AN INSTABILITY MECHANISM OF PULSATILE FLOW ALONG PARTICLE TRAJECTORIES FOR THE AXISYMMETRIC EULER EQUATIONS

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Abstract. The dynamics along the particle trajectories for the 3D axisymmetric Euler equations in an infinite cylinder are considered. It is shown that if the inflow-outflow is rapidly increasing in time, the corresponding laminar profile of the Euler flow is not (in some sense) stable provided that the swirling component is not small. This exhibits an instability mechanism of pulsatile flow. In the proof, Frenet-Serret formulas and orthonormal moving frame are essentially used.

1. Introduction

We study the dynamics along the particle trajectories for the 3D axisymmetric Euler equations. Such Lagrangian dynamics have already been studied in mathematics (see [1, 2, 3]). For example, in [2], Chae considered a blow-up problem for the axisymmetric 3D incompressible Euler equations with swirl. More precisely, he showed that under some assumption of local minima for the pressure on the axis of symmetry with respect to the radial variations along some particle trajectory, the solution blows up in finite time.

Although the blowup problem of 3D Euler equations is still an outstanding open problem, in this paper, we focus on a different problem in physics, especially, the cardiovascular system [6]. If the blood flow is in large and medium sized vessels, the flow is governed by the usual incompressible Navier-Stokes equations. In this study field, Womersley number is the key. The Womersley number comes from oscillating (in time) solutions to the Navier-Stokes equations in a tube. Let us explain more precisely. We define a pipe \( \Omega_R \) as \( \Omega_R := \{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < R, \ 0 < x_3 < \ell \} \) with its side-boundary \( \partial \Omega_R = \{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} = R, \ 0 < x_3 < \ell \} \). The incompressible Navier-Stokes equations are described as follows:

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u = -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega_R
\]

with \( u = u(x, t) = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t)) \) and \( p = p(x, t) \).

To give the Womersley number, we need to focus on the axisymmetric Navier-Stokes flow without swirl (see [10]). If \( p_1 \) and \( p_2 \) are the pressure at the ends of the pipe \( \Omega_R \), the pressure gradient can be expressed as \( (p_1 - p_2)/\ell \). If the pressure gradient is time-independent, \( (p_1 - p_2)/\ell =: p_s \), then we can find the stationary

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Navier-Stokes flow (Poiseuille flow):

\begin{equation}
    u_s = (u_1, u_2, u_3) = (0, 0, \frac{p_s}{4\nu \ell}(R^2 - r^2)),
\end{equation}

where \( r = \sqrt{x_1^2 + x_2^2} \). Note that \( u_s \) is also a solution to the linearized Navier-Stokes equations. Next we consider the oscillating pressure gradient case,

\begin{equation}
    \frac{p_1(t) - p_2(t)}{\ell} = p_o e^{i\lambda t}
\end{equation}

which is periodic in the time. Then its corresponding solution \( u_o \) can be written explicitly by using a Bessel function (see [10, (8)] and [9, (1)]) with \( u_1 = u_2 = 0 \). Thus \( u_o \) is also a solution to the linearized Navier-Stokes equations. Now we can give the Womersley number \( \alpha \) as follows:

\[ \alpha = \frac{R}{\nu} \sqrt{\frac{N}{\nu}}. \]

In [9], they also defined the oscillatory Reynolds number and the mean Reynolds number by using \( u_o \) and \( u_s \) respectively, and they investigated how the transition of pulsatile flow from the laminar to the turbulent (critical Reynolds number) is affected by the Womersley number and the oscillatory Reynolds number. According to their experiment, measurement at different Womersley numbers yield similar transition behavior, and variation of the oscillatory Reynolds number also appear to have little effect. Thus they conclude that the transition seems to be determined only by the mean Reynolds number. However it seems they did not investigate the effect of the non-small swirl component (azimuthal component), and thus our aim here is to show that the non-small swirl component induces an instability which is, at a glance, nothing to do with wall turbulence. Let us explain more precisely. Since we would not like to take the boundary layer into account, it is reasonable to consider a simpler model: the 3D axisymmetric Euler flow in an infinite cylinder \( \Omega := \{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < 1, x_3 \in \mathbb{R} \} \) (the setting \( \Omega \) is just for simplicity). The incompressible Euler equations are expressed as follows:

\begin{equation}
    \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega,
\end{equation}

\[ u|_{\partial \Omega} = u_0, \quad u \cdot n = 0 \quad \text{on} \quad \partial \Omega, \quad u(x, t) \to (0, 0, g(t)) \quad (x_3 \to \pm \infty) \]

with \( u = u(x, t) = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t)) \), \( p = p(x, t) \) and an uniform (in space) inflow-outflow condition \( g = g(t) \) (the uniform setting is just for simplicity, we can easily generalize it), where \( n \) is a unit normal vector on the boundary.

**Remark 1.1.** According to the boundary layer theory, outside the boundary layer the fluid motion is accurately described by the Euler flow. Thus the above simplification seems (more or less) valid. For the recent progress on the mathematical analysis of the boundary layer, see [8].

Roughly saying, the inflow-outflow \( g \) is a simplification of \( u_s + u_o \), namely, \( u_s \) is approximated by the mean-value of \( g \), \( \partial_t u_o \) and \( \partial_x^2 u_o \) are approximated by \( g' \) and \( g'' \) respectively. Since we consider the axisymmetric Euler flow, we can simplify the Euler equations (1.3). Let \( e_r := x_h/|x_h|, e_\theta := x_h^*/|x_h| \) and \( e_z = (0, 0, 1) \) with \( x_h = (x_1, x_2, 0), x_h^* = (-x_2, x_1, 0) \). The vector valued function \( u \) can be rewritten as \( u = v_r e_r + v_\theta e_\theta + v_z e_z \), where \( v_r = v_r(r, z, t), v_\theta = v_\theta(r, z, t) \) and \( v_z = v_z(r, z, t) \).
with \( r = |x_1| \) and \( z = x_3 \). Then the axisymmetric Euler equations can be expressed as follows:

\[
\begin{align*}
(1.5) \quad & \partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r - \frac{v_z^2}{r} + \partial_r p = 0, \\
(1.6) \quad & \partial_t v_\theta + v_r \partial_r v_\theta + v_z \partial_z v_\theta + \frac{v_r v_\theta}{r} = 0, \\
(1.7) \quad & \partial_t v_z + v_r \partial_r v_z + v_z \partial_z v_z + \partial_z p = 0, \\
(1.8) \quad & \frac{\partial_r (rv_r)}{r} + \partial_z v_z = 0.
\end{align*}
\]

In order to show that the non-small swirl component induces the instability, we need to measure appropriately the rate of laminar profile of the Euler flow.

