Abstract. First we give a new proof of Goto’s theorem for Lie algebras of compact semisimple Lie groups using Coxeter transformations. Namely, every $x$ in $L = \text{Lie}(G)$ can be written as $x = [a, b]$ for some $a, b$ in $L$. By using the same method, we give a new proof of the following theorem (thus avoiding the classification tables of fundamental weights): in compact semisimple Lie algebras, orthogonal Cartan subalgebras always exist (where one of them can be chosen arbitrarily). Some of the consequences of this theorem are the following. (i) If $L = \text{Lie}(G)$ is such a Lie algebra and $C$ is any Cartan subalgebra of $L$, then the $G$-orbit of $C^\perp$ is all of $L$. (ii) The consequence in part (i) answers a question by L. Florit and W. Ziller on fatness of certain principal bundles. It also shows that in our case, the commutator map $L \times L \rightarrow L$ is open at $(0,0)$. (iii) Given any regular element $x$ of $L$, there exists a regular element $y$ such that $L = [x, L] + [y, L]$ and $x, y$ are orthogonal.

Then we generalize this result about compact semisimple Lie algebras to the class of non-Hermitian real semisimple Lie algebras having full rank.

Finally, we survey some recent related results, and construct explicitly orthogonal Cartan subalgebras in $\mathfrak{su}(n), \mathfrak{sp}(n), \mathfrak{so}(n)$.

1. Introduction

The Lie algebra version of Goto’s Theorem [10] for compact semisimple Lie groups is not as well known as it ought to be. It says that

**Theorem 1.1.** Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group $G$. Then every element $x$ in $L$, can be written as $x = [a, b]$ for some $a, b$ in $L$.

The first proof of Theorem 1.1 (i.e. Goto’s Theorem, additive version) was proved by Karl-Hermann Neeb [13, p. 653] (that he has communicated to the authors of [13]). His proof was based on Kostant’s convexity Theorem [16, Thm. 8.1], or more precisely, the version of
Kostant’s convexity Theorem for compact semisimple Lie groups. Such proof will be presented in Appendix A, but slightly simplified using [1, Lemma 2.2]. In section 2, we prove Theorem 1.1 directly by using any Coxeter transformation of the Weyl group $W(L, C)$ where $C$ is a maximal toral subalgebra of $L$. We remark that certain Coxeter transformations were used to prove Goto’s Theorem (on the group level) as in Bourbaki’s book [4, Corollary, section 4 of chapter 9] or Hofmann and Morris’s book [13, Corollary 6.56]. To the best of our knowledge, this direct proof is new. In section 3, we prove the following theorem.

**Theorem 1.2.** Let $\text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group $G$. Then, with respect to the negative of its Killing form, $\text{Lie}(G)$ has orthogonal Cartan subalgebras (maximal toral subalgebras) where one of them can be chosen arbitrarily.

This interesting result was proved recently by d’Andrea and Maffei in [7, Lemma 2.2], via the classification tables of fundamental weights for all types of simple Lie algebras (except type $A_n$). Specifically, their proof uses the tables in [4] to verify that (in any root system of $(L, C)$ where $C$ is a Cartan subalgebra of $L$), the highest root is either equal or twice some fundamental weight (in all simple Lie algebras except of type $A_n$). However, our proof is a simple consequence of our methods in section 2. Theorem 1.2 has the following Corollary.

**Corollary 1.3.** Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group $G$. Let $C$ be a Cartan subalgebra of $L$, and let $C^\perp$ be the orthogonal complement of $C$ (with respect to negative of the Killing form on $L$). Then the $G$-orbit of $C^\perp$ is all of $L$.

We note that Corollary 1.3 can be easily obtained directly from Kostant’s convexity Theorem (see Appendix A). In fact, Corollary 1.3 was essentially the key step in Karl-Hermann Neeb’s proof of Goto’s Theorem, additive version. In section 4, we present two “applications” of Corollary 1.3. First, Corollary 1.3 answers a question by L. Florit and W. Ziller on fatness of certain principal bundles as follows.

**Corollary 1.4.** If $G$ is a compact semisimple Lie group, endowed with minus its Killing form, and $G \hookrightarrow P \rightarrow B$ is a principal $G$-bundle on $B$ with total space $P$, endowed with a connection 1-form $\theta : TP \to g$, then $\theta$ is fat if and only if $C^\perp$ is fat, where $C$ can be chosen to be any Cartan subalgebra of $g$.

Our second application of Corollary 1.3 is the following.

**Corollary 1.5.** Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group $G$. Then the commutator map $L \times L \to L$ is open at $(0, 0)$.
This fact was obtained in [7] using their Lemma 2.2 which was proved by using some tables about fundamental weights as explained above. (Recall that their Lemma 2.2 is our Theorem 1.2 above). However, after a careful study of their proof, we found out that one only needs the partial result stated in Corollary 1.3 above. In section 5.1, we make the following remark.

**Corollary 1.6.** Let \( L \) be a compact semisimple Lie algebra, and let \( a \) be a regular element of \( L \). Then there exists a regular element \( b \) in \( L \), such that

\[
L = [a, L] + [b, L]
\]

and \( a, b \) are orthogonal (in fact the centralizers of \( a \) and \( b \) in \( L \) are also orthogonal).

More generally, we have

**Corollary 1.7.** Let \( L = \mathfrak{k} \oplus \mathfrak{p} \) be a Cartan decomposition of a semisimple Lie algebra \( L \) (where \( \mathfrak{k} \) is a maximal compact Lie subalgebra) such that \( L \) is non-Hermitian of full rank. That is, \( \mathfrak{k} \) is semisimple and \( \text{rank}(\mathfrak{k}) = \text{rank}(L) \). Let \( a \) be a regular element of \( \mathfrak{k} \). Then there exists a regular element \( b \) in \( \mathfrak{k} \), such that \( L = [a, L] + [b, L] \) and \( a, b \) are orthogonal.

In section 5.1, we make the following trivial remark.

**Remark 1.8:** The following two conjectures are equivalent.

1) Every element \( x \) in a real semisimple Lie algebra \( L \) is the commutator of two elements where one element can be chosen to be regular (hence semisimple).

2) Every element \( x \) in a real semisimple Lie algebra \( L \) is orthogonal to some Cartan subalgebra of \( L \).

