Volume comparison theorems in Finsler spacetimes

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Abstract

In a $(1+n)$-dimensional Lorentz–Finsler manifold with $N$-Bakry–Émery Ricci curvature bounded from below where $N \in (n, \infty]$, using the Riccati equation techniques, we establish the Bishop–Gromov volume comparison theorem for the so-called standard sets for comparisons in Lorentzian volumes (SCLVs). We also establish the Günther volume comparison theorem for SCLVs when the flag curvature is bounded above.

Contents

1 Introduction 1

2 Lorentz–Finsler manifolds 3
   2.1 Metric ................................................. 3
   2.2 Causality ............................................ 4
   2.3 Geodesics and curvatures ............................ 6
   2.4 Jacobi tensor field .................................. 9

3 Bishop–Gromov’s theorem 10

4 Günther’s Theorem 12

5 The case $N = \infty$ 14

1 Introduction

Comparison theorems are classical subjects in Lorentzian geometry in connection with, for example, physical convergence conditions (nonnegative Ricci curvature), Raychaudhuri equations and singularity theorems. Recently Lorentzian comparison theory is attracting growing interests also from the synthetic geometric viewpoint, we refer to [AB, KuSa] for triangle comparison theorems, and to [Mc, MS] for connections with optimal transport theory and the curvature-dimension condition.

Motivated by them, we initiated comparison theory in the weighted Lorentz–Finsler setting in [LMO], where we studied Raychaudhuri equations and singularity theorems.
This article is a continuation and extend some volume comparison theorems to the Lorentz–Finsler case. The Bishop–Gromov comparison theorem plays an important role in global analysis, especially in comparison geometry of Riemannian manifolds. The original version of the theorem assumes a lower bound for the Ricci curvature of a Riemannian manifold. (See Chavel’s textbook \[Ch\], for example.) It has been developed by various papers in different aspects. Generalizations of the Bishop–Gromov comparison theorem for weighted Riemannian manifolds under the $N$-Bakry–Émery Ricci curvature condition were completed by Qian \[Qi\] for $N \in [n, \infty)$ and considered by Wei and Wylie \[WW\] for $N = \infty$ with some additional assumptions on the weight function. Shen \[Sh1\] extended this comparison theorem to the Finsler setting with unweighted Ricci curvature and $S$-curvature conditions. After Ohta \[Oh1\] and \[Yin\] gave proofs to the Bishop–Gromov comparison theorems under weighted Ricci curvature conditions for $N \in [n, \infty)$ and $N = \infty$ with some assumptions on the $S$-curvature of the manifold, respectively. Moreover, by using a differential inequality for an elliptic second order differential operator acting on distance functions, \[BQ\] deduced Bishop–Gromov comparison theorems and diameter bounds without use of the theory of Jacobi fields. In the Lorentzian settings, Ehrlich, Jung and Kim \[EJK\] studied the Bishop–Gromov theorems for compact geodesic wedges in globally hyperbolic spacetimes. Later on, Ehrlich and Sánchez \[ES\] defined a more natural set on which the assumption of global hyperbolicity was no more necessary in either the Bishop–Gromov’s or the Günther’s volume comparisons. For other generalizations, we refer to \[Lee, Zhu, Kim\] and \[Wei\] for a survey on this topic.

Throughout this paper the function $s_\kappa$ is the solution to the differential equation $f'' + \kappa f = 0$ with $f(0) = 0$ and $f'(0) = 1$, i.e.

$$s_\kappa(t) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin (\sqrt{\kappa}t), & \text{for } \kappa > 0; \\ t, & \text{for } \kappa = 0; \\ \frac{1}{\sqrt{\kappa}} \sinh (\sqrt{\kappa}t), & \text{for } \kappa < 0. \end{cases}$$

Our main theorems are as follows.

**Theorem 1.1** Let $(M, L, \rho)$ be a Finsler spacetime equipped with a measure $\rho$ on $M$. Let $U_x$ be a SCLV at $x \in M$. Assume that $\text{Ric}_N(v) \geq c$ for some $c \in \mathbb{R}$ and for all unit timelike radial vectors $v$, where $N \in (n, \infty)$. If $t_{U_x}$ is constant on $U_1$, then we have

$$\frac{\rho(U_x(r))}{\rho(U_x(R))} \geq \frac{\int_0^{t_{U_x}} s_{c/N}^N(t) \, dt}{\int_0^{t_{U_x}} s_{c/N}^N(t) \, dt},$$

for all $0 < r \leq R \leq 1$, where $T_x = t_x$ if $c \leq 0$; $T_x = \min\{t_x, \pi \sqrt{N/c}\}$ if $c > 0$.

We also refer to the flag curvature $K$ in Definition 2.14 and the weight function $\psi$ in Definition 2.16. The Günther volume theorem could be stated as follows with some notations introduced in the end of Section 3.
Theorem 1.2 Let \((M, L, \rho)\) be a Finsler spacetime equipped with a measure \(\rho\) on \(M\). Let \(U_x\) be a SCLV at \(x \in M\). Assume that the flag curvature \(K(\pi) \leq -c\) for some \(c \geq 0\) and for all radially timelike planes \(\pi\). If \(\psi \leq k\), then
\[
\rho(U_x) \geq e^{-k} \sigma \left( \tilde{U}_1 \right) \int_0^{t_x} s_c^{-1}(t) dt,
\]
where \(\sigma\) is the area measure on \(\text{Fut}_1(x)\) induced from \(\rho\).

In the next section, we give some preliminaries of Lorentz–Finsler manifolds. Theorems 1.1 and 1.2 are shown in Sections 3 and 4, respectively. We also consider the case \(N = \infty\) in Section 5 along the lines of [WW].

2 Lorentz–Finsler manifolds

Let \(M\) be a connected paracompact \(C^\infty\)-manifold of dimension \(1 + n\). We also refer to [BCS, Sh2] for Finsler geometry in the positive-definite case.

2.1 Metric

In this article, Beem’s definition of Lorentz–Finsler manifolds [Be] is employed.

Definition 2.1 (Lorentz–Finsler structure) A Lorentz–Finsler structure of \(M\) will be a function \(L : TM \longrightarrow \mathbb{R}\) satisfying the following conditions:

1. \(L \in C^\infty(TM \setminus \{0\})\);
2. \(L(cv) = c^2 L(v)\) for all \(v \in TM\) and \(c > 0\);
3. For any \(v \in TM \setminus \{0\}\), the symmetric matrix
\[
g_{\alpha\beta}(v) := \frac{1}{2} \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}(v), \quad \alpha, \beta = 0, 1, \ldots, n,
\]
is non-degenerate with signature \((- , + , \ldots , + )\).

A pair \((M, L)\) is then said to be a Lorentz–Finsler manifold or a Lorentz–Finsler space.