**Definition 1.2.** (Axis-length streamline in \( z \).) For a unilateral flow \( v_z = u \cdot e_z > 0 \) in \( \Omega \), we can define an axis-length streamline \( \gamma(z) \). Let \( t \) be fixed, and let \( \gamma(z) \) be such that

\[ \gamma(\bar{r}_0, z, t) = \gamma(z) := (\bar{R}(z) \cos \bar{\Theta}(z), \bar{R}(z) \sin \bar{\Theta}(z), z) \]

with \( \bar{R}(z) = \bar{R}(\bar{r}_0, z, t) = \bar{r}_0, \bar{\Theta}(z) = \bar{\Theta}(z, t) \) and we choose \( \bar{R} \) and \( \bar{\Theta} \) in order to satisfy

\[ \partial_z \gamma(z) = \left( \frac{u}{u \cdot e_z} \right) (\gamma(z), t) \]

We easily see

\[ \partial_z \gamma \cdot e_z = 1, \quad \partial_z \gamma \cdot e_r = \partial_z \bar{R} = \frac{v_r}{v_z}, \quad \partial_z \gamma \cdot e_\theta = \bar{R} \partial_z \bar{\Theta} = \frac{v_\theta}{v_z} \]

In this paper we always assume existence of a unique smooth solution to the Euler equations. Since \( \partial_{\bar{r}_0} \bar{R} > 0 \) due to the smoothness, we have its inverse \( r_0 = \bar{R}^{-1}(r, z, t) \). We now give the key definition.

**Definition 1.3.** (Rate of laminar profile.) Let \( \partial = \partial_z \) or \( \partial_{\bar{r}_0} \), and let \( \bar{\partial} = \partial_z \) or \( \partial_r \). We define “rate of laminar profile” \( L^x \) and \( L^t \) as follows:

\[
L^x(\bar{r}_0, z, t) := \sum_{\ell=1}^{3} |\partial^\ell \bar{R}(\bar{r}_0, z, t)| + \sum_{\ell=1}^{3} |(\partial^\ell \bar{R}^{-1})(\bar{R}(\bar{r}_0, z, t), z, t)|
\]

and

\[
L^t(\bar{r}_0, z, t) = |(\partial_{\bar{r}_0} \bar{R}^{-1})(\bar{R}(\bar{r}_0, z, t), z, t)| + |(\partial_{\bar{r}_0} \partial z \bar{R})(\bar{r}_0, z, t)| + |(\partial_{\bar{r}_0} \partial z \bar{R})(\bar{r}_0, z, t)|.
\]

Later we deal with the curvature and torsion of the particle trajectory, thus it is natural to see up to three derivatives.

**Remark 1.4.** As we already assumed that solutions to the Euler equations are always unique and smooth enough, thus, when the transition of the Euler flow from the laminar to the turbulent regime occurs, then \( L^x \) and/or \( L^t \) must tend to infinity.

**Remark 1.5.** Minumum value of \( L^x \) is 2, since \( |\partial_{\bar{r}_0} \bar{R}^{-1}| = 1/|\partial_{\bar{r}_0} \bar{R}| \).

**Remark 1.6.** We easily see that \( u = (0, 0, g) \) is one of the solution to \( \text{[1.2]} \). This flow is the typical laminar flow. In this case

\[ L^x = 2 \quad \text{and} \quad L^t = 0. \]

Now we give the main theorem.
Theorem 1.7. Assume there is a unique smooth solution to the Euler equations (1.4) with smooth initial data satisfying \(|L^2| \leq \beta\) for some positive constant \(\beta\) (we will determine \(\beta\) later). For any \(x \in \Omega\) satisfying \(u_0(x) \cdot e_\theta \approx 1\) and \(x \cdot e_r > 1/\beta\), and any \(\epsilon > 0\), then there is \(\delta > 0\) such that

\[
L^t(\bar{r}_0, z, 0) \geq 1/\epsilon
\]

with any smooth inflow-outflow \(g(t)\) satisfying

\[
1/\beta^5 \leq g(0) \leq 1/\epsilon^5 \quad \text{and} \quad \frac{1}{\delta^3} < \frac{g'(0)}{\delta^2} < g''(0),
\]

where \((\bar{r}_0, z) = \gamma^{-1}(x, 0)\).

Remark 1.8. In contrast with [9], the transition of the pulsatile flow from the laminar to the turbulent may be affected by the Womersley number and the oscillatory Reynolds number provided by the non-small swirl component.

Remark 1.9. This instability mechanism is (at a glance) different from wall turbulence. It would be interesting to consider the interaction between this instability mechanism and wall turbulence, and this is our future work.

In the next section, we prove the main theorem.

2. Proof of the main theorem.

Notations “\(\approx\)” and “\(\lesssim\)” are convenient. The notation “\(a \approx b\)” means there is a positive constant \(C > 0\) such that

\[
C^{-1}a \leq b \leq Ca,
\]

and “\(a \lesssim 1\)” means there is a positive constant \(C > 0\) such that

\[
0 \leq a \leq C.
\]

This constant \(C\) is not depending on neither \(\epsilon\) nor \(\delta\). Throughout this paper we use \(C(\beta)\) (different from the above \(C\)) as a positive constant depending on \(\beta\). Now we define the particle trajectory. The associated Lagrangian flow \(\eta(t)\) is a solution of the initial value problem

\[
\frac{d}{dt} \eta(x, t) = u(\eta(x, t), t),
\]

\[
\eta(x, 0) = x.
\]

To prove the main theorem, it is enough to show the following lemma:

