Baxter’s Q-operators for supersymmetric spin chains

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Journal reference: Nucl. Phys. B 805 [FS] (2008) 451-516
DOI: 10.1016/j.nuclphysb.2008.06.025
Report number: OIQP-07-13

Abstract

We develop Yang-Baxter integrability structures connected with the quantum affine superalgebra $U_q(\hat{sl}(2|1))$. Baxter’s Q-operators are explicitly constructed as supertraces of certain monodromy matrices associated with ($q$-deformed) bosonic and fermionic oscillator algebras. There are six different Q-operators in this case, obeying a few fundamental fusion relations, which imply all functional relations between various commuting transfer matrices. The results are universal in the sense that they do not depend on the quantum space of states and apply both to lattice models and to continuous quantum field theory models as well.
1 Introduction

The method of the Q-operator, introduced by Baxter in his seminal paper [1] on the exact solution of the eight-vertex model, finds many applications in the theory of integrable quantum systems. Its relationship to the algebraic structure of quantum groups [2, 3] was unraveled in [4, 5]. This method does not require the existence of the “bare” vacuum state and therefore has wider applicability than the traditional approaches to integrable systems such the coordinate [6] or algebraic Bethe Ansatz [7].

Here we consider integrable models of statistical mechanics and quantum field theory associated with the quantum affine superalgebra \( \mathcal{U}_{q}(\mathfrak{sl}(2|1)) \). The fundamental \( R \)-matrix serving these models was found by Perk and Schultz [8],

\[
R(x) = q^{\frac{1}{2}e_{13} - x^{\frac{1}{2}e_{11} + \frac{1}{2}e_{21}}} \begin{pmatrix}
(q^{-1} - q)q^{-\frac{1}{2}x e_{21}} & (q^{-1} - q)q^{-\frac{1}{2}x e_{31}} \\
(q^{-1} - q)e_{12} & q^{e_{22} - x^{\frac{1}{2}e_{22} + \frac{1}{2}}} & (q^{-1} - q)q^{-\frac{1}{2}x e_{32}} \\
(q^{-1} - q)e_{13} & (q^{-1} - q)e_{23} & q^{e_{33} - x q^{e_{33} - \frac{1}{2}}}
\end{pmatrix}, \tag{1.1}
\]

where \( e_{ij} \) is a \( 3 \times 3 \) matrix whose \((k, l)\) element is \( \delta_{ik}\delta_{jl} \). It defines an “interaction-round-a-vertex” model on the square lattice with three different states, \( s = 1, 2, 3 \), for each lattice edge. The states “1” and “2” will be referred to as bosonic (even) states and the state “3” as a fermionic (odd) state. We will call this model as the 3-state \( gl(2|1) \)-Perk-Schultz model or just as the “3-state model”. It is worth noting that the paper [8] contains more general \( R \)-matrices with an arbitrary number of states per edge associated with the \( \mathcal{U}_{q}(\mathfrak{sl}(m|n)) \) superalgebras. We also remark that closely related \( R \)-matrices for the non-graded case of \( \mathcal{U}_{q}(\mathfrak{sl}(m)) \) were previously given by Cherednik [9].

The edge configurations of the whole lattice in the 3-state \( gl(2|1) \)-Perk-Schultz model obey simple kinematic constraints, analogous to the “arrow conservation law” in the ordinary 6-vertex model. Here we consider the periodic boundary conditions in the horizontal direction. Then for every allowed edge configuration the numbers \( m_{1}, m_{2}, m_{3} \), counting the edges of the types “1”, “2” and “3” in a horizontal row are the same for all rows of the lattice. The row-to-row transfer matrix of the model reduces to a block-diagonal form, where the blocks are labeled by these conserved numbers. Note that, \( m_{1} + m_{2} + m_{3} = L \), where \( L \) is the horizontal size of the lattice.

The spin chain Hamiltonian connected with the \( gl(2|1) \)-Perk-Schultz model describes the trigonometric generalization [8, 10, 11] of the supersymmetric \( t-J \) model [12]. Both models (rational and trigonometric, and also their multicomponent analogs) were studied by many authors (see for example, [13]-[48]). Owing to the edge-type conservation properties discussed above these models can be solved via the “nested” Bethe Ansatz [12, 13]. The problem of the diagonalization of the Hamiltonian is then reduced to the solution of certain algebraic equations, called the Bethe Ansatz equations, where the number of unknowns depends edge occupation numbers \( m_{1}, m_{2}, m_{3} \). It is important to note that the integrability of the model is not affected by an introduction of two arbitrary “horizontal fields” (or “boundary twists”) which requires only simple modifications to the transfer matrix and to the Hamiltonian. We found this generalization to be extremely useful. In the following we will always consider the non-zero field case.

It was remarked many times [23, 24, 34, 39, 44, 45, 49–51] that there are equivalent, but different, forms of the Bethe Ansatz in the model. In fact, it is easy to argue that there are precisely 3! = 6 different Bethe Ansätze in this case. They are related by permutations of the occupation numbers \( m_{1}, m_{2}, m_{3} \). Indeed, there are three ways choose the bare vacuum state and then two ways to proceed on the second “nested” stage of the Bethe Ansatz. Of course, the super-symmetry does
not play any special role in this respect. Exactly the same counting also takes place for all models related with $U_q(\widehat{sl}(3))$ algebra, see [52, 53]. It is important to realize, however, that the above arguments fully apply only to a generic non-zero field case. If the fields are vanishing (or take some special values) then only a few of these Bethe Ansätze are well defined, while the other suffer from the “beyond the equator” problem, first encountered in [54] for the XXX-model.

The three-state $gl(2|1)$-Perk-Schultz $R$-matrix is just one representative of an infinite set of $R$-matrices associated with the $U_q(\widehat{sl}(2|1))$ algebra. These $R$-matrices can be constructed as different specializations of the universal $R$-matrix. In particular, there are the so-called fusion [55] $R$-matrices related to the matrix representations of the finite-dimensional algebra $U_q(gl(2|1))$. Other important $R$-matrices connected with the ($q$-deformed) oscillator algebras and continuous quantum field theory. There are two ways this variety could be used. First, one can consider models with different quantum spaces of states. Second, for the same quantum space there are “higher” or “fusion” transfer matrices corresponding to different finite-dimensional representation of $U_q(gl(2|1))$ in the “auxiliary” space. All these transfer-matrices are operator-valued functions of a (multiplicative) spectral variable “$x$”. We will call them the $T$-operators. The $T$-operators with different values of $x$ form a commuting family of operators. They satisfy a number of important functional relations, called the fusion relations (see eqs.(3.12) below). For the case of $U_q(\widehat{sl}(2|1))$ related models a complete set of these relations was proposed in [33–35] (see also, [28, 30, 45, 46, 56, 57]). However, a direct algebraic proof of these relations in full generality, i.e., for an arbitrary quantum space and a generic value of the deformation parameter $q$, was not hitherto known. Here we fill this gap.

The precise form of the functional relations, obviously, depends on the normalization of the $T$-operators. Here we use a distinguished normalization determined by the universal $R$-matrix (see Eqs. (2.16) and (2.39) below). The functional relations then take a universal form, which do not contain various model-dependent scalar factor. Such factors are usually present in the transfer matrix relations in lattice theory. To restore these factors in our approach one needs to explicitly calculate the specializations of the universal $R$-matrix for particular models. Here we compute these factors for the 3-state lattice model and in the case of the continuous conformal field theory, arising in quantization of the AKNS soliton hierarchy [58].

An important part in the theory of integrable quantum systems is played by the so-called $Q$-operators, introduced by Baxter in his pioneering work on the eight-vertex model of lattice statistics [1]. The $Q$-operators belong to the same commuting family of operators, as the $T$-operators. In this paper we present a complete algebraic theory of the $Q$-operators for the models related with $U_q(\widehat{sl}(2|1))$ algebra. There are six different $Q$-operators in this case. We denote them as $Q_i(x)$ and $\overline{Q}_i(x)$, $i = 1, 2, 3$. They are single-valued functions in the whole complex plane of the variable of $x$, except the origin $x = 0$, where they have simple algebraic branching points,

$$Q_k(e^{2\pi i} x) = e^{2\pi i S_k} Q_k(x), \quad \overline{Q}_k(e^{2\pi i} x) = e^{-2\pi i S_k} \overline{Q}_k(x), \quad k = 1, 2, 3, \quad (1.2)$$

Here $S_1$, $S_2$ and $S_3$, such that $S_1 + S_2 + S_3 = 0$, are constant operators (acting in the quantum space) given by certain linear combinations of the Cartan generators of $U_q(\widehat{sl}(2|1))$ and external field parameters. They commute among themselves and with all the other operators in the commuting family. Their eigenvalues $S_i$ are conserved quantum numbers, which in the case of the 3-state lattice model reduce to the edge occupation $m_1$, $m_2$ and $m_3$, mentioned above (see Eq. (1.4) below).

From an algebraic point of view the $Q$-operators are very similar to the transfer matrices. They are constructed as traces of certain (in general, infinite-dimensional) monodromy matrices, arising as specializations of the universal $R$-matrix to the representations of the fermionic
and bosonic q-oscillator algebras. Using some special decomposition properties of products of these infinite-dimensional representations we show that the Q-operators satisfies a few fundamental functional relations. There are four independent Wronskian-type relations between the Q-operators,

\[
    c_{12} = c_{13} Q_1(q^{+\frac{1}{2}}) \overline{Q}_1(q^{-\frac{1}{2}}) - c_{23} Q_2(q^{+\frac{1}{2}}) \overline{Q}_2(q^{-\frac{1}{2}}),
\]

\[
    c_{12} = c_{13} Q_1(q^{-\frac{1}{2}}) \overline{Q}_1(q^{+\frac{1}{2}}) - c_{23} Q_2(q^{-\frac{1}{2}}) \overline{Q}_2(q^{+\frac{1}{2}}),
\]

\[
    c_{21} Q_3(x) = \overline{Q}_1(qx) \overline{Q}_2(q^{-1}x) - \overline{Q}_1(q^{-1}x) \overline{Q}_2(qx),
\]

\[
    c_{12} \overline{Q}_3(x) = Q_1(qx) Q_2(q^{-1}x) - Q_1(q^{-1}x) Q_2(qx),
\]

where \( c_{ij} = (z_i - z_j)/\sqrt{z_i z_j} \), with \( z_1 = q^{2s_1}, z_2 = q^{2s_2} \) and \( z_3 = z_1 z_2 = q^{-2s_3} \).

The fusion transfer matrices are expressed as polynomial combinations of the Q-operators. In particular, the fundamental transfer matrix (corresponding to the three-dimensional representation in the auxiliary space) is given by

\[
    T(x) = c_{13} Q_1(q^{\frac{3}{2}}) \overline{Q}_1(q^{-\frac{3}{2}}) - c_{23} Q_2(q^{\frac{3}{2}}) \overline{Q}_2(q^{-\frac{3}{2}})
\]

(1.4)

With an account of \([133]\) the last formula can be transformed to any of the six equivalent forms

\[
    T(x) = p_i \overline{Q}_1(q^{-p_i - \frac{1}{2}}) + p_j Q_1(q^{-p_i - \frac{3}{2}}) \overline{Q}_1(q^{-p_j - \frac{1}{2}}) + p_k Q_k(q^{-p_i + p_j + 2p_k - \frac{3}{2}})
\]

(1.5)

where \( p_1 = p_2 = -p_3 = 1 \) and \( (i, j, k) \) is any permutation of \((1, 2, 3)\), which are the standard Bethe Ansatz type expressions for the transfer matrix. All the above functional relations are written in the normalization of the universal R-matrix (used for both T and Q operators). They can be easily adjusted for the traditional normalization in the lattice theory, where the corresponding eigenvalues \( T(x) \) and \( Q_i(x) \) and \( \overline{Q}_i(x) \) become finite polynomials of \( x \). For instance, for the 3-state lattice model

\[
    T(x) = p_i f(q^{2p_i - \frac{1}{2}}) \overline{Q}_i(q^{-p_i - \frac{1}{2}}) + p_j f(q^{-\frac{1}{2}}) \overline{Q}_i(q^{p_j - \frac{1}{2}}) Q_k(q^{p_i - p_j - \frac{1}{2}}) + p_k f(q^{-\frac{1}{2}}) \overline{Q}_k(q^{p_i + p_j - \frac{3}{2}}) Q_k(q^{p_i - p_j - \frac{1}{2}})
\]

(1.6)

where \( f(x) = (1 - x)^L \) and

\[
    Q_i(x) = x^{S_i} \prod_{\ell=1}^{m_i} (1 - x/x^{(i)}_\ell), \quad \overline{Q}_i(x) = x^{-S_i} \prod_{\ell=1}^{L-m_i} (1 - x/x^{(i)}_\ell).
\]

(1.7)

The zeroes \( \{x^{(i)}_\ell\}, \ell = 1, 2, \ldots, m_i \) and \( \{x^{(i)}_\ell\}, \ell = 1, 2, \ldots, L - m_i, i = 1, 2, 3 \), satisfy the Bethe Ansatz equations,

\[
    -\frac{p_i}{p_j} \frac{f(q^{+p_i} x^{(i)}_\ell)}{f(q^{-p_i} x^{(i)}_\ell)} = \frac{\overline{Q}_i(q^{+2p_j} x^{(i)}_\ell)}{\overline{Q}_i(q^{-2p_j} x^{(i)}_\ell)} \frac{Q_k(q^{-p_j} x^{(i)}_\ell)}{Q_k(q^{+p_j} x^{(i)}_\ell)}, \quad \ell = 1, 2, \ldots, L - m_i
\]

(1.8)

\[
    -\frac{p_j}{p_k} = \frac{\overline{Q}_i(q^{-p_j} x^{(k)}_\ell)}{\overline{Q}_i(q^{+p_j} x^{(k)}_\ell)} \frac{Q_k(q^{+2p_j} x^{(k)}_\ell)}{Q_k(q^{-2p_j} x^{(k)}_\ell)}, \quad \ell = 1, 2, \ldots, m_k
\]
where, as before, \((i, j, k)\) denotes an arbitrary permutation of \((1, 2, 3)\). Further, the eigenvalues of \(S_1, S_2\) and \(S_3\), entering the exponents in \((1.7)\), are given by

\[
2S_1 = L - m_1 + b_1, \quad 2S_2 = L - m_2 + b_2, \quad 2S_3 = -L - m_3 - b_1 - b_2,
\]

where \(m_1\) and \(m_2\) and \(m_3 = L - m_1 - m_2\) are the edge occupation numbers, and \(b_1\) and \(b_2\) are arbitrary field parameters.

Eqs. \((1.8)\) provide six self-contained sets of the Bethe Ansatz equations only involving zeros, belonging to a pair of the eigenvalues \((A_i(x), \overline{A}_j(x)), \ i \neq j\). Once any such pair is determined, the remaining zeroes can be found from the functional equations.

The organization of the paper is as follows. The algebraic definitions of the \(R\)-matrices, transfer matrices and \(Q\)-operators are given in Section \(2\). This section also contains necessary information about the representation theory of \(U_q(\widehat{sl}(2|1))\). The functional relations are presented in Section \(3\). Their applications in continuous quantum field theory and their connections to the spectral theory of ordinary differential equations are considered in Section \(4\). A direct algebraic proof of the functional relations is given in Section \(5\). Technical details of calculations are removed to four Appendices.

Some of our results in Sect. \(3\) partially overlap with those in [44] devoted to some models in the rational case \(q = 1\). Our approach to the \(Q\)-operators is different from that of [44]; in particular, it is applicable for an arbitrary quantum space and to generic values of \(q\).

## 2 Yang-Baxter equation, transfer matrices and \(Q\)-operators

### 2.1 The universal \(R\)-matrix

The quantum affine algebra \(A = U_q(\widehat{sl}(2|1))\) [59] (see also [60]) is generated by the elements \(h_0, h_1, h_2, e_0, e_1, e_2\) and \(f_0, f_1, f_2\), which are of two types: “fermionic” and “bosonic”. The elements \(e_0, e_2, f_0, f_2\) are fermionic, while all the other generating elements are bosonic. It is convenient to assign the parity

\[
p(X) = \begin{cases} 
1, & X = e_0, e_2, f_0, f_2, \\
0, & X = e_1, f_1, h_0, h_1, h_2,
\end{cases}
\]

such that

\[
p(XY) = p(X) + p(Y) \pmod{2}, \quad X, Y \in A,
\]

and introduce the generalized commutator

\[
[X, Y]_q = XY - (-1)^{p(X)p(Y)} q YX.
\]

Note, in particular, that \([X, Y] \equiv [X, Y]_1\) is reduced to the ordinary commutator when at least one of the elements \(X, Y\) is even and to the anti-commutator when both of them are odd. The algebra \(U_q(\widehat{sl}(2|1))\) is defined by the following commutation relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},
\]

where \(i, j = 0, 1, 2\), the Serre relations

\[
e_j^2 = f_j^2 = [e_1, [e_1, e_j]_{q^{-1}}]_q = [f_1, [f_1, f_j]_{q^{-1}}]_q = 0, \quad j = 0, 2,
\]

6
and the extra Serre relations
\[ [e_0, [e_2, [e_0, [e_2, e_1]_{q^{-1}}]]]_q = [e_2, [e_0, [e_2, e_0, e_1]_{q^{-1}}]]_q, \]  
\[ [f_0, [f_2, [f_0, [f_2, f_1]_{q^{-1}}]]]_q = [f_2, [f_0, [f_2, f_0, f_1]_{q^{-1}}]]_q . \]  
(2.6)

(2.7)

As usual, \((a_{ij})\) denotes the Cartan matrix
\[
(a_{ij})_{0 \leq i,j \leq 2} = \begin{pmatrix}
0 & -1 & 1 \\
-1 & 2 & -1 \\
1 & -1 & 0
\end{pmatrix},
\]  
(2.8)

Note, that the sum
\[ k = h_0 + h_1 + h_2, \]  
(2.9)

is a central element, commuting with all other elements of the algebra. In this paper, we consider the case \(k = 0\).

The algebra \(A = U_q(\widehat{sl}(2|1))\) is a Hopf algebra with the co-multiplication
\[
\Delta : A \longrightarrow A \otimes_s A
\]  
(2.10)

defined as
\[
\Delta(h_i) = h_i \otimes_s 1 + 1 \otimes_s h_i, \\
\Delta(e_i) = e_i \otimes_s 1 + q^{h_i} \otimes_s e_i, \\
\Delta(f_i) = f_i \otimes_s q^{-h_i} + 1 \otimes_s f_i,
\]  
(2.11)

where \(i = 0, 1, 2\) and \(\otimes_s\) denotes the graded tensor product, such that
\[
(A \otimes_s B)(C \otimes_s D) = (-1)^{p(B)p(C)} AC \otimes_s BD
\]  
(2.12)

There is another co-multiplication \(\Delta'\) obtained from (2.11) by interchanging factors of the direct products,
\[
\Delta' = \sigma \circ \Delta, \quad \sigma \circ (X \otimes_s Y) = (-1)^{p(X)p(Y)} Y \otimes_s X, \quad X,Y \in A.
\]  
(2.13)

The Borel subalgebras \(B_+ \subset A\) and \(B_- \subset A\) are generated by \(h_0, h_1, h_2, e_0, e_1, e_2\) and \(h_0, h_1, h_2, f_0, f_1, f_2\), respectively. There exists a unique element \([61, 62]\)
\[
\mathcal{R} \in B_+ \otimes B_-,
\]  
(2.14)

satisfying the following relations
\[
\Delta'(a) \mathcal{R} = \mathcal{R} \Delta(a) \quad (\forall a \in A),
\]  
(2.15)

\[
(\Delta \otimes_s 1) \mathcal{R} = \mathcal{R}^{13} \mathcal{R}^{23},
\]
\[
(1 \otimes_s \Delta) \mathcal{R} = \mathcal{R}^{13} \mathcal{R}^{12}
\]

where \(\mathcal{R}^{12}, \mathcal{R}^{13}, \mathcal{R}^{23} \in A \otimes_s A \otimes_s A\) and \(\mathcal{R}^{12} = \mathcal{R} \otimes 1, \mathcal{R}^{23} = 1 \otimes \mathcal{R}, \mathcal{R}^{13} = (\sigma \otimes 1) \mathcal{R}^{23}.\) The element \(\mathcal{R}\) is called the universal \(R\)-matrix. It satisfies the Yang-Baxter equation
\[
\mathcal{R}^{12} \mathcal{R}^{13} \mathcal{R}^{23} = \mathcal{R}^{23} \mathcal{R}^{13} \mathcal{R}^{12},
\]  
(2.16)
which is a simple corollary of the definitions (2.15). The universal $R$-matrix is understood as a formal series in generators in $B_+ \otimes B_-$. Its dependence on the Cartan elements can be isolated as a simple factor,

$$\mathcal{R} = \overline{\mathcal{R}} \, q^\mathcal{K}, \quad \mathcal{K} = -h_0 \otimes h_2 - h_2 \otimes h_0,$$

where the “reduced” universal $R$-matrix

$$\overline{\mathcal{R}} = \text{series in } (e_j \otimes 1) \text{ and } (1 \otimes f_j),$$

is a series in $(e_j \otimes 1) \in B_+ \otimes 1$ and $(1 \otimes f_j) \in 1 \otimes B_-$, $j = 0, 1, 2$, and does not contain Cartan elements. Remind that we assume $k = h_0 + h_1 + h_2 = 0$. A few first terms in (2.18) can be readily calculated directly from the definitions (2.14) and (2.15),

$$\overline{\mathcal{R}} = 1 - (q - q^{-1}) \sum_{j=0}^{2} (-1)^{p(j)} e_j \otimes_s f_j + \frac{(q - q^{-1})^2}{q^2 + 1} (e_1)^2 \otimes_s (f_1)^2$$

$$+ (q - q^{-1}) \sum_{i \neq j} \left\{ e_i e_j \otimes_s f_i f_j - (-1)^{p(i)p(j)} q^{-\alpha_i \epsilon_j} e_i e_j \otimes_s f_i f_j \right\} + \cdots,$$  

(2.19)

The symbol $p(j)$ denotes the parity of the corresponding element $e_j$, namely

$$p(0) = p(2) = 1, \quad p(1) = 0.$$  

(2.20)

The higher terms in (2.19) soon become very complicated and their general form is unknown. This complexity should not be surprising, since the universal $R$-matrix contains infinitely many nontrivial solutions of the Yang-Baxter equation associated with $U_q(\widehat{sl}(2|1))$. Fortunately, for applications one only needs certain specializations of universal $R$-matrix, which can be calculated explicitly. Almost all these specializations are associated with the evaluation homomorphisms from the infinite-dimensional algebra $U_q(\widehat{sl}(2|1))$ (and from its Borel subalgebras) into finite-dimensional algebras. The most important case is the evaluation map to the finite-dimensional quantum algebra $U_q(gl(2|1))$. This algebra is generated by the elements $E_{ii}$, $i = 0, 1, 2$ and $E_{ij}$, $(i, j) = (1, 2), (2, 1), (2, 3), (3, 2)$, for which we also use the notations

$$E_\alpha = E_{12}, \quad E_\beta = E_{23}, \quad F_\alpha = E_{21}, \quad F_\beta = E_{32},$$

(2.21)

and

$$H_\alpha = E_{11} - E_{22}, \quad H_\beta = E_{22} + E_{33}, \quad H_{\alpha + \beta} = E_{11} + E_{33}.$$  

(2.22)

The elements $E_\beta$ and $F_\beta$ are odd, $p(E_\beta) = p(F_\beta) = 1$, all other generators are even. They satisfy the following relations (written with the generalized commutator (2.23))

$$[E_{ii}, E_{jj}] = 0, \quad [E_{ii}, E_{kl}] = (\delta_{ik} - \delta_{il})E_{kl}, \quad [E_{\alpha i}, F_{\alpha j}] = \delta_{\alpha_i, \alpha_j} \frac{q^{H_{\alpha_i}} - q^{-H_{\alpha_i}}}{q - q^{-1}},$$

$$E_\beta^2 = F_\beta^2 = [E_\alpha, [E_\alpha, E_\beta]_{q^{-1}}]_q = [F_\alpha, [F_\alpha, F_\beta]_{q^{-1}}]_q = 0,$$  

(2.23)

where the Greek indices $\alpha_i$ and $\alpha_j$ take two values $\alpha$ or $\beta$. Introduce the following elements

$$E_{13} = q^{E_{22} + 2E_{33}} [E_{12}, E_{23}]_{q^{-1}}, \quad E_{31} = [E_{32}, E_{21}]_{q^{-1}} q^{-E_{22} - 2E_{33}}$$

$$\overline{E}_{13} = q^{-E_{22} - 2E_{33}} [E_{12}, E_{23}]_{q^{-1}}, \quad \overline{E}_{31} = [E_{32}, E_{21}]_q q^{E_{22} + 2E_{33}}$$  

(2.24)

$$E_{13} = q^{E_{22} + 2E_{33}} [E_{12}, E_{23}]_{q^{-1}}, \quad E_{31} = [E_{32}, E_{21}]_{q^{-1}} q^{-E_{22} - 2E_{33}}$$

(2.25)
Let $x$ be a complex (spectral) parameter. Define the evaluation map

$$\text{Ev}_x : \quad U_q(\widehat{sl}(2|1)) \longrightarrow U_q(gl(2|1))$$

as follows,

$$\text{Ev}_x(h_0) = -E_{11} - E_{33}, \quad \text{Ev}_x(h_1) = E_{11} - E_{22}, \quad \text{Ev}_x(h_2) = E_{22} + E_{33},$$

$$\text{Ev}_x(e_0) = -x E_{31}, \quad \text{Ev}_x(e_1) = E_{12}, \quad \text{Ev}_x(e_2) = E_{23},$$

$$\text{Ev}_x(f_0) = x^{-1} E_{13}, \quad \text{Ev}_x(f_1) = E_{21}, \quad \text{Ev}_x(f_2) = E_{32}. \quad (2.26)$$

One can check that this map is an algebra homomorphism as all the defining relations \((2.4) - (2.7)\) becomes corollaries of \((2.23)\).

A brief introduction into the representation theory of $U_q(gl(2|1))$ is given in the Appendix A. We also summarize some important facts here. Let $\pi_{\mu}$, with $\mu = (\mu_1, \mu_2, \mu_3)$, such that $\mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}$, denotes the irreducible finite-dimensional representation of the $U_q(gl(2|1))$ with the highest weight $\mu$ and the highest weight vector $|0\rangle >$ defined as

$$E_{12} |0\rangle >= E_{23} |0\rangle >= 0, \quad E_{ii} |0\rangle >= \mu_i |0\rangle, \quad i = 1, 2, 3. \quad (2.27)$$

Any such representation is realized by linear transformations $\text{End}(V)$ of some graded vector space $V = V_0 \oplus V_1$, where $p(V_0) = 0$ and $p(V_1) = 1$. The latter is always a subspace of the vector space generated by a free action of the elements $E_{21}$ and $E_{32}$ on the highest weight vector (note that the action of $E_{32}$ changes the parity, while the action of $E_{21}$ leaves it unchanged). There are bases of $V$, called homogeneous, where all basis vectors $|v_i\rangle$ have definite parities, i.e., $v_i \in V_0$ or $v_i \in V_1$ for any $v_i$. Let $A$ be an arbitrary matrix $A \in \text{End}(V)$, and $A_{ij}$ denote its matrix elements in a homogeneous basis $A v_k = \sum_j v_j A_{jk}$. The supertrace of $A$ over $V$ is defined as \(\text{Str}_V A = \sum_j (-1)^{q(v_j)} A_{jj}\).

Further, let $\pi_{\mu}(x)$ be the representation of $U_q(\widehat{sl}(2|1))$ obtained by the composition of $\pi_{\mu}$ with the evaluation map \((2.26)\),

$$\pi_{\mu}(x) = \pi_{\mu} \circ \text{Ev}_x. \quad (2.28)$$

Then the finite-dimensional $R$-matrices, obtained from the universal $R$-matrix,

$$R_{\mu_1\mu_2}(x_1/x_2) = (\pi_{\mu_1}(x_1) \otimes \pi_{\mu_2}(x_2))[R] \quad (2.29)$$

satisfy appropriate specializations of the Yang-Baxter equation \((2.16)\),

$$R_{\mu_1\mu_2}(x_1/x_2) R_{\mu_1\mu_3}(x_1/x_3) R_{\mu_2\mu_3}(x_2/x_3) = R_{\mu_2\mu_3}(x_2/x_3) R_{\mu_1\mu_3}(x_1/x_3) R_{\mu_1\mu_2}(x_1/x_2) \quad (2.30)$$

### 2.2 T-operators (transfer matrices)

First, let us review standard definitions of the transfer matrices in lattice models. The $R$-matrix $R_{\mu\nu}(x)$, defined in \((2.29)\), acts in the graded product of two representation spaces $V_\mu \otimes_s V_\nu$. It is convenient to consider the first of these spaces as an “auxiliary space” and the second one as a “quantum space”. The transfer matrix for a homogeneous periodic chain of the length $L$ is defined as follows,

$$T_{\mu}(x|\nu, y, L) = \text{Str}_{\pi_{\mu}} \left( D \underbrace{R_{\mu\nu}(x/y) \otimes_s R_{\mu\nu}(x/y) \otimes_s \cdots \otimes_s R_{\mu\nu}(x/y)}_{L-\text{times}} \right), \quad (2.31)$$

\(^1\text{Note there is another (non-equivalent) evaluation map obtained from \((2.26)\), if } E_{31} \text{ and } E_{13} \text{ are replaced with } F_{31} \text{ and } F_{13}.\)
where the tensor product is taken with respect to the quantum spaces $V_\nu = \pi_\nu(y)$, while the matrix product and the supertrace is taken with respect to the auxiliary space $V_\mu = \pi_\mu(x)$.

The boundary operator $\bf D$ reads

$$\bf D = q^{b_1 E_{11} + b_2 E_{22} + (b_1 + b_2) E_{33}} = \mathcal{E} v_x \left[ q^{-b_1 h_0 + b_2 h_2} \right],$$

where $E_{11}, E_{22}, E_{33}$ are defined in (2.23) and $b_1, b_2$ denotes two arbitrary horizontal field parameters. The transfer matrix (2.31) acts in a Hilbert space

$$\bf T_\mu(x|\nu, y, L) : \mathcal{H}^{(\nu)} \rightarrow \mathcal{H}^{(\nu)}, \quad \mathcal{H}^{(\nu)} = \underbrace{V_\nu \otimes \cdots \otimes V_\nu}_{L\text{-times}},$$

which is the graded product of $L$ copies of the space $V_\nu = \pi_\nu(y)$.

The symbols $\mu$ and $\nu$ in the notation for the transfer matrix $\bf T_\mu(x|\nu, y, L)$, obviously, refer to the auxiliary and quantum spaces respectively. For the same quantum space $\mathcal{H}^{(\nu)}$ there is an infinite number of different transfer matrices, corresponding to different choices of the representation $\mu$ in the auxiliary space. The Yang-Baxter equation (2.30) implies that these matrices form a commuting family

$$[\bf T_{\mu_1}(x|\nu, y, L), \bf T_{\mu_2}(x|\nu, y, L)] = 0, \quad \text{for all } \mu_1, \mu_2, x_1, x_2. \quad (2.34)$$

Note that due to the invariance property of the $R$-matrix (which trivially follows from (2.11) and the first relation in (2.15))

$$(\bf D_{\mu} \otimes \mathcal{S} \bf D_\nu) \bf R_{\mu\nu}(x) = \bf R_{\mu\nu}(x) (\bf D_{\mu} \otimes \mathcal{S} \bf D_\nu), \quad \bf D_\mu = \pi_\mu[\bf D],$$

the commutativity (2.34) is not affected by the presence of non-zero fields in definition (2.31).

Below we will derive algebraic relation between different transfer matrices, using decomposition properties of products of representations of the quantum affine superalgebra $U_q(\hat{sl}(2|1))$ in the auxiliary space. We would like to stress that our results are independent on the quantum space of the model. To facilitate these considerations it is useful to make a model-independent definition of the transfer-matrices,

$$\bf T_\mu(x) = \left( \mathcal{S} \text{tr}_{\pi_\mu(x)} \otimes 1 \right) \left[ (q^{-b_1 h_0 + b_2 h_2} \otimes 1) \mathcal{R} \right],$$

where $\mathcal{R}$ is the universal $R$-matrix, and $b_1$ and $b_2$ are the external field parameters. This formula defines a “universal” $\bf T$-operator which is an element of the Borel subalgebra $\mathcal{B}_-$ associated with the quantum space. To specialize it for a particular model one needs to choose an appropriate representation of $\mathcal{B}_-$. For example, choosing the latter to be the product $\pi_\nu(y) \otimes \cdots \otimes \pi_\nu(y)$, where $\pi_\nu(y)$ is defined by (2.28), one immediately obtains\footnote{To evaluate the RHS of (2.37) one needs to repeatedly use the third equation in (2.15).} the lattice transfer matrix (2.31)

$$\bf T_\mu(x|\nu, y, L) = \left( \pi_\nu(y) \otimes \cdots \otimes \pi_\nu(y) \right) \left[ \bf T_\mu(x) \right], \quad (2.37)$$

Another important example of the specialization of (2.30), related with the continuous superconformal field theory associated with $U_q(\hat{sl}(2|1))$ algebra, is considered in Section 4.

It is convenient to define new operators

$$z_1 = q^{2S_1} = q^{h_2 + b_1}, \quad z_2 = q^{2S_2} = q^{-h_0 + b_2}, \quad z_3 = q^{-2S_3} = z_1 z_2, \quad (2.38)$$

where the tensor product is taken with respect to the quantum spaces $V_\nu = \pi_\nu(y)$, while the matrix product and the supertrace is taken with respect to the auxiliary space $V_\mu = \pi_\mu(x)$.\footnote{To evaluate the RHS of (2.37) one needs to repeatedly use the third equation in (2.15).}
which are elements of the same Borel subalgebra $B_-$. The definition (2.36) can be then rewritten as
\[
T_\mu(x) = \left( \text{Str}_{\pi_\mu(x)} \otimes 1 \right) \left[ (1 \otimes z_1)^{-(h_0 \otimes 1)} (1 \otimes z_2)^{(h_2 \otimes 1)} \overline{R} \right]
\]
where $\overline{R}$ is now the reduced universal $R$-matrix from (2.17). The last formula looks a bit cumbersome, so in the following we will use shorthand notations and simply write
\[
T_\mu(x) = \text{Str}_{\pi_\mu(x)} \left[ z_1^{-h_0} z_2^{h_2} \overline{R} \right]
\]
but assume the same meaning as in (2.39).

