GENERALIZED MOONSHINE IV: MONSTROUS LIE ALGEBRAS

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ABSTRACT. We prove an existence theorem for constructing new vertex algebras given compatible one-dimensional spaces of intertwining operators. As an application to generalized moonshine, we construct a class of abelian intertwining algebras from direct sums of irreducible twisted modules of the monster vertex operator algebra. We apply a bosonic string quantization functor to construct Borcherds-Kac-Moody Lie algebras equipped with actions of large finite groups, and we determine explicit graded dimensions of all irreducible twisted modules of the monster vertex algebra. We prove that for Fricke classes of prime order, together with 4B, all characters of centralizers acting on the corresponding irreducible twisted modules yield Hauptmoduln, thereby establishing the generalized moonshine conjecture for those cases.

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1. Introduction

In this paper, we introduce new homological techniques for constructing vertex algebras and infinite dimensional Lie algebras equipped with actions of finite groups. We apply these methods to establish concrete facts about characters of automorphisms of twisted modules of the monster vertex algebra $V^2$, and verify that some cases of Norton's generalized moonshine conjecture hold for irreducible twisted $V^2$-modules.

1.1. Generalized Moonshine. At about the same time as Conway and Norton formulated the Monstrous Moonshine conjecture [Conway-Norton-1979], Queen did some computations that suggested that moonshine-like phenomena, i.e., the appearance of modular functions from characters of finite groups, is not limited to the monster simple group $\mathbb{M}$. In particular, the low-order coefficients of many modular functions can be assembled from representations of sporadic groups other than the monster. For example, the irreducible representations of the Baby monster sporadic simple group have dimension 1, 4371, 96256, ..., while the power series $q^{-1} + 4372q + 96256q^2 + \cdots$ is the Fourier expansion of the normalized Hauptmodul of $\Gamma_0(2)^+ \subset PSL_2(\mathbb{R})$, i.e., by setting $q = e^{2\pi i \tau}$ as $\tau$ ranges over points in the complex upper half-plane $\mathcal{H}$, this function generates the field of meromorphic functions on the genus zero quotient $\mathcal{H}/\Gamma_0(2)^+$. Norton employed these computations and many of his own to formulate the following conjecture:

**Conjecture.** (Generalized Moonshine [N87], revised in [N01]) There exists a rule that assigns to each element $g$ of the monster simple group $\mathbb{M}$ a $\mathbb{Q}$-graded projective representation $V(g) = \bigoplus_{n \in \mathbb{Q}} V(g)_n$ of the centralizer $C_{\mathbb{M}}(g)$, and to each pair $(g, h)$ of commuting elements of $\mathbb{M}$ a one dimensional complex vector space $P(g, h)$, satisfying the following conditions:

1. Elements of $P(g, h)$ are holomorphic functions on the complex upper half plane that are either zero, or of the form $Z(\tau) = \sum_{n \in \mathbb{Q}} \text{Tr}(\hat{h}V(g)_n)q^{n-1}$, where $\hat{h}$ is a lift of $h$ to a linear transformation on $V(g)$. The nonzero functions are either constant, or are Hauptmoduln for genus zero groups.

2. $P(g, h)$ is invariant under simultaneous conjugation of the pair $(g, h)$ in $\mathbb{M}$.

3. If $(a, b, c, d) \in SL_2(\mathbb{Z})$ and $Z(\tau) \in P(g^a h^c, g^b h^d)$, then $Z(\tau + \frac{c + h}{c + d}) \in P(g, h)$.

4. $P(g, h)$ contains $J(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = q^{-1} + 196884q + \cdots$ if and only if $g = h = 1 \in \mathbb{M}$.

This is a generalization of the Monstrous moonshine conjecture, in the sense that fixing $g = 1$ yields an assertion that one has a graded representation of $\mathbb{M}$ whose characters form Hauptmoduln. It also subsumes the example observation above, in the sense that if $q^{-1} + 4372q + 96256q^2 + \cdots$ is the Moonshine character of an element $g$ in class 2A of the monster (i.e., the function lies in $P(1, g)$), then $q^{-1/2} + 4372q^{1/2} + 96256q + \cdots \in P(g, 1)$.

Shortly after Norton published this conjecture, Dixon, Ginsparg, and Harvey found a physical interpretation: The spaces $V(g)$ are twisted Hilbert spaces in a conformal field theory with $\mathbb{M}$ symmetry, and $P(g, h)$ is spanned by the genus one partition function for a torus twisted in “space” by $g$ and in “time” by $h$ [Dixon-Ginsparg-Harvey-1988]. In algebraic language, for each $g \in \mathbb{M}$, $V(g)$ is the irreducible $g$-twisted module of the monster vertex operator algebra. Specifically, by [Dong-Li-Mason-2000] Theorem 10.3, $g$-twisted modules are unique up to isomorphism, and by general nonsense, they admit canonical projective actions of $C_{\mathbb{M}}(g)$. Under this interpretation of the conjecture, it remains to prove suitable modularity properties for the traces of lifts of centralizing elements on these twisted modules.

In contrast to the situation with the original Monstrous Moonshine conjecture, we do not have a list of conjugacy classes of commuting pairs of elements in $\mathbb{M}$, or a list of character tables of central
extensions of centralizers of elements at our disposal. Conversations with group theorists suggest that these explicit data are still beyond current technology, so at the moment, we cannot solve the conjecture by computational brute force. Instead, one may seek general methods for proving that a trace on a twisted module is a Hauptmodul, and for showing that traces are \( SL_2(\mathbb{Z}) \)-compatible.

In [Carnahan-2010], we described a sufficient condition for a power series to be the \( q \)-expansion of a Hauptmodul, and showed that it could be applied to shorten both Borcherds’s proof of the Monstrous Moonshine conjecture, and Hoehn’s proof of generalized moonshine for the case where \( g \) lies in conjugacy class 2A. This condition amounts to the existence of a group action on a certain type of Lie algebra, and in [Carnahan-2012], we constructed Lie algebras of the correct form for each “Fricke type” element of the monster. The remaining task for proving that the Hauptmodul condition holds for \( g \) of Fricke type is to construct canonical projective actions of \( C_M(g) \) on these Lie algebras, and that is the primary goal of this paper.

In order to construct actions on Lie algebras that inform us about actions on twisted modules, we need a functorial way to pass from twisted modules to Lie algebras. We will employ a bosonic string quantization functor that has been in use for about 40 years. The quantization functor takes conformal vertex algebras of central charge 26 to Lie algebras, and one may exploit the phenomenon of cancellation of oscillators to describe the root spaces of the Lie algebras explicitly in terms of the input vertex algebra. This theory was one of the main engines behind Frenkel’s bound on root multiplicities of the Lie algebra whose simple roots form the Leech Lattice as its Dynkin diagram, Borcherds’s proof of the Monstrous Moonshine conjecture, and Hoehn’s proof of generalized moonshine in the 2A case.

The remaining difficulty in this paper lies in the construction of a suitable vertex algebra of central charge 26 from \( g \)-twisted modules. It is evident from examination of root multiplicities that we cannot construct our Lie algebra using a single \( g \)-twisted module. Instead, we need to make a vertex algebra using twisted modules for all powers of \( g \). The irreducible twisted modules do not have known explicit constructions except in some low-order cases (e.g., Lam-Yamauchi-2010 for 4A seems to be the state of the art), so we must rely on abstract existential methods to describe our vertex algebras. The procedure is rather elaborate, using not only the homological obstruction theory worked out in the first half of this paper, but also a theory of conformal blocks on twisted nodal curves developed in Carnahan≥2012a.

In order to complete the proof of generalized moonshine, we need to resolve the root-of-unity ambiguities that appear when considering elements of composite order. We need to prove that the characters for non-Fricke \( g \) also are constant or Hauptmoduln, and we need to prove \( SL_2(\mathbb{Z}) \)-compatibility. I believe I have methods for resolving these difficulties, but such work is in a preliminary stage.

1.2. Overview. In section 2, we fix notation and definitions. We have made some nonstandard notational choices for describing vertex algebra structure, in hopes of making the idea of iterated multiplication easier to see. We apologize to anyone who is confused by this departure from the rest of the literature. We introduce a notion of vertex algebra object in the braided tensor category \( \text{Vect}_{\mathbf{A}, F, \Omega}^A \). It is a mild generalization of the notions of abelian intertwining algebra [Dong-Lepowsky-1993] and para-algebra [Feingold-Frenkel-Ries-1991] that is missing some of their finiteness conditions. On a fundamental level, it is not particularly new, but we feel that the definition is natural enough to merit consideration, and it is particularly well-suited to the techniques employed in this paper.

In section 3, we describe the obstruction theory for generalized locality, and we describe the automorphism group that arises from projective actions on modules. For the obstruction theory, we build up enough structure to produce an object resembling the abelian intertwining algebras introduced in [Dong-Lepowsky-1993], and then we show that under ideal conditions, one can make adjustments to annihilate all obstructions - this is similar to an argument found in Höhn-2003a. We
apply this theory to generalized moonshine, by using previous results about equivariant intertwining operators for irreducible twisted modules of holomorphic $C_2$-cofinite vertex operator algebras. This section is expanded from a single sentence in my dissertation, and I am thankful to Gerald Höhn for pointing out that the problem is not nearly as trivial as I had convinced myself it was.

In section 4, we construct Lie algebras from abelian intertwining algebras, using a generalization of Borcherds’s method for constructing the monster Lie algebra. We show that the resulting Lie algebras are necessarily Borcherds-Kac-Moody.

In section 5, we determine the characters of all irreducible twisted $V^2$ modules. For Fricke-invariant elements of prime order, this then yields isomorphisms with the Lie algebras constructed in [Carnahan-2012], and we find that characters of centralizers are Hauptmoduln.

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2. Notation and definitions

We will use the notation $e(s)$ to denote the normalized exponential $e^{2\pi is}$.

2.1. Formal power series. We will only describe basic aspects of power series with fractional exponents. The development of this theory can be subtle, because when we refer to the coordinate $z$, we are implicitly manipulating the formal disc, while fractional powers naturally live on branched covers. When multiple variables are in play, we may need to choose a branched cover of the polydisc from a nontrivial collection of possibilities.

Concretely, we will encounter situations similar to the following: we are given maps from a vector space $W$ to the formal power series vector spaces $z^s V((z))((w))$ and $w^s V((w))((z))$ for $s$ a complex number, and we would like to say that both maps factor through a common subspace $(z - w)^s V[[z, w]][(z - w)^{-1}]$. To make sense of such a claim, we need to make an unambiguous choice of inclusions from the latter vector space to the former two.

The methods of formal calculus (see e.g., [Frenkel-Lepowsky-Meurman-1988] Chapter 8) provide one way to make such a selection: To expand in the first space, we write $(z - w)^s$ as $z^s (1 - w/z)^s$, and use the binomial theorem to get the power series $z^s \sum_{k \geq 0} \binom{s}{k} w^k / z^k$. To expand in the second space, we identify $(z - w)^s$ with $(e^{s/2} (w - z))^s$, expand as $e(s/2) w^s (1 - z/w)^s$, and apply the binomial theorem as before. It is worth noting that we have made a choice of logarithm of $-1$ in order to apply fractional powers, and that other choices differ from this one by factors of $e(ns)$ as $n$ ranges over integers.

Because we only need power series in three variables in this paper, we will not formulate a completely general theory. Instead, we will explicitly enumerate the embeddings we will need, and we will check that our conventions are consistent.

Definition 2.1.1. (1) We consider power series only in the 6 fundamental coordinates $z, w, t$, $z - w, z - t, \text{ and } w - t$.

(2) Given a fundamental coordinate such as $z$, we use the notation $V \mapsto V\{z\}$, $f \mapsto f\{z\}$ to denote the endofunctor on the category of complex vector spaces that takes a vector space $V$ to the space $V\{z\}$ of fractional formal power series in $z$ with coefficients in $V$, and takes a complex linear transformation $f$ to the corresponding map of power series induced by applying the transformation to each coefficient. Elements of $V\{z\}$ can be identified with set-theoretic maps $\mathbb{C} \to V$, by identifying the coefficient of $z^s$ with the image of $s \in \mathbb{C}$ under the map.

(3) Given an algebraically independent set of fundamental coordinates, we may iterate this functor to get spaces like $V\{z, z - w, w - t\}$, in this case identifiable with the space of
set-theoretic maps $\mathbb{C}^3 \to V$. All other spaces we consider will be identified as subspaces of these.

(4) The formal Taylor series functor $V \mapsto V[[z]]$ takes a vector space $V$ to the space of formal Taylor series in $z$ with coefficients in $V$, which can be identified with the set of maps $\mathbb{C} \to V$ with support in $\mathbb{Z}_{\geq 0}$. Similarly, the polynomial functor $V \mapsto V[z]$ yields the subspace whose elements correspond to maps that are finitely supported, with support in $\mathbb{Z}_{\geq 0}$.

(5) The formal Laurent series functor $V \mapsto V((z))$ yields the space whose elements correspond to maps $\mathbb{C} \to V$ supported on $\mathbb{Z}$, with support that is bounded from below. Note that $V((z))((w))$ and $V((w))((z))$ do not form equal subspaces of $V\{z,w\}$.

(6) If $s \in \mathbb{C}$, we have a shifted Laurent series functor $V \mapsto z^sV((z))$, where $z^sV((z))$ corresponds to the space of maps $\mathbb{C} \to V$ supported on the coset $s + \mathbb{Z}$, with support whose real part is bounded from below. Similarly, shifted formal Taylor series $z^sV[[z]]$ correspond to functions $\mathbb{C} \to V$ supported on $s + \mathbb{Z}_{\geq 0}$. Shifting commutes with taking formal series, e.g., $(z^sV[[z]])([w]) = z^s(V[[z]][[w]])$, when viewed as subfunctors of a fractional formal power series functor.

(7) If $k$ is a positive integer, we write $V((z^{1/k}))$ to denote the direct sum of all $z^{r/k}V((z))$ as $r$ ranges over coset representatives of $\mathbb{Z}$ in $\frac{1}{k}\mathbb{Z}$. For algebraically independent fundamental coordinates, we may iterate this construction to obtain a finite direct sum of shifted iterated Laurent series spaces, e.g., $V((z^{1/N}))((w^{1/N})) = \bigoplus_{k,\ell=1}^N z^{k/N}w^{\ell/N}V((z))((w))$.

(8) Partial derivative operators on series spaces act on individual fundamental coordinates. To clarify by example, the operator $\partial_w$, as defined on $V((w))((z-w))$, takes $w^m(z-w)^n$ to $mw^{m-1}(z-w)^n$, not to $mw^{m-1}(z-w)^n - nw^m(z-w)^{n-1}$.

The reader should note that we may simplify the notation for maps when it is convenient to do so. For example, if $L(-1) : V \to V$ is a linear endomorphism of a vector space, we may denote the corresponding endomorphism on $z^aV((z))$ by $L(-1)$ instead of $z^aL(-1)((z))$. We apologize for any confusion this may cause.

**Definition 2.1.2.** Using the ordering $z > w > t$, we choose the following conventions for embeddings:

1. $(z-w)^s = z^s(1-w/z)^s \in z^s\mathbb{C}[z^{-1}][[w]] \subset z^s\mathbb{C}((z))((w))$, but we expand $(z-w)^s$ in $w^s\mathbb{C}[w^{-1}][[z]] \subset w^s\mathbb{C}(w)((z))$ as $e(s/2)w^s(1-z/w)^s$. The cases of $z-t$ and $w-t$ are treated similarly.

2. Whenever the expressions $(-z)^s$, $(-w)^s$, $(-t)^s$, $(w-z)^s$, $(t-z)^s$, or $(t-w)^s$ are used, they are to be interpreted as $e(s/2)z^s$, $e(s/2)w^s$, $e(s/2)t^s$, $e(s/2)(z-w)^s$, $e(s/2)(z-t)^s$, and $e(s/2)(w-t)^s$, respectively. In particular, notation like $V((w-z))$ and $V((-t))$ will not be used.

3. We expand $w^s$ in $z^s\mathbb{C}[[z]][[w]] \subset z^s\mathbb{C}((z))((w-z))$ as $z^s(1-\frac{z-w}{z-w})^s$, but we expand $z^s$ in $w^s\mathbb{C}[[w]][[z]] \subset w^s\mathbb{C}(w)((z-w))$ as $w^s(1+\frac{z-w}{z-w})^s$. The cases of $z-t$ and $w-t$ are treated similarly.

4. We expand $(z-t)^s$ in $(w-t)^s\mathbb{C}[[w-t]][[z-w]] \subset (w-t)^s\mathbb{C}((w-t))((z-w))$ as $(w-t)^s(1+\frac{z-w}{z-w})^s$. Similarly, we expand $(z-w)^s$ in $(z-t)^s\mathbb{C}[[z-t]][[w-t]] \subset (z-t)^s\mathbb{C}((z-t))((w-t))$ as $(z-t)^s(1-\frac{w-t}{z-t})^s$.

5. We expand $(w-t)^s$ as $e(-s/2)(z-w)^s(1-\frac{z-w}{z-w})^s$ in $(z-w)^s\mathbb{C}[[z-w]][[z-t]] \subset (w-t)^s\mathbb{C}(w)((z-w))((z-t))$. The minus sign in the exponential may seem arbitrary here, but we justify it with the equation $e(-s/2)(t-w)^s = e(-s/2)((z-w) - (z-t))^s$. The sign will be important in Lemma 2.1.7.

**Definition 2.1.3.** For any complex numbers $a, b, c$ and any vector space $V$, we define $z^aw^b(z-w)^cV[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ as the subspace of $z^aw^bV((z))((w))$ in which multiplication by
the product of $z^{-a}w^{-b}(z - w)^{-c}$ with a sufficiently large integer power of $zw(z - w)$ yields an element of $V[[z, w]]$. This yields a subfunctor of $V \mapsto z^{a+c}w^bV((z))(w))$.

**Lemma 2.1.4.** The conventions listed above yield the following embeddings:

1. $z^aw^b(z - w)^cV[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \to z^{a+c}w^bV((z))(w))$
2. $z^aw^b(z - w)^cV[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \to w^{b+c}z^aV((w))(z))$
3. $z^aw^b(z - w)^cV[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \to z^{a+b}(z - w)^cV((z))(w))$
4. $(z - w)^a(z - t)^b(w - t)^cV[[z - w, w - t]][z^{-1}, (z - t)^{-1}, (w - t)^{-1}] \to (z - t)^{a+b}(w - t)^cV((z - t))(w - t))$
5. $(z - w)^a(z - t)^b(w - t)^cV[[z - w, w - t]][z^{-1}, (z - t)^{-1}, (w - t)^{-1}] \to (w - t)^{b+c}(z - w)^aV((w - t))(z - w))$

**Proof.** Omitted. □

**Remark 2.1.5.** There are some additional expansions we need to consider when working with three variables at a time.

1. We expand $(z - t)^s$ in $(w - t)^sC[[w, t]][w^{-1}, t^{-1}, (w - t)^{-1}](z - w))$ as $(z - w) + (w - t)^s = (w - t)^s(1 + \frac{z - w}{w - t})^s$. Our choice of $(w - t)$ instead of $w$ or $t$ as the distinguished invertible fundamental coordinate is due to the fact that it yields a binomial.
2. Some trinomial expansions are equal but not obviously so. For example $(1 - \frac{w}{z})^s$ is equal to both $(1 - \frac{t}{s})(1 - \frac{w}{z(1 - t/z)})^s$ and $(1 - \frac{t(1 + w/t)}{z})^s$. These more complicated expressions may appear when composing two embeddings.

**Lemma 2.1.6.** Given a vector space $V$ and complex numbers $a, b, c, d, f, g$, define the following vector spaces:

1. $V_{ijkl} = z^aw^b(t)(z - w)^d(z - t)^f(w - t)^gV[[z, w, t]][z^{-1}, w^{-1}, t^{-1}, (z - w)^{-1}, (z - t)^{-1}, (w - t)^{-1}]$
2. $V_{ijkl} = z^{a+d-f}w^b(t)(z - w)^dV(z, w, t)[z^{-1}, w^{-1}, t^{-1}, (z - w)^{-1}]
3. $V_{ijkl} = z^{a+f}w^{b+g}(z - w)^dV(z, w, t)[z^{-1}, w^{-1}, t^{-1}, (z - w)^{-1}](t)$
4. $V_{ijkl} = z^{a+b+c}(z - w)^d(z - t)^f(w - t)^gV(z, w, t)[z^{-1}, w^{-1}, t^{-1}, (z - w)^{-1}](w - t))$
5. $V_{ijkl} = z^{a+b+c}(z - w)^d(z - t)^f(w - t)^gV((t))[[z - w, w - t]][z^{-1}, (z - t)^{-1}, (w - t)^{-1}]
6. $V_{ijkl} = z^{a+b+c}(z - w)^d(z - t)^f(w - t)^gV((t))[[z - w, w - t]][z^{-1}, (z - t)^{-1}, (w - t)^{-1}]
7. $V_{ijkl} = z^{a+d+f}w^{b+g}t^eV((z))(t)
8. $V_{ijkl} = z^{a+b+c}(z - w)^dV((z))(w - t))
9. $V_{ijkl} = z^{a+b+c}(z - w)^dV((w))(t)
10. $V_{ijkl} = z^{a+b+c}(z - w)^dV((t))(z - w))
11. $V_{ijkl} = z^{a+b+c}(w - t)^f+g(z - w)^dV((t))((w - t))(z - w))$
Then, our conventions for expansions yield the following commutative diagram of functorial inclusions of vector spaces:

![Diagram](https://via.placeholder.com/150)

**Proof.** To check the commutativity of all quadrilaterals, it suffices to expand \( z^a w^b t c (z - w)^d (z - t)^f (w - t)^g \) along each of the compositions.

The inner five arrows yield:

1. \( z^{a+d+f} (1 - w/z)^d (1 - t/z)^f w^b t c (w - t)^g \in V_{i(jk)\ell} \)
2. \( z^{a+f} (1 - t/z)^f w^{b+g} (1 - t/w)^g (z - w)^d t^c \in V_{ij\ell} \)
3. \( z^{a+b+c} (1 - w/z)^b (z - t)^d + f (1 - w/z)^d (w - t)^g \in V_{i(jk)\ell} \)
4. \( w^{a+b} (1 + \frac{z-w}{w}) a t c (w - t)^f g (1 + \frac{z-w}{w}) t (z - w)^d \in V_{ij\ell} \)
5. \( t^{a+b+c} (1 + \frac{z-t}{t}) a (1 + \frac{w-t}{w}) b (z - w)^d (z - t)^f (w - t)^g \in V_{i(jk)\ell} \)

We obtain five comparisons:

1. The top diamond yields \( z^{a+d+f} w^{b+g} t c (1 - w/z)^d (1 - t/z)^f (1 - t/w)^g \).
2. The right diamond compares \( w^{a+b+f+g} c (z - w)^d (1 + \frac{z-w}{w}) a f (1 - t/w)^g (1 - \frac{t}{w(1+\frac{t}{w})}) f \) to \( w^{a+b+f+g} (1 + \frac{z-w}{w}) a t c (1 - t/w)^f g (1 + \frac{z-w}{w(1-t/w)}) (z - w)^d \). This amounts to comparing \( (1 - \frac{w}{w(1+\frac{t}{w})}) f \) to \( (1 - t/w)^f (1 + \frac{z-w}{w(1-t/w)}) f \), and both are expansions of \( 1 - \frac{w}{w(1+\frac{t}{w})} \) in \( C[w^{-1}] \).
3. The left diamond compares \( z^{a+d+f} t^{b+c} (w - t)^g (1 - t/z)^d (1 - t/w)^g b (1 - \frac{w-t}{z(1-t/z)})^d \) to \( z^{a+d+f} b^{c} (w - t)^g (1 - t/z)^d (1 - t/w)^g b (1 - \frac{t}{z(1-t/z)})^d \). This amounts to comparing \( (1 - \frac{t}{z(1-t/z)})^d \) to \( (1 - \frac{t}{z(1-t/z)})^d \), and both are expansions of \( 1 - \frac{t}{z} \) in \( C[z^{-1}] \).
4. The lower right diamond compares \( t^{a+b+c} (w - t)^f + g (z - w)^d (1 + \frac{w-t}{t}) a + b (1 + \frac{z-w}{l(1+\frac{w-t}{t})}) a (1 + \frac{z-w}{w-t}) f \) to \( t^{a+b+c} (w - t)^f + g (z - w)^d (1 + \frac{z-w}{t}) a (1 + \frac{w-t}{t}) b (1 - \frac{w-t}{z-w}) \). This amounts to comparing \( 1 + \frac{z-w}{t} \) to \( 1 + \frac{z-w}{t} \), and both are expansions of \( 1 + \frac{z-w}{t} \) in \( C[t^{-1}] \).
5. The lower left diamond yields \( t^{a+b+c} (z - t)^d f (w - t)^g (1 + \frac{z-t}{t}) a (1 + \frac{w-t}{t}) b (1 - \frac{w-t}{z-w}) d \).

\[\square\]
Lemma 2.1.7. In addition to the vector spaces from Lemma 2.1.6, we define the following spaces:

1. \( V_{j((ki)ℓ)} = w^{b+d+g}z^{a+c}(z-t)fV((w))(z)((z-t)) \)
2. \( V_{j((ikj)ℓ)} = z^{a+b+c}(z-t)d+g(w-t)gV((z))(z)((w-t)) \)
3. \( V_{j((ki))ℓ} = z^{a+b+c}(z-w)d+g(z-t)fV((z))(z-w)((z-t)) \)
4. \( V_{j((ik))ℓ} = w^{b+d+g}z^{a+c}(z-t)fV((w))(z)((z-t)) \)
5. \( V_{j((ik))ℓ} = w^{b+d+g}z^{a+c}(z-t)fV((w))(z)((z-t)) \)
6. \( V_{j((ik))ℓ} = t^{a+b+c}(w-t)d+g(z-t)fV((t))(w-t)((z-t)) \)
7. \( V_{j((jki))ℓ} = t^{a+b+c}(z-t)f+g(z-w)dV((t))((z-w)) \)
8. \( V_{j((kij))ℓ} = w^{a+b+c}(w-t)f+g(z-w)dV((w))((z-w)) \)
9. \( V_{j((jki))ℓ} = w^{a+b+c}(w-t)f+g(z-w)dV((w))((z-w)) \)
10. \( V_{j((ikj))ℓ} = w^{a+b+c}(w-t)f+g(z-w)dV((w))((z-w)) \)
11. \( V_{j((kij))ℓ} = w^{a+b+c}(w-t)f+g(z-w)dV((w))((z-w)) \)
12. \( V_{j((ikj))ℓ} = w^{a+b+c}(w-t)f+g(z-w)dV((w))((z-w)) \)
13. \( V_{j((ikj))ℓ} = w^{a+b+c}(w-t)f+g(z-w)dV((w))((z-w)) \)
14. \( V_{j((j ki))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
15. \( V_{j((j ki))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
16. \( V_{j((j ki))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
17. \( V_{j((j ki))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
18. \( V_{j((kij))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
19. \( V_{j((kij))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
20. \( V_{j((j ki))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)
21. \( V_{j((j ki))ℓ} = w^{a+b+c}(z-w)d+g(z-t)fV((z))[z-w, z-t][z-w, t-z][z-1, t-1, (z-w)^{-1}] \)

Then, we have the following commutative diagrams of embeddings:
Proof. It suffices to expand the expression $z^n w^b t^c (z - w)^d (z - t)^f (w - t)^g$ along each of the compositions. We will check one of the quadrilaterals, and leave the remaining for the reader who wishes to repeat similar arguments.

Going up from $V_{ijk\ell}$ in the first diagram, we get $e(d/2) w^{b+d+g} z^a t^c (z - t)^f (1 - z/w)^d (1 - t/w)^g \in V_{j[i][k][\ell]}$. Going to the upper left, we get $e(-g/2) z^a c w^b (z - w)^d + g (1 - z/w)^g (1 - z/t)^g (z - t)^f \in V_{j[i][k]}$. Expanding again, we compare $e(d/2) w^{b+d+g} z^a t^c (z - t)^f (1 - z/w)^d (1 - z/(1-w)) (1 - z/t)^g (1 - z/t)^c$ with $e(d/2) w^{b+d+g} z^a t^c (z - t)^f (1 - z/w)^d (1 - z/w)^d (1 - z/w)^d (1 - z/w)^d$ in $V_{j[i][k][\ell]}$.

Both evaluate to $e(d/2) w^{b+d+g} z^a t^c (z - t)^f (1 - z/w)^d (1 - z/w)^d (1 - z/w)^d (1 - z/w)^d$.

**Lemma 2.1.8. (Formal Taylor theorem)** If $f \in V\{z\}$, then we have the following equalities in $V\{z\}[[w]]$:

1. $f(z + w) = e^{w \partial_z} f(z) = \sum_{n \geq 0} \left( \frac{w}{n!} \right)^n \partial_z^n f(z)$.
2. $f(ze^w) = e^{w \partial_z} f(z) = \sum_{n \geq 0} \left( \frac{zw}{n!} \right)^n \partial_z^n f(z)$.

The analogous result holds for any algebraically independent pair of fundamental coordinates.

**Proof.** See [Frenkel-Lepowsky-Meurman-1988] Proposition 8.3.1. 

**Remark 2.1.9.** One needs to be cautious about where elements live when using the formal Taylor theorem. For example, a naïve expansion of $\sum_{n \geq 0} \left( \frac{z-w}{n!} \right)^n \partial_w^n (w^{-1})$ in $\mathbb{C}\{z, w\}$ yields a divergent sum for the coefficient of $w^{-1}$. However, as an element of $\mathbb{C}((w))[[z - w]] \subseteq \mathbb{C}((z-w))$, we obtain the expansion of $z^{-1}$ as $\frac{1}{w} \sum_{n \geq 0} \left( \frac{w-z}{w} \right)^n$.

**Lemma 2.1.10.** (Frenkel-Ben-Zvi-2004, Remark 5.1.4) The map $f(z, z - w) \mapsto e^{(z-w) \partial_w} f(w, z - w) = \sum_{n \geq 0} \left( \frac{z-w}{n!} \right)^n \partial_w^n f(w, z - w)$ describes an isomorphism $V((z))((z - w)) \to V((w))((z - w))$. Furthermore, it forms the horizontal arrow in the following commutative diagram:

**Proof.** This follows straightforwardly from Lemma 2.1.8. 

□
2.2. Cohomology of $K(A, 2)$. In [MacLane-1952], Mac Lane introduces a cohomology theory of abelian groups that is distinct from ordinary group cohomology, and describes the explicit computation of the cohomology in low degree. Eilenberg explains in [Eilenberg-1952] that this cohomology theory is in fact the cohomology of a specific cell complex whose cohomology is isomorphic in low degree to a shift of the cohomology of the second Eilenberg-Mac Lane space. This claim is proved in [Eilenberg-Mac Lane-1953] Theorem 20.4 (but with the definition of $A(\Pi, n)$ shifted in degree from previous papers), where they do an explicit construction of $K(A, 2)$ via the diagonal of a two-fold Bar resolution. There are other methods for explicitly describing $K(A, 2)$, e.g., applying Dold-Kan correspondence to a homologically shifted abelian group (a freely available reference for this is [Durov-2007] section 8.5.2), but the Eilenberg-Mac Lane complex happens to be the most convenient for our purposes.

**Definition 2.2.1.** ([MacLane-1952]) Let $A$ be an abelian group. The Eilenberg-Mac Lane cochain complex with coefficients in $\mathbb{C}^\times$ has the following description in low degree:

$$
K^1 \xrightarrow{d^1} K^2 = \{ f : A \times A \to \mathbb{C}^\times \} \xrightarrow{d^2} K^3 = \{ (F, \Omega) : A^{\otimes 3} \times A^{\otimes 2} \to \mathbb{C}^\times \} \xrightarrow{d^3} \ldots
$$

where:

1. $K^1 = \{ \phi : A \to \mathbb{C}^\times \}$ and $K^2 = \{ f : A \times A \to \mathbb{C}^\times \}$ are groups of set-theoretic maps.
2. $K^3 = \{ (F, \Omega) : A^{\otimes 3} \times A^{\otimes 2} \to \mathbb{C}^\times \}$ is a group of pairs of set-theoretic maps.
3. The map $d^1 : K^1 \to K^2$ is given by the usual group cohomology coboundary $\phi \mapsto d^1 \phi$, defined by $d^1 \phi(i, j) = \phi(j) - \phi(i + j) + \phi(i)$.
4. The map $d^2 : K^2 \to K^3$ is given by the group cohomology coboundary $d^2 f(i, j, k) = f(j, k) - f(i + j, k) + f(i, j + k) - f(i, j)$ together with the antisymmetrizer: $(i, j) \mapsto f(i, j) - f(j, i)$.
5. The map $d^3 : K^3 \to K^4$ vanishes if and only if the following conditions are satisfied:
   - $(a)$ $F(i, j, k)F(i, j + k, \ell)F(j, k, \ell) = F(i + j, k, \ell)F(i, j, k + \ell)$ for all $i, j, k, \ell \in A$
   - $(b)$ $F(i, j, k)^{-1} \Omega(i, j + k)F(j, i, k)^{-1} = \Omega(i, j)F(j, i, k)^{-1} \Omega(i, k)$
   - $(c)$ $F(i, j, k)\Omega(i + j, k)F(k, i, j) = \Omega(j, k)F(i, k, j)\Omega(i, k)

Elements annihilated by $d^n$ are called abelian $n$-cocycles, and an abelian $n$-cocycle lies in the $n$th abelian cohomology class.

**Remark 2.2.2.** The reader should be aware that elements of $n$-th abelian cohomology are identified with elements of $H^{n+1}(K(A, 2), \mathbb{C}^\times)$. That is, there is a shift in degrees.

**Lemma 2.2.3.** ([MacLane-1952] Theorem 3) If $(F, \Omega)$ is an abelian 3-cocycle, then the map $Q : A \to \mathbb{C}^\times$ defined by $i \mapsto \Omega(i, i)$ is a quadratic function, i.e., $Q(i) = Q(-i)$ for all $i \in A$, and $\frac{Q(i + j + k)Q(i)Q(j)Q(k)}{Q(i + j)Q(i + k)Q(j + k)} = 1$ for all $i, j, k \in A$. Furthermore, the trace map $(F, \Omega) \mapsto (i \mapsto \Omega(i, i))$ induces a bijection from $H^4(K(A, 2), \mathbb{C}^\times)$ to the set of $\mathbb{C}^\times$-valued quadratic functions on $A$.

**Proof.** This is proved as [Eilenberg-Mac Lane-1954], theorem 26.1.

**Definition 2.2.4.** Let $A$ be an abelian group.

1. Let $\eta : A \otimes A \to \mathbb{C}^\times$ be a function, viewed as either a group cohomology 2-cochain or an abelian 2-cochain. We say that $\eta$ is normalized if $\eta(0, i) = \eta(i, 0) = 1$ for all $i \in A$.
2. Let $F : A \otimes A \otimes A \to \mathbb{C}^\times$ be a function, viewed as a group cohomology 3-cochain. We say that $F$ is normalized if $F(0, i, j) = F(i, 0, j) = F(i, j, 0) = 1$ for all $i, j \in A$.
3. We say that an abelian 3-cochain $(F, \Omega)$ is normalized if $F$ is normalized as a group cohomology 3-cochain, and $\Omega$ is normalized as a 2-cochain.

