$K \to \pi\pi$ decay, $\varepsilon'$ and the RBC-UKQCD kaon physics program

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Abstract. An overview of the kaon physics program of the RBC and UKQCD collaborations will be presented with a focus on the lattice calculation of $K \to \pi\pi$ decay and the direct CP violation parameter $\varepsilon'$. We will describe substantial improvements to our earlier 2015 $K \to \pi\pi$ calculation, including the use of three independent $\pi\pi$ interpolating operators and the results we obtain for $I = 0$ $\pi\pi$ scattering for energies at and below the kaon mass. While the new result for $\varepsilon'$ is not yet complete, the enhanced statistics and improved analysis that underlies our expanded calculation will be presented.

1. Introduction
The study of weak interactions of kaons both within and beyond the Standard Model remains one of the central activities of the RBC and UKQCD Collaborations. The project to compute the $K_L - K_S$ mass difference is now focused on the analysis of 152 configurations with a $64^3 \times 128$ lattice volume and four valence quark flavors with physical masses [1]. A similar calculation of the long-distance contribution to $K^+ \to \pi^+\nu\overline{\nu}$ [2] has collected results from 37 configurations with the final analysis now underway. The status of the latest RBC/UKQCD calculation of $K^+ \to \pi^+\ell^+\ell^-$ [3] was described by Fionn Ó hÓgáin in this conference.

As the precision of lattice calculations increases, electromagnetic (E&M) effects grow in importance and their direct calculation using lattice QCD is now a major research activity. The RBC and UKQCD Collaborations are now calculating the electromagnetic corrections to $Kl^2$ and $\pi l^2$ decays [4] and $K \to \pi\pi$ decay [5]. Of particular interest is a new method for performing E&M corrections which is free from power-law, finite-volume errors [6]. In this talk we will discuss two further projects: The calculation of $\pi^0 \to e^+e^-$ as a first step toward the calculation of the two-photon component of $K_L \to \mu^+\mu^-$ decay and calculation of $K \to \pi\pi$ decay and $\varepsilon'$.

2. $K_L \to \mu^+\mu^-$ and $\pi^0 \to e^-e^+$
In the Standard Model the decay $K_L \to \mu^+\mu^-$ is a highly suppressed, “strangeness changing neutral current process” that requires the exchange of two weak bosons with an accurately measured branching fraction $B(K_L \to \mu^+\mu^-) = (6.84 \pm 0.11) \times 10^{-9}$ [7]. For this measurement to become an important short-distance test of the Standard Model, the competing $O(\alpha^2_{EM}G_F)$ two-photon contribution must be computed. While the imaginary part of this contribution can be obtained from the $K \to \gamma\gamma$ decay rate and the optical theorem, the real part must be computed in QCD [8]. Depending on a relative sign, a 10% calculation of the real part of the $\alpha^2_{EM}G_F$ amplitude would lead to a 6 to 17% test of the Standard Model.
Figure 1 shows a schematic representation of the $O(\alpha^2_{EM}G_F)$ contribution to $K \to \mu^+\mu^-$ decay. Instead of the usual Feynman diagram showing quark lines, we show only the photon and muon lines and a particular time ordering for the three hadronic operators. The solid line represents the QCD part of the amplitude.

Figure 1. Schematic diagram showing the two-photon contribution to $K \to \mu^+\mu^-$ decay. The dark solid line represents the hadronic part of the amplitude connecting the two E&M currents, the weak Hamiltonian and the initial kaon, in a specific time order.

The real part of the amplitude represented in Fig. 1 can be obtained in Euclidean space. However, such an Euclidean space evaluation will contain other unphysical terms which will often dominate the limit of large time separation needed in such an Euclidean space calculation to project onto the kaon ground state. These additional terms are referred to as exponentially growing with increasing time separation because they correspond to possible intermediate states with energy less than that of the kaon.

For example, in the time order shown in Fig. 1 a state composed of two pions and a photon can appear between $H_W$ and the left-most E&M current $J_\mu$. The energy of such a state can be smaller than the kaon mass $M_K$. As a result as one increases the time separation $T$ between the operator which creates the $K_L$ and the operators which absorb the final muons, a large contribution will come when $H_W$ and $J_\nu$ quickly convert the kaon into a $\gamma\pi\pi$ state with energy $E_{\gamma\pi\pi}$. This light state propagates from the location of the kaon to the leptonic E&M current which absorbs that photon. For the component of interest a state with energy $M_K$ propagates over the time interval $T$ (as either the initial kaon or the final $\mu^+\mu^-$ pair), with a short-range interaction causing the $K \to \mu^+\mu^-$ transition of interest over a time interval of length $1/\Lambda_{QCD}$. The former dominates over the latter by the exponentially growing factor $\exp\{-M_{\gamma\pi\pi}(E_{\gamma\pi\pi} - M_K)/T\}$.

