Asymmetric Systematic Errors

Roger Barlow

Department of Physics
Manchester University
England

Abstract

Asymmetric systematic errors arise when there is a non-linear dependence of a result on a nuisance parameter. Their combination is traditionally done by adding positive and negative deviations separately in quadrature. There is no sound justification for this, and it is shown that indeed it is sometimes clearly inappropriate. Consistent techniques are given for this combination of errors, and also for evaluating $\chi^2$, and for forming weighted sums.
1. Introduction

Although most errors on physics results are Gaussian, there are occasions where the Gaussian form no longer holds, and indeed when the distribution is not even symmetric.

This can occur for statistical errors, when the one-$\sigma$ interval is read off a log likelihood curve which is not well described by a parabola [1]. It can also arise in evaluating systematic errors: if a ‘nuisance parameter’ $a$ which affects the result $x$ has an uncertainty described by a Gaussian distribution with mean $\mu_a$ and standard deviation $\sigma_a$, then the uncertainty in $a$ produces an uncertainty in $x$ given to first order by the standard combination of errors formula:

$$\sigma_x^2 = \left( \frac{dx}{da} \right)^2 \sigma_a^2.$$

The uncertainty in $a$ may be frequentist (for example, a Monte Carlo parameter determined by another experiment) or Bayesian (for example, a Monte Carlo parameter set by judgement of theorists.) Bayesian probabilities may be admissible even in basically frequentist analyses if the effects are small [2]. The assumption that $a$ has a Gaussian probability distribution may be questioned, but that brings in further complications we do not wish to consider here.

If the differential is not known analytically a numerical evaluation can be done, most conveniently by evaluation of $x(\mu_a + \sigma_a)$ and $x(\mu_a - \sigma_a)$. See [3] for a discussion of the procedure and some issues that may arise.

Both $x(\mu_a + \sigma_a) - x(\mu_a)$ and $x(\mu_a) - x(\mu_a - \sigma_a)$ give estimates of the uncertainty $\sigma_x$. If they are different then this is a sign that the dependence is non-linear and the symmetric distribution in $a$ gives an asymmetric distribution in $x$.

The questions that can be asked are:

- How should asymmetric errors be combined?
- How should a $\chi^2$ be formed?
- How should a weighted mean be formed from results with asymmetric errors?

Current practice is to combine such errors separately, i.e. to add the $\sigma^+$ values together in quadrature, and then do the same thing for the $\sigma^-$ values. This is not, to my knowledge, documented anywhere and, as will be shown, is certainly wrong.

2. Models

The analysis gives 3 co-ordinate pairs: $(a - \sigma_a, x - \sigma_x^-), (a, x)$ and $(a + \sigma_a, x + \sigma_x^+)$.

In practice there are errors on these points, and one might be well advised to assume a straight line dependence and take the error as symmetric, however we will assume that this is not a case where this is appropriate. Again, faced with a real non-linear dependence one might well be advised to map out more than three points; we will likewise assume that this is not done. We consider cases where a non-linear effect is not small enough to be ignored entirely, but not large enough to justify a long and intensive investigation. Such cases are common in practice.

For simplicity we transform $a$ to the variable $u$ described by a unit Gaussian, and work with $X(u) = x(u) - x(0)$. For future convenience it is useful to define the mean $\sigma$,
the difference $\alpha$, and the asymmetry $A$:

$$\sigma = \frac{\sigma^+ + \sigma^-}{2} \quad \alpha = \frac{\sigma^+ - \sigma^-}{2} \quad A = \frac{\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-} \quad \quad (1)$$

There are infinitely many non-linear relationships between $a$ and $X$ that will go through these three points. We consider two.

**Model 1 : Two straight lines**

Two straight lines are drawn, meeting at the central value

$$X = \sigma^+ u \quad u \geq 0$$
$$= \sigma^- u \quad u \leq 0. \quad \quad (2)$$

**Model 2 : A quadratic function**

The parabola through the three points is

$$X = \sigma u + \alpha u^2 = \sigma u + A\sigma u^2. \quad \quad (3)$$

These forms are shown in Figure 1 for a small asymmetry of 0.1, and a larger asymmetry of 0.4.

