GLOBAL BEHAVIOR OF FINITE ENERGY SOLUTIONS TO THE $d$-DIMENSIONAL FOCUSING NONLINEAR SchröDINGER EQUATION

CRISTI GUEVARA

ABSTRACT. We study the global behavior of finite energy solutions to the $d$-dimensional focusing nonlinear Schrödinger equation (NLS), $i\partial_t u + \Delta u + |u|^{p-1}u = 0$, with initial data $u_0 \in H^1$, $x \in \mathbb{R}^d$. The nonlinearity power $p$ and the dimension $d$ are such that the scaling index $s = \frac{d}{2} - \frac{2}{p-1}$ is between 0 and 1, thus, the NLS is mass-supercritical ($s > 0$) and energy-subcritical ($s < 1$).

For solutions with $\mathcal{ME}[u_0] < 1$ ($\mathcal{ME}[u_0]$ stands for an invariant and conserved quantity in terms of the mass and energy of $u_0$), a sharp threshold for scattering and blowup is given. Namely, if the renormalized gradient $\mathcal{G}_u$ of a solution $u$ to NLS is initially less than 1, i.e., $\mathcal{G}_u(0) < 1$, then the solution exists globally in time and scatters in $H^1$ (approaches some linear Schrödinger evolution as $t \to \pm \infty$); if the renormalized gradient $\mathcal{G}_u(0) > 1$, then the solution exhibits a blowup behavior, that is, either a finite time blowup occurs, or there is a divergence of $H^1$ norm in infinite time.

This work generalizes the results for the 3d cubic NLS obtained in a series of papers by Holmer-Roudenko and Duyckaerts-Holmer-Roudenko with the key ingredients, the concentration compactness and localized variance, developed in the context of the energy-critical NLS and Nonlinear Wave equations by Kenig and Merle.

1. INTRODUCTION

In this paper, we consider the focusing Cauchy problem for the nonlinear Schrödinger equation (NLS), denoted by $\text{NLS}_p(\mathbb{R}^d)$, with finite energy initial data (i.e., $u_0 \in H^1(\mathbb{R}^d)$),

$$
\begin{cases}
  i\partial_t u + \Delta u + |u|^{p-1}u = 0 \\
  u(x,0) = u_0(x) \in H^1(\mathbb{R}^d),
\end{cases}
$$

where $u = u(x,t)$ is a complex-valued function in space-time $\mathbb{R}^d \times \mathbb{R}$, $p \geq 1$.

For a fixed $\lambda \in (0, \infty)$, the rescaled function $u_\lambda(x,t) := \lambda^{\frac{2}{p-1}}u(\lambda x, \lambda^2 t)$ is a solution of $\text{NLS}_p(\mathbb{R}^d)$ in (1.1) if and only if $u(x,t)$ is. This scaling property gives rise to the scale-invariant norms. Sobolev norm $\dot{H}^s(\mathbb{R}^d)$ with $s := \frac{d}{2} - \frac{2}{p-1}$. Ginibre-Velo [GV79a, GV79b] showed that the initial-value problem $\text{NLS}_p(\mathbb{R}^d)$ with initial data $u(x,0) = u_0(x) \in H^1(\mathbb{R}^d)$, $1 \leq p < 1 + \frac{4}{d-2}$ is locally well-posed in $\dot{H}^s(\mathbb{R}^d)$ with $s \geq 1$. Later, Cazenave-Weissler [CW90] showed that for small initial data in $\dot{H}^s(\mathbb{R}^d)$, with $0 \leq s < \frac{d}{2}$ and $0 < p \leq \frac{d+2}{d-2}$, there exists a unique solution to $\text{NLS}_p(\mathbb{R}^d)$ defined for all times. If the data is not small, we can define the maximal interval of existence of solutions to $\text{NLS}_p(\mathbb{R}^d)$ and denote it by $(T_*, T^*)$. We say a solution is global in forward time if $T^* = +\infty$. Similarly, if $T_* = -\infty$, the solution is global in backward time. A solution is global if $(T_*, T^*) = \mathbb{R}$. 

1
On their maximal interval of existence solutions to (1.1) have three conserved quantities: mass, energy and momentum, where

\[ M[u](t) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx = M[u_0], \]

\[ E[u](t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dx - \frac{\mu}{p+1} \int_{\mathbb{R}^d} |u(x,t)|^{p+1} dx = E[u_0], \]

\[ P[u](t) = \text{Im} \int_{\mathbb{R}^d} \bar{u}(x,t) \nabla u(x,t) dx = P[u_0]. \]

The following quantities are scaling invariant:

\[ E[u]^{1-\gamma_c} M[u]^{1-\gamma_c}, \quad \text{and} \quad \|u\|_{L^2(\mathbb{R}^d)}^{\frac{1-\gamma_c}{\gamma_c}} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\gamma_c}. \]

These quantities were first introduced in [HR07] in the context of mass-supercritical NLS (0 < s < 1) and used to classify the global behavior of solutions.

We say that a global solution \( u(t) \) to NLS\(_p(\mathbb{R}^d) \) scatters in \( H^s(\mathbb{R}^d) \) as \( t \to +\infty \) if there exists \( \psi^+ \in H^s(\mathbb{R}^d) \) such that

\[ \lim_{t \to +\infty} \|u(t) - e^{it\Delta} \psi^+\|_{H^s(\mathbb{R}^d)} = 0. \]

Similarly, we can define scattering in \( H^s(\mathbb{R}^d) \) for \( t \to -\infty \).

For the \( L^2 \)-critical NLS equation (i.e. \( s = 0 \)) with \( u_0 \in H^1(\mathbb{R}^d) \), Weinstein in [Wei82] established a sharp threshold for global existence, namely, the condition \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \), where \( Q \) is the ground state solution (see Section 2.4), guarantees a global existence of evolution \( u_0 \to u(t) \). Solutions at the threshold mass, i.e., when \( \|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)} \), may blowup in finite time. Such solutions are called the minimal mass blowup solutions. Merle in [Mer93] characterized the minimal mass blowup \( H^1 \) solutions showing that all such solutions are pseudo-conformal transformations of the ground state (up to \( H^1 \) symmetries), that is,

\[ \psi_\tau(x,t) = e^{i/(T-t)} e^{i|x|^2/(T-t)} Q \left( \frac{x}{T-t} \right). \]

In the energy-critical case \( s = 1 \), Kenig-Merle [KM06] studied global behavior of solutions with \( u_0 \in H^1(\mathbb{R}^d) \) in dimensions \( d = 3, 4, \) and 5 and showed that under a certain energy threshold (namely, \( E[u_0] < E[W] \), where \( W \) is the positive solution of \( \Delta W + W^p = 0 \), decaying at \( \infty \)), it is possible to characterize global existence versus finite blowup depending on the size of \( \|\nabla u_0\|_{L^2(\mathbb{R}^d)} \), and also prove scattering for globally existing solutions. To obtain the last property, they applied the concentration-compactness and rigidity technique. The concentration-compactness method appears in the context of wave equation in Gérard [Ger96] and NLS in Merle-Vega [MV98], which was later followed by Keraani [Ker01], and dates back to works of P-L. Lions [Lio84] and Brezis-Coron [BC85]. The rigidity argument (estimates on a localized variance) is the technique of Merle from mid 1980’s.

The mass-supercritical and energy-subcritical case (0 < s < 1) is discussed in detail in the next section, and the energy-supercritical case (s > 1) is largely open.

In the mass- supercritical and energy-subcritical case (0 < s_c < 1) the 3d cubic NLS equation with \( u_0 \in H^1 \) was studied in a series of papers [HR08], [DHR08], [DR10], [HR10c].
and [HPR10]. The authors obtained a sharp scattering threshold for radial initial data in [HR08], under a so called mass-energy threshold \( M[u]E[u] < M[Q]E[Q] \), where \( Q \) is the ground state solution. The extension of these results to the nonradial data is in [DHR08]. Behavior of solutions and characterization of all solutions at the mass-energy threshold \( M[u]E[u] = M[Q]E[Q] \) is in [DR10]. For infinite variance nonradial solutions Holmer-Roudenko in [HR10c] introduced a first application of concentration-compactness and rigidity arguments to prove the existence of a “weak blowup”\(^1\). In addition, Holmer-Platte-Roudenko [HPR10] consider (both theoretically and numerically) solutions to the 3d cubic NLS above the mass-energy threshold and give new blowup criteria in that region. They also predict the asymptotic behavior of solutions for different classes of initial data (modulated ground state, Gaussian, super-Gaussian, off-centered Gaussian, and oscillatory Gaussian) and provide several conjectures in relation to the threshold for scattering.

In the spirit of [DHR08], [HR08], [HR10c], Carreon-Guevara [CG11] study the long-term behavior of solutions for the 2d quintic NLS equation with \( u_0 \in H^1(s = \frac{1}{2}) \). This equation is important to study since it has a higher power of nonlinearity (higher than cubic), and recently a nontrivial blowup result (a standing ring) was exhibited by Raphaël in [Rap06] (there are further extensions of [Rap06] to higher dimensions and different nonlinearities in [RS09], [HR10b], [HR10a]).

1.1. Statement of the results. Throughout this document, unless otherwise specified, we assume that \( 0 < s < 1 \) and \( s = \frac{d}{2} - \frac{2}{p-1} \), \( \alpha := \frac{\sqrt{d(p-1)}}{2} \), and \( \beta := 1 - \frac{(d-2)(p-1)}{4} \). Let

\[
u_{\alpha}(x, t) := e^{i\beta t} Q(\alpha x).
\]

Then \( u_\alpha(x, t) \) solves the equation (1.1), provided \( Q \) solves\(^2\)

\[-\beta Q + \alpha^2 \Delta Q + Q^p = 0, \quad Q = Q(x), \quad x \in \mathbb{R}^d.
\]

The theory of nonlinear elliptic equations (Berestycki-Lions [BL83a], [BL83b]) shows that (1.1) has an infinite number of solutions in \( H^1(\mathbb{R}^d) \), but a unique solution of minimal \( L^2 \)-norm, which we denote by \( Q(x) \). It is positive, radial, exponentially decaying (for example, [Tao06, Appendix B]) and is called the ground state solution.

We introduce the following notation:

- the renormalized gradient

\[
\mathcal{G}_u(t) := \frac{\|u\|^{1-s}_{L^2(\mathbb{R}^d)} \|\nabla u(t)\|_s^{s}}{\|u_\alpha\|^{1-s}_{L^2(\mathbb{R}^d)} \|\nabla u_\alpha\|_s^{s}},
\]

- the renormalized momentum

\[
\mathcal{P}[u] := \frac{P[u]^s \|u\|^{1-2s}_{L^2(\mathbb{R}^d)}}{\|u_\alpha\|^{1-s}_{L^2(\mathbb{R}^d)} \|\nabla u_\alpha\|_s^{s}},
\]

- the renormalized Mass-Energy

\[
\mathcal{M}[u] := \frac{M[u]^{1-s} E[u]^{s}}{M[u_\alpha]^{1-s} E[u_\alpha]^{s}} \quad \text{for} \ E[u_\alpha] > 0.
\]

\(^1\) See Section 4 for exact formulation and discussion.

\(^2\) Here, in the equation (1.1) and definition of \( Q \), we use the notation from Weinstein [Wei82]. Rescaling \( Q(x) \mapsto \beta^{\frac{1}{p-1}} Q(\sqrt{\frac{2}{\pi}} x) \) will solve a the nonlinear elliptic equation \(-Q + \Delta Q + Q^p = 0.\)
Note that [1.7] we only consider $E[u] > 0$, since for $E[u] < 0$ the blowup is known (see [VPT71], [Zak72], [Gla77], [GM95]).

**Remark 1.1 (Negative energy).** Note that it is possible to have initial data with $E[u] < 0$ and the blowup from the dichotomy in Theorem A Part II (a) below applies. (It follows from the standard convexity blow up argument and the work of Glangetas-Merle [GM95]). Therefore, we only consider $E[u] \geq 0$ in the rest of the paper.

The main result of this paper is

**Theorem A.** Consider NLS$_p(\mathbb{R}^d)$ such that $0 < s < 1$, $u_0 \in H^1(\mathbb{R}^d)$, $d \geq 1$ and let $u(t)$ be the corresponding solution on its maximal time interval of existence $(T_*, T^*)$. Assume

$$(\mathcal{M}E[u])^{\frac{1}{2}} - \frac{d}{2s} (\mathcal{P}[u])^{\frac{2}{2}} < 1. \quad (1.8)$$

I. If

$$(\mathcal{G}_u(0))^{\frac{1}{2}} - (\mathcal{P}[u])^{\frac{2}{2}} < 1, \quad (1.9)$$

then

(a) $[\mathcal{G}_u(t)]^{\frac{1}{2}} - (\mathcal{P}[u])^{\frac{2}{2}} < 1$ for all $t \in \mathbb{R}$, and thus, the solution is global in time ($T_* = -\infty$, $T^* = +\infty$) and

(b) $u$ scatters in $H^1(\mathbb{R}^d)$, i.e., there exists $\phi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} \phi_\pm\|_{H^1(\mathbb{R}^d)} = 0.$$

II. If

$$(\mathcal{G}_u(0))^{\frac{1}{2}} - (\mathcal{P}[u])^{\frac{2}{2}} > 1, \quad (1.10)$$

then $[\mathcal{G}_u(t)]^{\frac{1}{2}} - (\mathcal{P}[u])^{\frac{2}{2}} > 1$ for all $t \in (T_*, T^*)$ and

(a) if $u_0$ is radial (for $d \geq 3$ and in $d = 2$, $3 < p \leq 5$) or $u_0$ is of finite variance, i.e., $|r|u_0 \in L^2(\mathbb{R}^d)$, then the solution blows up in finite time ($T^* < +\infty$, $T_* > -\infty$).

(b) If $u_0$ is non-radial and of infinite variance, then either the solution blows up in finite time ($T^* < +\infty$, $T_* > -\infty$) or there exists a sequence of times $t_n \to +\infty$ (or $t_n \to -\infty$) such that $\|\nabla u(t_n)\|_{L^2(\mathbb{R}^d)} \to \infty$.

We say there is a “weak blowup” occurs if $\mathcal{M}E[1] < 1$, and $u(t)$ exists globally for all positive time (or negative times) and there exists a sequence of times $t_n \to \pm\infty$ such that $\|\nabla u(t_n)\|_{L^2} \to \infty$. In other words, $L^2$ norm of the gradient diverges along at least one infinite time sequence.

Our arguments follow [DHR08] [HR07] [HR08] [HR10c] [CG11] which considered the focusing NLS$_3(\mathbb{R}^3)$ and NLS$_6(\mathbb{R}^2)$, i.e., the integer powers of the nonlinearity. However for the general case we need to consider the fractional powers $p$ as well. To deal with them our innovation is to use Besov spaces to treat the local theory, the long term perturbation and the $H^1$ scattering (see Propositions 2.13, 2.14 and 2.15). In particular, the range of the Strichartz exponents is adjusted for the $d$–dimensional case, as well as the range of admissible pairs for the Kato-type estimates. And using interpolation tricks on admissible pairs $(p, r)$ with
r < +∞ for the Strichartz and Kato-type estimates to avoid the pair (2, ∞) which is not \( \dot{H}^s \)-admissible.

The key argument to obtain scattering\(^3\) and “weak blowup” is the concentration compactness technique together with a rigidity theorem. Note that for \( 2 < q < \frac{2d}{d-2} \) the embedding \( H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \) is not compact; however, a profile decomposition allows to manage this lack of compactness and to produce a “critical element”. Then a localization principle proves scattering or “weak blowup”, depending on the initial assumptions.

The structure of this paper is as follows: Section 2 reviews the local theory, the properties of the ground state and reduction of the problem with nonzero momentum to the case \( P[u] = 0 \) via Galilean transformation for the equation (1.1). In Section 3 we present the outline of concentration compactness machinery and localized virial identity. We include the detailed proofs for the linear and nonlinear profile decompositions, these are the key to prove scattering and to obtain the “weak blowup” in Section 4.

1.2. Notation. The space-time norms are

\[
\|u\|_{L^q_tL^r_x(\mathbb{R} \times \mathbb{R}^d)} = \|u\|_{L^q_tL^r_x} := \left( \int_\mathbb{R} \left( \int_{\mathbb{R}^d} |u(x,t)|^r \, dx \right)^\frac{q}{r} \, dt \right)^\frac{1}{q},
\]

with the corresponding changes when either \( q = \infty \) or \( r = \infty \).

Consider the Littlewood-Paley projection operators: if \( \varphi \in C^\infty_{comp}(\mathbb{R}^d) \) be such that \( \varphi(\xi) = \begin{cases} 1 & |\xi| \leq 1 \\ 0 & |\xi| \geq 2 \end{cases} \). For each dyadic number \( N \in 2^\mathbb{Z} \) and a Schwartz function \( f \), define the Littlewood-Paley operators

\[
\hat{P}_{\leq N} f(\xi) := \varphi \left( \frac{\xi}{N} \right) \hat{f}(\xi), \quad \hat{P}_{> N} f(\xi) := \left( 1 - \varphi \left( \frac{\xi}{N} \right) \right) \hat{f}(\xi),
\]

\[
\hat{P}_N f(\xi) := \left( \varphi \left( \frac{\xi}{N} \right) - \varphi \left( \frac{2\xi}{N} \right) \right) \hat{f}(\xi).
\]

For \( 1 \leq p, q \leq \infty \) and \( \sigma > \frac{d}{p} \), the inhomogeneous Besov space \( \dot{B}^{\sigma}_{p,q}(\mathbb{R}^d) = \{ u \in S'(\mathbb{R}^d) : \|u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^d)} < \infty \} \), where

\[
\|u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^d)} := \|P_{\leq N} u\|_{L^p_x} + \left( \sum_{j=1}^{\infty} \left( 2^j \|P_{2^j} u\|_{L^p_x} \right)^q \right)^\frac{1}{q} = \|P_{\leq N} u\|_{L^p} + \left( \sum_{N \in 2^\mathbb{Z}} \left( N^\sigma \|P_N u\|_{L^p_x} \right)^q \right)^\frac{1}{q}.
\]

\(^3\)When writing this paper, we got aware of the paper [FXC11] proving the scattering for general case using the embedding of \( \{ u \in H^1, \mathcal{M}^c[u] < 1 \text{ and } G_u(t) < 1 \} \) to \( L^{2\left(\frac{n-1}{n}(\frac{n}{p-1})\right)}([0, \infty), \mathcal{L}^p([-1, 1])) \) instead of the Besov spaces as we do to treat the nonlinearity. Although, the scattering result in both paper is the same, we provide a different approach (via Besov spaces). Furthermore, our approach lets us also obtain the “weak blowup” result.

\(^4\)In fact, given any \( f \in H^1(\mathbb{R}^d) \), the sequence \( f_n(x) = f(x - x_n) \), where the sequence \( x_n \to \infty \) in \( \mathbb{R}^d \), is uniformly bounded in \( H^1(\mathbb{R}^d) \), but has no convergent sequence on \( L^q \).
and the homogenous Besov space \( \dot{\beta}^\sigma_{p,q}(\mathbb{R}^d) = \{ u \in S'(\mathbb{R}^d) : \| u \|_{\dot{\beta}^\sigma_{p,q}(\mathbb{R}^d)} < \infty \} \), where
\[
\| u \|_{\dot{\beta}^\sigma_{p,q}(\mathbb{R}^d)} := \left( \sum_{N \in 2^\mathbb{Z}} (N^\sigma \| P_N u \|_{L^p_x})^q \right)^{\frac{1}{q}}.
\]

Note that most of the \( L^p, H^s, \dot{H}^s, \beta^\sigma_{p,q} \) and \( \dot{\beta}^\sigma_{p,q} \) norms are defined on \( \mathbb{R}^d \), thus, we will omit the symbol \( \mathbb{R}^d \) unless we need a specific space dimension.

1.3. Acknowledgments. This project was as a part of doctoral research of the author and was partially supported by grants from the National Science Foundation (NSF - Grant DMS - 080808; PI Roudenko), the Alfred P. Sloan Foundation. The author would like to thank Gustavo Ponce for discussions on the subject and Svetlana Roudenko for guidance on this topic.

2. Preliminaries

In this section, we review the Strichartz estimates (e.g., see Cazenave, Keel-Tao, Foschi, Visciglia), fractional calculus tools and local theory; these are the instruments to treat the nonlinearity \( F(u) = |u|^{p-1}u \), in particular, when \( p \) is fractional. In addition, we survey the ground state properties and the reduction to the zero momentum which allows us to restate Theorem A into a simpler form.

2.1. Fractional calculus tools. For Lemmas 2.1, 2.3, 2.2 assume \( p, p_i \in (1, \infty), \frac{1}{p} = \frac{1}{p_i} + \frac{1}{p_{i+1}} \), with \( i = 1, 2, 3 \).

Lemma 2.1 (Chain rule [KPV93]). Suppose \( F \in C^1(\mathbb{C}) \). Let \( \sigma \in (0, 1) \), then
\[
\| D^\sigma F(u) \|_{L^p} \lesssim \| F'(u) \|_{L^{p_1}} \| D^\sigma f \|_{L^{p_2}}.
\]

Lemma 2.2 (Leibniz rule [KPV93]). Let \( \sigma \in (0, 1) \), then
\[
\| D^\sigma (fg) \|_{L^p} \lesssim \left( \| f \|_{L^{p_1}} \| D^\sigma g \|_{L^{p_2}} + \| g \|_{L^{p_3}} \| D^\sigma f \|_{L^{p_4}} \right).
\]

Lemma 2.3 (Chain rule for Hölder-continuous functions [VIs07]). Let \( F \) be a Hölder-continuous function of order \( 0 < \rho < 1 \), then for every \( 0 < \sigma < \rho \), and \( \frac{\sigma}{\rho} < \nu < 1 \) we have
\[
\| D^\sigma F(u) \|_{L^p} \lesssim \| \| u \|^{\rho - \frac{\sigma}{\rho}} \|_{L^{p_1}} \| D^\nu u \|_{L^{\frac{\rho}{\nu}} L^{p_2}},
\]
provided \( (1 - \frac{\sigma}{\rho})p_1 > 1 \).

2.2. Strichartz type estimates. We say the pair \( (q, r) \) is \( \dot{H}^s -\)admissible if
\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \text{with} \quad 2 \leq q, r \leq \infty \quad \text{and} \quad (q, r, d) \neq (2, \infty, 2);
\]
and the pair \( (q, r) \) is \( \frac{d}{2} -\)acceptable if
\[
1 \leq q, r \leq \infty, \quad \frac{1}{q} < \frac{1}{d} \left( \frac{1}{2} - \frac{1}{r} \right), \quad \text{or} \quad (q, r) = (\infty, 2).
\]
As usual we denote by \( q' \) and \( r' \) the Hölder conjugates of \( q \) and \( r \), respectively (i.e., \( \frac{1}{q} + \frac{1}{r} = 1 \)). Note that any \( L^2 \)-admissible pair is also a \( \frac{d}{2} \)-acceptable, but not vice versa.

2.2.1. Strichartz estimates. The Strichartz estimates (e.g., see Cazenave [Caz03], KT98, Foschi [Fos05]) are

\[
\left\| e^{it\Delta} \phi \right\|_{L_t^q L_x^r} \lesssim \| \phi \|_{L^2}, \quad \left\| \int e^{-it\tau} f(\tau) d\tau \right\|_{L^2} \lesssim \| \phi \|_{L_t^{q'} L_x^{r'}},
\]

(2.1)

\[
\left\| \int_{t < \tau} e^{i(t-\tau)\Delta} f(\tau) d\tau \right\|_{L_t^q L_x^r} \lesssim \| f \|_{L_t^{q'} L_x^{r'}},
\]

(2.2)

where \((q, r)\) is an \( L^2 \)-admissible pair. The retarded estimate (2.2) have a wider range of admissibility (not only \( L^2 \)-admissible) and holds when the pair \((q, r)\) is \( \frac{d}{2} \)-acceptable [Kat94].

In order upgrade the estimates (2.1) and (2.2) to the \( \dot{H}^s \) level, define the Strichartz space \( S(\dot{H}^s) = S(\dot{H}^s(\mathbb{R}^d \times I)) \) as the closure of all test functions under the norm \( \| \cdot \|_{S(\dot{H}^s)} \) with

\[
\| u \|_{S(\dot{H}^s)} = \begin{cases} 
\sup \left\{ \| u \|_{L_t^q L_x^r} \left( \frac{q}{1+q} \right)^+ \leq q \leq \infty, \frac{2d}{d-2s} \leq r \leq \left( \frac{2d}{d-2s} \right)^- \right\} & \text{if } d \geq 3 \\
\sup \left\{ \| u \|_{L_t^q L_x^r} \left( \frac{2}{1-s} \right)^+ \leq q \leq \infty, \frac{2}{1-s} \leq r \leq \left( \frac{2}{1-s} \right)^- \right\} & \text{if } d = 2 \\
\sup \left\{ \| u \|_{L_t^q L_x^r} \frac{1}{1-2s} \leq q \leq \infty, \frac{2}{1-2s} \leq r \leq \infty \right\} & \text{if } d = 1.
\end{cases}
\]

Here, \((a^+)^r\) is defined as \((a^+)^r := \frac{a^+ - a}{a^+ - a^-}\), so that \( \frac{1}{a} = \frac{1}{(a^+)^r} + \frac{1}{a^+} \) for any positive real value \( a \), with \( a^+ \) being a fixed number slightly larger than \( a \). Likewise, \( a^- \) is a fixed number slightly smaller than \( a \).

