Entanglement of distant atoms by projective measurement: the role of detection efficiency

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Abstract. We assess proposals for entangling two distant atoms by measurement of emitted photons, analyzing how their performance depends on the photon detection efficiency. We consider schemes based on measurement of one or two photons and compare them in terms of the probability to obtain the detection event and of the conditional fidelity with which the desired entangled state is created. Based on an unraveling of the master equation, we quantify the parameter regimes in which one or the other scheme is more efficient, including the possible combination of the one-photon scheme with state purification. In general, protocols based on one-photon detection are more efficient in setups characterized by low photon detection efficiency, while at larger values two-photon protocols are preferable. We give numerical examples based on current experiments.
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### 1. Introduction

Quantum networks based on atom–photon interfaces require full control of the atom–photon interactions and correlations [1]. A major issue, related to the realization of quantum repeaters [1, 2], is the entanglement of distant nodes. Existing proposals for establishing entanglement by deterministic interactions are usually very demanding on the technological side [3, 4]. Present experimental effort is therefore partially focused on the realization of entanglement of the internal degrees of freedom of distant atoms by projective measurement through photodetection [5]–[7]. These protocols are essentially based on first entangling atoms with their emitted photons, and then performing entanglement swapping from the photons onto the atoms by photodetection in a Bell measurement setup. In this context, several experiments have analyzed the coherence properties of photons emitted by distant atoms [8]–[16]. Atom–photon entanglement was demonstrated in [17, 18]. More recently, in [19] the protocol described in [7] was applied, thereby realizing entanglement of two distant atoms.

These investigations with single atomic systems are complementary to experimental studies which aim at realizing photonic interfaces with atomic ensembles. In this context, entanglement of the collective spins of distant atomic ensembles by photodetection was demonstrated in the experiments of [20]–[23]. Theoretical investigations are also studying interface implementations with different types of systems, such as solid-state based quantum bits [24]. More complex constructions have been recently theoretically discussed, which in principle allow for generating many-particle entangled states [25, 26] and performing quantum communication [27].

An important issue in protocols based on quantum state projection by photodetection is the role of the detection efficiency: detector quantum efficiency and finite photon collection efficiency of the optical setup constitute intrinsic limits to the fidelity of the produced entanglement, as well as to the rate at which a protocol can be successfully realized.
In this work, we focus on protocols for entangling two distant individual atomic systems [5]–[7] and analyze quantitatively how the photon detection efficiency affects their performance. We use two criteria to assess the protocols, the success probability with which their execution produces the required detection event, and the conditional fidelity, corresponding to the probability of finding the target entangled state after that event. We calculate these criteria from an unraveling of the quantum optical master equation describing the respective protocols. In their comparison, we find that schemes based on detection of two photons exhibit larger fidelity, while schemes requiring detection of a single photon have usually larger success probability. The latter ones therefore lend themselves to applying purification protocols. In particular, we study application of the purification protocol proposed in [28] to the entanglement scheme of [5] and identify the regimes of detection efficiency in which proposals based on single-photon detection are more efficient than proposals requiring coincidence detection of two photons. In our treatment, we do not consider dark counts in the photodetection; their effect, as well as that of imperfections of the optical setup, have been analyzed in [29].

This paper is organized as follows. In section 2, we summarize the basic features of the considered protocols and introduce the criteria we apply in order to determine their efficiency. In section 3, we introduce the theoretical tools that we use for calculating these criteria and which account explicitly for the detector efficiency. In section 4, we analyze and compare the efficiency of the various entanglement protocols. Experimental considerations are reported in section 5, before we conclude in section 6. In the appendix, we report some additional details.

2. Entanglement schemes

In this section, we summarize the basic features of recently proposed protocols [5]–[7] which exploit atom–photon correlations in order to achieve entanglement between the internal states of distant atoms. The discussion is focused on entanglement schemes, but teleportation generally follows analogous procedures (see, for instance [30]). In all the considered proposals, entanglement of the atomic internal states is achieved by interference of photons emitted from distant atoms and subsequent photo-detection projecting the atoms into a Bell state. These protocols are probabilistic in the sense that the expected result is conditioned on the successful detection of a photon.

In order to characterize the schemes we will apply two main criteria, the success probability, \( P_{\text{suc}} \), and the conditional fidelity, \( F \). The success probability of a protocol measures which fraction of its executions leads to the desired photon detection,

\[
P_{\text{suc}} = \frac{\text{#detection events}}{\text{#scheme executions}}.
\]

After a successful detection event, the conditional fidelity measures with which probability the density matrix of the system, \( \rho_M \), matches the density matrix of the desired target state, \( \rho_T \),

\[
F = \text{Tr} \{ \rho_T \rho_M \}.
\]

We will evaluate the state after the measurement, \( \rho_M \), using an unraveling of the master equation, where the detection event is explicitly introduced as an operator determining the system evolution. This formalism is developed in the next section. A more general (although less practical) quantity providing an overall comparison of the schemes is the average fidelity, \( \bar{F} \), which we define as

\[
\bar{F} = P_{\text{suc}} F,
\]

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and which corresponds to the total probability to obtain the desired state in one execution of the entanglement protocol. We will apply these efficiency criteria qualitatively when we present the various schemes in the reminder of this section; their calculation and a detailed evaluation will be presented in later sections.

Since all of the considered protocols are conditioned on the detection of photons, the fidelity and the success probability will depend on the efficiency, $\eta$, with which these photons are detected, as well as on the probability, $p$, with which they are emitted. Assuming that all optical elements perform perfect unitary local operation, and that the initial atomic state can be prepared with unit fidelity, the detection efficiency is determined by the finite photon-collection efficiency of the optical apparatus, $\chi$, i.e. the probability that an emitted photon reaches the detector, and by the quantum efficiency of the detector itself, $\eta_d$, through

$$\eta = \chi \eta_d. \quad (4)$$

The basic system of all considered proposals are two atoms with Lambda-shaped level configuration, confined at two distant positions in space. Information is encoded in the (meta-)stable internal states $|e\rangle_j$ and $|g\rangle_j$ which are coupled to a common excited state $|r\rangle_j$, where $j = 1, 2$ labels the atoms. Both atoms, which may be placed in free space or coupled to a resonator field, are initially prepared in the ground state $|e\rangle_j$. They are then either excited by a laser pulse to an intermediate state which thereafter evolves during a given detection time interval, or they are continuously laser driven during the detection time interval. We distinguish protocols which rely on the detection of one or of two photons during this detection time interval.

