Generalisation of Fractional Cox-Ingersoll-Ross Process

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Abstract
In this paper, we define a generalised fractional Cox-Ingersoll-Ross process \((X_t)_{t\geq 0}\) as a square of singular stochastic differential equation with respect to fractional Brownian motion with Hurst parameter \(H \in (0,1)\) taking the form \(dZ_t = (f(t, Z_t)Z_t^{-1}dt + \sigma dW^H_t) / 2\), where \(f(t, z)\) is a continuous function on \(\mathbb{R}_+^2\). Firstly, we show that this differential equation has a unique solution \((Z_t)_{t\geq 0}\) which is continuous and positive up to the time of the first visit to zero. In addition, we prove that the stochastic process \((X_t)_{t\geq 0}\) satisfies the differential equation \(dX_t = f(t, \sqrt{X_t})dt + \sigma \sqrt{X_t} \circ dW^H_t\) where \(\circ\) refers to the Stratonovich integral. Moreover, we prove that the process \((X_t)\) is strictly positive everywhere almost surely for \(H > 1/2\). In the case where \(H < 1/2\), we consider a sequence of increasing functions \((f_n)\) and we prove that the probability of hitting zero tends to zero as \(n \to \infty\). These results are illustrated with some simulations using the generalisation of the extended Cox-Ingersoll-Ross process.

Keywords: Fractional Brownian motion, Fractional Cox-Ingersoll-Ross process, Hitting times, Stratonovich integral.

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1 Introduction

In mathematics of finance, the Cox-Ingersoll-Ross (CIR) process is a diffusion process that was initially introduced by Cox et al. (1985) to model the dynamics of interest rates. In a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), the CIR process satisfies the following stochastic differential equation:

\[
dX_t = \theta(\mu - X_t)dt + \sigma \sqrt{X_t}dW_t,
\]

where \(\theta\) is a positive parameter that represents the speed of reversion of the stochastic process \((X_t)_{t\geq 0}\) towards its long-run mean \(\mu > 0\), \(\sigma > 0\) is the volatility of \((X_t)_{t\geq 0}\) and \((W_t)_{t\geq 0}\) is the standard Brownian motion.

The CIR process has several interesting properties: its sample paths are strictly positive provided that the condition \(2\theta \mu > \sigma^2\) holds, it is mean reverting in the sense that the process is pulled towards its long-run mean \(\mu\) when it goes higher or lower than \(\mu\). Moreover, the CIR process admits a stationary distribution and it is ergodic. For more details, see e.g. ? and ? with references therein. These properties were the main motivations of using the CIR process in modeling the dynamics of interest rates (Cox et al.; 1985) and the random behavior of spot volatility (Heston; 1993).

Since the standard CIR process is driven by a Brownian motion, it does not display memory. Recently, it was shown that there is a certain range of dependency within financial data. For example, spot volatilities may display long-range dependency as discussed by Comte and Renault (1998) and Chronopoulou and Viens (2010), or short range dependency known as “rough volatility” as demonstrated by Gatheral et al. (2018) and Livieri et al. (2018) with references therein. This was a motivation of replacing the standard Brownian motion in (1.1) by a fractional Brownian motion (fBm) as source of randomness.

Although the empirical definition of fractional Cox-Ingersoll-Ross (fCIR) process can be now formulated as a CIR process driven by a fBm, Mishura et al. (2018) define the fCIR process as a stochastic process \((X_t)_{t\geq 0}\) given by

\[
X_t(\omega) = Z_t^2(\omega)1_{[0,\tau(\omega)]}, \quad \forall t \geq 0, \ \omega \in \Omega,
\]

where \((Z_t)_{t\geq 0}\) is a fractional Ornstein–Uhlenbeck process that satisfies the stochastic differential equation \(dZ_t = -\frac{1}{2}\theta Z_t dt + \frac{1}{2}\sigma dW_t^H\), with \(W_t^H\) a fBm of Hurst parameter \(H \in (0,1)\), and where \(\tau\) is the first time the process \((Z_t)_{t\geq 0}\) hits zero. On the other hand, Mishura and Yurchenko-Tytarenko (2018) consider the process \((Z_t)_{t\geq 0}\) defined by the equation

\[
dZ_t = \frac{1}{2}\left(\mu - \theta Z_t^2\right)Z_t^{-1}dt + \frac{1}{2}\sigma dW_t^H,
\]

(1.2)
(where $\mu$, $\theta$ and $\sigma > 0$ are parameters) and the corresponding process
\[ X_t(\omega) = Z_t^2(\omega)1_{[0,\tau(\omega)]}, \quad \forall t \geq 0, \quad \omega \in \Omega. \]

Mishura and Yurchenko-Tytarenko (2018) proved that the fCIR process $(X_t)_{t \geq 0}$ given by (1.2) verifies the equation given by
\[ X_t = X_0 + \int_0^t (\mu - \theta X_s)ds + \sigma \int_0^t \sqrt{X_s} \circ dW_s^H, \tag{1.3} \]
where $\int_0^t \sqrt{X_s} \circ dW_s^H$ is a Stratonovich integral with respect to the fBm $(W_s^H)_{s \geq 0}$, $H \in (0, 1)$. Moreover, they proved that the process $(X_t)_{t \geq 0}$ is strictly positive and will never hit zero for $H > \frac{1}{2}$. For $H < \frac{1}{2}$, they obtained that the probability of hitting zero converges to 0 when the speed of reversion $\theta$ tends to infinity. In addition, Hong et al. (2019) investigated strong convergence of some numerical approximations for fCIR process in the case where $H \geq \frac{1}{2}$.

In this paper, we extend the results of Mishura and Yurchenko-Tytarenko (2018) to a general process $(Z_t)_{t \geq 0}$ defined by the stochastic differential equation
\[ dZ_t = \frac{1}{2} f(t, Z_t)Z_t^{-1}dt + \frac{1}{2} \sigma dW_t^H, \quad Z_0 > 0, \tag{1.4} \]
where $f : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$, $(t, x) \mapsto f(t, x)$, is a continuous drift function. We consider as previously a fCIR process defined by
\[ X_t(\omega) = Z_t^2(\omega)1_{[0,\tau(\omega)]}, \quad \forall t \geq 0, \quad \omega \in \Omega. \]

This general case has been previously investigated by Hu et al. (2008) and Nualart and Ouknine (2002) under some additional assumptions on the drift function $f$. Hu et al. (2008) proved that if $H > 1/2$ and if the drift function $f(t, x)$ is such that the function $g$ given by $g(t, x) = f(t, x)/(2x)$ satisfies the following conditions,

(C1) $g : [0, \infty) \times (0, \infty) \to [0, \infty)$ is a nonnegative continuous function which has a continuous partial derivative $\partial g(t, x)/\partial x \leq 0$ for all $(t, x) \in (0, \infty) \times (0, \infty)$,

(C2) There exist $x_1 > 0$, $a > \frac{1}{H} - 1$ and a continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) > 0$ for all $t > 0$ such that $g(t, x) \geq \varphi(t)x^{-a}$ for all $t \geq 0$ and $0 < x < x_1$,

then (1.4) has a strictly positive solution $(Z_t)_{t \geq 0}$ that is, almost surely $Z_t > 0$ for all $t > 0$. (See Theorem 2.1 and Theorem 3.1 in Hu et al. (2008)). In addition, they also showed that

