Abstract

The Kakimizu complex, named after Osamu Kakimizu, is usually defined in the context of knots. Several recent results describe the geometric structure of this complex. In particular, Johnson, Pelayo and Wilson showed that the Kakimizu complex of a knot is quasi-Euclidean (see [11]). Prior to this, Przytycki and the author extended the definition of the Kakimizu complex to the context of 3-manifolds and showed that, even in this broader context, the Kakimizu complex is contractible (see [10]).

The goal of this paper is to adapt the definition of the Kakimizu complex to the setting of 2-manifolds in order to illustrate the techniques used in its study. The resulting complex is closely related to the homology curve complexes defined by Hatcher (see [5]) and Irmer (see [4]) and related to a complex studied by Bestvina-Bux-Margalit (see [2]) and Hatcher-Margalit (see [6]). Their insights translate into a geometric picture of the Kakimizu complex of a 2-manifold and can be promoted to the setting of the Kakimizu complex of certain 3-manifolds via product constructions.

Hatcher proved that the homology curve complex is contractible and computed its dimension. Irmer studied geodesics of the homology curve complex and exhibited quasi-flats. These insights translate into a geometric picture of the Kakimizu complex of a 2-manifold. Specifically, we prove similar, and in some cases analogous, results in the setting of the Kakimizu complex of a 2-manifold. We do so using the techniques used in the setting of the Kakimizu complex of a 3-manifold, most notably infinite cyclic covers corresponding to homology classes and a projection map on the set of vertices that can be defined on the Kakimizu complex via cut-and-paste in the infinite cyclic cover of the surface. As in the setting of 3-manifolds, the projection map, \( \pi_v \), provides the key ingredient for establishing contractibility of the Kakimizu complex.
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1 The Kakimizu complex of a surface

The work here follows in the footsteps of [10]. Whereas the setting for [10] is surfaces in 3-manifolds, the setting here is 1-manifolds in 2-manifolds. Recall that an element of a finitely generated free abelian group $G$ is \textit{primitive} if it is an element of a basis for $G$. In the following we will always assume: 1) $S$ is a compact (possibly closed) connected oriented 2-manifold; 2) $\alpha$ is a primitive element of $H_1(S, \partial S, \mathbb{Z})$.

\textbf{Definition 1.} A \textit{Seifert curve} for $(S, \alpha)$ is a union, $c$, of pairwise disjoint oriented simple closed curves and arcs in $S$ that represents $\alpha$. Moreover, we require that $S \setminus c$ is connected.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{seifert_curve.png}
\caption{A Seifert curve}
\end{figure}

Our definition of Seifert curve disallows null homologous subsets. Indeed, a null homologous subset would bound a component of $S \setminus c$ and would hence be separating. In fact, $c$ contains no bounding subsets. Conversely, if a representative $d$ of $\alpha$ contains no bounding subsets, then $S \setminus d$ is connected. Note that a Seifert curve, by definition, possesses an orientation (on each component).

\textbf{Definition 2.} For each pair $(S, \alpha)$, the isomorphism between $H_1(S, \partial S)$ and $H^1(S)$ identifies an element $a^*$ of $H^1(S)$ corresponding to $\alpha$ that lifts to a homomorphism $h_a : \pi_1(S) \to \mathbb{Z}$. We denote the covering space corresponding to $N_a = \ker(h_a)$ by $(p_a, \hat{S}_a, S)$, or simply $(p, \hat{S}, S)$, and call it the infinite cyclic covering space associated with $\alpha$.

We now describe the Kakimizu complex of $(S, \alpha)$. As \textit{vertices} we take isotopy classes of Seifert curves for $(S, \alpha)$. Consider a pair of vertices $v, v'$ and representatives $c, c'$. Here $S \setminus c$ and $S \setminus c'$ are connected, hence path-connected. It follows that lifts of
$S \setminus c$ and $S \setminus c'$ to the covering space associated with $\alpha$ are simply path components of $p^{-1}(S \setminus c)$ and $p^{-1}(S \setminus c')$. We span an edge $e = (v, v')$ on the vertices $v, v'$ if and only if there are representatives $c, c'$ of $v = [c], v' = [c']$ such that a lift of $S \setminus c$ to the covering space associated with $\alpha$ intersects exactly two lifts of $S \setminus c'$. (Note that in this case $c$ and $c'$ are necessarily disjoint.) See Figure 2.

**Definition 3.** Let $X$ be a simplicial complex. If, whenever the 1-skeleton of a simplex $\sigma$ is in $X$, the simplex $\sigma$ is also in $X$, then $X$ is said to be flag.

**Definition 4.** The Kakimizu complex of $(S, \alpha)$, denoted by $\text{Kak}(S, \alpha)$ is the flag complex with this graph as its 1-skeleton.

**Remark 5.** The Kakimizu complex is defined for a pair $(S, \alpha)$. For simplicity we use the expression “the Kakimizu complex of a surface” in general discussions, rather than the more cumbersome “the Kakimizu complex of a pair $(S, \alpha)$, where $S$ is a surface and $\alpha$ is a primitive element of $H_1(S, \partial S, \mathbb{Z})$”.

Figure 2: Two Seifert curves corresponding to vertices of distance 1

Figure 3 provides an example of a pair $(c, c')$ of disjoint (disconnected) Seifert curves that do not span an edge. The arc from one side of $c$ to the other side of $c$ intersects $c'$ twice with the same orientation and a lift of $S \setminus c$ will hence meet at least three distinct lifts of $S \setminus c'$. For a 3-dimensional analogue of Figure 3, see [1].

**Remark 6.** Let $S$ be a compact orientable hyperbolic surface with geodesic boundary. Let $\sigma$ be a simplex in $\text{Kak}(S, \alpha)$ of dimension $n$. Denote the vertices of $\sigma$ by $v_0, \ldots, v_n$ and let $c_0, \ldots, c_n$ be geodesic representatives of $v_0, \ldots, v_n$ such that arc components of $c_0, \ldots, c_n$ are perpendicular to $\partial S$. It is a well known fact that closed geodesics that can be isotoped to be disjoint must be disjoint. The same is true for the geodesic arcs considered here, and combinations of closed geodesics and geodesic arcs, because their doubles are closed geodesics in the double of $S$. I.e., $c_i \cap c_j = \emptyset \; \forall i, j, i \neq j$.

