Construction of a Natural Transformation from a Classical to a Quantum 0-Species

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2010 Mathematics Subject Classification. Primary 46M99, 46H35, 47D06, 46Kxx; Secondary 46F05

Key words and phrases. dynamical patterns, natural transformations, topological *-algebras of linear operators, C_0-semigroups on locally convex spaces, topological *-algebras of test functions.

Abstract. A natural transformation \( \mathcal{J} \) between functors valued in the category \( \mathcal{Chdv}_0 \) is assembled. \( \mathcal{Chdv}_0 \) is obtained by replacing both the categories ptsl and ptsa with the category of topological linear spaces in the defining properties of the category \( \mathcal{Chdv} \) introduced in one of our previous papers. By letting a \( dp \)-valued functor be (classical) quantum whenever every its value is a dynamical pattern whose set map takes values in the set of (commutative) noncommutative topological unital *-algebras, and letting a (classical) quantum 0-species be a \( \mathcal{Chdv}_0 \)-valued functor factorizing through the canonical functor from \( dp \) to \( \mathcal{Chdv}_0 \) into a (classical) quantum \( dp \)-valued functor, we have that the domain and codomain of \( \mathcal{J} \) are a classical and a quantum 0-species respectively.
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Notation

In this section we fix the notation and collect general facts we shall use often without any further mention in the paper including the Introduction. If \( X, Y \) are topological spaces, then \( \mathcal{C}(X, Y) \) is the set of continuous maps from \( X \) to \( Y \). If the set underlying \( X \) is a subset of the set underlying \( Y \), then \( X \hookrightarrow Y \) means \( i^Y_X \in \mathcal{C}(X, Y) \) with \( i^Y_X \) the inclusion map of \( X \) into \( Y \). \( \text{vct} \) is the category of complex vector spaces and linear maps, \( \text{top} \) is the category of topological spaces and continuous maps.

All topological vector spaces are considered over \( \mathbb{C} \), \( \text{tls} \) is the category of topological vector spaces and continuous linear maps. If \( E, F \in \text{tls} \), then we let \( \mathcal{U}(E, F) := \text{Mor}_{\text{tls}}(E, F) = \mathcal{C}(E, F) \cap \text{Mor}_{\text{vct}}(E, F) \), \( \mathcal{U}(E) := \mathcal{U}(E, E) \) and \( E' := \mathcal{U}(E, \mathbb{C}) \). If \( E \) and \( F \) are locally convex spaces and \( \mathfrak{G} \) is a family of bounded subsets of \( E \), then \( \mathcal{U}_\mathfrak{G}(E, F) \) is the locally convex space \( \mathcal{U}(E, F) \) endowed with the topology of uniform convergence over the sets in \( \mathfrak{G} \). \( \mathcal{U}_\mathfrak{G}(E, F), \mathcal{U}_\mathfrak{b}(E, F) \) and \( \mathcal{U}_{\text{pc}}(E, F) \) stand for \( \mathcal{U}_\mathfrak{b}(E, F) \) with \( \mathfrak{G} \) respectively the family of finite, bounded and precompact subsets of \( E \). If \( p \) is a continuous seminorm on \( F \) and \( B \) is a bounded set of \( E \), then \( p_B : T \mapsto \sup_{x \in B} p(Tx) \) is a continuous seminorm of \( \mathcal{U}_b(E, F) \). \( \text{ptls} \) is the subcategory of \( \text{tls} \) of preordered topological vector spaces and linear continuous positive maps, \( r \in \text{Fct}(\text{ptls}, \text{tls}) \) is the forgetful functor.

For any unital \( \ast \)-algebra \( A \), \( F_{\text{max}} \) (relative to a fixed wedge in the hermitian elements of \( A \)) is defined in \([6 \, \text{p.24}]\), while \( A \)-invariant non-empty subsets of \( F_{\text{max}} \) are defined in \([6 \, \text{p.25}]\). If \( \mathcal{H} \) and \( \mathcal{G} \) are Hilbert spaces, then for every densely defined linear operator in \( \mathcal{H} \) with values in \( \mathcal{G} \), let \( S^\dagger \) denote the adjoint of \( S \). The concept of \( O^\ast \)-algebra \( A \) on a dense linear subspace \( D \) of a Hilbert space is provided in \([6 \, \text{Def. 2.1.6}]\), the locally convex space \( \mathcal{L}(D, A) \) is defined in \([6 \, \text{Def. 2.1.1}]\) and its topology is called the graph topology of \( A \) on \( D \). The linear space \( \mathcal{L}(D_B, D_B^\ast) \) is defined in \([6 \, \text{Def. 3.2.1}]\) and the bounded topology \( \tau_b \) on it in \([6 \, \text{p.76}]\).

\( \text{tsa}_0 \) is the category of topological \( \ast \)-algebras and continuous \( \ast \)-morphisms, \( \text{tsa} \) is the category of topological unital \( \ast \)-algebras and continuous unit preserving \( \ast \)-morphisms. Let \( \tilde{\mathfrak{q}}_0 \in \text{Fct}(\text{tsa}_0, \text{tls}) \), \( q_0 \in \text{Fct}(\text{tsa}, \text{tls}) \) and \( q_1 \in \text{Fct}(\text{tsa}, \text{top}) \) be the forgetful functors. If \( A \) is a subcategory of \( B \), then we let \( I^B_A \) denote the inclusion functor. Given an object \( A \) with a structure we often use, as we did above, the common abuse of language of denoting by \( A \) each of its underlying structure. So for instance if \( A \) and \( B \) are topological unital \( \ast \)-algebras, \( \mathcal{U}(A, B) \) stands for \( \mathcal{U}(q_0(A), q_0(B)) \) while \( \mathcal{C}(A, B) \) stands for \( \mathcal{C}(q_1(A), q_1(B)) \). A topological unital sub \( \ast \)-algebra \( A \) of \( B \in \text{tsa} \) here always means that \( A \in \text{tsa} \) so that \( A \) is a topological subspace of \( B \), \( A \) is a sub \( \ast \)-algebra of \( B \) and \( A \) holds the same unit of \( B \). Given \( A \in \text{tsa}_0 \) we let \( A_1 \in \text{tsa} \) be the unitization of \( A \) \([2 \, \text{p.38}]\) whose topology by definition is the product topology. In case \( A \) is a locally convex \( \ast \)-algebra whose topology is generated by the set \( S \) of seminorms, then \( \hat{S} = \{ \hat{r} | r \in S \} \) generates the locally convex topology of \( A_1 \), where \( \hat{p}(a, \lambda) := p(\hat{a}) + |\lambda| \) for every seminorm \( p \) on \( A \). In particular if \( p \) is a continuous seminorm on \( A \), then \( \hat{p} \) is continuous on \( A_1 \). Thus if \( A \in \text{tsa}_0 \), and \( B \in \text{tsa} \) are both locally convex and \( T \in \text{Mor}_\text{tls}(\mathfrak{q}_0(A), q_0(B)) \), then

\[
(1) \quad (a, \lambda) \mapsto Ta + \lambda 1_B) \in \text{Mor}_\text{tls}(q_0(A_1), q_0(B));
\]
since for every continuous seminorm \( q \) of \( B \) there exists a continuous seminorm \( p \) of \( A \) such that for all \((a, \lambda) \in A_1\), we have \( q(Ta + \lambda 1_B) \leq q(Ta) + k|\lambda| \leq p(a) + k|\lambda| \) with \( k = q(1_B) \), thus if \( k \neq 0 \), then \( q(Ta + \lambda 1_B) \leq kp'(a, \lambda) \) with \( p' = k^{-1}p \), otherwise \( q(Ta + \lambda 1_B) \leq \tilde{p}(a, \lambda) \).

Given two top-quasi enriched categories \( A \) and \( B \), let \( \text{Fct}_{\text{top}}(A, B) \) denote the set of functors of top-quasi enriched categories from \( A \) to \( B \) [7 Section 1.2].

If \( X \) is a locally compact space, then \( \mathcal{K}(X) \) denotes the locally convex space of complex valued continuous maps on \( X \) with compact support endowed with the usual inductive limit topology [1] Ch. 3, §1, n. 1]. In this paper a measure on \( X \) always means an element of \( \mathcal{K}(X) \) [1, Ch. 3, §1, n. 3, Def. 2].

All manifolds are smooth and finite dimensional, hence locally compact. All vector fields are smooth. Let \( M \) be a manifold. \( \mathcal{E}^\infty(M) \) denotes the locally convex space of complex valued smooth maps on \( M \) endowed with the Frechet topology [8 p.12] here denoted by \( \tau^\infty(M) \) or simply \( \tau^\infty \). \( \mathcal{D}(M) \) denotes the locally convex space of complex valued smooth maps on \( M \) with compact support endowed with the usual inductive limit topology [3 p.13] here denoted by \( \tau^\infty_c(M) \) or simply \( \tau^\infty_c \). We have \( \mathcal{D}(M) \hookrightarrow \mathcal{E}^\infty(M) \) [8 Rmk. 1.1.13]. \( \mathcal{D}(M) \) is a Montel space [3 example 6, p.241] so barrelled, sequentially complete [8 Thm. 1.1.11(i)] topological \( \ast \)-algebra [2, 28.7, 28.12] such that \( \mathcal{D}(M) \hookrightarrow \mathcal{K}(M) \) [3 p.241]. Let \( \langle \mathcal{D}(M), \cdot, \cdot \rangle \) denote the natural left \( \mathcal{E}^\infty(M) \)-module.

If \( N \) and \( M \) are manifolds and \( \phi : N \to M \) is a smooth proper map [8 Def. 1.1.16], then we shall consider \( \phi^* \) defined on \( \mathcal{D}(M) \), thus \( \phi^* \in \mathcal{L}(\mathcal{D}(M), \mathcal{D}(N)) \) [8 Prp. 1.1.17]. For any \( k \in \mathbb{Z}^+ \) let \( \text{DiffOp}^k(M, N) \) be the set of the restrictions at \( \mathcal{D}(M) \) of the elements in \( \text{DiffOp}^k_0(M \times \mathbb{C}, N \times \mathbb{C}) \) where for every vector bundle \( A \) on \( M \) and \( B \) on \( N \), we let \( \text{DiffOp}^k_0(A, B) \) be the set of differential operators of order \( k \) from \( A \) to \( B \) [8 Def. 1.2.1].

Set \( \text{DiffOp}^k(M) = \text{DiffOp}^k_0(M, M) \). \( \langle \text{DiffOp}^k(M, N), \cdot, \cdot \rangle \) is naturally a left \( \mathcal{E}^\infty(N) \)-module, since \( \text{DiffOp}^k_0(A, B) \) it is so, where for every \( F \in \mathcal{E}^\infty(N) \) and \( T \in \text{DiffOp}^k(M, N) \) we set \( (F \circ T) : \mathcal{D}(M) \to \mathcal{D}(N) \), \( h \mapsto F \cdot T(h) \). If \( T \in \text{DiffOp}^k(M, N) \), then \( \text{supp}(F) \subseteq \text{supp}(f) \) for every \( f \in \mathcal{D}(M) \) [8 Rmk.1.2.2(iv)]. We have \( \text{DiffOp}^k(M, N) \subset \mathcal{L}(\mathcal{D}(M), \mathcal{D}(N)) \) [8 Thm. 1.2.10].

Let \( \mathfrak{X}(M) \) be the set of vector fields of \( M \), if \( U \in \mathfrak{X}(M) \), then let \( \mathcal{L}_U \) be the restriction at \( \mathcal{D}(M) \) of the Lie derivative on \( \mathcal{E}^\infty(M) \) associated with \( U \) here denoted by \( \mathcal{L}^u \); so \( \mathcal{L}_U f = U f \) for every \( f \in \mathcal{D}(M) \) and in particular \( \mathcal{L}_U \in \text{DiffOp}^1(M) \). If \( V \in \mathfrak{X}(N), U \in \mathfrak{X}(M), \phi : N \to M \) is smooth and \( V \) and \( U \) are \( \phi \)-related, then \( \mathcal{L}_V \circ \phi^* = \phi^* \circ \mathcal{L}_U \). Whenever \( U \) is complete, we let \( \theta_U : \mathbb{R} \to \text{Diff}(M) \) be the flow on \( M \) generated by \( U \) and for every \( t \in \mathbb{R} \) let \( \eta^U_M(t) := (\theta^U_M(-t))^* \in \mathcal{L}(\mathcal{D}(M)) \) namely the map \( f \mapsto f \circ \theta^U_M \).

