Asymptotic Analysis of Branched Willmore Surfaces

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Abstract: We consider a closed Willmore surface properly immersed in $\mathbb{R}^{m\geq 3}$ with square-integrable second fundamental form, and with one point-singularity of finite arbitrary integer order. Using the “conservative” reformulation of the Willmore equation introduced in [Ri1], we show that, in an appropriate conformal parametrization, the gradient of the Gauss map of the immersion has bounded mean oscillations if the singularity has order one, and is bounded if the order is at least two. We develop around the singular point local asymptotic expansions for the immersion, its first and second derivatives, and for the mean curvature vector. Finally, we exhibit an explicit condition ensuring the removability of the point-singularity.

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I Overview

The Willmore energy of an immersed closed surface $\Phi: \Sigma \rightarrow \mathbb{R}^{m\geq 3}$ is given by

$$W(\Phi) := \int_{\Sigma} |\vec{H}|^2 \ d\text{vol}_g,$$

where $\vec{H}$ denotes the weak mean curvature vector, and $d\text{vol}_g$ is the area form of the metric $g$ induced on $\Phi(\Sigma)$ by the canonical Euclidean metric on $\mathbb{R}^m$. Critical points of the Lagrangian $W$ for perturbations of the form $\Phi + t \xi$, where $\xi$ is an arbitrary compactly supported smooth map on $\Sigma$ into $\mathbb{R}^m$, are known as Willmore surfaces. Not only is the Willmore functional invariant under reparametrization, but more importantly, it is invariant under the group of Möbius transformations of $\mathbb{R}^m \cup \{\infty\}$. This remarkable property prompts the use of the Willmore energy in various fields of science. A survey of the Willmore functional, of its properties, and of the relevant literature is available in [Ri3].

The study of singular points of Willmore immersions is primarily motivated by the fact that sequences of Willmore immersions with uniformly bounded energy converge everywhere except on a finite set of points where the energy concentrates (cf. [BR2] and the references therein). Such point singularities of Willmore surfaces also occur as blow-ups of the Willmore flow (cf. [KS1]). In their seminal paper [KS1], Ernst Kuwert and Reiner Schätzle initiated the

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Owing to the Gauss-Bonnet theorem, we note that the Willmore energy \( E \) may be equivalently expressed as
\[
W(\vec{\Phi}(\Sigma)) = \int_\Sigma |\vec{I}|^2_g d\mu_g + \pi \chi(\Sigma),
\]
where \( \vec{I} \) is the second fundamental form, and \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \), which is a topological invariant for a closed surface. From the variational point of view, Willmore surfaces are thus critical points of the energy
\[
\int_\Sigma |\vec{I}|^2_g d\text{vol}_g.
\]
It then appears natural to restrict our attention on immersions whose second fundamental forms are locally square-integrable.

We assume that the point-singularity lies at the origin, and we localize the problem by considering a map \( \vec{\Phi} : D^2 \to \mathbb{R}^{m \geq 3} \), which is an immersion of \( D^2 \setminus \{0\} \), and satisfying
\[
\begin{align*}
(i) & \quad \vec{\Phi} \in C^0(D^2) \cap C^\infty(D^2 \setminus \{0\}); \\
(ii) & \quad \mathcal{H}^2(\vec{\Phi}(D^2)) < \infty; \\
(iii) & \quad \int_{D^2} |\vec{I}|^2_g d\text{vol}_g < \infty.
\end{align*}
\]
By a procedure detailed in \cite{KS2}, it is possible to construct a parametrization \( \zeta \) of the unit-disk such that \( \vec{\Phi} \circ \zeta \) is conformal. To do so, one first extends \( \vec{\Phi} \) to all of \( \mathbb{C} \setminus \{0\} \) while keeping a bounded image and the second fundamental form square-integrable. One then shifts so as to have \( \vec{\Phi}(0) = \vec{0} \), and inverts about the origin so as to obtain a complete immersion with square-integrable second fundamental form. Calling upon a result of Huber \cite{Hu} (see also \cite{MS} and \cite{To}), one deduces that the image of the immersion is conformally equivalent to \( \mathbb{C} \). Inverting yet once more about the origin finally gives the desired conformal immersion \( \vec{\Phi} \), which we shall abusively continue to denote \( \vec{\Phi} \). It has the aforementioned properties (i)-(iii), and moreover,
\[
\vec{\Phi}(0) = \vec{0} \quad \text{and} \quad \vec{\Phi}(D^2) \subset B_R^m(0) \quad \text{for some} \ 0 < R < \infty.
\]
\footnote{which degenerates at the origin in a particular way, see \cite{I3}.}
Hence, \( \tilde{\Phi} \in L^\infty \cap W^{1,2}(D^2 \setminus \{0\}) \). Away from the origin, we define the Gauss map \( \tilde{n} \) via
\[
\tilde{n} = \star \left( \frac{\partial \tilde{\Phi} \wedge \partial \tilde{\Phi}}{|\partial \tilde{\Phi} \wedge \partial \tilde{\Phi}|} \right),
\]
where \((x_1, x_2)\) are standard Cartesian coordinates on the unit-disk \(D^2\), and \(\star\) is the Euclidean Hodge-star operator. The immersion \(\tilde{\Phi}\) is conformal, i.e.
\[
|\partial \tilde{\Phi}| = e^\lambda = |\partial \tilde{\Phi}| \quad \text{and} \quad \partial \tilde{\Phi} \cdot \partial \tilde{\Phi} = 0 ,
\]
where \(\lambda\) is the conformal parameter. An elementary computation shows that
\[
dvol_2 = e^{2\lambda} dx \quad \text{and} \quad |\nabla \tilde{n}|^2 dx = e^{2\lambda} |\tilde{n}|^2 dy = |\tilde{n}|^2 dvol_2 .
\]
Hence, by hypothesis, we see that \(\tilde{n} \in W^{1,2}(D^2 \setminus \{0\})\). In dimension two, the 2-capacity of isolated points is null, so we actually have \(\tilde{\Phi} \in W^{1,2}(D^2)\) and \(\tilde{n} \in W^{1,2}(D^2)\) (note however that \(\tilde{\Phi}\) remains a non-degenerate immersion only away from the singularity). Rescaling if necessary, we shall henceforth always assume that
\[
\int_{D^2} |\nabla \tilde{n}|^2 dx < \varepsilon_0 ,
\]
where the adjustable parameter \(\varepsilon_0\) is chosen to fit our various needs (in particular, we will need it to be “small enough” in Proposition A.1).

For the sake of the following paragraph, we consider a conformal immersion \(\tilde{\Phi} : D^2 \to \mathbb{R}^m\), which is smooth across the unit-disk. We introduce the local coordinates \((x_1, x_2)\) for the flat metric on the unit-disk \(D^2 = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 \}\). The operators \(\nabla = (\partial_{x_1}, \partial_{x_2})\), \(\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})\), \(\text{div} = \nabla \cdot\), and \(\Delta = \nabla \cdot \nabla\) will be understood in these coordinates. The conformal parameter \(\lambda\) is defined as in (I.2). We set
\[
\varepsilon_j := e^{-\lambda} \partial_{x_j} \tilde{\Phi} \quad \text{for} \quad j \in \{1,2\}.
\]

As \(\tilde{\Phi}\) is conformal, \(\{\varepsilon_1(x), \varepsilon_2(x)\}\) forms an orthonormal basis of the tangent space \(T_{\tilde{\Phi}(x)} \tilde{\Phi}(D^2)\). Owing to the topology of \(D^2\), there exists for almost every \(x \in D^2\) a positively oriented orthonormal basis \(\{\tilde{n}_1, \ldots, \tilde{n}_{m-2}\}\) of the normal space \(N_{\tilde{\Phi}(x)} \tilde{\Phi}(D^2)\), such that \(\{\varepsilon_1, \varepsilon_2, \tilde{n}_1, \ldots, \tilde{n}_{m-2}\}\) forms a basis of \(T_{\tilde{\Phi}(x)} \mathbb{R}^m\). From the Plücker embedding, realizing the Grassmannian \(Gr_{m-2}(\mathbb{R}^m)\) as a submanifold of the projective space of the \((m-2)\)th exterior power \(\mathbb{P}(\wedge^{m-2} \mathbb{R}^m)\), we can represent the Gauss map as the \((m-2)\)-vector \(\tilde{n} = \bigwedge_{a=1}^{m-2} \tilde{n}_a\). Via the Hodge operator \(\star\), we identify vectors and \((m-1)\)-vectors in \(\mathbb{R}^m\), namely:
\[
\star (\tilde{n} \wedge \varepsilon_1) = \varepsilon_2 , \quad \star (\tilde{n} \wedge \varepsilon_2) = -\varepsilon_1 , \quad \star (\varepsilon_1 \wedge \varepsilon_2) = \tilde{n} .
\]

In this notation, the second fundamental form \(\overline{I}\), which is a symmetric 2-form on \(T_{\tilde{\Phi}(x)} \tilde{\Phi}(D^2)\) into \(N_{\tilde{\Phi}(x)} \tilde{\Phi}(D^2)\), is expressed as
\[
\overline{I} = \sum_{a,i,j} e^{-2\lambda} h_{ij}^a \tilde{n}_a \, dx_i \otimes dx_j \equiv \sum_{a,i,j} h_{ij}^a \tilde{n}_a (\varepsilon_i^*) \otimes (\varepsilon_j^*) ,
\]
where
\[ h_{ij}^\alpha = -e^{-\lambda} \vec{e}_i \cdot \partial_{x_j} \vec{n}_\alpha. \]

The mean curvature vector is
\[ \vec{H} = \sum_{\alpha=1}^{m-2} H^\alpha \vec{n}_\alpha = \frac{1}{2} \sum_{\alpha=1}^{m-2} (h_{11}^\alpha + h_{22}^\alpha) \vec{n}_\alpha. \]

The Willmore equation \([\text{We}]\) is cast in the form
\[ \Delta_{\perp} \vec{H} + \sum_{\alpha,\beta,i,j} h_{ij}^\alpha h_{ij}^\beta H^\beta \vec{n}_\alpha - 2 |\vec{H}|^2 \vec{H} = 0, \tag{I.6} \]
with
\[ \Delta_{\perp} \vec{H} := e^{-2\lambda} \pi_{\vec{n}} \operatorname{div} (\pi_{\vec{n}} (\nabla \vec{H})), \]
and \(\pi_{\vec{n}}\) is the projection onto the normal space spanned by \(\{\vec{n}_\alpha\}_{\alpha=1}^{m-2}\).

The Willmore equation \([\text{We}]\) is a fourth-order nonlinear equation (in the coefficients of the induced metric, which depends on \(\vec{\Phi}\)). With respect to the coefficients \(H^\alpha\) of the mean curvature vector, it is actually a strongly coupled nonlinear system whose study is particularly challenging. In codimension 1, there is one equation for the scalar curvature; in higher codimension however, the situation becomes significantly more complicated, and one must seek different techniques to approach the problem. Fortunately, in a conformal parametrization, it is possible to recast the system \([\text{We}]\) in an equivalent, yet analytically more suitable form \([\text{Ri1}]\). Namely, there holds
\[ \operatorname{div} (\nabla \vec{H} - 3 \pi_{\vec{n}} (\nabla \vec{H}) + \ast (\nabla \vec{n} \wedge \vec{H})) = 0. \tag{I.7} \]

This reformulation in divergence form of the Willmore equation is the starting point of our analysis. In our singular situation, \([\text{I.7}]\) holds only away from the origin, on \(D^2 \setminus \{0\}\). In particular, we can define the constant \(\vec{c}_0 \in \mathbb{R}^m\), called residue, by
\[ \vec{c}_0 := \int_{\partial D^2} \vec{\nu} \cdot \left(\nabla \vec{H} - 3 \pi_{\vec{n}} (\nabla \vec{H}) + \ast (\nabla \vec{n} \wedge \vec{H})\right), \tag{I.8} \]
where \(\vec{\nu}\) denotes the unit outward normal vector to \(\partial D^2\). We will see that the residue appears in the local asymptotic expansion of the mean curvature vector around the singularity (cf. Propositions \([\text{I.2}]\) and \([\text{I.3}]\).

We next state the main result of the present paper. It concerns the regularity of the Gauss map around the point-singularity.

**Theorem I.1**: Let \(\vec{\Phi} \in W^{1,2} \cap C^\infty (D^2 \setminus \{0\}) \cap C^0 (D^2)\) be a conformal Willmore immersion of the punctured disk into \(\mathbb{R}^m\), and whose Gauss map \(\vec{n}\) lies in \(W^{1,2}(D^2)\). Then \(\nabla^2 \vec{n} \in L^{2,\infty}(D^2)\), and thus in particular \(\nabla \vec{n}\) is an element of \(\text{BMO}\). Furthermore, \(\vec{n}\) satisfies the pointwise estimate
\[ |\nabla \vec{n}(x)| \lesssim |x|^{-\epsilon} \quad \forall \ \epsilon > 0. \]

\(^2\)This procedure requires to choose the normal frame \(\{\vec{n}_\alpha\}\) astutely. See \([\text{Ri1}]\) for details.
If the order of degeneracy of the immersion $\vec{\Phi}$ at the origin is at least two, then in fact $\nabla \vec{n}$ belongs to $L^\infty(B_1(0))$.

A conformal immersion of $D^2 \setminus \{0\}$ into $\mathbb{R}^m$ such that $\vec{\Phi}$ and its Gauss map $\vec{n}$ both extend to maps in $W^{1,2}(D^2)$ has a distinct behavior near the point-singularity located at the origin. One can show (cf. [MS], and Lemma A.5 in [Ri2]) that there exists a positive natural number $\theta_0$ with

$$|\vec{\Phi}(x)| \simeq |x|^{\theta_0} \quad \text{and} \quad |\nabla \vec{\Phi}(x)| \simeq |x|^{\theta_0-1} \quad \text{near the origin.} \tag{1.9}$$

In addition, there holds

$$\lambda(x) := \frac{1}{2} \log \left( \frac{1}{2} |\nabla \vec{\phi}(x)|^2 \right) = (\theta_0 - 1) \log |x| + u(x),$$

where $u \in W^{2,1}(D^2)$; and one has

$$\left\{ \begin{array}{ll}
\nabla \lambda \in L^2(D^2) , & \text{when } \theta_0 = 1 \\
\nabla \lambda(x) \lesssim |x|^{-1} \in L^{2,\infty}(D^2) , & \text{when } \theta_0 \geq 2. \tag{1.10}
\end{array} \right.$$ 

The integer $\theta_0$ is the density of the current $\vec{\Phi}_*[D^2]$ at the image point $0 \in \mathbb{R}^m$.

When such a conformal immersion is Willmore on $D^2 \setminus \{0\}$, it is possible to refine the asymptotics (1.9). The following result describes the behavior of the immersion $\vec{\Phi}$ locally around the singularity at the origin.

**Proposition I.1** Let $\vec{\Phi}$ be as in Theorem I.1 with conformal parameter $\lambda$, and let $\theta_0$ be as in (I.9). There exists a constant vector $\vec{A} = \vec{A}^1 + i\vec{A}^2 \in \mathbb{R}^2 \otimes \mathbb{R}^m$ such that

$$\vec{A}^1 \cdot \vec{A}^2 = 0 , \quad |\vec{A}^1| = |\vec{A}^2| = \theta_0^{-1} \lim_{x \to 0} \frac{e^{\lambda(x)}}{|x|^{\theta_0-1}}, \quad \pi_{\vec{n}(0)} \vec{A} = \vec{0},$$

and

(i) when $\theta_0 = 1$,

$$\vec{\Phi}(x) = \Re(\vec{A} x) + \vec{\zeta}(x), \tag{I.11}$$

with $\vec{\zeta} \in \bigcap_{p<\infty} W^{2,p}(D^2)$ and

$$\vec{\zeta}(x) = O(|x|^{2-\epsilon}), \quad \nabla \vec{\zeta}(x) = O(|x|^{1-\epsilon}), \quad \forall \ \epsilon > 0.$$

(ii) when $\theta_0 \geq 2$,

$$\vec{\Phi}(x) = \Re(\vec{A} x^{\theta_0} + \vec{B} x^{\theta_0+1} + \vec{C} |x|^{2} x^{\theta_0-1}) + |x|^{\theta_0-1} \vec{\xi}(x), \tag{I.12}$$

where $\vec{B}$ and $\vec{C}$ are constant vectors in $\mathbb{C}^m$. And for all $\epsilon > 0$:

$$\vec{\xi}(x) = O(|x|^{3-\epsilon}), \quad \nabla \vec{\xi}(x) = O(|x|^{2-\epsilon}), \quad \nabla^2 \vec{\xi}(x) = O(|x|^{1-\epsilon}).$$

Roughly speaking, $\nabla \vec{\Phi}(0) = \vec{0}$. The notion of "order of degeneracy" is made precise below.
The plane $\text{span}\{\vec{A}^1, \vec{A}^2\}$ is tangent to the surface at the origin. If $\theta_0 = 1$, this plane is actually $T_0\Sigma$. One can indeed show that the tangent unit vectors $\vec{e}_j$ spanning $T_0\Sigma$ (defined in (I.5)) satisfy $\vec{e}_j(0) = \vec{A}^j/|\vec{A}^j|$. In contrast, when $\theta_0 \geq 2$, the tangent plane $T_0\Sigma$ does not exist in the classical sense, and the vectors $\vec{e}_j(x)$ “spin” as $x$ approaches the origin (cf. (II.21). More precisely, $T_0\Sigma$ is the plane $\text{span}\{\vec{A}^1, \vec{A}^2\}$ covered $\theta_0$ times.

**Remark I.1** when $\theta_0 = 1$, the immersion $\vec{\Phi}$ belongs to $C^{1,\alpha}(D^2)$ for all $\alpha \in [0, 1)$. In general however, $\vec{\Phi}$ need not be $C^{1,1}(D^2)$, as the following example shows. A conformal parametrization of the catenoid is

$$(r, \varphi) \mapsto \left((r + r^{-1})\cos(\varphi), \ (r + r^{-1})\sin(\varphi), \log r\right).$$

Inverting the catenoid about the origin gives a Willmore surface whose behavior near the origin consists of two identical graphs (mirror-symmetric about the $(x_1, x_2)$-plane) of order $\theta_0 = 1$ at the origin. One computes the inverted parametrization (for one graph) to be

$$\vec{\Phi}(r, \varphi) = (r\cos(\varphi), \ r\sin(\varphi), \ r^2 \log r) + \mathcal{O}(r^3 \log^2 r).$$

Identifying (through $x = r e^{i\varphi}$) with (I.11) shows $\vec{A} = (1, -i, 0)$ and $\vec{\zeta}(x) = \mathcal{O}(|x|^2 \log |x|)$.

Thus, we cannot expect in general $\epsilon = 0$ in (I.11). Moreover, $\vec{\Phi} \notin C^{1,1}(D^2)$, and a computation reveals that

$$|\nabla \vec{n}(x)| \simeq \log |x| \in \text{BMO} \setminus L^\infty(D^2).$$

It is also possible to obtain information on the local asymptotic behavior of the mean curvature vector near the origin. This is the object of the next proposition.

