Angular analysis of bremsstrahlung in $\alpha$-decay

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Received: 9 March 2006
Published online: 14 July 2006 — Societa Italiana di Fisica / Springer-Verlag 2006
Communicated by G. Orlandini

This paper is dedicated to the memory of Dr. Ivan Egorovich Kashuba — a brilliant scientist with bright nature who worked in science till his last days.

Abstract. A new quantum electrodynamical method of calculations of bremsstrahlung spectra in the $\alpha$-decay of heavy nuclei taking into account the angle between the directions of $\alpha$-particle motion (or its tunneling) and photon emission is presented. The angular bremsstrahlung spectra for $^{210}$Po have been obtained for the first time. According to calculations, the bremsstrahlung in the $\alpha$-decay of this nucleus depends extremely weakly on the angle. Taking into account nuclear forces, such dependence is not changed visibly. An analytical formula of the angular dependence of the bremsstrahlung spectra is proposed and gives its harmonic behavior. The extremal values of the angle, at which the bremsstrahlung has maximal and minimal values, has been found.

Key words. Alpha-decay, photon bremsstrahlung, angular spectra for $^{210}$Po, sub-barrier, tunneling

PACS. 23.60.+e Alpha decay – 41.60.-m Radiation by moving charges – 23.20.Js Multipole matrix elements (in electromagnetic transitions) – 03.65.Xp Tunneling, traversal time, quantum Zeno dynamics

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1 Introduction

Experiments [1,2,6] with measurements of bremsstrahlung (Br) spectra in the α-decay of the nuclei $^{210}$Po, $^{214}$Po, $^{226}$Ra and $^{241}$Cm have caused an increased interest. One of the key ideas of the fulfillment of such experiments consists in finding a method of extraction of a new information about α-decay dynamics from the Br spectra (and a detailed information about the dynamics of tunneling). One can note a certain difference between the Br spectra [1] and [2,3] for $^{210}$Po, obtained experimentally for the values 90° and 25° of the angle between the directions of the α-particle propagation and the photon emission (these experiments and the difference of their spectra are discussed in [4,5]). One can explain such difference between the Br angular spectra on the basis of the following idea: the Br intensity depends on the directions of emission of the photons and motion (with possible tunneling) of the α-particle relatively the daughter nucleus. In such a way, a three-dimensional picture of the α-decay with the accompanying Br in the spatial region of nuclear boundaries has been devised.

However, if the Br intensity varies enough visibly with changing the angle value, then one can suppose, that the photon emission is able to influence essentially on the α-decay dynamics and, therefore, to change all of its characteristic. From this point of view, the discussions in [4,5] open a way for obtaining a new information about the α-decay — through the angular analysis of the Br during the α-decay. But for such researches a model describing the Br in the α-decay, which takes into account the value of the angle between the directions of the α-particle propagation (or tunneling) and the photon emission, is needed.

In theoretical aspect, some progress has been made here. One can note models of calculations of the Br spectra in the α-decay, developed on the basis of quantum electrodynamics with use of perturbation theory: the first paper [6] where a general quantum-mechanical formalism of the calculation of the Br spectra in the α-decay is proposed and the Br spectrum for $^{210}$Po inside the photons energy region up to 200 keV was estimated (even until the fulfillment of the first experiments); essentially improved models in the dipole approximation [7,8] and in the multipolar expansion of photons current (wave function) with application of the Fermi golden rule: an approach [10] of the calculation of the Br spectra with realistic barriers of the α-decay), models [11,12,13,8] developed in semiclassical approximation (see also the Br spectra calculations in [8], instant accelerated models [10,9] constructed on the basis of classical electrodynamics (see also [12]), methods [14,19,15,11,12], directed on a nonstationary description of the α-decay with the accompanying Br and the calculations of such non-stationary characteristics as tunneling time. One can recall also papers [16,17,15,19] with study of dynamics of subbarrier tunneling in the α-decay; an effect, opened in [20] and named Munchhausen effect, which increases the barrier penetrability due to charged-particle emission during its tunneling and which can be extremely interesting for further study of the photon bremsstrahlung during subbarrier tunneling in the α-decay). However, one needs to say that at this stage the calculations of the Br spectra by all these approaches are reduced to obtaining their integral (or averaged by angles) values and, therefore, they do not allow to fulfill an angular analysis of the experimental Br spectra (here, one can quote an approach in [10,9] based on classical electrodynamics, which shows a way for obtaining the angular spectra (see (15) and (17), p. 999), however here we shall use the direct quantum-mechanical approach of the Br spectra calculation, which describes the quantum effect of the subbarrier Br more precisely).

In [21] we had developed a multipolar method which takes into account the angle between the directions of the α-particle propagation and the photon emission. But the angular integrals used in this method are difficult enough to be obtained and some approximations are used, there was a convergence problem with the calculations of multipoles of larger orders, only the angular dependences of matrix elements of the selected multipoles E1 and M1 were found, while it is interesting to know also the angular dependence of the total Br spectra. Moreover, computer calculations of the Br spectra and their angular analysis will be essentially more complicated, if one passes from the type of the potential used in this paper (and also in [7,9]) to realistic potentials. In this sense, the approach proposed in [21] is not convenient enough.

In this paper we present a new method for the angular calculations of the Br spectra in the α-decay of nuclei (started in [22], with the resolution of a convergence problem in the Br spectra calculations existing in [22]). In our approach we introduce a simplified transformation, which reduce the complicated angular formalism of the calculation of the Br spectra (presented in [21]) to a maximally simple form (this makes the method clearer and more comprehensible), with keeping the calculating accuracy as good as possible, where only one angle from all angular parameters is used — the angle used in experiments [4,5] for $^{210}$Po. We show, that this proposed transformation works in the low-energy region of photons and, therefore, can be applied to the analysis of all up to now existing experimental data of Br spectra in the α-decay of spherical nuclei (we relate $^{210}$Po to them). In this paper we present the results of calculations of the angular spectra of the Br in the α-decay for $^{210}$Po (the Br angular spectra in the α-decay have been obtained for the first time). We show, how the Br spectra are changed after deformation of the form of the α-decay barrier as a result of the correction of a component of a α-nucleus potential of nuclear forces (we did not find such calculations in other papers). An analysis of the problem of the calculations convergence of the Br spectra in the α-decay is included in the paper, whose resolution plays a key role for obtaining of the reliable values of the spectra.
2 A formalism of the calculations of the bremsstrahlung spectra in the stationary approach