Remark 2.4. Note that \( \frac{2d}{d-2s} < \left( \frac{2d}{d-2} \right)^- \) if \( d \geq 3 \). Additionally, when \( d = 2 \) and \( s \neq \frac{1}{2} \), the quantity \( r = \frac{2d}{d-2s} \) might be very large, but \( \frac{2d}{d-2s} < \left( \frac{2}{1-s} \right)^+ \). Similarly, define the dual Strichartz space \( S'(\dot{H}^{-s}) = S'(\dot{H}^{-s}(\mathbb{R}^d \times I)) \) as the closure of all test functions under the norm \( \| \cdot \|_{S'(\dot{H}^{-s})} \) with

\[
\| u \|_{S'(\dot{H}^{-s})} = \begin{cases} 
\inf \left\{ \| u \|_{L_t^q L_x^r} \left( \frac{2}{1+s} \right)^+ \leq q \leq \left( \frac{1}{s} \right)^-, \left( \frac{2d}{d-2s} \right)^+ \leq r \leq \left( \frac{2d}{d-2s} \right)^- \right\} & \text{if } d \geq 3 \\
\inf \left\{ \| u \|_{L_t^q L_x^r} \left( \frac{2}{1+s} \right)^+ \leq q \leq \left( \frac{1}{s} \right)^-, \left( \frac{2}{1+s} \right)^+ \leq r \leq \left( \frac{2}{1+s} \right)^+ \right\} & \text{if } d = 2 \\
\inf \left\{ \| u \|_{L_t^q L_x^r} \frac{2}{1+2s} \leq q \leq \left( \frac{1}{s} \right)^-, \left( \frac{2}{1+s} \right)^+ \leq r \leq \infty \right\} & \text{if } d = 1.
\end{cases}
\]
Remark 2.5. Note that $S(L^2) = S(\dot{H}^0)$ and $S'(L^2) = S' (\dot{H}^{-0})$. In this dissertation, if $(q, r)$ is $\dot{H}^{-0}$ admissible we say a pair $(q', r')$ is $L^2-$dual admissible.

Under the above definitions, the Strichartz estimates (2.1) become

$$\|e^{it\Delta} \phi\|_{S(L^2)} \leq c\| \phi \|_{L^2} \quad \text{and} \quad \left\| \int_{s=t} e^{i(t-s)\Delta} f(s)ds \right\|_{S(L^2)} \leq c\| f \|_{S'(L^2)}$$

and in this paper, we refer to them as the (standard) Strichartz estimates.

Combining (2.3) with the Sobolev embedding $W^{s,r}_x(\mathbb{R}^d) \hookrightarrow L^{\frac{n}{s}, \frac{n}{r}}(\mathbb{R}^d)$ for $s < \frac{n}{r}$ and interpolating yields the Sobolev Strichartz estimates

$$\|e^{it\Delta} \phi\|_{S(\dot{H}^s)} \leq c\| \phi \|_{\dot{H}^s} \quad \text{and} \quad \left\| \int_0^t e^{i(t-s)\Delta} f(s)ds \right\|_{S(\dot{H}^s)} \leq c\| D^s f \|_{S'(L^2)},$$

and in similar fashion (2.2) leads to the Kato’s Strichartz estimate [Kat87, Fos05]

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s)ds \right\|_{S(\dot{H}^{-s})} \leq c\| f \|_{S'(L^2)}.$$ (2.5)

Kato’s Strichartz estimate along with the Sobolev embedding imply the inhomogeneous estimate (second estimate in (2.4) but not vice versa) and it is the key estimate in the long term perturbation argument (Proposition 2.14).

2.2.2. Besov Strichartz estimates. We address the question of non-integer nonlinearities for the NLS$_p(\mathbb{R}^d)$. The following remark is due

Remark 2.6. The complex derivative of the nonlinearity $F(u) = |u|^{p-1}u$ is $F_z(z) = \frac{p+1}{2} |z|^{p-1}$ and $F_z(z) = \frac{p-1}{2} |z|^{p-1} \frac{1}{z}$. They are Hölder-continuous functions of order $p$, and for any $u, v \in \mathbb{C}$, we have

$$F(u) - F(v) = \int_0^1 \left[ F_z(v + t(u - v))(u - v) + F_z(v + t(u - v))\overline{(u - v)} \right] dt,$$ (2.6)

thus,

$$|F(u) - F(v)| \lesssim |u - v|(|u|^{p-1} + |v|^{p-1}).$$ (2.7)

Hence, the nonlinearity $F(u)$ satisfies

(a) $F \in C^2(\mathbb{C})$, if $2 \leq d < 5$, or $d = 5$ when $\frac{1}{2} < s < 1$,

(b) $F \in C^1(\mathbb{C})$, if $d \geq 6$, or $d = 5$ when $0 < s \leq \frac{1}{2}$.

When estimating the fractional derivatives of (2.6), in the case (b), there is a lack of smoothness. This issue is resolved by using the Besov spaces.
Define the Besov Strichartz space \( \dot{\beta}^\sigma_{S(H^s)} = \dot{\beta}^\sigma_{S(H^s)}(\mathbb{R}^d \times I) \) as the closure of all test functions under the semi-norm \( \| \cdot \|_{\dot{\beta}^\sigma_{S(H^s)}} \) with

\[
\| u \|_{\dot{\beta}^\sigma_{S(H^s)}} = \begin{cases} 
\sup \left\{ \| u \|_{L_t^q L_x^r} \left| (q, r) \dot{H}^s \right. \right. & \text{- admissible with} \\
\left. \left. \left( \frac{2d}{1-s} \right)^+ \leq q \leq \infty, \quad \frac{2d}{d-s} \leq r \leq \left( \frac{2d}{d-s} \right)^- \right\} \right. \quad \text{if } d \geq 3
\end{cases}
\]

Similary, define the dual Besov Strichartz space \( \dot{\beta}^\sigma_{S'(H^{-s})} = \dot{\beta}^\sigma_{S'(H^{-s})}(\mathbb{R}^d \times I) \) as the closure of all test functions under the semi-norm \( \| \cdot \|_{\dot{\beta}^\sigma_{S'(H^{-s})}} \) with

\[
\| u \|_{\dot{\beta}^\sigma_{S'(H^{-s})}} = \begin{cases} 
\inf \left\{ \| u \|_{L_t^{q'} L_x^{r'}} \left| (q, r) \dot{H}^{-s} \right. \right. & \text{- admissible with} \\
\left. \left. \left( \frac{2d}{1+s} \right)^+ \leq q \leq \left( \frac{1}{s} \right)^-, \quad \frac{2d}{d+s} \leq r \leq \left( \frac{2d}{d+s} \right)^- \right\} \right. \quad \text{if } d \geq 3
\end{cases}
\]

Lemma 2.7. If \( u \in \dot{\beta}^\sigma_{S(H^s)} \) and \( \sigma \geq 0, s \in \mathbb{R} \), then

\( \| D^\sigma u \|_{S(H^s)} \lesssim \| u \|_{\dot{\beta}^\sigma_{S(H^s)}} \).

Proof. Let \( (q, r) \) be \( \dot{H}^s \)-admissible pair, then

\[
\| D^\sigma u \|_{L_t^q L_x^r} \lesssim \left( \sum_{N \in 2^\mathbb{Z}} \| P_N D^\sigma u \|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{N \in 2^\mathbb{Z}} \| P_N D^\sigma u \|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \lesssim \| u \|_{\dot{\beta}^\sigma_{\dot{L}_t^q \dot{L}_x^r}}.
\]

Taking sup over all \( (q, r) \) \( \dot{H}^s \)-admissible pairs, yields the claim. \( \square \)

Lemma 2.8 (Embedding). For any compact time interval \( I \), assume \( 0 \leq \sigma < \rho, 1 \leq r, r_1, q \leq \infty \). Then

\[
\| D^\sigma u \|_{L_t^q L_x^r} \lesssim \| D^\rho u \|_{L_t^{q_1} L_x^{r_1}},
\]

where \( r_1 = \frac{r d}{(\rho-\sigma) r d} \) and \( q_1 = q_2 \).
Proof. The Sobolev embedding $\hat{W}^{\rho,1}_x(\mathbb{R}^d) \hookrightarrow \hat{W}^{\rho,r}_x(\mathbb{R}^d)$ yields the inequality (2.8). \hfill \Box

Remark 2.9. If $q', r'$ and $r_i'$ are the Hölder’s conjugates of $r, q$ and $r_i$, respectively, then we have

$$
\|D^\rho u\|_{L^q_t L^{r_i'}_x} \lesssim \|D^\rho u\|_{L^{q'}_t L^{r_i'}_x}.
$$

Lemma 2.10 (Linear Besov-Strichartz). Let $u \in \hat{\beta}^\sigma_{S(\mathcal{L}^2)}$ be a solution to the forced Schrödinger equation

$$
i u_t + \Delta u = \sum_{m=1}^M F_m
$$

for some functions $F_1, \ldots, F_M$ and $\sigma = 0$ or $\sigma = s$. Then on $\mathbb{R}^d \times I$ we have

$$
\|u\|_{\hat{\beta}^\sigma_{S(\mathcal{H}^s)}} \lesssim \|u_0\|_{\mathcal{H}^s} + \sum_{m=1}^M \|F_m\|_{\hat{\beta}^\sigma_{S(\mathcal{L}^2)}}.
$$

Lemma 2.11 (Inhomogeneous Besov Strichartz estimate). If $F \in \hat{\beta}^\sigma_{S(\mathcal{H}^{-s})}$, then

$$
\left\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \right\|_{\hat{\beta}^\sigma_{S(\mathcal{H}^s)}} \lesssim \|F\|_{\hat{\beta}^\sigma_{S(\mathcal{H}^{-s})}}.
$$

Proofs of Lemma 2.10 and 2.11 can be found in [Tao06].

Lemma 2.12 (Interpolation inequalities for Besov spaces [Tri78]). Let $1 \leq p_i, q_i \leq \infty$ and $u \in \beta^\sigma_{p_i,q_i}(\mathbb{R}^d)$, where $i = 1, 2, 3$. Then

$$
\|u\|_{\beta^\sigma_{p_1,q_1}(\mathbb{R}^d)} = \|u\|_{\beta^\sigma_{p_2,q_2}(\mathbb{R}^d)} \|\theta\|_{\beta^\sigma_{p_3,q_3}(\mathbb{R}^d)}
$$

provided that

$$
\sigma_1 = (1 - \theta)\sigma_2 + \theta\sigma_3,
$$

$$
\frac{1}{p_1} = \frac{1 - \theta}{p_2} + \frac{\theta}{p_3}
$$

and

$$
\frac{1}{q_1} = \frac{1 - \theta}{q_2} + \frac{\theta}{q_3}.
$$

2.3. Local Theory. In this subsection the global existence and scattering in $H^1(\mathbb{R}^d)$ for small data in $\hat{H}^s$ (Propositions 2.13 and 2.15), and a long perturbation argument (Proposition 2.14) are examined. The proofs rely on Besov spaces which allow us to treat the lack of smoothness of the nonlinearity $F(u) = |u|^{p-1}u$ (see Remark 2.6).

Although it may appear that the small data theory (Proposition 2.13) is by now a straightforward argument, we write out its proof carefully to show how we deal with the non-integer nonlinearities. For the same reason we include full proofs of the long-term perturbation (Proposition 2.14) and the $H^1$ scattering (Proposition 2.15).

Proposition 2.13 (Small data). Suppose $\|u_0\|_{\hat{H}^s} \lesssim A$. There exists $\delta_{sd} = \delta_{sd}(A) > 0$ such that if $\|e^{it\Delta} u_0\|_{\hat{\beta}^0_{S(\mathcal{H}^s)}} \lesssim \delta_{sd}$, then $u(t)$ solving the NLS$_p(\mathbb{R}^d)$ is global in $\hat{H}^s(\mathbb{R}^d)$ and

$$
\|u\|_{\hat{\beta}^0_{S(\mathcal{H}^s)}} \lesssim 2\|e^{it\Delta} u_0\|_{\hat{\beta}^0_{S(\mathcal{H}^s)}},
$$

$$
\|u\|_{\hat{\beta}^s_{S(\mathcal{L}^2)}} \lesssim 2c\|u_0\|_{\hat{H}^s}.
$$
Proof. Using a fixed point argument in a ball $B$, the existence of solutions to (1.1) and continuous dependence on the initial data is proven as follows.

Let

$$B = \left\{ \|u\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^p)} \lesssim 2\|e^{it\Delta} u_0\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^p)}, \; \|u\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)} \lesssim 2\|u_0\|_{\dot{\mathcal{H}}^s} \right\}.$$ 

Assume $F(u) = |u|^{p-1}u$ and the map $u \mapsto \Phi_{u_0}(u)$ defined via

$$\Phi_{u_0}(u) := e^{it\Delta} u_0 + i \int_0^t e^{i(t-\tau)\triangle} F(u(\tau)) d\tau.$$ 

Combining the triangle inequality and the Linear Besov Strichartz estimates (2.10) and the fact that $F(u) \in C^1$, we obtain

$$\|\Phi_{u_0}(u)\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^p)} \lesssim \|e^{it\Delta} u_0\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^p)} + \|F(u)\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)},$$

$$\|\Phi_{u_0}(u)\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)} \lesssim \|u_0\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)} + \|F(u)\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)}.$$ 

For each dyadic number $N \in 2^\mathbb{Z}$, the fractional chain rule (Lemma 2.1) and Hölder’s inequality lead to

$$\|D^s F(u)\|_{S^0(\mathcal{L}^2)} \lesssim \|D^s(|u|^{p-1}u)\|_{L_t^2 L_x^{\frac{2p}{p-1}}} \lesssim \|u\|^{p-1}_{L_t^\infty L_x^{\frac{2p}{p-1}}} \|D^s u\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{2p-1}}} \lesssim \|u\|^{p-1}_{S(\dot{\mathcal{H}}^s)} \|D^s u\|_{S(\mathcal{L}^2)},$$

thus, Littlewood-Paley theory yields

$$\|\|u\|^{p-1}u\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)} \lesssim \|u\|^{p-1}_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} \|u\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)}.$$  

Therefore,

$$\|\Phi_{u_0}(u)\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} \lesssim \|e^{it\Delta} u_0\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} + \|u\|^{p-1}_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} \|u\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)},$$

$$\|\Phi_{u_0}(u)\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)} \lesssim \|u_0\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)} + \|u\|^{p-1}_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} \|u\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)}.$$ 

and if $\|e^{it\Delta} u_0\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} \leq \delta_1$ with $\delta_1 = \min\left\{ \frac{1}{2p c_1^p A_p^s}, \frac{1}{\sqrt[2p]{2p c_1^{2p-1} A_p^s}} \right\}$ leads to $\Phi_{u_0}(u) \in B$.

To complete the proof, we need to show that the map $u \mapsto \Phi_{u_0}(u)$ is a contraction. Take $u, v \in B$, and note that the triangle inequality and Besov Strichartz estimates yield

$$\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)} \lesssim \left\| \int_0^t e^{i(t-\tau)\triangle} \left( F(u(\tau)) - F(v(\tau)) \right) d\tau \right\|_{\dot{\mathcal{B}}^0_{s}(\dot{\mathcal{H}}^s)}$$

$$\lesssim \|D^s (F(u) - F(v))\|_{\dot{\mathcal{B}}^0_{s}(\mathcal{L}^2)} \approx \|F(u) - F(v)\|_{\dot{\mathcal{B}}^s_{s}(\mathcal{L}^2)},$$
Here, we used the Hölder split
\[
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{\dot{S}^{s}(L^2)} \approx \|D^s(\Phi_{u_0}(u) - \Phi_{u_0}(v))\|_{\dot{S}^{s}(L^2)}
\]
\[
\lesssim \left\| \int_0^t e^{i(t-\tau)\Delta} D^s\left(F(u(\tau)) - F(v(\tau))\right) \, d\tau \right\|_{\dot{S}^{s}(L^2)}
\lesssim \|D^s(F(u) - F(v))\|_{\dot{S}^{s}(L^2)} \approx \|F(u) - F(v)\|_{\dot{S}^{s}(L^2)}.
\]

For each dyadic number \( N \in 2^\mathbb{Z} \), we estimate \( \|D^s(F(u) - F(v))\|_{S^s(L^2)} \). Recall that we are considering the mass-supercritical energy-subcritical NLS, i.e., \( 0 < s < 1 \) and \( p = 1 + \frac{4}{d-2s} \). Due to the lack of smoothness of the nonlinearity (Remark 2.6), we consider two (complementary) cases:

(a) The function \( F(u) \) is at least in \( C^2(\mathbb{C}) \).

(b) The nonlinearity \( F(u) \) is at most in \( C^1(\mathbb{C}) \).

In the rest of the proof we examine these cases separately. After the proof we refer to the specific examples to illustrate our approach.

Case (a). \( F(u) \) is at least in \( C^2(\mathbb{C}) \): this case occurs when \( 1 \leq d \leq 4+2s \), i.e., dimensions 2, 3, and 4 for \( 0 < s < 1 \), or dimension 5 when \( \frac{1}{2} \leq s < 1 \). Combining (2.7), chain rule (Lemma 2.1), Hölder’s inequality, gives
\[
\|D^s(F(u) - F(v))\|_{S^s(L^2)} \lesssim \|D^s(|u|^{p-1}u - |v|^{p-1}v)\|_{L_t^2 L_x^2} \lesssim \|D^s|u - v|\|_{S^s(L^2)} \lesssim \|D^s|u - v|\|_{S^s(H^s)}(\|u|^{p-1}_S + \|v|^{p-1}_S).
\]
Here, we used the Hölder split
\[
\frac{2d^2(p-1)}{d^2(p-1) + 16} = \frac{d^2p - 8s}{2d^2p} + (p-1)\frac{2(d+4)}{d^2p(p-1)}
\] (2.13)
together with the fact that the pair \( \left( \frac{d}{d^2}, \frac{2d^2(p-1)}{d^2(p-1) + 16} \right) \) is \( L^2 \)– dual admissible, the pair \( \left( \frac{dp}{d^2}, \frac{d^2p(p-1)}{2(d+4)} \right) \) is \( \dot{H}^s \)– admissible.

Therefore, \( \|F(u) - F(v)\|_{\dot{S}^s(L^2)} \lesssim \|u - v\|_{\dot{S}^s(L^2)} \left( \|u\|^{p-1}_{\dot{S}^s(S(H^s))} + \|v\|^{p-1}_{\dot{S}^s(S(H^s))} \right) \). If \( \|e^{it\Delta}u_0\|_{\dot{S}^0(S(H^s))} \leq \delta_2 \) with \( \delta_2 = \min \left\{ \sqrt{\frac{1}{2pC}}, \frac{1}{2pA^p - 2C} \right\} \) implies that \( \Phi_{u_0} \) is a contraction.

Case (b). \( F(u) \) is at most in \( C^1(\mathbb{C}) \): this corresponds to dimensions higher than \( 4 + 2s \), i.e., \( d = 5 \) with \( 0 < s < \frac{1}{2} \) or \( d \geq 6 \) with \( 0 < s < 1 \). Let \( w = u - v \), therefore (2.6) and the
triangle inequality imply

\[\|D^s(F(u) - F(v))\|_{S'(L^2)} \lesssim \|D^s(|u|^{p-1}u - |v|^{p-1}v)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \]

\[\lesssim \|D^sF_z(v + w)w\|_{L_t^4 L_x^{2d^2(p-1)+16}} + \|D^sF_z(v + \bar{w})\bar{w}\|_{L_t^4 L_x^{2d^2(p-1)+16}}. \quad (2.14)\]

To estimate (2.14), we consider the subcases (i) \(s \leq p-1\) and (ii) \(s > p-1\).

(i) If dimensions \(4 + 2s < d \leq \frac{4+2s^2}{s}\), then \(s \leq p-1 < 1\), thus,

\[\|D^sF_z(u)w\|_{L_t^4 L_x^{2d^2(p-1)+16}} \lesssim \|D^sF_z(u)w\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.15)\]

\[\|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \lesssim \|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.16)\]

\[\|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \lesssim \|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.17)\]

\[\|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \lesssim \|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.18)\]

\[\|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \lesssim \|D^sF_z(u)\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.19)\]

where, Remark 2.9 yields (2.15), since \(\frac{4d^2(p-1)}{(d+4)(d-dp+8)+d^2p(p-1)} \) and \(\frac{4d^2(p-1)}{d^2(p-1)+16}\) are Hölder conjugates and \(\frac{s(p-1)}{2} \leq s \leq 1\). Leibniz rule gives (2.16) and (2.17). Then applying chain rule for Hölder-continuous functions (Lemma 2.3) with \(\rho := p - 1\), \(\sigma := \frac{s(p-1)}{2}\) and \(\nu := s\) to (2.16), we obtain (2.18). Noticing that \(L_x^{2d^2(p-1)+4s} \hookrightarrow L_x^{2d^2(p-1)+16}\), Lemma 2.8 implies (2.19). The last line comes from the fact that the pairs \((\frac{dp}{2s}, \frac{2d^2p}{2(d+4)}), (\frac{dp}{2s}, \frac{2d^2p}{2(d+4)}), (\frac{dp}{2s}, \frac{2d^2p}{2(d+4)}), (\frac{dp}{2s}, \frac{2d^2p}{2(d+4)})\) are \(\bar{H}^s\)–admissible, and the pairs \((\frac{dp}{2s}, \frac{2d^2p}{2d^2-4s}), (\frac{dp}{2s}, \frac{2d^2p}{2d^2-4s})\) are \(L^2\)–admissible. In a similar fashion, we obtain the estimate for the conjugate

\[\|D^sF_z(v + \bar{w})\bar{w}\|_{L_t^4 L_x^{2d^2(p-1)+16}} \lesssim \|D^sF_z(v + \bar{w})\bar{w}\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.18)\]

Thus, Littlewood-Paley theory implies that

\[\|F(u) - F(v)\|_{\bar{H}^s(L^2)} \lesssim 2\|u - v\|_{\bar{H}^s(L^2)} \left(\|u\|_{L_t^4 L_x^{2d^2(p-1)+16}}^{\frac{p-1}{s}} + \|u\|_{L_t^4 L_x^{2d^2(p-1)+16}}^{\frac{p-1}{s}}\right), \quad (2.19)\]

and letting \(\delta \leq \frac{\sqrt{\frac{1}{2(4+2s^2)}}}{CA^{\frac{4+2s^2}{s}}\sqrt{\frac{1}{2}}} \) gives that \(\Phi_{u_0}\) is a contraction.

(ii) If the dimensions \(d > \frac{4+2s^2}{s}\), then \(p-1 < s\). Therefore, we make an estimate for

\[\|D^sF_z(u)w\|_{L_t^4 L_x^{2d^2(p-1)+16}} \quad (2.20)\]

as follows
\[ \left\| D^s F_z(u) \right\|_{L_t^p L_x^{d(p-1)+16}} \lesssim \left\| D_{(p-1)}^2 F_z(u) \right\|_{L_t^p L_x^{2(d+4)+d(p-1)+16}} \lesssim \left\| D_{(p-1)}^2 F_z(u) \right\|_{L_t^p L_x^{2(d+4)+d(p-1)}} \]  
\[ \lesssim \left\| D_{(p-1)}^2 F_z(u) \right\|_{L_t^p L_x^{2(d+4)+d(p-1)+16}} + \left\| D_{(p-1)}^2 F_z(u) \right\|_{L_t^p L_x^{2(d+4)+d(p-1)}} \]  
\[ \lesssim \left\| D^s u \right\|_{S(L^2)} \left( \left\| D^s u \right\|_{S(H^s)} + \left\| D_{(p-1)}^2 u \right\|_{S(H^s)} \right) \]  

Therefore, Littlewood-Paley theory produces

\[ \left\| F(u) - F(v) \right\|_{\beta^s_{S(L^2)}} \lesssim 2 \left\| u - v \right\|_{\beta^s_{S(L^2)}} \left( \left\| u \right\|_{\beta^{(p-1)(1+s-p)}_{S(S(H^s))}} + \left\| u \right\|_{\beta^{(p-1)}_{S(S(H^s))}} \right) \]

and taking \( \delta_4 \leq \frac{(p-1)(1+s-p)}{s} \sqrt{\frac{1}{2^{(p+1)} C A_{\beta^s_{S(L^2)}}} } \) implies that \( \Phi_{u_0} \) is a contraction.

From cases (a) and (b) choosing \( \delta_{ad} \leq \min \{ \delta_1, \delta_2, \delta_3, \delta_4 \} \) implies that the map \( u \mapsto \Phi_{u_0}(u) \) is a contraction which concludes the proof. \( \square \)

To better understand the difference for the above cases (a), (b)(i) and (b)(ii), we refer to the reader to [Gue11] Examples 2.14, 2.15 and 2.16] were we give examples of \( \dot{H}^{\frac{1}{4}} \)-criticalNLS_2(R^4), NLS_2(R^7) and NLS_\frac{13}{2}(R^{10}) and demonstrate how the estimates work.

**Proposition 2.14** (Long term perturbation). For each \( A > 0 \), there exist \( \epsilon_0 = \epsilon_0(A) > 0 \) and \( c = c(A) > 0 \) such that the following holds. Let \( u = u(x,t) \in H^1(R^d) \) solve NLS_p(R^d). Let \( v = v(x,t) \in H^1(R^d) \) for all \( t \) and satisfy \( \tilde{e} = iv_t + \Delta v + |v|^{p-1}v \). If \( \left\| v \right\|_{\beta^0_{S(H^s)}} \leq A \), \( \left\| e_i \right\|_{\beta^0_{S(H^s)}} \leq \epsilon_0 \) and \( \left\| e^{(t-t_0)}\Delta(u(t_0) - v(t_0)) \right\|_{\beta^0_{S(H^s)}} \leq \epsilon_0, \) then \( \left\| u \right\|_{\beta^0_{S(H^s)}} \leq c. \)
Proof. Let $F(u) = |u|^{p-1}u$, $w = u - v$, and $W(v, w) = F(u) - F(v) = F(v + w) - F(v)$. Therefore, $w$ solves the equation

$$iw_t + \Delta w + W(v, w) + \tilde{\epsilon} = 0.$$  

Since $\|v\|_{S_{\dot{H}^s}(I_j)} \leq A$, split the interval $[t_0, \infty)$ into $K = KA$ intervals $I_j = [t_j, t_{j+1}]$ such that for each $j$, $\|v\|_{S_{\dot{H}^s}(I_j)} \leq \delta$ with $\delta$ to be chosen later. Recall that the integral equation of $w$ at time $t_j$ is given by

$$w(t) = e^{i(t-t_j)\Delta}w(t_j) + i \int_{t_j}^{t} e^{i(t-\tau)\Delta}(W + \tilde{\epsilon})(\tau)d\tau.$$  

(2.25)

Applying Kato Besov Strichartz estimate (2.14) on (2.25) for each $I_j$, we obtain

$$\|w\|_{S_{\dot{H}^s}(I_j)} \lesssim \|e^{i(t-t_j)\Delta}w(t_j)\|_{S_{\dot{H}^s}(I_j)} + \| \int_{t_j}^{t} e^{i(t-\tau)\Delta}(W + \tilde{\epsilon})(\tau)d\tau\|_{S_{\dot{H}^s}(I_j)}$$

$$\lesssim \|e^{i(t-t_j)\Delta}w(t_j)\|_{S_{\dot{H}^s}(I_j)} + c\|W(v, w)\|_{S_{\dot{H}^s}(I_j)} + c\|\tilde{\epsilon}\|_{S_{\dot{H}^s}(I_j)}$$

$$\lesssim \|e^{i(t-t_j)\Delta}w(t_j)\|_{S_{\dot{H}^s}(I_j)} + c\|W(v, w)\|_{S_{\dot{H}^s}(I_j)} + c\epsilon_0.$$  

Thus, for each dyadic number $N \in 2\mathbb{Z}$, the following estimate holds

$$\|W(v, w)\|_{S_{\dot{H}^s}(I_j)} \lesssim \|F(v + w) - F(v)\|_{L_{t_j}^{12(d-2a)}L_{x_j}^{p_i}}^{(5s-2)\frac{6}{d-2}}$$

$$\lesssim \|w\|_{L_{t_j}^{\frac{6}{d-2}}L_{x_j}^{\frac{6d}{d-2}}}^{\frac{6}{d-2}} + \frac{1}{(8+3d-6s)(1-s)} + \frac{6d}{d-2}$$  

(2.26)

$$\lesssim \|w\|_{S_{\dot{H}^s}(I_j)} \left( \|v\|_{S_{\dot{H}^s}(I_j)}^{p-1} + \|w\|_{S_{\dot{H}^s}(I_j)}^{p-1} \right)$$

$$\lesssim \|w\|_{S_{\dot{H}^s}(I_j)} \left( \delta_{N}^{p-1} + \|w\|_{S_{\dot{H}^s}(I_j)}^{p-1} \right).$$  

(2.27)

where we first observed that the pairs $(\frac{6}{1-s}, \frac{6d}{d-2}), (\frac{4}{1-s}, \frac{2d}{d-1})$ are $\dot{H}^s-$admissible; the pair $(\frac{\frac{12(d-2a)}{3d-4s-2}}{3(d^2+2s^2)+9d(1-s)-2(5s+4)}, \frac{6d}{d-2})$ is $\dot{H}^s-$admissible. Thus, we used (2.27) and Hölder’s inequality to obtain (2.26). Since $\|v\|_{S_{\dot{H}^s}(I_j)} \leq \delta$ for each dyadic interval, there exists $\delta_N = \delta(N)$, so we obtain (2.27).

