In this study, the solutions of $(2 + 1)$-dimensional nonlinear Date–Jimbo–Kashiwara–Miwa (DJKM) equation are characterized, which can be used in mathematical physics to model water waves with low surface tension and long wavelengths. The integration scheme, namely, the extended direct algebraic method, is used to extract complex trigonometric, rational and hyperbolic functions. The complex-valued solutions represent traveling waves in different structures, such as bell-, V-, and W-shaped multiwaves. The results obtained in this article are novel and more general than those contained in the literature (Wang et al., 2014, Yuan et al., 2017, Pu and Hu 2019, Singh and Gupta 2018). Furthermore, the mechanical features and dynamical characteristics of the obtained solutions are demonstrated by three-dimensional graphics.

1. Introduction

Nonlinear evolution equations (NLEEs) can represent various nonlinear problems that occur in a wide range of scientific fields such as nonlinear optics, mathematical physics, superconductivity, biophysics, optical fiber, modern optics, solid state physics, fluid mechanics, fluid dynamics, plasma physics, chemical physics, and chemical kinetics. In the literature, various effective approaches have been proposed to calculate the exact solutions for NLEEs [1–3], such as the Hereman–Nuseir method [4], inverse scattering transformation [5], Painlevé technique [6], Bäcklund transformation [7], extended modified auxiliary equation mapping method [8], Darboux transformation [9], Exp-function method [10], binary-bell-polynomial scheme [11], modified Khater method [12], ansatz method [13], sine-Gordon expansion method [14], trial equation method [15, 16], extended direct algebraic method [17], and auxiliary equation method [18].

In this study, the nonlinear DJKM equation [19] is investigated to construct various solitary wave solutions. In the integrable systems of KP hierarchy, the Jimbo–Miwa equation is the second equation used to explain such interesting $(2 + 1)$-dimensional waves in physics. The DJKM equation can be used in mathematical physics to model water waves with low surface tension and long wavelengths with weakly nonlinear restoring forces and frequency dispersion. Firstly, Hu and Li [19] applied bilinear Bäcklund transformations and nonlinear superposition formula for nonlinear DJKM equations, and after a gap of more than two decades, Wang et al. [20] used the bell polynomials to study the integrable properties of nonlinear DJKM equations such as Lax system, Bäcklund transformations, and infinite conservation laws along with multishock wave. Yuan et al. [21] presented Grammian- and Wronskian-type solutions by the Hirota method, and other types of solution are also obtained like auxiliary variables, the bilinear Bäcklund transformation, and N-soliton. Pu and Hu [22] employed
the sine-Gordon expansion method in finding the traveling wave solutions of nonlinear DJKM equations and obtained hyperbolic, trigonometric, and complex solutions. Singh and Gupta [23] used the direct method and nonlinear self-adjointness to find the Painlevé analysis, symmetric properties, and conservation laws of the nonlinear DJKM equation. Sajid and Akram [24] utilized exp(−Φ(ξ))-expansion method and derived some exact traveling wave solutions including trigonometric, hyperbolic, and rational functions and W-shaped soliton of the DJKM equation. The proposed research analyzes some more new exact solutions such as bell-, V-, and W-shaped multiwave types of the nonlinear DJKM equation which are not yet found in the literature. To our utmost understanding, the DJKM equations were not analyzed using the extended direct algebraic method. Furthermore, this beneficial and powerful approach can be used to investigate other NLEEs which frequently emerge in different scientific real-world applications.

The novelty of this paper lies in the following: (i) complex-valued solutions and solitons are in different shapes and (ii) 3-dimensional figures are first presented by the extended direct algebraic method. The limitations of this work include that the solution methods for the construction of exact solutions to the equation involve various parameters. Such parameters show up in the final precise solution expressions and create hurdles in some physical situations. These are resolved with a careful selection of appropriate parametric values which is possible through graphical interpretation and testing of the solution expressions.

The structure of this paper is organized as follows: in Section 2, detailed explanation of the extended direct algebraic method has been presented. Section 3 illustrates the method to solve the (2 + 1)-dimensional DJKM equation. In Section 3.1, the physical explanation of the solutions by mechanical features and dynamical characteristics is demonstrated. Finally, conclusion is given in Section 4.

1.1. Governing Model. Considering the governing model, (2 + 1)-dimensional nonlinear DJKM equation, as

\[ Q(U, \omega U', \omega U, -k U'' - \omega^2 U''') = 0, \]

where \( \Phi = \Phi(x, y, t) \) is the real-valued function. The DJKM equation belongs to the well-known KP hierarchy [25, 26] which can be obtained from the first two bilinear equations using transformation \( u = 2(\log \tau)_x \). The KP hierarchy is an infinite set of nonlinear PDEs.

2. Extended Direct Algebraic Method [27]

According to extended direct algebraic method, we have the following.

Step 1. Consider NLEE in three independent variables \( x, y, \) and \( t \) of the form, as

\[ Q(U, \omega U', \omega U, -kU'' - \omega^2 U''') = 0, \]

where prime denotes the derivatives w.r.t. \( \xi \).