Lemma 2.1. Assume there is a unique smooth solution to the Euler equations (1.4) with smooth initial data satisfying \(|L^2| \leq \beta\) for some positive constant \(\beta\). For any \(x \in \Omega\) satisfying \(u_0(x) \cdot e_\theta \approx 1\) and \(x \cdot e_r > 1/\beta\), and any \(\epsilon > 0\), then there is \(\delta > 0\) such that for any small time interval \(I\) with \(|I| < \beta^2 \epsilon^3\), at least either of the following four cases must happen:

- \(L^t(\bar{r}_0, z, t) > 1/\beta\),
- \(L^t(\bar{r}_0, z, t) \gtrsim 1/\epsilon\),
- \(|\eta(x, t) \cdot e_r| < \beta\),
- \(\bar{r}_0 < \beta\),

where \((\bar{r}_0, z) = \gamma^{-1}(x, 0)\).
for some $t \in I$, with any inflow-outflow $g(t)$ satisfying

$$1/\beta^3 \leq g(t) \leq 1/\epsilon^3$$

and

$$\frac{1}{\delta^3} < \frac{g'(t)}{\delta^2} < g''(t) \quad \text{in} \quad t \in I,$$

where $(\tilde{r}_0, z) = (\gamma^{-1} \circ \eta)(x, t)$ (in this case $\tilde{r}_0$ and $z$ are depending on $t$).

Since the time interval $I$ is arbitrary, we see that $L^x$ or $\eta \cdot e_r$ or $\tilde{r}_0$ is not continuous at the initial time $t = 0$, or $L^t \gtrsim 1/\epsilon$ at the initial time. The discontinuity contradicts the smoothness assumption, thus

$$L^t \gtrsim 1/\epsilon$$

only occurs. In what follows, we prove the above lemma. For any time interval $I$ with $|I| \leq \beta^2 \epsilon^3$, assume that the axisymmetric smooth Euler flow satisfies the following conditions:

- $L^x(\tilde{r}_0, z, t) \leq 1/\beta$ and $L^t(\tilde{r}_0, z, t) \lesssim 1/\epsilon$ for any $t \in I$,
- $|\eta(x, t) \cdot e_r| \geq \beta$ and $\bar{r}_0 \geq \beta$ for any $t \in I$,

where $(\tilde{r}_0, z) = (\gamma^{-1} \circ \eta)(x, t)$. First we express $v_z$ and $v_r$ by using $\bar{R}$ and $\tilde{R}^{-1}$.

To do so, we define the cross section of the stream-tube (annulus). Let $B_{-\epsilon}(\tilde{r}_0) = \{x \in \mathbb{R}^3 : |x_h| < \tilde{r}_0, x_3 = -\infty\}$ and let

$$A(\tilde{r}_0, z, \epsilon, t) := \bigcup_{x \in B_{-\epsilon}(\tilde{r}_0) \setminus B_{-\epsilon}(\tilde{r}_0)} \gamma(x, z, t).$$

We see that its measure is

$$|A(\tilde{r}_0, z, \epsilon, t)| = \pi \left( \bar{R}(\tilde{r}_0, \epsilon, z, t)^2 - \tilde{R}(\tilde{r}_0, z, t)^2 \right).$$

**Definition 2.2.** (Inflow propagation.) Let $\rho$ be such that

$$\rho(\tilde{r}_0, z, t) := \lim_{\epsilon \to 0} \frac{|A(\tilde{r}_0, -\infty, \epsilon, t)|}{|A(\tilde{r}_0, z, \epsilon, t)|}.$$

We see that

$$\rho(\tilde{r}_0, z, t) = \frac{\partial_{\tilde{r}_0} \bar{R}(\tilde{r}_0, -\infty, t)}{\partial_{\tilde{r}_0} \bar{R}(\tilde{r}_0, z, t)} \frac{\bar{r}_0}{\bar{R}(\tilde{r}_0, z, t)} = \frac{2\bar{r}_0}{\bar{R}(\tilde{r}_0, z, t)^2}.$$

**Remark 2.3.** By the assumption on $L^x$, we have the estimates of the inflow propagation $\rho$:

$$|\partial_z \rho|, |\partial_z^2 \rho|, |\partial_{\tilde{r}_0} \rho|, |\partial_{\tilde{r}_0}^2 \rho| \lesssim C(\beta) \quad \text{for} \quad t \in I$$

with some positive constant $C(\beta)$. Note that $\bar{R}(\tilde{r}_0, z, t) > \beta$ for $(\tilde{r}_0, z) = (\gamma^{-1} \circ \eta)(x, t)$ due to $\eta(x, t) \cdot e_r > \beta$.

Since

$$2\pi \int_{\bar{R}(\tilde{r}_0, z, t)} u_z(r', z, t) r' dr' = 2\pi \int_{\tilde{r}_0}^{\tilde{r}_0 + \epsilon} u_z(r', -\infty, t) r' dr'$$

by divergence-free and Gauss’s divergence theorem, we can figure out $v_z$ by using the inflow propagation $\rho$,

$$v_z(r, z, t) = \lim_{\epsilon \to 0} \frac{2\pi}{|A(\tilde{r}_0, z, \epsilon, t)|} \int_{\bar{R}(\tilde{r}_0, z, t)} v_z(r', z, t) r' dr'$$

$$= \lim_{\epsilon \to 0} \frac{|A(\tilde{r}_0, -\infty, \epsilon, t)|}{|A(\tilde{r}_0, z, \epsilon, t)|} \frac{2\pi}{|A(\tilde{r}_0, -\infty, \epsilon, t)|} \int_{\tilde{r}_0}^{\tilde{r}_0 + \epsilon} v_z(r', -\infty, t) r' dr'$$

$$= \rho(\tilde{r}_0, z, t) u_z(\tilde{r}_0, -\infty, t).$$
Thus we have the following proposition.

