Uniform convergence for sequences of best $L^p$ approximation

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Abstract

Let $f$ be a continuous monotone real function defined on a compact interval $[a, b]$ of the real line. Given a sequence of partitions of $[a, b]$, $\Delta_n$, $\|\Delta_n\| \to 0$, and given $l \geq 0, m \geq 1$, let $S^l_m(\Delta_n)$ be the space of all functions with the same monotonicity of $f$ that are $\Delta_n$-piecewise polynomial of order $m$ and that belong to the smoothness class $C^l[a, b]$.

In this paper we show that, for any $m \geq 2l + 1$,

- sequences of best $L^p$-approximation in $S^l_m(\Delta_n)$ converge uniformly to $f$ on any compact subinterval of $(a, b);$  
- sequences of best $L^p$-approximation in $S^0_m(\Delta_n)$ converge uniformly to $f$ on the whole interval $[a, b]$.  

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1. Introduction

It is well known that the $L^p$-convergence of a sequence $(f_n)$ to a continuous function $f : [a, b] \to \mathbb{R}$ doesn’t imply the uniform convergence. However J.T. Lewis and O. Shisha, in [8], proved that in the case of increasing sequences of functions $(f_n)$, the $L^p$-convergence of $(f_n)$ to a continuous function $f$ implies the uniform convergence on any compact interval strictly included in $(a, b)$.

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The problem of the uniform convergence of \((f_n)\) on the whole interval \([a, b]\) was proved by E. Kimchi and N. Richter-Dyn in [5] by considering sequences that have first-order divided differences uniformly bounded from above and from below.

In this paper we study sequences \((f_n)\) of piecewise polynomial functions obtained from the \(L^p\) minimization under monotonicity and smoothness constraints. More precisely, we will investigate the local and the global uniform convergence of \((f_n)\).

Our result comes also from an idea of E. Passow [11].

Given a nondecreasing and continuous function \(f : [a, b] \to \mathbb{R}\), a real \(p \in [1, \infty)\), a partition \(\Delta\) of \([a, b]\) and integers \(m, l\) with \(m \geq 1\) and \(0 \leq l \leq m - 1\), let us consider the projection, with respect to the \(L^p\) norm, of \(f\) onto the class of all nondecreasing \(\Delta\)-piecewise polynomial functions with degree less than or equal to \(m\) that belongs to the smoothness class \(C^l[a, b]\). Such a projection, called the best \(L^p\) approximation of \(f\) with respect to the partition \(\Delta\), there exists and it is unique (Theorem 1). Then, given a sequence \((\Delta_n)_{n \geq 1}\) of partitions of \([a, b]\) whose norm tends to zero, by [11, Theorem 1] it is possible to generate a sequence of \((\Delta_n)\)-piecewise polynomial functions that converges to \(f\) in the \(L^p\) norm whenever \(m \geq 2l + 1\). Combining this results with the one of Lewis and Shisha mentioned earlier, we will prove that the sequence of best \(L^p\) approximations of \(f\) along the partitions’ sequence \((\Delta_n)_{n \geq 1}\) converges uniformly to \(f\) on any compact interval included in \((a, b)\), whenever \(m \geq 2l + 1\). Finally, in the special case \(l = 0\) and \(m\) arbitrary, we extend this results to the uniform convergence of such a sequence on the whole interval \([a, b]\) (Main Theorem). It is obvious that the most stringent question is whether such result also holds for other values of \(l\) if not in general (that is, higher order of smoothness), as well as for the cases when the condition \(m \geq 2l + 1\) could be dropped or replaced by a weaker one. Clearly, this is an important open question that worth further investigation. The proof of the Main Theorem which is long and uses many auxiliary results (among them, the main results in [8], [11] and [12]), however, it seems to work only for the case \(l = 0\).

This paper is organized as follows: in Section 2 we present some basic notions about piecewise polynomial functions and about \(L^p\)-type approximations for some given real \(p \in [1, \infty)\). In the same section, we present, also, some properties of the projection operator onto the set of monotone piecewise polynomial functions from the class \(C^l[a, b]\) of degree less than or equal to \(m \geq 1\) on any compact subinterval of a given partition. In Section 3 we
give some useful results on the uniform convergence of the sequence of best $L^p$ approximation of $f$ and we enunciate the Main Theorem. In Section 4 we prove the Main Theorem. Finally, in Section 5 some open questions, with the purpose to complete and generalize the result of this paper, are presented.

In conclusion, remark that, since piecewise polynomial functions are used as tools in approximation or interpolation problems, data analysis and many others (see [3], [4], [9], · · · , for example), this note could be interesting also from the point of view of approximation theory even if the sequences of piecewise polynomial functions, used in the paper, cannot be expressed analytically.

2. Piecewise polynomial approximation of $L^p$-integrable functions

For $p \in [1, +\infty)$ and for a compact interval $[a, b]$, let $L^p[a, b]$ be the space of all $L^p$-integrable functions on $[a, b]$, with the norm $\| \cdot \|_p : L^p[a, b] \to [0, \infty)$,

$$\| f \|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p}.$$ 

It is well known that $(L^p[a, b], \| \cdot \|_p)$ is a uniformly convex Banach space for any $p \in (1, +\infty)$.

