Posterior Probabilities: Dominance and Optimism

Sergiu Hart† Yosef Rinott‡

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Abstract

The Bayesian posterior probability of the true state is stochastically dominated by that same posterior under the probability law of the true state. This generalizes to notions of “optimism” about posterior probabilities.

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Alice is a classical Bayesian decision-maker, in a setup with a state that is unknown, a prior probability distribution over the states, and signals whose distribution depends on the state. Alice is interested in a particular “good” state $\gamma$, and so she computes, by Bayes’ rule, the posterior probability of $\gamma$ for each possible signal; call this posterior probability $Q_\gamma$. Before getting the signal, Alice thus believes that $Q_\gamma$ will be either higher or lower than the prior probability $p$ of $\gamma$, with an expectation exactly equal to $p$.

Now suppose that Carroll (an outside observer, say) knows that the state is in fact $\gamma$, and he considers what will be the posterior probability $Q_\gamma$ computed by Alice (who does not know that the state is $\gamma$). How does Carroll’s (probabilistic) belief on $Q_\gamma$ compare to Alice’s belief (both beliefs are of course before the signal)?

It turns out that Carroll always assigns higher probability than Alice to high values of $Q_\gamma$, and lower probability to low values; formally, we will show that Carroll’s probability distribution of $Q_\gamma$ stochastically dominates Alice’s in the so-called “likelihood ratio” order, which is stronger than first-order stochastic dominance. What may come as a surprise here is that this fact, whose proof is straightforward, is unknown. Even its immediate consequence that Carroll’s expectation of $Q_\gamma$ is higher than the prior $p$ (which is equal to Alice’s expectation of $Q_\gamma$) seems to be still widely unknown despite having appeared in the literature.

The Supplementary Material contains full proofs and additional comments.

## 1 Dominance

The Bayesian setup that Alice faces is standard; for simplicity, all sets are assumed to be finite. The set of states of nature is $\Theta$, and $\pi$ is the prior probability distribution on $\Theta$. There is a set $\Gamma \subset \Theta$ of states of nature that is of interest; $\Gamma$ could well consist of a single state $\gamma$. To avoid trivialities, assume that its prior probability $p \equiv \pi(\Gamma)$ satisfies $0 < p < 1$. Let $S$ be the set of signals and $\sigma(s|\theta)$ the probability of signal $s$ in state $\theta$ (thus $\sum_{s \in S} \sigma(s|\theta) = 1$ for all $\theta$); we will refer to $(S, \sigma)$ as a signaling structure (on $\Theta$). The prior probability $\pi$ and the signaling probabilities $\sigma$ induce a probability $P \equiv P_{\pi,\sigma}$ on states and signals, i.e., on $\Theta \times S$.

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1Lest there be any misunderstanding, we are not considering Carroll’s posterior probability of $\gamma$ (which equals 1), but rather Carroll’s view of Alice’s posterior probability $Q_\gamma$ of $\gamma$. For simple setups where this is a natural question, e.g., Alice is the uninformed market and Carroll is an informed agent, see Section II below.

2A standard reference on stochastic orders is the book of Shaked and Shanthikumar (2010).

3Mailath and Samuelson (2006); Francetich and Kreps (2014). None of the colleagues whom we asked knew this result, just as those whom Francetic and Kreps asked didn’t.

4Available at [http://www.ma.huji.ac.il/hart/abs/posterior.html](http://www.ma.huji.ac.il/hart/abs/posterior.html)
For each signal $s$ in $S$, Alice’s posterior probability of $\Gamma$, which we denote by $Q_\Gamma(s)$, is given by Bayes’ formula:

$$Q_\Gamma(s) \equiv \mathbb{P}(\Gamma|s) = \frac{\mathbb{P}(\Gamma) \cdot \mathbb{P}(s|\Gamma)}{\mathbb{P}(s)} = p \cdot \frac{\mathbb{P}^\Gamma(s)}{\mathbb{P}(s)},$$

(1)

where $\mathbb{P}^\Gamma$ denotes the conditional-on-$\Gamma$ probability, i.e., $\mathbb{P}^\Gamma(s) \equiv \mathbb{P}(s|\Gamma) = (1/p) \sum_{\theta \in \Gamma} \pi(\theta) \sigma(s|\theta)$, and $\mathbb{P}(s) = \sum_{\theta \in \Theta} \pi(\theta) \sigma(s|\theta)$. Alice’s posterior of $\Gamma$ is a random variable that takes the value $Q_\Gamma(s)$ with probability $\mathbb{P}(s)$; its expectation equals the prior, i.e., $\mathbb{E}[Q_\Gamma] = \pi(\Gamma) = p$.

Carroll knows that the state is in fact in $\Gamma$. He thus assigns probability $\mathbb{P}^\Gamma(s)$ to the signal $s$, and so he believes that Alice’s posterior of $\Gamma$ takes the value $Q_\Gamma(s)$ with probability $\mathbb{P}(s)$ (instead of $\mathbb{P}(s)$). We denote this random variable by $(Q_\Gamma, \mathbb{P}^\Gamma)$, whereas Alice’s posterior random variable, which takes the value $Q_\Gamma(s)$ with probability $\mathbb{P}(s)$, is denoted by $(Q_\Gamma, \mathbb{P})$.

