On the congruences of Eisenstein series with polynomial indexes

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Abstract
In this paper, based on Serre’s $p$-adic family of Eisenstein series, we prove a general family of congruences for Eisenstein series $G_k$ in the form

$$\sum_{i=1}^{n} g_i(p) G_{f_i(p)} \equiv g_0(p) \pmod{p^N},$$

where $f_1(t), \ldots, f_n(t) \in \mathbb{Z}[t]$ are non-constant integer polynomials with positive leading coefficients and $g_0(t), \ldots, g_n(t) \in \mathbb{Q}(t)$ are rational functions. This generalizes the classical von Staudt–Clausen and Kummer congruences of Eisenstein series, and also yields some new congruences.

Keywords Eisenstein series · Congruences · Serre’s $p$-adic family of Eisenstein series · $p$-adic analysis · Bernoulli numbers

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1 Introduction

1.1 Motivation

For any even integer \( k \geq 4 \), let \( E_k \) be the normalized Eisenstein series of weight \( k \) for the modular group \( SL_2(\mathbb{Z}) \) given by the following \( q \)-expansion:

\[
E_k = 1 - \frac{2k}{B_k} \sum_{j=1}^{\infty} \sigma_{k-1}(j) q^j,
\]

where \( q = e^{2\pi i \tau} \), \( B_k \) is the \( k \)th Bernoulli number and \( \sigma_{k-1}(j) = \sum_{d \mid j} d^{k-1} \). \( E_k \) can be regarded as formal power series in the indeterminate \( q \). If \( f, g \in \mathbb{Q}[[q]] \) are power series and \( N \) is a natural number, \( f \equiv g \mod N \) means that \( f \) and \( g \) are both \( N \)-integral and the congruence holds coefficientwise (see [2, p. 132]).

In what follows, we assume that \( p \) is an odd prime.

Several well-known congruences of \( E_k \) have been given in [4, p. 164, Theorem 7.1]. For example, from von Staudt–Clausen’s and Kummer’s congruences for Bernoulli numbers, one can easily obtain (see [2, Equation (1.3)])

\[
E_k \equiv 1 \pmod{p^r} \quad \text{if} \quad k \equiv 0 \pmod{(p - 1)p^{r-1}}
\]

and

\[
E_k \equiv E_l \pmod{p^r} \quad \text{if} \quad k \equiv l \pmod{(p - 1)p^{r}},
\]

for \( k, l \geq r + 1 \), and \( (k, p), (l, p) \) are regular. The last condition means that \( p \) does not divide (the numerator of) \( B_k \). Since this condition depends only on the residue class of \( k \pmod{p - 1} \), it holds simultaneously for \( k \) and \( l \) (see [2, Equation (1.3)]). By using Serre’s theory of \( p \)-adic modular forms [6] and viewing the coefficients of Eisenstein series as Iwasawa functions [2, Theorem 4.7], Gekeler [2] proved several congruences of the shape \( E_{k+l} \equiv E_k \cdot E_l \) modulo prime power.

In this paper, we study congruence relations of Serre’s normalized Eisenstein series (see [6, p. 194])

\[
G_k = -\frac{B_k}{2k} + \sum_{j=1}^{\infty} \sigma_{k-1}(j) q^j, \quad k \geq 4 \text{even.} \tag{1.1}
\]

For further deductions, we make a convention that

\[
G_k = 0, \quad \text{if} \, k \in \mathbb{Z} \text{ but } k \text{ is not even greater than 2.} \tag{1.2}
\]

Let \( f(t) \in \mathbb{Z}[t] \) have positive leading coefficient and satisfy \( f(1) = 0 \). Then, von Staudt–Clausen’s congruence of Bernoulli numbers in polynomial index (see [5,
Equation (1.2)) implies that

$$2pf(p)G_{f(p)} \equiv 1 \pmod{p}$$

(1.3)

for every sufficiently large prime $p$ (note that $f(p)$ is even because $f(1) = 0$).

Besides, let $f(t), g(t) \in \mathbb{Z}[t]$ be distinct non-constant polynomials with positive leading coefficient, and suppose that $f(1) = g(1) \neq 0$. Let $d$ be the largest power of $t$ dividing $f(t) - g(t)$. Then, by using Kummer’s congruence of Bernoulli numbers in polynomial index (see [5, Equation (1.3)]), one can obtain

$$G_{f(p)} \equiv G_{g(p)}(\text{mod } p^{d+1})$$

(1.4)

for every sufficiently large prime $p$.

In order to generalize (1.3) and (1.4), we consider the following problem:

**Question 1.1** Given polynomials $f_1(t), \ldots, f_n(t) \in \mathbb{Z}[t]$ with positive leading coefficients, rational functions $g_0(t), g_1(t), \ldots, g_n(t) \in \mathbb{Q}(t)$ and a positive integer $N$, determine whether the congruence

$$\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv g_0(p)(\text{mod } p^N)$$

is true for any sufficiently large prime $p$.

