INVARIANT SCHWARZIAN DERIVATIVES OF HIGHER ORDER

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ABSTRACT. We argue relations between the Aharonov invariants and Tamaøi’s Schwarzian derivatives of higher order and give a recursion formula for Tamaøi’s Schwarzians. Then we propose a definition of invariant Schwarzian derivatives of a nonconstant holomorphic map between Riemann surfaces with conformal metrics. We show a recursion formula also for our invariant Schwarzians.

1. INTRODUCTION

The Schwarzian derivative $S_f$ of a non-constant meromorphic function $f$ on a plane domain is defined by

$$S_f = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 = \left( \frac{f'''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = T_f' - \frac{1}{2} T_f^2,$$

where $T_f = f''/f'$ is the pre-Schwarzian derivative of $f$. Note that $S_f(z)$ is holomorphic at $z = z_0$ when $f(z)$ is locally univalent at $z = z_0$, whereas $S_f(z)$ has a pole of order 2 at $z = z_0$ when $f(z)$ has a branch at $z = z_0$.

It is well recognized that the pre-Schwarzian derivative is crucially used to construct a conformal mapping of the upper half-plane onto a polygonal domain. (This is the so-called Schwarz-Christoffel mapping.) The Schwarzian derivative was introduced by Schwarz to construct further a conformal mapping of the upper half-plane onto a simply connected domain bounded by finitely many circular arcs. After Nehari discovered univalence criteria of meromorphic functions in terms of the Schwarzian derivative in the late 1940’s, Bers and Ahlfors found an intimate connection with quasiconformal mappings and utilized it to embed Teichmüller spaces onto bounded domains in complex Banach spaces (see, for example, [9]). Thus one may be tempted to define higher-order analogues of the Schwarzian derivative. Indeed, Aharonov [1] and Tamaøi [14] gave definitions of higher-order analogues of the Schwarzian derivative. Aharonov gave a necessary and sufficient condition for a nonconstant meromorphic function on the unit disk to be univalent in terms of his Schwarzians, whereas Tamaøi studied combinatorial structures of his Schwarzians. In Section 2, we briefly recall their definitions and argue the relation between them. We should also note here that Schippers [12] proposed yet another definition of Schwarzian derivatives of higher order. They fit the Löwner theory and have nice properties. However, as he noted in his paper, his Schwarzians have a different nature from those of Aharonov and Tamaøi. Thus we do not treat with Schipper’s Schwarzians in this note.

One reason why the Schwarzian derivative is so useful is that it has a nice invariance property. On the other hand, Peschl and Minda introduced a sort of invariant derivatives

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We will use later a concrete form of the inverse map $N$. For $f$ be meromorphic in the derivatives of higher order. We generalize this idea to define invariant Schwarzian derivatives of higher order in terms of the Peschl-Minda derivatives analogously. One of our main results is a recursion formula of the invariant Schwarzian derivatives of higher order (see Theorem 4.4). These invariant Schwarzian derivatives and the recursion formula have applications to univalence criteria (see \[8\] for details).

2. SCHWARZIAN DERIVATIVES OF HIGHER ORDER

Let $f$ be a nonconstant meromorphic function on a domain $D$ in the complex plane. For $z \in D$ with $f(z) \neq \infty, f'(z) \neq 0$, we consider the quantity

$$G(\zeta, z) = \frac{f''(z)}{f'(\zeta) - f(z)}.$$ 

We now expand it in the power series

$$G(z + w, z) = \frac{1}{w} - \sum_{n=1}^{\infty} \psi_n[f](z) w^{n-1}$$

for small enough $w$. The quantities $\psi_n[f](z)$ were introduced by Aharonov [1] and called the Aharonov invariants by Harmelin [2]. Since the quantity

$$\frac{\partial G}{\partial \zeta}(\zeta, z) = -\frac{f'(z)f''(\zeta)}{(f(\zeta) - f(z))^2} = \frac{-1}{(\zeta - z)^2} - \sum_{n=1}^{\infty} (n-1) \psi_n[f](z)(\zeta - z)^{n-2}$$

