Some observations on a Kapteyn series

Diego Dominici ∗
Department of Mathematics
State University of New York at New Paltz
75 S. Manheim Blvd. Suite 9
New Paltz, NY 12561-2443
USA
Phone: (845) 257-2607
Fax: (845) 257-3571

Abstract
We study the Kapteyn series \( \sum_{n=1}^{\infty} t^n J_n (nz) \). We find a series representation in powers of \( z \) and analyze its radius of convergence.

Keywords: Kapteyn series, power series, Bessel functions.
MSC-class: 42C10 (Primary) 30B50, 33C10 (Secondary).

1 Introduction

Series of the form
\[
\sum_{n=1}^{\infty} \alpha_n J_{n+\nu} (z) , \quad \nu \in \mathbb{C}
\] (1)
where \( J_n (\cdot) \) is the Bessel function of the first kind, are called Kapteyn series. The series (1) is convergent and represents an analytic function throughout the domain in which \( \Omega (z) \) < \( \lim_{n \to \infty} |\alpha_n|^{-(\nu+n)} \), (2)
where
\[
\Omega(z) = \left| \frac{z \exp \left[ \sqrt{1 - z^2} \right]}{1 + \sqrt{1 - z^2}} \right|. \tag{3}
\]

Kapteyn series first appeared in Lagrange’s solution [7]
\[
E = M + 2 \sum_{n=1}^{\infty} \frac{1}{n} J_n (\varepsilon n) \sin (M n) \tag{4}
\]
of Kepler’s equation [3]
\[
M = E - \varepsilon \sin(M).
\]
The solution (4) was independently discovered by Friedrich Bessel in [2], where he introduced the functions which now bear his name.

Kapteyn series were systematically studied by Willem Kapteyn in his article [5], where he proved the following expansion theorem.

**Theorem 1** Let \( f(z) \) be a function which is analytic throughout the region
\[
D_a = \{ z \in \mathbb{C} \mid \Omega(z) \leq a \},
\]
with \( a \leq 1 \). Then,
\[
f(z) = \alpha_0 + 2 \sum_{n=1}^{\infty} \alpha_n J_n (nz), \quad z \in D_a
\]
where
\[
\alpha_n = \frac{1}{2\pi i} \oint_{\Theta_n(z)} f(z) dz
\]
and the path of integration is the curve on which \( \Omega(z) = a \). The function \( \Theta_n(z) \) is the Kapteyn polynomial defined by
\[
\Theta_0(z) = \frac{1}{z}
\]
\[
\Theta_n(z) = \frac{1}{4} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n - 2k)^2 (n - k - 1)!}{k! \left( \frac{n z}{2} \right)^{2k-n}}, \quad n \geq 1,
\]
where \( \lfloor \cdot \rfloor \) denotes the integer-part function.
Proof. See [8, 17.4]. ■

As a corollary [8], one finds that if the Taylor series for \( f(z) \) is

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

then,

\[
\alpha_0 = a_0
\]

\[
\alpha_n = \frac{1}{4} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-2k)^2(n-k-1)!}{k! \left( \frac{n}{2} \right)^{n-2k+1}} a_{n-2k}, \quad n \geq 1.
\] (5)

In [6], Kapteyn established the fundamental formula

\[
\frac{1}{1-z} = 1 + 2 \sum_{n=1}^{\infty} J_n(nz), \quad \Omega(z) < 1,
\] (6)

using the integral representation [8]

\[
J_n(nz) = \frac{1}{2\pi i} \int_{\mathcal{H}} \left\{ \frac{\exp \left[ \frac{z}{2} (s - \frac{1}{s}) \right]}{s} \right\}^n \frac{ds}{s},
\]

where the Hankel contour \( \mathcal{H} \) encircles the origin once counterclockwise.

