On estimation of solutions of neutral type systems on nonclosed sets*

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Abstract

We consider a differential system of neutral type with distributed delay. We obtain a precise norm estimation of solutions of the system in question on some nonclosed set. Our result is based on a spectral analysis of the operator and Riesz basis theory. We also discuss the stability properties of corresponding solutions.

Keywords delay systems, neutral type systems, asymptotic behavior of solutions

1 Introduction

The important problem in the theory of differential equations is to determine the asymptotic behavior of the solutions. Even in the case when solutions tend to the equilibrium it may be important to see how fast they approach it. If the equilibrium is zero then those questions relate directly to the stability analysis of a system. In [6] we gave an estimation of growth speed of all solutions of neutral type systems of some class. Recently the estimations of the solutions growth which are not valid on whole state space but for initial states from some nonclosed set were described. A. Borichev and Y. Tomilov characterized in [1] the decay rate of solutions of an abstract linear differential equation in the Hilbert space when initial states were from a certain nonclosed dense set. Their characterization was given in resolvent terms. This encouraged us to study the problem of the rate of growth (or decay) of solutions of neutral type equations in the same context. We give an extension of those result in the case of neutral type equations without an assumption about boundedness of corresponding semigroup. Our concept is based on the theorems on maximal asymptotics [4, 5] and our previous results concerning some class of delay systems of neutral type [6].

2 Preliminaries

In the present work, following [3], we consider the delay systems of neutral type of the form

\[ \dot{z}(t) = A_{-1} \dot{z}(t-1) + \int_{-1}^{0} A_2(\theta) \dot{z}(t+\theta) d\theta + \int_{-1}^{0} A_3(\theta) z(t+\theta) d\theta, \tag{1} \]

where \( A_{-1} \) is a \( n \times n \) invertible complex matrix, \( A_2 \) and \( A_3 \) are \( n \times n \) matrices of functions from \( L_2(-1,0) \). We rewrite equation (1) in the operator form

\[ \dot{x} = Ax, \quad x \in M_2, \tag{2} \]

*This work was partially supported by Polish National Science Centre grant No. N N514 238438.
where \( M_2 = C^n \times L_2(-1, 0; C^n) \), the operator \( A \) is then given by
\[
A \left( \begin{array}{c} y(t) \\ z(t) \end{array} \right) = \left( \int_{-1}^{0} A_2(\theta) z(\theta) d\theta + \int_{-1}^{0} A_3(\theta) y(\theta) d\theta \right),
\]
and the domain of \( A \) is as follows:
\[
D(A) = \{(y, z(\cdot)) : z \in H^1(-1, 0; C^n), y = z(0) - A_{-1} z(-1) \} \subset M_2.
\]

Before we prove Theorem 3.1 we give two lemmas

Theorem 3.1

The sequence of \( W \)-dimensional subspaces \( W \) is then given by
\[
W = \sum_{k=1}^{m} P_m^{(k)} M_2, \quad P_m^{(k)} x = \frac{1}{2\pi} \int_{L_m^{(k)}} R(A, \lambda) x d\lambda
\]
and some \( 2(2N+1) \)-dimensional subspaces \( W_N \) constitute \( A \)-invariant Riesz basis of space \( M_2 \).

Notice that the \( A \)-invariant subspaces \( W \) are quadratically close to the previous one.

Theorem 7 states in particular that for each \( \tilde{\lambda}_m^{(k)} \in \sigma(\tilde{A}) \setminus \{0\} \) and each Jordan chain of eigen- and rootvectors of the matrix \( A_{-1} \) corresponds the Jordan chain of \( \tilde{A} : \tilde{V}^{(k)}_{m,j} = P_m^{(k)} M_2, \quad P_m^{(k)} x = \frac{1}{2\pi} \int_{L_m^{(k)}} R(A, \lambda) x d\lambda \)

where \( P_m^{(k)}(\theta) \) is some polynomial independent of \( k, s = 0, \ldots, p_{m,j} - 1; m = 1, \ldots, \ell; k \in \mathbb{Z}; j = 1, \ldots \).