**Proposition 2.4.** We have the following formula of $v_z$ and $v_r$:

\begin{equation}
(2.3) \quad v_z(r, z, t) = \rho(\bar{R}^{-1}(r, z, t), z, t) v_z(\bar{R}^{-1}(r, z, t), -\infty, t) = \rho(\bar{R}^{-1}, z, t) g(t)
\end{equation}

and

\begin{equation}
(2.4) \quad v_r(r, z, t) = (\partial_2 \bar{R})(\bar{R}^{-1}(r, z, t), z, t) v_z(r, z, t).
\end{equation}

By the above proposition, we see

$$\beta^{-6} \lesssim |v_z| \lesssim \beta^{-2} \epsilon^{-5} \quad \text{and} \quad |v_r| \lesssim \beta^{-3} \epsilon^{-5}.$$  

We now define the Lagrangian flow along $r, z$-direction. Let

\begin{equation}
(2.5) \quad \frac{d}{dt} Z(t) = v_z(R(t), Z(t), t),
\end{equation}

and

\begin{equation}
(2.6) \quad \frac{d}{dt} R(t) = v_r(R(t), Z(t), t),
\end{equation}

with $Z(t) = Z(r_0, z_0, t)$ and $R(t) = R(r_0, z_0, t)$. By the second assumption: $|\eta(x, t) - e_r| \geq \beta$, $R$ satisfies the following:

$$R(t) > \beta \quad \text{for} \quad t \in I.$$  

Since $v_z > 0$, then we can define the inverse of $Z$ in $t$: $t = Z^{-1}(z, r_0, z_0)$. In this case we can estimate $\partial_z Z^{-1} = 1/\partial_t Z = 1/v_z \lesssim \beta^6$ and $\partial^2 Z^{-1} = -\frac{2v_z}{\epsilon^2}$. First we show the following estimates.

**Lemma 2.5.** For $t \in I$, we have the following estimates along the axis-length trajectory:

\begin{equation}
(2.7) \quad \begin{cases}
\partial_z v_z(R(Z_t^{-1}(z)), z, Z_t^{-1}(z)) \approx (\partial_t Z)^{-1} \rho g' + \text{remainder}, \\
\partial^2 Z_t^{-1} \approx -(\partial_t Z)^{-3} \rho g' + \text{remainder}, \\
\partial^2 v_z(R(Z_t^{-1}(z)), z, Z_t^{-1}(z)) \approx (\partial_t Z)^{-2} \rho g'' + \text{remainder}, \\
|\partial_z v_z(R(Z_t^{-1}(z)), z, Z_t^{-1}(z))| \lesssim (\partial_t Z)^{-1} \rho g' + \text{remainder}, \\
|\partial^2 v_z(R(Z_t^{-1}(z)), z, Z_t^{-1}(z))| \lesssim (\partial_t Z)^{-2} \rho g'' + \text{remainder},
\end{cases}
\end{equation}

where “remainder” is small compared with the corresponding main term provided by small $\delta > 0$. Moreover, we have

\begin{equation}
(2.8) \quad v_\theta(R(Z_t^{-1}(z)), z, Z_t^{-1}(z)) \approx 1
\end{equation}

for some $z$ sufficiently close to $z_0$,

\begin{equation}
(2.9) \quad \begin{cases}
|\partial_z v_\theta(R(Z_t^{-1}(z)), z, Z_t^{-1}(z))| \lesssim C(\beta), \\
|\partial^2 v_\theta(R(Z_t^{-1}(z)), z, Z_t^{-1}(z))| \lesssim (\partial_t Z)^{-3} \rho g' + \text{remainder},
\end{cases}
\end{equation}

and

$$|\partial_t |u(\eta(x, t), t)| | \lesssim C(\beta) g' + \text{remainder}.$$
Proof: The estimates of \( v_r \) and \( v_z \) with several derivatives are just direct calculation using the formulas (2.3) and (2.4). The point is just extracting the main terms composed by \( g' \) or \( g'' \). Here we control \( v_\theta \) by using (2.14). By (1.6) we see that
\[
\partial_t v_\theta (R(t), Z(t), t) = v_\theta (r_0, z_0, 0) - \frac{v_r (R(t), Z(t), t) v_\theta (R(t), Z(t), t)}{R(t)}.
\]
Applying the Gronwall equality, we see
\[(2.10)\]
\[
v_\theta (R(Z^{-1}_t(z)), z, Z^{-1}_t(z)) = v_\theta (r_0, z_0, 0) \exp \left\{ - \int_0^{Z^{-1}_t(z)} \frac{v_r (R(\tau), Z(\tau), \tau)}{R(\tau)} d\tau \right\}.
\]

Remark 2.6. To obtain the formula of \( v_r \) and \( v_z \), we are only using the function \( R, g \) and Gauss’s divergence theorem. Namely, we do not need to use the Euler equations. However, to obtain the above formula of \( v_\theta \), we essentially use the Euler equation (1.6). Thus, to pursue the the Navier-Stokes flow case, we need some new idea.

Since \( |v_r| \lesssim \beta^{-3} \varepsilon^{-5} \), \( R(t) > \beta \) and choose \( z \) sufficiently close to \( z_0 \), we have (2.8). More precisely, we choose a point \( Z(t) \) such that
\[
|Z(t) - z_0| \lesssim \beta^{-4} \varepsilon^5.
\]
In this case, by \( Z(t) = z_0 + \int_t^t v_z (R(\tau), Z(\tau), \tau) d\tau \) and \( |v_z| \gtrsim \beta^{-6} \), the time interval \( I \) always satisfies \( |I| \lesssim \beta^2 \varepsilon^5 \). Just taking derivatives to (2.10) in \( z \)-valueable, then we also have (2.9). Now we estimate \( \partial_t |u(\eta, t)| \). We set the usual trajectory \( \eta(x, t) \) using smooth functions \( R, Z \) and \( \Theta \):
\[
\eta(x, t) = (R(t) \cos \Theta(t), R(t) \sin \Theta(t), Z(t))
\]
with \( e_\theta = (-\sin \Theta(t), \cos \Theta(t), 0) \) and \( e_r = (\cos \Theta(t), -\sin \Theta(t), 0) \). Then, by a direct calculation with \( u = v_r e_r + v_\theta e_\theta + v_z e_z \), we see that
\[(2.11)\]
\[
\frac{1}{2} \partial_t |u(\eta(x, t), t)|^2 = \partial_t u \cdot u = \partial_t v_r v_r + \partial_t v_\theta v_\theta + \partial_t v_z v_z
\]
along the trajectory. In fact, since
\[
\partial_t \eta = \partial_t R(\cos \Theta, \sin \Theta, Z) + \partial_t \Theta (-R \sin \Theta, R \cos \Theta, Z),
\]
and
\[
v_\theta = \partial_\theta \eta \cdot e_\theta = \partial_\theta R,
\]
we see \( \partial_t \Theta = v_\theta / R \). We multiply \( u = v_r e_r + v_\theta e_\theta + v_z e_z \) to
\[
\partial_t u = \partial_t v_r e_r + \partial_\theta v_\theta e_\theta + \partial_t v_z e_z + v_r \partial_r \Theta e_\theta - v_\theta \partial_\theta \Theta e_r,
\]
then we have (2.11). Just take a time derivative to \( v_z \) along the trajectory, then we have
\[
\partial_t (v_z (R(t), Z(t), t)) = \rho g' + \text{remainder}.
\]
Here we used the fact that \( L^t \lesssim 1/\varepsilon \). Thus
\[
\partial_t v_z v_z = \rho^2 g' g + \text{remainder} \quad \text{for} \quad t \in I.
\]
The remainder becomes small compare with the main term provided by small \( \delta > 0 \). By the similar calculation,
\[
\partial_t v_r v_r = (\partial_\tau R)^2 \rho^2 g' g + \text{remainder} \quad \text{for} \quad t \in I.
\]
Here we also used the fact that $L^t \lesssim 1/\epsilon$. Clearly $|\partial_t v_\theta| \lesssim \beta^{-4} \epsilon^{-5}$ and then $|\partial_t v_\theta v_\phi| \lesssim \beta^{-4} \epsilon^{-5}$. Thus we have

