Phase amplitude conformal symmetry in Fourier transforms

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Abstract. For the Fourier transform \( F : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) of a complex-valued even or odd function \( \varphi \), it is found that the amplitude invariance \( |F\varphi| = |\varphi| \) leads to a phase invariance or inversion as \( \arg(F\varphi) = \pm \arg \varphi + \theta \) (\( \theta \) = constant). The converse holds unless \( \arg \varphi \) = constant.

The condition \( |\varphi| = |F\varphi| \) is required in dealing with, for example, the minimum uncertainty relation between position and momentum. Without the evenness or oddness of \( \varphi \), \( |F\varphi| = |\varphi| \) does not necessarily imply \( \arg(F\varphi) = \pm \arg \varphi + \theta \), nor is the converse.

1. Introduction

The relation between the phase and amplitude in the Fourier transform of a function \( \varphi : \mathbb{R}^n \to \mathbb{C} \) has been studied when we deal with, for example, the problem of whether the function \( \varphi \) can be obtained from the amplitude (or magnitude) of its Fourier transform \( \hat{\varphi} \) by retrieving its phase \( \arg(\hat{\varphi}) \). This kind of phase retrieval problem [1, 2] has arisen in many fields, such as optics, electron microscopy, X-ray crystallography. However, due to the lack of uniqueness of \( \arg(\hat{\varphi}) \), some conditions should be imposed on \( \hat{\varphi} \) so as to reconstruct \( \arg(\hat{\varphi}) \) uniquely.

One assumption, which is adopted in many authors [3, 4], is that \( \hat{\varphi} \) is band limited, that is, \( \varphi \) has a compact support \( \Omega \subset \mathbb{R}^n \). In this case, \( \hat{\varphi} \) turns out to be an entire function of a finite order by Paley-Wiener theorem [5]. However, this assumption on \( \varphi \) does not imply the uniqueness of \( \arg(\hat{\varphi}) \) (except a trivial factor). So some further restriction is necessary for the phase retrieval. As such a condition, the irreducibility of \( \hat{\varphi} \) has been known to date [6].

For \( n = 1 \), on the one hand, the irreducibility of \( \hat{\varphi} \) may be too strong, because the entire function of one variable with a finite order can always be factorized using its complex zeros by Hadamard factorization theorem [7]. So the irreducibility indicates that the function \( \hat{\varphi}(z) \) is restricted to such that has only one simple zero, as in the form \( (z - z_0) e^{\omega(z)} \), where \( z_0 \in \mathbb{C} \) and \( \omega(z) \) represents a polynomial. For \( n \geq 2 \), on the other hand, the irreducibility of \( \hat{\varphi} \) is not so strong a condition. This is because the entire function \( \hat{\varphi}(z_1, \ldots, z_n) \) for \( n \geq 2 \) cannot always be factorized using their zeros, so that the \( \hat{\varphi}(z_1, \ldots, z_n) \) in itself is likely to be irreducible.

It is to be mentioned that the analogous characterization of the uniqueness problem of the phase retrieval holds for discrete Fourier transform for \( \varphi : \mathbb{Z}^n \to \mathbb{C} \) and \( \hat{\varphi} : T^n \to \mathbb{C} \) (where \( \mathbb{Z} \) and \( T^n \) represent the set of integers and the \( n \)-dimensional torus, respectively); more generally, for Fourier transform on groups as \( \varphi : G \to \mathbb{C} \) and \( \hat{\varphi} : \Gamma \to \mathbb{C} \) [8] (where \( G \) and \( \Gamma \) represent a (locally) compact abelian group and its dual group, respectively). As in the case of
continuous Fourier transform, \( \text{arg}(\mathfrak{F}\varphi) \) for discrete Fourier transform can be determined from the irreducibility of \( \mathfrak{F}\varphi \) [9, 10] up to a trivial factor.

In this paper, however, we do not assume the irreducibility nor the band limitedness of \( \mathfrak{F}\varphi \). Instead, we will deal with a symmetric relation under the exchange of \( |\varphi| \leftrightarrow |\mathfrak{F}\varphi| \). To be more precise, let \( \mathcal{M} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{R} \) be a functional of \( |\varphi| \) and \( |\mathfrak{F}\varphi| \), and consider the relation of \( \mathcal{M}[|\varphi|, |\mathfrak{F}\varphi|] = 0 \). Such a relation is found, for example, in the position-momentum uncertainty relation (whether it is based on the variance [11, 12] or on the Shannon entropy [13]). In the case of minimum uncertainty, it is required further that the functional form of \( |\mathfrak{F}\varphi| \) should be the same as \( |\varphi| \) (after an appropriate dilation of argument). Although the condition of \( |\mathfrak{F}\varphi| = |\varphi| \) in itself does not necessarily lead to the uniqueness of \( \text{arg}(\mathfrak{F}\varphi) \) (hence that of \( \text{arg} \varphi \)), we can relate \( \text{arg}(\mathfrak{F}\varphi) \) to \( \text{arg} \varphi \) under a certain condition.