The purpose of this paper is to generalize (6) by means of the function

\[
F(z, t) = \sum_{n=1}^{\infty} t^n J_n(nz), \quad z \in \mathbb{C}, \quad t \in \mathbb{R},
\] (7)

so that (6) corresponds to the particular case \( t = 1 \). In Section 2 we write \( F(z, t) \) as a series in powers of \( z \)

\[
F(z, t) = \sum_{n=1}^{\infty} A_n(t) z^n
\] (8)

and find the coefficients \( A_n(t) \). Although one could replace \( \alpha_n = t^n \) in (5) and solve the problem

\[
t^n = \frac{1}{4} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-2k)^2(n-k-1)!}{k! \left( \frac{n}{2} \right)^{n-2k+1}} A_{n-2k}(t),
\]
we take a different approach that only uses the differential equation satisfied by the Bessel functions. In Section 3 we analyze the radius of convergence of (5) for different ranges of \( t \). Finally, in Section 4 we solve the inversion problem (5) for arbitrary \( \alpha_n \).

## 2 Power series

Since the Bessel function \( J_n(z) \) is a solution of \( 8 \)

\[
 z^2 J_n'' + z J_n' + (z^2 - n^2) J_n = 0,
\]

the function \( y_n(z) = J_n(nz) \) satisfies the ODE

\[
 z^2 y_n'' + z y_n' + n^2 (z^2 - 1) y_n = 0. \tag{9}
\]

We also observe that

\[
 t^2 (t^n)''' + t (t^n)' = n^2 t^n. \tag{10}
\]

Using (9) and (10) in (7), we have

\[
 z^2 F_{zz} + z F_z = (1 - z^2) \left( t^2 F_{tt} + t F_t \right), \tag{11}
\]

where the subscripts denote partial derivatives.

Replacing (8) in (11), we get the equation

\[
 t^2 A_n'' + t A_n' - n^2 A_n = t^2 A_{n-2}'' + t A_{n-2}', \quad n \geq 1, \tag{12}
\]

where we define \( A_n \equiv 0 \) for \( n = -1, 0 \).

From (7) and (8), we have

\[
 0 = F(z, 0) = \sum_{n=1}^{\infty} A_n(0) z^n
\]

and, from (6), we see that

\[
 \sum_{n=1}^{\infty} \frac{1}{2} z^n = \frac{z}{2 (1 - z)} = F(z, 1) = \sum_{n=1}^{\infty} A_n(1) z^n,
\]

which together imply that

\[
 A_n(0) = 0, \quad A_n(1) = \frac{1}{2}, \quad n \geq 1. \tag{13}
\]
Solving (12) with the boundary conditions (13), we obtain
\[
A_1(t) = \frac{t}{2}, \quad A_2(t) = \frac{t^2}{2}, \quad A_3(t) = \frac{9}{16}t^3 - \frac{1}{16}t
\]
\[
A_4(t) = \frac{2}{3}t^4 - \frac{1}{6}t^2, \quad A_5(t) = \frac{625}{768}t^5 - \frac{81}{256}t^3 + \frac{1}{384}t, \ldots
\]
Thus, the function \( A_n(t) \) is a polynomial in \( t \) of degree \( n \). Writing
\[
A_n(t) = \sum_{k=0}^{n} C_n^k t^k
\] (14)
and using (12), we get the recurrence
\[
(n^2 - k^2) C_n^k = -k^2 C_{n-2}^k, \qquad \text{for } k = 0, 1, \ldots, n - 2
\] (15)
\[
(n^2 - k^2) C_n^k = 0, \qquad \text{for } k = n - 1, n.
\]
To solve (15), we recall that the double factorial function satisfies
\[
(n - k)!! = (n - k) (n - k - 2)!!, \quad (n + k)!! = (n + k) (n + k - 2)!!
\]
which gives
\[
\frac{1}{(n-k)!! \times (n+k)!!} = \frac{1}{(n-k-2)!! \times (n+k-2)!!}
\]
This suggests the solution
\[
C_n^k = \frac{\cos \left( \frac{n-k}{2} \pi \right) k^n}{(n-k)!! \times (n+k)!!}, \quad 0 \leq k \leq n.
\] (16)
Therefore,
\[
A_n(t) = \sum_{k=0}^{n} \frac{\cos \left( \frac{n-k}{2} \pi \right) k^n}{(n-k)!! \times (n+k)!!} t^k
\] (17)
or, after rearranging terms,
\[
A_n(t) = \frac{(-1)^n}{n!} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{k} \left( k - \frac{n}{2} \right)^n t^{n-2k},
\] (18)
It is clear from (17) that \( A_n(0) = 0 \). To show that \( A_n(1) = \frac{1}{2} \), we use the identity \[ 1.054, #5 \]

