Research Article

Harmonic Maps and Stability on $f$-Kenmotsu Manifolds

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The purpose of this paper is to study some submanifolds and Riemannian submersions on an $f$-Kenmotsu manifold. The stability of a $\varphi$-holomorphic map from a compact $f$-Kenmotsu manifold to a Kählerian manifold is proven.

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1. Introduction

In Section 2, we give preliminaries on $f$-Kenmotsu manifolds. The concept of $f$-Kenmotsu manifold, where $f$ is a real constant, appears for the first time in the paper of Jannsens and Vanhecke [1]. More recently, Olszak and Rosca [2] defined and studied the $f$-Kenmotsu manifold by the formula (2.3), where $f$ is a function on $M$ such that $df \wedge \eta = 0$. Here, $\eta$ is the dual 1-form corresponding to the characteristic vector field $\xi$ of an almost contact metric structure on $M$. The condition $df \wedge \eta = 0$ follows in fact from (2.3) if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$.

A 1-Kenmotsu manifold is a Kenmotsu manifold (see Kenmotsu [3, 4]. Theorem 2.1 provides a geometric interpretation of an $f$-Kenmotsu structure.

In Section 3, we initiate a study of harmonic maps when the domain is a compact $f$-Kenmotsu manifold and the target is a Kähler manifold.

Ianus and Pastore [5, 6] defined a $(\varphi, J)$-holomorphic map between an almost contact metric manifold $M(\varphi, \eta, \xi, g)$ and an almost Hermitian manifold $N(J, h)$ as a smooth map $F : M \to N$ such that the condition $F_* \circ \varphi = J \circ F_*$ is satisfied. Then, the formula $J(\tau(F)) = F_*(\text{div} \varphi) - \text{Tr}_g \beta$ holds, where $\tau(F)$ is the tension field of $F$ and $\beta(X, Y) = (\tilde{\nabla}_X J)(F, Y)$, $\tilde{\nabla}$ being the connection induced in the pull-back bundle $F^*(TN)$ (see [7]). It is easy to see that in our assumptions $\text{div} \varphi = 0$ and $\text{Tr}_g \beta = 0$ so that a $(\varphi, J)$-holomorphic map between an
$f$-Kenmotsu manifold $M$ and a Kähler manifold $N$ is a harmonic map. If $M$ is a compact manifold, a second-order elliptic operator $J_F$, called the Jacobi operator, is associated to the harmonic map $F$. It is well known that the spectrum of $J_F$ consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities, bounded by the first one. We define the Morse index of the harmonic map $F$ as the sum of multiplicities of negative eigenvalues of the Jacobi operator $J_F$ [8, 9]. A harmonic map is called stable if the Morse index is zero. We have proven that any $(\varphi, f)$-holomorphic map from a compact $f$-Kenmotsu manifold to a Kähler manifold is a stable harmonic map (see [10]).

2. $f$-Kenmotsu manifolds

A differentiable $(2n + 1)$-dimensional manifold $M$ is said to have a $(\varphi, \xi, \eta)$-structure or an almost contact structure if there exist a tensor field $\varphi$ of type $(1, 1)$, a vector field $\xi$, and a 1-form $\eta$ on $M$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

(2.1)

where $I$ denotes the identity transformation.

It seems natural to include also $\varphi \xi = 0$ and $\eta \circ \varphi = 0$; both can be derived from (2.1).

Let $g$ be an associated Riemannian metric on $M$ such that

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y).$$

(2.2)

Putting $Y = \xi$ in (2.2) and using (2.1), we get $\eta(X) = g(X, \xi)$, for any vector field $X$ on $M$.

In this paper, we denote by $C^\infty(M)$ and $\Gamma(E)$ the algebra of smooth functions on $M$ and the $C^\infty(M)$-module of smooth sections of a vector bundle $E$, respectively. All manifolds are assumed to be connected and of class $C^\infty$. Tensors fields, distribution, and so on are assumed to be of class $C^\infty$ if not stated otherwise.

We say that $M$ is an $f$-Kenmotsu manifold if there exists an almost contact metric structure $(\varphi, \xi, \eta, g)$ on $M$ satisfying

$$(\bar{\nabla}_X \varphi)Y = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X)$$

(2.3)

for $X, Y \in \Gamma(TM)$, where $f$ is a smooth function on $M$ such that $df \wedge \eta = 0$.