Let \( M = (M, g) \) be a semi-Riemannian manifold, thus for every \( f \in \mathcal{E}^\infty(M) \) let \( \text{grad}_g(f) \) be the gradient of \( f \) w.r.t. \( g \), thus \( \text{grad}_g(f) \in \mathfrak{X}(M) \) such that \( \langle \text{grad}_g(f), Y \rangle_M = \mathcal{L}_Y(f) \) for every \( Y \in \mathfrak{X}(M) \) where \( \langle \cdot, \cdot \rangle_M : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{E}^\infty(M) \) is the \( \mathcal{E}^\infty(M) \)-bilinear map corresponding to the metric \( g \). Let \( \mu_g \) denote the measure on \( M \) associated via [4 Thm. 4.7] with the density relative to \( g \) [8 Prp. 2.1.15(ii)], set \( \mathcal{H}_g := L^2(M, d\mu_g) \). We have \( \mathcal{K}(M) \hookrightarrow \mathcal{H}_g \) since for instance [8 Thm. 1.11(iv)], since \( \mu_g \) is continuous on \( \mathcal{K}(M) \) and since \( ||f^* f|| = ||f||^2 \) for every compact \( K \) and every \( f \in \mathcal{E}(X, K) \), being a \( C^* \)-algebra
the normed space $\mathcal{C}(X,K)$ of complex valued continuous maps on $X$ with support in $K$ endowed with the sup − norm. The inclusion $\mathcal{K}(M) \hookrightarrow \mathcal{H}_g$ is dense \[ \text{[1] Ch. 4, \$3, n. 4, Def. 2]} \] as well as the inclusion $\mathcal{D}(M) \hookrightarrow \mathcal{H}_g$. If $(N, g')$ is a semi-Riemannian manifold and $D \in \text{DiffOp}^\infty(M, N)$, then $D^\tau$ is well-set since $\mathcal{D}(M)$ is dense in $\mathcal{H}_g$, moreover by \[ \text{[8] Thm. 1.2.15]} \] we deduce that $\mathcal{D}(N) \subseteq \text{Dom}(D^\tau)$ and $D^\tau \mathcal{D}(N) \subseteq \mathcal{D}(M)$. If $\phi : N \rightarrow M$ is a smooth diffeomorphism such that $\phi^* g = g'$, thus $\phi^*$ (on $\mathcal{D}(M)$) extends to a unitary operator from $\mathcal{H}_g$ onto $\mathcal{H}_{g'}$, still denoted $\phi^*$ such that $(\phi^*)^\tau = (\phi^{-1})^*$. Here by a $C_0$-semigroup on a topological vector space $Y$ is meant a map $U \in \mathcal{C}(\mathbb{R}_+, \mathcal{L}_s(Y))$ such that $U(s + t) = U(s)U(t)$ for all $s, t \in \mathbb{R}_+$ and $U(0) = 1$. In addition by letting $\tau$ the topology of $Y$, $U$ is called $\tau$-equicontinuous or simply equicontinuous if $\{U(t) \mid t \in \mathbb{R}_+\}$ is a $(\tau, \tau)$-equicontinuous set. Similar definitions for a $C_0$-group by replacing $\mathbb{R}_+$ with $\mathbb{R}$.

If $X$ is a sequentially complete locally convex space with topology $\tau$ and $T \in \mathcal{L}(X)$ such that $\{T^n \mid n \in \mathbb{Z}_+\}$ is $(\tau, \tau)$-equicontinuous, then it is well-known that there exists a $C_0$-semigroup $\exp^T_X$ on $X$ such that:

1. $T$ is the infinitesimal $\tau$-generator of $\exp^T_X$;
2. $\exp^T_X(t)x = \sum_{k=0}^\infty \frac{(t\tau)^k}{k!}x$ convergence in $X$ for every $t \in \mathbb{R}_+$ and $x \in X$;
3. by letting $\overline{\exp^T_X} : \mathbb{R}_+ \ni t \mapsto \exp(-t)\exp^T_X(t)$ we have that $\overline{\exp^T_X}$ is an equicontinuous $C_0$-semigroup on $X$.

Since the equicontinuity hypothesis it is clear that the series in (2) extends to $t \in \mathbb{R}$, so $\exp^T_X$ extends to a $C_0$-group on $X$ still denoted by the same symbol, moreover $\overline{\exp^T_X}$ is an equicontinuous $C_0$-semigroup on $X$, where $\exp^T_X : \mathbb{R}_+ \ni t \mapsto \exp(-t)\exp^T_X(-t)$.

Introduction

In \[ \text{[7] Cor.1.6.43}] \] and the discussion after we have shown that the existence of a natural transformation, from the classical gravity species $\mathcal{A}$ to a strict quantum gravity species, satisfying certain constraints would render the dark energy hypothesis unnecessary in explaining the actual cosmic acceleration. This paper is one step toward a better understanding of the way to construct such a natural transformation.

In order to describe our results we need some additional terminology.

First of all we recall that an object of the category $\mathcal{DP}$ of dynamical patterns (\[ \text{[7] Cor. 1.4.5 and Def. 1.4.1}] \) is a functor of top-quasi enriched categories (i.e. a functor whose morphism map is continuous) valued in the top-quasi enriched category $\mathcal{ TSA}$ of unital topological $\ast$-algebras (enriched by endowing the morphism set of every two objects of $\mathcal{ TSA}$ with the topology of simple convergence). A morphism of dynamical patterns is a couple $(f, T)$ formed by a functor $f$ of top-quasi enriched categories from the domain of the second dynamical pattern to the domain of the first one, and by a natural transformation $T$ from the composition of the first dynamical pattern with $f$ to the second dynamical pattern.

Next let $\mathcal{CHDV}_0$ be the category introduced in Prp. \[ \text{[1.26]} \] and obtained by replacing in the defining properties of the category $\mathcal{CHDV}$ (\[ \text{[7] Cor. 1.4.18 and Def. 1.4.17}] \) both the
categories $\text{ptls}$ and $\text{ptsa}$ with the category of topological linear spaces $\text{tls}$. Similarly at $\Psi$ there exists the (canonical) functor $\Psi_0$ from $\mathcal{V}$ to $\mathcal{C}_{\mathcal{V}0}$. A 0-species is a functor valued in $\mathcal{C}_{\mathcal{V}0}$ which factorizes through $\mathcal{V}$ (Def. 1.27) in particular a 0-species is a 1-cell of the 2-category $\mathcal{V}$. A dynamical pattern is called quantum (respectively classical) if all its values are noncommutative (respectively commutative) algebras. A functor valued in $\mathcal{V}$ is called quantum (respectively classical) if all its values are quantum (respectively classical) dynamical patterns. Finally a 0-species is called quantum (respectively classical) if it factorizes through $\Psi_0$ into a quantum (respectively classical) functor valued in $\mathcal{V}$. Thus we have what follows.

In Thm. 2.2 and Thm. 2.4 we construct two functors valued in $\mathcal{V}$, the first $x$ classical and the second $z$ quantum.

Then in Thm. 3.10 we establish our main result: The existence of the natural transformation $\mathcal{Z}$ from the classical 0-species $x$ to the quantum 0-species $z$, where $x$ factorizes to the left and to the right through $x$ and $z$ factorizes to the left and to the right through $\mathcal{Z}$.

Now the following observation is worthwhile.

Since in the present paper we are decisively dealing with the categories $\mathcal{V}$ and $\mathcal{C}_{\mathcal{V}0}$, specifically with the construction of the functors $x$ and $z$ and the construction of the natural transformation $\mathcal{Z}$, statements concerning continuity acquire a distinctive value. Specifically we refer to:

1. The $C_0$-continuity of the semigroup $\Gamma_M^U$ (Thm. 1.24(2)) at the core of the object map of $\mathcal{Z}$.
2. The continuity of the $*$-morphism $T(\phi)$ (Thm. 1.15(2)) at the core of the morphism map of $\mathcal{Z}$.
3. The continuity of the map $f \mapsto \mathcal{E}_{\text{grad}_M(f)}$ (Cor. 3.6) at the core of $\mathcal{Z}$.

In the remaining of this introduction we shall briefly outline the main steps to arrive at our main result.

Thm. 1.15 and Thm. 1.24 are the main results of section 1. In Thm. 1.15(1) we prove that $\mathcal{B}(M)$ is a unital topological $*$-algebra and in Thm. 1.15(2) we prove that $\phi$ implements via $T$ a morphism of unital topological $*$-algebras. In Thm. 1.24(2) we establish the existence of $\Gamma_M^U$ a $C_0$-group on $\mathcal{B}(M)$ of $*$-automorphisms and in Thm. 1.24(3) we prove that $\Gamma$ and $T$ are equivariant namely (8) holds true. Here $M = (M, g)$ and $N = (N, g')$ are semi-Riemannian manifolds and $\phi : N \to M$ is a smooth diffeomorphism such that $\phi^* g = g'$. Our construction of $\mathcal{B}$ is calibrated to ensure that $T$ and $\Gamma$ possess the above properties.

We start by defining $\exp_M^U$ as the exponential $C_0$-group, on the sequentially complete locally convex space $\mathcal{D}(M)$ (remember $\mathcal{D}(M)$ is endowed with the inductive limit topology $\tau^\infty_\mathcal{U}(M)$), generated by $\mathcal{E}_U$ provided $\{\mathcal{E}_k\}_{k \in \mathbb{Z}}$ be $(\tau^\infty_\mathcal{U}, \tau^\infty_\mathcal{U})$-equicontinuous, and let $\Lambda_M^U$ be the corresponding action on $\mathcal{B}(\mathcal{D}(M))$ namely

$$\Lambda_M^U : t \mapsto (T \mapsto \exp_M^U(t) \circ T \circ \exp_M^U(-t)).$$
Next we define the set $\mathcal{B}(M)$ underlying $\mathfrak{B}(M)$ in Def. 1.7 as the subset of those linear and continuous operators on $\mathcal{D}(M)$ whose Hilbert space adjoint in $\mathcal{H}_g$ is such that its domain contains $\mathcal{D}(M)$, maps $\mathcal{D}(M)$ into itself and its restriction to $\mathcal{D}(M)$ is continuous:

$$\mathcal{B}(M) := \{ T \in \mathfrak{L}(\mathcal{D}(M)) \mid \mathcal{D}(M) \subseteq \text{Dom}(T^\dagger),$$

$$T^\dagger \mathcal{D}(M) \subseteq \mathcal{D}(M), T^\dagger := T^\dagger \upharpoonright \mathcal{D}(M) \in \mathfrak{L}(\mathcal{D}(M)) \};$$

where $T^\dagger$ is the $\mathcal{H}_g$-adjoint of the operator $T$. In Prp. 1.9 we show that $\mathcal{B}(M)$ is a $O^*$-algebra on $\mathcal{D}(M)$, $\mathcal{D}(M)$ seen in this context as a dense linear subspace of $\mathcal{H}_g$, in particular $\mathcal{B}(M)$ is a unital $*$-algebra. Then in Def. 1.11 we define a set of functionals $F_M$ over $\mathcal{B}(M)$ and prove in Prp. 1.12 that $F_M$ is a $\mathcal{B}(M)$-invariant non-empty subset of $\mathcal{F}_\text{max}$ (relative to the wedge of finite sums of positive elements of $\mathcal{B}(M)$). This result along with the general result [6, Lemma 1.5.7] applied to our $*$-algebra $\mathcal{B}(M)$ and our set $F_M$, enables us to show in Thm. 1.15(1) that

$$(2) \quad \mathfrak{B}(M) := \mathcal{B}(M)[\tau_M] \in \text{tsa};$$

where $\tau_M$ (Def. 1.13) is the locally convex topology on $\mathcal{B}(M)$ generated by the following set of seminorms

$$\{ \{ q^B \mid B \in \text{Bounded}(\mathcal{D}(M)) \};$$

$$\{ q^B : \mathcal{B}(M) \ni T \mapsto \sup_{f \in B} \langle f, Tf \rangle_{\mathcal{H}_g} \}. $$

There is another tsa-structure over $\mathcal{B}(M)$, indeed $\mathcal{B}(M)$ endowed with the topology relative to the bounded topology on $\mathfrak{L}(\mathcal{D}(M)_{B(\mathcal{M})}, \mathcal{D}(M)_{B(\mathcal{M})}^*)$ is a unital topological $*$-algebra and this topology is stronger than $\tau_M$ (Prp. 1.17). Next by letting

$$(3) \quad T(\phi)(T) := \phi^* \circ T \circ (\phi^{-1})^*;$$

we prove in Thm. 1.15(2) that

$$(4) \quad T(\phi) \in \text{Mor}_{\text{tsa}}(\mathfrak{B}(M), \mathfrak{B}(N)).$$

In Cor. 1.6 we prove that $\Lambda^U_M$ is a $C_0$-group on the locally convex space $\mathfrak{L}(\mathcal{D}(M))$ of continuous linear maps on $\mathcal{D}(M)$ endowed with the topology of uniform convergence over the bounded subsets of $\mathcal{D}(M)$ namely

$$(5) \quad \begin{cases} \Lambda^U_M \in \mathfrak{C}(\mathfrak{L}(\mathcal{D}(M))); \\ \Lambda^U_M \text{ is a one-parameter group.} \end{cases}$$

This result is a consequence of Lemma 1.5, a more general result important in its own, enlightening the twofold essential role played by the Montel space $\mathcal{D}(M)$ in obtaining Cor. 1.6 the first directly by its definition, the second permitting to use of the Banach-Steinhaus Thm. since any Montel space is barrelled.

In Def. 1.10 we define the category $\mathfrak{M}_0$ of the couples $(M, U)$ with the following properties: $M = (M, g)$ is a semi-Riemannian manifold, $U$ is a vector field of $M$ such that
\{L^\nu_n \mid n \in \mathbb{Z}_+^*\} is \((\tau^e, \tau^e)\)-equicontinuous, and the following property of invariance holds true

\begin{equation}
\mu_\sigma \circ E_U = 0.
\end{equation}

While \(\phi\) is a morphism from \((M, U)\) to \((N, V)\) iff \(\phi : N \to M\) is smooth, \(\phi^*g = g'\), and \(U\)
and \(V\) are \(\phi\)-related with \(N = (N, g')\). \(\mathcal{V}\) is the subcategory of \(\mathcal{V}_0\) with the same object set and diffeomorphisms as morphisms. Now the reason of introducing the above categories stands on Thm. [1.21][23] establishing that whenever \((M, U), (N, V) \in \mathcal{V}\) and \(\phi \in \text{Mor}_{\mathcal{V}}((M, U), (N, V))\), \(\Lambda^U_M\) restricts to a \(C_0\)-group \(\Gamma^U_M\) on \(\mathcal{B}(M)\) of \(*\)-automorphisms such that \(\mathcal{J}\) and \(\Gamma\) are equivariant, namely

\begin{equation}
\left\{
\begin{aligned}
\Gamma^U_M &\in \mathcal{C}(\mathbb{R}, \mathcal{L}_c(\mathcal{B}(M))), \\
\Gamma^U_M &\text{ is a one-parameter group of } \ast\text{-automorphisms;}
\end{aligned}
\right.
\end{equation}

and for every \(t \in \mathbb{R}\)

\begin{equation}
\mathcal{J}(\phi) \circ \Gamma^U_M(t) = \Gamma^\mathcal{J}_\mathcal{N}(t) \circ \mathcal{J}(\phi).
\end{equation}

Let us outline the essential steps yielding to (7). Firstly, in Cor. [1.19] we show that whenever \((M, U) \in \mathcal{V}_0\), the group \(\text{exp}^U_M\) extends to a unique \(C_0\)-group \(\text{exp}^U_M\) on \(H_\mathcal{N}\) of unitary operators whose infinitesimal generator extends \(E_U\).

