Evaluation of Fifth Degree Elliptic Singular Moduli

Nikos Bagis
Stenimahou 5 Edessa, Pellas 58200
Edessa, Greece
nikosbagis@hotmail.gr

Keywords: Modular equations; Ramanujan; Continued Fractions; Elliptic Functions; Modular Forms; Polynomials; Algebraic Numbers

Abstract

Our main result in this article is a formula for the extraction of the solution of the fifth degree modular polynomial equation i.e. the value of \( k_{25n_0} \), when we know only two consecutive values \( k_{r_0} \) and \( k_{r_0/25} \). By this way we reduce the problem of solving the depressed equation if we known two consecutive values of the Elliptic singular moduli \( k_r \).

1 Introductory Definitions

For \(|q| < 1\), the Rogers Ramanujan continued fraction (RRCF) is defined as

\[
R(q) := \frac{q^{1/5}}{1+ \frac{q^1}{1+ \frac{q^2}{1+ \frac{q^3}{1+ \cdots}}}}
\]

(1)

From the Theory of elliptic functions the complete elliptic integral of the first kind is (see [3],[4],[5]):

\[
K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin^2(t)}} dt
\]

(2)

It is known that the inverse elliptic nome (singular moduli), \( k = k_r, k_r^2 = 1 - k_r^2 \) is the solution of:

\[
\frac{K (k_r')}{K (k)} = \sqrt{r}
\]

(3)

In what it follows we assume that \( r \in R_+^* \), (when \( r \) is positive rational then \( k_r \) is algebraic). The function \( k_r \) can be evaluated in certain cases exactly.

Continuing we define:

\[
f(-q) := \prod_{n=1}^{\infty} (1 - q^n)
\]

(4)
Also holds the following relation of Ramanujan (see [1],[2],[8]):

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}$$  (5)

We can write the function $f$ using elliptic functions. It holds

$$f(-q)^8 = \frac{2^{8/3}}{\pi^4} q^{-1/3}(k_r)^2/3(k'_r)^8/3 K(k_r)^4$$  (6)

also holds (see [3]):

$$f(-q^2)^6 = \frac{2k_r k'_r K(k_r)^3}{\pi^3 q^{1/2}}$$  (7)

2 Known results on Rogers-Ramanujan continued fraction

**Theorem 2.1** (see also [6])

If $q = e^{-\pi \sqrt{r}}$ and $r$ real positive, then we define

$$a = a_r := \left(\frac{k'_r}{k'^2_{25r}}\right)^2 \sqrt{\frac{k_r}{k'_{25r}}} M_5(r)^{-3}$$  (8)

Then

$$R(q) = \left(-\frac{11}{2} - a_r + \frac{1}{2} \sqrt{125 + 22a_r + a^2_r}\right)^{1/5},$$  (9)

where $M_5(r)$ is root of: $(5Y - 1)^5(1 - Y) = 256(k_r)^2(k'_r)^2 Y$.

**Proof.**

Suppose that $N = n^2\mu$, where $n$ is positive integer and $\mu$ is positive real then holds that

$$K[n^2\mu] = M_n(\mu)K[\mu],$$  (10)

where $K[\mu] := K(k_\mu)$

The following formula for $M_5(r)$ is known

$$(5M_5(r) - 1)^5(1 - M_5(r)) = 256(k_r)^2(k'_r)^2 M_5(r)$$  (11)

Thus if we use (6),(7),(11),(12), we get:

$$R^{-5}(q) - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)} = a = a_r = \left(\frac{k'_r}{k'^2_{25r}}\right)^2 \sqrt{\frac{k_r}{k'_{25r}}} M_5(r)^{-3}$$  (12)

Solving with respect to $R(q)$ we get the result.
Theorem 2.2 (see [6],[7])

\[ R(q) = \left( -\frac{11}{2} - \frac{a_r}{2} + \frac{1}{2} \sqrt{125 + 22a_r + a_r^2} \right)^{1/5}. \]  (11)

with \( w^2 = k_r k_{25r}, (w')^2 = k'_r k'_{25r} \).

\[ k_r^6 + k_r^4(-16 + 10k_r^2)w + 15k_r^4w^2 - 20k_r^3w^3 + 15k_r^2w^4 + k_r(10 - 16k_r^2)w^5 + w^6 = 0 \]  (14)

Once we know \( k_r \) we can evaluate \( w \) from the above equation (15) and hence the \( k_{25r} \). Hence the problem reduces to solve the 6-th degree equation (15), which under the change of variable \( w = \sqrt{k_r k_{25r}} \) reduces to the 'depressed equation' (see [4]):

\[ u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0 \]  (15)

with \( u^6 = k_r^2, v^6 = k_{25r}^2 \).

