RANK CONJECTURE REVISITED

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Abstract. The rank conjecture says that rank of an elliptic curve is one less the arithmetic complexity of the corresponding non-commutative torus. We prove the conjecture for elliptic curves over a number field $K$. As a corollary, one gets a simple estimate for the rank in terms of the length of a continued fraction attached to the non-commutative torus. As an illustration, we consider a family of elliptic curves with complex multiplication and a family of rational elliptic curves.

1. Introduction

Let $E(K)$ be an elliptic curve over the field $K$, where $K \cong \mathbb{Q}$ or a finite extension of $\mathbb{Q}$. The Mordell-Weil Theorem says that the $K$-rational points of $E(K)$ is a finitely generated abelian group, see e.g. [Silverman & Tate 1992] [15, Chapter 1]. Little is known about the rank $\text{rk} E(K)$ of such a group, except for (a part of) the Birch and Swinnerton-Dyer Conjecture comparing the $\text{rk} E(K)$ with the order of zero at $s = 1$ of the Hasse-Weil $L$-function $L_{E(K)}(s)$ of the $E(K)$ [Tate 1974] [16, p.198].

Recall that the non-commutative torus is a $C^*$-algebra $A_\theta$ generated by the unitary operators $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta}uv$ for a real constant $\theta$ [Rieffel 1990] [13]. The algebra $A_\theta$ is said to have real multiplication if $\theta$ is a quadratic irrationality [Manin 2004] [5]; we denote such an algebra by $A_{RM}$. There exists a functor $F$ from elliptic curves to non-commutative tori, such that $F(E(K)) = A_{RM}$ [7], [9]. The $F$ maps elliptic curves $E(K)$ which are isomorphic over $K$ (over the algebraic closure $\bar{K} \cong \mathbb{C}$, resp.) to isomorphic (Morita equivalent, resp.) non-commutative tori $A_{RM}$; we refer the reader to remark 3.2.

The aim of our note is a simple formula for the $\text{rk} E(K)$ in terms of an invariant of the algebra $A_{RM}$. Namely, any quadratic irrationality can be written in the form $\theta_d = \frac{a + b\sqrt{d}}{c}$, where $a, b, c \in \mathbb{Z}$ and $d > 0$ is a square-free integer. The continued fraction of $\theta_d$ is eventually periodic, i.e. $\theta_d = [g_1(x), \ldots, g_m(x), k_1(x), \ldots, k_{n-m}(x)]$, where $(k_1(x), \ldots, k_{n-m}(x))$ is the minimal period and $n \geq m + 1 \geq 1$. Consider a family $\theta_x$ of irrational numbers of the form:

$$\theta_x := \left\{ \frac{a + b\sqrt{x}}{c} \mid a, b, c = \text{Const} \right\},$$

where $x$ runs through a set $U_d$ of the square-free positive integers containing $d$. By the Euler equations we understand a system of polynomial relations in the ring $\mathbb{Z}[g_1, \ldots, g_m; k_1, \ldots, k_{n-m}]$ which are necessary and sufficient for each $\theta_x$ to have the form:

$$\theta_x = [g_1(x), \ldots, g_m(x), k_1(x), \ldots, k_{n-m}(x)],$$

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where \( g_i(x) \) and \( k_i(x) \) are integer valued functions of \( x \). (For an immediate example of the Euler equations in the case \( a = 0 \) and \( b = c = 1 \) we refer the reader to [Perron 1954] [12, p.88] or Section 3.) The Euler equations define an affine algebraic set; by the Euler variety \( \mathcal{V}_E \) we understand the projective closure of an irreducible affine variety containing the point \( x = d \) of this set. An arithmetic complexity \( c(A_{RM}) \) of the algebra \( A_{RM} \) is defined as the Krull dimension of the Euler variety \( \mathcal{V}_E \).