We shall consider the decay of the nucleus as the decay of the compound quantum system: α-particle and daughter nucleus. The α-particle is the electrically charged particle and during its motion inside the electromagnetic field of the daughter nucleus it emits photons. The spontaneous emission of the photon changes a state of the compound system, which is described by its wave function. For a quantitative estimation of the photon Br we use a transition of the system from its state before the photon emission (we name such state as the initial \( i \)-state) into its state after the photon emission (we name such state as the final \( f \)-state). One can define a matrix element of such transition of the system and on its basis find the Br probability during the α-decay (for convenience, we denote it as \( W(w) \)). According to [21], we obtain:

\[
W(w) = N_0 k_f w |p(w)|^2, \quad N_0 = \frac{Z_{\text{eff}}^2 e^2}{(2\pi)^4 m}, \quad k_{i,f} = \sqrt{2mE_{i,f}}, \quad w = E_i - E_f,
\]

where \( p(w) \) has a form

\[
p(w) = \sum_{\alpha=1,2} \sum_{\beta=1,2} e^{(\alpha)(\beta)} \int_0^\infty dr \int \left| r^2 \psi^*_f(r) e^{-ikr} \frac{\partial}{\partial r} \psi_i(r) \right| d\Omega.
\]

Here \( Z_{\text{eff}} \) and \( m \) are effective charge and reduced mass of the system, \( E_{i,f}, k_{i,f} \) and \( \psi_{i,f}(r) \) are total energy, wave vector and wave function of the system in the initial \( i \)-state or in the final \( f \)-state (in dependence on the index \( i \) or \( f \) in use), \( \psi_i(r) \) and \( \psi_f(r) \) are the wave function of the system in the initial \( i \)- and the final \( f \)-states, \( e^{(\alpha)} \) is polarization vector of the photon emitted, \( k \) is photon wave vector, \( w = k = |k| \) is photon frequency (energy). The vector \( e^{(\alpha)} \) is perpendicular to \( k \) in Coulomb calibration. We use such system of units: \( h = 1 \) and \( c = 1 \). Notations are used in accordance with [21]. Similar expressions for the Br probability are used in [7,8,9] with further application of Fermi golden rule.

In accordance with main statements of quantum mechanics, the wave functions of the system in the states before and after the photon emission are defined inside all space region, including the region of the subbarrier tunneling. A definition of a matrix element of the transition of this system requires an account of all space region of the definition of the wave functions of this system in two states. Therefore, we should include the tunneling region into the definition of the matrix element of the Br, irrespective of we know, whether the photons emission is possible during tunneling or not.

Let’s consider a subintegral expression in (2). Here, the wave function \( \psi_i(r) \) for the initial \( i \)-state and the wave function \( \psi_f(r) \) for the final \( f \)-state take into account the directions of propagation (or tunneling) of the α-particle before the photon emission and after it, correspondingly; the photons wave function (its main part consists in the exponent \( \exp(-ikr) \)) points to the direction of propagation of the photon emitted. We see, that the quantum mechanical approach for calculation of the Br spectra initially has a detailed angular information about the process of α-decay with the accompanying Br.

However, we see that a further development of the approach for the calculations of the Br spectra in the α-decay on the basis of the formulas (1) and (2) by other authors (which consist in the calculations of \( p(w) \)) gives rise to angular averaging of the spectra. And a necessity has been arising in construction of an approach, which allows simply enough to calculate the Br spectra (with possible resolving the convergence problem in the computer calculations) with taking into account of the angle between the directions of the α-particle propagation (with possible tunneling) and the photon emission and without (essential) decreasing of the accuracy.

3 A simplified angular method of the calculations of the matrix element

In [22] the approach for the calculation of the Br spectra, allowing to find a dependence of the total Br spectra on the angle between the directions of the α-particle propagation (or tunneling) and the photon emission, was proposed. However, further research has shown, that it is extremely difficult to achieve a convergence in the computer calculations of the Br spectra by such approach and, therefore, such a method requires an essential development. Here, we propose a consecutive statement of such approach with a resolution of the convergence problem.

Let’s rewrite vectors \( e^\alpha \) of polarization through the vectors \( \xi_{-1} \) and \( \xi_{+1} \) of circular polarization with opposite directions of rotation (see [23], p. 42):

\[
\xi_{-1} = \frac{1}{\sqrt{2}}(e^1 - ie^2), \quad \xi_{+1} = -\frac{1}{\sqrt{2}}(e^1 + ie^2).
\]
Substituting these values into (2), we obtain:

\[ p(w) = \sum_{\mu = -1,1} h_{\mu} \xi_{\mu}^{\nu} \int_{0}^{+\infty} dr \int r^2 \psi'_{j}(r)e^{-ikr} \frac{\partial}{\partial r} \psi_{i}(r) d\Omega, \]  

(4)

where

\[ h_{-1} = \frac{1}{\sqrt{2}}(1 - i), \ h_{1} = -\frac{1}{\sqrt{2}}(1 + i), \ h_{-1} + h_{1} = -i\sqrt{2}. \]  

(5)

Using the following properties (see [23] p. 44–46, [21]):

\[ \frac{\partial}{\partial r} \psi_{i}(r) = -\frac{d\psi_{i}(r)}{dr} T_{01,0}(n_{i}), \]

\[ T_{01,0}(n_{i}) = \sum_{\mu = -1,1} (110| - \mu \mu 0)Y_{1,\mu}(n_{i})\xi_{\mu}, \]

(6)

where \((110| - \mu \mu 0)\) are Clebsch-Gordan coefficients and \(T_{\nu',\mu}(n)\) are vector spherical harmonics (see [23], p. 45 and we use quantum numbers \(l = m = 0\) in the initial \(i\)-state), \(Y_{l,\mu}(n_{i,f})\) are normalized spherical functions (see [24], p. 118–121 (28.7), p. 752–755), we obtain:

\[ \frac{\partial}{\partial r} \psi_{i}(r) = -\frac{d\psi_{i}(r)}{dr} \sqrt{\frac{1}{3}} \sum_{\mu = -1,1} Y_{1,-\mu}(n_{i})\xi_{\mu}. \]

(7)

Taking into account (4), (7) and the orthogonality condition of the vectors \(\xi_{\pm 1}\) and \(\xi_{\mp 1}\) of polarization, we find:

\[ p(w) = -\sqrt{\frac{1}{3}} \sum_{\mu = -1,1} h_{\mu} \int_{0}^{+\infty} dr \int r^2 \psi'_{j}(r) \frac{\partial}{\partial r} \psi_{i}(r) \int Y_{\nu,m}^{*}(n_{f})Y_{1,-\mu}(n_{i})e^{-ikr} d\Omega, \]

(8)

where \(\psi_{j}(r) = \psi_{j}(r)Y_{\nu',m'}(n_{f})\).

