### Citation
Khan, Kamil A., Vibhu P. Saxena, and Paul I. Barton. "Sensitivity Analysis of Limit-Cycle Oscillating Hybrid Systems." SIAM Journal on Scientific Computing 33 (2011): 1475. © 2011 Society for Industrial and Applied Mathematics.

### As Published
http://dx.doi.org/10.1137/100804632

### Publisher
Society for Industrial and Applied Mathematics

### Version
Final published version

### Citable link
http://hdl.handle.net/1721.1/67305

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SENSITIVITY ANALYSIS OF LIMIT-CYCLE OSCILLATING HYBRID SYSTEMS∗

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Abstract. A theory is developed for local, first-order sensitivity analysis of limit-cycle oscillating hybrid systems, which are dynamical systems exhibiting both continuous-state and discrete-state dynamics whose state trajectories are closed, isolated, and time-periodic. Methods for the computation of initial-condition sensitivities and parametric sensitivities are developed to account exactly for any jumps in the sensitivities at discrete transitions and to exploit the time-periodicity of the system. It is shown that the initial-condition sensitivities of any limit-cycle oscillating hybrid system can be represented as the sum of a time-decaying component and a time-periodic component so that they become periodic in the long-time limit. A method is developed for decomposition of the parametric sensitivities into three parts, characterizing the influence of parameter changes on period, state variable amplitudes, and relative phases, respectively. The computation of parametric sensitivities of period, amplitudes, and different types of phases is also described. The methods developed in this work are applied to particular models for illustration, including models exhibiting state variable jumps.

Key words. hybrid system, limit cycle, amplitude sensitivity, period sensitivity, phase sensitivity, boundary-value problem

AMS subject classifications. 37C27, 65L10, 34K34

DOI. 10.1137/100804632

1. Introduction. Hybrid discrete/continuous systems are dynamical systems which exhibit both discrete-state and continuous-state dynamics. Over the time evolution of such systems, time may be partitioned into epochs in which different continuous-state dynamics apply. The dynamic behavior during a particular epoch may be described, for example, by a system of ordinary differential equations (ODEs) or a system of differential-algebraic equations (DAEs). Successive epochs are separated in time by instantaneous events at which state variables may jump, and the formulation of the continuous-state dynamics may change discretely from one mode to another.

As defined rigorously in the next section, hybrid limit-cycle oscillators (HLCOs) are time-periodic hybrid systems whose state trajectories are closed and isolated. Examples of HLCOs arise in biological systems, such as models of the cell-cycle [3, 4] in which a cell’s mass drops instantaneously upon mitosis. Other examples include models of walking and hopping robots [10, 11, 14], in which state variables change discontinuously whenever a robot’s foot touches the ground.

Sensitivity analysis is useful in providing information about the influence of infinitesimal parameter changes on a system’s behavior. Applications of sensitivity analysis include numerical optimization, parameter estimation, experimental design, analysis of biochemical pathways, and model reduction. The sensitivity analysis of HLCOs, however, is complicated both by the hybrid nature and the oscillatory behav-
ior of the system. Sensitivities of hybrid systems will in general jump at events, even when all state variables are continuous \[9\]. As has been shown for continuous-state systems \[15, 21\], initial conditions for parametric sensitivities also cannot be set to zero when dealing with systems confined to the periodic orbits of limit cycles. This is because the periodic orbit is sensitive to the parameters, and the initial state is confined to this orbit. In general, the initial state must therefore be a nontrivial function of the system parameters, yielding nontrivial initial parametric sensitivities.

Established proofs of properties of limit-cycle oscillators \[13\] and the development of sensitivity analysis of limit-cycle oscillators \[15, 21\] do not extend directly to HLCOs. In the current work, a theory for sensitivity analysis of HLCOs is developed by bridging and extending previous work on the sensitivity analysis of hybrid systems \[9\] and continuous-state limit-cycle oscillators \[15, 21\]. Exact expressions are developed for the local, first-order sensitivities of state variables with respect to parameters and initial conditions. Additionally, expressions are derived for parametric sensitivities of quantities characterizing the periodic orbit as a whole, namely, period, amplitude, and two types of phase. Several properties from the sensitivity analysis of continuous-state oscillators are demonstrated to hold for HLCOs as well, including the existence of a monodromy matrix.

2. System definition. Similar to the hybrid automaton model presented in previous work \[2, 9\], the oscillating hybrid systems analyzed in this work are represented by the following definition.

**Definition 2.1.** An oscillating hybrid system is a nondimensionalized 12-tuple $H = (\mathcal{M}, p, T, E, \mathcal{T}_M, x, x_0, f, \theta, \sigma, \Theta)$, with the elements of $H$ defined as follows:

1. $\mathcal{M} = \{1, 2, \ldots, n_m\}$ for some $n_m \in \mathbb{N}$,
2. $p \in P$, where $P$ is an open subset of $\mathbb{R}^{n_p}$ for some $n_p \in \mathbb{N}$,
3. $T: P \to \mathbb{R}^+$,
4. $E = \{1, 2, \ldots, n_e + 1\}$ for some $n_e \in \mathbb{N}_0$,
5. $\mathcal{T}_M$ is a sequence $\{m_i\}_{i \in E}$, where $m_i \in \mathcal{M}$ for each $i \in E$,
6. $x: E \times P \times \mathbb{R}^+_0 \to X$, where $X$ is an open subset of $\mathbb{R}^{n_x}$ for some $n_x \in \mathbb{N}$,
7. $x_0: P \to X$,
8. $f: \mathcal{M} \times X \times P \to \mathbb{R}^{n_x}$,
9. $\theta \in \mathcal{M}^2$,
10. $\mathcal{L}: \theta \times X \times P \to \mathbb{R}$,
11. $\sigma: P \to (\mathbb{R}^+_0)^{n_e+2}$, and
12. $\Theta: \theta \times X \times P \to X$.

The elements of $H$ in the above definition refer to the following components of the hybrid system:

1. $\mathcal{M}$ is a set of indices for the discrete modes of the system, arbitrarily enumerated.
2. $p$ is a vector of constant parameters influencing the dynamic evolution of the system, the initial conditions of the system, and the period of oscillation.
3. $T(p)$ is the period of oscillation of the hybrid system.
4. $E$ is the set of indices for the time epochs visited by the system during each period, enumerated in chronological order. Here, $n_e$ is the number of events occurring over one period of oscillation at which one epoch ends and the next epoch begins. For convenience, define $E^+$ to be $E \cup \{n_e + 2\}$, and define $E^-$ to be $E \setminus \{n_e + 1\}$.
5. The mode trajectory $\mathcal{T}_M$ is the order in which modes $m_i$ are visited by the system’s state during each period. Individual modes may be visited multiple times during each period, and the system may remain in the same mode after an event. As described below, the system is defined so that $m_{n_e+1} = m_1$. 

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6. \( x(i, p, t) \) is the vector of continuous state variables of the system, given parameters \( p \), and time \( t \), in the \( i \)th epoch visited during the current period.

7. \( x_0(p) \) is the initial condition of the system at an initial time of \( t_0 = 0 \). As described in the next section, the initial condition must lie on the periodic orbit implicitly defined by the parameters \( p \) and is therefore dependent on \( p \).

8. \( f(m_i, z, p) \) is the vector field describing the time evolution of state variables when in a particular mode \( m_i \in \mathcal{M} \), given the current vector \( z \) of state variable values and parameters \( p \).

9. \( \theta \) is the set of pairs \((m_i, m_j) \in \mathcal{M}^2 \) such that the system’s discrete state may exhibit a transition from mode \( m_i \) into mode \( m_j \) at a particular time.

10. \( \mathcal{L}(m_i, m_j, z, p) \) is a discontinuity function implicitly defining the time of transition from mode \( m_i \) into mode \( m_j \) for each pair \((m_i, m_j) \in \theta \), in the manner described below.

11. For each \( i \in \mathcal{E} \), the \( i \)th element of \( \sigma(p) \), \( \sigma_i(p) \), is the time at which the \( i \)th epoch begins during the first period. \( \sigma_{n+2}(p) \) is defined to be \( T(p) \) for convenience. \( \sigma(p) \) is defined implicitly by \( \mathcal{L} \). For each \( i \in \mathcal{E} \), define \( \tau_i(p) := \sigma_{i+1}(p) \) to be the time at which the \( i \)th epoch ends during the first period.

12. \( \Theta(m_i, m_j, z, p) \) is a transition function reinitializing the continuous state variables after a transition from mode \( m_i \) into mode \( m_j \) for each pair \((m_i, m_j) \in \theta \).

The considered hybrid oscillators are autonomous in that \( f, \Theta, \) and \( \mathcal{L} \) do not depend explicitly on time.

2.1. Dynamic behavior. The time evolution of the system’s state proceeds as follows. The system starts at time \( \sigma_1(p) := 0 \) at the initial condition
\[
(2.1) \quad x(1, p, 0) = x_0(p).
\]

As described in section 2.4, \( x_0(p) \) is chosen so as not to coincide with the location of any event on the periodic trajectory. For each \( i \in \mathcal{E} \), supposing \( x(i, p, \sigma_i(p)) \) has already been defined, the continuous state variables evolve for times \( t \in (\sigma_i(p), \tau_i(p)] \) as the solution to the following system of ODEs:
\[
(2.2) \quad \frac{dx}{dt}(i, p, t) = f(m_i, x(i, p, t), p).
\]

For each \( i \in \mathcal{E}^- \), there exists at least one discontinuity function \( \mathcal{L}(m_i, m_j, \cdot, \cdot, \cdot) \) defined for mode \( m_i \). The event ending a particular epoch occurs at time \( \tau_i(p) \), defined as the earliest time such that \( \tau_i(p) > \sigma_i(p) \), and such that the transition condition
\[
\mathcal{L}(m_i, m_j, x(i, p, \tau_i(p)), p) \leq 0
\]
is satisfied for some \( m_j \in \mathcal{M} \).

Once \( \tau_i(p) \) is determined, the next mode is then defined to be \( m_{i+1} = m_j \), with the system state reinitialized in the new mode according to
\[
x(i + 1, p, \sigma_{i+1}(p)) = \Theta(m_i, m_{i+1}, x(i, p, \tau_i(p)), p),
\]
whereupon the state evolves in the new mode according to (2.2). The state variables may thus take two values at an event time; the epoch index \( i \in \mathcal{E} \) is an explicit argument of \( x \) in order to resolve any ambiguity when referring to state variable values at events.
It is assumed throughout this work that for each \( i \in \mathcal{E} \), there is no pair \((m_k, m_\ell) \in M^2\) such that \( \mathcal{L}(m_i, m_k), \mathbf{x}(i, p, \tau_i(p)), \) and \( \mathcal{L}(m_i, m_\ell), \mathbf{x}(i, p, \tau_\ell(p)) \) are both zero, and that this remains true when different initial conditions and parameters are chosen from some respective neighborhoods \( X_0 \subset X \) and \( P_0 \subset P \) of \( \mathbf{x}_0(p) \) and \( p \). It is further assumed that \( \frac{\partial}{\partial t} \mathcal{L}(m_i, m_{i+1}), \mathbf{x}(i+1, p, t_e), \mathbf{p}) \) is nonzero. These assumptions are sufficient to ensure that perturbations in parameters \( p \) within some neighborhood of \( p \) will not change the mode trajectory \( T_M \) [9].

To avoid pathological deadlock phenomena in which hybrid systems can exhibit infinitely many transitions at a single time [2], it is assumed throughout this work that for each \( i \in \mathcal{E} \), if a transition from mode \( m_i \) to mode \( m_{i+1} \) occurs at some time \( t = t_e \), then it must be true that \( \mathcal{L}(m_{i+1}, m_j), \mathbf{x}(i+1, p, t_e), p) \) is greater than zero for every \( m_j \in M \) such that \( (m_{i+1}, m_j) \in \theta \). This assumption prohibits the next event from being triggered instantaneously upon entering the new mode and therefore ensures that the system stays in each visited mode for a nonzero duration.

Due to the above assumptions, \( \{(m_i, m_{i+1}) : i \in \mathcal{E}^- \} \) is the subset of \( \theta \) corresponding to actual discrete transitions taken by the system. All other elements of \( \theta \) are not accessed on the system’s periodic orbit and may therefore be neglected without loss of generality. Discontinuity and transition functions describing the system’s discrete transitions may therefore be expressed in terms of epochs instead of modes without ambiguity. Throughout the rest of this work, therefore, the following expressions are used for each \( i \in \mathcal{E}^- \) in order to simplify notation:

\[
\mathcal{L}(i, \tau) := \mathcal{L}(m_i, m_{i+1}, \cdot, \cdot), \quad \Theta(i, \tau) := \Theta((m_i, m_{i+1}), \cdot, \cdot).
\]

Since the state trajectory of the system is \( T(p) \)-periodic, the system eventually reaches mode \( m_{n_e+1} = m_1 \), where \( \tau_{n_e+1}(p) = \sigma_{n_e+2}(p) = T(p) \) is defined so that

\[
\mathbf{x}(n_e+1, p, T(p)) = \mathbf{x}(1, p, 0), \quad \dot{\mathbf{x}}(n_e+1, p, T(p)) = \dot{\mathbf{x}}(1, p, 0).
\]

The ith epoch during the \((N+1)\)th period spans times \([\sigma_{i,N}(p), \tau_{i,N}(p)]\), where

\[
\sigma_{i,N}(p) := NT(p) + \sigma_i(p), \quad \tau_{i,N}(p) := NT(p) + \tau_i(p).
\]

Hence, for each \( i \in \mathcal{E} \), each \( t \in [\sigma_i(p), \tau_i(p)] \), and each \( N \in \mathbb{N}_0 \),

\[
\mathbf{x}(i, p, t + NT(p)) = \mathbf{x}(i, p, t), \quad \dot{\mathbf{x}}(i, p, t + NT(p)) = \dot{\mathbf{x}}(i, p, t).
\]

If \( t \) is an event time in the above equation, then \( \dot{\mathbf{x}} \) denotes the right-hand or left-hand time derivative of \( \mathbf{x} \), as appropriate.

