A Logarithmic Additive Integrality Gap for Bin Packing

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Abstract

For bin packing, the input consists of \( n \) items with sizes \( s_1, \ldots, s_n \in [0, 1] \) which have to be assigned to a minimum number of bins of size 1. Recently, the second author gave an LP-based polynomial time algorithm that employed techniques from discrepancy theory to find a solution using at most \( OPT + O(\log OPT \cdot \log OPT) \) bins.

In this paper, we present an approximation algorithm that has an additive gap of only \( O(\log OPT) \) bins, which matches certain combinatorial lower bounds. Any further improvement would have to use more algebraic structure. Our improvement is based on a combination of discrepancy theory techniques and a novel 2-stage packing: first we pack items into containers; then we pack containers into bins of size 1. Apart from being more effective, we believe our algorithm is much cleaner than the one of Rothvoss.

1 Introduction

One of the classical combinatorial optimization problems that is studied in computer science is Bin Packing. It appeared as one of the prototypical \( \mathbf{NP} \)-hard problems already in the book of Garey and Johnson [GJ79] but it was studied long before in operations research in the 1950’s, for example by [Eis57]. We refer to the survey of Johnson [CGJ84] for a complete historic account. Bin packing is a good example to study the development of techniques in approximation algorithms as well. The 1970’s brought simple greedy heuristics such as First Fit, analyzed by Johnson [Joh73] which requires at most \( 1.7 \cdot OPT + 1 \) bins and First Fit Decreasing [DU74], which yields a solution with at most \( \frac{11}{9} OPT + 4 \) bins (see [Dós07] for a tight bound of \( \frac{11}{9} OPT + \frac{6}{9} \)). Later, an asymptotic PTAS was developed by Fernandez de la Vega and Luecker [FdlVL81]. One of their main technical contributions was an item grouping technique to reduce the number of different item types. The algorithm of De la Vega and Luecker finds solutions using at most \( (1 + \varepsilon)OPT + O(\frac{1}{\varepsilon}) \) bins, while the running time is either of the form \( O(n^{f(\varepsilon)}) \) if one uses dynamic programming or of the form \( O(n \cdot f(\varepsilon)) \) if one applies linear programming techniques.

A big leap forward in approximating bin packing was done by Karmarkar and Karp in 1982 [KK82]. First of all, they argue how a certain exponential size LP can be approximately solved in polynomial time; secondly they provide a sophisticated rounding scheme which

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produces a solution with at most $OPT + O(\log^2 OPT)$ bins, corresponding to an asymptotic FPTAS.

It will be convenient throughout this paper to allow a more compact form of input, where $s \in [0, 1]^n$ denotes the vector of different item sizes and $b \in \mathbb{N}^n$ denotes the multiplicity vector, meaning that we have $b_i$ copies of item type $i$. In this notation we say that $\sum_{i=1}^n b_i$ is the total number of items. The linear program that we mentioned earlier is called the Gilmore-Gomory LP relaxation \cite{Eis57,GG61} and it is of the form

$$\min \{ 1^T x \mid Ax \geq b, x \geq 0 \}.$$  \hspace{1cm} (1)

Here, the constraint matrix $A$ consists of all column vectors $p \in \mathbb{Z}_{\geq 0}^n$ that satisfy $\sum_{i=1}^n p_i s_i \leq 1$. The linear program has variables $x_p$ that give the number of bins that should be packed according to the pattern $p$. We denote the value of the optimal fractional solution to (1) by $OPT_f$, and the value of the best integral solution by $OPT$. As we mentioned before, the linear program (1) does have an exponential number of variables, but only $n$ constraints. A fractional solution $x$ of cost $1^T x \leq OPT_f + \delta$ can be computed in time polynomial in $\sum_{i=1}^n b_i$ and $1/\delta$ \cite{KK82} using the Grötschel-Lovász-Schrijver variant of the Ellipsoid method \cite{GLS81}. An alternative and simpler way to solve the LP approximately is via the Plotkin-Shmoys-Tardos framework \cite{PST95} or the multiplicative weight update method. See the survey of \cite{AHK12} for an overview.

The best known lower bound on the integrality gap of the Gilmore-Gomory LP is an instance where $OPT = \lceil OPT_f \rceil + 1$; Scheithauer and Terno \cite{ST97} conjecture that these instances represent the worst case additive gap. While this conjecture is still open, it is understandable that the best approximation algorithms are based on rounding a solution to this amazingly strong Gilmore Gomory LP relaxation. For example, the Karmarkar-Karp algorithm operates in $\log n$ iterations in which one first groups the items such that only $\frac{1}{2} \sum_{i=1}^n s_i$ many different item sizes remain; then one computes a basic solution $x$ and buys $\lfloor x_p \rfloor$ times pattern $p$ and continues with the residual instance. The analysis provides a $O(\log^2 OPT)$ upper bound on the additive integrality gap of (1).

The rounding mechanism in the recent paper of the second author \cite{Rot13} uses an algorithm by Lovett and Meka that was originally designed for discrepancy minimization. The Lovett-Meka algorithm \cite{LM12} can be conveniently summarized as follows:

**Theorem 1** (Lovett-Meka ’12). Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be vectors with $x_{\text{start}} \in [0, 1]^n$ and parameters $\lambda_1, \ldots, \lambda_m \geq 0$ so that $\sum_{j=1}^m e^{-\lambda_j^2/16} \leq \frac{n}{16}$. Then in randomized polynomial time one can find a vector $x_{\text{end}} \in [0, 1]^n$ so that $|\langle x_{\text{end}} - x_{\text{start}}, v_j \rangle| \leq \lambda_j \cdot \| v_j \|_2$ for all $j \in \{1, \ldots, m\}$ and at least half of the entries of $x_{\text{end}}$ are in $0/1$.

Intuitively, the points $x_{\text{end}}$ satisfying the linear constraints $|\langle x_{\text{end}} - x_{\text{start}} \rangle| \leq \lambda_j \cdot \| v_j \|_2$ form a polytope and the distance of the $j$th hyperplane to the start point is exactly $\lambda_j$. Then the condition $\sum_{j=1}^m e^{-\lambda_j^2/16} \leq \frac{n}{16}$ essentially says that the polytope is going to be “large enough”. The algorithm of \cite{LM12} itself consists of a random walk through the polytope. For more details, we refer to the very readable paper of \cite{LM12}. 


The bin packing approximation algorithm of Rothvoss [Rot13] consists of logarithmically many runs of Lovett-Meka. To be able to use the Lovett-Meka algorithm effectively, Rothvoss needs to rebuild the instance in each iteration and “glue” clusters of small items together to larger items. His procedure is only able to do that for items that have size at most \( \frac{1}{\text{polylog}(n)} \) and each of the iterations incurs a loss in the objective function of \( O(\log \log n) \). In contrast we present a procedure that can even cluster items together that have size up to \( \Omega(1) \). Moreover, Rothvoss’ algorithm only uses two types of parameters for the error parameters, namely \( \lambda_j \in \{0, O(\sqrt{\log \log n})\} \). In contrast, we use the full spectrum of parameters to achieve only a constant loss in each of the logarithmically many iterations.

