On the Complexity of Computing the Topology of Real Algebraic Space Curves

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ABSTRACT
In this paper, we present a deterministic algorithm to find a strong generic position for an algebraic space curve. We modify our existing algorithm for computing the topology of an algebraic space curve and analyze the bit complexity of the algorithm. It is $O(N^3)$, where $N = \max\{d, \tau\}$, $d$, $\tau$ are the degree bound and the bit size bound of the coefficients of the defining polynomials of the algebraic space curve. To our knowledge, this is the best bound among the existing work. It gains the existing results at least $N^2$.

Categories and Subject Descriptors
I.1.2 [SYMBOLIC AND ALGEBRAIC MANIPULATION]: Algorithms—Algebraic algorithms

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Algorithm, Complexity

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algebraic space curve, topology, bit complexity.

1. INTRODUCTION
Algebraic space curves are used in computer aided (geometric) design, and geometric modeling. Computing the topology of an algebraic curve is also a basic step to compute the topology of algebraic surfaces. There have been many papers studied the guaranteed topology and meshing for plane algebraic curves [13, 14, 15, 16, 19, 23, 28, 33]. But there are only a few papers which studied the guaranteed topology of space algebraic curves [2, 11, 12, 15, 22, 25]. The complexity analysis for computing the topology of an algebraic space curve is also not deeply studied. In this paper, we will deal with this problem, which is one contribution of the paper.

Most of the existing work [2, 11, 12, 15, 25] of computing the topology of algebraic space curves require the space curve to be in a generic position. But checking whether an algebraic space curve is in a generic position or not is not a trivial task, see [2, 12, 16]. In this paper, we will give a deterministic algorithm to find a generic position for an algebraic space curve, which is another contribution of the paper.

Related work In [2], the space curve is assumed to be without singular points and in a generic position related to both the $xy$-plane and $xz$-plane, and any point of the algebraic space curve will not correspond to a singularity of both projection curves. Their generic position checking involves mainly resultant computation. In [25], ElKahoui considers the algebraic space curve defined by the $n$ tri-variate polynomials. They give a generic position definition (See Definition 4.1 in [25]) which is stronger than Definition 5 in this paper and provide a method to check it, which involves computation of Gröbner basis. In [15], the authors require the space curve to be in a generic position and any apparent singularities should be a node. They check the generic position by computing the subresultant sequence related to the defining polynomials of the space curve. The checking of another condition involves testing whether the Hessian matrix of the projection curve of the algebraic space curve at a plane algebraic point is regular or not. In [12], it requires only the algebraic space curve to be in a weak generic position. In [11] the authors provide a deterministic and easy way to compute a weak generic position for an algebraic space curve.

For the complexity of computing the topology of algebraic space curves, Diatta et. al [15] give a descriptive bit complexity of $O(d^{21}\tau)$ for the topology computation of an algebraic space curve implied the assumption of that the input algebraic space curve is in a generic position. Cheng et. al [12] give a complexity analysis which is bounded by $O(d^{27}\tau)$ also implied the assumption that the space curve is in a weak generic position hypothesis. Both of the algorithms imply the assumption of generic position since they provide only the way to check the generic position but not to find a generic position for the algebraic space curve.

Our contributions The main contributions of this paper are as follows. Firstly, we find a method which deterministically puts the original algebraic space curve in a generic position. It is easy to be implemented. As we know, all of the methods mentioned above except [11] provide only a way to check the generality of an algebraic space curve, they do not present a method to find a certified generic position deterministically. Secondly, we present an algorithm to compute the topology of an algebraic space curve, which is modified
2. NOTATIONS AND PRELIMINARIES

In this section, notations and known results needed in this paper are given. Let \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \) be the fields of rational numbers, real numbers and complex numbers respectively. \( \mathbb{Z} \) is the set of integers.

For the simplicity of notations, we use \( [q], q \in \mathbb{Z}_+ \) to denote the set of positive integers not greater than \( q \).

Let \( h(x, y) \in \mathbb{R}[x, y] \). We denote the plane algebraic curve defined by \( h = 0 \) as \( \mathcal{C}_h \). Let \( p = (x_0, y_0) \) be a point on \( \mathcal{C}_f \). We call \( p \) as an \emph{x-critical point} if \( f(p) = \frac{\partial f}{\partial x}(p) = 0 \), a \emph{singular point} if \( f(p) = \frac{\partial f}{\partial y}(p) = 0 \), \( x_0 \) is called an \emph{x-critical value} of \( f \) if \( p \) is an \emph{x-critical point} of \( f \).

Let \( \mathcal{C}_{f, g} \) denote the algebraic space curve defined by polynomials \( f(x, y, z), g(x, y, z) \in \mathbb{R}[x, y, z] \). We always use \( \mathcal{C} \) to replace \( \mathcal{C}_{f, g} \) when no ambiguity exists.

A point on an algebraic space curve \( \mathcal{C} \) is called an \emph{x-critical point} of \( \mathcal{C} \) if its projection point is an \emph{x-critical point} of the projection plane curve of the algebraic space curve.