### 2.2. Model assumptions.

It is assumed that the state trajectories \( \mathbf{x}(i, p, t) \) defined in the previous section exist in \( X \) and are unique for each \( i \in \mathcal{E} \), each \( N \in \mathbb{N}_0 \), and each \( t \in [\sigma_{i,N}(p), \tau_{i,N}(p)] \). For each \( i \in \mathcal{E} \), the functions \( f(m_i, \cdot, \cdot), \mathcal{L}(i, \cdot, \cdot), \) and \( \Theta(i, \cdot, \cdot) \) are assumed to be continuously differentiable on \( X \times P \). It follows from (2.2) that for each \( i \in \mathcal{E} \) and for each \( N \in \mathbb{N}_0 \), \( \mathbf{x}(i, p, \cdot) \) is continuously differentiable on \([\sigma_{i,N}(p), \tau_{i,N}(p)]\), with right-hand and left-hand derivatives defined at \( \sigma_{i,N}(p) \) and \( \tau_{i,N}(p) \), respectively. Moreover, \( \mathbf{x}(i, \cdot, t) \) is continuously differentiable on some neighborhood of \( p \) for each \( t \in [\sigma_{i,N}(p), \tau_{i,N}(p)] \).

When combined with the assumptions concerning discontinuity functions in the previous section, the above assumptions ensure that parametric sensitivities of state variables exist, are unique, and are continuous in each particular mode visited [9].

To be a limit-cycle oscillator, a system must admit a periodic state trajectory tracing out a closed and isolated orbit in state space. For systems confined to such
periodic orbits, we define the orbit to be a stable limit cycle if \( \frac{\partial x}{\partial n}(n_e + 1, p, T(p)) \) has a single eigenvalue equal to one and has the magnitudes of all of its other eigenvalues being less than one. This definition is analogous to a definition [13] for limit cycles in continuous-state systems, and it is shown in the present work that hybrid systems satisfying this definition exhibit useful properties that are similar to those of continuous-state limit cycles. This work deals exclusively with hybrid systems whose states are confined to the periodic orbits of stable limit cycles.

### 2.3. Special partial derivatives

Throughout this work, \( \frac{\partial x}{\partial p}(i, p, t) \) refers to the partial derivative of \( x(i, \cdot, \cdot) \) with respect to its second argument, evaluated at parameters \( p \) and time \( t \). Thus, for example, the total derivative of \( x(1, p, T(p)) \) with respect to \( p \) is

\[
\frac{\partial}{\partial p} [x(1, p, T(p))] = \frac{\partial x}{\partial p}(1, p, T(p)) + x(1, p, T(p)) \cdot \frac{\partial T}{\partial p}(p).
\]

Let \( \tilde{x}(i, p, t; z) \) denote the trajectory of the system described in section 2.1, except with (2.1) replaced by \( \tilde{x}(1, p, 0; z) = z \). Let \( e_k \) denote the \( k \)th unit vector in \( \mathbb{R}^{n_x} \). The initial-condition sensitivity \( \frac{\partial x}{\partial x_0} \) of the system described in section 2.1 is then defined so that for each \( i \in \mathcal{E} \), each \( N \in \mathbb{N}_0 \), each \( t \in [\sigma_{i,N}, \tau_{i,N}(p)] \), and each pair \( (j, k) \in \{1, 2, \ldots, n_x\}^2 \),

\[
\frac{\partial x_j}{\partial x_{0,k}}(i, p, t) := \lim_{\epsilon \to 0} \left( \frac{\tilde{x}_j(i, p, t; x_0(p) + \epsilon e_k) - \tilde{x}_j(i, p, t)}{\epsilon} \right).
\]

The quantity \( \frac{\partial x}{\partial p}(i, p, t) \) is defined only on intervals \( [\sigma_{i,N}(p), \tau_{i,N}(p)] \) in accordance with the chain rule as follows:

\[
\left( \frac{\partial x}{\partial p}(i, p, t) \right)_{x_0=\text{const}} := \frac{\partial x}{\partial p}(i, p, t) - \frac{\partial x}{\partial x_0}(i, p, t) \cdot \frac{\partial x_0}{\partial p}(p).
\]

Many quantities in this work are defined on closed time intervals. For example, for each \( i \in \mathcal{E} \), the quantity \( \frac{\partial x}{\partial p}(i, p, \cdot) \) is defined only on intervals \( t \in [\sigma_{i,N}(p), \tau_{i,N}(p)] \) for \( N \in \mathbb{N}_0 \). Whenever a time derivative of any such quantity is evaluated at a boundary of any such time interval, this derivative refers to the left-hand or right-hand time derivative as appropriate.

### 2.4. Determining the initial condition and period

The initial condition \( x_0(p) \) and the oscillation period \( T(p) \) are defined implicitly through a boundary-value problem (BVP), as are the event times \( \sigma_i(p) \) for each \( i \in \mathcal{E} \). Similarly to the BVP developed in [21], this BVP enforces the periodicity of state variables on the limit cycle and also specifies a phase-locking condition (PLC) to confine the initial state to an isolated point on the periodic orbit which is not visited at any event. The choice of PLC is arbitrary as long as it satisfies this condition: in the examples covered in this work, the PLC is often chosen to enforce that a particular element \( \dot{x}_j \) of \( \dot{x} \) is zero.

Hence, the BVP defining the initial conditions \( x_0(p) \) and the period \( T(p) \) is as follows. For each \( N \in \mathbb{N}_0 \),

\[
\begin{align*}
0 &= x(n_e + 1, p, NT(p)) - x_0(p), \\
0 &= \dot{x}_j(1, p, \sigma_{1,N}(p)).
\end{align*}
\]

Here \( x(n_e + 1, p, NT(p)) \) is defined as the vector of state variables at time \( NT(p) \) with initial condition \( x_0(p) \). The dynamic evolution of the state variables in the intervening time proceeds as in section 2.1, with the transition conditions implicitly defining each event time.
3. Solutions to a general class of difference equations. As shown in later sections of this paper, several key quantities in the sensitivity analysis of limit-cycle oscillating hybrid systems are defined inductively through difference equations of a common form. In Theorem 3.2, a general solution is obtained for this class of difference equations.

Definition 3.1. Given any sequence \( \{X_n\}_{n \in \mathbb{N}} \) of matrices and any sequence \( \{Y_n\}_{n \in \mathbb{N}} \) of square matrices, wherever \( p, q \in \mathbb{N} \) are such that \( p < q \), define the following otherwise undefined sums and products for convenience:

\[
\begin{align*}
\sum_{i=q}^{p} X_i &:= 0; & \prod_{i=q}^{p} Y_i &:= 1.
\end{align*}
\]

Theorem 3.2. Let \( n_1, n_2 \in \mathbb{N} \), and let \( E \) be either \( \mathbb{N}_0 \) or \( \{0,1,\ldots,N\} \) for some \( N \in \mathbb{N} \). Define \( E^+ \) to be \( E \) if \( E = \mathbb{N}_0 \) and to be \( E \cup \{N+1\} \) if \( E = \{0,1,\ldots,N\} \). Let matrices \( X_0 \in \mathbb{R}^{n_1 \times n_2} \), \( A_i \in \mathbb{R}^{n_1 \times n_1} \), and \( B_i \in \mathbb{R}^{n_1 \times n_2} \) be defined for all \( i \in E \). Suppose that matrices \( X_i \) are defined inductively through the difference equation

\[
X_{i+1} = A_i X_i + B_i \quad \forall i \in E.
\]

Then, for each \( i \in E^+ \setminus \{0\} \),

\[
X_i = \left( \prod_{j=0}^{i-1} A_{(i-1)-j} \right) X_0 + \sum_{j=1}^{i} \left[ \prod_{k=j}^{i-1} A_{j+(i-1)-k} \right] B_{j-1}.
\]

Proof. Proceed by induction on \( i \in E^+ \setminus \{0\} \) as follows.

Base case: By (3.2) and (3.1),

\[
X_1 = A_0 X_0 + B_0 = \left( \prod_{j=0}^{0} A_{-j} \right) X_0 + \left( \prod_{k=1}^{0} A_{1+(1-1)-k} \right) B_0,
\]

as required.

Inductive step: Suppose that the proposition is true for some \( i = m \in E \setminus \{0\} \):

\[
X_m = \left( \prod_{j=0}^{m-1} A_{(m-1)-j} \right) X_0 + \sum_{j=1}^{m} \left[ \prod_{k=j}^{m-1} A_{j+(m-1)-k} \right] B_{j-1}.
\]

Applying (3.2) to the above equation yields the following expression for \( X_{m+1} \):

\[
X_{m+1} = A_m \left( \prod_{j=0}^{m-1} A_{(m-1)-j} \right) X_0 + \sum_{j=1}^{m} A_m \left( \prod_{k=j}^{m-1} A_{j+(m-1)-k} \right) B_{j-1} + B_m.
\]

Changing the indices of multiplication and rearranging yields

\[
X_{m+1} = \left( \prod_{j=0}^{m} A_{m-j} \right) X_0 + \sum_{j=1}^{m+1} \left[ \prod_{k=j}^{m} A_{j+m-k} \right] B_{j-1},
\]

as required.

This completes the induction. \( \Box \)
COROLLARY 3.3. Let \( n_1, n_2 \in \mathbb{N} \), and define matrices \( X_0 \in \mathbb{R}^{n_1 \times n_2} \), \( A \in \mathbb{R}^{n_1 \times n_1} \), and \( B \in \mathbb{R}^{n_1 \times n_2} \). Suppose that matrices \( X_i \) are defined inductively through the difference equation

\[
X_i = AX_{i-1} + B \quad \forall i \in \mathbb{N}.
\]

Then, for each \( i \in \mathbb{N}_0 \),

\[
X_i = A^iX_0 + \sum_{j=0}^{i-1} A^jB.
\]

Proof. The result is trivial for \( i = 0 \), since in this case the sum over \( j \) vanishes by (3.1). For \( i \in \mathbb{N} \), the result follows directly from Theorem 3.2 by setting \( E = \mathbb{N}_0 \) and assigning \( A_i = A \) and \( B_i = B \) for each \( i \in E \).

COROLLARY 3.4. Let \( n_1, n_2 \in \mathbb{N} \), and define \( \mathcal{E} \) and \( \mathcal{E}^+ \) as in section 2. Suppose that matrices \( X_0 \in \mathbb{R}^{n_1 \times n_2} \), \( A'_\ell \in \mathbb{R}^{n_1 \times n_1} \), and \( B'_\ell \in \mathbb{R}^{n_1 \times n_2} \) are defined for all \( \ell \in \mathcal{E} \), and suppose that matrices \( X_{\ell,N} \) are defined inductively through the difference equations

\[
\begin{align*}
X_{1,0} &= X_0, \\
X_{\ell+1,N} &= A'_\ell X_{\ell,N} + B'_\ell \quad \forall \ell \in \mathcal{E}, \forall N \in \mathbb{N}_0, \\
X_{1,N+1} &= X_{n_e+2,N} \quad \forall N \in \mathbb{N}_0
\end{align*}
\]

and that matrices \( C \) and \( D \) are defined as follows:

\[
C := \prod_{j=1}^{n_e+1} A'_{(n_e+2)-j}; \quad D := \sum_{j=1}^{n_e+2} \left[ \prod_{k=j}^{n_e+1} A'_{j+(n_e+1)-k} B'_{j-1} \right].
\]

Then, for each \( \ell \in \mathcal{E}^+ \) and each \( N \in \mathbb{N}_0 \),

\[
X_{\ell,N} = \left( \prod_{j=1}^{\ell-1} A'_{(\ell-1)-j} \right) C^N X_0 + \sum_{j=0}^{N-1} C^j D + \sum_{j=2}^{\ell} \left[ \prod_{k=j}^{\ell-1} A'_{j+(\ell-1)-k} B'_{j-1} \right].
\]

Proof. Apply Theorem 3.2 to (3.4) for arbitrary \( N \in \mathbb{N} \), setting \( E = \mathcal{E} \), \( i = \ell - 1 \), \( X_{\ell-1} = X_{\ell,N} \), \( A_{\ell-1} = A'_{\ell} \), and \( B_{\ell-1} = B'_{\ell} \) for all \( \ell \in \mathcal{E} \):

\[
X_{\ell,N} = \left( \prod_{j=0}^{\ell-2} A'_{(\ell-1)-j} \right) X_{1,N} + \sum_{j=1}^{\ell-1} \left[ \prod_{k=j}^{\ell-2} A'_{j+(\ell-1)-k} B'_{j-1} \right] \quad \forall \ell \in \mathcal{E}^+ \setminus \{1\}.
\]

By (3.1), the above equation trivially holds for \( \ell = 1 \) as well. Changing the indices of summation and multiplication then yields

\[
\begin{align*}
X_{\ell,N} &= \left( \prod_{j=1}^{\ell-1} A'_{\ell-j} \right) X_{1,N} + \sum_{j=2}^{\ell} \left[ \prod_{k=j}^{\ell-1} A'_{j+(\ell-1)-k} B'_{j-1} \right] \quad \forall \ell \in \mathcal{E}^+. \\
X_{1,N+1} &= X_{n_e+2,N} = CX_{1,N} + D \quad \forall N \in \mathbb{N}_0
\end{align*}
\]
Applying Corollary 3.3 to (3.3) and (3.8), with $A = C$ and $B = D$, yields

$$X_{1,N} = C^N X_0 + \sum_{j=0}^{N-1} C^j D$$ \quad \forall N \in \mathbb{N}_0.$$ 

Hence (3.7) implies that (3.6) holds for each $\ell \in \mathcal{E}^+$ and each $N \in \mathbb{N}_0$. \qed

4. Initial-condition sensitivities. In this section, a theory is developed to describe the time evolution of initial-condition sensitivities $\frac{\partial x}{\partial x_0}$ for HLCOs. Note that the initial-condition sensitivities $\frac{\partial x}{\partial x_0}(i, p, \cdot)$ are not periodic in general.

For each $i \in \mathcal{E}$, each $N \in \mathbb{N}_0$, and each $t \in [\sigma_i, N(p), \tau_i, N(p)]$, differentiating (2.2) with respect to initial conditions $x_0$ yields

$$\frac{d}{dt} \left( \frac{\partial x}{\partial x_0} \right)(i, p, t) = A(i, p, t) \cdot \frac{\partial x}{\partial x_0}(i, p, t), \quad \text{where } A(i, p, t) := \frac{\partial f}{\partial x}(m_i, x(i, p, t), p).$$

Since $x(i, p, \cdot)$ is $T(p)$-periodic wherever it is defined, and since $f(i, \cdot, \cdot)$ is continuously differentiable on $X \times P$, it follows that $A(i, p, \cdot)$ is $T(p)$-periodic and continuous wherever it is defined as well.