**Lemma 2.2.5.** All cohomology classes for the Eilenberg-Mac Lane model of $K(A, 2)$ in degree at most 4 are represented by normalized cochains. Normalized cocycles in an abelian cohomology class of degree at most 4 have a transitive action by differentials of normalized cochains.
Proof. Omitted - this is a straightforward computation. □

Joyal and Street [Joyal-Street-1986] gave a category theoretic interpretation of this cohomology. If we consider the abelian category $\text{Vect}^A$ of complex vector spaces graded by an abelian group $A$, any abelian 3-cocycle $(F, \Omega)$ defines the associator and commutor for a braided tensor structure on this category. We denote by $\text{Vect}_{F,\Omega}^A$ the braided tensor category corresponding to such a cocycle. Braided tensor categories are useful in our context, because they are essentially the minimal structure necessary for defining a reasonable notion of commutative algebra:

**Definition 2.2.6.** (see e.g. [Baez-1992]) Let $A$ be an abelian group, and let $(F, \Omega)$ be an abelian 3-cocycle with coefficients in $\mathbb{C}^\times$. Then a commutative algebra in $\text{Vect}_{F,\Omega}^A$ is an $A$-graded vector space $V = \oplus_{i\in A} V^i$, equipped with a multiplication map $m : V \otimes V \to V$ that restricts to maps $V^i \otimes V^j \to V^{i+j}$, such that for all $i, j, k \in A$, the following diagram commutes:

$$
\begin{array}{ccc}
(V^i \otimes V^j) \otimes V^k & \xrightarrow{\Omega(i,j)} & V^i \otimes (V^j \otimes V^k) \\
\downarrow m \otimes 1 & & \downarrow 1 \otimes m \\
V^{i+j} \otimes V^k & \xrightarrow{m} & V^{i+j+k}
\end{array}
$$

Here, the decorations $F(i, j, k)$, $\Omega(i, j)$, and $F(j, i, k)$ over the arrows near the top indicate multiplication by the corresponding constants, or equivalently, application of the corresponding associator and commutor equivalences in the braided structure on $\text{Vect}_{F,\Omega}^A$.

The abelian 3-cocycle condition is precisely what is necessary for the multiplication of any finite number of objects under any choice of reordering to be consistently definable. The change in multiplication under such reorderings yields a system of one dimensional representations of braid groups.

Given an abelian 2-cochain $f : A \times A \to \mathbb{C}^\times$, one may define perturbed versions of such an algebra, by rescaling the restriction of $m$ to $V^i \otimes V^j$ by $f(i, j)$ for each $i, j \in A$. This yields a commutative algebra in $\text{Vect}_{F',\Omega'}^A$, where $(F', \Omega') = d^2 f + (F, \Omega)$. A more thorough analysis shows that the algebras are isomorphic if $f$ represents the zero class in degree 2 abelian cohomology, and the braided tensor categories are equivalent if and only if $f$ is an abelian 2-cocycle. In particular, it is a theorem from [Joyal-Street-1986] that equivalence classes of braided structures on $\text{Vect}^A$ (for which the braiding is determined by homogeneous one-dimensional spaces) are parametrized by $H^4(K(A, 2), \mathbb{C}^\times)$.

A rough homotopy theoretic justification is that homotopy classes of maps to $K(\mathbb{C}^\times, 2)$ classify local systems of categories with fiber $\text{Vect}$, and two-fold loop maps from $A$ describe such local systems on $A$ with $E_2$ extension structure. Delooping twice yields a parametrization of equivalence classes of such two-fold loop maps by homotopy classes of maps $K(A, 2) \to K(\mathbb{C}^\times, 4)$.

One may replace the rather straightforward theory of commutative rings with singular commutative rings, i.e., vertex algebras (see e.g., [Borcherds-2001] for another perspective on this). In the first half of this paper, we shall use the homological machinery described above to construct new vertex algebras in $\text{Vect}_{F,\Omega}^A$ using partially defined multiplication data.

2.3. Structured vertex algebras.

**Definition 2.3.1.** Given a tensor product $A_1 \otimes \cdots \otimes A_n$ of complex vector spaces and a permutation $\sigma \in S_n$, we write $\tau_\sigma$ to indicate the linear isomorphism $A_1 \otimes \cdots \otimes A_n \to A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}$ determined by the assignment $a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$, where $a_i \in A_i$ for all $1 \leq i \leq n$. Given
a set $S$ of maps $A_1 \otimes \cdots \otimes A_n \rightarrow B$, we write $\tau_\sigma S$ to denote the set whose elements are maps $A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)} \rightarrow B$ given by precomposing elements of $S$ with $\tau_\sigma$.

**Definition 2.3.2.** A vertex algebra is a tuple $(V, 1, L(-1), m_z)$, where:

1. $V$ is a vector space.
2. $1 \in V$ is an “identity” or “vacuum” element.
3. $L(-1) \in \text{End}(V)$ is a “translation” operator.
4. $m_z : V \otimes V \rightarrow V((z))$ is a linear “multiplication” map.

These data are required to satisfy the following conditions:

1. $m_z(1 \otimes v) = vz^0$ and $m_z(v \otimes 1) \in vz^0 + zV[[z]]$ for all $v \in V$.
2. $L(-1)m_z(u \otimes v) - m_z(u \otimes L(-1)v) = \frac{d}{dz}m_z(u \otimes v)$ for all $u, v \in V$. Here, the first $L(-1)$ is an abbreviation for $L(-1)((z))$.
3. The following diagram commutes:

$$
\begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{\tau_{(12)}} & V \otimes V \otimes V \\
1 \otimes m_w & & 1 \otimes m_z \\
V \otimes V((w)) & \xrightarrow{m_z((w))} & V[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}] \\
& & \xrightarrow{m_w((z))} V((w))((z)) \\
V((w))((w)) & & V((w))((z))
\end{array}
$$

**Remark 2.3.3.** This looks slightly different from the definitions presented elsewhere (e.g., [Kac-1997] section 4.9). One typically has a map $Y : V \rightarrow (\text{End}(V)[[z^\pm 1]])$ in place of the map $m_z : V \otimes V \rightarrow V((z))$ - it is straightforward to show that we obtain the original axioms by replacing $m_z(u \otimes v)$ with $Y(u, z)v$.

**Definition 2.3.4.** A weighted (or $\mathbb{Z}$-graded) vertex algebra is a vertex algebra equipped with an operator $L(0) : V \rightarrow V$ satisfying:

1. $L(0)$ acts semisimply on $V$ with integer eigenvalues, so that $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is an eigenvalue decomposition, where $L_0v = nv$ for all $v \in V_n$.
2. $L(0)m_z(u \otimes v) - m_z(u \otimes L(0)v) = m_z(L(0)u \otimes v) + zm_z(L(-1)u \otimes v)$ for all $u, v \in V$.
3. $[L(-1), L(0)] = -L(-1)$.

A Möbius (resp., quasi-conformal) vertex algebra is a weighted vertex algebra equipped with an operator $L(1)$ (resp., operators $\{L(i)\}_{i=1}^\infty$) such that:

1. For all $i, j \geq -1$, $[L(i), L(j)] = (i-j)L(i+j)$. In particular, the linear span of the operators $L(-1), L(0), L(1)$ (resp., $\{L(i)\}_{i=1}^\infty$) has a Lie algebra structure.
2. For all $i \geq -1$, $L(i)m_z(u \otimes v) - m_z(u \otimes L(i)v) = \sum_j z^j \partial_m z(L(i-j)u \otimes v)$.
3. The sub-Lie algebra spanned by $\{L_i\}_{i \geq 1}$ acts locally nilpotently on $V$.

A conformal vertex algebra of central charge $c \in \mathbb{C}$ is a weighted vertex algebra equipped with a distinguished element $\omega \in V_2$ such that the operators $\{L(n) : V \rightarrow V\}_{n \in \mathbb{Z}}$ defined by $m_z(\omega \otimes v) = \sum_{n \in \mathbb{Z}} L(n)vz^{-n-2}$ satisfy the Virasoro relations:

$$
[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12} \delta_{m+n,0}c
$$

**Lemma 2.3.5.** For future reference, we include the following facts:
(1) For any element \( v \) in a vertex algebra \( V \), \( m_z(v \otimes 1) = e^{zL(-1)v} \). More generally, we have the antisymmetry property: \( m_z(u \otimes v) = e^{zL(-1)m_{-z}(v \otimes u)} = e^{zL(-1)m_{-z}(v \otimes u)} \). Also, we have \( m_z(L(-1)u \otimes v) = \frac{d}{dz}m_z(u \otimes v) \).

(2) Given a vertex algebra \( V \), the following diagram commutes:

\[
\begin{array}{ccc}
V((z-w)) \otimes V & \rightarrow & V \otimes V((w)) \\
m_{-w} \otimes 1 \downarrow & & \downarrow 1 \otimes m_w \\
V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}] & \rightarrow & m_z((w)) \\
m_w((z-w)) \downarrow & & m_z((w)) \\
V((w))((z-w)) & \rightarrow & V((z))((w))
\end{array}
\]

This is called the "associativity property".

(3) Let \( V \) be a vertex algebra, let \( n \geq 2 \) be an integer, and let \( \sigma \in S_n \) be a permutation. Then, the map \( m_z \circ (1 \otimes m_z) \circ \cdots \circ (1 \otimes \cdots \otimes 1 \otimes m_z) \) from \( V^\otimes n \otimes V \) to \( V((z_0)) \cdots ((z_1)) \) and the map \( m_{\sigma(1)} \circ (1 \otimes m_{\sigma(2)}) \circ \cdots \circ (1 \otimes \cdots \otimes 1 \otimes m_{\sigma(n)}) \circ (\tau_\sigma \otimes 1) \) from \( V^\otimes n \otimes V \) to \( V((z_{\sigma(1)})) \cdots ((z_{\sigma(1)})) \) factor through equal maps from \( V^\otimes n \otimes V \) to \( V[[z_1, \ldots, z_n]](\prod_{i=1}^n (z_i \prod_{j>i}(z_i - z_j))^{-1}) \).

Proof. The first claim is proved in [Frenkel-Ben-Zvi-2004], Proposition 3.2.5 and Corollary 3.1.6.

The second claim is proved in, e.g., [Frenkel-Ben-Zvi-2004] Theorem 3.2.1.

The third claim is proved in [Frenkel-Ben-Zvi-2004] Theorem 4.5.1.

\( \square \)

Definition 2.3.6. Given a group \( G \), a \( G \)-action on a vertex algebra \( V \) is a \( G \)-action on the underlying vector space that fixes 1, commutes with \( L(-1) \), and satisfies \( m_z(gu \otimes gv) = gm_z(u \otimes v) \) for all \( g \in G \) (where we have tacitly extended the endomorphism \( g \) to \( V((z)) \) by the \((-((z))\)) functor).

A \( G \)-action on a weighted (resp., Möbius, quasi-conformal) vertex algebra is a \( G \)-action on the underlying vertex algebra that commutes with all defined operators \( L(i) \). A \( G \)-action on a conformal vertex algebra is a \( G \)-action on the underlying vertex algebra that fixes \( \omega \).

Lemma 2.3.7. If \( (V,1,L(-1),m_z) \) and \( (V',1',L(-1)',m_z') \) are vertex algebras, then \( (V \otimes V',1 \otimes 1',L(-1) \otimes \text{id}_{V'} + \text{id}_V \otimes L(-1)',m_z \otimes m_z') \) is a vertex algebra. If \( V \) and \( V' \) are weighted (resp., Möbius, quasi-conformal, conformal), then so is \( V \otimes V' \), and central charges add.

Proof. Omitted.

\( \square \)

2.4. Projectively equivariant (twisted) modules.

Definition 2.4.1. Let \( V \) be a vertex algebra. A \( V \)-module is a vector space \( M \) equipped with an action map \( act_z : V \otimes M \rightarrow M((z)) \) and an operator \( L(-1)^M \), satisfying the following conditions:

1. \( act_z(1 \otimes x) = xz^0 \) for all \( x \in M \)
2. \( L(-1)^M act_z(u \otimes x) - act_z(u \otimes L(-1)^M x) = act_z(L(-1)u \otimes x) \) for all \( u \in V, x \in M \).
The following diagram commutes:

\[
\begin{array}{ccc}
V \otimes V \otimes M & \xrightarrow{1 \otimes \text{act}_w} & V \otimes M((w)) \\
\downarrow m_{z-w} \otimes 1 & & \downarrow \text{act}_z((w)) \\
V((z-w)) \otimes M & \xrightarrow{\text{act}_w((z-w))} & M[[z,w][z^{-1},w^{-1},(z-w)^{-1}]] \\
\downarrow & & \downarrow \\
M((z)((z-w)) & \xrightarrow{\text{act}_z((z-w))} & M((z))((z-w)) \\
\end{array}
\]

If \( V \) is weighted (resp., Möbius, quasi-conformal), a \( V \)-module is a module \( M \) for the underlying vertex algebra, equipped with an operator \( L(0)^M \) (resp., operators \( L(0)^M \) and \( L(1)^M \), operators \( \{L(i)^M\}_{i \geq -1} \)), satisfying the following conditions:

1. \( L(0)^M \) acts semisimply on \( M \), so that \( M = \bigoplus_{n \in \mathbb{C}} M_n \) is an eigenvalue decomposition.
2. For all \( i, j \geq -1 \), \( [L(i)^M, L(j)^M] = (i - j)L(i + j)^M \).
3. For all \( i \geq -1 \), any \( u \in V \), and any \( v \in M \), \( L(i)^M \text{act}_z(u \otimes v) - \text{act}_z(u \otimes L(i)^M v) = \sum_{j=0}^{i+1} \binom{i+1}{j} z^j m_z(L(i-j)u \otimes v) \).

If \( V \) is a conformal vertex algebra of central charge \( c \), a \( V \)-module is a module \( M \) for the underlying weighted vertex algebra, satisfying the additional condition that the operators \( \{L(n)^M : M \rightarrow M\}_{n \in \mathbb{Z}} \) defined by \( \text{act}_z(\omega \otimes x) = \sum_{n \in \mathbb{Z}} (L(n)^M x \omega^{-n-2} \) yield a representation of the Virasoro algebra of central charge \( c \). If \( V \) is weighted, than a \( V \)-module \( M \) has weight \( k \) (mod \( \mathbb{Z} \)) for some \( k \in \mathbb{C} \) if the \( L(0) \)-spectrum of \( M \) lies in the coset \( k + \mathbb{Z} \subset \mathbb{C} \).

**Definition 2.4.2.** Let \( V \) be a vertex algebra, and let \( G \) be a group acting on \( V \) by automorphisms. A projectively \( G \)-equivariant module is a \( V \)-module \( M \) equipped with a projective \( G \)-action on the underlying vector space \( M \), such that:

1. For any \( g \in G \) and any lift \( \tilde{g} \) of \( g \) to a linear transformation of \( M \), \( \text{act}_z(gu \otimes \tilde{g}v) = \tilde{g} \cdot \text{act}_z(u \otimes v) \) for all \( u \in V \), \( v \in M \).
2. The projective \( G \)-action commutes with \( L(-1)^M \), i.e., for any \( g \in G \) and any lift \( \tilde{g} \) of \( g \) to a linear transformation on \( M \), \( \tilde{g}L(-1)^M = L(-1)^M \tilde{g} \).

If \( V \) is weighted (resp., Möbius, quasiconformal, conformal), we also demand that the projective \( G \)-action commute with \( L(i)^M \) for all applicable operators \( L(i)^M \).

**Remark 2.4.3.** An alternative definition is that \( \text{act} \) induces a \( G \)-equivariant projective linear map between the projective spaces of \( V \otimes M \) and \( M((z)) \). This may appear to be weaker, because \( \text{act}_z(gu \otimes \tilde{g}v) \text{ and } \tilde{g} \cdot \text{act}_z(u \otimes v) \) seem to agree only up to a global constant, but this constant is forced to be unity by the case \( u = 1 \). Note also that when \( V \) is conformal, the condition that all \( L(i) \) commute with \( \tilde{g} \) is redundant, because \( g \) fixes \( \omega \).

**Definition 2.4.4.** Let \( V \) be a vertex algebra, and let \( g \) be an automorphism of \( V \) of order \( N \). A \( g \)-twisted module is a vector space \( M \) equipped with a map \( \text{act}_z : V \otimes M \rightarrow M((z^{1/N})) \) and an operator \( L(-1)^M \) satisfying the following conditions:

1. Identity: \( \text{act}_z(1 \otimes x) = xz^0 \) for all \( x \in M \).
2. Translation: \( L(-1)^M \text{act}_z(u \otimes x) - \text{act}_z(u \otimes L(-1)^M x) = \text{act}_z(L(-1)u \otimes x) \).
Remark 2.4.5

If we define $V_{k,\ell}$, then for all $G$ be a group acting on $M$, the following diagram commutes:

$$
\begin{array}{ccc}
V((z-w)) \otimes M & \xrightarrow{act_w((z-w))} & M((w^{1/N}))((z-w)) \\
V \otimes V \otimes M & \xrightarrow{\text{compatibility axiom}} & V \otimes M((w^{1/N})) \otimes M((z-w)) \\
& \xrightarrow{\text{monodromy}} & M[[z^{1/N}, w^{1/N}]](z^{-1}, w^{-1}, (z-w)^{-1})
\end{array}
$$

(4) Monodromy: For all $u \in V$ and $v \in M$, $act_z(gu \otimes v) = act_z(u \otimes v)$. In other words, if $g = e^{2\pi ik/N}u$, then $act_z(u \otimes v) \in z^{k/N}M((z))$.

If $V$ is weighted (resp., Möbius, quasi-conformal), a $g$-twisted $V$-module is a $g$-twisted module $M$ for the underlying vertex algebra, equipped with an operator $L(0)^M$ (resp., operators $L(0)^M$ and $L(1)^M$, operators $\{L(i)^M\}_{i \geq -1}$), satisfying the following conditions:

1. $L(0)^M$ acts semisimply on $M$, so that $M = \bigoplus_{n \in \mathbb{C}} M_n$ is an eigenvalue decomposition.
2. For all $i, j \geq -1$, $[L(i)^M, L(j)^M] = (i-j)L(i+j)^M$.
3. For all $i \geq -1$, any $u \in V$, and any $v \in M$, $L(i)^M act_z(u \otimes v) - act_z(u \otimes L(i)^Mv) = \sum_{j=0}^{i+1} \frac{1}{i+1-j} z^j m_z(L(i-j)u \otimes v)$.

If $V$ is a conformal vertex algebra of central charge $c$, a $V$ module is a module $M$ for the underlying vertex algebra, satisfying the additional condition that the operators $\{L(n)^M : M \rightarrow M\}_{n \in \mathbb{N}}$ defined by $act_z(\omega \otimes x) = \sum_{n \in \mathbb{N}} L(n)^M x z^{-n-2}$ yield a representation of the Virasoro algebra of central charge $c$. If $V$ is weighted, then a $g$-twisted $V$-module $M$ has weight $k \pmod{\frac{1}{N}\mathbb{Z}}$ for some $k \in \mathbb{C}$ if the $L(0)$-spectrum of $M$ lies in the coset $k + \frac{1}{N}\mathbb{Z} \subset \mathbb{C}$.

Remark 2.4.5. By using the monodromy axiom, we can make the compatibility axiom look slightly simpler: If we define $V^k = \{v \in V : gv = e(k/N)v\}$, then the axiom is equivalent to the condition that for all $k, \ell \in \mathbb{Z}/N\mathbb{Z}$, the following diagram commutes:

$$
\begin{array}{ccc}
V^k \otimes V^\ell \otimes M & \xrightarrow{m_{z-w} \otimes 1} & V^{k+\ell}((z-w)) \otimes M \\
& \xrightarrow{\text{compatibility axiom}} & V^{k} \otimes w^{\ell/N}M((w)) \otimes M((z-w)) \\
& \xrightarrow{z^{k/N}w^{\ell/N}M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]} & z^{k/N}w^{\ell/N}M((z))((z-w))
\end{array}
$$

Definition 2.4.6. Let $V$ be a vertex algebra, let $g$ be a finite order automorphism of $V$, and let $G$ be a group acting on $V$ by automorphisms. A projectively $G$-equivariant $g$-twisted module is a $g$-twisted $V$-module $M$ equipped with a projective $G$-action on the underlying vector space $M$, such that:

1. For any $h \in G$ and any lift $\tilde{h}$ of $h$ to a linear transformation of $M$, $act_z(hu \otimes \tilde{h}v) = \tilde{h} \cdot act_z(u \otimes v)$ for all $u \in V$, $v \in M$. 

1. For any $h \in G$ and any lift $\tilde{h}$ of $h$ to a linear transformation of $M$, $act_z(hu \otimes \tilde{h}v) = \tilde{h} \cdot act_z(u \otimes v)$ for all $u \in V$, $v \in M$. 

(2) The projective $G$-action commutes with $L(-1)^M$, i.e., for any $h \in G$ and any lift $\tilde{h}$ of $h$ to a linear transformation on $M$, $\tilde{h}L(-1)^M = L(-1)^M\tilde{h}$.

If $V$ is weighted (resp., Möbius, quasiconformal, conformal), we also demand that the projective $G$-action commute with $L(i)^M$ for all applicable operators $L(i)^M$.

**Lemma 2.4.7.** Let $V$ be a weighted vertex algebra, and let $g$ be a finite order automorphism of $V$. Then any $g$-twisted $V$-module admits a canonical projectively $(g)$-equivariant structure, where the endomorphism $e(jL(0))$ is a lift of $g^j$ for all integers $j$.

**Proof.** We check that the conditions are satisfied:

1. To prove that $e(jL(0))act_z(u \otimes v) = act_z(g^j u \otimes e(jL(0))v)$ for all integers $j$, it suffices to consider the case when $u$ is an eigenvector for $g$ (say, with eigenvalue $e(\ell)$), and $v$ is an eigenvector for $L(0)$, say with eigenvalue $\ell$. Then $e(jL(0))v = e(j\ell)v$, and $act_z(g^j u \otimes v) = e(j\ell)act_z(u \otimes v)$. Since the monodromy axiom implies $act_z(u \otimes v) \in z^*M((z))$, we have:
\[
e(jL(0))act_z(u \otimes v) = e(j(r + \ell))act_z(u \otimes v) = e(j\ell)act_z(g^j u \otimes v) = act_z(g^j u \otimes e(jL(0))v)
\]

2. To check commutativity with $L(i)$ for all applicable operators $L(i)$, we note that if we restrict to $L(0)$-eigenvectors, $e(L(0))$ is multiplication by a constant. Any operator $L(i)$ takes $L(0)$-eigenvectors to $L(0)$-eigenvectors with eigenvalue shifted by the integer $i$, so the restriction of $e(L(0))$ to $L(0)$-eigenvectors commutes with all $L(i)$. Since $L(0)$-eigenvectors span $M$, $e(L(0))$ commutes with all $L(i)$ in general.

\[\square\]

**Lemma 2.4.8.** Let $G$ be a group that acts on vertex algebras $V_1$ and $V_2$ by automorphisms, let $g_1, g_2 \in G$ be finite order commuting elements, let $M_1$ be a $g_1$-twisted module of $V$, and let $M_2$ a $g_2$-twisted module of $V_2$. Then:

1. $M_1 \otimes M_2$ is a $g_1 g_2$-twisted $V_1 \otimes V_2$-module. In particular, if $g_1 g_2 = 1$, then the module is untwisted.
2. If $M_1$ and $M_2$ are projectively $G$-equivariant, then $M_1 \otimes M_2$ is projectively $G$-equivariant.
3. If $V_1$ and $V_2$ are weighted (resp., Möbius, quasi-conformal, conformal) and $M_1$ and $M_2$ are modules in this enhanced sense, then so is $M_1 \otimes M_2$. Furthermore, the $L(0)$-spectra add.

**Proof.** The first claim follows from a straightforward application of the character decomposition of $V_1 \otimes V_2$ under the group $H = \langle g_1, g_2 \rangle$. For an $H$-eigenvector $v_1 \otimes v_2 \in V_1 \otimes V_2$, the fractional exponents arising from $act_z(v_1 \otimes v_2 \otimes m_1 \otimes m_2)$ are determined by the sum of fractional exponents arising from $act_z(v_1 \otimes m_1)$ and $act_z(v_2 \otimes m_2)$.

For the second claim, we note that for any $h \in G$ and any lift $\tilde{h}$ to linear transformations on $V_1$ and $V_2$, the power series $act_z(hu \otimes \tilde{h}v' \otimes \tilde{hv'})$ is formed from are tensor products of the terms in $act_z(hu \otimes \tilde{hv'}) = \tilde{h}act_z(u \otimes v)$ and act$_z(hu \otimes \tilde{hv'}) = \tilde{h}act_z(u \otimes v')$. We therefore have equality with $\tilde{h}act_z(u \otimes v \otimes v')$, as claimed.

The third claim is straightforward to verify. \[\square\]

2.5. **Intertwining operators.**

**Definition 2.5.1.** Let $V$ be a vertex algebra, and let $M_1, M_2, M_3$ be $V$-modules. An intertwining operator of type $(\frac{M_3}{M_1,M_2})$ is a map $I_z : M_1 \otimes M_2 \to M_3\{z\}$ satisfying the following conditions:

1. Translation covariance: $L(-1)^{M_3}I_z(u \otimes v) - I_z(u \otimes L(-1)^{M_3}v) = \frac{d}{dz}I_z(u \otimes v)$.
In the conformal case, this is Proposition 5.4.7 in \cite{Frenkel-Huang-Lepowsky-1993}. In the proof.

Lemma 2.5.5. Let $L$ be a weighted vertex algebra, and let $M_1$, $M_2$, and $M_3$ be $V$-modules of weight $k_1, k_2, k_3 \pmod{Z}$, respectively. Then any intertwining operator $I_z : M_1 \otimes M_2 \rightarrow M_3\{z\}$ takes values in $z^{k_3-k_2-k_3}M_3\{z\}$.

Proof. This is a well-known result. One may find proofs in \cite{Frenkel-Huang-Lepowsky-1993} Remark 5.4.4, and \cite{Huang-Lepowsky-Zhang-2007} Corollary 3.23, under some superfluous hypotheses. □

Lemma 2.5.3. Let $V$ be a vertex algebra. Given an intertwining operator $I_z : M_1 \otimes M_2 \rightarrow M_3\{z\}$, the map $I^*_z : M_2 \otimes M_1 \rightarrow M_3\{z\}$ given by $I^*_z(v \otimes u) = e^{zL(-1)}I_{-z}(u \otimes v)$ is an intertwining operator. In particular, for any $V$-module $M$, the map $\text{act}_*: M \otimes V \rightarrow M\{z\}$ defined by $\text{act}_*(x \otimes v) = e^{zL(-1)}\text{act}_{-z}(v \otimes x)$ for all $v \in V$ and $x \in M$, is a projectively $G$-equivariant intertwining operator. The same statements hold if $V$ is weighted, Möbius, or (quasi-)conformal.

Proof. In the conformal case, this is Proposition 5.4.7 in \cite{Frenkel-Huang-Lepowsky-1993}. In the Möbius case, this is a special case of Proposition 3.4.4 in \cite{Huang-Lepowsky-Zhang-2007}, where $I_z$ is non-logarithmic, and $r = 0$. The argument in their proof does not use the Möbius structure that is assumed to exist, except where it pertains to $L(i)$-compatibility conditions of $I_z^*$. Projective $G$-equivariance is straightforward. □

Remark 2.5.4. Lemma 2.5.3 implies the intertwining operator $\text{act}_z$, together with $\text{act}_z$, induces a “split square-zero extension” vertex algebra structure on $V \otimes M$.

Lemma 2.5.5. Let $V$ be a weighted vertex algebra, and let $M_1$ and $M_2$ be $V$-modules of weight $k_1, k_2 \pmod{Z}$, respectively. If $I_z : M_1 \otimes M_1 \rightarrow M_2\{z\}$ is a nonzero intertwining operator that is a constant multiple of $I^*_z$, then $I_z(u \otimes v) = \pm e^{(k_2-k_1)}I_z^*(u \otimes v)$ for all $u, v \in M_1$.

Proof. We consider two cases, depending on whether $I_z$ is alternating.

First case: Suppose there exists $u \in M_1$ such that $I_z(u \otimes u) \neq 0$. By Lemma 2.5.2 we may write $I_z(u \otimes u) = \sum_{n=0}^{\infty} g_{n+r}z^{n+r}$, with $g_r$ a nonzero element of $M_2$, and $r \equiv k_2 - 2k_1 \pmod{Z}$. Since $I^*_z(u \otimes u) = e^{zL(-1)}I_{-z}(u \otimes u)$, it suffices to check that the lowest-order term in the latter series is $e^{\pi ir}g_rz^r$. By assumption, $I_z$ is proportional to $I^*_z$, and our calculation shows that the constant of proportionality is $e^{(k_2-k_1)}$.

Second case: If for all $u \in M_1$, $I_z(u \otimes u) = 0$, then bilinearity implies $I_z(u \otimes v) = -I_z(v \otimes u)$ for all $u, v \in M_1$. Write $I_z(u \otimes v) = \sum_{n=0}^{\infty} g_{n+r}z^{n+r}$, with $g_r$ a nonzero element of $M_2$, and $r \equiv k_2 - 2k_1 \pmod{Z}$. We note that:

$$I^*_z(u \otimes v) = e^{zL(-1)}I_{-z}(v \otimes u) = -e^{zL(-1)}\sum_{n=0}^{\infty} g_{n+r}(e^{\pi iz})^{n+r}$$
so the lowest order term is \(-e^{\pi i r_g z^r}\). By assumption, \(I_z\) is proportional to \(I^*_z\), and our calculation shows that the constant of proportionality is \(\pm e^\left(\frac{r_g}{2}\right) = \pm e^\left(\frac{k_2}{2} - k_1\right)\).

\[\square\]

**Definition 2.5.6.** Let \(V\) be a vertex algebra with an action of a group \(G\), and let \(M_1, M_2, M_3\) be projectively \(G\)-equivariant modules. An intertwining operator \(I_z : M_1 \otimes M_2 \to M_3\) is projectively \(G\)-equivariant if it satisfies the following condition: For any \(g \in G\), and any lifts \(g_1, g_2, g_3\) of \(g\) to linear transformations on \(M_1, M_2, M_3\), there exists \(\delta_{1,2,3} \in \mathbb{C}^x\) such that \(I_z(g_1 u \otimes g_2 v) = \delta_{1,2,3} g_3 I_z(u \otimes v)\) for all \(u \in M_1\) and \(v \in M_2\).

**Lemma 2.5.7.** If \(M_1, M_2, M_3\) are projectively \(G\)-equivariant \(V\)-modules, and \(I_z : M_1 \otimes M_2 \to M_3\) is a projectively \(G\)-equivariant intertwining operator, then \(I_z^* : M_2 \otimes M_1 \to M_3\) is also projectively \(G\)-equivariant.

**Proof.** Omitted.

**Definition 2.5.8.** ([Dong-Li-Mason-1995], [Xu-1995]) Let \(V\) be a vertex algebra with an action of a group \(G\), and let \(g_1\) and \(g_2\) be commuting finite order elements of \(G\). Let \(M_1\) be a \(g_1\)-twisted \(V\)-module, let \(M_2\) be a \(g_2\)-twisted \(V\)-module, and let \(M_3\) be a \(g_1 g_2\)-twisted \(V\)-module. An intertwining operator of type \((M_1, M_2, M_3)\) is a map \(I_z : M_1 \otimes M_2 \to M_3\) satisfying the following conditions:

1. Translation covariance: \(L(-1)^{M_1} I_z(u \otimes v) - I_z(u \otimes L(-1)^{M_2} v) = \frac{d}{dz} I_z(u \otimes v)\).
2. \(V\)-compatibility: For integers \(r_1, r_2\), let \(V^{r_1,r_2}\) denote the subspace of \(V\) on which \(g_1\) acts by \(e(r_1)\) and \(g_2\) acts by \(e(r_2)\). Then the map \(I_z \circ (1 \otimes I_w) : V^{r_1,r_2} \otimes M_1 \otimes M_2 \to z^{r_1 + r_2} M_3(z)\) and the map \(I_w \circ (act_z \otimes 1) : V^{r_1,r_2} \otimes M_1 \otimes M_2 \to (z-w)^{r_1 + r_2} M_3(z)\) are \(G\)-equivariant, where \(\delta_{1,2,3} M_3(z)\) factor through the space \(z^{r_1 + r_2} M_3(z)\) where they coincide.
3. Truncation: For any \(u \in M_1, v \in M_2\), and any \(c \in \mathbb{C}\), there exists \(n_c \in \mathbb{Z}\) such that for all \(n < n_c\), the \(z^{c+n}\) term in \(I_z(u \otimes v)\) vanishes.

If \(V\) is weighted (resp., Möbius, quasi-conformal, conformal), and \(M_1, M_2, M_3\) are \(V\)-modules in the weighted (resp., Möbius, quasi-conformal, conformal) sense, then an intertwining operator of type \((M_3, M_1, M_2)\) is an intertwining operator for the underlying vertex algebra modules, satisfying the compatibility:

\[L(i)^{M_3} I_z(u \otimes v) - I_z(u \otimes L(i)^{M_2} v) = \sum_{j=0}^{i+1} \binom{i+1}{j} z^j I_z(L(i-j)^{M_1} u \otimes v)\]

for all applicable operators \(L(i)\).

**Lemma 2.5.9.** Let \(g_1, g_2, g_3\) be commuting elements of a group \(G\) of automorphisms of a vertex algebra \(V\), and for each \(i = 1, 2, 3\), let \(M_i\) and \(M'_i\) be \(g_i\)-twisted \(V\)-modules. Let \(I_z\) be an intertwining operator of type \((M_3, M_1, M_2)\) and let \(I'_z\) be an intertwining operator of type \((M'_3, M'_1, M'_2)\). Then \(I_z \otimes I'_z\) in an intertwining operator of type \((M_3 \otimes M'_3, M_1 \otimes M'_1, M_2 \otimes M'_2)\). Furthermore, there is a canonical linear injection from the tensor product of the space of all intertwining operators of type \((M_3, M_1, M_2)\) with the space of all intertwining operators of type \((M'_3, M'_1, M'_2)\) to the space of all intertwining operators of type \((M_3 \otimes M'_3, M_1 \otimes M'_1, M_2 \otimes M'_2)\).

**Proof.** Omitted.

**Definition 2.5.10.** An intertwining operator between twisted \(V\)-modules \(M_1, M_2, M_3\) is projectively \(G\)-equivariant if it satisfies the following condition: For any \(g \in G\), and any lifts \(g_1, g_2, g_3\) of \(g\) to linear transformations on \(M_1, M_2, M_3\), there exists \(\delta_{1,2,3} \in \mathbb{C}^x\) such that \(I_z(g_1 u \otimes g_2 v) = \delta_{1,2,3} g_3 I_z(u \otimes v)\) for all \(u \in M_1\) and \(v \in M_2\).
Lemma 2.5.11. Let $g_1, g_2, g_3$ be commuting elements of a group $G$ of automorphisms of a vertex algebra $V$, and for each $i = 1, 2, 3$, let $M_i$ and $M'_i$ be $g_i$-twisted $V$-modules. Let $I_z$ be a projectively $G$-equivariant intertwining operator of type $\left( \frac{M_3}{M_1 M_2} \right)$ and let $I'_z$ be a projectively $G$-equivariant intertwining operator of type $\left( \frac{M'_3}{M'_1 M'_2} \right)$. Then the intertwining operator $I_z \otimes I'_z$ of type $\left( \frac{M_3 \otimes M'_3}{M_1 \otimes M'_1 M_2 \otimes M'_2} \right)$ is projectively $G$-equivariant.

Proof. Omitted. □

2.6. Generalized vertex algebras. We concretely describe a notion of vertex algebra object in the braided tensor category $\text{Vect}_F^A$. Fundamentally, this class of objects is not new, and we will describe precise comparisons with existing literature after the definition. In future work, we intend to replace $\text{Vect}_F^A$ with more general braided categories.

