Research on Improving Kinematics Equation of Traditional Orbit Parameters Based on Orbital Dynamics

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Abstract. Complex space missions have higher and higher requirements for orbit representation, orbit prediction and orbit control. Traditional orbit parameters have many shortcomings in orbit representation and orbit extrapolation. From the point of view of improving the shortcomings of traditional orbital parameters, this paper analyzes several groups of orbital parameters including unit quaternions. Based on a group of quaternion orbital elements composed of unit quaternion and vector parameters such as radial distance, radial velocity and angular momentum, the orbital dynamics and kinematics equations are derived, and the transformation relationship between quaternion orbital elements and traditional orbital parameters is given. The simulation example proves the effectiveness of improving the traditional kinematic equation method of orbit parameters.

Keywords: Orbit of spacecraft; Quaternion; Orbit optimization design.

1. Introduction

More and more complex space missions put forward higher requirements for orbit determination and control. For example, real-time missions such as autonomous navigation and autonomous rendezvous require high accuracy and speed of orbit prediction, which requires researchers to study methods that can improve the accuracy of orbit determination and control from various angles [1]. In addition to the complexity brought by the refinement of the corresponding mechanical model, the pure analysis method for solving the equation of motion can no longer meet the requirements, and the numerical method becomes the main tool for solving the problem.

Aerospace technology involves a huge systematic project, in which orbit design, orbit control and orbit determination are important components, and its core is orbit problem [2-3]. Spacecraft is actually a kind of artificial small celestial body, which includes various artificial earth satellites, lunar probes (including artificial lunar satellites) and planetary probes (including planetary orbiters), etc. This paper mainly focuses on orbit modeling based on quaternion orbit elements, which includes giving a set of orbit parameters based on quaternion to describe the orbit state; On this basis, the orbital dynamics equation and kinematics equation of spacecraft are established.
2. Perturbed equation of motion

Because artificial earth satellites fly in near-earth space, especially some low-orbit satellites, which are only a few hundred kilometers away from the ground, the damping effect of the atmosphere is obvious, which makes the elliptical orbit of satellites become smaller and rounder, which often plays a decisive role in the life of satellites [4]. Therefore, the atmosphere is also a major perturbation factor affecting satellite motion, especially for those low-orbit satellites with large surface quality.

At present, J-77, CIRA-72, CIRA-86 and other atmospheric models are widely used in satellite orbit determination. because the physical mechanism of atmospheric state and its change is not very clear, and due to mathematical difficulties, all models have two common problems: First, there is a 5% ~ 10% error in the inner character, and the outer character is even worse; Another problem is that the calculation of mass density is complicated. If the atmospheric drag perturbation must be considered in the precise orbit determination of artificial satellite, the corresponding ephemeris can only be calculated by numerical method [5-6].

Perturbed equation of motion considering the influence of atmospheric drag on the semi-long axis of satellite orbit in spherical static atmosphere model [7];

\[
\frac{da}{dt} = -\frac{A n a^2}{(1 - e^2)^{3/2}} \left(1 + 2e\cos f + e^2\right)^{3/2} \rho \]

(1)

In which \( A = \frac{C_D S}{m} \), \( S \) are the effective surface-mass ratio of satellite, \( C_D \) is the drag coefficient, \( m \) is the angular velocity of satellite motion, \( n = \sqrt{\frac{\mu}{a^3}} \), \( \mu = GM \) is the geocentric gravitational constant, \( f \) is the true near point angle, and \( \rho \) is the atmospheric density.

The above equation shows that the semi-long axis \( a \) of satellite orbit is decreasing under the perturbation of static atmospheric resistance. If the analytical solution is given to this equation, the form of the solution is very complex, which is not convenient to obtain the change of semi-long axis \( a \) affected by atmospheric resistance. The numerical method can obtain high precision results, which is convenient to analyze the change of semi-long axis \( a \). A numerical acceleration algorithm is given below.