For solving the depressed equation and the Rogers-Ramanujan Continued fraction we need a relation of the form

\[ k_{25r} = \Phi(k_r). \]  (16)

But such a construction of the root of the depressed equation is not found yet. In the next section we give a relation of the form

\[ k_{25r} = \Phi(k_r, k_r/25), \]  (17)

which is the main result of this paper.

3 The Evaluation of the 5th Degree Modular Equation

Let \( q = e^{-\pi\sqrt{r}}, r > 0 \) and \( v_r = R(q) \), then it have been proved by Ramanujan that

\[ v_{r/25}^5 = v_r \frac{1 - 2v_r + 4v_r^2 - 3v_r^3 + v_r^4}{1 + 3v_r + 4v_r^2 + 2v_r^3 + v_r^4}, \]  (18)

also is

\[ a_r = \frac{f(-q)^6}{qf(-q^5)^6}. \]  (19)

Then from Theorem 2.1

\[ R(q) = \left( -\frac{11}{2} - \frac{a_r}{2} + \frac{1}{2} \sqrt{125 + 22a_r + a_r^2} \right)^{1/5}. \]  (20)

Set

\[ t_r = \frac{11}{2} + \frac{a_r}{2} \text{ and } t_r = i \sin(\theta_r), \]  (21)
then
\[ a_r = 2i \sin(\theta_r) - 11 \text{ and } R(q) = e^{-i\theta_r/5} \] (22)

The function \( \theta_r \) can take complex values.

From the above and (19) we get the following modular equation of \( a_r \):
\[ a_r/25 = Q(a_r) \] (23)

where
\[ y = \text{arcsinh} \left( \frac{11 + x}{2} \right) \]

and
\[ Q(x) = \frac{(-1 - e^{\frac{1}{2}y} + e^{\frac{3}{2}y})^5}{(e^{\frac{1}{2}y} - e^{\frac{3}{2}y} + 2e^{\frac{5}{2}y} - 3e^{\frac{7}{2}y} + 5e^{\frac{9}{2}y} + 2e^{\frac{11}{2}y} + e^{\frac{13}{2}y} + e^{\frac{15}{2}y})} \] (24)

Assume now the equation
\[ \frac{X^2}{\sqrt{5}Y} - \frac{\sqrt{5}Y}{X^2} = \frac{1}{\sqrt{5}} (Y^3 - Y^{-3}) \] (25)

with the solutions
\[ Y = U(X) = \sqrt{-\frac{5}{3x^2} + \frac{25}{3x^2h(x)} + \frac{x^4}{h(x)} + \frac{h(x)}{3x^2}} \] (26)

where
\[ h(x) = \left(-125 - 9x^6 + 3\sqrt{3}\sqrt{-125x^6 - 22x^{12} - x^{18}}\right)^{1/3} \]

\[ U^*(Y) = X = \sqrt{-\frac{1}{2Y^2} + \frac{Y^4}{2} + \frac{\sqrt{1 + 18Y^6 + Y^{12}}}{2Y^2}}. \] (27)

From [8] we have the following:

**Theorem 3.1**

If
\[ A = \frac{f(-q^2)}{q^{1/3}f(-q^{10})} = a_{4r}^{1/6} \text{ and } V' = \frac{G_{25r}}{G_r} \]

then
\[ \frac{A^2}{\sqrt{5}V'} = \frac{1}{\sqrt{5}} \left( V'^3 - V'^{-3} \right) \] (28)

**Theorem 3.2**

\[ \frac{G_r}{G_{r/25}} = U \left[ Q^{1/6} \left[ U^{*6} \left[ \frac{G_{25r}}{G_r} \right] \right] \right] \] (29)
Proof.