$$\partial_t |u(\eta(x,t),t)|^2 \approx (1 + (\partial_z \tilde{R})^2)g'g + \text{remainder} \quad \text{for} \quad t \in I$$

and then

$$\partial_t |u(\eta(x,t),t)| \approx \rho g' + \text{remainder} \quad \text{for} \quad t \in I.$$  

\[\square\]

Our strategy is to estimate the curvature and torsion of the arc-length particle trajectory. To do so, we need to define the axis-length trajectory $\tilde{\eta}$ in $z$.

**Definition 2.7.** (Axis-length trajectory.) Let $\tilde{\eta}$ be such that

$$\tilde{\eta}(z) := (r(z) \cos \theta(z), r(z) \sin \theta(z), z)$$

and we choose $r(z)$ and $\theta(z)$ in order to satisfy $\tilde{\eta}(z) = \eta(x, Z_t^{-1}(z))$.

For $t \in I$, we see

$$\partial_z \tilde{\eta} \cdot e_\theta = \frac{\partial \tilde{\eta} \cdot e_\theta}{v_z} = r \theta' = \frac{v_\theta(R(\tilde{Z}_t^{-1}(z)), z, Z_t^{-1}(z))}{v_z(R(\tilde{Z}_t^{-1}(z)), z, Z_t^{-1}(z))} \lesssim \beta^6,$$

$$\partial_z \tilde{\eta} \cdot e_r = \frac{v_r}{v_z} = (\partial_z \tilde{R})(\tilde{R}^{-1}, Z, t) = r', \quad |r'| \lesssim C(\beta, \epsilon) \quad \text{and} \quad |r''| \lesssim C(\beta, \epsilon)$$

with some positive constant $C(\beta, \epsilon)$ depending on $\beta$ and $\epsilon$. In particular we need the estimates of $\theta''$ and $\theta'''$. These estimates specify the curvature and torsion of the particle trajectory. By Lemma 2.5 we can immediately obtain the following proposition.

**Proposition 2.8.** By Lemma 2.5 we have (just see the highest order term)

$$\theta''(z) = -\frac{v_\theta \partial_z v_z}{v_z^2} + \text{remainder} = -C(\beta)g' + \text{remainder}$$

and

$$\theta'''(z) = -\frac{v_\theta \partial_z v_z}{v_z^2} + \frac{2v_\theta (\partial_z v_z)^2}{v_z^3} + \text{remainder}$$

$$= -C(\beta)g'' + \text{remainder} \quad \text{for} \quad t \in I,$$

where “remainder” is small compared with the corresponding main term provided by small $\delta > 0$.

From the trajectory $\eta(x,t)$, we define the arc-length trajectory $\eta^*(s) = \eta^*(x,s)$.

**Definition 2.9.** (Arc-length trajectory.) Let $\eta^*$ be such that

$$\eta^*(s) = \eta^*(x,s) := \eta(x,t(s)) \quad \text{and} \quad \eta^*(x,0) = \eta(x,0)$$

with $\partial_s t(s) = |u|^{-1}$.

In this case we see $|\partial_s \eta^*(s)| = 1$. We define the unit tangent vector $\tau$ as

$$\tau(s) = \partial_s \eta^*(x,s),$$

the unit curvature vector $n$ as $kn = \partial_s \tau$ with a curvature function $\kappa(s) > 0$, the unit torsion vector $b$ as $b(s) := \pm \tau(s) \times n(s)$ ($\times$ is an exterior product) with a torsion function to be positive $T(s) > 0$ (once we restrict $T$ to be positive, then the direction of $b$ can be uniquely determined), that is,

$$Tb := \partial_s n + \kappa \tau, \quad |b| = 1$$
due to the Frenet-Serret formula. By the estimates of $\theta''$ and $\theta'''$ in Proposition 2.8, we obtain the following key estimates.

**Lemma 2.10.** For any $\epsilon > 0$, we have

\[
(1/6)|u|^2|\partial_s \kappa| > \kappa \partial_t |u|, (1/2)|\partial_s \kappa| > |\kappa T b \cdot e_\theta| \quad \text{and} \quad -1 \leq (n \cdot e_\theta) < -1/2
\]

for sufficiently small $\delta > 0$.