**Example 1.** The Kakimizu complexes of the disk and sphere are empty. The annulus has a non-empty but trivial Kakimizu complex $\text{Kak}(A, \alpha)$ consisting of a single
vertex. Specifically, let $A = \text{annulus}$, and $\alpha$ a generator of $H_1(A, \partial A, \mathbb{Z}) = \mathbb{Z}$. Then $\alpha$ is represented by a spanning arc. The spanning arc is, up to isotopy, the only representative of its homology class. Thus $Kak(A, \alpha)$ consists of a single vertex.

Similarly, the torus has non-empty but trivial Kakimizu complexes, each consisting of a single vertex. Specifically, let $T = \text{torus}$, and $\beta$ a primitive element of $H_1(T, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. Again, all representatives of $\beta$ are isotopic. There are infinitely many choices for $\beta$, but in each case, $Kak(T, \beta)$ consists of a single vertex.

**Definition 7.** Consider $Kak(S, \alpha)$. Let $(p, \hat{S}, S)$ be the infinite cyclic cover of $S$ associated with $\alpha$. Let $\tau$ be the generator of the group of covering transformations of $(p, \hat{S}, S)$ (which is $\mathbb{Z}$) corresponding to $1$. Note that $\tau$ is canonical up to sign.

Let $c, c'$ be Seifert curves in $(S, \alpha)$. Let $S_0$ denote a lift of $S \setminus c$ to $\hat{S}$, i.e., a path component of $p^{-1}(S \setminus c)$. Set $S_i = \tau^i(S_0)$, $c_i = \text{closure}(S_i) \cap \text{closure}(S_{i+1})$. We will assume, in what follows, that $\tau$ has been chosen so that $S_i$ lies to the left of $c_i$. Let $S'_0$ be a lift of $S \setminus c'$ to $\hat{S}$. Set $d_K(c, c) = 0$ and for $c \neq c'$, set $d_K(c, c')$ equal to one less than the number of translates of $S_0$ met by $S'_0$. Let $v, v'$ be vertices in $Kak(S, \alpha)$. Set $d_k(v, v) = 0$ and for $v \neq v'$ set $d_K(v, v')$ equal to the minimum of $d_K(c, c')$ for $c, c'$ representatives of $v, v'$.

**Remark 8.** Here $d_K(c, c')$ is finite. Indeed, $[c] = [c'] = \alpha$ and so $[c], [c']$ are in the kernel, $N_{\alpha}$, of the cohomology class dual to $\alpha$. Thus the homotopy classes of $c, c'$ lie in $N_{\alpha}$. More specifically, since the cohomology class dual to $\alpha$ is represented by the intersection pairing with $c$ and also the intersection pairing with $c'$, it is clear that each component of $c, c'$ lies in the kernel of this homomorphism and hence in $N_{\alpha}$. Thus lifts of $c, c'$ are homeomorphic to $c, c'$, respectively, in particular, they are compact 1-manifolds. It follows that $d_K(c, c')$ is finite, whence $d_K(v, v')$ is also finite.

It is not hard to verify, but important to note, the following theorem (see [7, Proposition 1.4]):

**Proposition 1.** (Kakimizu) The function $d_K$ is a metric on the vertex set of $Kak(S, \alpha)$.
Definition 9. Consider the setting of Definition 7. Let \( C, D \) be disjoint separating subsets of \( \hat{S} \). We say that \( D \) lies above \( C \) if \( D \) lies in the component of \( \hat{S} \setminus C \) containing \( \tau(C) \). Otherwise, we say that \( D \) lies below \( C \).

2 Relation to homology curve complexes

In [5], Hatcher introduces the cycle complex of a surface:

“By a cycle in a closed oriented surface \( S \) we mean a nonempty collection of finitely many disjoint oriented smooth simple closed curves. A cycle \( c \) is reduced if no subcycle of \( c \) is the oriented boundary of one of the complementary regions of \( c \) in \( S \) (using either orientation of the region). In particular, a reduced cycle contains no curves that bound disks in \( S \), and no pairs of circles that are parallel but oppositely oriented.

Define the cycle complex \( C(S) \) to be the simplicial complex having as its vertices the isotopy classes of reduced cycles in \( S \), where a set of \( k + 1 \) distinct vertices spans a \( k \)-simplex if these vertices are represented by disjoint cycles \( c_0, \ldots, c_k \) that cut \( S \) into \( k + 1 \) cobordisms \( C_0, \ldots, C_k \) such that the oriented boundary of \( C_i \) is \( c_{i+1} - c_i \), subscripts being taken modulo \( k + 1 \), where the orientation of \( C_i \) is induced from the given orientation of \( S \) and \( -c_i \) denotes \( c_i \) with the opposite orientation. The cobordisms \( C_i \) need not be connected. The faces of a \( k \)-simplex are obtained by deleting a cycle and combining the two adjacent cobordisms into a single cobordism. One can think of a \( k \)-simplex of \( C(S) \) as a cycle of cycles. The ordering of the cycles \( c_0, \ldots, c_k \) in a \( k \)-simplex is determined up to cyclic permutation. Cycles that span a simplex represent the same element of \( H_1(S) \) since they are cobordant. Thus we have a well-defined map \( \pi_0 : C(S) \to H_1(S) \). This has image the nonzero elements of \( H_1(S) \) since on the one hand, every cycle representing a nonzero homology class contains a reduced subcycle representing the same class (subcycles of the type excluded by the definition of reduced can be discarded one by one until a reduced subcycle remains), and on the other hand, it is an elementary fact, left as an exercise, that a cycle that represents zero in \( H_1(S) \) is not reduced. For a nonzero class \( x \in H_1(S) \) let \( C_x(S) \) be the subcomplex of \( C(S) \) spanned by vertices representing \( x \), so \( C_x(S) \) is a union of components of \( C(S) \).” See [5, Page 1].