3. Quaternion representation of track features

Traditional orbital description parameters, such as Kepler orbital elements and Cartesian coordinates, can uniquely determine a close orbit and the position of spacecraft in the orbit. Orbital parameters in Cartesian coordinates can conveniently represent the motion of particles in three-dimensional space, and Kepler orbital equation established on this basis is also very concise. The track parameters in Cartesian coordinates can not directly reflect the geometric characteristics of the track. However, it is often necessary to change the geometric parameters such as the semi-major axis and eccentricity of the track in track control. It is inconvenient to choose the track parameters in Cartesian coordinates. For example, when the orbit is a circular orbit in the equatorial plane, the right ascension of ascending intersection point and the angular distance of near center point are not defined, and the definition of true near angle should be changed to the angular distance from particle to vernal equinox [9].

Referring to the application of Euler parameters in attitude dynamics, the unit quaternion is used to replace Euler angle to represent the angular position of spacecraft relative to inertial coordinate system, and the ascending intersection point right ascension, orbit inclination angle and angle distance near the center point in Kepler orbit elements are replaced, thus obtaining a new orbit element:

\[
a, e, \theta, \lambda_0, \lambda_1, \lambda_2, \lambda_3 \]

(2)
In the above formula is the four elements of quaternion, which are not independent and satisfy the following constraints:
\[ \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \]  \hspace{1cm} (3)

From the relationship between Euler axis angle and Euler parameters, the specific expressions of the above four quaternion elements can be obtained as follows

\[ \lambda_0 = \cos \frac{\Phi}{2} \]
\[ \lambda_1 = e_x \sin \frac{\Phi}{2} \]
\[ \lambda_2 = e_y \sin \frac{\Phi}{2} \]
\[ \lambda_3 = e_z \sin \frac{\Phi}{2} \]  \hspace{1cm} (4)

In the above formula, \( \Phi \) is Euler axis, and \( e_x, e_y, e_z \) is the three components of unit vector \( e \) of Euler axis in reference system (here, geocentric inertial system). The above-mentioned quaternion-based orbital elements are only an attempt to introduce quaternion into orbital parameters, but it fully expresses the general method of applying quaternion in orbital parameters. Same as the essence of this method, domestic scholars put forward a new quaternion orbital element, which is:

\[ q_0, q_1, q_2, q_3 \]  \hspace{1cm} (5)

In the above formula, \( e_x, e_y \) is the two components of eccentricity vector \( e \) on the coordinate axis in the orbital plane, and \( q_i (i = 0,1,2,3) \) is a new set of quaternion parameters. Their relationship with the orbital elements in formula (2) is as follows:

\[ q_i = p^{\frac{1}{2}} \lambda_i = a(1-e^2)\lambda_i, (i = 0,1,2,3) \]
\[ e_x = e \cos \theta \]
\[ e_y = e \sin \theta \]  \hspace{1cm} (6)

It can be seen from formula (6) that the relationship between the two groups of orbital elements is very close. Orbital parameters in the form of position and velocity describe the orbital motion of spacecraft particles in geocentric inertial coordinate system, and orbital motion can also be described in orbital coordinate system. Quaternion is introduced to describe the attitude position of orbital coordinate system relative to inertial coordinate system. For convenience of description, firstly, inertial coordinate system \( OX_1X_2X_3 \) and orbital coordinate system \( Gg_1g_2g_3 \) are defined, as shown in Figure 1.

![Figure 1. Inertial coordinate system and orbital coordinate system](image)
$O$ is the center of the earth, the $OX_1$ axis points to the vernal equinox, and $OX_2$ is perpendicular to the equatorial plane in the same direction as the earth's autobiographical angular velocity.

4. Selection of variable for numerical solution of motion equation
The numerical solution itself will bring calculation errors, so we try our best to improve the accuracy of calculation in order to achieve the purpose of calculation accuracy. However, when solving the spacecraft motion equation in practice, there are more data and a large amount of calculation, so the calculation efficiency is extremely important and even directly affects the calculation accuracy. The research shows that the selection of basic variables will directly affect the whole calculation efficiency [10].