is invariant under the Möbius transformations of $f$, we obtain $\psi_n[M \circ f] = \psi_n[f]$ for $n \geq 2$ and a Möbius transformation $M(z) = (az + b)/(cz + d)$. Thus these quantities can be defined even when $f(z) = \infty$ as long as $f$ is locally univalent at $z$. Note that

$$\psi_1[f](z) = \frac{f''(z)}{2f'(z)}$$

and

$$\psi_2[f](z) = \frac{1}{6} \left[ \frac{f''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right].$$

Thus $2! \psi_1[f]$ and $3! \psi_2[f]$ are the pre-Schwarzian derivative $T_f$ and the Schwarzian derivative $S_f$ of $f$, respectively. Thus, $\psi_n[f], n = 2, 3, \ldots$, can be regarded as Schwarzian derivatives of higher order.

Tamanoi [14] proposed another definition of Schwarzian derivatives of higher order. Let $f$ be meromorphic in $D$. Fix a point $z \in D$ with $f(z) \neq \infty, f'(z) \neq 0$, and take a Möbius transformation $M_z$ so that

$$M_z(0) = f(z), M'_z(0) = f'(z), M''_z(0) = f''(z).$$

We will use later a concrete form of the inverse map $N_z = M_z^{-1}$ of $M_z$:

$$N_z(t) = \frac{f'(z)(t - f(z))}{\frac{1}{2}f''(z)(t - f(z)) + f'(z)^2}.$$
Then we expand the function $V = (M_z^{-1} \circ f)(z + w) = N_z(f(z + w))$ as a power series
\begin{equation}
V = \frac{f'(z)(f(z + w) - f(z))}{2f''(z)(f(z + w) - f(z)) + f'(z)^2} = \sum_{n=0}^{\infty} S_n[f](z) \frac{w^{n+1}}{(n+1)!}
\end{equation}
around $w = 0$. The quantity $S_n[f]$ is called the Schwarzian derivative of virtual order $n$ for $f$ (see [14]). By the choice of $M_z$, we see that $S_0[f] = 1, S_1[f] = 0$ and $S_2[f]$ is the classical Schwarzian derivative $S_I = f''/f' - 3(f''/f')^2/2$. Also, by construction, $V$ and thus $S_n[f]$ are Möbius invariant. In particular, $S_n[f](z)$ can be defined even when $f(z) = \infty$ as long as $f$ is locally univalent at $z$.

Aharonov [1] (see also [2]) proved the recursion formula
\begin{equation}
(n + 1)\psi_n[f] = \psi_{n-1}[f'] + \sum_{k=2}^{n-2} \psi_k[f]\psi_{n-k}[f].
\end{equation}

We now show a similar formula for Tamanoi’s Schwarzians.

**Proposition 2.1.**
\[ S_n[f] = S_{n-1}[f'] + \frac{1}{2}S_2[f] \sum_{k=1}^{n-1} \binom{n}{k} S_{k-1}[f]S_{n-k-1}[f], \quad n \geq 3. \]

**Proof.** We denote by $\hat{N}_z(t)$ the partial derivative of $N_z(t)$ with respect to $z$, namely, $\hat{N}_z(t) = \partial_z N_z(t)$. The following formula is easily verified by a direct computation:
\begin{equation}
\hat{N}_z(t) = -1 - \frac{1}{2}S_2[f](z)N_z(t)^2.
\end{equation}

We now compute partial derivatives of $V = N_z(f(z + w))$
\[ \partial_{z}V = N_z'(f(z + w))f'(z + w), \quad \partial_{w}V = N_z'(f(z + w))f'(z + w) + \hat{N}_z(f(z + w)). \]

By [22], we have
\begin{equation}
\partial_{z}V - \partial_{w}V = \hat{N}_z(f(z + w)) = -1 - \frac{1}{2}S_2[f](z)V^2.
\end{equation}

