I. INTRODUCTION

Skyrme’s old idea [1] that baryons are chiral solitons has been successful in describing the static nucleon properties [2] since Witten’s illustration that the soliton picture of baryons is consistent with QCD in the large $N_c$ approximation [3]. The Skyrme model has been widely used to discuss baryons and baryonic-system properties. The Skyrme model with the product Ansatz has also been applied to the nucleon-nucleon interaction [4] and to the static properties of the deuteron [5]. At the classical level, the deuteron mass has been successfully calculated in $ab$ initio approach [6]. Atiyah and Manton [7] revealed a mathematically elegant connection between the Skyrme soliton and the instantons in the SU(2) gauge field, in which the holonomy of the instanton along all lines parallel to the Euclidean time axis generates the skyrmion with the same topological charge. Using the Ansatz generated in this approach, Leese et al. [8] reinvestigated the deuteron properties and gave an experimentally more agreeable result. However, since self-dual and antiself-dual equations allow only trivial solutions of zero topological charge, it is difficult to apply the Skyrme model to the interaction between skyrmion and anti-skyrmion in such a manner. The nucleon-antinucleon static potential was studied in the Skyrme model using the product Ansatz by Lu and Amado [10], who showed that the Skyrmie picture with the product Ansatz is the reasonable first step to obtain the real part of nucleon-antinucleon interaction.

Recently, the BES Collaboration observed a near-threshold enhancement in the proton-antiproton ($p\bar{p}$) mass spectrum from the radiative decay $J/\psi \rightarrow \gamma p\bar{p}$ [11]. This enhancement can be fitted with either an $S$- or $P$-wave Breit-Wigner resonance function. In the case of $S$-wave fit, the peak mass is at $M = 1859^{+3}_{-10}$(stat)$^{+5}_{-25}$(sys) with a total width $\Gamma < 30$ MeV/$c^2$ at 90% percent confidence level. The corresponding spin and parity are $J^{PC} = 0^{-+}$. Moreover, the Belle Collaboration also reported similar observations of the decays $B^+ \rightarrow K^+ p\bar{p}$ [12] and $B^0 \rightarrow D^0 p\bar{p}$ [13], showing enhancements in the $p\bar{p}$ invariant mass distributions near $2m_p$. These observations could be interpreted as signals for baryonium $p\bar{p}$ bound states [14] or flavorless gluon states [15]. There are also suggestions that the near-threshold enhancement of $p\bar{p}$ is due to final state interactions [16] or as a result of the quark fragmentation process [17].

In this paper, we provide a possible explanation that the enhancement might be explained as a baryonium $p\bar{p}$ bound state in a phenomenological potential inspired by the investigation of the static energy in the ungroomed $S^S$ channel. In Sec. 2, we scrutinize the nucleon-antinucleon static energy in the ungroomed $S^S$ channel from the SU(2) Skyrme model. In Sec. 3, inspired by this energy, we construct a phenomenological skyrmion-type potential, investigate the bound state in this potential and calculate the width by WKB approximation through quantum tunnelling effect. Then, in Sec. 4, we give our conclusion and discussion, with emphasis on the significant implication on the decay mode of the baryonium annihilation at rest.

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II. THE STATIC ENERGY IN THE UNGROOMED $S\bar{S}$ CHANNEL

The Lagrangian for the SU(2) Skyrme model is

$$\mathcal{L} = \frac{1}{16} F_{\rho}^2 \text{Tr}(\partial_{\mu} U \partial^\mu U) + \frac{1}{32e^2} \text{Tr}([\partial_{\mu} U]U^\dagger, [\partial_{\mu} U]U^1)|^2| + \frac{1}{8} m_{\rho}^2 F_{\rho}^2 \text{Tr}(U - 1),$$

where $U(t, x)$ is the SU(2) chiral field, expressed in terms of the pion fields:

$$U(t, x) = \sigma(t, x) + i\pi(t, x) \cdot \tau.$$  \hfill (2)

