New private types for the cubic-quartic optical solitons in birefringent fibers in its four forms of nonlinear refractive index

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Abstract
From the point of view of the extended simple equation method, multiple new private distinct types for the cubic-quartic optical soliton birefringent fibers with four forms nonlinear refractive index have been extracted. The suggested model has vital effective effect in all modern telecommunications process. The suggested method which has invited for this purpose examined previously for many other nonlinear problems arising in various branches of science and continuously gives good results. We will implement this method to extracting multiple new private types for the cubic-quartic soliton with its four different forms of the NLSE which are, the cubic-quartic in polarization-preserving fibers with the kerr-low nonlinearity, quadratic-cubic law nonlinearity, parabolic law nonlinearity and non-local law nonlinearity. Actual comparison between our achieved results and that realized previously by other authors has been established.

Keywords The cubic-quartic optical solitons in birefringent fibers · The extended simple equation method · The optical soliton solutions

1 Introduction
The propagations of waves in optical fiber is one of important operations in all telecommunications process, specially the propagations in birefringence fiber and improve this process by develop the model governed this process will give future studies to this phenomena. Some studies have been established in the last few decays through good efforts via little number of authors to discuss the cubic-quartic of the birefringence for which the optical property of a material having a refractive index that depends on the polarization and propagation direction. Recently, it is important to study the transition of light through optical fibers which responsible the propagation of waves in all modern telecommunications,
The birefringence fiber is a material which possesses refractive index related to the polarization and propagation direction of light. This optical property is responsible for the phenomenon of double refraction whereby a ray of light, when incident upon a birefringent material, is split by polarization into two rays taking slightly different paths. Refractive indices due to orthogonally polarized materials vary effectively in the case of fiber birefringence in optical modes. Moreover, it often quantified as the maximum difference between refractive indices exhibited by the material in general. Crystals with non-cubic crystal structures are often birefringent, as are plastics under mechanical stress. In related subject Birefringence is used in many optical devices. Liquid–crystal displays, which are used utilized in medical diagnostics. In fact the double refraction phenomenon depends on Birefringence, because when a light ray incident upon a birefringent material it will split into two rays which have slightly various paths under the polarization property. In the literature this effect has been observed in calcite, a crystal having one of the strongest birefringence by the Danish scientist Rasmus Bartholin (1669). Moreover, Augustin-Jean Fresnel at 19th described the phenomenon in terms of polarization he can understand that the light can be considered as a wave with field components in transverse polarization. In addition the birefringence for which the optical property of a material having a refractive index that depends on the polarization and propagation direction of light has been discussed by Neves (1998); Amos 2013). Some studies have been demonstrated to study various forms of the fractional and non-fractional NLSE which describes important nonlinear problems in different branches of sciences see for example Cevikel et al. (2014) who applied the functional variable method for finding the periodic wave and solitary wave solutions of the generalized Zakharov equation and higher-order nonlinear Schrödinger equations, Cevikel and Bekir (2013) who constructed periodic and soliton solutions for the (2+1)-dimensional Davey-Stewartson (DS) equations by using the sine–cosine, tanh-coth, and exp-function method, Bekir et al. (2014) who obtained the 1-soliton solutions of the (2+1)-dimensional Boussinesq equation and the Camassa–Holm–KP equation by using a solitary wave ansatz method, Güner, et al. (2013) who extracted the dark (topological) soliton solutions of the Kudryashov-Sinelshchikov and Jimbo–Miwa equations by using the sine–cosine method, Cevikel (2018) who extracted new exact solutions of the space–time fractional KDV Burgers and nonlinear fractional Foam-Drainage equation by using the fractional sub-equation method and the first integral method, Bekir et al. (2017) who applied the Exp-function method to extract the exact solutions of the time fractional Fitzhugh-Nagumo equation, the time fractional KDV equation, Bekir et al. (2017) who established new exact solutions for fractional differential-difference equations in the sense of modified Riemann–Liouville derivative by using (G'/G)-expansion method, Aksoy et al. (2019) who applied the Kudryashov method to find exact solutions he space–time fractional Symmetric Regularized Long wave equation and the space–time fractional generalized Hirota–Satsuma coupled KdV equation, Cevikel and Aksoy (2021) who applied the generalized Kudryashov method extract a certain type of soliton solutions to the time-fractional Chan-Allen equation, the space–time fractional Klein–Gordon equation and the space–time fractional ZK-BBM equation, Kudryashov (2020) who achieved highly dispersive solitary wave solutions of perturbed nonlinear Schrödinger equations, Shehata et al. (2020) who extracted the optical solitons to a perturbed Gerdjikov-Ivanov equation using two different techniques and Bekir and Zahran (2021a) who achieved new visions of the soliton solutions to the modified nonlinear Schrödinger equation. Recently, big numbers of authors discussed various forms of the cubic-quartic NLSE through good efforts to extract the soliton solutions via different techniques (Bekir and Zahran 2021a, 2020; Biswas et al. 2017a, 2017b, 2017c; Bansal et al. 2018;
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Biswas and Arshed 2018; Das et al. 2019; Gonzalez-Gaxiola et al. 2020; Kohl et al. 2019). Relevant studies to other various forms of Schrödinger equation have been demonstrated see for example Hosseini et al. (2021a) who applied the modified Kudryashov method and symbolic computations are adopted to successfully retrieve optical solitons of a high-order nonlinear Schrödinger equation in a non-Kerr law media with the weak non-local nonlinearity and extracting some real and imaginary parts of the model using a wave variable transformation, Hosseini et al. (2021b) who studies a nonlinear (2+1)-dimensional evolution model describing the propagation of nonlinear waves in Heisenberg ferromagnetic spin chain system and a number of solitons and Jacobi elliptic function solutions to the Heisenberg ferromagnetic spin chain equation are formally derived, Hosseini et al. (2021c) who studied the high-order nonlinear Schrödinger equations in non-Kerr law media with different laws of nonlinearities using new Kudryashov method and extracted the soliton solutions of this model. The propagations of waves in optical fiber is one of important operations in all telecommunications process, specially the propagations in birefringence fiber and improve this process by develop the model governed this process will give future studies to this phenomena. The propagations of waves in optical fiber is one of important operations in all telecommunications process, specially the propagations in birefringence fiber and improve this process by develop the model governed this process will give future studies to this phenomena. Some studies have been established in the last few decays through good efforts via little number of authors to discuss the cubic-quartic of the birefringence for which the optical property of a material having a refractive index that depends on the polarization and propagation direction. The recent articles which investigate the Kerr-law, polynomial law, parabolic law, power law, dual-power law, Quadratic Law, Non-local law of this model have been demonstrated (Biswas et al. 2017a, 2017d; Hosseini et al. 2021c; Amsa et al. 2018; Wazwaz and Xu 2020; Yildirim et al. 2020a).

Our aimed apply the ESEM which is the famous ansatze method that governed by the auxiliary equation \( \phi'(\zeta) = a_0 + a_1 \phi + a_2 \phi^2 \) to achieve new types of optical solitons to cubic-quartic NLSE with its special forms which are mentioned in the abstract. The suggested method has been examined previously for many other NLPDE and continuously gives good result see for example (Shehata et al. 2020; Kohl et al. 2019; Bekir and Zahran 2020; Yildirim et al. 2020b; Lu et al. 2017; Zhao et al. 2013; Bekir and Zahran 2021a, b).

This article is organized as follow, the introduction section which gives short survey of the NLSE forms as well as the methods that used to solve it, in section two we will give the description of the ESEM and its application to find the soliton solution of the suggested model. In section three the application of the ESEM schema to find the optical solitons of the cubic-quartic for the Kerr-Low NLSE in polarization-preserving fibers. In section four, the application the ESEM schema to extract the optical solitons of the cubic-quartic for the quadratic-low NLSE in polarization-preserving fibers. In section five, the application the ESEM schema to extract the optical solitons of the cubic-quartic for the parabolic-low NLSE in polarization-preserving fibers. In section six the application the ESEM schema to extract optical solitons of the cubic-quartic for the Non-local-low NLSE in polarization-preserving fibers. In section seven brief conclusion of our work has been established.
2 Description of the ESEM

Any nonlinear evolution equation can be written in the form,

\[ R(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{tt}, ...) = 0. \]  \hspace{1cm} (1)

where \( R \) is a function of \( \psi(x,t) \) and its partial derivatives that involves the highest order derivatives and nonlinear terms, according to the transformation \( \psi(x,t) = \psi(\zeta), \ \zeta = x - C_0 t \). Eq. (1) can be reduced to the following ODE:

\[ S(\psi, \psi', \psi'', .......) = 0 \]  \hspace{1cm} (2)

where \( S \) is a function in \( \psi(\zeta) \) and its total derivatives, while \( \zeta' = \frac{d}{d\zeta} \).

The constructed solution according to this method is

\[ \psi(\zeta) = \sum_{i=-N}^{N} A_i \varphi^i(\zeta). \]  \hspace{1cm} (3)

where the positive integer \( N \) in Eq. (3) can be located by balancing the highest order derivative term and the nonlinear term, while the arbitrary constants \( A_i \) could be calculated later, the function \( \varphi(\zeta) \) satisfies the following new ansatz equation

\[ \varphi'(\zeta) = a_0 + a_1 \varphi + a_2 \varphi^2 \]  \hspace{1cm} (4)

where \( a_0, a_1 \) and \( a_2 \) other arbitrary constants which admit these two cases;

1. If \( a_1 = a_3 = 0 \) it will transform to the Riccati equation (Biswas et al. 2017b, 2017c), which has the following solutions;

\[ \varphi(\zeta) = \frac{\sqrt{a_0 a_2}}{a_2} \tan(\sqrt{a_0 a_2}(\zeta + \zeta_0), a_0 a_2 > 0 \]  \hspace{1cm} (5)

\[ \varphi(\zeta) = \frac{-\sqrt{-a_0 a_2}}{a_2} \tanh(\sqrt{-a_0 a_2}(\zeta - \frac{\rho \ln \zeta_0}{2}), a_0 a_2 < 0, \zeta > 0, \rho = \pm 1 \]  \hspace{1cm} (6)

2. If \( a_0 = a_3 = 0 \), it will transform the Bernoulli equation (Biswas et al. 2017c), which has the following solutions;

\[ \varphi(\zeta) = \frac{a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 - a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 > 0 \]  \hspace{1cm} (7)

\[ \varphi(\zeta) = \frac{-a_1 \text{Exp}[a_1(\zeta + \zeta_0)]}{1 + a_2 \text{Exp}[a_1(\zeta + \zeta_0)]}, a_1 < 0 \]  \hspace{1cm} (8)

And the general solution to ansatz Eq. (4) is as follows:
\[ \varphi(\zeta) = -\frac{1}{a_2} \left( a_1 - \sqrt{4a_1a_2 - a_1^2} \tan \left( \frac{\sqrt{4a_1a_2 - a_1^2}}{2} (\zeta + \zeta_0) \right) \right), \quad 4a_1a_2 > a_1^2, \ a_2 > 0, \] (9)

\[ \varphi(\zeta) = \frac{1}{a_2} \left( a_1 + \sqrt{4a_1a_2 - a_1^2} \tanh \left( \frac{\sqrt{4a_1a_2 - a_1^2}}{2} (\zeta + \zeta_0) \right) \right), \quad 4a_1a_2 > a_1^2, \ a_2 < 0. \] (10)

where \( \zeta_0 \) denotes to integration constancy.