2.1. Single-photon schemes

We first focus on the proposal by Cabrillo et al [5], based on the projective measurement of a single photon emitted from either of two continuously excited atoms in free space. The typical setup is sketched in figure 1(a).

The two atoms are initially prepared in the state $|e\rangle_1|e\rangle_2$, from where they are weakly driven during the detection interval by a laser resonant with the $|e\rangle_j \rightarrow |r\rangle_j$ transition. Once an atom has been excited to the state $|r\rangle_j$, the protocol requires spontaneous emission along the transition $|r\rangle_j \rightarrow |g\rangle_j$ into one of the modes $\lambda$ of the free space electromagnetic field corresponding to this transition. Denoting by $\varepsilon$ the probability amplitude associated with this event, the atom–photon system is in the state

$$|\Psi^I\rangle = \alpha^2|e_1, e_2; 0\rangle + \varepsilon \alpha(|e_1, g_2; 1_\lambda\rangle + |g_1, e_2; 1_\lambda\rangle) + \varepsilon^2|g_1, g_2; 1_\lambda, 1_\lambda\rangle, \quad (5)$$

where $|\alpha|^2 + |\varepsilon|^2 = 1$, and the states $|0\rangle, |1_\lambda\rangle$ represent zero and one photon in the mode $\lambda$.

To obtain an entangled state, one mixes the collected photons on a 50:50 BS, such that at its outputs one cannot determine which atom emitted the photon, i.e. all which-path information is erased. When additionally a stable phase relation between the two scattering paths is established, i.e., single-photon interference of the probability amplitudes is warranted, then a click at the detector $D_{\pm}$ at one of the output ports projects the atoms onto one of the Bell states

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|e_1, g_2\rangle \pm |e_2, g_1\rangle). \quad (6)$$

For simplicity, we neglect the dependence of the probability amplitude on the mode (direction of space) in which the photon is emitted.

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Figure 1. Setups for entangling the internal degrees of freedom of two distant atoms by detection of one photon. The transition $|e⟩ \rightarrow |r⟩$ is driven by a laser. Photons emitted on $|r⟩ \rightarrow |g⟩$ are mixed on a 50 : 50 beam splitter (BS). A click of a detector $D_\pm$ ideally projects the atoms into the Bell state of equation (6). In (a) the photon is emitted by spontaneous Raman scattering into free space [5], while in (b) it is emitted by coherent Raman scattering into the mode of a cavity, which then decays into free space [6].

In order to obtain this target state with high fidelity, one has to suppress the emission of two photons, which would project the atoms into a product state. This may be achieved by only very weakly exciting the atoms on the $|e⟩ \rightarrow |r⟩$ transition, which implies $|\varepsilon|^2 \ll 1$ and hence entails low success probability. The success probability is also limited by the finite solid angle within which photons emitted into free space are collected, which leads to low detection efficiency. The latter problem can be partially solved by confining each atom inside a resonator, such that the photons are emitted preferably into the resonator modes which strongly couple to the atomic transition $|r⟩ \rightarrow |g⟩$ [6].

Instead of continuously exciting the atoms, one may apply a laser pulse to atoms placed in cavities, like in the teleportation protocol proposed by Browne et al [6], whose setup is sketched in figure 1(b). During the initial pulse, the states $|e⟩_j$ and $|g⟩_j$ are resonantly coupled via a stimulated Raman transition driven by the laser and the cavity field, such that the atom–cavity system is prepared in the state

$$|\Psi^I⟩ = (\alpha |e_1; 0⟩_1 + \varepsilon |g_1; 1⟩_1) (\alpha |e_2; 0⟩_2 + \varepsilon |g_2; 1⟩_2),$$

where $|n_j⟩$ ($n = 0, 1$) corresponds to $n$ photons in the cavity $j$. During the detection interval, starting after the preparatory laser pulse, photons may leak out from the cavities via the mirrors of finite transmittance, and are then mixed on a 50 : 50 BS to erase any which-path information. Again, single photon interference and subsequent detection of one photon projects the atoms
Figure 2. Setup for entangling the internal states of two distant atoms by measurement of two photons [7]. The atoms are prepared in the state $|r_1, r_2\rangle$ and then decay spontaneously. The photon wavepackets overlap at a 50 : 50 BS. The detection apparatus at the output ports of the BS involves two polarizing beam splitters (PBS) and four detectors. Coincident clicks at $D_{+e}$ and $D_{+g}$ or $D_{-e}$ and $D_{-g}$ project the atoms into the state $|\Psi_1^+\rangle$. The state $|\Psi_1^-\rangle$ is found by coincident clicks at $D_{+e}$ and $D_{-g}$ or $D_{+g}$ and $D_{-e}$.

into the entangled target state $|\Psi^{\pm}\rangle$ with a certain fidelity and success probability. Although the detection efficiency may be high in this scheme due to the enhanced emission probability into the cavity modes, the protocol is still limited by the finite probability that two photons are emitted of which only one is detected during the measurement interval, leaving the atoms in a product state.

2.2. Two-photon schemes

The conceptually slightly different approach discussed in [7] relies on a (partial) Bell state measurement of two photons, instead of single photon detection. In this protocol, the atoms are located in free space and are initially excited to the state $|r_1, r_2\rangle$. Each atom can decay along the two possible decay channels $|r\rangle \rightarrow |e\rangle$ and $|r\rangle \rightarrow |g\rangle$, which are assumed to produce photons of orthogonal polarizations, $\xi^e$ and $\xi^g$, at equal decay constants. The corresponding state into which the atoms decay reads

$$|\Psi^I\rangle = \frac{1}{2} \left( |e_1; 1_{\lambda_1, \xi^e}\rangle + |g_1; 1_{\lambda_1, \xi^g}\rangle \right) \left( |e_2; 1_{\lambda_2, \xi^e}\rangle + |g_2; 1_{\lambda_2, \xi^g}\rangle \right),$$

(8)

where the states $|1_{\lambda_j, \xi}\rangle$ represent one photon in the spatial mode of emission corresponding to atom $j$ and polarization $\xi^\lambda$. After the emission, each atom is maximally entangled with its corresponding photon [17, 18], and in case the two emitted photons have orthogonal polarizations, the two atoms must be in different ground states. A partial Bell state measurement performed on the two photons with the apparatus shown in figure 2 projects the atoms into the corresponding entangled state.