4.1. State transition matrix. As shown by Farkas [7], for any $i \in \mathcal{E}$, any $N \in \mathbb{N}_0$, and any times $t, s \in [\sigma_i, N(p), \tau_i, N(p)]$, there exists a unique state transition matrix $H_N(i, p, t, s)$ solving the ODE system

$$\frac{dH_N}{dt} (i, p, t, s) = A(i, p, t) \cdot H_N(i, p, t, s), \quad H_N(i, p, s, s) = I,$$

so that $H_N(i, p, t, s)$ also satisfies

$$\frac{\partial x}{\partial x_0}(i, p, t) = H_N(i, p, t, s) \cdot \frac{\partial x}{\partial x_0}(i, p, s).$$

The matrix-valued functions $H_N(i, p, \cdot, \cdot)$ exhibit the following useful property.

**Theorem 4.1.** Let $i \in \mathcal{E}$, let $N \in \mathbb{N}$, and choose times $t, s \in [\sigma_i, N(p), \tau_i, N(p)]$. Then $H_N(i, p, t, s) = H_0(i, p, t - NT(p), s - NT(p))$.

**Proof.** For any $i \in \mathcal{E}$ and any $N \in \mathbb{N}_0$, choose $s \in [\sigma_i(p), \tau_i(p)]$. Noting that $A(i, p, \cdot)$ is $T(p)$-periodic, (4.1) yields the following for each $t \in [\sigma_i(p), \tau_i(p)]$:

$$\frac{d}{dt} [H_N(i, p, NT(p) + t, NT(p) + s) = A(i, p, t) \cdot H_N(i, p, NT(p) + t, NT(p) + s),$$

with $H_N(i, p, NT(p) + s, NT(p) + s) = I$.

By uniqueness [7] of solutions to (4.1), comparison of the above equation with (4.1) yields the following for any $t, s \in [\sigma_i(p), \tau_i(p)]$:

$$H_N(i, p, NT(p) + t, NT(p) + s) = H_0(i, p, t, s).$$

This is equivalent to $H_N(i, p, t, s) = H_0(i, p, t - NT(p), s - NT(p))$ for any choice of $t, s \in [NT(p) + \sigma_i(p), NT(p) + \tau_i(p)]$. \qed

4.2. Explicit expression for initial-condition sensitivities. For each $i \in \mathcal{E}^-$ and each $N \in \mathbb{N}_0$, the event time $\tau_i, N(p)$ is the earliest time after $\sigma_i, N(p)$ satisfying

$$\mathcal{L}(i, x(i, p, \tau_i, N(p)), p) = 0.$$
Differentiating this equation with respect to $x_0$ yields
\begin{equation}
0 = \frac{\partial L}{\partial x_0}(i, x(i, p, \tau_i, N(p)), p) \left( \frac{\partial \tau_i, N}{\partial x_0}(i, p, \tau_i, N(p)) \right) - \frac{\partial x}{\partial x_0}(i, p, \tau_i, N(p)) \frac{\partial \tau_i, N}{\partial x_0}(i, p, \tau_i, N(p)) + \frac{\partial x}{\partial x_0}(i, p, \tau_i, N(p)).
\end{equation}

This linear equation can be solved for a unique value of \( \frac{\partial \tau_i, N}{\partial x_0}(i, x(i, p, \tau_i(p)), p) \cdot \frac{\partial x}{\partial x_0}(i, p, \tau_i(p)) \) provided that \( \frac{\partial L}{\partial x_0}(i, x(i, p, \tau_i(p)), p) \cdot \frac{\partial x}{\partial x_0}(i, p, \tau_i(p)) \) is nonzero, which was assumed in section 2. In this case, noting that \( \sigma_i+1, N(p) = \tau_i, N(p) \) and that \( x(i, p, \cdot) \) and \( \dot{x}(i, p, \cdot) \) are \( T(p) \)-periodic for each \( i \in \mathcal{E} \), (4.4) can be rearranged to yield
\begin{equation}
\frac{\partial \sigma_i+1, N}{\partial x_0}(i, p) = -\frac{\partial L}{\partial x_0}(i, x(i, p, \tau_i(p)), p) \cdot \frac{\partial x}{\partial x_0}(i, p, \tau_i(p)) \cdot \frac{\partial \tau_i, N}{\partial x_0}(i, p, \tau_i(p)),
\end{equation}

Jumps in initial-condition sensitivities at events can be evaluated as follows. For each \( i \in \mathcal{E}^- \) and each \( N \in \mathbb{N}_0 \), the transition function \( \Theta \) is defined so that
\begin{equation}
x(i + 1, p, \sigma_i+1, N(p)) = \Theta(i, x(i, p, \tau_i(N(p)), p).
\end{equation}

Recalling that \( \sigma_i+1, N(p) = \tau_i, N(p) \), that \( \sigma_i, N(p) = \sigma_i(p) + NT(p) \), and that \( x(i, p, \cdot) \) and \( \dot{x}(i, p, \cdot) \) are each \( T(p) \)-periodic, differentiating (4.6) with respect to initial conditions \( x_0 \) and substituting (4.5) into the result yields the following equation for each \( i \in \mathcal{E}^- \) and each \( N \in \mathbb{N}_0 \):
\begin{equation}
\frac{\partial x}{\partial x_0}(i + 1, p, \sigma_i+1, N(p)) = C(i, p) \cdot \frac{\partial x}{\partial x_0}(i, p, \tau_i, N(p)),
\end{equation}

where \( C(i, p) \) is defined for each \( i \in \mathcal{E}^- \) as
\begin{equation}
C(i, p) = \frac{\partial \Theta}{\partial x}(i, x(i, p, \tau_i(p)), p)
- \left[ \frac{\partial \Theta}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \dot{x}(i, p, \tau_i(p)) - \dot{x}(i, p, \tau_i(p)) \right]
\cdot \left( \frac{\partial L}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \dot{x}(i, p, \tau_i(p)) \right).
\end{equation}

Let \( C(n_e + 1, p) := I \). Invoking Theorem 4.1, define matrices \( \mathcal{A}(i, p) \) as follows for each \( i \in \mathcal{E} \) and each \( N \in \mathbb{N}_0 \):
\begin{equation}
\mathcal{A}(i, p) := C(i, p) \cdot H_N(i, p, \tau_i, N(p), \sigma_i, N(p)) = C(i, p) \cdot H_0(i, p, \tau_i(p), \sigma_i(p)).
\end{equation}

For notational convenience, define the following quantity for each \( N \in \mathbb{N}_0 \):
\begin{equation}
\frac{\partial x}{\partial x_0}(n_e + 2, p, \sigma_{n_e + 2, N}(p)) := \frac{\partial x}{\partial x_0}(n_e + 1, p, NT(p)).
\end{equation}

Hence, initial-condition sensitivities \( \frac{\partial x}{\partial x_0}(i, p, t) \) are inductively described by the following equations for all \( i \in \mathcal{E}, \ N \in \mathbb{N}_0, \) and \( t \in (\tau_i, N(p), \tau_i, N(p)) \):
\begin{align}
\frac{\partial x}{\partial x_0}(1, p, 0) &= I, \\
\frac{\partial x}{\partial x_0}(i, p, t) &= H_N(i, p, t, \tau_i, N(p)) \cdot \frac{\partial x}{\partial x_0}(i, p, \tau_i, N(p)), \\
\frac{\partial x}{\partial x_0}(i + 1, p, \sigma_i+1, N(p)) &= C(i, p) \cdot \frac{\partial x}{\partial x_0}(i, p, \tau_i, N(p)), \\
\frac{\partial x}{\partial x_0}(1, p, \sigma_{n_e + 1}(p)) &= \frac{\partial x}{\partial x_0}(n_e + 2, p, \sigma_{n_e + 2, N}(p)).
\end{align}
Here (4.11) follows from the statement that $\dot{x}(1, p, NT(p)) = x(n_e + 1, p, NT(p))$ and $\dot{x}(1, p, NT(p)) = x(n_e + 1, p, NT(p))$ for each $N \in \mathbb{N}_0$, since no event occurs at time $\tau_{i,N}(p) = NT(p)$.

Combining (4.9) and (4.10) for each $i \in \mathcal{E}$ and each $N \in \mathbb{N}_0$, setting $t = \tau_{i,N}(p)$ yields

$$
\frac{\partial x}{\partial x_0}(i + 1, p, \sigma_{i+1,N}(p)) = A(i, p) \cdot \frac{\partial x}{\partial x_0}(i, p, \sigma_{i,N}(p)).
$$

Applying Corollary 3.4 to (4.8), (4.12), and (4.11) yields, with each $B'_k = D = 0_{n_e \times n_e}$,

$$
\frac{\partial x}{\partial x_0}(i, p, \sigma_{i,N}(p)) = \left( \prod_{j=1}^{i-1} A(i-j, p) \right) \cdot \left( \prod_{k=1}^{n_e+1} A((n_e+2)-k, p) \right)^N.
$$

Hence, if a monodromy matrix $M(p)$ is defined as

$$
M(p) := \frac{\partial x}{\partial x_0}(n_e + 1, p, T(p)) = \prod_{k=1}^{n_e+1} A((n_e+2)-k, p),
$$

then (4.9), (4.14), and Theorem 4.1 imply that for each $i \in \mathcal{E}$, each $N \in \mathbb{N}_0$, and each $t \in [\sigma_{i,N}(p), \tau_{i,N}(p)]$, the sensitivity of state variables with respect to initial conditions at time $t$ is given by

$$
\frac{\partial x}{\partial x_0}(i, p, t) = H_0(i, p, t-NT(p), \sigma_i(p)) \cdot \left( \prod_{j=1}^{i-1} A(i-j, p) \right) \cdot [M(p)]^N.
$$

### 4.3. Decomposition of initial-condition sensitivities

In this section, it is demonstrated that for any HLCO, for each $i \in \mathcal{E}$, $\frac{\partial x}{\partial x_0}(i, p, \cdot)$ can be decomposed as the sum of a time-decaying part and a nondecaying periodic part, wherever it is defined. This result is claimed by Rosenwasser and Yusupov for limit-cycle oscillating continuous-state systems [15] with reference to Russian language sources [5, 12], and it is not clear from Rosenwasser and Yusupov alone whether the result extends to HLCOs as well.

First, it follows from (4.9), (4.13), and (4.14) that for each $i \in \mathcal{E}$ and each $N \in \mathbb{N}_0$,

$$
\frac{\partial x}{\partial x_0}(i, p, t+T(p)) = \frac{\partial x}{\partial x_0}(i, p, t) \cdot M(p) \quad \forall t \in [\sigma_{i,N}(p), \tau_{i,N}(p)].
$$

The matrix $M(p)$ is thus analogous to the monodromy matrix defined for continuous-state oscillators [15].

The following theorem presents a useful property of $M(p)$ for oscillating hybrid systems, which has been demonstrated to hold for oscillating continuous-state systems [6].

**Theorem 4.2.** If $M(p)$ is defined according to (4.14) such that (4.16) holds, then $\dot{x}(1, p, 0)$ is a right eigenvector of $M(p)$ with a corresponding eigenvalue of unity.

**Proof.** For any times $s \in [0, \tau_{1}(p)]$ and $t \in [T(p), T(p) + \tau_{1}(p)]$, it is clear that $t \geq s$, so a forward state transition function $F(t, s, p, a)$ can be defined such that

$$
\dot{x}(1, p, t) = F(t, s, p, x(1, p, s)) \quad \forall s \in [0, \tau_{1}(p)], \forall t \in [T(p), T(p) + \tau_{1}(p)].
$$
Noting that \( x(1, p, 0) = x_0(p) \), setting \( s \) to zero in the above equation yields
\[
(4.18) \quad x(1, p, t) = F(t, 0, p, x_0(p)) \quad \forall t \in [T(p), T(p) + \tau_1(p)].
\]
Differentiating this equation with respect to \( x_0 \) yields
\[
\frac{\partial x}{\partial x_0}(1, p, t) = \frac{\partial F}{\partial a}(t, 0, p, x_0(p)) \quad \forall t \in [T(p), T(p) + \tau_1(p)].
\]
Hence, by (4.11) and (4.14), setting \( t \) to \( T(p) \) in the above equation yields the following relation between \( M(p) \) and \( F \):
\[
(4.19) \quad M(p) = \frac{\partial x}{\partial x_0}(n_c + 1, p, T(p)) = \frac{\partial x}{\partial x_0}(1, p, T(p)) = \frac{\partial F}{\partial a}(T(p), 0, p, x_0(p)).
\]
If the system were initialized at some time \( t' \in [0, \tau_1(p)] \) instead of \( t \) at time \( 0 \) without any alteration of parameters or initial conditions, then the state variable trajectories for this new system would be the same as the old trajectories, only translated forward in time by duration \( t' \). Thus, for any \( t' \in [0, \tau_1(p)] \) and for initial conditions \( a \in X \),
\[
F(T(p), 0, p, a) = F(t' + T(p), t', p, a).
\]
Differentiating this equation with respect to \( t' \) yields
\[
0 = \frac{\partial F}{\partial t'}(t' + T(p), t', p, a) + \frac{\partial F}{\partial s}(t' + T(p), t', p, a).
\]
Using this result, setting \( s \) to \( t' \) and \( t \) to \( (t' + T(p)) \) in (4.17), and then differentiating with respect to \( t' \) yields
\[
\dot{x}(1, p, t') = \dot{x}(1, p, t' + T(p))
\]
\[
= \frac{d}{dt'} \left[ F \right](t' + T(p), t', p, x(1, p, t'))
\]
\[
= \frac{\partial F}{\partial t'}(t' + T(p), t', p, x(1, p, t')) + \frac{\partial F}{\partial s}(t' + T(p), t', p, x(1, p, t'))
\]
\[
+ \frac{\partial F}{\partial a}(t' + T(p), t', p, x(1, p, t')) \cdot \dot{x}(1, p, t')
\]
\[
= \frac{\partial F}{\partial a}(t' + T(p), t', p, x(1, p, t')) \cdot \dot{x}(1, p, t').
\]
Setting \( t' \) to \( 0 \) in the above equation and using (4.19) yields
\[
\dot{x}(1, p, 0) = M(p) \cdot \dot{x}(1, p, 0),
\]
so that \( \dot{x}(1, p, 0) \) is a right eigenvector of \( M(p) \) with a corresponding eigenvalue of unity. \( \square \)

Let \( n_i \leq n_x \) be the number of linearly independent eigenvectors of the monodromy matrix \( M(p) \). Then Theorem 4.2 implies that \( M(p) \) can be expressed in Jordan form [17] as
\[
(4.20) \quad M(p) = S(p) \cdot \Lambda(p) \cdot [S(p)]^{-1},
\]
where $\mathbf{S} := [\mathbf{x}(1, \mathbf{p}, 0) \; \mathbf{s}_2(\mathbf{p}) \; \mathbf{s}_3(\mathbf{p}) \; \cdots \; \mathbf{s}_{n_\ell}(\mathbf{p})]$ is an invertible matrix with linearly independent, generalized right eigenvectors of $\mathbf{M}(\mathbf{p})$ as columns, and where

$$
\Lambda(\mathbf{p}) :=
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & J_2(\mathbf{p}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_{n_\ell}(\mathbf{p})
\end{bmatrix}
$$

with $\lambda_k(\mathbf{p})$ being the eigenvalue of $\mathbf{M}(\mathbf{p})$ corresponding to the Jordan block $\mathbf{J}_k(\mathbf{p})$ for each $k \in \{2, 3, \ldots, n_\ell\}$. Each $\mathbf{J}_k(\mathbf{p})$ is a square matrix with dimension one greater than the degeneracy of the corresponding eigenvector. By definition of a limit cycle [13], $\lambda_k(\mathbf{p})$ lies strictly within the unit circle for each $k \in \{2, 3, \ldots, n_\ell\}$.