Compare with the similar formula (2.8) in [1]. We now substitute (2.1) into the last formula to obtain
\begin{align*}
\sum_{n=0}^{\infty} S_n[f]'(z) \frac{w^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} S_n[f](z) \frac{w^n}{n!} &= -1 + \sum_{n=1}^{\infty} \{ S_{n-1}[f]'(z) - S_n[f](z) \} \frac{w^n}{n!} \\
&= -1 - \frac{1}{2}S_2[f](z) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} S_{k-1}[f](z)S_{l-1}[f](z) \frac{w^{k+l}}{k!l!} \\
&= -1 - \frac{1}{2}S_2[f](z) \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \binom{n}{k} S_{k-1}[f](z)S_{n-k-1}[f](z) \frac{w^n}{n!}.
\end{align*}

By comparing the coefficients of $w^n$, we obtain the required relation. \qed

In particular, we have the following result. Here and hereafter, $\mathbb{Z}$ denotes the ring of integers. Note that a similar result was obtained by Tamanoi (see [14, Theorem 6-4]).
Corollary 2.2. \( S_n[f] \) is expressed in the form \( S_{n-1}[f] + P(S_2[f], \ldots, S_{n-4}[f], S_{n-2}[f]) \) for \( n \geq 3 \), where \( P(x_2, x_3, \ldots, x_{n-5}, x_{n-4}, x_{n-2}) \) is a polynomial in \( x_2, x_3, \ldots, x_{n-5}, x_{n-4}, x_{n-2} \) with non-negative coefficients in \( \mathbb{Z} \).

Proof. When \( n \) is odd, we can write
\[
S_n[f] = S_{n-1}[f] + S_2[f] \sum_{k=1}^{n+1} \binom{n}{k} S_{k-1}[f] S_{n-k-1}[f].
\]
When \( n \) is even, we have
\[
S_n[f] = S_{n-1}[f] + S_2[f] \sum_{k=1}^{n-1} \binom{n}{k} S_{k-1}[f] S_{n-k-1}[f] + \frac{1}{2} \binom{n}{\frac{n}{2}} S_2[f] S_{\frac{n}{2}-1}[f]^2.
\]
Since
\[
2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2 \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} + \binom{n}{\frac{n}{2}},
\]
we see that \( \binom{n}{\frac{n}{2}} \) is an even number for \( n \geq 3 \). Since \( S_1[f] = 0 \), the assertion follows. \( \square \)

Let us write down first several nontrivial Schwarzians:
\[
\begin{align*}
S_3[f] &= S_2[f]' \\
S_4[f] &= S_3[f]' + 4S_2[f]^2 \\
S_5[f] &= S_4[f]' + 5S_2[f]S_3[f] \\
S_6[f] &= S_5[f]' + 6S_2[f]S_4[f] + 10S_2[f]^3.
\end{align*}
\]

It should be noted here that \( S_n[f] = 0 \) does not necessarily imply \( S_{n+1}[f] = 0 \). For example, consider the function \( f(z) = e^{az} \) for a constant \( a \neq 0 \). Then \( S_2[f] = -a^2/2 \). Therefore, \( S_3[f] = 0 \) but \( S_4[f] = a^4 \).

We conclude the present section with the relationship between the Aharonov invariants and Tamanoi’s Schwarzian derivatives. For convenience, we set
\[
\sigma_n[f] = \frac{S_n[f]}{(n+1)!} \quad \text{for } n = 0, 1, 2, \ldots.
\]
By the elementary formula
\[
(G + \psi_1[f])V = \left( 1 - \sum_{n=2}^{\infty} \psi_n[f](z)w^n \right) \sum_{n=0}^{\infty} \sigma_n[f](z)w^n = 1,
\]
we obtain the following.

