TIGHT CLOSURE WITH RESPECT TO A MULTIPLICATIVELY CLOSED SUBSET OF AN F-PURE LOCAL RING

RODNEY Y. SHARP

ABSTRACT. Let $R$ be a (commutative Noetherian) local ring of prime characteristic that is $F$-pure. This paper studies a certain finite set $I$ of radical ideals of $R$ that is naturally defined by the injective envelope $E$ of the simple $R$-module. This set $I$ contains $0$ and $R$, and is closed under taking primary components. For a multiplicatively closed subset $S$ of $R$, the concept of tight closure with respect to $S$, or $S$-tight closure, is discussed, together with associated concepts of $S$-test element and $S$-test ideal. It is shown that an ideal $a$ of $R$ belongs to $I$ if and only if it is the $S'$-test ideal of $R$ for some multiplicatively closed subset $S'$ of $R$. When $R$ is complete, $I$ is also ‘closed under taking test ideals’, in the following sense: for each proper ideal $c$ in $I$, it turns out that $R/c$ is again $F$-pure, and if $g$ and $h$ are the unique ideals of $R$ that contain $c$ and are such that $g/c$ is the (tight closure) test ideal of $R/c$ and $h/c$ is the big test ideal of $R/c$, then both $g$ and $h$ belong to $I$. The paper ends with several examples.

0. Introduction

This paper is concerned with a (commutative Noetherian) local ring $R$ having maximal ideal $m$ and prime characteristic $p$; the Frobenius homomorphism $f : R \rightarrow R$, for which $f(r) = r^p$ for all $r \in R$, will play a central rôle. Let us (temporarily) use $R_f$ to denote the Abelian group $R$ considered as an $(R, R)$-bimodule with $r_1 \cdot r_2 = r_1 r_2^p$ for all $r, r_1, r_2 \in R$.

The ring $R$ is said to be $F$-pure if and only if, for every $R$-module $M$, the map $\alpha_M : M \rightarrow R_f \otimes_R M$ for which $\alpha_M(m) = 1 \otimes m$, for all $m \in M$, is injective. This property can be reformulated in terms of certain left modules over the skew polynomial ring $R[x, f]$ associated to $R$ and $f$ in the indeterminate $x$ over $R$, which we refer to as the Frobenius skew polynomial ring over $R$. Recall that $R[x, f]$ is, as a left $R$-module, freely generated by $(x^i)_{i \in \mathbb{N}_0}$ (we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers), and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \in \mathbb{N}_0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule $xr = f(r)x = r^p x$ for all $r \in R$. Note that $R[x, f]$ can be considered as a positively-graded ring $R[x, f] = \bigoplus_{n=0}^{\infty} R[x, f]_n$, with $R[x, f]_n = Rx^n$ for $n \in \mathbb{N}_0$. If we endow $Rx^n$ with its natural structure as an $(R, R)$-bimodule (inherited from its being a graded component of $R[x, f]$), then $Rx^n$ is isomorphic (as $(R, R)$-bimodule)
to $R$ viewed as a left $R$-module in the natural way and as a right $R$-module via $f^n$, the $n$th iterate of the Frobenius ring homomorphism. In particular, $Rx \cong R_f$ as $(R, R)$-bimodules.

A left $R[x, f]$-module $G$ is said to be $x$-torsion-free precisely when $xg = 0$, where $g \in G$, implies that $g = 0$. We can say that $R$ is $F$-pure if and only if, for every $R$-module $M$, the graded left $R[x, f]$-module $R[x, f] \otimes_R M = \bigoplus_{n \in \mathbb{N}_0} Rx^n \otimes_R M$ is $x$-torsion-free. We shall also use the following alternative characterization of $F$-purity.

0.1. Theorem ([9] Theorem 3.2]). The local ring $(R, m)$ is $F$-pure if and only if the $R$-module structure on $E_R(R/m)$, the injective envelope of the simple $R$-module, can be extended to an $x$-torsion-free left $R[x, f]$-module structure.

In the paper [7], some useful properties of $x$-torsion-free left $R[x, f]$-modules were developed, and it is appropriate to recall some of them at this point.