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (k + \alpha)^n = (-1)^n n!,
\]

(19)

where \( \alpha \) is arbitrary. From (18) we have

\[
A_n(t) + A_n\left(\frac{1}{t}\right) = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(k - \frac{n}{2}\right)^n t^{n-2k}.
\]

(20)

Setting \( t = 1 \) in (20), we have

\[
2A_n(1) = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(k - \frac{n}{2}\right)^n
\]

and from (19), we conclude that \( 2A_n(1) = 1 \).

3 Radius of convergence

We shall now find the radius of convergence \( R(t) \) of the power series (8). From (7) we observe that \( R(t) \to 0 \) as \( t \to \infty \) and \( R(t) \to \infty \) as \( t \to 0 \), while (6) gives \( R(1) = 1 \) (see Figure 1). Since from (18) we have \( A_n(-t) = (-1)^n A_n(t) \), we shall limit our analysis to \( t > 0 \).

When \( t \gg 1 \), the largest term in the sum (18) corresponds to \( k = 0 \), and therefore

\[
A_n(t) \sim \frac{1}{n!} \left(\frac{nt}{2}\right)^n, \quad t \to \infty.
\]

(21)

Using Stirling’s formula \( 1.054 \)

\[
n! \sim \sqrt{2\pi n} n^n e^{-n}, \quad n \to \infty
\]

we obtain

\[
A_n(t) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{et}{2}\right)^n, \quad t \to \infty.
\]

(22)

Thus,

\[
R(t) = \lim_{n \to \infty} |A_n(t)|^{-\frac{1}{n}} \sim \frac{2e^{-1}}{t}, \quad t \to \infty.
\]

(23)
Figure 1: A sketch of $|A_{500}(t)|^{-\frac{1}{200}} \simeq R(t)$. 
When $t < 1$, (18) is highly oscillatory and there is no approximation like (21) valid in this range.

To find a formula for $R(t)$ valid for all $t$, we observe that, for a fixed value $t > 0$, we have

$$
\ln |A_n(t)| \simeq \beta(t)n,
$$

(24)

for some function $\beta(t)$ (see Figure 2).

Thus, we consider an asymptotic expansion of the form

$$
A_n(t) \sim \frac{1}{2} \theta_n(t) [\psi(t)]^n, \quad n \to \infty,
$$

(25)

where $\theta_n(t)$ denotes the sign of $A_n(t)$ and $\psi(1) = 1$. Replacing (25) in (12) and setting

$$
\theta_n'(t) = \theta_n''(t) = 0, \quad \theta_{n-2}'(t) = \theta_{n-2}''(t) = 0,
$$
we obtain, to leading order in $n$,
\[(t\psi')^2 (\theta_n \psi^2 - \theta_{n-2}) - \theta_n \psi^4 = 0.\] (26)

Since
\[\theta_n = \pm 1, \quad \theta_{n-2} = \pm 1,\]
and $\psi(t)$ is increasing for $t > 0$, we get
\[\psi' = \frac{\psi^2}{t \sqrt{\psi^2 \pm 1}}.\] (27)

Solving (27) subject to $\psi(1) = 1$, we obtain the implicit solutions
\[\ln(t) + \left[\frac{\psi^2(t) + 1}{\psi(t)}\right]^\frac{3}{2} - \psi(t) \sqrt{\psi^2(t) + 1} - \text{arcsinh}[\psi(t)] - \sqrt{2 + \ln(\sqrt{2} + 1)} = 0,\] (28)
for $0 < t \leq 1$ and
\[\ln(t) - \left[\frac{\psi^2(t) - 1}{\psi(t)}\right]^\frac{3}{2} + \psi(t) \sqrt{\psi^2(t) - 1} - 1 - \ln\left[\psi(t) + \sqrt{\psi^2(t) - 1}\right] = 0\] (29)
for $t \geq 1$.

