RECENT RESULTS ON THE TRANSVERSE LATTICE

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Abstract
We review recent progress of field theory on a transverse lattice including:
realistic calculations of pion structure in large-Nc QCD; fermion doubling and
chiral symmetry; a new strong coupling limit; supersymmetry on the transverse
lattice; lightcone zero-mode analysis.

1. INTRODUCTION
The transverse lattice formulation of gauge theories [1] has been developed in recent years as a tool
for hadronic physics [2]. It has particularly been applied to phenomenological studies of pure glue and
glueballs [3] and mesons [5][6] in the large Nc limit on coarse lattices, in the context of the colour-
dielectric expansion. More recently, as well as refinements of those studies [8], a number a new directions
have been explored for the transverse lattice [9][10][11][12].

2. THE PION AND BEYOND
We first introduce the basic structure common to all transverse lattice approaches. In spacetime we
introduce a square lattice of spacing a in the ‘transverse’ directions \( x = \{x^1, x^2\} \) and a continuum
in the \( \{x^0, x^3\} \) directions. The lightcone coordinates are \( x^\pm = (x^0 \pm x^3)/\sqrt{2} \) and \( x^+ \) is treated as
canonical time; we use indices \( r, s \in \{1, 2\} \) and \( \alpha, \beta \in \{+, -\} \). \( SU(N) \) gauge field degrees of freedom are represented by continuum Hermitian gauge potentials \( A_\alpha(x, x^+, x^-) \) and \( N \times N \) link matrices \( M_r(x, x^+, x^-) \). \( A_\alpha(x) \) and Dirac fermions \( \Psi(x) \) reside on the plane \( x = \) constant, while \( M_r(x) \) is associated with a link from \( x \) to \( x + a\hat{r} \), where \( \hat{r} \) is a unit vector in direction \( r \). \( M_r(x) \) goes from \( x + a\hat{r} \)
to \( x \). These variables map under transverse lattice gauge transformations \( V(x, x^+, x^-) \in SU(N) \) as

\[
\begin{align*}
A_\alpha(x) &\to V(x)A_\alpha(x)V^\dagger(x) + i (\partial_\alpha V(x)) V^\dagger(x) \\
M_r(x) &\to V(x)M_r(x)V^\dagger(x + a\hat{r}) \\
\Psi(x) &\to V(x)\Psi(x). 
\end{align*}
\]

From these fields one can write down gauge invariant actions of the form

\[
L = \sum_x \int dx^- \sum_{\alpha, \beta = +, -} \sum_{r=1,2} - \frac{1}{2G^2} \text{Tr}\{F^{\alpha\beta}(x)F_{\alpha\beta}(x)\} \\
+ \text{Tr}\{\overline{\nabla}_a M_r(x)(\overline{\nabla}^a M_r(x))^\dagger\} \\
- \mu_b^2 \text{Tr}\{M_r M_r^\dagger\} + i \overline{\Psi}(x)\gamma^\alpha(\partial_\alpha + i A_\alpha)\Psi - \mu_f \overline{\Psi}\Psi \\
+ i\kappa_s (\overline{\Psi}(x)\gamma^r M_r(x)\Psi(x + a\hat{r}) - \overline{\Psi}(x)\gamma^r M^\dagger_r(x - a\hat{r})\Psi(x - a\hat{r})) \\
+ \kappa_s (\overline{\Psi}(x)M_r(x)\Psi(x + a\hat{r}) + \overline{\Psi}(x)M^\dagger_r(x - a\hat{r})\Psi(x - a\hat{r})) - \\
\frac{\beta}{N_c a^2} \sum_{r \neq s} \text{Tr}\{M_r(x)M_s(x + a\hat{s})M^\dagger_r(x + a\hat{s})M^\dagger_s(x)\} + \cdots,
\]

where \( F^{\alpha\beta}(x) \) is the continuum field strength in the \( \{x^0, x^3\} \) planes at each \( x \),

\[
\overline{\nabla}_a M_r(x) = (\partial_\alpha + i A_\alpha(x))M_r(x) - i M_r(x)A_\alpha(x + a\hat{r}) 
\]

and \( \overline{\nabla}_a \) is the continuum gauge covariant derivative.
and · · · indicates other gauge-invariant terms, typically at higher order in powers of the fields $M_e$ and $\Psi$.

If we allow only kinetic terms that are quadratic then, in the lightcone gauge $A_\perp = 0$, it is straightforward to formally derive the most general light-cone hamiltonian allowed for this system. One way to truncate this most general hamiltonian for practical applications is to work on coarse lattices, using the colour-dielectric expansion. This assumes the link variables $M_e$ are highly disordered and take values in the space of all complex matrices. This has the advantage that linearized variables are easy to quantize and hadrons are small in lattice units, but leads to complicated effective hamiltonians. Alternatively, as the continuum limit $a \to 0$ is approached, the link variables are forced into the $SU(N_c)$ group manifold and all but a few terms in the action are irrelevant. However, it is difficult to quantize these non-linear variables and hadrons are large in lattice units.

