Normalized additional velocity distribution: a fast sample analysis for dark matter or modified gravity models

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ABSTRACT

Here we propose a fast and complementary approach to study galaxy rotation curves directly from the sample data, instead of first performing individual rotation curve fits. The method is based on a dimensionless difference between the observational rotation curve and the expected one from the baryonic matter \((\delta V^2)\). It is named as Normalized Additional Velocity (NAV). Using 153 galaxies from the SPARC galaxy sample, we find the observational distribution of \(\delta V^2\). This result is used to compare with the model-inferred distributions of the same quantity. We consider the following five models to illustrate the method, which include a dark matter model and four modified gravity models: Burkert profile, MOND, Palatini \(f(R)\) gravity, Eddington-inspired-Born-Infeld (EiBI) and general relativity with renormalization group effects (RGGR). We find that the Burkert profile, MOND and RGGR have reasonable agreement with the observational data, the Burkert profile being the best model. The method also singles out specific difficulties of each one of these models. Such indications can be useful for future phenomenological improvements. The NAV method is sufficient to indicate that Palatini \(f(R)\) and EiBI gravities cannot be used to replace dark matter in galaxies, since their results are in strong tension with the observational data sample.

Key words: galaxies: disc – dark matter – gravitation – methods: data analysis – methods: analytical

1 INTRODUCTION

Galaxies constitute one important and unique piece of information about dark matter phenomena. Considering the several different physical probes that indicate the dark matter existence, the standard approach is to consider that dark matter is here in the universe and it is more abundant than standard baryonic matter (Bergström 2000; Courteau et al. 2014; Bertone & Hooper 2018). On the other hand, it has not been directly detected and there are several different open possibilities for its microphysics (Bertone et al. 2005; Profumo 2017). If gravity in the Milky Way, or in other galaxies, is not essentially Newtonian, then there are more possibilities: either leading to different dark matter particles or perhaps replacing dark matter (e.g., Capozziello et al. 2007; Martins & Salucci 2007; Famaey & McGaugh 2012; Rodrigues et al. 2014; de Almeida et al. 2018; Green & Moffat 2019; Naik et al. 2019; Moffat & Toth 2021; Henrichs et al. 2021).

There are currently several dark matter halo and several modified gravity models that are candidates to explain dark matter-like phenomena in galaxies. Besides galaxy rotation curves (RCs), the possible difficulties faced by modified gravity without dark matter include solar system constraints, galaxy-galaxy lensing and cosmology. Nonetheless, it is important to understand which models can work well inside galaxies and which ones cannot. A model with good properties for explaining RCs, even if it has issues, can unveil new phenomenological correlations and can suggest new research directions.\(^1\) For this purpose, the development of fast sample tests is welcome. This is especially so for modified gravity, since these theories are harder to test due to their nontrivial dependence on the standard matter distribution.

In the following work we develop and apply a method that tests a given model with a sample of galaxies, instead of doing galaxy by galaxy fits. The method is based on the normalized additional velocity (NAV), which is denoted by \(\delta V^2\). By additional velocity we mean the additional component to the circular velocity beyond the Newtonian one due to the baryons. In the context of dark matter, the additional velocity will be the dark matter contribution to the RC. For modified gravity without dark matter, the additional velocity is the non-Newtonian contribution.

This method focus on a particular, but relevant, characteristic: the shape of the additional velocity, that is, how the additional velocity changes along the galactic radius, but neglecting the absolute amplitude of the changes. To this end, one normalizes the additional velocity, hence arriving at the NAV. This normalization commonly

\(^1\) An example of the latter comes from the Radial Acceleration Relation (RAR) (McGaugh et al. 2016), which is a phenomenological relation whose development was clearly inspired by MOND (Milgrom 2016). Afterwards, other theories aimed to understand and explain it (e.g., Ludlow et al. 2017; O’Brien et al. 2018; Ren et al. 2019; Green & Moffat 2019; Stone & Courteau 2019). The usefulness of the RAR is independent from whether MOND is or is not a good gravitational theory (Famaey & McGaugh 2012; Rodrigues et al. 2018a; Pardo & Spergel 2020; Skordis & Zlosnik 2021).

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elminates one model parameter and leads to a plane where the observational data is densely distributed and clearly correlated. Instead of either maximizing certain likelihood with priors, or minimizing a $\chi^2$ quantity for each galaxy, the aim is to compare the observational data distribution with the model data distribution. In order to find the distribution, we use a kernel density estimation (KDE), as detailed in Appendix A.

This work is organized as follows: in the next section we present the method, explaining how to compute $\delta V^2$ from the observational and the model perspectives. Section 3 is devoted to presenting the five models that are here considered. Section 4 details the approximations that are used for modified gravity models. In Secs. 5 and 6 we present respectively the results and the conclusions. Further details on some data and procedures can be found in the appendices A and B. Further specific details on the computations can be found in the NAVanalysis code.2

2 THE METHOD

2.1 Outline

The method deals with dark matter or modified gravity models. For a given dark matter model, its purpose is to compare the dark matter RC contribution with the distribution found from the observed RC minus the baryonic contribution. This comparison is done under a normalization that eliminates the dependence on the dark matter RC magnitude, thus focusing on the RC shape. For the case of modified gravity, the essence is the same but the “baryonic” component is understood as the Newtonian RC inferred from the baryons. In this work, we only consider modified gravity models that try to fully replace the need for dark matter.

Two essential definitions that are used here are the observational additional (squared) velocity, $\Delta V^2_{\text{obs}}$, which is defined as

$$
\Delta V^2_{\text{obs}} \equiv V^2_{\text{obs}} - V^2_{\text{bar}},
$$

and the model additional (squared) velocity, $\Delta V^2_{\text{mod}}$,

$$
\Delta V^2_{\text{mod}} \equiv V^2_{\text{mod}} - V^2_{\text{bar}}.
$$

In the above, $V^2_{\text{obs}}$ is the square of the observational RC, $V^2_{\text{mod}}$ is the square of the model RC and $V^2_{\text{bar}}$ is the square of the expected baryonic RC (which is inferred from the stars and the HI regions radiation). To be more explicit,

$$
V^2_{\text{bar}} = V^2_{\text{st}} + V^2_{\text{sh}} + V^2_{\text{gas}},
$$

where $V^2_{\text{st}}$, $V^2_{\text{sh}}$ and $V^2_{\text{gas}}$ are the circular velocity squared of the following components respectively: stellar disk, stellar bulge and the gas (atomic hydrogen and helium). The mass-to-light ratios are included in these quantities.

Although it is a common convention in this context, it is important to clarify that these squared velocities are not truly the square of a given physical velocity, namely

$$
V^2_x(r) \equiv a_x(r) r,
$$

that is, the component $x$ contribution is the centripetal acceleration due to the component $x$ times the (cylindrical) radial position $r$. Hence, since the matter distribution is not spherical, $V^2_x$ can be either positive or negative. A negative $V^2_x$ means that the component $x$, in a given radius, reduces the total centripetal acceleration.3 It is common to use this convention and this detail is relevant depending on the particular model and the particular galaxy considered.

The velocities $\Delta V^2_{\text{obs}}$ and $\Delta V^2_{\text{mod}}$ express the dark matter or modified gravity effects in velocity space. They are here called additional velocities. The former being the observational and the latter the model additional velocity.

The method is based on the observational and model normalized additional velocities (NAVs) ($\Delta V^2_{\text{obs}}$, $\Delta V^2_{\text{mod}}$) as a function of the normalized radius ($r_n$), which are given by

$$
\delta V^2_{\text{NAV}}(r_n) \equiv \frac{\Delta V^2_{\text{NAV}}(r_n, r_{\max})}{\Delta V^2_{\text{NAV}}(r_{\max})},
$$

with $r_n \in [0, 1]$. In the above, $r_{\max}$ is the largest radial value with RC data and $r_n \equiv r/r_{\max}$ is the normalized galaxy radial coordinate.

The purpose of the method is to compare the distributions of $\delta V^2_{\text{obs}}$ and $\delta V^2_{\text{mod}}$. Good models should yield distributions that are similar to the observational one. The comparisons are performed both graphically and numerically, always with focus on the sample results, not on the individual galaxies. There are different methods to compare distributions, here we consider comparisons between the 1$\sigma$ and 2$\sigma$ highest density regions (HDRs) between observational and model data. These concepts are presented in detail in the next subsections.

At last, we stress that the this method was not built to test all the relevant aspects of galaxy rotation curves, but to focus on a relevant subset of such data. A model with a good compatibility with observational data in the NAV plane ($r_n$, $\delta V^2$) may be a poor realistic model due to other reasons that are not captured by the NAV plane. On the other hand, the NAV plane is a relevant and fast sample test that can be easily applied to modified gravity or dark matter in situations where individual fits may be unfeasible. A model that is not compatible with the observational NAV plane cannot be a realistic model, and certain model systematics can be promptly unveiled by this method.

2.2 The $\delta V^2_{\text{obs}}$ distribution

Figure 1 shows the $\delta V^2_{\text{obs}}$ distribution due to 153 galaxies from the SPARC catalogue (Lelli et al. 2016). These are the same 153 galaxies that were used to find the radial acceleration relation (RAR) (McGaugh et al. 2016)3 This figure was derived using the galaxy distance and inclination central values, as provided by SPARC. It is also assumed that the stellar mass-to-light ratios for the disk and the bulge in the 3.6 $\mu$m band from Spitzer are given respectively by $\Upsilon_d = 0.5$ and $\Upsilon_b = 0.6$. For the disk, this is exactly the same value used by McGaugh et al. (2016), but for the bulge we opted for a slight variation (0.6 instead of 0.7), which is in agreement with Meidt et al. (2014). This difference is not essential for our results, but since the value 0.7

3 Negative contributions within Newtonian gravity can commonly be found due to the gas component, this since the highest gas densities can be commonly found far from the centre of the galaxy.

4 The SPARC catalogue includes 175 galaxies, after eliminating galaxies with low inclinations ($i < 30^\circ$), to reduce the impact of inclination errors, and galaxies that are not sufficiently symmetric (“quality flag 3”), one is left with 153 galaxies. The main RAR plot of McGaugh et al. (2016) also removes data points from some galaxies (those whose relative uncertainty in $V_{\text{obs}}$ is larger than 10%), this last step is not here considered since its effects for this analysis are small overall, and they are larger for $r_n < 0.2$, which is a region that will not be considered. The NAVanalysis code includes a brief explicit evaluation of this case.
was in part motivated by reducing the RAR dispersion, which is not our motivation here, and since 0.6 is in agreement with Meidt et al. (2014) and at the same time it slightly reduces the negative values for \(\delta V_{\text{abs}}\), we opted for \(Y_0 = 0.6\). The most relevant differences between these choices in Fig. 1 appear only for small radii, \(r_n < 0.2\).

