On Some Algebraic Structures Arising in String Theory

Michael Penkava
University of California
Davis, CA 95616
michae@math.ucdavis.edu

Albert Schwarz
University of California
Davis, CA 95616
asschwarz@math.ucdavis.edu

Abstract

Lian and Zuckerman proved that the homology of a topological chiral algebra can be equipped with the structure of a BV-algebra; i.e., one can introduce a multiplication, an odd bracket, and an odd operator $\Delta$ having the same properties as the corresponding operations in Batalin-Vilkovisky quantization procedure. We give a simple proof of their results and discuss a generalization of these results to the non-chiral case. To simplify our proofs we use the following theorem giving a characterization of a BV-algebra in terms of multiplication and an operator $\Delta$:

If $A$ is a supercommutative, associative algebra and $\Delta$ is an odd second order derivation on $A$ satisfying $\Delta^2 = 0$, one can provide $A$ with the structure of a BV-algebra.

The concept of a super commutative, associative algebra equipped with an additional anti-bracket (odd bracket) structure $\{\cdot,\cdot\}$ which satisfies the following relations:

\[
\{g, f\} = (-1)^{(f+1)(g+1)+1}\{f, g\} \quad \text{(super anticommutativity)} \quad (1)
\]

\[
\{f\{g, h\}\} = \{\{f, g\}h\} + (-1)^{(f+1)(g+1)+1}\{g\{f, h\}\} \quad \text{(super Jacobi identity)} \quad (2)
\]

as well as

\[
\{f, gh\} = \{f, g\}h + (-1)^{(f+1)g+1}\{f, h\} \quad \text{(super derivation rule)} \quad (3)
\]

was first seen in Gerstenhaber’s 1963 article on cohomology of associative rings [3]. (Here we use the convention that in superscripts $f$ stands for the parity of
where \( f \) is assumed to be homogeneous. By an odd bracket, we mean that the parity of \( \{f, g\} \) is opposite to the parity of \( fg \). Note that if we consider the grading with the reversed parity, the first two properties yield the structure of a super Lie algebra, but the multiplication is an additional structure on the algebra, so we shall call an algebra satisfying relations (1), (2) and (3) a G-algebra (Gerstenhaber algebra). Later such algebras began appearing in different problems in physics and mathematics, in particular, in the Batalin-Vilkovisky approach to quantization of gauge theories\[1, 9, 6, 5\]. In \[4\], it was shown that for every topological chiral conformal algebra, where all elements have integral weights (conformal dimensions), the corresponding homology naturally has the structure of a G-algebra. In these two latter cases, the G-algebra is also equipped with an odd operator \( \Delta \), which satisfies
\[
\Delta(fg) = (\Delta f)g + (-1)^f f \Delta g + (-1)^f \{f, g\}
\]
and \( \Delta^2 = 0 \). We will call a G-algebra equipped with such an operator \( \Delta \) a BV-algebra (Batalin-Vilkovisky algebra). The term used for such an algebra in \[4\] is coboundary Gerstenhaber algebra or CGA.

One of the goals of the present paper is to simplify the constructions and proofs of \[4\]. These constructions are closely related to the papers \[10\] and \[8\] that are devoted formally to the strings in the \( c = 1 \) background, but actually contain some general considerations. We consider also the case of a topological conformal algebra containing the two Virasoro subalgebras (left and right Virasoro algebras) without the assumption that all weights are integral. We prove that in this case the corresponding BV-algebra also can be constructed and discuss briefly the relation between this construction and string theory.

To prove that our constructions really lead to BV-algebras we use a theorem essentially giving a description of BV-algebras in terms of multiplication and the
operator $\Delta$. (The expressability of \{f, g\} in these terms follows immediately from equation (4); see [9].) We will begin with the formulation and proof of the theorem, that permits us to simplify the verification of the axioms of a BV-algebra.