Lemma 1. When both are defined, i.e., when \( S \) is closed, of genus at least 2, and \( \alpha \) is primitive, \( \text{Kak}(S, \alpha) \) is a subcomplex of \( C_\alpha(S) \).

Proof: The vertex set of \( \text{Kak}(S, \alpha) \) is a subset of the vertex set of \( C_\alpha(S) \): Indeed, the requirement on Seifert curves that \( S \setminus c \) be connected implies the requirement that multi-curves be reduced.

Let \((p, \hat{S}, S)\) be the covering space corresponding to \( \alpha \) and let \( \sigma \) be a simplex in \( \text{Kak}(S, \alpha) \). Denote the vertices of \( \sigma \) by \( v_0, \ldots, v_n \) and let \( c_0, \ldots, c_n \) be geodesic representatives of \( v_0, \ldots, v_n \) such that arc components of \( c_0, \ldots, c_n \) are perpendicular to \( \partial S \). By Remark \[6] \( c_i \cap c_j = \emptyset \) \( \forall i, j, i \neq j \). Consider a lift \( S_0 \) of \( S \setminus c_0 \) to \( \hat{S} \). For
each $j \neq 0$, $c_j$ lifts to a separating collection $\hat{c}_j$ of simple closed curves and simple arcs. Moreover, since $S_0$ is homeomorphic to $S \setminus c_0$, the lifts $\hat{c}_i, \hat{c}_j$ are disjoint as long as $i \neq j$. By reindexing $c_1, \ldots, c_n$ if necessary, we can thus ensure that $\hat{c}_i$ lies above $\hat{c}_j$ for $i > j$. This ensures that $c_0, \ldots, c_n$ form a cycle of cycles so that $\sigma$ is a simplex in $C_\alpha$.

**Corollary 2.** Let $\sigma$ be a $n$-simplex in $Kak(S, \alpha)$ and denote the vertices of $\sigma$ by $v_0, \ldots, v_n$. Then there are representatives $c_0, \ldots, c_n$ of $v_0, \ldots, v_n$ such that the following hold:

1. $c_i \cap c_j = \emptyset \ \forall i \neq j;
2. S \setminus (c_0 \cup \cdots \cup c_n)$ is partitioned into subsurfaces $P_0, \ldots, P_n$ such that $\partial P_i = c_i - c_{i-1}$.

**Proof:** The conclusion restates portions of Hatcher’s definition of simplex, hence this follows from the proof of Lemma 1.

Hatcher proves that for each $x \in H_1(S)$, $C_x(S)$ is contractible. (In particular, it is therefore connected and hence constitutes just one component of $C(S)$.) In Section 4 we prove an analogous result for $Kak(S, \alpha)$, using a technique from the 3-dimensional setting.

The cyclic cycle complex and the Kakimizu complex are simplicial complexes. The complex defined by Bestvina-Bux-Margalit (see [2]) is not simplicial, but can be subdivided to obtain a simplicial complex. See the final comments in Section 2. There is a subcomplex of the cyclic cycle complex that equals this subdivision of the complex defined by Bestvina-Bux-Margalit. Bestvina-Bux-Margalit and Hatcher-Margalit employ this complex to gain insights into the Torelli group.

In [4], Irmer defines the homology curve complex of a surface:

“Suppose $S$ is a closed oriented surface. $S$ is not required to be connected but every component is assumed to have genus $g \geq 2$.

Let $\alpha$ be a nontrivial element of $H_1(S, \mathbb{Z})$. The homology curve complex, $HC(S, \alpha)$, is a simplicial complex whose vertex set is the set of all homotopy classes of oriented multicurves in $S$ in the homology class $\alpha$. A set of vertices $m_1, \ldots, m_k$ spans a simplex if there is a set of pairwise disjoint representatives of the homotopy classes.

The distance, $d_H(v_1, v_2)$, between two vertices $v_1$ and $v_2$ is defined to be the distance in the path metric of the one-skeleton, where all edges have length one.” (See [4] Page 1.)

It is not hard to see the following (cf, Remark 6 and Figure 3):

**Lemma 2.** When both are defined, i.e., when $S$ is closed, each component of $S$ has genus at least 2 and $\alpha$ is primitive, $Kak(S, \alpha)$ is a subcomplex of $HC(S, \alpha)$. Moreover, for vertices $v, v'$ of $Kak(S, \alpha)$,

$$d_K(v, v') \geq d_H(v, v')$$
Irmer shows that distance between vertices of $\mathcal{HC}(S,\alpha)$ is bounded above by a linear function of the intersection number of representatives. The same is true for vertices of the Kakimizu complex. Irmer also constructs quasi-flats in $\mathcal{HC}(S,\alpha)$. Her construction carries over to the setting of the Kakimizu complex. See Section 7.

3 The projection map, distances and geodesics

In [10], Kakimizu defined a map on the vertices of the Kakimizu complex of a knot. He used this map to prove several things, for instance that the metric, $d_K$, on the vertices of the Kakimizu complex equals graph distance. (Quoted and reproved here as Theorem 3) In [10], Kakimizu’s map was rebranded as a projection map.

We wish to define

$$\pi_{\text{Vert}(\text{Kak}(S,\alpha))} : \text{Vert}(\text{Kak}(S,\alpha)) \to \text{Vert}(\text{Kak}(S,\alpha))$$

on the vertex set of $\text{Kak}(S,\alpha)$. Let $(p,\hat{S},S)$ be the infinite cyclic covering space associated with $\alpha$. Let $v,v'$ be vertices in $\text{Kak}(S,\alpha)$ such that $v \neq v'$. Here $v = [c]$ for some compact oriented 1-manifold $c$ and $v' = [c']$ for some compact oriented 1-manifold $c'$. We may assume that $c$ and $c'$ have been chosen so that $d_K(c,c') = d_K(v,v')$. Define $\tau, S_i, S_i', c_i$ and, by analogy, $c_i'$, as in Definition 7. Note that $S_i'$ also lies on to the left of $c_i'$.