To better understand Fig. 1 meaning, we start by considering Newtonian gravity with dark matter. In this context, \(\delta V_{\text{mod}}^2\) should be strictly positive (dark matter can only increase the total circular velocity). The observational data, however, is not expected to always yield \(\Delta V_{\text{obs}}^2 > 0\) due to different sources of observational error. The main sources of error come from distance and stellar mass-to-light ratio. The observational NAV plane clearly displays a correlation, and therefore \(\Delta V_{\text{mod}}^2\) in this case should have random oscillations about zero. Therefore, the expected \(\delta V_{\text{obs}}^2\) and \(\delta V_{\text{mod}}^2\). The latter is clearly not the case for the observational NAV plane inferred from the SPARC data (Fig. 1), hence the displayed correlation demonstrates the need for dark matter or modified gravity.

To proceed with the method, we need to find a smoothed distribution from the data in the observational NAV plane. This is necessary since the essence of the method is to compare the \(\delta V_{\text{obs}}^2\) distribution with the \(\delta V_{\text{mod}}^2\) one. A standard procedure to find such distribution is by using a kernel density estimator (KDE). Whenever we use a KDE in this work, for either 1D or 2D data, we use a simple and well known one, namely a Gaussian kernel with the Silverman rule for the bandwidths (Silverman 1998; Scott 2014). Appendix A covers further KDE details.

Once the KDE for the observational NAV plane is known, one can find the 1\(\sigma\) and 2\(\sigma\) highest density regions. These regions are also displayed in Fig. 1. It is important to stress that these 1\(\sigma\) and 2\(\sigma\) regions are regions where most of the rotation curves pass by, these regions are neither credible nor confidence regions. There is an intrinsic diversity in galaxy rotation curve data that should be captured by a good model.

A good model should be such that the predicted \(\delta V_{\text{mod}}^2\) has a distribution close to \(\delta V_{\text{obs}}^2\). A perfect covering of \(\delta v_{\text{mod}}\) is not a necessity because of the presence of observational and KDE errors. About the latter ones, close to \(r_n = 0\) and \(r_n = 1\), there are errors due to the KDE resolution. The first case is actually dominated by the baryonic errors, as previously commented, but for \(r_n \sim 1\) the observational errors should be negligible, thus the KDE resolution is expected to be the major error source. A simple way to mitigate these boundary errors is to exclude extrema \(r_n\) values. Since bandwidth values for the \(r_n\) axis are about \(\sim 0.05\), we do not consider data with \(5 r_n < 0.1\) and \(r_n > 0.9\). Due to the observational errors, for low \(r_n\) values we adopt a stronger limit, hence the \(r_n\) range of especial concern is \(0.2 < r_n < 0.9\).

2.3 The \(\delta V_{\text{mod}}^2\) distribution

There are two approaches to compare \(\delta V_{\text{mod}}^2\) with the \(\delta V_{\text{obs}}^2\) distribution: one for \(\delta V_{\text{mod}}^2\) independent of baryonic parameters, which is the typical case of dark matter halos; and the other if the baryonic matter distribution directly influences \(\delta V_{\text{mod}}^2\), which is typically the case of modified gravity.

For both cases, the method can do the following: i) eliminate models that are far from the \(\delta V_{\text{mod}}^2\) distribution (i.e., those that can only cover a small part of the 1\(\sigma\) or 2\(\sigma\) \(\delta V_{\text{obs}}^2\) regions); ii) find the model parameters that correspond to the most dense \(\delta V_{\text{obs}}^2\) regions, and this is done directly from the sample and commonly with one less parameter; iii) find the types of galaxies that are harder for the model to cover; among other possibilities. The method is not supposed to be a very precise quantitative one. On the other hand, all the major methods to compare galaxy fits have drawbacks and qualitative aspects.

2.3.1 Dark matter-like case: \(\delta V_{\text{mod}}^2\) is independent from baryonic parameters

Commonly, one of the parameters that \(\Delta V_{\text{mod}}^2\) depends on is a multiplicative parameter that adjusts the model contribution magnitude (e.g., \(\rho_c\) for the Burkert profile, or \(\rho_0\) for the NFW profile, Navarro et al. 1996; Navarro et al. 1997). We call this parameter linear magnitude parameter. It should be clear from its definition that \(\delta V_{\text{mod}}^2\) cannot depend on it. Whenever there is a linear magnitude parameter, \(\Delta V_{\text{mod}}^2\) depends on one less parameter than \(\Delta V_{\text{mod}}^2\).

After computing \(\delta V_{\text{mod}}^2\), for a given model, one can in principle select any of its remaining free parameters (e.g., \(r_n\) for the Burkert profile) in order to plot the highest and lowest possible curves in the NAV plane, which allows one to determine if the model has any chance of reproducing the observational data. These extremum cases can be sufficient as a quick test.

A less qualitative approach is to use the model parameters to try to mimic the \(\delta V_{\text{obs}}^2\) distribution as much as possible, with focus on the 1\(\sigma\) and the 2\(\sigma\) HDRs. To this end, one can fit the remaining model parameters (e.g., \(r_n\) to the observational 1\(\sigma\) and the 2\(\sigma\) HDRs. It should be stressed that this is a direct fit to the sample data, not a fit to individual galaxies. The lower and upper limits of the HDRs can be found by minimizing the following integral

\[
I^{(\text{mod})}(p_i) = \int_0^1 \left( \delta V_{\text{mod}}^2(p_i, r_n) - \lambda_{\text{obs}}^{(1)}(r_n) \right)^2 \omega(r_n) \, dr_n,
\]

where \(\lambda_{\text{obs}}^{(1)}(r_n)\) is the curve that delimits the \(k\sigma\) region of \(V_{\text{obs}}^2\) with \(+k\) corresponding to the upper limit, and \(-k\) with the lower limit. The \(\lambda_{\text{obs}}^{(1)}\) and \(\lambda_{\text{obs}}^{(2)}\) functions in tabular form are provided in Table 1. \(\omega(r_n)\) is a weight function that establishes the confidence one has on the NAV plane as a function of \(r_n\). Two simple and relevant cases are \(\omega = 1\) (i.e., all the \(r_n\) values are equally important) and \(\omega = 0.2\Theta(r_n - 0.2)\Theta(0.9 - r_n)\) (i.e., only the data in the range \(0.2 < r_n < 0.9\) should be considered). A smooth weight function can be more realistic than the latter case, but it requires further developments that are outside the purpose of this work.

\footnote{The most used methods currently are based on \(\chi^2_{\text{red}}\) which yields, for such applications, a qualitative model comparison (e.g., Andrae et al. 2010; Cameron et al. 2020; Rodrigues et al. 2020). The correspondence between \(\chi^2_{\text{red}}\) and a good fit cannot be trusted and it is a qualitative indication.}
2.3.2 Modified gravity-like case: $\delta V^2_{\text{mod}}$ depends on baryonic parameters

Likewise the previous case, if $\Delta V^2_{\text{mod}}$ depends on a linear magnitude parameter, this parameter will not appear in $\delta V^2_{\text{obs}}$ and this commonly simplifies the analysis. If $\delta V^2_{\text{mod}}$ is fully determined from the baryonic parameters, one can use the baryonic parameters of each galaxy in order to find $\delta V^2_{\text{mod}}$ for each one of them. Then one can randomly discretize these curves, find the model $\sigma$ and $2\sigma$ HDRs and compare such regions with the observational ones. These steps are slightly more complex than the previous case, however, these steps save more work than the previous case, since modified gravity theories are typically harder to compare with observational data than dark matter profiles.

For most of the modified gravity models, finding $\delta V^2_{\text{mod}}$ for each galaxy requires solving a modified Poisson equation for each galaxy, which in turn depends on the knowledge of the 3D mass distributions of the stellar and the gaseous components. Not only solving hundreds of modified Poisson equations can be time-consuming, but detailed data on those 3D mass distributions is not commonly available (although $V^2_{\text{gas}}$ and $V^2_{\text{kin}}$ are commonly provided). For instance, this has lead Green & Moffat (2019) and Naik et al. (2019) to develop their own models for these 3D mass distributions.

MOND is a notorious exception of the issue commented above. This since MOND (in its original form Milgrom 1983) allows one to express $\delta V^2_{\text{mod}}$ as a function of $V^2_{\text{obs}}$ alone, hence there is no need to solve modified Poisson equations. This peculiarity of MOND will allow us to develop the analysis of MOND by two different ways, which will be shown to be compatible.

In order to circumvent the need to know the 3D mass distributions of each galaxy and the need to solve modified Poisson equations, we use a set of approximations. Since we are directly working with a sample of galaxies, we can bypass most of the difficulties of the individual cases and use averages that should be reasonable for a sample of galaxies, even if they may fail at individual cases. The idea is to find simple 3D models whose stellar and gaseous velocity contributions coincide with those from the original data sample (SPARC in this case).

For the stellar component, there is a well known approximation that works remarkably well for a large number of galaxies: that of an exponential disk (Freeman 1970; van der Kruit & Freeman 2011). The profiles for the gas component are more irregular, making them harder to describe with simple models, hence several different models can be found (e.g., Tonini et al. 2006; Wang et al. 2014; Green & Moffat 2019). A very simple approach is to add another exponential disk to model the gas contribution, as used by Naik et al. (2019) in order to compute the non-Newtonian contribution from a $f(R)$ model. There are two important advantages of this approach: i) simplicity, since it is a simple model that only depends on 2 parameters; and ii) existence of an analytical expression for the circular velocity: hence it is possible to find its parameters by fitting the provided gas velocity contribution. We will show that MOND, which is the single model that can be easily studied either with the original or the approximated profiles, yields essentially the same results, does providing support for these approximations.

These approximations are further detailed in Sec. 4 and Appendix B.

2.4 Model efficiency in the NAV plane

Besides a graphical comparison between the $V^2_{\text{obs}}$ and $V^2_{\text{mod}}$ distributions, it is convenient to introduce a numerical comparison. No single number can capture all the data from the 2D plots, but a convenient one, since we are focusing on the 1$\sigma$ and 2$\sigma$ curves, is based on the areas related to these quantities as below described.

First we introduce three areas:
perfect model should yield

There should be some penalization for models that predict regions

efficiency. Hence, this definition for the efficiency is not satisfactory.

a model that has a very large NAV region that fully includes the
observational and the model NAV regions. This is interesting, but

correspond to the case in which there is no intersection between the
model regions in Fig. 1.

\[ A_{\text{obs}}(n) \] is the area of the observational NAV region at \( n \sigma \) level.

\[ A_{\text{mod}/\text{obs}}(n) \] is the area of the intersection between the model
and observational NAV regions at \( n \sigma \) level. Hence, a perfect model
is expected to satisfy \( A_{\text{mod}/\text{obs}}(n) = A_{\text{obs}}(n) \).

\[ A_{\text{mod}/\text{obs}}(n) \] is the area of the region difference between the
model NAV and the observational NAV regions, both at \( n \sigma \) level. A
perfect model should yield \( A_{\text{mod}/\text{obs}}(n) = 0 \).