First of all we recall that an operator $\alpha$ acting on a $\mathbb{Z}_2$-graded algebra $A$ is called a (super) derivation if for any two elements $a, b \in A$, we have

$$\begin{align*}
[\alpha, \hat{a}] b &= \alpha(a)b \\
&= \alpha(\hat{a}b)
\end{align*} \tag{5}$$

Here $\hat{a}$ stands for the operator of multiplication by $a$, i.e. $\hat{a}b = ab$, and $[\alpha, \beta]$ stands for the (super)commutator of $\alpha$ and $\beta$, i.e. $[\alpha, \beta] = \alpha\beta - (-1)^{\alpha\beta} \beta\alpha$. If $\alpha$ is an even (i.e. parity preserving) operator, it follows from (5) that

$$\alpha(ab) = \alpha(a)b + a\alpha(b). \tag{6}$$

If $\alpha$ is an odd (parity reversing) operator, then

$$\alpha(ab) = \alpha(a)b + (-1)^{\alpha\beta}a\alpha(b). \tag{7}$$

An operator $\alpha$ acting on $A$ will be called a second order (super)derivation if the expression

$$[\alpha, \hat{a}] b - \alpha(a)b$$

is a (super)derivation on $A$ for fixed $a$. It follows immediately from (4) and (3) that the operator $\Delta$ in a BV-algebra is a second order derivation. This remark leads to a description of BV-algebras, which is given by the following theorem:

**Suppose that $A$ is a super commutative, associative algebra and that an odd operator $\Delta$ on $A$ is an second order derivation and obeys $\Delta^2 = 0$. Define the bracket $\{f, g\}$ by the formula**

$$\{f, g\} = (-1)^f \Delta(fg) + (-1)^{f+1}(\Delta f)g - f\Delta g \tag{9}$$
Then the operation $\{f, g\}$ satisfies (4) and (5) (super anticommutativity and the super Jacobi identity), so that $A$, equipped with this bracket, is a BV-algebra.

**proof:** To show equation (4), we use the super commutativity of $A$ and equation (4) to yield:

$$(-1)^g \{g, f\} = \Delta(gf) - (\Delta g)f + (-1)^{g+1}g \Delta f$$

$$= (-1)^g \Delta(fg) + (-1)^{g+1}f \Delta g + (-1)^{g+1}(\Delta f)g$$

$$= (-1)^g(\Delta f)g + (-1)^{g+1}f \Delta g + (-1)^{g+1}f \{f, g\}$$

$$+ (-1)^{g+1}f \{f, \Delta g\}$$

$$= (-1)^{g+1}f \{f, g\}$$

from which the desired equality follows immediately. To facilitate the remainder of the proof, we first prove a lemma

*The operator $\Delta$ is a super derivation operator on its bracket, i.e.*

$$\Delta \{f, g\} = \{\Delta f, g\} + (-1)^{f+1} \{f, \Delta g\} \tag{10}$$

**proof:** This depends only on the defining property in equation (4) of the bracket and the condition $\Delta^2 = 0$. To wit:

$$0 = \Delta^2(fg) = \Delta[(\Delta f)g + (-1)^f f \Delta g + (-1)^f \{f, g\}]$$

$$= (\Delta^2 f)g + (-1)^{f+1}(\Delta f) \Delta g + (-1)^{f+1} \{f, \Delta g\}$$

$$+ (-1)^f(\Delta f) \Delta g + f \Delta^2 g + \{f, \Delta g\} + (-1)^f \Delta \{f, g\}$$

$$= (-1)^{f+1} \{\Delta f, g\} + \{f, \Delta g\} + (-1)^f \Delta \{f, g\}$$

$$\square$$

The proof of equation (5) follows from the above lemma and the following:

$$(-1)^g \{f, \{g, h\}\} = \{f, (-1)^g \{g, h\}\}$$

$$= \{f, \Delta gh - (\Delta g)h + (-1)^{g+1}g \Delta h\}$$

4
\[ = \{f, \Delta (gh)\} - \{f, \Delta g\}h + (-1)^{g+1}\{f, g\Delta h\} \quad (11) \]
\[
\Delta \{f, gh\} = \{\Delta f, g\} + (-1)^{f+1}\{f, \Delta (gh)\} \quad (12) \]
\[
\{f, \Delta (gh)\} = (-1)^{f+1}\Delta \{f, gh\} + (-1)^{f}\{\Delta f, gh\} \quad (13) \]
\[
(-1)^g\{f, \{g, h\}\} = (-1)^{f+1}\Delta \{f, gh\} + (-1)^f\{\Delta f, gh\} \quad (14) \]
\[
\Delta \{f, gh\} = \Delta \{\{f, g\}h + (-1)^{f+1}g\{f, h\} \}
\]
\[= (\Delta \{f, g\})h + (-1)^{f+g+1}\{f, g\} \Delta h \]
\[+ (-1)^{f+g+1}\{f, g\}h + (-1)^{(f+1)g}\{\Delta g\}\{f, h\} \]
\[+ (-1)^{fg}g\Delta \{f, h\} + (-1)^{fg}\{g, f\} \]
\[= \{\Delta f, g\}h + (-1)^{fg+1}\{f, \Delta g\}h \]
\[+ (-1)^{fg+1}\{f, g\} \Delta h + (-1)^{fg+1}\{f, g\}h \]
\[+ (-1)^{(f+1)g}\{\Delta g\}\{f, h\} + (-1)^{fg}\{\Delta f, h\} \]
\[+ (-1)^{fg+1}g\{f, \Delta h\} + (-1)^{fg}\{g, f, h\} \] (15)
\[
(-1)^f\{\Delta f, gh\} - \{f, \{\Delta g\}h\} + (-1)^{g+1}\{f, g\} \Delta h \]
\[= (-1)^f\{\Delta f, gh\} + (-1)^{(f+1)g}\{\Delta f, h\} - \{f, \Delta g\}h \]
\[+ (-1)^{(f+1)(g+1)+1}\{\Delta g\}\{f, h\} + (-1)^{g+1}\{f, g\} \Delta h \]
\[+ (-1)^{fg+1}g\{f, \Delta h\} \] (16)
\[
(-1)^{f+1}\Delta \{f, gh\} = (-1)^{f+1}\{\Delta f, g\}h + \{f, \Delta g\}h + (-1)^g\{f, g\} \Delta h + \]
\[(-1)^g\{\{f, g\}h\} + (-1)^{(f+1)(g+1)}\{\Delta g\}\{f, h\} \]
\[+ (-1)^{fg+1}\{f, \Delta h\} \]
\[+ (-1)^{fg}\{g, \{f, h\}\} \] (17)
\[
(-1)^g\{f, \{g, h\}\} = (-1)^g\{\{f, g\}h\} + (-1)^g(-1)^{(f+1)(g+1)}\{g\{f, h\}\} \] (18)
\[
\{f, \{g, h\}\} = \{\{f, g\}h\} + (-1)^{(f+1)(g+1)}\{g\{f, h\}\} \] (19)
Note that the proof of equation (3) depends only on the lemma and the
derivation property given in equation (3), while the lemma does not depend on
super anticommutativity of the bracket. The proof of the super anticommutativ-
ity of the bracket is the only place where the fact that \(A\) is super commutative,
not merely a graded algebra, is used. Therefore, we see that for a general asso-
ciative, \(\mathbb{Z}_2\)-graded algebra \(A\), if the operator \(\Delta\) is defined as above, satisfying
equations (1) and (3) as well as \(\Delta^2 = 0\), then the super Jacobi identity equa-
tion (2) holds. Thus we may in this case extend the construction to the case
of arbitrary graded algebras. Of course, the form of the super Jacobi identity
used here will not in general be equivalent to other standard forms unless super
anticommutativity of the bracket holds. However, it is clear that super anti-
commutativity of the bracket is sufficient for both conditions of the theorem to
hold. In this case we note that the bracket also satisfies the right derivation rule
\(\{fg, h\} = (-1)^{(h+1)g} \{f, h\} g + f \{g, h\}\). This is a simple consequence of (3) and
(1). All versions of the super Jacobi identity will also hold, since they derive
from the super anticommutativity and any version of the super Jacobi identity.