This is inspired by a recent work of Julian Rosen [5]. He investigated a similar problem for Bernoulli numbers [5, Question 1.1], and he also obtained a very general criterion (see [5, Theorem 1.2]). The main tool is a Taylor expansion for the Kubota–Leopoldt $p$-adic zeta functions (see [5, Proposition 2.1]). As pointed out in [5, p. 1896], the well-known Kummer and von Staudt–Clausen congruences of Bernoulli numbers in polynomial index which have been given in [1, Sections 9.5 and 11.4.2] can be deduced from this criterion.

### 1.2 Main results

From now on, let $N$ be a fixed positive integer. Let $f_1(t), \ldots, f_n(t) \in \mathbb{Z}[t]$ be non-constant integer polynomials with positive leading coefficients, and let $g_0(t), \ldots, g_n(t) \in \mathbb{Q}(t)$ be rational functions. Write $v_t$ for the $t$-adic valuation on $\mathbb{Q}(t)$, and set

$$M = \min_{i=1, \ldots, n} \{v_t(g_i(t))\}.$$

Here, we fix a convention that $v_t(0) = \infty$. For the $p$-adic valuation $v_p$, as usual we fix the convention $v_p(0) = \infty$.

We define the following four conditions:
C1: 
\[ v_t \left( g_0(t) + \frac{1}{2} (1 - \frac{1}{t}) \sum_{i=1}^{n} g_i(t) f_i(t)^{-1} \right) + \frac{1}{2} \sum_{i=1}^{n} \frac{B_{f_i(1)}}{f_i(1)} (1 - t^{f_i(1)-1}) g_i(t) \geq N; \]

C2: for every even integer \( l \leq 2 \) and every \( 0 \leq m \leq N - M - 1 \),
\[ v_t \left( \sum_{i=1}^{n} g_i(t) f_i(t)^m \right) \geq N - m; \]

C3: for every even integer \( l \geq 4 \) and every \( 1 \leq m \leq N - M - 1 \),
\[ v_t \left( \sum_{i=1}^{n} g_i(t)(f_i(t)^m - l^m) \right) \geq N - m; \]

C4: for every even integer \( l \geq 4 \),
\[ v_t \left( \sum_{i=1}^{n} g_i(t) \right) \geq N. \]

We remark that if \( N - M < 1 \), then the condition C2 automatically holds; and if \( N - M \leq 1 \), the condition C3 also holds automatically. Besides, if \( f_i(1) \leq 3 \) for each \( 1 \leq i \leq n \), then the conditions C3 and C4 hold automatically.

In order to make our main result effective, let \( P \) be a positive integer satisfying:

- \( P \geq N - M + 3 \);
- \( P \geq |f_i(1)| + 1 \) for each \( 1 \leq i \leq n \);
- for each \( 1 \leq i \leq n \) and any integer \( j > P \), \( f_i(j) > \max\{3, N\} \);
- for each \( 1 \leq i \leq n \), if \( g_i(t) \neq 0 \), we write \( g_i(t) = \sum_{i=1}^{n} a_i(t)/b_i(t) \) for some integer \( d_i \) and \( a_i(t), b_i(t) \in \mathbb{Z}[t] \) satisfying \( a_i(0)b_i(0) \neq 0 \) and \( \gcd(a_i(t), b_i(t)) = 1 \), and then \( P \geq |a_i(0)| \) and \( P \geq |b_i(0)| \); (Under this condition, for any prime \( p > P \), we have \( v_p(g_i(p)) = v_t(g_i(t)) \) for each \( 1 \leq i \leq n \); see Proposition 3.1.)
- for each valued function, say \( g(t) \), in the \( v_t \) valuation in the conditions C1, C2, C3 and C4 (for instance, in C4, \( g(t) = \sum_{i=1}^{n} g_i(t) \)) if \( g(t) \neq 0 \), then we write \( g(t) = t^da(t)/b(t) \) for some integer \( d \) and \( a(t), b(t) \in \mathbb{Z}[t] \) satisfying

\[ \deg(a(t)) = \max\{\deg(a(t)) + 1, d\} \approx P; \]

\[ \deg(b(t)) = \max\{\deg(b(t)) + 1, d\} \approx P; \]

\[ \deg(g(t)) = \max\{\deg(g(t)) + 1, d\} \approx P; \]
a(0)b(0) \neq 0 \text{ and } \gcd(a(t), b(t)) = 1, \text{ and then } P \geq |a(0)| \text{ and } P \geq |b(0)|;
(Under this condition, for any prime } p > P, \text{ we have } v_p(g(p)) = v_t(g(t)); \text{ see Proposition 3.1.)}

When the initial data } N, f_1, \ldots, f_n, g_0, \ldots, g_n \text{ are given, the verification of the conditions } C_1, C_2, C_3 \text{ and } C_4 \text{ is in fact a finite computation, and also it is easy to get an explicit choice for the integer } P.

Our main result is the following congruence relation of Eisenstein series } G_k.

**Theorem 1.2** The congruence
\[
\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv g_0(p)(mod\ p^N)
\]
holds for every odd prime } p > P \text{ if all the conditions } C_1, C_2, C_3 \text{ and } C_4 \text{ hold.}

Theorem 1.2 is an analogue of [5, Theorem 1.2] and moreover in an effective manner.