Let $\mathbf{v}_j(\mathbf{p})$ denote the $j$th row of $[\mathbf{S}(\mathbf{p})]^{-1}$ for each $j \in \{1, 2, \ldots, n_\ell\}$. It then follows from (4.20) that $\mathbf{v}_1(\mathbf{p})$ is the left eigenvector of $\mathbf{M}(\mathbf{p})$ corresponding to the eigenvalue of unity. Define the following quantities for each $i \in \mathcal{E}$:

\begin{align}
(4.21) \quad & \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}(i, \mathbf{p}, t) := \mathbf{x}(i, \mathbf{p}, t) \cdot \mathbf{v}_i^T(\mathbf{p}), \\
(4.22) \quad & \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}(i, \mathbf{p}, t) := \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}(i, \mathbf{p}, t) - \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}(i, \mathbf{p}, t),
\end{align}

so that for each $i \in \mathcal{E}$, each $N \in \mathbb{N}_0$, and each $t \in [\sigma_i, N \mathbf{p}, \tau_i, N \mathbf{p}]$,

\begin{equation}
(4.23) \quad \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}(i, \mathbf{p}, t) = \left( \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0} \right)_1(i, \mathbf{p}, t) + \left( \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0} \right)_2(i, \mathbf{p}, t).
\end{equation}

Then $\frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}_1(i, \mathbf{p}, \cdot)$ is $T(\mathbf{p})$-periodic wherever it is defined, due to the periodicity of $\mathbf{x}(i, \mathbf{p}, \cdot)$.

With $t$ restricted to $(\sigma_i, \mathbf{p}, \tau_i, \mathbf{p})$ for each $i \in \mathcal{E}$, differentiating (2.2) with respect to $t$ yields

\begin{equation}
(4.24) \quad \frac{d\mathbf{x}}{dt}(i, \mathbf{p}, t) = \mathbf{x}(i, \mathbf{p}, t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p}) \cdot \mathbf{x}(i, \mathbf{p}, t) = \mathbf{A}(i, \mathbf{p}, t) \cdot \mathbf{x}(i, \mathbf{p}, t).
\end{equation}

Moreover, postmultiplying each side of (4.7) by $\mathbf{x}(i, \mathbf{p}, \tau_i(\mathbf{p}))$ yields the following equation for each $i \in \mathcal{E}^-$:

$$
\mathbf{x}(i + 1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) = C(i, \mathbf{p}) \cdot \mathbf{x}(i, \mathbf{p}, \tau_i(\mathbf{p})).
$$

This equation holds for $i = n_\ell + 1$ as well, since $\mathbf{x}(1, \mathbf{p}, T(\mathbf{p})) = \mathbf{x}(n_\ell + 1, \mathbf{p}, T(\mathbf{p}))$ and since $C(n_\ell + 1, \mathbf{p}) = \mathbf{I}$. It then follows from (4.24) and (4.21) that $\frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}$ satisfies (4.9), (4.10), and (4.11) in place of $\frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}$. Hence, (4.23) implies that $\frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}_2$ also satisfies (4.9), (4.10), and (4.11) in place of $\frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}_2$, and therefore satisfies (4.12), (4.13), and (4.16) as well. Proceeding as in section 4.2, Corollary 3.4 yields

$$
\left( \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0} \right)_2(i, \mathbf{p}, t + NT(\mathbf{p})) = \mathbf{H}_0(i, \mathbf{p}, t, \sigma_i(\mathbf{p})) \cdot \left( \prod_{j=1}^{i-1} A(i-j, \mathbf{p}) \right) \cdot \left( \frac{\partial \mathbf{x}}{\partial \mathbf{X}_0} \right)_2(1, \mathbf{p}, NT(\mathbf{p})).
$$

(4.25)

It follows from (4.8) and the identity $\mathbf{S}(\mathbf{p}) \cdot [\mathbf{S}(\mathbf{p})]^{-1} = \mathbf{I}$ that

$$
\frac{\partial \mathbf{x}}{\partial \mathbf{X}_0}(1, \mathbf{p}, 0) = \mathbf{I} = \mathbf{x}(1, \mathbf{p}, 0) \cdot \mathbf{v}_1^T(\mathbf{p}) + \sum_{j=2}^{n_\ell} [\mathbf{s}_j(\mathbf{p}) \cdot \mathbf{v}_j^T(\mathbf{p})].
$$
Substituting the above equation and (4.21) into (4.22) yields

\[
(4.26) \quad \left( \frac{\partial x}{\partial x_0} \right)_2 (1, p, 0) = \sum_{j=2}^{n_p} [s_j(p) \cdot v_j^T(p)] = S_{\text{red.}}(p) \cdot [S(p)]^{-1},
\]

where \(S_{\text{red.}}(p)\) is the same as \(S(p)\), but with its first column replaced with a zero vector. It follows from (4.20) that for any \(N \in \mathbb{N}_0\),

\[
\left( \frac{\partial x}{\partial x_0} \right)_2 (1, p, 0) \cdot [M(p)]^N = S_{\text{red.}}(p) \cdot [A(p)]^N \cdot [S(p)]^{-1} = S(p) \cdot [A_{\text{red.}}(p)]^N \cdot [S(p)]^{-1},
\]

(4.27)

where \(A_{\text{red.}}(p)\) is the same as \(A(p)\), but with its (1,1)-element reassigned to be zero instead of unity. Since \(\lambda_k(p)\) lies strictly within the unit circle for each \(k \in \{2, 3, \ldots, n_1\}\), and since it was shown above that \(\left( \frac{\partial x}{\partial x_0} \right)_2\) satisfies (4.16) in place of \(\frac{\partial x}{\partial x_0}\), then (4.27) and (4.23) imply that \(\left( \frac{\partial x}{\partial x_0} \right)_2 (1, p, NT(p))\) tends to zero in the limit of large \(N\).

Hence, (4.25) implies that \(\left( \frac{\partial x}{\partial x_0} \right)_2 (i, p, t + NT(p))\) tends to zero in the limit of large \(N\) as well. As a result, (4.23) implies that for any \(i \in E\) and any \(t \in [\sigma_i(p), \tau_i(p)]\),

\[
\frac{\partial x}{\partial x_0} (i, p, t + NT(p)) \to \left( \frac{\partial x}{\partial x_0} \right)_1 (i, p, t + NT(p)) \quad \text{as} \quad N \to \infty.
\]

The initial-condition sensitivities therefore tend toward the periodic steady-state solution \(\left( \frac{\partial x}{\partial x_0} \right)_1 (i, p, t)\) in the long-time limit.

To evaluate \(\left( \frac{\partial x}{\partial x_0} \right)_1\) in practice, \(v_1(p)\) may be computed as follows. If \(w_1(p)\) is a right eigenvector of \([M(p)]^T\) corresponding to the eigenvalue of unity, then it follows from the definition of \(v_1(p)\), Theorem 4.2, and the identity \([S(p)]^{-1} : S(p) = I\) that

\[
v_1(p) = \frac{w_1(p)}{w_1^T(p) \cdot x(1, p, 0)}.
\]

5. Parametric sensitivities. In this section, a theory is developed to describe the time evolution of parametric sensitivities \(\frac{\partial x}{\partial p}\). Similar to the initial-condition sensitivities, these parametric sensitivities are not periodic in general.

For each \(i \in E\), each \(N \in \mathbb{N}_0\) and each \(t \in [\sigma_i,N(p), \tau_i,N(p)]\), let \(B(i, p, t) := \frac{\partial x}{\partial p}(m_i, x(i, p, t), p)\). Differentiating (2.2) with respect to parameters \(p\) then yields

\[
(5.1) \quad \frac{d}{dt} \left( \frac{\partial x}{\partial p} \right) (i, p, t) = A(i, p, t) \cdot \frac{\partial x}{\partial p}(i, p, t) + B(i, p, t),
\]

where \(A(i, p, t)\) is defined as in section 4. Since \(x(i, p, \cdot)\) is \(T(p)\)-periodic wherever it is defined, the definition of \(B\) implies that \(B(i, p, \cdot)\) is \(T(p)\)-periodic wherever it is defined as well.

Let state transition matrices \(H_N(i, p, t, s)\) be defined as in section 4.1. Invoking Theorem 4.1, there is a unique solution to the above differential equation within any particular epoch \([\tau]\), which can be expressed as follows for any \(i \in E\), any \(N \in \mathbb{N}_0\), and any times \(t, s \in [\sigma_i,N(p), \tau_i,N(p)]\):

\[
\frac{\partial x}{\partial p} (i, p, t) = H_0(i, p, t - NT(p), s - NT(p)) \cdot \frac{\partial x}{\partial p}(i, p, s)
\]

\[
+ \int_{s - NT(p)}^{t - NT(p)} H_0(i, p, t - NT(p), t') \cdot B(i, p, t') dt'.
\]
5.1. Explicit expression for parametric sensitivities. For any \( i \in \mathcal{E}^{-} \) and any \( N \in \mathbb{N}_0 \), differentiation of (4.3) with respect to parameters \( p \) yields
\[
\frac{\partial \mathcal{C}(i, p, \sigma_{i,N}(p))}{\partial p} \cdot \left( \mathcal{X}(i, p, \sigma_{i,N}(p)) \cdot \frac{\partial \sigma_{i,N}(p)}{\partial p} + \frac{\partial \mathcal{X}(i, p, \sigma_{i,N}(p))}{\partial p} \right)
\]
\[+ \frac{\partial \mathcal{C}(i, p, \sigma_{i,N}(p))}{\partial p} \cdot \mathcal{X}(i, p, \sigma_{i,N}(p)) = 0.
\]
(5.2)

Since it was assumed in section 2 that \( \frac{\partial \mathcal{C}}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \dot{x}(i, p, \tau_i(p)) \) is nonzero, the above linear equation can be solved for a unique value of \( \frac{\partial \sigma_{i,N}(p)}{\partial p} = \frac{\partial \sigma_{i+1,N}(p)}{\partial p} \).

Noting that \( \sigma_{i+1,N}(p) = \sigma_{i,N}(p) \) and that \( x(i, p, \cdot) \) is \( T(p) \)-periodic for each \( i \in \mathcal{E} \), (5.2) can then be rearranged to yield
\[
\frac{\partial \sigma_{i+1,N}(p)}{\partial p} = - \left( \frac{\partial \mathcal{C}}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \dot{x}(i, p, \tau_i(p)) + \frac{\partial \mathcal{C}}{\partial p}(i, x(i, p, \tau_i(p)), p) \right).
\]
(5.3)

Jumps in parametric sensitivities at events can be evaluated as follows. Recalling that \( \sigma_{i+1,N}(p) = \tau_{i,N}(p) \), that \( \sigma_{i,N}(p) = \sigma_i(p) + NT(p) \), and that \( x(i, p, \cdot) \) and \( \dot{x}(i, p, \cdot) \) are each \( T(p) \)-periodic, differentiating (4.6) with respect to parameters \( p \) and substituting (5.3) into the resulting yields the following equation for each \( i \in \mathcal{E}^{-} \) and each \( N \in \mathbb{N}_0 \):
\[
\frac{\partial x}{\partial p}(i+1, p, \sigma_{i+1,N}(p)) = \mathcal{C}(i, p) \cdot \frac{\partial x}{\partial p}(i, p, \sigma_{i,N}(p)) + \mathcal{D}(i, p),
\]
where \( \mathcal{C}(i, p) \) is defined by (4.7), and \( \mathcal{D}(i, p) \) is defined for each \( i \in \mathcal{E}^{-} \) as
\[
\mathcal{D}(i, p) := - \left( \frac{\partial \Theta}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \dot{x}(i, p, \tau_i(p)) - \dot{x}(i+1, p, \sigma_{i+1,N}(p)) \right)
\]
\[+ \left( \frac{\partial \mathcal{C}}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \dot{x}(i, p, \tau_i(p)) \right) + \frac{\partial \Theta}{\partial p}(i, x(i, p, \tau_i(p)), p).
\]
The \( \frac{\partial \mathcal{C}}{\partial x}(i, x(i, p, \tau_i(p)), p) \cdot \frac{\partial x}{\partial p}(i, p, \tau_i(N(p))) \) term appearing in (5.3) enters (5.4) via the first term on the right-hand side. For each component \( p_j \) of \( p \), it is therefore possible for \( \frac{\partial \mathcal{C}}{\partial p_j} \) to be zero while the other component of \( \mathcal{C}(i, p) \) not to be zero, depending on \( p_j \) explicitly.

Define \( \mathcal{D}(n_e+1, p) := 0 \), and define \( \mathcal{C}(n_e+1, p) \) as in section 4.2. Invoking Theorem 4.1, define the following matrices for each \( i \in \mathcal{E}^+ \), each \( N \in \mathbb{N}_0 \), and each \( t \in (\sigma_{i,N}(p), \tau_{i,N}(p)) \):
1. \( \mathcal{I}(i, p, t - NT(p)) := \int_{t - NT(p)}^{t} \mathcal{H}_0(i, p, t - NT(p), t') \cdot \mathcal{B}(i, p, t') dt' \)
2. \( \mathcal{B}(i, p) := \mathcal{C}(i, p) \cdot \mathcal{I}(i, p, \tau_i(p)) + \mathcal{D}(i, p) \).