For the simpler case of the $K_L - K_S$ mass difference where only two intermediate operators with two possible time orders appear, such exponentially growing terms can be evaluated and accurately removed. However, the $K \to \mu^+\mu^-$ process is much more difficult with five operators and 120 time orders. It is natural to seek a better method.

As a step in this direction we have developed a new method that can be directly applied to the decay $\pi^0 \to e^+e^-$ and used it to compute both the real and imaginary parts of this simpler decay process [9]. We begin with the conventional Minkowski-space decay amplitude, factor it into leptonic and hadronic pieces, explicitly integrate over both the average position of the two hadronic and the two leptonic E&M currents to impose four-momentum conservation and write the result as a combination of position and momentum integrals:

$$A_{\pi^0 \to e^+e^-} = \int d^4w \, \tilde{L}(k_-, k_+, w)_{\mu\nu} \langle 0| T\{J_\mu(w)J_\nu(-w)\}|\pi^0(\vec{P} = 0)\rangle,$$

where the leptonic part of this amplitude is given by

$$\tilde{L}(k_-, k_+, w)_{\mu\nu} = \int dp_0 \int d^3p \, \bar{\nu}(k_-)\gamma_\mu \frac{p - p_+ + m_e}{(p - k_+)^2 + m_e^2 - i\epsilon} \gamma_\nu \nu(k_+) \frac{1}{(p - \frac{P}{2})^2 - i\epsilon} \frac{1}{(p + \frac{P}{2})^2 - i\epsilon} e^{-ip\cdot w}.$$ (2)
In order to evaluate the hadronic Green’s function in Eq. (1) using lattice QCD, we must change from Minkowski to Euclidean space. This can be done within the Minkowski space amplitude $A_{\pi^0 \to e^+ e^-}$ by performing a simultaneous rotation of the phases of the integration variables $p_0 \rightarrow p_0 e^{i\phi}$ in Eq. (2) and $w_0 \rightarrow w_0 e^{-i\phi}$ in Eq. (1), increasing $\phi$ from zero to $\pi/2$. This change of integration contour is an application of Cauchy’s theorem which does not change the quantity $A_{\pi^0 \to e^+ e^-}$. The amplitude $A_{\pi^0 \to e^+ e^-}$ remains a complex, Minkowski-space quantity. Note, the analytic dependence of the hadronic amplitude in Eq. (1) on $w_0$ can be easily established by inserting a sum over intermediate states between the two currents. As shown in Fig. 2 the deformation of the $p_0$ contour must be more elaborate than the simple phase rotation suggested above. The positive energy of the $\pi^0$ implies singularities in the first and third quadrants of the complex $p_0$ plane which require the contour deformations in Fig. 2.

The two loops in the deformed $p_0$ contour in Fig. 2 that lie along the real axis with real parts as large as $\pm m_{\pi}/2$ introduce exponential growth for large imaginary $w_0$ caused by the factor $e^{\mp i\omega_0 w_0}$ in Eq. (2). However, the hadronic Green’s function in Eq. (1) falls for large imaginary $w_0$ as $e^{-3m_{\pi}|w_0|}/2$ insuring that the Euclidean-time $w_0$ integration converges.

Thus, the complex Minkowski-space decay amplitude $A_{\pi^0 \to e^+ e^-}$ can be computed directly on the lattice. Preliminary results [9] with physical quark masses from 31, $48^3 \times 96$ gauge configurations with an inverse lattice spacing $1/a = 1.73$ GeV are

$$\text{Im}A = 35.26(57)(1.83)\text{eV} \quad \text{Re}A = 19.68(52)(1.10)\text{eV}$$

(3)

where the first error is statistical and the second systematic. Their experimental values are $35.07(37)$ and $21.51(2.02)$. We are now attempting to apply this new approach to $K_L \rightarrow \mu^+ \mu^-$. 

### 3. $K \rightarrow \pi\pi$ decay and $\epsilon'$

The second topic addressed in this talk is the calculation of $K \rightarrow \pi\pi$ decay and $\epsilon'$ in the Standard Model. In 2015 RBC/UKQCD published results for the $I = 2$ decay amplitude $A_2$ [10] and the $I = 0$ amplitude $A_0$ and $\epsilon'$ [11]. The more difficult calculation of $A_0$ was performed on 216 gauge configurations from a $32^3 \times 64$ ensemble with $1/a = 1.38$ GeV and G-parity boundary conditions. These boundary conditions determined that the lowest energy two-pion state would have an energy close to that of the kaon, allowing an energy-conserving calculation of $A_0$.