![Figure 1: X (vertically) against u (horizontally)]
Model 1 (two straight lines) is shown in red, and Model 2 \((x\) as a quadratic function of \(u\)) in green. Both go through the 3 specified points. The differences between them within the range \(-1 \leq u \leq 1\) are not large; outside that range they diverge considerably.

We have no knowledge of whether either of them is better than the other in a particular case. Model 1 has kink at \(u = 0\) which is unphysical. Model 2 has a turning point, which may well be unrealistic (though it only gets into the relevant region if \(A\) is fairly large.) The practitioner may select one of the two - or some other model - on the basis of their knowledge of the problem, or preference and experience. Working with asymmetric errors at all involves the assumption of some model for the non-linearity. The ‘correctness’ of any model may be arguable, but once chosen it must be used consistently.

The distribution in \(u\) is a unit Gaussian, \(G(u)\), and the distribution in \(X\) is obtained from \(P(X) = \frac{G(u)}{|dX/du|}\). For Model 1 this gives a dimidated Gaussian - two Gaussians with different standard deviation for \(X > 0\) and \(X < 0\) †. For model 2 with small asymmetries the curve is a distorted Gaussian, given by \(\frac{G(u)}{|\sigma + 2\alpha u|}\) with \(u = \sqrt{\sigma^2 + 4\alpha X - \sigma}\). For larger asymmetries and/or larger \(|X|\) values, the second root also has to be considered. Examples are shown in Figure 2.

![Figure 2: Examples of the distributions from combined asymmetric errors.](image)

† This is sometimes called a ‘bifurcated Gaussian’, but this is inaccurate. ‘Bifurcated’ means ‘split’ in the sense of forked. ‘Dimidated’ means ‘cut in half’, with the subsidiary meaning of ‘having one part much smaller than the other’[4].
It can be seen that the Model 1 dimidated Gaussian and Model 2 distorted Gaussian are not dissimilar if the asymmetry is small, but are very different if the asymmetry is large. Again, in a particular case there is no unique reason for choosing one above the other in the absence of further information.

3. Bias

If a nuisance parameter $u$ is distributed with a Gaussian probability distribution, and the quantity $X(u)$ is a nonlinear function of $u$, then the expectation $\langle X \rangle$ is not $X(\langle u \rangle)$.

For model 1 one has

$$< X >= \int_{-\infty}^{0} \sigma^+ u e^{-u^2/2} \frac{du}{\sqrt{2\pi}} + \int_{0}^{\infty} \sigma^- u e^{-u^2/2} \frac{du}{\sqrt{2\pi}} = \frac{\sigma^+ - \sigma^-}{\sqrt{2\pi}} \tag{4}$$

For model 2 one has

$$< X >= \int_{-\infty}^{\infty} \alpha u^2 e^{-u^2/2} \frac{du}{\sqrt{2\pi}} = \frac{\sigma^+ - \sigma^-}{2} = \alpha \tag{5}$$

Hence in these models, or others, if the result quoted is $X(0)$, it is not the mean. It is perhaps defensible as a number to quote as the result as it is still the median - there is a 50% chance that the true value is below it and a 50% chance that it is above.

4. Adding Errors

If a derived quantity $z$ contains parts from two quantities $x$ and $y$, so that $z = x + y$, the distribution in $z$ is given by the convolution:

$$f_z(z) = \int dx f_x(x) f_y(z-x) \tag{6}$$

With Model 1 the function for $z \geq 0$ can be written:

$$f(z) = \int_{-\infty}^{0} dx f_x-(x) f_y+(z-x) + \int_{0}^{z} dx f_x+(x) f_y+(z-x) + \int_{z}^{\infty} dx f_x+(x) f_y-(z-x)$$