### 1.1 Our contribution

Our main contribution is the following theorem:

**Theorem 2.** For any Bin Packing instance \((s, b)\) with \(s_1, \ldots, s_n \in [0, 1]\), one can compute a solution with at most \( \text{OPT} + O(\log \text{OPT}) \) bins, where \( \text{OPT} \) denotes the optimal value of the Gilmore-Gomory LP relaxation. The algorithm is randomized and the expected running time is polynomial in \( \Sigma b_i \).

The recent book of Williamson and Shmoys [WS11] presents a list of 10 open problems in approximation algorithms. Problem #3 in the list is whether the Gilmore-Gomory LP has a constant integrality gap; hence we make progress towards that question.

We want to remark that the original algorithm of Karmarkar and Karp has an additive approximation ratio of \( O(\log \text{OPT} \cdot \log(\max_i j \frac{s_i}{s_j})) \). For 3-partition instances where all item sizes are strictly between \( \frac{1}{4} \) and \( \frac{1}{2} \), this results in an \( O(\log n) \) guarantee, which coincides with the guarantees of Rothvoss [Rot13] and this paper if applied to those instances. A paper of Eisenbrand et al. [EPR11] gives a reduction of those instances to minimizing the discrepancy of 3 permutations. Interestingly, shortly afterwards Newman and Nikolov [NNN12] showed that there are instances of 3 permutations that do require a discrepancy of \( \Omega(\log n) \). It seems unclear how to realize those permutations with concrete sizes in a bin packing instance — however any further improvement for bin packing even in that special case with item sizes in \([\frac{1}{4}, \frac{1}{2}][\) would need to rule out such a realization as well. The second author is willing to conjecture that the integrality gap for the Gilmore Gomory LP is indeed \( \Theta(\log n) \).
2 A 2-stage packing mechanism

It is well-known that for the kind of approximation guarantee that we aim to achieve, one can assume that the items are not too tiny. In fact it suffices to prove an additive gap of \( O(\log \max(n, \frac{1}{s_{\min}})) \) where \( n \) is the number of different item sizes and \( s_{\min} \) is a lower bound on all item sizes. Note that in the following, “polynomial time” means always polynomial in the total number of items \( \sum_{i=1}^{n} b_i \).

**Lemma 3.** Assume for a monotone function \( f \), there is a polynomial time \( \text{OPT}_f + f(\max(n, \frac{1}{s_{\min}})) \) algorithm for Bin Packing instances \((s, b)\) with \( s \in [0, 1]^n \) and \( s_1, \ldots, s_n \geq s_{\min} > 0 \). Then there is a polynomial time algorithm that for all instances finds a solution with at most \( \text{OPT}_f + f(\text{OPT}_f) + O(\log \text{OPT}_f) \) bins.

For a proof, we refer to Appendix A. From now on we assume that we have \( n \) different item sizes with all sizes satisfying \( s_i \geq s_{\min} \) for some given parameter \( s_{\min} \) (as a side remark, the reduction in Lemma 3 will choose \( s_{\min} = \Theta(\frac{1}{\text{OPT}_f}) \)). Starting from a fractional solution \( x \) to \( \{1\} \) our goal is to find an integral solution of cost \( 1^T x + O(\log \max(n, \frac{1}{s_{\min}})) \). Another useful standard argument is as follows:

**Lemma 4.** Any bin packing instance \((s, b)\) can be packed in polynomial time into at most \( 2 \sum_{i=1}^{n} s_i b_i + 1 \) bins.

**Proof.** Simply assign the items greedily and open new bins only if necessary. If we end up with \( k \) bins, then at least \( k - 1 \) of them are at least half full, which means that \( \sum_{i=1}^{n} s_i b_i \geq \frac{1}{2} \cdot (k - 1) \). Rearranging gives the claim.

Now, we come to the main mechanism that allows us the improvement over Rothvoss [Rot13].

Consider an instance \((s, b)\) and a fractional LP solution \( x \). We could imagine the assignment of items in the input to slots in \( x \) as a fractional matching in a bipartite graph, where we have nodes \( i \in [n] \) on the left hand side, each with demand \( b_i \) and nodes \((p, i)\) on the right hand side with supply \( x_{pi} \cdot p_i \). Instead, our idea is to employ a 2-stage packing: first we pack items into containers, then we pack containers into bins. Here, a container is a multiset of items. Before we give the formal definition, we want to explain our construction with a small example that is visualized in Figure 1. The example has \( n = 3 \) items of size \( s = (0.3, 0.2, 0.1) \) and multiplicity vector \( b = (2, 1, 7) \). Those items are assigned into containers \( C_1, C_2, C_3 \) which also have multiplicities. In this case we have \( y_{C_1} = y_{C_2} = 1 \) copies of the first two containers and \( y_{C_3} = 2 \) copies of the third container. Moreover, in our example we have 3 patterns \( p_1, p_2, p_3 \) each with fractional value \( x_{p_1} = x_{p_2} = x_{p_3} = \frac{1}{2} \). For example, item 2 is packed into container \( C_1 \) and that container is assigned with a fractional value of \( \frac{1}{2} \) each to pattern \( p_2 \) and \( p_3 \). The reader might have noticed that we do allow that some copies of item \( i_3 \) are assigned to slots of a larger item \( i_2 \). On the other hand, we have \( b_3 = 7 \) copies of item 3, but only 6 slots in containers that we could use. So there will be 1 unit that we won’t be able to pack. Similarly, we have \( y_{C_3} = 2 \) copies of container \( C_3 \), but only \( \frac{2}{3} \) slots in the patterns. Later we will say that the deficiency of the 2-stage packing is \( 1 \cdot s_3 + \frac{2}{3} \cdot s(C_3) \) where \( s(C_3) \) is the size of container \( C_3 \).

Now, we want to give the formal definitions. We call any vector \( C \in \mathbb{Z}_{\geq 0}^n \) with \( \sum_{i=1}^{n} s_i C_i \leq 1 \) a container. Here \( C_i \) denotes the number of copies of item \( i \) that are in the container. The size of the container is denoted by \( s(C) := \sum_{i=1}^{n} C_i s_i \). Let \( \mathcal{C} := \{ C \in \mathbb{Z}_{\geq 0}^n \mid s(C) \leq 1 \} \) be the set of all containers. As we will pack containers into bins, we want to define a pattern as a vector of
the form $p \in \mathbb{Z}_{\geq 0}^C$ where $p_C$ denotes the number of times that the pattern contains container $C$. Of course the sum of the sizes of the containers should be at most 1, thus

$$\mathcal{P} := \left\{ p \in \mathbb{Z}_{\geq 0}^C \mid \sum_{C\in\mathcal{C}} p_C \cdot s(C) \leq 1 \right\}$$

is set of all (valid) patterns.