Now define special notations for the most important $T$-operators (2.40), associated with rectangular Young diagrams. We denote them as $T_m^a(x)$, where $m \geq 0$ is the length and $a \geq 1$ is the height of the corresponding Young diagram, namely,

(i) the operator $T_m^1(x)$ corresponding to the $(2m + 1)$-dimensional class-2 atypical representations with the highest weight $\mu = (m, 0, 0)$,
\[
T_m^1(x) \equiv T_{(m,0,0)}(x), \quad m \in \mathbb{Z}_{\geq 0}.
\]

(ii) the operator $T_m^{-1}(x)$ corresponding to the $(2m + 1)$-dimensional class-1 atypical representations with the highest weight $\mu = (-1, -m, 0)$,
\[
T_m^{-1}(x) \equiv T_{(-1,-m,0)}(x), \quad m \in \mathbb{Z}_{\geq 0}.
\]

(iii) special notations to the operators $T(x)$ and $T(x)$ corresponding to the 3-dimensional representations ($m = 0$ case in (i) and (ii) above)
\[
T(x) \equiv T_1^1(x) \equiv T_{(1,0,0)}(x), \quad \overline{T}(x) \equiv T_{-1}^1(x) \equiv T_{-1}^{(-1,0,0)}(x) \equiv -T_{(-1,-1,0)}(x)
\]

(iv) the operator $T_c^2(x)$ corresponds to the 4-dimensional typical representation $\pi_{\mu}(x)$ with the highest weight $\mu = (c, c, 0)$, where the parameter $c \in \mathbb{C}$ is not necessarily an integer
\[
T_c^2(x) = T_{(c,c,0)}(x q^{c+1}), \quad c \in \mathbb{C}.
\]

It is convenient to define also
\[
T_{c}^{(2)}(x) = T_{(-c-1)}^{(2)}(x)
\]
For $c = 0$ or $c = -1$, the representation $\pi_{(c,c,0)}(x)$ becomes reducible (but still indecomposable). As a result one obtains
\[
T_{0}^{(2)}(x) = T_{-1}^{(2)}(x) = 1 - T_{1}^{(1)}(x), \quad T_{0}^{(2)}(x) = T_{-1}^{(2)}(x) = 1 - T_{1}^{(1)}(x).
\]
Note also that the case $m = 0$ in (2.41) and (2.42) corresponds to trivial one-dimensional representations, so that
\[
T_0^{(1)}(x) \equiv 1, \quad \overline{T}_0^{(1)}(x) \equiv 1.
\]
2.3 Q-operators

An important part in the theory of integrable quantum systems is played by the so-called Q-operators, introduced by Baxter in his pioneering work on the eight-vertex model of lattice statistics [1]. From the algebraic point of view these operators are not much different from the T-operators. They are also constructed as supertraces of certain (in general, infinite-dimensional) monodromy matrices. This is essentially the idea of Baxter’s original work [1], which has been further developed in [4, 5]. In order to construct the Q-operators this way one needs to find alternatives to the evaluation map (2.26). Each side of the Yang-Baxter equation (2.16) is an element of $B_+ \otimes_s \mathcal{A} \otimes_s B_-$. Therefore to obtain a specialization of this relation it is not necessary to have a realization of the full quantum affine superalgebra in all three factors of the direct product. For example, in the first factor one only needs to construct a realization of the Borel subalgebra $B_+$. The evaluation map (2.26) is a particular, but not the only example of such map. Below we construct several maps from $B_+$ into the graded direct products (2.12) of oscillator algebras.

Define the bosonic $q$-oscillator algebra $H_q$,

$$H_q : \quad [b^\pm, b^\pm] = \pm b^\pm, \quad q b^+ b^- - q^{-1} b^- b^+ = \frac{1}{q - q^{-1}}, \quad [b^\pm, b^\mp] = 0$$

and the fermionic oscillator algebra $F$,

$$F : \quad [f^\pm, f^\pm] = \pm f^\pm, \quad f^+ f^- + f^- f^+ = 1, \quad (f^+)^2 = (f^-)^2 = 0,$$

The Fermi operators $f^+$ and $f^-$ are odd, $p(f^+) = p(f^-) = 1$; all the other generators are even.

Define the following three maps. The first one is

$$\rho_1(x) : \quad B_+ \rightarrow H_q \otimes F$$

where $x$ is the spectral parameter,

$$\rho_1(x) : \quad \begin{cases} e_0 \rightarrow x f_2^-, & e_1 \rightarrow b_1^+ q^{\mathcal{H}_2}, & e_2 \rightarrow -q^{\frac{3}{2}} q^{-\mathcal{H}_2} b_1^- f_2^+, \\ h_0 \rightarrow -\mathcal{H}_1, & h_1 \rightarrow 2\mathcal{H}_1 + \mathcal{H}_2, & h_2 \rightarrow -\mathcal{H}_1 - \mathcal{H}_2. \end{cases}$$

The second one

$$\rho_2(x) : \quad B_+ \rightarrow F \otimes H_q$$

is given by

$$\rho_2(x) : \quad \begin{cases} e_0 \rightarrow -q^\frac{1}{2} x q^{-\mathcal{H}_2} f_1^+ b_2^+, & e_1 \rightarrow b_2^-, & e_2 \rightarrow f_1^+ q^{\mathcal{H}_2}, \\ h_0 \rightarrow \mathcal{H}_1 + \mathcal{H}_2, & h_1 \rightarrow -\mathcal{H}_1 - 2\mathcal{H}_2, & h_2 \rightarrow \mathcal{H}_2. \end{cases}$$

And the third one

$$\rho_3(x) : \quad B_+ \rightarrow F \otimes_s F$$

is given by

$$\rho_3(x) : \quad \begin{cases} e_0 \rightarrow x f_1^+ q^{-\mathcal{H}_2}, & e_1 \rightarrow q^{-\frac{1}{2}} q^{\mathcal{H}_2} f_1^- f_2^+ / (q - q^{-1}), & e_2 \rightarrow f_2^-, \\ h_0 \rightarrow -\mathcal{H}_2, & h_1 \rightarrow -\mathcal{H}_1 + \mathcal{H}_2, & h_2 \rightarrow \mathcal{H}_1. \end{cases}$$
The indices 1 and 2 above refer respectively to the first and second factors in the tensor products (2.50), (2.52) and (2.54).

The operators $\tilde{Q}_i(x)$ are defined similarly to the $\mathbf{T}$-operators (2.40),

$$\tilde{Q}_i(x) = x^{S_i} \tilde{A}_i(x), \quad \tilde{A}_i(x) = Z_i^{-1} \mathrm{Str}_{\rho_i(x)} \left[ z_1^{-h_0} z_2^{h_2} \mathcal{R} \right], \quad i = 1, 2, 3, \quad (2.56)$$

where $S_i$, $i = 1, 2, 3$ are defined in (2.35) and the normalization constants read

$$Z_i = \mathrm{Str}_{\rho_i(x)} \left[ z_1^{-h_0} z_2^{h_2} \right]. \quad (2.57)$$

The trace is now taken over the Fock space representations of the $q$-oscillator superalgebras involved in the maps $\rho_i(x)$. An important property of the definition (2.56) is that the supertrace therein (normalized by the constants $Z_i$) is completely determined by the commutation relations (2.48) and (2.49) and the cyclic property of the supertrace, so the specific choice of the representations in (2.56) is not important as long as the supertrace exist. (Notice that the representations of the bosonic $q$-oscillator algebra (2.48) are infinite-dimensional so the question of convergence should be kept in mind. There is no real problem here, the convergence can always be achieved with a proper choice of the external field parameters $b_{1,2}$. See Sect.5.2.3 for further details).

The remaining three operators $\overline{Q}_i(x)$ are defined in a similar way

$$\overline{Q}_i(x) = x^{-S_i} \overline{A}_i(x), \quad \overline{A}_i(x) = Z_i^{-1} \mathrm{Str}_{\overline{\rho}_i(x)} \left[ z_1^{-h_0} z_2^{h_2} \mathcal{R} \right], \quad i = 1, 2, 3, \quad (2.58)$$

where normalization constants read

$$\overline{Z}_i = \mathrm{Str}_{\overline{\rho}_i(x)} \left[ z_1^{-h_0} z_2^{h_2} \right], \quad (2.59)$$

and the corresponding maps $\overline{\rho}_i(x)$ are defined as follows. The first one

$$\overline{\rho}_1(x) : \quad \mathbf{B}_+ \rightarrow H_q \otimes \mathbf{F} \quad \mathrm{(2.60)}$$

is given by

$$\overline{\rho}_1(x) : \begin{cases} e_0 \rightarrow x f_2^+, & e_1 \rightarrow b_1^+ q^{H_2}, & e_2 \rightarrow -q^{-\frac{1}{2}} q^{-H_2} b_1^+ f_2^-, \\ h_0 \rightarrow H_1, & h_1 \rightarrow -2H_1 - H_2, & h_2 \rightarrow H_1 + H_2. \end{cases} \quad (2.61)$$

The second one

$$\overline{\rho}_2(x) : \quad \mathbf{B}_+ \rightarrow \mathbf{F} \otimes H_q \quad \mathrm{(2.62)}$$

is given by

$$\overline{\rho}_2(x) : \begin{cases} e_0 \rightarrow -q^{-\frac{1}{2}} x q^{-H_2} f_1^+ b_2^+, & e_1 \rightarrow b_2^+, & e_2 \rightarrow f_1^+ q^{H_2}, \\ h_0 \rightarrow -H_1 - H_2, & h_1 \rightarrow H_1 + 2H_2, & h_2 \rightarrow -H_2. \end{cases} \quad (2.63)$$

And the third one

$$\overline{\rho}_3(x) : \quad \mathbf{B}_+ \rightarrow \mathbf{F} \otimes \mathbf{F} \quad \mathrm{(2.64)}$$

is given by

$$\overline{\rho}_3(x) : \begin{cases} e_0 \rightarrow x f_1^+ q^{-H_2}, & e_1 \rightarrow -q^{\frac{1}{2}} q^{H_2} f_2^+ f_2^-/(q - q^{-1}), & e_2 \rightarrow f_2^+, \\ h_0 \rightarrow H_2, & h_1 \rightarrow H_1 - H_2, & h_2 \rightarrow -H_1. \end{cases} \quad (2.65)$$
2.4 Lattice $R$-matrices

As an illustration of the above construction consider all specializations of the universal $R$-matrix related to the 3-state model \((1.1)\). We postpone details of calculations to a separate publication \([63]\), but present the final results here.

Below we will fix the (local) quantum space at each lattice site to be the 3-dimensional evaluation representation $\pi_{(1,0,0)}(q^{1/2})$ with the weight $\mu = (1,0,0)$ and the spectral parameter $q^{1/2}$, as defined in \((2.28)\). First consider the $L$-operator, obtained as

$$L(x) = N'(x) \left( \text{Ev}_x \otimes_s \pi_{(1,0,0)}(q^{1/2}) \right) \left[ R \right].$$

(2.66)

where $\text{Ev}_x$ is the evaluation map \((2.26)\) and $N'$ is a normalization factor. It is convenient to present this operator

$$L = \sum_{i,j=1}^3 L_{ij} \otimes_s e_{ij}, \quad L_{ij} \in U_q(gl(2|1))$$

(2.67)

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ is the matrix unit, in the form of a $3 \times 3$ matrix, acting in the quantum space,

$$L = \begin{pmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
-L_{31} & -L_{32} & L_{33}
\end{pmatrix},$$

(2.68)

whose entries are operators belonging to the finite-dimensional algebra $U_q(gl(2|1))$ (the reader should pay attention to the signs in \((2.68)\)). Remind that the relevant parities are $p(L_{jk}) = p(e_{jk}) = p(j) + p(k) \pmod{2}$, and $p(1) = p(2) = 0$, $p(3) = 1$. The operators $L_{ij}$ act in the auxiliary space. With a suitable choice of the normalization factor $N'(x)$ one obtains (cf. \([64]\))

$$L(x) = \begin{pmatrix}
Cq^{-E_{11}} - q^{1/2}xq^{E_{11}} & -q^{1/2}a_q x q^{E_{11}} E_{21} & -q^{1/2}a_q x q^{E_{11}} E^{33} \\
-a_q C q^{E_{12}} q^{-E_{11}} & Cq^{-E_{22}} - q^{1/2}xq^{E_{22}} & -q^{1/2}a_q x q^{E_{22}} E_{32} \\
-q a_q q^{-E_{33}} E_{13} & -a_q Cq^{E_{23}} q^{-E_{22}} & Cq^{E_{33}} - q^{1/2}x q^{-E_{33}}
\end{pmatrix}$$

(2.69)

where $a_q = (q - q^{-1})$ and

$$C = q^{E_{11} + E_{22} + E_{33}}$$

(2.70)

is a central element of the algebra $U_q(gl(2|1))$. Note that matrix elements of $L(x)$ are first order polynomials in the spectral parameter $x$. Such normalization is especially convenient for lattice models. The factor $N'(x)$ depends on central elements of $U_q(gl(2|1))$, so it depends on the representation. It can be thought of as a diagonal operator also acting in the auxiliary space. In general it is a meromorphic function of $x$. The formula \((2.69)\) can be further specialized by choosing some particular representation $\pi_{\mu}$ of $U_q(gl(2|1))$ in the auxiliary space. The resulting operator is, essentially, a particular case of \((2.29)\) with $\nu = (1,0,0)$ and $y = q^{1/2}$, differing from the later merely by a scalar factor

$$L_{\mu}(x) = \pi_{\mu}[L(x)] = N_{\mu}(x) R_{\mu \nu}(x q^{-\frac{1}{2}}) \bigg|_{\nu = (1,0,0)}$$

(2.71)

where

$$N(\mu_1, \mu_2, \mu_3)(x) = \frac{\psi(xq^{-1/2} + \mu_1 - \mu_2 - \mu_3) \psi(xq^{-1/2} - \mu_1 + \mu_2 - \mu_3)}{\psi(xq^{-1/2} - \mu_1 - \mu_2 - 3\mu_3)}, \quad \psi(x) = 1 - x.$$
Similarly to (2.31) it is convenient to define lattice transfer matrices constructed with this \( L \)-operator,

\[
T^{(L)}_{\mu}(x) = \text{Str}_{\pi}(D \underbrace{L(x) \otimes \cdots \otimes L(x)}_{\text{\(L\)-times}})
\]

\[
= (N_{\mu}(x))^{L} \left( \underbrace{\pi_{\nu}(q^{\frac{1}{L}}) \otimes \cdots \otimes \pi_{\nu}(q^{\frac{1}{L}})}_{\text{\(\nu\)-times}} \right) T_{\nu}(x)
\]

(2.73)

which, to within the scalar factor \( (N_{\mu})^{L} \), coincide with \( T_{\mu}(x)|_{\nu=1,0,0} = \psi(xq^{\frac{3}{L}}) = (1 - xq^{\frac{3}{L}}) \).

(2.74)

Now define additional \( L \)-operator associated with the oscillator algebras (2.48), (2.49).

\[
L_{j}(x) = N_{j}(x) \left( \rho_{j}(x) \otimes \pi_{(1,0,0)}(q^{\frac{1}{L}}) \right) \left[ R \right], \quad j = 1, 2, 3
\]

(2.77)

\[
\overline{L}_{j}(x) = N_{j}(x) \left( \overline{\rho}_{j}(x) \otimes \pi_{(1,0,0)}(q^{\frac{1}{L}}) \right) \left[ R \right], \quad j = 1, 2, 3
\]

(2.78)

where \( R \) is the universal \( R \)-matrix and the maps \( \rho_{j}(x) \) and \( \overline{\rho}_{j}(x) \) defined in the previous subsection. Explicitly, one obtains [63]

\[
L_{1}(x) = \begin{pmatrix}
q^{-\mathcal{H}_{1}-\mathcal{H}_{2}} - qx^{\mathcal{H}_{1}+\mathcal{H}_{2}}C_{b_{1}}C_{f_{2}} & -aq_{x}q^{\mathcal{H}_{1}+\mathcal{H}_{2}}b_{1}^{-}C_{f_{2}} & q^{-\frac{3}{2}}aq_{x}q^{-\mathcal{H}_{2}}f_{2}^{-}
-q_{b_{1}^{-}}q^{-\mathcal{H}_{1}} & q^{\mathcal{H}_{1}} & 0
q^{\mathcal{H}_{2}} & q^{\mathcal{H}_{1}-\mathcal{H}_{2}}f_{2}^{+} & q^{-\mathcal{H}_{2}}
\end{pmatrix}
\]

(2.79)

\[
L_{2}(x) = \begin{pmatrix}
q^{\mathcal{H}_{2}} & -aq_{x}q^{\mathcal{H}_{1}-\mathcal{H}_{2}}b_{2}^{+}C_{f_{1}} & -q^{-1}aq_{x}q^{-\mathcal{H}_{1}-\mathcal{H}_{2}}f_{2}^{-}b_{2}^{+}
-qaq_{x}^{\mathcal{H}_{1}}b_{2}^{-} & q^{-\mathcal{H}_{1}-\mathcal{H}_{2}} - qx^{\mathcal{H}_{1}+\mathcal{H}_{2}}C_{f_{1}}C_{b_{2}} & -qx^{-1}q^{-\mathcal{H}_{1}+\mathcal{H}_{2}}f_{1}^{-}C_{b_{2}}
0 & -qaq^{-\mathcal{H}_{1}}f_{1}^{+} & q^{-\mathcal{H}_{1}}
\end{pmatrix}
\]

(2.80)

\[
L_{3}(x) = \begin{pmatrix}
q^{\mathcal{H}_{1}} & 0 & q^{-\frac{3}{2}}aq_{x}q^{\mathcal{H}_{1}}f_{1}^{+}
-q^{\mathcal{H}_{2}} & q^{-1}aq_{x}q^{-\mathcal{H}_{1}+\mathcal{H}_{2}}f_{2}^{+}C_{f_{1}} & q^{-1}aq_{x}q^{-\mathcal{H}_{1}+\mathcal{H}_{2}}f_{2}^{-}C_{f_{1}}^{-1}
-qaq_{x}^{\mathcal{H}_{2}}f_{2}^{-} & q^{\mathcal{H}_{1}+\mathcal{H}_{2}} - qx^{\mathcal{H}_{1}-\mathcal{H}_{2}}C_{f_{1}}C_{b_{2}}^{-1} & q^{\mathcal{H}_{2}}
\end{pmatrix}
\]

(2.81)
where

\[ \psi = \frac{\psi_{N+1} } {xq^N} \]

and

\[ \mathbf{L}_1(x) = \begin{pmatrix}
q^{\mathcal{H}_1 + \mathcal{H}_2} & -a_q xq^{-\mathcal{H}_1 - \mathcal{H}_2 b^+_1} & q^{-\frac{3}{2} a_q xq^{\mathcal{H}_2 f^+_2}} \\
-a_q b^+_1 q^{\mathcal{H}_1 + 2\mathcal{H}_2} & q^{\mathcal{H}_1} - xq^{\mathcal{H}_1 C_{b_1}} & -q^{-\frac{3}{2} a_q xq^{2\mathcal{H}_2 b^-_1} f^+_2} \\
-q^2 a_q q^{\mathcal{H}_1 - \mathcal{H}_2 b^+_2} & q^2 a_q q^{\mathcal{H}_1 - \mathcal{H}_2 b^+_2} f^+_2 & q^{\mathcal{H}_2} - xq^{-\mathcal{H}_2 C_{f_2}^{-1}}
\end{pmatrix}, \tag{2.82} \]

\[ \mathbf{L}_2(x) = \begin{pmatrix}
q^{-\mathcal{H}_1 - xq^{\mathcal{H}_2 C_{b_2}}} & -a_q xq^{\mathcal{H}_1 + \mathcal{H}_2 b^-_2} & -q^{-2 a_q xq^{\mathcal{H}_1 - \mathcal{H}_2 f^+_1 b^-_2}} \\
-q a_q xq^{-2\mathcal{H}_2 b^+_2} & q^{\mathcal{H}_1 + \mathcal{H}_2} & xq^{-1} q^{\mathcal{H}_1 - \mathcal{H}_2 f^+_1} \\
q a_q f^+_1 b^-_2 & -q a_q xq^{\mathcal{H}_1 + 2\mathcal{H}_2 f^-_1} & q^{\mathcal{H}_1} - xq^{-\mathcal{H}_1 C_{f_1}^{-1}}
\end{pmatrix}, \tag{2.83} \]

\[ \mathbf{L}_3(x) = \begin{pmatrix}
q^{-\mathcal{H}_1} - xq^{\mathcal{H}_1 C_{f_1}} & -q^3 a_q xq^{-2\mathcal{H}_2 f^-_1 f^+_2} & q^{-\frac{3}{2} a_q xq^{-\mathcal{H}_1 - 2\mathcal{H}_2 f^-_1}} \\
q^2 q^{\mathcal{H}_1 + \mathcal{H}_2 f^+_1 f^-_2} & q^{\mathcal{H}_2} - xq^{\mathcal{H}_2 C_{f_2}} & q^{-1} xq^{-\mathcal{H}_1 - \mathcal{H}_2 f^-_2} \\
q^2 a_q q^{\mathcal{H}_1 + \mathcal{H}_2 f^+_1} & -q a_q f^+_1 q^{-\mathcal{H}_2} & q^{\mathcal{H}_1} - \mathcal{H}_2
\end{pmatrix}, \tag{2.84} \]

where \( a_q = (q - q^{-1}) \) and the central charges for the bosonic and fermionic oscillator algebras (2.48) and (2.49) are given by

\[ C_b = q q^{-2\mathcal{H}_b} \left( 1 - (q - q^{-1})^2 b^+ b^- \right), \tag{2.85} \]

\[ C_f = q^{-1} q^{-2\mathcal{H}_f} \left( 1 - f^+ f^- \right). \tag{2.86} \]

The normalization factors read

\[ N_1(x) = \psi(x), \quad N_2(x) = \psi(x), \quad N_3(x) = 1/\psi(x), \quad N_1(x) = 1, \quad N_2(x) = 1, \quad N_3(x) = \psi(xq) \psi(xq^{-1}). \tag{2.87} \]

where \( \psi(x) = 1 - x \). For the 3-state model the eigenvalues of the diagonal operators (2.38) are expressed in terms of the external field parameters \( b_{1,2} \) and the edge occupation numbers (or the “magnon numbers”),

\[ z_1 = q^{2S_1}, \quad z_2 = q^{2S_2}, \quad z_3 = q^{-2S_3}, \tag{2.88} \]

where

\[ 2S_1 = L - m_1 + b_1, \quad 2S_2 = L - m_2 + b_2, \quad 2S_3 = -2S_1 - 2S_2 \tag{2.89} \]

are expressed in terms of the external field parameters \( b_{1,2} \) and the edge occupation numbers (or the “magnon numbers”),

\[ m_i = \sum_{\ell=1}^{L} e_i^{(\ell)}, \quad m_1 + m_2 + m_3 = L, \tag{2.90} \]
where \( c_{ij}^{(\ell)} \) are three by three matrices \( ||c_{ij}^{(\ell)}||_{ab} = \delta_{ia}\delta_{jb}, a, b = 1, 2, 3 \) acting at the \( \ell \)-th site of the chain.

We can now explicitly define lattice Q-operators for the 3-state model,

\[
A_j(x) = Z_j^{-1} \text{Str}_{\rho_j(x)} \left\{ q^{-b_1 h_0 + b_2 h_2} L_j(x) \otimes_s L_j(x) \otimes_s \cdots \otimes_s L_j(x) \right\}
\]

and

\[
A_j(x) = Z_j^{-1} \text{Str}_{\rho_j(x)} \left\{ q^{-b_1 h_0 + b_2 h_2} L_j(x) \otimes_s L_j(x) \otimes_s \cdots \otimes_s L_j(x) \right\}
\]

where \( j = 1, 2, 3 \), the generators \( h_0, h_1 \) in the exponents act in the auxiliary space and the quantities \( Z_j \) and \( \bar{Z}_j \) are given in \( (2.57) \). A word caution: one has to use there the same representations of the oscillator algebras \( (2.48) \) and \( (2.49) \) as in the corresponding formula \( (2.91) \) or \( (2.92) \). Then these definitions become independent on a choice of these representations. The quantities \( Z_j \) and \( \bar{Z}_j \), however, do depend on this choice (see Sect 5.2.3 for further details).

It is evident from \( (2.79, 2.84) \) that \( A_j(x) \) and \( \bar{A}_j(x) \) are operator-valued polynomials in \( x \) of the degree \( L \). They are simply connected with the specializations of the universal Q-operators \( (2.56) \) and \( (2.58) \) to the 3-state model, namely,

\[
A_j(x) = (N_j(x))^L \left( \otimes_{\nu=1}^{\nu=0} \pi_{\nu}(q^{\frac{1}{2}}) \otimes_s \pi_{\nu}(q^{\frac{1}{2}}) \otimes_s \cdots \otimes_s \pi_{\nu}(q^{\frac{1}{2}}) \right) \left[ A_j(x) \right], \quad \nu = (1, 0, 0), \quad (2.93)
\]

and

\[
\bar{A}_j(x) = (\bar{N}_j(x))^L \left( \otimes_{\nu=1}^{\nu=0} \pi_{\nu}(q^{\frac{1}{2}}) \otimes_s \pi_{\nu}(q^{\frac{1}{2}}) \otimes_s \cdots \otimes_s \pi_{\nu}(q^{\frac{1}{2}}) \right) \left[ \bar{A}_j(x) \right], \quad \nu = (1, 0, 0), \quad (2.94)
\]

where \( j = 1, 2, 3 \).

### 3 Functional relations

As is well known \( [65] \), the analyticity of the transfer matrices becomes an extremely powerful condition when combined with the functional relations which the transfer matrices satisfy, and, in principle, allows one to determine all their eigenvalues. Therefore the functional relations for the T- and Q-operators are of a primary interest. We present these relations here, but postpone their proof (which is purely algebraic) to Section 5. It is convenient to split these relations into three groups, (i) the Wronskian-type relations, (ii) the T-Q relations and (iii) the fusion relations. We remark that some of our results in this section overlap with those obtained in \( [44–46, 50] \) for the rational case \( q = 1 \).

#### 3.1 Wronskian-type relations

These relations only involve the Q-operators, defined in \( (2.50) \) and \( (2.58) \). First, there are four independent relations, quoted in the Introduction,

\[
c_{12} = c_{13} Q_1(q^{\frac{1}{2}}x) Q_1(q^{-\frac{1}{2}}x) - c_{23} Q_2(q^{\frac{1}{2}}x) Q_2(q^{-\frac{1}{2}}x)
\]

\[
c_{12} = c_{13} Q_1(q^{\frac{1}{2}}x) Q_1(q^{-\frac{1}{2}}x) - c_{23} Q_2(q^{\frac{1}{2}}x) Q_2(q^{-\frac{1}{2}}x)
\]

\[
(3.1)
\]
and
\[ c_{21} Q_3(x) = \overline{Q}_1(xq)\overline{Q}_2(xq^{-1}) - \overline{Q}_1(xq^{-1})\overline{Q}_2(xq), \]  
\[ c_{12} \overline{Q}_3(x) = Q_1(xq)Q_2(xq^{-1}) - Q_1(xq^{-1})Q_2(xq), \]  
where
\[ c_{ij} = \frac{z_i - z_j}{(z_i z_j)^{\frac{1}{2}}} \]  
with the operators \( z_1, z_2 \) and \( z_3 \) defined in (2.35). Combining these relations one easily obtains
\[ c_{13} Q_3(x)Q_1(x) = \overline{Q}_2(xq) - \overline{Q}_2(xq^{-1}), \]  
\[ c_{23} Q_2(x)Q_3(x) = \overline{Q}_1(xq) - \overline{Q}_1(xq^{-1}), \]  
\[ c_{13} \overline{Q}_3(x)\overline{Q}_1(x) = Q_2(xq^{-1}) - Q_2(xq), \]  
\[ c_{23} \overline{Q}_2(x)\overline{Q}_3(x) = Q_1(xq^{-1}) - Q_1(xq). \]  

For example, substituting (3.2b) into the LHS of (3.4) and using both relations (3.1) in the resulting expression, one gets
\[ c_{13} Q_3(x)Q_1(x) = \frac{c_{13}}{c_{21}}\left\{ \overline{Q}_1(xq)\overline{Q}_2(q^{-1}x)Q_1(x) - \overline{Q}_1(q^{-1}x)\overline{Q}_2(xq)Q_1(x) \right\} 
= \frac{1}{c_{21}} \left\{ \overline{Q}_2(q^{-1}x)\left[ c_{12} + c_{23} Q_2(x)\overline{Q}_2(qx) \right] - \overline{Q}_2(xq)\left[ c_{12} + c_{23} Q_2(x)\overline{Q}_2(q^{-1}x) \right] \right\} 
= \overline{Q}_2(q^{-1}x). \]  

### 3.2 T-Q relations

The next group of relations connects the T- and Q-operators. An important part of this group consists of relations, which express the T-operators as polynomial combinations of the Q-operators. For the operators (2.41), (2.42) and (2.44), introduced above, these relations read
\[ c_{12} T_m^{(1)}(x) = c_{13} Q_1(xq^{m+\frac{1}{2}})\overline{Q}_1(xq^{-m-\frac{1}{2}}) - c_{23} Q_2(xq^{m+\frac{1}{2}})\overline{Q}_2(xq^{-m-\frac{1}{2}}) \]  
\[ c_{12} \overline{T}_m^{(1)}(x) = c_{13} Q_1(xq^{-m-\frac{1}{2}})\overline{Q}_1(xq^{m+\frac{1}{2}}) - c_{23} Q_2(xq^{-m-\frac{1}{2}})\overline{Q}_2(xq^{m+\frac{1}{2}}) \]  
where \( m \in \mathbb{Z}_{\geq 0} \) and
\[ T_m^{(2)}(x) = c_{23} c_{13} Q_3(xq^{-c+\frac{1}{2}})\overline{Q}_3(xq^{c+\frac{1}{2}}) \]  
\[ \overline{T}_m^{(2)}(x) = c_{23} c_{13} Q_3(xq^{c+\frac{1}{2}})\overline{Q}_3(xq^{-c+\frac{1}{2}}) \]  
Taking into account the normalization conditions (2.37) it is easy to see that Eqs. (3.9) with \( m = 0 \) simply reduce to the Wronskian-type relations (3.1). The relation (3.10b) is just a corollary of the definition (2.45).

Using (3.1)-(3.7) one can transform the expressions (3.9) to a more familiar Bethe-Ansatz type form. For example the operators \( T(x) \) and \( \overline{T}(x) \), defined in (2.43), corresponding to the
3-dimensional representations in auxiliary space, can be transformed to any of the six equivalent forms

\[
T(x) = p_1 \frac{Q_i(q^{-p_i-\frac{1}{2}}x)}{Q_k(q^{-p_i+\frac{1}{2}}x)} + p_j \frac{Q_i(q^{p_j+2p_j-\frac{1}{2}}x)}{Q_k(q^{p_j-\frac{1}{2}}x)} + p_k \frac{Q_i(q^{p_j+p_j+2p_k-\frac{1}{2}}x)}{Q_k(q^{p_j+p_j+p_k-\frac{1}{2}}x)},
\]

where \(p_1 = p_2 = -p_3 = 1\) and \((i, j, k)\) is an arbitrary permutation of \((1, 2, 3)\). The most general transfer matrix expression of this type is considered in the Appendix C (see, Eq. (C.8) therein).

### 3.3 Fusion relations

These relations only involve the \(T\)-operators \(T_m(x)\), corresponding to the rectangular Young diagrams of the length \(m \geq 0\) and the height \(a \geq 1\). A complete set of the fusion relations for \(U_q(s\ell(r + 1)\mathfrak{s} + 1)\) involving the \(T\)-operators (3.9) and (3.10) has been previously proposed in [33–35]. These relations were deduced there from the Bethe Ansatz solution of the \((r+s+2)\)-state lattice model. In particular for \(U_q(s\ell(2|1))\) case, they take the form

\[
\begin{align*}
T_m^{(1)}(q^{-1}x)T_m^{(1)}(q^x) &= T_{m-1}^{(1)}(x)T_{m+1}^{(2)}(x) + T_m^{(2)}(x) & \text{for } m \in \mathbb{Z}_{\geq 1},
T_1^{(2)}(q^{-1}x)T_1^{(2)}(q^x) &= T_2^{(2)}(x) + T_1^{(3)}(x),
T_m^{(2)}(q^{-1}x)T_m^{(2)}(q^x) &= T_{m-1}^{(2)}(x)T_{m+1}^{(2)}(x) & \text{for } m \in \mathbb{Z}_{\geq 2},
T_1^{(a)}(q^{-1}x)T_1^{(a)}(q^x) &= T_{1}^{(a-1)}(x)T_{1}^{(a+1)}(x) & \text{for } a \in \mathbb{Z}_{\geq 3},
\end{align*}
\]

where \(T_0^{(1)}(x) = 1\). There is also a duality relation:

\[
T_a^{(2)}(x) = (-1)^{a-1}T_1^{(1+a)}(x) & \text{ for } a \in \mathbb{Z}_{\geq 1},
\]

which maps last two relation in (3.12) into one another. The fusion relations for conjugate representations have also the same form\(^3\)

\[
\begin{align*}
\overline{T}_m^{(1)}(q^{-1}x)\overline{T}_m^{(1)}(q^x) &= \overline{T}_{m-1}^{(1)}(x)\overline{T}_{m+1}^{(1)}(x) + \overline{T}_m^{(2)}(x) & \text{for } m \in \mathbb{Z}_{\geq 1},
\overline{T}_1^{(2)}(q^{-1}x)\overline{T}_1^{(2)}(q^x) &= \overline{T}_2^{(2)}(x) + \overline{T}_1^{(3)}(x),
\overline{T}_m^{(2)}(q^{-1}x)\overline{T}_m^{(2)}(q^x) &= \overline{T}_{m-1}^{(2)}(x)\overline{T}_{m+1}^{(2)}(x) & \text{for } m \in \mathbb{Z}_{\geq 2},
\overline{T}_1^{(a)}(q^{-1}x)\overline{T}_1^{(a)}(q^x) &= \overline{T}_{1}^{(a-1)}(x)\overline{T}_{1}^{(a+1)}(x) & \text{for } a \in \mathbb{Z}_{\geq 3},
\end{align*}
\]

where \(\overline{T}_0^{(1)}(x) = 1\). The corresponding duality relation reads

\[
\overline{T}_a^{(2)}(x) = (-1)^{a-1}\overline{T}_1^{(1+a)}(x) & \text{ for } a \in \mathbb{Z}_{\geq 1},
\]

Note that our results (3.11)–(3.10) imply all the fusion relations (3.12) and (3.14) as trivial corollaries. Additional functional relations for \(T\)-operators are discussed in the Appendix C (see also [57]).