Therefore,

$$\|F(v + w) - F(v)\|_{S_{\dot{H}^s}(I_j)} \lesssim \|w\|_{S_{\dot{H}^s}(I_j)} \left( \|v\|_{S_{\dot{H}^s}(I_j)}^{p-1} + \|w\|_{S_{\dot{H}^s}(I_j)}^{p-1} \right)$$

$$\lesssim \|w\|_{S_{\dot{H}^s}(I_j)} \left( \delta_{N}^{p-1} + \|w\|_{S_{\dot{H}^s}(I_j)}^{p-1} \right).$$

Choosing $\delta = \sum_{N \in 2\mathbb{Z}} \delta_N < \min \left\{ 1, \frac{1}{4c_1} \right\}$ and $\|e^{i(t-t_j)\Delta}w(t_j)\|_{S_{\dot{H}^s}(I_j)} + c_1 \epsilon_0 \leq \min \left\{ 1, \frac{1}{2\sqrt{4c_1}} \right\}$, we have

$$\|w\|_{S_{\dot{H}^s}(I_j)} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{S_{\dot{H}^s}(I_j)} + 2c_1 \epsilon_0.$$
Taking \( t = t_{j+1} \), applying \( e^{i(t-t_{j+1})\Delta} \) to both sides of (2.25) and repeating the Kato estimates (2.5), we obtain

\[
\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{\dot{\mathcal{B}}^0_{S(H^s)}} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{\dot{\mathcal{B}}^0_{S(H^s,t_j)}} + 2c_1\epsilon_0.
\]

Iterating this process until \( j = 0 \), we obtain

\[
\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{\dot{\mathcal{B}}^0_{S(H^s)}} \leq 2^j\|e^{i(t-t_0)\Delta}w(t_0)\|_{\dot{\mathcal{B}}^0_{S(H^s,t_j)}} + (2^j - 1)2c_1\epsilon_0 \leq 2^{j+2}c_1\epsilon_0.
\]

These estimates must hold for all intervals \( I_j \) for \( 0 \leq j \leq K - 1 \), therefore,

\[
2^{K+2}c_1\epsilon_0 \leq \min \left\{ 1, \frac{1}{2\sqrt{4c_1}} \right\},
\]

which determines how small \( \epsilon_0 \) has to be taken in terms of \( K \) (as well as, in terms of \( A \)). \( \square \)

An illustration of specific cases (the nonlinearity \( F(u) \) is (a) at least in \( C^2(C) \) and (b) at most in \( C^1(C) \)) of the estimate \( \|W(v,w)\|_{S(H^s,t_j)} \) is given in [Gue11, Examples 2.18, 2.19 and 2.20].

**Proposition 2.15** (\( H^1 \) scattering). Assume \( u_0 \in H^1(\mathbb{R}^d) \). Let \( u(t) \) be a global solution to NLS\(_p(\mathbb{R}^d) \) with the initial condition \( u_0 \), globally finite \( \mathring{H}^s \) Besov Strichartz norm \( \|u\|_{\mathcal{B}^0_{S(H^s)}} < +\infty \) and uniformly bounded \( \dot{H}^1(\mathbb{R}^d) \) norm \( \sup_{t \in [0,\infty)} \|u(t)\|_{H^1} \leq B \). Then there exists \( \phi_+ \in H^1(\mathbb{R}^d) \) such that (1.2) holds, i.e., \( u(t) \) scatters in \( H^1(\mathbb{R}^d) \) as \( t \to +\infty \). Similar statement holds for negative time.

**Proof.** Suppose \( u(t) \) solves NLS\(_p(\mathbb{R}^d) \) with the initial datum \( u_0 \), and satisfies the integral equation

\[
u(t) = e^{it\Delta}u_0 + i\mu \int_0^t e^{i(t-\tau)\Delta} (|u|^{p-1}u(\tau)) \, d\tau.
\]

The assumption \( \|u\|_{\dot{\mathcal{B}}^0_{S(H^s)}} < +\infty \) implies that for each dyadic \( N \in 2^\mathbb{Z} \) there exists \( M = \|u\|_{\mathcal{B}^\infty_{L^p_xL^{2(d+1)}}} < \infty \) and let \( \tilde{M} \sim M^{\frac{2d}{p}} \). Decompose \( [0,\infty) = \cup_{j=1}^M I_j \), such that for each \( j \),

\[
\|u\|_{\mathcal{B}^\infty_{L^p_xL^{2(d+1)}}} < \delta.
\]

Hence, the triangle inequality and Strichartz estimates yield

\[
\|u\|_{S(L^2)} \lesssim \|e^{it\Delta}u_0\|_{S(L^2)} + \|F(u)\|_{S'(L^2)};
\]

\[
\|\nabla u\|_{S(L^2)} \lesssim \|e^{it\Delta}u_0\|_{S(L^2)} + \|\nabla F(u)\|_{S'(L^2)}.
\]

Therefore, the integral equation (2.28) on \( I_j \), combined with the above inequalities, leads to

\[
\|\nabla u\|_{S(L^2,t_j)} \lesssim B + \|u|^{p-1}\nabla u\|_{S'(L^2,t_j)} \lesssim B + \|u|^{p-1}\nabla u\|_{\mathcal{B}^\infty_{L^p_xL^{2(d+1)}}} \lesssim B + \|u|^{p-1}\nabla u\|_{\mathcal{B}^\infty_{L^p_xL^{2(d+1)}}} \lesssim B + \delta^{p-1}\|\nabla u\|_{S(L^2,t_j)}.
\]
The pairs \( \left( \frac{d}{2s}, \frac{2d^2p}{d^2(p-1)+16} \right) \) and \( \left( \frac{d}{2s}, \frac{2d^2p}{d^2(p-1)} \right) \) are \( L^2 \)-admissible and the pair \( \left( \frac{d}{2s}, \frac{2d^2(p-1)}{d^2(p-1)+16} \right) \) is \( L^2 \)-dual admissible; we obtain (2.30) applying Hölder’s inequality to (2.29). Similarly, by dropping the gradient, it follows

\[
\|u\|_{S(L^2;I)} \lesssim B + \delta^{p-1} \|u\|_{S(L^2;I)} .
\]

(2.32)

Combining (2.31) and (2.32) and using the fact that \( \delta \) can be chosen appropriately small, gives that \( \|(1 + |\nabla|)u\|_{S(L^2;I)} \lesssim 2B \). Summing over the \( M \) intervals, leads to

\[
\|(1 + |\nabla|)u\|_{S(L^2)} \lesssim BM^\frac{np}{2}. \]

Define the wave operator

\[
\phi_+ = u_0 + i \int_0^{+\infty} e^{-i\tau\Delta} F(u(\tau)) d\tau,
\]

note that \( \phi_+ \in H^1 \), thus, Strichartz estimates and the hypothesis lead to

\[
\|\phi_+\|_{H^1} \lesssim \|u_0\|_{H^1} + \|u\|_{S'(L^2)} \lesssim \|u_0\|_{H^1} + \|u\|_{S'(L^2)} \lesssim B + BM^\frac{p(d+2)+2p}{2} .
\]

(2.33)

Additionally,

\[
u(t) - e^{it\Delta} \phi^+ = -i \int_t^{+\infty} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau.
\]

(2.34)

Therefore, estimating the \( L^2 \) norm of (2.34), Strichartz estimates and Hölder’s inequality give

\[
\|u(t) - e^{it\Delta} \phi^+\|_{L^2} \lesssim \left\| \int_t^{+\infty} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \right\|_{S(L^2)} \lesssim \|F(u(\tau))\|_{S'(L^2;[t,\infty))} \lesssim \|u\|_{S'(L^2;[t,\infty))} \lesssim \|u\|_{S'(L^2;[t,\infty))} \lesssim \|u\|_{S'(L^2;[t,\infty))} .
\]

(2.35)

and similarly, estimating the \( \dot{H}^1 \) norm of (2.34), we obtain

\[
\|\nabla (u(t) - e^{it\Delta} \phi^+)\|_{L^2} \lesssim \left\| \int_t^{+\infty} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \right\|_{S(L^2)} \lesssim \|F(u(\tau))\|_{S'(L^2;[t,\infty))} \lesssim \|u\|_{S'(L^2;[t,\infty))} \lesssim \|u\|_{S'(L^2;[t,\infty))} .
\]

(2.36)

Using the Leibniz rule (Lemma 2.2) to estimate (2.35) and (2.36), yields

\[
\|u\|_{S'(L^2;[t,\infty))} \lesssim B + \delta^{p-1} \|u\|_{S'(L^2;[t,\infty))} .
\]

By (2.33) the term \( \|u\|_{S'(L^2;[t,\infty))} \) is bounded. Then as \( t \to \infty \) the term

\[
\|u\|_{S'(L^2;[t,\infty))} \to 0,
\]

thus, summing over all dyadic \( N \), (1.2) is obtained. \( \square \)
2.4. Properties of the Ground State. Weinstein [Wei82] proved the Gagliardo-Nirenberg inequality
\[ \|u\|^p \leq C_{GN}\|\nabla u\|^q \] (2.37)
with the sharp constant
\[ C_{GN} = \frac{p+1}{2\|Q\|^{p+1}} \] (2.38)
where \( Q \) is as in (1.4).

This inequality (2.37) is optimized by \( Q \), i.e., \( \|Q\|^p \leq \frac{p+1}{2}\|\nabla Q\|^q \). Multiplying (1.4) by \( Q \) and integrating, gives \( \|Q\|^p \leq \alpha^2\|\nabla Q\|^2 + \beta\|Q\|^2 \), thus, Pohozaev identities yield \( \|\nabla Q\| = \|Q\| \), and, \( \|Q\|^p = \frac{p+1}{2}\|Q\|^2 \).

In addition,
\[ \|u_Q\|^2 = \alpha^{-d}\|Q\|^2, \quad \|\nabla u_Q\|^2 = \alpha^{-2d}\|\nabla Q\|^2, \quad \text{and} \quad \|u_Q\|^p = \alpha^{-d}\|Q\|^p \]
therefore, the scale invariant quantity becomes
\[ \|u_Q\|^2 \|\nabla u_Q\|^2 = \alpha^{-2d}\|Q\|^2 \] (2.40)
and the mass-energy scale invariant quantity is
\[ M[u_Q]^{-s}E[u_Q] = \left( \alpha^{-d}\|Q\|^2 \right)^{-s}\left( \frac{\alpha^{-2d}}{2}\|\nabla Q\|^2 - \frac{\alpha^{-d}}{p+1}\|Q\|^p \right)^s \] (2.41)
\[ = \frac{\alpha^{-d}}{2^s}\left( \frac{p-1}{2} \right)^s \|Q\|^2 \] (2.42)
\[ = \left( \frac{s}{d} \right)^s \left( \|u_Q\|^2 \|\nabla u_Q\|^2 \right)^2 \] (2.43)
since the energy definition yields (2.41), Pohozaev identities (2.39) and (2.40) imply (2.42) and (2.43).

Notice that
\[ M[u]^{-s}E[u] = \left( \|u\|^2 \right)^{-s}\left( \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\|u\|^p \right)^s \] (2.44)
\[ \geq \left( \|u\|^2 \|\nabla u\|^2 \right)^2\left( \frac{1}{2} - \frac{C_{GN}}{p+1}\left( \|u\|^s \|\nabla u\|^p \right) \right)^s \]
\[ \geq \frac{1}{2^s}\left( \|u\|^2 \|\nabla u\|^2 \right)^2\left( 1 - \alpha^{-2}\left( \frac{\|u\|^2 \|\nabla u\|^2}{\|u_Q\|^2 \|\nabla u_Q\|^2} \right) \right)^s \]
therefore,
\[ \frac{d}{2s}\left[ G_u(t) \right]^2 \left( 1 - \frac{[G_u(t)]^{p-1}}{\alpha^2} \right) \leq \left( \mathcal{M}[E[u]] \right)^{\frac{1}{s}} \leq \frac{d}{2s}\left[ G_u(t) \right]^2. \] (2.44)

Summarizing, the upper bound in (2.44) is obtained by bounding the energy \( E[u] \) above by the kinetic energy part, and the lower bound is achieved using the definition of energy and the sharp Gagliardo-Nirenberg inequality (2.37) to bound the potential term.
2.5. Properties of the Momentum. Let $u$ be a solution of $NLS_p(\mathbb{R}^d)$ and assume that $P[u] \neq 0$. Let $\xi_0 \in \mathbb{R}^d$ to be chosen later and $w$ be the Galilean transformation of $u$

$$w(x, t) = e^{ix\xi_0 e^{-it|\xi_0|^2}} u(x - 2\xi_0 t, t).$$

Then

$$\nabla w(x, t) = i\xi_0 \cdot e^{ix\xi_0 e^{-it|\xi_0|^2}} u(x - 2\xi_0 t, t) + e^{ix\xi_0 e^{-it|\xi_0|^2}} \nabla u(x - 2\xi_0 t, t),$$

therefore,

$$\|\nabla w\|_{L^2}^2 = |\xi_0|^2 M[u] + 2\xi_0 \cdot P[u] + \|\nabla u\|_{L^2}^2.$$ (2.45)

Observe that $M[w] = M[u], P[w] = \xi_0 M[u] + P[u], and

$$E[w] = \frac{1}{2} |\xi_0|^2 M[u] + \xi_0 \cdot P[u] + E[u].$$ (2.46)

Note that the value $\xi_0 = -\frac{P[u]}{M[u]}$ minimizes the expressions (2.45) and (2.46), with $P[u] = 0,$ that is,

$$E[w] = E[u] - \frac{(P[u])^2}{2M[u]} \quad \text{and} \quad \|\nabla w\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \frac{(P[u])^2}{M[u]}.$$ (2.47)

Thus, the conditions (1.8), (1.9) and (1.10) in Theorem A become

$$(\mathcal{M}\mathcal{E}[u])^{\frac{1}{2}} = (\mathcal{M}\mathcal{E}[u]) - \frac{d}{2s} (P[u])^{\frac{2}{s}} < 1, \quad [G_w(0)]^{\frac{1}{2}} = [G_u(0)]^{\frac{1}{2}} - P^{\frac{1}{2}}[u] < 1$$

and $[G_w(0)]^{\frac{1}{2}} > 1$, hence we restate Theorem A as

Theorem A* (Zero momentum). Let $u_0 \in H^1(\mathbb{R}^d), d \geq 1,$ with $P[u_0]$ and $u(t)$ be the corresponding solution to (1.1) in $H^1(\mathbb{R}^d)$ with maximal time interval of existence $(T_*, T^*)$ and $s \in (0, 1).$ Assume $\mathcal{M}\mathcal{E}[u] < 1.$

I. If $G_u(0) < 1,$ then

(a) $G_u(t) < 1$ for all $t \in \mathbb{R},$ thus, the solution is global in time $(T_* = -\infty, T^* = +\infty)$

and

(b) $u$ scatters in $H^1(\mathbb{R}^d),$ this means, there exists $\phi_{\pm} \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} \phi_{\pm}\|_{H^1(\mathbb{R}^d)} = 0.$$ (2.48)

II. If $G_u(0) > 1,$ then $G_u(t) > 1$ for all $t \in (T_*, T^*)$ and if

(a) $u_0$ is radial (for $d \geq 3$ and in $d = 2, 3 < p \leq 5$) or $u_0$ is of finite variance, i.e., $|x|u_0 \in L^2(\mathbb{R}^d),$ then the solution blows up in finite time $(T^* < +\infty, T_* > -\infty).$

(b) $u_0$ non-radial and of infinite variance, then either the solution blows up in finite time $(T^* < +\infty, T_* > -\infty)$ or there exists a sequence of times $t_n \to +\infty$ (or $t_n \to -\infty$) such that $\|\nabla u(t_n)\|_{L^2(\mathbb{R}^d)} \to \infty.$

Thus, in the rest of the paper, we will assume that $P[u] = 0$ and prove only Theorem A*.

To illustrate the scenarios for global behavior of solutions given by Theorem A* we provide Figure II. We plot $y = (\mathcal{M}\mathcal{E}[u])^{\frac{1}{2}}$ vs. $x = [G_u(t)]^{\frac{1}{2}}$ using the (2.44) restriction in Figure 1.
2.6. Energy bounds and Existence of the Wave Operator.

**Lemma 2.16** (Comparison of Energy and Gradient). Let \( u_0 \in H^1(\mathbb{R}^d) \) such that \( \mathcal{G}_u(0) < 1 \) and \( \mathcal{M}\mathcal{E}[u] < 1 \). Then

\[
\frac{s}{d} \| \nabla u(t) \|^2_{L^2} \leq E[u] \leq \frac{1}{2} \| \nabla u(t) \|^2_{L^2}.
\]

**Proof.** The energy definition combined with \( \mathcal{G}(0) < 1 \) (and thus, by Theorem A* part I (a) \( \mathcal{G}_u(t) < 1 \)), the Gagliardo-Nirenberg inequality \((2.37)\) and Pohozaev identities \((2.39)\) and
\[ \theta = \text{Lower bound on the convexity of the variance} \]

which gives the middle inequality in (2.49).

**Proof.**

The first inequality in (2.47) yields then the relation (2.40) and recalling that \( \alpha \)

Next, considering the right side of (2.49), applying Gagliardo-Nirenberg inequality (2.37),

(2.40) yield

\[ \frac{\alpha^2 - 1}{2\alpha^2} \| \nabla u(t) \|_{L^2}^2 = \frac{s}{d} \| \nabla u(t) \|_{L^2}^2, \quad (2.48) \]

where the equality (2.48) is obtained from combining (2.40), the sharp constant (2.38) and \( \alpha = \sqrt{\frac{d(p-1)}{2}} \). The second inequality of (2.47) follows directly from the definition of energy. \( \square \)

**Lemma 2.17** (Lower bound on the convexity of the variance). Let \( u_0 \in H^1(\mathbb{R}^d) \) satisfy \( G_u(0) < 1 \) and \( \mathcal{M}[u] < 1 \). Then \( G_u(t) \leq \omega := \sqrt{\mathcal{M}[u]} \) for all \( t \), and

\[ 16(1 - \omega^{p-1})E[u] \leq 8(1 - \omega^{p-1})\| \nabla u \|_{L^2}^2 \leq 8\| \nabla u \|_{L^2}^2 - \frac{4d(p-1)}{p+1} \| u \|_{L^{p+1}}^{p+1}. \quad (2.49) \]

**Proof.** The first inequality in (2.47) yields \( \| \nabla u \|_{L^2}^2 \leq \frac{4}{s}E[u] \), multiplying it by \( M^\theta[u] \), where \( \theta = \frac{1-s}{s} \), normalizing by \( \| \nabla u_Q \|_{L^2}^2 \| u_Q \|_{L^2}^{2s} \) and using the fact that \( \| \nabla u_Q \|_{L^2}^2 \leq \frac{4}{s}E[u_Q] \) leads to

\[ [G_u(t)]^2 \leq \mathcal{M}[u], \quad \text{i.e.,} \quad G_u(t) \leq \omega. \]

Next, considering the right side of (2.49), applying Gagliardo-Nirenberg inequality (2.37), then the relation (2.40) and recalling that \( \alpha = \frac{\sqrt{d(p-1)}}{2} \), we obtain

\[ 8\| \nabla u \|_{L^2}^2 - \frac{4d(p-1)}{p+1} \| u \|_{L^{p+1}}^{p+1} \geq \| \nabla u \|_{L^2}^2 \left( 8 \frac{2d(p-1)}{\alpha^2} [G_u(t)]^{p-1} \right) \geq 8\| \nabla u \|_{L^2}^2 (1 - \omega^{p-1}), \quad (2.50) \]

which gives the middle inequality in (2.49).

Finally, combining (2.50) with the second inequality in (2.47), completes the proof. \( \square \)

**Proposition 2.18** (Existence of Wave Operators). Let \( \psi \in H^1(\mathbb{R}^d) \).

1. Then there exists \( v_+ \in H^1 \) such that for some \( -\infty < T_* < +\infty \) it produces a solution \( v(t) \) to NLS\(_p(\mathbb{R}^d) \) on time interval \([T_*, \infty)\) such that

\[ \| v(t) - e^{it\Delta} \psi \|_{H^1} \to 0 \quad \text{as} \quad t \to +\infty \quad (2.51) \]

Similarly, there exists \( v_- \in H^1 \) such that for some \( -\infty < T_* < +\infty \) it produces a solution \( v(t) \) to NLS\(_p(\mathbb{R}^d) \) on time interval \(( -\infty, T_*) \) such that

\[ \| v(t) - e^{-it\Delta} \psi \|_{H^1} \to 0 \quad \text{as} \quad t \to +\infty \quad (2.52) \]
II. Suppose that for some $0 < \sigma \leq \left(\frac{2}{d}\right)^{\frac{1}{2}} < 1$

\[\|\psi\|_{L^2}^{2(1-s)}\|\nabla \psi\|_{L^2}^{2s} < \sigma^2 \left(\frac{d}{s}\right)^s M[u_0]^{1-s} E[u_0]^s. \tag{2.53}\]

Then there exists $v_0 \in H^1$ such that $v(t)$ solving $\text{NLS}_p(\mathbb{R}^d)$ with initial data $v_0$ is global in $H^1$ with

\[M[v] = \|\psi\|_{L^2}^2, \quad E[v] = \frac{1}{2}\|\nabla \psi\|_{L^2}^2, \quad \mathcal{G}_v(t) \leq \sigma < 1 \tag{2.54}\]

and

\[\|v(t) - e^{it\Delta} \psi\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{2.55}\]

Moreover, if $\|e^{it\Delta} \psi\|_{\dot{H}^s} \leq \delta_{sd}$, then $\|v_0\|_{\dot{H}^s} \leq 2\|\psi\|_{\dot{H}^s}$ and $\|v\|_{\dot{H}^s} \leq 2\|e^{it\Delta} \psi\|_{\dot{H}^s}$.

**Proof.** I. This is essentially Theorem 2 part (a) of [Str81a] adapted to the case $0 < s < 1$ (see his Remark (36) and [Str81b, Theorem 17]).

II. For this part, we consider the integral equation

\[v(t) = e^{it\Delta} \psi - i \int_t^\infty e^{i(t-t')\Delta} (|v|^{p-1}v) dt'. \tag{2.56}\]

We want to find a solution to (2.56) which exists for all $t$. Note that for $T > 0$ from the small data theory (Proposition 2.13) there exists $\delta_{sd} > 0$ such that $\|e^{it\Delta} \psi\|_{\dot{H}^s} \leq \delta_{sd}$. Thus, repeating the argument of Proposition 2.13, we first show that we can solve the equation (2.56) in $\dot{H}^s$ for $t \geq T$ with $T$ large. So this solution will estimate $\|\nabla v\|_{S(L^2;[T,\infty))}$, which will also show that $v$ is in $H^1$.

Observe that for any $v \in H^1$

\[\|\nabla |v|^{p-1}v\|_{S'(L^2)} \lesssim \|\nabla |v|^{p-1}v\|_{L^\infty} \frac{2d^2(p-1)}{2d^2(p-1) + 16} \lesssim \|v\|_{L^2}^{p-1} \|\nabla v\|_{L^2} \frac{2d^2(p-1) + 16}{d^2(p-1)} \lesssim \|v\|_{\dot{H}^s}^{p-1} \|\nabla v\|_{S(L^2)}. \tag{2.57}\]

Note that the pairs $\left(\frac{d}{2}, \frac{d^2(p-1)}{2(d+4)}\right)$ and $\left(\frac{d^2}{4}, \frac{2d^2(p-1)}{d^2(p-1) + 16}\right)$ are $L^2$-admissible and the pair $\left(\frac{d}{2}, \frac{d^2(p-1)}{2(d+4)}\right)$ is $L^2$-dual admissible. Thus, the Hölder’s inequality yields (2.57). Now, the Strichartz (2.3) and Kato Strichartz (2.5) estimates imply

\[\|\nabla v\|_{S([T,\infty),L^2)} \lesssim c_1 \|\nabla \psi\|_{S([T,\infty),L^2)} + c \|\nabla (|v|^{p}v)\|_{S'([T,\infty),L^2)} \lesssim c_1 \|\psi\|_{\dot{H}^s} + c_3 \|v\|_{\dot{H}^s} \|\nabla v\|_{S([T,\infty),L^2)}.\]

Taking $T$ large enough, so that $c_3 \|v\|_{S([T,\infty),\dot{H}^s)} \leq \frac{1}{2}$, we obtain

\[\|\nabla v\|_{S([T,\infty),L^2)} \leq 2c_1 \|\psi\|_{\dot{H}^1}.\]

It now follows

\[\|\nabla (v - e^{it\Delta} \psi)\|_{S([T,\infty),L^2)} \leq \|\nabla (|v|^{p-1}v)\|_{S'([T,\infty),L^2)} \leq \|\nabla v\|_{S([T,\infty),L^2)} \|v\|_{S([T,\infty),\dot{H}^s)} \|\nabla v\|_{S([T,\infty),\dot{H}^s)} \leq c \|\psi\|_{\dot{H}^1},\]

hence, $\|\nabla (v - e^{it\Delta} \psi)\|_{S(L^2([T,\infty))))} \to 0$ as $T \to \infty$. 