The aim of the present paper is to examine how the amplitude invariance under Fourier transform reflects on the phase invariance (or inversion), and vice versa. In Sect. 2, we give some preliminaries of the Fourier transform \( \mathfrak{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) of a function \( \varphi : \mathbb{R} \to \mathbb{C} \). From a physical point of view, we assume that \( \varphi \) is continuous on the real axis, so that \( \varphi \) can be expanded in terms of the Hermite polynomials. In Sect. 3, we show that for \( \varphi \) being an even or odd function, the condition of \( |\mathfrak{F}\varphi| = |\varphi| \) leads to \( \text{arg}(\mathfrak{F}\varphi) = \pm \text{arg} \varphi + \text{constant} \). In Sect. 4, we show the converse holds in an analogous way. In Sec. 5, we make a simple generalization of the phase amplitude relation obtained in Sects. 3 and 4. Conclusion is given in Sec. 6.

2. Preliminaries

Let \( \mathfrak{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be the Fourier transform of a function \( \varphi : \mathbb{R} \to \mathbb{C} \) by

\[
(\mathfrak{F}\varphi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(k) e^{ikx} dk,
\]

from which

\[
\mathfrak{F}^2 = \mathfrak{R}, \quad \mathfrak{F}^3 = \mathfrak{F}^{-1} = \mathfrak{C}\mathfrak{F}\mathfrak{C}, \tag{1}
\]

where \( \mathfrak{R} : \varphi(x) \mapsto \varphi(-x) \) and \( \mathfrak{C} : \varphi(x) \mapsto \varphi^*(x) \) represent the space inversion and complex conjugate, respectively. For later convenience, we introduce some following sets:

\[
A := \{ \varphi(x) \in Q | |\mathfrak{F}\varphi| = |\varphi| \}, \\
P_\pm := \{ \varphi(x) \in Q | \arg(\mathfrak{F}\varphi) = \pm \arg \varphi + \theta \ (\theta \ \text{constant}) \}, \\
E_\pm := \{ \varphi(x) \in Q | \mathfrak{F}\varphi = \lambda \mathfrak{C}^{(1\pm)} \varphi \ (\lambda \in \mathbb{C}) \},
\]

where \( Q := \{ \varphi(x) \in L^2(\mathbb{R}) | \varphi(x) \neq 0 \ \text{almost everywhere} \} \). From (1), it is found that \( |\lambda| = 1 \) for \( \varphi \in E_+ \), and \( \lambda \in \{1, -1, i, -i\} \) for \( \varphi \in E_- \), so that \( A \cap P_\pm = E_\mp \). Hence, we get

\[
A \cap P = E, \tag{2}
\]

where \( P = P_+ \cup P_- \) and \( E = E_+ \cup E_- \).

We assume that \( \varphi \in L^2(\mathbb{R}) \) is continuous on the real axis. This implies that \( \varphi(x) \) can be expanded using the Hermite polynomial \( H_n(x) \) as

\[
\varphi = \sum_{n=0}^{\infty} \alpha_n h_n \quad (\alpha_n \in \mathbb{C}), \quad h_n(x) = H_n(x) e^{-x^2/2}.
\]

Noticing that \( h_n \) is an eigenfunction for \( \mathfrak{F} \) (with its eigenvalue being \( i^n \)):

\[
(\mathfrak{F}h_n)(x) = i^n h_n(x),
\]

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we find that \( \varphi \in A, P \) can be rewritten as

\[
\begin{align*}
\varphi \in A & \iff \sum_{n,m \in \mathbb{N}} \beta_{nm} H_n(x) H_m(x) \equiv 0, \\
\varphi \in P & \iff \sum_{n,m \in \mathbb{N}} \gamma_{nm} H_n(x) H_m(x) \equiv 0,
\end{align*}
\]

