A NOTE ON THE EIGHTFOLD WAY

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Abstract. Assuming the existence of a Mahlo cardinal, we construct a model in which there exists an \(\omega_2\)-Aronszajn tree, the \(\omega_1\)-approachability property fails, and every stationary subset of \(\omega_2 \cap \text{cof}(\omega)\) reflects. This solves an open problem of [1].

Cummings, Friedman, Magidor, Rinot, and Sinapova [1] proved the consistency of any logical Boolean combination of the statements which assert the \(\omega_1\)-approachability property, the tree property on \(\omega_2\), and stationary reflection at \(\omega_2\). For most of these combinations, they assumed the existence of a weakly compact cardinal in order to construct the desired model. This is a natural assumption to make, since the \(\omega_2\)-tree property implies that \(\omega_2\) is weakly compact in \(L\). On the other hand, Harrington and Shelah [4] proved that stationary reflection at \(\omega_2\) is equiconsistent with the existence of a Mahlo cardinal. Cummings et al. [1] asked whether a Mahlo cardinal is sufficient to prove the consistency of the existence of an \(\omega_2\)-Aronszajn tree, the failure of the \(\omega_1\)-approachability property, and stationary reflection at \(\omega_2\). In this article we answer this question in the affirmative.

We begin by reviewing the relevant definitions and facts. We refer the reader to [1] for a more detailed discussion of these ideas and their history. A stationary set \(S \subseteq \omega_2 \cap \text{cof}(\omega)\) is said to reflect at an ordinal \(\beta \in \omega_2 \cap \text{cof}(\omega_1)\) if \(S \cap \beta\) is a stationary subset of \(\beta\). If \(S\) does not reflect at any such ordinal, \(S\) is non-reflecting.

We say that stationary reflection holds at \(\omega_2\) if every stationary subset of \(\omega_2 \cap \text{cof}(\omega)\) reflects to some ordinal in \(\omega_2 \cap \text{cof}(\omega_1)\).

An \(\omega_2\)-Aronszajn tree is a tree of height \(\omega_2\), whose levels have size less than \(\omega_2\), and which has no cofinal branches. The \(\omega_2\)-tree property is the statement that there does not exist an \(\omega_2\)-Aronszajn tree. A well-known fact is that if the \(\omega_2\)-tree property holds, then \(\omega_2\) is a weakly compact cardinal in \(L\). Therefore, if one starts with a Mahlo cardinal \(\kappa\) which is not weakly compact in \(L\) (for example, if \(\kappa\) is the least Mahlo cardinal in \(L\)), then in any subsequent forcing extension in which \(\kappa\) equals \(\omega_2\), there exists an \(\omega_2\)-Aronszajn tree.

The \(\omega_1\)-approachability property is the statement that there exists a sequence \(\bar{a} = \langle a_i : i < \omega_2 \rangle\) of countable subsets of \(\omega_2\) and a club \(C \subseteq \omega_2\) such that for all limit ordinals \(\alpha \in C\), \(\alpha\) is approachable by \(\bar{a}\) in the following sense: there exists a cofinal set \(c \subseteq \alpha\) with order type equal to \(\text{cf}(\alpha)\) such that for all \(\beta < \alpha\), \(c \cap \beta\) is a member of \(\{a_i : i < \alpha\}\). Essentially, this property is a very weak form of the square principle \(\square_{\omega_1}\). The failure of the \(\omega_1\)-approachability property is known to

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hold in Mitchell’s model [6] in which there does not exist a special ω2-Aronszajn tree, which he constructed using a Mahlo cardinal.

A solution to the problem of [11] addressed in this article was originally discovered by the first author, using a mixed support forcing iteration similar to the forcings appearing in [11] and [2]. Later, the second author found a different proof using the idea of a disjoint stationary sequence. The latter proof is somewhat easier, since it avoids the technicalities of mixed support iterations, and also can be easily adapted to arbitrarily large continuum. In this article we present the second proof.

In Section 1, we discuss the idea of a disjoint stationary sequence, which was originally introduced by the second author in [5]. In Section 2, we prove the main result of the paper. In Section 3, we adapt our model to arbitrarily large continuum using an argument of I. Neeman, which we include with his kind permission.

1. DISJOINT STATIONARY SEQUENCES

Recall that for an uncountable ordinal α ∈ ω2, Pω1(α) denotes the set of all countable subsets of α. A set c ⊆ Pω1(α) is a club if it is cofinal in Pω1(α) and closed under unions of countable increasing sequences. A set s ⊆ Pω1(α) is stationary if it has non-empty intersection with every club in Pω1(α).

Let α be an uncountable ordinal in ω2. Fix an increasing and continuous sequence ⟨b_i : i < ω1⟩ of countable sets with union equal to α (for example, fix a bijection f : ω1 → α and let b_i := f[i]). Note that the set {b_i : i < ω1} is club in Pω1(α).

A set s ⊆ Pω1(α) is stationary in Pω1(α) iff the set x := {i < ω1 : b_i ∈ s} is a stationary subset of ω1. Indeed, if C ⊆ ω1 is a club which is disjoint from x, then the set {b_i : i ∈ C} is a club subset of Pω1(α) which is obviously disjoint from s. On the other hand, if c ⊆ Pω1(α) is a club which is disjoint from s, then the set {i < ω1 : b_i ∈ c} is a club in ω1, and this club is clearly disjoint from x.

Definition 1.1. A disjoint stationary sequence on ω2 is a sequence ⟨s_α : α ∈ S⟩, where S is a stationary subset of ω2 ∩ cof(ω1), satisfying:

1. for all α ∈ S, s_α is a stationary subset of Pω1(α);
2. for all α < β in S, s_α ∩ s_β = ∅.

As we will show below, the existence of a disjoint stationary sequence ⟨s_α : α ∈ S⟩ on ω2 implies the failure of the ω1-approachability property (more specifically, that the set S is not in the approachability ideal I[ω2]). In our main result, the failure of the ω1-approachability property will follow from the existence of a disjoint stationary sequence.