Finally, via inserting Eq. (3) into Eq. (4) and equating the coefficients of different powers of \( \varphi' \) to zero, we get a system of algebraic equations, by solving it we obtain the values of the unknown variable mentioned in these equations. Moreover, inserting these achieved variables into Eq. (4) then the aimed solutions have been extracted.

### 3 The Cubic-Quartic NLSE for the Kerr-Low nonlinearity using the ESEM

In the framework of Hosseini et al. (2021a, b) the cubic-quartic for the Kerr-low NLSE in polarization-preserving fibers can be proposed in the form,

\[ iQ_t + iaQ_{xxx} + bQ_{xxxx} + cQ^{2m}Q = 0 \] (11)

where \( a, b \) and \( c \) denote respectively to the 3-th order dispersion index, the 4-th order dispersion index, Kerr-Low refractive coefficient while \( i = \sqrt{-1} \).

The cubic-quartic NLSE in birefringent fiber for Kerr-Low are

\[ iu_t + ia_1u_{xxx} + b_1u_{xxxx} + [c_1|u|^2 + d_1|v|^2]u = 0 \]
\[ iv_t + ia_2v_{xxx} + b_2v_{xxxx} + [c_2|v|^2 + d_2|u|^2]v = 0 \] (12)

In the framework of the ESEM the solution is

\[ u(x, t) = \psi_1(\zeta) e^{i\eta_1(x,t)}, \ \zeta = k_1x + w_1t, \ \eta = q_1x + \delta_1t \] (13)

\[ v(x, t) = \psi_2(\zeta) e^{i\eta_2(x,t)}, \ \zeta = k_2x + w_2t, \ \eta = q_2x + \delta_2t \] (14)

Hence,

\[ u_t = i\delta_1\psi_1 e^{i\eta_1} + w_1\psi_1' e^{i\eta_1} \] (15)
\[ u_x = iq_1\psi_1 e^{i\eta_1} + k_1\psi_1' e^{i\eta_1} \] (16)
\[ u_{xx} = -q_1^2\psi_1 e^{i\eta_1} + 2ik_1q_1\psi_1' e^{i\eta_1} + k_1^2\psi_1'' e^{i\eta_1} \] (17)
\[ u_{xxx} = -i q_1^4 \psi_1 e^{in_1} - 3k_1 q_1^3 \psi'_1 e^{in_1} + 3i q_1 k_1^2 \psi''_1 e^{in_1} + k_1^3 \psi'''_1 e^{in_1} \]  
(18)

\[ u_{xxxx} = q_1^4 \psi_1 e^{in_1} - 4i k_1 q_1^3 \psi'_1 e^{in_1} - 6k_1 q_1^2 \psi''_1 e^{in_1} + 4i q_1 k_1^2 \psi'''_1 e^{in_1} + k_1^3 \psi''''_1 e^{in_1} \]  
(19)

\[ |\mu| = \psi_1^2 \]  
(20)

\[ v_i = i \delta_{2} \psi_2 e^{in_2} + w_2 \psi'_2 e^{in_2} \]  
(21)

\[ v_x = i q_2 \psi_2 e^{in_2} + k_2 \psi''_2 e^{in_2} \]  
(22)

\[ v_{xx} = -q_2^2 \psi_2 e^{in_2} + 2i k_2 q_2 \psi'_2 e^{in_2} + k_2^2 \psi''_2 e^{in_2} \]  
(23)

\[ v_{xx} = -i q_2^2 \psi_2 e^{in_2} - 3k_2 q_2 \psi'_2 e^{in_2} + 3i q_2 k_2 \psi''_2 e^{in_2} + k_2^3 \psi'''_2 e^{in_2} \]  
(24)

\[ v_{xxxx} = q_2^4 \psi_2 e^{in_2} - 4i k_2 q_2 \psi'_2 e^{in_2} - 6k_2 q_2 \psi''_2 e^{in_2} + 4i q_2 k_2 \psi'''_2 e^{in_2} + k_2^3 \psi''''_2 e^{in_2} \]  
(25)

\[ |\nu| = \psi_2^2 \]  
(26)

Via inserting the relations (13) and (15–20) into the first part of Eq. (12) and the relations (14) and (21–26) into the second part of Eq. (12), the following real and imaginary parts must be emerged respectively,

\[ \text{Re} \Rightarrow b_j k_j^4 \psi_j'''' = (3a_j q_j^3 + 6b_j q_j^2 k_j + b_j q_j^3) \psi_j'''' + (b_j q_j^3 + 6a_j q_j^2 - \delta_j) \psi_j + (c_j + d_j) \psi_j^3 = 0. \]  
(27)

\[ \text{Im} \Rightarrow (a_j k_j^4 + 4b_j q_j^3 \psi_j'''' - (3k_j a_j q_j^2 + 4k_j b_j q_j^3 - w_j) \psi_j' = 0. \]  
(28)

Equation (28) implies \( a_j = -4b_j q_j, \ w_j = 3k_j a_j q_j^2 + 4k_j b_j q_j^3, \) hence under this constrain Eq. (27) will be converted to,

\[ b_j k_j^4 \psi_j'''' + 6b_j q_j^2 k_j^2 \psi_j'' - (3b_j q_j^3 + \delta_j) \psi_j + (c_j + d_j) \psi_j^3 = 0. \]  
(29)

Now, via integrating Eq. (29) with respect to \( \xi \) it becomes,

\[ b_j k_j^4 \psi_j'''' + 6b_j q_j^2 k_j^2 \psi_j' = \frac{1}{2} (3b_j q_j^4 + \delta_j) \psi_j^2 + \frac{1}{4} (c_j + d_j) \psi_j^4 = 0. \]  
(30)

Equation (30) represents either the first or the second parts of Eq. (12) which are the same, for similarity we will implement the solution for the first part \( j = 1 \) which is

\[ b_1 k_1^4 \psi_1'''' + 6b_1 q_1^2 k_1^2 \psi_1' = \frac{1}{2} (3b_1 q_1^4 + \delta_1) \psi_1^2 + \frac{1}{4} (c_1 + d_1) \psi_1^4 = 0. \]  
(31)

Via balancing \( \psi_1'''' \), \( \psi_1^4 \) at Eq. (31) leads to \( 4N = N + 3 \) which implies that \( N = 1 \), hence the solution is
where \( \phi' = a_0 + a_1 \phi + a_2 \phi^2 + a_3 \phi^3 \).

**Case 1:** The first family, in which \( a_1 = a_3 = 0 \Rightarrow \phi' = a_0 + a_2 \phi^2 \), consequently

\[
\psi'_{1} = -\frac{a_0 A_{-1}}{\phi^2} + A_1 a_0 + A_1 a_2 \phi^2 - a_2 A_{-1}
\]  

(33)

\[
\psi''_{1} = -\frac{6a_0^2 A_{-1}}{\phi^4} - \frac{8a_2^2 A_{-1}}{\phi^2} - 2A_{-1}a_0 a_2^2 + 2A_1 a_2 a_0^2 + 8A_1 a_0 a_2^2 \phi^2 + 6A_1 a_2^3 \phi^4
\]  

(34)

Via inserting \( \psi''_{1}, \psi'_{1}, \psi_{1}^2 \) and \( \psi_{1}^4 \) into Eq. (31), collecting and equating the coefficients of various powers of \( \phi' \) to zero leads to system of equations, by solving it we get these results,

\[
a_0 = -\frac{\sqrt{3} A_{-1} q_1}{2k_1 \sqrt{A_1}}, \quad a_2 = \frac{\sqrt{3} A_{-1} q_1}{2k_1 \sqrt{A_{-1}}}, \quad \delta_1 = \frac{3\left( \frac{6\sqrt{3} A_{-1} q_1}{\sqrt{A_{-1}}} - A_1 b_1 q_1^4 \right)}{A_1}, \quad c_1 = \frac{-d_1 A_1^3 - \left( \frac{6\sqrt{3} A_{-1} q_1}{\sqrt{A_{-1}}} \right)}{A_1^3}, \quad A_0 = 0
\]

(35)

\[
a_0 = \frac{\sqrt{3} A_{-1} q_1}{2k_1 \sqrt{A_1}}, \quad a_2 = -\frac{\sqrt{3} A_{-1} q_1}{2k_1 \sqrt{A_{-1}}}, \quad \delta_1 = \frac{3\left( \frac{6\sqrt{3} A_{-1} q_1}{\sqrt{A_{-1}}} - A_1 b_1 q_1^4 \right)}{A_1}, \quad c_1 = \frac{-d_1 A_1^3 + \left( \frac{6\sqrt{3} A_{-1} q_1}{\sqrt{A_{-1}}} \right)}{A_1^3}, \quad A_0 = 0
\]

(36)

\[
a_0 = -\frac{3 q_1^2}{a_2 k_1^2}, \quad \delta_1 = \frac{3\left( 2a_2 b_1 k_1^2 q_1^2 + A_1 b_1 q_1^4 \right)}{A_1}, \quad c_1 = \frac{-d_1 A_1^3 - 24a_2^3 b_1 q_1^4}{A_1^3}, \quad A_{-1} = 0, A_0 = 0
\]

(37)

\[
a_0 = -\frac{3 q_1^2}{a_2 k_1^2}, \quad \delta_1 = \frac{3\left( -2a_2 b_1 k_1^2 q_1^2 + A_{-1} b_1 q_1^4 \right)}{A_{-1}}, \quad c_1 = \frac{d_1 A_1^3 + 24a_2^3 b_1 q_1^4}{A_{-1}^3}, \quad A_1 = 0, A_0 = 0
\]

(38)

These 4-different results will achieve 4-various types of solutions, from which we will plot the extracted solutions for the first and the third results.