Definition 2.6.1. Let $A$ be an abelian group, and let $(F, \Omega)$ be an abelian 3-cocycle. A vertex algebra in $\text{Vect}_F^A$ is a quadruple $(V, 1, m_z, L(-1))$, where $V = \bigoplus_{i \in A} V^i$ is an $A$-graded vector space, $1 \in V^0$ is an identity element, $m_z : V \otimes V \to V\{z\}$ is a multiplication map, and $L(-1) \in \text{End} V$ is an operator, such that

1. For any $i, j \in A$, the restriction of $m_z$ to $V^i \otimes V^j \subset V \otimes V$ has image in $z^{b(i,j)}V^{i+j}(z) \subset V\{z\}$, where $b : A \times A \to \mathbb{C}$ is any map satisfying $e(-b(i, j)) = \Omega(i, j) \Omega(j, i)$.
2. For all $v \in V$, $m_z(1 \otimes v) = v$ and $m_z(v \otimes 1) = e^{z L(-1)} v$.
3. For all $u, v \in V$, $L(-1) m_z(u \otimes v) - m_z(u \otimes L(-1)v) = \frac{d}{dz} m_z(u \otimes v)$.
4. For any $i, j, k \in A$, the following diagram commutes:

$$
\begin{array}{ccc}
V^i \otimes (V^j \otimes V^k) & \xrightarrow{m_z \otimes 1} & V^j \otimes (V^i \otimes V^k) \\
\downarrow{1 \otimes m_w} & & \downarrow{1 \otimes m_z} \\
(z - w)^{b(i,j)}V^{i+j}(z - w) \otimes V^k & \xrightarrow{m_w} & V^j \otimes z^{b(i,k)}V^{i+k}(z) \\
\end{array}
$$

where the $F(i, j, k)$, $\Omega(i, j)$, and $F(i, j, k)$ over the arrows near the top indicate multiplication by the constants, and the arrows labeled $m_z$, $m_w$, and $m_w$ near the bottom are abbreviations of $w^{b(i,j)}m_z((w))$, $(z - w)^{b(i,j)}m_w((z - w))$, and $z^{b(i,k)}m_w((z))$, respectively. We will adhere to the following conventions:

1. We will use the vague term “generalized vertex algebra” to refer to vertex algebras in $\text{Vect}_F^A$ when $A, F$, and $\Omega$ are unspecified.
A weighted (resp., Möbius, quasi-conformal, conformal) vertex algebra in $\text{Vect}_{F,\Omega}^A$ is a vertex algebra in $\text{Vect}_{F,\Omega}^A$ equipped with a weighted (resp., Möbius, quasi-conformal, conformal) vertex algebra structure on $V^0$ such that:

1. For each $i \in A$, $V^i$ is a $V^0$-module, in the weighted (resp., Möbius, quasi-conformal, conformal) sense.
2. For each $i, j \in A$, the restriction of $m_z$ to $V^i \otimes V^j$ is an intertwining operator in the weighted (resp., Möbius, quasi-conformal, conformal) sense.

A weighted (resp., Möbius, quasi-conformal, conformal) vertex algebra in $\text{Vect}_{F,\Omega}^A$ is even if for all $i \in A$, $V^i$ has weight $k_i$ for some $k_i \in \mathbb{C}$, and $\Omega(i, i)$ is equal to $e(k_i)$. In other words, $V$ is even if it has type defined by the normalized exponential of weights.

Remark 2.6.2. As we mentioned earlier, the notion of vertex algebra in $\text{Vect}_{F,\Omega}^A$ is not particularly new, but notation may vary between authors. One often encounters the situation where $F$ is identically 1, making $\Omega$ bimultiplicative by the hexagon axiom. This corresponds to the situation where the underlying monoidal structure in $\text{Vect}_{F,\Omega}^A$ is strict. The structure one tends to see in the literature is a bimultiplicative function $\eta : A \times A \to \mathbb{C}^\times$ that appears as a constant correction to locality, and a map $b : A \times A \to \mathbb{C}$ that describes the fractional power of $w - z$ necessary for locality to hold. The map $b$ is expected to be symmetric and bilinear modulo $2\mathbb{Z}$, and satisfy a compatibility like $\eta(i, j)\eta(j, i) = e(-(b(i, j)))$, so everything is derived from setting $\eta = \Omega$ and possibly choosing $b$ modulo $2\mathbb{Z}$. We have the following precise comparisons with previous literature:

- A vertex operator para-algebra (in the sense of [Feingold-Frenkel-Ries-1991]) is equipped with a bilinear form $\eta : A \times A \to \mathbb{C}^\times$ and a function $\Delta : A \to \mathbb{Q}$ induced by the $L(0)$-grading, that induces a map $\Delta : A \times A \to \mathbb{Q}$ that is bilinear modulo $\mathbb{Z}$. By setting $\Omega$ equal to $\eta$, we find that a vertex operator para-algebra is the same as an even conformal vertex algebra in $\text{Vect}_{F,\Omega}^A$ such that:
  1. The abelian group $A$ is finite.
  2. For each $i \in A$, the $L(0)$-spectrum of $V^i$ is bounded below.
  3. Each $L(0)$-eigenvalue has finite multiplicity.
  4. The function $F$ is identically 1.

- An abelian intertwining algebra (in the sense of [Dong-Lepowsky-1993]) is an even conformal vertex algebra in $\text{Vect}_{F,\Omega}^A$, such that the $\mathbb{C}^\times$-valued quadratic function $i \mapsto \Omega(i, i)$ takes values in roots of unity of uniformly bounded order. The data $(F, \Omega)$ coincide with ours, and their $\hat{b}$ is equal to our $b$, but there is a sign error, so their $b$ should be the mod $\mathbb{Z}$ reduction of $-\hat{b}$ as opposed to the mod $\mathbb{Z}$ reduction of their $\hat{b}$ as they claim.

- A vertex algebra (in the sense of [Mossberg-1994]), or equivalently, a semi-vertex algebra satisfying property $\mathcal{E}$, is equipped (when the characteristic zero base field $\mathbb{F}$ is specified to $\mathbb{C}$) with a symmetric map $\langle -, - \rangle : A \times A \to \mathbb{C}$ that is biadditive modulo $2\mathbb{Z}$ that determines the fractional power on $w - z$ in locality, and one assumes the existence of a bimultiplicative anti-symmetric map $c : A \times A \to \mathbb{C}^\times$ that yields a multiplicative locality correction. One may obtain a weighted vertex algebra in $\text{Vect}_{F,\Omega}^A$ from these data, i.e., setting $F = 1$ and $\Omega = c$. Going back, one needs a vertex algebra in $\text{Vect}_{1,\Omega}^A$ where $\Omega$ is anti-symmetric, together with a choice of map $b$ that yields $\langle -, - \rangle$. A VOA in the sense of loc. cit. is a conformal vertex algebra in $\text{Vect}_{F,\Omega}^A$ for which $F$ is identically 1 and $\Omega$ is anti-symmetric, equipped with a map $b$. 

(2) A vertex algebra in $\text{Vect}_{F,\Omega}^A$ has type $Q$ for a quadratic map $Q : A \to \mathbb{C}^\times$ if $Q(i) = \Omega(i, i)$ for all $i \in A$. We may say “generalized vertex algebra of type $Q$” when referring to a vertex algebra in $\text{Vect}_{F,\Omega}^A$, when the precise form of $F$ and $\Omega$ are not specified.
• A generalized vertex algebra (in the sense of [Bakalov-Kac-2006]) is equipped with a symmetric bilinear function $\Delta : A \times A \to \mathbb{C}$ that describes the fractional power of $w - z$ necessary for locality to hold, together with an assumption that there exists a bimultiplicative map $\eta : A \times A \to \mathbb{C}^\times$ that gives a scalar correction to locality. Because $\Delta$ is determined by $\eta$, we find that by setting $\eta = \Omega$, the definition of generalized vertex algebra in this sense with a fixed choice of $\eta$ coincides with our definition of vertex algebra in $\text{Vect}_{F}^{A}$ such that $F$ is identically 1.

There are many other potentially useful generalizations of the notion of vertex algebra that do not invite straightforward comparisons to what we introduce here. Key phrases that may aid one’s search include “quantum vertex algebras”, “nonlocal vertex operator algebras”, “quantum operator algebras”, “OPE algebras”, and “chiral algebras”.

**Definition 2.6.3.** Let $V$ be a vertex algebra in $\text{Vect}_{F}^{A}$, and let $\eta : A \oplus A \to \mathbb{C}^\times$ be a normalized 2-cochain from Definition 2.2.4. We define a new multiplication $\eta m_z : V \otimes V \to V\{z\}$ by specifying that on graded pieces: $\eta m_z(u \otimes v) = \eta(i,j)m_z(u \otimes v)$ for $u \in V^i$ and $v \in V^j$. We write $\eta V$ to denote the tuple $(V, \eta m_z)$.

**Lemma 2.6.6.** For any normalized 2-cocycle $\eta$, the multiplication $\eta m_z$ endows $\eta V$ with the structure of a vertex algebra in $\text{Vect}_{d\eta(F,\Omega)}^{A}$. In particular, if $\eta$ satisfies the (abelian) cocycle condition, then $\eta V$ is a vertex algebra in $\text{Vect}_{F}^{A}$. If $\eta_1$ and $\eta_2$ differ by a coboundary $\alpha \cdot A \to \mathbb{C}^\times$, then the operation of rescaling $V^i$ by the constant $\alpha(i)$ for all $i \in A$ induces an isomorphism between $\eta_1 V$ and $\eta_2 V$ as vertex algebras in $\text{Vect}_{d\eta(F,\Omega)}^{A}$. In particular, the type of a vertex algebra is invariant under the rescaling action of (abelian) 2-cocycles.

**Proof.** This is essentially the content of [Dong-Lepowsky-1993] Remark 12.23 (for finite order cocycles), and [Bakalov-Kac-2006] section 3.4 (for the case of trivial $F$). It suffices to verify that the cocycle $(F, \Omega)$ is shifted by the abelian differential of $\eta$. The last claim, concerning additional symmetry, is clear. □

**Definition 2.6.5.** (Graded tensor product) Let $V$ be a vertex algebra in $\text{Vect}_{F}^{A}$ and let $W$ be a vertex algebra in $\text{Vect}_{F}^{A}$, then the graded tensor product $V \otimes^{L} W$ is the vertex algebra in $\text{Vect}_{F,F',\Omega,\Omega'}^{A}$ given by the following structure:

1. The underlying $A$-graded vector space is $\bigoplus_{i \in A} V^i \otimes W^i$.
2. Multiplication is induced by tensor product maps on graded components.
3. $1_{V \otimes^{L} W} = 1_{V} \otimes 1_{W}$.
4. $L(-1)_{V \otimes^{L} W} = 1 \otimes L(-1)_{W} + L(-1)_{V} \otimes 1$

If $V$ and $W$ are graded (resp., Möbius, quasi-conformal), then the action of $L(i)$ is defined like $L(-1)$. If $V$ and $W$ are conformal, then $\omega_{V \otimes^{L} W} = 1 \otimes \omega_{W} + \omega_{V} \otimes 1$.

**Lemma 2.6.6.** Graded tensor product induces addition of types in $H^4(K(A,2),\mathbb{C}^\times)$, and preserves evenness.

**Proof.** Omitted. □

**Definition 2.6.7.** (Modules for generalized vertex algebras - [Dong-Lepowsky-1993], Remark 6.15) If $V$ is a vertex algebra in $\text{Vect}_{F}^{A}$ with $F$ trivial, and $S$ is an $A$-set, an $S$-graded $V$-module is an $S$-graded vector space $M = \bigoplus_{k \in S} M^k$, equipped with a map $\text{act}_{z} : V \otimes M \to M\{z\}$, such that:
(1) Monodromy: For any \(i \in A\) and \(k \in S\), the restriction of \(\text{act}_z\) to \(V^i \otimes M^k \subset V \otimes M\) has image in \(z^{b(i,j)}M^{i+k}(z) \subset M\{z\}\), where \(b' : A \times S \to \mathbb{C}\) is any map satisfying \(b'(a_1 + a_2, a_3 + s) \equiv b(a_1, a_3) + b(a_2, a_3) + b'(a_1, s) + b'(a_2, s) \mod \mathbb{Z}\), for any \(a_1, a_2, a_3 \in A\), any \(s \in S\), and \(b\) defined as in the beginning of Definition 2.6.1.

(2) Identity: \(\text{act}_z(1 \otimes x) = xz^0\) for all \(x \in M\).

(3) Translation: \(L(-1)^{M} \text{act}_z(u \otimes x) - \text{act}_z(u \otimes L(-1)^{M}x) = \text{act}_z(L(-1)^{M}u \otimes x)\).

(4) Compatibility with multiplication: for each \(i, j \in A\) and \(k \in S\), the diagram commutes:

\[
\begin{array}{ccc}
(V^i \otimes V^j) \otimes M^k & \xrightarrow{\text{act}_z} & V^i \otimes (V^j \otimes M^k) \\
(z-w)^{b(i,j)}V^{i+j}(z-w) \otimes M^k & \xrightarrow{m_{z-w} \otimes 1} & V^i \otimes w^{b(j,k)}M^{j+k}((w)) \\
\end{array}
\]

A strict \(V\)-module is an \(A\)-graded \(V\)-module such that we may take \(b' = b\).

**Lemma 2.6.8.** (*Twisted modules as graded modules*) Let \(V\) be a vertex algebra, let \(g\) be a finite order automorphism of \(V\), and let \(M\) be a \(g\)-twisted \(V\)-module.

1. If we let \(A = \text{Hom}(\langle g \rangle, \mathbb{C}^\times)\), and let \(F\) and \(\Omega\) be identically unity, then the character decomposition of \(V\) with respect to \(\langle g \rangle\) induces the structure of a vertex algebra in \(\text{Vect}_F^{\tilde{A}}\) on \(V\).
2. If we let \(S\) be a singleton \(\{s\}\) with \(b'\) satisfying \(e(b'(a,s)) = a(g)\), then \(M\) is an \(S\)-graded \(V\)-module.
3. If \(M\) is irreducible, the character decomposition of \(M\) with respect to any fixed lift of \(g\) to a linear automorphism endows \(M\) with an \(S\)-grading for \(S\) an \(A\)-torsor. Any map \(b' : A \times S \to \mathbb{C}^\times\) satisfying \(e(b'(a,s)) = a(g)\) for all \(s \in S\) endows \(M\) with the structure of an \(S\)-graded \(V\)-module.

**Proof.** The first two claims are clear from the definition. The third claim follows from the first claim in Proposition 6.1.3. \(\square\)

**Definition 2.6.9.** Let \(V\) be a vertex algebra in \(\text{Vect}_F^{\tilde{A}}\) with \(F\) trivial, and let \(g\) be a finite order homogeneous automorphism of \(V\). A strict \(g\)-twisted \(V\)-module is an \(S\)-graded \(V\)-module, where

1. \(\tilde{V}\) with its grading group enhanced to \(\tilde{A} = A \times \text{Hom}(\langle g \rangle, \mathbb{C}^\times)\), with \(\tilde{F}\) and \(\tilde{\Omega}\) pulled back from \(A\) along the projection, so that the restriction of the action of \(g\) to any graded piece of \(\tilde{V}\) is by scalar multiplication.
2. \(S\) is a trivial \(A\)-torsor with \(A\)-equivariant isomorphism \(e : S \to A\), such that \(\bar{b} : \tilde{A} \times S \to \mathbb{C}^\times\) satisfies \(e(\bar{b}(a, f, s)) = e(b(a, e(s))) \cdot f(g)\). Here, the action of \(\tilde{A}\) on \(S\) is given by projection through \(A\).

**Remark 2.6.10.** The definition of strict \(g\)-twisted \(V\)-module can be simplified slightly by setting \(S = A\), but this seems to make the proof of the next lemma more confusing.

**Lemma 2.6.11.** Let \(V = \bigoplus_{i \in \hat{A}} V^i\) be a vertex algebra in \(\text{Vect}_F^{\tilde{A}}\) with \(F\) trivial, and let \(V'\) be the vertex subalgebra \(\bigoplus_{i \in \hat{A}'} V^i\), for some subgroup \(A' \subset A\). Suppose that there is a surjective homomorphism \(\pi : A \rightarrow \langle h \rangle\) with kernel \(A'\), for some finite order faithful homogeneous automorphism \(h\) of \(V\), so that for each \(k \in \mathbb{Z}\), \(\bigoplus_{i \pi^{-1}(h^k)} V^i\) is a strict \(h^k\)-twisted \(V'\)-module. If \(g\) is a finite
order homogeneous automorphism of $V$ whose restriction to $V'$ commutes with $h$, then any strict $g$-twisted $V$-module canonically decomposes as a sum of strict $gh^i$-twisted $V'$-modules.

**Proof.** Let $M$ be a strict $g$-twisted $V$-module, and let $\varepsilon : S \to A$ be the specified $A$-equivariant isomorphism such that the pullback $\tilde{b} : A \times \text{Hom}(\langle g \rangle, \mathbb{C}^\times) \times S \to \mathbb{C}^\times$ of $b$ satisfies

$$e(\tilde{b}(a, f, s)) = e(b(a, \varepsilon(s))) \cdot f(g)$$

for all $a \in A$, $f \in \text{Hom}(\langle g \rangle, \mathbb{C}^\times)$, and $s \in S$. We let $\rho : S \to \langle h \rangle$ denote $\pi \circ \varepsilon$.

Let $\eta_1 : \text{Hom}(\langle g, h \rangle, \mathbb{C}^\times) \to \text{Hom}(\langle g \rangle, \mathbb{C}^\times)$ and $\eta_2 : \text{Hom}(\langle g, h \rangle, \mathbb{C}^\times) \to \text{Hom}(\langle h \rangle, \mathbb{C}^\times)$ be duals of inclusion. We let $\tilde{A}' = A' \times \text{Hom}(\langle g, h \rangle, \mathbb{C}^\times)$, and define the maps: $b' : \tilde{A}' \times A \to \mathbb{C}^\times$ and $\tilde{b}' : \tilde{A}' \times S \to \mathbb{C}^\times$ by pulling $b$ and $\tilde{b}$ back along the canonical maps $\tilde{A}' \to A$ and $\tilde{A}' \to A \times \text{Hom}(\langle g \rangle, \mathbb{C}^\times)$.

Because $\bigoplus_{i \in \pi^{-1}(h^k)} V^i$ is a strict $h^k$-twisted $V'$-module for each $h^k \in \langle h \rangle$, we have $A'$-equivariant isomorphisms $\epsilon'_k : \pi^{-1}(h^k) \to A'$, such that $e(b'(a', f, a)) = e(b'(a', \epsilon'_k(a))) \cdot \eta_2(f)(h^k)$ for all $a' \in A$, $a \in \pi^{-1}(h^k)$, and $f \in \text{Hom}(\langle g, h \rangle, \mathbb{C}^\times)$. Equivalently, $e(b'(a', f, a)) = e(b(a', \epsilon'_k(a))) \cdot f(\pi(a))$. Because $b'$ and $\tilde{b}'$ are pullbacks of $b$ and $\tilde{b}$, the previous identity $e(\tilde{b}(a, f, s)) = e(b(a, \varepsilon(s))) \cdot f(g)$ pulls back to

$$e(\tilde{b}'(a', f, s)) = e(\tilde{b}(a', \eta_1(f), \epsilon'_k(\varepsilon(s)))) \cdot f(\rho(s)),$$

which holds for all $a' \in A'$, $f \in \text{Hom}(\langle g, h \rangle, \mathbb{C}^\times)$, and $s \in \rho^{-1}(h^k) \subset S$.

We let $\tilde{F}'$ and $\tilde{\Omega}'$ describe the pullback cocycle on $\tilde{A}'$. This makes the forgetful functor from $\text{Vect}_F^A$ to $\text{Vect}^A_{F, \Omega}$ a braided tensor functor. We define $\tilde{V}'$ to be the unique vertex algebra in $\text{Vect}^A_{F, \Omega}$ whose image under this forgetful functor is $V'$, such that the $\tilde{A}'$-grading reflects the character decomposition with respect to the action of $(g, h)$. Fixing $h^k \in \langle h \rangle$, we set $S' = \rho^{-1}(h^k)$. This is an $A'$-torsor under the restricted action of $A$ on $S'$, trivialized by $\epsilon'_k \circ \varepsilon$, and the action of $\tilde{A}'$ is by projection through $A'$. Then $M(h^k) = \bigoplus_{i \in \rho^{-1}(h^k)} M^i$ is naturally an $S'$-graded $\tilde{V}'$-module. To show that $M(h^k)$ is isomorphic to a strict $gh^i$-twisted $V'$-module, it suffices to show that $e(\tilde{b}'(a', f, s')) = e(b(a', \epsilon'_k \circ \varepsilon(s'))) \cdot f(gh^k)$ for all $a' \in A'$, $f \in \text{Hom}(\langle g, h \rangle, \mathbb{C}^\times)$, and $s' \in S'$. However, we have already established the following chain of equalities:

$$e(\tilde{b}'(a', f, s')) = e(\tilde{b}(a', \eta_1(f), \epsilon'_k(\varepsilon(s')))) \cdot f(\rho(s')) = e(b(a', \epsilon'_k \circ \varepsilon(s'))) \cdot f(g) \cdot f(\rho(s')) = e(b(a', \epsilon'_k \circ \varepsilon(s'))) \cdot f(gh^k).$$

\[\square\]

### 2.7. Lattice vertex algebras.

An important source of conformal vertex algebras in $\text{Vect}^A_{F, \Omega}$ comes from the lattice construction. If $L$ is an even lattice, one chooses a double cover (unique up to isomorphism) and constructs from it a lattice vertex algebra $V_L$, defined in [Borcherds-1986]. Dong and Lepowsky showed in [Dong-Lepowsky-1993] Chapter 12 that this construction can be generalized, so that $L$ can be any rational lattice with bounded denominators, but that one needs to weaken the locality axiom to admit specified fractional powers. Bakalov and Kac further generalized this construction in [Bakalov-Kac-2000] section 5, so that $L$ can be any abelian subgroup of a complex vector space.

For any complex vector space $V$ equipped with a nondegenerate symmetric bilinear form $(-, -)$, we write $\pi^V_0$ to denote the corresponding Heisenberg vertex algebra, which can be described as the Fock space $\text{Sym}(t^{-1}V[t^{-1}])$ with multiplication described by normally-ordered compositions of operators $v_n = vt^n$ arising from the Heisenberg Lie algebra $V[t^\pm 1] \otimes \mathbb{C}K$. The irreducible $V$-modules are parametrized by $V$, i.e., for each $\lambda \in V$, we write $\pi^V_\lambda$ to denote the irreducible $V$ module for which $vt^0$ acts by $(v, \lambda)$ for each $v \in V$. The weight one subspace is naturally identified with $V$, and...
i.e., $(\pi_0^V)_1 = V$. If $V$ is the complexification of $\mathbb{R}^r \otimes \lambda$, and $\lambda \in \mathbb{R}^r \otimes \lambda$, then we write $\pi_\lambda^{r \otimes \lambda}$ to denote the corresponding module equipped with real structure.

$\pi_0^V$ has a family of conformal vectors parametrized by $V$: If we choose bases $\{a^\mu u\}$ and $\{b^\nu\}$ such that $(a^\mu, b^\nu) = \delta_{\mu,\nu}$, then for any $v \in V$, the vector $v_{-2} + \frac{1}{2} \sum_\mu a^\mu b_1 - 1$ is conformal of central charge $\dim V - 12(v,v)$ and independent of the choice of bases. If we set $v = 0$, then the character of $\pi_\lambda^V$ is $Tr(q^{L(0)-c/24}\pi_\lambda^V) = q^{(\lambda, \lambda)/2}\eta(\tau)\dim V$, where $c = e(\tau)$, and $\eta(\tau) = q^{1/24}\prod_{i=1}^{\infty}(1 - q^i)$. Let $L$ be a subgroup of $V$ and let $\tilde{L} \to L$ be a double cover with kernel generated by $\zeta$. Given a set-theoretic section $e : L \to \tilde{L}$, we obtain a 2-cocycle $\epsilon$ from associativity, and we write $\mathbb{C}(\tilde{L}) = \mathbb{C}[\tilde{L}] / (\zeta + 1)$ to denote the twisted group ring. From the section $e$, we get a decomposition $\bigoplus_{\alpha \in L} C e^\alpha$ in terms of basis elements. We write $V_L$ to denote the $\pi_\lambda^V$-module $\text{Sym}(t^{-1}V[t^{-1}]) \otimes \mathbb{C}(\tilde{L}) \cong \bigoplus_{\alpha \in L} \pi_\lambda^V \otimes C e^\alpha$. This space may be equipped with a multiplication structure as described in section 5.1 of [Bakalov-Kac-2006]: The left multiplication map $m_z(1 \otimes e^\alpha, -)$ on elements of $\pi_\lambda^V$ is a classical example of a vertex operator, often written as $\Gamma_{\epsilon}(z)$ or $V_{\epsilon}(z)$ in the literature, and is defined by $\exp(\sum_{n>0} \frac{z^n}{n} a_n) \exp(\sum_{n<0} \frac{z^n}{n} a_n) = e^\alpha e^\beta$, where $\lambda e^\beta = \delta(\alpha, \beta) e^\beta$, and $e^\alpha e^\beta = e(\alpha, \beta) e^{\alpha + \beta}$. This extends naturally to left multiplication $m_z(f \otimes e^\alpha, -)$ for $f \in \pi_\lambda^V$ by the use of normally ordered products as in, e.g., [Kac-1997] theorem 5.5. As shown in [Bakalov-Kac-2006], the multiplication defines the structure of a conformal vertex algebra in $\text{Vec}^{L}_{1,\Omega}$, where $\Omega(\alpha, \beta) = e^{\pi(\alpha, \beta)}$.

We will be concerned with the following conformal lattice algebras:

1. The generalized vertex algebra attached to a rank 2 rational Lorentzian lattice. For any positive integer $n$, and any divisor $h|n$, we identify the even lattice $H_{1,1}(-nh)$ with $nh\mathbb{Z} \times nh\mathbb{Z}$, and its dual lattice $H_{1,1}(-1/nh)$ with $\mathbb{Z} \times \mathbb{Z}$, where both are given the quadratic norm function $(a, b) \mapsto \frac{a^2}{nh}$. Under this identification, we define $L_{n|h}^+$ as the sublattice $nh\mathbb{Z} \times nh\mathbb{Z} + (n, n)\mathbb{Z}$ and $L_{n|h}^-$ as the sublattice $nh\mathbb{Z} \times nh\mathbb{Z} + (n, n)\mathbb{Z}$. Then the dual lattices are identified by $L_{n|h}^y = h\mathbb{Z} \times h\mathbb{Z} + (1, -1)\mathbb{Z}$, and $L_{n|h}^- = h\mathbb{Z} \times h\mathbb{Z} + (1, 1)\mathbb{Z}$. Note that $L_{n|h}^y / L_{n|h}^\pm \cong \mathbb{Z} / (\frac{n}{h}n)\mathbb{Z} \times \mathbb{Z} / (\frac{n}{h}n, 2)\mathbb{Z}$. We define $V_{n|h}^+$ to be the generalized vertex algebra attached to the lattice $L_{n|h}^y$ and we define $V_{n|h}^-$ to be the generalized vertex algebra attached to the lattice $L_{n|h}^-$. Both may be viewed as generalized vertex subalgebras of $V_{H_{1,1}(-1/nh)}$.

2. The Leech lattice vertex algebra $V_{\text{Leech}}$, attached to the unique positive definite even unimodular lattice of rank 24 with no roots. This vertex algebra has central charge 24 and character $q^{-1} + 24 + 196884q + \cdots$. It plays an essential role in the construction of the monster vertex algebra $V^{\natural}$ in [Frenkel-Lepowsky-Meurman-1988].

3. The ghost vertex algebra $V_{\text{ghost}}$. This is the lattice vertex superalgebra $V_{\mathbb{Z}}$, where the odd lattice $\mathbb{Z}$ has the inner product induced by multiplication. Choosing a generator $b \in \mathbb{Z}$, we endow $V_{\text{ghost}}$ with the special conformal vector $\frac{1}{2}b_1^2 + \frac{3}{2}b_{-2}$, giving it central charge $-26$ (cf. [Frenkel-Ben-Zvi-2004] 5.2.8). By boson-fermion correspondence (c.f. [Kac-1997] or [Green-Schwarz-Witten-1987] chapter 3), it is also isomorphic to the ghost $bc$-system of weight 2, which is a fermionic Fock space. Since the notation varies through the literature, we offer the following brief dictionary for the $bc$-system of weight $\frac{1}{2} + \lambda$ and central charge $1 - 12\lambda^2$ (taken from [Kac-1997], [Green-Schwarz-Witten-1987], [Frenkel-Lepowsky-Meurman-1988], and [Frenkel-Ben-Zvi-2004]):

| Source   | $b$     | $c$     | Virasoro vector                                                                 |
|----------|---------|---------|---------------------------------------------------------------------------------|
| Fermion  | $\psi^+(z)$ | $\psi^-(z)$ | $\left( \left( \frac{1}{2} - \lambda \right) \psi^+(z) + \frac{3}{2} \psi^-(z) \right) 0$ |
| GSW      | $b(z)$     | $c(z)$     | (not specified)                                                              |
| Boson (FLM) | $X(1, z)$ | $X(-1, z)$ | $\frac{1}{2} h(-1)^2 + \lambda h(-2)$                                    |
| Boson (Kac) | $\Gamma_1(z)$ | $\Gamma_{-1}(z)$ | $\left( \frac{3}{2} a_{-1} b_{-1} + \lambda b_{-2} \right) 0$            |
| Boson (FBZ) | $V(z)$     | $V_{-1}(z)$ | $\frac{1}{2} b_{-1}^2 + \lambda b_{-2}$                                   |
Lemma 2.7.1. Let $h$ and $n$ be positive integers, with $h|n$, and identify $II_{1,1}(-1/nh)$ with $\mathbb{Z} \times \mathbb{Z}$ with quadratic norm function $(a,b) \mapsto \frac{ab}{nh}$ as above, so that $II_{1,1}(-1)$ is identified with $h\mathbb{Z} \times n\mathbb{Z}$. Let $\sigma^+$ be the automorphism of $V_{II_{1,1}}$ that multiplies the Heisenberg module at $(a,b)$ by the scalar $e(\frac{ab}{nh})$, and let $\sigma^-$ be the automorphism of $V_{II_{1,1}}$ that multiplies the Heisenberg module at $(a,b)$ by the scalar $e(\frac{ab}{nh})$. Then the fixed-point vertex subalgebra of $\sigma^+$ is $L_{n|h^+}$, and the fixed-point vertex subalgebra of $\sigma^-$ is $V_{n|h^-}$. Furthermore, $V_{n|h^+}$ is isomorphic to the sum of irreducible twisted modules $\bigoplus_{i=0}^{n-1} V_{II_{1,1}}(\sigma^{+i})$, and $V_{n|h^-}$ is isomorphic to the sum of irreducible twisted modules $\bigoplus_{i=0}^{n-1} V_{II_{1,1}}(\sigma^{-i})$.

Proof. To prove the first claim, we note that the subgroup of $h\mathbb{Z} \times n\mathbb{Z}$ on which $e(\frac{ab}{nh})$ is equal to 1 is $L_{n|h^\pm}$.

The fact that the sums in the second claim are twisted modules follows from Lemma 2.6.11. The fact that they are irreducible follows from the decomposition of $II_{1,1}(-1/nh)$ as a sum of irreducible $II_{1,1}(nh)$-modules.

Definition 2.7.2. Let $h$ and $n$ be positive integers, with $h|n$. We define the following quadratic maps from $\mathbb{Z}/\frac{nh}{(h,n)}\mathbb{Z} \times \mathbb{Z}/\frac{n(h,2)}{h}\mathbb{Z}$ to $\mathbb{C}^x$:

1. $\rho_{n|h^+}$ takes $(a,b)$ to $e(\frac{abh-a^2}{nh})$.
2. $\rho_{n|h^-}$ takes $(a,b)$ to $e(\frac{abh+a^2}{nh})$.
3. $\tilde{\rho}_{n|h^+}$ takes $(a,b)$ to $e(-\frac{abh+a^2}{nh})$.
4. $\tilde{\rho}_{n|h^-}$ takes $(a,b)$ to $e(-\frac{abh-a^2}{nh})$.

Remark 2.7.3. If $h=1$, we may pull $\rho_{n|1}$ back along the shearing transformation $(a,b) \mapsto (a,b+a)$ to get the more familiar hyperbolic form $(a,b) \mapsto e(\frac{ab}{n})$. The same holds for $\tilde{\rho}_{n|1}$ with $(a,b) \mapsto e(-\frac{ab}{n})$.

Lemma 2.7.4. The generalized vertex algebra $V_{n|h^+}$ has type $\rho_{n|h^+}$, and the generalized vertex algebra $V_{n|h^-}$ has type $\rho_{n|h^-}$. In particular, if $V$ is an even generalized vertex algebra of type $\rho_{n|h^+}$ (resp. $\rho_{n|h^-}$), then the graded tensor product of $V$ with $V_{n|h^+}$ (resp. $V_{n|h^-}$) is equivalent to a vertex algebra (in the ordinary sense).

Proof. For $V_{n|h^+}$, it suffices to choose an identification of $L_{n|h^+}/L_{n|h^+}$ with $\mathbb{Z}/\frac{nh}{(h,n)}\mathbb{Z} \times \mathbb{Z}/\frac{n(h,2)}{h}\mathbb{Z}$ and verify that the conformal weights match properly. We choose the isomorphism $(a,bh-a)+L_{n|h^+} \mapsto (a,b)$. Similarly, for $V_{n|h^-}$, we choose $(a,bh+a)+L_{n|h^-} \mapsto (a,b)$.

To verify that the conformal weights match properly, we restrict the quadratic form $(x,y) \mapsto e(xy/nh)$ on $\mathbb{Z} \times \mathbb{Z}$ to see that the conformal weight of the $(a,bh-a)$-Heisenberg module is $\frac{-abh-a^2}{nh}$ (mod $\mathbb{Z}$), and the conformal weight of the $(a,bh+a)$-Heisenberg module is $\frac{-abh-a^2}{nh}$ (mod $\mathbb{Z}$).

Remark 2.7.5. If $h=1$, we see by the previous remark that a composite of two shearing automorphisms on the grading group induces an isomorphism between $V_{n|1}$ and $V_{n|1}$. Similarly, the shearing automorphism $(a,b) \mapsto (a,a+b)$ induces an isomorphism between $V_{n|2}$ and $V_{n|2}$, since the quadratic form passes from $e(\frac{2ab+a^2}{2n})$ to $e(\frac{2ab-a^2}{2n})$. In particular, if $h=1$ or $h=2$, then we may remove the $1/n|h\pm$ when we are not referring to the explicit presentation of the lattices in terms of coordinates.

3. Obstruction Theory

Let $A$ be an abelian group, $(V,1,m_z,(L(-1),L(0)))$ a weighted vertex algebra, and $\{M_i\}_{i \in A}$ a set of $V$-modules with integral weight, such that $V = M_0$. Suppose we are given one-dimensional vector spaces $Z_{i,j}^{i+j}$ whose elements are intertwining operators $m_{z,i}^{i,j} : M_i \otimes M_j \to M_{i+j}(\{z\})$ for all
of $V$ to be a set $\{i, j, k\}$. The following diagram for each $i, j, k \in A$:

\[
\begin{array}{ccc}
M_i \otimes (M_j \otimes M_k) & \xrightarrow{\tau_{(12)}} & M_j \otimes (M_i \otimes M_k) \\
M_{i+j+k}((z-w)) \otimes M_k & \xrightarrow{m_{i,j+k}^i \otimes 1} & M_i \otimes M_{j+k}((w)) \\
M_{i+j+k}((w))((z-w)) & \xrightarrow{m_{i,j+k}^j} & M_{i+j+k}((z))((w)) \\
\end{array}
\]

One way to approach the associativity condition is to assume the canonical composition operation on $\mathcal{T}_{i,j}^{i+j+k} \otimes \mathcal{T}_{j,k}^{i+j}$ and $\mathcal{T}_{i,j+k}^{i+j+k} \otimes \mathcal{T}_{i,j}^{i+j}$ yields elements in the same one dimensional space $\mathcal{T}_{i,j,k}^{i+j+k}$ of maps:

\[M_i \otimes M_j \otimes M_k \rightarrow M_{i+j+k}([z, w])[z^{-1}, w^{-1}, (z-w)^{-1}]\]

If this holds, then for any choice of nonzero elements in each $\mathcal{T}_{i,j}^{i+j}$, the failure of the associativity diagram to commute is encoded by a set of nonzero constants $\{F(i, j, k) \in \mathbb{C}^\times\}_{i,j,k \in A}$. More specifically, we have the following definition:

**Definition 3.1.1.** Let $V$ be a vertex algebra, let $A$ be an abelian group, and let $\{M_i\}_{i \in A}$ be a set of $V$-modules, such that $M_0 = V$. We define a one-dimensional integral weight associativity datum to be a set $\{\mathcal{T}_{i,j}^{i+j}\}_{i,j \in A}$ of one-dimensional vector spaces, whose elements are intertwining operators $M_i \otimes M_j \rightarrow M_{i+j}((z))$, such that:

1. For each $i, j, k \in A$, there exists a one dimensional vector space $\mathcal{T}_{i,j,k}^{i+j+k}$ whose elements are maps $M_i \otimes M_j \otimes M_k \rightarrow M_{i+j+k}([w, z])[w^{-1}, z^{-1}, (w-z)^{-1}]$, such that composition of intertwining operators, combined with the canonical injections from Lemma 2.1.4, induces isomorphisms from $\mathcal{T}_{i,j+k}^{i+j+k} \otimes \mathcal{T}_{j,k}^{j+k}$ to $\mathcal{T}_{i,j,k}^{i+j+k}$ and from $\mathcal{T}_{i,j+k}^{i+j+k} \otimes \mathcal{T}_{i,j}^{i+j}$ to $\mathcal{T}_{i,j,k}^{i+j+k}$.
(2) For any $i, j, k, \ell \in A$, there exists a one-dimensional space $T_{i,j,k,\ell}^{i+j+k+\ell}$ whose elements are maps:

$$M_i \otimes M_j \otimes M_k \otimes M_\ell \to M_{i+j+k+\ell}[[z, w, t]][z^{-1}, w^{-1}t^{-1}, (z-w)^{-1}, (z-t)^{-1}, (w-t)^{-1}]$$

such that composition of intertwining operators, combined with one of the canonical injections from Lemma 2.1.6 induces an isomorphism from $T_{i,j,k,\ell}^{i+j+k+\ell} \otimes T_{i,k,\ell}^{j+k+\ell} \otimes T_{k,\ell}^{k+\ell}$ to $T_{i,j,k,\ell}^{i+j+k+\ell}$.

