On a Class Of $h$-Fourier Integral Operators With The Complex Phase

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ABSTRACT: In this work, we study the $L^2$-boundedness and $L^2$-compactness of a class of $h$-Fourier integral operators with the complex phase. These operators are bounded (respectively compact) if the weight of the amplitude is bounded (respectively tends to 0).

Key Words: $h$-Fourier integral operators, $h$-pseudodifferential operators, complex function, Symbol and phase.

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1. Introduction

Since 1970, numerous mathematicians are interested in these types of operators:

$$F\varphi(x) = (2\pi h)^{-n} \int \int_{\mathbb{R}^n} e^{i(S(x, \theta) - y\theta)} a(x, \theta) \varphi(y) dyd\theta, \varphi \in \mathcal{S}(\mathbb{R}^n).$$

like [6,12,1,9,7,18]. The integral operators (1.1) appear naturally in the expression of the solutions of the semiclassical hyperbolic partial differential equations and when expressing the $C^\infty$ solution of the associated Cauchy’s problem. Two $C^\infty$ functions appear in (1.1): the phase function $\phi(x, y, \theta) = S(x, \theta) - y\theta$ and the amplitude $a$.

In 1974 Melin and Sjostrand [15] studied an extension of the computation of the Fourier integral in the case where the phase functions assume complex values.

Our work consist a spectral study the $L^2$-boundedness and $L^2$-compactness of a class of $h$-Fourier integral operators with the complex phase; we’re more particularly interested in continuity studies and on compactness on $L^2(\mathbb{R}^n)$.

It was proven in [1] by a very elaborate demonstration and under certains conditions (relatively strong) on the phase function $\phi$ and the amplitude $a$ that all operators of the form:

$$ (I(a, \phi; h) \psi)(x) = (2\pi h)^{-n} \int_{\mathbb{R}_y^n} e^{i\phi(x, \theta, y)} a(x, \theta, y) \psi(y) dyd\theta $$

are bounded on $L^2$, where $\psi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$.

The used technique is to show that $I(a, \phi) \Gamma^* (a, \phi)$, $\Gamma^* (a, \phi) I(a, \phi)$ are $h$–pseudodifferential and apply the Calderón-Vaillancourt’s theorem (here $\Gamma^* (a, \phi)$ is the adjoint of $I(a, \phi)$).

In this paper, we will apply the same technic of [1] to establish $L^2$-boundedness and $L^2$-compactness of form (1.1) operators. That’s why we will give brief demonstrations.
We mainly prove the continuity of the operator $F_h$ on $L^2(\mathbb{R}^n)$ when the weight of the amplitude $a$ is bounded. Moreover, $F_h$ is compact on $L^2(\mathbb{R}^n)$ if this weight tends to zero. Using the estimate given in [17,19] for $h$--pseudodifferential ($h$--admissible) operators, we also establish an $L^2$-estimate of $\|F_h\|$.

We note that if the amplitude $a$ is just bounded, the Fourier integral operator $F$ is not necessarily bounded on $L^2(\mathbb{R}^n)$.

2. A general class of $h$-Fourier integral operators with the complex phase

We consider the following integral transformations

$$
(I (a, \phi; h) \psi)(x) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n_y \mathbb{R}^n} e^{i\pi \phi(x,\theta,y)} a(x, \theta, y) \psi(y) dy d\theta
$$

(2.1)

for $\psi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ ((if $N = 0$, $\theta$ doesn’t appear in (2.1)).

In general, the integral (2.1) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander [13]. The phase function and the amplitude $a$ are assumed to satisfy the following hypothesis:

\begin{itemize}
  \item [(H1)] \quad $\phi \in C^\infty (\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mathbb{C})$ when $\phi$ is a complex function, $\text{Im}(\phi)$ is non negative.
  
  \item [(H2)] \quad For all $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, there exists $C_{\alpha,\beta,\gamma} > 0$, such that:
  \begin{equation}
  \left| \partial_y^\alpha \partial_\theta^\beta \partial_x^\gamma \phi (x, \theta, y) \right| \leq C_{\alpha,\beta,\gamma} \lambda^{(2-|\alpha|-|\beta|-|\gamma|)} (x, \theta, y)
  \end{equation}

where

$$
\lambda(x, \theta, y) = \left( 1 + |x|^2 + |\theta|^2 + |y|^2 \right)^{1/2},
$$

$$
(2 - |\alpha| - |\beta| - |\gamma|)_+ = \max (2 - |\alpha| - |\beta| - |\gamma|, 0)
$$

\item [(H3)] \quad There exists $K_1, K_2 > 0$, such that:
  
  $$
  K_1 \lambda(x, \theta, y) \leq \lambda(\partial_y \phi, \partial_\theta \phi, y) \leq K_2 \lambda(x, \theta, y), \quad \text{for all} \quad (x, \theta, y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y
  $$

\item [(H3)'] \quad There exists $K_1^*, K_2^* > 0$, such that:
  
  $$
  K_1^* \lambda(x, \theta, y) \leq \lambda(x, \partial_\theta \phi, \partial_x \phi) \leq K_2^* \lambda(x, \theta, y), \quad \text{for all} \quad (x, \theta, y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y
  $$

\end{itemize}

For any open subset $\Omega$ of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0, 1]$; we set:

$$
\Gamma_\mu^\rho (\Omega) = \left\{ a \in C^\infty (\Omega) : \forall (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists C_{\alpha,\beta,\gamma} > 0; \quad \left| \partial_y^\alpha \partial_\theta^\beta \partial_x^\gamma a(x, \theta, y) \right| \leq C_{\alpha,\beta,\gamma} \lambda^{\mu - \rho (|\alpha| + |\beta| + |\gamma|)} (x, \theta, y) \right\}
$$

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, we denote $\Gamma_\mu^\rho (\Omega) = \Gamma_\mu^\rho$. To give a meaning to the right hand side of (2.1), we consider $g \in \mathcal{S}\left( \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y \right)$, $g(0) = 1$.