Let \(x \in M\) and \(v \in T_x M \setminus \{0\}\). The matrix (2.1) induces a Lorentzian metric \(g_v\) called the fundamental tensor by
\[
g_v \left( \sum_{\alpha=0}^{n} a^\alpha \frac{\partial}{\partial x^\alpha}, \sum_{\beta=0}^{n} b^\beta \frac{\partial}{\partial x^\beta} \right) = \sum_{\alpha, \beta=0}^{n} a^\alpha b^\beta g_{\alpha\beta}(v).
\]

Similar to the positive-definite case, the tensor \(g_v\) for \(v \in TM \setminus \{0\}\) satisfies that
\[
g_v(v, v) = \sum_{\alpha, \beta=0}^{n} v^\alpha v^\beta g_{\alpha\beta}(v) = L(v)
\]
2.2 Causality

**Definition 2.2** Let \((M, L)\) be a Lorentz–Finsler manifold. A vector \(v \in TM\) is said to be a **timelike vector** if \(L(v) < 0\) and a **null vector** if \(L(v) = 0\). A vector \(v\) is said to be **lightlike** if it is null and non-zero. The **spacelike vectors** are those for which \(L(v) > 0\) or \(v = 0\). The **causal** (or **non-spacelike**) vectors are those which are lightlike or timelike \((L(v) \leq 0 \text{ and } v \neq 0)\). The set of timelike vectors will be denoted by

\[
\Omega'_x := \{ v \in T_xM \mid L(v) < 0 \}, \quad \Omega' := \bigcup_{x \in M} \Omega'_x.
\]

**Definition 2.3** Let \((M, L)\) be a Lorentz–Finsler manifold. A continuous vector field \(X\) on \(M\) is said to be **timelike** if \(L(X(x)) < 0\) for all \(x \in M\). If \((M, L)\) admits a timelike smooth vector field \(X\), then \((M, L)\) is said to be **time oriented** by \(X\), or simply **time oriented**. A time oriented Lorentz–Finsler manifold is then said to be a **Finsler spacetime**.

Let \(X_M\) be a fixed timelike smooth vector field. A causal vector \(v \in T_xM\) is said to be **future-directed** (with respect to \(X_M\)) if it lies in the same connected component of \(\Omega'_x \setminus \{0\}\) as \(X_M(x)\). Denote by \(\Omega_x \subset \Omega'_x\) the set of all future-directed timelike vectors that is a connected component of \(\Omega'_x\) and make the following notations,

\[
\Omega := \bigcup_{x \in M} \Omega_x, \quad \Omega := \bigcup_{x \in M} \Omega_x, \quad \Omega \setminus \{0\} := \bigcup_{x \in M} (\Omega_x \setminus \{0\}).
\]

A \(C^1\)-curve in \((M, L)\) is said to be **timelike** (resp. **causal**, **lightlike**, **spacelike**) if its tangent vectors are always timelike (resp. causal, lightlike, spacelike). A causal curve is said to be **future-directed** if its tangent vectors are always future-directed.

**Remark 2.1** It is well-known that, in general, the number of connected components of \(\Omega'_x\) may be larger than 2, but in a reversible Lorentz–Finsler manifold of dimension larger than 2, \(\Omega'_x\) has exactly two connected components. See [Be, Min]. In this special case, one may define past-directed vectors as well.

Given distinct points \(x, y \in M\), we write \(x \ll y\) if there is a future-directed timelike curve from \(x\) to \(y\). Similarly, \(x < y\) means that there is a future-directed causal curve from \(x\) to \(y\), and \(x \leq y\) means that \(x = y\) or \(x < y\). The **chronological past** and **future** of \(x\) are defined by

\[
I^-(x) := \{ y \in M \mid y \ll x \}, \quad I^+(x) := \{ y \in M \mid x \ll y \},
\]

and the **causal past** and **future** by

\[
J^-(x) := \{ y \in M \mid y \leq x \}, \quad J^+(x) := \{ y \in M \mid x \leq y \}.
\]

For a general set \(S \subset M\), we define \(I^+(S), I^-(S), J^+(S)\) and \(J^-(S)\) analogously.

**Definition 2.4 (Causal convexity)** Let \((M, L)\) be a Finsler spacetime. An open set \(U \subset M\) is said to be **causally convex** if no causal curve intersects \(U\) in a disconnected set of its domain.
Using these terminologies, several causality conditions may be defined as follows.

**Definition 2.5 (Causality conditions)** Let \((M, L)\) be a Finsler spacetime.

1. \((M, L)\) is said to be **chronological** if \(x \notin I^+(x)\) for all \(x \in M\).
2. We say that \((M, L)\) is **causal** if there is no closed causal curve.
3. \((M, L)\) is said to be **strongly causal at a point** \(x \in M\), if \(x\) has arbitrarily small causally convex neighborhoods. \((M, L)\) is said to be **strongly causal** if it is strongly causal at every point \(x \in M\).
4. We say that \((M, L)\) is **globally hyperbolic** if it is strongly causal and, for any \(x, y \in M\), \(J^+(x) \cap J^-(y)\) is compact.

**Remark 2.2** From the definitions, it is clear that strong causality implies causality and a causal spacetime is chronological. The chronological condition ensures that the spacetime is not compact. Indeed, if we assume the contrary, since \(\{I^+(x)\}_{x \in M}\) forms an open cover of \(M\), the existence of a finite subcover of \(M\) and the fact that for any \(x \in M\) we have \(x \notin I^+(x)\) indicate that there must be a point in the future set of itself.

**Remark 2.3** In volume comparison theorems in the positive-definite case, we usually assume the completeness and compare the volumes of concentric balls. In the Lorentzian case, however, since \(\text{Fut}_1(x)\) is noncompact and “balls” can have infinite volume, we need to restrict ourselves to a compact set (like a SCLV). In [EJK], the authors assumed the global hyperbolicity and the positivity of the injectivity radius within relevant directions, however, the global hyperbolicity is not fulfilled even by some standard examples like anti-de Sitter spacetimes. Then the SCLV was introduced in [ES] as a more direct notion suitable for volume comparison theorems in the Lorentzian setting, and we followed this line.

The definition of **standard for comparisons of Lorentzian volumes (SCLV)** was originally introduced in [ES]. (See Remark 2.3 for more explanations.)