Proposition 2.3.
\[
\sigma_n[f] = \psi_n[f] + \sum_{k=2}^{n-2} \psi_k[f] \sigma_{n-k}[f], \quad n \geq 2.
\]

Corollary 2.4. \( \sigma_n[f] \) can be expressed as a polynomial of \( \psi_2[f], \ldots, \psi_n[f] \) with non-negative coefficients in \( \mathbb{Z} \).
For example, we have
\[
\begin{align*}
\sigma_2[f] &= \psi_2[f], \\
\sigma_3[f] &= \psi_3[f], \\
\sigma_4[f] &= \psi_4[f] + \psi_2[f]^2, \\
\sigma_5[f] &= \psi_5[f] + 2\psi_2[f]\psi_3[f], \\
\sigma_6[f] &= \psi_6[f] + \psi_2[f]^3 + \psi_3[f]^2 + 2\psi_2[f]\psi_4[f].
\end{align*}
\]

3. Expression of $S_n[f]$ in terms of the quotients $f^{(k)}/f'$

Let $R = \mathbb{Z}[rac{1}{2}]$ be the ring generated by $1/2$ over $\mathbb{Z}$. We consider the ring $R[x_1, x_2, \ldots]$ of polynomials of infinitely many indeterminates $x_1, x_2, \ldots$ over $R$. The weight of a monomial $x_{j_1} \cdots x_{j_k}$ is defined to be the number $j_1 + \cdots + j_k$. Let $\mathcal{P}_m$ be the sub $R$-module of $R[x_1, x_2, \ldots]$ generated by monomials of weight $m$. A polynomial $P \in R[x_1, x_2, \ldots]$ is said to be of weight $m$ if $P \in \mathcal{P}_m$. It is easy to see that $\mathcal{P} = \sum_{m=0}^{\infty} \mathcal{P}_m$ becomes a graded ring. We denote by $\mathcal{E}$ the $R$-derivation on $R[x_1, x_2, \ldots]$ defined by

\[
\mathcal{E}P = \sum_{n=1}^{\infty} (x_{n+1} - x_1 x_n) \frac{\partial P}{\partial x_n}.
\]

Note that $\mathcal{E}$ maps $\mathcal{P}_m$ into $\mathcal{P}_{m+1}$. We now define polynomials $P_n \in R[x_1, x_2, \ldots]$, $n = 0, 1, 2, \ldots$, inductively by $P_0 = 1, P_1 = 0, P_2 = x_2 - (3/2)x_1^2$ and

\[
(3.1) \quad P_n = \mathcal{E}P_{n-1} + \frac{1}{2} P_2 \sum_{k=1}^{n-1} \binom{n}{k} P_{k-1} P_{n-k-1}, \quad n \geq 3.
\]

For instance,
\[
\begin{align*}
P_3 &= x_3 - 4x_1 x_2 + 3x_1^3, \\
P_4 &= x_4 - 5x_1 x_3 + 5x_1^2 x_2, \\
P_5 &= x_5 - 6x_1 x_4 + \frac{15}{2} x_1^2 x_3 - 10x_1 x_2^2 + 30x_3 x_2 - \frac{45}{2} x_1^5, \\
P_6 &= x_6 - 7x_1 x_5 + \frac{21}{2} x_1^2 x_4 - 35x_1 x_2 x_3 + \frac{105}{2} x_1^3 x_3 + 105x_1^2 x_2^2 - 210x_1 x_2^2 - \frac{315}{4} x_1^6, \\
P_7 &= x_7 - 8x_1 x_6 + 14x_1^2 x_5 - 56x_1 x_2 x_4 + 84x_3 x_4 - 35x_1^2 x_3 + 420x_1^2 x_2 x_3 \\
&\quad - 420x_1^3 x_2 - 420x_1^2 x_3^2 + 420x_1^5 x_2.
\end{align*}
\]

**Lemma 3.1.** The above polynomials $P_n$, $n = 0, 1, 2, \ldots$, satisfy the following properties:

(i) $P_n$ is of weight $n$.
(ii) $P_n \in R[x_1, \ldots, x_n]$.
(iii) $\sum_{k=1}^{n} k x_k \frac{\partial P_n}{\partial x_k} = n P_n$.