\(^3\)See Appendix B in [33].
Note also that the operator (3.10) satisfies the relation
\[ T_c^{(2)}(q^{-d}x)T_c^{(2)}(q^{d}x) = T_{c-d}^{(2)}(x)T_{c+d}^{(2)}(x) \quad \text{for} \quad c, d \in \mathbb{C}, \] (3.16)
which generalizes the third relation in (3.12). Finally, quote again useful relations (2.46) connecting two types of the T-operators for the atypical and typical representations.

3.4 Eigenvalue equations

The T- and Q-operators with different values of the spectral parameter \( x \) form a commuting family of operators and can be simultaneously diagonalized by an \( x \)-independent similarity transformation. Therefore, the above functional equations are satisfied by eigenvalues of these operators, corresponding to the same eigenstate. The equations are the same for all eigenstates. Here we presented these equations in a universal model-independent form. They are written in the normalization of the universal R-matrix, which is uniquely defined by its series expansion (2.19). This is a distinguished normalization where the functional equations does not contain any \( x \)-dependent scalar factors (however, the eigenvalues in this case are, in general, meromorphic functions of \( x \)). From analytic point of view it is more convenient to work with a normalization where the eigenvalues are entire functions of \( x \) (it is commonly used in the lattice theory since Baxter’s pioneering work [1]). With this “analytic” normalization the functional equations acquire some non-universal scalar factors, depending on the quantum space of the model.

As an example consider the 3-state lattice model. All relevant definition are already given in Section 2.4. First consider the operators \( A_j(x) \) and \( \overline{A}_j(x) \), defined in (2.91) and (2.92), respectively. Let \( A_j(x) \) and \( \overline{A}_j(x) \) denote a set eigenvalues of their eigenvalues, corresponding to the same eigenstate. All these eigenvalues are \( L \)-th degree polynomials in \( x \), where \( L \) is the length of the chain. Moreover, the definitions (2.91) and (2.92) imply
\[ A(0) = \overline{A}(0) = 1. \] (3.17)
Remind that the constants \( z_1, z_2, z_3 \), defined in (2.88), are expressed through the conserved edge occupation numbers \( m_1, m_2 \) and \( m_3 \) and the external field parameters \( b_{1,2} \). Now write the eigenvalues in the product form (cf. (1.7))
\[ A_i(x) = \prod_{\ell=1}^{m_i} (1 - x/x_\ell^{(i)}), \quad \overline{A}_i(x) = \prod_{\ell=1}^{L-m_i} (1 - x/x_\ell^{(i)}). \] (3.18)
where the numbers of zeroes are uniquely determined from elementary considerations of the leading \( x \to \infty \) asymptotics in (2.91) and (2.92). Define the scalar function
\[ f(x) = (1 - x)^L. \] (3.19)
Substituting (2.93) and (2.94) into (3.17) and using the expressions (2.87) for the normalization factors, one obtains,
\[ c_{12} f(q^{\frac{1}{2}}x) = c_{13} z_1^{\frac{1}{2}} A_1(q^{\frac{1}{2}}x) A_1(q^{\frac{1}{2}}x) - c_{23} z_2^{\frac{1}{2}} A_2(q^{\frac{1}{2}}x) A_2(q^{\frac{1}{2}}x), \] (3.20)
\[ c_{12} f(q^{-\frac{1}{2}}x) = c_{13} z_1^{-\frac{1}{2}} A_1(q^{-\frac{1}{2}}x) A_1(q^{\frac{1}{2}}x) - c_{23} z_2^{-\frac{1}{2}} A_2(q^{-\frac{1}{2}}x) A_2(q^{\frac{1}{2}}x) \] (3.21)
Eq.(3.16) is a special case of more general functional relation given in (3.33) of [35].
\begin{align}
c_{21} f(x) A_3(x) &= \left( \frac{z_2}{z_1} \right)^{\frac{1}{2}} \overline{A}_1(xq) \overline{A}_2(xq^{-1}) - \left( \frac{z_1}{z_2} \right)^{\frac{1}{2}} A_1(xq^{-1}) A_2(xq), \quad (3.22) \\
c_{12} \overline{A}_3(x) &= \left( \frac{z_1}{z_2} \right)^{\frac{1}{2}} A_1(xq) A_2(xq^{-1}) - \left( \frac{z_2}{z_1} \right)^{\frac{1}{2}} A_1(xq^{-1}) A_2(xq), \quad (3.23)
\end{align}

and

\begin{align}
c_{13} A_1(x) A_3(x) &= \left( \frac{z_1}{z_3} \right)^{\frac{1}{2}} \overline{A}_2(xq) - \left( \frac{z_3}{z_1} \right)^{\frac{1}{2}} \overline{A}_2(xq^{-1}), \quad (3.24) \\
c_{23} A_2(x) A_3(x) &= \left( \frac{z_2}{z_3} \right)^{\frac{1}{2}} A_1(xq) - \left( \frac{z_3}{z_2} \right)^{\frac{1}{2}} A_1(xq^{-1}), \quad (3.25) \\
c_{13} \overline{A}_1(x) \overline{A}_3(x) &= \left( \frac{z_1}{z_3} \right)^{\frac{1}{2}} f(qx) A_2(xq^{-1}) - \left( \frac{z_3}{z_1} \right)^{\frac{1}{2}} f(q^{-1}x) A_2(xq), \quad (3.26) \\
c_{23} \overline{A}_2(x) \overline{A}_3(x) &= \left( \frac{z_2}{z_3} \right)^{\frac{1}{2}} f(qx) A_1(xq^{-1}) - \left( \frac{z_3}{z_2} \right)^{\frac{1}{2}} f(q^{-1}x) A_1(xq), \quad (3.27)
\end{align}

The eigenvalues \( T_m^{(1)}(x) \) of the \( T \)-operators (3.9) can be written in a similar form

\[ c_{12} T_m^{(1)}(x) = c_{13} z_1^{m+\frac{1}{2}} A_1(xq^{m+\frac{1}{2}}) \overline{A}_1(xq^{-m-\frac{1}{2}}) - c_{23} z_2^{m+\frac{1}{2}} A_2(xq^{m+\frac{1}{2}}) \overline{A}_2(xq^{-m-\frac{1}{2}}). \quad (3.28) \]

The eight relations (3.20)–(3.27) can be solved in different ways. For example, for the coefficients of the polynomials (3.18). Altogether there are exactly 3L unknown coefficients. Let us count the number of equations. Begin by dropping the last four relation among the eight, since they are simple corollaries of the first four. The remaining four relations are of the degrees \( L, L, L + m_3 \) and \( L - m_3 \), respectively. They are trivially satisfied at \( x = 0 \), so we are left with only 4L equations. It is easy to see that \( L \) of them are dependent. For example, expressing \( A_1(x) \) and \( A_2(x) \) from (3.20) and (3.21) and substituting them into (3.23) one immediately concludes that the RHS of (3.22) is always divisible by \( f(x) \). So that \( L \) out of 4L remaining equation are automatically satisfied. Thus the there are exactly 3L polynomial equations for 3L unknown coefficients.

The equations (3.20)–(3.27) can also be solved for the zeroes of the polynomials (3.18). Let us rewrite these equation in the Bethe Ansatz form. Substitute \( x = q^{-\frac{1}{2}} x_\ell^{(1)} \) into (3.20) and \( x = q^{\frac{1}{2}} x_\ell^{(1)} \) into (3.21) and then divide the resulting two equation,

\[ c_{12} f(x_\ell^{(1)}) = -c_{23} z_2^{x_\ell^{(1)}} A_2(q^{-\frac{1}{2}} x_\ell^{(1)} A_2(x_\ell^{(1)}), \quad c_{12} f(x_\ell^{(1)}) = -c_{23} z_2^{x_\ell^{(1)}} A_2(q^{\frac{1}{2}} x_\ell^{(1)} A_2(x_\ell^{(1)}), \quad (3.29) \]

by one another. It follows that

\[ z_2^{-1} = \frac{\overline{A}_2(q^{-1} x_\ell^{(1)})}{\overline{A}_2(q^{1} x_\ell^{(1)})}, \quad \ell = 1, \ldots, m_1. \quad (3.30) \]

where \( x_\ell^{(1)} \) denotes the zeroes of \( A_1(x) \). Performing similar manipulation for other zeroes one obtains the complete set of the Bethe Ansatz type equations.

Zeroes of \( A_1(x) \):

\[ -z_1^{-2} z_3 = \frac{A_1(q^{+2} x_\ell^{(1)})}{A_1(q^{-2} x_\ell^{(1)})} \overline{A}_3(q^{-1} x_\ell^{(1)}) \overline{A}_3(q^{-1} x_\ell^{(1)})^2, \quad z_2^{-1} = \frac{A_2(q^{-1} x_\ell^{(1)})}{A_2(q^{1} x_\ell^{(1)})}, \quad \ell = 1, \ldots, m_1, \quad (3.31) \]
Zeros of $A_2(x)$:

$$-z_2^2z_3 = \frac{A_2(q^{+2}x^{(2)}_\ell)}{A_2(q^{-2}x^{(2)}_\ell)} \frac{A_3(q^{-1}x^{(2)}_\ell)}{A_3(q^{+1}x^{(2)}_\ell)}, \quad z_1^{-1} = \frac{A_1(q^{-1}x^{(2)}_\ell)}{A_1(q^{+1}x^{(2)}_\ell)}, \quad \ell = 1, \ldots, m_2 , \quad (3.32)$$

Zeros of $A_3(x)$:

$$z_1^{-1} = \frac{A_1(q^{-1}x^{(3)}_\ell)}{A_1(q^{+1}x^{(3)}_\ell)}, \quad z_2^{-1} = \frac{A_2(q^{-1}x^{(3)}_\ell)}{A_2(q^{+1}x^{(3)}_\ell)}, \quad \ell = 1, \ldots, m_3 , \quad (3.33)$$

Zeros of $\overline{A}_1(x)$:

$$-z_1^2z_3^{-1} \frac{f(q^{+1}x^{(1)}_\ell)}{f(q^{-1}x^{(1)}_\ell)} = \frac{\overline{A}_1(q^{+2}x^{(1)}_\ell)}{\overline{A}_1(q^{-2}x^{(1)}_\ell)} \frac{A_3(q^{-1}x^{(1)}_\ell)}{A_3(q^{+1}x^{(1)}_\ell)}, \quad z_2^2 \frac{f(q^{-1}x^{(1)}_\ell)}{f(q^{+1}x^{(1)}_\ell)} = \frac{A_2(q^{-1}x^{(1)}_\ell)}{A_2(q^{+1}x^{(1)}_\ell)}, \quad \ell = 1, \ldots, L - m_1 , \quad (3.34)$$

Zeros of $\overline{A}_2(x)$:

$$-z_2^2z_3^{-1} \frac{f(q^{+1}x^{(2)}_\ell)}{f(q^{-1}x^{(2)}_\ell)} = \frac{\overline{A}_2(q^{+2}x^{(2)}_\ell)}{\overline{A}_2(q^{-2}x^{(2)}_\ell)} \frac{A_3(q^{-1}x^{(2)}_\ell)}{A_3(q^{+1}x^{(2)}_\ell)}, \quad z_1^{-1} \frac{f(q^{-1}x^{(2)}_\ell)}{f(q^{+1}x^{(2)}_\ell)} = \frac{A_1(q^{-1}x^{(2)}_\ell)}{A_1(q^{+1}x^{(2)}_\ell)}, \quad \ell = 1, \ldots, L - m_2 , \quad (3.35)$$

Zeros of $\overline{A}_3(x)$:

$$z_1 \frac{f(q^{-1}x^{(3)}_\ell)}{f(q^{+1}x^{(3)}_\ell)} = \frac{A_1(q^{-1}x^{(3)}_\ell)}{A_1(q^{+1}x^{(3)}_\ell)}, \quad z_2 \frac{f(q^{-1}x^{(3)}_\ell)}{f(q^{+1}x^{(3)}_\ell)} = \frac{A_2(q^{-1}x^{(3)}_\ell)}{A_2(q^{+1}x^{(3)}_\ell)}, \quad \ell = 1, \ldots, L - m_3 , \quad (3.36)$$

All factors $z_1, z_2, z_3$, in the above equations can be absorbed into redefined eigenvalues

$$Q_i(x) = x^{z_i}, A_i(x), \quad Q_i(x) = x^{-z_i}, A_i(x), \quad (3.37)$$

The equations (3.31)-(3.36) then become identical to the system (1.8), quoted in the introduction.

There are six self-contained sets of the Bethe Ansatz equations involving only subsets of zeros, belonging to any of the six pairs of the eigenvalues $(A_i(x), \overline{A}_j(x)), \ i \neq j$. Once any such pair is determined, the remaining eigenvalues can be easily found by using the above functional equations.

Finally note that for small chains the eigenvalues can, of course, be found from direct diagonalization the of $Q$-operators (thanks to that we now have their explicit definitions (2.91) and (2.92)). As an illustration consider the simplest, but still interesting, case of a 1-site chain, $L = 1$. Evaluating the trace in (2.91) and (2.92) one obtains the operators $A_i(x)$ and $\overline{A}_i(x)$ in the form of diagonal 3x3 matrices. We list their eigenvalues below.

---

For instance, if $(A_1(x), \overline{A}_2(x))$ is known then: (i) $A_3(x)$ and $\overline{A}_3(x)$ are explicitly expressed from (3.31) and (3.27), (ii) (3.22) and (3.23) then become linear equations for the coefficients of $A_2(x)$ and $\overline{A}_1(x)$.
1. \( m_1 = 1, m_2 = 0, m_3 = 0, z_1 = q^{b_1}, z_2 = q^{b_2 + 1}, z_3 = q^{b_1 + b_2 + 1}, \)

\[
A_1(x) = 1 - x \frac{(z_1 - z_2)(q^{-2}z_2 - 1)}{(z_1 - q^{-2}z_2)(z_2 - 1)}, \quad A_2(x) = 1, \quad A_3(x) = 1, \tag{3.38}
\]

\[
\overline{A}_1(x) = 1, \quad \overline{A}_2(x) = 1 - q^{-1}x \frac{z_1 - z_2}{z_1 - q^{-2}z_2}, \quad \overline{A}_3(x) = 1 - q^{-1}x \frac{q^{-2}z_2 - q^2}{z_2 - 1}, \tag{3.39}
\]

2. \( m_1 = 0, m_2 = 1, m_3 = 0, z_1 = q^{b_1 + 1}, z_2 = q^{b_2}, z_3 = q^{b_1 + b_2 + 1}, \)

\[
A_1(x) = 1, \quad A_2(x) = 1 - x \frac{(z_1 - z_2)(z_1 - q^2)}{(z_1 - q^2 z_2)(z_1 - 1)}, \quad A_3(x) = 1, \tag{3.40}
\]

\[
\overline{A}_1(x) = 1 - qx \frac{z_1 - z_2}{z_1 - q^2 z_2}, \quad \overline{A}_2(x) = 1, \quad \overline{A}_3(x) = 1 - qx \frac{q^{-2}z_1 - 1}{z_1 - 1}, \tag{3.41}
\]

3. \( m_1 = 0, m_2 = 0, m_3 = 1, z_1 = q^{b_1 + 1}, z_2 = q^{b_2 + 1}, z_3 = q^{b_1 + b_2 + 2}, \)

\[
A_1(x) = 1, \quad A_2(x) = 1, \quad A_3(x) = 1 - x \frac{(q^{-2}z_1 - 1)(z_2 - q^2)}{(z_1 - 1)(z_2 - 1)}, \tag{3.42}
\]

\[
\overline{A}_1(x) = 1 - qx \frac{q^{-2}z_2 - 1}{z_2 - 1}, \quad \overline{A}_2(x) = 1 - q^{-1}x \frac{z_1 - q^2}{z_1 - 1}, \quad \overline{A}_3(x) = 1, \tag{3.43}
\]

It is easy to check that these eigenvalues satisfy all the functional and Bethe Ansatz type equations given above, as they, of course, should do.

### 4 Applications in continuous quantum field theory.

In this section, we explain how the general results of the previous sections can be specialized to the problems of the continuous quantum field theory in two dimensions.

#### 4.1 T- and Q-operators in conformal field theory

The Borel subalgebra \( \mathcal{B}_+ \) of \( U_q(\hat{sl}(2|1)) \), defined after (2.13), can be realized with two chiral Bose fields

\[
\phi^{(k)}(u) = X^{(k)} + P^{(k)}u + \sum_{n \neq 0} a_n^{(k)} e^{inu}, \quad k = 1, 2, \tag{4.1}
\]

where \( P^{(k)} \) and \( X^{(k)} \) and \( a_n^{(k)} (n = \pm 1, \pm 2, \pm 3, \ldots) \) are operators which satisfy the commutation relations of the Heisenberg algebra,

\[
[X^{(k)}, P^{(l)}] = i \delta_{kl}, \quad [a^{(k)}_m, a^{(l)}_n] = n \delta_{kl} \delta_{m+n,0} . \tag{4.2}
\]

The variable \( u \) is interpreted as the coordinate on the 2D cylinder of the circumference \( 2\pi \). The field \( \phi(u) = (\phi^{(1)}(u), \phi^{(2)}(u)) \) is a quasi-periodic function of \( u \),

\[
\phi(u + 2\pi) = \phi(u) + 2\pi \mathbf{P}, \quad \mathbf{P} = (P^{(1)}, P^{(2)}) . \tag{4.3}
\]

Let \( \alpha_1, \alpha_2, \alpha_3 \) be 2-dimensional vectors

\[
\alpha_0 = (\beta, \gamma), \quad \alpha_1 = (-2\beta, 0), \quad \alpha_2 = (\beta, -\gamma), \quad \alpha_0 + \alpha_1 + \alpha_2 = 0, \tag{4.4}
\]

23
\[
\beta^2 + \gamma^2 = 1.
\] (4.5)

Introduce the vertex operators
\[
V_j(u) = e^{i\alpha_j \phi(u)} : j = 0, 1, 2,
\] (4.6)
where \( \alpha_j \phi(u) = \alpha_j^{(1)}(u) + \alpha_j^{(2)}(u) \) denote the Euclidean scalar product of two-dimensional vectors \( \alpha_j = (\alpha_j^{(1)}, \alpha_j^{(2)}) \) and \( \phi(u) \), and the symbol \( : \ldots : \) denotes the following normal ordering
\[
e^{i\alpha_j \phi(u)} := \exp \left( i \sum_{n=1}^{\infty} \frac{\alpha_j a_n}{n} e^{-i\beta^n} \right) \exp \left( -i \sum_{n=1}^{\infty} \frac{\alpha_j a_n}{n} e^{i\gamma^n} \right),
\] (4.7)
where \( a_n = (a_n^{(1)}, a_n^{(2)}) \), \( X = (X^{(1)}, X^{(2)}) \). It is easy to show that
\[
P V_j(u) = V_j(u) (\alpha_j + P),
\] (4.8)
and
\[
V_j(u) V_j(v) = (-1)^{p(j)q} V_j(v) V_j(u) \quad \text{for} \quad u > v
\] (4.9)
where the Cartan matrix and the parity \( p(j) \) are the same as in (2.8) and (2.20) and
\[
q = e^{-2i\pi \beta^2}.
\] (4.10)

Let \( \mathcal{F}_p \) be the Fock space generated by the free action of the operators \( a^{(k)}_n \), with \( n < 0, k = 1, 2 \) on the vacuum state \( |p\rangle, \quad p = (p^{(1)}, p^{(2)}) \), defined as
\[
P |p\rangle = p |p\rangle, \quad a_n |p\rangle = 0, \quad n > 0.
\] (4.11)

The vertex operators act invariantly in the extended Fock space,
\[
V_j(u) : \hat{\mathcal{F}}_p \rightarrow \hat{\mathcal{F}}_p, \quad \hat{\mathcal{F}}_p = \bigoplus_{(n_1, n_2, n_3) \in \mathbb{Z}^3} \mathcal{F}_{p + n_1 \alpha_1 + n_2 \alpha_2 + n_3 \alpha_3}.
\] (4.12)

This space supports the action of the Borel subalgebra \( \mathcal{B}_- \subset U_q(\hat{sl}(2|1)) \), if one realizes its generators as
\[
h_0 = \frac{p^{(1)} - p^{(2)}}{2\beta}, \quad h_1 = -\frac{p^{(1)}}{\beta}, \quad h_2 = \frac{p^{(1)}}{2\beta} + \frac{p^{(2)}}{2\gamma}, \quad h_0 + h_1 + h_2 = 0,
\] (4.13)
\[
f_j = -\frac{(-1)^{p(j)}}{q - q^{-1}} \int_0^{2\pi} V_j(u) du, \quad j = 0, 1, 2.
\] (4.14)

The commutation relations (2.4) between (4.13) and (4.14) trivially follow from (4.8). The generators (4.14) satisfy the Serre relations (2.5) and (2.7). To see this one needs to rewrite the products of \( f_j \)'s in the form of ordered integrals. An example of such calculations is given in the Appendix [D].

Consider now the specialization of the reduced universal \( R \)-matrix (2.19) to the representation (4.14) in the quantum space (it is the second space in (2.14)). It has an extremely elegant form
\[
\mathcal{Z} = \mathcal{R}_{l_j} = \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{Z}(u) du \right),
\] (4.15)
where
\[ \mathcal{Z}(u) = e_0 \otimes_s V_0(u) + e_1 \otimes_s V_1(u) + e_2 \otimes_s V_2(u). \] (4.16)

It was first discovered in [66] for finite-dimensional quantized algebras and then generalized in [67] for the case of the quantum affine algebra \( U_q(sl(2)). \) A proof can be found in [53]. For the consistency of the fermionic grading we also assume\(^6\)
\[ (e_i \otimes_s V_i(u))(e_j \otimes V_j(v)) = (-1)^{p(i)p(j)}(e_i e_j \otimes V_i(u)V_j(v)). \] (4.17)

It follows then that the following “universal” \( L \)-operator
\[ \mathcal{L} = \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{Z}(u) du \right) q^\mathcal{K}, \quad \mathcal{K} = -\frac{p^{(1)}}{2\beta}(h_0 + h_2) + \frac{p^{(2)}}{2\gamma}(h_2 - h_0), \] (4.18)

obtained from (4.15) and (2.17) satisfy the Yang-Baxter equation
\[ \mathcal{R}_{12} \mathcal{L}_1 \mathcal{L}_2 = \mathcal{L}_2 \mathcal{L}_1 \mathcal{R}_{12}. \] (4.19)

where \( \mathcal{R}_{12} \) is the universal \( R \)-matrix. The operator (4.18) is an element of the Borel subalgebra \( \mathcal{B}_+ \) whose coefficients are operators acting in the quantum space (4.12).

Now we can define commuting \( T \)- and \( Q \)-operators, acting in the Fock space of the Bose fields (4.1), using our universal formulae (2.40), (4.11) and (2.59). For the reason explained below we make a special choice of the external parameters \( b_{1,2} \) in (2.38) such that
\[ z_1 = q^{p^{(2)}/\gamma + p^{(1)}/\beta}, \quad z_2 = q^{p^{(2)}/\gamma - p^{(1)}/\beta}, \quad z_3 = z_1 z_2 \] (4.20)

The definition (2.40) then become\(^7\)
\[ T_\mu^{(CFT)}(x) = \text{Str}_{\pi_\mu(x)} \left[ z_1^{-h_0} z_2^{-h_2} \mathcal{L} \right] \] (4.21)
\[ = \text{Str}_{\pi_\mu(x)} \left[ q^{2\mathcal{K}} \mathcal{P} \exp \left( \int_0^{2\pi} (e_0 \otimes_s V_0(u) + e_1 \otimes_s V_1(u) + e_2 \otimes_s V_2(u)) du \right) \right] \]

where \( \mathcal{K} \) is given by (4.18). An important case is the 3-dimensional representations
\[ T^{(CFT)}(x) \equiv T^{(CFT)}_{(1,0,0)}(x), \quad \overline{T}^{(CFT)}(x) \equiv -T^{(CFT)}_{(-1,-1,0)}(x). \] (4.22)

Similarly define the corresponding \( Q \)-operators
\[ K_i^{(CFT)}(x) = Z_i^{-1} \text{Str}_{\pi_i(x)} \left[ q^{2\mathcal{K}} \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{Z}(u) du \right) \right], \] (4.23)
\[ \overline{K}_i^{(CFT)}(x) = Z_i^{-1} \text{Str}_{\pi_i(x)} \left[ q^{2\mathcal{K}} \mathcal{P} \exp \left( \int_0^{2\pi} \mathcal{Z}(u) du \right) \right], \]

where \( i = 1, 2, 3, \mathcal{Z}(u) \) is defined in (4.17) and \( Z_i, \overline{Z}_i \) are given by (2.57) with \( z_1, z_2 \) and \( z_3 \) defined in (4.20).

\(^6\)The only effect of the extra signs arising from this relation is the negation of the spectral parameter \( x \).

\(^7\)Note that the choice (4.20) leads to an extra factor \( q^\mathcal{K} \) in (4.22) in addition to the one which comes from universal \( R \)-matrix.
For the choice \( (4.20) \) all the operators \( (4.22) \) and \( (4.23) \) commute with an infinite series of the local integral of motion (LIM), \( \mathbb{I}_n, n = 1, 2, 3, \ldots, \infty, \)

\[
[\mathbb{I}_n, T^{(\text{CFT})}_\mu(x)] = [\mathbb{I}_n, A^{(\text{CFT})}_\mu(x)] = [\mathbb{I}_n, \overline{A}^{(\text{CFT})}_\mu(x)] = 0
\]

(4.24)
of a two-dimensional CFT which arise in quantization of the AKNS soliton hierarchy \([58]\). They are defined as integrals

\[
\mathbb{I}_n = \int_0^{2\pi} W_{n+1}(u) \, du.
\]

(4.25)
where the local densities are \( W_{n+1}(u) \) are some polynomials in the \( u \)-derivatives of the Bose fields \( (4.1) \) (the \( k \)-th density \( W_k(u) \) is of the degree \( k \) in the derivative \( \partial_u \)). The first non-trivial densities read [58],

\[
W_2(u) = -\frac{1}{2} (\partial_u \phi^{(1)}(u))^2 - \frac{1}{2} (\partial_u \phi^{(2)}(u))^2 + \frac{1}{\sqrt{2}n} \partial_u^2 \phi^{(1)}(u),
\]

(4.26)
\[
W_3(u) = \frac{6n + 4}{6\sqrt{2}} (\partial_u \phi^{(1)}(u))^3 + \frac{n}{\sqrt{2}} (\partial_u \phi^{(1)}(u))^2 \partial_u \phi^{(2)}(u) + \frac{n\sqrt{n}}{2} \partial_u^2 \phi^{(1)}(u) \partial_u \phi^{(2)}(u)
\]

\[
- \frac{(n + 2)\sqrt{n}}{2} \partial_u \phi^{(1)}(u) \partial_u^2 \phi^{(2)}(u) + \frac{n + 2}{6\sqrt{2}} \partial_u^3 \phi^{(2)}(u),
\]

(4.27)
where \( \sqrt{n} = i\sqrt{2}/\beta \) is a parameter used in \([58]\) instead of our \( \beta \) in \( (4.10) \). Note also that their chiral Bose fields \( X(u) \) and \( Y(u) \) are related to \( (4.1) \) as \( X(u) = \phi^{(1)}(u)/\sqrt{2} \) and \( Y(u) = \phi^{(2)}(u)/\sqrt{2} \).

The CFT versions of the \( \mathbf{T} \)- and \( \mathbf{Q} \)-operators defined above act invariantly in the single Fock space \( \mathcal{F}_p \) (even though the \( \mathbf{L} \)-operator \( (4.18) \) acts in the extended space \( (4.12) \)). They satisfy all the functional relation given in Sections 3.1–3.3, without any modifications\(^8\). One just need to supply the superscript \( (CFT) \) to all the \( \mathbf{T} \)- and \( \mathbf{Q} \)-operators and identify the quantities \( z_1, z_2, z_3 \) therein with those defined in \( (2.38) \). Note that the vacuum state \( (4.11) \) is an eigenstate for all these operators

\[
T^{(\text{CFT})}_\mu(x)|p\rangle = T^{(\text{vac})}_\mu(x)|p\rangle >, \quad A^{(\text{CFT})}_i(x)|p\rangle = A^{(\text{vac})}_i(x)|p\rangle >, \quad \overline{A}^{(\text{CFT})}_i(x)|p\rangle = \overline{A}^{(\text{vac})}_i(x)|p\rangle >.
\]

(4.28)
(4.29)
The eigenvalues satisfy the functional relations

\[
c_{12} = c_{13} \, z_1^{\frac{1}{2}} A^{(\text{vac})}_1(q + \frac{i}{2} x) \overline{A}^{(\text{vac})}_1(q - \frac{i}{2} x) - c_{23} z_2^{\frac{1}{2}} A^{(\text{vac})}_2(q + \frac{i}{2} x) \overline{A}^{(\text{vac})}_2(q - \frac{i}{2} x),
\]

(4.30)
\[
c_{12} = c_{13} \, z_1^{-\frac{1}{2}} A^{(\text{vac})}_1(q - \frac{i}{2} x) \overline{A}^{(\text{vac})}_1(q + \frac{i}{2} x) - c_{23} z_2^{-\frac{1}{2}} A^{(\text{vac})}_2(q - \frac{i}{2} x) \overline{A}^{(\text{vac})}_2(q + \frac{i}{2} x)
\]

(4.31)
which are corollaries of \( (3.1) \).

The operators \( (4.22) \) and \( (4.23) \) understood as series in spectral parameter \( x \). The first nontrivial terms in their expansions can be obtained from the third order term in the expansion of the universal \( R \)-matrix given in the Appendix D

\[
T^{(\text{CFT})}(x) = z_1 + z_2 - z_3 + xG_1 + O(x^2),
\]

(4.32)
\[
\overline{T}^{(\text{CFT})}(x) = -z_3^{-1} + z_2^{-1} + z_1^{-1} - x(z_1 z_2 z_3)^{-1}G_1 + O(x^2),
\]

(4.33)

\(^8\)Let us stress that in the considered case of CFT no additional scalar factors arise in the functional relations (unlike the lattice models).
where \( T^{(\text{CFT})}(x) \) and \( \overline{T}^{(\text{CFT})}(x) \) are defined in \([4,22]\). The quantities \( G_1 \) and \( \overline{G}_1 \) are **non-local** integrals of motion defined as linear combinations of ordered integrals of the vertex operators,

\[
G_1 = z_1 J(1, 2, 0) + z_2 J(2, 0, 1) - z_3 J(0, 1, 2), \quad (4.33)
\]

\[
\overline{G}_1 = -z_3 z_2 J(0, 2, 1) - z_1 z_3 J(1, 0, 2) + z_2 z_1 J(2, 1, 0), \quad (4.34)
\]

where

\[
J(i_1, i_2, \ldots, i_n) = \int_{u_1 \geq u_2 \geq \cdots \geq u_n} V_{i_1}(u_1) V_{i_2}(u_2) \cdots V_{i_n}(u_n) du_1 du_2 \cdots du_n. \quad (4.35)
\]

Similarly for the \( Q \)-operators \([4,23]\) one has

\[
\begin{align*}
\overline{T}_{1}^{(\text{CFT})}(x) &= 1 - \frac{q^2 x (q^{-1} z_1 G_1 + \overline{G}_1)}{(q - q^{-1})(q^2 z_2 - z_1)(z_3 - z_1)} + O(x^2), \\
\overline{T}_{2}^{(\text{CFT})}(x) &= 1 - \frac{q^2 x (q^{-1} z_2 G_1 + \overline{G}_1)}{(q - q^{-1})(q^2 z_1 - z_2)(z_3 - z_2)} + O(x^2), \\
\overline{T}_{3}^{(\text{CFT})}(x) &= 1 - \frac{q^{-\frac{1}{2}} x (q z_3 G_1 + \overline{G}_1)}{(q - q^{-1})(z_1 - z_3)(z_2 - z_3)} + O(x^2), \\
\overline{A}_{1}^{(\text{CFT})}(x) &= 1 + \frac{q^{-\frac{3}{2}} x (q z_1 G_1 + \overline{G}_1)}{(q - q^{-1})(q^{-2} z_2 - z_1)(z_3 - z_1)} + O(x^2), \\
\overline{A}_{2}^{(\text{CFT})}(x) &= 1 + \frac{q^{-\frac{3}{2}} x (q z_2 G_1 + \overline{G}_1)}{(q - q^{-1})(q^{-2} z_1 - z_2)(z_3 - z_2)} + O(x^2), \\
\overline{A}_{3}^{(\text{CFT})}(x) &= 1 + \frac{q^2 x (q^{-1} z_3 G_1 + \overline{G}_1)}{(q - q^{-1})(z_1 - z_3)(z_2 - z_3)} + O(x^2). 
\end{align*} \tag{4.36}
\]

Note, that the \( T \)-operators similar to \([4,22]\) but with a different realization of the vertex operators (through two Bose and two Fermi free fields) were introduced in \([68]\) in connection with a CFT with an extended (super)symmetry which arises in quantization of a supersymmetric extension of the KdV hierarchy (also related with the \( U_q(\hat{sl}(2|1)) \) algebra). As stated in \([68]\) their analog of the \( L \)-operator \([4,18]\) satisfy the same Yang-Baxter equation \([4,19]\). Our results then imply that their \( T \)-operators obey exactly the same functional relations as in Section 3, provided one uses the same definitions \([4,22]\), but with their realization of the vertex operators and also define by \([4,23]\) the corresponding \( Q \)-operators, which were not considered in \([68]\). It would be interesting to further clarify these connections.

