Exact solution of the $O(n)$ model on a random lattice

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Abstract

We present an exact solution of the $O(n)$ model on a random lattice. The coupling constant space of our model is parametrized in terms of a set of moment variables and the same type of universality with respect to the potential as observed for the one-matrix model is found. In addition we find a large degree of universality with respect to $n$; namely for $n \in ]-2,2[$ the solution can be presented in a form which is valid not only for any potential, but also for any $n$ (not necessarily rational). The cases $n = \pm 2$ are treated separately. We give explicit expressions for the genus zero contribution to the one- and two-loop correlators as well as for the genus one contribution to the one-loop correlator and the free energy. It is shown how one can obtain from these results any multi-loop correlator and the free energy to any genus and the structure of the higher genera contributions is described. Furthermore we describe how the calculation of the higher genera contributions can be pursued in the scaling limit.

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1 Introduction

The $O(n)$ model on a random lattice is a matrix model which regarding its complexity can be placed somewhere in between the one-matrix model and the two-matrix model. It is therefore a natural intermediate step if one wants to study the generalization of 1-matrix model techniques and results to the two- and eventually the multi-matrix case. The model is also interesting in its own right having an appealing geometrical interpretation and a very rich phase structure. In particular when $n = 2\cos(\nu\pi)$ with $\nu = l/k$, $0 < l < k$ and $l, k \in \mathbb{Z}$ the model has critical points for which the associated scaling behaviour is that characteristic of 2D gravity interacting with rational conformal matter fields of the type $(p, q) = (k, (2m + 1)k \pm l)$ and with $\nu$ general any central charge between $c = -\infty$ ($\nu = 1$) and $c = 1$ ($\nu = 0$) can be reached. However, the continuum theories that one obtains from the $O(n)$ model in the rational case contain only a subset of the operators of the corresponding minimal models.

In the present paper we will solve the model exactly, i.e. without any assumption of being close to a critical point. The genus zero contribution to the 1-loop correlator will be calculated solving the saddle point equation of the model, following the idea of references and the higher genera contributions by a generalization of the moment technique of reference. As usual this technique will allow us to find from the 1-loop correlator any multi-loop correlator as well as the free energy. The parametrization of the coupling constant space of the model in terms of moment variables reveals that the model possesses the same kind of universality with respect to the potential as the one-matrix model. In addition there appears a large degree of universality with respect to $n$.

In the case of the one-matrix model the moment description facilitated the analysis of the double scaling limit. For example the result that the continuum 1-matrix model partition function is a $\tau$-function of the kdV hierarchy could easily be understood in this description, the analysis relying on a representation of the $\tau$-function as a matrix model, namely the Kontsevich model, and the moment description of this model. The $\tau$-functions of the kdV$_p$ hierarchies with $p > 2$ can also be represented as matrix models, namely as generalized Kontsevich models and recently the appropriate moment description of these models has been found. Hence it should be possible to determine which is the precise relation between the continuum partition function of the $O(n)$ model, for $n$ rational, and the $\tau$-functions of the kdV$_p$ hierarchies by comparing the moment description of the $O(n)$ model with the moment description of the generalized Kontsevich models. This requires of course that a d.s.l. relevant version of the moment description is developed for the $O(n)$ model. A part of
our paper will be devoted to the development of such a description.

We will start by, in section 2, presenting the model and the most important equations needed for its solution. Then we will proceed with the exact solution, for \( n \in \mathbb{Z} - 2, 2 \) in section 3, and for \( n = \pm 2 \) in section 4. Section 5 is devoted to the study of the double scaling limit and section 6 contains our conclusion and a discussion of possible future directions of investigation.

2 The Model

In the following we will consider the \( \mathrm{O}(n) \) model on a random lattice, given by the partition function

\[
Z = e^{N^2 F} = \int_{N \times N} dM \prod_{i=1}^{n} dA_i \exp \left( -N \text{Tr} \left[ V(M) + M \sum_{i=1}^{n} A_i^2 \right] \right)
\]  

where \( M \) and \( A_i, i = 1, \ldots, n \) are hermitian \( N \times N \) matrices and

\[
V(M) = \sum_{j=1}^{\infty} g_j M^j.
\]

In the language of Feynman diagrams the model describes a gas of \( n \) different types of self-avoiding loops; non-interacting and living on a random surface\(^2\). To begin with \( n \) is an integer but by analytical continuation the model can be defined also for non integer values of \( n \). We will restrict ourselves to the case \(|n| \leq 2\) and we will use the following parametrization

\[
n = 2 \cos(\nu \pi), \quad 0 \leq \nu \leq 1.
\]

We note that for \( n = 0 \) the model is identical to the usual hermitian 1-matrix model. Furthermore for \( n = 1 \) and a special cubic potential the model describes the Ising model on a random lattice\(^3\). We shall in particular be concerned with the calculation of the free energy, \( F \), and correlators of the \( M \)-field of the following type

\[
W(p_1, \ldots, p_s) = N^{s-2} \left< \text{Tr} \frac{1}{p_1 - M} \text{Tr} \frac{1}{p_2 - M} \ldots \text{Tr} \frac{1}{p_s - M} \right>_{\text{conn}}
\]

The genus expansion of these objects reads

\[
F = \sum_{g=0}^{\infty} N^{-2g} F_g, \quad W(p_1, \ldots, p_s) = \sum_{g=0}^{\infty} N^{-2g} W_g(p_1, \ldots, p_s).
\]

\(^2\)Strictly speaking, to have this interpretation, we should include mass terms for the \( A \)-fields and exclude the term linear in \( M \) in our action. However, this rearrangement can be obtained by a linear shift of the matrix \( M \) and since we will work with a generic potential such a shift can always be performed in the final result.
and we have
\[ W^g(p_1, \ldots, p_s) = \frac{d}{dV(p_1)} \cdots \frac{d}{dV(p_s)} F_g, \quad g \geq 1 \text{ or } s \geq 2 \] (2.6)

where
\[ \frac{d}{dV(p)} = -\sum_{j=1}^{\infty} \frac{j}{p^{j+1}} \frac{d}{dg_j}. \] (2.7)

In the remaining part of this section we shall introduce the tools which will allow us to determine, for any potential \( V(M) \) and any \( n \in [-2, 2] \), \( W^g(p_1, \ldots, p_s) \) and \( F_g \) for (in principle) any \( g \) and any \( s \). Eventually it will be convenient to treat separately the cases \( n \in ]-2, 2[ \) and \( n = \pm 2 \) but here we shall address the aspects which are common to all values of \( n \).

### 2.1 The saddle point equation

The integration over the \( A \) matrices in our partition function (2.1) is gaussian and can directly be carried out. This leads to the appearance of a 1-matrix integral in which we can diagonalize the matrices and integrate out the angular degrees of freedom. By this procedure our partition function (up to a constant) turns into the following integral over the eigenvalues of the matrix \( M \) \[ Z \propto \int_{-\infty}^{\infty} d\lambda_i e^{-N \sum_i V(\lambda_i)} \prod_{i,j} (\lambda_i + \lambda_j)^{-n/2} \prod_{i<j} (\lambda_i - \lambda_j)^2. \] (2.8)

In the limit \( N \to \infty \) the eigenvalue configuration is determined by the saddle point of the integral above \( \[ \] \). The corresponding saddle point equation reads \[ V'(\lambda_i) = \frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} - \frac{n}{N} \sum_j \frac{1}{\lambda_i + \lambda_j}. \] (2.9)

As usual this discrete equation can be transformed into a continuous one by introducing corresponding to the matrix \( M \) an eigenvalue density \( \rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \) which in the limit \( N \to \infty \) becomes a continuous function \( \[ \] \). When one of the eigenvalues approaches the origin, the integral (2.8) ceases to exist (cf. to equation (2.9)). Therefore we will always assume that the eigenvalues are confined to the positive real axis. More precisely we will consider the situation where the eigenvalue density has support only on one interval \([a, b]\) on the positive real axis and is normalized to one, i.e.

\[ \text{supp } \rho(\lambda) = [a, b], \quad a > 0, \] (2.10)
\[ \int_a^b \rho(\lambda) d\lambda = 1. \] (2.11)
Of course the results obtained in this situation will allow an analysis of the case $a \to 0$. In terms of the eigenvalue distribution the saddle point equation (2.9) reads

$$V'(\lambda) = 2 \int_a^b d\mu \frac{\rho(\mu)}{\lambda - \mu} - n \int_a^b d\mu \frac{\rho(\mu)}{\lambda + \mu}. \quad (2.12)$$

The saddle point equation can also be written in terms of the genus zero one-loop correlator $[3]$. One has

$$W^0(p) = \int_a^b d\mu \frac{\rho(\mu)}{p - \mu} \quad (2.13)$$

and the conditions (2.10) and (2.11) on $\rho(\lambda)$ are equivalent to demanding that $W(p)$ is analytic in the complex plane except from a cut $[a, b]$ and that

$$W(p) \to \frac{1}{p}, \quad p \to \infty. \quad (2.14)$$

The inverse relation to (2.13) reads

$$\rho(\lambda) = \frac{1}{2\pi i} \left\{ W^0(\lambda - i0) - W^0(\lambda + i0) \right\} \quad (2.15)$$

and the saddle point equation for $\rho(\lambda)$ turns into the following equation for the genus zero contribution the one-loop correlator

$$V'(p) = W^0(p + i0) + W^0(p - i0) + n W^0(-p), \quad p \in [a, b]. \quad (2.16)$$

### 2.2 The loop equations

The loop equations of the model can be derived in various ways $[4]$. Here let us use a formulation which exposes very clearly the analogy with the 1-matrix model case. First we exploit the invariance of the partition function (2.1) under the following redefinition of the field $M$

$$M \to M + \epsilon \frac{1}{p - M}. \quad (2.17)$$

Introducing this shift in (2.1) gives rise to the following equation

$$\oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p - \omega} W(\omega) + n\chi(p) = (W(p))^2 + \frac{1}{N^2} \frac{d}{dV(p)} W(p) \quad (2.18)$$

where

$$\chi(p) = \frac{1}{N} \langle \text{Tr} \frac{1}{p - M} A_i^2 \rangle \quad (2.19)$$

and where the contour $C_1$ encloses the cut $[a, b]$ of $W(\omega)$ but not the point $\omega = p$. We will use the convention that all contours are oriented counterclockwise. Next, let us consider the following redefinition of the field $A_i$

$$A_i \to A_i + \epsilon \frac{1}{p - M} A_i \frac{1}{p - M}. \quad (2.20)$$
Inserted into (2.1) this shift leads to the following identity
\[ -\chi(p) - \chi(-p) = W(p)W(-p) + \frac{1}{N^2 dV(p)}W(-p). \]  
(2.21)

From (2.18) and (2.21) we can obtain a closed equation for the 1-loop correlator of the \( M \)-field
\[
2 \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)\omega}{p^2 - \omega^2} W(\omega) = \left( W(p) \right)^2 + \left( W(-p) \right)^2 + nW(p)W(-p) + \frac{1}{N^2} \left\{ \frac{d}{dV(p)} W(p) + \frac{d}{dV(-p)} W(-p) + n \frac{d}{dV(p)} W(-p) \right\}. 
\]  
(2.22)

This equation exhibits a strong similarity with the equation for the 1-loop correlator of the hermitian 1-matrix model but as opposed to the latter it is non local. However, as we shall see in section 3.1, 4.1.1 and 4.2.1 there exists an efficient way to deal with this non-locality.

### 3 The case \( n \in ] -2, 2] \)

#### 3.1 Reformulation of the loop equation

With the aim of reformulating (2.22) as a local equation, let us introduce a function \( W_r(p) \) by
\[
W_r(p) = \frac{2V'(p) - nV'(-p)}{4 - n^2}. 
\]  
(3.1)

Furthermore, let us corresponding to a function or an operator \( h(p) \) define \( h_\pm(p) \) by
\[
h_+(p) = \frac{e^{i\nu\pi/2}h(p) + e^{-i\nu\pi/2}h(-p)}{2\sin(\nu\pi)}, \quad h_-(p) = h_+(-p). 
\]  
(3.2)

Inversely we then have
\[
h(p) = -i \left( e^{i\nu\pi/2}h_+(p) - e^{-i\nu\pi/2}h_-(p) \right). 
\]  
(3.3)

Introducing the transformation (3.2) into the loop equation (2.22) one gets
\[
\oint_{C_2} \frac{d\omega}{2\pi i} \frac{\omega}{p^2 - \omega^2} \left\{ W_+(\omega)W_-(\omega) + W_-(\omega)W_+(\omega) \right\} = W_+(p)W_-(p) + \frac{1}{N^2 dV_+(p)}W_-(p) 
\]  
(3.4)

where the contour, \( C_2 \), now encircles \([a, b]\) as well as \([-b, -a]\) but not the point \( \omega = p \) and where \( d/dV_+(p) \) is shorthand notation for \( (d/dV(p))_+ \). Introducing two linear operators \( \hat{K}_+ \) and \( \hat{K}_- \) by
\[
\hat{K}_+ f(p) = \oint_{C_2} \frac{d\omega}{2\pi i} \frac{\omega W_+(\omega)}{p^2 - \omega^2} f(\omega), \quad \hat{K}_- f(p) = \oint_{C_2} \frac{d\omega}{2\pi i} \frac{\omega W_-(\omega)}{p^2 - \omega^2} f(\omega) 
\]  
(3.5)
and inserting the genus expansion of the correlators in (3.4) we find
\[
\left\{ \hat{K}_+ - W^0_+(p) \right\} W^g_+(p) + \left\{ \hat{K}_- - W^0_-(p) \right\} W^g_-(p) = \sum_{g'=1}^{g-1} W^g_+(p) W^{g-g'}_-(p) + \frac{d}{dV_+(p)} W^{g-1}_-(p), \quad g \geq 1.
\] (3.6)
The similarity with the corresponding equation of the hermitian 1-matrix model appearing in reference [7] is striking and we will later show how the strategy of reference [7] for solving the loop equation genus by genus can be generalized to the present case. Of course an iterative procedure for solving (3.6) requires the knowledge of \( W^0(p) \).

In the next section we will show how one can write down a closed expression for this correlator, i.e. an expression which is valid for any potential \( V(M) \) and any \( n \in [-2, 2] \).

