Statistical Inference of Sine Inverse Rayleigh Distribution

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Abstract: We study in this manuscript a new one-parameter model called sine inverse Rayleigh (SIR) model that is a new extension of the classical inverse Rayleigh model. The sine inverse Rayleigh model is aiming to provide more fitting for real data sets of purposes. The proposed extension is more flexible than the original inverse Rayleigh (IR) model and it has many applications in physics and medicine. The sine inverse Rayleigh distribution can have a uni-model and right skewed probability density function (PDF). The hazard rate function (HRF) of sine inverse Rayleigh distribution can be increasing and J-shaped. Several of the new model’s fundamental characteristics, namely quantile function, moments, incomplete moments, Lorenz and Bonferroni Curves are studied. Four classical estimation methods for the population parameters, namely least squares (LS), weighted least squares (WLS), maximum likelihood (ML), and percentile (PC) methods are discussed, and the performance of the four estimators (namely LS, WLS, ML and PC estimators) are also compared by numerical implementations. Finally, three sets of real data are utilized to compare the behavior of the four employed methods for finding an optimal estimation of the new distribution.

Keywords: Sine generated family; inverse Rayleigh distribution; classic estimation methods; applications

1 Introduction

Reference [1] investigated a significant distribution in analysis of lifetime, namely the inverse Rayleigh (IR) model. The IR model’s considered probability density function (PDF) and the corresponding distribution function (CDF) are given by

\[ g(y) = 2\theta y^{-3} e^{-y^2/\theta^2}, \quad \theta > 0, \quad y > 0. \]  

and

\[ G(y) = e^{-y^2/\theta^2}, \quad \theta > 0, \quad y > 0. \]  

Much effort has been invested in the literature on estimating the IR model; see, for instance, [2–10]. In last years, several extensions for the IR model were established by means of various generalization methods such as beta IR, transmuted IR (TIR), modified IR, transmuted modified IR,
Kumaraswamy exponentiated IR, weighted IR and odd Fréchet IR and half logistic IR models as mentioned in [2–9].

In the last years, many different statisticians are attracted by generated families of distributions as: sine generated (S-G) by [10], Type II half logistic-G by [11], odd Fréchet-G by [12], truncated Cauchy power-G by [13], transmuted odd Fréchet-G by [14], exponentiated M-G by [15], Topp-Leone odd Fréchet-G studied in [16], among others.

For instance, for S-G the CDF and the PDF are

\[
F(y; \xi) = \sin \left( \frac{\pi}{2} G(y; \xi) \right), \quad x \in R. \tag{3}
\]

and

\[
f(y; \xi) = \frac{\pi}{2} g(y; \xi) \cos \left( \frac{\pi}{2} G(y; \xi) \right), \quad x \in R. \tag{4}
\]

where \( g(y; \xi) \) considers a PDF of baseline distribution.

Now, we put forward a novel lifetime model with one parameter named sine inverse Rayleigh (SIR) distribution, whose CDF with parameter \( \theta \) is obtained by employing (2) in (3) as

\[
F(y) = \sin \left( \frac{\pi}{2} e^{-\frac{y}{\theta}} \right), \quad y > 0, \quad \theta > 0, \tag{5}
\]

Likewise, by combining (1), (2) and (4), one obtains the corresponding PDF to (5) as

\[
f(y) = \frac{\pi \theta}{2 y^3} e^{-\frac{y}{\theta}} \cos \left( \frac{\pi}{2} e^{-\frac{y}{\theta}} \right), \quad y > 0, \quad \theta > 0, \tag{6}
\]

where \( \theta \) is a scale parameter.

When the random variable \( Y \) has an SIR model, one can define \( X \)’s hazard rate function (HRF), reversed HRF, cumulative HRF, and survival function (SF) as

\[
R(y) = 1 - \sin \left( \frac{\pi}{2} e^{-\frac{y}{\theta}} \right),
\]

\[
h(y) = \frac{\pi \theta}{2 y^3} e^{-\frac{y}{\theta}} \cos \left( \frac{\pi}{2} e^{-\frac{y}{\theta}} \right),
\]

\[
\tau(y) = \frac{\pi \theta}{2 y^3} e^{-\frac{y}{\theta}} \cot \left( \frac{\pi}{2} e^{-\frac{y}{\theta}} \right),
\]

and

\[
H(y) = -\ln \left( 1 - \sin \left( \frac{\pi}{2} e^{-\frac{y}{\theta}} \right) \right).
\]

Figs. 1 and 2 present the PDF and HRF plots of the SIR distribution, respectively, for various \( \theta \) values.