It is worthwhile remarking that the unitary extension is essentially due to (6) (proof of Lemma [1.18]). Thus Cor. [1.19] ensures that \(\Lambda^U_M\) restricts to a \(C_0\)-group \(\Gamma^U_M\) on \(\mathcal{B}(M)\) of \(*\)-automorphisms (Cor. [1.20]).

Now in the fundamental Lemma [1.22] we prove that the topology \(\tau_M\) is generated by a collection of seminorms extending to \(\mathcal{L}_b(\mathcal{D}(M))\)-continuous seminorms. While \(\Gamma^U_M(t)\) is a continuous linear map on \(\mathcal{B}(M)\) since \(\text{exp}^U_M(t)\) maps bounded sets into bounded sets. Therefore (5) implies that \(\Gamma^U_M\) is a \(C_0\)-group on \(\mathcal{B}(M)\). We remark that in showing Lemma [1.22] the fact that \(\mathcal{D}(M)\) is barrelled is essential.

In addition to the above results, by an application of the Banach-Steinhaus Thm. and of the fact that \(\mathcal{D}(M)\) is specifically a Montel space, in Thm. [1.21][1] we prove that \(\text{exp}^U_M : \mathbb{R} \to \mathcal{U}(M)\) is a continuous morphism of groups, where \(\mathcal{U}(M)\) is the group of unitary elements of \(\mathcal{B}(M)\) endowed with the relative topology.

In conclusion of section 1 we determine in Prp. [1.26] the category \(\mathcal{C}_0\mathcal{D}_0\) and the functor \(\Psi_0\), while 0-species are introduced in Def. [1.27].

Thm. [2.2] and Thm. [2.4] are the main results of section 2, where by using Thm. [1.15] and Thm. [1.24], we construct two functors \(x\) and \(\mathcal{J}\) from the category \(\mathcal{V}\) to the category of dynamical patterns \(\mathcal{D}\), classical \(x\) and quantum \(\mathcal{J}\).

Let us delineate what above said for the more interesting quantum functor \(\mathcal{J}\), but first of all we outline the main structures involved.

For every \((M, U) \in \mathcal{V}\) let \((M, U)\) be the top-quasi enriched category of subsets of \(M\) such that for all \(X, Y \in (M, U)\) we have \(\text{Mor}_{(M, U)}(X, Y) = \{(X, Y) \times \text{mor}_{(M, U)}(X, Y)\} \times \mathcal{D}(M, Y)\) endowed with the topology inherited by \(\mathbb{R}\) where

\[\text{mor}_{(M, U)}(X, Y) = \{t \in \mathbb{R} \mid \text{exp}^U_M(t)\mathcal{D}(M, X) = \mathcal{D}(M, Y)\}\]
and \(\mathcal{D}(M, X)\) is the topological sub *-algebra of \(\mathcal{D}(M)\) of those maps whose support is contained in \(X\). Next let \(\mathcal{B}(M, X)\) be the topological unital sub *-algebra of \(\mathcal{B}(M)\) of those \(T\) such that \(T\mathcal{D}(M, X) \subseteq \mathcal{D}(M, X)\) and \(T^*\mathcal{D}(M, X) \subseteq \mathcal{D}(M, X)\). Thus we can define the maps \((\mathcal{F}_{(M, U) \circ})\) and \((\mathcal{F}_{(M, U) \circ})\) on the object and morphism set of \(\langle M, U \rangle\) respectively as

\[
\begin{align*}
(F_{(M, U) \circ}) : X &\mapsto \mathcal{B}(M, X), \\
(F_{(M, U) \circ}) : ((X, Y), t) &\mapsto \left(\mathcal{B}(M, X) \rightarrow \mathcal{B}(M, Y), \quad T \mapsto \Gamma^U_M (t) T\right).
\end{align*}
\]

While for every \((M, U), (N, V) \in \mathcal{V}\) and \(\phi \in \text{Mor}_\mathcal{V}((M, U), (N, V))\) we can set the maps \(f_o^\phi\) and \(f_m^\phi\) over the object and the morphism set of \(\langle N, V \rangle\) respectively such that

\[
\begin{align*}
f_o^\phi : Y &\mapsto \phi(Y); \\
f_m^\phi : (Y, Z, s) &\mapsto ((\phi(Y), \phi(Z)), s).
\end{align*}
\]

and define the map \(\mathcal{T}\) over the morphism set of \(\mathcal{V}\) such that

\[
\mathcal{T}^\phi : Y \mapsto \left(\mathcal{B}(M, \phi(Y)) \rightarrow \mathcal{B}(N, Y), \quad T \mapsto \mathcal{T}(\phi) T\right).
\]

Thus we are able to define \(3\) on the category \(\mathcal{V}\) such that

\[
\begin{align*}
3_o : (M, U) &\mapsto \langle \langle M, U \rangle, \mathcal{F}_{(M, U) \circ} \rangle; \\
3_m : \phi &\mapsto (f_o^\phi, \mathcal{T}^\phi).
\end{align*}
\]

Now we have to see that effectively

\(3 \in \text{Fct}(\mathcal{V}, \mathcal{V})\).

What happens is that \(\mathcal{7}\) is the core of the proof that the object map \(3_o\) is well-set namely

\(\mathcal{F}_{(M, U) \circ} \in \text{Fct}_{\text{top}}(\langle M, U \rangle, \text{tsa})\); and

Lemma \(\mathcal{1.4}\) implies that the first component of the morphism map \(3_m\) is well-set, namely

\(f_o^\phi \in \text{Fct}_{\text{top}}(\langle N, V \rangle, \langle M, U \rangle)\); that \(\mathcal{4}\) and the equivariance \(\mathcal{8}\) are the core of the proof that also the second component of \(3_m\) is well-set, namely

\(\mathcal{T}^\phi \in \text{Mor}_{\text{Fct}_{\langle N, V \rangle}, \text{tsa}}(\mathcal{F}_{(M, U) \circ} \circ f_o^\phi, \mathcal{F}_{(N, V) \circ})\); finally that \(\mathcal{T}(\phi \circ \psi) = \mathcal{T}(\psi) \circ \mathcal{T}(\phi)\) implies that \(3_m\) preserves the morphism composition, and \(\mathcal{9}\) follows.

About the classical functor \(x\) the main novelties and advantages with respect to the functor \(a\) constructed in \(\mathcal{7}\) Thm. 1.6.24 are represented by two facts: Firstly in order to construct a group associated with a vector field \(U\), here \(U\) needs not to be complete, rather we require \(\{L_k^U | k \in \mathbb{Z}_+\}\) to be \((\tau_c^\infty, \tau_c^\infty)\)-equicontinuous, by obtaining in this way the additional \(C_0\)-property of \(\exp^U\). Secondly here we select a specific topology on \(\mathcal{D}(M, X)\), then by force on its unitization \(\mathcal{D}_1(M, X)\), by de facto avoiding the problem of introducing what in \(\mathcal{7}\) Def. 1.6.18] we called a \(\mathcal{V}\)-topology. This because for what just above said
and since $\phi^*$ is $(\tau_\ast^\infty, \tau^\infty)$-continuous, the $\tau^\infty$-topology satisfies the requirements of a $\text{vf}$-topology provided $\eta_{M}^{U}$, the adjoint on $\mathcal{D}(M)$ of the flow on $M$ generated by a complete vector field $U$ of $M$, be replaced by the group $\exp_{M}^{U}$. Said that the construction of $x$ mimics the one of $a$, by replacing $\eta_{M}^{U}$ with $\exp_{M}^{U}$. At the end of this Introduction we shall see that in special cases $\exp_{M}^{U}$ equals $\eta_{M}^{U}$.

**Thm. 3.10** establishes the main result of section 3 and of the entire work, namely the existence of the natural transformation

$$
\exists \in \text{Mor}_{\mathcal{C}t(\mathfrak{Bf}, \mathfrak{Bbb})}(X, Z);
$$

between the classical 0-species $X := \Psi_0 \circ x \circ \tau_{\psi}^{0}$ and the quantum 0-species $Z := \Psi_0 \circ \exists \circ \tau_{\psi}^{0}$, uniquely determined by

$$
\exists : \text{Obj}(\mathfrak{Bf}) \ni (M, U) \mapsto \exists(M, U) = (1_{(M, U)}, \mathfrak{g}^t_{(M, U)}, \mathfrak{g}_{(M, U)});
$$

$$
\mathfrak{g}_{(M, U)} : \text{Obj}((M, U)) \ni X \mapsto \mathfrak{g}^X_{(M, U)};
$$

$$
\mathfrak{g}^t_{(M, U)} : \text{Obj}((M, U)) \ni X \mapsto (\mathfrak{g}^X_{(M, U)})^t;
$$

where

$$
\mathfrak{g}^X_{(M, U)} : \mathcal{D}(M, X) \to \mathfrak{B}(M, X)
$$

$$(f, \lambda) \mapsto \mathcal{L}_{[\mathfrak{g}_{\mathfrak{f}}(f), U]} + \lambda 1;
$$

and where $\mathfrak{Bf}$ is the full subcategory of $\mathfrak{vf}$ of those $(M, U)$ for which there exists a frame $\{E_i\}$ of orthonormal fields of $M$ such that

$$
(\forall i \in [1, \text{dim}M] \cap Z)((U, E_i) = 0).
$$

Three are the fundamental steps to establish (11).

First of all Cor. 3.6 by stating that

$$
\left( f \mapsto \mathcal{L}_{[\mathfrak{g}_{\mathfrak{f}}(f)]} \right) \in \mathcal{U}(\mathcal{D}(M, X), \mathfrak{B}(M, X));
$$

ensures that $\mathfrak{g}_{(M, U)}(X)$ is a continuous linear map.

Then what right now stated and Thm. 3.1 by establishing that

$$
\exp_{M}^{U} \circ \mathcal{L}_{[\mathfrak{g}_{\mathfrak{f}}(f), U]} = \mathcal{L}_{[(\mathfrak{g}_{\mathfrak{f}} \circ \exp_{M}^{U}(f), U)]} \circ \exp_{M}^{U}(f);
$$

ensure that

$$
\exists(M, U) \in \text{Mor}_{\mathcal{C}t((M, U), \mathfrak{Bbb})}(\mathfrak{a}(M, U), \mathfrak{z}(M, U));
$$

which together its adjoint imply

$$
\exists(M, U) \in \text{Mor}_{\mathcal{C}t(\mathfrak{Bf}, \mathfrak{Bbb})}(X(M, U), Z(M, U)).
$$

It is in order to determine (13) that we require the use of the category $\mathfrak{Bf}$ rather than $\mathfrak{vf}$. Specifically hypothesis (12) ensures that the following term

$$
\sum_{i=1}^{n} \varepsilon_i \cdot \mathcal{L}_{E_i}(f) \mathcal{E}_{[E_i, U]}
$$

in the right side of (24) vanishes.
Finally Lemma 3.3 states that
\[ \phi^* \circ \mathcal{L}_{\text{grad}_M(f)} = \mathcal{L}_{(\text{grad}_M \circ \phi^*)} \circ \phi^*. \]
which together (14) represent the core of the proof of the commutativity of the following diagram in the category \( C \).

\[ \begin{array}{ccc}
\mathfrak{x}(N, V) & \xrightarrow{\alpha(N, V)} & \mathfrak{z}(N, V) \\
\mathfrak{x}(\phi) & \downarrow & \mathfrak{z}(\phi) \\
\mathfrak{x}(M, U) & \xrightarrow{\alpha(M, U)} & \mathfrak{z}(M, U)
\end{array} \]

and (11) follows.

