A general framework of nondistortion quantum interrogation

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Abstract

We present a general framework to study nondistortion quantum interrogation which preserves the internal state of the quantum object being detected. We obtain the necessary and sufficient condition for successful performing nondistortion interrogation for unknown quantum object when the interaction between the probe system and the detected system takes place only once. When the probe system and interrogation process have been limited we develop a mathematical frame to determine whether it is possible to realize NQI processes only relying on the choice of the original probe state. We also consider NQI process in iterative cases. A sufficient criterion for NQI is obtained.

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I. INTRODUCTION

One of the counterintuitive effects of quantum mechanics is the “negative result measurement”, i.e. the nonobservance of a result represents additional information and hence modifies the wave function. This notion is first prescribed in 1960 by Renninger [1], and later Dicke who analyzed the change of an atomic wave function by the nonscattering of a photon [2]. In 1993, Elitzur and Vaidman proposed the novel concept of interaction-free measurements (IFMs), in which the presence of an absorbing object in a Mach-Zehnder interferometer can be inferred without apparent interaction with the probe photon [3]. In recent years, EV IFMs led to numerous investigations and several experiments have been performed [1, 3]. In the original EV scheme, the measurement is interaction-free at most half of the time. However, when combined with quantum zeno effect IFMs can in principle be done with unity efficiency, in an asymptotic sense [14].

Aside from their conceptual significance, IFMs are of obvious application interest too. As emphasized by Vaidman, the paradoxical feature of the EV IFM is that it obtains information about a region in space without anything coming in, out, or through this place [12]. This feature allows people to monitor radiation sensitive objects without exciting them, such as the quantum nondemolition measurement of the ground state atom number [13], noninvasive testing of materials [14, 15], localization of atom beams without physical interaction [17], interaction-free imaging [18] and so on. People also applied the idea of IFMs in quantum information science, such as quantum cryptography [16] and counterfactual computation [19].

In the field of quantum information science, a physical system may be regarded as an information carrier if it can be prepared in some distinguishable quantum states. The advantage of processing information quantum mechanically is that, contrary to the classical case, information can be stored in quantum superposition states. However the quantum superposition is quite fragile. It is interesting to ask whether the similar treatments on IFMs can be applied in the nondemolition interrogation of the internal state of quantum object.
Suppose there is a black box, in which a quantum object characterized by its quantum superposition can be present (or absent). Our task is to determine if the quantum object in it, without disturbing the internal state of the object. We may call such an interrogation a nondistortion quantum interrogation (abbreviated NQI). Unfortunately, IFMs in general are not internal state preserving measurements. Since quantum superposition of the object is subject to measurement dependent decoherence. Obtaining of “which way” information will cause an unavoidable change of the internal state of the detected object, even though the measurement is seemingly “interaction free”.

In this paper we will concentrate on some fundamental limits for the NQI process, under some definite physical conditions. In Section II, we present general physical assumptions that we use to formulate the NQI process. Section III is devoted to single-shot NQI process, in which the interaction between the detected object and the probe system takes place only once. We provide the necessary and sufficient condition for successful performing a single-shot nondistortion interrogation of an unknown quantum state. We also make a primary attempt to test this criterion when the probe system and the interaction are restricted. We find that in some simple cases it reduces to a solvable problem. In Section IV we discuss iterative NQI processes. A sufficient criterion for Zeno type NQI processes is obtained. We conclude in Section V by summarizing all the results.

II. FUNDAMENTAL FRAMEWORK OF NQI

“Black box” problems have attracted people in many contexts. The aim of the nondistortion quantum interrogation is to find out whether there is a quantum object in a black box (a region in space) without disturbing the internal state of the object, even if its original quantum state is unknown.

According to Von Neumann’s treatments of quantum measurement, any observable can be coupled to a pointer. The information about the observable is obtained by detecting the pointer [20]. All the possible measurement results about this pointer can be described by a
set of orthogonal projectors \( \{P_a\} \) in Hilbert space which satisfy

\[
P_a = P_a^+, P_a P_b = \delta_{ab} P_a, \sum_a P_a = 1.
\] (1)

A Von Neumann measurement will take the probed pure state \( |\Phi\rangle \langle \Phi| \) to \( \frac{P_a |\Phi\rangle \langle \Phi| P_a}{(\langle \Phi| P_a |\Phi\rangle} \) with probability \( \text{Pr}(a) = \langle \Phi| P_a |\Phi\rangle \), if the final detected pointer is in the \( P_a \). The original quantum state will be modified by the measurement process unless the state \( |\Phi\rangle \) is the eigenstate of the projector \( P_a \). Obviously, Von Neumann measurement is not an effective way to perform nondistortion interrogation, especially when the quantum state in the black box is unknown to us.

For the purpose of NQI, we use a probe wave function \( |\Psi_{probe}\rangle = \alpha |\Psi_r\rangle + \beta |\Psi_d\rangle \). We let part of it \( |\Psi_d\rangle \) go through the black box and interact with the object in the box. After the interaction \( |\Psi_r\rangle \) and \( |\Psi_d\rangle \) are recombined and a measurement is done on the probe wave function, possibly after a unitary operation on it (see Fig.1).
FIGURES

FIG. 1. Schematic of the nondistortion interrogation of the quantum object in the black box. One branch of the probe wave function ($|\Psi_d\rangle$) goes through the black box and the other branch $|\Psi_r\rangle$ is under free evolution. The detector $D_1$ registers decay signals from the black box. If $D_1$ does not fire the two branches are recombined and a Von Neumann measurement is done on the probe wave function. The final measurement outcomes will be recorded by $D_2$.