Now suppose we have an instance $(s, b)$ and a fractional vector $x \in \mathbb{R}^P_{\geq 0}$. To keep track of which containers should be used in the intermediate packing step, we also need to maintain an integral vector $y \in \mathbb{Z}^B_{\geq 0}$.

We say that a bipartite graph $G = (V_L \cup V_R, E)$ is a packing graph if each $v \in V_L \cup V_R$ has an associated size $s(v) \in [0,1]$ and multiplicity $\text{mult}(v) \in \mathbb{R}_{\geq 0}$, and the edge set is given by $E = \{(u, v) \in V_L \times V_R \mid s(u) \leq s(v)\}$. An assignment in a packing graph is a function $a : E \to \mathbb{R}_{\geq 0}$ so that for any $v \in V$, we have $\sum_{e \in \delta(v)} a(e) \leq \text{mult}(v)$, where $\delta(v)$ denotes the set of edges incident to $v$. The deficiency of a packing graph is the total size of left nodes that fail to be packed in an optimal assignment. That is,

$$\text{def}(G) := \min_{a \text{ assignment of } G} \left\{ \sum_{v \in V_L} s(v) \cdot (\text{mult}(v) - a(\delta(v))) \right\}.$$ 

The edge set of those graphs is extremely simple, so that one can directly obtain the deficiency as follows:

**Observation 1.** For any packing graph, an optimal assignment $a : E \to \mathbb{R}_{\geq 0}$ which attains $\text{def}(G)$ can be obtained as follows: go through the nodes $v \in V_R$ in any order. Take the node $u \in V_L$ of maximum size that has some capacities left and satisfies $s(u) \leq s(v)$. Increase $a(u, v)$ as much as possible.
We then define the deficiency of the pair \((x, y)\) with item multiplicities \(b\) to be the sum
\[
def_b(x, y) := \text{def}(G_1(b, y)) + \text{def}(G_2(x, y)).
\]
In later sections we will often leave off the \(b\) to simplify notation.

We should discuss why the 2-stage packing via the containers is useful. First of all, it is easy to find some initial configuration.

**Lemma 5.** For any bin packing instance \((s, b)\), one can compute a “starting solution” \(x \in \mathbb{R}_{\geq 0}^n\) and \(y \in \mathbb{Z}_{\geq 0}^C\) in polynomial time so that \(1^T x \leq \text{OPT}_f + 1\) and \(\text{def}(x, y) = 0\) with \(|\text{supp}(x)| \leq n\).

**Proof.** As we already argued, one can compute a fractional solution \(x\) for (1) in polynomial time that has cost \(1^T x \leq \text{OPT}_f + 1\). We simply use singleton containers \([i]\) for all items \(i \in \{1, \ldots, n\}\) and set \(y_{[i]} := b_i\).

Next, we argue that our notation of deficiency was actually meaningful in recovering an assignment of items to bins.

**Lemma 6.** Suppose that \(x \in \mathbb{Z}_{\geq 0}^P, y \in \mathbb{Z}_{\geq 0}^C\) are both integral. Then there is a packing of all items into at most \(1^T x + 2\text{def}(x, y) + 1\) bins.

**Proof.** Since \(x\) and \(y\) are both integral, all multiplicities in \(G_1\) and \(G_2\) will be integral and we can find two integral assignments \(a_1, a_2\) attaining \(\text{def}(x, y)\). Buy all the patterns suggested by \(x\). Use \(a_2\) to pack the containers in \(y\). Then use \(a_1\) to map the items to containers. There are some items that will not be assigned — their total size is \(\text{def}(G_1(b, y))\). Moreover, there might also be containers in \(y\) that have not been assigned; their total size is \(\text{def}(G_2(x, y))\). We pack items and containers greedily into at most \(2\text{def}(x, y) + 1\) many extra bins using Lemma 4.

In each iteration of our algorithm, it will be useful for us to be able fix the integral part of \(x\) and focus solely on the fractional part.

**Lemma 7.** Suppose \(x \in \mathbb{R}_{\geq 0}^P, y \in \mathbb{Z}_{\geq 0}^C,\) and \(b \in \mathbb{Z}_{\geq 0}^n\). If \(\hat{x}_p = \lfloor x_p \rfloor\) for all patterns \(p\), then there exist vectors \(\hat{y} \in \mathbb{Z}_{\geq 0}^C, \hat{b} \in \mathbb{Z}_{\geq 0}^n\) with \(\hat{b} \leq b\) so that \(\text{def}_b(\hat{x}, \hat{y}) = 0\) and \(\text{def}_{\hat{b} - b}(x - \hat{x}, y - \hat{y}) = \text{def}_b(x, y)\).
Proof. Let us imagine that we replace each node \((C, p)\) in \(G_2(x, y)\) with two copies, a “red” node and a “blue” node. The red copy receives an integral multiplicity of \(\text{mult}_{\text{red}}(C, p) = \hat{x}_p \cdot p_C\) while the blue copy receives a fractional multiplicity of \(\text{mult}_{\text{blue}}(C, p) = (x_p - \hat{x}_p) \cdot p_C\). Now we apply Observation\(^\dagger\) to find the best assignment \(a\). Crucially, we set up the order of the right hand side nodes so that we first process the red integral nodes and then the blue fractional ones. Note that the assignment that this greedy procedure computes is optimal and moreover, the assignments for red nodes will be integral. For each container \(C\) on the left, we define \(\hat{y}_C\) to be the total red multiplicity of its targets under this optimal assignment. Then \(\text{def}(G_2(\hat{x}, \hat{y})) = 0\) and \(\text{def}(G_2(x - \hat{x}, y - \hat{y})) = \text{def}(G_2(x, y))\). In the graph \(G_1(b, y)\), all multiplicities are integral anyway, so we can trivially find an integral vector \(\hat{b}\) so that \(\text{def}(G_1(\hat{b}, \hat{y})) = 0\) and \(\text{def}(G_1(b - \hat{b}, y - \hat{y})) = \text{def}(G_1(b, y))\). \(\square\)

Define \(\text{supp}(x) := \{p \in \mathcal{P} : x_p > 0\}\) as the support of \(x\) and \(\text{frac}(x) := \{p \in \mathcal{P} : 0 < x_p < 1\}\) as the patterns in \(p\) that are still fractional. Now we have enough notation to state our main technical theorem:

**Theorem 8.** Let \((s, b)\) be an instance with \(s_1, \ldots, s_n \geq s_{\text{min}} > 0\). Let \(y \in \mathbb{Z}_{\geq 0}^s\) and \(x \in \{0, 1\}^\mathcal{P}\) with \(|\text{supp}(x)| \geq L \log \left(\frac{1}{s_{\text{min}}}\right)\), where \(L\) is a large enough constant. Then there is a randomized polynomial time algorithm that finds \(\tilde{y} \in \mathbb{Z}_{\geq 0}^s\) and \(\tilde{x} \in \mathbb{R}_{\geq 0}^\mathcal{P}\) with \(1^T \tilde{x} = 1^T x\) and \(\text{def}(\tilde{x}, \tilde{y}) \leq \text{def}(x, y) + O(1)\) while \(|\text{frac}(\tilde{x})| \leq \frac{1}{2} |\text{frac}(x)|\).

