}^2 + \frac{1}{2} \Phi^2 - (\lambda + y_0) \Phi^3 - \sum_{i=1}^{\infty} (y_i C_i + z_i S_i) \right) \right]}
\]

Performing the matrix integral, we get

\[
\log \int \mathcal{D}\Phi(t) e^{-NS} = 2\pi R N^2 (-a_1 x + \frac{1}{2} a_2 x^2) + F_0(x, N^2) + F_1(x, y_n, z_n, N^2),
\]

\[
F_0(x, N^2) = R N^2 \left( \frac{1}{2} a_3 x^2 / \log x + \ldots \right) - \frac{1}{24} \left( R + \frac{1}{R} \right) \log x + \ldots
\]

\[
F_1(x, y_n, z_n, N^2) = \pi R N^2 \sum_{n=1}^{\infty} \left( y_n^2 + \frac{2}{n} \right) (b_n + b_n'(x/|\log x|)^{n/R} + \ldots) + \ldots
\]

We have exhibited the terms in \( F_1 \) that come from the two-point functions of \( C_n \) and \( S_n \). These quadratic terms determine whether the variables \( y_n \) and \( z_n \) become critical simultaneously with the variable \( x = c_2 - \lambda - y_0 \).
From a calculation of the momentum dependence of the puncture two-point function \cite{22}, we have

\[ b_n = \int_{-\infty}^{\infty} ds \frac{s^2}{s^2 + (n/R)^2} f^2(s) \]

where \( f(s) \) is proportional to the Fourier transform of the classical trajectory at the top of the critical potential. Since \( b_n \) is a decreasing function of \( n \), and \( b_0 = a_2 \), we conclude that \( b_n < a_2 \) for all \( n > 1 \). This crucial finding implies that, as the variable \( x \) becomes critical for \( g = 1/a_2 \), all the other baby universe variables are still away from criticality. Since their propagators are massive, their fate is the same as of the variable \( v \) in the Ising case: integrating them out makes no effect on the relevant terms in the effective action for \( x \). For \( g = 1/a_2 \), the important integral over \( x \) reduces to

\[ \bar{F}(\Delta, N^2) = \log \int_{-\infty}^{\infty} dx Ne^{2\pi R N^2 \Delta x + F_0(x, N^2)} \]  

(2.21)

where \( \Delta = a_2(c_2 - \frac{a_1}{a_2} - \lambda) \). While in other models we could express this integral directly in terms of scaling variables, for \( c = 1 \) this is impossible because of the logarithmic scaling violations in \( F_0(x, N^2) \). Actually, the situation turns out to be even simpler than for \( c < 1 \).

In the leading saddle point approximation,

\[ \bar{F}(\Delta, N^2) = 2\pi R N^2 \Delta x_s + F_0(x_s, N^2) , \]

\[ - \frac{\partial F_0}{\partial x}(x = x_s) = 2\pi R N^2 \Delta . \]  

(2.22)

Thus, \( \bar{F} \) is the Legendre transform of \( -F_0 \), with \( 2\pi R N^2 \Delta \) being the conjugate variable of \( x \). The leading order relation between \( \Delta \) and \( x \) is

\[ \Delta \log \Delta \sim x , \]

in agreement with ref. \cite{15}. Our analysis of integration around the saddle point, based on eq. (2.6), indicates that, remarkably, all such corrections are suppressed by powers of \( \log \Delta \). Thus, in the double scaling limit where \( N \Delta \) is kept fixed, eq. (2.22) is exact. This
Legendre transform was introduced in ref. [21] to calculate the sum over “one puncture irreducible” surfaces. It was shown to satisfy a simple equation,

\[
\frac{\partial^2 F}{\partial \Delta^2} = 2\pi R N^2 \tilde{\rho}(\Delta) = RN^2 \left[ -\ln \Delta + \sum_{m=1}^{\infty} \left( 2N \Delta \sqrt{R} \right)^{-2m} f_m(R) \right],
\]