### 4.2 Connections with the spectral theory of differential equations.

Here we briefly illustrate the remarkable correspondence between the spectral theory of the Schrödinger equation and the integrable structure of the conformal field theory, which attracted much attention recently \([69–73]\). This correspondence relates some spectral characteristics of certain ordinary differential equations (and, more generally, integro-differential equations \([73]\)) to the eigenvalues of the continuous analogs of the Baxter’s \( Q \)-operators in quantum field theory with the conformal symmetry (in general, with an extended (super)conformal symmetry).

The relevant differential equation in our case is, in fact, a one-dimensional Schrödinger equation on the half-line,

\[
\left\{ - \frac{d^2}{dy^2} + \frac{\ell(\ell + 1)}{y^2} + r y^{\alpha - 1} + y^{2\alpha} - E \right\} \Psi(y) = 0, \quad 0 < y < +\infty. \tag{4.37}
\]
with real \(0 < \alpha < \infty\) and arbitrary \(r\) and \(\ell\). It was first considered in the case \(l = 0\) and \(\alpha > 0\) by Suzuki \[72\], who pointed out its connection to the quantum affine superalgebra \(U_q(\hat{sl}(2|1))\) and to the corresponding Bethe Ansatz equations. Eq.(4.37) with \(l \neq 0\) appeared in \[74\]. The full equation with arbitrary values of \(\ell\) and \(r\), in the regime \(\alpha < -1\), was recently considered in \[58\] in connection with the quantization of the integrable AKNS soliton hierarchy.

Here we consider the case \(\alpha > 0\) with arbitrary values of \(\ell\) and \(r\). To simplify our considerations we will assume that \(\alpha > 1\), however the results apply to the full range \(0 < \alpha < \infty\). For \(\text{Re} \, \ell > -\frac{1}{2}\), Eq.(4.37) has a unique solution, satisfying the condition

\[
\psi(y,E,r,\ell) = \left(\frac{2}{\alpha + 1}\right)^{\frac{2\ell + 2 + \alpha}{2(\alpha + 1)}} \Gamma \left(-\frac{2\ell + 1}{\alpha + 1}\right) y^{\ell + 1} + O(y^{\ell + 3}), \quad \text{as} \quad y \to 0. \tag{4.38}
\]

This solution can be analytically continued outside the domain \(\text{Re} \, \ell > -\frac{1}{2}\). Obviously, the function \(\psi(y,E,r,-\ell - 1)\), defined in this way, satisfy the same equation (4.37) and for generic values of \(\ell\) the two solutions

\[
\psi_1(y) = \psi(y,E,r,\ell), \quad \psi_2(y) = \psi(y,E,r,-\ell - 1), \tag{4.39}
\]

are linearly independent, since

\[
(4\pi i)^{-1} \text{Wr} [\psi_1, \psi_2] = \left(q^{\ell + \frac{1}{2}} - q^{-\ell - \frac{1}{2}}\right)^{-1}, \tag{4.40}
\]

where \(\text{Wr}[f,g] = f\partial_y g - \partial_y fg\) denotes the usual Wronskian and

\[
q = \exp \left(\frac{2\pi i}{\alpha + 1}\right). \tag{4.41}
\]

From now on we will make the \(\ell\)-dependence implicit, considering \(\ell\) as a fixed parameter. Further, for all values of \(E\) the equation (4.37) has a unique solution \(\chi(y,E,r)\) which decays at \(y \to +\infty\). We normalize this solution as

\[
\chi(y,E,r) \to \left(\frac{2}{\alpha + 1}\right)^{-\frac{2\ell + 2 + \alpha}{2(\alpha + 1)}} y^{\frac{\alpha + 1}{\alpha + 1}} \exp\left(-\frac{y^{\alpha + 1}}{\alpha + 1}\right), \quad y \to +\infty. \tag{4.42}
\]

It can be expanded in the basis (4.39)

\[
\chi(y,E,r) = D_2(E,r) \frac{\psi_1(y)}{\Gamma \left(\frac{\alpha + r - 2\ell}{2(\alpha + 1)}\right)} + D_1(E,r) \frac{\psi_2(y)}{\Gamma \left(\frac{\alpha + r + 2\ell + 2}{2(\alpha + 1)}\right)}, \tag{4.43}
\]

where the connection coefficients \(D_{1,2}(E,r)\), which are entire functions of \(E\), are of our primary interest. They normalized by the condition

\[
D_{1,2}(E,r) = 1 + O(E), \quad E \to 0. \tag{4.44}
\]

This follows from the fact that for \(E = 0\) the substitution

\[
\Psi(y) = y^{\ell + 1} \exp\left(-\frac{y^{\alpha + 1}}{\alpha + 1}\right) w \left(\frac{2 y^{\alpha + 1}}{\alpha + 1}\right) \tag{4.45}
\]

brings Eq.(4.37) to the Kummer equation

\[
z \frac{d^2}{dz^2} w(z) + (b - z) \frac{d}{dz} w(z) - a w(z) = 0 \tag{4.46}
\]
It is easy to see then that Eq. (4.43) with \( E = 0 \) reduces to the relation between Kummer’s functions given in §13.1.3 of ref. [75].

The connection coefficients \( D_{1,2}(E, r) \) in (4.43) can be interpreted as the spectral determinants. Indeed, at certain isolated values of \( E \) one of the solutions \( (4.39) \) will decay for \( x \to +\infty \) and, thus, becomes proportional to \( \chi(y, E, r) \). One of the terms in the RHS of (4.43) then vanish. Let \( \{ E_n^{(i)}(r) \}_{n=1}^\infty, \ i = 1, 2 \) denotes ordered spectral sets such that

\[
\psi_1(y, E_n^{(1)}(r)) \to 0, \quad \psi_2(y, E_n^{(2)}(r)) \to 0, \quad y \to \infty .
\]

It is easy to see then that

\[
D_i(E, r) = \prod_{n=1}^\infty \left( 1 - \frac{E}{E_n^{(i)}(r)} \right), \quad i = 1, 2 .
\]

Simple WKB analysis shows that at large \( n \) the eigenvalues \( E_n^{(1,2)}(r) \) accumulate along positive real axis and that

\[
E_n^{(1,2)}(r) \sim n^{2\alpha} \ , \quad n \to \infty .
\]

Therefore for \( \alpha > 1 \) the infinite products (4.49) converge as written. It follows then

\[
\log D_{1,2}(E, r) \simeq \text{const}(-E)^{\frac{\alpha+1}{\alpha}}, \quad E \to \infty, \quad |\text{arg}(-E)| < \pi .
\]

Strictly speaking, the spectral conditions (4.48) only define (4.49) up to the multiplication by an entire function without zeroes. However, comparing (4.51) with the large \( E \) asymptotics, which follows from the quasi-classical approximation to (4.43), one concludes that this function is a constant and then from (4.44) that it is equal to one.

The spectral determinants \( D_{1,2}(E, r) \) satisfy certain functional equation, which we will now derive. The key observation is that Eq. (4.37) is invariant under the transformation

\[
\hat{\Omega}: \quad y \to q^{1/2} y, \quad r \to -r, \quad \ell \to \ell, \quad E \to q^{-1} E ,
\]

where \( q \) is the same as in (4.31). Therefore the functions

\[
\chi_k(y) = (iq^{-\frac{1}{2}})^k \hat{\Omega}^k \left[ \chi(y, E, r) \right], \quad k = 0, 1, 2, \ldots, \infty ,
\]

also satisfy (4.37). It is easy to check that

\[
\text{Wr} \{ \chi_0, \chi_1 \} = 2 .
\]

The solutions (4.39) are simply transformed under (4.52),

\[
\hat{\Omega}[\psi_1(y)] = q^{(\ell+1)/2} \psi_1(y), \quad \hat{\Omega}[\psi_2(y)] = q^{-\ell/2} \psi_2(y)
\]

Introduce the constants,

\[
(z_1)_{\frac{1}{2}} = iq^{\frac{\ell-1}{4}} - q^{-\frac{\ell+1}{4}}, \quad (z_2)_{\frac{1}{2}} = -iq^{-\frac{\ell+1}{4}} - q^{\frac{\ell-1}{4}}, \quad (z_3)_{\frac{1}{2}} = (z_1 z_2)_{\frac{1}{2}} = q^{-\frac{\ell}{4}},
\]

and also \( c_{ij} = (z_i - z_j)/(z_i z_j)^{\frac{1}{2}} \),

\[
c_{12} = q^{-\ell} - q^{\ell+4}, \quad c_{13} = iq^{\frac{\ell}{4} + \frac{\ell+1}{4}} + iq^{\frac{\ell}{4} - \frac{\ell+1}{4}}, \quad c_{23} = -iq^{\frac{\ell}{4} + \frac{\ell+1}{4}} - iq^{\frac{\ell}{4} - \frac{\ell+1}{4}},
\]

\[
a = \frac{\alpha + r + 2\ell + 2}{2(\alpha + 1)}, \quad b = \frac{\alpha + 2\ell + 2}{\alpha + 1} . \quad (4.47)
\]
It follows from \((4.43)\) and \((4.53)-(4.57)\) that
\[
c_{12} = c_{13} \frac{1}{3} D_1(qE, r) D_2(E, -r) - c_{23} \frac{1}{2} D_1(E, -r) D_2(qE, r) .
\] (4.58)

Negating \(r\) in the last relation one also gets
\[
c_{12} = c_{13} \frac{1}{3} D_1(E, r) D_2(qE, -r) - c_{23} \frac{1}{2} D_1(qE, -r) D_2(E, r) .
\] (4.59)

We now want to identify the functions \(D_{1,2}(E, \pm r)\) with the vacuum eigenvalues \(\mathcal{A}_{1,2}^{(\text{vac})}(x)\) and \(\mathcal{A}_{1,2}^{(\text{vac})}(x)\) of the \(Q\)-operators of the super-conformal field theory considered in Section 4.1. First let us identify the parameters \(q\) and \(z_1, z_2, z_3\) of this Section defined in \((4.11)\) and \((4.16)\) with those in \((4.10)\) and \((4.20)\). We expect the following exact correspondence
\[
D_1(E, r) = \mathcal{A}_{1}^{(\text{vac})}(\rho E), \quad D_1(E, -r) = \mathcal{A}_{3}^{(\text{vac})}(\rho E), \quad i = 1, 2,
\] (4.60)
where \(\rho\) is a scalar factor depending on \(\alpha\). For an elementary consistency check one can easily verify that these quantities obey the same normalization conditions \((4.36)\) and \((4.11)\) and satisfy the identical functional relations \((4.30)\), \((4.31)\) and \((4.58)\), \((4.59)\). A complete proof of \((4.60)\) (and, in particular, the calculation of the constant \(\rho\)) requires much deeper considerations which (hopefully) will be presented elsewhere. Here we only mention that the vacuum eigenvalues of some other commuting operators plays the role of the Stokes multipliers describing the monodromy properties of the differential equation near its irregular singular point \(y = \infty\). Every three solutions of \((4.37)\) satisfy a linear relation, in particular,
\[
\chi_n(y, E, r) = X_n(E, r) \chi_0(y, E, r) + Y_n(E, r) \chi_1(y, E, r), \quad n \in \mathbb{Z},
\] (4.61)
where \(\chi_n\) is defined in \((4.53)\). Using \((4.40)\) and \((4.43)\) and assuming the correspondence \((4.60)\) it is not difficult to show that
\[
X_{2k}(E, r) = (z_3)^{\frac{1}{2}} \mathcal{T}_{-k}^{(1)(\text{vac})}(q^{-k} x), \quad Y_{2k+1}(E, r) = \mathcal{T}_{k}^{(1)(\text{vac})}(q^{-k} x), \quad k \in \mathbb{Z}
\] (4.62)
and (omitting unimportant factors here)
\[
X_{2k+1}(E, r) \sim \left(\frac{z_1}{z_2}\right)^{\frac{1}{2}} \mathcal{A}_1^{(\text{vac})}(q^{-2k-1} x) \mathcal{A}_2^{(\text{vac})}(q^{-k} x) - \left(\frac{z_2}{z_1}\right)^{\frac{1}{2}} \mathcal{A}_1^{(\text{vac})}(q^{-k} x) \mathcal{A}_2^{(\text{vac})}(q^{-2k-1} x)
\]
\[
Y_{2k}(E, r) \sim \left(\frac{z_1}{z_2}\right)^{\frac{1}{2}} \mathcal{A}_1^{(\text{vac})}(x) \mathcal{A}_2^{(\text{vac})}(q^{-2k} x) - \left(\frac{z_2}{z_1}\right)^{\frac{1}{2}} \mathcal{A}_1^{(\text{vac})}(q^{-2k} x) \mathcal{A}_2^{(\text{vac})}(x)
\] (4.63)
where \(x \equiv \rho E\). It is interesting to note that
\[
Y_2(E) \sim \mathcal{A}_3^{(\text{vac})}(q^{-1} x), \quad X_3(E) \sim \mathcal{A}_3^{(\text{vac})}(q^{-2} x).
\] (4.64)

5 Algebraic proof of the functional relations

In this section we will prove all the functional relations among the \(Q\)-operators and \(T\)-operators, given in Sect. 3. Fortunately, due to symmetry transformations (see below) a set of functional relations, which require a separate proof, reduces to only three relations \((3.2a)\), \((3.9a)\) and \((3.10a)\). Their proof is presented below.
5.1 Symmetry transformations

The $T$- and $Q$-operators possess a number of simple, but important symmetry relations. Consider the following automorphism of the algebra $U_q(sl(2|1))$,

$$
\sigma_{02} : \begin{cases} 
  e_0 \rightarrow e_2, & e_1 \rightarrow e_1, & e_2 \rightarrow e_0, \\
  f_0 \rightarrow f_2, & f_1 \rightarrow f_1, & f_2 \rightarrow f_0, \\
  h_0 \rightarrow h_2, & h_1 \rightarrow h_1, & h_2 \rightarrow h_0.
\end{cases}
$$

(5.1)

Note that this is an involution $(\sigma_{02})^2 = 1$. It is easy to see that all the defining relations (2.4), (2.5), (2.6) are invariant with respect to (5.1). This transformation also preserves the parities of the elements of the algebra and the co-multiplication (2.11),

$$
\Delta(\sigma_{02}) = \sigma_{02} \otimes \sigma_{02}.
$$

(5.2)

It follows then that the universal $R$-matrix is invariant with respect to the diagonal action of $\sigma_{02}$,

$$
(\sigma_{02} \otimes \sigma_{02})[R] = R.
$$

(5.3)

Further, let the external field parameters $b_1$ and $b_2$ in (2.36) are also replaced

$$
\sigma_{02} : \quad b_1 \rightarrow -b_2, \quad b_2 \rightarrow -b_1,
$$

(5.4)

simultaneously with (5.1). Note that the combined transformation (5.1), (5.4) acts on the operators $z_{1,2,3}$ from (2.38) as follows

$$
\sigma_{02} : \quad z_1 \rightarrow 1/z_2, \quad z_2 \rightarrow 1/z_1, \quad z_3 \rightarrow 1/z_3.
$$

(5.5)

Further, the substitution $\sigma_{02}$ is also an automorphism of the Borel subalgebra $B_+$ ($B_-$) and, therefore, transforms its representations into each other. Namely, for the maps $\rho_i(x)$ and $\overline{\rho}_i(x)$, introduced in Section 2.3, such transformation leads to the following relations

$$
\rho_i(x) \rightarrow \rho_i(x) \cdot \sigma_{02} \simeq \overline{\rho}_{3-i}(x), \quad \rho_3(x) \rightarrow \rho_3(x) \cdot \sigma_{02} \simeq \overline{\rho}_2(x),
$$

$$
\overline{\rho}_i(x) \rightarrow \overline{\rho}_i(x) \cdot \sigma_{02} \simeq \rho_{3-i}(x), \quad \overline{\rho}_3(x) \rightarrow \overline{\rho}_3(x) \cdot \sigma_{02} \simeq \rho_2(x), \quad i = 1, 2.
$$

(5.6)

By definition, the $Q$-operators (2.54) and (2.55) are elements of the Borel subalgebra $B_-$, associated with the quantum space. It is easy to see that the action of the automorphism $\sigma_{02}$ on this subalgebra (together with the substitution (5.4)) will induce some permutation of the $Q$-operators. Namely, from (5.3), (5.5) and (5.6) it follows that this action is

$$
Q_i(x) \rightarrow \sigma_{02}[Q_i(x)] = \overline{Q}_{3-i}(x), \quad Q_3(x) \rightarrow \sigma_{02}[Q_3(x)] = \overline{Q}_3(x), \quad i = 1, 2.
$$

(5.7)

Similarly, taking into account (3.9), (5.5) and (5.7) one gets

$$
T_m^{(1)}(x) \rightarrow \sigma_{02}[T_m^{(1)}(x)] = T_{-m-1}^{(1)}(x), \quad m \in \mathbb{Z}.
$$

(5.8)

The stated equivalences hold up to certain similarity transformations of the products of the oscillator algebras (2.48), (2.49), which does affect the (super)trace.
Finally, the proof of the functional relations, given below, is based on decomposition properties of products of the representations of $B_+(U_q(\hat{sl}(2|1)))$ with respect to the co-multiplication (2.11). Owing to (5.2), the whole set of the functional relations splits into pairs of relations following from each other under the substitution (5.7), (5.8).

This symmetry leaves only three independent functional relations: (3.2a), (3.9a) with $m \in \mathbb{Z}_{\geq 0}$ and (3.10a), while all other relations become their simple corollaries. To proceed further with an algebraic proof of the remaining three relations we need some new notations.

5.2 Additional notations

5.2.1 Shifted modules

For any $j_0, j_2 \in \mathbb{C}$, let $p_{[j_0, j_2]}$ be a shift automorphism of the Borel subalgebra $B_+ \subset U_q(\hat{sl}(2|1))$ such that

$$p_{[j_0, j_2]}(e_i) = e_i, \quad p_{[j_0, j_2]}(h_0) = h_0 + j_0, \quad p_{[j_0, j_2]}(h_1) = h_1 - j_0 - j_2, \quad p_{[j_0, j_2]}(h_2) = h_2 + j_2.$$ (5.9)

For any representation $\pi$ of $B_+$ define shifted representation

$$\pi[j_0, j_2] = \pi \cdot p_{[j_0, j_2]}.$$

We will often use the following identity

$$\text{Str}_{\pi[j_0, j_2]}(z_1^{-h_0 - j_0 z_2} \mathcal{R}) = \text{Str}_{\pi}(z_1^{-h_0 - j_0 z_2} \mathcal{R}) = z_1^{-j_0 j_2} \text{Str}_{\pi}(z_1^{-h_0 z_2} \mathcal{R}),$$ (5.10)

where the super trace is understood as $\text{Str}_{\pi[j_0, j_2]} \otimes 1$ or as $\text{Str}_\pi \otimes 1$, see the note after (2.40). Here we used the fact that the reduced universal $R$-matrix $\mathcal{R} \in B_+ \otimes B_-$, defined in (2.17) and (2.18), does not contain powers of the Cartan elements $h_i \otimes 1$.

5.2.2 Modified versions of the maps $\rho_i$ and $\overline{\rho}_i$

From now on and to the rest of the paper (including all appendices) we will use a slightly modified version of the maps introduced in Sect. 2.3

$$\rho'_1(x) = x^{+(1 \otimes \mathcal{H}')} \cdot \rho_1(x) \cdot x^{-(1 \otimes \mathcal{H}')}, \quad \overline{\rho}'_1(x) = x^{-(1 \otimes \mathcal{H}')} \cdot \overline{\rho}_1(x) \cdot x^{+(1 \otimes \mathcal{H}')},$$

$$\rho'_2(x) = \rho_2(x), \quad \overline{\rho}'_2(x) = \overline{\rho}_2(x),$$

$$\rho'_3(x) = x^{-(\mathcal{H}' \otimes 1)} \cdot \rho_3(x) \cdot x^{+(\mathcal{H}' \otimes 1)}, \quad \overline{\rho}'_3(x) = x^{+(\mathcal{H}' \otimes 1)} \cdot \overline{\rho}_3(x) \cdot x^{-(\mathcal{H}' \otimes 1)}.$$ (5.11)

They contain additional similarity transformation in some fermionic Fock spaces. Obviously, these transformations do not affect the definition of the $Q$-operators, which only involve the super-trace.

5.2.3 Fock spaces for oscillator algebras

It was already remarked that the definitions (2.56) and (2.58) does not depend on a choice of representations of oscillator algebras (2.48) and (2.49). For definiteness, assume that $|q| = 1$, but not a root of unity $q^N \neq 1$ (the reasonings can also be repeated when $|q| \neq 1$). Consider, for
example, the simplest non-trivial trace, $\text{Tr}(e^{-\omega H_b} b^+ b^-)$, for the bosonic algebra (2.48). Using
the commutation relation and the cyclic property of the trace, one obtains

$$
\text{Tr}(e^{\omega H_b} b^+ b^-) = q^{-2} \text{Tr}(e^{\omega H_b} b^- b^+) + \frac{1}{q(q - q^{-1})} \text{Tr}(e^{\omega H_b})
$$

(5.12)

$$
= q^{-2} e^{-\omega} \text{Tr}(e^{\omega H_b} b^+ b^-) + \frac{1}{q(q - q^{-1})} \text{Tr}(e^{\omega H_b}).
$$

(5.13)

It follows then

$$
\frac{\text{Tr}(e^{\omega H_b} b^+ b^-)}{\text{Tr}(e^{\omega H_b})} = \frac{1}{q(q - q^{-1})(1 - q^{-2}e^{-\omega})}.
$$

(5.14)

The only assumption made in this calculation is the existence of the trace.

The same quantity (5.14) can also be calculated by using the highest weight representations (Fock representations) of the algebra (2.48). This algebra has only two non-equivalent Fock representations $w_{\pm}(\mathcal{H}_q)$, acting on the bases $|k\rangle_{\pm}, k \in \mathbb{Z}_{\geq 0},$

$$
w_{\pm}(b^\mp)|k\rangle_{\pm} = |k + 1\rangle_{\pm}, \quad w_{\pm}(b^\pm)|k\rangle_{\pm} = \frac{1 - q^{\mp 2k}}{(q - q^{-1})^2} |k - 1\rangle_{\pm}, \quad w_{\pm}(H_b)|k\rangle_{\pm} = \mp k|k\rangle_{\pm},
$$

(5.15)

where one needs to take all upper or all lower signs. Using these definitions, one easily obtains two expressions

$$
\begin{align*}
\text{Tr}_{w_{\pm}}(e^{\omega H_b}) &= \sum_{k=0}^{\infty} e^{-k\omega} = \frac{1}{1 - e^{-\omega}}, \\
\text{Tr}_{w_{\pm}}(e^{\omega H_b} b^+ b^-) &= \sum_{k=0}^{\infty} e^{-k\omega} \frac{(1 - q^{-2}e^{-\omega})^k}{(q - q^{-1})^2} = \frac{1}{q(q - q^{-1})(1 - q^{-2}e^{-\omega})},
\end{align*}
$$

(5.16)

which imply formula (5.14). For $|q| = 1$ the above series converge for $\text{Re} \omega > 0$. Thus, the last calculation only implies (5.14) in the half-plane $\text{Re} \omega > 0$. Of course, the final answer is a meromorphic function of $\omega$ and can be analytically continued to the whole complex $\omega$-plane.

One can perform a similar calculation using the second Fock representation, $w_-$, which requires $\text{Re} \omega < 0$. Note that a replacement of $w_{\pm}$ with $w_-$ in (5.16) changes the values of the trace. However, the final result for the ratio (5.14) remains the same, as expected. To summarize, for explicit calculations one can use any of the Fock representations (5.14), depending on convenience.

For the fermionic algebra (2.49) there is only one two-dimensional Fock representation (up to the shifts of the $\mathcal{H}_f$). However, to streamline the notations we define two representations

$$
w_{\pm}(f^\pm)|0\rangle_{\pm} = 0, \quad w_{\pm}(f^\pm)|1\rangle_{\pm} = |0\rangle_{\pm}, \\
w_{\pm}(f^\mp)|0\rangle_{\pm} = |1\rangle_{\pm}, \quad w_{\pm}(f^\mp)|1\rangle_{\pm} = 0, \\
w_{\pm}(H_f)|k\rangle_{\pm} = \mp k|k\rangle_{\pm}, \quad k \in \{0, 1\},
$$

(5.17)

differing by an exchange of the basis vectors $|0\rangle$ and $|1\rangle$ and a shift of $\mathcal{H}_f$.

Each of the definitions (2.56) and (2.58) involves the super-trace over some representation of the direct product of two oscillator algebras, entering the corresponding map $\rho_i(x)$ or $\overline{\rho}_i(x)$. According to the above discussion this representation in each case can be chosen in four ways $W_{\xi_1,\xi_2} = w_{\xi_1} \otimes w_{\xi_2}$, labeled by two sign variables $\xi_1, \xi_2 = \pm$. For the map $\rho_a(x)$ (resp. $\overline{\rho}_a(x)$),

33
we will denote such representation as \( W_{a_1, a_2}^\pm(x) \) (resp. \( \overline{W}_{a_1, a_2}^\pm(x) \)). Explicit form of the action of the Borel subalgebra \( \mathcal{B}_+ \) in the basis of these Fock representation is given in Appendix [B.3]. Here we will use the following 6 representations: \( \overline{W}_{1}^{++,}(x) \), \( \overline{W}_{2}^{--}(x) \), \( 
abla_{3}^{++}(x) \), \( W_{1}^{--}(x) \), \( W_{2}^{++}(x) \) and \( W_{3}^{+-}(x) \), completely presented in [B.9]-[B.14]. The normalization constants (2.57) and (2.59) for these representations are

\[
Z_1 = Z_1 = \frac{z_2(z_1 - z_3)}{z_3(z_1 - z_2)}, \quad Z_2 = Z_2 = \frac{z_3 - z_2}{z_1 - z_2}, \quad Z_3 = Z_3 = \frac{(z_3 - z_1)(z_2 - z_3)}{z_2 z_3},
\]

where \( z_1 z_2 z_3^{-1} = 1 \).

As an example, we give here the representation \( W_{3}^{+-}(x) \). It is a 4-dimensional representation spanned on the vectors

\[
|m, n >_{+-} = |m >_+ \otimes_s |n >_+ = (f_1^-m |0 >_+ \otimes_s (f_2^+)^n |0 >_- = (f_1^-m (f_2^+)^n |0 >_+ , \quad (5.19)
\]

where \( m, n = 0, 1 \), with the following action of the generators of \( \mathcal{B}_+ \)

\[
e_0 |m, n >_{+-} = q^{-n} |m - 1, n >_{+-},
\]

\[
e_1 |m, n >_{+-} = \frac{(-1)^m q^{m+1} x}{q - q^{-1}} |m + 1, n + 1 >_{+-},
\]

\[
e_2 |m, n >_{+-} = (-1)^m |m, n - 1 >_{+-},
\]

\[
(h_0, h_1, h_2) |m, n >_{+-} = (-n, m + n, -m) |m, n >_{+-},
\]

where \( |m, n >_{+-} \) vanishes if either of the indices \( m, n \) take values \(-1\) or \(+2\). The parities of the vectors (important for the super trace) are equal to

\[
p( |m, n >_{+-} ) = (m + n) \pmod{2} . \quad (5.21)
\]

### 5.3 Wronskian-type relation (3.2) for \( Q \)-operators

Below we will prove the relation (3.2a). Using (2.56) and (2.58) we will write it in the form

\[
-c_{12} \mathcal{H}_3(x) = \left( \frac{z_2}{z_1} \right)^{1/2} \mathcal{H}_1(xq) \mathcal{H}_2(xq^{-1}) - \left( \frac{z_1}{z_2} \right)^{1/2} \mathcal{H}_1(xq^{-1}) \mathcal{H}_2(xq). \quad (5.22)
\]

Making now a particular choice of representations for oscillator algebras (which utilizes the freedom explained in the previous subsection) one can rewrite this functional relation as

\[
\frac{z_2}{z_2 - z_1} \text{Str}_{W_{3}^{+-}(x)} \left( z_1^{-h_0} z_2 h_2 \mathcal{R} \right) = \text{Str}_{\overline{W}_{1}^{++}(xq) \otimes_s \overline{W}_{2}^{--}(xq^{-1})} \left( z_1^{-h_0} z_2 h_2 \mathcal{R} \right) - \frac{z_1}{z_2} \text{Str}_{\overline{W}_{1}^{++}(xq^{-1}) \otimes_s \overline{W}_{2}^{--}(xq)} \left( z_1^{-h_0} z_2 h_2 \mathcal{R} \right). \quad (5.23)
\]

We will split the proof into two steps.

**Step 1.** First, consider the tensor product module \( \overline{W}_{1}^{++}(xq) \otimes_s \overline{W}_{2}^{--}(xq^{-1}) \). Let us write its basis vectors as

\[
w_{j_1, j_2, j_3, j_4} = |j_1, j_2 >_{++} \otimes_s |j_3, j_4 >_{--}, \quad j_1, j_2 \in \mathbb{Z}_{\geq 0}, \quad j_2, j_3 = 0, 1 \quad (5.24)
\]
where \(|j_1,j_2 >_+\) and \(|j_3,j_4 >_-\) denote bases in \(\overline{W}_1^+ (xq)\) and \(\overline{W}_2^- (xq^{-1})\), defined in \((B.9)\) and \((B.10)\), respectively. We will assume that \(w_{j_1,j_2,j_3,j_4} = 0\), if the indices \(j_1,j_2,j_3,j_4\) lie outside the domain specified in \((5.24)\). Note that the parity \(p(w_{j_1,j_2,j_3,j_4}) = (j_2 + j_3) \mod 2\). Taking into account \((B.9)\), \((B.10)\) and the formula for the co-multiplication \((2.11)\), one can calculate the action of the generators of \(B_+\) on the basis \((5.24)\),

\[
e_0 w_{j_1,j_2,j_3,j_4} = w_{j_1,j_2-1,j_3,j_4} + \frac{(-1)^{j_2} (q^{i-j_2} - q^{-j_2}) q^{-j_1 - \frac{1}{2}}}{(q - q^{-1})^2} w_{j_1,j_2,j_3+1,j_4-1},
\]

\[
e_1 w_{j_1,j_2,j_3,j_4} = q^{-j_2} w_{j_1+1,j_2,j_3,j_4} + q^{2j_1+j_2} w_{j_1,j_2,j_3,j_4+1},
\]

\[
e_2 w_{j_1,j_2,j_3,j_4} = -\frac{x (1 - q^{-2j_1}) q^{j_2 + \frac{3}{2}}}{(q - q^{-1})^2} w_{j_1-1,j_2+1,j_3,j_4} + (-1)^{j_1} q^{-j_1 - j_2 + j_4} w_{j_1,j_2,j_3-1,j_4},
\]

\[
h_0 w_{j_1,j_2,j_3,j_4} = -(j_1 + j_3 + j_4) w_{j_1,j_2,j_3,j_4},
\]

\[
h_1 w_{j_1,j_2,j_3,j_4} = (2j_1 + j_2 + j_3 + 2j_4) w_{j_1,j_2,j_3,j_4},
\]

\[
h_2 w_{j_1,j_2,j_3,j_4} = -(j_1 + j_2 + j_4) w_{j_1,j_2,j_3,j_4}.
\]

It is convenient to define vectors, with the same weights,

\[
w_{m,j}^{(1)} = w_{j,0,0,m-j}, \quad 0 \leq j \leq m,
\]

\[
w_{m,j}^{(2)} = w_{j,1,1,m-j-1}, \quad w_{m,j}^{(3)} = w_{j,1,0,m-j-1}, \quad w_{m,j}^{(4)} = w_{j,0,1,m-j-1}, \quad 0 \leq j \leq m-1,
\]

and introduce the following vectors

\[
v_{00}^{(m)} = \sum_{j=0}^{m} q^{(m-j-2)j} \left[ \begin{array}{c} m \\ j \end{array} \right] w_{m,j}^{(1)} - \frac{x q^{-j - \frac{1}{2}} (m - j)}{q - q^{-1}} w_{m,j}^{(2)},
\]

\[
v_{10}^{(m)} = \sum_{j=0}^{m} q^{(m-j-2)j} \left[ \begin{array}{c} m \\ j \end{array} \right] w_{m+1,j}^{(3)},
\]

\[
v_{01}^{(m)} = \sum_{j=0}^{m} q^{(m-j)j-m} \left[ \begin{array}{c} m \\ j \end{array} \right] w_{m+1,j}^{(4)},
\]

\[
v_{11}^{(m)} = \sum_{j=0}^{m+1} q^{(m-j)j-m} \left( \frac{q^{-\frac{1}{2}} (q - q^{-1})^2}{x} \right) \left[ \begin{array}{c} m \\ j \end{array} \right] w_{m+1,j}^{(1)} + q^{-2j-1} \left[ \begin{array}{c} m \\ j \end{array} \right] w_{m+1,j}^{(2)},
\]

where

\[
\left[ \begin{array}{c} m \\ j \end{array} \right] = \frac{[m]!}{[j]! [m-j]!}, \quad j = 0, 1, \ldots, m, \quad m \geq 0,
\]

are the \(q\)-binomial coefficients,

\[
[m]! = [1][2] \cdots [m], \quad m \in \mathbb{Z}_{\geq 1}, \quad [0]! = 1,
\]

is the \(q\)-factorial and

\[
[r] = (q^r - q^{-r})/(q - q^{-1}).
\]
Here we put \( [m]_{-1} = 0 \) for \( m \in \mathbb{Z}_{\geq 0} \). The action of generators of \( \mathcal{B}_+ \) on these vectors is as follows

\[
e_{0}v_0^{(m)} = 0, \quad e_1v_0^{(m)} = \frac{q^2 - q}{q - q}v_1^{(m)} + v_0^{(m+1)}, \quad e_2v_0^{(m)} = 0,
\]

\[
e_0v_{10}^{(m)} = v_0^{(m)}, \quad e_1v_{10}^{(m)} = qv_0^{(m+1)}, \quad e_2v_{10}^{(m)} = 0,
\]

\[
e_0v_{01}^{(m)} = 0, \quad e_1v_{01}^{(m)} = qv_0^{(m+1)}, \quad e_2v_{01}^{(m)} = v_0^{(m)} ,
\]

\[
e_0v_{11}^{(m)} = q^{-1}v_{01}^{(m)}, \quad e_1v_{11}^{(m)} = q^2v_0^{(m+1)}, \quad e_2v_{11}^{(m)} = -v_{10}^{(m)} ,
\]

and

\[
(h_0, h_1, h_2)_m v_0^{(m)} = ( -m, 2m, -m ) v_0^{(m)}
\]

\[
(h_0, h_1, h_2)_m v_{10}^{(m)} = ( -m, 2m + 1, -m - 1 ) v_{10}^{(m)}
\]

\[
(h_0, h_1, h_2)_m v_{01}^{(m)} = ( -m - 1, 2m + 1, -m ) v_{01}^{(m)}
\]

\[
(h_0, h_1, h_2)_m v_{11}^{(m)} = ( -m - 1, 2m + 2, -m - 1 ) v_{11}^{(m)} .
\]

For each \( m \in \mathbb{Z}_{\geq 0} \), let \( W^{(m)} \) be the vector space spanned by the vectors \( \{ v_0^{(k)}, v_{10}^{(k)}, v_{01}^{(k)}, v_{11}^{(k)} \}_k \). By construction,

\[
\overline{W}_1^{++}(xq) \otimes_s \overline{W}_2^{-}(xq^{-1}) \supset W^{(0)} \supset W^{(1)} \supset W^{(2)} \supset \ldots .
\]

Examining (5.31), it is easy to conclude that

(i) each \( W^{(m)} \) is an invariant subspace with respect to the action of \( \mathcal{B}_+ \),

(ii) for each \( m \in \mathbb{Z}_{\geq 0} \) the factor module \( W^{(m)} / W^{(m+1)} \) is isomorphic to the shifted module \( W_3^{+}(x)[-m, -m] \). To see this one needs to drop all vectors \( v_{jk}^{(m+1)} \) in the RHS of (5.31b) and identify the vectors \( v_{jk}^{(m)} \) therein with \( |j,k| > +\ ) in (5.20).