**Proof.** Recall the arc-length trajectory ($z = z(s)$):

\[
\eta^*(x, s) = \tilde{\eta}(x, z) = (r(z) \cos \theta(z), r(z) \sin \theta(z), z) \quad \text{with} \quad \theta' > 0.
\]

Thus $\tau$ and $\kappa n$ are expressed as

\[
\tau = (\partial_z \tilde{\eta})z', \quad \kappa n = \partial_s^2 \eta^* = \partial_z^2 \tilde{\eta}(z')^2 + \partial_z \tilde{\eta}z''.
\]

Also we immediately have the following formula:

\[
\partial_z^3 \eta^* = \partial_s (\kappa n) = \partial_z^3 \tilde{\eta}(z')^3 + 3 \partial_z^2 \tilde{\eta}z'\tilde{\eta}z''.
\]

We recall that

\[
\partial_z \tilde{\eta} \cdot e_z = 1, \quad \partial_z \tilde{\eta} \cdot e_\theta = r \theta' = \frac{u_\theta}{u_z} \lesssim \beta^0
\]

and then we see that

\[
\theta' \lesssim \beta^0 \quad \text{and} \quad \theta'' \approx -C(\beta)g'.
\]

Also recall that

\[
\partial_z \tilde{\eta} \cdot e_r = r' = \frac{u_r}{u_z} = (\partial_z \tilde{R})(\tilde{R}^{-1}, Z, t).
\]

Direct calculations yield (just see the highest order term composed by $\theta''$ or $\theta'''$, and neglect the small terms composed by $\theta'$)

\[
\partial_z \tilde{\eta}(x, z) = (-r \theta' \sin \theta, r \theta' \cos \theta, 1) + (r' \cos \theta, r' \sin \theta, 0),
\]

\[
\partial_z^2 \tilde{\eta}(x, z) = -r(\theta')^2 (\cos \theta, \sin \theta, 0) + (-r \theta'' \sin \theta, r \theta'' \cos \theta, 0)
\]

\[
+ r''(\cos \theta, \sin \theta, 0) + 2r \theta' (-\sin \theta, \cos \theta, 0)
\]

\[
= r \theta''' (-\sin \theta, \cos \theta, 0) \quad \text{remainder,}
\]

\[
\partial_z^3 \tilde{\eta}(x, z) \approx r \theta''' (-\sin \theta, \cos \theta, 0) \quad \text{remainder,}
\]

\[
z'(s) = |\partial_z \tilde{\eta}|^{-1} = (1 + (r')^2 + (r')^2)^{-1/2} = (1 + (r')^2)^{-1/2} \quad \text{remainder,}
\]

\[
z''(s) = -(1 + (r')^2 + (r')^2)^{-2}(r' r'' + r\theta'(r' \theta' + r\theta''))
\]

\[
= -(1 + (r')^2)^{-2} r^2 \theta' \theta'' \quad \text{remainder,}
\]

\[
z'''(s) \approx C(\beta) \theta''' \quad \text{remainder.}
\]

Therefore

\[
\partial_z^2 \eta^* \cdot e_\theta = \kappa n \cdot e_\theta
\]

\[
= r \theta''' (1 + (r')^2)^{-1} - r \theta' (1 + (r')^2)^{-2} r^2 \theta' \theta'' \quad \text{remainder,}
\]

\[
\kappa^2 = |\kappa n|^2 = |\partial_z^2 \tilde{\eta}(z')^4 + 2(\partial_z \tilde{\eta} \cdot \partial_z^2 \tilde{\eta})(z')^2 z'' + |\partial_z \tilde{\eta}|^2 (z'')^2
\]

\[
= (r \theta'')^2 (1 + (r')^2)^{-2} + ((r')^2 + 1)(1 + (r')^2)^{-4} (r^2 \theta' \theta'')^2 \quad \text{remainder}
\]

and

\[
n \cdot e_\theta = \frac{\kappa n \cdot e_\theta}{\kappa}
\]

\[
= -\frac{r(1 + (r')^2)^{-1} - (1 + (r')^2)^{-2} r^2 \theta'}{(r^2 (1 + (r')^2)^{-2} + (1 + (r')^2)^{-3} r^4 (\theta')^2)^{1/2}} \quad \text{remainder.}
\]
The remainders are small provided by small $\delta$. We choose $\beta$ (which is independent of $\epsilon$ and $\delta$) such that the main term in $n \cdot e_{\theta}$ to be strictly smaller than $-1/2$ (by taking small $\beta$, then $\theta'$ becomes small). Thus we have

\begin{equation}
(2.12) \quad n \cdot e_{\theta} = \frac{\kappa n \cdot e_{\theta}}{\kappa} < -1/2.
\end{equation}

Also recall that $\theta''' \approx -C(\beta)g''$. The dominant term of $\partial_{s}(\kappa'^{2})$ is composed by $\theta''$ and $\theta''''$, more precisely,

\begin{equation}
\partial_{s}(\kappa'^{2}) = 2(\partial_{s}\kappa)\kappa = 2r\theta''''(r\theta'')(1 + (r')^2)^{-5/2} + \text{remainder} \approx C(\beta)\theta''\theta''''.
\end{equation}

We also see that

\begin{align*}
\kappa &= |r\theta''|(1 + (r')^2)^{-1} + \text{remainder}, \\
\kappa n \cdot e_{\theta} &= r\theta''(1 + (r')^2)^{-1} + \text{remainder}, \\
\partial_{s}(\kappa n) \cdot e_{\theta} &= \partial_{s}\bar{\eta} \cdot e_{\theta} = r\theta''''(1 + (r')^2)^{-3/2} + \text{remainder}, \\
\partial_{s}\kappa &= \frac{r\theta''''(r\theta'')(1 + (r')^2)^{-5/2}}{\kappa} + \text{remainder} \\
&= -r\theta''''(1 + (r')^2)^{-3/2} + \text{remainder}
\end{align*}

in $t \in I$. “remainder” is small compare with the main terms provided by small $\delta > 0$. We immediately obtain $(1/6)|u|^2|\partial_{s}\kappa| > \kappa\bar{\eta}|u|$ for sufficiently small $\delta > 0$, since the left hand side is $\delta^{-3}$-order, while, the right hand side is $\delta^{-2}$-order. In order to show $(1/2)|\partial_{s}\kappa| > |\kappa T b \cdot e_{\theta}|$, we use (2.12). By the Frenet-Serret formula,

\[ T b = \partial_{s} n + \kappa \tau, \]

we see that

\[ \kappa T b \cdot e_{\theta} = \partial_{s}(\kappa n) \cdot e_{\theta} - (\partial_{s}\kappa)n \cdot e_{\theta} + \kappa^{2}\tau \cdot e_{\theta}. \]

Thus, by the direct calculation with (2.12), we can find a cancellation on the highest order term composed by $\theta''''$, and then we have

\[ |\kappa T b \cdot e_{\theta}| < (1/2)|\partial_{s}\kappa| \]

for sufficiently small $\delta > 0$.