One of the advantages of disjoint stationary sequences over other methods for obtaining the failure of approachability, such as using the ω1-approximation property, is their upward absoluteness.

Lemma 1.2. Suppose that ⟨s_α : α ∈ S⟩ is a disjoint stationary sequence. Let P be a forcing poset which preserves ω1 and ω2, preserves the stationarity of S, and preserves stationary subsets of ω1. Then P forces that ⟨s_α : α ∈ S⟩ is a disjoint stationary sequence.

Proof. For each α ∈ S, fix an increasing, continuous, and cofinal sequence ⟨b_α,i : i < ω1⟩ in Pω1(α). Observe that by upward absoluteness and the fact that ω1 is preserved, these sequences have the same property in V_P. Define x_α := {i < ω1 : b_α,i ∈ s_α}. Then x_α is a stationary subset of ω1. By the assumptions on P, each x_α
remains a stationary subset of $\omega_1$ in $V^P$, and hence by the discussion above, each $s_\alpha$ is still stationary in $P_{s_\alpha}(\alpha)$ in $V^P$. Obviously, in $V^P$ we have that $s_\alpha \cap s_\beta = \emptyset$ for all $\alpha < \beta$ in $S$, since this is true in $V$. Also, $S$ remains stationary in $\omega_2$ by the assumptions on $P$, and since $\omega_1$ is preserved, $S$ is still a subset of $\omega_2 \cap \text{cof}(\omega_1)$.

**Corollary 1.3.** Assume that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence. Let $P$ be a forcing poset which is either c.c.c., or $\omega_2$-distributive and preserves the stationarity of $S$. Then $P$ forces that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence.

The next result describes a well-known consequence of approachability; we include a proof for completeness.

**Proposition 1.4.** Assume that the $\omega_1$-approachability property holds. Then for any stationary set $S \subseteq \omega_2 \cap \text{cof}(\omega_1)$, there exists an $\omega_2$-distributive forcing which adds a club subset of $S \cup (\omega_2 \cap \text{cof}(\omega))$.

**Proof.** Fix a sequence $\vec{a} = \langle a_i : i < \omega_2 \rangle$ of countable subsets of $\omega_2$ and a club $C \subseteq \omega_2$ such that for all limit ordinals $\alpha \in C$, there exists a set $e \subseteq \alpha$ which is cofinal in $\alpha$, has order type $\text{cf}(\alpha)$, and for all $\beta < \alpha$, $e \cap \beta \in \{a_i : i < \alpha\}$.

Define $P$ as the forcing poset consisting of all closed and bounded subsets of $S \cup (\omega_2 \cap \text{cof}(\omega))$, ordered by end-extension. We will show that $P$ is $\omega_2$-distributive. Observe that if $c, \gamma < \omega_2$, then there is $d \leq c$ with $\sup(d) \geq \gamma$ (for example, $d := c \cup \min(S \setminus \max(\{\sup(c), \gamma\}))$). Using this, a straightforward argument shows that, if $P$ is $\omega_2$-distributive, then $P$ adds a club subset of $S \cup (\omega_2 \cap \text{cof}(\omega))$.

To show that $P$ is $\omega_2$-distributive, fix $c \in P$ and a family $\{D_i : i < \omega_1\}$ of dense open subsets of $P$. We will find $d \leq c$ in $\bigcap\{D_i : i < \omega_1\}$.

Fix a regular cardinal $\theta$ large enough so that all of the objects mentioned so far are members of $H(\theta)$. Fix a well-ordering $\leq$ of $H(\theta)$. Since $S$ is stationary, we can find an elementary substructure $N$ of $(H(\theta), \in, \leq)$ such that $\vec{a}, C, S, P, c$, and $\langle D_i : i < \omega_1 \rangle$ are members of $N$ and $\alpha := N \cap \omega_2 \in S$. In particular, $\alpha \in C \cap \text{cof}(\omega_1)$. Fix a cofinal set $e \subseteq \alpha$ with order type $\omega_1$ such that for all $\beta < \alpha$, $e \cap \beta \in \{a_i : i < \alpha\}$. Enumerate $e$ in increasing order as $\langle \gamma_i : i < \omega_1 \rangle$. Note that since $\langle a_i : i < \alpha \rangle$ is a subset of $N$ by elementarity, for all $\beta < \alpha$, $e \cap \beta \in N$. Consequently, for each $\delta < \omega_1$, the sequence $\langle \gamma_i : i < \delta \rangle$ is a member of $N$.

We define by induction a strictly descending sequence of conditions $\langle c_i : i < \omega_1 \rangle$, starting with $c_0 := c$, together with some auxiliary objects. We will maintain that for each $\delta < \omega_1$, the sequence $\langle c_i : i < \delta \rangle$ is definable in $H(\theta)$ from parameters in $N$, and hence is a member of $N$.

Given a limit ordinal $\delta < \omega_1$, assuming that $c_i$ is defined for all $i < \delta$, we define $c_{\delta, 0}$ to be equal to $\bigcup\{c_i : i < \delta\}$. Then clearly $\sup(c_{\delta, 0})$ is an ordinal of cofinality $\omega$. Hence, $c_\xi := c_{\delta, 0} \cup \{\sup(c_{\delta, 0})\}$ is a condition and is a strict end-extension of $c_i$ for all $i < \delta$. Now assume that $\xi < \omega_1$ and $c_i$ is defined for all $i < \xi$. Let $c_{\xi, 0}$ be the $\leq$-least strict end-extension of $c_\xi$ such that $\max(c_{\xi, 0}) \geq \gamma_\xi$. Now let $c_{\xi + 1}$ be the $\leq$-least condition in $D_\xi$ which is below $c_{\xi, 0}$. This completes the construction. Define $d_0 := \bigcup\{c_i : i < \omega_1\}$.

