Dynamics of a nonlinear energy harvester with subharmonic responses

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Abstract. A frequency transmission band of the nonlinear energy harvester shall be studied numerically. For the analysis, the nonlinear piezoelectric energy harvesting system based on a cantilever elastic beam has been applied to harvest the kinetic energy of the moving frame. We used a double-well potential induced by permanent magnets for a ferromagnetic beam resonator. A piezoelectric patch attached along the beam was used as a transducer of the mechanical into electrical energy. It occurred that the system could work in a wide interval of frequency beyond the linear resonance. Besides the response with a period of excitation, solutions with dominating sub-harmonics of the harmonic inertial force excitation have been found. Particular solutions were illustrated, classified, and discussed using phase portraits and Fourier spectra of the output signals.

1. Introduction
Within the last decade, there was a strong development of energy harvesting (EH) systems, as they could be used as an efficient power supply for miniaturised low-power consumption sensors. Most of them are based on the conversion of energy accumulated in mechanical vibrations into electrical energy and they are called vibratory energy harvesters (VEHs) [1,2]. Harvesters can be classified depending on the type of physical phenomena applied in energy conversion. Namely, electromagnetic induction by Lenz’s law [3,4], piezoelectric [5,6], and electrostatic [7,8] effects were considered. Over the recent years, researchers studied and discussed various EH systems providing a broad amount of information on their designs as energy conversion efficiency, the influence of nonlinear effects, etc. collecting it in review articles [9-13]. One of the noteworthy types of EH systems is the piezoelectric one, usually consisting of the cantilever beam with one (unimorph) or two (bimorph) piezo-ceramic layers. In the excited beam, the voltage is induced in the electrodes as the result of strains occurring in the deflected beam. The simple design of the system can be presented in the form of the mathematical model. The configuration of the system mainly differs from each other by the number of potential wells in the system and the following configurations can be defined: monostable (one potential well), bistable (two potential wells), tristable (three potential wells), and multistable [14-16]. Specific potential well can be introduced by the application of additional magnets interacting with the beam and arisen equilibrium points can be stable or unstable [12,17].

The piezoelectric EH with two potential wells can be described with Duffing’s oscillator equation of motion, which can operate as intra-well (inside one of the wells) or inter-well (full oscillation between wells) [18-20]. The inter-well operation is desirable, as the output voltage is higher this state is activated after overcoming one of the potential barriers. Moreover, the analysed system has a strong nonlinear
character and its advantage over a linear system is that it can operate in a wider range of excitation frequencies, called the broadband effect [21-23]. The careful design of the nonlinear system can adapt it to the various excitation in the real environment vibration sources [24]. Namely, nonlinearities of the system lead to inclinations of the resonance curve adjusting the amplitude to a frequency in the resonance region [16]. Furthermore, additional resonances and multiple periodic and non-periodic (chaotic) solutions resulting occur. In the frequency spectra, except the main hardening or softening resonance curve, additional sub- or super- harmonics are present [16, 25-27]. Its origin is miscellaneous, they are related to higher periods, resonant or non-resonant operation or during activated inter- or intra-well behaviour. The motivation of the paper is to explore such solutions, which are dominated by sub-harmonic components. To indicate their occurrence, we have studied phase portraits of the mechanical resonator with the corresponding Poincaré points and apply the Fast Fourier Transform (FFT) to analyse the spectra for the specific frequency of excitation. Applied methods allowed to identify the dynamics of the system, and also to find optimal operating conditions.

The paper is organised as follows. After the present introduction, the scheme and mathematical description of the Piezoelectric Energy Harvester based on cantilever shall be provided (Section 2). Next, the frequency-sweep of the system is analysed by specific system’s behaviour (Section 3). Finally, Section 4 summarises the paper with proposed conclusions.

2. Nonlinear piezoelectric energy harvesting system

The energy harvester applied to the analysis is based on the piezoelectric cantilever beam with a bimorph structure mounted in the closed frame. In Figure 1, the tested system is presented. Additional permanent magnets mounted on the frame, create stable equilibrium points representing the double potential well. The concept of the ferromagnetic beam was studied in the series of Erturk et al. articles [12, 28-30] dealing with the experimental and modelling approach. In our analysis, the EH is subjected to the kinematic harmonic excitation with a random selection of initial conditions. Hereby, the responses with increasing frequency of excitation have been studied.