We fix the parameters $F_{\rho}$ and $e$ as in [2], and our units are related to conventional units via

$$\frac{F_{\rho}}{4e} = 5.58 \text{ MeV}, \quad \frac{2}{eF_{\rho}} = 0.755 \text{ fm}.$$  \hfill (3)

Skyrme avoided the obstacle of minimizing the energy by invoking the hedgehog Ansatz

$$U_H(r) = \exp[i \tau \cdot \hat{r} f(r)].$$  \hfill (4)

The product Ansatz describing the behavior of skyrmion and antiskyrmion system with relatively arbitrary rotation in the iso-space is of the form

$$U_s = C U_H(r + \frac{\rho}{2} \hat{z}) C^\dagger U_H^\dagger(r - \frac{\rho}{2} \hat{z}),$$  \hfill (5)

where $C$ is an element in the isospin SU(2) group. The skyrmion and the antiskyrmion are separated along the $\hat{z}$-axis by a distance $\rho$. The interactions between various groomed skyrmion and antiskyrmion have been studied in Ref. [10], in which the configuration lies in a manifold of relatively higher dimension. We find that the static potential in the ungroomed $S\bar{S}$ channel is physically more interesting with $C = I$, which satisfies

$$U \to 1 \text{ when } \rho \to 0.$$  \hfill (6)

Fixing our Ansatz as above, we can get the static energy from the skyrmion Lagrangian:

$$M(\rho) = \int d^3 r \left[ -\frac{1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}([R_i, R_j]^2) - m_{\rho}^2 \text{Tr}(U - 1) \right],$$  \hfill (7)

where $i, j = 1, 2, 3$. The right-currents $R_{\mu}$ are defined via

$$R_{\mu} = (\partial_{\mu} U)U^\dagger,$$  \hfill (8)

and we express the energy in the units defined above. In this picture, the binding energies for $S\bar{S}$ (which correspond to the classical binding energies of $p\bar{p}$) are

$$\Delta E_B = 2m_p - M(\rho),$$  \hfill (9)

where $m_p = 867 \text{ MeV}$ is the mass of a classical nucleon (or classical skyrmion). A stable or quasi-stable $p\bar{p}$-binding state corresponds to the skyrmion configuration $U_s(r, \rho_B) = U_H(r + \frac{\rho_B}{2} \hat{z}) U_H^\dagger(r - \frac{\rho_B}{2} \hat{z})$ with $\Delta E_B(\rho_B) < 0$ and $\frac{d}{d\rho_B}(\Delta E_B(\rho)|_{\rho = \rho_B}) = 0$.

The numerical result of the static energy as a function of $\rho$ is shown in Fig. 1. From it, we find that there is a quasi-stable $p\bar{p}$-binding state:

$$\rho_B \approx 2.5 \text{ fm},$$  \hfill (10)

$$\Delta E_B(\rho_B) \approx 10 \text{ MeV}.$$  \hfill (11)

In the above investigation, we actually conjecture the binding energy between $S$ and $\bar{S}$ under the Ansatz of $U_s(r, \rho_B) = U_H(r + \frac{\rho_B}{2} \hat{z}) U_H^\dagger(r - \frac{\rho_B}{2} \hat{z})$ being approximately the binding energy of $p\bar{p}$. This conjecture is based on the considerations in the studies of deuteron as a soliton in the Skyrme model done by Braaten and Carson [3]. In their works, the two-skyrmion configuration with $B = 2$ has been treated as a single soliton $U_2(r)$ by using the product Ansatz, and the spin- and isospin-quantum numbers of the physical states arise from the semiclassical quantization of the collective coordinates of the soliton. By this way, the deuteron state with $(I = 0, J = 1)$ has been
identified, and its mass (or its corresponding binding energy) has been obtained. In this formulation, the deuteron’s mass (or energies) is divided into two parts: (1) the classical energies of statical soliton configuration \( U_2(\mathbf{r}) \); (2) the semiclassical corrections. The latter one is suppressed in large \( N_c \)-expansion limit. Namely, the classical part of the energies of the skyrmion, \( i.e., \) the toroidal configuration is qualitatively dominant. Moreover, the detailed analysis by Forest et al. does indeed exhibit the toroidal configuration in their quantum mechanical deuteron wave function, which, though making up only a very small component, could have a revealing relation to certain aspects of QCD \( \text{III} \).