Take $m = \max\{i \mid S_{i+1} \cap S_0' \neq \emptyset\}$. Consider a connected component $C$ of $S_{m+1} \cap S_0'$. Its frontier consists of a subset of $c_0'$ and a subset of $c_m$. The subset of $c_0'$ lies above the subset of $c_m$. In particular, $C$ lies to the right of $c_m$ and to the left of $c_0'$, hence the orientations of the subset of $c_0'$ are opposite those of the subset of $c_m$. See Figure 4. Because the subset of $c_0'$ and the subset of $c_m$ cobound $C$, they are homologous.

Replace the subset of the frontier of $S_{m+1} \cap S_0'$ that lies in $c_0'$ by the subset of the frontier of $S_{m+1} \cap S_0'$ that lies in $c_m$. Denote the result by $u(c_m,c_0')$. After a small isotopy, $u(c_m,c_0')$ no longer meets $S_{m+1}$ and lies in the interior of $S_0'$. Abusing notation slightly, we continue to denote the result of this small isotopy by $u(c_m,c_0')$. See Figure 5. Since $u(c_m,c_0')$ lies in $S_0'$, which is homeomorphic to $S\backslash c'$, $u(c_m,c_0')$ is homeomorphic to $p(u(c_m,c_0'))$ via $p|_{S_0'}$ and thus $p(u(c_m,c_0'))$ consists of pairwise disjoint oriented simple closed curves and arcs. Moreover, $u(c_m,c_0')$ is homologous to $c_0'$ via a cobordism that lies in $S_0'$. Each component $C$ of $S_{m+1} \cap S_0'$ also descends to a cobordism $p|_{S_0'}(C)$ and hence $c'$ is homologous to $p(u(c_m,c_0'))$. This proves the following:

**Lemma 3.** The homology class $[[p(u(c_m,c_0'))]] = \alpha$.

We make two observations: 1) A result of Oertel, see [9], shows that the isotopy class of $p(u(c_m,c_0'))$ does not depend on the choices made; 2) It is important to realize that $p(u(c_m,c_0'))$ may not be a Seifert curve, because $S\backslash p(u(c_m,c_0'))$ is not necessarily connected.
To choose a subset \( \hat{u}(c_m, c'_0) \) of \( u(c_m, c'_0) \) such that \( S \setminus p(\hat{u}(c_m, c'_0)) \) is connected, we proceed as follows: Below \( u(c_m, c'_0) \) lies one unbounded component of \( \hat{S} \setminus u(c_m, c'_0) \) and, possibly, bounded components. Suppose that \( C' \) is a bounded component of \( \hat{S} \setminus u(c_m, c'_0) \) that lies below \( u(c_m, c'_0) \). Its frontier consists of a subset of \( c'_0 \) and a subset of \( c_m \). Since \( C' \) lies to the left of \( c_m \) and to the left of \( c'_0 \), the subset of \( u(c_m, c'_0) \) lying in the frontier of \( C' \) is null homologous. See Figure 6. Furthermore, since \( C' \) lies in \( S'_0 \), the frontier of \( p|_{S'_0}(C') \) is also null homologous.

Denote by \( \hat{u}(c_m, c'_0) \) the subset of \( u(c_m, c'_0) \) consisting of those curves that lie in the frontier of the unbounded component of \( \hat{S} \setminus u(c_m, c'_0) \) that lies below \( u(c_m, c'_0) \). Then,

\[
\left[ [\hat{u}(c_m, c'_0)] \right] = \left[ [u(c_m, c'_0)] \right].
\]

This proves the following:

**Lemma 4.** The homology class \( [[p(\hat{u}(c_m, c'_0))]] \) = \( \alpha \) and \( S \setminus p(\hat{u}(c_m, c'_0)) \) is connected.

**Definition 10.** We denote the isotopy class \( [p(\hat{u}(c_m, c'_0))] \) by \( \pi_v(v') \).

**Lemma 5.** For \( v \neq v' \), the following hold:

\[
d_K(\pi_v(v'), v') = 1
\]

and

\[
d_K(\pi_v(v'), v) \leq d_K(v', v) - 1.
\]
It will follow from Theorem 3 below that the inequality is in fact an equality.

Proof: By construction, \( \hat{u}(c_m, c'_0) \) lies strictly between \( c'_0 \) and \( c'_{-1} \). So \( \tau(\hat{u}(c_m, c'_0)) \) lies strictly between \( c'_1 \) and \( c'_0 \). Thus the lift of \( S \setminus \hat{p}(\hat{u}(c_m, c'_0)) \) with frontier \( \hat{u}(c_m, c'_0) \cup \tau(\hat{u}(c_m, c'_0)) \) meets \( S'_0 \) and \( S'_1 \) and is disjoint from \( S'_i \) for \( i \neq 0, 1 \). Hence

\[
d_K(\pi_v(v'), v') = 1.
\]

In addition, suppose that \( c'_0 \cap S_i \neq \emptyset \) if and only if \( i \in \{n, \ldots, m+1\} \). Then \( c'_1 \cap S_i \neq \emptyset \) if and only if \( i \in \{n+1, \ldots, m+2\} \). Hence the lift of \( S \setminus c' \) that lies strictly between \( c'_0 \) and \( c'_1 \) meets exactly \( S_n, \ldots, S_{m+2} \).

By construction, \( \hat{u}(c_m, c'_0) \cap S_i \) can be non-empty only if \( i \in \{n, \ldots, m\} \) and thus \( \tau(\hat{u}(c_m, c'_0)) \cap S_i \) can be non-empty only if \( i \in \{n+1, \ldots, m+1\} \). Hence the lift of \( S \setminus \hat{p}(\hat{u}(c_m, c'_0)) \) with frontier \( \hat{u}(c_m, c'_0) \cup \tau(\hat{u}(c_m, c'_0)) \) can meet \( S_i \) only if \( i \in \{n, \ldots, m+1\} \). Whence

\[
d_K(\pi_v(v'), v) \leq m - n = d_K(v', v) - 1.
\]

Definition 11. The graph distance on a complex \( \mathcal{C} \) is a function that assigns to each pair of vertices \( v, v' \) the least possible number of edges in an edge path in \( \mathcal{C} \) from \( v \) to \( v' \).