The condition that all the polynomials } f_i \text{ are non-constant is for simplicity. But it is not essential, that is, for each } f_i \text{ that is constant, we can move the term } g_i(p)G_{f_i(p)} \text{ to the right-hand side of the congruence in Theorem 1.2.}

**Remark 1.3** Using Theorem 1.2, we can directly recover the congruences (1.3) and (1.4). For proving (1.3), we choose } N = 1, n = 1, f_1(t) = f(t) \text{ satisfying } f(1) = 0, g_0(t) = 1 \text{ and } g_1(t) = 2tf(t) \text{ in Theorem 1.2; while for proving (1.4), we choose } n = 2, f_1(t) = f(t), f_2(t) = g(t), g_0(t) = 0, g_1(t) = 1, g_2(t) = -1 \text{ and } N = v_t(f - g) + 1 \text{ in Theorem 1.2 and notice the condition } f(1) = g(1) \neq 0.

The following corollary is a generalization of (1.4).

**Corollary 1.4** In Theorem 1.2, choose
\[
N = 1 + \min_{1 \leq i, j \leq n} v_t(f_i - f_j),
\]
and assume that } f_1(1) = \cdots = f_n(1) \neq 0, \ g_0 = 0, \ g_1 + \cdots + g_n \equiv 0 (mod\ t^N). Then, for any odd prime } p > P, \text{ we have}
\[
\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv 0 (mod\ p^N).
\]

The following corollary is a direct consequence of Theorem 1.2.

**Corollary 1.5** Assume that } f_i(1) \leq 3 \text{ for each } 1 \leq i \leq n. Then, The congruence
\[
\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv g_0(p)(mod\ p^N)
\]
holds for every odd prime \( p > P \) if the following two conditions hold:

\[ v_l \left( g_0(t) + \frac{1}{2} \left( 1 - \frac{1}{t} \right) \sum_{i=1}^{n} g_i(t) f_i(t)^{-1} \right) \geq N; \]

(2) for every even integer \( l \leq 2 \) and every \( 0 \leq m \leq N - M - 1 \),

\[ v_l \left( \sum_{i=1}^{n} g_i(t) f_i(t)^m \right) \geq N - m. \]

From Corollary 1.5, one can get some more examples about congruence of Eisenstein series.

**Example 1.6** In Corollary 1.5, we choose \( n = 2, g_0(t) = 0, g_1(t) = 1, g_2(t) = -1 \) and \( N = v_l(f_1 - f_2) - 1 \) such that \( f_1(0) f_2(0) \neq 0, f_1(1) = f_2(1) = 0 \) and \( N \geq 1 \), then we have

\[ G f_1(p) \equiv G f_2(p) \pmod{p^N} \]

for any sufficiently large prime \( p \). This example can be viewed as a complement to the congruence (1.4), where \( f(1) = g(1) \neq 0 \).

**Example 1.7** In Corollary 1.5, we choose \( n \geq 2, f_i(t) = a_i(t - 1) \) for each \( 1 \leq i \leq n \), \( g_0(t) = \frac{1}{2} \left( \frac{1}{a_2} + \cdots + \frac{1}{a_n} - \frac{n-1}{a_1} \right), g_1(t) = (n-1)t, g_i(t) = -t \) for each \( 2 \leq i \leq n \), and \( N = 2 \), then we have for any prime \( p > \text{max}\{4, n-1\} \),

\[ (n-1) p G_{a_1(p-1)} - \sum_{i=2}^{n} p G_{a_i(p-1)} \equiv \frac{1}{2} \left( \frac{1}{a_2} + \cdots + \frac{1}{a_n} - \frac{n-1}{a_1} \right) \pmod{p^2}. \]

In particular, we have for any prime \( p > 4 \),

\[ p G_{a_1(p-1)} - p G_{a_2(p-1)} \equiv \frac{1}{2} \left( \frac{1}{a_2} - \frac{1}{a_1} \right) \pmod{p^3}. \]

For the proof of Theorem 1.2, the approach is similar as in [5], but it indeed needs some extra considerations in the setting of Eisenstein series. For example, we need a new ingredient, that is, a Taylor expansion for the non-constant coefficients of \( p \)-adic Eisenstein series in Proposition 3.4, which will play a key role in the proof.

Our paper will be organized as follows. In Sect. 2, we give a brief recall of Serre’s \( p \)-adic family of Eisenstein series. In Sect. 3, we prove Theorem 1.2 and Corollary 1.4.
**2 p-Adic Eisenstein series**

In this section, we recall some facts about Serre’s $p$-adic family of Eisenstein series [6]; see also [3]. Recall that $p$ is an odd prime.