It is also well known that, for each $p \in [1, +\infty)$, $L^p[a, b]$ includes the class of all continuous functions on $[a, b]$ and the class of all monotone functions on $[a, b]$. Moreover it is also well known that, for some $j \in \mathbb{N}$, the class of all functions that are continuously differentiable of order $j$ on $[a, b]$ is denoted by $C^j[a, b]$. In particular, $C[a, b]$ denotes the class of all continuous functions defined on $[a, b]$, with the norm $\| \cdot \| : C[a, b] \to [0, \infty)$,

$$\| f \| = \max_{x \in [a, b]} |f(x)|.$$ 

Let $\mathcal{D}[a, b]$ be the set of all partitions of $[a, b]$. If $a = \alpha_1 < \alpha_2 < \ldots < \alpha_n = b$ is a partition of $[a, b]$, then we identify it with the vector $\Delta_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. As usual, we denote by $\| \Delta_n \|$ the norm of this partition, where $\| \Delta_n \| = \max\{\alpha_{i+1} - \alpha_i : 1 \leq i \leq n-1\}$. A function $f : [a, b] \to \mathbb{R}$ is called $\Delta_n$-piecewise polynomial of order $m$, with $m \geq 1$, if the restriction of $f$ on any interval $[\alpha_i, \alpha_{i+1}]$, $i = 1, n-1$ is a polynomial function of degree less than or equal to $m$. Let $P_{m, \Delta_n}$ be the space of all functions that are
$\Delta_n$-piecewise polynomial of order $m$ and let $S_m^l(\Delta_n) = P_m,\Delta_n \cap C_l[a, b]$ be the class of all $\Delta_n$-piecewise polynomials of order $m$ which are continuously differentiable up to order $l$, for some nonnegative integer $l \leq m - 1$. Moreover let $SI_m^l(\Delta_n)$ and $SD_m^l(\Delta_n)$ be the set of all nondecreasing and nonincreasing functions in $S_m^l(\Delta_n)$, respectively. Clearly, $SD_m^l(\Delta_n) = -SI_m^l(\Delta_n)$. In all that follows in this paper we will consider only the case of best $L^p$ approximation in $SI_m^l(\Delta_n)$ and in the main results we will consider only nondecreasing functions to be approximated. Obviously the case of approximation of nondecreasing functions in $SD_m^l(\Delta_n)$ has the same treatment and therefore we omit the details that the reader can easily deduce from the other case.

It is easy to observe that $SI_m^l(\Delta_n)$ is a finite dimensional linear subspace of $L^p[a, b]$ and that $SI_m^l(\Delta_n)$ and $SD_m^l(\Delta_n)$ are nonempty closed cones in $S_m^l(\Delta_n)$. Therefore, by the existence and uniqueness of the projection onto a closed convex subset of a Banach uniformly convex space, it follows the existence and uniqueness of the best approximation with respect to $\|\cdot\|_p$, with $p \in (1, +\infty)$, in the spaces $SI_m^l(\Delta_n)$ and $SD_m^l(\Delta_n)$, respectively. In additions, even if $L^1[a, b]$ is no longer a Banach uniformly convex space (nor even reflexive), the best approximation with respect to $\|\cdot\|_1$ always exists and in the particular case $f \in C[a, b]$, the projection of $f$ onto $SI_m^l(\Delta_n)$ and $SD_m^l(\Delta_n)$ respectively, not only there exists but also it is unique (see Theorem 2.3 in [12] and see also [10]). Note that in [12] the results is obtained in a more general setting for piecewise polynomial functions satisfying generalized convexity constraints. Consequently, we can state the following result:

**Theorem 1.** If $p \in [1, \infty)$, $m \geq 1$, $0 \leq l \leq m - 1$, $f \in C[a, b]$ and $\Delta_n \in \mathcal{D}[a, b]$, then there exists uniquely $SI_m^l(\Delta_n, f, p) \in SI_m^l(\Delta_n)$ so that

$$\|f - SI_m^l(\Delta_n, f, p)\|_p = \min_{g \in SI_m^l(\Delta_n)} \|f - g\|_p.$$  

Taking into account the facts from above, for some given real $p \in [1, +\infty)$, we consider the approximation operator $U_{m, \Delta_n}^l : C[a, b] \to SI_m^l(\Delta_n)$ defined by $U_{m, \Delta_n}^l(f) = SI_m^l(\Delta_n, f, p)$.

The following properties are immediate and therefore we omit their proofs.

**Proposition 2.** Let $f \in C[a, b]$. For some $p \in [1, +\infty)$, $m \geq 1$, $0 \leq l \leq m - 1$ and $\Delta_n \in \mathcal{D}[a, b]$ with $\Delta_n = (\alpha_1, \alpha_2, ..., \alpha_n)$, we have that:

(i) $SI_m^l(\Delta_n, f - c, p) = SI_m^l(\Delta_n, f, p) - c$, for all $c \in \mathbb{R}$;

(ii) $SI_m^l(\Delta_n, cf, p) = c\cdot SI_m^l(\Delta_n, f, p)$ and $SD_m^l(\Delta_n, cf, p) = c\cdot SD_m^l(\Delta_n, f, p)$, for all $c \in [0, \infty]$;
In all that follows, we no longer impose that $\Delta_n \in \mathcal{D}[a, b]$ has exactly $n$ knots. Indeed, later on we construct sequences of partitions of $[a, b]$ such that any partition of the sequence has an arbitrary number of knots. Therefore, in general, we will consider a vector of the form $\Delta_n = (\alpha_1, \alpha_2, \ldots, \alpha_{k_n})$.