While it will take the remainder of this paper to prove the theorem, the algorithm behind the statement can be split into the following two steps:

(I) **Rebuilding the container assignment:** We will change the assignments for the pair \((x, y)\) so that for every container in size class \(\sigma\) the patterns in \(\text{supp}(x)\) use, they use nearly \((\frac{1}{7})^{1/2}\) copies, while no individual pattern in \(\text{supp}(x)\) contains more than \((\frac{1}{7})^{1/4}\) copies of the same container.

(II) **Application of Lovett-Meka:** We will apply the Lovett-Meka algorithm to sparsify the fractional solution \(x\). Here, the vectors \(v_j\) that comprise the input for the LM-algorithm will correspond to sums over intervals of rows of the constraint matrix \(A\). Recall that the error bound provided by Lovett-Meka crucially depends on the lengths \(\|v_j\|_2\). The procedure in (I) will ensure that the Euclidean length of those vectors is small.

Once we have proven Theorem\(^\ddagger\) the main result easily follows:

**Proof of Theorem\(^\ddagger\)** We compute a fractional solution \(x\) to \((\dagger)\) of cost \(1^T x \leq \text{OPT}_f + 1\). In fact, we can assume that \(x\) is a basic solution to the LP and hence \(|\text{supp}(x)| \leq n\). We construct a container assignment \(y\) consisting only of singletons, see Lemma\(^\ddagger\). Then for \(\log(n)\) iterations, we first use Lemma\(^\ddagger\) to split the current solution \(x\) as \(x = x^\text{int} + x^\text{frac}\) where \(x^\text{int} = \lfloor x_p \rfloor\) and obtain a corresponding split \(y = y^\text{int} + y^\text{frac}\). Then we run Theorem\(^\ddagger\) with input \((x^\text{frac}, y^\text{frac})\) and denote the result by \((\tilde{x}^\text{frac}, \tilde{y}^\text{frac})\). Finally we update \(x := x^\text{int} + \tilde{x}^\text{frac}\) and \(y := y^\text{int} + \tilde{y}^\text{frac}\).

As soon as \(|\text{frac}(x)| \leq O(\log \frac{1}{s_{\text{min}}} )\), we can just buy every pattern in \(\text{frac}(x)\). In each iteration the deficiency increases by at most \(O(1)\). At the end, we use Lemma\(^\ddagger\) to actually pack the items into bins. We arrive at a solution of cost \(\text{OPT}_f + O(\log \max\{n, \frac{1}{s_{\text{min}}}\})\) which is enough, using Lemma\(^\ddagger\) \(\square\)

We will describe the implementation of (I) in Section\(^\ddagger\) and then (II) in Section\(^\ddagger\).
3 Rebuilding the container assignment

In this section we assume that we are given \( x \in \{0, 1\}^P \) with \(|\text{supp}(x)| = m\). To ease notation, we will only write the nonzero parts of \( x \), so that if \( \text{supp}(x) = \{p_1, p_2, \ldots, p_m\} \), then \( x = (x_{p_i})_{i=1}^m \).

We update \( x \) by altering the patterns that make up its support. Even though some patterns could become identical, we continue to treat them as separate patterns.

Originally, we had defined \( A \) as the incidence matrix of the Gilmore Gomory LP in (1) where the rows correspond to items. Due to our 2-stage packing, we actually consider the patterns to be multi-sets of containers, not items anymore. Hence, let us for the rest of the paper redefine the meaning of \( A \). Now, the rows of \( A \) correspond to the containers in \( C \) ordered from largest to smallest, and columns represent the patterns in \( \text{supp}(x) \). As we perform the grouping and container-forming operations, we update the columns of the matrix. The resulting columns then yield a new fractional solution \( \tilde{x} \) by taking \( x_{p_i} \) copies of the pattern now in column \( i \).

We will now describe our grouping and container reassignment operations, keeping track of what happens to the fractional solution as well as to the corresponding matrix.

First, we need a lemma that tells us how rebuilding the fractional solution affects the deficiency. To have some useful notation, define \( \text{mult}(C, x) := \sum_{p \in P} \text{mult}(C, p) = \sum_{p \in P} x_p p_C \) to be the number of times that the patterns cover container \( C \in C \).

Now, if \( \sum_{s(C) \geq s} y_C \leq \sum_{s(C) \geq s} y_C \) for all \( s \geq 0 \), then we write \( y \leq \tilde{y} \). Moreover, if \( \sum_{s(C) \geq s} \text{mult}(C, \tilde{x}) \geq \sum_{s(C) \geq s} \text{mult}(C, x) \) for all \( s \geq 0 \), then we write \( \tilde{x} \geq x \). Observe that if \( y \leq \tilde{y} \) and \( \tilde{x} \geq x \), then \( \text{def}(G_2(\tilde{x}, \tilde{y})) \leq \text{def}(G_2(x, y)) \).

**Lemma 9.** Now suppose that \( t_\sigma \geq 0 \) is such that

\[
\sum_{s(C) \geq s} \text{mult}(C, \tilde{x}) \geq \begin{cases} 
\sum_{s(C) \geq s} \text{mult}(C, x) & \text{if } s > \sigma \\
\sum_{s(C) \geq s} \text{mult}(C, x) - t_\sigma & \text{if } s \leq \sigma
\end{cases}
\]

Then \( \text{def}(\tilde{x}, y) \leq \text{def}(x, y) + \sigma \cdot t_\sigma \).

**Proof.** Let \( C_0 \) be the largest container of size at most \( \sigma \), and let \( x' \) be the vector representing \( t_\sigma \) copies of the pattern containing a single copy of \( C_0 \). Then \( \tilde{x} + x' \geq x \), and so \( \text{def}(G_2(\tilde{x} + x', y)) \leq \text{def}(G_2(x, y)) \). But if \( y' \) is the vector representing \( t_\sigma \) copies of \( C_0 \), then \( \text{def}(G_2(\tilde{x}, y)) = \text{def}(G_2(\tilde{x} + x', y + y')) \), since we can find an optimal assignment taking the containers in \( y' \) to those of \( x' \). Since the total size of \( y' \) is at most \( \sigma \cdot t_\sigma \), we have \( \text{def}(G_2(\tilde{x}, y)) \leq \text{def}(G_2(\tilde{x} + x', y + y')) + \sigma t_\sigma \leq \text{def}(G_2(x, y)) + \sigma t_\sigma \), and therefore \( \text{def}(\tilde{x}, y) \leq \text{def}(x, y) + \sigma t_\sigma \).