We say a graph \( \mathcal{G} \) is \emph{isotopic} to an algebraic space curve \( \mathcal{C} \subset \mathbb{R}^3 \) if there exists a continuous mapping \( \gamma : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3 \) which, for any fixed \( t \in [0, 1] \), is a homeomorphism \( \gamma(\cdot, t) \) from \( \mathbb{R}^3 \) to itself, and which continuously deforms \( \mathcal{G} \) into \( \mathcal{C} \) : \( \gamma(\cdot, 0) = id, \gamma(\cdot, 1) = \mathcal{C} \).

We use a set \( \mathcal{R}, \mathcal{B} \) as the \emph{topology (information)} of a curve \( \mathcal{C} \) (or \( \mathcal{C}_h \)) if \( \mathcal{R} \) and \( \mathcal{B} \) satisfy the following two conditions:

- \( \mathcal{R} \) is the set of points which are located on \( \mathcal{C} \) (or \( \mathcal{C}_h \)), \( \mathcal{B} \) is the set of the connection relationship for the points in \( \mathcal{R} \).

- If we connect the points in \( \mathcal{R} \) as the connection relationship in \( \mathcal{B} \), then the obtained graph, denoted as \( \mathcal{G} \), is isotopic to \( \mathcal{C} \) (or \( \mathcal{C}_h \)).

We list some necessary definitions and results which have been introduced in \((11, 12, 28)\) below for the consistency of the paper.

**Definition 2.1** (Intersection of two plane curves). Let \( u, v \in \mathbb{Q}[x, y] \), \( \gcd(l(u, v), l(c, v)) = 1 \). \( \mathbb{N}_0 \) \( \in \mathbb{V} \mathcal{C}(\mathcal{W}), u(\alpha, y), v(\alpha, y) \) have only one common zero in \( \mathbb{C} \).

**Definition 2.2.** Let \( h \) be a squarefree polynomial in \( \mathbb{Q}[x, y] \). The real algebraic plane curve defined by \( h \), say \( \mathcal{C}_h \), is in generic position w.r.t. \( y \) if the following two conditions are satisfied:

1. The leading coefficient of \( h \) w.r.t. \( y \) is a constant.
2. \( \forall \alpha \in \mathbb{R}, h(\alpha, y), h_y(\alpha, y) \) have at most one common distinct zero in \( \mathbb{C} \).

According to the above definitions, the following corollary is immediate.

**Corollary 2.3.** Let \( f \in \mathbb{Q}[x, y] \) be a square free polynomial, \( \text{lcf}(h, y) \) is a constant, then the plane curve \( \mathcal{C}_f \) in a generic position w.r.t. \( y \) if and only if the intersections of \( h, h_y \) is in a generic position.

For a plane curve \( \mathcal{C}_h \), we can get a set \( S \subset \mathbb{R} \) s.t. the plane curve \( \mathcal{C}_h \) is in a generic position for \( \forall s \in S \) by resultant computation, where \( h = h(x + sy, y) \). It can be proceed as below: \( h = \text{sqrfree}(h), \text{let} q(x, s) = \text{sqrfree}(\text{Res}_s(h, h_y)), \text{Q}(s) = \text{sqrfree}([q(s)], \text{then} S = \mathbb{R} \setminus \text{V}_S(Q), \text{see} [9, 24] \).

The method above can be generalized as the following algorithm. We can simply call the algorithm \( \text{PCGP}(s) \).

**PCGP(s):** Given a plane curve with a parameter \( s \), say \( h(x,y,s) = 0 \), our aim is to find a proper \( \alpha, \beta \in \mathbb{R} \) s.t. the plane curve \( h(x, y, s) = 0 \) is in a generic position.

We will use this technique to compute a similar set \( S \subset \mathbb{Q} \) s.t. the sheared algebraic space curve is in a generic position under the shearing related to any number in \( S \) later. First, we give some relevant definitions.

**Definition 2.4** (11, 12). Let \( f, g \in \mathbb{Q}[x,y,z] \) such that \( \gcd(f,g) = 1 \), \( h = \text{sqrfree}(\text{Res}_z(f, g)) \), we say \( f, g \) are in a weak generic position w.r.t. \( z \) if

1. \( \gcd(\text{lcf}(f, z), \text{lcf}(g, z)) = 1 \).
2. There are only a finite number of \( (\alpha, \beta) \in \mathbb{V} \mathcal{C}(h) \subset \mathbb{C}^2 \) such that \( f(\alpha, \beta, z), g(\alpha, \beta, z) \) have more than one distinct common zeros in \( \mathbb{C} \).

The definition of generic position for an algebraic space curve here is weaker than that appears in [15], where not only the plane projection curve should be in a generic position, but the apparent singular point at each fiber in the plane curve should be a node. It is close to the definition Pseudo-generic position in [15]. In the following, we will provide a simple method to check weak generic position for an algebraic space curve.

**Theorem 2.5** (11). If \( \gcd(\text{lcf}(f, z), \text{lcf}(g, z)) = u \) be a constant, then \( f(x, y, z), g(x, y, z) \) is in a weak generic position w.r.t. \( z \) if and only if \( \forall \alpha \in \mathbb{C} \), such that \( f(\alpha, y, z), g(\alpha, y, z) \) is in a generic position w.r.t. \( z \).

According to the above theorem, we can transpose the problem of weak generic position checking of an algebraic space curve into the generic position checking of the intersection of two plane algebraic curves.