Firstly, for the first result Eq. (35) which is

\[
a_0 = -\frac{\sqrt{3} A_{-1} q_1}{2k_1 \sqrt{A_1}}, \quad a_2 = \frac{\sqrt{3} A_{-1} q_1}{2k_1 \sqrt{A_{-1}}}, \quad \delta_1 = \frac{3\left( \frac{6\sqrt{3} A_{-1} q_1}{\sqrt{A_{-1}}} - A_1 b_1 q_1^4 \right)}{A_1}, \quad c_1 = \frac{-d_1 A_1^3 - \left( \frac{6\sqrt{3} A_{-1} q_1}{\sqrt{A_{-1}}} \right)}{A_1^3}, \quad A_0 = 0
\]

This result can be simplified to be,

\[
A_{-1} = 1, A_0 = 0, A_1 = 1, a_0 = -0.9, a_2 = 0.9, \\
k_1 = b_1 = d_1 = q_1 = C_0 = 1, \quad c_1 = -16.6, \quad \delta_1 = -34.2, \quad \zeta_0 = 2, \quad \rho = -1
\]

From the point of view of the proposed method the solution is
\[ \phi(\zeta) = -\frac{a_0a_2}{a_2} \tanh(-\frac{a_0a_2\zeta - \rho \ln \zeta_0}{2}), \quad a_0a_2 < 0, \quad \zeta > 0, \quad \rho = \pm 1 \]  

(39)

\[ \phi(\zeta) = \tanh(0.9\zeta + 0.3) \]

\[ \phi(\zeta) = \tanh(0.9x - 7.2t + 0.3) \]

\[ \psi_1(\zeta) = \frac{A_{-1}}{\phi} + A_0 + A_1 \phi \]

(40)

\[ \psi_1(\zeta) = \coth(0.9x - 7.2t + 0.3) + \tanh(0.9x - 7.2t + 0.3) \]

\[ u(x, t) = \psi_1(\zeta) e^{i(\zeta x + \omega_1 t)}, \quad \zeta = k_1 x + \omega_1 t, \quad \eta_1 = q_1 x + \delta_1 t \]

\[ u(x, t) = (\coth[0.9x - 7.2t + 0.3] + \tanh[0.9x - 7.2t + 0.3]) e^{i(x - 34.2t)} \]

(41)

\[ \text{Re } u = (\coth[0.9x - 7.2t + 0.3] + \tanh[0.9x - 7.2t + 0.3]) \times \cos(x - 34.2t) \]

(42)

\[ \text{Im } u = (\coth[0.9x - 7.2t + 0.3] + \tanh[0.9x - 7.2t + 0.3]) \times \sin(x - 34.2t) \]

(43)

Secondly, for the third result Eq. (36) which is

\[ a_0 = \frac{-3q_1^2}{a_2^2k_1^3}, \delta_1 = \frac{-3k_1b_1q_1^2(-12a_2 + A_1q_1^2)}{A_1}, c_1 = \frac{-d_1A_1^3 - 24a_2^2b_1q_1^4}{A_1^3}, A_{-1} = 0, A_0 = 0 \]

This result can be simplified to be,

\[ A_{-1} = A_0 = 0, A_1 = 1, a_0 = 0.9 + 0.13i, a_2 = 0.3 + 0.01i, \]

\[ k_1 = b_1 = d_1 = q_1 = c_1 = C_0 = 1, \delta_1 = -4.8 - 0.6i, \zeta_0 = 2, \rho = -1 \]

From the point of view of the proposed method the solution is,

\[ \phi(\zeta) = \frac{\sqrt{-a_0a_2}}{a_2} \tan(\sqrt{-a_0a_2}(\zeta + \zeta_0)), \quad a_0a_2 > 0 \]

(44)

\[ \phi(\zeta) = 5.7 \tan[1.7x - 13.6t + 1.7] \]

\[ \psi(\zeta) = 5.7 \tan[1.7x - 13.6t + 1.7] \]

(45)

\[ u(x, t) = 5.7 \tan[1.7x - 13.6t + 1.7] \times e^{i(x - 4.8 + 0.6t)} \]

(46)

\[ \text{Re } u = 5.7 \tan[1.7x - 13.6t + 1.7] \times e^{0.6t} \times \cos(x - 4.8t) \]

(47)

\[ \text{Im } u = 5.7 \tan[1.7x - 13.6t + 1.7] \times e^{0.6t} \times \sin(x - 4.8t) \]

(48)

Via the same manner, we can plot the other two cases.

**Case 2:** The second family, in which \( a_0 = a_3 = 0 \) \( \Rightarrow \phi' = a_1 \phi + a_2 \phi^2 \), consequently

\[ \psi_1' = A_1a_2 \phi^2 + a_1A_1 \phi - \frac{A_{-1}a_1}{\phi} - A_{-1}a_2, \]

(49)
\[ \psi'''_1 = 6A_1 a_2^3 \varphi^4 + 12a_1 A_1 a_2^2 \varphi^3 + 7A_1 a_1^2 a_2^2 \varphi^2 + A_1 a_1^3 \varphi - A_{-1} a_2 a_1 - \frac{a_1^3 A_{-1}}{\varphi}, \]  

(50)

Via inserting \( \psi''_1, \psi'_1, \psi^2_1 \) and \( \psi^4_1 \) into Eq. (31), collecting and equating the coefficients of various powers of \( \varphi \) to zero implies a system of equations, by solving it many solutions will be achieved from which only 8-results are suitable while the remaining are refused,

\[ a_1 = \frac{2\sqrt{3} q_1}{k_1}, a_2 = \frac{\sqrt{3} A_1 q_1}{A_0 k_1}, \delta_1 = -3 \left( \frac{12\sqrt{3} b_1 k_1 q_1^3 + A_0 b_1 q_1^4}{A_0} \right), c_1 = -d_1 = \frac{72 b_1 q_1^2 \sqrt{3}}{\sqrt{k_1 A_0^3}} , A_{-1} = 0 \]  

(51)

\[ a_1 = \frac{-2\sqrt{3} q_1}{k_1}, a_2 = \frac{\sqrt{3} A_1 q_1}{A_0 k_1}, \delta_1 = -3 \left( \frac{-12\sqrt{3} b_1 k_1 q_1^3 + A_0 b_1 q_1^4}{A_0} \right), c_1 = -d_1 = \frac{72 b_1 q_1^2 \sqrt{3}}{\sqrt{k_1 A_0^3}} , A_{-1} = 0 \]  

(52)

\[ a_1 = \frac{i q_1 \sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1 q_1^4, c_1 = -d_1 \]  

(53)

\[ a_1 = \frac{-i q_1 \sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1 q_1^4, c_1 = -d_1 \]  

(54)

\[ a_1 = \frac{-i q_1 \sqrt{6}}{k_1}, \delta_1 = -3b_1 q_1^4, c_1 = -d_1, A_1 = 0 \]  

(55)

\[ a_1 = \frac{i q_1 \sqrt{6}}{k_1}, \delta_1 = -3b_1 q_1^4, c_1 = -d_1, A_{-1} = 0 \]  

(56)

\[ a_1 = \frac{i q_1 \sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1 q_1^4, c_1 = -d_1, A_{-1} = 0 \]  

(57)

\[ a_1 = \frac{-i q_1 \sqrt{6}}{k_1}, a_2 = 0, \delta_1 = -3b_1 q_1^4, c_1 = -d_1, A_{-1} = 0 \]  

(58)

From which we can get 8-various solutions, we will plot some cases say, the first, the fifth and the eighth results.

For the 1-st result which is

\[ a_1 = \frac{2\sqrt{3} q_1}{k_1}, a_2 = \frac{\sqrt{3} A_1 q_1}{A_0 k_1}, \delta_1 = -3 \left( \frac{12\sqrt{3} b_1 k_1 q_1^3 + A_0 b_1 q_1^4}{A_0} \right), c_1 = -d_1 = \frac{72 b_1 q_1^2 \sqrt{3}}{\sqrt{k_1 A_0^3}} , A_{-1} = 0 \]  

This result can be simplified to be,

\[ A_{-1} = 0, A_0 = 3, A_1 = 1, a_1 = 3.5, a_2 = 0.6, k_1 = q_1 = d_1 = \zeta_0 = 1, c_1 = -5.6, \delta_1 = -23.8 \]

From the point of view of the suggested method the solution is,
\[ \varphi(\zeta) = \frac{3.5 \exp[3.5(x - 8t + 1)]}{1 - 0.6\exp[3.5(x - 8t + 1)]} \]  
(59)

\[ \psi_1(\zeta) = 3 + \frac{3.5 \exp[3.5(x - 8t + 1)]}{1 - 0.6\exp[3.5(x - 8t + 1)]} \]  
(60)

\[ u(x, t) = \psi_1(\zeta) e^{i\eta_1(x, t)}, \quad \zeta = k_1x + w_1t, \quad \eta_1 = q_1x + \delta_1t \]

\[ u(x, t) = \left( 3 + \frac{3.5 \exp[3.5(x - 8t + 1)]}{1 - 0.6\exp[3.5(x - 8t + 1)]} \right) e^{i(x-23.8t)} \]  
(61)

\[ \text{Re } u = \left( 3 + \frac{3.5 \exp[3.5(x - 8t + 1)]}{1 - 0.6\exp[3.5(x - 8t + 1)]} \right) \times \cos(x - 23.8t) \]  
(62)

\[ \text{Im } u = \left( 3 + \frac{3.5 \exp[3.5(x - 8t + 1)]}{1 - 0.6\exp[3.5(x - 8t + 1)]} \right) \times \sin(x - 23.8t) \]  
(63)

For the fifth result which is, \( A_1 = 1, A_0 = 1, A_1 = 0, a_1 = -2.4i, a_2 = 0, k_1 = w_1 = q_1 = d_1 = -1, \delta_1 = -3 \)

\[ \varphi(\zeta) = -2.4i\exp[-2.4i(x - 8t + 1)] \]

\[ \psi_1(\zeta) = \frac{1}{-2.4i\exp[-2.4i(x - 8t + 1)]} + 1 \]  
(64)

\[ \psi_1(\zeta) = 0.4i\exp[2.4i(x - 8t + 1)] + 1 \]  
(65)

\[ u(x, t) = \psi_1(\zeta) e^{i\eta_1(x, t)}, \quad \zeta = k_1x + w_1t, \quad \eta_1 = q_1x + \delta_1t \]

\[ u(x, t) = \left( [1 - 0.4 \sin(2.4x - 19.2t + 2.4)] + 0.4i \cos(2.4x - 19.2t + 2.4) \right) e^{i(x - 3t)} \]  
(66)

\[ \text{Re } u = ((1 - 0.4 \sin(2.4x - 19.2t + 2.4)) \cos(x - 3t) - 0.4 \cos(2.4x - 19.2t + 2.4) \sin(x - 3t)) \]  
(67)

\[ \text{Im } u = ((1 - 0.4 \sin(2.4x - 19.2t + 2.4)) \sin(x - 3t) + 0.4 \cos(2.4x - 19.2t + 2.4) \cos(x - 3t)) \]  
(68)

For the 8-th result which is, \( A_1 = 0, A_0 = 1, A_1 = 1, a_1 = 2.4i, a_2 = 0, k_1 = q_1 = d_1 = \zeta_0 = 1, c_1 = -1, \delta_1 = -3 \)

\[ \varphi(\zeta) = 2.4i\exp[2.4i(x - 8t + 1)] \]

\[ \psi_1(\zeta) = [1 - 2.4 \sin(2.4x - 19.2t + 2.4)] + 2.4i \cos(2.4x - 19.2t + 2.4) \]  
(69)

\[ u(x, t) = \left( [1 - 2.4 \sin(2.4x - 19.2t + 2.4)] + 2.4i \cos(2.4x - 19.2t + 2.4) \right) e^{i(x - 3t)} \]  
(70)
Re \( u = (1 - 2.4 \sin(2.4x - 19.2t + 2.4)) \times \cos(x - 3t) - 2.4 \cos(2.4x - 19.2t + 2.4) \times \sin(x - 3t) \) \)
\[ (73) \]

Im \( u = (2.4 \cos(2.4x - 19.2t + 2.4)) \times \cos(x - 3t) + 2.4((1 - 2.4 \sin(2.4x - 19.2t + 2.4)) \times \sin(x - 3t)) \)
\[ (74) \]

In the same manner, we can plot the other cases.

