On the Exact Convex Hull of IFS Fractals

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Abstract

Bounding sets to IFS fractals are useful largely due to their property of iterative containment, in both theoretical and computational settings. The tightest convex bounding set, the convex hull has revealed itself to be particularly relevant in the literature. The problem of its exact determination may have been overshadowed by various approximation methods, so our aim is to emphasize its relevance and beauty. The finiteness of extrema is examined a priori from the IFS parameters – a property of the convex hull often taken for granted. Former methods are surveyed and improved upon, and a new “outside-in” approach is introduced and crystallized for practical applicability. Periodicity in the address of extremal points will emerge to be the central idea.*

MSC class: 28A80 (primary); 52A10, 52A27 (secondary).
Keywords: IFS fractals, convex hull, extremal points.

*Some but not all of these results appeared in the author's doctoral dissertation [17] and were presented at the 2014 Winter Meeting of the Canadian Mathematical Society, though some updates have been made since then. This research was started in the Fall of 2009 and the paper was finalized in February 2015.
# Contents

1 Introduction 3  
1.1 Former Research 3  
1.2 Overview 4  

2 IFS Fractals 5  
2.1 Definition and Existence 5  
2.2 The Address Set 6  
2.3 Lemmas 7  

3 The Number of Extrema 9  
3.1 Extremal Points 9  
3.2 Rational vs. Irrational Fractals 11  
3.3 Fractals of Unity 13  

4 Methods for the Convex Hull 17  
4.1 A Method for Equiangular Fractals of Unity 17  
4.2 A General Method for Fractals of Unity 18  
4.3 Linear Optimization over IFS Fractals 20  
4.4 The Armadillo Method for Regular Fractals 24  

5 The Principal Direction 26  
5.1 The Normal Form of Bifractals 26  
5.2 A Candidate Direction for C-IFS Fractals 27  
5.3 The Armadillo Method Adopted for C-IFS Fractals 30  
5.4 Examples 31  

6 Concluding Remarks 37  

References 38
1 Introduction

1.1 Former Research

The paper investigates the following intentionally vague problem for planar IFS fractals – defined in Section 2.1 – its vagueness allowing greater freedom of discussion.

Problem 1.1 (Convex Hull Problem) Devise a general practical method for finding the convex hull of IFS fractals, meaning the exact locations of extremal points. Determine a priori from the IFS parameters, if the generated fractal will have a finite or infinite number of extremal points.

Various numerical methods have been devised to find an approximation to the convex hull of IFS fractals such as [13], but the goal here is to find the exact convex hull as specified above, which limits the focus and scope of this brief survey.

The following theorem is an intuitive result, which may serve as an inspirational spark for further contemplation of the problem, to one who is already familiar with IFS fractals – see Section 2.1 for an introduction.

Theorem 1.1 (Berger [2]) For the extremal points \( E_F \) of an IFS fractal \( F \) invariant under the Hutchinson operator \( H(F) = F \), we have that \( E_F \subset H(E_F) \).

Corollary 1.1 (Deliu, Geronimo, Shonkwiler [5]) For any IFS fractal with a finite number of extremal points, there exists an IFS that generates it, and any extremal address is eventually periodic with respect to this IFS.

This theorem seems to suggest a recursive property for the set of extremal points, if the above containment is applied repeatedly. If this set is to have a finite cardinality, then the extremal addresses must clearly possess some sort of periodicity, as the corollary states. The authors employ this corollary to resolve the Inverse Problem of IFS fractals, meaning they devise a theoretical method for finding IFS parameters that generate a given IFS fractal. Their a priori assumption that the fractal has a finite number of extrema seems to be an innocent one. As we will see in Section 3 it certainly is not, and finiteness will emerge as an intriguing subproblem worthy of our special attention.

The significance of the Convex Hull Problem 1.1 is underlined by the work of Pearse and Lapidus [15] where the convex hull is an integral tool of their theoretical framework. Furthermore, its relevance for the Fractal-Line Intersection Problem (a.k.a. “slicing”) is illustrated by the following theorem, shown for a special case by Mendivil and Taylor [14].

Theorem 1.2 [19] An IFS fractal is hyperdense if any hyperplane that intersects its convex hull also intersects the Hutchinson of its convex hull. A hyperplane intersects a hyperdense fractal if and only if it intersects its convex hull. This equivalence holds only if the fractal is hyperdense.
To highlight a stimulating approach to the Convex Hull Problem, Wang and Strichartz et al. [16, 10, 20] examine the convex hull of self-affine tiles. Among other intriguing results, they derive an important theorem below for the finiteness of extrema in a special case, which is to be generalized here by Theorems 3.4 and 3.12. Note that their work on self-affine tiles is in a sense more general, since we only consider IFS with scaled rotation matrices, which however can be unequal unlike in the self-affine case. Their work reveals the intriguing connection between the outward normals to the convex hull, and the eigenvectors of powers of the IFS matrices. This paper restricts IFS to affine similarities in the complex plane, and as the results progress it will become clear why, though generalizations will be suggested.

**Theorem 1.3** (Strichartz and Wang [16]) *If the IFS factors are all equal to a scaled rotation matrix, then the extremal points of the generated fractal are finite in number if and only if this matrix to some power is a multiple of the identity matrix.*

Mandelbrot and Frame [12] in the process of characterizing “self-contacting binary trees” – with two-map IFS fractal canopies – determine a periodic address that leads to a particular extremal point. Their result is equivalent to the explicit formula for a periodic point given in Corollary 2.1 of this paper. Their method indicates the relevance of finding the exact convex hull to resolving the Connectedness Problem.

The work most relevant to this paper was done by Kırat and Koçyiğit [8, 9] focusing essentially on Problem 1.1. They devise a constructive terminating algorithm for finding the convex hull of IFS fractals when the factors are equal, and a non-terminating method when they are not. Their methods are presented in a reworked and simplified form in Sections 4.1 and 4.2, with the important improvement of guaranteed termination.

### 1.2 Overview

After some preliminary definitions and lemmas in Section 2, our investigation begins in Section 3 with the exclusion of a broad class of IFS fractals – called “irrational fractals” – for which the cardinality of extremal points is deduced to be infinite, interestingly only dependent on the rotation angles of the IFS maps. The “rational” class of fractals on the other hand, contains the broad class of “fractals of unity” which as shown in Section 3.3 are guaranteed to have a finite number of extremal points. Furthermore by continuity in angular parameters – shown in Section 3.2 – fractals of unity can approximate irrational fractals arbitrarily.

Two methods are detailed for finding the convex hull, and contrasted in terms of efficiency and robustness. The first is a “general method” that works for any fractal of unity is presented in Section 4.2, with its shortcomings assessed. The more efficient though less general “Armadillo Method” is presented in Section 4.4 for the subclass of “regular fractals”, which hinges on the linear optimization algorithm of Section 4.3. Examples are given for the method’s practical implementation in Section 5.4.
2 IFS Fractals

2.1 Definition and Existence

The attractors of Iterated Function Systems – IFS Fractals – were pioneered by Hutchinson [7] and may well be the most elementary fractals possible, occurring in Nature as the Romanesco Broccoli. They are the attractors of a finite set of affine linear contraction mappings – the Iterated Function System (IFS) – which when combined and iterated to infinity, converges to an attracting limit set, the IFS fractal itself. IFS fractals as introduced here, can be considered the linearization of Julia fractals.

Definition 2.1 Let a planar similarity affine contractive mapping (briefly similarity contraction, contraction map, or similitude) \( T : \mathbb{C} \to \mathbb{C} \) be defined for all \( z \in \mathbb{C} \) as \( T(z) := p + \varphi (z - p) \) where \( p \in \mathbb{C} \) is the fixed point of \( T \), and \( \varphi = \lambda e^{\Theta i} \in \mathbb{C} \) is the factor of \( T \), with \( \lambda \in (0, 1) \) the contraction factor of \( T \), and \( \Theta \in (-\pi, \pi] \) the rotation angle of \( T \).

An equivalent definition may be given using rotation matrices \( R \in \mathbb{R}^{d \times d} \), \( R^T R = I \), \( d \in \mathbb{N} \) corresponding to \( e^{\Theta i} \) when \( d = 2 \). Then contraction maps take the form \( T(z) = p + \lambda R(z - p) \) for \( z \in \mathbb{R}^d \), \( p \in \mathbb{R}^d \), \( \lambda \in (0, 1) \).

Definition 2.2 Let a planar similarity affine contractive \( n \)-map iterated function system (briefly IFS or \( n \)-map IFS, \( n \in \mathbb{N} \)) be defined as a finite set of contractions, and denoted as \( \mathcal{T} := \{T_1, \ldots, T_n\} \). Denote the index set as \( \mathcal{N} := \{1, \ldots, n\} \), the respective fixed points as \( p_1, \ldots, p_n \) and their set as \( \mathcal{P} \), and the respective factors as \( \varphi_1, \ldots, \varphi_n \). The fixed points and the factors are called collectively as the IFS parameters.

Definition 2.3 Let \( \mathcal{T} = \{T_1, \ldots, T_n\} \) be an IFS. Define the Hutchinson operator \( \mathcal{H} \) belonging to \( \mathcal{T} \) as

\[
\mathcal{H}(S) = \mathcal{H}_\mathcal{T}(S) := \bigcup_{k=1}^n T_k(S) \quad \text{where} \quad T_k(S) := \{T_k(z) : z \in S\}, \; S \subset \mathbb{C}.
\]

Theorem 2.1 (Hutchinson [7]) For any IFS \( \mathcal{T} \) there exists a unique compact set \( F_\mathcal{T} \subset \mathbb{C} \) such that \( \mathcal{H}_\mathcal{T}(F_\mathcal{T}) = F_\mathcal{T} \). Furthermore, for any nonempty compact set \( S_0 \subset \mathbb{C} \), the recursive iteration \( S_{n+1} := \mathcal{H}_\mathcal{T}(S_n) \) converges to \( F_\mathcal{T} \) in the Hausdorff metric.

Proof The proof follows from the Banach Fixed Point Theorem, since it can be shown that \( \mathcal{H}_\mathcal{T} \) is contractive in the Hausdorff metric over compact sets. □

Definition 2.4 Let the set \( F_\mathcal{T} \) in the above theorem be called a fractal generated by an IFS \( \mathcal{T} \) (briefly IFS fractal; note that multiple IFS can generate the same fractal). Denote \( \langle \mathcal{T} \rangle = \langle T_1, \ldots, T_n \rangle := F_\mathcal{T} \). If \( n = 2, 3 \) or \( n > 3 \) we say that \( F_\mathcal{T} \) is a bifractal, trifractal, or polyfractal respectively (see [18] for reasons). If all rotation angles are congruent modulo \( 2\pi \) then the IFS fractal is equiangular, and if they are all congruent to zero then it is a Sierpiński fractal.
Figure 1: Generation of an IFS fractal by iterating a square, with the following IFS parameters \( p_1 = 1 + \frac{1}{2}i, \ p_2 = i, \ \varphi_1 = \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}i}, \ \varphi_2 = \frac{1}{2}e^{0i} \) (figure by S. Draves).

### 2.2 The Address Set

**Definition 2.5** Let \( N^L := N \times \ldots \times N \) be the index set to the \( L \)-th Cartesian power, and call this \( L \in \mathbb{N} \) the iteration level. Then define the address set as

\[
A := \{0\} \cup \bigcup_{L=1}^{\infty} N^L \cup \mathbb{N}^\mathbb{N}.
\]

For any \( a \in A \) denote its \( k \)-th coordinate as \( a(k) \), \( k \in \mathbb{N} \). Let its dimension or length be denoted as \( |a| \in \mathbb{N} \) so that \( a \in N^{|a|} \) and let \( |0| := 0 \). Define the map with address \( a \in A \) acting on any \( z \in \mathbb{C} \) as the function composition \( T_a(z) := T_{a(1)} \circ \ldots \circ T_{a(|a|)}(z) \). Let the identity map be \( T_0 := \text{Id} \). Further denote

\[
A_{\text{fin}} := \{a \in A : |a| < \infty\}, \ A_\infty := \{a \in A : |a| = \infty\} = \mathbb{N}^\mathbb{N}.
\]

For the weights \( w_1, \ldots, w_n \in (0,1) \) let \( w_a := w_{a(1)} \cdot \ldots \cdot w_{a(|a|)} \), for the factors \( \varphi_1, \ldots, \varphi_n \) let \( \varphi_a := \varphi_{a(1)} \cdot \ldots \cdot \varphi_{a(|a|)} \), and for the angles \( \vartheta_1, \ldots, \vartheta_n \) let \( \vartheta_a := \vartheta_{a(1)} + \ldots + \vartheta_{a(|a|)} \).

**Definition 2.6** Let \( ab \) denote the concatenation \((a, b)\) for any \( a, b \in A_{\text{fin}} \) so that \( T_{ab} = T_aT_b := T_a \circ T_b \) and \( a^k := a \ldots a \) \( k \)-times, \( k \in \mathbb{N} \). Let a periodic address be denoted \( \bar{a} := aa \ldots \) where \( a \in A_{\text{fin}} \) repeats ad infinitum. Denote the inverse of an address \( a \in A_{\text{fin}} \) with \( a^{-1} \) so that \( T_{a^{-1}} = T_a^{-1} \).

We say that the address \( a \in A \) is a truncation of \( b \in A \) if \( |a| < \infty \) and there is a \( c \in A \) such that \( b = ac \), denoted as \( a < b \) (note that this includes \( a = 0 \)). Furthermore \( a \in A_{\text{fin}} \) is a strict truncation of \( b \in A \) if \( a < b \) and \( a \neq b \) denoted as \( a \leq b \).
Theorem 2.2 For any fixed point \( p_k \in \mathcal{P} \) we have

\[
\langle T_1, \ldots, T_n \rangle = \lim_{L \to \infty} H^L(\{p\}) = \text{Cl}\{T_a(p_k) : a \in A_{\text{fin}}\}.
\]

We call this the address generation of the IFS fractal \( F = \langle T_1, \ldots, T_n \rangle \) from the seed \( p_k \). We call \( T_a(F), \ a \in A_{\text{fin}} \) a subfractal of \( F = \bigcup_{|a| = L} T_a(F) \) for any \( L \in \mathbb{N} \).