Serre’s normalized Eisenstein series has been defined in (1.1). We pass to the $p$-adic limit. Let $\mathbb{X} = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. The integers $\mathbb{Z}$ are embedded into $\mathbb{X}$ naturally by $j \mapsto (j, j)$. For $k \in \mathbb{X}$ and $j \geq 1$, define

$$\sigma_{k-1}^*(j) = \sum_{d | j \pmod{\mathbb{Z}_p}, (p,d)=1} d^{-1}$$

(see [6, p. 205], and see [6, p. 201] for the definition of $d^{-1}$). If $k$ is even (that is, $k \in 2\mathbb{X}$), there exists a sequence of even integers $\{k_i\}_{i=1}^{\infty}$ such that $|k_i| \to \infty$ and $k_i \to k$ (in $\mathbb{X}$) when $i \to \infty$. Then, the sequence $G_{k_i} = -\frac{B_{k_i}}{2k_i} + \sum_{j=1}^{\infty} \sigma_{k_i-1}(j)q^j$ has a limit: (see [6, p. 206])

$$G_k^* = a_0(G_k^*) + \sum_{j=1}^{\infty} a_j(G_k^*)q^j$$

(2.1)

with $a_0(G_k^*) = \frac{1}{2}\zeta^*(1-k)$ by defining $\zeta^*(1-k) = \lim_{i \to \infty} \zeta(1-k_i)$ and $a_j(G_k^*) = \sigma_{k-1}^*(j)$, where $\zeta(s)$ is the Riemann zeta function. The function $\zeta^*$ is thus defined on the odd elements of $\mathbb{X} \setminus \{1\}$.

Let $\chi$ be a Dirichlet character on $\mathbb{Z}_p$, and let $L_p(s, \chi)$ be the $p$-adic $L$-function. We have the following result on $\zeta^*$.

**Theorem 2.1** (see [6, p. 206, Théorème 3]) If $(s, u) \neq 1$ is an odd element of $\mathbb{X} = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$, then

$$\zeta^*(s, u) = L_p(s, \omega^{1-u}),$$

where $\omega$ is the Teichmüller character.

For $k = (s, u) \in \mathbb{X}$ and $u$ is even, by Theorem 2.1 the coefficients of $G_k^* = G_{s,u}^*$ are given by (see [6, p. 245])

$$a_0(G_{s,u}^*) = \frac{1}{2}\zeta^*(1-s, 1-u) = \frac{1}{2}L_p(1-s, \omega^u),$$

$$a_j(G_{s,u}^*) = \sum_{d | j \pmod{\mathbb{Z}_p}, (p,d)=1} d^{-1} \omega(d)^u \langle d \rangle^s,$$

(2.2)

where $\langle d \rangle = d / \omega(d) \equiv 1 \pmod{p}$.

Thus, the assignment

$$(s, u) \mapsto G_{s,u}^*$$
gives a family of $p$-adic modular forms parametrized by the group of weights $X$.

For any even integer $k \geq 4$, we first write

$$G_k = a_0(G_k) + \sum_{j=1}^{\infty} a_j(G_k)q^j,$$  \hspace{1cm} (2.3)

where $a_0(G_k) = -\frac{B_{2k}}{2k}$, $a_j(G_k) = \sigma_{k-1}(j)$; and then from (2.2), we have

$$a_0(G_k^\ast) = a_0(G_{k,k}^\ast) = \frac{1}{2}g^\ast(1-k, 1-k) = \frac{1}{2}L_p(1-k, \omega^k) = -\frac{1-p^{k-1}B_k}{2}$$

$$= (1-p^{k-1})a_0(G_k),$$

where we also use the relation between $L_p$ and Bernoulli numbers (see, for instance, the first paragraph in the proof of [5, Proposition 2.1]), and

$$a_j(G_k^\ast) = a_j(G_{k,k}^\ast) = \sum_{d|j, (p,d)=1} d^{-1}\omega(d)^k(d)^k = \sum_{d|j, (p,d)=1} d^{k-1}.$$ \hspace{1cm} (2.5)

The proof of our main result is based on the following relationship between the $p$-adic Eisenstein Series $G_k^\ast$ and the Eisenstein series $G_k$:

$$G_k \equiv G_k^\ast \mod p^{k-1}, \quad k \geq 4 \text{ even},$$ \hspace{1cm} (2.6)

which can be easily deduced from (2.1), (2.3), (2.4) and (2.5).

As in (1.2), we also make a convention that

$$G_k^\ast = 0, \quad \text{if } k \in \mathbb{Z} \text{ but } k \text{ is not even greater than } 2.$$ \hspace{1cm} (2.7)

### 3 Proofs of the main results

Recall that $p$ is an odd prime. For the proofs, we need some preparations. We start with an explicit version of [5, Proposition 3.1].

**Proposition 3.1** Suppose that $g(t) \in \mathbb{Q}(t)$ is non-zero, and write $g(t) = t^d a(t)/b(t)$ with $a(t), b(t) \in \mathbb{Z}[t]$ satisfying $a(0)b(0) \neq 0$ and $\gcd(a(t), b(t)) = 1$. Then,

$$v_p(g(p)) = v_t(g(t))$$

for any prime $p$ satisfying $p > |a(0)|$ and $p > |b(0)|$.  

\[ Springer \]
Proof For any prime $p$ satisfying $p > |a(0)|$ and $p > |b(0)|$, since $a(p) \equiv a(0) \pmod{p}$ and $b(p) \equiv b(0) \pmod{p}$, we must have that $p \nmid a(p)$ and $p \nmid b(p)$. So, we obtain

$$v_p(g(p)) = v_p(p^d a(p) / b(p)) = d = v_t(g(t)).$$

⊓⊔

The following proposition follows from (2.4) and [5, Proposition 2.1] directly.