In the next result we will provide sufficient conditions which will imply that the sequence $\left( \text{SI}_m^l(\Delta_n, f, p) \right)_{n \geq 1}$ converges to $f$ in the $L^p$ norm $\| \cdot \|_p$ for some given $p \in [1, \infty)$. Combining this result with several auxiliary results, in the next section we will be able to prove a local uniform convergence for this sequence and even a global uniform convergence in the special case $l = 0$. At the moment, the next result will not suffice to obtain these uniform convergence properties since we will rely on a rate of convergence for an estimation in the $L^p$ norm (see relation (1) in the proof) and it is well known that in general such estimations will not necessarily imply convergence in the uniform norm based on the simple fact that these two norms are not equivalent.

**Proposition 3.** Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and consider in $\mathcal{D}[a, b]$ a sequence $(\Delta_n)_{n \geq 1}$ with $\| \Delta_n \| \to 0$. Moreover, let $l$ be an arbitrary nonnegative integer. Then if $f$ is nondecreasing, for any $p \in [1, \infty)$ and $m \geq 2l + 1$ we have:

$$
\left\| f - \text{SI}_m^l(\Delta_n, f, p) \right\|_p \leq (b - a)^{1/p} \omega_f(\| \Delta_n \|), \text{ for all } n \geq 1,
$$

where

$$
\omega_f(\delta) = \sup\{ |f(x) - f(y)| : x, y \in [a, b], |x - y| \leq \delta \}
$$

denotes the modulus of uniform continuity of $f$.

So, in particular, we have:

$$
\lim_{n \to \infty} \left\| f - \text{SI}_m^l(\Delta_n, f, p) \right\|_p = 0.
$$

**Proof.** Let us choose arbitrary $n \geq 1$ and suppose that $\Delta_n = (\alpha_1, \alpha_2, \ldots, \alpha_{k_n})$. In [11] (see Theorem 1 there) it is proved that if $(x_i, y_i)_{i=1}^s$ are in $\mathbb{R}^2$ with $x_1 < x_2 < \cdots < x_s$ then there exists $P \in \mathcal{S}_{2l+1}^l(\Delta_s)$, where $\Delta_s = (x_1, \ldots, x_s)$, such that $P(x_i) = y_i$, $i = 1, s$ and $P$ is monotone on any interval $[x_i, x_{i+1}]$, $i = 1, s - 1$. Then, applying this result to $(\alpha_i, f(\alpha_i))_{i=1}^{k_n}$ we get $P \in \mathcal{S}_{2l+1}^l(\Delta_n)$ that interpolates $f$ on the knots of $\Delta_n$, and since $m \geq 2l + 1$, it follows that $P \in \text{SI}_m^l(\Delta_n)$.
So, since $f$ and $P$ are both nondecreasing and since $f(\alpha_i) = P(\alpha_i)$, $i = 1, k_n$, it easily results that for any $i \in \{1, \ldots, k_n - 1\}$ and $x \in [\alpha_i, \alpha_{i+1}]$ we have $|f(x) - P(x)| \leq f(\alpha_{i+1}) - f(\alpha_i)$. Moreover, since $f(\alpha_{i+1}) - f(\alpha_i) \leq \omega f(\|\Delta_n\|)$ for all $i \in \{1, \ldots, k_n - 1\}$, we have $|f(x) - P(x)| \leq \omega f(\|\Delta_n\|)$, for all $x \in [a, b]$.

This easily implies

$$\|f - P\|_p \leq (b - a)^{1/p} \omega f(\|\Delta_n\|).$$

Then, by the definition of $\text{SI}_m^f(\Delta_n, f, p)$ and since $P \in \text{SI}_m^f(\Delta_n)$, it results that

$$\|f - \text{SI}_m^f(\Delta_n, f, p)\|_p \leq f - P\|_p \leq (b - a)^{1/p} \omega f(\|\Delta_n\|).$$

(1)

The continuity of $f$ and the fact that $\|\Delta_n\| \to 0$ implies $\omega f(\|\Delta_n\|) \to 0$, hence by (1) we have $\lim_{n \to \infty} \|f - \text{SI}_m^f(\Delta_n, f, p)\|_p = 0$ and the proof is complete.

3. Uniform convergence of best piecewise polynomial approximations

This section is dedicated to the study of the uniform convergence of the sequence of all piecewise polynomial functions $(f_n)_{n \geq 1}$ to a continuous nondecreasing function $f$ of the sequence $(\text{SI}_m^f(\Delta_n, f, p))_{n \geq 1}$ whenever $\|\Delta_n\| \to 0$.

It is well known that if $f$ and $(f_n)_{n \geq 1}$ are continuous functions on $[a, b]$ such that $(f_n)_{n \geq 1}$ converges uniformly to $f$ on $[a, b]$ then $(f_n)_{n \geq 1}$ converges towards $f$ in the $L^p$ norm for any $p \in [1, \infty)$. But it is also well known that the converse property does not hold in general. However, if $(f_n)_{n \geq 1}$ is a sequence of nondecreasing functions then the following theorem provides us a local convergence property in the uniform norm.

**Theorem 4.** (see [8, Theorem 1]) Let $f$ be a real, continuous function on the finite interval $(a, b)$, and let $(f_n)_{n \geq 1}$ be a sequence of nondecreasing functions on $(a, b)$ such that $\lim_{n \to \infty} \|f_n - f\|_p = 0$, where $1 \leq p < \infty$. Then for any $c$ and $d$ with $a < c < d < b$, the sequence $(f_n)_{n \geq 1}$ converges uniformly to $f$ on $[c, d]$. 6
Combining Proposition 3 with Theorem 4, we obtain the following result.