If \( \sigma \) is a power of 2, say \( \sigma = 2^{-\ell} \) for \( \ell \in \mathbb{Z}_{\geq 0} \), then we say the size class of \( \sigma \) is the set of items with sizes between \( \frac{1}{2} \sigma \) and \( \sigma \). In this next lemma, we round containers in patterns down so that each container type in size class \( \sigma \) is either not used at all or is used at least \( \frac{\sigma}{\delta} \) times.

**Lemma 10 (Grouping).** Let \((s, b)\) be a bin packing instance with \( y \in \mathbb{Z}^C_{\geq 0} \) and \( x \in \{0, 1\}^P \). For any size class \( \sigma \) and \( \delta > 0 \), we can find \( \tilde{x} \in \{0, 1\}^P \) so that

1. \( 1^T \tilde{x} = 1^T x \)
2. \(|\text{supp}(\tilde{x})| \leq |\text{supp}(x)|\)
3. For each container type \( C \) in size class \( \sigma \), either \( \text{mult}(C, \tilde{x}) = 0 \) or \( s(C) \cdot \text{mult}(C, \tilde{x}) \geq \delta \). In all other size classes, the multiplicities of containers in patterns do not change.
4. $\text{def}(\tilde{x}, y) \leq \text{def}(x, y) + O(\delta)$.

**Proof.** Assume containers are sorted by size, from largest to smallest. Define $S_\delta$ to be the set of containers in size class $\sigma$ not satisfying condition (3) above. In other words, $S_\delta := \{C \in \text{size class } \sigma \mid 0 < s(C) \cdot \text{mult}(C, x) < \delta\}$.

For a subset $H \subset S_\delta$, define the weight of $H$ to be $w(H) := \sum_{C \in H} s(C) \cdot \text{mult}(C, x)$. Note that the weight of a single container is at most $\delta$. Hence we can partition $S_\delta = H_1 \cup H_2 \cup \ldots \cup H_r$ so that:

1. $w(H_k) \in [2\delta, 3\delta], \forall k = 1, \ldots, r - 1.$
2. $w(H_r) \leq 3\delta.$
3. $C \in H_k, C' \in H_{k+1}$ implies $s(C) \geq s(C').$

For each $k = 1, \ldots, r - 1$ and container $C \in H_k$, we replace containers of type $C$ in all patterns $p \in \text{frac}(x)$ with the smallest container type appearing in $H_k$. For all $C \in H_r$, remove containers of type $C$ from all patterns $p \in \text{frac}(x)$. Call the updated vector $\tilde{x}$. We see immediately that $1^T \tilde{x} = 1^T x$ and $|\text{supp}(\tilde{x})| \leq |\text{supp}(x)|$.

Moreover, since every container type $C$ appearing in $\tilde{x}$ now has an entire group using it, and the weight of each container didn't change by more than a factor of 2, we have $s(C) \cdot \text{mult}(C, \tilde{x}) \geq \delta$, and so condition (3) is satisfied. To complete the proof, it remains to show that $\text{def}(G_2(\tilde{x}, y)) \leq \text{def}(G_2(x, y)) + O(\delta)$.

Now, for any $i$, there is at most one group $H_k$ whose containers (partly) changed from being larger than $s(C_i)$ to smaller. The weight of this group is at most $3\delta$, so $\sum_{j \leq i} \text{mult}(C_j, x) - \sum_{j < i} \text{mult}(C_j, \tilde{x}) \leq \frac{6\delta}{\delta}$. Since this holds for all $i$, we can therefore apply Lemma 9 to conclude that $\text{def}(G_2(\tilde{x}, y)) \leq \text{def}(G_2(x, y)) + O(\delta)$.

We now remark what happens to the associated matrix $A$ under this grouping operation. Write $A, \tilde{A}$ as our original and updated matrices, and $A_C, \tilde{A}_C$ as the rows for container $C$. For container types $C$ in size class $\sigma$, either $\tilde{A}_C x = 0$ or $s(C) \cdot \tilde{A}_C x \geq \delta$. For all other size classes, $\tilde{A}_C = A_C$. In particular, notice that we have either $\|\tilde{A}_C\|_1 = 0$ or $s(C) \cdot \|\tilde{A}_C\|_1 \geq \delta$.

Before we introduce the next main lemma — how to reassign containers — we prove a useful result about decomposing packing graphs in a nice way. For a visualization of the following lemma, see Figure 2.

**Lemma 11.** Suppose $G = (V_r \cup V_r, E)$ is a left-integral packing graph as in Section 2, and that for every $v \in V_r$, we are given red and blue multiplicities so that $\text{mult}(v) = \text{mult}_{\text{red}}(v) + \text{mult}_{\text{blue}}(v)$. Suppose further that all nodes $v \in V_r$ of size greater than $\sigma$ have $\text{mult}_{\text{red}}(v) = 0$. Then we can find left-integral packing graphs $G_{\text{red}}$ and $G_{\text{blue}}$ with the same edges, nodes, and sizes of $G$ but with multiplicities satisfying $\text{mult}_{\text{red}} + \text{mult}_{\text{blue}} = \text{mult}$. Moreover, we have $\text{def}(G_{\text{red}}) = 0$ and $\text{def}(G_{\text{blue}}) \leq \text{def}(G) + \sigma$.

**Proof.** By allowing fractional red and blue multiplicities, we can find initial values for the red and blue multiplicities of left nodes so that $\text{def}(G_{\text{red}}) = 0$ and $\text{def}(G_{\text{blue}}) = \text{def}(G)$. To enforce integrality, we will update these multiplicities by swapping (fractional parts of) larger red nodes for smaller blue nodes.

Suppose nodes on the left with positive red multiplicity are ordered by size, so that $\sigma \geq s(v_1) \geq s(v_2) \geq \ldots \geq s(v_\ell)$. While the multiplicities are not all integral, let $i$ be the index of
Proof. Consider the graph \( G_2(x, y) \) as in section 2. For every right node \((C, p)\), we assign 
\[ \text{mult}_{\text{red}}(C, p) = k \cdot \left\lceil \frac{p}{k} \right\rceil \cdot x_p \] 
for \( C \) in size class \( \sigma \), and \( \text{mult}_{\text{red}}(C, p) = 0 \) for all other \( C \). We set

The additional blue nodes we fail to pack will therefore all have size at most \( \sigma \) and their total multiplicity will be at most 1, so the deficiency of the blue graph increases by at most \( \sigma \). □

A key technical ingredient for our algorithm is to be able to replace sets of identical copies of a container in patterns of \( x \) by a bigger container that contains the union of the smaller containers.

**Lemma 12.** Given a pair \((x, y)\) with \( x \in \mathbb{R}^D_\geq 0 \) and \( y \in \mathbb{Z}^C_\geq 0 \). Let \( k \in \mathbb{N} \) and \( 0 < \sigma \leq 1 \) be two parameters. Let \( \tilde{x} \in \mathbb{R}^D_{\geq 0} \) be the vector that emerges if for all containers \( C \) with \( \frac{1}{k}\sigma \leq s(C) \leq \sigma \) and all patterns \( p \) we replace \( k \cdot \left\lceil \frac{p}{k} \right\rceil \) copies of \( C \) by \( \left\lceil \frac{p}{k} \right\rceil \) copies of the container that is \( k \cdot C \). Then there is a \( \tilde{y} \in \mathbb{Z}^C_\geq 0 \) so that \( \text{def}(\tilde{x}, \tilde{y}) \leq \text{def}(x, y) + O(\sigma k) \).