Proof The proof follows from Theorem 2.1 with the initial compact sets \( \{p\} \) or \( \mathcal{P} \). \( \square \)

Definition 2.7 Let the address \( \text{adr}(f) \) of a fractal point \( T_a(p_k) \) in the address generation be \( a \) is \( |a| = \infty \) or \( a \bar{k} \) otherwise (if two such addresses exist, then take the lexicographically lower one).

Definition 2.8 We say that a fractal point \( f \in F \) is a periodic point if its address \( a \in A \) is periodic, meaning it is infinite with a finite repeating part \( a = \bar{x}, \ x \in A_{\text{fin}} \). Denote it as \( p_x := f \) and note that \( T_x(f) = f \). Let the set of all periodic points be denoted as \( \text{Per}(F) := \{p_x : x \in A_{\text{fin}}\} \). Let the cycle of a finite address \( x \) be the set \( \text{Cyc}(x) := \{p_{ba} : x = ab\} \).

Note that \( p_x \) is an abuse of notation that is consistent with the fixed points of the IFS for which \( p_k = T_k(p_k), \ k \in \mathcal{N} \). Also note that \( p_x \in \text{Cyc}(x) \) since \( a = 0 < x \).

Definition 2.9 A fractal point \( f \in F \) with address \( a \in A_\infty \) is eventually periodic (briefly: eventual) if \( T_b^{-1}(f) \in \text{Per}(F) \) is periodic for some \( b < a \). Let the set of all eventual points be denoted as \( \text{Eve}(F) \). (Clearly \( \text{Per}(F) \subset \text{Eve}(F) \) with \( b = 0 \).)

Lemma 2.1 A fractal point \( f \in F \) is eventually periodic iff \( \exists a, b \in A_{\text{fin}} : f = T_a T_b T_a^{-1}(f) \).

Proof \( f = T_a T_b T_a^{-1}(f) \iff f = \lim_{L \to \infty}(T_a T_b T_a^{-1})^L(f) = \lim_{L \to \infty} T_a T_b T_a^{-1}(f) = T_a(p_b) \iff f \in \text{Eve}(F) \). \( \square \)

2.3 Lemmas

Lemma 2.2 (Containment Lemma) If for a nonempty compact set \( S \subset \mathbb{C} : H(S) \subset S \) then \( F \subset S \). Also if \( H(S) \subset \text{Conv}(S) \) (where Conv denotes the convex hull), then \( F \subset \text{Conv}(S) \). On the other hand, if \( F \subset S \) then \( F \subset H^L(S) \) for any \( L \in \mathbb{N} \).

Proof The first part of the lemma follows directly from Theorem 2.1 by observing that \( S \supset H^L(S) \to F \) as \( L \to \infty \).

The second part follows by applying the containment inductively, utilizing that \( H \) is defined via affine maps. \( H^1(S) \subset \text{Conv}(S) \) holds for \( l = 1 \) and supposing that it holds for \( l \leq L - 1 \) we show it for \( l = L \).

\[ H^L(S) = H^{L-1}(H(S)) \subset H^{L-1}(\text{Conv}(S)) \subset \text{Conv}(H^{L-1}(S)) \subset \text{Conv}(\text{Conv}(S)) = \text{Conv}(S) \].
This implies $\text{Conv}(S) \supset H^L(S) \to F$ as $L \to \infty$ by Theorem 2.1 since $S$ is compact.

The last part follows by observing that $F = H(F)$ implies $F = H^L(F)$ for any $L \in \mathbb{N}$, and that $F \subset S$ implies $F = H^L(F) \subset H^L(S)$. □

Let us proceed to showing another important lemma, which will prove useful in some unexpected situations – the noted corollaries already highlighting its relevance.

**Lemma 2.3 (Slope Lemma)** The slope of the map $T_a$, $a \in \mathcal{A}_{\text{fin}}$ is the constant $\varphi_a \in \mathbb{C}$ i.e.

$$\frac{T_a(z_1) - T_a(z_2)}{z_1 - z_2} = \varphi_a \quad \text{for any distinct} \quad z_1, z_2 \in \mathbb{C}.$$

**Proof** We show the property by induction with respect to $|a|$. For any $|a| = 1$ address, i.e. $k \in \mathcal{N}$ we have

$$T_k(z_1) - T_k(z_2) = p_k + \varphi_k(z_1 - p_k) - p_k - \varphi_k(z_2 - p_k) = \varphi_k(z_1 - z_2).$$

Now let us suppose the property holds for $|a| \leq L$ and we show it for length $L + 1$. Taking any $k \in \mathcal{N}$ we need the property for $(k,a)$.

$$T_{(k,a)}(z_1) - T_{(k,a)}(z_2) = T_k(T_a(z_1)) - T_k(T_a(z_2)) = \varphi_k(T_a(z_1) - T_a(z_2)) = \varphi_k \varphi_a(z_1 - z_2) = \varphi_{(k,a)}(z_1 - z_2). \quad \square$$

**Corollary 2.1** A periodic point can be evaluated as $p_x = T_x(0)/(1 - \varphi_x)$, $x \in \mathcal{A}_{\text{fin}}$.

**Proof** Follows from the lemma with $z_1 = p_x$ and $z_2 = 0$. Note that the identity still holds if $p_x = 0$ since then $T_x(0) = 0$. □

**Corollary 2.2** The action of a finite map composition is centered at a periodic point as $T_x(z) = p_x + \varphi_x(z - p_x) = T_x(0) + \varphi_x z$ where $z \in \mathbb{C}$, $x \in \mathcal{A}_{\text{fin}}$.

**Proof** The first equality follows from the lemma with $z_1 = z$ and $z_2 = p_x$, and note that it still holds if $z = p_x$. The second equality follows by applying the previous corollary. □

Many beautiful algebraic properties can be shown for periodic points using the Slope Lemma, which we omit for their lack of utility in this paper. Note also that in the first corollary, 0 is just a part of the identity, and not necessarily a fixed point of the IFS.

**Lemma 2.4 (Affine Lemma)** For any affine map $M(z) = cz + d$ ($c, d, z \in \mathbb{C}$) we have $MT_kM^{-1}(z) = M(p_k) + \varphi_k(z - M(p_k))$ for $k \in \mathcal{N}$. Furthermore $M\langle T \rangle = \langle M TM^{-1} \rangle$.

**Proof** The first identity follows considering that an affine map $M$ preserves affine combinations. So the factors of the IFS are the original $\varphi_k$, so the Hutchinson operator $H_{MTM^{-1}}$ is contractive in the Hausdorff metric and it has a unique attractor $\langle M TM^{-1} \rangle$ by Theorem 2.1. Taking any $f \in F$ and letting $a := \text{adr}(f)$ with any seed $p \in \mathcal{P}$ we have

$$M(f) = \lim_{L \to \infty} M \circ T_{a(1)} \circ \ldots \circ T_{a(L)}(p) = \lim_{L \to \infty} MT_{a(1)}M^{-1} \circ \ldots \circ MT_{a(L)}M^{-1}(M(p))$$

so any $M(f) \in M\langle T \rangle$ is generated by the IFS $M TM^{-1}$ from the seed $M(p)$. □
3 The Number of Extrema

3.1 Extremal Points

Some preliminary observations are deduced about the extremal points of polyfractals $F = \langle T_1, \ldots, T_n \rangle$, leading up to the Extremal Theorem at the end of this section. These ideas will prove fundamental in upcoming sections.

**Definition 3.1** We say that a point $s$ in a compact set $S$ is **extremal** if it is a vertex of the convex hull $\text{Conv}(S)$ meaning $\exists s_{1,2} \in S$ ($s_1 \neq s_2$), $\lambda \in (0,1) : s = \lambda s_1 + (1-\lambda)s_2$ and call it an **extremal point** or extremum. Denote the set of all inverse iterates as $\text{Inv}(e)$. We say that an **address** is extremal in an IFS fractal, if the corresponding fractal point is extremal.

**Theorem 3.1** (Inverse Iteration of Extrema) For any extremal point $e \in \text{Ext}(F)$ with address $a = \text{adr}(e)$, taking any truncation $b < a$, $b \in \mathcal{A}_{\text{fin}}$ the **inverse iterate** $T_b^{-1}(e)$ will also be extremal. Denote the set of all inverse iterates as $\text{Inv}(e) \subset \text{Ext}(F)$.

**Proof** Suppose indirectly that $T_b^{-1}(e) \in F$, $b < a$ is not extremal, meaning there are two points $f_{1,2} \in F(f_1 \neq f_2)$ and some $\lambda \in (0,1)$ for which $T_a^{-1}(e) = (1-\lambda)f_1 + \lambda f_2$. Since $T_a$ is an affine map, we have that $e = T_a(T_a^{-1}(e)) = (1-\lambda)T_a(f_1) + \lambda T_a(f_2)$ where $T_a(f_{1,2}) \in F$ contradicting that $e$ is extremal. (Note that $T_a(f_1) \neq T_a(f_2)$ since $T_a(z) = p_a + \varphi_a(z - p_a)$, $z \in \mathbb{C}$ is invertible.) $\square$

**Corollary 3.1** The cycle of a periodic extremum $p_x \in \text{Ext}(F)$, $x \in \mathcal{A}_{\text{fin}}$ is also extremal

\[
\text{Cyc}(x) = \{ p_{ba} : x = ab \} = \{ T_a^{-1}(p_x) : a \leq x \} = \{ p_{a^{-1}x} : a \leq x \} \subset \text{Ext}(F).
\]

**Proof** This follows trivially from the observation that $p_{ba} = T_a^{-1}(p_x) \in \text{Inv}(p_x)$. $\square$

**Corollary 3.2** Inverse iteration of a non-eventual extremal point $e \in \text{Ext}(F) \setminus \text{Eve}(F)$ generates an infinite number of distinct extrema.

**Proof** Letting $a := \text{adr}(e)$, suppose indirectly that the inverse iterates of $e$ are non-distinct, meaning that for some truncations $b \leq c < a$ we have that $T_b^{-1}(e) = T_c^{-1}(e)$. This implies that $\exists d \neq 0 : c = bd$ and $T_c^{-1} = T_d^{-1}T_b^{-1}$ so $T_b^{-1}(e) = T_c^{-1}(e) = T_d^{-1}(e)$ contradicting by Lemma 2.1 that $e$ is non-eventual. $\square$

**Definition 3.2** Define the **value** of a finite address $a \in \mathcal{A}_{\text{fin}} \setminus \{ 0 \}$ with respect to an IFS $\mathcal{T}$ as the number $\nu(a) = \nu_{\mathcal{T}}(a) := \vartheta_a \equiv \vartheta_{a(1)} + \ldots + \vartheta_{a(|a|)} \mod 2\pi \in [0,2\pi)$ and $\nu(0) := 0$, while the value of a set $\mathcal{A}_0 \subset \mathcal{A}_{\text{fin}}$ as $\nu(\mathcal{A}_0) = \{ \nu(a) : a \in \mathcal{A}_0 \}$. We say that an address is **focal** if its value is zero, and **strictly focal** if it is focal and each index in $\mathcal{N}$ occurs as a coordinate. We say that a fractal point is focal, if its address is a focal periodic one, in other words it is the fixed point of a focal address. Denote the set of all focal points of $F = \langle \mathcal{T} \rangle$ as $\text{Foc}(F) = \text{Foc}_{\mathcal{T}}(F)$. We say that a fractal point is **eventually focal** if it is the finite iterate of a focal point, and denote the set of all eventually focal points as $\text{EFoc}(F) = \text{EFoc}_{\mathcal{T}}(F)$. 

9
Theorem 3.2 A periodic extremal point must be focal, meaning $\Ext(F) \cap \Per(F) \subset \Foc(F)$. Furthermore $\mathcal{P} \cap \Ext(F) \subset \Foc(F)$, and if $\mathcal{P} \subset \Ext(F)$ then $F$ is Sierpiński. Lastly, if $F$ is a Sierpiński fractal then $\Conv(F) = \Conv(\mathcal{P})$ meaning $\Ext(F) \subset \mathcal{P}$.

Proof Suppose that an extremal point $e$ is periodic, meaning there is an $x \in \mathcal{A}_{fin}$ for which $e = p_x$. Then by Corollary 2.2 we have $T_x(z) = p_x + \varphi_x(z - p_x)$, $z \in \mathbb{C}$. Suppose indirectly that $\nu(x) \neq 0$ meaning $\varphi_x \in \mathbb{C} \setminus \mathbb{R}$. Then the trajectory $t \mapsto T_x^t(z_0) = p_x + \varphi_x^t(z_0 - p_x)$ traces out a logarithmic spiral for any $z_0 \in \mathbb{C}$, $t \in \mathbb{R}$. In the case $\nu(x) \neq \pi$ taking any fractal point $f \in F \setminus \{p_x\}$ the iterates $T_x^k(f) \in F$, $k \in \mathbb{N} \cup \{0\}$ envelop $p_x$ in their convex hull, contradicting that it is extremal. When $\nu(x) = \pi$ the planar segment $[f, T_x(f)]$ contains $p_x$ in its interior, again contradicting its extremality. Thus we must have $\nu(x) = 0$ implying by definition that $e = p_x \in \Foc(F)$.

The property $\mathcal{P} \cap \Ext(F) \subset \Foc(F)$ follows from the previous one, since $\mathcal{P} \subset \Per(F)$. Furthermore if $\mathcal{P} \subset \Ext(F)$ then $\mathcal{P} = \mathcal{P} \cap \Ext(F) \subset \Foc(F) \subset \Per(F) \cap \Ext(F) \subset \Foc(F)$ implying that $\nu(k) = 0$ meaning $F$ is Sierpiński by definition.