4.1. Relationship between track number and position vector and velocity vector
Generally, position vector $\mathbf{r}$ and velocity vector $\mathbf{v}$ are used in the calculation of orbital motion equation, and the corresponding problem boils down to solving the following ordinary differential initial value problem:

$$\begin{align*}
\mathbf{r} &= F = F_0(\mathbf{r}) + F_\varepsilon(\mathbf{r}, \mathbf{v}, t, \varepsilon) \\
\mathbf{v}(t_0) &= \mathbf{r}(t_0) = r_0
\end{align*}$$

(7)

In which $F_0$ is the gravitational acceleration of the earth center (that is, the acceleration of the two-body problem), $F_\varepsilon$ is various perturbed accelerations, and $\varepsilon$ is a small parameter, which is generally the order of $10^{-3}$ of the earth oblateness factor $J_2$. If the number of orbital elements $\sigma$ is used instead of $\mathbf{r}$ and $\mathbf{v}$, the corresponding problem (7) will be transformed into the following form:

$$\begin{align*}
\mathbf{r} &= f(\sigma, t, \varepsilon) \\
\sigma(t_0) &= \sigma_0
\end{align*}$$

(8)

In which the orbital number $\sigma$ is the commonly used Kepler ellipse number $a, e, i, \Omega, \omega, M$.

4.2. Selection of variable for numerical solution of motion equation
The reason for adopting equation (7) is obvious. The form of the right function of the equation is simple, and the calculation efficiency often depends on whether the right function is simple or not. The disadvantage of this choice is that the right function contains an unperturbed part and changes rapidly. Under certain precision requirements, the integration step size is often limited to a small extent [11-12]. Using equation (8), if $\sigma$ is taken as six commonly used Kepler roots: $a, e, i, \Omega, \omega, M$, and adopts the form of general perturbation equation, in this case, if the right function $f$ is expressed as the form of orbital roots $f(\sigma, t, \varepsilon)$. Then, its concrete expression is obviously complicated, especially when there are many perturbation factors. This is the reason why people always think that it is too complicated to use the number of orbital elements as the basic variable when using numerical method, but this formal complexity can be eliminated by a simple way.

It can be clearly seen from the above processing that the calculation of right function is not very complicated, especially when there are many perturbation factors. However, under the same accuracy requirement, the integral step size of this equation can be made larger, which reduces the whole calculation amount. If the same step size is taken, the local truncation error will be obviously reduced and the calculation accuracy will be improved. Therefore, when the perturbed acceleration itself is complex, choosing the number of orbits as the basic variable is not only uncomplicated, but also can improve the calculation efficiency, which has practical value, especially in the precise orbit determination of artificial earth satellite, and can also simplify the calculation of state transition matrix [13].
5. Orbit optimization model and solution

5.1. Dynamical equation

For an ideal geostationary orbit, the inclination angle $i = 0^\circ$, eccentricity $e = 0$, and semi-major axis $a$ are the radius of the geostationary orbit, while the values of ascending intersection point right longitude $\Omega$, perigee angle $\omega$ and near point angle $M$ are only singular values in the mathematical sense. In order to avoid singular values in orbit calculation, this paper adopts the vernal equinox orbit elements defined in reference [3]:

$$x = [a, l, e_x, e_y, i_x, i_y]^T$$ (9)

In which: $l = \omega + M + \Omega - \Theta, \Theta = \Theta_0 + \omega t, \Theta_0$ is Greenwich mean time at time $t = 0$ and $\omega_e$ is the angular velocity of the earth's rotation; $e_x = e \sin(\omega + \Omega), e_y = e \cos(\omega + \Omega), i_x = i \sin \Omega, i_y = i \cos \Omega$.

Under the epoch true equatorial coordinate system, considering the earth's non-spherical gravity perturbation (fourth-order main term), the sun-moon gravity perturbation and the solar light pressure perturbation, the differential equation is established:

$$\frac{\delta x}{\delta t} = f_K(x) + f_L(x,t) + f_G(x,t)u$$ (10)