**Proposition I.2** Let $\vec{\Phi}$ be as in Theorem I.1, $\lambda$ be its conformal parameter, and $\theta_0$ be as in (I.9). Locally around the singularity, the mean curvature vector satisfies

(i) when $\theta_0 = 1$,

$$\vec{H}(x) + \frac{\vec{C}_0}{4\pi} \log |x| \in \bigcap_{p < \infty} W^{1,p}(D^2),$$

where $\vec{C}_0$ is the residue defined in (I.8).

(ii) when $\theta_0 \geq 2$,

$$e^{\lambda(x)} \vec{H}(x) = f(x) \Re \left[ \tilde{C} \left( \frac{x}{|x|} \right)^{\theta_0 - 1} \right] + \mathcal{O}(|x|^{1-\epsilon}) \quad \forall \ \epsilon > 0,$$

where $\tilde{C} \in \mathbb{C}^m$ is the same constant vector as in Proposition I.1-(ii), and

$$f(x) := 2 \theta_0 |x|^{\theta_0 - 1} e^{-\lambda(x)} \in C^0(D^2,(0, \infty)) .$$

In particular, since $\vec{H}$ is a normal vector, we note that $\pi_{\vec{n}(0)} \vec{C} = \tilde{C}$.4

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4 for it is the image of a minimal (thus Willmore) surface under a Möbius transformation.
When $\theta_0 \geq 2$, the weighted mean curvature vector $e^\lambda \vec{H}$ is thus bounded across the singularity (unlike in the case $\theta_0 = 1$, where it behaves logarithmically). But its limit may not exist: $e^\lambda(x) \vec{H}(x)$ is a “spinning vector” as $x$ approaches the origin. However, when this limit exists (and is thus necessarily zero), an interesting phenomenon occurs: both the mean curvature vector and the Gauss map undergo a “leap of regularity”. More precisely,

**Proposition I.3** Let $\vec{\Phi}, \vec{n}, \vec{H}$, and $\theta_0 \geq 2$ be as in Proposition [I.2](#).

If $\lim_{x \to 0} e^\lambda(x) \vec{H}(x)$ exists (i.e. if the vector $\vec{C}$ from Proposition [I.2](#) vanishes), then there holds

(i) $\nabla^{\theta_0+1} \vec{n} \in L^{2,\infty}(D^2)$, and hence $\nabla^{\theta_0} \vec{n} \in BMO$. Furthermore,

$$\nabla^j \vec{n}(x) = O(|x|^{\theta_0-j-\epsilon}) \quad \forall \epsilon > 0, \ j \in \{0, \ldots, \theta_0\}.$$ (I.13)

(ii) locally around the singularity,

$$\vec{\Phi}(x) = \sum_{j=0}^{\theta_0-1} \Re(\alpha_j \vec{A} x^{\theta_0+j}) + \vec{\zeta}(x),$$

where $\vec{A}$ is as in Proposition [I.1](#), $\alpha_0 = 1$, and $\alpha_j \in \mathbb{C}^m$ are constants. The function $\vec{\zeta}$ satisfies

$$\nabla^j \vec{\zeta}(x) = O(|x|^{2\theta_0-j-\epsilon}) \quad \forall \epsilon > 0, \ j \in \{0, \ldots, \theta_0\};$$

(iii) the mean curvature vector satisfies

$$\vec{H}(x) + \frac{\vec{c}_0}{4\pi} \log |x| \in \bigcap_{p<\infty} W^{\theta_0,p}(D^2),$$

where $\vec{c}_0$ is the residue defined in [I.3](#).

This apparent “leap of regularity” is in some cases mildly surprising. It can indeed happen that the Willmore surface under consideration has been “poorly” parametrized by $\vec{\Phi}$ (namely, $\vec{\Phi}$ parametrizes the same surface covered $\theta_0$ times), and therefore the mean curvature vector is just as regular as in the case when the point-singularity has order $\theta_0 = 1$. The following example sheds some light onto this phenomenon: we exhibit an “unclever” conformal parametrization of the inverted catenoid, which degenerates at the origin with order $\theta_0 \geq 2$. As expected for the inverted catenoid (cf. Remark [I.1](#)), the mean curvature behaves logarithmically near the singularity, regardless of the order of degeneracy of the immersion.

**Remark I.2** The result from the last proposition is sharp, as the following example shows. We may conformally parametrize the $\theta_0$-times covered inverted catenoid by composing the parametrization of the single-covered inverted catenoid given in Remark [I.1](#) with $x^{\theta_0}$. Namely, the parametrization (for one graph) is

$$\vec{\Phi}(r, \varphi) = (r^{\theta_0} \cos(\theta_0 \varphi), r^{\theta_0} \sin(\theta_0 \varphi), r^{2\theta_0} \log r^{\theta_0}) + O(r^{3\theta_0} \log^2 r).$$

Note however that the function $f(x)$ does have (positive) a limit at $x = 0$, as shown in [MS].
Identifying the latter (through \(x = re^{i\varphi}\)) with (1.12) shows that
\[
\vec{A} = (1, -i, 0) \quad \text{and} \quad \vec{B} = \vec{0} = \vec{C},
\]
so this example fits indeed within the context of Proposition I.3. One computes explicitly the residue \(\vec{c}_0\) in this case, namely
\[
\vec{c}_0 = -16\pi \theta_0 (0, 0, 1).
\]
Moreover, there holds
\[
|\nabla \vec{n}(x)| \simeq |x|^\theta_0 - 1 \log |x|,
\]
thereby confirming that
\[
|\nabla \vec{n}(x)| \lesssim |x|^\theta_0 - 1 - \epsilon \quad \text{for all } \epsilon > 0, \text{ but not for } \epsilon = 0.
\]

It is currently unknown to the authors whether there exist Willmore immersions which degenerate at the origin with the order \(\theta_0\), for which \(\phi^3 \vec{H}\) has a limit at the origin, and which do not parametrize a \(\theta_0\)-times-covered Willmore surface whose immersion degenerates at the origin with order 1. It seems never to be the case for branched inverted minimal surface (in \(\mathbb{R}^3\) at least). Admittedly however, inverted minimal surfaces are a very special kind of Willmore surfaces.

Finally, when the residue \(\vec{c}_0 = \vec{0}\) (and in the case \(\theta_0 \geq 2\) the constant vector \(\vec{C}\) from Proposition I.2(ii) vanishes: \(\vec{C} = \vec{0}\)), the singularity at the origin is removable. Namely,

**Theorem I.2** Under the hypotheses of Proposition I.2, if \(\theta_0 = 1\) and \(\vec{c}_0 = \vec{0}\), or if \(\theta_0 = 2\) and \(\vec{c}_0 = \vec{0} = \vec{C}\), then the immersion \(\vec{\Phi}\) is smooth across the unit-disk.

This is in particular the case for branched minimal immersions.

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II Proof of Theorems

II.1 Fundamental Results and Reformulation

We place ourselves in the situation described in the introduction. Namely, we have a Willmore immersion \(\vec{\Phi}\) on the punctured disk which degenerates at the origin in such a way that
\[
|\vec{\Phi}(x)| \simeq |x|^\theta_0 \quad \text{and} \quad |\nabla \vec{\Phi}(x)| = \sqrt{2} \psi^\lambda(x) \simeq |x|^\theta_0 - 1,
\]
for some \( \theta_0 \in \mathbb{N} \setminus \{0\} \).

Amongst the analytical tools available to the study of weak Willmore immersions with square-integrable second fundamental form, the most important one is certainly the “\( \varepsilon \)-regularity”. The version appearing in Theorem 2.10 and Remark 2.11 from [KS3] (see also Theorem I.5 in [Ri1]) states that there exists \( \varepsilon_0 > 0 \) such that, if
\[
\int_{B_1(0)} |\nabla \vec{n}|^2 \, dx < \varepsilon_0,
\]
then there holds
\[
\|\nabla \vec{n}\|_{L^\infty(B^g_2)} \leq C \|\nabla \vec{n}\|_{L^2(B^g_{2\sigma})} \quad \forall \ B^g_{2\sigma} \subseteq \Omega := D^2 \setminus \{0\},
\]
where \( B^g_2 \) is a geodesic disk of radius \( \sigma \) for the induced metric \( g = \Phi^*g_{\mathbb{R}^m} \), and \( C \) is a universal constant.

The \( \varepsilon \)-regularity enables us to obtain the following result (already observed in [KS2]), decisive to the remainder of the argument.

**Lemma II.1** The function \( \delta(r) := r \sup_{|x| = r} |\nabla \vec{n}(x)| \) satisfies
\[
\lim_{r \searrow 0} \delta(r) = 0 \quad \text{and} \quad \int_0^1 \delta^2(r) \frac{dr}{r} < \infty.
\]

**Proof.** From (I.3) and (I.9), the metric \( g \) satisfies
\[
g_{ij}(x) \simeq |x|^{2(\theta_0 - 1)}\delta_{ij} \quad \text{on} \quad B_{2r}(0) \setminus B_{r/2}(0) \quad \forall \ r \in (0, 1/2).
\]
A simple computation then shows that
\[
B^g_{2\sigma}(x) \subset B_{2r}(0) \setminus B_{r/2}(0) \quad \forall \ x \in \partial B_r(0),
\]
where \( 0 < 2\theta_0 \epsilon < 1 - 2^{-\theta_0} \).
Since the metric \( g \) does not degenerate away from the origin, given \( r < 1/2 \), we can always cover the flat circle \( \partial B_r(0) \) with finitely many metric disks:
\[
\partial B_r(0) \subset \bigcup_{j=1}^N B^g_{cr}(x_j) \quad \text{with} \quad x_j \in \partial B_r(0).
\]
Hence, per the latter, (II.2), and (II.3), we obtain
\[
\sup_{|x| = r} |\nabla \vec{n}(x)| \leq \sup_{|x| = r} \|\nabla \vec{n}\|_{L^\infty(B^g_{c\rho_0}(x))} \leq \sup_{|x| = r} \|\nabla \vec{n}\|_{L^2(B^g_{2\sigma}(x))}
\]
\[
\leq \|\nabla \vec{n}\|_{L^2(B_{2r}(0) \setminus B_{r/2}(0))}.
\]
As \( \nabla \vec{n} \) is square-integrable by hypothesis, letting \( r \) tend to zero in the latter yields the first assertion.
The second assertion follows from (II.4), namely,
\[
\int_0^{1/2} \delta^2(r) \frac{dr}{r} \leq \int_0^{1/2} \|\nabla \vec{n}\|^2_{L^2(B_{2r}(0) \setminus B_{r/2}(0))} \frac{dr}{r} = \log(4) \|\nabla \vec{n}\|^2_{L^2(B_1(0))},
\]



which is by hypothesis finite.

Recalling (I.3) linking the Gauss map to the mean curvature vector and the fact that \( e^{\lambda(x)} \simeq |x|^{\theta_0 - 1} \), we obtain from Lemma II.1 that

\[
\sup_{|x|=r} |\hat{H}(x)| \leq \sup_{|x|=r} e^{-\lambda(x)} |\nabla \bar{n}(x)| \lesssim \delta(r) . \tag{II.5}
\]

The Willmore equation (I.7) may be alternatively written

\[
\text{div} \left( \nabla \hat{H} - 3 \pi \bar{n} (\nabla \hat{H}) \ast (\nabla \bar{n} \wedge \hat{H}) \right) = 0 \quad \text{on} \quad \Omega := B_1(0) \setminus \{0\}.
\]

It is elliptic [RI1]. Using the information on the gradient of \( \bar{n} \) given by (II.2), and some standard analytical techniques for second-order elliptic equations in divergence form (cf. [GW]), one deduces from (II.5) that

\[
\sup_{|x|=r} |\nabla \hat{H}(x)| \lesssim \delta(r) . \tag{II.6}
\]

These observations shall be helpful in the sequel.

The equation (I.7) implies that for any ball \( B_\rho(0) \) of radius \( \rho \) centered on the origin and contained in \( \Omega \), there holds

\[
\int_{\partial B_\rho(0)} \bar{v} \cdot \left( \nabla \hat{H} - 3 \pi \bar{n} (\nabla \hat{H}) \ast (\nabla \bar{n} \wedge \hat{H}) \right) = 0 , \quad \forall \, \rho \in (0,1) , \tag{II.7}
\]

where \( \bar{c}_0 \) is the residue defined in (I.8). Here \( \bar{v} \) denotes the unit outward normal vector to \( \partial B_\rho(0) \). An elementary computation shows that

\[
\int_{\partial B_\rho(0)} \bar{v} \cdot \nabla \log |x| = 2\pi , \quad \forall \, \rho > 0 .
\]

Thus, upon setting

\[
\bar{X} := \nabla \hat{H} - 3 \pi \bar{n} (\nabla \hat{H}) \ast (\nabla \bar{n} \wedge \hat{H}) - \frac{\bar{c}_0}{2\pi} \nabla \log |x| , \tag{II.8}
\]

we find

\[
\text{div} \bar{X} = 0 \quad \text{on} \quad \Omega , \quad \text{and} \quad \int_{\partial B_\rho(0)} \bar{v} \cdot \bar{X} = 0 \quad \forall \, \rho \in (0,1) .
\]

As \( \bar{X} \) is smooth away from the origin, the Poincaré Lemma implies now the existence of an element \( \bar{L} \in C^\infty(\Omega) \) such that

\[
\bar{X} = \nabla^+ \bar{L} \quad \text{on} \quad \Omega . \tag{II.9}
\]

We deduce from Lemma II.1 and (II.5)-(II.9) that

\[
\int_{B_1(0)} |x|^{2\theta_0} |\nabla \bar{L}|^2 \, dx \lesssim \int_0^1 \delta^2(s) \frac{ds}{s} < \infty . \tag{II.10}
\]
A classical Hardy-Sobolev inequality gives the estimate
\[
\theta_0^2 \int_{B_1(0)} |x|^{2(\theta_0-1)} |\vec{L}|^2 \, dx \leq \int_{B_1(0)} |x|^{2\theta_0} |\nabla \vec{L}|^2 \, dx + \theta_0 \int_{\partial B_1(0)} |\vec{L}|^2 \, dx, \quad (II.11)
\]
which is a finite quantity, owing to \((II.10)\) and to the smoothness of \(\vec{L}\) away from the origin. The immersion \(\vec{\Phi}\) has near the origin the asymptotic behavior
\[|\nabla \vec{\Phi}(x)| \simeq |x|^{\theta_0 - 1} .\]
Hence \((II.11)\) yields that
\[
\vec{L} \cdot \nabla \vec{\Phi} , \vec{L} \wedge \nabla \vec{\Phi} \in L^2(B_1(0)) .
\]
(II.12)

We next set \(\vec{F}(x) := \frac{c_0}{2\pi} \log |x|\), and define the functions \(g\) and \(\vec{G}\) via
\[
\begin{cases}
\Delta g = \nabla \vec{F} \cdot \nabla \vec{\Phi} , & \Delta \vec{G} = \nabla \vec{F} \wedge \nabla \vec{\Phi} \quad \text{in} \quad B_1(0) \\
g = 0 , & \vec{G} = \vec{0} \quad \text{on} \quad \partial B_1(0) .
\end{cases}
\]
(II.13)

Since \(|\nabla \vec{\Phi}(x)| \simeq |x|^{\theta_0 - 1}\) near the origin and \(\vec{F}\) is the fundamental solution of the Laplacian, by applying Calderon-Zygmund estimates to \((II.13)\), we find
\[
\nabla^2 g , \nabla^2 \vec{G} \in \begin{cases}
L^2,\infty(B_1(0)) , & \theta_0 = 1 \\
BMO(B_1(0)) , & \theta_0 \geq 2 .
\end{cases}
\]
(II.14)

In the paper \([BR1]\) (cf. Lemma A.2), the authors derive the identities
\[
\begin{cases}
\nabla \vec{\Phi} \cdot (\nabla \perp \vec{L} + \nabla \vec{F}) = 0 \\
\nabla \vec{\Phi} \wedge (\nabla \perp \vec{L} + \nabla \vec{F}) = -2 \nabla \vec{\Phi} \wedge \nabla \vec{H} .
\end{cases}
\]
(II.15)

Accounted into \((II.13)\), the latter yield that there holds in \(\Omega\):
\[
\begin{cases}
\text{div}(\vec{L} \cdot \nabla \vec{\Phi} - \nabla g) = 0 \\
\text{div}(\vec{L} \wedge \nabla \vec{\Phi} - 2 \vec{H} \wedge \nabla \vec{\Phi} - \nabla \vec{G}) = \vec{0} ,
\end{cases}
\]
(II.16)

where we have used the fact that
\[
\Delta \vec{\Phi} \wedge \vec{H} = 2 e^{2\lambda} \vec{H} \wedge \vec{H} = \vec{0} .
\]

Note that the terms under the divergence symbols in \((II.16)\) both belong to \(L^2(B_1(0))\), owing to \((II.12)\) and \((II.14)\). The distributional equations \((II.16)\), which are \textit{a priori} to be understood on \(\Omega\), may thus be extended to all of \(B_1(0)\). Indeed, a classical result of Laurent Schwartz states that the only distributions supported on \(\{0\}\) are linear combinations of derivatives of the Dirac delta mass. Yet, none of these (including delta itself) belongs to \(W^{-1,2}\). We shall thus understand \((II.16)\) on \(B_1(0)\). It is not difficult to verify (cf. Corollary IX.5 in

\textsuperscript{6}The weak-\(L^2\) Marcinkiewicz space \(L^{2,\infty}(B_1(0))\) is defined as those functions \(f\) which satisfy \(\sup_{a>0} a^2 \left\{ x \in B_1(0) : |f(x)| \geq a \right\} < \infty\). In dimension two, the prototype element of \(L^{2,\infty}\) is \(|x|^{-1}\). The space \(L^{2,\infty}\) is also a Lorentz space, and in particular is a space of interpolation between Lebesgue spaces, which justifies the first inclusion in \((II.14)\). See \([He]\) or \([Al]\) for details.

\textsuperscript{7}Observe that \(\nabla \perp \vec{L} + \nabla \vec{F}\) is exactly the divergence-free quantity appearing in \([L7]\).
that a divergence-free vector field in $L^2(B_1(0))$ is the curl of an element in $W^{1,2}(B_1(0))$. We apply this observation to (II.16) so as to infer the existence of two functions $S$ and of $\vec{R}$ in the space $W^{1,2}(B_1(0)) \cap C^\infty(\Omega)$, with

$$
\begin{align*}
\nabla \perp S &= \vec{L} \cdot \nabla \perp \Phi - \nabla g \\
\nabla \perp \vec{R} &= \vec{L} \wedge \nabla \perp \Phi - 2 \vec{H} \wedge \nabla \vec{G}.
\end{align*}
$$

Moreover, $S$ and $\vec{R}$ may be chosen to be constant on the boundary of the unit disk; without loss of generality, we shall assume that $S|_{\partial B_1(0)} = 0$ and $\vec{R}|_{\partial B_1(0)} = \vec{0}$.