Lastly we show that for a Sierpiński fractal $\Conv(F) = \Conv(\mathcal{P})$ (which is clearly equivalent to $\Ext(F) \subset \mathcal{P}$). The $\Conv(\mathcal{P}) \subset \Conv(F)$ containment holds since $\mathcal{P} \subset F$, and for the reverse containment $\Conv(F) \subset \Conv(\mathcal{P})$ we show that $F \subset \Conv(\mathcal{P})$. All rotation angles are congruent to zero in a Sierpiński fractal, so the IFS contractions are of the form $T_k(z) = (1 - \lambda_k)p_k + \lambda_k z$, $\lambda_k \in (0, 1)$. Further denoting $F_L := \{T_a(p) : |a| = L\} \subset F$ for some $p \in \mathcal{P}$ fixed seed, we show that $F_L \subset \Conv(\mathcal{P})$ for all $L \in \mathbb{N}$, which implies that $F \subset \Conv(\mathcal{P})$ as desired. We show this by induction over $L$. For $L = 1$ and any $k \in \mathcal{N}$ we have $T_k(p) = (1 - \lambda_k)p_k + \lambda_k p \in \Conv(\mathcal{P})$ so $F_1 \subset \Conv(\mathcal{P})$. Now suppose that $F_L \subset \Conv(\mathcal{P})$ for some $L \in \mathbb{N}$, and take any $f \in F_L$. Then for any $k \in \mathcal{N}$ we have $T_k(f) = (1 - \lambda_k)p_k + \lambda_k f \in \Conv(\mathcal{P})$ thus $F_{L+1} = \bigcup_{k \in \mathcal{N}} T_k(F_L) \subset \Conv(\mathcal{P})$. □

Definition 3.3 We say that $\tau \in \mathbb{C} \setminus \{0\}$ is a target direction (briefly target) if our aim is to maximize the linear target function $z \mapsto \langle \tau, z \rangle$ (where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{C}$) over some compact set $z \in S \subset \mathbb{C}$. The number $\langle \tau, z \rangle$ is called the target value of $z \in \mathbb{C}$. Since IFS fractals are compact, a target function attains its maximum in an extremal point by the Extreme Value Theorem, which we refer to as a maximizer of $\tau$ (or as a maximizing extremal point). Let the set of maximizers be denoted as $M(\tau)$ (note that $|M(\tau)| \in \{1, 2\}$). We call the address of a maximizer, a maximizing address of $\tau$. A truncation of a maximizing address is called a maximizing truncation.

Theorem 3.3 (Extremal Theorem; ET) If $a < \adr M(\tau)$ then $M(\tau) = T_a(M(e^{-\nu(a)\tau}))$. Furthermore, if $bx < \adr M(\tau)$, $\nu(x) = 0$, $x \neq 0$ then $M(\tau) = \{T_b(p_x)\}$. Lastly, if an extremal address begins with $bx \in \mathcal{A}_{fin}$, $\nu(x) = 0$, $x \neq 0$ then that address is $b\bar{x}$.

Proof Since $a < \adr M(\tau)$ the maximum is attained over $T_a(F)$ so we have that

$$M(\tau) = \arg\max_{f \in F} \langle \tau, f \rangle = \arg\max_{f \in T_a(F)} \langle \tau, f \rangle = \arg\max_{T_a(f_0) \in T_a(F)} \langle \tau, T_a(f_0) \rangle = T_a \left( \arg\max_{f_0 \in F} \langle \tau, p_a + \varphi_a(f_0 - p_a) \rangle \right) = T_a \left( \arg\max_{f_0 \in F} \langle \tau, e^{\nu(a)b_0} f_0 \rangle \right) =$$
Thus the first statement holds. Now suppose $bx < \text{adr } M(\tau), \nu(x) = 0, \ x \neq 0$. Then

$$M(\tau) \subset T_b T_x(F) \Rightarrow S_0 := M(e^{-\nu(b)\tau}) = T_b^{-1}(M(\tau)) \subset T_x(F) \Rightarrow x < \text{adr } M(e^{-\nu(b)\tau}).$$

Letting $\tau' := e^{-\nu(b)\tau}$ and considering that $|S_0| = |M(\tau')| \in \{1, 2\}$ we have that $S_0 = \{p_x\}$ meaning $\{p_x\} = S_0 = M(\tau') = M(e^{-\nu(b)\tau}) = T_b^{-1}(M(\tau))$ implying $M(\tau) = \{T_b(p_x)\}$.

The last statement of the theorem follows trivially from the second, considering that a fractal point is extremal iff it maximizes some target direction uniquely. □

### 3.2 Rational vs. Irrational Fractals

The objective of this section is to show that the cardinality of extrema is infinite for any “irrational”, while potentially finite for “rational” polyfractals $F = \langle T_1, \ldots, T_n \rangle$.

**Definition 3.4** We say that a polyfractal is **irrational** with respect to the IFS $\mathcal{T}$, if it has no focal points $\text{Foc}_\mathcal{T}(F) = \emptyset$, and **rational** if it does.

**Theorem 3.4** (Cardinality of Extrema I) If a polyfractal has a finite number of extrema, then all must be eventual. An irrational polyfractal has no eventual extrema, and has at least a countably infinite number of extremal points.

**Proof** If there was at least one non-eventual extremal point, then by Corollary 3.2 an infinite number of distinct extrema could be generated.

Consider an irrational fractal, and suppose indirectly that some extremal point $e \in \text{Ext}(F)$ is eventual, meaning $\exists b, x \in A_{\text{fin}} : e = T_b(p_x)$. Then by Theorem 3.1 the inverse iterate $p_x = T_b^{-1}(e) \in \text{Ext}(F)$ so $p_x \in \text{Ext}(F) \cap \text{Per}(F) \neq \emptyset$. By the irrationality of $F$ however $\text{Foc}(F) = \emptyset$ so by Theorem 3.2 we get the empty intersection $\text{Ext}(F) \cap \text{Per}(F) = \emptyset$ which is a contradiction.

Taking any extremal point, it must be non-eventual by the above, so Corollary 3.2 implies that inverse iterating this extremal point will generate a countably infinite number of distinct extrema, so we must have $|\text{Ext}(F)| \geq \aleph_0$. □

The question remains whether the converse is true, that is do rational polyfractals have a finite number of extrema? We show this for the broad subclass of “fractals of unity” in Section 3.3. But first, we reason why it is sufficient to consider only rational fractals in practice.

Due to the lack of extremal focality in an irrational fractal, the extrema tend to infinitesimally cluster around the vertices of the convex hull. Thus the corners are “rounded off self-similarly” in an infinite number of extrema, so finding the exact convex hull of an irrational
fractal seems hopelessly difficult. We may however turn our efforts to finding the potentially finite convex hull of rational fractals, and fortunately by the theorem below, they approximate the irrational case via the subclass of “fractals of unity”.

**Theorem 3.5** (Continuity in Angular Parameters) Keeping the rotation angles variable as a vector \( \vartheta := (\vartheta_1, \ldots, \vartheta_n) \) while the other parameters of the IFS constant, and denoting the resulting attractor as \( F[\vartheta] \), the map \( \vartheta \mapsto F[\vartheta] \) is continuous in the following domain and range metrics respectively

\[
\delta_\infty(\vartheta, \vartheta') := \max_{k \in \mathcal{N}} |\vartheta_k - \vartheta'_k| \quad \text{and} \quad d_\infty(F, F') := \sup_{a \in \mathcal{A}} |T_a(p) - T_a'(p)|
\]

where \( p \in \mathcal{P} \) is any seed.

**Proof** It is clear that both \( \delta_\infty \) and \( d_\infty \) are metrics. Let \( F[\vartheta] \) be a generate of the IFS \( T_k[\vartheta](z) = p_k + \lambda_k e^{i\vartheta_k}(z - p_k) \), \( k \in \mathcal{N} \) leaving only the rotation angles variable among the IFS parameters. Then we refer to some composition of maps variable in \( \vartheta \) as \( T_a[\vartheta], a \in \mathcal{A} \). Take any fixed point \( p \in \mathcal{P} \) to be the seed. By the compactness of \( F[\vartheta] \) for a fixed \( \vartheta \) we may reasonably restrict the approximating domain in a bounded manner to \( D_\vartheta := \{ \vartheta' \in (-\pi, \pi]^n : \text{diam}(F[\vartheta']) < D[\vartheta] \} \) where \( D[\vartheta] > \text{diam}(F[\vartheta]) \) is fixed and arbitrary. We need to show the following statement for continuity:

\[
\forall \vartheta \in (-\pi, \pi]^n \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \vartheta' \in D_\vartheta \ (\delta_\infty(\vartheta, \vartheta') < \delta) : \ d_\infty(F[\vartheta], F[\vartheta']) < \varepsilon.
\]

In our quest, we first observe that for any finite address \( a \in \mathcal{A}, |a| \leq L \) and seed \( p \in \mathcal{P} \) the map \( \vartheta \mapsto T_a[\vartheta](p) \) is continuous, meaning

\[
\forall L \in \mathbb{N} \ \forall a \in \mathcal{A} \ (|a| \leq L) \ \forall \vartheta \in (-\pi, \pi]^n \ \forall \varepsilon > 0 \ \exists \delta_a > 0
\]

\[
\forall \vartheta' \in D_\vartheta \ (\delta_\infty(\vartheta, \vartheta') < \delta) : \ |T_a[\vartheta](p) - T_a[\vartheta'](p)| < \varepsilon.
\]

Then with the notations \( d_L^\infty(F, F') := \max_{|a| \leq L} |T_a(p) - T_a'(p)| \) and \( \delta := \min_{|a| \leq L} \delta_a \) we get the following reinterpretation of the above logical statement (since \( \forall \) terms are logically commutative) which means continuity in \( d_L^\infty \).

\[
\forall L \in \mathbb{N} \ \forall \vartheta \in (-\pi, \pi]^n \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \vartheta' \in D_\vartheta \ (\delta_\infty(\vartheta, \vartheta') < \delta) : \ d_L^\infty(F[\vartheta], F[\vartheta']) < \varepsilon.
\]

We now sidetrack briefly, and derive an inequality between \( d_\infty \) and \( d_L^\infty \). First we make the observation using the Slope Lemma that for any \( a, b \in \mathcal{A}_{\text{fin}}, |a| = L \) with the notation \( \lambda_* := \max_{k \in \mathcal{N}} \lambda_k \) we can rewrite and estimate the quantity \( |T_{ab[\vartheta]}(p) - T_{ab[\vartheta']}((p))| \) for \( \vartheta \in (-\pi, \pi]^n, \vartheta' \in D_\vartheta \) as follows

\[
|T_{ab[\vartheta]}(p) - T_{ab[\vartheta']}(p)| =
\]

\[
= |(T_{ab[\vartheta]}(p) - T_a[\vartheta](p)) + (T_a[\vartheta](p) - T_a[\vartheta'](p)) + (T_a[\vartheta'](p) - T_{ab[\vartheta']}(p))| \leq
\]

\[
\leq \lambda_a |T_b[\vartheta](p) - p| + |T_a[\vartheta](p) - T_a[\vartheta'](p)| + \lambda_a |p - T_b[\vartheta'](p)| \leq
\]

\[
\leq |T_a[\vartheta](p) - T_a[\vartheta'](p)| + 2\lambda_* D[\vartheta] \ \Rightarrow
\]

12
Proof of \(N = (\vartheta, L, \varphi, \epsilon)\) being fixed numbers that ensure desired continuity.

Finally, we proceed to showing the desired continuity in angular parameters. Let the locus of rotation angles of the IFS be of the form \(\vartheta = k\pi/\nu \mod (\pi, \pi)\) for some \(L \in \mathbb{N}\) and \(\varphi \in \varphi_k\). Theorem 3.5 implies that \(\lambda^*_L D[\vartheta] < \frac{\epsilon}{4}\). Then, by the inequality in \(d_{\infty}\) above, we have for these \(L, \varphi, \epsilon\) fixed numbers that

\[
\exists \delta > 0 \forall \varphi' \in D_{\varphi} \left( \delta_{\infty}(\varphi, \varphi') < \delta \right) : d_{\infty}(F[\varphi], F[\varphi']) < \varepsilon
\]

and with this same \(\delta\) and any such \(\varphi'\) we have by the above inequalities that

\[
d_{\infty}(F[\varphi], F[\varphi']) \leq d_{\infty}(F[\varphi], F[\varphi']) + 2\lambda^*_L D[\vartheta] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

implying the desired property of continuity. \(\square\)

### 3.3 Fractals of Unity

Since only rational fractals can have a finite number of extrema – and thus a determinable convex hull – we focus on the rational subclass of “fractals of unity”, aiming to show that they indeed have a finite number of extrema at the end of this section. Theorem 3.5 implies that ir/rational fractals can be approximated continuously by fractals of unity.

**Theorem 3.6** If all normalized factors \(\frac{\phi_k}{\varphi_k} = e^{\alpha_k}\), \(k \in N\) of the IFS are roots of unity, meaning \(\frac{\phi_k}{\varphi_k} \in \mathbb{Q}\), then the fractal is rational. We refer to such a case as a fractal of unity.

**Proof** If \(\frac{\phi_k}{\varphi_k} \in \mathbb{Q}\), \(k \in N\) then clearly there is a non-trivial linear combination \(k_1 \varphi_1 + \ldots + k_n \varphi_n \equiv 0 \mod (-\pi, \pi)\) with \(k_j \in \{0\} \cup \mathbb{N}\). So for any finite address \(x \in A_{fin}\) for which \(\varphi = k_1 \varphi_1 + \ldots + k_n \varphi_n\) we have \(\nu(x) = 0\) meaning \(p_x \in \text{Foc}(F)\) which implies that \(\text{Foc}(F) \neq \emptyset\). \(\square\)

**Lemma 3.1** If the rotation angles of the IFS are of the form \(\varphi_k = 2\pi N_k / M\) where \(M \in \mathbb{N}, N_k \in [0, M) \cap \mathbb{N}, k \in N = \{1, \ldots, n\}\) then the number of possible address values is \(|\nu(A_{fin})| = M/\gcd(N_1, \ldots, N_n, M)\).

**Proof**

\[
\nu(A_{fin}) = \left\{ \frac{2\pi}{M} (k_1 N_1 + \ldots + k_n N_n) \mod 2\pi : k_j \in \{0\} \cup \mathbb{N}, j \in N \right\} = \]
\[
\frac{2\pi}{M} \left\{ (k_1N_1 + \ldots + k_nN_n) \mod M : k_j \in \{0\} \cup \mathbb{N}, \ j \in \mathcal{N} \right\}.
\]

Bézout’s Lemma implies that the possible values of \((k_1N_1 + \ldots + k_nN_n) \mod M\) are limited to \(DN \mod M (N \in \mathbb{N})\) where \(D = \gcd(N_1, \ldots, N_n)\). Since \(|\{DN \mod M : N \in \mathbb{N}\}| = M/\gcd(D, M)\) we have that \(|\nu(A_{fin})| = M/\gcd(N_1, \ldots, N_n, M)|. □

**Lemma 3.2** In a fractal of unity, any infinite address \(a \in A_{\infty}\) begins with a truncation \(bx < a\) for which \(b, x \in A_{fin}, \nu(x) = 0, \ x \neq 0\).