We argue that when one uses similar product Ansatz to deal with the skyrmion-antiskyrmion system, the situation should be similar to the case of skyrmion-skyrmion. Namely, the \((SS)\)-binding energy eq. \( \text{II} \) should roughly be the \((pp)\)-binding energy. And the spin- and isospin-quantum numbers of the \((pp)\)-bound state arise from the semiclassical quantization of the collective coordinates of \( U_s(\mathbf{r},\rho_B) \). Since \( U_s(\mathbf{r},\rho_B) \) serves as a classical soliton Ansatz of \((pp)\) (rather than \((NN)\) for deuteron case), it could be expected that the state with spin-0 and isospin-0 would be the lowest semi-classical state. However, the real situation might be complicated and we leave this subject to be one of the topics for future studies.

Of course, we should not take this potential too seriously for two reasons. On one hand, the product Ansatz is an exact solution to the equation of motion of the system only when \( \rho_B \to \infty \). On the other hand, even in the case of Deuteron \( \text{III} \), the toroidal configuration gives a binding energy of the order of the strong interaction scale, which is \( \sim \) pion decay constant \( (\sim 100 \text{ MeV}) \), not of the nuclear scale, \( \sim \) a few MeV, and the size and quadrupole moment are simply too small. However, the potential in Fig. 1 seems suggestive to give the observed near-threshold \( pp \) enhancement in experiments \( \text{III} \) a possible phenomenological explanation, as simple as possible, just like that in Ref. \( \text{II} \) based on the inspiration from it. Moreover, by such a phenomenological potential reflecting the character of Fig. 1, we can also obtain a picture about how the \( pp \) bound state decays: there is a \( pp \) bound state in a well, which will mostly annihilate through a barrier penetration by tunnel effect, and this will be discussed in detailed in the next section.

![FIG. 1](image_url) The static energy of skyrmion-antiskyrmion system, where \( m_c^p \) is the classical single skyrmion mass without quantum correction.

### III. A PHENOMENOLOGICAL MODEL WITH A SKYRMION-TYPE POTENTIAL

In this section, we employ the phenomenological model induced from the skyrmion picture of \((pp)\)-interactions for the nucleons. We favor such a potential as showed in Fig. 1 because it seems that the potential can be physically substantiated. For over fifty years there has been a general understanding of the nucleon-nucleon interaction as one in which there is, in potential model terms, a strong repulsive short distance core together with a longer range weaker attraction. The attractive potential at the middle range binds the neutron and the proton to form a deuteron. In comparison with the skyrmion result on the deuteron \( \text{III} \), we notice several remarkable features of the static energy \( M(\rho) \) and the corresponding \((pp)\)-potential \( V(\rho) \) . Firstly, the potential is attractive at \( \rho > 2.0 \text{ fm} \), similar to the deuteron case. This is due to the reason that the interaction via pseudoscalar \( \pi \)-meson exchange is attractive for both quark-quark \( qq \) and quark-antiquark \( q\overline{q} \) pairs. Physically, the attractive force between \( p \) and \( \overline{p} \) should be stronger than that of \((pp)\). Therefore the fact that our result of the \((pp)\)-binding energy (see eq. \( \text{II} \)) \( \Delta E_B(\rho_B) (\approx 10 \text{ MeV}) \) is larger than that of deuteron \( (2.225 \text{ MeV}) \) is quite reasonable physically. Secondly, there is a static skyrmion energy peak at \( \rho \sim 1 \text{ fm} \) in Fig. 1. This means that the corresponding potential between \( p \) and \( \overline{p} \) is repulsive at that range. This is an unusual and also an essential feature. The possible explanation for it is that the skyrmions are extended objects, and there would emerge a repulsive force to counteract the deformations of their configuration shapes when they close to each other. Similar repulsive potential has also been found in previous numerical calculation \( \text{III} \). Thirdly,
a well potential at middle $\rho$-range is formed due to the competition between the repulsive and attractive potentials mentioned above, similar to the deuteron case. But the depth should be deeper than that in the deuteron case, as argued in a QCD based discussion \[14\]. The $p$ and $\bar{p}$ will be bound to form a baryonium in this well potential. Finally, the potential turns to decrease quickly from $\sim 2000$ MeV to zero when $\rho \to 0$. This means that there is a strong attractive force at $\rho \sim 0$. Physically, $p\bar{p}$ are annihilated.