In section 5.2, we survey most of the results of D. Akhiezer in his recent interesting paper [1] that shows that the above conjecture is valid in many real simple Lie algebras (see Theorem 5.3). In section 5.3, we survey one result by G. Bergman and N. Nahlus in [2] and another result in preparation by the second author in [19] related to 1.5 generators of simple Lie algebras. For example, in any simple Lie algebra \( L \) over a field of characteristic 0, and for any \( x \) in \( L \setminus \{0\} \), there exists a regular (hence semisimple) element \( y \) in \( L \) such that \( L = [L, x] + [L, y] \) ([19]). This last property is related to the concept of 1.5 generators as follows: if \( x \) and \( y \) generate \( L \) as a Lie algebra then \( L = [L, x] + [L, y] \) by [3, Lemma 25c].

Finally in section [3], we construct explicit examples of orthogonal Cartan subalgebras in the cases of \( \mathfrak{su}(n) \), \( \mathfrak{sp}(n) \), based on the idea of circulant matrices (cf. [17]), and in the case of \( \mathfrak{so}(n) \).
Acknowledgements. The authors are grateful to Anthony Knapp for his help in answering some questions about the general theory of non-compact semisimple Lie algebras. We are also indebted to Karl-Hermann Neeb for his encouragement and interesting discussions about two results in the paper. Finally, we are also indebted to Wolfgang Ziller for kindly pointing out that our Corollary 1.3 answers a question about fatness in principal bundles.

2. Goto’s theorem by Coxeter transformations

In this section, we give a proof of Goto’s theorem, additive version, by using Coxeter transformations.

Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group. Let $C$ be a maximal toral subalgebra of $L$. Let $\Sigma \subseteq C^*$ be the root system of $G$ with respect to $C$ and we fix an ordering of the roots, with $\Delta$ as the set of positive roots, and let $\{a_1, \ldots, a_n\}$ be the set of simple positive roots.

Lemma 2.1. Let $\{s_1, \ldots, s_n\}$ be the Weyl reflections corresponding to the simple positive roots $\{a_1, \ldots, a_n\}$. Then, for any permutation of $\{1, \ldots, n\}$, the Coxeter transformation $c = s_1s_2\cdots s_n$ has no fixed points, other than $0$. That is, $1$ is not an eigenvalue of $c$.

See the very short proof in [14], p.76.

Lemma 2.2. (cf. [12], Lemma 6.53) Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group. Let $C = \text{Lie}(T)$ be a maximal toral subalgebra. Let $c$ be any Coxeter transformation with respect to $(L, C)$ as in lemma 2.1. $c = \text{Ad}(n)$ for some $n \in N(T)$, the normalizer of $T$ in $G$, since $c$ is in the Weyl group of $(G, T)$. Then

$$\text{Ad}(n)|_C - \text{Id} : C \to C$$

is an isomorphism.

Proof. Immediate from lemma 2.1. \qed

Theorem 2.3. In the setting of lemma 2.2, suppose $n = \exp(N)$ for some $N$ in $L = \text{Lie}(G)$ (which exists since the exponential map is surjective in connected compact semisimple Lie groups). Then $C$ is contained in $[L, N]$.

Proof. Let $x \in C = \text{Lie}(T)$. Then by lemma 2.2, $x = \text{Ad}(n)(t) - t$ for some $t$ in $C$. But the exponential map is surjective in compact connected semisimple Lie groups, so $n = \exp(N)$, for some $N \in L$. Hence

$$x = \text{Ad}(n)(t) - t = \text{Ad}(\exp(N))(t) - t = \exp(\text{ad}(N))(t) - t$$
But \( x = \exp(\text{ad}(N))(t) - t \) can be written as \([N, y(x)]\), for some \( y(x) \in L \) (by factorization) through writing the exponential series as the limit of its partial sums. Hence \( x = [N, y(x)] \).

Next we strengthen Theorem 2.3 to replace \( N \) by a regular element of \( L \).

**Theorem 2.4.** Let \( L = \text{Lie}(G) \) be the Lie algebra of a compact semisimple Lie group \( G \), and let \( C \) be a Cartan subalgebra of \( L \). Then there exists a regular element \( a \) of \( L \) such that \( C \subseteq [a, L] \). In particular, every element \( x \) of \( L \) can be written as \( x = [a, b] \) where \( a \) can be chosen to be regular.

**Comment:** for comparison, the proof by Kostant’s convexity Theorem in the Appendix shows that \( C^\perp \subseteq [x, L] \) for some regular element \( x \) of \( L \).

**Proof.** By Theorem 2.3 \( C \subseteq [N, L] \). Since \( N \) is semisimple, \( N \) is contained in a Cartan subalgebra \( C' \) of \( L \), so \( C' \subseteq Z(N) \). Now \( C' = Z(a) \) for some regular element \( a \) of \( L \). So \( Z(a) \subseteq Z(N) \). Hence \( Z(N)^\perp \subseteq Z(a)^\perp \). But \( Z(N)^\perp = [N, L] \) and \( Z(a)^\perp = [a, L] \) by Remark 5.2. Hence \([N, L] \subseteq [a, L] \). So \( C \subseteq [a, L] \) as desired. The last conclusion follows from the fact that every element of \( L \) belongs to a Cartan subalgebra, and all Cartan subalgebras of \( L \) are conjugate. □

**Example 2.5:** Consider \( \text{su}(2k+1) \), which is a compact semisimple Lie algebra. It can be shown by a simple calculation that

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

is an element of \( N(T) \) which gives modulo \( T \) a Coxeter transformation of \( SU(2k+1) \). Consider the unitary change of basis matrix \( g = (g_{ab}) \) for \( 1 \leq a, b \leq 2k+1 \), with

\[
g_{ab} = \frac{1}{\sqrt{2k+1}} \gamma_k^{(a-1)(b-1)}
\]

where \( \gamma_k = \exp\left(\frac{2\pi i}{2k+1}\right) \). Let \( D = \text{diag}(1, \gamma, \gamma^2, \cdots, \gamma^{2k}) \). Then a simple calculation shows that \( n = gDg^{-1} \). So we let

\[
\Lambda = \text{diag}(0, -ic_k, -2ic_k, \cdots, -kic_k, kic_k, (k-1)ic_k, \cdots, ic_k),
\]
where $c_k = \frac{2\pi}{2k+1}$. It is clear that $\Lambda$ is an element of $\mathfrak{su}(2k + 1)$ and that $\exp(\Lambda) = D$. Hence we obtain that
\[ N = g\Lambda g^{-1} \]
is an element of $\mathfrak{su}(2k + 1)$ which satisfies $\exp(N) = n$.