**Proof** By the above Lemma 3.1, the truncations of an infinite address \(a \in A_{\infty}\) cannot all be distinct in value, since that would require an infinite number of possible values. Thus
\[
\exists b, x \in A_{fin} : bx < a, \ x \neq 0, \nu(b) = \nu(bx) = \nu(b) + \nu(x)
\]
implies that \(\nu(x) = 0\). □

**Theorem 3.7** In a fractal of unity, all extremal points are eventually focal.

**Proof** Take any \(e \in \text{Ext}(F)\). Then by the above Lemma 3.2 we have \(\exists b, x A_{fin} (\nu(x) = 0, \ x \neq 0) : bx < \text{adr}(e)\). By the Extremal Theorem 3.3 we get that \(e = T_b(p_x) \in \text{EFoc}(F)\). □

**Corollary 3.3** A fractal of unity has at least one focal extremal point.

**Proof** This follows from the above theorem by inverse iteration, see Theorem 3.1. □

A finite number of extrema would imply that all are eventual by Theorem 3.4, thus Theorem 3.7 is a weaker statement that hints at the possibility of finiteness of extrema, which will soon be shown. The existence of a focal periodic extremal point by the corollary, will be exploited in Sections 4.4 and 5 to generate the entire set of extremal points. We proceed to introducing the concept of “irreducibility” of extremal addresses, which by the above theorem are necessarily eventually focal for fractals of unity. This concept will prove to be critical for showing the finiteness of extrema.

**Definition 3.5** Define the **forms** of an eventually focal fractal point \(f \in F\) with respect to the IFS \(T\ (F = \langle T \rangle)\), as the set
\[
\text{Frm}(f) = \text{Frm}_{\mathcal{T}}(f) = \left\{ (b, x) \in A_{fin}^2 : \text{adr}(f) = b\bar{x}, \nu_{\mathcal{T}}(x) = 0, \ x \neq 0 \right\}
\]
and note that \((b, x)\) also conveniently represents the concatenation \(bx \in A_{fin}\). A form \((b, x) \in \text{Frm}(f)\) is **irreducible** if there is no shorter form that represents \(f\), meaning \(\nexists (c, y) \in \text{Frm}(f) : |cy| < |bx| (\iff cy \leq bx)\). If a shorter \((c, y)\) form does exist, it is called a **reduction** of \((b, x)\). An eventually focal point \(f = T_b(p_x) \in F\) is said to be in **irreducible form** if \((b, x) \in \text{Frm}(f)\) is irreducible (note that \(b = 0\) is permitted).
This definition becomes relevant in light of the Extremal Theorem 3.3 (ET), which is essentially an “inside-out blow-up property” for extremal points \( e \in \text{Ext}(F) \). Regardless of what comes after the reduction \( cy < \text{adr}(e) \), the ET implies that \( \text{adr}(e) = cy \) meaning \( e = T_c(p_y) \). Irreducibility implies that no such reduction or simplification of the \( \text{adr}(e) = b\bar{x} \) representation is possible.

**Theorem 3.8** Any extremal point \( e \in \text{Ext}(F) \) in a fractal of unity has a unique irreducible form, and it is a reduction of any other form representing \( e \). Denote it as \( \text{ifrm}(e) \).

**Proof** Clearly \( \forall e \in \text{Ext}(F) : \text{Frm}(e) \neq \emptyset \) by Theorem 3.7. We first prove the existence of the irreducible form by infinite descent. Suppose indirectly that \( e \in \text{Ext}(F) \), \( a := \text{adr}(e) \) has no irreducible form, meaning any form \( (b, x) \in \text{Frm}(e) \) has a reduction \( (c, y) \in \text{Frm}(e) \). Clearly \( \text{Frm}(e) \neq \emptyset \) by Theorem 3.7, so starting with a form it cannot be reduced ad infinitum, since any form must have at least a length of one. So for the address \( a = \text{adr}(e) \in \mathcal{A}_\infty \) the reductions lead inductively to \( (0, a(1)) \in \text{Frm}(e) \) which is irreducible, leading to a contradiction of the assumption that \( \#\text{ifrm}(e) \).

We proceed to showing the uniqueness of an irreducible form. Suppose indirectly that there are two distinct irreducible forms \( (b, x), (c, y) \in \text{Frm}(e) \). Then \( |bx| = |cy| \) otherwise one would be a reduction of the other. Since both \( bx, cy < \text{adr}(e) \) we have that \( bx = cy \). If \( b = c \) then \( x = y \) leading to a contradiction, so without the restriction of generality, we may suppose that \( b \leq c \) meaning \( \exists z \neq 0 : c = bz \). Then \( \nu(b) = \nu(bz) = \nu(cy) = \nu(c) = \nu(b) + \nu(z) \) implying \( \nu(z) = 0 \), therefore \( bz = c \leq cy < a \) so by the ET \( (b, z) \in \text{Frm}(e) \) and it is a reduction of \( (c, y) \), contradicting the assumption that the latter is irreducible.

Lastly, the irreducible form is shorter than any other form, since if there were another form of the same length, then that too would need to be irreducible (since no shorter form exists), which would contradict the uniqueness of the irreducible form. So it must be a truncation and thus a reduction of any other form. \( \square \)

**Lemma 3.3** If \( e = T_b(p_x) \in \text{Ext}(F) \) is in irreducible form, then so is \( p_x \in \text{Inv}(e) \subset \text{Ext}(F) \).

**Proof** Suppose indirectly that \( \exists (c, y) \in \text{Frm}(p_x) : |cy| < |0x| = |x| \). Then \( e = T_b(p_x) = T_{bc}(p_y) \) and \( |bcy| = |b| + |cy| < |b| + |x| = |bx| \) implying that \( (bc, y) \in \text{Frm}(e) \) and that it is a reduction of \( (b, x) \), contradicting that the latter form is irreducible. \( \square \)

**Theorem 3.9** If \( e = T_b(p_x) \in \text{Ext}(F) \) is in irreducible form, then \( |\text{Inv}(e)| = |b| + |x| \) where \( \text{Inv}(e) = \{ T_{c}^{-1}(e) : c \leq b \} \cup \text{Cyc}(x) \) and this union is disjoint.

**Proof** First we show that if \( p_x \in \text{Ext}(F) \cap \text{Foc}(F) \) is in irreducible form, then \( |\text{Cyc}(x)| = |x| \). Clearly \( |\text{Cyc}(x)| \leq |x| \) and suppose indirectly that \( |\text{Cyc}(x)| < |x| \). Then the cycle must have two identical elements \( \exists b, c (b \leq c \leq x) : T_{b}^{-1}(p_x) = T_{c}^{-1}(p_x) \). Then \( \exists y \neq 0 : c = by \) so \( T_{c}^{-1} = T_{y}^{-1}T_{b}^{-1} \) implying that \( T_{y}(T_{c}^{-1}(p_x)) = T_{b}^{-1}(p_x) = T_{c}^{-1}(p_x) \) meaning \( p_y = T_{c}^{-1}(p_x) \in \text{Cyc}(x) \subset \text{Ext}(F) \) which by Theorem 3.2 implies \( \nu(y) = 0 \). Thus we have by \( c \leq x, \nu(y) = 0, y \neq 0 \) so \( |by| < |x| \) which by the ET implies that \( p_x = T_b(p_y) \) and so \( (b, y) \in \text{Frm}(p_x) \) is a reduction of \( (0, x) \), contradicting that the latter is irreducible. Thus \( |\text{Cyc}(x)| = |x| \).
For the general case, take an $e = T_b(p_x) \in \text{Ext}(F)$ in irreducible form, and first note that $\text{Inv}(e) = \{T_c^{-1}(e) : c \leq b\} \cup \text{Cyc}(x)$. Since by Lemma 3.3 the extremal point $p_x$ is also in irreducible form, we have by the above that $|\text{Cyc}(x)| = |x|$, so we only need to show that $|\{T_c^{-1}(e) : c \leq b\}| = |b|$ and $\{T_c^{-1}(e) : c \leq b\} \cap \text{Cyc}(x) = \emptyset$ to arrive at $|\text{Inv}(e)| = |b| + |x|$. Clearly $|\{T_c^{-1}(e) : c \leq b\}| \leq |b|$ holds, and suppose indirectly that $|\{T_c^{-1}(e) : c \leq b\}| < |b|$. Then $\exists c, d \leq b (c \neq d) : T_c^{-1}(e) = T_d^{-1}(e)$ and we can assume that $c \leq d$ without restricting generality. Thus $\exists y \neq 0 : d = cy$ so $T_y(T_d^{-1}(e)) = T_c^{-1}(e) = T_d^{-1}(e)$ implying that $p_y = T_d^{-1}(e)$ but since $e \in \text{Ext}(F)$ and $d \leq b < \text{adr}(e)$ we have that $p_y \in \text{Inv}(e) \subset \text{Ext}(F) \cap \text{Per}(F) \subset \text{Foc}(F)$ by Theorem 3.10, so we can conclude that $\nu(y) = 0$. Since $p_y = T_d^{-1}(e) = T_c^{-1}(e)$ we have that $T_c(p_y) = e$, $\nu(y) = 0$ implying $(c, y) \in \text{Frm}(e)$ and $|cy| = |d| < |b| < |bx|$, meaning that $(c, y)$ is a reduction of $(b, x)$ contradicting that the latter is irreducible. So we have that $|\{T_c^{-1}(e) : c \leq b\}| = |b|$.

Lastly we show that $\{T_c^{-1}(e) : c \leq b\} \cap \text{Cyc}(x) = \emptyset$. Suppose indirectly that $\exists c \leq b, x \leq x : e_0 := T_c^{-1}(T_b(p_x)) = T_d^{-1}(p_x)$. Then clearly $\exists y, z \neq 0 : b = cy, x = dz$ so we have $T_b^{-1} = T_y^{-1}T_c^{-1}$ which implies $T_d^{-1}(p_x) = T_c^{-1}(T_b(p_x)) = T_b(p_x)$ and thus $T_b(p_x) = p_x$. Therefore $p_{dy} = p_x \in \text{Ext}(F)$ which by Theorem 3.2 implies $\nu(yd) = \nu(dy) = 0$. Furthermore $yd \neq 0$ since $y \neq 0$, and also $cyd = bd \leq bx$ so by the ET we have $e = T_b(p_x) = T_c(p_{yd})$ meaning $(c, yd) \in \text{Frm}(e)$ is a reduction of $(b, x)$ which contradicts that the latter is irreducible. \Box

**Theorem 3.10** If an eventually focal extremal point $e = T_b(p_x) \in \text{Ext}(F)$ is in irreducible form, then so are its inverse iterates $\text{Inv}(e) = \{T_c^{-1}(e) : c \leq b\} \cup \text{Cyc}(x)$.

**Proof** First we show this for focal extrema, meaning if $e = p_x \in \text{Ext}(F) \cap \text{Foc}(F)$ is in irreducible form, then for any $a \leq x$ the form $(0, a^{-1}xa) \in \text{Frm}(p_{a^{-1}xa})$ is irreducible. Suppose indirectly that for some $a \leq x$ it is reducible, meaning $\exists (b, y) \in \text{Frm}(p_{a^{-1}xa}) : |by| < |a^{-1}xa| = |x|$ and we may assume that this $(b, y)$ is the irreducible form of $p_{a^{-1}xa}$ by Theorem 3.8. Then by the above Theorem 3.9 we have that $|by| = |\text{Inv}(T_b(p_y))| = |\text{Inv}(p_{a^{-1}xa})| = |\text{Cyc}(a^{-1}xa)| = |\text{Cyc}(x)| = |x|$ where the latter equality holds since $p_x$ was assumed to be irreducible. Therefore $|by| = |x|$ which contradicts $|by| < |x|$ above. For the general case, take an $e = T_b(p_x) \in \text{Ext}(F)$ in irreducible form. Lemma 3.3 implies that $p_x$ is also in irreducible form, so by the above the elements of $\text{Cyc}(x)$ are as well. Thus we only need to show that $(c^{-1}b, x) \in \text{Frm}(T_c^{-1}(e))$ is irreducible for any $c \leq b$. Clearly $\exists d \neq 0 : b = cd$ meaning $d = c^{-1}b$, so we need to show that $(d, x) \in \text{Frm}(T_c^{-1}(e))$ is irreducible. Suppose indirectly that a reduction exists $\exists (a, y) \in \text{Frm}(T_c^{-1}(e)) : T_a(p_y) = T_d(p_x)$, $|ay| < |dx|$. Then $e = T_b(p_x) = T(T_d(p_x) = T_c(p_y)$ and $|cay| = |c| + |ay| < |c| + |dx| = |cdx| = |cd| + |x| = |b| + |x| = |bx|$ implying that $(ca, y) \in \text{Frm}(e)$ is a reduction of $(b, x) \in \text{Frm}(e)$ contradicting that the latter form is irreducible. \Box

These theorems show that Definition 3.5 is not merely an intuitive definition, but the proper way to define irreducibility, in order to arrive at the expected properties above.

**Theorem 3.11** A focal extremal point cannot have an irreducible form longer than $|\nu(A_{\text{fin}})|$, and an eventually focal extremal point cannot have an irreducible form longer than $2|\nu(A_{\text{fin}})|$. 
Proof First consider indirectly a focal \( e = p_x \in \text{Ext}(F) \) in irreducible form, for which \( |x| > |\nu(A_{\text{fin}})| \). Then the truncations of \( x \) cannot all be different in value, so \( \exists b, c \leq x \) \( (b \leq c) : \nu(b) = \nu(c) \). Thus \( \exists y \neq 0 : c = by \) and \( \nu(b) = \nu(c) = \nu(b) + \nu(y) \) which implies \( \nu(y) = 0 \), so since \( by = c < x < \text{adr}(e) \) by the ET \( e = T_b(p_y) \) meaning \( (b, y) \in \text{Frm}(e) \) and \( |by| = |c| < |x| \) which contradicts that \((0, x) \in \text{Frm}(e)\) is irreducible.

Now consider indirectly an eventually focal \( e = T_b(p_x) \in \text{Ext}(F) \) in irreducible form, for which \( |bx| > 2|\nu(A_{\text{fin}})| \). Then either \( |x| > |\nu(A_{\text{fin}})| \) or \( |b| > |\nu(A_{\text{fin}})| \) must hold (if neither held, then \( |bx| \leq 2|\nu(A_{\text{fin}})| \)). But by Lemma 3.3 we know that \( p_x \) is also in irreducible form, so by the above \( |x| \leq |\nu(A_{\text{fin}})| \) thus necessarily \( |b| > |\nu(A_{\text{fin}})| \) must hold. Then the truncations of \( b \) cannot all be different in value, so \( \exists c, d \leq b \) \( (c \leq d) : \nu(c) = \nu(d) \).