As a first proposal for an efficiency coefficient one may consider
\( A_{\text{mod}/\text{obs}}/A_{\text{obs}} \). For this case, a perfect model would have an efficiency of 1, while the lowest possible efficiency would be zero, which would
correspond to the case in which there is no intersection between the
observational and the model NAV regions. This is interesting, but
a model that has a very large NAV region that fully includes the
observational region would be characterized as having the maximum
efficiency. Hence, this definition for the efficiency is not satisfactory.

There should be some penalization for models that predict regions
that are outside the observational region.

Considering the above, we define the model efficiency at \( n \sigma \) level

\[ E(n) \equiv \frac{A_{\text{mod}/\text{obs}}(n) - A_{\text{mod}/\text{obs}}(n)}{A_{\text{obs}}(n)}. \] (7)

Hence the highest efficiency value is \( E = 1 \), and \( E \) can be arbitrarily
negative. Any reasonable model should have positive values for \( E \).
Models with negative values are those whose predicted area that
agrees with the observational data is smaller than the predicted area
that is outside the observational region (at the same \( n \sigma \) level).

\( E(n) \) provides a 1D comparison between the model and observa-
tional NAV distributions, which is a simplification with respect to
the original 2D data. Since we focus on the 1\( \sigma \) and 2\( \sigma \) HDRs, the
above model efficiency can be conveniently summarized in a single
number as follows,

\[ E_M = \frac{E(1) + E(2)}{2}, \] (8)

which is here called the mean efficiency for a given model. This is a
number that we will compute and use it to compare the models. To
3 THE MODELS

3.1 Burkert profile

The Burkert profile (Burkert 1995) is a two-parameter cored dark matter halo that has been extensively studied and which has achieved phenomenological success (e.g., Salucci & Burkert 2000; Salucci et al. 2007; Rodrigues et al. 2017; Li et al. 2020). Its density profile is given by

$$\rho_{\text{Bur}}(r) = \frac{\rho_c}{(1 + r/r_c)(1 + r^2/r_c^2)}.$$  

In the above, $r$ is the spherical radial coordinate, while $r_c$ and $\rho_c$ are constants that can change from galaxy to galaxy.

The internal mass of the Burkert profile reads,

$$M_{\text{Bur}}(r) = 4\pi \int_0^r \rho_{\text{Bur}}(r') r'^2 \, dr = 2\pi \rho_c r_c^3 \xi \left( \frac{r}{r_c} \right),$$

where

$$\xi(x) \equiv \ln \left(1 + x\sqrt{1 + x^2} \right) - \tan^{-1}(x).$$

It is convenient to introduce the normalized core radius as

$$r_{\text{cn}} \equiv \frac{r_c}{r_{\text{max}}}. \quad (12)$$

Using that, for a spherical mass distribution, $V^2(r) = GM(r)/r$, we can now compute $\Delta V^2_{\text{mod}}$ and $\delta V^2_{\text{mod}}$ as

$$\Delta V^2_{\text{Bur}}(r_n) = 2\pi G \rho_c r_c^3 \frac{r_{\text{max}}}{r_n} \xi \left( \frac{r_n}{r_{\text{cn}}} \right), \quad (13)$$

$$\delta V^2_{\text{Bur}}(r_n) = \frac{1}{2} \frac{\xi(r_n/r_{\text{cn}})}{r_{\text{cn}} \xi(1/r_{\text{cn}})}. \quad (14)$$

We note that, apart from $r_{\text{cn}}, \Delta V^2_{\text{Bur}}(r_n)$ depends on a multiplicative constant composed by $r_c, \rho_c, r_{\text{max}}$ and $G$. This multiplicative constant is a linear magnitude parameter, which is absent from $\delta V^2_{\text{Bur}}$, as expected.

3.2 Modified Newtonian Dynamics (MOND)

MOND is a well-known and simple modified gravity model that, contrary to several other proposals, has achieved certain degree of success in the context of galaxy rotation curves (Milgrom 1983; Famaey & McGaugh 2012). Following the original MOND formulation, as well as the more recent motivation from the Radial Acceleration Relation (RAR) (McGaugh et al. 2016; Li et al. 2018), this model imposes that the observational acceleration can be expressed as a function of the Newtonian acceleration:

$$a_S = \mu \left( \frac{a}{a_0} \right) a,$$  

where $a_0$ is a constant introduced by MOND, $\mu$ is the interpolation function, $a_N$ is the Newtonian acceleration inferred from baryonic matter and $a$ the observational one. Within axisymmetric disk galaxies, the acceleration vectors $a$ and $a_N$ are along the radial direction in the plane of the disk ($z = 0$), hence $a = -\frac{V_z^2}{r} \hat{r}$ and $a_N = |a| = \frac{V_z^2}{r}$.

The interpolation function $\mu$ must satisfy $\mu(x) = 1$ for $x \gg 1$ and $\mu(x) = x$ for $x \ll 1$. For $x \sim 1$, the interpolation function behaviour should be fixed from the observational data. The RAR suggests the following relation between $a$ and $a_N$ (McGaugh et al. 2016),

$$a = \frac{a_N}{1 - e^{-V_{\text{mod}}/(a_0)}}$$  

and the same data lead to $a_0 = 1.2 \times 10^{-10}$ m/s$^2$ as the best fit for $a_0$.

Rodrigues et al. (2018a); Marra et al. (2020) use Bayesian inference to show that the observational uncertainties of the same dataset are not compatible with the assumption that there is a common $a_0$ value for all the galaxies. The issue was found with a high significance for different interpolating functions and different quality cuts, always with significance larger than 5$\sigma$. These results were further discussed by McGaugh et al. (2018); Kroupa et al. (2018); Rodrigues et al. (2018b); Cameron et al. (2020); Rodrigues et al. (2020); Rodrigues & Marra (2020); Li et al. (2021), see also Stone & Courteau (2019); Zhou et al. (2020); Eriksen et al. (2021). Nonetheless, although this is a problem for MOND as a fundamental theory for galaxies, it poses no problem on the use of MOND as an effective theory that works under certain approximations (see also Navarro et al. 2017; Stone & Courteau 2019; Dutton et al. 2019, for interpreting the RAR within CDM).

From eq. (16) with $a_N = a_{\text{bar}} = V_{\text{bar}}^2/r$, the $\Delta V^2_{\text{mod}}$ MOND expression reads

$$\Delta V^2_{\text{MOND}} = V_{\text{mod}}^2 - V_{\text{bar}}^2 = \frac{V_{\text{bar}}^2}{e^{\sqrt{a_{\text{bar}}/a_0}} - 1},$$

hence

$$\delta V^2_{\text{MOND}}(r_n) = \frac{V_{\text{bar}}^2}{V_{\text{mod}}^2(1)} e^{\sqrt{a_{\text{mod}}(1)/a_0}} - 1.$$  

Since MOND is nonlinear with respect to the constant that parametrizes the intensity of its effects ($a_0$), this constant is present in $\delta V^2_{\text{mod}}$.

The Newtonian limit of MOND is found in the limit $a_0 \rightarrow 0$. Assuming that $V_{\text{bar}}^2(r_n) > 0$, then for a given $r_n \in (0, 1)$,

$$\lim_{a_0 \rightarrow 0} \delta V^2_{\text{MOND}}(r_n) = \begin{cases} 0, & \text{if } a_{\text{bar}}(1) < a_{\text{bar}}(r_n) \\ \infty, & \text{if } a_{\text{bar}}(1) > a_{\text{bar}}(r_n). \end{cases}$$

Hence, the Newtonian limit of MOND leads to $\delta V^2_{\text{mon}} = 0$ for either all or most of the $r_n$ values. We recall that, in general, $V^2_{\text{mod}}$ is ill defined for Newtonian gravity without dark matter. The limit of a modified gravity theory towards Newtonian gravity displays different features depending on the considered theory.

3.3 Palatini $f(R)$ gravity

Extended reviews on Palatini $f(R)$ gravity were done by Sotiriou & Faraoni (2010); De Felice & Tsujikawa (2010); Olmo (2011). Here we use a notation very similar to that of Tonniato et al. (2020).

7 Considering the observational uncertainties of the individual galaxies, the best value for $a_0$ is lower than the commonly used one, namely $9.6 \times 10^{-11}$ m/s$^2$ (Marra et al. 2020). The latter is the value that minimizes the tensions between the $a_0$ values for individual galaxies, which is different from the value that best fits the RAR.

8 The original MOND version, which is based on eq. (15).
The (torsionless) Palatini $f(R)$ gravity action, in the presence of matter fields $\Psi$, reads,

$$S[g, \Gamma, \Psi] = \frac{1}{2\kappa} \int f(R(\Gamma, g)) \sqrt{-g} \, d^4x + S_{\text{matter}}[g, \Psi],$$  \hspace{1cm} (20)

where $\kappa$ is a constant, $g_{\mu\nu}$ is the spacetime metric, $\Gamma^\mu_{\rho\nu}$ is an independent and torsionless affine connection. That is, in the action, $\Gamma^\mu_{\rho\nu}$ enters as an independent field from the spacetime metric and $\Gamma^\mu_{\rho\nu} = \Gamma^\mu_{\rho\nu}$. The Riemann tensor is such that, for any vector field $A^\nu$, $R^\alpha_{\rho\sigma}A^\rho = (\nabla_\sigma - \Gamma^\lambda_{\rho\sigma}A^\lambda)A^\rho$, where $\nabla_\sigma$ is the covariant derivative with respect to the affine connection $\Gamma^\mu_{\rho\nu}$. The Ricci tensor and Ricci scalar are respectively $R^\rho_{\mu\nu}(\Gamma) \equiv R^\rho_{\mu\nu\lambda}(\Gamma)$ and $\kappa \equiv g^{\mu\nu}R^\rho_{\mu\nu}(\Gamma)$. Let us notice that in the case of Palatini gravity (and EiBI discussed in the further part) the matter fields couple only to the spacetime metric, not to the connection $\Gamma$; the situation, however, would be different if fermionic fields were present Alfonso et al. (2017).