(C3) if there exists a function $h : [0, \infty) \to [0, \infty)$ which is nonnegative and locally bounded such that $g(t, x) \leq h(t)(1 + 1/x)$ for all $t \geq 0$ and $x > 0$, then the solution $(Z_t)_{t \geq 0}$ is such that for any fixed $T > 0$,
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |Z_t|^p \right) < \infty, \quad \forall p > 0. \]
Our objective is to study the solution to the stochastic differential equation (1.4) under some mild conditions weaker than conditions than (C1) and (C2). We shall consider the following conditions:

\[(D1)\] The function \(g : [0, \infty) \times (0, \infty) \to (-\infty, \infty)\) defined by \(g(t, x) = f(t, x)/(2x)\) is continuous and admits a continuous partial derivative with respect to \(x\) on \((0, \infty)\). In addition, there exists a number \(x^* > 0\) such that for every \(x > x^*, g(t, x) < 0, \) for all \(t \geq 0\).

\[(D2)\] for any \(T > 0\), there exists \(x_T > 0\) such that \(f(t, x) > 0\) for all \(0 < t \leq T, 0 \leq x \leq x_T\).

Condition \((D2)\) implies that for all \(S > 0\) and \(T > 0\), there exists \(x_T > 0\) such that \(\inf\{f(t, x) : S \leq t \leq T, 0 \leq x \leq x_T\} > 0\).

We shall first show that condition \((D1)\) and the initial condition \(Z_0 > 0\) guarantee the existence, uniqueness, continuity and positiveness of a solution \((Z_t)\) to equation \((1.4)\) up to the first time it hits zero. In addition, We will show that the square stochastic process \((X_t)_{t \geq 0}\) (which is also defined up to the first time it hits zero) satisfies the stochastic differential equation

\[
dX_t = f(t, \sqrt{X_t})dt + \sigma \sqrt{X_t} \circ dW^H_t, \ X_0 > 0, H \in (0, 1)\]

We shall also prove that in the case where \(H > 1/2\), the solution to the stochastic differential equation \((1.4)\) is not only positive up to the time of the first visit to zero but it is strictly positive everywhere. In other words, almost surely it never hits zero on the whole line \([0, \infty)\). It is remarkable that this result is true under mild conditions \((D1)\) and \((D2)\).

In the case where \(H < 1/2\), we obtain that the probability of the process \((X_t)_{t \geq 0}\) hitting zero is small if the drift function \(f\) is sufficiently large. More precisely, if \((f_n)_{n \in \mathbb{N}}\) is an increasing sequence of continuous functions \(f_n\) defined on \([0, \infty) \times [0, \infty)\) and taking values in \(\mathbb{R}\) and satisfying conditions \((D1)\) and \((D2)\), such that \(\lim_{n \to \infty} f_n = \infty\) and \((X^n_t)\) is the solution to equation \((1.4)\) corresponding to \(f_n\) (up to the first time it hits zero), then the probability of \((X^n_t)\) hitting zero converges to 0 as \(n \to \infty\). Our results generalize the results recently by Mishura and Yurchenko-Tytarenko (2018) for the function \(f(t, x) = \frac{1}{2}(\mu - \theta x^2)\) for constants \(\mu > 0\) and \(\theta > 0\). We provide some illustrating examples using simulation.

More recently? studied the stochastic differential equation

\[
dX_t = g(X_t)dt + \sigma X_t^\beta dW^H_t\]  \(1.5\)
for $1/2 < H < 1$, $1/2 \leq \beta < 1$ and where the function $g$ is such that there exists a continuously differentiable function $f$ defined on $(0, \infty)$ such that: (1) $g(x) = x^\beta f(x^{1-\beta})$, (2) there exist $a > 0$ and $\alpha \geq 0$ such that $f(x) > ax^{-(1+\alpha)}$ for sufficiently small $x$ and (3) there exists $K \in \mathbb{R}$ such that $f'(x) \leq K$. Under these conditions, it is proven that equation (1.5) has a unique and positive solution and derived an important estimator of the $H$ for the solution. In some sense our model (1.4) extends (1.5). It would be interesting to carry out an analysis of the $H$ parameter of the solution to equation (1.4) as in ?.

The rest of this paper is organised as follows. Section 2 discusses the existence and uniqueness of the generalised $fCIR$ processes. In Section 3 we show such processes satisfy a stochastic differential equation with respect to Stratonovich integral. The positiveness of these processes for $H > 1/2$ is given in section 4 and for $H < 1/2$ in section 5. Section 6 contains some illustrations of the main results using simulation and finally the last section contains some concluding remarks.

2 The generalised $fCIR$ processes

In this section, we consider a more general process $(Z_t)_{t \geq 0}$ defined by the differential stochastic equation:

$$dZ_t = \frac{f(t, Z_t)}{2Z_t} dt + \frac{\sigma}{2} dW_t^H, \quad Z_0 > 0$$

where $f : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$, $(t, z) \mapsto f(t, z)$ is a continuous function satisfying conditions (D1) and (D2). We shall first discuss the existence and uniqueness of the solution to (2.1).

**Theorem 2.1.** If the drift function $f(t, x)$ satisfies condition (D1), then for all $0 < H < 1$, equation (2.1) has a unique solution $(Z_t)_{t \geq 0}$ which is continuous and positive up to the time of the first visit to 0.

**Proof.** Let $\ell > 0$ be a small number such that $\ell < Z_0$ and $\ell < x^*$. For fixed $T > 0$, consider the sequence of processes $(Z_n(t))$ defined on $[0, T]$ by

$$Z_0(t) = Z_0, \quad \text{for all} \quad t \in [0, T]$$

and for all $n \in \mathbb{N}$,

$$Z_{n+1}(t) = \left\{ \begin{array}{ll} Z_0 + \int_0^t g(s, Z_n(s)) ds + \frac{\sigma}{2} W_t^H, & \text{if} \quad t \leq \tau_{n, \ell} \\ \ell & \text{otherwise} \end{array} \right.$$ 

where $g(t, x) = f(t, x)/(2x)$ and $\tau_{n, \ell} = \inf\{0 \leq t \leq T : Z_n(t) = \ell\}$ (the first time the process $(Z_n(t))$ reaches the level $\ell$ with $\inf(\emptyset) = +\infty$). Clearly, if $Z_n(t)$ does
not reach the level \( \ell \) on \([0, T]\), then \( Z_{n+1}(t) \) is defined by
\[
Z_{n+1}(t) = Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2} W_t^H
\]
for all \( t \in [0, T] \). For instance
\[
Z_1(t) = Z_0 + \int_0^t g(s, Z_0)ds + \frac{\sigma}{2} W_t^H, \quad t \in [0, T].
\]