If one instead uses a plane-wave decomposition, the direction of the emitted photons is distributed over the whole solid angle according to the corresponding dipole pattern of emission.
In contrast to the single-photon schemes, this protocol is based on two-photon interference \[33\] of photon wavepackets overlapping on a BS, which makes it indistinguishable which atom decayed into which ground state.

The success probability for this scheme is low, as it relies on the detection of two photons, both of which are recorded with finite probability, although the detection efficiency could be enhanced by coupling the atoms to cavities. On the other hand, one finds that the conditional fidelity in an ideal setup approaches unity for coincident detection of two photons on certain detector combinations, see figure 2. This protocol has been applied in the experiment of Moehring et al \[19\].

3. The theoretical model

In order to derive systematically the efficiency of the entanglement schemes under the effect of finite detection probability, we use a master equation description for the density matrix \(\rho\) of photons and atoms. At time \(t = 0\), the system is in a given initial state, which depends on the specific protocol as explained in the previous section. Its dynamics are governed by the master equation

\[
\frac{\partial}{\partial t} \rho = \left[\mathcal{L} + J\right] \rho(t),
\]

where \(\mathcal{L}\) and \(J\) are superoperators that describe the dynamics of photon emission by the atomic system. They take the generic form

\[
\begin{align*}
\mathcal{L} \rho &= -R \sum_{j=1,2} \sum_{\xi} \left[ A_{\xi}(j) A_{\xi}^\dagger(j) \rho + \rho A_{\xi}^\dagger(j) A_{\xi}(j) \right], \\
J \rho &= 2R \sum_{j=1,2} \sum_{\xi} A_{\xi}(j) \rho A_{\xi}^\dagger(j),
\end{align*}
\]

where \(2R\) is the photon emission rate and the operators \(A_{\xi}(j)\) and \(A_{\xi}^\dagger(j)\) describe, respectively, the atomic dipole lowering and rising operators, when the photon is emitted by a spontaneous decay of the atom \(j = 1, 2\) \[5, 7\]. In protocols where the atoms are inside a resonator \[6, 30\], they correspond instead to the annihilation and creation operator of a cavity photon. The subscript \(\xi\) labels the photon polarization, which is relevant in the two-photon protocol \[7\].

In equation (9), the superoperator \(\mathcal{L}\) describes the damping process, while the superoperator \(J\) is the so-called jump operator, describing the quantum-state projection associated with the emission process. It can be decomposed into the sum

\[
J \rho = C \rho + (1 - \eta) J \rho,
\]

where \(\eta\) is the detection efficiency, defined in equation (4), and \(C = \eta J\) is the so-called click operator, describing quantum-state projection associated with a click of the detection apparatus (for a review, see \[32\] and references therein). The second term on the right-hand side (RHS) of equation (11) describes quantum-state projection associated with the photons which are emitted but not detected. Using the unraveling of the master equation \[32\]–\[34\], we write the solution of equation (9) as

\[
\rho(t) = U(t) \rho(0) + \int_0^t \! d\tau \! U(t - \tau) C U(\tau) \rho(0) + \int_0^t \! d\tau \! \int_0^\tau \! d\tau_1 U(t - \tau) C U(\tau - \tau_1) C U(\tau_1) \rho(0),
\]

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where the operator
\[ U(\tau) = \exp \left[ (\mathcal{L} + (1 - \eta)J)\tau \right] \] (13)
gives the evolution conditioned on no click at the detector in the interval of time \([0, \tau]\). Each term on the RHS of equation (12) can be identified with different detection scenarios over the time \(t\) in which the experiment is run. The first term corresponds to the case in which no click is recorded at the detector. The probability that this occurs is given by
\[ P_0 = \text{Tr}\{U(t)\rho(0)\} \] (14)
and the density matrix, conditioned on this event, is \(\rho^{(0)} = U(t)\rho(0)/P_0\). The second term corresponds to the case in which a click is recorded at time \(\tau\) and no further click until time \(t\). The probability of this event is
\[ P_1 = \text{Tr} \left\{ \int_0^t d\tau U(t - \tau) C U(\tau)\rho(0) \right\} \] (15)
and the corresponding density matrix is
\[ \rho_1(t) = \int_0^t d\tau U(t - \tau) C U(\tau)\rho(0)/P_1. \] (16)
Finally, the last term on the RHS of equation (12) describes events in which two photons are revealed at the detector at the instants \(\tau_1\) and \(\tau > \tau_1\), which occur at probability
\[ P_2 = \text{Tr} \left\{ \int_0^t d\tau \int_0^\tau d\tau_1 U(t - \tau) C U(\tau - \tau_1) C U(\tau_1)\rho(0) \right\} \] (17)
and produce the corresponding density matrix.

Photo-detection occurs in the schemes at the outputs of a BS whose two input ports are the modes into which atoms 1 and 2 (or cavities 1 and 2) emit. We now express the click operator \(C\) through the projectors of the corresponding von-Neumann measurement. This corresponds to decomposing the click operator into a sum accounting for the possible clicks one can record,
\[ C \rho = \sum_\xi \left[ C^\pm_\xi + C^-_\xi \right] \rho, \] (18)
where \(C^\pm_\xi\) describes quantum-state projection by recording a photon at the output port \(D^\pm_\xi\). Denoting by \(d^\pm_\xi\) and \(d^{\dagger}_\xi\) the operators for the output field after the BS, then
\[ C^\pm_\xi \rho = 2R \eta d^\pm_\xi \rho d^{\dagger}_\xi, \]
with
\[ d^\pm_\xi = \frac{1}{\sqrt{2}} \left( A^{(1)}_\xi \pm A^{(2)}_\xi \right), \] (19)
where we have used the input–output formalism to connect the output field to the system operators [35] and omitted the input noise terms\(^6\).

Let us now identify the density matrix of the system after a particular detection event which is to be used in the evaluation of the fidelity in equation (2). We consider the case in which one

\(^6\) The input noise terms give rise to the dark counts discussed in [29]. They can be discarded in our formalism as we assume that the external electromagnetic field is in the vacuum. Hence, their contribution to the photo-detection signal vanishes.