### 3.2 \( W^0(p) \) in terms of an auxiliary function \( G(p) \)

To determine the 1-loop correlator at genus zero we follow the idea of reference [3]. As mentioned earlier we restrict ourselves to the one-cut situation. Our starting point will be the saddle point equation (2.16) which together with the boundary condition (2.14) determines uniquely \( W^0(p) \). Let us split \( W(p) \) in a regular part \( W_r(p) \) and a singular part \( W_s(p) \)
\[
W(p) = W_r(p) - W_s(p).
\] (3.7)
From (2.16) it follows that \( W_r(p) \) is given by (3.1) while \( W^0_s(p) \) obeys the homogeneous saddle point equation and the boundary equation
\[
W^0_s(p) \sim W_r(p) - \frac{1}{p}, \quad p \to \infty.
\] (3.8)
In the language of the rotated functions \( W_\pm(p) \) we have the following situation
\[
W^0_\pm(p) = W^0_{r \pm}(p) - W^0_{s \pm}(p).
\] (3.9)
with \( W_{r \pm}(p) \) being given by
\[
W_{r \pm}(p) = i \frac{e^{-i\nu\pi/2} V'(p) - e^{i\nu\pi/2} V'(-p)}{4 - n^2}
\] (3.10)
and with \( W^0_{s \pm}(p) \) obeying the equations [3]
\[
W^0_{s \pm}(p - i0) = e^{\pm i\nu\pi} W^0_{s \mp}(p + i0).
\] (3.11)
The boundary equation (3.8) translates into
\[
W^0_{s +}(p) \sim W_{r +}(p) - \frac{i}{2\cos(\nu\pi/2)} \frac{1}{p}.
\] (3.12)
In order to obtain a closed expression for \( W^0_{s \pm}(p) \) we introduce an auxiliary function \( G(p) \) with the following properties
1. $G_{\pm}(p)$ fulfill the equations (3.11).

2. $G(p)$ is analytic in the complex plane except from a cut $[a, b]$ and behaves as $(p - a)^{-1/2}(p - b)^{-1/2}$ in the vicinity of $a$ and $b$.

3. $G_{\pm}(p) \sim \pm \frac{i}{p}$, $p \to \infty$.

In section 3.3 we will show that these requirements are enough to fix $G(p)$ uniquely.

For the moment let us note that 1–3 imply that the function $R(p) = G_{+}(p)G_{-}(p)$ is even, behaves as $1/p^2$ as $p \to \infty$ and can have no singularities except for single poles for $p = \pm a, \pm b$, i.e.

$$R(p) = \frac{(p^2 - e^2)}{(p^2 - a^2)(p^2 - b^2)}.$$  \hfill (3.13)

We will choose the convention that $+e$ is a root of $G_{+}(p)$ while $-e$ is a root of $G_{-}(p)$. We will later write down an equation which determines $e$ in terms of $a$ and $b$. Now if we write a generic solution of (3.11), $S_{\pm}$, as

$$S_{\pm}(p) = \overline{S}_{\pm}(p)G_{\pm}(p)$$  \hfill (3.14)

we have

$$\overline{S}_{\pm}(p - i0) = \overline{S}_{\pm}(p + i0)$$  \hfill (3.15)

which means that the even function $\overline{S}_{+}(p) + \overline{S}_{-}(p)$ is a regular function while the odd function $\overline{S}_{+}(p) - \overline{S}_{-}(p)$ has a square root branch cut $[a, b]$. Hence we have

$$\overline{S}_{+}(p) = A(p^2) + pB(p^2)\sqrt{p}, \quad \sqrt{p} = \sqrt{(p^2 - a^2)(p^2 - b^2)}$$  \hfill (3.16)

with $A(p^2)$ and $B(p^2)$ regular but not necessarily entire functions. Since $+e$ is a root of $G_{+}(p)$, $A(p^2)$ and $B(p^2)$ may have a pole for $p = e$ without $S_{+}(p)$ becoming singular there provided the accompanying pole for $p = -e$ is cancelled (cf. equations (3.13) and (3.14)). Since we are not particularly interested in solutions which vanish for $p = e$ a more convenient parametrization is

$$S_{+}(p) = A(p^2)G_{+}(p) + pB(p^2)g_{+}(p)G_{+}(p)$$  \hfill (3.17)

where

$$g_{+}(p) = \frac{\sqrt{p} + \frac{p}{e}\sqrt{e}}{p^2 - e^2}$$  \hfill (3.18)

and where $A$ and $B$ are again regular but not necessarily entire functions. We can also write

$$S(p) = A(p^2)G(p) + pB(p^2)\tilde{G}(p)$$  \hfill (3.19)
\[ \tilde{G}(p) = - \left( e^{i\nu\pi/2} g_+(p) G_+(p) + e^{-i\nu\pi/2} g_-(p) G_-(p) \right) \] (3.20)

We draw the attention of the reader to the equation (3.19). This equation will play a key role throughout the paper. It says that any solution of the saddle point equation (3.11) can be parametrized in terms of the two functions \( G(p) \) and \( \tilde{G}(p) \). In particular one has that any such solution can be parametrized in terms of any two other independent solutions. A study of the analyticity properties of \( \tilde{G}(p) \) reveals an interesting symmetry of the model. Let us for a moment write the function \( G(p) \) as \( G_\nu(p) \) where the index \( \nu \) is the parameter which enters the relation \( n = 2 \cos(\nu \pi) \). Then we have

\[ \tilde{G}_\nu(p) = G_{1-\nu}(p) \] (3.21)

This follows from the fact that \( \tilde{G}(p) \) is a solution of the saddle point equation (3.11) with \( \nu \) being replaced by \( 1-\nu \). Furthermore from (3.21) it follows that the parameter \( e = e_\nu \) entering the relation (3.13) for \( G_\nu(p) \) is related to the corresponding parameter, \( e_{1-\nu} \), for \( G^{1-\nu}(p) \) by \(^3\)

\[ e_{1-\nu} = -\frac{ab}{e_\nu} \] (3.22)

Let us now specialize to the 1-loop correlator. Since we want this function to be finite in the limits \( p \to a, b \) we choose in this case a slightly different but equivalent parametrization, namely

\[ W_{s+}^0(p) = \sqrt{p} \left( A(p^2) g_+(p) G_+(p) + pB(p^2) G_+(p) \right) . \] (3.23)

Due to the assumptions concerning the analyticity properties of the 1-loop correlator \( A(p^2) \) and \( B(p^2) \) must here be entire functions and from the boundary condition (3.12) it follows that they are necessarily polynomials. Using the relation (3.13) one easily concludes that \( A(p^2) \) and \( B(p^2) \) can be expressed in the following way

\[
A(p^2) = \frac{1}{2} \left( G_-(p) W_{s+}^0(p) + G_+(p) W_{s-}^0(p) \right), \\
pB(p^2) = \frac{1}{2} \left( g_-(p) G_-(p) W_{s+}^0(p) - g_+(p) G_+(p) W_{s-}^0(p) \right). \] (3.24) (3.25)

The fact that \( A(p^2) \) and \( B(p^2) \) are polynomials and that \( W_{s+}^0(p) \sim W_{r+}(p) + O(1/p) \) allows one to conclude

\[
A(p^2) = \text{Even polynomial part of } W_{r-}(p) G_+(p) \\
= \int_{\infty} d\omega \frac{\omega W_{r-}(\omega)}{2\pi i p^2 - \omega^2} G_+(\omega) \] (3.26)

\(^3\)The choice of sign in this relation is a rather technical point. It relies on the expression (3.34) in the next section. We note, however, that all physical quantities depend only on \( e^2 \).
\[ p \mathcal{B}(p^2) = -\text{Odd polynomial part of } W_{r-}(p)g_+(p)G_+(p) \]
\[ = -p \oint_{\infty} \frac{d\omega}{2\pi i} \frac{W_{r-}(\omega)}{p^2 - \omega^2} g_+(p)G_+(\omega) \]  
\[ (3.27) \]

where \( \oint_{\infty} \) means integration along a contour which encircles \( \infty \). In total one can write \( W^0_+ (p) \) as the following contour integral

\[ W^0_+ (p) = \sqrt{p} G_+ (p) \oint_{C_2} \frac{d\omega}{2\pi i} \frac{W_{r-}(\omega)}{p^2 - \omega^2} G_+(\omega) \{ \omega g_+(p) - pg_+(\omega) \} \]  
\[ (3.28) \]

where the contour \( C_2 \) encircles the cuts \([a, b]\) and \([-b, -a]\) of \( G_+ (\omega) \) but not the point \( \omega = p \). The points \( a \) and \( b \) are determined by the following two equations

\[ \oint_{C_2} \frac{d\omega}{2\pi i} W_{r-}(\omega)g_+(\omega)G_+(\omega) = 0, \]  
\[ (3.29) \]
\[ \oint_{C_2} \frac{d\omega}{2\pi i} W_{r-}(\omega)G_+(\omega)\omega = -\frac{1}{2 \cos(\nu \pi/2)} \]  
\[ (3.30) \]

which follow from the boundary condition (2.14). We note that by using the analyticity properties of the various functions entering the integrand (3.28) one can replace \( \oint_{C_2} \frac{d\omega}{2\pi i} W_{r-}(\omega) \{ \ldots \} \) by \( -\frac{e^{-i\omega \nu/2}}{2 \sin(\nu \pi)} \oint_{C_1} \frac{d\omega}{2\pi i} V'(\omega) \{ \ldots \} \) in the expressions (3.28)–(3.30). It is a matter of taste which expression one prefers to work with. The former reflects more clearly the structure of the loop equation while the latter expression resembles more the one of the hermitian 1-matrix model.

### 3.3 Determination of the auxiliary function

#### 3.3.1 General case

One can derive a differential equation for \( G_+(p) \). To do so one first observes that the function \( \frac{\partial}{\partial p} \left( \sqrt{p} G_+(p) \right) \) will fulfill the equation (3.11) and hence have a parametrization of the type (3.17). When supplemented by the boundary condition for \( G_+(p) \) this observation allows one to conclude

\[ \frac{\partial}{\partial p} \left( \sqrt{p} G_+(p) \right) = \left( \alpha - \frac{\sqrt{e}}{e} + pg_+(p) \right) G_+(p) \]  
\[ (3.31) \]

where \( \alpha \) is some yet not determined constant which has the following role

\[ G_+(p) = \frac{i}{p} \left( 1 - \frac{\alpha}{p} + \mathcal{O} \left( \frac{1}{p^2} \right) \right), \quad p \to \infty. \]  
\[ (3.32) \]

For given \( e \) and \( \alpha \) the equation (3.31) determines \( G_+(p) \) uniquely. It is easy to see that \( G_+(p) \) given by the following elliptic integral is the unique solution we seek

\[ \log \left( \sqrt{p} G_+(p) \right) = \int_0^p dx \sqrt{x} \left( \frac{e \sqrt{e}}{x^2 - e^2} + \alpha \right) \]  
\[ (3.33) \]
provided the following two equations hold
\[
\int_b^\infty \frac{dx}{\sqrt{x}} \left( \frac{e^{\sqrt{e}}}{x^2 - e^2} + \alpha \right) = \frac{i(1 - \nu)\pi}{2}, \quad \int_0^a \frac{dx}{\sqrt{x}} \left( \frac{e^{\sqrt{e}}}{x^2 - e^2} + \alpha \right) = \frac{i\nu\pi}{2}.
\] (3.34)

These equations ensure that \(G_+(p)\) has the correct asymptotic behaviour as \(p \to \infty\) and that \(G_+(a + i0) = -e^{i\nu\pi/2}G_-(a - i0)\) (cf. to equation (3.11)). Together they determine the unknowns \(e\) and \(\alpha\). In particular it can be shown that \(e\) must necessarily lie on the positive imaginary axis and behave as \(a^\nu\) when \(a \to 0\). One can derive another set of equations which determines these two quantities and which will be of importance for the analysis in the following sections. Using the same strategy as for the derivation of (3.31) one finds the following expression for the derivative of \(G_+(p)\) with respect to \(a^2\)
\[
\frac{\partial}{\partial a^2} G_+(p) = \frac{1}{p^2 - a^2} \left( \lambda_a p g_+(p) + \frac{1}{2} (1 - \rho_a) \right) G_+(p)
\] (3.35)
where
\[\lambda_a = -\frac{1}{2} \frac{\partial e^2}{\partial a^2} \frac{e^2 - a^2}{e^{\sqrt{e}}}, \quad \rho_a = \frac{a^2 \partial e^2}{e^2 \partial a^2}\] (3.36)

Now comparing the expressions for \(\frac{\partial}{\partial a^2} \frac{\partial}{\partial p} G_+(p)\) and \(\frac{\partial}{\partial a^2} \frac{\partial}{\partial a} G_+(p)\) that one obtains from (3.31) and (3.33) respectively, one finds the following relation between \(a, b, e\) and \(\alpha\)
\[
\alpha = -\frac{e^{\sqrt{e}}}{a^2 - e^2} + \frac{\partial e^2}{\partial a^2} \frac{a^2(b^2 - a^2)}{e^{\sqrt{e}}},
\] (3.37)
\[
\frac{\partial \alpha}{\partial a^2} = -\lambda_a.
\] (3.38)

In particular these two equations allow one to write down a second order differential equation for \(e(a,b)\). We shall refrain from doing so since we have not been able to extract any further information about the model from the resulting equation. Let us for later convenience note that we have also the relation
\[
\frac{\partial}{\partial p} \left( \sqrt{p} g_+(p) G_+(p) \right) = (\alpha g_+(p) + p) G_+(p)
\] (3.39)
as well as
\[
\frac{\partial}{\partial a^2} (pG_+(p)) = \frac{e^{\sqrt{e}}}{e^2 - b^2} \frac{\partial}{\partial a^2} (g_+(p) G_+(p)) + \lambda_a g_+(p) G_+(p)
\] (3.40)
Here (3.38) follows immediately from (3.21) by noting that for \(g_+(p) G_+(p)\) the parameter \(\alpha\) entering (3.32) is replaced by \(\tilde{\alpha}\),
\[
-\tilde{\alpha} = \alpha - \frac{\sqrt{e}}{e}
\] (3.41)
and to derive the relation (3.40) one makes use of the fact that any two solutions of the saddle point equation can be parametrized in terms of any two other independent solutions. The detailed nature of the parametrization follows from the analyticity properties and the asymptotic behaviour of the functions involved. Needles to say that relations similar to (3.35) and (3.40) concerning the differentiation with respect to $b^2$ follow from these by the interchangelgse $a \leftrightarrow b$ and that $e$ and $\alpha$ depend on $a$ and $b$ in a symmetrical manner. As we will show in the next section when $\nu$ is rational $G_+(p)$ can be further explicited.

