Light propagation in a Cole-Cole nonlinear medium via Burgers-Hopf equation.

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Abstract

Recently, a new model of propagation of the light through the so-called weakly three-dimensional Cole-Cole nonlinear medium with short-range nonlocality has been proposed. In particular, it has been shown that in the geometrical optics limit, the model is integrable and it is governed by the dispersionless Veselov-Novikov (dVN) equation. Burgers-Hopf equation can be obtained as 1+1-dimensional reduction of dVN equation. We discuss its properties in the specific context of nonlinear geometrical optics. An illustrative explicit example is considered.

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Many recent studies concerning the dispersionless integrable systems have shown their relevance in a broad variety of fields in physics such as Laplacian growth, topological field theory, nonlinear optics [11]-[14], as well as their deep connection with various fields in mathematics, in particular, with the theory of asymptotic expansions, conformal and quasiconformal mappings and the study of integrable deformations of complex algebraic curves [11]-[14].

In the present paper we are interested in the application of the theory of dispersionless integrable systems in nonlinear geometrical optics. In particular, it has been shown in [11] that the Maxwell equations describing the propa-

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gation of the light through the Cole-Cole weakly three-dimensional nonlinear medium admit an integrable geometrical optics limit.

In this limit the system is governed by the dispersionless Veselov-Novikov (dVN) hierarchy, which is amenable by the quasiclassical $\bar{\partial}$-dressing method [10].

A reduction method based on the symmetry constraints [15] appears to be an efficient approach to calculate solutions of the dVN equation [16].

In what follows we will focus on a 1+1-dimensional reduction when the refractive index does not depend on one coordinate. In this case dVN hierarchy is reduced to the so-called Burgers-Hopf (BH) hierarchy. For sake of simplicity we will focus on the first equations of the hierarchy, the so-called dVN and Burgers-Hopf equations, respectively.

In this regard, it is quite interesting to interpret some of the properties of BH equation within the specific context of nonlinear geometrical optics. One of them concerns the existence of breaking wave solutions. They could be useful to model dielectrics which present a kind of “impurities” inducing abrupt variations of the refractive index. Indeed, it can be seen that, at so-called breaking points, the curvature of the light rays blows up just like it happens at interface between different media.

Let us start with the Maxwell equations in a dielectric medium

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \quad \nabla \cdot \mathbf{D} = 0$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \nabla \cdot \mathbf{B} = 0$$

(1)

along with the constitutive equations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}.$$  

(2)

Looking for monochromatic solutions

$$\mathbf{E} (x, y, z, t) = \mathbf{E}_0 (x, y, z) e^{-i\omega t}$$
$$\mathbf{H} (x, y, z, t) = \mathbf{H}_0 (x, y, z) e^{-i\omega t},$$

(3)

one gets from (12) the following well-known second order equations

$$\nabla^2 \mathbf{E}_0 + \omega^2 \mu \varepsilon \mathbf{E}_0 + (\nabla \log \mu) \times (\nabla \times \mathbf{E}_0) + \nabla (\mathbf{E}_0 \cdot \nabla \log \varepsilon) = 0$$
$$\nabla^2 \mathbf{H}_0 + \omega^2 \mu \varepsilon \mathbf{H}_0 + (\nabla \log \varepsilon) \times (\nabla \times \mathbf{H}_0) + \nabla (\mathbf{H}_0 \cdot \nabla \log \mu) = 0.$$  

(4)

Medium under study is characterized by the following set of properties [9][10]:
1. $\varepsilon$ and $\mu$ obeys the Cole-Cole dispersion law

$$\varepsilon = \varepsilon_0 + \frac{\tilde{\varepsilon}}{1 + (i\omega\tau_0)^{2\nu}}, \quad 0 < \nu < \frac{1}{2}$$

$$\mu = \mu_0 + \frac{\tilde{\mu}}{1 + (i\omega\tau_0)^{2\nu}},$$

We highlight that the range of values of the exponent $\nu$ plays a crucial role in the construction of an integrable high frequency limit.