Cor. 4.6 is the main result of the closing section 4 where under the hypothesis that \( U \) is complete and an additional equicontinuity condition on \( \mathcal{L}_U \), we answer in Cor. 4.6(2) the natural question in the affirmative on whether \( \exp^U \) equals the adjoint action on \( \mathcal{D}(M) \) of the flow on \( M \) generated by \( U \). In the same section we also prove in Lemma 4.2 that under the obvious additional equicontinuity request over \( \mathcal{L}_U \) the Lie derivative of \( U \) on \( \mathcal{C}^\infty(M) \), the exponential one-parameter group \( \text{Exp}^U \) generated by \( \mathcal{L}_U \) extends \( \exp^U \). As a result in Prp. 4.1(2) we obtain that \( \Lambda^U_M(t) \) restricts to a morphism \( \mathfrak{A} \to \mathfrak{A}_t \) of left \( \mathcal{C}^\infty(M) \)-modules where \( \mathfrak{A} \subset \mathcal{U}(\mathcal{D}(M)) \) is naturally a left \( \mathcal{C}^\infty(M) \)-module such that \( \Lambda^U_M(t) \mathfrak{A} \subseteq \mathfrak{A} \) while \( \mathfrak{A}_t \) is the left \( \mathcal{C}^\infty(M) \)-module whose underlying group is \( \mathfrak{A} \) and external law is given by \( F \cdot Q \mapsto \text{Exp}^U_M(t)(F) \cdot Q. \)

1. Construction of the topological \( * \)-algebra \( \mathcal{B}(M) \) and the \( C_0 \)-group \( \Gamma^U_M \)

Definition 1.1. Define \( \text{vf}^* \) to be the category such that its object set is the set of couples \( (M, U) \) where \( M \) is a manifold and \( U \) is a vector field on \( M \) such that \( \{ \mathcal{L}_U^n \ | \ n \in \mathbb{Z}_+ \} \) is \( (\tau^\infty_c, \tau^\infty_c) \)-equicontinuous. For every \( (M, U), (N, V) \in \text{vf}^* \), \( \text{Mor}_{\text{vf}^*}( (M, U), (N, V)) \) is the set of proper smooth maps \( \phi : N \to M \) so that \( U \) and \( V \) are \( \phi \)-related, while for every \( (Q, K) \in \text{vf}^* \) and \( \psi \in \text{Mor}_{\text{vf}^*}( (N, V), (Q, K)) \) we set \( \psi \circ_{\text{vf}^*} \phi := \phi \circ \psi. \)

Since \( \mathcal{D}(M) \) is sequentially complete we can set the following

Definition 1.2. Let \( (M, U) \in \text{vf}^* \) define
\[ \exp^U_M := \exp^\mathcal{L}_M \]
set \( \exp^\mathcal{L}_M : \mathbb{R}_+ \ni t \mapsto \exp(-t) \exp^U_M(t) \) and \( \exp^U : \mathbb{R}_+ \ni t \mapsto \exp(-t) \exp^U_M(-t) \). Moreover define \( \Lambda^U_M : \mathbb{R} \to \text{End}_{\text{vf}^*}(\mathcal{U}(\mathcal{D}(M))) \) such that
\[ \Lambda^U_M : t \mapsto \exp^U_M(t) \circ T \circ \exp^\mathcal{L}_M(-t). \]
Remark 1.3. Let $(M, U) \in \mathfrak{vf}^*$. Thus for every constant map $c$ on $M$ and $f \in \mathcal{D}(M)$ we have $\exp^U_M(t)(c \cdot f) = c \cdot \exp^U_M(t)(f)$, since $\mathcal{L}_U(c \cdot f) = c \cdot \mathcal{L}_U(f)$ being $\mathcal{L}_U(c) = 0$, by $\mathcal{D}(M) \leftrightarrow \mathcal{C}^\infty(M)$ and since $\mathcal{C}^\infty(M)$ is a topological algebra. Moreover $\exp^U_M$ is a group of $*$-automorphisms of $\mathcal{D}(M)$ indeed its infinitesimal $\tau^\infty_c$-generator $\mathcal{L}_U$ is a $*$-preserving derivation on $\mathcal{D}(M)$ then the statement follows.

Lemma 1.4. Let $(M, U), (N, V) \in \mathfrak{vf}$ and $\phi \in \text{Mor}_{\mathfrak{vf}^*}((M, U), (N, V))$ thus $\phi^* \circ \exp^U_M(t) = \exp^V_N(t) \circ \phi^*$ and $\mathcal{L}_U \circ \exp^U_M(t) = \exp^U_M(t) \circ \mathcal{L}_U$ for every $t \in \mathbb{R}$.

Proof. Since $U$ and $V$ are $\phi$-related we have that $\phi^* \circ \mathcal{L}_U = \mathcal{L}_V \circ \phi^*$, thus the first equality follows since $\phi^*$ is $\tau^\infty_c$-continuous, the second equality follows since $\mathcal{L}_U$ is $\tau^\infty_c$-continuous. □

Lemma 1.5. Let $X$ be a Montel space, $Y$ a topological space, $U, V : Y \to \mathcal{U}_c(Y)$ continuous at $t_0 \in Y$ and such that $\{U(t) \mid t \in Y\}$ is equicontinuous. If the neighbourhood filter of $t_0$ in $Y$ admits a countable basis, then $Z^{U,V} : t \mapsto (T \mapsto U(t) \circ T \circ V(t))$.

Proof. In this proof we let $Z$ denote $Z^{U,V}$ which is well-defined namely $Z(t) \in \mathcal{U}_c(\mathcal{U}_b(X))$ for every $t \in Y$. Indeed let $p$ be a continuous seminorm on $X$, $B$ a bounded subset of $X$ and $T \in \mathcal{U}_c(X)$, thus $p_p(Z(t)T) = (q^I_cT)$ with $q^I = p \circ U(t)$ and $C^I = V(t)B$. But $q^I$ is a continuous seminorm of $X$ while $C^I$ is bounded in $X$ since $V(t)$ is linear and continuous, thus $q^I_c$ is a continuous seminorm of $\mathcal{U}_b(X)$ and then $Z(t) \in \mathcal{U}_c(\mathcal{U}_b(X))$. Next assume that the neighbourhood filter of $t_0$ in $Y$ admits a countable basis. Now $X$ is a Montel space thus it is sufficient to prove that for every sequence $\{t_n\}$ in $Y$ converging at $t_0$ and every $T \in \mathcal{U}_c(X)$, we have that $Z_{t_n}T$ converges at $T$ in $\mathcal{U}_c(\mathcal{U}_b(X))$. Now since the equicontinuity hypothesis, there exists a continuous seminorm $q$ of $X$ such that for all $x \in X$ and $n \in \mathbb{Z}^*_+$ we have

$$p(Z_{t_n}(T)x - Z_{t_0}(T)x) \leq p(U_{t_n}(TV_{t_n}x - TV_{t_0}x)) + p((U_{t_n} - U_{t_0})TV_{t_0}x)$$

so $p(Z_{t_n}(T)x - Z_{t_0}(T)x)$ converges at 0, but $p$ is an arbitrary continuous seminorm on $X$, thus $Z_{t_n}T$ converges at $Z_{t_0}T$ in $\mathcal{U}_c(X)$. Finally $X$ is barrelled being Montel, thus by the Banach-Steinhaus Thm. we deduce that $Z_{t_n}T$ converges at $Z_{t_0}T$ in $\mathcal{U}_c(\mathcal{U}_b(X))$ which is what we claimed to prove. □

Corollary 1.6. Let $(M, U) \in \mathfrak{vf}^*$, thus $\Lambda^U_M$ is a $C_0$-group on $\mathcal{U}_b(\mathcal{D}(M))$.

Proof. By letting $U_+ = \overline{\exp^U_M} : \mathbb{R}_+ \ni t \mapsto \exp(t) \exp^U_M(-t)$, $U_- = \exp^U_M$ and $V_- : \mathbb{R}_+ \ni t \mapsto \exp(t) \exp^U_M(t)$ the statement follows by Lemma 1.5 and since $\Lambda^U_M \upharpoonright \mathbb{R}_+ = Z^{U_+V_-}$ and $\Lambda^U_M \upharpoonright \mathbb{R}_- = Z^{U_-V_+}$. □

Since $\mathcal{D}(M)$ is dense in $\mathcal{H}_g$ we can set the following

Definition 1.7 (The set $\mathcal{B}(M)$). Let $M = (M, g)$ be a semi-Riemannian manifold, define $\mathcal{B}(M) := \{T \in \mathcal{U}(\mathcal{D}(M)) \mid \mathcal{D}(M) \subseteq \text{Dom}(T^*), T^* \mathcal{D}(M) \subseteq \mathcal{D}(M), T^* := T^* \upharpoonright \mathcal{D}(M) \in \mathcal{U}(\mathcal{D}(M))\}$. 

where $T^\ast$ is the $H_g$-adjoint of the operator $T$.

**Remark 1.8.** Let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold, thus since the discussion in Notation we deduce that $\text{DiffOp}^k(M) \subset B(M)$ for every $k \in \mathbb{Z}^+$. 

**Proposition 1.9** ($B(M)$ is a $O^*$-algebra on $D(M)$). Let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold, thus $B(M)$ is a $O^*$-algebra on $D(M)$ in particular it is a unital $*$-algebra with involution $(\cdot)^\dagger$.

**Proof.** Let $S, T \in B(M)$, thus $T$ is closable since $T^\ast$ is densely defined. Since $(S + T)^\ast \supseteq S^\ast + T^\ast$ we obtain $D(M) \subseteq \text{Dom}((S + T)^\ast)$ and $(S + T)^\dagger = S^\dagger + T^\dagger \in \mathcal{L}(D(M))$. Next since $(ST)^\ast \supseteq S^\ast T^\ast$ we obtain $D(M) \subseteq \text{Dom}((ST)^\ast)$ and $(ST)^\dagger = T^\dagger S^\dagger \in \mathcal{L}(D(M))$. Finally $\overline{T} = (T^\ast)^\dagger \subseteq (T^\ast)^\ast$ since $T^\ast \subseteq T^\dagger$, so $D(M) \subseteq \text{Dom}((T^\ast)^\dagger)$ and $(T^\ast)^\dagger = T \in \mathcal{L}(D(M))$. □

**Definition 1.10** (The Categories $vf_0$ and $vf$). Define $vf_0$ to be the unique category whose object set consists of the couples $(\mathcal{M}, U)$ where $\mathcal{M} = (M, g)$ is a semi-Riemannian manifold, $(M, U) \in vf^*$ and

$$\mu_g \circ \mathcal{E}_U = 0.$$ 

The morphism set of $vf_0$ is such that $\text{Mor}_{vf_0}((\mathcal{M}, U), (N, V))$ consists of those $\phi \in \text{Mor}_{vf}(\mathcal{M}, U), (N, V))$ such that $\phi^* g = g^\prime$ where $N = (N, g^\prime)$, and whose composition is given by $\phi \circ_{vf_0} \psi = \psi \circ_{vf} \phi$ with $\circ$ the map composition. Let $vf$ be the subcategory of $vf_0$ with the same object set and $\text{Mor}_{vf}(\mathcal{M}, U), (N, V))$ consisting of those $\phi \in \text{Mor}_{vf_0}(\mathcal{M}, U), (N, V))$ such that $\phi$ is a diffeomorphism.

**Definition 1.11.** Let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold, define

$$\begin{cases} \omega_f : B(M) \rightarrow C, T \mapsto \langle f, Tf \rangle_{\mathcal{H}_g}, f \in D(M); \\ \omega_f^Q : B(M) \rightarrow C, T \mapsto \omega_f(Q^\dagger TQ), f \in D(M), Q \in B(M); \\ F_M := \{\omega_f | f \in B) | B \in \text{Bounded}(D(M)) \}. \\
\end{cases}$$

**Proposition 1.12.** Let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold, $B \in \text{Bounded}(D(M))$, $T, Q \in B(M)$. Thus $\|\omega_f(T)\| f \in B$ is bounded and $\{\omega_f^Q | f \in B \} \in F_M$.

**Proof.** $T(B)$ is $\tau^\infty_\omega$-bounded since $T$ is $(\tau^\infty_\omega, \tau^\infty_\omega^\prime)$-continuous, thus $B$ and $T(B)$ are $\| \cdot \|_{\mathcal{H}_g}$-bounded since $D(M) \rightarrow \mathcal{H}_g$, so the first part of the statement follows since $\sup_{f \in B} \langle f, Tf \rangle_{\mathcal{H}_g} \leq \sup_{f \in B} \|f\|_{\mathcal{H}_g} \sup_{f \in B} \|Tf\|_{\mathcal{H}_g}$. The second part follows since $\omega_f^Q = \omega_Q f$ and $Q(B)$ is $\tau^\infty_\omega$-bounded. □

The first part of Prp. [L.T2] allows us to give the following

**Definition 1.13** (The Topology $\tau_\mathcal{M}$ on $B(M)$). Let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold, define $\tau_\mathcal{M}$ to be the locally convex topology on $B(M)$ generated by the set of seminorms $\{q^B | B \in \text{Bounded}(D(M)) \}$ where $q^B : B(M) \rightarrow \mathbb{R}$. $T \mapsto \sup_{f \in B} |\omega_f(T)|$.

**Remark 1.14.** Let $\mathcal{M} = (M, g)$ be a semi-Riemannian manifold, thus by the polarization formula and since for any locally convex space $X$, $\lambda, \mu \in C$ and $B, C$ bounded subsets of $X$, the set $\lambda B + \mu C$ is bounded in $X$ we deduce that the topology $\tau_\mathcal{M}$ is
generated by the following set of seminorms \( \{q_{B,C} \mid B, C \in \text{Bounded}(D(M))\} \), where \( q_{B,C}(T) := \sup_{(f,g)\in B \times C} |\langle f, Tg \rangle_{\gamma_G}| \).

**Theorem 1.15** (The topological \(*\)-algebra \( \mathfrak{B}(M) \) and the morphism \( \mathcal{T}(\phi) \)). Let \( M = (M, g) \) be a semi-Riemannian manifold, thus

1. \( \mathfrak{B}(M)[\tau_M] \in \text{tsa} \)
2. if \( N = (N, g') \) is a semi-Riemannian manifold and \( \phi : N \to M \) is a smooth diffeomorphism such that \( \phi^* g = g' \), then by letting

\[
\begin{align*}
\mathcal{T}(\phi) &: \mathfrak{B}(M) \to \mathfrak{B}(N), \\
T &\mapsto \phi^* \circ T \circ (\phi^{-1})^*;
\end{align*}
\]

we have that

\( \mathcal{T}(\phi) \in \text{Mor}_{\text{tsa}}(\mathfrak{B}(M)[\tau_M], \mathfrak{B}(N)[\tau_N]) \).