**Proposition 3.2** Let $l$ be an even residue class modulo $p - 1$. Then, there exist coefficients $a_m^{(0)}(p, l) \in \mathbb{Q}_p$, think $m = 0, 1, 2, \ldots$, such that for every even integer $k \geq 4$ with $k \equiv l \pmod{p - 1}$, there is a convergent $p$-adic series identity

$$a_0(G_k^*) = -\frac{1 - p^{k-1}}{2} B_k = -\frac{1}{2} \sum_{m=0}^{\infty} a_m^{(0)}(p, l) k^{m-1}. \quad (3.1)$$

The coefficients $a_m^{(0)}(p, l)$ satisfy the following conditions:

1. \[ a_0^{(0)}(p, l) = \begin{cases} 1 - \frac{1}{p} & \text{if } l \equiv 0 \pmod{p-1}, \\ 0 & \text{otherwise}, \end{cases} \]

2. for all $m$, $p$ and $l$,

$$v_p(a_m^{(0)}(p, l)) \geq \frac{p - 2}{p - 1} m - 2,$$

3. for $p \geq m + 2$ and all $l$,

$$v_p(a_m^{(0)}(p, l)) \geq m - 1.$$

Using Proposition 3.2, we obtain a congruence relation for the coefficient $a_0(G_k^*)$ in polynomial index.

**Proposition 3.3** The congruence

$$\sum_{i=1}^{n} g_i(p) a_0(G_{f_i(p)}^*) \equiv g_0(p) \pmod{p^N}$$

holds for every odd prime $p > P$ if the conditions $\text{C1}$, $\text{C2}$ and $\text{C3}$ hold.

Proof We extend the proof of [5, Theorem 1.2] to our case.
Since \( p > P \) and noticing the choice of \( P \), we know that \( f_i(p) \geq 4 \) for each \( 1 \leq i \leq n \). In view of the convention (2.7), we consider the quantity

\[
A^{(0)}(p) = g_0(p) - \sum_{i=1}^{n} g_i(p) a_0(G^*_f(p)).
\]

Then, it is equivalent to prove that \( A^{(0)}(p) \equiv 0 \mod p^N \). By Proposition 3.2, we have

\[
A^{(0)}(p) = g_0(p) + \sum_{f_i(p) \text{ even}}^{n} g_i(p) \left( \frac{1}{2} \sum_{m=0}^{\infty} a_m^{(0)}(p, f_i(p)) f_i(p)^{m-1} \right)
\]

\[
= g_0(p) + \sum_{h \in \mathbb{Z} / (p-1) \mathbb{Z}}^{n} \sum_{i=1}^{h \text{ even}, m \geq 0, f_i(p) \equiv h \text{ (mod } p-1)}^{n} \frac{1}{2} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, h).
\]

Since \( f_i(p) \equiv f_i(1) \text{ (mod } p-1) \) for each \( 1 \leq i \leq n \), we have

\[
A^{(0)}(p) = g_0(p) + \sum_{l \in \mathbb{Z}}^{\text{even}} \sum_{m \geq 0, f_i(1) = l}^{n} \frac{1}{2} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l)
\]

\[
= g_0(p) + \frac{1}{2} \sum_{l \leq 2}^{\text{even}} \sum_{i=1}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l)
\]

\[
+ \frac{1}{2} \sum_{l \geq 4}^{\text{even}} \sum_{m \geq 0, f_i(1) = l}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, 0)
\]

\[
+ \frac{1}{2} \sum_{l \leq 2}^{\text{even}} \sum_{m \geq 0, f_i(1) = l}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l),
\]

which, by Proposition 3.2 (1), becomes

\[
A^{(0)}(p) = g_0(p) + \frac{1}{2} \sum_{l \leq 2}^{\text{even}} \sum_{i=1}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l)
\]

\[
+ \frac{1}{2} \left( 1 - \frac{1}{p} \right) \sum_{f_i(1) = 0}^{n} g_i(p) f_i(p)^{-1} + \frac{1}{2} \sum_{m \geq 1}^{n} \sum_{f_i(1) = 0}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, 0)
\]
+ \frac{1}{2} \sum_{l \geq 4 \text{ even}} \sum_{m \geq 0} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l).