Reviewing the inductive definition of the sequence $\langle c_i : i < \omega_1 \rangle$, we see that for all $\delta < \omega_1$, $\langle c_i : i < \delta \rangle$ is definable in $H(\theta)$ from parameters in $N$, including specifically the sequence $\langle \gamma_i : i < \delta \rangle$. Therefore, each $c_i$ is in $N$. In addition, for each $i < \omega_1$, $\max(c_{i+1}) \geq \gamma_i$. Since $\{\gamma_i : i < \omega_1\}$ is cofinal in $\alpha$, $\sup(d_0) = \alpha$. Let $d := d_0 \cup \{\alpha\}$. Then $d$ is a condition since $\alpha \in S$, and $d \leq c_i$ for all $i < \omega_1$, and in particular, $d \leq c$. For each $i < \omega_1$, $c_{i+1} \in D_i$, so $d \in D_i$. \[\square\]
Proposition 1.5. Suppose that \( \langle s_\alpha : \alpha \in S \rangle \) is a disjoint stationary sequence. Then \((\omega_2 \cap \text{cof}(\omega_1)) \setminus S\) is stationary.

Proof. Let \( C \) be club in \( \omega_2 \). By induction, it is easy to define an increasing and continuous sequence \( \langle N_i : i < \omega_1 \rangle \) satisfying:

1. each \( N_i \) is a countable elementary substructure of \( H(\omega_3) \) containing the objects \( \langle s_\alpha : \alpha \in S \rangle \) and \( C \);
2. for each \( i < \omega_1 \), \( N_i \in N_{i+1} \).

Let \( N := \bigcup \{ N_i : i < \omega_1 \} \). Then by elementarity, \( \omega_1 \subseteq N \) and \( \beta := N \cap \omega_2 \) has cofinality \( \omega_1 \) and is in \( C \).

We claim that \( \beta \notin S \), which completes the proof. Suppose for a contradiction that \( \beta \in S \). Then \( \beta \) is defined and is a stationary subset of \( P_{\omega_1}(\beta) \). On the other hand, \( \langle N_i \cap \omega_2 : i < \omega_1 \rangle \) is a club subset of \( P_{\omega_1}(\beta) \). So we can fix \( i < \omega_1 \) such that \( N_i \cap \omega_2 \in s_\beta \).

Now the sequence \( \langle s_\alpha : \alpha \in S \rangle \) is a member of \( N \), and also \( N_i \cap \omega_2 \in N \cap s_\beta \). So by elementarity, there exists \( \alpha \in N \cap S \) such that \( N_i \cap \omega_2 \in s_\alpha \). Then \( \alpha \in N \cap \omega_2 = \beta \), so \( \alpha < \beta \). Thus, we have that \( N_i \cap \omega_2 \) is a member of both \( s_\alpha \) and \( s_\beta \), which contradicts that \( s_\alpha \cap s_\beta = \emptyset \).

Corollary 1.6. Assume that there exists a disjoint stationary sequence on \( \omega_2 \). Then the \( \omega_1 \)-approachability property fails.

Proof. Suppose for a contradiction that \( \langle s_\alpha : \alpha \in S \rangle \) is a disjoint stationary sequence and the \( \omega_1 \)-approachability property holds. By Proposition 1.4, fix an \( \omega_2 \)-distributive forcing \( \mathbb{P} \) which adds a club subset of \( S \cup (\omega_2 \cap \text{cof}(\omega)) \). In particular, \( \mathbb{P} \) forces that \( (\omega_2 \cap \text{cof}(\omega)) \setminus S \) is non-stationary in \( \omega_2 \). By Proposition 1.5, the sequence \( \langle s_\alpha : \alpha \in S \rangle \) is not a disjoint stationary sequence in \( V^\mathbb{P} \).

Now \( \mathbb{P} \) is \( \omega_2 \)-distributive, and it preserves the stationarity of \( S \) because it adds a club subset of \( S \cup (\omega_2 \cap \text{cof}(\omega)) \). By Corollary 1.3, \( \langle s_\alpha : \alpha \in S \rangle \) is a disjoint stationary sequence in \( V^\mathbb{P} \), which is a contradiction. \( \square \)

2. The main result

Assume for the rest of the section that \( \kappa \) is a Mahlo cardinal. Without loss of generality, we may also assume that \( 2^\kappa = \kappa^+ \), since this can be forced while preserving Mahloness. Define \( S \) as the set of inaccessible cardinals below \( \kappa \).

We will define a two-step forcing iteration \( \mathbb{P} * \mathbb{A} \) with the following properties. The forcing \( \mathbb{P} \) collapses \( \kappa \) to become \( \omega_2 \) and adds a disjoint stationary sequence on \( S \). In \( V^\mathbb{P} \), \( \mathbb{A} \) is an iteration for destroying the stationarity of non-reflecting subsets of \( \kappa \cap \text{cof}(\omega) \). The forcing \( \mathbb{A} \) will be \( \kappa \)-distributive and preserve the stationarity of \( S \), which implies by Corollary 1.3 that there exists a disjoint stationary sequence in \( V^{P \mathbb{A}} \). Thus, in \( V^{P \mathbb{A}} \) we have that stationary reflection holds at \( \omega_2 \) and the \( \omega_1 \)-approachability property fails. If, in addition, we assume that the Mahlo cardinal \( \kappa \) is not weakly compact in \( L \), then there exists an \( \omega_2 \)-Aronszajn tree in \( V^{P \mathbb{A}} \) as discussed above.

The remainder of this section is divided into two parts. In the first part we will develop the forcing \( \mathbb{P} \), and in the second we will handle the forcing \( \mathbb{A} \) in \( V^\mathbb{P} \). We will use the following theorem of Gitik [3]. Suppose that \( V \subseteq W \) are transitive models of \( \text{ZFC} \) with the same ordinals and the same \( \omega_1 \) and \( \omega_2 \). If \( (\mathbb{P}(\omega) \cap W) \setminus V \) is non-empty, then in \( W \) the set \( \mathbb{P}_{\omega_1}(\omega_2) \setminus V \) is stationary in \( \mathbb{P}_{\omega_1}(\omega_2) \).
We define by induction a forcing iteration

$$\langle P_\alpha, Q_\beta : \alpha \leq \kappa, \beta < \kappa \rangle.$$ 

This iteration will be a countable support forcing iteration of proper forcings. We will then let $P := P_\kappa$.