**Example 2.6**: The case of $\mathfrak{su}(2k)$ can be dealt with in a similar fashion. We obtain that
\[ n = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \]
is an element of $N(T)$ which gives modulo $T$ a Coxeter transformation of $SU(2k)$. Consider the unitary change of basis given by $g = (g_{ab})$, where
\[ g_{ab} = \frac{1}{\sqrt{2k^2}} \beta_k^{(a-1)(2b-1)} \]
where $\beta_k = \exp\left(\frac{\pi i}{2k}\right)$. Also, let $D = (D_{ab})$, where
\[ D_{ab} = \exp\left(-\frac{(2a-1)\pi i}{2k}\right) \delta_{ab} \]
Then a simple calculation shows that $n = gDg^{-1}$. Finally, let $\Lambda = \text{diag}(-id_k, -3id_k, \ldots, -(2k-1)id_k, (2k-1)id_k, (2k-3)id_k, \ldots, id_k)$ where $d_k = \frac{\pi}{2k}$. As in example 2.5, $N = g\Lambda g^{-1}$ is an element in $\mathfrak{su}(2k)$ satisfying $n = \exp(N)$.

### 3. Orthogonal Cartan subalgebras

In this section, we prove the following non-trivial theorem which was proved in [7] lemma 2.2, via the classification tables of all fundamental weights. Their proof relies on the observation found in the classification tables [4], that the highest root is equal or twice some fundamental weight, in all simple Lie algebras except those of type $A_n$. It would be nice to have a more direct argument for this nice result, and this is what we shall present in this section.

**Theorem 3.1.** Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group. Then $L$ has orthogonal Cartan subalgebras (where one of them can be chosen arbitrarily)
Proof. Let $C$ be a Cartan subalgebra of $L$. Then by Theorem 2.4, $C \subseteq [a, L]$ for some regular element $a$ of $L$. Since $a$ regular in $L$, $Z(a)$ is a Cartan subalgebra of $L$. Moreover, $Z(a)^\perp = [a, L]$ by Remark 5.2. Hence $C \subseteq Z(a)^\perp$. Thus the Cartan subalgebras $C$ and $Z(a)$ are orthogonal. Finally, $C$ can be chosen arbitrarily since all Cartan subalgebras (in Lie algebras of compact semisimple Lie groups) are conjugate and the Killing form is invariant under automorphisms of $L$ [15, Prop. 1.119]. This proves the theorem. □

Theorem 3.2. Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group. Let $C$ be a given Cartan subalgebra, and let $C^\perp$ be the orthogonal of $C$ (with respect to the negative of the Killing form on $L$). Then for every element $x$ of $L$, there exists an element $g$ in $G$ such that $g.x$ belongs to $C^\perp$. That is, $C^\perp$ intersects every $G$-orbit of $L$.

Proof. By Theorem 3.1, there exists a Cartan subalgebra $C'$ which is orthogonal to $C$. For every element $x$ in $L$, we know that there exist a $g$ in $G$ such that $g.x \in C'$. Hence $g.x$ is orthogonal to $C$, so $g.x$ belongs to $C^\perp$. Thus, $C^\perp$ intersects every $G$-orbit in $L$. (Or equivalently, $L$ is the union of all $g.C^\perp$ as $g$ varies over $G$). □

As noted in the introduction, Theorem 3.2 can be obtained almost immediately from Kostant’s Convexity Theorem (See Appendix, Theorem A.3).

Corollary 3.3. Let $L$ be a compact semisimple Lie algebra and let $a$ be a given regular element of $L$. Then there exists a regular element $b$ in $L$, such that $L = [a, L] + [b, L]$ and $b$ is orthogonal to $a$ (under the negative of the Killing form of $L$).

Proof. By Theorem 3.1, there exist orthogonal Cartan subalgebras $C$ and $D$ in $L$, so $D \subseteq C^\perp$. Since $L = D \oplus D^\perp$, it follows that $L = C^\perp + D^\perp$. Let $x$ be any regular element of $L$ in $C$ and let $y$ be any regular element of $L$ in $D$, so $C = Z(x)$ and $D = Z(y)$. Thus $L = Z(x)^\perp + Z(y)^\perp$. But $Z(x)^\perp = [x, L]$ and $Z(y)^\perp = [y, L]$ by Remark 5.2. Hence $L = [x, L] + [y, L]$. Note that $x$ and $y$ are regular in $L$ and orthogonal (since they belong to $C$ and $D$ which are orthogonal).

The given regular (semisimple) element $a$ belongs to a Cartan subalgebra and all Cartan subalgebras in $L$ are conjugate. Hence there exists an inner automorphism $f$ such that $f(a)$ is in $C$. Since $a$ and hence $f(a)$ is regular in $L$, the element $x$ in the previous paragraph can be chosen to be $f(a)$. Hence $L = [f(a), L] + [y, L]$. Now we apply $f^{-1}$ on both sides to get $L = [a, L] + [f^{-1}(y), L]$. Since $x = f(a)$ and $y$ are orthogonal, then so are $a$ and $f^{-1}(y)$ since automorphisms preserve...
orthogonality [15] Prop. 1.119. Finally, since $y$ is regular, then so is $f^{-1}(y)$. □

Corollary 3.3 can be easily generalized to the case of non-Hermitian real semisimple Lie algebras of full rank.

**Corollary 3.4.** Let $L = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of a semisimple Lie algebra $L$ (where $\mathfrak{k}$ is the compact Lie algebra) such that $L$ is non-Hermitian of full rank. That is, $\mathfrak{k}$ is semisimple and $\text{rank}(\mathfrak{k}) = \text{rank}(L)$. Let $a$ be a regular element of $\mathfrak{k}$. Then there exists a regular element $b$ in $\mathfrak{k}$, such that $L = [a, L] + [b, L]$ and $b$ is orthogonal to $a$. Moreover the centralizers $Z(a)$ and $Z(b)$ of $a$ and $b$ in $L$ are also orthogonal.