Although we cannot solve (28) and (29) exactly, we can consider the limiting cases as $t \to 0$ and $t \to \infty$. Since $\psi(t) \to 0$ as $t \to 0$, we obtain from (28)
\[\psi(t) \sim \frac{1}{-\ln(t) + \sqrt{2 + \ln(\sqrt{2} - 1)}}, \quad t \to 0.\] (30)
When $t \to \infty$, we have $\psi(t) \to \infty$, and we get from (29)
\[\psi(t) \sim \frac{e}{2} t, \quad t \to \infty,\] (31)
which agrees with (22).

Since $R(t) = \frac{1}{\psi(t)}$, we have from (28) and (29), after exponentiating
\[e^{-\sqrt{2}} \left(1 + \sqrt{2}\right) \frac{R(t) \exp\left[\sqrt{1 + R^2(t)}\right]}{1 + \sqrt{1 + R^2(t)}} t = 1, \quad 0 < t \leq 1,\] (32)
Figure 3: A comparison of $R(t)$ (solid curve) and $|A_{500}(t)|^{-\frac{1}{100}}$ (ooo).
and
\[
\frac{R(t) \exp \left[ \sqrt{1 - R^2(t)} \right]}{1 + \sqrt{1 - R^2(t)}} t = 1, \quad t \geq 1.
\] (33)

In Figure 3 we graph the solutions of (32), (33) and the approximate value of \( R(t) \) given by \( |A_{500}(t)|^{\frac{1}{500}} \).

Using (2) with \( \alpha_n = t^n \), we conclude that the Kapteyn series (7) converges for those \( t \in \mathbb{R} \) and \( z \in \mathbb{C} \) such that
\[
\left| \frac{z \exp \left[ \sqrt{1 - t^2} \right]}{1 + \sqrt{1 - t^2}} t \right| < 1.
\] (34)

Replacing \( z = re^{i\omega} \) in (34), we find that the minimum value of \( r \) corresponds to \( \omega = \pm \frac{\pi}{2} \). Thus, for \( t > 0 \), the Kapteyn series (7) will converge inside the circle \(|z| < r(t)\), with
\[
\frac{r(t) \exp \left[ \sqrt{1 + r(t)^2} \right]}{1 + \sqrt{1 + r(t)^2}} t = 1.
\] (35)

As it was observed in [8], the circles of convergence of the Kapteyn series (7) and the power series (8) are not equal, the former being slightly smaller than the latter. However, the radius \( r(t) \) given by (35) and \( R(t) \) are asymptotically equal as \( t \to 0 \) and \( t \to \infty \) (see Figure 4).

We summarize our results in the following theorem.

**Theorem 2** The Kapteyn series

\[
F(z, t) = \sum_{n=1}^{\infty} t^n J_n(nz),
\]

converges for those \( t \in \mathbb{R} \) and \( z \in \mathbb{C} \) such that
\[
\left| \frac{z \exp \left[ \sqrt{1 - t^2} \right]}{1 + \sqrt{1 - t^2}} t \right| < 1.
\]