Let’s consider the vectors \(k\) and \(r\). The vector \(k\) is an impulse of the photon, pointed out the direction of its propagation. The vector \(r\) is a radius-vector, pointed out a position of the \(\alpha\)-particle relatively a center of mass of the daughter nucleus and (because of mass of the daughter nucleus is larger sufficiently than mass of the \(\alpha\)-particle) pointed out the direction of its motion (or tunneling). Then an angle between the vectors \(k\) and \(r\) (let’s denote it as \(\beta\)) is the angle between the direction \(n_{r} = r/r\) of motion (or tunneling) of the \(\alpha\)-particle and the direction \(n_{ph} = k/k\) of a propagation of the photon emitted, i. e. it is the angle used in the experiments [3, 4, 11]. One can write

\[ \exp(-ikr) = \exp(-ikr \cos \beta), \ k = |k|, \ r = |r|. \]

(9)

Now we make such assumptions:

– the photon emission process does not change the direction of motion (or tunneling) of the \(\alpha\)-particle:

\[ n_{i}^{f} = n_{i}^{f}, \]

(10)

– the angle \(\beta\) is not depended on the direction of the outgoing \(\alpha\)-particle motion from the nucleus region.

Then, taking into account these assumptions and the orthogonality property of the functions \(Y_{\nu,m}(n_{r})\), we obtain the following expression for \(p(w, \beta)\):

\[ p(w, \beta) = -\sqrt{\frac{1}{3}} \sum_{\mu = -1,1} h_{\mu} \int_{0}^{+\infty} dr \int r^2 \psi'_{j}(r) \frac{\partial}{\partial r} \psi_{i}(r) e^{-ikr \cos \beta} dr, \]

(11)

and selection rules for quantum numbers \(l\) and \(m\) of the final \(f\)-state:

the initial state: \(l_{i} = 0, \ m_{i} = 0; \)

the final state: \(l_{f} = 1, \ m_{f} = -\mu = \pm 1. \)

(12)
4 Spherical wave expansion

For further computer calculations of the integral \(11\), let’s use an expansion of the plane wave in the spherical waves (for example, see [24], p. 144, (34.1)):

\[
e^{ikz} = \sum_{l=0}^{+\infty} (-i)^l (2l + 1) P_l(\cos \beta) \left( \frac{r}{k} \right)^l \left( \frac{1}{r} \frac{d}{dr} \right)^l \sin kr,
\]

where \(z = r \cos \beta\). Introducing spherical Bessel functions (see [24], p. 139, (33.9), (33.10) and (33.11)):

\[
j_l(kr) = (-1)^l \left( \frac{r}{k} \right)^l \left( \frac{1}{r} \frac{d}{dr} \right)^l \sin kr,
\]

we obtain:

\[
e^{-ikr \cos \beta} = \left( e^{ikr \cos \beta} \right)^* = \sum_{l=0}^{+\infty} i^l (-1)^l (2l + 1) P_l(\cos \beta) j_l(kr)
\]

and from \(11\) we find:

\[
p(w, \beta) = -\sqrt{\frac{1}{3}} \sum_{l=0}^{+\infty} i^l (-1)^l (2l + 1) P_l(\cos \beta) \sum_{\mu=1,1} h_{\mu} J_{m_f}(l, w),
\]

where

\[
J_{m_f}(l, w) = \int_0^{+\infty} r^2 \psi^*_f(r) \frac{\partial \psi_i(r)}{\partial r} j_l(kr) dr.
\]

5 The bremsstrahlung in the Coulomb field

Practically, in the numerical calculation of the Br spectra it is convenient to divide the whole region of the integration into two parts: the region 1 of a joint action of the Coulomb and nuclear forces not far from the nucleus and the region 2, in which one can neglect by the action of the nuclear forces in comparison with the action of the Coulomb forces. Our analysis has shown, that an attainment of the convergence of the Br spectra calculations (which determines their accuracy, reliability of the found Br spectra) is reached first of all by correctness of the calculations in the region 2. Namely, in this region it needs to solve a problem with definition of the external boundary of integration (its increasing leads to increasing of the accuracy of the obtained spectra, but to increasing of difficulty of the calculations and analysis), to choose the most effective method of the numerical integration (of an improper integral with an oscillating and weakly damping sub-integral function), to solve a problem with attainment of needed accuracy and convergence of the calculations. It defines time of the calculations, minimization of which appears extremely important for fulfillment of the real analysis of the obtained Br spectra in dependence on needed parameters. Therefore, maximal simplification of the formulas for the Br spectra in the region 2 is useful.

Let’s assume, that the potential, used in the radial integral \(17\), in spatial region of \(r\) is Coulomb since the value \(R_c\). We accept \(R_c\) as the internal boundary of the region 2. One can write the radial integral \(J(l, w)\) in \(17\) so:

\[
J_{m_f}(l, w) = J_{in,m_f}(l, w) + J_c(l, w),
\]

where

\[
J_{in,m_f}(l, w) = \int_0^{R_c} r^2 \psi^*_f(r, m_f) \frac{\partial \psi_i(r)}{\partial r} j_l(kr) dr,
\]

\[
J_c(l, w) = \int_{R_c}^{+\infty} r^2 \psi^*_f(r) \frac{\partial \psi_i(r)}{\partial r} j_l(kr) dr.
\]
The radial integral $J_c(l, w)$ does not depend on the quantum number $m$ of the systems in the final $f$-state. Then, one can write $p(w, \beta)$ so (with taking into account \ref{5} for the Coulomb component):

$$p(w, \beta) = p_{in}(w, \beta) + p_c(w, \beta), \quad (20)$$

where

$$p_{in}(w, \beta) = -\sqrt{\frac{1}{3}} \sum_{l=0}^{+\infty} \epsilon^l (-1)^l (2l + 1) P_l(\cos \beta) \sum_{\mu = -1, 1} h_{\mu} J_{in, \mu}(l, w),$$

$$p_c(w, \beta) = \sqrt{\frac{2}{3}} \sum_{l=0}^{+\infty} \epsilon^{l+1} (-1)^l (2l + 1) P_l(\cos \beta) J_c(l, w). \quad (21)$$

We see, that there is no any interference between the components $p_{in}(w, \beta)$ and $p_c(w, \beta)$ in the calculations of the total value of $p(w, \beta)$, but it exists in calculations of the total Br spectra.