**Definition 2.6 (SCLV)** Let \((M, L)\) be a Finsler spacetime. Let \(x \in M\) and \(U_x \subset M\) be a neighborhood of \(x\). \(U_x\) is said to be a SCLV at \(x\) if there is a set \(\tilde{U}_x \subset T_x M\) satisfying that

1. \(\tilde{U}_x\) is an open set in the causal future of \(x\), (see Subsection 2.2 for the definition of causal future);
2. \(\tilde{U}_x\) is **star-shaped** from the origin, i.e. if \(v \in \tilde{U}_x\), then \(tv \in \tilde{U}_x\), for all \(t \in (0, 1)\);
3. \(\tilde{U}_x\) is contained in a compact set in \(T_x M\);
4. the exponential map at \(x\) is defined on \(\tilde{U}_x\), and the restriction of \(\exp_x\) to \(\tilde{U}_x\) is a diffeomorphism onto its image \(U_x = \exp_x(\tilde{U}_x)\).
2.3 Geodesics and curvatures

Let \((M, L)\) be a Finsler spacetime. Let \(\eta : [a, b] \rightarrow M\) be a future-directed \(C^1\)-causal curve. Define the action
\[
S(\eta) := \int_a^b L(\dot{\eta}(t)) \, dt.
\]
The Euler–Lagrange equation for \(S\) provides the geodesic equation (with the help of homogeneous function theorem)
\[
\ddot{\eta}^\alpha + \sum_{\beta, \gamma=0}^n \check{\Gamma}^\alpha_{\beta\gamma}(\dot{\eta}) \dot{\eta}^\beta \dot{\eta}^\gamma = 0,
\]
(2.3)
where we define
\[
\check{\Gamma}^\alpha_{\beta\gamma}(v) := \frac{1}{2} \sum_{\delta=0}^n g^{\alpha\delta}(v) \left( \frac{\partial g_{\beta\gamma}}{\partial x^\delta} + \frac{\partial g_{\beta\delta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\delta}}{\partial x^\beta} \right)(v)
\]
(2.4)
for \(v \in TM \setminus \{0\}\) and \((g^{\alpha\beta}(v))\) denotes the inverse matrix of \((g_{\alpha\beta}(v))\). The equation (2.3) implies that \(L(\dot{\eta})\) is constant.

Definition 2.7 (Causal geodesics) Let \(\eta : [a, b] \rightarrow M\) be a \(C^\infty\)-causal curve. It is said to be a (future-directed) causal geodesic if (2.3) holds for all \(t \in (a, b)\).

Remark 2.4 Since \(L(\dot{\eta})\) is constant, a causal geodesic is indeed either a timelike geodesic or a lightlike geodesic. By the basic ordinary differential equation theory, given arbitrary \(v \in \Omega_x\), there exists some \(\varepsilon > 0\) and a unique \(C^\infty\)-geodesic \(\eta : (-\varepsilon, \varepsilon) \rightarrow M\) satisfying \(\dot{\eta}(0) = v\).

Definition 2.8 (Radial vector) Let \((M, L)\) be a Finsler spacetime and \(x \in M\). A tangent vector \(v \in T_xM\) is said to be radial if there is a geodesic \(\eta : [0, T] \rightarrow M\) with \(\eta(T) = x\) such that \(v = \dot{\eta}(T)\). Let \(U_x\) be a SCLV at \(x\). A tangent plane \(\pi\) to \(U_x\) is said to be radially timelike if it contains a timelike radial vector.

Definition 2.9 (Exponential map) Given \(v \in \Omega_x\), if there is a geodesic \(\eta : [0, 1] \rightarrow M\) with \(\dot{\eta}(0) = v\), then we define \(\exp_x(v) := \eta(1)\).

We now define the geodesic spray coefficients and the nonlinear connection as
\[
G^\alpha(v) := \sum_{\beta, \gamma=0}^n \check{\Gamma}^\alpha_{\beta\gamma}(v) v^\beta v^\gamma, \quad N^\alpha_\beta(v) := \frac{1}{2} \frac{\partial G^\alpha}{\partial v^\beta}(v)
\]
(2.5)
for \(v \in TM \setminus \{0\}\), and \(G^\alpha(0) = N^\alpha_\beta(0) := 0\). Note that \(G^\alpha\) is positively 2-homogeneous and \(N^\alpha_\beta\) is positively 1-homogeneous, and \(2G^\alpha(v) = \sum_{\beta=0}^n N^\alpha_\beta(v) v^\beta\) holds by the homogeneous function theorem. The geodesic equation (2.3) is now written as \(\ddot{\eta}^\alpha + 2G^\alpha(\dot{\eta}) = 0\). In order to define the covariant derivative, we need to modify \(\check{\Gamma}^\alpha_{\beta\gamma}\) in (2.4) as
\[
\Gamma^\alpha_{\beta\gamma}(v) := \check{\Gamma}^\alpha_{\beta\gamma}(v) - \frac{1}{2} \sum_{\delta, \mu=0}^n g^{\alpha\delta}(v) \left( \frac{\partial g_{\delta\gamma}}{\partial v^\mu} N^\mu_\beta + \frac{\partial g_{\beta\delta}}{\partial v^\mu} N^\mu_\gamma - \frac{\partial g_{\beta\gamma}}{\partial v^\mu} N^\mu_\delta \right)(v)
\]
(2.6)
for $v \in TM \setminus \{0\}$. Note that $\Gamma^\alpha_{\beta\gamma}$ could be used for defining the Chern connection in Finsler spacetimes.

**Definition 2.10 (Covariant derivative)** Let $X$ be a $C^1$-vector field on $M$, $x \in M$ and $v, w \in T_x M$ with $w \neq 0$. Define the covariant derivative of $X$ by $v$ with reference (support) vector $w$ by

$$D^w_v X := \sum_{\alpha, \beta=0}^n \left\{ v^\beta \partial X^\alpha \left( x \right) \right\} \frac{\partial}{\partial x^\alpha} |_x. \quad (2.7)$$

**Definition 2.11 (Parallel vector field)** A vector field $V$ along a curve $\eta : I \rightarrow M$ is said to be $g_{\eta}$-parallel if

$$D^\eta_{\dot{\eta}} \dot{\eta} V(t) = 0,$$

for all $t \in I$.

As usual, one can define Jacobi fields along a causal geodesic by considering a variation. We here just employ the needful notations for completeness.

Define

$$R^\alpha_{\beta\gamma}(v) := \frac{\partial G^\alpha}{\partial x^\beta}(v) - \sum_{\gamma=0}^n \left[ \frac{\partial N^\alpha_\beta}{\partial x^\gamma}(v) v^\gamma - \frac{\partial N^\alpha_\gamma}{\partial v^\beta}(v) G^\gamma(v) \right] - \sum_{\gamma=0}^n N^\alpha_\gamma(v) N^\gamma_\beta(v),$$

for $v \in \overline{\Omega} \ (R^\alpha_\beta(0) = 0)$. Then, the Riemannian curvature is defined as

$$R_v(w) := \sum_{\alpha, \beta=0}^n R^\alpha_{\beta\gamma}(v) w^\beta \left. \frac{\partial}{\partial x^\alpha} \right|_x, \quad (2.8)$$

for $v \in \overline{\Omega}_x$ and $w \in T_x M$.

**Definition 2.12 (Jacobi fields)** Let $\eta : [a, b] \rightarrow M$ be a causal geodesic. A smooth vector field $Y : [a, b] \rightarrow TM$ along $\eta$ is said to be a Jacobi field if $Y$ satisfies the Jacobi equation,

$$D^\eta_{\dot{\eta}} D^\eta_{\dot{\eta}} Y + R_\eta(Y) = 0. \quad (2.9)$$

**Definition 2.13 (Conjugate points)** Let $\eta : [a, b] \rightarrow M$ be a nonconstant causal geodesic. If there is a nontrivial Jacobi field $Y$ along $\eta$ such that $Y(a) = Y(t) = 0$ for some $t \in (a, b]$, then we call $\eta(t)$ a conjugate point of $\eta(a)$ along $\eta$.