What is the relation between $(Q_\Gamma, \mathbb{P}^\Gamma)$ and $(Q_\Gamma, \mathbb{P})$? It turns out that high values of the posterior $Q_\Gamma$ are more probable under $\mathbb{P}^\Gamma$ than under $\mathbb{P}$, whereas low values of the posterior are more probable under $\mathbb{P}$ than under $\mathbb{P}^\Gamma$; formally, the former is always higher\(^6\) than the latter, in the following precise sense.

**Proposition 1** $(Q_\Gamma, \mathbb{P}^\Gamma)$ dominates $(Q_\Gamma, \mathbb{P})$ in the likelihood ratio order.

The likelihood ratio stochastic order is stronger than first-order stochastic dominance. Formally, let $X$ and $Y$ be two random variables. Then $X$ first-order stochastically dominates $Y$ if $\mathbb{P}(X \geq y) \geq \mathbb{P}(Y \geq y)$ for every $y$; and $X$ dominates $Y$ in the likelihood ratio order, denoted by $X \geq_{lr} Y$, if $\mathbb{P}(X = y)/\mathbb{P}(Y = y)$ is an increasing function of $y$. The former is equivalent to $\mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)]$ for every increasing function $\varphi$, and the latter to $\mathbb{E}[\varphi(X)|X \in A] \geq \mathbb{E}[\varphi(Y)|Y \in A]$ for every increasing function $\varphi$ and every measurable set of values $A \subseteq \mathbb{R}$.

**Proof.** Immediate, since $Q_\Gamma$ is proportional to the likelihood ratio $\mathbb{P}^\Gamma/\mathbb{P}$ by Bayes’ formula (1). Formally, $Q_\Gamma(s) = q$ if and only if $\mathbb{P}^\Gamma(s) = (q/p) \cdot \mathbb{P}(s)$, and so summing over all $s$ with $Q_\Gamma(s) = q$ yields $\mathbb{P}^\Gamma(Q_\Gamma = q) = (q/p) \cdot \mathbb{P}(Q_\Gamma = q)$. The likelihood ratio thus equals $q/p$, which is an increasing function of $q$. \(\blacksquare\)

The dominance is in fact strict (i.e., the likelihood ratio is a strictly increasing function of $Q_\Gamma$), except when the signal is completely uninformative about $\Gamma$ (i.e., for every signal the posterior of $\Gamma$ equals its prior $p$).

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\(^5\)Superscripts (on $\mathbb{P}$, $\mathbb{E}$, $\pi$) are used throughout to denote conditioning. When $\Gamma$ consists of a single state $\gamma$ we have $\mathbb{P}^\gamma(s) = \sigma(s|\gamma)$.

\(^6\)All comparisons such as “higher” and “increasing” are meant in the weak-inequality sense.
Despite its simplicity, this result appears not to be known. One easy implication that does appear in the literature (see footnote 3) is the resulting inequality on expectations:

**Corollary 2** \( E^\Gamma [Q_\Gamma] \geq E [Q_\Gamma] = \pi(\Gamma). \)

Thus, the conditional-on-\( \Gamma \) expectation of the posterior of \( \Gamma \) is higher than the prior of \( \Gamma \); that is, the posterior probability of \( \Gamma \) is a submartingale relative to \( P^\Gamma \). Moreover, the inequality is strict except when the signal is completely uninformative about \( \Gamma \).

When does this stochastic domination matter? For a simple setup, consider an uncertain asset whose market price is determined by the estimated probability that the state is in a set of “good” states \( \Gamma \); specifically, the price is an increasing function \( \varphi \) of the probability of \( \Gamma \). Suppose that a signal whose distribution depends on the state (such as a quarterly report or a management announcement) is forthcoming. Before the signal the expectation of the price according to the uninformed market (represented by Alice) is thus \( E [\varphi(Q_\Gamma)] \), whereas for an informed trader, Carroll, who knows that the state is in fact in \( \Gamma \), this expectation is \( E^\Gamma [\varphi(Q_\Gamma)] \). Our result says that Carroll’s expectation of the price is always higher than that of the market. Moreover, this holds even if one conditions on a certain set of values of the posterior, such as the posterior being within certain bounds: \( E^\Gamma [\varphi(Q_\Gamma)|Q_\Gamma \in A] \geq E [\varphi(Q_\Gamma)|Q_\Gamma \in A] \) for every \( A \subseteq [0, 1] \). Thus, no matter what the information structure is, the informed trader who knows that the state is in \( \Gamma \) is ex-ante (i.e., before the signal) more “optimistic” than the market about the ex-post (i.e., after the signal) price. He will therefore buy the asset now.

A general setup consists of players that possess different information (such as uninformed players, like Alice, and informed players, like Carroll), with actions that depend monotonically on beliefs (such as “threshold strategies”), and these actions may be unobserved or imperfectly monitored. One specific such class is that of reputation models with imperfect monitoring (see Fudenberg and Levine 1992, and Mailath and Samuelson 2006, Section 15.4): a sequence of short-lived players \( P_2 \) face a long-lived player \( P_1 \), who may be of a type \( \gamma \) that is committed to playing a fixed action. If the non-committed type of \( P_1 \) plays like the committed type \( \gamma \) then \( P_2 \)'s posterior belief that \( P_1 \) is of type \( \gamma \) becomes higher in

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We emphasize that the optimism is not about \( \Gamma \) per se (which goes without saying), but rather about the distribution of the market’s posterior of \( \Gamma \) after the signal, and thus about the market’s behavior then.
the likelihood ratio order, thus strengthening P1’s “reputation,” and potentially increasing his payoff.