Figs. 1 and 2 exhibit that the SIR distribution can have a uni-model and right skewed PDF, while its HRF can be J-shaped and increasing.
The remaining parts of this manuscript are presented as follows. Section 2 introduces structural characteristics of SIR distribution including; quantile function, moments, incomplete moments, and Lorenz and Bonferroni curves. Section 3 discusses some estimators for SIR distribution parameters on the basis of four different methods of estimations of least squares (LS), weighted least squares (WLS), maximum likelihood (ML), and percentile (PC). Simulation schemes are performed in Section 4. Three sets of data for real-life application are utilized for comparing the behavior of the four methods of estimating the new distribution in Section 5. Several concluding remarks close the work.

**Figure 1:** Plots of the PDF the SIR distribution

**Figure 2:** Plots of the HRF the SIR distribution
2 Fundamental Properties

We study in this part several statistical characteristics of the SIR model.

2.1 Quantile Function

If \( Y \sim SIR \) then, the quantile function of SIR is

\[
Q(u) = \sqrt{\ln \left( \frac{\pi}{2\text{Arcsin}(u)} \right)}^{-1}.
\]

and by taking \( u = 0.5 \) we get the median (M) as \( M = \sqrt{\ln(3)}^{-1} \).

2.2 Moments

**Theorem 1:** Assume that \( Y \) is an r.v. from SIR, thus the \( r \)th moment of SIR distribution is

\[
\mu_r = \sum_{i=0}^{\infty} \frac{\theta^{r-1} \Gamma \left( 1 - \frac{r}{2} \right)}{2(2i+1)^{1-\frac{r}{2}}} , \quad \frac{r}{2} < 1.
\] (7)

**Proof:** Assume that \( Y \) is an r.v. with pdf (6). One can determine the \( r \)th moments of SIR distribution from

\[
\mu_r = \int y^r f(y) dy = \frac{\pi \theta}{2} \int_0^\infty y^{r-3} e^{-\frac{\theta}{2} y^2} \cos \left( \frac{\pi}{2} y \right) dy,
\]

By inserting the expansion \( \cos[G(y)] = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} [G(y)]^{2i} \), in the previous equation then,

\[
\mu_r = \sum_{i=0}^{\infty} \frac{\theta (-1)^i}{(2i)!} \left( \frac{\pi}{2} \right)^{2i+1} \int_0^\infty y^{r-3} e^{-(2i+1) \left( \frac{\theta}{2} y^2 \right)} dy,
\]

The last equation can be rewritten as

\[
\mu_r = \sum_{i=0}^{\infty} C \int_0^\infty y^{r-3} e^{-(2i+1) \left( \frac{\theta}{2} y^2 \right)} dy.
\]

where \( C = \sum_{i=0}^{\infty} \frac{\theta (-1)^i}{(2i)!} \left( \frac{\pi}{2} \right)^{2i+1} \).

Let \( Z = \frac{\theta}{2} z \), then

\[
\mu_r = \sum_{i=0}^{\infty} \theta^{r-1} C \int_0^\infty z^{-\frac{r}{2}} e^{-(2i+1)z} dz.
\]