Some of the consequences of this characterization for the norm of semigroup \( e^{At} \) are given in [6]

3 Growth bound estimation

With assumption that for all eigenvalues of operator \( A \) their real parts lie at the left side of \( \tilde{\omega} \) it was proven that
\[
\|e^{At}\| \leq M_p e^{\tilde{\omega}t} (t^{p-1}), \quad t \geq 1,
\]
where \( p \) is the sum of sizes of all Jordan blocks corresponding to the maximal eigenvalue of \( A_{-1} \). Such assumption implies that there exist a sequence of eigenvalues of \( A \) whose real parts tend to \( \tilde{\omega} \). If this tending is arbitrary slow or initial state is sufficiently smooth then the power of \( t \) in the estimation (6) can be decreased. Namely we have the following

Theorem 3.1 We consider system (2). If \( \text{Re} \lambda < \omega \) for all \( \lambda \in \sigma(A) \) and \( \text{Re} \tilde{\lambda}^{(k)}_{i,j} \leq \omega - |k|^{-s}, k \in \mathbb{Z}, i = 1, \ldots, p_1, \) where \( \tilde{\lambda}^{(k)}_{i,j} \) then
\[
\|e^{At} A^{-n} x\| \leq C e^{\omega t} t^{p-1} |x|.
\]

Before we prove Theorem 3.1 we give two lemmas
Lemma 3.2 We consider system (2). If $\Re\lambda < \omega$ for all $\lambda \in \sigma(A)$ and for some $s \in \mathbb{N}$ holds $\Re\lambda^{(k)}_{ij} \leq \omega - |k|^{-s}$, $k \in \mathbb{Z}, i = 1, \ldots, p$, where $\{\lambda^{(k)}_{ij}\}_{i=1} \subset L^{(k)}$ then there exist a constant $C_1 > 0$ independent of $k$ such that

$$\|e^{-\lambda^{(k)}_{ij}}\| \leq C_1 e^{(\omega - |k|^{-s})t^{p-1}}, \quad t > 1.$$  

Proof Because the families of subspaces $\{V_{m}^{(k)}\}_{k \in \mathbb{Z}, m = 1, \ldots, \ell}$ and $\{\tilde{V}_{m}^{(k)}\}_{k \in \mathbb{Z}, m = 1, \ldots, \ell}$ constitute quadratically close Riesz basis, there exists a bounded operator $T_N$, with bounded inverse, which is close to identity and transforms almost all subspaces $V_{m}^{(k)}$ to $\tilde{V}_{m}^{(k)}$. Such operator $T_N$ can be defined on every basis subspace $V_{m}^{(k)}$ by the formula

$$T_N|_{V_{m}^{(k)}} x = B^{(k)}_{m} x, \text{ for } |k| > N, m = 1, 2, \ldots, \ell,$$

and $T_N|_{W_N} x = x$. It is easy to see that the operator $T_N$ is bounded on $M_2$ and close to identity. Therefore $T_N$ is invertible, its inverse $T^{-1}_N$ is bounded and transforms all but finitely many subspaces $V_{m}^{(k)}$ onto $\tilde{V}_{m}^{(k)}$. Now we denote $A^0_{k} = A^{(k)} - \tilde{\lambda}^{(k)} I, |k| > N$ and using $T_N$ we define the operator $B^0_{k} : V_k \to \tilde{V}_k$ close to $A^0_{k} : V_k \to V_k$ for each $|k| > N$ by the formula $B^0_{k} = T_N A^0_{k} T^{-1}_N$. Let $x = T_N x$. Then we get

$$\|e^{A^0_{k} t}\| = \|e^{\tilde{\lambda}^{(k)} t}\| \cdot \|T^{-1}_N e^{\delta t} T_N\|,$$

which can be rewritten as

$$\|e^{A^0_{k} t}\| \leq C e^{\omega t} \|e^{\delta t}\|,$$

where $C$ is a positive constant. It was shown in Lemma 2.2 [6] that for each $\delta > 0$ there exist $k_0$ large enough such that

$$\|B^0_{k} - A^0_{k}\| \leq \delta \text{ for any } |k| > k_0,$$

where $A^0_{k} = A^{(k)} - \tilde{\lambda}^{(k)}$.

The eigen- and rootvectors of operator $\tilde{A}$ are given by (5), we can see that the number and lengths of all Jordan chains of operators $\tilde{A}_{k} = \tilde{A}_{1_{V_k}}$ are independent of $k$. Thus all operators $\tilde{A}^0_{k} := \tilde{A}_{k} - \tilde{\lambda}_k I$ have the same matrix, say $A_0$ in the basis $\{e^{(k) s}\}_{s=1}^{p_1}$. Moreover if $A^0_{k} = S^{-1}_k A_0 S_k$, where $S_k : V^{(k)}_k \to C^{p_1}$ then $\|S_k\|, \|S_k^{-1}\|$ are uniformly bounded. Therefore denoting the matrix of operator $B^0_{k}$ by $B^0_{k}$ we have

$$\|e^{B^0_{k} t}\| \leq C_1 \|e^{B^0_{k} t}\|,$$

which can be rewritten as

$$\|e^{B^0_{k} t}\| \leq C_1 e^{-|k|^{s_1}} \|e^{(B^0_{k} + |k|^{s_1}) t}\|.$$  

Inequality (10) is also satisfied for matrices of operators $B^0_{k}, \tilde{A}^0_{k}$ and it is easy to see that for any $\delta > 0$ there exists $k_0$ such that