### 3.3.2 Rational case

Let us parametrize $\nu$ in the following way

$$\nu = \frac{l}{q}, \quad 0 < l < q, \quad l, q \in \mathbb{Z}_+ (3.42)$$

and let us following reference [5] introduce the function

$$T(p) = \frac{1}{2} \left\{ (G_+(p))^q + (-1)^{q+l}(G_-(p))^q \right\}. \quad (3.43)$$

From the requirements 1–3 on $G(p)$ it follows that $T(p)$ is a rational function with poles at $\pm a$ and $\pm b$ of order $[q/2]$ (the integer part of $q/2$). Furthermore from (3.13) and (3.43) it follows that $(G_+(p))^q$ can be expressed via the two rational functions $T(p)$ and $R(p)$ in the following way

$$(G_+(p))^q = T(p) - \sqrt{T(p)^2 - (-1)^{l+q}R(p)^q} \quad (3.44)$$

where the negative sign in front of the square root ensures the correct asymptotic behaviour of $G_+(p)$ as $p \to \infty$. Now the requirement that $G(p)$ must be analytic in the complex plane except from a cut $[a, b]$ implies that the the square root term above can have singularities only at $a$ and $b$ and therefore must decompose as $\tilde{T}(p)\sqrt{p}$ with $\tilde{T}(p)$ another rational function. Hence we can parametrize $G_+(p)$ in the following way

$$G_+(p) = \frac{i}{\sqrt{p}} \left\{ (p^2 - e^2) \left[ A(p)g_-(p) + B(p) \right] \right\}^{1/q} \quad (3.45)$$

where $A(p)$ and $B(p)$ are polynomials of degree less than or equal to $q-2$ and where we have made use of the function $g_-(p)$ in order to obtain the property $G_+(e) = 0$ assumed earlier. Noting that in the relation (3.44) both $T(p)$ and the function appearing under the square root, for given $l$ and $q$, are functions of a definite parity one finds that the same must be true for $A(p)$ and $B(p)$. More precisely

$$A(-p) = (-1)^{l+1}A(p), \quad B(-p) = (-1)^lB(p). \quad (3.46)$$
To determine the polynomials $A(p)$ and $B(p)$ as well as the parameter $e$ it suffices to evoke the relation (3.13) which implies

$$(-1)^{q+l} (p^2 - e^2)^{q-1} = (p^2 - e^2)B^2(p) - \left(p^2 - \frac{a^2b^2}{e^2}\right) A^2(p) - 2\sqrt{e} p A(p) B(p).$$

(3.47)

and where we note that the number of equations exactly matches the number of unknowns. However, this set of algebraic equations may have many different solutions and we must add some boundary condition to select the correct one. Let us note that equations (3.45), (3.46) and (3.47) do not depend on $l$ but only on its parity. We claim that the different solutions of equation (3.47) correspond to different values of $l$. For a given $l$ the correct solution can be identified for instance by its asymptotic behaviour in the $a \to 0$ limit. As mentioned in the previous section $e(a,b)$ always lies on the positive imaginary axis and in the limit $a \to 0$, it behaves as $a^\nu$. More precisely as we shall see in section 5 one has

$$e(a,b) \sim 2ib \left(\frac{a}{4b}\right)^{1/q}$$

(3.48)

and this is the criterion which allows us to pick out a unique solution of equation (3.47). One might prefer evaluating the logarithmic derivative $\rho_a$ introduced in (3.36) which must behave as

$$\rho_a \sim \frac{l}{q}.$$  

(3.49)

Let us close this section by considering some explicit examples. In each case the function $G(p)$ is determined by the equations (3.45), (3.46), (3.47) and (3.48) or (3.49).

**The case l=1, q=2, i.e. n=0:** Here equations (3.45), (3.47) and the condition that the degree of $A$ and $B$ is less than $q - 2$ imply that

$$A(p) = 1, \quad B(p) = 0 \quad \text{and} \quad e^2 = \frac{a^2b^2}{e^2} \quad \text{i.e.} \quad e = +i\sqrt{ab}.$$  

(3.50)

The expression (3.45) for $G_+(p)$ hence reads

$$G_+(p) = \frac{i\sqrt{p}}{\sqrt{\sqrt{p} - p\sqrt{e}}}^{1/2} = \frac{i}{\sqrt{p}} \left(\sqrt{\sqrt{p} - ip(b-a)}\right)^{1/2}.$$  

(3.51)

After performing the transformation (3.3) one finds the familiar form of the solution of the saddle point equation of the hermitian 1-matrix model with behaviour $G(p) \to \frac{\sqrt{2}}{p}$, $p \to \infty$

$$G(p) = \frac{\sqrt{2}}{\sqrt{(p-a)(p-b)}}.$$  

(3.52)
Furthermore one easily verifies that with the expression (3.51) for \( G_+(p) \) the formulas in section (3.2) correctly reproduce the usual contour integral representation of the solution of the 1-matrix model.

**The case \( l=1, q=3, \text{i.e. } n=1: \)** Let us emphasize that this set of models contains the Ising model on a random lattice as a special case. Since the polynomials \( A(p) \) and \( B(p) \) are of degree less than or equal to \( q-2 = 1 \) and obey the parity condition (3.46) we write them in the following way:

\[
A(p) = c, \quad B(p) = p
\]  

(3.53)

The constant \( c \) and the parameter \( e \) are determined by (3.47). For \( c \) one finds

\[
c = \frac{-2e\sqrt{e}}{e^2 - a^2b^2/e^2}
\]  

(3.54)

while \( e \) is given by

\[
e^2 = -\epsilon ab, \quad \epsilon^4 - 6\epsilon^2 - 4\epsilon \left( \frac{a}{b} + \frac{b}{a} \right) - 3 = 0
\]  

(3.55)

According to (3.48) and (3.49) we have to choose the branch of the solution of this fourth degree equation for which \( \epsilon > 0 \) and \( \rho_a \to 1/3 \) when \( a \to 0 \), i.e. when \( \epsilon \to \infty \). For the solution which matches these criteria one has

\[
\rho_a = \frac{1}{2} \left( 1 - \frac{1}{3} \sqrt{\frac{\epsilon^2 - 9}{\epsilon^2 - 1}} \right)
\]  

(3.56)

where it is understood that the positive square root should be taken.

### 3.4 The two-loop correlator at genus zero

One way to calculate the two-loop correlator is to use the directly the recipe

\[
W(p, q) = \frac{d}{dV(p)} W(q).
\]  

(3.57)

However, there exists a less work demanding method. The two-loop correlator at genus zero must satisfy the following equation

\[
W^0(p + i0, q) + W^0(p - i0, q) + nW^0(-p, q) = -\frac{1}{(p - q)^2}, \quad p \in [a, b]
\]  

(3.58)

which appears when one applies the loop insertion operator \( d/dV(q) \) to the saddle point equation (2.16). This is an equation of the same type as (2.16). One can split
the two-loop correlator in a regular and a singular part. The regular part is easily
found and coincides with what one finds by acting with the loop insertion operator on
the regular part of the one-loop correlator. The singular part fulfills the homogeneous
version of the equation (3.58). To solve this equation it is convenient to perform a
rotation like (3.2) for each of the two variables of $W(p,q)$ so that one has

$$W(p,q) = -e^{i\nu\pi}W_{++}(p,q) + W_{+-}(p,q) + W_{-+}(p,q) - e^{-i\nu\pi}W_{-+}(p,q).$$

with

$$W_{+-}(p,q) = W_{-+}(-p,q) = W_{-+}(-p,-q) = W_{++}(p,-q).$$

We note that

$$W_{i,j}(p,q) = \frac{d}{dV_i(p)}W_j(q), \quad i,j \in \{+, -\}.$$  (3.61)

Now the singular part of $W^0_{++}(p,q)$ fulfills an equation similar to (3.11) in each of
the variables and a parametrization of the most general solution can be written down
using the functions $G_+(p)$ and $g_+(p)G_+(p)$ introduced in section (3.2). The following
requirements on $W(p,q)$ single out a unique solution.

- $W(p,q) = \frac{d}{dV(p)}\frac{d}{dV(q)}F$ must be symmetrical in $p$ and $q$ and regular when $p = q$
- $W^0(p,q)$ can have a singularity of the form $((p - a)(p - b))^{-1/2}$ but no additional
  poles at $a$ or $b$ since $W^0(p)$ has only a singularity of the type $((p - a)(p - b))^{1/2}$.
- $W(p,q)$ has the following asymptotic behaviour

$$W(p,q) \sim O(1/p^2), \quad p \to \infty.$$  (3.62)

The unique solution reads

$$W^0_{++}(p,q) = \frac{1}{4 - n^2} \left\{ G_+(p)G_+(q) \left[ -1 - \alpha \frac{pg_+(p) - qg_+(q)}{p^2 - q^2} \right. \\
+ \frac{(pg_+(p) - qg_+(q)) \left( p\sqrt{p} - q\sqrt{q} \right)}{(p^2 - q^2)^2} \left. \right] - \frac{1}{(p + q)^2} \right\}.$$  (3.63)

We see that the result does not show any explicit dependence of the matrix model
potential. Hence the universality of the two-loop function observed for the 1-matrix
model [10, 17, 18] extends to the $O(n)$ model on a random lattice. In addition there is
a large degree of universality with respect to $n$. (We remind the reader that the result
above is valid for any $n$, but that different values of $n$ give rise to different functions
$G_+(p)$.)
In accordance with the fact that \( W_0^+ (p) \) depends on the potential \( V(p) \) only via \( W_{r-}(p) \) and that
\[
\frac{d}{dV_+(q)} W_{r-}(p) = \frac{1}{4 - n^2} \frac{\partial}{\partial q} \left( \frac{1}{p - q} \right)
\]  
we find that the two-loop correlator can be written as a total derivative
\[
W_{++}^0 (p, q) = \frac{1}{4 - n^2} \frac{\partial}{\partial q} \left\{ -\frac{1}{p + q} + G_+(p)G_+(q) \sqrt{q} \frac{pg_+(p) - qg_+(q)}{p^2 - q^2} \right\}.
\]  

To proceed with the solution of the loop equation we need to know \( W_{0-}^0 (p, p) \). To determine this quantity we must analyse carefully the limit \( p \to q \) of \( W_{++}^0 (p, -q) \) (which is a rather time consuming task). The outcome of the analysis is
\[
W_{+-}^0 (p, p) = \frac{1}{4 - n^2} \left\{ \frac{e^2 - \alpha^2}{2 (p^2 - a^2)(p^2 - b^2)} + \frac{1}{4} (a^2 - b^2)^2 \frac{p^2}{(p^2 - a^2)^2(p^2 - b^2)^2} \right\}. \]  

We draw the attention of the reader to the fact that \( W_{+-}^0 (p, p) \) is a rational even function with poles at \( p = \pm a \) and \( p = \pm b \). This will be of importance for the following.

### 3.5 The one-loop correlator at genus 1

#### 3.5.1 The structure of the 1-loop correlator

For genus 1 the loop equation reduces to
\[
\hat{K} \frac{4}{4 - n^2} W^{(1)}(p) = W_{+-}^0 (p, p)
\]  
where \( \frac{\hat{K}}{4 - n^2} \) is the linear operator entering the left hand side of the loop equation (3.6), i.e.
\[
\frac{\hat{K}}{4 - n^2} f(p) = \left\{ \hat{K}_+ - W_+^0 (p) \right\} f_-(p) + \left\{ \hat{K}_- - W_-^0 (p) \right\} f_+(p).
\]  

Let us note for later convenience that using the decomposition (3.7) we can write the action of the operator \( \hat{K} \) on a function \( f(p) \) as
\[
\hat{K} f(p) = (4 - n^2) \int_{c_2} \frac{d\omega}{2\pi i} \frac{\omega}{p^2 - \omega^2} \left\{ W_{s+}(\omega) f_-(\omega) + W_{s-}(\omega) f_+(\omega) \right\}
\]  
Noticing the structure of expression (3.66) for \( W_{+-}^0 (p, p) \) and bearing in mind the strategy for calculating higher genera contributions in the case of the hermitian 1-matrix model [7] we will seek to express \( W^{(1)}(p) \) in the following way
\[
W^{(1)}(p) = A_1^{(2)} \chi_a^{(2)}(p) + B_1^{(2)} \chi_b^{(2)}(p) + A_1^{(1)} \chi_a^{(1)}(p) + B_1^{(1)} \chi_b^{(1)}(p)
\]  

(3.70)
the idea being that the $\chi$-functions should allow us to invert the operator $\hat{K}$, i.e.
\[
\hat{K}\chi_a^{(m)}(p) = \frac{1}{(p^2 - a^2)^m}, \quad \hat{K}\chi_b^{(m)}(p) = \frac{1}{(p^2 - b^2)^m}.
\] (3.71)

We will require that the $\chi$-functions have the following asymptotic behaviour
\[
\chi_a^{(m)}(p), \chi_b^{(m)}(p) \sim \mathcal{O} \left( \frac{1}{p^2} \right)
\] (3.72)

where the possibility of a $1/p$ term has been excluded in order to ensure that the relation (2.14) remains true. The coefficients $A_1^{(i)}$, $B_1^{(i)}$, $i = 1, 2$ follow from the decomposition of $W_0^+(p,p)$ into fractions of the type $(p^2 - a^2)^{-m}$ and $(p^2 - b^2)^{-m}$, \(m = 1, 2\) and read
\[
A_1^{(1)} = \frac{1}{2} \frac{1}{a^2 - b^2} \left( e^2 - \alpha^2 - \frac{1}{2} (a^2 + b^2) \right), \quad A_1^{(2)} = \frac{1}{4} a^2, \quad B_1^{(1)} = \frac{1}{2} \frac{1}{b^2 - a^2} \left( e^2 - \alpha^2 - \frac{1}{2} (b^2 + a^2) \right), \quad B_1^{(2)} = \frac{1}{4} b^2.
\] (3.73)

\[
A_2^{(1)} = \frac{1}{2} \frac{1}{a^2 - b^2} \left( e^2 - \alpha^2 - \frac{1}{2} (a^2 + b^2) \right), \quad A_2^{(2)} = \frac{1}{4} a^2, \quad B_2^{(1)} = \frac{1}{2} \frac{1}{b^2 - a^2} \left( e^2 - \alpha^2 - \frac{1}{2} (b^2 + a^2) \right), \quad B_2^{(2)} = \frac{1}{4} b^2.
\] (3.74)

### 3.5.2 Determination of the $\chi$-functions

Since the analyticity structure of the $\chi$-functions should be compatible with that of the 1-loop correlator it is natural to try to construct these functions using as starting point the functions $G(p)$ and $\tilde{G}(p)$ (cf. to equation (3.19)). From (3.69) it follows that
\[
\hat{K}G(p) = \hat{K}\tilde{G}(p) = 0.
\] (3.75)

Next, let us consider the following functions
\[
\phi_a^{(k)}(p) = \frac{G(p)}{(p^2 - a^2)^k}, \quad \tilde{\phi}_a^{(k)}(p) = \frac{p\tilde{G}(p)}{(p^2 - a^2)^k}.
\] (3.76)