(3) The space $T_{0,0}^i$ is spanned by the module structure map $act_{\frac{1}{2}}^i : V \otimes M_i \to M_i((z))$.

(4) The space $T_{i,0}^i$ is spanned by the intertwining operator $act_{\frac{1}{2}}^{i,*} : M_i \otimes V \to M_i((z))$ described in Lemma 2.5.3.

Given a one-dimensional integral weight associativity datum, and any choice of nonzero intertwining operators $\{m_{i,j}^{i,j} \in T_{i,j}^{i+j}\}_{i,j \in A}$, we define the function $F : A^\otimes 3 \to \mathbb{C}^\times$ by

$$m_{i,j}^{i,j} \circ (1 \otimes m_{w}^{w}) = F(i, j, k) m_{i,j}^{i,j} \circ (m_{z-w}^{z,j} \otimes 1)$$

Remark 3.1.2. In order for this definition to make sense, it is necessary to choose certain embeddings of vector spaces (or more precisely, natural transformations of power series endofunctors on $\text{Vect}$), and Lemma 2.1.6 ensures this can be done consistently.

Remark 3.1.3. It does not seem to be necessary, for the results in this paper, to assume the existence of the spaces $T_{i,j,k,\ell}^{i+j+k+\ell}$, if we assume that all of the modules $\{M_i\}_{i \in A}$ are irreducible. However, without irreducibility, it is possible for the composition of nonzero intertwining operators to be zero, and at the time of this writing, we cannot prove that the modules that we will use are in fact irreducible.

Lemma 3.1.4. Given a one-dimensional integral weight associativity datum $\{T_{i,j}^{i+j}\}_{i,j \in A}$, the composition of intertwining operators induces isomorphisms between the following one-dimensional vector spaces, for all $i, j, k, \ell \in A$:

$$T_{i,j,k,\ell}^{i+j+k+\ell} \cong T_{i,j,k,\ell}^{i+j+k+\ell} \otimes T_{j,k,\ell}^{j+k+\ell} \otimes T_{k,\ell}^{k+\ell} \cong T_{i,j,k,\ell}^{i+j+k+\ell} \otimes T_{i,k,\ell}^{j+k+\ell} \otimes T_{k,\ell}^{k+\ell} \cong T_{i,j,k,\ell}^{i+j+k+\ell} \otimes T_{i,j,k}^{j+k+\ell} \otimes T_{j,k}^{k+\ell} \cong T_{i,j,k,\ell}^{i+j+k+\ell} \otimes T_{i,k,\ell}^{j+k+\ell} \otimes T_{j,k}^{k+\ell} \cong T_{i,j,k,\ell}^{i+j+k+\ell} \otimes T_{i,j,k}^{j+k+\ell} \otimes T_{j,k}^{k+\ell}$$

In particular, any element in one of the above spaces describes a composite of intertwining operators that takes elements of $M_i \otimes M_j \otimes M_k \otimes M_\ell$ and factors through $M_{i+j+k+\ell}[[z, w, t]][z^{-1}, w^{-1}t^{-1}, (z-w)^{-1}, (z-t)^{-1}, (w-t)^{-1}]$.

Proof. The first isomorphism is the condition defining $T_{i,j,k,\ell}^{i+j+k+\ell}$. The other isomorphisms follow from the conditions defining the set of spaces $\{T_{i,j,k}^{i+j+k}\}_{i,j,k \in A}$. The fact that all of the maps factor through $M_{i+j+k+\ell}[[z, w, t]][z^{-1}, w^{-1}t^{-1}, (z-w)^{-1}, (z-t)^{-1}, (w-t)^{-1}]$, and hence the corresponding commutative diagram of embeddings given in Lemma 2.1.6 follows from the standard fact that if $f : U \to V$ is a linear map of vector spaces that factors as a composite of linear maps $U \to W \to V$, and $g$ is a constant multiple of $f$, then $g$ factors through $W$. 

Lemma 3.1.5. Given a one-dimensional integral weight associativity datum, and any choice of nonzero intertwining operators $\{m_{i,j}^{i,j} \in T_{i,j}^{i+j}\}_{i,j \in A}$, the function $F$ satisfies the pentagon identity:

$$F(i, j, k) F(i, j + k, \ell + \ell) F(j, k, \ell) = F(i + j, k, \ell) F(i, j, k + \ell)$$

for any $i, j, k, \ell \in A$. In other words, the function $F$, viewed as a group cohomology 3-cochain, is in fact a 3-cocycle.
Lemma 2.1.8 applied to \( M \) together with the relation (1).

By the existence of \( I \) to deduce statements about the composition of other intertwining operators:

For any \( z \in SCOTT HUAI-LEI CARNAHAN \)

\[
F(i,j,k) = (i(j)k)_\ell
\]

\[
F(i,j + k,\ell) = (ij)(k\ell)
\]

\[
i(i(j)k\ell) = F(j,k,\ell) = i(j(k)\ell)
\]

(1) The composite \( m_{i,j+k,\ell}^i \circ (1 \otimes m_{w,\ell}^j) \circ (m_{z,w}^{i,j} \otimes 1 \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_\ell \) to \( F(i,j,k,\ell)u' \).

(2) The composite \( m_{i,j+k,\ell}^i \circ (1 \otimes m_{w,\ell}^j) \circ (1 \otimes m_{w,t}^j \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_\ell \) to \( F(j,k,\ell)u' \).

(3) The composite \( m_{i,j+k,\ell}^i \circ (m_{z,t}^{i,j} \otimes 1) \circ (1 \otimes m_{w,t}^j \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_\ell \) to \( F(i,j+k,\ell) \cdot F(j,k,\ell)u' \).

(4) The composite \( m_{i,j+k,\ell}^i \circ (m_{z,t}^{i,j} \otimes 1 \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_\ell \) to both \( F(i,j,k) \cdot F(i,j+k,\ell) \cdot F(j,k,\ell)u' \) and \( F(i,j+k,\ell)u' \).

We find that \( F(i,j,k) \cdot F(i,j+k,\ell) \cdot F(j,k,\ell)u' = F(i+j,k,\ell)u' \). Since \( u' \) is a nonzero element of a complex vector space, the result follows.

\[ \square \]

Definition 3.1.6. Given a one-dimensional integral weight associativity datum, a choice \( \{ m_{i,j}^z \in T_{i,j}^z \}_{i,j \in A} \) of nonzero intertwining operators is normalized if for all \( i \in A \), \( m_{i,i}^0 = actz_i \) and \( m_{i,0}^i = actz_i \).

Lemma 3.1.7. Suppose we are given a one-dimensional integral weight associativity datum. Then, for any normalized choice \( \{ m_{i,j}^z \in T_{i,j}^z \}_{i,j \in A} \) of nonzero intertwining operators, we have \( F(0,i,j) = F(i,j,0) = 1 \) for all \( i, j \in A \). In other words, \( F \) describes a normalized group cohomology 3-cocycle in the sense of Definition 2.2.4.

Proof. Fix elements \( u_i \in M_i \) and \( v \in M_j \) such that \( m_{i,j}^z(u \otimes v) \neq 0 \), and set \( g(w) = \sum_{j \geq r} g_j w^j = m_{i,j}^z(u \otimes v) \in M_{i+j}((u)) \). By the defining property of \( F(i,0,j) \), \( m_{i,j}^z \circ (1 \otimes m_{0,j}^0) \) and \( F(i,0,j)m_{i,j}^z \circ (m_{z,0}^i \otimes 1) \) are given by a single map from \( M_i \otimes M_0 \otimes M_j \) to \( M_{i+j}([z,w][z^{-1},w^{-1},(z-w)^{-1}],(z-w)^{-1}] \), followed by the standard injections. Applying this map to \( u \otimes 1 \otimes v \), we find that \( m_{i,j}^z(\mu \otimes v) \) and \( F(i,0,j)m_{i,j}^z(e(z-w)^{L(-1)}u \otimes v) \) are expansions of the same element of \( M_{i+j}([z,w][z^{-1},w^{-1},(z-w)^{-1}] \). The left side is \( g(z) \), while the right side is equal to \( F(i,0,j)(z-w)^{L(-1)}g(w) \) by Lemma 2.3.5. By Lemma 2.3.8 applied to \( M_{i,j}((u))([z-w]) \), the right side is also equal to \( F(i,0,j)g(z) \), so \( F(i,0,j) = 1 \).

Now, we use Lemma 3.1.5 If \( k = \ell = 0 \), then \( F(i,j,0)f(i,j,0)F(j,k,0)F(i,j,0) \) together with the relation \( F(j,0,0) = F(i+j,0,0) = 1 \) established in the previous paragraph.
implies $F(i, j, 0) = 1$. If $i = j = 0$, then $F(0, 0, k)F(0, k, \ell)F(0, k, \ell) = F(0, k, \ell)F(0, 0, k + \ell)$ together with the relation $F(0, 0, k) = F(0, 0, k + \ell) = 1$ established in the previous paragraph implies $F(0, k, \ell) = 1$.

\section{Commutativity.}

\textbf{Definition 3.2.1.} Let $V$ be a vertex algebra with an action of a group $G$, let $A$ be an abelian group, and let \{\(M_i\)\}_{i \in A} be a set of $G$-equivariant $V$-modules, such that $M_0 = V$. We define a one-dimensional integral weight commutativity datum to be a one-dimensional integral weight associativity datum that satisfies the additional condition that for any $i, j \in A$, and any nonzero $m_{i,j}^{i,j} \in T_{i,j}^{i,j}$, the intertwining operator $m_{i,j}^{i,j}$ is an element of $T_{j,i}^{j,i}$. In other words, for any nonzero $m_{i,j}^{i,i} \in T_{j,i}^{j,i}$, there exists $\lambda \in \mathbb{C}^\times$ such that for any $u \otimes v \in M_i \otimes M_j$, the elements $\lambda m_{i,j}^{i,j}(u \otimes v)$ and $\epsilon z L(-1)m_{i,j}^{i,j}(v \otimes u)$ of $M_{i+j}(z)$ are equal.

\textbf{Remark 3.2.2.} We note that when $V$ is conformal and the modules $M_i$ satisfy some finiteness conditions, then a commutativity datum is a special case of Huang’s notion of “locally grading-restricted conformal intertwining algebra” [Huang-2004], where the set $A$ is our group $A$, and the vector spaces $V_{i,j}^k$ are $I_{i,j}^{i,j}$ when $k = i + j$ and 0 otherwise. The reader should be aware that although the word “algebra” appears in the name, Huang’s notion of intertwining algebra does not have a distinguished choice of multiplication operator in the defining data.

\textbf{Lemma 3.2.3.} Suppose we are given a one-dimensional integral weight commutativity datum. Then for any choice of nonzero intertwining operators \(m_{i,j}^{i,j} \in T_{i,j}^{i,j}\), and any $i, j, k \in A$, the compositions $m_{i,j}^{i+j} \circ (m_{i,j}^{i,j} \otimes 1)$ and $m_{i,j}^{i+j} \circ (m_{i,j}^{i,j} \otimes 1) \circ \tau_{12}$, factor as maps $M_i \otimes M_j \otimes M_k \to M_{i+j+k}[\{z, w\}[z^{-1}, w^{-1}, (z - w)^{-1}]$ that differ by a constant multiple that does not depend on $k$, followed by inclusions $\iota_{w,z-w}$ and $\iota_{z,z-w}$. In particular, the vector spaces $\tau_{12}^*(T_{i,j,k}^{i+j+k})$ and $T_{j,i,k}^{j+i+k}$ are equal.

\textbf{Lemma 3.2.4.} Given a one-dimensional integral weight commutativity datum, for any choice of nonzero operators \(m_{i,j}^{i,j} \in T_{i,j}^{i,j}\) we define the function $\Omega : A^{\otimes 2} \to \mathbb{C}^\times$ by

\[ m_{i,j}^{i,j} = \Omega(i, j) \epsilon z L(-1)m_{i,j}^{i,j} \circ \tau_{12}. \]

\textbf{Remark 3.2.5.} We note that by Lemma 3.2.3 the following holds for all $k \in A$:

\[ m_{i,j}^{i+j,k} \circ (m_{i,j}^{i,j} \otimes 1) = \Omega(i, j)m_{i,j}^{i+j,k} \circ (m_{i,j}^{i,j} \otimes 1) \circ \tau_{12}. \]

\textbf{Remark 3.2.6.} One may alternatively define $\Omega$ by setting $B(i, j, k)$ to be the defect in the original locality diagram, and setting $\Omega(i, j) = F(i, j, k)^{-1}B(i, j, k)F(j, i, k)$ (see Dong-Lepowsky-1993).
Chapter 12. In terms of intertwining operators, we have:

\[ m^{i,j+k}_z \circ (1 \otimes m^{i,k}_w) \sim F(j, i, k)^{-1} \Omega(i, j)F(i, j, k)m^{j,i+k}_w \circ (1 \otimes m^{i,k}_z) \circ \tau_{(12)}. \]

Diagrammatically, \( \Omega(i, j) \) can be represented as the defect obstructing commutativity in the outer pentagon:

\[
(M_i \otimes M_j) \otimes M_k \xrightarrow{F(i,j,k)^{-1}} M_i \otimes (M_j \otimes M_k) \xrightarrow{B(i,j,k)} M_j \otimes (M_i \otimes M_k) \xrightarrow{F(j,i,k)} (M_j \otimes M_i) \otimes M_k
\]

\[
M_{i+j+k}[z, w][z^{-1}, w^{-1}, (z - w)^{-1}]
\]

**Lemma 3.2.7.** Given a one dimensional integral weight commutativity datum, for any choice of nonzero elements \( \{m^{i,j}_{i,j} \in T^{i+j}_{i,j}\}_{i,j \in A} \), the functions \( F \) and \( \Omega \) satisfy the hexagon conditions:

1. \( F(i, j, k)^{-1} \Omega(i, j + k) F(j, k, i)^{-1} = \Omega(i, j)F(i, j, k)^{-1} \Omega(i, k) \)
2. \( F(i, j, k) \Omega(i + j, k) F(k, i, j) = \Omega(j, k)F(i, j, k)\Omega(i, k) \)

**Proof.** As in Lemma 3.1.5 for any \( i, j, k \in A \), let \( u_i \in M_i, u_j \in M_j, u_k \in M_k \) and \( u_0 \in M_0 = V \) satisfy:

\[ m^{i,j+k}_z \circ (1 \otimes m^{j,k}_w) \circ (1 \otimes 1 \otimes m^{0,k}_t)(u_i \otimes u_j \otimes u_k \otimes u_0) \neq 0 \in M_{i+j+k}(z)((w))((t)). \]

For convenience, we write \( M' = M_{i+j+k}[z, w, t][z^{-1}, w^{-1}t^{-1}, (z - w)^{-1}, (z - t)^{-1}, (w - t)^{-1}] \). As before, by the existence of \( T^{i+j+k}_{i,j,k,0} \), a satisfactory choice of elements \( (u_i, u_j, u_k, u_0) \) exists, and the map \( m^{i,j+k+k}_{i} \circ (1 \otimes m^{k,k+\ell}_w) \circ (1 \otimes 1 \otimes m^{k,\ell}_t) \) is the composite of a map to \( M' \) followed by the injection \( u_{i,w,z} \). We let \( u' \) be the image of \( u_i \otimes u_j \otimes u_k \otimes u_0 \) in \( M' \) in this factorization.

To prove the first equation, we follow the following diagram:

\[
\begin{array}{ccc}
j((ik)0) & \xrightarrow{\Omega(i,k)_{(i,k)0}} & j((ik)0) \\
\downarrow & & \downarrow \\
F(j, i+k, 0)^{-1} & \xrightarrow{F(j, i+k, 0)^{-1}} & (j(ik))0 \\
\downarrow & & \downarrow \\
(j(ik))0 & \xrightarrow{F(j, i+k, 0)^{-1}} & (j(ik))0 \\
\downarrow & & \downarrow \\
\Omega(i, j+k) & \xrightarrow{F(i, j+k, 0)^{-1}} & (i(ik))0 \\
\end{array}
\]

To deduce statements about the composition of the corresponding intertwining operators. The left side of the octagon is traversed as follows:

1. \((ij)k)0\): By Lemma 3.1.5, the composite \( m^{i+j+k,0}_{i} \circ (m^{i+j+k}_w \otimes 1) \circ (m^{i,j}_{i,z-w} \otimes 1 \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_0 \) to \( F(i, j, k)u' \).
2. \((i)k)0\): The composite \( m^{i+j+k,0}_{i} \circ (m^{i,j+k}_{i,w} \otimes 1 \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_0 \) to \( u' \).
3. \((jk)i)0\): By Lemma 3.2.3, the composite \( m^{i+k+j,0}_{i} \circ (m^{i+k,j}_{i,w} \otimes 1 \otimes 1) \circ (m^{j,k}_{j,w-i} \otimes 1 \otimes 1) \) factors through \( M' \) and takes \( u_i \otimes u_j \otimes u_k \otimes u_0 \) to \( \Omega(i, j + k)u' \).
4. \((ji)k)0\): The composite \( m^{i+k+j,0}_{i} \circ (m^{i,k+j}_{i,w} \otimes 1 \otimes 1) \circ (m^{j,k}_{j,w-i} \otimes 1 \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_0 \) to \( F(j, k, i)^{-1} \Omega(i, j + k)u' \).
5. \((jk)i)0\): By Lemma 3.1.7, the composite \( m^{i+k+j,0}_{i} \circ (1 \otimes m^{i+k}_w \otimes 1 \otimes 1) \circ (1 \otimes m^{k,i}_{i,w} \otimes 1) \) factors through \( M' \), and takes \( u_i \otimes u_j \otimes u_k \otimes u_0 \) to \( F(j, i + k, 0)^{-1} F(j, i + k) \Omega(i, j + k)u' = F(j, i + k)^{-1} \Omega(i, j + k)u' \).

The right side of the octagon is traversed as follows:
i((kj)0): The composite $m^j_{i+k,0} \circ (m^j_{j,0} \otimes 1) \circ (m^i_{i,0} \otimes 1 \otimes 1)$ factors through $M'$, and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $\Omega(i,j)F(i,j,k)u'$. 

(j(ik)0): The composite $m^j_{i+k,0} \circ (m^j_{j,0} \otimes 1) \circ (1 \otimes m^i_{l,0} \otimes 1)$ factors through $M'$, and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $F(j,i,k)\Omega(i,j)F(i,j,k)u'$. 

(j(k)0): The composite $m^j_{l,0} \circ (1 \otimes m^j_{j,0} \otimes 1)$ factors through $M'$, and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $\Omega(i,k)\Omega(j,i)F(i,j,k)u'$. 

We see that $F(j,i,k)\Omega(i,j+k)u' = \Omega(i,k)F(j,i,k)\Omega(i,j)F(i,j,k)u'$, and because $u'$ is a nonzero element of a complex vector space, the first hexagon equation holds.

To prove the second equation, we follow the following diagram:

\[
\begin{array}{c}
\Omega(i+j,k) \\
(k(ij))0 \downarrow \\
((ij)k)0 \\
F(i,j,k) \downarrow \\
\Omega(j,k) \\
F((ij)k)0 \leftarrow ((ik)j)0 \\
F(i,k) \leftarrow (i(k)j)0 \\
(i(k))0 \\
\Omega(i,k) \leftarrow F(i,k) \overset{(i(k)+j,0)=1}{\overleftarrow{F(i,k)0}} \\
F((k)j)0 \leftarrow (i(k))0 \\
((k)i)0 \\
\Omega(i,j+k) \leftarrow F(i,j+k) \overset{F(i,j+k,0)=1}{\overleftarrow{F(i+j,0)=1}} \\
(k(i))0 \\
\Omega(i+j,k) \leftarrow F(i,j,k) \overset{(i(k)+j,0)=1}{\overleftarrow{F(i,j,k)0}} \\
((ij)k)0 \\
\end{array}
\]

to deduce statements about the composition of the corresponding intertwining operators. After omitting previous evaluations, the left side of the octagon is traversed as follows:

i((ij)k)0: The composite $m^j_{i+j,k,0} \circ (m^j_{j,k} \otimes 1) \circ (m^i_{i,0} \otimes 1 \otimes 1)$ factors through $M'$ and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $F(i,j,k)u'$. 

(k(i)j)0: The composite $m^j_{k+i+j,0} \circ (m^j_{k+i,j} \otimes 1) \circ (1 \otimes m^i_{i,0} \otimes 1)$ factors through $M'$, and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $\Omega(i+i,k)F(i,j,k)u'$. 

((ki)j)0: The composite $m^j_{k+i+j,0} \circ (1 \otimes m^i_{i,0} \otimes 1)$ factors through $M'$ and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $F(i,j,k)\Omega(i+j,k)F(i,j,k)u'$. 

The right side of the octagon is traversed as follows:

i((kj))0: The composite $m^j_{i+k,j} \circ (m^j_{j,0} \otimes 1 \otimes 1)$ factors through $M'$, and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $\Omega(i,j,k)u'$. 

(i(k)j)0: The composite $m^j_{i+k,j} \circ (m^j_{i+k,j} \otimes 1 \otimes 1)$ factors through $M'$, and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $F(i,k+j,0)\Omega(i,j,k)u' = \Omega(i,j,k)u'$. 

((ik)j)0: The composite $m^j_{i+k,j} \circ (m^j_{i+k,j} \otimes 1 \otimes 1)$ factors through $M'$ and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $\Omega(i,k)F(i,j,k)\Omega(i,j,k)u'$. 

((kj)i)0: The composite $m^j_{i+k,j} \circ (m^j_{i+k,j} \otimes 1 \otimes 1)$ factors through $M'$ and takes $u_i \otimes u_j \otimes u_k \otimes u_0$ to $\Omega(i,k)F(i,k,j)\Omega(i,j,k)u'$. 

We see that $F(k,i,j)\Omega(i+j,k)F(i,j,k)u' = \Omega(i,k)F(i,k,j)\Omega(i,j,k)u'$, and because $u'$ is a nonzero element of a complex vector space, the second hexagon equation holds.

**Proposition 3.2.8.** Let $V$ be a vertex algebra with an action of a group $G$, let $A$ be an abelian group, and let $\{M_i\}_{i \in A}$ be a set of $G$-equivariant $V$-modules, such that $M_0 = V$. Given a one dimensional integral weight commutativity datum, and any normalized choice of nonzero elements $\{m_{i,j}^{i,j} \in T_{i,j}^{i,j}\}_{i,j \in A}$, the following hold:
(1) The pair \((F, \Omega)\) derived from \(\{m^{i,j}\}\) forms a normalized abelian 3-cocycle for \(A\) with coefficients in \(\mathbb{C}^\times\).

(2) The intertwining operators \(\{m^{i,j}\}\) define on \(\bigoplus_{i \in A} M_i\) the structure of a vertex algebra in \(\text{Vect}_{F, \Omega}^A\).

(3) The function \(A \rightarrow \mathbb{C}^\times\) defined by \(i \mapsto \Omega(i, i)\) is a quadratic function invariant under alteration by 2-cocycles. In particular, the abelian cohomology class of \((F, \Omega)\) is canonically attached to the commutativity datum.

(4) The quadratic function \(i \mapsto \Omega(i, i)\) is identically one if and only if there exists a normalized 2-cochain \((i, j) \mapsto \lambda_{i,j} \in \mathbb{C}^\times\) such that \(\{\lambda_{i,j}m^{i,j}_s\}\) describe a vertex algebra structure on \(\bigoplus_{i \in A} M_i\).

**Proof.** The first statement follows from the fact that the abelian 3-cocycle condition is given by the pentagon condition (proved in Lemma 3.1.5), and the hexagon condition (proved in Lemma 3.2.7).

To prove the second statement, it suffices to check that the associativity-commutativity diagram \(\square\) commutes. This is straightforward, because \(F\) and \(\Omega\) were defined to make the corresponding parts of the diagram commute.

In the third statement, the invariance of the abelian cohomology class follows from Lemma 2.6.4 together with the fact that normalized 2-cocycles act simply transitively on the set of normalized choices of intertwining operators. Both the fact that \(i \rightarrow \Omega(i, i)\) is quadratic and the fact that it is invariant follow from Lemma 2.2.3.

To prove the fourth statement, we note that by Lemma 2.6.4 the action of the normalized 2-cochain \(\lambda\) takes the vertex algebra in \(\text{Vect}_{\Omega}^A\) to a vertex algebra in \(\text{Vect}_{\lambda, (F, \Omega)}^A\) and by Lemma 2.2.5 the abelian differential defines a transitive action of normalized 2-cochains on the normalized abelian cocycles in the abelian cohomology class of \((F, \Omega)\). In other words there exists such \(\lambda\) if and only if the pair \((F, \Omega)\) is an abelian cocycle in the zero cohomology class. By Lemma 2.2.3, the abelian cohomology class is determined by its trace, and is trivial if and only if \(\Omega(i, i) = 1\) for all \(i\).

\[\square\] 3.3. **Evenness.** The condition that \(\Omega(i, i) = 1\) for all \(i \in A\) that appears in the last claim of Proposition 3.2.8 is nontrivial. Indeed, any supercommutative ring \(R\) whose odd part has nonzero three-fold products can be written as a vertex algebra in \(\text{Vect}_{F, \Omega}^A\) as follows: We set \(A = \mathbb{Z}/2\mathbb{Z}\), \(V = M_0 = R_{\text{even}}, M_1 = R_{\text{odd}}, L(-1)\) is the zero map, and \(m^{i,j}\) are the restrictions of multiplication on \(R\). We find that the spans of \(\{m^{i,j}\}\) form a one-dimensional integral weight commutativity datum, but the abelian 3-cocycle is nontrivial: \(F\) is the zero map, but \(\Omega(i, j) = (-1)^{ij}\). This counterexample works in the graded, Möbius, quasi-conformal, and conformal cases, because we may define a conformal element by setting \(\omega = 0\), causing \(L(i)\) to act by zero for all \(i \in \mathbb{Z}\).

**Definition 3.3.1.** Let \(V\) be a weighted vertex algebra with a \(G\)-action, and let \(M_1\) and \(M_2\) be weighted \(V\)-modules. An intertwining operator \(I_z : M_1 \otimes M_1 \rightarrow z^s M_2(\{z\})\) is called even if one of the following conditions holds:

1. For any \(u \in M_1\), \(I_z(u \otimes u) = 0\), and for any \(L(0)\)-eigenvectors \(u, v \in M_1\) with eigenvalues \(k, \ell\), such that \(I_z(u \otimes v) \in z^s M_2[[z]] \setminus z^{s+1} M_2[[z]]\) for some \(s \in \mathbb{C}\), we have \(s + 1 \equiv k + \ell \pmod{2}\).

2. There exists \(u \in M_1\) such that \(I_z(u \otimes u) \in z^s M_2[[z]] \setminus z^{s+1} M_2[[z]]\), for some \(s \in \mathbb{C}\) satisfying \(s/2 \equiv k_1 \pmod{2}\).

An equivariant intertwining operator \(I_z : M_1 \otimes M_1 \rightarrow M_2(\{z\})\) between integral weight equivariant modules is called odd if one of the following conditions holds:

1. For any \(u \in M_1\), \(I_z(u \otimes u) = 0\), and for any \(L(0)\)-eigenvectors \(u, v \in M_1\) with eigenvalues \(k, \ell\), such that \(I_z(u \otimes v) \in z^s M_2[[z]] \setminus z^{s+1} M_2[[z]]\), we have \(s \equiv k + \ell \pmod{2}\).

2. There exists \(u \in M_1\) such that \(I_z(u \otimes u) \in z^s M_2[[z]] \setminus z^{s+1} M_2[[z]]\) for some \(s \in \mathbb{C}\) satisfying \(s + 1 \equiv 2k_1 \pmod{2}\).
A commutativity datum is even if for all \( i \in A \), any nonzero \( m_{z}^{i,i} \in T_{z}^{i,i} \) is even.

**Lemma 3.3.2.** Given a one dimensional integral weight commutativity datum, for any \( i \in A \), any nonzero \( m_{z}^{i,i} \in T_{z}^{i,i} \) is either even or odd, and this does not change when \( m_{z}^{i,i} \) is rescaled in \( T_{z}^{i,i} \).

**Proof.** This follows from the fact that the definition of even and odd give an exhaustive enumeration of possible forms of \( m_{z}^{i,i} \).

**Lemma 3.3.3.** Suppose we are given a one dimensional integral weight commutativity datum, and suppose we are given nonzero \( m_{z}^{i,j} \in T_{z}^{i+j} \) for all \( i, j \in A \). Then for all \( i \in A \), \( m_{z}^{i,i} \) is even if and only if \( \Omega(i, i) = 1 \), and \( m_{z}^{i,i} \) is odd if and only if \( \Omega(i, i) = -1 \). In particular, the commutativity datum is even if and only if \( \Omega(i, i) = 1 \) for all \( i \in A \).

**Proof.** We apply the technique used in the proof of Lemma 2.5.5. First case: If \( m_{z}^{i,i}(u \otimes u) = 0 \) for all \( u \in M_{i} \), then bilinearity implies that for any \( u, v \in M_{i} \), \( m_{z}^{i,i}(u \otimes v) = -m_{z}^{i,i}(v \otimes u) \). Since \( m_{z}^{i,i} \) is nonzero, there exist \( u, v \in M_{1} \) such that \( m_{z}^{i,i}(u \otimes v) \in z^{n}M_{2}[[z]] \setminus z^{n+1}M_{2}[[z]] \) for some integer \( n \), so we write \( m_{z}^{i,i}(u \otimes v) = g_{n}z^{n} + g_{n+1}z^{n+1} + \cdots \) with \( g_{n} \) nonzero. Then by the defining property of commutativity datum:

\[
g_{n}z^{n} + \cdots = m_{z}^{i,i}(u \otimes v) = \Omega(i, i)e^{zL(-1)}m_{z}^{i,i}(v \otimes u) = -\Omega(i, i)e^{zL(-1)}g_{n}(-z)^{n} + \cdots = (-1)^{n+1}\Omega(i, i)g_{n}z^{n} + \cdots
\]

If \( n \) is odd (i.e., if \( m_{z}^{i,i} \) is even), then \( g_{n} = \Omega(i, i)g_{n} \), and \( \Omega(i, i) = 1 \). If \( n \) is even (i.e., if \( m_{z}^{i,i} \) is odd), then \( g_{n} = -\Omega(i, i)g_{n} \), and \( \Omega(i, i) = -1 \).

Second case: If there exists \( u \in M_{1} \) such that \( m_{z}^{i,i}(u \otimes u) \neq 0 \), then \( m_{z}^{i,i}(u \otimes u) \in z^{n}M_{2}[[z]] \setminus z^{n+1}M_{2}[[z]] \) for some integer \( n \). We write \( m_{z}^{i,i}(u \otimes u) = g_{n}z^{n} + g_{n+1}z^{n+1} + \cdots \) with \( g_{n} \) nonzero. By the defining property of commutativity datum:

\[
g_{n}z^{n} + \cdots = m_{z}^{i,i}(u \otimes u) = \Omega(i, i)e^{zL(-1)}m_{z}^{i,i}(u \otimes u) = \Omega(i, i)e^{zL(-1)}g_{n}(-z)^{n} + \cdots = (-1)^{n}\Omega(i, i)g_{n}z^{n} + \cdots
\]

If \( n \) (hence \( m_{z}^{i,i} \)) is even, then \( g_{n} = \Omega(i, i)g_{n} \), and \( \Omega(i, i) = 1 \). If \( n \) (hence \( m_{z}^{i,i} \)) is odd, then \( g_{n} = -\Omega(i, i)g_{n} \), and \( \Omega(i, i) = -1 \).

**Proposition 3.3.4.** Let \( V \) be a weighted vertex algebra, let \( A \) be an abelian group, and let \( \{ M_{i} \}_{i \in A} \) be a set of \( V \)-modules of integral weight, such that \( M_{0} = V \). Given a one dimensional integral weight even commutativity datum, there exists a choice of nonzero elements \( \{ m_{z}^{i,j} \in T_{z}^{i+j} \}_{i, j \in A} \) that defines a weighted vertex algebra structure on \( \bigoplus_{i \in A} M_{i} \). If in addition \( V \) is Möbius (resp., quasi-conformal, conformal), all \( M_{i} \) are \( V \)-modules in the Möbius (resp., quasi-conformal, conformal) sense, and the intertwining operators in the commutativity datum respect this structure, then \( \bigoplus_{i \in A} M_{i} \) is Möbius (resp., quasi-conformal, conformal). If \( V \) is a vertex operator algebra and \( A \) is finite, then \( \bigoplus_{i \in A} M_{i} \) is also a vertex operator algebra.

**Proof.** By Proposition 3.2.8 to prove the first assertion, it suffices to show that \( \Omega(i, i) = 1 \) for all \( i \in A \). This follows from Lemma 3.3.3 because the commutativity datum is even.

To define Möbius (resp., quasi-conformal, conformal) structures on \( \bigoplus_{i \in A} M_{i} \), we use the operators \( L(i)^{M_{i}} \) defined on each module, and take the unique linear extension.

\[\square\]
Corollary 3.3.5. Let $V$ be a weighted vertex algebra, let $A$ be an abelian group, let $2 \cdot A$ denote the subgroup of $A$ whose elements are even multiples, and let $\{ M_i \}_{i \in A}$ be a set of $V$-modules of integral weight, such that $M_0 = V$. Given a one dimensional integral weight commutativity datum, there exists a choice of nonzero elements $\{ m_z^{i,j} \in \mathcal{T}_{i,j}^{i+j+k} \}_{i,j \in 2 \cdot A}$ that defines a weighted vertex algebra structure on $\bigoplus_{i,j \in 2 \cdot A} M_i$. If $V$ is Möbius (resp., quasi-conformal, conformal) and all $M_i$ are $V$-modules in the Möbius (resp., quasi-conformal, conformal) sense, then $\bigoplus_{i,j \in 2 \cdot A} M_i$ is Möbius (resp., quasi-conformal, conformal). In particular, evenness is automatic for 2-divisible groups.

Proof. By 3.3.3 the function $i \mapsto \Omega(i,i)$ takes values in $\pm 1$, and by Lemma 2.2.3 it is a quadratic function, so $i \mapsto \Omega(2i,2i)$ is identically 1. Then the result follows from Proposition 3.3.4. \hfill \Box

3.4. Fractional conformal weight. We would like to extend our results to construct vertex algebras in $\text{Vect}^A_{F, \Omega}$ from modules with non-integer conformal weight and intertwining operators with shifted exponents.