Now we can turn to the proof of the results of [4]. Let us begin with a chiral
algebra \(V\). Recall that the structure of a chiral algebra on a linear superspace
\(V\) is determined by a linear map which assigns to each element \(\varphi \in V\) a formal
power series \(\varphi(z) = \sum \varphi(n) z^{-n-1}\), where the \(\varphi(n)\) are linear operators on \(V\) and
\(\varphi(0) = \varphi\). The elements of \(V\) are interpreted as states, the series \(\varphi(z)\) as a field.
It is assumed that there is an operator expansion (OPE) of the product of two
fields, and that OPE satisfies the associativity condition. It is also assumed that
there are even elements 1 and \(L\) in \(V\) such that the corresponding fields \(1(z)\) and
\(L(z)\) can be considered as the unit operator and the energy-momentum tensor,
in other words, these operators satisfy the appropriate OPE conditions.
We consider the case when $V$ is $\mathbb{Z}$ graded, $V = \sum V^k$. We say that an element $\varphi \in V^k$ and its corresponding field $\varphi(z)$ have the weight (conformal dimension) $k$, denoting the weight of $\varphi$ by $\Delta_\varphi$. We assume that for $\varphi \in V^k$ the operator $\varphi(n)$ acts from $V^r$ to $V^{r-n+k}$. The weight of 1 is equal to zero, the weight of $L$ is equal to 2. If $\varphi$ is a field of weight $k$, it is convenient to introduce the notation $\varphi_n = \varphi(n+k-1)$. Then the field $\varphi$ can be represented in the form $\varphi = \sum \varphi_n z^{-n-k}$. We do not give a detailed definition of the chiral algebra, referring to [4] or [2].

Let us write down, however, some relations that will be used later:

$$L(z) = \sum L_n z^{-n-2}$$ (20)

where the $L_n$ are the generators of the Virasoro algebra.

$$(L_0 \psi)(z) = \Delta_\psi \psi(z), \quad (L_{-1} \psi)(z) = \frac{d\psi(z)}{dz}$$ (21)

$$\psi(1) = \psi_0 = 1,$$
$$[\psi(m), \varphi(n)] = \sum_{i \in \mathbb{N}} \binom{m}{i} (\psi(i) \varphi)(m+n-i)$$ (22)

In particular,

$$[\psi(0), \varphi(n)] = (\psi(0), \varphi)(n)$$ (23)
$$[\psi(1), \varphi(n)] = (\psi(1) \varphi)(n) + (\psi(0) \varphi)(n+1)$$ (24)

One can derive (22) from the "Jacobi identity" of [4], by taking $f(z, w) = z^m w^n$ (see [3]). It follows from (23) that

$$[\psi(0), \varphi(z)] \xi = (\psi(0) \varphi)(z) \xi$$ (25)

In other words, $\psi(0)$ acts as a (super) derivation on the product $\psi(z) \xi$. Analogously, we get from (24) that $\psi(1)$ acts as a second order derivation of $\psi(z) \varphi$. 7
More precisely,

\[ [\psi(1), \varphi(z)] \xi - (\psi(1) \varphi)(z)(\xi) = z(\psi(0) \varphi)(z)\xi \]  

(26)

Let us suppose now that an odd operator \( Q \) satisfying \( Q^2 = 0 \), acts on the chiral algebra \( V \), and that \( Q \) satisfies the condition

\[ [Q, \psi(z)] \varphi = (Q \psi)(z) \varphi \]  

(27)

Usually \( Q \) can be represented in the form \( Q = \oint \sigma(z) \, dz = \sigma_0 \), where \( \sigma(z) \) is an odd field of weight 1; then (27) follows from (25). Suppose that there is an odd field \( b \) of weight 2 obeying

\[ L(z) = [Q, b(z)] \]  

(28)

\[ [b(z), b(z')] = 0 \]

Hence \([L, Q] = 0\) and \( Q \) preserves the weight. We say then that \( V \) is a topological chiral algebra.