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click is recorded in the time interval \([0, t]\) at the detector \(D_{+,-}\), while no click has been recorded otherwise at any detector. This occurs with probability

\[
P_{+,-}(t) = \int_0^t d\tau \text{Tr}[U(t - \tau)C_{+,-}U(\tau)\rho(0)]
\]

and the system is projected into the state

\[
\rho_{+,-}(t) = \int_0^t d\tau U(t - \tau)C_{+,-}U(\tau)\rho(0) / P_{+,-}(t),
\]

which is the state \(\rho_{M}\) in a one-photon scheme. This state is the weighted sum of all trajectories which involve a single-photon detection in the interval \([0, t]\), at time \(\tau\). In appendix A, we show that the state at time \(t\), corresponding to a single trajectory involving a photon detection at the instant \(\tau\), does not depend on \(\tau\).

4. Efficiency of the protocols

The formalism developed so far allows us to evaluate the efficiency of the entanglement protocols described above, in terms of their success probability and fidelity.

4.1. Single-photon schemes

We first analyze the schemes based on single-photon detection [5, 6], beginning with the setup without cavities sketched in figure 1(a), and considering the situation that during a detection time interval \([0, T_{cw}]\) the two atoms are weakly driven by a laser which excites spontaneous Raman transitions from \(|e\rangle_j\) to \(|g\rangle_j\) via the excited level \(|r\rangle_j\). Simultaneously, the detectors monitor photon emission on the transition \(|r\rangle_j \rightarrow |g\rangle_j\). The atoms thus form effective two-level systems, with state \(|e\rangle\) decaying to \(|g\rangle\) by spontaneous Raman scattering; the rate of this process will be denoted by \(\Gamma_{eg}\). Then, the master equation governing the atomic dynamics is of the form given in equation (9), with \(A^{(j)} = |g\rangle_j \langle e|\) and with the dissipation constant \(R\) corresponding to \(\Gamma_{eg}/2\).

The probability that a photon is emitted by one atom at any time during \([0, T_{cw}]\) is

\[
p_1 = 1 - e^{-\Gamma_{eg} T_{cw}}.
\]

The corresponding success probability, that exactly one photon is recorded at the detectors, is calculated from equation (15),

\[
P_{\text{succ:1cw}} = 2\eta p_1 (1 - \eta p_1).
\]

We note that \(P_{\text{succ:1cw}} \approx 2\eta p_1\) for \(p_1 \ll 1\) or \(\eta \ll 1\). The conditional fidelity with which the atoms are in the target state \(|\Psi^-\rangle\) reads

\[
F_{\text{1cw}} = \frac{1 - p_1}{1 - \eta p_1},
\]

and \(F_{\text{1cw}} \approx 1 - (1 - \eta) p_1\) for \(p_1 \ll 1\) or \(\eta \ll 1\). Success probability and conditional fidelity are displayed in figures 3(a) and (b), respectively, as functions of \(T_{cw}\) and for various values of the detection efficiency \(\eta\). The fidelity is maximum for short times, i.e. for small excitation probability, and then decreases with increasing \(T_{cw}\), because the probability for emission of two photons increases with time. In the regime of small \(p_1\), both success probability and conditional
Figure 3. Efficiency of the protocol by Cabrillo et al [5] when spontaneous Raman scattering on $|e\rangle \rightarrow |g\rangle$ is continuously driven during the detection time interval $[0, T_{cw}]$. (a) Success probability, (b) conditional fidelity and (c) average fidelity, versus $T_{cw}$ and for $\eta = 0.05$ (solid line), $\eta = 0.5$ (dotted line) and $\eta = 0.95$ (dashed line).

Fidelity increase with the detection efficiency $\eta$. The increase of $F_{1cw}$ with $\eta$ is due to the fact that higher $\eta$ makes it more likely that a second emitted photon is also detected, such that this event would be discarded.

Figure 3(c) displays the average fidelity $\bar{F}_{1cw}$ defined in equation (3), which for this case reads

$$\bar{F}_{1cw} = 2\eta p_1 (1 - p_1).$$  \hfill (25)

One can see that it increases with the detection efficiency $\eta$, and it always reaches a maximum at time $T_{cw} = \ln(2)/\Gamma_{eg}$.

The figures give an indication how to choose $p_1$, i.e. the detection time interval $T_{cw}$, at a given detection efficiency $\eta$. Thereby one can optimize the desired efficiency parameters, putting emphasis on either $F$, $P_{suc}$, or $\bar{F}$. From equation (25), we find the maximal value for the average fidelity at $p_1 = 0.5$. However, with a setup like the one shown in figure 1(a), realistic values of $\eta$ are usually in the %-range or below (see the experimental considerations in section 5), which results in relatively low conditional fidelity when $\bar{F}$ is optimal. Larger values of $\eta$ have been included in figure 3 to account for the case that the same scheme of simultaneous excitation and detection may be applied to a system where the photon collection efficiency is enhanced by placing a cavity around each atom. In this regime of $\eta$ close to 1, one may achieve high fidelity at success probabilities close to 0.5, as the dashed curves in figure 3 show.

It should also be mentioned that the assumption of slow Raman excitations, leading to the scattering probability described by equation (22), is not strictly necessary. At stronger...
excitations, Rabi oscillations on $|e\rangle \leftrightarrow |r\rangle$ appear and equation (22) for $p_1$ has to be replaced by one that describes the atomic dynamics in this regime. Nevertheless, the results for conditional fidelity and success probability as functions of $p_1$ in equations (23) and (24) remain valid.