The field equations are

$$f'(R)R_{\mu\nu} - \frac{1}{2} f'(R)g_{\mu\nu} = \kappa T_{\mu\nu},$$  \hspace{1cm} (21)

$$\nabla_\lambda \left( \sqrt{-g} f'(R)(g^{\mu\nu}) \right) = 0.$$  \hspace{1cm} (22)

The prime above means the derivative with respect to the function argument while $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}.$$  \hspace{1cm} (23)

However, $\nabla_\lambda T_{\mu\nu} \neq 0$. Indeed, the action variation with respect to $\Psi$, together with diffeomorphism invariance of $S_{\text{matter}}$, implies that

$$\nabla^\nu T_{\mu\nu} = 0,$$  \hspace{1cm} (24)

where $\nabla^\nu$ is the covariant derivative with respect to the connection $C^\lambda_{\mu\nu} = \frac{1}{2} S^{\lambda\sigma\rho} (g_{\mu\lambda\sigma} + g_{\sigma\nu\rho} - g_{\mu\nu,\rho})$. In the above, a comma is used to denote a partial derivative. In particular, this implies that test particles will follow geodesics from the connection $C^\lambda_{\mu\nu}$, not $\Gamma^\lambda_{\mu\nu}$!

From the trace of eq. (21),

$$f'(R)R - 2 f(R) = \kappa T,$$  \hspace{1cm} (26)

which shows that $R$ can be algebraically expressed as a function of $T$. This is in contrast to the metric $f(R)$ case.

In order to proceed with the weak field limit, and similarly to Toniato et al. (2020), we consider the following $f(R)$ case,

$$f(R) = R + \alpha R^2 + O(R^3),$$  \hspace{1cm} (27)

where $\alpha$ is a constant. That is, we are assuming that $f(R)$ is either an analytical function about $R = 0$, with $f(0) = 0$, or at least that it can be approximated by one up to $O(R^2)$.

Hence, eq. (26) implies that

$$R = -\kappa T + O(R^3).$$  \hspace{1cm} (28)

It is convenient to express the connection $\Gamma^\mu_{\rho\nu}$ as a function of a “metric”, and indeed this is well known to be possible. Equation (22) points out that $\Gamma^\mu_{\rho\nu}$ can be written as a function of a tensor $\bar{g}_{\mu\nu}$, that is conformally related to the spacetime metric $g_{\mu\nu}$; indeed, let

$$\bar{g}_{\mu\nu} = f'(R) g_{\mu\nu},$$  \hspace{1cm} (29)

and hence the independent connection results as

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2} \bar{g}^{\nu\sigma} \left( \bar{g}_{\mu\sigma,\nu} + \bar{g}_{\sigma\nu,\mu} - \bar{g}_{\mu\nu,\sigma} \right).$$  \hspace{1cm} (30)

As usual for the weak field limit, the background is taken as Minkowski $\eta_{\mu\nu}$.

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}.$$  \hspace{1cm} (31)

Using eq. (28), up to the first nontrivial order, the field equation (21) does not depend on $\alpha$ and it is equivalent to the Einstein field equation for the metric $\bar{g}_{\mu\nu}$, coupling constant $\kappa$ and energy-momentum tensor $\Gamma^\mu_{\rho\nu}$.

The relevant energy momentum tensor for modeling galaxies, is that of a perfect fluid with energy density $\rho$ and pressure $p$,

$$T_{\mu\nu} = p U^\mu U^\nu + g(\mu \nu^\nu + g^\mu \nu^\nu).$$  \hspace{1cm} (32)

Since the fluid particles are not assumed to move at relativistic velocities, we consider $\rho \gg p$ and thus $T \approx -\rho$.

Therefore, using $\bar{h}_{00} = -2\bar{\Phi}$,

$$\nabla^2 \bar{\Phi} \approx \frac{k}{2} \rho.$$  \hspace{1cm} (33)

The difference with respect to GR comes from the Euler equation, which is derived from eq. (24) and depends on the connection $C^\lambda_{\mu\nu}(g)$, instead of $\Gamma^\lambda_{\mu\nu}(\bar{g})$. This implies that test particles will accelerate due to a gravitational potential $\Phi$, with $g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$ and $h_{00} = -2\Phi$. The Poisson equation becomes

$$\nabla^2 \Phi \approx \frac{k}{2} \left( \rho + 2\alpha \nabla^2 \rho \right).$$  \hspace{1cm} (34)

To find the above, it is only necessary to express $\bar{\Phi}$ as a function of $\Phi$. Hence, in the galaxy plane,

$$\frac{v_c^2}{c} = \frac{\partial_r \Phi}{r} = \frac{\partial_r \bar{\Phi} + \kappa \alpha \partial_r \rho}{r}.$$  \hspace{1cm} (35)

In the above, we use $r$ for the radial cylindrical coordinate and $v_c$ for the circular velocity.

Toniato et al. (2020), in a post-Newtonian context, found no differences between Palatini $f(R)$ and GR in the Solar system (apart from certain pressure and internal energy redefinitions that are possible and realistic in the Solar System context). In the above, gravity seems to be strongly modified if $\alpha \neq 0$, and indeed we will follow this interpretation here, in which eq. (33) will be the Newtonian Poisson equation, with $k = 8\pi G$, and $\alpha \neq 0$ will parametrize Newtonian gravity deviations. Our approach in the case of galaxies corresponds to the assumption that the matter pressure $p$ is indeed negligibly small, had we opted for using an effective pressure (as $p_{\text{eff}}$ in the mentioned reference), then there would be no deviations with respect to Newtonian gravity. It is important to stress two context differences: i) here we are not doing post-Newtonian analysis; instead, we are explicitly considering a model in which it is assumed to deal with a change in Newtonian gravity, ii) the galaxy components are modeled as fluids that generate the galaxy internal structure. In the Solar System case, the Sun and the planets are modeled as fluids, but any Palatini correction is only relevant for their internal forces, as Toniato et al. (2020) have shown. Therefore the motion of the planets about the Sun is not affected by Palatini gravity, even when considering the planets and Sun’s internal structure being affected by the modifications as shown by Wojnar (2021); Kozak & Wojnar (2021a,b,c), however a galaxy dynamics can be affected.

Since razor-thin disks (Binney & Tremaine 1988) will be considered to model galaxies, it is important to check if the use of such disk with $\rho \propto \delta(z)$ is compatible with Palatini $f(R)$ gravity. The razor-thin disk generates an infinite force in the disk and perpendicular to it. For negative $\alpha$, it is impossible for any nonrelativistic matter to scape...
Combining those field equations one may write
\[ \Delta V^2_{\text{mod}}(r) = 8\pi G r \rho'(r), \quad (36) \]
and
\[ \delta V^2_{\text{Palatini}}(r_n) = r_n \rho'(r_n) \quad \text{or} \quad (37) \]

In particular, from the above, the non-Newtonian contribution from Palatini \( f(R) \) in the galactic plane \( (z = 0) \) does not depend on the galaxy’s thickness. Thick disks are commonly modelled with a density profile \( \rho \) that can be split as follows \( \rho(r_n, z) = \Sigma(r_n) Z(z) \). Therefore, for a thick disk, \( \delta V^2_{\text{Palatini}} \) is precisely the same as above [with \( \Sigma(r_n) \) in place of \( \rho(r_n) \)].

### 3.4 Eddington inspired Born-Infeld gravity (EiBI)

Another simple metric-affine model is the Eddington-inspired Born-Infeld gravity (EiBI) (Bañados & Ferreira 2010), see also the review Beltran Jimenez et al. (2018).

The EiBI action reads
\[ S[g, \Gamma, \psi_m] = \frac{1}{k^2} \int \! d^4x \left[ \sqrt{g} \left[ \frac{1}{2} R - \frac{\epsilon}{4} (R^2 - 2 R_{\mu\nu} R^{\mu\nu}) \right] - A \sqrt{\gamma} \right] + S_m[g, \Gamma, \psi_m] + O(\epsilon^2) \quad (38) \]
where \( \epsilon \) is the typical scale of this theory and one introduces parenthesis in the Ricci tensor to stress the symmetric character of this object.

It can be shown (Pani et al. 2013) that an expansion of the action (38) for fields \( R_{\mu\nu} \ll \epsilon^{-1} \) provides
\[ S[g, \Gamma, \psi_m] = \frac{1}{k} \int \! d^4x \left[ R^\Gamma - 2\Lambda + \frac{\epsilon}{4} (R^2 - 2 R_{\mu\nu} R^{\mu\nu}) \right] + S_m[g, \Gamma, \psi_m] + O(\epsilon^2) \quad (39) \]
which turns out to be the Lagrangian of GR, with an effective cosmological constant \( \Lambda = \frac{4\Lambda}{k} \), and with additional added quadratic curvature corrections. Essentially, it is a particular \( f(R) \) case that fits the Palatini \( f(R) \) case that was presented in the previous section, together with a squared Ricci tensor term (i.e., \( R_{\mu\nu} R^{\mu\nu} \)).

Variation of the action (38) with respect to metric \( g_{\mu\nu} \) and an independent connection \( \Gamma \) gives the field equations
\[ \sqrt{g + \epsilon R} \left[ (g + \epsilon R)^{-1} g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa \Gamma_{\mu\nu}, \right. \quad (40) \]
\[ g_{\mu\nu} + \epsilon R_{\mu\nu} \equiv q_{\mu\nu}. \quad (41) \]
Here, the introduced metric \( q_{\mu\nu} \) is a metric compatible with \( \Gamma \). Combining those field equations one may write
\[ \sqrt{q} q^{\mu\nu} = \sqrt{g} g^{\mu\nu} - \epsilon \sqrt{\gamma} T^{\mu\nu} \quad (42) \]
which together with (41) is the simplest set of equations in order to examine the EiBI theory. Similarly as it happens in Palatini \( f(R) \) gravity, the EiBI field equations can be written in Einstein-like form with respect to the metric \( q \) as (Bañados & Ferreira 2010)
\[ G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = \Lambda^{-1} T^\mu_\nu - \Lambda \delta^\mu_\nu, \quad (43) \]

where the apparent stress tensor \( T^\mu_\nu \) is
\[ T^\mu_\nu = \tau T^{\mu}_{\nu} + \mathcal{P} \delta^\mu_\nu, \quad \Lambda^{-1} \mathcal{P} = \tau - 1 - \frac{1}{2} \tau \Lambda^{-1} \epsilon. \quad (44) \]

Here, the quantity \( \mathcal{P} \) plays a role of an isotropic pressure while the scalar \( \tau \) is obtained from the original stress energy tensor \( T^\mu_\nu \) by \( \tau = (\det(\delta^\mu_\nu - \epsilon \Lambda^{-1} T^\mu_\nu))^{-\frac{1}{2}}. \) The “bar” quantities are objects totally constructed with the metric \( q \), in a similar manner as it was done for \( f(R) \) gravity models.