It is well known that one can obtain a topological chiral algebra from any chiral algebra with central charge 26 by adding ghosts, and from any \( N = 2 \) superconformal field theory.

It is proved in [4] that the superspace \( H = \ker Q/ImQ \) (the homology of \( Q \)), can be equipped with the structure of a BV-algebra. We will give a simple proof of this result. Let us first note that, as follows from (27), for \( \psi, \varphi \in \ker Q \), the product \( \psi(z) \varphi \) satisfies \( Q(\psi(z) \varphi) = 0 \), and therefore generates an element \( \psi \circ \varphi \) of the space \( H \left[z, z^{-1}\right] \) of formal Laurent series with coefficients in \( H \). This element is unchanged if we replace \( \psi \) and \( \varphi \) by \( \psi + Q \alpha \) and \( \varphi + Q \beta \) (again, this follows from (27)). Using (28) and (21), we obtain

\[ \frac{d}{dz} (\psi(z) \varphi) = (L_{-1} \psi)(z) \varphi = ((Q, b_{-1}) \psi)(z) \varphi = (Q b_{-1} \psi)(z) \varphi = Q((b_{-1} \psi)(z) \varphi) \]  

(29)
It follows from (29) that \( \frac{d}{dz}(\psi \circ \varphi) = 0 \), and therefore \( \psi \circ \varphi \) can be considered as an element of \( H \). Thus we can define a product \( a \circ b \) of elements \( a, b \in H \), as the homology class of \( \psi \circ \varphi \), where \( \psi \) and \( \varphi \) are representatives of the homology classes of \( a \) and \( b \), resp. It is easy to verify that the product \( a \circ b \) coincides with the dot product \( a \cdot b \) defined in [4], as the homology class of

\[
\int_0^1 \psi(z)e^{2\pi it} \varphi dt
\]

where \( \psi \in a \) and \( \varphi \in b \). To prove this statement, we note that \( \int_0^1 \psi(z)e^{2\pi it} \varphi dt \) is homologous to \( \psi(z) \varphi \). This is a particular case of a more general assertion: if \( Q\alpha(t) = 0 \) for \( 0 \leq t \leq 1 \) and \( \alpha'(t) = Q\beta(t) \), then \( \int_0^1 \alpha(t) dt - \alpha(0) = Q \int_0^1 \beta(t) dt \), and therefore \( \int_0^1 \alpha(t) dt \) is homologous to \( \alpha(0) \). (Of course, we have to assume that \( \beta(t) \) is an integrable function of \( t \).) To apply this assertion to the case at hand we take \( \alpha(t) = \psi(z)e^{2\pi it} \varphi \); it follows from (29) that then one can take \( \beta(t) = (b - 1 \psi)(ze^{2\pi it}) \varphi \). In this manner we have shown that the homology class of \( \int_0^1 \psi(z)e^{2\pi it} \varphi dt \) coincides with the homology class of \( \psi(z) \varphi \), in particular, it does not depend on \( z \). We see that this homology class coincides with \( \psi(z) \varphi \), because \( \psi(z) \varphi \) can be obtained by means of substitution of \( z = 1 \) into \( \int_0^1 \psi(z)e^{2\pi it} \varphi dt \).

The multiplication \( (a, b) \mapsto a \circ b \) in \( H \) is both associative and distributive. The distributivity is evident, the associativity follows from the associativity of OPE. To be more exact, OPE is not associative in the standard sense; the associativity is expressed by the equation

\[
(u(z_1)v)(z_2)w = u(z_1 + z_2)(v(z_2)w)
\]

If \( u, v, w \) are representatives of the homology classes \( \varphi, \psi, \zeta \) resp., then the left hand side belongs to the homology class \( (\varphi \circ \psi) \circ (\zeta) \), and the right hand side belongs to the homology class \( \varphi \circ (\psi \circ \zeta) \). (To check this one should use (29).)
Therefore, associativity follows from (31).