According to the identities (A.13) from the Appendix, the functions $S$ and $\vec{R}$ satisfy on $B_1(0)$ the following system of equations, called conservative conformal Willmore system:

$$
\begin{align*}
\Delta S &= -\nabla (\star \vec{n}) \cdot \nabla \perp \vec{R} - \text{div}((\star \vec{n}) \cdot \nabla \vec{G}) \\
\Delta \vec{R} &= -\nabla (\star \vec{n}) \cdot \nabla \perp \vec{R} + \nabla (\star \vec{n}) \cdot \nabla \perp S \\
&\quad - \text{div}((\star \vec{n}) \cdot \nabla \vec{G} - \star \vec{n} \nabla g).
\end{align*}
$$

Not only is this system independent of the codimension, but it further displays two fundamental advantages. Analytically, (II.18) is uniformly elliptic. This is in sharp contrast with the Willmore equation (I.6) whose leading operator $\Delta_\perp$ degenerates at the origin, owing to the presence of the conformal factor $e^{\lambda(x)} \simeq |x|^\theta_0 - 1$. Structurally, the system (II.18) is in divergence form. We shall in the sequel capitalize on this remarkable feature to develop arguments of “integration by compensation”.

A priori however, since $\vec{n}$, $S$, and $\vec{R}$ are elements of $W^{1,2}$, the leading terms on the right-hand side of the conservative conformal Willmore system (II.18) are critical. This difficulty is nevertheless bypassed using the fact that the $W^{1,2}(B_1(0))$-norm of the Gauss map $\vec{n}$ is chosen small enough (cf. (I.4)).

### II.2 The general case when $\theta_0 \geq 1$

We have gathered enough information about the functions involved to apply to the system (II.18) (a slightly extended version of) Proposition (A.1) and thereby obtain that

$$
\nabla S, \nabla \vec{R} \in L^p(B_1(0)) \quad \text{for some } p > 2.
$$

It is shown in the Appendix (cf. (A.14)) that

$$
-2 \Delta \vec{\Phi} = (\nabla S - \nabla \perp g) \cdot \nabla \perp \vec{\Phi} - (\nabla \vec{R} - \nabla \perp \vec{G}) \cdot \nabla \perp \vec{\Phi}.
$$

Hence, as $|\nabla \vec{\Phi}(x)| \simeq e^{\lambda(x)} \simeq |x|^\theta_0 - 1$ around the origin, using (II.14) and (II.19), we may call upon Proposition (A.2) with the weight $|\mu| = e^\lambda$ and $a = \theta_0 - 1$ to conclude that

$$
\nabla \vec{\Phi}(x) = \vec{P}(x) + e^{\lambda(x)} \vec{T}(x),
$$

$S$ is a scalar while $\vec{R}$ is $\Lambda^2(\mathbb{R}^m)$-valued.

$^9$Refer to the Appendix for the notation and the operators used.
where $\vec{F}$ is a $\mathbb{C}^m$-valued polynomial of degree at most $(\theta_0 - 1)$, and $\vec{T}(x) = O(|x|^{1-\delta-\epsilon})$ for every $\epsilon > 0$. Because $e^{-\lambda} \vec{\nabla} \vec{\Phi}$ is a bounded function, we deduce more precisely that $\vec{F}(x) = \theta_0 \vec{A}^\dagger \vec{F}(x)$, for some constant vector $\vec{A} \in \mathbb{C}^m$ (we denote its complex conjugate by $\vec{A}^\dagger$), so that

$$\vec{n} \Phi(x) = \left( \frac{\Re}{-3} \right) (\theta_0 \vec{A} x^{\theta_0-1}) + e^{\lambda(x)} \vec{T}(x). \quad (\text{II.21})$$

Equivalently, upon writing $\vec{A} = \vec{A}^1 + i \vec{A}^2$, where $\vec{A}^1$ and $\vec{A}^2$ are two vectors in $\mathbb{R}^m$, the latter may be recast as

$$\begin{cases}
\partial_{x_1} \vec{\Phi}(x) = \theta_0 |x|^{\theta_0-1} \left[ \vec{A}^1 \cos((\theta_0 - 1)\varphi) - \vec{A}^2 \sin((\theta_0 - 1)\varphi) \right] + e^{\lambda(x)} \Re(\vec{T}(x)) \\
-\partial_{x_2} \vec{\Phi}(x) = \theta_0 |x|^{\theta_0-1} \left[ \vec{A}^2 \cos((\theta_0 - 1)\varphi) + \vec{A}^1 \sin((\theta_0 - 1)\varphi) \right] - e^{\lambda(x)} \Im(\vec{T}(x)).
\end{cases}$$

The conformality condition of $\vec{\Phi}$ shows easily that

$$|\vec{A}^1| = |\vec{A}^2| \quad \text{and} \quad \vec{A}^1 \cdot \vec{A}^2 = 0. \quad (\text{II.22})$$

Yet more precisely, as $|\vec{n} \Phi|^2 = 2 e^{2\lambda}$, we see that

$$|\vec{A}^1| = |\vec{A}^2| = \frac{1}{\theta_0} \lim_{x \to 0} \frac{e^{\lambda(x)}}{|x|^{\theta_0-1}} \in ]0, \infty[. \quad (\text{II.23})$$

Because $\vec{\Phi}(0) = \vec{0}$, we obtain from (II.21) the local expansion

$$\vec{\Phi}(x) = \Re(\vec{A} x^{\theta_0}) + O(|x|^{\theta_0-\delta-\epsilon}).$$

Since $\pi_R \vec{\nabla} \vec{\Phi} \equiv \vec{0}$, we deduce from (II.21) that

$$\pi_R \vec{A} = -\theta_0^{-1} x_1^{1-\theta_0} e^\lambda \pi_R \vec{T}^a(x) = O(|x|^{1-\delta-\epsilon}) \quad \forall \epsilon > 0. \quad (\text{II.24})$$

Let now $\delta := 1 - \frac{2}{p} \in (0, 1)$, and let $0 < \eta < p$ be arbitrary. We choose some $\epsilon$ satisfying

$$0 < \epsilon < \frac{2 \eta}{p(p - \eta)} \equiv \delta + 1 + \frac{2}{p - \eta}.$$

We have observed that $\pi_R \vec{A} = O(|x|^{\delta-\epsilon})$, hence $\pi_R \vec{A} = o(|x|^{1-\delta-\eta})$, and in particular, we find

$$\frac{1}{|x|} \pi_{R(x)} \vec{A} \in L^{p-\eta}(B_1(0)) \quad \forall \eta > 0. \quad (\text{II.25})$$

This fact shall come helpful in the sequel.

When $\theta_0 = 1$, one directly deduces from the standard Calderon-Zygmund theorem applied to (II.20) that $\vec{\nabla}^2 \vec{\Phi} \in L^p$. In that case, $e^\lambda$ is bounded from above and below, and thus the identity (cf. (A.3) in the Appendix)

$$|\vec{n} \vec{\Phi}| = e^{-\lambda} |\pi_R \vec{\nabla}^2 \vec{\Phi}| \quad (\text{II.26})$$

$^{10}\varphi$ denotes the argument of $x$. 

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yields that $\nabla \hat{n} \in L^p$. When now $\theta_0 \geq 2$, we must proceed slightly differently to obtain analogous results. From (I.10), we know that $|x| \nabla \lambda(x)$ is bounded across the unit-disk. We may thus apply Proposition A.2 (ii) to (II.20) with the weight $|\mu| = e^\lambda$ and $a = \theta_0$. The required hypothesis (A.27) is fulfilled, and we so obtain

$$\nabla^2 \hat{\Phi}(x) = \theta_0 (1 - \theta_0) \left( - \frac{\Re}{\Im} \left( A x^{\theta_0 - 2} \right) + e^\lambda(x) \hat{Q}(x) \right),$$

where $\hat{A}$ is as in (II.21), while $\hat{Q}$ belongs to $R^4 \otimes L^{p-\epsilon}(B_1(0))$ for every $\epsilon > 0$.

According to (II.28), the first summand on the right-hand side of the latter belongs to $L^{p-\eta}$ for all $\eta > 0$. Moreover, we have seen that $\pi n \hat{Q}$ lies in $L^{p-\epsilon}$ for all $\epsilon > 0$. Whence, it follows that $e^{-\lambda} |\pi n \nabla^2 \hat{\Phi}| \lesssim |x|^{-1} |\pi n \hat{A}| + |\pi n \hat{Q}|$.

In light of this new fact, we may now return to (II.18). In particular, recalling (II.14), we find

$$\Delta S \equiv - \nabla (\nabla \hat{R} + \nabla \hat{G}) - (\nabla \hat{n}) \cdot \Delta \hat{G} \in L^q(B_1(0)),$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{p - \epsilon}$. 

We attract the reader’s attention on an important phenomenon occurring when $\theta_0 = 1$. In this case, if the aforementioned value of $q$ exceeds 2 (i.e. if $p > 4$), then $\Delta S \not\in L^q$, but rather only $\Delta S \in L^{q/2}$. This integrability “barrier” stems from that of $\Delta \hat{G}$, as given in (II.14). The same considerations apply naturally with $\hat{R}$ and $\hat{G}$ in place of $S$ and $\hat{G}$, respectively.

Our findings so far may be summarized as follows:

$$\nabla S, \nabla \hat{R} \in \begin{cases} W^{1,(2,\infty)}, & \text{if } \theta_0 = 1 \text{ and } p > 4 \\ W^{1,q}, & \text{otherwise}. \end{cases}$$

With the help of the Sobolev embedding theorem, we infer

$$\nabla S, \nabla \hat{R} \in \begin{cases} W^{1,(2,\infty)} \subset BMO, & \text{if } \theta_0 = 1 \text{ and } p > 4 \\ L^{\infty}, & \text{if } \theta_0 \geq 2 \text{ and } p > 4 \\ L^s, & \text{if } \theta_0 \geq 1 \text{ and } p \leq 4, \end{cases}$$

with $\frac{1}{s} = \frac{1}{q} - \frac{1}{2} = \frac{1}{p} + \frac{1}{p - \epsilon} - \frac{1}{2} < \frac{1}{p}$. 

\[11\] we also use a result of Luc Tartar \[15\] stating that $W^{1,(2,\infty)} \subset BMO$. 

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Comparing (II.30) to (II.19), we see that the integrability has been improved. The process may thus be repeated until reaching that

\[ \nabla S, \nabla \vec{R} \in L^b(B_1(0)) \quad \forall \ b < \infty \]

holds in all configurations. With the help of this newly found fact, we reapply Proposition A.2 so as to improve (II.29) and (II.28) to

\[ \nabla S, \nabla \vec{R} \in \{ W^1, (2, \infty)(B_1(0)) \} \]

and

\[ \nabla \vec{n} \in L^b(B_1(0)) \quad \forall \ b < \infty . \]

The \( \epsilon \)-regularity in the form (II.4) then yields pointwise estimates for the Gauss map. Namely, in a neighborhood of the origin,

\[ |\nabla \vec{n}(x)| \lesssim |x|^{-\epsilon} \quad \forall \ \epsilon > 0 . \]

II.3 The case \( \theta_0 = 1 \)

We shall now investigate further the case \( \theta_0 = 1 \), when \( |\nabla \vec{\Phi}| \simeq e^{\lambda} \) is bounded from both above and below around the origin. Setting

\[ \vec{F}_1 := \nabla \perp \vec{R} + \nabla \vec{G} \quad \text{and} \quad F_2 := \nabla \perp S + \nabla g \]

in (II.20) gives

\[ 2 \Delta \vec{\Phi} = F_2 \cdot \nabla \vec{\Phi} - \vec{F}_1 \cdot \nabla \vec{\Phi} . \]

According to (II.14) and (II.29), the right-hand side of the latter has bounded mean oscillations. Hence \( \nabla^2 \vec{\Phi} \in \bigcap_{p<\infty} L^p \). Using the fact that \( 2 e^{2\lambda} \vec{H} = \Delta \vec{\Phi} \), we differentiate (II.33) to obtain

\[ 4 \nabla (e^{2\lambda} \vec{H}) = \nabla F_2 \cdot \nabla \vec{\Phi} - \nabla \vec{F}_1 \cdot \nabla \vec{\Phi} + F_2 \cdot \nabla^2 \vec{\Phi} - \vec{F}_1 \cdot \nabla^2 \vec{\Phi} \in L^{2, \infty} . \]

This shows that \( \vec{H} \in BMO \). Moreover, since \( \nabla \lambda \in L^2 \), it follows that \( \nabla \vec{H} \in L^{2, \infty} \cap \bigcap_{1 \leq p < 2} L^p \). We shall now obtain an asymptotic expansion for \( \vec{H}(x) \) near the origin. To achieve this, we use a “generic” procedure, which will be called upon again in section II.4.1.

Proposition II.1 Let the immersion \( \vec{\Phi} \) satisfy an expansion of the type (II.27), for all \( p < \infty \). Suppose that \( \vec{n} \in \bigcap_{p<\infty} W^{1,p}(B_1(0)) \) and \( \vec{H} \in \bigcap_{p<2} W^{1,p}(B_1(0)) \). Then locally around the origin,

\[ \vec{H}(x) + \frac{\vec{c}_0}{4\pi} \log |x| \in \bigcap_{p<\infty} W^{1,p}(B_1(0)) , \]

where \( \vec{c}_0 \) is the residue defined in (II.7).

Proof. In order to demonstrate this result, one must return to the formalism developed in [R1], where it is shown that

\[ \mathcal{L}(\vec{H}) := \text{div} \left( \nabla \vec{H} - 3 \pi \vec{n} \nabla \vec{H} + * (\nabla \perp \vec{n} \wedge \vec{H}) \right) = 0 \quad \text{on} \ B_1(0) \ \{ 0 \} . \]
Owing to the hypotheses on $\vec{n}$ and $\vec{H}$, this equation has a distributional sense. Since $L(\vec{H})$ is supported on the origin and it belongs to $W^{-1,p}$ for $p < 2$, it must be proportional to the Dirac mass $\delta_0$. From (II.7), we deduce that

$$L(\vec{H}) = -\vec{c}_0 \delta_0.$$ 

Let $\vec{A} \in \mathbb{C}^m$ be the constant vector appearing in the expansion (II.21). Since $\pi_\vec{n}(0) \cdot \vec{A} = 0$ (cf. (II.24)), an elementary computation gives

$$\vec{A} \cdot \vec{c}_0 \delta_0 = -\pi_T \vec{A} \cdot \vec{c}_0 \delta_0 = -\pi_T \vec{A} \cdot L(\vec{H})$$

$$= -\text{div}(-\vec{H} \cdot \nabla \pi_T A + \pi_T \vec{A} \cdot \star (\nabla^\perp \vec{n} \wedge \vec{H}))$$

$$+ \nabla \pi_T \vec{A} \cdot \left(\nabla \vec{H} - 3 \pi_\vec{n} \nabla \vec{H} + \star (\nabla^\perp \vec{n} \wedge \vec{H})\right),$$ (II.35)

where we have used the fact that $\pi_T \vec{H} \equiv \vec{0}$.

Because $\vec{A}$ is constant and $\nabla \vec{n} \in \bigcap_{p < \infty} L^p$, it follows from (II.55) that $\nabla \pi_\vec{n} \vec{A}$ and thus $\nabla \pi_T \vec{A}$ lie in $\bigcap_{1 < p < 2} L^p$. Moreover, $\nabla \vec{H} \in \bigcap_{1 < p < 2} L^p$ by hypothesis. Introducing this information into (II.35), we note that its right-hand side belongs to $W^{-1,p}$ for all $p < \infty$. Yet, its left-hand side is proportional to the Dirac mass, which does not belong to any $W^{-1,p}$ for $p \geq 2$. We accordingly conclude that $\vec{A} \cdot \vec{c}_0 = 0$. Returning to the expansion (II.21) reveals now that

$$\vec{c}_0 \cdot \left(\begin{array}{c} \vec{e}_1(x) \\ \vec{e}_2(x) \end{array} \right) \simeq \vec{c}_0 \cdot \vec{T}(x) = O(|x|^{1-\epsilon}) \quad \forall \epsilon > 0,$$

whence

$$|x|^{-1} \pi_T(\vec{c}_0) \in \bigcap_{p < \infty} L^p(B_1(0)).$$ (II.36)

A direct computation gives

$$L(\vec{c}_0 \log |x|) = 4\pi \vec{c}_0 \delta_0 + \text{div}(3 \pi_T(\vec{c}_0) \nabla \log |x| + \star (\nabla^\perp \vec{n} \wedge \vec{c}_0) \log |x|)$$

$$= -4\pi L(\vec{H}) + \text{div}(3 \pi_T(\vec{c}_0) \nabla \log |x| + \star (\nabla^\perp \vec{n} \wedge \vec{c}_0) \log |x|).$$

Using the fact that $\nabla \vec{n} \in \bigcap_{p < \infty} L^p$ and (II.36) shows that

$$L\left(\vec{H} + \frac{\vec{c}_0}{4\pi \log |x|}\right) \in \bigcap_{p < \infty} W^{-1,p}.$$ 

It is established in [Ri1] that the operator $L$ is elliptic and in particular that it satisfies $L^{-1} W^{-1,p} \subset W^{1,p}$. The desired claimed therefore ensues:

$$\vec{H}(x) + \frac{\vec{c}_0}{4\pi \log |x|} \in \bigcap_{p < \infty} W^{1,p},$$

We end our study of the case $\theta_0 = 1$ by a slight improvement on the regularity of the Gauss map $\vec{n}$. In the Appendix (cf. (A.7)), it is shown that the
The case \( \theta \geq 2 \)

We now return to (II.20) in the case when \( \theta \geq 2 \). Setting

\[
\vec{F}_1 := \nabla^\perp \vec{R} + \nabla \vec{G} \quad \text{and} \quad F_2 := \nabla^\perp S + \nabla g,
\]

it reads

\[
2 \Delta \vec{\Phi} = F_2 \cdot \nabla \vec{\Phi} - \vec{F}_1 \cdot \nabla \vec{\Phi}.
\] (II.39)

Owing to (II.14) and (II.29), the functions \( \vec{F}_1 \) and \( F_2 \) are Hölder continuous of any order \( \alpha \in (0, 1) \). It thus makes sense to define the constants

\[
\vec{f}_1 := \vec{F}_1(0) \quad \text{and} \quad f_2 := F_2(0).
\]

They are elements of \( \mathbb{R}^2 \otimes \bigwedge^2(\mathbb{R}^m) \) and of \( \mathbb{R}^2 \), respectively. We will in the sequel view \( \vec{f}_1 \) as an element of \( \bigwedge^2(\mathbb{C}^m) \) and \( f_2 \) as an element of \( \mathbb{C} \).