6 The first approximation at $l = 0$

Legendre’s polynomial of the order $l$ equals (for example, see \ref{24}, p. 752 (c.1)):

$$P_l(\theta) = \frac{1}{2^l l!} \frac{d^l}{d\theta^l} (\theta^2 - 1)^l, \quad P_0(\theta) = 1, \quad P_1(\theta) = \theta, \quad \theta = \cos \beta. \quad (22)$$

Then at $l = 0$ we find:

$$p_{in}^{(l=0)}(w, \beta) = -\sqrt{\frac{1}{3}} \sum_{\mu = -1, 1} h_{\mu} J_{in, \mu}(0, w),$$

$$p_c^{(l=0)}(w, \beta) = i \sqrt{\frac{2}{3}} J_c(0, w). \quad (23)$$

If for nuclei $^{210}\text{Po}$, $^{214}\text{Po}$, $^{226}\text{Ra}$ to use the potential with parameters as in \ref{24} (and as in \ref{7,9} also), then we find, that Br from the internal spatial region till to $R_c$ is extremely small ($p_{in}(w, \beta) << p_c(w, \beta)$). According to our estimations, for such potential it is smaller in $10^{-22} - 10^{-24}$ times then Br from the external region. Therefore, one can neglect by Br from the internal region, and the total Br can be determined by the Coulomb field inside the barrier region and the external region. From \ref{1} we write down the Br probability in the first approximation at $l = 0$:

$$W_{l=0}(w) = N_0 k_f w |p_c^{(l=0)}(w, \beta)|^2 = \frac{2}{3} N_0 k_f w |J_c(0, w)|^2. \quad (24)$$

One can conclude (it has obtained for the first time):

- The Br probability in the first approximation at $l = 0$, formed by the Coulomb field both with taking into account of the nuclear forces of any shape, and without such forces, does not depend on a value of the angle $\beta$ between the directions of the $\alpha$-particle propagation (or its tunneling) and the photon emission.
- The Coulomb field is degenerated by the quantum number $m$. This property distinguishes the Coulomb field from the nuclear forces at their account in the model. This difference is shown in the matrix elements \ref{23}. The nuclear forces participate in formation of the decay barrier and, therefore, one can consider approximately them as forces working in the spatial region of the barrier, where there is a tunneling. One can assume, that one can divide the emissions from the barrier region and from the external region on the basis of the quantum number $m_f$. It can be interesting to find a possible way of extraction of the Br spectrum from the barrier region (or from the external region) from experimental Br spectrum on the basis of this property.

7 The second approximation at $l = 1$

Taking into account \ref{22}, we find the Br probability in the second approximation at $l = 1$:

$$p_{in}^{(l=1)}(w, \beta) = i \sqrt{3} \cos \beta \sum_{\mu = -1, 1} h_{\mu} J_{in, \mu}(1, w),$$

$$p_c^{(l=1)}(w, \beta) = \sqrt{6} \cos \beta J_c(1, w). \quad (25)$$
Neglecting by Br from the internal region, we obtain the following expressions for the component of the Br probability of the second approximation at \( l = 1 \):

\[
W^{(l=1)}(w, \beta) = N_0 k_f \omega \left| p^{(l=1)}_e(w, \beta) \right|^2 = 6N_0 k_f \omega \left| J_c(1, w) \right|^2 \cos^2 \beta
\]  

(26)

and for the total Br probability in the second approximation at \( l = 1 \):

\[
W_{l=1}(w, \beta) = W_{l=0}(w) \left| 1 - N(w) \cos \beta \right|^2, \quad N(w) = 3i \frac{J_c(1, w)}{J_c(0, w)}.
\]

(27)

One can conclude (it has found for the first time):

- The dependence of the Br probability in the \( \alpha \)-decay on the value of the angle \( \beta \) between the directions of the \( \alpha \)-particle propagation (or its tunneling) and the photon emission has harmonic type \( (27) \).
- The account of the nuclear forces does not change the dependence of the Br probability in the second approximation at \( l = 1 \) on such angle value.
- Exp. \( (27) \) allows to find analytically maximums and minimums in the Br spectra in dependence on the angle \( \beta \).

8 Convergence of calculations in asymptotic region

There is an essential difficulty in the calculations of the Br spectra for the given nucleus, concerned with obtaining of the radial integrals \( (19) \) (or \( (17) \)). This difficulty is caused by that such integral is improper, and its sub-integral function is oscillated and damped slowly with increasing of \( r \). The function damps weaker with increasing of \( r \), the larger region of integration should be taken into account in the numerical integration. For \( ^{210} \text{Po} \) the damping degree of the sub-integral function is such as for reliable values of the first 2–3 digits for the Br spectrum it needs to take into account (with the higher accuracy of calculations) 1 million of oscillations of this function.

As an evident demonstration of this problem, let’s consider an one-dimensional integral:

\[
\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx.
\]

(28)

An exact analytical value of this integral at \( a = 0 \) is known from theory of functions of complex variables, equal to \( \pi / 2 \).

The numerical calculation of the integral (with use of simple method of trapeziums, method of Gauss or other methods of the numerical integration) allows to obtain quickly the same result also, but with a given degree of accuracy (which determines a region of the numerical integration). It proves a convergence of the computer calculations of such integral with a concrete choice of the parameter \( a \). But at weak increasing of the parameter \( a \) the region of the numerical integration for obtaining of the same calculating accuracy for the integral \( (28) \) increases essentially, and, therefore, a difficulty to calculate this integral numerically increases essentially. However, the application of methods of theory of functions of complex variables makes the calculation of such integral as simple again. So, on the example of the simple integral \( (28) \) one can meet with the numerical problem of the convergence of the calculations of the improper integrals with the damping slowly, oscillating sub-integral functions.

We fulfill an analysis of the convergence of the calculation of the integral \( (19) \) on the basis of the analysis of damping of its sub-integral function in the asymptotic region, which is defined by wave functions in the initial \( i \)-, final \( f \)-states and the spherical Bessel function of order \( l \).

For enough large values of \( r \) one can use an asymptotic representation of the spherical Bessel function of order \( l \):

\[
J_{l}^{(as)}(kr) = \frac{1}{kr} \sin \left( kr - \frac{\pi l}{2} \right)
\]

(29)

or

\[
J_{2n}^{(as)}(kr) = (-1)^n J_{0}^{(as)}(kr) = (-1)^n \frac{\sin kr}{kr},
\]

\[
J_{2n+1}^{(as)}(kr) = (-1)^n J_{1}^{(as)}(kr) = (-1)^{n+1} \frac{\cos kr}{kr}.
\]

(30)

where \( n \) is a natural number.