Theorem 3. (Kakimizu) The function \( d_K \) equals graph distance.
**Proof:** Denote the graph distance between $v'$ and $v$ by $d(v', v)$. If $d_K(v', v) = 1$, then $d(v', v) = 1$ and vice versa by definition. So suppose $d_K(v', v) = m > 1$ and consider the path

$$v', \pi_v(v'), \pi_v^2(v'), \ldots, \pi_v^{m-1}(v'), \pi_v^m(v') = v.$$  

By Remark \[5\] $d_K(\pi_v(v'), v') = 1$ and $d_K(\pi_v^i(v'), \pi_v^{i-1}(v')) = 1$. Thus the existence of this path guarantees that $d(v', v) \leq m$. Hence $d(v', v) \leq d_K(v', v)$. Let $v' = v_0, v_1, \ldots, v_n = v$ be the vertices of a path realizing $d(v', v)$. By the triangle inequality and the fact that $d(v_{i-1}, v_i) = d_K(v_{i-1}, v_i)$:

$$d_K(v', v) \leq d_K(v_0, v_1) + \cdots + d_K(v_{n-1}, v_n) = 1 + \cdots + 1 = d(v_0, v_1) + \cdots + d(v_{n-1}, v_n) = d(v', v)$$

The following theorem is a reinterpretation of a theorem of Scharlemann and Thompson, see [12]:

**Theorem 4.** The Kakimizu complex is connected.

**Proof:** For vertices $v, v'$ in $Kak(S, \alpha)$, $d_K(v, v')$ is finite. Hence $d(v, v')$ is finite. Thus there is a path between $v$ and $v'$.

**Definition 12.** A geodesic between vertices $v, v'$ in a Kakimizu complex is an edge-path that realizes $d(v, v')$.  

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Figure 6: The setup with $c_m, c'_0$
Theorem 5. The path \( v', \pi_v(v'), \pi_v^2(v'), \ldots, \pi_v(v')^{m-1}, \pi_v^m(v') = v \) is a geodesic.

Proof: This follows from Theorem 3 because the path

\[ \pi_v'(v'), \pi_v^2(v'), \ldots, \pi_v^m(v') = v \]

realizes \( d(v', v) \).

Remark 13. Theorem 5 tells us that geodesics in the Kakimizu complex joining two given vertices are thus, at least theoretically, constructible. It follows that distances between two given vertices are also, theoretically, computable, as the length of the geodesic constructed.

Note that, typically, \( \pi_v(v') \neq \pi_{v'}(v) \). See Figure 9 for a step in the construction of \( \pi_{v'}(v) \).

4 Contractibility

The proof of contractibility presented here is a streamlined version of the proof given in the 3-dimensional case in [10]. Those familiar with Hatcher’s work in [5], will note certain similarities with his first proof of contractibility of \( C_\alpha(S) \) in the case that \( \alpha \) is primitive.

Lemma 6. Suppose that \( v, v^1, v^2 \) are vertices in \( \text{Kak}(S, \alpha) \). Then there are representatives \( c, c^1, c^2 \) with \( v = [c], v^1 = [c^1], \) and \( v^2 = [c^2] \) that realize \( d_K(v, v^1), d_K(v, v^2), \) and \( d_K(v^1, v^2) \).
Proof: Let \( c, c^1, c^2 \) be geodesic representatives of \( v, v^1, v^2 \) such that arc components of \( c, c^1, c^2 \) are perpendicular to \( \partial S \). Lifts of \( c, c^1, \) and \( c^2 \) to \( (p, \hat{S}, S) \), the infinite cyclic covering of \( S \) associated with \( \alpha \), are also geodesics. Points of intersection lift to points of intersection. Geodesics that intersect can’t be isotoped to be disjoint. Hence \( c, c^1, c^2 \) realize \( d_K(v, v^1), d_K(v, v^2), \) and \( d_K(v^1, v^2) \).

Lemma 7. Suppose that \( v, v^1, v^2 \) are vertices in \( \text{Kak}(S, \alpha) \) and \( d_K(v^1, v^2) = 1 \). Then \( d_K(\pi_v(v^1), \pi_v(v^2)) \leq 1 \).

Proof: By Lemma 6, we may choose representatives \( c, c^1, \) and \( c^2 \) of \( v, v^1 \) and \( v^2 \) to realize \( d_K(v, v^1), d_K(v, v^2), \) and \( d_K(v^1, v^2) \). Let \( (p, \hat{S}, S) \) be the infinite cyclic cover of \( S \) associated with \( \alpha \). Define \( \tau, S_i, S^1_i, S^2_i, c_i, c^1_i, c^2_i \) as in Definition 7 but this time label \( S^1_i, S^2_i \) so that \( S^1_0, S^2_0 \) meet \( S_1 \) and meet \( S_j \) only if \( j \leq 1 \).

Since \( c^1, c^2 \) realize \( d_K(v^1, v^2) \), they must be disjoint. Since \( c^1_0 \) is separating, \( c^2_0 \) lies either above or below \( c^1_0 \). Without loss of generality, we will assume that \( c^2_0 \) lies above \( c^1_0 \). See Figures 10 and 11. Proceeding as in the discussion preceding Lemma 6 construct \( u(c_0, c^1_0) \) and \( u(c_0, c^2_0) \). Then

\[
u(c_0, c^1_0) = (c^1_0 - (c^1_0 \cap Fr(S_1 \cap S^1_0))) \cup (c_0 \cap Fr(S_1 \cap S^1_0))
\]