For future purposes, let us define \( \vec{\Gamma} \) via

\[
\Delta \vec{\Gamma} = 4 \theta_0 \Re(\vec{C} |x|^{\theta_0 - 1}) \quad \text{with} \quad 8 \vec{C} := f_2 \cdot \vec{A} - \vec{f}_1 \cdot \vec{A},
\] (II.40)

where \( \vec{A} \) is the constant vector in (II.21). This equation is solved explicitly (up to an unimportant harmonic function):

\[
\vec{\Gamma}(x) = \Re(\vec{C} |x|^2 |x|^{\theta_0 - 1}).
\] (II.41)

Note next that (II.39) and (II.40) give

\[
2 \Delta(\vec{\Phi} - \vec{\Gamma}) = (F_2 - f_2) \cdot \nabla \vec{\Phi} - (\vec{F}_1 - \vec{f}_1) \cdot \nabla \vec{\Phi} + e^\lambda [f_2 \cdot \vec{T} - \vec{f}_1 \cdot \vec{T}],
\] (II.42)

where we have used the representation (II.21). We have seen (compare (II.21) to (II.27)) that \( \partial_{x_j}(e^\lambda \vec{T}) = e^\lambda \vec{Q}_j \), where \( \vec{Q}_j \) belongs to \( L^p \) for all \( p < \infty \). Differentiating (II.42) throughout with respect to \( x_j \) gives

\[
2 \Delta \partial_{x_j}(\vec{\Phi} - \vec{\Gamma}) = \partial_{x_j} F_2 \cdot \nabla \vec{\Phi} - \partial_{x_j} \vec{F}_1 \cdot \nabla \vec{\Phi} + e^\lambda [f_2 \cdot \vec{Q}_j - \vec{f}_1 \cdot \vec{Q}_j]
\]

\[
+ (F_2 - f_2) \cdot \nabla \partial_{x_j} \vec{\Phi} - (\vec{F}_1 - \vec{f}_1) \cdot \nabla \partial_{x_j} \vec{\Phi}.
\] (II.43)
Since $\partial \xi \vec{F}_i, \partial x F_2$, and $Q_j$ belong to $L^p$ for every finite $p$, while $|\nabla \vec{F}| \simeq e^\lambda \simeq |x|^{\theta_0-1}$, we may apply Proposition [2](i) to the first three summands on the right-hand side of (II.43). Moreover, $|\vec{F}_i(x) - \vec{f}_i| + |F_2(x) - f_2| \lesssim |x|^{\alpha}$ for all $\alpha \in (0, 1)$ while $|\nabla \partial \xi \vec{F}(x)| \simeq |x|^{\theta_0-2}$, so that the last two summands on the right-hand side of (II.43) fit within the frame of Corollary [3]. Accordingly, we deduce from (II.45) and (II.41) the expansion (recall that $|\nabla \vec{F}(0)| = 0$ = $|\vec{F}(0)|$).

\[
\nabla \partial \xi (\vec{F} - \vec{F})(x) = \vec{F}_j(x) + e^{\lambda(x)} \vec{U}_j(x),
\]

where $\vec{F}_j$ is a polynomial of degree at most $(\theta_0 - 1)$, and $\vec{U}_j(x) = O(|x|^{1-\epsilon})$, for every $\epsilon > 0$.

One sees in (II.41) that $\nabla \partial \xi \vec{F}(x) = O(|x|^{\theta_0-1})$. Hence, from (II.44) and the fact that $|\nabla \partial \xi \vec{F}(x)| \simeq |x|^{\theta_0-2}$, it follows that the polynomial $\vec{F}_j$ contains exactly one monomial of degree of $(\theta_0 - 2)$ and one monomial of degree of $(\theta_0 - 1)$. More precisely, identifying the representation (II.22) with (II.44) yields

\[
\nabla^2 \vec{F}(x) = \left( -\Re \begin{array}{c} 3 \\ 3 \end{array} \right) \left( \theta_0(1 - \theta_0) \bar{A} x^{\theta_0-2} - \theta_0(1 + \theta_0) \bar{A} x^{\theta_0-1} \right) + \nabla^2 \vec{F}(x) + e^{\lambda(x)} \vec{U}(x),
\]

where $\bar{A} \in \mathbb{C}^m$ is a constant vector and $\vec{U}(x) = O(|x|^{1-\epsilon})$ for all $\epsilon > 0$. The constant vector $\bar{A}$ is as in (II.21).

We deduce from (II.45) and (II.44) the expansion (recall that $|\nabla \vec{F}(0)| = 0$ = $|\vec{F}(0)|$):

\[
\vec{F}(x) = \Re \left( \bar{A} x^{\theta_0} + \bar{B} x^{\theta_0+1} + \bar{C} x^2 x^{\theta_0-1} \right) + |x|^{\theta_0-1} \vec{\xi}(x),
\]

where

\[
\vec{\xi}(x) = O(|x|^{3-\epsilon}), \quad \nabla \vec{\xi}(x) = O(|x|^{2-\epsilon}), \quad \nabla^2 \vec{\xi}(x) = O(|x|^{1-\epsilon}) \quad \forall \epsilon > 0.
\]

Moreover, as $2 e^{2x} \vec{H} = \Delta \vec{F}$, the representation (II.45) along with (II.40) gives the local asymptotic expansion

\[
e^{\lambda(x)} \vec{H}(x) = f(x) \Re \left[ \bar{C} \left( \frac{x}{|x|} \right)^{\theta_0-1} \right] + O(|x|^{1-\epsilon}),
\]

where $\bar{C}$ is as above, and $f(x) := 2 \theta_0 |x|^{\theta_0-1} e^{-\lambda(x)}$, which is known to have a positive limit at the origin. This shows that $e^{\lambda(x)} \vec{H}(x)$ is a bounded function. However, it “spins” as $x$ approaches the origin: its limit need not exist; and, if it does exist, it must be zero (i.e. $\vec{C} = 0$). This possibility is studied in details below.

We close this section by proving that $\nabla^2 \vec{n} \in L^{2, \infty}$, and that $\nabla \vec{n} \in L^\infty$. We have seen that $e^{\lambda} \vec{H}$ is bounded. Applying standard elliptic techniques to (II.7) then yields that $|x| e^{\lambda} \nabla \vec{H}$ is bounded as well, and hence that $e^{\lambda} \nabla \vec{H} \in L^{2, \infty}$. Going back to the perturbed harmonic map equation (II.37) satisfied by the Gauss map $\vec{n}$, and using the fact that $e^{\lambda} \vec{n}$ inherits the regularity of $\nabla \vec{n} \in \bigcap_{2 < p < \infty} L^p$, we deduce that $\Delta \vec{n}$ lies in $L^{2, \infty}$, and therefore indeed that $\nabla^2 \vec{n} \in L^{2, \infty}$. In particular, this implies that $\nabla \vec{n} \in BMO$. It is actually
possible to show that $\nabla \vec{n} \in L^\infty(B_1(0))$. To see this, we first note that (II.46) yields
\[
\nabla \vec{\Phi}(x) = \left( \begin{array}{c} \nabla \\
-3 \end{array} \right) (\theta_0 \vec{A} x^{\theta_0-1}) + \nabla (|x|^{|\theta_0-1|} \vec{\xi}(x)) + O(|x|^{|\theta_0|}) .
\]

Since $\pi_\vec{n} \nabla \vec{\Phi} \equiv 0$, the latter and the estimates on $\vec{\xi}$ give
\[
|\pi_\vec{n}(x) \vec{A}| = O(|x|) .
\]

A quick inspection of the identity (II.45) then reveals that
\[
|\pi_\vec{n} \nabla^2 \vec{\Phi}(x)| \lesssim |\pi_\vec{n}(\vec{A})| |x|^{\theta_0-2} = O(|x|^{|\theta_0-1|}) .
\]

Combining this to (II.26) gives thus that $\nabla \vec{n}$ is bounded across the singularity.

II.4.1 When $e^\lambda \vec{H}$ has a limit at the origin

We shall now consider the case when $\lim_{|x| \to 0} e^\lambda \vec{H}(x)$ exists. Then, as seen in (II.47), we automatically have $\vec{C} = \vec{0}$, and accordingly
\[
e^\lambda \vec{H}(x) = O(|x|^{-\epsilon}) \quad \forall \epsilon > 0 . \tag{II.48}
\]

We draw the reader’s attention on the fact that when $\theta_0 = 2$, the latter implies $\vec{H}(x) = O(|x|^{-\epsilon})$ for all $\epsilon > 0$.

We now show that $\vec{C} = \vec{0}$ implies that the constants $f_1$ and $f_2$ are also trivial. To see this, recall (II.38) and (II.17), namely
\[
f_1 = (\nabla^\perp \vec{H} + \nabla \vec{G})(0) = (\vec{L} \nabla^\perp \vec{\Phi} - 2 \vec{H} \nabla \vec{\Phi})(0) \quad \text{and}
\]
\[
f_2 = (\nabla^\perp S + \nabla g)(0) = (\vec{L} \cdot \nabla^\perp \vec{\Phi})(0) . \tag{II.49}
\]

From $|\nabla \vec{\Phi}| \simeq e^\lambda$ and (II.48), we know that $(\vec{H} \nabla^\perp \vec{\Phi})(0) = \vec{0}$. To obtain $f_1 = 0 = f_2$, it thus suffices to show that $\lim_{|x| \to 0} e^\lambda \vec{H}(x) = \vec{0}$. This is what we shall do. Using a standard argument from elliptic analysis (identical to that enabling to deduce (II.6) from (II.3)), it follows from (II.38) that $e^\lambda \nabla \vec{H}(x) = O(|x|^{-\epsilon})$ for all $\epsilon > 0$. Bringing this information into (II.8) and (II.9), along with the fact that $\nabla \vec{n} \in L^p$ for all $p < \infty$, shows now that $e^\lambda \nabla \vec{L} \in L^p$ for all finite $p$. The Hardy-Sobolev inequality (II.11) with $\theta_0 = 1$ in place of $\theta_0$ implies in particular that $|x|^{-1} e^\lambda \vec{L} \in L^2$. Owing to (II.49), the limit $f_2 = \lim_{|x| \to 0} (\vec{L} \cdot \nabla^\perp \vec{\Phi})(x)$ exists. Yet, we have seen the function $|x|^{-1} (\vec{L} \cdot \nabla^\perp \vec{\Phi})(x)$ is square-integrable near the origin. This is only possible if $f_2 = 0$. We proceed mutatis mutandis to show that $\lim_{|x| \to 0} (\vec{L} \cdot \nabla^\perp \vec{\Phi})(x) = \vec{0}$, thereby yielding $f_1 = \vec{0}$.

In the Appendix (cf. (A.7)), it is shown that the $A^{m-2}(S^{m-1})$-valued Gauss map $\vec{n}$ satisfies a perturbed harmonic map equation:
\[
\Delta \vec{n} + |\nabla \vec{n}|^2 \vec{n} = 2 \cdot e^\lambda [\vec{e}_1 \wedge \pi_\vec{n} \partial_{x_2} \vec{H} - \vec{e}_2 \wedge \pi_\vec{n} \partial_{x_2} \vec{H}] - 2 \cdot e^{2\lambda} \vec{h}_{12} \wedge (\vec{h}_{11} - \vec{h}_{22}) . \tag{II.50}
\]
Moreover, as explained at the end of Section II.3, $e^\lambda \vec{h}_{ij}$ inherits the regularity of $\nabla \vec{n}$. Namely,

$$|\nabla \vec{n}| = e^{-\lambda} |\pi_\vec{n} \nabla^2 \vec{\Phi}| = e^\lambda \left| \begin{array}{cc} \vec{h}_{11} - \vec{h}_{12} \\ \vec{h}_{21} \end{array} \right|. \quad (\text{II.51})$$

We then deduce from (II.50) the estimate

$$|\Delta \vec{n}| \lesssim |\nabla \vec{n}|^2 + e^\lambda |\pi_\vec{n} \nabla \vec{H}|.$$ 

In proving (II.49), we have seen that $e^{\lambda(x)} \nabla \vec{H}(x) = O(|x|^{-\epsilon})$ for all $\epsilon > 0$. Furthermore, from (II.32), we know that $\nabla \vec{n}$ has as much integrability as we please. The right-hand side of the equation (II.50) thus belongs to $L^p$ for all finite $p$, thereby showing that (II.50) is subcritical, and thus yielding

$$\nabla^2 \vec{n} \in L^p(B_1(0)) \quad \forall \ p < \infty. \quad (\text{II.52})$$

When $\theta_0 = 2$, the argument comes to a halt at this point. However, if $\theta_0 \geq 3$, we note from (II.13) that the regularity of $\nabla g$ and $\nabla \vec{G}$ improves to $W^{2,p}$ for all $p < \infty$. Introducing this new information along with (II.31) and (II.52) into (II.18) shows that $\nabla S$ and $\nabla \vec{R}$ are elements of $W^{2,p}$ for all finite $p$. Hence the functions $\vec{F}_1$ and $\vec{F}_2$ defined in (II.38) now lie in $W^{2,p}$ for all $p < \infty$. Moreover, we have seen that they both vanish at the origin, so that

$$|\vec{F}_1(x)| + |\vec{F}_2(x)| \lesssim |x|^{1+\alpha} \quad \forall \ \alpha \in (0,1).$$

Returning to (II.39) and applying Corollary A.2 with $\mu = e^\lambda$, $a = \theta_0 - 1$, $n = 1 = J$, and $r = \alpha$ gives

$$\nabla^2 \vec{\Phi}(x) = \nabla \vec{F}(\overline{\tau}) + e^{\lambda(x)} \vec{V}(x), \quad (\text{II.53})$$

where $\vec{P}$ is a polynomial of degree at most $(\theta_0 + 1)$ and $\vec{V}(x) = O(|x|^{2-\epsilon})$ for all $\epsilon > 0$. Note that since $|\nabla \vec{\Phi}(x)| \sim |x|^\theta_0 - 1$ by hypothesis, $\vec{P}$ has no terms of degree smaller than $(\theta_0 - 1)$. Being a (nonlinear) polynomial of the variable $\overline{\tau}$, the polynomial $P$ has traceless gradient. Whence we deduce from (II.53) that

$$2 e^{\lambda(x)} \vec{H}(x) \equiv e^{-\lambda(x)} \text{Tr} \nabla^2 \vec{\Phi}(x) = \text{Tr} \vec{V}(x) = O(|x|^{2-\epsilon}).$$

In particular, when $\theta_0 = 3$, we arrive at $\vec{H}(x) = O(|x|^{-\epsilon})$ for all $\epsilon > 0$.

As we did in the paragraph following (II.39), we deduce from the asymptotics of $\vec{H}$ those of $\nabla \vec{H}$, namely $e^{\lambda(x)} \nabla \vec{H}(x) = O(|x|^{1-\epsilon})$. To further improve on the regularity of the mean curvature, we may differentiate (II.7) throughout with respect to $x_j$. We obtain an equation for $\partial_{x_j} \vec{H}$ in divergence form valid on $B_1(0) \setminus \{0\}$. The coefficients involve $\vec{n}$, its first and its second derivatives, all of which belong to $L^p$ for every $p < \infty$. As previously done, we can now deduce local asymptotics for $\nabla \partial_{x_j} \vec{H}$ from those of $\partial_{x_j} \vec{H}$. More precisely, $e^{\lambda(x)} \nabla^2 \vec{H}(x) = O(|x|^{1-\epsilon})$ for all $\epsilon > 0$. Since $\nabla \lambda(x) \lesssim |x|^{-1}$, we find

$$\nabla (e^\lambda \partial_{x_j} \vec{H}) \equiv e^\lambda (\nabla \partial_{x_j} \vec{H} + \nabla \lambda \partial_{x_j} \vec{H}) \lesssim |x|^{-\epsilon} \quad \forall \ \epsilon > 0. \quad (\text{II.54})$$
In general for a vector \( \vec{V} \), there holds
\[
|\nabla \pi \vec{V}| \lesssim |\nabla \vec{V}| + |\vec{V}| |\nabla \vec{n}|.
\] (II.55)

Since \( \nabla^2 \vec{n} \in \bigcap_{p < \infty} L^p \), (II.54) gives in particular
\[
|\nabla(e^\lambda \pi \partial_x \vec{H})| \lesssim |x|^{-\epsilon} \quad \forall \epsilon > 0.
\] (II.56)

In addition,
\[
\nabla \vec{G}_k \equiv e^{-\lambda}(\partial_x \vec{\Phi} - \nabla \lambda \partial_x \vec{\Phi}) \lesssim |x|^{-1}.
\]
Combining the latter to (II.56) shows that
\[
|\nabla(e^\lambda \vec{G}_k \wedge \pi \partial_x \vec{H})| \lesssim |x|^{-\epsilon} \quad \forall \epsilon > 0.
\]

We now introduce this information along with (II.51) and (II.52) into (II.50) to obtain
\[
|\Delta \nabla \vec{n}| \lesssim |\nabla \vec{n}|^2 + |\nabla \vec{n}| |\nabla^2 \vec{n}| + |x|^{-\epsilon} \in \bigcap_{p < \infty} L^p,
\]
so that
\[
\nabla^3 \vec{n} \in L^p(B_1(0)) \quad \forall \ p < \infty.
\]

Note also that
\[
|\nabla^3 \vec{\Phi}| = |\nabla^2 |\nabla \vec{\Phi}|| \simeq |\nabla^2 e^\lambda| = e^\lambda |\nabla^2 \lambda + (\nabla \lambda)^2|.
\]

The expansion (II.53) thus gives
\[
|x|^2 |\nabla^2 \lambda| \lesssim |x| |\nabla \lambda|^2 + |x|^2 e^{-\lambda} |\nabla^2 \vec{P}| + |x|^2 e^{-\lambda} |\nabla(e^\lambda \vec{V})|.
\]

We know that \( |x| \lambda \) is a bounded function. Moreover, since \( e^{-\lambda} \simeq |x|^{1-\theta_0} \) and \( \vec{P} \) is a polynomial containing no terms of degree less than \( (\theta_0 - 1) \), we get
\[
|x|^2 |\nabla^2 \lambda| \leq C + |x|^2 e^{-\lambda} |\nabla(e^\lambda \vec{V})|
\]
for some constant \( C \).