In which $u = [u_\alpha, u_\beta, u_\gamma]^T$ is thrust vector; $f_K(x)$ is the change equation of GEO orbital state under ideal conditions, and $f_K(x) = [0, n - \omega_e]^T$. $n - \omega_e$ is the difference between the rotation speed of satellite around the earth and the rotation speed of the earth. $f_L(x,t)$ is a perturbed acceleration equation. According to the definition of vernal equinox orbital elements, it can be obtained from Lagrange planetary equation:

$$f_G(x,t) = \left[ \begin{array}{c} \frac{2}{na} \frac{\partial R}{\partial l} - \frac{2}{na} \frac{\partial R}{\partial \alpha} \frac{1}{na^2} \frac{\partial R}{\partial \epsilon_x} - \frac{1}{na^2} \frac{\partial R}{\partial \epsilon_y} \frac{1}{na^2} \frac{\partial R}{\partial \epsilon_x} - \frac{1}{na^2} \frac{\partial R}{\partial \epsilon_y} \end{array} \right]$$ (11)

In the formula, $R$ is a perturbation function, and the derivation is shown in reference [1]. Because the orbit maintenance considers long-term operation, it only needs to control the orbit roots within a certain range, so the short-term term in formula (11) is ignored, and only the period term and long-term term of half a month or more are retained. The specific form is limited by space and is not listed here. $f_G(x,t)$ is the thrust equation, which can be obtained from Gaussian planetary equation:

$$f_G(x,t) = \left[ \begin{array}{c} \frac{2}{n} & 0 & 0 \\ 0 & -\frac{2}{na} & 0 \\ \frac{2 \sin l}{na \sin l} & \frac{2 \cos l}{na \cos l} & 0 \\ 0 & \frac{\sin l}{na} & 0 \\ 0 & \frac{\cos l}{na} & 0 \end{array} \right]$$ (12)

Therefore establishing a state variable equation

$$\frac{\delta x}{\delta t} = A(t)y + D(t) + B(t)u$$ (13)
In which \( y = x - x_0, A(t) = \left( \frac{df}{dx} \right)_{x=x_0}, D(t) = f_1(x_0, t), B(t) \) is the matrix form of thrust equation.

The true position angle of satellite relative to the earth in space is reflected by true longitude \( \lambda \) and true latitude \( \varphi \):

\[
\lambda = l + 2e_x \sin(l + \Theta) - 2e_y \cos(l + \Theta) \quad (14)
\]

\[
\varphi = i_x \sin(\lambda + \Theta) - i_y \cos(\lambda + \Theta) \quad (15)
\]

5.2. Optimization model

Optimize objective function

In the central gravitational field, the optimal control problem of the orbital rendezvous of two spacecraft can choose fuel consumption and transfer time as the index of optimizing performance. For the intersection problem of finite thrust control, the module of thrust acceleration can be selected as fuel consumption, while considering time and fuel consumption, the objective function expression is:

\[
J = \int_{t_0}^{t_f} (1 + \alpha p^2(t)) dt, \alpha = \text{const} \geq 0
\]

In the above formula, \( \alpha \) is the weighting coefficient, \( t_f \) and \( t_0 \) are the starting and ending time of intersection, and \( P \) is the modulus of thrust acceleration \( P \). When \( \alpha = 0 \), the objective function is transformed into the following form:

\[
J = t_f - t_0
\]

The above formula only contains the beginning and end time of rendezvous, which is the objective function of the orbital rendezvous problem with the shortest time; When \( \alpha > 0 \), equation (16) considers the intersection time and fuel consumption of orbital transfer at the same time, and transforms the multi-objective optimization problem into a single-objective optimization problem through the weighting coefficient \( \alpha \). At this time, the weight of fuel and time in the optimization objective function can be changed by changing the value of weighting coefficient \( \alpha \).

Of course, the orbit optimization can also take fuel consumption as the optimization target, and at this time, Equation (16) is rewritten as:

\[
J = \int_{t_0}^{t_f} p^2 dt
\]

Boundary conditions

For the orbital rendezvous problem, the boundary conditions in the optimization model mainly refer to the terminal conditions of rendezvous. "Soft" rendezvous requires equal position and velocity of two spacecraft at the end time, while "hard" rendezvous only requires equal position parameters. The boundary conditions are expressed by the orbital parameters in Cartesian coordinates in inertial frame.

\[
x_A = x_B
\]

\[
v_A = v_B
\]

According to equation (19), the boundary conditions expressed by quaternion orbital elements are:

\[
r_A = r_B, \mathcal{E}_A = \mathcal{E}_B, h_A = h_B
\]

\[
\lambda_A = \lambda_B
\]