In Bayesian statistics, where $\theta$ is an unknown parameter with prior distribution $\pi$, the interpretation of our results for, say, $\Gamma = \{\theta_0\}$ is that data generated under $\theta_0$ (i.e., an observation $s$ that is distributed according to $\mathbb{P}(\theta_0)$) increases a Bayesian’s belief in $\theta_0$ in the likelihood order sense, and thus also in expectation: $\mathbb{E}_\theta[\mathbb{P}(\theta_0|s)] \geq \pi(\theta_0)$. See Hart and Rinott (2020) for a study of sequences of observations.

Finally, a word of caution. If Carroll knows more than $\Gamma$—for instance, if $\Gamma = \{\beta, \gamma\}$ and Carroll knows that the state is in fact $\gamma$—then some of these inequalities may well be reversed: while knowing $\Gamma$ makes one optimistic (relative to Alice, or the market) about the posterior of $\Gamma$, knowing strictly more than $\Gamma$ may well turn an optimist into a pessimist. For a simple example, suppose that the set of states is $\Theta = \{\alpha, \beta, \gamma\}$, the prior $\pi$ is uniform on $\Theta$ (i.e., each state has probability $1/3$), and $\Gamma = \{\beta, \gamma\}$. Let $S = \{0, 1\}$ be the set of signals, with signaling probabilities $\sigma(s|\theta)$ as given in the table below (the last two columns then give the total probability $\mathbb{P}(s)$ of each signal $s$ and the posterior $Q_\Gamma(s) = \mathbb{P}(\Gamma|s)$):

| $s$ | $\sigma(s|\alpha)$ | $\sigma(s|\beta)$ | $\sigma(s|\gamma)$ | $\mathbb{P}(s)$ | $\mathbb{P}_\Gamma(s)$ | $Q_\Gamma(s)$ |
|-----|---------------------|---------------------|---------------------|----------------|----------------------|-------------|
| 1   | $3/8$               | $3/4$               | $1/2$               | $13/24$        | $5/8$                | $10/13$     |
| 0   | $5/8$               | $1/4$               | $1/2$               | $11/24$        | $3/8$                | $6/11$      |

We have $(Q_\Gamma, \mathbb{P}_\Gamma) \succ lr (Q_\Gamma, \mathbb{P})$ (by Proposition 1). However, if Carroll knows that the state is $\gamma$, then the dominance is reversed: $(Q_\Gamma, \mathbb{P}) \prec lr (Q_\Gamma, \mathbb{P})$. In the asset example, if Carroll knows that the state is $\beta$ or $\gamma$ then he buys the asset, but if he knows more, specifically, that it is $\gamma$, then he sells it.

## 2 Optimism

The result of the previous section suggests the following question: under what circumstances—besides knowing that the state is in $\Gamma$—is Carroll’s belief about $Q_\Gamma$ guaranteed to be higher than Alice’s belief, no matter what the signaling structure is? Suppose that Carroll has a prior distribution $\tilde{\pi}$ over the states of nature in $\Theta$ that may be different from Alice’s prior $\pi$ (in the previous section, $\tilde{\pi}$ is $\pi^\Gamma$). Let $\mathbb{P}_\tilde{\pi} \equiv \mathbb{P}_{\tilde{\pi},\sigma}$ denote Carroll’s probability on $\Theta \times S$ that is induced by $\tilde{\pi}$ and $\sigma$. Before the signal $s$, Carroll believes that Alice’s posterior probability of $\Gamma$

\footnote{Lemma 15.4.2 in Mailath and Samuelson (2006) gives this in expectation (i.e., Corollary 2).}
will be \( Q_\Gamma(s) \) with probability \( \tilde{P}(s) \); we denote this random variable by \( (Q_\Gamma, \tilde{P}) \).

We introduce two concepts that compare posteriors and priors, respectively.

- Carroll is \( \Gamma \)-**optimistic** if \((Q_\Gamma, \tilde{P})\) dominates \((Q_\Gamma, P)\) in the likelihood ratio order for every signaling structure \((S, \sigma)\) on the state space \(\Theta\).
- Carroll’s prior \( \tilde{\pi} \) is a \( \Gamma \)-**strengthening** of the prior \( \pi \) if \( \tilde{\pi} \) is an average of \( \pi \) and \( \pi^\Gamma \), i.e., \( \tilde{\pi} = a\pi^\Gamma + (1 - a)\pi \) for some \( a \in [0, 1] \) (where \( \pi^\Gamma(\theta) \equiv \pi(\theta|\Gamma) \) is the prior probability of \( \theta \) conditional on \( \Gamma \)).

It is readily seen that \( \Gamma \)-strengthening is equivalent to starting with the same prior \( \pi \) and then receiving (before the signal \( S \)) a specific signal \( t_0 \in T \) from another signaling structure \((T, \tau)\) that satisfies \( \tau(t_0|\theta) = b \geq c = \tau(t_0|\theta') \) for every \( \theta \in \Gamma \) and \( \theta' \not\in \Gamma \). A special case is where Carroll knows exactly \( \Gamma \), i.e., \( \tilde{\pi} = \pi^\Gamma \).