Thus \( \exists y \neq 0 : d = cy \) and \( \nu(c) = \nu(d) = \nu(c) + \nu(y) \) which implies \( \nu(y) = 0 \), so since \( cy = d < b < \text{adr}(e) \) by the ET \( e = T_c(p_y) \) meaning \( (c, y) \in \text{Frm}(e) \) and \( |cy| = |d| < |b| < |bx| \) contradicting that \((b, x) \in \text{Frm}(e)\) is irreducible. □

**Theorem 3.12** (Cardinality of Extrema II) A fractal of unity \( F = \langle T_1, \ldots, T_n \rangle \) has a finite number of extremal points, specifically \( |\text{Ext}(F)| \leq n^{2|\nu(A_{\text{fin}})|} \).

**Proof** By Theorem 3.7 all extremal points are eventually focal, and by Theorem 3.8 all can be written in irreducible form. By Theorem 3.11 the irreducible form of an eventually focal extremal point cannot be longer than \( 2|\nu(A_{\text{fin}})| \), so the maximum number of extremal points is \( n^{2|\nu(A_{\text{fin}})|} \) since there are \( n \) possible choices for each coordinate (or IFS map) up to that length. □

## 4 Methods for the Convex Hull

We proceed to the main results of this paper in determining the exact convex hull of polyfractals. “Exactness” is emphasized throughout to make it clear that the resulting extrema are the actual explicit extremal points, and not merely approximate or theoretical, as often is the case in the literature. The introduced “Armadillo Method” can be carried out in practice, as shown in the examples of Section 5.4.

As reasoned earlier, the case of irrational fractals seems hopelessly difficult, so we restrict our attention to fractals of unity, which by continuity in parameters approximate the irrational case.

### 4.1 A Method for Equiangular Fractals of Unity

Corollary 3.3 and Theorems 3.2 and 3.7 hinted at the relevance of focal periodic extrema for the convex hull, but their fundamental role will only become clear in this section. We derive the convex hull of perhaps the simplest type of polyfractals, relevant to self-affine fractals.
Theorem 4.1 An $F = \langle T_1, \ldots, T_n \rangle$, $\varphi_k = 2\pi N/M$, $M \in \mathbb{N}$, $N \in [0, M) \cap \mathbb{N}$ $(k \in \mathbb{N})$ equiangular fractal of unity can be generated as a Sierpiński fractal $F = \langle T_x : |x| = M' \rangle$ where $M' = M/\gcd(N, M)$, and furthermore $\text{Ext}(F) = \text{Ext}(p_x : |x| = M')$.

Proof Applying the fixed point property $F = H(F)$ repeatedly, we get that $F = H(F) = H^{M'}(F) = \bigcup_{|x|=M'} T_x(F)$ which implies that $F = \langle T_x : |x| = M' \rangle$. For any $x \in \mathcal{A}_{\text{fin}}$, $|x| = M'$ we have that $\nu(x) = 0$ since $M'\varphi_k = (2\pi N/M)(M/\gcd(N, M)) \equiv 0 \pmod{2\pi}$. Thus for every $T_x$ in the IFS $\mathcal{T}' = \{T_x : |x| = M'\}$ with fixed points $\mathcal{P}' = \{p_x : |x| = M'\}$ we have $\nu(x) = 0$, meaning $\mathcal{T}'$ generates a Sierpiński fractal $F = \langle \mathcal{T}' \rangle$, so Theorem 3.2 ensures that Conv($F$) = Conv($\mathcal{P}'$) implying that Ext($F$) = Ext($\mathcal{P}'$). □

The theorem implies that having equal IFS rotation angles $2\pi N/M$, it is sufficient to generate all $M'$-th level periodic points (computed using Corollary 2.1), as their convex hull will be that of the fractal. This is a simplified version of the method presented by Kırat and Koçyiğit [8] for the special case of planar equiangular IFS fractals.

To generalize the above to higher dimensions, define a focal address $x \in \mathcal{A}_{\text{fin}}$ in terms of matrix IFS factors $\varphi_k \in \mathbb{R}^{d \times d}$ as one for which $\varphi_x$ is a scalar multiple of the identity matrix. Then the part of Theorem 3.2 guaranteeing Conv($F$) = Conv($\mathcal{P}$) for Sierpiński fractals generalizes accordingly, to support the above proof.

4.2 A General Method for Fractals of Unity

A general method for finding the convex hull of fractals of unity $F = \langle \mathcal{T} \rangle$, $\mathcal{T} = \{T_1, \ldots, T_n\}$ is presented, which have a finite number of extrema according to Theorem 3.12, though the method also works for rational fractals in general. The presented method is a reworked simplification of the one by Kırat and Koçyiğit [8] for the case of fractals of unity based on Section 3.3, the significant improvement being that termination is hereby guaranteed by Theorem 3.11.

Heavily utilizing the Extremal Theorem 3.3 (ET), we take an “inside-out blow-up approach”, meaning we attempt to find the convex hull by address generation. Proving rather inefficient, we take an “outside-in” approach with the Armadillo Method in Section 4.4, using the linear optimization algorithm of Section 4.3, making the two approaches philosophical antitheses.

According to Theorem 3.11, the extremal points of a fractal of unity have irreducible forms no longer than $2|\nu(\mathcal{A}_{\text{fin}})|$ implying

$$\text{Ext}(F) = \text{Ext}(T_b(p_x) : |bx| \leq 2|\nu(\mathcal{A}_{\text{fin}})|, \nu(x) = 0, x \neq 0)$$

so the challenge becomes to compute the latter set efficiently, considering that its cardinality could reach $n^{2|\nu(\mathcal{A}_{\text{fin}})|}$ making its computation potentially exponential in runtime. So we take a shortcut via irreducibility and a containment property.
Definition 4.1 An address \(a \in \mathcal{A}_{\text{fin}}\) is blowable if \(\exists b, x : bx < a, \nu(x) = 0, x \neq 0\) denoted as \(\exists \text{blo}(a)\), and taking the shortest such \(bx\) denote its blow-up as \(\text{blo}(a) := T_b(p_x)\) (note that \((b, x) \in \text{Frm}(\text{blo}(a))\) is then irreducible). Denote the blow-up of a set \(\mathcal{A}_0 \subset \mathcal{A}_{\text{fin}}\) as \(\text{blo}(\mathcal{A}_0) := \{\text{blo}(a) : a \in \mathcal{A}_0, \exists \text{blo}(a)\}\). A set \(\mathcal{A}_0 \subset \mathcal{A}_{\text{fin}}\) is blowable if all of its elements are, denoted as \(\exists \text{blo}(\mathcal{A}_0)\). Define the set of eventually focal points of a level \(L \in \mathbb{N}\) as the blow-up of all blowable \(L\)-long addresses \(\text{EFoc}_L(F) := \{\text{blo}(a) : |a| = L, \exists \text{blo}(a)\}\). (Note the property \(\text{EFoc}_L(F) \subset \text{EFoc}_L(F) \forall L \leq L_0\).

The set \(\text{EFoc}_L(F)\) can be easily generated via an algorithm which examines longer-and-longer addresses for eventual focality, and if a \((b, x)\) form is found (meaning \(\nu(b) = \nu(bx), x \neq 0\)) it will remain irreducible for higher iteration levels (see Section 3.3) so no further addresses \(c > bx\) need to be examined (corresponding to sub-fractals of \(T_bT_x(F)\)), according to the ET. Iterating up to level \(2|\nu(\mathcal{A}_{\text{fin}})|\), the truncations of the extremal addresses necessarily emerge, due to Theorems 3.11, 3.7.

According to the Containment Lemma 2.2, if the containment property \(H(S) \subset \text{Conv}(S)\) holds for some compact \(S \subset \mathbb{C}\), then \(F = \langle \mathcal{T} \rangle \subset \text{Conv}(S)\) implying that \(\text{Conv}(F) \subset \text{Conv}(S)\). Furthermore if \(S \subset F\) also holds, then \(\text{Conv}(S) \subset \text{Conv}(F)\) so we can conclude that \(\text{Conv}(F) = \text{Conv}(S)\) or equivalently that \(\text{Ext}(F) = \text{Ext}(S)\). So we attempt to find such a set \(S\) among the sets \(\text{EFoc}_L(F) \subset F\) for increasing iteration levels \(L \in \mathbb{N}\). Theorem 3.11 guarantees that such a level \(L_\nu \leq 2|\nu(\mathcal{A}_{\text{fin}})|\) will be found, implying that the method below is guaranteed to terminate with an output satisfying \(\text{Ext}(F) = \text{Ext}(\text{EFoc}_{L_\nu}(F))\).

Method 4.1 A method for finding the exact convex hull of IFS fractals of unity.

1. Let the initial level be \(L := 1\).

2. Compute the set \(\text{EFoc}_L(F)\) and test whether the containment property \(H(\text{EFoc}_L(F)) \subset \text{Conv}(\text{EFoc}_L(F))\) holds. If it does, then let \(L_\nu := L\) and go to Step 3. If it does not, then increase the level \(L\) by one and repeat Step 2.

3. Conclude that \(\text{Ext}(F) = \text{Ext}(\text{EFoc}_{L_\nu}(F))\).

The specific simplifications relative to the method of Kirat and Koçyigit [8] are:

- The main difference is the increased efficiency due to the introduced idea of “focality” and the ET. Their method blows up addresses of the form \(bx^c, l \in \mathbb{N}\) (not considering the focality of \(x\)), while here a \(bx\) \(\nu(x) = 0\), \(x \neq 0\) form is sufficient for blow-up due to the ET. Computationally this is a significant improvement, since searching for a repeating part \(x\) in some \(a = bx^c\), \(|a| = L\) is more costly than reaching a point when \(\nu(b) = \nu(a), |a| = L\) for some \(b \leq a\) in the address generation \(n\)-ary tree. This also excludes redundant eventually periodic addresses where the periodic part is not focal, in light of Theorem 3.7.

- The above change guarantees the termination of this method at some \(L_\nu \leq 2|\nu(\mathcal{A}_{\text{fin}})|\) due to Theorem 3.11, which was not guaranteed by their method.
Instead of the containment property of Step 2, their method essentially checks whether the condition \( (H(E_L) \setminus E_L) \cup \text{Ext}(F_L) \subset \text{Conv}(E_L) \) holds, where \( F_L := H^L(\{p\}) \), \( p \in \mathcal{P} \), \( E_L := \text{Efoc}_L(F) \). Here \( E_L \subset \text{Conv}(E_L) \) necessarily, so this condition can be simplified to \( H(E_L) \cup \text{Ext}(F_L) \subset \text{Conv}(E_L) \). It is even less restrictive and sufficient to just require the containment property \( H(E_L) \subset \text{Conv}(E_L) \) as explained prior to the above method, since \( S = E_L \subset F \) is compact. This is the containment checked in Step 2, eliminating the redundancy of finding \( \text{Ext}(F_L) \) and further set operations.

Method 4.1 readily generalizes to higher dimensions as well, according to the remarks made after Theorem 4.1 about defining focality with matrix IFS factors. However appealing for its generality, the method is inefficient for the following reasons:

- The computational cost of evaluating the sets \( \text{Efoc}_L(F) \) can be high in Step 2, potentially carried out for each \( L \leq 2|\nu(A_{fin})| \). Though the property \( \text{Efoc}_l(F) \subset \text{Efoc}_L(F) \) \( (l \leq L) \) and the remarks after Definition 4.1 can increase the efficiency of evaluation cumulatively.

- Checking whether \( H(\text{Efoc}_L(F)) \subset \text{Conv}(\text{Efoc}_L(F)) \) holds can carry a significant risk of computational error. If the cardinality \( |\text{Ext}(F)| \) is large, then a program may arrive at the erroneous conclusion that the containment property is true prematurely, not at the actual sought level \( L_* \) since the extrema tend to cluster, as noted in the remarks after Theorem 3.4.

- Figures 7 and 10 generated from a seed, show why an “inside-out” address generation approach (Theorem 2.2) cannot possibly tackle the convex hull problem robustly. For certain IFS parameters, the extremal period length \( |x| \) can be large (in these figures 262 and 41 respectively), potentially causing an exponential runtime for the iterative address generation of \( \text{Efoc}_L(F) \) (see Theorems 3.11, 3.12).

As projected earlier, due to the above issues with this “inside-out” method, an alternative “outside-in” approach is presented in Section 4.4.

### 4.3 Linear Optimization over IFS Fractals

A method is introduced for maximizing a linear target function over a fractal of unity \( F = \langle T \rangle \), \( T = \{T_1, \ldots, T_n\} \) utilizing bounding circles and the Containment Lemma 2.2.

**Definition 4.2** We say that a closed ball \( C = B(c, r) = \{z \in \mathbb{C} : |z - c| \leq r\} \) with center \( c \in \mathbb{C} \) and radius \( r > 0 \) is a bounding circle to the fractal \( F \) if it contains it \( C \supset F \). It is an ideal bounding circle if it is invariant with respect to the Hutchinson operator \( H_T(C) \subset C \) and \( c \in \text{int Conv}(F) \).

The following \( B(c, r) \) is an ideal bounding circle for any IFS fractal [18]:

\[
c = \frac{1}{n} \sum_{k \in \mathcal{N}} p_k, \quad r = \frac{\mu_*}{1 - \lambda_*} \max_{k \in \mathcal{N}} |p_k - c| \quad \text{where} \quad \lambda_* = \max_{k \in \mathcal{N}} |\phi_k|, \quad \nu_* = \max_{k \in \mathcal{N}} |1 - \phi_k|.
\]
The method will search for maximizing truncation(s) recursively over the iterates \(T_a(C)\). To make the search efficient, it compares and discards iterates which are “dominated” by others, with respect to a target vector \(\tau\) and the corresponding target function.

**Definition 4.3** We say that a finite address \(a \in A_{\text{fin}}\) dominates another \(b \in A_{\text{fin}}\) with respect to the target \(\tau \in \mathbb{C}\), the IFS \(\mathcal{T} = \{T_1, \ldots, T_n\}\), and the ideal bounding circle \(C = B(c, r) \subseteq \mathbb{C}\), if \(|a| = |b|\) and \(\langle \tau, T_a(c) - T_b(c) \rangle \geq \lambda_b r |\tau|\). Denoted as \(a \succ_{\tau,\mathcal{T},C} b\) or just \(a \succ b\).