An alternative to this scheme is a protocol where the atoms have been initially prepared in the state $|\Psi_1\rangle_I$ by a short laser pulse. The excitation pulse is followed by a detection time interval $[0, T]$. This version of the protocol has been discussed in detail in [6] for a setup including cavities, as shown in figure 1(b). Preparation in the state $|\Psi_1\rangle$ of equation (7) is assumed to happen with unit fidelity. The subsequent dynamics are described by equations (9) and (10), where $R$ is replaced by the cavity decay constant $\kappa/2$, and $A_j$ and $A_j^\dagger$ are replaced by the annihilation and creation operators $a_j$ and $a_j^\dagger$ of photons in cavity $j$. Then the success probability, i.e. the probability to detect exactly one photon in $[0, T]$, is given by

$$P_{\text{succ:1pls}} = 2\eta p_{\text{cav}} (1 - \eta p_{\text{cav}}).$$  \hspace{1cm} (26)

where we have set

$$p_{\text{cav}} = |\epsilon|^2(1 - e^{-\kappa T}).$$ \hspace{1cm} (27)

The conditional fidelity reads

$$F_{1\text{pls}} = \frac{1 - |\epsilon|^2}{1 - \eta p_{\text{cav}}},$$ \hspace{1cm} (28)

and the average fidelity for this scheme is calculated as

$$\bar{F}_{1\text{pls}} = 2\eta p_{\text{cav}} (1 - |\epsilon|^2).$$ \hspace{1cm} (29)

These quantities are displayed in figure 4 as functions of $T$, $|\epsilon|^2$ and $\eta$. Like in the previously discussed scheme, large entanglement ($F$ close to 1) is obtained for small probability that one photon is created, $|\epsilon|^2 \ll 1$, see figures 4(b), (e) and (h), because this suppresses events where two photons are produced in the initial pulse. In contrast to the previous scheme, however, one observes in figures 4(b) and (e) that $F$ increases with increasing $T$. This is easily understood, as the number of photons created in the two cavities is fixed at $t = 0$, and with increasing time the absence of a second detection enhances the probability that no second photon is present in the cavities. This increase, on the other hand, depends critically on the detection efficiency, as can be seen in figure 4(e), because $\eta < 1$ allows for a second photon not being detected although it was initially created. In the limit $\eta \rightarrow 1$, when every created photon is also detected, high fidelity may be achieved at large success probability for any value of $|\epsilon|^2 < 1$, see figures 4(d) and (e); the time for reaching a given large fidelity, nevertheless, becomes longer the closer $|\epsilon|$ is to unity (figure 4(b)). The values for $T \rightarrow \infty$ are shown in figures 4(g)–(k). The average fidelity of this variant of the protocol is shown in figures 4(c), (f) and (k), and is not substantially improved over the scheme based on continuous excitation and detection.

4.2. Two-photon schemes

We now analyze the efficiency of the two-photon entanglement schemes [7], in a setup like the one sketched in figure 2. We assume that the initial state preparation in $|r_1, r_2\rangle$ is performed with unit fidelity. Out of this state the atoms decay spontaneously into the state $|\Psi_i\rangle$ of equation (8); the individual atomic decay rates are taken to be $\Gamma/2$ on both possible transitions, and photons are collected during the detection time interval $[0, T]$. With proper mode matching at the BS (but without the need for phase stability), coherence effects allow for eight possible two-photon
detections (including detection of two photons on the same detector) [31], of which four lead to the projection into the target Bell states. The success probability for this protocol is hence found by multiplying the probability of two-photon detection, equation (17), by the factor 1/2, yielding

\[ P_{\text{suc:2}} = \frac{1}{2} \eta^2 p_2^2, \] (30)

where \( p_2 = 1 - \exp(-\Gamma T) \) is the probability for each atom to emit a photon at any instant during the detection time interval \([0, T]\). In this protocol, the entanglement fidelity after detection of two clicks is unity,

\[ F_2 = 1, \] (31)

showing that once the successful event occurred, the atoms are found in the corresponding Bell state independently of \( \eta \) and \( p_2 \). The main limitation to the efficiency of the two-photon scheme is hence the photon detection probability, which typically gives small values of \( P_{\text{suc:2}} \); this might be substantially enhanced in a setup in which each atom is coupled to the mode of a resonator. The average fidelity for the two-photon scheme is equal to the success probability, and reads

\[ \bar{F}_2 = \frac{1}{2} \eta^2 p_2^2. \] (32)

4.3. Single-photon versus two-photon protocols

We now compare the efficiency of one- and two-photon protocols, focusing on the setups where the atoms emit in free space. First, we look at the average fidelity \( \bar{F} \): in the one-photon scheme it
is maximized to $\tilde{F}_{1\text{cw}} = \eta/2$ by using $p_1 = 0.5$, while in the two-photon scheme the maximum value is $\tilde{F}_2 = \eta^2/2$ for $p_2 \to 1$. Thus, in this respect a low detection efficiency $\eta$ favors the one-photon scheme. A practical assessment also needs to consider the real time requirements, i.e. how many trials the schemes allow to be carried out during some given experimental time. In this respect, Raman scattering at rate $\Gamma_{eg}$ is necessarily slower than spontaneous decay at rate $\Gamma$; on the other hand, this is offset by the fact that $p_2 \to 1$ requires $T \gg 1/\Gamma$, whereas $p_1 = 0.5$ only needs time $T_{1\text{cw}} \approx 1/\Gamma_{eg}$.

If the emphasis is on high fidelity, in the one-photon scheme this can only be reached for small probability of photon emission, $p_1 \ll 1$. In the two-photon scheme, on the contrary, $F_2 = 1$ independently of the various parameters. In order to explore in a practical manner which scheme has larger success probability for equal fidelity, we fix a threshold value $F_{\text{th}}$, such that if $F_{1\text{cw}} > F_{\text{th}}$, we operationally consider the experiment successful. Using equation (24), the threshold condition is

$$\frac{1 - p_1}{1 - \eta p_1} > \frac{1}{1 - \eta F_{\text{th}}} \iff p_1 < \frac{1 - F_{\text{th}}}{1 - \eta F_{\text{th}}}.$$  \hspace{1cm} (33)

For the corresponding success probabilities, we obtain $P_{\text{suc:1\text{cw}}} > P_{\text{suc:2}}$ when

$$2\eta p(1 - \eta p) > \frac{\eta^2}{2} \iff p - \eta p^2 > \frac{\eta}{4},$$  \hspace{1cm} (34)

where we have used equation (23), and equation (30) in the limit $p_2 \to 1$. Figure 5 shows these conditions for various values of $F_{\text{th}}$. In the practically relevant regime $\eta \ll 1$, one finds that fidelity values $F \leq 1 - \eta/4$ are reached with higher success probability by the one-photon scheme, while for $F > 1 - \eta/4$, the two-photon scheme shows higher $P_{\text{suc}}$. As mentioned before, a full comparison, i.e. for real experimental settings, also needs to take into account the time per execution which the considered protocols require, see section 5.

4.4. Purification

In general, one may consider to increase the success probability $P_{\text{suc:1}}$ by increasing the probability of photon emission $p_1$. Although this occurs at the expense of fidelity, one can then apply a purification protocol, in order to generate one highly entangled state out of several states with lower fidelity. A one-photon scheme will then result more efficient than the corresponding two-photon scheme, if the purification protocol converges over times shorter than the typical time for a successful two-photon event.