(3.2)

Due to the choice of $P$ and $p > P$, we have $p > |f_i(1)| + 1$ for each $1 \leq i \leq n$. So, for any non-zero even number $l$ satisfying $l = f_i(1)$ for some $1 \leq i \leq n$, it cannot happen that $l \equiv 0 \pmod{p - 1}$, which together with Proposition 3.2 (1) implies that

$$a_0^{(0)}(p, l) = 0. \quad (3.3)$$

Thus, from (2.4), (3.1) and (3.3), for any even $l \geq 4$ satisfying $l = f_i(1)$ for some $1 \leq i \leq n$, we have

$$a_i^{(0)}(p, l) = -2a_0(G_i^l) - \sum_{m \geq 2} a_m^{(0)}(p, l)l^{m-1}$$

$$= (1 - p^{l-1}) \frac{B_l}{l} - \sum_{m \geq 2} a_m^{(0)}(p, l)l^{m-1}. \quad (3.4)$$

Substituting (3.3) and (3.4) into (3.2), we have

$$A^{(0)}(p) = g_0(p) + \frac{1}{2} \sum_{l \leq 2 \text{ even}} \sum_{m \geq 0} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l)$$

$$+ \frac{1}{2} \left( 1 - \frac{1}{p} \right) \sum_{i=1}^{n} g_i(p) f_i(p)^{-1}$$

$$+ \frac{1}{2} \sum_{m \geq 1} \sum_{i=1}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, 0)$$

$$+ \frac{1}{2} \sum_{l \geq 4 \text{ even}} \sum_{i=1}^{n} \frac{B_l}{l} (1 - p^{l-1}) g_i(p)$$

$$+ \frac{1}{2} \sum_{l \geq 4 \text{ even}} \sum_{m \geq 2} g_i(p) (f_i(p)^{m-1} - l^{m-1} a_m^{(0)}(p, l). \quad (3.5)$$

Under the condition $C_1$ and noticing the choices of $p$ and $P$, we have

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\[ g_0(p) + \frac{1}{2} \left( 1 - \frac{1}{p} \right) \sum_{i=1}^{n} g_i(p) f_i(p)^{-1} + \frac{1}{2} \sum_{l \geq 4 \text{ even}} \sum_{i=1}^{n} \frac{B_i}{l} (1 - p^{l-1}) g_i(p) \equiv 0 \pmod{p^N}. \]  

(3.6)

For every even integer \( l \leq 2 \) satisfying \( l = f_i(1) \) for some \( 1 \leq i \leq n \), under the condition \( C2 \) and due to the choices of \( p \) and \( P \), for any \( 1 \leq m \leq N - M \), we have

\[ v_p \left( \sum_{i=1}^{n} g_i(p) f_i(p)^{m-1} \right) = v_t \left( \sum_{i=1}^{n} g_i(t) f_i(t)^{m-1} \right) \geq N - (m - 1); \]

and by Proposition 3.2 (3) and noticing \( p > P \geq N - M + 3 \geq m + 3 \) due to the choice of \( P \), we have

\[ v_p(a_m^{(0)}(p, l)) \geq m - 1; \]

and so we obtain

\[ v_p \left( \sum_{i=1}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l) \right) \geq N, \quad 1 \leq m \leq N - M. \]  

(3.7)

If \( m \geq N - M + 1 \) and \( p \geq m + 2 \), then by Proposition 3.2 (3), we have \( v_p(a_m^{(0)}(p, l)) \geq m - 1 \), which together with \( v_p(g_i(p)) = v_t(g_i(t)) \geq M \) for each \( 1 \leq i \leq n \) (due to the choices of \( p \) and \( P \)) implies that for \( m \geq N - M + 1 \) and \( p \geq m + 2 \), for some \( j \) with \( f_j(1) = l \),

\[ v_p \left( \sum_{i=1}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l) \right) \geq v_p(g_j(p)) + v_p(a_m^{(0)}(p, l)) \geq M + m - 1 \geq N. \]  

(3.8)
If $m \geq N - M + 1$ and $p \leq m + 1$, then by Proposition 3.2 (2) and noticing $v_p(g_i(p)) \geq M$ and $p > P \geq N - M + 3$, we obtain

$$v_p \left( \sum_{i=1 \atop f_i(l)=l}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l) \right) \geq M + v_p(a_m^{(0)}(p, l)) \geq M + \frac{p-2}{p-1} m - 2 \geq M + p - 4 \geq N.$$  \hfill (3.9)

Thus, under the condition $C2$ and combining (3.7), (3.8) with (3.9), for any even integer $l \leq 2$ satisfying $l = f_i(1)$ for some $1 \leq i \leq n$ and any $m \geq 1$, we have

$$\sum_{i=1 \atop f_i(l)=l}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l) \equiv 0 \pmod{p^N}. \hfill (3.10)$$

As the above, under the condition $C3$ and noticing the choices of $p$ and $P$, for every even $l \geq 4$ and $m \geq 2$, we have

$$\sum_{i=1 \atop f_i(l)=l}^{n} g_i(p) (f_i(p)^{m-1} - l^{m-1}) a_m^{(0)}(p, l) \equiv 0 \pmod{p^N}. \hfill (3.11)$$

Finally, by (3.5), (3.6), (3.10) and (3.11) we conclude that $A^{(0)}(p) \equiv 0 \pmod{p^N}$ for any odd prime $p > P$. This completes the proof. 