Applying the identity (5.10) one obtains

\[
\text{Str}_{W^{(0)}}(z_1^{-h_0}z_2^{-h_2}R) = \sum_{m=0}^{\infty} \text{Str}_{W^{(m)}/W^{(m+1)}}(z_1^{-h_0}z_2^{-h_2}R) = \sum_{m=0}^{\infty} \text{Str}_{W_3^{++}(x)[-m, -m]}(z_1^{-h_0}z_2^{-h_2}R)
\]

\[
= \sum_{m=0}^{\infty} \text{Str}_{W_3^{+}(x)}(z_1^{-h_0+m}z_2^{-h_2-m}R) = \frac{z_2}{z_2 - z_1} \text{Str}_{W_3^{+}(x)}(z_1^{-h_0}z_2^{-h_2}R).
\]

(5.32)

**Step 2.** Next we want to show that the factor module \( \overline{W}_1^{++}(xq) \otimes \overline{W}_2^{-}(xq^{-1}) / W^{(0)} \) is isomorphic to \( \overline{W}_1^{++}(xq^{-1}) \otimes_s \overline{W}_2^{-}(xq) \) up to a shift automorphism.
Introduce additional vectors in $\overline{W}_{1}^{+}(xq) \otimes_{v} \overline{W}_{2}^{-}(xq^{-1})$,

\[
\begin{align*}
u_{m,j}^{(1)} &= \frac{q^{j-m+2} m + 1}{m + 1} \sum_{k=j+1}^{m+1} q^{k(m+1-k)} \left[ \begin{array}{c} m+1 \\ j \end{array} \right] w_{m+1,k}^{(1)} & m \geq 0, \ 0 \leq j \leq m, \\
u_{m,j}^{(2)} &= \frac{q^{j-m+2} m + 1}{m - j} \sum_{k=j+1}^{m+1} q^{k(m-k)} \left\{ \frac{q^{-\frac{1}{2}}(q-q^{-1})^{2}[k-j-1]}{[k]} x \right\} \left[ \begin{array}{c} m \\ k-1 \end{array} \right] w_{m+1,k}^{(1)} \\
&\quad + q^{-2k+j} \left[ \begin{array}{c} m \\ k \end{array} \right] w_{m+1,k}^{(2)} & m \geq 1, \ 0 \leq j \leq m-1, \\
u_{m,j}^{(3)} &= \frac{q^{j-m+2} m + 1}{m - j} \sum_{k=j+1}^{m+1} q^{k(m-2-k)} \left[ \begin{array}{c} m \\ k \end{array} \right] w_{m+1,k}^{(3)} & m \geq 1, \ 0 \leq j \leq m-1, \\
u_{m,j}^{(4)} &= \frac{q^{j-m+2} m + 1}{m - j} \sum_{k=j+1}^{m+1} q^{k(m-k)} \left[ \begin{array}{c} m \\ k \end{array} \right] w_{m+1,k}^{(4)} & m \geq 1, \ 0 \leq j \leq m-1.
\end{align*}
\]

Note that the union of the three sets of vectors
\[
\{v_{m,j}^{(m)}\}_{m \in \mathbb{Z}_{\geq 0}} \cup \{u_{m,j}^{(1)}\}_{0 \leq j \leq m, m \in \mathbb{Z}_{\geq 0}} \cup \{u_{m,j}^{(2)}, u_{m,j}^{(3)}, u_{m,j}^{(4)}\}_{0 \leq j \leq m-1, m \in \mathbb{Z}_{\geq 1}}
\]
completely span the vector space $\overline{W}_{1}^{+}(xq) \otimes_{v} \overline{W}_{2}^{-}(xq^{-1})$. The action of $B_{+}$ on the vectors \([5.33]\) is as follows,

\[
\begin{align*}
e_{0}u_{m,j}^{(1)} &= \frac{q^{j+m+2} m + 1}{m} \frac{q^{-j+\frac{1}{2} m + 1}}{q^{-1}} u_{m,j}^{(4)}, & e_{0}u_{m,j}^{(2)} = u_{m,j}^{(4)}, \\
e_{1}u_{m,j}^{(1)} &= q^{j} u_{m+1,j}^{(4)} + u_{m+1,j+1}^{(4)}, & e_{1}u_{m,j}^{(2)} = q^{j} u_{m+1,j}^{(4)} + q^{-1} u_{m+1,j+1}^{(2)}, \\
e_{2}u_{m,j}^{(1)} &= -q^{-j+\frac{1}{2} m + 1} \frac{q^{m+2} x}{q^{-1}} v_{10}^{(m)}, & e_{2}u_{m,j}^{(2)} = -q^{-j+m-2} u_{m,j}^{(3)}, \\
e_{0}u_{m,j}^{(3)} &= u_{m-1,j}^{(1)} + \frac{q^{j+m+2} m + 1}{m} \frac{q^{-j+\frac{1}{2} m + 1}}{q^{-1}} u_{m-1,j}^{(2)}, \\
e_{1}u_{m,j}^{(3)} &= q^{j} u_{m+1,j}^{(3)} + q^{-1} u_{m+1,j+1}^{(3)}, \\
e_{2}u_{m,j}^{(3)} &= 0, \\
e_{0}u_{m,j}^{(4)} &= 0, \\
e_{1}u_{m,j}^{(4)} &= q^{j} u_{m+1,j}^{(4)} + u_{m+1,j+1}^{(4)}, \\
e_{2}u_{m,j}^{(4)} &= q^{-2j+m-1} u_{m-1,j}^{(1)} - \frac{q^{j+m+2} m + 1}{q^{-1}} \frac{q^{-j+\frac{1}{2} m + 1}}{q^{-1}} u_{m-1,j-1}^{(2)} - q^{m+2} x q^{m-\frac{5}{2}} v_{10}^{(m-1)}, \\
\end{align*}
\]

and

\[
\begin{align*}
(h_{0}, h_{1}, h_{2}) v_{m,j}^{(1)} &= ( -m-1, \ 2m+2, \ -m-1 ) u_{m,j}^{(1)}, \\
(h_{0}, h_{1}, h_{2}) v_{m,j}^{(2)} &= ( -m-1, \ 2m+2, \ -m-1 ) u_{m,j}^{(2)}, \\
(h_{0}, h_{1}, h_{2}) v_{m,j}^{(3)} &= ( -m, \ 2m+1, \ -m-1 ) u_{m,j}^{(3)}, \\
(h_{0}, h_{1}, h_{2}) v_{m,j}^{(4)} &= ( -m-1, \ 2m+1, \ -m ) u_{m,j}^{(4)}.
\end{align*}
\]

37
where \( u_{0,j}^{(2)} = u_{m,-1}^{(2)} = u_{0,j}^{(3)} = u_{m,-1}^{(3)} = 0 \).

Similarly to \((5.25)\) write down the action of \( B_+ \) on the basis vectors

\[
\{u_{j_1,j_2,j_3,j_4} \mid j_1,j_2 >+ \otimes_s j_3,j_4 >-\}
\]
of the tensor product module \( \overline{W}^{++}_1(xq^{-1}) \otimes_s \overline{W}^{-}_2(xq) \),

\[
e_0 u_{j_1,j_2,j_3,j_4} = u_{j_1,j_2-1,j_3,j_4} + \frac{(-1)^{j_2} q^{-j_1 + \frac{1}{2}} x(q^{j_4} - q^{-j_4})}{(q - q^{-1})^2} u_{j_1,j_2,j_3+1,j_4-1},
\]

\[
e_1 u_{j_1,j_2,j_3,j_4} = q^{-j_2} u_{j_1+1,j_2,j_3,j_4} + q^{2j_1+j_2} u_{j_1,j_2,j_3,j_4+1},
\]

\[
e_2 u_{j_1,j_2,j_3,j_4} = -\frac{q^{2j_2 - \frac{1}{2}} x(1 - q^{-2j_1})}{(q - q^{-1})^2} u_{j_1-1,j_2+1,j_3,j_4} + (-1)^{j_2} q^{-j_1-j_2+j_4} u_{j_1,j_2,j_3-1,j_4},
\]

\[
(\mathcal{h}_0, \mathcal{h}_1, \mathcal{h}_2) u_{j_1,j_2,j_3,j_4} = (-j_1 - j_3 - j_4, 2j_1 + j_2 + j_3 + 2j_4, -j_1 - j_2 - j_4) u_{j_1,j_2,j_3,j_4}
\]

where \( j_1,j_4 \in \mathbb{Z}_{\geq 0} \) and \( j_2,j_3 = 0,1 \), otherwise \( w_{j_1,j_2,j_3,j_4} = 0 \). Also define

\[
\overline{u}^{(1)}_{m,j} = u_{j,0,0,m-j}, \quad 0 \leq j \leq m,
\]

\[
\overline{u}^{(2)}_{m,j} = u_{j,1,1,m-j-1}, \quad \overline{u}^{(3)}_{m,j} = u_{j,1,0,m-j-1}, \quad \overline{u}^{(4)}_{m,j} = u_{j,0,1,m-j-1}, \quad 0 \leq j \leq m - 1.
\]

Comparing \((5.34)\) with \((5.35)\) one concludes that

(iii) the action of generators \( e_k \) on vectors \( \{\overline{u}^{(a)}_{m,j}\} \), defined in \((5.36)\), is exactly the same as that on the vectors \( \{u^{(a)}_{m,j}\} \) in \( \overline{W}^{++}_1(xq) \otimes_s \overline{W}^{-}_2(xq^{-1})/W^{(0)} \). To obtain the latter one needs to omit the terms containing \( v^{(m)}_{ij} \) in the RHS of \((5.34)\).

(iv) the action of \( h_k \) on \( \{\overline{u}^{(a)}_{m,j}\} \) coincides with that for the vector \( \{u^{(a)}_{m,j}\} \) in \( \overline{W}^{++}_1(xq) \otimes_s \overline{W}^{-}_2(xq^{-1})/W^{(0)} \) up to the shift \( m \to m + 1 \).

Thus one concludes that

\[
\overline{W}^{++}_1(xq) \otimes_s \overline{W}^{-}_2(xq^{-1})/W^{(0)} \simeq (\overline{W}^{++}_1(xq^{-1}) \otimes_s \overline{W}^{-}_2(xq))[{-1,-1}].
\]

Using the this formula and applying the identity \((5.10)\) one obtains

\[
\text{Str}_{\overline{W}^{++}_1(xq) \otimes_s \overline{W}^{-}_2(xq^{-1})/W^{(0)}}(z_1^{-h_0} z_2^{-h_2} \mathcal{R}) = \frac{z_1}{z_2} \text{Str}_{\overline{W}^{++}_1(xq^{-1}) \otimes_s \overline{W}^{-}_2(xq)}(z_1^{-h_0} z_2^{-h_2} \mathcal{R}).
\]

Combining this with \((5.32)\) one obtains \((5.23)\). This completes the proof of the relation \((5.24)\).

### 5.4 Factorization formula \((3.10)\) for a typical representation

Here we will prove \((3.10)\), namely

\[
\mathcal{T}^{(2)}_c(x) = (z_1 - z_3)(z_2 - z_3) z_3^{-1} A_3(x q^{-c - \frac{1}{2}}) A_3(x q^{c + \frac{1}{2}}),
\]

where \( \mathcal{T}^{(2)}_c(x) \) is defined in \((2.44)\),

\[
\mathcal{T}^{(2)}_c(x) = \text{Str}_{\pi(c,c,0)}(z_1^{-h_0} z_2^{h_2} \mathcal{R}).
\]

38
The 4-dimensional typical representation $\pi_{(c,c,0)}(x)$ of the Borel subalgebra $B_\pm$ is described in the Appendix \[1.2\].

Our proof of (5.39) is based on the decomposition of the 16-dimensional tensor product module $W_3^+(xq^{-c-\frac{1}{2}}) \otimes_s W_3^-(xq^{c+\frac{1}{2}})$ into four 4-dimensional modules. With respect to the actions of $B_\pm$ each of these four 4-dimensional modules is isomorphic to a shifted typical representation $\pi_{(c,c,0)}(xq^c)[s_0^{(a)}, s_2^{(a)}]$ ($a \in \{1, 2, 3, 4\}$), where the shift constants $s_0^{(a)}$ and $s_2^{(a)}$ are given in (5.47).

Introduce a basis in the 16-dimensional tensor product module

$$w_{j_1, j_2, j_3, j_4} = |j_1, j_2 > _\pm \otimes_s |j_3, j_4 > _\pm \in W_3^+(xq^{-c-\frac{1}{2}}) \otimes_s W_3^-(xq^{c+\frac{1}{2}}), \quad (5.41)$$

where $j_1, j_2, j_3, j_4 = 0, 1$, otherwise $w_{j_1, j_2, j_3, j_4} \equiv 0$. The parity of these vectors is $p(w_{j_1, j_2, j_3, j_4}) = j_1 + j_2 + j_3 + j_4 \pmod 2$. Using (B.14), (15.11) and the formula for the co-multiplication (2.11) one can calculate the action of $B_\pm$ in this tensor product module

$$e_0 w_{j_1, j_2, j_3, j_4} = q^{-j_2} w_{j_1-1, j_2, j_3, j_4} + (-1)^{j_1+j_2} q^{j_1-j_2} w_{j_1, j_2, j_3-1, j_4},$$
$$e_1 w_{j_1, j_2, j_3, j_4} = x \left(q^{-c}w_{j_1+1, j_2, j_3, j_4} - q^{c+j_1+j_2} w_{j_1, j_2, j_3+1, j_4+1}\right)/(q - q^{-1}),$$
$$e_2 w_{j_1, j_2, j_3, j_4} = (-1)^{j_1+j_2+j_3} q^{-j_1} w_{j_1, j_2, j_3, j_4-1} + (-1)^{j_1} w_{j_1-1, j_2, j_3, j_4},$$

$$(h_0, h_1, h_2) w_{j_1, j_2, j_3, j_4} = (-j_2 - j_4, j_1 + j_2 + j_3 + j_4, -j_1 - j_3) w_{j_1, j_2, j_3, j_4}. \quad (5.42)$$

We shall change the basis of the module from $\{w_{j_1, j_2, j_3, j_4}\}$ to $\{v_j^{(a)}\}$, where $j, a \in \{1, 2, 3, 4\}$:

$$v_1^{(1)} = -q^{2c-1}(q - q^{-1})w_{0,0,1,1}, \quad v_2^{(1)} = q^{-1}xw_{0,0,1,1},$$
$$v_3^{(1)} = -q^{c-1}(q - q^{-1})w_{0,1,0,0}, \quad v_4^{(1)} = w_{0,1,0,1},$$
$$v_1^{(2)} = xq^{c-1}w_{0,0,1,0}, \quad v_2^{(2)} = -x \left(q^c w_{0,0,1,1} - w_{1,1,0,0}\right)/(q^2 - 1),$$
$$v_3^{(2)} = w_{0,0,0,0}, \quad v_4^{(2)} = q^{-c} \left(q^{2c+2}w_{0,0,0,1} - w_{0,1,0,0}\right)/(q^2 - 1),$$
$$v_1^{(3)} = q^{-c}xw_{1,1,1,0}, \quad v_2^{(3)} = -wx_{1,1,1,1}/(q^2 - 1),$$
$$v_3^{(3)} = -w_{0,0,1,1} +qw_{1,0,0,1}, \quad v_4^{(3)} = q^{-c} \left(w_{0,1,1,1} + qw_{1,0,1,0}\right)/(q^2 - 1),$$
$$v_1^{(4)} = -q^{c-1}xw_{1,0,0,1}, \quad v_2^{(4)} = -x \left(q^{2c+1}w_{1,0,1,1} + w_{1,1,0,0}\right)/(q^2 - 1),$$
$$v_3^{(4)} = w_{1,0,0,0} - w_{0,0,1,0}, \quad v_4^{(4)} = \left(q^{c-2}(w_{0,0,1,1} - w_{1,1,0,0}) + q^{-c}(w_{0,1,1,0} + w_{1,1,0,0})\right)/(q^2 - 1).$$

The action of the generators $e_k$, $k = 0, 1, 2$ in this basis is

$$e_k v_j^{(a)} = \sum_{i=1}^{4} A_{ik}^{(a)} v_i^{(a)} + \sum_{b=0+1}^{4} \sum_{i=1}^{4} B_{ijk}^{(b)} v_i^{(b)}, \quad i, j, a \in \{1, 2, 3, 4\} \quad (5.44)$$

where the only non-zero coefficients $A_{ik}^{(a)}$ are

$$A_{31}^{0} = -A_{21}^{0} = q^{-c-1}x, \quad A_{12}^{1} = q^{1-c}, \quad A_{12}^{2} = [c], \quad A_{34}^{2} = [c+1],$$

where the symbol $[c]$ is defined in (5.39). The coefficients $B_{ijk}^{(b)}$ are known explicitly, but their exact form is not essential in the following. Note that, if the second term in the formula (5.44)
is omitted, then for any fixed value of \( a \) it becomes identical to the corresponding formula for the 4-dimensional representation \( \pi_{(c,c,0)}(xq^{a+1}) \), given in (5.6) up to a similarity transformation. The action of the generators \( h_k \) in this basis is

\[
h_k v_j^{(a)} = (v_{j,k} + s_k^{(a)}) v_j^{(a)}
\]

where \( a \in \{1, 2, 3, 4\} \) and the weights \( \nu_{j,k} \) are exactly the same as in the action of \( h_k \) in the 4-dimensional representation \( \pi_{(c,c,0)}(xq^{c+1}) \), see (5.6) in the Appendix. The shift constants, satisfy the relation \( s_0^{(a)} + s_1^{(a)} + s_2^{(a)} = 0 \); explicitly they read

\[
(s_0^{(a)}, s_1^{(a)}, s_2^{(a)}) = (c - \alpha(a), \alpha(a) + \beta(a), -c - \beta(a)), \quad a \in \{1, 2, 3, 4\}, \tag{5.47}
\]

where

\[
\beta^{(1)} = \beta^{(2)} = 1, \quad \beta^{(3)} = \beta^{(4)} = 2, \quad \alpha^{(1)} = \alpha^{(3)} = 1, \quad \alpha^{(2)} = \alpha^{(4)} = 0.
\]

Let \( W^{(a)}, a \in \{1, 2, 3, 4\} \) denotes the vector space spanned by the vectors \( \{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}, v_4^{(j)}\}_{j=\text{a}} \). By construction, the original tensor product module

\[
W^+_3(xq^{-c-\frac{1}{2}}) \otimes_s W^+_3(xq^{c+\frac{1}{2}}) \simeq W^{(1)}
\]

is isomorphic to \( W^{(1)} \). Examining (5.44) and (5.46) one easily finds that

(i) each \( W^{(a)} \) is an invariant space with respect to the action of \( \mathcal{B}_+ \),

(ii) the following isomorphisms with respect to the action of \( \mathcal{B}_+ \) take place

\[
W^{(a)}/W^{(a+1)} \simeq \pi_{(c,c,0)}(xq^{a+1})[s_0^{(a)}, s_2^{(a)}], \quad a = 1, 2, 3 \tag{5.48}
\]

\[
W^{(4)} \simeq \pi_{(c,c,0)}(xq^{c+1})[s_0^{(4)}, s_2^{(4)}], \tag{5.49}
\]

except that for \( a = 2, 3 \) the parities of all vectors on one side of the correspondence need to be inverted, see below.

Using these results, the definitions (2.56) and (2.58) and the identity (5.10) one obtains

\[
Z_3 \mathcal{A}_3(xq^{-c-\frac{1}{2}}) \mathcal{A}_3(xq^{c+\frac{1}{2}}) = \text{Str}_{W^+_3(xq^{-c-\frac{1}{2}}) \otimes_s W^+_3(xq^{c+\frac{1}{2}})}(z_1^{-h_0} z_2^{h_2} \mathcal{R})
\]

\[
= \text{Str}_{W^{(1)}}(z_1^{-h_0} z_2^{h_2} \mathcal{R}) + \sum_{a=1}^{3} \text{Str}_{W^{(a)}/W^{(a+1)}}(z_1^{-h_0} z_2^{h_2} \mathcal{R}) + \text{Str}_{W^{(4)}}(z_1^{-h_0} z_2^{h_2} \mathcal{R})
\]

\[
= \sum_{a=1}^{4} (-1)^{\gamma_a} \text{Str}_{\pi_{(c,c,0)}(xq^{c+1})}(z_1^{-h_0 - s_0^{(a)}} z_2^{h_2 + s_2^{(a)}} \mathcal{R})
\]

\[
= \sum_{a=1}^{4} (-1)^{\gamma_a} \text{Str}_{\pi_{(c,c,0)}(xq^{c+1})}(z_1^{-h_0 - s_0^{(a)}} z_2^{h_2 + s_2^{(a)}} \mathcal{R})
\]

\[
= \sum_{a=1}^{4} \text{Str}_{\pi_{(c,c,0)}(xq^{c+1})}(z_1^{-h_0} z_2^{h_2} \mathcal{R})
\]

\[
= \prod_{a=1}^{4} \text{Str}_{\pi_{(c,c,0)}(xq^{c+1})}(z_1^{-h_0} z_2^{h_2} \mathcal{R})
\]

\[
= \mathcal{A}_3(xq^{c+1}) \mathcal{A}_3(xq^{-c-\frac{1}{2}})
\]

The sign factor \((-1)^{\gamma_a}, \gamma_1 = \gamma_4 = 0, \gamma_2 = \gamma_3 = 1\), takes into account the parity of the vectors \( p(v_1^{(a)}) = \gamma_a \) (which needs to be compared with the (even) parity of the highest weight vector in the representation \( \pi_{(c,c,0)} \)). Using the expressions for the constants (5.18) one finally arrives to (5.50).
5.5 Wronskian type T-Q relation (5.9)

In this section it will be more convenient to use the Fock representations $W_{1}^{-}(x), W_{2}^{+}(x)$ $W_{1}^{++}(x)$ and $W_{1}^{--}(x)$ defined in (B.15), (B.16), (B.17) and (B.19). Note that these are different from those used in Sect. 5.2.3, 5.3. The normalization constants (2.57) and (2.59) for these new representations read

$$Z_{1} = Z_{1} = \frac{z_{1}(1-z_{2})}{z_{1}-z_{2}}, \quad Z_{2} = Z_{2} = \frac{z_{1}-1}{z_{1}-z_{2}}.$$  (5.51)

We will prove (3.9a),

$$T_{m}^{(1)}(x) = \frac{c_{13}}{c_{12}} z_{1}^{m+\frac{1}{2}} A_{1}(x q^{m+\frac{1}{2}}) A_{1}(x q^{-m-\frac{1}{2}}) - c_{23}^{2} z_{2}^{m+\frac{1}{2}} A_{2}(x q^{m+\frac{1}{2}}) A_{2}(x q^{-m-\frac{1}{2}}).$$  (5.52)

for $m \in \mathbb{Z}_{\geq 0}$. For this purpose, we introduce more general T-operators, corresponding to the infinite-dimensional representations of $U_{q}(\widehat{\mathfrak{sl}}(2|1))$,

$$T_{m}^{+}(x) = \text{Str}_{\pi_{(m,0,0)}^{+}(x)}(z_{1}^{m} x q^{m} z_{2}^{-h_{2} \mathcal{R}}), \quad T_{m}^{-}(x) = \text{Str}_{\pi_{(m,0,0)}^{-}(x)}(z_{1}^{m} x q^{m} z_{2}^{h_{2} \mathcal{R}}),$$  (5.53)

where $m \in \mathbb{C}$. The representations $\pi_{(m,0,0)}^{+}(x)$ and $\pi_{(m,0,0)}^{-}(x)$ are defined in the Appendix B. For integer values of $m \in \mathbb{Z}$ one of these representations becomes reducible and can be decomposed into a semi-direct sum of a finite-dimensional representation and an infinite-dimensional atypical representation. These properties are studied in details in the Appendix B. Here we quote just one relevant formula (5.4),

$$\pi_{(m,0,0)}^{+}(x)/\pi_{(m,0,0)}^{[0]}(x) \simeq \pi_{(m+1,0,0)}^{-}(x), \quad m \in \mathbb{Z}_{\geq 0}$$  (5.54)

The finite-dimensional representations $\pi_{(m,0,0)}^{[0]}(x)$, which appears above is precisely that entering in the definition of the T-operators (2.41), namely,

$$T_{m}^{(1)}(x) = \text{Str}_{\pi_{(m,0,0)}^{[0]}(x)}(z_{1}^{m} x q^{m} z_{2}^{-h_{2} \mathcal{R}}), \quad m \in \mathbb{Z}_{\geq 0},$$  (5.55)

$$T_{m}^{(1)}(x) = -\text{Str}_{\pi_{(m+1,0,0)}^{[0]}(x)}(z_{1}^{m+1} x q^{m+1} z_{2}^{-h_{2} \mathcal{R}}), \quad m \in \mathbb{Z}_{\geq -2},$$  (5.56)

$$T_{-1}^{(1)}(x) = -\text{Str}_{\pi_{(m+1,0,0)}^{[0]}(x)}(z_{1}^{m+1} x q^{m+1} z_{2}^{-h_{2} \mathcal{R}})$$

On the level of supertraces Eq.(5.54) implies

$$T_{m}^{(1)}(x) = T_{m}^{+}(x) - T_{m}^{-}(x), \quad m \in \mathbb{Z}_{\geq 0},$$  (5.57)

where the operators $T_{m}^{\pm}(x)$ are defined by (5.53). As we shall see below, these operators factorize into products of two Q-operators

$$T_{m}^{+}(x) = \frac{c_{13}}{c_{12}} z_{1}^{m+\frac{1}{2}} A_{1}(x q^{m+\frac{1}{2}}) A_{1}(x q^{-m-\frac{1}{2}}),$$  (5.58)

$$T_{m}^{-}(x) = \frac{c_{23}}{c_{12}} z_{2}^{m+\frac{1}{2}} A_{2}(x q^{m+\frac{1}{2}}) A_{2}(x q^{-m-\frac{1}{2}}).$$  (5.59)

The superscript “$p$” in the notation $\pi_{\mu}^{[p]}(x)$ denotes the parity of the highest weight vector. The representations $\pi_{\mu}^{[p]}(x)$ with $p = 0$ and $p = 1$ only differ by an overall sign of the supertrace, but otherwise equivalent.
These factorization properties hold for an arbitrary value of \( m \in \mathbb{C} \) (even though for \((5.56)\) one needs only integer values of \( m \)). Thus, the Wronskian-like formula \((5.52)\) in question is a corollary of the factorization relations \((5.57)\) and \((5.58)\). Their proof is presented below. Actually, one needs to prove one of these relations, since they follow from one another under the symmetry transformation \((5.5)\), \((5.7)\).

### 5.5.1 Factorization formula \((5.57)\) for infinite-dimensional representations

The formula \((5.57)\) reflects rather special properties of tensor product module \( W_1^{++}(xq^{m+\frac{1}{2}}) \otimes_s W_1^{--}(xq^{-m-\frac{1}{2}}) \). Below we will show that this infinite-dimensional module can be decomposed into an infinite number of infinite-dimensional modules, each of which is isomorphic to a shifted evaluation module \( \pi_{(m,0,0)}^+(x)[j_0,j_2] \) with \( j_0,j_2 \in \mathbb{Z} \).

Let us write the basis in \( W_1^{++}(xq^{m+\frac{1}{2}}) \otimes_s W_1^{--}(xq^{-m-\frac{1}{2}}) \) as

\[
w_{j_1,j_2,j_3,j_4} = |j_1,j_2 > \otimes_s |j_3,j_4 >, \quad j_1,j_2,j_3,j_4 \in \mathbb{Z}_{\geq 0}, \quad j_2,j_4 \in \{0,1\},
\]

\((5.59)\)

where \( |j_1,j_2 > \) and \( |j_3,j_4 > \) denote the basis vectors in \( W_1^{++}(xq^{m+\frac{1}{2}}) \) and \( W_1^{--}(xq^{-m-\frac{1}{2}}) \), defined in \((B.15)\) and \((B.17)\), respectively. The parity of these vectors is

\[
p(w_{j_1,j_2,j_3,j_4}) = j_2 + j_4 \mod 2.
\]

As usual, we assume that \( w_{j_1,j_2,j_3,j_4} = 0 \), if the indices \( j_1,j_2,j_3,j_4 \) lie outside the domain specified in \((5.59)\). Taking into account \((B.15)\), \((B.17)\) and the formula for the co-multiplication \((2.11)\), one can find the action of the generators of \( B_+ \),

\[
e_0 w_{j_1,j_2,j_3,j_4} = w_{j_1,j_2+1,j_3,j_4} + (-1)^{j_2} q^{j_1} w_{j_1,j_2,j_3+1,j_4+1},
\]

\[
e_1 w_{j_1,j_2,j_3,j_4} = \frac{1}{q - q^{-1}} (q^{-j_1-j_2}[j_1] w_{j_1-1,j_2,j_3,j_4} - q^{-2j_1-j_2+j_3+j_4}[j_3] w_{j_1,j_2,j_3-1,j_4}),
\]

\[
e_2 w_{j_1,j_2,j_3,j_4} = -x (q^{m+1} w_{j_1+1,j_2-1,j_3,j_4} + (-1)^{j_2} q^{j_1+j_2-m-1} w_{j_1,j_2,j_3+1,j_4-1}),
\]

\((h_0,h_1,h_2) w_{j_1,j_2,j_3,j_4} = (j_1 + j_3, -2j_1 - j_2 - 2j_3 - j_4, j_1 + j_2 + j_3 + j_4) w_{j_1,j_2,j_3,j_4},
\]

\((5.60)\)

It is convenient to define vectors, with the same weights,

\[
w^{(1)}_{n,j} = w_{j,0,n-j,0}, \quad w^{(2)}_{n,j} = w_{j,0,n-j,1}, \quad w^{(3)}_{n,j} = w_{j,1,n-j,0}, \quad w^{(4)}_{n,j} = w_{j,1,n-j,1},
\]

\((5.61)\)

where \( 0 \leq j \leq n, \quad n \in \mathbb{Z}_{\geq 0} \). Introduce the following vectors,

\[
v^{(1)}_{n,j} = q^{-j/2-m} \lambda_j x \sum_{k=0}^{n} q^{-k(k-n-2j-2)} \left[ \begin{array}{c} n \\ k \end{array} \right] w^{(1)}_{j+n,k},
\]

\[
v^{(2)}_{n,j} = q^{-3j/2}(q - q^{-1}) \lambda_j \sum_{k=0}^{n} q^{-k(k-n-2j-3)} \left[ \begin{array}{c} n \\ k \end{array} \right] w^{(2)}_{j+n,k},
\]

\[
v^{(3)}_{n,j} = q^{+j/2-m} \lambda_j x \sum_{k=0}^{n} q^{-k(k-n-2j-2)} \left[ \begin{array}{c} n \\ k \end{array} \right] (w^{(3)}_{j+n,k} + q^{2m-2j+k} w^{(2)}_{j+n,k}),
\]

\[
v^{(4)}_{n,j} = -q^{-j/2}(q - q^{-1}) \lambda_j \sum_{k=0}^{n} q^{-k(k-n-2j-3)} \left[ \begin{array}{c} n \\ k \end{array} \right] w^{(4)}_{j+n,k},
\]

\((5.62)\)
where $n, j \in \mathbb{Z}_{\geq 0}$ and $\lambda_j = q^{-j^2/2} (q^{-1} - q)^j$. Their parities are given by

$$p(v^{(2)}_{n,k}) = p(v^{(3)}_{n,k}) = 1, \quad p(v^{(1)}_{n,k}) = p(v^{(4)}_{n,k}) = 0.$$  

The action of $\mathcal{B}_+$ on these vectors is as follows

$$e_0 v^{(1)}_{n,j} = -x[m - j]v^{(2)}_{n,j} + q^{-j}v^{(3)}_{n,j}, \quad e_1 v^{(1)}_{n,j} = [j]v^{(1)}_{n,j-1}, \quad e_2 v^{(1)}_{n,j} = 0,$$

$$e_0 v^{(2)}_{n,j} = -q^{-j}v^{(4)}_{n,j}, \quad e_1 v^{(2)}_{n,j} = [j]v^{(2)}_{n,j-1}, \quad e_2 v^{(2)}_{n,j} = v^{(1)}_{n,j+1},$$

$$e_0 v^{(3)}_{n,j} = -x[m - j]v^{(4)}_{n,j}, \quad e_1 v^{(3)}_{n,j} = [j]v^{(3)}_{n,j-1}, \quad e_2 v^{(3)}_{n,j} = -xq^{m-j-1}v^{(1)}_{n+1,j},$$

$$e_0 v^{(4)}_{n,j} = 0, \quad e_1 v^{(4)}_{n,j} = [j]v^{(4)}_{n,j-1}, \quad e_2 v^{(4)}_{n,j} = v^{(3)}_{n,j+1} + xq^{m-j-2}v^{(2)}_{n+1,j},$$

and

$$\begin{align*}
(h_0, h_1, h_2) v^{(1)}_{n,j} &= (n + j, -2n - 2j, n + j) v^{(1)}_{n,j}, \\
(h_0, h_1, h_2) v^{(2)}_{n,j} &= (n + j, -2n - 2j - 1, n + j + 1) v^{(2)}_{n,j}, \\
(h_0, h_1, h_2) v^{(3)}_{n,j} &= (n + j, -2n - 2j - 1, n + j + 1) v^{(3)}_{n,j}, \\
(h_0, h_1, h_2) v^{(4)}_{n,j} &= (n + j, -2n - 2j - 2, n + j + 2) v^{(4)}_{n,j}.
\end{align*}$$

Introduce vector spaces

$$W_{2n}, \quad \text{spanned by the vectors } \left\{ \begin{array}{c} v^{(1)}_{p,j}, \ v^{(2)}_{p,j}, \ v^{(3)}_{p,j}, \ v^{(4)}_{p,j} \end{array} \right\}_{p=n, j=0}^{\infty},$$

$$W_{2n+1}, \quad \text{spanned by the vectors } \left\{ \begin{array}{c} v^{(1)}_{p+1,j}, \ v^{(2)}_{p+1,j}, \ v^{(3)}_{p,j}, \ v^{(4)}_{p,j} \end{array} \right\}_{p=n, j=0}^{\infty},$$

where $n \in \mathbb{Z}_{\geq 0}$. By construction,

$$W_{1}^{++} (xq^{m+\frac{1}{2}}) \otimes_s \overline{W}_{1}^{--} (xq^{-m-\frac{1}{2}}) = W^{(0)} \supset W^{(1)} \supset W^{(2)} \ldots$$

Examining (5.63), we find that

(i) for any $n \in \mathbb{Z}_{\geq 0}$, $W^{(n+1)}$ is an invariant subspace of $W^{(n)}$ with respect to the action of $\mathcal{B}_+$.