The wave functions \( \psi_i(r) \) and \( \psi_f(r) \) of the initial \( i \)- and the final \( f \)-states are linear combinations of the Coulomb functions \( F_i(\eta, \rho) \) and \( G_i(\eta, \rho) \) (divided on \( \rho_{c,f} \), with quantum number \( l = 0 \) or \( l = 1 \) for the initial \( i \)- or the final \( f \)-state, correspondingly). One can write the Coulomb functions for \( l \) in the asymptotic region so:

\[
F_i(\eta, \rho) = \sin \theta_i, \quad G_i(\eta, \rho) = \cos \theta_i.
\]

(31)
where
\[ \theta_l = \rho - \eta \log 2\rho + \frac{1}{2} \pi l + \sigma_l(\eta), \quad \rho_{i,f} = \frac{k_{i,f} r}{\nu}. \] (32)

where \( \Gamma(x) \) is Gamma function with argument \( x \), \( \nu \) is Sommerfeld parameter.

Now one can conclude:
- The spherical Bessel function \( j_l^{(as)}(kr) \) in the asymptotic region damp (and oscillates) with increasing of \( r \) equally for any order \( l \).
- The Coulomb functions \( F_0(\eta_i, \rho_i) \) and \( G_0(\eta_i, \rho_i) \) of order 0 for the initial \( i \)-state and the Coulomb functions \( F_l(\eta_f, \rho_f) \) and \( G_l(\eta_f, \rho_f) \) of order \( l = 1 \) for the final \( f \)-state damp in the asymptotic region with increasing of \( r \) equally, oscillate equally and are shifted at a phase between each other.
- The total sub-integral function of the integral (19) in the asymptotic region damp with increasing of \( r \) equally for any order \( l \).

Taking into account (19) and (30), we obtain:
\[ J_c^{(as)}(2n, w) = (-1)^n J_c^{(as)}(0, w), \quad J_c^{(as)}(2n + 1, w) = (-1)^n J_c^{(as)}(1, w). \] (33)

I.e. one can reduce any integral inside the asymptotic region to one of two integrals \( J^{(as)}(0, w) \) or \( J^{(as)}(1, w) \).

Let’s find the matrix element \( p_c^{(as)}(w, \beta) \) in the asymptotic region:
\[ p_c^{(as)}(w, \beta) = i \sqrt{\frac{2}{3}} j_c^{(as)}(0, w) \sum_{n=0}^{\infty} (4n + 1) P_{2n}(\cos \beta) + \sqrt{\frac{2}{3}} j_c^{(as)}(1, w) \sum_{n=0}^{\infty} (4n + 3) P_{2n+1}(\cos \beta). \] (34)

Thus, we reduce the formula (11) for the Br spectra to the linear combination of two radial integrals, which are convergent (one can calculate them with a desirable accuracy limited by the calculations accuracy of a concrete computer) and do not depend on the angle, and factors - sums on \( n \), into which the problem of convergence is carried out (one can meet with it in (11)).

It should seem, that one can cut off the region of the numerical integration in one boundary \( R \) for calculation of the integral \( J_c(l, w) \) from (19) for any \( l \). However, the calculation convergence of the integral is determined not only by the damping of the sub-integral function at large \( r \), but by its behavior on the whole integration region also. An analysis has shown, that the sub-integral function inside the barrier region and inside the external region closer to the barrier behaves so, that the calculation of the total integral becomes more and more sensible to it with increasing of \( l \) and the calculation convergence becomes worse. Therefore, for obtaining the reliable values of the integrals \( J_c(l, w) \) (for the same accuracy) it needs to increase the external boundary \( R \) of the integration region for larger \( l \) (such a conclusion has obtained by us is calculating of the angular Br spectra for \( ^{210}\text{Po} \) also).

9 Angular calculations for the Br spectra in the \( \alpha \)-decay of \( ^{210}\text{Po} \)

As a demonstration of the described above method, let’s calculate the angular Br spectra in the \( \alpha \)-decay of \( ^{210}\text{Po} \). For a comparison of results obtained in such approach, with results obtained by models (7-9), we shall choose the potential parameters as in (11) (they coincide with the parameters of the potential with the external Coulomb field in (11) and in (19)).

In spite of the fact that there are methods allowed to calculate absolute values of the Br spectra, in this paper at first we shall find the relative values of the Br spectrum for the given nucleus and then we shall normalize obtained spectra at one selected point of the Br experimental spectrum for the given angle value. This approach as against the previous one allows with a larger accuracy to analyze a behavior of the Br spectra in dependence on the angle (besides, it is more easy in application).

In the beginning we calculate the total Br probability in the second approximation at \( l = 1 \) for the angle \( 90^\circ \) by (11) with taking into account (24) (because the component of the Br probability (26) in the second approximation at \( l = 1 \) equals to zero at such angle). Then we normalize the obtained spectrum by the third point of the experimental data (1) (we have such values \( w = 0.179 \) keV and \( W = 10.1 \cdot 10^{-10} \) 1 / keV / decay), which were obtained for the angle \( \beta = 90^\circ \) also. Knowing the normalized factor and using formulas (27), we find the Br probability in the second approximation for the other values of the angle \( \beta \).
The angular values of the Br probability in the second approximation at $l = 1$ are shown in the Table 1. Here, one can see a variation of the Br probability in dependence on the angle $\beta$, however this change is extremely small. The Br probability in the first approximation at $l = 0$ coincides with the Br probability in the second approximation at $l = 1$ for the angle 90°. One can see that a contribution of the Br probability in the first approximation into the total spectrum is the largest for any angle value, i.e. it is extremely larger than the contribution of the component of the Br probability in the second approximation (for the angle values which are distinct from 90°) increases the total Br probability. Absolute and relative variations of the Br probability relatively its maximal and minimal values

$$\Delta W_1(w) = W_{l=1}(w, \beta = 0°) - W_{l=1}(w, \beta = 90°);$$

$$\Delta W_2(w) = \frac{[W_{l=1}(w, \beta = 0°) - W_{l=1}(w, \beta = 90°)]}{W_{l=1}(w, \beta = 90°)} \cdot 100$$

are included into the table also.