Corollary A.2 states that \( |x|^{\epsilon-1} e^{-\lambda} \nabla(e^\lambda \vec{V}) \) belongs to \( L^p \) for all \( p < \infty \) and all \( \epsilon > 0 \). However, by tracking the way this estimate is obtained, it is not difficult to verify that \( |x|^2 e^{-\lambda} \nabla(e^\lambda \vec{V}) \) tends to zero as \( x \) moves towards the origin. Hence,
\[
|x|^2 \nabla^2 \lambda(x) \in L^\infty(B_1(0)).
\]

This procedure continues on. As \( \theta_0 \) increases, so does the regularity of \( g \) and \( \vec{G} \), thereby improving that of \( S \) and \( \vec{R} \). Repeating the above argument through Corollary A.2 yields that
\[
\nabla^{j+1} \vec{\Phi}(x) = \nabla^j \vec{P}(x) + e^{\lambda(x)} \vec{V}_j(x), \quad \forall j \in \{0, \ldots, \theta_0 - 1\},
\]
12 namely, every time \( \theta_0 \) increases by an increment of one, so does the parameter \( n \) in Corollary A.2, and we increase accordingly the parameter \( J \) by one. The procedure allows up to \( n = \theta_0 - 2 \) and \( J = \theta_0 - 1 \).
where \( \tilde{P} \) is a polynomial of degree at most \((2\theta_0 - 2 - j)\) and \( \tilde{V}_j(x) = O(|x|^{\theta_0 - j - \epsilon}) \) for all \( \epsilon > 0 \). In particular, using the hypothesis that \(|\nabla \tilde{\Phi}(x)| \simeq |x|^\theta_{0-1}\), we deduce that

\[
\tilde{P}(x) = \theta_0 \tilde{A} x^{\theta_0-1} + \sum_{j=1}^{\theta_0-1} (\theta_0 + j) \tilde{A}_j x^{\theta_0-1+j},
\]

where \( \tilde{A} \) is as in (II.21) and \( \tilde{A}_j \in \mathbb{C}^m \) are constant vectors. Altogether, since \(|\Phi(0)| = 0 = |\nabla \Phi(0)| \) by hypothesis, we obtain the representation

\[
\tilde{\Phi}(x) = \Re \left( \tilde{A} x^{\theta_0} + \sum_{j=1}^{\theta_0-1} \tilde{A}_j x^{\theta_0+j} \right) + \tilde{\zeta}(x),
\]

where the function \( \tilde{\zeta} \) satisfies

\[
|\nabla^j \tilde{\zeta}(x)| = e^{\lambda(x)} |\tilde{V}_{j-1}(x)| = O(|x|^{2\theta_0 - j - \epsilon}) \quad \forall \; \epsilon > 0, \; j \in \{0, \ldots, \theta_0\}.
\]

The last item from Corollary A.2 also gives

\[
|\nabla^{\theta_0-1} \Delta \tilde{\zeta}(x)| \lesssim |x|^\theta_{0-1-\epsilon} \quad \forall \; \epsilon > 0.
\]

From the above we obtain inductively that

\[
|x|^j \nabla^j \lambda(x) \in L^\infty(B_1(0)) \quad \forall \; j \in \{0, \ldots, \theta_0 - 1\}.
\]

Moreover,

\[
\tilde{H}(x) = \frac{1}{2} e^{-2\lambda(x)} \Delta \tilde{\zeta}(x).
\]

Combining (II.58) - (II.60) then yields

\[
\nabla^j \tilde{H}(x) = O(|x|^{-j-\epsilon}) \quad \forall \; \epsilon > 0 \text{ and } j \in \{0, \ldots, \theta_0 - 1\}.
\]

In particular, we have \( \nabla \tilde{H}(x) = O(|x|^{-1-\epsilon}) \in \bigcap_{p<2} L^p \). With this fact at our disposal and (II.28), we call upon Proposition II.1 and obtain the local expansion

\[
\tilde{H}(x) + \frac{\tilde{c}_0}{4\pi} \log |x| \in \bigcap_{p<\infty} W^{1,p}(B_1(0)),
\]

where \( \tilde{c}_0 \) is the residue defined in (II.7).

In addition, the above procedure implies

\[
\nabla^{\theta_0} \tilde{n} \in L^p(B_1(0)) \quad \forall \; p < \infty.
\]

To obtain pointwise information about the Gauss map, we use a “higher order” version of the \( \epsilon \)-regularity which appears in [Ri1] (cf. Theorem I.5) along with the same technique as in the proof of Lemma II.1. Under the hypothesis (II.1), there holds

\[
r^j \sup_{|x|=r} |\nabla^j \tilde{n}(x)| \lesssim \|\nabla \tilde{n}\|_{L^2(B_{2r}(0) \setminus B_{r/2}(0))} \quad \forall \; r \in (0, 1/2), \; j \in \mathbb{N}^+ .
\]
Then, from (II.65), we deduce
\[ |\nabla^j \tilde{n}(x)| \lesssim |x|^{\theta_0 - j - \epsilon} \quad \forall \quad \epsilon > 0, \quad j \in \{0, \ldots, \theta_0\}. \tag{II.65} \]

We have seen that
\[ \tilde{\Phi}(x) = \Re \left( \tilde{A} x^{\theta_0} + \sum_{j=1}^{\theta_0-1} \tilde{A}_j x^{\theta_0+j} \right) + \tilde{\zeta}(x). \tag{II.66} \]

From \( \pi_\tilde{n} \nabla \tilde{\Phi} \equiv \tilde{0} \), we obtain after a few computations
\[
(\theta_0 + 1) \pi_\tilde{n} \tilde{A}_1 = x^{1-\theta_0} \pi_\tilde{n} \left( \partial_{x_1 x_1} \tilde{\Phi} - i \partial_{x_1 x_2} \tilde{\Phi} \right) + x^{1-\theta_0} \pi_\tilde{n} \left( \partial_{x_1 x_1} \tilde{\zeta} - i \partial_{x_1 x_2} \tilde{\zeta} \right) + x^{-\theta_0} \pi_\tilde{n} \left( \partial_{x_1 x_1} \tilde{\Phi} + i \partial_{x_1 x_2} \tilde{\Phi} \right) - \sum_{j=2}^{\theta_0-1} j(\theta_0 + j) \pi_\tilde{n} \tilde{A}_j x^{j-1}.
\]

Hence,
\[
|\pi_\tilde{n} \tilde{A}_1| \lesssim |x|^{1-\theta_0} |\pi_\tilde{n} \nabla^2 \tilde{\Phi}| + |x|^{1-\theta_0} |\pi_\tilde{n} \nabla^2 \tilde{\zeta}| + |x|^{-\theta_0} |\pi_\tilde{n} \nabla \tilde{\zeta}| + O(|x|).
\]

Using (II.58) and the fact that \( |\nabla \tilde{n}| = e^{-\lambda} |\pi_\tilde{n} \nabla^2 \tilde{\Phi}| \approx |x|^{1-\theta_0} |\pi_\tilde{n} \nabla^2 \tilde{\Phi}| \) yields
\[
|\pi_\tilde{n}(x) \tilde{A}_1| \lesssim |\nabla \tilde{n}(x)| + O(|x|^{\theta_0-1-\epsilon}) \quad \forall \quad \epsilon > 0.
\]

Then, from (II.65), we deduce
\[
\pi_\tilde{n}(0) \tilde{A}_1 = \tilde{0}.
\]

This process is repeated by taking successive derivatives and using (II.58). Namely, for each \( k \in \{0, \ldots, \theta_0 - 1\} \):
\[
|\pi_\tilde{n} \tilde{A}_k| \lesssim \sum_{j=1}^{k+1} |x|^{j-k-\theta_0} \left( |\pi_\tilde{n} \nabla^j \tilde{\Phi}| + |\pi_\tilde{n} \nabla^j \tilde{\zeta}| \right) + O(|x|) \\
\lesssim \sum_{j=1}^{k+1} |x|^{j-k-\theta_0} |\pi_\tilde{n} \nabla^j \tilde{\Phi}| + O(|x|^{\theta_0-k-\epsilon}) \quad \forall \quad \epsilon > 0.
\]

Because \( |\nabla \tilde{n}| = e^{-\lambda} |\pi_\tilde{n} \nabla^2 \tilde{\Phi}| \), we obtain through a simple calculation that
\[
|\pi_\tilde{n} \nabla^j \tilde{\Phi}| \lesssim e^{\lambda} \sum_{q=1}^{j-1} |\nabla^{j-1-q} \lambda| |\nabla^q \tilde{n}| = O(|x|^{2\theta_0-j-\epsilon}) \quad \forall \quad \epsilon > 0,
\]

where we have used (II.60) and (II.65). Combining altogether the latter two estimates yields
\[
|\pi_\tilde{n}(x) \tilde{A}_k| = O(|x|^{\theta_0-k-\epsilon}) \quad \forall \quad \epsilon > 0, \quad k \in \{0, \ldots, \theta_0 - 1\}.
\]

Hence in particular,
\[
\pi_\tilde{n}(0) \tilde{A}_k = \tilde{0} \quad \forall \quad k \in \{0, \ldots, \theta_0 - 1\}. \tag{II.67}
\]

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We have seen in general (cf. (II.22)-(II.23)) that \( \{\vec{A}^1, \vec{A}^2\} \) forms an orthogonal basis of a plane through the origin (this plane may a priori not be viewed as the tangent plane \( T_0\Sigma \) which need not exist). Moreover, from the expansion (II.21), we obtain that

\[
\star \vec{\pi} := \frac{\partial_{\vec{x}_1} \vec{\Phi} \wedge \partial_{\vec{x}_2} \vec{\Phi}}{|\partial_{\vec{x}_1} \vec{\Phi} \wedge \partial_{\vec{x}_2} \vec{\Phi}|} \simeq -\frac{\vec{A}^1 \wedge \vec{A}^2}{|A^1 \wedge A^2|},
\]

so that the Gauss map is well-defined at the origin. Hence, any constant vector \( \vec{V} \in \mathbb{R}^m \) has a representation

\[
\vec{V} = \frac{1}{a} (\vec{V} \cdot \vec{A}) \vec{A}^1 + \frac{1}{a} (\vec{V} \cdot \vec{A}) \vec{A}^2 + \pi_{\vec{\pi}(0)} \vec{V},
\]

where \( a := |\vec{A}^1| = |\vec{A}^2| \). For two vectors \( \vec{U} = \vec{U}^1 + i \vec{U}^2 \) and \( \vec{V} = \vec{V}^1 + i \vec{V}^2 \) in \( \mathbb{R}^2 \otimes \mathbb{R}^m \simeq \mathbb{C}^m \), we define the product

\[
\langle \vec{U}, \vec{V} \rangle_{\mathbb{C}^m} := (\vec{U}^1 \cdot \vec{V}^1 - \vec{U}^2 \cdot \vec{V}^2) + i (\vec{U}^1 \cdot \vec{V}^2 + \vec{U}^2 \cdot \vec{V}^1) \in \mathbb{C}.
\]

The conformality condition applied to (II.66) then easily yields that for every \( s \in \{0, \ldots, \theta_0 - 1\} \) there holds

\[
2 \sum_{0 \leq j < s} \langle \vec{A}_j, \vec{A}_{s-j} \rangle_{\mathbb{C}^m} = \langle \vec{A}_{s/2}, \vec{A}_{s/2} \rangle_{\mathbb{C}^m} = 0,
\]

where \( \vec{A}_0 := \vec{A} \), and last term is to be ignored when \( s \) is odd.

For \( s = 0 \), we recover what we have previously observed, namely \( \langle \vec{A}, \vec{A} \rangle_{\mathbb{C}^m} = 0 \), thereby yielding (II.22). Using \( s = 1 \) in (II.68) gives \( \langle \vec{A}, \vec{A}_1 \rangle_{\mathbb{C}^m} = 0 \), so that from (II.67) we deduce \( \vec{A}_1 = \alpha_1 \vec{A} \) for some \( \alpha_1 \in \mathbb{C} \). Putting now \( s = 2 \) in (II.68) gives

\[
2 \langle \vec{A}, \vec{A}_2 \rangle_{\mathbb{C}^m} = \langle \vec{A}_1, \vec{A}_1 \rangle_{\mathbb{C}^m} = \alpha_1^2 \langle \vec{A}, \vec{A} \rangle_{\mathbb{C}^m} = 0,
\]

so that \( \vec{A}_2 = \alpha_2 \vec{A} \) for some \( \alpha_2 \in \mathbb{C} \). Proceeding inductively reveals

\[
\vec{A}_j = \alpha_j \vec{A} \quad \text{for some } \alpha_j \in \mathbb{C} \quad \forall j \in \{0, \ldots, \theta_0 - 1\}.
\]

The representation (II.66) thus becomes

\[
\vec{\Phi}(x) = \sum_{j=0}^{\theta_0 - 1} \Re(\alpha_j \vec{A} x^{\theta_0 + j}) + \vec{\zeta}(x), \quad \text{with } \alpha_0 = 1. \quad (II.69)
\]

We will use this formulation to obtain a result describing the behavior of the mean curvature near the singularity and improving (II.62). To do so, we return to the proof of Proposition (II.3). Setting

\[
\mathcal{L}(\vec{H}) := \text{div} \left( \nabla \vec{H} - 3 \pi_{\vec{\pi}} \nabla \vec{H} + \star (\nabla \vec{\pi} \wedge \vec{H}) \right),
\]

we saw that

\[
\mathcal{L}(\vec{H}) = -c_0 \delta_0,
\]

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where \( \vec{c}_0 \) is the residue defined in (I.8). We also proved that
\[
\vec{c}_0 \cdot \vec{A} = 0.
\]
Introducing this fact into (II.69) then yields
\[
\pi T(\vec{c}_0) := e^{-2\lambda(x)}(\vec{c}_0 \cdot \partial_{x_1} \Phi(x)) \partial_{x_1} \Phi(x) + e^{-2\lambda(x)}(\vec{c}_0 \cdot \partial_{x_2} \Phi(x)) \partial_{x_2} \Phi(x).
\]
Hence, calling upon (II.58) and (II.60) gives the estimate
\[
|\nabla^k \pi T(\vec{c}_0)| \lesssim |x|^\theta_0 - k - \epsilon \quad \forall \epsilon > 0, \ k \in \{0, \ldots, \theta_0 - 1\}.
\]
Once combined to (II.65), the latter implies
\[
|\nabla^{\theta_0 - 2} \text{div}
\left[
3 \pi T \vec{c}_0 \nabla \log |x| + \ast (\nabla^\perp \vec{n} \wedge \vec{c}_0) \log |x|
\right]|
\lesssim |x|^{-\epsilon} \in \bigcap_{p<\infty} L^p(B_1(0)).
\]
We saw in the course of the proof of Proposition II.1 that
\[
\mathcal{L}
\left(4 \pi \vec{H} + \vec{c}_0 \log |x|ight) = \text{div}
\left[
3 \pi T \vec{c}_0 \nabla \log |x| + \ast (\nabla^\perp \vec{n} \wedge \vec{c}_0) \log |x|
\right].
\]
Hence
\[
\mathcal{L}
\left(\nabla^{\theta_0 - 2} (4 \pi \vec{H} + \vec{c}_0 \log |x|)\right) \in \bigcap_{p<\infty} L^p,
\]
so that (since \( \mathcal{L} \) is second-order elliptic and essentially behaves like the Laplacian [RII]),
\[
\vec{H}(x) + \frac{\vec{c}_0}{4\pi} \log |x| \in \bigcap_{p<\infty} W^{\theta_0, p}.
\]
Finally, to obtain that \( \nabla^{\theta_0 + 1} \vec{n} \in L^{2, \infty} \), and thus that \( \nabla^{\theta_0} \vec{n} \in BMO \), we return to (II.37). Since \( e^H \vec{n}_{ij} \) inherits the regularity of \( \nabla \vec{n} \), it is not hard to obtain
\[
|\Delta \nabla^{\theta_0 - 1} \vec{n}| \lesssim \sum_{k=0}^{\theta_0 - 1} |\nabla^{k+1} \vec{n}| |\nabla^{\theta_0 - k} \vec{n}| + |\nabla^{k+1} \Phi| |\nabla^{\theta_0 - k} \vec{H}|
\]
Bringing (II.57), (II.58), (II.65), and (II.70) into the latter shows that for all \( \epsilon > 0 \),
\[
|\Delta \nabla^{\theta_0 - 1} \vec{n}| \lesssim |x|^\theta_0 - 1 - \epsilon + |x|^{-1} + \text{terms in} \bigcap_{p<\infty} L^p \in L^{2, \infty},
\]
thereby yielding the sought out result.

II.5 When the residue \( \vec{c}_0 \) vanishes: point removability

This last section is devoted to proving Theorem I.2. We shall assume that the residue defined in (I.8) satisfies \( \vec{c}_0 = 0 \), and furthermore when \( \theta_0 \geq 2 \) that the constant vector appearing in Proposition II.1(ii) is also null: \( \vec{C} = 0 \). 25
When $\vec{c}_0 = \vec{0}$, the functions $g$ and $\vec{G}$ vanish identically, so the conservative conformal Willmore system (II.18)-(II.20) reads

$$
\begin{cases}
- \Delta S = \nabla (\star \vec{n}) \cdot \nabla \perp \vec{R} \\
- \Delta \vec{R} = \nabla (\star \vec{n}) \bullet \nabla \perp \vec{R} - \nabla (\star \vec{n}) \cdot \nabla \perp S \\
-2 \Delta \vec{\Phi} = \nabla S \cdot \nabla \perp \vec{\Phi} - \nabla \vec{R} \bullet \nabla \perp \vec{\Phi}
\end{cases}
$$

(II.71)

When $\theta_0 = 1$, the immersion $\vec{\Phi}$ is non-degenerate at the origin: its gradient is bounded from above and below. In this case, it was shown in [BR1] that the system (II.71) yields that $\vec{\Phi}$ is smooth across the unit disk. In the case when $\theta_0 \geq 2$, we have shown in the previous section that $\nabla \theta_0 \vec{n}, \nabla \theta_0 S, \nabla \theta_0 \vec{R} \in \bigcap_{p<\infty} L^p$.

It immediately follows from the first two equations in (II.71) that $S$ and $\vec{R}$ lie in $W^{\theta_0+1,p}$ for all $p < \infty$. We are thus in the position of applying the procedure given in the previous section (since $g$ and $\vec{G}$ no longer obstruct) and deduce that $\vec{n} \in W^{\theta_0+1,p}$ for all $p < \infty$. A bootstrapping argument is then implemented to increase the regularity of all functions involved up to $C^\infty(B_1(0))$. This is in particular the case for the immersion $\vec{\Phi}$, thereby completing the proof of Theorem I.2.

A Appendix

A.1 Notational Conventions

We append an arrow to all the elements belonging to $\mathbb{R}^m$. To simplify the notation, by $\vec{\Phi} \in X(D^2)$ is meant $\vec{\Phi} \in X(D^2, \mathbb{R}^m)$ whenever $X$ is a function space. Similarly, we write $\nabla \vec{\Phi} \in X(D^2)$ for $\nabla \vec{\Phi} \in \mathbb{R}^2 \otimes X(D^2, \mathbb{R}^m)$.