Hence, in the weak field limit, the EiBI potential reads (Bañados & Ferreira 2010; Pani et al. 2011; Olmo et al. 2021)
\[ \nabla^2 \Phi_{\text{EiBI}} \equiv \frac{k}{2} (\rho + 2\epsilon \nabla^2 \rho). \quad (45) \]

Apart from notation differences, this is the same result found for Palatini \( f(R) \) (34). Therefore, using \( k = 8\pi G \), EiBI yields the same result for \( \delta V^2_{\text{mod}} \), as in eq. (37). Since in this context there is no need to distinguish between Palatini \( f(R) \) and EiBI, any theory that leads to eq. (37) we will label it “Palatini”, regardless if it is indeed EiBI or other extensions that do not change the \( \delta V^2_{\text{Palatini}} \) expression.

### 3.5 Renormalization Group improved General Relativity (RGGR)

The Renormalization Group improved General Relativity (RGGR) (Rodríguez et al. 2010) model is based on a GR correction due to the scale-dependent \( G \) and \( \Lambda \) couplings. It was developed based on approaches of Reuter & Weyer (2004); Shapiro et al. (2005) in particular. This scale dependence is such that it modifies GR in the low energy limit. Such a possible dependence is provided by quantum field theory in curved spacetime (e.g., Shapiro & Sola 2009), by the quantum-gravity asymptotic safety approach (e.g., Reuter & Weyer 2004), or by other approaches (e.g., Basilakos et al. 2020; Alvarez et al. 2021).

Rodríguez et al. (2015a) have shown that the field equations of these scale-dependent gravities can be fully described by an action that extends the Einstein-Hilbert one (see also Koch & Ramirez 2011; Koch et al. 2015 for related approaches). Such action depends on a spacetime metric \( g_{\mu\nu} \) and on a second-rank tensor without derivatives (a non-dynamical field, \( \gamma_{\mu\nu} \)). The latter fixes how the spacetime perturbations source the scale-dependent corrections. A natural choice for \( \gamma_{\mu\nu} \) is the Minkowski metric, and we use this choice here.\(^9\) For the latter setting, the Minkowski spacetime is not affected by the scale-dependent corrections, but perturbations beyond Minkowski will modify \( G \) and \( \Lambda \).

The RGGR action also depends on a scalar function \( f \) that specifies the nature of the scale \( \phi \) that \( G(\phi) \) and \( \Lambda(\phi) \) depends on (this scale is commonly denoted by \( \mu \) or \( k \), but here, to be closer to the notation of other modified gravity actions, we used \( \phi \) to represent such scale). The field \( \phi \) not an independent degree of freedom, it is an auxiliary scalar field that can be eliminated from the action by using the constraint \( \phi = f(g, \gamma, \Psi) \) (Rodríguez et al. 2015a), where \( \Psi \) represent arbitrary matter fields.

The RGGR action, together with an action for matter \( (S_m) \), reads,
\[ S[g, \gamma, \phi, \Lambda, \Psi] = \int \left[ \frac{R - 2\Lambda(\phi)}{16\pi G(\phi)} + \Lambda \left[ \phi - f(g, \gamma, \Psi) \right] \right] \sqrt{-g} d^4x + S_m[g, \gamma]. \quad (46) \]

above, \( \Psi \) is a symbol for any matter fields, that is, any fields beyond \( g_{\mu\nu}, \gamma_{\mu\nu}, \phi \) and \( \Lambda \).

\(^9\) See Bertini et al. (2020) for \( \gamma_{\mu\nu} \) given by the cosmological background.
In the context of galaxies, we use:

• $S_m$ is the action of a fluid, with $U^\mu$ its 4-velocity,
• $f = U^\mu U^\nu (g_{\mu\nu} - \gamma_{\mu\nu})$, \[ (47) \]
• $G(\phi) = \frac{G_0}{1 + 2\epsilon \ln \phi}$,
• $\gamma_{\mu\nu} = \eta_{\mu\nu}$ (field equations level),

where $\eta_{\mu\nu}$ is the Minkowski metric, $G_0$ and $\epsilon$ are constants. The constant $\epsilon$ sets the strength of the RG corrections, with $\epsilon = 0$ corresponding to general relativity.

The expression of $G(\phi)$ above is a simple and natural one: different renormalization group considerations also lead to it (e.g., Fradkin & Tseytlin 1982; Nelson & Panangaden 2012; Bauer 2005; Solà 2008).

The $f$ expression in eq. (47) is explicitly a scalar and it can be straightforwardly computed in the context of weak fields and small velocities. In a reference frame where the fluid is slowly moving, $f \approx U^0 U^0 (g_{00} + 1) \approx -2\Phi$, \[ (48) \]

where $g_{00} = -1 - 2\Phi$. With this result, at the level of the field equations, one can write $G = G(\Phi)$.

The RGGR field equations found from the variation with respect to the metric read (Rodrigues et al. 2015a; Bertini et al. 2020)

$$G_{\mu\nu} + g_{\mu\nu} G G^{-1} - \nabla_{\mu} \nabla_{\nu} G^{-1} + A g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{49}$$

The action variation with respect to $\gamma_{\mu\nu}$ yields,

$$0 = \lambda \frac{\delta f}{\delta \gamma_{\mu\nu}} = -\lambda U^\mu U^\nu \implies \lambda = 0. \tag{50}$$

The variation with respect to $\phi$, together with the previous result, leads to

$$\partial_{\phi} \left( \frac{\Lambda}{G} \right) = \frac{1}{2} R \partial_{\phi} G^{-1} \implies \nabla_{\mu} \left( \frac{\Lambda}{G} \right) = \frac{1}{2} R \nabla_{\mu} G^{-1}. \tag{51}$$

The latter guarantees energy momentum conservation, since (Koch et al. 2015; Rodrigues et al. 2015a)

$$\nabla_{\mu} T^\mu_{\nu} = 0 \iff \nabla_{\mu} \left( \frac{\Lambda}{G} \right) = \frac{1}{2} R \nabla_{\mu} G^{-1}. \tag{52}$$

Since the energy-momentum tensor is conserved, the acceleration of test particles satisfy, $\dot{x}^i \approx -\partial_i \Phi$.

The variation with respect to $\lambda$, together with eq. (48), yields the constraint

$$\phi = f(g, \gamma, \Psi) \approx -2\Phi. \tag{53}$$

Finally, the variation with respect to $\Psi$ yields the standard matter field equations, without any changes made by RGGR. This is so since we have found that $\lambda = 0$ at the level of the field equations.

As previously pointed out, $G(\phi)$ is specified a priori. However, $\Lambda(\phi)$ is an unknown at the action level, but it can be derived from the field equations. Indeed, for vacuum (i.e., $T_{\mu\nu} = 0$), the $\Lambda$ solution is (Rodrigues et al. 2015a)

$$\Lambda = \Lambda_0 G_0 G^{-1} + O(\epsilon^2), \tag{54}$$

where $\Lambda_0$ is a constant. Inside matter, and in the context of matter perturbations over a Minkowski background, eq. (51) guarantees that $\Lambda$ stays as small as $\Lambda_0$ (Rodrigues et al. 2015a). In this context, this justifies neglecting $\Lambda$ whenever $\Lambda_0$ is negligible. We also stress that $\epsilon \to 0$ always implies $G \to G_0$ and $\Lambda \to \Lambda_0$.

In the stationary, weak field and small velocities limit, the field equation (49) can be written as

$$\nabla^2 \Phi + \frac{1}{2} G_0 \nabla^2 G^{-1} = 4\pi G_0 \rho, \tag{55}$$

hence,

$$\nabla^2 (\Phi + \epsilon \ln \Phi) \approx 4\pi G_0 \rho. \tag{56}$$

By identifying the Newtonian potential due to baryons $\Phi_{\text{bar}}$ as $\Phi_{\text{bar}} = \Phi + \epsilon \ln \Phi$, and using that $V_{\text{RGGR}}^2 / r = -\partial_r \Phi$, it is found that

$$V_{\text{RGGR}}^2 \approx V_{\text{bar}}^2 \left( 1 - \frac{\epsilon}{\Phi_{\text{bar}}} \right), \tag{57}$$

where $V_{\text{bar}}^2 = -r \partial_r \Phi_{\text{bar}}$. This is the same relation found by Rodrigues et al. (2010), but without using a complete action and using a conformal transformation.

Hence, within the approximations above,

$$\Delta V_{\text{RGGR}}^2(r) = -\epsilon \frac{V_{\text{bar}}^2(r)}{\Phi_{\text{bar}}(r)}. \tag{58}$$

The above result shows that $\Delta V_{\text{RGGR}}^2$ and $\delta V_{\text{RGGR}}^2$ are linear functions of $V_{\text{bar}}^2 / \Phi_{\text{bar}}$. This should be contrasted with MOND, which is a nonlinear function of $V_{\text{bar}}^2$, and the Palatini case, which is a linear function of the baryonic density derivative $\rho'$.

Rodrigues et al. (2010); Rodrigues (2012); Rodrigues et al. (2014) tested the relation (57) with several disk and elliptical galaxy data, without dark matter. A good agreement was found even without considering dark matter. Considering $\chi^2$ or $y_{\text{tot}}$ values, the fits quality were about as good as the NFW profile, although with one less free parameter. Also, these fit results were significantly better than those of MOND (nonetheless, MOND has one less free parameter that changes from galaxy to galaxy).

In spite of the good rotation and dispersion curve results, we briefly comment here that, if one wants to completely remove dark matter in place of RGGR effects, there are other issues that need to be answered as well, including galaxy-galaxy lensing (Rodrigues et al. 2015b), cosmology (Bertini et al. 2020) and solar system bounds (Farina et al. 2011; Zhao & Xie 2015; Rodrigues et al. 2016; Tonioni et al. 2017).
Figure 2. The 1σ and 2σ HDRs of the observational data are shown in shades of grey (see Fig. 1, left plot). The two rectangular regions are the less-reliable NAV regions. The two black dashed lines show Burkert extremum cases; the upper and lower curves correspond to $r_{\text{cn}} = 10^{-3}$ and $r_{\text{cn}} = 10^{3}$ respectively. The two bluish regions are the best approximations for the observational 1σ and 2σ regions (60).

scale length, however those results are approximations for the stellar component at large radii (and uncorrected for inclination) (see also, Green & Moffat 2019; Naik et al. 2019). We derive the best-fit stellar exponential disk for the SPARC galaxies from the stellar circular velocity, as detailed in Appendix B.