Fix $\alpha < \kappa$ and assume that $P_\alpha$ has been defined. We split the definition of $Q_\alpha$ into three cases. If $\alpha$ is an inaccessible cardinal, then let $Q_\alpha$ be a $P_\alpha$-name for the forcing $\text{Add}(\alpha)$. If $\alpha = \beta + 1$ where $\beta$ is inaccessible, then let $Q_\alpha$ be a $P_\alpha$-name for $\text{Add}(\omega)$. For all other cases, let $Q_\alpha$ be a $P_\alpha$-name for $\text{Col}(\omega_1, \omega_2)$. Note that in any case, $Q_\alpha$ is forced to be proper. Now let $P_{\alpha+1}$ be the set of all functions $p$ whose domain is a subset of $\alpha + 1$ such that $p \upharpoonright \alpha \in P_\alpha$, and if $\alpha \in \text{dom}(p)$ then $p(\alpha)$ is a $P_\alpha$-name for a member of $Q_\alpha$. The ordering on $P_{\alpha+1}$ is defined by letting $q \leq p$ if $q \upharpoonright \alpha \leq p \upharpoonright \alpha$ in $P_\alpha$ and $q \upharpoonright \alpha$ forces that $q(\alpha) \leq p(\alpha)$ in $Q_\alpha$. So $P_{\alpha+1}$ is forcing equivalent to $P_\alpha \ast Q_\alpha$.

At limit stages $\delta \leq \kappa$, assuming that $P_\alpha$ is defined for all $\alpha < \delta$, we let $P_\delta$ denote the countable support limit of these forcings. Specifically, a condition in $P_\delta$ is any function $p$ whose domain is a countable subset of $\delta$ such that for all $\alpha < \delta$, $p \upharpoonright \alpha \in P_\alpha$, ordered by $q \leq p$ if for all $\alpha < \delta$, $q \upharpoonright \alpha \leq p \upharpoonright \alpha$ in $P_\alpha$.

This completes the construction. For each $\alpha < \kappa$, $P_\alpha$ is a countable support iteration of proper forcings, and hence is proper. Also, by standard facts, if $\beta < \alpha$, then $P_\beta$ is a regular suborder of $P_\alpha$, and in $V^{P_\beta}$, the quotient forcing $P_\alpha / G_{P_\beta}$ is forcing equivalent to a countable support iteration of proper forcings, and hence is itself proper. We let $P_{\beta, \alpha}$ be a $P_\beta$-name for this proper forcing iteration which is equivalent to $P_\alpha / G_{P_\beta}$ in $V^{P_\beta}$. One can show by well-known arguments that for all inaccessible cardinals $\alpha < \kappa$, $P_\alpha$ has size $\alpha$, is $\alpha$-c.c., and forces that $\alpha = \omega_2$.

Let $P := P_\kappa$. In $V^P$, let us define a disjoint stationary sequence. Recall that $S$ is the set of inaccessible cardinals in $\kappa$ in the ground model $V$. Since $\kappa$ is Mahlo, $S$ is a stationary subset of $\kappa$ in $V$. As $P$ is $\kappa$-c.c., $S$ remains stationary in $V^P$. And since $P$ is proper and forces that $\kappa = \omega_2$, each member of $S$ has cofinality $\omega_1$ in $V^P$.

The set $S$ will be the domain of the disjoint stationary sequence in $V^P$. Consider $\alpha \in S$. Then $P_\alpha$ forces that $\alpha = \omega_2$. We have that $P_{\alpha+1}$ is forcing equivalent to $P_\alpha \ast \text{Add}(\alpha)$ and $P_{\alpha+2}$ is forcing equivalent to $P_\alpha \ast \text{Add}(\alpha) \ast \text{Add}(\omega)$.

Clearly, $\alpha$ is still equal to $\omega_2$ after forcing with $P_{\alpha+1}$ or $P_{\alpha+2}$.

Since there exists a subset of $\omega$ in $V^{P_{\alpha+2}} \setminus V^{P_{\alpha+1}}$, in $V^{P_{\alpha+2}}$ the set

$$s_\alpha := P_{\omega_1}(\alpha) \setminus V^{P_{\alpha+1}}$$

is a stationary subset of $P_{\omega_1}(\alpha)$ by Gitik’s theorem. Now the tail of the iteration $P_{\alpha+2} = P_{\alpha}$ is proper in $V^{P_{\alpha+2}}$. Therefore, $s_\alpha$ remains stationary in $P_{\omega_1}(\alpha)$ in $V^P$.

Observe that if $\alpha < \beta$ are both in $S$, then by definition $s_\alpha \subseteq V^{P_{\alpha+2}} \subseteq V^{P_\beta}$, whereas $s_\beta \cap V^{P_\beta} = \emptyset$. Thus, $s_\alpha \cap s_\beta = \emptyset$. It follows that in $V^P$, $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence on $\omega_2$.

For the second part of our proof, we work in $V^P$ to define a forcing iteration $A$ of length $\kappa^+$ which is designed to destroy the stationarity of any subset of $\omega_2 \cap \text{cof}(\omega)$ which does not reflect to an ordinal in $\omega_2 \cap \text{cof}(\omega_1)$. This forcing will be shown to be $\kappa$-distributive and preserve the stationarity of $S$. It follows from Corollary 1.3 that $A$ preserves the fact that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence. Note that since $P$ is $\kappa$-c.c. and has size $\kappa$, easily $2^\kappa = \kappa^+$ in $V^P$. 