(3)

where \( \beta_{nm} = [\alpha_n^* \alpha_m - (i^n \alpha_n)^* i^m \alpha_m] + (m \leftrightarrow n) \) and

\[
\gamma_{nm} = \begin{cases} [e^{i\theta} (i^n \alpha_n)^* \alpha_m - (c.c.)] + (m \leftrightarrow n) & (\text{for } \varphi \in P_+) , \\
[e^{i\theta} (i^n \alpha_n)^* \alpha_m - (c.c.)] + (m \leftrightarrow n) & (\text{for } \varphi \in P_-),
\end{cases}
\]

with \( \mathbb{R} \ni \theta = \arg(\mathbf{F}\varphi) \mp \arg \varphi \) for \( \varphi \in P \pm \), and (c.c.) representing the complex conjugate. Here we have symmetrized \( \beta_{nm} \) and \( \gamma_{nm} \) under \( m \leftrightarrow n \), due to the invariance of the sum over \( n \) and \( m \) in (3) under \( n \leftrightarrow m \).

The followings are basic properties concerning \( \beta_{nm} \). By its definition, it is easily found that

\[
\beta_{nm} \equiv 0 \quad (\text{for } n - m \in 4\mathbb{N}).
\]

(4)

In addition, it is found that for \( |n - m|, |n - m'| \notin 4\mathbb{N}, \)

\[
(\beta_{nm} = \beta_{nm'} = 0, \alpha_n \neq 0) \implies \beta_{nm'} = 0.
\]

(5)

To show (5), it is sufficient to deal with the case \( \alpha_m, \alpha_{m'} \neq 0 \); otherwise, \( \beta_{nm'} \) would vanish by its definition. For \( |n - m| \notin 4\mathbb{N} \) (with \( \alpha_n, \alpha_m \neq 0 \)), the condition of \( \beta_{nm} = 0 \) indicates the relative direction between \( \alpha_n \in \mathbb{C} \) and \( \alpha_m \in \mathbb{C} \) is determined in the complex plane. Thus the condition of \( \beta_{nm} = \beta_{nm'} = 0 \) implies that the relative direction between \( \alpha_m \) and \( \alpha_{m'} \) is determined, in which the relation of \( \beta_{nm'} = 0 \) is satisfied.

Moreover, we obtain

\[
\begin{align*}
\beta_{nm} = 0 \quad (\forall n, m \in \mathbb{N}) & \iff \varphi \in E, \\
\gamma_{nm} = 0 \quad (\forall n, m \in \mathbb{N}) & \iff \varphi \in E \cup C,
\end{align*}
\]

(6)

where \( C \) represents the set such that

\[
C := \{ \varphi \in Q \mid \arg(\mathbf{F}\varphi), \arg \varphi = \text{constant} \}.
\]

To show (6), it may be useful to classify the set \( \Phi = \{ n \in \mathbb{N} \mid \alpha_n \neq 0 \} \) into three following cases (although in solving \( \beta_{nm} = 0 \), the case (b) is not necessary):

\[
\Phi \subset \begin{cases} 4\mathbb{N} + k \quad (k = 0, 1, 2, 3), & (a) \\
2\mathbb{N} + k \quad (k = 0, 1), & (b) \\
\text{otherwise}, & (c)
\end{cases}
\]

The case where \( \varphi \in C \) is realized for the cases (a) and (b). In the case of (b) but not (a), \( \varphi \in C \) does not necessarily implies \( \varphi \in A \), as is found in the following example:

\[
\varphi(x) = (\alpha_0 + \alpha_2 H_2(x)) e^{-x^2/2} \quad (\alpha_0, \alpha_2 \in \mathbb{R} \setminus \{0\}),
\]

(7)

from which \( \mathbf{F}\varphi(x) = (\alpha_0 - \alpha_2 H_2(x)) e^{-x^2/2} \). Thus the condition \( \gamma_{nm} = 0 \) (for all \( n, m \in \mathbb{N} \)) does not necessarily implies that \( \varphi \in A \).
Table 1. Examples of $\varphi \in A \setminus E$, where $N = \deg_x p(x) + \deg_x q(x)$, and $p_1, p_2$ represent the real, imaginary parts of $p$, respectively. For the second row for $N = 7$, $x_0 = q^{1/3} - q^{-1/3}$ ($q = 2 + \sqrt{3}$), $c = \frac{1}{2}x_0(5 + x_0^2)$, $d = -\frac{1}{2}x_0(-12 + 9x_0^2 + x_0^4)$, $b_2 = 1 + i$, $b_1 = \frac{1}{2}x_0[4 + i(4 + 5x_0^2 + 4x_0^4)]$, and $b_0 = \frac{1}{2}[(4 - 7x_0^2 - 3x_0^4) + i(4 + 5x_0^2 + x_0^4)]$. From the property $\varphi' \in A \setminus E \iff \varphi \in A \setminus E$ under (10), and the linear independence of $p$ and $p^*$ and that of $q$ and $q^*$, it is required that $|\deg_x p_2(x) - \deg_x p_1(x)| \in 4N \setminus \{0\}$, and $\deg_x q(x) \geq 2$. Thus $N \geq 6$ is necessary for $\varphi \in A \setminus E$.