We want to show that there exists a number \( \eta > 0 \) independent of \( n \) and such that \( \tau_{n, \ell} \geq \eta \) for all \( n \). It is clear that
\[
\tau_{n, \ell} \geq \tau_{n+1, \ell}
\]
because \( Z_{n+1}(t) = \ell \) for all \( t \geq \tau_{n, \ell} \). The function \( t \mapsto g(t, Z_n(t)) \) is bounded on \( t \in [0, \tau_{n, \ell}] \). Indeed, for every \( t \in [0, \tau_{n, \ell}] \), write \([0, t] = I_1 \cup I_2\) where \( I_1 \) is the union of sub-intervals of \([0, \tau_{n, \ell}]\) where \( Z_n \leq x^* \) and \( I_2 \) is the union of sub-intervals of \([0, \tau_{n, \ell}]\) where \( Z_n > x^* \). Then
\[
\int_0^t g(s, Z_n(s))ds = \int_{I_1} g(s, Z_n(s))ds + \int_{I_2} g(s, Z_n(s))ds \leq \int_{I_1} g(s, Z_n(s))ds
\]
because \( g(s, Z_n(s)) < 0 \) for \( s \in I_2 \) by condition (D1). Therefore
\[
Z_{n+1}(t) = Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2} W_t^H
\]
\[
\leq Z_0 + \int_{I_1} g(s, Z_n(s))ds + \frac{\sigma}{2} W_t^H.
\]
Let
\[
A = \sup(\{|g(s, z)| : s \in [0, T] \text{ and } z \in [\ell, x^*]\}).
\]
Clearly \( A < \infty \) because \( g \) is continuous on \([0, +\infty) \times (0, +\infty)\). Because for \( s \in I_1 \), \( Z_n(s) < x^* \), then
\[
Z_{n+1}(t) \leq Z_0 + At + \frac{\sigma}{2} W_t^H \leq B
\]
where
\[
B = Z_0 + AT + \frac{\sigma}{2} \sup_{0 \leq t \leq T} |W_t^H|.
\]
(Here the bound \( B \) is independent of \( n \)). Therefore for all \( t \in [0, \tau_{n, \ell}] \), we have that \( Z_{n+1}(t) \in [\ell, B] \) for all \( n \in \mathbb{N} \). Since \( \tau_{n+1, \ell} \leq \tau_{n, \ell} \), it follows in particular that
\[
Z_{n+1}(t) \in [\ell, B] \text{ for all } 0 \leq t \leq \tau_{n+1, \ell}.
\]
Moreover, since by definition,
\[
Z_{n+1}(t) = Z_0 + \int_0^t g(s, Z_n(s))ds + \frac{\sigma}{2} W_t^H,
\]
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taking \( t = \tau_{n+1,\ell} \) yields

\[
\ell = Z_0 + \int_0^{\tau_{n+1,\ell}} g(s, Z_n(s))ds + \frac{\sigma}{2} W^{H}_{\tau_{n+1,\ell}}.
\]

Set

\[
K = \sup(\{|g(s, z)| : s \in [0, T] \text{ and } z \in [\ell, B]\}),
\]

then

\[
\ell \geq Z_0 - K\tau_{n+1,\ell} + \frac{\sigma}{2} W^{H}_{\tau_{n+1,\ell}}.
\]

That is

\[
\frac{\sigma}{2} W^{H}_{\tau_{n+1,\ell}} \leq \ell - Z_0 + K\tau_{n+1,\ell}.
\]

This implies that

\[
\tau_{n+1,\ell} \geq \inf\{t \geq 0 : \frac{\sigma}{2} W^{H}_t \leq \ell - Z_0 + Kt\}.
\]

Set

\[
\eta = \inf\{t \geq 0 : \frac{\sigma}{2} W^{H}_t \leq \ell - Z_0 + Kt\}.
\]

Clearly \( \eta > 0 \) because obviously the fractional Brownian motion \( (W^{H}_t) \) starts at 0, that is, \( W^{H}_0 = 0 \) and \( \ell < Z_0 \). Hence, \( \tau_{n+1,\ell} \geq \eta > 0 \) uniformly for \( n \) (and \( \eta \) is independent of \( n \)).

Let \( \tau_{\ell} = \inf_{n \geq 0} \tau_{n,\ell} \), then \( \tau_{\ell} \geq \eta > 0 \). We will then show that the problem has a positive solution on the interval \([0, \tau_{\ell}]\). For all \( n \) and all \( t \in [0, \tau_{\ell}] \), \( Z_n(t) \geq \ell \) and \( Z_n(t) \leq B \).

Since the function \( g(t, x) \) admits a partial derivative with respect to \( x \) on \((0, \infty)\), then in particular for fixed \( t \), the function \((t, x) \mapsto g(t, x)\) is uniformly Lipschitz for \( x \) in a bounded closed interval away from 0.

Since for all \( t \in [0, \tau_{\ell}] \), \( Z_n(t) \in [\ell, B] \), then there exists \( C > 0 \) such that

\[
|(g(t, Z_n(t)) - g(t, Z_{n-1}(t)))| \leq C|Z_n(t) - Z_{n-1}(t)|
\]

for all \( t \in [0, \tau_{\ell}] \). Therefore,

\[
|Z_{n+1}(t) - Z_n(t)| \leq \int_0^t |(g(s, Z_n(s)) - g(s, Z_{n-1}(s)))| ds
\]

\[
\leq C \int_0^t |Z_n(s) - Z_{n-1}(s)| ds.
\]

Then an application of Grönwall’s lemma implies that the sequence \((Z_n(t))\) converges uniformly on the interval \([0, \tau_{\ell}]\) and hence its limit is a positive continuous
solution to (1.4) on $[0, \tau]$. Therefore, equation (1.4) admits a positive solution up to the first time it hits the level $\ell$. For the uniqueness of the solution, if $(Z_t)$ and $(Y_t)$ are two solutions on some interval $[0, \tau_\ell)$ starting at the same point $Z_0$, then for any $t < \tau_\ell$,

$$|Z_t - Y_t| \leq \int_0^t |(g(s, Z_s) - g(s, Y_s))| ds \leq C \int_0^t |Z_s - Y_s| ds.$$  

Again Grönwall’s lemma implies that $Z_t = Y_t$ everywhere in $[0, \tau_\ell)$. Since $\ell > 0$ can be taken arbitrary small, this implies the existence of a solution up to the first time it hits 0.  

**Definition 2.1.** The stochastic process $(X_t)_{t \geq 0}$ defined by

$$X_t = Z_t^2 1_{[0, \tau)}(t), \quad t \geq 0, \quad \tau = \inf \{t > 0 : Z_t = 0\}$$  

where $(Z_t)_{t \geq 0}$ is the solution to (2.1) will be called the generalised $fCIR$ process defined by the function $f$.

**Remark.** When $f(t, z) = (\mu - \theta z^2)$ where $\theta$ and $\mu$ are constants, the generalised $fCIR$ process $(X_t)_{t \geq 0}$ coincides with the $fCIR$ process given by Mishura and Yurchenko-Tytarenko (2018). In addition, when the speed of reversion or the long-run mean are time dependent, that is $\theta = \theta_t$ or $\mu = \mu_t$ with $f(t, z) = \theta_t(\mu_t - z^2)$, the process $(X_t)_{t \geq 0}$ can be regarded as an extended $fCIR$ process (or a fractional Hull-White model that has been used by ? for pricing options). The latter process is very important not only because of the mean-reverting and positiveness properties but also because of the possibility of a perfect calibration of parameters to the market data.

### 3 Connection to Stratonovich integral

We recall that given two stochastic processes $(X_t)_{t \in [0,T]}$ and $(Y_t)_{t \in [0,T]}$, the pathwise Stratonovich integral $\int_0^T Y_s \circ dX_s$ is defined as a pathwise limit (when it exists) given by

$$\lim_{n \to \infty} \sum_{i=1}^n \left( \frac{Y_{t_i} + Y_{t_{i-1}}}{2} \right) (X_{t_i} - X_{t_{i-1}}),$$  

where $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ is a partition of the interval $[0, T]$ such that $\sup_{0 \leq i \leq n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$. We have the following result.