For notational convenience, define the following quantity for each \( N \in \mathbb{N}_0 \):
\[
\frac{\partial x}{\partial p}(n_e+2, p, \sigma_{n_e+2,N}(p)) := \frac{\partial x}{\partial p}(n_e+1, p, NT(p)).
\]

Hence, parametric sensitivities \( \frac{\partial x}{\partial p}(i, p, t) \) are inductively described by the following equations for all \( i \in \mathcal{E}, N \in \mathbb{N}_0 \), and \( t \in (\sigma_{i,N}(p), \tau_{i,N}(p)) \):
\[
\frac{\partial x}{\partial p}(1, p, 0) = \frac{\partial x_0}{\partial p}(p),
\]
(5.5)

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Substituting (2.3) and (4.14) into the above equation yields

\[ (5.11) \]

\[ \frac{\partial x}{\partial p}(i, p, t) = H_N(i, p, t, \sigma_{i,N}(p)) + \frac{\partial x}{\partial p}(i, p, \sigma_{i,N}(p)) + \mathcal{I}(i, p, t - NT(p)), \]

\[ (5.12) \]

\[ \frac{\partial x}{\partial p}(i + 1, p, \sigma_{i+1,N}(p)) = C(i, p) \cdot \frac{\partial x}{\partial p}(i, p, \tau_{i,N}(p)) + D(i, p), \]

\[ (5.13) \]

\[ \frac{\partial x}{\partial p}(1, p, \sigma_{1,N}(p)) = \frac{\partial x}{\partial p}(n_e + 2, p, \sigma_{n_e+2,N}(p)). \]

Here (5.8) follows from the statement that \( x(1, p, NT(p)) = x(n_e + 1, p, NT(p)) \) and \( \dot{x}(1, p, NT(p)) = \dot{x}(n_e + 1, p, NT(p)) \) for each \( N \in \mathbb{N}_0 \), since no event occurs at time \( \sigma_{1,N}(p) = NT(p) \). The computation of \( \frac{\partial x}{\partial p}(p) \) in (5.5) is described in section 5.2 and depends on theory from the current section.

Combining (5.6) and (5.7) for each \( i \in \mathcal{E} \) and each \( N \in \mathbb{N}_0 \), setting \( t = \tau_{i,N}(p) \) yields

\[ (5.9) \]

\[ \frac{\partial x}{\partial p}(i + 1, p, \sigma_{i+1,N}(p)) = A(i, p) \cdot \frac{\partial x}{\partial p}(i, p, \sigma_{i,N}(p)) + B(i, p). \]

Applying Corollary 3.4 to (5.5), (5.9), and (5.8) yields the following expression, with each \( A_i = A(\ell, p) \), each \( B_i = B(\ell, p) \), \( C = M(p) = \prod_{k=1}^{n_e+1} A((n_e+2)-k, p) \) by (4.14), and \( D = P(p) := \sum_{j=2}^{n_e+2} [\prod_{k=j}^{n_e+1} A(j+(n_e+1)-k, p)] \cdot B(j-1, p) \):

\[ (5.10) \]

\[ \frac{\partial x}{\partial p}(\ell, p, \sigma_{\ell,N}(p)) = \left( \prod_{j=1}^{\ell-1} A(\ell-j, p) \right) \left( [M(p)]^N \frac{\partial x_0}{\partial p}(p) + \sum_{j=0}^{N-1} [M(p)]^j P(p) \right) \]

\[ + \sum_{j=2}^{\ell} \left[ \prod_{k=j}^{\ell-1} A(j+(\ell-1)-k, p) \right] \cdot B(j-1, p) \].

Hence, for any \( i \in \mathcal{E} \), any \( N \in \mathbb{N}_0 \), and any \( t \in [\sigma_{i,N}(p), \tau_{i,N}(p)] \), the sensitivity of state variables to parameters at time \( t \) is given by (5.6), with \( \frac{\partial x}{\partial p}(i, p, \sigma_{i,N}(p)) \) evaluated according to (5.10).

Now, by (5.8) and (5.10),

\[ (5.11) \]

\[ \frac{\partial x}{\partial p}(n_e + 1, p, T(p)) = \frac{\partial x}{\partial p}(1, p, \sigma_1(p)) = M(p) \cdot \frac{\partial x_0}{\partial p}(p) + P(p). \]

Substituting (2.3) and (4.14) into the above equation yields

\[ (5.12) \]

\[ P(p) = \left( \frac{\partial x}{\partial p}(n_e + 1, p, T(p)) \right)_{x_0=const}. \]

\( P(p) \) may therefore be evaluated in practice by solving the parametric sensitivity equations (5.1) and (5.4), with (5.5) replaced by \( \frac{\partial x}{\partial p}(1, p, 0) \mid x_0=const = 0 \).

**5.2. Sensitivities of initial conditions and oscillation period.** When performing sensitivity analysis on a continuous-state system confined to a periodic trajectory, it is incorrect in general to assume that \( \frac{\partial x}{\partial p}(p) = 0 \) [21]. This concern naturally extends to periodic hybrid systems as well, and arises because the trajectory of the periodic orbit depends on \( p \), and initial conditions are constrained to lie on this orbit by definition. It follows that \( x_0 \) is a nontrivial function of \( p \) in general and is defined throughout the current work by a PLC such as (2.5). Hence, in this section,
a boundary-value formulation is developed to describe the parametric sensitivities of the initial conditions of the system and of the period of oscillation.

Differentiating (2.4) and (2.5) with respect to parameters \( p \) and applying (4.14) and (5.12) yields the following system of equations, expressed in matrix form:

\[
\begin{bmatrix}
(M(p) - I) & \dot{x}(n_e + 1, p, T(p)) \\
\frac{\partial f}{\partial p}(m_1, x_0(p), p) & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial p}(p) \\
\frac{\partial T}{\partial p}(p)
\end{bmatrix}
= \begin{bmatrix}
-P(p) \\
-\frac{\partial f}{\partial p}(m_1, x_0(p), p)
\end{bmatrix}.
\]

For any limit cycle with \( M(p) \) defined as in (4.14) such that the result of Theorem 4.2 holds, \( [M(p) - I \ \dot{x}(n_e + 1, p, T(p))] \) has rank \( n_e \). Note that the bottom row of (5.13) should be replaced with the equation obtained by differentiating each side of the PLC with respect to \( p \). Since (2.4) and the PLC together specify both \( x_0(p) \) and \( T(p) \), it follows that the PLC is linearly independent of all rows of (2.4). Hence, the bottom row of (5.13), altered to account for an alternative PLC if necessary, is linearly independent of the other rows. This system of linear equations may therefore be solved uniquely for the parametric sensitivities \( \frac{\partial x}{\partial p}(p) \) and \( \frac{\partial T}{\partial p}(p) \) of the initial conditions and the period of oscillation, respectively.

### 5.3. Decomposition of parametric sensitivities

In this section, the matrix of parametric sensitivities \( \frac{\partial x}{\partial p}(i, p, t) \) is shown to be the sum of three parts characterizing the influence of parameters on the period of oscillation, amplitudes of particular state variables, and phase, respectively. A similar result has been derived for continuous-state limit-cycle oscillators [21], but this derivation does not extend to HLCOs directly.

First, it will be shown that for each \( i \in \mathcal{E} \), each \( N \in \mathbb{N}_0 \), and each \( t \in [\sigma_{i,N}(p), \tau_{i,N}(p)] \), the parametric sensitivities \( \frac{\partial x}{\partial p}(i, p, t) \) can be written in the form

\[
\frac{\partial x}{\partial p}(i, p, t) = \frac{t}{T(p)} \cdot R(i, p, t) + Z(i, p, t),
\]

where \( R(i, p, \cdot) \) is \( T(p) \)-periodic, and where \( Z(i, p, \cdot) \) is both \( T(p) \)-periodic and independent of the influence of parameters on the period of oscillation. Define matrices \( R(i, p, t) \) and \( Z(i, p, t) \) as follows:

\[
R(i, p, t) := -\dot{x}(i, p, t) \cdot \frac{\partial T}{\partial p}(p), \quad Z(i, p, t) := \frac{\partial x}{\partial p}(i, p, t) - \frac{t}{T(p)} \cdot R(i, p, t),
\]

so that (5.14) is trivially satisfied and so that \( R(i, p, \cdot) \) is periodic due to the periodicity of \( \dot{x}(i, p, \cdot) \). Setting \( t = 0 \) in the above definition of \( Z(i, p, t) \) yields

\[
Z(1, p, 0) = \frac{\partial x_0}{\partial p}(p).
\]

### 5.3.1. Periodicity of \( Z \)

Since the state variables \( x(i, p, \cdot) \) are \( T(p) \)-periodic, for each \( i \in \mathcal{E} \), each \( N \in \mathbb{N}_0 \), and each \( t \in [\sigma_{i,N}(p), \tau_{i,N}(p)] \),

\[
x(i, p, t) = x(i, p, t + T(p)).
\]

Since \( \dot{x}(i, p, \cdot) \) is \( T(p) \)-periodic, differentiating the above equation with respect to parameters \( p \) and applying (5.15) yields

\[
\frac{\partial x}{\partial p}(i, p, t) = \frac{\partial x}{\partial p}(i, p, t + T(p)) + \dot{x}(i, p, t) \cdot \frac{\partial T}{\partial p}(p) = \frac{\partial x}{\partial p}(i, p, t + T(p)) - R(i, p, t).
\]

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Using the above equation and the $T(p)$-periodicity of $R(i,p,\cdot)$ to simplify (5.15) at time $(t + T(p))$ yields

$$Z(i,p,t+T(p)) = \frac{\partial x}{\partial p}(i,p,t+T(p)) - \frac{t + T(p)}{T(p)} \cdot R(i,p,t)$$

$$= \frac{\partial x}{\partial p}(i,p,t) - \frac{t}{T(p)} \cdot R(i,p,t)$$

$$= Z(i,p,t).$$

$Z(i,p,t)$ is therefore $T(p)$-periodic wherever it is defined.

### 5.3.2. Isolation of influence of parameters on period

It is demonstrated in this section that $\frac{1}{\tau(p)} \cdot R(i,p,t)$ is the part of $\frac{\partial x}{\partial p}(i,p,t)$ containing the influence of parameters on the period of oscillation. For each time $t \in \mathbb{R}^+$, define a cyclic time $	au(t, p) := \frac{t}{T(p)}$, so that $	au(t, T(p))$ represents the number of periods that have elapsed at time $t$.

For each $i \in \mathcal{E}$, each $N \in \mathbb{N}_0$, and each time $t \in [\sigma_{i,N}(p), \tau_{i,N}(p)]$, define $\hat{x}(i,p,\tau)$ to be the system’s state at cyclic time $\tau$, so that for each cyclic time $t$,

$$\hat{x}(i,p,\tau(t,T(p))) \equiv x(i,p,t).$$

Let $i$, $N$, and $t$ be held constant throughout this section. Differentiating the above equation with respect to $p$,

$$\frac{\partial x}{\partial p}(i,p,t) = \left( \frac{\partial}{\partial t} [x(i,p,\tau(t,T(p))) \right)_{T(p)=\text{const}}$$

$$+ \frac{\partial x}{\partial \tau}(i,p,\tau(t,T(p))) \cdot \frac{\partial \tau}{\partial t}(t,T(p)) \cdot \frac{\partial T}{\partial p}(p).$$

Also, since $\tau(t, p) = \frac{t}{T(p)}$, taking partial derivatives of this definition at $T = T(p)$ yields

$$\frac{\partial \tau}{\partial t}(t,T(p)) = \frac{1}{T(p)}, \quad \frac{\partial \tau}{\partial T}(t,T(p)) = -\frac{t}{(T(p))^2}.$$

Differentiation of the definition of $\hat{x}(i,p,\tau)$ therefore yields the following:

$$\frac{\partial \hat{x}}{\partial \tau}(i,p,\tau(t,T(p))) = \frac{\partial}{\partial \tau} [x(i,p,t)] = \frac{\partial x}{\partial t}(i,p,t) \cdot \frac{\partial t}{\partial \tau}(T(p)) = T(p) \cdot \hat{x}(i,p,t).$$

Since $\hat{x}(i,p,\tau(t,T(p))) \equiv x(i,p,t)$, if total derivatives of each side of this equation are taken with respect to $p$ with $T(p)$ held constant, both sides must remain equal. Thus, substituting (5.17) and (5.18) into (5.16) and applying (5.15) yields

$$\frac{\partial x}{\partial p}(i,p,t) = \left( \frac{\partial x}{\partial p}(i,p,t) \right)_{T(p)=\text{const}} + \frac{t}{T(p)} \cdot R(i,p,t).$$

Comparing (5.19) with (5.14) yields

$$Z(i,p,t) = \left( \frac{\partial x}{\partial p}(i,p,t) \right)_{T(p)=\text{const}} \forall i \in \mathcal{E}, \forall N \in \mathbb{N}_0, \forall t \in [\sigma_{i,N}(p), \tau_{i,N}(p)],$$

so that by (5.14), $\frac{1}{T(p)} \cdot R(i,p,t)$ is the part of $\frac{\partial x}{\partial p}(i,p,t)$ containing the influence of parameters on the period of oscillation.

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5.3.3. **Decomposition of Z.** For each \( i \in \mathcal{E} \), each \( N \in \mathbb{N}_0 \), and each \( t \in [\sigma_i(N), \tau_i(N)] \), define a row vector \( \delta(i, p, t) \) and a matrix \( W(i, p, t) \) as follows:

\[
(5.20) \quad \delta(i, p, t) := \frac{\dot{x}(i, p, t)^T \cdot Z(i, p, t)}{\|x(i, p, t)\|^2}, \quad W(i, p, t) := Z(i, p, t) - \dot{x}(i, p, t) \cdot \delta(i, p, t).
\]

Since \( \dot{x}(i, p, t) \) and \( Z(i, p, t) \) have each been shown to be \( T(p) \)-periodic, it follows that \( W(i, p, t) \) and \( \delta(i, p, t) \) are each \( T(p) \)-periodic as well.