### 4 The cubic-quartic nlse for the quadratic-low nonlinearity using the ESEM

The cubic-quartic NLSE in polarization-preserving fibers with quadratic-cubic law can be proposed in the form,

\[ iQ_t + iQ_{xxx} + bQ_{xxxx} = \left( c|Q| + d|Q|^2 \right) Q \]
\[ (75) \]

The cubic-quartic NLSE in birefringent fiber for quadratic-cubic law is

\[ iu_t + ia_1u_{xxx} + b_1u_{xxxx} + c_1u\sqrt{|u|^2 + |v|^2 + uv^* + vu^* + (d_1|u|^2 + e_1|v|^2)}u = 0 \]
\[ (76) \]

\[ iv_t + ia_2v_{xxx} + b_2v_{xxxx} + c_2v\sqrt{|v|^2 + |u|^2 + uv^* + vu^* + (d_2|v|^2 + e_2|u|^2)}v = 0 \]
\[ (77) \]

Via inserting the relations (13–26) into Eq. (76) and (77) will transform them to the following real and imaginary equations respectively:

\[ \text{Re} \Rightarrow b_jk^4\psi_j'''' - (3a_jq_jk^2 + 6k_j^2q_j^2)\psi_j'' + (b_jq_j^4 + a_jq_j^2 - \delta_j)\psi_j + 2c_j\psi_j^2 + (d_j + e_j)\psi_j^3 = 0 \]
\[ (78) \]

\[ \text{Im} \Rightarrow k_j^3(a_j + 4b_jq_j)\psi_j'''' + (w_j - 3a_jk_jq_j^2 - 4k_jq_j^2)\psi_j' = 0 \]
\[ (79) \]

Equation (79) implies \( \psi_j = -4b_jq_j, w_j = 3a_jq_j^2 + 4b_jq_j^4, \) hence under this constrain Eq. (78) will be converted to,

\[ b_jk^4\psi_j'''' + 6b_jk_j^2q_j^2\psi_j'' - (3b_jq_j^2 + \delta_j)\psi_j + 2c_j\psi_j^2 + (d_j + e_j)\psi_j^3 = 0 \]
\[ (80) \]

Equation (80) represents either the first part Eq. (76) or the second part Eq. (77) which is the same, for similarity we will implement the solution for the first part \( j = 1 \) which is,

\[ b_1k^4\psi_1'''' + 6b_1k_1^2q_1^2\psi_1'' - (3b_1q_1^2 + \delta_1)\psi_1 + 2c_1\psi_1^2 + (d_1 + e_1)\psi_1^3 = 0 \]
\[ (81) \]

Now, via integrating Eq. (81) with respect to \( \zeta \) it becomes,

\[ b_1k^4\psi_1'''' + 6b_1k_1^2q_1^2\psi_1'' - \frac{1}{2}(3b_1q_1^2 + \delta_1)\psi_1^2 + \frac{2}{3}c_1\psi_1^3 + \frac{1}{4}(d_1 + e_1)\psi_1^4 = 0 \]
\[ (82) \]

Via balancing \( \psi_1'''' \), \( \psi_1^A \) at Eq. (82) leads to \( 4N = N + 3 \) which implies that \( N = 1 \), hence the solution is

\[ \psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi \]
\[ (83) \]
where \( \varphi' = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3 \).

**Case 1:** The first family, in which \( a_1 = a_3 = 0 \Rightarrow \varphi' = a_0 + a_2 \varphi^2 \), consequently

\[
\psi'_1 = -\frac{a_0 A_{-1}}{\varphi^2} + A_1 a_0 + A_1 a_2 \varphi^2 - a_2 A_{-1}
\]  
(84)

\[
\psi''_1 = -\frac{6a_0^2 A_{-1}}{\varphi^4} - \frac{8a_0^2 a_2 A_{-1}}{\varphi^2} - 2A_{-1} a_0 a_2^2 + 2A_1 a_2 a_0^2 + 8A_1 a_0 a_2^2 \varphi^2 + 6A_1 a_2^3 \varphi^4
\]  
(85)

Via inserting \( \psi'''_1, \psi'_1, \psi^2_1 \) and \( \psi^4_1 \) into Eq. (82), collecting and equating the coefficients of various powers of \( \varphi' \) to zero leads to system of equations from which we construct that there are no solutions achieved because either \( a_0 = 0 \) or \( a_2 = 0 \) or both or \( A_{-1} = A_1 = 0 \).

**Case 2:** The second family, in which \( d_0 = a_3 = 0 \Rightarrow \varphi' = a_1 \varphi + a_2 \varphi^2 \), consequently

\[
\psi'_1 = A_1 a_2 \varphi^2 + a_1 A_1 \varphi - \frac{A_{-1} a_1}{\varphi} - A_{-1} a_2
\]

\[
\psi''_1 = 6A_1 a_2^3 \varphi^4 + 12a_1 A_1 a_2^2 \varphi^3 + 7A_1 a_2^2 a_1 \varphi^2 + A_1 a_1^3 \varphi - A_{-1} a_2 a_1^2 - \frac{a^3 A_{-1}}{\varphi}
\]  
(86)

Via inserting \( \psi'''_1, \psi'_1, \psi^2_1, \psi^3_1 \) and \( \psi^4_1 \) into Eq. (82), collecting and equating the coefficients of various powers of \( \varphi' \) to zero leads to system of equations which proposed big number of very complicated solutions from which only 12-solutions are valued and the other achieved solutions are refused because either \( a_1 = 0 \) or \( A_{-1} = A_1 = 0 \), we will choose only 8-solutions of them where the remaining are very long which are

\[
A_{-1} = 0, a_1 = \frac{i \sqrt{6} q_1}{k_1}, a_2 = \frac{i \sqrt{6} A_1 q_1}{A_0 k_1}, b_1 = \frac{-i A^3_0 (d_1 + e_1)}{144 \sqrt{6 k_1 q_1^3}}, c_1 = -\frac{3}{4} A_0 (d_1 + e_1),
\]

\[
\delta_1 = \frac{i [144 i A^2_0 k_1 (d_1 + e_1) + \sqrt{6} A^3_0 q_1 (d_1 + e_1)]}{288 k_1}
\]  
(87)

\[
A_{-1} = 0, a_1 = \frac{-i \sqrt{6} q_1}{k_1}, a_2 = \frac{-i \sqrt{6} A_1 q_1}{A_0 k_1}, b_1 = \frac{-i A^3_0 (d_1 + e_1)}{144 \sqrt{6 k_1 q_1^3}}, c_1 = -\frac{3}{4} A_0 (d_1 + e_1),
\]

\[
\delta_1 = \frac{i [-144 i A^2_0 k_1 (d_1 + e_1) + \sqrt{6} A^3_0 q_1 (d_1 + e_1)]}{288 k_1}
\]  
(88)

\[
A_{-1} = 0, A_0 = 0, a_1 = \frac{i \sqrt{6} q_1}{k_1}, a_2 = \frac{(-1)^{\frac{1}{3}} A_1 (d_1 + e_1)^{\frac{1}{3}}}{2 (b_1)^{\frac{1}{3}} (k_1)^{\frac{1}{3}}}, c_1 = -\frac{(-1)^{\frac{1}{3}} (3 b_1 k_1)^{\frac{1}{3}} [3 i \sqrt{6 q_1} (d_1 + e_1)]}{2 (d_1 + e_1)^{\frac{1}{3}}},
\]

\[
\delta_1 = \frac{3 [4 - (-1)^{\frac{1}{3}} (3 b_1 k_1)^{\frac{1}{3}} d_1 q_1^3 - 8 (-1)^{\frac{1}{3}} (3 b_1 k_1)^{\frac{1}{3}} d_1 e_1 q_1^3 - 4 (-1)^{\frac{1}{3}} (3 b_1 k_1)^{\frac{1}{3}} e_1^2 q_1^3]}{(d_1 + e_1)^{\frac{1}{3}}} - b_1 q_1^4
\]  
(89)
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\[
A_{-1} = 0, A_0 = 0, a_1 = \frac{-i \sqrt[4]{6} q_1}{k_1}, a_2 = \frac{(-1)^i A_1 (d_1 + e_1)^i}{2 (b_1)^{\frac{1}{4}} (k_1)^{i}}, c_1 = \frac{(-1)^i (3b_1 k_1) \frac{1}{2} [3 \sqrt[4]{6} q_1 (d_1 + e_1)]}{2 (d_1 + e_1)^{\frac{3}{4}}}, \delta_1 = \frac{3 \left[ -4 (-1)^i (3b_1 k_1)^{\frac{1}{2}} d_1^2 q_1^2 - 8 (-1)^i (3b_1 k_1)^{\frac{1}{2}} d_1 e_1 q_1^2 - 4 (-1)^i (3b_1 k_1)^{\frac{1}{2}} e_1^2 q_1^2 \right]}{(d_1 + e_1)^{\frac{3}{4}}} - b_1 q_1^4
\] (90)

\[
A_{-1} = 0, A_0 = 0, a_1 = \frac{i \sqrt[4]{6} q_1}{k_1}, a_2 = \frac{-A_1 (d_1 + e_1)^{\frac{1}{4}}}{2 (b_1)^{\frac{1}{4}} (k_1)^{i}}, c_1 = \frac{(3b_1 k_1)^{\frac{1}{2}} [3 \sqrt[4]{6} q_1 (d_1 + e_1)]}{2 (d_1 + e_1)^{\frac{3}{4}}}, \delta_1 = \frac{3 \left[ 4 (3b_1 k_1)^{\frac{1}{2}} d_1^2 q_1^2 + 8 (3b_1 k_1)^{\frac{1}{2}} d_1 e_1 q_1^2 + 4 (3b_1 k_1)^{\frac{1}{2}} e_1^2 q_1^2 \right]}{(d_1 + e_1)^{\frac{3}{4}}} - b_1 q_1^4
\] (91)

\[
A_{-1} = 0, A_0 = 0, a_1 = \frac{-i \sqrt[4]{6} q_1}{k_1}, a_2 = \frac{-A_1 (d_1 + e_1)^{\frac{1}{4}}}{2 (b_1)^{\frac{1}{4}} (k_1)^{i}}, c_1 = \frac{(-3b_1 k_1)^{\frac{1}{2}} [3 \sqrt[4]{6} q_1 (d_1 + e_1)]}{2 (d_1 + e_1)^{\frac{3}{4}}}, \delta_1 = \frac{3 \left[ 4 (-3b_1 k_1)^{\frac{1}{2}} d_1^2 q_1^2 + 8 (-3b_1 k_1)^{\frac{1}{2}} d_1 e_1 q_1^2 + 4 (-3b_1 k_1)^{\frac{1}{2}} e_1^2 q_1^2 \right]}{(d_1 + e_1)^{\frac{3}{4}}} - b_1 q_1^4
\] (92)

\[
A_{-1} = 0, A_0 = 0, a_1 = \frac{i \sqrt[4]{6} q_1}{k_1}, a_2 = \frac{-A_1 (d_1 + e_1)^{\frac{1}{4}}}{2 (b_1)^{\frac{1}{4}} (k_1)^{i}}, c_1 = \frac{(-3b_1 k_1)^{\frac{1}{2}} [3 \sqrt[4]{6} q_1 (d_1 + e_1)]}{2 (d_1 + e_1)^{\frac{3}{4}}}, \delta_1 = \frac{3 \left[ 4 (-3b_1 k_1)^{\frac{1}{2}} d_1^2 q_1^2 + 8 (-3b_1 k_1)^{\frac{1}{2}} d_1 e_1 q_1^2 + 4 (-3b_1 k_1)^{\frac{1}{2}} e_1^2 q_1^2 \right]}{(d_1 + e_1)^{\frac{3}{4}}} - b_1 q_1^4
\] (93)

For simplicity we will choose only one of these results say the first one, to extract its own solution and plotted it.