Definition 3.4.1. Let $V$ be a vertex algebra, let $A$ be an abelian group, let $\{ M_i \}_{i \in A}$ be a set of $V$-modules, and let $b : A \times A \to \mathbb{C}$ be a map of sets - we shall view two such maps as equivalent if their reductions modulo $\mathbb{Z}$ are equal. A one-dimensional fractional weight associativity datum is a set $\{ \mathcal{T}_{i,j}^{i+j} \}_{i,j \in A}$ of one-dimensional vector spaces whose elements are intertwining operators $M_i \otimes M_j \to z^{b(i,j)} M_{i+j}((z))$, such that:

1. For each $i, j, k \in A$, there exists a one dimensional vector space $\mathcal{T}_{i,j,k}^{i+j+k}$ whose elements are maps $M_i \otimes M_j \otimes M_k \to z^{b(i,k)} w^{b(j,k)} (z-w)^{b(i,j)} M_{i+j+k}([w,z][w^{-1},z^{-1},(w-z)^{-1}])$, and composition of intertwining operators induces isomorphisms from $\mathcal{T}_{i,j+k}^{i+j+k} \otimes \mathcal{T}_{j,k}^{j+k}$ to $\mathcal{T}_{i,j,k}^{i+j+k}$.

2. For any $i, j, k, \ell \in A$, there exists a one-dimensional space $\mathcal{T}_{i,j,k,\ell}^{i+j+k+\ell}$ whose elements are maps:

$$M_i \otimes M_j \otimes M_k \otimes M_\ell \to z^{b(i,\ell)} w^{b(j,\ell)} (z-w)^{b(i,j)} M_{i+j+k+\ell}([z,w,t][z^{-1},w^{-1}t^{-1},(z-w)^{-1},(z-t)^{-1},(w-t)^{-1}])$$

such that composition of intertwining operators induces an isomorphism from $\mathcal{T}_{i,j,k+\ell}^{i+j+k+\ell} \otimes \mathcal{T}_{j,k}^{j+k+\ell}$ to $\mathcal{T}_{i,j,k,\ell}^{i+j+k+\ell}$.

3. The space $\mathcal{T}_{0,0}^{0}$ is spanned by the module structure map $\text{act}^0_z : V \otimes M_i \to M_i((z))$.

4. The space $\mathcal{T}_{0,0}^{1}$ is spanned by the intertwining operator $\text{act}^1_z : M_i \otimes V \to M_i((z))$ described in Lemma 2.5.3.

Given a one-dimensional fractional weight associativity datum, and any choice of nonzero intertwining operators $\{ m_z^{i,j} \in \mathcal{T}_{i,j}^{i+j} \}_{i,j \in A}$, we define the function $F : A^\otimes 3 \to \mathbb{C}^\times$ by

$$m_z^{i,j+k} \circ (1 \otimes m_z^{j,k}) = F(i,j,k) m_z^{i+j+k} \circ (m_z^{i,j} \otimes 1)$$

Remark 3.4.2. Note that the composition rule on $\mathcal{T}_{i,j,k}^{i+j+k}$ implies the modulo $\mathbb{Z}$ reduction of $b$ is bilinear.

Lemma 3.4.3. Given a one-dimensional fractional weight associativity datum, and any choice of nonzero intertwining operators $\{ m_z^{i,j} \in \mathcal{T}_{i,j}^{i+j} \}_{i,j \in A}$, the function $F$ satisfies the pentagon identity:

$$F(i,j,k) F(i,j+k,\ell) F(j,k,\ell) = F(i+j,k,\ell) F(i,j,k+\ell)$$

for any $i, j, k, \ell \in A$. In other words, the function $F$, viewed as a group cohomology 3-cocycle, is in fact a 3-cocycle. Furthermore, for any $i, j \in A$, $F(0,i,j) = F(i,0,j) = F(i,j,0)$, i.e., $F$ is a normalized 3-cocycle.
Proof. The proof of the claim that $F$ is a 3-cocycle is essentially the same as in Lemma 3.1.5, but with some fractional powers of basic coordinates. The one additional difficulty is in checking that the pentagonal diagram commutes, but that follows from Lemma 2.1.6. The proof of the claim that $F$ is normalized is essentially the same as the proof of Lemma 3.1.7.

**Definition 3.4.4.** Let $V$ be a vertex algebra, let $A$ be an abelian group, and let $\{M_i\}_{i \in A}$ be a set of $V$-modules, such that $M_0 = V$. We define a one-dimensional fractional weight commutativity datum to be a one-dimensional fractional weight associativity datum that satisfies the additional condition that for any $i, j \in A$, and any nonzero $m_{ij} \in T_{ij}^+$, the intertwining operator $m_{ij}^{*}$ is an element of $T_{ji}^{*}$. Given a one dimensional fractional weight commutativity datum and a choice of nonzero operators $\{m_{ij} \in T_{ij}^+\}_{i, j \in A}$, we define the function $\Omega : A \times A \to \mathbb{C}^\times$ by

$$m_{ij} = \Omega(i, j)e^{zL(-1)}m_{ij}^{*} \circ \tau_{(12)}$$

**Lemma 3.4.5.** The condition defining commutativity datum implies $\Omega(i, j)\Omega(j, i) = e(-b(i, j))$ for all $i, j \in A$. In particular, the modulo $\mathbb{Z}$ reduction of $b$ is symmetric and bilinear.

Proof. Because $m_{ij}$ takes elements of $M_i \otimes M_j$ to $z^{b(i, j)}M_{i+j}(z)$, $m_{ij}$ is equal to $e^{2\pi i b(i, j)}m_{ij}$. Applying the half-twist twice, we see that:

$$m_{ij} = \Omega(i, j)e^{2\pi i (b+1)}m_{ij}^{*} \circ \tau_{(12)}$$

$$= \Omega(j, i)e^{-2\pi i (b+1)}\Omega(i, j)\Omega(j, i)e^{2\pi i (b+1)}m_{ij}^{*}$$

$$= \Omega(i, j)\Omega(j, i)e^{2\pi i b}m_{ij}^{*}.$$

**Lemma 3.4.6.** Given a one dimensional fractional weight commutativity datum, for any choice of nonzero elements $\{m_{ij} \in T_{ij}^+\}_{i, j \in A}$, the functions $F$ and $\Omega$ satisfy the hexagon conditions:

1. $F(i, j, k)^{-1}\Omega(i, j + k)F(j, k, i)^{-1} = \Omega(i, j)F(j, i, k)^{-1}\Omega(i, k)$
2. $F(i, j, k)^{-1}\Omega(i + j, k)F(k, i, j)^{-1} = \Omega(j, k)F(i, k, j)^{-1}\Omega(i, k)$

Proof. The proof is essentially the same as in Lemma 3.2.7 but with some fractional powers of basic coordinates. The octagonal diagrams still commute, by Lemma 2.1.7.

**Proposition 3.4.7.** Let $V$ be a vertex algebra, let $A$ be an abelian group, and let $\{M_i\}_{i \in A}$ be a set of $V$-modules, such that $M_0 = V$. Given a one dimensional fractional weight commutativity datum, and any choice of nonzero elements $\{m_{ij} \in T_{ij}^+\}_{i, j \in A}$, the following hold:

1. The pair $(F, \Omega)$ derived from $\{m_{ij}\}$ forms an abelian 3-cocycle for $A$ with coefficients in $\mathbb{C}^\times$.
2. The intertwining operators $\{m_{ij}\}$ define on $\bigoplus_{i \in A} M_i$ the structure of a vertex algebra in $\text{Vect}^A_{F, \Omega}$.
3. The function $A \to \mathbb{C}^\times$ defined by $i \mapsto \Omega(i, i)$ is a quadratic function invariant under alteration by 2-cocycles. In particular, the abelian cohomology class of $(F, \Omega)$ is canonically attached to the commutativity datum.
4. For all $i \in A$, $\Omega(i, i) = 1$ if and only if there exists a normalized 2-cochain $(i, j) \mapsto \lambda_{i, j} \in \mathbb{C}^\times$ such that $\{\lambda_{i, j} m_{ij}^{*}\}$ describe a vertex algebra structure on $\bigoplus_{i \in A} M_i$.

If $V$ is weighted (resp., Möbius, quasi-conformal, conformal), then so is the generalized vertex algebra we get.

Proof. The proof is essentially the same as in Proposition 3.2.8 but with some fractional powers of basic coordinates.
The third claim in the proposition allows us to make the following definition:

**Definition 3.4.8.** We say that a one dimensional fractional weight commutativity datum has type $Q$ for a quadratic map $Q : A \to \mathbb{C}^\times$ if there exists a (equivalently, if any) choice of nonzero intertwining operators yields a vertex algebra of type $Q$.

We now consider the weighted case, where each of the modules has $L(0)$-spectrum is supported on a single coset of $Z$ in $\mathbb{C}$. Recall that by Lemma 2.5.5, the type of a commutativity datum is constrained by the relation that $\Omega(i, i) = \pm e(\frac{k_i}{2} - k_i)$ when the modules $M_i$ have weight $k_i$.

**Definition 3.4.9.** Let $V$ be a weighted vertex algebra, and suppose we are given a one dimensional fractional weight commutativity datum. The commutativity datum is called even if all nonzero elements $\{m_{i,j}^z \in \mathcal{I}_{i,j} \}i,j \in A$ are even.

**Lemma 3.4.10.** Let $V$ be a weighted vertex algebra, and suppose we are given a collection of weighted $V$-modules $\{M_i\}i \in A$, such that for each $i \in A$, the module $M_i$ has weight $k_i$. A one dimensional fractional weight commutativity datum is even if and only if the map $A \to \mathbb{C}^\times$ defined by $i \mapsto e(k_i)$ is quadratic and the the datum is of this type, i.e., if $\Omega(i, i) = e(k_i)$ for all $i \in A$.

**Lemma 3.4.11.** Let $V$ be a weighted vertex algebra, and suppose we are given a one dimensional fractional weight commutativity datum attached to modules $M_i$ of weight $k_i$, such that the map $A \to \mathbb{C}^\times$ defined by $i \mapsto e(k_i)$ is quadratic. Then the restriction of the commutativity datum to the subgroup $2 \cdot A = \{i + i : i \in A\} \subset A$ is even. In particular, if $A$ is 2-divisible, then the whole commutativity datum is even.

**Proof.** By Lemma 2.5.5 we know that $\Omega(i, i) = \pm e(\frac{k_i}{2} - k_i)$ for all $i \in A$. Because $i \mapsto e(k_i)$ is quadratic, $e(k_{2j}/2) = e(k_j)$, so $\Omega(j, j) = \pm e(k_j)$. For any $2j \in 2 \cdot A$, we have $\Omega(2j, 2j) = (2k_j)^4 = e(k_j)^4 = e(k_j)$.

**Proposition 3.4.12.** Let $V$ be a conformal vertex algebra, let $A$ be an abelian group, and let $\{M_i\}i \in A$ be a set of $V$-modules such that:

1. $M_0 = V$
2. For each $i \in A$, the $L(0)$-spectrum lies in a single coset of $Z$.
3. The $L(0)$-eigenvalues of all $M_i$ are rational with globally bounded denominator.

Given a one dimensional fractional weight even commutativity datum, there exists a choice of nonzero elements $\{m_{i,j}^z \in \mathcal{I}_{i,j} \}i,j \in A$, endowing $\bigoplus_{i \in A} M_i$ with the structure of an abelian intertwining algebra.

**Proof.** By Proposition 3.4.7, any choice of nonzero intertwining operators yields a conformal vertex algebra in $\mathit{Vect}_{F,\Omega}^A$, for some cocycle $(F, \Omega)$. The evenness implies the quadratic form defined by $\Omega$ coincides with conformal weight. In order to get an intertwining algebra, it remains to show that $F$ and $\Omega$ take values in a finite order group of roots of unity. This follows from the global bound on denominators in the $L(0)$-spectrum.

3.5. **Enumeration of automorphisms.** We wish to describe the automorphisms of a vertex algebra in $\mathit{Vect}_{F,\Omega}^A$ that arise from lifts of automorphisms on projectively equivariant modules. To avoid pathologies where modules or multiplication maps are zero, we assume the multiplication in each vertex algebra spans a commutativity datum.

**Definition 3.5.1.** Let $G$ be a group, let $A$ be an abelian group, and let $V = \bigoplus_{i \in A} M_i$ be a vertex algebra in $\mathit{Vect}_{F,\Omega}^A$ whose multiplication maps $m_{i,j}^z : M_i \otimes M_j \to z^{b(i,j)}M_{i+j}(z)$ are nonzero elements in a commutativity datum. A projectively $G$-equivariant structure on $V$ is the following data:

1. A $G$-action on the vertex algebra $M_0$ by automorphisms.
A projectively $G$-equivariant structure on the module $M_i$, for each $i \in A$.

These data must satisfy the condition that for each $i, j \in A$, the intertwining operator $m_{ij}^{i,j} : M_i \otimes M_j \rightarrow z^{b(i,j)}(z)$ is projectively $G$-equivariant.

**Definition 3.5.2.** Let $V = \bigoplus_{i \in A} M_i$ be a vertex algebra in $\text{Vect}_F$ equipped with a projectively $G$-equivariant structure, and let $g \in G$. A lift of $g$ to $V$ is an $A$-homogeneous automorphism of $V$ whose restriction to each $M_i$ is a lift $g_i$ of the projective transformation $g$ to a linear transformation, such that the intertwining operators satisfy $g_i m_{ij}^{i,j}(u \otimes v) = m_{ij}^{i,j}(g_i u \otimes g_i v)$ for all $u \in M_i, v \in M_j$, $i, j \in A$.

**Definition 3.5.3.** Let $V = \bigoplus_{i \in A} M_i$ be a vertex algebra in $\text{Vect}_F$ equipped with a projectively $G$-equivariant structure, and let $g \in G$. Suppose we are given a collection of lifts $\{g_i\}_{i \in A}$ of $g$ applied to the $M_0$-modules $M_i$. For each $i, j \in A$, we define $\gamma_{i,j} \in \mathbb{C}^\times$ to be the unique constants satisfying the equation:

$$g_{i,j} m_{ij}^{i,j}(u \otimes v) = \gamma_{i,j} m_{ij}^{i,j}(g_i u \otimes g_i v)$$

for all $u \in M_i, v \in M_j$. We call the function $A \times A \rightarrow \mathbb{C}^\times$ defined by $(i, j) \mapsto \gamma_{i,j}$ the 2-cochain attached to $\{g_i\}_{i \in A}$.

**Lemma 3.5.4.** Let $V = \bigoplus_{i \in A} M_i$ be a vertex algebra in $\text{Vect}_F$ equipped with a projectively $G$-equivariant structure, and let $g \in G$. Suppose we are given a collection of lifts $\{g_i\}_{i \in A}$ of $g$ applied to the $M_0$-modules $M_i$. Then for all $i, j, k \in A$

$$\gamma_{i,j+k} \gamma_{j,k} = \gamma_{i,j} \gamma_{i,j}$$

In other words, the 2-cochain attached to $\{g_i\}_{i \in A}$ is a 2-cocycle.

**Proof.** For any $u_i \otimes u_j \otimes u_k \in M_i \otimes M_j \otimes M_k$, we have:

$$g_{i,j+k} m_{ij}^{i,j+k}(u_i \otimes m_{ij}^{i,j}(u_j \otimes u_k)) = \gamma_{i,j+k} \gamma_{j,k} m_{ij}^{i,j+k}(g_i u_i \otimes m_{ij}^{i,j}(g_j u_j \otimes g_k u_k))$$

and

$$g_{i,j+k} m_{ij}^{i,j+k}(m_{ij}^{i,j}(u_i \otimes u_j) \otimes u_k) = \gamma_{i,j+k} \gamma_{j,k} m_{ij}^{i,j+k}(g_i u_i \otimes m_{ij}^{i,j}(g_j u_j \otimes g_k u_k)).$$

By equation [1], the maps $m_{ij}^{i,j+k}(1 \otimes m_{ij}^{i,j})$ and $F(i, j, k)m_{ij}^{i,j+k}(m_{ij}^{i,j+k} \otimes 1)$ factor through a single map from $M_i \otimes M_j \otimes M_k$ to $z^{b(i,j)}(z-w)^{b(i,j)} M_{i,j+k}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$. □

**Lemma 3.5.5.** Let $V = \bigoplus_{i \in A} M_i$ be a vertex algebra in $\text{Vect}_F$ equipped with a projectively $G$-equivariant structure, and let $g \in G$. Suppose we are given a collection of lifts $\{g_i\}_{i \in A}$ of $g$ applied to the $M_0$-modules $M_i$. Then for all $i, j \in A$, $\gamma_{i,j} = \gamma_{j,i}$.

**Proof.** Since we assume the multiplication maps span a commutativity datum, we use the identity $m_{ij}^{i,j} = \Omega(i,j) e^{z\tau} m_{e_{ij}}^{i,j} \circ \tau(12)$ as follows:

$$\gamma_{i,j} m_{ij}^{i,j}(g_i u \otimes g_j v) = g_{i,j} m_{ij}^{i,j}(u \otimes v)$$

$$= g_{i,j} \Omega(i,j) e^{z\tau} m_{e_{ij}}^{i,j}(v \otimes u)$$

$$= \gamma_{j,i} \Omega(i,j) e^{z\tau} m_{e_{ij}}^{i,j}(g_j v \otimes g_i u)$$

$$= \gamma_{j,i} m_{ij}^{i,j}(g_i u \otimes g_j v)$$

□

**Proposition 3.5.6.** Let $V = \bigoplus_{i \in A} M_i$ be a vertex algebra in $\text{Vect}_F$ equipped with a projectively $G$-equivariant structure, and let $g \in G$. The set of lifts of $g$ to $V$ is a torsor under $\text{Hom}(A, \mathbb{C}^\times)$. Furthermore, the set of all lifts of elements of $G$ forms a group that is a central extension of $G$ by $\text{Hom}(A, \mathbb{C}^\times)$. 
Proof. Choose any collection of lifts \( \{g_i\}_{i \in A} \) of \( g \) applied to the \( M_0 \)-modules \( M_i \). Then the 2-cocycle attached to \( \{g_i\}_{i \in A} \) defines a central extension \( 0 \to \mathbb{C}^\times \to E \to A \to 0 \). By Lemma 3.5.5, the 2-cocycle is symmetric, so \( E \) is abelian. Because \( \mathbb{C}^\times \) is divisible, the exact sequence splits, and \( E \cong A \oplus \mathbb{C}^\times \). In particular, the cocycle is a coboundary, and the set of 1-cochains in the preimage of \( \gamma \) under the differential is a torsor under the group Hom\((A, \mathbb{C}^\times)\) of 1-cocycles.

The set of all lifts is a Hom\((A, \mathbb{C}^\times)\)-torsor over \( G \), so in order to prove it is a central extension, we need to show that it is closed under multiplication, the torsor structure map is a homomorphism, and all lifts of identity are central. All lifts of identity are central, because the restriction of a lift of identity to any \( M_i \) is a scalar, and all lifts of elements are sums of linear automorphisms of the modules \( M_i \). The first two claims follow from the following calculation: if \( \tilde{g} = \{g_i\}_{i \in A} \) and \( \tilde{h} = \{h_i\}_{i \in A} \) are lifts of \( g, h \in G \), then for all \( i, j \in A \),

\[
g_{i+j}h_{i+j}m_{z}^{ij}(u \otimes v) = g_{i+j}m_{z}^{ij}(h_i u \otimes h_j v) = m_{z}^{ij}(g_h u \otimes g_j h_v).
\]

Therefore, \( \tilde{g} \tilde{h} \) is a lift of \( gh \) to \( V \). \( \square \)

Corollary 3.5.7. Let \( V = \bigoplus_{i \in A} M_i \) be a vertex algebra in \( \text{Vect}_F^A, \Omega \) equipped with a projectively \( G \)-equivariant structure, let \( B \) be a subgroup of \( A \). The group of all lifts of elements of \( G \) to \( V \) is a central extension of the group of all lifts of \( G \) to \( V = \bigoplus_{i \in B} M_i \) by Hom\((A/B, \mathbb{C}^\times)\).

Proof. We claim that restriction of \( A \)-homogeneous lifts to \( \bigoplus_{i \in B} M_i \) induces a surjection to the set of \( B \)-homogeneous lifts. Restriction to \( M_0 \) yields a surjection to \( G \), so it suffices to show that any lift of the identity of \( G \) to \( \bigoplus_{i \in B} M_i \) lifts further to \( V \). However, lifts of identity are naturally identified with homomorphisms to \( \mathbb{C}^\times \), so by injectivity of \( \mathbb{C}^\times \), all homomorphisms extend.

To identify the central extension, we note that Hom\((A/B, \mathbb{C}^\times)\) is isomorphic to the kernel of the canonical map Hom\((A, \mathbb{C}^\times) \to \text{Hom}(B, \mathbb{C}^\times)\), so the pointwise stabilizer of \( \bigoplus_{i \in B} M_i \) is isomorphic to Hom\((A/B, \mathbb{C}^\times)\). \( \square \)

3.6. Twisted modules. We now have a theory that allows us to construct abelian intertwining algebras from ordinary modules and intertwining operators, but we wish to apply the techniques to irreducible twisted modules of a holomorphic \( C_2 \)-cofinite vertex operator algebra \( V \) (such as the Monster Vertex Operator Algebra \( V^2 \), by [Dong-1994]). We therefore need to split the twisted modules into ordinary modules for a fixed-point subalgebra of \( V \), and acquire enough information about their intertwining operators to assemble a commutativity datum.

For the duration of this subsection, \( V \) will denote a holomorphic \( C_2 \)-cofinite vertex operator algebra, and \( G \) will denote a finite group of automorphisms of \( V \).

Definition 3.6.1. Let \( g \in G \) have order \( n \), and let \( h|n \). We say that \( g \) has type \( n|h+ \) if \( V(g) \) has \( L(0) \)-spectrum in \( -\frac{1}{nh} + \frac{1}{n} \mathbb{Z} \). We say \( g \) has type \( n|h- \) if \( V(g) \) has \( L(0) \)-spectrum in \( \frac{1}{nh} + \frac{1}{n} \mathbb{Z} \).

Lemma 3.6.2. Let \( g \) be an element of \( G \) of order \( n \). Then there exists an abelian group \( A \), an exact sequence \( 0 \to \mu_n(\mathbb{C}) \to A \xrightarrow{\pi} \langle g \rangle \to 0 \), and an assignment of \( V^{(g)} \)-modules \( M_a \) to elements of \( A \), such that:

(1) For any \( i \in \mathbb{Z}/n\mathbb{Z} \), \( \bigoplus_{a \pi^{-1}(i)} M_a \) is a decomposition of \( V(g^i) \) into eigenspaces for any fixed lift of \( g \) to a linear transformation on \( V(g^i) \). In particular, the modules assigned to the kernel of \( \pi \) are the eigenspaces of \( V \) with respect to the \( \langle g \rangle \)-action.

(2) If \( M_a \subset V(g^a) \) and \( M_b \subset V(g^b) \), then the restriction of any intertwining operator \( V(g^a) \otimes V(g^b) \to V(g^{a+b})(z^{1/n}) \) to \( M_a \otimes M_b \) has coefficients in \( M_{a+b} \).

Furthermore, the extension class of \( A \) is uniquely defined by these properties, and the assignment of modules to elements of \( A \) is unique up to isomorphism.
Proof. By the first claim in Proposition \[6.1.3\] for any \(0 \leq i < |g|\), we have a canonical \(\mu_n(\mathbb{C})\)-torsor \(P_i\) attached to the irreducible \(g^i\)-twisted module \(V(g^i)\) and a decomposition \(V(g^i) \cong \bigoplus_{j \in P_i} M_{i,j}\) of \(V^{(g)}\)-modules according to the eigenvalues of any fixed lift of \(g\) to a linear transformation. We define \(A\) to be the union of these torsors, and we have a natural set-theoretic surjection \(\pi: A \to \langle g \rangle\), sending any element of \(P_i\) to \(g^i\), and the preimage of 1 is the group \(\mu_n(\mathbb{C})\) of eigenvalues of the action of \(g\) on \(V\).

We wish to define an abelian group structure on \(A\). By the second claim in Proposition \[6.1.3\] we have a binary composition law on \(A\): For any two ordered pairs \((i_1, j_1)\) and \((i_2, j_2)\) there is a unique \((i_3, j_3)\) such that restriction of a nonzero intertwining operator \(V(g^{i_1}) \otimes V(g^{i_2}) \to V(g^{i_1+i_2})(z^{1/n})\) to \(M_{i_1,j_1} \otimes M_{i_2,j_2}\) has coefficients in \(M_{i_3,j_3}\), where \(i_3 = i_1 + i_2\). By the second claim in Proposition \[6.1.1\] the composition is associative - that is, a failure of associativity would ensure the failure of composed intertwining operators to agree up to a constant multiple. Every element has an inverse given by the contragradient module, and the identity is \(V^{(g)}\). The abelian property is a consequence of the skew-symmetry condition in the first claim of Proposition \[6.1.2\]. Since the fusion structure is additive on twistings, the set map \(\pi\) is then a homomorphism of abelian groups, and the extension is uniquely determined by the fusion law on twisted modules. \(\square\)

Lemma 3.6.3. With notation from the previous lemma, the extension class of \(A\) is uniquely determined by the \(L(0)\)-spectrum of \(V(g)\). In particular, if \(g\) is an element of \(G\) of type \(n|h|\pm\), then \(A \cong \mathbb{Z}/\frac{nh}{(h,2)}\mathbb{Z} \times \mathbb{Z}/\frac{nh}{2(h,2)}\mathbb{Z}\).

Proof. Because elements of \(\text{Ext}^1(\langle g \rangle, \mu_n(\mathbb{C}))\) are uniquely determined by the \(n\)th power of any element in the preimage of the generator \(g\), it suffices to describe the \(n\)th power of \((1, j) \in A\). By the third claim in Proposition \[6.1.3\] the decomposition of \(V(g)\) into \(\bigoplus_{j \in P_i} M_{i,j}\) is given by the eigenvalues of \(e(L(0))\). Fix an element \(j \in P_i\), and suppose \(e(L(0))\) acts on \(M_{i,j}\) by the scalar \(\zeta \in \mu_{n^2}(\mathbb{C})\). By the fourth claim in Proposition \[6.1.3\] the multiplication of \(n\) elements of \(M_{i,j}\) yields a series with coefficients in the subspace of \(V^2 \otimes \mathbb{C}_{\zeta^{-n}}\) on which \(g\) acts by \(\zeta^n \in \mu_n(\mathbb{C})\). Therefore, \(\zeta^{2n}\) is the \(n\)th power of \((1, j)\) in \(A\).

If \(g\) has type \(n|h|\pm\), then the \(L(0)\)-eigenvalues of \(V(g)\) lie in the coset \(\pm \frac{1}{nh} + \frac{1}{n}\mathbb{Z}\), so we identify \(P_i\) with the exponential of this coset. Any element \((1, j) \in A\) has \(n\)th power equal to \(e(\mp 2/h)\), so any \((1, j)\) has order \(\frac{nh}{(h,2)}\). \(\square\)

Proposition 3.6.4. The restriction of intertwining operators \(V(g^{i_1}) \otimes V(g^{i_2}) \to V(g^{i_1+i_2})(z^{1/n})\) to the modules \(M_{i,j}\) yields a one dimensional fractional weight commutativity datum.

Proof. We need to check the axioms for commutativity datum.

1. For any \(i, j \in A\), we define \(T_{i,j}^{i+j}\) to be the one dimensional space of maps \(M_i \otimes M_j \to M_{i+j}((z^{1/n}))\) induced by restriction of the intertwining operators \(V(\pi(i)) \otimes V(\pi(j)) \to V(\pi(i+j))((z^{1/n}))\) given in Proposition \[6.1.1\].

2. For any \(i, j \in A\), any intertwining operator \(M_i \otimes M_j \to M_{i+j}((z^{1/n}))\) that comes from twisted modules takes values in \(z^{b(i,j)} M_{i+j}(z)\) for some fixed \(b(i,j) \in \frac{1}{n}\mathbb{Z}\). This defines an equivalence class of maps \(b: A \times A \to \mathbb{C}\) that takes integer values when \(i = 0\) or \(j = 0\), where two maps are equivalent if and only if they differ by a map to \(\mathbb{Z}\).

3. For any \(i, j, k \in A\), we consider the one-dimensional space of maps

\[V(\pi(i)) \otimes V(\pi(j)) \otimes V(\pi(k)) \to V(\pi(i+j+k))[[w^{1/n}, z^{1/n}]]z^{-1}, w^{-1}, (z - w)^{-1}]\]

given in the second assertion of Proposition \[6.1.1\] and let \(T_{i,j,k}^{i+j+k}\) be the one dimensional space of maps

\[M_i \otimes M_j \otimes M_k \to z^{b(i,k)} w^{b(j,k)} (z - w)^{b(i,j)} M_{i+j+k}[[w, z]][w^{-1}, z^{-1}, (w - z)^{-1}]\]
induced by restriction (we note that the appearance of $z^{b(i,k)}$ in the target is not entirely trivial). The second assertion of Proposition 6.1.1 implies composition induces isomorphisms from $T^{i+j+k}_{i,j,k} \otimes T^{j+k}_{j,k}$ to $T^{i+j+k}_{i,j,k}$ and from $T^{i+j+k}_{i,j,k} \otimes T^{i+j+k}_{i,j,k}$ to $T^{i+j+k}_{i,j,k}$.

(4) For any $i, j, k, \ell \in A$, we define $T^{i+j+k+\ell}_{i,j,k,\ell}$ to be the one-dimensional space whose elements are the maps $M_i \otimes M_j \otimes M_k \otimes M_\ell \rightarrow z^{b(i,\ell)} w^{b(j,\ell)} p^{b(k,\ell)} (z - w)^{b(i,j)} (z - t)^{b(i,k)} (w - t)^{b(j,k)}$.

\[ M_{i+j+k+\ell}[[z, w, t]] \rightarrow \prod_{\ell=0}^{i+j+k+\ell} (z^{-1}, w^{-1}, t^{-1}, (z - w)^{-1}, (z - t)^{-1}, (w - t)^{-1}) \]

induced by restriction from the one-dimensional space of maps $V(\pi(i)) \otimes V(\pi(j)) \otimes V(\pi(k)) \otimes V(\pi(\ell))$ given in the third assertion of Proposition 6.1.1. Again, restriction yields the claim that composition of intertwining operators induces an isomorphism from $T^{i+j+k+\ell}_{i,j,k,\ell} \otimes T^{j+k+\ell}_{j,k,\ell} \otimes T^{k+\ell}_{k,\ell}$ to $T^{i+j+k+\ell}_{i,j,k,\ell}$.

(5) The first claim of Proposition 6.1.2 includes the statement that for any $m \in T^{a+b}_{a,b}$, we have $m^* \in T^{a+b}_{a,b}$.

(6) The action of $V$ on irreducible twisted modules yields distinguished intertwining operators $\alpha z^i : M_0 \otimes M_i \rightarrow M_i((z))$ that span $T^0_{i,i}$. This yields a one dimensional fractional weight commutativity datum.

**Corollary 3.6.5.** For any holomorphic $C_2$-cofinite vertex operator algebra $V$ and any finite order isomorphism $g$, there exists a generalized conformal vertex algebra, which is isomorphic to $\bigoplus_{i=0}^{[g]-1} V(g^i)$ as a sum of irreducible twisted $V$-modules, such that the multiplication operation restricted to any $V(g^i) \otimes V(g^j)$ is a nonzero intertwining operator.

**Proof.** By Proposition 3.6.4, the twisted modules and spaces of intertwining operators decompose into modules for $V$ with a one dimensional fractional weight commutativity datum. By Proposition 3.2.8, any normalized choice of nonzero intertwining operators in the constituent spaces of the commutativity datum yield a generalized vertex algebra structure on the direct sum of twisted modules. Naturally, we are free to choose these intertwining operators to arise from the nonzero intertwining operators between the twisted modules.

**Lemma 3.6.6.** Let $g \in G$, and let $\bigoplus_{a \in A} M_a$ be the decomposition of $\bigoplus_{i=0}^{[g]-1} V(g^i)$ given in Lemma 3.6.2. There exists a unique homogeneous linear transformation on $\bigoplus_{a \in A} M_a$ satisfying:

1. For each $a \in A$, the preferred lift of $g$ acts by a nonzero complex scalar on $M_a$.
2. For any $a, b \in A$, if the preferred lift of $g$ acts on $M_a$ as the scalar $\zeta$ and on $M_b$ as $\tau$, then it acts on $M_{a+b}$ as $\zeta \tau$.
3. If $M_a \subset V(g)$, then the preferred lift of $g$ acts as the scalar $e(L(0))_{M_a}$.
4. If $M_a \subset V$, then the preferred lift of $g$ acts as the restriction of the automorphism $g$.

For any representative of the equivalence class of generalized vertex algebras defined in Corollary cor:generalized-vertex-algebra, this linear transformation is an automorphism.

**Proof.** The proof of the first claim amounts to checking that there is a unique homomorphism $A \rightarrow \mathbb{C}^*$ such that the restriction to $\pi^{-1}(1)$ describes the eigenvalue decomposition on $V$, and the restriction to $\pi^{-1}(g)$ is given by $e(L(0))$. Since $\pi^{-1}(1)$ and $\pi^{-1}(g)$ generate $A$, uniqueness follows from existence.

In order to prove existence, we need to show that the last two conditions specify a well-defined homomorphism. By Lemma 3.6.2, any $b \in \pi^{-1}(g)$ has order equal to the order of the root of unity.
e(L(0))_{M_b}. The remaining obstruction is showing that if \(a \in \pi^{-1}(1)\) and \(b \in \pi^{-1}(g)\), then the scalar \(e(L(0))_{M_{ab}}\) is equal to the product of the scalar \(g_{M_b}\) with the scalar \(e(L(0))_{M_a}\). However, this is precisely the monodromy condition in the definition of \(g\)-twisted module.

To prove the second claim, we note that restriction of \(g\) to \(M_0\) is the identity, so by Proposition 3.5.6 the set of lifts of \(g\) to \(\bigoplus_{a \in A} M_a\) is equal to \(\text{Hom}(A, \mathbb{C}^\times)\). In other words, by specifying a homomorphism, we have already described an automorphism of any generalized vertex algebra in the equivalence class we are given.

\[\square\]

**Definition 3.6.7.** For any fixed \(g \in G\), we define the preferred lift of \(g\) on \(\bigoplus_{a \in A} M_a\) to be the linear transformation given in Lemma 3.6.6. For any integer \(i\), we define the preferred lift of \(g\) on \(V(g^i)\) to be the restriction of the preferred lift on \(\bigoplus_{a \in A} M_a\) to \(V(g^i)\).

**Lemma 3.6.8.** Let \(k \in \mathbb{Z}_{\geq 0}\) such that \(k < |g|\). Then the action of the preferred lift of \(g^k\) on \(V(g^k)\) is given by the \(k\)th power of the preferred lift of \(g\). In particular, if \(v \in V(g^k)\), and the preferred lift of \(g\) acts by \(gv = \zeta v\), then the preferred lift of \(g^k\) acts by \(g^k v = \zeta^k v\), i.e., \(e(L(0))v = \zeta^k v\).

**Proof.** We identify the eigenvalues of the preferred lift of \(g\) on \(V(g^k)\) with the \(\mu_n\)-torsor \(P_k\). Restriction to the \(k\)th power of this lifts yields an identification of the eigenvalues of a lift of \(g^k\) with the \(\mu_{n/kh}\)-torsor that naturally parametrizes sub-\(V(g^k)\)-modules of \(V(g^k)\). By the first, third, and fourth claims in Proposition 6.1.3, the \(e(L(0))\)-eigenvalues yield a second \(\mu_{n/kh}\)-torsor that naturally parametrizes the decomposition of \(V(g^k)\) into eigenspaces of the preferred lift of \(g^k\). To show that these lifts coincide, it suffices to identify the eigenvalues attached to a single element of each torsor.