**Remark 1.1.** The number \( c(A_{RM}) \) counts algebraically independent entries in the continued fraction of \( \theta_d \); such a number is an isomorphism invariant of the algebra \( A_{RM} \). In particular, the following inequality is true:

\[
1 \leq c(A_{RM}) \leq n. \tag{1.3}
\]

**Theorem 1.2.** \( \text{rk} \mathcal{E}(K) = c(A_{RM}) - 1, \) where \( A_{RM} = F(\mathcal{E}(K)) \).

Theorem 1.2 and formula (1.3) imply an estimate of the rank of elliptic curve \( \mathcal{E}(K) \).

**Corollary 1.3.** \( \text{rk} \mathcal{E}(K) \leq n - 1. \)

The article is organized as follows. We introduce notation and review the known facts in Section 2. Theorem 1.2 is proved in Section 3. An illustration of theorem 1.2 is given in Section 4.

2. Preliminaries

In this section we briefly review the \( C^* \)-algebras and introduce the Mordell AF-algebras. For a general account of the \( C^* \)-algebras we refer the reader to [Murphy 1990] [6]. The AF-algebras were introduced in [Bratteli 1972] [1] and the \( K \)-theory of the AF-algebras can be found in [Effros 1981] [2].

2.1. **AF-algebras.** A \( C^* \)-algebra is an algebra \( A \) over \( \mathbb{C} \) with a norm \( a \mapsto ||a|| \) and an involution \( a \mapsto a^* \) such that it is complete with respect to the norm and \( ||ab|| \leq ||a|| ||b|| \) and \( ||a^*a|| = ||a^2|| \) for all \( a, b \in A \). Any commutative \( C^* \)-algebra is isomorphic to the algebra \( C_0(X) \) of continuous complex-valued functions on some locally compact Hausdorff space \( X \); otherwise, \( A \) represents a non-commutative topological space.

An AF-algebra (Approximately Finite \( C^* \)-algebra) is defined to be the norm closure of a dimension-increasing sequence of finite dimensional \( C^* \)-algebras \( M_n \), where \( M_n \) is the \( C^* \)-algebra of the \( n \times n \) matrices with entries in \( \mathbb{C} \). Here the index \( n = (n_1, \ldots, n_k) \) represents the semi-simple matrix algebra \( M_n = M_{n_1} \oplus \cdots \oplus M_{n_k} \).

The ascending sequence mentioned above can be written as

\[
M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots, \tag{2.1}
\]

where \( M_i \) are the finite dimensional \( C^* \)-algebras and \( \varphi_i \) the homomorphisms between such algebras. The homomorphisms \( \varphi_i \) can be arranged into a graph as follows. Let \( M_i = M_{i_1} \oplus \cdots \oplus M_{i_k} \) and \( M_{i'} = M'_{i_1} \oplus \cdots \oplus M'_{i_k} \) be the semi-simple \( C^* \)-algebras and \( \varphi_{i'} : M_i \to M_{i'} \) the homomorphism. (To keep it simple, one can assume that \( i' = i + 1 \).) One has two sets of vertices \( V_{i_1}, \ldots, V_{i_k} \) and \( V'_{i_1}, \ldots, V'_{i_k} \) joined by \( b_{rs} \) edges whenever the summand \( M_{i_r} \) contains \( b_{rs} \) copies of the summand \( M_{i_s} \) under the embedding \( \varphi_i \). As \( i \) varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The matrix \( B = (b_{rs}) \) is known as a partial multiplicity matrix; an infinite sequence of \( B_i \) defines a unique AF-algebra.
For a unital $C^*$-algebra $A$, let $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$; projections $p, q \in V(A)$ are equivalent if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$; the equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called the $K_0$-group of the algebra $A$. The functor $A \rightarrow K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+ \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0(A)$. The ordered abelian group $(K_0, K_0^+, u)$ with an order unit is called a dimension group; an order-isomorphism class of the latter we denote by $(G, G^+)$. The dimension group $(K_0(\mathbb{A}), K_0^+(\mathbb{A}), u)$ of an AF-algebra $\mathbb{A}$ is a complete invariant the isomorphism class of algebra $\mathbb{A}$ [Elliott 1976] [3].