If $a \in \Gamma_0^{\mu}$, we define

$$
a_\sigma (x, \theta, y) = g(x/\sigma, \theta/\sigma, y/\sigma) a(x, \theta, y), \quad \sigma > 0
$$

**Theorem 2.1.** If $\phi$ satisfies (H1), (H2), (H3), (H3)' and if $a \in \Gamma_0^{\mu}$ then:

1. For all $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{\sigma \to +\infty} [I (a_\sigma, \phi; h) \psi] (x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function $g$. We define:

$$
(I (a, \phi; h) \psi)(x) := \lim_{\sigma \to +\infty} (I (a_\sigma, \phi; h) \psi)(x),
$$

2. \ldots
2. \( I(\phi;h) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n)) \) and \( I(\phi;h) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n)) \) (here \( \mathcal{L}(\mathcal{S}(\mathbb{R}^n)) \) (resp. \( \mathcal{L}(\mathcal{S}'(\mathbb{R}^n)) \)) is the space of bounded linear mapping from \( \mathcal{S}(\mathbb{R}^n) \) to \( \mathcal{S}(\mathbb{R}^n) \) (resp. \( \mathcal{S}'(\mathbb{R}^n) \)) to \( \mathcal{S}'(\mathbb{R}^n) \)) and \( \mathcal{S}(\mathbb{R}^n) \) the space of all distributions with temperate growth on \( \mathbb{R}^n \).

**Proof.** Let \( \eta \in C^\infty(\mathbb{R}^n) \) such that \( \text{supp} \eta \subseteq [-1,2] \) and \( \eta \equiv 1 \) on \([0,1]\). For all \( \epsilon > 0 \), we set

\[
\omega_\epsilon(x, \theta, y) = \eta \left( \frac{|\partial_x \phi|^2 + |\partial_y \phi|^2}{\epsilon \lambda(x, \theta, y)^2} \right)
\]

The hypothesis \((H3)\) implies that there exists \( C > 0 \) such that we have on the support of \( \omega_\epsilon \)

\[
\lambda(x, \theta, y) \leq C \left[ (1 + |y|^2)^{\frac{3}{2}} + \epsilon \xi \lambda(x, \theta, y) \right]
\]

Therefore, there exists \( \epsilon_0 \) and a constant \( C_0 \), such that \( \forall \epsilon \leq \epsilon_0 \) we have on the support of \( \omega_\epsilon \)

\[
\lambda(x, \theta, y) \leq C_0 (1 + |y|^2)^{\frac{3}{2}}.
\]

In the sequel, we fix \( \epsilon = \epsilon_0 \). Then it is immediate that \( I(\omega_\epsilon, \phi; h) \psi \) is an absolutely convergent integral and we have

\[
I(\omega_\epsilon, \phi; h) \psi = \lim_{\sigma \to +\infty} I(\omega_\epsilon, \phi; h) \psi. \tag{2.2}
\]

Using \((H2)\) we prove also that \( I(\omega_\epsilon, \phi; h) \psi \) is a continuous operator from \( \mathcal{S}(\mathbb{R}^n) \) into itself. To study \( \lim_{\sigma \to +\infty} I((1 - \omega_\epsilon) a_\sigma, \phi; h) \psi \) we introduce the operator

\[
L = -i \hbar \left( |\partial_y \phi|^2 + |\partial_\theta \phi|^2 \right)^{-1} \sum_{l=1}^{n} (|\partial_y \phi| \partial_{y_l} - (\partial_{\theta_l} \phi) \partial_{\theta_l}) \].
\]

Clearly we have

\[
L(e^{\frac{i}{\hbar} \phi}) = e^{\frac{i}{\hbar} \phi}. \tag{2.3}
\]

Let \( \Omega_0 \) be the open subset of \( \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n \) defined by

\[
\Omega_0 = \left\{ (x, \theta, y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n; |\partial_y \phi|^2 + |\partial_\theta \phi|^2 > \frac{C_0}{2} \lambda(x, \theta, y)^2 \right\}.
\]

We need the following lemma.

**Lemma 2.2.** For all integer \( q \geq 0 \), and \( b \in C^\infty(\mathbb{R}^n \times \mathbb{R}_y^N) \), we have

\[
(\cdot L)^q ((1 - \omega_\epsilon_0) b) = \sum_{|\alpha| + |\beta| \leq q} g_{\alpha, \beta}^q \partial_\alpha \partial_\beta \left( (1 - \omega_\epsilon_0) b \right),
\]

\( \cdot L \) designates the transpose of \( L \), \( g_{\alpha, \beta}^q \in \Gamma_0^{-q}(\Omega_0) \) and depend only on \( \phi \).