The two concepts turn out to be equivalent.

**Proposition 3** Carroll is \( \Gamma \)-optimistic if and only if Carroll’s prior \( \tilde{\pi} \) is a \( \Gamma \)-strengthening of the prior \( \pi \).

**Proof outline.** \( \Gamma \)-strengthening implies that the likelihood ratio \( \tilde{P}/P \) equals \( (a/p)Q_\Gamma + 1 - a \), which is monotonic in \( Q_\Gamma \), and so \( \Gamma \)-optimism follows. For the converse take a signaling structure with \( S = \{0, 1\} \), \( \sigma(0|\theta) = 1/2 - x_\theta \), \( \sigma(1|\theta) = 1/2 + x_\theta \), \( \sum_{\theta \in \Theta} \pi(\theta)x_\theta = 0 \), and \( \sum_{\theta \in \Gamma} \pi(\theta)x_\theta > 0 \); by \( \Gamma \)-optimism, we have \( \sum_{\theta \in \Theta} \tilde{\pi}(\theta)x_\theta \geq 0 \). Thus, \( \pi \cdot x = 0 \) and \( \pi^\Gamma \cdot x > 0 \) imply \( \tilde{\pi} \cdot x \geq 0 \). Using a standard theorem of the alternative we then show that \( \tilde{\pi} = a\pi^\Gamma + b\pi \) for some \( a, b \geq 0 \); evaluating at \( \Theta \) yields \( 1 = a + b \). □

### 2.1 Monotonic Optimism

Suppose now that there is more than one state (or one set of states) that matters. For example, the price of the asset may well be determined by the entire probability distribution on \( \Theta \) that is the market’s belief, and not just by the market’s probability of one specific set \( \Gamma \) of “good” states. What does it take to be optimistic in this case? Unfortunately, Proposition 3 implies that already for two distinct sets \( \Gamma_1 \) and \( \Gamma_2 \), Carroll cannot be both \( \Gamma_1 \)-optimistic and \( \Gamma_2 \)-optimistic unless \( \tilde{\pi} = \pi \) (that is, Carroll has the same information as Alice).

The definition of optimism requires taking into account all possible signaling structures, which suggests restricting them to a natural subclass of interest. We consider the commonly used class of signaling structures that have the **monotone likelihood ratio property** (MLRP); i.e., \( \sigma(s'|\theta)/\sigma(s|\theta) \) is increasing in \( \theta \) for \( s' > s \), where \( \Theta \) and \( S \) are subsets of, say, the real line. The interpretation is that high \( \theta \)'s
represent “good” states and low \( \theta \)’s, “bad” states, and the higher a signal \( s \) is, the more indicative \( s \) is of higher states \( \theta \). In such setups one considers upper sets of states \( \Gamma \subseteq \Theta \), for which \( \theta \in \Gamma \) implies \( \theta' \in \Gamma \) for all higher \( \theta' > \theta \) in \( \Theta \). Returning to the asset example, it is only natural for its price \( R \) to increase as the probability of the good states increases (more precisely, as probability mass is moved from low \( \theta \) to high \( \theta \)). This means that \( R \), as a function of probability distributions on the state space \( \Theta \), is increasing with respect to (first-order) stochastic dominance; that is, \( R = \varphi(Q_{\Gamma_1}, Q_{\Gamma_2}, ..., Q_{\Gamma_m}) \) with \( \varphi \) increasing in each coordinate and all the \( \Gamma_i \) upper sets.

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We thus define the monotonic versions of our two concepts:

- Carroll is monotonic-optimistic if \( (Q_{\Gamma_1}, \tilde{P}) \geq_{lr} (Q_{\Gamma}, P) \) for every upper set \( \Gamma \subseteq \Theta \) and every MLRP signaling structure \( (S, \sigma) \) on the state space \( \Theta \).
- Carroll’s prior \( \tilde{\pi} \) is a monotonic strengthening of the prior \( \pi \) if \( \tilde{\pi} \) is obtained by a finite sequence of \( \Gamma_i \)-strengthenings that starts from \( \pi \) and all the \( \Gamma_i \subset \Theta \) are upper sets.

**Proposition 4** Carroll is monotonic-optimistic if and only if Carroll’s prior \( \tilde{\pi} \) is a monotonic strengthening of the prior \( \pi \).

Moreover, these are also equivalent to each one of the following statements:

(i) \( \tilde{\pi} \geq_{lr} \pi \) (on \( \Theta \));
(ii) \( \tilde{P} \geq_{lr} P \) (on \( S \)) for every MLRP signaling structure \( (S, \sigma) \); and
(iii) \( (R, \tilde{P}) \geq_{lr} (R, P) \) for every MLRP signaling structure \( (S, \sigma) \) and every increasing function \( R : S \to \mathbb{R} \).

MLRP implies that higher signals make upper sets more probable, and so \( Q_{\Gamma}(s) \) increases in \( s \) for every upper set \( \Gamma \). Therefore, the functions \( R \) in condition (iii) include all these \( Q_{\Gamma} \), as well as all their increasing functions, as discussed above. Monotonicity with respect to stochastic dominance is thus equivalent, by Proposition 4, to the concept of monotonic optimism that we defined (as optimism for upper sets).