Step 2. Consider that the formal solution of equation (4) has a form, as follows:

\[ U(\xi) = \sum_{j=0}^{N} b_j \phi_j(\xi), \quad b_N \neq 0, \]

where \( b_0, b_1, \ldots, b_N \) are constants and \( Q(\xi) \) satisfies the auxiliary equation, as

\[ Q'(\xi) = L_n(A)(\alpha + \beta Q(\xi) + \sigma Q^2(\xi)), \quad A \neq 0, 1, \]

where \( \sigma, \alpha, \) and \( \beta \) are constants. The solutions of equation (6) are given in the following.
Family 1. If $\beta^2 - 4\sigma < 0$ and $\sigma \neq 0$, then the solutions are given as

\[
Q_1(\xi) = \frac{\beta}{2\sigma} + \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2\sigma} \tan_A \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2} \xi \right),
\]

\[
Q_2(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2\sigma} \cot_A \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2} \xi \right),
\]

\[
Q_3(\xi) = \frac{\beta}{2\sigma} + \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2\sigma} \right) \left( \tan_A \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2} \xi \right) \pm \sqrt{pq \sec_A} \left( \sqrt{- (\beta^2 - 4\sigma)} \right) \right),
\]

\[
Q_4(\xi) = -\frac{\beta}{2\sigma} \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2\sigma} \right) \left( \cot_A \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{2} \xi \right) \mp \sqrt{pq \csc_A} \left( \sqrt{- (\beta^2 - 4\sigma)} \right) \right),
\]

\[
Q_5(\xi) = -\frac{\beta}{2\sigma} + \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{4\sigma} \right) \left( \tan_A \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{4} \xi \right) - \cot_A \left( \frac{\sqrt{- (\beta^2 - 4\sigma)}}{4} \xi \right) \right).
\]

Family 2. If $\beta^2 - 4\sigma > 0$ and $\sigma \neq 0$, then the solutions are given as

\[
Q_6(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\sigma}}{2\sigma} \tanh_A \left( \frac{\sqrt{\beta^2 - 4\sigma}}{2} \xi \right),
\]

\[
Q_7(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\sigma}}{2\sigma} \coth_A \left( \frac{\sqrt{\beta^2 - 4\sigma}}{2} \xi \right),
\]

\[
Q_8(\xi) = -\frac{\beta}{2\sigma} \left( \frac{\sqrt{\beta^2 - 4\sigma}}{2\sigma} \right) \left( \tanh_A \left( \sqrt{\beta^2 - 4\sigma} \xi \right) \mp i \sqrt{pq} \sec A \left( \sqrt{\beta^2 - 4\sigma} \xi \right) \right),
\]

\[
Q_9(\xi) = -\frac{\beta}{2\sigma} \left( \frac{\sqrt{\beta^2 - 4\sigma}}{2\sigma} \right) \left( \coth_A \left( \sqrt{\beta^2 - 4\sigma} \xi \right) \mp i \sqrt{pq} \csc A \left( \sqrt{\beta^2 - 4\sigma} \xi \right) \right),
\]

\[
Q_{10}(\xi) = -\frac{\beta}{2\sigma} \left( \frac{\sqrt{\beta^2 - 4\sigma}}{4\sigma} \right) \left( \tanh_A \left( \sqrt{\beta^2 - 4\sigma} \xi \right) \pm \coth_A \left( \sqrt{\beta^2 - 4\sigma} \xi \right) \right).
\]
Family 3. If \( \beta = 0 \), and \( \alpha \sigma \) > 0, then the solutions are given as
\[
\begin{align*}
Q_{11}(\xi) &= \sqrt{\frac{\alpha}{\sigma}} \tan_A \left( \sqrt{\alpha \sigma} \xi \right), \\
Q_{12}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}} \cot_A \left( \sqrt{\alpha \sigma} \xi \right), \\
Q_{13}(\xi) &= \sqrt{\frac{\alpha}{\sigma}} \tan_A \left( 2 \sqrt{\alpha \sigma} \xi \right) \pm \sqrt{pq} \sec \left( 2 \sqrt{\alpha \sigma} \xi \right), \\
Q_{14}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}} \cot_A \left( 2 \sqrt{\alpha \sigma} \xi \right) \mp \sqrt{pq} \csc \left( 2 \sqrt{\alpha \sigma} \xi \right), \\
Q_{15}(\xi) &= \frac{1}{2} \sqrt{\frac{\alpha}{\sigma}} \left( \tan_A \left( \frac{\sqrt{\alpha \sigma}}{2} \xi \right) - \cot_A \left( \frac{\sqrt{\alpha \sigma}}{2} \xi \right) \right).
\end{align*}
\]
(9)

Family 4. If \( \beta = 0 \) and \( \alpha \sigma < 0 \), then the solutions are given as
\[
\begin{align*}
Q_{16}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}} \tanh_A \left( \sqrt{-\alpha \sigma} \xi \right), \\
Q_{17}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}} \coth_A \left( \sqrt{-\alpha \sigma} \xi \right), \\
Q_{18}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}} \tanh_A \left( 2 \sqrt{-\alpha \sigma} \xi \right) \mp i \sqrt{pq} \sec h \left( 2 \sqrt{-\alpha \sigma} \xi \right), \\
Q_{19}(\xi) &= -\sqrt{\frac{\alpha}{\sigma}} \coth_A \left( 2 \sqrt{-\alpha \sigma} \xi \right) \mp \sqrt{pq} \csc h \left( 2 \sqrt{-\alpha \sigma} \xi \right), \\
Q_{20}(\xi) &= \frac{1}{2} \sqrt{\frac{\alpha}{\sigma}} \left( \tanh_A \left( \frac{\sqrt{-\alpha \sigma}}{2} \xi \right) + \coth_A \left( \frac{\sqrt{-\alpha \sigma}}{2} \xi \right) \right).
\end{align*}
\]
(10)