Then,

\[
\mu_r = \sum_{i=0}^{\infty} \frac{\theta^{r-1} \Gamma \left( 1 - \frac{r}{2} \right)}{2(2i+1)^{1-\frac{r}{2}}}.
\]
The mgf of $Y$ is

$$
M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r = \sum_{r,i=0}^{\infty} \frac{t^r \theta^{r-1} \Gamma\left(1 - \frac{r}{2}\right)}{2r!(2i + 1)^{1-\frac{r}{2}}}.
$$

obtain the incomplete moments, denoted by $\phi_s(t)$, of the SIR distribution as follows, where $\phi_s(t)$, defined by

$$
\phi_s(t) = \int f(y; \varphi) dy.
$$

Using (8), $\phi_s(t)$ will be as given

$$
\phi_s(t) = \sum_{i=0}^{\infty} C \int y^{s-3} e^{-(2i+1)\left(\frac{\varphi}{2}\right)} dy = \sum_{i=0}^{\infty} \frac{\theta^{-1} \Gamma\left(1 - \frac{s}{2}, (2i + 1)\left(\frac{\varphi}{2}\right)\right)}{2(2i + 1)^{1-\frac{s}{2}}},
$$

where $\Gamma(s, t) = \int_0^t y^{s-1} e^{-y} dy$ denotes the lower incomplete gamma function.

The Lorenz curve and the Bonferroni curve are generated, respectively, from following equations

$$
L_F(x) = \frac{\phi_1(y)}{E(Y)} = \frac{\sum_{i=0}^{\infty} \frac{\theta^{-1} \Gamma\left(\frac{1}{2}, (2i + 1)\left(\frac{\varphi}{2}\right)\right)}{2(2i + 1)^{\frac{1}{2}}}}{\sum_{i=0}^{\infty} \frac{\theta^{-1} \Gamma\left(\frac{1}{2}\right)}{2(2i + 1)^{\frac{1}{2}}}} = \frac{\sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}, (2i + 1)\left(\frac{\varphi}{2}\right)\right)}{(2i + 1)^{\frac{3}{2}}}}{\sum_{i=0}^{\infty} \frac{C\sqrt{\pi}}{(2i + 1)^{\frac{3}{2}}}},
$$

and

$$
B_F(x) = \frac{L_F(y)}{F(y; \varphi)} = \frac{\sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}, (2i + 1)\left(\frac{\varphi}{2}\right)\right)}{(2i + 1)^{\frac{3}{2}}}}{\sum_{i=0}^{\infty} \frac{C\sqrt{\pi}}{(2i + 1)^{\frac{3}{2}}} \left(\sin\left[\frac{\pi}{2} e^{-\frac{\varphi}{2}}\right]\right)}.
$$

The Zenga curves are given by

$$
A_F(y) = 1 - \frac{\mu^{-}(y)}{\mu^{+}(y)}
$$

where

$$
\mu^{-}(x) = L_F(y)
$$

and

$$
\mu^{+}(x) = \int_{y}^{\infty} f(y; \varphi) dy.
$$
3 Statistical Inference

The population parameters involved in the SIR model can be estimated by using four different methods of estimation namely; ML, LS, WLS and PC methods.

3.1 ML Estimators

To obtain the MLEs of the SIR model with a parameter $\theta$, let $Y_1, \ldots, Y_n$ be observed values of this model. The log-likelihood function denoted by $\ell$, can be expressed as

$$
\ell = n \log \frac{\pi}{2} + \log 2 - \theta \sum_{i=1}^{n} (y_i)^{-2} + \sum_{i=1}^{n} \log \left( \cos \left[ \frac{\pi}{2} e^{-\theta(y_i)^2} \right] \right).
$$

The ML equation of the SIR model then becomes

$$
\frac{\partial \ell}{\partial \theta} = n - \sum_{i=1}^{n} (y_i)^{-2} + \frac{\pi}{2} \sum_{i=1}^{n} (y_i)^{-2} e^{-\theta(y_i)^2} \tan \left[ \frac{\pi}{2} e^{-\theta(y_i)^2} \right].
$$

The MLEs of $\theta$ are then obtained by equating $\frac{\partial \ell}{\partial \theta}$ with zero and solving simultaneously these equations.