**Proof.** The proof is very similar to the proof of Corollary 3.3 with the evident modifications as follows. By Theorem 3.1, there exist orthogonal Cartan subalgebras $C$ and $D$ in $\mathfrak{k}$. In fact, $C$ and $D$ are also orthogonal with respect to the Killing form on $L$ (see Remark 3.5 below). Since $\text{rank}(\mathfrak{k}) = \text{rank}(L)$, $C$ and $D$ are also Cartan subalgebras of $L$. Since the restriction of the Killing form of $L$ on every Cartan subalgebra is non-degenerate, we have $L = D + D^\perp$. But $D$ is a subset of $C^\perp$ in $L$. Hence $L = C^\perp + D^\perp$. Let $x$ be any regular element of $L$ in $C$ and let $y$ be any regular element of $L$ in $D$. Then the rest of the proof is verbatim as in the proof of Corollary 3.3, leading to $L = [x, L] + [y, L]$ and finally, $L = [a, L] + [f^{-1}(y), L]$ where $a$ and $f^{-1}(y)$ are regular and orthogonal in $L$. □

Apart from the Lie algebras of compact semisimple Lie groups, some other examples of Lie algebras satisfying the hypotheses of Corollary 3.4 are $\mathfrak{so}(2j, 2k)$ with $j$ and $k$ both $> 1$, $\mathfrak{so}(2j, 2k + 1)$ if $j > 1$ and $\mathfrak{sp}(p, q)$ where $p$ and $q$ are both positive.

**Remark 3.5:** Over a field of characteristic 0, let $S$ be a semisimple Lie subalgebra of a semisimple Lie algebra $L$, and let $C$ be a Cartan subalgebra of $S$. Then the orthogonal space of $C$ in $S$ (with respect to the Killing form of $S$) is contained in the orthogonal space of $C$ in $L$ (with respect to the Killing form of $L$).

**Proof.** We know that $C = Z(a)$ for some regular element $a$ of $S$. Moreover, the restriction of the Killing form of $L$ to $S$ is also non-degenerate by a simple application of Cartan’s solvability criterion by [8], chapter 1, section 6.1, Proposition 1]. Hence, by [11, Lemma 2.2] (stated after Remark 5.2 below), the orthogonal space of $C = Z(a)$ in $S$ coincides
with \([a, S]\) independently of the invariant non-degenerate symmetric bilinear form on \(S\).

\[\square\]

4. Applications

4.1. Openness of commutator maps. We start with the following immediate application of Theorem 3.2.

**Lemma 4.1.** Let \(L = \text{Lie}(G)\) be the Lie algebra of a compact semisimple Lie group, and let \(C = \text{Lie}(T)\) be a maximal toral subalgebra of \(L\). Let \(V\) be a \(G\)-stable neighborhood of \(0 \in L\). Then

\[G.(V \cap C^\perp) = V\]

**Proof.** This follows immediately from Theorem 3.2 as follows. Since \(V\) is \(G\)-stable, \(G.(V \cap C^\perp) \subseteq V\). Conversely, if \(v \in V\), then by Theorem 3.2 there exists a \(g \in G\) such that \(g.v \in C^\perp\). Hence \(g.v \in V \cap C^\perp\); so \(v \in G.(V \cap C^\perp)\). \(\square\)

Using lemma 4.1 one can prove the following theorem.

**Theorem 4.2.** ([7], theorem 2.1). Let \(L = \text{Lie}(G)\) be the Lie algebra of a compact semisimple Lie group. Then the commutator map \(\text{Comm} : L \times L \to L\) given by \(\text{Comm}(x, y) = [x, y]\) is open at \((0, 0)\).

The proof of this theorem as given in [7] relies on their lemma 2.2, that we proved in Theorem 3.1 above (without the classification tables of the fundamental weights versus the highest roots).

However, a careful study of their proof of Theorem 4.2 only requires the partial result in our lemma 4.1.

**Remark 4.3:** It is worth noting that the authors in [7] were able to use the fact that the commutator map \(\text{Comm} : L \times L \to L\) given by \(\text{Comm}(x, y) = [x, y]\) is open at \((0, 0)\) to prove their main result that \(\text{Comm}_G : G \times G \to G\) given by \(\text{Comm}_G(g_1, g_2) = [g_1, g_2] = g_1g_2g_1^{-1}g_2^{-1}\) is open at \((\text{Id}, \text{Id})\) in \(G \times G\) ([7], Corollary 3.1).

4.2. Fatness of certain connections on principal fiber bundles. We first review the basic definition of fatness, in the context of principal fiber bundles with connection. Such material can easily be found in the literature (see the original paper [22] by A. Weinstein, or later works such as [9] or [18]). Let \(G \to P \to B\) be a principal \(G\)-bundle, with base manifold \(B\), and total space \(P\). Let \(\theta : TP \to \mathfrak{g}\) be a connection 1-form on \(P\), where \(TP\) denotes the tangent bundle of \(P\). We shall denote the curvature of \(\theta\) by \(\Omega(-, -) : \mathcal{H} \times \mathcal{H} \to \mathfrak{g}\), where \(TP = \mathcal{H} \oplus \mathcal{V}\) is the decomposition into horizontal and vertical subbundles induced by \(\theta\).
Fix a bi-invariant (positive-definite) inner product $B(-, -) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g}$.

**Definition 4.4.** In the setting above of a principal $G$-bundle $P$ with connection $\theta$, an element $u \in \mathfrak{g}$ is said to be fat if $\Omega_u : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is non-degenerate, where $\Omega_u$ is defined by

$$
\Omega_u(Y_1, Y_2) = B(\Omega(Y_1, Y_2), u)
$$

We also say that a subspace $V \subseteq \mathfrak{g}$ is fat if every non-zero element of $V$ is fat. The connection $\theta$ is said to be fat if $\mathfrak{g}$ is fat.

We can easily deduce the following theorem, answering a question raised in [9, Remark 3.13]

**Theorem 4.5.** If $G$ is a compact semisimple Lie group, $B(-, -)$ is minus its Killing form, and $G \hookrightarrow P \to B$ is a principal $G$-bundle on $B$ with total space $P$, endowed with a connection 1-form $\theta : TP \to \mathfrak{g}$, then $\theta$ is fat if and only if $C^\perp$ is fat, where $C$ can be chosen to be any Cartan subalgebra of $\mathfrak{g}$.