Inserting the appropriate Gaussian functions and using

$$\sigma^2_+ = \sigma^2_x + \sigma^2_y \quad \sigma^2_- = \sigma^2_x - \sigma^2_y \quad \sigma^2_\pm = \sigma^2_x + \sigma^2_y \quad \sigma^2_\mp = \sigma^2_x - \sigma^2_y$$

this gives

$$\sqrt{2\pi} f(z) = \frac{1}{\sigma_+} e^{\frac{-z^2}{2\sigma_+^2}} g(\frac{z\sigma_+}{\sigma_y \sigma_+}) + \frac{1}{\sigma_-} e^{\frac{-z^2}{2\sigma_-^2}} g(\frac{z\sigma_-}{\sigma_y \sigma_-}) + \frac{1}{\sigma_\pm} e^{\frac{-z^2}{2\sigma_\pm^2}} g(\frac{z\sigma_\pm}{\sigma_y \sigma_\pm})$$

where $g(x)$ is the cumulative Gaussian, equivalent to $\frac{1}{2}(1 + erf(x))$, and $g(x) = 1 - g(x)$. 

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For \( z \leq 0 \) the limits are different and the second region covers the case where \( x \) and \( y \) are both negative, giving

\[
\sqrt{2\pi} f(z) = \frac{1}{\sigma_+} e^{\frac{-z^2}{2\sigma_+^2}} g\left(\frac{z\sigma_+}{\sigma_x \sigma_+}\right) + \frac{1}{\sigma_-} e^{\frac{-z^2}{2\sigma_-^2}} \left( g\left(\frac{-z\sigma_-}{\sigma_y \sigma_-}\right) - g\left(\frac{z\sigma_+}{\sigma_x \sigma_-}\right) \right) + \frac{1}{\sigma_\pm} e^{\frac{-z^2}{2\sigma_\pm^2}} g\left(\frac{-z\sigma_\pm}{\sigma_y \sigma_\pm}\right)
\]

Figure 3: Examples of the distributions from combined asymmetric errors.

Figure 3 shows the distributions from some typical cases. The blue line shows the convolution, the black line is obtained by adding the positive and negative standard deviations separately in quadrature (the ‘usual procedure’).

The agreement is not good. It is apparent that the skew of the distribution obtained from the convolution is smaller than that obtained from the usual procedure. This is obvious: if two distributions with the same asymmetry are added then the ‘usual procedure’ will give a distribution with the same asymmetry. This violates the Central Limit Theorem, which says that convoluting identical distributions must result in a combined distribution which is more Gaussian, and therefore more symmetric, than its components. This shows that the ‘usual procedure’ for adding asymmetric errors is inconsistent. Even though, as stated earlier, there is no guarantee that Model 1 or any model is correct, once a model has been adopted it should be handled in a consistent fashion, and the ‘usual procedure’ fails to do this.
5. A consistent addition technique

If a distribution for $x$ is described by some 3 parameter function, $f(x; x_0, \sigma^+, \sigma^-)$, which is a Gaussian transformed according to Model 1 or Model 2 or anything else, then ‘combination of errors’ involves a convolution of two such functions according to Equation 6. This combined function is not necessarily a function of the same form. It is a special property of the Gaussian that the convolution of two Gaussians gives a third. Figure 3 is a demonstration of this. The convolution of two dimidated Gaussians is not a dimidated Gaussian.

Although the form of the function is changed by a convolution, some things are preserved. The semi-invariant cumulants of Thièlle (the coefficients of the power series expansion of the log of the Fourier Transform) add under convolution. The first two of these are the usual mean and variance. The third is the unnormalised skew:

$$\gamma = <x^3> - 3 <x> <x^2> + 2 <x>^3$$  \hspace{1cm} (7)

Within the context of any model, a rational approach to the combination of errors is to find the mean, variance and skew: $\mu, V$ and $\gamma$, for each contributing function separately. Adding these up gives the mean variance and skew of the combined function. Working within the model one then determines the values of $\sigma^-, \sigma^+$, and $x_0$ that give this mean, variance and skew.