Although this custom may seem at first odd, we allow the differential operators classically acting on scalars to act on elements of $\mathbb{R}^m$. Thus, for example, $\nabla \vec{\Phi}$ is the element of $\mathbb{R}^2 \otimes \mathbb{R}^m$ that can be written ($\partial_{x_1} \vec{\Phi}, \partial_{x_2} \vec{\Phi}$). If $S$ is a scalar and $\vec{R}$ an element of $\mathbb{R}^m$, then we let

$$
\vec{R} \cdot \nabla \vec{\Phi} := \left( \vec{R} \cdot \partial_{x_1} \vec{\Phi}, \vec{R} \cdot \partial_{x_2} \vec{\Phi} \right)
$$

$$
\nabla \perp S \cdot \nabla \vec{\Phi} := \partial_{x_1} S \partial_{x_2} \vec{\Phi} - \partial_{x_2} S \partial_{x_1} \vec{\Phi}
$$

$$
\nabla \perp \vec{R} \cdot \nabla \vec{\Phi} := \partial_{x_1} \vec{R} \cdot \partial_{x_2} \vec{\Phi} - \partial_{x_2} \vec{R} \cdot \partial_{x_1} \vec{\Phi}
$$

$$
\nabla \perp \vec{R} \land \nabla \vec{\Phi} := \partial_{x_1} \vec{R} \land \partial_{x_2} \vec{\Phi} - \partial_{x_2} \vec{R} \land \partial_{x_1} \vec{\Phi}.
$$

Analogous quantities are defined according to the same logic.

Two operations between multivectors are useful. The interior multiplication $\mathbf{L}$ maps a pair comprising a $q$-vector $\gamma$ and a $p$-vector $\beta$ to a $(q-p)$-vector. It is defined via

$$
\langle \gamma \mathbf{L} \beta, \alpha \rangle = \langle \gamma, \beta \land \alpha \rangle \quad \text{for each} \ (q-p)\text{-vector } \alpha.
$$

13 cf. last paragraph of Section III.2.1.
Let $\alpha$ be a $k$-vector. The first-order contraction operation $\bullet$ is defined inductively through
\[
\alpha \bullet \beta = \alpha \mathbf{L} \beta \quad \text{when } \beta \text{ is a 1-vector},
\]
and
\[
\alpha \bullet (\beta \land \gamma) = (\alpha \bullet \beta) \land \gamma + (-1)^{pq} (\alpha \bullet \gamma) \land \beta,
\]
when $\beta$ and $\gamma$ are respectively a $p$-vector and a $q$-vector.

A.2 Miscellaneous Facts

A.2.1 On the Gauss Map

Let $\tilde{\Phi}$ be a conformal immersion of the unit-disk into $\mathbb{R}^{m}$. By definition, for $j \in \{1, 2\}$,
\[
\tilde{e}_j := e^{-\lambda} \partial_{x_j} \tilde{\Phi} \quad \text{with } 2e^{2\lambda} = |\nabla \tilde{\Phi}|^2.
\]
One easily verifies (cf. details in [BR1] Section III.2.2) that
\[
\pi_T \nabla \tilde{e}_j = (\nabla^\perp \lambda) \tilde{e}_j' \quad \text{where } (\tilde{e}_1', \tilde{e}_2') := (\tilde{e}_2, -\tilde{e}_1).
\]
Moreover
\[
\pi_\mathcal{H} \nabla \tilde{e}_j = e^{-\lambda} \pi_\mathcal{H} \nabla \partial_j \tilde{\Phi} =: e^{\lambda} \left( \begin{array}{c} \tilde{h}_{1j} \\ \tilde{h}_{2j} \end{array} \right).
\]

With this notation, the mean curvature vector takes the form
\[
\vec{H} = \frac{1}{2} (\tilde{h}_{11} + \tilde{h}_{22}).
\]
The $(m - 2)$-vector $\vec{n}$ satisfies $\vec{n} := \ast (\vec{e}_1 \land \vec{e}_2)$. Accordingly, using (A.1), there holds
\[
\nabla \vec{n} = \ast \left[ (\pi_\mathcal{H} \nabla \vec{e}_1) \land \vec{e}_2 + \vec{e}_1 \land (\pi_\mathcal{H} \nabla \vec{e}_2) \right].
\]
so that
\[
\Delta \vec{n} = \ast \left[ \text{div}(\pi_\mathcal{H} \nabla \vec{e}_1) \land \vec{e}_2 + \vec{e}_1 \land \text{div}(\pi_\mathcal{H} \nabla \vec{e}_2) \right] + 2 \ast \left[ \pi_\mathcal{H} \nabla \vec{e}_1 \land \pi_\mathcal{H} \nabla \vec{e}_2 \right] + \ast \left[ \pi_\mathcal{H} \nabla \vec{e}_1 \land \pi_T \nabla \vec{e}_2 + \pi_T \nabla \vec{e}_1 \land \pi_\mathcal{H} \nabla \vec{e}_2 \right].
\]
The identities (A.1) yield
\[
\pi_T \nabla \vec{e}_k \land \pi_\mathcal{H} \nabla \vec{e}_l = (\nabla^\perp \lambda) \cdot (\vec{e}_{k'} \land \pi_\mathcal{H} \nabla \vec{e}_l),
\]
and thus
\[
\Delta \vec{n} = \ast \left[ \text{div}(\pi_\mathcal{H} \nabla \vec{e}_1) \land \vec{e}_2 + \vec{e}_1 \land \text{div}(\pi_\mathcal{H} \nabla \vec{e}_2) \right] + 2 \ast \left[ \pi_\mathcal{H} \nabla \vec{e}_1 \land \pi_\mathcal{H} \nabla \vec{e}_2 \right] + \ast \left[ (\nabla^\perp \lambda) \cdot (\vec{e}_1 \land \pi_\mathcal{H} \nabla \vec{e}_1 + \vec{e}_2 \land \pi_\mathcal{H} \nabla \vec{e}_2) \right].
\]

Footnotes:
14. $\pi_T$ denotes projection onto the tangent space spanned by $\{\vec{e}_1, \vec{e}_2\}$.
15. $\pi_\mathcal{H}$ denotes projection onto the normal space, namely $\pi_\mathcal{H} = \text{id} - \pi_T$.  

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Next, using the definition of $\vec{e}_k$ and again (A.1), we obtain\footnote{implicit summations over repeated indices are understood.}

$$\text{div } \pi_n \nabla \vec{e}_k \equiv \pi_n \text{div } \pi_n \nabla \vec{e}_k + \pi_T \text{div } \pi_n \nabla \vec{e}_k$$

$$= \pi_n \text{div } \pi_n \nabla (e^{-\lambda} \partial_k \vec{\Phi}) + (\vec{e}_l \cdot \text{div } \pi_n \nabla \vec{e}_k) \vec{e}_l$$

$$= e^{-\lambda} \pi_n \text{div } \pi_n \nabla \partial_k \vec{\Phi} - e^{-\lambda} \pi_n (\nabla \lambda \cdot \nabla \partial_k \vec{\Phi})$$

$$- (\pi_n \nabla \vec{e}_l \cdot \pi_n \nabla \vec{e}_k) \vec{e}_l$$

$$= e^{-\lambda} \pi_n \text{div } \pi_n \nabla \partial_k \vec{\Phi} - (\pi_n \nabla \vec{e}_l \cdot \pi_n \nabla \vec{e}_k) \vec{e}_l - \nabla \lambda \cdot \pi_n \nabla \vec{e}_k.$$  

Introducing the latter into (A.5) gives after a few elementary manipulations,

$$\Delta \vec{n} = * e^{-\lambda} \left[ \pi_n \text{div} (\pi_n \nabla \partial_{x_1} \vec{\Phi}) \wedge \vec{e}_2 + \vec{e}_1 \wedge \pi_n \text{div} (\pi_n \nabla \partial_{x_2} \vec{\Phi}) \right]$$

$$- \left[ \left| \pi_n \nabla \vec{e}_1 \right|^2 + \left| \pi_n \nabla \vec{e}_2 \right|^2 \right] \ast (\vec{e}_1 \wedge \vec{e}_2) + 2 \ast \left[ \pi_n \nabla \vec{e}_1 \wedge \pi_n \nabla \vec{e}_2 \right]$$

$$+ \ast (\nabla \parallel \lambda) \cdot \left[ \vec{e}_1 \wedge \pi_n (\nabla \vec{e}_1 - \nabla \perp \vec{e}_2) - \pi_n (\nabla \parallel \vec{e}_1 + \nabla \perp \vec{e}_2) \wedge \vec{e}_2 \right].$$

Owing to (A.2) and (A.3), we find

$$\Delta \vec{n} + |\nabla \vec{n}|^2 \vec{n} = * e^{-\lambda} \left[ \pi_n \text{div} (\pi_n \nabla \partial_{x_1} \vec{\Phi}) \wedge \vec{e}_2 + \vec{e}_1 \wedge \pi_n \text{div} (\pi_n \nabla \partial_{x_2} \vec{\Phi}) \right]$$

$$+ 2 \ast e^{-2\lambda} \left[ \pi_n \nabla \partial_{x_1} \vec{\Phi} \wedge \pi_n \nabla \partial_{x_2} \vec{\Phi} \right] + 2 \ast e^3 \vec{H} \wedge \left[ \partial_{x_2} \lambda \vec{e}_1 - \partial_{x_1} \lambda \vec{e}_2 \right].$$

Equivalently,

$$\Delta \vec{n} + |\nabla \vec{n}|^2 \vec{n} = * \vec{e}_1 \wedge \pi_n \left[ e^{-\lambda} \text{div} (\pi_n \nabla \partial_{x_1} \vec{\Phi}) - 2 e^3 \vec{H} \partial_{x_2} \lambda \right]$$

$$- * \vec{e}_2 \wedge \pi_n \left[ e^{-\lambda} \text{div} (\pi_n \nabla \partial_{x_1} \vec{\Phi}) - 2 e^3 \vec{H} \partial_{x_1} \lambda \right]$$

$$+ 2 \ast e^{-2\lambda} \left[ \pi_n \nabla \partial_{x_1} \vec{\Phi} \wedge \pi_n \nabla \partial_{x_2} \vec{\Phi} \right].$$

Moreover, (A.1) gives $\pi_T \nabla \partial_{x_j} \vec{\Phi} = \nabla (e^\lambda) \vec{e}_j + \nabla \parallel (e^\lambda) \vec{e}_j$. Hence, calling upon (A.2) implies

$$\pi_n \text{div } \pi_T \nabla \partial_{x_j} \vec{\Phi} = \nabla (e^\lambda) \cdot \pi_n \nabla \vec{e}_j + \nabla \parallel (e^\lambda) \cdot \pi_n \nabla \vec{e}_j = \vec{H} \partial_{x_j} e^{2\lambda},$$

and thus, as $\Delta \vec{\Phi} = 2e^{2\lambda} \vec{H}$,

$$\pi_n \text{div } \pi_n \nabla \partial_{x_j} \vec{\Phi} \equiv \pi_n \partial_{x_j} \Delta \vec{\Phi} - \pi_n \text{div } \pi_T \nabla \partial_{x_j} \vec{\Phi}$$

$$= 2 \pi_n \partial_{x_j} (e^{2\lambda} \vec{H}) - \vec{H} \partial_{x_j} e^{2\lambda}.$$ 

The interested reader will note that this equation is equivalent to the Codazzi-Mainardi identities. Substituted into (A.6), the latter gives

$$\Delta \vec{n} + |\nabla \vec{n}|^2 \vec{n} = 2 \ast e^3 \left[ \vec{e}_1 \wedge \pi_n \partial_{x_2} \vec{H} - \vec{e}_2 \wedge \pi_n \partial_{x_2} \vec{H} \right]$$

$$+ 2 \ast e^{-2\lambda} \left[ \pi_n \nabla \partial_{x_1} \vec{\Phi} \wedge \pi_n \nabla \partial_{x_2} \vec{\Phi} \right]$$

$$= 2 \ast e^3 \left[ \vec{e}_1 \wedge \pi_n \partial_{x_2} \vec{H} - \vec{e}_2 \wedge \pi_n \partial_{x_2} \vec{H} \right]$$

$$- 2 \ast e^{2\lambda} \vec{h}_{12} \wedge (\vec{h}_{11} - \vec{h}_{22}).$$

(A.7)
A.2.2 Conservative Conformal Willmore System

We establish in this section a few general identities. As before, we let \( \Phi \) be a (smooth) conformal immersion of the unit-disk into \( \mathbb{R}^m \), and set \( \bar{e}_j := e^{-\lambda} \partial_x^j \Phi \), where \( \lambda \) is the conformal parameter. Since \( \Phi \) is conformal, \( \{ \bar{e}_1, \bar{e}_2 \} \) forms an orthonormal basis of the tangent space. As \( \bar{n} = * (\bar{e}_1 \wedge \bar{e}_2) \), if \( \bar{V} \) is a 1-vector, we find

\[
* \bar{n} \cdot (\bar{V} \wedge \partial_x^j \bar{n}) = e^{-\lambda} (\bar{e}_1 \wedge \bar{e}_2) \cdot (\bar{V} \wedge \bar{e}_j) = -e^{-\lambda} \bar{e}_j' \cdot \bar{V} = -\partial_x^j \bar{\Phi} \cdot \bar{V},
\]

where \( (\bar{e}_1', \bar{e}_2') := (\bar{e}_2, -\bar{e}_1) \).

Whence,

\[
\begin{align*}
(* \bar{n}) \cdot (\bar{V} \wedge \nabla \bar{\Phi}) &= \bar{V} \cdot \nabla^\perp \bar{\Phi}, \\
(* \bar{n}) \cdot (\bar{V} \wedge \nabla^\perp \bar{\Phi}) &= -\bar{V} \cdot \nabla \bar{\Phi}.
\end{align*}
\]

(A.8)

We choose next an orthonormal basis \( \{ \bar{n}_\alpha \}_{\alpha = 1}^{m-2} \) of the normal space such that \( \{ \bar{e}_1, \bar{e}_2, \bar{n}_1, \ldots, \bar{n}_{m-2} \} \) is a positive oriented orthonormal basis of \( \mathbb{R}^m \).

Recalling the definition of the interior multiplication operator \( \mathbf{L} \) given in Section A.1, it is not hard to obtain

\[
(* \bar{n}) \mathbf{L} \bar{e}_j = (\bar{e}_1 \wedge \bar{e}_2) \mathbf{L} \bar{e}_j = \delta_{j2} \bar{e}_1 - \delta_{j1} \bar{e}_2,
\]

and

\[
(* \bar{n}) \mathbf{L} \bar{n}_\alpha = 0.
\]

Hence,

\[
(* \bar{n}) \cdot (\bar{e}_j \wedge \bar{n}_\alpha) = ((* \bar{n}) \mathbf{L} \bar{e}_j) \wedge \bar{n}_\alpha + ((* \bar{n}) \mathbf{L} \bar{n}_\alpha) \wedge \bar{e}_j = \delta_{j2} \bar{e}_1 \wedge \bar{n}_\alpha - \delta_{j1} \bar{e}_2 \wedge \bar{n}_\alpha.
\]

Moreover, there holds trivially

\[
(* \bar{n}) \cdot (\bar{e}_j \wedge \bar{e}_k) = (\bar{e}_j \mathbf{L} \bar{v}) \wedge \bar{e}_k + (\bar{e}_j \mathbf{L} \bar{v}) \wedge \bar{e}_k = (\bar{e}_j \cdot \bar{v}) \bar{e}_k + \delta_{j} \bar{v}.
\]

From this one easily deduces for every 1-vector \( \bar{V} \), one has

\[
\begin{align*}
(* \bar{n}) \cdot (\bar{V} \wedge \nabla \bar{\Phi}) &= \pi_\lambda \bar{V} \wedge \nabla^\perp \bar{\Phi} \\
(* \bar{n}) \cdot (\bar{V} \wedge \nabla^\perp \bar{\Phi}) &= -\pi_\lambda \bar{V} \wedge \nabla \bar{\Phi}.
\end{align*}
\]

(A.9)

There holds furthermore

\[
(\bar{V} \wedge \bar{e}_j) \cdot \bar{e}_i = (\bar{e}_i \mathbf{L} \bar{V}) \wedge \bar{e}_j + \bar{V} \wedge (\bar{e}_i \mathbf{L} \bar{e}_j) = (\bar{e}_i \cdot \bar{V}) \bar{e}_j + \delta_{ij} \bar{V}.
\]

From this, and \( \bar{e}_i := e^{-\lambda} \partial_x^i \bar{\Phi} \), it follows that whenever \( \bar{V} = V^i \bar{e}_i + V^\alpha \bar{n}_\alpha \) then

\[
\begin{align*}
(\bar{V} \wedge \nabla \bar{\Phi}) \cdot \nabla^\perp \bar{\Phi} &= e^{2\lambda} (3V^i \bar{e}_i + 2V^\alpha \bar{n}_\alpha) \\
(\bar{V} \wedge \nabla^\perp \bar{\Phi}) \cdot \nabla \bar{\Phi} &= e^{2\lambda} (V^2 \bar{e}_1 - V^1 \bar{e}_2) = (\bar{V} \cdot \nabla \bar{\Phi}) \cdot \nabla^\perp \bar{\Phi}.
\end{align*}
\]

(A.10)

We are now sufficiently geared to prove
**Lemma A.1** Let \( \Phi \) be a smooth conformal immersion of the unit-disk into \( \mathbb{R}^m \) with corresponding mean curvature vector \( \vec{H} \), and let \( \vec{L} \) be a 1-vector. We define \( A \in \mathbb{R}^2 \otimes \wedge^0(\mathbb{R}^m) \) and \( \vec{B} \in \mathbb{R}^2 \otimes \wedge^2(\mathbb{R}^m) \) via

\[
\begin{align*}
A &= \vec{L} \cdot \nabla \Phi \\
\vec{B} &= \vec{L} \wedge \nabla \Phi + 2 \vec{H} \wedge \nabla^\perp \Phi .
\end{align*}
\]

Then the following identities hold:

\[
\begin{align*}
A &= -(* \vec{n}) \cdot \vec{B}^\perp \\
\vec{B} &= -(* \vec{n}) \bullet \vec{B}^\perp + (* \vec{n}) A^\perp ,
\end{align*}
\]

(A.11)

where \( * \vec{n} := (\partial_{x_1} \Phi \wedge \partial_{x_2} \Phi) / |\partial_{x_1} \Phi \wedge \partial_{x_2} \Phi| \).

Moreover, we have

\[
-2 \Delta \vec{\Phi} = A \cdot \nabla^\perp \vec{\Phi} - \vec{B} \bullet \nabla^\perp \vec{\Phi} .
\]

(A.12)

**Proof.** The identities (A.8) give immediately (recall that \( \vec{H} \) is a normal vector, so that \( \vec{H} \cdot \nabla^\perp \vec{\Phi} = 0 \)) the required

\[
(* \vec{n}) \cdot \vec{B}^\perp = -\vec{L} \cdot \nabla \vec{\Phi} + 2 \vec{H} \cdot \nabla^\perp \vec{\Phi} = -\vec{L} \cdot \nabla \vec{\Phi} = -A .
\]

Analogously, the identities (A.9) give (again, \( \vec{H} \) is normal, so \( \pi \vec{n} \vec{H} = \vec{H} \)),

\[
(* \vec{n}) \bullet \vec{B}^\perp = -\pi \vec{H} \wedge \nabla \vec{\Phi} - 2 \vec{H} \wedge \nabla^\perp \vec{\Phi} = -\vec{B} + \pi T \vec{H} \wedge \nabla \vec{\Phi}
\]

\[
= -\vec{B} + e^\lambda((\vec{L} \cdot \vec{e}_1) \vec{e}_1 + (\vec{L} \cdot \vec{e}_2) \vec{e}_2) \wedge \left( \frac{\vec{e}_1}{\vec{e}_2} \right)
\]

\[
= -\vec{B} + e^\lambda \left( \frac{-\vec{L} \cdot \vec{e}_2}{\vec{L} \cdot \vec{e}_1} \right) \vec{e}_1 \wedge \vec{e}_2
\]

\[
= -\vec{B} + (\vec{L} \cdot \nabla^\perp \vec{\Phi})(* \vec{n}) = -\vec{B} + (* \vec{n}) A^\perp ,
\]

which is the second equality in (A.11).