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The definition of and arguments involving $\mathbb{A}$ are essentially the same as in the original construction of Harrington and Shelah [1]. The main differences are that we are using $\mathbb{P}$ to collapse $\kappa$ to become $\omega_2$ instead of $\text{Col}(\omega_1, < \kappa)$, and that we are now required to show that $\mathbb{A}$ preserves the stationarity of $S$. We will sketch the main points of the construction, but leave some of the technical details to be checked by the reader in consultation with [1].

Before giving the definition of $\mathbb{A}$ in $V^\mathbb{P}$, let us begin with some general discussion. We will assume in what follows that $2^{<\omega_1} = \omega_2$. We can define abstractly the idea of a suitable iteration

\[
(\mathbb{A}_i, \mathbb{T}_j : i \leq \alpha, j < \alpha),
\]

where $\alpha \leq \omega_3$. Roughly speaking, this is a forcing iteration of length $\alpha$ with supports of size less than $\omega_2$. For each $i < \alpha$, $\mathbb{T}_i$ is an $\mathbb{A}_i$-name for a subset of $\omega_2 \cap \text{cof}(\omega)$. And the iteration $\mathbb{A}_{i+1}$ is forcing equivalent to $\mathbb{A}_i * Q_i$, where $Q_i$ is an $\mathbb{A}_i$-name for the poset whose conditions are all closed and bounded subsets of $\omega_2$ disjoint from $T_i$, ordered by end-extension.

More precisely, instead of conditions being functions whose values are names, we will use functions whose values are actual closed and bounded sets. The motivation for this is that, in our context, each $\mathbb{A}_i$ will be $\kappa$-distributive, and hence all closed and bounded subsets of $\omega_2$ in $V^{\mathbb{A}_i}$ will be in the ground model $V$. Specifically, a condition in $\mathbb{A}_\alpha$ is any function $p$ whose domain is a subset of $\alpha$ of size less than $\omega_2$ such that for all $i \in \text{dom}(p)$, $p(i)$ is a non-empty closed and bounded subset of $\omega_2$ such that $p \upharpoonright i$ forces in $\mathbb{A}_i$ that $p(i) \cap T_i = \emptyset$. Defined in this simpler way, it is easy to check that for any transitive model $M$ of ZFC$^-$ which is closed under $\omega_1$-sequences, if $M$ models that $(\mathbb{A}_i, \mathbb{T}_j : i \leq \alpha, j < \alpha)$ is a suitable iteration, then in fact it is.

Observe that if $\alpha < \omega_3$, then the fact that $2^{<\omega_1} = \omega_2$ holds implies that $\mathbb{A}_\alpha$ has size $\omega_2$. On the other hand, if $\alpha = \omega_3$, then a straightforward application of the $\Delta$-system lemma shows that $\mathbb{A}_{\omega_3}$ is $\omega_3$-c.c. Using a covering and nice name argument, it then follows that if $\mathbb{A}_\alpha$ is $\omega_2$-distributive for all $\beta < \omega_3$, then so is $\mathbb{A}_{\omega_3}$.

Another important fact is that if for all $i < \alpha$, $\mathbb{A}_i$ forces that $\mathbb{T}_i$ is non-stationary, then $\mathbb{A}_\alpha$ contains an $\omega_2$-closed dense subset. Specifically, for each $i$ let $E_i$ be an $\mathbb{A}_i$-name for a club disjoint from $T_i$. Define $D$ as the set of conditions $p$ such that for all $i \in \text{dom}(p)$, $p \upharpoonright i$ forces that $\max(p(i)) \in E_i$. Using a standard argument, one can prove that $D$ is dense and $\omega_2$-closed. Also, $\mathbb{A}_\alpha$ is separative and every condition in it has $\omega_2$-many incompatible extensions. It follows by well-known facts that $\mathbb{A}_\alpha$, and in fact, $\mathbb{A}_\alpha/q$ for any $q \in \mathbb{A}_\alpha$, is forcing equivalent to Add$(\omega_2)$.

Suppose that $2^{<\omega_2} = \omega_3$ and assume that each $\mathbb{T}_i$ is forced to be non-reflecting. Assuming that we are able to prove that such an iteration is $\omega_2$-distributive, it is straightforward to construct such a forcing iteration which forces that stationary reflection holds at $\omega_2$. Namely, if $\mathbb{A}_i$ is defined for some $i < \omega_3$, then using $2^{<\omega_2} = \omega_3$ and the fact that $\mathbb{A}_i$ has size $\omega_2$, we can list out all $\mathbb{A}_i$-names for subsets of $\omega_2 \cap \text{cof}(\omega)$ in order type $\omega_3$. Using a bookkeeping function, we choose $\mathbb{T}_i$ as the first name listed in some stage less than or equal to $i$ which $\mathbb{A}_i$ forces to be non-reflecting, as specified by the bookkeeping function. We can thereby arrange that after $\omega_3$-many stages, all names which arise during the iteration are handled, and thus that the iteration destroys the stationarity of all non-reflecting sets.

This completes the abstract description of a suitable iteration and how it will be used to achieve stationary reflection at $\omega_2$. Returning to our construction, fix
a generic filter \(G\) on \(P\). Then in \(V[G]\) we have that \(\kappa = \omega_2, 2^{\omega_1} = \omega_2, \) and \(2^{\omega_2} = \omega_3 = \kappa^+\). Working in \(V[G]\), we define a suitable iteration \(\langle A_i, \dot{T}_j : i \leq \kappa^+, j < \kappa^+ \rangle\).

We will prove that each \(A_i\) is \(\omega_2\)-distributive and preserves the stationarity of \(S\).

By the discussion above, this will complete the proof of our main result.

Fix \(\alpha < \kappa^+\). We would like to prove that \(A_{\alpha}\) is \(\kappa\)-distributive and preserves the stationarity of \(S\). As one of our two inductive hypotheses, we will assume that for all \(\beta < \alpha, A_\beta\) is \(\kappa\)-distributive and preserves the stationarity of \(S\). In \(V\), fix \(P\)-names \(\dot{A}_i\) for all \(i \leq \alpha\) and \(\dot{T}_j\) for all \(j < \alpha\) which are forced to satisfy the definitions above (we will abuse notation by writing \(\dot{T}_j\) for the \(P\)-name for the \(A_j\)-name \(\dot{T}_j\)).