(2.23)

where \(\tilde{\rho}(\Delta)\) is the temperature corrected density of states, and \(\Delta\) is the distance of the Fermi level from the top of the potential. Integrating eq. (2.23), we find

\[
F = \frac{1}{8} \left\{ -(2N \Delta \sqrt{R})^2 \ln \Delta - 2 f_1(R) \ln \Delta + \sum_{m=1}^{\infty} \frac{f_{m+1}(R)}{m(2m+1)} (2N \Delta \sqrt{R})^{-2m} \right\},
\]

(2.24)

where we have exhibited only the terms that survive in the double-scaling limit. It is remarkable that the \(c = 1\) model with fine-tuned wormhole weights directly generates the “one puncture irreducible” surfaces. This model, which has no scaling violations as a function of the area [15], is in many ways simpler and more natural than the conventional \(c = 1\) model.

3. New Gravitational Dimensions

One lesson we can draw from the preceding section is that, even though a given model may have many baby universe integration variables, it is usually the case that only the integral over the lowest dimension coupling affects the critical behavior. It is clear, however, that this cannot be the most general situation. In this section we show how to make other integrations relevant by changing the type of trace-squared terms added to the action.

As an instructive example, let us consider the modified Ising model of section 2.2 with a more general class of trace-squared terms,

\[
g \left( \text{Tr} \left( \Phi_1^4 + \Phi_2^4 \right) \right)^2 + g' \left( \text{Tr} \left( \Phi_1^4 - \Phi_2^4 \right) \right)^2.
\]

(3.1)

For \(g \neq g'\) this introduces a term of the form \(\text{Tr} \Phi_1^4 \text{Tr} \Phi_2^4\) which generates wormholes with opposite values of spin at the two ends. It is not surprising that such processes
can make the integration over the spin field coupling constant relevant. We may, for instance, set $g = 0$ (actually, any $g < 1/a^2$ will do), while fine tuning $g'$ to its critical value. The partition function becomes

$$Z \sim \int_{-\infty}^{\infty} dv e^{-\frac{N^2v^2}{2\sigma}} \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-NS},$$

$$S = \text{Tr} \left[ \Phi_1^2 + \Phi_2^2 + k \text{Tr} \Phi_1 \Phi_2 - \lambda (\Phi_1^4 + \Phi_2^4) - (v + \tau_\sigma)(\Phi_1^4 - \Phi_2^4) \right]$$

where we have introduced coupling constant $\tau_\sigma$ in order to study correlation functions of $\text{Tr} (\Phi_1^4 - \Phi_2^4)$. Defining a shifted variable $u = v + \tau_\sigma$, we perform the matrix integral first and reduce the modified free energy to

$$\bar{F} = F_0(t) + \log \int du f(u),$$

$$f(u) = -\frac{N^2}{2g'}(u^2 - 2u\tau_\sigma + \tau_\sigma^2) + F_1(\Delta, u, N^2),$$

$$F_1 = \frac{1}{2}N^2u^2(b + b'\Delta^{2d-\gamma} + \ldots) + N^2u^4b''\Delta^{4d-2-\gamma} + \ldots$$

where $t \sim \Delta N^2/(2-\gamma)$ and $\Delta = c_2 - \lambda$. The universal part of $F_1$ is a function of $t$ and $t_\sigma = uN^2(1-d)/(2-\gamma)$. In our specific case $\gamma = -1/3$ and $d = 1/6$. If we now fine tune $g' = 1/b$ and introduce the scaling variable

$$\bar{t}_\sigma = b\tau_\sigma N^{2(1-\bar{d})/(2-\gamma)},$$

$$\bar{d} = \gamma - d,$$

then the universal part of the modified free energy is given by

$$\bar{F}(t, \bar{t}_\sigma) = \log \int_{-\infty}^{\infty} dt_\sigma e^{t_\sigma \bar{t}_\sigma + F(t, t_\sigma)}$$

Here $F(t, t_\sigma)$ is the universal part of the conventional sum over surfaces.