Following this strategy, we now compare the efficiency of the one-photon protocol in [5], to which purification is applied, with the two-photon scheme in [7] under otherwise equal conditions. We require, like before, a threshold fidelity to be reached in the preparation of a single entangled pair by means of the creation and subsequent purification of $n$ identical lower-fidelity copies, $F_{\text{pur}} > F_{\text{th}}$. Then we compare the total success probability $P_{\text{pur}}$ with the success probability of the two-photon scheme, $\eta^2/2$. The total success probability $P_{\text{pur}}$ is given by the product of the success probability for the preparation of $n$ entangled pairs, $P_{\text{suc:1}}/n$, and the probability for a successful realization of the purification protocol with $n$ prepared entangled pairs, $P_{\text{pur}}(n)$:

$$P_{\text{pur}} = \frac{1}{n} P_{\text{suc:1}} P_{\text{pur}}(n).$$
Figure 5. Comparison of the efficiency of the one-photon protocol by Cabrillo et al [5] and the two-photon scheme by Simon et al [7] when a threshold fidelity $F_{\text{th}}$ is set. The horizontal axis is the emission probability $p_1$ of the one-photon scheme, while $p_2$ has been set to 1. The vertical axis is the detection efficiency $\eta$ which affects the efficiency of both protocols. The regions above the dotted, dashed and dash-dotted lines correspond to $F_{\text{lew}} > F_{\text{th}} = 0.95, 0.98$ and 0.99, respectively. The region below the solid line corresponds to $P_{\text{suc}:1} > P_{\text{suc}:2}$. In the shaded parameter regimes both conditions are satisfied, and under the threshold criterion the one-photon protocol is more efficient than the two-photon scheme. The inset enlarges the area about the origin.

We evaluate $p_{\text{pur}}(n)$ for the purification protocol proposed in [28]. It makes use of $n = 2^J$ entangled pairs, with $J$ number of iteration steps of the protocol. The pairs are divided into $n/2 = 2^{J-1}$ couples of entangled pairs. Unitary local operation and local measurements (for which the details are found in [28]) are applied to each couple leading to $n/2$ entangled pairs with fidelity higher than that of the initial pairs; the remaining $n/2$ pairs are discarded. The protocol is repeated $J$ times until a single highly entangled pair remains. Therefore, if $N_{j-1}$ gives the probability of having purified one entangled pair at step $j$ of the protocol [28], then the probability for the successful realization of the protocol consisting of $J$ steps is

$$p_{\text{pur}}(n = 2^J) = \prod_{j=1}^{J} N_{j-1}^{2^{J-j}},$$

(35)

whereby $2^{J-j}$ is the number of couples of entangled pairs at the $j$th purification step.

We note that the purification protocol in [28] allows for the preparation of the Bell state $|\Phi^+\rangle$. In order to apply it to the entanglement scheme in [5], which allows one to create the Bell
Figure 6. Efficiency with purification. The area on the left of each dashed curve corresponds to the region of parameters \( p_1 \) and \( \eta \), where \( F_{\text{pur}} > F_{\text{th}} = 0.99 \), when the state is prepared with \( J = 0,1,2,3,4 \) iteration steps of the purification protocol (note that \( J = 0 \) corresponds to the result obtained without purification, see figure 5). Similarly, the areas below the solid lines correspond to the parameters where \( P_{\text{pur}} > P_{\text{suc:2}} \) after \( J \) purification steps.

state \( |\Psi^+\rangle \), one must perform the unitary transformation \( |\Psi^+\rangle \rightarrow |\Phi^+\rangle \) before the application of the purification\(^7\). Then, at the end of the protocol one inverts the transformation.

In figure 6, we compare the efficiency of the purified single-photon scheme with the efficiency of the corresponding two-photon scheme. The threshold fidelity is set to \( F_{\text{th}} = 0.99 \). The dark gray shaded area displays the parameter region where the single-photon scheme without purification is more efficient than the two-photon scheme, i.e. where \( F_{1\text{cw}} > 0.99 \) and \( P_{\text{suc:1}} > P_{\text{suc:2}} \), see also figure 5. The light gray shaded area, on the other hand, displays the parameter regime where the purified single-photon scheme is more efficient than the two-photon protocol, \( F_{\text{pur}} > 0.99 \) and \( P_{\text{pur}} > P_{\text{suc:2}} \), showing that this is also the case for larger values of \( p_1 \) and of \( \eta \). As \( p_1 \) is increased, one needs a larger number of iteration steps in order to reach fidelities \( F_{\text{pur}} \geq F_{\text{th}} \), and the corresponding success probability \( P_{\text{pur}} \) decreases.

5. Experimental considerations

To estimate what fidelities and success probabilities can be realistically achieved in experimental implementations of the proposals discussed mentioned, we will consider typical parameters of current ion trapping experiments.

\(^7\) It can be realized, for example, by the local unitary operation \( \sigma_x^{(1)}|\psi^+\rangle = |\phi^+\rangle \), where \( \sigma_x^{(1)} = |e\rangle_1\langle g| + |g\rangle_1\langle e| \) operates only on the first atom.
Benchmark values for the single-photon scheme proposed by Cabrillo et al [5] are derived for the ion trapping experiment with $^{40}$Ca$^+$ ions which is being operated by Hennrich et al [36]. Here, the three states $|e\rangle$, $|g\rangle$ and $|r\rangle$, are identified with the levels $S_{1/2}(m = -\frac{1}{2})$, $S_{1/2}(m = +\frac{1}{2})$ and $P_{1/2}(m = -\frac{1}{2})$, respectively. The transitions connecting the P-state with the two S-states are distinguished by their polarization, which allows for preparing the initial state $|e\rangle$ by optical pumping, as well as for identifying a photon emitted on $|r\rangle \rightarrow |g\rangle$. To reach high entanglement fidelities, the probability of exciting both atoms rather than just one has to be kept small, which is the case for low emission probability of a photon on $|r\rangle \rightarrow |g\rangle$. The probability for this spontaneous Raman process to happen during the detection time interval $[0, T_{cw}]$ is given by