Equivalently, $\eta(t)$ is conjugate to $\eta(a)$ if $d(\exp_{\eta(a)})_{(t-a)\eta(a)} : T_{(t-a)\eta(a)}(T_{(t-a)\eta(a)}M) \rightarrow T_{\eta(t)} M$ does not have full rank.

The flag and Ricci curvatures are defined by using $R_v$ in (2.8) as follows. The flag curvature corresponds to the sectional curvature in the Riemannian context.
Definition 2.14 (Flag curvature) For \( v \in \Omega_x \) and \( w \in T_x M \) linearly independent of \( v \), define the flag curvature of the plane \( v \wedge w \) spanned by \( v, w \) with flagpole \( v \) as
\[
K(v, w) := -\frac{g_v(R_v(w), w)}{g_v(v, v)g_v(w, w) - g_v(v, w)^2}.
\]
(2.10)

We remark that this is the opposite sign to [BEE], while the Ricci curvature will be the same. Note that, for \( v \) timelike, the denominator in the right-hand side of (2.10) is negative. The flag curvature is not defined for \( v \) lightlike, for in this case the denominator could vanish. We define the Ricci curvature directly as the trace of \( R_v \) in (2.8).

Definition 2.15 (Ricci curvature) For \( v \in \Omega_x \backslash \{0\} \), the Ricci curvature or Ricci scalar is defined by
\[
\text{Ric}(v) := \text{trace}(R_v).
\]

Since \( \text{Ric}(v) \) is positively 2-homogeneous, we can set \( \text{Ric}(0) := 0 \) by continuity. We say that \( \text{Ric} \geq K \) holds in timelike directions for some \( K \in \mathbb{R} \) if we have \( \text{Ric}(v) \geq K F(v)^2 = -KL(v) \) for all \( v \in \Omega \), where \( F(v) := \sqrt{-g_v(v, v)} = \sqrt{-L(v)} \).

For a normalized timelike vector \( v \in \Omega_x \) with \( F(v) = 1 \), \( \text{Ric}(v) \) can be given as
\[
\text{Ric}(v) = \sum_{i=1}^n K(v, e_i),
\]
where \( \{v\} \cup \{e_i\}_{i=1}^n \) is an orthonormal basis with respect to \( g_v \), namely \( g_v(e_i, e_j) = \delta_{ij} \) and \( g_v(v, e_i) = 0 \) for all \( i, j = 1, \ldots, n \).

Definition 2.16 (Weighted Ricci curvature) Let \((M, L, \rho)\) be a Finsler spacetime with \( \dim M = 1 + n \), where \( \rho \) is an arbitrary positive \( C^\infty \) measure on \( M \). Given a smooth causal vector field \( V \) such that all integral curves are geodesic, we can always decompose \( \rho \) in local coordinates as
\[
d\rho = e^{-\psi(V(x))} \sqrt{- \det \left( (g_{\alpha\beta}(V(x)))_{\alpha,\beta=0}^n \right)} \, dx_0 \cdots dx_n,
\]
where \( \psi : TM \backslash \{0\} \rightarrow \mathbb{R} \) is a positively 0-homogeneous smooth function called the weight function corresponding to the measure \( \rho \). For a causal geodesic \( \eta(t) = \exp_x(tv) \) with \( t \in (-\varepsilon, \varepsilon) \) and \( v \in \Omega_x \backslash \{0\} \), denote the weight function along \( \eta \) as \( \psi_\eta(t) := \psi(\dot{\eta}(t)) \).

Then, for \( N \in \mathbb{R} \backslash \{n\} \), we define the weighted Ricci curvature by
\[
\text{Ric}_N(v) := \text{Ric}(v) + \frac{\psi_\eta''(0)}{\sqrt{- L(v)}} - \frac{\psi_\eta''(0)^2}{N - n}.
\]
(2.11)

For the cases of \( N \rightarrow +\infty \) and \( N = n \), we also define
\[
\text{Ric}_\infty(v) := \text{Ric}(v) + \psi_\eta''(0),
\]
\[
\text{Ric}_n(v) := \begin{cases} \text{Ric}(v) + \psi_\eta''(0) & \text{if } \psi_\eta'(0) = 0, \\ -\infty & \text{if } \psi_\eta'(0) \neq 0. \end{cases}
\]
Remark 2.5 The weighted Ricci curvature $\text{Ric}_\mathcal{N}$ might also be called the Bakry–Èmery–Ricci curvature, due to the pioneering work by Bakry and Èmery [BE] in the Riemannian situation (we refer to the book [BGL] for further information). The Finsler version was introduced in [Oh1] as we mentioned, and we refer to [Ca] for the case of Lorentzian manifolds. The weighted version is essential in our Lorentz–Finsler setting, due to the possible lack of a canonical measure (like the volume measure in the Lorentzian case), see [Oh2] for the positive-definite case.

### 2.4 Jacobi tensor field

**Definition 2.17 (Jacobi tensor field)** Let $(M, L)$ be a Finsler spacetime. Let $\eta : I \rightarrow M$ be a timelike geodesic of unit speed. Define

$$N_\eta(t) := \{ v \in T_{\eta(t)}M : g_\eta(t)(v, \dot{\eta}(t)) = 0 \}.$$ 

An endomorphism $J(t) : N_\eta(t) \rightarrow N_\eta(t)$ for each $t \in I$ is called a *Jacobi tensor field* along $\eta$ if

1. for any $g_\eta$-parallel vector field $P$ along $\eta$, $Y_P^J(t) := J(t)(P(t))$ is a Jacobi field; and
2. for any $t \in I$, ker$(J(t)) \cap \ker(J'(t)) = \{ 0 \}$, where $J'(t)$ is defined by $J'(t)(P(t)) := D^2_\eta Y_P^J(t)$ for any $g_\eta$-parallel vector field $P$ along $\eta$.

Furthermore, a Jacobi tensor field $J$ is said to be $g_\eta$-symmetric if, for any $g_\eta$-parallel vector fields $P_1$ and $P_2$, it holds that $g_\eta(P_1, J(P_2)) = g_\eta(J(P_1), P_2)$.

**Remark 2.6** It is easy to show that $J(t)(w) \in N_\eta(t)$ if $w \in N_\eta(t)$. Therefore, a Jacobi tensor field is well defined. Besides, the derivative is also well defined in the sense that $J'(t) : N_\eta(t) \rightarrow N_\eta(t)$, since, for any $g_\eta$-parallel vector field $P$,

$$g_\eta(J'(t)(P(t)), \dot{\eta}) = \frac{d}{dt} (g_\eta(J(t)(P(t)), \dot{\eta})) - g_\eta(J(t)(P(t)), D^2_\eta \dot{\eta}(t)) = 0.$$ 

**Remark 2.7** With the above definitions, we can deduce that, for any $g_\eta$-parallel vector field $P$ along $\eta : I \rightarrow M$, and any $t \in I$,

$$J''(t)(P(t)) + R_{\eta(t)}(J(t)(P(t))) = (Y_P^J)'(t) + R_{\eta(t)}(Y_P^J(t)) = 0,$$

which is simplified in some references as $J'' + RJ = 0$. See [BEE].