Hence, (5.15) implies that for each \( i \in \mathcal{E} \), \( N \in \mathbb{N}_0 \), and \( t \in [\sigma_i(N), \tau_i(N)] \), the parametric sensitivities \( \frac{\partial x}{\partial p}(i, p, t) \) have been decomposed as the following sum:

\[
(5.21) \quad \frac{\partial x}{\partial p}(i, p, t) = \frac{t}{T(p)} \cdot R(i, p, t) + W(i, p, t) + \dot{x}(i, p, t) \cdot \delta(i, p, t).
\]

In the next section, it will be shown that \( W(i, p, t) \) and \( \delta(i, p, t) \) characterize the respective influence of parameters on the amplitudes and phase of the state trajectory.

6. **Amplitude and phase sensitivities.** In this section, methods are developed for evaluating parametric sensitivities of amplitude, relative phase, and peak-to-peak phase for the periodic orbit of an HLCO. Unlike the sensitivities \( \frac{\partial x}{\partial p}(i, p, t) \) described in section 5, these are all time-independent properties of the oscillator as a whole, rather than properties of a single point on the periodic orbit.

6.1. **Amplitude sensitivities.** Let \( x_j \) denote the \( j \)th element of \( x \) for any \( j \in \{1, 2, \ldots, n_x\} \). The amplitude of the \( j \)th state variable is defined as follows:

\[
(6.1) \quad \Omega_j(p) := x_j(\mu_j(p), t_{j,\max}(p)) - x_j(\nu_j(p), t_{j,\min}(p)),
\]

where the supremum and infimum values of \( x_j(\cdot, p, \cdot) \) are attained at times \( t_{j,\max}(p) \in [\sigma_{\mu_j}(p), \tau_{\mu_j}(p)] \) and \( t_{j,\min}(p) \in [\sigma_{\nu_j}(p), \tau_{\nu_j}(p)] \), respectively. Such times necessarily exist [16] since \( \mathcal{E} \) is finite, and since for each \( i \in \mathcal{E} \), \( x(i, p, \cdot) \) is continuous on the compact set \( [\sigma_i(p), \tau_i(p)] \).

It is assumed in the current work that \( t_{j,\max} \) and \( t_{j,\min} \) are continuously differentiable on some neighborhood of \( p \) (or are chosen to be continuously differentiable if there are multiple candidate values of \( t_{j,\max} \) and \( t_{j,\min} \)). This assumption will not hold if, for example, the supremum value of \( x_j(\cdot, p, \cdot) \) is attained at two distinct values of \( t_{j,\max}(p) \in [0, T(p)] \) in such a way that small perturbations in \( p \) can make either supremum dominate the other. Amplitude sensitivities will generally not be defined if this assumption does not hold, since \( \Omega_j \) will be a locally nonsmooth function of \( p \).

It is also assumed that if \( t_{j,\max}(p) \) coincides with an event \( \sigma_i(p) \) for some \( i \in \mathcal{E} \), then there exists some neighborhood \( P^* \) of \( p \) such that \( t_{j,\max}(p^*) = \sigma_i(p^*) \) for each \( p^* \in P^* \). Again, \( \Omega_j \) will in general be a nondifferentiable function of \( p \) if this assumption is violated. An analogous assumption is made for \( t_{j,\min}(p) \).

For each \( i \in \mathcal{E} \) and each \( t \in [\sigma_i(p), \tau_i(p)] \), let \( \xi_j(i, p, t), z_j(i, p, t), \) and \( w_j(i, p, t) \) denote the \( j \)th rows of \( \frac{\partial x}{\partial p}(i, p, t) \), \( Z(i, p, t) \), and \( W(i, p, t) \), respectively. Differentiating (6.1) with respect to parameters \( p \) then yields the following equation:

\[
(6.2) \quad \frac{\partial \Omega_j}{\partial p}(p) = \xi_j(\mu_j(p), \sigma_j(p), t_{j,\max}(p)) + \dot{x}_j(\mu_j(p), t_{j,\max}(p)) \cdot \frac{\partial t_{j,\max}}{\partial p}(p)
\]

\[
- \left( \xi_j(\nu_j(p), t_{j,\min}(p)) + \dot{x}_j(\nu_j(p), t_{j,\min}(p)) \right) \cdot \frac{\partial t_{j,\min}}{\partial p}(p).
\]
The first and third terms on the right-hand side of the above equation can be evaluated using (5.6) and (5.10), and the second and fourth terms can be evaluated as follows.

If \( t_{j,\text{max}}(\mathbf{p}) \) coincides with an event, then \( t_{j,\text{max}}(\mathbf{p}) \) is equal to either \( \sigma_{\mu_j}(\mathbf{p}) \) or \( \tau_{\mu_j}(\mathbf{p}) \). Due to the second assumption above regarding \( t_{j,\text{max}}(\mathbf{p}) \), \( \frac{\partial t_{j,\text{max}}}{\partial \mathbf{p}}(\mathbf{p}) \) must then equal either \( \frac{\partial \sigma_{\mu_j}}{\partial \mathbf{p}}(\mathbf{p}) \) or \( \frac{\partial \tau_{\mu_j}}{\partial \mathbf{p}}(\mathbf{p}) \), which can both be evaluated using (5.3). An analogous situation occurs when \( t_{j,\text{min}}(\mathbf{p}) \) coincides with an event.

If \( t_{j,\text{max}}(\mathbf{p}) \) does not coincide with any event, then it follows from (2.2) and the continuity of \( \beta_{i,j}(\mathbf{p}) \) which is a local maximum of \( x_j(\mu_j,\mathbf{p},t) \) on the right-hand side of (6.2) therefore vanishes in this case. If it is true that \( \dot{x}_j(\mu_j,\mathbf{p},t_{j,\text{max}}(\mathbf{p})) = 0 \), then (5.15) and (5.21) together imply that

\[
\xi_j(\mu_j,\mathbf{p},t_{j,\text{max}}(\mathbf{p})) = \mathbf{z}_j(\mu_j,\mathbf{p},t_{j,\text{max}}(\mathbf{p})) = \mathbf{w}_j(\mu_j,\mathbf{p},t_{j,\text{max}}(\mathbf{p})).
\]

Analogous results hold for \( t_{j,\text{min}}(\mathbf{p}) \). It follows from the above equation and from (6.2) that out of the decomposition of \( \frac{\partial x}{\partial \mathbf{p}}(i,\mathbf{p},t) \) described by (5.21), \( \mathbf{W}(i,\mathbf{p},t) \) is the part containing information about the shape in \( \mathbb{R}^n \) of the periodic orbit of the system.

**6.2. Phase sensitivities.** When describing an oscillator, phase refers to any measure of the progress along a periodic orbit either between two points on a single trajectory or between two corresponding points on different trajectories. In this section, two types of phase are defined for HLCOs, and methods are developed for evaluating their respective parametric sensitivities in an analogous manner to the phase sensitivities of continuous-state oscillators [21].

**6.2.1. Relative phase sensitivities.** Relative phase is a measure of the difference in time between two specified points on a periodic orbit and can be quantified as follows for periodic orbits of hybrid limit-cycle oscillators.

Consider two periodic state trajectories of a given HLCO, initialized at (possibly) distinct points on the same periodic orbit due to different PLCs (\( \text{PLC}^{(1)} \) and \( \text{PLC}^{(2)} \), respectively) replacing (2.5). Let quantities relating to the first trajectory be labeled with the superscript (1), and let quantities relating to the second trajectory be labeled with the superscript (2). Consider any indices \( i,j \in \mathcal{E} \) and times \( \beta(\mathbf{p}) \in (\sigma^{(1)}_i(\mathbf{p}),\tau^{(1)}_i(\mathbf{p})) \) and \( \alpha(\mathbf{p}) \in (\sigma^{(2)}_j(\mathbf{p}),\tau^{(2)}_j(\mathbf{p})) \) such that \( m^{(1)}_i = m^{(2)}_j \) and

\[
(6.3) \quad x^{(1)}(i,\mathbf{p},\beta(\mathbf{p})) = x^{(2)}(j,\mathbf{p},\alpha(\mathbf{p})),
\]

so that the same state is visited in the same discrete mode at time \( \beta(\mathbf{p}) \) along the first trajectory as at time \( \alpha(\mathbf{p}) \) along the second trajectory. Then the relative phase between the initial conditions specified by the two PLCs is defined as the quantity \( (\beta(\mathbf{p}) - \alpha(\mathbf{p})) \). Note that \( \beta(\mathbf{p}) \) and \( \alpha(\mathbf{p}) \) can always be defined so as not to coincide with events, since adding any constant \( \Delta t \) to both \( \beta(\mathbf{p}) \) and \( \alpha(\mathbf{p}) \) does not alter \( (\beta(\mathbf{p}) - \alpha(\mathbf{p})) \).

Assume that \( \text{PLC}^{(1)} \) and \( \text{PLC}^{(2)} \) each continue to specify an isolated point on the periodic orbit of the system for any choice of parameters within some neighborhood of \( \mathbf{p} \). If this assumption does not hold, then the parametric sensitivity of relative phase will not be defined in general, since there may not exist any neighborhood \( P^* \) of \( \mathbf{p} \) such that \( \alpha(\mathbf{p}^*) \) and \( \beta(\mathbf{p}^*) \) are both defined for each \( \mathbf{p}^* \in P^* \). Differentiating (6.3)
with respect to parameters \( p \) yields

\[
\frac{\partial x^{(1)}}{\partial p}(i, p, \beta(p)) + x^{(1)}(i, p, \beta(p)) \cdot \frac{\partial \beta}{\partial p}(p) = \frac{\partial x^{(2)}}{\partial p}(j, p, \alpha(p)) + x^{(2)}(j, p, \alpha(p)) \cdot \frac{\partial \alpha}{\partial p}(p).
\]

(6.4)

Since \( x^{(1)}(i, p, \beta(p)) = x^{(2)}(j, p, \alpha(p)) \) and \( m_{i}^{(1)} = m_{j}^{(2)} \), it follows that

\[
f^{(m_{i}^{(1)}, x^{(1)}(i, p, \beta(p)), p)} = f^{(m_{j}^{(2)}, x^{(2)}(j, p, \alpha(p)), p)},
\]

which is equivalent to

\[
\dot{x}^{(1)}(i, p, \beta(p)) = \dot{x}^{(2)}(j, p, \alpha(p)).
\]

(6.5)

Moreover, it follows from (5.15) and (5.20) that \( W(i, p, t) \) is obtained from \( \frac{\partial W}{\partial p}(i, p, t) \) by projection of each column onto the plane in \( \mathbb{R}^{n_{x}} \) perpendicular to \( \dot{x}(i, p, t) \). Thus, projecting each column of (6.4) onto the plane in \( \mathbb{R}^{n_{x}} \) perpendicular to \( \dot{x}^{(1)}(i, p, \beta(p)) \) and applying (6.5) yields

\[
W^{(1)}(i, p, \beta(p)) = W^{(2)}(j, p, \alpha(p)).
\]

(6.6)

Applying (5.21) and (5.15) to each trajectory yields

\[
\frac{\partial x^{(1)}}{\partial p}(i, p, t) = -\frac{t}{T(p)} \dot{x}^{(1)}(i, p, t) \cdot \frac{\partial T}{\partial p}(p) + W^{(1)}(i, p, t) + \dot{x}^{(1)}(i, p, t) \cdot \delta^{(1)}(i, p, t),
\]

\[
\frac{\partial x^{(2)}}{\partial p}(j, p, s) = -\frac{s}{T(p)} \dot{x}^{(2)}(j, p, s) \cdot \frac{\partial T}{\partial p}(p) + W^{(2)}(j, p, s) + \dot{x}^{(2)}(j, p, s) \cdot \delta^{(2)}(j, p, s).
\]

Substituting the above equations and (6.5) into (6.4) yields

\[
\dot{x}^{(1)}(i, p, \beta(p)) \cdot \left( \frac{\partial \beta}{\partial p}(p) - \frac{\beta(p)}{T(p)} \cdot \frac{\partial T}{\partial p}(p) + \delta^{(1)}(i, p, \beta(p)) \right) + W^{(1)}(i, p, \beta(p))
\]

\[
= \dot{x}^{(1)}(i, p, \beta(p)) \cdot \left( \frac{\partial \alpha}{\partial p}(p) - \frac{\alpha(p)}{T(p)} \cdot \frac{\partial T}{\partial p}(p) + \delta^{(2)}(j, p, \alpha(p)) \right) + W^{(2)}(j, p, \alpha(p)).
\]

Now, \( \dot{x}^{(1)}(i, p, \beta(p)) \) must be nonzero; otherwise \( x^{(1)}(i, p, \beta(p)) \) would be a fixed point of (2.2). If this were true, the system would be unable to complete its periodic orbit upon reaching this fixed point, which is absurd. Applying this observation and (6.6) to the above equation yields the following expression for the parametric sensitivity of relative phase:

\[
\frac{\partial(\beta - \alpha)}{\partial p}(p) = \frac{\beta(p) - \alpha(p)}{T(p)} \cdot \frac{\partial T}{\partial p}(p) - \delta^{(1)}(i, p, \beta(p)) + \delta^{(2)}(j, p, \alpha(p)).
\]

(6.7)

The above equation permits evaluation of the relative phase sensitivity and also highlights two separate contributions to this sensitivity. The first term on the right-hand side describes the contribution of period sensitivity to relative phase sensitivity, while the remainder of the right-hand side is independent of period sensitivity (recollecting from section 5.3.2 and (5.20) that the \( \delta \)'s are each independent of period sensitivity). This remainder instead describes the contribution of the influence of parameters on the particular points on the orbit specified by PLC\(^{(1)}\) and PLC\(^{(2)}\). It follows that out of the decomposition of \( \frac{\partial W}{\partial p}(i, p, t) \) described by (5.21), \( \delta(i, p, t) \) is the part containing information about relative phases in the system.
If \( \alpha(p) \) and \( \beta(p) \) are equal, then PLC\(^{(1)}\) and PLC\(^{(2)}\) both specify the same initial condition on the periodic orbit of the limit cycle, and the first term vanishes on the right-hand side of (6.7). In this case, the relative phase sensitivity can only arise from any differences between PLC\(^{(1)}\) and PLC\(^{(2)}\). If these PLCs are identical, then the relative phase sensitivity will be 0; if not, then the relative phase sensitivity will be nonzero in general, and some small deviation of the parameters \( p^* \) from \( p \) will break the equality of \( \alpha(p^*) \) and \( \beta(p^*) \).