To prove super commutativity of the product in \( H \) we use the equivalence of our definition of this product and the definition in [4], and refer the reader to the six line proof in [4]. Later we will give a geometric interpretation of the product, which will make the super commutativity almost obvious. Thus we have provided \( H \) with the structure of a super commutative, associative algebra.

Let us note that the operator \( b_0 \) can be considered as an operator acting on \( H \). Really, using 

\[
L_0 = [Q, b_0]
\]

we obtain from \( Q\psi = 0 \), and \( L_0\psi = \lambda\psi \), with \( \lambda \neq 0 \), that \( \psi = Q(\lambda^{-1}b_0\varphi) \). This means that each homology class \( x \in H \) has a representative \( \xi \in A \) satisfying 

\[
L_0\xi = 0.
\]

Using (32) once more we see that \( Q(b_0\xi) = 0 \) and therefore one can define \( b_0x \) as the homology class of \( b_0\xi \). It is evident that \( b_0^2 = 0 \). Using (26) we see that \( b_0 = b_{(1)} \) acts as a second order derivation. Therefore we can apply the theorem above, with \( b_0 \) as \( \Delta \). In this manner we obtain the structure of a BV-algebra on \( H \). It follows from (1) and (24) that the element \( \{x, y\} \), where \( x, y \in H \) can be written as the homology class of 

\[
z(b_{-1}\xi)(z)\eta
\]

where \( \xi \) and \( \eta \) are representatives of the classes \( x \) and \( y \) satisfying \( L_0\xi = 0 \) and \( L_0\eta = 0 \). Integrating (23) over the contour \( |z| = 1 \), we get that (23) is homologous to \( Res_z(b_{-1}\xi)(z)\eta \); this form of the bracket is obtained in [4].

Note that the assumption about integrality of weight was used in our proof of the coincidence of our definition of the product in \( H \) with the definition in [4]. (If the weight is not integral, the integrand in (30) is multivalued.) However, we have seen that with the calculations in \( H \) we can restrict ourselves to elements
of \( A \) having zero weight. Thus we can exclude fields with non-integral weight from our discussion.

Let us consider now a conformal algebra \( A \) with left and right Virasoro subalgebras. This means that to every element \( \varphi \in A \), we assign a field \( \varphi(z, \overline{z}) \), considered as a formal series

\[
\varphi(z, \overline{z}) = \sum \varphi_{m,n} z^{-(m+1)} \overline{z}^{-(n+1)}
\]

where the \( \varphi_{m,n} \) are operators acting on \( A \), and \( \varphi_{(0,0)} = \varphi \). The numbers \( m, n \) in \( (34) \) are not necessarily integers. We assume that for fields \( \varphi(z, \overline{z}), \psi(z, \overline{z}) \) corresponding to the elements \( \varphi, \psi \) in \( A \), one can write OPE

\[
\varphi(z + \zeta, \overline{z} + \overline{\zeta}) \circ \psi(z, \overline{z}) = \sum \zeta^m \overline{\zeta}^n C_{m,n}(z, \overline{z})
\]

The coefficients of OPE are given by the formula

\[
C_{m,n}(z, \overline{z}) = (\varphi_{(-m-1, -n-1)}(z, \overline{z})).
\]

We assume the associativity of OPE. We suppose also that there are elements \( 1, L, \overline{L} \) in \( A \) such that \( L(z) \) and \( \overline{L}(\overline{z}) \) are holomorphic and anti-holomorphic parts of the energy-momentum tensor resp., and that \( 1(z) \) is the unit operator. Standard OPE’s for products of these fields and other fields are assumed. Therefore representing \( L(z) \) and \( \overline{L}(\overline{z}) \) as \( L(z) = \sum L_n z^{-n-2} \) and \( \overline{L}(\overline{z}) = \sum \overline{L}_n \overline{z}^{-n-2} \), we obtain two commuting Virasoro algebras acting on \( A \) (with generators \( L_n \) and \( \overline{L}_n \), resp.) We have

\[
(L_{-1} \varphi)(z, \overline{z}) = \frac{\partial \varphi(z, \overline{z})}{\partial z}
\]

\[
(\overline{L}_{-1} \varphi)(z, \overline{z}) = \frac{\partial \varphi(z, \overline{z})}{\partial \overline{z}}.
\]