Table 1. Angular values of Br probability in $\alpha$-decay of $^{210}$Po in approximation $l = 1$

| $w$ (keV) | $\beta = 0°$ | $\beta = 15°$ | $\beta = 30°$ | $\beta = 45°$ | $\beta = 60°$ | $\beta = 75°$ | $\beta = 90°$ |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 50        | 1.641E-08    | 1.635E-08    | 1.587E-08    | 1.535E-08    | 1.511E-08    | 1.476E-08    | 1.735E-09    |
| 100       | 4.974E-09    | 4.933E-09    | 4.796E-09    | 4.673E-09    | 4.531E-09    | 4.381E-09    | 5.928E-10    |
| 150       | 1.897E-09    | 1.880E-09    | 1.838E-09    | 1.793E-09    | 1.744E-09    | 1.692E-09    | 2.047E-10    |
| 200       | 8.021E-10    | 7.993E-10    | 7.833E-10    | 7.618E-10    | 7.427E-10    | 7.236E-10    | 7.949E-11    |
| 250       | 3.548E-10    | 3.534E-10    | 3.493E-10    | 3.394E-10    | 3.295E-10    | 3.196E-10    | 3.609E-11    |
| 300       | 1.611E-10    | 1.605E-10    | 1.554E-10    | 1.515E-10    | 1.469E-10    | 1.421E-10    | 1.901E-11    |
| 350       | 7.628E-11    | 7.590E-11    | 7.480E-11    | 7.335E-11    | 7.193E-11    | 7.067E-11    | 9.494E-11    |
| 400       | 3.251E-11    | 3.236E-11    | 3.194E-11    | 3.127E-11    | 3.042E-11    | 2.943E-11    | 1.108E-11    |
| 450       | 1.278E-11    | 1.275E-11    | 1.266E-11    | 1.234E-11    | 1.213E-11    | 1.191E-11    | 8.716E-13    |
| 500       | 6.094E-12    | 6.077E-12    | 6.030E-12    | 5.956E-12    | 5.860E-12    | 5.749E-12    | 4.615E-13    |
| 550       | 3.198E-12    | 3.179E-12    | 3.124E-12    | 3.038E-12    | 2.928E-12    | 2.802E-12    | 2.671E-13    |
| 600       | 1.624E-12    | 1.612E-12    | 1.578E-12    | 1.524E-12    | 1.455E-12    | 1.378E-12    | 2.197E-14    |
| 650       | 5.731E-13    | 5.712E-13    | 5.656E-13    | 5.560E-13    | 5.456E-13    | 5.326E-13    | 4.428E-14    |
| 700       | 2.198E-13    | 2.186E-13    | 2.150E-13    | 2.094E-13    | 2.032E-13    | 1.943E-13    | 3.983E-14    |
| 750       | 9.515E-14    | 9.445E-14    | 9.240E-14    | 8.920E-14    | 8.512E-14    | 8.050E-14    | 1.944E-14    |
| 800       | 2.409E-14    | 2.411E-14    | 2.418E-14    | 2.430E-14    | 2.446E-14    | 2.467E-14    | 2.490E-14    |

10 An analyses of the maximums and minimums in the angular Br spectra

Let’s find the values of the angle $\beta$ between the directions of the $\alpha$-particle propagation (or tunneling) and the photon emission, at which the Br probability has the maximal and minimal values. Using a derivative

$$\frac{dW_{l=1}(w, \beta)}{d\beta} = W_{l=0}(w)\left(N^*(w) + N(w) - 2|N(w)|^2 \cos \beta\right) \sin \beta,$$

we find conditions of extremal values of the function $W_{l=1}(w, \beta)$:

$$\sin(\beta) = 0;$$

$$\cos(\beta) = \frac{N^*(w) + N(w)}{2|N(w)|^2} = \frac{Re(N(w))}{Re(N(w))^2 + Im(N(w))^2}.$$  (37)

Calculations for $^{210}$Po for the given potential have shown, that the second condition in (37) in a range $w = 50–800$ keV is not carried out. One can explain this by that the integral $J_{c}(1, w)$ is smaller than the integral $J_{c}(0, w)$ in $10^2–10^4$
times (that is in agreement with a condition of the convergence of the Br spectra at increasing of $l$). From the first condition in (37) we obtain such extremal values for the angle $\beta$:

$$\beta = 0, \pi.$$  

(38)

The Br probability with such angle values has the maximal and minimal values, and between them it varies monotonously for any energy of the photon emitted in the range $w = 50–800$ keV. One can see this also from the Table 1.

11 Inclusion of Woods-Saxon potential into the model

Now let’s analyze, how the Br spectrum in the $\alpha$-decay of the studied nucleus $^{210}$Po is changed, if in the approach for calculations of the Br spectra from the potential of interaction between the $\alpha$-particle and the daughter nucleus, pointed out in Sec. 9 and having the simplified barrier, to pass to a potential with the barrier, constructed on the basis of the account of realistic nuclear forces of interaction between the $\alpha$-particle and the daughter nucleus which is used in realistic nuclear models.

With such a purpose we shall take a potential, proposed in [25] for description of the $\alpha$-decay and synthesis of nuclei. Among extensive set of literature giving us different types of the $\alpha$-nucleus potentials, we have given preference to such a paper because of here we see universal and clear approach for calculation of parameters of the potential after choice of needed such nucleus. In result, we suppose to obtain universal recipe for calculation of the Br spectra after choosing the nucleus.

So, according to [25] (see (6)–(10)), we use such potential of interaction:

$$V(r, \theta, l, Q) = V_C(r, \theta) + V_N(r, \theta, Q) + V_l(r),$$  

(39)

where

$$V_C(r, \theta) = \frac{2Z^2e^2}{r^2} \left(1 + \frac{3R^2}{5r^2} \beta Y_{20}(\theta)\right),$$

$$V_N(r, \theta, Q) = \frac{v(A, Z, Q)}{1 + \exp \left(\frac{r - r_0}{d}\right)},$$

$$V_l(r) = \frac{l(l+1)}{2mr^2}.  \hspace{1cm} (40)$$

Fig. 1. The Br spectra in $\alpha$-decay of $^{210}$Po: $1$ is a curve, extracted from [10] by the instant accelerated model; $2$ is the experimental data [11]; $3$ is the experimental data [3]; $4$ is a curve of the Br probability $W_{l=1}$ in the first approximation $l = 1$ by our approach at $\beta = 90^\circ$; $5$ is a curve of the Br probability component $W_{l=1}$ in the first approximation $l = 1$ at the angle $45^\circ$ by our approach; $6$ is a curve calculated by us with radial integral (6) and formula $dv_i(r)/dr = -\psi_i(r)/w dV(r)/dr$ (it coincides with radial integral (7) in [10] with factor $-1/w$) and further normalization at the third point of data [11]; $7$ is a curve calculated on the basis of the potential (39)–(41) times (that is in agreement with a condition of the convergence of the Br spectra at increasing of $l$). From the first condition in (37) we obtain such extremal values for the angle $\beta$:

$$\beta = 0, \pi.$$  

(38)
At current stage (with a purpose to simplify the numerical calculations of the Br spectra), for determination of the component $V_C(r, \theta)$ we use the formula (7) in [25] on whole region of $r$ (without use of (8) in [25]).