We first show that \( u(c_0, c^1_0) \) and \( u(c_0, c^2_0) \) can be made disjoint. Here \( c^1_0 \cap Fr(S_1 \cap S^1_0) \) is disjoint from \( c^2_0 \cap Fr(S_1 \cap S^2_0) \), but \( c_0 \cap Fr(S_1 \cap S^1_0) \) is not necessarily disjoint from \( c_0 \cap Fr(S_1 \cap S^2_0) \). However, since \( c^2_0 \) lies above \( c^1_0 \), subsets of \( c_0 \cap Fr(S_1 \cap S^1_0) \) coincident
with subsets of $c_0 \cap Fr(S_1 \cap S_0^2)$ can be isotoped to lie below such subsets of $c_0 \cap Fr(S_1 \cap S_0^2)$. It then follows that $u(c_0, c_1^0)$ lies below $u(c_0, c_2^0)$, whence $\hat{u}(c_0, c_1^0)$ lies below $\hat{u}(c_0, c_0^2)$. In particular $\hat{u}(c_0, c_1^0)$ and $\hat{u}(c_0, c_2^0)$ are disjoint. Moreover, $\tau^{\pm 1}(\hat{u}(c_0, c_i^0))$ can be constructed from $c_{i+1}, c_{i-1}$ in an analogous fashion. This illustrates that the lift of $S \setminus p(\hat{u}(c_0, c_1^0))$ with frontier $\hat{u}(c_0, c_1^0) \cup \tau(\hat{u}(c_0, c_2^0))$ lies above $\tau^{-1}(\hat{u}(c_0, c_2^0))$ and below $\tau(\hat{u}(c_0, c_2^0))$ and hence meets at most two lifts of $S \setminus p(u(c_0, c_0^2))$. Since $\pi_v(v^1) = [p(\hat{u}(c_0, c_1^0))]$ and $\pi_v(v^2) = [p(\hat{u}(c_0, c_2^0))]$,

$$d_K(\pi_v(v^1), \pi_v(v^2)) \leq 1.$$

\[ \square \]

**Lemma 8.** If $d_K(v^1, v^2) = m$, then $d_K(\pi_v(v^1), \pi_v(v^2)) \leq m$.

**Proof:** Let $v^1 = v_0, v_1, \ldots, v_{m-1}, v_m = v^2$ be the vertices of a path from $v^1$ to $v^2$ that realizes $d_K(v^1, v^2)$. By Lemma \[\square\] $d_K(\pi_v(v_i), \pi_v(v_{i+1})) \leq d_K(v_i, v_{i+1}) = 1$ for $i = 0, \ldots, m - 1$. Hence

$$d_K(\pi_v(v^1), \pi_v(v^2)) \leq d_K(\pi_v(v_0), \pi_v(v_1)) + \cdots + d_K(\pi_v(v_{m-1}), \pi_v(v_m)) \leq d_K(v_0, v_1) + \cdots + d_K(v_{m-1}, v_m) \leq m.$$

\[ \square \]
Theorem 6. The Kakimizu complex of a surface is contractible.

Proof: Let $Kak(S, \alpha)$ be a Kakimizu complex of a surface. By a theorem of Whitehead, it suffices to show that every finite subcomplex of $Kak(S, \alpha)$ is contained in a contractible subcomplex of $Kak(S, \alpha)$. Let $\mathcal{C}$ be a finite subcomplex of $Kak(S, \alpha)$. Choose a vertex $v$ in $\mathcal{C}$ and denote by $\mathcal{C}'$ the smallest flag complex containing every geodesic of the form given in Theorem 5 for $v'$ a vertex in $\mathcal{C}$. Since $\mathcal{C}$ is finite, it follows that $\mathcal{C}'$ is finite.

Define $c : Vert(\mathcal{C}') \rightarrow Vert(\mathcal{C}')$ on vertices by $c(v') = \pi_v(v')$. By Lemma 7, this map extends to edges. Since $\mathcal{C}'$ is flag, the map extends to simplices and thus to all of $\mathcal{C}'$. By Lemma 8 this map is continuous. It is not hard to see that $c$ is homotopic to the identity map. Repeated application of $c$ yields a contraction map. (Specifically, $c^d$, where $d$ is the diameter of $\mathcal{C}'$, is a contraction map.)

5 Dimension

In [5], Hatcher proves that the dimension of $C_\alpha(S)$ is $2g(S) - 3$, where $g(S)$ is the genus of the closed oriented surface $S$. An analogous argument derives the same result in the context of $Kak(S, \alpha)$.

Lemma 9. Let $S$ be a closed orientable surface with genus($S$) $\geq 2$ and $\alpha$ a primitive class in $H_1(S, \partial S)$. The dimension of $Kak(S, \alpha)$ is $-\chi(S) - 1 = 2\text{genus}(S) - 3$. 

Figure 10: $c_0^1$ and $c_0^2$
Proof: It is not hard to build a simplex of $Kak(S, \alpha)$ of dimension $2\text{genus}(S) - 3$. See for example Figure 12. Thus the dimension of $Kak(S, \alpha)$ is greater than or equal to $2\text{genus}(S) - 3$.

Conversely, let $\sigma$ be a simplex of maximal dimension in $Kak(S, \alpha)$. Label the vertices of $\sigma$ by $v_0, \ldots, v_n$ and let $c_0, \ldots, c_n$ be geodesic representatives of $v_0, \ldots, v_n$. By Corollary 2, $S \setminus (c_0 \cup \cdots \cup c_n)$ consists of subsurfaces $P_0, \ldots, P_n$ with boundaries $c_0 - c_n, c_1 - c_0, \ldots , c_n - c_{n-1}$. Since $c_i$ and $c_{i-1}$ are not isotopic, no $P_i$ can consist of annuli. In addition, no $P_i$ can be a sphere, hence each must have negative Euler characteristic. Thus the number of $P_i$'s is at most $-\chi(S)$. I.e.,

$$n \leq -\chi(S) = 2\text{genus}(S) - 2.$$ 

In other words, the dimension of $\sigma$ and hence the dimension of $Kak(S, \alpha)$ is less than or equal to $2\text{genus}(S) - 3$. \hfill \qed

We can extend this argument to compact surfaces, by introducing the following notion of complexity:

**Definition 14.** Let $S$ be a compact surface and let $P$ be an open subset of $S$ whose boundary consists of open subarcs of $\partial S$ and, possibly, components of $\partial S$. Define

$$c(P, S) = -2\chi(P) + \text{number of open subarcs in } \partial P$$

The following lemma is immediate:
Figure 12: A simplex in a genus 3 surface

**Lemma 10.** Let $C$ be a union of simple closed curves and simple arcs in $S$. Then

$$c(S,S) = c(S \setminus C, S)$$

**Theorem 7.** Let $S$ be a compact orientable surface with $\chi(S) \leq -1$ and $\alpha$ a primitive class in $H_1(S, \partial S)$. The dimension of $Kak(S, \alpha)$ is $-2\chi(S) - 1 = 4\text{genus}(S) + 2b - 5$, where $b$ is the number of boundary components of $S$.