Family 5. If \( \beta = 0 \) and \( \sigma = \alpha \), then the solutions are given as
\[
\begin{align*}
Q_{21}(\xi) &= \tan_A \left( \alpha \xi \right), \\
Q_{22}(\xi) &= -\cot_A \left( \alpha \xi \right), \\
Q_{23}(\xi) &= \tan_A \left( 2 \alpha \xi \right) \pm \sqrt{pq} \sec \left( 2 \alpha \xi \right), \\
Q_{24}(\xi) &= -\cot_A \left( 2 \alpha \xi \right) \mp \sqrt{pq} \csc \left( 2 \alpha \xi \right), \\
Q_{25}(\xi) &= \frac{1}{2} \left( \tan_A \left( \frac{\alpha}{2} \xi \right) - \cot_A \left( \frac{\alpha}{2} \xi \right) \right).
\end{align*}
\]
(11)

Family 6. If \( \sigma = -\alpha \) and \( \beta = 0 \), then the solutions are given as
\[
\begin{align*}
Q_{26}(\xi) &= -\tanh_A \left( \alpha \xi \right), \\
Q_{27}(\xi) &= -\coth_A \left( \alpha \xi \right), \\
Q_{28}(\xi) &= -\tanh_A \left( 2 \alpha \xi \right) \pm i \sqrt{pq} \sec h \left( 2 \alpha \xi \right), \\
Q_{29}(\xi) &= -\coth_A \left( 2 \alpha \xi \right) \pm \sqrt{pq} \csc h \left( 2 \alpha \xi \right), \\
Q_{30}(\xi) &= -\frac{1}{2} \left( \tanh_A \left( \frac{\alpha}{2} \xi \right) + \coth_A \left( \frac{\alpha}{2} \xi \right) \right).
\end{align*}
\]
(12)

Family 7. If \( \beta^2 = 4 \alpha \sigma \), then the solution is given as
\[
Q_{31}(\xi) = \frac{-2 \alpha (\beta \xi \log A + 2)}{\beta^2 \xi \log A} 
\]
(13)

Family 8. If \( \beta = l, \sigma = 0, \) and \( \alpha = ml (m \neq 0) \), then the solution is given as
\[
Q_{32}(\xi) = A^{\xi - m}. 
\]
(14)

Family 9. If \( \beta = 0 \) and \( \sigma = \alpha \), then the solution is given as
\[
Q_{33}(\xi) = \alpha \xi \log A. 
\]
(15)

Family 10. If \( \beta = \alpha = 0 \), then the solution is given as
\[
Q_{34}(\xi) = -\frac{1}{\alpha \xi \log A}. 
\]
(16)

Family 11. If \( \beta \neq 0 \) and \( \alpha = 0 \), then the solutions are given as
\[
\begin{align*}
Q_{35}(\xi) &= \frac{p \beta}{\sigma (\cosh_A (\beta \xi) - \sinh_A (\beta \xi) + p)}, \\
Q_{36}(\xi) &= -\frac{\beta (\sinh_A (\beta \xi) + \cosh_A (\beta \xi))}{\sigma (\sinh_A (\beta \xi) + \cosh_A (\beta \xi) + q)}. 
\end{align*}
\]
(17)

Family 12. If \( \beta = l \), \( \alpha = 0 \), and \( \sigma = ml (m \neq 0) \), then the solution is given as
\[
Q_{37}(\xi) = \frac{p A^{\xi l}}{q - m p A^{\xi l}}. 
\]
(18)

Remark 1. The generalized triangular functions and hyperbolic functions [28] are defined as follows:
\[
\sinh_A(\xi) = \frac{pA^\xi - qA^{-\xi}}{2},
\]
\[
\cosh_A(\xi) = \frac{pA^\xi + qA^{-\xi}}{2},
\]
\[
\tanh_A(\xi) = \frac{pA^\xi - qA^{-\xi}}{pA^\xi + qA^{-\xi}},
\]
\[
\coth_A(\xi) = \frac{pA^\xi + qA^{-\xi}}{pA^\xi - qA^{-\xi}},
\]
\[
\sec h_A(\xi) = \frac{2}{pA^\xi + qA^{-\xi}},
\]
\[
\csc h_A(\xi) = \frac{2}{pA^\xi - qA^{-\xi}},
\]
\[
\sin_A(\xi) = \frac{pA^\xi - qA^{-\xi}}{2i},
\]
\[
\cos_A(\xi) = \frac{pA^\xi + qA^{-\xi}}{2},
\]
\[
\tan_A(\xi) = -\frac{pA^\xi - qA^{-\xi}}{pA^\xi + qA^{-\xi}},
\]
\[
\cot_A(\xi) = \frac{pA^\xi + qA^{-\xi}}{pA^\xi - qA^{-\xi}},
\]
\[
\sec A(\xi) = \frac{2}{pA^\xi + qA^{-\xi}},
\]
\[
\csc A(\xi) = \frac{2i}{pA^\xi - qA^{-\xi}},
\]

where \(p, q > 0\) and \(\xi\) is an independent variable.