**Theorem 3.1.** Assume that the function $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is continuous and satisfies (D1). Then the corresponding generalised $fCIR$ process $(X_t)$ defined by (2.2) up to the first time it hits zero satisfies the equation:

$$X_t = X_0 + \int_0^t f(s, \sqrt{X_s}) ds + \sigma \int_0^t \sqrt{X_s} \circ dW^H_s,$$  

where $\int_0^t \sqrt{X_s} \circ dW^H_s$ is the Stratonovich integral.
Proof. Our proof is a generalisation of a proof given by Mishura and Yurchenko-Tytarenko (2018) applied to the particular function \( f(t, x) = f(t, x) = \frac{1}{2} (\mu - \theta x^2) \). As already discussed condition (D1) implies the uniqueness of the solution \((Z_t)\) up to the first time it hits zero. For \( \tau := \inf \{ s > 0 : Z_s = 0 \} \) and \( t \in [0, \tau) \) fixed, we have from (2.1) and (2.2) that

\[
X_t = Z_t^2 = \left( Z_0 + \frac{1}{2} \int_0^t f(s, Z_s) Z_s^{-1} ds + \frac{\sigma}{2} dW_t^H \right)^2, (3.3)
\]

where \( Z_0 \) is an initial value of the stochastic process \((Z_t)_{t \in [0, \tau)}\). In discrete time, assume that the interval \([0, t]\) is subdivided into \( N \) equal subintervals with \( 0 < t_1 < \cdots < t_N = t \), the time-steps \( \delta t = t/N \), and \( t_i = i\delta t, \ i = 0, \cdots, N \). Then it follows that

\[
X_t = X_0 + \sum_{i=1}^{N} (X_{t_i} - X_{t_{i-1}})
\]

\[
= X_0 + \sum_{i=1}^{N} \left( \left[ Z_0 + \frac{1}{2} \int_0^{t_i} f(s, Z_s) Z_s^{-1} ds + \frac{\sigma}{2} dW_{t_i}^H \right]^2 - \left[ Z_0 + \frac{1}{2} \int_0^{t_{i-1}} f(s, Z_s) Z_s^{-1} ds + \frac{\sigma}{2} dW_{t_{i-1}}^H \right]^2 \right).
\]

Then

\[
X_t = X_0 + \sum_{i=1}^{N} \left[ \frac{1}{2} \int_{t_{i-1}}^{t_i} f(s, Z_s) Z_s^{-1} ds + \frac{\sigma}{2} \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \right]
\]

\[
\times \left[ 2Z_0 + \frac{1}{2} \left( \int_0^{t_i} f(s, Z_s) Z_s^{-1} ds + \int_0^{t_{i-1}} f(s, Z_s) Z_s^{-1} ds \right) + \frac{\sigma}{2} \left( W_{t_i}^H + W_{t_{i-1}}^H \right) \right].
\]

The last equation above is obtained by factorising the difference of two squares. After some expansions, we obtain that
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Let

\[ X_t = X_0 + Z_0 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s, Z_s) Z_s^{-1} ds \]

\[ + \frac{1}{4} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s, Z_s) Z_s^{-1} ds \left( \int_{0}^{t_i} f(s, Z_s) Z_s^{-1} ds + \int_{0}^{t_{i-1}} f(s, Z_s) Z_s^{-1} ds \right) \]

\[ + \frac{\sigma^2}{4} \sum_{i=1}^{N} \left( W_{t_i}^H + W_{t_{i-1}}^H \right) \left( W_{t_i}^H + W_{t_{i-1}}^H \right) \]

\[ + \frac{\sigma}{4} \sum_{i=1}^{N} \left( \int_{0}^{t_i} f(s, Z_s) Z_s^{-1} ds + \int_{0}^{t_{i-1}} f(s, Z_s) Z_s^{-1} ds \right) \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \]

\[ + \sigma Z_0 \sum_{i=1}^{N} \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \]

where

\[
\begin{align*}
\mathcal{I}_1(N, t, Z_t) &= Z_0 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s, Z_s) Z_s^{-1} ds \\
\mathcal{I}_2(N, t, Z_t) &= \frac{1}{4} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(s, Z_s) Z_s^{-1} ds \left( \int_{0}^{t_i} f(s, Z_s) Z_s^{-1} ds + \int_{0}^{t_{i-1}} f(s, Z_s) Z_s^{-1} ds \right) \\
\mathcal{I}_3(N, t, Z_t) &= \frac{\sigma^2}{4} \sum_{i=1}^{N} \left( W_{t_i}^H + W_{t_{i-1}}^H \right) \int_{t_{i-1}}^{t_i} f(s, Z_s) Z_s^{-1} ds \\
\mathcal{I}_4(N, t, Z_t) &= \sigma Z_0 \sum_{i=1}^{N} \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \\
\mathcal{I}_5(N, t, Z_t) &= \frac{\sigma^2}{4} \sum_{i=1}^{N} \left( \int_{0}^{t_i} f(s, Z_s) Z_s^{-1} ds + \int_{0}^{t_{i-1}} f(s, Z_s) Z_s^{-1} ds \right) \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \\
\mathcal{I}_6(N, t, Z_t) &= \frac{\sigma}{4} \sum_{i=1}^{N} \left( W_{t_i}^H + W_{t_{i-1}}^H \right) \left( W_{t_i}^H - W_{t_{i-1}}^H \right).
\end{align*}
\]

and

Set

\[ I(t) = \int_{0}^{t} f(s, Z_s) Z_s^{-1} ds. \]
Generalisation of Fractional Cox-Ingersoll-Ross Process

Then it follows that

$$
\sum_{k=1}^{3} \mathcal{I}_k(N, t, Z_t) = \sum_{i=1}^{N} \left( I(t_i) - I(t_{i-1}) \right) Z_0 \\
+ \left( I(t_i) - I(t_{i-1}) \right) \left( \frac{(I(t_i) + I(t_{i-1}))}{4} + \frac{\sigma(W_{t_i}^H + W_{t_{i-1}}^H)}{4} \right).
$$

Then

$$
\lim_{N \to \infty} \sum_{k=1}^{3} \mathcal{I}_k(N, t, Z_t) = Z_0 I(t) + \frac{1}{2} \int_0^t \left( I(s) + \sigma W_s^H \right) \circ dI(s).
$$

Since $I(s)$ is differentiable, then it follows that

$$
\lim_{N \to \infty} \sum_{k=1}^{3} \mathcal{I}_k(N, t, Z_t) = Z_0 I(t) + \frac{1}{2} \int_0^t \left( I(s) + \sigma W_s^H \right)dI(s) \\
= \left( \int_0^t f(s, Z_s)Z_s^{-1}ds \right) Z_0 \\
+ \frac{1}{2} \int_0^t \left( \left( \int_0^s f(u, Z_u)Z_u^{-1}du \right) + \sigma W_s^H \right) f(s, Z_s)Z_s^{-1}ds \\
= \int_0^t f(s, Z_s)Z_s^{-1} \left( Z_0 + \frac{1}{2} \int_0^s f(u, Z_u)Z_u^{-1}du + \frac{\sigma}{2} W_s^H \right) ds \\
= \int_0^t f(s, Z_s)Z_s^{-1}Z_s ds = \int_0^t f(s, Z_s)ds.
$$