![Figure 2: Illustration of a finite address dominating another for the parameters: \(\varphi_1 = 0.7 \exp \left(\frac{5\pi i}{15}\right), \varphi_2 = 0.6 \exp \left(\frac{3\pi i}{15}\right)\) (the fixed points only affect scaling; see Section 5.1).](image)

Note that the inequality in this definition is equivalent to the following

\[
t_1 := \langle \tau, T_a(c) \rangle \geq t_2 := \left\langle \tau, T_b(c) + \lambda_b r \frac{\tau}{|\tau|} \right\rangle
\]

which is illustrated by Figure 2. This implies that the centre of the iterated circle \(T_a(C) \subseteq H^L(C)\), \(L \in \mathbb{N}\) has a greater-than-or-equal target value \(t_1\) than any point (including the maximizing value \(t_2\)) over \(T_b(C) \subseteq H^L(C)\). Due to \(c \in \text{int Conv}(F)\), we can infer that there must be a point in the subfractal \(T_a(F) \subseteq H^L(F) = F\) which has a strictly larger target value than any of the points in the subfractal \(T_b(F) \subseteq H^L(F) = F\), and therefore the maximizing algorithm can discard \(T_b(F)\) and thus \(T_b(C)\). This greatly increases the efficiency of seeking maximizers of \(\tau\).

**Theorem 4.2** The relation \(\succ\) is a strict partial order over finite addresses, meaning it is an ordering relation that is irreflexive, transitive, and asymmetric.
Proof. We see that the relation is irreflexive, since $a \succ a$ would imply that $0 = \langle \tau, T_a(c) - T_a(c) \rangle \geq \lambda_b r|\tau| > 0$ which is a contradiction.

To show transitivity, assuming that $a_1 \succ a_2 \succ a_3$, $a_{1,2,3} \in A_{fin}$ we need $a_1 \succ a_3$.

$$\langle \tau, T_{a_1}(c) - T_{a_3}(c) \rangle = \langle \tau, T_{a_1}(c) - T_{a_2}(c) \rangle + \langle \tau, T_{a_2}(c) - T_{a_3}(c) \rangle \geq \lambda_{a_3} r|\tau| \Rightarrow a_1 \succ a_3.$$  

Lastly for asymmetry, we need that if $a \succ b$ then $b \succ a$ cannot hold. Clearly $a \succ b$ implies that $\langle \tau, T_a(c) - T_b(c) \rangle$ is strictly positive, but if $b \succ a$ also held then $\langle \tau, T_b(c) - T_a(c) \rangle = -\langle \tau, T_a(c) - T_b(c) \rangle$ would also be strictly positive, which is a contradiction. □

Definition 4.4. We say that an element is maximal among the finite addresses $A_0 \subset A_{fin}$ if no other element dominates it. Furthermore, denote the subset of maximal addresses as $\text{Argmax}(A_0) = \text{Argmax}_{\tau, T, C}(A_0) = \{ a \in A_0 : \not\exists b \in A_0 : b \succ_{\tau, T, C} a \}$.

Figure 3: Illustration of Argmax at some iteration level.

The above definition for the “maximal element(s)” in a subset of finite addresses is the standard way for partial ordering relations, illustrated in Figure 3. It depicts in the highlighted circle iterates the four maximal addresses with respect to the target $\tau$ and a bounding circle $C$, at some iteration level $L \in \mathbb{N}$ by the Hutchinson operator $H$. These addresses are considered to be maximal, since no other address dominates them at that iteration level. In the iterative maximization, we keep these addresses for the next iteration.

Finally, we arrive at the algorithm employing the above concepts. The algorithm uses the invariance of the ideal bounding circle $C \supset H(C)$ to eliminate the redundant non-maximal iterated circles of the form $T_a(C)$. The recursiveness lies in the step from potential maximizing truncations $A_0$ to $A_0 \times N$, corresponding to the subdivision of each circle $T_a(C)$, $a \in A_0$ into the sub-circles $T_a(T_k(C))$, $k \in N$ to be compared for maximality. Only the maximal
ones survive, and can again be subdivided iteratively for comparison. At each iteration, we have the properties
\[
\arg\max_{f \in F} \langle \tau, f \rangle \subset \bigcup_{a \in A_0} T_a(F) \subset \bigcup_{a \in A_0} T_a(C).
\]
So upon iteration, a set \(A_0 \subset A_{\text{fin}}\) of potential maximizing truncations are accumulated, and then the circles \(T_a(C), a \in A_0\) are subdivided further. According to Lemma 3.2 and Theorem 3.11, as the subdivisions progress and the addresses in \(A_0\) get longer, all addresses in \(A_0 \subset A_{\text{fin}}\) eventually become blowable (see Definition 4.1), since extremal irreducible forms cannot be longer than \(2|\nu(A_{\text{fin}})|\) according to Theorem 3.11. The blow-ups can then be verified to maximize the target function \(z \mapsto \langle \tau, z \rangle\) \((z \in \mathbb{C})\). So the stopping criterion of the algorithm will be whether \(\exists \text{blo}(A_0)\) holds, which must eventually occur within \(2|\nu(A_{\text{fin}})|\) iterations. The returned output will be
\[
\text{ifrm}
\left(\arg\max_{f \in \text{blo}(A_0)} \langle \tau, f \rangle\right)
\]
representing the irreducible forms belonging to the maximizing blow-ups of \(A_0\). A target is maximized in either one or two extrema in \(\text{blo}(A_0) \cap \text{Ext}(F)\), so the above argmax set has either one or two elements.

**Algorithm 1** (Linear Optimization Algorithm for IFS Fractals of Unity; LOAF)
Determines the irreducible forms of the maximizing extrema of a target direction \(\tau \in \mathbb{C}\) over a fractal of unity \(F = \langle T \rangle\) (see Section 3.3), using an ideal bounding circle \(C = B(c, r)\). To be called with \(A_0 = \mathcal{N}\).

1: function \text{FractalLinOpt}(A_0, \tau, T, C)
2: \quad if \exists \text{blo}(A_0) \then
3: \quad \quad return ifrm \left(\arg\max_{f \in \text{blo}(A_0)} \langle \tau, f \rangle\right)
4: \quad else
5: \quad \quad \quad \quad \quad \quad A_0 := \text{Argmax}_{\tau, T, C}(A_0 \times \mathcal{N}) \quad \triangleright \text{Subdivision of } H^L(C) \text{ into } H^L(H(C)).
6: \quad \quad return \text{FractalLinOpt}(A_0, \tau, T, C)
7: \quad end if
8: end function

To ensure that the pseudocode can be implemented in a more efficient way as a program, we relate the elements of \(A_0\) to the calculation of \(\text{Argmax}_{\tau, T, C}(A_0 \times \mathcal{N})\). This corresponds to the subdivision of circles into their local iterates, and finding the maximal among them. Thus the calculation of Argmax is simplified via the following equivalences:
\[
a \in \text{Argmax}_{\tau, T, C}(A_0 \times \mathcal{N}) \iff \exists b \in A_0 \times \mathcal{N} : b \succ a \iff
\]
\[
\iff \forall b \in A_0 \times \mathcal{N} : \left(\tau, T_a(c) + \lambda_a r \frac{\tau}{|\tau|}\right) > \langle \tau, T_b(c) \rangle \iff
\]
\[
\left(\tau, T_a(c) + \lambda_a r \frac{\tau}{|\tau|}\right) > \max_{b \in A_0 \times \mathcal{N}} \langle \tau, T_b(c) \rangle \quad \text{for each } a \in A_0 \times \mathcal{N}.
\]
So we see that the next-level iterates of $C$ can be tested for maximality amongst one another, by first calculating the constant max in the last inequality, and then comparing it to each new target value on the left, for $a \in A_0 \times N$. In order to make the above calculations even more efficient, we can keep track of the iterated centers $T_a(c)$ and the iterated fixed points $T_a(p_j), j \in N$, since they imply the next-level iterates as follows:

$$T_aT_k(c) = T_a(p_k) + \varphi_k(T_a(c) - T_a(p_k)) \quad (a \in A_0, \ k \in N)$$

$$T_aT_k(p_j) = T_a(p_k) + \varphi_k(T_a(p_j) - T_a(p_k)) \quad (a \in A_0, \ k,j \in N).$$

These equations follow from the fact that $T_a$ is an affine map (Corollary 2.2).

### 4.4 The Armadillo Method for Regular Fractals

The method to be introduced utilizes certain ideal target directions – called “regular directions” – to locate one focal extremal point, which is exploited via its cycle to generate a set of neighboring extrema. These extrema when iterated by each IFS map alone, will map out the entire convex hull. This method works if such a target direction exists, so in the next section we discuss a general heuristic candidate for bifractals, called the “principal direction”. We continue to discuss fractals of unity $F = \langle \mathcal{T} \rangle$, $\mathcal{T} = \{T_1, \ldots, T_n\}$.

**Definition 4.5** We say that two distinct extremal points are **neighbors** if the fractal is a subset of a closed half-space determined by the line connecting them. The right/left **neighbor** of an extremal point is the neighbor which comes counter/clockwise around the vertices of the convex hull. A set of extrema is **consecutive** if each element has a neighbor in the set, and we call such a set a **plate**. A focal address $x \in A_{fin}, \nu(x) = 0$ is **consecutive** if $p_x \in \text{Ext}(F)$ and $\text{Cyc}(x) \subset \text{Ext}(F)$ is consecutive around $\text{Conv}(F)$; furthermore we say that $p_x$ is a **consecutive focal point**.

**Definition 4.6** A target direction is **regular** if its maximizing extremal point is unique, strictly focal, and consecutive. A polyfractal is a **regular fractal** if it has a regular target direction, meaning if it has a strictly focal extremal point whose cycle is consecutive.

The regularity of a target direction $\tau \in \mathbb{C}$ can be verified by running Algorithm 1, which results in the maximizing irreducible form(s). If $(b,x), b \neq 0$ is one of these forms, then by the ET $p_x \in \text{Ext}(F)$ is the maximizer of $\tau' := e^{-\nu(b)i}\tau$, so we can still check whether $x$ is strictly focal and $\text{Cyc}(x)$ is consecutive, and we may conclude that $\tau'$ is regular instead.

Consecutiveness of the cycle can be verified as follows. To test if $e_{1,2} \in \text{Cyc}(x) \subset \text{Ext}(F)$ are neighbors, take an outward normal $\tau_0 \in \mathbb{C}$ to the line connecting them, and run Algorithm 1 to see what irreducible form(s) maximize(s) this target. If those of $e_1$ and $e_2$ do, then they must be neighbors. But how does one determine the order of the elements of $\text{Cyc}(x) \subset \text{Ext}(F)$ on the boundary of the convex hull? This would be a prerequisite for testing the cycle for consecutiveness. Simply connect each element $e \in \text{Cyc}(x)$ to some fixed interior point of the convex hull $f \in \text{int} \text{Conv}(F)$, and the values $\text{arg}(e - f)$ imply the sought order.
**Theorem 4.3** Let \( p_x \in \text{Ext}(F) \cap \text{Foc}(F) \) be the maximizer of a regular target direction. Then the iterates of the plate \( \text{Cyc}(x) \) by each \( T_k \in \mathcal{T} \) generate all the extrema of \( F \), meaning

\[
\text{Conv}(F) = \text{Conv} \bigcup_{k=1}^{n} \bigcup_{l=0}^{\lceil \pi/|\vartheta_k| \rceil} T_k^l (\text{Cyc}(x)).
\]

**Proof** Due to regularity we have that \( \text{Cyc}(x) \subset \text{Ext}(F) \) are consecutive extrema. By the strict focality of \( x \) we have that each \( k \in \mathcal{N} = \{1, \ldots, n\} \) occurs in \( x \) so \( \forall k \in \mathcal{N} \exists e_{1,2} \in \text{Cyc}(x) : e_1 = T_k(e_2) \) therefore the angle with vertex at \( p_k \) and spanned by \( \text{Cyc}(x) \) is at least \( |\vartheta_k| \) for each \( k \in \mathcal{N} \). So the iterated plate \( T_k^l(\text{Cyc}(x)), l \in \mathbb{N} \) produces more (potentially overlapping) plates of extrema. Since each \( T_k \) traces a logarithmic spiral trajectory, we only need to iterate up to a \( \lceil \pi/|\vartheta_k| \rceil \) number of times for each \( k \in \mathcal{N} \). This gives the convex hull identity in the theorem. The identity also readily implies that \( F \) has a finite number of extrema, since on the left of the identity we have a finite union of finite sets. \( \Box \)

**Method 4.2** (Armadillo Method) **Determines the extremal points of a fractal of unity.**

1. Find a reasonable candidate direction \( \tau \in \mathbb{C} \setminus \{0\} \) which is potentially regular.

2. Run Algorithm 1 (LOAF) and see if it gives a unique strictly focal maximizing form \((0,x)\). If it does, then proceed to Step 3. Otherwise go to Step 1 and try a different direction.

3. Evaluate the cycle \( \text{Cyc}(x) \subset \text{Ext}(F) \) and deduce the order of its elements on the boundary of the convex hull, as discussed above.

4. Determine if \( \text{Cyc}(x) \) is a plate, by connecting its elements with lines (ordered in the previous step), and running the LOAF. If the algorithm returns the two connected cycle extrema, then they are neighbors on the convex hull.

5. Iterate the cycle \( \text{Cyc}(x) \) by each IFS map according to Theorem 4.3 to find the rest of the extremal points.

In the last step, the iterates of the plate \( \text{Cyc}(x) \) can overlap – an easily resolvable redundancy – similarly to the plates of the armored mammal armadillo. The above method is potentially more efficient and robust than the general method of Section 4.2. A candidate target direction can be readily verified for regularity with the LOAF. While the general method blows up all finite addresses up to a level (inefficient, since it may run up to an address length of \( 2|\nu(A_{\text{fin}})| \) making it exponential) and checks a containment property for the blow-ups (may not be robust), the LOAF efficiently eliminates most subfractals recursively.