According to Theorem 2.5, it is equivalent to choose an \( \alpha \) and \( S_1 \) s.t. \( f(\alpha, y + sz, z) \land g(\alpha, y + sz, z) \) is in a generic position w.r.t. \( z, \forall s \in S_1 \). It can be proceed as follow: Let \( q_1(y, s) = \text{sqrfree}([f(\alpha, y + sz, z), G(\alpha, y + sz, z) \}), \) and compute \( Q_1(s) = \text{sqrfree}(q_1(s, \frac{\partial f}{\partial z}(s, \alpha))) \). Then the set \( S_1 \) can be chosen as the complementary set of \( V_3(Q_1(s)) \) in \( \mathbb{R} \). In the practical computation, the default value of \( \alpha \) is 0.
3. CERTIFYING GENERIC POSITION

In this section, we will present an algorithm to compute a deterministic generic position for an algebraic space curve.

Definition 3.1. Let \( f, g \in \mathbb{Q}[x, y, z] \) be squarefree polynomials. The algebraic space curve defined by \( f, g \) denoted as \( C \), is called in a generic position w.r.t. \( z \) if

1) \( f, g \) are in a weak generic position w.r.t. \( z \)
2) The projected plane curve \( C_{\bar{h}} \) is in a generic position w.r.t. \( y \), where \( h = \text{sqrfree}(\text{Res}_z(f, g)) \).

For an algebraic space curve \( C \) in a generic position, two \( x \)-critical points of \( C \) may correspond to the same projection \( x \)-critical point of \( C_{\bar{h}} \). We prefer to remove this case. So we give the following definition.

Definition 3.2. Let \( f, g \in \mathbb{Q}[x, y, z] \) be squarefree polynomials. The algebraic space curve defined by \( f, g \) denoted as \( C \), is called in a strong generic position w.r.t. \( z \) if

1) \( f, g \) are in a generic position w.r.t. \( z \)
2) For each \( x \)-critical point \( p \) of \( C_{\bar{h}} \), there is at most one \( x \)-critical points of \( C \) whose projection is \( p \).

We give an algorithm below to compute a deterministic strong generic position for an algebraic space curve.

Algorithm 1: Find-strongGP

Input: \( f(x, y, z), g(x, y, z) \in \mathbb{Q}[x, y, z] \).
Output: \( F \wedge G \) (isotopic to \( f \wedge g \)) is in a strong generic position.

1. If \( \deg(f) \neq \deg(f, z) \) and \( \deg(g) \neq \deg(g, z) \),
   \( f := f(x + z, y + z, z), g := g(x + z, y + z, z), \)
2. Let \( h_1(x, y, s_1) := \text{Res}_s(f(x + s_1 y, y, z), g(x + s_1 y, y, z)) \).
   Call \( \text{PCGP}(s_1) \) (see Section 2) to find \( s_1 \in \mathbb{Q} \) s.t. the projected curve of \( f(x + s_1 y, y, z) \wedge g(x + s_1 y, y, z) \) is in a generic position w.r.t. \( y \). Still denote the obtain new algebraic space curve as \( f \wedge g \);
3. Let \( h_2(x, y, s_2) := \text{Res}_s(f(x + s_2 z, z, y), g(x + s_2 z, y, z)) \).
   Call \( \text{PCGP}(s_2) \) to find \( s_2 \in \mathbb{Q} \) s.t. the \( x \)-coordinates of the \( x \)-critical points of the algebraic space curve \( f(x + s_2 z, z, y) \wedge g(x + s_2 z, y, z) \) are distinct. Still denote the obtain algebraic space curve as \( f \wedge g \);
4. Let \( h_3(x, y, s_3) := \text{Res}_s(f(x, y + s_3 z, z), g(x, y + s_3 z, z)) \).
   For \( s \in \mathbb{Q} \), find \( s_3 \in \mathbb{Q} \) s.t.
   1. the intersection of \( f(a, y + s_3 z, z), g(a, y + s_3 z, z) \) is in a generic position w.r.t. \( z \).
   2. Call \( \text{PCGP}(s_3) \) to find \( s_3 \in \mathbb{Q} \) s.t. the plane projected curve of \( f(x, y + s_3 z, z) \wedge g(x, y + s_3 z, z) \) is in a generic position w.r.t. \( y \).

Denote the space curve as \( F \wedge G \);
Return \( F, G \).

Theorem 3.3. Algorithm \[ \] is correct and it terminates in finite steps.

Proof. The termination of the Algorithm \[ \] is clear. We need only to prove the correctness of the algorithm. The first linear coordinate transformation makes the gcd of the leading coefficients w.r.t. \( z \) of the defining polynomials to be a constant. The second coordinate transformation of Algorithm \[ \] puts the projected plane curve in a generic position w.r.t. \( y \); After the transformation, the \( x \)-critical points of the new space curve are with different \( x \)-coordinates except for the case their \( (x, y) \) coordinate pair are the same. The third coordinate transformation makes the \( x \)-coordinates of all the \( x \)-critical points of the sheared algebraic space curve to be distinct. Note that the projection curve of the space curve after the second or third coordinate transformations both are in a generic position. The last coordinate transformation ensures that the space curve is in a weak generic position w.r.t. \( z \) and the projection curve is in a generic position. According to the definition \[ \] we know the output space curve \( F \wedge G \) is in a strong generic position w.r.t. \( z \).