We prove the lemma by recurrence. It is obvious for \( q = 0 \). Now we see easily that

\[
\cdot L = \sum_i (F_i \partial_{y_i} + G_i \partial_{\theta_i}) + H, \tag{2.4}
\]

where \( F_i, G_i \) in \( \Gamma_0^{-1}(\Omega_0) \), and \( H \in \Gamma_0^{-2}(\Omega_0) \) (wich results from \((H2)\)). Therefore, the recurrence is immediately proved.

We have from \((2.3)\), \( \forall q \geq 0 \)

\[
I((1 - \omega_\epsilon_0) a_\sigma, \phi; h) \psi(x) = \frac{1}{(2\pi \hbar)^{n}} \int_{\mathbb{R}_y^n} e^{\frac{i}{\hbar} \phi(x, \theta, y)} (\cdot L)^q ((1 - \omega_\epsilon_0) a_\sigma \psi; h) (x, \theta, y) dy d\theta. \tag{2.5}
\]
Now \((^tL)^q ((1 - \omega_{\alpha}) a_{\sigma} \psi)\) described (when \(q\) varies) a bound of \(\Gamma_0^{\mu-q}\), and for all \((x, \theta, y) \in \mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n\)

\[
\lim_{\sigma \to \infty} (^tL)^q ((1 - \omega_{\alpha}) a_{\sigma} \psi) (x, \theta, y) = (^tL)^q ((1 - \omega_{\alpha}) a \psi) (x, \theta, y).
\]  

(2.6)

Finally, \(\forall s > n + N\) we have

\[
\iint_{\mathbb{R}_x^N \times \mathbb{R}_\theta^N} \lambda^{-s} (x, \theta, y) \, dy \, d\theta \leq C_n \lambda^{n+N-s} (x).
\]  

(2.7)

So it results from (2.5), (2.7) and using Lebesgue’s theorem we have

\[
\lim_{\sigma \to \infty} I ((1 - \omega_{\alpha}) a_{\sigma}, \phi; h) \psi(x) = (2\pi h)^{-n} \iint_{\mathbb{R}_x^N \times \mathbb{R}_\theta^N} e^{\frac{i}{h} \phi(x, \theta, y)} (^tL)^q ((1 - \omega_{\alpha}) a \psi; h) (x, \theta, y) \, dy \, d\theta.
\]  

(2.8)

where \(q > n + N + \mu\). From (2.2) and (2.8) we can prove the first part of the theorem.

Now let us show that \(I ((1 - \omega_{\alpha}) a_{\sigma}, \phi; h)\) is continuous. Taking account of (2.4) and (2.8), we get

\[
I ((1 - \omega_{\alpha}) a, \phi; h) \psi(x) = (2\pi h)^{-n} \sum_{|\gamma| \leq q} \iint_{\mathbb{R}_x^N \times \mathbb{R}_\theta^N} e^{\frac{i}{h} \phi(x, \theta, y)} b_\gamma^{(q)} (x, \theta, y) \partial_y^\gamma \psi(y) \, dy \, d\theta,
\]  

(2.9)

with \(b_\gamma^{(q)} \in \Gamma_0^{\mu-q}\). On the other hand, we have

\[
x^\alpha \partial_x^\beta \left( e^{\frac{i}{h} \phi(x, \theta, y)} b_\gamma^{(q)} (x, \theta, y) \right) \in \Gamma_0^{\mu-q+|\alpha|+|\beta|}.
\]  

(2.10)

We deduce from (2.9) and (2.10) that, for all \(q > n + N + \mu + |\alpha| + |\beta|\), there exists a constant \(C_{\alpha, \beta, q}\) such that

\[
|\partial_x^\alpha \partial_y^\beta \left( I ((1 - \omega_{\alpha}) a, \phi; h) \psi(x) \right) | \leq C_{\alpha, \beta, q} \sup_{x \in \mathbb{R}_x^n} \left| \partial_y^\gamma \psi(x) \right|,
\]

which proves the continuity of \(I ((1 - \omega_{\alpha}) a, \phi; h)\).

\[\square\]

**Example 2.3.** Let us give two examples of operators of the form (1.1) which satisfy (H1) to (H3)*:

1. **The Fourier transform**

   \[S(\mathbb{R}^n) \ni \psi \mapsto \mathcal{F} \psi (x) = \int_{\mathbb{R}^n} e^{-\frac{i}{h} \pi \theta y} \psi (y) \, dy,\]

2. **Pseudodifferential operators**

   \[S(\mathbb{R}^n) \ni \psi \mapsto \text{Op} \psi (x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h} \pi (x - y) \theta} a (x, y, \theta) \psi (y) \, dy \, d\theta,\]

   \[a \in \Gamma_0^{\mu} (\mathbb{R}^{3n}).\]
3. Special form of the phase function

We consider the phase function $\phi(x, y, \theta) = S(x, \theta) - y\theta$. Where

$$S(x, \theta) = f(x, \theta) + iT(x, \theta),$$

(3.1)

and $S$ satisfies: (G1) $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\theta, \mathbb{C})$, ($S$ is a complex function)

(G2) For all $(\alpha, \beta) \in \mathbb{N}^{2n}$, there exists $C'_{\alpha, \beta} > 0$,

$$\left| \partial^\alpha_x \partial^\beta_\theta f(x, \theta) \right| \leq C'_{\alpha, \beta} \lambda(x, \theta)^{2-|\alpha|-|\beta|}$$