5.1 Model 1

For Model 1, for which $\langle x^3 \rangle = \sqrt{2/\pi} (\sigma^3 + \sigma^-)$ we have

$$\mu = x_0 + \frac{1}{\sqrt{2\pi}} (\sigma^+ - \sigma^-)$$

$$V = \frac{1}{2} (\sigma^+^2 + \sigma^-^2) - \frac{1}{2\pi} (\sigma^+ - \sigma^-)^2 = \sigma^2 + \alpha^2 \left( 1 - \frac{2}{\pi} \right)$$

$$\gamma = \frac{1}{\sqrt{2\pi}} \left[ 2(\sigma^3 + \sigma^-^3) - 3 \frac{3}{2} (\sigma^+ - \sigma^-)(\sigma^+^2 + \sigma^-^2) + \frac{1}{\pi} (\sigma^+ - \sigma^-)^3 \right]$$  \hspace{1cm} (8)

So given a set of error contributions then the equations (8) give the cumulants $\mu, V$ and $\gamma$. The first three cumulants of the combined distribution are given by adding up the individual contributions. Then one can find the set of parameters $\sigma^-, \sigma^+, x_0$ which give these values by using Equations (8) in the other sense.

It is convenient to work with $\Delta$, where $\Delta$ is the difference between the final $x_0$ and the sum of the individual ones. The parameter is needed because of the bias mentioned earlier. Even though each contribution may have $x_0 = 0$, i.e. it describes a spread about the quoted result, it has non-zero $\mu_i$ through the bias effect (c.f. Equation 4). The $\sigma^+$ and $\sigma^-$ of the combined distribution, obtained from the total $V$ and $\gamma$, will in general not give the right $\mu$ unless a location shift $\Delta$ is added. The value of the quoted result will shift.

Recalling section 3, for a dimensioned Gaussian one could defend quoting the central value as it was the median, even though it was not the mean. The convoluted distribution not only has a non-zero mean, it also (as can be seen in Figure 2) has non-zero median. Consider two dimensioned Gaussians with, say, $\sigma^+ > \sigma^-$, which are convoluted. There is
a 25% chance that both will contribute a negative value, a similar 25% chance that both will be positive, and a 50% chance of getting one positive and one negative contribution - which will probably be positive overall (as $\sigma^+ > \sigma^-$). So for two combined distributions the zero value may lie as far away as the 25th percentile.

If you want to combine asymmetric errors then you have to accept that the quoted value will shift. To make this correction requires a real belief in the asymmetry of the error values. At this point the practitioner, unless they are really sure that their errors really do have a significant asymmetry, may be persuaded to revert to quoting symmetric errors.

Solving the Equations (8) for $\sigma^-, \sigma^+, x_0$ given $\mu, V$ and $\gamma$ has to be done numerically. If we write $D = \sigma^+ - \sigma^-$ and $S = \sigma^- + \sigma^+$ then the equations

$$S = 2V + D^2 / \pi$$
$$D = \frac{2}{3S} \left( \sqrt{2\pi \gamma} - D^3 \left( \frac{1}{\pi} - 1 \right) \right)$$

(9)

can be solved by repeated substitution (starting with $D = 0$). Then $\Delta$ is given by

$$\Delta = \mu - \frac{D}{\sqrt{2\pi}}$$

(10)

A program for this is available on \url{http://www.slac.stanford.edu/~barlow}. Some results are shown in Figure 4 and Table 1.

Figure 4: Examples of combined errors with the correct first 3 cumulants using Model 1.
Comparing Figure 4 and Figure 3 (note that the blue curves are the same in both figures; the consistent technique is shown in purple), it is apparent that the new technique does a very much better job than the old. It is not an exact match, but does an acceptable job given that there are only 3 adjustable parameters in the function.

5.2 Model 2

In terms of the difference \( \alpha = (\sigma^+ - \sigma^-)/2 \) and the mean \( \sigma = (\sigma^+ + \sigma^-)/2 \) the moments are

\[
<x> = \alpha \\
<x^2> = \sigma^2 + 3\alpha^2 \\
<x^3> = 9\alpha\sigma^2 + 15\alpha^3
\]

Giving

\[
\mu = x_0 + \alpha \\
V = \sigma^2 + 2\alpha^2 \\
\gamma = 6\sigma^2\alpha + 8\alpha^3
\]

As with Method 1, these are used to find the cumulants of each contributing distribution, which are summed to give the three totals, and then Equation 11 is used again to find the parameters of the distorted Gaussian with this mean, variance and skew. There is only one equation to be solved numerically, again by iteration

\[
\alpha = \frac{\gamma}{6V - 4\alpha^2}
\]

after which, \( \sigma = \sqrt{V - 2\alpha^2} \) and \( \Delta = \mu - \alpha \).