2.2. Mordell AF-algebras. Denote by $S^1$ a unit circle in the complex plane; let $G$ be a multiplicative subgroup of $S^1$ given by a finite set of generators $\{\gamma_j\}_{j=1}^s$.

**Lemma 2.1.** There exists a bijection between groups $G \subset S^1$ and the dimension groups $(\Lambda_G, \Lambda_G^+, u)$ given by the formula:

\[
\begin{align*}
\Lambda_G & := \mathbb{Z} + \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_s \subset \mathbb{R}, \quad \omega_j = \frac{1}{\pi} \text{Arg} (\gamma_j), \\
\Lambda_G^+ & := \Lambda_G \cap \mathbb{R}^+, \\
u & := \text{an order unit.}
\end{align*}
\]

The rank of $(\Lambda_G, \Lambda_G^+, u)$ is equal to $s - t + 1$, where $t$ is the total number of roots of unity among $\gamma_j$.

**Proof.** If $\gamma_j$ is a root of unity, then $\omega_j = \frac{2\pi}{q_j}$ is a rational number. We have

\[
\begin{align*}
\Lambda_G & \cong \mathbb{Z} + \mathbb{Z}\frac{2\pi}{q_1} + \cdots + \mathbb{Z}\frac{2\pi}{q_s} + \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{s-t} \cong \\
& \cong \mathbb{Z} + \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{s-t},
\end{align*}
\]

where $\omega_i$ are linearly independent irrational numbers. Since the set $\Lambda_G$ in dense in $\mathbb{R}$, the triple $(\Lambda_G, \Lambda_G^+, u)$ is a dimension group [Effros 1981] [2, Corollary 4.7]. Clearly, the rank of such a group is equal to $s - t + 1$. A converse statement is proved similarly. Lemma 2.1 follows. \hfill \Box

Let $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{E}(\mathbb{C})$) be the real (complex, resp.) points of $\mathcal{E}(K)$; we have the following inclusions:

\[
\mathcal{E}(K) \subset \mathcal{E}(\mathbb{R}) \subset \mathcal{E}(\mathbb{C}).
\]

In view of the Mordell-Ne"eron Theorem, we shall write $\mathcal{E}(K) \cong \mathbb{Z}^r \oplus \mathcal{E}_{\text{tors}}(K)$, where $r = rk \mathcal{E}(K)$ and $\mathcal{E}_{\text{tors}}(K)$ are the torsion points of $\mathcal{E}(K)$. It is well known that the Weierstrass $\wp$-function maps $\mathcal{E}(\mathbb{C})$ (resp. $\mathcal{E}(\mathbb{R})$) to a complex torus $\mathbb{C}/\Lambda$ (the one-dimensional compact connected Lie group $S^1$, resp.) In view of inclusion (2.4) the points of $\mathcal{E}(K)$ map to an abelian subgroup $G$ of $S^1$, such that the $\mathcal{E}_{\text{tors}}(K)$ consists of the roots of unity, see e.g. [Silverman & Tate 1992] [15, p.42]. Denote by $t$ the total number of generators of $\mathcal{E}_{\text{tors}}(K)$; then $\mathcal{E}(K)$ has $s = r + t$ generators. Lemma 2.1 implies a bijection between elliptic curves $\mathcal{E}(K)$ modulo their torsion points and dimension groups $(\Lambda_G, \Lambda_G^+, u)$ of rank $r + 1$; the following definition is natural.
Definition 2.2. By the Mordell AF-algebra $\mathcal{A}_E(K)$ of an elliptic curve $E(K)$ we understand an AF-algebra given by the Elliott isomorphism:

$$K_0(\mathcal{A}_E(K)) \cong (\Lambda_G, \Lambda^+_G, u),$$

(2.5)

where $(\Lambda_G, \Lambda^+_G, u)$ is a dimension group of the rank $r + 1$.

3. Proof

Theorem 1.2 follows from an observation that the Mordell AF-algebra $\mathcal{A}_E(K)$ is a non-commutative coordinate ring of the Euler variety $\mathcal{V}_E$ defined in Section 1. We shall split the proof in a series of lemmas.