The graded two-sided ideals of $R[x, f]$ are just the subsets of the form $\bigoplus_{n \in \mathbb{N}_0} a_nx^n$, where $(a_n)_{n \in \mathbb{N}_0}$ is an ascending sequence of ideals of $R$. (Of course, such a sequence $a_0 \subseteq a_1 \subseteq \cdots \subseteq a_n \subseteq \cdots$ is eventually stationary.) Let $H$ be a left $R[x, f]$-module. An $R[x, f]$-submodule of $H$ is said to be a special annihilator submodule of $H$ if it has the form

$$\text{ann}_H(\mathfrak{A}) := \{h \in H : \theta h = 0 \text{ for all } \theta \in \mathfrak{A}\}$$

for some graded two-sided ideal $\mathfrak{A}$ of $R[x, f]$.

We shall use $\mathcal{A}(H)$ to denote the set of special annihilator submodules of $H$.

The graded annihilator $\text{gr-ann}_{R[x,f]} H$ of $H$ is defined to be the largest graded two-sided ideal of $R[x, f]$ that annihilates $H$. Note that, if $\text{gr-ann}_{R[x,f]} H = \bigoplus_{n \in \mathbb{N}_0} a_nx^n$, then $a_0 = (0 :_R H)$, which we shall sometimes refer to as the $R$-annihilator of $H$.

We shall use $\mathcal{I}(H)$ (or $\mathcal{I}_R(H)$ when it is desirable to specify the ring $R$) to denote the set of $R$-annihilators of $R[x, f]$-submodules of $H$.

0.2. The Basic Correspondence ([7] §1, §3]). Let $G$ be an $x$-torsion-free left module over $R[x, f]$.

(i) By [7] Lemma 1.9, the members of $\mathcal{I}(G)$ are all radical ideals of $R$; they are referred to as the $G$-special $R$-ideals; in fact, the graded annihilator of an $R[x, f]$-submodule $L$ of $G$ is equal to $\bigoplus_{n \in \mathbb{N}_0} a_nx^n$, where $a = (0 :_R L)$.

(ii) By [7] Proposition 1.11, there is an order-reversing bijection, $\Theta : \mathcal{A}(G) \longrightarrow \mathcal{I}(G)$ given by

$$\Theta : N \mapsto (\text{gr-ann}_{R[x,f]} N) \cap R = (0 :_R N).$$

The inverse bijection, $\Theta^{-1} : \mathcal{I}(G) \longrightarrow \mathcal{A}(G)$, also order-reversing, is given by

$$\Theta^{-1} : b \mapsto \text{ann}_G(bR[x, f]).$$

(iii) By [7] Theorem 3.6 and Corollary 3.7, the set of $G$-special $R$-ideals is precisely the set of all finite intersections of prime $G$-special $R$-ideals (provided one includes the empty intersection, $R$, which corresponds under the bijection of (ii) to the zero special annihilator submodule of $G$). In symbols,

$$\mathcal{I}(G) = \{p_1 \cap \cdots \cap p_t : t \in \mathbb{N}_0 \text{ and } p_1, \ldots, p_t \in \mathcal{I}(G) \cap \text{Spec}(R)\}.$$ 

(iv) By [7] Corollary 3.11, when $G$ is Artinian as an $R$-module, the sets $\mathcal{I}(G)$ and $\mathcal{A}(G)$ are both finite.
Thus, if \((R, \mathfrak{m})\) is \(F\)-pure and we endow \(E_R(R/\mathfrak{m})\) with a structure as \(x\)-torsion-free left \(R[x, f]\)-module (this is possible, by Theorem 0.1), then the resulting set \(\mathcal{I}(E_R(R/\mathfrak{m}))\) is finite. In fact, rather more can be said.

0.3. **Theorem** ([2] Corollary 4.11). Suppose that \((R, \mathfrak{m})\) is \(F\)-pure and local. Then the left \(R[x, f]\)-module \(R[x, f] \otimes_R E_R(R/\mathfrak{m})\) is \(x\)-torsion-free, and, furthermore, the set \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\) of \((R[x, f] \otimes_R E_R(R/\mathfrak{m}))\)-special \(R\)-ideals, is finite.

In fact, for any \(x\)-torsion-free left \(R[x, f]\)-module structure on \(E_R(R/\mathfrak{m})\) that extends its \(R\)-module structure (and such exist, by Theorem 0.1), we have

\[ \mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m})) \subseteq \mathcal{I}(E_R(R/\mathfrak{m})), \]

and the latter set is finite.

It turns out that, in the situation of Theorem 0.3, all the minimal prime ideals of \(R\) belong to \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\), and the smallest ideal of positive height in \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\) is the big test ideal of \(R\), that is, the ideal generated by all big test elements for \(R\). (A reminder about big test elements is included in the next section.)