**Corollary 5.** Suppose that \( f : [a, b] \to \mathbb{R} \) is a continuous nondecreasing function and consider in \( D[a, b] \) a sequence \( (\Delta_n)_{n \geq 1} \), such that \( \|\Delta_n\| \to 0 \). Then for any nonnegative integer \( l \) and for any \( m \geq 2l + 1 \), the sequence \( (SI^l_m(\Delta_n, f, p))_{n \geq 1} \) is uniformly convergent to \( f \) on any interval \([c, d]\), with \( a < c < d < b \).

**Proof.** By Proposition 3, we have \( \lim_{n \to \infty} \|SI^l_m(\Delta_n, f, p) - f\|_p = 0 \). Hence, we obtain the conclusion taking \( f_n = SI^l_m(\Delta_n, f, p) \) for any \( n \) in Theorem 4.

Now let us consider the function \( f \equiv 0 \) and the sequence \( (f_n)_{n \geq 1} \) defined on \([0, 1]\) by \( f_n(x) = x^n, n = 1, 2, \ldots \). It is trivial to observe that \( \|f_n\|_p \to 0 \), for any \( p \in [1, \infty) \) and that \( (f_n)_{n \geq 1} \) is uniformly convergent to zero on any \([c, d]\), with \( 0 < c < d < 1 \). However, \( (f_n)_{n \geq 1} \) is not uniformly convergent to zero on \([0, 1]\).

Let us note that in the previous corollary as well as in Proposition 3 which implies this corollary, we need to assume that \( m \geq 2l + 1 \) because otherwise we cannot use Theorem 1 in [11] used to prove Proposition 3. Actually, in the case \( m \leq 2l \) the interpolant used to prove Proposition 3 may not exist (see Remark 2 in [11]). Of course, it is an open question whether these results would hold also for \( m \leq 2l \) but then another proof is needed which does not use Theorem 1 in [11].

In this paper we prove that:

**Main Theorem** If \( f : [a, b] \to \mathbb{R} \) is a continuous nondecreasing function and if \( (\Delta_n)_{n \geq 1} \in D[a, b] \) such that \( \|\Delta_n\| \to 0 \), then, for any real \( p \in [1, \infty) \) and \( m \geq 1 \), the sequence \( (SI^0_m(\Delta_n, f, p))_{n \geq 1} \) is uniformly convergent to \( f \) on \([a, b] \); that is \( \lim_{n \to \infty} \|SI^0_m(\Delta_n, f) - f\| = 0 \).

To prove this theorem we need some auxiliary results.

**Lemma 6.** (see [4, Lemma 4.3]) Suppose that \( f : [a, b] \to \mathbb{R} \) is a continuous monotone function and suppose that \( (f_n)_{n \geq 1} \) is a sequence of monotone functions, all with the same monotonicity as \( f \), such that for some \( p \geq 1 \) we have \( \lim_{n \to \infty} \|f_n - f\|_p = 0 \). If \( \lim_{n \to \infty} f_n(a) = f(a) \) and \( \lim_{n \to \infty} f_n(b) = f(b) \), then \( \lim_{n \to \infty} \|f_n - f\| = 0 \).

First of all we give a simple sufficient condition to obtain the pointwise convergence needed in the previous lemma.
Lemma 7. Suppose that \( f : [a, b] \rightarrow \mathbb{R} \) is a nondecreasing continuous function and suppose that \((f_n)_{n \geq 1}\) is a sequence of nondecreasing functions, such that for some \( p \geq 1 \) we have \( \lim_{n \rightarrow \infty} \| f_n - f \|_p = 0 \). If \((x_n)_{n \geq 1}\) is a sequence in \([a, b]\) such that \( x_n \rightarrow a \) and \( f_n(x_n) \geq f(x_n) \) for all \( n \geq 1 \), then \( \lim_{n \rightarrow \infty} f_n(x_n) = \lim_{n \rightarrow \infty} f(x_n) = f(a) \). In particular, if \( f_n(a) \geq f(a) \) for all \( n \geq 1 \), then \( \lim_{n \rightarrow \infty} f_n(a) = f(a) \).

Proof. To avoid unnecessary reasonings with subsequences, we may assume that \( \lim_{n \rightarrow \infty} f_n(x_n) \) exists, including the case when it is unbounded. By way of contradiction, suppose that \( \lim_{n \rightarrow \infty} f_n(x_n) = f(a) \) does not hold. Then, since \( f(x_n) \rightarrow f(a) \), there exists \( \varepsilon > 0 \) such that for sufficiently large \( n \) we have \( f_n(x_n) - f(x_n) > \varepsilon \). Letting \( n \rightarrow \infty \) we get \( \lim_{n \rightarrow \infty} f_n(x_n) - f(a) > \varepsilon/2 \). From the continuity of \( f \) it results the existence of \( y_0 \in [a, b] \) such that \( \lim_{n \rightarrow \infty} f_n(x_n) - f(x) > \varepsilon/2 \) for all \( x \in [a, y_0] \). In particular we get \( \lim_{n \rightarrow \infty} f_n(x_n) - f(y_0) > \varepsilon/2 \). This implies the existence of \( n_1 \in \mathbb{N} \) such that

\[
    f_n(x_n) - f(y_0) > \varepsilon/2, \text{ for all } n \geq n_1.
\]

Let us choose arbitrary \( y_1 \in (a, y_0) \). Since \( x_n \rightarrow a \) there is \( n_2 \in \mathbb{N} \) (which may depends only on \( y_1 \)) such that \( x_n \leq y_1 \) for all \( n \geq n_2 \). By the monotonicity of \( f \) and \( f_n \), for each \( n \), and by (2), it results

\[
    f(x) \leq f(y_0) < f_n(x_n) \leq f_n(x), \text{ for all } n \geq \max\{n_1, n_2\} \text{ and } x \in [y_1, y_0].
\]

This implies

\[
    \int_a^b |f_n(x) - f(x)|^p \, dx
\]

\[
    \geq \int_{y_1}^{y_0} |f_n(x) - f(x)|^p \, dx \geq (y_0 - y_1) |f_n(x_n) - f(y_0)|^p > (y_0 - y_1) \cdot \left( \frac{\varepsilon}{2} \right)^p
\]

for all \( n \geq \max\{n_1, n_2\} \).