Before describing the second inductive hypothesis, we need to develop some ideas and notation. For each \(\beta \leq \alpha\), define in \(V\) the set \(\mathcal{X}_\beta\) to consist of all sets \(N\) satisfying:

1. \(N \prec H(\kappa^+)\);
2. \(N\) contains as members \(P\) and \(\langle \dot{A}_i, \dot{T}_j : i \leq \beta, j < \beta \rangle\);
3. \(\kappa_N := |N| = N \cap \kappa\) and \(N^{<\kappa_N} \subseteq N\);
4. \(\kappa_N \in S\).

An easy application of the stationarity of \(S\) and the inaccessibility of \(\kappa\) shows that each \(\mathcal{X}_\beta\) is a stationary subset of \(P_\kappa(H(\kappa^+))\). Also note that if \(N \in \mathcal{X}_\beta\) and \(\gamma \in N \cap \beta\), then \(N \in \mathcal{X}_\gamma\).

Consider \(N\) in \(\mathcal{X}_\alpha\). Since \(P\) is \(\kappa\)-c.c., the maximal condition in \(P\) is \((N, P)\)-generic. So if \(G\) is a \(V\)-generic filter on \(P\), then \(N[G] \cap V = N\). In particular, \(N[G] \cap \kappa = N \cap \kappa = \kappa_N \in S\). Let \(\pi : N[G] \rightarrow N[G]\) be the transitive collapsing map of \(N[G]\) in \(V[G]\). Let \(G^* := G \cap P_{\kappa_N}\), which is a \(V\)-generic filter on \(P_{\kappa_N}\).

Straightforward arguments show that \(\pi \downarrow N : N \rightarrow \overline{N}\) is the transitive collapsing map of \(N\) in \(V\), \(\pi(P) = P_{\kappa_N}, \pi(G) = G^*, \) and \(\overline{N[G]} = \overline{N(G^*)}\). In particular, \(\overline{N[G]}\) is a member of \(V[G^*]\). Since \(\overline{N[G]} \subseteq \overline{N}\) in \(V\) by the closure of \(N\) and \(P_{\kappa_N}\) is \(\kappa_N\)-c.c., it follows that \(\overline{N[G]} = \overline{N(G^*)}\) is closed under \(< \kappa_N\)-sequences in \(V[G^*]\) as well.

Now we are ready to state our second inductive hypothesis: for all \(\beta < \alpha\) and for all \(N \in \mathcal{X}_\beta\), letting \(\pi : N[G] \rightarrow N[G]\) be the transitive collapsing map of \(N[G]\) and \(G^* := \pi(G)\), for all \(q \in \pi(A_{\beta})\), the forcing poset \(\pi(A_{\beta})/q\) is forcing equivalent to \(\text{Add}(\omega_2)\) in \(V[G^*]\).

Let \(N\) be in \(\mathcal{X}_\alpha\). Our main claim is that in \(V[G]\), for all \(a \in N[G] \cap A_\alpha\) there exists \(b \leq a\) in \(A_\alpha\) satisfying that for any dense open subset \(D\) of \(A_\alpha\) in \(N[G]\), there exists \(x \in N[G] \cap D\) such that \(b \leq x\). If this claim is true, then we can easily argue that \(A_\alpha\) is \(\kappa\)-distributive and preserves the stationarity of \(S\). Namely, given a family of fewer than \(\kappa\) many dense open subsets of \(A_\alpha\), we may pick \(N\) so that this collection is in \(N[G]\). But then \(b\) is in the intersection of that family. And given a name for a club subset of \(\kappa\), we may choose \(N\) to contain that name. Then since \(b\) is \((N[G], A_\alpha)\)-generic, \(b\) will force that \(N[G] \cap \kappa = N \cap \kappa = \kappa_N\) is in the intersection of \(S\) with that club.

We will use the following fact, which can be easily checked: if \(I\) is a filter on \(N[G] \cap A_\alpha\) in \(V[G]\) which meets every dense subset of \(A_\alpha\) in \(N[G]\), then there is a lower bound of \(I\) in \(A_\alpha\). Thus, in order to verify the first inductive hypothesis for \(\alpha\) it suffices to show that for all \(a \in N[G] \cap A_\alpha\), there exists such a filter \(I\) which contains \(a\).

We now prove the inductive hypotheses for \(\alpha\), assuming that they hold for all \(\beta < \alpha\). Let \(N \in \mathcal{X}_\alpha\). Let \(G^* := G \cap P_{\kappa_N}\). Since \(\pi\) is an isomorphism, by the
absoluteness of suitable iterations we have that in $V[G^*], \langle \mathcal{A}_\gamma, T_\gamma^* : i \leq \pi(\alpha), j < \pi(\alpha) \rangle := \pi((\mathcal{A}_i, T_j : i \leq \alpha, j < \alpha))$ is a suitable iteration of length $\pi(\alpha) < \omega_3$. So by the general discussion regarding suitable iterations above, in order to prove the second inductive hypothesis for $\alpha$, it suffices to show that for all $\gamma \in N \cap \alpha$, $\pi(\mathcal{A}_\gamma)$ forces over $V[G^*]$ that $\pi(T_\gamma)$ is non-stationary in $\kappa_N$.