In short, to each \(F\)-pure local ring \((R, \mathfrak{m})\) of characteristic \(p\), there is naturally associated a finite set of radical ideals \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\) of \(R\), and the smallest member of positive height in this set has significance for tight closure theory. The purpose of this paper is to address the following question: what can be said about the other members of \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\)? We shall show that, for each multiplicatively closed subset \(S\) of \(R\), one can define reasonable concepts of \(S\)-tight closure, \(S\)-test element and \(S\)-test ideal; it turns out that an ideal \(\mathfrak{b}\) of \(R\) is a member of \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\) if and only if it is the \(S\)-test ideal of \(R\) for some choice of multiplicatively closed subset \(S\) of \(R\).

We shall also show that, when \(R\) is complete, \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\) is 'closed under taking (big) test ideals', in the following sense: it turns out that, for each proper ideal \(\mathfrak{c} \in \mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\), the ring \(R/\mathfrak{c}\) is again \(F\)-pure, and the set \(\mathcal{I}(R[x, f] \otimes_R E_R(R/\mathfrak{m}))\) has among its membership the unique ideals \(\mathfrak{g}\) and \(\mathfrak{h}\) of \(R\) such that \(\mathfrak{g}, \mathfrak{h} \supseteq \mathfrak{c}\) and \(\mathfrak{g}/\mathfrak{c} = \overline{\tau}(R/\mathfrak{c})\), the big test ideal of \(R/\mathfrak{c}\), and \(\mathfrak{h}/\mathfrak{c} = \tau(R/\mathfrak{c})\), the test ideal of \(R/\mathfrak{c}\).

I am grateful to Mordechai Katzman for helpful discussions about the material in this paper.

### 1. Internal \(S\)-tight closure

1.1. **Notation.** From this point onwards in the paper, \(R\) will denote a commutative Noetherian ring of prime characteristic \(p\). We shall only assume that \(R\) is local when this is explicitly stated; then, the notation ‘\((R, \mathfrak{m})\)’ will denote that \(\mathfrak{m}\) is the maximal ideal of \(R\). As in tight closure theory, we use \(R^\circ\) to denote the complement in \(R\) of the union of the minimal prime ideals of \(R\). The Frobenius homomorphism on \(R\) will always be denoted by \(f\).

We shall use \(\Phi\) (or \(\Phi_R\) when it is desirable to specify which ring is being considered) to denote the functor \(R[x, f] \otimes_R \cdot\) from the category of \(R\)-modules (and all \(R\)-homomorphisms) to the category of all \(N_0\)-graded left \(R[x, f]\)-modules (and all homogeneous \(R[x, f]\)-homomorphisms). For an \(R\)-module \(M\), we shall identify \(\Phi(M)\)
with $\bigoplus_{n \in \mathbb{N}_0} R x^n \otimes_R M$, and (sometimes) identify its 0th component $R \otimes_R M$ with $M$, in the obvious ways.

For $n \in \mathbb{Z}$, we shall denote the $n$th component of a $\mathbb{Z}$-graded module $L$ by $L_n$.

Throughout the paper, $S$ will denote a multiplicatively closed subset of $R$. (We require that each multiplicatively closed subset of $R$ contains 1.)

Also throughout the paper, $H$ will denote a left $R[x, f]$-module and $G$ will denote an $x$-torsion-free left $R[x, f]$-module.

1.2. Definitions. We define the internal $S$-tight closure of zero in $H$, denoted $\Delta^S(H)$ (or $\Delta^S_R(H)$), to be the $R[x, f]$-submodule of $H$ given by

$$\Delta^S(H) = \{ h \in H : \text{there exists } s \in S \text{ with } sx^n h = 0 \text{ for all } n \gg 0 \}.$$ 

Note that if the left $R[x, f]$-module $H$ is $\mathbb{Z}$-graded, then $\Delta^S(H)$ is a graded submodule.

Let $M$ be an $R$-module, and consider $\Phi(M)$, as in [1]. Now $\Delta^S(\Phi(M))$ is a graded $R[x, f]$-submodule of $\Phi(M)$; we refer to the 0th component of $\Delta^S(\Phi(M))$ as the $S$-tight closure of 0 in $M$, or the tight closure with respect to $S$ of 0 in $M$, and denote it by $0^*_M$ (or by $0^S$ when it is clear what $S$ is).

Thus $0^S_M$ is the set of all elements $m$ of $M$ for which there exists $s \in S$ such that, for all $n \gg 0$, we have $sx^n \otimes m = 0$ in $Rx^n \otimes_R M$. In particular, $0^{*R} = \text{the usual tight closure of 0 in } M$. (See M. Hochster and C. Huneke [2, §8].)