The above process is a bit similar to the case described by EV scheme. Indeed, it will be shown that EV IFM is a special case of our "black box problem". While, to perform the NQI in the general case some more exquisite designs need to be considered. To formulate general NQI processes we make the following assumptions:

a) $S$, the quantum system under interrogation (the quantum object in the black box), is a metastable system whose Hilbert space is denoted as $H_S$ with dimension $n$.

b) The Hilbert space of the probe system $D$, $H_D$, is composed of two orthogonal subspaces $H_r$ and $H_d$: $H_D = H_r \oplus H_d$. Here, $H_r$ and $H_d$ are the Hilbert spaces of $|\Psi_r\rangle$ and $|\Psi_d\rangle$ respectively and the dimension of the space $H_d$ is $m$. We let $|\Psi_d\rangle$ go through the black box and interact with $S$ (if the object is in the box), while $|\Psi_r\rangle$ is under free evolution.

c) The interaction between $D$ and $S$ is governed by a unitary operator. We assume the time of the interaction is some known $t$. When the interaction is over, any state driven out of space $H_S$ will quickly decay in an irreversible way to some stable ground state $|g\rangle$ which is out of space $H_S$. The decay signal will be registered by some properly arranged sensitive detectors, if the decay event actually happens.

In assumption a) we note that the physical property of the interrogated quantum system should be known to us although the original internal state of this system may be unknown. More exactly, a NQI only refers to an interrogation for a definite Hilbert space. In assumption b), the introduction of $H_r$ might appear to be redundant since it does not interact with $S$. But it is actually vital for the purpose of NQI, because the interaction between $|\Psi_d\rangle$ and $|\Psi_S\rangle$ changes the interference between $|\Psi_r\rangle$ and $|\Psi_d\rangle$ which provides information on what is
in the black box. Assumption c) can be named as “dissipation postulation”, which makes sure that the state out of the space $H_S$ can not travel back into the original space $H_S$ after the interaction between the probe system and the detected system. We assume that the interrogation process will drive the initial sytem $S$ into an extended space $H_S \oplus H_S^\perp$, while the states in the space $H_S^\perp$ will rapidly decay into a stable state $|g\rangle$ ( which is out of the space $H_S \oplus H_S^\perp$) in an irreversible way. Based on the above assumptions, the following major steps are followed to find out if there is an object in the black box ( see Fig.1):

i) Let the probe wave function $|\Psi_d\rangle$ go through the black box and interact with the quantum object if it is in the box.

ii) The decay signal detectors are used to register any decay event. If a decay is detected, we conclude that the quantum object was in the black box but its initial state is destroyed. Otherwise, go to the next step. This is equivalent to a partial projection measurement on the whole system $\rho_{tot}$:

$$\rho_{out} = P\rho_{tot}P + P_{\perp}\rho_{tot}P_{\perp}. \tag{2}$$

where the operators $P$ and $P_{\perp}$ refer to unity operators of Hilbert space $H_S \otimes H_D$ and its complementary space $H_S \otimes H_D$ respectively.

iii)Perform a Von Neummann measurement on the probe wave function (in $H_D$). Possible measurement outcomes are characterized by some projectors in an orthogonal projector set $O$.

In step iii), we keep in mind that if nothing is in the black box the probe will be in some definite final state corresponding to the free evolution of the initial state. We designate the projector to that state as $P_e$. If a successful interrogation of the object can be done, the probe will end up in some different state. For the consideration of universality we require that the probability of successful nondistortion interrogations of the quantum object be independent on its (unknown) initial state.

We approximately divide the nondistortion interrogation into two types: single-shot NQI and iterative NQI according to how many times the interaction happens within an
interrogation process. A scheme of single-shot NQI will follow the above steps i)-iii). In iterative NQI process, step i) and ii) are repeated. After a certain number of interactions the final measurement (step iii)) will be executed. Of course, some suitable operations on the probe wave function are allowed in between the iterations. Some results about two types of NQI processes are shown in Section III and IV.

III. SINGLE-SHOT NQI PROCESS

Let the initial state of the probe wave function be $|\Psi_{probe}\rangle = \alpha |\Psi_r\rangle + \beta |\Psi_d\rangle$, where $|\Psi_r\rangle \in H_r$ and $|\Psi_d\rangle \in H_d$. When the probe passes through the black box the time evolution of the whole system is as follows:

$$|\Psi_{probe}\rangle |\Psi_S\rangle \rightarrow \alpha e^{-i/\hbar (H^S+H^D)t} |\Psi_r\rangle |\Psi_S\rangle + \beta e^{-i/\hbar \int_0^t(H^S+H^D+H^I)dt'} |\Psi_d\rangle |\Psi_S\rangle$$