(G3) For all $(x, \theta) \in \mathbb{R}^{2n}$, $T(x, \theta)$ is nonnegative, and for all $(\alpha, \beta) \in \mathbb{N}^{2n}$, there exists $C''_{\alpha, \beta} > 0$,

$$\left| \partial^\alpha_x \partial^\beta_\theta T(x, \theta) \right| \leq C''_{\alpha, \beta} \lambda(x, \theta)^{2-|\alpha|-|\beta|}$$

(G4) There exists $\delta_0 > 0$,

$$\inf_{x, \theta \in \mathbb{R}^n} \left| \det \frac{\partial^2 f}{\partial x \partial \theta}(x, \theta) \right| \geq \delta_0.$$

Lemma 3.1. [16] If $S$ satisfies (G1), (G2), (G3) and (G4). Then the function $\phi(x, y, \theta) = S(x, \theta) - y\theta$ satisfies (H1), (H2), (H3) and (H3')

Lemma 3.2. [16] If $S$ satisfies (G1), (G2), (G3) and (G4), then there exists $C_2 > 0$, such that for all $(x, \theta, (x', \theta')) \in \mathbb{R}^{2n}$

$$|x - x'| + |\theta - \theta'| \leq C_2 \left[ \left| (\partial_y f)(x, \theta) - (\partial_y f)(x', \theta') \right| + |\theta - \theta'| \right]$$

(3.2)

Lemma 3.3. [8] If $S$ satisfies (G1), (G2) et (G3). Then there exists a constant $\varepsilon_0 > 0$, such that the phase function $\phi$ belongs to $\Gamma^1_2(\Omega_{\phi, \varepsilon_0})$, where

$$\Omega_{\phi, \varepsilon_0} = \left\{ (x, \theta, y) \in \mathbb{R}^{3n}; \left| \partial_y \phi(x, \theta, y) \right|^2 < \varepsilon_0 \left( |x|^2 + |y|^2 + |\theta|^2 \right) \right\}$$

Proposition 3.4. [8] If $(x, \theta) \mapsto a(x, \theta)$ belongs to $\Gamma^m_k(\mathbb{R}^n_x \times \mathbb{R}^n_\theta)$, then the function $(x, \theta, y) \mapsto a(x, \theta)$ belongs to $\Gamma^m_k(\mathbb{R}^n_x \times \mathbb{R}^n_\theta \cap \Gamma^1_k(\Omega_{\phi, \varepsilon_0}), k \in \{0, 1\}$.

4. $L^2$-boundedness and $L^2$-compactness of $F_h$ with the complex phase

Theorem 4.1. Let $F_h$ be the integral operator of distribution kernel

$$K(x, y) = \int_{\mathbb{R}^n} e^{\frac{i}{\pi}f(x, \theta)} \frac{2\pi i}{h} a(x, \theta) \, d\theta$$

(4.1)

where $d\theta = (2\pi)^{-n} d\theta$, $a \in \Gamma^m_k(\mathbb{R}^{2n}_x)$, $k = 0, 1$ and $S$ satisfies (G1), (G2), (G3) and (G4). Then $FF^*$ and $F^*F$ are $h$-pseudodifferential operators with symbol in $\Gamma^m_k(\mathbb{R}^{2n})$, $k = 0, 1$, given by

$$\sigma(FF^*)(x, \partial_x f(x, \theta)) = e^{-\frac{2\pi i}{h} f(x, \theta)} \left| a(x, \theta) \right|^2 \left| \left( \det \frac{\partial^2 f}{\partial \theta \partial x}(x, \theta)^{-1} \right) \right|$$

$$\sigma(F^*F)(\partial_y f(x, \theta), \theta) = e^{-\frac{2\pi i}{h} f(x, \theta)} \left| a(x, \theta) \right|^2 \left| \left( \det \frac{\partial^2 f}{\partial \theta \partial x}(x, \theta)^{-1} \right) \right|$$

We denote here $a \equiv b$ for $a, b \in \Gamma^p_k(\mathbb{R}^{2n})$ if $(a - b) \in \Gamma^{2p-2}_k(\mathbb{R}^{2n})$ and $\sigma$ stands for the symbol.
Proof. If $u \in S(\mathbb{R}^n)$, we have

$$F_h u(x) = \int_{\mathbb{R}^n} K(x, y) u(y) \, dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i \frac{1}{h} (f(x, \theta) - f(\tilde{x}, \theta))} a(x, \theta) u(y) \, dy \, d\theta$$

$$= \int_{\mathbb{R}^n} e^{i \frac{1}{h} f(x, \theta) - \frac{T(x, \theta)}{h}} a(x, \theta) (\int_{\mathbb{R}^n} e^{-\frac{1}{h} y^T \theta} u(y) \, dy) d\theta$$

$$= \int_{\mathbb{R}^n} e^{i \frac{1}{h} f(x, \theta) - \frac{T(x, \theta)}{h}} a(x, \theta) F u(\theta) d\theta, \quad (4.2)$$

where $F$ the Fourier transform and for all $v \in S(\mathbb{R}^n)$,

$$< F_h u, v >_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} e^{i \frac{1}{h} f(x, \theta) - \frac{T(x, \theta)}{h}} a(x, \theta) F u(\theta) d\theta) v(x) \, dx$$

$$< F_h u, v >_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \hat{u}(\theta)(\int_{\mathbb{R}^n} e^{i \frac{1}{h} f(x, \theta) - \frac{T(x, \theta)}{h}} a(x, \theta) v(x) \, dx) d\theta,$$

then

$$< Fu(x), v(x) >_{L^2(\mathbb{R}^n)} = (2\pi h)^{-n} < F u(\theta), F ((F^* v)(\theta)) >_{L^2(\mathbb{R}^n)}$$

and,

$$\mathcal{F}((F^* v)(\theta)) = \int_{\mathbb{R}^n} e^{-i \frac{1}{h} f(\tilde{x}, \theta) - \frac{T(\tilde{x}, \theta)}{h}} a(\tilde{x}, \theta) v(\tilde{x}) d\tilde{x}. \quad (4.3)$$