![FIG. 2: The skyrmion-type potential of $p\bar{p}$-system.](image)

The qualitative features of the proton-neutron potential for the deuteron can be well described by a simple phenomenological model of a square well potential \[17,18,19\] with a depth which is sufficient to bind the $pn$ $^3S_1$-state with a binding energy of $-2.225$ MeV. Numerically, the potential width $\rho_{pm}$ is about 2.0 fm, and the depth is about $V_{pn} = 36.5$ MeV. Similarly, from the above illustration on the features of the potential between $p$ and $\bar{p}$ based on the Skyrmion picture, we now construct a phenomenological potential model for the $p\bar{p}$ system, as shown in Fig. 2, and it will be called as the skyrmion-type potential hereafter.

We take the width of the square well potential, denoted as $\rho_{p\bar{p}}$, as close to that of the deuteron, i.e., $\rho_{p\bar{p}} \sim \rho_{pn} \simeq 2.0$ fm. According to QCD inspired considerations \[14,20,21\], the well potential between $q$ and $\bar{q}$ should be double attractive than the $qq$-case, i.e., the depth of the $p\bar{p}$ square well potential is $V_{p\bar{p}} \simeq 2V_{pn} = 73$ MeV. The width for the repulsive force revealed by the Skyrme model can be fitted by the decay width of the baryonium, and we take it to be $\lambda = 1/(2m_p) \sim 0.1$ fm, the Compton wave length of the bound state of $p\bar{p}$. The square barrier potential begins from $\rho \sim \lambda$, and the height of the potential barrier, which should be constrained by both the decay width and the binding energy of the baryonium, is taken as $2m_p + h$, where $h \sim m_p/4$. At $\rho \sim 0$, $V(p\bar{p})(\rho) \sim -c \delta(\rho)$ with a constant $c > 0$.

Analytically, the potential $V(\rho)$ is expressed as follows

$$V(\rho) = 2m_p - c \delta(\rho) + V_c(\rho),$$

(12)

where

$$V_c(\rho) = \begin{cases} 
  h = m_p/4, & 0 < \rho < \lambda, \\
  -V_{p\bar{p}} = -73 \text{ MeV}, & \lambda < \rho < \rho_{p\bar{p}}, \\
  0, & \rho > \rho_{p\bar{p}}.
\end{cases}$$

(13)

With this potential, the Schrödinger equation for $S$-wave bound states is

$$-\frac{1}{2(m_p/2)} \frac{\partial^2}{\partial \rho^2} u(\rho) + [V(\rho) - E] u(\rho) = 0,$$

(14)

where $u(\rho) = \rho \psi(\rho)$ is the radial wave function, $m_p/2$ is the reduced mass. This equation can be solved analytically, and there are two bound states $u_1(\rho)$ and $u_2(\rho)$: $u_1(\rho)$ with binding energy $E_1 < -V_{p\bar{p}} = -73$ MeV is due to $-c \delta(\rho)$-function potential mainly, and $u_2(\rho)$ with binding energy $E_2 > -73$ MeV is due to the attractive square well potential at middle range mainly. $u_1(\rho)$ is the vacuum state, and, clearly, $u_2(\rho)$ should correspond to a deuteron-like molecule state and it may be interpreted as the new $p\bar{p}$ resonance reported by BES \[11\]. It is also expected that corresponding binding energies $-E_2$ in the potential model provided in above $\Delta E_B(\rho_B)$ (see Eq. \[11\]) are all in agreement of the data within errors of BES \[11\]. By fitting experimental data, we have

$$E_1 = -(2m_p - m_{0\pi}) \simeq -976 \text{ MeV},$$

(15)

$$E_2 = -17.2 \text{ MeV}.$$

(16)
Considering its decay width which will be derived soon (see Eq. (20)), we conclude that the near-threshold narrow enhancement in the $p\bar{p}$ invariant mass spectrum from $J/\psi \to \gamma p\bar{p}$ might be interpreted as a state of protonium in this potential model.