For the 1-th result which is

\[
A_{-1} = 0, A_0 = 0, a_1 = \frac{i \sqrt[4]{6} q_1}{k_1}, a_2 = \frac{i \sqrt[4]{6} A_1 q_1}{A_0 k_1}, b_1 = \frac{-i A_0^2 (d_1 + e_1)}{144 \sqrt[4]{6} k_1 q_1^3}, c_1 = \frac{-3 A_0 (d_1 + e_1)}{4 A_0 (d_1 + e_1)}, \delta_1 = \frac{-A_0^2 (d_1 + e_1)}{2} + \frac{i \sqrt[4]{6} A_0^2 (d_1 + e_1)}{288 k_1}.
\]

This result will simplified to be,
$A_{-1} = 0, A_0 = 4, A_1 = 1, a_1 = 0.6i, a_2 = 0.6i, b_1 = -0.4i,$
$e_1 = d_1 = k_1 = q_1 = \zeta_0 = 1, c_1 = -6, \delta_1 = -16 + 1.1i$

Via these values of constants the solution in the framework of the proposed method is

$$\varphi(\zeta) = \frac{i\sqrt{6} \exp[i\sqrt{6}(x + t + 1)]}{1 - i\sqrt{6}\exp[i\sqrt{6}(x + t + 1)]}$$

$$\psi_1(\zeta) = 4 + \frac{i\sqrt{6} \exp[i\sqrt{6}(x + t + 1)]}{1 - i\sqrt{6}\exp[i\sqrt{6}(x + t + 1)]}$$

$$\psi_1(\zeta) = \frac{4 - 3i\sqrt{6} \exp[i\sqrt{6}(x + t + 1)]}{1 - i\sqrt{6}\exp[i\sqrt{6}(x + t + 1)]}$$

$$u(x, t) = \left(\frac{4 - 3i\sqrt{6} \exp[i\sqrt{6}(x + t + 1)]}{1 - i\sqrt{6}\exp[i\sqrt{6}(x + t + 1)]}\right) \times e^{(x + 1.1i - 16t)}$$

$$u(x, t) = e^{-1.1t} \left(\frac{4 - 3i\sqrt{6} \exp[i\sqrt{6}(x + t + 1)]}{1 - i\sqrt{6}\exp[i\sqrt{6}(x + t + 1)]}\right) \times e^{(x - 16t)}$$

$$\text{Re } u = e^{-1.1t} \left\{ \frac{(22 + 7\sqrt{6} \sin[\sqrt{6}(x + t + 1)]) \cos(x - 16t)}{7 + 2\sqrt{6} \sin[\sqrt{6}(x + t + 1)]} \right\}$$

$$\text{Im } u = e^{-1.1t} \left\{ \frac{(22 + 7\sqrt{6} \sin[\sqrt{6}(x + t + 1)]) \sin(x - 16t)}{7 + 2\sqrt{6} \sin[\sqrt{6}(x + t + 1)]} \right\}$$

5 The Cubic-Quartic NLSE for the Parabolic-Low nonlinearity using the ESEM

The cubic-quartic NLSE in polarization-preserving fibers with parabolic law can be proposed in the form,

$$iQ_t + i\alpha Q_{xxx} + bQ_{xxxx} = \left( c|Q|^2 + d|Q|^4 \right)Q$$ (100)
The cubic-quartic NLSE in birefringent fiber for parabolic-law is,

\begin{align}
  iu_t + ia_1 u_{xxx} + b_1 u_{xxxx} + c_1 |u|^2 u + d_1 |v|^2 u + \left( e_1 |u^4| + g_1 |u^2|^2 + h_1 |v^4| \right) u &= 0 \\
  iv_t + ia_1 v_{xxx} + b_1 v_{xxxx} + c_1 |v|^2 v + d_1 |u|^2 v + \left( e_1 |v^4| + g_1 |v^2|^2 + h_1 |u^4| \right) v &= 0
\end{align}

(V1)

Via inserting the relations (13–26) into the Eqs. (101) and (102) will transform them to the following real and imaginary equations respectively

Re \rightarrow b_j k_j^4 \psi_j''' - (3a_j q_j^2 + 6b_j k_j^2 q_j^2) \psi_j'' + (b_j q_j^4 + a_j q_j^3 - \delta_j) \psi_j + (c_j + d_j) \psi_j^3 + (e + g + h) \psi_j^5 = 0

(103)

Im \rightarrow k_j^2 (a_j + 4b_j q_j) \psi_j''' + (w_j - 3a_j k_j q_j^2 - 4b_j k_j q_j^3) \psi_j' = 0

(104)

Equation (104) implies \( a_j = -4b_j q_j \), \( w_j = 3k_j a_j q_j^2 + 4k_j b_j q_j^3 \), hence under this constrain Eq. (103) will be converted to,

\begin{align}
  b_j k_j^4 \psi_j''' + 6b_j q_j k_j^2 \psi_j'' - (3k_j b_j q_j^2 + \delta_j) \psi_j + (c_j + d_j) \psi_j^3 + (e + g + h) \psi_j^5 = 0
\end{align}

Equation (105) represents either the first part Eq. (101) or the second part Eq. (102) which is the same, for similarity we will implement the solution for the first part \( j = 1 \) which is

\begin{align}
  b_1 k_1^4 \psi_1''' + 6b_1 q_1 k_1^2 \psi_1'' - (3b_1 q_1^2 + \delta_1) \psi_1 + (c_1 + d_1) \psi_1^3 + (e + g + h) \psi_1^5 = 0
\end{align}

(V2)

Equation balancing \( \psi_1''' \), \( \psi_1^5 \) at Eq. (106) leads to \( 5N = N + 4 \) which implies that \( N = 1 \), hence the solution is

\begin{equation}
  \psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi
\end{equation}

where \( \varphi' = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3 \).

**Case 1:** The first family, in which \( a_1 = a_3 = 0 \) \( \Rightarrow \varphi' = a_0 + a_2 \varphi^2 \), consequently

\begin{align}
  \psi_1' &= -\frac{a_0 A_{-1}}{\varphi^2} + A_1 a_0 + A_1 a_2 \varphi^2 - a_2 A_{-1} \\
  \psi_1'' &= \frac{2a_0^2 A_{-1}}{\varphi^5} + \frac{2a_0 a_2 A_{-1}}{\varphi} + 2A_1 a_0 a_2 \varphi + 2A_1 a_2 \varphi^3
\end{align}

(V3)

\begin{align}
  \psi_1''' &= \frac{24a_0^4 A_{-1}}{\varphi^5} + \frac{40a_0^3 a_2 A_{-1}}{\varphi^3} + \frac{16a_0^2 a_2^2 A_{-1}}{\varphi} + 16A_1 a_0^2 a_2 \varphi + 40A_1 a_0 a_2^2 \varphi^3 + 24A_1 a_2^3 \varphi^5
\end{align}

(V4)

Via inserting \( \psi_1''' \), \( \psi_1'' \), \( \psi_1' \), \( \psi_1^3 \) and \( \psi_1^5 \) into Eq. (106), collecting and equating the coefficients of various powers of \( \varphi' \) to zero leads to system of equations, by solving this system many solutions will be achieved from which only 8-results are acceptable while the remaining are refused. These acceptable results are
\[ A_0 = 0, a_2 = \frac{a_0 A_1}{A^{-1}}, \delta_1 = \frac{b_1 k_1 (16a_0^4 A_1^2 k_1^3 - 24a_0^2 A_1 A_{-1} q_1^2 - 3A_{-1}^2 q_1^2)}{A_{-1}^2}, \]
\[ c_1 = \frac{-A_{-1}^3 d_1 + 80a_0^4 A_1 b_1 k_1^4 - 12a_0^2 A_{-1} b_1 q_1^2 k_1^2}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (111)

\[ A_0 = 0, a_2 = \frac{-a_0 A_1}{A^{-1}}, \delta_1 = \frac{-b_1 (-256a_0^4 A_1^2 k_1^4 + 48a_0^2 A_1 A_{-1} k_1^2 q_1^2 + 3A_{-1}^2 q_1^2)}{A_{-1}^2}, \]
\[ c_1 = \frac{-A_{-1}^3 d_1 + 160a_0^4 A_1 b_1 k_1^4 - 12a_0^2 A_{-1} b_1 k_1^2 q_1^2}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (112)

\[ A_0 = 0, a_2 = \frac{ia_0 A_1}{A^{-1}}, \delta_1 = \frac{532b_1 a_0^4 A_1^2 k_1^4}{3A_{-1}^2}, q_1 = \frac{(-1)^{\frac{3}{2}} k_1 a_0 \sqrt{10A_{-1}}}{\sqrt{A_{-1}}} \]
\[ c_1 = \frac{-A_{-1}^3 d_1 + 120a_0^4 A_1 b_1 k_1^4}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (113)

\[ A_0 = 0, a_2 = \frac{-ia_0 A_1}{A^{-1}}, \delta_1 = \frac{532b_1 a_0^4 A_1^2 k_1^4}{3A_{-1}^2}, q_1 = \frac{(-1)^{\frac{3}{2}} k_1 a_0 \sqrt{10A_{-1}}}{\sqrt{A_{-1}}} \]
\[ c_1 = \frac{-A_{-1}^3 d_1 + 120a_0^4 A_1 b_1 k_1^4}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (114)

\[ A_0 = 0, a_2 = \frac{-ia_0 A_1}{A^{-1}}, \delta_1 = \frac{532b_1 a_0^4 A_1^2 k_1^4}{3A_{-1}^2}, q_1 = \frac{(-1)^{\frac{3}{2}} k_1 a_0 \sqrt{10A_{-1}}}{\sqrt{A_{-1}}} \]
\[ c_1 = \frac{-A_{-1}^3 d_1 + 120a_0^4 A_1 b_1 k_1^4}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (115)