If \( \alpha(p) \) is set to zero, then (6.7) describes the sensitivity of the time at which PLC\(^{(2)}\) is first satisfied, as measured in a time scale for which PLC\(^{(1)}\) is satisfied at time \( t = 0 \).

### 6.2.2. Peak-to-peak phase sensitivities.

As in [21], peak-to-peak phase sensitivity is defined as the duration between the occurrence of two extrema in (not necessarily distinct) state variables \( x_m \) and \( x_n \). Assume that these extrema are each isolated and that they continue to exist for any choice of parameters within some neighborhood \( P^* \) of \( p \). It is of course possible for neither, one, or both of these extrema to occur at events, and each of these cases will be considered separately in the remainder of this section.

If neither extremum occurs at an event, then the peak-to-peak phase sensitivity is a particular type of relative phase sensitivity and can be evaluated using the results of the previous section. A simpler alternative is as follows, which avoids evaluation of \( \delta \)'s by exploiting the fact that both PLCs correspond to extrema of certain state variables. PLC\(^{(1)}\) may be chosen in the form of (2.5) such that the system is initialized at the extremum of \( x_m \) at time \( t = 0 \). Then for some \( i \in E \), the peak-to-peak phase is \( \beta(p) \in (\sigma_i(p), \tau_i(p)) \) such that

\[
\dot{f}_n(m_i^{(1)}(i, p, \beta(p)), p) = \dot{x}_n^{(1)}(i, p, \beta(p)) = 0.
\]

Differentiating this equation with respect to \( p \) yields

\[
0 = \frac{\partial f_n}{\partial x} \left( m_i^{(1)}(i, p, \beta(p)), p \right) \cdot \left( \dot{x}_i^{(1)}(i, p, \beta(p)) \right) + \frac{\partial \beta}{\partial p}(p) + \frac{\partial x_i^{(1)}}{\partial p}(i, p, \beta(p))
\]

which can be solved for the peak-to-peak phase sensitivity \( \frac{\partial \beta}{\partial p}(p) \) provided that \( \frac{\partial f_n}{\partial x}(m_i^{(1)}(i, p, \beta(p)), p) \cdot \dot{x}_i^{(1)}(i, p, \beta(p)) \) is nonzero. If this condition does not hold, then either the PLC is invalid or calculation of \( \frac{\partial \beta}{\partial p}(p) \) may require higher-order derivatives of the above equation with respect to \( p \). This eventuality is not discussed further in the current work.

If exactly one of the two extrema occurs at an event, the PLC (2.5) can be chosen to initialize the system at time \( t = 0 \) to the extremum occurring away from any event. The other extremum then occurs at time \( \sigma_k(p) \) for some \( k \in E \). The peak-to-peak phase sensitivity is then \( \frac{\partial \sigma_k}{\partial p}(p) \), which may be evaluated using (5.3) and (5.10).

If both of the extrema occur at events, then neither may be chosen as the initial state, as discussed in section 2.4. Instead, if some arbitrary PLC is chosen, then there exist \( k, \ell \in E \) such that the peak-to-peak phase is \( (\sigma_k(p) - \sigma_{k}(p)) \). The peak-to-peak phase sensitivity is then \( \frac{\partial (\sigma_k - \sigma_{k})}{\partial p}(p) \), which may be evaluated using (5.3) and (5.10).
7. Examples. In this section, the theory of the previous sections is applied to conduct sensitivity analyses for several example systems. Throughout this section, liberties are taken with notation where there will be no chance of confusion for the sake of readability. Components of \( \mathbf{x} \) are referred to by the symbols of the individual state variables of the system which they represent. The element of \( \mathbf{x}_0(\mathbf{p}) \) representing the initial condition for a particular state variable \( y \) will be referred to as \( y_0 \). Moreover, \( \mathbf{x}(t) \) refers to \( \mathbf{x}(i, \mathbf{p}, t) \) evaluated at each \( i \) such that \( t \in [\tau_i, N(\mathbf{p}), \tau_i, N(\mathbf{p})] \) exists for some \( N \in \mathbb{N}_0 \), so that \( \mathbf{x}(t) \) may take two values at event times. \( Z_{y,q}(t) \), for example, refers to the element of \( \mathbf{Z}(i, \mathbf{p}, t) \) corresponding to state variable \( y \) and parameter \( q \) and may similarly take two values at event times.

7.1. Numerical methods. In each example, the following procedure was used to conduct a sensitivity analysis of the corresponding system. All numerical integration of states and sensitivities over \( t \) was performed using DSL48SE \([8, 18, 19]\). DSL48SE performs robust event detection and location for hybrid systems and was modified both to permit direct input of \( \mathbf{f}, \mathbf{L} \), and \( \Theta \) and to compute jumps in sensitivities according to (5.4) when state variables jump. DAEPACK \([20]\) was employed to compute all necessary partial derivatives of \( \mathbf{f}, \mathbf{L} \), and \( \Theta \) using automatic differentiation and to provide further necessary information to DSL48SE.

First, the BVP defined by (2.4) and the PLC (2.5) was solved for consistent initial conditions \( \mathbf{x}_0(\mathbf{p}) \) and \( T(\mathbf{p}) \) by Newton’s method as follows. The dynamic evolution of the system was first simulated, when initialized within its limit cycle’s range of attraction. Once the simulation had qualitatively settled into periodic behavior, a solution of the system was first simulated, when initialized within its limit cycle’s range. The original system defining \( \mathbf{x}(i, \mathbf{p}, \tau; \mathbf{x}_0, T) \), in which \( \mathbf{x}_0 \) and \( T \) are treated as parameters. Here (2.2) and (2.4) were replaced by the following equations:

\[
\frac{d\mathbf{x}}{dT}(i, \mathbf{p}, \tau; \mathbf{x}_0, T) = T \cdot \mathbf{f}(m_i, \dot{\mathbf{x}}(i, \mathbf{p}, \tau; \mathbf{x}_0, T), \mathbf{p}),
\]

\[
\mathbf{x}_0 = \mathbf{x}(n_e + 1, \mathbf{p}, 1; \mathbf{x}_0, T).
\]

The Jacobian \( \mathbf{J}^{(k)} \) for the \( k \)th iteration step was then as follows, with \( \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} \) and \( \frac{\partial \mathbf{x}}{\partial \mathbf{p}} \) evaluated using DSL48SE. Dealing with \( \dot{\mathbf{x}} \) instead of \( \mathbf{x} \) ensured that sensitivities with respect to \( T \) could be evaluated in the same manner as any parametric sensitivity:

\[
\mathbf{J}^{(k)} := \begin{bmatrix}
\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e + 1, \mathbf{p}, 1; \mathbf{x}_0^{(k)}, T^{(k)}) - \mathbf{I} & \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e + 1, \mathbf{p}, 1; \mathbf{x}_0^{(k)}, T^{(k)}) \\
T^{(k)} \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}}(m_1, \mathbf{x}_0^{(k)}, \mathbf{p}) & f_j(m_1, \mathbf{x}_0^{(k)}, \mathbf{p})
\end{bmatrix}.
\]

The \( (k+1) \)th estimates of \( \mathbf{x}_0(\mathbf{p}) \) and \( T(\mathbf{p}) \) were then defined so as to solve the following system of equations:

\[
\mathbf{J}^{(k)} \cdot \begin{bmatrix}
\mathbf{x}_0^{(k+1)} \\
T^{(k+1)}
\end{bmatrix} = \begin{bmatrix}
\mathbf{x}_0^{(k)} \\
T^{(k)}
\end{bmatrix} + \begin{bmatrix}
\mathbf{x}(n_e + 1, \mathbf{p}, 1; \mathbf{x}_0^{(k)}, T^{(k)}) - \mathbf{x}_0^{(k)} \\
T^{(k)} \cdot f_j(m_1, \mathbf{x}_0^{(k)}, \mathbf{p})
\end{bmatrix}.
\]

If a PLC other than (2.5) was used, the bottom row of this system of equations was altered accordingly. This system of equations was solved for \( \mathbf{x}_0^{(k+1)} \) and \( T^{(k+1)} \) using the \texttt{dgeqv} subroutine of LAPACK \([1]\). The iterative method was determined to have converged to a solution once (7.1) and the PLC were satisfied to within an absolute
tolerance of $10^{-8}$ and a relative tolerance of $10^{-6}$. Convergence was reached within ten iterations for each example covered in the current work.

The monodromy matrix $M(p)$ and the matrix $P(p)$ were calculated by integrating $\frac{\partial x}{\partial x_0}$ and $(\partial x_0)_{x_0=\text{const}}$ together over one period using DSL48SE initializing these quantities to $x_0(p)$, $I$, and $0$, respectively.

$M(p)$ and $P(p)$ were both used to compute $\frac{\partial x_0}{\partial p}(p)$ and $\frac{\partial T}{\partial p}(p)$ by solving (5.13) in MATLAB. As described in section 5.2, the bottom row of (5.13) was altered whenever a PLC other than (2.5) was used. $x(i, p, t), \dot{x}(i, p, t), \frac{\partial x}{\partial x_0}(i, p, t)$, and $\frac{\partial x}{\partial p}(i, p, t)$ were then computed at all desired times with DSL48SE, using the definition of $f$ to determine $\dot{x}$, and initializing $x$ and $\frac{\partial x}{\partial x_0}$ to $x_0(p)$ and $\frac{\partial x_0}{\partial p}(p)$, respectively.

$R(i, p, t), Z(i, p, t), W(i, p, t), \text{and } d(i, p, t)$ were evaluated at all desired times using MATLAB, according to (5.15) and (5.20). All desired amplitude and phase sensitivities were then computed using MATLAB by applying the results of section 6 to the relevant quantities computed in previous steps.

The theory of sensitivity analysis developed in the current work is exact so that the accuracies of all quantities evaluated through the above procedure are limited only by the accuracy of the underlying model, the accuracy to which (2.4) and (2.5) are solved, and the working precision to which matrix operations and integration steps are performed.

7.2. Pressure relief valve. A simple one-dimensional hybrid model describes an isothermal tank into which an ideal gas is fed at a constant molar flow rate, and which has a single outlet stream fitted with a pressure relief valve. The model has two modes, and follows the mode trajectory: $T_M := \{1, 2, 1\}$. State variables and parameters are defined, respectively, to be

$$x := [P_{\text{tank}}], \quad p := [R \ T_f \ V \ k \ P_a \ P_s \ P_r \ F_{in}]^T$$

$$= [8.3145 \times 10^{-5} \ 300 \ 1.0221 \ 20 \ 1.01325 \ 10 \ 9 \ 40]^T,$$

where $P_{\text{tank}}$ denotes the gas pressure within the tank, $R$ denotes the gas constant, $T_f$ denotes the feed temperature, $V$ denotes the volume of the tank, $k$ denotes the valve constant, $P_a$ denotes atmospheric pressure, and $F_{in}$ denotes the molar rate of inflow. The outlet valve opens whenever the tank pressure $P_{\text{tank}}$ rises to $P_s$, and closes again when $P_{\text{tank}}$ drops to $P_r$.

Dynamic evolution of the system in each epoch is described as follows, with $f$, $L$, and $\Theta$ each defined as in section 2:

$$f(m_i, x, p) := \begin{cases} \dot{P}_{\text{tank}} := \left[ \frac{R T_f}{P_{\text{tank}}} \cdot F_{in} \right] & \text{for } m_i = 1, \\ \left[ \frac{R T_f}{P_{\text{tank}}} \cdot (F_{in} - k \sqrt{P_{\text{tank}} - P_a}) \right] & \text{for } m_i = 2, \end{cases}$$

$$L(i, x, p) := \begin{cases} P_s - P_{\text{tank}} & \text{for } i = 1, \\ P_{\text{tank}} - P_r & \text{for } i = 2, \end{cases} \quad \Theta(i, x, p) := x \ \forall i \in \{1, 2\}.$$ 

If $P_{\text{tank}}$ is initialized so that $P_{\text{tank}}(0) \in (P_r, P_s)$, if each parameter is positive, and if $k \sqrt{P_r - P_a} > F_{in}$ (as is the case for the choice of parameters considered), then it is impossible to have $P_{\text{tank}} = 0$ at any time. As a result, the PLC (2.5) is not appropriate for this model and was replaced with the alternative condition $P_{\text{tank},0} = 9.5$. Equation (2.4) and this PLC were solved simultaneously to yield $T(p) = 3.2757$. As described in section 5.2, (5.13) was modified accordingly. The dynamic evolution of $P_{\text{tank}}$ is illustrated in Figure 7.1(a).
The monodromy matrix in this example was computed to be $M(p) = [1]$. The system is a trivial limit cycle, since each choice of initial conditions $P_{tank,0}$ within $(P_r, P_s)$ lies on the periodic orbit. It follows from the results of section 4.3 that $\frac{\partial P_{tank}}{\partial P_{tank,0}}(i, p, t)$ may be written as the sum of a time-decaying part and a periodic part. Moreover, it follows from (4.26), (4.9), and (4.10) that this time-decaying part is identically zero at all times, so that $\frac{\partial P_{tank}}{\partial P_{tank,0}}(i, p, t)$ is in fact periodic (as illustrated in Figure 7.1(b)). A similar situation arises for any one-dimensional hybrid oscillator.

A sensitivity analysis was performed for this model as described in section 7.1, so that parametric sensitivities were computed for the period of oscillation and the amplitude $\Omega$ of $P_{tank}$. The amplitude sensitivity may be calculated analytically in this case, since $\Omega = P_s - P_r$, which is readily differentiated by $p$. The resulting expression for $\frac{\partial \Omega}{\partial p}$ agrees with (6.2). The results of this sensitivity analysis are presented in Table 7.1, and the recovery of periodic components $Z_{P_{tank,k}}(i, p, t)$, $W_{P_{tank,k}}(i, p, t)$, and $\delta_k(i, p, t)$ from $\frac{\partial P_{tank}}{\partial k}(i, p, t)$ according to (5.15) and (5.20) is illustrated in Figure 7.2.
7.3. Compass-biped robot model. A model developed by Goswami, Thuilot, and Espiau [10] describes the passive gait of a hypothetical compass-like biped robot down a slope of fixed gradient. During each step in the robot’s gait, one leg (the support leg) touches the ground, while the other leg (the nonsupport leg) swings through the air. Whenever the nonsupport leg touches the ground ahead of the support leg, the nonsupport leg becomes the new support leg and vice-versa. Immediately after such events, angular velocities of the legs are recalculated so as to conserve the robot’s total angular momentum.