On the other hand, Proposition $2.15$ ($H^1$ Scattering) implies $v(t) \to e^{it\Delta} \psi$ in $H^1$ as $t \to \infty$, and the decay estimate together with the embedding and $H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for $q \leq \frac{2d}{d-2}$ when $3 \leq d$, $q < \infty$ when $d = 2$ and $q \leq \infty$ when $d = 1$ imply
\[ \|e^{it\Delta} \psi\|_{L^q_{t+1}} \leq |t|^{-\frac{(p-1)d}{2(p+1)}} \|\psi\|_{H^1}, \]
thus, $\|e^{it\Delta} \psi\|_{L^q_{t+1}} \to 0$ as $t \to \infty$. Since $\|\nabla e^{it\Delta} \psi\|_{L^2} = \|\nabla \psi\|_{L^2}$, it follows
\[ E[v] = \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1}, \]
and
\[ M[v] = \lim_{t \to \infty} \|v(t)\|_{L^2}^2 = \lim_{t \to \infty} \|e^{it\Delta} \psi\|_{L^2}^2 = \|\psi\|_{L^2}^2. \]
From the hypothesis (2.53), we obtain
\[ M[v]^{1-s} E[v] = \frac{1}{2} \|\nabla \psi\|_{L^2}^{2(1-s)} \|\nabla \psi\|_{L^2}^{2s} < \sigma^2 \left( \frac{d}{2s} \right)^s M[u_\sigma]^{1-s} E[u_\sigma] \]
and thus, $M \mathcal{E}[v] < 1$, since $\sigma^2 < \left( \frac{d}{2s} \right)^s$. Furthermore,
\[ \lim_{t \to \infty} \|v(t)\|_{L^2}^{2(1-s)} \|\nabla v(t)\|_{L^2}^{2s} = \lim_{t \to \infty} \|e^{it\Delta} \psi\|_{L^2}^{2(1-s)} \|\nabla (e^{it\Delta} \psi)\|_{L^2}^{2s} = \|\psi\|_{L^2}^{2(1-s)} \|\nabla \psi\|_{L^2}^{2s} < \sigma^2 \left( \frac{d}{s} \right)^s M[u_\sigma]^{1-s} E[u_\sigma] \]
where, the inequality is due to (2.53) and the last equality is obtained using (2.43). Hence,
\[ \lim_{t \to \infty} \mathcal{G}_v(t) \leq \sigma < 1. \]
We can take $T > 0$ large so that $\mathcal{G}_v(T) \leq 1$. Then applying Theorem A* part I (a) (global existence of solutions with $M \mathcal{E}[v] < 1$ and $\mathcal{G}_v(t) < 1$), we evolve $v$ from time $T$ back to time $0$ (we automatically get $\mathcal{G}_v(t) \leq 1$ for all $t \in [0, +\infty)$). Thus, we obtain $v$ with initial data $v_0 \in H^1$ and properties (2.54) and (2.55) as desired. \qed

3. Scattering via Concentration Compactness

Theorem 2.1 and Corollary 2.5 of Holmer-Roudenko [HR07] proved the general case for the mass-supercritical and energy-subcritical NLS equations with $H^1$ initial data, thus, establishing Theorem A* I(a) and II(a) for finite variance data. In addition, [CCG11] included the proof of the blow up in finite time when $d = 2$ and $p = 5$ for the radial initial data (i.e., Theorem A* part II(a)), since it was not included in [HR07], the authors considered $p < 5$.

The goal of this section is to prove scattering in $H^1(\mathbb{R}^d)$ of global solutions of NLS$_p(\mathbb{R}^d)$ from Theorem A* part I (a).

**Definition 3.1.** Suppose $u_0 \in H^1(\mathbb{R}^d)$ and let $u$ be the corresponding $H^1(\mathbb{R}^d)$ solution to (1.1) on $[0, T^*)$, the maximal (forward in time) interval of existence. We say that $SC(u_0)$ holds if $T^* = +\infty$ and $\|u\|_{S(H^1)} < \infty$. 


3.1. Outline of Scattering via Concentration Compactness. Notice that $H^1$-scattering of $u(t) = \text{NLS}(t)u_0$ is obtained when $SC(u_0)$ holds by Proposition 2.15. Therefore, to establish Theorem A* part I (b), it will be enough to verify that the global-in-time $\dot{H}^s$ Besov-Strichartz norm is finite, i.e., $\|u\|_{\dot{S}^s_{(\dot{H}^s)^+}} < \infty$, since the hypotheses provides an a priori bound for $\|\nabla u(t)\|_{L^2}$ (by Theorem A* part I a), thus, the maximal forward time of existence is $T = +\infty$. In other words, it remains to show

**Proposition 3.2.** If $G_u(0) < 1$ and $\mathcal{M}\mathcal{E}[u] < 1$, then $SC(u_0)$ holds.

The technique to achieve the scattering property above (Proposition 3.2) is the induction argument on the mass-energy threshold introduced in [KM08] and based on [HR08]. We describe it in steps 1, 2, 3.

**Step 1:** Small Data. The equivalence of energy with the gradient (Lemma 2.16) yields

$$
\|u_0\|_{L^2}^{p+1} \leq (\|u_0\|_{L^2}^{1-s}\|\nabla u_0\|_{L^2}^s)^{\frac{2s}{p+1}} \leq \left(\left(\frac{d}{s}\right)^s M[u][1-s E[u]^s\right]^{\frac{p+1}{4}}.
$$

If $G_u(0) < 1$ and $M[u][1-s E[u]^s < (\frac{d}{s})^s s^4$, then using the above inequality one obtains $\|u_0\|_{H^s} \leq \delta_{sd}$ and by Strichartz estimates $\|e^{it\Delta}u_0\|_{\dot{S}^s_{(\dot{H}^s)^+}} \leq c\delta_{sd}$. Hence, the small data (Proposition 2.13) yields $SC(u_0)$ property.

Observe that Step 1 gives the basis for induction: Assume $G_u(0) < 1$. Then for small $\delta > 0$ such that $M[u_0][1-s E[u]^s < \delta$, we have that $SC(u_0)$ holds.

Let $(ME)_c$ be the supremum of all such $\delta$ for which $SC(u_0)$ holds, namely,

$$(ME)_c = \sup\{\delta \mid u_0 \in H^1(\mathbb{R}^d) \text{ with the property: } G_u(0) < 1 \text{ and } M[u][1-s E[u]^s < \delta \Rightarrow SC(u_0) \text{ holds}\}.$$

Thus, the goal is to show that $(ME)_c = M[u_q][1-s E[u_q]^s$.

**Remark 3.3.** In the definition of $(ME)_c$, it should be considered $G_u(0) \leq 1$ instead of the strict inequality $G_u(0) < 1$. However, $G_u(0) = 1$ only when $\mathcal{M}\mathcal{E}[u] = 1$ (see Figure 1 point D). In other words, $u_0 = u_q(x)$ is a soliton solution to (1.1) and does not scatter, thus, it suffices to consider the strict inequality $G_u(0) < 1$.

**Step 2:** Induction on the scattering threshold and construction of the “critical” solution. Assume that $(ME)_c < M[u_q][1-s E[u_q]^s$. This means that, there exists a sequence of initial data $\{u_{n,0}\}$ in $H^1(\mathbb{R}^d)$ which will approach the threshold $(ME)_c$ from above and produce solutions which do not scatter, i.e., there exists a sequence $u_{n,0} \in H^1(\mathbb{R}^d)$ with

\begin{align*}
G_{u_n}(0) &< 1 \text{ and } M[u_{n,0}][1-s E[u_{n,0}^s] \gtrsim (ME)_c \text{ as } n \to \infty \quad (3.1) \\
\text{and } \|u\|_{\dot{S}^s_{(\dot{H}^s)^+}} &= +\infty,
\end{align*}

i.e., $SC(u_{n,0})$ does not hold (this is possible by definition of supremum of $(ME)_c$).
Using a nonlinear profile decomposition on the sequence \( \{u_{n,0}\} \) will allow us to construct a “critical” solution of NLS\(_p\)(\( \mathbb{R}^d \)), denoted by \( u_c(t) \), that will lie exactly at the threshold \((ME)_c\) and will not scatter, see Existence of the Critical solution (Proposition 3.11).

**Step 3:** Localization properties of the critical solution.

The critical solution \( u_c(t) \) will have the property that \( K = \{u_c(t)|t \in [0, +\infty)\} \) is precompact in \( H^1(\mathbb{R}^d) \) (Lemma 3.12). Hence, its localization implies that for given \( \epsilon > 0 \), there exists an \( R > 0 \) such that \( \|\nabla u(x,t)\|^2_{L^2(|x-x(t)|>R)} \leq \epsilon \) uniformly in \( t \) (Corollary 3.13); this combined with the zero momentum will give control on the growth of \( x(t) \) (Lemma 3.14).

On the other hand, the rigidity theorem (Theorem 3.15) implies that such compact in \( H^1 \) solutions with the control on \( x(t) \), can only be zero solutions, which contradicts the fact that \( u_c \) does not scatter. As a consequence, such \( u_c \) does not exist and the assumption that \((ME)_c < M[u_0]E[u_0]\) is not valid. This finishes the proof of scattering in Theorem A*, Part 1(b).

In the rest of this section we proceed with the linear and nonlinear profile decomposition and the proof of the existence and properties of the critical solution described in Step 2 and Step 3.

### 3.2. Profile decomposition. This section contains the profile decomposition for linear and nonlinear flows for NLS\(_p^+\)(\( \mathbb{R}^d \)). The important point to make here is that these are general profile decompositions for bounded sequences on \( H^1 \).

**Proposition 3.4** (Linear Profile decomposition). Let \( \phi_n(x) \) be a uniformly bounded sequence in \( H^1(\mathbb{R}^d) \). Then for each \( M \in \mathbb{N} \) there exists a subsequence of \( \phi_n \) (also denoted \( \phi_n \)), such that, for each \( 1 \leq j \leq M \), there exist, fixed in \( n \), a profile \( \psi^j \) in \( H^1(\mathbb{R}^d) \), a sequence \( t_n^j \) of time shifts, a sequence \( x_n^j \) of space shifts and a sequence \( W_n^M(x) \) of remainders,\(^{11}\) in \( H^1(\mathbb{R}^d) \), such that

\[
\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j} \Delta \psi^j(x - x_n^j) + W_n^M(x)
\]

with the properties:

- **Pairwise divergence for the time and space sequences.** For \( 1 \leq k \neq j \leq M \),
  \[
  \lim_{n \to \infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = +\infty.
  \]
  \[
  (3.2)
  \]

- **Asymptotic smallness for the remainder sequence**
  \[
  \lim_{M \to \infty} \left( \lim_{n \to \infty} \|e^{it_n^j} W_n^M\|_{S(H^s)} \right) = 0.
  \]
  \[
  (3.3)
  \]

- **Asymptotic Pythagorean expansion.** For fixed \( M \in \mathbb{N} \) and any \( 0 \leq s \leq 1 \), we have
  \[
  \|\phi_n\|_{H^s}^2 = \sum_{j=1}^{M} \|\psi^j\|_{H^s}^2 + \|W_n^M\|_{H^s}^2 + o_n(1).
  \]
  \[
  (3.4)
  \]

\(^{11}\)Here, in Proposition 3.4 and Proposition 3.6 \( W_n^M(x) \) and \( \tilde{W}_n^M(x) \) represent the remainders for the linear and nonlinear decompositions, respectively.
Proof. Let $\phi_n$ be uniformly bounded in $H^1$, and $c_1 > 0$ such that $\|\phi_n\|_{H^1} \leq c_1$.

For each dyadic $N \in 2^\mathbb{N}$, given $(q, r)$ an $\dot{H}^s$ admissible pair, pick $\theta = \frac{4d(d+2r) - 2d^2 r}{r^2 - 2d^2 r + 2d + 2s - 4}$, so $0 < \theta < 1$.

Let $r_1 = \frac{r(d-2)+2d}{2(d-2)}$, and $q_1 = \frac{8(d-2)+4dr}{r(d-2)(d-2)-2dr(d+2s-4)}$, so $(q_1, r_1)$ is $\dot{H}^s$ admissible pair, for $0 < s < 1$ and $d \geq 2$. Interpolation and Strichartz estimates \cite{2,3} yield

$$\|e^{it\Delta} W_n^M\|_{L_t^q L_x^r} \leq \|e^{it\Delta} W_n^M\|_{L_t^q L_x^r}^{1-\theta} \|e^{it\Delta} W_n^M\|_{L_t^q L_x^r}^\theta \leq c \|W_n^M\|_{H^s}^{1-\theta} \|e^{it\Delta} W_n^M\|_{L_t^q L_x^r}^\theta,$$

(3.5)

The goal is to write the profile $\phi_n$ as $\sum_{j=1}^M e^{-it\Delta} \psi^j(x-x_n^j) + W_n^M(x)$ with $\|W_n^M(x)\|_{H^s} \leq c_1$, for some constant $c_1$. By (3.5), it suffices to show

$$\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \sup_{\theta} \left\| e^{it\Delta} W_n^M \right\|_{L_t^q L_x^r}^{\theta} \right] = 0.$$

We have $d \geq 2s$, since we are considering

$$\begin{cases}
  (i) & 0 \leq s \leq 1 \quad \text{in} \quad d \geq 3 \\
  (ii) & 0 < s < 1 \quad \text{in} \quad d = 2 \\
  (iii) & 0 < s < \frac{1}{2} \quad \text{in} \quad d = 1.
\end{cases}$$

(3.6)

Construction of $\psi^1_n$:
Let $A_1 = \limsup_{n \to +\infty} \left\| e^{it\Delta} \phi_n \right\|_{L_t^\infty L_x^{2d/2s}}$. If $A_1 = 0$, taking $\psi^j = 0$ for all $j$ finishes the construction.

Suppose that $A_1 > 0$, and let $c_1 = \limsup_{n \to +\infty} \|\phi_n\|_{H^1} < \infty$. Passing to a subsequence $\phi_n$, we show that there exist sequences $t_n^1$ and $x_n^1$ and a function $\psi^1 \in H^1$, such that

$$e^{it\Delta} \phi_n(\cdot + x_n^1) \to \psi^1 \quad \text{in} \quad H^1,$$

and a constant $K > 0$, independent of all parameters, with

$$K c_1^{\frac{d+2s-4s^2}{2s}} \|\psi^1\|_{H^s} \geq A_1^{\frac{d+4s-4s^2}{2s}}.$$

(3.7)

Note that $d + 2s - 4s^2 = d + 2s(1 - 2s) > 0$ by (3.6).

Let $\chi_r$ be a radial Schwartz function such that $\text{supp } \chi_r \subset \left[\frac{1}{2r}, 2r\right]$ and $\hat{\chi}_r(\xi) = 1$ for $\frac{1}{r} \leq |\xi| \leq r$. Note that $|1 - \hat{\chi}_r| \leq 1$ and $\dot{H}^s \hookrightarrow L^{\frac{2d}{d-s}}$ in $\mathbb{R}^d$ with $2s < d$, then

$$\left\|e^{it\Delta} \phi_n - \chi_r * e^{it\Delta} \phi_n\right\|_{L_t^\infty L_x^{2d/2s}}^2 \leq \left|\int |\xi| (1 - \hat{\chi}_r(\xi))^2 |\hat{\phi}_n(\xi)|^2 d\xi\right|$$

$$\leq \int_{|\xi| \leq \frac{1}{2r}} |\xi||\hat{\phi}_n|^2 d\xi + \int_{|\xi| \geq r} |\xi||\hat{\phi}_n(\xi)|^2 d\xi \leq \frac{\|\phi_n\|_{L_x^2}^2 + \|\phi_n\|_{H^s}^2}{r} \leq c_1^2 \frac{1}{r}.$$
Take \( r = \frac{4\epsilon^2}{A_1^2} \), then \( A_1 = \frac{2\epsilon}{\sqrt{r}} \). Using the definition of \( A_1 \), triangle inequality and the previous calculation, for large \( n \) we have

\[
\frac{A_1}{2} \leq \| \chi_r * e^{it\Delta} \phi_n \|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}. \tag{3.8}
\]

Therefore, interpolation implies

\[
\| \chi_r * e^{it\Delta} \phi_n \|_{L_t^\infty L_x^{\frac{2d}{d-2s}}} \leq \| \chi_r * e^{it\Delta} \phi_n \|_{L_t^\infty L_x^{\frac{d-2s}{2}}} \| \chi_r * e^{it\Delta} \phi_n \|_{L_t^\infty L_x^{2s}} \leq \| \phi_n \|_{L_t^\infty L_x^{d-2s}} \| \chi_r * e^{it\Delta} \phi_n \|_{L_t^\infty L_x^{2s}}, \tag{3.9}
\]

where the second inequality follows from the fact that \(|\vec{\chi}_r| \leq 1\) and \( L^2 \) isometry property of the linear Schrödinger operator. Using the definition of \( c_1 \), combining (3.8) and (3.9), we get

\[
\left( \frac{A_1}{2c_1 \epsilon} \right)^{\frac{d}{2}} \leq \| \chi_r * e^{it\Delta} \phi_n \|_{L_t^\infty L_x^{\infty}}. \]

Thus, there exists a sequence of \( (x_1^n, t_1^n) \in \mathbb{R}^d \times \mathbb{R} \) satisfying

\[
\left( \frac{A_1}{2c_1 \epsilon} \right)^{\frac{d}{2}} \leq | \chi_r * e^{it\Delta} \phi_n (x_1^n) |. \]

Since \( e^{it\Delta} \) is an \( H^1 \) isometry and translation invariant, it follows that \( \{ e^{it\Delta} \phi_n (\cdot + x_1^n) \} \) is uniformly bounded in \( H^1 \) (with the same constant as \( \phi_n \)'s) and along a subsequence \( \{ e^{it\Delta} \phi_n (\cdot + x_1^n) \} \rightarrow \psi^1 \) with \( \| \psi^1 \|_{H^1} \leq c_1 \).

Observe that

\[
\left( \frac{A_1}{2c_1 \epsilon} \right)^{\frac{d}{2}} \leq \left| \int_{\mathbb{R}^2} \chi_r (x_1^n - y) \psi^1 (y) dy \right| \leq \| \chi_r \|_{H^{-s}} \| \psi^1 \|_{H^s} \leq r^{1-s} \| \psi^1 \|_{H^s},
\]

since \( \| \chi_r \|_{H^{-s}} \lesssim r^{1-s} \) (by converting to radial coordinates) and the Hölder’s inequality produces (3.7) with \( K = \frac{2^{d+4s-4\epsilon^2}}{2s} \).

Define \( W_{n}^1(x) = \phi_n(x) - e^{-it\Delta} \psi^1 (x - x_1^n) \). Note that \( e^{it\Delta} \phi_n (\cdot + x_1^n) \rightarrow \psi^1 \) in \( H^1 \), therefore, for any \( 0 \leq s \leq 1 \), we have

\[
\langle \phi_n, e^{-it\Delta} \psi^1 (\cdot - x_1^n) \rangle_{H^s} = \langle e^{it\Delta} \phi_n, \psi^1 (\cdot - x_1^n) \rangle_{H^s} \rightarrow \| \psi^1 \|_{H^s}^2,
\]

and since \( \| W_{n}^1 \|_{H^s}^2 = \langle \phi_n - e^{-it\Delta} \psi^1 (\cdot - x_1^n), \phi_n - e^{-it\Delta} \psi^1 (\cdot - x_1^n) \rangle_{H^s} \), we obtain

\[
\lim_{n \to \infty} \| W_{n}^1 \|_{H^s}^2 = \lim_{n \to \infty} \| e^{it\Delta} \phi_n \|_{H^s}^2 \rightarrow \| \psi^1 \|_{H^s}^2.
\]

Taking \( s = 1 \) and \( s = 0 \), yields \( \| W_{n}^1 \|_{H^1} \leq c_1 \).

**Construction of \( \psi^j \) for \( j \geq 2 \)** (Inductively we assume that \( \psi^{j-1} \) is known and construct \( \psi^j \):)

Let \( M \geq 2 \). Suppose that \( \psi^j, x_j^n, t_j^n \) and \( W_{n}^j \) are known for \( j \in \{1, \cdots, M - 1\} \). Consider

\[
A_M = \limsup_n \| e^{it\Delta} W_{n}^{M-1} \|_{L_t^\infty L_x^{\frac{2d}{d-2s}}}.
\]

If \( A_M = 0 \), then taking \( \psi^j = 0 \) for \( j \geq M \) will end the construction.
Assume $A_M > 0$, we apply the previous step to $W_n^{M-1}$, and let $c_M = \limsup_n \|W_n^{M-1}\|_{H^1}$, thus, obtaining sequences (or subsequences) $x_n^M$, $t_n^M$ and a function $\psi^M \in H^1$ such that

$$
e^{it_n^M}W_n^{M-1}(\cdot + x_n^M) \rightarrow \psi^M \quad \text{in} \quad H^1 \quad \text{and} \quad Kc_M\frac{d+2s-4\xi^2}{2x}\|\psi^M\|_{H^s} \geq A_M\frac{d+2s-4\xi^2}{2x}.$$

(3.10)

Define

$$W_n^M(x) = W_n^{M-1}(x) - e^{-it_n^M}M(x - x_n^M).$$

Then (3.2) and (3.3) follow from induction, i.e., we assume (3.3) holds at rank $M - 1$, then expanding

$$\|W_n^M(x)\|_{H^s}^2 = \|e^{it_n^M}M(x + x_n^M) - \psi^M\|_{H^s}^2,$$

the weak convergence yields (3.3) at rank $M$.

In the same fashion, we assume (3.2) is true for $j, k \in \{1, \ldots, M - 1\}$ with $j \neq k$, that is $|t_n^j - t_n^k| + |x_n^j - x_n^k| \rightarrow +\infty$ as $n \rightarrow \infty$. Take $k \in \{1, \ldots, M - 1\}$ and show that

$$|t_n^M - t_n^k| + |x_n^M - x_n^k| \rightarrow +\infty.$$

Passing to a subsequence, assume $t_n^M - t_n^k \rightarrow t^M_k$ and $x_n^M - x_n^k \rightarrow x^M_k$ finite, then as $n \rightarrow \infty$

$$e^{it_n^M}M(x + x_n^M) = e^{i(t_n^M - t_n^k)}M(x_n^j - x_n^j) - \sum_{k=j+1}^{M-1} e^{i(t_n^j - t_n^k)}\psi^M(x_n^j - x_n^k).$$

The orthogonality condition (3.2) implies that the right hand side goes to 0 weakly in $H^1$, while the left side converges weakly to a nonzero $\psi^M$, which is a contradiction. Note that the orthogonality condition (3.2) holds for $k = M$, and since (3.3) holds for all $M$, we have

$$\|\phi_n\|_{H^s}^2 \geq \sum_{j=1}^{M} \|\psi^j\|_{H^s}^2 + \|W_n^M\|_{H^s}^2$$

and $c_M \leq c_1$. Fix $s$. If for all $M$, $A_M > 0$, then (3.10) yields

$$\sum_{M \geq 1} \left(\frac{A_M\frac{d+2s-4\xi^2}{2x}}{Kc_1}\right)^2 \leq \sum_{n \geq 1} \|\psi^M\|_{H^s}^2 \leq \limsup_n \|\phi_n\|_{H^s}^2 < \infty,$$

therefore, $A_M \rightarrow 0$ as $M \rightarrow \infty$, and consequently, $\|e^{it_n^M}M\|_{H^s} \rightarrow 0$ as $n \rightarrow \infty$. Finally, summing over all dyadic $N$, yields (3.3).

**Proposition 3.5** (Energy Pythagorean expansion). Under the hypothesis of Proposition 3.4, we have

$$E[\phi_n] = \sum_{j=1}^{M} E[e^{-it_n^j}\psi^j] + E[W_n^M] + o_n(1).$$

(3.11)
Thus, combining (3.14) and (3.15), we obtain (3.12).

Step 1. Pythagorean expansion of a sum of orthogonal profiles. Fix \( M \geq 1 \). We want to show that the condition (3.2) yields

\[
\|e^{-it_n \Delta} \psi^j (\cdot - x^j_n)\|_{L^{p+1}_{x, t}} = \sum_{j=1}^{M} \|e^{-it_n \Delta} \psi^j\|_{L^{p+1}_{x, t}} + o_n(1).
\]  

By rearranging and reindexing, we can find \( M_0 \leq M \) such that

(a) \( t^j_n \) is bounded in \( n \) whenever \( 1 \leq j \leq M_0 \),

(b) \( |t^j_n| \to \infty \) as \( n \to \infty \) if \( M_0 + 1 \leq j \leq M \).

For the case (a) take a subsequence and assume that for each \( 1 \leq j \leq M_0 \), \( t^j_n \) converges (in \( n \)), then adjust the profiles \( \psi^j \)'s such that \( t^j_n = 0 \). From (3.2) we have \( |x^j_n - x^k_n| \to +\infty \) as \( n \to \infty \), which implies

\[
\left\| \sum_{j=1}^{M_0} \psi^j (\cdot - x^j_n) \right\|_{L^{p+1}_{x, t}} = \sum_{j=1}^{M_0} \|\psi^j\|_{L^{p+1}_{x, t}} + o_n(1).
\]  

For the case (b), i.e., for \( M_0 \leq j \leq M \), \( |t^j_n| \to \infty \) as \( n \to \infty \) and for \( \tilde{\psi} \in \dot{H}^p_{p+1} \cap L^p_{p+1} \), thus, the Sobolev embedding and the \( L^p \) space-time decay estimate yield

\[
\|e^{-it_n \Delta} \psi^k\|_{L^{p+1}_{x, t}} \leq C\|\psi^k - \tilde{\psi}\|_{\dot{H}^p_{p+1}} + C \sup_{n} \|t^k_n\|_{\dot{H}^p_{p+1}}\|\tilde{\psi}\|_{L^{p+1}_{x, t}},
\]

and approximating \( \psi^k \) by \( \tilde{\psi} \in C_{\text{comp}}^\infty \) in \( \dot{H}^p_{p+1} \), we have

\[
\|e^{-it_n \Delta} \psi^k\|_{L^{p+1}_{x, t}} \to 0 \text{ as } n \to \infty.
\]

Thus, combining (3.14) and (3.15), we obtain (3.12).