Eq. (3.4) implies that the gravitational dimension of the spin field has changed from $d = 1/6$ to $\bar{d} = -1/2$. Thus, in modified matrix models negative dimensions arise naturally. Although we have discussed a specific example, the change of gravitational
dimension given by eq. (3.4) is general. As shown in section 1.1, this formula agrees with the change in dimension caused by changing the branch of Liouville dressing. Therefore, there are serious reasons to believe that such operators, which were previously thought not to exist, are in fact present in the spectra of modified matrix models.

Eq. (3.5) shows that, by a fine-tuning of \( g' \), the coupling constant corresponding to the spin field has been driven to criticality. It is not hard to see that a simultaneous tuning of \( g \) to \( 1/a_2 \) also makes the coupling constant \( t \), corresponding to the puncture operator, critical, so that

\[
\bar{F}(\bar{t}, \bar{t}_\sigma) = \log \int_{-\infty}^{\infty} dt dt_\sigma e^{t \bar{t} + t_\sigma \bar{t}_\sigma + F(t, t_\sigma)} .
\]  

(3.6)

It is now clear that a fine-tuning of \( n \) parameters in the trace-squared terms can result in integration over \( n \) coupling constants, giving the general formula (1.12).

4. Sum over Surfaces of Genus One

In this section we focus on the torus contribution to the free energy which, in any string theory, is directly related to the spectrum. For all conventional matrix models the torus free energy has been successfully reproduced by path integration in Liouville theory [23,24]. For this reason it is particularly interesting to calculate the corresponding quantity in modified matrix models and ask for its continuum interpretation.

For any \((p, q)\) minimal model coupled to gravity the torus free energy is*

\[
F^{G=1}(t) = -\frac{(p - 1)(q - 1)}{24(p + q - 1)} \log t .
\]  

(4.1)

This result was reproduced [23] in Liouville theory with the interaction term of eq. (1.2). Let us now study the modifications to this result due to the integration over coupling constants. Consider, for instance, a model where the gravitational dimension of operator \( O \) has been changed from \( d_O \) to \( \gamma - d_O \). Here the modified sum over surfaces is given

* We are quoting the answer for matrix models with non-symmetric potentials. For corresponding models with symmetric potentials the free energy is doubled.
by eq. (1.11). Setting $\bar{t}_O = 0$ and performing gaussian integration around the saddle point, we find

$$F^{G=1}(t) = F^{G=1}(t) + \frac{1}{2} (\gamma - 2d_O) \log t .$$

(4.2)

This result is puzzling from the point of view of the simplest Liouville approach. Since we have not changed the string susceptibility exponent, it would seem that the Liouville action is still (1.2), and that the calculation of ref. [23] with the result (4.1) should still apply. The matrix model tells us otherwise: the moment we change the gravitational dimension of an operator, the torus free energy receives a correction. We may speculate that in Liouville theory this correction originates from a boundary term in the modular integral, but at the moment we do not know how to derive it directly. In the following, however, we will give a plausibility argument for the presence of the correction found in eq. (4.2).

Our argument is based on the interpretation [25] of the torus free energy in $(p, q)$ models as the sum over zero-point energies of an infinite number of one-dimensional particles (harmonic oscillators). Each oscillator corresponds to an operator in Liouville theory of the form $O e^{\beta \phi}$, where $O$ has dimension $h$ and

$$\beta = -\frac{Q}{2} + \omega ,$$

$$\omega = \sqrt{\frac{Q^2}{4} - 2 + 2h} .$$

$\omega$, which is the “Liouville energy”, gives the frequency of the oscillator. A priori there is a sign ambiguity for $\omega$, but in conventional Liouville theory all $\omega$ are taken to be positive. Each operator contributes zero-point energy $\frac{1}{2} \omega$ to the coefficient of the Liouville volume, $-\log t/\alpha$, where $\alpha = \alpha_+$ from eq. (1.2). Thus,

$$F^{G=1}(t) = -\frac{\log t}{\alpha} \sum_i \frac{1}{2} \omega_i .$$

(4.3)

For the $(p, q)$ model the spectrum of energies is given by [25]

$$\omega_i = \frac{s_i}{\sqrt{2pq}}$$

where $s_i$ are the positive integers not divisible by either $p$ or $q$. After substituting this
into (4.3) and using zeta function regularization of the infinite sum, ref. [25] recovered eq. (4.1).