| $N$ | $p_1(x)$ | $p_2(x)$ | $q(x)$ | $\text{rank } \beta_{\text{m}}$ |
|-----|-----|-----|-----|-----|
| 6   | $x(x^3 - 7x + 4)$ | $x(x^3 - 9x + 4\sqrt{3})$ | $iH_2(x) - 2(1 - i)H_1(x) - 4H_0(x)$ | 4 |
| 7   | $x - x_0$ | $x(x^3 - 9x + c)(x - x_0) + d$ | $(1 + i)H_3(x) + 4\sqrt{3}H_2(x) - 12(1 - i)H_1(x) - 8\sqrt{3}H_0(x)$ | 4 |

3. Amplitude to phase

In this section, we obtain a function $\varphi \in L^2(\mathbb{R})$ such that $\varphi \in A$ implies that $\varphi \in P$. Before proceeding further, it should be remarked that the relation of $\varphi \in A$ does not necessarily implies that $\varphi \in P$. The element of $\varphi \in A$ can be rewritten using polynomials $p, q : \mathbb{R} \to \mathbb{C}$ such that (Hadamard decomposition)

$$\varphi(x) = p(x)q(x) e^{-x^2/2}, \quad (\overline{\varphi}(x)) = p(x)q^*(x) e^{-x^2/2}.$$  

(8)

In this case, the condition of $\varphi \notin P$ can be rephrased by the linear independence of $p(x)$ and $p^*(x)$ and that of $q(x)$ and $q^*(x)$. For $\deg_x p(x) + \deg_x q(x) \geq 6$ (where “deg” represents the degree), such linear independence can be realized, as summarized in Table 1, where $p_1$ and $p_2$ represent the real and imaginary parts of $p$, respectively.

It should be noticed that from (1), there exist two following transformations $T : \varphi \mapsto \varphi'$ such that

$$\varphi \in A \setminus E \iff \varphi' \in A \setminus E.$$  

(9)

(i) $T = \mathcal{R}$, where $\mathcal{R} : \varphi(x) \mapsto \varphi'(x) = \varphi(-x)$ represents the space inversion. Due to the commutativity of $[\mathcal{R}, \mathfrak{F}] = [\mathcal{R}, \mathfrak{E}] = 0$ and the isomorphism $\mathfrak{R}(\varphi_1\varphi_2) = (\mathfrak{R}\varphi_1)(\mathfrak{R}\varphi_2)$, the statement of (9) holds.

(ii) $T = \mathfrak{E}$, where $\mathfrak{E}$ is defined by

$$\mathfrak{E} : \varphi \mapsto \varphi' = (1 \ i) M \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \varphi, \quad (M \in \text{GL}_2(\mathbb{R})), $$

where $f_1 = \frac{1}{2}(1 + \mathfrak{F} \mathfrak{E})$ and $f_2 = \frac{1}{2}(1 - \mathfrak{F} \mathfrak{E})$. For $\varphi \in A$, where (8) is realized, $\mathfrak{E}$ is reduced to

$$\mathfrak{E} : \begin{pmatrix} p_1 \\ p_2 \\ q \end{pmatrix} \mapsto \begin{pmatrix} p_1' \\ p_2' \\ q' \end{pmatrix} = (M \oplus 1) \begin{pmatrix} p_1 \\ p_2 \\ q \end{pmatrix}. $$

(10)

Considering the relation

$$\langle f_1 \varphi, if_2 \varphi \rangle = 0 \quad \text{(for } \varphi \in A),$$

where $\langle z_1, z_2 \rangle = \frac{1}{2}(z_1^* z_2 + (c.c.))$ represents the Euclidean inner product, we find that $\varphi \in A \implies \varphi' \in A$. The linear independence of $p$ and $p^*$ is conserved due to $\det M \neq 0$. Hence (9) is satisfied for $T = \mathfrak{E}$.
In both transformations $T = R, L$, the rank of $\beta_{nm}$ remains invariant:

$$\text{rank } \beta'_{nm} = \text{rank } \beta_{nm}.$$ 

This is because $\beta_{nm}$ is transformed as

$$\beta_{nm} \rightarrow \beta'_{nm} = \begin{cases} (-1)^{n+m} \beta_{nm} & (\text{for } T = R), \\ \det M \cdot \beta_{nm} & (\text{for } T = L). \end{cases}$$

For $N = 6, 7$, there is no other solution for $\varphi \in A \setminus E$ than those listed in Table 1, up to the solutions obtained under the transformations $T = R, L$, so that it can be concluded that $\text{rank } \beta_{nm} = 4$ for all $\varphi \in A \setminus E$. From this finding, it is conjectured that

$$\text{rank } \beta_{nm} = 4 \quad (\text{for all } \varphi \in A \setminus E), \quad (11)$$

despite the value of $N \geq 6$.