We have,

$$(F F^* v)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i \frac{1}{h} (f(x, \theta) - f(\tilde{x}, \theta))} e^{-\frac{(T(x, \theta) + T(\tilde{x}, \theta))}{h} \theta} a(x, \tilde{x}) v(\tilde{x}) d\tilde{x} d\theta, \quad (4.4)$$

for all $v \in S(\mathbb{R}^n)$. The main idea to show that $F F^*$ is a $h-$pseudodifferential operator, is to use the fact that $f(x, \theta) - f(\tilde{x}, \theta)$ can be expressed by the scalar product $< x - \tilde{x}, \xi (x, \tilde{x}, \theta) >$ after considering the change of variables

$$(x, \tilde{x}, \theta) \rightarrow (x, \tilde{x}, \xi = \xi (x, \tilde{x}, \theta)).$$

The distribution kernel of $F F^*$ is

$$K(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{i \frac{1}{h} (f(x, \theta) - f(\tilde{x}, \theta))} e^{-\frac{(T(x, \theta) + T(\tilde{x}, \theta))}{h} \theta} a(x, \theta) v(\tilde{x}) d\theta.$$ 

We obtain from (3.2) that if

$$|x - \tilde{x}| \geq \frac{\varepsilon}{2} \lambda (x, \tilde{x}, \theta) \quad \text{(where } \varepsilon > 0 \text{ is sufficiently small)}$$

Then

$$|(\partial_\theta f)(x, \theta) - (\partial_\theta f)(\tilde{x}, \theta)| \geq \frac{\varepsilon}{2C_2} \lambda (x, \tilde{x}, \theta). \quad (4.5)$$

Choosing $C^\infty (\mathbb{R})$ such that

$$\left\{ \begin{array}{l} \omega (x) \geq 0, \quad \forall x \in \mathbb{R} \\ \omega (x) = 1 \quad \text{si} \quad x \in [-\frac{1}{2}, \frac{1}{2}] \\ \text{supp} \omega \subset [-1, 1] \end{array} \right.$$ 

and setting

$$\left\{ \begin{array}{l} b(x, \tilde{x}, \theta) := e^{-\frac{(T(x, \theta) + T(\tilde{x}, \theta))}{h} \theta} a(x, \theta) v(\tilde{x}, \theta) = b_{1, \varepsilon} (x, \tilde{x}, \theta) + b_{2, \varepsilon} (x, \tilde{x}, \theta) \\ b_{1, \varepsilon} (x, \tilde{x}, \theta) = \omega \left( \frac{|x - \tilde{x}|}{\varepsilon \lambda (x, \tilde{x}, \theta)} \right) b(x, \tilde{x}, \theta) \\ b_{2, \varepsilon} (x, \tilde{x}, \theta) = \left[ 1 - \omega \left( \frac{|x - \tilde{x}|}{\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \right] b(x, \tilde{x}, \theta). \end{array} \right.$$
We have

\[ K(x, \tilde{x}) = K_{1, \varepsilon}(x, \tilde{x}) + K_{2, \varepsilon}(x, \tilde{x}), \]

where

\[ K_{j, \varepsilon}(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{\varepsilon \pi (f(x, \theta) - f(\tilde{x}, \theta))} b_{j, \varepsilon}(x, \tilde{x}, \theta) \, d\theta, \quad j = 1, 2. \]

We will study separately the kernels \( K_{1, \varepsilon} \) and \( K_{2, \varepsilon} \).

The study of \( K_{2, \varepsilon} \). We shall show that for all \( h \), we have

\[ K_{2, \varepsilon}(x, \tilde{x}) \in S(\mathbb{R}^n \times \mathbb{R}^n). \]

Indeed, let

Then \( L \) is a linear partial differential operator \( L \) of order 1 such that

\[ L \left( e^{\varepsilon \pi (f(x, \theta) - f(\tilde{x}, \theta))} \right) = e^{\varepsilon \pi (f(x, \theta) - f(\tilde{x}, \theta))}, \]

where \( L = -ih \left| (\partial_{\theta} f)(x, \theta) - (\partial_{\theta} f)(\tilde{x}, \theta) \right|^2 \sum_{l=1}^{n} \left| (\partial_{\theta} f)(x, \theta) - (\partial_{\theta} f)(\tilde{x}, \theta) \right| \, d\theta. \)