2. $\varepsilon_0$ and $\mu_0$ depend only on the coordinates $x$, $y$ and $z$, while $\tilde{\varepsilon}$ and $\tilde{\mu}$ are assumed to be depending on the coordinates, the fields and the spatial derivatives of the fields. The latter can be explained in terms of a mechanism of short-range non-locality by means of an integral constitutive relation among the electric field $E$ and the displacement vector $D$ [10].

3. All quantities, in high frequency limit show slow dependence on the variable $z$ formally given by

$$\frac{\partial}{\partial z} = \omega^{-\nu} \frac{\partial}{\partial \xi}.$$

where $\xi$ is a “slow” variable defined by $z = \omega^\nu \xi$.

Moreover, any of such a quantity $f(x, y, z)$ can be expanded in asymptotic series on the parameter $\omega^\nu$

$$f(x, y, z) = f(x, y, \xi) + \omega^{-\nu} f_1(x, y, \xi) + \omega^{-2\nu} f_2(x, y, \xi) + \ldots.$$

Representing

$$E_0 = \tilde{E}_0 e^{i\omega S}, \quad H_0 = \tilde{H}_0 e^{i\omega S},$$

and using the previous assumptions in one of equations (4), in high frequency limit ($\omega \to \infty$) one gets in the orders $\omega^2$ and $\omega^{2-2\nu}$ the main non-trivial contributions $[9, 10]$

$$S_x^2 + S_y^2 = 4u,$$

$$S_\xi = \varphi(x, y, \xi, S_x, S_y).$$

Equation (7) is the standard eikonal equation in two dimensions, $\varphi$ is a certain function and $4u = \varepsilon_0 \mu_0$. As discussed in the paper [10] the equations (7)
and constitute an overdetermined system for the phase $S$. The compatibility condition imposes specific restrictions on the possible forms of the function $\varphi$ and the admissible refractive indices $u$. If $\varphi$ is a polynomial differential of $S$, the first non-trivial case is given by the third degree polynomial, i.e.

$$
\varphi = \frac{1}{4} S_x^3 - \frac{3}{4} S_x S_y^2 + V_1 S_x + V_2 S_y.
$$

(9)

The compatibility condition gives

$$
u_\xi = (V_1 u)_x + (V_2 u)_y$$

$$V_1 x - V_2 y = -3 u_x$$

$$V_2 x + V_1 y = 3 u_y,$$

which is the dVN equation. In the paper [16] it has been shown how symmetry constraints allow us to construct its 1+1-dimensional reductions of hydrodynamic type. A number of explicit solutions for the complex dVN equation have been discussed in [10].

Particular situation in which refractive index depend only on one coordinate on the plane $x$-$y$ is of physical interest too. So, let us assume that

$$
u_y = 0.
$$

(11)

As a consequence of the eikonal equation (7) the phase function $S$ must satisfy the condition $S_y = c$, with $c = \text{const}$, then

$$S = cy + \tilde{S}(x, \xi)$$

(12)

where $\tilde{S}$ does not depend on $y$. Eikonal equation (7) becomes

$$\tilde{S}_x^2 = 4u - c^2,$$

(13)

while, choosing $V_2 = 0$, one gets

$$V_1 = -3u,$$

and the dVN equation (10) is reduced to the Burgers-Hopf (BH) equation

$$u_\xi + 6uu_x = 0.$$  

(14)

Solutions of BH equation can be expressed in the following hodograph form [17]

$$x - 6u \xi + \psi(u) = 0,$$

(15)
where $\psi$ is an arbitrary function of its argument. Moreover, once calculated $u(x, \xi)$ by the (15) the phase function is given explicitly

$$S = cy \pm \int \sqrt{4u - c^2} dx.$$  \hfill (16)

Shock structure of the Burgers-Hopf equation is connected to the existence of a value $\xi^*$ (breaking point), for which

$$u_x \to \infty.$$ \hfill (17)

In the theory of partial differential equations these solutions are called breaking waves. It is a textbook exercise to calculate the breaking point [17]. For instance, choosing the initial datum $u(x_0, 0) = (1 - \tanh x_0)/6$ the breaking point is $\xi^* = 1$ (see figure 1).