Now we go back to obtain a function $\varphi$ such that $\varphi \in A$ implies that $\varphi \in P$ (so that by (2), $\varphi \in P$ can be replaced by $\varphi \in E$). This corresponds to obtaining a set $X$ whose element $\chi \subset \mathbb{N}$ is given by

$$\forall \chi \in X, \quad \left\{ (\beta_{nm})_{n,m \in \chi} \right\} \sum_{n,m \in \chi \subset \mathbb{N}} \beta_{nm} H_n(x) H_m(x) \equiv 0 \right\} = \{(0_{nm})_{n,m \in \chi}\},$$

where $(0_{nm})$ represents a zero matrix. Although it seems to be somewhat challenging to obtain all the elements in $X$, the followings are the examples of those:

$$S_5, \ T_0 \cup T_2, \ T_1 \cup T_3 \in X,$$

where $S_n := \{0, 1, \ldots, n\}$ and $T_k := 4N + k$. The relation of $T_k \cup T_{k+2} \in X$ (for $k = 0, 1$) indicates that for $\varphi$ an even or odd function, $\varphi \in A$ implies $\varphi \in P$. The above elements in $X$ are slightly generalized to such that

$$S_6 \setminus \{n\}, \ T_k \cup T_\ell \cup T_m \in X \quad (12)$$

for all $n \in S_6$ and $k, \ell, m \in S_3$. It should be recalled that

$$S_6 \notin X, \quad (13)$$

as is found from the example for $N = 6$ in Table 1.

To show (12), the subsequent identity may be useful.

$$\sum_{n,m \in \mathbb{N}} \beta_{nm} H_n(x) H_m(x) = \sum_{k \in \mathbb{N}} \hat{\beta}_k H_k(x), \quad (14)$$

where the coefficients $\hat{\beta}_k$’s are given by multiplying both sides of (14) with $H_k(x) e^{-x^2/2}$ and integrating over $x \in \mathbb{R}$, with the result that (for the integral formula, see, for example [14])

$$\hat{\beta}_k = \sum_{n,m \in \mathbb{N}} \frac{2^{s-k} n! m!}{(s-k)! (s-m)! (s-n)!} \beta_{nm}.$$
with $s = \frac{k+m+n}{2}$. Notice that the lengths $k, m, n$ form a triangle, due to $s - k, s - m, s - n \geq 0$. For the proof of $S_6 \in X$, see Appendix A. In a similar way, we can prove $S_6 \setminus \{n\} \in X$ for all $n \in S_6$.

Moreover, we can show $T_k \cup T_\ell \cup T_m \in X$ in an analogous way. In this case, we first restrict $\varphi = \sum_{n \in T_k \cup T_\ell \cup T_m} \alpha_n h_n$ to such that the sum over $n$ is bounded as $n \leq N$. Solving $\beta_k = 0$ for $k = 2N$ to 0 in a descending order, where the repeated application of (5) is made, we obtain $\beta_{nm} = 0$ inductively for all $n, m \leq N$. Considering that $N$ can be chosen as arbitrary in $\mathbb{N}$, we eventually find that $T_k \cup T_\ell \cup T_m \in X$.

To summarize, we obtain

$$A |_{I_p} \subset P |_{I_p},$$

where $|_{I_p}$ represents $\cdot \cap I_p$, with $I_p := \{ \sum_{n \in \chi} \alpha_n h_n(x) | \alpha_n \in \mathbb{C}, \chi \in X \}$.

4. Phase to amplitude

In the previous section, we have found that for $\varphi \in I_p$, the relation of $\varphi \in A$ implies $\varphi \in P$. In this section, we deal with the converse. For this aim, we introduce a set $Y$, which corresponds to $X$, such that

$$\forall v \in Y, \quad \{ (\gamma_{nm})_{n,m \in v} | \sum_{n,m \in u \subset \mathbb{N}} \gamma_{nm} H_n(x) H_m(x) \equiv 0 \} = \{ (0_{nm})_{n,m \in v} \}.$$  

As in (12) and (13), we can show that

$$S_6 \setminus \{n\}, T_k \cup T_\ell \cup T_m \in Y; \quad S_6 \notin Y,$

for all $n \in S_6$ and $k, \ell, m \in S_3$. This suggests that the set $Y$ might be identical to $X$:

$$Y = X,$$  

although a rigorous proof has not been made.