Clearly this implies that we cannot have \( \lim_{n \rightarrow \infty} \| f_n - f \|_p = 0 \) and this is a contradiction. In conclusion, we have \( \lim_{n \rightarrow \infty} f_n(x_n) = \lim_{n \rightarrow \infty} f(x_n) = f(a) \).

The following lemma is inspired by the inequality of Markov for real polynomials.
Lemma 8. Let \( P : [a, b] \to \mathbb{R} \) be a nondecreasing polynomial function of degree at most \( m \geq 1 \). Then,

\[
P \left( a + \frac{b - a}{2m^2 + 1} \right) \leq P(a) + \frac{2m^2}{2m^2 + 1} \cdot (P(b) - P(a)).
\]

Proof. We start with the special case \( a = 0, b = 1, P(0) = 0 \) and \( P(1) = 1 \). Let us define on \([-1, 1]\) the polynomial \( Q(x) = P\left(\frac{x + 1}{2}\right) \). From the Markov's inequality we have

\[
\max_{-1 \leq x \leq 1} |Q'(x)| \leq m^2 \cdot \max_{-1 \leq x \leq 1} |Q(x)|
\]

Since \( Q'(x) = \frac{1}{2} P'\left(\frac{x+1}{2}\right) \), this implies

\[
\max_{-1 \leq x \leq 1} \left| P'\left(\frac{x+1}{2}\right) \right| \leq 2m^2 \cdot \max_{-1 \leq x \leq 1} \left| P\left(\frac{x+1}{2}\right) \right|
\]

hence

\[
\max_{0 \leq x \leq 1} |P'(x)| \leq 2m^2 \cdot \max_{0 \leq x \leq 1} |P(x)| = 2m^2.
\]

Then, by the mean value theorem, we have

\[
P \left( \frac{1}{2m^2 + 1} \right) = \int_0^{1/(2m^2+1)} P'(x)dx = \frac{1}{2m^2 + 1} P'(x_m) \leq \frac{2m^2}{2m^2 + 1}.
\]

Now, let us consider the general case. It is immediate that \( P_1 : [0, 1] \to \mathbb{R} \), defined as \( P_1(x) = (P(a+(b-a)x) - P(a))/(P(b)-P(a)) \), is a nondecreasing polynomial of degree at most \( m \), such that \( P_1(0) = 0 \) and \( P_1(1) = 1 \). It means that \( P_1 \left( \frac{1}{2m^2 + 1} \right) \leq \frac{2m^2}{2m^2 + 1} \), from which the desired conclusion easily follows.

4. Proof of the Main Theorem

If \( f \) is a constant, then the conclusion is immediate since \( \text{SI}_m^0(\Delta_n, f, p) = f \) for any \( n \). Therefore, in what it follows, we assume that \( f \) is not a constant. By Corollary 4 it results that \( \left( \text{SI}_m^0(\Delta_n, f, p) \right)_{n \geq 1} \) converges uniformly to \( f \) on any interval \([c, d] \) with \( a < c < d < b \). In order to prove the desired uniform convergence on \([a, b] \), it is enough to prove that \( \lim_{n \to \infty} \text{SI}_m^0(\Delta_n, f, p)(a) = f(a) \)
and \( \lim_{n \to \infty} \text{SI}_m^0(\Delta_n, f, p)(b) = f(b) \), respectively (see, e. g. Lemma 4.3 in [1]). Without loss of generality we may assume that \( f(a) = 0 \). In fact, supposed that the thesis holds for such functions, given a continuous nondecreasing generic function \( f \) we set \( g(x) = f(x) - f(a) \). Then \( g \) is continuous nondecreasing and \( g(a) = 0 \). So \( \lim_{n \to \infty} \| \text{SI}_m^0(\Delta_n, g, p) - g \| = 0 \). Now, by Proposition [2](i), we have \( \text{SI}_m^0(\Delta_n, g, p) = \text{SI}_m^0(\Delta_n, f, p) - f(a) \), for all \( n \geq 1 \). Therefore

\[
\begin{align*}
\lim_{n \to \infty} \| \text{SI}_m^0(\Delta_n, f, p) - f \| & = \lim_{n \to \infty} \| \text{SI}_m^0(\Delta_n, f, p) - f(a) + f(a) - f \| \\
& = \lim_{n \to \infty} \| \text{SI}_m^0(\Delta_n, g, p) - g \| \\
& = 0.
\end{align*}
\]

From now on, until the end of the proof, we will set \( f_n = \text{SI}_m^0(\Delta_n, f, p) \) for all \( n \geq 1 \). To avoid the use of subsequences we may suppose that there are only two cases: either \( f_n(a) < f(a) = 0 \) for all \( n \geq 1 \), or \( f_n(a) \geq f(a) = 0 \) for all \( n \geq 1 \). But in this latter one, by Lemma 7 it results that \( \lim_{n \to \infty} f_n(a) = f(a) \) and therefore in this case there is nothing to be proved. So, in all that follows we will suppose that \( f_n(a) < f(a) = 0 \) for all \( n \geq 1 \).