Step 3. Using homogeneous balancing principle in equation (4), the value of \(N\) can be determined. Substituting equation (6) along with equation (5) into equation (4), collecting the coefficients of each power \(Q^j(\xi)\) \((j = 0, 1, 2, \ldots)\), and then setting each coefficient to zero give a system of equations.

Step 4. Unknowns can be found by calculating the system of equations. Putting the unknowns in equation (6), the required solutions of equation (2) are obtained.

3. Application to the DJKM Equation

The extended direct algebraic scheme is presented to obtain the optical solitons and other solutions to equation (1). After utilizing the transformation \(\Phi(x, y, t) = V(\xi)\), where \(\xi = \omega(x + \mu y - kt)\), to equation (1), we obtain nonlinear ODE as follows:

\[
\mu \omega^3 V''' + 6\mu \omega V''V' + \left(\mu^3 + 2k\right)V'' = 0.
\]

Setting \(U = V'\), we obtain

\[
\mu \omega^2 U''' + 6\mu \omega U''U' + \left(\mu^3 + 2k\right)U'' = 0.
\]

Balancing \(U''\) with \(UU'\) in equation (21) gives \(N = 2\). Thus, the solution can be written as

\[
U(\xi) = b_0 + b_1 Q(\xi) + b_2 Q^2(\xi),
\]

where \(b_0, b_1, b_2\) are constants to be determined. Substituting equations (22) into (21), collecting all terms with the same power of \(Q(\xi)^i\) \((i = 0, 1, 2, 3, 4, 5)\), and equating the coefficients of each polynomial to zero will yield a set of algebraic equations for \(b_0, b_1, b_2,\) and \(\omega\) as follows:

\[
2kb_1\ln(Ln(\alpha) + 2\mu \omega b_1(Ln(\alpha))^3 \sigma^2 + 6\mu \omega b_1(Ln(\alpha))\sigma + \mu \omega b_1(Ln(\alpha))\sigma + \mu \omega b_1(Ln(\alpha))\sigma + 16\mu \omega b_1(Ln(\alpha))\sigma + 4kb_1(Ln(\alpha))\sigma + \mu \omega b_1(Ln(\alpha))\sigma + 2kb_1(Ln(\alpha))\sigma + 8\mu \omega b_1(Ln(\alpha))\beta + \mu \omega b_1(Ln(\alpha))\beta + 2kb_1(Ln(\alpha))\beta + 8\mu \omega b_1(Ln(\alpha))\beta + \mu \omega b_1(Ln(\alpha))\beta + 4kb_1(Ln(\alpha))\beta + 8\mu \omega b_1(Ln(\alpha))\beta + \mu \omega b_1(Ln(\alpha))\beta + 2kb_1(Ln(\alpha))\beta + 8\mu \omega b_1(Ln(\alpha))\beta + \mu \omega b_1(Ln(\alpha))\beta + 16\mu \omega b_1(Ln(\alpha))\beta = 0, \]
\[
38\mu \omega b_1(Ln(\alpha))\beta + 18\mu \omega b_1(Ln(\alpha))\beta + 4kb_1(Ln(\alpha))\beta + 8\mu \omega b_1(Ln(\alpha))\beta + \mu \omega b_1(Ln(\alpha))\beta + 12\mu \omega b_1(Ln(\alpha))\beta = 0, \]
\[
18\mu \omega b_1(Ln(\alpha))\beta + 54\mu \omega b_1(Ln(\alpha))\beta = 0, \]
\[
24\mu \omega b_1(Ln(\alpha))\beta = 0.
\]

Solving system (23) for \(b_0, b_1, b_2,\) and \(\omega\) gives

\[
b_0 = \frac{-1}{6(\mu \omega^2(Ln(\alpha))^2 \beta^2 + 8\mu \omega^2(Ln(\alpha))^2 \sigma \alpha + 2k + \mu^3 \omega)},
\]
\[
b_1 = -2\omega(Ln(\alpha))^2 \beta \sigma, \quad b_2 = -2\omega(Ln(\alpha))^2 \sigma^2, \quad \omega = \omega.
\]

Five families of traveling wave solutions of the DJKM equation can be obtained, as shown in the following.

Family 13. When \(\beta^2 - 4\alpha \sigma < 0\) and \(\sigma \neq 0\), the dark, combined dark-bright, singular, combined dark-singular, and combined singular solutions are obtained, as follows:
$$\Phi_1 = \frac{-1}{6(\xi(-5\mu\omega (\ln(A))^2 \beta^2 + 2k + 8\mu\omega (\ln(A))^2 \sigma\alpha + \mu^3)/\mu\omega)}$$
$$+ (\ln(A))^2 \tanh \left( \frac{1}{2\sqrt{\xi}} \right) \sqrt{\gamma} + (\ln(A))^2 \xi(-\beta^2 + 2\alpha\sigma),$$

$$\Phi_2 = \frac{-1}{6(\xi(-5\mu\omega (\ln(A))^2 \beta^2 + 2k + 8\mu\omega (\ln(A))^2 \sigma\alpha + \mu^3)/\mu\omega)}$$
$$+ (\ln(A))^2 \coth \left( \frac{1}{2\sqrt{\xi}} \right) \sqrt{\gamma}$$
$$+ 2\left(\frac{(\ln(A))^2(-\beta^2 + 2\alpha\sigma) \arctan h_A(\coth A(1/2\sqrt{\xi}))}{\sqrt{\gamma}}\right),$$