Now we are ready to examine whether or not the converse of (15) holds. From the second relation of (6), it is found that $\varphi \in P \cap I_p$ (assuming (16); if not, we should simply replace $X$ in the definition of $I_p$ by $X \cap Y$) implies either $\varphi \in E$ or $\varphi \in C$. In the case of $\varphi \in E$, the relation of $\varphi \in A$ is indeed satisfied. However, in the case of $\varphi \in C$, the relation of $\varphi \in A$ does not necessarily hold, as in an example of (7). If we eliminate $C$ from $I_p$, we obtain $P |_{I'_p} \subset A |_{I'_p}$, where $I'_p = I_p \setminus C$. This, together with $A |_{I'_p} \subset A |_{I_p}$, leads to

$$P |_{I'_p} \subset A |_{I'_p}.  

From (15) and (17), it is found that

$$A |_{I_p} \neq P |_{I_p}, \quad A |_{I'_p} = P |_{I'_p}.$$  

The first inequality comes from $P |_{I'_p} \subset P |_{I_p}$.

At the end of this section, we make a remark about rank $\gamma_{nm}$. From an explicit solution of $P \setminus (E \cup C) \ni \varphi = \sum_{n=0}^{N} \alpha_n h_n$ for $N = 6, 7, \ldots$, it is conjectured that

$$\text{rank } \gamma_{nm} = 4 \quad \text{(for all } \varphi \in P \setminus (E \cup C)),$$

which is analogous to (11).
5. Simple generalization
We have found that for $\varphi \in I_P$, the condition of $\varphi \in A$ implies the relation of $\varphi \in P$. In this section, we relax the condition of $\varphi \in A$ in such a way that the relation between $\arg(F\varphi)$ and $\arg \varphi$ still holds in an analogous way. Such a generalization can be given by a (one parameter) modification which is commutative with $\mathfrak{F}$ and with $\mathcal{C}$. Let $\varphi_\epsilon(x)$ denote $\varphi(x, \epsilon)$ (where $x \in \mathbb{R}, \epsilon \in I := [-A,A] \subset \mathbb{R}$). Suppose that $\varphi_\epsilon$ is continuous with respect to $\epsilon$ on $I$, and assume the uniform convergence of $\mathfrak{F}\varphi_\epsilon$ with respect to $\epsilon$. Then the modification $\tau_\delta : \varphi_\epsilon \mapsto \varphi_{\epsilon+\delta}$ is commutative with $\mathfrak{F}$ and $\mathcal{C}$ (for all $\epsilon, \epsilon + \delta \in I$). Denote by $A^{(A)}$ and $P^{(A)}$ the sets such that

$$A^{(A)} := \{ \varphi_\epsilon \in Q \mid \forall \epsilon \in I, \ |\mathfrak{F}\varphi_\epsilon| = |\varphi_{-\epsilon}| \},$$

$$P^{(A)} := \{ \varphi_\epsilon \in Q \mid \forall \epsilon \in I, \ \arg(\mathfrak{F}\varphi_\epsilon) = \pm \arg \varphi_{-\epsilon} + \theta \quad (\theta = \text{constant}) \}.$$ 

It is apparent that $A = A^{(0)}$ and $P = P^{(0)}$. In what follows, we first show that

$$A^{(A)}|_{I_P} \subset P^{(A)}|_{I_P}. \quad (18)$$

In deriving (18), notice that there is no loss of generality that $\varphi_\epsilon$ and $\varphi_{-\epsilon}$ (for $\epsilon \neq 0$) are linearly independent of each other. Otherwise, the condition of $|\mathfrak{F}\varphi_\epsilon| = |\varphi_{-\epsilon}|$ would turn out to be $|\mathfrak{F}\varphi_\epsilon| = |\varphi_\epsilon|$, the same problem dealt with in Sect. 3.