**Proof outline.** First, monotonic strengthening is equivalent to (i): for one direction, every upper-set strengthening yields \( \tilde{\pi} \geq_{lr} \pi \), and \( \geq_{lr} \) is transitive; for the converse, express the increasing function \( \tilde{\pi}/\pi \) as a positive linear combination of indicators of upper sets, which translates to \( \tilde{\pi} \) being a convex combination of \( \pi_{\Gamma} \)'s for upper sets \( \Gamma \), and from which we then obtain a sequence of upper-set strengthenings from \( \pi \) to \( \tilde{\pi} \).

Since \( \pi_1 \) first-order stochastically dominates \( \pi_2 \) if and only if \( \pi_1(\Gamma) \geq \pi_2(\Gamma) \) for all upper sets \( \Gamma \).
Second, standard composition arguments show that (i) implies (ii); (ii) implies (iii); and (iii) includes monotonic optimism (as we saw above).

Finally, monotonic optimism implies (i): for any two adjacent elements \( \theta_1 < \theta_2 \) of \( \Theta \) we construct an MLRP \((S, \sigma)\) and an upper set \( \Gamma \) such that \((Q_\Gamma, \tilde{P}) \geq_{lr} (Q_\Gamma, \mathbb{P})\) yields \( \tilde{\pi}(\theta_2)/\pi(\theta_2) \geq \tilde{\pi}(\theta_1)/\pi(\theta_1) \).

Suppose that there is another agent, Bob, and that Carroll’s prior \( \tilde{\pi} \) is a monotonic strengthening of Bob’s prior \( \hat{\pi} \) (and we make no assumptions relating to Alice). Then Proposition 4 (iii) implies that \( 10 \) \((Q_\Gamma, \tilde{P}) \geq_{lr} (Q_\Gamma, \hat{P}) \) for any upper set \( \Gamma \) and any MLRP signaling structure: Carroll is more optimistic than Bob about Alice’s posterior probabilities—and thus, about the market price.

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10 Apply Proposition 4 with \( \hat{\pi} \) instead of \( \pi \) and use (iii) with \( R = Q_\Gamma \) (which is Alice’s posterior probability of \( \Gamma \), and is increasing in \( s \)). We need to use (iii) because monotonic optimism here means \((\hat{Q}_\Gamma, \hat{P}) \geq_{lr} (\hat{Q}_\Gamma, \hat{P})\), where \( \hat{Q}_\Gamma \) denotes Bob’s posterior probability of \( \Gamma \).
Supplementary Material

1 Dominance

We provide some details on the example at the end of Section 1 of the paper. The simple remark below is useful here (as well as in the next section).

Remark. In the binary case where $X$ and $Y$ take only two values, say $v_1 < v_2$, likelihood ratio dominance is easily seen to be equivalent to the high value having a higher probability; i.e., $X \geq_{lr} Y$ if and only if $P(X = v_2) \geq P(Y = v_2)$ (which is also equivalent to first-order stochastic dominance).

Example. Suppose that the set of states is $\Theta = \{\alpha, \beta, \gamma\}$, the prior $\pi$ is uniform on $\Theta$ (i.e., each state has probability $1/3$), and $\Gamma = \{\beta, \gamma\}$. Let $S = \{0, 1\}$ be the set of signals, with signaling probabilities $\sigma(s|\theta)$ as given in the table below (the last two columns then give the total probability $P(s)$ of each signal $s$ and the posterior $Q_\Gamma(s) \equiv P(\Gamma|s)$):

| $s$ | $\sigma(s|\alpha)$ | $\sigma(s|\beta)$ | $\sigma(s|\gamma)$ | $P(s)$ | $P^\Gamma(s)$ | $Q_\Gamma(s)$ |
|-----|-------------------|-------------------|-------------------|--------|--------------|-------------|
| 1   | 3/8               | 3/4               | 1/2               | 13/24  | 5/8          | 10/13       |
| 0   | 5/8               | 1/4               | 1/2               | 11/24  | 3/8          | 6/11        |

Proposition 1 gives $(Q_\Gamma, P^\Gamma) >_{lr} (Q_\Gamma, P)$. We see this here (recall the Remark above) in the posterior $Q_\Gamma$ being higher after signal $s = 1$ than after signal $s = 0$.

\[\text{E-mail: } \text{hart@huji.ac.il} \quad \text{Web site: } \text{http://www.ma.huji.ac.il/hart}\]

\[\text{E-mail: } \text{yosef.rinott@mail.huji.ac.il} \quad \text{Web site: } \text{http://pluto.huji.ac.il/~rinott}\]
i.e., $Q_\Gamma(1) = 10/13 > 6/11 = Q_\Gamma(0)$) together with $\mathbb{P}^\Gamma$ giving a higher probability than $\mathbb{P}$ to $s = 1$ (i.e., $\mathbb{P}^\Gamma(1) = 5/8 > 13/24 = \mathbb{P}(1)$). However, if Carroll knows that the state is $\gamma$, then the dominance is reversed: $(Q_\Gamma, \mathbb{P}^\gamma) <_\text{lr} (Q_\Gamma, \mathbb{P})$ (indeed, the high-posterior signal $s = 1$ then has a lower probability under $\mathbb{P}^\gamma$ than under $\mathbb{P}$ (i.e., $\mathbb{P}^\gamma(1) = 1/2 < 13/24 = \mathbb{P}(1)$). In the asset example, if Carroll knows that the state is $\beta$ or $\gamma$ then he buys the asset, but if he knows more, specifically, that it is $\gamma$, then he sells it.