**Proof.** St. (1) follows by Prp. 1.12 and [6] Lemma 1.5.7 applied to the unital \(*\)-algebra \( \mathfrak{B}(M) \) and the set \( F_M \). Next let \( T \in \mathfrak{B}(M) \) thus we have what follows. \( \mathcal{T}(\phi)(T) \in \mathcal{L}(D(N)) \) since \( \phi^*, T \) and \( (\phi^{-1})^* \) are all linear and \( \tau_c^\infty \)-continuous operators. Next by recalling the property \( (\phi^*)^* = (\phi^{-1})^* \) discussed in Notation, we have

\[
\mathcal{T}(\phi)(T)^* = ((\phi^{-1})^*)^* \circ T^* \circ (\phi^*)^* = \phi^* \circ T^* \circ (\phi^{-1})^*.
\]

Thus \( D(N) \subseteq \text{Dom}(\mathcal{T}(\phi)(T)) \), \( \mathcal{T}(\phi)(T)^* D(N) \subseteq D(N) \) and \( \mathcal{T}(\phi)(T)^* = \mathcal{T}(\phi)(T)^* \in \mathcal{L}(D(N)) \). Therefore \( \mathcal{T}(\phi) \) is well-set namely \( \mathcal{T}(\phi)(T) \in \mathfrak{B}(N) \), moreover \( \mathcal{T}(\phi) \) is a \(*\)-morphism. Finally let us prove the continuity of \( \mathcal{T}(\phi) \). For every \( f \in D(N) \) we have that

\[
\langle f, \mathcal{T}(\phi)(T)f \rangle_{\gamma_G} = \langle (\phi^*)^* f, T((\phi^{-1})^* f) \rangle_{\gamma_G} = \langle (\phi^{-1})^* f, T((\phi^{-1})^* f) \rangle_{\gamma_G};
\]

but \( (\phi^{-1})^* \) is \( \tau_c^\infty (N), \tau_c^\infty (M) \)-continuous, therefore \( (\phi^{-1})^* (B) \) is \( \tau_c^\infty (M) \)-bounded for every \( \tau_c^\infty (N) \)-bounded set \( B \) hence \( \mathcal{T}(\phi) \) is \( (\tau_M, \tau_N) \)-continuous and st. (2) follows. \( \square \)

The above result justifies the following

**Definition 1.16.** Let \( M = (M, g) \) be a semi-Riemannian manifold, define \( \mathfrak{B}(M) \) to be the unital topological \(*\)-algebra \( \mathfrak{B}(M)[\tau_M] \).

**Proposition 1.17.** Let \( M = (M, g) \) be a semi-Riemannian manifold, thus \( \mathfrak{B}(M) \subset \mathcal{L}(D(M)_{\mathfrak{B}(M)}, D(M)_{\mathfrak{B}(M)}^+) \) and \( \mathfrak{B}(M)[\tau_B] \) is a unital topological \(*\)-algebra such that \( \mathfrak{B}(M)[\tau_B] \hookrightarrow \mathfrak{B}(M) \).

**Proof.** Let \( T \in \mathfrak{B}(M) \), thus \( (f, h) \mapsto \langle Tf, h \rangle_{\gamma_G} \) is clearly jointly continuous w.r.t. the graph topology of \( \mathfrak{B}(M) \) on \( D(M) \) so the inclusion in the statement follows, in particular \( \mathfrak{B}(M)[\tau_B] \) is well-set and it is a topological \(*\)-algebra since [6] Prp. 3.3.10. Next it is well-known that the graph topology of a \( O^\ast \)-algebra \( A \) on a dense subspace \( D \) of a
Hilbert space $\mathcal{H}$ is the weakest among all the locally convex topologies $\tau$ on $D$ such that $A \subseteq \mathcal{L}(D[\tau],\mathcal{H})$. But $\mathcal{B}(M) \subseteq \mathcal{L}(D(M),\mathcal{H}_g)$, therefore $D(M) \hookrightarrow D(M)_{\mathcal{B}(M)}$. In particular $\text{Bounded}(D(M)) \subseteq \text{Bounded}(D(M)_{\mathcal{B}(M)})$ which implies $\mathcal{B}(M)[\tau_{1b}] \hookrightarrow \mathcal{B}(M)$. \hfill \Box

**Lemma 1.18.** Let $(M, U) \in \mathcal{V}_0$ with $M = (M, g)$ and $t \in \mathbb{R}$, then $\exp^U_M(t)$ extends uniquely to a unitary operator $\exp^U_M(t)$ on $\mathcal{H}_g$ such that $\exp^U_M(t)^\dagger = \exp^U_M(-t)$.

**Proof.** $D(M) \hookrightarrow \mathcal{H}(M)$ and $\mu_g \cdot \mathcal{L}_U = 0$ imply that $\mu_g \circ \exp^U_M(t) = \mu_g$ for all $t \in \mathbb{R}$, therefore for every $f, h \in (D(M)$ we have

$$\langle \exp^U_M(t)f, \exp^U_M(t)h \rangle_{\mathcal{H}_g} = \mu_g(\exp^U_M(t)f \exp^U_M(t)h)
\begin{align*}
&= \left(\mu_g \circ \exp^U_M(t)\right)(fh) \\
&= \mu_g(fh) \\
&= \langle f, h \rangle_{\mathcal{H}_g},
\end{align*}
$$

(15)

where the second equality follows since Rmk. 1.3. Thus $\exp^U_M(t)$ is a unitary operator on the dense subspace $D(M)$ of $\mathcal{H}_g$, which then extends uniquely to a unitary operator $\exp^U_M(t)$ on $\mathcal{H}_g$. Next since $\exp^U_M(t)$ is unitary and since $\exp^U_M(t)^{-1} = \exp^U_M(-t)$, we have that $\exp^U_M(t)^\dagger \upharpoonright D(M) = \exp^U_M(-t)^\dagger \upharpoonright D(M)$ and the equality in the statement follows. \hfill \Box

**Corollary 1.19.** Let $(M, U) \in \mathcal{V}_0$ with $M = (M, g)$, then there exists a unique $C_0$-group $\exp^U_M$ on $\mathcal{H}_g$ of unitary operators extending $\exp^U_M$ and whose infinitesimal generator $\mathcal{L}_U$ extends $\mathcal{L}_U$.

**Proof.** Since Lemma 1.18 it remains only to prove the $C_0$-property and $\mathcal{L}_U \supseteq \mathcal{L}_U$. To this end let $f \in D(M)$, then $t \mapsto \exp^U_M(t)f$ is $\| \cdot \|_{\mathcal{H}_g}$-continuous since $t \mapsto \exp^U_M(t)f$ is $\tau^\infty_\mathcal{L}$-continuous by construction and since $D(M) \hookrightarrow \mathcal{H}_g$. Next $\exp^U_M(\mathbb{R})$ is $\| \cdot \|_{\mathcal{H}_g}$-continuous since isometric, while $D(M)$ is dense in $\mathcal{H}_g$. Therefore since over equicontinuous sets the uniform structure of simple convergence coincides with the uniform structure of simple convergence over a total set, we conclude that for every $h \in \mathcal{H}_g$ the map $t \mapsto \exp^U_M(t)h$ is $\| \cdot \|_{\mathcal{H}_g}$-continuous namely $\exp^U_M$ is a $C_0$-group on $\mathcal{H}_g$. Finally $\mathcal{L}_U$ extends the infinitesimal $\tau^\infty_\mathcal{L}$-generator $\mathcal{L}_U$ of $\exp^U_M$ since $\exp^U_M$ extends $\exp^U_M$ and since $D(M) \hookrightarrow \mathcal{H}_g$. \hfill \Box

**Corollary 1.20.** Let $(M, U) \in \mathcal{V}_0$ with $M = (M, g)$ and $t \in \mathbb{R}$, then $\exp^U_M(t) \in \mathcal{B}(M)$ such that $\exp^U_M(t)^\dagger = \exp^U_M(-t)$, in particular $\Lambda^U_M(t) \mathcal{B}(M) \subseteq \mathcal{B}(M)$.

**Proof.** Since Cor. 1.19. \hfill \Box

Cor. 1.20 allows to give the following

**Definition 1.21 (The Group $\Gamma^U_M$).** Let $(M, U) \in \mathcal{V}_0$, define $\Gamma^U_M : \mathbb{R} \to \text{End}_{\text{ct}}(\mathcal{B}(M))$ such that

$$\Gamma^U_M : t \mapsto (T \mapsto \Lambda^U_M(t)(T)),$$

where $M$ is the manifold underlying $\mathcal{M}$. 
Lemma 1.22. Let $M$ be a semi-Riemannian manifold, thus the topology $\tau_M$ is generated by a collection of seminorms extending to $\mathcal{U}_b(D(M))$-continuous seminorms.

Proof. Let $B$ and $C$ be bounded subsets of $D(M)$ and let $\mathcal{B}^B_{{\mathcal{D}(M)}} \to \mathbb{R}$, $h \mapsto \sup_{f \in B} |\langle f, h \rangle|_{\mathcal{N}_B}$ finite since $D(M) \hookrightarrow \mathcal{K}_g$. Now since $D(M) \hookrightarrow \mathcal{K}(M)$ and $h \cdot \mu_g \in \mathcal{K}(M)'$ for every $h \in \mathcal{K}(M)$ we have $\langle f, \cdot \rangle_{\mathcal{N}_B} \circ i_{D(M)} = \langle f \cdot \mu_g \rangle_{D(M)} \circ i_{D(M)} \in \mathcal{D}(M)'$. Therefore $\mathcal{B}^B_{{\mathcal{D}(M)}}$ is lower $\tau$-continuous. So the statement follows by the fact that $(\mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M))$ so the statement follows by the fact that $(\mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M))$ is barrelled since $\mathcal{D}(M)$ is barrelled it follows by the Banach-Steinhaus Thm. that $\exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M) \circ (\mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M))$ is $\tau$-continuous since Cor. 1.20, we have $\mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M) = \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M) \circ (\mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M))$ is barrelled it follows by the Banach-Steinhaus Thm. that $\exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M) \circ (\mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M))$ is $\tau$-continuous since Cor. 1.20, so st. (1) follows by Lemma 1.22. Next let $B$ be a $\tau_c^\infty$-bounded set and $t \in \mathbb{R}$, thus since $\exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M) = \exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M)$ by Cor. 1.20 we have $q^B \circ \Gamma_{\mathcal{B}^B_{{\mathcal{D}(M)}}} = \exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M)$. Thus $\Gamma_{\mathcal{B}^B_{{\mathcal{D}(M)}}} = \exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M)$ is $\tau_c^\infty$-continuous, therefore $\Gamma_{\mathcal{B}^B_{{\mathcal{D}(M)}}} = \exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M)$ is $\tau_c^\infty$-continuous since Cor. 1.6 and Lemma 1.22. Finally $\Gamma_{\mathcal{B}^B_{{\mathcal{D}(M)}}} = \exp \mathcal{B}^B_{{\mathcal{D}(M)}} \circ_\mathcal{D}(M)$ is $\tau_c^\infty$-continuous since Cor. 1.20 so st. (2) is proven. St. (3) follows since Lemma 1.4.

Next we set the following definition of $\dagger$ here used instead of the analog one given in [7, Def. 1.4.14]. Clearly the corresponding of [7, Cor. 1.4.16] holds.

Definition 1.25. Let $D$ be a category, $a, b \in \text{Fct}(D, \text{tls})$ and $T \in \prod_{d \in D} \text{Mor}_{\text{tls}}(a(d), b(d))$, then define $T^* \in \prod_{d \in D} \text{Mor}_{\text{tls}}(b(d)', \alpha(d)')$ such that $T^*(e) := (T(e))^*$ for all $e \in D$, where $S^*(\alpha) := \alpha^* \circ S$.

We conclude this section with the existence of the category $\mathcal{C}^\dagger_{\text{tls}}$ uniquely determined in [7, Cor. 1.4.18] the category pts into the category tls and the category ptsa into the category tls. Then we obtain the corresponding functor $\Psi_{\dagger}$ in analogy with the functor $\Psi$ in [7, Thm. 1.4.19] Before the next result let us recall that for any $\alpha \in \text{tls}$, $\alpha^*$ is defined in [7, Def. 1.4.13] and that since $\Psi^* \circ \alpha^* = \alpha^* \circ \Psi^*$ is well-set.
Proposition 1.26 (The category $\mathcal{C} \mathfrak{nd}_{0}$). There exists a unique category $\mathcal{C} \mathfrak{nd}_{0}$ whose object set equals the object set of $\mathcal{D} \mathfrak{p}$ and whose morphism set is such that for every $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{C} \mathfrak{nd}_{0}$ we have
\[
\text{(16)} \quad \text{Mor}_{\mathcal{C} \mathfrak{nd}_{0}}(\mathfrak{A}, \mathfrak{B}) = \prod_{f \in \text{Fct}_{\text{top}}(G_{\mathfrak{A}}, G_{\mathfrak{B}})} \text{Mor}_{\text{Fct}(G_{\mathfrak{A}}^{\mathfrak{op}}, G_{\mathfrak{B}})}( \tau \circ \sigma_{\mathfrak{A}}, \tau \circ \sigma_{\mathfrak{B}} \circ f) \times \text{Mor}_{\text{Fct}(G_{\mathfrak{B}}, G_{\mathfrak{A}})}(q_{0} \circ \sigma_{\mathfrak{B}} \circ f, q_{0} \circ \sigma_{\mathfrak{A}})
\]
and
\[
\text{(17)} \quad (\circ) : \text{Mor}_{\mathcal{C} \mathfrak{nd}_{0}}(\mathfrak{B}, \mathfrak{C}) \times \text{Mor}_{\mathcal{C} \mathfrak{nd}_{0}}(\mathfrak{A}, \mathfrak{B}) \to \text{Mor}_{\mathcal{C} \mathfrak{nd}_{0}}(\mathfrak{A}, \mathfrak{C}),
\]
\[
(g, L, S) \circ (f, H, T) := (f \circ g, (H \circ 1_{\mathfrak{A}}) \circ L, S \circ (T \circ 1_{\mathfrak{C}})).
\]
Moreover the maps $\mathfrak{A} \mapsto \mathfrak{A}$ and $(f, T) \mapsto (f, (1_{\mathfrak{A}} \circ T) \circ 1_{\mathfrak{A}}, 1_{\mathfrak{A}} \circ T)$ determine uniquely an element $\Psi_{0} \in \text{Fct}(\mathcal{D} \mathfrak{p}, \mathcal{C} \mathfrak{nd}_{0})$.