We suppose that the space \( A \) can be represented as a direct sum of subspaces \( A_{m,n} \) consisting of eigenvectors of the commuting operators \( L_0, \overline{L}_0 \), with
eigenvalues \( m \) and \( n \), resp; the pair \((m, n) = (\Delta_\psi, \Delta_\phi)\) is called a weight (or conformal dimension) of \( \varphi \in A_{m,n} \). If \( \varphi \in A_{k,l} \), then the operator \( \varphi_{(m,n)} \) has to act from \( A_{r,s} \) to \( A_{r-m+k,s-n+l} \). It follows from our assumptions that

\[
(\mathcal{L}_0 \varphi)(z, \overline{z}) = \Delta_\psi \varphi(z, \overline{z})
\]

(39)

Let us emphasize that we don’t assume that the eigenvalues of \( \mathcal{L}_0 \) and \( \overline{\mathcal{L}}_0 \) are integers, however \( \Delta_\psi - \overline{\Delta}_\phi \) must be integral to guarantee that the fields are single valued.

We say that a conformal algebra \( A \) is a topological conformal algebra if there are odd elements \( \sigma, \overline{\sigma}, b, \overline{b} \) in \( A \) having weights \((1,0), (0,1), (2,0), (0,2)\) resp. and satisfying the conditions

\[
\mathcal{T}_n \sigma = -L_n \mathbf{\overline{\sigma}} = \mathcal{T}_n b = -L_n \mathbf{\overline{b}} = 0.
\]

(40)

(In other words, we have holomorphic fields \( \sigma, b \) and anti-holomorphic fields \( \overline{\sigma}, \overline{b} \).) We define operators \( Q, \mathcal{Q} \) by the formulas \( Q = \oint \sigma(z) dz = \sigma_{(0,0)}, \mathcal{Q} = -\oint \overline{\sigma}(\overline{z}) \overline{dz} = \overline{\sigma}_{(0,0)} \) and suppose that \( Q^2 = \mathcal{Q}^2 = 0, [Q, \mathcal{Q}(z)] = 0, [\mathcal{Q}, \mathcal{Q}(\overline{z})] = 0, \mathcal{L}(z) = [Q, b(z)], \mathcal{T}(\overline{z}) = [\mathcal{Q}, \mathbf{\overline{b}}(\overline{z})] \). Standard OPE for \( b, \overline{b}, L \) and \( \mathcal{T} \) are assumed.

Let us consider the homology \( H = \ker(Q + \mathcal{Q})/\text{Im}(Q + \mathcal{Q}) \) constructed by means of the operator \( Q + \mathcal{Q} \) acting on \( A \). One can define a multiplication in \( H \) by repeating the construction given in the chiral case.

It is possible to give a geometric interpretation of this multiplication. Let us take as a starting point an axiomatic approach to conformal field theory. Consider a closed, oriented Riemann surface \( \Sigma \), i.e. a complex curve, and \( m \) holomorphic and \( n \) anti-holomorphic maps of a disc into this surface. In axiomatic field theory, these data permit us to construct a map \( A^{\otimes m} \rightarrow A^{\otimes n} \). (Here, \( A \)
stands for the space of states.) In particular, if $\Sigma = S^2$, $m = 2$ and $n = 1$, we obtain a map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, which can be interpreted as a multiplication in $\mathcal{A}$. For every conformal algebra $\mathcal{A}$ one can construct a conformal field theory at least on the surfaces of genus zero. The multiplication $(u, v) \to u(z)v$ coincides with the map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ for a specific choice of the maps of the disc into $S^2$. If $\mathcal{A}$ is a topological chiral algebra, it is easy to check that the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ corresponding to an arbitrary choice of discs in $S^2$ determines a multiplication in the homology group $H$, and this multiplication does not depend on the choice of the discs. (The proof is similar to the proof of independence of the homology class of $u(z)v$ on the choice of $z$.) The geometric interpretation makes obvious the associativity and super commutativity of this multiplication. A slight modification of the considerations applied in the chiral case shows that the operator $b_0 + \overline{b}_0$ determines an odd second order derivation in $H$ and that $(b_0 + \overline{b}_0)^2 = 0$. This means that this operator specifies the structure of a BV algebra in $H$. Thus we have shown the following theorem:

Let $\mathcal{A}$ be a topological conformal algebra. Then the corresponding homology $H$ of $\mathcal{A}$ with respect to $Q + \overline{Q}$ naturally has the structure of a BV-algebra.

The constructions above are closely related to the constructions of string field theory. We considered here the operations in the homology group $H$. One can consider the corresponding operations in $\mathcal{A}$, but the properties of the operations in $H$ are valid in $\mathcal{A}$ only up to homotopy. The consideration of higher homotopies in $\mathcal{A}$ should lead to interesting algebraic structures, including the structure of a homotopy Lie algebra, as was constructed in [11] (see also [7]). However the most interesting structures are connected with the relative homology $H_k = \ker(Q + \overline{Q}) \cap \mathcal{A}_k/(Q + \overline{Q})\mathcal{A}_k$ where $\mathcal{A}_k$ denotes the subspace of
\[ \mathcal{A} \text{ consisting of all elements } \varphi \in \mathcal{A} \text{ satisfying} \]
\[
    (L_k - \overline{L}_k)\varphi = 0 \quad (41)
\]
\[
    (b_k - \overline{b}_k)\varphi = 0
\]

It is well known that one can construct string amplitudes considering \( \mathcal{A} \) as a conformal background (see [10]). (More precisely, to construct string amplitudes one has to extend \( \mathcal{A} \) to a topological conformal field theory, defined on surfaces of arbitrary genus.) Then \( H_0 \) is closely related to the space of physical states. Note that we don’t impose any conditions on the ghost number of states in \( H_0 \). (Moreover, we don’t need to introduce the notion of ghost number.) It would be interesting to prove that \( H_0 \) can be equipped with the structure of a BV-algebra. This can be done in certain circumstances (see [8], [11]), but we don’t know a general proof. However, slight modifications of the considerations above can be used to introduce a BV-algebra structure in \( H_{-1} \).

We are indebted to M. Kontsevich, J. Stasheff, E. Verlinde, G. Zuckerman, and B. Zwiebach for useful discussions.

**References**

[1] Batalin, I. & Vilkovisky, G.: *Gauge Algebra and Quantization*. Physics Letters, 102B, 27 (1981).

[2] Frenkel, I., Lepowsky, J., Meurmann, A.: *Vertex Operator Algebras and the Monster*. Boston: Academic Press (1988)

[3] Gerstenhaber, M.: *The Cohomology Structure of an Associative Ring*. Annals of Mathematics (2), 78 (1963).

[4] Lian, B., Zuckerman, G.: *New Perspectives on the BRST-Algebraic Structure of String Theory*. Preprint.
[5] Schwarz, A.:The Geometry of Batalin-Vilkovisky Quantization. Commun. Math. Phys. (To appear).

[6] Schwarz, A.:Semiclassical Approximation in Batalin-Vilkovisky Formalism. Preprint.

[7] Stasheff, J.:Towards a Closed String Field Theory: Topology and Convolution Algebra. UNC-MATH 90/1.

[8] Verlinde, E.:The Master Equation of 2D String Theory. Preprint. IASSNS-HEP-9214.

[9] Witten, E.:A Note on the Antibracket Formalism. Mod. Phys. Lett. A5,487(1990).

[10] Witten, E.:Chern-Simons Gauge Theory as a String Theory. Preprint. IASSNS-HEP-92145.

[11] Zwiebach, B.:Closed String Field Theory: Quantum Action and the BV Master Equation. Preprint. IASSNS-HEP-92141.