3.2 LS and WLS Estimators

Let $Y_1, Y_2, \ldots, Y_n$ be an n-sized random sample (RS) from SIR model and denote the ordered samples in the RS by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$. The expectation and variance of this model do not depend the unknown parameter given by

$$
E(F(Y_{(i)})) = \frac{i}{n+1} \quad \text{and} \quad \text{var}(F(Y_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.
$$

In the above equations, $F(Y_{(i)})$ represents the CDF of the model while $Y_{(i)}$ represents the $i^{th}$ order statistic (OS). Thus, the LSEs can be determined by obtaining the least sum of squared errors as follows,

$$
\sum_{i=1}^{n} \left[ F_{(i)}(y) - \frac{i}{n+1} \right]^2,
$$

regarding $\theta$. Thus, the LSEs of the population parameter $\theta$ of SIR follow by calculating the minimum of the sum

$$
\sum_{i=1}^{n} \left[ \sin \left( \frac{\pi}{2} e^{-\theta(y_i)^2} \right) - \frac{i}{n+1} \right]^2,
$$

regarding $\theta$. Moreover, the LSEs of $\theta$ can be determined from

$$
\sum_{i=1}^{n} \left[ \sin \left( \frac{\pi}{2} e^{-\theta(y_i)^2} \right) - \frac{i}{n+1} \right] \left( \frac{1}{(y_{(i)})^2} e^{-\theta(y_{(i)})^2} \cos \left[ \frac{\pi}{2} e^{-\theta(y_{(i)})^2} \right] \right) = 0.
$$

The WLSEs of $\theta$ can then be derived by calculating the minimum of the sum

$$
\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ \sin \left( \frac{\pi}{2} e^{-\theta(y_i)^2} \right) - \frac{i}{n+1} \right]^2,
$$
regarding $\theta$. Moreover, one can determine the WLSEs of $\theta$ from

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ \sin \left( \frac{\pi}{2} e^{-\frac{n}{\nu_0(i)}} \right) - \frac{i}{n+1} \right] \left( \frac{1}{(y(i))^2} e^{-\frac{n}{\nu_0(i)}} \cos \left( \frac{\pi}{2} e^{-\frac{n}{\nu_0(i)}} \right) \right) = 0.$$

### 3.3 PC Estimator (PCEs)

Let $Y_1, \ldots, Y_n$ denote an RS taken from SIR and assume that $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ is the corresponding OS. The PCEs of parameter $\theta$ are calculated by minimizing the next

$$\sum_{i=1}^{n} \left[ \ln \left( \frac{i}{n+1} \right) - \ln \left( \sin \left( \frac{\pi}{2} e^{-\frac{i}{\nu_0(i)}} \right) \right)^2 \right],$$

regarding $\theta$.

### 4 Numerical Results

We generate 3000 RS $Y_1, \ldots, Y_n$ of sizes $n = 10, 20, 30$ and $50$ from SIR were generated. Three different values of the parameter $\theta$ are chosen:

The parameter $\theta$’s ML, LS, WLS and PC estimates are calculated. Subsequently, the MSEs of the estimate of the unknown parameter are determined. Numerical results are mentioned in Tabs. 1–3 and the following observations can be made:

- The MSEs of ML estimates of $\theta$ are the lowest among all determined MSEs in almost every case.
- The MSEs of all the estimates decrease with increasing sample sizes.

#### Table 1: Statistical inference of SIR model for ML, LS, WLS and PC estimates when $\theta = 0.5$

| $n$ | MLEs | LSEs | WLSEs | PCEs |
|-----|------|------|-------|------|
|     | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs |
| 10  | 0.527   | 0.018 | 0.521   | 0.023 | 0.519   | 0.022 | 0.458   | 0.020 |
| 20  | 0.513   | 0.008 | 0.509   | 0.009 | 0.511   | 0.009 | 0.473   | 0.010 |
| 30  | 0.509   | 0.005 | 0.506   | 0.006 | 0.503   | 0.006 | 0.475   | 0.007 |
| 50  | 0.506   | 0.003 | 0.505   | 0.004 | 0.504   | 0.0032 | 0.477   | 0.004 |

#### Table 2: Statistical inference of SIR model for ML, LS, WLS and PC estimates when $\theta = 0.8$

| $n$ | MLEs | LSEs | WLSEs | PCEs |
|-----|------|------|-------|------|
|     | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs |
| 10  | 0.841   | 0.047 | 0.836   | 0.061 | 0.834   | 0.057 | 0.742   | 0.052 |
| 20  | 0.821   | 0.021 | 0.821   | 0.027 | 0.810   | 0.0213 | 0.745   | 0.026 |
| 30  | 0.814   | 0.013 | 0.811   | 0.015 | 0.808   | 0.014 | 0.751   | 0.017 |
| 50  | 0.807   | 0.008 | 0.805   | 0.009 | 0.807   | 0.0084 | 0.763   | 0.010 |
5 Applications to Real Data

Further we employ three real data sets for assessing the goodness-of-fit of the SIR model with the purpose of comparing the performance of the considered estimation methods.