A 1-Kenmotsu manifold is a Kenmotsu manifold [2, 3].

The following theorem provides a geometric interpretation of any $f$-Kenmotsu structure.

**Theorem 2.1** (Olszak-Rośc). Let $M$ be an almost contact metric manifold. Then, $M$ is $f$-Kenmotsu if and only if it satisfies the following conditions:

(a) the distribution $D = \text{Ker} \eta$ is integrable and any leaf of the foliation $\mathcal{F}$ corresponding to $D$ is a totally umbilical hypersurface with constant mean curvature;

(b) the almost Hermitian structure $(J, g)$ induced on an arbitrary leaf is Kähler;

(c) $\nabla \xi = 0$ and $L_\xi \varphi = 0$. 

Moreover, we have
\[ \nabla_X \xi = f(X - \eta(X)) \xi \] (2.4)
which gives \( \text{div} \xi = 2nf \).

The characteristic vector field of an \( f \)-Kenmotsu manifold also satisfies
\[ R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X). \] (2.5)

Levy proven that a second-order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor [11]. On the other hand, Sharma proven that there is no nonzero skew-symmetric second-order parallel tensor on a Sasakian manifold [12]. For an \( f \)-Kenmotsu manifold we have the following theorem.

**Theorem 2.2.** There is no nonzero parallel 2-form on an \( f \)-Kenmotsu manifold.

**Proof.** We omit it. \( \square \)

A plane section \( p \) in \( T_x \tilde{M}, x \in \tilde{M} \), of a Kenmotsu manifold \((f = 1)\) is called a \( \varphi \)-section if it spanned by a vector \( X \) orthogonal to \( \xi \) and \( \varphi X \). A connected Kenmotsu manifold \( \tilde{M} \) is called a Kenmotsu space form and it is denoted by \( \tilde{M}(c) \) if it has the constant \( \varphi \)-sectional curvature \( c \).

The curvature tensor of a Kenmotsu space form \( \tilde{M}(c) \) is given by
\[ 4R(X, Y)Z = (c - 3)\{g(Y, Z)X - g(X, Z)Y\} \]
\[ + (c + 1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\eta \}
\[ + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z \}
\] (2.6)
for any \( X, Y, Z \in \Gamma(T\tilde{M}) \).

Now, let \( M(f, g') \) be a \( 2m \)-dimensional almost Hermitian manifold. A surjective map \( \pi : \tilde{M} \rightarrow M \) is called a contact-complex Riemannian submersion if it is a Riemannian submersion and satisfies [10]
\[ \pi_* \circ \varphi = J \circ \pi_* . \] (2.7)

In [13], we have proven the following theorem.

**Theorem 2.3.** Let \( \pi : \tilde{M} \rightarrow M \) be a contact-complex Riemannian submersion from a \((2m + 1)\)-dimensional Kenmotsu manifold \( \tilde{M} \) to a \( 2m \)-dimensional almost Hermitian manifold \( M \). Then, \( M \) is a Kählerian manifold. Moreover, \( \tilde{M} \) is a Kenmotsu space form if and only if \( M \) is a complex space form.

3. Harmonic maps and stability

Let \( (M, g) \) and \( (N, h) \) be two Riemannian manifolds and \( F : M \rightarrow N \) a differentiable map. Then, the second fundamental form \( \alpha_F \) of \( F \) is defined by
\[ \alpha_F(X, Y) = \nabla_X F Y - F_* (\nabla_X Y) , \] (3.1)
where $\nabla$ is the Levi-Civita connection on $M$ and $\tilde{\nabla}$ is the connection induced by $F$ on the bundle $F^{-1}(TN)$, which is the pull-back of the Levi-Civita connection $\nabla'$ on $N$, and satisfies the following formula (see [8]):

$$\tilde{\nabla}_XF,Y - \tilde{\nabla}_YF,X = F_*([X,Y]),\quad X,Y \in \Gamma(TM).$$

(3.2)

The tension field $\tau(F)$ of $F$ is defined as the trace of the second fundamental form $\alpha_F$, that is $\tau(F) = \sum \alpha_F(e_i,e_i)(x)$, where $(e_1,\ldots,e_m)$ is an orthonormal basis for $T_xM$ at $x \in M$.