We apply the fifth claim in Proposition 6.1.3 under the following choice of input: We set the \(k\) integers \(a_1, \ldots, a_k\) equal to \(k\), and take any \(v_1, \ldots, v_k \in M_b\), for \(\phi(b) = (1, \mp 1) + L_{n|h^\pm}\). The proposition yields the conclusion that since \(M_b\) has \(L(0)\)-weight \(\frac{\mp 1}{n} + \mathbb{Z}\), then \(M_{kb}\) has \(L(0)\)-weight \(\frac{\mp k^2}{nh}\). The preferred lift of \(g\) acts on \(M_b\) with the eigenvalue \(e(\mp 1/nh)\), so by multiplicativity, the preferred lift of \(g\) acts on \(M_{kb}\) with the eigenvalue \(e(\mp k/nh)\). We conclude that the \(kth\) power of the preferred lift of \(g\) acts on \(M_{kb}\) with the eigenvalue \(e(\mp k^2/nh)\), and this is equal to the eigenvalue of \(e(L(0))\).

\[\square\]

**Lemma 3.6.9.** Let \(g \in G\) have type \(n|h^\pm\), and let \(\phi : A \rightarrow \Pi_{1,1}(-1/nh)/L_{n|h^\pm}\) be the map that assigns the subspace of \(V(g^i)\) on which the preferred lift of \(g\) acts by \(e(\frac{i}{m})\) to the coset \((i, r)+L_{n|h^\pm}\). Then \(\phi\) is injective, its image is equal to \(L_{n|h^\pm}/L_{n|h^\pm}\), and under this identification of \(A\) with \(L_{n|h^\pm}/L_{n|h^\pm}\), the \(e(L(0))\)-eigenvalues of the \(V(g)\)-modules \(M_a\) give a quadratic form of type \(\rho_{n|h^\pm}\) on \(A\).

**Proof.** The fact that \(\phi\) is injective follows from the \(\mu_n\)-torsor property of \(P_t\), given in the first claim of Proposition 6.1.3. The claim about the image follows from the fact that the image of \(\pi^{-1}(g)\) in \(\mathbb{C}^\times\), namely \(e(\frac{\mp 1}{nh})\mu_n\), corresponds under \(\phi\) to the subset \(\{(1, \mp 1 + h\mathbb{Z}) + L_{n|h^\pm}\} \subset \Pi_{1,1}(-1/nh)/L_{n|h^\pm}\).

The last claim translates to the claim that for each \(a \in A\), the \(e(L(0))\)-eigenvalue of the module \(M_a\) for \(\phi(a) = (k, r) + L_{n|h^\pm}\) is \(e(\frac{\pm kr}{nh})\). This follows from Lemma 3.6.8, since the condition on \(\phi(a)\) implies the preferred lift of \(g\) acts on \(M_a\) by \(e(r/nh)\).

\[\square\]

**Lemma 3.6.10.** Let \(g \in G\), and suppose we are given a generalized vertex algebra \(W\) that arises from the commutativity datum given in Proposition 3.6.4. Then \(W\) admits a \(C_G(g)\)-equivariant structure.

**Proof.** By the definition of equivariant structure, it suffices to show that the modules \(M_{i,j}\) are projectively \(C_G(g)\)-equivariant, and the intertwining operators are projectively \(C_G(g)\)-equivariant. This in turn follows from Proposition 6.1.2 since the decomposition of each irreducible twisted module into eigenspaces for \(\tilde{g}\) is projectively \(C_G(g)\)-equivariant.

\[\square\]
We now wish to show that our obstruction theory yields an abelian intertwining algebra structure on $\bigoplus_{i=0}^{|g|-1} V(g^i)$, especially for the case $V = V^\natural$. The remaining obstacle is showing that the intertwining operators are even, i.e., eliminating the possibility that elements have square-zero. We first give a general method for eliminating many cases, then in the next subsection, we work out the case of $V^\natural$ by appealing to the lattice orbifold structure.

**Definition 3.6.11.** Let $g \in G$. We say that $g$ is even if the commutativity datum given in Proposition 3.6.4 for $\bigoplus_{i=0}^{|g|-1} V(g^i)$ is even.

**Proposition 3.6.12.** Let $g$ be even. Then there exists an abelian intertwining algebra structure on $\bigoplus_{i=0}^{|g|-1} V(g^i)$ whose multiplication map restricts to the vertex operator algebra structure on $V$, the twisted module structures on $V^\natural(g^i)$, and nonzero intertwining operators between twisted modules. If $g$ has type $n|\hbar\pm$, then the abelian intertwining algebra has type $\rho_n|\hbar\pm$. Furthermore, there is a canonical projective action of $C_G(g)$ on this abelian intertwining algebra by automorphisms, and this action restricts to the canonical projective action on each twisted module.

**Proof.** By Proposition 3.6.4 there is a one-dimensional fractional weight commutativity datum on $\bigoplus_{i=0}^{|g|-1} V(g^i)$, and under our hypothesis, this commutativity datum is even. Therefore, by Proposition 3.4.12 there is an abelian intertwining algebra structure on $\bigoplus_{i=0}^{|g|-1} V(g^i)$, and by Lemma 3.6.9 if $g$ has type $n|\hbar\pm$, then it has type $\rho_n|\hbar\pm$. The action of the central extension is given by Proposition 3.5.6 from the projectively $C_G(g)$-equivariant structure given in the first claim of Proposition 6.1.2.

**Definition 3.6.13.** Let $g$ be even. We write $V_g$ for any object in the equivalence class of the abelian intertwining algebra described in Proposition 3.6.12. We define $\widehat{C_G(g)}$ to be the central extension of $C_G(g)$ by $\mu|\hbar|\Omega$ whose elements are lifts to $V_g$ of elements of $C_G(g)$ acting on $V$, as specified in Corollary 3.5.4.

**Corollary 3.6.14.** Let $g$ be even of type $n|\hbar\pm$. There exists an $A$-graded conformal vertex algebra structure on $V_g \otimes^A V_n|\hbar\pm$, and a natural homogeneous action of $\widehat{C_G(g)}$.

**Proof.** Because $V_g$ has type $\rho_n|\hbar\pm$ and $V_n|\hbar\pm$ has type $\tilde{\rho}_n|\hbar\pm$, the graded tensor product has type 1, i.e., it is a conformal vertex algebra in $Vect^A_{f,\Omega}$ for $(F,\Omega)$ an abelian 3-coboundary. By Lemma 2.6.4 there exists a 2-cochain $\eta$ such that $\eta(V_g \otimes^A V_n|\hbar\pm)$ is a conformal vertex algebra in $Vect^A_{(1,1)}$, i.e., an $A$-graded conformal vertex algebra. The action of $\widehat{C_G(g)}$ is induced from the action on $V_g$, together with the trivial action on $V_n|\hbar\pm$.

**Definition 3.6.15.** Let $g$ be even of type $n|\hbar\pm$. We write $W_g$ to denote any $A$-graded conformal vertex algebra equivalent to $V_g \otimes^A V_n|\hbar\pm$.

We now describe some situations where we can prove evenness.

**Lemma 3.6.16.** The property of $g$ being even is invariant under conjugation in $G$.

**Proof.** For any $h \in G$, conjugation $g^k \mapsto hg^kh^{-1}$ induces a set of natural isomorphisms of categories between $g^k$-twisted modules and $hg^kh^{-1}$-twisted modules, with the same underlying vector space but with conjugated action. These isomorphisms yield isomorphisms between spaces of intertwining operators, preserving the alternating property and the leading exponents of intertwining operators.

**Lemma 3.6.17.** Let $a$ be the odd part of $|g|$, i.e., the unique odd number such that $|g|/a$ is a power of 2. Then $g$ is even if and only if $g^a$ is even. Also, if there exists $h \in G$ such that $g = h^2$, then $g$ is even. In summary, if all automorphisms $V$ whose order is a power of 2 are even, then all finite-order automorphisms of $V$ are even.
Proof. By the third claim in Proposition 3.6.7, $\Omega \circ \Delta : A \to \mathbb{C}^\times$ is a quadratic form, and by Lemma 3.6.9, $e(L(0))$ is another quadratic form. Therefore, the pointwise quotient is a quadratic form, and by Lemma 2.5.5, it takes values in $\pm 1$.

The claims then follow straightforwardly from the quadratic property and the structure of $A$. □

Lemma 3.6.18. Let $V = V^+$, and let $G = G$. Any element of $M$ is even, with the possible exception of elements in the following 25 conjugacy classes: $8B, 8C, 8D, 8F, 16A, 24A, 24D, 24E, 24F, 24G, 24H, 24J, 32A, 32B, 40A, 40B, 40C, 40D, 48A, 56B, 56C, 88A, 88B, 104A, 104B$. In particular, if $g$ is even for $g$ in the classes $8B, 8C, 8D, 8F, 16A, 24A, 24D, 24E, 24F, 24G, 24H, 24J, 32A, 32B, 40A, 40B, 40C, 40D, 48A, 56B, 56C, 88A, 88B, 104A, 104B$. Then any finite order automorphism $g$ of $V$ is even.

Proof. Lemma 3.6.17 allows us to eliminate any classes for which an odd power is even, and any classes that are squares. The remaining classes are obtained by examining the power maps in [Conway-Norton-1979] Table 2 (also freely available in [GAP]). The second list is given by those classes in the first list that have order a power of 2. □

Lemma 3.6.19. Let $V$ be a vertex algebra attached to a positive definite even unimodular lattice. Then any finite order automorphism $g$ of $V$ is even.

Proof. By Proposition 3.6.21, any nonzero intertwining operator $m_z : V(g) \otimes V(g) \to V(g)((z^{1/|g|}))$ satisfies the property that for $u$ an $L(0)$-eigenvector with eigenvalue $k$ in the $L(0)$-spectrum, $m_z(u \otimes u) \in z^sV_L(g)((z)) \setminus z^{s+1}V_L(g)((z))$ for some $s \in \mathbb{Q}$ satisfying $s/2 \equiv k \pmod{\mathbb{Z}}$. Specializing to the case $u \in M_i$, the restriction of $m_z$ to the $V(g)$-module $M_i$ containing $u$ is even. In particular, $e(L(0))|_{M_i}$ coincides with $\Omega(i, i)$ for all $i$. □

3.7. Huang-type abelian intertwining algebras. We describe a method for transferring evenness from twisted modules of lattice vertex algebras to certain orbifolds. In the case of the monster, the transfer takes place using an object first described by Dixon, Ginsparg and Harvey [Dixon-Ginsparg-Harvey-1988] as a sum of twisted modules, and given an abelian intertwining algebra structure by Huang [Huang-1996].

Lemma 3.7.1. Let $V$ be a holomorphic $C_2$-cofinite vertex operator algebra, and let $\sigma, g$ be a commuting pair of finite order automorphisms of $V$, where $\sigma$ is an even involution of type $2|1$. Let $V^+$ and $V^-$ be the subspaces of $V$ on which $\sigma$ acts as $1$, resp. $-1$. Then there exists a canonical splitting of the irreducible $g$-twisted module $V(g)$ into a sum of two $g$-twisted $V^+$-modules $V(g)_1$ and $V(g)_2$. Furthermore, the restriction of the action map induces intertwining operators $V^- \otimes V(g)_1 \to V(g)_2((z^{1/|g|}))$ and $V^- \otimes V(g)_2 \to V(g)_1((z^{1/|g|}))$.

Proof. Because $g$ commutes with $g$, the automorphism $\sigma$ on $V$ induces an automorphism $\tilde{\sigma}$ of the abelian category of $g$-twisted $V$-modules, preserving the unique isomorphism class of the irreducible object $V(g)$. The image of $V(g)$ under this functor is the same underlying vector space, but conjugated $V$-action. By Schur’s Lemma, the set of twisted module isomorphisms $\tilde{\sigma}(V(g)) \to V(g)$ is a torsor under $\mathbb{C}^{\times}$. Let $\tilde{\sigma}$ be one such isomorphism. Schur’s Lemma implies $\tilde{\sigma} \circ \sigma(\tilde{\sigma}) : \tilde{\sigma}^2V(g) = V(g) \to V(g)$ is some nonzero scalar $\lambda^2$. On the underlying vector space, this composition is $\tilde{\sigma}^2$, so $\tilde{\sigma}$ has eigenvalues $\pm \lambda$. Rescaling $\tilde{\sigma}$ in the $\mathbb{C}^{\times}$-torsor of lifts changes the eigenvalues, but preserves the decomposition into eigenspaces. This yields a canonical decomposition of the underlying vector space of $V(g)$ into subspaces $V(g)_1$ and $V(g)_2$.

We claim that $V(g)_1$ and $V(g)_2$ are $g$-twisted $V^+$-modules. To show this, it suffices to check that the restriction of the action map $V \otimes V(g) \to V(g)((z^{1/|g|})) \to V^+ \otimes V(g)_1$ lands in $V(g)_1((z^{1/|g|})) \subset V(g)((z^{1/|g|}))$, and similarly for $V(g)_2$. If we extend $\tilde{\sigma}$ to $V \otimes V(g)$ by taking $V$ to itself via $\sigma$, then the action map is equivariant by the definition of the conjugated $V$-action. If $\tilde{\sigma}$ acts on $V(g)_1$ by $\lambda$, then by applying $\tilde{\sigma}$ to the restriction of the action map to $V^+ \otimes V(g)_1$, we find that $\tilde{\sigma}$ acts on any vector in the image by $\lambda$, i.e., all image vectors lie in $V(g)_1((z^{1/|g|}))$. Similarly, $\tilde{\sigma}$ acts on vectors in
the image of $V^+ \otimes V(g)_2$ by $-\lambda$. The analogous argument for $V^-$ shows the restriction of the action map to $V^- \otimes V(g)_1$ takes values in $V(g)_2((z^{1/|g|}))$ and similarly for $V^- \otimes V(g)_2 \rightarrow V(g)_1((z^{1/|g|}))$.

The fact that these maps are intertwining operators follows from the fact that they are restrictions of action maps.

\[ \square \]

**Proposition 3.7.2.** Let $V$ be a holomorphic $C_2$-cofinite vertex operator algebra, and let $\sigma$ be an even involution of type $2|1$. Fix an abelian intertwining algebra $V_\sigma$ in $\text{Vect}^A_{F,\Omega}$, where $F = 1$ and $\Omega$ is the nonsymmetric bimultiplicative form on $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given by $\Omega((a,b),(c,d)) = (-1)^{ad}$ (see section 3.4 of [Huang-1996]). Then for any finite order automorphism $g$ of $V$ that commutes with $\sigma$, there exists a lift of $g$ to a homogeneous automorphism $\tilde{g}$ of $V_\sigma$ such that $V(g) \otimes V(g\sigma)$ admits the structure of an irreducible strict $\tilde{g}$-twisted $V_\sigma$-module.

**Proof.** Let $n$ denote the order of $g$. By Proposition 6.1.1 there are one dimensional vector spaces $\mathcal{I}_{\sigma,g}$ and $\mathcal{I}_{\sigma,g\sigma}$ of intertwining operators $V(\sigma) \otimes V(g) \rightarrow V(g\sigma)((z^{1/2n}))$, and $V(\sigma) \otimes V(g\sigma) \rightarrow V(g)((z^{1/2n}))$, and the composition of any fixed elements $I_z^{\sigma,g}$ and $I_z^{\sigma,g\sigma}$ in those spaces yields maps from $V(\sigma) \otimes V(\sigma) \otimes V(g)$ to $V(g)[[z^{1/2n}, w^{1/2n}][z^{-1}, w^{-1}, (z-w)^{-1}]$ and from $V(\sigma) \otimes V(\sigma) \otimes V(g\sigma)$ to $V(g\sigma)[[z^{1/2n}, w^{1/2n}][z^{-1}, w^{-1}, (z-w)^{-1}]$ that are proportional to the multiplication in $V_\sigma$ followed by the corresponding action maps. To describe this more precisely, we fix $I_z^{\sigma,g}$, and $I_z^{\sigma,g\sigma}$, and say that in the following diagram:

The composition on the right side is $c_g \in \mathbb{C}^\times$ times the composition on the left side, and the analogous diagram with $g$ replaced with $g\sigma$ yields a constant $c_{g\sigma} \in \mathbb{C}^\times$. The pentagon diagram of compositions on $V(\sigma) \otimes V(\sigma) \otimes V(\sigma) \otimes V(g)$ then yields:

so $c_g = c_{g\sigma}$. In particular, scaling these intertwining operators by suitable constants (which are far from unique) yields associativity of composition, i.e., taken together, they define a map $\bar{\text{act}}_z : V_\sigma \otimes (V(g) \oplus V(g\sigma))((z^{1/2n}))$ such that the following diagram commutes:
It remains to show that $\tilde{\text{act}}_z$ describes a strict $\tilde{g}$-twisted $V_\sigma$-module structure on $V(g) \oplus V(g\sigma)$.

We have a canonical decomposition $V_\sigma = V^+ \oplus V^- \oplus V(\sigma)^+ \oplus V(\sigma)^-$, where $V(\sigma)^-$ is the $V^+$-submodule of $V(\sigma)$ on which the preferred lift of $\sigma$ acts by $-1$. Following the notation established in Lemma 6.1.1, the projective $(\sigma)$-equivariance of intertwining operators given in Proposition 3.7.1 implies there is a unique bijection $\tau : \{1, 2\} \to \{1, 2\}$ such that the restriction of $\tilde{\text{act}}_z$ to $V(\sigma)^+ \oplus V(g)_i$ has image in $V(g\sigma)_{\tau(i)}((z^{1/2n}))$ for any $i \in \{1, 2\}$. By associativity, we also have:

1. The restriction of $\tilde{\text{act}}_z$ to $V(\sigma)^- \otimes V(g)_i$ has image in $V(g\sigma)_{3-\tau(i)}((z^{1/2n}))$ for any $i \in \{1, 2\}$.
2. The restriction of $\tilde{\text{act}}_z$ to $V(\sigma)^+ \otimes V(g\sigma)_i$ has image in $V(g\sigma)_{\tau(i)}((z^{1/2n}))$ for any $i \in \{1, 2\}$.
3. The restriction of $\tilde{\text{act}}_z$ to $V(\sigma)^- \otimes V(g\sigma)_i$ has image in $V(g\sigma)_{3-\tau(i)}((z^{1/2n}))$ for any $i \in \{1, 2\}$.

We set $S$ to be the set $\{g, g\sigma\} \times \{1, 2\}$, and define an $A$-torsor structure on $S$ by setting translation by $(0, 1) \in A$ to be given by $(g, 1) \mapsto (g, 2)$, and translation by $(1, 0)$ to be given by $(g, 1) \mapsto (g\sigma, \tau(1))$. We assign modules to elements of $A$ by $(0, 0) \mapsto V^+$, $(0, 1) \mapsto V^-$, $(1, 0) \mapsto V(\sigma)^+$, and $(1, 1) \mapsto V(\sigma)^-$. We assign twisted modules to elements of $S$ by $(g, 1) \mapsto V(g)_1$, $(g, 2) \mapsto V(g)_2$, $(g\sigma, 1) \mapsto V(g\sigma)_{\tau(1)}$, and $(g\sigma, 2) \mapsto V(g\sigma)_{\tau(2)}$. We see that $\tilde{\text{act}}_z$ induces maps of modules compatible with the $A$-action on $S$.

It remains to choose a lift $\tilde{g}$ of $g$ to $V_\sigma$, and a map $\hat{b} : A \times \text{Hom}(\{\tilde{g}\}, C^\times) \times S \to \frac{1}{2n}\mathbb{Z}$ so that the monodromy condition is satisfied. We note that the restriction of $\hat{b}$ to the subset where the first coordinate is $\mathbb{Z}/2\mathbb{Z} \times \{0\} \subset A$ is already determined up to integer translation and satisfies the monodromy condition, because $V(g)$ and $V(g\sigma)$ are suitably twisted $V$-modules, so we need only extend $\hat{b}$ to inputs corresponding to $V(\sigma)$. We shall calculate this extension by determining the exponents appearing in the map $\tilde{\text{act}}_z$.

By Proposition 3.5.6 there are two lifts $g'$ and $g'\theta$ of $g$ to $V_\sigma$, where $\theta$ is the automorphism on $V_\sigma$ that fixes $V$ and is $-1$ on $V(\sigma)$. In particular, the decomposition of $V(\sigma)$ into eigenspaces under a lift of $g$ is invariant under change of lift, but the eigenvalues are multiplied by $-1$.

Choose a nonzero eigenspace $V^i \subset V(\sigma)^+$ for $g'$, and let $e(r)$ be its eigenvalue. $V^i$ is also the eigenspace for $g'\theta$ with eigenvalue $e(r+1/2)$. Multiplication in $V_\sigma$ sends $V^i \otimes V^j$ to $V^{2i}((z))$, where $V^{2i}$ is a subspace of $V^+$ on which $g$ acts by $e(2r) = e(2(r+1/2))$. We claim that the restriction of $\tilde{\text{act}}_z$ to $V^i \otimes V(g)_1$ has image contained in either $z^rV(g\sigma)_{\tau(1)}((z))$ or $z^{r+1/2}V(g\sigma)_{\tau(2)}((z))$. Indeed,
it follows from the fact that the arrows down the left side of the following diagram:

\[
\begin{array}{ccc}
V^i \otimes V^i \otimes V(g)_1 & \rightarrow & V^i \otimes w^i V\langle(g\sigma)_{\tau(1)}\rangle((w)) \\
V^{2i}((z-w)) \otimes V(g)_1 & \rightarrow & z^2 w^i V\langle(g\sigma)_{\tau(1)}\rangle[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] \\
& \rightarrow & w^{2i} V\langle(g\sigma)_{\tau(1)}\rangle((z-w)) \\
& \rightarrow & z^2 w^i V\langle(g\sigma)_{\tau(1)}\rangle((z))((w))
\end{array}
\]

are defined by the multiplication in \(V_\sigma\) and the action of \(V\) on \(V\langle(g\sigma)\rangle\). By associativity of the action, the question marks must indicate either \(r\) or \(r+1/2\). If the restriction of \(\tilde{\text{act}}_z\) to \(V^i \otimes V(g)_1\) has image in \(z^r V\langle(g\sigma)_{\tau(1)}\rangle((z))\), we set \(\tilde{g} = g^r\). Otherwise, we set \(\tilde{g} = g^r\theta\).

Now that we have fixed a lift \(\tilde{g}\) of \(g\), we decompose \(V\langle\sigma\rangle^+\) into eigenspaces under the action of \(\langle\tilde{g}\rangle\). We have chosen \(\tilde{g}\) so that there exists an eigenspace \(V^i\) of \(V\langle\sigma\rangle^+\) such that \(\tilde{g}\) acts by \(e(s)\) for some \(s\), and \(\tilde{\text{act}}_z(V^i \otimes V(g)_1) \subset z^s V\langle(g\sigma)_{\tau(1)}\rangle((z))\). Now, let \(V^j\) be a \(\tilde{g}\)-eigenspace in \(V\langle\sigma\rangle^+\) with eigenvalue \(e(s')\). There exists a \(g\)-eigenspace \(V^{j-i} \subset V^+\) with eigenvalue \(e(s' - s)\) such that multiplication in \(V_\sigma\) takes \(V^{j-i} \otimes V^i\) to \(V^j((z))\). Then associativity implies the diagram

\[
\begin{array}{ccc}
V^{j-i} \otimes V^i \otimes V(g)_1 & \rightarrow & V^{j-i} \otimes w^s V\langle(g\sigma)_{\tau(1)}\rangle((w)) \\
V^j((z-w)) \otimes V(g)_1 & \rightarrow & z^2 w^s V\langle(g\sigma)_{\tau(1)}\rangle[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] \\
& \rightarrow & w^2 V\langle(g\sigma)_{\tau(1)}\rangle((z-w)) \\
& \rightarrow & z^{s'-s} w^s V\langle(g\sigma)_{\tau(1)}\rangle((z))((w))
\end{array}
\]

can be filled in on the bottom left with \(w^s V\langle(g\sigma)_{\tau(1)}\rangle((w))((z-w))\). This implies the restriction of \(\tilde{\text{act}}_z\) to \(V^j \otimes V(g)_1\) has image in \(w^s V\langle(g\sigma)_{\tau(1)}\rangle((z))\).

We may use essentially the same argument for an eigenspace \(V^j\) of \(V\langle\sigma\rangle^-\) under \(\tilde{g}\) with eigenvalue \(e(s')\), using an eigenspace \(V^{j-i}\) of \(V^-\) to enforce the compatibility. The diagram in this case has some additional complexity due to the fact that \(g\sigma\) acts on \(V^{j-i}\) by \(e(s' - s + 1/2)\):

\[
\begin{array}{ccc}
V^{j-i} \otimes V^i \otimes V(g)_1 & \rightarrow & V^{j-i} \otimes w^s V\langle(g\sigma)_{\tau(1)}\rangle((w)) \\
(z-w)^{1/2} V^j((z-w)) \otimes V(g)_1 & \rightarrow & z^2 w^s (z-w)^{1/2} V\langle(g\sigma)_{\tau(2)}\rangle[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] \\
& \rightarrow & w^2 (z-w)^{1/2} V\langle(g\sigma)_{\tau(2)}\rangle((z-w)) \\
& \rightarrow & z^{s'-s+1/2} w^s V\langle(g\sigma)_{\tau(2)}\rangle((z))((w))
\end{array}
\]

We find that the space on the bottom left must be \(w^s (z-w)^{1/2} V\langle(g\sigma)_{\tau(2)}\rangle((z-w))\), so the restriction of \(\tilde{\text{act}}_z\) to \(V^j \otimes V(g)_1\) has image in \(w^s V\langle(g\sigma)_{\tau(2)}\rangle((z))\).
We have finished defining $\tilde{b}(a, f, s)$ for all $a \in A$, all $f : \langle \tilde{g} \rangle \to \mathbb{C}^\times$, and $s = (g, 1) \in S$. Furthermore, we have verified that under the trivialization $\epsilon : S \to A$ uniquely determined by $(g, 1) \mapsto (0, 0)$, the identity $e(\tilde{b}(a, f, s)) = e(b(a, \epsilon(s))) \cdot f(\tilde{g})$ holds for $b : A \times A \to \frac{1}{2} \mathbb{Z}$ any map satisfying $e(-b(i, j)) = \Omega(i, j)\Omega(j, i)$.

To finish the proof of strictness, it remains to determine the corresponding exponents for the actions on the remaining subspaces of the twisted module: $V(g)_2$, $V(g\sigma)_1$ and $V(g\sigma)_2$. The arguments are essentially identical to the ones we’ve given, with minor adjustments for signs.

Irreducibility follows from the following facts:

1. Restriction to the $V$-action gives a sum of irreducible twisted modules - this restricts the possible nonzero proper submodules to individual twisted $V$-modules.

2. The “compatibility with multiplication” axiom implies the constituent intertwining maps are nonzero - this implies individual twisted $V$-modules are not $V_\sigma$-modules.

\[ \square \]

**Lemma 3.7.3.** Let $V$ be the vertex algebra attached to an even unimodular lattice, and let $\sigma$ be an even involution of type $2|1$. Then for any finite order homogeneous automorphism $g$ of $V_\sigma$, any irreducible $g$-twisted $V_\sigma$-module decomposes under the twisted $V$-action as $V(g) \oplus V(g\sigma)$.

**Proof.** Applying Lemma 2.6.11, we see that any $g$-twisted $V_\sigma$-module $M$ splits into a sum of a $g$-twisted $V$-module and a $g\sigma$-twisted $V$-module. By Theorem 4.2 of Bakalov-Kac-2004, lattice twisted modules are completely reducible, so we have $M \cong V(g)^{\oplus k} \oplus V(g\sigma)^{\oplus \ell}$ for some non-negative integers $k$ and $\ell$. In fact, both $k$ and $\ell$ are positive, because the compatibility with the product structure in $V_\sigma$ means the intertwining operators from $V(\sigma) \otimes V(g)$ and $V(\sigma) \otimes V(g\sigma)$ obtained by restriction of the twisted $V_\sigma$-module structure are necessarily nonzero.

By Schur’s Lemma, the vector space of $g$-twisted $V$-module homomorphisms $V(g) \to V(g)^{\oplus k}$ is isomorphic to $\mathbb{C}^k$, and the vector space of $g\sigma$-twisted $V$-module homomorphisms $V(g\sigma)^{\oplus \ell} \to V(g\sigma)$ is isomorphic to $\mathbb{C}^\ell$. Furthermore, the identity morphism on $V(g)^{\oplus k}$ (resp. $V(g\sigma)^{\oplus \ell}$) is given by a sum of $k$ (resp. $\ell$) such homomorphisms. In other words, by fixing nonzero intertwining operators $I^g_{\sigma} : V(\sigma) \otimes V(g) \to V(g\sigma)((z^{1/2|g|}))$ and $I^{g\sigma}_{\sigma} : V(\sigma) \otimes V(g\sigma) \to V(g)((z^{1/2|g|}))$, the module structure on $M$ is completely described by pair of complex matrices:

1. $B^g_{\sigma,g}$: a $k \times \ell$ matrix whose $(i, j)$ entry is the multiple of $I^g_{\sigma}$ given by the composition

$$V(\sigma) \otimes V(g) \xrightarrow{i} V(\sigma) \otimes V(g)^{\oplus k} \xrightarrow{act_{\sigma}} V(g\sigma)^{\oplus \ell} \xrightarrow{I^g_{\sigma}} V(g)((z^{1/2|g|}))$$

2. $B^{g\sigma}_{\sigma,g}$: an $\ell \times k$ matrix whose $(i, j)$ entry is the multiple of $I^{g\sigma}_{\sigma}$ given by the composition

$$V(\sigma) \otimes V(g\sigma) \xrightarrow{i} V(\sigma) \otimes V(g)^{\oplus \ell} \xrightarrow{act_{g\sigma}} V(g\sigma)^{\oplus k} \xrightarrow{I^{g\sigma}_{\sigma}} V(g)((z^{1/2|g|}))$$

By associativity, $B^g_{\sigma,g}B^{g\sigma}_{\sigma,g}$ is the $k \times k$ identity matrix, and $B^{g\sigma}_{\sigma,g}B^g_{\sigma,g}$ is the $\ell \times \ell$ identity matrix. In particular, $k = \ell$, and by suitably changing the basis on $\mathbb{C}^\ell$ (e.g., by using $B^g_{\sigma,g}$ to pull back the basis on $\mathbb{C}^k$ to $\mathbb{C}^\ell$), we can change $B^g_{\sigma,g}$ and $B^{g\sigma}_{\sigma,g}$ to identity matrices. In particular, Proposition 3.7.2 implies we obtain a decomposition of $M$ into a direct sum of $g$-twisted $V_\sigma$-modules of the form $V(g) \oplus V(g\sigma)$. By our irreducibility assumption, $k = \ell = 1$. \[ \square \]

**Lemma 3.7.4.** Let $V$ be a holomorphic $C_2$-cofinite vertex operator algebra, and let $\sigma$ be an even involution of type $2|1$. Suppose there is some positive definite even unimodular lattice $L$ such that $V_L \cong V^+ \oplus V(\sigma)^+$. Then any automorphism $g$ of $V$ that commutes with $\sigma$ is even.

**Proof.** It suffices to show that there exists an $L(0)$-eigenvector $u \in V(g)$ with eigenvalue $k$, such that $m_z(u \otimes u)$ has leading exponent $s$ satisfying $s/2 \equiv k \pmod{\mathbb{Z}}$. By Lemma 3.7.3, $V(g)$ naturally embeds as a subspace of a $\tilde{g}$-twisted module of $V_\sigma$ for some lift $\tilde{g}$, and under such an embedding, it is spanned by vectors in irreducible twisted modules of $V_L$. By Lemma 3.6.19, the restriction of any lift $\tilde{g}$ of $g$ to $V_L$ is even, so choosing $u$ to be any vector of $V_L(\tilde{g})$ yields the result. \[ \square \]
We restrict our attention to the case $V = V^\natural$ and $\sigma$ lies in class 2B in the monster. Under these new assumptions, $V_\sigma$ is isomorphic to the abelian intertwining algebra described in \cite{Dixon-Ginsparg-Harvey-1988} and constructed more precisely in \cite{Huang-1996}. We note the following features:

1. The subalgebra supported in $\mathbb{Z}/2\mathbb{Z} \times \{0\}$ is $V^\natural$ decomposed according to eigenvalues of $\sigma$.
2. The subspace supported in $\mathbb{Z}/2\mathbb{Z} \times \{1\}$ is an irreducible $\sigma$-twisted $V^\natural$-module $V^\natural(\sigma)$ (unique up to isomorphism).
3. The subalgebra supported in $\{0\} \times \mathbb{Z}/2\mathbb{Z}$ is $V_{\text{Leech}}$ decomposed according to eigenvalues of the involution $\theta$ induced by the $-1$ automorphism of the Leech lattice (described in \cite{Frenkel-Lepowski-Meurman-1988}).
4. The subspace supported in $\{1\} \times \mathbb{Z}/2\mathbb{Z}$ is an irreducible $\theta$-twisted $V_{\text{Leech}}$-module $V_{\text{Leech}}(\theta)$.

Lemma 3.7.5. Any element in class 2B is even of type $2|1$.

Proof. Elements in class 2B are even because elements in class 4A have squares in class 2B, and they have type $2|1$ because the 2B McKay-Thompson series is equal to $\frac{\Delta(\tau)}{\Delta(2\tau)} + 24$, which is invariant under $\Gamma_0(2)$.

Lemma 3.7.6. Let $g$ be an element of $\mathbb{M}$ that commutes with some element $\sigma$ in class 2B. Then there exists a lift of $g$ to a homogeneous automorphism of $V_\sigma$, and an embedding of $V(g)$ into an irreducible $\tilde{g}$-twisted $V_\sigma$-module, such that the image of any element $u \in V(g)$ is a sum of elements in $V_{\text{Leech}}(\tilde{g})$ and $V_{\text{Leech}}(\tilde{g}\theta)$.

Proof. The embedding follows from Proposition 3.7.4 and the decomposition of the $\tilde{g}$-twisted module follows from Lemma 3.7.3.

Proposition 3.7.7. Let $g$ be an element of $\mathbb{M}$ that commutes with an element in class 2B. Then $g$ is even.

Proof. This follows from 3.7.4.

Corollary 3.7.8. For any element $g \in \mathbb{M}$, $g$ is even.

Proof. By Lemma 3.6.18, it suffices to consider the cases where $g$ lies in the conjugacy classes 8B, 8C, 8D, 8F, 16A, 32A, and 32B. Under this additional hypothesis, $g$ commutes with an element in class 2B, since the centralizer of any 2B element in $\mathbb{M}$ contains a 2-Sylow subgroup of $\mathbb{M}$, and these classes have 2-power order. By Proposition 3.7.7, $g$ is even.

Remark 3.7.9. We would be interested to see a general proof of evenness that does not require knowledge of lattices.

Theorem 3.7.10. For any $g \in \mathbb{M}$, there exists an abelian intertwining algebra structure $V_g$ on $\bigoplus_{i=0}^{|g|-1} V^\natural(g^i)$ whose multiplication map restricts to the vertex algebra structure on $V^\natural$, the twisted module structures on $V^\natural(g^i)$, and nonzero intertwining operators between the irreducible twisted modules. If $g$ has type $n|h\pm$ for some $n$ and $h$, then $V_g$ has type $\rho_{n|h\pm}$. Furthermore, there is a canonical projective action of $C_{\text{M}}(g)$ on this abelian intertwining algebra by automorphisms, and this action restricts to the canonical projective action on each twisted module.

Proof. By Corollary 3.7.8 all $g \in \mathbb{M}$ are even, and by Proposition 3.6.12, the claims follow under the hypothesis that $g$ is even.

4. Lie algebras

In this section we construct the monstrous Lie algebras. Our construction employs the physical principle first articulated in \cite{Lovelace-1971} and demonstrated in \cite{Goddard-Thorn-1972} and \cite{Brower-1972}, that in the critical dimension $d = 26$ of bosonic string theory, Virasoro-type Ward
identities cancel two full sets of oscillators. This principle now has at least three methods of realization in physics (cf. Green-Schwarz-Witten-1987 chapters 2,3): Covariant quantization, passage to the light-cone gauge (also called “construction of the transversal space”), and BRST cohomology. The BRST method was pioneered in [Kato-Ogawa-1983].

The use of this cancellation principle in mathematics is also far from new. For example, BRST cohomology was used in [Frenkel-Garland-Zuckerman-1986] to compute the character of the space of physical states for a bosonic string propagating in Lorentzian 26-space. The covariant quantization method is also used in the analysis of three Lie algebras, each of which was named the “monster Lie algebra” upon construction:

1. The first example is the Kac-Moody Lie algebra whose simple roots are those of the reflection group of the even unimodular Lorentzian lattice \(II_{1,25}\). The existence of a monster action was suggested in [Borcherds-Conway-Queen-Sloane-1984], and Frenkel [Frenkel-1985] used the no-ghost theorem to bound its root multiplicities by coefficients of the modular form \(1/\Delta\).