The transpose operator of \( L \) is

\[ {^t}L = \sum_{l=1}^{n} F_l(x, \tilde{x}, \theta) \, \partial_{\theta_l} + G(x, \tilde{x}, \theta) \]

where \( F_l(x, \tilde{x}, \theta) \in \Gamma_0^{-1}(\Omega_{\varepsilon}), G(x, \tilde{x}, \theta) \in \Gamma_0^{-2}(\Omega_{\varepsilon}): \)

\[ F_l(x, \tilde{x}, \theta) = ih \left| (\partial_{\theta} f)(x, \theta) - (\partial_{\theta} f)(\tilde{x}, \theta) \right|^2 \left( (\partial_{\theta} f)(x, \theta) - (\partial_{\theta} f)(\tilde{x}, \theta) \right) \]

\[ G(x, \tilde{x}, \theta) = ih \sum_{l=1}^{n} \partial_{\theta_l} \left[ \left| (\partial_{\theta} f)(x, \theta) - (\partial_{\theta} f)(\tilde{x}, \theta) \right|^2 \left( (\partial_{\theta} f)(x, \theta) - (\partial_{\theta} f)(\tilde{x}, \theta) \right) \right] \]

\[ \Omega_{\varepsilon} = \left\{ (x, \tilde{x}, \theta) \in \mathbb{R}^{3n}; \left| \partial_{\theta} f(x, \theta) - \partial_{\theta} f(\tilde{x}, \theta) \right| > \frac{\varepsilon}{2C_2} \lambda(x, \tilde{x}, \theta) \right\}. \]

On the other hand we prove by induction on \( q \) that

\[ \left( {^t}L \right)^q b_{2, \varepsilon}(x, \tilde{x}, \theta) = \sum_{|\gamma| \leq q} g^{(q)}_\gamma(x, \tilde{x}, \theta) \partial_{\theta}^\gamma b_{2, \varepsilon}(x, \tilde{x}, \theta), \quad g^{(q)}_\gamma \in \Gamma_0^{-q}, \quad \forall q \in \mathbb{N}, \]

and so,

\[ K_{2, \varepsilon}(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{\varepsilon \pi (f(x, \theta) - f(\tilde{x}, \theta))} \left( {^t}L \right)^q b_{2, \varepsilon}(x, \tilde{x}, \theta) \, d\theta. \]

Using Leibnitz’s formula, \((G2)\) and the form \( \left( {^t}L \right)^q \), we can choose \( q \) large enough such that

\[ \forall \alpha, \alpha', \beta, \beta' \in \mathbb{N}^n, \exists C_{\alpha, \alpha', \beta, \beta'} > 0; \sup_{x, \tilde{x} \in \mathbb{R}^n} x^{\alpha} \tilde{x}^{\alpha'} \partial^\beta_x \partial^\beta_{\tilde{x}} K_{2, \varepsilon}(x, \tilde{x}) \leq C_{\alpha, \alpha', \beta, \beta'}. \]

Next, we study \( K_{1, \varepsilon} \). This is more difficult and depends on the choice of the parameter \( \varepsilon \). It follows from Taylor’s formula that

\[ f(x, \theta) - f(\tilde{x}, \theta) = \langle x - \tilde{x}, \xi(x, \tilde{x}, \theta) \rangle_{\mathbb{R}^n} \]

\[ \xi(x, \tilde{x}, \theta) = \int_0^1 (\partial_{x} f) (\tilde{x} + t \, (x - \tilde{x}), \theta) \, dt. \]
We define the vectorial function:
\[
\tilde{\xi}_\varepsilon(x, \tilde{x}, \theta) = \omega \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \xi(x, \tilde{x}, \theta) + \left( 1 - \omega \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \right) (\partial_x f)(\tilde{x}, \theta).
\]

We have
\[
\tilde{\xi}_\varepsilon(x, \tilde{x}, \theta) = \xi(x, \tilde{x}, \theta) \text{ on supp} b_{1,\varepsilon},
\]
Moreover, for \( \varepsilon \) sufficiently small,
\[
\lambda(x, \theta) \simeq \lambda(\tilde{x}, \theta) \simeq \lambda(x, \tilde{x}, \theta) \text{ on supp} b_{1,\varepsilon}.
\]
Let us consider the mapping
\[
\mathbb{R}^{3n} \ni (x, \tilde{x}, \theta) \mapsto \left( x, \tilde{x}, \tilde{\xi}_\varepsilon(x, \tilde{x}, \theta) \right); \tag{4.7}
\]
for which Jacobian matrix is
\[
\begin{pmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
\partial_x \tilde{\xi}_\varepsilon & \partial_\tilde{x} \tilde{\xi}_\varepsilon & \partial_\theta \tilde{\xi}_\varepsilon
\end{pmatrix}.
\]
We have
\[
\frac{\partial \tilde{\xi}_{\varepsilon,j}}{\partial \theta_i}(x, \tilde{x}, \theta) = \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) + \omega \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \left( \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right)
\]
\[
- \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \frac{\partial \lambda}{\partial \theta_i}(x, \tilde{x}, \theta) \lambda^{-1}(x, \tilde{x}, \theta)
\]
\[
\times \omega' \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \left( \xi_j(x, \tilde{x}, \theta) - \frac{\partial f}{\partial x_j}(\tilde{x}, \theta) \right).
\]
Thus, using that supp \( \omega' \subset \text{supp} \omega \subset ]-1,1[ \) and \( \frac{\partial \lambda}{\partial \theta_i}(x, \tilde{x}, \theta) \leq 1 \), we obtain
\[
\left| \frac{\partial \tilde{\xi}_{\varepsilon,j}}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \leq \omega \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \left| \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right|
\]
\[
+ \lambda^{-1}(x, \tilde{x}, \theta)
\]
\[
\times \omega' \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda (x, \tilde{x}, \theta)} \right) \left| \xi_j(x, \tilde{x}, \theta) - \frac{\partial f}{\partial x_j}(\tilde{x}, \theta) \right|.
\]
Now it follows from (G2), (4.6) and Taylor’s formula that
\[
\left| \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \leq \int_0^1 \left| \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x} + t(x - \tilde{x}), \theta) - \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| dt
\]
\[
\leq C_5 |x - \tilde{x}| \lambda^{-1}(x, \tilde{x}, \theta), \quad C_5 > 0 \tag{4.8}
\]
\[
\left| \xi_j(x, \tilde{x}, \theta) - \frac{\partial f}{\partial x_j}(\tilde{x}, \theta) \right| \leq \int_0^1 \left| \frac{\partial f}{\partial x_j}(\tilde{x} + t(x - \tilde{x}), \theta) - \frac{\partial f}{\partial x_j}(\tilde{x}, \theta) \right| dt
\]
\[
\leq C_6 |x - \tilde{x}|, \quad C_6 > 0. \tag{4.9}
\]
From (4.8) and (4.9), there exists a positive constant \( C_7 > 0 \), such that
\[
\left| \frac{\partial \tilde{\xi}_{\varepsilon,j}}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 f}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \leq C_7 \varepsilon, \quad \forall i, j \in \{1, \ldots, n\}. \tag{4.10}
\]
If \( \varepsilon < \frac{\delta_0}{2C} \), then (4.10) and (G4) yields the estimate
\[
\delta_0/2 \leq -C\varepsilon + \delta_0 \leq -C\varepsilon + \det \frac{\partial^2 f}{\partial x \partial \theta}(x, \theta) \leq \det \partial_\theta \tilde{\xi}_\varepsilon (x, \theta, \theta), \text{ with } C > 0.
\] (4.11)