According to (14)–(20) in [25], we calculate the parameters as follows:

$$v(A, Z, Q) = -(30.275 - 0.45838Z/A^{1/3} + 58.270I - 0.24244Q),$$

$$r_m = 1.5268 + R,$$

$$R = R_p(1 + 3.0909/R_p) + 0.1243t,$$

$$R_p = 1.24A^{1/3}(1 + 1.646/A - 0.191I),$$

$$t = I - 0.4A/(A + 200),$$

$$d = 0.49290.$$

According to [26], we see, that the parameter $\beta$ for $^{210}$Po is very small, that points out to a high degree of sphericity of this nucleus. Therefore, for calculation of the Br spectra we note the following:

- In definition of $r_m(\theta)$ and $R(\theta)$ we do not use (21) and (22) of [25] (using (15) from [24]).
- The formalism for calculation of the Br spectra, presented in Sec. 3 and 10, is constructed on the basis of division of the total wave function into its radial and angular components, i.e. in the assumption of spherical symmetry of the decaying nucleus. Therefore, for the nucleus $^{210}$Po our approach for calculation of the Br spectra is applicable with account of realistic nuclear forces also.

Further, we calculate the radial wave function of the decaying system for the potential (39)–(41). This gives us the general solution for the wave function in dependence on the selected energy level for $\alpha$-decay. To achieve, that the found solutions describe the states of the decaying system before and after the spontaneous photon emission, we should take into account boundary conditions in initial and final states. Here, we use such conditions:

$$\chi_i(r \to +\infty) \to G(r) + iF(r),$$
$$\chi_f(r = 0) = 0,$$

where $\chi_{i,f}(r) = \frac{\varphi_{i,f}(r)}{r}, F$ and $G$ are Coulomb functions.

One note, that in absence of a scattering of the $\alpha$-particle on the nucleus, where as the boundary condition for the initial $i$-state a finiteness of the radial wave function $\varphi_i(r)$ should be used at point $r = 0$ ($\chi_i(r = 0) = 0$), for decay we choose a natural requirement, that the radial wave function tend to a spherical divergent wave in asymptotic region ($\chi_i(r \to \infty)$ tend to a plane divergent wave). This condition gives us inevitably the divergence of the total radial wave function in the initial i-state at point $r = 0$ (which real and imaginary parts consist from regular and singular solutions)! One can make sure in this by requiring a fulfillment of the continuity condition for the radial wave function on the whole region on its definition; or by requiring the constancy of the radial flux density, which is distinct from zero and is directed outside in the asymptotic region (and, therefore, it should be not zero near point $r = 0$, that is impossible to execute with null wave function at any chosen point of $r$). This peculiarity complicates essentially the calculations of the Br spectra for the $\alpha$-decay in comparison with the problems of the calculation of the Br spectra for the scattering of charged particles on nuclei (where the essential progress has been achieved and a lot of papers are published).

It turns out, that real and imaginary parts of a sub-integral function of (17) for the calculation of the matrix elements, constructed on the basis of the found solutions for the wave function for the initial and final states, tend to zero at point $r \to 0$! This interesting peculiarity provides the convergence of the matrix elements near point $r = 0$ and, therefore, in the whole region of $r$ (in the asymptotic region the convergence of the wave function is determined by the convergence of the Coulomb functions, considered above). Thus, we resolve the divergence problem in the calculations of the Br spectra in the $\alpha$-decay of the nucleus $^{210}$Po.

The Br spectrum for the nucleus $^{210}$Po with the $\alpha$-nucleus potential (39)–(41) by our approach and calculations is shown by the curve with number 7 in Fig. 4. From here one can see, that the new curve 7 of the Br probability is located very close to the curve 4 for the Br probability with the potential from Sec. 4 with the simplified barrier (and also close to the curve 6 by the approaches 7, 9).

Conclusions:

- The account of nuclear forces, essentially changed a shape of the barrier in its internal part (and essentially changed Br from this internal region), changes very slowly the spectrum of the total Br in the $\alpha$-decay of the nucleus $^{210}$Po (in comparison with the earlier obtained Br spectrum on the basis of the potential from Sec. 9 with the simplified barrier).
- This point confirms the result (obtained early on the basis of the $\alpha$-nucleus potential pointed out in Sec. 4 with the barrier of the simplified shape) that the Br emission from the internal region till point $r$ for barrier maximum
gives the very small contribution into the total Br spectrum. This conclusion coincides logically with a property (found on the basis of microscopic models of the nuclei with their α-decay) of leaving of the α-particle from the nuclear surface during the first decay stage.

12 Conclusions and perspectives

We present the new method of calculation of the Br spectra in the α-decay, where the angle between the directions of the α-particle motion (with tunneling) and the photon emission (used in the experiments [15]) is taken into account. Using it, the angular spectra for the nucleus 210Po are obtained for the first time. Now let us formulate the main conclusions and perspectives:

- The method gives such a dependence of the bremsstrahlung spectrum in the α-decay of 210Po on the angle (this has been obtained for the first time):
  - the first approximation at \( l = 0 \) gives independence of the spectrum on the angle;
  - the second approximation at \( l = 1 \) gives a slow monotonous variation of the slope of the spectrum curve with changing the angle and without a visible change of the shape of the spectrum curve (i.e. without the appearance of humps and holes in the spectrum);
  - for arbitrary energy of the photon emitted in the range of \( w = 50–750 \text{ keV} \) the bremsstrahlung probability is maximal at the angle 0° and is minimal at the angle 180°, between these angular values the bremsstrahlung probability varies monotonously.

- Results for 210Po have obtained on the basis of these approximations:
  - the bremsstrahlung process does not depend on the direction of the leaving the α-particle relatively to the shape of the daughter nucleus before the photon emission (this supposition has been fulfilled for 210Po, because, in accordance with [23] (see Fig. 5 on p. 33), coefficients \( \beta \) of the shape deformation for this nucleus at \( \lambda = 2, 4, 6, 8 \) are extremely close to zero in comparison with other nuclei with other numbers of protons and neutrons, i.e. 210Po is one of the most spherical nuclei);
  - the photon emission does not change the direction of the α-particle propagation (this supposition is suitable for the low-energy photons, and, therefore, it is applicable for analysis of all existing experimental data of the bremsstrahlung spectra, where one can select a region with smaller photons energies for increasing accuracy);
  - the bremsstrahlung spectra have been calculated by means of the α-nucleus potential with the simplified barrier pointed out in Sec. 9 (they coincide with the α-nucleus potential in [21], and also with the potential with the external Coulomb field in [2] and in [7]) and with use of the α-nucleus potential with the barrier, pointed out in Sec. 11 (see [25]) and constructed on the basis of realistic nuclear forces.