In what follows, we will use Einstein summation convention, so we will omit the sigma symbol.

We say that a map $F : M \to N$ is a harmonic map $\tau(F) x \in M$.

Examples. (1) If $M$ is the circle $S^1$, a map $F : S^1 \to (N,g)$ is harmonic if and only if it is a geodesic parametrized proportionally to arc length. (2) If $N = \mathbb{R}$, a harmonic map $F : (M,g) \to \mathbb{R}$ is a harmonic function. (3) A holomorphic map between two Kähler manifolds is harmonic [8]. For examples in the contact metric geometry, see [5, 6, 14].

Now let us consider a variation $F_{s,t} \in C^\infty(M,N)$, with $s,t \in (-\varepsilon,\varepsilon)$ and $F_{0,0} = F$. If the corresponding variation vector fields are denoted by $V$ and $W$, the Hessian of $F$ is given by

$$H_F(V,W) = \int_M h(J_F(V),W)\mathcal{U}_g,$$

where $\mathcal{U}_g$ is the canonical measure associated to the Riemannian metric $g$ and $J_F(V)$ is a second-order self-adjoint operator acting on $\Gamma(F^{-1}(TN))$ by

$$J_F(V) = \sum_i (\tilde{\nabla}_Ve_i - \tilde{\nabla}_eV)e_i - \sum_i R'(V,F_*e_i)F_*e_i,$$

(3.4)

where $R'$ is the curvature operator on $(N,h)$.

We say that a map $f : (M,\varphi,\xi,\eta,g) \to (N,J,h)$ from an almost contact metric manifold to an almost Hermitian manifold is a $(\varphi,J)$-holomorphic map if and only if $F \circ \varphi = f \circ F_*$. If $M(\varphi,\xi,\eta,g)$ is a Sasaki manifold and $N(J,h)$ is a Kähler manifold, then any $(\varphi,J)$-holomorphic map from $M$ to $N$ is a harmonic map [14].

Then, we can prove the same result for any $(\varphi,J)$-holomorphic map from an $f$-Kenmotsu manifold to a Kähler manifold (see also [15]).

Our main result is the following.

**Theorem 3.1.** Let $M(\varphi,\xi,\eta,g)$ be a compact $f$-Kenmotsu manifold and let $N(J,h)$ be a Kähler manifold. Then, any $(\varphi,J)$-holomorphic map $F : M \to N$ is stable.

If $M$ is compact, the spectrum of $J_F$ consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities, bounded below by the first one. We define the Morse index of the harmonic map $F : M \to N$ as the sum of multiplicities of negative eigenvalues of the Jacobi operator $J_F$. Equivalently, the Morse index of $F$ equals the dimension of the largest subspace of $\Gamma(F^{-1}(TN))$ on which the Hessian $H_F$ is negative definite (see [8, 9]).
We recall the following formula (see [5, 9]):

\[ H_F(V, W) = \int_M (h(\nabla_{e_i} V, \nabla_{e_i} W) + h(R(F_{e_i} V)F_{e_i} V + h(R(F_{f_i} V)F_{f_i} V)) \mathcal{U}_g, \]

(3.5)

where we omitted the summation symbol for repeated indices \( a = 1, \ldots, n, \ n = \dim M \) [5].

Now, let \((e_1, \ldots, e_m; f_1, \ldots, f_m, \xi)\) be a local orthonormal \(\varphi\)-basis on \(M(\varphi, \xi, \eta, g)\) such that \(f_i = \varphi e_i, \ i = 1, \ldots, m\).

From the \((\varphi, \xi)\)-holomorphicity of \(F\) and by \(\varphi \xi = 0\), we have \(F_\xi = 0\). Thus, from (3.5), we obtain the following.