Path constraint and control constraint

In order to reduce the complexity of the optimization problem and ensure the feasibility of the solution, it is necessary to give the effective value range of each state variable and control variable.
according to the specific conditions of the problem, that is, path constraint and control constraint. For the target spacecraft $A$, which runs in Kepler orbit, the range of each state variable can be determined by the geometric properties of the orbit. In addition, a constraint with quaternion modulus of 1 needs to be added. Path constraints of spacecraft $A$ and $B$ during rendezvous are as follows:

$$\|r_A\|^2 = \|r_B\|^2 = 1$$

$$r_A, r_B > R_e$$

(21)

For chasing spacecraft $B$, due to the limited thrust $P$, it runs in non-Kepler orbit, and the range of orbital parameters cannot be effectively determined. It can be given with reference to the range of $A$ orbital parameters of spacecraft. For example, the lower limit of the distance $r_B$ from the chasing spacecraft $B$ to the gravity center can be taken as the average radius of the earth, and the upper limit can be taken as twice as long as the distance between the distant places of the two spacecraft.

Control constraint means that the amplitude of continuous thrust exerted on chasing spacecraft $B$ is limited, and its maximum value is $p_{max}$. According to different optimization objective functions, the control constraint is different. For the orbit optimization problem with the shortest time as the optimization goal, the necessary condition for obtaining the optimal solution according to the maximum principle is that the thruster needs to output with the maximum thrust, that is:

$$p_1^2 + p_2^2 + p_3^2 = p_{max}^2$$

(22)

For the orbit optimization problem aiming at the minimum fuel consumption, the output of thruster is not a fixed value, and its value range is:

$$0 \leq p_1^2 + p_2^2 + p_3^2 \leq p_{max}^2$$

(23)

6. Example analysis
Design parameter $i = 40^\circ$: in October 2020, Beijing (longitude and latitude are 116.388e and 39.7396n) will be accurately observed for 7 days. According to the requirements of ground resolution, the orbit height should be within 280 ~ 700km, and the sun height angle on the ground should not be less than $30^\circ$ for the camera. The initial time obtained by simulation software is 7:00:00 on October 1, 2020, and the orbital parameters at the initial time are as follows:

$$a = 6801.714km, e = 0.025, i = 40^\circ$$

$$\Omega = 153.670^\circ, \omega = 55.241^\circ, i = 30^\circ$$

The altitude of satellite over Beijing in the next 7 days is shown in Figure 2. Seven points in the figure indicate the satellite over Beijing for 7 times respectively. The abscissa indicates the time of arrival in Beijing (in one year), and the ordinate indicates the time at this time. It can be seen that the altitude of the satellite when it arrived over Beijing for the first seven times did not exceed 500km, which met the requirement of ground resolution.
Figure 2. Height of satellite over Beijing in the next 7 days

Every time a satellite arrives in Beijing, it is earlier than the previous one, and the requirements can be met as long as the sun height angle on the ground in Beijing is not less than 30°. The simulation results are shown in Figure 3.

Figure 3. The time when the satellite arrived over Beijing the first seven times

It can be seen from the figure that in the next seven days from October 1, 2020, the decline degree of each time the satellite arrives in Beijing is higher than that of the critical time when the ground solar altitude angle is 30° above Beijing, and the seventh day is still within the limit of the critical time of that day. Therefore, the initial time of the orbit is selected as 7:00:00 on October 1, 2020, which meets the requirements.

7. Summary
Based on quaternion, the application of orbit representation and new orbit parameters in orbit extrapolation and optimization is studied. The angular position of the orbit plane relative to the inertial system is described by unit quaternion by using the similarity between the rotating motion part of the orbit and the attitude of the rigid body. Quaternion is combined with velocity vector, angular velocity vector and angular momentum vector to form a complete orbit description. Based on a group of
quaternion orbital elements, the corresponding orbital dynamics and kinematics equations are derived, the transformation relationship between quaternion orbital elements and traditional orbital parameters is given, and the orbital map using quaternion orbital elements is drawn. Finally, the design method of small elliptical orbit for emergency observation is summarized, and a simulation example is given to prove the effectiveness of the method.

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