We consider criteria including maximum likelihood (denoted by $-\hat{\ell}_n$) (B1), the Akaike information criterion (AIC) (B2), the consistent AIC (B3), the Schwarz criterion (B4) and the Hannan-Quinn information criterion (B5). The model that has lowest B1-B5 values is deemed the best one with respect to fitting the real data.

**The first data set:** contains the survival time (days) of 72 guinea pigs with virulent tubercle bacilli infection, originally presented in [17].

**The second data set:** consists the waiting time (minutes) of 100 bank customers before they were served, as presented in [18].

**The third data set:** contains 100 results on the breaking stress (Gba) of carbon fibers, as presented in [19].

The parameters of SIR are estimated by the four estimation methods, namely the ML, LS, WLS and PC estimation methods. The efficiency of the estimation methods is the same in the three data sets as shown in Tabs. 4–6. We see that among the estimation methods adopted, ML provides the best results for the included data sets.

### Table 3: Statistical inference of SIR model for ML, LS, WLS and PC estimates when $\theta = 1.5$

| n  | MLEs Estimates | MLEs MSEs | LSEs Estimates | LSEs MSEs | WLSEs Estimates | WLSEs MSEs | PCEs Estimates | PCEs MSEs |
|----|----------------|-----------|----------------|-----------|-----------------|-----------|---------------|-----------|
| 10 | 1.585          | 0.169     | 1.555          | 0.201     | 1.553           | 0.182     | 1.386         | 0.177     |
| 20 | 1.543          | 0.077     | 1.533          | 0.080     | 1.521           | 0.078     | 1.404         | 0.083     |
| 30 | 1.525          | 0.048     | 1.522          | 0.061     | 1.519           | 0.052     | 1.413         | 0.064     |
| 50 | 1.516          | 0.027     | 1.512          | 0.034     | 1.510           | 0.030     | 1.434         | 0.037     |

### Table 4: The goodness of fit statistics for parameters estimated under various methods for the first data set

| Method | $\hat{\theta}$ | B1   | B2   | B3   | B4   | B5   |
|--------|----------------|------|------|------|------|------|
| MLE    | 0.783          | 264.046 | 530.092 | 529.95 | 562.968 | 530.149 |
| LSE    | 2.377          | 388.36 | 778.72 | 780.435 | 782.533 | 778.894 |
| WLSE   | 2.3774         | 388.394 | 778.789 | 780.503 | 782.601 | 778.963 |
| PCE    | 0.051          | 517.874 | 1038 | 1039 | 1042 | 1038 |

### Table 5: The goodness of fit statistics for parameters estimated under various methods for the second data set

| Method | $\hat{\theta}$ | B1   | B2   | B3   | B4   | B5   |
|--------|----------------|------|------|------|------|------|
| MLE    | 21.51          | 523.491 | 1049 | 684.806 | 686.915 | 1050 |
| LSE    | 66.133         | 693.104 | 1388 | 1390 | 1392 | 1388 |
| WLSE   | 66.097         | 692.908 | 1388 | 1390 | 1392 | 1388 |
| PCE    | 4.117          | 684.116 | 1370 | 1051 | 1053 | 1370 |
We study a new model called SIR. Some fundamental characteristics of the model are investigated. We estimate the model parameters according to the ML, LS, WLS and PC methods. Numerical experiments are carried out in order to comparatively explore the performance of the four different estimation methods. Numerical results show that ML performs better than LS, WLS and PC in almost all considered situations. Applications on three sets of real data indicate that the ML method is superior in terms of the fits to the LS, WLS and PC methods.

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