(3)

where $H^S$ and $H^D$ are the free Hamiltonian of the systems $S$ and $D$ respectively, and $H^I$ characterizes the interaction between the two systems. If there is nothing in the black box the interaction Hamiltonian $H^I$ vanishes and the above process reduces to the free evolution of two separate systems. In contrast, when the black box is occupied with the quantum object, $|\Psi_S\rangle$ may be driven into a larger space $H_S \oplus H_S^\perp$ due to the interaction Hamiltonian $H^I$. In view of the assumption c) a decay could happen with certain probability. While, if no decay signal is detected, the quantum state of the whole system is collapsed into the following (a projection on $H_S \otimes H_D$):

$$\alpha e^{-i/\hbar (H^S+H^D)t} |\Psi_r\rangle |\Psi_S\rangle + \beta (I_d \otimes I_S) e^{-i/\hbar \int_0^t(H^S+H^D+H^I)dt'} |\Psi_d\rangle |\Psi_S\rangle$$

$$= \alpha |\Psi'_r\rangle |\Psi'_S\rangle + \beta (I_d \otimes I_S) e^{-i/\hbar \int_0^t(H^S+H^D+H^I)dt'} e^{i/\hbar (H^S+H^D)t} (I_d \otimes I_S) |\Psi'_d\rangle |\Psi'_S\rangle$$

(4)

where $I_d$ and $I_S$ are the unity operators in the Hilbert spaces $H_d$ and $H_S$, $|\Psi'_r(d)\rangle = e^{-i/\hbar H^D t} |\Psi_r(d)\rangle$ and $|\Psi'_S\rangle = e^{-i/\hbar H^S t} |\Psi_S\rangle$. To simplify the future calculations, the above wave function is unnormalized. In Eq(4), we see that the evolution of the system is fully specified by the following interrogation operator $D$:

$$D = (I_d \otimes I_S) e^{-i/\hbar \int_0^t(H^S+H^D+H^I)dt'} e^{i/\hbar (H^S+H^D)t} (I_d \otimes I_S).$$

(5)
A. necessary and sufficient condition for single-shot NQI

In the subsection we will review some results in ref [24].

To obtain the necessary and sufficient condition for single-shot NQI the following lemma is necessary.

Lemma1: The necessary condition that a single-shot NQI can be done is that there exist a pair of vectors $|\chi\rangle, |\Psi'_d\rangle \in H_d$ which satisfy $\langle \chi | D |\Psi'_d\rangle = c I_S$, where $|c| \leq 1$.

Proof: If the black box is empty no interaction takes place in it. Thus, the probe ends up in the state $\alpha |\Psi'_r\rangle + \beta |\Psi'_d\rangle$. In term of the promise in above section, this process will give rise to an exact detecting outcome $P_e = (\alpha |\Psi'_r\rangle + \beta |\Psi'_d\rangle)(\alpha^* \langle \Psi'_r | + \beta^* \langle \Psi'_d |) \otimes I_S$. On the contrary, the evolution of the probe wave function will be modified when a quantum object occupies the black box. To make sure that a successful NQI can be done, there must exist a projector $P_I = |\Psi_I\rangle \langle \Psi_I | \otimes I_S$ (orthogonal to $P_e$) in the set $O$ which corresponds to the outcome component of NQI with nonzero probability $|\Delta|^2$. This relation can be embodied in the following equations:

\[ P_I P_e = 0 \tag{6} \]

\[ P_I (\alpha |\Psi'_r\rangle |\Psi'_S\rangle + \beta D |\Psi'_d\rangle |\Psi'_S\rangle) = \Delta |\Psi_I\rangle |\Psi'_S\rangle \tag{7} \]

where $|\Psi_I\rangle \in H_D$ is some normalized vector of the probe. Relying on the Eqs(6,7), we may obtain:

\[ \langle \Psi_I | D |\Psi'_d\rangle |\Psi'_S\rangle = \left( \frac{\Delta}{\beta} + \langle \Psi_I | \Psi'_d \rangle \right) |\Psi'_S\rangle \tag{8} \]

In terms of the definition of the operator $D$ the vector $D |\Psi'_d\rangle |\Psi'_S\rangle$ is a vector in Hilbert space $H_d \otimes H_S$. Thereby, setting a wave vector $|\chi\rangle = I_d |\Psi_I\rangle$, which is the projection of $|\Psi_I\rangle$ on $H_d$, we find $\langle \Psi_I | D |\Psi'_d\rangle |\Psi'_S\rangle = \langle \chi | D |\Psi'_d\rangle |\Psi'_S\rangle$. Similarly, we also have the relation $\langle \Psi_I | \Psi'_d \rangle |\Psi'_S\rangle = \langle \chi | \Psi'_d \rangle |\Psi'_S\rangle$. Since $\Delta$ and $\beta$ are nonzero we deduce that $||\chi|| \neq 0$. Hence, the Eq(8) can be rewritten as:
\[ \langle \chi | D | \Psi' \rangle | \Psi'_S \rangle = (\Delta + \langle \chi | \Psi'_d \rangle) | \Psi'_S \rangle = c | \Psi'_S \rangle \]  

(9)

where \( c = \Delta + \langle \chi | \Psi'_d \rangle \). Since \( | \Psi'_S \rangle \) is an arbitrary vector in the space \( H_S \) the following must be satisfied:

\[ \langle \chi | D | \Psi'_d \rangle = c I_S. \]  

(10)

We thus complete our proof.

Once the necessary condition (10) is satisfied the interrogation operator \( D \) will modify the wavefunction component \( | \Psi'_d \rangle | \Psi'_S \rangle \) in the following way:

\[ D | \Psi'_d \rangle | \Psi'_S \rangle = c | \chi \rangle | \Psi'_S \rangle + \sum_{j=1}^{m-1} | \bar{\chi}_j \rangle | \bar{m}_{S(j)} \rangle. \]  

(11)

Here, \( | \bar{m}_{S(j)} \rangle = \langle \bar{\chi}_j | D | \Psi'_d \rangle | \Psi'_S \rangle \) and the vectors \( \{| \chi \rangle, | \bar{\chi}_j \rangle; j = 1, ..., m - 1 \} \) form the orthogonal complete bases in Hilbert space \( H_d \).