$$\Phi_3 = \frac{1}{6(\xi(-\mu\omega (\ln(A))^2 \beta^2 - 2k + 4\mu\omega (\ln(A))^2 \sigma\alpha - \mu^3)/\mu\omega)}$$
$$+ \frac{1}{2(\ln(A))^2 \tanh ((\sqrt{\xi}) \sqrt{\gamma}(1 + pq))} - \frac{i(\ln(A))^2 \sqrt{pq}}{\cosh ((\sqrt{\xi})},$$

$$\Phi_4 = \frac{1}{6(\xi(-\mu\omega (\ln(A))^2 \beta^2 - 2k + 4\mu\omega (\ln(A))^2 \sigma\alpha - \mu^3)/\mu\omega)}$$
$$+ \frac{1}{2(\ln(A))^2 \sqrt{\gamma} \coth ((\sqrt{\xi})(1 + pq))} - \frac{(\ln(A))^2 \sqrt{pq}}{\sinh ((\sqrt{\xi})},$$

$$\Phi_5 = \frac{1}{6(\xi(-\mu\omega (\ln(A))^2 \beta^2 - 2k + 4\mu\omega (\ln(A))^2 \sigma\alpha - \mu^3)/\mu\omega)}$$
$$+ \frac{1}{2(\ln(A))^2 \tanh (1/4 \sqrt{\xi}) \sqrt{\gamma}} + \frac{1}{2(\ln(A))^2 \coth (1/4 \sqrt{\xi}) \sqrt{\gamma}}$$

Family 14. When $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, the singular, dark, and combined dark-singular solutions are obtained, as follows:

$$\Phi_6 = \frac{-1}{6(\xi(-5\mu\omega (\ln(A))^2 \beta^2 + 2k + 8\mu\omega (\ln(A))^2 \sigma\alpha + \mu^3)/\mu\omega)}$$
$$+ \sqrt{\gamma}(\ln(A))^2 \tanh \left( \frac{1}{2\sqrt{\xi}} \right)$$
$$\left(\frac{(\ln(A))^2(-\beta^2 + 2\alpha\sigma)(\ln(\tanh A(1/2\sqrt{\xi}) - 1) - \ln(\tanh A(1/2\sqrt{\xi}) + 1))}{\sqrt{\gamma}}\right).$$
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When $\alpha \sigma > 0$ and $\beta = 0$, the periodic-singular solutions are obtained as

$$\Phi_7 = \frac{-1}{6(\xi (\frac{1}{2}) \beta^2 + 2k + 8\mu \omega (\frac{1}{2}) \sigma \alpha + \mu^3) / \mu \omega)}$$
$$+ \sqrt{\gamma} (\frac{1}{2}) \coth \alpha \left( \frac{1}{2} \sqrt{\gamma} \xi \right)$$
$$- \left( \frac{1}{2} \sqrt{\gamma} \right)^2 (\beta^2 + 2\alpha \sigma) (\frac{1}{2} \sqrt{\gamma} \xi - 1) - \frac{\ln (\coth \alpha (2 \sqrt{\gamma} \xi + 1))}{\sqrt{\gamma}}$$

$$\Phi_8 = \frac{1}{6(\xi (\frac{1}{2}) \beta^2 - 2k + 4\mu \omega (\frac{1}{2}) \sigma \alpha - \mu^3) / \mu \omega)}$$
$$- (\frac{1}{2} \sqrt{\gamma} \xi) \coth \alpha \left( \frac{1}{2} \sqrt{\gamma} \xi \right)$$
$$- (\frac{1}{2} \sqrt{\gamma} \xi) \coth \alpha \left( \frac{1}{2} \sqrt{\gamma} \xi \right)$$
$$+ \frac{1}{2} \frac{\ln (\coth \alpha (2 \sqrt{\gamma} \xi + 1))}{\sqrt{\gamma}}$$

$$\Phi_9 = \frac{1}{6(\xi (\frac{1}{2}) \beta^2 - 2k + 4\mu \omega (\frac{1}{2}) \sigma \alpha - \mu^3) / \mu \omega)}$$
$$- (\frac{1}{2} \sqrt{\gamma} \xi) \coth \alpha \left( \frac{1}{2} \sqrt{\gamma} \xi \right)$$
$$+ \frac{1}{2} \frac{\ln (\coth \alpha (2 \sqrt{\gamma} \xi + 1))}{\sqrt{\gamma}}$$

$$\Phi_{10} = \frac{1}{6(\xi (\frac{1}{2}) \beta^2 - 2k + 4\mu \omega (\frac{1}{2}) \sigma \alpha - \mu^3) / \mu \omega)}$$
$$+ \frac{1}{2} \frac{\ln (\coth \alpha (2 \sqrt{\gamma} \xi + 1))}{\sqrt{\gamma}}$$