Some results are shown in Figure 5 and Table 2. The true convolution cannot be done analytically but can be done by a Monte Carlo calculation.

\[
\begin{array}{cccccccc}
\sigma_x^- & \sigma_x^+ & \sigma_y^- & \sigma_y^+ & \sigma^- & \sigma^+ & \Delta \\
1.0 & 1.0 & 0.8 & 1.2 & 1.32 & 1.52 & 0.08 \\
0.8 & 1.2 & 0.8 & 1.2 & 1.22 & 1.61 & 0.16 \\
0.5 & 1.5 & 0.8 & 1.2 & 1.09 & 1.78 & 0.28 \\
0.5 & 1.5 & 0.5 & 1.5 & 0.97 & 1.93 & 0.41 \\
\end{array}
\]

Table 2: The values used for the curves with correct cumulants in Figure 5.
Again the true curves (blue) are not well reproduced by the ‘usual procedure’ (black) whereas the curves with the correct cumulants (purple) do a very reasonable job. (The sharp behaviour at the lower edge of the curves is due to the minimum value of $y$.)

The web program mentioned earlier will also do the calculations for Model 2.

6. Evaluating $\chi^2$

For Model 1 the $\chi^2$ contribution from a discrepancy $\delta$ is just $\delta^2/\sigma^2$ or $\delta^2/\sigma^{-2}$ as appropriate. This is manifestly inelegant, especially for minimisation procedures as the value goes through zero.

For Model 2 one has

$$\delta = \sigma u + A\sigma u^2$$

This can be considered as a quadratic for $u$ with solution

$$u = \frac{\sqrt{1 + 4 \frac{\delta}{\sigma} A} - 1}{2A}$$

Squaring gives $u^2$, the $\chi^2$ contribution, as

$$u^2 = \frac{2 + 4A\frac{\delta}{\sigma} - 2(1 + 4A\frac{\delta}{\sigma})^{\frac{1}{2}}}{4A^2}$$
This is not really exact, in that it only takes one branch of the solution, the one approximating to the straight line, and does not consider the extra possibility that the $\delta$ value could come from an improbable $u$ value the other side of the turning point of the parabola. Given this imperfection it makes sense to expand the square root as a Taylor series, which, neglecting correction terms above the second power, leads to

$$
\chi^2 = \left(\frac{\delta}{\sigma}\right)^2 \left(1 - 2A\left(\frac{\delta}{\sigma}\right) + 5A^2\left(\frac{\delta}{\sigma}\right)^2\right).
$$

(13)

The first order approximation to this is

$$
\chi^2 = \left(\frac{\delta}{\sigma}\right)^2(1 - 2A\left(\frac{\delta}{\sigma}\right)).
$$

(14)

This can be modified to a form forced to give $\chi^2 = 1$ for deviations of $+\sigma^+$ and $-\sigma^-$. \[
\chi^2 = \delta^2\left(\frac{\sigma^+ + 3\sigma^-}{\sigma^+ \sigma^- (\sigma^+ + \sigma^-)}\right)\left(1 - \frac{\delta\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-}\right)
\]

(15)

Figure 6: $\chi^2$ approximations

Figure 6 shows these forms. The black line is the simplest $\chi^2 = (\frac{\delta}{\sigma})^2$ form. The green is the full form involving the square root. It goes to $+\infty$ for values beyond the turning point which in principle can never happen. The blue line is the third order form of Equation 14 and the red line is the higher order Equation 13. The yellow is Equation 15, the first order form constrained to go through unity at $+\sigma^+$ and $-\sigma^-$, shown by the
two crosses. For a 10% asymmetry all the approximations are pretty well equivalent and a significantly better form than the simplest one. For a larger 20% asymmetry the lower order forms show undesirable behaviour, turning over for a moderate (2σ) deviation.