(ii) for any $n \in \mathbb{Z}_{\geq 0}$, the following isomorphisms with respect to the action of $\mathcal{B}_+$ take place

$$W^{(2n)}/W^{(2n+1)} \simeq \pi_{(m,0,0)}^{+}(x)[n + m, n],$$

$$W^{(2n+1)}/W^{(2n+2)} \simeq \pi_{(m,0,0)}^{+}(x)[n + m, n + 1],$$

except for that the parities of all the vectors on one side of the second formula, (5.67), need to be inverted (note that $v^{(1)}_{n,j}$ is even and $v^{(3)}_{n,j}$ is odd). Remind that the representation $\pi_{(m,0,0)}^{+}(x)$ is defined in (15.2).
Using these results and the identity (5.10), we obtain

\[ \text{Str}_{W_{1}^{++}}(x_q^{m+\frac{1}{2}} \otimes W_{1}^{-}(x_q^{-m-\frac{1}{2}}) (z_1^{-h_0} z_2^{h_2}) = \text{Str}_{W^{(0)}}(z_1^{-h_0} z_2^{h_2}) } \]

\[ = \sum_{n=0}^{\infty} \text{Str}_{W^{(n)}/W^{(n+1)}}(z_1^{-h_0} z_2^{h_2}) \]

\[ = \sum_{n=0}^{\infty} \text{Str}_{\pi^{+}_{(m,0,0)}(x)[n+m,n]}(z_1^{-h_0} z_2^{h_2}) - \sum_{n=0}^{\infty} \text{Str}_{\pi^{+}_{(m,0,0)}(x)[n+m,n+1]}(z_1^{-h_0} z_2^{h_2}) \]

\[ = \sum_{n=0}^{\infty} \text{Str}_{\pi^{+}_{(m,0,0)}(x)}(z_1^{-h_0-n-m} z_2^{h_2+n}) - \sum_{n=0}^{\infty} \text{Str}_{\pi^{+}_{(m,0,0)}(x)}(z_1^{-h_0-n-m} z_2^{h_2+n+1}) \]

\[ = (1 - z_2)z_1^{1-m} \text{Str}_{\pi^{+}_{(m,0,0)}(x)}(z_1^{-h_0} z_2^{h_2}) \].

Then using the definitions (2.56), (2.57), (2.58), (2.59), (3.3), (5.51), (5.53) one finally arrive to (5.57). This completes the proof of the relation (5.52).

6 Concluding remarks

The Baxter’s Q-operators find many important applications in the theory of integrable quantum systems. In this paper we have developed an algebraic theory of the Q-operators for solvable models associated with the quantized affine algebra \( U_q(\hat{sl}(2|1)) \), extending previously known results for \( U_q(\hat{sl}(2)) \) [4,5] and \( U_q(\hat{sl}(3)) \) [53] (see also [76] and [77] for \( U_q(\hat{sl}(n)) \) case).

Our general formalism has been illustrated by two representative cases: the 3-state lattice model and a continuous quantum field theory, associated with the AKNS soliton hierarchy. Here we assumed a generic value of the deformation parameter \( q \neq 1 \). The isotropic case \( q = 1 \) can be obtained by a more or less straightforward limiting procedure (though requires additional considerations, similar to [78]) and will be considered elsewhere.

Finally, note very useful connections between functional relations in solvable models with the theory of the (super) characters and symmetric functions. Namely we refer to the Weyl first and second formulae for the Schur functions. Actually, there two different but related, “second” formulae, often called the Jacobi-Trudy and Giambelli formulae, respectively. These formulae have super-symmetric generalizations. They are referred to by adding the adjective “super-symmetric” to the respective name (except that the super-symmetric analog of the first Weyl formula is usually called the Sergeev-Pragacz formula). These connections are discussed in the Appendix C.

Acknowledgments

We thank M. Bortz, P.G. Bouwknegt, M. Jimbo, S. M. Khoroshkin and H. Yamane for useful discussions. We also thank V. Kazakov and M. Staudacher for their interest to this work and numerous stimulating discussions. ZT would like to thank the members of the integrable system group at Australian National University, especially, M.T.Batchelor and X.W. Guan, for their kind hospitality during his stay at the ANU, where a part of this work was done. ZT was supported by Grant-in-Aid for Scientific Research from JSPS, #16914018, Bilateral Joint Projects “Solvable models and their thermodynamics in statistical mechanics and field theory”
Appendix A. Highest weight representations of $U_q(gl(2|1))$

Here we briefly discuss the representation theory of $U_q(gl(2|1))$. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$ graded vector space with the parity $p$ such that $p(v) = 0$ (even) for $v \in V_0$ and $p(v) = 1$ (odd) for $v \in V_1$. There is a basis $\{v_i\}$ of $V$, called a homogeneous basis, such that $v_i \in V_0$ or $v_i \in V_1$. For any linear operator $A \in \text{End}(V)$, its matrix expression in this basis reads $Av_k = \sum_j v_j A_{jk}$. The supertrace of $A$ is defined as $\text{Str} A = \sum_j (-1)^{p(v_j)} A_{jj}$.

A.1 Finite-dimensional representations

In this paper we only need the highest weight representations of $U_q(gl(2|1))$. Any such representation can be constructed by the so-called induced module construction, which was proposed by Kac [79] for the $q = 1$ case and generalized to the $q$-generic case in [80]. It is built on the highest weight vector $|0\rangle$

$$E_{12} |0\rangle = E_{23} |0\rangle = 0, \quad E_{ii} |0\rangle = \mu_i |0\rangle, \quad i = 1, 2, 3. \quad (A.1)$$

with the weight $\mu = (\mu_1, \mu_2, \mu_3)$, where $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$. The module is finite-dimensional when $\mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}$ and will be denoted, in this case, as $\hat{\pi}_\mu$.

The pair of numbers $[b_1, b_2]$, where $b_1 = \mu_1 - \mu_2$ and $b_2 = \mu_2 + \mu_3$, is called the Kac-Dynkin label of $\hat{\pi}_\mu$. Note that two modules $\hat{\pi}_{(\mu_1, \mu_2, \mu_3)}$ and $\hat{\pi}_{(\mu_1+\eta, \mu_2+\eta, \mu_3-\eta)}$, where $\eta$ is arbitrary, have the same Kac-Dynkin label. In general, $\hat{\pi}_\mu$ is not an irreducible representation. The corresponding irreducible representation, obtained by factoring out a maximal proper invariant subspace of $\hat{\pi}_\mu$, will be denoted as $\pi_\mu$. Sometimes, we will use the notation $\pi_\mu^{[p]}$ to indicate the parity $p$ of the highest weight vector of $\pi_\mu$. There are three types of finite dimensional representations:

(i) typical representations

$$\dim \pi_\mu = 4(\mu_1 - \mu_2 + 1), \quad (\mu_2 + \mu_3)(\mu_1 + \mu_3 + 1) \neq 0, \quad \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}, \quad (A.2)$$

(ii) class-1 atypical representation,

$$\dim \pi_\mu = 2(\mu_1 - \mu_2) + 3, \quad \mu_1 + \mu_3 + 1 = 0, \quad \mu_1 - \mu_2 \in \mathbb{Z}_{\geq -1}, \quad (A.3)$$

(iii) class-2 atypical representation,

$$\dim \pi_\mu = 2(\mu_1 - \mu_2) + 1, \quad \mu_2 + \mu_3 = 0, \quad \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}. \quad (A.4)$$

11 “Workshop and Summer School: From Statistical Mechanics to Conformal and Quantum Field Theory”, the university of Melbourne, January, 2007 [http://www.smft2007.ms.unimelb.edu.au/program/LectureSeries.html]; meeting of the Physical Society of Japan, March, 2007; "Physics and Mathematics of interacting quantum systems in low dimensions", the University of Tokyo (Kashiwa), 24-26 May, 2007 [http://oshikawa.issp.u-tokyo.ac.jp/pmiqs/poster.html]; La 79eme Rencontre entre physiciens theo-riciens et mathemati-ciens “Supersymmetry and Integrability”, IRMA Strasbourg, June, 2007 [http://wwwirma.u-strasbg.fr/article383.html]; Annual Statistical Mechanics Meeting, Australian National University, December, 2007.
Note that the case $\mu_1 - \mu_2 = -1$ in (A.3) is special; see (A.13) below.

### A.1.1 Typical representations

Below we present the action of the generators of $U_q(\mathfrak{gl}(2|1))$ on a basis of $\tilde{\pi}_\mu$. Let $2l = \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}$. Here we use a (slightly modified) basis from [81]. There are 4 sets of vectors

\[
\{w_j^{(1)}\}_{j=0}^{2l} \subset V^{(1)}, \quad \{w_j^{(2)}\}_{j=0}^{2l+1} \subset V^{(2)}, \quad \{w_j^{(3)}\}_{j=0}^{2l-1} \subset V^{(3)}, \quad \{w_j^{(4)}\}_{j=0}^{2l} \subset V^{(4)}, \quad \text{(A.5)}
\]

where $V^{(a)}$, $a = 1, 2, 3, 4$ denote the vector spaces spanned by these sets. Below we allow the index $j$ to take arbitrary integer values and assume that $w_j^{(a)} \equiv 0$, if $j$ lies outside the intervals specified in (A.5) for each value of $a = 1, 2, 3, 4$. The parities are $p(V^{(1)}) = p(V^{(4)}) = 0$, $p(V^{(2)}) = p(V^{(3)}) = 1$. For $j \in \mathbb{Z}_{\geq 0}$, the action of the generators of $U_q(\mathfrak{gl}(2|1))$ on these vectors reads

\[
(E_{11}, E_{22}, E_{33}) \ w_j^{(1)} = (\mu_1 - j, \mu_2 + j, \mu_3) \ w_j^{(1)},
\]

\[
(E_{11}, E_{22}, E_{33}) \ w_j^{(2)} = (\mu_1 - j, \mu_2 - 1 + j, \mu_3 + 1) \ w_j^{(2)},
\]

\[
(E_{11}, E_{22}, E_{33}) \ w_j^{(3)} = (\mu_1 - 1 - j, \mu_2 + j, \mu_3 + 1) \ w_j^{(3)},
\]

\[
(E_{11}, E_{22}, E_{33}) \ w_j^{(4)} = (\mu_1 - 1 - j, \mu_2 - 1 + j, \mu_3 + 2) \ w_j^{(4)},
\]

\[
E_{12} w_j^{(1)} = [j] w_j^{(1)} - 1, \quad E_{12} w_j^{(2)} = [j] w_j^{(2)},
\]

\[
E_{12} w_j^{(3)} = [j] w_j^{(3)} - 1, \quad E_{12} w_j^{(4)} = [j] w_j^{(4)},
\]

\[
E_{21} w_j^{(1)} = [2l - j] w_j^{(1)} - 1, \quad E_{21} w_j^{(2)} = [2l - 1 - j] w_j^{(2)} - 1,
\]

\[
E_{21} w_j^{(3)} = [2l - 1 - j] w_j^{(3)} - 1, \quad E_{21} w_j^{(4)} = [2l - j] w_j^{(4)} - 1,
\]

\[
E_{23} w_j^{(1)} = 0, \quad E_{23} w_j^{(2)} = [\mu_2 + \mu_3][2l + 1 - j] w_j^{(1)},
\]

\[
E_{23} w_j^{(3)} = [\mu_1 + \mu_3 + 1][2l + 1] w_j^{(1)} - [\mu_2 + \mu_3][2l - j] w_j^{(3)},
\]

\[
E_{32} w_j^{(1)} = w_j^{(2)} + [j] w_j^{(3)} - 1, \quad E_{32} w_j^{(2)} = [j] w_j^{(4)} - 1, \quad E_{32} w_j^{(3)} = -w_j^{(4)}, \quad E_{32} w_j^{(4)} = 0,
\]

\[
E_{31} w_j^{(1)} = q^{-2\mu_3} \left\{ -q^{-1-\mu_1} w_j^{(2)} + q^{-\mu_2}[2l - j] w_j^{(3)} \right\}, \quad \text{(A.6)}
\]

\[
E_{31} w_j^{(2)} = q^{-1+\mu_2 - 2\mu_3} [2l + 1 - j] w_j^{(4)},
\]

\[
E_{31} w_j^{(3)} = q^{-2+\mu_2 - 2\mu_3} w_j^{(4)} + 1, \quad E_{31} w_j^{(4)} = 0,
\]

\[
E_{13} w_j^{(1)} = 0, \quad E_{13} w_j^{(2)} = -q^{1+\mu_1 + 2\mu_3}[j][\mu_2 + \mu_3] w_j^{(1)},
\]

\[
E_{13} w_j^{(3)} = q^{\mu_2 + 2\mu_3}[\mu_1 + \mu_3 + 1] w_j^{(1)},
\]

\[
E_{13} w_j^{(4)} = q^{1+2\mu_3} \left\{ q^{\mu_2}[\mu_1 + \mu_3 + 1] w_j^{(2)} + q^{1+\mu_1}[j][\mu_2 + \mu_3] w_j^{(3)} \right\}.
\]
The vector \( w_0^{(1)} \) is the highest weight vector \(^{12}\) (denoted as \(|0\rangle \) in (A.1)). When the weights \((\mu_1, \mu_2, \mu_3)\) meet the conditions (A.2) the above formulae define a typical irreducible representation \( \pi_\mu \) of the dimension \( \dim \pi_\mu = 4(\mu_1 - \mu_2 + 1) \).

**A.1.2 The 4-dimensional typical representation**

The typical representation \( \pi_{(\mu_1, \mu_1, \mu_3)} \) is 4-dimensional; it is spanned by the vectors \( w_0^{(1)}, w_0^{(2)}, w_1^{(2)}, w_0^{(4)} \). From (A.6), one obtains

\[
\begin{align*}
(E_{11}, E_{22}, E_{33}) w_0^{(1)} &= (\mu_1, \mu_1, \mu_3) w_0^{(1)}, \\
(E_{11}, E_{22}, E_{33}) w_0^{(2)} &= (\mu_1, \mu_1 - 1, \mu_3 + 1) w_0^{(2)}, \\
(E_{11}, E_{22}, E_{33}) w_1^{(2)} &= (\mu_1 - 1, \mu_1, \mu_3 + 1) w_1^{(2)}, \\
(E_{11}, E_{22}, E_{33}) w_0^{(4)} &= (\mu_1 - 1, \mu_1 - 1, \mu_3 + 2) w_0^{(4)}, \\
E_{12} w_1^{(2)} &= w_0^{(2)}, \\
E_{23} w_0^{(2)} &= [\mu_1 + \mu_3] w_0^{(1)}, \\
E_{32} w_0^{(1)} &= w_0^{(2)}, \\
E_{31} w_0^{(1)} &= -q^{-1-\mu_1-2\mu_3} w_1^{(2)}, \\
E_{13} w_1^{(2)} &= -q^{1+\mu_1+2\mu_3} [\mu_1 + \mu_3] w_0^{(1)}, \\
E_{13} w_0^{(4)} &= q^{1+\mu_1+2\mu_3} [\mu_1 + \mu_3 + 1] w_0^{(2)}.
\end{align*}
\]

where all other matrix element vanish.

**A.1.3 Class-1 atypical representation**

When the weights fall to the case (A.3) the module \( \hat{\pi}_{(\mu_1, \mu_2, -1-\mu_1)} \) has an invariant subspace \((V^{(3)} + V^{(4)})\), where \( V^{(a)} \) is defined in (A.4). The factor space \( \hat{\pi}_{(\mu_1, \mu_2, -1-\mu_1)}/(V^{(3)} + V^{(4)}) \) corresponds to the \((2(\mu_1 - \mu_2) + 3)\)-dimensional class-1 atypical representation \( \pi_0^{[0]}_{(\mu_1, \mu_2, -1-\mu_1)} \).

The action of the generators of \( U_q(\mathfrak{gl}(2|1)) \) is obtained from (A.6) by dropping the vectors \( w_j^{(3)} \) and \( w_j^{(4)} \). For \( j \in \mathbb{Z}_{\geq 0} \) one obtains

\[
\begin{align*}
(E_{11}, E_{22}, E_{33}) w_j^{(1)} &= (\mu_1 - j, \mu_2 + j, -1 - \mu_1) w_j^{(1)}, \\
(E_{11}, E_{22}, E_{33}) w_j^{(2)} &= (\mu_1 - j, \mu_2 - 1 + j, -\mu_1) w_j^{(2)}, \\
E_{12} w_j^{(1)} &= [j] w_{j-1}^{(1)}, \\
E_{21} w_j^{(1)} &= [2l - j] w_{j+1}^{(1)}, \\
E_{23} w_j^{(2)} &= -[2l + 1 - j] w_j^{(1)}, \\
E_{31} w_j^{(1)} &= -q^{1+\mu_1} w_j^{(2)}, \\
E_{13} w_j^{(2)} &= q^{1-\mu_1} [j] w_j^{(1)},
\end{align*}
\]

where \( 2l = \mu_1 - \mu_2 \) and all other matrix elements vanish. The basis of this representation consists of the vectors

\[
\{ w_j^{(1)} \}_{j=0}^{2l} \subset V^{(1)}, \quad \{ w_j^{(2)} \}_{j=0}^{2l+1} \subset V^{(2)}.
\]

\(^{12}\)The vector \( w_0^{(2)} \) is the highest weight vector when \( \mu_1 - \mu_2 = -1 \).
Note, that \( w^{(2)}_{2l+1} \) is the lowest weight vector
\[
E_{21} w^{(2)}_{2l+1} = E_{32} w^{(2)}_{2l+1} = 0, \quad (E_{11}, E_{22}, E_{33}) w^{(2)}_{2l+1} = (\mu_2 - 1, \mu_1, -\mu_1) w^{(2)}_{2l+1}.
\] (A.10)

Further, for \( \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 1} \) the invariant subspace \( V^{(3)} + V^{(4)} \) is isomorphic to \( \widehat{\pi}^{[1]}_{(\mu_1 - 1, \mu_2 - 1 - \mu_1)} \) (while in the special point \( \mu_1 - \mu_2 = 0 \) it is isomorphic to \( \pi^{[0]}_{(\mu_1 - 1, \mu_1 - 1, 1 - \mu_1)} \)). In this way one obtains
\[
\frac{\widehat{\pi}^{[1]}_{(\mu_1, \mu_2, 1 - \mu_1)}}{\pi^{[1]}_{(\mu_1 - 1, \mu_2 - 1 - \mu_1)}} \simeq \frac{\pi^{[0]}_{(\mu_1, \mu_2, 1 - \mu_1)}}{\pi^{[0]}_{(\mu_1 - 1, \mu_1 - 1, 1 - \mu_1)}}, \quad \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 1},
\] (A.11)
\[
\frac{\widehat{\pi}^{[0]}_{(\mu_1, \mu_1 - 1 - \mu_1)}}{\pi^{[0]}_{(\mu_1 - 1, \mu_1 - 1, 1 - \mu_1)}} \simeq \frac{\pi^{[0]}_{(\mu_1, \mu_1 - 1, 1)}}{\pi^{[0]}_{(\mu_1 - 1, \mu_1 - 1, 1 - \mu_1)}}, \quad \mu_1 - \mu_2 = 0.
\] (A.12)

In the special case \( \mu_1 - \mu_2 = -1 \) the representation \( \widehat{\pi}^{(\mu_1, \mu_1 + 1, -\mu_1 - 1)} \) is one-dimensional. It is spanned by the only vector \( w^{(2)}_0 \) whose weight is equal to \((\mu_1, \mu_1, -\mu_1)\) and the parity is odd,
\[
\frac{\widehat{\pi}^{[1]}_{(\mu_1, \mu_1, 1 - \mu_1)}}{\pi^{[1]}_{(\mu_1, \mu_1, 1)}},
\] (A.13)

### A.1.4 Class-2 atypical representation

As before, let \( V^{(a)} \), \( a = 1, 2, 3, 4 \), be the vector spaces, defined in [A.5]. In the case [A.4] the module \( \widehat{\pi}^{(\mu_1, \mu_2, -\mu_2)} \) is reducible; it contains an invariant subspace \( V^{(2)} \oplus V^{(4)} \). The factor space \( \pi^{[0]}_{(\mu_1, \mu_2, -\mu_2)} / (V^{(2)} \oplus V^{(4)}) \) corresponds to the \( (2\mu_1 - \mu_2 + 1) \)-dimensional class-2 atypical representation \( \pi^{[0]}_{(\mu_1, \mu_2, -\mu_2)} \). To get explicit expressions for its matrix elements, one simply drops all vectors \( w^{(2)}_j \) and \( w^{(4)}_j \) from (A.6). For \( j \in \mathbb{Z}_{\geq 0} \) one obtains
\[
(E_{11}, E_{22}, E_{33}) w^{(3)}_j = \begin{cases} 
(\mu_1 - j, \mu_2 + j, -\mu_2) w^{(1)}_j, & \text{if } j \leq 0, \\
(\mu_1 - 1 - j, \mu_2 + j, -\mu_2 + 1) w^{(3)}_j, & \text{if } j > 0.
\end{cases}
\] (A.14)

E.g.\[
E_{12} w^{(1)}_j = [j] w^{(1)}_{j-1}, \quad E_{12} w^{(3)}_j = [j] w^{(3)}_{j-1},
\]
\[
E_{21} w^{(1)}_j = [2l - j] w^{(1)}_{j+1}, \quad E_{21} w^{(3)}_j = [2l - 1 - j] w^{(3)}_{j+1},
\]
\[
E_{32} w^{(3)}_j = w^{(1)}_{j+1}, \quad E_{32} w^{(1)}_j = [j] w^{(3)}_{j-1},
\]
\[
E_{31} w^{(1)}_j = q^{-\mu_2} [2l - j] w^{(3)}_j, \quad E_{31} w^{(3)}_j = q^{-\mu_2} w^{(1)}_j,
\]
where all other matrix elements vanish. The basis of this representation consists of the vectors
\[
\{w^{(1)}_j\}_{j=0}^{2l} \subset V^{(1)}, \quad \{w^{(3)}_j\}_{j=0}^{2l-1} \subset V^{(3)},
\] (A.15)
of the parities are \( p(w^{(1)}_j) = 0 \) and \( p(w^{(3)}_j) = 1 \). For \( 2l = \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 1} \) the vector \( w^{(3)}_{2l-1} \) is the lowest weight vector
\[
E_{21} w^{(3)}_{2l-1} = E_{32} w^{(3)}_{2l-1} = 0, \quad (E_{11}, E_{22}, E_{33}) w^{(3)}_{2l-1} = (\mu_2, \mu_1 - 1, 1 - \mu_2) w^{(3)}_{2l-1}.
\] (A.16)
A simple inspection shows that the invariant subspace \( \langle V^{(2)} \otimes V^{(4)} \rangle \) of \( \hat{\pi}_{(\mu_1, \mu_2, -\mu_2)} \) is isomorphic to \( \pi^{[1]}_{(\mu_1, \mu_2, 1, -1, -\mu_2)} \) just defined in \( \text{(A.14)} \). Thus, one obtains

\[
\hat{\pi}_{(\mu_1, \mu_2, -\mu_2)}/\pi^{[1]}_{(\mu_1, \mu_2, 1, -1, -\mu_2)} \simeq \pi^{[0]}_{(\mu_1, \mu_2, -\mu_2)}, \quad \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}.
\]  
\( \text{(A.17)} \)

**A.2 Infinite-dimensional representations of \( U_q(gl(2|1)) \)**

Below we will not assume any integrality conditions for the difference \( \mu_1 - \mu_2 \), unless otherwise is explicitly stated. Introduce two types of infinite-dimensional representations by the formulae \( \text{(A.8)} \) and \( \text{(A.14)} \) assuming that the weights \( \mu_1, \mu_2 \in \mathbb{C} \) are now arbitrary and the index \( j \) takes an infinite number of integer values \( j = 0, 1, 2, \ldots, \infty \). With these conventions

(iv) Eq. \( \text{(A.8)} \) defines a class-1 infinite-dimensional representation \( \pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)} \),

(v) Eq. \( \text{(A.14)} \) defines a class-2 infinite-dimensional representations \( \pi^{+}_{(\mu_1, \mu_2, -\mu_2)} \).

**A.2.1 Class-1 infinite-dimensional representation \( \pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)} \)**

The action of generators for this representation is defined by \( \text{(A.8)} \), the same formulae as for the finite dimensional representation \( \pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)} \), except that the index \( j \) now runs infinitely many values \( j \in \mathbb{Z}_{\geq 0} \). The basis consists of the vectors \( \{ w^{(1)}_j \}_{j=0}^{\infty} \) and \( \{ w^{(2)}_j \}_{j=0}^{\infty} \) with the parities \( p(w^{(1)}_j) = 0, p(w^{(2)}_j) = 1 \). The highest weight vector is \( w^{(1)}_0 \). We assume also that \( w^{(1)}_j \equiv 0, w^{(2)}_j \equiv 0, \) if \( j < 0 \).

For generic values of \( \mu_1, \mu_2 \) the representation \( \pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)} \) is irreducible. It becomes reducible iff \( \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 1} \). First consider the main case \( \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0} \). In this case \( \pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)} \) contains the class-1 atypical finite dimensional representation \( \pi^{[0]}_{(\mu_1, \mu_2, -1 - \mu_1)} \) as an invariant subspace, while the factor-module

\[
\pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)}/\pi^{[0]}_{(\mu_1, \mu_2, -1 - \mu_1)} \simeq \pi^{+}_{(\mu_1, \mu_2, -1 - \mu_1)}, \quad \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0},
\]  
\( \text{(A.18)} \)

is isomorphic to the infinite-dimensional class-2 representation \( \pi^{+}_{(\mu_2 - 1 - \mu_1, \mu_1, -1 - \mu_1)} \), briefly described above in the beginning of Sect. \( \text{A.2} \) (see Sect. \( \text{A.2.2} \) for more details). To see this let us write basis vectors in the factor module \( \pi^{-}_{(\mu_1, \mu_2, -1 - \mu_1)}/\pi^{[0]}_{(\mu_1, \mu_2, -1 - \mu_1)} \) as \( v^{(1)}_j = w^{(1)}_{j+2l+1} \) and \( v^{(2)}_j = w^{(2)}_{j+2l+2} \), \( j = 0, 1, 2, \ldots, \infty \). Then from \( \text{(A.8)} \) one obtains

\[
(E_{11}, E_{22}, E_{33}) v^{(1)}_j = (\mu_2 - j - 1, \mu_1 + j + 1, -1 - \mu_1) v^{(1)}_j,
\]

\[
(E_{11}, E_{22}, E_{33}) v^{(2)}_j = (\mu_2 - j - 2, \mu_1 + j + 1, -\mu_1) v^{(2)}_j,
\]

\[
E_{12} v^{(1)}_j = [j + 2l + 1] v^{(1)}_{j-1}, \quad E_{12} v^{(2)}_j = [j + 2l + 2] v^{(2)}_{j-1},
\]

\[
E_{21} v^{(1)}_j = -[j + 1] v^{(1)}_{j+1}, \quad E_{21} v^{(2)}_j = -[j + 1] v^{(2)}_{j+1},
\]

\[
E_{23} v^{(2)}_j = [j + 1] v^{(1)}_{j+1}, \quad E_{32} v^{(1)}_j = v^{(2)}_{j-1},
\]

\[
E_{31} v^{(1)}_j = -q^{1+\mu_1} v^{(2)}_j, \quad E_{13} v^{(2)}_j = q^{-1-\mu_1} [j + 2l + 2] v^{(1)}_j.
\]  
\( \text{(A.19)} \)

\( 2l = \mu_1 - \mu_2 \) and all other matrix elements vanish. It is not difficult to check that \( \text{(A.19)} \) becomes identical to \( \text{(A.14)} \), provided the quantities \( \mu_1, \mu_2 \) and \( 2l \) in \( \text{(A.14)} \) are replaced with \( \mu_2 - 1, \)
\[ w^{(1)}_j = \frac{[2l + 2]}{[j + 2l + 1]} v^{(1)}_j, \quad w^{(3)}_j = \frac{u^{(2)}_j}{[j + 2l + 2]} . \quad (A.20) \]

In the special case \( \mu_1 - \mu_2 = -1 \) the invariant subspace of \( \pi^-_{(\mu_1, \mu_2, -1, \mu_1)} \) is one-dimensional; the analog of \( (A.18) \) reads

\[ \pi^-_{(\mu_1, \mu_1 + 1, -\mu_1 - 1)} \simeq \pi^+_{(\mu_1, \mu_1 + 1, -\mu_1 - 1)}, \quad \mu_1 - \mu_2 = -1, \quad (A.21) \]

**A.2.2 Class-2 infinite-dimensional representation \( \pi^+_{(\mu_1, \mu_2, -\mu_2)} \)**

This representation is defined by \( (A.14) \), i.e. by the same formulae as for the finite dimensional representation \( \pi_{(\mu_1, \mu_2, -\mu_2)} \), except that the index \( j \) now runs infinitely many values \( j \in \mathbb{Z}_{\geq 0} \). The basis consists of the vectors \( \{ w^{(1)}_j \}_{j=0}^\infty \) and \( \{ w^{(3)}_j \}_{j=0}^\infty \) with the parities \( p(w^{(1)}_j) = 0 \), \( p(w^{(3)}_j) = 1 \). The highest weight vector is \( w^{(1)}_0 \). We assume also that \( w^{(1)}_j \equiv 0, w^{(3)}_j \equiv 0 \), if \( j < 0 \).

For generic values of \( \mu_1, \mu_2 \) the representation \( \pi^+_{(\mu_1, \mu_2, -\mu_1)} \) is irreducible. It becomes reducible iff \( \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0} \). In this case it contains the class-2 atypical finite dimensional representation \( \pi^{[0]}_{(\mu_1, \mu_2, -\mu_1)} \) as an invariant subspace, while the factor-module

\[ \pi^+_{(\mu_1, \mu_2, -\mu_2)}/\pi^{[0]}_{(\mu_1, \mu_2, -\mu_2)} \simeq \pi_{(\mu_2 - 1, \mu_1 + 1, -\mu_2)}, \quad 2l = \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}, \quad (A.22) \]

is isomorphic to the infinite-dimensional class-1 representation \( \pi^-_{(\mu_2 - 1, \mu_1 + 1, -\mu_2)} \), considered above in Sect.\( \text{A.2.1} \). To see this let us write basis vectors in the factor module \( \pi^+_{(\mu_1, \mu_2, -\mu_2)}/\pi^{[0]}_{(\mu_1, \mu_2, -\mu_2)} \) as \( u^{(1)}_j = w^{(1)}_{j + 2l + 1} \) and \( u^{(3)}_j = w^{(3)}_{j + 2l}, \quad j = 0, 1, 2, \ldots \infty \). Then from \( (A.14) \) one obtains

\[ (E_{11}, E_{22}, E_{33}) \ u^{(1)}_j = ( \mu_2 - j - 1, \mu_1 + j + 1, -\mu_2 ) \ u^{(1)}_j , \]
\[ (E_{11}, E_{22}, E_{33}) \ u^{(3)}_j = ( \mu_2 - 1 - j, \mu_1 + j, -\mu_2 + 1 ) \ u^{(3)}_j , \]
\[ E_{12} u^{(1)}_j = [j + 2l + 1] u^{(1)}_{j+1}, \quad E_{12} u^{(3)}_j = [j + 2l] u^{(3)}_{j+1}, \]
\[ E_{21} u^{(1)}_j = -[j + 1] u^{(1)}_{j+1}, \quad E_{21} u^{(3)}_j = -[j + 1] u^{(3)}_{j+1} , \]
\[ E_{23} u^{(3)}_j = u^{(1)}_j , \quad E_{32} u^{(1)}_j = [j + 2l] u^{(1)}_{j+1} , \]
\[ E_{31} u^{(1)}_j = -q^{\mu_2} [j + 1] u^{(3)}_{j+1}, \quad E_{13} u^{(3)}_j = q^{-\mu_2} u^{(1)}_{j+1} . \quad (A.23) \]

\( 2l = \mu_1 - \mu_2 \) and all other matrix elements vanish. It is not difficult to check that \( (A.23) \) becomes identical to \( (A.8) \), provided the quantities \( \mu_1, \mu_2 \) and \( 2l \) in \( (A.8) \) are replaced with \( \mu_2 - 1, \mu_1 + 1 \) and \(-2 - 2l\), respectively, and the basis in \( (A.8) \) is re-scaled as

\[ w^{(1)}_j = u^{(1)}_j/[j + 2l + 1], \quad w^{(2)}_j = [2l + 1] u^{(3)}_j/[j + 2l] . \quad (A.24) \]

**Appendix B. Representations of the Borel subalgebra \( \mathcal{B}_+(U_q(sl(2|1))) \)**
First we will consider the evaluation representations of $B_+$, based on a composition of the evaluation map (2.28) and the representations of the (non-affine) algebra $U(gl(2|1))$, described in the previous subsections. Next we will consider representations obtained by the composition of the maps (5.11) with the Fock representations of the oscillator algebras (2.48) and (2.49). This leads to the following set of representations (which are all used in this paper)

- **Finite dimensional representations**
  
  (I) typical $\pi_{(\mu_1, \mu_2, \mu_3)}(x)$, \quad $\dim = 4$,  
  
  (II) class-1 atypical $\pi_{(\mu_1, \mu_2, -\mu_1-1)}(x)$, \quad $\dim = 2(\mu_1 - \mu_2) + 3$, $\mu_1 - \mu_2 \in \mathbb{Z}_{\geq -1}$,  
  
  (III) class-2 atypical $\pi_{\mu_1, \mu_2, -\mu_2}(x)$, \quad $\dim = 2(\mu_1 - \mu_2) + 1$, $\mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}$,  
  
  (IV) oscillator-type $W_3(x)$ and $\overline{W}_3(x)$, \quad $\dim = 4$.