The idea of the proof is to show that any subsequence of \( (f_n(a))_{n \geq 1} \) contains a subsequence that converges to 0. Clearly, this will imply that \( \lim_{n \to \infty} f_n(a) = 0 \). For this reason, without losing generality we may suppose that there exists the limit \( \lim_{n \to \infty} f_n(a) \) (including the case when this limit is \( -\infty \)). For any \( n \geq 1 \) let \( \Delta_n = (\alpha_1(n), \alpha_2(n), ..., \alpha_{k_n}(n)) \). We must observe that for each \( n \) the function \( f_n \) cannot be a constant function. Indeed, if \( f_n \) would be constant (with strictly negative constant value according to our assumption) then, taking \( h(x) = 0, x \in [a, b] \), we obviously have \( h \in \text{SI}_m^0(\Delta_n) \) and in addition one can easily prove that \( \| h - f \|_p < \| f_n - f \|_p \), which contradicts the fact that \( f_n \) is the best approximation of \( f \) in \( \text{SI}_m^0(\Delta_n) \) with respect to \( \| \cdot \|_p \). Therefore, since \( f_n \) is not constant, it results that there exists \( j \in \{1, ..., k_n-1\} \) such that \( f_n(\alpha_j(n)) = f_n(a) < f_n(\alpha_{j+1}(n)) \). In addition, we notice that necessarily \( f_n \) is strictly increasing on \( [\alpha_j(n), \alpha_{j+1}(n)] \) (otherwise \( f_n \) is not polynomial on this interval). Since \( f \) is not a constant function let \( a_1 \in [a, b] \) such that \( f(a_1) = f(a) = 0 < f(x) \), for all \( x \in (a_1, b] \). Without any loss of generality, suppose that \( (\alpha_j(n))_{n \geq 1} \) is convergent and denote its limit with \( u_0 \). If \( u_0 > a_1 \) then for some fixed value \( x_0 \in (a_1, u_0) \) and sufficiently large \( n \) we have \( f_n(x_0) \leq f_n(\alpha_j(n)) < 0 < f(x_0) \). It means that \( f_n(x_0) \) does
Therefore, we have \( f(x) = f(a) = 0 \). Now, let us prove that \( f_n(\alpha_{j+1}(n)) > f(a) = 0 \) for all \( n \geq 1 \). By way of contradiction, let us consider, for some \( n \geq 1 \), the function \( h_n : [a, b] \to \mathbb{R}, h_n(x) = f_n(\alpha_{j+1}(n)) \) if \( x \in [a, \alpha_{j+1}(n)] \) and \( h_n(x) = f_n(x) \) elsewhere. Then \( h_n \in \text{SI}^0_m(\Delta_n) \) and \( f_n(x) < h_n(x) \leq f(x) \), for all \( x \in [a, \alpha_{j+1}(n)] \) (here it is important that \( f_n \) is strictly increasing on \( [\alpha_j(n), \alpha_{j+1}(n)] \), that \( f_n(\alpha_{j+1}(n)) \leq 0 \) as well as the monotonicity of \( f \)). Consequently, this implies that

\[
\int_a^{\alpha_{j+1}(n)} |f(x) - h_n(x)|^p \, dx < \int_a^{\alpha_{j+1}(n)} |f(x) - f_n(x)|^p \, dx
\]

that easily implies

\[
\int_a^b |f(x) - h_n(x)|^p \, dx < \int_a^b |f(x) - f_n(x)|^p \, dx.
\]

Thus, \( \|h_n - f\|_p < \|f_n - f\|_p \), which, since \( h_n \in \text{SI}^0_m(\Delta_n) \), contradicts the fact that \( f_n \) is the best approximation of \( f \) in \( \text{SI}^0_m(\Delta_n) \) with respect to \( \|\cdot\|_p \). Therefore, we have \( f_n(\alpha_{j+1}(n)) > f(a) = 0 \) for all \( n \geq 1 \).

Now let us show that \( \lim_{n \to \infty} f_n(\alpha_{j+1}(n)) = 0 \). By way of contradiction we suppose that this is not true. In this case we may assume that there exists \( \gamma_1 > 0 \) such that \( f_n(\alpha_{j+1}(n)) > \gamma_1 \) for sufficiently large \( n \). Since \( f(u_0) = 0 \) and by the continuity of \( f \), there exists \( \delta > u_0 \) such that \( f(x) < \gamma_1 / 2 \) for all \( x \in [u_0, \delta] \). Since \( \lim_{n \to \infty} \alpha_{j+1}(n) = u_0 \), for sufficiently large \( n \), we have \( \alpha_{j+1}(n) < (u_0 + \delta) / 2 \). Therefore, by the monotonicity of \( f_n \), for sufficiently large \( n \) we get

\[
f(x) < \gamma_1 / 2 < \gamma_1 < f_n(\alpha_{j+1}(n)) \leq f_n(x), \text{ for all } x \in [(u_0 + \delta) / 2, \delta].
\]

This implies that, for some \( x_0 \in [(u_0 + \delta) / 2, \delta] \), \( f_n(x_0) \) does not converge to \( f(x_0) \); a contradiction. Therefore, we have \( \lim_{n \to \infty} f_n(\alpha_{j+1}(n)) = 0 \).