The function \( F(z, t) \) admits the power series representation
\[
F(z, t) = \sum_{n=1}^{\infty} A_n(t) z^n,
\]
Figure 4: A comparison of $r(t)$ (solid curve) and $R(t)$ (ooo).
with coefficients

\[ A_n(t) = \frac{(-1)^n}{n!} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{k} \left( k - \frac{n}{2} \right)^n t^{n-2k} \]

and radius of convergence \( R(t) \) defined by the implicit equations

\[
e^{-\sqrt{2}} \left( 1 + \sqrt{2} \right) \frac{R(t) \exp \left[ \sqrt{1 + R^2(t)} \right]}{1 + \sqrt{1 + R^2(t)}} t = 1, \quad 0 < |t| \leq 1,
\]

\[
e^{-\sqrt{1-R^2}} \left( 1 + \sqrt{1-R^2} \right) \frac{R(t) \exp \left[ \sqrt{1 - R^2(t)} \right]}{1 + \sqrt{1 - R^2(t)}} t = 1, \quad |t| \geq 1.
\]

4 The general problem

We shall now consider the problem of finding \( a_n \) in terms of \( \alpha_n \), where

\[
\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \alpha_n J_n(nz),
\]

for arbitrary \( a_n \) and \( \alpha_n \). Rewriting the power series of \( J_n(nz) \) in the form

\[
J_n(nz) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (n+j)!} \left( \frac{nz}{2} \right)^{n+2j}
\]

in the form

\[
J_n(nz) = \left( \frac{n}{2} \right)^n \sum_{j=0}^{\infty} \frac{\cos \left( \frac{\pi}{2} j \right) \left( \frac{n}{2} \right)^j}{(\frac{j}{2})! (n+\frac{j}{2})!} z^{n+j},
\]

we have,

\[
\sum_{n=1}^{\infty} \alpha_n J_n(nz) = \sum_{n=1}^{\infty} \alpha_n \left( \frac{n}{2} \right)^n \sum_{j=0}^{\infty} \frac{\cos \left( \frac{\pi}{2} j \right) \left( \frac{n}{2} \right)^j}{(\frac{j}{2})! (n+\frac{j}{2})!} z^{n+j}
\]

\[
= \sum_{k=1}^{\infty} \sum_{n=1}^{k} \alpha_n \left( \frac{n}{2} \right)^n \frac{\cos \left[ \frac{\pi}{2} (k-n) \right] \left( \frac{n}{2} \right)^{k-n}}{(k-n-n)!(\frac{k+n}{2})!} z^k.
\]
Therefore, we obtain from (36) that
\[
a_n = \sum_{n=1}^{k} \alpha_n \cos \left[ \frac{\pi}{2} (k - n) \right] \left( \frac{n}{2} \right)^k \left( \frac{k-n}{2} \right)! \left( \frac{k+n}{2} \right)!
\]
or
\[
a_n = \sum_{n=1}^{k} \alpha_n \cos \left[ \frac{\pi}{2} (k - n) \right] \frac{n}{(k-n)!! (k+n)!!} n^k.
\tag{37}
\]
In particular, setting \( \alpha_n = t^n \) in (37), we recover (17).

Acknowledgement 3 We would like to thank our colleague Michael Adams and the anonymous referees, for extremely useful comments on earlier versions of this paper.

References

[1] M. Abramowitz and I. A. Stegun. Handbook of Mathematical Functions. Dover, New York, 9th ed., 1972.

[2] F. W. Bessel. Analytische Aufläsung der Keplerschen Aufgabe. Berliner Abh., pages 49–55, 1819.

[3] P. Colwell. Solving Kepler’s equation over three centuries. Willmann-Bell Inc., Richmond, VA, 1993.

[4] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. Academic Press Inc., San Diego, CA, sixth edition, 2000.

[5] W. Kapteyn. Recherches sur les fonctions de Fourier–Bessel. Ann. Sci. École Norm. Sup., x(3):91–120, 1893.

[6] W. Kapteyn. Over Bessel’sche Functiën. Nieuw Archief voor Wiskunde, xx:116–127, 1893.

[7] J. L. Lagrange. Sur le problème de Kepler. Hist. de l’Acad. R. des Sci. de Berlin, xxv:204–233, 1771.

[8] G. N. Watson. A treatise on the theory of Bessel functions. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.