- Taking into account nuclear forces in the method gives the following:
  - it does not change dependences on the angle of bremsstrahlung probability in the first and second approximations;
  - it essentially changes the shape of the barrier in its internal region (sufficiently changes the Br from such internal region) and changes very little the spectrum of the total Br in the α-decay for the nucleus 210Po at selected angle value.

From here a question naturally arises: which improvement should be made in the method that results in enough visible changes of the Br spectrum curve, to achieve a better description of the experimental data? Note the following:

- According to our calculations, in consideration of the possibility of the α-particle leaving at the energy of the exited state of the decaying system, the angle of the slope of the Br spectrum curve increases (monotonously). Apparently, it allows to displace the calculated Br curve (for example, by our method) essentially closer to the experimental data [1].

- For obtaining reliable values of the Br spectra for the α-decay one needs to achieve in the calculations the convergence of integrals for such spectra. This leads to the necessity to consider wave functions inside a large space region with the external boundary far enough from the nucleus. From here a new question naturally arises as to taking into account electrons shells of the atom with such nuclear α-decay in the calculation of the Br spectra in the α-decay and one can formulate the following hypothesis about the visible influence of the electrons shells of the atom on the total Br spectrum in the α-decay. One can note that an essential progress has been made by M. Amusia in the study of the Br in atomic physics [27,28]. Moreover, according to [29] (see p.20–21), there is an inevitable influence of the α-decay process at its starting time stage on the electrons shells of the atom whose nucleus decays. So, the α-particle during its leaving (with tunneling) deforms and polarizes these electrons shells. In one’s turn, the changed electrons shells can correct our understanding of the real α-nucleus potential in the model, which should be used for the calculations of the Br spectra (one should note that these effects are still not studied in details). Note that these effects (partially) take place in the same space region, where it is necessary to use the wave functions for the calculation of the matrix elements to achieve converging values of the Br spectra. Therefore, it is desirable to study these effects to obtain for more accurately the total Br spectra of the α-decay.
The inclusion of the $\alpha$-nucleus potential from \cite{25} in our method, deforming the decay barrier, does not essentially displace a point, where the $\alpha$-particle starts to tunnel through the barrier. It turns out that the displacement of this point is much smaller in comparison with the tunneling region and even with a "mixed region", introduced in \cite{8}. Therefore, after taking into account the realistic nuclear forces in the method, the interest to analyze the Br from these regions (with detailed study of tunneling) remains.

We assume that further development of the time formalism for the description of the Br in the $\alpha$-decay at its first stage will give new abilities in the accurate description of the Br. One cannot exclude the assumption about appearance of "holes" in the Br spectra (see \cite{2,8}), that can allow to better describe the experimental data \cite{2,8}. However, in such a case it is not clear how to connect this with the available experimental data \cite{1} without of "holes" in the logical basis of our method.

Now let’s formulate the conclusions, which have a physical sense and on the basis of the calculations for $^{210}$Po by our model:

- The bremsstrahlung in the $\alpha$-decay of the spherical nuclei depends on the angle extremely weakly. In taking into account of the nuclear forces, such dependence is not changed visibly.
- It is not enough to take into account only one angle for the explanation of the difference between the experimental spectra \cite{1} and \cite{2,3} for $^{210}$Po (which equals to 90° and 25°, correspondingly) on the basis of our model and for the explanation of the difference between these experimental spectra and the calculated curves averaged by angle values in the approaches \cite{7,8,9} (that can be supposed in \cite{1,5}).
- The small visible change of the Br spectra after taking into account of the realistic nuclear forces in our method confirms the result (obtained on the basis of the $\alpha$-nucleus potential from Sec. 3 with the barrier of the simplified shape, also see \cite{21}) that the Br from the internal region till point $r$ for barrier maximum gives the very small contribution into the total Br spectrum. This conclusion become natural if to take into account such a property (found on the basis of microscopic models of the $\alpha$-decay) as the $\alpha$-decay starts from leaving of the $\alpha$-particle from the nuclear surface.

In closing, supposing that the Br spectra in the $\alpha$-decay must to change essentially at change of the angle value, we note, how this point can be explained, analyzing this question in theoretical and experimental aspects:

- One can explain such angular change of the bremsstrahlung spectrum so:
  - In the $\alpha$-decay of the (initially) spherical nuclei — by strong angular deformation of the decay barrier and continuous redistribution of the electromagnetic charge (or "nuclear polarization" likely the polarization of the electrons shells during tunneling of the $\alpha$-particle, according to \cite{29} (see p. 20–21)). One can suppose, that here non-central forces between the $\alpha$-particle and nucleons of the daughter nucleus play essential role, which exist in the barrier region mainly. Note, that a serious progress was achieved early in the microscopic approach in study of the bremsstrahlung in scattering of the nucleons and the $\alpha$-particles on the light nuclei (see \cite{30,31,32,33,34,35}), in study of the bremsstrahlung in collisions between heavy ions and the nuclei (see \cite{36}), and in the non-microscopic approaches in study of bremsstrahlung induced by protons during their collisions on heavier nuclei (see \cite{37}).
  - In the $\alpha$-decay of the deformed nuclei — by essential appearance of the angular anisotropy of the $\alpha$-nucleus potential. Then one can extract an information about the shape of the nucleus from the angular bremsstrahlung spectra.
  - Experimental confirmation of the change the Br spectrum in the $\alpha$-decay of the spherical nuclei at change of the angle value gives the following:
    - It will prove an existence of essential microscopic forces between the $\alpha$-particle and the nucleons of the daughter nucleus, reinforcing the angular deformation of the barrier.
    - It will prove a visible influence of the bremsstrahlung on dynamics of the $\alpha$-decay. In accordance with our model, it will be an experimental confirmation of the effect of variation of the barrier penetrability in result of the emission during tunneling of a charged particle, proposed in \cite{20}.

The angular analysis of the bremsstrahlung spectra gives the new additional information about the $\alpha$-decay. Therefore, further angular experimental measurements of the bremsstrahlung spectra will have the interest.

**Acknowledgements**

Authors express their deep gratitude to Dr. I. E. Kashuba for his valuable assistance in development of numerical algorithms for calculation of Coulomb functions with higher accuracy, that has allowed to obtain a convergence in calculations of the bremsstrahlung spectra for $^{210}$Po.
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