As an analogue of Proposition 3.2, we obtain a convergent $p$-adic series identity for each coefficient $a_j(G_k^*)$, $j \geq 1$. The approach here is different from the one in [5, Proposition 2.1].

**Proposition 3.4** Let $p$ be an odd prime, $l$ an even residue class modulo $p - 1$ and $j$ a positive integer. Then, there exist coefficients $a_m^{(j)}(p, l) \in \mathbb{Q}_p$, $m = 0, 1, 2, \ldots$, such that for every even integer $k \geq 4$ with $k \equiv l \pmod{p - 1}$, there is a convergent $p$-adic series identity

$$a_j(G_k^*) = \sum_{m=0}^{\infty} a_m^{(j)}(p, l) k^m.$$ 

The coefficients $a_m^{(j)}(p, l)$ satisfy the following conditions:

(1) for all $m$, $p$, $l$,

$$v_p(a_m^{(j)}(p, l)) \geq \frac{p-2}{p-1} m,$$
(2) for \( p \geq m + 2 \) and all \( l \),
\[
v_{p}(a_{m}^{(j)}(p, l)) \geq m.
\]

**Proof** For \((s, u) \in X\) and \( u \) is even, by (2.2), we have
\[
a_{j}(G_{s,u}^{*}) = \sum_{d | j} d^{-1} \omega(d)^{u} \langle d \rangle^{s}. \quad (3.12)
\]
Write \( \langle d \rangle = 1 + pq_{d} \) with \( q_{d} \in \mathbb{Z}_{p} \), we have
\[
\langle d \rangle^{s} = \sum_{m=0}^{\infty} \binom{s}{m} p^{m} q_{d}^{m}.
\]
Substituting the above into (3.12), we have
\[
a_{j}(G_{s,u}^{*}) = \sum_{d | j} d^{-1} \omega(d)^{u} \sum_{m=0}^{\infty} \binom{s}{m} p^{m} q_{d}^{m} \sum_{d | j} d^{-1} \omega(d)^{u}.
\]
Thus, we obtain
\[
a_{j}(G_{k}^{*}) = a_{j}(G_{k,k}^{*}) = \sum_{m=0}^{\infty} \binom{k}{m} p^{m} \sum_{d | j} q_{d}^{m} d^{-1} \omega(d)^{k}
\]
\[
= \sum_{m=0}^{\infty} \binom{k}{m} p^{m} \sum_{d | j} q_{d}^{m} d^{-1} \omega(d)^{l}, \quad (3.13)
\]
where the last equality comes from the fact that \( \omega(a)^{k} = \omega(a)^{l} \) if \( k \equiv l \pmod{p-1} \).

For each \( 1 \leq m \leq k \), we have
\[
\binom{k}{m} = \frac{k(k-1) \cdots (k-m+1)}{m!} \quad (3.14)
\]
\[
= \frac{1}{m!} (k^{m} + b_{m,m-1} k^{m-1} + \cdots + b_{m,1} k)
\]
for some integers \( b_{m,1}, \ldots, b_{m,m-1} \in \mathbb{Z} \) depending only on \( m \).
Substituting (3.14) into (3.13), we obtain

\[ a_j(G^*_k) = \sum_{m=0}^{\infty} a_m^{(j)}(p, l)k^m \]

for some \( a_m^{(j)}(p, l) \in \mathbb{Q}_p \) satisfying

\[ v_p(a_m^{(j)}(p, l)) \geq \min\{v_p(p^m/m!), v_p(p^{m+1}/(m+1)!), \ldots\} \geq m - \frac{m}{p-1}, \]

where the number of terms in the \( \min \) function is finite and the last inequality follows from the fact that \( v_p(m!) \leq m/(p-1) \). This gives the conclusion (1) of the proposition. The conclusion (2) (in the case \( p \geq m+2 \)) follows from (1) directly by noticing \( v_p(a_m^{(j)}(p, l)) \in \mathbb{Z} \).

Applying Proposition 3.4, we can also obtain a congruence relation for the coefficient \( a_j(G^*_k), j \geq 1 \), in polynomial index.

**Proposition 3.5** For any integer \( j \geq 1 \), the congruence

\[ \sum_{i=1}^{n} g_i(p)a_j(G^*_f_i(p)) \equiv 0 \pmod{p^N} \]

holds for every odd prime \( p > P \) if the conditions \( C_2, C_3 \) and \( C_4 \) hold.