Next in analogy with [7, Def. 1.5.8] we give the following

Definition 1.27 (The fibered category of 0–species). Define $\mathcal{S} \mathfrak{p}_{0}$ the fibered category over $2 - \mathcal{D} \mathfrak{p}$ such that for all $\mathfrak{D} \in 2 - \mathcal{D} \mathfrak{p}$
\[
\mathcal{S} \mathfrak{p}_{0}(\mathfrak{D}) = 2 - \mathcal{D} \mathfrak{p}(\mathfrak{D}, \mathcal{C} \mathfrak{nd}_{0}),
\]
moreover set
\[
\mathcal{S} \mathfrak{p}_{0}^{c} := \{(a, b) \in \mathcal{S} \mathfrak{p}_{0} \times \mathcal{S} \mathfrak{p}_{0} \mid d(a) = d(b)\}.
\]

2. Construction of the classical and quantum 0-species $x$ and $z$

Since $\exp_{M}^{U}$ is a $C_{0}$-group on $\mathcal{D}(M)$ we immediately obtain the following

Proposition 2.1. Let $(\mathfrak{M}, U) \in \mathcal{W}_{0}$. Thus there exists a unique $(\mathfrak{M}, U, F_{(\mathfrak{M}, U)}) \in \mathcal{D} \mathfrak{p}$ with the following properties. $(\mathfrak{M}, U)$ is the unique top-quasi enriched category with the following properties. The object set of $(\mathfrak{M}, U)$ is the set of all subsets of $\mathfrak{M}$, the morphism set of $(\mathfrak{M}, U)$ is such that for every $X, Y \in (\mathfrak{M}, U)$ we have $\text{Mor}_{(\mathfrak{M}, U)}(X, Y) = \{(X, Y) \times \text{mor}_{(\mathfrak{M}, U)}(X, Y),
\]
\[
\text{mor}_{(\mathfrak{M}, U)}(X, Y) = \{t \in \mathbb{R} \mid \exp_{(\mathfrak{M}, U)}^{U}(t) \mathcal{D}(M, X) = \mathcal{D}(M, Y)\},
\]
where we let $\mathfrak{M}$ be the manifold underlying $\mathfrak{M}$ and $\mathcal{D}(M, X)$ be the topological sub-$*$-algebra of $\mathcal{D}(M)$ of those maps whose support is contained in $X$. The topology on $\text{Mor}_{(\mathfrak{M}, U)}(X, Y)$ is that induced by the topology on $\mathbb{R}$, while the composition is that inherited by the addition in $\mathbb{R}$. While $F_{(\mathfrak{M}, U)} \in \text{Fct}_{\text{top}}((\mathfrak{M}, U), \text{tsa})$ such that for every $t \in \text{mor}_{(\mathfrak{M}, U)}(X, Y)$ we have
\[
\begin{cases}
F_{(\mathfrak{M}, U)}(X) = \mathcal{D}_{1}(M, X), \\
F_{(\mathfrak{M}, U)}((X, Y), t) : \mathcal{D}_{1}(M, X) \to \mathcal{D}_{1}(M, Y), \\
(f, \lambda) \mapsto (\exp_{(\mathfrak{M}, U)}^{U}(t)f, \lambda);
\end{cases}
\]
with $\mathcal{D}_{1}(M, X) \in \text{tsa} the unitization of $\mathcal{D}(M, X)$.
2. CONSTRUCTION OF THE CLASSICAL AND QUANTUM 0-SPECIES x AND z

**Theorem 2.2.** There exists a unique \( \phi \in \text{Fct}(\phi, \text{dy}) \) such that for all \((M, U), (N, V) \in \phi\) and \( \phi \in \text{Mor}_{\phi}(\langle M, U \rangle, (N, V)) \)

1. \( x((M, U)) = \langle (M, U), F_{(M, U)} \rangle, \)
2. \( x(\phi) = (f^\phi, T^\phi_1); \)

where \( f^\phi \in \text{Fct}_{\phi}(\langle N, V \rangle, \langle M, U \rangle) \) and

\[
T^\phi_1 \in \text{Mor}_{\text{Fct}(\langle N, V \rangle, \text{tsa})}(F_{(M, U)} \circ f^\phi, F_{(N, V)});
\]

such that for all \( Y, Z \in \langle N, V \rangle \) and \( t \in \text{mor}_{\langle N, V \rangle}(Y, Z) \)

1. \( f^\phi(Y) = \phi(Y); \)
2. \( f^\phi((Y, Z), t) = ((\phi(Y), \phi(Z)), t); \)
3. \( T^\phi(Y) : D(M, \phi(Y)) \rightarrow D(N, Y) \quad h \mapsto \phi^* h; \)
4. \( T^\phi_1(Y) : D_1(M, \phi(Y)) \rightarrow D_1(N, Y) \quad (h, \lambda) \mapsto (\phi^* h, \lambda); \)

where \( M \) and \( N \) are the manifolds underlying \( M \) and \( N \) respectively. In particular \( \Psi \circ x \in \text{Se}_p(\phi) \) and \( \Psi_0 \circ x \in \text{Se}_p(\phi). \)

**Proof.** Let us take the properties of the statement as definition of \( x \). Let \( t \in \text{mor}_{\langle N, V \rangle}(Y, Z) \) and \( f \in D(M, \phi(Y)) \), so since Lemma 1.4 we have

\[
\phi^* (\exp^U_M(t)f) = \exp^V_N(t)(\phi^* f) \\
\in \exp^V_N(t)(D(N, Y)) \subseteq D(N, Z);
\]

namely

\[
\exp^U_M(t)f \in (\phi^{-1})^* D(N, Z) = (\phi^{-1})^* D(N, \phi^{-1}(\phi(Z))) \subseteq D(M, \phi(Z)).
\]

Therefore \( f^\phi((Y, Z), t) \in \text{Mor}_{(M, U)}(\phi(Y), \phi(Z)), \) next \( f^\phi \) is clearly continuous and composition preserving so \( f^\phi \in \text{Fct}_{\phi}(\langle N, V \rangle, \langle M, U \rangle) \). Next \( T^\phi_1(Y) \) is continuous since \( \phi^* \) is \((\tau^\phi_{\phi}(M), \tau^\phi_{\phi}(N))-\text{continuous}, moreover for every \( f \in D(M, \phi(Y)) \) and \( \lambda \in C, \) since Lemma 1.4 we have

\[
(T^\phi_1(Z) \circ F_{(M, U)})(\phi(Y), \phi(Z), t))(f, \lambda) = ((\phi^* \circ \exp^U_M(t))f, \lambda) \\
= ((\exp^V_N(t) \circ \phi^* f), \lambda) \\
= (F_{(N, V)}(Y, Z, t) \circ T^\phi_1(Y))(f, \lambda);
\]

which proves (18). Finally \( x(\psi \circ_{\phi} \phi) = x(\psi) \circ_{\phi} x(\phi) \) (18) follows by the same line of reasoning we use in [7] Thm. 1.6.24 to prove that \( a(\psi \circ_{\phi} \phi) = a(\psi) \circ_{\phi} a(\phi). \) \( \square \)

**Theorem 2.3.** Let \((M, U) \in \phi_0, \) thus there exists a unique \( \langle (M, U), \mathcal{F}_{(M, U)} \rangle \in \text{dp} \) with the following properties. \( \mathcal{F}_{(M, U)} \in \text{Fct}_{\phi}(\langle M, U \rangle, \text{tsa}) \) such that for every subset \( X \) and \( Y \) of \( M \) and every \( t \in \text{mor}_{(M, U)}(X, Y) \) we have

\[
\begin{align*}
\mathcal{F}_{(M, U)}(X) &= \mathcal{B}(M, X), \\
\mathcal{F}_{(M, U)}((X, Y), t) : \mathcal{B}(M, X) &\rightarrow \mathcal{B}(M, Y) \quad T \mapsto \Gamma^U_M(t)T;
\end{align*}
\]
where we let $M$ be the manifold underlying $M$ and $\mathcal{B}(M, X)$ be the topological unital sub $*$-algebra of $\mathcal{B}(M)$ of those $T$ such that $T \mathcal{D}(M, X) \subseteq \mathcal{D}(M, X)$ and $T^* \mathcal{D}(M, X) \subseteq \mathcal{D}(M, X)$.

**Proof.** Since Thm. 1.24(2). \hfill \Box

**Theorem 2.4.** There exists a unique $\zeta \in \text{Fct}(\mathfrak{v}, \mathfrak{d}p)$ such that for all $(M, U), (N, V) \in \mathfrak{v}$ and $\phi \in \text{Mor}_{\text{sf}}((M, U), (N, V))$

(1) $\zeta(M, U) = \langle \langle M, U \rangle, \mathcal{F}_M(U) \rangle$,

(2) $\zeta(\phi) = (f^\phi, \mathcal{F}^\phi)$;

where

$$\tag{19} \mathcal{I}^\phi \in \text{Mor}_{\text{Fct}(\mathcal{N}, \mathcal{V})_{\text{lsa}}}((\mathcal{F}_M(U) \circ f^\phi, \mathcal{F}_N(V))$$

such that for all $Y, Z \in \langle N, V \rangle$ we have

$$\mathcal{I}^\phi(Y) : \mathcal{B}(M, \phi(Y)) \rightarrow \mathcal{B}(N, Y) \quad T \mapsto \mathcal{I}(\phi)T.$$

In particular $\Psi \circ \zeta \in \mathcal{E}(\mathfrak{v})$ and $\Psi_0 \circ \zeta \in \mathcal{E}(\mathfrak{p}_0(\mathfrak{v}))$.

**Proof.** Let us take the properties of the statement as definition of $\zeta$. Let $Q \in \mathcal{B}(M, \phi(Y))$ thus since Thm. 2.2(3) we obtain

$$\mathcal{I}^\phi(Y)(Q) \upharpoonright \mathcal{D}(N, Y) = \mathcal{T}^\phi(Y) \circ Q \circ \mathcal{T}^{-1}(\phi(Y)),$$

moreover $\mathcal{F}(\phi)\mathcal{B}(M) \subseteq \mathcal{B}(N)$ since Thm. 1.15(2), thus we obtain that $\mathcal{I}^\phi(Y)\mathcal{B}(M, \phi(Y)) \subseteq \mathcal{B}(N, Y)$ and then $\mathcal{I}^\phi$ is well-set. Next $\mathcal{I}^\phi(Y)$ is continuous since it is so $\mathcal{I}(\phi)$ according to Thm. 1.15(2). Next for every $t \in \text{mor}_{\langle \mathcal{N}, \mathcal{V} \rangle}(Y, Z)$ we have by Thm. 1.24(3)

$$(\mathcal{I}^\phi(Z) \circ \mathcal{F}_M(U))(\phi(Y), \phi(Z), t))Q = \langle \mathcal{I}(\phi) \circ \Gamma_M(t) \rangle Q$$

$$= \langle \Gamma^Y_M(t) \circ \mathcal{I}(\phi) \rangle Q$$

$$= \langle \mathcal{F}_{\langle \mathcal{N}, \mathcal{V} \rangle}(Y, Z, t) \circ \mathcal{I}^\phi(Y) \rangle Q;$$

which proves (19). Finally for every $\psi \in \text{Mor}_{\mathfrak{v}}$ which is $\mathfrak{v}$–composable to the left with $\phi$ we have $\zeta(\psi \circ_{\mathfrak{v}} \phi) = \zeta(\psi) \circ_{\mathfrak{d}p} \zeta(\phi)$ since $\mathcal{I}(\phi \circ \psi) = \mathcal{I}(\psi) \circ \mathcal{I}(\phi)$. \hfill \Box

### 3. Construction of the natural transformation $\zeta$ from $x$ to $z$

**Theorem 3.1.** Let $M$ be a semi–Riemannian manifold with underlying manifold $M$, thus for every $f, h \in \mathcal{D}(M)$ we have

$$\mathcal{L}_{\text{grad}_M(f)}(h) = \mathcal{L}_{\text{grad}_M(h)}(f).$$

Next let $U$ be such that $(M, U) \in \mathfrak{v}^*$. If there exists a frame $\{E_i\}_{i=1}^{n=\dim M}$ of orthonormal fields of $M$ such that $[U, E_i] = 0$ for every $i \in \{1, n\} \cap \mathbb{Z}$, then for every $f \in \mathcal{D}(M)$ and $t \in \mathbb{R}$ we have

$$\exp^U_M(t) \circ \mathcal{L}_{\text{grad}_M(f)}(U) = \mathcal{L}_{\text{grad}_M, \exp^U_M(t)(f), U} \circ \exp^U_M(t).$$
3. CONSTRUCTION OF THE NATURAL TRANSFORMATION \( J \) FROM \( x \) TO \( z \)

Proof. By the same symbol \([\cdot, \cdot]\) we shall denote the Lie bracket of vector fields on \( M \) and the Lie bracket induced by the associative product on \( \mathfrak{L}(\mathcal{D}(M)) \), namely \([T, Q] = T \circ Q - Q \circ T\) for \( T, Q \in \mathfrak{L}(\mathcal{D}(M)) \). Next let \( \{E_i\}_{i=1}^{n=\dim M} \) be a frame of orthonormal fields of \( M \), let \( \varepsilon_i = \langle E_i, E_i \rangle_M \) for every \( i \in \{1, n\} \cap \mathbb{Z} \). Thus for every vector field \( W \) we have \( \mathcal{L}_W = \sum_{i=1}^{n} \varepsilon_i \langle W, E_i \rangle_M \cdot \mathcal{L}_{E_i} \). Next let \( f, h \in \mathcal{D}(M) \) thus

\[
\mathcal{L}_{\operatorname{grad}_M(f)}(h) = \sum_{i=1}^{n} \varepsilon_i \langle \operatorname{grad}_M(f), E_i \rangle_M \cdot \mathcal{L}_{E_i}(h)
\]