In what follows, we use a differential geometric idea. See Chan-Czubak-Y [4 Section 2.5], more originally, see Ma-Wang [7 (3.7)]. They considered 2D separation phenomena using fundamental differential geometry. The key idea here is “local pressure estimate” on a normal coordinate in $\bar{\theta}, \bar{r}$ and $\bar{z}$ valuables. Two derivatives to the scalar function $p$ on the normal coordinate is commutative, namely, $\partial_{s}\partial_{\bar{r}} p(\bar{\theta}, \bar{r}, \bar{z}) - \partial_{\bar{r}}\partial_{s} p(\bar{\theta}, \bar{r}, \bar{z}) = 0$ (Lie bracket). This fundamental observation is the key to extract the local effect of the pressure.

**Remark 2.11.** It should be noticed that Enciso and Peralta-Salas [3] considered the existence of Beltrami fields $u$ with a nonconstant proportionality factor $f$:

\begin{equation}
(2.13) \quad \nabla \times u = fu, \quad \nabla \cdot u = 0 \quad \text{in} \quad \mathbb{R}^3.
\end{equation}

It is well known that a Beltrami field is also a solution of the steady Euler equation in $\mathbb{R}^3$. They showed that for a generic function $f$, the only vector field $u$ satisfying (2.13) is the trivial one $u \equiv 0$. See (2.12), (3.4) and (3.6) in [4] for the specific condition on $f$. Note that $\gamma_{ij}$ (induced metric of the level set of $f$) is the fundamental component of the condition. It would be also interesting to consider whether we can apply their method to our unsteady flow problem, and compare with our method.
Lemma 2.13. We see

\[ \frac{\partial \hat{g}}{\partial x} = \tau + \hat{r}(Tb - \kappa \tau) + \hat{z}Kn, \]
\[ \frac{\partial \hat{e}}{\partial x} = n, \]
\[ \frac{\partial \hat{z}}{\partial x} = b. \]

This means that

\[ \begin{pmatrix} \frac{\partial \hat{g}}{\partial t} \\ \frac{\partial \hat{e}}{\partial t} \\ \frac{\partial \hat{z}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 - \kappa \hat{r} & \hat{z} \kappa & \hat{r}T(1 - \kappa \hat{r})^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ n \\ b \end{pmatrix}. \]

Remark 2.12. For any smooth scalar function \( f \), we have

\[ \frac{\partial \hat{g}}{\partial f}(x) = \nabla f \cdot \frac{\partial \hat{g}}{\partial x}. \]

\( \nabla f \) itself is essentially independent of any coordinates, thus we can regard a partial derivative as a vector.

By the fundamental calculation, we have the following inverse matrix:

\[
\begin{pmatrix} \tau \\ n \\ b \end{pmatrix} = \begin{pmatrix} (1 - \kappa \hat{r})^{-1} & -\hat{z}T(1 - \kappa \hat{r})^{-1} & -\hat{r}T(1 - \kappa \hat{r})^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{g}}{\partial t} \\ \frac{\partial \hat{e}}{\partial t} \\ \frac{\partial \hat{z}}{\partial t} \end{pmatrix}.
\]

Therefore we have the following orthonormal moving frame: \( \frac{\partial \hat{e}}{\partial t} = n, \frac{\partial \hat{z}}{\partial t} = b \) and \( (1 - \kappa \hat{r})^{-1}\frac{\partial \hat{g}}{\partial s} - \hat{z}T(1 - \kappa \hat{r})^{-1}\frac{\partial \hat{e}}{\partial s} - \hat{r}T(1 - \kappa \hat{r})^{-1}\frac{\partial \hat{z}}{\partial s} = \tau. \)

In order to abbreviate the complicated indexes, we re-define the absolute value of the velocity along the trajectory. Let (the indexes are \( x \) and \( t \) respectively)

\[ |u| := |u(\eta(x', t), t)| \text{ with } x' = \eta^{-1}(x, t) \]

and

\[ \left. \frac{\partial}{\partial t} |u| := \frac{\partial}{\partial t} |u(\eta(x', t'), t')| \right|_{t'=t} \text{ with } x' = \eta^{-1}(x, t). \]

Lemma 2.13. We see \(-\nabla p \cdot \tau = \frac{\partial}{\partial t} |u| \) along the trajectory.

Proof. Let us define a unit tangent vector \( \hat{\tau} \) (in time \( t' \)) as follows:

\[ \hat{\tau}_{x,t}(t') := \frac{u}{|u|}(\eta(x', t'), t') \text{ with } x' = \eta^{-1}(x, t). \]

Note that there is a re-parametrize factor \( s(t') \) such that

\[ \tau(s(t')) = \hat{\tau}(t'). \]

Since \( u \cdot \partial_u \tau = \tau \), we see that

\[ \left. \frac{\partial}{\partial t} |u(\eta(x', t'), t')| \right|_{t'=t} = \frac{\partial}{\partial t} (u(\eta(x', t'), t') \cdot \hat{\tau}_{x,t}(t')) = \left. \frac{\partial}{\partial t} (u(\eta(x', t'), t') \cdot \hat{\tau}_{x,t}(t')) + u(\eta(x', t'), t') \cdot \frac{\partial}{\partial t} \tau \frac{\partial}{\partial s} \right|_{t'=t} = \left. \frac{\partial}{\partial t} (u(\eta(x', t'), t')) \cdot \hat{\tau}_{x,t}(t') \right|_{t'=t}. \]

By the above calculation we have

\[ -\nabla p \cdot \tau = \left. \frac{\partial}{\partial t} (u(\eta(x', t'), t') \cdot \tau) \right|_{t'=t} = \left. \frac{\partial}{\partial t} (u(\eta(x', t'), t')) \cdot \hat{\tau}_{x,t}(t') \right|_{t'=t} = \left. \frac{\partial}{\partial t} |u| \right|_{t'=t}. \]

\( \square \)
Lemma 2.14. Along the arc-length trajectory, we have
\[3\kappa \partial_t |u| + \partial_s \kappa |u|^2 = \partial_z \partial_t |u|\]
and
\[T \kappa |u|^2 = \partial_z \partial_t |u|\].