**Proof.** The “only if” part is trivial. The other direction follows from the fact that $G.C^\perp = \mathfrak{g}$ by [1.3] (which follows immediately from Kostant’s convexity theorem), and the known fact that if $u \in \mathfrak{g}$ is fat, then $g.u$ is also fat, for any $g \in G$ (see for example [9]). □

5. Commutators in semisimple Lie algebras and survey of some related results

5.1. Conjectures about commutators in semisimple Lie algebras.

**Remark 5.1:** The following two conjectures are equivalent.

1) Every element $x$ in a real semisimple Lie algebra $L$ is the commutator of two elements where one element can be chosen to be regular (hence semisimple).

2) Every element $x$ in a real semisimple Lie algebra $L$ is orthogonal to some Cartan subalgebra of $L$.

**Proof.** Note that every regular element of a semisimple Lie algebra (over a field of characteristic 0) is semisimple [4, Chapter 7, section 2, Corollary 2]. First we prove conjecture 1) implies conjecture 2). Suppose $x = [a, b]$ where $a$ is regular (semisimple). Then $x$ belongs to $[a, L]$. But $[a, L] = Z(a)^\perp$ by Remark 5.2. Hence $x$ belongs to $Z(a)^\perp$. Now $a$ is regular (semisimple), $Z(a)$ is a Cartan subalgebra $C$ of $L$. Consequently, $x$ belongs to $C^\perp$. Conversely, if $x$ belongs to $C^\perp$ where $C$ is a Cartan subalgebra of $L$. Then $C = Z(a)$ for some regular
element $a$ of $L$. So $x$ belongs to $Z(a)^{\perp}$ which coincides with $[a, L]$ by Remark 5.2. Hence $x$ belongs to $[a, L]$. □

Remark 5.2: Let $a$ be a semisimple element of a semisimple Lie algebra $L$ over a field of characteristic 0. Then $Z(a)^{\perp} = [a, L]$ with respect to the negative of the Killing form on $L$ (where $Z(a)$ is the centralizer of $a$ in $L$.

Remark 5.2 is part of Theorem 4.1.6 in [20] which can be checked directly. Roughly, $Z(a)$ is orthogonal to $[a, L]$, and both $Z(a)^{\perp}$ and $[a, L]$ have the same dimension. However, it is worth noting that Remark 5.2 (with essentially the same proof) is also valid for any element of $L$ and, for any invariant non-degenerate symmetric bilinear form on $L$ as shown in [1] Lemma 2.1.

5.2. Survey of some recent results on the commutator conjecture. In the recent paper [1], the commutator conjecture was proved in many cases.

Theorem 5.3. [1] Thms 1.1–1.4] Let $L = \text{Lie}(G)$ be a simple real Lie algebra. Let $L = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $L$ where $\mathfrak{k}$ is the reductive part. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$, and let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then every element $x$ of $L$ can be written as $x = [a, b]$ where $a$ can be chosen to be a regular (semisimple) element of $L$, in each of the following cases:

(i) $L$ is compact or split (i.e. $\mathfrak{m} = 0$)
(ii) $L$ is non-hermitian (i.e. $\mathfrak{k}$ is semisimple) and $\text{rank}(\mathfrak{k}) = \text{rank}(L)$
(iii) $\mathfrak{m}$ is a semisimple Lie algebra

The paper [1] relies on the Kostant convexity theorem and some interesting arguments by the author.

For more information about complex or split semisimple Lie algebras, see [7] Thm. 4.3 and [6] Lemma 2.

5.3. Survey on generators and 1.5 generators of simple Lie algebras.

Theorem 5.4. [2] Thm. 26] Let $L$ be a semisimple Lie algebra over a field of characteristic 0 (or characteristic $p > 3$). Then there exist $x$, $y$ in $L$ such that $L = [x, L] + [y, L]$.

Theorem 5.5. [19] Thm. 2] (on 1.5 generators). Let $L$ be a semisimple Lie algebra over any field of characteristic 0. For any fixed $x \in L \setminus \{0\}$, there exists a regular element $y \in L$ such that $L = [x, L] + [y, L]$.
We note that the above two theorems are related respectively to the concepts of generators and 1.5 generators in Lie algebras in view of the following fact: If \( x \) and \( y \) generate, in the Lie algebra sense, a Lie algebra \( L \), then \( L = [L, x] + [L, y] \) by [3, Lemma 25c].

6. Examples of orthogonal Cartan subalgebras

In the examples of orthogonal Cartan subalgebras we will see, corresponding to the classical groups, circulant matrices play a special role. We begin by reviewing circulant matrices, following the presentation in [17]. Given a vector \( \mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^n \), we define

\[
A = \begin{pmatrix}
  a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\
  a_{n-1} & a_0 & \cdots & a_{n-3} & a_{n-2} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_2 & a_3 & \cdots & a_0 & a_1 \\
  a_1 & a_2 & \cdots & a_{n-1} & a_0
\end{pmatrix}
\]

Let \( \epsilon = e^{2\pi i/n} \), which is a primitive \( n \)’th root of unity, and let

\[
x_l = \frac{1}{\sqrt{n}} (1, \epsilon^l, \epsilon^{2l}, \ldots, \epsilon^{(n-1)l}) \in \mathbb{C}^n
\]

for \( l = 0, \ldots, n \). We let

\[
U = (x_0, x_1, \ldots, x_{n-1})
\]

In other words, \( U \) is the square matrix having as columns \( x_0, x_1, \ldots, x_{n-1} \). It is easy to check that \( U \) is both unitary and symmetric. We introduce the numbers

\[
\lambda_l = \sum_{j=0}^{n-1} \epsilon^{lj} a_j
\]

for \( l = 1, 2, \ldots, n \). Let \( \Lambda = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \). A calculation shows that

\[
A = U\Lambda U^{-1}
\]

so that the \( \lambda_l \)s are the eigenvalues of \( A \). An important observation is that \( U \) does not depend on \( A \), so that the same unitary matrix \( U \) diagonalizes all complex circulant \( n \) by \( n \) matrices. In particular, this also implies that the space \( \text{Circ}(n) \) of all complex circulant \( n \) by \( n \) matrices is abelian (which can be seen directly too).
6.1. The Lie algebras $\mathfrak{su}(n)$.