As we know, this is the first time to find a deterministic (strong) generic position for an algebraic space curve, not just checking the generic position as presented in \[ \]

4. OUTLINE OF THE ALGORITHM

In this section, we consider the topology computation for an algebraic space curve \( f(x, y, z) \wedge g(x, y, z) \), \( f, g \in \mathbb{Z}[x, y, z] \). The main steps of the algorithm are presented in \[ \]. We modify the lifting step in the algorithm in \[ \] and add a preparatory step. Thus there is no assumption for the input algebraic space curve.

Overview of the algorithm.

1. Certified generic position: Call Algorithm \[ \] We denote \( C = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = G(x, y, z) = 0 \} \), where \( F \) and \( G \) are the output polynomials.
2. Projection: Compute the topology of the projection plane curve \( C_{\bar{h}} \).
3. Lifting: Get the space point candidates which may contain points of \( C \).
4. Compute \( s \): First, we compute a set \( S' \) such that every two sheared space point candidates (obtained from step 3) will not overlap after linear coordinate transformation and projection for any \( s \in S' \). Then we compute \( S'' \) s.t. \( \forall s \in S'' \) the sheared algebraic space curve \( F(x, y + s z, z) \wedge G(x, y + s z, z) \) is in a weak generic position w.r.t. \( z \). Let \( S = S' \cap S'' \), and we choose a rational number \( s \in S \).
5. Compute the topology of \( C_{\bar{h}} \): Let \( \mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid \mathcal{F} = F(x, y + s z, z) = 0, \mathcal{G} = G(x, y + s z, z) = 0 \} \) and project the space curve \( \mathcal{C} \) into the \( x y z \)-plane and compute the topology of the projected curve \( C_{\bar{h}} \), where \( \bar{h} = \text{sqrfree}(\text{Res}_z(\mathcal{F}, \mathcal{G})) \).
6. Computing the refined topologies and certifying: For the \( x \)-critical values of \( C_{\bar{h}} \) which does not appear in the \( x \)-critical values of \( C_{\bar{h}} \), we use the corresponding vertical lines to intersect \( C_{\bar{h}} \) and add the points in the topology information of \( C_{\bar{h}} \). We do the similar operation on \( C_{\bar{h}} \). Then we certify the space point candidates by comparing the topology information of \( C_{\bar{h}} \) and \( C_{\bar{h}} \).
7. Connection: Connect the space points by line segments in an appropriate manner (by comparing the connection of the points of \( C_{\bar{h}} \) and \( C_{\bar{h}} \)). Hence we get the topology of the algebraic space curve.

Remark: In Step 1, when we call Algorithm 1, the last step in Step 4.2 can be removed since \( F \wedge G \) in Step 1 can be not in a weak generic position.
4.1 Certified generic position

Please see Algorithm 1 and Remark above.

4.2 Computing the topology of $C_h$ ($C_h$)

In this step, we will compute the topology of algebraic plane curves $C_h$, $C_h$. We consider only $C_h$ for example. First, project the space curve $C$ onto the $xy$-plane to get a plane curve $C_{h}$, where $h = \text{sqfree}(\text{Res}_2(F, G))$. Then the topology information $\{\tilde{A}_h, \tilde{A}_h\}$ of $C_h$ can be easily computed. Actually, there are many papers [11, 13, 15, 18, 23, 26, 29, 33] dealing with the topology computation of a plane curve. We adopt the methods presented in [29] in this paper. In the process of topology computation of $C_h$, $p(x) = \text{sqfree}(\text{Res}_2(h, y))$ is also obtained. Assuming $\tilde{A}_h = \{\alpha_i, \beta_i\}, i = 1, \ldots, l$, we want to do is to certify which boxes have no intersections each other. But we can simplify the computation as below:

For $\forall i \in [l_0]$, we define a pair set $P_i$, accurately $P_i = \{(j, k) | j \in [l], k \in [l_i]\}$. Since $C_h$ is in a generic position, there is only one $x$-critical point of $C_h$ on the fibre related to $i$. We assume that the space point candidates related to the $x$-critical point of $C_h$ are $P_i = \{(j, k) | P_i \in [l]\}$. They may not be certified (Some of them may be certified, see [13]). But the space point candidates in $P_i \setminus P_i'$ are certified.

Let $S_i = \{s \in \mathbb{R}(J_{i,j_0} + s K_{i,j_0,k_0}) \cap (J_{i,j_1} + s K_{i,j_1,k_1}) = \emptyset, \forall (j_0, k_0) \in P_i, (j_1, k_1) \in P_i \setminus P_i' \}$, then the topology of $C_h$ is the isolated box $\tilde{A}_h$.

Now we choose a simple rational number $s \in S$ and define two maps $\phi_s$ and $\pi$.

\[
\phi_s : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \quad (x, y, z) \mapsto (x, y + sz, z)
\]

\[
\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad (x, y) \mapsto (x, y)
\]

\[\mathbb{F}(x, y, z) = \phi_s(F) = F(x, y + sz, z), \mathbb{G}(x, y, z) = \phi_s(G) = G(x, y + sz, z)\]

Then, we compute the topology of the plane curve $C_h$ as Section 4.2 In the next subsection, we will use the points on $C_h$ to certify the space point candidates in $SBS$.