**Family 15.**
The combined singular solutions are obtained, as follows:

\[ \Phi_{14} = \frac{1}{6\left(\xi(-2k + 4\mu\omega (\text{Ln}(A))\sigma\alpha - \mu_{1})/\mu\omega\right)} \]
\[ - 2\frac{(\text{Ln}(A))^{2}\sqrt{pq\sigma}}{\sin\alpha_{L}(2\sqrt{\alpha_{0} \xi})} + (\text{Ln}(A))^{2}\sqrt{\alpha_{0}\text{cot}\alpha_{L}(2\sqrt{\alpha_{0} \xi})(1 + pq)}, \]
\[ \Phi_{15} = \frac{1}{6\left(\xi(-2k + 4\mu\omega (\text{Ln}(A))\sigma\alpha - \mu_{1})/\mu\omega\right)} \]
\[ - (\text{Ln}(A))^{2}\sqrt{\alpha_{0}}\left(\tan_{L}\left(\frac{1}{2\sqrt{\alpha_{0}} \xi}\right) - \cot_{L}\left(\frac{1}{2\sqrt{\alpha_{0}} \xi}\right)\right). \] 

Family 16. When \( \alpha\sigma < 0 \) and \( \beta = 0 \), the singular, dark, combined dark-bright, combined dark-singular, and combined singular solutions are obtained, as follows:

\[ \Phi_{16} = \frac{-1}{6\left(\xi(2k + 8\mu\omega (\text{Ln}(A))\sigma\alpha + \mu_{1})/\mu\omega\right)} \]
\[ + (\text{Ln}(A))^{2}\sqrt{-\alpha_{0}}\left(\text{Ln}\left(\frac{\text{tanh}_{L}\left(\sqrt{-\alpha_{0}} \xi\right) - 1}{\text{tanh}_{L}\left(\sqrt{-\alpha_{0}} \xi\right) + 1}\right)\right) \]
\[ + 2(\text{Ln}(A))^{2}\sqrt{-\alpha_{0}}\text{tanh}_{L}\left(\sqrt{-\alpha_{0}} \xi\right), \]
\[ \Phi_{17} = \frac{-1}{6\left(\xi(2k + 8\mu\omega (\text{Ln}(A))\sigma\alpha + \mu_{1})/\mu\omega\right)} \]
\[ + (\text{Ln}(A))^{2}\sqrt{-\alpha_{0}}\left(\text{Ln}\left(\frac{\text{coth}_{L}\left(\sqrt{-\alpha_{0}} \xi\right) - 1}{\text{coth}_{L}\left(\sqrt{-\alpha_{0}} \xi\right) + 1}\right)\right) \]
\[ + 2(\text{Ln}(A))^{2}\sqrt{-\alpha_{0}}\text{coth}_{L}\left(\sqrt{-\alpha_{0}} \xi\right), \]
\[ \Phi_{18} = \frac{-1}{6\left(\xi(2k - 4\mu\omega (\text{Ln}(A))\sigma\alpha + \mu_{1})/\mu\omega\right)} \]
\[ + (\text{Ln}(A))^{2}\sqrt{-\alpha_{0}}\text{tanh}_{L}(2\sqrt{-\alpha_{0}} \xi)(1 + pq) + 2\frac{(\text{Ln}(A))^{2}\sqrt{-pq\sigma}}{\cosh_{L}(2\sqrt{-\alpha_{0}} \xi)}, \]
\[ \Phi_{19} = \frac{-1}{6\left(\xi(2k - 4\mu\omega (\text{Ln}(A))\sigma\alpha + \mu_{1})/\mu\omega\right)} + (\text{Ln}(A))^{2} \]
\[ \sqrt{-\alpha_{0}}\text{coth}_{L}(2\sqrt{-\alpha_{0}} \xi)(1 + pq) + 2\frac{(\text{Ln}(A))^{2}\sqrt{-pq\sigma}}{\sinh_{L}(2\sqrt{-\alpha_{0}} \xi)}, \]
\[ \Phi_{20} = \frac{-1}{6\left(\xi(2k - 4\mu\omega (\text{Ln}(A))\sigma\alpha + \mu_{1})/\mu\omega\right)} + (\text{Ln}(A))^{2} \]
\[ \sqrt{-\alpha_{0}}\text{tanh}_{L}\left(\frac{1}{2\sqrt{-\alpha_{0}} \xi}\right) + (\text{Ln}(A))^{2}\sqrt{-\alpha_{0}}\text{coth}_{L}\left(\frac{1}{2\sqrt{-\alpha_{0}} \xi}\right). \]
Family 17. When $\beta = 0$ and $\sigma = a$, the periodic-singular solutions are obtained as follows:

\[
\begin{align*}
\Phi_{21} &= \frac{-1}{6(\xi(2k + 8(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} - 2(Ln(A))^{2}a\left(\tan_{a}(a\xi) - a\xi\right), \\
\Phi_{22} &= \frac{-1}{6(\xi(2k + 8(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} + 2(Ln(A))^{2}a\left(\cot_{a}(a\xi) - \frac{1}{2\pi} + a\xi\right), \\
\Phi_{23} &= \frac{-1}{6(\xi(2k - 4(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} - (Ln(A))^{2}\tan_{a}(2a\xi)(1 + pq) \\
&- 2\frac{(Ln(A))^{2}a\sqrt{pq}}{\cos_{a}(2a\xi)}, \\
\Phi_{24} &= \frac{-1}{6(\xi(2k - 4(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} + (Ln(A))^{2}\cot_{a}(2a\xi)(1 + pq) \\
&- 2\frac{(Ln(A))^{2}a\sqrt{pq}}{\sin_{a}(2a\xi)}, \\
\Phi_{25} &= \frac{-1}{6(\xi(2k - 4(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} - (Ln(A))^{2}\tanh_{a}(2a\xi) \\
&\left(1 + pq\right) - 2\frac{(Ln(A))^{2}ai\sqrt{pq}}{\cosh_{a}(2a\xi)},
\end{align*}
\]  