As we know, the decomposition of a pure quantum state is roughly dependent on the structure of its Hilbert space. However, it should be noted that the representation of Eq(11) is not enough “compact”. It seems too extravagant for decomposing the state \( D | \Psi'_d \rangle | \Psi'_S \rangle \) in the Hilbert space \( H_d \otimes H_S \). Whether we can choose a more compact subspace in the space \( H_d \otimes H_S \) to achieve a full decomposition for it? This point is quite vital for obtaining the necessary and sufficient condition for the single-shot NQI. To express the Eq(11) in a more compact way the following steps should be done. We may define a set of operators \( Q^{(i)} = Tr_S [D | \Psi'_d \rangle | i \rangle \langle i | \Psi'_d | D^+ \rangle \) if the Eq(10) holds. Here, \( \{| i \rangle, i = 1, ..., n \} \) is a set of orthogonal bases in the Hilbert space \( H_S \). In the Hilbert space \( H_d \), the kernel operator of the operator \( Q^{(i)} \) is denoted as \( K_i \). Thus, the operator \( K_i \otimes I_S \) can be seen as the annihilation operator for the component \( D | \Psi'_d \rangle | i \rangle \):

\[ K_i \otimes I_S (D | \Psi'_d \rangle | i \rangle) = 0. \]  

(12)

If we set the intersection of all the \( n \) kernel operators as \( K = \bigcap_{i=1}^{n} K_i \) the operator \( K \otimes I_S \) will annihilate the quantum state \( D | \Psi'_d \rangle | \Psi'_S \rangle \). Because
\( K \otimes I_S(D |\Psi'_d) |\Psi'_S) = \sum_{i=1}^{n} c_i^S K \otimes I_S(D |\Psi'_d) |i) = 0 \)  

(13)

where \(|\Psi'_S) = \sum_{i=1}^{n} c_i^S |i)\). Therefore, a more compact space for decomposing the quantum state \(D |\Psi'_d) |\Psi'_S)\) is \(H_K \otimes H_S\). Here, \(H_K\) is the complementary space of kernel space \(H_S\) in \(H_d\). We denote the dimension of the space \(H_K\) by \(l(l \leq m)\). When we pick up some set of orthonormal states \(|\chi)\), \(|\chi_1)\), \(...\), \(|\chi_{l-1})\) spanning the space \(K\). Then, Eq(11) can be rewritten as follows:

\[
D |\Psi'_d) |\Psi'_S) = c |\chi) |\Psi'_S) + \sum_{j=1}^{l-1} |\chi_j) |m_{S(j)}\). 
\]

(14)

We may outline the above treatments into the following lemma.

Lemma2: Eq(14) is the equivalent representation of the Eq(11).

In light of the above preparations we may provide our main theorem.

Theorem 1: The necessary and sufficient condition for the single-shot NQI is that Eq(14) holds and \(|\Psi'_d) - c |\chi)\) is linearly independent of the state set \(|\chi_j)\); \(j = 1, ..., l - 1\).

Proof: Lemma 1 and 2 show that Eq(14) is the necessary condition for single-shot NQI. If Eq(14) holds, in Hilbert space \(H_S \otimes H_D\) the final state of the whole system will be:

\[
|\Psi_{probe}) |\Psi_S) \rightarrow \alpha |\Psi'_r) |\Psi'_S) + \beta c |\chi) |\Psi'_S) + \beta \sum_{j=1}^{l-1} |\chi_j) |m_{S(j)}\). 
\]

(15)

Reconsidering Eqs. (8) and (7), we may attain the following relations:

\[
\langle \Psi_I | (\alpha |\Psi'_r) + \beta |\Psi'_d) \rangle = 0 
\]

(16)

\[
\langle \Psi_I |\chi_j \rangle = 0 
\]

(17)

\[
\langle \Psi_I | (\alpha |\Psi'_r) + c\beta |\chi) \rangle = \Delta. 
\]

(18)

Subtracting Eq(18) from Eq(16) we obtain

\[
\langle \Psi_I | \beta (|\Psi'_d) - c |\chi) \rangle = -\Delta. 
\]

(19)
\( \Delta \neq 0 \) requires that \( |\Psi'_d \rangle - c |\chi\rangle \) be linearly independent of the set of vectors \( \{ |\chi_j \rangle : j = 1, \ldots, l-1 \} \). We thus confirm the necessity of the criterion.