(22)

and (20) follows. Next let \( W \in \mathfrak{x}(M) \), so \( \mathcal{L}_W \) is \((\tau^\infty_c, \tau^\infty_c)\)-continuous thus

\[
\exp_M^U(t) \circ \mathcal{L}_W = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{L}_W^k \circ \exp_M^U(t)
\]

(23)

both converging in \( \mathfrak{L}_s(\mathcal{D}(M)) \). Since \( \mathcal{L} : \mathfrak{x}(M) \to \operatorname{Der}(\mathcal{D}(M)) \) is a Lie algebra isomorphism onto the Lie algebra of derivations of \( \mathcal{D}(M) \), we have \([W, U] = 0 \Leftrightarrow \mathcal{L}_{[W, U]} = 0 \Leftrightarrow \mathcal{L}_W \mathcal{L}_U = 0 \Leftrightarrow (\forall n \in \mathbb{Z}_+)([\mathcal{L}_W, \mathcal{L}_U^n] = 0) \) therefore

Next by the defining property of derivations, the second equality of (22) and Rmk. 1.3 we deduce that

\[
\mathcal{L}_W \circ \exp_M^U(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{L}_W \mathcal{L}_U^k;
\]

Let \( A \) and \( B \) denote the left and right sides of the equality (21) respectively, thus

\[
A = \exp_M^U(t) \circ [\mathcal{L}_{\operatorname{grad}_M(f)}(f), \mathcal{L}_U];
\]

\[
B = [\mathcal{L}_{(\operatorname{grad}_M, \exp_M^U(t))(f)}, \mathcal{L}_U] \circ \exp_M^U(t).
\]
Now if $[U, E_i] = 0$ for every $i \in [1, n] \cap \mathbb{Z}$, then by (24) and Rmk. 1.3 we obtain

$$A = (-1) \sum_{i=1}^{n} \varepsilon_i \cdot (\exp_U(t) \circ \mathcal{L}_U \circ \mathcal{L}_E)(f)(\exp_U(t) \circ \mathcal{L}_E);$$

$$B = (-1) \sum_{i=1}^{n} \varepsilon_i \cdot (\mathcal{L}_U \circ \mathcal{L}_E \circ \exp_U(t))(\mathcal{L}_E \circ \exp_U(t));$$

then st. (21) follows since (23).

\[\square\]

**Remark 3.2.** Since (23) we have $[U, W] = 0 \Rightarrow (\forall t \in \mathbb{R})(\Lambda^U(t)(\mathcal{L}_W) = \mathcal{L}_W)$.

**Lemma 3.3.** Let $\mathcal{M} = (M, g)$, $\mathcal{N} = (N, g')$ be semi-Riemannian manifolds and $\phi : N \to M$ be a smooth diffeomorphism such that $\phi^*g = g'$. Thus for every $f \in \mathcal{D}(M)$ we have

$$\phi^* \circ \mathcal{L}_{\text{grad}_{\phi}}(f) = \mathcal{L}_{(\text{grad}_{\phi})}(f) \circ \phi^*.$$

**Proof.** Let $\{E_i\}_{i=1}^{\dim M}$ be a frame of orthonormal fields of $\mathcal{M}$, and set $G_i := d(\phi^{-1}) \circ E_i \circ \phi$, thus $E_i$ and $G_i$ are $\phi$-related, hence $\phi^* \circ \mathcal{L}_{E_i} = \mathcal{L}_{G_i} \circ \phi^*$ and $\{G_i\}_{i=1}^{\dim M}$ is a frame of orthonormal fields of $\mathcal{N}$ since [7] Lemma 1.6.6. Therefore since again [7] Lemma 1.6.6 we have that

$$\phi^* \circ \mathcal{L}_{\text{grad}_{\phi}}(f) = \sum_{i=1}^{\dim M} \phi^*(\langle E_i, E_i \rangle_M) \phi^*(\mathcal{L}_{E_i}(f))(\phi^* \circ \mathcal{L}_{E_i})$$

$$= \sum_{i=1}^{\dim M} \langle G_i, G_i \rangle_N \mathcal{L}_{G_i}(\phi^* f)(\mathcal{L}_{G_i} \circ \phi^*)$$

$$= \mathcal{L}_{(\text{grad}_{\phi})}(f) \circ \phi^*.$$

\[\square\]

**Definition 3.4 (The map $\mathbb{Z}$).** Let $\mathcal{M}$ be a semi-Riemannian manifold, $\mathcal{M}$ be the manifold underlying $\mathcal{M}$, $U \in \mathfrak{X}(\mathcal{M})$ and $X \subseteq M$. Define

$$\mathbb{Z}^X_{(\mathcal{M}, U)} : \mathcal{D}_1(M, X) \to \mathfrak{B}(\mathcal{M}, X)$$

$$(f, \lambda) \mapsto \mathcal{L}_{\text{grad}_{\lambda}}(f) + \lambda 1;$$

where 1 is the unit element of the unital algebra $\mathfrak{B}(\mathcal{M})$.

**Proposition 3.5.** Def. 3.4 is well-set namely $\mathbb{Z}^X_{(\mathcal{M}, U)}(f) \in \mathfrak{B}(\mathcal{M}, X)$ for every $f \in \mathcal{D}(M, X)$.

**Proof.** $\mathbb{Z}^X_{(\mathcal{M}, U)}(f) \in \mathfrak{B}(\mathcal{M})$ since Rmk. 1.8 and (1). Next by Notation we know that the elements in DiffOp$^1(M)$ are local, and that $D \in \text{DiffOp}^1(M) \Rightarrow D^* \in \text{DiffOp}^1(M)$, thus $l \in \mathcal{D}(M, X)$ implies supp($\mathbb{Z}^X_{(\mathcal{M}, U)}(f)(l)$) $\subseteq$ supp($l$) $\subseteq$ $X$ and supp($\mathbb{Z}^X_{(\mathcal{M}, U)}(f^*(l))$) $\subseteq$ $X$.

**Corollary 3.6.** Let $\mathcal{M}$ be a semi-Riemannian manifold, $\mathcal{M}$ be the manifold underlying $\mathcal{M}$ and $X \subseteq M$, thus

$$\left( f \mapsto \mathcal{L}_{\text{grad}_{\lambda}}(f) \right) \in \mathcal{L}(\mathcal{D}(M, X), \mathfrak{B}(\mathcal{M}, X)).$$
Proof. Let \( \xi \) be the map in the statement, \( \xi : \mathcal{D}(M) \to \mathfrak{B}(M) \) be the map \( f \mapsto \mathcal{L}_{\text{grad}_M(f)} \), and let \( \xi : = \mathcal{L}_{\mathfrak{B}(M), M} \circ \xi \). The above maps are well-set namely \( \xi(\mathcal{D}(M, X)) \subseteq \mathfrak{B}(M, X) \) since the proof of Prp. 3.5 while \( \xi(\mathcal{D}(M)) \subseteq \mathfrak{B}(M) \) since Rmk. 1.8. We claim to show that

\[
\xi \in \mathcal{U}(\mathcal{D}(M), \mathfrak{B}(M));
\]

which would prove our statement since \( \mathcal{L}_{\mathfrak{B}(M), M} \circ \xi = \xi \circ \mathcal{D}(M, X) \). Now since Lemma 1.22 we have that (25) would follow if we prove that \( \xi \in \mathcal{U}(\mathcal{D}(M), \mathfrak{B}(M)) \). But \( \mathcal{D}(M) \) is barrelled therefore by applying the Banach-Steinhaus Thm. the above statement would follow if \( \xi \in \mathfrak{B}(M) \), \( \mathfrak{B}(\mathcal{D}(M)) \) which at once follows since (20). \( \square \)

For any semi-Riemannian manifold \( M \) let \( \text{Onf}(M) \) denote the set of frames of orthonormal fields of \( M \).

Definition 3.7. Let \( \mathfrak{B} \) be the unique full subcategory of \( \mathfrak{V} \) such that

\[
\text{Obj}(\mathfrak{B}) = \{(M, U) \mid (\mathfrak{U}(E_i)_{i=0}^{\dim M} \in \text{Onf}(M)) (\forall i \in [1, n] \cap \mathbb{Z})(\mathfrak{U}(E_i) = 0)\}.
\]

Definition 3.8. Define \( x := \mathfrak{U}_0 \circ x \circ \mathfrak{I}^\mathfrak{B}_f \) and \( z := \mathfrak{U}_0 \circ z \circ \mathfrak{I}^\mathfrak{B}_f \).

Corollary 3.9 (The 0–species \( x \) and \( z \)). \( x \in \mathfrak{E}_p(\mathfrak{B}) \) and \( z \in \mathfrak{E}_p(\mathfrak{B}) \).

Proof. Since Thm. 2.2 and Thm. 2.4, \( \square \)

Theorem 3.10 (Main). A natural transformation from \( x \) to \( z \). There exists a natural transformation

\[
\mathfrak{J} \in \text{Mor}_{\mathfrak{F}^4(\mathfrak{B}_f, \mathfrak{B}_f)}(x, z)
\]

uniquely determined by the following properties:

\[
\mathfrak{J} : \text{Obj}(\mathfrak{B}_f) \ni (M, U) \mapsto \mathfrak{J}(M, U) = (1_{(M, U)}, \partial\mathfrak{B}_f(M, U), \partial\mathfrak{B}_f(M, U));
\]

\[
\mathfrak{J}_f(M, U) : \text{Obj}((M, U)) \ni X \mapsto \mathfrak{J}(X, (M, U));
\]

\[
\mathfrak{J}^\mathfrak{B}_f(M, U) : \text{Obj}((M, U)) \ni X \mapsto \mathfrak{J}(X, (M, U))^\mathfrak{B}.
\]

Proof. Let us take the properties of \( \mathfrak{J} \) given in the statement as its definition, then show that the definition is well-set and that \( \mathfrak{J} \) is a natural transformation from \( x \) to \( z \). Let \( (M, U) \in \text{Obj}(\mathfrak{B}_f) \) and \( X, Y \in (M, U) \). By \( \mathcal{L}_{\text{grad}_M(f), U} = [\mathcal{L}_{\text{grad}_M(f), U}] \) and since \( \mathfrak{B}(M, X) \) is a topological algebra we obtain by Cor. 3.6 that

\[
(\mathcal{L}_{\text{grad}_M(f), U}) \in \mathcal{U}(\mathcal{D}(M, X), \mathfrak{B}(M, X)).
\]

Since (26) and (1) we have

\[
\mathfrak{J}(M, U)(X) \in \text{Mor}_{\text{ls}}(\mathfrak{B}(\mathfrak{D}_1(M, X)), \mathfrak{B}(\mathfrak{M}(M, X)));
\]

Next (21) is equivalent to say that for all \( f \in \mathcal{D}(M) \) we have

\[
\Gamma^U_M(t)(\mathcal{L}_{\text{grad}_M(f), U}) = \mathcal{L}_{(\text{grad}_M \circ \exp^U_M(t))(f), U}.
\]
therefore by considering (27) we have that the following is a commutative diagram in the category tls

\[
\begin{array}{ccc}
q_0(D_1(M, Y)) & \xrightarrow{\mathcal{J}(M, U)(Y)} & q_0(\mathfrak{B}(M, Y)) \\
\downarrow q_0(\exp_U^{j_2}(t) \ast \text{id}_c) & & \downarrow q_0(\mathcal{J}(M, U)(t)) \\
q_0(D_1(M, X)) & \xrightarrow{\mathcal{J}(M, U)(X)} & q_0(\mathfrak{B}(M, X))
\end{array}
\]

namely

(28) \quad \mathcal{J}(M, U) \in \text{Mor}_{\text{Fct}(\langle M, U \rangle, \text{tls})}(q_0 \circ F_{\langle M, U \rangle}, q_0 \circ \mathcal{J}_{\langle M, U \rangle}).

Thus by Def. [1,25] and the corresponding of [7] Cor. 1.4.16

(29) \quad \mathcal{J}^+_{\langle M, U \rangle} \in \text{Mor}_{\text{Fct}(\langle M, U \rangle^{\text{op}}, \text{tls})}(r \circ \mathcal{J}^+_{\langle M, U \rangle}, r \circ F^+_{\langle M, U \rangle}).

(28) and (29) imply

(30) \quad \exists(M, U) \in \text{Mor}_{\text{End}_0}(x(M, U), z(M, U)).