- **Infinite-dimensional representations**
  
  (V) class-1 atypical $\pi_{(\mu_1, \mu_2, -\mu_1-1)}(x)$,  
  
  (VI) class-2 atypical $\pi_{\mu_1, \mu_2, -\mu_2}(x)$,  
  
  (VII) oscillator-type $W_1(x)$, $\overline{W}_1(x)$, $W_2(x)$, $\overline{W}_2(x)$.

**Remark:** There is a trivial isomorphism involving a shift of the spectral parameter for any evaluation representation $V_{(\nu_1, \nu_2, \nu_3)}(x)$ of $B_+$ with spectral parameter $x$ and the highest weight $(\nu_1, \nu_2, \nu_3)$

\[ V_{(\nu_1, \nu_2, \nu_3)}(x) \simeq V_{(\nu_1+\eta, \nu_2+\eta, \nu_3-\eta)}(xq^{-\eta}). \]  

(B.1)

**B.1 Atypical representations of $B_+$**

**B.1.1 Class 2 representations**

Let us start from the class-2 representations. Both the finite dimensional $\pi_{\mu_1, \mu_2, -\mu_2}(x)$, $\mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0}$, and the infinite-dimensional $\pi_{\mu_1, \mu_2, -\mu_2}^+(x)$, $\mu_1 - \mu_2 \in \mathbb{C}$, representations are defined by the same formulae (which follow from (2.28) and (A.14))

\[
(h_0, h_1, h_2) \; w_j^{(1)} = (j - 2l, \quad 2l - 2j, \quad j) \; w_j^{(1)}, \\
(h_0, h_1, h_2) \; w_j^{(3)} = (j - 2l, \quad 2l - 2j - 1, \quad j + 1) \; w_j^{(3)}, \\
e_0 w_j^{(1)} = -xq^{\mu_2}[2l - j]w_j^{(3)}, \quad e_1 w_j^{(1)} = [j]w_{j-1}^{(1)}, \quad e_2 w_j^{(1)} = 0, \\
e_0 w_j^{(3)} = 0, \quad e_1 w_j^{(3)} = [j]w_{j-1}^{(3)}, \quad e_2 w_j^{(3)} = w_{j+1}^{(1)}, \]  

\tag{B.2}

where $2l = \mu_1 - \mu_2$ and $w_k^{(1)} = w_k^{(3)} \equiv 0$ if $k < 0$. The parity of the vectors is $p(w_j^{(1)}) = 0$ and $p(w_j^{(3)}) = 1$. The index $j$ takes any non-negative integer values $j \in \mathbb{Z}_{\geq 0}$ in the infinite-dimensional case and a restricted finite set of values in finite dimensional case, corresponding to the basis vectors (A.15).

**B.1.2 Class 1 representations**

Both the finite dimensional $\pi_{(\mu_1, \mu_2, -\mu_1-1)}(x)$, with $\mu_1 - \mu_2 \in \mathbb{Z}_{\geq -1}$, and the infinite-dimensional $\pi_{(\mu_1, \mu_2, -\mu_1-1)}^+(x)$, with $\mu_1 - \mu_2 \in \mathbb{C}$, representations are defined by the same formulae
Explicitly, one obtains the map (2.28) with the relations (A.7). It is spanned by four basis vectors

\[ B \]

Typical representation of under the symmetry transformation \( \sigma \) where

\[ \pi \]

and \( p \)\( l \) \((\text{which follow from (2.28) and (A.8)})\)

\[ \begin{align*}
(h_0, h_1, h_2) \, w_j^{(1)} &= (j + 1, \quad 2l - 2j, \quad -2l + j - 1) \, w_j^{(1)}, \\
(h_0, h_1, h_2) \, w_j^{(2)} &= (j, \quad 2l - 2j + 1, \quad -2l + j - 1) \, w_j^{(2)}, \\
e_0 w_j^{(1)} &= xq^{\mu+1}w_{j+1}^{(2)}, \quad e_1 w_j^{(1)} = [j]w_{j-1}^{(1)}, \quad e_2 w_j^{(1)} = 0, \\
e_0 w_j^{(2)} = 0, \quad e_1 w_j^{(2)} = [j]w_{j-1}^{(2)}, \quad e_2 w_j^{(2)} = -[2l + 1 - j]w_j^{(1)},
\end{align*} \]  

(B.3)

where \( l = (\mu_1 - \mu_2)/2; \quad w_k^{(1)} = w_k^{(2)} = 0 \) if \( k \notin \mathbb{Z}_{>0} \). The parity of the vectors is \( p(w_j^{(1)}) = 0 \) and \( p(w_j^{(2)}) = 1 \). The index \( j \) takes any non-negative integer values \( j \in \mathbb{Z}_{\geq 0} \) in the infinite-dimensional case and a restricted finite set of values in finite dimensional case, corresponding to the basis vectors (A.9).

**B.1.3 Reductions**

For integer values of \( \mu_1 - \mu_2 \) the infinite dimensional representations \( \pi_{(\mu_1, \mu_2, -\mu_2)}^{+}(x) \) and \( \pi_{(\mu_1, \mu_2, -\mu_1 - 1)}(x) \) become reducible. Namely, it follows from (A.18), (A.21) and (A.22) that

\[ \begin{align*}
\pi_{(\mu_1, \mu_2, -\mu_2)}^{+}(x)/\pi_{(\mu_1, \mu_2, -\mu_2)}^{[0]}(x) &\simeq \pi_{(\mu_2 - 1, \mu_1 + 1, -\mu_2)}^{-}(x), \\
\pi_{(\mu_1, \mu_2, -\mu_1 - 1)}^{-}(x)/\pi_{(\mu_1, \mu_2, -\mu_1 - 1)}^{[0]}(x) &\simeq \pi_{(\mu_2 - 1, \mu_1 + 1, -\mu_1 - 1)}^{+}(x) , \\
\pi_{(\mu_1, \mu_1 + 1, -\mu_1 - 1)}^{-}(x)/\pi_{(\mu_1, \mu_1, -\mu_1)}^{[0]}(x) &\simeq \pi_{(\mu_1, \mu_1 + 1, -\mu_1 - 1)}^{+}(x),
\end{align*} \]  

(B.4)

where \( \mu_1, \mu_2 \in \mathbb{C} \), and \( \mu_1 - \mu_2 \in \mathbb{Z}_{\geq 0} \).

**B.1.4 Symmetry**

There is a correspondence

\[ \sigma_{02}[\pi_{(\mu_1, \mu_2, -\mu_2)}^{+}(x)] \simeq \pi_{(-1 - \mu_2, -\mu_1, \mu_2)}^{-}(q^{2\mu_2} x) \]  

(B.5)

under the symmetry transformation \( \sigma_{02} \) discussed in Section 5.1, \( p = 0, 1 \text{ (mod 2)} \).

**B.2 Typical representation of \( \mathcal{B}_{+} \)**

The 4-dimensional representation \( \pi_{(\mu_1, \mu_1, \mu_3)}(x), \mu_1, \mu_3 \in \mathbb{C}, \) is obtained by the composition of the map (2.28) with the relations (A.7). It is spanned by four basis vectors \( w_0^{(1)}, w_0^{(2)}, w_1^{(2)}, w_0^{(4)} \). Explicitly, one obtains

\[ \begin{align*}
(h_0, h_1, h_2) \, w_0^{(1)} &= (\quad -\mu_1 - \mu_3, \quad 0, \quad \mu_1 + \mu_3 \quad) \, w_0^{(1)}, \\
(h_0, h_1, h_2) \, w_0^{(2)} &= (\quad -\mu_1 - \mu_3 - 1, \quad 1, \quad \mu_1 + \mu_3 \quad) \, w_0^{(2)},  \\
(h_0, h_1, h_2) \, w_1^{(2)} &= (\quad -\mu_1 - \mu_3, \quad -1, \quad \mu_1 + \mu_3 + 1 \quad) \, w_1^{(2)}, \\
(h_0, h_1, h_2) \, w_0^{(4)} &= (\quad -\mu_1 - \mu_3 - 1, \quad 0, \quad \mu_1 + \mu_3 + 1 \quad) \, w_0^{(4)},
\end{align*} \]  

(B.6a)

\[ \begin{align*}
e_0 w_0^{(1)} &= xq^{-1-\mu_1-2\mu_3}w_1^{(2)}, \quad e_0 w_0^{(2)} = -xq^{-1-\mu_1-2\mu_3}w_0^{(4)}, \\
e_1 w_1^{(2)} = w_0^{(2)}, \quad e_2 w_0^{(2)} = [\mu_1 + \mu_3]w_0^{(1)}, \quad e_2 w_0^{(4)} = [\mu_1 + \mu_3 + 1]w_1^{(2)},
\end{align*} \]  

(B.6b)

52
where all omitted matrix elements vanish. The parities of the vectors read \( p(w_0^{(1)}) = p(w_0^{(4)}) = 0, \ p(w_0^{(2)}) = p(w_1^{(2)}) = 1 \). The representation is irreducible, except when \( (\mu_1 + \mu_3) = 0, -1 \). In the latter case one obtains \(^{13}\)

\[
\pi_{(\mu_1,\mu_1,-\mu_1)}(x)/\pi_{(\mu_1,\mu_1-1,-\mu_1)}(x) \simeq \pi_{(\mu_1,\mu_1,-\mu_1)}(x), \quad \mu_1 + \mu_3 = 0, \tag{B.7}
\]

\[
\pi_{(\mu_1,\mu_1,-\mu_1)}(x)/\pi_{(\mu_1-1,\mu_1-1,-\mu_1)}(x) \simeq \pi_{(\mu_1,\mu_1,-\mu_1)}(x), \quad \mu_1 + \mu_3 = -1. \tag{B.8}
\]

**B.3 Oscillator representations of \( \mathcal{B}_+ \)**

Here we list explicit forms of the basis for the Fock representations introduced in section \(^5\)

(a) module \( \overline{W}_1^{++}(x) \) based on \( \overline{\mathcal{F}}_1(x) \),

\[
|m,n>_{++} = (b_1^n)(f_2^n)|0>_{++}, \quad m \in \mathbb{Z}_{\geq 0}, \quad n \in \{0,1\},
\]

\[
e_0|m,n>_{++} = |m,n-1>_{++},
\]

\[
e_1|m,n>_{++} = q^{-n}|m+1,n>_{++},
\]

\[
e_2|m,n>_{++} = \frac{q^{n+\frac{1}{2}}(1-q^{-2m})x}{(q-q^{-1})^2}|m-1,n+1>_{++},
\]

\[
(h_0,h_1,h_2)|m,n>_{++} = (-m,2m+n,-m-n)|m,n>_{++},
\]

where \( | -1,n>_{++} = |m,-1>_{++} = |m,2>_{++} = 0 \) and \( p(|m,n>_{++}) = n \ (\text{mod} \ 2) \).

(b) module \( \overline{W}_2^- (x) \) based on \( \overline{\mathcal{F}}_2(x) \),

\[
|m,n>_{--} = (f_1^n)(b_2^n)|0>_{--}, \quad m \in \{0,1\}, \quad n \in \mathbb{Z}_{\geq 0},
\]

\[
e_0|m,n>_{--} = \frac{q^n(q^n-q^{-n})x}{(q-q^{-1})^2}|m+1,n-1>_{--},
\]

\[
e_1|m,n>_{--} = |m,n+1>_{--},
\]

\[
e_2|m,n>_{--} = q^n|m-1,n>_{--},
\]

\[
(h_0,h_1,h_2)|m,n>_{--} = (-m-n,m+2n,-n)|m,n>_{--},
\]

where \( | -1,n>_{--} = |2,n>_{--} = |m,-1>_{--} = 0 \) and \( p(|m,n>_{--}) = m \ (\text{mod} \ 2) \).

(c) module \( \overline{W}_3^{+-} (x) \) based on \( \overline{\mathcal{F}}_3(x) \),

\[
|m,n>_{+-} = (f_1^n)(f_2^n)|0>_{+-}, \quad m \in \{0,1\},
\]

\[
e_0|m,n>_{+-} = q^n|m-1,n>_{+-},
\]

\[
e_1|m,n>_{+-} = (-1)^{m+1}\frac{q^{n-\frac{1}{2}}x}{q-q^{-1}}|m+1,n+1>_{+-},
\]

\[
e_2|m,n>_{+-} = (-1)^m|m,n-1>_{+-},
\]

\[
(h_0,h_1,h_2)|m,n>_{+-} = (-n,m+n,-m)|m,n>_{+-},
\]

\(^{13}\)Note that we are using the symbol \( \pi \) instead of \( \hat{\pi} \), despite the representation is reducible at these special points.
where \( | -1, n >_{+-} = \{2, n >_{+-} = |m, -1 >_{+-} = |m, 2 >_{+-} = 0 \) and \( p(|m, n >_{+-}) = m + n \) (mod 2).

(d) module \( W_{1}^{--}(x) \) based on \( \rho_{1}'(x) \),

\[
\begin{align*}
|m, n >_{--} & = (b_{1}^{+})^{m}(f_{2}^{-})^{n}|0 >_{--}, \quad m \in \mathbb{Z}_{\geq 0}, \quad n \in \{0, 1\}, \\
e_{0}|m, n >_{--} & = |m, n - 1 >_{--}, \\
e_{1}|m, n >_{--} & = q^{n}|m + 1, n >_{--}, \\
e_{2}|m, n >_{--} & = \frac{q^{-n - \frac{1}{2}}(1 - q^{2m})x}{(q - q^{-1})^{2}}|m - 1, n + 1 >_{--}, \\
(h_{0}, h_{1}, h_{2})|m, n >_{--} & = (-m, 2m + n, -m - n)|m, n >_{--},
\end{align*}
\]

where \( | -1, n >_{--} = \{2, n >_{--} = |m, -1 >_{--} = |m, 2 >_{--} = 0 \) and \( p(|m, n >_{--}) = n \) (mod 2).

(e) module \( W_{2}^{++}(x) \) based on \( \rho_{2}'(x) \)

\[
\begin{align*}
|m, n >_{++} & = (f_{1}^{-})^{m}(b_{2}^{-})^{n}|0 >_{++}, \quad m \in \{0, 1\}, \quad n \in \mathbb{Z}_{\geq 0}, \\
e_{0}|m, n >_{++} & = \frac{q^{-\frac{1}{2}}(q^{n} - q^{-n})x}{(q - q^{-1})^{2}}|m + 1, n - 1 >_{++}, \\
e_{1}|m, n >_{++} & = |m, n + 1 >_{++}, \\
e_{2}|m, n >_{++} & = q^{-n}|m - 1, n >_{++}, \\
(h_{0}, h_{1}, h_{2})|m, n >_{++} & = (-m - n, m + 2n, -n)|m, n >_{++},
\end{align*}
\]

where \( | -1, n >_{++} = \{2, n >_{++} = |m, -1 >_{++} = |m, 2 >_{++} = 0 \) and \( p(|m, n >_{++}) = m \) (mod 2).

(f) module \( W_{3}^{+-}(x) \) based on \( \rho_{3}'(x) \)

\[
\begin{align*}
|m, n >_{+-} & = (f_{1}^{-})^{m}(f_{2}^{+})^{n}|0 >_{+-}, \quad m, n \in \{0, 1\}, \\
e_{0}|m, n >_{+-} & = q^{-n}|m - 1, n >_{+-}, \\
e_{1}|m, n >_{+-} & = (-1)^{m}\frac{q^{n + \frac{1}{2}}x}{q - q^{-1}}|m + 1, n + 1 >_{+-}, \\
e_{2}|m, n >_{+-} & = (-1)^{m}|m, n - 1 >_{+-}, \\
(h_{0}, h_{1}, h_{2})|m, n >_{+-} & = (-n, m + n, -m)|m, n >_{+-},
\end{align*}
\]

where \( | -1, n >_{+-} = \{2, n >_{+-} = |m, -1 >_{+-} = |m, 2 >_{+-} = 0 \) and \( p(|m, n >_{+-}) = m + n \) (mod 2).
(g) module $W_{1^-}^-(x)$ based on $\mathcal{P}_1'(x)$

\[
|m,n >_-= (b_1^+)^m(b_2^-)^n|0 >_-, \ m \in \mathbb{Z}_{\geq 0}, \ n \in \{0,1\},
\]

\[
e_0|m,n >_-= |m,n + 1 >_-,
\]

\[
e_1|m,n >_-= -\frac{q^{m+n}|m|}{q - q^{-1}}m - 1, n >_-,
\]

\[
e_2|m,n >_-= -q^{-\frac{1}{2}}x|m + 1, n - 1 >_-,
\]

\[
(h_0, h_1, h_2)|m,n >_-= (m, -2m - n, m + n)|m,n >_-,
\]

where $| - 1, n >_- = |m, -1 >_- = |m, 2 >_- = 0$ and $p(|m,n >_-) = n \pmod{2}$.

(h) module $W_{2^+}^+(x)$ based on $\mathcal{P}_2(x)$

\[
|m,n >_{++} = (f_1^-)^m(b_2^+)^n|0 >_{++}, \ m \in \{0,1\}, \ n \in \mathbb{Z}_{\geq 0},
\]

\[
e_0|m,n >_{++} = -q^{n+\frac{1}{2}}x|m - 1, n + 1 >_{++},
\]

\[
e_1|m,n >_{++} = \frac{q^{-n}|n|}{q - q^{-1}}m, n - 1 >_{++},
\]

\[
e_2|m,n >_{++} = q^{-n}|m + 1, n >_{++},
\]

\[
(h_0, h_1, h_2)|m,n >_{++} = (m + n, -m - 2n, m)|m,n >_{++},
\]

where $| - 1, n >_{++} = |2, n >_{++} = |m, -1 >_{++} = 0$ and $p(|m,n >_{++}) = m \pmod{2}$.

(i) module $W_{1^+}^+(x)$ based on $\mathcal{P}_1'(x)$

\[
|m,n >_{++} = (b_1^-)^m(f_2^-)^n|0 >_{++}, \ m \in \mathbb{Z}_{\geq 0}, \ n \in \{0,1\},
\]

\[
e_0|m,n >_{++} = |m,n + 1 >_{++},
\]

\[
e_1|m,n >_{++} = \frac{q^{m-n}|m|}{q - q^{-1}}m - 1, n >_{++},
\]

\[
e_2|m,n >_{++} = -q^{\frac{1}{2}}x|m + 1, n - 1 >_{++},
\]

\[
(h_0, h_1h_2)|m,n >_{++} = (m, -2m - n, m + n)|m,n >_{++},
\]

where $| - 1, n >_{++} = |m, -1 >_{++} = |m, 2 >_{++} = 0$ and $p(|m,n >_{++}) = n \pmod{2}$.

(j) module $W_{2^-}^-(x)$ based on $\mathcal{P}_2'(x)$

\[
|m,n >_- = (f_1^+)^m(b_2^-)^n|0 >_- , \ m \in \{0,1\}, \ n \in \mathbb{Z}_{\geq 0},
\]

\[
e_0|m,n >_- = -q^{-n-\frac{1}{2}}x|m - 1, n + 1 >_-,
\]

\[
e_1|m,n >_- = -\frac{q^n|n|}{q - q^{-1}}|m, n - 1 >_-,
\]

\[
e_2|m,n >_- = q^n|m + 1, n >_-,
\]

\[
(h_0, h_1, h_2)|m,n >_- = (m + n, -m - 2n, n)|m,n >_-,
\]

where $| - 1, n >_- = |2, n >_- = |m, -1 >_- = 0$ and $p(|m,n >_-) = m \pmod{2}$.
Appendix C. Quantum affine superalgebra analogue of the first and second Weyl formulae

The so-called Bazhanov-Reshetikhin formula [82] is a determinant expression of the eigenvalue of the transfer matrix for the fusion model for $U_q(\hat{sl}(m))$. This formula allows a supersymmetric generalization for $U_q(\hat{sl}(m|n))$, which may be called the "quantum supersymmetric Jacobi-Trudi and Giambelli formula" (C.11) and (C.12) [33–35] (see also [83] for $U_q(B_r^{(1)})$ case). This is a quantum affine superalgebra analogue of the second Weyl formula for the transfer matrices. It is natural to consider an analogue of the first Weyl formula. Weyl first formula for the superalgebra $gl(m|n)$ is often called "Sergeev-Pragacz formula" in mathematical literature [85, 86] (see (C.31)). Eqs.(C.15)-(C.18) are quantum affine superalgebra analogue of the Sergeev-Pragacz formula.

C.1 Partitions, Young diagrams and admissible tableaux.

Introduce notations for the integer partitions and Young diagrams (see, e.g., [87] for additional details). A partition is a sequence of integers $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_1 \geq \mu_2 \geq \cdots \geq 0$. Repeated entries $k, k, \ldots, k$ of the same integer $k$ in the partition can be abbreviated as $k_{m_k}$, where $m_k$ denotes the corresponding multiplicity. Two partitions are regarded equivalent if all their non-zero elements coincide. For example, $(3, 3, 2, 1, 1, 0, 0) = (3, 3, 2, 1, 1) = (3^2, 2, 1^2)$. Partitions can be visualized by Young diagrams, formed by rows of identical square boxes in the plane. The Young diagram $\mu$, corresponding to a partition $\mu$, has $\mu_k$ boxes in the $k$-th row, see Fig. 1. Individual boxes are referred to by integer coordinates $(i,j) \in \mathbb{Z}^2$, where the row index $i$ increases downwards while the column index $j$ increases from left to right. The top left corner of $\mu$ has coordinates $(1, 1)$. The partition $\mu' = (\mu'_1, \mu'_2, \ldots)$ is called conjugate of $\mu$, where $\mu'_j$ is defined as the maximal integer $k$ such that $\mu_k \geq j$. The Young diagrams for conjugated partitions are obtained from each other by the transposition of rows and columns, as in example in Fig. 2.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be two partitions such that $\mu_i \geq \lambda_i : i = 1, 2, \ldots$ and $\lambda_{\mu'_1} = \lambda_{\mu_1}' = 0$. We denote a skew-Young diagram defined by these two partitions as $\lambda \subset \mu$. This is the domain obtained by the subtraction $\mu - \lambda$ as in the example in Fig. 3. If $\lambda$ is an empty set $\phi$, then $\lambda \subset \mu$ coincides with $\mu$. Individual boxes on the skew-Young diagram $\lambda \subset \mu$ are referred to by their coordinates on $\mu$.

Next, define a space of admissible tableaux $\text{Tab}(\nu)$ on a (skew) Young diagram $\nu$. In each box $(i,j)$ of the diagram write an integer $t_{ij}$. An admissible tableau (or, simply, a tableau) $t \in \text{Tab}(\nu)$ is a set of integers $t = \{t_{jk}\}_{(j,k) \in \nu}$, where all $t_{jk} \in B = \{1,2,3\} = B_+ \cup B_-$, $B_+ = \{1,2\}$, $B_- = \{3\}$, satisfy the conditions...
Figure 2: The Young diagram for the partition $\mu' = (5, 3, 2)$, conjugated to $\mu = (3^2, 2, 1^2)$.

Figure 3: The skew Young diagram $\lambda \subset \mu$ with $\lambda = (2, 1)$ and $\mu = (3^2, 2, 1^2)$.

(i) $t_{jk} \leq t_{j+1,k} , t_{j,k+1}$

(ii) $t_{jk} < t_{j,k+1}$ if $t_{jk} \in B_-$ or $t_{j,k+1} \in B_-$

(iii) $t_{jk} < t_{j+1,k}$ if $t_{jk} \in B_+$ or $t_{j+1,k} \in B_+$.

Fig. 4 shows all admissible tableaux for the skew Young diagram $\lambda \subset \mu$ with $\lambda = (1^2)$ and $\mu = (2^3)$.

C.2 Q-operators

Now recall the $Q$-operators $A_i(x)$ and $\overline{A}_i(x)$ defined in (2.56) and (2.58). They satisfy the functional relations of Sect.5.3. When written in terms of $A_i(x)$ and $\overline{A}_i(x)$ these relations have

Figure 4: All admissible tableaux for the skew-Young diagram $\lambda \subset \mu$ with $\lambda = (1^2)$ and $\mu = (2^3)$.
the form
\[ c_{21} A_3(x) = \left( \frac{z_2}{z_1} \right)^{\frac{1}{2}} A_1(xq) A_2(xq^{-1}) - \left( \frac{z_1}{z_2} \right)^{\frac{1}{2}} A_1(xq^{-1}) A_2(x), \quad (C.1) \]
\[ c_{12} \overline{A}_3(x) = \left( \frac{z_1}{z_2} \right)^{\frac{1}{2}} \overline{A}_1(xq) \overline{A}_2(xq^{-1}) - \left( \frac{z_2}{z_1} \right)^{\frac{1}{2}} \overline{A}_1(xq^{-1}) \overline{A}_2(x), \quad (C.2) \]
\[ c_{13} \overline{A}_1(x) A_3(x) = \left( \frac{z_1}{z_3} \right)^{\frac{1}{2}} \overline{A}_2(x) - \left( \frac{z_3}{z_1} \right)^{\frac{1}{2}} \overline{A}_2(xq^{-1}), \quad (C.3) \]
\[ c_{23} \overline{A}_2(x) A_3(x) = \left( \frac{z_2}{z_3} \right)^{\frac{1}{2}} \overline{A}_1(xq^{-1}) - \left( \frac{z_3}{z_2} \right)^{\frac{1}{2}} \overline{A}_1(x), \quad (C.4) \]
\[ c_{13} \overline{A}_1(x) \overline{A}_3(x) = \left( \frac{z_1}{z_3} \right)^{\frac{1}{2}} A_2(xq^{-1}) - \left( \frac{z_3}{z_1} \right)^{\frac{1}{2}} A_2(x), \quad (C.5) \]
\[ c_{23} \overline{A}_2(x) \overline{A}_3(x) = \left( \frac{z_2}{z_3} \right)^{\frac{1}{2}} A_1(xq^{-1}) - \left( \frac{z_3}{z_2} \right)^{\frac{1}{2}} A_1(x). \quad (C.6) \]
where the quantities \( z_1, z_2, z_3 \), defined in (2.38), are constant (\( x \)-independent) operators, satisfying the relation \( z_1 z_2 z_3^{-1} = 1 \).

### C.3 Higher transfer matrices and sums over tableaux

Define operator-valued functions
\[ \mathcal{X}(1, x) = z_1 \frac{A_1(xq^{-\frac{1}{2}})}{A_1(xq^{\frac{1}{2}})}, \quad \mathcal{X}(2, x) = z_2 \frac{A_1(xq^{\frac{1}{2}}) A_3(xq^{-\frac{1}{2}})}{A_1(xq^{\frac{1}{2}}) A_3(xq^{\frac{1}{2}})}, \quad \mathcal{X}(3, x) = z_3 \frac{A_3(xq^{-\frac{1}{2}})}{A_3(xq^{\frac{1}{2}})}, \quad (C.7) \]
of the spectral variable \( x \in \mathbb{C} \). Remind, that all the operators, appearing in (C.1)-(C.7), are commutative. For any skew Young diagram \( \nu \) define an operator
\[ F_{\nu}(x) = \sum_{t \in \text{Tab}(\nu)} \prod_{(i,j) \in \nu} (-1)^{p(t_{i,j})} \mathcal{X}(t_{i,j}, xq^{-\nu_i + \nu'_j - 2i + 2j}), \quad (C.8) \]
where the sum is taken over all admissible tableaux, and the product is taken over all boxes of the Young diagram \( \nu \). The parities are \( p(1) = p(2) = 0, p(3) = 1 \) and the integers \( \nu_1, \nu'_j \) for a skew diagram \( \nu = \lambda \subset \mu \) are defined as \( \nu_1 = \mu_1, \nu'_j = \mu'_j \). For example for the diagram in Fig. 4 one has from (C.8)
\[ F_{(12) \subset (23)}(x) = \]
\[ -\mathcal{X}(1, xq^{-3}) \mathcal{X}(1, xq^3) \mathcal{X}(2, xq) \mathcal{X}(3, xq^{-1}) + \mathcal{X}(1, xq^{-3}) \mathcal{X}(1, xq^3) \mathcal{X}(3, xq) \mathcal{X}(3, xq^{-1}) + \]
\[ +\mathcal{X}(2, xq^{-3}) \mathcal{X}(2, xq^3) \mathcal{X}(3, xq) \mathcal{X}(3, xq^{-1}) - \mathcal{X}(1, xq^{-3}) \mathcal{X}(3, xq^3) \mathcal{X}(3, xq) \mathcal{X}(3, xq^{-1}) \quad (C.9) \]
\[ -\mathcal{X}(2, xq^{-3}) \mathcal{X}(1, xq^3) \mathcal{X}(2, xq) \mathcal{X}(3, xq^{-1}) + \mathcal{X}(2, xq^{-3}) \mathcal{X}(1, xq^3) \mathcal{X}(3, xq) \mathcal{X}(3, xq^{-1}) \]
\[ +\mathcal{X}(2, xq^{-3}) \mathcal{X}(2, xq^3) \mathcal{X}(3, xq) \mathcal{X}(3, xq^{-1}) - \mathcal{X}(2, xq^{-3}) \mathcal{X}(3, xq^3) \mathcal{X}(3, xq) \mathcal{X}(3, xq^{-1}). \]

Note, that since \( \overline{A}_1(0) = A_3(0) = 1 \), Eq. (C.8) with \( x = 0 \) reduces to the super-character formula (the supersymmetric Shur function)
\[ S_{\nu}(z_1, z_2|z_3) = \sum_{t \in \text{Tab}(\nu)} \prod_{(i,j) \in \nu} (-1)^{p(t_{i,j})} z_{t_{i,j}} \quad (C.10) \]
for the classical super-algebra $gl(2|1)$.

An important feature of Eq. (C.8) is that it defines an entire function of $x$ (the pole terms cancel out due to the Bethe Ansatz equations [18], see [33–35]). It is expected that this formula defines the most general (higher) transfer matrix, associated with $U_q(s\ell(2|1))$. For example, the quantity $F_{(1)}(x)$ (for the simplest diagram $\nu = (1)$, consisting of one square) exactly coincides with one of the six expressions (3.11) for the fundamental transfer matrix $T(x)$. The general statement will be formulated in the Conjecture [1] below, but before that let us discuss some other properties of (C.8).

Note that sum over the tableaux expressions, like (C.8), arise also in the theory of character for quantum affine algebras [88, 89].

C.4 Bazhanov-Reshetikhin formulae

The definition (C.8) implies the following determinant identities

\[
F_{\lambda \subset \mu}(x) = \det_{1 \leq i,j \leq \mu_1} \left( F_{(1)}(xq^{-\mu_1+\mu_1'-\lambda'_i+i+j-1})(xq^{-\mu_1+\mu_1'+\lambda_i-i-j+1}) \right)
\]

where $F_{(1^a)} = F_{(0)} \equiv 1$ and $F_{(1^a)} = F_{(a)} \equiv 0$ for $a < 0$. For example, for the same operator as in (C.9) one obtains

\[
F_{(2^a) \subset (2^a)}(x) = \det \left( \begin{array}{cc} F_{(1)}(xq^{-3}) & F_{(1)}(xq) \\ 1 & F_{(1)}(xq) \end{array} \right) = \det \left( \begin{array}{cc} F_{(1)}(xq^3) & 0 \\ F_{(0)}(x) & F_{(0)}(xq) \end{array} \right).
\]

The determinant representations (C.11), (C.12) for the algebra $U_q(s\ell(m))$ with $\lambda = \phi$ were first obtained by Bazhanov and Reshetikhin in [82]. For the relevant here case of $U_q(s\ell(m,n))$ these representations were generalized in [33] (see, also [34, 35]).

The fact that (C.8) implies the relations (C.11)-(C.12) is a combinatorial theorem which can be proven by induction on the size of the determinants $\mu_1$ or $\mu_1'$ (this theorem mentioned in [83] in the same context for $U_q(B_r^{(1)})$ case). In the case $x = 0$ the above determinant expressions reduce to the super-symmetric Jacobi-Trudi and Giambelli formulae for the characters of $gl(2|1)$ (or the second Weyl formula). On the other hand, the fact that the tableau sum (or the related determinant formulae (C.11) or (C.12)) exactly coincides with a higher transfer matrix is a non-trivial statement. Obviously, to prove this one needs to connect (C.8) with the definition of transfer matrices by using algebraic properties of representations of the quantum affine algebra (or the corresponding Yangian in the rational case $q = 1$) in the auxiliary space.

So far the formulae (C.11) or (C.12) have been proven only for a few cases: (i) for $U_q(s\ell(2))$ [84], (ii) for $U_q(s\ell(3))$ with an arbitrary quantum space [53] and (iii) for the Yangian case $Y(gl(m|n))$, corresponding to $q = 1$, but when the quantum space is a tensor product of the fundamental $(m + n)$-dimensional representations. In all cases only non-skew diagrams (with $\lambda = 0$) were considered.