Applying the operator $\hat{K}$ to these functions one finds
\[
\hat{K}\phi_a^{(k)} = \sum_{l=1}^{k} \frac{m_{a,l}^{(k)}}{(p^2 - a^2)^l}, \quad \hat{K}\tilde{\phi}_a^{(k)} = \sum_{l=1}^{k} \frac{\tilde{m}_{a,l}^{(k)}}{(p^2 - a^2)^l}.
\] (3.77)

where \(\{m_{a,l}^{(k)}\}\) and \(\{\tilde{m}_{a,l}^{(k)}\}\) are some constants. This means that from either of the two series of functions $\phi_a^{(k)}(p)$ and $\tilde{\phi}_a^{(k)}(p)$ we can construct functions $\chi_a^{(m)}(p)$ obeying (3.71) and (3.72). (We remind the reader that $G(p), \tilde{G}(p) \sim \mathcal{O}(1/p)$, $p \to \infty$.) However, neither of the two series alone can serve as building blocks for $\chi_a^{(m)}(p)$ since all the functions $\phi_a^{(k)}(p)$ and $\tilde{\phi}_a^{(k)}(p)$ have poles at $p = -a$ which contradicts the assumption concerning the analyticity structure of $W(p)$. We are hence forced to take linear combinations of $\phi$’s and $\tilde{\phi}$’s to kill these unwanted poles. One type of such linear combinations with
correct analyticity properties leaps to the eye; the functions \( \left( \frac{\partial}{\partial a^2} \right)^k G(p) \) (cf. to equation (3.33)). Let us try to build \( \chi_a^{(k)}(p) \) from such functions. The expression for \( \frac{\partial G(p)}{\partial a^2} \) appeared in equation (3.35) and for \( \frac{\partial^2 G(p)}{\partial (a^2)^2} \) we find using (3.35) and (3.40)

\[
\left( \frac{\partial}{\partial a^2} \right)^2 G(p) = \frac{3}{2} \frac{1}{p^2 - a^2} \left\{ \frac{\partial}{\partial a^2} G(p) + C_\infty G(p) \right\} + C_2 \frac{\partial}{\partial a^2} G(p) \tag{3.78}
\]

where the constants \( C_1 \) and \( C_2 \) are given by

\[
C_\infty = \frac{2}{3} \left\{ \frac{1}{4a^2} \rho_a (1 - \rho_a) - \frac{1}{2} \frac{\partial \rho_a}{\partial a^2} - \frac{1}{2} \frac{\lambda_a}{\partial a^2} \right\} \tag{3.79}
\]

\[
C_2 = \frac{1}{\lambda_a} \frac{\partial \lambda_a}{\partial a^2}. \tag{3.80}
\]

We note that we always have a recursive relation like (3.78) relating the \( (k + 1) \)th derivative of \( G(p) \) to the \( k \)th and the \( (k - 1) \)th. This follows from the fact, already evoked several times, that any solution of the saddle point equation (2.16) can be parametrized in terms of any two other independent solutions. The nature of the parametrization follows from an analysis of the analyticity structure and the asymptotic behaviour of the functions involved and a recursive relation for the expansion coefficients can be found. Since for the moment we will need only \( \frac{\partial G(p)}{\partial a^2} \) and \( \frac{\partial^2 G(p)}{\partial (a^2)^2} \) we shall not enter into a detailed discussion of this point, but we will make use of such considerations in section 3.7 concerning the calculation of higher genera contributions.

Now, let us consider the action of the operator \( \hat{K} \) on the functions above. One finds

\[
\hat{K} \left( \frac{\partial G(p)}{\partial a^2} \right) = \mathcal{M}_1 \frac{1}{p^2 - a^2}, \tag{3.81}
\]

\[
\hat{K} \left( \frac{\partial^2 G(p)}{\partial (a^2)^2} \right) = \mathcal{M}_2 \frac{1}{p^2 - a^2} + \frac{2}{3} \mathcal{M}_1 \frac{1}{(p^2 - a^2)^2} \tag{3.82}
\]

where the moments \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are defined by

\[
\mathcal{M}_i = (4 - n^2) \oint_{C_2} \frac{d\omega}{2\pi i} \left\{ W_{r_+}(\omega) \left( \frac{\partial}{\partial a^2} \right)^i G_-(\omega) \right\} + W_{r_-}(\omega) \left( \frac{\partial}{\partial a^2} \right)^i G_+(\omega) \right\} \]

\[
= 2 \oint_{C_1} \frac{d\omega}{2\pi i} V'(\omega) \left( \frac{\partial}{\partial a^2} \right)^i G(\omega), \quad i = 1, 2. \tag{3.83}
\]

This means that we can choose our \( \chi_a \)-functions in the following way

\[
\chi_a^{(1)}(p) = \frac{1}{\mathcal{M}_\infty} \frac{\partial}{\partial a^2} G(p) \tag{3.84}
\]

\[
\chi_a^{(2)}(p) = \frac{1}{\mathcal{M}_\infty} \left\{ \frac{2}{3} \left( \frac{\partial}{\partial a^2} \right)^2 G(p) - \frac{\mathcal{M}_\infty}{\mathcal{M}_\infty} \frac{\partial}{\partial a^2} G(p) \right\} \tag{3.85}
\]
Needless to say that \( \chi^{(m)}(p) \) appears from \( \chi^{(m)}(p) \) by the replacement \( a \leftrightarrow b \). Now combining the relations (3.73), (3.74), (3.84) and (3.85) one has an explicit expression for the 1-loop correlator at genus one.

3.6 The free energy at genus 1

To determine the free energy at genus one we use again the strategy of reference [7]; namely we seek to express the basis vectors \( \chi^{(m)}(p) \) and \( \chi^{(m)}(p) \) as total derivatives with respect to the loop insertion operator. The case \( m = 1 \) is relatively simple. Using the relation (3.64) as well as (3.31), (3.35), (3.39) and (3.40) one finds from the boundary equations (3.29) and (3.30) after a lengthy but in principle straightforward calculation

\[
\chi^{(1)}_a(p) = \frac{1}{4} \frac{d \log a^2}{dV(p)}; \quad \chi^{(1)}_b(p) = \frac{1}{4} \frac{d \log b^2}{dV(p)}. \quad (3.86)
\]

The case \( m = 2 \) is less simple but due to the appearance of the factor \( M_2/M_1 \) in the relation (3.85) is is obvious that \( \chi^{(2)}_{a}(p) \) must be closely related to \( \frac{\partial}{\partial b^2} \log M_1 \). By explicit computation one finds that this quantity can actually be expressed entirely in terms of \( \chi^{(2)}_a(p), \chi^{(1)}_a(p), \chi^{(1)}_b(p), a^2, b^2, e^2, \frac{\partial a^2}{\partial a^2} \) and \( \frac{\partial a^2}{\partial b^2} \) which is a non-trivial result. Let us briefly comment on the key relations which ensure this property. (We will also need these relations for our discussions in section 3.7.) Acting with the loop insertion operator on \( M_1 \) as usual implies performing an explicit differentiation after the matrix model coupling constants as well as an implicit differentiation after \( a^2 \) and \( b^2 \). The explicit differentiation leads to the appearance of the quantity \( \frac{\partial}{\partial p} \left( p \frac{\partial}{\partial a^2} G(p) \right) \) which using the relations (3.31), (3.35) and (3.40) can be written as

\[
\frac{\partial}{\partial p} \left( p \frac{\partial}{\partial a^2} G(p) \right) = -2a^2 \left( \frac{\partial}{\partial a^2} \right)^2 G(p) \quad \text{and} \quad \frac{1}{b^2 - a^2} \left( \frac{1}{e^2 - b^2} \frac{\partial}{\partial b^2} \right) \frac{\partial}{\partial a^2} G(p)

\]

\[
- \left( \frac{2a^2 - b^2}{a^2 - b^2} + \frac{b^2}{e^2 - a^2} \frac{\partial}{\partial a^2} \right) \frac{\partial}{\partial b^2} G(p). \quad (3.87)
\]

The implicit differentiations lead to the appearance of mixed double derivatives of \( G(p) \) which with the use of (3.37) and (3.40) can be expressed in the following way

\[
\frac{\partial}{\partial b^2} \frac{\partial}{\partial a^2} G(p) = \frac{1}{2} \left( \frac{1}{a^2 - b^2} + \frac{1}{e^2 - a^2} \right) \frac{\partial}{\partial a^2} G(p)

\]

\[
+ \frac{1}{2} \left( \frac{1}{b^2 - a^2} + \frac{1}{e^2 - b^2} \right) \frac{\partial}{\partial b^2} G(p). \quad (3.88)
\]

In total one ends up with the following expression for \( \chi^{(2)}_a(p) \)

\[
3a^2 \chi^{(2)}_a(p) = -\frac{1}{2} \frac{d \log M_1}{dV(p)} \quad \text{and} \quad -\frac{1}{2} \frac{d \log |a^2 - b^2|}{dV(p)} \quad \text{and} \quad -\frac{1}{2} \frac{d \log a^2}{dV(p)} + \frac{1}{4} \frac{d \log |a^2 - e^2|}{dV(p)} \quad (3.89)
\]
Evidently the relevant expression for $\chi^{(2)}_b(p)$ appears from (3.89) by the interchange $a^2 \leftrightarrow b^2$. Now inserting the here obtained expressions for the $\chi$-functions into the expression (3.70) for $W^1(p)$ we can do the integration and obtain $F_1$. The result reads

$$F_1 = -\frac{1}{24} \log M_1 - \frac{1}{24} \log J_1 - \frac{1}{6} \log |a^2 - b^2| + \frac{1}{48} \log |a^2 - e^2| + \frac{1}{48} \log a^2 + \frac{1}{48} \log |b^2 - e^2| + \frac{1}{48} \log b^2 + f_1 \left( \frac{a}{b} \right)$$

(3.90)

where $f_1 \left( \frac{a}{b} \right)$ obeys the following differential equation

$$xf_1'(x) = \frac{1}{4} \frac{e^2 - \alpha^2}{a^2 - b^2}$$

(3.91)

and where $J_1 = M_1(a \leftrightarrow b)$. We emphasize that this expression for $F_1$ holds for any potential $V(M)$ and any $n \in [1 - 2, 2]$. The first three terms of (3.90) have a structure similar to the terms which appeared in the case of the 1-matrix model and one can easily verify that the 1-matrix model ($n=0$) result is correctly recovered.

### 3.7 Higher genera and multi loops

Having calculated $W^1(p)$ we are in a position to further iterate the genus expanded version of the loop equation (3.6). While the moments and basis vectors introduced in section 3.5.2 certainly lead to simple expressions for the genus one quantities presented there, they do not give the optimal parametrization of the model when it comes to the representation of higher genera contributions. Let us describe now what we consider as the optimal parametrization of the model. We will still work with a set of $\chi$-functions satisfying the relations (3.71) and (3.72). However we will change the set of basis functions and moments.

As basis functions we shall use instead of the functions \{ \left( \frac{\partial}{\partial a} \right)^k G(p), \left( \frac{\partial}{\partial b} \right)^k G(p) \} a set of functions \{ $G^{(k)}_a(p), G^{(k)}_b(p)$ \} defined by

1. $G^{(k)}_a(p)$ and $G^{(k)}_b(p)$ satisfy the homogeneous saddle point equation (2.16)

$$G^{(k)}(p + i0) + G^{(k)}(p - i0) + nG^{(k)}(-p) = 0, \quad p \in [a, b].$$

2. $G^{(k)}_a(p)$ and $G^{(k)}_b(p)$ behave near the end points of the cut $[a, b]$ as

$$G^{(k)}_a(p) \sim (p - a)^{-k-1/2}(p - b)^{-1/2} \quad G^{(k)}_b(p) \sim (p - b)^{-k-1/2}(p - a)^{-1/2}. $$

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3. \( G^{(k)}_a(p) \) and \( G^{(k)}_b(p) \) are analytical outside the cut (especially near \(-a\) and \(-b\)).

4. \( G^{(k)}_a(p) \) and \( G^{(k)}_b(p) \) have the following asymptotic behaviour

\[
G^{(k)}_a(p), G^{(k)}_b(p) \sim \frac{1}{p^{k+1}}, \quad p \to \infty.
\]

Here the conditions 1 and 3 ensure that the analyticity properties of \( G^{(k)}_a(p) \) and \( G^{(k)}_b(p) \) are compatible with those of the one-loop correlator and that \( \hat{K}G^{(k)}_a(p) \) and \( \hat{K}G^{(k)}_b(p) \) will be even rational functions with poles at \( p = \pm a \) and \( p = \pm b \) respectively. The purpose of condition 2 is simply to relate the degree of the poles to the index \( k \). The conditions 1–3 are satisfied by many different families of functions, but only one family of functions fulfills all four conditions. We note that the set of functions \( \left\{ \left( \frac{\partial}{\partial \omega} \right)^k G(p), \left( \frac{\partial}{\partial \omega} \right)^k G(p) \right\} \) introduced in section 3.5 fulfills the conditions 1–3 but not condition 4. Furthermore we note that for \( n = 0 \) we reproduce exactly the basis functions used in reference [7]. Now for \( n = 2 \cos(\nu \pi) \), let us denote by \( \left\{ \tilde{G}^{(k)}_a(p), \tilde{G}^{(k)}_b(p) \right\} \) the basis functions corresponding to \( n = 2 \cos((1 - \nu)\pi) \), i.e.

\[
\left\{ \tilde{G}^{(k)}_a(p), \tilde{G}^{(k)}_b(p) \right\}_\nu = \left\{ G^{(k)}_a(p), G^{(k)}_b(p) \right\}_{1-\nu}
\]  (3.92)

These are the functions that will appear in our definition of moments, namely we define

\[
M_k = 2 \oint_{C_1} \frac{d\omega}{2\pi i} V''(\omega) \tilde{G}^{(k)}_a(\omega), \quad J_k = M_k(a \leftrightarrow b).
\]  (3.93)

In the case of the 1-matrix model (\( n=0 \)) we have \( \left\{ \tilde{G}^{(k)}_a(p), \tilde{G}^{(k)}_b(p) \right\} = \left\{ G^{(k)}_a(p), G^{(k)}_b(p) \right\} \) and hence we reproduce with our definition exactly the moments (up to a factor 2) used for the 1-matrix model [7]. Furthermore as in the one-matrix model case, we can write one of the boundary condition, \( (B.28) \), as

\[
M_0 = 0
\]  (3.94)

However, we stress that \( n = 0 \) is a very special case. In general we will have \( G^{(k)}(p) \neq \tilde{G}^{(k)}(p) \). We draw the attention of the reader to the importance of the condition number 4 in the definition of the basis functions. With a boundary condition of this type we will have for a potential of degree \( d \) that \( M_k = J_k = 0 \) for \( k > d - 1 \). This gives the parametrization of the model in terms of the smallest possible number of moments\footnote{One may argue that we still have one parameter too much since for a potential of degree \( d \) described by \( d \) coupling constants, we have \( d - 1 \) moment variables plus the two variables \( a \) and \( b \). Indeed there is a constraint that would allow us to reduce the number of parameters by one, namely the fact that the free energy has to be dimensionless. Hence, if we defined our moment variables to be dimensionless our results would depend on \( a \) and \( b \) only via \( a/b \). However, we have not found that such a redefinition leads to any simplification from a computational point of view.}.
The functions \( \left\{ \left( \frac{\partial}{\partial a} \right)^k G(p), \left( \frac{\partial}{\partial b} \right)^k G(p) \right\} \) behave as \( O(1/p^2) \) as \( p \to \infty \). Hence with these functions as basis functions we would have the unpleasant situation that even with a potential of finite order we would have an infinite number of moment variables. We have kept these less pleasant moment variables in section 3.5 and section 3.6 since they render the formulation of the idea of our iterative procedure more comprehensible and since they give a particularly simple representation of the free energy for genus 1. The nature of the prefactor of \( 1/p^{k+1} \) in the requirement on the asymptotic behaviour of the basis functions is not important for the argument above. However, it is convenient for the analysis of the critical behaviour of the model that this prefactor is independent of \( a \) and \( b \). We choose it equal to one for simplicity. Let us mention that we can also write

\[
M_k = (4 - n^2) \int_{C_2} \frac{d\omega}{2 \pi i} \left\{ W_{r+}(\omega) \tilde{G}_{a+}(\omega) - W_{r-}(\omega) \tilde{G}_{a-}(\omega) \right\}
\]

(3.95)

\[
= (4 - n^2) \int_{C_2} \frac{d\omega}{2 \pi i} \left\{ W_{s+}(\omega) \tilde{G}_{a+}(\omega) - W_{s-}(\omega) \tilde{G}_{a-}(\omega) \right\}
\]

(3.96)

The expression (3.96) is particularly appealing since the integrand does not have any cut but only singularities in the form of poles at \( \pm a \). This follows from the fact that \( W_{s \pm}(p) \) is a solution of the saddle point equation (3.11) while \( \tilde{G}_{a \pm}(p) \) is a solution of the same equation with \( \nu \to 1 - \nu \) and that \( \tilde{G}_{a \pm}^{(k)}(p) \sim (p - b)^{-1/2} \) for \( p \sim b \) while \( W_{s}(p) \sim (p - b)^{1/2} \) for \( p \sim b \). Since in addition the integrand is odd, the contour \( C_2 \) can be deformed to a small loop encircling the point \( a \). Similarly, in the case of the \( J_k \) moments the contour \( C_2 \) can be deformed into a small loop encircling the point \( b \).