Now let $N$ be an $R$-submodule of $M$. The inverse image of $0^S_{M/N}$ under the natural epimorphism $M \twoheadrightarrow M/N$ is defined to be the $S$-tight closure of $N$ in $M$, and is denoted by $N^*_M$ or $N^S$. Thus $N^*_M$ is the set of all elements $m$ of $M$ for which there exists $s \in S$ such that, for all $n \gg 0$, the element $sx^n \otimes m$ of $Rx^n \otimes_R M$ belongs to the image of $Rx^n \otimes_R N$ in $Rx^n \otimes_R M$. Note that $N^*_{M/N}$ is the usual tight closure of $N$ in $M$. (See Hochster–Huneke [2, §8].) I am grateful to the referee for pointing out that the $S$-tight closure of $N$ in $M$ is the tight closure of $N$ in $M$ with respect to $C$ in the sense of Hochster–Huneke [2 Definition (10.1)], where $C = \{ sR : s \in S \}$, the family of principal ideals generated by elements of $S$, directed by reverse inclusion $\supseteq$.

The $S$-tight closure of $a$ in $R$ is referred to simply as the $S$-tight closure of $a$ and is denoted by $a^{*S}$. The fact that there is a homogeneous $R[x, f]$-isomorphism

$$R[x, f] \otimes_R (R/a) \cong \bigoplus_{n \in \mathbb{N}_0} R/a[p^n]$$

(where the right-hand side has the left $R[x, f]$-module structure for which $x(r + a[p^n]) = r^p + a[p^{n+1}]$ for all $r \in R$) enables one to conclude that

$$a^{*S} = \{ r \in R : \text{there exists } s \in S \text{ with } srp^n \in a[p^n] \text{ for all } n \gg 0 \}.$$ 

Thus $a^{*R}$ is the usual tight closure $a^*$ of $a$. (See Hochster–Huneke [2, §3].)

1.3. Examples. In the situation of Definition [1] let $S, T$ be multiplicatively closed subsets of $R$ with $S \subseteq T$. Then clearly $\Delta^S(H) \subseteq \Delta^T(H)$.

(i) Note that $\{1\}$ is a multiplicatively closed subset of $R$, and

$$\Delta^{\{1\}}(H) = \{ h \in H : x^n h = 0 \text{ for all } n \gg 0 \} = \{ h \in H : \text{there exists } n \in \mathbb{N}_0 \text{ with } x^n h = 0 \},$$
the \textit{x-torsion submodule} \( \Gamma_x(H) \) of \( H \). Thus \( \Gamma_x(H) \subseteq \Delta^S(H) \) and therefore \((0 :_R \Delta^S(H)) \subseteq (0 :_R \Gamma_x(H)) \).

(ii) Let \( M \) be an \( R \)-module and let \( h, n \in \mathbb{N}_0 \). Endow \( R^{x^n} \) and \( R^{x^h} \) with their natural structures as \( (R, R) \)-bimodules (inherited from their being graded components of \( R[x, f] \)). Then there is an isomorphism of \( (R) \)-modules \( \phi : R^{x^{n+h}} \otimes R M \xrightarrow{\cong} R^{x^n} \otimes R (R^{x^h} \otimes R M) \) for which \( \phi(r x^{n+h} \otimes m) = r x^n \otimes (x^h \otimes m) \) for all \( r \in R \) and \( m \in M \).

One can use isomorphisms like that described in the above paragraph to see that

\[
\Delta^S(R[x, f] \otimes_R M) = 0^* \otimes 0^* \otimes R^{x^h} \otimes_R M \oplus \cdots \oplus 0^* \otimes R^{x^h} \otimes_R M \oplus \cdots .
\]

1.4. \textbf{Notation.} We define \( \mathcal{M}^S(G) \), for the \( x \)-torsion-free left \( R[x, f] \)-module \( G \), to be the set of minimal members (with respect to inclusion) of the set

\[
\{ p \in \text{Spec}(R) \cap \mathcal{I}(G) : p \subseteq S \neq \emptyset \}
\]

of prime \( G \)-special \( R \)-ideals that meet \( S \).