If \( \varepsilon \) is such that (4.6) and (4.11) are true, then the mapping given in (4.7) is a global diffeomorphism of \( \mathbb{R}^{3n} \). Hence there exists a mapping
\[ \theta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (x, \bar{x}, \xi) \rightarrow \theta (x, \bar{x}, \xi) \in \mathbb{R}^n \]
such that
\[
\begin{align*}
\tilde{\xi}_\varepsilon (x, \bar{x}, \theta (x, \bar{x}, \xi)) &= \xi \\
\theta \left( x, \bar{x}, \tilde{\xi}_\varepsilon (x, \bar{x}, \theta) \right) &= x \\
\partial^2 \theta (x, \bar{x}, \xi) &= 0 (1), \quad \forall \alpha \in \mathbb{N}^{3n} \setminus \{0\}.
\end{align*}
\] (4.12)

If we change the variable \( \xi \) by \( \theta (x, \bar{x}, \xi) \) in \( K_{1, \varepsilon} (x, \bar{x}) \) we obtain
\[
K_{1, \varepsilon} (x, \bar{x}) = \int_{\mathbb{R}^n} e^{ix - \bar{x}, \xi} \cdot b_{1, \varepsilon} (x, \bar{x}, \theta (x, \bar{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi} (x, \bar{x}, \xi) \right| d\xi
\] (4.13)

From (4.12) we have, for \( k = 0, 1 \), that \( b_{1, \varepsilon} (x, \bar{x}, \theta (x, \bar{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi} (x, \bar{x}, \xi) \right| \) belongs to \( \Gamma^m_k (\mathbb{R}^{3n}) \) if \( a \in \Gamma^m_k (\mathbb{R}^{2n}) \).

Applying the stationary phase theorem (cf. [20,17]) to (4.13), we obtain the expression of the symbol of the \( h \)-pseudodifferential operator \( FF^* \):
\[
\sigma(FF^*) = b_{1, \varepsilon} (x, \bar{x}, \theta (x, \bar{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi} (x, \bar{x}, \xi) \right|_{|\bar{x} = x} + R(x, \xi),
\]
where \( R(x, \xi) \in \Gamma^{2m-2}_k (\mathbb{R}^{2n}) \) if \( a \in \Gamma^m_k (\mathbb{R}^{2n}) \), \( k = 0, 1 \).

For \( \bar{x} = x \), we have
\[
b_{1, \varepsilon} (x, \bar{x}, \theta (x, \bar{x}, \xi)) = e^{-\frac{2T(x, \theta)}{\hbar}} \left| a (x, \theta (x, x, \xi)) \right|^2,
\]
where \( \theta (x, x, \xi) \) is the inverse of the mapping \( \theta \rightarrow \partial_x f (x, \theta) = \xi \). Thus
\[
\sigma(FF^*) (x, \partial_x f (x, \theta)) = e^{-\frac{2T(x, \theta)}{\hbar}} \left| a (x, \theta) \right|^2 \left| \det \frac{\partial^2 f}{\partial \theta \partial x} (x, \theta) \right|^{-1}.
\]