We remark that, as discussed in [18], the singular sector of BH hierarchy induces a stratification of the affine space of independent variables which gives rise to the integrable deformations of hyperelliptic curves.

The propagation of the light on the plane $\pi : \xi = \xi_0$ ($\xi_0 = \text{const}$) is governed by the standard eikonal equation (7). Denoting by $\Sigma$ the congruence on $\pi$ normal to the family of curves $S(x, y, \xi_0) = \lambda$, parametrized by $\lambda$, the curvature of $\gamma \in \Sigma$ can be represented by the well known formula [19]

$$\kappa = \frac{1}{2u} \nu \cdot \nabla u,$$ \hfill (18)

where $\nabla = (\partial_x, \partial_y)$ and $\nu = (\nu_1, \nu_2)$ is the unit principal normal to $\gamma$ on the plane $\pi$. Now, let us observe that the curve $\gamma$ coincides locally with the projection of the light ray on the plane $\pi$. 

Figure 1: Typical breaking solution of Burgers-Hopf equation. Beyond breaking point ($\xi > \xi^*$) function $u$ becomes multi-valued.
Figure 2: Density plot of the real part of the refractive index $n = n_1 + i n_2$. Dashed line delimits the region where there is an imaginary contribution to refractive index (we denoted $t \equiv \xi$).

Figure 3: -a- Three-dimensional visualization of the imaginary part of refractive index. -b- Three-dimensional visualization of the imaginary part of $S$. It vanishes everywhere except inside the absorption region.
In virtue of the condition (11), the formula (18) assumes the form
\[ \kappa = \frac{\nu_1}{2u} u_x. \] (19)

Then, if \( u_x \) blows up we conclude that the curvature of the light ray blows up as well. This is the typical behavior which light rays exhibit on the interface between different materials. This type of solutions could be useful to describe situations where light rays propagate in the non-homogeneous medium and at certain point they cross some kind of impurity. These impurities with drastically different optical properties could be responsible of the abrupt change of direction.

As well known in electrodynamics, complex-valued refractive indices also have a physical meaning since they can be used to describe absorption effects of radiation. In this specific case, complex values of the refractive index \( n \) are associated with a strong damping of the electromagnetic wave. We illustrate this fact by the following explicit example.

Let us consider a solution obtained from the hodograph relation (15) setting \( \psi(u) = u^2 \). Expliciting \( u \) from (15) and using it in the expression (16), one gets
\[ u = \frac{1}{2} \left( -6\xi + \sqrt{36\xi^2 - 4x} \right), \] (20)
\[ S = \frac{4}{15} \sqrt{2} \sqrt{-6\xi + \sqrt{36\xi^2 - 4x}} \left( 6\xi \left( -6\xi + \sqrt{36\xi^2 - 4x} \right) + 12x \right). \] (21)

We note that the refractive index \( n = 2\sqrt{u} \) (we do not consider negative refractive indices) is not real-valued on whole plane \( x-\xi \). Inside regions in
which the refractive index takes a complex contribution even the phase function $S$ becomes complex-valued. Thus, writing $S = S_1 + iS_2$, if $S_2 > 0$ the electric field acquires a damping factor

$$E = E_0 e^{-\omega S_2} e^{i\omega S_1}. \quad (22)$$

In this regions strong absorptions effects are occurring.

Figure 2 shows the real part of refractive index $n = 2\sqrt{u}$. The dashed line marks the region where the refractive index acquires an imaginary contribution. This has been visualized in the three-dimensional plot in the figure 3-a. Just where refractive index becomes complex-valued, the phase $S$ takes an imaginary contribution indicating a strong absorption (see figure 3-b). Figure 4 shows the wave fronts configuration in the plane $x-\xi$. Shaded region represents the absorption region. For $\xi \to -\infty$ we have an almost plane wave front. Towards absorption region, wave fronts deform itself following level lines of refractive index. Just before absorption wave fronts present a deviation of an almost right angle.

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