2. The second example is the Borcherds-Kac-Moody Lie algebra whose real simple roots are those given in the first example, but it has additional imaginary simple roots of multiplicity 24 on positive multiples of the Weyl vector. This was constructed in [Borcherds-1986], and is now known as the “fake monster Lie algebra”. In [Borcherds-1990], Borcherds showed that the root multiplicities saturated the \(1/\Delta\) bound given by Frenkel.

3. The third and last example is the generalized Kac-Moody Lie algebra constructed in [Borcherds-1992]. This is the only example known to have a monster action.

Supersymmetric variants of this construction have been used, e.g., in [Scheithauer-2000].

4.1. Preliminaries. Recall the Virasoro algebra \(\text{vir}\) is the complex vector space with a distinguished basis given by symbols \(K, \{L(n)\}_{n \in \mathbb{Z}}\), with the Lie bracket defined by \([L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} K\) for all \(m, n \in \mathbb{Z}\), and \([L(n), K] = 0\) for all \(n \in \mathbb{Z}\). We write \(\text{vir}^+\) for the subalgebra spanned by \(\{L(n)\}_{n \geq 1}\), \(\text{vir}^-\) for the subalgebra spanned by \(\{L(n)\}_{n \leq -1}\), and \(\text{vir}^0\) for the 2-dimensional subalgebra spanned by \(L(0)\) and \(K\).

Definition 4.1.1. Let \(V\) be a representation of the Virasoro algebra, i.e., a complex vector space equipped with operators \(\{L(n)\}_{n \in \mathbb{Z}}\) and \(K\) that satisfy the Virasoro relations.

1. We say that \(V\) is positive-energy if:
   (a) \(L(0)\) acts diagonally, i.e., \(V\) splits into a direct sum of \(L(0)\)-eigenspaces.
   (b) \(\text{vir}^+\) acts locally nilpotently, i.e., for all \(v \in V\), there exists \(N > 0\) such that for all sequences \(n_1, \ldots, n_k\) of positive integers satisfying \(n_1 + \cdots + n_k > N\), \(L(n_1) \cdots L(n_k)v = 0\).

2. We say that \(V\) has central charge \(c \in \mathbb{C}\) if \(Kv = cv\) for all \(v \in V\).

3. We say that \(V\) is unitarizable if it admits a positive-definite hermitian form, such that \(L(n)\) is adjoint to \(L(-n)\) for all \(n \in \mathbb{Z}\).

Given \(c, h \in \mathbb{C}\), we define the Verma module \(M_{h,c}\) as the induced module \(\text{vir} \otimes_{\text{vir}^+ \oplus \text{vir}^0} \mathbb{C}_{c,h}\), where \(\mathbb{C}_{c,h}\) is the one dimensional vector space with trivial action of \(\text{vir}^+\), such that \(L(0)\) has eigenvalue \(h\), and \(K\) has eigenvalue \(c\). We write \(L_{h,c}\) for the unique irreducible quotient of \(M_{h,c}\) with highest weight \(h\).

Lemma 4.1.2. Any Verma module \(M_{h,c}\) is positive-energy. The character of \(M_{h,c}\) is

\[
Tr(q^{L(0)}|M_{h,c}) = q^h \prod_{n=1}^{\infty} (1 - q^n)^{-1}
\]

so the span of vectors in \(M_{h,c}\) with \(L(0)\)-eigenvalue \(h + m\) has dimension equal to the number of partitions of \(m\).
Lemma 4.1.3. We set $c = 24$, because that is the case we will use. Similar results hold for other central charges in the open interval $(1, 25) \subset \mathbb{R}$.

1. If $h \neq 0$, then $M_{h,24}$ is irreducible (i.e., equal to $L_{h,c}$), and the category of lowest weight vir-modules with central charge 24 and lowest weight $h$ is completely reducible, with $L_{h,c}$ as its unique irreducible object.

2. For the case $h = 0$, $M_{0,24}$ has a unique proper submodule $M_{1,24}$, and $L_{0,24} = M_{0,24}/M_{1,24}$.

3. $M_{h,24}$ is unitarizable if and only if $h > 0$. $L_{h,24}$ is unitarizable if and only if $h \geq 0$.

4. If $V$ is a positive energy vir-module of central charge $c = 24$ whose $L(0)$-spectrum is positive, then $V$ is unitarizable, and is isomorphic to a direct sum of modules of the form $M(c,h)$ with $h > 0$.

Proof. The reducibility and unitarizability results follow from the Kac determinant formula, introduced in [Kac-1978]. See the first four sections of [Kac-Raina-1987] for detailed proofs. The last claim follows from induction in each coset of $Z$ in the spectrum of $L(0)$.

Lemma 4.1.4. Let $g$ be an automorphism of $V^2$. Then the irreducible $g$-twisted module $V^2(g)$ is unitarizable. If $g \neq 1$, then $V^2(g)$ decomposes as a direct sum of Virasoro submodules of the form $M(c,h)$ as $h$ ranges over a multiset of positive rational numbers of the form $\frac{k}{|g|^2}$.

Proof. For the case $g = 1$, it suffices to show that $V^2$ is unitarizable - this is well-known, and follows straightforwardly from its conformal structure.

Suppose $g \neq 1$. By Theorem 13.1 of [Dong-Li-Mason-2000], the character of $V^2(g)$ is a constant multiple of $T_g(-1/\tau)$, where $T_g(\tau) = \text{Tr}(g q^{L(0)-1} | V^2)$ is the McKay-Thompson series. If $T_g$ is regular at $\tau = 0$ (i.e., $g$ is non-Fricke), then the lowest-weight space of $V(g)$ has $L(0)$-eigenvalue at least 1. If $T_g$ has a pole at $\tau = 0$ (i.e., $g$ is Fricke), then the pole has order $1/N$, where $N$ is the level of $T_g$, and the lowest-weight space of $V^2(g)$ has $L(0)$-eigenvalue $1 - 1/N$. We conclude that as long as $g$ is nontrivial, the lowest weight space has positive $L(0)$-eigenvalue, and by the last claim in Lemma 4.1.3, $V^2(g)$ is therefore a sum of unitarizable Verma modules.

Lemma 4.1.5. Let $\lambda \in \mathbb{R}^{r,s}$, and let $\pi_{\lambda}^{r,s}$ denote the corresponding irreducible Heisenberg module, endowed with the Virasoro action arising from the conformal element $\frac{1}{2} \sum_{\mu \in \mathbb{R}^{r,s}} b^\mu_{-1} b^{-1}_{-1} \in \pi_{0}^{r,s}$. Then $\pi_{\lambda}^{r,s}$ is a positive-energy vir-module that admits a natural Hermitian form for which $L(n)$ is adjoint to $L(-n)$, and $b^n$ is adjoint to $b^{\kappa}_{-n}$, for all $n \in \mathbb{Z}$ and all vectors $b^\mu \in \mathbb{R}^{r,s}$.

Proof. By irreducibility, the conditions on the Hermitian form determine it up to a normalization. See, e.g., section 2.2 of [Frenkel-Garland-Zuckerman-1986].

Definition 4.1.6. Let $V$ be a Virasoro representation.

1. A vector $v \in V$ is called primary (or physical) if $L(i)v = 0$ for all $i > 0$, i.e., if $vir^+$ acts trivially. Given a positive-energy representation and a complex number $r$, we write $P^r_v$ to denote the subspace of primary vectors $v$ such that $L(0)v = rv$. We may write $P^r_v$ when there is more than one representation under consideration.

2. A vector $v \in V$ is spurious if it has the form $L_{-i}w$, for some $w \in V$ and some $i > 0$. A primary spurious vector is called null. We write $null^i_v$ to denote the subspace of null vectors of weight $i$ in $V$.

3. A symmetric bilinear form on $V$ is called Virasoro-invariant if for all $n$, $L_n$ is adjoint to $L_{-n}$.

Lemma 4.1.7. Let $V$ be a Virasoro representation equipped with a Virasoro-invariant bilinear form $(,)$.

1. The $L(0)$-eigenspaces in $V$ are pairwise orthogonal.
(2) \( L(-1)P^0 \subset P^1 \).
(3) Any null vector in \( P^1 \) is orthogonal to all other vectors. In particular, \( \text{null}_1^1 \) contains \( L(-1)P^0 \).
(4) Any Virasoro module admits an invariant bilinear form, called the Shapovalov form, which is nonsingular if and only if the module is irreducible.

Proof. All claims but the last are straightforward calculations. The last claim can be found in [Kac-Raina-1987].

We introduce a second notion of invariance for vertex algebras.

**Definition 4.1.8.** Let \( V \) be a Möbius vertex algebra over a characteristic zero field. A bilinear form \((, ) \) on \( V \) is invariant if for all \( u, v, w \in V \), the identity \((m_z(u \otimes v), w) = (v, m_{z^{-1}}(e^{zL(1)}(-z^{-2})L(0)u \otimes w)) \) holds. Equivalently, if \( u \) has conformal weight \( i \), then the adjoint of the operator \( u_n \) is \((-1)^i \sum_{j \geq 0} L_j^1(u)_{2j-j-n-2}/j! \).

**Lemma 4.1.9.** Let \( V \) be a conformal vertex algebra. Then any invariant bilinear form is Virasoro-invariant.

**Proof.** This follows from defining the identity with \( u \) as the Virasoro vector \( \omega \). \( \square \)

**Lemma 4.1.10.** Let \( g \in \mathbb{M} \), and let \( W_g \) be the \( A \)-graded conformal vertex algebra whose existence is given by Theorem 3.7.10. Then the tensor product of the contragradient pairing between \( V(g^j) \) and \( V(g^{-i}) \) for each \( g^j \in \mathbb{M} \) and the contragradient pairing between \( \pi^1_{\lambda,1} \) and \( \pi^1_{\lambda,1} \) for each \( \Lambda \in \mathbb{M}^{1,1} \) is an invariant bilinear form \((, ) \) on \( W_g \) satisfying the following conditions:

1. For any \( i, j \in A \), with \( i + j \neq 0 \), we have \((u, v) = 0 \) for any \( u \in W^i_g \) and \( v \in W^j_g \).
2. If \( L(0)u = ru \) and \( L(0)v = sv \), and \( r \neq s \), then \((u, v) = 0 \).
3. If \( h \in C_\mathbb{M}(g) \), then \((hu, hv) = (u, v) \).

**Proof.** By Theorem 2.13 in [Scheithauer-1998], \( \text{Hom}((W_g)_0/L(1)(W_g)_1, \mathbb{C}) \) is isomorphic to the space of invariant bilinear forms on \( W_g \), and an isomorphism is given by sending a map \( f \) to the form \((u, v) = (\pi^1 \circ (u^*_1, v) \), where \( u^*_n = (-1)^{n} \sum_{m \geq 0} (L_n^1 - m!)u_{2m-m-n-2} \) when \( L(0)u = su \). The contragradient pairing corresponds to the “dual to identity” map that sends the unit to 1, and annihilates all weight zero \( u \otimes v \) for \( u \in \pi^1_{\lambda,1} \) and \( v \in V^2(g^i) \) when \( \Lambda \neq 0 \). The enumerated properties follow straightforwardly. \( \square \)

### 4.2. Old covariant quantization

When quantizing a field theory, one must ensure that the space of states is ghost-free, i.e., that there are no negative-norm states. The existence of ghosts presents an obstruction to unitarity and causality. Indeed, this was one of the basic defects of Veneziano’s original 4-point function, during the birth of string theory. In the early 1970s, dual models showed promise as a theory of strings that can avoid this problem. Goddard and Thorn [Goddard-Thorn-1972], and independently Brower [Brower-1972] used covariant quantization to show that in dimension at most 26, the bosonic theory had no ghosts. Furthermore, they described an isomorphism between the states that couple, and the bosonic theory given by 24 oscillators (following the transversal space construction of Del Giudice, de Vecchia, and Fubini).

In covariant quantization, one begins with a naive space of states, that has redundancies due to internal symmetry. In the case of internal Virasoro symmetry, to obtain the relevant space of states that couple, one considers the “physical” subspace of states that are invariant under the action of \( \text{vir}^+ \), and quotients by the “spurious” subspace, which is spanned by the image of the augmentation ideal in \( U(\text{vir}^-) \) together with the radical of the induced Hermitian form.

An enhancement of the original covariant approach was used in [Borcherds-1992] to construct the monster Lie algebra as the space of physical states arising from the vertex algebra is \( V^2 \otimes V_{II,1,1}^1 \). We will use a variant of this method to construct our monstrous Lie algebras.
Lemma 4.2.2. We list some relatively straightforward facts:

1. If $W$ is a conformal vertex algebra that is positive-energy, then $P^1/L(-1)P^0$ has a Lie algebra structure induced by setting $[u,v] = u_0v$, i.e., the coefficient of $z^{-1}$ in $m_2(u\otimes v)$. If a group $G$ acts on $W$ by conformal vertex algebra automorphisms, then $G$ acts on $P^1/L(-1)P^0$ by Lie algebra automorphisms.

2. If $W$ is a positive-energy conformal vertex algebra equipped with an invariant bilinear form, such that for each $i \in \mathbb{Z}$, $L(i)$ is adjoint to $L(-i)$, then the Lie algebra $P^1/L(-1)P^0$ has an invariant inner product induced from $W$, and the radical of that inner product is an ideal. In particular, $\text{Quant}(W) = (P^1/L(-1)P^0)/\text{rad}(\cdot)$ is a Lie algebra equipped with a nondegenerate inner product. If a group $G$ acts on $W$ by conformal vertex algebra automorphisms, and the invariant bilinear form is $G$-equivariant, then $\text{Quant}(W)$ has a natural $G$ action by Lie algebra automorphisms.

Remark 4.2.3. These facts are implicitly used in the proof of the Monstrous Moonshine conjecture in [Borcherds-1992].

Proof. The first claim is Lemma 3.2 in [Jurisich-2009]. The second claim follows from the discussion after the proof in loc. cit. The claims about $G$-actions follow from the fact that automorphisms preserve the Virasoro action (hence the primitivity property, the grading, and the $L(-1)$ operator), and the product $(u,v) \mapsto u_0v$.

The following proposition seems to capture the core of Lovelace’s principle of oscillator cancellation. It is sometimes given the name “No Ghost Theorem”, because in its original form, it asserts that quantization yields a positive-definite space of states. Here, the word “ghost” means negative-norm state (which is important in physics, since the existence of such ghosts obstructs reasonable formulations of causality), and should not be confused with Faddeev-Popov ghosts, which get their name from their violation of the spin-statistics theorem.

Proposition 4.2.4. Let $V$ be a unitarizable Virasoro representation of central charge 24 equipped with a non-degenerate Virasoro-invariant bilinear form, let $\pi^1_{\alpha,1}$ be the Heisenberg module attached to the linear function $\alpha$ on $\mathbb{R}^{1,1}$, and let $W = V \otimes \pi^1_{\alpha,1}$. Then:

1. If $p \neq 0$, then $\text{Quant}(W)$ is isomorphic to the subspace of $V$ on which $L(0)$ acts by $1 - (p,p)$, where $(\cdot,\cdot)$ is the induced bilinear form on the dual space of $\mathbb{R}^{1,1}$.

2. If $p = 0$, then $\text{Quant}(W)$ is isomorphic to $V_0 \otimes (\pi^1_{0,1})_1 \oplus V_1 \otimes (\pi^1_{0,1})_0 \cong V_0 \oplus V_0 \oplus V_1$, where $V_1$ is the subspace of $V$ on which $L(0)$ acts by identity.

Proof. One may find a full proof of a slightly altered version of this claim in the appendix of [Jurisich-1998]. It is an expansion of the sketch of a proof for Theorem 5.1 in [Borcherds-1992], which is in turn a minor alteration of the argument in [Goddard-Thorn-1972]. In order to pass from our claim to the claim proved in Jurisich-1998, we need to make the following changes:

1. The vector $\alpha$ here is $r$ there, and we remove the restriction that it lie in the lattice $II_{1,1}$, since that restriction is never used in the proof.

2. We remove the unnecessary assumption that $V$ is a vertex operator algebra, as the proof does not use any extra structure on the vector space $V$ except the Virasoro action.

3. The space $W$ here is $H$ there.

4. Although it isn’t necessary for this paper, we remove the assumption that the weight 1 subspace $V_1$ of $V$, defined by the condition that $L(0)$ acts by identity, is zero. One should therefore ignore the paragraph concerning the case $\alpha = 0$ at the end of the proof in
Proposition 4.2.6. Let the following properties:

Definition 4.2.5. For each nonzero $g$ of type $n|h\pm$, $W_g$ is the vertex algebra $V_g \otimes^A V_{n|h\pm}$ in Definition 3.6.15

Proposition 4.2.6. Let $g \in \mathbb{M}$, and suppose $g$ has type $n|h\pm$. Then the Lie algebra $m_g$ satisfies the following properties:

1. $m_g$ is graded by $\Pi_{1,1}(n|h\pm)$, and its degree $(0,0)$ subspace is $\mathbb{C}^2$, identified with the weight one space $(\pi_0^{1,1})_1$ in the Heisenberg module.

2. For each nonzero $(a,b) \in \Pi_{1,1}(n|h\pm)$, the degree $(a,b)$ subspace is isomorphic as a $\widehat{C_M(g)}$-module to the subspace of $V^2(g^n)$ on which $g$ acts by $e^{2\pi i b/N}$ and $L(0)$ acts by $ab - 1$. Furthermore, the degree is given by the eigenvalue of the bracket with elements of the degree $(0,0)$-subspace.

3. $m_g$ has a nondegenerate invariant bilinear form, and if $u$ and $v$ are homogeneous vectors of degree $a,b \in \Pi_{1,1}(n|h\pm)$ such that $a + b \neq 0$, then $u$ and $v$ are orthogonal.

4. $m_g$ has a canonical action of $\widehat{C_M(g)}$ by homogeneous Lie algebra automorphisms that preserve the bilinear form.

Proof. We prove the claims in the order they are stated.

1. The grading property follows from the fact that $W_g$ is graded by $\Pi_{1,1}(n|h\pm)$. Because $(V^0_g)_1 = 0$ and $(V^0_g)_0 = \mathbb{C} \cdot 1$, the degree $(0,0)$ space is identified with $V_0 \otimes (\pi_0^{1,1})_1 = (\pi_0^{1,1})_1$.

2. The identification of modules follows from Proposition 4.2.4. The eigenvalue assertion follows from the definition of tensor product, together with the fact that for $u \in (\pi_0^{1,1})_1$ and $v \in \pi_0^{1,1}$, the $z^{-1}$-coefficient of $m_z(u \otimes v)$ is $\alpha(u)v$.

3. The fact that the form is nondegenerate and invariant is given in Lemma 4.2.2. The degree restriction follows from the fact that any lifts of $u$ and $v$ to elements in $W_g$ of degree $a$ and $b$ are orthogonal if $a + b \neq 0$.

4. By Theorem 3.7.10, there is a canonical homogeneous projective action of $C_M(g)$ on $V_0$, and this endows $W_g$ with a corresponding action. By the third claim of Lemma 4.2.2, homogeneous automorphisms of $W_g$ are taken to Lie algebra homomorphisms of $m_g$.

4.3. BRST cohomology. In addition to covariant quantization, there is the “modern” method of Brecchi, Rouet, and Stora (with Tyutin’s independent unpublished work often credited), where one extracts the relevant space of states as a cohomology group. This has the advantage of being “very functorial”, but the disadvantage of being rather complicated - one extracts the BRST current and the resulting differential by way of an super-variational principle. In the string-theoretic context, BRST quantization was pioneered in Kato-Ogawa-1983. One may find string-theoretic expositions of this method in chapter 3 of Green-Schwarz-Witten-1987, Chapter 4 of Polchinski-1998, and section 7 of D'Hoker-1997, the last of which is intended for a more mathematical audience.

In the mathematical world, BRST quantization of Virasoro representations was introduced using semi-infinite cohomology in Reigim-1984, and generalized in Frenkel-Garland-Zuckerman-1986.
Proofs can be found in [Lian-Zuckerman-1991]. Borchers’s use of the no-ghost theorem for moonshine was reinterpreted in terms of BRST cohomology in [Lian-Zuckerman-1995]. We will give a modified presentation that is applicable to the vertex algebra objects we have constructed in the previous section.

We briefly describe three closely related cohomology functors and some key properties. There is nothing original in this subsection except perhaps the method of presentation.

**Definition 4.3.1.** Recall from section 2.7 that $V_{\text{ghost}}$ denotes the vertex superalgebra of the odd lattice $\mathbb{Z}$, equipped with the conformal element $\omega_{\text{ghost}} = \beta_{-1}^2 + 3\beta_{-2}/2$, with central charge $-26$. Given a Virasoro (“matter”) representation $V^m$ of central charge $D$ and energy-momentum tensor $T^m = \sum_n L^n(n)z^{−n−2}$, we define the Virasoro module $C^{∞/2+}(\text{vir}, V^m) = V^m \otimes V_{\text{ghost}}$, which has central charge $D − 26$. The BRST current is defined to be $j^{BRST} = eT^m + \frac{1}{2} : eT^{bc} :$, and the $U(1)_{bc}$ current is $j^{bc} =: bc :$. We consider the following operations on $C^{∞/2+}(\text{vir}, V^m)$:

1. The BRST operator $Q = j^{BRST}_0 = \frac{1}{2\pi i} \oint dz j^{BRST}(z) = \frac{1}{2} \sum_n L^n(n)c_{−n} + \sum_{m<n}(m−n)$:

   $:b_{m+n}c_{−m}c_{−n}:$

2. The ghost number operator $U = j^{bc}_0 = \frac{1}{2}(c_0b_0−b_0c_0) + \sum_{n=1}(c_{−n}b_n−b_{−n}c_n)$. An eigenvector of $U$ with eigenvalue $k$ is said to have ghost number $k$.

When $Q^2 = 0$, we call $C^{∞/2+}(\text{vir}, V^m)$ the “BRST complex”. We also define the “relative subcomplex” ([Zuckerman-1989], section 4) to be the subspace of $V \otimes V_{\text{ghost}}$ annihilated by $b_0$.

**Remark 4.3.2.** We are using antiquated conventions for our BRST and ghost number currents and operators. That is to say, sources from the 1980s and early 1990s (e.g., [Kato-Ogawa-1983], [Frenkel-Garland-Zuckerman-1986], [Green-Schwarz-Witten-1987], [Lian-Zuckerman-1991]) use the BRST current, BRST operator, and ghost number operator that we described above, while newer work (e.g., [Polchinski-1998] section 4.3, [D’Hoker-1997] Lecture 4, [Lian-Zuckerman-1995]) adds an additional $\frac{1}{2}\partial^2c$ to the BRST current and a constant $3/2$ to the ghost number. This change impacts the output of various functors in the following ways:

1. The addition of the derivative term makes the new BRST current a “$(1,0)$-form as a quantum operator”. Concretely, this means the OPE with the total stress-energy tensor $T^m + T^{bc}$ has the form $T(z)j^{BRST}_0(0) \sim \frac{1}{z^2} j^{BRST}_0(0) + \frac{1}{2} \partial j^{BRST}_0(0)$. In the absence of the extra term, one has a pole of order 3 as the leading term. The extra simplicity makes many calculations easier, especially when quantizing in higher genus settings. However, it is irrelevant for the case at hand, since we just want to compute the cohomology spaces.

2. When we work out the space of physical states, we find that it lies in ghost degree $−1/2$, while the ghost degree in modern literature is 1. In particular, the space of interest was written $H_{BRST}^{−1/2}$ in older work, but is written $H_{BRST}^1$ now. The older convention has the convenient property that Poincaré duality gives a symmetry around zero (see [Frenkel-Garland-Zuckerman-1986] Theorem 1.6 and the remark before Theorem 2.5), while the newer convention is more manageable when considering Gerstenhaber or BV structures on cohomology ([Lian-Zuckerman-1993]).

The calculations relevant to us are unchanged, because a total derivative integrates to zero when evaluating charges. Because the advantages of the new convention are not relevant to our application, and because most of the theorems we use are proved with the old convention, we shall use the old convention. We offer the following dictionary relating [Green-Schwarz-Witten-1987], Polchinski, [Frenkel-Garland-Zuckerman-1986], and later work by Lian and Zuckerman such as [Lian-Zuckerman-1993]:
Lemma 4.3.3. Let $V^m$ be a positive-energy representation of Virasoro (called the “matter representation” in the physics literature). We have the following operator calculations in $V^m \otimes V_{\text{ghost}}$:

1. $Q^2 = 0$ if and only if $D = 26$. That is, $C^{\infty/2+\epsilon}(V^m)$ is a complex with differential $Q$ if and only if $V^m$ has central charge 26.
2. $[U, Q] = Q$, so $Q$ increases ghost number by one.
3. $[U, L(0)] = 0$, so ghost number and $L(0)$-eigenvalue provide a bigrading.

Now, suppose $V^m$ has central charge $D = 26$, so $Q$ gives $V^m \otimes V_{\text{ghost}}$ the structure of a complex of vector spaces.

1. $[Q, L(0)] = 0$, so the BRST complex and cohomology are graded by $L(0)$-eigenvalues.
2. $[Q, b_0] = L(0)$. This implies the relative subcomplex is in fact a subcomplex, and all BRST-closed states with nonzero $L(0)$-eigenvalue are BRST-exact. In particular, the cohomology is supported in weight zero.

Proof. These claims follow from standard OPE calculations. One may find them enumerated in section 4 of [Zuckerman-1989].

Lemma 4.3.4. If $V$ is a conformal vertex algebra of central charge 26, then $H^1_{\text{BRST}}(V)$ is a Lie algebra, and morphisms of conformal vertex algebras are taken to maps of Lie algebras under this functor. In particular, if a group $G$ acts on $V$ by conformal vertex algebra automorphisms, then $G$ naturally acts on $H^1_{\text{BRST}}(V)$ by Lie algebra automorphisms.

Proof. For the Lie algebra claim, see [Lian-Zuckerman-1993] Theorem 2.2. While the authors do not make the explicit claim that morphisms of conformal vertex algebras are taken to Lie algebra homomorphisms, the statement follows from the explicit formulas in their proof together with the faithfulness of the “underlying vector space” functors. The claim about group actions then follows from the functoriality.

Proposition 4.3.5. Let $\alpha \in \mathbb{R}^{1,1}$, and let $\pi^1_\alpha$ be the Heisenberg representation attached to $\alpha$. If $V$ is a unitarizable Virasoro representation of central charge 24, then $H^1_{\text{BRST}}(V \otimes \pi^1_\alpha)$ is isomorphic to the subspace $V_{1-\alpha^2}$ of $V$ on which $L(0)$ acts with eigenvalue $1 - \alpha^2$ (if $\alpha \neq 0$), and $\mathbb{C}^2 \oplus V_1$ (if $\alpha = 0$).

Proof. The case $\alpha \neq 0$ is covered in [Zuckerman-1989] Theorem 4.9 under the following translation of notation:

1. $\pi^1_\alpha$ is written $\mathcal{H}(D, p)$, with $D = 2$ and $p = \alpha$.
2. $V$ is written $\mathcal{K}$.
3. The BRST differential on $\mathcal{H}(D, p) \otimes \mathcal{K} \otimes V_{\text{ghost}}$ is written $Q_{\text{mod}}$.
4. $H^1_{\text{BRST}}(V \otimes \pi^1_\alpha)$ is written $H^u_{Q_{\text{mod}}}(p)$ for the same values of $u$, and $p = \alpha$.
5. The character $Tr(q^{L(0)}|V)$ of $\mathcal{K}$ is written $\chi(\mathcal{K}, q) = trq^{b_0}$.

The conclusion of the theorem is that $H^{-1/2}_{Q_{\text{mod}}}(p) \cong H^0_{r\text{rel}}(p, Q_{\text{mod}})$, and that dim $H^0_{r\text{rel}}(p, Q_{\text{mod}})$ is the coefficient of $q^{-p(p+1)/2}$ in the shifted character $Tr(q^{L(0)-1}|V)$. (The statement given in the paper contains an erroneous division by $\Delta$ rather than $q$ - this can be checked by comparison with the free boson case in Theorem 4.3, or using Corollary 2.27 in [Lian-Zuckerman-1991], which is proved in more detail.) This completes the case $\alpha \neq 0$. 

| Source | BRST current |
|--------|--------------|
| GSW    | $Q = \sum_{m=-\infty}^{\infty} : (L^m_0 + \frac{1}{2} L^c_0 - a \delta_m) c_m : , a = 1$ |
| P      | $Q = \frac{1}{\pi i} \int dz (cT^m_0 + \frac{1}{2} : bc \partial c : + \frac{3}{2} \partial^2 c)$ |
| FGZ    | $d = \frac{1}{2\pi i} \int dz (TV(z) + \frac{1}{2} T_h(z)) c(z) : dz$ |
| LZ     | $Q = J_0, J(z) = (LV(z) + \frac{1}{2} L^A(z)) c(z) + \frac{1}{2} \partial^2 c(z)$ |
The case \( \alpha = 0 \) follows from an explicit computation of cocycle representatives, following the example of Proposition 2.9 in [Frenkel-Garland-Zuckerman-1986]. One obtains a basis of BRST cohomology given by \( \{ x \otimes L'_{-1} \wedge L'_{-2} \wedge \cdots \} \) as \( x \) ranges over a basis of \( (\pi_{0,1}^1 \otimes V)_1 = ((\pi_{0,1}^1)_1 \otimes V_0) \oplus ((\pi_{0,1}^1)_0 \otimes V_1) \cong \mathbb{C}^2 \oplus V_1. \)

We conclude with a natural isomorphism between the two quantization functors.

Lemma 4.3.6. There is a natural isomorphism \( (W \mapsto H_{\text{BRST}}^1(W)) \Rightarrow (W \mapsto P_W^1/\text{null}^1(W)), \) as \( W \) ranges over the groupoid of Virasoro representations of the form \( V \otimes \pi_{\alpha,1}^1, \) where \( V \) is unitarizable of central charge 24, and \( \alpha \in \mathbb{R}^{1,1}. \)

Proof. An explicit natural isomorphism is given in [D’Hoker-1997] section 4.6 (which is in lecture 7 part F on the IAS web site), where the author constructs a map from \( P_W^1 \) to \( W \otimes V_{\text{ghost}}, \) and identifies the spurious states with BRST-exact vectors.

A different proof can be found in Lemma 6.7 in [Lian-Zuckerman-1995], which gives an isomorphism when \( V \) is irreducible. Since all unitarizable Virasoro representations of central charge 24 are direct sums of irreducible modules, the fact that both functors commute with direct sums yields the existence of a natural transformation. \( \square \)

4.4. Borcherds-Kac-Moody property. We discuss a class of infinite dimensional Lie algebras defined by Borcherds in [Borcherds-1988], where he called them generalized Kac-Moody algebras. They are quite similar to Kac-Moody Lie algebras, since they are defined by generators and relations encoded by a generalized Cartan matrix, and they admit representation-theoretic data like Weyl character formulas. They differ in the sense that the conditions defining a suitable generalized Cartan matrix are weaker, and most importantly, simple roots are allowed to be imaginary.

There is more than one definition of Borcherds-Kac-Moody algebra in the literature, but they are essentially equivalent, and the few claims we make will work for all of the definitions. We will say very little about the general theory, so the reader interested in an introduction should see the end of chapter 11 in [Kac-1990].

We will show that the Monstrous Lie algebras defined earlier this section are Borcherds-Kac-Moody algebras, using the characterization from [Borcherds-1995], Theorem 1.

Lemma 4.4.1. Any complex Lie algebra \( \mathfrak{g} \) satisfying the following conditions is Borcherds-Kac-Moody:

1. \( \mathfrak{g} \) admits a nonsingular invariant symmetric bilinear form, i.e., \( (x,y) = (y,x), (x,[y,z]) = ([x,y],z) \) for all \( x,y,z \in \mathfrak{g}, \) and if \( (x,y) = 0 \) for all \( y \in \mathfrak{g}, \) then \( x = 0. \)
2. \( \mathfrak{g} \) has a self-centralizing subalgebra \( H, \) called a Cartan subalgebra, such that \( \mathfrak{g} \) is the sum of eigenspaces under the action of \( H, \) and the nonzero eigenvalues, called roots, have finite multiplicity.
3. \( H \) has a totally real structure \( H_{\mathbb{R}} \subset H \) on which the bilinear form is real-valued, and the roots lie in the dual \( H_{\mathbb{R}}^*. \)
4. There exists an element \( h \in H_{\mathbb{R}}, \) such that:
   - (a) the centralizer of \( h \) in \( \mathfrak{g} \) is \( H, \)
   - (b) for any \( M \in \mathbb{R}, \) there exist at most finitely many roots \( \alpha \) such that \( |\alpha(h)| < M. \)
   This vector \( h \) is called a regular element. If \( \alpha(h) < 0 \) then we say that \( \alpha \) is negative, and if \( \alpha(h) > 0 \) then we say that \( \alpha \) is positive.
5. The norms of roots under the inner product \( (,) \) are bounded above.
6. Any two roots of non-positive norm that are both positive or both negative have inner product at most zero, and if the inner product is zero, then their root spaces commute.

Remark 4.4.2. Roots of non-positive norm are called imaginary roots, and roots of positive norm are called real roots.
Proof. This is essentially theorem 1 in [Borcherds-1995], but we remove the hypothesis that $g$ is defined over the real numbers. Borcherds’s proof still works, because the reconstruction of $g$ by recursively adding generators only uses the real structure on $H$, not on $g$.

Proposition 4.4.3. For each $g \in M$, the Lie algebra $m_g$ defined in 4.2.5 is a Borcherds-Kac-Moody Lie algebra.

Proof. It suffices to check that the conditions in Lemma 4.4.1 hold.

1. The bilinear form is induced from the duality form, as shown in Lemma 4.1.10.
2. We define $H$ to be the $(0,0)$-graded piece of $m_g$. By Proposition 4.2.6, this is isomorphic to $\mathbb{C}^2$, and is naturally identified with $I_{1,1}(n|h) \oplus \mathbb{Z}\mathbb{C}$, which has a natural real structure. Furthermore, the left multiplication by $H$ is given by the Heisenberg action on the tensor product vertex algebra, and their eigenvalues are given by degree. The fact that anything not in $H$ has nonzero eigenvalue implies the self-centralizing condition is satisfied.
3. If we identify $H$ with $L^\vee \mathfrak{h} \oplus \mathbb{Z}\mathbb{C}$, we let $h$ be any negative norm element whose inner product with any root is nonzero. A standard choice is $(2,1)$, under the identification $L^\vee \mathfrak{h} \cong h\mathbb{Z} \oplus h\mathbb{Z} + (1, \mp 1)\mathbb{Z} \subset \mathbb{R}_{1,1}$.
4. The norm of a root at $r \in I_{1,1}(n|h)$ is given by $r^2$. By Proposition 4.2.6, the root space at $r$ is identified with a subspace of $V_g$ whose vectors have $L(0)$-eigenvalue equal to $1 - r^2$. The $L(0)$-eigenvalues in $V_g$ are bounded below by zero, so the norms of roots are bounded above by $-1$.
5. Because $H$ has a real Lorentzian structure, a pair of imaginary positive roots is orthogonal if and only if the vectors are positive multiples of the same norm zero vector. If $g$ is Fricke, then there are no imaginary positive roots of norm zero. If $g$ is non-Fricke, the norm zero roots are given by elements of $L(0)$-eigenvalue 1 in $g^i$-twisted $V^\vee$-modules. By Lemma 3.7.6, any weight 1 vector in a $g^i$-twisted $V^\vee$-module is a sum of weight 1 vectors in twisted $V_{Leech}$-modules, and by Proposition 6.2.2, all $z^{-1}$ coefficients of intertwining operators on weight 1 vectors in twisted $V_{Leech}$-modules vanish. Because the Lie brackets are given by the $z^{-1}$-coefficients of intertwining operators, the root spaces commute.

The following lemmata will be useful in the next section, when we consider concrete quantitative questions. They are well-known to experts, but we couldn’t find suitably concise statements in the literature.

Lemma 4.4.4. If $g$ is a Borcherds-Kac-Moody algebra, then any real simple root has multiplicity one.

Proof. Suppose a real simple root space $\mathfrak{g}_\alpha$ has dimension more than one. Then there exists a pair of linearly independent elements in that root space whose inner product is positive. However, this contradicts a defining property of the generalized Cartan matrix, namely that the off-diagonal entries need to be non-positive.