**Theorem 6.1.** If $C$ is the Cartan subalgebra of $\mathfrak{su}(n)$ consisting zero trace skew-hermitian diagonal $n$ by $n$ complex matrices, and

$$C' = \text{Circ}(n) \cap \mathfrak{su}(n)$$

then $C$ and $C'$ are two orthogonal Cartan subalgebras of $\mathfrak{su}(n)$, with respect to minus the Killing form of $\mathfrak{su}(n)$.

**Proof.** We already know that $C'$ is abelian, since $\text{Circ}(n)$ is abelian, so it only remains to prove that $\dim \mathbb{R}(C') = n - 1$. This is so because $C'$ corresponds to vectors $a$ having

$$\begin{align*}
  a_0 &= 0 \\
  a_{n-l} &= -\bar{a}_l, \quad 1 \leq l \leq n - 1
\end{align*}$$

from which it follows indeed that the real dimension of $C'$ is $n - 1$. \hfill \Box

6.2. The Lie algebras $\mathfrak{sp}(n)$. It is well known that $\mathfrak{gl}(n, \mathbb{H}) \subseteq \mathfrak{gl}(2n, \mathbb{C})$. This relies on the observation that a quaternion $x$ can be written in a unique way as $x = u + jv$, where $u$ and $v$ are complex numbers. It can also be verified that the map

$$x \mapsto \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$$

is an injective algebra homomorphism from $\mathbb{H}$ into $\mathfrak{gl}(2, \mathbb{C})$.

The Lie algebras $\mathfrak{sp}(n)$ can be written as

$$\mathfrak{sp}(n) = \mathfrak{gl}(n, \mathbb{H}) \cap \mathfrak{u}(2n)$$

where both Lie algebras on the right-hand side are understood as Lie subalgebras of $\mathfrak{gl}(2n, \mathbb{C})$, using the remark above. We then have the following theorem.

**Theorem 6.2.** If $C$ is the Cartan subalgebra of $\mathfrak{sp}(n)$ consisting of complex diagonal pure imaginary $2n$ by $2n$ matrices, and

$$C' = \text{Circ}(2n) \cap \mathfrak{sp}(n)$$

where both Lie algebras on the right-hand side are thought of as Lie subalgebras of $\mathfrak{gl}(2n, \mathbb{C})$ in the usual way, then $C$ and $C'$ are two orthogonal Cartan subalgebras of $\mathfrak{sp}(n)$, with respect to minus the Killing form of $\mathfrak{sp}(n)$.

**Proof.** The only thing to check is that $\dim \mathbb{R}(C') = n$. This essentially follows from the fact that $\text{Circ}(2n)$ is a complex $2n$-dimensional subspace of $\mathfrak{gl}(2n, \mathbb{C})$, which admits the decomposition

$$\text{Circ}(2n) = C' \oplus iC' \oplus jC' \oplus kC'$$
the latter being a real vector space decomposition, from which it can be deduced that \( \dim_\mathbb{R}(C') = n \), as claimed.

6.3. The Lie algebras \( \mathfrak{so}(n) \). Define

\[
C_{2k} = \left\{ \begin{pmatrix} 0_k & \Lambda \\ -\Lambda & 0_k \end{pmatrix} : \Lambda \text{ is a real } k \times k \text{ diagonal matrix} \right\}
\]

\[
C_{2k+1} = \left\{ \begin{pmatrix} 0_k & \Lambda & 0_{k,1} \\ -\Lambda & 0_k & 0_{k,1} \\ 0_{1,k} & 0_{1,k} & 0 \end{pmatrix} : \Lambda \text{ is a real } k \times k \text{ diagonal matrix} \right\}
\]

Here \( 0_k \) denotes the zero \( k \times k \) matrix, while for instance \( 0_{j,k} \) denotes the \( j \) by \( k \) zero matrix. It is easy to check that \( C_n \) is a Cartan subalgebra of \( \mathfrak{so}(n) \) for all \( n \geq 3 \). Our plan in this section is to construct for each \( n \), a Cartan subalgebra \( C'_n \) which is orthogonal to \( C_n \). Our next lemma will allow us to restrict our attention to \( \mathfrak{so}(n) \), with \( n \) even.

**Lemma 6.3.** if \( C'_{2k} \) is a Cartan subalgebra of \( \mathfrak{so}(2k) \) orthogonal to \( C_{2k} \), and if \( \iota : \mathfrak{so}(2k) \to \mathfrak{so}(2k+1) \) denotes the inclusion of Lie algebras which satisfies \( \iota(C_{2k}) = C_{2k+1} \), then \( \iota(C'_{2k}) \) is a Cartan subalgebra of \( \mathfrak{so}(2k+1) \) which is orthogonal to \( C'_{2k+1} \).

**Proof.** This follows from the fact that \( \mathfrak{so}(2k) \) and \( \mathfrak{so}(2k+1) \) both have rank \( k \), and that the restriction of the Killing form of \( \mathfrak{so}(2k+1) \) to \( \mathfrak{so}(2k) \) is a constant multiple of the Killing form of \( \mathfrak{so}(2k) \).

The next lemma will enable us to restrict our attention further to just \( \mathfrak{so}(4) \) and \( \mathfrak{so}(6) \). But first, we introduce the following notation. If \( X \) is a \( k \) by \( k \) matrix, we denote by \( \tilde{X} \) the following \( k+2 \) by \( k+2 \) matrix

\[
\tilde{X} = X \oplus 0_2
\]

Also, if \( x \) is \( 2k \) by \( 2k \) matrix, one can write \( x \) as

\[
x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

where \( A, B, C \) and \( D \) are each \( k \) by \( k \) matrices. We then denote by \( \hat{x} \) the following \( 2k+4 \) by \( 2k+4 \) matrix

\[
\hat{x} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}
\]

**Lemma 6.4.** If \( \text{Span}_\mathbb{R}\{x_i : 1 \leq i \leq k\} \) is a Cartan subalgebra of \( \mathfrak{so}(2k) \) which is orthogonal to \( C_{2k} \), then \( \text{Span}_\mathbb{R}\{\hat{x}_i, y, z : 1 \leq i \leq k\} \) is a Cartan
subalgebra of \( \mathfrak{so}(2k+4) \) which is orthogonal to \( C_{2k+4} \), where

\[
y = \begin{pmatrix} 0_k & S \\ -S & 0_k \end{pmatrix}
\]

\[
z = \begin{pmatrix} 0_k & T \\ T & 0_k \end{pmatrix}
\]

where

\[
S = 0_k \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
T = 0_k \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