This observation will prove very useful for our considerations in section 5 concerning the scaling limit of the model.

3.7.1 Recursion relations for \( G^{(k)}_a(p), G^{(k)}_b(p) \)

From section 3.3 it follows that

\[
G^{(0)}_a(p) = G^{(0)}_b(p) = \frac{G(p)}{2 \cos(\nu \pi/2)}.
\]

(3.97)

Furthermore from the defining conditions 1–4 one can conclude

\[
G^{(k+1)}_a(p) = \frac{1}{\lambda^{(k)}_a} \frac{\partial G^{(k)}_a(p)}{\partial a^2}
\]

(3.98)

since \( \frac{\partial G^{(k)}_a(p)}{\partial a^2} \) fulfills the requirements 1–3 for \( G^{(k+1)}_a(p) \) and the appropriate asymptotic behaviour can be obtained by multiplication by a constant. The expressions for \( \lambda^{(0)}_a \) and \( \lambda^{(1)}_a \) can be extracted from the relations (3.35) and (3.79) respectively. One finds

\[
\lambda^{(0)}_a = i \tan(\nu \pi/2) \lambda_a,
\]

(3.99)

\[
\lambda^{(0)}_a \lambda^{(1)}_a = -\frac{1}{2} \frac{\partial \rho_a}{\partial a^2} + \frac{1}{4a^2} \rho_a \left( 1 - \rho_a \right) - \frac{1}{2} \frac{1}{\lambda_a} \left( 1 - \rho_a \right) \frac{\partial \lambda_a}{\partial a^2}
\]

(3.100)
We will now derive a set of recursion relations which allow us from the knowledge of $G^{(0)}_a(p)$, $\lambda^{(0)}_a$ and $\lambda^{(1)}_a$ to calculate any $G^{(k)}_a(p)$, $k > 2$. First we use the fact that any solution of the saddle point equation (2.16) can be parametrized in terms of any two other independent solutions to write

$$G^{(k+1)}_a(p) = \frac{1}{p^2 - a^2} \left\{ G^{(k-1)}_a(p) + c^{(k)}_a G^{(k)}_a(p) \right\}.$$  \hspace{1cm} (3.101)

Here the prefactor $1/(p^2 - a^2)$ generates the correct leading singularity as well as the correct asymptotic behaviour of $G^{(k+1)}_a(p)$. The constant $c^{(k)}_a$ is determined by the requirement that $G^{(k+1)}_a(p)$ should not have a pole at $p = -a$, i.e.

$$c^{(k)}_a = -\frac{G^{(k-1)}_a(-a)}{G^{(k)}_a(-a)}.$$  \hspace{1cm} (3.102)

Next, by combining (3.98) and (3.101) we obtain the following relations between the coefficients $c^{(k)}_a$ and $\lambda^{(k)}_a$

$$c^{(k)}_a \lambda^{(k)}_a = (k + \frac{1}{2}), \quad \lambda^{(k+1)}_a - \lambda^{(k-1)}_a = \frac{\partial c^{(k)}_a}{\partial a^2}.$$  \hspace{1cm} (3.103)

From our knowledge of $G^{(0)}_a(p)$, $\lambda^{(0)}_a$ and $\lambda^{(1)}_a$ we can now by means of (3.101) and (3.103) easily write down an explicit expression for any $G^{(k)}_a(p)$ (and similarly for $G^{(k)}_b(p)$). Furthermore it is obvious that the $\tilde{G}$-functions appear from the $G$-functions by the substitutions $\nu \to 1 - \nu$ and we will use for the relations involving $\tilde{G}$-functions the same notation as above just with all quantities being equipes with a tilde. We have in addition the following relation between the $G$ and $\tilde{G}$ functions

$$p G^{(k)}_a(p) = \tilde{G}^{(k-1)}_a(p) + s^{(k)}_a \tilde{G}^{(k)}_a(p)$$  \hspace{1cm} (3.104)

The argument goes as above and the constant $s^{(k)}_a$ is given by

$$s^{(k)}_a = -\frac{\tilde{G}^{(k-1)}_a(0)}{G^{(k)}_a(0)}.$$  \hspace{1cm} (3.105)

By inserting (3.104) in (3.98) and (3.101) we find the following expression for $s^{(k)}_a$

$$s^{(k+1)}_a = s^{(k)}_a \frac{\tilde{\lambda}^{(k)}_a}{\lambda^{(k)}_a}.$$  \hspace{1cm} (3.106)

Hence it suffices to calculate $s^{(1)}_a$. It reads

$$s^{(1)}_a = \frac{1}{2} \left( \frac{1 - \rho_a}{\lambda^{(0)}_a} \right).$$  \hspace{1cm} (3.107)
Now, if we define another set of moment variables by
\[ M_k = 2 \oint_{C_1} \frac{d\omega}{2\pi i} \omega V'(\omega) G_a^{(k)}(\omega), \quad J_k = M_k(a \leftrightarrow b) \quad (3.108) \]
we have
\[ M_k = M_{k-1} + s_a^{(k)} M_k \quad (3.109) \]
and inversely
\[ M_k = \frac{1}{s_a^{(k)}} M_k - \frac{1}{s_a^{(k)} s_a^{(k-1)}} M_{k-1} + \ldots + (-1)^{k-1} \frac{1}{s_a^{(k)} \ldots s_a^{(1)}} M_1. \quad (3.110) \]
To derive this equation one explicitly makes use of the fact that \( M_0 = 0 \). These two relations allow us to move freely between the two sets of variables. However, we stress that it is the \( M \)-moments which are the fundamental quantities since these, as mentioned earlier, give the parameterization of the model in terms of the smallest possible number of moments. Working with the \( \overline{M} \)-moments would for a given potential of finite degree (or for a given multi-critical point) lead to the appearance of one additional parameter. (For a potential of degree \( d \) we will have \( \overline{M}_k = 0 \) only for \( k > d \) while \( M_k = 0 \) for \( k > d - 1 \).)

### 3.7.2 Recursion relations for \( \chi^{(k)}_a(p) \) and \( \chi^{(k)}_b(p) \)

We remind the reader that the aim of introducing the basis functions was to be able to invert the operator \( \hat{K} \). Let us therefore examine the effect of acting with \( \hat{K} \) on such a function. One finds
\[ \hat{K} G_a^{(k)}(p) = \sum_{l=0}^{k-1} \frac{\mu_{k,l}}{(p^2 - a^2)^{l+1}}, \quad \hat{K} G_b^{(k)}(p) = \sum_{l=0}^{k-1} \frac{\tau_{k,l}}{(p^2 - b^2)^{l+1}} \quad (3.111) \]
where \( \mu_{k,l} \) and \( \tau_{k,l} \) are defined by
\[ \mu_{k,l} = (4 - n^2) \oint_{C_2} \frac{d\omega}{2\pi i} (\omega^2 - a^2)^l \left\{ W_{s+}(\omega) G_a^{(k)}(\omega) + W_{s-}(\omega) G_{a+}^{(k)}(\omega) \right\}, \quad (3.112) \]
\[ \tau_{k,l} = \mu_{k,l}(a \leftrightarrow b). \quad (3.113) \]
From (3.111) we can write down a recursive relation for the \( \chi \)-functions, namely
\[ \chi^{(k)}_a(p) = \frac{1}{\mu_{k,k-1}} \left\{ G_a^{(k)}(p) - \sum_{i=1}^{k-1} \mu_{k,i-1} \chi^{(i)}_a(p) \right\} \quad (3.114) \]
and similarly for \( \chi^{(k)}_b(p) \). The \( \mu \)-coefficients can be expressed in terms of the moment variables and the \( c_a^{(k)} \)'s. One has
\[ \mu_{k,l} = 0, \quad l \geq k, \quad \mu_{k,0} = \overline{M}_k. \quad (3.115) \]
and the remaining \( \mu \)-coefficients then follow from the recursion relation

\[
\mu_{k+1,l} = \mu_{k-1,l-1} + c^{(k)}_a \mu_{k,l-1}
\]

which is a simple consequence of (3.101). We note that in particular we have

\[
\mu_{k,k-1} = k - 1 \prod_{i=1}^{k-1} c^{(i)}_i M_1
\]

### 3.7.3 The one-loop correlator at genus \( g \)

In analogy with what was the case for the hermitian 1-matrix model we have the following representation for the genus \( g \) contribution to the 1-loop correlator.

\[
W^g(p) = \sum_{m=1}^{3g-1} \left\{ A^{(m)}_g \chi^{(m)}_a(p) + B^{(m)}_g \chi^{(m)}_b(p) \right\}
\]

where the \( \chi \)-functions are given by (3.114) and where the coefficients \( A^{(m)}_g \) take the form

\[
A^{(m)}_g = \sum f^{g,m}_{\beta_i,\gamma_j} (a,b) \frac{M_{\beta_1} \cdots M_{\beta_l} J_{\gamma_1} \cdots J_{\gamma_s}}{M^\beta_1 J^\gamma_1}
\]

with the indices being restricted by the conditions

\[
(l - \beta) + (s - \gamma) = 2 - 2g,
\]

\[
\sum_{i=1}^{s} (\beta_i - 1) + \sum_{j=1}^{l} (\gamma_j - 1) \leq 3g - m - 1.
\]

That the equation (3.118) holds can be proven by induction using as the starting point the expression obtained earlier for \( W^1(p) \). Obviously the proof consists in showing that with the representation (3.118) valid for \( g' = 1, \ldots, g-1 \) the right hand side of the loop equation (3.6) can be decomposed into fractions of the type \((p^2 - a^2)^{-m}, (p^2 - b^2)^{-m}\), \( m = 1, \ldots, 3g - 1 \) with appropriate coefficients. Let us just draw the attention of the reader to a few essential ingredients of the proof.

As regards the first term on the right hand side of the loop equation the existence of the above mentioned decomposition follows from the fact that the basis functions fulfill the homogeneous saddle point equation and the analyticity requirements 2 and 3 on page 20. This means that a function of the type \( G^{(k)}_+(p) G^{(m)}_- (p) \) can not have any cut but must be a rational fraction with poles at \( p = \pm a \) and \( p = \pm b \) of order less than or equal to \( k + m \).

The important step in proving that the second term on the right hand side of (3.6) indeed takes the desired form consists in showing that \( dM_k/dV(p) \) and \( dJ_k/dV(p) \) can again be expressed in terms of basis functions and \( M \)- and \( J \)-moments. However,
due to the relations (3.109) and (3.110) it is equivalent to show that \( d\Omega_k/dV(p) \) and \( dJ_k/dV(p) \) can be expressed in terms of basis functions and moments of the type \( \Omega_i \), \( J_i \). For simplicity we shall here take the latter line of action. From the definition (3.108) it follows that

\[
\frac{d\Omega_k}{dV(p)} = 2 \frac{\partial}{\partial p} \left( pG_a^{(k)}(p) \right) + \frac{da^2}{dV(p)} \lambda_a^{(k)} \Omega_{k+1} + \frac{db^2}{dV(p)} \left( 2 \int_{C_1} \frac{d\omega}{2\pi i} \omega V'(\omega) \frac{\partial G_a^{(k)}(\omega)}{\partial b^2} \right) \quad (3.122)
\]

where the first term comes from the explicit differentiation after coupling constants and the two others from the implicit differentiation after \( a^2 \) and \( b^2 \). Exploiting the fact that \( p^{k+1}G_a^{(k)}(p) \) depends on \( p \) only via \( p/a \) and \( p/b \) we can rewrite the first term of (3.122) as

\[
\frac{\partial}{\partial p} \left( pG_a^{(k)}(p) \right) = -kG_a^{(k)}(p) - 2a^2 \frac{\partial G_a^{(k)}(p)}{\partial a^2} - 2b^2 \frac{\partial G_a^{(k)}(p)}{\partial b^2} \quad (3.123)
\]

Furthermore the analyticity properties of \( \frac{\partial G_a^{(k)}(p)}{\partial b^2} \) allow us to conclude that we have a decomposition of the following type

\[
\frac{\partial G_a^{(k)}(p)}{\partial b^2} = v_{a,k}G_b^{(1)}(p) + \sum_{i=1}^{k} v_{a,k}^{(i)}G_a^{(k)}(p) \quad (3.124)
\]

where the \( v_{a,k}^{(i)} \)'s are some constants. From (3.88) it follows that for \( k = 1 \) we have

\[
v_{a,1}^{(0)} = -v_{a,1}^{(1)} = \frac{1}{2} \lambda_a^{(0)} \frac{\partial}{\partial a^2} \log \left( \frac{b^2 - c^2}{b^2 - a^2} \right) \quad (3.125)
\]

and the remaining \( v \)-coefficients can be found by repeatedly use of the \( k = 1 \) relation and the relation (3.98). In conclusion one can write \( d\Omega_k/dV(p) \) as

\[
\frac{d\Omega_k}{dV(p)} = -2kG_a^{(k)} - 4a^2 \lambda_a^{(k)} \left\{ G_a^{(k+1)}(p) - \frac{\Omega_{k+1}}{\Omega_1}G_a^{(1)}(p) \right\} - 4b^2 \sum_{i=1}^{k} v_{a,k}^{(i)} \left\{ G_a^{(i)}(p) - \frac{\Omega_i}{J_1}G_b^{(1)}(p) \right\} \quad (3.126)
\]

where we note that the \( v_{a,k}^{(0)} \) terms have cancelled. Collecting the here given information it is straightforward to complete the proof of the representation (3.110) for \( W^g(p) \).