Step 2. Finishing the proof. Note that

\[
\|W_n M_1\|_{L^{p+1}} \leq \|W_n M_1\|_{\dot{H}^{p+1}} \leq \|W_n M_1\|_{L^\infty}^{1/2} \|W_n M_1\|_{L^{p+1}}^{1/2} \|W_n M_1\|_{L^\infty}^{1/2} \|W_n M_1\|_{L^{p+1}}^{1/2}
\]

\[
\leq \|W_n M_1\|_{L^\infty}^{1/2} \|W_n M_1\|_{L^{p+1}}^{1/2} \|W_n M_1\|_{L^{p+1}}^{1/2} \sup_n \|\phi_n\|_{H^1}.
\]

By (3.3) it follows that

\[
\lim_{M_1 \to +\infty} \left( \lim_{n \to +\infty} \|e^{it \Delta} W_n M_1\|_{L^{p+1}} \right) = 0.
\]  

Let \( M \geq 1 \) and \( \epsilon > 0 \). The sequence of profiles \( \{\psi^n\} \) is uniformly bounded in \( H^1 \) and in \( L^{p+1} \). Hence, (3.16) implies that the sequence of remainders \( \{W_n M_1\} \) is also uniformly bounded in
$L^{p+1}_x$. Pick $M_1 \geq M$ and $n_1$ such that for $n \geq n_1$, we have

$$\left| \| \phi_n - W_n^{M_1} \|_{L^{p+1}_x}^{p+1} - \| \phi_n \|_{L^{p+1}_x}^{p+1} \right| + \left| \| W_n^M - W_n^{M_1} \|_{L^{p+1}_x}^{p+1} + \| W_n^M \|_{L^{p+1}_x}^{p+1} \right| \leq C \left( \sup_n \| \phi_n \|_{L^{p+1}_x}^p + \sup_n \| W_n^M \|_{L^{p+1}_x}^p + \| W_n^M \|_{L^{p+1}_x}^{p+1} + \| W_n^M \|_{L^{p+1}_x}^{p+1} \right) \leq \frac{\varepsilon}{3}. \\ (3.17)$$

Choose $n_2 \geq n_1$ such that $n \geq n_2$. Then (3.13) yields

$$\left| \| \phi_n - W_n^{M_1} \|_{L^{p+1}_x}^{p+1} - \sum_{j=1}^{M_1} \| e^{-it_n^j \Delta} \psi_j \|_{L^{p+1}_x}^{p+1} \right| \leq \frac{\varepsilon}{3}. \quad (3.18)$$

Since $W_n^M - W_n^{M_1} = \sum_{j=M+1}^{M_1} e^{-it_n^j \Delta} \psi_j (\cdot - x_n^j)$, by (3.13), there exist $n_3 \geq n_2$ such that $n \geq n_3$,

$$\left| \| W_n^M - W_n^{M_1} \|_{L^{p+1}_x}^{p+1} - \sum_{j=M+1}^{M_1} \| e^{-it_n^j \Delta} \psi_j \|_{L^{p+1}_x}^{p+1} \right| \leq \frac{\varepsilon}{3}. \quad (3.19)$$

Thus for $n \geq n_3$, (3.17), (3.18), and (3.19) yield

$$\left| \| \phi_n \|_{L^{p+1}_x}^{p+1} - \sum_{j=1}^{M} \| e^{-it_n^j \Delta} \psi_j \|_{L^{p+1}_x}^{p+1} - \| W_n^M \|_{L^{p+1}_x}^{p+1} \right| \leq \varepsilon, \quad (3.20)$$

which concludes the proof. \qed

**Proposition 3.6** (Nonlinear Profile decomposition). Let $\phi_n(x)$ be a uniformly bounded sequence in $H^1(\mathbb{R}^d)$. Then for each $M \in \mathbb{N}$ there exists a subsequence of $\phi_n$, also denoted by $\phi_n$, for each $1 \leq j \leq M$, there exist a (same for all $n$) nonlinear profile $\tilde{\psi}_j$ in $H^1(\mathbb{R}^d)$, a sequence of time shifts $t_n^j$, and a sequence of space shifts $x_n^j$ and in addition, a sequence (in $n$) of remainders $\tilde{W}_n^M(x)$ in $H^1(\mathbb{R}^d)$, such that

$$\phi_n(x) = \sum_{j=1}^{M} NLS(-t_n^j) \tilde{\psi}_j^j (x - x_n^j) + \tilde{W}_n^M (x), \quad (3.21)$$

where (as $n \to \infty$)

(a) for each $j$, either $t_n^j = 0, t_n^j \to +\infty$ or $t_n^j \to -\infty$,

(b) if $t_n^j \to +\infty$, then $\| NLS(-t) \tilde{\psi}_j \|_{S([0,\infty),H^s)} < +\infty$ and if $t_n^j \to -\infty$, then $\| NLS(-t) \tilde{\psi}_j \|_{S((-\infty,0),H^s)} < +\infty$,

(c) for $k \neq j$, then $|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty$.

The remainder sequence has the following asymptotic smallness property:

$$\lim_{M \to \infty} \left( \lim_{n \to \infty} \| NLS(t) \tilde{W}_n^M \|_{S(H^s)} \right) = 0. \quad (3.22)$$

For fixed $M \in \mathbb{N}$ and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion

$$\| \phi_n \|_{H^s}^2 = \sum_{j=1}^{M} \| NLS(-t_n^j) \tilde{\psi}_j \|_{H^s}^2 + \| \tilde{W}_n^M \|_{H^s}^2 + o_n(1) \quad (3.23)$$
and the energy Pythagorean decomposition (note that $E[NLS(-t_n^j)\tilde{\psi}^j] = E[\tilde{\psi}^j]$):

$$E[\phi_n] = \sum_{j=1}^{M} E[\tilde{\psi}^j] + E[\tilde{W}_n^M] + o_n(1).$$ \hspace{1cm} (3.24)

**Proof.** From Proposition 3.14 given that $\phi_n(x)$ is a uniformly bounded sequence in $H^1$, we have

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n\Delta} \psi^j(x - x_n^j) + \tilde{W}_n^M(x)$$ \hspace{1cm} (3.25)

satisfying (3.2), (3.3), (3.4) and (3.11). We will choose $M \in \mathbb{N}$ later. To prove this proposition, the idea is to replace a linear flow $e^{it\Delta} \psi^j$ by some nonlinear flow.

For each $\psi^j$ we can apply the wave operator (Proposition 2.18) to obtain a function $\tilde{\psi}^j \in H^1$, which we will refer to as the nonlinear profile (corresponding to the linear profile $\psi^j$) such that the following properties hold:

For a given $j$, there are two cases to consider: either $t_n^j$ is bounded, or $|t_n^j| \to +\infty$.

**Case $|t_n^j| \to +\infty$:**

If $t_n^j \to +\infty$, Proposition 2.18 Part I equation (2.51) implies that

$$\|NLS(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta} \psi^j\|_{H^1} \to 0 \text{ as } t_n^j \to +\infty$$

and so

$$\|NLS(-t)\tilde{\psi}^j\|_{L^2_{\text{rad}}(\mathbb{R}^+)} < +\infty.$$ \hspace{1cm} (3.26)

Similarly, if $t_n^j \to -\infty$, by (2.52) we obtain

$$\|NLS(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta} \psi^j\|_{H^1} \to 0 \text{ as } t_n^j \to -\infty,$$

and hence,

$$\|NLS(-t)\tilde{\psi}^j\|_{L^2_{\text{rad}}(\mathbb{R}^+)} < +\infty.$$ \hspace{1cm} (3.27)

**Case $t_n^j$ is bounded** (as $n \to \infty$): Adjusting the profiles $\psi^j$ we reduce it to the case $t_n^j = 0$. Thus, (3.2) becomes $|x_n^j - x_n^k| \to +\infty$ as $n \to \infty$, and continuity of the linear flow in $H^1$, leads to $e^{-it_n^j\Delta} \psi^j \to \psi^j$ strongly in $H^1$ as $n \to \infty$. In this case, we simply let

$$\tilde{\psi}^j = NLS(0)e^{-i(\lim_{n \to \infty} t_n^j)\Delta} \psi^j = e^{-it_n^j\Delta} \psi^j = \psi^j.$$

Thus, in either case of sequence $\{t_n^j\}$, we have a new nonlinear profile $\tilde{\psi}^j$ associated to each original linear profile $\psi^j$ such that

$$\|NLS(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta} \psi^j\|_{H^1} \to 0 \text{ as } n \to +\infty.$$ \hspace{1cm} (3.28)

Thus, we can substitute $e^{-it_n^j\Delta} \psi^j$ by $NLS(-t_n^j)\tilde{\psi}^j$ in (3.25) to obtain

$$\phi_n(x) = \sum_{j=1}^{M} NLS(-t_n^j)\tilde{\psi}^j(x - x_n^j) + \tilde{W}_n^M(x),$$ \hspace{1cm} (3.29)
where
\[
\tilde{W}_n^M(x) = W_n^M(x) + \sum_{j=1}^{M} \left\{ e^{-it_n^j} \psi_j(x-x_n^j) - \text{NLS}(-t_n^j) \tilde{\psi}_j(x-x_n^j) \right\} = W_n^M(x) + \sum_{j=1}^{M} \mathcal{T}_j. \tag{3.30}
\]

The triangle inequality yields \( \| e^{it \Delta} \tilde{W}_n^M \|_{\beta_0} \leq \| e^{it \Delta} W_n^M \|_{\beta_0} + c \sum_{j=1}^{M} \| \mathcal{T}_j \|_{\beta_0} \). By \( (3.28) \) we have that \( \| e^{it \Delta} \tilde{W}_n^M \|_{\beta_0} \leq \| e^{it \Delta} W_n^M \|_{\beta_0} + c \sum_{j=1}^{M} o_n(1) \), and thus,
\[
\lim_{M \to \infty} \left( \lim_{n \to \infty} \| e^{it \Delta} \tilde{W}_n^M \|_{\beta_0} \right) = 0.
\]

Now we are going to apply a nonlinear flow to \( \phi_n(x) \) and approximate it by a combination of “nonlinear bumps” \( \text{NLS}(t-t_n^j) \tilde{\psi}_j(x-x_n^j) \), i.e.,
\[
\text{NLS}(t) \phi_n(x) \approx \sum_{j=1}^{M} \text{NLS}(t-t_n^j) \tilde{\psi}_j(x-x_n^j).
\]

Obviously, this can not hold for any bounded in \( H^1 \) sequence \( \{ \phi_n \} \), since, for a example, a nonlinear flow can introduce finite time blowup solutions. However, under the proper conditions we can use the long term perturbation theory (Proposition 2.14) to guarantee that a nonlinear flow behaves basically similar to the linear flow.

To simplify notation, introduce the nonlinear evolution of each separate initial condition \( u_{n,0} = \phi_n; \ u_n(t, x) = \text{NLS}(t) \phi_n(x) \), the nonlinear evolution of each separate nonlinear profile (“bump”: \( \psi_j(t, x) = \text{NLS}(t) \tilde{\psi}_j(x) \), and a linear sum of nonlinear evolutions of “bumps”: \( \tilde{u}_n(t, x) = \sum_{j=1}^{M} \psi_j(t-t_n^j, x-x_n^j) \).

Intuitively, we think that \( \phi_n = u_{n,0} \) is a sum of bumps \( \tilde{\psi}_j \) (appropriately transformed) and \( u_n(t) \) is a nonlinear evolution of their entire sum. On the other hand, \( \tilde{u}_n(t) \) is a sum of nonlinear evolutions of each bump so we now want to compare \( u_n(t) \) with \( \tilde{u}_n(t) \).

Note that if we had just the linear evolutions, then both \( u_n(t) \) and \( \tilde{u}_n(t) \) would be the same.

Thus, \( u_n(t) \) satisfies
\[
i \partial_t u_n + \Delta u_n + |u_n|^{p-1} u_n = 0,
\]
and \( \tilde{u}_n(t) \) satisfies
\[
i \partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^{p-1} \tilde{u}_n = \tilde{e}_n^M,
\]
where
\[
\tilde{e}_n^M = |\tilde{u}_n|^{p-1} \tilde{u}_n - \sum_{j=1}^{M} |v_n^j(t-t_n^j, \cdot - x_n^j)|^{p-1} v_n^j(t-t_n^j, \cdot - x_n^j).
\]

**Claim** 3.7. There exists a constant \( A \) independent of \( M \), and for every \( M \), there exists \( n_0 = n_0(M) \) such that if \( n > n_0 \), then \( \| \tilde{u}_n \|_{\beta_0} \leq A \).

**Claim** 3.8. For each \( M \) and \( \epsilon > 0 \), there exists \( n_1 = n_1(M, \epsilon) \) such that if \( n > n_1 \), then \( \| \tilde{e}_n^M \|_{\beta_0} \leq \epsilon. \)
Note $\bar{u}_n(0, x) - u_n(0, x) = \tilde{W}_n^M(x)$. Then for any $\tilde{\epsilon} > 0$ there exists $M_1 = M_1(\tilde{\epsilon})$ large enough such that for each $M > M_1$ there exists $n_2 = n_2(M)$ with $n > n_2$ implying

$$\|e^{it\Delta}(\bar{u}_n(0) - u_n(0))\|_{\beta_0^{S(\dot{H}^{s})}} \leq \tilde{\epsilon}.$$  

Therefore, for $M$ large enough and $n = \max(n_0, n_1, n_2)$, since

$$e^{it\Delta}(\bar{u}_n(0)) = e^{it\Delta}\left(\sum_{j=1}^{M} \psi^j(-t^j_n, x - x^j_n)\right),$$

which are scattering by (3.28), Proposition 2.14 implies $\|u_n\|_{\beta_0^{S(\dot{H}^{s})}} < +\infty$, a contradiction.

Coming back to the nonlinear remainder $\tilde{W}_n^M$, we estimate its nonlinear flow as follows (recall the notation of $\tilde{W}_n^M$, $W_n^M$ and $T^j$ in (3.30)):

By Besov Strichartz estimates (2.11) and by the triangle inequality, we get

$$\|\text{NLS}(t)\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} \leq \|e^{it\Delta}\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} + \left\|\tilde{W}_n^M\right\|_{p-1} \left\|W_n^M\right\|_{p} \leq \|e^{it\Delta}\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} + \sum_{j=1}^{M} \|T^j\|_{p-1} \|T^j\|_{p},$$

(3.31)

By Besov Strichartz estimates (2.11) and by the triangle inequality, we get

$$\|\text{NLS}(t)\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} \leq \|e^{it\Delta}\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} + \sum_{j=1}^{M} \|T^j\|_{p-1} \|T^j\|_{p},$$

(3.32)

We used (2.12) to obtain (3.31) and since $s < 1$ we have $\dot{H}^1 \hookrightarrow \dot{H}^{s}$, so it yields (3.32). Hence,

$$\|\text{NLS}(t)\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} \leq \|e^{it\Delta}\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} + \sum_{j=1}^{M} \|e^{-it\Delta}\psi^j - \text{NLS}(-t^j_n)\psi^j\|_{H^1}$$

and by (3.28) and then applying (3.3), we obtain $\lim_{n \to \infty} \|e^{it\Delta}\tilde{W}_n^M\|_{\beta_0^{S(\dot{H}^{s})}} \to 0$ as $M \to \infty$. Thus we proved (3.29), (3.22). This also gives (3.23).

Next, we substitute the linear flow in Lemma 3.5 by the nonlinear and repeat the above long term perturbation argument to obtain

$$\|\phi_n\|_{L^{p+1}}^{p+1} = \sum_{j=1}^{M} \|\text{NLS}(-t^j_n)\psi^j\|_{L^{p+1}}^{p+1} + \|\tilde{W}_n^M\|_{L^{p+1}}^{p+1} + o_n(1),$$

(3.33)

which yields the energy Pythagorean decomposition (3.24). The proof will be concluded after we prove the Claims 3.7 and 3.8.

Proof of Claim 3.7 We show that for a large constant $A$ independent of $M$ and if $n > n_0 = n_0(M)$, then

$$\|\bar{u}_n\|_{S(\dot{H}^{s})} \leq A.$$

(3.34)
Let $M_0$ be a large enough such that $\|e^{it\Delta}V_{n_0}M_0\|_{S(H^s)} \leq \delta_{sd}$. Then, by (3.30), for each $j > M_0$, we have $\|e^{it\Delta}g^j\|_{S(H^s)} \leq \delta_{sd}$, thus, Proposition 2.18 yields $\|v^j\|_{S(H^s)} \leq 2\|e^{it\Delta}g^j\|_{S(H^s)}$ for $j > M_0$.

Assume both $s \neq \frac{1}{2}$ and $d \neq 2$, the pairs $(\frac{2(d+2)}{d-2s}, \frac{2(d+2)}{d-2s})$, $(\infty, \frac{2d}{d-2s})$, $(\frac{6}{1-s}, \frac{6d}{3d-4s-2})$ and $(\frac{4}{1-s}, \frac{2d}{d-s-1})$, are $\dot{H}^s$ admissible. Hence, we have

$$\|\tilde{u}_n\|_{L^\infty_t L^{\frac{2d}{d-2s}}_x} = \sum_{j=1}^{M_0} \|v^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \sum_{j=M_0+1}^{M} \|v^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \text{cross terms}$$

$$\leq \sum_{j=1}^{M_0} \|v^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \sum_{j=M_0+1}^{M} \|e^{it\Delta}g^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \text{cross terms}, \quad (3.35)$$

note that by (3.25) we have

$$\|e^{it\Delta}g_n\|_{L^\infty_t L^{\frac{2d}{d-2s}}_x} = \sum_{j=1}^{M_0} \|e^{it\Delta}g^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \sum_{j=M_0+1}^{M} \|e^{it\Delta}g^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \text{cross terms}. \quad (3.36)$$

Observe that by (3.2) and taking $n_0 = n_0(M)$ large enough, we can consider $\{u_n\}_{n>n_0}$ and thus, make “the cross terms” $\leq 1$. Then (3.36) and $\|e^{it\Delta}g_n\|_{L^\infty_t L^{\frac{2d}{d-2s}}_x} \leq c_1\|g_n\|_{H^s} \leq c_1$ imply $\sum_{j=M_0+1}^{M} \|e^{it\Delta}g^j\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x} + \text{cross terms}$ is bounded independent of $M$ provided $n > n_0$. If $n > n_0$, then $\|\tilde{u}_n\|_{L^\infty_t L^{\frac{2d}{d-2s}}_x}$ is also bounded independent of $M$ by (3.35).

In a similar fashion, one can prove that $\|\tilde{u}_n\|_{L^\infty_t L^{\frac{2d}{d-2s}}_x}$ is bounded independent of $M$ provided $n > n_0$. Interpolation between $\|\tilde{u}_n\|_{L^\infty_t L^{\frac{2(d+2)}{d-2s}}_x}$ and $\|\tilde{u}_n\|_{L^\infty_t L^{\frac{2d}{d-2s}}_x}$ gives $\|\tilde{u}_n\|_{L^\infty_t L^{\frac{6d}{3d-4s-2}}_x}$ and $\|\tilde{u}_n\|_{L^\infty_t L^{\frac{5d}{3d-4s-2}}_x}$ are both bounded independent of $M$ for $n > n_0$.

When $s = \frac{1}{2}$ and $d = 2$, the previous argument takes the pair $(2, \infty)$ which is not an admissible pair in dimension 2. Instead we estimate $\|\tilde{u}\|_{L^\infty_t L^8_x}$ and $\|\tilde{u}\|_{L^\infty_t L^4_x}$, and interpolate between them to get that $\|\tilde{u}\|_{L^2_t L^6_x}$ is bounded independent of $M$ provided $n > n_0$.

To close the argument, we apply Kato estimate (2.5) to the integral equation of

$$i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^{p-1}\tilde{u}_n = e^{it\Delta}g_n.$$

Claiming $\|\tilde{e}^M_n\|_{S(H^{-s})} \leq 1$ (see Claim 3.8), as in Proposition 2.14 we obtain that $\|\tilde{u}_n\|_{S(H^s)}$ is as well bounded independent of $M$ provided $n > n_0$. Thus, Claim 3.7 is proved.
Proof of Claim 3.8. Note that the pairs $\left(\frac{6}{1-s}, \frac{6d}{3d-4s-2}\right)$, $\left(\frac{4}{1-s}, \frac{2d}{3d-4s-2}\right)$ are $\dot{H}^s$ admissible and the pair $\left(\frac{12(d-2s)}{(8+3d-6s)(1-s)}, \frac{6d(d-2s)}{3(d^2+2s^2)+9d(1-s)-2(5s+4)}\right)$ is $\dot{H}^{-s}$ admissible. Recall the elemental inequality: for $a_j, a_k \in \mathbb{C}$,
\[
\left|\sum_{j=1}^{M} a_j\right|^p - \sum_{j=1}^{M} |a_j|^p \leq c_{p,M} \sum_{j=1}^{M} \sum_{k \neq j} |a_k|^p |a_j|,
\]
which combined with the Hölder’s inequality, for each dyadic number $N \in 2\mathbb{Z}$, leads to
\[
\|\tilde{e}_n^M\|_{S'(\dot{H}^{-s})} \leq \left\|\tilde{e}_n^M\right\|_{L_t^6(0,\infty) \cap L_x^{\frac{6d}{3d-4s-2}}} \leq \sum_{j=1}^{M} \sum_{k = 1 \atop k \neq j}^{M} \|v^k(t - t_n^k, x - x^k)\|^{p-1}_{L_t^6} \|v^j(t - t_n^j, x - x^j)\| \frac{4}{L_t^\frac{6d}{3d-4s-2}} \rightarrow 0.
\]

Here, we used the following Hölder splits:
\[
\frac{(p-1)(1-s)}{6} + \frac{1-s}{4} = \frac{(8+3d-6s)(1-s)}{12(d-2s)},
\]
\[
\frac{(p-1)(3d-4s-2)}{6d} + \frac{d-s-1}{2d} = \frac{3(d^2+2s^2)+9d(1-s)-2(5s+4)}{6d(d-2s)}.
\]

Note that either $\{t_n^k\} \rightarrow \pm \infty$ or $\{t_n^k\}$ is bounded.

If $\{t_n^j\} \rightarrow \pm \infty$, without loss of generality assume $|t_n^k - t_n^j| \rightarrow \infty$ as $n \rightarrow \infty$ and by adjusting the profiles that $|x_n^k - x_n^j| \rightarrow 0$ as $n \rightarrow \infty$. Since $v^k \in \dot{L}_t^\frac{6}{6} \cap L_x^\frac{6d}{3d-4s-2}$ and $v^j \in \dot{L}_t^\frac{6}{6} \cap L_x^\frac{6d}{3d-4s-2}$, then
\[
\|v^k(t - t_n^k, x - x^k)\|^{p-1}_{L_t^6} \|v^j(t - t_n^j, x - x^j)\| \frac{4}{L_t^\frac{6d}{3d-4s-2}} \rightarrow 0.
\]

If $\{t_n^j\}$ is bounded, without loss of generality, assume $|x_n^j - x_n^j| \rightarrow \infty$ as $n \rightarrow \infty$, then
\[
\|v^k(t - t_n^k, x - x^k)\|^{p-1}_{L_t^6} \|v^j(t - t_n^j, x - x^j)\| \frac{4}{L_t^\frac{6d}{3d-4s-2}} \rightarrow 0.
\]

Thus, in either case we obtain Claim 3.8. This finishes the proof of Proposition 3.6. 

Observe that (3.23) gives $\dot{H}^1$ asymptotic orthogonality at $t = 0$ and the following lemma extends it to the bounded NLS flow for $0 \leq t \leq T$.

**Lemma 3.9** ($\dot{H}^1$ Pythagorean decomposition along the bounded NLS flow). Suppose $\phi_n$ is a bounded sequence in $H^1(\mathbb{R}^d)$. Let $T \in (0, \infty)$ be a fixed time. Assume that $u_n(t) \equiv NLS(t)\phi_n$ exists up to time $T$ for all $n$, and $\lim_{n \rightarrow \infty} \|\nabla u_n(t)\|_{L_t^\infty L_x^2} < \infty$. Consider the nonlinear profile decomposition from Proposition 3.6. Denote $W_n^M(t) \equiv NLS(t)\tilde{v}_n^M$. Then for all $j$, the nonlinear profiles $v^j(t) \equiv NLS(t)\tilde{v}_n^j$ exist up to time $T$ and for all $t \in [0, T]$, 
\[
\|\nabla u_n(t)\|_{L_x^2}^2 = \sum_{j=1}^{M} \|\nabla v^j(t - t_n^j)\|_{L_x^2}^2 + \|\nabla W_n^M(t)\|_{L_x^2}^2 + o_n(1), \tag{3.37}
\]
where \( o_n(1) \to 0 \) uniformly on \( 0 \leq t \leq T \).

**Proof.** We use Proposition 3.6 to obtain profiles \( \{ \tilde{v}^j \} \) and the nonlinear profile decomposition (3.21). Note that \( \lim_{n \to \infty} \| \text{NLS}(t) W_n^M \|_{\tilde{\beta}_0(S(H^s))} \to 0 \) as \( M \to \infty \), so by choosing a large \( M \) we can make \( \| \text{NLS}(t) W_n^M \|_{\tilde{\beta}_0(S(H^s))} \) small.

Let \( M_0 \) be such that for \( M \geq M_0 \) (and for \( n \) large), we have \( \| \text{NLS}(t) W_n^M \|_{\tilde{\beta}_0(S(H^s))} \leq \delta_{sd} \) (recall \( \delta_{sd} \) from Proposition 2.13). Reorder the first \( M_0 \) profiles and let \( M_2, 0 \leq M_2 \leq M \), be such that

1. For each \( 1 \leq j \leq M_2 \), we have \( \| t_n^j \| = 0 \). Observe that if \( M_2 = 0 \), there are no \( j \) in this case.
2. For each \( M_2 + 1 \leq j \leq M_0 \), we have \( \| t_n^j \| \to \infty \). If \( M_2 = M_0 \), then it means that there are no \( j \) in this case.