An approximative method for finding a regular direction, could be to find one for a “simpler” fractal. Approximate the IFS rotation angles \( \vartheta_k/2\pi \) with low-numerator fractions, and find a regular direction for this simpler IFS (Section 5.2 reasons why numerators are relevant). By Theorem 3.5, the extrema of the simplified fractal approximate that of the original, so if the approximation is not too crude, then the fractals should share this regular direction.
5 The Principal Direction

In this section, we restrict our attention to “C-IFS” fractals of unity in our quest to find a reasonable candidate target direction – called the “principal direction” – the regularity of which can then be verified via the methods discussed in the previous section. Nevertheless a detailed heuristic reasoning is given for its probable regularity, while a solid proof may not even be possible, since there can be rare counterexamples.

5.1 The Normal Form of Bifractals

To simplify a discussion, it is often preferable to transform a bifractal $F = \langle T_1, T_2 \rangle$ to “normal form”, i.e. to normalize the fixed points $p_{1,2} \in P$ of the IFS to 0 and 1 (it is up to our preference which fixed point we normalize to 0). This can be accomplished with the affine similarity transform $N(z) := (z - p_1)/(p_2 - p_1)$.

**Theorem 5.1** With the above $N$ map, we have for $T'_k := N \circ T_k \circ N^{-1}$, $k = 1, 2$ that $T'_1(z) = \varphi_1 z$, $T'_2(z) = 1 + \varphi_2 (z - 1)$ and $N(F) = \langle T'_1, T'_2 \rangle$ which we call the normal form of $F$. Furthermore for any $x \in \mathcal{A}_{\text{fin}}$ using $T'_x = NT'_xN^{-1}$ we can express the periodic point $p_x = T_x(p_x)$ as

$$p_x = \frac{T'_x(0)}{1 - \varphi_x} = p_1 + \frac{T'_2(0)}{1 - \varphi_2}(p_2 - p_1).$$

**Proof** The first sentence follows from the Affine Lemma 2.4. To show the second property, we first observe that $T'_x = NT'_xN^{-1}$ is trivial by the cancellation of $N$ terms in the compositions of $T'_k$. So by Corollary 2.2 we have that

$$T'_x(0) = NT'_xN^{-1}(0) = NT_x(p_1) = N(p_x + \varphi_x(p_1 - p_x)) = \frac{(1 - \varphi_x)(p_x - p_1)}{p_2 - p_1} \Rightarrow$$

$$p_1 + \frac{T'_x(0)}{1 - \varphi_x}(p_2 - p_1) = p_1 + \frac{(1 - \varphi_x)(p_x - p_1)}{(p_2 - p_1)(1 - \varphi_x)}(p_2 - p_1) = p_x. \square$$

Essentially, the above theorem tells us that we can scale down the IFS fractal to normal form, examine its geometrical properties (such as its convex hull), and then transform back the results, since due to the Affine Lemma 2.4 $F = N^{-1}(N \circ T_1 \circ N^{-1}, N \circ T_2 \circ N^{-1})$. Therefore the factors $\varphi_{1,2}$ entirely characterize the geometry of a bifractal, since they remain unchanged under normalization. Thus we restrict the discussion to normalized bifractals and drop the apostrophes, so the IFS maps become $T_1(z) = \varphi_1 z$ and $T_2(z) = 1 + \varphi_2 (z - 1)$.

**Theorem 5.2** All finite compositions of $T_{1,2}$ ending in $T_2$ with seed $p_1 = 0$ take the following form when like terms are collected

$$T_{1}^{n_{0}}T_{2}T_{1}^{n_{1}}T_{2} \ldots T_{1}^{n_{L-1}}T_{2}(0) = (1 - \varphi_2) \sum_{j=0}^{L} \varphi_1^{s_j} \varphi_2^{j} \text{ where } L \in \mathbb{N}, n_j \geq 0, s_j := \sum_{k=0}^{j} n_k.$$
Proof It is clear that all such address compositions can be brought to the form on the left by collecting terms, so in order to prove the form on the right, we proceed to showing the following more general property

\[ T_1^{n_0}T_2^{n_1}T_2 \cdots T_1^{n_L}T_2(z) = (1 - \varphi_2) \left( \sum_{j=0}^{L} \varphi_1^s j \varphi_2^j \right) + \varphi_1^s L^0 \varphi_2^{L+1} z \]

from which the earlier one follows with \( z = 0 \). We show this by induction. For \( L = 0 \)

\[ T_1^{n_0}T_2(z) = \varphi_1^{n_0}(1 + \varphi_2(z - 1)) = \varphi_1^{n_0} + \varphi_1^{n_0} \varphi_2 z - \varphi_1^{n_0} \varphi_2 = (1 - \varphi_2) \varphi_1^{n_0} \varphi_2^0 + \varphi_1^{n_0} \varphi_2^2. \]

Now suppose that it holds up to \( L = l \) and we show it for \( L = l + 1 \). With \( z = T_1^{n_l+1}T_2(z_0) \) for any \( z_0 \in \mathbb{C} \) we have by the associativity of composition that

\[ T_1^{n_0}T_2T_1^{n_1}T_2 \cdots T_1^{n_l}T_2 \left( T_1^{n_l+1}T_2(z_0) \right) = \]

\[ = (1 - \varphi_2) \left( \sum_{j=0}^{l} \varphi_1^s j \varphi_2^j \right) + \varphi_1^{n_l+1} \varphi_2^{(l+1)}(1 + \varphi_2(z_0 - 1)) = \]

\[ = (1 - \varphi_2) \left( \sum_{j=0}^{l+1} \varphi_1^s j \varphi_2^j \right) + \varphi_1^{n_l+1} \varphi_2^{(l+1)+1} z. \]

\( \square \)

5.2 A Candidate Direction for C-IFS Fractals

Definition 5.1 We say that a bifractal is a C-IFS fractal (after the Lévy C curve), if its rotation angles satisfy \( \vartheta_1 \in (-\pi, 0), \vartheta_2 \in (0, \pi), |\vartheta_1| \leq \vartheta_2 \).

Heuristic 5.1 Let \( F = \langle T_1, T_2 \rangle \) be a normalized C-IFS fractal of unity with

\( \vartheta_1 = -\frac{2\pi P}{M}, \vartheta_2 = \frac{2\pi Q}{M} \) where \( P, Q, M \in \mathbb{N}, 0 < P \leq Q < \frac{M}{2} \).

Then the target direction \( \tau_* := i(1 - \varphi_2) \log \varphi_1 \) called the principal direction, is likely to be regular, and its maximizer is likely to be the strictly focal periodic point \( p_x \) where

\[ T_x = T_2T_1^{n_1} \cdots T_2T_1^{n_J}, \ |x| = \frac{P + Q}{\gcd(P, Q)} \]

\[ n_j = s_j - s_{j-1}, \ j = 1, \ldots, J := \frac{P}{\gcd(P, Q)}, \ n_0 = s_0 = 0 \]

\[ s_j := \text{argmax} \left( \lambda_s \cos(\vartheta_1 s + \vartheta_2 j + \alpha) : s = \left\lfloor \frac{Qj}{P} \right\rfloor, \left\lceil \frac{Qj}{P} \right\rceil \right), \ \alpha := \arctan \left( \frac{\ln \lambda_1}{\vartheta_1} \right). \]

Let a maximizer with respect to the principal direction be called a principal extremal point, and denote it as \( e_* \) if it is unique.
Reasoning First of all, we calculate the Euclidean inner product of the principal direction \( \tau_* \) with a general normalized fractal point given by Theorem 5.2, and connect it to \( s_j \). Our calculation is aided by the identity \( \forall u, v \in \mathbb{C} : \langle u, v \rangle = \text{Re}(u \overline{v}) \). Denoting \( t_j(s) := \lambda^*_s \cos(\vartheta_1 s + \vartheta_2 j + \alpha) \) we have

\[
\langle i(1 - \varphi_2) \log \varphi_1, (1 - \varphi_2) \sum_{j=0}^{L} \varphi_1^s \varphi_2^s \rangle = |1 - \varphi_2|^2 \sum_{j=0}^{L} \langle i \log \varphi_1, \varphi_1^s \varphi_2^s \rangle = \\
= |1 - \varphi_2|^2 |\log \varphi_1| \sum_{j=0}^{L} \lambda_1^s \lambda_2^s \cos \langle i \log \varphi_1, \varphi_1^s \varphi_2^s \rangle = |1 - \varphi_2|^2 |\log \varphi_1| \sum_{j=0}^{L} \lambda_2^s t_j(s_j)
\]

since \( \langle i \log \varphi_1, \varphi_1^s \varphi_2^s \rangle \equiv \text{Arg}(\varphi_1^s \varphi_2^s) - \text{Arg}(-\vartheta_1 + i \ln \lambda_1) \equiv \\
\equiv (\vartheta_1 s_j + \vartheta_2 j) - \arctan \left( -\frac{\ln \lambda_1}{\vartheta_1} \right) \equiv \vartheta_1 s + \vartheta_2 j + \alpha \pmod{2\pi} \).

So we deduced that in order to maximize \( z \mapsto \langle \tau_*, z \rangle \) over \( F = \langle T_1, T_2 \rangle \) we need to maximize the sum \( \sum_{j=0}^{L} \lambda_2^s t_j(s_j) \) over all \( L \in \mathbb{N} \) and \( 0 \leq s_j \leq s_{j+1} \ (j = 0, 1, 2, \ldots) \). We temporarily drop the latter optimization constraint, and just consider each \( s \mapsto t_j(s) \) function individually, and see where its local extrema are. Taking the derivative

\[
t'_j(s) = (\ln \lambda_1) \lambda^*_1 \cos(\vartheta_1 s + \vartheta_2 j + \alpha) - \vartheta_1 \lambda^*_1 \sin(\vartheta_1 s + \vartheta_2 j + \alpha) = \\
= (\ln \lambda_1) \lambda^*_1 [\cos(\vartheta_1 s + \vartheta_2 j) \cos(\alpha) - \sin(\vartheta_1 s + \vartheta_2 j) \sin(\alpha)] - \\
- \vartheta_1 \lambda^*_1 [\sin(\vartheta_1 s + \vartheta_2 j) \cos(\alpha) + \cos(\vartheta_1 s + \vartheta_2 j) \sin(\alpha)] = \\
= \lambda^*_1 (\cos(\vartheta_1 s + \vartheta_2 j)) [\ln \lambda_1 \cos \alpha - \vartheta_1 \sin \alpha] - \sin(\vartheta_1 s + \vartheta_2 j) [\ln \lambda_1 \sin \alpha + \vartheta_1 \cos \alpha] = \\
= -\lambda^*_1 \sin(\vartheta_1 s + \vartheta_2 j) [\ln \lambda_1 \sin \alpha + \vartheta_1 \cos \alpha] = \\
= |\vartheta_1| (\cos(\alpha)(1 + (\tan \alpha)^2) \sin(\vartheta_1 s + \vartheta_2 j) \\
\text{since} \ (\ln \lambda_1) \cos \alpha - \vartheta_1 \sin \alpha = (\cos \alpha) \vartheta_1 (\tan \alpha - \tan \alpha) = 0 \\
\text{and} \ (\ln \lambda_1) \sin \alpha + \vartheta_1 \cos \alpha = \vartheta_1 (\cos \alpha)((\tan \alpha)^2 + 1) \neq 0.
\]

Here \( \cos \alpha > 0 \) by \( \alpha = \arctan \left( \frac{\ln \lambda_1}{\vartheta_1} \right) \in (0, \frac{\pi}{2}) \) due to \( \frac{\ln \lambda_1}{\vartheta_1} > 0 \) so remarkably

\[
\text{sgn} \ t'_j(s) = \text{sgn} \ \sin(\vartheta_1 s + \vartheta_2 j) = -\text{sgn} \ \sin(|\vartheta_1| s - \vartheta_2 j)
\]

which ultimately followed from our special choice of \( \tau_* \). Since the \( \vartheta_1 < 0 \) factor flips the oscillation of sin, the local maxima occur at

\[
\vartheta_1 s + \vartheta_2 j = 2\pi k, \ k \in \mathbb{Z}
\]

\[
s \in \left\{-\frac{\vartheta_2 j}{\vartheta_1} + \frac{2\pi}{\vartheta_1} k : k \in \mathbb{Z}\right\} = \left\{ \frac{Qj}{P} + \frac{M}{P} k : k \in \mathbb{Z}\right\}.
\]
Denote by \( s_j \) the heuristically most reasonable sequence \( j = 0, 1, 2, \ldots \) for maximizing \( f \mapsto \langle \tau_*, f \rangle, f \in F \). For \( j = 0 \) the likely solution is \( s_0 = 0 \) with \( k = 0 \), since for \( k < 0 \) the number \( \frac{Q_j}{P} + \frac{M}{P} k \) is negative, and for \( k \geq 1 \) the maximal values of \( t_j \) decrease, since \( s \mapsto t_0(s) \) oscillates between the decreasing exponential curves \( s \mapsto \pm \lambda_1^s \). Therefore we conclude heuristically that \( s_0 = 0 \).

We further reason that \( s_j \in \{ \lfloor \frac{Q_j}{P} \rfloor, \lceil \frac{Q_j}{P} \rceil \} \) for each \( j \in \mathbb{N} \). Since \( \frac{Q_j}{P} \) occur with a spacing of \( \frac{Q_j}{P} < \frac{M}{P} \) while the local maxima of \( t_j \) occur with a spacing of \( \frac{M}{P} \) then assuming heuristically the somewhat stricter constraint \( s_j < s_{j+1}, j \in \mathbb{N} \) we necessarily have that \( s_j \) occurs in an integer near \( \frac{Q_j}{P} \).

We may thus conclude the heuristic statement that the maximizing address with respect to \( \tau_* \) is likely to have the collected exponents \( n_j = s_j - s_{j-1}, j \in \mathbb{N} \) where

\[
s_j = \argmax \left( t_j(s) : s = \left\lfloor \frac{Q_j}{P} \right\rfloor, \left\lceil \frac{Q_j}{P} \right\rceil \right).
\]

We show that this address is periodic according to the map

\[
T_x = T_2 T_1^{n_1} \ldots T_2 T_1^{n_J}, j = 1, \ldots, J = \frac{P}{\gcd(P,Q)}
\]

or equivalently that \( n_j \) is periodic by \( J \). If we showed that \( s_{j+J} = s_j + s_J \) then it would imply periodicity \( n_{j+J} = n_j \) since

\[
n_{j+J} = s_{j+J} - s_{j+J-1} = (s_j + s_J) - (s_{j-1} + s_J) = s_j - s_{j-1} = n_j.
\]

First of all, \( s_J = \frac{Q_J}{P} \) necessarily, so \( \vartheta_1 s_J + \vartheta_2 J = 0 \) implying

\[
t_{j+J}(s + s_J) = \lambda_1^{s_j+s_J} \cos(\vartheta_1 (s + s_J) + \vartheta_2 (j + J) + \alpha) = \lambda_1^s t_j(s).
\]

Since the optimum of \( t_j \) with respect to the global target \( \tau_* \) was deduced to likely be \( s_j \in \{ \lfloor \frac{Q_j}{P} \rfloor, \lceil \frac{Q_j}{P} \rceil \} \) and due to

\[
s_{j+J} \in \left\{ \left\lfloor \frac{Q(j + J)}{P} \right\rfloor, \left\lceil \frac{Q(j + J)}{P} \right\rceil \right\} = s_j + \left\{ \left\lfloor \frac{Q_j}{P} \right\rfloor, \left\lceil \frac{Q_j}{P} \rceil \right\}
\]

we necessarily have \( s_{j+J} = s_j + s_J \) implying the periodicity of \( n_j \) by \( J \).