In case of the ordinary one-matrix model one has

\[
f^{g,m}_{\beta_i,\gamma_i,\beta,\gamma}(a,b) = (a - b)^{-\delta}
\]

where \( \delta = 4g - 2 - m - \sum_{i=1}^{l} (\beta_i - 1) - \sum_{j=1}^{l} (\gamma_j - 1) \). In the general case this is no longer true. However, we emphasize that we still have that all explicit dependence on the matrix model coupling constants is hidden in the moment variables. The function \( f^{g,m}_{\beta_i,\gamma_i,\beta,\gamma}(a,b) \) is a function of the endpoints of the cut only and expressed in terms of the variables \( e \) and \( \alpha \) it takes the same form for all values of \( n \in [-2,2] \). Unfortunately we have not been able to write down the generic expression for \( f^{g,m}_{\beta_i,\gamma_i,\beta,\gamma}(e,\alpha) \).
3.7.4 Multi-loop correlators

From $W_g(p)$ we can obtain $W_g(p_1, p_2, \ldots, p_s)$ for any $s$ by repeatedly use of the loop insertion operator (cf. to equation (2.6)). Analyzing the structure of the loop insertion operator, one can write down formulas similar to (3.118) for the multi-loop correlators. We will not pursue this aim, but let us mention that from the discussion in the previous section it follow that the genus $g$ contribution to the $s$-loop correlator as in the 1-matrix model case depends on at most $2 \times (3g - 2 + s)$ moments for $g \geq 1$. The same statement is true for $g = 0$ provided $s \geq 3$. This can be seen from the expression (3.57) for the two-loop correlator at genus zero. We note that the expression (3.57) could also have been obtained by applying the loop insertion operator to the one-loop correlator at genus zero. However, this method of calculation is more time consuming than the one actually used.

3.7.5 The free energy

From $W_g(p)$ we can obtain $F_g$ by application of the inverse loop insertion operator, the inversion being possible due to the relation (3.126). One easily infers that as in the 1-matrix model case the genus $g$ contribution to the free energy for $g \geq 1$ depends on at most $2 \times (3g - 2)$ moments and that for $g \geq 2$, $F_g$ will be a sum of terms of the same type as those entering the relation (3.119) where the indices fulfill (3.120) as well as a relation like (3.121) where on the right hand side $3g - m - 1$ is replaced by $3g - 3$.

4 The cases $n = \pm 2$

The cases $n = \pm 2$ pose no particular problems. On the contrary they are in a certain sense easier to solve than the generic cases, namely the saddle point equation as well as the loop equations can be expressed in terms of functions of a definite parity and the generic solution to the saddle point equation can be parametrized using only one singular function.

4.1 $n = -2$

Let us start by noting that for $n = -2$, if we introduce $\Lambda_i = \lambda_i^2$ in the integral (2.8) we find

$$Z \propto \int_0^{\infty} d\Lambda_i e^{-N \sum_i V(\sqrt{\Lambda_i})} \prod_{i<j} (\Lambda_i - \Lambda_j)^2$$

(4.1)

Hence the partition function looks very similar to the one of the usual hermitian 1-matrix model. There are two important differences though. Firstly the interval of integration is restricted to the positive real axis. While this does not give rise to any
complications concerning the solution procedure it shows that the present model clearly contains other critical points than the usual hermitian 1-matrix model; namely points for which the eigenvalue distribution exactly touches the origin. In this respect the model is very similar to the complex matrix model which is given by an integral of the same type, the $\Lambda_i$’s playing the role of the positive eigenvalues of a matrix $\phi^*\phi$ \[13\]. However, there is an important feature which differentiates the $O(-2)$ model from both the complex and the hermitian one matrix model. The potential $V(\sqrt{\Lambda})$ might contain half integer powers of $\Lambda_i$. Likewise the correlation functions that one would be interested in calculating will typical involve half integer powers of $\Lambda_i$. Let us proceed to discussing how the usual iterative procedure can be adjusted to these circumstances.

4.1.1 The one-loop correlator at genus zero

As in the previous sections we will assume that the 1-loop correlator $W(p)$ (defined by (2.6)) is analytic in the complex plane and that it behaves as $1/p$ as $p \to \infty$. Let us decompose $W(p)$ as

$$W(p) = W_+(p) + pW_-(p)$$ \hspace{1cm} (4.2)

where the functions $W_+(p)$ and $W_-(p)$ are both even in $p$. Now $W_+(p)$ and $W_-(p)$ have in addition to the cut $[a,b]$ a cut $[-b,-a]$ and the analyticity requirement on $W(p)$ implies

$$W_+(p+i0) - W_+(p-i0) = p(W_-(p+i0) - W_-(p-i0)), \quad p \in [a,b] \hspace{1cm} (4.3)$$

In particular the eigenvalue density can be found from either one of the two functions $W_+(p)$ and $W_-(p)$ (cf. to equation (2.15))

$$\rho(p) = \frac{p}{i\pi} (W_-(p+i0) - W_-(p-i0)), \quad p \in [a,b]$$

$$= \frac{1}{i\pi} (W_+(p+i0) - W_+(p-i0)), \quad p \in [a,b] \hspace{1cm} (4.4)$$

The saddle point equation (2.16) becomes an equation for $W_-(p)$ and expressed in terms of the variable $p^2$ instead of $p$ it takes the same form as the saddle point equation of the hermitian 1-matrix model. Hence the solution of the present equation can be read off from the solution of the latter. One finds \[1\]

$$W_-(p) = \frac{1}{2} \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'\omega}{(\omega^2 - a^2)(\omega^2 - b^2)} \left\{ \frac{(p^2 - a^2)(p^2 - b^2)}{(\omega^2 - a^2)(\omega^2 - b^2)} \right\}^{1/2} \hspace{1cm} (4.5)$$

whew $a^2$ and $b^2$ are given by

$$\int_{C_1} \frac{d\omega}{2\pi i} \frac{V'\omega}{(\omega^2 - a^2)^{1/2}(\omega^2 - b^2)^{1/2}} = 0, \quad \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'\omega^2}{(\omega^2 - a^2)^{1/2}(\omega^2 - b^2)^{1/2}} = 2 \hspace{1cm} (4.6)$$
We note that from $W_−(p)$ we can find $W_+(p)$ by the following recipe

$$W_+(p) = 2 \oint_{C_1} \frac{d\omega}{2\pi i} W_+(\omega) \frac{\omega}{p^2 - \omega^2} = 2 \oint_{C_1} \frac{d\omega}{2\pi i} W_-(\omega) \frac{\omega^2}{p^2 - \omega^2}$$

(4.7)

4.1.2 Higher genera and multi loops

Let us introduce a decomposition of the loop insertion operator, namely

$$\frac{d}{dV(p)} = \frac{d}{dV_+(p)} + p \frac{d}{dV_-(p)}$$

(4.8)

where the operators $d/dV_+(p)$ and $d/dV_-(p)$ contain only even powers of $p$. Then we can rewrite the loop equation (2.22) as

$$\frac{1}{p^2} \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)\omega^2}{p^2 - \omega^2} W_-(\omega) = (W_-(p))^2 + \frac{1}{N^2} \frac{d}{dV_-(p)} W_-(p)$$

(4.9)

where we have explicitly made use of the relation (4.3). Instead of searching a solution of the equation (4.9) one can search a solution of the following equation

$$\oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p^2 - \omega^2} W_-(\omega) = (W_-(p))^2 + \frac{1}{N^2} \frac{d}{dV_-(p)} W_-(p)$$

(4.10)

since such a function will automatically fulfill

$$\oint_{C_1} d\omega V''(\omega) W_-(\omega) = 0$$

(4.11)

The genus $g$ contribution to the free energy of the $O(-2)$ model now takes the same form as the genus $g$ contribution of the free energy of the hermitian one-matrix model given in reference [7] provided the moments $M_k$ and $J_k$ are defined by

$$M_k = \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\omega^2 - a^2}$$

(4.12)

$$J_k = M_k(a^2 \leftrightarrow b^2)$$

(4.13)

and the parameter $d$ is replaced by

$$d = b^2 - a^2.$$

(4.14)

This statement is easily proven. First one rewrites the loop insertion operator $d/dV_-(p)$ in the moment parametrization and realizes that it takes the same form as the loop insertion operator of the hermitian one-matrix model (with the modifications given above) except for $p$ being replaced by $p^2$. This means that the analogy between the
loop equations of the two models holds to all orders in the genus expansion. Secondly one notes that for the $O(-2)$ model the following obvious relation holds

$$W_g - (p) = \frac{d}{dV_-(p)}F_g$$

and the correctness of the statement concerning the free energy becomes evident after a few moments thoughts.

We emphasize that the contour $C_1$ above encircles only the cut $[a, b]$. If the potential is even, however, one can immediately rewrite the integrals above as integrals along the contour $C_2$. Then performing the change of variable $\omega^2 \to \omega$ one reproduces exactly the expression for the free energy of the hermitian one matrix model (of course with the assumption that the support of the eigenvalue distribution lies on the positive real axis).

We will not pursue the explicit calculation of multi-loop correlators for the $O(-2)$ model in the present publication but let us emphasize that such calculations pose no particular difficulties. One simply rewrites the loop insertion operator in the moment parametrization, using the boundary equations (4.6) and applies it to the free energy. As mentioned above $d/dV_-(p)$ has a structure similar to the loop insertion operator of the hermitian 1-matrix model. The even part $d/dV_+(p)$, however, is less simple and involves elliptic integrals.

4.2 $n = +2$

4.2.1 The one-loop correlator at genus zero

Let us introduce again the decomposition of the 1-loop correlator given in equation (4.2). As before we then have the relation (4.3) between $W_+(p)$ and $W_-(p)$ and as before the eigenvalue density can be found from either of the two as described in equation (4.4). The saddle point equation turns into an equation for $W_0^+(p)$. This equation when expressed in terms of $p^2$ takes the same form as the saddle point equation for the hermitian 1-matrix model and the solution of the present equation can be found by exploiting the analogy with the latter. The result for $W_0^+(p)$ reads

$$W_0^+(p) = \frac{1}{2} \int_{C_1} d\omega V'_{\omega}(\omega)\omega \left\{ \frac{(p^2 - a^2)(p^2 - b^2)}{(\omega^2 - a^2)(\omega^2 - b^2)} \right\}^{1/2}. \quad (4.16)$$

Of the two boundary conditions which determine $a^2$ and $b^2$ one ensures the correct asymptotic behaviour, $W_+(p) \sim O(1/p^2)$ as $p \to \infty$, and can be written in the standard form

$$\int_{C_1} d\omega \frac{V'_{\omega}(\omega)}{2\pi i (\omega^2 - a^2)^{1/2}(\omega^2 - b^2)^{1/2}} = 0 \quad (4.17)$$
The other one expresses the fact that the eigenvalue distribution is normalized to one and reads

\[ \oint_{C_1} \frac{d\omega}{2\pi i} W^0_+(\omega) = \frac{1}{2} \] (4.18)

or

\[ \frac{1}{\pi} \int_a^b dp \frac{p^2 - a^2}{b^2 - p^2} \frac{1}{(p^2 - \omega^2)^{1/2}(\omega^2 - a^2)^{1/2}} W'(\omega) = 2 \] (4.19)

As opposed to what is normally the case this second condition cannot be written as a single contour integral. This is due to the fact that \( W_+(p) \) contains only the even powers of \( p \), i.e. the behaviour \( W(p) \sim \frac{1}{p} \) can not as usual be imposed by simply referring to the contour integral (4.16).

Even though the complexity of the second boundary equation does render the iterative calculation of the free energy and the multi-loop correlators more involved than for \( n = -2 \), the moment technique is still applicable. However, a detailed analysis of the structure of the free energy and the multi-loop correlators at higher genera is rather work demanding and we shall in the present publication restrict ourselves to exemplifying the applicability of the moment description by calculating the free energy at genus 1. Our line of action will follow closely the one taken for \( n \in [-2, 2] \).

4.2.2 The two-loop correlator at genus zero

Introducing the decomposition (4.8) of the loop insertion operator we can write the loop equation (2.22) as

\[ \oint_{C_1} \frac{d\omega}{2\pi i} \omega V'(\omega) W_+(p) = (W_+(p))^2 + \frac{1}{N^2} \frac{d}{dV_+(p)} W_+(p). \] (4.20)

which in its genus expanded version reads

\[ \{ \hat{K} - 2W^0_+(p) \} W^g_+(p) = \sum_{g'=1}^{g-1} W^g_+(p)W^{g-g'}_+(p) + \frac{d}{dV_+(p)} W^{g-1}_+(p) \] (4.21)

where

\[ \hat{K}f(p) = \oint_{C_1} \frac{d\omega}{2\pi i} \frac{\omega V'(\omega)}{p^2 - \omega^2} f(\omega) \] (4.22)

To proceed with the solution we need to calculate the following two-loop correlator

\[ W^0_{++}(p, p) = \frac{d}{dV_+(p)} W^0_+(p) \] (4.23)

The simplest way to do this is to proceed as in section 3.4. From (2.16) it follows that \( W^0_{++}(p, q) \) must fulfill the following saddle point equation

\[ W^0_{++}(p + i0, q) + W^0_{++}(p - i0, q) = -\frac{1}{2} \frac{p^2 + q^2}{(p^2 - q^2)^2}, \quad p \in [a, b] \] (4.24)
The complete solution of this equation, consisting of the sum of a particular solution
and the complete solution to the corresponding homogeneous equation, is easily written
down. Using the fact that $W_{++}^0(p, q)$ must be symmetric in $p$ and $q$, finite for $p = q$,
have the asymptotic behaviour $W_{++}^0(p, q) \sim O(1/p^2)$ as $p \to \infty$ and behave as
$((p - a)(p - b))^{1/2}$ in the vicinity of $a$ and $b$. one finds that it must necessarily take
the form

$$W_{++}^0(p, q) = \frac{1}{8} \frac{1}{\sqrt{p} \sqrt{q}} \left\{ C + (p^2 + q^2) \left( 1 - \left( \frac{\sqrt{p} - \sqrt{q}}{p^2 - q^2} \right)^2 \right) \right\}$$

(4.25)

where $C$ is some yet undetermined constant. Now the boundary equation (4.18) implies
that $W_{++}(p, q)$ must fulfill the following equation

$$\oint_{C_1} dp W_{++}(p, q) = 0, \quad \forall q$$
(4.26)

From this equation one can extract the value of $C$. This is most easily done evaluating
the integral at $q = 0$. The result for $C$ reads

$$C = a^2 + b^2 - 2b^2 \frac{E(k_a)}{K(k_a)} = a^2 + b^2 - 2a^2 \frac{E(k_b)}{K(k_b)}$$

(4.27)

where

$$k_a = \left( \frac{b^2 - a^2}{a^2} \right)^{1/2}, \quad k_b = k_a(a^2 \leftrightarrow b^2)$$

(4.28)

and where $K(k_a)$ and $E(k_a)$ are the complete elliptic integrals of the first and the
second kind respectively. To determine $W_{++}^0(p, p)$ which is the quantity which enters
the loop equation we must analyze carefully the limit $p \to q$ of the expression (4.25). One finds

$$W_{++}^0(p, p) = \frac{1}{16} \frac{p^2(a^2 - b^2)^2}{(p^2 - a^2)(p^2 - b^2)} \left\{ \frac{C}{16} \right\}$$

(4.29)

We note that the right hand side of the loop equation (4.21) for $g = 1$ takes the same
form as in the case $n \in \mathbb{Z} \quad -2, 2\mathbb{Z}$, the constant $C$ playing the role of $e^2 - \alpha^2$ (cf. to
equation (3.66)).