Consider $\gamma \in N \cap \alpha$. Then $\mathcal{A}_\gamma$ forces that $T_\gamma$ is a subset of $\kappa \cap \text{cof}(\omega)$ which does not reflect to any ordinal in $\kappa \cap \text{cof}(\omega_1)$. In particular, $\mathcal{A}_\gamma$ forces that $T_\gamma \cap \kappa_N$ is non-stationary in $\kappa_N$. Consider $q \in \pi(\mathcal{A}_\gamma)$. We will find a $V[G^*]$-generic filter $H$ on $\pi(\mathcal{A}_\gamma)$ which contains $q$ such that in $V[G^*][H]$, $\pi(T_\gamma)^H$ is non-stationary in $\kappa_N$. Because $q$ is arbitrary, this proves that $\pi(\mathcal{A}_\gamma)$ forces that $\pi(T_\gamma)$ is non-stationary. Since $N$ is in $\mathcal{X}_\gamma$ and $\gamma \in N \cap \alpha$, $N$ is in $\mathcal{X}_\gamma$. By the inductive hypotheses, $\pi(\mathcal{A}_\gamma)/q$ is forcing equivalent to $\text{Add}(\kappa_N)$ in $V[G^*]$. Hence, we can write $V[G \cap \mathbb{P}_{\kappa_N}]/\pi(\mathcal{A}_\gamma)/q$.

Now $\pi : \mathcal{X}_\gamma$ is an isomorphism between the posets $N[G] \cap \mathcal{A}_\gamma$ and $\pi(\mathcal{A}_\gamma)$. Therefore, $I := \pi^{-1}(H)$ is a filter on $N[G] \cap \mathcal{A}_\gamma$. The fact that $H$ is a $V[G^*]$-generic filter on $\pi(\mathcal{A}_\gamma)$ implies by a straightforward argument that $I$ meets every dense subset of $\mathcal{A}_\gamma$ in $N[G]$. So we can fix $t \in \mathcal{A}_\gamma$ such that $t \leq \gamma$ for all $s \in I$.

Let $h$ be a generic filter on $\mathcal{A}_\gamma$ which contains $t$. Now $\pi^{-1} : N[G^*] \rightarrow N[G]$ is an elementary embedding of $N[G^*]$ into $H(\kappa^+)^{V[G]}$ which satisfies that $\pi^{-1}(H) = I \subseteq h$. So by standard facts, we can extend $\pi^{-1}$ to an elementary embedding $\tau : N[G^*][H] \rightarrow N[G][h]$. Let $T^* := \pi(T_\gamma)^H$ and $T_\gamma := (T_\gamma)^h$. Then clearly, $\tau(T^*) = T_\gamma$.

Since $\kappa_N$ is the critical point of $\tau$, $T_\gamma \cap \kappa_N = T^*$. As $\mathcal{A}_\gamma$ forces that $T_\gamma$ does not reflect to $\kappa_N$, $T^*$ is a non-stationary subset of $\kappa_N$ in the model $V[G][h]$. By the first inductive hypothesis, $\mathcal{A}_\gamma$ is $\kappa$-distributive. Therefore, any club of $\kappa_N$ in $V[G][h]$ is actually in $V[G]$. Thus, $T^*$ is non-stationary in $V[G]$. But $V[G]$ is a generic extension of $V[G^*][H]$ by the proper forcing $\mathbb{P}_{\kappa_N, \kappa}$. So $T^*$ is non-stationary in $V[G^*][H]$.

We have proven in $V[G^*]$ that $\langle \mathcal{A}_\gamma, T_\gamma^* : i \leq \pi(\alpha), j < \pi(\alpha) \rangle$ is a suitable iteration where each $\mathcal{A}_\gamma^*$ forces that $T_\gamma^*$ is non-stationary in $\kappa_N$. As $2^{\omega_1} = \omega_2 = \kappa_N$ in $V[G^*]$, it follows that for each $q \in \mathcal{A}_\gamma^*(\alpha), \mathcal{A}_\gamma^*(\alpha)/q$ is forcing equivalent to $\text{Add}(\kappa_N)$. Thus, the second inductive hypothesis is maintained.

It remains to show that $\mathcal{A}_\alpha$ is $\kappa$-distributive and preserves the stationarity of $S$. As discussed above, it suffices to show that for all $N \in \mathcal{X}_\alpha$, for all $a \in N[G] \cap \mathcal{A}_\alpha$, there exists $b \leq a$ such that for any dense open subset $D$ of $\mathcal{A}_\alpha$ in $N[G]$, there exists $x \in N[G] \cap D$ such that $b \leq x$. By the second inductive hypothesis, $\pi(\mathcal{A}_\alpha)/\pi(\alpha)$ is forcing equivalent to $\text{Add}(\kappa_N)$ in $V[G^*]$, where $\pi$ is the transitive collapsing map of $N[G]$ and $G^* = G \cap \mathbb{P}_{\kappa_N}$. As above, there is a $V[G^*]$-generic filter $H$ on $\pi(\mathcal{A}_\alpha)/\pi(\alpha)$ in $V[G]$. Let $I := \pi^{-1}(H)$. Then $a \in I$ and $I$ is a filter on $N[G] \cap \mathcal{A}_\alpha$ which meets every dense open subset of $\mathcal{A}_\alpha$ in $N[G]$. Let $b$ be a lower bound of $I$, and we are done.

### 3. Arbitrarily Large Continuum

In the model of the previous section, $2^{\omega_1} = \omega_2$ holds. A violation of CH is necessary, since $\text{CH}$ implies the $\omega_1$-approximation property, as witnessed by any enumeration of all countable subsets of $\omega_2$ in order type $\omega_2$. In this section, we
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Theorem 3.1 (Neeman). Assume that stationary reflection holds at $\omega_2$. Then for any ordinal $\mu$, $\operatorname{Add}(\omega, \mu)$ forces that stationary reflection still holds at $\omega_2$.

Proof. We first prove the result in the special case that $\mu = \omega_2$. Let $p \in \operatorname{Add}(\omega, \omega_2)$, and suppose that $p$ forces that $\dot{S}$ is a stationary subset of $\omega_2 \cap \operatorname{cof}(\omega)$. We will find $q \leq p$ and an ordinal $\beta \in \omega_2 \cap \operatorname{cof}(\omega_1)$ such that $q$ forces that $\dot{S} \cap \beta$ is stationary in $\beta$.