When \( \mathcal{M}^S(G) \) is finite, we shall set \( \mathfrak{b} := \bigcap_{p \in \mathcal{M}^S(G)} p \), although we shall write \( \mathfrak{b}^S \) for \( \mathfrak{b} \) when it is desirable to indicate \( S \) and \( G \); in that case, it follows from \ref{0.2}(iii) that \( \mathfrak{b} \) is the smallest member of \( \mathcal{I}(G) \) that meets \( S \) (and, in particular, \( \mathfrak{b} \) is contained in every other member of \( \mathcal{I}(G) \) that meets \( S \)). (In the special case where \( \mathcal{M}^S(G) = \emptyset \), we interpret \( \mathfrak{b} \) as \( R \), the intersection of the empty family of prime ideals of \( R \).)

1.5. \textbf{Proposition.} Consider the \( x \)-torsion-free left \( R[x, f] \)-module \( G \). The set \( \mathcal{M}^S(G) \) (see \ref{1.3}) is finite if and only if \((0 :_R \Delta^S(G)) \cap S \neq \emptyset \).

When these conditions are satisfied, and \( \mathfrak{b} \) denotes the intersection of the prime ideals in the finite set \( \mathcal{M}^S(G) \), then

\[
\Delta^S(G) = \text{ann}_G(\mathfrak{b} R[x, f]) \quad \text{and} \quad (0 :_R \Delta^S(G)) = \mathfrak{b}.
\]

\textit{Proof.} \((\Leftarrow)\) Set \( \mathfrak{c} := (0 :_R \Delta^S(G)) \) and assume that there exists \( s \in \mathfrak{c} \cap S \). Since \( G \) is \( x \)-torsion-free and \( \Delta^S(G) \) is an \( R[x, f] \)-submodule of \( G \), it follows from \ref{0.2}(i) that \( \mathfrak{c} \) is radical and \( \text{gr-ann}_{R[x, f]} \Delta^S(G) = \mathfrak{c} R[x, f] \). Now

\[
\text{ann}_G(\mathfrak{c} R[x, f]) \subseteq \text{ann}_G((s R) R[x, f]) \subseteq \Delta^S(G) \subseteq \text{ann}_G(\mathfrak{c} R[x, f]),
\]

and so \( \text{ann}_G(\mathfrak{c} R[x, f]) = \Delta^S(G) \). Note that \( \mathfrak{c} \in \mathcal{I}(G) \).

If \( \mathfrak{c} = R \), then \( \Delta^S(G) = 0 \), so that \( \mathcal{M}^S(G) \) is empty because a \( p \in \mathcal{I}(G) \cap \text{Spec}(R) \) with \( p \cap S \neq \emptyset \) must satisfy \( \text{ann}_G(\mathfrak{p} R[x, f]) \subseteq \Delta^S(G) = 0 \), and this leads to a contradiction to \ref{0.2}(ii). We therefore suppose that \( \mathfrak{c} \neq R \).

Let \( \mathfrak{c} = p_1 \cap \cdots \cap p_t \) be the minimal primary decomposition of the (radical) ideal \( \mathfrak{c} \).

By \ref{7} Theorem 3.6 and Corollary 3.7, the prime ideals \( p_1, \ldots, p_t \) all belong to \( \mathcal{I}(G) \); they all meet \( S \). Since \( \text{Spec}(R) \) satisfies the descending chain condition, each member of \( \{ p' \in \text{Spec}(R) \cap \mathcal{I}(G) : p' \cap S \neq \emptyset \} \) contains a member of \( \mathcal{M}^S(G) \). In particular, each of \( p_1, \ldots, p_t \) contains a member of \( \mathcal{M}^S(G) \).

Now let \( p \in \mathcal{M}^S(G) \). Since \( p \) meets \( S \), we must have that \( \text{ann}_G(p R[x, f]) \subseteq \Delta^S(G) = \text{ann}_G(c R[x, f]) \). It now follows from the inclusion-reversing bijective correspondence of \ref{0.2}(ii) that \( p \supseteq c \), so that \( p \) contains one of \( p_1, \ldots, p_t \). We can therefore conclude that \( p_1, \ldots, p_t \) are precisely the minimal members of

\[
\{ p \in \mathcal{I}(G) \cap \text{Spec}(R) : p \cap S \neq \emptyset \}.
\]
Therefore $\mathcal{M}^S(G)$ is finite.