From Proposition 3.6 and the profile decomposition (3.21) we have that \( v^j(t) \) for \( j > M_0 \) are scattering, and for \( M_2 + 1 \leq j \leq M_0 \) we have \( \| v^j(t - t_n^j) \|_{S(H^s; [0, T])} \to 0 \) as \( n \to +\infty \).

In fact, taking \( t_n^j \to +\infty \) and \( \| v^j(-t) \|_{S(H^s; [0, +\infty])} < \infty \), dominated convergence leads to \( \| v^j(-t) \|_{L^1_{[0, +\infty]} L^\infty_x} < \infty \), for \( q < \infty \), where \( (r, q) \) is an \( \dot{H}^s \) admissible pair, and consequently, \( \| v^j(t - t_n^j) \|_{L^q_{[0, T]} L^\infty_x} \to 0 \) as \( n \to \infty \). As \( v^j(t) \) has been constructed via the existence of wave operators to converge in \( H^1 \) to a linear flow, the \( L^\infty_x \) decay of the linear flow

\[
\| v^j(t - t_n^j) \|_{L^\infty_{[0, T]} L^\infty_x} \to 0,
\]

with

\[
r = \begin{cases} 
\frac{2d}{d - 2s} & d \geq 3 \\
\frac{2}{1 - s} & d = 2 \\
\frac{1}{1 - 2s} & d = 1
\end{cases}
\quad \text{and } s \text{ as in (3.6)}.
\]

Let \( B = \max \{ 1, \lim_n \| \nabla u_n(t) \|_{L^\infty_{[0, T]} L^2_x} \} < \infty \). For each \( 1 \leq j \leq M_2 \), let \( T^j \leq T \) be the maximal forward time such that \( \| \nabla v^j \|_{L^1_{[0, T]} L^2_x} \leq 2B \), and \( \tilde{T} = \min_{1 \leq j \leq M_2} T^j \) or \( \tilde{T} = T \) if \( M_2 = 0 \). It is sufficient to prove that (3.37) holds for \( \tilde{T} = T \), since for each \( 1 \leq j \leq M_2 \), we have \( T^j = T \), and therefore, \( \tilde{T} = T \). Thus, let’s consider \([0, \tilde{T}]\). For each \( 1 \leq j \leq M_2 \), we have for \( d \geq 3 \):

\[
\| v^j(t) \|_{S(H^s; [0, \tilde{T}])} \lesssim \| v^j \|_{L^{2d}_{[0, \tilde{T}]} L^d_x}^{\frac{2d}{d - 2s}} + \| v^j \|_{L^\infty_{[0, \tilde{T}]} L^d_x}^{\frac{2d}{d - 2s}} \tag{3.38}
\]

\[
\lesssim \| v^j \|_{L^{2d}_{[0, \tilde{T}]} L^d_x}^{\frac{2d}{d - 2s}} \| v^j \|_{L^\infty_{[0, \til{T}]} L^2_x}^{\frac{2d}{d - 2s}} + \| v^j \|_{L^{2d}_{[0, \til{T}]} L^d_x}^{\frac{2d}{d - 2s}} \| v^j \|_{L^\infty_{[0, \til{T}]} L^2_x}^{\frac{2d}{d - 2s}} \tag{3.39}
\]

\[
\lesssim (\tilde{T}^{\frac{1}{d - 2s}} + c^{1-s}) \| \nabla v^j \|_{L^\infty_{[0, \til{T}]} L^2_x} \lesssim (\tilde{T}^{\frac{1}{d - 2s}}) B, \tag{3.40}
\]

note that (3.38) comes from the “end point” admissible \( S(H^s) \) Strichartz norms \( (L^{2d}_{[0, \til{T}]} L^2_x) \), and \( L^\infty_{[0, \til{T}]} L^2_x \), since all other \( S(H^s) \) norms will be bounded by interpolation; the Hölder’s inequality yields (3.39) and the Sobolev’s embedding \( \dot{H}^1(\mathbb{R}^d) \to L^{\frac{2d}{d - 2s}}(\mathbb{R}^d) \) together with \( \| v^j \|_{L^\infty_{[0, \til{T}]} L^2_x} = \| \phi_n \|_{L^2} \leq \| \phi_n \|_{L^2} \) from (3.23) with \( s = 0 \), gives (3.40).
For $d = 2$:

$$\|v^j(t)\|_{S(\dot{H}^s;[0,T])} \lesssim \|v^j\|_{L^\infty_{[0,T]} L^2_x} + \|v^j\|_{L^2_{[0,T]} L^\infty_x}$$ (3.41)

$$\lesssim \|v^j\|_{L^\infty_{[0,T]} L^2_{x}} \|v^j\|_{L^\infty_{[0,T]} L^\infty_{x}} + \|v^j\|_{L^2_{[0,T]} L^\infty_{x}} \|v^j\|_{L^\infty_{[0,T]} L^2_{x}}$$ (3.42)

$$\lesssim \|v^j\|_{L^\infty_{[0,T]} L^2_{x}} \|\nabla v^j\|_{L^\infty_{[0,T]} L^2_{x}} + \|v^j\|_{L^2_{[0,T]} L^\infty_{x}} \|v^j\|_{L^\infty_{[0,T]} H^{1-\frac{s}{2}}_x}$$ (3.43)

$$\lesssim \left( \|v^j\|_{L^\infty_{[0,T]} L^2_{x}} + \|v^j\|_{L^2_{[0,T]} L^\infty_{x}} \right) \|\nabla v^j\|_{L^\infty_{[0,T]} L^2_{x}}$$ (3.44)

$$\lesssim (\bar{T}^{\frac{1-2s}{4}} + c^{-1-s}) \|\nabla v^j\|_{L^\infty_{[0,T]} L^2_{x}} \lesssim \langle \bar{T}^{\frac{1-2s}{4}} \rangle B,$$ (3.45)

where $r = \left( \left( \frac{1}{2-s} \right)^{4} \right)^{1}$. Note that (3.44) comes from the “end point” admissible Strichartz norms ($L^\infty_x L^{1-\frac{s}{2}}_x$ and $L^2_x L^\infty_x$); Hölder’s inequality yields (3.42); the Sobolev’s embeddings $\dot{H}^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and $\dot{H}^{1-\frac{s}{2}}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$ leads to (3.43); since $r$ is large we have the Sobolev’s embedding $\dot{H}^1(\mathbb{R}^2) \hookrightarrow \dot{H}^{1-\frac{s}{2}}(\mathbb{R}^2)$, which implies (3.44), and finally, since $\|v^j\|_{L^\infty_{[0,T]} L^2_x} = \|\psi^j\|_{L^2_x} \leq \|\phi_n\|_{L^2}$ by (3.23) with $s = 0$ we get (3.45).

For $d = 1$:

$$\|v^j(t)\|_{S(\dot{H}^s;[0,T])} \lesssim \|v^j\|_{L^\infty_{[0,T]} L^\frac{2}{1-2s}_x} + \|v^j\|_{L^\frac{1}{1-2s}_{[0,T]} L^\infty_x}$$ (3.46)

$$\lesssim \|v^j\|_{L^\infty_{[0,T]} L^\frac{2}{1-2s}_x} \|v^j\|_{L^\infty_{[0,T]} L^\infty_x} + \|v^j\|_{L^\frac{1}{1-2s}_{[0,T]} L^\infty_x} \|v^j\|_{L^\infty_{[0,T]} L^\infty_x}$$ (3.47)

$$\lesssim \left( \|v^j\|_{L^\infty_{[0,T]} L^\frac{2}{1-2s}_x} + \|v^j\|_{L^\frac{1}{1-2s}_{[0,T]} L^\infty_x} \right) \|\nabla v^j\|_{L^\infty_{[0,T]} L^2_x}$$ (3.48)

note that (3.46) comes from the “end point” admissible Strichartz norms ($L^\infty_x L^{1-\frac{s}{2}}_x$ and $L^2_x L^\infty_x$); Hölder’s inequality yields (3.47); the Sobolev’s embeddings $\dot{H}^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ implies (3.47), and finally, $\|v^j\|_{L^\infty_{[0,T]} L^2_x} = \|v^j\|_{L^2_x} \leq \|\phi_n\|_{L^2}$ leads to (3.48).

As in the proof of Proposition 3.6 set $\tilde{u}_n(t,x) = \sum_{j=1}^{M} v^j(t - t^j_n, x - x^j_n)$ and a linear sum of nonlinear flows of nonlinear profiles $\tilde{\psi}^j$, $\tilde{e}_n^M = i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^{p-1} \tilde{u}_n$. Thus, for $M > M_0$ we have

Claim 3.7. There exist a constant $A = A(\bar{T})$ independent of $M$, and for every $M$, there exists $n_0 = n_0(M)$ such that if $n > n_0$, then $\|\tilde{u}_n\|_{S(\dot{H}^s)} \leq A$.

Claim 3.8. For each $M$ and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that if $n > n_1$, then $\|\tilde{e}_n^M\|_{S(\dot{H}^{s-\epsilon})}$.

Remark 3.10. Note since $u(0) - \tilde{u}_n(0) = \tilde{W}^M_n$, there exists $M' = M'(\epsilon)$ large enough so that for each $M > M'$ there exists $n_2 = n_2(M)$ such that $n > n_2$ implies

$$\|e^{i\Delta (u(0) - \tilde{u}_n(0))}\|_{S(\dot{H}^s;[0,T])} \leq \epsilon.$$
We will next apply the long term perturbation argument (Proposition 2.14); note that in Proposition 2.14, \( T = +\infty \), while here, it is not necessary. However, \( T \) does not form part of the parameter dependence, since \( \epsilon_0 \) depends only on \( A = A(T) \), not on \( T \), that is, there will be dependence on \( T \), but it is only through \( A \).

Thus, the long term perturbation argument (Proposition 2.14) gives us \( \epsilon_0 = \epsilon_0(A) \). Selecting an arbitrary \( \epsilon \leq \epsilon_0 \), and from Remark 3.10 take \( M' = M'(\epsilon) \). Now select an arbitrary \( M > M' \) and take \( n' = \max(n_0, n_1, n_2) \). Then combining claims 3.7 - 3.8 Remark 3.10 and Proposition 3.6 we obtain that for \( n > n'(M, \epsilon) \) with \( c = c(A) = c(T) \) we have

\[
\| u_n - \tilde{u}_n \|_{S([\dot{H}^1;[0,T])} \leq c(T)\epsilon. \tag{3.49}
\]

We will next prove (3.37) for \( 0 \leq t \leq T \). Recall that for each dyadic number \( N \in 2 \mathbb{Z} \), \( \| v^j(t - t^j_n) \|_{S([\dot{H}^s;[0,T])} \to 0 \) as \( n \to \infty \) and for each \( 1 \leq j \leq M_2 \), we have \( \| \nabla v^j \|_{L^{\infty}} L^2_{[0,T]} \leq 2B \).

Strichartz estimates imply \( \| \nabla v^j(t - t^j_n) \|_{L^{\infty}} L^2_{[0,T]} \leq \| \nabla v^j(-t^j_n) \|_{L^{\infty}} L^2_{[0,T]} \) then

\[
\| \nabla \tilde{u}(t) \|^2_{L^2_{[0,T]} L^2_x} = \sum_{j=1}^{M_2} \| \nabla v^j (t) \|^2_{L^2_{[0,T]} L^2_x} + \sum_{j=M_2+1}^{M} \| \nabla v^j (t - t^j_n) \|^2_{L^2_{[0,T]} L^2_x} + o_n(1) \\
\lesssim M_2 B^2 + \sum_{j=M_2+1}^{M} \| \nabla \phi^j \|^2_{L^2_x} + o_n(1) \lesssim M_2 B^2 + B^2 + o_n(1). 
\]

Using (3.49), we obtain for \( d \geq 3 \):

\[
\| u_n - \tilde{u}_n \|_{L^2_{[0,T]} L^{p+1}_x} \lesssim \| u_n - \tilde{u}_n \|_{S([\dot{H}^s;[0,T])} \| \nabla (u_n - \tilde{u}_n) \|_{L^2_{[0,T]} L^{2+2s}_x} \tag{3.50}
\]

\[
\lesssim \| u_n - \tilde{u}_n \|_{S([\dot{H}^s;[0,T])} \| \nabla (u_n - \tilde{u}_n) \|_{L^2_{[0,T]} L^{2+2s}_x} \tag{3.51}
\]

in this case, we used H"{o}lder's inequality to get (3.50) and the Sobolev embedding \( \dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2s}}(\mathbb{R}^d) \) to obtain (3.51).

For \( d = 2 \):

\[
\| u_n - \tilde{u}_n \|_{L^2_{[0,T]} L^{p+1}_x} \lesssim \| u_n - \tilde{u}_n \|_{S([\dot{H}^s;[0,T])} \| \nabla (u_n - \tilde{u}_n) \|_{L^{1+2s}_x} \tag{3.52}
\]

\[
\lesssim \| u_n - \tilde{u}_n \|_{S([\dot{H}^s;[0,T])} \| \nabla (u_n - \tilde{u}_n) \|_{L^{1+2s}_x} \tag{3.53}
\]

here, we used H"{o}lder's inequality to get (3.52) and the Sobolev embedding \( \dot{H}^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \) to obtain (3.53).
For $d = 1$:
\[
\|u_n - \tilde{u}_n\|_{L^\infty_{[0,T]} L^{p+1}_x} \lesssim \|u_n - \tilde{u}_n\|_{L^{3-2s}_{[0,T]} L^\frac{4}{3-2s}_x} \|u_n\|_{L^{\frac{1-2s}{3-2s}_{[0,T]} L^\infty_x}}^{1-2s} (3.54)
\]
\[
\lesssim \|u_n - \tilde{u}_n\|_{L^{3-2s}_{[0,T]} L^\frac{4}{3-2s}_x} \|\nabla (u_n - \tilde{u}_n)\|_{L^\infty_{[0,T]} L^2_x} (3.55)
\]
\[
\lesssim c(T) \frac{2}{2s} (M_2 B^2 + B^2 + o(1)) \frac{1-2s}{2s} \epsilon \frac{1-2s}{2s},
\]
here, we used Hölder’s inequality to get (3.54) and the Sobolev embedding $\dot{H}^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ to obtain (3.55).

Similar to the argument in the proof of (3.33), we establish that for $0 \leq t \leq \tilde{T}$
\[
\|u_n(t)\|_{L^{p+1}} = \sum_{j=1}^M \|v^j(t - t^j_n)\|_{L^{p+1}} + \|W^M_n(t)\|_{L^{p+1}} + o_n(1). (3.56)
\]
Energy conservation and (3.24) give us
\[
E[u_n(t)] = \sum_{j=1}^M E[v^j(t - t^j)] + E[W^M_n] + o_n(1) = \sum_{j=1}^M E[v^j] + E[W^M_n] + o_n(1). (3.57)
\]
Combining (3.56) and (3.57), completes the proof of (3.31).

We now have all the profile decomposition tools to apply to our particular situation in part I (a) of Theorem A*.

**Proposition 3.11** (Existence of a critical solution). There exists a global ($T = +\infty$) $H^1$ solution $u_c(t) \in H^1(\mathbb{R}^d)$ with initial datum $u_{c,0} \in H^1(\mathbb{R}^d)$ such that
\[
\|u_{c,0}\|_{L^2} = 1, \quad E[u_c]^s = (ME)_c < M[u_q]^{1-s} E[u_q]^s,
\]
\[
G_u(t) < 1 \quad \text{for all} \quad 0 \leq t < +\infty,
\]
\[
\|u_c\|_{\dot{S}^{(H^s)}} = +\infty. (3.58)
\]

**Proof.** Consider a sequence of solutions $u_n(t)$ to NLS$_p(\mathbb{R}^d)$ with corresponding initial data $u_{n,0}$ such that $G_{u_n}(0) < 1$ and $M[u_n]^{1-s} E[u_n]^s \searrow (ME)_c$ as $n \to +\infty$, for which $SC(u_{n,0})$ does not hold for any $n$.

Without loss of generality, rescale the solutions so that $\|u_{n,0}\|_{L^2} = 1$, thus,
\[
\|\nabla u_{n,0}\|_{L^2} < \|u_{Q,0}\|_{L^\frac{4}{\beta}} \|\nabla u_{Q,0}\|_{L^\frac{4}{\beta}}^\frac{2}{s} \quad \text{and} \quad E[u_n]^s \searrow (ME)_c.
\]
By construction, $\|u_n\|_{\dot{S}^{(H^s)}} = +\infty$. Note that the sequence $\{u_{n,0}\}$ is uniformly bounded on $H^1$. Thus, applying the nonlinear profile decomposition (Proposition 3.6), we have
\[
u_{n,0}(x) = \sum_{j=1}^M \text{NLS}(t^j_n) \tilde{v}^j(x - x^j_n) + \tilde{W}^M_n(x). (3.59)
\]
Now we will refine the profile decomposition property (b) in Proposition 3.6 by using part II of Proposition 2.18 (wave operator), since it is specific to our particular setting here.
Recall that in nonlinear profile decomposition we consider 2 cases when \( |t_n^j| \to \infty \) and \( |t_n^j| \) is bounded. In the first case, we can refine it to the following.

First note that we can obtain \( \tilde{\psi}^j \) (from linear \( \psi^j \)) such that

\[
\| \text{NLS}(-t_n^j)\tilde{\psi}^j - e^{-i t_n^j \Delta} \psi^j \|_{H^1} \to 0 \quad \text{as} \quad n \to +\infty
\]

with properties (2.54), since the linear profiles \( \psi^j \)'s satisfy

\[
\| \psi \|_{L^2}^{2(1-s)} \| \nabla \psi \|_{L^2}^{2s} < \sigma^2 \left( \frac{d}{s} \right)^s M[u_Q]^{1-s} E[u_Q]^s.
\]

We also have,

\[
\sum_{j=1}^{M} M[e^{-i t_n^j \Delta} \psi^j] + \lim_{n \to +\infty} M[W_n^M] = \lim_{n \to +\infty} M[u_{n,0}] = 1.
\]

\[
\sum_{j=1}^{M} \lim_{n \to +\infty} E[e^{-i t_n^j \Delta} \psi^j] + \lim_{n \to +\infty} E[W_n^M] = \lim_{n \to +\infty} E[u_{n,0}] = (ME)_c,
\]

thus, \( \frac{1}{2} \| \psi^j \|_{L^2}^{1-s} \| \nabla \psi^j \|_{L^2}^s \leq (ME)_c \).

The properties (2.54) for \( \psi^j \) imply that \( ME[\tilde{\psi}^j] < (ME)_c \), and thus, we get that

\[
\| \text{NLS}(t)\tilde{\psi}^j(\cdot - x_n^j) \|_{H^s(S(\mathbb{R}^+))} < +\infty.
\]  

(3.60)

This fact will be essential for case 1 below. Otherwise, in nonlinear decomposition (3.59) we also have the Pythagorean decomposition for mass and energy:

\[
\sum_{j=1}^{M} \lim_{n \to +\infty} E[\tilde{\psi}^j] + \lim_{n \to +\infty} E[W_n^M] = \lim_{n \to +\infty} E[u_{n,0}] = (ME)_c^{1/2}.
\]

Since each energy is greater than 0 (Lemma 2.16), for all \( j \) we obtain

\[
E[\tilde{\psi}^j]^s \leq (ME)_c.
\]  

(3.61)

Furthermore, \( s = 0 \) in (3.23) imply

\[
\sum_{j=1}^{M} M[\tilde{\psi}^j] + \lim_{n \to +\infty} M[W_n^M] = \lim_{n \to +\infty} M[u_{n,0}] = 1.
\]  

(3.62)

We show that in the profile decomposition (3.59) either more than one profiles \( \tilde{\psi}^j \) are non-zero, or only one profile \( \tilde{\psi}^j \) is non-zero and the rest \( (M - 1) \) profiles are zero. The first case will give a contradiction to the fact that each \( u_n(t) \) does not scatter, consequently, only the second possibility holds. That non-zero profile \( \tilde{\psi}^j \) will be the initial data \( u_{c,0} \) and will produce the critical solution \( u_c(t) = \text{NLS}(t)u_{c,0} \), such that \( \| u_c \|_{S(\mathbb{R}^+)} = +\infty \).

**Case 1:** More than one \( \tilde{\psi}^j \neq 0 \). For each \( j \), (3.62) gives \( M[\tilde{\psi}^j] < 1 \) and for a large enough \( n \), (3.61) and (3.62) yield

\[
M[\text{NLS}(t)\tilde{\psi}^j]^{1-s} E[\text{NLS}(t)\tilde{\psi}^j]^s = M[\tilde{\psi}^j]^{1-s} E[\tilde{\psi}^j]^s < (ME)_c.
\]
Recall (3.60), we have
\[ \| \text{NLS}(t - t^j) \tilde{\psi}^j (\cdot - x_n^j) \|_{S(H^s)} < +\infty, \]
for large enough \( n \),
and thus, the right hand side in (3.59) is finite in \( S(\dot{H}^s) \), since (3.22) holds for the remainder \( \tilde{W}_n^M(x) \). This contradicts the fact that \( \| \text{NLS}(t) u_n,0 \|_{S(H^s)} = +\infty \).

**Case 2:** Thus, we have that only one profile \( \tilde{\psi}^j \) is non-zero, renamed to be \( \tilde{\psi}^1 \),
\[ u_{n,0} = \text{NLS}(-t^1_n) \tilde{\psi}^1 (\cdot - x^1_n) + \tilde{W}_n^1, \quad (3.63) \]
with
\[ M[\tilde{\psi}^1] \leq 1, \quad E[\tilde{\psi}^1]^s \leq (ME)_c \quad \text{and} \quad \lim_{n \to +\infty} \| \text{NLS}(t) \tilde{W}^1_n \|_{S(H^s)} = 0. \]

Let \( u_c \) be the solution to \( \text{NLS}_p(\mathbb{R}^d) \) with the initial condition \( u_{c,0} = \tilde{\psi}^1 \). Applying \( \text{NLS}(t) \) to both sides of (3.63) and estimating it in \( \| \tilde{\psi}^1 \|_{S(H^s)} \), we obtain (by the nonlinear profile decomposition Proposition 3.6) that
\[ \| u_c \|_{S(H^s)} = \lim_{n \to +\infty} \| \text{NLS}(t - t^1_n) \tilde{\psi}^1 \|_{S(H^s)} = \lim_{n \to +\infty} \| \text{NLS}(t) u_{n,0} \|_{S(H^s)} = +\infty, \]
since by construction \( \| u_n \|_{S(H^s)} = +\infty \), completing the proof.

**Lemma 3.12** (Precompactness of the flow of the critical solution). Assume \( u_c \) as in Proposition 3.11. Then there exists a continuous path \( x(t) \) in \( \mathbb{R}^d \) such that
\[ K = \{ u_c (\cdot - x(t), t) | t \in [0,+\infty) \} \]
is precompact in \( H^1(\mathbb{R}^d) \).

**Proof.** Let a sequence \( \tau_n \to +\infty \) and \( \phi_n = u_c(\tau_n) \) be a uniformly bounded sequence in \( H^1 \); we want to show that \( u_c(\tau_n) \) has a convergent subsequence in \( H^1 \).

The nonlinear profile decomposition (Proposition 3.6) implies the existence of profiles \( \tilde{\psi}^j \), the time and space sequences \( \{ t_n^j, x_n^j \} \) and an error \( \tilde{W}_n^M(x) \) such that
\[ u_c(\tau_n) = \sum_{j=1}^M \text{NLS}(-t_n^j) \tilde{\psi}^j (x - x_n^j) + \tilde{W}_n^M(x), \quad (3.64) \]
with \( |t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty \) as \( n \to +\infty \) for fixed \( j \neq k \). In addition,
\[ \sum_{j=1}^M E[\tilde{\psi}^j] + E[\tilde{W}_n^M] = E[u] = (ME)_c, \]
since each energy is nonnegative, and we have
\[ \lim_{n \to +\infty} E[\text{NLS}(-t_n^j) \tilde{\psi}^j (x - x_n^j)] \leq (ME)_c. \]
Taking $s = 0$ in (3.23)

\[ \sum_{j=1}^{M} M[\tilde{\psi}^j(x - x_n^j)] + \lim_{n \to \infty} \|\tilde{W}_n^M\|_{L^2} = M[u_c] = 1. \]

Note that, in the decomposition (3.64) either we have more than one $\tilde{\psi}^j \neq 0$ or only one $\tilde{\psi}^j \neq 0$ and $\tilde{\psi}^j = 0$ for all $2 \leq j < M$. Following the argument of Proposition 3.11, we show that only the second case occurs:

\[ u_c(\tau_n) = \text{NLS}(-t_1^n \tilde{\psi}^1(x - x_n^1) + \tilde{W}_n^1(x) \]

such that

\[ M[\tilde{\psi}^1] = 1, \quad \lim_{n \to \infty} E[\text{NLS}(-t_1^n \tilde{\psi}^1(x - x_n^1))] = (ME)_c, \]

\[ \lim_{n \to \infty} M[\tilde{W}_n^M] = 0 \quad \text{and} \quad \lim_{n \to \infty} E[\tilde{W}_n^M] = 0. \]

Lemma 2.16 implies

\[ \lim_{n \to \infty} \|\tilde{W}_n^M\|_{H^1} = 0. \] (3.66)

The sequence $x_n^1$ will create a path $x(t)$ by continuity. We now show that $t_1^n$ has a convergence subsequence $\tilde{t}_n^1$.