The following results are fundamental and frequently used in our proofs. We here just give some simple explanations.

**Lemma 2.1** Let $A$ be a Jacobi tensor field along $\eta : [0, T] \rightarrow M$ with $A(0) = O_n$ and $A'(0) = I_n$. Then we have $A(t)(P_i(t)) = (d \exp_x)_{tw}(te_i)$, where $\{e_i\}_{i=1}^n \cup \{v\}$ forms an orthonormal basis of $(T_xM, g_x)$ with $\eta(0) = v$, and $P_i(t)$ is obtained by extending $e_i$ parallelly along with $P_i(0) = e_i$, for all $i = 1, \ldots, n$. Furthermore,

$$\det (d \exp_x)_{tw} = t^{-n} \det(A(t)),$$

under the basis $\{P_i(t)\}_{i=1}^n$ of $N_\eta(t)$.
Proof. Let $J_t(t) := (d\exp_x)_{tv}(te_i)$ and $Y_t(t) := A(t)(P_i(t))$, for any $t \in [0,T]$ and $i = 1, \ldots, n$. Since $Y_t(0) = J_t(0) = 0$ and $Y_t'(0) = J_t'(0) = e_i$, we know that $Y_t = J_t$. Then, for any $i = 1, \ldots, n$, 

$$A(t)(P_i(t)) = (d\exp_x)_{tv}(te_i) = t(d\exp_x)_{tv}(e_i),$$

which shows that the representation matrix of $A(t)$ is the same as $t(d\exp_x)_{tv}$. Therefore, 

$$\det A(t) = t^n \det (d\exp_x)_{tv},$$

which completes the proof.

3 Bishop–Gromov’s theorem

In this section, for $x \in M$ we denote by $\text{Fut}_1(x)$ the set of all unit, i.e. $F(v) := \sqrt{-g_v(v,v)} = \sqrt{-L(v)} = 1$, future-directed timelike vectors $v \in T_xM$ such that $\exp_x(v)$ is defined and by $\Gamma_1(x) \supset \text{Fut}_1(x)$ the set of all unit timelike geodesics starting from $x \in M$. The cut function of $U_x$ is defined by $t_{U_x}(v) := \sup \{ t \in (0,\infty) : tv \in \tilde{U}_x \}$, for all $v \in \text{Fut}_1(x)$. If $tv \notin \tilde{U}_x$ for any $t \in (0,\infty)$, then let $t_{U_x}(v) = \infty$. Define $\tilde{U}_1 := \{ v \in \text{Fut}_1(x) : t_{U_x}(v) < \infty \}$ and $t_x := \inf_{v \in \text{Fut}_1(x)} t_{U_x}(v)$ if there occurs no confusion. For all $r \in (0,1]$, we also define $\tilde{U}_x(r) := \{ rv : v \in \tilde{U}_x \}$, $U_x(r) := \exp_x(\tilde{U}_x(r))$.

The following results are useful in our proof.

Lemma 3.1 Suppose $f$ and $g$ are positive integrable functions, of a real variable $r$, for which $f/g$ is non-increasing with respect to $r$. Then the function

$$\frac{\int_0^r f(t) \, dt}{\int_0^r g(t) \, dt}$$

is also non-increasing with respect to $r$.

Proof. The proof is the same as that in [Ch, Lemma III.4.1]).

The following proposition is standard and useful in the proof of Theorem 1.1. Recall the definition of a $g_\eta$-symmetric Jacobi tensor in Definition 2.17.

Proposition 3.1 Let $A$ be a symmetric Jacobi tensor field along a geodesic $\eta$ with $\dot{\eta}(0) \in \Gamma_1(x)$ and satisfy that $\det(A(t)) > 0$ for $t \in (0,T)$, where $T > 0$. Let $N \in (-\infty,0) \cup (n,\infty)$ and

$$h(t) := e^{-\psi_\eta(t)/N} (\det(A(t)))^{1/N}.$$ 

If, for some $t \in (0,T)$, $\text{Ric}_N(\dot{\eta}(t)) \geq c$, where $c \in \mathbb{R}$, then

$$Nh''(t) + ch(t) \leq 0.$$
Proof. Let
\[ h_1(t) := e^{-\frac{\psi_\eta(t)}{N-n}}, \quad h_2(t) := (\det A(t))^{1/n}. \]
Then,
\[ h = h_1^{N-n} h_2^{\frac{n}{N}}, \]
which implies that
\[ Nh''h^{-1} = (N-n)h_1^{-1}h''_1 + nh_2^{-1}h''_2 - \frac{(N-n)n}{N} (h_1^{-1}h_1' - h_2^{-1}h_2')^2. \] (3.1)
It can be easily calculated that, for any \( t \in [0, T] \),
\[ \frac{h_1''(t)}{h_1(t)} = \left( \frac{\psi_\eta'(t)}{N-n} \right)^2 - \frac{\psi_\eta''(t)}{N-n}. \] (3.2)
Combining (3.1), (3.2) with \( N \in (-\infty, 0) \cup (n, \infty) \), we have
\[ Nh''(t)h^{-1}(t) \leq \left( \frac{\psi_\eta'(t)}{N-n} \right)^2 - \psi_\eta''(t) - \text{Ric}(\dot{\eta}(t)) = -\text{Ric}_N(\dot{\eta}(t)) \leq -c, \]
where we used the curvature bound in the last inequality and the unweighted Bishop inequality, i.e.
\[ (\text{tr}(A'A^{-1}))' \leq -\text{Ric}(\dot{\eta}) - \frac{((\text{tr}(A'A^{-1}))^2}{n}, \]
the proof of which is the same as that of (5.1). \( \square \)