2 Optimism

We provide the full proof of Proposition 3, followed by two remarks.

**Proof of Proposition 3.** If $\tilde{\pi} = a\pi^\Gamma + (1-a)\pi$ then $\tilde{\mathbb{P}}(s) = a\mathbb{P}^\Gamma(s) + (1-a)\mathbb{P}(s)$ for every $s$, and so, dividing by $\mathbb{P}(s)$ and recalling the Bayes formula (1), we have

$$\frac{\tilde{\mathbb{P}}(s)}{\mathbb{P}(s)} = \frac{a}{p}Q_\Gamma(s) + (1-a).$$

The likelihood ratio is thus increasing in $Q_\Gamma(s)$; this yields the likelihood ratio dominance as in the proof of Proposition 1.

Conversely, assume that Carroll is $\Gamma$-optimistic. Take $S = \{0, 1\}$ and put $\sigma(1|\theta) = 1/2 + x_\theta$ and $\sigma(0|\theta) = 1/2 - x_\theta$, where $|x_\theta| \leq 1/2$. If $\sum_{\theta\in\Theta} \pi(\theta)x_\theta = 0$ and $\sum_{\theta\in\Gamma} \pi(\theta)x_\theta > 0$ then $\mathbb{P}(1) = \mathbb{P}(0) = 1/2$ and $\mathbb{P}(\Gamma \cap 1) > \mathbb{P}(\Gamma \cap 0)$, and thus $Q_\Gamma(1) = \mathbb{P}(\Gamma|1) > \mathbb{P}(\Gamma|0) = Q_\Gamma(0)$. By $\Gamma$-optimism, this yields $\tilde{\mathbb{P}}(1) \geq \mathbb{P}(1)$, i.e., $\sum_{\theta} \tilde{\pi}(\theta)x_\theta \geq 0$ (see the Remark on binary random variables in the previous section).

We thus have the following: if $\sum_{\theta\in\Theta} \pi(\theta)x_\theta = 0$ and $\sum_{\theta\in\Theta} \pi^\Gamma(\theta)x_\theta > 0$ (we have divided the second sum by $\pi(\Gamma)$) then $\sum_{\theta\in\Theta} \tilde{\pi}(\theta)x_\theta \geq 0$; this holds for any $x = (x_\theta)_{\theta\in\Theta}$ even if some $x_\theta$ do not satisfy $|x_\theta| \leq 1/2$: just rescale $x$ so that they all do, which does not affect any of the above three homogeneous relations. Equivalently, there is no $x$ such that $\sum_{\theta\in\Theta} \pi(\theta)x_\theta = 0$, $\sum_{\theta\in\Theta} \pi^\Gamma(\theta)x_\theta > 0$, and $\sum_{\theta\in\Theta} \tilde{\pi}(\theta)x_\theta < 0$, i.e., no $x$ such that $\pi \cdot x = 0$, $\pi^\Gamma \cdot x > 0$, and $(\tilde{\pi}) \cdot x > 0$. Motzkin’s Theorem of the Alternative (see, e.g., Mangasarian 1969) therefore yields $y, z, w$ such that

$$y\pi + z\pi^\Gamma + w(-\tilde{\pi}) = 0 \quad (1)$$

and $z, w \geq 0$ with at least one of them strictly positive.

For $\theta \notin \Gamma$ with $\pi(\theta) > 0$ (such a $\theta$ exists since $\pi(\Gamma) < 1$) we have $y\pi(\theta) = w\tilde{\pi}(\theta)$ by (1), and so $y \geq 0$ (because $w \geq 0$). If $w = 0$ then we must have $z > 0$, but then
the sum in (1) is strictly positive for \( \theta \in \Gamma \) with \( \pi(\theta) > 0 \) (such a \( \theta \) exists since \( \pi(\Gamma) > 0 \)), a contradiction. Therefore \( w > 0 \), and dividing by \( w \) yields

\[
\tilde{\pi} = a\pi^\Gamma + a'\pi,
\]

where \( a = z/w \geq 0 \) and \( a' = y/w \geq 0 \). Evaluating at \( \Theta \) gives \( 1 = a + a' \), and the proof is complete. ■

**Remarks.** (a) The proof of the second direction shows that when defining optimism it suffices to require first-order stochastic dominance (instead of likelihood ratio dominance) to hold for all binary signaling structures, i.e., \(|S| = 2\) (instead of all signaling structures).

(b) Applying Proposition 1 to \( \Gamma^C = \Theta \setminus \Gamma \), the complement of \( \Gamma \), and using \( P(\Gamma^C|s) = 1 - P(\Gamma|s)) \) yields the following: “\( \Gamma \)-pessimism” (which is defined by reversing the direction of the dominance relation in \( \Gamma \)-optimism) is equivalent to \( \Gamma^C \)-strengthening (which, in turn, is nothing other than “\( \Gamma \)-weakening”); in terms of the additional signal \( t_0 \) from the signaling structure \((T, \tau)\) that Carroll receives, it means that \( \tau(t_0|\theta) = b \leq c = \tau(t_0|\theta) \) for every \( \theta \in \Gamma \) and \( \theta' \notin \Gamma \).