![Figure 1. Scheme of the cantilever beam piezoelectric EH. The bidirectional arrow refers to the external excitation of the system. On the right-hand side the schematic double-well potential is displaced. The lines indicate the position of the equilibrium points with green colour for stable and red for unstable, respectively.](image)

Derivation of the mathematical model [28] demands a coupling of mentioned parts conforming electromechanical differential equations of motion (Eqs.1 and 2):

\[
\ddot{x} + 2\zeta \dot{x} - \frac{1}{2}x^2 (1-x^2) - \chi v = F \cos(\omega t),
\]

(1)

\[
\dot{v} + \lambda v + \kappa \dot{x} = 0,
\]

(2)
where: $\zeta$ – mechanical damping ratio, $\chi$ – piezoelectric coupling term in the mechanical circuit equation, $\nu$ – voltage across the load resistance, $F\cos(\omega t) = A\omega^2 \cos(\omega t)$ – excitation force (inertial), $A$ – frame displacement amplitude, $\lambda$ – reciprocal of the time constant, $\kappa$ – piezoelectric coupling term in the electrical circuit equation, $\omega$ – angular frequency of excitation, $t$ – time.

All terms in the above equations have a dimensionless form. They are based on the equation of motion for the Duffing oscillator (Eq. 1) and Kirchoff laws regarding the electrical circuit (Eq. 2). The initial conditions (ICs) taken for simulation are random and they are selected from a determined range of values. As the final system’s performance, the root mean square (RMS) of an induced voltage is calculated. By the frequency sweep, the constant amplitude of excitation is considered $A=0.183$. The adopted value of $A$ enables to observe multiple solutions. For the simulation, the Runge-Kutta algorithm is applied with the system parameters presented in Table 1. Finally, the results of calculations are presented in a kind of a bifurcation diagram with the voltage-frequency relation in Figure 2.

### Table 1. Values of system parameters for the electromechanical model.

| Parameter | Value |
|-----------|-------|
| $\zeta$   | 0.01  |
| $\chi$    | 0.05  |
| $\lambda$ | 0.01  |
| $A$       | 0.183 |
| $\kappa$  | 0.5   |
| $\omega$  | (0.1 – 4.0) |

3. Results and discussion

Results voltage output (RMS) versus increasing frequency are presented in Figures 2(a) and (b) for the same initial conditions presented in Figure 3.

![Figure 2(a)](image1)
![Figure 2(b)](image2)

**Figure 2.** Voltage (RMS value) versus excitation frequency $\omega$ (angular frequency). For specific behaviour observed in the spectra. The particular cases (a-w) are collected in Table 2. Note that Figure 2(a) corresponds solutions for shorter time simulations with visible transients at $\omega > 2$ and stationary solutions while Figure 2(b) for longer time simulations with stationary solutions. Shorter and longer computation times correspond to 80 and 800 excitation cycles, respectively.
Depending on the simulation time, in Figure 2a results after a short interval simulation with possible transient solutions are presented, while Figure 2b corresponds to stationary solutions. Interestingly, several lines grouping (branches) vibration solutions of the given properties as periodic, non-periodic, intra- and inter-well solutions may be observed. Comparing, Figures. 2(a) and (b), one can notice that the branches are better visible in Figure 2(b). This is because they reach their stationary solutions, which show specific properties. On the other hand, for \( \omega > 2 \), one can observe some singular points beyond the branches, which are related to non-periodic transients. Note that in practice, in the ambient variable source of vibrations, transient behaviour can play an important role. Figure 2(b) shows, that such transient solutions converge to upper or lower branches representing intra- and inter-well oscillations, respectively.

![Figure 3. Initial velocity versus angular frequency \( \omega \). Note that for each \( \omega \), the value of initial \( x_0 \) was adopted as one of the potential minimum (\( x_0 =1 \) see Figure 1 for a schematic plot of the potential) and the nodal value of the initial voltage (\( v_0 =0 \)), while the initial velocity \( \dot{x}_0 \) was chosen as a random value from the interval [-1.5,1.5] with a uniform distribution. Cases (a)-(w) are indicated additionally by arrows (to identify the cases see also Figure 2 and Table 2).](image)