$$\|(B^0_{k} + |k|^{s_1}) - A_0\| \leq \delta, |k| > k_0.$$  

Hence we can apply Lemma 2.3 [6] to the family of matrices $\{B^0_{k} + |k|^{s_1}\}_{|k| > k_0}$, we get

$$\|e^{(B^0_{k} + |k|^{s_1}) t}\| \leq C_2 t^{p_1 - 1}, \quad t > 1, |k| > k_0,$$

where we used the fact that $\sigma(B^0_{k} + |k|^{s_1}) \subset C^-$. Combining (9), (11), (12) we get

$$\|e^{A^0_{k} t}\| \leq C_1 e^{(\omega - |k|^{-s})t^{p_1 - 1}}, \quad t > 1, |k| > k_0,$$

which ends the proof of Lemma 3.2. □

Lemma 3.3 We consider system (2). There exist $k_0 \in \mathbb{N}$ and constant $C_2 > 0$ such that

$$\|A^{-1}_{k} \| \leq \frac{C_2}{|k|}, \quad |k| \geq k_0.$$
Proof Using uniformly bounded operators \( T_N, S_k \) described in the proof of Lemma 3.2 we define operator \( B_k : \mathcal{V}_k \to \mathcal{V}_k \) by the formula \( B_k = T_N A_k T_N^{-1}, |k| > N \). Notice that the matrix of operator \( B_k \) is given by \( B_k = S_k^{-1} B_k S_k \). Hence

\[
\| A_k^{-1} \| \leq C \| B_k^{-1} \|
\]

for \(|k|\) large enough. Moreover for corresponding matrices of operators \( B_k, \tilde{A}_k \) and some constant \( C_4 \) we have \( \| B_k - \tilde{A}_k \| \leq C_4 \| B_k - A_k \|, |k| > N \). Lemma 2.2 [6] implies that there exists \( k_0 \in \mathbb{N} \) such that \( \| B_k - \tilde{A}_k \| \leq C_4^{-1} \) for \(|k| > k_0\), thus we have \( \| B_k - \tilde{A}_k \| \leq 1 \). Matrix \( \tilde{A}_k \) consist of Jordan blocks with eigenvalue \( \tilde{\lambda}_i^{(k)} = \tilde{\omega} + i(\arg \mu_1 + 2k\pi) \). Therefore we are able to use Statement 4.2 and we obtain

\[
\| A_k^{-1} \| \leq \frac{C C_3}{|\omega + i(\arg \mu_1 + 2k\pi)|} \leq \frac{C_2}{|k|}, \quad |k| > k_0,
\]

for some constant \( C_2 \).

Proof of Theorem 3.1 Without loss of generality we assume that all eigenvalues of matrix \( A_{-1} \) have the same modulus, it means that all eigenvalues of operator \( \tilde{A} \) lie on the line \( x = \tilde{\omega} \). Therefore we omit index \( m \) and write \( \lambda_i^{(k)}, L_k, V_k \) instead of \( \lambda_{m_i}^{(k)}, L_m, V_m \). We decompose \( x \) in the Riesz Basis \{\( V_k \)\}_{|k| > N} \( \cup \) \( W_N \) i.e. \( x = \sum_{|k| > N} x_k, x_k \in \mathcal{V}_k \) and we have

\[
\| e^{\tilde{A}^*} A_k^n x_k \|^2 \leq \sum_{|k| > N} \| e^{\tilde{A}^*} A_k^n x_k \|^2 \leq \sum_{|k| > N} \| e^{\tilde{A}^*} A_k^n \|^2 \| A_k^{-1} \|^2 \| x_k \|^2.
\]

We apply Lemma 3.2 and Lemma 3.3 in above inequality and we obtain

\[
\| e^{\tilde{A}^*} A_k^n x_k \|^2 \leq C_1^2 C_2^2 \sum_{|k| > N} e^{2(\omega - |k|^{-1}) \mu_1 t^2(p-1)|k|^{-2n}} \| x_k \|^2.
\]

Gathering the terms independent of \( k \) and additional term \( t^{-2} \) we get

\[
\| e^{\tilde{A}^*} A_k^n x_k \|^2 \leq \left( C_1 C_2 e^{\omega t^2(p-1)} \right)^2 \sum_{|k| > N} \left( e^{-|k|^{-1} \left| \left( \left( \omega - |k|^{-1} \right) \mu_1 t \right) \right|} \right)^2 \| x_k \|^2.
\]