We may further prove the converse by constructing a projector \( P_I \) satisfying Eq(3) and Eq(4). This procedure can appeal to the Schmidt orthogonalization method. First we define the state set \( N \) consisting of \( l + 1 \) normalized vectors \( \{ \alpha |\Psi'_r \rangle + \beta |\Psi'_d \rangle, \gamma (\alpha |\Psi'_r \rangle + c\beta |\chi\rangle), |\chi_j \rangle : j = 1, \ldots, l-1 \} \), where \( \gamma = \frac{1}{\| \alpha |\Psi'_r \rangle + c\beta |\chi\rangle \|} \) is the normalization coefficient for \( \alpha |\Psi'_r \rangle + c\beta |\chi\rangle \). We assume that \( \alpha \neq 0 \). Since \( |\chi\rangle, |\chi_1\rangle, \ldots, |\chi_{l-1}\rangle \) and \( |\Psi'_d \rangle \) are orthogonal to each other and \( |\Psi'_r \rangle - c |\chi\rangle \) is linearly independent of \( \{ |\chi_j \rangle : j = 1, \ldots, l-1 \} \), we may deduce that all vectors in the state set \( N \) are linearly independent. To construct an orthonormal set out of \( N \), we let the first \( l - 1 \) vectors be \( \{ |\chi_j \rangle : j = 1, \ldots, l-1 \} \). We then calculate the \( l \)th vector using \( \alpha |\Psi'_r \rangle + \beta |\Psi'_d \rangle : |\tilde{\Psi} \rangle = \gamma' (\alpha |\Psi'_r \rangle + \beta |\Psi'_d \rangle - \sum_{i=1}^{l-1} \langle \chi_i | (\alpha |\Psi'_r \rangle + \beta |\Psi'_d \rangle) |\chi_i \rangle ), \) where \( \gamma' \) is the normalization coefficient. Similarly, the last vector is \( |\Psi'_I \rangle = \gamma'' \left( \gamma (\alpha |\Psi'_r \rangle + c\beta |\chi\rangle) - \langle \tilde{\Psi} | \gamma (\alpha |\Psi'_r \rangle + c\beta |\chi\rangle |\tilde{\Psi} \rangle \right) \) with the normalization coefficient \( \gamma'' \). It is then obvious that the projector \( P_I = |\Psi'_I \rangle \langle \Psi'_I | \otimes I_S \) satisfies Eqs(3) and (4). In addition, once the coefficient \( \alpha \) is fixed the projector \( P_I \) obtained by using Schmidt orthogonalization will maximize the success probability \( |\Delta|^2 \) [24]. We thus complete our proof.

From our proof it is clear that \( |\Psi'_r \rangle \) is not redundant, even though it does not interact with the object. If \( \alpha = 0 \) the capability of performing an NQI will be severely limited. For instance, if \( c \) is zero in Eq(14) (This corresponds to an IFM of an opaque object in which the probe wave function is blocked by the absorbing object), no nondistortion interrogation of the object can be done without the introduction of \( |\Psi'_r \rangle \). We may elucidate the novel NQI phenomena in the sense of quantum interference. The quantum object in the black box can be seen as a scattering object corresponding to the probe wave. The scattering process between the object and the probe wave makes each scattering wave component entangle different evolution of the object. If all the information on the evolution of the whole composite system is known it possibly allows us to choose a proper probe wave such that a successful scattering wave component is produced. For the purpose of NQI this
component gets entangled with the free evolution of the object. Now, it seems clearly for
the meaning of the above theorem. Once the necessary and sufficient condition for single-
shot NQI holds the form of this successful probe wave component $|\Psi_I\rangle$ can be obtained by
using Schmidt orthogonalization steps outlined in the proof of theorem 1.

We note that the vectors $|\Psi'_d\rangle$ and $|\chi\rangle$ satisfying Eq(10) may not be unique. However,
once $|\chi\rangle$ and $|\Psi'_d\rangle$ are chosen we may obtain the optimal success probability for these two
vectors, by following the steps outlined in the proof of theorem 1. If the initial state of the
probe is $\alpha |\Psi_r\rangle + \beta |\Psi_d\rangle$, the success probability is

$$\text{Pr obs}(\alpha) = |\langle \Psi_I | (\alpha |\Psi'_r\rangle + c\beta |\chi\rangle)|^2. \quad (20)$$

Therefore we have the following corollary.

**Corollary 1:** Under the condition that the wave function $|\Psi'_d\rangle$ and $|\chi\rangle$ are given the
optimal success probability of the NQI is as follows:

$$P_{opt} = \max_{|\alpha| \in [0,1]} \text{Pr obs}(\alpha). \quad (21)$$

**B. Verifying the criterion of NQI**

In the former context we have pointed out the operator $D$ fully characterizes the inter-
action between two systems and the dissipation process. In this subsection, starting from
the operator $D$ we will devote ourself to explore the existance of the pair of vectors $|\Psi'_d\rangle$
and $|\chi\rangle$ satisfying the criterion of NQI.

The operator $D$ can be written as:

$$D = \sum_{i,j,k,l} d_{i,j;k,l} |i_d\rangle |k_S\rangle \langle l_S| \langle j_d|. \quad (22)$$

For the simplicity of discussion we will omit the subscripts of the vectors. Unless pointed out
otherwise $|i\rangle$ ($|j\rangle$) will indicate the vectors in the Hilbert Space $H_d$ and $|k\rangle$ ($|l\rangle$) will indicate
the vectors in the Hilbert Space $H_S$. 