Therefore the statement will follow if we prove that for every (\mathcal{N}, V) \in \text{Obj}(\mathfrak{B}!) and every \phi \in \text{Mor}_{\mathfrak{B}!}(\langle M, U \rangle, (\mathcal{N}, V)) the following is a commutative diagram in the category \text{End}_0

\[
\begin{array}{ccc}
x(N, V) & \xrightarrow{\exists(N, V)} & z(N, V) \\
\downarrow \psi_0(f^\phi, T^\phi_1) & & \downarrow \psi_0(f^\phi, T^\phi) \\
x(M, U) & \xrightarrow{\exists(M, U)} & z(M, U)
\end{array}
\]

namely

\[
(f^\phi, (1_{a_0} \ast \mathfrak{T}^\phi)^\dagger, 1_{a_0} \ast \mathfrak{T}^\phi) \circ (1_{\langle M, U \rangle}, \mathcal{J}^+_{\langle M, U \rangle}, \mathcal{J}(M, U)) = \big(1_{\langle N, V \rangle}, \mathcal{J}^+_{\langle N, V \rangle}, \mathcal{J}(N, V)\big) \circ \big(f^\phi, (1_{a_0} \ast T^\phi_1)^\dagger, 1_{a_0} \ast T^\phi_1\big);
\]

that according to (17) is equivalent to

\[
\big(1_{\langle M, U \rangle} \circ f^\phi, (\mathcal{J}^+_{\langle M, U \rangle} \ast 1_{f^\phi}) \circ (1_{a_0} \ast \mathfrak{T}^\phi)^\dagger, (1_{a_0} \ast \mathfrak{T}^\phi) \circ (\mathcal{J}(M, U) \ast 1_{f^\phi})\big) = \\
\big(f^\phi \circ 1_{\langle N, V \rangle}, ((1_{a_0} \ast T^\phi_1)^\dagger \ast 1_{1_{\langle N, V \rangle}}) \circ \mathcal{J}^+_{\langle N, V \rangle}, \mathcal{J}(N, V) \circ ((1_{a_0} \ast T^\phi_1) \ast 1_{1_{\langle N, V \rangle}})\big),
\]

which reduces to the following equality of morphisms of the category Fct(\langle \mathcal{N}, V \rangle, \text{tls})

\[
(1_{a_0} \ast \mathfrak{T}^\phi) \circ (\mathcal{J}(M, U) \ast 1_{f^\phi}) = \mathcal{J}(N, V) \circ ((1_{a_0} \ast T^\phi_1) \ast 1_{1_{\langle N, V \rangle}}).
\]

Which is equivalent to say that for every Z \in \langle \mathcal{N}, V \rangle we have the following equality of morphisms of the category tls

\[
\mathfrak{T}^\phi(Z) \circ \mathcal{J}(M, U)(\phi(Z)) = \mathcal{J}(N, V)(Z) \circ T^\phi_1(Z);
\]
4. ADDITIONAL EQUICONTINUITY REQUESTS

Definition 4.1. Let \( M \) be a manifold and \( U \in \mathfrak{X}(M) \) such that \( \{(E^*_U)^k \mid k \in \mathbb{Z}_+\} \) is \((\tau^\infty, \tau^\infty)\)-equicontinuous, define

\[
\text{Exp}_M^U := \exp^{C_0(M)}_{(E^*_U)^k}.
\]

Under the same reasoning used in Rmk. [1.3] we have that \( \text{Exp}_M^U \) is a group of unit preserving \( * \)-automorphisms of \( C_0(M) \).

Lemma 4.2. Let \( M \) be a manifold and \( U \in \mathfrak{X}(M) \) such that \( \{(E^*_U)^k \mid k \in \mathbb{Z}_+\} \) is \((\tau^\infty, \tau^\infty)\)-equicontinuous and \( \{E^*_U \mid k \in \mathbb{Z}_+\} \) is \((\tau^\infty \uparrow \mathcal{D}(M), \tau^\infty)\)-equicontinuous. Thus

\[
\text{Exp}_M^U(t) \circ t_{\mathcal{D}(M)}^{(\tau^\infty)} = t_{\mathcal{D}(M)}^{(\tau^\infty)} \circ \text{Exp}_M^U(t);
\]

in particular \( \exp^U_M(t) \in \mathcal{L}(\mathcal{D}(M)[\tau^\infty]) \).

Proof. By the hypothesis and since \( \mathcal{D}(M) \hookrightarrow C_0(M) \) we have that \( \{E^*_U \mid k \in \mathbb{Z}_+\} \) is \((\tau^\infty, \tau^\infty)\)-equicontinuous. Thus \( \exp^U_M(t) \) is well-set and for every \( f \in \mathcal{D}(M) \) we have

\[
\exp^U_M(t)f = \sum_{k=0}^{\infty} \frac{t^k}{k!} E^*_U(f) \text{ w.r.t. the } \tau^\infty_c\text{-topology}
\]

\[
= \sum_{k=0}^{\infty} \frac{t^k}{k!} (E^*_U)^k(f) \text{ w.r.t. the } \tau^\infty\text{-topology}
\]

\[
= \text{Exp}_M^U(t)f.
\]

\( \square \)

Proposition 4.3. Under the hypothesis of Lemma 4.2, for every \( t \in \mathbb{R} \), \( F \in C_0(M) \) and \( f \in \mathcal{D}(M) \) we have \( \exp^U_M(t)(F \cdot f) = \exp^U_M(t)(F) \cdot \exp^U_M(t)(f) \).

Proof. Since Lemma 4.2 and since \( \text{Exp}_M^U(t) \) is a \( * \)-automorphism of \( C_0(M) \). \( \square \)
Proposition 4.4. Under the hypothesis of Lemma 4.2 let $\mathcal{A} \subset \mathcal{U}(\mathcal{D}(M))$ such that $\mathcal{A}$ is naturally a left $C^\infty(M)$-module namely w.r.t. the external product defined by $(F \cdot Q) : \mathcal{D}(M) \to \mathcal{D}(M)$, $h \mapsto F \cdot Q(h)$ for every $Q \in \mathcal{A}$ and $F \in C^\infty(M)$. Thus for every $t \in \mathbb{R}$ we have

1. if $h \in \mathcal{D}(M)$, then $\Lambda^U_M(t)(F \cdot Q)h = \exp^U_M(t)(F) \cdot \Lambda^U_M(t)(Q)h;$
2. if $\Lambda^U_M(t)A \subseteq \mathcal{A}$, then $\Lambda^U_M(t)(F \cdot Q) = \exp^U_M(t)(F) \cdot \Lambda^U_M(t)(Q)$ with the $(\cdot)$ operation in $\mathcal{A}$.

Proof. St. (2) follows since st. (1). Next we have

$$\Lambda^U_M(t)(F \cdot Q)h = \exp^U_M(t)\left( F \cdot (Q \exp^U_M(-t)h) \right)$$
$$= \exp^U_M(t)(F) \cdot \Lambda^U_M(t)(Q)h;$$

where the second equality follows by Prp. 4.3.

Corollary 4.5. Let $\mathcal{M}$ be a semi-Riemannian manifold, $M$ its underlying manifold and $\{E_i\}_{i}^{n=\dim M}$ be a frame of orthonormal fields of $\mathcal{M}$. Thus under the hypothesis of Lemma 4.2 we have for every $V \in \mathfrak{X}(M)$, $t \in \mathbb{R}$ and $h \in \mathcal{D}(M)$ that

$$\Gamma^U_M(t)(E_V)h = \sum_{i=1}^{n} \varepsilon_i \exp^U_M(t)(\langle V, E_i \rangle_M) \cdot \Gamma^U_M(t)(E_{E_i})h.$$  \hspace{1cm} (32)

If in addition $[U, E_i] = 0$ for every $i \in [1, n] \cap \mathbb{Z}$, then

$$\Gamma^U_M(t)(E_V) = E_V,$$  \hspace{1cm} (33)

where $V_i := \sum_{i=1}^{n} \varepsilon_i \exp^U_M(t)(\langle V, E_i \rangle_M)E_i$. Moreover the left $C^\infty(M)$-submodule $\mathcal{A}_M$ of $\text{DiffOp}^1(M)$ generated by the set $\{E_W \mid W \in \mathfrak{X}(M)\}$ is such that $\Gamma^U_M(t)\mathcal{A}_M \subseteq \mathcal{A}_M$, and

$$\Gamma^U_M(t)(E_V) = \sum_{i=1}^{n} \varepsilon_i \exp^U_M(t)(\langle V, E_i \rangle_M) \cdot E_{E_i};$$  \hspace{1cm} (34)

with the $(\cdot)$ operation in $\mathcal{A}_M$.

Proof. (32) follows since Prp. 4.4(1) applied to the natural left $C(M)^\infty$-module $\text{DiffOp}^1(M)$. Next if $[U, E_i] = 0$ for every $i \in [1, n] \cap \mathbb{Z}$, then by (32) and Rmk. 3.2 we obtain

$$\Gamma^U_M(t)(E_V)h = \sum_{i=1}^{n} \varepsilon_i \exp^U_M(t)(\langle V, E_i \rangle_M) \cdot E_{E_i}h$$
$$= \frac{E_V}{h}. \hspace{1cm} (35)$$

Hence (33) follows, which together Prp. 4.4(1) imply $\Gamma^U_M(t)\mathcal{A}_M \subseteq \mathcal{A}_M$. (34) follows since the first equality in (35) and the fact that $\mathcal{A}_M$ is a left $C(M)^\infty$-module; alternatively by $\Gamma^U_M(t)\mathcal{A}_M \subseteq \mathcal{A}_M$ and Prp. 4.4(2).

We conclude this section with a result providing sufficient conditions on a complete vector field $U$ on $\mathcal{M}$ ensuring that $\exp^U_M = \eta^U_M$. 


Corollary 4.6. Let \((M, U) \in \mathfrak{VF}^\ast\) with \(M = (M, \rho)\) such that \(U\) is complete and
\[(\forall t \in \mathbb{R})(\mu_\rho \circ \eta^U_M(t) = \mu_\rho \circ i^{\mathfrak{K}(M)}_U).\]

Thus

1. \((M, U) \in \mathfrak{VV}_0\) and \(\eta^U_M\) extends to a \(C_0\)-group \(\eta^U_M\) on \(\mathcal{H}_g\) of unitary operators whose infinitesimal generator extends \(E_U\).

2. If \(\{\varepsilon_k^U | k \in \mathbb{Z}_+\}\) is \((\|\cdot\|_{\mathcal{H}_g}, \|\cdot\|_{\mathcal{H}_g}^\ast)\)-equicontinuous, then \(\mathcal{D}(M)\) is a core for both the infinitesimal generators of \(\eta^U_M\) and \(\exp^U_M\). Thus \(\exp^U_M = \eta^U_M\), in particular \(\exp^U_M = \eta^U_M\).

Proof. Let \(t \in \mathbb{R}\). \(\eta^U_M(t)\) is an isometry of the \(\mathcal{H}_g\)-dense subspace \(\mathcal{D}(M)\) since \(\eta^U_M\) is a morphism of \(*\)-algebras and since \((36)\), thus \(\eta^U_M(t)\) extends to a unitary operator \(\eta^U_M(t)\) on \(\mathcal{H}_g\). Next let \(\{t_n\}_{n \in \mathbb{N}}\) be a sequence in \(\mathbb{R}\) converging at 0 and \(f \in \mathcal{D}(M)\), thus \(\lim_{n \to \infty} \eta^U_M(t_n)(f) = f\) pointwise since the flow \(\theta^U\) is pointwise continuous. But \(\eta^U_M(t_n)(f) \in \mathcal{D}(M)\) as well \(f \in \mathcal{D}(M)\) and \(\mathcal{D}(M) \subseteq \mathcal{H}_g\) therefore by applying the Lebesgue Thm. \(\lim_{n \to \infty} \eta^U_M(t_n)(f) = f\) w.r.t. the norm topology of \(\mathcal{H}_g\). But \(\eta^U_M\) is a semigroup of norm continuous operators, therefore we conclude that \(\eta^U_M\) is a \(C_0\)-group on \(\mathcal{H}_g\) and we let \(m_U\) denote its infinitesimal generator. Next by definition of \(\theta^U\) we deduce that for every \(f \in \mathcal{D}(M)\), \(E_Uf\) is the pointwise derivative at \(t = 0\) of the map \(t \mapsto \eta^U_M(t)f\). Thus by \(\mathcal{D}(M) \subseteq \mathcal{H}_g\) and by applying the Lebesgue Thm. we conclude that \(E_Uf\) is the derivative at \(t = 0\) of the map \(t \mapsto \eta^U_M(t)f\) w.r.t. the norm topology of \(\mathcal{H}_g\), namely \(m_U\) is an extension of \(E_U\). Now \(\eta^U_M(t)f \in L^1(M, dm_\rho)\) since \(\mu_\rho (\eta^U_M(t)f) = \mu_\rho (\eta^U_M(t)|f|) = \mu_\rho (|f|) < \infty\) where the first equality follows at once by the definition of \(\eta^U_M\), the second equality follows by \((36)\), the inequality follows since \(\mathcal{D}(M) \subseteq \mathfrak{K}(M) \subseteq L^1(M, dm_\rho)\). Moreover \(E_Uf \in \mathcal{D}(M) \subseteq L^1(M, dm_\rho)\) thus by applying the Lebesgue Thm. similarly as above, we obtain that \(E_Uf\) is the derivative at \(t = 0\) of the map \(t \mapsto \eta^U_M(t)f\) w.r.t. the norm topology of \(L^1(M, dm_\rho)\). Next \(\mu_\rho\) extends to an element of \(L^1(M, \mu_\rho)\), therefore what right now proven and \((36)\) imply that \(\mu_\rho \circ E_U = 0\) and st. \((1)\) follows. Now st. \((1)\) and Cor. 1.19 imply that there exists a unique \(C_0\)-group \(\exp^U_M\) on \(\mathcal{H}_g\) of unitary operators extending \(\exp^U_M\) and whose infinitesimal generator \(l_U\) extends \(E_U\), in particular
\[(37)\quad m_U \uparrow \mathcal{D}(M) = l_U \uparrow \mathcal{D}(M).\]

Therefore the additional equicontinuity hypothesis in st. \((2)\) implies that \(\mathcal{D}(M)\) is a set of analytic elements for both the generators \(m_U\) and \(l_U\), moreover \(\mathcal{D}(M)\) is dense in \(\mathcal{H}_g\) w.r.t. the norm topology thus also w.r.t. the \(\sigma(\mathcal{H}_g, \mathcal{H}_g^\ast)\)-topology, and \(E_U \mathcal{D}(M) \subseteq \mathcal{D}(M)\), thus we conclude that \(\mathcal{D}(M)\) is a core of \(m_U\) and \(l_U\) by applying well-known general results about the core of generators of \(C_0\)-semigroups, and then the first sentence of st. \((2)\) follows.

The first sentence of st. \((2)\) and \((37)\) imply \(m_U = l_U\) and then the second sentence of st. \((2)\) follows by the well-known uniqueness of the generator of a equicontinuous \(C_0\)-semigroup. 

\(\square\)
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