On the other hand

$$
\sum_{k=4}^{6} \mathcal{I}_k(N, t, Z_t) = \sigma Z_0 \sum_{i=1}^{N} \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \\
+ \sum_{i=1}^{N} \frac{\sigma I(t_i) + I(t_{i-1})}{2} \left( W_{t_i}^H - W_{t_{i-1}}^H \right) \\
+ \frac{\sigma^2}{4} \sum_{i=1}^{N} \left( W_{t_i}^H + W_{t_{i-1}}^H \right) \left( W_{t_i}^H - W_{t_{i-1}}^H \right).
$$
Therefore,

$$\lim_{N \to \infty} \sum_{k=4}^{6} \mathcal{I}_k(N, t, Z_t) = \sigma Z_0 W_t^H + \frac{\sigma}{2} \int_0^t I(s) \circ dW_s^H + \frac{\sigma^2}{2} \int_0^t W_s^H \circ dW_s^H$$

$$= \sigma Z_0 W_t^H + \frac{\sigma}{2} \int_0^t \left( \int_0^s f(u, Z_u)Z_u^{-1} du \right) \circ dW_s^H + \frac{\sigma^2}{2} \int_0^t W_s^H \circ dW_s^H$$

$$= \sigma Z_0 W_t^H + \frac{\sigma}{2} \int_0^t (2Z_s - 2Z_0 - \sigma W_s^H) \circ dW_s^H + \frac{\sigma^2}{2} \int_0^t W_s^H \circ dW_s^H$$

$$= \sigma \int_0^t Z_s \circ dW_s^H.$$

The third equality follows the fact that

$$\int_0^s f(u, Z_u)Z_u^{-1} du = 2Z_s - 2Z_0 - \sigma W_s^H$$

because

$$Z_s = Z_0 + \frac{1}{2} \int_0^s f(u, Z_u)Z_u^{-1} du + \frac{\sigma}{2} W_s^H.$$

Now taking $N \to \infty$, that is, $\delta t \to 0$, yields

$$\lim_{N \to \infty} X_{\delta t N} = \lim_{\delta t \to 0} X_{\delta t N} = X_0 + \lim_{N \to \infty} \sum_{k=4}^{6} \mathcal{I}_k(N, t, Z_t)$$

$$= X_0 + \int_0^t f(s, Z_s) ds + \sigma \int_0^t Z_s \circ dW_s^H$$

$$= X_0 + \int_0^t f(s, \sqrt{X_s}) ds + \sigma \int_0^t \sqrt{X_s} \circ dW_s^H.$$

It follows that $dX_t = f(t, \sqrt{X_t}) dt + \sigma \sqrt{X_t} \circ dW_t^H$, which concludes the proof. □

4 Analysis of positiveness of $(X_t)_{t \geq 0}$ for $H > 1/2$

**Theorem 4.1.** Assume that $H > \frac{1}{2}$. Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a continuous function satisfying conditions (D1) and (D2). Then the process $(Z_t)_{t \geq 0}$ defined by

$$dZ_t = \frac{f(t, Z_t)}{2Z_t} dt + \frac{\sigma}{2} dW_t^H, \quad Z_0 > 0, \quad (4.1)$$

is strictly positive everywhere almost surely.

In the proof we shall make use of the following Hölder continuous property of fractional Brownian motion of index $H$. In the probability space $(\Omega, \mathcal{F}, P)$, $\exists \mathcal{Y} \subset$
\[ \Omega, \quad \mathbb{P}(\Omega') = 1, \text{ such that } \forall \omega \in \Omega', \]
\[ \forall 0 \leq s \leq t \text{ and } \forall \alpha > 0, \quad \exists c = c(\omega, \alpha) : \]
\[ |W_t^H(\omega) - W_s^H(\omega)| \leq c|t - s|^{H - \alpha}. \quad (4.2) \]

For more background on fBm, we refer the reader to \cite{Nourdin2012} and Mishura and Yurchenko-Tytarenko (2013).

**Proof.** We have proven that condition (D1) guarantees the existence, uniqueness and positiveness of a solution up to the first time it hits zero. We shall now prove that the mere condition (D2) that the function \( f(t, x) > 0 \) on \([0, T] \times (0, x_T] \) for any \( T > 0 \) and \( x_T \) depending on \( T \) implies that the process \((Z_t)_{t \geq 0}\) never hits zero almost surely. We shall indeed prove that
\[ \mathbb{P}\{\omega \in \Omega : \tau(\omega) = \infty\} = 1, \] where \( \tau(\omega) = \inf\{t \geq 0 : Z_t(\omega) = 0\} \).

Let us assume that
\[ \mathbb{P}\{\omega \in \Omega : \tau(\omega) = \infty\} < 1 \text{ or equivalently } \mathbb{P}\{\tau < T\} > 0, \]
for some fixed real \( T > 0 \), and prove that this leads to a contradiction. From now on we fix a real number \( x_T \) depending on \( T \) for which condition (D2) holds. Since the sample paths of fBm \((W_t^H)_{t \geq 0}\) are (almost surely) locally Hölder continuous of order \( H - \alpha \) (for each small number \( \alpha > 0 \)), then we can fix as in (Mishura and Yurchenko-Tytarenko; 2018) a subset \( \Omega_1 \) of the underlying sample space \( \Omega \) with \( \mathbb{P}(\Omega_1) = 1 \) such that for each \( \omega \in \Omega_1, \alpha > 0, \)
\[ |W_t^H(\omega) - W_s^H(\omega)| \leq c|t - s|^{H - \alpha}, \quad \forall s, t \in [0, T] \]
where \( c = c(T, \omega, \alpha) \) is a random constant depending on \( T, \omega \) and \( \alpha \). Our assumption \( \mathbb{P}(\tau < T) > 0 \) implies
\[ \mathbb{P}(\tau < T) = \mathbb{P}\{\omega \in \Omega_1 : \tau(\omega) < T\} > 0. \]

Now choose \( \omega \in \Omega_1 \) with \( \tau(\omega) < T \). It is given that the process \((Z_t)\) starts at the point \( Z_0 > 0 \). Using condition (D2), for fixed \( T > 0 \), we take a point \( x_T \) small enough such that \( 0 < x_T < Z_0 \). Let \( S \) be the first time \((Z_t)\) hits the value \( x_T \), that is, \( S = \inf\{t : Z_t = x_T\} \). Consider a small number \( \varepsilon \) such that \( 0 < \varepsilon < x_T \). Since \( f(t, x) > 0 \) for all \( 0 < t \leq T \) and \( 0 \leq x \leq x_T \), then in particular that \( f(t, x) > 0 \) for all \( S \leq t \leq T \) and \( 0 \leq x \leq \varepsilon \). Let \( A = \inf\{f(t, x) : S \leq t \leq T, 0 \leq x \leq x_T\} \).