**Proof.** Property (i) can easily be checked by induction on $n$. Property (ii) follows from (i). We thus prove only property (iii).
For a monomial \( A = x_1^{e_1} \cdots x_n^{e_n} \), we compute
\[
 x_k \frac{\partial A}{\partial x_k} = e_k A.
\]
Therefore,
\[
 \sum_{k=1}^{n} j x_j \frac{\partial A}{\partial x_k} = (e_1 + 2e_2 + \cdots + ne_n) A.
\]
Note here that the weight of \( A \) is \( e_1 + 2e_2 + \cdots + ne_n \).

By property (ii) above, we may think of \( P_n \) as a function \( P_n(x_1, \ldots, x_n) \) of \( x_1, \ldots, x_n \).

Let
\[
 q_n[f] = \frac{f^{(n+1)}}{f'}, \quad n = 1, 2, \ldots.
\]
Then the principal result in this section is the following.

**Theorem 3.2.**

\[
 S_n[f] = P_n(q_1[f], q_2[f], \ldots, q_n[f]), \quad n \geq 0.
\]

**Proof.** For \( n = 0, 1, 2, \) this is clear by definition. If this is true for \( 1, 2, \ldots, n - 1 \), then
\[
 S_{n-1}[f]' = P_{n-1}(q_1[f], \ldots, q_{n-1}[f])' = \sum_{k=1}^{n-1} \frac{\partial P_{n-1}}{\partial x_k}(q_1[f], \ldots, q_{n-1}[f]) q_k[f]' .
\]
Since
\[
 (3.2) \quad q_k[f]' = q_{k+1}[f] - q_1[f]q_k[f],
\]
the relation \( S_{n-1}[f]' = \mathcal{E} P_{n-1}(q_1[f], \ldots, q_n[f]) \) holds. Now we use Proposition 2.1 to show the assertion by induction. \( \square \)

4. **INVARIANT SCHWARZIAN DERIVATIVES**

In this section, we first recall the definition of a sort of invariant derivatives for holomorphic maps between plane domains (or, more generally, Riemann surfaces) with (smooth) conformal metrics. These were introduced by Peschl [11] when the domains are either the unit disk, the complex plane or the Riemann sphere with canonical metrics. Later the notion was generalized by Minda for general conformal metrics. We call those derivatives the Peschl-Minda derivatives and detailed accounts were recently supplied by Schippers [13] and the authors [7].

For simplicity, we consider only plane domains in the present note. However, the notions below can easily be extended for Riemann surfaces as we will make a remark on it later.

Let \( \Omega \) and \( \Omega' \) be plane domains with (smooth) conformal metrics \( \rho = \rho(z)|dz| \) and \( \sigma = \sigma(w)|dw| \), respectively. We first define the \( \rho \)-derivative of a smooth function \( \varphi \) on \( \Omega \) by
\[
 \partial_\rho \varphi = \frac{1}{\rho(z)} \frac{\partial \varphi(z)}{\partial z}.
\]
For a holomorphic map \( f : \Omega \to \Omega' \), we define invariant differential operators \( D^n f = D^n_{\sigma,\rho} f \) of order \( n \) with respect to \( \rho \) and \( \sigma \) inductively by
\[
D^1_{\sigma,\rho} f = \frac{\sigma \circ f'}{\rho},
\]
\[
D^{n+1}_{\sigma,\rho} f = \left[ \partial_\rho - n \partial_\rho (\log \rho) + (\partial_\sigma \log \sigma) \circ f \cdot D^1_{\sigma,\rho} f \right] D^n_{\sigma,\rho} f \quad (n \geq 1).
\]

See [7] or [13] for details.

The quotient \( Q f = D^2 f / D^1 f \) with variable metrics was effectively used by Ma, Minda and others in the geometric study of analytic maps between plane domains (see, for instance, [5] and [6] and references therein). We also consider its higher-order analogues:
\[
Q^n f = D^{n+1} f / D^1 f, \quad n \geq 1.
\]

When we need to indicate the metrics, we write \( Q^n_{\sigma,\rho} f \) instead of \( Q^n f \).