Lemma 3.1. The $\mathcal{V}_E$ is a flat family of abelian varieties $\mathcal{A}_E$ over the projective line $\mathbb{CP}^1$.

Proof. To prove lemma 3.1 we shall adopt and generalize an argument of [Perron 1954] [12, Proposition 3.17]. Let us show that there exists a flat morphism $p : \mathcal{V}_E \to \mathbb{CP}^1$ of the Euler variety $\mathcal{V}_E$ into $\mathbb{CP}^1$ with the fiber $p^{-1}(x)$ an abelian variety $\mathcal{A}_E$.

Indeed, recall that the algebras $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are said to be Morita equivalent if $\mathcal{A}_\theta \otimes \mathcal{K} \cong \mathcal{A}_{\theta'} \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators. The $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are Morita equivalent if and only if $\theta' = \frac{a\theta + b}{c\theta + d}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ satisfy the equality $\alpha\delta - \beta\gamma = \pm 1$ [Rieffel 1990] [13]. In other words, the continued fractions of $\theta$ and $\theta'$ must coincide everywhere but a finite number of entries [Perron 1954] [12, Section 13]. On the other hand, $\mathcal{A}_{\theta'} \cong \mathcal{A}_\theta$ are isomorphic algebras if and only if $\theta' = \theta$ or $\theta' = 1 - \theta$ [Rieffel 1990] [13].

Remark 3.2. Functor $F : \mathcal{E}(K) \to \mathcal{A}_{RM}$ maps elliptic curves which are isomorphic over the field $K$ (over the algebraic closure $\bar{K} \cong \mathbb{C}$, resp.) to isomorphic (Morita equivalent, resp.) non-commutative tori. In particular, twists of the $\mathcal{E}(K)$ correspond to the isomorphism classes of the algebra $\mathcal{A}_{RM}$. This fact follows from an isomorphism $\mathcal{A}(K) \otimes M_n \cong \mathcal{A}(K^{(n)})$, where $\mathcal{A}(K)$ is an algebra over the field $K$, $M_n$ is the matrix algebra of rank $n$ and $K^{(n)}$ is an extension of degree $n$ of $K$.

Let $\theta_d = \frac{a + b\sqrt{d}}{c} = [g_1, \ldots, g_m, \frac{k_1}{k_2}, \ldots, \frac{k_{n-m}}{k_n}]$, where $(k_1, \ldots, k_{n-m})$ is the minimal period of the continued fraction of $\theta_d$. To find the Euler equations for $\theta_d$, let $D = b^2d$ and write $\theta_D$ in the form:

$$\theta_D = \frac{A_n \theta_D + A_{n-1}}{B_n \theta_D + B_{n-1}},$$

(3.1)

where $\frac{A_n}{B_n}$ is the $i$-th partial fraction of $\theta_D$ and $A_n B_{n-1} - A_{n-1} B_n = \pm 1$ [Perron 1954] [12, Section 19]; in particular,

$$\begin{cases} A_n = k_n A_{n-1} + A_{n-2} \\ B_n = k_n B_{n-1} + B_{n-2}. \end{cases}$$

(3.2)

From equation (3.1) one finds that

$$\theta_D = \frac{A_n - B_{n-1} + \sqrt{(A_n - B_{n-1})^2 + 4A_{n-1}B_n}}{2B_n}.$$    \hfill (3.3)
Comparing equation (3.3) with conditions (1.1), we conclude that the Euler equations for $\theta_d$ have the form:

$$
\begin{align*}
A_n - B_{n-1} &= c_1 \\
2B_n &= c_2 \\
D &= c_1^2 + 2c_2A_{n-1},
\end{align*}
$$

(3.4)

where $c_1, c_2 \in \mathbb{Z}$ are constants. Using formulas (3.2) one can rewrite $c_1$ and $c_2$ in the form:

$$
\begin{align*}
c_1 &= k_nA_{n-1} + A_{n-2} - B_{n-1} \\
c_2 &= 2k_nB_{n-1} + 2B_{n-2}.
\end{align*}
$$