\[ A_0 = 0, a_2 = \frac{-ia_0 A_1}{A^{-1}}, \delta_1 = \frac{532b_1 a_0^4 A_1^2 k_1^4}{3A_{-1}^2}, q_1 = \frac{(-1)^{\frac{3}{2}} k_1 a_0 \sqrt{10A_{-1}}}{\sqrt{A_{-1}}} \]
\[ c_1 = \frac{-A_{-1}^3 d_1 + 120a_0^4 A_1 b_1 k_1^4}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (116)

\[ A_{-1} = 0, A_0 = 0, \delta_1 = -b_1 (-16a_0^2 a_2^2 k_1^4 - 12a_0 a_2 k_1^2 q_1^2 + 3q_1^4), \]
\[ c_1 = \frac{-A_{-1}^2 d_1 - 40a_0 b_1 a_2^2 k_1^4 - 12a_0^2 b_1 k_1^2 q_1^2}{A_{-1}^2}, g = \frac{-(e + h)A_{-1}^4 - 24b_1 a_0^4 k_1^4}{A_{-1}^4} \] (117)
\[
A_1 = 0, A_0 = 0, \delta_1 = -b_1(-16a_0^2a_2^2k_4^4 - 12a_0a_2k_7^2q_1^2 + 3q_1^4),
\]

\[
c_1 = \frac{-A_{-1}^2d_1 - 40a_2b_1a_0^3k_4^4 - 12a_0^2b_1k_7^2q_1^2}{A_{-1}^2}, g = \frac{-(e + h)A_{-1}^4}{A_{-1}^4} - 24b_1a_0^4k_4^4
\] (118)

\[
A_1 = 0, A_0 = 0, a_2 = 0, \delta_1 = -3b_1q_1^2c_1 = \frac{-A_{-1}^2d_1 - 12b_1a_0^2k_7^2q_1^2}{A_{-1}^2}, g = \frac{-(e + h)A_{-1}^4}{A_{-1}^4} - 24b_1a_0^4k_4^4
\] (119)

Now we will extract the different solutions corresponding to these 9-acceptable results, for simplicity we will choose only one result say the first one.

For the 1-st result which is

\[
A_0 = 0, a_2 = \frac{a_0A_1}{A_{-1}}, \delta_1 = b_1k_1(16a_0^2A_1^2k_1^3 - 24a_0^2A_1A_{-1}q_1^2 - 3A_{-1}^2q_1^2),
\]

\[
c_1 = \frac{-A_{-1}^3d_1 + 80a_0A_1b_1k_1^4 - 12a_0^2A_{-1}b_1q_1^2k_1^2}{A_{-1}^3}, g = \frac{-(e + h)A_{-1}^4}{A_{-1}^4} - 24b_1a_0^4k_4^4
\] (120)

This result will generate 4- sub results which are

\[
A_0 = 0, A_{-1} = A_1 = 1, a_0 = a_2 = k_1 = q_1 = b_1 = c_1 = d_1 = \zeta_0 = 1, g + h + e = -24, \delta_1 = -9
\]

\[
A_0 = 0, A_{-1} = A_1 = 1, a_0 = a_2 = -1, k_1 = q_1 = b_1 = c_1 = d_1 = \zeta_0 = 1, g + h + e = -24, \delta_1 = -9
\]

\[
A_0 = 0, A_{-1} = A_1 = 1, a_0 = a_2 = 0.4, k_1 = q_1 = b_1 = c_1 = d_1 = \zeta_0 = 1, g + h + e = -0.8, \delta_1 = -2.3
\]

\[
A_0 = 0, A_{-1} = A_1 = 1, a_0 = a_2 = -0.4, k_1 = q_1 = b_1 = c_1 = d_1 = \zeta_0 = 1, g + h + e = -0.8, \delta_1 = -2.3
\]

For simplicity we will take only one of these results say the first, and extracting the corresponding solution according to the point of view of the suggested method which is

\[
\varphi(\zeta) = \frac{\sqrt{a_0}d_2}{a_2} \tan(\sqrt{a_0}d_2(\zeta + \zeta_0)), a_0d_2 > 0
\] (121)

\[
\varphi(\zeta) = \tan[x - 8t + 1]
\]

\[
\psi_1(\zeta) = \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi
\] (122)

\[
\psi_1(\zeta) = \cot[x - 8t + 1] + \tan[x - 8t + 1]
\]

\[
u(x, t) = \psi_1(\zeta) e^{\eta_1(x, t)}, \zeta = k_1x + w_1t, \eta_1 = q_1x + \delta_1t
\]

\[
u(x, t) = (\cot[x - 8t + 1] + \tan[x - 8t + 1]) e^{(x-9t)}
\] (123)

\[
\text{Re } u = (\cot[x - 8t + 1] + \tan[x - 8t + 1]) \times \cos(x - 9t)
\] (124)

\[
\text{Im } u = (\cot[x - 8t + 1] + \tan[x - 8t + 1]) \times \sin(x - 9t)
\] (125)

Via the same manner, we can plot the other solutions.

**Case 2:** The second family, in which \(a_0 = a_3 = 0 \Rightarrow \varphi' = a_1 \varphi + a_2 \varphi^2\), consequently
\[ \psi_1' = A_1 a_2 \varphi^2 + a_1 A_1 \varphi - \frac{A_{-1} a_1}{\varphi} - A_{-1} a_2 \] (126)

\[ \psi_1'' = 2A_1 a_2^2 \varphi^3 + 3A_1 a_1 a_2 \varphi^2 + A_1 a_1^2 \varphi + A_{-1} a_2 a_1 + \frac{a_1^2 A_{-1}}{\varphi} \] (127)

\[ \psi_1''' = \frac{a_1^4 A_{-1}}{\varphi} + A_{-1} a_2 a_1^3 + 14A_1 a_2 a_1^3 \varphi^2 + 50A_1 a_2^2 a_1^2 \varphi^3 \]

\[ + A_1 a_1 (1 + 60a_2^2) \varphi^4 + A_1 (a_1 + 24a_2^4) \varphi^5 \] (128)

Via inserting \( \psi_1'''', \psi_1'', \psi_1, \psi_1^3 \) and \( \psi_1^4 \) into Eq. (106), collecting and equating the coefficients of various powers of \( \varphi \) to zero leads to system of equations in which the coefficients of \( \varphi^{-5}, \varphi^{-4} \) imply that \( e + g + h = 0 \) and the coefficients of \( \varphi^{-3}, \varphi^{-2} \) imply \( c_1 = -d_1 \) and by using these new constrain at the other equations of this system it will be reduced to the following system,

\[ a_1^4 A_{-1} b_1 k_1^4 + 6a_1^2 A_{-1} b_1 k_1^2 q_1^2 - 3A_{-1} b_1 k_1 q_1^4 - A_1 \delta_1 = 0 \]

\[ a_1 A_1 b_1 k_1^4 + 24a_1^2 A_1 b_1 q_1^4 = 0 \]

\[ a_1 A_1 b_1 k_1^4 + 60a_1 a_2^2 A_1 b_1 q_1^4 = 0 \]

\[ 50a_1 a_2^2 A_1 b_1 k_1^4 + 12a_1^2 A_1 b_1 k_1^2 q_1^2 = 0 \]

\[ 14a_1^3 A_1 b_1 k_1^4 + 18a_1 a_2 A_1 b_1 k_1^2 q_1^2 = 0 \]

\[ 6a_1^2 A_1 b_1 k_1^2 q_1^2 - 3A_1 b_1 q_1^4 - A_1 \delta_1 = 0 \]

\[ a_1^2 a_2 A_{-1} b_1 k_1^4 + 6a_1 a_2 A_{-1} b_1 k_1^2 q_1^2 - 3A_0 b_1 q_1^4 - A_0 \delta_1 = 0 \] (129)

The solution of the reduced system (129) gives the following three results,

\[ (1) A_1 = 0, \delta_1 = \frac{-b_1 k_1^4 q_1^4}{12}, q_1 = \frac{-ia_1 k_1}{\sqrt{6}} \]

\[ (2) A_1 = 0, \delta_1 = \frac{-b_1 k_1^4 q_1^4}{12}, q_1 = \frac{ia_1 k_1}{\sqrt{6}} \] (130)

\[ (3) A_1 = 0, A_0 = \frac{a_1^2 A_{-1}}{d_1}, \delta_1 = b_1 k_1^4 a_1^4 + 6a_1^2 b_1 k_1^2 q_1^2 - 3b_1 q_1^4 \]

We will extract the solutions in the framework of the suggested method for the first and the third result.

For the first result which is, \( A_1 = 0, \delta_1 = \frac{-b_1 k_1^4 q_1^4}{12}, q_1 = \frac{-ia_1 k_1}{\sqrt{6}} \).

This result can be simplified to be,

\[ A_0 = A_{-1} = 1, A_1 = 0, a_1 = \sqrt{6} i, a_2 = 0, k_1 = q_1 = d_1 = \xi_0 = 1, \delta_1 = -0.1, c_1 = -d_1, e + g + h = 0 \]

From the point of view of the suggested method the solution is
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\begin{align}
\varphi(\zeta) &= \frac{a_1 \exp[a_1(\zeta + \zeta_0)]}{1 - a_2 \exp[a_1(\zeta + \zeta_0)]}, a_1 > 0 \tag{131} \\
\varphi(\zeta) &= 2.45 i \exp[2.45 i (x - 8t + 1)]
\end{align}

\begin{align}
\psi_1(\zeta) &= \frac{-i \exp[-2.45 i (x - 8t + 1)]}{2.45} + 1 \tag{132} \\
\psi_1(\zeta) &= (1 - 0.4 \sin[2.45 x - 19.6 t + 2.45]) - 0.4 i \cos[2.45 x - 19.6 t + 2.45]
\end{align}

\begin{align}
u(x, t) &= \psi_1(\zeta) e^{i \eta_i(x, t)}, \quad \zeta = k_1 x + w_1 t, \quad \eta_1 = q_1 x + \delta_1 t \\
u(x, t) &= (1 - 0.4 \sin[2.45 x - 19.6 t + 2.45]) - 0.4 i \cos[2.45 x - 19.6 t + 2.45] \times e^{i(x - 0.1t)} \tag{133}
\end{align}

\begin{align}
\text{Re } u &= (1 - 0.4 \sin[2.45 x - 19.6 t + 2.45]) \cos(x - 0.1t) \\
+ (0.4 \cos[2.45 x - 19.6 t + 2.45]) \sin(x - 0.1t) \tag{134} \\
\text{Im } u &= (1 - 0.4 \sin[2.45 x - 19.6 t + 2.45]) \sin(x - 0.1t) \\
- (0.4 \cos[2.45 x - 19.6 t + 2.45]) \cos(x - 0.1t) \tag{135}
\end{align}

For the third result which is, \( A_1 = 0, A_0 = \frac{a_0 a_1}{a_1}, \delta_1 = b_1 k_1^4 a_1^4 + 6 a_1^2 b_1 k_1^2 q_1^2 - 3 b_1 q_1^4 \).