Recent work using the colour-dielectric expansion combined with DLCQ has allowed realistic calculations of the structure of the pion at large $N_c$ \cite{Chakrabarti:2015vct}. The eigenstates of the lightcone hamiltonian are in the form of wavefunctions that can be directly related to many experimentally accessible observables (modulo some possible problems with final state interactions \cite{Broniowski:2015yxa}). The structure function $V_\pi(x)$, elastic form factor $F_E(Q^2)$ and distribution amplitude $\phi_\pi(x)$ of the pion are in reasonable agreement with experimental data for even the simplest truncation of the colour-dielectric expansion. There is an interesting comparison of these results with other theoretical techniques discussed at this workshop. Chiral quark models, which have an essentially structureless pion at low resolution scales, can produce even closer agreement with Drell-Yan data on $V_\pi(x)$ (see talk of Ruiz Arriola and Ref. \cite{RuizArriola:2015tva}). Dyson-Schwinger calculations are somewhat in disagreement with data, but seem more consistent with perturbative QCD arguments for the $x \to 1$ behaviour (see talk of Roberts and Ref. \cite{Roberts:2015yaa}). It is important to clear up this discrepancy — who is right, if anyone?

The corresponding distribution of valence quark helicity, \textit{i.e.} projection of quark spin on the direction of motion of the fast-moving hadron, was calculated for the first time in the pion \cite{Chakrabarti:2015vct}. It was found to have the same shape as $V(x)$ but there is a large probability to find the quark and anti-quark helicities aligned with one another, which is completely different from the picture of the non-relativistic quark model. To investigate simultaneously the transverse and longitudinal structure one can use the impact parameter dependent parton distributions $I(x, 0, b)$, where $b$ is impact parameter, that measure the probability of finding a quark with given Bjorken $x$ and transverse position $b$. These were calculated for the transverse lattice \cite{Broniowski:2015yxa} at particular lattice values of $b$, and are shown in Figure 3. This corresponds to a sharp fall-off of the hadron wavefunction in transverse space at a particular ($x$-dependent) radius. The talk of Broniowski in these proceedings and Ref. \cite{Broniowski:2015yxa} shows that chiral quark models predict something qualitatively similar. It should be possible to calculate this in the Dyson-Schwinger approach also.

It is a challenge to extend this work to the baryons and other mesons. It the case of baryons, since large $N_c$ no longer restricts the number of quarks, one needs a suitable truncation of the Fock space commensurate with the colour-dielectric expansion. In the case of mesons, one needs to understand particularly how the pion-rho splitting arises. The coarse lattice hamiltonian contains spin flip operators, involving link-field emission, that do split the pion and rho, and are presumably a consequence of spontaneous chiral symmetry breaking, but at the expense of also splitting the rho multiplet itself \textit{i.e.} breaking Lorentz invariance. One needs further operators that improve both explicit chiral and Lorentz symmetry. One possibility allowed is four-fermi interactions, since these are renormalisable with respect to the two continuum dimensions $x^\alpha$. They have the potential to split the pion from the rho without splitting the rho itself \cite{Chakrabarti:2015vct}. These would bring the transverse lattice wavefunctions to a form more closely resembling those of the chiral quark models; in particular, they would not vanish at Bjorken $x = 0, 1$.

### 3. FERMION DOUBLING AND CHIRALITY

As with other lattice formulations, the transverse lattice suffers from the fermion doubling problem. This has been recently studied in detail by Chakrabarti \textit{et al.} \cite{Chakrabarti:2015vct}. They find that with symmetric lattice deriva-
Fig. 1: Valence distribution functions $xV_\pi(x)$ compared to pion-nucleon Drell-Yan data. Solid line: transverse lattice result evolved to 6.6 GeV. Data points: E615 experiment [17]. Short-dashed line: NA10 experiment fit to $x^\alpha(1 - x)^\beta$ form [18]. Long-dashed line: NA3 experiment fit to $x^\alpha(1 - x)^\beta$ form [19].
Fig. 2: Pion form factor $F(Q^2)$. Solid curves are the transverse lattice result, gray curve for wavefunctions calculated within a one-link truncation, black curve for a three-link truncation. The experimental data points are from Ref. [20].
tives, in the free field limit fermions on even and odd transverse lattice sites decouple. As a result, four species of fermions appear on the (two-dimensional) transverse lattice as excitations around zero transverse momentum. This is quite different from Euclidean lattice theory, where doublers have at least one momentum component near the edge of the Brillouin zone. They propose a staggered fermion formulation on the lightfront transverse lattice that eliminates two of the species, with the other two interpreted as different flavours. Previous coarse transverse lattice calculations have used Wilson fermions, which also removes doublers [5].