Proof of Theorem 1.1. Fix \( v \in \tilde{U}_1 \), then \( t_{U_x}(v) = t_x \) by our assumption. Put \( \eta(t) := \exp_x(tv) \) for all \( t \in [0, t_x] \). Choose a \( g_\eta \)-orthonormal basis \( \{e_i\}_{i=1}^n \cup \{v\} \) of \( T_xM \).
Let \( \{E_i(t)\}_{i=0}^n \) be a frame along \( \eta \) such that \( E_i \) is \( g_\eta \)-parallel with \( E_0(0) = v \) and \( E_i(0) = e_i \) for \( i = 1, \ldots, n \). Then, \( \{E_i(t)\}_{i=0}^n \) forms an orthonormal basis of \( T_{\eta(t)}M \), and hence, for all \( t \in [0, t_x] \),
\[ \det[g_{\eta(t)}(E_i(t), E_j(t))] = -1. \] (3.3)
Moreover, \( \{E_i(t)\}_{i=1}^n \) is a basis of \( N_\eta(t) \). For any \( w \in N_\eta(t) \), we extend it to the \( g_\eta \)-parallel vector field \( P \) along \( \eta \) such that \( P(t) = w \) and define
\[ A(t)(w) := (d(\exp_x))_{t_x}(tP(0)). \]
Then, for any \( g_\eta \)-parallel vector field \( Q \) along \( \eta \), we have
\[ Y^A_Q(t) := A(t)(Q(t)) \]
is a Jacobi field and if there is some \( t_0 \in [0, t_x] \) such that \( Y^A_Q(t_0) = 0 \) and \( (Y^A_Q)'(t_0) = 0 \), then \( Q(t) \equiv 0 \) for all \( t \in [0, t_x] \). This shows that \( A \) is a Jacobi tensor field.
On the other hand, it is easy to check that \( A(0) = O_n, A'(0) = I_n \), and under the basis \( \{ E_i(t) \}_{i=1}^n \) of \( N_\eta(t) \), \( \det(A(t)) > 0 \) for \( t \in (0, t_x) \). Let \( \psi_\eta \) be the weight function of \( \rho \) along \( \eta \) and \( h_v(t) := e^{-\psi_\eta(t)/N}(\det(A(t)))^{1/N} \). We use Proposition \( 3.1 \) for \( \text{Ric}_N \geq c \), to obtain that
\[
 h''_v(t) + \frac{c}{N}h_v(t) \leq 0,
\]
for all \( t \in (0, t_x) \), which together with \( s''_{c/N}(t) + (c/N)s_{c/N}(t) = 0 \), and \( s_{c/N}(t) \geq 0 \) for \( t \in [0, T_x] \), we can infer that
\[
 \frac{d}{dt} [h'_v(t)s_{c/N}(t) - h_v(t)s'_v(t)] \leq 0.
\]
Since \( h_v(0) = 0 \) and \( N \in (n, \infty) \), we know that
\[
 \lim_{t \to 0^+} h'_v(t)s_{c/N}(t) = \lim_{t \to 0^+} \frac{h_v(t)s_{c/N}(t)}{t} = \lim_{t \to 0^+} e^{-\psi_\eta(t)/N} \lim_{t \to 0^+} \left( \frac{\det(A(t))}{t^{N-n}} \right)^{1/N} \frac{s_{c/N}(t)}{t} \t
 = \lim_{t \to 0^+} e^{-\psi_\eta(t)/N} \lim_{t \to 0^+} \left( \frac{\det(A(t)/t)}{t^{N-n}} \right)^{1/N} \frac{s_{c/N}(t)}{t} \t
 = \lim_{t \to 0^+} e^{-\psi_\eta(t)/N} \lim_{t \to 0^+} (\det(A'(0)))^{1/N} t^n/N = 0,
\]
which implies that the function \( h_v/s_{c/N} \) is non-increasing in \( (0, t_x) \). By Lemma \( 3.1 \), we have
\[
 \rho(U_x(r)) \frac{\rho(U_x(R))}{\rho(U_x(R))} = \frac{\int_{U_1} \int_0^{rt} h_v^N(t) \, dtd\sigma(v)}{\int_{U_1} \int_0^{Rt} h_v^N(t) \, dtd\sigma(v)} \geq \frac{\int_0^{rt} s_{c/N}(t) \, dt}{\int_0^{Rt} s_{c/N}(t) \, dt},
\]
where we employ Lemma \( 2.1 \) and (3.3), together with that \( t_{U_x} \) is constant on \( U_1 \), to deduce that
\[
 \rho(U_x(r)) = \int_{(\exp_x)^{-1}(U_x(r))} e^{-\psi(u)} \det(d(\exp_x))_u \, du \t
 = \int_{U_1} \int_0^{rt} e^{-\psi_\eta_0(t)} \det(A(t)) \, dtd\sigma(v) = \int_{U_1} \int_0^{rt} (h_v(t))^N \, dtd\sigma(v),
\]
which completes the proof. \( \square \)

4 Günther’s Theorem

Proof of Theorem 1.2. For any future-directed timelike unit vector \( v \in \tilde{U}_1 \), and a geodesic \( \eta_v \) starting at \( x \) with \( \eta_v(0) = v \), note that \( \eta_v \) has no conjugate points in \( [0, t_{U_x}(v)) \), we can define, for all \( t \in [0, t_{U_x}(v)) \),
\[
 \mathbb{A}_v(t) := \eta_v(t) \mapsto \eta_v(t) \t
 w \mapsto (d\exp_x)_{t_v} (tP_w(0)),
\]
where \( t_v = \frac{\rho(U_x(r))}{\rho(U_x(R))} \). For any future-directed timelike unit vector \( v \in \tilde{U}_1 \), and a geodesic \( \eta_v \) starting at \( x \) with \( \eta_v(0) = v \), note that \( \eta_v \) has no conjugate points in \( [0, t_{U_x}(v)) \), we can define, for all \( t \in [0, t_{U_x}(v)) \),
\[
 \mathbb{A}_v(t) := \eta_v(t) \mapsto \eta_v(t) \t
 w \mapsto (d\exp_x)_{t_v} (tP_w(0)),
\]
where $P_w$ is the $g_{\eta_v}$-parallel vector field along $\eta_v$ with $P_w(t) = w$. Then $A_v$ is a Jacobi tensor field for the same reason as in the proof of Theorem 1.1. Now let $\{e_i\}_{i=1}^n \cup \{v\}$ be orthonormal and

$$f(t) := \frac{\det A_v(t)}{s_{-c}(t)},$$

for all $t \in (0, t_{U_v}(v))$. We claim that $f \geq 1$ in $(0, t_{U_v}(v))$. Indeed, by noting that

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow 0^+} \frac{\det A_v(t)}{t^n s_{-c}(t)} = \det A_v'(0) = 1,$$

we need only to prove $f' \geq 0$, which is equivalent to

$$\frac{(\det A_v(t))'}{\det A_v(t)} \geq \frac{n s_{-c}'(t)}{s_{-c}(t)}.$$

Let $B_v := A_v^T A_v$. Then

$$\text{tr}(B_v B_v^{-1}) = \frac{(\det B_v(t))'}{\det B_v(t)} = 2 \frac{(\det A_v(t))'}{\det A_v(t)}.$$

Since $B_v$ is symmetric, by Gram–Schmidt process, we can find another orthonormal basis, without loss of generality, still denoted by $\{e_i(t)\}_{i=1}^n$ of $N_{\eta_v}(t)$ for any $t \in (0, t_{U_v}(v))$ such that $B_v(t) e_i(t) = \lambda_i(t) e_i(t)$. Observe that

$$\lambda_i = g_{\eta_v}(B_v e_i, e_i) = g_{\eta_v}(A_v e_i, A_v e_i) = g_{\eta_v}(Y_{A_v}^{e_i}, Y_{A_v}^{e_i}),$$

with the notations in Definition 2.17, we can obtain that

$$\text{tr} \left( B_v B_v^{-1} \right) = \sum_{i=1}^n \lambda_i' \lambda_i^{-1} = \sum_{i=1}^n \frac{(g_{\eta_v}(Y_{A_v}^{e_i}, Y_{A_v}^{e_i}))'}{g_{\eta_v}(Y_{A_v}^{e_i}, Y_{A_v}^{e_i})}.$$