Proof. Consider the graph \( G_2(x, y) \) as in section 2. For every right node \((C, p)\), we assign 
\[ \text{mult}_{\text{red}}(C, p) = k \cdot \left\lceil \frac{p}{k} \right\rceil \cdot x_p \] 
for \( C \) in size class \( \sigma \), and \( \text{mult}_{\text{red}}(C, p) = 0 \) for all other \( C \). We set
Figure 3: Visualization of the reassignment in Lemma 12 for $k = 3$. The upper packing graph is the red part of $G_2(x, y)$ with the optimal assignment $a$, assuming that each container has multiplicity 1. The lower graph gives the red part of $G_2(\tilde{x}, \tilde{y})$ with the constructed assignment $a$ that we give in the analysis. Darker colors indicate larger containers.
Lemma 13 (Reassigning containers). Suppose $x \in \mathbb{R}_{\geq 0}^D, y \in \mathbb{Z}_{\geq 0}^C$, and $\sigma < 2^{-4}$. Then we can combine containers in size class $\sigma$ in $x$ and $y$ into larger containers, yielding new solutions $\tilde{x}, \tilde{y}$ satisfying the following conditions.

1. $1^T \tilde{x} = 1^T x$.

2. $|\text{supp}(\tilde{x})| \leq |\text{supp}(x)|$.

3. For all patterns $p \in \text{supp}(\tilde{x})$ and containers $C$ in size class $\sigma$, $p_C \leq (\frac{1}{\sigma})^{1/4}$.

4. Multiplicities of small containers in patterns in $\text{supp}(x)$ are not affected.

5. $\text{def}(\tilde{x}, \tilde{y}) \leq \text{def}(x, y) + O(\sigma^{3/4})$.

Proof. We apply Lemma 12 with parameter $k = [\frac{1}{\sigma}]^{1/4}$ and obtain a pair $(\tilde{x}, \tilde{y})$ so that $\text{def}(\tilde{x}, \tilde{y}) \leq O(k \sigma) \leq O(\sigma^{3/4})$, and so condition (5) is satisfied. Since we have updated $x$ by altering the patterns in its support, conditions (1) and (2) are also satisfied. In the process of Lemma 12, we decreased $p_C$ for $C$ in size class $\sigma$ to at most $k$. Since $\sigma < 2^{-4}$, we know that $k \geq 2$, and so the containers we created are in strictly larger size classes. Therefore conditions (3) and (4) are satisfied. \qed
Let us say briefly what the container reassignment does to the associated matrix $A$. If $\tilde{A}_C$ is any row of the updated matrix corresponding to a container in size class $\sigma$, we know $\tilde{A}_C$ is entrywise less than or equal to $A_C$ and $\|\tilde{A}_C\|_{\infty} \leq \left(\frac{1}{\sigma}\right)^{1/4}$. In all rows corresponding to smaller size classes, $\tilde{A}_C = A_C$.

Before we talk about applying Lovett-Meka, we want to summarize the results of our grouping and container reassignment. We summarize the procedure:

1. For size classes $s_{\text{min}} \leq \sigma \leq 2^{-72}$, starting with the smallest, do:
   
   2. Group the containers in size class $\sigma$ with $\delta = \sqrt{\sigma}$.
   
   3. Whenever we find more than $\left(\frac{1}{\sigma}\right)^{1/4}$ copies of the same container in one pattern, we put them together in a larger container.

4. For $\sigma > 2^{-72}$, group the containers in size class $\sigma$ with $\delta = 64$.

In the following we will call a size class $\sigma$ small if $\sigma \leq 2^{-72}$ and large otherwise. First note that the increase in deficiency of the entire procedure is at most

$$\sum_{\sigma \in 2^{-N}} (O(\sigma^{1/2}) + O(\sigma^{3/4})) + 72 \cdot 64 = O(1).$$

Let $A$ be the matrix we obtain at the end of this procedure. In addition, we would like to keep much of the group structure that was created during the procedure. Define the shadow incidence matrix $\tilde{A}$ to be the matrix that agrees with $A$ on large size classes, but for small size classes represents the incidences after step (2), but before step (3). We can imagine that whenever a container is put into a larger container, its incidence entry remains in $\tilde{A}$. In particular a container might be put into containers iteratively and hence it may contribute to several incidences in $\tilde{A}$ but only one in $A$. Note that $\tilde{A}$ is entrywise at least as large as $A$.

For all containers $C \in \mathcal{C}$, let $A_C$ denote the row of $A$ corresponding to $C$, and $\tilde{A}_C$ the corresponding row of $\tilde{A}$. Recall that $A$ and $\tilde{A}$ contain columns for patterns in $\text{frac}(x)$. Now, let us summarize the properties that the container-forming procedure provides:

(A) For a container $C$ in size class $\sigma$ one has $\|\tilde{A}_C\|_1 \geq \left(\frac{1}{\sigma}\right)^{1/2}$ if $\sigma$ is small, and $\|\tilde{A}_C\|_1 = \|A_C\|_1 \geq 64$ if $\sigma$ is large.

(B) For a container $C$ in a small size class $\sigma$, and column $j = 1, ..., m$, one has $A_{Cj} \leq \left(\frac{1}{\sigma}\right)^{1/4}$.

(C) One has

$$\sum_{i=1}^{s} \|\tilde{A}_{Ci}\|_1 \cdot s(C_i)^{17/16} \leq 24 \sum_{i=1}^{s} \|A_{Ci}\|_1 \cdot s(C_i).$$

Here (A) follows from the fact that after step (2), we have $\left(\frac{1}{\sigma}\right)^{1/2}$ incidences for each container. (B) follows since after step (3), there are at most $\left(\frac{1}{\sigma}\right)^{1/4}$ containers of each type in a pattern. The condition in (C) can be understood as follows: if we have a container of size $s(C)$, then the containers in it may appear many times in $\tilde{A}$ but only in smaller size classes. By discounting smaller incidences, we can upper-bound the contribution of the shadow incidences by the contribution of the actual containers.