**Table 2. Analysed cases of excitation frequency \( \omega \).**

| Case | Frequency \( \omega \) (angular frequency) | Case | Frequency \( \omega \) (angular frequency) |
|------|------------------------------------------|------|------------------------------------------|
| a    | 0.245                                    | l    | 2.345                                    |
| b    | 0.505                                    | m    | 2.435                                    |
| c    | 0.605                                    | n    | 2.595                                    |
| d    | 0.755                                    | o    | 2.605                                    |
| e    | 0.895                                    | p    | 2.675                                    |
| f    | 1.145                                    | q    | 2.850                                    |
| g    | 1.595                                    | r    | 3.075                                    |
| h    | 1.645                                    | s    | 3.175                                    |
| i    | 1.995                                    | t    | 3.455                                    |
| j    | 2.095                                    | u    | 3.855                                    |
| k    | 2.205                                    | w    | 3.935                                    |

Similar branches were discussed in some previous papers on energy harvesting with double-well potentials [25,26,31]. Some of the periodic nature with a higher value of voltage output was identified as sub-harmonic solutions [25,26]. To shed more light on their nature and to classify the branches, we marked the specific solutions for selected frequencies (see frequencies in Table 2). For such frequencies, we provided a time series for transient and stationary solutions together with the corresponding phase portraits (including Poincaré points) and Fourier spectra. The results are provided in Figure 4.
Figure 4. Displacement and voltage time series (first and second column after shorter and longer computation time corresponding to 80 and 800 excitation cycles, respectively); phase plots with Poincaré stroboscopic points (third column) and the corresponding FFT spectra (fourth column) of longer computation time for particular cases (a)-(w) indicated in Figures 2 and 3. The case frequencies are provided in Table 2. To indicate, the response frequency in the relation to the excitation one, there are additional vertical red (dashed) lines in all Fourier spectra informing about the excitation frequency $\omega$. In the first two columns, the $x$ axis was scaled in $t/\omega'$ to compare the time series of system responses to different fixed omega excitations for the same excitation cycles.

Examining particular solutions, we start our analysis from the smallest angular frequency 0.1 and end up at 4.0. For fairly small $\omega$ case (in case ‘a’, $\omega=0.245$ see Figures 3 and 4(a)), the system oscillations synchronised with the excitation frequency are limited to a single potential well around the stable equilibrium point ($x=1$). It is important to remind, that the single well natural frequency of the linearised system is 1. Therefore, we observe the system escape from the potential well for $\omega\approx1$. The cases ‘c’ and ‘d’ belong to the inter-well oscillations with large orbits [32]. The responses are synchronised with the excitation (note that we have a single Poincaré point per close loop in the phase portraits (Figures 4(c) and (d)). The deviations of the phase plot orbits from the circular one inform about an additional higher periodic present in the Fourier spectra. Between the cases ‘a’ and ‘c’, we observe an interesting super-harmonic solution in the case ‘b’. Note that this case occurs for $\omega=0.5$, 

\begin{align}
\text{(s)}
\end{align}
which is half of the linearised natural frequency. In this case, we observe the single Poincaré point per two loops, and the dominating frequency is two times higher than the linear natural frequency. In the case ‘e’ (Figure 4(e)), we observe the chaotic solution. Its nature is indicated in non-periodic time series, where we see a nontrivial combination of intra and inter well oscillations, but also in the phase portrait, where the trajectory does not form to a closed orbit. Poincaré points are distributed forming a strange attractor and the frequency response is forming the frequency band. The similar chaotic solution is also visible in the case ‘h’ (Figure 4h) and between these two cases (frequency, $\omega \in [0.89, 1.70]$) there is the chaotic branch of solutions. Note that this kind of solution frequently appear in the nonlinear resonance frequency region, where a large response is present. The case ‘f’ (Figure 4(f)) is related to the intra-well resonance solution with important sub- and super-harmonic components visible in the Fourier spectrum. The phase portrait indicates two loops closed orbit and two Poincaré points. Note that this solution frequency, $\omega=1.145$ is relatively close to the natural frequency of the single potential well ($\omega=1$ for the linearized system).