To begin with, it may be convenient to decompose $\varphi_{\pm \epsilon} \in A^{(A)} \cap I_P$ into

$$\varphi_{\pm \epsilon} = \varphi^+ \pm \varphi^-,$$

where $\varphi^+ = \frac{1}{2}(\varphi_\epsilon \pm \varphi_{-\epsilon})$ (double-sign corresponds). Then the condition of $\varphi_{\pm \epsilon} \in A^{(A)}$ can be rewritten as

$$|\mathfrak{F}\varphi^+|^2 - |\varphi^+|^2 = |\varphi^-|^2 - |\mathfrak{F}\varphi^-|^2, \quad (19)$$

$$\langle \mathfrak{F}\varphi^+, \mathfrak{F}\varphi^- \rangle = -\langle \varphi^+, \varphi^- \rangle. \quad (20)$$

Considering that the left-hand side and right-hand side of (19) are even and odd function of $\epsilon$, respectively, and that the relation of (19) holds for all $\epsilon \in I$, we find that the both sides of (19) should be constant $c$ with respect to $\epsilon$:

$$c = |\mathfrak{F}\varphi^+|^2 - |\varphi^+|^2.$$

By taking the above relation in the limit of $\epsilon \rightarrow 0$, which is commutative with $\mathcal{C}$ and $\mathfrak{F}$, and by recalling that $\lim_{\epsilon \rightarrow 0} \varphi_\epsilon = \varphi_0 \in A$, it is found that $c = 0$, so that

$$\varphi^\pm \in A. \quad (21)$$

Recall also the condition of $\varphi_{\pm \epsilon} \in I_P$, namely, $\varphi^\pm \in I_P$. This, together with (21), leads to

$$\varphi^\pm \in E \quad (22)$$

by $A|_{I_P} \subset P|_{I_P}$.

The remaining thing is to derive $\varphi_\epsilon \in P^{(A)}$ from (22). Depending upon the case where $\varphi^+, \varphi^-$ belong to $E_+$ or $E_-$, we deal with the four following cases:
(i) For $\varphi^\pm \in E_+$, we can write \[
abla \varphi^\pm = \left(\frac{\lambda_+ \varphi^+}{\lambda_- \varphi^-}\right), \quad \text{where} \quad (\lambda^\pm)^4 = 1.
\]
Substituting this relation into (20), we obtain \[
\langle \lambda_+ \lambda_-^* + 1, (\varphi^+)\varphi^-\rangle = 0,
\] (23)
where use has been made of $(z_1, z_2 z_3) = (z_1 z_2^*, z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$. Recall that $\varphi_\epsilon$ and $\varphi_{-\epsilon}$ can be chosen as linearly independent, in which $\arg \varphi^+ - \arg \varphi^-$ is not constant. In this case, the relation of (23) indicates that $\lambda_+ \lambda_-^* + 1 = 0$, or equivalently, $\lambda_+ + \lambda_- = 0$ by $|\lambda_-| = 1$. As a consequence, we obtain
\[
\nabla \varphi_\epsilon = \nabla (\varphi^+ + \varphi^-)
= \lambda_+ \varphi^+ + \lambda_- \varphi^-
= \lambda_+ \varphi_{-\epsilon},
\]
so that $\varphi_\epsilon \in P^{(A)}$.

(ii) For $\varphi^\pm \in E_-$, we can write \[
\nabla \varphi^\pm = \left(\frac{\epsilon^{i\eta_\pm} (\varphi^+)^*}{\epsilon^{i\eta_-} (\varphi^-)^*}\right), \quad \text{where} \quad \eta_\pm \in \mathbb{R}.
\]
In this case, we can obtain $\nabla \varphi_\epsilon = \epsilon^{i\eta_\pm} (\varphi_{-\epsilon})^*$ in an analogous way to the case (i), so that $\varphi_\epsilon \in P^{(A)}$.

(iii) For $\varphi^\pm \in E_\pm$, we can write \[
\nabla \varphi^\pm = \left(\frac{\lambda_+ \varphi^+}{\epsilon^{i\eta_-} (\varphi^-)^*}\right).
\]
Substituting this relation into (20), we obtain either $\arg \varphi^+$ or $\arg \varphi^-$ is constant, which corresponds to a special case of (ii) or (i), respectively. Thus in either case, it follows that $\varphi_\epsilon \in P^{(A)}$.

(iv) For $\varphi^\pm \in E_\mp$, we also obtain $\varphi_\epsilon \in P^{(A)}$ in quite an analogous way to the case (iii).

In all cases (i)–(iv), it is found that $\varphi_\epsilon \in P^{(A)}$. This ends the proof of (18).

The converse of (18) does not necessarily hold, as in the case of $\Lambda = 0$. However, if we eliminate the case of $\varphi_\epsilon \in C$, the converse holds as in the form
\[
P^{(A)}|_{I_p} \subset A^{(A)}|_{I_p},
\] (24)
which is reduced to (17) in the limit of $\Lambda \to 0$.