To make this more concrete, consider a container $C$ appearing in $A$ in some size class. If this container came from $k$ smaller containers, then those smaller containers are size at most
2 \cdot \frac{s(C)}{k}$. Here the factor 2 comes from the fact that during grouping our container could have been rounded down by a factor of 2. Therefore the contribution of the shadow incidences of these smaller containers to the left hand side is \((\frac{2s(C)}{k})^{17/16} \cdot k = s(C)^{17/16} \cdot 2^{17/16} k^{-1/16}\). But we chose the parameters so that whenever we combine \(k\) containers we have \(k \geq 2^{18}\) and so the contribution is at most \(2^{-1/16} \cdot s(C)^{17/16}\). The shadow incidences \(\ell\) levels down similarly contribute \((2^{-1/16})^\ell \cdot s(C)^{17/16}\). Then the total contribution of the shadows of \(C\) to the left hand side of property (C) is at most

\[
\sum_{\ell \geq 0} (2^{-1/16})^\ell s(C)^{17/16} \leq 24 \cdot s(C)^{17/16} \leq 24 \cdot s(C).
\]

4 Applying the Lovett-Meka algorithm

Using the grouping and container reassignment above, we can replace \(y\) with \(\bar{y}\) and \(x\) with \(\bar{x}\) so that the incidence matrix \(A\) and shadow matrix \(\bar{A}\) satisfy properties (A)–(C). We now want to create intervals of the rows of \(A\) and \(\bar{A}\) in a nice way so that we can apply Lovett-Meka and make \(x\) more integral. Formally, we will argue the following:

**Claim 14.** Suppose \(x \in [0,1]^P, y \in \mathbb{Z}_{\geq 0}^C\), \(A\) is the incidence matrix of \(x\), and \(\bar{A}\) is a matrix so that \(A\) and \(\bar{A}\) satisfy conditions (A) + (B) + (C). Then there is a randomized polynomial time algorithm to find a vector \(\bar{x}\) satisfying

\[
\begin{align*}
&\bullet 1^T \bar{x} = 1^T x \\
&\bullet \text{def}(\bar{x}, y) \leq \text{def}(x, y) + O(1) \\
&\bullet |\text{frac}(\bar{x})| \leq \frac{1}{4} |\text{frac}(x)|
\end{align*}
\]

Suppose the containers appearing in the patterns in \(\text{supp}(x)\) are \(C_1, ..., C_s\), ordered from largest to smallest. As we fix the fractional solution \(x\) for now, let us denote \(n(C_i) := \sum_{p \in \text{frac}(x)} p_{C_i} = \|A_{C_i}\|_1\) as the number of incidences of container \(C_i\) in \(A\). Similarly, let \(\bar{n}(i) = \|\bar{A}_{C_i}\|_1\) be the number of incidences in the shadow matrix \(\bar{A}\). Again, we have \(n(i) \leq \bar{n}(i)\) for all \(i\). Finally, let us denote \(\bar{n}_\sigma := \sum_{i \in \text{class } \sigma} \bar{n}(i)\) as the total number of shadow incidences that occur for size class \(\sigma\).

For a fixed constant \(K > 0\), and for each small size class \(\sigma\), we first create level 0 intervals of the rows as follows. For any row \(i\) satisfying \(\bar{n}(i) > \frac{1}{2} K (\frac{1}{\sigma})^{17/16}\), we let \(\{i\}\) be its own interval. We then subdivide the remaining rows into intervals so that \(\bar{n}(I) \leq K (\frac{1}{\sigma})^{17/16}\) for each interval \(I\). We need a total of at most \(\frac{4}{\sigma} K^{17/16} \bar{n}_\sigma + 1\) intervals on level 0.

Now, given an interval \(I\) on level \(\ell\) with \(|I| > 1\), we will subdivide \(I\) into at most 3 intervals on level \(\ell + 1\). First, for any row \(i \in I\) with \(\bar{n}(i) > \frac{1}{2} (\frac{1}{\sigma})^{\ell+1} K (\frac{1}{\sigma})^{17/16}\), let \(\{i\}\) be its own interval. We then subdivide the remaining rows into intervals so that \(\bar{n}(I) \leq (\frac{1}{2})^\ell K (\frac{1}{\sigma})^{17/16}\). Since none of the rows \(i \in I\) became its own interval on level \(\ell\), we also know that \(\bar{n}(I) \leq (\frac{1}{2})^\ell K (\frac{1}{\sigma})^{17/16}\), and so in fact this bound holds for every interval on level \(\ell + 1\). The number of intervals on level \(\ell\) is at most \(3^\ell \cdot (\frac{K}{\sigma})^{17/16} \bar{n}_\sigma + 1\).

For large size classes \(\sigma\), create an interval for each row \(\{i\}\). Due to the grouping procedure, the size of each interval is at least 64. All such intervals are level zero, and we do not create any higher levels.

Let us abbreviate all intervals on level \(\ell\) for size class \(\sigma\) as \(\mathcal{I}_{\sigma, \ell}\). We denote \(\mathcal{I}_\sigma := \bigcup_{\ell \geq 0} \mathcal{I}_{\sigma, \ell}\) as the whole family for size class \(\sigma\) and \(\mathcal{I} := \bigcup_{\sigma} \mathcal{I}_\sigma\) as the union over all size classes.
For an interval $I$, we define the vector
$$v_I := \sum_{i \in I} A_i$$
as the sum of the corresponding rows in the incidence matrix.

For an interval $I \in \mathcal{I}_{\sigma, \ell}$, we define $\lambda_I := \ell$ (that means the parameter just denotes the level on which it lives). The input for the Lovett-Meka algorithm will consist of the pairs $\{(v_I, \lambda_I)\}_{I \in \mathcal{I}}$ where we use $\lambda_I \geq 0$ as the parameter for a constraint with normal vector $v_I$. Additionally, we add a single vector $v_{\text{obj}} := 1$ with parameter $\lambda_{\text{obj}} := 0$ to control the objective function. There are two things to show. First we argue that the parameters are chosen so that the condition of the Lovett-Meka algorithm is actually satisfied:

**Lemma 15.** Suppose that $|\text{supp}(x)| \geq L \log(\frac{1}{\min})$. For $K, L$ large enough constants, one has
$$\sum_{I \in \mathcal{I}} e^{-\lambda_I^2/16} + 1 \leq \frac{1}{16} |\text{supp}(x)|$$

*Proof. On level 0, we have $|\mathcal{I}_{\sigma, 0}| \leq 4 \sigma^{17/16} \cdot \tilde{n}_\sigma + 1$ many intervals and hence on level $\ell \geq 0$ there are $|\mathcal{I}_{\sigma, \ell}| \leq 3^\ell \cdot (\frac{4}{K} \sigma^{17/16} \cdot \tilde{n}_\sigma + 1)$ many. We can calculate that
$$\sum_{I \in \mathcal{I}} e^{-\lambda_I^2/16} \leq \sum_{\sigma \text{ small}} e^{-\lambda_I^2/16} \cdot |\mathcal{I}_{\sigma, \ell}| \leq \sum_{\sigma \text{ small}} e^{-\lambda_I^2/16} \cdot (\frac{4}{K} \sigma^{17/16} \cdot \tilde{n}_\sigma + 1) + \sum_{\sigma \text{ large}} |\mathcal{I}_{\sigma, 0}|$$
for $K,L$ large enough
$$\leq \frac{1}{64} |\text{supp}(x)| + \frac{1}{128} \cdot 24 \cdot \sum_{\sigma \text{ small}} \sigma^{17/16} \cdot \tilde{n}_\sigma + \frac{1}{64} |\text{supp}(x)|$$
by property (C)
$$\leq \frac{3}{64} |\text{supp}(x)| \leq \frac{1}{16} |\text{supp}(x)| - 1.$$
We used that the total size for each pattern is at most 1, and so the sum of the sizes of all incidences in the matrix $A$ is at most $|\text{supp}(x)|$. \qed