On the other hand, the solution ‘g’ (Figure 4(g)) indicates the closed orbit of two loops and four Poincaré points. The consisting loops are of similar sizes. This phase portrait and the dominating frequency indicate that this solution can be classified as the sub-harmonic period 2 solution as the dominating period of the response is two times larger than the excitation period. In other words, the dominating frequency is two times smaller than the excitation frequency. The small frequency band components in the Fourier spectrum indicates that the convergence of the solution to stationary response is slow and probably some transients are still present in the time interval used to estimate the Fourier spectrum. On the other hand, the case ‘i’ (Figure 4(i)) belongs to a similar sub-harmonic period two solution branch. However, here we have an orbit, which is formed by a single closed loop and two Poincaré points, note the appearance of this branch in Figure 2(b). Formally, the transition from case ‘i’ to case ‘g’ can be performed by a period-doubling bifurcation. According to earlier papers, [25,26], such a class of solutions with even period sub-harmonic solutions are more pronounced in the non-symmetric potentials. Then, the next cases, i.e. ‘j’, ‘n’, ‘o’ ‘p’, and ‘r’ presented in Figures 4, (j),(n),(o),(p), and (r) are discussed. These cases represent large orbital solutions of similar nature. One can see a single loop and 3 Poincaré points in each of the phase portraits and the frequency responses dominated by $\omega/3$. Therefore, they are sub-harmonic period 3 solutions. Their corresponding branch is visible in Figure 2(b), where covers the fairly long frequency interval $\omega \in [1.7, 3.1]$. 

One can notice that for $\omega \in [2.3, 4.0]$ non-resonant intra-well solutions are present (Figure 2(b)). These solutions are represented by cases ‘l’, ‘m’, and ‘s’ (see also Figure 4(l),(m), and (s)). They show small orbits localised in one of the potential wells. Consequently, they provide low voltage outputs. The frequency spectra indicate a single frequency corresponding to the excitation. Interesting resonant solutions occur in cases ‘k’, ‘q’, and ‘t’ (Figures 4(k), (q), and (t)). They represent higher number loop solutions but in the single potential well. They are members of small branches of moderate voltage outputs. Note that they are logical continuations of the case ‘k’. In particular, the number of Poincaré points are increasing in the following way: for the case, ‘k’ – 2 points, while for ‘q’ and ‘t’ – 3 Poincaré points. However, the number of loops is increased in the case ‘t’ comparing to the case ‘q’. These loops are not comparable in size, and therefore it is not easy to indicate the correct integer number of them. Looking for the frequency spectra, we notice that the cases are dominating by the excitation frequency, however, sub-harmonic components play also important roles ($\omega/3$ in the case ‘q’, while $\omega/4$ in the case ‘t’). Within the studied frequency region, there are more sub-harmonic solutions. Namely, they are period 5 and 7 solutions represented by the cases ‘u’ and ‘w’ (see Figures. 4(u) and (w)). These solutions are inter-well large orbits solutions dominated by odd sub-harmonic components $\omega/5$ and $\omega/7$. They are also characterised by the closed orbits with 5 and 7 Poincaré points, respectively. These solutions are also members of the sub-harmonic solution branches (sub-harmonics of period 5 for $\omega \in [3.0, 4.0]$ and period 7 for $\omega \in [3.5, 4.0]$). In the graphs, velocity versus displacement (Figure 4), multiple periodic solutions are presented. Among the presented distinctions, there are chaotic solutions c) and h). In Figure 2(b), periodic solutions are arranged along the linear sections (linear systems in the graph RMS (voltage) vs. frequency), while the chaotic solutions outside the linear sections.
4. Conclusions
We have analysed the dynamics of a simple nonlinear model for the piezo-magneto-elastic energy harvesting system described by the Duffing double-well potential of the mechanical beam resonator. Note that in Figure 2 we used one initial condition set per each excitation frequency \( \omega \). Therefore, we had only one solution for any \( \omega \). However, higher mesh in the frequency combined with a random choice of initial condition for each frequency shed more light on bifurcations. Namely, solutions are grouped in particular branches.

As expected, higher voltage responses occurred for the larger amplitude of the resonator. In contrast to the linear system, where the higher voltage is localised in frequency around the resonance value, \( \omega_o \approx 1 \), in the case of the conducted research the higher voltage appears fairly beyond this value forming the band response. This is not only related to sub-harmonic and super-harmonic resonances of intra-well responses, but also to the more complex inter-well oscillations including large orbits of one and multiple excitation periods (sub-harmonic) solutions or chaotic solutions formed on the frequency voltage diagram the characteristic sub-harmonic branches. Furthermore, we also observed the voltage amplitude increase for transient solutions. Such transients can be also used in real applications. In general, our stationary and transient results show possibilities of nonlinearities exploitation in nonlinear energy harvesters, however, there is still a problem of coexisting solutions with small voltage outputs. This needs special procedures or conditions to favour the solution with higher kinetic energy. Consequently, further analysis is necessary to determine the stability of the indicated large orbit solutions.

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