Proof. By using the orthonormal moving frame, we have the following gradient of the pressure,
\[\nabla p = (\partial_r p)\tau + (\partial_n p)n + (\partial_b p)b.\]
Recall that
\[\partial_s t = |u|^{-1} \text{ and } \partial_t \eta \cdot \tau = |u|.\]
By the unit normal vector with the curvature constant, we see
\[\kappa n = \partial_s^2 \eta^* = \partial_s (\partial_t \kappa \partial_s t) = \partial_s^2 \kappa (\partial_s t)^2 + \partial_t \kappa \partial_s^2 t.\]
Thus we have
\[-(\nabla p \cdot n) = (\partial_s^2 \eta \cdot n) = \kappa |u|^2,\]
\[-\partial_s (\nabla p \cdot n) = \partial_s (\kappa (\partial_s t)^2) = \partial_s \kappa (\partial_s t)^2 - 2\kappa (\partial_s t)^3 (\partial_s^2 t),\]
\[-\nabla p \cdot \tau = -|u|^3 \partial_s^2 t,\]
\[-\nabla p \cdot b = 0.\]
By Lemma 2.13 with \(0 = \kappa n \cdot \tau = \partial_s^2 \eta \cdot \tau (\partial_s t)^2 + \partial_t \kappa \cdot \tau (\partial_s^2 t),\) we also have
\[\partial_s^2 t = -|u|^{-3} \partial_t |u|.\]
Recall that
\[\partial_r = (1 - \kappa \bar{r})^{-1} \bar{\partial}_\theta - \bar{z} T(1 - \kappa \bar{r})^{-1} \bar{\partial}_\bar{r} - \bar{r} T(1 - \kappa \bar{r})^{-1} \bar{\partial}_\bar{z}.\]
Along the arc-length trajectory, we have
\[-\partial_r (\nabla p \cdot \tau) = -\partial_z \partial_r p\]
\[= -\kappa \bar{\partial}_\theta p - \partial_r \bar{\partial}_\theta p - T \partial_z p\]
(commute \(\partial_r \) and \(\partial_\theta\))
\[= -\kappa (\nabla p \cdot \tau) - \bar{\partial}_\theta (\nabla p \cdot n) - T (\nabla p \cdot b)\]
\[= -\kappa |u|^3 \partial_s^2 t + \partial_s \kappa (\partial_s t)^2 - 2\kappa (\partial_s t)^3 (\partial_s^2 t)\]
\[= 3\kappa \partial_t |u| + \partial_s \kappa |u|^2.\]
Since \(\nabla p \cdot b = \partial_z p \equiv 0\) along the trajectory, then
\[-\partial_z (\nabla p \cdot \tau) |_{\bar{r}, \bar{z} = 0} = -\partial_z \partial_r p |_{\bar{r}, \bar{z} = 0} = -\partial_z \bar{\partial}_\theta p - T \partial_r p = -T (\nabla p \cdot n) = T \kappa |u|^2.\]
By Lemma 2.13 along the arc-length trajectory \(\eta^*\), we have
\[3\kappa \partial_t |u| + \partial_s \kappa |u|^2 = -\partial_r (\nabla p \cdot \tau) |_{\bar{r}, \bar{z} = 0} = \partial_r \partial_t |u|\]
and
\[T \kappa |u|^2 = -\partial_z (\nabla p \cdot \tau) |_{\bar{r}, \bar{z} = 0} = \partial_z \partial_t |u|.\]

By using the above lemma we can finally prove the main theorem. Since
\[\partial_\theta = (e_\theta \cdot n)\partial_r + (e_\theta \cdot b)\partial_z\]
and the axisymmetric flow is rotation invariant (along the \(e_\theta\)-direction),
\[0 = \partial_\theta \partial_t |u| = (e_\theta \cdot n) \partial_r \partial_t |u| + (e_\theta \cdot b) \partial_z \partial_t |u|\]
\[= 3(e_\theta \cdot n) (\kappa \partial_t |u| + \partial_s \kappa |u|^2) + (e_\theta \cdot b) T \kappa |u|^2.\]
However, by Lemma 2.10 we have

\[ |3(e_\theta \cdot n)(\kappa \partial_t |u| + \partial_s \kappa |u|^2) + (e_\theta \cdot b) T \kappa |u|^2| \]
\[ \geq \frac{3 |\partial_s \kappa|^2 |u|^2}{2} - 3 \kappa |\partial_t |u|| - T \kappa |u|^2 \]
\[ \geq \frac{|\partial_s \kappa|^2 |u|^2}{2} > 0, \]

and it is in contradiction.

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References

1. D. Chae, On the Lagrangian dynamics for the 3D incompressible Euler equations, Comm. Math. Phys., 269, (2007), 557-569.
2. D. Chae, On the blow-up problem for the axisymmetric 3D Euler equations, Nonlinearity, 21, (2008), 2053-2060.
3. D. Chae, On the Lagrangian dynamics of the axisymmetric 3D Euler equations, J. Diff. Eq., 249 (2010), 571-577.
4. C-H. Chan, M. Czubak and T. Yoneda, An ODE for boundary layer separation on a sphere and a hyperbolic space, Physica D, 282 (2014), 34-38.
5. A. Enciso and D. Peralta-Salas, Beltrami fields with a nonconstant proportionality factor are rare, Arch. Rational Mech. Anal., 220 (2016), 243-260.
6. L. Formaggia, A. Quarteroni and A. Veneziani, Cardiovascular mathematics, modeling and simulation of the circulatory system, Springer-Verlag, Italia, Milano, 2009.
7. T. Ma and S. Wang, Boundary layer separation and structural bifurcation for 2-D incompressible fluid flows. Partial differential equations and applications, Discrete Contin. Dyn. Syst., 10 (2004), 459-472.
8. Y. Maekawa and A. Mazzucato, Inviscid limit and boundary layers for Navier-Stokes flows, to appear in Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Y. Giga and A. Novotný Ed., Springer; arXiv:1610.05372.
9. R. Trip, D.J. Kuik, J. Westerweel and C. Poelma, An experimental study of transitional pulsatile pipe flow, Phys. Fluids, 24 (2012), 014103.
10. J. R. Womersley, Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known, J. Physiol., 127 (1955), 553-563.

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