C.5 Quantum affine analogue of the first Weyl formula

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\[\text{The evaluation representations considered here obviously connected with non-skew diagrams as well.}\]
Here we consider quantum affine analogues of the first Weyl formula for the characters. Such representations for the T-operators were first introduced in [4] for \( U_q(\hat{sl}(2)) \) and in [53] for \( U_q(\hat{sl}(3)) \). Examples of such representations for \( U_q(\hat{sl}(2|1)) \) are the Wronskian-like expressions (3.9) and (3.10), considered in the main text. Here it is convenient to rewrite them in terms of \( A_i(x) \) and \( \hat{A}_i(x) \),

\[
T^{(1)}_m(x) = \frac{c_{13}}{c_{12}} z_1^{m+\frac{1}{2}} A_1(x q^{m+\frac{1}{2}}) \hat{A}_1(x q^{-m-\frac{1}{2}}) - \frac{c_{23}}{c_{12}} z_2^{m+\frac{1}{2}} A_2(x q^{m+\frac{1}{2}}) \hat{A}_2(x q^{-m-\frac{1}{2}}), \tag{C.15}
\]

\[
T^{(2)}_m(x) = c_{13} c_{23} z_3^{m+\frac{1}{2}} A_3(x q^{-m-\frac{1}{2}}) \hat{A}_3(x q^{m+\frac{1}{2}}). \tag{C.16}
\]

For any \( m_1, m_2, m_3 \in \mathbb{C} \), define the following functions

\[
\mathcal{T}_{(m_1,m_2,m_3)}(x) = (-1)^{m_3+1} \frac{c_{31} c_{23}}{c_{12}} \frac{m_1+m_3}{z_3} 
\times \left( z_1^{m_1+m_3+1} z_2^{-m_1+2m_2+m_3} A_1(x q^{m_1-2m_2-m_3+\frac{3}{2}}) \hat{A}_2(x q^{m_1-2m_2-m_3+\frac{1}{2}}) \right) \hat{A}_3(x q^{m_1+m_3+\frac{1}{2}}), \tag{C.17}
\]

\[
\mathcal{T}_{(m_1,m_2,m_3)}(x) = (-1)^{m_3+1} \frac{c_{13} c_{23}}{c_{12}} \frac{m_1+m_3}{z_3} 
\times \left( z_1^{m_1+m_3+1} z_2^{-m_1+2m_2+m_3} A_1(x q^{m_1+2m_2+m_3+\frac{1}{2}}) \hat{A}_2(x q^{m_1+2m_2+m_3+\frac{3}{2}}) \right) \hat{A}_3(x q^{m_1-m_3-\frac{1}{2}}), \tag{C.18}
\]

where \( c_{ij} = (z_i - z_j)/(z_i z_j)^{\frac{1}{2}} \). They obey the following symmetry relations

\[
\mathcal{T}_{(m_1,m_2,m_3)}(x) = (-1)^{m_3} \mathcal{T}_{(m_1+m_3,m_2+m_3,0)}(x), \tag{C.19}
\]

\[
\mathcal{T}_{(m_1,m_2,m_3)}(x) = (-1)^{m_3} \mathcal{T}_{(m_1+m_3,m_2+m_3,0)}(x). \tag{C.20}
\]

From (C.1)–(C.6) it follows, that in a particular case, when \( m_1 = m_2 = m \in \mathbb{C} \) the expressions (C.17) and (C.18) reduce to

\[
\mathcal{T}_{(m,m,m)}(x) = \mathcal{T}_{(m,m,m)}(x) = (-1)^{m_3+1} c_{23} c_{31} z_3^{m_3+\frac{1}{2}} A_3(x q^{-m-3} \frac{1}{2}) \hat{A}_3(x q^{m+3+\frac{1}{2}}). \tag{C.21}
\]

Comparing this with (3.10) one concludes,

\[
T_m^{(2)}(x) = \mathcal{T}_{(m,0,0)}(x). \tag{C.22}
\]

Similarly,

\[
T_m^{(1)}(x q^{-1}) - T_m^{(1)}(x) = \mathcal{T}_{(m,0,0)}(x) \tag{C.23}
\]

\[
T_m^{(1)}(x) - T_m^{(1)}(x) = \mathcal{T}_{(m,0,0)}(x). \tag{C.24}
\]

For the general case we have the following
**Conjecture 1** For \( m_1, m_2, m_3, m_1 - m_2 \in \mathbb{Z}_{\geq 0} \) and \( (m_2 + m_3) \neq 0 \), the expressions (C.17) and (C.18) can be written as tableau sums

\[
\mathcal{T}_{(m_1,m_2,m_3)}(x) = \mathcal{F}_{(m_1-1)m_3,m_1-m_2)\subset(m_1^{m_3+2})}(x), \tag{C.25}
\]

\[
\mathcal{T}_{(m_1,m_2,m_3)}(x) = \mathcal{F}_{(m_1,m_2,1^{m_3})}(x), \tag{C.26}
\]

and, similarly,

\[
\mathcal{T}_{m}^{(1)}(x) = \mathcal{F}_{(m)}(x), \quad m \in \mathbb{Z}_{\geq 0}. \tag{C.27}
\]

We convinced ourselves in the validity of the these relations by numerous checks for particular values of \( m, m_1, m_2, m_3 \). Apparently there exist an elementary general proof, which we postpone for the future work. For example for \( (m_1, m_2, m_3) = (2, 1, 1) \), one has

\[
\mathcal{T}_{(2,1,1)}(x) = \mathcal{F}_{(1^{1})\subset(2^1)}(x) = \mathcal{F}_{(1^{2})\subset(2^1)}(x), \tag{C.28}
\]

\[
\mathcal{T}_{(2,1,1)}(x) = \mathcal{F}_{(2,1,1)}(x) = \mathcal{F}_{(2,1)}(x). \tag{C.29}
\]

From (C.22) and (C.26) it follows that

\[
\mathcal{T}_{m}^{(2)}(x) = \mathcal{F}_{(m,m,0)}(x), \quad m \in \mathbb{Z}_{\geq 0}. \tag{C.30}
\]

The relations and (C.27) and (C.30) provide one with the tableau sum (and, thus, the determinant expressions (C.11), (C.12)) for the \( T \)-operators (C.15) and (C.16). Remind that the later were derived form the \textit{ab initio} definition of Sect. 2.2. This is the main result in this section.

The \( gl(m|n) \) analog of the first Weyl formula is called the Sergeev-Pragacz formula [85, 86]. It gives an alternative representation of the supersymmetric Shur function, in addition to the tableau sum formula of the type (C.10). Let \( \mu \) be a (non-skew) partition, then [15]

\[
S_{\mu}(x_1, \ldots, x_m|y_1, \ldots, y_n) = \sum_{\sigma \in S_m \times S_n} \text{sgn}(\sigma) \left[ \prod_{i=1}^{m-1} x_i^{m-i} \prod_{j=1}^{n-1} y_j^{n-j} \prod_{(i,j) \in \mu} (x_i - y_j) \right] \prod_{i<j} (x_i - x_j) \prod_{i<j} (y_i - y_j), \tag{C.31}
\]

where the third product in the numerator is taken over all boxes \((i,j) \in \mu\) of the Young diagram \( \mu \). It is assumed that \( x_i \equiv 0 \) if \( i \geq m + 1 \) and \( y_j \equiv 0 \) if \( j \geq n + 1 \). The symbol \( S_m \) (resp. \( S_n \)) denotes the symmetric group of order \( m \) (resp. \( n \)). The notation \( \sigma[\ldots] \) stands for the action of the permutation \( \sigma \) on the variables \( x_1, \ldots, x_m, y_1, \ldots, y_n \) inside the square brackets.

When the spectral parameter vanishes, \( x = 0 \), the Wronskian type relations (C.15) and (C.16), connecting \( T \)- and \( Q \)-operators, reduce to the \( gl(2|1) \) Sergeev-Pragacz formula with \( x_1 = z_1, x_2 = z_2 \) and \( y_1 = z_3 \). Note that a similar statement [63] holds also for any quantum affine superalgebra \( U_q(sl(m|n)) \) with arbitrary \( m \) and \( n \).

[15] The sign of the variables \( y_1, \ldots, y_n \) in [85, 86] is opposite to our definition.
Appendix D. Expansion of the universal R-matrix

The third order in \( e_j \) term in the expansion (2.19) of the reduced universal \( R \)-matrix read

\[
O(e_j^3) \text{ term in } (2.19) = \]
\[
- \frac{q^{-3}(q - q^{-1})^3}{[2][3]} e_1^3 \otimes s f_1^3 + q(q - q^{-1})^2 (e_0 e_1 e_0 \otimes s f_0 f_1 f_0 + e_2 e_1 e_2 \otimes s f_2 f_1 f_2)
+ q^{-1}(q - q^{-1})^2 (e_0 e_0 e_0 \otimes s f_0 f_0 f_0 + e_2 e_0 e_2 \otimes s f_2 f_0 f_2)
+ \frac{(q - q^{-1})^2}{q[2]} \sum_{j \in \{0, 2\}} (q e_j e_2^2 \otimes s f_j f_1^2 + q e_1^2 e_j \otimes s f_j f_j^2 - [2] e_1 e_j e_1 \otimes s f_1 f_j f_1)
+ (q - q^{-1}) \left\{ e_0 e_1 e_2 \otimes s (q[2] f_0 f_1 f_2 - q f_0 f_2 f_1 - q f_1 f_0 f_2 + f_1 f_2 f_0 + f_0 f_1 f_2 - [2] f_2 f_1 f_0)ight.
+ e_2 e_0 e_1 \otimes s (q f_0 f_1 f_0 - q f_2 f_0 f_1 - q f_1 f_0 f_0 + f_0 f_1 f_2 + f_1 f_2 f_0 - [2] f_2 f_1 f_0)
+ e_0 e_2 e_1 \otimes s (q f_0 f_1 f_0 - q f_2 f_1 f_0 - q f_1 f_0 f_0 + f_0 f_1 f_0 + f_1 f_2 f_0 - [2] f_2 f_1 f_0)
+ e_1 e_2 e_0 \otimes s (-q f_0 f_1 f_2 + f_0 f_2 f_0 - q f_1 f_2 f_0 + f_2 f_0 f_0 - [2] f_2 f_1 f_0)
\].
\]

On the other hand the expansion of the CFT \( L \)-operator (4.15) reads

\[
\mathcal{Z} = \mathcal{P} \exp \left( \int \mathcal{Z}(u) du \right) \quad \text{(D.2)}
\]
\[
= 1 + \int \mathcal{Z}(u) du + \int_{u_1 \geq u_2} \mathcal{Z}(u_1) \mathcal{Z}(u_2) du_1 du_2 + \cdots \quad \text{(D.3)}
\]
\[
\mathcal{Z}(u) = e_0 \otimes s V_0(u) + e_1 \otimes s V_1(u) + e_2 \otimes s V_2(u) \quad \text{(D.4)}
\]

where \( \mathcal{Z}(u) \) is defined in (4.13). Introduce the ordered integrals

\[
J(i_1, i_2, \cdots, i_n) = \int_{u_1 \geq u_2 \geq \cdots \geq u_n} V_{i_1}(u_1) V_{i_2}(u_2) \cdots V_{i_n}(u_n) du_1 du_2 \cdots du_n. \quad \text{(D.5)}
\]

Then the products of the vertex operators (4.11) can be written as

\[
f_1 = - \frac{1}{q - q^{-1}} J(1), \quad \text{(D.6)}
\]
\[
f_1^2 = \frac{q[2]}{(q - q^{-1})^2} J(1, 1), \quad \text{(D.7)}
\]
\[
f_1^3 = - \frac{q^3[2][3]}{(q - q^{-1})^3} J(1, 1, 1), \quad \text{(D.8)}
\]
\[ f_{0}f_{1}f_{0} = -\frac{q^{-1}}{(q - q^{-1})^2} J(0, 1, 0), \] (D.9)

\[ f_{2}f_{1}f_{2} = -\frac{q^{-1}}{(q - q^{-1})^2} J(2, 1, 2), \] (D.10)

\[ f_{0}f_{2}f_{0} = -\frac{q}{(q - q^{-1})^2} J(0, 2, 0), \] (D.11)

\[ f_{2}f_{0}f_{2} = -\frac{q}{(q - q^{-1})^2} J(2, 0, 2) \] (D.12)

\[
\begin{pmatrix}
  f_{0}f_{1}^2 \\
  f_{1}f_{0}f_{1} \\
  f_{1}^2f_{0}
\end{pmatrix}
= \frac{[2]}{(q - q^{-1})^3}
\begin{pmatrix}
  q & 1 & q^{-1} \\
  1 & \frac{2}{q} & 1 \\
  q^{-1} & 1 & q
\end{pmatrix}
\begin{pmatrix}
  J(0, 1, 1) \\
  J(1, 0, 1) \\
  J(1, 0, 0)
\end{pmatrix},
\] (D.13)

\[
\begin{pmatrix}
  f_{2}f_{1}^2 \\
  f_{1}f_{2}f_{1} \\
  f_{1}^2f_{2}
\end{pmatrix}
= \frac{[2]}{(q - q^{-1})^3}
\begin{pmatrix}
  q & 1 & q^{-1} \\
  1 & \frac{2}{q} & 1 \\
  q^{-1} & 1 & q
\end{pmatrix}
\begin{pmatrix}
  J(2, 1, 1) \\
  J(1, 2, 1) \\
  J(1, 1, 2)
\end{pmatrix},
\] (D.14)

Determinant of the above matrix is 0. The Serre relations (2.5) for \( \{f_j\} \) follows immediately. Then one can derive

\[ qf_{0}f_{1}^2 - f_{1}f_{0}f_{1} = \frac{q[2]}{(q - q^{-1})^2} \left( J(0, 1, 1) + \frac{J(1, 0, 1)}{[2]} \right), \] (D.15)

\[ qf_{1}^2f_{0} - f_{1}f_{0}f_{1} = \frac{q[2]}{(q - q^{-1})^2} \left( J(1, 1, 0) + \frac{J(1, 0, 1)}{[2]} \right), \] (D.16)

\[ qf_{2}f_{1}^2 - f_{1}f_{2}f_{1} = \frac{q[2]}{(q - q^{-1})^2} \left( J(2, 1, 1) + \frac{J(1, 2, 1)}{[2]} \right), \] (D.17)

\[ qf_{1}^2f_{2} - f_{1}f_{2}f_{1} = \frac{q[2]}{(q - q^{-1})^2} \left( J(1, 1, 2) + \frac{J(1, 2, 1)}{[2]} \right). \] (D.18)

Thanks to the Serre relations (2.5) for \( \{e_j\} \) one can derive

\[
\begin{align*}
  &e_{j}e_{j}^2 \otimes_s J(j, 1, 1) + e_{1}e_{j}e_{1} \otimes_s J(1, j, 1) + e_{1}e_{j} \otimes_s J(1, 1, j) \\
  = &e_{j}e_{j}^2 \otimes_s \left( J(j, 1, 1) + \frac{J(1, j, 1)}{[2]} \right) + e_{1}e_{j} \otimes_s \left( J(1, 1, j) + \frac{J(1, j, 1)}{[2]} \right) \\
  = &\frac{(q - q^{-1})^2}{q[2]} \left\{ e_{j}e_{j}^2 \otimes_s (qf_{j}f_{1}^2 - f_{1}f_{j}f_{1}) + e_{1}e_{j} \otimes_s (qf_{1}^2f_{j} - f_{1}f_{j}f_{1}) \right\} \\
  = &\frac{(q - q^{-1})^2}{q[2]} (qe_{j}e_{j}^2 \otimes_s f_{j}f_{1}^2 + qe_{1}e_{j} \otimes_s f_{1}^2f_{j} - [2]e_{1}e_{j}e_{1} \otimes_s f_{1}f_{j}f_{1}), \tag{D.19}
\end{align*}
\]

where \( j \in \{0, 2\} \). We also have

\[
\begin{pmatrix}
  f_{0}f_{1}f_{2} \\
  f_{0}f_{2}f_{1} \\
  f_{1}f_{0}f_{2} \\
  f_{1}f_{2}f_{0} \\
  f_{2}f_{0}f_{1} \\
  f_{2}f_{1}f_{0}
\end{pmatrix}
= \frac{1}{q^2(q - q^{-1})^3}
\begin{pmatrix}
  -q^2 & -q & -q & q^2 & q & q \\
  -q & -q^2 & -1 & q & q^3 & q^2 \\
  -q & -1 & -q^2 & q^3 & q & q^2 \\
  q^2 & q & q^3 & -q^2 & -1 & -q \\
  q^2 & q^3 & q & -1 & -q^2 & -q \\
  q & q^2 & q^2 & -q & -q & -q^2
\end{pmatrix}
\begin{pmatrix}
  J(0, 1, 2) \\
  J(0, 2, 1) \\
  J(1, 0, 2) \\
  J(1, 2, 0) \\
  J(2, 0, 1) \\
  J(2, 1, 0)
\end{pmatrix}. \tag{D.20}
\]
This relation is invertible. Now one can show that the third order term in (D.4)

\[
\int_{u_1 \geq u_2 \geq u_3} \mathcal{Z}(u_1)\mathcal{Z}(u_2)\mathcal{Z}(u_3)du_1du_2du_3 = \sum_{j_1=0}^2 \sum_{j_2=0}^2 \sum_{j_3=0}^2 (-1)^{p(j_1)p(j_2)+p(j_1)p(j_3)+p(j_2)p(j_3)}
\times e_{j_1} e_{j_2} e_{j_3} \otimes_s \int_{u_1 \geq u_2 \geq u_3} V_{j_1}(u_1)V_{j_2}(u_2)V_{j_3}(u_3)du_1du_2du_3,
\]

(D.21)

exactly coincide with (D.1).

References

[1] R. J. Baxter, Partition function of the eight-vertex lattice model. Ann. Physics 70 (1972) 193-228.

[2] V. G Drinfel’d, Quantum groups, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820 Amer. Math. Soc., Providence, RI, 1987.

[3] M. Jimbo, A q-difference analogue of U(G) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985) 63-69.

[4] V.V. Bazhanov, S.L. Lukyanov, A.B. Zamolodchikov, Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation, Commun.Math.Phys. 190 (1997) 247-278 [arXiv:hep-th/9604044].

[5] V.V. Bazhanov, S.L. Lukyanov, A.B. Zamolodchikov, Integrable Structure of Conformal Field Theory III. The Yang-Baxter Relation, Commun.Math.Phys. 200 (1999) 297-324 [arXiv:hep-th/9805008].

[6] H. A. Bethe, Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette, Z. Physik 71 (1931) 205-231.

[7] E. K. Sklyanin, L. A. Tahtadžjan, and L. D. Faddeev, Quantum inverse problem method. I. Teoret. Mat. Fiz. 40 (1979) 194-220.

[8] J. H. H. Perk and C. L. Schultz, New families of commuting transfer matrices in q-state vertex models. Phys. Lett. A84 (1981) 407–410.

[9] I. V. Cherednik, On a method of constructing factorized S matrices in elementary functions, Theor. Math. Phys. 43 (1980) 356-358.

[10] A. Foerster and M. Karowski, The supersymmetric t − J model with quantum group invariance, Nucl. Phys. B 408 (1993) 512-534.

[11] A. Gonzalez-Ruiz, Integrable open-boundary conditions for the supersymmetric t − J model the quantum-group-invariant case, Nucl. Phys. B 424 (1994) 468-486.

[12] B. Sutherland, Model for a multicomponent quantum system, Phys. Rev. B 12 (1975) 3795-3805.

[13] C. K. Lai, Lattice gas with nearest-neighbor interaction in one dimension with arbitrary statistics, J. Math. Phys. 15 (1974) 1675-1676.
[14] P. P. Kulish and E. K. Sklyanin, On the solution of the Yang-Baxter equation, J. Sov. Math. 19 (1982) 1596-1620.

[15] C. L. Schultz, Eigenvectors of the multicomponent generalization of the six-vertex model, Physica A122 (1983) 71-88.

[16] P. P. Kulish, Integrable graded magnets, J. Sov. Math. 35 (1986) 2648.

[17] P. Schlottmann, Integrable narrow-band model with possible relevance to heavy-fermion systems, Phys. Rev. B36 (1987) 5177-5185.

[18] P. A. Bares and G. Blatter, Supersymmetric t-J model in one dimension: Separation of spin and charge, Phys. Rev. Lett. 64 (1990) 2567-2570.

[19] N. Kawakami and S.-K. Yang, Correlation functions in the one-dimensional t-J model, Phys. Rev. Lett. 65 (1990) 2309-2311.

[20] J. Suzuki, On a one-dimensional system associated with a $gl(m|n)$ vertex model, J. Phys. A 25 (1992) 1769-1779.

[21] T. Deguchi and P. P. Martin, An Algebraic Approach to Vertex Models and Transfer-Matrix Spectra, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 165-196.

[22] P. P. Martin and V. Rittenberg, A Template for Quantum Spin Chain Spectra, Int. J. Mod. Phys. A7 Suppl. 1B (1992) 707-730.

[23] P. A. Bares, I. M. P. Carmelo, J. Ferrer and P. Horsch, Charge-spin recombination in the one-dimensional supersymmetric t-J model, Phys. Rev. B46 (1992) 14624-14654.

[24] F. H. L. Essler and V. E. Korepin, Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric $t-J$ model, Phys. Rev. B 46 (1992) 9147-9162.

[25] A. Foerster and M. Karowski, Algebraic properties of the Bethe Ansatz for an spl(2,1)-supersymmetric $t-J$ model, Nucl. Phys. B 396(1993) 611-638.

[26] F. H. L. Essler, V. E. Korepin and K. Schoutens, Exact Solution of an Electronic Model of Superconductivity in 1+1 Dimensions, Int. J. Mod. Phys. B 8 (1994) 3205-3242 arXiv:cond-mat/9211001.

[27] A. J. Bracken, M. D. Gould, J. R. Links, Y.-Z. Zhang, A New Supersymmetric and Exactly Solvable Model of Correlated Electrons, Phys. Rev. Lett. 74 (1995) 2768 arXiv:cond-mat/9411026.

[28] Z. Maassarani, $U_{q}{osp(2,2)}$ Lattice Models, J.Phys. A28 (1995) 1305-1324 arXiv:hep-th/9407032.

[29] P.B. Ramos, M.J. Martins, One parameter family of an integrable spl(2|1) vertex model : Algebraic Bethe Ansatz approach and ground state structure, Nucl.Phys. B474 (1996) 678-714 arXiv:hep-th/9604072.

[30] M. P. Pfannmuller, H. Frahm, Algebraic Bethe Ansatz for $gl(2,1)$ Invariant 36-Vertex Models, Nucl.Phys. B479 (1996) 575-593 arXiv:cond-mat/9604082.
[31] Y.-K. Zhou, M. T. Batchelor, Spin excitations in the integrable open quantum group invariant supersymmetric $t-J$ model, Nucl. Phys. B 490 (1997) 576-594 [arXiv:cond-mat/9611013].

[32] A. Klümper, T. Wehner and J. Zittartz, Thermodynamics of the quantum Perk - Schultz model, J. Phys. A: Math. Gen. 30 (1997) 1897-1912.

[33] Z. Tsuboi: Analytic Bethe Ansatz and functional equations for Lie superalgebra $sl(r+1|s+1)$, J.Phys.A: Math. Gen. 30 (1997) 7975-7991.

[34] Z. Tsuboi: Analytic Bethe Ansatz and functional equations associated with any simple root systems of the Lie superalgebra $sl(r+1|s+1)$, Physica A 252 (1998) 565-585.

[35] Z. Tsuboi: Analytic Bethe Ansatz related to a one-parameter family of finite-dimensional representations of the Lie superalgebra $sl(r+1|s+1)$, J.Phys.A: Math. Gen. 31 (1998) 5485-5498.

[36] G. Jüttner, A. Klümper and J. Suzuki, From fusion hierarchy to excited state TBA, Nucl. Phys. B512 (1998) 581-600 [arXiv:hep-th/9707074].

[37] F. Göhmann and S. Murakami, Fermionic representations of integrable lattice systems, J. Phys. A 31 (1998) 7729-7749 [arXiv:cond-mat/9805129].

[38] H. Saleur, The continuum limit of $sl(N/K)$ integrable super spin chains, Nucl.Phys. B578 (2000) 552-576, [arXiv:solv-int/9905007].

[39] F. Göhmann, A. Seel: A note on the Bethe Ansatz solution of the supersymmetric $t$-J model, contributed to the 12th Int. Colloquium on Quantum Groups and Integrable Systems, Prague (2003), Czech.J.Phys. 53 (2003) 1041-1046 [arXiv:cond-mat/0309138].

[40] M. Bortz, A. Klümper, Lattice path integral approach to the one-dimensional Kondo model, J. Phys. A: Math. Gen. 37 (2004) 6413-6436 [arXiv:cond-mat/0405085].

[41] D. Arnaudon, J. Avan, N. Crampe, A. Doikou, L. Frappat, E. Ragoucy, General boundary conditions for the $sl(N)$ and $sl(M|N)$ open spin chains, J.Stat.Mech. 0408 (2004) P005 [arXiv:math-ph/0406021].

[42] Z. Tsuboi, Nonlinear integral equations and high temperature expansion for the $U_q(sl(r+1|s+1))$ Perk-Shultz Model, Nucl. Phys. B737 (2006) 261-290 [arXiv:cond-mat/0510458].

[43] Shao-You Zhao, Wen-Li Yang, Yao-Zhong Zhang, Determinant Representations of Correlation Functions for the Supersymmetric $t – J$ Model, Commun.Math.Phys. 268 (2006) 505-541 [arXiv:hep-th/0511028].

[44] A.V. Belitsky, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov: Baxter Q-operator for graded $SL(2|1)$ spin chain, J.Stat.Mech. 0701 (2007) P005 [arXiv:hep-th/0610332].

[45] V. Kazakov, A. Sorin, A. Zabrodin, Supersymmetric Bethe Ansatz and Baxter Equations from Discrete Hirota Dynamics, Nucl. Phys. B 790 (2008) 345-413 [arXiv:hep-th/0703147].

[46] A. Zabrodin, Backlund transformations for the difference Hirota equation and the supersymmetric Bethe Ansatz, Theor. Math. Phys. 155 (2008) 567-584 [arXiv:0705.4006[hep-th]].
[47] E. Ragoucy, G. Satta, Analytical Bethe Ansatz for closed and open $gl(M|N)$ super-spin chains in arbitrary representations and for any Dynkin diagram, JHEP 09 (2007) 001 [arXiv:0706.3327[hep-th]].

[48] O. Foda, M. Wheeler, M. Zuparic, Factorized domain wall partition functions in trigonometric vertex models, J. Stat. Mech. (2007) P10016 [arXiv:0709.4540[math-ph]].

[49] N. Beisert, V. A. Kazakov, K. Sakai, K. Zarembo, Complete Spectrum of Long Operators in N=4 SYM at One Loop, JHEP 0507 (2005) 030 [arXiv:hep-th/0503200].

[50] N. Gromov, P. Vieira, Complete 1-loop test of AdS/CFT, JHEP 04 (2008) 046, [arXiv:0709.3487[hep-th]].

[51] F. Woynarovich, Low-energy excited states in a Hubbard chain with on-site attraction, J.Phys.C: Solid State Phys. 16 (1983) 6593-6604.

[52] G.P. Pronko and Yu.G. Stroganov, Families of solutions of the nested Bethe Ansatz for the $A_2$ spin chain, J. Phys. A: Math. Gen. 33 (2000) 8267-8273 [arXiv:hep-th/9902085].

[53] V. V. Bazhanov, A. N. Hibberd and A. N. Khoroshkin, Integrable structure of $W_3$ Conformal Field Theory. Quantum Boussinesq Theory and Boundary Affine Toda Theory. Nucl. Phys. B 622 (2002) 475–547 [arXiv:hep-th/0105177].

[54] G.P. Pronko, Yu.G. Stroganov, Bethe Equations “On the Wrong Side of Equator”, J.Phys. A32 (1999) 2333-2340 [arXiv:hep-th/9808153].

[55] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, Yang-Baxter equations and representation theory. I, Lett. Math. Phys. 5 (1981) 393-403.

[56] A.Kuniba, T. Nakanishi, J.Suzuki, Functional Relations in Solvable Lattice Models I: Functional Relations and Representation Theory, Int.J.Mod.Phys. A9 (1994) 5215-5266 [arXiv:hep-th/9309137].

[57] V. Kazakov and P. Vieira, From Characters to Quantum (Super)Spin Chains via Fusion, JHEP10(2008)050 [arXiv:0711.2470[hep-th]].

[58] V. A. Fateev, S. L. Lukyanov, Boundary RG Flow Associated with the AKNS Soliton Hierarchy, J.Phys. A39 (2006) 12889-12926 [arXiv:hep-th/0510271].

[59] H. Yamane, On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras, Publ. Res. Inst. Math. Sci. 35 (1999) 321-390; errata: Publ. Res. Inst. Math. Sci. 37 (2001) 615–619 [arXiv:q-alg/9603015].

[60] S. Khoroshkin, V. Tolstoy, Twisting of quantum (super)algebras. Connection of Drinfeld’s and Cartan-Weyl realizations for quantum affine algebras [arXiv:hep-th/9404036].

[61] V. Drinfeld, Hopf algebras and the quantum Yang-Baxter equations, Sov. Math. Dokl. 32 (1985) 264-268.

[62] S. M. Khoroshkin and V. N. Tolstoy, The uniqueness theorem for the universal $R$-matrix, Lett. Math. Phys. 24 (1992) 231–244.

[63] work in progress.
[64] R. B. Zhang, Invariants of the quantum supergroups $U_q(gl(m|1))$, J. Phys. A: Math. Gen. 24 (1991) L1327-L1332.

[65] R. J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press Inc., London (1982) [http://tpsrv.anu.edu.au/research/mathphys/Book.pdf].

[66] V. A. Fateev and S. L. Lukyanov, Vertex operators and representations of quantum universal enveloping algebras. Internat. J. Modern Phys. A 7 (1992) 1325-1359.

[67] V.V. Bazhanov, S.L. Lukyanov, A.B. Zamolodchikov, Integrable Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz, Commun.Math.Phys. 177 (1996) 381-398 [arXiv:hep-th/9412229].

[68] P. P. Kulish, A. M. Zeitlin, Quantization of N=1 and N=2 SUSY KdV models, in Problems of Mathematical Physics and Applied Mathematics, Ioffe Physical-Technical Institute, St. Petersburg, 2006, pp. 80-100, [arXiv:hep-th/0601019]. A. M. Zeitlin, Quantization of the N=2 Supersymmetric KdV Hierarchy, Theor. Math. Phys. 147 (2006) 698-708 [arXiv:hep-th/0606129].

[69] A. Voros, Exact quantization condition for anharmonic oscillators (in one dimension), J. Phys. A27 (1994) 4653-4661.

[70] P. Dorey, R. Tateo, Anharmonic oscillators, the thermodynamic Bethe Ansatz, and nonlinear integral equations, J. Phys. A32 (1999) L419-L425 [arXiv:hep-th/9812211].

[71] V. Bazhanov, S. Lukyanov, A. Zamolodchikov, Spectral determinants for Schroedinger equation and Q-operators of Conformal Field Theory, J.Statist.Phys. 102 (2001) 567-576 [arXiv:hep-th/9812247].

[72] J. Suzuki: Functional relations in Stokes multipliers — Fun with $x^6 + \alpha x^2$ potential, J.Statist.Phys. 102 (2001) 1029-1047 [arXiv:quant-ph/0003066].

[73] P. Dorey, C. Dunning, D. Masoero, J. Suzuki, R. Tateo, Pseudo-differential equations, and the Bethe Ansatz for the classical Lie algebras, Nucl. Phys. B 772 (2007) 249-289 [arXiv:hep-th/0612298].

[74] P. Dorey, C. Dunning, R. Tateo: Spectral equivalences, Bethe Ansatz equations, and reality properties in PT-symmetric quantum mechanics, J.Phys.A: Math. Gen. 34 (2001) 5679-5704 [arXiv:hep-th/0103051].

[75] M. Abramowitz, I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover Publications Inc.,New York, (1992).

[76] K. Hikami, Baxter Equation for Quantum Discrete Boussinesq Equation, Nucl.Phys. B604 (2001) 580-602 [arXiv:nlin/0102021].

[77] T. Kojima, The Baxter’s Q-operator for the W-algebra $W_N$, J.Phys.A: Math. Theor. 41 (2008) 355206 [arXiv:0803.3505 [nlin.SI]].

[78] V.V. Bazhanov, Solving the Heisenberg magnet without a Bethe Ansatz, in preparation (2008)

[79] V. Kac, Representations of Classical Lie superalgebras, Lecture Notes in Mathematics, 676 (1978) 597-626.
[80] R. B. Zhang, Finite dimensional irreducible representations of the quantum supergroup $U_q(gl(m|n))$, J. Math. Phys. 34 (1993) 1236-1254.

[81] Nguyen Anh Ky, Nguyen thi Hong Van, Finite-dimensional representations of $U_q[gl(2/1)]$ in a basis of $U_q[gl(2) \oplus gl(1)]$. [arXiv:math/0305195 [math.QA]].

[82] V. V. Bazhanov and N. Reshetikhin, Restricted solid-on-solid models connected with simply laced algebras and conformal field theory, J. Phys. A Math. Gen. 23 (1990) 1477-1492.

[83] A. Kuniba, Y. Ohta, J. Suzuki, Quantum Jacobi-Trudi and Giambelli Formulæ for $U_q(B_r^{(1)})$ from Analytic Bethe Ansatz, J. Phys. A28 (1995) 6211-6226.

[84] A. N. Kirillov and N. Yu. Reshetikhin, Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum, J. Phys. A: Math. Gen. 20 (1987) 1565-1585.

[85] P. Pragacz, Algebro-Geometric applications of Schur S- and Q-polynomials, Lecture Notes in Mathematics 1478 (1991) 130-191.

[86] J. Van der Jeugt, J. W. B. Hughes, R. C. King, J. Thierry-Mieg, Character formulas for irreducible modules of the Lie superalgebras $sl(m/n)$, J. Math. Phys. 31 (1990) 2278-2304.

[87] I. G. Macdonald, Symmetric Functions and Hall Polynomials (2nd edition), Oxford University Press (1995).

[88] E. Frenkel and N. Reshetikhin, The $q$-characters of representations of quantum affine algebras and deformations of W-algebras, Contemporary Math. 248 (1999) 163-205 [arXiv:math/9810055 [math.QA]].

[89] E. Frenkel and E. Mukhin, Combinatorics of $q$-characters of finite-dimensional representations of quantum affine algebras, Commun. Math. Phys. 216 (2001) 23-57 [arXiv:math/9911112 [math.QA]].