The model is represented here as a hybrid oscillator with two modes. State variables and parameters are defined, respectively, to be

\[ x := [\theta_{ns}, \theta_s, \omega_{ns}, \omega_s]^T, \quad p := [a, \beta, \mu, g, \phi]^T = [0.3, 1, 2, 9.8, \frac{g}{120}]^T, \]

where \( \theta_{ns} \) and \( \theta_s \) denote the respective angles of the nonsupport and support legs relative to the vertical, \( \omega_{ns} \) and \( \omega_s \) denote the legs’ respective angular velocities, \( a, \beta, \) and \( \mu \) characterize the robot’s size and mass distribution, \( g \) denotes the gravitational field strength, and \( \phi \) denotes the angle of the incline’s slope relative to the horizontal.

Dynamic evolution of the system is as described by Goswami, Thuilot, and Espiau [10], with the state vector \( x \) jumping whenever the nonsupport leg touches the ground ahead of the support leg. An extra dummy mode was introduced with the same dynamics as the single mode in the original model [10] to ensure that immediately after any event, the transition condition determining the time of the next event is not already satisfied.

With \( j \) set to 1 in (2.5), consistent initial conditions satisfying (2.4) and (2.5) were found to be \( x_0(p) = [0.2717, -0.1207, 0, -0.9427]^T. \) The period of oscillation was found to be \( T(p) = 0.5517. \) The resulting phase portrait of the system’s periodic orbit is illustrated in Figure 7.1(c), and the eigenvalues of the corresponding monodromy
Parametric sensitivities of initial conditions and derived oscillation quantities for the compass-biped robot model. \( T \) denotes the period, \( \Omega_{\theta_n} \) and \( \Omega_p \) denote the respective amplitudes of variables \( \theta_n \) and \( \theta \), and \( \beta \) denotes the peak-to-peak phase between maxima of \( \theta \) and \( \theta_n \). “\( FD \)” denotes quantities approximated using finite differencing for comparison, with \( \epsilon = 0.001 p \). \( \frac{\partial \theta_n}{\partial p} \) is identically 0 due to the employed PLC: \( \theta_n(1, p, 0) = \omega_{n,0}(p) = 0 \).

| Parameter \( p \) | \( a \) | \( \beta \) | \( \mu \) | \( g \) | \( \phi \) |
|------------------|-----|-----|-----|-----|-----|
| \( \frac{\partial \theta_n}{\partial p} \) (p) | 0.0000 | 0.0128 | 0.0131 | 0.0000 | 4.0621 |
| \( \frac{\partial \theta_n}{\partial p} \) (FD) | 0.0000 | 0.0128 | 0.0131 | 0.0000 | 4.062 |
| \( \frac{\partial \theta}{\partial p} \) (p) | 0.0000 | 0.0130 | -0.0159 | 0.0000 | -1.1598 |
| \( \frac{\partial \theta}{\partial p} \) (FD) | 0.0000 | 0.0130 | -0.0159 | 0.0000 | -1.1598 |
| \( \frac{\partial \omega}{\partial p} \) (p) | 1.5711 | 0.4165 | -0.0016 | -0.0481 | -9.9220 |
| \( \frac{\partial \omega}{\partial p} \) (FD) | 0.9195 | 0.3135 | 0.0266 | -0.0281 | 0.7089 |
| \( \frac{\partial \theta_n}{\partial p} \) (p) | 0.0000 | 0.0429 | 0.0284 | 0.0000 | 7.3897 |
| \( \frac{\partial \theta_n}{\partial p} \) (FD) | 0.0000 | 0.0430 | 0.0283 | 0.0000 | 7.391 |
| \( \frac{\partial \omega_n}{\partial p} \) (p) | 0.0000 | 0.0143 | 0.0303 | 0.0000 | 5.5263 |
| \( \frac{\partial \omega_n}{\partial p} \) (FD) | 0.0000 | 0.0143 | 0.0303 | 0.0000 | 5.5263 |
| \( \frac{\partial \beta}{\partial p} \) (p) | 0.1667 | 0.0507 | 0.0017 | -0.0051 | 0.9382 |

matrix \( M(p) \) were calculated to be 1.0000, 0.2359 ± 0.5449i, and 0.1822. The system is therefore a stable limit cycle for the choice of \( p \) considered, by the definition in section 2. It follows from the results of section 4.3 that \( \frac{\partial \theta_n}{\partial p}(i, p, t) \) tends toward the \( T(p) \)-periodic expression (4.21) in the long-time limit. This is illustrated for \( \frac{\partial \theta_n}{\partial p} \) and for \( \frac{\partial \theta}{\partial p} \) in Figure 7.3.

A sensitivity analysis was performed for this model according to the procedure in section 7.1, so that parametric sensitivities were computed for the initial conditions, the period \( T \), the respective amplitudes \( \Omega_{\theta_n} \) and \( \Omega_\theta \) of \( \theta_n \) and \( \theta \), and the peak-to-peak phase \( \beta \) between maxima of \( \theta \) and \( \theta_n \). The results of this analysis are presented in Table 7.2, and the recovery of periodic components \( Z_{\theta_n,\phi}(i, p, t) \), \( W_{\theta_n,\phi}(i, p, t) \), and \( \delta_\phi(i, p, t) \) from \( \frac{\partial \theta_n}{\partial p}(i, p, t) \) according to (5.15) and (5.20) is illustrated in Figure 7.4. Parametric sensitivities of \( \theta_n,0 \), \( T \), and \( \Omega_{\theta_n} \) were also approximated using
Fig. 7.4. Sensitivity trajectories of nonsupport leg angle $\theta_{ns}$ with respect to incline slope $\phi$ for the compass-biped robot model: (a) full sensitivity $\frac{\partial \theta_{ns}}{\partial \phi}(i, p, t)$; (b) periodic part $Z_{\theta_{ns}, \phi}(i, p, t)$; (c) periodic part $W_{\theta_{ns}, \phi}(i, p, t)$; and (d) periodic part $\delta, \phi(i, p, t)$ for the PLC: $\dot{\theta}_{ns}(1, p, 0) = 0$.

finite differencing for comparison, with elements of $p$ perturbed by $\pm 0.1\%$. These finite difference estimates agree with the computed sensitivities to within $1\%$. $\dot{\theta}_s$ is always negative for the choice of parameters considered, as illustrated in Figure 7.1(c), so that $\theta_s$ attains both its supremum and infimum values at events. As a result, no terms of (6.2) can be dropped when computing $\frac{\partial \Omega_{\theta_s}}{\partial p}(p)$.

7.4. Hopping robot model. A model developed by Koditschek and Bühler [11] describes the vertical hopping of a robot developed by Raibert [14]. The robot comprises a body on top of a single leg. These together form a pneumatic cylinder, with the body acting as the piston. To control the body’s vertical motion, this cylinder can be either sealed or connected to external pressure to provide thrust.

The model is represented here as a hybrid oscillator with five modes and follows the mode trajectory: $T_M := \{1, 2, 3, 4, 5, 1\}$. Modes 1 and 2 refer to the robot’s flight phase, mode 3 to the compression phase, mode 4 to the thrust phase, and mode 5 to the decompression phase. State variables and parameters are defined as follows, using the same parameter values as Koditschek and Bühler [11]:

$$\mathbf{x} := \begin{bmatrix} x & v & \xi & \dot{t} \end{bmatrix}^T, \quad \mathbf{p} := \begin{bmatrix} \lambda & g & \tau & d & \eta & \gamma \end{bmatrix}^T = [0.5, 10, 41.86, 0.01, 5.81, 2.33]^T,$$

where $x$ denotes the vertical displacement of the robot’s body, $v$ denotes the body’s vertical velocity, $\lambda$ denotes the leg extension at which the robot leaves the ground, $g$ denotes the gravitational field strength, $\tau$ denotes the thrust force, $d$ denotes the thrust duration, $\eta$ denotes the initial spring constant of the robot’s leg, and $\gamma$ denotes the friction constant for the leg. $\xi$ and $\dot{t}$ are auxiliary state variables which are described below.
Parametric sensitivities of initial conditions and derived oscillation quantities for the hopping robot model. $T$ denotes the period, $\Omega_x$ and $\Omega_v$ denote the respective amplitudes of variables $x$ and $v$, and $\beta$ denotes the peak-to-peak phase between maxima and minima of $x$. "FD" denotes quantities approximated using finite differencing for comparison, with $\epsilon = 0.001p$. $\frac{\partial}{\partial p}(p)$ is identically $0$ due to the employed PLC: $\dot{x}(1,p,0) = v_0(p) = 0$.

| Parameter $p$ | $\lambda$ | $T$ | $\gamma$ | $d$ | $\eta$ | $\tau$ |
|---------------|-----------|-----|-----------|-----|-------|-------|
| $\frac{\partial x_1}{\partial p}(p)$ | 1.3182 | $-0.0612$ | 0.0119 | 0.7283 | 0.0250 | $-0.0259$ |
| $\frac{\partial x_1}{\partial p}(p)$ (FD) | 1.318 | $-0.0612$ | 0.0119 | 0.730 | 0.0250 | $-0.0259$ |
| $\frac{\partial x_2}{\partial p}(p)$ | 0.9749 | $-0.0617$ | 0.0072 | 0.5459 | $-0.0147$ | $-0.0296$ |
| $\frac{\partial x_2}{\partial p}(p)$ (FD) | 0.975 | $-0.0617$ | 0.0072 | 0.455 | $-0.0147$ | $-0.0296$ |
| $\frac{\partial x_3}{\partial p}(p)$ | 1.1544 | $-0.0588$ | 0.0162 | 0.9935 | $-0.0039$ | $-0.0579$ |
| $\frac{\partial x_3}{\partial p}(p)$ (FD) | 1.154 | $-0.0587$ | 0.0163 | 0.980 | $-0.0039$ | $-0.0579$ |
| $\frac{\partial x_4}{\partial p}(p)$ | 2.4466 | $-0.2103$ | 0.0918 | 5.6001 | 0.1923 | $-0.1993$ |
| $\frac{\partial x_5}{\partial p}(p)$ | 0.5477 | $-0.0302$ | 0.0035 | 0.2138 | $-0.0102$ | $-0.00968$ |

Dynamic evolution of the system is as described by Koditschek and Bühler [11]:

$$f(m_i, x, p) = \begin{bmatrix} \dot{x} & \dot{v} & \dot{\xi} & \dot{\lambda} \end{bmatrix}^T := \begin{cases} [v \ -g \ 0 \ 1]^T & \forall m_i \in \{1, 2\}, \\ [v \ (\frac{\tau}{2} - \gamma v - g) \ 0 \ 1]^T & \text{for } m_i = 3, \\ [v \ (\tau - \gamma v - g) \ 0 \ 1]^T & \text{for } m_i = 4, \\ [v \ (\frac{\tau}{2} - \gamma v - g) \ 0 \ 1]^T & \text{for } m_i = 5, \end{cases}$$

$$L(i, x, p) := \begin{cases} v + 1 & \text{for } i = 1, \\ x - \lambda & \text{for } i = 2, \\ -v & \text{for } i = 3, \\ d - \dot{t} & \text{for } i = 4, \\ \lambda - x & \text{for } i = 5, \end{cases}$$

$$\Theta(i, x, p) := \begin{cases} [x \ v \ 0 \ 0]^T & \forall i \in \{1, 2, 3, 5\}, \\ [x \ v \ x \ 0]^T & \text{for } i = 4. \end{cases}$$

The dummy mode 2 was introduced so that the transition condition triggering the compression phase would not be satisfied immediately upon entering the flight phase. The robot's dynamics during the decompression phase depend on the final value of $x$ in the thrust phase, so the state variable $\xi$ was introduced to store this value throughout the decompression phase. The decompression phase is triggered once a duration $d$ has elapsed in the thrust phase, so the state variable $\dot{t}$ was introduced to measure the time spent in each epoch. Note that $\xi$ and $\dot{t}$ jump at certain events, even though $x$ and $v$ remain continuous.

With $j$ set to 1 in (2.5), consistent initial conditions satisfying (2.4) and (2.5) were found to be $x_0(p) = [0.83825 \ 0 \ 0 \ 0.26009]^T$. The period of oscillation was found to be $T(p) = 0.87796$. The resulting phase portrait of the system's periodic orbit is illustrated in Figure 7.1(d), and the eigenvalues of the corresponding monodromy matrix $M(p)$ were calculated to be $1.0000, -0.3150, 0.0000,$ and $0.0000$. The system is therefore a stable limit cycle for the choice of $p$ considered, by the definition in section 2.

According to the procedure in section 7.1, parametric sensitivities were computed for the initial conditions, the period $T$, the respective amplitudes $\Omega_x$ and $\Omega_v$ of $x$ and $v$, and the peak-to-peak phase $\beta$ between maxima and minima of $x$. The results of this analysis are presented in Table 7.3. Parametric sensitivities of $x_0$, $T$, and $\Omega_x$ were
also approximated using finite differencing, with elements of $p$ perturbed by $\pm 0.1\%$. These finite difference estimates agree with the computed sensitivities to within $2\%$.

8. Conclusion. This work develops a method for the sensitivity analysis of hybrid systems confined to the periodic orbits of limit cycles and shows that several known properties of continuous-state limit-cycle oscillators extend to HLCOs.

A BVP is solved to yield a consistent set of initial conditions and the oscillation period of an HLCO. A second BVP is solved to yield the parametric sensitivities of each of these. Explicit expressions are provided for both parametric sensitivities and initial-condition sensitivities of the system state, showing that evaluation of these quantities over the first period is sufficient to calculate these quantities at any future time. Methods have been presented for the computation of sensitivities of derived quantities describing the oscillation of such a system, including period, amplitude, and relative phase. All expressions for these quantities in the current work are exact, so that all described sensitivities can be computed to accuracies limited only by the numerical errors implicit in solving the BVPs and in evaluating integrals and event times.

It has been shown that a monodromy matrix satisfying both (4.14) and (4.16) exists for all hybrid oscillators. For HLCOs in particular, the initial-condition sensitivities of the system states can be written as the sum of a time-periodic part and a time-decaying part, with explicit expressions provided for each. It has been shown that parametric sensitivities can be decomposed into three parts, containing the respective influences of the period sensitivity, the relative phase sensitivity, and the influence of parameters on the periodic orbit’s shape in state space.

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