4.5 Computing the two refined topologies and Certifying

Now, we will compute the refined topology information for the plane curves of $C_h$ and $C_h$ such that the numbers of $x$-fibers in their refined topology information are equal.

With the above preparation, we can show how to certify all the candidate boxes in $SBS$. The main idea for certification is comparing the points on the topology of $C_h$ and $C_h$. We just give a simple introduction for the certification of the elements in $SBS$, for more details, please refer to [11].

For each space point candidate $I_{i,j,k} = I_i \times J_{j,k} \times K_{i,j,k}$ in $SBS$, we assume that its image is $I_i \times J_{j,k} \times K_{i,j,k}$ under $\phi_s$, where $J_{j,k} = J_{j,k} - sK_{i,j,k}$. Then we project its image onto the $xy$-plane which leads to a box $I_i \times J_{j,k}$. Meanwhile, for the same $i$ we can get a disjoint interval set from the topology information of the curve $C_h$, that is, the isolating intervals of the real roots of $h(\alpha_i, y)$. Let this interval set be $J_i = \{J_{j,k} | j \in [l_i]\}$, where $l_i$ is the number of the distinct real roots of $h(\alpha_i, y)$. The certification of space point candidates are achieved by comparing each $J_{j,k}$ with the interval set $J_i$. See [11, 13] for more details.

After this operation, the superfluous candidate boxes are deleted and the left space point candidates have unique correspondence with the intervals in $J_i$. Then the left thing is to determine the connection relationship of space points between each two adjacent $x$-fibers, and it will be discussed in the next subsection.

4.6 Connecting and space curve recovery
Let us recall the information of what we have obtained. That is, the topology information of two plane curves \( C_a \) and \( C_b \), the space points lifted from the plane points on \( C_a \) and the correspondence relationship between these 3-d points and the 2-points on \( C_b \). As a matter of fact, this information is enough to recover the topology of \( C \) accurately.

Recovering the topology of the space curve means that we should find the connection relationship between the space points on each two adjacent parallel planes with the form \( x - \alpha = 0 \). Both \( C_a \) and \( C_b \) are in a generic position w.r.t. \( y \), \( C \) is in a generic position w.r.t. \( z \), and the certified space points have a one to one correspondence with the points on \( C_b \). Thus the connection of the space points can be obtained by the connection of \( C_b \) and \( C_a \). If necessary, we can add a vertical plane between two adjacent \( x \)-fibers to get the connection. In a word, we can get the connection relationship of the space points. For more details, please refer to [11, 12].

Thus, we get the topology of \( C \).

5. COMPLEXITY ANALYSIS

In this section, we will analyze the complexity for computing the topology of an algebraic space curve. In this paper, all the complexity means the bit complexity, and the \( \mathcal{O} \)-notation means that we ignore logarithmic factors. The complexity for computing the topology of a plane curve has been studied by several of previous algorithms. See [3, 27, 41, 17, 26], and the best complexity is \( \mathcal{O}(N^{10}) \) which is given in [26], where \( N = \max\{d, \tau\} \), and \( d \) is the degree bound for the input polynomial while \( \tau \) is the bit size bound for the coefficients of the polynomials. In the following part, we will show that the bit complexity of our algorithm is bounded by \( \tilde{O}(d^{3\nu}(d + \tau)) \) but without generic position assumption. This is the best complexity that improves the previous best known complexity bounds by a factor of \( d^3 \). First, we will give some notations and results for complexity analysis.

5.1 Basic results for complexity

For a univariate polynomial \( f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x] \) with roots \( z_1, \ldots, z_n \in \mathbb{C} \) and \( a_n \neq 0 \), \( n = \deg(f) \) denotes its degree. The separation bound of \( f \) is defined as \( \text{sep}(f) = \min_{z_i \neq z_j} |z_i - z_j| \). We take the conventions in [26] that an integer polynomial is called of magnitude \( (d, \tau) \) if its total degree is bounded by \( d \), and each integer coefficient is bounded by \( 2^\tau \) in its absolute value. For simplicity, we mostly ignore the logarithmic factors in \( d \) and \( \tau \) in the complexity bounds. \( \tilde{O} \) indicates that we omit logarithmic factors. For \( a \in \mathbb{Q} \), \( \mathcal{L}(a) \) is the maximum bit size of the numerator and the denominator. First, we list the complexity of several basic operations on univariate and bivariate polynomials.

**Definition 5.1.** For an interval \( I = [a, b] \), we define its bit size to be the maximal bit size of the endpoints of the interval, that is, \( \mathcal{L}(I) = \max(\mathcal{L}(a), \mathcal{L}(b)) \). Moreover, if a real number \( \xi \) is represented by an interval \( I = [a, b] \), then the bit size of \( \xi \) is defined to be \( \mathcal{L}(I) \).

**Lemma 5.2** (see [31, 32]). For a square-free polynomial \( f \) of degree \( n \) with integer coefficients of modulus less than \( 2^\nu \), we can compute isolating intervals for all real roots of \( f \) (for a single real root of \( f \)) of width less than \( 2^{-\nu} \) using no more than \( \tilde{O}(n^3 \tau + n^2 \nu) / (\mathcal{O}(n^2 \tau + n \tau) / \mathcal{O}(n \nu \tau + nu)) \) bit operations.