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Once a successful NQI process can be put in practice the operator $D$, as a subblock in a unitary matrix (see Eq (5)), satisfies the following relation:

$$\langle \chi | D | \Psi'_d \rangle = cI_S. \quad (23)$$

We set $|\chi\rangle = \sum_{i=1}^m a_i^* |i\rangle$ and $|\Psi'_d\rangle = \sum_{i=1}^m b_i |i\rangle$ the relation (23) can be parameterized as:

$$\sum_{i,j} a_i d_{i,j,k,l} b_j = c \delta_{kl}, (k,l = 1, 2, ..., n) \quad (24)$$

Furthermore, we may define $\sum_j d_{i,j,k,l} b_j$ as a matrix $\mathcal{R}_{i,k,l}$ with a set of parameters $\{b_j\}$. Thus, whether the equation (24) holds is equivalent to whether there are a set of parameters $\{a_i\}$ satisfying the following equation:

$$\sum_i a_i \mathcal{R}_{i,k,l} = c \delta_{kl}, (k,l = 1, 2, ..., n) \quad (25)$$

Since the above equations are linear, in principle, it is possible to identify the existence of the unknown parameters $\{a_i\}$ satisfying the equation set (25).

Theorem 2: the necessary and sufficient condition for the existence of the solution is that the matrix $(\mathcal{R})_{m \times n^2}$ and its augmented matrix $(\tilde{\mathcal{R}})_{(m+1) \times n^2}$ has the same rank (see ref [25]):

$$\text{rank}(\mathcal{R}) = \text{rank}(\tilde{\mathcal{R}}) \quad (26)$$

where the augmented matrix $(\tilde{\mathcal{R}})$ is

$$\tilde{\mathcal{R}}_{i,k,l} = \mathcal{R}_{i,k,l} (i \leq m), \tilde{\mathcal{R}}_{m+1,k,l} = c \delta_{kl}. \quad (27)$$

Unfortunately, since the matrix $\mathcal{R}_{i,k,l}$ includes the uncertain parameters $\{b_j\}_{j=1}^m$, how to develop a general method to determine the existence of the solution of the equation set (25) remains unknown. Numerically, we may simply try all possible $b_j$’s ($\sum_j |b_j|^2 = 1$) and $c$ to see if Eq (25) is satisfied. When the total number of the equations in (25) is not larger than $m$ it is a readily solvable problem. In principle, if the solutions exist, it can be formally described as:

$$a_i = a_i (b_1, b_2, ..., b_m, c); i = 1, ..., m. \quad (28)$$
Furthermore, according to the steps shown in the above context we may test the criterion of the NQI. The solutions of the equation set (25) may not be unique, that is, for different $b_i$'s and $c$ we may get different $a_i$'s. In practice, we may choose the set of solutions which maximize the success probability. Here, let us consider a concrete example.

As in Fig. 2, the model we consider is a multi-level atom. The atom is prepared in an arbitrary superposition of the two degenerate metastable states $|m+\rangle$ and $|m-\rangle$. By absorbing a $+(-)$ circularly polarized photon with the frequency $\omega$ the atom can make a resonant transition from $|m+\rangle (|m-\rangle)$ to the corresponding excited state $|e+\rangle (|e-\rangle)$. Here, we may view the transition between $|m+\rangle (|m-\rangle)$ and $|e+\rangle (|e-\rangle)$ caused by the $+(-)$ circularly polarized photon as a resonant Jaynes-Cummings model. Clearly, there are no correlations between the two processes. We assume that the atom will decay rapidly from the excited state $|e+\rangle (|e-\rangle)$ to the ground state $|g\rangle$ in an irreversible way when the electromagnetic field with the frequency $\omega$ fades away. At last, the decay signals can be recorded by some sensitive detectors. To study the criterion of NQI process we start with the Hamiltonian of the whole system:

$$H_{tot} = H_{atom} + H_{photon} + H_{interaction}$$

$$= \sum_{k=+,-} \left[ \hbar \omega a_k^{\dagger} a_k + \hbar \omega |ek\rangle \langle ek| + \hbar g_k \left( \sigma_k^+ a_k + a_k^+ \sigma_k^- \right) \right]$$

FIG. 2. Level structure of the atom. The atom can make a transition to the excited state $|e+\rangle$ ($|e-\rangle$) from $|m+\rangle (|m-\rangle)$ by absorbing a $+(-)$ circular polarized photon. It then decays rapidly to the stable ground state $|g\rangle$.

where $a_k^{\dagger}$ and $a_k$ are the creation and annihilation operators of the $k$ polarized photon. the operators $\sigma^+_k$ and $\sigma^-_k$ separately refer to $|ek\rangle \langle mk|$ and $|mk\rangle \langle ek|$. Here, we define $|mk\rangle$ as the zero point of the energy. To formulate the operator $D$ our main task is to obtain the representation of the unitary operator $U(t) = e^{-i/\hbar H_{tot}t}$. For that purpose, we may bring in the total particle number operator $N = \sum_{k=+,-} (\hbar \omega a_k^{\dagger} a_k + \hbar \omega |ek\rangle \langle ek|)$. Since the operator $N$ commutes with the Hamiltonian $H_{tot}$, i.e. $[N, H_{tot}] = 0$, the unitary operator $U(t)$ can be
decomposed into the direct summands of a series of subspaces labelled by the eigenvalues of the operator $N$. As a prior assumption of this NQI process, we assume that only one photon comes into the system and interacts with the atom prepared in the superposition of the metastable states. Due to the fact that $(N - \hbar \omega) |\text{one\_photon}\rangle |\text{metastable\_state}\rangle = 0$, during the whole interaction process, the quantum state of the whole system can not escape from the eigenspace of the operator $N$ with the eigenvalue $\hbar \omega$. This subspace is spanned by six basis vectors: $|ek\rangle |0\rangle$, $|mk\rangle |k'\rangle$ ($k, k' = +, -$). Here, $|k\rangle$ and $|0\rangle$ refer to the state of single $k$ (circularly) polarized photon and the vacuum state respectively. Thus, in this problem, we need only give the representation of the unitary operator $U(t)$ in the eigenspace of the operator $N$ labelled by eigenvalue $\hbar \omega$.