Another possibility, discussed in ref. [9], is to use separate forward and backward derivatives. Introducing the left and right moving components of the fermion field $\psi^\pm = \gamma^\pm \gamma^\pm \Psi$, the transverse gauge kinetic term is

$$i\bar{\psi}^- \gamma_r D^r_l \psi^+ + i\bar{\psi}^+ \gamma_r D^b_r \psi^- ,$$

(4)

where

$$D^l_r \psi(x) \equiv \frac{1}{a} [ M_r(x) \psi(x + a\hat{r}) - \psi(x)] ,$$

$$D^b_r \psi(x) \equiv \frac{1}{a} [ \psi(x) - M_r(x - a\hat{r}) \psi(x - a\hat{r})] .$$

(5)

In the free field limit, for bare fermion mass $m$ with transverse momentum $k = (k_1, k_2)$, this leads to an invariant mass

$$\mathcal{M}^2 = m^2 + \frac{4}{a^2} \sum_r \sin^2 \frac{k_r a}{2} \pm \frac{4m}{a} \sqrt{\sum_r \sin^4 \frac{k_r a}{2}} .$$

(6)

$\mathcal{M}^2 = m^2$ only for the case $k = 0$ and there is no doubling. (The last term above occurs due to a helicity flip term in the hamiltonian).
In all the above cases the question of explicit chiral symmetry breaking arises; the usual theorems lead one to expect it if doublers are removed. One must distinguish between conventional chiral transformations, which act on $\Psi$, and lightcone chiral transformations that act on the physical component $\psi^+$ only. It is often the case that the latter can be a symmetry even in theories that are obviously not chirally symmetric in the usual sense (e.g. free massive fermions), and may even be useful for classifying hadrons in the quark model \[24\]. How to check for conventional explicit chiral symmetry? Chakrabarti et al. suggest an ‘even-odd’ helicity flip transformation on the physical component

$$\psi^+(x^1, x^2) \to (\gamma_1^2 x^1 (\gamma_2^2)^2) \psi^+(x_1, x_2)$$  \(7\)

that is always violated in the cases above. Alternatively one might try to directly check conservation of the usual chiral current \[8\]. Conservation of the ‘1+1’ current

$$j_5^\alpha = \sum_x \Psi \gamma_5 \gamma^\alpha \Psi$$  \(8\)

is a necessary condition for conservation of the four-dimensional current. One cannot check $\partial^\alpha j_5^\alpha = 0$ directly because it contains a bad current $j^-\gamma$ that leads to normal-ordering ambiguities. Suppose however we consider the vacuum-to-one-pseudoscalar-meson matrix element\[1\]:

$$\langle 0 | j_5^\alpha (0) | \text{meson}(P^+, P) \rangle = f P^\alpha \delta(P)$$  \(9\)

The form of the RHS follows from exact 1+1 symmetries. $\partial^\alpha j_5^\alpha = 0$ implies that either $P^- = 0$ or $f = 0$, depending on the meson. The first condition is the massless pion condition. In addition there are an infinite number of further conditions — excitations of the pion must have zero decay constant $f = 0$. By tuning irrelevant operators, for example within the colour-dielectric expansion, one might hope to gradually realise these conditions. Other operators that improve Lorentz symmetry might then gradually move the conditions towards sufficiency for chiral symmetry. This is an interesting idea yet to be attempted in practice.

Some applications of the asymmetric derivatives to meson calculations appeared after the workshop \[21\].

4. STRONG COUPLING LIMIT

An interesting limit of transverse lattice gauge theory has been formulated by Patel and Ratabole \[10\], which is an analogue of the strong coupling limit in conventional lattice theories. They take the link-field $M_r$ in the $SU(N)$ group manifold, as in the continuum limit. $M_r$ can be totally disordered if one turns off all its gauge and self interactions and its kinetic term. Provided the coupling to fermions is linear in $M_r$, which is the case with the simplest lattice discretization of the Wilson-Dirac operator, the link field can be integrated out of the path integral exactly at large $N_c$. This allows derivation of the effective action, quark propagator, chiral condensate, and a boundstate equation for mesons consisting of the ’t Hooft equation modified by a meson transverse hopping via four-fermi interaction:

$$\left[ M^2 - \frac{m^2 - q^2}{2x} \right] \Phi(x) = \frac{1}{2x(1-x)} \left[ \frac{m^2 - q^2}{2x} \gamma^+ + x\gamma^- + m + \frac{g^2 \gamma^+}{2\pi x} \right]$$

$$\times \int \frac{dy}{2\pi} \left\{ g^2 P \frac{\gamma^+ \Phi(x) \gamma^+}{(x - y)^2} - i\kappa^2 \left[ 2\Phi(x) + \sum_{n=1}^{2} \gamma^n \Phi(x) \gamma^n \right] \right\}$$

$$\times \left[ - \left( \frac{M^2}{2} - \frac{m^2 - q^2}{2x} \right) \gamma^+ - (1 - x)\gamma^- + m - \frac{g^2 \gamma^+}{2\pi(1-x)} \right]$$  \(10\)

\[1\]This approach arose in discussions with M. Burkardt during the workshop
The above equation is for states of overall zero transverse momentum, \( m \) is the bare mass, \( g \) the longitudinal gauge coupling, \( \kappa \) the Dirac hopping parameter on the lattice, and the Wilson parameter \( r = 1 \).

The wavefunction \( \Phi \) in this case is a matrix in Dirac space and covers all the possible spin structures of the quark anti-quark pair. This equation has yet to be solved in detail.

A possible generalisation of this idea would treat the link fields \( M_r \) as complex matrices, so that they are already disordered, and include also a potential \( V[M] \) localised on each link that does not propagate links in either transverse or longitudinal directions. Integrating out the link variables would produce a class of generalisations of the 't Hooft model with transverse hopping.

5. SUPERSYMMETRY

It is known that DLCQ in 2 dimensions preserves supersymmetry \(^{25}\). A recent attempt has been made to extend this to higher dimensions using a transverse lattice. Of course, one may not expect to recover exact supersymmetry, but Harada and Pinsky \(^{11}\) have shown, using the SDLCQ technique, that one may preserve half of an \( N = (1, 1) \) supersymmetry in \( 2 + 1 \) dimensions. The fermion \( \Psi \) is now in the adjoint representation. In the lightcone SUSY algebra

\[
\{Q^\pm, Q^\pm\} = 2\sqrt{2} P^\pm, \quad \{Q^+, Q^-\} = 2 P
\] (11)

they propose a charge \( Q^- \) that preserves gauge invariance, whose square defines the hamiltonian \( P^- \):

\[
Q^- = 2^{3/4} g \sum_x \int dx^- \text{Tr}\{J^+(x) \frac{1}{\partial_-} \psi^+(x)\}
\] (12)

\[
J^+(x) = \frac{i}{2g^2 a^2} \left( M_r(x) \partial_- M^\dagger_r(x) + M^\dagger_r(x - a\hat{r}) \partial_- M_r(x - a\hat{r}) \right) + 2\psi^+(x)\psi^+(x)
\] (13)

(In this case \( r = 1 \) is the single transverse direction.) The continuum limit \( a \to 0 \) of \( P^- \sim (Q^-)^2 \) reproduces the correct Super Yang-Mills result. The existence of \( Q^- \) ensures fermion-boson degeneracy in the massive spectrum, at least for \( P = 0 \), but the absence of a suitable gauge-invariant \( Q^+ \) means that the number of massless states do not match. Numerical diagonalisations of the lightcone hamiltonian problem exhibit interesting features, such as a winding mode spectrum around compact transverse directions that varies inversely with winding number. This seems to be a feature of the exact supersymmetry in this case since one might naively expect mass to increase with winding number, as the gauge strings get longer. It may be that the limit of small soft SUSY breaking gives a different result from working exactly at the SUSY point.

6. ZERO MODES

It is well known in the context of the DLCQ regulator, which makes \( x^- \) periodic, that it is not possible to completely reach the lightcone gauge \( A_- = 0 \). The \( x^- \) zero mode of \( A_- \) remains as a dynamical degree of freedom. In all explicit transverse lattice calculations performed so far, this zero mode has simply been dropped as a dynamical approximation. Paston et al. have studied the inclusion of this zero mode in the formalism \(^{12}\). The zero mode \( A^0_- \) can be diagonalised in colour space by a residual gauge choice, then the link fields are expanded in eigen modes of the operator

\[
D_- M_r(x) = [\partial_- - i A^0_-(x) + i A^0_- (x - a\hat{r})] M_r(x)
\] (14)

The state space can then be written as a product of the action of the above eigen modes on a Fock vacuum \( |0\rangle \) and normalisable functionals \( F[A^0_-] \) of classical functions \( A^0_- (x) \). It is not yet clear whether this causes any substantive effect on physical observables compared to a calculation based on the Fock vacuum \( |0\rangle \) alone with suitable renormalisation of couplings.
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