## 2.1 Monotonic Optimism

We provide the full version of the statement of Proposition 4 and its proof.

Let \( \pi \) and \( \tilde{\pi} \) be two prior probability distributions on \( \Theta \). Given a signaling structure \((S, \sigma)\) on \( \Theta \), we denote by \( P \) and \( \tilde{P} \) the probability distributions that are induced by the priors \( \pi \) and \( \tilde{\pi} \), respectively; for each \( s \in S \) and \( \Gamma \subset \Theta \), we denote by \( Q_{\Gamma}(s) = P(\Gamma|s) \) the posterior probability of \( \Gamma \) given the signal \( s \).

**Proposition 4** The following statements are equivalent:

(a) \( \tilde{\pi} \) is a monotonic strengthening of \( \pi \);

(b) \( \tilde{\pi} \in \text{conv}\{\pi^\Gamma : \Gamma \subseteq \Theta \text{ upper set}\} \);

(c) \( \tilde{\pi} \geq_{lr} \pi \) (on \( \Theta \));

(d) \( \tilde{P} \geq_{lr} P \) (on \( S \)) for every MLRP signaling structure \((S, \sigma)\);

(e) \( (R, \tilde{P}) \geq_{lr} (R, P) \) for every MLRP signaling structure \((S, \sigma)\) and every increasing function \( R : S \to \mathbb{R} \); and

(f) \( (Q_{\Gamma}, \tilde{P}) \geq_{lr} (Q_{\Gamma}, P) \) for every MLRP signaling structure \((S, \sigma)\) and every upper set \( \Gamma \).

Statements (a) and (f) provide the main equivalence of Proposition 4 in the paper; (c), (d), and (e) here are, respectively, (i), (ii), and (iii) there.
Proof. The proof contains two cycles: first, (a) ⇒ (c) ⇒ (b) ⇒ (a), and then (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (e).

Monotonicity of ratios, i.e., \( a'/b' \geq a/b \), is always taken to mean \( a'b \geq ab' \), which applies also when denominators equal zero. We assume without loss of generality that \( \pi(\theta) > 0 \) for all \( \theta \in \Theta \).

(a) implies (c). If \( \pi_2 = a\pi_1^1 + (1-a)\pi_1 \) is a \( \Gamma \)-strengthening of \( \pi_1 \) then

\[
\frac{\pi_2(\theta)}{\pi_1(\theta)} = \frac{a}{\pi_1(\Gamma)}1_{\Gamma}(\theta) + 1 - a,
\]

where \( 1_{\Gamma} \) denotes the indicator of the set \( \Gamma \). When \( \Gamma \) is an upper set this ratio is increasing in \( \theta \), and thus \( \pi_2 \geq_{\Gamma} \pi_1 \). Using the transitivity of the likelihood ratio dominance (because \( \pi_3/\pi_1 = (\pi_3/\pi_2) \cdot (\pi_2/\pi_1) \)) completes the proof.

(c) implies (b). Let \( \theta_1 < \theta_2 < \ldots < \theta_n \) be the elements of \( \Theta \). If \( \pi \geq_{\Gamma} \pi \) then the increasing function \( h(\theta) := \pi(\theta)/\pi(\theta) \) can be represented as \( \sum_{i=1}^n a_i1_{\Gamma_i}(\theta) \) where \( \Gamma_i := \{\theta_i, \ldots, \theta_n\} \) (an upper set) and \( a_i \geq 0 \) for all \( i \). Therefore \( \tilde{\pi}(\theta) = \sum_{i=1}^n a_i1_{\Gamma_i}(\theta)\pi(\theta) = \sum_{i=1}^n a_i\pi(\Gamma_i)\pi^{\Gamma_i}(\theta) \), showing that \( \tilde{\pi} \) is a convex combination of the \( \pi^{\Gamma_i} \) (summing over all \( \theta \) shows that the coefficients add up to 1, because \( \tilde{\pi} \) and the \( \pi^{\Gamma_i} \) are all probability distributions).

(b) implies (a). Take \( \pi_0 = \sum_{i=1}^m a_i\pi_{\Gamma_i} \), where \( a_i \geq 0 \), \( \sum_{i=1}^m a_i = 1 \), and the \( \Gamma_i \) are in decreasing order, i.e., \( \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_m \). Then

\[
\pi_0 = a_m\pi_{\Gamma_m} + (1-a_m)\pi_1,
\]

where \( \pi_1 = \sum_{i=1}^{m-1} b_i\pi_{\Gamma_i} \) and \( b_i = a_i/(1-a_m) \). Now

\[
\pi_1^{\Gamma_m}(\theta) = \frac{1}{\pi_1(\Gamma_m)} \sum_{i=1}^{m-1} b_i\pi_{\Gamma_i}(\theta \cap \Gamma_m) = \frac{1}{\pi_1(\Gamma_m)} \sum_{i=1}^{m-1} b_i\pi(\Gamma_m)\pi^{\Gamma_i}(\theta)
\]

\[
= \left( \sum_{i=1}^{m-1} c_i \right) \pi^{\Gamma_m}(\theta) = \pi^{\Gamma_m}(\theta),
\]

by \( \Gamma_i \supset \Gamma_m \) for \( i = 1, \ldots, m-1 \), and \( \sum_{i=1}^{m-1} c_i = 1 \) since \( \pi_1^{\Gamma_m} \) and \( \pi^{\Gamma_m} \) are both probability measures. Substituting in (2) gives

\[
\pi_0 = a_m\pi_1^{\Gamma_m} + (1-a_m)\pi_1,
\]

and so \( \pi_0 \) is a \( \Gamma_m \)-strengthening of \( \pi_1 \). Induction on \( m \) completes the proof.