In order to prove (A.12), we will use (A.10). Namely, since \( \vec{H} = H^\alpha \vec{n}_\alpha \), we find

\[
\vec{B} \bullet \nabla^\perp \vec{\Phi} = (\vec{L} \cdot \nabla \vec{\Phi}) \cdot \nabla^\perp \vec{\Phi} + 4 e^{2\lambda} \vec{H} = A \cdot \nabla^\perp \vec{\Phi} + 4 e^{2\lambda} \vec{H} .
\]

Hence,

\[
\vec{B} \bullet \nabla^\perp \vec{\Phi} - A \cdot \nabla^\perp \vec{\Phi} = 4 e^{2\lambda} \vec{H} .
\]

Finally, there remains to recall that \( \Delta \vec{\Phi} = 2 e^{2\lambda} \vec{H} \) to reach the desired identity.

We choose now

\[
A = \nabla S - \nabla^\perp g \quad \text{and} \quad \vec{B} = \nabla \vec{R} - \nabla^\perp \vec{G} ,
\]

where \( S \) and \( g \) are scalars and \( \vec{R} \) and \( \vec{G} \) are 2-vectors. Then Lemma A.1 yields

\[
\begin{align*}
\nabla S &= -(* \vec{n}) \cdot (\nabla^\perp \vec{R} + \nabla \vec{G}) + \nabla^\perp g \\
\nabla \vec{R} &= -(* \vec{n}) \bullet (\nabla^\perp \vec{R} + \nabla \vec{G}) + (* \vec{n}) (\nabla^\perp S + \nabla g) + \nabla^\perp \vec{G} ,
\end{align*}
\]

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thereby giving
\[
\begin{align*}
- \Delta S &= \nabla(\star \mathbf{n}) \cdot \nabla^\perp \mathbf{R} + \text{div}(\star \mathbf{n}) \cdot \nabla \mathbf{\tilde{G}} \\
- \Delta \mathbf{R} &= \nabla(\star \mathbf{n}) \cdot \nabla^\perp \mathbf{R} - \nabla(\star \mathbf{n}) \cdot \nabla^\perp S \\
&\quad + \text{div}(\star \mathbf{n}) \cdot \nabla \mathbf{\tilde{G}} - \star \mathbf{n} \nabla g.
\end{align*}
\]

(A.13)

Furthermore, there holds,
\[
- 2 \Delta \mathbf{\phi} = (\nabla S - \nabla^\perp g) \cdot \nabla^\perp \mathbf{\phi} - (\nabla \mathbf{R} - \nabla^\perp \mathbf{G}) \cdot \nabla^\perp \mathbf{\phi}.
\]

(A.14)

A.3 Nonlinear and weighted elliptic results

Proposition A.1 Let \( u \in W^{1,2}(B_1(0)) \cap C^1(B_1(0) \setminus \{0\}) \) satisfy the equation
\[
- \Delta u = \nabla b \cdot \nabla^\perp u + \text{div}(b \nabla f) \quad \text{on} \quad B_1(0),
\]

(A.15)

where \( f \in W^{2,2}(B_1(0)) \), and moreover
\[
b \in W^{1,2} \cap L^\infty(B_1(0)) \quad \text{with} \quad \|\nabla b\|_{L^2(B_1(0))} < \varepsilon_0,
\]

(A.16)

for some \( \varepsilon_0 \) chosen to be “small enough”. Then
\[
\nabla u \in L^p(B_{1/4}(0)) \quad \text{for some} \quad p > 2.
\]

Proof. Before delving into the proof of the statement, one important remark is in order. Let \( D \) be any disk included (properly or not) in \( B_1(0) \). From the very definition of the space \( L^{2,\infty} \) (cf. [Ta]), there holds
\[
\|\Delta f\|_{L^{2}(D)} \leq |D|^{\frac{1}{2}} \|\Delta f\|_{L^{2,\infty}(D)} \lesssim |D|^{\frac{1}{2}} \|\nabla^2 f\|_{L^{2,\infty}(D)}.
\]

(A.17)

Moreover, an embedding result of Luc Tartar [Ta] states that \( \nabla f \) has bounded mean oscillations. Whence in particular
\[
\|\nabla f\|_{L^{2,\infty}(D)} \lesssim |D|^{\frac{1}{2}-\epsilon} \quad \forall \ \epsilon > 0.
\]

(A.18)

These inequalities shall come helpful in the sequel.

We now return to the proof of the proposition. Let us fix some point \( x_0 \in B_{1/2}(0) \) and some radius \( \sigma \in (0, \frac{1}{2}) \), and we let \( k \in (0, 1) \). Note that \( B_{k\sigma}(x_0) \) is properly contained in \( B_1(0) \). To reach the desired result, we decompose the solution to (A.15) as the sum \( u = u_0 + u_1 \), where
\[
\begin{align*}
- \Delta u_0 &= \text{div}(b \nabla f) & - \Delta u_1 &= \nabla b \cdot \nabla^\perp u & \text{in} \ B_{\sigma}(x_0) \\
u_0 &= u & u_1 &= 0 & \text{on} \ \partial B_{\sigma}(x_0).
\end{align*}
\]

Accounting for the hypotheses (A.16) and (A.18) into standard elliptic estimates (cf. Proposition 4 in [Al]) yields
\[
\|\nabla u_0\|_{L^2(B_{k\sigma}(x_0))} \lesssim \|b \nabla f\|_{L^2(B_{k\sigma}(x_0))} + k \|\nabla u\|_{L^2(B_{k\sigma}(x_0))} \lesssim (k\sigma)^{1-\epsilon} + k \|\nabla u\|_{L^2(B_{k\sigma}(x_0))},
\]

(A.19)
up to some unimportant multiplicative constants. On the other hand, applying Wente’s inequality (see [He] Theorem 3.4.1) gives

\[
\|\nabla u_1\|_{L^2(B_{\sigma}(x_0))} \leq \|\nabla u_1\|_{L^2(B_\sigma(x_0))} \\
\lesssim \|\nabla b\|_{L^2(B_\sigma(x_0))} \|\nabla u\|_{L^2(B_\sigma(x_0))} \\
\leq \epsilon_0 \|\nabla u\|_{L^2(B_\sigma(x_0))},
\] (A.20)

again up to some multiplicative constant without bearing on the sequel. Hence, combining (A.19) and (A.20), we obtain the estimate

\[
\|\nabla u\|_{L^2(B_{\sigma}(x_0))} \leq \|\nabla u_0\|_{L^2(B_{\sigma}(x_0))} + \|\nabla u_1\|_{L^2(B_\sigma(x_0))} \\
\lesssim (k + \epsilon_0) \|\nabla u\|_{L^2(B_\sigma(x_0))} + (k\sigma)^{1-\epsilon}.
\]

Because \(\epsilon_0\) and \(\epsilon\) are small adjustable parameters, we may always choose \(k\) so as to arrange for \((k + \epsilon_0)\) to be less than 1. A standard “controlled-growth” argument (see e.g. Lemma 3.5.11 in [He]) enables us to conclude that there exists some \(\beta \in (0, 1)\) for which

\[
\|\nabla u\|_{L^2(B_\sigma(x))} \leq C_0 \sigma^\beta, \quad \forall \sigma \in \left(0, \frac{1}{2}\right), \ x \in B_{1/2}(0),
\] (A.21)

and for some constant \(C_0\).

With the help of the Poincaré inequality, this estimate may be used to show that \(u\) is locally Hölder continuous. We are however interested in another implication of (A.21). Consider the maximal function

\[
M_{2-\beta} g(x) := \sup_{\sigma > 0} \sigma^{-\beta} \int_{B_\sigma(x)} |g(y)| \, dy.
\] (A.22)

We recast the equation (A.15) in the form

\[
- \Delta u = b \Delta f + \nabla b \cdot (\nabla^\perp u + \nabla f).
\]

Calling upon (A.16)–(A.18) and upon the estimate (A.21), we derive that for \(x \in B_{1/2}(0)\), there holds

\[
M_{2-\beta} \left(\chi_{B_{1/2}(0)} \Delta u\right)(x) \leq \|b\|_{L^\infty(B_{1/0}(0))} \sup_{0 < \sigma < \frac{1}{2}} \sigma^{-\beta} \|\Delta f\|_{L^2(B_\sigma(x))} \\
+ \|\nabla b\|_{L^2(B_{1/0}(0))} \sup_{0 < \sigma < \frac{1}{2}} \sigma^{-\beta} \left(\|\nabla u\|_{L^2(B_\sigma(x))} + \|\nabla f\|_{L^2(B_\sigma(x))}\right) \\
\lesssim \sup_{0 < \sigma < \frac{1}{2}} \sigma^{-\beta+1} + \epsilon_0 \sup_{0 < \sigma < \frac{1}{2}} (\sigma^{-\beta+1} + \sigma^{-\beta+1-\epsilon}) < \infty,
\] (A.23)

for all \(0 < \epsilon \leq 1 - \beta\). Moreover, it is clear that \(\Delta u\) is integrable on \(B_{1/2}(0)\). We may thus use Proposition 3.2 from [Ad] to deduce that

\[
\frac{1}{|x|} \chi_{B_{1/2}(0)} \Delta u \in L^{r, \infty}(B_{1/2}(0)) \quad \text{with} \quad r := \frac{2 - \beta}{1 - \beta} > 2.
\]

\[\text{namely,} \quad \|\frac{1}{|x|} \cdot g\|_{L^{r, \infty}} \lesssim \|M_{2-\beta} g\|_{L^{1, \infty}} \|g\|_{L^1}\]

for \(r = \frac{2 - \beta}{1 - \beta}\) and \(\beta \in (0, 1)\).
A classical estimate about Riesz kernels states there holds in general
\[ |\nabla u(y)| \lesssim \frac{1}{|x|} * \chi_{B_{1/2}(0)} \Delta u + C, \quad \forall \ y \in B_{1/4}(0), \]
where \( C \) is a constant depending on the \( C^1 \)-norm of \( u \) on \( \partial B_{1/2}(0) \), hence finite by hypothesis. It follows in particular that
\[ \nabla u \in L^p(B_{1/4}(0)) \quad \text{for all} \quad p < r, \]
as announced.

\[ \blacksquare \]

**Proposition A.2** Let \( u \in C^2(B_1(0) \setminus \{0\}) \) solve
\[ \Delta u(x) = \mu(x) f(x) \quad \text{in} \ B_1(0), \tag{A.24} \]
where \( f \in L^p(B_1(0)) \) for some \( p > 2 \). The weight \( \mu \) satisfies
\[ |\mu(x)| \simeq |x|^a \quad \text{for some} \ a \in \mathbb{N}. \tag{A.25} \]
Then
\[ (i) \ there \ hold \] \[ \nabla u(x) = P(\overline{x}) + |\mu(x)| T(x), \tag{A.26} \]
where \( P(\overline{x}) \) is a complex-valued polynomial of degree at most \( a \), and near the origin \( T(x) = O(|x|^{1-\frac{a}{p}-\epsilon}) \) for every \( \epsilon > 0 \).

\[ (ii) \ furthermore, \ if \ \mu \in C^1(B_1(0) \setminus \{0\}), \ if \ a \neq 0, \ and \ if \]
\[ |x|^{1-a} \nabla \mu(x) \in L^\infty(B_1(0)), \tag{A.27} \]
there holds
\[ \nabla^2 u(x) = \nabla P(\overline{x}) + |\mu(x)| Q(x), \tag{A.28} \]
where \( P \) is as in (i), and
\[ Q \in L^{p-\epsilon}(B_1(0), C^2) \quad \forall \ \epsilon > 0. \]
As a \((2 \times 2)\) real-valued matrix, \( Q \) satisfies in addition
\[ \text{Tr} \ Q \in L^p(B_1(0)). \]

Naturally, if \( a = 0 \), the standard Calderon-Zygmund theorem yields that \( u \in W^{2,p}(B_1(0)) \). The hypothesis \((A.27)\) becomes unnecessary, and \((A.28)\) holds with \( P \) being constant and \( \epsilon = 0 \).

\[ \overline{x} \] is the complex conjugate of \( x \). Namely, we parametrize \( B_1(0) \) by \( x = x_1 + i x_2 \), and then \( \overline{x} = x_1 - i x_2 \). With this notation, \( \nabla u \) on the left-hand side of \((A.26)\) is understood as \( \partial_{x_1} u + i \partial_{x_2} u \).
Proof. Using Green’s formula for the Laplacian, an exact expression for the solution $u$ may be found and used to obtain

$$\nabla u(x) = \frac{1}{2\pi} \int_{\partial B(0)} \left[ \frac{x-y}{|x-y|^2} \partial_y u(y) - u(y) \partial_y \frac{x-y}{|x-y|^2} \right] d\sigma(y)$$

$$- \frac{1}{2\pi} \int_{\partial B(0)} \frac{x-y}{|x-y|^2} \nu(y) f(y) dy$$

$$= : J_0(x) + J_1(x), \quad \forall \; x \in B_1(0), \quad (A.29)$$

where $\nu$ is the outer normal unit-vector to the boundary of $B_1(0)$. Without loss of generality, and to avoid notational clutter, because $u$ is twice differentiable away from the origin, we shall henceforth assume that $|x| < 1/2$.

We will estimate separately $J_0$ and $J_1$, and open the discussion by noting that when $|y| > |x|$, we have the expansion

$$\frac{x-y}{|x-y|^2} = - \sum_{m \geq 0} P^m(x, y) \quad \text{with} \quad P^m(x, y) := x^m y^{-(m+1)}.$$ 

Hence, we deduce the identity

$$J_0(x) = - \frac{1}{2\pi} \sum_{m \geq 0} \int_{\partial B_1(0)} [P^m(x, y) \partial_y u(y) - u(y) \partial_y P^m(x, y)] dS(y)$$

$$= - \frac{1}{2\pi} \sum_{m \geq 0} x^m \int_0^{2\pi} [(m+1) u(e^{i\varphi}) - (\partial_{\varphi} u)(e^{i\varphi})] e^{i(m+1)\varphi} d\varphi$$

$$= \sum_{m \geq 0} C_m x^m, \quad (A.30)$$

where $C_m$ are (complex-valued) constants depending only on the $C^1$-norm of $u$ along $\partial B_1(0)$. As $u$ is continuously differentiable on the boundary of the unit disk by hypothesis, and $|x| < 1$, it is clear that $|J_0(x)|$ is bounded above by some constant $C$ for all $x \in B_1(0)$. Since $|C_m|$ grows sublinearly in $m$, we can surely find two constants $\gamma$ and $\delta$ such that

$$|C_m| < \gamma \delta^m \quad \forall \; m \geq 0.$$ 

Hence, when $|x| \leq R < \delta^{-1}$, there holds

$$\left| \sum_{m \geq a+1} C_m x^m \right| \leq \gamma \delta^{a+1} |x|^{a+1} \sum_{m \geq 0} (\delta R)^m \lesssim |x|^{a+1}.$$ 

And because $J_0$ is bounded, when $R < |x| < 1$, we find some large enough constant $K = K(C, a, \gamma, \delta)$ such that

$$\left| \sum_{m \geq a+1} C_m x^m \right| \leq |J_0(x)| + \sum_{0 \leq m \leq a} C_m |x|^m \leq C + (a+1) \gamma \delta^a$$

$$\leq K \delta^{a+1} \leq K \left(R^{-1}\delta^{a+1} |x|^{a+1} \right) \lesssim |x|^{a+1}.$$
As by hypothesis $|\mu(x)| \simeq |x|^a$, we may now return to (A.30) and write

$$J_0(x) = P_0(\mathbf{\check{x}}) + |\mu(x)| T_0(x),$$

(A.31)

where $P_0$ is a polynomial of degree at most $a$, and the remainder $T_0$ is controlled by some constant depending on the $C^1$-norm of $u$ on $\partial B_1(0)$. Moreover, $T_0(x) = O(|x|)$ near the origin.

We next estimate the integral $J_1$. To do so, we proceed as above and write

$$J_1(x) = I_1(x) + \sum_{m=a+1}^{\infty} I_1^m(x) - \sum_{m=0}^{a} I_1^m(x) + \sum_{m=0}^{a} I_2^m(x),$$

(A.32)

where we have put

$$I_1(x) := \frac{1}{2\pi} \int_{B_1(0) \cap B_{2|x|}(0)} \frac{x - y}{|x - y|^2} \mu(y) f(y) \, dy,$$

$$I_1^m(x) := \frac{1}{2\pi} \int_{B_1(0) \cap B_{2|x|}(0)} P^m(x, y) \mu(y) f(y) \, dy,$$

$$I_2^m(x) := \frac{1}{2\pi} \int_{B_1(0) \setminus B_{2|x|}(0)} P^m(x, y) \mu(y) f(y) \, dy.$$

We first observe that the last sum in (A.32) may be written

$$P_1(x) := \sum_{0 \leq m \leq a} I_1^m(x) + I_2^m(x) = \sum_{0 \leq m \leq a} \int_{B_1(0)} P^m(x, y) \mu(y) f(y) \, dy$$

$$= \sum_{0 \leq m \leq a} A_m \pi^m,$$

where

$$A_m := - \int_{B_1(0)} y^{-(m+1)} \mu(y) f(y) \, dy.$$

From the fact that $f \in L^p(B_1(0))$ for $p > 2$, and the hypothesis $|\mu(y)| \simeq |y|^a$, it follows easily that $|A_m| < \infty$ for $m \leq a$, and thus that $P_1$ is a polynomial of degree at most $a$.

We have next to handle the other summands appearing in (A.32), beginning with $I_1$. We find

$$|I_1(x)| \lesssim |\mu(x)| \int_{B_{2|x|}(0)} \frac{|f(y)|}{|x - y|} \, dy \lesssim |\mu(x)| \int_{B_{2|x|}(x)} \frac{|f(y)|}{|x - y|} \, dy$$

$$\lesssim |\mu(x)| |x| \tilde{M}_0f(x) \lesssim |x|^{1 - \frac{a}{2}} |\mu(x)|,$$

(A.33)

where we have used the fact that $B_{2|x|}(0) \subset B_{3|x|}(x)$, and a classical estimate bounding convolution with the Riesz kernel by the maximal function (cf. Proposition 2.8.2 in [Zi]). We have also used the simple estimate $\tilde{M}_0f(x) \lesssim |x|^{1 - \frac{a}{2}} \|f\|_{L^p}$.