Now, suppose we do run the Lovett-Meka algorithm and obtain a solution $\tilde{x}$ with $|\text{frac}(\tilde{x})| \leq \frac{1}{2} |\text{frac}(x)|$ so that
$$|\langle v_I, x - \tilde{x} \rangle| \leq \lambda_I \cdot \| v_I \|_2 \quad \forall I \in \mathcal{I} \quad \text{and} \quad 1^T x = 1^T \tilde{x}.$$The following is crucial to our error analysis: the lengths $\| v_I \|_2$ that appear in the error bound are not too long and in particular the ratio $\|v_I\|_2 / n(I)$ decreases with smaller container sizes.

**Lemma 16.** Fix an interval $I \in \mathcal{I}_{\sigma, \ell}$ where $\sigma$ is small. Then $\| v_I \|_2 \leq \tilde{n}(I) \cdot \sigma^{1/8}$.

*Proof. Recall that $v_I = \sum_{i \in I} A_{C_i}$ where each row $A_{C_i}$ has a row-sum of $\|A_{C_i}\|_1 \leq \|A_{C_i}\|_1$. We have $\tilde{n}(i) = \|A_{C_i}\|_1 \geq (\frac{1}{\sigma})^{1/2}$, while $\|A_{C_i}\|_\infty \leq (\frac{1}{\sigma})^{1/4}$. Therefore, we have
$$\|A_{C_i}\|_2 \leq \sqrt{\|A_{C_i}\|_1 \cdot \|A_{C_i}\|_\infty} \leq \sqrt{\|A_{C_i}\|_1 \cdot \|A_{C_i}\|_\infty} = \|A_{C_i}\|_1 \sqrt{\frac{\|A_{C_i}\|_\infty}{\|A_{C_i}\|_1}} \leq \tilde{n}(i) \cdot \sigma^{1/8}.$$Then by the triangle inequality $\| v_I \|_2 \leq \sum_{i \in I} \| A_{C_i} \|_2 \leq \tilde{n}(I) \cdot \sigma^{1/8}$. \qed
The next step should be to argue that the error in terms of the deficiency will be small. Recall that we still assume that containers are sorted so that $1 \geq s(C_1) \geq s(C_2) \geq \ldots \geq s(C_s) > 0$.

**Lemma 17.** Let $C_i$ be a container in small size class $\sigma$. Then

$$\left| \sum_{j \leq i} A_{C_j}(x - \tilde{x}) \right| \leq O\left( \frac{1}{\sigma} \right)^{15/16}.$$ 

If $C_i$ is a large container, then $\sum_{j \leq i} A_{C_j}(x - \tilde{x}) = 0$.

**Proof.** If $C_i$ is a container in small size class $\sigma$, we can write the interval $\{1, \ldots, i\} = \bigcup_{I \in I(i)} I$ as the disjoint union of intervals $\mathcal{I}(i) \subseteq \mathcal{I}$ from our collection so that the only intervals $I \in \mathcal{I}(i)$ with $\lambda_I > 0$ that we are using are from class $\sigma$ and we only take at most three intervals from each level; for all three such intervals on level $\ell$, we have $\|v_I\|_2 \leq \tilde{n}(I)\sigma^{1/8} \leq K \cdot 2^{-\ell} \left( \frac{1}{\sigma} \right)^{15/16}$.

Consequently, we can bound

$$\left| \sum_{j \leq i} A_{C_j}(\tilde{x} - x) \right| \leq \sum_{I \in \mathcal{I}(i)} \lambda_I \cdot \|v_I\|_2 \leq \sum_{\ell \geq 0} 3\ell \cdot K \cdot 2^{-\ell} \left( \frac{1}{\sigma} \right)^{15/16} = O(1) \cdot \left( \frac{1}{\sigma} \right)^{15/16}.$$ 

If $C_i$ is a large container, we can write $\{1, \ldots, i\}$ as a disjoint union of intervals with $\lambda = 0$, and so the statement holds.

It remains to argue why $\text{def}(\tilde{x}, y) \leq \text{def}(x, y) + O(1)$ for one application of Lovett-Meka. First notice that $A_{C_j} \tilde{x} = \text{mult}(C_j, \tilde{x})$ and $A_{C_j} x = \text{mult}(C_j, x)$. Therefore by Lemmas 9 and 17 the rounding of each size class $\sigma$ increases the deficiency by at most $O(1) \cdot \left( \frac{1}{\sigma} \right)^{15/16} \cdot \sigma = O(1) \cdot \sigma^{1/16}$. Summing over all size classes gives a total increase in deficiency

$$O(1) \cdot \sum_{\sigma \in 2^{-N}} \sigma^{1/16} \leq O(1).$$

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Here we give the proof of Lemma \ref{lem:opt-bounds}. Recall that for our result we would need $f(k) = \Theta(\log(k))$.

**Proof.** Let $(s, b)$ be any bin packing instance with $s \in [0, 1]^n$ and $b \in \mathbb{N}^n$. Let $U := \sum_{i=1}^n b_i s_i$ be the total size. Note that $U \leq OPT_f \leq 2U + 1$, so $U$ is a good estimate on the value of the LP
optimum. We split items into large ones \( L := \{i \in [n] \mid s_i \geq \frac{1}{U}\} \) and small ones \( S := \{i \in [n] \mid s_i < \frac{1}{U}\} \).

Now, we perform the geometric grouping from \([\text{KK82}]\) to the large items as follows: sort items consecutively and form groups of total size between 2 and 3. Then for each group, round all items to the largest item type in its group. This procedure allows to reduce the number of different item types to \( U \) while the optimal fractional value increases to at most \( \text{OPT}' \leq \text{OPT} + O(\log U) \). Now we run the assumed algorithm to assign items in \( L \) to at most \( \text{OPT}' + f(U) \leq \text{OPT} + f(\text{OPT}) + O(\log U) \) bins. Here we are using that \( \text{OPT} \geq U \) is an upper bound on the number of items in the modified instance and \( s_{\text{min}} := \frac{1}{U} \) is a lower bound on the item sizes in \( L \).

Then we “sprinkle” the small items greedily over those bins. If no new bin needs to be opened, we are done. Otherwise, we know that the solution consists of \( k \) bins such that \( k - 1 \) bins are at least \( 1 - \frac{1}{U} \) full. This implies \( U \geq (k - 1) \cdot (1 - \frac{1}{U}) \), and hence \( k \leq U + 3 \leq \text{OPT} + 3 \) assuming \( U \geq 2 \). \( \square \)