Since function \( f(x) = e^{-x^2/2}, \ x \geq 0 \) is bounded by some constant \( M > 0 \), therefore \( e^{-|k|^{-1} \left| \left( \omega - |k|^{-1} \right) \mu_1 t \right|} \leq M \) for any \( k \in \mathbb{Z} \) and we have

\[
\| e^{\tilde{A}^*} A_k^n x_k \|^2 \leq \left( M C_1 C_2 e^{\omega t^2(p-1)} \right)^2 \sum_{|k| > N} \| x_k \|^2.
\]

Subspaces \{\( V_k \)\}_{|k| > N} \( \cup \) \( W_N \) constitute Riesz basis, thus there exists constant \( C_3 > 0 \) such that \( \sum_{|k| > N} \| x_k \|^2 \leq C_3^2 \| x \|^2 \) and we finaly obtain

\[
\| e^{\tilde{A}^*} A_k^n x \| \leq C e^{t^2(p-1)} \| x \|,
\]

where \( C = M C_1 C_2 \) is a new constant. \( \square \)

4 Appendix

Statement 4.1 Let \( A_0 = [a_{i,j}] \in M_n(\mathbb{C}) \) be a Jordan block of eigenvalue \( \lambda \), where \( |\lambda| \geq 1 \) and \( B_0 = [b_{i,j}] \in M_n(\mathbb{C}) \in M_n(\mathbb{C}) \) be such that \( \| B_0 - A_0 \| \leq 1 \), where \( \| A_\| = \sum |a_{i,j}| \), then there exist constant \( M \) such that

\[
| \det B_0 \| \leq M |\lambda|^n,
\]

and for \( |\lambda| \geq 2M \) we have also

\[
| \det B_0 \| \geq \frac{1}{2} |\lambda|^n.
\]
Proof Let us define \( \varepsilon_{i,j} = b_{i,j} - a_{i,j} \), it is easy to see that
\[
\det B_\lambda = |\lambda^n + f_1(\varepsilon_1, \ldots, \varepsilon_n)\lambda^{n-1} + \ldots + f_n(\varepsilon_1, \ldots, \varepsilon_n)\lambda|,
\]
where \( f_1, \ldots, f_n \) are polynomials. From the assumption that \( \|B_\lambda - A_\lambda\| \leq 1 \) we have \( \sum |\varepsilon_{i,j}| \leq 1 \) and therefore
\[
M_0 := \sup \{|f_1(\varepsilon_1, \ldots, \varepsilon_n)|, \ldots, |f_n(\varepsilon_1, \ldots, \varepsilon_n)| : \sum |\varepsilon_{i,j}| \leq 1\}
\]
is finite. Then from (13) we get
\[
|\det B_\lambda| \leq |\lambda|^n + M_0 n|\lambda|^n,
\]
where we used triangle inequality and \( |\lambda|^i \leq |\lambda|^n, i = 0, 1, \ldots, n \). Taking \( M = (n + 1)M_0 \) we get the first inequality. To prove the second one we also use triangle inequality in (13) and obtain
\[
|\det B_\lambda| \geq |\lambda|^n - M_0 n|\lambda|^{n-1}.
\]
Hence for \( |\lambda| \geq 2M \) we have
\[
|\det B_\lambda| \geq \frac{1}{2}|\lambda|^n,
\]
which ends the proof of the statement.

Remark 1 With the same assumptions we can prove similarly the first inequality of Statement 4.1 for the cofactors of matrix \( B_\lambda \). Namely, if we define cofactors of matrix \( B_\lambda \in M_n(\mathbb{C}) \) by \( B_{i,j} \) then \( |B_{i,j}| \leq M|\lambda|^{n-1} \).

Statement 4.2 Let \( A_\lambda, B_\lambda \in M_n(\mathbb{C}) \) such that \( A_\lambda \) consist of Jordan blocks of eigenvalue \( \lambda \), where \( |\lambda| \) is sufficiently large and \( \|B_\lambda - A_\lambda\| \leq 1 \), where \( \|A\| = \sum |a_{i,j}| \), then there exist constant \( C \) such that
\[
\|B_\lambda\| \leq \frac{C}{|\lambda|}.
\]
Proof Without loss of generality we assume that \( A_\lambda \) is a Jordan block. Using inversion formula to matrix \( B_\lambda \) we get
\[
\|B_\lambda\| = |\det B_\lambda|^{-1} \sum |B_{i,j}|,
\]
where \( B_{i,j} \) are cofactors of matrix \( B_\lambda \). Using Statement 4.1 and Remark 1 to estimate \( |\det B_\lambda| \) and \( |B_{i,j}| \) we obtain
\[
\|B_\lambda\| \leq \frac{2}{|\lambda|^n} \cdot M|\lambda|^{n-1} = \frac{2M}{|\lambda|},
\]
which ends the proof.
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