To simplify our notations, we denote that $Y_i := Y_{A_v}^{e_i}$ and $\eta := \eta_v$. Since $Y_i(t) \neq 0$ for all $t \in (0, t_{U_v}(v))$, we have

$$\frac{d^2}{dt^2} \left[ g_\eta(Y_i, Y_i) \right] = \frac{d}{dt} \left[ g_\eta(Y_i', Y_i') \right] = \frac{g_\eta(Y_i'', Y_i) + g_\eta(Y_i', Y_i')}{\sqrt{g_\eta(Y_i, Y_i)}} - \frac{[g_\eta(Y_i, Y_i')]^2}{[g_\eta(Y_i, Y_i)]^{3/2}}$$

$$= -\frac{g_\eta(R_\eta(Y_i), Y_i)}{\sqrt{g_\eta(Y_i, Y_i)}} + \frac{g_\eta(Y_i', Y_i') g_\eta(Y_i, Y_i) - [g_\eta(Y_i, Y_i')]^2}{[g_\eta(Y_i, Y_i)]^{3/2}}.$$  

By the Cauchy–Schwarz inequality, the second term is nonnegative, thus we have the following inequalities by using the curvature bound condition $K(\eta, Y_i) \leq -c$ for $c \geq 0$:

$$\frac{d^2}{dt^2} \left[ \sqrt{g_\eta(Y_i, Y_i)} \right] \geq -\frac{g_\eta(R_\eta(Y_i), Y_i)}{\sqrt{g_\eta(Y_i, Y_i)}} = \frac{K(\eta, Y_i) [g_\eta(\eta, \eta) g_\eta(Y_i, Y_i) - g_\eta(\eta, Y_i)^2]}{\sqrt{g_\eta(Y_i, Y_i)}}$$

$$\geq c \frac{[g_\eta(Y_i, Y_i) + g_\eta(\eta, Y_i)^2]}{\sqrt{g_\eta(Y_i, Y_i)}} \geq c \sqrt{g_\eta(Y_i, Y_i)},$$
which, together with the definition of $s_{-c}$, implies that
\[
\frac{d}{dt} \left[ \frac{d}{dt} \left( \sqrt{g_\eta(Y_i(t),Y_i(t))} s_{-c}(t) - \sqrt{g_\eta(Y_i(t),Y_i(t))s'_{-c}(t)} \right) \right] = \frac{d^2}{dt^2} \left( \sqrt{g_\eta(Y_i(t),Y_i(t))} s_{-c}(t) - c\sqrt{g_\eta(Y_i(t),Y_i(t))s_{-c}(t)} \geq 0, \right.
\]
for all $t \in (0, t_{U_x}(v))$. Combining this with $Y_i(0) = 0$, we obtain that
\[
\frac{(\det A_v(t))'}{\det A_v(t)} = \frac{1}{2} \text{tr} \left( B_v(t)B^{-1}(t) \right) = \frac{1}{2} \sum_{i=1}^n \frac{g_\eta(Y_i,Y_i)'(t)}{g_\eta(Y_i,Y_i)(t)} \geq \sum_{i=1}^n \frac{s'_{-c}(t)}{s_{-c}(t)} = \frac{ns'_{-c}(t)}{s_{-c}(t)},
\]
which completes the proof of the claim. Finally,
\[
\rho(U_x) = \int_{U_1} \int_0^{t_{U_x}(v)} e^{-\psi_\eta(t)} \det (A_v(t)) \, dt \, d\sigma(v)
\geq e^{-k} \int_{U_1} \int_0^{t_x} s^n_{-c}(t) \, dt \, d\sigma(v) = e^{-k} \sigma(\tilde{U}_1) \int_0^{t_x} s^n_{-c}(t) \, dt,
\]
where we used $\psi_\eta(t) \leq k$ to complete the proof. \hfill \Box

5 The case $N = \infty$

For $N = \infty$, following the lines in [Sh1, WW], we obtain the following volume comparison theorem.

**Theorem 5.1** Let $(M,L,\rho)$ be a Finsler spacetime satisfying $\text{Ric}_\infty(v) \geq nc$ for some $c \in \mathbb{R}$ and for all unit timelike radial vectors $v \in T_xM$ tangent to some SCLV subset $U_x \subset M$ at $x \in U$. Suppose that, for any causal geodesic $c(t) := \exp_x(tv)$ with $v \in \tilde{U}_1$, it holds that $\eta_v'' \geq -a$ along $\eta_v$, for some $a \in \mathbb{R}$ and for all $t \in (0, t_{U_x}(v))$. If $t_{U_x}$ is constant on $U_1$, then
\[
\frac{\rho(U_x(r))}{\rho(U_x(R))} \geq \int_0^{rT_x} e^{a t} s^n(t) \, dt / \int_0^{rT_x} e^{a t} s^n(t) \, dt,
\]
for all $0 < r \leq R \leq 1$, where $T_x := t_x$ if $c \leq 0$ and $T_x := \min\{t_x, \pi/(2\sqrt{c})\}$ if $c > 0$.

**Proof.** Define $\{E_i(t)\}_{i=0}^n$, $N_\eta(t)$ and $A(t)$ as in the proof of Theorem 1.1. Let $R(t) : N_\eta(t) \rightarrow N_\eta(t)$ be an endomorphism such that $R(t)(w) := R_\eta(t)(w)$, where $R_\eta(t)$ is the curvature operator as in (2.8). Then, $R(t)$ is in fact a linear map, and hence a matrix under the basis $\{E_i(t)\}_{i=1}^n$. Let $C(t) := A'(t)A^{-1}(t)$ for $t \in (0, t_{U_x}(v))$. $C$ is well defined since $A$ is non-degenerate for that $\eta$ has no conjugate points in $(0, t_{U_x}(v))$. Since $A$ is a Jacobi tensor field, we have
\[
A'' + RA = 0.
\]
Right multiply $A^{-1}$ to both sides in the above equation, we deduce that $A''A^{-1} + R = 0$, which gives
\[
C' + C^2 + R = A''A^{-1} + A'(A^{-1})' + A'A^{-1}A' + R = A''A^{-1} + R = 0.
\]