Let $T$ be the set of ordinals $\alpha < \omega_2$ such that for some $s \leq p$, $s$ forces that $\alpha \in \dot{S}$. Then $T \subseteq \omega_2 \cap \operatorname{cof}(\omega_1)$. An easy observation is that $p$ forces that $\dot{S} \subseteq T$, and consequently $T$ is a stationary subset of $\omega_2$. For each $\alpha \in T$, fix a witness $s_\alpha \leq p$ which forces that $\alpha \in \dot{S}$, and define

$$a_\alpha := s_\alpha \upharpoonright (\alpha \times \omega) \text{ and } b_\alpha := s_\alpha \upharpoonright ([\alpha, \omega_2) \times \omega).$$

Using Fodor’s lemma, we can find a stationary set $U \subseteq T$ and a set $x$ satisfying that for all $\alpha \in U$, $a_\alpha = x$. Observe that $q := x \cup p$ is a condition which extends $p$. Applying the fact that stationary reflection holds in the ground model together with an easy closure argument, we can fix $\beta \in \omega_2 \cap \operatorname{cof}(\omega_1)$ such that $U \cap \beta$ is stationary in $\beta$ and for all $\alpha < \beta$, $\operatorname{dom}(s_\alpha) \subseteq \beta \times \omega$.

We claim that $q$ forces that $\dot{S} \cap \beta$ is stationary in $\beta$, which finishes the proof. Suppose for a contradiction that there is $r \leq q$ which forces that $\dot{S} \cap \beta$ is non-stationary in $\beta$. Using the fact that $\operatorname{Add}(\omega, \omega_2)$ is c.c.c. and $\operatorname{cf}(\beta) = \omega_1$, there exists a club $D \subseteq \beta$ in the ground model such that $r$ forces that $D \cap \dot{S} = \emptyset$. As $r$ is finite, we can fix $\delta < \beta$ such that $\operatorname{dom}(r) \cap (\delta \times \omega) \subseteq \delta \times \omega$.

Since $U \cap \beta$ is stationary in $\beta$, fix $\alpha \in U \cap D$ larger than $\delta$. We claim that $s_\alpha$ and $r$ are compatible. By the choice of $U$, $s_\alpha \upharpoonright (\alpha \times \omega) = x$, and by the choice of $\beta$, $\operatorname{dom}(s_\alpha) \subseteq \beta \times \omega$. Suppose that $(\xi, n) \in \operatorname{dom}(s_\alpha) \cap \operatorname{dom}(r)$. Then $\xi < \beta$, so $(\xi, n) \in \operatorname{dom}(r) \cap (\beta \times \omega) \subseteq \delta \times \omega$. Thus, $\xi < \delta < \alpha$. So $(\xi, n) \in \alpha \times \omega$, and hence $s_\alpha(\xi, n) = a_\alpha(\xi, n) = x(\xi, n)$. On the other hand, $r \leq q \leq x$, and so $r(\xi, n) = x(\xi, n) = s_\alpha(\xi, n)$.

This proves that $r$ and $s_\alpha$ are compatible. Fix $t \leq r, s_\alpha$. Since $t \leq s_\alpha$, $t$ forces that $\alpha \in \dot{S}$. On the other hand, $\alpha \in D$, and $r$ forces that $\dot{S} \cap D = \emptyset$. So $r$, and hence $t$, forces that $\alpha \notin \dot{S}$, which is a contradiction.

Now we prove the result for arbitrary ordinals $\mu$. If $\mu < \omega_2$, then $\operatorname{Add}(\omega, \omega_2)$ is isomorphic to $\operatorname{Add}(\omega, \mu) \times \operatorname{Add}(\omega, \omega_2 \setminus \mu)$, since stationary reflection holds in $V^{\operatorname{Add}(\omega, \omega_2)}$, it also holds in the submodel $V^{\operatorname{Add}(\omega, \mu)}$, since a non-reflecting stationary set in the latter model would remain a non-reflecting stationary set in the former model.

Suppose that $\mu > \omega_2$. Let $p$ be a condition in $\operatorname{Add}(\omega, \mu)$ which forces that $\dot{S}$ is a stationary subset of $\omega_2 \cap \operatorname{cof}(\omega)$, for some nice name $\dot{S}$, and $\omega_2$. Since $X$ has size $\omega_2$, $\operatorname{Add}(\omega, X)$ is isomorphic to $\operatorname{Add}(\omega, \omega_2)$. By the first result above, we can find $q \leq p$ in $\operatorname{Add}(\omega, X)$ and $\beta \in \omega_2 \cap \operatorname{cof}(\omega_1)$ such that $q$ forces in $\operatorname{Add}(\omega, X)$ that $\dot{S} \cap \beta$ is stationary in $\beta$. Since $\operatorname{Add}(\omega, \mu)$ is isomorphic to $\operatorname{Add}(\omega, X) \times \operatorname{Add}(\omega, \mu \setminus X)$ and $\operatorname{Add}(\omega, \mu \setminus X)$ is c.c.c. in $V^{\operatorname{Add}(\omega, X)}$, an easy argument shows that $q$ forces in $\operatorname{Add}(\omega, \mu)$ that $\dot{S} \cap \beta$ is stationary in $\beta$. \qed
Now start with the model $W := V^{|V|^+}$ from the previous section. Then $\omega_2$ is not weakly compact in $L$, there exists a disjoint stationary sequence in $W$, and stationary reflection holds at $\omega_2$ in $W$. Let $\mu$ be any ordinal and let $H$ be a $W$-generic filter on $\text{Add}(\omega, \mu)$. Since $\text{Add}(\omega, \mu)$ is c.c.c., Corollary 1.3 implies that there exists a disjoint stationary sequence in $W[H]$. As $\omega_2$ is not weakly compact in $L$, there exists an $\omega_2$-Aronszajn tree in $W[H]$. And stationary reflection holds in $W[H]$ by Theorem 3.1.

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