4.2.3 The one-loop correlator at genus one

We shall try to express $W_{++}^1(p)$ as in equation (3.70) with the function $\chi_a^{(i)}(p)$ and $\chi_b^{(i)}(p)$
obeys again the relations (3.71) and (3.8) with $K$ given by (4.22). The corresponding
$A$ and $B$ coefficients read

$$A_1^{(1)} = \frac{1}{16} \frac{1}{a^2 - b^2}, \quad A_1^{(2)} = \frac{1}{16} a^2,$$
$$B_1^{(1)} = \frac{1}{16} \frac{1}{b^2 - a^2}, \quad B_1^{(2)} = \frac{1}{16} b^2$$
In analogy with the case \( n \in \mathbb{Z} - 2,2 \) we will express the \( \chi \)-functions in terms of a set of basis functions \( \{G_a^{(k)}, G_b^{(k)}\} \). To begin with let us introduce

\[
\phi^{(0)}(p) = \frac{1}{(p^2 - a^2)^{1/2}(p^2 - b^2)^{1/2}} \tag{4.32}
\]

This function clearly fulfill the following identity

\[
\hat{K}\phi^{(0)}(p) = 0. \tag{4.33}
\]

We now define \( G_a^{(k)}(p) \) and \( G_b^{(k)}(p) \) for \( k \geq 1 \) by the following requirements

1. \( G_a^{(k)}(p) \) and \( G_b^{(k)}(p) \) are even in \( p \) and fulfill the homogeneous saddle point equation

\[
G(p + i0) + G(p - i0) = 0, \quad p \in [a, b]. \tag{4.34}
\]

2. In the vicinity of the endpoints of the cuts the functions \( G_a^{(k)}(p) \) and \( G_b^{(k)}(p) \) behave as

\[
G_a^{(k)}(p) \sim (p^2 - a^2)^{-k-1/2}(p^2 - b^2)^{-1/2}, \quad G_b^{(k)}(p) \sim (p^2 - a^2)^{-1/2}(p^2 - b^2)^{-k-1/2}. \tag{4.35}
\]

3. \( G_a^{(k)}(p) \) and \( G_b^{(k)}(p) \) are analytic everywhere else

4. They fulfill the conditions

\[
\oint_{C_1} dp G_a^{(k)}(p) = \oint_{C_1} dp G_b^{(k)}(p) = 0. \tag{4.36}
\]

5. They have the following asymptotic behaviour

\[
G_a^{(k)}(p), G_b^{(k)}(p) = \text{const} \cdot \phi^{(0)}(p) + \frac{1}{p^{2k+2}} + O\left(\frac{1}{p^{2k+4}}\right), \quad p \to \infty. \tag{4.37}
\]

The role of the three first requirements is the same as in the generic case. Condition number 4 ensures that the eigenvalue distribution stays normalized to all orders in the genus expansion (cf. to equation (4.18)). In the generic case this could simply be taken care of by demanding that \( W_g(p) \sim O(1/p^2), \quad p \to \infty \) for \( g > 1 \), i.e. by excluding the possibility of terms of order \( 1/p \) in the \( G \)-functions. However, in the present case we are calculating only the even part of \( W(p) \) so the normalization condition must be imposed in a different way. Condition 5 is chosen with the aim of rendering the moment variables as simple as possible. This should become clear shortly. The conditions 1–5 determine the \( G \)-functions uniquely. One has

\[
G_a^{(k)}(p) = \phi_a^{(k)}(p) + S_a^{(k)}(p) \tag{4.38}
\]
where
\[ \phi_a^{(k)}(p) = \frac{1}{(p^2 - a^2)^{k+1/2}(p^2 - b^2)^{1/2}} \] (4.39)
and
\[ S_a^{(k)} = -\frac{\int C_1 d\omega \phi_a^{(k)}(\omega)}{\int C_1 d\omega \phi^{(0)}(\omega)} \] (4.40)
and similarly for \( G_b^{(k)}(p) \). For our considerations in the following section we will need the explicit expressions for \( S_a^{(1)} \) and \( S_a^{(2)} \). They read
\[
S_a^{(1)} = -\frac{1}{a^2 - b^2} \left\{ 1 - \frac{E(k_b)}{K(k_b)} \right\},
\]
\[
S_a^{(2)} = -\frac{1}{(a^2 - b^2)^2 a^2} \left\{ a^2 - \frac{1}{3} b^2 + \frac{2}{3} (b^2 - 2a^2) \frac{E(k_b)}{K(k_b)} \right\}.
\] (4.42)

From the basis functions it is straightforward to construct the \( \chi \)-functions. For that purpose let us consider the action of the operator \( \hat{K} \) on the \( G \)-functions. One finds
\[
\hat{K} G_a^{(k)}(p) = \sum_{l=1}^{k-1} M_l \frac{1}{(p^2 - a^2)^l},
\]
\[
\hat{K} G_b^{(k)}(p) = \sum_{l=1}^{k-1} J_l \frac{1}{(p^2 - b^2)^l}
\] (4.43)
where the moments \( M_l \) and \( J_l \) are given by
\[
M_k = \oint_{C_1} \frac{d\omega}{2\pi i} \omega V'(\omega) G_a^{(k)}(\omega) = \oint_{C_1} \frac{d\omega}{2\pi i} \frac{\omega V'(\omega)}{(\omega^2 - a^2)^{k+1/2}(\omega^2 - b^2)^{1/2}}.
\]
\[
J_k = M_k (a^2 \leftrightarrow b^2)
\] (4.44)

The advantage of imposing the requirement 5 on the \( G \)-functions should be clear by now. One could have taken \( G_a^{(k)}(p) \) as a linear combination of \( \phi_a^{(k)}(p) \) with any \( \phi_a^{(l)}(p) \) with \( l < k \). However, due to the condition (4.17) we obtain a particularly simple expression for the moments by choosing \( l = 0 \). From (4.43) it follows that the \( \chi \)-functions are given by
\[
\chi_a^{(k)}(p) = \frac{1}{M_1} \left\{ G_a^{(k)}(p) - \sum_{l=1}^{k-1} M_{k-l+1} G_a^{(l)}(p) \right\},
\]
\[
\chi_b^{(k)}(p) = \chi_a^{(k)}(p) (a \leftrightarrow b)
\] (4.46)

We note that for a potential of degree \( p \) one has \( M_q = J_q = 0 \) for \( q > p \). Now all the elements in the representation (3.70) of the 1-loop correlator at genus one have been determined and it is easy, collecting the results of the present section, to write down a completely explicit expression for \( W_1^1(p) \).

### 4.2.4 The free energy at genus one

To determine the free energy at genus one we use the usual strategy of expressing the \( \chi \)-functions as total derivatives with respect to the loop insertion operator \( d/dV_+(p) \). The
key point in this procedure consists in determining $da^2/dV_+(p)$ and $db^2/dV_+(p)$. These quantities can as usual be extracted from the boundary conditions (4.17) and (2.14). The actual calculation is more involved than usual but after the use of various relations between elliptic integrals one arrives at the following pleasant result

$$\chi_a^{(1)}(p) = \frac{d \log a^2}{dV_+(p)}, \quad \chi_b^{(1)}(p) = \frac{d \log b^2}{dV_+(p)}$$

(4.47)

Having obtained the expressions for $da^2/dV_+(p)$ and $db^2/dV_+(p)$ it is relatively straightforward to show that

$$a^2 \chi_a^{(2)}(p) = -\frac{2}{3} \frac{d \log M_1}{dV_+(p)} - \frac{1}{3} \frac{d \log (a^2 - b^2)}{dV_+(p)} - \frac{2}{3} \frac{d \log a^2}{dV_+(p)}$$

and similarly for $\chi_b^{(2)}(p)$. Now combining the $A$ and $B$ coefficients given in (4.30) and (4.31) with the here obtained expressions for the $\chi$-functions one finds that $W_1^1(p)$ indeed takes the form of a total derivative. The free energy at genus one can hence be extracted and reads

$$F_1 = -\frac{1}{24} \log M_1 - \frac{1}{24} \log J_1 - \frac{1}{6} \log (b^2 - a^2)$$

$$+ \frac{1}{48} a^2 + \frac{1}{48} \log b^2 - \frac{1}{4} \log (K(k_a)) - \frac{1}{4} \log (K(k_b))$$

(4.49)

It is interesting to note the similarity of (4.49) with the expression (3.90) obtained for $n \in [2, 2]$. 

5 The critical regime

5.1 The critical points

As mentioned earlier the matrix integral defining the $O(n)$ model ceases to exist when the support of the eigenvalue distribution approaches zero, i.e. when $a \to 0$. This gives rise to a new set of critical points for which no analogues exist for the 1-matrix model $[3, 4, 5]$. These are the critical points that we will consider in the following. We will take $\nu$ to be in the interval $0 < \nu < 1$. Then we always have $a \ll |e| \ll b$ (since as we shall see very soon $e \sim a^\nu$) which simplifies the analysis. Although not more complicated the cases $\nu = 0, 1$ require special treatment.

At the singular points the eigenvalue distribution vanishes at one endpoint of its support (here $a = 0$) with a critical exponent, $\mu$, or equivalently

$$W_s(p) \sim p^\mu, \quad p \to 0.$$

(5.1)

Let us recall the possible values of $\mu$ for the $O(n)$ model $[1, 2]$. These can be read of from the expression (3.23) for $W_s^0(p)$. Obviously the possibility of new types of critical
behaviour is due to the presence of the function \( G(p) \). For \( a = 0 \) (or equivalently \( a \ll p \sim b \)) the function \( G(p) \) takes the form

\[
G(p) \sim -\frac{2}{b} \cosh \psi \frac{\sinh(1-\nu)\psi}{\sinh \psi}, \quad \cosh \psi = -\frac{b}{p}.
\] (5.2)

This is most easily seen by verifying that the function (5.2) satisfies the criteria 1–3 on page \( \text{8} \). Now letting \( p \to 0 \) we find that

\[
G(p) \sim p^{\nu - 1} + \mathcal{O}(p^{\nu + 1}) \quad \text{and hence} \quad \tilde{G}(p) \sim p^{-\nu} + \mathcal{O}(p^{\nu}).
\] (5.3)

It then follows from (3.23), (3.20) and (3.3) that by fine tuning the potential of our model (i.e. the polynomials \( \mathcal{A}(p^2) \) and \( \mathcal{B}(p^2) \)) we can reach for a given value of \( \nu \) (or \( n \)) the following two series of critical points

\[
\mu_{2m+1} = 2m + 1 - \nu : \quad \mathcal{A}(p^2) \sim p^{2m}, \quad \mathcal{B}(p^2) \sim \mathcal{O}(p^{2m}) \quad \text{(5.4)}
\]

\[
\mu_{2m+2} = 2m + 1 + \nu : \quad \mathcal{B}(p^2) \sim p^{2m}, \quad \mathcal{A}(p^2) \sim \mathcal{O}(p^{2m+2}) \quad \text{(5.5)}
\]

The possible values of \( \mu \) are exactly those for which \( n = -2 \cos(\mu \pi) \). Furthermore it can be shown that \( \gamma_{\text{str}} = -2\nu/(\mu + 1 + \nu) \) \([3, 4, 5]\). When \( \nu = \frac{l}{q} \), with \( 0 < l < q \) and \( l, q \in \mathbb{Z} \), the critical points being characterized by the exponents \( \mu = 2m+1 \pm \nu \) exhibit the scaling behaviour characteristic of 2D gravity interacting with rational conformal matter fields of the type \((q, (2m+1)q \pm l)\). However, the continuum theories that one obtains from the \( \mathcal{O}(n) \) model do not contain all the operators of the corresponding minimal models \([3, 4, 5]\).

For later book-keeping purposes, let us arrange all critical points into one series where the \( M \)'th multi-critical point is characterized by

\[
\mu_M = M - \eta_{M+1}, \quad \eta_{2k} = \nu, \quad \eta_{2k+1} = 1 - \nu
\] (5.6)

We note that this definition reproduces the usual notion of a \( M \)'th critical point of the 1-matrix model (\( \nu = \frac{1}{2} \)) case.

### 5.2 Scaling at a \( M \)'th critical point

In this section we will calculate the scaling behaviour of the basic elements of our description, i.e. the functions \( G_{a,b}(p) \) and the moments \( \{M_k, J_k\} \). Knowing the scaling properties of these objects we can easily extract continuum results from our exact results or develop a procedure for calculating directly continuum quantities.

The most fundamental quantity of our description is the function \( G(p) \). From \( G(p) \) all other quantities can be derived. One can show that in the scaling region (\( a \sim p \ll b \))

\[
G(p) \sim -\frac{2ie}{ab} \frac{\sinh \nu \phi}{\sinh \phi}, \quad \cosh \phi = -\frac{p}{a}
\] (5.7)
The prefactor comes from the relation \((3.13)\). Now matching the expressions \((5.2)\) and \((5.7)\) in the intermediate region \(a \ll p \ll b\) one can determine \(e\) to leading order in \(a\). The result reads

\[
e = 2ib \left( \frac{a}{4b} \right) \nu, \quad \text{i.e.} \quad \rho_a = \nu \tag{5.8}
\]

which we note justifies our statements concerning \(e\) made in section 3.3. Hence for \(p \sim a\) we have in accordance with the analysis of the previous section

\[
G(p) \sim a^{-\nu - 1}, \quad \tilde{G}(p) \sim a^{-\nu} \tag{5.9}
\]

Knowing the scaling of \(e\) we can furthermore determine the scaling of all \(G_a^{(k)}(p)\) and \(G_b^{(k)}(p)\). Namely, from the relations \((3.99)\) and \((3.100)\) we see that \(\lambda^{(0)}_a \sim a^{2\nu - 2}\) and \(\lambda^{(1)}_a \sim a^{-2\nu}\) and then the recursion relations \((3.103)\) tells us that

\[
\lambda_a^{(k)} \sim \frac{1}{c_a^{(k)}} \sim a^{-2\eta_k + 1} \tag{5.10}
\]

In particular \(G_a^{(1)}(p) \sim a^{-1 - \nu}\) and in general

\[
G_a^{(k)}(p) \sim a^{-k - \eta_k + 1}, \quad \tilde{G}_a^{(k)} \sim a^{-k - \eta}, \quad p \sim a \tag{5.11}
\]

while all the \(G_b^{(k)}(p)\) - and \(\tilde{G}_b^{(k)}\) -functions for \(p \sim a\) become proportional to \(G(p)\) and \(\tilde{G}(p)\) respectively.