\footnote{cf. (A.22) for the definition of $M_0f$.}
Next, let \( q \in [1, 2) \) be the conjugate exponent of \( p \). We immediately deduce for \( 0 \leq m \leq a \) that
\[
|I^m_1(x)| \lesssim |x|^m \int_{B_{2i}(0)} |y|^{-1-m+a} |f(y)| \, dy
\lesssim |x|^a \left\| |y|^{-1} \right\|_{L^q(B_{2i}(0))} \left\| f \right\|_{L^p(B_i(0))} \lesssim |x|^{1-\frac{2}{p}} |\mu(x)|. \tag{A.34}
\]

We next estimate \( I^m_2 \). As \( m \geq a + 1 \), we note that for any \( \epsilon > 0 \), there holds
\[
a + 1 - m - \epsilon \frac{2}{p} < 0.
\]

With again \( q \) being the conjugate exponent of \( p \), we find thus
\[
|I^m_2(x)| \lesssim |x|^m \int_{B_i(0) \setminus B_{2i}(0)} |y|^{a+1-m-\epsilon - \frac{2}{p}} |y|^{-\frac{2}{p}} |f(y)| \, dy
\leq 2^{a+1-m-\epsilon - \frac{2}{p}} |x|^{1-\frac{2}{p} - \epsilon} \left\| |y|^{-\frac{2}{p}} \right\|_{L^q(B_i(0))} \left\| f \right\|_{L^p(B_i(0))}
\lesssim 2^{a+1-m-\epsilon - \frac{2}{p}} |x|^{1-\frac{2}{p} - \epsilon} |\mu(x)|. \tag{A.35}
\]

Combining altogether in (A.32) our findings (A.33)-(A.35), we obtain that
\[
J_1(x) = P_1(\overline{x}) + |\mu(x)|T_1(x), \tag{A.36}
\]
where \( P_1 \) is a polynomial of degree at most \( a \), and the remainder \( T_1 \) satisfies the estimate
\[
|T_1(x)| \lesssim |x|^{1-\frac{2}{p} - \epsilon}, \quad \forall \ \epsilon > 0. \tag{A.37}
\]

Altogether, (A.31) and (A.36) put into (A.29) show that there holds
\[
\nabla u(x) = P(\overline{x}) + |\mu(x)|T(x), \tag{A.38}
\]
where \( P := P_0 + P_1 \) is a polynomial of degree at most \( a \), and the remainder \( T := T_0 + T_1 \) satisfies the same estimate (A.37) as \( T_1 \). The announced statement (i) ensues immediately.

We prove next statement (ii). Comparing (A.28) to (A.38), we see that
\[
|\mu(x)| Q(x) = \nabla (|\mu(x)| T(x)) \tag{A.39}
= \nabla (|\mu(x)| T_0(x)) + \nabla I_1(x) + \sum_{m \geq a+1} \nabla I^m_2(x) - \sum_{0 \leq m \leq a} \nabla I^m_1(x).
\]

By definition,
\[
|\mu(x)| T_0(x) = \sum_{m \geq a+1} C_m \overline{x}^m,
\]
with the constants \( C_m \) depending only on the \( C^1 \)-norm of \( u \) along \( \partial B_1(0) \) and growing sublinearly in \( m \). Using similar arguments to those leading to (A.31), it is clear from (A.25) that
\[
|\mu(x)|^{-1} \nabla (|\mu(x)| T_0(x)) \in L^\infty(B_1(0)). \tag{A.40}
\]

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Controlling the gradients of $I_m^1$ and $I_m^2$ is done mutatis mutandis the estimates (A.34) and (A.35). For the sake of brevity, we only present in details the case of $I_m^1$. Namely,

$$
\nabla I_m^1(x) = \frac{1}{2\pi} \int_{B_1(0) \cap B_{2|x|}(0)} \nabla x P^m(x, y) \mu(y) f(y) \, dy 
+ \frac{1}{2\pi} \frac{x}{|x|} \otimes \int_{\partial B_{2|x|}(0)} P^m(x, y) \mu(y) f(y) \, dy.
$$

(A.41)

After some elementary computations, and using the hypothesis $|\mu(y)| \simeq |y|^a$, we reach

$$
|\nabla I_m^1(x)| \lesssim m |x|^{a-2} \int_{B_1(0) \cap B_{2|x|}(0)} |f(y)| \, dy + |x|^{a-1} \int_{\partial B_{2|x|}(0)} |f(y)| \, dy,
$$

so that immediately

$$
\| |x|^{-a} \nabla I_m^1(x) \|_{L^{p-\epsilon}(B_1(0))} < \infty, \quad \forall \, \epsilon > 0.
$$

Proceeding analogously for $\nabla I_m^2$, we reach that for any $\epsilon > 0$ there holds

$$
\sum_{m \geq a+1} \| |x|^{-a} \nabla I_m^2(x) \|_{L^{p-\epsilon}(B_1(0))} < \sum_{0 \leq m \leq a} \| |x|^{-a} \nabla I_m^1(x) \|_{L^{p-\epsilon}(B_1(0))} < \infty.
$$

(A.42)

Hence, there remains only to estimate $\nabla I_1$. This is slightly more delicate. For notational convenience, we write

$$
\nabla I_1(x) = \frac{1}{2\pi} \nabla \int_{B_1(0) \cap B_{2|x|}(0)} \frac{x-y}{|x-y|^2} \mu(y) f(y) \, dy 
= \frac{1}{2\pi} \left( L(x) + K(x) \right),
$$

with

$$
K(x) = \chi_{B_{2|x|}(0)}(x) \frac{x}{|x|} \otimes \int_{\partial B_{2|x|}(0)} \frac{x-y}{|x-y|^2} \mu(y) f(y) \, dy,
$$

and the convolution

$$
L(x) = \left( \Omega \ast f(y) \right) \mu(y) \chi_{B_1(0) \cap B_{2|x|}(0)}(y),
$$

where $\Omega$ is the $(2 \times 2)$-matrix made of the Calderon-Zygmund kernels:

$$
\Omega(z) := \frac{|z|^2 \mathbb{1}_2 - 2z \otimes z}{|z|^4}.
$$

The boundary integral $K$ is easily estimated:

$$
|x|^{-a} |K(x)| \lesssim \frac{1}{|x|} \int_{\partial B_{2|x|}(0)} |f(y)| \, dy.
$$

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Using (A.27), we deduce easily
\[
\| |x|^{-a}K(x) \|_{L^p(B_1(0))} \lesssim \| f \|_{L^p(B_1(0))} .
\]

To estimate \( L \), we proceed as follows
\[
L(x) - \mu(x) (\Omega * f \chi_{B_1(0)} \cap B_{|x|} (0))(x)
= \int_{B_1(0) \cap B_{|x|} (0)} \Omega(x-y) f(y) (\mu(y) - \mu(x)) \, dy .
\] (A.45)

Let \( S_x \) be the cone with apex the point \( x/2 \) and such that the disk \( B_{|x|/4}(0) \) is inscribed in it. Note that for \( y \in S_x \), there holds \( 2|x-y| > |x| \). Hence, we find
\[
\int_{S_x \cap B_1(0) \cap B_{|x|} (0)} \Omega(x-y) f(y) (\mu(y) - \mu(x)) \, dy 
\lesssim |\mu(x)| \, |x|^{-2} \int_{B_{|x|} (0)} |f(y)| \, dy .
\] (A.46)

By hypothesis, the function \( \mu \) is continuously differentiable away from the origin. Thus, to each point \( y \) in the complement of the cone \( S_x \), there corresponds some \( \alpha \equiv \alpha(x,y) \in [0,1] \) with
\[
\mu(y) - \mu(x) = (x-y) \cdot \nabla \mu (\alpha x + (1-\alpha)y) .
\]
Using (A.27), we deduce easily
\[
|\mu(y) - \mu(x)| \lesssim |x|^{\alpha - 1} |x-y| \quad \forall \ y \in S_x^c \cap B_1(0) \cap B_{|x|} (0) .
\]

Accordingly, there holds
\[
\int_{S_x^c \cap B_1(0) \cap B_{|x|} (0)} \Omega(x-y) f(y) (\mu(y) - \mu(x)) \, dy 
\lesssim |x|^{\alpha - 1} \int_{B_{|x|} (0)} \frac{|f(y)|}{|x-y|} \, dy \lesssim |\mu(x)| \, M_0 f(x) ,
\] (A.47)

where we have used the same estimate as in (A.33). Bringing (A.46) and (A.47) into (A.35) and using the fact that \( |\mu(x)| \simeq |x|^\alpha \) yields
\[
|\mu(x)|^{-1} L(x) \lesssim (\Omega * f(y) \chi_{B_1(0) \cap B_{|x|} (0)}(y))(x)
+ \frac{1}{|x|^2} \int_{B_{|x|} (0)} |f(y)| \, dy + M_0 f(x) .
\]

Because \( f \in L^p \), standard estimates on Calderon-Zygmund operators, on the maximal function, and a classical Hardy inequality then give us
\[
\| |\mu|^{-1} L \|_{L^p(B_1(0))} \lesssim \| f \|_{L^p(B_1(0))} < \infty .
\]

Owing to the latter and to (A.44), we obtain from (A.43) that \( |\mu|^{-1} \nabla I_1 \in L^p(B_1(0)) \). With (A.40) and (A.42), the identity (A.38) thus implies that \( Q \)
belongs to $L^{p-\epsilon}$ for all $\epsilon > 0$. This completes the first part of statement (ii).

We shall now prove the second part of (ii), and show that the trace of $Q$ is in $L^p$. To this end, let us note that

$$\text{Tr } \nabla \pi = \text{Tr } \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = 0.$$  \hfill (A.48)

We have seen in (A.39) that

$$|\mu| Q = \nabla (|\mu| T_0) + \nabla I_1 + \sum_{m \geq a+1} \nabla I^m_2 - \sum_{0 \leq m \leq a} \nabla I^m_1.$$  \hfill (A.49)

By definition, $|\mu(x)| T_0(x) = \sum_{m \geq a+1} C_m \Phi^m$, so that (A.48) gives

$$\text{Tr } \nabla (|\mu(x)| T_0(x)) = 0.$$  \hfill (A.50)

Owing to the fact that $P^m(x,y) = \Phi^m \Phi^{-(m+1)}$, it then easily follows from (A.48) and (A.41) that

$$\text{Tr } \nabla I^m_1(x) = \frac{1}{2\pi} \text{Tr } x \frac{x}{|x|} \otimes \int_{\partial B_{2\epsilon}(0)} P^m(x,y) \mu(y) f(y) \, dy;$$

whence the estimate

$$|\mu(x)|^{-1} \text{Tr } \nabla I^m_1(x) \lesssim 2^{a-m-1} \frac{1}{|x|} \int_{\partial B_{2\epsilon}(0)} |f(y)| \, dy,$$

and thus

$$\| |\mu|^{-1} \text{Tr } \nabla I^m_1 \|_{L^p} \lesssim 2^{a-m-1} \|f\|_{L^p}.$$  \hfill (A.51)

In exactly the same fashion, one finds

$$\| |\mu|^{-1} \text{Tr } \nabla I^m_2 \|_{L^p} \lesssim 2^{a-m-1} \|f\|_{L^p}.$$  \hfill (A.52)

Finally, there remains to handle the term $|\mu|^{-1} \text{Tr } \nabla I_1$. But this term belongs to $L^p$, as we have shown that $|\mu|^{-1} \nabla I_1$ does. Combining this altogether with (A.50) into (A.49) yields the announced result.

\[\blacksquare\]

**Corollary A.1** Let $u \in C^2(B_1(0) \setminus \{0\})$ solve

$$\Delta u(x) = \mu(x) f(x) \quad \text{ in } B_1(0),$$

where

$$|f(x)| \lesssim |x|^{n+r} \quad \text{and} \quad |\mu(x)| \simeq |x|^a,$$

for two non-negative natural numbers $n$ and $a$; and $r \in (0,1)$.

Then

$$\nabla u(x) = P(\pi) + |\mu(x)| T(x),$$  \hfill (A.53)

where $P$ is a complex-valued polynomial of degree at most $(a + n + 1)$, and near the origin $T(x) = O(|x|^{n+1+r-\epsilon})$ for every $\epsilon > 0$. If in addition $\mu$ satisfies (A.37), then

$$x^{-n-r} |\mu|^{-1} \text{Tr } \nabla (|\mu| T) \text{ belongs to } L^p$$

for all finite $p$. Furthermore, there holds the estimate

$$\| \text{Tr } \nabla (|\mu(x)| T(x)) \| \lesssim |x|^{n+r} |\mu(x)|.$$  \hfill (A.54)
Proof. The argument goes along the same lines as that of Proposition A.2. We set

\[ \omega(x) := |x|^{n+r} \mu(x) \quad \text{and} \quad h(x) := |x|^{-(n+r)} f(x). \]

From the given hypotheses, we see that \( h \in L^\infty \), and \( \omega \) satisfies (A.26) with \((a+n+r)\) in place of \( a \). If \( \mu \) satisfies (A.27), then so does \( \omega \), again with \((a+n+r)\) in place of \( a \).

Using the representation (A.29) gives

\[
\nabla u(x) = \frac{1}{2\pi} \int_{\partial B_1(0)} \left[ \frac{x - y}{|x - y|^2} \partial \nu u(y) - u(y) \partial \nu \frac{x - y}{|x - y|^2} \right] d\sigma(y)
\]

\[
- \frac{1}{2\pi} \int_{B_1(0)} \frac{x - y}{|x - y|^2} \omega(y) h(y) dy
\]

\[=: J_0(x) + J_1(x), \quad \forall \ x \in B_1(0), \]

where \( \nu \) is the outer normal unit-vector to the boundary of \( B_1(0) \).

The integral \( J_0 \) is estimated as in (A.31) so as to yield

\[ J_0(x) = P_0(\mathcal{F}) + |\mu(x)| T_0(x), \]

where \( P_0 \) is a polynomial of degree at most \((a + n + 1)\), and \( T_0(x) = O(|x|^{n+2}) \) with \(|\mu|^{-1} \nabla (|\mu| T_0) = O(|x|^{n+1})\).

We next estimate the integral \( J_1 \). We proceed again as in the proof of Proposition A.2. Namely,

\[ J_1(x) = I_1(x) + \sum_{m=a+n+2}^{\infty} P_m^m(x) - \sum_{m=0}^{a+n+1} I_1^m(x) + \sum_{m=0}^{a+n+1} I_2^m(x), \]

where we have put

\[ I_1(x) := \frac{1}{2\pi} \int_{B_1(0) \cap B_2|x|} \frac{x - y}{|x - y|^2} \omega(y) h(y) dy, \]

\[ I_1^m(x) := \frac{1}{2\pi} \int_{B_1(0) \cap B_2|x|} P^m(x,y) \omega(y) h(y) dy, \]

\[ I_2^m(x) := \frac{1}{2\pi} \int_{B_1(0) \setminus B_2|x|} P^m(x,y) \omega(y) h(y) dy. \]

As before, \( P^m(x,y) := \mathcal{F}^m \bar{y}^{-(m+1)} \). We first observe that the last sum in the expression for \( J_1 \) may be written

\[ P_1(x) := \sum_{0 \leq m \leq a+n+1} I_1^m(x) + I_2^m(x) \]

\[ = \sum_{0 \leq m \leq a+n+1} \int_{B_1(0)} P^m(x,y) \omega(y) h(y) dy = \sum_{0 \leq m \leq a+n+1} A_m \mathcal{F}^m, \]

where

\[ A_m := \int_{B_1(0)} \bar{y}^{-(m+1)} \omega(y) h(y) dy. \]
From the boundedness of \( h \) and the hypothesis \(|\omega(y)| \simeq |y|^{a+n+r}\), it follows easily that \(|A_m| < \infty\) for \( m < a + n + 1 + r \), and thus since \( r > 0 \), that \( P_1 \) is a polynomial of degree at most \((a + n + 1)\). Once this has been observed, the remainder of the proof is found mutatis mutandis that of Proposition \( \text{A.2} \).

Namely, we write
\[
J_1(x) = P_1(\mathcal{T}) + |\omega(x)| T_1(x),
\]
with \( T_1(x) = O(|x|^{-1}) \) for all \( \epsilon > 0 \). Moreover, and \(|\omega|^{-1}\nabla(|\omega| T_1) \in L^p\) for all \( p < \infty \); and \(|\omega|^{-1}\text{Tr} \nabla(|\omega| T_1) \in L^\infty\).

Finally, setting \( P = P_0 + P_1 \) and \( T = T_0 + |x|^{n+r} T_1 = O(|x|^{n+r+1-\epsilon}) \) gives the desired representation \( \text{A.39} \). Clearly, from \( \text{A.27} \) and the above, there holds
\[
|\mu|^{-1}\nabla(|\mu| T) \lesssim |\mu|^{-1}\nabla(|\mu| T_0) + |x|^{n+r}|\omega|^{-1}\nabla(|\omega| T_1)|
\]
so that indeed \( |x|^{-(n+r)}|\mu|^{-1}\nabla(|\mu| T) \) belongs to \( L^p \) for all finite \( p \). Furthermore, we have
\[
|\text{Tr} \nabla(|\mu| T)| \leq |\text{Tr} \nabla(|\mu| T_0)| + |\text{Tr} \nabla(|\omega| T_1)| \lesssim |x|^{n+1}|\mu| + |\omega| \lesssim |x|^{n+r}|\mu|,
\]
as announced.

We may further iterate the previous result to obtain the next one.

**Corollary A.2** Let \( u \in C^\infty(B_1(0) \setminus \{0\}) \) solve
\[
\Delta u(x) = \mu(x)f(x) \quad \text{in } B_1(0),
\]
where \(|\mu(x)| \simeq |x|^a\), for some \( a \in \mathbb{N}^* \). In addition, we assume that
\[
|\nabla^j f(x)| \lesssim |x|^{n-j+r} \quad \text{and} \quad |\nabla^j \mu(x)| \lesssim |x|^{n-j},
\]
for some \( n \in \mathbb{N} \) and \( r \in (0,1) \), and for all \( j \) satisfying
\[
0 \leq j \leq J \leq \min\{a, n+1\}, \quad \text{for some } J \in \mathbb{N}^*.
\]
Then there holds for all \( j \leq J \):
\[
\nabla^{j+1} u(x) = \nabla^j P(x) + |\mu(x)| V_j(x),
\]
where \( P \) is a two-component real-valued polynomial of degree at most \((a+n+1)\), and near the origin \( V_j(x) = O(|x|^{n+r-j+1-\epsilon}) \) for every \( \epsilon > 0 \).

Furthermore\(^{20}\),
\[
|x|^{-(a+n+r-j)} \nabla(|\mu(x)| V_j(x)) \in \bigcap_{p<\infty} L^p(B_1(0)).
\]
and
\[
|\text{Tr} \nabla(|\mu(x)| V_j(x))| \lesssim |x|^{a+n+r-j}.
\]

\(^{20}\)note that \(|\mu| V_{j+1} = \nabla(|\mu| V_j)|
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