Lemma 3.2. Let \(F: M \to N\) be a \((\varphi, \xi)\)-holomorphic map from an \(f\)-Kenmotsu manifold \(M\) to a Kähler manifold \(N\). Then, one has

\[
H_F(V, V) = \int_M (h(\nabla_{e_i} V, \nabla_{e_i} V) + h(\tilde{\nabla}_{f_i} V, \tilde{\nabla}_{f_i} V)) \mathcal{U}_g + \int_M (h(R(F_{e_i} V)F_{e_i} V + h(R(F_{f_i} V)F_{f_i} V)) \mathcal{U}_g. \\
(3.6)
\]

Lemma 3.3. Let \(T\) be a vector field on \(M\) such that

\[
g(T, X) = h(\tilde{\nabla}_\varphi X, JV) \]

(3.7)

for any \(X \in \Gamma(D)\), where \(D = \text{Ker} \eta\) and \(g(T, \xi) = 0\). Then,

\[
\text{div} (T) = h(R(F_{e_i} V)F_{f_i} V, JV) + 2h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{f_i} V). \\
(3.8)
\]

Proof. Let

\[
h(R(F_{e_i} V)F_{f_i} V, JV) = h(\tilde{\nabla}_{e_i} \tilde{\nabla}_{f_i} V - \tilde{\nabla}_{f_i} \tilde{\nabla}_{e_i} V - \tilde{\nabla}_{[e_i, f_i]} V, JV) \\
= e_i h(\tilde{\nabla}_{f_i} V, JV) - h(\tilde{\nabla}_{f_i} V, \tilde{\nabla}_{e_i} JV) - f_i h(\tilde{\nabla}_{e_i} V, JV) \\
+ h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{f_i} JV) - h(\tilde{\nabla}_{f_i, e_i} V, JV) + h(\tilde{\nabla}_{f_i, e_i} V, JV). \\
(3.9)
\]

By using (3.7) and (2.3), we obtain

\[
\text{div} (T) = g(\nabla_{e_i} T, e_i) + g(\nabla_{f_i} T, f_i) + g(\nabla_{\xi} T, \xi) \\
= e_i g(T, e_i) - g(T, \nabla_{e_i} e_i) + f_i g(T, f_i) - g(T, \nabla_{f_i} f_i) \\
= e_i h(\tilde{\nabla}_{f_i} V, JV) - f_i h(\tilde{\nabla}_{e_i} V, JV) + h(\tilde{\nabla}_{f_i, e_i} V, JV) + h(\tilde{\nabla}_{e_i, f_i} V, JV) \\
(3.10)
\]

and (3.8) follows. \(\square\)

Proposition 3.4. Let \(M(\varphi, \xi, \eta, g)\) be a compact \(f\)-Kenmotsu manifold. Then, the function \(f\) satisfies

\[
\int_M f \mathcal{U}_g = 0. \\
(3.11)
\]
Proof. We have

\[ \text{div} (\xi) = g(e_i, \nabla e_i \xi) + g(f_i, \nabla f_i \xi) + g(\xi, \nabla \xi \xi). \quad (3.12) \]

Using (2.1)–(2.4), we obtain \( \text{div}(\xi) = -2nf \). Since \( M \) is a compact manifold (without boundary), using Stokes’s theorem, we have

\[ \int_M \text{div} (\xi) \mathcal{U}_g = 0, \quad (3.13) \]

so that (3.11) follows from (3.13).

Now we are ready to prove Theorem 3.1. Since \( F \) is a \((\varphi, J)\)-holomorphic map, by using the curvature Kähler identity \( R'(U, V) JW = JR'(U, V) W \) on \( N(J, h) \) and Bianchi’s identity, we have

\[ R'(F*e_i, V) F*e_i + R'(F*f_i, V) F*f_i = -JR'(F*e_i, F*f_i V). \quad (3.14) \]

For any \( V \in \Gamma(F^{-1}(TN)) \), we define the operator

\[ DV : \Gamma(TM) \rightarrow \Gamma(F^{-1}(TN)) \quad (3.15) \]

by the formula

\[ DV(X) = \tilde{\nabla}_X V - J\tilde{\nabla}_X V, \quad (3.16) \]

for any \( X \in \Gamma(TM) \) (see [5]).

Using Lemmas 3.2, 3.3, and (3.14), by a straightforward calculation, we obtain

\[ H_F(V, V) = \frac{1}{2} \int_M \left( h(DV(e_i), DV(e_i)) + h(DV(f_i), DV(f_i)) \right) \mathcal{U}_g \quad (3.17) \]

because \( \int_M \text{div} (T) \mathcal{U}_g = 0 \).

Thus, we have \( H_F(V, V) \geq 0 \) for any \( V \in \Gamma(F^{-1}(TN)) \), so that \( F \) is a stable harmonic map.

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