**Proof** We apply the same strategy as in the proof of Proposition 3.3.

Since \( p > P \) and noticing the choice of \( P \), we know that \( f_i(p) \geq 4 \) for each \( 1 \leq i \leq n \). In view of the convention (2.7), we consider the quantity

\[ A^{(j)}(p) = \sum_{i=1}^{n} g_i(p)a_j(G^*_f_i(p)). \]

By Proposition 3.4, we have

\[ A^{(j)}(p) = \sum_{i=1}^{n} g_i(p) \sum_{m=0}^{\infty} a_m^{(j)}(p, f_i(p))f_i(p)^m \]

\[ = \sum_{h \in \mathbb{Z}/(p-1)\mathbb{Z}} \sum_{\substack{h \text{even, } m \geq 0, \ f_i(p) \equiv h \mod{p-1}}} g_i(p)f_i(p)^m a_m^{(j)}(p, h). \]

Since \( f_i(p) \equiv f_i(1) \mod{p-1} \), we have
$$A^{(j)}(p) = \sum_{m \geq 0}^{\text{even}} \sum_{i=1}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l)$$

$$= \sum_{l \leq 2 \text{ even}}^{\text{even}} \sum_{i=1}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l)$$ 

$$+ \sum_{l \geq 4 \text{ even}}^{\text{even}} \sum_{m \geq 0}^{\text{even}} g_i(p) f_i(p)^m a_m^{(j)}(p, l).$$

(3.15)

For any even integer \( l \geq 4 \) satisfying \( l = f_i(1) \) for some \( 1 \leq i \leq n \), by Proposition 3.4, we have

$$a_0^{(j)}(p, l) = a_j(G_i^*) - \sum_{m=1}^{\infty} a_m^{(j)}(p, l) l^m.$$

Substituting the above equation into (3.15), we have

$$A^{(j)}(p) = \sum_{l \leq 2 \text{ even}}^{\text{even}} \sum_{i=1}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l)$$

$$+ \sum_{l \geq 4 \text{ even}}^{\text{even}} a_j(G_i^*) \sum_{i=1}^{n} g_i(p)$$ 

$$+ \sum_{l \geq 4 \text{ even}}^{\text{even}} \sum_{m \geq 1}^{\text{even}} g_i(p)(f_i(p)^m - l^m) a_m^{(j)}(p, l).$$

(3.16)

As in the proof of Proposition 3.3, under the condition \( C2 \) and the choices of \( p \) and \( P \) and using Proposition 3.4, for every even integer \( l \leq 2 \) and \( m \geq 0 \), we obtain

$$\sum_{i=1}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l) \equiv 0 \text{(mod } p^N).$$

(3.17)

Similarly, under the condition \( C3 \), for every even integer \( l \geq 4 \) and \( m \geq 1 \), we have

$$\sum_{i=1}^{n} g_i(p)(f_i(p)^m - l^m) a_m^{(j)}(p, l) \equiv 0 \text{(mod } p^N).$$

(3.18)
Also, under the condition \( \textbf{C4} \) and noticing \( a_j(G^*_l) \in \mathbb{Z}_p \) by (2.5), for every even integer \( l \geq 4 \), we have

\[
a_j(G^*_l) \sum_{i=1}^{n} g_i(p) \equiv 0 \pmod{p^N}. \quad (3.19)
\]

Finally, by (3.16), (3.17), (3.18) and (3.19), we conclude that \( A^{(j)}(p) \equiv 0 \pmod{p^N} \) for any odd prime \( p > P \). This completes the proof. \( \square \)

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2** Since \( p > P \) and noticing the choice of \( P \), we have \( f_i(p) > N \) for each \( 1 \leq i \leq n \). Thus, by (2.6), for any \( 1 \leq i \leq n \) with even \( f_i(p) \), we have

\[
G_{f_i(p)} \equiv G^*_{f_i(p)} \pmod{p^N}. \quad (3.20)
\]

Otherwise if \( f_i(p) \) is odd, then by the conventions (1.2) and (2.7), we have \( G_{f_i(p)} = G^*_{f_i(p)} = 0 \), and so (3.20) still holds. On the other hand, by Propositions 3.3 and 3.5, we directly obtain

\[
\sum_{i=1}^{n} g_i(p)G^*_{f_i(p)} \equiv g_0(p) \pmod{p^N} \quad (3.21)
\]

for every odd prime \( p > P \) if all the conditions \( \textbf{C1}, \textbf{C2}, \textbf{C3} \) and \( \textbf{C4} \) hold. The desired result now follows from (3.20) and (3.21). \( \square \)

Finally, we prove Corollary 1.4.

**Proof of Corollary 1.4** First, by assumptions, it is easy to see that the conditions \( \textbf{C1} \) and \( \textbf{C4} \) hold.

Since \( f_1(t) = \ldots = f_n(t) = g_1 + \ldots + g_n \equiv 0 \pmod{t^N} \), for verifying the conditions \( \textbf{C2} \) and \( \textbf{C3} \), it suffices to show that for any \( m \geq 1 \),

\[
v_t \left( \sum_{i=1}^{n} g_i(t) f_i(t)^m \right) \geq N - 1. \quad (3.22)
\]

By the choice of \( N \), we have \( f_i(t) \equiv f_1(t) \mod{t^{N-1}} \) for any \( 1 \leq i \leq n \). So, combining this with the assumption \( g_1 + \ldots + g_n \equiv 0 \pmod{t^N} \), we have

\[
\sum_{i=1}^{n} g_i(t) f_i(t)^m \equiv f_1(t)^m \sum_{i=1}^{n} g_i(t) \equiv 0 \pmod{t^{N-1}}.
\]

Then, (3.22) follows, and this completes the proof. \( \square \)

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