Taking the traces of both sides, with the help of Cauchy–Schwarz inequality, we get a Riccati inequality for $\text{tr}(C)$ as
\[
[\text{tr}(C)]' + \frac{[\text{tr}(C)]^2}{n} + \text{Ric}(\dot{\eta}) \leq 0. \quad (5.1)
\]
For $t \in (0, t_{U_x}(v))$, let
\[
\lambda(t) := [\log(\det A(t))]',
\]
\[
\lambda_\psi(t) := \lambda(t) - \psi_\eta'(t) = (\log[e^{-\psi_\eta(t)} \det A(t)])',
\]
\[
\lambda_c(t) := n\frac{s_c'(t)}{s_c(t)}.
\]
Recall that $\text{Ric}_\infty(\dot{\eta}) = \text{Ric}(\dot{\eta}) + \psi_\eta'' \geq nc$. By (5.1), we obtain that
\[
\lambda' + \frac{\lambda^2}{n} + \text{Ric}(\dot{\eta}) \leq 0,
\]
which, together with
\[
\lambda_c' + \frac{\lambda_c^2}{n} + nc = 0,
\]
implies that
\[
[s_c^2(\lambda - \lambda_c)]' = 2s_c s_c'(\lambda - \lambda_c) + s_c^2(\lambda' - \lambda_c') = s_c^2 \left[ \frac{2s_c'}{s_c} (\lambda - \lambda_c) + (\lambda' - \lambda_c') \right]
\]
\[
\leq s_c^2 \left[ \frac{2\lambda_c}{n} (\lambda - \lambda_c) - \frac{\lambda^2 - \lambda_c^2}{n} + \psi_\eta'' \right] = s_c^2 \left[ -\frac{(\lambda - \lambda_c)^2}{n} + \psi_\eta'' \right] \leq s_c^2 \psi_\eta''. \quad (5.2)
\]
Integrating (5.2) from 0 to some $t \in (0, t_{U_x}(v))$ yields that
\[
s_c^2(t)(\lambda(t) - \lambda_c(t)) \leq \int_0^t s_c^2(\tau)\psi_\eta''(\tau)d\tau = s_c^2(t)\psi_\eta'(t) - \int_0^t (s_c^2)'(\tau)\psi_\eta'(\tau)d\tau.
\]
Since $\psi_\eta'(\tau) \geq -a$ and $(s_c^2)'(\tau) > 0$, for $\tau \in (0, \pi/(2 \sqrt{c}))$, we see that
\[
\lambda_\psi(t) \leq \lambda_c(t) + \frac{a}{s_c^2(t)} \int_0^t (s_c^2)'(\tau)d\tau = \lambda_c(t) + a,
\]
which is equivalent to
\[
[\log(e^{-\psi_\eta(t)} \det(A(t)))]' \leq [\log(e^{at} s_c^n(t))]' .
\]
By again Lemma 3.1, we have
\[
\frac{\rho(U_x(r))}{\rho(U_x(R))} \geq \frac{\int_0^{rT_x} e^{at} s_c^n(t)dt}{\int_0^{rT_x} e^{at} s_c^n(t)dt},
\]
for $0 < r \leq R \leq 1$. \hfill \Box
Using a similar method, we can deduce the following estimate when \( N = \infty \). Let us mention Sturm’s original results for metric measure spaces, see [St, Theorem 4.26]. If \( U \) is a SCLV at \( x \), then define \( \mathfrak{B}_U(x, r) := \exp_x(\{ v \in \tilde{U}_x : F(v) < r \}) \).

**Theorem 5.2** Let \((M, L, \rho)\) be a Finsler spacetime and satisfy that \( \text{Ric}_\infty \geq c \). If \( U \) is a SCLV at \( x \in M \), then

\[
\rho(\mathfrak{B}_U^+(x, r)) \leq \rho(\mathfrak{B}_U^+(x, 4\varepsilon)) + \sigma(\tilde{U}_1) \int_{4\varepsilon}^r e^{C_0 t - \frac{c t^2}{2}} dt,
\]

sufficiently small \( \varepsilon > 0 \) and all \( r > 4\varepsilon \), where \( C_0 > 0 \) is a constant depending on \( c, \varepsilon \) and \( x \). In particular, when \( c = 0 \), i.e. the timelike \( \infty \)-convergence condition (see [LMO, Definition 9.10]) holds,

\[
\rho(\mathfrak{B}_U^+(x, r)) \leq \rho(\mathfrak{B}_U^+(x, 4\varepsilon)) + \frac{\sigma(\tilde{U}_1)}{C_0} e^{C_0 r}.
\]

**Proof.** Let \( x \in M \) and \( v \in \tilde{U}_1 \). Define \( \eta \) and \( A \) as in the above proof. Let \( f(t) := e^{-\psi_0(t)} \det A(t) \) and \( h(t) := (\det A(t))^{1/n} \). Then, by again the Bishop inequality, see [LMO, (5.4)]

\[
\left( \log \left( f(t) e^{ct^2/2} \right) \right)'' = \left[ -\psi'(t) + \frac{ct^2}{2} + \log(\det A(t)) \right]''
\]

\[
= \left[ -\psi'(t) + ct + \frac{nh'(t)}{h(t)} \right]'
\]

\[
= -\psi''(t) + c + \frac{nh''(t)h(t) - [h'(t)]^2}{[h(t)]^2}
\]

\[
\leq -\psi''(t) + c - \text{Ric}(\dot{\eta}(t)) = -\text{Ric}_\infty(\dot{\eta}(t)) + c \leq 0.
\]

Therefore, for all \( t > 4\varepsilon > 0 \),

\[
\log(f(2\varepsilon)e^{2c\varepsilon^2}) \geq \frac{t - 2\varepsilon}{t - \varepsilon} \log \left( f(\varepsilon) e^{c\varepsilon^2/2} \right) + \frac{\varepsilon}{t - \varepsilon} \log \left( f(t) e^{ct^2/2} \right) \quad \Rightarrow \quad \log(f(t)e^{ct^2/2}) \leq \frac{(t - 2\varepsilon) \log \left( f(\varepsilon) e^{c\varepsilon^2/2} \right)}{t \varepsilon} t \leq -\frac{\log \left( f(\varepsilon) e^{c\varepsilon^2/2} \right)}{\varepsilon} t =: C_0 t.
\]

Since \( \lim_{t \to 0} f(t) = 0 \), there exists \( \delta \) such that, for all \( \varepsilon \in (0, \delta) \), we have \( \log(f(2\varepsilon)e^{2c\varepsilon^2}) < 0 \). For such \( \varepsilon > 0 \), we can deduce from (5.3) that

\[
\log \left( f(t) e^{ct^2/2} \right) \leq \frac{(t - 2\varepsilon) \log \left( f(\varepsilon) e^{c\varepsilon^2/2} \right)}{t \varepsilon} t \leq -\frac{\log \left( f(\varepsilon) e^{c\varepsilon^2/2} \right)}{\varepsilon} t =: C_0 t.
\]

Finally, from the above estimates, we get that

\[
\rho(\mathfrak{B}_U^+(x, r)) \leq \rho(\mathfrak{B}_U^+(x, 4\varepsilon)) + \int_{4\varepsilon}^r f(t) dt d\sigma(v)
\]

\[
\leq \rho(\mathfrak{B}_U^+(x, 4\varepsilon)) + \sigma(\tilde{U}_1) \int_{4\varepsilon}^r e^{C_0 t - \frac{c t^2}{2}} dt,
\]

which is exactly what we need.

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