$$ p_1 = \alpha_{eg}(1 - e^{-\gamma' T_{cw}}), $$

where $\gamma'$ is the effective rate at which state $|e\rangle$ is emptied by the laser excitation, and $\alpha_{eg}$ is the branching ratio for an atom to decay into $|g\rangle$ (see appendix B). Any desired value of the emission probability $p_1$ is easily achieved in the experiment through control of the Rabi frequency, i.e. by adjusting the laser intensity. The detection time interval in which the laser is applied and the detectors are gated can be as short as 12.5 ns for the considered apparatus. Setting the Rabi frequency to $\Omega_{cw} = 2\pi \times 5.4$ MHz, one achieves an emission probability of $p_1 \approx 0.15$ for $T_{cw} \approx 200$ ns. The detection efficiency of the setup is $\eta = \eta_0 \chi \approx 0.5\%$, estimated from the detector quantum efficiency, $\eta_0 \approx 0.25$, for photons emitted on the $P_{1/2} \rightarrow S_{1/2}$ transition (397 nm), and from the photon collection efficiency $\chi = (\Delta \Omega / 4\pi) L \approx 2\%$. The latter is determined by the solid angle within which photons are collected with a high numerical aperture (NA = 0.4) lens and the transmission losses in optical elements and fiber coupling, $L \approx 0.5$. For these values of detection efficiency and emission probability, one expects to obtain a conditional fidelity $F_{1cw} = 85\%$ at a success probability $P_{suc;1cw} = 1.5 \times 10^{-3}$, leading to an average fidelity $\bar{F}_{1cw} = 1.3 \times 10^{-3}$, see equations (23)–(25). It seems feasible to run the protocol at a rate of $10^5$ experimental sequences per second, which provides sufficient time for state preparation, detection and cooling. Hence, about 150 entangled pairs per second of the above fidelity may be generated. Since the scheme relies on single-photon interference for the creation of entanglement, interferometric stability of the setup is required, which is experimentally challenging.

One may hope to obtain better values when increasing the detection efficiency by placing the atoms in cavities and preparing entangled atom–cavity states by a short initial laser pulse, as described in section 2. Here, we consider such a system on the basis of an experimental implementation with a single trapped calcium ion coupled to a high-finesse cavity [37]. The cavity, in this case, couples strongly to the $P_{1/2} \rightarrow D_{3/2}$ transition, while the ion is laser-excited on the $S_{1/2} \rightarrow P_{1/2}$ transition. For the given setup, the decay constant of the cavity field was determined to be $\kappa / 2 = 2\pi \times 54$ kHz and photons were detected at a rate of 33 kHz for a mean intra-cavity photon number of $\langle n \rangle = 1$, from which we infer the detection efficiency $\eta = 0.31$. We identify the $S_{1/2}$, $P_{1/2}$ and $D_{1/2}$ states with the three levels of the $\Lambda$-system, $|e\rangle$, $|r\rangle$ and $|g\rangle$, respectively. Initially, the atoms are assumed to be prepared in the state $|\Psi'\rangle$ of equation (7) by a laser pulse, which drives a cavity-assisted coherent Raman transition on $|e\rangle \rightarrow |g\rangle$. The Rabi frequency and the detunings of laser and cavity are chosen such that the probability to make

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8 See also http://www.icfo.es/groups/eschner.
the Raman transition is $|\epsilon|^2 = 0.15$, comparable to the previous scenario. For a given initial state and a fixed value of the cavity decay rate, the probability that a photon is emitted by the atom–cavity system, $p_{\text{cav}}$ of equation (27), is determined only by the detection time interval, $T$. Since the conditional fidelity, equation (28), for this set of parameters varies only in the range of 85% to 89% with detection time (see also figure 4(e)), one may choose to optimize the number of detection events per time interval, which is the experimental rate of scheme executions times the success probability of equation (26). The optimum value of about 2000 events per second is found for detection time intervals of approximately 10 $\mu s$, which results in $p_{\text{cav}} = 0.1$. Allowing for additional 20 $\mu s$ for cooling and for the preparation of the initial atom–cavity state, we obtain a rate of $3.3 \times 10^4$ experimental sequences per second. From these values, we calculate the expected success probability, $P_{\text{suc:1pls}} = 6.0 \times 10^{-2}$, the fidelity, $F_{\text{1pls}} = 0.88$ and the average fidelity, $\bar{F}_{\text{1pls}} = 5.3 \times 10^{-2}$. As expected, the higher detection efficiency due to the increased collection efficiency of fluorescence photons emitted from an atom coupled to a cavity, leads to higher success probabilities at similar values of the conditional fidelities as compared to the free-space scheme considered above. However, the experimental implementation of a single ion coupled to a high-finesse cavity is considerably more demanding than the free-space case.

Creation of entanglement through a two-photon scheme [7] has been experimentally realized in [19, 38]. Two remotely located trapped $^{171}\text{Yb}^+$ ions were entangled with fidelity $F_2 = 81\%$, and a Bell inequality violation by more than three standard deviations was reported. We base our evaluation of the efficiency of the scheme on the experiment of [38]. In this work, each ion is initially prepared with near unit efficiency in its $^2\text{P}_{1/2}$, $|F = 1, m_F = 0\rangle$ state from where it decays with equal probability to either of the three $^2\text{S}_{1/2}$ states, emitting a 369.5 nm photon. Observing along the quantization axis, one detects $\sigma^-$ and $\sigma^+$-polarized photons from the decay to the $|1, 1\rangle$ and $|1, -1\rangle$ states, respectively, while $\pi$-polarized photons emitted in the decay to the $|0, 0\rangle$ state do not propagate along this direction. For detection time intervals significantly longer than the excited state lifetime ($\approx 8$ ns), the emission probability consequently reads $p_2 = 2/3$. Just as in the theoretical proposal, the polarization of each emitted photon is then entangled with the state of its respective ion. The photons are collected with high numerical aperture lenses and coupled into single mode fibers before they are overlapped on a free-space BS. Interference of the single photon wavepackets and coincidence detection of one photon at each output port of the BS projects the two ions into one of the four Bell states. Considering the quantum efficiency of the detectors, transmission through optical elements, fiber coupling, and the solid angle of photon collection, the detection efficiency was $\eta = 6.7 \times 10^{-4}$. At a typical experiment repetition rate of $5.2 \times 10^5$ per second, the generation of one entangled atom pair of the above fidelity every 39 s was reported, from which we infer a success probability of $P_{\text{suc:2}} = \frac{1}{4} \eta^2 p_2^2 = 4.9 \times 10^{-8}$. The factor 1/4 accounts for the fact that one out of all four Bell states was detected. With a conditional fidelity $F_2$ of 81%, estimated from the density matrix obtained by state tomography, the average fidelity reads $\bar{F}_2 = 4 \times 10^{-8}$. The measured fidelity was limited by experimental imperfections in the generation of ion–photon entanglement and imperfect interference contrast of the photon modes at the BS. It has to be pointed out that this scheme has the advantage of relying on two-photon interference, which significantly relaxes the conditions on the experimental setup as compared to the single-photon schemes, which require interferometric stability. This has certainly contributed to making the two-photon protocol the first one with which distant entanglement of single atoms was demonstrated.