(29)

Family 18. When $\beta = 0$ and $\sigma = -a$, the singular dark, combined dark-bright, combined dark-singular, and combined singular solutions are obtained, as follows:

\[
\begin{align*}
\Phi_{26} &= \frac{-1}{6(\xi(2k - 8(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} + (Ln(A))^{2} \\
&\left(2\tanh_{a}(a\xi) + Ln\left(\frac{\tanh_{a}(a\xi) - 1}{\tanh_{a}(a\xi) + 1}\right)\right), \\
\Phi_{27} &= \frac{-1}{6(\xi(2k - 8(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} + (Ln(A))^{2} \\
&\left(2\coth_{a}(a\xi) + Ln\left(\frac{\coth_{a}(a\xi) - 1}{\coth_{a}(a\xi) + 1}\right)\right), \\
\Phi_{28} &= \frac{-1}{6(\xi(2k + 4(Ln(A))^{2}a^{3}\mu \omega + \mu^{3})/\mu \omega)} + (Ln(A))^{2}\tanh_{a}(2a\xi) \\
&\left(1 + pq\right) - 2\frac{(Ln(A))^{2}ai\sqrt{pq}}{\cosh_{a}(2a\xi)},
\end{align*}
\]
Family 19. When $\beta^2 = 4\alpha\sigma$, the rational solution is obtained, as

\[
\Phi_{31} = -\frac{1}{6\left((x+\mu y-kt)(2k+\mu)^2/\mu\omega\right)} + \frac{2\omega}{(x+\mu y-kt)}.
\]  

Family 20. When $\beta = l$, $\sigma = 0$, and $\alpha = ml (m \neq 0)$, the rational solution is obtained, as follows:

\[
\Phi_{32} = -\frac{1}{6\left((x+\mu y-kt)(\ln(A)^2\mu\omega+2k+\mu^2)/\mu\omega\right)}
\]  

Family 21. When $\beta = \sigma = 0$, the rational solution is obtained as

\[
\Phi_{33} = -\frac{1}{6\left(2k+\mu^2\right)(x+\mu y-kt)/\mu\omega)}
\]  

Family 22. When $\beta = \alpha = 0$, the rational solution is obtained as

\[
\Phi_{34} = -\frac{1}{6\left(2\xi^2 k + \mu \xi^2 - 12\mu\omega/\mu\xi\right)}
\]  

Family 23. When $\alpha = 0$ and $\beta \neq 0$, the singular and dark-singular combo solitons solutions are obtained, as follows:
Family 24. When $\beta = l$, $\alpha = 0$, and $\sigma = ml \ (m \neq 0)$, the rational solution is obtained, as follows:

$$\Phi_{37} = \frac{-1}{6(\xi((\text{Ln}(A))^2T\mu\omega + 2k + \mu^3)/\mu\omega)} + 2\text{Ln}(A)\text{Ln}\left(\frac{q - mpA^e}{-q + mpA^e}\right) + 2\frac{\text{Ln}(A)lq}{-q + mpA^e}$$

(36)

where $\gamma = \beta^2 - 4\alpha\sigma$ and $\xi = \omega(x + \mu y - kt)$.

3.1. Physical Description of Solutions. A solitary wave is a restricted gravity wave that maintains its finite amplitude and propagates with consistent speed and constant shape. Solitons are the solitary wave with an elastic dispersive property. Solitons are the consequence of a delicate balance between nonlinear and dispersive impact in the medium. If the solution is in the form of tangent, secant, cotangent, and cosecant hyperbolic, then the solution is called dark, bright, singular, singular-soliton solutions, respectively. The solution of hyperbolic tangent plus hyperbolic secant form is called combined dark-bright soliton solution. The solution of hyperbolic cotangent plus hyperbolic cosecant form is called combined singular soliton solution, and the solution of hyperbolic tangent plus hyperbolic cotangent form is called dark-singular combo soliton solution.