Clearly \( A > 0 \). Let \( \tau_\varepsilon \) be the last time the process \((Z_t)\) hits \( \varepsilon \) before reaching zero, that is,
\[ \tau_\varepsilon(\omega) = \sup\{t \in (0, \tau(\omega)) : Z_t(\omega) = \varepsilon\}. \]

Clearly \( 0 < S < \tau_\varepsilon < \tau < T \). The equality
\[ Z_t = Z_0 + \frac{1}{2} \int_0^t f(s, Z_s)Z_s^{-1}ds + \frac{\sigma}{2} W_t^H, \]
implies in particular that
\[ Z_\tau - Z_{\tau_\epsilon} = \frac{1}{2} \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds + \frac{\sigma}{2} (W^H_\tau - W^H_{\tau_\epsilon}). \]

Since \( Z_\tau = 0 \) and \( Z_{\tau_\epsilon} = \epsilon \), then
\[ \frac{1}{2} \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds + \frac{\sigma}{2} (W^H_\tau - W^H_{\tau_\epsilon}) = -\epsilon \]
or equivalently,
\[ \frac{\sigma}{2} (W^H_\tau - W^H_{\tau_\epsilon}) = -\epsilon - \frac{1}{2} \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds. \]

Since for all \( s \in [\tau_\epsilon, \tau) \subset [S, T], \) it is the case that \( Z_s \in [0, \epsilon] \subset [0, x_T], \) then by condition (D2),
\[ f(s, Z_s) > 0 \text{ for all } s \in [\tau_\epsilon, \tau]. \]

This implies that
\[ \frac{\sigma}{2} |W^H_\tau - W^H_{\tau_\epsilon}| = \epsilon + \frac{1}{2} \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds \]
or equivalently
\[ \sigma |W^H_\tau - W^H_{\tau_\epsilon}| = 2\epsilon + \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds. \]

Since \( \omega \in \Omega_1, \) and \( \tau_\epsilon, \tau \in [0, T], \) then
\[ |W^H_\tau - W^H_{\tau_\epsilon}| < c|\tau - \tau_\epsilon|^{H-\alpha}. \]
Hence
\[ 2\epsilon + \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds \leq \sigma c|\tau - \tau_\epsilon|^{H-\alpha}. \]

On the other hand
\[ \int_{\tau_\epsilon}^\tau f(s, Z_s)Z_s^{-1}ds \geq \int_{\tau_\epsilon}^\tau A\epsilon^{-1}ds = A\epsilon^{-1}(\tau - \tau_\epsilon). \tag{4.3} \]

Therefore
\[ 2\epsilon + A\epsilon^{-1}(\tau - \tau_\epsilon) \leq \sigma c|\tau - \tau_\epsilon|^{H-\alpha} \]
from which it follows that
\[ A\epsilon^{-1}(\tau - \tau_\epsilon) - c\sigma|\tau - \tau_\epsilon|^{H-\alpha} + 2\epsilon \leq 0. \tag{4.4} \]

Consider the function \( F_\epsilon \) defined by
\[ F_\epsilon(x) = A\epsilon^{-1}x - c\sigma x^{H-\alpha} + 2\epsilon, \]
that is, \( F_\varepsilon(x) \) is obtained by replacing \( \tau - \tau_\varepsilon \) with \( x \). Then the inequality (4.4) yields
\[
F_\varepsilon(\tau - \tau_\varepsilon) \leq 0,
\]
for every \( \varepsilon > 0 \). The next step in this proof is to show that the inequality in (4.5) does not hold. In fact we shall construct a number \( \varepsilon^* > 0 \) such that uniformly for all \( 0 < \varepsilon < \varepsilon^* \), \( F_\varepsilon(x) > 0 \) for all \( x \geq 0 \). This will conclude the proof of the theorem. We will see that the conditions \( H > 1/2 \) and \( A > 0 \) (based on (D2)) are necessary. First of all, it is clear that \( F_\varepsilon(0) = 2 \varepsilon > 0 \).

Let us find all critical points of \( F_\varepsilon(x) \). Clearly, the first and second derivatives with respect to \( x \) are respectively given by
\[
F'_\varepsilon(x) = A\varepsilon^{-1} - c\sigma(H - \alpha)x^{H - \alpha - 1}
\]
and
\[
F''_\varepsilon(x) = -c\sigma(H - \alpha)(H - \alpha - 1)x^{H - \alpha - 2}.
\]
It is clear that \( F_\varepsilon(x) \) is convex as \( F''_\varepsilon(x) > 0 \). Moreover, the critical point \( \hat{x} \) of \( F_\varepsilon(x) \) is given by
\[
\hat{x} = \left( \frac{A\varepsilon^{-1}}{c\sigma(H - \alpha)} \right)^{\frac{1}{H - \alpha - 1}}.
\]
Note that \( \hat{x} \) is well defined since \( A > 0 \). Hence,
\[
F_\varepsilon(\hat{x}) = A\varepsilon^{-1}\hat{x} - c\sigma\hat{x}^{H - \alpha} + 2\varepsilon
\]
\[
= \hat{x} \left( A\varepsilon^{-1} - c\sigma\hat{x}^{H - \alpha - 1} \right) + 2\varepsilon
\]
\[
= \hat{x} \left( A\varepsilon^{-1} - \frac{A\varepsilon^{-1}}{H - \alpha} \right) + 2\varepsilon
\]
\[
= \hat{x}A\varepsilon^{-1}(H - \alpha - 1) + 2\varepsilon
\]
\[
= \left( \frac{A^{H-\alpha}}{c\sigma(H - \alpha)^{2+\alpha-H}} \right)^{\frac{1}{H-\alpha-1}} \varepsilon^{\frac{H-\alpha}{1-H+\alpha}}(H - \alpha - 1) + 2\varepsilon.
\]
Since \( H - \alpha - 1 < 0 \), then
\[
F_\varepsilon(\hat{x}) \geq \left( \frac{A^{H-\alpha}}{c\sigma(H - \alpha)^{2+\alpha-H}} \right)^{\frac{1}{H-\alpha-1}} \varepsilon^{\frac{H-\alpha}{1-H+\alpha}}(H - \alpha - 1) + 2\varepsilon.
\]
Set
\[
\kappa = - \left( \frac{A^{H-\alpha}}{c\sigma(H - \alpha)^{2+\alpha-H}} \right)^{\frac{1}{H-\alpha-1}}(H - \alpha - 1)
\]
and
\[
q = \frac{H - \alpha}{1 - H + \alpha}.
\]
Clearly, since $H > 1/2$, we can choose $\alpha$ so small that $H > \frac{1}{2} + \alpha$ and obtain that $q \geq 1$. Then it follows that

$$F_\varepsilon(\hat{x}) \geq -\kappa \varepsilon^q + 2\varepsilon.$$  

It is now an easy matter to show that there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, it is the case that

$$F_\varepsilon(\hat{x}) \geq -\kappa \varepsilon^q + 2\varepsilon > 0.$$  

Indeed, choosing $\varepsilon^* \leq \left(\frac{2}{\kappa}\right)^{\frac{1}{q-1}}$ yields $F_\varepsilon(\hat{x}) > 0$. (Note that $\varepsilon^*$ because $A \neq 0$.) Hence $F_\varepsilon(x) > 0$ for all $x \geq 0$. This concludes the proof of the theorem. \qed

5 Analysis of positiveness of $(X_t)_{t \geq 0}$ for $H < 1/2$

We shall consider a sequence of continuous functions

$$f_k(t, z) : [0, \infty) \times [0, \infty) \rightarrow (-\infty, +\infty), \quad k \in \mathbb{N}$$

such that each function $f_k$ satisfies conditions (D1) and (D2). Moreover for each point $(t, z) \in [0, \infty) \times [0, \infty)$,

$$f_k(t, z) \leq f_{k+1}(t, z) \quad \text{and} \quad \lim_{k \to \infty} f_k(t, z) = \infty.$$  