In the skyrmion-type potential of $p\bar{p}$, there are two attractive potential wells: one is at $\rho \sim 0$ and the other is at middle scale, together with a potential barrier between them. At $\rho \sim 0$, the baryon- and anti-baryon pair annihilates. The baryon-antibaryon annihilation has been studied in the Skyrme model in literature, see, e.g., Refs. [22, 23]. Naturally, we postulate that the bound states decay dominantly by annihilation and, therefore, we can derive the width of protonium state $u_2(\rho)$ by calculating the quantum tunnelling effect for $u_2(\rho)$ passing through the potential barrier. By WKB-approximation, the tunnelling coefficient (i.e., barrier penetrability) reads [19]

$$T_0 = \exp \left[ -2 \int_0^\lambda \frac{m_p}{\sqrt{m_p(h - E_2)}} \right]$$

In the square well potential from $\lambda$ to $a_{p\bar{p}}$, the time-period $\theta$ of round trip for the particle is

$$\theta = \frac{2 [a_{p\bar{p}} - \lambda]}{v} = [a_{p\bar{p}} - \lambda] \sqrt{\frac{m_p}{V_{p\bar{p}} + E_2}}.$$  \hspace{1cm} (17)

Thus, the state $u_2(r)$’s life-span is $\tau = \theta T_0^{-1}$, and hence the width of that state reads

$$\Gamma \equiv \frac{1}{\tau} = \frac{1}{a_{p\bar{p}} - \lambda} \sqrt{\frac{V_{p\bar{p}} + E_2}{m_p}} \exp \left[ -2\lambda \sqrt{m_p(h - E_2)} \right].$$  \hspace{1cm} (19)

Numerically, substituting $E_2 = -17.2$ MeV, $a_{p\bar{p}} = 2.0$ fm into (19), we obtain the prediction of $\Gamma$:

$$\Gamma \approx 15.5 \text{ MeV},$$

which is compatible with the experimental data [11].

IV. CONCLUSION

In conclusion, we investigated the $S\bar{S}$ static potential in the Skyrme model, and similar to the phenomenological potential model of the deuteron, we constructed a skyrmion-type potential model to study the recent discovery of a narrow $N\bar{N}$-resonance in the decay $J/\psi \to \gamma p\bar{p}$ by BES, and also in the decays $B^+ \to K^+ p\bar{p}$ and $B^0 \to D^0 p\bar{p}$ by Belle. The parameters in the model are guided by the parameters in the deuteron model and by QCD inspired considerations. We found that this skyrmion-type potential model has one baryonium solution, which might be explained as the $p\bar{p}$ bound state. By fitting the mass, we found that the width can be compatible with the experimental data.

We have learned from the studies in this paper that based on the skyrmion considerations, the baryonium decays are mainly due to the annihilations of nucleon-antinucleon. This is a significant feature for the particle decay of the baryonium. It has been well known that the nucleon-antinucleon annihilation at rest mostly favors processes with 4 to 7 pions in the final states over those with two or three pions [23]. Considering that the binding energy of the nucleon (or antinucleon) is rather small (compared with the mass of nucleon), the annihilation of the baryonium occurs nearly at rest. Therefore, we would predict that the baryonium hadronic decay should also mostly favor processes with 4 to 7 pseudoscalar mesons in the final states over those with two or three mesons, even though the phase spaces for the latter are larger than the former one. This prediction is non-trivial and needs to be tested by experiments.

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