Proof. It suffices to prove that the \( \hat{x}_i \)'s, \( y \) and \( z \) all mutually commute, which is easy to check. \( \Box \)

Hence, using lemmas \([6.3] \) and \([6.4] \), it suffices for our purposes to construct a Cartan subalgebra of \( \mathfrak{so}(4) \), respectively \( \mathfrak{so}(6) \) which is orthogonal to \( C_4 \), respectively \( C_6 \). Then we can, using the constructions in these two lemmas, construct orthogonal Cartan subalgebras of \( \mathfrak{so}(n) \) for any \( n \geq 4 \) (the case of \( \mathfrak{so}(3) \) is trivial).

Let us consider first \( \mathfrak{so}(4) \). Define

\[
x_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\]

\[
x_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

It is easy to check that \( x_1 \) and \( x_2 \) span a Cartan subalgebra of \( \mathfrak{so}(4) \) which is orthogonal to \( C_4 \).
We finally consider the case of $\mathfrak{so}(6)$. Define

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We then define $x_1 = A \oplus 0_3$, and

$$x_2 = \begin{pmatrix} 0_3 & B \\ -B^T & 0_3 \end{pmatrix}$$

as well as $x_3 = 0_3 \oplus C$.

Then a straightforward computation shows that $x_1, x_2$ and $x_3$ mutually commute, so that they span a Cartan subalgebra of $\mathfrak{so}(6)$, which is also orthogonal to $C_6$ (since each of $x_1, x_2$ and $x_3$ is orthogonal to $C_6$). This finishes our description of orthogonal Cartan subalgebras for $\mathfrak{so}(n)$, for $n \geq 4$, with the case of $n = 3$ being trivial.

Appendix A. Some consequences of Kostant’s convexity theorem

As noted in the introduction, we shall present, in this Appendix, a slight simplification of Karl-Hermann Neeb’s proof of Goto’s Theorem, additive version, using Kostant’s Convexity Theorem [13, p. 653]. His proof essentially passes through the proof of Theorem 3.2.

**Theorem A.1** (Kostant’s convexity theorem). Let $L = \text{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group. Let $C = \text{Lie}(T)$ be a maximal toral subalgebra of $L$, corresponding to a maximal torus $T \subseteq G$. Let $p : L \to C$ be the orthogonal projection of $L$ onto $C$ with respect to the negative of the Killing form on $L$. Let $x$ be an element of $C$, and let $W$ be the Weyl group of $(G, T)$ or $(L, C)$. Then $p(G.x) = \text{Conv}(W.x)$.

This version of Kostant’s Convexity Theorem follows very easily from the original version [13, Theorem 8.2] by taking the complexification $L(G)_\mathbb{C}$ of $L(G)$, viewed as a real semisimple Lie algebra. Then $L(G)_\mathbb{C}$ has a Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = L(G)$ and $\mathfrak{p} = iL(G)$. So
we apply Theorem 8.2 of [16] to \( L(G)_C \) viewed as a real semisimple Lie algebra, with \( \mathfrak{k} \) and \( \mathfrak{p} \) as above.

**Lemma A.2.** [1, lemma 2.2] Let \( V \) be a real vector space and \( W \subset GL(V) \) a finite linear group acting without fixed vectors (other than 0). Then the convex hull of any \( W \)-orbit contains \( 0 \in V \).

For the reader’s convenience, we explain the idea of its short proof in [1] which is that the centroid \( z \) of the points of any \( W \)-orbit in \( V \) is fixed under the action of \( W \). But \( z = 0 \) is the only fixed point under \( W \). Hence \( z = 0 \) belongs to the convex hull of any \( W \)-orbit.

Now we give another proof of Theorem 3.2.

**Theorem A.3.** (= Theorem 3.2) In the setting of theorem A.1, let \( C^\perp \) be the orthogonal complement of \( C \) (with respect to negative the Killing form on \( L \)). Then for every element \( x \) of \( L \), there exists an element \( g \in G \) such that \( g.x \) belongs to \( C^\perp \).

**Proof.** Combine A.1 and A.2. \( \square \)

**Theorem A.4** (Goto’s theorem, additive version). In the Lie algebra \( L \) of a compact semisimple Lie group, every element \( x \) is a bracket. That is, \( x = [a, b] \) for some \( a, b \) in \( L \). Moreover, \( a \) or \( b \) can be chosen to be regular.

**Proof.** By A.3 there exists an element \( g \) in \( G \) such that \( g.x \) belongs to \( C^\perp \). But \( C = Z(a) \) for some regular element \( a \) of \( L \). Moreover, \( Z(a)^\perp = [a, L] \) by 5.2. Hence \( g.x \) belongs to \( [a, L] \). Consequently, \( x \) belongs to \( [g^{-1}.a, L] \), and \( g^{-1}.a \) is regular. \( \square \)

**Appendix B. Alternative proof of Cor. 1.3 by \( SU(2) \)-rotations**

In this section, we give a more direct proof of Cor. 1.3 without using Kostant’s convexity theorem. The proof is based on \( SU(2) \) rotations in the orthogonal “root” space decomposition of the Lie algebra \( L = \text{Lie}(G) \) where \( G \) is a compact semisimple Lie group.

We equip \( L \) with its natural inner product (negative the Cartan Killing form of \( L \)). Let \( C \) be a maximal toral subalgebra of \( L \). Let \( \Sigma \subset C^* \) (the dual of \( C \)) be the root system of \( G \) with respect to \( C \) and we fix an ordering of the roots with \( \Delta \) as the set of positive roots.

Then under the adjoint action of \( C \), \( L \) has an orthogonal decomposition

\[
L = C \oplus \sum_{\alpha \in \Delta} L_{\alpha}
\]
Let \( a \) be a positive root, and consider \( \text{Ker}(a) = \{ x \in C : a(x) = 0 \} \), and its orthogonal space \( \text{Ker}(a)^\perp \) in \( C \). Then we have

\[
L = \text{Ker}(a) \oplus \text{Ker}(a)^\perp \oplus \sum_{\alpha \in \Delta} L_\alpha
\]

where \( \text{Ker}(a)^\perp \oplus L_a \simeq \mathfrak{su}(2) \).