Consider, for each $k$, the stochastic process $(Z_t^{(k)})_{t \geq 0}$ defined by

$$Z_t^{(k)} = \begin{cases} 
 Z_0 + \int_0^t \frac{1}{2} f_k(t, Z_s^{(k)}) \left(Z_s^{(k)}\right)^{-1} ds + \frac{a}{2} W_t^H & \text{if } t < \tau^{(k)}(\omega) \\
 0 & \text{otherwise,} 
\end{cases}$$

where $\tau^{(k)}(\omega) = \inf\{t \geq 0 : Z_t^{(k)}(\omega) = 0\}$. We have the following result:

**Theorem 5.1.** For any $T > 0$,

$$P(\tau^{(k)}(\omega) > T) \to 1 \quad \text{as } k \to \infty.$$  

**Proof.** The case where $f_k(t, z) = k - az^2$ for some $k, a > 0$ is studied by Mishura and Yurchenko-Tytarenko (2018). Their proof is based on the observation that for $k_1 < k_2$,

$$\tau^{(k_1)}(\omega) \leq \tau^{(k_2)}(\omega) \quad \text{and} \quad Z_t^{(k_1)}(\omega) < Z_t^{(k_2)}(\omega)$$

for all $t$ such that $0 < t < \tau^{(k_2)}(\omega)$. It is easy to see that this extends immediately to our general case. We assume that there exist $T > 0$, an increasing sequence $(k_n)_{n \geq 1}$ and $p > 0$ such that

$$P(\tau^{(k_n)} \leq T) \to p, \quad k_n \to \infty.$$  

(5.1)
As in the previous proof, for fixed $T > 0$, consider a point $x_T$ small enough such that $0 < x_T < Z_0$ and $S > 0$ be the first time $(Z_t)$ hits the value $x_T$. Take $0 < \varepsilon < x_T$. Then uniformly for all $k \in \mathbb{N}$, $f_k(t, x) > 0$ for all $S \leq t \leq T$ and $0 \leq x \leq \varepsilon$. Let $A = \inf \{ f(t, x) : S \leq t \leq T, 0 \leq x \leq x_T \}$. Clearly $A > 0$. Also let $\tau_{\varepsilon}^{(k_n)} = \sup \{ t \in (0, \tau) : Z_t^{(k_n)} = \varepsilon \}$ be the last hitting time of $\varepsilon$ before reaching zero. Let

$$A_k = \inf \{ f_k(t, z) : S \leq t \leq T, 0 \leq z \leq Z_0 \}, \quad k > 0.$$ 

Moreover, consider for a small number $\alpha > 0$, (by the Hölder continuity) the subspace $\Omega_1$ of probability 1 such that

$$|W_t^H(\omega) - W_s^H(\omega)| \leq c|t - s|^{H - \alpha}, \quad \text{for all } s, t \in [0, T]$$

where $c = c(T, \omega, \alpha)$ is a constant depending on $T$, $\omega$ and $\alpha$. Let

$$\Omega_T^{(k_n)} = \{ \omega \in \Omega_1 : \tau_{\varepsilon}^{(k_n)} \leq T \}. \quad \text{ (5.2)}$$

Then, for all $\omega \in \Omega_T^{(k_n)}$, similar arguments as in the proof of Theorem 4.1 yield

$$Z_{\tau_{\varepsilon}^{(k_n)}} - Z_{\tau_{\varepsilon}^{(k_n)}} = -\varepsilon = \frac{1}{2} \int_{\tau_{\varepsilon}^{(k_n)}}^{\tau_{\varepsilon}^{(k_n)}} f_{k_n}(t, Z_t^{(k_n)}) \left( Z_t^{(k_n)} \right)^{-1} ds + \frac{\sigma}{2} (W_{\tau_{\varepsilon}^{(k_n)}} - W_{\tau_{\varepsilon}^{(k_n)}}).$$

In a similar way as in the previous proof,

$$f_{k_n}(t, Z_t^{(k_n)}) \left( Z_t^{(k_n)} \right)^{-1} \geq A_{k_n} \varepsilon^{-1}, \quad \forall s \in [\tau_{\varepsilon}^{(k_n)}, \tau_{\varepsilon}^{(k_n)}].$$

Since

$$\left| W_{\tau_{\varepsilon}^{(k_n)}} - W_{\tau_{\varepsilon}^{(k_n)}} \right| \leq c \left| \tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)} \right|^{H - \alpha},$$

it follows (as in the previous proof) that

$$c\sigma \left( \tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)} \right)^{H - \alpha} \geq A_{k_n} \varepsilon^{-1} (\tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)}) + 2\varepsilon.$$

This implies in particular that

$$\begin{cases} 
\frac{c\sigma}{\varepsilon} (\tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)})^{H - \alpha} \geq 2\varepsilon \\
\frac{c\sigma}{\varepsilon} (\tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)})^{H - \alpha} \geq A_{k_n} (\tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)}) \varepsilon^{-1}.
\end{cases} \quad \text{(5.3)}$$

We shall show that the two inequalities are contradictory. Elementary calculations show that the second inequality in (5.3) is equivalent to

$$\left( \tau_{\varepsilon}^{(k_n)} - \tau_{\varepsilon}^{(k_n)} \right) \leq \left( \frac{1}{c\sigma} A_{k_n} \varepsilon^{-1} \right)^{H - \alpha}.$$
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Taking both sides with power $H - \alpha$ and thereafter multiplying both sides by $c\sigma$ yields

$$
c\sigma \left( \tau^{(k_n)} - \tau^{(k_n)}_{\varepsilon} \right)^{H-\alpha} \leq c\sigma \left( \frac{1}{c\sigma} A_{k_n} \varepsilon^{-1} \right)^{\frac{H-\alpha}{H+\alpha}} = \left( c^{\frac{1}{1-H+\alpha}} \sigma^{\frac{1}{1-H+\alpha}} \right)^{\varepsilon^{\frac{H-\alpha}{1-H+\alpha}}} (A_{k_n})^{-\frac{H-\alpha}{H+\alpha}}.
$$

In the right hand side, the Hölder constant $c = c(\omega)$ is random depending on the path $\omega$ of fBm. As in Mishura and Yurchenko-Tytarenko (2018), it is well-known that $c(\omega)$ is finite almost surely and hence since $P\left( \bigcap_{n>1} \Omega_T^{(k_n)} \right) = p > 0$, then there exists a (non-random) constant $M$ and a subset $E$ of $\bigcap_{n>1} \Omega_T^{(k_n)}$ with $P(E) > 0$ such that $c = c(\omega) \leq M$ for all $\omega \in E$. Hence, everywhere in $E$,

$$
c\sigma \left( \tau^{(k_n)} - \tau^{(k_n)}_{\varepsilon} \right)^{H-\alpha} \leq \left( M^{\frac{1}{1-H+\alpha}} \sigma^{\frac{1}{1-H+\alpha}} \varepsilon^{\frac{H-\alpha}{1-H+\alpha}} (A_{k_n})^{-\frac{H-\alpha}{H+\alpha}} \right).
$$

Clearly $M$ and $\sigma$ are constants. Moreover, since $f_n(t, z) \to \infty$ as $n \to \infty$ (for every $(t, z)$) then clearly also $A_{k_n} \to \infty$ for $k_n \to \infty$. Hence

$$
\lim_{k_n \to \infty} (A_{k_n})^{-\frac{H-\alpha}{1-H+\alpha}} = 0,
$$

because $-\frac{H-\alpha}{1-H+\alpha} < 0$. Then clearly, for any given $\varepsilon > 0$, we can choose $k_n$ very large (depending on $\varepsilon$) such that

$$
\left( M^{\frac{1}{1-H+\alpha}} \sigma^{\frac{1}{1-H+\alpha}} \varepsilon^{\frac{H-\alpha}{1-H+\alpha}} (A_{k_n})^{-\frac{H-\alpha}{H+\alpha}} \right) < 2\varepsilon.
$$

This yields

$$
c\sigma \left( \tau^{(k_n)} - \tau^{(k_n)}_{\varepsilon} \right)^{H-\alpha} < 2\varepsilon,
$$

which contradicts the first inequality in (5.3). This concludes the proof of the theorem.