**Lemma 5.3** (Kerber 2012 [26]). Let \( g \in \mathbb{Z}[x] \) be a polynomial with degree \( d \) and bit size \( \lambda \). Its square-free part \( g^* \) can be computed in \( \tilde{O}(d^2 \lambda) \) and it has degree at most \( d \). Its bit size of each coefficient of \( g^* \) is bounded by \( \tilde{O}(d + \lambda) \).

**Lemma 5.4** ([11]). If both the magnitude of the univariate polynomials \( u(x) \) and \( w_1(x) \) are \( (d, \tau) \), then the magnitude of the product polynomial \( u(x) \cdot w_1(x) \) is \( (d, \tau) \).

**Lemma 5.5** (univariate polynomial evaluation [4]). If \( g \in \mathbb{Z}[x] \) is of magnitude \( (d, \tau) \), and a rational number \( a \) with bit size \( \sigma \), then evaluating \( g(a) \) has a complexity of \( \tilde{O}(d(\tau + \sigma)) \), and the bit size of \( g(a) \) is \( \tilde{O}(d\tau + \sigma) \).

According to Lemma 5.5 we have the following corollary:

**Corollary 5.6** (multivariate polynomial evaluation). Let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be of magnitude \( (d, \tau) \), and \( m \geq 1 \) rational numbers \( a_1, \ldots, a_m \) all with bit size \( \sigma \), then evaluating \( g(a_1, \ldots, a_m) \) has a complexity of \( \tilde{O}(d^m(\tau + \sigma)) \), and the resulting univariate polynomial is of magnitude \( (d, \tau + \sigma) \).

**Proof.** Its correctness follows easily that there are \( \tilde{O}(d^m) \) arithmetic operations and the bit size of each number is bounded by \( \tilde{O}(d\tau + \sigma) \). Thus the bit complexity is \( \tilde{O}(d^m(\tau + \sigma)) \).

**Corollary 5.7.** Let \( g \in \mathbb{Z}[x_1, \ldots, x_m] \) be a multivariate polynomial with degree \( d \) and coefficients bounded by \( 2^{\nu} \), and \( m - 1, (m \geq 2) \) rational numbers \( a_1, \ldots, a_{m-1} \) all with bit size \( \sigma \), then evaluating \( g(a_1, \ldots, a_m) \) has a bit complexity of \( \tilde{O}(d^m(\tau + \sigma)) \), and the resulting univariate polynomial is of magnitude \( (d, \tau + \sigma) \).

**Lemma 5.8** ([17]). Let \( f(x) \) be a polynomial in \( \mathbb{Z}[x] \) and \( \deg_{\nu}(f) \leq d \). Then the separation bound of \( f \) is

\[
\text{sep}(f) \geq d^{-\nu} (d + 1)^{\frac{1}{\nu} - d},
\]

thus \( \log(\text{sep}(f)) = \tilde{O}(d\tau) \). The latter provides a bound on the bit size of the endpoints of the isolating intervals.

**Lemma 5.9.** ([17]) Let \( f, g \in (\mathbb{Z}[y_1, \ldots, y_k])[x] \) with \( \deg_{\nu}(f) = p \geq q = \deg_{\nu}(g) \). \( \deg_{\nu}(f) \leq p \) and \( \deg_{\nu}(g) \leq q \). \( \mathcal{L}(f) = \tau \geq \sigma = \mathcal{L}(g) \). We can compute \( \text{Res}_{y_1}(f, g) \) in \( \mathcal{O}(p^\tau + q^\tau) \). And \( \deg_{\nu}(\text{Res}_{y_1}(f, g)) \leq 2pq \), and the bit size of coefficients for the resultant is \( \tilde{O}(p\nu + q\nu) \).

**Theorem 5.10** ([7, 20]). Let \( f_1(x, y), f_2(x, y) \in \mathbb{Z}[x, y] \) be of magnitude \( (d, \tau) \), and they have only trivial common factor in \( \mathbb{C}[x, y] \). Then their real roots can be computed with \( \tilde{O}(d^2(\tau + \sigma)) \).

We remark that in [13], we also present an efficient algorithm for solving zero-dimensional polynomial systems. The bit complexity for bivariate case is \( \tilde{O}(d^3(\tau + \sigma)) \). For the bit complexity of an algebraic plane curve, we have the following:

**Theorem 5.11** ([26]). If \( f(x, y) \in \mathbb{Q}[x, y] \) with magnitude of \( (d, \tau) \), then we can compute the topology of plane curve \( C_f \) in \( \tilde{O}(d^3(\tau + \sigma)) \) bit operations.

5.2 Main results for the complexity of the algorithm

In this subsection, we will give the complexity analysis for the algorithm step by step.
Theorem 5.12. Let \( f, g \in \mathbb{Z}[x, y, z] \) with magnitude of \((d, \tau)\). The bit complexity of Algorithm 1 is bounded by \( \tilde{O}(N^{20}) \), where \( N = \max(d, \tau) \). That is, we can find a deterministic strong generic position for an algebraic space curve in \( \tilde{O}(N^{20}) \) bit complexity.