This result split into four sub results which are

\begin{align}
A_0 &= A_{-1} = 1, A_1 = 0, a_1 = a_2 = \sqrt{0.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0 \\
A_0 &= A_{-1} = 1, A_1 = 0, a_1 = a_2 = -\sqrt{0.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0 \\
A_0 &= A_{-1} = 1, A_1 = 0, a_1 = a_2 = i\sqrt{0.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0 \\
A_0 &= A_{-1} = 1, A_1 = 0, a_1 = a_2 = -i\sqrt{0.6}, k_1 = q_1 = d_1 = \delta_1 = \zeta_0 = 1, c_1 = -d_1, e + g + h = 0
\end{align}

We will plot only one say the first, from the point of view of the suggested method the solution is

\begin{align}
\varphi(\zeta) &= \frac{a_1 \exp[a_1(\zeta + \zeta_0)]}{1 - a_2 \exp[a_1(\zeta + \zeta_0)]}, a_1 > 0 \tag{136} \\
\varphi(\zeta) &= \frac{0.8 \exp[0.8 x - 6.4 t + 0.8]}{1 - 0.8 \exp[0.8 x - 6.4 t + 0.8]}
\end{align}

\begin{align}
\psi_1(\zeta) &= 1.3 \exp[-0.8 x + 6.4 t - 0.8] - 1 \tag{137} \\
u(x, t) &= (1.3 \exp[-0.8 x + 6.4 t - 0.8] - 1) \times e^{i(x + t)} \tag{138}
\end{align}

\begin{align}
\text{Re } u &= (1.3 \exp[-0.8 x + 6.4 t - 0.8] - 1) \times \cos(x + t) \tag{139} \\
\text{Im } u &= (1.3 \exp[-0.8 x + 6.4 t - 0.8] - 1) \times \sin(x + t) \tag{140}
\end{align}

By the same manner we can plot the other remained cases.
6 The Cubic-Quartic NLSE for the Non-Local Low nonlinearity using the ESEM

The cubic-quartic NLSE in polarization-preserving fibers with non-local low be proposed in the form,

\[ iQ_t + iaQ_{xxx} + bQ_{xxxx} + \left( c|Q|^2 \right)_{xx} Q = 0 \]  \hspace{1cm} (141)

The cubic-quartic NLSE in birefringent fiber non-local low is,

\[ iu_t + ia_1 u_{xxx} + b_1 u_{xxxx} + \left( c_1 |u|^2 \right)_{xx} u = 0 \]  \hspace{1cm} (142)

\[ iv_t + ia_2 v_{xxx} + b_1 v_{xxxx} + \left( c_2 |v|^2 \right)_{xx} v = 0 \]  \hspace{1cm} (143)

Via inserting the relations (13–26) into Eq. (142) and (143) will transform them to the following real and imaginary equations respectively:

\[ \text{Re} \Rightarrow b_1 k_j^4 \psi_j''' - (3a_j q_j^4 + 6b_1 k_j^2 q_j^2)\psi_j'' + (b_j q_j^4 + a_j q_j^2 - \delta_j)\psi_j + (2c_j + 2d_j)\psi_j |^2 + (2c_j + 2d_j)\psi_j' |^2 = 0 \]  \hspace{1cm} (144)

\[ \text{Im} \Rightarrow k_j^2 (a_j + 4b_j q_j) \psi_j''' + (w_j - 3a_j k_j q_j^2 - 4b_j k_j q_j^2)\psi_j' = 0 \]  \hspace{1cm} (145)

Equation (145) implies \( a_j = -4b_j q_j \), \( w_j = 3k_j a_j q_j^2 + 4k_j b_j q_j^3 \), hence under this constrain Eq. (144) will be converted to,

\[ b_1 k_j^4 \psi_j''' + 6b_1 k_j^2 q_j^2 \psi_j'' - (3b_j q_j^4 + \delta_j)\psi_j + (2c_j + 2d_j)\psi_j |^2 + (2c_j + 2d_j)\psi_j' |^2 = 0 \]  \hspace{1cm} (146)

Equation (146) represents either the first part Eq. (142) or the second part Eq. (143) which is the same, for similarity we will implement the solution for the first part \( j = 1 \) which is,

\[ b_1 k_1^4 \psi_1''' + 6b_1 k_1^2 q_1^2 \psi_1'' - (3b_1 q_1^4 + \delta_1)\psi_1 + (2c_1 + 2d_1)\psi_1 |^2 + (2c_1 + 2d_1)\psi_1' |^2 = 0 \]  \hspace{1cm} (147)

Via balancing \( \psi_1''' \), \( \psi_1'' \) at Eq. (147) leads to \( 4N = 2N + N + 2 \) which implies that \( N = 2 \), hence the solution is

\[ \psi_1(\zeta) = \frac{A_{-2}}{\varphi^2} + \frac{A_{-1}}{\varphi} + A_0 + A_1 \varphi + A_2 \varphi^2 \]  \hspace{1cm} (148)

where \( \zeta = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3 \).

Case 1: The first family, in which \( a_1 = a_3 = 0 \) \( \Rightarrow \varphi' = a_0 + a_2 \varphi^2 \), consequently

\[ \psi_1' = \frac{-2a_0 A_{-2}}{\varphi^3} - \frac{a_0 A_{-1}}{\varphi^2} - \frac{2a_2 A_{-2}}{\varphi} + (A_{1} a_0 - a_2 A_{-1}) + 2A_2 a_0 \varphi + (A_1 a_2 + 2a_2 A_2) \varphi^2. \]  \hspace{1cm} (149)

\[ \psi_1'' = \frac{6a_0^2 A_{-2}}{\varphi^3} + \frac{2a_0 A_{-1}}{\varphi^2} + \frac{8a_0 a_2 A_{-2}}{\varphi} + \frac{2a_0 A_{-1}}{\varphi} + 2(A_{2} a_0^2 + A_{2} a_2^2) + 2a_0 a_2 A_1 \varphi + +8a_0 a_2 A_2 \varphi^2 + 2a_2^3 A_1 \varphi^3 + 6a_2^3 A_2 \varphi^4. \]  \hspace{1cm} (150)
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\[
\psi'''_1 = \frac{120a_0^4 A_{-2}}{\varphi^6} + \frac{24a_0^4 A_{-1}}{\varphi^5} + \frac{240a_0^4 a_2 A_{-2}}{\varphi^4} + \frac{40a_0^3 a_2 A_{-1}}{\varphi^3} \\
+ \frac{136a_0^2 a_2^2 A_{-2}}{\varphi^2} + \frac{16a_0^2 a_2^2 A_{-1}}{\varphi} + 16a_0^3 a_2 A_{-2} + 16a_0^3 a_2 A_{-1} + 16a_0^2 a_2^2 A_1 \varphi \\
+ 136a_0^2 a_2^2 A_2 \varphi^2 + 40a_0 a_2 A_1 \varphi^3 + 240a_0 a_2^3 A_2 \varphi^4 + 24a_0^4 A_1 \varphi^5 + 120a_0^2 A_2 \varphi^6
\]

(151)

Via inserting \(\psi'''_1, \psi''_1, \psi'_1, \psi^2_1\) and \(\psi^3_1\) into Eq. (147), collecting and equating the coefficients of various powers of \(\varphi\) to zero leads to system of equations, by solving this system rejected results will be achieved because in all these results either \(a_0 = 0\) or \(a_2 = 0\) or \(A_{-2} = A_{-1} = A_1 = A_2 = 0\), hence all these results will be refused.

**Case 2:** The second family, in which \(a_0 = a_3 = 0 \Rightarrow \varphi' = a_1 \varphi + a_2 \varphi^2\), consequently

\[
\psi'_1 = \frac{-2a_1 A_{-2}}{\varphi^2} = \frac{2a_2 A_{-2} + A_{-1} a_1}{\varphi} - a_2 A_{-1} + A_1 a_1 \varphi + (+A_1 a_2 + 2a_1 A_2) \varphi^2 + 2A_2 a_2 \varphi^3
\]

(152)

\[
\psi''_1 = \frac{4a_1^2 A_{-2}}{\varphi^2} + \frac{6a_1 a_2 A_{-2} + A_{-1} a_1^2}{\varphi} + A_{-1} a_1 a_2 + 2A_{-2} a_2^2 + a_2^2 A_1 \varphi \\
+ (4a_2 A_2 + 3a_1 a_2 A_1) \varphi^2 + (10a_1 a_2 A_2 + 2a_2^2 A_1) \varphi^3 + 6a_2^2 A_2 \varphi^4.
\]

(153)

\[
\psi'''_1 = \frac{16a_1^3 A_{-2}}{\varphi^2} + \frac{30a_1^2 a_2 A_{-2}}{\varphi} + A_{-1} a_1^3 + 14A_{-2} a_1^2 a_2 + A_1 a_1^4 \varphi \\
+ (15A_1 a_2 a_1^3 + 16A_2 a_1^4) \varphi^2 + (50a_2^2 a_1^2 a_2 + 130A_2^2 a_1^2 a_2) \varphi^3 \\
+ (330A_2^2 a_1^2 a_2^2 + 60A_1 a_2 a_1 a_2^2) \varphi^4 + (24a_2^4 A_1 + 336a_2^3 a_1 a_2) \varphi^5 + 120a_2^4 A_2 \varphi^6
\]

(154)

Via inserting \(\psi'''_1, \psi''_1, \psi'_1, \psi^2_1\) and \(\psi^3_1\) into Eq. (147), collecting and equating the coefficients of various powers of \(\varphi\) zero, will lead to a system of equations, by solving this system only six acceptable solutions are achieved and the remain are refused because either \(a_1 = 0\) for which there are no solution according to the suggested method or \(A_{-2} = A_{-1} = A_1 = A_2 = 0\) or \(b_1 = 0\).

The acceptable results are

(1) \(A_{-1} = A_1 = A_2 = 0, a_2^2 = \frac{-5A_0 (3b_1 q_1^4 + \delta_1)}{108A_{-2} b_1 k_1 g_1^2}, a_1 = -1.1 \frac{i q_1}{k_1}, c_1 = -d_1\)

(2) \(A_{-1} = A_1 = A_2 = 0, a_2^2 = \frac{-5A_0 (3b_1 q_1^4 + \delta_1)}{108A_{-2} b_1 k_1 g_1^2}, a_1 = 1.1 \frac{i q_1}{k_1}, c_1 = -d_1\)

(3) \(A_{-1} = A_1 = A_2 = a_2 = 0, a_1 = -1.1 \frac{i q_1}{k_1}, b_1 = \frac{-\delta_1}{3 q_1^4}, c_1 = -d_1\)

(4) \(A_{-1} = A_1 = A_2 = a_2 = 0, a_1 = 1.1 \frac{i q_1}{k_1}, b_1 = \frac{-\delta_1}{3 q_1^4}, c_1 = -d_1\)

(5) \(A_{-2} = A_2 = a_2 = 0, A_{-1} = 0.3A_1, a_1 = -1.1 \frac{i q_1}{k_1}, c_1 = -d_1\)

(6) \(A_{-2} = A_2 = a_2 = 0, A_{-1} = 0.3A_1, a_1 = 1.1 \frac{i q_1}{k_1}, c_1 = -d_1\)
For simplicity and repetition we will establish the solutions corresponding to the first and the six results and plot them.