6. Conclusion
We have shown that under the Fourier transform $\tilde{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of an even or odd complex-valued function $\varphi$, the invariance of the amplitude ($\varphi \in A$) leads to either the invariance or inversion of the phase up to an additive constant ($\varphi \in P$). We have also shown that the converse holds in an analogous way, unless $\arg \varphi$ is constant. Without the evenness or oddness of $\varphi$, this phase amplitude relation does not hold in general; such an example is found in Table 1. It does not seem to be so simple a matter to obtain a necessary and sufficient condition for $A \subset P$ and that for $P \subset A$; we have just presented a sufficient condition, as in the form of (12).

The generalization of this phase amplitude relation can be easily made if the modification $\tau$ of $\varphi$ is commutative with $\tilde{F}$ and with the complex conjugate $\tilde{C}$, as is summarized in (18) and (24). To make further application, we should relax the commutativity of $\tau$ with $\tilde{F}$ and $\tilde{C}$.

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Appendix A. Proof of $S_5 \in X$

To prove $S_5 \in X$, we show that the condition $\sum_{n,m \in S_5} \beta_{nm} H_n(x) H_m(x) \equiv 0$ leads to $\beta_{nm} = 0$ for all $n, m \in S_5$. In this Appendix, we deal with the case where $\alpha_5 \neq 0$.

The case of $\alpha_5 = 0$ can be dealt with in an analogous way. To begin with, the relation of $\sum_{n,m \in S_5} \beta_{nm} H_n(x) H_m(x) \equiv 0$ can be rewritten using (14) as

\[
0 = \beta_{45} \quad (A.2)
= \beta_{35} \quad (A.3)
= \beta_{25} + \beta_{34} \quad (A.4)
= \beta_{24} \quad (A.5)
= \beta_{05} + \beta_{14} + \beta_{23} + 20 \beta_{25} + 24 \beta_{34} \quad (A.6)
= \beta_{13} \quad (A.7)
= \beta_{03} + \beta_{12} + 8 \beta_{14} + 12 \beta_{23} + 80 \beta_{25} + 144 \beta_{34} \quad (A.8)
= \beta_{02} \quad (A.9)
= \beta_{01} + 4 \beta_{12} + 24 \beta_{23} + 192 \beta_{34}, \quad (A.10)
\]

each of which corresponds to $\hat{\beta}_k = 0$ for $k = 9, 8, \ldots , 1$, respectively, where use has been made of the first relation of (4). For the above relations corresponding to $\hat{\beta}_\ell = 0$ (for $\ell = 7, 6, \ldots , 1$), some relations of $\hat{\beta}_m = 0$ (for $m = \ell + 2, \ell + 4, \ldots , 8$ or 9) have been taken into account so as to simplify the relation of $\hat{\beta}_\ell = 0$. Under the condition of (A.1), we obtain from (A.2) and (A.3)

\[
\beta_{34} = 0, \quad (A.11)
\]

which, together with (A.4), leads to

\[
\beta_{25} = 0. \quad (A.12)
\]

In deriving (A.11), use has been made of (5). Using (5) again, we obtain from (A.3) and (A.12)

\[
\beta_{23} = 0. \quad (A.13)
\]

At this stage, the relations of (A.6), (A.8), and (A.10) are reduced to

\[
\begin{cases}
\beta_{05} + \beta_{14} = 0, \\
\beta_{03} + \beta_{12} + 8 \beta_{14} = 0, \\
\beta_{01} + 4 \beta_{12} = 0.
\end{cases} \tag{A.14}
\]

The rest thing is to show that each of the terms in the left-hand side of (A.14) is vanishing. Here we classify the case where $(\alpha_2, \alpha_3)$ is equal to $(0, 0)$ or not.

(i) For $\alpha_2 \neq 0$, it is found from (A.9), (A.12), and (A.13) that $\beta_{05} = \beta_{03} = 0$ by using (5). The remaining $\beta_{nm}$’s are all vanishing by (A.14).

(ii) For $\alpha_3 \neq 0$, we similarly obtain $\beta_{14} = \beta_{12} = 0$ from (A.7), (A.11), and (A.13), so that the remaining $\beta_{nm}$’s are all vanishing by (A.14).

(iii) For $\alpha_2 = \alpha_3 = 0$, it follows that $\beta_{03} = \beta_{12} = 0$ by definition, so that the remaining $\beta_{nm}$’s are all vanishing by (A.14).

In either case, we obtain $\beta_{nm} = 0$ for all $m, m \in S_5$. 


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