More specifically, each \( L_\alpha \) has an orthogonal basis \( \{ u_\alpha, v_\alpha \} \) and \( \text{Ker}(\alpha)^\perp \) has a basis \( \{ h_\alpha \} \) such that

(i) \( [h, u_\alpha] = \alpha(h) v_\alpha \) for all \( h \) in \( C \),
(ii) \( [h, v_\alpha] = -\alpha(h) u_\alpha \) for all \( h \) in \( C \), and \( \{ \alpha, u_\alpha, v_\alpha \} \),
(iii) \( [u_\alpha, v_\alpha] = h_\alpha \) in \( C \),
(iv) \( \text{Ker}(\alpha)^\perp \oplus L_\alpha = \text{Span}_\mathbb{R} \{ h_\alpha, u_\alpha, v_\alpha \} \simeq \mathfrak{su}(2) \).

(cf. [12], 6.48, 6.49, [15], p. 353, [23], p. 59)

Let \( S_a = \text{Ker}(a)^\perp \oplus L_a \), and let \( m_a = \sum_{\alpha \in \Delta \setminus \{ \alpha \}} L_\alpha \). Then

\[
L = \text{Ker}(a) \oplus S_a \oplus m_a
\]

where \( S_a \simeq \mathfrak{su}(2) \).

**Lemma B.1.** (cf. [21], lemma 2.2) In the above setting where

\[
L = \text{Ker}(a) \oplus S_a \oplus m_a
\]

where \( a \in \Delta \) is a positive root, let \( x \in C \). Then there exists a \( g \) in the Lie subgroup \( G_a \simeq SU(2) \) corresponding to \( S_a \simeq \mathfrak{su}(2) \), such that

(i) \( g \cdot x \) has no \( \text{Ker}(a)^\perp \) component
(ii) \( x \) and \( g \cdot x \) have the same \( \text{Ker}(a) \) components
(iii) the \( m_a \) components of \( x \) and \( g \cdot x \) have the same norms

**Proof.** Let \( x' \) be the component of \( x \) in \( \text{Ker}(a)^\perp \). So \( x' \in S_a = \text{Ker}(a)^\perp \oplus L_a \simeq \mathfrak{su}(2) \). Hence there exists a \( g \) in \( G_a \simeq SU(2) \) such that \( g \cdot x' \) has no \( \text{Ker}(a)^\perp \) component. We do this via rotations in \( SU(2) \) (or the fact that all Cartan subalgebras of \( \mathfrak{su}(2) \) are conjugate). Because we are dealing with \( \text{Ker}(a) \), the decomposition

\[(B.1) \quad L = \text{Ker}(a) \oplus S_a \oplus m_a \]

is easily checked to be invariant under the action of \( S_a = \text{Ker}(a)^\perp \oplus L_a \). Hence \( g \cdot x \) has no \( \text{Ker}(a)^\perp \) component, which proves (i). Moreover, \( [S_a, \text{Ker}(a)] = [L_a, \text{Ker}(a)] = 0 \). Consequently, the group \( G_a \simeq SU(2) \) fixes \( \text{Ker}(a) \). Hence \( x \) and \( g \cdot x \) have the same \( \text{Ker}(a) \) components, thus proving (ii). Finally, since the action of \( G_a \) preserves the decomposition (B.1) and every element of \( G_a \) acts by orthogonal transformations (with respect to minus the Killing form), (iii) follows. □
Theorem B.2 (equiv. to Cor. [13]). Let \( L = \text{Lie}(G) \) be the Lie algebra of a compact semisimple Lie group. Let \( C \) be a maximal toral subalgebra of \( L \), and let \( x \in L \). Then there exists a \( g \in G \), such that \( g.x \) belongs to \( C^\perp \) (in particular, the \( G \)-orbit of any \( C^\perp \) is all of \( L \)).

Proof. The following is a proof by contradiction. Assume not. Then there is an \( x \in L \), necessarily non-zero, such that \( g.x \) is not in \( C^\perp \), for all \( g \in G \). But \( G \) is compact, so that \( G.x \) is also compact (being the continuous image of \( G \)), so there is a \( y \in G.x \) with the property that its \( C \)-component has minimal norm, among all points in \( G.x \). By our assumption, the norm of the \( C \)-component of \( y \) is positive. In order to obtain a contradiction, we shall exhibit a \( y' \in G.x \) whose \( C \)-component has norm which is strictly less than that of \( y \). This will follow by an easy application of lemma [B.1]. Indeed, since \( y \) is not in \( C^\perp \), it follows that there is a positive root \( a \in \Delta \) such that the \( C \)-component of \( y \) is not in the kernel of \( a \). Apply lemma [B.1] to \( y \) and such a positive root \( a \). Thus, there exists an \( y' = g.y \in G.y = G.x \), for some \( g \in G \), such that

(i) \( y' \) has no \( \text{Ker}(a)^\perp \) component
(ii) \( y \) and \( y' \) have the same \( \text{Ker}(a) \) components
(iii) the \( m_a \)-components of \( y \) and \( y' \) have the same norms

Writing

\[
y = y_1 + y_2 + y_3
\]
\[
y' = y_1' + y_2' + y_3'
\]

where \( y_1, y_1' \) are elements of \( \text{Ker}(a)^\perp \), \( y_2, y_2' \) are in \( \text{Ker}(a) \), and \( y_3, y_3' \) are elements of \( C^\perp \), we have that \( y_1' = 0 \) and that \( y_2' = y_2 \).

\[
|y_1' + y_2'|^2 = |y_2|^2 < |y_1|^2 + |y_2|^2 = |y_1 + y_2|^2
\]

The previous inequality is strict since \( y_1 \) is non-zero, by our choice of positive root \( a \). Since \( y_1 + y_2 \) is the \( C \)-component of \( y \) (and similarly \( y_1' + y_2' \) is the \( C \)-component of \( y' \)), this contradicts the property of \( y \) having minimal \( C \)-component norm among all points in \( G.x \), thus finishing the proof. \( \square \)

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