(c) implies (d). This is shown by a standard composition argument (\( \mathbb{P} \) and \( \mathbb{P} \) are the compositions of \( \sigma \) with \( \pi \) and \( \pi \), respectively; see, e.g., Karlin 1968,
formulas (0.1) and (1.1), which are versions of the Cauchy–Binet formula. We provide here (and in the following arguments as well) a direct proof. We have

$$\tilde{P}(s')P(s) - \tilde{P}(s)P(s') = \sum_{\theta' > \theta} \left[ \tilde{\pi}(\theta') \pi(\theta) - \tilde{\pi}(\theta) \pi(\theta') \right] \left[ \sigma(s'|\theta') \sigma(s|\theta) - \sigma(s|\theta') \sigma(s'|\theta) \right],$$

where the double sum is over all $\theta' > \theta$. When $s' > s$ the terms on the right-hand side are all $\geq 0$ by $\tilde{\pi} \geq \pi$ and MLRP.

(d) implies (e). We have

$$\tilde{P}(R = r')P(R = r) - \tilde{P}(R = r)P(R = r') = \sum_{s' \in S : R(s') = r'} \sum_{s \in S : R(s) = r} \left[ \tilde{P}(s')P(s) - \tilde{P}(s)P(s') \right].$$

When $r' > r$ the terms on the right-hand are all $\geq 0$ because $R(s') = r' > r = R(s)$ implies $s' \geq s$.

(e) implies (f). We need to show that $Q_\Gamma$ is increasing for every upper set $\Gamma$.

We have

$$P(\Gamma \cap s')P(s) - P(\Gamma \cap s)P(s') = P(\Gamma \cap s')P(\Gamma^C \cap s) - P(\Gamma \cap s)P(\Gamma^C \cap s') = \sum_{\theta' \in \Gamma^C} \sum_{\theta \in \Gamma} \pi(\theta') \pi(\theta) \left[ \sigma(s'|\theta') \sigma(s|\theta) - \sigma(s|\theta') \sigma(s'|\theta) \right].$$

When $s' > s$ the terms on the right-hand side are $\geq 0$ by MLRP, because $\theta' \in \Gamma$ and $\theta \in \Gamma^C$ implies $\theta' > \theta$ (since $\Gamma$ is an upper set).

(f) implies (c). Let $\theta_1 < \theta_2$ be two adjacent elements of $\Theta$ (i.e., there is no $\theta \in \Theta$ between $\theta_1$ and $\theta_2$), and take the following signaling structure: $S = \{0, 1, 2, 3\}$; for $\theta < \theta_1$ the signal is $s = 0$, for $\theta > \theta_2$ it is $s = 3$, for $\theta = \theta_1$ it is $s = 1$ with probability 2/3 and $s = 2$ with probability 1/3, and for $\theta = \theta_2$ it is $s = 1$ with probability 1/3 and $s = 2$ with probability 2/3. Clearly, $(S, \pi)$ is MLRP. Take the upper set $\Gamma = \{\theta \in \Theta : \theta \geq \theta_2\}$. The posterior probabilities of $\Gamma$ are

$$Q_\Gamma(0) = 0 < Q_\Gamma(1) < Q_\Gamma(2) < 1 = Q_\Gamma(3),$$

where

$$Q_\Gamma(1) = \frac{1}{2} \pi(\theta_2) \quad \text{and} \quad \frac{2}{3} \pi(\theta_1) + \frac{1}{3} \pi(\theta_2) + 1,$$

$$Q_\Gamma(2) = \frac{2}{3} \pi(\theta_2) \quad \text{and} \quad \frac{1}{2} \pi(\theta_1) + \frac{1}{3} \pi(\theta_2) + 1.$$
Therefore
\[
\frac{1}{3} \pi(\theta_1) + \frac{2}{3} \pi(\theta_2) = \frac{\bar{P}(Q_\Gamma = \alpha_2)}{\bar{P}(Q_\Gamma = \alpha_1)} \geq \frac{\bar{P}(Q_\Gamma = \alpha_1)}{\bar{P}(Q_\Gamma = \alpha_2)} = \frac{\frac{2}{3} \pi(\theta_1) + \frac{1}{3} \pi(\theta_2)}{\frac{2}{3} \pi(\theta_1) + \frac{1}{3} \pi(\theta_2)},
\]
where the inequality is by (f). Simplifying yields \( \bar{\pi}(\theta_2)/\pi(\theta_2) \geq \bar{\pi}(\theta_1)/\pi(\theta_1) \); since this holds for any adjacent \( \theta_1 < \theta_2 \), we have \( \bar{\pi} \geq l_r \pi \). ■

References

Karlin, S. (1968), *Total Positivity*, Stanford University Press.

Mangasarian, O. L. (1969), *Nonlinear Programming*, McGraw-Hill.