(3.5)

We substitute $c_1$ and $c_2$ given by (3.5) into the last equation of system (3.4) and get an equation:

$$
D - k_n^2A_{n-1}^2 - 2(A_n - A_{n-2} - A_{n-1}B_{n-1} + 2B_{n-1})k_n = (A_{n-2} + B_{n-1})^2 \pm 4.
$$

(3.6)

But equation (3.6) is a linear diophantine equation in variables $D - k_n^2A_{n-1}^2$ and $k_n$; since (3.6) has one solution in integers numbers, it has infinitely many solutions. In other words, the projective closure $\mathcal{V}_E$ of an affine variety defined by the Euler equations (3.4) is a fiber bundle over the projective line $\mathbb{C}P^1$.

Let us show that the fiber $p^{-1}(x)$ of a flat morphism $p : \mathcal{V}_E \to \mathbb{C}P^1$ is an abelian variety $\mathcal{A}_E$. Indeed, the fraction $[g_1, \ldots, g_m, k_1, \ldots, k_{n-m}]$ corresponding to the quadratic irrationality $\theta_d$ can be written in a normal form for which the Euler equations $A_n - B_{n-1} = c_1$ and $2B_n = c_2$ in system (3.4) become a trivial identity. (For instance, if $c_1 = 0$, then the normal fraction has the form $[\gamma_1, \kappa_1, \kappa_2, \ldots, \kappa_2, \gamma_1, \kappa_1, \kappa_2 \gamma_1]$, see e.g. [Perron 1954] [12, Section 24].) Denote by $[\gamma_1, \ldots, \gamma_m, k_{n-m}, \ldots, k_{n-1}k_1, \ldots, k_{n-m}]$ a normal form of the fraction $[g_1, \ldots, g_m, k_1, \ldots, k_{n-m}]$. Then the Euler equations (3.4) reduce to a single equation $D = c_1^2 + 2c_2A_{n-1}$, which is equivalent to the linear equation (3.6). Thus the quotient of the Euler variety $\mathcal{V}_E$ by $\mathbb{C}P^1$ corresponds to the projective closure $\mathcal{A}_E$ of an affine variety defined by $\gamma_i$ and $\kappa_j$. But the $\mathcal{A}_E$ has an obvious translation symmetry, i.e. $\{\gamma_i' = \gamma_i + c_i, \kappa_j' = \kappa_j + c_j \mid c_i, c_j \in \mathbb{Z}\}$ define an isomorphic projective variety. In other words, the fiber $p^{-1}(x) \cong \mathcal{A}_E$ is an abelian group, i.e. the $\mathcal{A}_E$ is an abelian variety. Lemma 3.1 follows.

**Corollary 3.3.** $\dim_{\mathbb{C}}\mathcal{V}_E = 1 + \dim_{\mathbb{C}}\mathcal{A}_E$

**Proof.** Since the morphism $p : \mathcal{V}_E \to \mathbb{C}P^1$ is flat, we conclude that

$$
\dim_{\mathbb{C}}\mathcal{V}_E = \dim_{\mathbb{C}}(\mathbb{C}P^1) + \dim_{\mathbb{C}}p^{-1}(x).
$$

(3.7)

But $\dim_{\mathbb{C}}(\mathbb{C}P^1) = 1$ and $p^{-1}(x) \cong \mathcal{A}_E$. Corollary 3.3 follows from formula (3.7).

**Lemma 3.4.** The algebra $k_{E(K)}$ is a non-commutative coordinate ring of the abelian variety $\mathcal{A}_E$.

**Proof.** This fact follows from an equivalence of four categories $E_0 \cong A_0 \cong M \cong V$, where $E_0 \subset E$ is a subcategory of the category $E$ of all non-singular elliptic curves consisting of the curves defined over a number field $K$, $A_0 \subset A$ is a subcategory of the category $A$ of all non-commutative tori consisting of the tori with real multiplication, the $M$ is a category of all Mordell $AF$-algebras and the $V$ is a category of all Euler varieties. The morphisms in the categories $E$ and $V$ are isomorphisms between the projective varieties and morphisms in the categories $A$ and $M$ are
Morita equivalences (stable isomorphisms) between the corresponding $C^\ast$-algebras [Murphy 1990] [6]. Let us pass to a detailed argument.