$$U(\hbar \omega)(t) = e^{-i\omega t} \left\{ \sum_{k=+, -} (\cos g_k t |mk\rangle \langle k| (k| \langle mk| - i \sin g_k t |ek\rangle \langle 0| \langle k| \langle mk|) + \sum_{k=+, -} (\cos g_k t |ek\rangle \langle 0| \langle ek| - i \sin g_k t |mk\rangle \langle 0| \langle ek|) + |m-\rangle |+\rangle \langle m-| + |m+\rangle |-\rangle \langle m+| \right\}. \quad (30)$$

Based on the equation (3) the operator $D$ can be obtained:

$$D = \sum_{k=+, -} \cos g_k t |mk\rangle \langle k| (k| \langle mk| + |m-\rangle |+\rangle \langle m-| + |m+\rangle |-\rangle \langle m+| . \quad (31)$$

We may further deduce that the space $\mathcal{K}$ is just $H_d$ spanned by two vectors $|+\rangle$ and $|-\rangle$.

By setting $|\chi\rangle = \sum_{i=+, -} a_i^* |i\rangle$ and $|\Psi'_d\rangle = \sum_{i=+, -} b_i |i\rangle$ we can rephrase the equation (33) as follows:

$$\begin{bmatrix} p_+ b_+ & b_- \\ b_+ & p_- b_- \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix} \quad (32)$$

where $p_\pm = \cos g_\pm t$. To make sure that there are rational solutions of $a_+$ and $a_-$ it is essential that the determinant of the matrix on the left hand of the above equation is non-zero: $(p_+ p_- - 1)b_+ b_- \neq 0$ i.e. $p_+ p_- \neq 1$ and $b_+ b_- \neq 0$. The solutions of the equation set (32) are that: $a_+ = \frac{(p_+ - 1)c}{(p_+ p_- - 1)b_+}$, $a_- = \frac{(p_- - 1)c}{(p_+ p_- - 1)b_-}$ the normalizaton of the vector $|\chi\rangle$ provides the limit of the norm of the constant $c$: $|c| = \frac{|b_+ b_- [(1-p_+ p_-)]}{\sqrt{(p_- - 1)^2 |b_-|^2 + (p_- - 1)^2 |b_+|^2}}.$
For the purpose of simplicity we set \( p_+ = p_- = p \). Thus, the above parameters should be separately reduced to: \( a_+ = \frac{c}{(p+1)b_+} \), \( a_- = \frac{c}{(p+1)b_-} \), and \( |c| = |b_+ b_-| (1+p) \). Starting from the state vector \( |\chi\rangle = \frac{c^*}{(p+1)b_+} |+\rangle + \frac{c^*}{(p+1)b_-} |-\rangle \) we can deduce \( |\chi_1\rangle = \frac{c}{(p+1)b_-} |+\rangle - \frac{c}{(p+1)b_+} |-\rangle \).

By comparing \( |\Psi_d'\rangle - c |\chi\rangle \) with \( |\chi_1\rangle \) we may find that the two state vectors are linearly independent unless \( p = 1 \). In other words, for the atomic system as depicted in Fig. 2, the criterion of NQI process can be satisfied in most cases. Therefore, it allows us to obtain the projector \( P_I \) by taking advantage of the steps outlined in section III A and to devise feasible protocols of nondistortion interrogation. One such example is given in [21], where \( |\chi\rangle = |\psi_d'\rangle = \frac{1}{\sqrt{2}} \left( a_{i,-}^+ - a_{i,+}^+ \right) |0\rangle \), and the maximum success probability is \( \frac{1}{16} \).

IV. ITERATIVE NQI PROCESS

People have recently presented several protocols of NQI [21,22,23]. However, it is quite difficult to increase the interrogation efficiency in single-shot NQI processes. We proved that the optimal success probability in the protocol investigated by Pötting et. al. [21] can only reach \( \frac{1}{16} \) [24]. By expanding the Hilbert space of the probe system, the success probability for NQI can be raised to \( \frac{1}{8} \) in an advanced scheme [22]. The design for the previous two schemes is just in terms of the fundamental steps i)-iii) described in section II. As we know, in most protocols of the interaction-free measurements, it is possible to make the probability of IFMs arbitrarily close to one by taking advantage of the quantum Zeno effect [4,5,7,11]. Similar treatments can also be considered in NQI. A primary attempt was presented in ref [23].

For the iterative NQI processes, a basic consideration is that when the probe wavefunction comes out of the detected system one makes it come in the detected system again after an appropriate manipulation on it, instead of performing the Von Neumann measurement on it. A final measurement can be done on the probe after the certain number of iterations. When we denote the quantum manipulation between the loops by the operator \( L^{(i)} \) the final state in \( H_D \otimes H_S \) is:
\[ | \Psi \rangle_{\text{non-decay}} = \prod_{i=0}^{N-1} \left( L^{(N-i)} ((I_r \otimes I_S) \oplus D) U_{\text{free}} \right) | \Psi_D \rangle | \Psi_S \rangle \]  

(33)

where \( U_{\text{free}} = e^{-i/\hbar (H_D + H_S)t} \) and \( N \) refers to the number of the iterations. Because we allow to insert the proper operation \( L^{(i)} \) in between the iterations it becomes hard to obtain the necessary and sufficient for the iterative NQI processes. Here, we only consider the sufficient condition for the iterative cases.