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6. Conclusions

Based on recent proposals and experimental progress, we have studied the efficiency of protocols for entangling distant atoms by projective measurement of emitted photons, focusing on the role of the photon detection efficiency. We distinguish schemes based on one-photon and two-photon detection. For their comparison, we have calculated, using an unraveling of the master equation, their success probability and conditional fidelity. We conclude that for low detection efficiency, which is typical for current experiments, and when the fidelity is assumed to be similar in both methods, protocols based on the detection of a single photon exhibit larger success probability. This includes the possibility of applying state purification. We calculated the efficiency criteria for concrete situations encountered in recent or ongoing experiments. Beyond these criteria, the decision on which method to choose in a given experimental setting will also have to take into account that the experimental difficulty of the schemes may be significantly different (the only experimental result so far is based on two-photon detection [19]), and that there may be a detrimental effect of dark counts (see [29]). With marginal differences, the results of our analysis can be extended to the efficiency of teleportation protocols based on projective measurement [7, 30].

While the calculation of the efficiency criteria may vary with the physical system, we believe that these concepts are generally applicable to all systems that may be considered for the creation of distant entanglement, including atomic-ensemble, photonic and solid state implementations. In all such systems the detection efficiency will have a similar, important role for the use of the entanglement as a resource in quantum technologies.

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Appendix A

In this appendix, we show that, when a single photo-detection event occurs at the instant $\tau$ in the time interval $[0, t]$, the corresponding state of the system at time $t$ is the same independent of the specific instant $\tau$ in which the photon was measured. This implies that the state $\rho_{+, \xi}(t)$ in equation (21) satisfies the equation $\rho_{+, \xi}(t) = \hat{\rho}_{+, \xi}(t, \tau)$, where

$$\hat{\rho}_{+, \xi}(t, \tau) = U(t - \tau)C_{+, \xi}U(\tau)\rho(0)/\mathcal{N}_{+, \xi}(t, \tau)$$

is the state corresponding to a single trajectory when a single photon is measured at time $\tau$ in the detection interval $[0, t]$, and $\mathcal{N}_{+, \xi}(t, \tau) = \text{Tr}(U(t - \tau)C_{+, \xi}U(\tau)\rho(0))$. For simplicity, we just restrict to single-photon schemes and therefore we do not use the index $\xi$ for the field polarization in the operators $A^{(j)}_{\xi}$. In our demonstration, we will use the
following relations. Since in the considered setup each atom can emit at most a single photon then
\[ A^{(j)} A^{(j)} \rho(0) = \rho(0) A^{(j)} A^{(j)} = 0. \] (A.2)
This relation implies
\[ e^{-Rt} A^{(j)} A^{(j)} \rho(0) = A^{(j)} \rho(0), \] (A.3)
\[ A^{(j)} e^{-Rt} A^{(j)} A^{(j)} \rho(0) = e^{-Rt} A^{(j)} \rho(0). \] (A.4)
Using relation (A.2), we rewrite the evolution operator \( U(t) \) in equation (A.1) as
\[ U(t) \rho = [U_0(t) + (1 - \eta)U_1(t)] \rho, \] (A.5)
with
\[ U_0(t) \rho = e^{-Rt} \sum_j A^{(j)} A^{(j)} \rho e^{-Rt} \sum_j A^{(j)} A^{(j)}, \]
\[ U_1(t) \rho = \int_0^t \mathrm{d}t' U_0(t - t') JU_0(t') \rho. \] (A.6)
We use equations (A.3) and (A.4) in (A.5), and consider the term \( U(t - \tau)C_+ U(\tau) \rho(0) \) in equation (A.1). After some algebra, we get
\[ U(t - \tau)C_+ U(\tau) \rho(0) = e^{-2Rt} \left[ \chi(t) \rho(0) \chi^\dagger(t) + 2(1 - e^{-2Rt}) A^{(1)} A^{(2)} \rho(0) A^{(1)} A^{(2)} \right], \] (A.7)
with
\[ \chi(t) = A^{(1)} e^{-Rt} A^{(2)} A^{(2)} + A^{(2)} e^{-Rt} A^{(1)} A^{(1)}. \] (A.8)
Substituting equation (A.7) in (A.1), we obtain \( \tilde{\rho}_{+ \xi}(t, \tau) = \rho_{+ \xi}(t) \).

Appendix B

The effective rate \( \gamma' \) at which population is removed from the initial state \( |e\rangle \) is given by the rates at which population is transferred either to the state \( |g\rangle \) or out of the considered three-level system to the \( D_{3/2} \)–state, \( |D\rangle \), to which the P-level can also decay. It reads
\[ \gamma' = \frac{\Omega_{er}^2}{\Gamma_r^2} (\Gamma_{rg} + \Gamma_{rd}), \] (B.1)
where \( \Omega_{er} \) is the Rabi frequency of the laser driving the transition \( |e\rangle \rightarrow |r\rangle \), \( \Gamma_r = 21.7 \text{ MHz} \) is the total decay rate of the excited state, and \( \Gamma_{rg} = 13.3 \text{ MHz} \) and \( \Gamma_{rd} = 1.7 \text{ MHz} \) are the decay rates from state \( |r\rangle \) into states \( |g\rangle \) and \( |D\rangle \), respectively.

The branching ratio \( \alpha_{rg} \) is, correspondingly,
\[ \alpha_{rg} = \frac{\Gamma_{rg}}{\Gamma_{rd} + \Gamma_{rg}} = 89\%. \] (B.2)
These modifications to the formalism in section 3 must be applied because of the presence of the additional decay channel to state \( |D\rangle \).
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