Lemma 4.4.5. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Borcherds-Kac-Moody Lie algebras with finite dimensional Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$. Given an inner-product-preserving isomorphism $f : \mathfrak{h}_1 \to \mathfrak{h}_2$, there exists an extension to a Lie algebra isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ if and only if the root multiplicities are identical under the isometry $\mathfrak{h}_1 \to \mathfrak{h}_2$.

Proof. It is clear that an isomorphism of Lie algebras that restricts to an isometry of Cartan subalgebras preserves root multiplicities.

Now, assume $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have identical root multiplicities under the isometry $\mathfrak{h}_1 \to \mathfrak{h}_2$. The Weyl-Kac-Borcherds denominator formula implies the simple roots of a Borcherds-Kac-Moody algebra
are determined by the root multiplicities, so under a choice of matching regular elements, \( g_1 \) and \( g_2 \) have the same simple roots. This implies both \( g_1 \) and \( g_2 \) are isomorphic to the Borcherds-Kac-Moody Lie algebra given by generators and relations from the same generalized Cartan matrix. By matching the simple roots through the isometry, we obtain a Lie algebra isomorphism that extends the isometry of Cartan subalgebras.

\[ \begin{align*}
\text{Remark 4.4.6. Theorem 7.2 of } & \text{Borcherds-1992} \text{ is essentially an application of the previous lemma to identify the Monster Lie algebra } M \text{ defined by applying the no-ghost theorem to } V^2 \text{ with the abstract Lie algebra } N \text{ defined by a specification of simple roots.}
\end{align*} \]

5. First comparison theorems

5.1. Multiplicity hypotheses. In \[ \text{Dong-Li-Mason-2000}, \] the authors showed that for each element \( g \) of the monster simple group, the monster vertex algebra \( V^2(g) \) admits a unique isomorphism class of irreducible representation \( V^2(g) \) (Theorem 10.3). Furthermore, they showed that for each \( i, j \in \mathbb{Z} \), and any lift \( \tilde{g} \) of \( g \) acting on \( V^1(g^i) \), there is a constant \( C^\gamma(i, j) \in \mathbb{C}^\times \) such that

\[ Tr(\tilde{g}q^{-i-1}|V^2(g^i)) = C^\gamma(i, j)T_{g^i(v, j)} \left( \frac{\tau + s}{(\tau + t)(\tau + u)} \right) \] (Theorem 13.1). In particular, they showed that the character of identity on \( V(g) \) is a constant multiple of \( T_g(-1/\tau) \).

One may organize these constants by lifting the cyclic group generated by \( g \) to a central extension that is more amenable to treatment with modular functions.

**Definition 5.1.1.** For any \( g \in \mathbb{M} \), set \( N \) to be the level of \( T_g(\tau) \), and construct a \( \mathbb{C}[\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}] \)-valued modular function \( \hat{F} \) by setting \( \hat{F}_{i,j} = T_{g^i(v, j)} \left( \frac{s + t}{(r + s)(r + t)} \right) \) by Lemma 3.6 in \[ \text{Carnahan-2012}, \] this satisfies \( \hat{F}_{ai+cj, bi+dj}(\tau) = \hat{F}_{i,j}(\frac{a \tau + b}{c \tau + d}) \) for all \( (a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z}) \).

We introduce a new convention for twisted modules as follows: If \( g \) has type \( n|h \pm \), then for \( 0 \leq i < n \) and \( 0 \leq k < h \), we define \( V(g_{i+n}) = V(g^i) \otimes \mathbb{C}e(\pm k/h) \), where \( \mathbb{C}e(\pm k/h) \) is the one-dimensional \( \langle g \rangle \)-module on which \( g \) acts by the scalar \( e(\pm k/h) \).

For each \( i, j \in \mathbb{Z} \cap [0, N-1] \) we define the constant \( C^\gamma(i, j) \) as the constant multiple of \( \hat{F}_{i,j} \) that yields the character of the preferred lift of \( g^i \) on \( V(g^i) \).

In section 4.4 of \[ \text{Carnahan-2012}, \] we introduced three classes of hypotheses concerning the projective action of \( C_M(g) \) on \( V(g) \). We shall partially solve them as we list them.

**Hypothesis (A).** \( C^\gamma(1,0) = 1 \). That is, \( \sum_{n \in \mathbb{Q}} \dim V(g)nq^{n-1} = T_g(-1/\tau) \).

Recall that an element \( g \) in the monster is called Fricke if its McKay-Thompson series \( T_g(\tau) \) has a pole at zero. Note that this is a definition that depends on the validity of the monstrous moonshine conjectures, proved in \[ \text{Borcherds-1992}, \] The Fricke property is conjugation-invariant, since the McKay-Thompson series is a trace. The conjectural enumeration of functions in \[ \text{Conway-Norton-1979} \] (proved to be valid by Borcherds) shows that 120 of the 171 distinct McKay-Thompson series are Fricke, and 143 of the 194 conjugacy classes in the Monster are Fricke.

**Lemma 5.1.2.** If \( g \) is a Fricke element of the monster, then Hypothesis A holds. In particular, \( V^2(g) \) has character \( T_g(-1/\tau) \).

**Proof.** If \( g \) has type \( n|h \pm \), then \( m_g \) contains a nontrivial real simple root space in degree \( (1,-1) \), because the lowest weight space in \( V^2(g) \) has \( L(0) \)-eigenvalue \( 1 - \frac{1}{nh} \), and the preferred lift of \( g \) acts on vectors in that space by \( e(\frac{-1}{nh}) \). Equivalently, \( T_g(-1/\tau) \) has a pole at infinity with residue one. By Theorem 5.1.8 the simple root space is isomorphic to the lowest weight space of \( V^2(g) \), which is a vector space of dimension \( C^\gamma(1,0) \). By Lemma 4.4.4 any real simple root space must have dimension one, so \( C^\gamma(1,0) = 1 \). \( \square \)
Lemma 5.1.3. If \( g \in \mathbb{M} \) commutes with an element \( \sigma \) in class 2B, then Hypothesis \( A_g \) holds.

Proof. By Proposition 3.7.2, \( V^\natural(g) \oplus V^\natural(g\sigma) \) admits the structure of an irreducible \( \tilde{g} \)-twisted module for \( V_g \), and is the direct sum of the irreducible twisted \( V^\natural \)-modules \( V^\natural(g) \) and \( V^\natural(g\sigma) \). The irreducible twisted modules for any lattice vertex algebra satisfy strict modular compatibility with Hypothesis \( 5.1.3. \) If \( \tau \) are equal to the \( \tau \) for \( V \), we find that the character of \( V \) in \( V \) transforms of the characters of \( g \) and \( g\sigma \) on \( V^\natural \). By applying the transformation to the following chain of equalities:

\[
\begin{align*}
\text{Tr}(\tilde{g}q^{L(0)-1}|V^\natural) + \text{Tr}(\theta gq^{L(0)-1}|V^\natural) &= 2\text{Tr}(\tilde{g}q^{L(0)-1}|V^\natural) \\
&= 2\text{Tr}(gq^{L(0)-1}|V^\natural) \\
&= T_g(\tau) + T_{g\sigma}(\tau)
\end{align*}
\]

we find that the character of \( V^\natural(g) \oplus V^\natural(g\sigma) \) is equal to \( T_g(-1/\tau) + T_{g\sigma}(-1/\tau) \). This is consistent with Hypothesis \( A_g \), but we still need to separate the summands.

If \( g \) and \( \sigma g \) are conjugate, or if the characters of \( V^\natural(g) \) and \( V^\natural(g\sigma) \) are linearly independent, then the characters of the irreducible twisted modules are uniquely determined by the character of the sum. Therefore, the remaining cases to consider are those elements \( g \) such that \( \sigma g \) has the same McKay-Thompson series as \( g \), but is not conjugate to \( g \). In particular, \( \sigma \) cannot be a power of \( g \), and \( g \) must have even order. By checking the list of classes in Table 2 of \([\text{Conway-Norton-1979}]\), we find that only type 46CD is not immediately eliminated. However, the classes 46C and 46D are Fricke, so we may invoke Lemma 5.1.2 in this case.

Theorem 5.1.4. Hypothesis \( A_g \) holds for all \( g \in \mathbb{M} \).

Proof. By checking Table 2 of \([\text{Conway-Norton-1979}]\), we find that any non-Fricke element in \( \mathbb{M} \) commutes with an element of type 2B. Therefore, all cases are covered by Lemma 5.1.2 or Lemma 5.1.3.

Hypothesis \((B_g)\). \( C^g(i, j) = 1 \) for all \( i, j \in \mathbb{Z} \).

In \([\text{Carnahan-2012}]\) section Theorem 4.2, we defined for each \( g \in \mathbb{M} \) a Borcherds-Kac-Moody Lie algebra \( L_g \) (written \( W_g \) in Proposition 4.4 of that paper), whose Weyl-Kac-Borcherds denominator identity is and infinite product expansion of \( T_g(\sigma) - T_g(-1/\tau) \).

Lemma 5.1.5. Let \( g \in \mathbb{M} \). If Hypothesis \( B_g \) holds, then there exists a homogeneous Lie algebra isomorphism \( m_g \to L_g \).

Proof. By Proposition 4.4.3, \( m_g \) is a Borcherds-Kac-Moody Lie algebra whose Cartan subalgebra has a real Lorentzian structure. By Theorem 4.2 of \([\text{Carnahan-2012}]\), \( L_g \) is a Borcherds-Kac-Moody Lie algebra whose Cartan subalgebra has a real Lorentzian structure. By Lemma 4.4.5 if we are given two Borcherds-Kac-Moody Lie algebras and an inner-product-preserving isomorphism between their Cartan subalgebras, this isomorphism can be extended to an isomorphism of the full Lie algebras if and only if the root multiplicities are equal. If \( g \) has type \( n|h \), the root lattices of both Lie algebras are identified with \( II_{1,1}(n|h) \), so it suffices to compare dimensions of the homogeneous spaces of nonzero degree.

By \([\text{Carnahan-2012}]\) Proposition 4.4.4, the \((a, b/N)\) root space of \( L_g \) has dimension equal to the Fourier coefficient \( c_{a,b}(ab/N) \) of the vector-valued modular form \( F \) defined in \([\text{Carnahan-2012}]\) Corollary 3.25.

Hypothesis \( B_g \) implies that for all \( i, j \), the character of the preferred lift of \( g^i \) on \( V^\natural(g^j) \) is given by \( \hat{F}_{ij}(\tau) \), so the dimensions of eigenspaces are given by the coefficients of the vector-valued form \( F \) which is defined as the Fourier transform of \( \hat{F} \). In particular, the dimension of the eigenspace in \( V^\natural(g^a) \) on which the preferred lift of \( g \) (given in Lemma 3.6.7) acts by \( e(b/N) \) and \( L(0) \) acts by...
$1 + ab/N$ is equal to the coefficient $c_{a,b}(ab/N)$. By Proposition 4.2.6, the dimension of the $(a,b/N)$ part of $m_g$ is equal to the dimension of this eigenspace. \hfill\Box

**Lemma 5.1.6.** If $g$ has prime order, then Hypothesis $B_g$ holds.

*Proof.* For elements of type $pA$ and $pB$, this is Lemma 4.9 in [Carnahan-2012].

The remaining case is for $g$ of type $3C$, where the function $F$ is given in the first table in section 1.5 of [Carnahan-2012]. In particular, when $(9,i) = 1$, $\hat{F}_{i,j} = T_g \left( \frac{-1}{7 + ji} \right)$, for $i$ satisfying $ii \equiv 1 \pmod{9}$, and when $i = 3k$, then $\hat{F}_{i,j} = e(jk/3)T_g(\tau)$.

Recall that we define $V^2(g^{3k+i})$ to be $V(g^i) \otimes \mathbb{C}_{e(k/3)}$ for $k = 1, 2$ and $i = 0, 1, 2$. By Hypothesis $A_g$, $Tr(g^{L_0-1}|V^2(g)) = T_g(-1/\tau)$, and because the preferred lift of $g$ on $V^2(g)$ is $e(L(0))$, composition with translation yields $Tr(g^i q^{L_0-1}|V^2(g)) = T_g \left( \frac{-1}{7 + ji} \right)$. Any power of $g$ coprime to 3 is conjugate to $g$, so we can extract the preferred lift of $g$ on $V(g^i)$ from the preferred lift of $g^3$ on $V(g)$ to obtain:

$$Tr(g^i q^{L_0-1}|V^2(g^i)) = T_g \left( \frac{-1}{7 + ji} \right).$$

The remaining cases are $V^2(g^{3k})$, which are copies of $V^2 \otimes \mathbb{C}_{e(k/3)}$. As expected, $Tr(g^i q^{L_0-1}|V^2 \otimes \mathbb{C}_{e(k/3)}) = e(jk/3)T_g(\tau)$. \hfill\Box

**Lemma 5.1.7.** If $g$ is in class $4B$, then Hypothesis $B_g$ holds.

*Proof.* The element $g$ is Fricke of type $4[2+]$, so by Lemma 5.1.2 the lowest $L(0)$-weight space of $V^2(g)$ is one dimensional with eigenvalue $7/8$, and the preferred lift of $g$ acts on this one dimensional space by the scalar $e(-1/8)$. By Proposition 4.2.6 this one-dimensional space corresponds to a root of $m_g$ in degree $(1,-1)$. The root has positive norm, i.e., it is real.

By the power maps in Table 2 of [Conway-Norton-1979], $g^2$ lies in conjugacy class $2A$, which is Fricke. By Lemma 5.1.2 the lowest $L(0)$-weight space of $V^2(g^2)$ is one dimensional with eigenvalue $1/2$, and the preferred lift of $g^2$ acts on this one-dimensional space by the scalar $-1$. This implies the preferred lift of $g$ acts on this by either $e(1/4)$ or $e(3/4)$. If the preferred lift of $g$ acts by $e(3/4)$, then Proposition 4.2.6 implies $m_g$ has a real in degree $(2,-2)$. This yields a contradiction, because if $\alpha$ is a simple root of a Borcherds-Kac-Moody Lie algebras then the only simple roots collinear with $\alpha$ are $\pm \alpha$. We conclude that the preferred lift of $g$ acts on this one dimensional space as the scalar $e(1/4)$.

It remains to match components of $\hat{F}$ with the characters of powers of $g$ acting on twisted modules. Because all powers of $g$ are Fricke, it suffices to compare the polar terms of $\hat{F}_{i,j}$ with the character of $g$ acting on lowest $L(0)$-weight spaces. For $i = 0$, we have $\hat{F}_{0,1}(\tau) = T_g(\tau)$, as expected. For $i = 4$, $V^2(g^4) = V^2 \otimes \mathbb{C}_{e(1/2)}$, so $\hat{F}_{1,1}(\tau) = -T_g(\tau) = -q^{-1} + O(1)$, as expected. For $i = 1$, $\hat{F}_{1,1}(\tau) = T_g(\frac{-1}{7+1}) = e(-1/8)q^{-1/8} + O(1)$. Since the preferred lift of $g$ is by $e(L(0))$ on $V^2(g)$, this agrees as well. The same is true for $\hat{F}_{3,1}(\tau) = e(5/8)q^{-1/8} + O(1)$, $\hat{F}_{5,1}(\tau) = e(3/8)q^{-1/8} + O(1)$, and $\hat{F}_{7,1}(\tau) = e(1/8)q^{-1/8} + O(1)$. For $i = 2$, $\hat{F}_{2,1}(\tau) = T_g(\frac{7}{11+27}) = e(1/4)q^{-1/2} + O(1)$, so this agrees with the polar term determined in the previous paragraph. Finally, for $i = 6$, $\hat{F}_{6,1}(\tau) = T_g(\frac{-1}{11+27}) = e(3/4)q^{-1/2} + O(1)$, and this is equal to $-1 \cdot \hat{F}_{2,1}(\tau)$. Therefore, the polar term agrees with the character of $g$ on $V^2(g^2) \otimes \mathbb{C}_{e(1/2)}$. \hfill\Box

**Hypothesis (C_g).** There is an action of $\widehat{C_{\mathbb{M}}(g)}$ on $L_g$ by Lie algebra automorphisms, such that the action on the $(a,b)$ root space coincides with the action on the subspace of $V^2(g^a)$ on which the preferred lift of $g$ acts by $e(b/N)$ and $L(0)$ acts by $1-\lambda b/N$.

**Theorem 5.1.8.** For any $g \in \mathbb{M}$, hypothesis $B_g$ implies hypothesis $C_g$. 
Proof. This follows by transport of structure: we identify $L_g$ with $m_g$ by the isomorphism in Lemma 5.1.5. By Proposition 4.2.6, $m_g$ has an action of $\widehat{C_M(g)}$ by Lie algebra automorphisms, such that the action on root spaces coincides with the action on the corresponding homogeneous spaces of the abelian intertwining algebra $V_g$. □

5.2. Main theorem.

Theorem 5.2.1. If $g$ is a Fricke element and Hypothesis $B_g$ holds, then for any $h \in C_M(g)$ and any lift $\hat{h}$ of $h$ to a linear transformation of $V(g)$, there exists a discrete group $\Gamma_{g,h} \subset PSL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$, such that for any lift $\hat{h}$ of $h$, the power series $\sum_{n \in \mathbb{Q}} Tr(\hat{h}|V^\natural(g)_n) q^n$ is the $q$-expansion of a modular function on $\mathcal{H}$ that is invariant under $\Gamma_{g,h}$, and generates the function field of the quotient $\mathcal{H}/\Gamma_{g,h}$.

Proof. By [Carnahan-2012], Lemma 4.10, Hypothesis $C_g$ implies the positive subalgebra $E_g$ of $m_g$ is Fricke-compatible, and by Proposition 6.3 in [Carnahan-2010], if $E_g$ is Fricke-compatible, then the power series $\sum_{n \in \mathbb{Q}} Tr(\hat{h}|V^\natural(g)_n) q^n$ is the $q$-expansion of a Hauptmodul.

By Theorem 5.1.8, Hypothesis $B_g$ implies Hypothesis $C_g$. □

Corollary 5.2.2. If $g$ is an element of the monster in class $pA$ for any prime $p$, or $3C$ or $4B$, then for all $h$ commuting with $g$, the characters of lifts of $h$ on $V^\natural(g)$ are Hauptmoduln. That is, there exists a discrete group $\Gamma_{g,h} \subset PSL_2(\mathbb{R})$ containing some $\Gamma(N)$, such that for any lift $\hat{h}$ of $h$, the power series $\sum_{n \in \mathbb{Q}} Tr(\hat{h}|V^\natural(g)_n) q^n$ is the $q$-expansion of a modular function on $\mathcal{H}$ that is invariant under $\Gamma_{g,h}$, and generates the function field of the quotient $\mathcal{H}/\Gamma_{g,h}$.

Proof. By Corollary 5.2.1, the conclusion of the theorem follows when Hypothesis $B_g$ holds. By Lemmata 5.1.6 and 5.1.7, the conjugacy classes listed satisfy Hypothesis $B_g$. □

Remark 5.2.3. Hypothesis $B_g$ seems to be rather tricky for general $g$ of composite order. The problem is that we only have concrete information about the preferred lift of $g^i$ on $V(g^i)$, when we need to know the precise preferred lift of $g$. When some $g^i$ is Fricke, whence the action is determined by a single character on the one-dimensional lowest-weight space, Hypothesis $B_g$ amounts to a root-of-unity ambiguity, but the ambiguity is rather well-embedded in the structure of intertwining operators.

There should be a strict modular invariance result that holds for vector-valued characters, but it seems to be beyond reach at the moment. Such a result would allow us to avoid the use of special properties of the functor $Quant$ together with explicit properties of lattices to prove Hypothesis $A_g$. I think I know a way to solve the cases of Hypothesis $B_g$ I need by appealing to explicit structures in $V_{Leech}$, but the work is in a very preliminary state.

6. This section will be deleted in version 2

Because [Carnahan-2012a] is not yet completely written, we summarize the results that we use in this paper. Naturally, the cautious reader should feel free to treat all theorems in this paper that depend on these claims as conditional results.

6.1. Equivariant intertwining operators.

Proposition 6.1.1. Let $V$ be a holomorphic $C_2$-cofinite vertex operator algebra, and let $G$ abelian group of order $n \in \mathbb{Z}_{>0}$ whose elements are automorphisms of $V$.

1. For any $g_1, g_2 \in G$, the space $V_{g_1}^{g_2}$ of intertwining operators $V(g_1) \otimes V(g_2) \to V(g_1 + g_2)((z^{1/n}))$ between the irreducible twisted $V$-modules is one-dimensional, and its elements are projectively $C_{Aut V}(g_1, g_2)$-equivariant.
Proposition 6.1.2. Let $V$ be a holomorphic $C_2$-cofinite vertex operator algebra, and let $g$ be an automorphism of $V$ of order $n$.

1. For each $i \in \mathbb{Z}/n\mathbb{Z}$, there is a canonical $\mu_i(\mathbb{C})$-torsor $P_i$, such that the irreducible $g^i$-twisted module $V(g^i)$ canonically decomposes into a sum $\bigoplus_{j \in P_i} M_i,j$ of $g^j$-modules, such that the lift $\hat{g}$ of the projective action of $g$ to order $n$ linear transformations on $V(g^i)$ are in canonical bijection with the trivializations of $P_i$. Specifically, if $\phi : P_i \rightarrow \mu_n(\mathbb{C})$ is a trivialization, then the corresponding lift $\hat{\phi}$ acts on $M_i,j$ by the $n$-th root of unity $\phi(j)$.

2. For any $i_1, i_2 \in \mathbb{Z}/n\mathbb{Z}$, and any $j_1 \in P_{i_1}$ and $j_2 \in P_{i_2}$, there exists a unique $j_3 \in P_{i_1+i_2}$ such that the restriction of any nonzero intertwining operator $V(g^{i_1}) \otimes V(g^{i_2}) \rightarrow V(g^{i_1+i_2})((z^{1/n}))$ to $M_{i_1,j_1} \otimes M_{i_2,j_2} \rightarrow M_{i_1,i_2,j_3}((z^{1/n}))$ is nonzero.

3. The $\mu_n$-torsor $P_1$ admits a distinguished identification with the set of $L(0)$-eigenvalues of $V(g)$ modulo $\mathbb{Z}$. That is, exponentiating yields a $\mu_n$-equivariant injective map $\psi : P_1 \rightarrow \mu_n(\mathbb{C}) \subset \mathbb{C}^\times$, such that for each $j \in P_1$, any element of $M_{1,j}$ is an eigenvector of the operator $e(L(0))$ with eigenvalue $\psi(j)$.

4. For each $i \in \mathbb{Z}/n\mathbb{Z}$, $V(g^i)$ admits a distinguished lift of $g^i$ to a linear transformation, given by $e(L(0))$.

5. Any intertwining operator $m_{z_1,\ldots,z_k}$ with $k$ inputs from twisted modules $V(g^{i_1}),\ldots,V(g^{i_k})$ satisfies the following equivariance condition: If there are integers $a_1,\ldots,a_k$ such that $a_1i_1 + \cdots + a_ki_k \equiv \sum_{j=1}^k a_ji_j \pmod{n}$, then for any $v_1 \in V(g^{i_1}),\ldots,v_k \in V(g^{i_k})$,

\[ m_z(e(a_1L(0))v_1 \otimes \cdots \otimes e(a_kL(0))v_k) = e(\sum_{j=1}^k a_jL(0))m_z(v_1 \otimes \cdots \otimes v_k). \]

6. Given an intertwining operator $m_{z_1,\ldots,z_n}$ with $n$ inputs from $V(g)$, and given $n$ elements $v_1,\ldots,v_n \in M_{1,j}$ for any fixed $j \in P_1$, the coefficients of $m_{z_1,\ldots,z_n}(v_1 \otimes \cdots \otimes v_n)$ lie in the subspace of $V \otimes \mathbb{C}_{\psi(j)^n}$ on which $g$ acts by $\psi(j)^n$ in $\mu_n(\mathbb{C})$. Here, $V \otimes \mathbb{C}_{\psi(j)^n}$ is isomorphic to $V$ as a $V$-module, but the action of $(g)$ is twisted by the character that takes $g$ to $\psi(j)^n$. 
6.2. Intertwining operators for lattice twisted modules.

Proposition 6.2.1. Let \( L \) be a positive definite even unimodular lattice, and let \( V_L \) be the vertex algebra canonically attached to the double cover of \( L \). If \( g \) is a finite order automorphism of \( V_L \), then for any \( L(0) \)-eigenvector \( u \in V_L(g) \) and any nonzero intertwining operator \( I_z : V_L(g) \otimes V_L(g) \to V_L(g^2) \), we have \( I_z(u \otimes u) \in z^sV_L(g^2)[[z]] \setminus z^{s+1}V_L(g^2)[[z]] \) for some \( s \in \mathbb{Q} \) satisfying \( s/2 \equiv k \) (mod \( \mathbb{Z} \)), where \( L(0)u = ku \).

Proposition 6.2.2. Let \( L \) be a positive definite even unimodular lattice, let \( V_L \) be the vertex algebra canonically attached to the double cover of \( L \), and let \( g \) be a finite order automorphism of \( V_L \). Then for any \( i, j \in \mathbb{Z}/|g|\mathbb{Z} \), any \( u \in V_L(g^i) \) and \( v \in V_L(g^j) \) satisfying \( L(0)u = u \) and \( L(0)v = v \), and any intertwining operator \( m_z : V_L(g^i) \otimes V_L(g^j) \to V_L(g^{i+j})(z^{1/|g|}) \), the \( z^{-1} \)-term of \( m_z(u \otimes v) \) is zero.

References

[Baez-1992] J. Baez, R-commutative geometry and quantization of Poisson algebras Adv. Math. 95 (1992), 61–91.

[Bakalov-Kac-2004] B. Bakalov, V. Kac, Twisted modules over lattice vertex algebras ArXiv preprint, http://arxiv.org/abs/0402315

[Bakalov-Kac-2006] B. Bakalov, V. Kac, Generalized vertex algebras ArXiv preprint, http://arxiv.org/abs/math.QA/0602072

[Borcherds-Conway-Queen-Sloane-1984] R. E. Borcherds, J. H. Conway, L. Queen, N. J. A. Sloane, A monster Lie algebra? Adv. in Math. 53 (1984) 75–79.

[Borcherds-1986] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster Proc. Nat. Acad. Sci. USA, 83 no. 10 (1986) 3068–3071.

[Borcherds-1988] R. Borcherds, Generalized Kac-Moody algebras J. Algebra, 115 no. 2 (1988) 501–512.

[Borcherds-1990] R. Borcherds, The monster Lie algebra Adv. Math. 83 no. 1 (1990) 30–47.

[Borcherds-1992] R. Borcherds, Monstrous moonshine and monstrous Lie superalgebras Invent. Math. 109 (1992), 405–444.

[Borcherds-1995] R. Borcherds, A characterization of generalized Kac-Moody algebras J. Algebra, 174 no. 3 (1995) 1073-1079.

[Borcherds-2001] R. Borcherds, Quantum vertex algebras Taniguchi Conference on Mathematics Nara ’98, 51–74, Adv. Stud. Pure Math., 31, Math. Soc. Japan, Tokyo, (2001).

[Brower-1972] R. Brower, Spectrum generating algebra and the no-ghost theorem for the dual model Phys. Rev. D6 (1972) 1655–1662.

[Carnahan-2010] S. Carnahan, Generalized Moonshine I: Genus zero functions Algebra and Number Theory 4 no.6 (2010) 649–679. ArXiv preprint http://arxiv.org/abs/0812.3440

[Carnahan-2012] S. Carnahan, Generalized Moonshine II: Borcherds products Duke Math J. 161 no. 5 (2012) 893–950. ArXiv preprint http://arxiv.org/abs/0908.4223

[Carnahan≥2012a] S. Carnahan, Generalized Moonshine III: Equivariant intertwining operators In preparation.

[Conway-Norton-1979] J. Conway, S. Norton, Monstrous Moonshine Bull. Lond. Math. Soc. 11 (1979) 308–339.

[D’Hoker-1997] E. D’Hoker, String theory Quantum Fields and Strings: a course for mathematicians, volume 2, ed. Deligne, et al, AMS, Providence, RI (1999) 807–1012.

[Dixon-Ginsparg-Harvey-1988] L. Dixon, P. Ginsparg, J. Harvey, Beauty and the Beast: Superconformal Symmetry in a Monster Module Commun. Math. Phys. 119 (1988) 221–241.

[Dong-1994] C. Dong, Representations of the moonshine module vertex operator algebra Contemporary Math. 175 (1994) 27–36.

[Dong-Lepowsky-1993] C. Dong, J. Lepowsky, Generalized vertex algebras and relative vertex operators Progress in Mathematics 112 Birkhuser Boston, Inc., Boston, MA, (1993).
C. Dong, H. Li, G. Mason, Simple currents and extensions of vertex operator algebras Comm. Math. Phys. 180 no. 3 (1996) 671–707.

C. Dong, H. Li, G. Mason, Modular invariance of trace functions in orbifold theory Comm. Math. Phys. 214 (2000) no. 1, 1–56. ArXiv preprint http://arxiv.org/abs/q-alg/9703016.

N. Durov, A new approach to Arakelov geometry ArXiv preprint http://arxiv.org/abs/0704.2030.

S. Eilenberg, Homotopy groups and algebraic homology theories Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R.I., (1952) 350–353.

S. Eilenberg, S. Mac Lane, On the groups $H(\pi, n)$, I Ann. Math. (2) 58 no. 1 (1953), 55–106.

S. Eilenberg, S. Mac Lane, On the groups $H(\pi, n)$, II, Methods of computation Ann. Math. (2) 60 no. 1 (1954), 49–139.

B. L. Feigin, The semi-infinite homology of Kac-Moody and Virasoro Lie algebras Uspekhi Mat. Nauk, 39:2 (236) (1984), 195–196. Available at http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=rm&paperid=2354.

I. Frenkel, Representations of Kac-Moody algebras and dual resonance models Applications of group theory in theoretical physics, Lect. Appl. Math. 21 A.M.S. (1985) 325–353.

E. Frenkel, D. Ben-Zvi, Vertex algebras and Algebraic Curves Mathematical Surveys and Monographs 88 American Mathematical Society, Providence, RI, (2004).

I. Frenkel, H. Garland, G. Zuckerman, Semi-infinite cohomology and string theory Proc. Nat. Acad. Sci. USA, 83 (1986) 8442–8446.

I. Frenkel, Y. Huang, J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules Mem. Amer. Math. Soc. 104 (1993) no. 494.

I. Frenkel, J. Lepowsky, A. Meurman, Vertex operator algebras and the Monster Pure and Applied Mathematics 134 Academic Press, Inc., Boston, MA, (1988).

P. Goddard, C. Thorn, Compatibility of the dual pomeron with unitarity and the absence of ghosts in the dual resonance model Physics Letters 40B no. 2, (1972) 235–238.

M. Green, J. Schwarz, E. Witten, Superstring theory Cambridge University Press, Cambridge, UK (1987).

G. Höhn, Genera of vertex operator algebras and three dimensional topological quantum field theories Vertex operator algebras in mathematics and physics (Toronto, ON 2000), 89–107, Fields Inst. Commun., 39 Amer. Math. Soc., Providence, RI, 2003. ArXiv preprint http://arxiv.org/abs/math/0209333.

G. Höhn, Generalized moonshine for the baby monster Preprint (2003)

Y. Huang, Conformal-field-theoretic analogues of codes and lattices Kac-Moody Lie Algebras and Related Topics, Proc. Ramanujan International Symposium on Kac-Moody Lie algebras and applications, ed. N. Sthanumoorthy and K. C. Misra, Contemp. Math., Vol. 343 Amer. Math. Soc., Providence, RI, 2004, 131–145.

Y. Huang, A non-meromorphic extension of the moonshine module vertex operator algebra Contemporary Math., 193 AMS (1996), 123–148.

Y. Huang, J. Lepowsky, L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra ArXiv preprint http://arxiv.org/abs/0710.2687.
[Jurisich-1998] E. Jurisich, *Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the monster Lie algebra* J. Pure and Appl. Alg. **126** (1998) 233–266.

[Jurisich-2009] E. Jurisich, *Borcherds’ proof of the Conway-Norton conjecture* Arxiv preprint: [http://arxiv.org/abs/0903.4456](http://arxiv.org/abs/0903.4456)

[Joyal-Street-1986] A. Joyal, R. Street, *Braided tensor categories* Macquarie Mathematics report no. 860061, available at [http://rutherglen.ics.mq.edu.au/~street/JS86.pdf](http://rutherglen.ics.mq.edu.au/~street/JS86.pdf)

[Kac-1978] V. Kac, *Highest weight representations of infinite dimensional Lie algebras* Proc. ICM 1978, Helsinki 299–304.

[Kac-1990] V. Kac, *Infinite Dimensional Lie Algebras* third edition, Cambridge University Press, Cambridge (1990)

[Kac-1997] V. Kac, *Vertex Algebras for Beginners* University Lecture Series **10**, Amer. Math. Soc., Providence, RI (1997)

[Kac-Peterson-1985] V. Kac, D. Peterson, *112 constructions of the basic representation of the loop group of $E_8$* Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), World Sci. Publishing, Singapore (1985) 276–298.

[Kac-Raina-1987] V. Kac, A. Raina, *Bombay Lectures on highest weight representations of infinite dimensional Lie algebras* Advanced Series in Mathematical Physics **2**, World Scientific, Singapore, (1987).

[Kato-Ogawa-1983] M. Kato, K. Ogawa, *Covariant quantization of string based on BRS invariance* Nucl. Phys. B212 (1983) 443–460.

[Lam-Yamauchi-2010] C. Lam, H. Yamauchi, *On the Structure of Framed Vertex Operator Algebras and Their Pointwise Frame Stabilizers* Comm. Math. Phys. **277** no. 1 (2008) 237–285.

[Lian-Zuckerman-1991] B. Lian, G. Zuckerman, *BRST Cohomology and highest weight vectors I* Comm. Math. Phys. **135** (1991) 547–580.

[Lian-Zuckerman-1993] B. Lian, G. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory* Comm. Math. Phys. **154** (1993) 613–646. Available as [http://arxiv.org/abs/hep-th/9211072](http://arxiv.org/abs/hep-th/9211072)

[Lian-Zuckerman-1995] B. Lian, G. Zuckerman, *Moonshine Cohomology* RIMS Proceedings on Vertex Operator Algebras and the Monster (1995). Available as [http://arxiv.org/abs/q-alg/9501025](http://arxiv.org/abs/q-alg/9501025)

[Lovelace-1971] C. Lovelace *Pomeron form factors and dual regge cuts* Phys. Lett. **34B** (1971) 500–506.

[MacLane-1952] S. Mac Lane, *Cohomology theory of abelian groups* Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R.I., (1952) 8–14.

[Mossberg-1994] G. Mossberg, *Axiomatic Vertex Algebras and the Jacobi Identity* J. Algebra, **170** (1994) 956–1010.

[N87] S. Norton, *Generalized moonshine* Proc. Sympos. Pure Math. **47** Part 1, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 209–210, Amer. Math. Soc., Providence, RI (1987).

[N01] S. Norton, *From Moonshine to the Monster* Proceedings on Moonshine and related topics (Montréal, QC, 1999), 163–171, CRM Proc. Lecture Notes **30**, Amer. Math. Soc., Providence, RI (2001).

[Polchinski-1998] J. Polchinski, *String Theory* Cambridge University Press, Cambridge, UK (1998).

[Q81] L. Queen, *Modular Functions arising from some finite groups* Mathematics of Computation **37** (1981) No. 156, 547–580.

[Scheithauer-1998] N. Scheithauer, *Vertex Algebras, Lie Algebras, and Superstrings* J. Algebra **200** (1998) 363–403.

[Scheithauer-2000] N. Scheithauer, *The Fake Monster Superalgebra* Adv. Math. **151** (2000) 226–269.

[Xu-1995] X. Xu, *Intertwining operators for twisted module for a colored vertex superalgebra* J. Algebra **175** (1995) 241–273.

[Zuckerman-1989] G. Zuckerman, *Modular forms, strings, and ghosts* Proceedings of Symposia in Pure Mathematics, **49** I (1989) 273–284.