We therefore suggest that Equation 13 be used. The even power ensures that χ^2 does not turn over but increases at large deviations, which is desirable. It does not go to infinity when δ approaches the turning point, which is probably a good feature. A poor determination of the parameters of Equation 3 could give an unrealistic minimum value which could be exceeded by an experimental value, and one would not want this to give an undefined χ^2.

Higher order (5th, 6th...) terms do not significantly improve the agreement with the full (green) curve.

7. Weighted means

Suppose a value x has been measured several times, x_1, x_2...x_N, each measurement having its own σ_i^+ and σ_i^- . For the usual symmetric errors the ‘best’ estimate (i.e. unbiassed and with smallest variance) is given by the weighted sum

$$\hat{x} = \frac{\sum w_i x_i}{\sum w_i}$$

with $w_i = 1/\sigma_i^2$. We wish to find the equivalent for asymmetric errors.

As noted in Section 3, when sampling from an asymmetric distribution the result is biassed towards the high tail. The expectation value $\langle x \rangle$ is not the location parameter $x$. So for an an unbiassed estimator one has to take

$$\hat{x} = \frac{\sum w_i (x_i - b_i)}{\sum w_i}$$

where

$$b = \frac{\sigma^+ - \sigma^-}{\sqrt{2\pi}}$$ (Model 1)  \quad b = \alpha \quad (Model 2) \quad (16)$$

The variance of this is given by

$$V = \frac{\sum w_i^2 V_i}{(\sum w_i)^2}$$

where $V_i$ is the variance of the $i^{th}$ measurement about its mean.

Differentiating with respect to $w_i$ to find the minimum gives

$$\frac{2 w_i V_i}{(\sum w_j)^2} - \frac{2 \sum w_j^2 V_j}{(\sum w_j)^3} = 0 \quad \forall i$$

which is satisfied by $w_i = 1/V_i$. This is the equivalent of the familiar weighting by $1/\sigma^2$. The weights are given by (see Equations 8 and 11)

$$V = \sigma^2 + (1 - \frac{2}{\pi})\alpha^2 \quad (Model 1) \quad V = \sigma^2 + 2\alpha^2 \quad (Model 2) \quad (17)$$

Note that this is not the ML estimator - writing down the likelihood in terms of the $\chi^2$ of Equation 13 and differentiating does not give to a nice form - so in principle there may be better estimators, but they will not have the simple form of a weighted sum.
8. Asymmetric statistical errors

When the estimated value and range are obtained using a maximum likelihood estimate and the shape of the log likelihood is not parabolic, the one standard deviation limits are taken as the points at which the log likelihood falls by 0.5 from its peak [1].

The treatment of these errors will be given in a subsequent publication. Although treatment of asymmetric errors involves, for both systematic and statistical errors, the mapping of the actual distribution onto a Gaussian one, there is a considerable difference of interpretation. It is, however, worth pointing out that if two separate statistical effects are combined - say two backgrounds from different sources - then the combined background is the simple arithmetic sum of the two with no shift to the central value. This is because, for these statistical errors, the value quoted is the mean.

9. Summary

The treatment of asymmetric systematic errors cannot be based on secure foundations, and if they cannot be avoided they need careful handling. The practitioner needs to choose a model for the dependence, which could be one of the two proposed here.

In combining asymmetric errors, the traditional procedure of adding positive and negative values separately in quadrature is unjustifiable. Instead, values should be determined which, within the limitations of the model, give the correct mean, variance, and skew. A program is available to do this on http://www.slac.stanford.edu/~barlow.

The $\chi^2$ contribution for a value with asymmetric errors can be represented by

$$\chi^2 = \left(\frac{\delta}{\sigma}\right)^2 \left(1 - 2A\frac{\delta}{\sigma} + 5A^2\left(\frac{\delta}{\sigma}\right)^2\right)$$

where

$$\sigma = \frac{\sigma^+ + \sigma^-}{2} \quad A = \frac{\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-}$$

In forming a weighted sum one should use

$$\hat{x} = \sum \frac{(x_i - b_i)}{V_i} / \sum \frac{1}{V_i}$$

where the bias $b$ and Variance $V$ are given by Equations 17 and 18 above.

References

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