Proof. In Algorithm 1 the bit complexity of Step 1 and Step 4.1 can be ignored compared to other steps. It is not difficult to find that the bit complexity of Step 2, Step 3 and Step 4.2 are similar. So we just consider the complexity of one coordinate transformation, say \( \{x, y, z\} \rightarrow \{x, y + sz, z\} \). The sheared polynomials \( f(x, y + sz, z), g(x, y + sz, z) \) are with magnitude \( (2d, \tilde{O}(\tau)) \). Then we compute \( h(x, y, s) = \text{sqrfree}(\text{Res}(f(x, y + sz, z), g(x, y + sz, z))) \). This computation needs \( \tilde{O}(d(2d + d)^{2s+1} \tau) = \tilde{O}(d^7 \tau) \) bit complexity by Lemma 5.10 and \( h(x, y, s) \) is of magnitude \((d^2, \tau)\) when ignoring the logarithmic factors. According to the plane generic position checking in [13] [17], we know it costs
\[
\tilde{O}(d^{310} + (d^2)^{3}\tau) = \tilde{O}(d^{20} + d^{19} \tau)
\]
bit complexity, which is bounded by \( \tilde{O}(N^{20}) \), where \( N = \max(d, \tau) \). So we prove the theorem.

It is obvious that the bitsize of \( s \) is \( O(\log d) \). Thus the magnitude of the sheared polynomials are of \((2d, \tilde{O}(d \log d + \tau)) = (O(d), \tilde{O}(d + \tau))\).

Lemma 5.13. The complexity of the lifting process is bounded by \( \tilde{O}(d^{11}(d + \tau)) \), and the bitsize for the endpoints of the space candidates is bounded by \( \tilde{O}(d^{13}) \).

Proof. We consider the \( x \)-critical value \( \alpha_i \in \mathbb{I}_i = [a, b] \) as an illustration. As discussed in Subsection 5.1 we divide it into two steps. First, we consider the isolating boxes for the real root of the system \( \Sigma = \{h(x, y) = h_i(x, y) = 0\} \), solving this system needs
\[
\tilde{O}((d^2)^9(d^2 + d\tau)) = \tilde{O}(d^{19}(d + \tau))
\]
bit complexity and the bitsize of the endpoints of those isolating boxes can be controlled by \( \tilde{O}(\tau) \) using the results in [13] since the univariate polynomials are with magnitude of \((d, d^2 \tau)\). Then these boxes are substituted into the polynomials \( F(x, y, z) \) and \( G(x, y, z) \), and the resulting interval polynomials \( F_i, G_i \) and \( G_i \) are of degree \( d \) and coefficients bounded by \( \tilde{O}(d^7) \). Thus the bitsize of the \( z \)-coordinate is bounded by \( \tilde{O}(d^7 \tau) \), and the evaluation of \( F_i, G_i \) yields \( \tilde{O}(d^3(d^2 + \tau)) = \tilde{O}(d^{19} \tau) \) bit complexity according to the Corollary 5.1. By Lemma 5.2 we know the complexity of isolating the interval polynomials is \( \tilde{O}(d^3d^3 \cdot d^2 \tau) = \tilde{O}(d^{17} \tau) \). Note that there are only one \( x \)-critical point in each \( x \) fiber. So we get the space points candidates lifted from \( x \)-critical value \( \alpha_i \) of the plane curve \( C_n \).

In order to get the space candidates lifted from the regular points of the topology information of \( C_n \) we do the following consideration. First we choose a rational value \( t_i \in \mathbb{I}_i \), and solve the bivariate system \( \{F(t_i, y, z) = G(t_i, y, z) = 0\} \). Both the magnitudes of the polynomials \( F(t_i, y, z) \) and \( G(t_i, y, z) \) are of \((d, d^2 \tau)\), solving this system requires
\[
\tilde{O}(d^3(d + d^8 \tau)) = \tilde{O}(d^{15} \tau)
\]
bit complexity according to Theorem 5.10 and the bitsize of the obtained isolating boxes is bounded by \( \tilde{O}(d^{13}) \). By comparing the solutions of \( \{F(t_i, y, z) = G(t_i, y, z) = 0\} \) and the branch numbers at the \( \alpha_i \) fiber of \( C_n \), we can easily get the space candidates lifted from the regular points, and the bitsizes of these boxes are bounded by \( \tilde{O}(d^{17} \tau) \).

So far, we get all the space candidate at the \( \alpha_i \) fiber and the total bit complexity is bounded by \( \tilde{O}(d^{15} \tau + d^8 \tau) \). There are \( \tilde{O}(d^7) \) fibers. Thus, the total bit complexity to get all the space point candidates is
\[
\tilde{O}(d^3) \cdot \tilde{O}(d^{15} \tau) = \tilde{O}(d^{19} \tau),
\]
and the bitsizes of the candidates are \( \tilde{O}(d^{11} \tau) \), this completes the lemma.

Now we analyze the complexity of computing \( S \).

Lemma 5.14. The complexity of computing \( S \) is bounded by \( \tilde{O}(N^{20}) \), where \( N = \max(d, \tau) \).

Proof. According to the Theorem 5.12 we know the complexity of computing \( S'' \) is \( \tilde{O}(d^{19}(d + \tau)) \). Now we consider only the process of computing \( S' \). In each \( x = \alpha_i \) fiber, there are \( \tilde{O}(d^7) \) candidates at most. Then for every two candidates, we just need \( \tilde{O}(1) \) multiplications. So the complexity is \( \tilde{O}(1) \tilde{O}(d^{11} \tau) = \tilde{O}(d^{11} \tau) \) if we use fast Fourier transform since the bitsize of the endpoint is bounded by \( \tilde{O}(d^{11} \tau) \).