**Proof:** To build a simplex of $Kak(S, \alpha)$ of dimension $4\text{genus}(S) + 2b - 5$, see for example Figure 13. Thus the dimension of $Kak(S, \alpha)$ is greater than or equal to $4\text{genus}(S) + 2b - 5$.

Conversely, let $\sigma$ be a simplex of maximal dimension in $Kak(S, \alpha)$. Label the vertices of $\sigma$ by $v_0, \ldots, v_n$ and let $c_0, \ldots, c_n$ be geodesic representatives of $v_0, \ldots, v_n$ such that arc components of $c_0, \ldots, c_n$ are perpendicular to $\partial S$. By Corollary \[ S \setminus (c_0 \cup \cdots \cup c_n) \] consists of subsurfaces $P_0, \ldots, P_n$ with boundaries $c_0 - c_n, c_1 - c_0, \ldots, c_n - c_{n-1}$. Since $c_i$ and $c_{i-1}$ are not isotopic, $P_i$ can’t consist of annuli or disks with exactly two open subarcs of $\partial S$ in their boundary. In addition, no $P_i$ can be a sphere or a disk with exactly one open subarc of $\partial S$ in its boundary, hence each must have positive complexity. Thus the number of $P_i$’s is at most $c(S, S \setminus (c_0 \cup \cdots \cup c_n))$. I.e.,

$$n \leq c(S, S \setminus (c_0 \cup \cdots \cup c_n)) = c(S, S) = -2\chi(S).$$

In other words, the dimension of $\sigma$ and hence the dimension of $Kak(S, \alpha)$ is less than or equal to $-2\chi(S) - 1 = 4\text{genus}(S) + 2b - 5$. \[
\]

6 A computation

We consider the example of a closed orientable surface $S$ of genus 2. A non trivial primitive homology class $\alpha$ is represented by either a non separating simple closed curve (type 1) or by a pair of non separating disjoint simple closed curves (type 2). Distinct representatives of $\alpha$ of type 1 must intersect. A representative of type 1 can
be disjoint from a representative of type 2. Suppose that $c$ is a representative of type 1 that is disjoint from $d$, a representative of type 2. Then the three disjoint simple closed curves $c \cup d$ cut $S$ into pairs of pants. Any representative of $\alpha$ that is disjoint from $c \cup d$ must be parallel to either $c$ or $d$. Thus there are no other representatives that are simultaneously disjoint from both $c$ and $d$. Distinct representatives of $\alpha$ of type 2 must also intersect. It follows that $Kak(S, \alpha)$ has dimension $1 = (2)(2) - 3$, as prescribed by Theorem [9]. Note that, since $Kak(S, \alpha)$ is contractible, it is a tree.

Consider a representative $c$ of type 1. The link of $[c]$ in $Kak(S, \alpha)$ consists of equivalence classes of representatives of type 2. These representatives of type 2 are isotopy classes of pairs of curves that lie in $S \setminus c$, aren’t parallel to $c$, and are separating in $S \setminus c$ but not in $S$. There are infinitely many such pairs of curves. More specifically, $S \setminus c$ is a twice punctured torus, so the curves are parallel curves that separate the two punctures and can be parametrized by $Q$. Note that distinct such pairs of curves can’t be isotoped to be disjoint and hence have distance two in $Kak(S, \alpha)$.

Now consider a representative $d$ of type 2. Then $[d]$ consists of a pair of curves $d^1 \sqcup d^2$. We consider $S \setminus (d^1 \sqcup d^2)$, a sphere with four punctures. The link of $[d^1 \sqcup d^2]$ consists of isotopy classes of essential curves that are separating in $S \setminus (d^1 \sqcup d^2)$ but not in $S$ and that partition the punctures of $S \setminus (d^1 \sqcup d^2)$ as does $d$. There are infinitely many such curves. They too can be parametrized by $Q$. Note that distinct such curves can’t be isotoped to be disjoint and hence correspond to distance two vertices.
of $Kak(S, \alpha)$. In summary, $Kak(S, \alpha)$ is a tree each of whose vertices has a countably infinite discrete (i.e., 0-dimensional) link.

Recall that Johnson, Pelayo and Wilson showed that the Kakimizu complex of a knot in the 3-sphere is quasi-Euclidean. As the above computation shows, the Kakimizu complex of the genus 2 surface is hyperbolic and certainly not quasi-Euclidean.

7 quasi-flats

In this section we explore an idea of Irmer. See [4, Section 7].

Consider Figure 14. Denote the surface depicted by $S$ and the homology class of $c$ by $\alpha$. The curves $t_1$ and $t_2$ are homologous as are $t_3$ and $t_4$. Denote by $v$ the vertex corresponding to $c$, by $w_1$ the vertex corresponding to the result, $d_1$, obtained from $c$ by Dehn twisting $n$ times around $t_1$ and $-n$ times around $t_2$, and by $w_2$ the vertex corresponding to the result, $d_2$, obtained from $c$ by Dehn twisting $n$ times around $t_3$ and $-n$ times around $t_4$. Then $w_1, w_2$ are homologous to $v$, so we obtain three vertices $v, w_1, w_2$ in $Kak(S, \alpha)$. Note the following:

$$d(v, w_i) = d_K(v, w_i) = n$$
$$d(w_1, w_2) = d_K(w_1, w_2) = n$$

For $i = 1, 2$, we consider the geodesics $g_i$ with vertices $w_1, \pi_v(w_1), \ldots, \pi_v^n(w_1) = v$. In addition, consider the geodesic $g_3$ with vertices $w_2, \pi_w(w_2), \ldots, \pi_w^n(w_2) = w_1$ and note that $\pi_w^n(w_2)$ is represented by a curve obtained from $c$ by Dehn twisting $i$ times around $t_1, -i$ times around $t_2, n - i$ times around $t_3$ and $-(n - i)$ times around $t_4$.