By (4.2) and (4.3) we have:
\[
(F(F^*F)F^{-1}) \psi (\theta) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} f(x, \theta) - \frac{T(x, \theta)}{\hbar}} a (x, \theta) F(F^{-1} \psi) (x) dx
\]
\[
= \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} S(x, \theta) - \frac{T(x, \theta)}{\hbar}} a(x, \theta) \times \left( \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} f(x, \theta) - \frac{T(x, \theta)}{\hbar}} a \left( x, \theta \right) F(F^{-1} \psi) \left( \theta \right) \bar{\theta} d\theta dx \right)
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} \left( f(x, \theta) - f(x, \theta) \right) - \frac{T(x, \theta)+T(x, \bar{\theta})}{\hbar}} e^{-\frac{T(x, \theta)+T(x, \bar{\theta})}{\hbar}} \times a(x, \theta) a \left( x, \theta \right) v \left( \bar{\theta} \right) d\theta dx, \forall \psi \in S (\mathbb{R}^n).
The distribution kernel of the integral operator $\mathcal{F}(F^*F)^{-1}$ is

$$
\mathcal{K}(\theta, \tilde{\theta}) = \int_{\mathbb{R}^n} e^{-\frac{\pi}{h} (f(x, \theta) - f(x, \tilde{\theta}))} e^{-\frac{(T(x, \theta) + T(x, \tilde{\theta}))}{\pi} a(x, \theta)} a(x, \theta) \, dx.
$$

Observe that we can deduce $K(x, \tilde{x})$ from $\mathcal{K}(\theta, \tilde{\theta})$ by replacing $x$ by $\theta$. On the other hand, all assumptions used here are symmetrical on $x$ and $\theta$ so therefore $\mathcal{F}(F^*)^T = \mathcal{I}$ is a nice $h-$pseudodifferential operator with symbol

$$
\sigma(\mathcal{F}(F^*)^T) = e^{-\frac{2T(x, \theta)}{h}} |a(x, \theta)|^2 \left| \frac{\partial^2 f}{\partial x \partial \theta}(x, \theta) \right|^{-1}. 
$$

Thus the symbol of $F^*F$ is given by (cf. [14])

$$
\sigma(F^*F)(\partial_\theta f(x, \theta), \theta) = e^{-\frac{2T(x, \theta)}{h}} |a(x, \theta)|^2 \left| \frac{\partial^2 f}{\partial x \partial \theta}(x, \theta) \right|^{-1}. 
$$

\[ \square \]

**Corollary 4.2.** Let $F_h$ be the integral operator with the distribution kernel

$$
K(x, y) = \int_{\mathbb{R}^n} e^{\frac{\pi}{h} (f(x, \theta) + T(x, \theta))} a(x, \theta) \, d\theta
$$

where $a \in \Gamma^m_0(\mathbb{R}^{2n})$ and $S$ satisfies (G1), (G2), (G3) and (G4).

Then, we have:

1) If $m \leq 0$, $F_h$ can be extended to a bounded linear mapping on $L^2(\mathbb{R}^n)$.

2) If $m < 0$, $F_h$ can be extended to a compact operator on $L^2(\mathbb{R}^n)$.

**Proof.** It follows from Theorem 4.1 that $F_h^*F_h$ is a $h-$pseudodifferential operator with symbol in $\Gamma^m_0(\mathbb{R}^{2n})$.

1) If $m \leq 0$, the weight $\lambda^m(x, \theta)$ is bounded, so we can apply the Caldérón-Vaillancourt theorem (cf. [3,17,19]) for $F_h^*F_h$ and obtain the existence of a positive constant $\gamma(n)$ and an integer $k(n)$ such that

$$
\| (F_h^*F_h) u \|_{L^2(\mathbb{R}^n)} \leq \gamma(n) Q_{k(n)}(\sigma(F_h^*F_h)) \| u \|_{L^2(\mathbb{R}^n)}, \forall u \in \mathcal{S}(\mathbb{R}^n),
$$

where

$$
Q_{k(n)}(\sigma(F_h^*F_h)) = \sum_{|\alpha| + |\beta| \leq k(n)} \sup_{(x, \theta) \in \mathbb{R}^{2n}} \left| \partial_\theta^\alpha \partial_x^\beta \sigma(F_h^*F_h)(\partial_\theta f(x, \theta), \theta) \right|.
$$

Hence, we have for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$
\| F_h u \|_{L^2(\mathbb{R}^n)} \leq \| F_h^*F_h \|_{L^2(\mathbb{R}^n)}^{1/2} \| u \|_{L^2(\mathbb{R}^n)} \leq \left( \gamma(n) Q_{k(n)}(\sigma(F_h^*F_h)) \right)^{1/2} \| u \|_{L^2(\mathbb{R}^n)}.
$$

Thus $F_h$ is also a bounded linear operator on $L^2(\mathbb{R}^n)$.

2) If $m < 0$, $\lim_{|x| + |\theta| \to \infty} \lambda^m(x, \theta) = 0$, and the compactness theorem (see. [17,19]) show that the operator $F_h^*F_h$ can be extended to a compact operator on $L^2(\mathbb{R}^n)$. Thus, the Fourier integral operator $F_h$ is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$
\left\| F_h^*F_h - \sum_{j=1}^n \varphi_j, \cdot, F_h^*F_h \varphi_j \right\|_{n \to +\infty} \to 0.
$$
Since $F_h$ is bounded, we have for all $l \in L^2(\mathbb{R}^n)$

\[ \left\| F_h l - \sum_{j=1}^{n} < \varphi_j, l > F_h \varphi_j \right\|^2 \leq \left\| F_h^* F_h l - \sum_{j=1}^{n} < \varphi_j, l > F_h^* F_h \varphi_j \right\| \left\| l - \sum_{j=1}^{n} < \varphi_j, l > \varphi_j \right\| . \]

Hence

\[ \left\| F_h - \sum_{j=1}^{n} < \varphi_j, l > F_h \varphi_j \right\| \longrightarrow 0. \]

\[ \square \]

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