When any two space point candidates on a fiber \( x = \alpha \) do a similar operation, there are at most \( \tilde{O}(d^7) \) combinations, which implies the bit complexity for computing \( S' \) is bounded by \( \tilde{O}(d^7) \tilde{O}(d^{11} \tau) = \tilde{O}(d^{19} \tau) \).

There are \( \tilde{O}(d^3) \) fibers at most, thus the total complexity for computing \( S' \) is bounded by
\[
\tilde{O}(d^3) \tilde{O}(d^{19} \tau) = \tilde{O}(d^{19} \tau).
\]
So the total complexity is dominated by \( \tilde{O}(d^{19}(d + \tau)) \), which is as desired.

As to the connection process, we have the following lemma.

Lemma 5.15. The bit complexity for the connection step is bounded by \( \tilde{O}(d^{19} \tau) \).

Proof. In the process of the connection determination, for the most cases, we can connect correctly easily by comparing the connection of the points in \( C_n \) and \( C_{n+1} \). As to the difficult case, we take the strategy which has been used in [12] [15] [10]. That is, adding an auxiliary number between each two \( x \)-critical values to certify the connection. It can be achieved by solving bivariate polynomials system \( \{F(t_i, y, z) = G(t_i, y, z) = 0\} \), where \( \alpha_i < t_i < \alpha_{i+1} \) and the bitsize of \( t_i \) is \( \tilde{O}(d^7 \tau) \), both \( F(t_i, y, z) \) and \( G(t_i, y, z) \) are with magnitudes of \((d, d^2 \tau)\). So it needs
\[
\tilde{O}(d^3(d + d^8 \tau)) = \tilde{O}(d^{15} \tau)
\]
bit complexity by Theorem 5.10. There are \( \tilde{O}(d^7) \) fibers, thus the total complexity for connection process is bounded by \( \tilde{O}(d^7) \tilde{O}(d^{15} \tau) = \tilde{O}(d^{19} \tau) \).

Theorem 5.16. Let \( f, g \in \mathbb{Z}[x, y, z] \) be with magnitude of \((d, \tau)\), then we can compute the topology of the algebraic space curve defined by the polynomials \( f(x, y, z) \) and \( g(x, y, z) \) in \( \tilde{O}(N^{20}) \) bit operations, where \( N = \max\{d, \tau\} \).

Proof. First we do coordinate transformation to \( f \land g \) s.t. the sheared space curve is in a generic position w.r.t. \( z \), and its complexity is bounded by \( \tilde{O}(N^{20}) \) according to Theorem 5.12. Moreover, the magnitude of \( F \) and \( G \) are also \((d, \tau)\) if we ignore the logarithmic factors.
Let \( h = \text{sqrfree}(\text{Res}_h(F, G)) \), and we compute \( h \) using
\[
\tilde{O}(d(d + d)^{2 + 1}d^2) = \tilde{O}(d^7)\]
bit operations, and the magnitude of \( h \) is \( (d^2, 2d^2) \) according
to Lemma 5.14. Then we compute the topology of the plane curve \( C_h \) in
\( \tilde{O}(d^7) \) bit operations, and the magnitude of \( h \) is \( (d^2, 2d^2) \).
This resultant computation needs \( \tilde{O}(d(d + d)^{2 + 1}d^2) \) bit operations
due to Theorem 5.11.

Assume the points in \( \mathbb{R}_h \) are \( \{i \times l_j, i \in [l_0], j \in [l] \} \) and
the bitsizes of the endpoints of \( I_i \) and \( J_i, j \) are bounded by
\( O(d^7) \) and \( O(d^{11}) \) respectively.

Now it turns to the processes of lifting and computing \( s \),
both complexity are \( \tilde{O}(d^{19}) \) by Lemma 5.13 and Lemma 5.11.

Then we choose a rational number (usually integer) \( s \in S \)
and consider the sheared space curve \( \mathcal{C}_s \), obviously, the bitsize
of \( s \) is \( \log(d^4 \cdot d^2) = 8 \log(d) \) according to Lemma 24 of
\[21\]. Moreover, the polynomials \( F, G \) have magnitude of
\( (d, d \log d + \tau) \). Then we compute \( h = \text{sqrfree}(\text{Res}_h(F, G)) \)
which has magnitude of \( (d^2, O(d^2 \log d)) \). This resultant
computation needs \( \tilde{O}(d(d + d)^{2 + 1}d^2 d \log d + \tau)) = \tilde{O}(d^9 \log d + \tau) \)
bit complexity. Then we analyse the topology of the
plane curve \( C_s \) which yields
\[
\tilde{O}((d^8(d^2 + O(d^2 \log d + dr))) = \tilde{O}(d^{19}(d + \tau))
\]
bit complexity. Theorem 5.11 this finishes the certifying step.

Due to Lemma 5.14, the complexity of connection step is
also bounded by \( \tilde{O}(d^{19}) \).

In conclusion, the total complexity for all steps are bounded by
\( O(N^{20}) \), where \( N = \max\{d, \tau\} \). Thus, we prove the theorem.

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