Let \( P_n(x_1, \ldots, x_n) \) be the polynomial defined in the previous section. We define the invariant Schwarzian derivative \( \Sigma^n f \) of virtual order \( n \) for \( f \) by
\[
\Sigma^n f = P_n(Q^1 f, \ldots, Q^n f).
\]

To indicate the metrics involved, we sometimes write \( \Sigma^n_{\sigma,\rho} f \) instead of \( \Sigma^n f \).

Note that \( \Sigma^n f \) reduces to \( S^n f \) when \( \rho = \sigma = |dz| \).

We have the following invariance property for these quantities.

**Lemma 4.1.** Let \( \Omega, \hat{\Omega}, \Omega', \hat{\Omega}' \) be plane domains with smooth conformal metrics \( \rho, \hat{\rho}, \sigma, \hat{\sigma} \), respectively. Suppose that locally isometric holomorphic maps \( g : \hat{\Omega} \to \Omega \) and \( h : \Omega' \to \hat{\Omega}' \) are given. Then, for a non-constant holomorphic map \( f : \Omega \to \Omega' \), the formulae
\[
Q^n_{\hat{\sigma},\hat{\rho}} (h \circ f \circ g) = (Q^n_{\sigma,\rho} f) \circ g \cdot \left( \frac{g'}{|g'|} \right)^n
\]
\[
\Sigma^n_{\hat{\sigma},\hat{\rho}} (h \circ f \circ g) = (\Sigma^n_{\sigma,\rho} f) \circ g \cdot \left( \frac{g'}{|g'|} \right)^n
\]
are valid on \( \hat{\Omega} \).

**Proof.** By [7, Lemma 3.6], we have
\[
D^n_{\hat{\sigma},\hat{\rho}} (h \circ f \circ g) = \left( \frac{h'}{|h'|} \right) \circ f \circ g \cdot (D^n_{\sigma,\rho} f) \circ g \cdot \left( \frac{g'}{|g'|} \right)^n.
\]

Thus the assertion for \( Q^n \) follows immediately. To prove that for \( \Sigma^n \), it is enough to observe the identity
\[
P_n(\alpha x_1, \alpha^2 x_2, \ldots, \alpha^n x_n) = \alpha^n P_n(x_1, x_2, \ldots, x_n)
\]
for \( \alpha \in \mathbb{C} \), which can be seen easily.

**Remark 4.2.** By the above lemma, the quantities \( Q^n f \) and \( \Sigma^n f \) can be defined as \( (\frac{\Omega}{2}, -\frac{\Omega}{2}) \)-forms on the Riemann surface \( R \) for a non-constant holomorphic map \( f : R \to R' \) between Riemann surfaces \( R \) and \( R' \) with conformal metrics. In particular, \( |Q^n f| \) and \( |\Sigma^n f| \) are independent of the particular choices of local coordinates and thus can be regarded as functions on \( R \).

The next result is an analogue of (3.2).
Lemma 4.3. 
\[ \partial_{\rho}(Q^n f) = Q^{n+1} f - [Q^1 f - n\partial_{\rho} \log \rho] Q^n f. \]

Proof. Recall that 
\[ D^{n+1} f = \partial_{\rho}(D^n f) + [- n\partial_{\rho} \log \rho + (\partial_{\sigma} \log \sigma) \circ f \cdot D^1 f] D^n f \]
for \( n \geq 1 \). By dividing both sides by \( D^1 f \), we have
\[ Q^n f = \frac{\partial_{\rho}(D^n f)}{D^1 f} + [- n\partial_{\rho} \log \rho + (\partial_{\sigma} \log \sigma) \circ f \cdot D^1 f] Q^{n-1} f. \]
Since 
\[ \partial_{\rho}(D^n f) = \partial_{\rho}(Q^{n-1} f \cdot D^1 f) \]
\[ = \partial_{\rho}(Q^{n-1} f) + Q^{n-1} f \cdot \frac{\partial_{\rho} D^1 f}{D^1 f} \]
\[ = \partial_{\rho}(Q^{n-1} f) + Q^{n-1} f [Q^1 f + \partial_{\rho} \log \rho - (\partial_{\sigma} \log \sigma) \circ f \cdot D^1 f], \]
we obtain the assertion for \( n - 1 \). \( \square \)