Assume that $\tilde{t}_n^1 \to -\infty$, apply NLS($t$) to (3.65) implies then triangle inequality yields

\[ \|\text{NLS}(t)u_c(\tau_n)\|_{j^0_{\delta(S(H^s,[0,+)\}}) \leq \|\text{NLS}(t - \tilde{t}_n^1)\tilde{\psi}^1(x - x_n^1)\|_{j^0_{\delta(S(H^s,[0,+)\}}} + \|\text{NLS}(t)\tilde{W}_n^M(x)\|_{j^0_{\delta(S(H^s,[0,+)\})}. \]

Note

\[ \lim_{n \to +\infty} \|\text{NLS}(t - \tilde{t}_n^1)\tilde{\psi}^1(x - x_n^1)\|_{j^0_{\delta(S(H^s,[0,+)\}}} = \lim_{n \to +\infty} \|\text{NLS}(t)\tilde{\psi}^1(x - x_n^1)\|_{j^0_{\delta(S(H^s,[0,+)\}} = 0, \]

and

\[ \|\text{NLS}(t)\tilde{W}_n^M\|_{j^0_{\delta(S(H^s)}} \leq \frac{1}{2} \delta_{sd}, \]

thus, taking $n$ sufficiently large, the small data scattering theory (Proposition 2.13) implies

\[ \|u_c\|_{j^0_{\delta(S(H^s,(-\infty,\tau_n)))}} \leq \delta_{sd} \]

a contradiction.

In a similar fashion, assuming that $\tilde{t}_n^1 \to +\infty$, we obtain that for $n$ large,

\[ \|\text{NLS}(t)u_c(\tau_n)\|_{j^0_{\delta(S(H^s,(-\infty,0))}} \leq \frac{1}{2} \delta_{sd}, \]

and thus, the small data scattering theory (Proposition 2.13) shows that

\[ \|u_c\|_{j^0_{\delta(S(H^s,(-\infty,\tau_n)))) \leq \delta_{sd}. \] (3.67)

Taking $n \to +\infty$ implies $\tau_n \to +\infty$, thus (3.67) becomes $\|u_c\|_{j^0_{\delta(S(H^s,(-\infty,+)}}} \leq \delta_{sd}$, a contradiction. Thus, $\tilde{t}_n^1$ must converge to some finite $t^1$.

Since (3.66) holds and $\text{NLS}(t_1^n)\tilde{\psi}^1 \to \text{NLS}(t^1)\tilde{\psi}^1$ in $H^1$, (3.65) implies $u_c(\tau_n)$ converges in $H^1$. \hfill \Box
Corollary 3.13. (Precompactness of the flow implies uniform localization.) Assume $u$ is a solution to (1.1) such that

$$K = \{ u(\cdot - x(t), t) | t \in [0, +\infty) \}$$

is precompact in $H^1(\mathbb{R}^d)$. Then for each $\epsilon > 0$, there exists $R > 0$, so that for all $0 \leq t < \infty$

$$\int_{|x + x(t)| > R} |\nabla u(x, t)|^2 + |u(x, t)|^2 + |u(x, t)|^{p+1} dx < \epsilon. \tag{3.68}$$

Furthermore, $\|u(t, \cdot - x(t))\|_{H^1(\{|x| > R\})} < \epsilon$.

Proof. Assume (3.68) does not hold, i.e., there exists $\epsilon > 0$ and a sequence of times $t_n$ such that for any $R > 0$, we have

$$\int_{|x + x(t_n)| > R} |\nabla u(x, t_n)|^2 + |u(x, t_n)|^2 + |u(x, t_n)|^{p+1} dx \geq \epsilon.$$

Changing variables, we get

$$\int_{|x| > R} |\nabla u(x - x(t_n), t_n)|^2 + |u(x - x(t_n), t_n)|^2 + |u(x - x(t_n), t_n)|^{p+1} dx \geq \epsilon. \tag{3.69}$$

Note that since $K$ is precompact, there exists $\phi \in H^1$ such that, passing to a subsequence of $t_n$, we have $u(\cdot - x(t_n), t_n) \to \phi$ in $H^1$. For all $R > 0$, (3.69) implies

$$\forall R > 0, \int_{|x| > R} |\nabla \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^4 \geq \epsilon,$$

which is a contradiction with the fact that $\phi \in H^1$. Thus, (3.68) and $\|u(t, \cdot - x(t))\|_{H^1(\{|x| > R\})} < \epsilon$ hold.

Lemma 3.14. Let $u(t)$ be a solution of $\text{NLS}_p(\mathbb{R}^d)$ defined on $[0, +\infty)$ such that $P[u] = 0$ and either

(a) $K = \{ u(\cdot - x(t), t) | t \in [0, +\infty) \}$ is precompact in $H^1(\mathbb{R}^d)$, or

(b) for all $0 < t$,

$$\|u(t) - e^{i\theta(t)} u_Q(\cdot - x(t))\|_{H^1} \leq \epsilon_1 \tag{3.70}$$

for some continuous function $\theta(t)$ and $x(t)$. Then

$$\lim_{t \to +\infty} \frac{x(t)}{t} = 0. \tag{3.71}$$

Proof of this Lemma can be found in [Gue11] or adjusted from [DHR08].

Theorem 3.15. (Rigidity Theorem.) Let $u_0 \in H^1$ satisfy $P[u_0] = 0$, $\mathcal{M}E[u_0] < 1$ and $G_0(0) < 1$. Let $u$ be the global $H^1(\mathbb{R}^d)$ solution of $\text{NLS}_p(\mathbb{R}^d)$ with initial data $u_0$ and suppose that $K = \{ u_0(\cdot - x(t), t) | t \in [0, +\infty) \}$ is precompact in $H^1$, then $u_0 \equiv 0$.

Proof. Let $\phi \in C_0^\infty$ be radial, with

$$\phi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$
For $R > 0$ define
\[
z_R(t) = \int R^2 \phi \left( \frac{x}{R} \right) |u(x, t)|^2 dx. \tag{3.72}
\]
Then
\[
z_R'(t) = 2 \text{Im} \int R \nabla \phi \left( \frac{x}{R} \right) \cdot \nabla u(t) \bar{u}(t) dx,
\tag{3.73}
\]
and Hölder’s inequality yields
\[
|z_R'(t)| \leq cR \int_{\{|x| \leq 2R\}} |\nabla u(t)||u(t)| dx \leq cR \|u(t)\|_{L^2}^{2(1-s)} \|\nabla u(t)\|_{L^2}^{2s}.
\tag{3.74}
\]
Note that,
\[
z_R''(t) = 4 \sum_j \int \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left( \frac{|x|}{R} \right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} - \frac{1}{R^2} \int \Delta^2 \phi \left( \frac{|x|}{R} \right) |u|^2
- 4 \left( \frac{1}{2} - \frac{1}{p+1} \right) \int \Delta \phi \left( \frac{|x|}{R} \right) |u|^{p+1}.
\tag{3.75}
\]
Since $\phi$ is radial, we have
\[
z_R''(t) = 8 \int |\nabla u|^2 - \frac{4d(p-1)}{p+1} \int |u|^{p+1} + A_R(u(t)), \tag{3.76}
\]
where
\[
A_R(u(t)) = 4 \sum_j \int \left( \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left( \frac{|x|}{R} \right) - 2 \right) \left| \frac{\partial u}{\partial x_j} \right|^2 + 4 \sum_{j \neq k} \int_{R \leq |x| \leq 2R} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left( \frac{|x|}{R} \right)
- \frac{1}{R^2} \int \Delta^2 \phi \left( \frac{|x|}{R} \right) |u|^2 - 4 \left( \frac{1}{2} - \frac{1}{p+1} \right) \int \left( \Delta \phi \left( \frac{|x|}{R} \right) - 2d \right) |u|^{p+1}.
\tag{3.77}
\]
Thus,
\[
|A_R(u(t))| = c \int_{|x| \geq R} \left( |\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^{p+1} \right) dx. \tag{3.78}
\]
Choosing $R$ large enough, over a suitably chosen time interval $[t_0, t_1]$, with $0 \ll t_0 \ll t_1 < \infty$, combining (3.76) and (2.49), we obtain
\[
|z_R''(t)| \geq 16(1 - \omega^{p-1}) E[u] - |A_R(u(t))|. \tag{3.79}
\]
From Corollary 3.13, letting $\epsilon = \frac{1-\omega^{p-1}}{c}$, with $c$ as in (3.78), we can obtain $R_0 \geq 0$ such that for all $t$,
\[
\int_{|x+x(t)| > R_0} \left( |\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^{p+1} \right) \leq \frac{1 - \omega^{p-1}}{c} E[u]. \tag{3.80}
\]
Thus combining (3.78), (3.79) and (3.80), and taking $R \geq R_0 + \sup_{t_0 \leq t \leq t_1} |x(t)|$, gives that for all $t_0 \leq t \leq t_1$,
\[
|z''(t)| \geq 8(1 - \omega^{p-1}) E[u]. \tag{3.81}
\]
SCATTERING AND BLOW-UP

By Lemma 3.14 there exists \( t_0 \geq 0 \) such that for all \( t \geq t_0 \), we have \( |x(t)| \leq \gamma t \). Taking \( R = R_0 + \gamma t_1 \), we have that (3.81) holds for all \( t \in [t_0, t_1] \). Thus, integrating it over this interval, we obtain

\[
|z'_R(t_1) - z'_R(t_0)| \geq 8(1 - \omega^{p-1})E[u](t_1 - t_0).
\]

(3.82)

In addition, for all \( t \in [t_0, t_1] \), combining (3.74), \( G_u(0) < 1 \), and Lemma 2.17 we have

\[
|z'_R(t)| \leq cR\|u(t)\|^{2(1-s)}\|\nabla u(t)\|^2_{L^2} \leq 2cR\|u_Q\|^{2(1-s)}\|\nabla u_Q\|^2_{L^2}.
\]

(3.83)

\[
\leq c\|u_Q\|^{2(1-s)}\|\nabla u_Q\|^2_{L^2}(R_0 + \gamma t_1).
\]

Combining (3.82) and (3.83) yields

\[
8(1 - \omega^{p-1})E[u](t_1 - t_0) \leq 2c\|u_Q\|^{2(1-s)}\|\nabla u_Q\|^2_{L^2}(R_0 + \gamma t_1).
\]

(3.84)

Observe that, \( \omega \), and \( R_0 \) are constants depending on \( \mathcal{M}E[u] \), and \( t_0 = t(\gamma) \). Let \( \gamma = \frac{(1-\omega^{p-1})E[u]}{c\|u_Q\|^{2(1-s)}\|\nabla u_Q\|^2_{L^2}} > 0 \). Then (3.84) yields

\[
6(1 - \omega^{p-1})E[u]t_1 \leq 2c\|u_Q\|^{2(1-s)}\|\nabla u_Q\|^2_{L^2}R_0 + 8(1 - \omega^{p-1})E[u]t_0.
\]

(3.85)

Now sending \( t_1 \to +\infty \), implies that the left hand side of (3.85) goes to \( \infty \) and the right hand side is bounded, which is a contradiction, unless \( E[u] = 0 \) which implies \( u \equiv 0 \). □

4. Weak blowup via Concentration Compactness

In this section, we complete the proof of Theorem A\* part II (b), i.e., if under the mass-energy threshold \( \mathcal{M}E[u] < 1 \), a solution \( u(t) \) to NLS\(_p(\mathbb{R}^d) \) with the initial condition \( u_0 \in H^1 \) such that \( G_u(0) > 1 \) exists globally for all positive time, then there exists a sequence of times \( t_n \to +\infty \) such that \( G_u(t_n) \to +\infty \). We call this solution a “weak blowup” solution.

**Definition 4.1.** Let \( \lambda > 0 \). The horizontal line for which

\[
M[u] = M[u_Q] \quad \text{and} \quad \frac{E[u]}{E[u_Q]} = \frac{d}{2s} \lambda^{\frac{2}{\lambda}} \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right)
\]

is called the “mass-energy” line for \( \lambda \).

Notice that in Definition 4.1 the renormalized energy definition comes naturally by expressing the energy in terms of the gradient which is assumed to be \( \lambda \). We illustrate the mass-energy line notion in Figure 2.

4.1. Outline for Weak blowup via Concentration Compactness. Suppose that there is no finite time blowup for a nonradial and infinite variance solution (from Theorem A\* part II), thus, the existence on time (say, in forward direction) is infinite \( (T^* = +\infty) \). Now, under the assumption of global existence, we study the behavior of \( G_u(t) \) as \( t \to +\infty \), and use a concentration compactness type argument for establishing the divergence of \( G_u(t) \) in \( H^1 \)-norm as it was developed in [HR10d], note that the concentration compactness and the rigidity argument is not used here to prove scattering but to prove for a blowup property. The description of this argument is in steps 1, 2 and 3.

**Step 1:** Near boundary behavior.
Theorem A* II part (a) yields $G_u(t) > 1$ for all $t \in (T_*, T^*)$ whenever $G_u(0) > 1$ on the “mass-energy” line for some $\lambda > 1$. We illustrate this in Figure 2: given $u_0 \in H^1$, we first determine $M[u_0]$ and $E[u_0]$ which specifies the “mass-energy” line GH. Then the gradient $G_u(t)$ of a solution $u(t)$ lives on the line GH. Note that $G_u(t) > \lambda_2 > 1$ if $G_u(0) > 1$. A natural question is whether $G_u(t)$ can be, with time, much larger than 1 or $\lambda_2$. Proposition 4.6 shows that it can not. Thus, we prove that the renormalized gradient $\mathcal{G}_u(t)$ can not forever remain near the boundary if originally $\mathcal{G}_u(0)$ is very close to it, that is, if $\lambda_0 > 1$, there exists $\rho_0(\lambda_0) > 0$ such that for all $\lambda > \lambda_0$ there is NO solution at the “mass-energy”
\[ x = [\mathcal{G}_u(t)]^2 \]

\[ y = (\mathcal{M}[\mathcal{E}[u]])^{\frac{1}{4}} \]

**Figure 3.** Near boundary behavior of \( \mathcal{G}(t) \). We investigate whether the solution can remain close to the boundary (see the dash dot line KL) for all time.

line for \( \lambda \) satisfying

\[ \lambda \leq \mathcal{G}_u(t) \leq \lambda(1 + \rho_0). \]

Using the Figure 3, this means that the solution \( u(t) \) would have a gradient \( \mathcal{G}_u(t) \) very close to the boundary DF (for all times), i.e., between the boundary DF and the dashed line KL. We will show that \( \mathcal{G}_u(t) \) on any “mass-energy” line with \( \mathcal{M}[u] < 1 \) and \( \mathcal{G}_u(0) > 1 \) will escape to infinity (along this line). By contradiction, assume that all solutions (starting from some mass-energy line corresponding to the initial renormalized gradient \( \mathcal{G}_u(0) = \lambda_0 > 1 \)) are bounded in renormalized gradient for all \( t > 0 \).

Step 1 gives the basis for induction, giving that when \( \lambda > 1 \), any solution \( u(t) \) of NLS\(_p\)(\( \mathbb{R}^d \)) at the “mass-energy” line for this \( \lambda \) can not have a renormalized gradient \( \mathcal{G}_u(t) \) bounded near the boundary DF for all time (see Figure 3). We will show that \( \mathcal{G}_u(t) \), in fact, will tend to \( +\infty \) (at least along an infinite time sequence).

**Definition 4.2.** Let \( \lambda > 1 \). We say the property GBG(\( \lambda, \sigma \)) holds if there exists a solution \( u(t) \) of NLS\(_p\)(\( \mathbb{R}^d \)) at the mass-energy line \( \lambda \) (i.e., \( M[u] = M[u_Q] \) and \( \frac{E[u]}{E[u_Q]} = \frac{d}{2s} \lambda^2 \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right) \)) such that \( \lambda \leq \mathcal{G}_u(t) \leq \sigma \) for all \( t \geq 0 \). Figure 4 illustrates this definition.

In other words, GBG(\( \lambda, \sigma \)) is not true if for every solution \( u(t) \) of NLS\(_p\)(\( \mathbb{R}^d \)) at the “mass-energy” line for \( \lambda \), such that \( \lambda \leq \mathcal{G}_u(t) \) for all \( t > 0 \), there exists \( t^* \) such that \( \sigma < \mathcal{G}_u(t^*) \). Iterating, we conclude that, there exists a sequence \( \{t_n\} \to \infty \) with \( \sigma < \mathcal{G}_u(t_n) \) for all \( n \).

---

GBG stands for *globally bounded gradient.*
Figure 4. On this graph the statement “GBG(λ, σ) holds” shows that \( G(t) \) is only on the segment GJ.

Note that, if GBG(λ, σ) does not hold, then for any σ′ < σ, GBG(λ, σ′) does not hold either. This will allow us induct on the GBG notion.

**Definition 4.3.** Let \( λ_0 > 1 \). We define the critical threshold \( σ_c \) by

\[
σ_c = \sup \{ σ | σ > λ_0 \text{ and } \text{GBG}(λ, σ) \text{ does NOT hold for all } λ \text{ with } λ_0 \leq λ \leq σ \}.
\]

Note that \( σ_c = σ_c(λ_0) \) stands for “σ-critical”.

From the step 1 (Proposition 4.6) we have that GBG(λ, λ(1 + \( ρ_0(λ_0) \) )) does not hold for all \( λ \geq λ_0 \).

Step 2: Induction argument.

Let \( λ_0 > 1 \). We would like to show that \( σ_c(λ_0) = +∞ \). Arguing by contradiction, we assume \( σ_c(λ_0) \) is finite.

Let \( u(t) \) be a solution to NLS\(_p(\mathbb{R}^d) \) with initial data \( u_{n,0} \) at the “mass-energy” line for \( λ > λ_0 \), i.e., \( \frac{E[u]}{E[u_Q]} = \frac{d}{2σ} λ^2 \left( 1 - \frac{λ^p - 1}{α^2} \right) \), \( M[u] = M[u_Q] \) and \( G_u(0) > 1 \). We want to show that there exists a sequence of times \( \{ t_n \} \to +∞ \) such that \( G_u(t_n) \to ∞ \). Suppose the opposite, that is, such sequence of times does not exist.

Then there exists \( σ < ∞ \) satisfying \( λ \leq G_u(t) \leq σ \) for all \( t \geq 0 \), i.e., GBG(λ, σ) holds with \( σ_c(λ_0) ≤ σ < ∞ \). At this point we can apply Proposition 3.6 (the nonlinear profile decomposition).

The nonlinear profile decomposition of the sequence \( \{ u_{n,0} \} \) and profile reordering will allow us to construct a “critical threshold solution” \( u(t) = u_c(t) \) to NLS\(_p(\mathbb{R}^d) \) at the “mass-energy” line \( λ_c \), where \( λ_0 < λ_c < σ_c(λ_0) \) and \( λ_c < G_{u_c}(t) < σ_c(λ_0) \) for all \( t > 0 \) (see Existence of threshold solution Lemma 4.8).
Step 3: Localization properties of critical threshold solution.
By construction, the critical threshold solution \( u_c(t) \) will have the property that the set
\( K = \{ u(\cdot - x(t), t) | t \in [0, +\infty) \} \) has a compact closure in \( H^1 \) (Lemma 4.9). Thus, we will have uniform concentration of \( u_c(t) \) in time, which together with the localization property (Corollary 3.13) implies that for a given \( \epsilon > 0 \), there exists an \( R > 0 \) such that
\[ \| \nabla u(x, t) \|_{L^2(|x + x(t)| > R)}^2 \leq \epsilon \] uniformly in \( t \) ; as a consequence, \( u_c(t) \) blows up in finite time (Lemma 4.10), that is, \( \sigma_c = +\infty \), which contradicts the fact that \( u_c(t) \) is bounded in \( H^1 \). Thus, \( u_c(t) \) can not exist since our assumption that \( \sigma_c(\lambda_0) < \infty \) is false, and this ends the proof of the “weak blowup”.

In the rest of this chapter we proceed with the proof of claims described in Step 1, 2 and 3.

First, recall variational characterization of the ground state.

4.2. Variational Characterization of the Ground State. Proposition 4.4 is a restate-
ment of Proposition 4.4 [HR10c] adjusted for our general case, and shows that if a solution
\( u(t, x) \) is close to \( u_Q(t, x) \) in mass and energy, then it is close to \( u_Q \) in \( H^1(\mathbb{R}^d) \), up to a phase and shift in space. The proof is identical so we omit it.

Proposition 4.4. There exists a function \( \epsilon(\rho) \) defined for small \( \rho > 0 \) with \( \lim_{\rho \to 0} \epsilon(\rho) = 0 \), such that for all \( u \in H^1(\mathbb{R}^d) \) with
\[ \| u \|_{L^p} - \| u_Q \|_{L^p} + \| u \|_{L_2} - \| u_Q \|_{L_2} + \| \nabla u \|_{L^2} - \| \nabla u_Q \|_{L^2} \leq \rho, \]
there is \( \theta_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^d \) such that
\[ \| u - e^{i\theta_0}u_Q(\cdot - x_0) \|_{H^1} \leq \epsilon(\rho). \] (4.1)

The Proposition 4.5 is a variant of Proposition 4.1 [HR10c], rephrased for our case.

Proposition 4.5. There exists a function \( \epsilon(\rho) \) such that \( \epsilon(\rho) \to 0 \) as \( \rho \to 0 \) satisfying the following: Suppose there exists \( \lambda > 0 \) such that
\[ (\mathcal{M}[u])^\frac{1}{2} - \frac{d}{2s} \lambda^\frac{2}{s} \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right) \leq \rho \lambda^{\frac{2(p-1)}{s}} \] (4.2)

and
\[ [G_u(t)]^\frac{1}{2} - \lambda \leq \rho \begin{cases} \lambda^\frac{2}{s} & \text{if} \quad \lambda \leq 1 \\ \lambda & \text{if} \quad \lambda \geq 1 \end{cases}. \] (4.3)

Then there exist \( \theta_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^d \) with \( \kappa = \left( \frac{M[u]}{M[u_Q]} \right)^\frac{1}{1-s} \) such that
\[ \left\| u(x) - e^{i\theta_0} \lambda \kappa^{-\frac{1}{1-s}} u_Q \left( \lambda \kappa^{-\frac{1}{1-s}} x - x_0 \right) \right\|_{L^2} \leq \kappa^{\frac{1}{2(1-s)} + \epsilon(\rho)}, \]
and
\[ \left\| \nabla \left[ u(x) - e^{i\theta_0} \lambda \kappa^{-\frac{1}{1-s}} u_Q \left( \lambda \kappa^{-\frac{1}{1-s}} x - x_0 \right) \right] \right\|_{L^2} \leq \lambda \kappa^{-\frac{1}{2(1-s)} + \epsilon(\rho)}. \]
Proof. Set \( v(x) = \kappa^{\frac{2}{p-1}} u(\kappa^{\frac{2}{p-1}} x) \), hence \( M[v] = \kappa^{\frac{2}{p-1}} M[u] \). Assume \( M[v] = M[u_Q] \), Then there exists \( \lambda > 0 \) such that (4.2) and (4.3) become
\[
\left| \frac{E[v]}{E[u_Q]} - \frac{d}{2s} \lambda^2 \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right) \right| \leq \rho_0 \lambda^{\frac{2(p-1)}{s}} ,
\]
and
\[
\left| \frac{\|\nabla v\|_{L^2}}{\|\nabla u_Q\|_{L^2}} - \lambda \right| \leq \rho_0 \left\{ \begin{array}{ll}
\lambda^\gamma & \text{if } \lambda \leq 1 \\
1 & \text{if } \lambda \geq 1
\end{array} \right.
\]
(4.5)
Letting \( \tilde{u}(x) = \lambda^{\frac{2((p-1)(d-2)-d)}{(p-1)(d-2)-4s}} v(\lambda^{\frac{2((p-1)(d-2)-d)}{(p-1)(d-2)-4s}} x) \), we have
\[
\left| \frac{\|\nabla \tilde{u}\|_{L^2}}{\|\nabla u_Q\|_{L^2}} - 1 \right| \leq \rho_0 \left\{ \begin{array}{ll}
\lambda^\gamma & \text{if } \lambda \leq 1 \\
1 & \text{if } \lambda \geq 1
\end{array} \right. \leq \rho_0 .
\]
(4.6)
Combining Pohozhaev identities, (4.4) and (4.5), gives
\[
\frac{d}{2s\alpha^2} \left| \frac{\|v\|_{L_{p+1}^p}}{\|u_Q\|_{L_{p+1}^p}} - \lambda^{\frac{2(p-1)}{s}} \right| \leq \left| \frac{E[v]}{E[u_Q]} - \left( \frac{d}{2s} \lambda^2 \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right) \right) \right| + \frac{d}{2s} \left| \frac{\|\nabla v\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} - \lambda^2 \right|
\leq \rho_0 \left( \lambda^{\frac{2(p-1)}{s}} + \frac{d}{2s} \left\{ \begin{array}{ll}
\lambda^\gamma & \text{if } \lambda \leq 1 \\
1 & \text{if } \lambda \geq 1
\end{array} \right. \right) \leq \frac{d + 2s}{2s} \rho_0 \lambda^{\frac{2(p-1)}{s}} .
\]
This yields
\[
\left| \frac{\|\tilde{u}\|_{L_{p+1}^p}}{\|u_Q\|_{L_{p+1}^p}} - 1 \right| \leq \frac{\alpha^2(d + 2s)}{d} \rho_0 ,
\]
(4.7)
From (4.6) and (4.7) we have
\[
\left| \|\tilde{u}\|_{L_{p+1}^p} - \|u_Q\|_{L_{p+1}^p} \right| + \left| \|\tilde{u}\|_{L^2} - \|u_Q\|_{L^2} \right| + \left| \|\nabla \tilde{u}\|_{L^2} - \|\nabla u_Q\|_{L^2} \right| \leq C(\|u_Q\|_{L^2}) \rho_0 .
\]
Let \( \rho = \frac{\rho_0}{C(\|u_Q\|_{L^2})} \), then by Proposition 4.3 there exist \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^d \) such that (4.4) holds for \( \tilde{u} \). Rescaling to \( v \) and then to \( u \), completes the proof. \( \Box \)

Next proposition is “close to the boundary” behavior.

**Proposition 4.6.** Fix \( \lambda_0 > 1 \). There exists \( \rho_0 = \rho_0(\lambda_0) > 0 \) (with the property that \( \rho_0 \to 0 \) as \( \lambda_0 \to 1 \)) such that for any \( \lambda \geq \lambda_0 \), there is NO solution \( u(t) \) of NLS\( _p(\mathbb{R}^d) \) with \( P[u]=0 \) satisfying \( \|u\|_{L^2} = \|u_Q\|_{L^2} \) and \( \frac{E[u]}{E[u_Q]} = \frac{d}{2s} \lambda^2 \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right) \) (i.e., on any “mass-energy” line corresponding to \( \lambda \geq \lambda_0 \) and \( ME < 1 \)) with \( \lambda \leq \mathcal{G}_u(t) \leq \lambda(1 + \rho_0) \) for all \( t \geq 0 \). A similar statement holds for \( t \leq 0 \).