Figure 1 demonstrates the solutions of $|\Phi_1|$, $|\Phi_2|$, $|\Phi_3|$, $|\Phi_4|$, and $|\Phi_5|$ for the particular parameters $\alpha = 1.5$, $\beta = 2$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.75$, and $A = 2.5$. (a) $|\Phi_1|$, (b) $|\Phi_2|$, (c) $|\Phi_3|$, (d) $|\Phi_4|$, (e) $|\Phi_5|$.
parameters such as \( \alpha = 1.5, \beta = 2, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = 0.5, \) and \( A = 2.5. \) In Figure 3, the solutions of \( |\Phi_{11}|, |\Phi_{12}|, |\Phi_{13}|, |\Phi_{14}|, \) and \( |\Phi_{15}| \) are plotted in the finite domain for the parameters \( \alpha = 1.5, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = 0.75, \) and \( A = 2.5. \) Figure 4 demonstrates the solutions of \( |\Phi_{16}|, |\Phi_{17}|, |\Phi_{18}|, |\Phi_{19}|, \) and \( |\Phi_{20}| \) for the particular parameters \( \alpha = 1.5, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = -0.75, \) and \( A = 2.5. \) Figure 5 demonstrates the solutions of \( |\Phi_{21}|, |\Phi_{22}|, |\Phi_{23}|, |\Phi_{24}|, \) and \( |\Phi_{25}| \) for the particular parameters \( \alpha = 1.5 = \sigma, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = -1.5, \) and \( A = 2.5. \) Figure 6 demonstrates the 3D graphics of the solutions of \( |\Phi_{26}|, |\Phi_{27}|, |\Phi_{28}|, |\Phi_{29}|, \) and \( |\Phi_{30}| \) for the particular parameters \( \alpha = 1.5, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = -1.5, \) and \( A = 2.5. \) Figure 7 demonstrates the 3D graphics for the particular values \( \alpha = 1.5, k = y = 1, \omega = 0.5, \mu = 0.25, \) and \( A = 2.5 \) for \( |\Phi_{31}|, |\Phi_{32}|, \) and \( |\Phi_{33}| \) with \( \beta = 1 \) and \( \sigma = 0.5. \) Figure 8 demonstrates the 3D graphics for the particular values \( \alpha = 1.5, \beta = 0 = \sigma, k = y = 1, \omega = 0.5, \mu = 0.25, \) and \( A = 2.5. \)
Figure 3: 3D profile of Family 15 with $\alpha = 1.5$, $\beta = 0$, $k = y = 1$, $\omega = 0.5$, $p = q = 0.9$, $\mu = 0.25$, $\sigma = 0.75$, and $A = 2.5$. (a) $|\Phi_{11}|$. (b) $|\Phi_{12}|$. (c) $|\Phi_{13}|$. (d) $|\Phi_{14}|$. (e) $|\Phi_{15}|$.

Figure 4: Continued.
Figure 4: 3D profile of Family 16 with $\alpha = 1.5, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = -0.75$, and $A = 2.5$. (a) $|\Phi_{16}|$, (b) $|\Phi_{17}|$, (c) $|\Phi_{18}|$, (d) $|\Phi_{19}|$, (e) $|\Phi_{20}|$.

Figure 5: Continued.
Figure 5: 3D profile of Family 17 with \( \alpha = 1.5 = \sigma, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \) and \( A = 2.5. \) (a) \(|\Phi_{21}|. \) (b) \(|\Phi_{22}|. \) (c) \(|\Phi_{23}|. \) (d) \(|\Phi_{24}|. \) (e) \(|\Phi_{25}|. \)

Figure 6: 3D profile of Family 18 with \( \alpha = 1.5, \beta = 0, k = y = 1, \omega = 0.5, p = q = 0.9, \mu = 0.25, \sigma = -1.5, \) and \( A = 2.5. \) (a) \(|\Phi_{26}|. \) (b) \(|\Phi_{27}|. \) (c) \(|\Phi_{28}|. \) (d) \(|\Phi_{29}|. \) (e) \(|\Phi_{30}|. \)
Figure 7: 3D graphics of Families 19, 20, and 21 under the values of $\alpha = 1.5$, $k = y = 1$, $\omega = 0.5$, $\mu = 0.25$, and $A = 2.5$ for (a) $\Phi_{31}$ with $\beta = 1$ and $\sigma = 0.5$, (b) $\Phi_{32}$ with $\beta = 1$, $\sigma = 0$, $l = 1$, and $m = 0.5$, and (c) $\Phi_{33}$ with $\beta = 0 = \sigma$.

Figure 8: Continued.
of the solutions $|\Phi_{34}|, |\Phi_{35}|, |\Phi_{36}|$, and $|\Phi_{37}|$ under the particular values $\alpha = 0, k = y = 1, \omega = 0.5, \sigma = 0.5, \mu = 0.25$, and $A = 2.5$ for $|\Phi_{34}|$, $|\Phi_{35}|$, $|\Phi_{36}|$, and $|\Phi_{37}|$ with $\beta = 0.5$, and $|\Phi_{37}|$ with $\beta = 0.5 = l, \sigma = 1$, and $m = 2$. Families 13, 14, 15, and 16 represent the singular, dark, combined dark-bright, combined dark-singular, combined singular, and solitary wave solutions. Families 15 and 17 represent the exact periodic traveling wave solutions, whereas the Families 19, 20, 21, 22, 23, and 24 show the rational solutions.

4. Conclusion

To investigate the $(2 + 1)$-dimensional DJKM equation for exact solutions, the extended direct algebraic method is applied. By the extended direct algebraic method, many new exact solitary wave solutions are constructed including the singular, dark, combined dark-bright, periodic-singular, combined dark-singular, combined singular, and rational kinds. Such observations show that the suggested approaches are highly helpful and efficient in solving the NEEs. The complex-valued solutions represent traveling waves in different structures. Even though some are of the well-known forms such as bell-, V-, and W-shaped multiwaves, the shape of some others are completely different from them which were not found in the previous literature. The results of this investigation can be useful in illustrating the physical meaning of the studied model by 3D graphics. The performance of the method is reliable and a computerized mathematical approach to conduct other NLEEIs in the field of mathematical physics and applied sciences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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