We are now able to show the following result, which is a generalization of Proposition 2.1.

Theorem 4.4. Let \( f \) be a non-constant holomorphic map between plane domains \( \Omega \) and \( \Omega' \) with conformal metrics \( \rho \) and \( \sigma \), respectively. Then
\[ \Sigma^n f = (\partial_{\rho} - (n - 1)\partial_{\rho} \log \rho) \Sigma^{n-1} f + \frac{1}{2} \Sigma^2 f \sum_{k=1}^{n-1} \binom{n}{k} \Sigma^{k-1} f \Sigma^{n-k-1} f, \quad n \geq 3. \]

Proof. By definition, we compute
\[ \partial_{\rho} \Sigma^{n-1} f = \partial_{\rho} P_{n-1}(Q^1 f, \ldots, Q^{n-1} f) = \sum_{k=1}^{n-1} \frac{\partial P_{n-1}}{\partial x_k}(Q^1 f, \ldots, Q^{n-1} f) \cdot \partial_{\rho} Q^k f. \]
We now substitute the relation in Lemma 4.3 into the above to get
\[ \partial_{\rho} \Sigma^{n-1} f = \sum_{k=1}^{n-1} \frac{\partial P_{n-1}}{\partial x_k}(Q^1 f, \ldots, Q^{n-1} f) [Q^{k+1} f - [Q^1 f - k\partial_{\rho} \log \rho] Q^k f] \]
\[ = \sum_{k=1}^{n-1} \frac{\partial P_{n-1}}{\partial x_k}(Q^1 f, \ldots, Q^{n-1} f) [Q^{k+1} f - Q^1 f Q^k f] \]
\[ + \partial_{\rho} \log \rho \sum_{k=1}^{n-1} kQ^k f \cdot \frac{\partial P_{n-1}}{\partial x_k}(Q^1 f, \ldots, Q^{n-1} f) \]
\[ = (E P_{n-1})(Q^1 f, \ldots, Q^n f) + (n - 1)\partial_{\rho} \log \rho \cdot P_{n-1}(Q^1 f, \ldots, Q^{n-1} f), \]
where we used the definition of $E$ and property (iii) in Lemma 3.1. We finally recall the defining relation (3.1) of $P_n$ to obtain
\[
∂ρΣ^{n-1}f = \left\{ P_n - \frac{1}{2} P_2 \sum_{k=1}^{n-1} \binom{n}{k} P_{k-1} P_{n-k-1} \right\} (Q^1 f, \ldots, Q^n f) \\
+ (n - 1)∂ρ log ρ \cdot P_{n-1} (Q^1 f, \ldots, Q^n f) \\
= Σ^n f - \frac{1}{2} Σ^2 f \sum_{k=1}^{n-1} \binom{n}{k} Σ^{k-1} f Σ^{n-k-1} f + (n - 1)∂ρ log ρ \cdot Σ^{n-1} f.
\]
Thus the assertion has been shown.

We remark that the first author [3] gives a generating function of the invariant Schwarzian derivatives $Σ^n f$ and proves Theorem 4.4 based on a relation similar to (2.3) for $f : C_δ → C_ε$. Here $δ, ε = +1, 0, or −1, and C_{+1} = C, C_0 = C and C_{-1} = D$ and the domain $C_δ$ is equipped with the standard metric $λ_δ = |dz|/(1 + δ|z|^2)$ for $δ = +1, 0, −1$.

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