When we fine tune the theory as in section 4, we replace $\omega_O$ by $-\omega_O$ in the Liouville dressing of operator $O$. If we make the same replacement in eq. (4.3), we arrive at

$$F^{G=1}(t) = F^{G=1}(\bar{t}) + \frac{|\omega_O|}{\alpha} \log \bar{t}.$$  \hspace{0.5cm} (4.4)

Remarkably, since $\gamma - 2d_O = 2\frac{|\omega_O|}{\alpha}$, this agrees with the matrix model result, eq. (4.2)! This suggests a connection between the operator content and the torus free energy in modified matrix models. Our basic premise is that a fine-tuned operator with negative Liouville energy, $-|\omega|$, contributes a negative zero-point energy, $-\frac{1}{2}|\omega|$. This suggests that, from the space-time point of view, a negatively dressed operator is a fermion. It would be interesting to find an explanation for this effect.

Proceeding to other modified matrix models we note that, for the models described by relation (1.9),

$$\bar{F}^{G=1}(\bar{t}) = -\frac{(p - 1)(q - 1)}{24(p + q + 1)} \log \bar{t} - \frac{1}{p + q + 1} \log \bar{t}.$$  \hspace{0.5cm} (4.5)

A naive Liouville calculation with potential (1.5) would give only the first term in the above [16]. However, applying eq. (4.3) with $\alpha = \alpha_-$ from eq. (1.5), and including negative zero-point energy for $O_{\text{min}}$, reproduces the matrix model result.

The second terms in eqs. (4.2) and (4.5) are due to integration around the saddle point in eqs. (1.11) and (1.9). These corrections can be eliminated if we treat the baby universe variables classically, i.e. if we freeze them at their saddle point values. In such a theory $\bar{F}$ would simply be the Legendre transform of $-F$. Of course, this theory does not correspond to the original matrix model with trace-squared terms, but it does have a simple geometric interpretation. It calculates the sum over trees of touching random surfaces (bubbles), with each bubble allowed to have arbitrary genus. In other words, the wormholes are present, but they are not allowed to increase the overall genus. Since from the world sheet point of view this constraint is highly non-local, we do not regard such a theory as natural. Eq. (4.5) shows, however, that the relative importance of surfaces where a wormhole closes the loop decreases with increasing $p$ and $q$. For $c = 1$
the quantum effects associated with the baby universe variables become completely negligible, so that the modified sum over surfaces is simply the Legendre transform, eq. (2.22).

5. Conclusions

Our non-perturbative solutions of matrix models modified by various trace-squared terms strongly suggest that there exists a continuum Liouville formulation of these models. All the modified scaling exponents agree with the idea [16] that a fine-tuning of trace-squared terms changes the branch of Liouville dressing of some operators. Since the new branch of dressing does not have a semiclassical limit, the resulting Liouville theory is more complicated and more interesting than the conventional one. We hope that our matrix model results will provide a useful guide towards such a theory.

The solution of the modified matrix models is also quite interesting in itself because of its connection with general wormhole phenomena in quantum gravity. The microscopic wormholes, introduced by the trace-squared terms, lead to integration over coupling constants, as expected on general grounds. Such integration arises in any theory with bilocal operators in the action. Physical effects of integration over coupling constants have even been found in theory of elasticity, where they change the order of phase transitions* [27]. Our work provides another example of a system where integration over coupling constants introduces a profound change, affecting even the order of the phase transition for planar surfaces. It is interesting to look for other physical systems where coupling constants turn into dynamical variables.

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