An equivalence $\mathcal{E}_0 \cong \mathcal{A}_0$ follows from the results of [7]. It remains to show that $\mathcal{E}_0 \cong \mathcal{M}$ and $\mathcal{A}_0 \cong \mathcal{V}$.

(i) Let us prove that $\mathcal{E}_0 \cong \mathcal{M}$. Let $\mathcal{E}(K) \in \mathcal{E}_0$ be an elliptic curve and $\mathcal{A}_{\mathcal{E}(K)} \in \mathcal{M}$ be the corresponding Mordell $\mathcal{AF}$-algebra, see Section 2.2. Then $K_0(\mathcal{A}_{\mathcal{E}(K)}) \cong \mathcal{E}(K)$ by the Elliott isomorphism (2.5). Note that the order unit $u$ in (2.5) depends on the choice of generators of the abelian group $\mathcal{E}(K)$ and, therefore, an isomorphism class of $\mathcal{E}(K)$ corresponds to the Morita equivalence class of the $\mathcal{AF}$-algebra $\mathcal{A}_{\mathcal{E}(K)}$. Thus formula (2.5) defines an equivalence between the categories $\mathcal{E}_0$ and $\mathcal{M}$.

(ii) Let us prove that $\mathcal{A}_0 \cong \mathcal{V}$. Let $\mathcal{A}_{RM} \in \mathcal{A}_0$ be a non-commutative torus with real multiplication and let $(k_1, \ldots, k_n)$ be the minimal period of continued fraction corresponding to the quadratic irrational number $\theta_d$. The period is a Morita invariant of the algebra $\mathcal{A}_{RM}$ and a cyclic permutation of $k_i$ gives an algebra $\mathcal{A}'_{RM}$ which is Morita equivalent to the $\mathcal{A}_{RM}$. On the other hand, it is not hard to see that the period $(k_1, \ldots, k_n)$ can be uniquely recovered from the coefficients of the Euler equations (3.4) and a cyclic permutation of $k_i$ in (3.4) defines an Euler variety $\mathcal{V}'_E$ which is isomorphic to the $\mathcal{V}'_E$. In other words, the categories $\mathcal{A}_0$ and $\mathcal{V}$ are isomorphic.

Comparing (i) and (ii) with the equivalence $\mathcal{E}_0 \cong \mathcal{A}_0$, one concludes that $\mathcal{V} \cong \mathcal{M}$ are equivalent categories, where a functor $F : \mathcal{V} \to \mathcal{M}$ acts by the formula $\mathcal{V}_E \mapsto \mathcal{A}_{\mathcal{E}(K)}$ given by the closure of arrows of the commutative diagram in Figure 1. It remains to recall that the $\mathcal{V}_E$ is a fiber bundle over the $\mathbb{C}P^1$; therefore all geometric data of $\mathcal{V}_E$ is recorded by the fiber $\mathcal{AF}_{\mathcal{E}}$ alone. We conclude that the algebra $\mathcal{A}_{\mathcal{E}(K)}$ is a coordinate ring of the abelian variety $\mathcal{AF}_{\mathcal{E}}$, i.e. an isomorphism class of $\mathcal{AF}_{\mathcal{E}}$ corresponds to the Morita equivalence class of the algebra $\mathcal{A}_{\mathcal{E}(K)}$. Lemma 3.4 follows.