Now, suppose that \( n \geq 1 \) is fixed and, to simplify the notations, let us denote \( v_n = f_n(\alpha_{j+1}(n)) \). Let \( g_n : [a, b] \to \mathbb{R} \), defined by

\[
g_n(x) = \begin{cases} 
0, & \text{if } x \in [a, \alpha_j(n)]; \\
\frac{(x-\alpha_j(n))v_n}{\alpha_{j+1}(n)-\alpha_j(n)}, & \text{if } x \in [\alpha_j(n), \alpha_{j+1}(n)]; \\
f_n(x), & \text{if } x \in [\alpha_{j+1}(n), b].
\end{cases}
\]
It is trivial to observe that $g_n \in \mathbf{SI}_m(\Delta_n)$, so
\[ \|f_n - f\|_p \leq \|g_n - f\|_p. \]

We also easily notice that
\[ \alpha_j(n) \int_a^{\alpha_j(n)} |f(x) - f_n(x)|^p \, dx > \alpha_j(n) \int_a^{\alpha_j(n)} |f(x) - g_n(x)|^p \, dx \]
and
\[ \alpha_j(n) \int_{\alpha_j(n)}^{\alpha_{j+1}(n)} |f(x) - f_n(x)|^p \, dx = \alpha_{j+1}(n) \int_{\alpha_j(n)}^{\alpha_{j+1}(n)} |f(x) - g_n(x)|^p \, dx. \]

Hence
\[ \alpha_j(n) \int_{\alpha_j(n)}^{\alpha_{j+1}(n)} |f(x) - f_n(x)|^p \, dx \leq \alpha_{j+1}(n) \int_{\alpha_j(n)}^{\alpha_{j+1}(n)} |f(x) - g_n(x)|^p \, dx. \]
This further implies that
\[ \int_{\alpha_j(n)}^{u_n} |f(x) - f_n(x)|^p \, dx < \int_{\alpha_j(n)}^{\alpha_{j+1}(n)} |f(x) - g_n(x)|^p \, dx, \tag{3} \]
where $u_n = \alpha_j(n) + \frac{1}{2m^2+1} (\alpha_{j+1}(n) - \alpha_j(n))$.

Applying the mean value theorem in both integrals in (3), there exist $c_n \in (\alpha_j(n), u_n)$ and $d_n \in (\alpha_j(n), \alpha_{j+1}(n))$, such that
\[ \frac{\alpha_{j+1}(n) - \alpha_j(n)}{2m^2+1} \cdot |f(c_n) - f_n(c_n)|^p < (\alpha_{j+1}(n) - \alpha_j(n)) |f(d_n) - g_n(d_n)|^p, \]
which by simple calculations gives
\[ |f(c_n) - f_n(c_n)| < (2m^2 + 1)^{1/p} |f(d_n) - g_n(d_n)|. \tag{4} \]
As $f_n$ is nondecreasing, by Lemma 8, we get that
\[ f_n(\alpha_j(n)) \leq f_n(c_n) \leq f_n(u_n) \leq f_n(\alpha_j(n)) + \frac{2m^2}{2m^2+1} \cdot (f_n(\alpha_{j+1}(n)) - f_n(\alpha_j(n))). \]
This implies that there exists \( t_n \in \left[0, \frac{2m^2}{2m^2 + 1}\right] \), such that

\[
\begin{align*}
f_n(c_n) &= f_n(\alpha_j(n)) + t_n \cdot (f_n(\alpha_{j+1}(n)) - f_n(\alpha_j(n))) \\
&= (1 - t_n) \cdot f_n(\alpha_j(n)) + t_n \cdot f_n(\alpha_{j+1}(n)) \\
&= (1 - t_n) \cdot f_n(a) + t_n \cdot v_n.
\end{align*}
\]

Returning to inequality (4), we get

\[
|(1 - t_n) \cdot f_n(a) + t_n \cdot v_n - f(c_n)| < \left(2m^2 + 1\right)^{1/p} |f(d_n) - g_n(d_n)|. \tag{5}
\]

As \( c_n, d_n \in (\alpha_j(n), \alpha_{j+1}(n)) \), and \( \lim_{n \to \infty} \alpha_{j+1}(n) \leq a_1 \) and \( f(a_1) = f(a) = 0 \), by the monotonicity of \( f \), it easily results \( f(c_n) \to 0 \) and \( f(d_n) \to 0 \). Then, by the construction of \( g_n \), we have \( 0 \leq g_n(x) \leq f_n(\alpha_{j+1}(n)) \), for all \( x \in [\alpha_j(n), \alpha_{j+1}(n)] \). In particular we have \( 0 \leq g_n(d_n) \leq f_n(\alpha_{j+1}(n)) \). Hence, since \( \lim_{n \to \infty} f_n(\alpha_{j+1}(n)) = 0 \), it follows that \( \lim_{n \to \infty} g_n(d_n) = 0 \). This means that both expressions in the inequality (5) converge to 0.

Next, since \( 0 \leq t_n < 1 \), it follows that \( t_n \cdot f_n(\alpha_{j+1}(n)) \to 0 \). On the other hand, as \( 1 - t_n \geq \frac{1}{2m^2 + 1} \), we have that \( f_n(a) \to 0 \). In fact, if \( (1 - t_n) \cdot f_n(a) + t_n \cdot f_n(\alpha_{j+1}(n)) - f(c_n) \) would not converge to 0 we have a contradiction, since the absolute value of the expression in (5) is bounded by an expression converging to 0. In conclusion, we just proved that any subsequence of \( (f_n(a))_{n \geq 1} \) contains a subsequence that converges to 0 (please note again that all subsequences were denoted the same to avoid too complicated notations). Clearly this implies that \( \lim_{n \to \infty} f_n(a) = 0 \).