For the gas component, we also use an exponential thin disk as an approximation. For individual galaxies, commonly this is not a good approximation. However, it is a simple approximation that is significantly better than neglecting the gas and, for the combined data of galaxies, the impact of such approximation is much smaller than for individual galaxies. We add that Green & Moffat (2019) developed their own analytical gas profiles (which includes a dependence on the morphological type and leads to some systematical differences with respect to the original SPARC analysis); while Naik et al. (2019) use an exponential profile for the gas, similarly to our case, but the galaxies are considered individually.

The procedures and results for these exponential approximations are detailed in Appendix B.

5 RESULTS

5.1 Burkert profile application

From eq. (14), it is straightforward to plot $\Delta V_{\text{bul}}^2$ for a given $r_{\text{cn}}$ value. The questions to be addressed here are: i) is there a $r_{\text{cn}}$ interval that can cover a large part of the observational regions given in Fig. 1? ii) Which ranges of $r_{\text{cn}}$ best approximate the 1σ and 2σ HDRs? iii) Are there any systematic issues with the Burkert profile that can be uncovered and improved?

The Burkert profile is well known to be a good phenomenological dark matter halo, thus the answer to the first question is expected to be positive. Indeed, one can consider extrema $r_{\text{cn}}$ cases, like $r_{\text{cn}} = 10^{-3}$ and $r_{\text{cn}} = 10^{3}$ and the corresponding curves clearly cover a large part of the observational data, as shown in Fig. 2.

To answer the second question, from eq. (6) with $\omega = 1$ we find the $r_{\text{cn}}$ intervals that best describe the observational HDRs,

$$1\sigma \text{ range: } 0.17 < r_{\text{cn}} < 0.61,$$

$$2\sigma \text{ range: } 0.10 < r_{\text{cn}} < \infty .$$

The above regions are the bluish regions in Fig. 2. By using $\omega = 1$ if $0.2 < r_{\text{cn}} < 0.9$ and $\omega = 0$ otherwise, one finds no relevant change for this application. Namely: $0.20 < r_{\text{cn}} < 0.58$ and $0.12 < r_{\text{cn}} < \infty$, for the 1σ and 2σ regions respectively. The previous result with $\omega = 1$ will be useful for a comparison with the standard individual fits (where one does not neglect data in a given radial range).

To answer the third question, we focus the analysis in the region $0.2 < r_{\text{cn}} < 0.9$. It is curious that the lowest possible $\Delta V_{\text{bul}}^2$ curve (which requires $r_{\text{cn}} \gg 1$) almost coincides with the lower 2σ observational bound. Considering the upper limits, there is certain systematical imbalance: the model curve is too high at $r_{\text{cn}} \sim 0.3$ and too low at $r_{\text{cn}} \sim 0.8$. In principle there could be an halo model that could improve these aspects. We plan to investigate further dark matter halo profiles in a future work.

Besides answering the three questions above, we point out that the $r_{\text{cn}}$ regions found in eq. (60) can be used as approximations for the $r_{\text{cn}}$ values found from individual galaxy fits. We stress that the latter values were derived completely independently from the other Burkert parameter ($r_{\text{cn}}$).

We have performed individual fits on the 153 RAR galaxies, with two different set of priors, using the MAGMA code,\footnote{github.com/davi-rodrigues/MAGMA.} which automatically performs $\chi^2$ minimizations for each of the sample galaxies. We considered individual fits with $\Upsilon_{\cdot D} = 0.5M_\odot/L_\odot$ and $\Upsilon_{\cdot B} = 0.6M_\odot/L_\odot$, fixed for all the galaxies, together with the two free Burkert profile parameters ($r_{\text{cn}}$ and $\rho_c$). The results for each one of the galaxies are in Table 2. With the $r_{\text{cn}}$ results from the individual fits, one can find the $r_{\text{cn}}$ distribution. And, from a KDE, it is possible to infer the corresponding 1σ and 2σ HDRs, which read

$$1\sigma : \ 0.10 < r_{\text{cn}} < 0.47,$$

$$2\sigma : \ 0.10 < r_{\text{cn}} < 1.23.$$  

The 1σ and 2σ regions from the individual fits are not the same from those of the NAV plane, but they are similar, and the infinity found in the NAV plane is expected. Indeed, large $r_{\text{cn}}$ values in the
NAV plane accumulate in the lower part of the plot in Fig. 2, which will be a dense region for the KDE analysis. On the other hand, a $r_{cn}$ histogram will make the cases $r_{cn} = 1$ and $r_{cn} = 10$ appear as far apart cases, thus the region with $r_{cn} > 1$ will be much sparser then the corresponding region in the NAV plane. Considering the physical meaning, the NAV plane yields a more sensible result: it does not differentiate between $r_{cn} = 10$ and $r_{cn} = 100$ because the physical implications of these cases are essentially the same.

These sample results are also robust under several variations. As a relevant example, we have also repeated the same procedure for $Y_d$ and $Y_s$ with gaussian priors [with a standard deviation of 0.1 dex (Meidt et al. 2014)]. We found that, as expected, the sample behaviour does not change appreciably with respect to the fixed case. These results are in Table 3 and in Fig. 3. Below we cite the 1$\sigma$ and 2$\sigma$ limits,

$$1\sigma : 0.11 < r_{cn} < 0.49,$$
$$2\sigma : 0.11 < r_{cn} < 1.16.$$

These limits are essentially the same limits found with fixed $Y_d$ and $Y_s$.

5.2 MOND application

MOND is a particularly simple type of modified gravity. Two different methods to find its corresponding $\delta V^2_{\text{mod}}$ results can be used: one straight from the observational data and another with the approximated results explained in Sec. 4. Both cases are here developed, and they are shown to lead to essentially the same results.

From eq. (18) and the knowledge of $V_{\text{bar}}^2$, as provided by SPARC, one can plot the $\delta V^2_{\text{mod}}$ curves of MOND for each of the selected 153 SPARC galaxies, which are the same galaxies that were considered suitable for the RAR analysis (McGaugh et al. 2016). The magnitude of the non-Newtonian effects depends on the constant $a_0$, which appears nonlinearly in $\Delta V^2_{\text{mod}}$ and in $\delta V^2_{\text{mod}}$. Figure 4 shows the $\delta V^2_{\text{mod}}$ plots for three different $a_0$ values: i) 1 km/s$^2$, ii) $1.2 \times 10^{-13}$ km/s$^2$, iii) $10^{-15}$ km/s$^2$. The first case is clearly unrealistic and it has an excessively large non-Newtonian contribution, but such magnitude issue does not appear in the NAV plane, due to the normalization. The corresponding $\delta V^2_{\text{mod}}$ result has a single issue: the $\delta V^2_{\text{mod}}$ curves are excessively concentrated with respect to the observational limits.

Before commenting on the second case, the third $a_0$ case corresponds to a very low $a_0$ value, it is so low that the majority of the 1$\sigma$ HDR is not covered. Hence, the $\delta V^2_{\text{mod}}$ result alone can eliminate the third case as a realistic one.

About the second case in Fig. 4, whose $a_0$ value is a commonly used one (e.g., McGaugh et al. 2016), the resulting curves are placed inside the observational region, but they have an even lower dispersion than the first case. So, the model distribution (which is inferred from baryonic data) cannot reproduce the observational distribution. The study of individual fits alone does not unveils this issue: one can (at least for some galaxies) use the individual uncertainties to find a good match to the individual observational curves, this without realizing that the overall result is a systematic (non-random) change of parameters, which is such that the NAV plane is systematically modified.

In Fig. 5, we repeat everything we did in Fig. 4, but using the approximations detailed in Sec. 4. These approximations have impact on the individual galaxy rotation curves, but the $\delta V^2_{\text{mod}}$ distribution is essentially the same. The issue with dispersion is slightly stronger (i.e., there is lower dispersion) when these approximation is used.

5.3 Palatini application

Here we are concerned with the results for any theory whose $\delta V^2_{\text{mod}}$ has the same form of eq. (37), which includes $f(R)$ Palatini and EIBI gravities.

Since $\delta V^2_{\text{Palatini}}$ directly depends on the baryonic matter distribution, we use the approximations detailed in Sec. 4 to generate the Palatini plot in Fig. 6.

From the plot, one sees that Palatini has difficulties on covering the bottom part of the observational NAV region, while there is a large excess of data above the observational NAV region. It is important to stress that these results are for any $\alpha$ values, independently on whether $\alpha$ is the same for every galaxy or if it changes from galaxy to galaxy.

We conclude that neither Palatini $f(R)$ gravity, nor EIBI, or any other theory with $\delta V^2_{\text{mod}} = \delta V^2_{\text{Palatini}}$ (37) is compatible with the observational data. That is, these theories cannot replace dark matter in galaxies (even if $\alpha$ changes from galaxy to galaxy).

It is interesting to estimate how would be Fig. 6 if we had not used the approximation of Sec. 4. Similarly to the MOND case in the previous section, the $\delta V^2_{\text{mod}}$ curves will display oscillations. These oscillations for Palatini will be significantly larger than those of MOND, since the former directly depends on the derivative of the density ($\rho'$) [see eq. (37)], while the MOND correction depends on the derivative of the Newtonian potential, i.e. $\delta V^2_{\text{mod}}$. Such large oscillations can only increase the dispersion of $V^2_{\text{mod}}$ and worsen the concordance with the observational data.

5.4 RGGR application

Since $\delta V^2_{\text{RGGR}}$ directly depends on the baryonic matter distribution, we use the approximations detailed in Sec. 4 to generate the RGGR plot in Fig. 7.

From the latter figure, one sees that most of the RGGR curves have a large intersection with the observational regions, but a few are problematic. The shape of the RGGR 1$\sigma$ region is in general close to the observational 1$\sigma$ region, and it becomes closer at large radii ($r_{\text{cn}} \geq 0.5$). Apart from the peculiar protuberance in the 2$\sigma$ region, this region is in general closer to the observational 2$\sigma$ limits than MOND or Burkert.

It is interesting to consider how would be Fig. 7 if we had not used the approximation of Sec. 4. The $\delta V^2_{\text{mod}}$ curves without the exponential approximations will have additional oscillations, but they are expected to be very smooth, similarly to the MOND case or even smoother. This since $\delta V^2_{\text{RGGR}}$ depends linearly on $V^2_{\text{bar}}/\Phi_{\text{bar}}$, $\Phi_{\text{bar}}$ is even smoother than $V^2_{\text{bar}}$, and $|\Phi_{\text{bar}}|$ is smaller at large radii, where $V^2_{\text{bar}}$ does not change abruptly. Indeed, plots for individual galaxies show that RGGR induce gentle rotation curve oscillations (Rodrigues et al. 2010, 2014).

In conclusion, the RGGR results display relevant features in accordance with observational data, the concordance being clearly better than Palatini gravity but worse than the Burkert profile. One important issue is that some galaxies have too large $\delta V^2_{\text{mod}}$ gravities.