For the first result which is

\[ A_{-2} = A_2 = a_2 = 0, A_{-1} = 0.3A_1, a_1 = 1.1, c_1 = -d_1 \]

This result can be simplified to be,

\[ A_{-1} = A_1 = A_2 = 0, |a_1| = 1.1, |a_2| = 0.44, A_0 = A_{-2} = q_1 = k_1 = \delta_1 = 1, c_1 = -d_1 \]  
(156)

The solution in the framework of the suggested method is where,

\[ \psi_1(\zeta) = \frac{A_{-2}}{\phi^2} + \frac{A_{-1}}{\phi} + A_0 + A_1 \phi + A_2 \phi^2 \]

\[ \phi(\zeta) = \frac{1.1 \exp[1.1(\zeta + \zeta_0)]}{1 - 0.44\exp[1.1(\zeta + \zeta_0)]} \]  
(157)

\[ \psi_1(x, t) = 0.8\exp[-2.2(\zeta + \zeta_0)] - 0.7\exp[-1.1(\zeta + \zeta_0)] + 1.2 \]  
(158)

\[ \psi_1(x, t) = (1.2 + 0.8\exp[-2.2(\zeta + \zeta_0)] - 0.7\exp[-1.1(\zeta + \zeta_0)]) \times e^{i(x+t)} \]  
(159)

Re \[ \psi_1(x, t) = (1.2 + 0.8\exp[-2.2(\zeta + \zeta_0)] - 0.7\exp[-1.1(\zeta + \zeta_0)]) \times \cos(x + t) \]  
(160)

Im \[ \psi_1(x, t) = (1.2 + 0.8\exp[-2.2(\zeta + \zeta_0)] - 0.7\exp[-1.1(\zeta + \zeta_0)]) \times \sin(x + t) \]  
(161)

For the six result which is,

\[ A_{-1} = A_1 = A_2 = 0, a_2^2 = \frac{-5A_0(3b_1q_1^4 + \delta_1)}{108A_{-2}b_1k_1^2q_1^2}, a_1 = -1.1 \frac{q_1}{k_1}, c_1 = -d_1 \]  
(162)

This result can be simplified to be,

\[ A_{-2} = A_0 = A_2 = a_2 = 0, A_{-1} = 0.3, |a_1| = 1.1, A_1 = q_1 = k_1 = \delta_1 = 1, c_1 = -d_1 \]  
(162)

The solution in the framework of the suggested method is,

\[ \phi(\zeta) = 1.1 \exp[1.1(\zeta + \zeta_0)] \]  
(163)

\[ \psi_1(\zeta) = 0.3\exp[-1.1(\zeta + \zeta_0)] + 1.1\exp[1.1(\zeta + \zeta_0)] \]  
(164)

\[ \psi_1(x, t) = (0.3\exp[-1.1(\zeta + \zeta_0)] + 1.1\exp[1.1(\zeta + \zeta_0)]) \times e^{i(x+t)} \]  
(165)

Re \[ \psi_1(x, t) = (0.3\exp[-1.1(\zeta + \zeta_0)] + 1.1\exp[1.1(\zeta + \zeta_0)]) \times \cos(x + t) \]  
(166)
\[ \text{Im} \psi_1(x, t) = (0.3 \text{Exp}[-1.1(\zeta + \zeta_0)] + 1.1 \text{Exp}[1.1(\zeta + \zeta_0)]) \times \sin(x + t) \] (167)

By the same manner we can plot the other results.
In this paper, we already extracted new multiple accurate soliton types to the cubic-quartic NLSE in polarization-preserving fibers with its four various forms via the ESEM. Firstly, through overview the plotted Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 it is clear that the ability of the suggested method to gives good new description of soliton types which never
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achieved before to the cubic-quartic for the Kerr-Low NLSE in polarization-preserving fibers. Furthermore, through overview the plotted Figs. 11 and 12 we also already documented other new soliton types to the cubic-quartic for the Quadratic-Low NLSE in polarization-preserving fibers coupled equations by the same method. In related subject, the suggested method has been used effectively to established other new soliton types to the cubic-quartic for the Parabolic-Low NLSE in polarization-preserving fibers coupled

Fig. 7 The cubic-quartic soliton of the Re. part of kerr-low Eq. (68) in 2D and 3D with values: \( w_1 = -8 \times 10^{-6} \), \( A_{-1} = 1, A_0 = 1, A_1 = 0, a_1 = -2.4i, a_2 = 0, k_1 = d_1 = \xi_0 = 1, c_1 = -1, \delta_1 = -3 \)

Fig. 8 The cubic-quartic soliton of the Im. part of kerr-low Eq. (69) in 2D and 3D with values: \( w_1 = -8 \times 10^{-6} \), \( A_{-1} = 1, A_0 = 1, A_1 = 0, a_1 = -2.4i, a_2 = 0, k_1 = d_1 = \xi_0 = 1, c_1 = -1, \delta_1 = -3 \)

Fig. 9 The cubic-quartic soliton of the Re. part of kerr-low Eq. (73) in 2D and 3D with values: \( w_1 = -8 \times 10^{-6} \), \( A_{-1} = 1, A_0 = 1, A_1 = 1, a_1 = 2.4i, a_2 = 0, k_1 = d_1 = \xi_0 = 1, c_1 = -1, \delta_1 = -3 \)
Fig. 10  The cubic-quartic soliton of the Im. part of kerr-low Eq. (74) in 2D and 3D with values: $w_1 = -8$
values: $A_{-1} = 0, A_0 = 1, A_1 = 1, a_1 = 2.4i, a_2 = 0, k_1 = q_1 = d_1 = \xi_0 = 1, c_1 = -1, \delta_1 = -3$

Fig. 11  The cubic-quartic soliton of the Re. part of quadratic-low Eq. (98) in 2D and 3D at: $w_1 = -8$,
values: $A_{-1} = 0, A_0 = 4, A_1 = 1, a_1 = 0.6i, a_2 = 0.6i, b_1 = -0.4i, e_1 = d_1 = k_1 = q_1 = \xi_0 = 1, c_1 = -6, \delta_1 = -16 + 1.1i$

Fig. 12  The cubic-quartic soliton of the Im. part of quadratic-low Eq. (99) in 2D and 3D at: $w_1 = -8$,
values: $A_{-1} = 0, A_0 = 4, A_1 = 1, a_1 = 0.6i, a_2 = 0.6i, b_1 = -0.4i, e_1 = d_1 = k_1 = q_1 = \xi_0 = 1, c_1 = -6, \delta_1 = -16 + 1.1i$
Fig. 13 The cubic-quartic soliton of the Re. part of parabolic low Eq. (124) in 2D and 3D with values: $A_0 = 0, A_{-1} = A_1 = 1, a_0 = a_2 = k_1 = q_1 = b_1 = c_1 = d_1 = \zeta_0 = 1, g + h + e = -24, \delta_1 = -9, w_1 = -8$

Fig. 14 The cubic-quartic soliton of the Im. part of parabolic low Eq. (125) in 2D and 3D with values: $A_0 = 0, A_{-1} = A_1 = 1, a_0 = a_2 = k_1 = q_1 = b_1 = c_1 = d_1 = \zeta_0 = 1, g + h + e = -24, \delta_1 = -9, w_1 = -8$

Fig. 15 The cubic-quartic soliton of the Re. part of parabolic low Eq. (134) in 2D and 3D with values: $A_0 = A_{-1} = 1, A_1 = 0, a_1 = \sqrt{6} i, a_2 = 0, k_1 = q_1 = d_1 = \zeta_0 = 1, \delta_1 = -0.1, c_1 = -d_2, w_1 = -8, e + g + h = 0$
Fig. 16 The cubic-quartic soliton of the Im. part of parabolic low Eq. (135) in 2D and 3D with values:
\[ A_0 = A_{-1} = 1, A_1 = 0, a_1 = \sqrt{6} i, a_2 = 0, k_1 = q_1 = d_1 = \zeta_0 = 1, \delta_1 = -0.1, c_1 = -d_1, w_1 = -8, e + g + h = 0 \]

Fig. 17 The cubic-quartic soliton of the Re. part of parabolic low Eq. (139) in 2D and 3D with values:
\[ A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = 0.8, k_1 = q_1 = d_1 = \zeta_0 = \delta_1 = 1, c_1 = -d_1, w_1 = -8, e + g + h = 0 \]

Fig. 18 The cubic-quartic soliton of the Im. part of parabolic low Eq. (140) in 2D and 3D with values:
\[ A_0 = A_{-1} = 1, A_1 = 0, a_1 = a_2 = 0.8, k_1 = q_1 = d_1 = \zeta_0 = \delta_1 = 1, c_1 = -d_1, w_1 = -8, e + g + h = 0 \]
Fig. 19 The cubic-quartic soliton of the Re. part of non-local low Eq. (160) in 2D and 3D with values: $A_{-1} = A_{1} = A_{2} = 0, |a_{1}| = 1.1, |a_{2}| = 0.44, A_{0} = A_{-2} = q_{1} = k_{1} = \delta_{1} = 1, c_{1} = -d_{1}$

Fig. 20 The cubic-quartic soliton of the Im. part of non-local low Eq. (161) in 2D and 3D with values: $A_{-1} = A_{1} = A_{2} = 0, |a_{1}| = 1.1, |a_{2}| = 0.44, A_{0} = A_{-2} = q_{1} = k_{1} = \delta_{1} = 1, c_{1} = -d_{1}$

Fig. 21 The cubic-quartic soliton of the Re. part of non-local low Eq. (160) in 2D and 3D with values: $A_{-2} = A_{0} = A_{2} = a_{2} = 0, A_{-1} = 0.3, |a_{1}| = 1.1, A_{1} = q_{1} = k_{1} = \delta_{1} = 1, c_{1} = -d_{1}$
equations which appeared clearly in the plotted Figures 13, 14, 15, 16, 17 and 18). Finally, we also achieved new types of soliton to the cubic-quartic for the Non-local Low NLSE in polarization-preserving fibers in the framework of the suggested method, we can detect it through the plotted Figs. 19, 20, 21 and 22). All the new types of soliton that we documented for each case the four forms mentioned above individually "which weren't achieved before by any other authors" denote to the novelty of these results, especially compared with that achieved previously by Bekir and Zahran (2020); Bekir and Zahran 2021b; Hosseini et al. 2021a; Hosseini et al. 2021b; Hosseini et al. 2021c) who used various manners to study these forms that emerged from this model. Consequently, new distinct and impressive visions of the solitons to the quartic-cubic NLSE for birefringent fibers of these four different forms have been constructed and the achieved solitons will add improved extended studies for all modern telecommunication engineering and all related phenomena.

Data availability  Not available.

Code availability  Not available.

Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

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