Set
\[ A = \left( a_{4r/25} \right)^{1/6} \text{ and } V' = \frac{G_r}{G_{r/25}}, \]

then from Theorem 3.1 and (24), (25), (27), (28) we have
\[ \frac{G_r}{G_{r/25}} = U\left[\left( a_{4r/25} \right)^{1/6} \right] = U\left[ Q^{1/6} (a_{4r}) \right] = U \left[ Q^{1/6} \left( U^* \left( \frac{G_{25r}}{G_r} \right) \right) \right], \]

which completes the proof.

Continuing we have
\[ G_r = 2^{-1/12} (k_r k'_r)^{-1/12} \]
and
\[ \left( \frac{k_r k'_r}{k_{r/25} k'_{r/25}} \right)^{-1/12} = U \left[ Q \left( U^* \left( \left( \frac{k_{25r} k'_{25r}}{k_r k'_r} \right)^{-1/12} \right)^6 \right) \right]^{1/6} \]
from
\[ k_{1/r} = k'_r, \]
we get
\[ \left( \frac{k_{1/r} k'_{1/r}}{k_{25/r} k'_{25/r}} \right)^{-1} = U \left[ Q \left( U^* \left( \left( \frac{k_{1/(25r)} k'_{1/(25r)}}{k_{1/r} k'_{1/r}} \right)^{-1/12} \right)^6 \right) \right]^{1/12} \]
and setting \( r \to 1/r \):

Theorem 3.3

\[ \sqrt[12]{\frac{k_{25r} k'_{25r}}{k_r k'_r}} = U \left[ Q^{1/6} \left[ U^* \left[ \left( \frac{k_{1/r} k'_{1/r}}{k_{25/r} k'_{25/r}} \right)^{1/12} \right] \right] \right] \]

Once we know \( k_{r_0} \) and \( k_{r_0/25} \) we can evaluate in closed form the \( k_{25r_0} \). Hence if we repeat the process we can find any higher or lower order of \( k_{25^n r_0} \) in closed radicals form, for \( n \in \mathbb{Z} - \{0, -1\} \).

Example 1.

\[ R(e^{-\pi})^{-5} - 11 - R(e^{-\pi})^5 = \frac{125}{2} \left( 1147 + 513\sqrt{5} - \sqrt{2 \left( 1315405 + 588267\sqrt{5} \right)} \right) \]
Example 2.

\[k_{1/5} = \sqrt[5]{\frac{9 + 4\sqrt{5} + 2\sqrt{38 + 17\sqrt{5}}}{18 + 8\sqrt{5}}}\]
\[k_5 = \sqrt[5]{\frac{9 + 4\sqrt{5} - 2\sqrt{38 + 17\sqrt{5}}}{18 + 8\sqrt{5}}}\]
\[k_{125} = \sqrt[25]{\frac{1}{2} - \frac{1}{2} \sqrt{1 - (9 - 4\sqrt{5})P[1]^2}}, \quad (34)\]

where

\[P(x) = P[x] = U^{12}(Q^{1/6}(U^{x^{1/12}}))\] \(\quad (35)\)

Hence

\[R(e^{-x\sqrt{5}})^5 - 11 - R(e^{-x\sqrt{5}})^5 = A \left( \frac{9 + 4\sqrt{5} - 2\sqrt{38 + 17\sqrt{5}}}{9 + 4\sqrt{5}} \right)^{3/4}\]
\[\quad \left[- \left( 9 + 4\sqrt{5} + 2\sqrt{38 + 17\sqrt{5}} \right)^{1/4} \sqrt{P[1]^2} + \right.\]
\[\left. + \sqrt{9 + 4\sqrt{5} + 2\sqrt{38 + 17\sqrt{5}}} \left( 1 - \sqrt{1 + (-9 + 4\sqrt{5})P[1]^2} \right)^{1/4} \right.\]
\[\left. + (9 + 4\sqrt{5})^{1/4} \left( 1 + \sqrt{1 + (-9 + 4\sqrt{5})P[1]^2} \right)^{1/4} \right)^3, \quad (36)\]

where

\[A = \frac{\sqrt{405 + 47\sqrt{5}} + \sqrt{\frac{1}{2} \left( 11111 + 4969\sqrt{5} \right)}}{P[1]^2} \left( 1 - \sqrt{1 + (-9 + 4\sqrt{5})P[1]^2} \right)^{3/4}\]

Example 3.