5.5 Model efficiency comparison

Following Sec. 2.4, Table 4 shows the efficiency parameters results for each model.
Table 3. Burkert profile results for 153 SPARC galaxies with Gaussian priors on $Y_d$ and $Y_b$, which centered on $0.5$ M$_\odot$/L$_\odot$ and $0.6$ M$_\odot$/L$_\odot$ respectively, and with a standard deviation of 0.1 dex. The complete table is provided as a supplementary material. The fitted parameters are $r_c$, $\rho_c$, $Y_d$ and $Y_b$. Col. (1): galaxy name. Col (2): best-fit $r_c$. Col (3): best-fit log$_{10}\rho_c$. Col (4): best-fit $Y_d$. Col (5): best-fit $Y_b$. Col (6): minimum chi-squared value. Col (7): the number of data points (DP) of the galaxy. Since for some cases $r_c$ can be arbitrarily large, we used the constraint $r_c < 1000$ kpc. Galaxies without a relevant bulge have “–” in the $Y_b$ column. All mass-to-light ratios are for the 3.6 $\mu$m band.

| Galaxy  | $r_c$ (kpc) | log$_{10}\rho_c$ (M$_\odot$/kpc$^2$) | $Y_d$ (M$_\odot$/L$_\odot$) | $Y_b$ (M$_\odot$/L$_\odot$) | $\chi^2$ | DP |
|---------|-------------|-----------------------------------|-----------------------------|-----------------------------|-----------|----|
| CamB    | 1000.       | 6.507                             | 0.300                       | –                           | 21.75     | 9  |
| D512-2  | 1.50        | 7.645                             | 0.500                       | –                           | 0.12      | 4  |
| D564-8  | 2.15        | 7.120                             | 0.497                       | –                           | 0.27      | 6  |
| D631-7  | 7.00        | 7.017                             | 0.388                       | –                           | 12.48     | 16 |
| DDO064  | 1.91        | 7.757                             | 0.498                       | –                           | 4.09      | 14 |

Figure 3. Normalized histograms, together with the corresponding PDF’s, for the $r_c$ distributions from individual galaxy fits. These fits use a Burkert halo, with Gaussian priors on the stellar mass-to-light ratios centered on $Y_d = 0.5$ and $Y_b = 0.6$, with 0.1 dex as the standard deviation. **Left plot:** The vertical black dashed lines show the 1σ region limits from eq. (60), while the vertical red dot-dashed lines delimit the 1σ highest probability region from the PDF inferred from the individual fits (the PDF is estimated from the $r_c$ fit results and using a KDE with the Silverman rule). See also eq. (62). **Right plot:** The same as the left plot, but it refers to the 2σ regions. The two different method results are qualitatively similar, but they do not yield the same regions since they answer different questions about the $r_c$ distribution.

Figure 4. The NAV plane for MOND, as given by eq. (16), for three different $a_0$ values. Each transparent blue curve corresponds to one of the 153 SPARC galaxies, the same ones that were used for the RAR (McGaugh et al. 2016). The central plot assumes the commonly used $a_0$ value, while the other two plots assume extrema cases. The central plot includes the limits of the 1σ and 2σ HDRs, which are shown in dashed purple curves. See also Fig. 2.
Figure 5. Same as Fig. 4, but the blue curves are based on exponential approximations for the stellar and gas disks. Moreover, the galaxies with a relevant bulge are not considered, thus each of the three plots has 122 blue curves. The sample results are essentially the same of those in Fig. 4, but with a slightly smaller dispersion.

Figure 6. The NAV plane for Palatini gravity (37). Each transparent blue curve corresponds to one of the 122 bulgeless SPARC galaxies, as described in Sec. 4. The plot includes the 1σ and 2σ HDRs limits, which are shown in dashed purple curves. Since for several galaxies the $\delta V^2_{\text{Palatini}}$ curve is much above the observational $\delta V^2_{\text{obs}}$ regions (in grey), the vertical axis goes up to $\delta V^2 = 4$, which is not sufficiently high to see the complete 2σ region. See also Fig. 2.

Figure 7. The NAV plane for RGGR (59). Each transparent blue curve corresponds to one of the 122 bulgeless SPARC galaxies, as described in Sec. 4. The plot includes the 1σ and 2σ HDRs limits, which are shown in dashed purple curves. The majority of $\delta V^2_{\text{RGGR}}$ curves stay in the observational region, but some of them go beyond it and generate a peculiar protuberance in the 2σ region, which extends up to $\delta V^2 \sim 2.0$. See also Fig. 2.

It is no surprise that the Burkert profile has the best score in Table 4. It is a model known to be capable of providing good fits to a large number of galaxies, usually being better (i.e., with lower $\chi^2$ values) than NFW profile and modified gravity approaches (e.g., Rodrigues et al. 2014).

MOND appears twice in Table 4, one for the case without approximations, Fig. 4, and the other with the exponential approximations for the stellar and gaseous components, Fig. 5. For both cases, the mean efficiency ($E_M$) values are about $\sim 0.5$. This shows that the concordance is reasonable. As previously commented, the issue with MOND is that the predicted model region is excessively dense, thus a significant part of the observational data is outside the model predicted region. This type of issue makes the MOND with approximated baryonic contributions slightly worse than the case without approximations. The exponential approximations increase the regu-

Table 4. Model efficiency results. $E(1)$, $E(2)$ and $E_M$ refer respectively to the 1σ, 2σ and mean efficiencies, as defined in Sec. 2.4.

| Model                  | $E(1)$ | $E(2)$ | $E_M$ |
|------------------------|--------|--------|-------|
| Burkert profile        | 0.66   | 0.82   | 0.74  |
| MOND (no approx.)      | 0.52   | 0.54   | 0.53  |
| MOND (with approx.)    | 0.33   | 0.50   | 0.41  |
| Palatini ($f(R)$ and EiBI) | −1.94   | < −2   | < −2  |
| RGGR                   | 0.55   | 0.67   | 0.61  |
larity of the rotation curves and hence, as expected, slightly reduce the dispersion in the NAV plane.

As clearly shown in Fig. 6, the agreement between Palatini gravity and the observational NAV region is very low. Palatini gravity, without dark matter, both fails to intercept a large observational region and it predicts a much larger region outside the observational one. This leads to a strong penalization in the \( E(1) \) and \( E(2) \) values, due to large \( \Delta_{\text{mod/obs}}(1) \) and \( \Delta_{\text{mod/obs}}(2) \) contributions. The \( E(2) \) result is not explicitly shown in Table 4 since Palatini 2\( \sigma \) region cannot be reliably determined, that is, the shape of its 2\( \sigma \) HDR region (which can achieve \( \delta V^2_{\text{mod}} > 10 \)) depends strongly on how to discretize the \( \delta V^2_{\text{mod}} \) curves for the purpose of applying a KDE. Nonetheless, it is safe to state that \( E(2) < -2 \).

At last, the efficiency results for RGGR indicate a reasonable correspondence with the observational data. Its efficiency score is slightly higher than that of MOND, while slightly lower than that of Burkert. The main issue of RGGR, within this analysis, is an excess of dispersion.

6 CONCLUSIONS

We developed a new method, based on the normalized additional velocity (NAV) (5), to study galaxy RCs directly from a sample of galaxies. This approach is inspired on an analysis briefly used by Rodrigues et al. (2010, 2012). The method was not developed to be a complete data assessment, but to provide an efficient sample comparison of a particular aspect of galaxy RCs: their shape. This method alone is not sufficient to point out which is the best model for galaxy rotation curves, but it can be used to eliminate proposals and to uncover trends that are hard to be spotted from standard individual fits.

Besides being faster than performing hundreds of individual galaxy fits, one relevant practical advantage of the NAV method appears for modified gravity. The method does not depend on the detailed knowledge of the baryonic matter distribution, it is based on sample approximations that reproduce the Newtonian baryonic rotation curve contribution. The approximations proposed in Sec. 4 are not necessarily good approximations for testing individual galaxies, but are good approximations for a large sample. This point was illustrated in detail for MOND (Figs. 4, 5).

It is feasible to extend the approximations of Sec. 4 by assuming that the baryons are in a disk of finite thickness. For most of the models, disk thickness should have negligible consequences to the sample overall results, but there may be a modified gravity model that is strongly sensitive to disk thickness, thus requiring some care on this point. Let us remark that Palatini gravity (Secs. 3.3, 3.4) lies in the opposite extremum: its non-Newtonian contribution is insensitive to disk thickness.

Five models were here considered: one is a dark matter profile, the Burkert profile, and the four others are modified gravity models without dark matter: MOND, Palatini \( f(R) \), EiBI and RGGR. We summarize the NAV results for these models as follows:

**Burkert.** The Burkert profile is well known for being a phenomenological satisfactory model for a large number of galaxies, thus, as expected, its \( r_b \) parameter can be chosen such that a large part of the observational NAV region can be covered, without including large regions that lack observational data (as shown in Fig. 2). The \( r_b \) values that best fit the NAV regions can be used to estimate the most common \( r_b \) values found from the individual fits (Fig. 3) (and, contrary to the individual fits approach, the \( r_b \) regions can be found independently on \( r_b \)). The NAV results also suggest a possible enhancement for this profile, since its rotation curve contribution approaches \( \delta V^2 = 1 \) too steeply. Different dark matter profiles will be studied in a future work.

**MOND.** Although it is probably the most successful modified gravity model that looks for replacing dark matter, it is well-known that its best fit results commonly yield \( \chi^2 \) values significantly larger than those from dark matter profiles (e.g., Gentile et al. 2004; Rodrigues et al. 2014). Therefore, the NAV results are not expected to be as good as those from the Burkert profile. This model has two peculiar features for the NAV analysis: i) the single parameter introduced by MOND (\( \alpha_0 \)) is not absent from \( \delta V^2_{\text{MOND}} \), due to MOND’s nonlinear dependence on it. ii) Contrary to other modified gravity models, its rotation curve can be written as a function of \( V^2_{\text{out}} \) alone. Hence, its analysis can be done with or without the approximations from Sec. 4.

MOND’s issue is peculiar: \( \delta V^2_{\text{MOND}} \) regions are well inside the densest observational NAV regions, but they lack the proper dispersion. The three main sources of observational error: stellar mass-to-light ratios, distance and inclination may be tuned to make individual galaxies display acceptable fits (Li et al. 2018), but this suggests a systematical change in the NAV plane, not random changes, which is expected to happen if there are unknown systematical observational errors. Further data analysis on this topic can be considered in a future work.