Remark 3.5. The diagram in Figure 1 implies an equivalence of the categories: (i) $\mathcal{E}_0 \cong \mathcal{V}$ and (ii) $\mathcal{A}_0 \cong \mathcal{M}$. The bijection (i) is realized by the formula $\mathcal{E}(K) \mapsto \mathcal{AF}_{\mathcal{E}} \cong \text{Jac} (X(N))$, where $X(N)$ is an Eichler-Shimura-Taniyama modular curve of level $N$ over the $\mathcal{E}(K)$ and $\text{Jac} (X(N))$ is the Jacobian of the $X(N)$. The bijection (ii) is given by the formula $\mathcal{A}_{RM} \mapsto \mathcal{A}_{\mathcal{E}(K)}$, such that the number field

\begin{figure}
\centering
\begin{tikzpicture}
\node (E) {$\mathcal{E}(K)$};
\node (V) [right of=E] {$\mathcal{V}_E \cong (\mathcal{AF}_{\mathcal{E}}, \mathbb{C}P^1, p)$};
\node (A) [below of=E] {$\mathcal{A}_{RM}$};
\node (K) [below of=V] {$\mathcal{A}_{\mathcal{E}(K)}$};
\node (F) [above of=V, left of=E] {\large $F$};
\draw [->] (E) to (V);
\draw [->] (E) to (A);
\draw [->] (V) to (K);
\end{tikzpicture}
\caption{Functor $F : \mathcal{V} \to \mathcal{M}$.}
\end{figure}
that a coordinate ring of the elliptic curve and a definition of the non-commutative torus for the primes $1 < p < 100$. (The respective calculations of the rank can be found in [Gross 1980] [4] and of the arithmetic complexity in [11].) It is easy to see, that in all examples of Table 1 the rank of elliptic curves and the corresponding arithmetic complexity satisfy the equality:

$$rk_{\mathbb{Q}} \left( \mathcal{E}_{CM}^{(p,1)} \right) = c \left( A_{RM}^{(p,1)} \right) - 1,$$

where $A_{RM}^{(p,1)} = F(\mathcal{E}_{CM}^{(p,1)})$. 

### 4. Examples

We shall illustrate theorem 1.2 by two series of examples; the first are the so-called Q-curves introduced in [Gross 1980] [4] and the second is a family of the rational elliptic curves.

#### 4.1. Q-curves

Denote by $\mathcal{E}_{CM}^{(-d,f)}$ an elliptic curve with complex multiplication by an order $R_f$ of conductor $f$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, see e.g. [Silverman 1994] [14, pp.95-96]. It is known that the non-commutative torus corresponding to $\mathcal{E}_{CM}^{(-d,f)}$ will have real multiplication by the order $R_f$ of conductor $f$ in the quadratic field $\mathbb{Q}(\sqrt{-d})$; such a torus we denote by $A_{RM}^{(-d,f)}$. The conductor $f$ is defined by equation $[Cl (R_f)] = [Cl (R_1)]$, where $Cl$ is the class group of the respective orders [9]. Let $(\mathcal{E}_{CM}^{(-d,f)})^\sigma$, $\sigma \in Gal (k|\mathbb{Q})$ be the Galois conjugate of the curve $\mathcal{E}_{CM}^{(-d,f)}$; by a Q-curve one understands $\mathcal{E}_{CM}^{(-d,f)}$, such that there exists an isogeny between $(\mathcal{E}_{CM}^{(-d,f)})^\sigma$ and $\mathcal{E}_{CM}^{(-d,f)}$ for each $\sigma \in Gal (k|\mathbb{Q})$. Let $\mathfrak{P}_3 \mod 4$ be the set of all primes $p = 3 \mod 4$; it is known that $\mathcal{E}_{CM}^{(-p,1)}$ is a Q-curve whenever $p \in \mathfrak{P}_3 \mod 4$ [Gross 1980] [4, p.33]. The rank of $\mathcal{E}_{CM}^{(-p,1)}$ is always divisible by $2h_K$, where $h_K$ is the class number of field $K := \mathbb{Q}(\sqrt{-p})$ [Gross 1980] [4, p.49]; by a Q-rank of $\mathcal{E}_{CM}^{(-p,1)}$ we understand the integer $rk_{\mathbb{Q}}(\mathcal{E}_{CM}^{(-p,1)}) := \frac{1}{2\pi K} rk (\mathcal{E}_{CM}^{(-p,1)})$. Table 1 below shows the Q-ranks of the Q-curves and the arithmetic complexity of the corresponding non-commutative tori for the primes $1 < p < 100$. (The respective calculations of the rank can be found in [Gross 1980] [4] and of the arithmetic complexity in [11].) It is easy to see, that in all examples of Table 1 the rank of elliptic curves and the corresponding arithmetic complexity satisfy the equality:

$$rk_{\mathbb{Q}} \left( \mathcal{E}_{CM}^{(-p,1)} \right) = c \left( A_{RM}^{(p,1)} \right) - 1,$$

where $A_{RM}^{(p,1)} = F(\mathcal{E}_{CM}^{(-p,1)})$. 

Lemma 3.6. $rk_{\mathbb{Q}} \mathcal{E}(K) = dim_{\mathbb{C}} A_{E}$. 

**Proof.** Let $A_{E}$ be an abelian variety over the field $\mathbb{Q}$ and let $n = dim_{\mathbb{C}} A_{E}$. In this case it is known that the coordinate ring of $A_{E}$ is isomorphic to the algebra $\mathcal{A}_{RM}^n$, where $\mathcal{A}_{RM}^n$ is a $2n$-dimensional non-commutative torus with real multiplication [8, Section 3.2]. (Notice that the proof can be easily extended from the case of complex multiplication to the case of $A_{E}$ over $\mathbb{Q}$.) Moreover,

$$K_0(\mathcal{A}_{E}(K)) \otimes \mathbb{Q} = dim_{\mathbb{C}} A_{E},$$

see [8, Section 1.1]. But we know from lemma 3.4 that a coordinate ring of the abelian variety $A_{E}$ is isomorphic to the algebra $k_{E}(K)$, such that

$$K_0(\mathcal{A}_{RM}^n) \cong \mathbb{Z} + \mathbb{Z} \theta_1 + \cdots + \mathbb{Z} \theta_n,$$

(3.8) see Section 2.2. Comparing formulas (3.8) and (3.9), we conclude that $rk_{\mathbb{Q}} \mathcal{E}(K) = n = dim_{\mathbb{C}} A_{E}$. Lemma 3.6 follows. □

Proof. Theorem 1.2 follows from corollary 3.3, lemma 3.6 and a definition of the arithmetic complexity $c(A_{RM}) = dim_{\mathbb{C}} \mathcal{F}_{E}$. □
4.2. Rational elliptic curves. For an integer $b \geq 3$ we shall consider a family of the rational elliptic curves given by the equation:

$$E_b(Q) \cong \{ (x, y, z) \in \mathbb{C}P^2 \mid y^2z = x(x - z)(x - b-2z) \}. \quad (4.2)$$

It was shown in [10] that $F(E_b(Q)) = A_{\theta_d}$, where $\theta_d = \frac{b+\sqrt{b^2-4}}{2}$. The following evaluation of the rank of elliptic curves in the family $E_b(Q)$ is true.

**Corollary 4.1.** $rk E_b(Q) \leq 2$.

**Proof.** It is easy to see that

$$\theta_d = \frac{b+\sqrt{b^2-4}}{2} = [b-1,1,b-2]. \quad (4.3)$$

Thus $m = 1$ and $n = 3$. From corollary 1.3, one gets $rk E_b(Q) \leq 2$. $\square$

**Corollary 4.2.** The elliptic curve $E_3(Q)$ has a twist of rank 0.

**Proof.** From equation (4.3), one gets $\theta_d = \frac{1}{2}(3+\sqrt{5}) = [2,1]$. But the $A_{\theta_d}$ is Morita equivalent to an algebra $A_{\theta'_d}$, where

$$\theta'_d = \frac{1+\sqrt{5}}{2} = [1] = 1+\frac{1}{1+\frac{1}{1+\ldots}} \quad (4.4)$$

is a purely periodic continued fraction. (The latter is known as the “golden mean” continued fraction.) In particular, we have $m = 0$ and $n = 1$. From corollary 1.3 and remark 3.2, one concludes that the rank of a twist of $E_3(Q)$ is equal to 0. $\square$

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