It is

\[k_1 = \frac{1}{\sqrt{2}}\]
\[k_{25} = \frac{1}{\sqrt{2 \left( 51841 + 23184\sqrt{5} + 12\sqrt{37325880 + 16692641\sqrt{5}} \right)}}\]

Hence

\[k_{625}k'_{625} = \frac{1}{2(161 + 72\sqrt{5})} P \left[ 161 - 72\sqrt{5} \right]\]

and hence

\[k_{625} = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{1 - \left( \frac{P \left[ 161 - 72\sqrt{5} \right]}{161 + 72\sqrt{5}} \right)^2}} \quad (36)\]
From Theorem 2.2 we have

\[ R(e^{-5\pi})^{-5} - 11 - R(e^{-5\pi})^5 = B \cdot \sqrt[p]{1 + (161 - 72\sqrt{5}) \sqrt{q}}^{1/4} + \]

\[ \sqrt[p]{2p - 1} \left(51841 + 23184\sqrt{5} - (161 + 72\sqrt{5}) \sqrt[4]{q} \right)^{1/4} s - \]

\[ (51841 - 23184\sqrt{5})^{1/4} P^{3/4} \sqrt[4]{q} \]

\[ \times \left[ (51841 + 23184\sqrt{5})^{1/4} \sqrt{2p - 1} \right]^{-3}, \]

where we have set

\[ p = 51841 + 23184\sqrt{5} + 12\sqrt{37325880 + 16692641\sqrt{5}} \]

\[ q = 51841 + 23184\sqrt{5} - P^{161 - 72\sqrt{5}} \]

\[ s = (1 - 22x + 14x^2 + 2x^3 + x^4)_1 \]

\[ B = 2(1 + 2x + 14x^2 - 22x^3 + x^4)_2 \cdot (1 + 414728x + 414744x^2 + 32x^3 + 16x^4)_1 \times \]

\[ j^{-1/4}(1 - 4x + 103686x^2 - 207364x^3 + x^4)_{-3/2} \times \]

\[ \times \left[ j : (1 - 22x + 14x^2 + 2x^3 + x^4)_1^2 + (16 + 1658912x + 414744x^2 + 8x^3 + x^4)_2 \right]^{-1} \]

\[ j = 1 + (-161 + 72\sqrt{5}) \sqrt[10]{51841 + 23184\sqrt{5} - P^{161 - 72\sqrt{5}}} \]

The index \((H(x))_h\) means the \(h\)-th root of \(H(x) = 0\) with the notation of Mathematica program.

**Example 4.**

If

\[ P^{(n)}(x) = (P \circ \ldots \circ P)(x), \]

with \(P\) that of (35) then:

\[ k_{25^n} = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{1 - \left( \frac{P^{(n-1)}(t_0)}{t_0} \right)^2}}, \quad (37) \]

in full closed form expansion, where \(t_0 = 161 - 72\sqrt{5}\) and \(t_0^{-} = 161 + 72\sqrt{5}\).
References

[1]: B.C.Berndt. 'Ramanujan's Notebooks Part III'. Springer Verlang, New York (1991).
[2]: B.C.Berndt. 'Ramanujan's Notebooks Part V'. Springer Verlang, New York (1998).
[3]: E.T.Whittaker and G.N.Watson. 'A course on Modern Analysis'. Cambridge U.P. (1927).
[4]: J.V. Armitage W.F. Eberlein. 'Elliptic Functions'. Cambridge University Press. (2006).
[5]: J.M. Borwein and P.B. Borwein. 'Pi and the AGM'. John Wiley and Sons, Inc. New York, Chichester, Brisbane, Toronto, Singapore. (1987).
[6]: Nikos Bagis. 'The complete evaluation of Rogers-Ramanujan and other continued fractions with elliptic functions'. arXiv:1008.1304v1 [math.GM] 7 Aug 2010.
[7]: Nikos Bagis. 'Parametric Evaluations of the Rogers Ramanujan Continued Fraction'. International Journal of Mathematics and Mathematical Sciences. Vol. 2011
[8]: Bruce C. Berndt, Heng Huat Chan, Sen-Shan Huang, Soon-Yi Kang, Jaebum Sohn and Seung Hwan Son. 'The Rogers-Ramanujan Continued Fraction'. (page stored in the Web).
[9]: D. Broadhurst: Solutions by radicals at Singular Values $k_N$ from New Class Invariants for $N = 3 \ mod \ 8$. arXiv:0807.2976 (math-phy).