**Palatini.** We recall that we refer here by Palatini gravity either Palatini \( f(R) \) or EiBI gravity. From Fig. 6, it is clear that Palatini gravity cannot be used as a dark matter replacement. It is not hard to qualitatively conclude the same from the analytical expression in eq. (37), since \( \rho^2_{\text{out}} \) is expected to decrease at large radii, which is where dark matter effects are expected to be larger. Thus, Fig. 6 shows a large relative RC enhancement at small radii (\( r_b \sim 0.2 \)), going much beyond the observational region. We have not explored the possibility of combining Palatini effects with a dark matter halo.

**RGGR.** This is the first time that RGGR was tested with the SPARC data. So far, several disk galaxies have been studied within the original RGGR formulation without dark matter (e.g., Rodrigues et al. 2010, 2014), and generally, good results were found. Considering the previous results, it is not unexpected that the majority of galaxy results match well the most dense observational region, as seen in Fig. 7. However, there is a number of outliers, and these are enough to insert a large protuberance in the model 2\( \sigma \) limits.

The main aim of this work was to systematically apply the NAV concept to study a galaxy sample. When studying a particular model, the NAV will not provide the most complete information about it, but it can provide a quick and useful analysis. It can show that the model is not viable, highlight model trends, and suggest possible model improvements. We plan to study further this method and its applications in a future work.

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APPENDIX A: KDE BANDWIDTH SELECTION

In order to find the distributions used in this work, it was necessary to convert data points into distributions. A well-established procedure is to use a kernel density estimation (KDE). To use it, one needs to specify a kernel and a bandwidth. It is well known that a Gaussian kernel is both convenient and it has a high efficiency, with respect to the mean integrated square error (MISE) (Silverman 1998; Scott 2014). The Gaussian kernel is always assumed here.

Although the bandwidth for several problems can be chosen by eye, for this work it is important to use a single algorithm to select it, thus avoiding any human bias on the bandwidth selection. Here we opted for the Silverman rule (Silverman 1998), which is a simple and standard bandwidth selection rule. For consistency, every time a KDE is used here, we use the Gaussian kernel with the Silverman rule.

We start by considering the unidimensional problem. This brief review is mainly based on Silverman (1998). Let \( \{ x_i \} \) be a set of \( n \) observations whose probability density is denoted by \( f(x) \) and it is a priori unknown. A KDE estimator \( \hat{f} \) for the distribution \( f \) is given by

\[
\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right),
\]

where \( h \) is the bandwidth and \( K \) is the kernel. The latter satisfies

\[
\int_{-\infty}^{\infty} K(x) \, dx = 1.
\]

This condition ensures the normalization of \( \hat{f} \). Therefore, a KDE with a Gaussian kernel is a sum of normalized Gaussians, one Gaussian for each data point.

The bandwidth \( (h) \) selection is the trickiest part. This since the variation of \( \hat{f} \) decreases with \( h \), while its bias increases with \( h \). A common and natural approach to find the optimum \( h \) value starts by considering the MISE minimization, that is,

\[
\text{MISE}(\hat{f}) = \left( \int (\hat{f}(x) - f(x))^2 \, dx \right) = \int \left( (\hat{f}(x) - f(x))^2 \, dx + \int \text{var}(\hat{f}(x)) \, dx \right)
\]

where \( (\quad) \) denotes the expected value. There is no simple general solution to perform the above minimization, since the MISE depends on the unknown distribution \( f(x) \).

For simplicity, let both \( K \) and \( f(x) \) be normal distributions. The kernel can be fixed as such without relevant loss of efficiency, as previously comment. For this, the second, this is a particular case, but it can be useful as a starting point. In this context, let \( n \) be the number of data points and \( \text{var}(f(x)) = \sigma^2 \), then the MISE minimization provides the optimum \( h \) as

\[
h_{\text{opt}} = \left( \frac{4}{3} \right)^{1/5} \sigma n^{-1/5}.
\]

Therefore, if \( f(x) \) can be assumed to be a normal distribution, from the data standard deviation (SD) one can estimate \( \sigma \) and use the above formula to find \( h_{\text{opt}} \).

For distributions that are close to Gaussian, the rule (A4) should be sufficient. The Silverman rule improves this setting by considering small changes that make it more robust against distributions that are not so close to Gaussian. In particular, the SD-based \( h_{\text{opt}} \) formula (A4) tends to generate too large bandwidths for multimodal distributions. The latter case behaves better with bandwidths based on a more robust dispersion estimate, namely the interquartile range (IQR), which can be defined by the difference between the third and first quartile limits, thus the IQR determines a range that contains about 50% of the data. For Gaussian data, SD and \( \sigma \) are equal, but the IQR is larger than the SD by a factor about 1.34. On the other hand, SD works better than IQR for some distributions, like the binomial one. Following Silverman proposal, the idea is to use an \( h_{\text{opt}} \) rule based on the quantity \( A \), with

\[
A = \min(\text{SD}, \text{IQR}/1.34).
\]

By considering MISE evaluations with this \( A \)-based rule for several distributions, including some mildly multimodal ones, it was found that on average it is better to replace the factor \((4/3)^{1/5} \sim 1.06\) by 0.9. Putting all together, we find the one-dimensional Silverman rule as (Silverman 1998)

\[
h_{\text{Silverman}} = 0.9 \, A \, n^{-1/5}.
\]

The multidimensional case is an extension of the above for \( d \) dimensions. It reads

\[
h_{\text{Silverman}} = \frac{9}{10} \left( \frac{3}{4} \right)^{1/5} \left( \frac{4}{d+2} \right)^{1/4} \pi^{1/4} A \, n^{-1/5}. \tag{A7}
\]

We use the expression above for deriving all the bandwidths, either for 1D or 2D data.

As a final improvement here implemented, since the probability of finding data with \( r_a < 0 \) or \( r_a > 1 \) is necessarily zero, the smooth PDF found from the KDE method is cropped at \( r_a = 0 \) and \( r_a = 1 \), and then normalized. The complete implementation of such technical details can be found in our code NAVanalysis.\(^{11}\)

APPENDIX B: STELLAR AND GASEOUS DISKS APPROXIMATIONS

In order to do the approximations detailed in Sec. 4, we derived the exponential profiles that best-fit the stellar and the gaseous circular velocity components. The galaxy sample is constituted by the 122 SPARC galaxies that satisfy the RAR quality cuts\(^{12}\) and that do not have a relevant bulge.

The circular velocity due to an axisymmetric exponential profile of negligible thickness has a well-known analytical expression (Freeman 1970; Binney & Tremaine 1988), which reads

\[
V^2 = 4\pi G \Sigma_0 b y^2 [I_0(y) K_0(y) - I_1(y) K_1(y)]. \tag{B1}
\]

In the above, the exponential profile is given by

\[
\Sigma(r) = \Sigma_0 e^{-r/h}, \tag{B2}
\]

where \( y \equiv r/(2h) \), and \( I_n \) and \( K_n \) are modified Bessel functions of the first and the second kind, respectively.

The best fits for each galaxy are derived by minimizing the following \( \chi^2 \) quantity,

\[
\chi^2 = \sum_i \left( \frac{V(r_i) - V_i}{\sigma_i} \right)^2. \tag{B3}
\]

The sum on \( i \) runs through all the data points for a given galaxy. The

\(^{11}\) github.com/davi-rodrigues/NAVanalysis

\(^{12}\) This excludes the galaxies that are not sufficiently symmetric (“quality flag 3”) and whose inclination is smaller than 30°.
velocity $V(r_i)$ is given by eq. (B1), while $V_i$ corresponds to either the stellar or the gaseous components. The latter data are commonly provided by several different data samples, including SPARC. The minimization procedure yields the best fit values for $h$ and $\Sigma_0$.

The uncertainty on the values of $V_i^2$ depends on the galaxy distance and, for the stellar part, the mass-to-light ratio. These two uncertainties contribute with a global constant to $\text{chi}^2$.

On the other hand, this 10% error may lead to unrealistically small errors for a few data points that are too close to zero. This is mainly relevant for the gas component. It is unrealistic since there are other sources of error that were not considered. Hence, to avoid such issue with a simple procedure, we use a minimum error $\sigma_{\text{min}} = 2$ km/s for all the galaxies and all the data points. In conclusion,

$$\sigma_i = \text{max}(0.10V_i, 2 \text{ km/s}) . \quad \text{(B4)}$$

The complete fit results are shown in tables B1 and B2. The average gas fit results are not as reasonable as the stellar fits. Although there are some particular cases in which the exponential profile fits very well, the gas profile is more diverse than the stellar density profile. There are several cases with very large $\chi^2$ values or with $h_0 > 1$. On the other hand, as detailed in Sec. 4, there is no need for a high quality gas model, but this simple model is significantly better than nothing and it is sufficient for a sample analysis (even though it may not be satisfactory for detailed studies of individual galaxies).

### Table B1

The best fit results of the exponential model for the stellar component (one result for each of the selected 122 SPARC galaxies without bulge). The model is given by eq. (B2). Each line corresponds to a galaxy and they are ordered alphabetically. Col.(1): best-fit disk scale length. Col.(2): the logarithm of the best-fit central surface brightness. Col.(3): normalized disk scale length ($h_n \equiv h/r_{max}$). Col.(4): the minimum chi-squared. Col.(5): the number of galaxy data points that were used for the fit. The complete table is provided as a supplementary material.

| $h$ (kpc) | $\log_{10} \Sigma_0$ ($M_\odot$/pc$^2$) | $h_n$ | $\chi^2$ | Data points |
|-----------|-------------------------------|-------|---------|-------------|
| 0.75      | 7.50                          | 0.42  | 0.929   | 9           |
| 1.29      | 7.51                          | 0.34  | 0.425   | 4           |
| 0.72      | 7.03                          | 0.23  | 0.013   | 6           |
| 0.85      | 7.63                          | 0.12  | 3.369   | 16          |
| 1.05      | 7.51                          | 0.35  | 2.237   | 14          |
| ...       | ...                           | ...   | ...     | ...         |

### Table B2

The same of Table B1, but for the gas component. Since for some galaxies $h_n$ can be arbitrarily large, we used the constraint $h_n < 100$ (the precise value of this constraint, say 50, 100 or 1000, has no impact on our results). The complete table is provided as a supplementary material.

| $h$ (kpc) | $\log_{10} \Sigma_0$ ($M_\odot$/pc$^2$) | $h_n$ | $\chi^2$ | Data points |
|-----------|-------------------------------|-------|---------|-------------|
| 0.77      | 6.91                          | 0.43  | 1.052   | 9           |
| 2.66      | 6.42                          | 0.69  | 0.000   | 4           |
| 1.34      | 6.55                          | 0.44  | 0.062   | 6           |
| 1.50      | 7.43                          | 0.21  | 0.062   | 16          |
| 298.      | 8.44                          | 100   | 39.691  | 14          |
| ...       | ...                           | ...   | ...     | ...         |

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