Theorem 3: The sufficient condition that an iterative NQI can be done with certainty is that there exists a vector \( | \chi \rangle \in H_d \) which satisfies \( \langle \chi | D | \chi \rangle = c I_S \), where \( c \) is a known constant and is not equal to one.

Proof: Suppose that the original probe wavefunction is: \( | \Psi_D \rangle = e^{i/\hbar H_D t} (\cos \theta | \psi_r \rangle + \sin \theta | \chi \rangle) \). If the condition \( \langle \chi | D | \chi \rangle = c I_S (c \neq 1) \) holds, for any state vector \( | \psi'_S \rangle \in S (H_S) \), we have the following relation:

\[ D | \chi \rangle | \psi'_S \rangle = c | \chi \rangle | \psi'_S \rangle + \sum_{j=1}^{m-1} | \chi_j \rangle | m_{S(j)} \rangle. \]  

(34)

Here, \( \{ | \chi_j \rangle_{j=1}^{m-1}, | \chi \rangle \} \) form the bases of the Hilbert space \( H_d \) and \( | m_{S(j)} \rangle = \langle \chi_i | D | \chi \rangle | \psi'_S \rangle \).

For the iterative NQI, the main task is to elaborately devise the operations between the adjacent loops. Considering the condition(34) we may set the operator \( L^{(i)} \) as follows:

\[ L^{(i)} = e^{i/\hbar H_D t} U M \]  

(35)

where \( M \) is a projector which projects the probe wavefunction into the subspace \( (| \psi_r \rangle \langle \psi_r | + | \chi \rangle \langle \chi |) \otimes I_S \) with some probability. The unitary operator \( U \) has the following form:

\[ U = \begin{pmatrix} \cos \delta & \sin \delta \\ -\sin \delta & \cos \delta \end{pmatrix} \]  

(36)

where the matrix is in the bases\( | \psi_r \rangle \) and \( | \chi \rangle \). We choose \( \delta = \theta - \theta' \), where \( \cos \theta' = \cos \theta/\sqrt{\cos^2 \theta + c^2 \sin^2 \theta} \). Thus, if there is a quantum object characterized in the state \( | \psi_S \rangle \) in the black box, after the probe wavefunction initially prepared in the state \( | \Psi_D \rangle \) undergoes
N iterations, if we choose \( N \) such that \( N\delta = \pi/2 \), the non-decay quantum fraction of the whole system will be transformed into:

\[
|\Psi\rangle_{\text{non-decay}} = e^{i\mathcal{H}_D t/\hbar} \gamma^N \left( \cos \theta |\psi_r\rangle + \sin \theta |\chi\rangle \right) e^{-i\mathcal{H}_S Nt/\hbar} |\psi_S\rangle
\]

(37)

where \( \gamma = 1/\sqrt{\cos^2 \theta + \sin^2 \theta} \). As the final result, the probe will be end up in the state

\[
|\Psi\rangle = e^{i\mathcal{H}_D t/\hbar} \left( \cos \left( \theta + \pi/2 \right) |\psi_r\rangle + \sin \left( \theta + \pi/2 \right) |\chi\rangle \right)
\]

(38)

with certainty. Due to the orthogonality and (asymptotic) certainty of the two outcomes we may perfectly perform the NQI in the iterative way. This completes the proof of theorem 3.

V. DISCUSSION AND CONCLUSION

In the quantum interrogation, the interaction between the probe wave and the quantum object modifies the interference among the original probe wave components. Meanwhile, each of the probe wave components entangles the corresponding evolution of the scattered object. For the NQI processes, there must exist a successful probe wave component, which should entangle with the free evolution of the object, at the same time, be orthogonal to any other scattering wave components. Quantum measurement theory ensures that once this component is registered the corresponding quantum branch will come into reality. In this case, we may not only obtain information on the location of a quantum object, but also make the evolution of the internal state of the detected object free from the disturbance from the interrogation process. Here, we should emphasize that before an experimentist provides a practical NQI scheme all the physical conditions about the interaction in the black box should be considered carefully. In addition, As far as some complicated interactions are concerned, it is quite difficult to test the criterion of NQI. Especially, how to obtain the sufficient and necessary criterion for the iterative NQI processes is still an open question.
Nevertheless, NQI may provide some miraculous applications. It should be noted that the interrogated target is not a quantum state, but a space of quantum states. In other words, quantum information in Hilbert space $H_S$ will not be contaminated by the probing process. This way to manipulate quantum objects is of potential application in the recently developed quantum information science. Since the detected system need not be restricted to pure states we may also cast our interests on mixed states. As pointed out by Pötting et al [21] the nondistortion interrogation provides a tool to monitor a subsystem in a many-particle system without destroying the entanglement between the particles.

In conclusion, we have studied the process of NQI under some physical assumptions. We proved the necessary and sufficient condition for single-shot NQI process in our formulation. Furthermore, we consider the NQI processes in iterative cases and obtain a sufficient condition for the iterative nondistortion quantum interrogation. As a novel method to manipulate quantum systems NQI may be applied in future quantum information processing.

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Figure 1
Fig 2. The level structure of the atom