**Proof.** To the contrary, assume that there exists a solution \( u(t) \) of (1.1) with \( \|u\|_{L^2} = \|u_Q\|_{L^2} \),
\[
\frac{E[u]}{E[u_Q]} = \frac{d}{2s} \lambda^2 \left( 1 - \frac{\lambda^{p-1}}{\alpha^2} \right) \text{ and } \mathcal{G}_u(t) \in [\lambda, \lambda(1 + \rho_0)] .
\]
By continuity of the flow $u(t)$ and Proposition 4.5, there are continuous $x(t)$ and $\theta(t)$ such that

$$\|u(x) - e^{i\theta_0 \lambda Q(x - x_0)}\|_{L^2} \leq \epsilon(\rho), \quad (4.8)$$

and

$$\|\nabla \left[u(x) - e^{i\theta_0 \lambda Q(x - x_0)}\right]\|_{L^2} \leq \lambda \epsilon(\rho). \quad (4.9)$$

Define $R(T) = \max \left\{ \max_{0 \leq t \leq T} |x(t)|, \log \epsilon(\rho)^{-1} \right\}$. Consider the localized variance (3.72). Note

$$\frac{d}{s} \lambda^\frac{2}{p} E[u_Q] = \lambda^\frac{2}{p} \|\nabla u_Q\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2,$$

then,

$$z''_R = 4d(p - 1) E[u] - (2d(p - 1) - 8) \|\nabla u\|_{L^2}^2 + A_R(u(t))$$

$$= 16\alpha^2 E[u] - 8(\alpha^2 - 1) \|\nabla u\|_{L^2}^2 + A_R(u(t)) \leq -8\frac{d}{s} \lambda^\frac{2}{p} (\lambda^p - 1) E[u_Q] + A_R(u(t)),$$

where $A_R(u(t))$ is given by (3.77).

Let $T > 0$ and for the local virial identity (3.75) assume $R = 2R(T)$. Therefore, (4.8) and (4.9) assure that there exists $c_1 > 0$ such that

$$|A_R(u(t))| \leq c_1 \lambda^2 (\epsilon(\rho) + e^{-R(T)}) \leq \tilde{c}_1 \lambda^2 \epsilon(\rho)^2.$$

Taking a suitable $\rho_0$ small (i.e., $\lambda > 1$ is taken closer to 1), such that for $0 \leq t \leq T$, $\epsilon(\rho)$ is small enough, we get

$$z''(t) \leq -8\frac{d}{s} \lambda^\frac{2}{p} (\lambda^p - 1) E[u_Q].$$

Integrating $z''_R(t)$ in time over $[0, T]$ twice, we obtain

$$\frac{z_R(T)}{T^2} \leq \frac{z_R(0)}{T^2} + \frac{z'_R(0)}{T} - 8\frac{d}{s} \lambda^\frac{2}{p} (\lambda^p - 1) E[u_Q].$$

Note $\sup_{x \in \mathbb{R}^d} \phi(x)$ from (3.72), is bounded, say by $c_2 > 0$. Then from (3.72) we have

$$|z_R(0)| \leq c_2 R^2 \|u_0\|_{L^2}^2 = c_2 R^2 \|u_Q\|_{L^2}^2,$$

and by (3.74)

$$|z'_R(0)| \leq c_3 R \|u_0\|_{L^2}^{2(1-s)} \|\nabla u_0\|_{L^2}^{2s} \leq c_3 R \|u_Q\|_{L^2}^{2(1-s)} \|\nabla u_Q\|_{L^2}^{2s} \lambda^\frac{1}{s}(1 + \rho_0).$$

Taking $T$ large enough so that by Lemma 3.14 we have $\frac{R(T)}{T} < \epsilon(\rho)$, we estimate

$$\frac{z_{2R(T)}(T)}{T^2} \leq c_4 \left( \frac{R(T)^2}{T^2} + \frac{R(T)}{T} \right) - 4\frac{d}{s} \lambda^\frac{2}{p} (\lambda^p - 1) E[u_Q]$$

$$\leq C(\epsilon(\rho)^2 + \epsilon(\rho)) - 4\frac{d}{s} \lambda^\frac{2}{p} (\lambda^p - 1) E[u_Q].$$

We can initially choose $\rho_0$ small enough (and thus, $\epsilon(\rho_0)$) such that $C(\epsilon(\rho)^2 + \epsilon(\rho)) < 4\frac{d}{s} \lambda^\frac{2}{p} (\lambda^p - 1) E[u_Q]$. We obtain $0 \leq z_{2R(T)}(T) < 0$, which is a contradiction, showing that
our initial assumption about the existence of a solution to (1.1) with bounded $G_u(t)$ does not hold. □

Before we exhibit the existence of a critical element/solution, we return to the nonlinear profile decomposition (Proposition 3.6) and introduce reordering.

**Lemma 4.7** (Profile reordering). Suppose $\phi_n = \phi_n(x)$ is a bounded sequence in $H^1(\mathbb{R}^d)$. Let $\lambda_0 > 1$. Assume that $M[\phi_n] = M[u_Q]$ and $E[\phi_n] = \frac{d}{2s} \lambda_n^\frac{2}{s}(1 - \frac{\lambda_n^{p-1}}{\alpha^2})$ such that $1 < \lambda_0 \leq \lambda_n$ and $\lambda_n \leq G_{\phi_n}(t)$ for each $n$. Apply Proposition 3.6 to the sequence $\{\psi_n\}$ and obtain nonlinear profiles $\{\tilde{\psi}^j\}$. Then, these profiles $\tilde{\psi}^j$ can be reordered so that there exist $1 \leq M_1 \leq M_2 \leq M$ and

(1) For each $1 \leq j \leq M_1$, we have $t_n^j = 0$ and $v^j(t) \equiv \text{NLS}(t)\tilde{\psi}^j$ does not scatter as $t \to +\infty$. (In particular, there is at least one such $j$)

(2) For each $M_1 + 1 \leq j \leq M_2$, we have $t_n^j = 0$ and $v^j(t)$ scatters as $t \to +\infty$. (If $M_1 = M_2$, there are no $j$ with this property.)

(3) For each $M_2 + 1 \leq j \leq M$, we have $|t_n^j| \to \infty$ and $v^j(t)$ scatters as $t \to +\infty$. (If $M_2 = M$, there are no $\tilde{j}$ with this property.)

**Proof.** Pohozaev identities (2.39) and energy definition yield

$$\left(\frac{\|\phi_n\|_{L^{p+1}}}{\|u_Q\|_{L^{p+1}}}\right)^{p+1} = \frac{d}{2s} M[\phi_n(t)]^\frac{2}{s} - \frac{2s}{2s} E[\phi_n] \geq \lambda_n \frac{2(p-1)}{s} \geq \lambda_0 \frac{2(p-1)}{s} > 1.$$ 

Notice that if $j$ is such that $|t_n^j| \to \infty$, then $\|\text{NLS}(-t_n^j)\tilde{\psi}^j\|_{L^{p+1}} \to 0$, and by (3.33) we have that $\frac{\|\phi_n\|_{L^{p+1}}}{\|u_Q\|_{L^{p+1}}} \to 0$. Therefore, there exists at least one $j$ such that $t_n^j$ converges. Without loss of generality, assume $t_n^j = 0$, and reorder the profiles such that for $1 \leq j \leq M_2$, we have $t_n^j = 0$ and for $M_2 + 1 \leq j \leq M$, we have $|t_n^j| \to 0$.

It is left to prove that there exists at least one $j$, $1 \leq j \leq M_2$ such that $v^j(t)$ is not scattering. Assume that for all $1 \leq j \leq M_2$ we have that all $v^j$ are scattering, and thus, $\|v^j(t)\|_{L^{p+1}} \to 0$ as $t \to +\infty$. Let $\epsilon > 0$ and $t_0$ large enough such that for all $1 \leq j \leq M_2$ we have $\|v^j(t)\|_{L^{p+1}} \leq \epsilon/M_2$. Using $L^{p+1}$ orthogonality (3.56) along the NLS flow, and letting $n \to +\infty$, we obtain

$$\lambda_0 \frac{2(p-1)}{s} \|u_Q\|_{L^{p+1}}^{p+1} \leq \|u_n(t)\|_{L^{p+1}}^{p+1} = \sum_{j=1}^{M_2} \|v^j(t_0)\|_{L^{p+1}}^{p+1} + \sum_{j=M_2+1}^{M} \|v^j(t_0 - t_n^j)\|_{L^{p+1}}^{p+1} + \|W_n^M(t)\|_{L^{p+1}}^{p+1} + o_n(1)$$

$$\leq \epsilon + \|W_n^M(t)\|_{L^{p+1}}^{p+1} + o_n(1).$$

The last line is obtained since $\sum_{j=M_2+1}^{M} \|v^j(t_0 - t_n^j)\|_{L^{p+1}}^{p+1} \to 0$ as $n \to \infty$, and gives a contradiction. □

Recall that we have a fixed $\lambda_0 > 1$. 


Lemma 4.8 (Existence of a threshold solution). There exists initial data $u_{c,0} \in H^1(\mathbb{R}^d)$ and $1 < \lambda_0 \leq \lambda_c \leq \sigma_c(\lambda_0)$ such that $u_c(t) \equiv \text{NLS}(t)u_{c,0}$ is a global solution with $M[u] = M[u_Q]$, $\frac{E[u_c]}{E[u_Q]} = \frac{d}{2s} \lambda_c^2 \left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$ and, moreover, $\lambda_c \leq \mathcal{G}_{u_c}(t) \leq \sigma_c$ for all $t \geq 0$.

Proof. Definition of $\sigma_c$ implies the existence of sequences $\{\lambda_n\}$ and $\{\sigma_n\}$ with $\lambda_0 \leq \lambda_n \leq \sigma_n$ and $\sigma_n \searrow \sigma_c$ such that $\text{GBG}(\lambda_n, \sigma_n)$ is false. This means that there exists $u_{n,0}$ with $M[u] = M[u_Q]$, $\frac{E[u_{n,0}]}{E[u_Q]} = \frac{d}{2s} \lambda_n^2 \left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$ and $\lambda_n \leq \frac{\|\nabla u\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} = [\mathcal{G}_u(t)]^{\frac{1}{2}} \leq \sigma_c$, such that $u_n(t) = \text{NLS}(t)u_{n,0}$ is global.

Note that the sequence $\{\lambda_n\}$ is bounded, thus, we pass to a convergent subsequence $\{\lambda_{n_k}\}$. Assume $\lambda_{n_k} \to \lambda'$ as $n_k \to \infty$, thus $\lambda_0 \leq \lambda' \leq \sigma_c$.

We apply the nonlinear profile decomposition (Proposition 3.6) and reordering (Lemma 4.7).

In Lemma 4.7 let $\phi_n = u_{n,0}$. Recall that $v^j(t)$ scatters as $t \to \infty$ for $M_1 + 1 \leq j \leq M_2$, and by Proposition 3.6 $v^j(t)$ also scatter in one or the other time direction for $M_2 + 1 \leq j \leq M$ and $E[\tilde{\psi}^j] = E[\tilde{\psi}] \geq 0$. Thus, by the Pythagorean decomposition for the nonlinear flow (3.24) we have

$$\sum_{j=1}^{M_1} E[\tilde{\psi}^j] \leq E[\phi_n] + o_n(1).$$

For at least one $1 \leq j \leq M_1$, we have $E[\tilde{\psi}^j] \leq \max\{\lim_n E[\phi_n], 0\}$. Without loss of generality, we may assume $j = 1$. Since $1 = M[\tilde{\psi}^1] \leq \lim_n M[\phi_n] = M[u_Q] = 1$, it follows $\left(\mathcal{M}\mathcal{E}[\tilde{\psi}^1]\right)^{\frac{1}{2}} \leq \max\left(\lim_n \frac{E[\phi_n]}{E[u_Q]}\right)$, thus, for some $\lambda_1 \geq \lambda_0$, we have $\left(\mathcal{M}\mathcal{E}[\tilde{\psi}^1]\right)^{\frac{1}{2}} = \frac{d}{2s} \lambda_1^2 \left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right)$.

Recall $\tilde{\psi}^1$ is a nonscattering solution, thus $[\mathcal{G}_{\tilde{\psi}^1}(t)]^{\frac{1}{2}} > \lambda$, otherwise it will contradict Theorem A Part 1 (b). We have two cases: either $\lambda_1 \leq \sigma_c$ or $\lambda_1 > \sigma_c$.

Case 1. $\lambda_1 \leq \sigma_c$. Since the statement “$\text{GBG}(\lambda_1, \sigma_c - \delta)$ is false” implies for each $\delta > 0$, there is a nonincreasing sequence $t_k$ of times such that $\lim[\mathcal{G}_{\tilde{\psi}^1}(t_k)]^{\frac{1}{2}} \geq \sigma_c$, thus,

$$\sigma_c^2 - o_k(1) \leq \lim[\mathcal{G}_{\tilde{\psi}^1}(t_k)]^{\frac{1}{2}} \leq \frac{\|\nabla v^1(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \frac{\sum_{j=1}^{M} \|\nabla v^1(t_k - t_n)\|_{L^2}^2 + \|W_{M}^n(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \frac{\|\nabla u_n(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} + o_n(1) \leq \sigma_c^2 + o_n(1).$$

Taking $k \to \infty$, we obtain $\sigma_c^2 - o_n(1) = \sigma_c^2 + o_k(1)$. Thus, $\|W_{M}^n(t_k)\|_{H^1} \to 0$ and $M[v^1] = M[u_Q]$. Then, Lemma 3.9 yields that for all $t$,

$$\frac{\|\nabla v^1(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \lim_n \frac{\|u_n(t)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \sigma_c.$$
Take \( u_{c,0} = v^1(0)(= \psi^1) \), and \( \lambda_c = \lambda_1 \).

**Case 2.** \( \lambda_1 \geq \sigma_c \). Note that

\[
\lambda_1^2 \leq \lim[G_{v^1}(t_k)]^2. \tag{4.11}
\]

Replacing the first line of (4.10) by (4.11), taking \( t_k = 0 \) and sending \( n \to +\infty \), we obtain

\[
\lambda_1^2 \leq \frac{\|v^1(t_k)\|_{L^2}^2 \|\nabla \psi(t_k)\|_{L^2}^2}{\|u_Q\|_{L^2}^2 \|\nabla u_Q\|_{L^2}^2} \leq \frac{\|\nabla v^1(t_k)\|_{L^2}^2}{\|\nabla u_Q\|_{L^2}^2} \leq \sum_{j=1}^{M} \|\nabla v^1(t_k - t_n)\|_{L^2}^2 + \|W_n^M(t_k)\|_{L^2}^2 \|
\]

Thus, we have \( \lambda_1 \leq \sigma_c \), which is a contradiction. Thus, this case cannot happen. \( \square \)

**Lemma 4.9.** Assume \( u(t) = u_c(t) \) to be the critical solution provided by Lemma 4.8. Then there exists a path \( x(t) \) in \( \mathbb{R}^d \) such that

\[
K = \{u(\cdot - x(t), t) | t \geq 0\}
\]

has a compact closure in \( H^1(\mathbb{R}^d) \).

**Proof.** As we proved in Lemma 3.13, it suffices to show that for each sequence of times \( t_n \to \infty \), passing to a subsequence, there exists a sequence \( x_n \) such that \( u(\cdot - x_n, t_n) \) converges in \( H^1 \). Let \( \phi_n = u(t_n) \) as in Proposition 4.7 and apply the proof of Lemma 4.8 It follows for \( j \geq 2 \) we have \( \psi_j = 0 \) and \( \tilde{W}_n \to 0 \) in \( H^1 \) as \( n \to \infty \). And thus, \( u(\cdot - x_n, t_n) \to \psi^1 \) in \( H^1 \). \( \square \)

**Lemma 4.10** (Blow up for a priori localized solutions). Suppose \( u \) is a solution of the \( NLS_p(\mathbb{R}^d) \) at the mass-energy line \( \lambda > 1 \), with \( G_{\psi}(0) > 1 \). Select \( \kappa \) such that \( 0 < \kappa < \min(\lambda - 1, \kappa_0) \), where \( \kappa_0 \) is an absolute constant. Assume that there is a radius \( R \geq \kappa^{-1/2} \) such that for all \( t \), we have \( G_{u_R}(t) := \frac{\|u\|_{L^2(|x|\geq R)}^2 \|\nabla u(t)\|_{L^2(|x|\geq R)}^2}{\left\| u_Q \right\|_{L^2(|x|\geq R)}^2 \|\nabla u_Q\|_{L^2(|x|\geq R)}^2} \leq \kappa \). Define \( \tilde{r}(t) \) to be the scaled local variance:

\[
\tilde{r}(t) = \frac{z(t)}{32 \alpha^4 E[u_Q]} \left( \frac{d}{2s} \lambda^2 \left( 1 - \frac{\alpha^2}{\alpha^2} - \kappa \right) \right).
\]

Then blowup occurs in forward time before \( t_b \) (i.e., \( T^* \leq t_b \)), where \( t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)} \).

**Proof.** By the local virial identity (3.76),

\[
r''(t) = \frac{16 \alpha^2 E[u] - 8(\alpha^2 - 1) \|\nabla u\|_{L^2}^2 + A_R(u(t))}{16 \alpha^2 E[u_Q]} \left( \frac{d}{2s} \lambda^2 \left( 1 - \frac{\alpha^2}{\alpha^2} - \kappa \right) \right),
\]

where

\[
|A_R(u(t))| = \|\nabla u(t)\|_{L^2(|x|\geq R)}^2 + \frac{1}{R^2} \|u(t)\|_{L^2(|x|\geq R)}^2 + \|u(t)\|_{L^{p+1}(|x|\geq R)}^{p+1}.
\]
Note that, $E[u] = \frac{d}{s} \| \nabla u \|_{L^2}^2$ and definition of the mass-energy line yield

$$\frac{16\alpha^2 E[u] - 8(\alpha^2 - 1) \| \nabla u \|_{L^2}^2}{16\alpha^2 E[u]} = \frac{E[u]}{E[u]} - \frac{d \| \nabla u \|_{L^2}^2}{sE[u]} - \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}^2} = \frac{E[u]}{E[u]} - \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}^2}$$

(4.12)

In addition, we have the following estimates

$$\| \nabla u(t) \|_{L^2(\{x|\geq R\})}^2 \lesssim \kappa, \quad \frac{\| u(t) \|_{L^2(\{x|\geq R\})}}{R^2} = \frac{\| u_Q \|_{L^2}}{R^2} \lesssim \kappa,$$

$$\| u(t) \|_{L^{p+1}(\{x|\geq R\})} \lesssim \| \nabla u \|_{L^2(\{x|\geq R\})}^{\frac{d(p-1)}{2}} \| u \|_{L^2(\{x|\geq R\})}^{\frac{d(p-2)(p-1)}{2}} \lesssim \| G_u(t) \|_{L^2}^{\frac{2}{p-1}} \| u_Q \|_{L^2}^{\frac{2}{p-1}} \lesssim \kappa. \quad (4.14)$$

We used the Gagliardo-Nirenberg to obtain (4.14) and noticing that $\| \nabla u_Q \|_{L^2}$ and $\| u_Q \|_{L^2}$ are constants, the last expression is estimated by $\kappa$ (up to a constant). In addition, $G_u(t) > 1$, then $\kappa \lesssim \kappa [G_u(t)]^2$. Applying the above estimates, it follows

$$r''(t) \lesssim \frac{4}{25} \lambda^\frac{2}{s} \left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right) - \left[ G_u(t) \right]^2 \left(1 - \frac{\lambda^{p-1}}{\alpha^2}\right) - \kappa.$$ 

Since $G_u(t) \geq \lambda$, we obtain $r''(t) \leq -1$, which is a contradiction. Now integrating in time twice gives $r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0)$.

The positive root of the polynomial on the right-hand side is $t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}$. 

This concludes all the claims in steps 1, 2 and 3 in subsection [4.1] and finishes the proof of Theorem A* part II (b).

**REFERENCES**

[BC85] H. Brezis and J.-M. Coron, *Convergence of solutions of $H$-systems or how to blow bubbles*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 21–56. MR 784102 (86g:53007)

[BL83a] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345. MR MR695535 (84h:35054a)

[BL83b], *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 347–375. MR MR695536 (84h:35054b)

[Caz03] Thierry Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003. MR MR2002047 (2004j:35266)

[CG11] Fernando Carreon and Cristi Guevara, *Scattering and blow up for the two dimensional focusing quintic nonlinear Schrödinger equation*, Submitted, 2011.

[CW90] Thierry Cazenave and Fred B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$*, Nonlinear Anal. **14** (1990), no. 10, 807–836. MR MR1055532 (91j:35252)
[DHR08] Thomas Duyckaerts, Justin Holmer, and Svetlana Roudenko, *Scattering for the non-radial 3D cubic nonlinear Schrödinger equation*, Math. Res. Lett. 15 (2008), no. 6, 1233–1250. MR MR2470397

[DR10] Thomas Duyckaerts and Svetlana Roudenko, *Threshold solutions for the focusing 3D cubic Schrödinger equation*, Rev. Mat. Iberoam. 26 (2010), no. 1, 1–56. MR 2662148 (2011c:35533)

[Fos05] Damiano Foschi, *Inhomogeneous Strichartz estimates*, J. Hyper. Diff. Eq. 2 (2005), no. 1, 1–24. MR MR2134950 (2006a:35043)

[FXC11] Daoyuan Fang, Jian Xie, and Thierry Cazenave, *Scattering for the focusing energy-subcritical nls*.

[Ger96] Patrick Gerard, *Oscillations and concentration effects in semilinear dispersive wave equations*, J. Funct. Anal. 141 (1996), no. 1, 60–98. MR MR1414374 (97k:35171)

[Gla77] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18 (1977), no. 9, 1794–1797. MR MR0460850 (57 #842)

[GM95] L. Glangetas and F. Merle, *A geometrical approach of existence of blow up solutions in h^1 for nonlinear schrödinger equation*, Rep. No. R95031, Laboratoire d’Analyse Numérique, Univ. Pierre and Marie Curie. (1995).

[Gue11] Cristi Guevara, *Global behavior of finite energy solutions to the focusing nonlinear Schrödinger equation in d-dimensions*, Ph.D. thesis, Arizona State University, April 2011.

[GV79a] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. 32 (1979), no. 1, 1–32. MR MR533218 (82c:35057)

[GV79b], *On a class of nonlinear Schrödinger equations. II. Scattering theory, general case*, J. Funct. Anal. 32 (1979), no. 1, 33–71. MR MR533219 (82c:35058)

[HPR10] Justin Holmer, Rodrigo Platte, and Svetlana Roudenko, * Blow-up criteria for the 3D cubic nonlinear Schrödinger equation*, Nonlinearity 23 (2010), no. 4, 977–1030.

[HR07] Justin Holmer and Svetlana Roudenko, *On blow-up solutions to the 3D cubic nonlinear Schrödinger equation*, Appl. Math. Res. Express. AMRX 15 (2007), no. 1, Art. ID abm004, 31. MR MR2354447 (2008i:35227)

[HR10a], *A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation*, Comm. Math. Phys. 282 (2008), no. 2, 435–467. MR 2424184 (2009h:35043)

[HR10b], *Blow-up solutions on a sphere for the 3d quintic NLS in the energy space*, to appear in Analysis & PDE (2010).

[HR10c], *Divergence of infinite-variance nonradial solutions to the 3D NLS equation*, Comm. PDE 35 (2010), no. 5, 878–905.

[Kat87] Tosio Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), no. 1, 113–129. MR MR877998 (88f:35133)

[Kat94] , *An L^q,r.-theory for nonlinear Schrödinger equations*, Spectral and scattering theory and applications, Adv. Stud. Pure Math., vol. 23, Math. Soc. Japan, Tokyo, 1994, pp. 223–238. MR MR1275405 (95i:35276)

[Ker01] Sahbi Keraani, *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*, J. Differential Equations 175 (2001), no. 2, 353–392. MR MR1855973 (2002j:35081)

[KM06] Carlos E. Kenig and Frank Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. 166 (2006), no. 3, 645–675. MR MR2257393 (2007g:35322)

[KPV93] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620. MR MR1211741 (94h:35229)

[KT98] Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), no. 5, 955–980. MR MR1646048 (2000d:35018)

[lio84] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 4, 223–238. MR MR778974 (87e:49035b)
[Mer93] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, Duke Math. J. 69 (1993), no. 2, 427–454. MR MR1203233 (94b:35262)

[MV98] F. Merle and L. Vega, Compactness at blow-up time for $L^2$ solutions of the critical nonlinear Schrödinger equation in 2D, Internat. Math. Res. Notices (1998), no. 8, 399–425. MR MR1628235 (99d:35156)

[Rap06] Pierre Raphaël, Existence and stability of a solution blowing up on a sphere for an $L^2$-supercritical nonlinear Schrödinger equation, Duke Math. J. 134 (2006), no. 2, 199–258. MR MR2248831 (2007k:35472)

[RS09] Pierre Raphaël and Jérémie Szeftel, Standing ring blow up solutions to the N-dimensional quintic nonlinear Schrödinger equation, Comm. Math. Phys. 290 (2009), no. 3, 973–996. MR MR2525647

[Str81a] Walter A. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. 41 (1981), no. 1, 110–133. MR MR614228 (83b:47074a)

[Str81b] ———, Nonlinear scattering theory at low energy: sequel, J. Funct. Anal. 43 (1981), no. 3, 281–293. MR MR636702 (83b:47074b)

[Tao06] Terence Tao, Nonlinear dispersive equations, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006, Local and global analysis. MR MR2233925 (2008i:35211)

[Tri78] Hans Triebel, Interpolation theory, function spaces, differential operators, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam, 1978. MR MR503903 (80i:46032b)

[Vis07] Monica Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, Duke Math. J. 138 (2007), no. 2, 281–374. MR MR2318286 (2008f:35387)

[VPT71] S.N. Vlasov, V.A. Petrishchev, and V.I. Talanov, Averaged description of wave beams in linear and nonlinear media (the method of moments), Radiophysics and Quantum Electronics (translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika) 14 (1971), 1062–1070.

[Wei82] Michael I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1982), no. 4, 567–576. MR MR691044 (84d:35140)

[Zak72] V. E. Zakharov, Collapse of langmuir waves, Soviet Physics JETP (translation of the Journal of Experimental and Theoretical Physics of the Academy of Sciences of the USSR) 35 (1972), 908–914.

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona, 85287

E-mail address: Cristi.guevara@asu.edu