Thus the maximizer of \( \tau_* \) is heuristically likely to be \( p_x \) which is strictly focal since \( \nu(x) = \vartheta_1 s_J + \vartheta_2 J = 0 \) and \( T_x \) contains both \( T_1 \) and \( T_2 \).

Using Algorithm 1 we can verify if this \( p_x \) is indeed the maximizer of \( \tau_* \), in which case then necessarily \( \text{Cyc}(x) \subset \text{Ext}(F) \). With this algorithm, we can also verify the consecutiveness of the cycle (see the remarks after Definition 4.6), which has mostly been the case in our computational experiments, meaning \( \tau_* \) is usually a regular target direction. □

Note that due to \( P, Q \leq M/2 \) we can estimate

\[
|x| = \frac{P + Q}{\gcd(P,Q)} \leq \frac{M}{\gcd(P,Q,M)} \frac{\gcd(P, Q, M)}{\gcd(P, Q)} \leq \frac{M}{\gcd(P, Q, M)} = |\nu(A_{fin})|
\]

according to Lemma 3.1, so Theorem 3.11 suggests that \( p_x \) could be in irreducible form.
5.3 The Armadillo Method Adopted for C-IFS Fractals

**Method 5.1 (Exact)** An exact method for finding the convex hull of C-IFS fractals of unity, based on the heuristic prediction that the principal direction is likely to be regular.

1. Normalize the IFS maps to $T_1(z) = \varphi_1 z$, $T_2(z) = 1 + \varphi_2(z - 1)$.
2. Calculate the principal direction $\tau_* = i(1 - \varphi_2) \log \varphi_1$.
3. Run Algorithm 1 (LOAF) and see if the maximizer of $\tau_*$ is a unique $p_x$ strictly focal point. If it is then proceed to Step 4, otherwise try a different candidate direction.
4. Next deduce the order of $e \in \text{Cyc}(x) \subset \text{Ext}(F)$ on the convex hull from the values $\arg(e - f)$ where $f \in \text{int Conv}(F)$ is fixed. Verify if the cycle is consecutive by connecting each cycle point with its likely neighbor, and check using the LOAF in the normal direction to the connecting line, whether the maximizers are these two extrema. If the cycle turns out not to be consecutive, then try a different candidate direction in Step 3.
5. Iterate the cycle $\text{Cyc}(x)$ by each IFS map according to Theorem 4.3 to find the rest of the extrema.
6. Lastly, map the determined extrema back by the inverse of the normalizing map $N^{-1}(z) = p_1 + z(p_2 - p_1)$ to get the extrema of the original fractal.

If all steps can be carried out, then it results in the convex hull of a C-IFS fractal of unity.

**Method 5.2 (Heuristic)** A heuristic method based on Heuristic 5.1 for finding the convex hull of C-IFS fractals of unity.

1. Normalize the IFS maps to $T_1(z) = \varphi_1 z$, $T_2(z) = 1 + \varphi_2(z - 1)$.
2. Calculate the principal direction $\tau_* = i(1 - \varphi_2) \log \varphi_1$ and $J = P / \gcd(P, Q)$.
3. Calculate $s_j = \arg\max (\lambda^s \cos(\vartheta_1 s + \vartheta_2 j + \alpha) : s = \left\lfloor \frac{Qj}{P} \right\rfloor, \left\lceil \frac{Qj}{P} \right\rceil)$, $1 \leq j \leq J$.
4. Calculate $n_j = s_j - s_{j-1}$ and $x$ such that $T_x = T_2 T_1^{n_1} \ldots T_2 T_1^{n_J}$ and the resulting $p_x$ using Corollary 2.1.
5. Iterate the cycle $\text{Cyc}(x)$ by each IFS map, according to the heuristic application of the Armadillo Method, to find the rest of the extrema.
6. Lastly, map the determined extrema back by the inverse of the normalizing map to get the extrema of the original fractal.

This method results in the vertices of a polygon, which is heuristically predicted to be the convex hull of the C-IFS fractal.

According to Theorem 3.5, for irrational bifractals one may resort to finding the extrema of increasingly better approximations by fractals of unity, varying the angular parameters. Considering Method 5.2, this means that the exponents $n_j$ will be generated ad infinitum.
5.4 Examples

Example 5.1 (Lévy C Curve [3, 6, 11, 1], Figure 4)

Find the extrema of the C-IFS fractal of unity in normal form with IFS factors

\[ \varphi_1 = \frac{1}{\sqrt{2}} \exp \left( -\frac{\pi}{4} i \right), \quad \varphi_2 = \frac{1}{\sqrt{2}} \exp \left( \frac{\pi}{4} i \right). \]

Solution

Applying Method 5.1, we maximize in the principal direction

\[ \tau_* = i(1 - \varphi_2) \log \varphi_1 \approx 0.2194 - 0.5660i \]

and using Algorithm 1 we arrive at the unique strictly focal maximizing truncation \( x = (2, 1) \) resulting in the principal extremal point and its cycle

\[
\begin{align*}
e_* &= p_x = \frac{T_x(0)}{1 - \varphi_x} = \frac{1 - \varphi_2}{1 - \varphi_1 \varphi_2} = 1 - i \\
\text{Cyc}(x) &= \{e_*, T_1(e_*)\} = \left\{ \frac{1 - \varphi_2}{1 - \varphi_1 \varphi_2}, \frac{\varphi_1(1 - \varphi_2)}{1 - \varphi_1 \varphi_2} \right\} = \{1 - i, -i\}.
\end{align*}
\]

Next deduce using Algorithm 1 that the cycle is consecutive (meaning \( e_*, T_1(e_*) \) are neighbors). Lastly, we iterate the cycle according to Theorem 4.3 and get the rest of the extremal points \( \text{Ext}(F) \subset C_x \cup T_1(C_x) \cup T_2(C_x) \). \( \square \)

Example 5.2 (Twindragon / Davis-Knuth Dragon [4], Figure 5)

Find the extrema of the C-IFS fractal of unity in normal form with IFS factors

\[ \varphi_1 = \frac{1}{\sqrt{2}} \exp \left( -\frac{\pi}{4} i \right), \quad \varphi_2 = \frac{1}{\sqrt{2}} \exp \left( \frac{3\pi}{4} i \right). \]

Solution

Applying Method 5.1, we maximize in the principal direction

\[ \tau_* = i(1 - \varphi_2) \log \varphi_1 \approx 1.0048 - 0.9126i \]

and using Algorithm 1 we arrive at the unique strictly focal maximizing truncation \( x = (2, 1, 1, 1) \) or \( T_x = T_2 T_1^3 \) resulting in the principal extremal point and its cycle

\[
\begin{align*}
e_* &= p_x = \frac{T_x(0)}{1 - \varphi_x} = \frac{1 - \varphi_2}{1 - \varphi_1^3 \varphi_2} = 2 - \frac{2}{3} i \\
C_x := \text{Cyc}(x) &= \{e_*, T_1(e_*) , T_1^2(e_*), T_1^3(e_*)\} = \\
&= \left\{ \frac{1 - \varphi_2}{1 - \varphi_1^3 \varphi_2}, \frac{\varphi_1(1 - \varphi_2)}{1 - \varphi_1^3 \varphi_2}, \frac{\varphi_1^2(1 - \varphi_2)}{1 - \varphi_1^3 \varphi_2}, \frac{\varphi_1^3(1 - \varphi_2)}{1 - \varphi_1^3 \varphi_2} \right\} = \\
&= \left\{ 2 - \frac{2}{3} i, \frac{2}{3} - \frac{4}{3} i, \frac{1}{3} - i, -\frac{2}{3} - \frac{1}{3} i \right\}.
\end{align*}
\]

Next deduce using Algorithm 1 that the cycle is consecutive. Lastly, we iterate the cycle according to Theorem 4.3 and get the rest of the extremal points \( \text{Ext}(F) \subset C_x \cup T_1(C_x) \cup T_2(C_x) \). \( \square \)
Example 5.3 (Figure 6)
Find the extrema of the C-IFS fractal of unity in normal form with IFS factors

\[ \varphi_1 = 0.65 \exp \left( -\frac{2\pi}{6} i \right), \quad \varphi_2 = 0.65 \exp \left( \frac{3\pi}{6} i \right). \]

Solution Applying Method 5.1, we maximize in the principal direction

\[ \tau_* = i(1 - \varphi_2) \log \varphi_1 \approx 0.7672 - 1.1115i \]

and using Algorithm 1 we arrive at the unique strictly focal maximizing truncation \( x = (2, 1, 1, 2, 1) \) or \( T_x = T_2T_1^2T_2T_1 \) resulting in the principal extremal point and cycle

\[ e_* = p_x = \frac{T_x(0)}{1 - \varphi_x} = \frac{(1 - \varphi_2)(1 + \varphi_1^2\varphi_2)}{1 - \varphi_1^2\varphi_2} \approx 1.2993 - 1.0655i \]

\[ C_* := \text{Cyc}(x) = \{ e_*, T_1(e_*), T_2T_1(e_*), T_1T_2T_1(e_*), T_2^2T_1T_2(e_*) \}. \]

Next deduce using Algorithm 1 that the cycle is consecutive. Then iterate the cycle according to Theorem 4.3 and get the rest of the extremal points \( \text{Ext}(F) \subset C_* \cup \tau_1(C_*) \cup T_1^2(C_*) \cup T_2(C_*) \cup T_2^2(C_*) \). □

In the figures that follow, the principal extremal point \( e_* \) is marked by a blue dot, and the blue line is perpendicular to the principal direction. The fixed points \( p_{1,2} \) are plotted with red \( \diamond \) and the iterates \( T_1(p_2), T_2(p_1) \) with magenta. The cycle vertices are circled in red.

Figures 7, 10, and 11 clearly illustrate the utility of the Armadillo Method 4.2.

The IFS factors and the principal extrema for the rest of the figures:

- Figure 7: \( \varphi_1 = 0.6177 \exp \left( -\frac{99\pi}{180} i \right), \quad 0.8594 \exp \left( \frac{163\pi}{180} i \right) \)
  \( T_x = T_2(T_a^4T_b^3)(T_a^5T_b^4T_aT_b^{-1}), \quad e_* = p_x \approx 2.7130 - 0.5959i \)
  with \( T_a = T_1^2T_2T_1T_2^2T_2, \quad T_b = T_1T_2T_1^2T_2 \) and
  \[ |x| = \frac{99 + 163}{\gcd(99,163)} = 1 + 2(4|a| + |b|) + 4(5|a| + |b|) + |a| - 1 = 29|a| + 6|b| = 262. \]

- Figure 8: \( \varphi_1 = 0.6 \exp \left( -\frac{8\pi}{12} i \right), \quad 0.8 \exp \left( \frac{9\pi}{12} i \right) \)
  \[ |x| = \frac{8 + 9}{\gcd(8,9)} = 17, \quad T_x = (T_2T_1)^4T_1(T_2T_1)^4, \quad e_* = p_x \approx 2.9439 - 0.5767i. \]

- Figure 9: \( \varphi_1 = 0.5479 \exp \left( -\frac{5\pi}{45} i \right), \quad 0.9427 \exp \left( \frac{12\pi}{45} i \right) \)
  \[ |x| = \frac{5 + 12}{\gcd(5,12)} = 17, \quad T_x = (T_2T_1)^3T_1(T_2T_1)^3T_2T_1^2, \quad e_* = p_x \approx 0.5203 - 0.9244i. \]

- Figure 10: \( \varphi_1 = 0.9 \exp \left( -\frac{6\pi}{45} i \right), \quad 0.5 \exp \left( \frac{35\pi}{45} i \right) \)
  \[ |x| = \frac{6 + 35}{\gcd(6,35)} = 41, \quad T_x = (T_2T_1)^3T_2T_1^5(T_2T_1)^2, \quad e_* = p_x \approx 1.8720 - 0.4808i. \]

- Figure 11: \( \varphi_1 = 0.9421 \exp \left( -\frac{2\pi}{180} i \right), \quad 0.9561 \exp \left( \frac{17\pi}{180} i \right) \)
  \[ |x| = \frac{2 + 17}{\gcd(2,17)} = 19, \quad T_x = T_2T_1^9T_2T_1^8, \quad e_* = p_x \approx 0.1958 - 0.6532i. \]
Figure 4: Lévy C Curve, Example 5.1.

Figure 5: Twindragon, Example 5.2.
Figure 6: Example 5.3.

Figure 7: Illustration of the subtlety of convex hull determination: $|\text{Ext}(F)| \geq 262$. 

34
Figure 8: A random C-IFS fractal.

Figure 9: A random C-IFS fractal.
Figure 10: Illustration of the method’s predictive nature; iteration level $L = 20$.

Figure 11: Illustration of the method’s predictive nature; iteration level $L = 20$. 
6 Concluding Remarks

Methods have been introduced to determine the exact (not approximate) convex hull of certain classes of IFS fractals, and it has been reasoned how this classification prevents such efforts beyond the class of “rational” IFS fractals. A general though inefficient method was discussed for fractals of unity, and a more efficient method for regular fractals. In the latter “Armadillo Method”, a certain periodic extremal point was exploited to generate the rest of the convex hull.

These methods were designed with potential generalization to 3D IFS fractals in mind – as remarked after Theorem 4.1 – so that the convex hull of plants like L-system trees or the Romanesco Broccoli can be found. In fact with those remarks, Sections 3 and 4 can be generalized to higher dimensions. The primary aim of this paper was to find the most natural approach to the Convex Hull Problem in the plane.

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