Let us now examine the scaling properties of the moment variables. We remind the reader of the fact that the integrals defining \(M_k\) and \(J_k\) when written in the form \((3.96)\) reduce to local integrations around \(a\) and \(b\) respectively. Since for \(a \ll p \sim b\), \(G^{(k)}(p)\) as well as \(W_s(p)\) are independent of \(a\) we have

\[
J_k \sim \mathcal{O}(a^0). \tag{5.12}
\]

The \(M\)-moments, on the contrary, have a non trivial scaling. By definition of a \(M\)'th multi-critical point one has at such a point

\[
W_s^{\mu_M}(p) \sim p^{M - \eta_M + 1} \sim a^{M - \eta_M + 1} W_s^{\mu_M} \left( \frac{p}{a} \right). \tag{5.13}
\]

It now follows that

\[
M_k \sim a^{M - k + (\eta_M - \eta_k)} \tag{5.14}
\]

We see that for \(k < M\), \(M_k\) scales with a positive power of \(a\) and that \(M_M \sim a^0\). The moments \(M_k\) with \(k > M\) are equal to zero. This can be seen by deforming the contour to infinity.

Having determined the scaling properties of basis functions and moments it is easy to pass to the continuum limit. For instance to determine the genus one contribution
to the 1-loop correlator in the scaling limit it suffices to note that in this limit the right hand side of the loop equations for $W^1(p)$ reduces to

$$
\hat{K}W^1(p) = \frac{1}{4-n^2} \left\{ \frac{a^2}{4} \frac{1}{(p^2-a^2)^2} - \frac{2\nu^2 - 4\nu + 1}{4} \right\}. \tag{5.15}
$$

This immediately tells us that

$$
W^1(p) = \frac{a^2}{4} \chi_a^{(2)}(p) - \frac{2\nu^2 - 4\nu + 1}{4} \chi_a^{(1)}(p). \tag{5.16}
$$

We note that there is no simplification of the $\chi$-functions in the scaling limit. All terms in the relation (3.114) are of the same order in $a$. Now using the relations (3.84) and (3.85) bearing in mind that $M_1 = \lambda_a^{(0)} s^{(1)}_a M_1$ we find the following expression for $F_1$ in the scaling limit.

$$
F_1 = -\frac{1}{24} \log \left( M_1 a^{-6\nu^2+13\nu-5} \right). \tag{5.17}
$$

The exponent of $a$ vanishes if and only if $\nu = \frac{1}{2}$. Hence we reproduce correctly the 1-matrix model result and we see once again that this case is very particular. From the expression (5.16) of the scaling relevant part of $W^1(p)$ one can pursue the iterative solution of the loop equation directly in the continuum. This only requires that one writes down a continuum version of the loop insertion operator. Let us mention a few properties of this operator. First of all one finds that the loop insertion operator in the scaling limit reduces to a differentiation after $a^2$ and the moments $M_k$. This implies that, not surprisingly, no $J$-moments will appear in the scaling limit. Furthermore the dimension of the loop insertion operator can easily be extracted. It equals $a^{-\mu M - 2}$.

Let us stress that the expressions (5.16) and (5.17) as well as all results that one would obtain by further iterations of the loop equations are valid in the vicinity of any $M$’th critical point and independent of which detailed prescription one might choose for approaching such a point. However, whenever needed one can easily specialize to a given scaling prescription. In section 5.4 we will show how one can calculate explicitly the moments when one approaches the critical point by tuning an overall coupling constant of the potential.

## 5.3 The basis functions in the continuum

As explained in the previous section the $G^{(k)}_b$-functions do not play any role in the scaling limit. Let us write the $G^{(k)}_a$-functions in this limit as

$$
G^{(k)}_a(p) = \frac{1}{2 \cos (\nu \pi/2)} \left( \frac{\nu k + 1}{b^k+1} \right) f_k(\phi) \tag{5.18}
$$
where we use again the parametrization \( p = -a \cosh \phi \). Now \( f_k(\phi) \) is a dimensionless function and all dependence on the scaling parameter \( a \) is hidden in the prefactor \( e^{\nu_k} \) where

\[
\epsilon = \frac{a}{4b}.
\] (5.19)

The value of the exponent \( \nu_k \) follows from the relation (5.11) and reads

\[
\nu_k = -k - \eta_{k+1}.
\] (5.20)

The expression for the function \( f_0(\phi) \) can be read off from (5.7). One has

\[
f_0(\phi) = \frac{\sinh \nu \phi}{\sinh \phi}.
\] (5.21)

The remaining \( f_k \)-functions can be found from the continuum versions of the recursion relations (3.98), (3.101) and (3.103). They read

\[
l_k f_{k+1}(\phi) = \nu_k f_k(\phi) - \frac{\cosh \phi}{\sinh \phi} f'_k(\phi),
\] (5.22)

\[
f_{k+1}(\phi) = \frac{1}{16 \sinh^2 \phi} \left( f_{k-1}(\phi) + \gamma_k f_k(\phi) \right)
\] (5.23)

and

\[
l_k \gamma_k = 32(k + 1/2), \quad l_{k+1} - l_{k-1} = 2 \gamma_k \eta_{k+1}.
\] (5.24)

The new dimensionless parameters \( \gamma_k \) and \( l_k \) are related to the original ones \( c_a^{(k)} \) and \( \lambda_a^{(k)} \) by

\[
\gamma_k = \lim_{a \to 0} \frac{c_a^{(k)}}{b} e^{-2\eta_{k+1}}, \quad l_k = \lim_{a \to 0} 32b \lambda_a^{(k)} e^{2\eta_{k+1}}.
\] (5.25)

From the expressions (3.99) and (3.100) we can determine \( l_0 \) and \( l_1 \) and this enables us to solve exactly the recursion relations (5.24). We find

\[
l_k = 4 \tan \left( \frac{\eta_k \pi}{2} \right) \frac{\eta_k(1 + \eta_k)(2 + \eta_k) \ldots (k + \eta_k)}{(1 - \eta_k)(2 - \eta_k) \ldots (k - \eta_k)}.
\] (5.26)

One can derive additional interesting properties of the quantities appearing above. For instance one has

\[
l_k l_{k+1} = 16 \nu_k (\nu_k - 1)
\] (5.27)

and it appears that the \( f \)-functions satisfy the following differential equation

\[
f''_k(\phi) + 2(k + 1) \frac{\cosh \phi}{\sinh \phi} f'_k(\phi) + \left( (k + 1)^2 - \eta_k^2 \right) f_k(\phi) = 0.
\] (5.28)

As usual the relevant expressions for the \( \tilde{G} \)-functions appear from those of the \( G \)-functions by the substitution \( \nu \to 1 - \nu \) and we will use for the relations involving
\( \tilde{G} \) functions the same notation as above just with all quantities being equipped with a tilde. Equation (5.104) relating \( G \)- and \( \tilde{G} \)-functions translates to the scaling limit as

\[
- 4 \cosh \phi f_k(\phi) = \cot \left( \frac{\nu \pi}{2} \right) \left\{ \tilde{f}_{k-1}(\phi) + \sigma_k \tilde{f}_k(\phi) \right\}
\]

where the dimensionless parameter \( \sigma_k \) is related to \( s_a^{(k)} \) by

\[
\sigma_k = \lim_{\eta \to 0} e^{-2
u \frac{1}{b} s_a^{(k)}}.
\]

Using the relations (3.106), (3.107), (5.25) and (5.26) one can determine \( \sigma_k \) explicitly. It is given by

\[
\sigma_k (k + \eta) = l_k.
\]

### 5.4 Explicit calculations at a M’th multi critical point

In this section we specialize to a particular prescription for approaching a \( M’ \)th multicritical point. We replace the potential \( V(p) \) of our model by \( V_c(p) \) where \( V_c(p) \) is a critical potential corresponding to the critical point in question and where \( T \) plays the role of the cosmological constant (or the temperature). We now approach the critical point by letting \( T \to T_c = 1 \) and define a renormalized cosmological constant \( \Lambda_R \) by

\[
T - T_c = a^{\mu M + 1 - \nu} \Lambda_R.
\]

where \( \mu_M = M - \eta_{M+1} \). That the power of \( a \) appearing above is indeed what is needed to make \( \Lambda_R \) dimensionless can be seen by expanding \( W(p) \) around \( W_c(p) \), considering \( p \sim a \) and using that

\[
\frac{\partial (TW(p))}{\partial T} = G_0(p).
\]

This relation follows from the fact that the expression on the left hand side fulfills the conditions that determined uniquely the function \( G_0(p) \). Now, using the relation (5.32) it follows from (5.14) and (5.17) that

\[
F_1 = -\frac{1}{24} \left\{ 1 - \frac{6 \nu^2 - 15 \nu + 7}{\mu_M + 1 - \nu} \right\} \log \Lambda_R.
\]

This case is particularly simple. Due to the logarithm we do not need to know the explicit expressions for the moments in the scaling limit. However, to determine the continuum version of any other quantity such expressions are needed. We shall now proceed to deriving these. Our starting point will be the relation (3.96). We remind the reader that the contour integral appearing in this relation reduces to a local integration around the point \( a \). Hence we only need to know the integrands in the scaling limit. The relevant expressions for the \( \tilde{G} \) functions appear from the previous section. However,
we shall not make use of their explicit form. It suffices to know that they fulfill the following relation with \( x = p/a = -\cosh \phi \)

\[
\tilde{v}_k \tilde{f}_k(x) - x \tilde{f}_k'(x) = \tilde{l}_k \tilde{f}_{k+1}(x). \tag{5.35}
\]

The scaling limit of the function \( W_s(p) \) has been determined explicitly in reference \[3\]. In the vicinity of a \( M' \)th multi-critical point one has

\[
W_s(p) = \frac{1}{b} \epsilon^{M-\eta M+1} F_M(x) \tag{5.36}
\]

where

\[
F_M(x) = \text{const} \cdot x^{M-\eta M+1} \left( B + \int_{-\infty}^{\phi} d\alpha \frac{\sinh(\nu \alpha)}{(\cosh \alpha)^{M-\eta M}} \right) \tag{5.37}
\]

The constant \( B \) can be completely explicitized as a B Euler function but its precise form will not be of importance for the following. The prefactor is non-universal and depends on the critical potential chosen. From the explicit expression for \( F_M(x) \) one easily verifies that the following relation holds

\[
(M - \eta M)F_M(x) - x F'_M(x) = \text{const} \cdot f_0(x) \tag{5.38}
\]

Now our moments take the form

\[
\hat{M}_k = 4^{\epsilon^{M+\eta M+\hat{v}_k}} \hat{M}_k \tag{5.39}
\]

with

\[
\hat{M}_k = (4 - n^2) \oint \frac{dx}{2\pi i} \left\{ F_{M+}(x) \tilde{f}_{k+}(x) - F_{M-}(x) \tilde{f}_{k-}(x) \right\} \equiv \langle F_M, \tilde{f}_k \rangle \tag{5.40}
\]

where the contour encircles the point \( x = 1 \). From the relations (5.38) and (5.35) it follows that

\[
\tilde{l}_k \hat{M}_{k+1} = (M - k + \eta_M - \eta_k) \hat{M}_k - \text{const} \cdot \langle f_0, \tilde{f}_k \rangle \tag{5.41}
\]

and since \( \langle f_0, \tilde{f}_k \rangle \propto \delta_{k,0} \) we see that in accordance with the analysis of section 5.2 the moments \( M_k \) with \( k > M \) will vanish. For \( 1 \leq k \leq M \) we have

\[
\frac{\hat{M}_{k+1}}{\hat{M}_k} = \frac{M - k + \eta_M - \eta_k}{l_k} \tag{5.42}
\]

Hence we can express all our moments in terms of only one, say \( M_1 \). This allows us to determine any continuum quantity up to a non-universal constant. For instance we find for \( W_1(\phi) \) by means of (5.14), (5.22), (5.31) and (5.42)

\[
W_1(\phi) = \text{const} \cdot \left\{ l_1 f_2(\phi) - (\mu_M + 1 + 6 \nu^2 - 12 \nu + 3) f_1(\phi) \right\}. \tag{5.43}
\]

Using the relations (5.22) and (5.28) one can easily verify that this results agrees with the one obtained in the unitary case within the framework of strings with discrete target spaces by S. Higuchi and I.K. Kostov \[20\].

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6 Conclusion and outlook

One interesting conclusion which can be drawn from the obtained exact solution of the $O(n)$ on a random lattice is that the model exhibits the same kind of universality with respect to the potential as the hermitian 1-matrix model. One must expect this kind of universality to occur also for two- and multi-matrix models and the present work can be taken as an indicator of how one could make use of this universality in the solution of these more complicated models.

It is also interesting to note that our solution provides an exact solution of the Ising model on a random lattice. This gives the possibility of studying spin excitations of this model away from criticality. Unfortunately the representation of the Ising model on a random surface that one obtains from the $O(n)$ model has vanishing magnetic field. However, it is possible to include a magnetic field by adding a $1/M$ term to the action appearing in equation (2.1). In analogy with this one would expect that in general the addition of terms with negative powers of $M$ would enlarge the operator content of the continuum theories obtained from the model. It would hence be interesting to generalize the moment technique to this situation.

As mentioned in the introduction our d.s.l. relevant moment description of the $O(n)$ model should allow us by comparison with the corresponding moment description of the generalized Kontsevich models to determine which is the precise relation between the continuum partition function of the $O(n)$ model for $n$ rational and the $\tau$-functions of the generalized kdv hierarchies. We have not completed this analysis but let us mention a few observations. First of all we see that for the $O(n)$ model on a random lattice we have in the double scaling limit two series of moments with different scaling properties. In general for a $\tau$-function of the kdv$_p$ hierarchy describing the interaction of 2D gravity with matter fields of the type $(p, pm - 1), \ldots, (p, pm - (p - 1))$ there will appear $(p - 1)$ series of moments with different scaling properties \cite{14, 15}. Hence the only models for which we could hope for an exact equivalence are the models $(p, q) = (3, 3m - 1), (3, 3m - 2)$. However, as the example with the Ising model clearly shows, not even in this case will the equivalence be exact.

Another interesting aspect concerning the double scaling limit is the interpretation of the continuum theories corresponding to non-rational values of $\nu$. For instance, one might wonder what the topological interpretation of these models is and if there exist integrable hierarchies describing them.

Finally one can remark that the results that we have obtained are actually analytical in $\nu$. This might open the possibility of attributing a meaning to the model for $n > 2$ and maybe approaching the question of interaction of 2D gravity with matter fields with $c > 1$. 

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