□

6 Some illustrating examples with simulations

In this section, we provide some examples of generalised fCIR processes to illustrate the results of this paper using simulations. The process that will be used represents a generalisation of the classical “extended CIR” process.

The classical extended CIR process is defined by

$$
dX_t = \theta_t(\mu_t - X_t)dt + \sigma \sqrt{X_t}dW_t, \quad X_0 > 0
$$

where $\theta_t$ is the time-depending speed of reversion towards its time-depending long run mean $\mu_t$ of the process $(X_t)_{t \geq 0}$ and $\sigma$ a positive parameter. This model was
initially introduced by \( ? \) and it is widely used in both short interest rates and spot volatilities modelling. The choice of parameters \( \theta_t \) and \( \mu_t \) are done through market calibration. As already discussed, we shall consider the general case where the Brownian motion is replaced with a \( fBm \). The process is called "Extended fCIR" and takes the form

\[
X_t = Z_t^2 1_{[0, \tau)}, \quad t \geq 0
\]  \hspace{1cm} (6.2)

where

\[
dZ_t = \frac{f(t, Z_t)}{2Z_t} dt + \frac{\sigma^2}{2} dW^H_t, \quad Z_0 > 0
\]  \hspace{1cm} (6.3)

with

\[
f(t, x) = \theta_t (\mu_t - x^2).
\]  \hspace{1cm} (6.4)

We shall then simulate the corresponding process \( (X_t) \) on a finite interval \([0, T]\) using the well-known Euler-Maruyama method. (See e.g. \( ? \) for more details about the method.) Subdivide the interval \([0, T]\) into \( N \) subintervals of equal length \( \delta t = T/N \) with end points \( 0 = t_0, t_1, t_2, \ldots, t_N = T \). The corresponding discrete version of the process \( (X_t)_{t \geq 0} \) is given by

\[
X_{tn} = Z_{tn}^2,
\]

where \( Z_0 > 0 \) and for \( n = 1, 2, \ldots, N \),

\[
Z_{tn} = \begin{cases} 
Z_{n-1} + \frac{f(t_{n-1}, Z_{n-1})}{2Z_{n-1}} \delta t + \frac{\sigma^2}{2} \delta W^H_{tn} & \text{if } Z_{tn-1} > 0, \\
0 & \text{otherwise}
\end{cases}
\]

with

\[
\delta W^H_{tn} = W^H_{tn} - W^H_{tn-1}.
\]

In what follows, we shall consider two different drift functions for simulation of the process (6.2).

**Illustration I**

We consider \( \theta_t = \theta > 0 \) and

\[
\mu_t = c + \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta t} \right)
\]

where \( c > 0 \) is a constant. This yields the drift function

\[
f(t, x) = \frac{\sigma^2}{2} \left( 1 - e^{-2\theta t} \right) + \theta (c - x^2), \quad t \geq 0, x \geq 0.
\]  \hspace{1cm} (6.5)

It is clear that the function \( f(t, x) \) satisfies conditions (D1) and (D2) and hence for We simulate 1000 sample paths of the process \( (X_t)_{t \in [0, T]} \) where \( T = 10 \), volatility
\( \sigma = 0.4 \) starting at \( X_0 = 1 \) with time-step \( \delta t = 0.001 \) and the results are given in Figures 4.1 to 4.4 (with given parameters \( c, \theta \) and \( H \)). All the sample paths in Figures 4.1 and 4.2 where \( H > 0.5 \) are strictly positive (do not hit zero) in line with Theorem 3.3.

\[ \theta = 1, \ c = 2, \ H = 0.6 \]

\[ \theta = 1, \ c = 2, \ H = 0.8 \]

**Illustration II**

In the second illustration, we consider again \( \theta_t = \theta > 0, \ \sigma > 0 \) and

\[ \mu_t = \left( 1 + \frac{c}{\theta} \right) e^{ct} + \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta t} \right), \]

where \( c > 0 \) is a constant. This yields the function

\[ f(t, x) = \left( \theta + c \right) e^{ct} + \frac{\sigma^2}{2} \left( 1 - e^{-2\theta t} \right) - \theta x^2, \]  

(6.6)

It is again clear that \( f(t, x) \) satisfies conditions (D1) and (D2). We considered 1000 realisations of the sample paths of the stochastic process \( (X_t)_{t \in [0,1]} \) with volatility \( \sigma = 0.4 \) starting at \( X_0 = 1 \) with time-step \( \delta t = 0.001 \). We have observed similar results compared to Simulation I and the output is given from Figures 4.6 to 4.9.
In this work, we analysed the general fCIR processes of the form $X^2_t = Z^2_t 1_{[0, \tau)}$ with

$$dZ_t = \frac{1}{2} f(t, Z_t) Z_t^{-1} dt + \frac{1}{2} \sigma dW^H_t, \quad Z_0 > 0,$$

where $f(t, x)$ is a continuous function on $\mathbb{R}_+^2$ under two mild conditions on the function $f(t, x)$. We proved that the process $(X_t)$ satisfies the equation $dX_t = f(t, \sqrt{X_t}) dt + \sigma \sqrt{X_t} \circ dW^H_t$. Moreover if the Hurst parameter $H > 1/2$, the process $(X_t)_{t \geq 0}$ processes will never hit zero, that is, it remains strictly positive everywhere almost surely. The conditions (D1) and (D2) imposed on $f(t, x)$ are very weak so that the class of functions to which our results apply is clearly larger

Concluding remarks

In this work, we analysed the general fCIR processes of the form $X^2_t = Z^2_t 1_{[0, \tau)}$ with

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than previously understood. In the case, $H < 1/2$, we considered a sequence of increasing drift functions $(f_n)$ that tends to infinity and we proved that the probability of hitting zero converges to zero as $n$ goes to infinity. These results are illustrated with some simulations. The generalised fCIR process may take several forms and one of them is given as an extended $fCIR$ or fractional Hull-White process. This process belongs to the class of mean-reverting processes and may yield perfect calibrations of time-dependent parameters. Calibration under $fCIR$ process constitutes an important area of further investigations. Another line of further research is to study the properties of moments of the process $(X_t)$ in order to see if results that have been obtained under more stronger conditions remain valid under the mild conditions (D1) and (D2). We hope the results and discussions in this paper will be of some help in that direction. It is important to note that our results generalise previous results obtained by Mishura and Yurchenko-Tytarenko (2018) to the particular function $f(t, x) = \frac{1}{2}(\mu - \theta x^2)$ for constants $\mu > 0$ and $\theta > 0$.

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