It remains to prove that \( \lim_{n \to \infty} f_n(b) = f(b) \). First, let us notice that that \( \lim_{n \to \infty} f_n(a) = f(a) \) if \( f \) is continuous and nonincreasing with \( f_n = SD^1_m(\Delta_n, f, p) \), for all \( n \geq 1 \). Indeed, this is immediate from the first part of the proof taking into account that obviously we have \( S_1(\Delta_n, -f, p) = -SD^0_m(\Delta_n, f, p) \), for all \( n \geq 1 \). Now, suppose again that \( f \) is nondecreasing. Considering the function \( g(x) = f(a + b - x) \), it is easily seen that \( S_1(\Delta_n, f, p)(b) = SD^0_m(\Delta_n, g, p)(a) \), where \( \Delta_n = (\beta_1(n), \beta_2(n), ..., \beta_k(n)) \) and \( \beta_i = a + b - \alpha_{k_i - i + 1}, i = 1, k_n \). Since \( g \) is continuous and \( \|\Delta_n\| \to 0 \), the previous result implies \( \lim_{n \to \infty} SD^0_m(\Delta_n, g, p)(a) = g(a) \) which implies that \( \lim_{n \to \infty} S^1_m(\Delta_n, f, p)(b) = f(b) \). It is clear now that we can apply Lemma 6, which means that \( \lim_{n \to \infty} \|S^1_m(\Delta_n, f, p) - f\| = 0 \), and the proof is complete. ■

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5. Some open questions

It remains, of course, to extend the results obtained here for \( l = 0 \), to an arbitrary value of \( l \). At this end, let us briefly discuss the case \( l \geq 1 \). To do this, let us notice that the crucial point in the proof of the Main Theorem is that the best \( L^p \) approximation is constantly negative until a given knot \( \alpha_j(n) \) and positive on the following knot \( \alpha_{j+1}(n) \). In the case \( l \geq 1 \) this property may not hold, or at least a different type of reasoning is needed to prove it. Actually, for important classes of partitions (see the equidistant partitions or even the partitions based on the Chebyshev knots of the first kind, for instance), it would be sufficient to prove that if the function is constantly negative until a given knot \( \alpha_j(n) \) then it is positive on the knot \( \alpha_{j+k}(n) \) where \( k \) is a constant that does not depend on \( n \).

Another interesting problem is to consider instead of the \( L^p \) norms the more general approach with monotone norms.

Moreover, it would interesting to find an estimation for the rate of the uniform convergence of the sequences \( (\text{SI}_m^0(\Delta_n, f, p))_{n \geq 1} \) and of the sequence \( (\text{SD}_m^0(\Delta_n, f, p))_{n \geq 1} \), respectively.

Then of course, as we already mentioned this problem earlier, it would be interesting to study whether the assumption \( m \geq 2l + 1 \) can be relaxed in the the statements of Proposition 3 and Corollary 5 by only assuming that \( m \geq l + 1 \). Of course, in this case another approach is needed that does not use Theorem 1 in [11] which necessarily implies the limitation \( m \geq 2l + 1 \).

Finally, it remains the question whether the main theorem remains true if we drop the assumptions on the monotonicity of \( f \) and of the best piecewise polynomial approximations. But again, these limitations are necessarily implied by Theorem 1 in [3] and Theorem 1 in [11]. Therefore, in order to generalize the main theorem for arbitrary not necessarily monotone functions, a new approach is needed that does not use the aforementioned theorems.
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References

[1] Bogachev L. V. and Zarbaliev S. M., Universality of the limit shape on convex lattice polygonal lines, The Annals of Probability, 39 (2011) 2271-2317.

[2] Caravantes J., Gomez-Molleda M.A. and Gonzales-Vega L., A canonical form for the continuous piecewise polynomial functions, Journal of Computational and Applied Mathematics, 283 (2015) 17-27.

[3] De Boor C. and Swartz B., Piecewise monotone interpolation, Journal of Approximation Theory, 21 (1977) 411-416.

[4] Greiner H., A survey on univariate data interpolation and approximation by splines of given shape, Mathematical and Computer Modelling, 15 (1991) 97-106.

[5] Kimchi E. and Richter-Dyn N., Convergence Properties of Sequences of Functions with Application to Restricted Derivative Approximation, Journal of Approximation Theory, 22 (1978) 289-303.

[6] Kopotun K. A., Leviatan D. and Shevchuk I. A., Interpolatory pointwise estimates for monotone polynomial approximation, Journal of Mathematical Analysis and Applications, 459 (2018) 1260-1295.

[7] Kopotun K. A., Leviatan D. and Shevchuk I. A., Interpolatory estimates for convex piecewise polynomial approximation, Journal of Mathematical Analysis and Applications, 474 (2019) 467-479.

[8] Lewis J. T. and Shisha O., $L_p$ convergence of Monotone Functions and their Uniform Convergence, Journal of Approximation Theory, 14 (1975) 281-284.

[9] Li J., Wang R., Xu M. and Fang Q., Piecewise linear approximation methods with stochastic sampling sites, Journal of Computational and Applied Mathematics, 329 (2018) 173-178.
[10] Lorentz R. A., Uniqueness of best approximation by monotone polynomials, Journal of Approximation Theory, 4 (1971) 401-418.

[11] Passow E., Piecewise monotone spline interpolation, Journal of Approximation Theory, 12 (1974), 240-241.

[12] Pence D. D., Best mean approximation by splines satisfying generalized convexity constraints, Journal of Approximation Theory, 28 (1980) 333-348.

[13] Richards F. B., A Gibbs phenomenon for spline functions, Journal of Approximation Theory, 66 (1991) 344-351.

[14] Saff E. B. and Tachev S., Gibbs phenomenon for best $L^p$ approximation by polygonal lines, East Journal of Approximation, 5 (1999) 235-251.