**Definition 15.** Let $(X, d)$ be a metric space. A triangle is a 6-tuple $(v_1, v_2, v_3, g_1, g_2, g_3)$, where $v_1, v_2, v_3$ are vertices and the edges $g_1, g_2, g_3$ satisfy the following: $g_1$ is a distance minimizing path between $v_1$ and $v_2$, $g_2$ is a distance minimizing path between $v_2$ and $v_3$, $g_3$ is a distance minimizing path between $v_3$ and $v_1$.

A triangle $(v_1, v_2, v_3, g_1, g_2, g_3)$ is $\delta$-thin if each $e_i$ lies in a $\delta$-neighborhood of the other two edges. A metric space $(X, d)$ is $\delta$-hyperbolic if every triangle in $(X, d)$ is $\delta$-thin. It is hyperbolic if there is a $\delta > 0$ such that $(X, d)$ is $\delta$-hyperbolic.
For \( n \) even, the midpoint, \( m_1 \), of the geodesic \( g_1 \) is the vertex corresponding to the result, \( d'_1 \), obtained from \( c \) by Dehn twisting \( \frac{n}{2} \) times around \( t_1 \) and \(-\frac{n}{2} \) times around \( t_2 \). Likewise, the midpoint, \( m_2 \), of the geodesic \( g_2 \) is the vertex corresponding to the result, \( d'_2 \), obtained from \( c \) by Dehn twisting \( \frac{n}{2} \) times around \( t_3 \) and \(-\frac{n}{2} \) times around \( t_4 \). The midpoint, \( m_3 \), of \( g_3 \) is represented by a curve obtained from \( c \) by Dehn twisting \( \frac{n}{2} \) times around \( t_1 \) and around \( t_3 \) and \(-\frac{n}{2} \) times around \( t_2 \) and \( t_4 \).

**Lemma 11.** Let \( S \) be a compact oriented surface with genus 4. Then \( \text{Kak}(S, \alpha) \) is not hyperbolic.

**Proof:** The triangle \((v, w_1, w_2, g_1, g_2, g_3)\) described depends on \( n \), so we will denote it by \( T_n \). In \( T_n \) we have the following:

\[
\begin{align*}
    d(v, m_3) &= d_K(v, m_3) = n \\
    d(w_1, m_2) &= d_K(w_1, m_2) = n \\
    d(w_2, m_1) &= d_K(w_2, m_1) = n
\end{align*}
\]

In particular, \( g_3 \) is contained in a \( \delta \)-neighborhood of the two geodesics \( g_1, g_2 \) only if \( n \) is less than \( \delta \). Thus the triangle \( T_n \) in \( \text{Kak}(S, \alpha) \) is not \( \delta \)-thin for \( n \geq \delta \). It follows that \( \text{Kak}(S, \alpha) \) is not hyperbolic. \( \square \)

**Definition 16.** Let \((X, d)\) be a metric space. A quasi-flat in \((X, d)\) is a quasi-isometry from \( \mathbb{R}^n \) to \((X, d)\), for \( n \geq 2 \).

Note the following:

\[
\begin{align*}
    d(m_1, m_2) &= d_K(m_1, m_2) = \frac{n}{2} \\
    d(m_1, m_3) &= d_K(m_1, m_3) = \frac{n}{2} \\
    d(m_2, m_3) &= d_K(m_2, m_3) = \frac{n}{2}
\end{align*}
\]

Thus the triangle \( T_n \) scales like a Euclidean triangle. It is not too hard to see that a triangle with this property can be used to construct a quasi-isometry between \( \mathbb{R}^2 \) and an infinite union of such triangles lying in \( \text{Kak}(S, \alpha) \). Thus \( \text{Kak}(S, \alpha) \) contains quasi-flats. It is also not hard to adapt this construction to show that, for \( S \) an oriented surface, \( \text{Kak}(S, \alpha) \) is not hyperbolic and contains quasi-flats if the genus of \( S \) is greater than or equal to 4, or the genus of \( S \) is greater than or equal to 2 and \( \chi(S) \leq -6 \).
8 3-manifolds

Let $S$ be a compact oriented surface. Take $M = S \times I$. Then $M$ is homotopy equivalent to $S$ and hence has the same homology as $S$. Incompressible surfaces in a product manifold are either horizontal or vertical. Vertical surfaces have the form $c \times I$, where $c$ is a multi-curve in $S$. It follows that $\text{Kak}(M, [[c \times I]]) = \text{Kak}(S, [[c]])$, where $[[\cdot]]$ denotes the homology class of $\cdot$.

**Theorem 8.** There exist 3-manifolds with hyperbolic Kakimizu complex.

**Proof:** Let $S$ be the closed oriented surface of genus 2, $\alpha$ a primitive homology class in $H_1(S)$ and $c$ a compact 1-manifold representing $\alpha$. Then $\text{Kak}(S, \alpha)$ is the graph discussed in Section 6. In particular, $\text{Kak}(S, \alpha)$ is hyperbolic. Take $M = S \times I$. Then $\text{Kak}(M, [[c \times I]]) = \text{Kak}(S, \alpha)$ is also hyperbolic.

By the same type of reasoning, the product of a closed oriented surface of genus 4 with an interval can be shown to contain quasi-flats. However, the existence of 3-manifolds whose Kakimizu complex contains quasi-flats is already known, since the Kakimizu complex of a knot is quasi-Euclidean.
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