This 2015 calculation of $A_0$ was performed with essentially physical kinematics ($M_{\pi} = 143.1(2.0)$ MeV, $M_K = 490.6(2.2)$ MeV and $E_{\pi\pi} = 498(11)$ MeV) and gave:

$$\text{Re}A_0 = 4.66(1.00)(1.26) \times 10^7\text{GeV}, \quad \text{Im}A_0 = 1.90(1.23)(1.08) \times 10^{11}\text{GeV},$$

(4)

$$\text{Re} \left( \frac{\epsilon'}{\epsilon} \right) = 1.38(5.15)(4.59) \times 10^{-4}, \quad \delta_{0}^{'f=0} = 23.8(2.2)^{o}.$$  

(5)
where the result for $\varepsilon'$ made use of our earlier result for $\text{Im}A_2$ and the $I = 0$, $s$-wave $\pi\pi$ scattering phase shift $\delta_0^{I=0}$ was determined from Lüscher’s finite volume quantization condition and our result for $E_{\pi\pi}$. The result for $\text{Re}A_0$ agrees with the experimental value $3.3201(18) \times 10^7$ GeV.

These 2015 results raise two important questions:

- Is this $2.1\sigma$ difference from the experimental result $\text{Re}(\varepsilon'/\varepsilon) = (16.6 \pm 2.3) \times 10^{-4}$ real?
- Why is $\delta_0^{I=0}$ so different from the dispersive result $\delta_0^{I=0} = 36^\circ$ [12]?

We have addressed these questions by increasing statistics from 216 to 741 gauge configurations analyzed and including correlation functions with two additional $\pi\pi$ interpolating operators: one a $u\bar{u} + d\bar{d}$, $\sigma$-like operator and the second a $\pi\pi$, $q\bar{q}q\bar{q}$ operator where the relative momentum of the two pions is a cubically symmetric combination of momenta of the sort $2\pi(3,1,1)/L$. These are referred to as $\sigma$ and $\pi\pi(311)$ while the original $\pi\pi$ operator with each pion given the minimum momentum allowed by our G-parity boundary conditions is labeled $\pi\pi(111)$.

The additional operators resolve the first question. In Fig. 3 we show the fitted energy of the $\pi\pi$ ground state as a function of the smallest time separation $t_{\min}$ used in the fit. Including these extra operators systematically lowers the ground state energy, extends the plateau to smaller times and results in a $\pi\pi$ ground state energy which implies $\delta_0 = 31.7(6)^\circ$, not far from the dispersive prediction of $36^\circ$, considering that only the statistical error is given. Using only the $\pi\pi(111)$ operator and performing one- or two-state fits, we are not able to resolve this lower energy, even using an extended ensemble of 1438 configurations. Using multiple operators is necessary to obtain an accurate result, given our current level of precision.

We are not yet ready to present an updated result for the $I = 0$ decay amplitude $A_0$ and $\varepsilon'$. However, the increased accuracy of our new calculation can be seen by comparing our 2015 results for the $K \to \pi\pi$ matrix element of one of the relevant four-quark operators $Q$ in Fig. 4 with the same matrix element from our new calculation in Fig. 5, both in arbitrary units. The upper points in Fig. 5 come from those combinations of two- and three-pion operators which project onto the ground state. These upper points show a more convincing plateau starting at smaller times while the enhanced statistics allow an accurate result for the matrix element to be obtained from a fit beginning at the more conservative value of $t_{\min} = 5$ instead of 4.

A final important ingredient in our new calculation is a method to correctly calculate the $p$-values for our fits: the probability that a generic Monte Carlo sample would give an equal or larger value of $\chi^2$ than that found for our fit, assuming our theoretical expectation for the functional form of our data is correct, e.g., that a two-state description is correct. If the covariance matrix used to define the $\chi^2$ were determined from uncorrelated Gaussian samples, we could determine a $p$-value from Hotelling’s $T^2$ distribution. However, our Monte Carlo samples are correlated and not Gaussian. Fortunately, this difficulty can be overcome by a bootstrap
Figure 4. Results from 2015 for the $K^0-Q^0$ correlation function showing the dependence on the separation between the operator $Q$ and the $\pi\pi(111)$ operator, normalized so that the physical signal should be independent of $t_{\pi\pi} - t_Q$.

Figure 5. New results for the same operator with $t = t_{\pi\pi} - t_Q$. The lower data come from the $\pi\pi(111)$ operator. The two series of upper points come from the optimal combination of two and three pion operators and show a much better plateau.

4. Conclusion

The weak interactions of kaons, especially their rare decays and mixings, provide an important opportunity to search with high sensitivity for phenomena not predicted by the Standard Model. First-principles calculations of these Standard Model predictions using the methods of lattice QCD are becoming increasingly accurate and applicable to a wider range of phenomena. Lattice methods augment traditional theoretical studies and will continue to extend the reach of low energy, high-precision searches for physics beyond the Standard Model.

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