(\Rightarrow) Assume that $\mathcal{M}^S(G)$ is finite. Then $b$ is the smallest member of $\mathcal{I}(G)$ that meets $S$. Consequently, $\text{ann}_G(bR[x,f]) \subseteq \Delta^S(G)$. Let $g \in \Delta^S(G)$, so that there exist $s' \in S$ and $n_0 \in \mathbb{N}_0$ such that $s'x^ng = 0$ for all $n \geq n_0$. Thus $g \in \text{ann}_G(\bigoplus_{n \geq n_0} Rs'x^n) =: J$, a special annihilator submodule of $G$. Let $b'$ be the $G$-special $R$-ideal that corresponds to this special annihilator submodule (in the bijective correspondence of [7, (ii)]). Since $\bigoplus_{n \geq n_0} Rs'x^n \subseteq \text{gr-ann}_{R[x,f]} J = b'R[x,f]$, we have $s' \in b'$, so that $b' \cap S \neq \emptyset$. Therefore $b' \supseteq b$, and $g \in J = \text{ann}_G(b'R[x,f]) \subseteq \text{ann}_G(bR[x,f])$. Therefore $\Delta^S(G) \subseteq \text{ann}_G(bR[x,f])$. We conclude that $\Delta^S(G) = \text{ann}_G(bR[x,f])$, and that $(0 :_R \Delta^S(G)) = (0 :_R \text{ann}_G(bR[x,f])) = b$. All the claims in the statement of the proposition have now been proved. \hfill \Box

1.6. **Definition.** An $S$-test element for $R$ is an element $s \in S$ such that, for every $R$-module $M$ and every $j \in \mathbb{N}_0$, the element $sx^j$ annihilates $1 \otimes m \in (\Phi(M))_0$ for every $m \in 0^*_M$. The ideal of $R$ generated by all the $S$-test elements for $R$ is called the $S$-test ideal of $R$, and denoted by $\tau^S(R)$.

One of the aims of this paper is to show that $S$-test elements for $R$ exist when $R$ is $F$-pure and local.

This is a suitable point at which to remind the reader of some of the classical concepts related to tight closure test elements.

1.7. **Reminder.** Recall that a test element for modules for $R$ is an element $c \in R^e$ such that, for every finitely generated $R$-module $M$ and every $j \in \mathbb{N}_0$, the element $cx^j$ annihilates $1 \otimes m \in (\Phi(M))_0$ for every $m \in 0^*_M$. The phrase ‘for modules’ is inserted because Hochster and Huneke have also considered a concept of a test element for ideals for $R$, which is defined to be an element $c \in R^e$ such that, for every cyclic $R$-module $M$ and every $j \in \mathbb{N}_0$, the element $cx^j$ annihilates $1 \otimes m \in (\Phi(M))_0$ for every $m \in 0^*_M$. When $R$ is reduced and excellent, the concepts of test element for modules and test element for ideals for $R$ coincide: see [2, Discussion (8.6) and Proposition (8.15)].

Hochster and Huneke define the test ideal $\tau(R)$ of $R$ to be $\bigcap_M(0 :_R 0^*_M)$, where the intersection is taken over all finitely generated $R$-modules $M$. In [2, Proposition (8.23)(b)] they show that, if $R$ has a test element for modules, then $\tau(R)$ is the ideal generated by the test elements for modules, and $\tau(R) \cap R^e$ is the set of test elements for modules.

1.8. **Reminder.** Recall also that a big test element for $R$ is an element $c \in R^e$ such that, for every $R$-module $M$ and every $j \in \mathbb{N}_0$, the element $cx^j$ annihilates $1 \otimes m \in (\Phi(M))_0$ for every $m \in 0^*_M$. We let $\bar{\tau}(R)$ denote the ideal generated by all big test elements for $R$, and call this the big test ideal of $R$.

Note that an $R^e$-test element for $R$, in the sense of Definition 1.6, is just a big test element for $R$.

Recall that an injective cogenerator of $R$ is an injective $R$-module $E$ with the property that, for every $R$-module $M$ and every non-zero element $m \in M$, there exists an $R$-homomorphism $\phi : M \rightarrow E$ such that $\phi(m) \neq 0$. As $R$ is Noetherian, $\bigoplus_{m \in \text{Max}(R)} E_R(R/m)$, where $\text{Max}(R)$ denotes the set of maximal ideals of $R$, is one injective cogenerator of $R$. 
1.9. **Reminders.** Let $E := \bigoplus_{m \in \text{Max}(R)} E_R(R/m)$, an injective cogenerator of $R$.

(i) Recall from Hochster–Huneke [2] Definition (8.19) that the finitistic tight closure of $0$ in $E$, denoted by $0^e_{E}$, is defined to be $\bigcup_M 0^M_{E}$, where the union is taken over all finitely generated $R$-submodules $M$ of $E$. It was shown in [2 Proposition (8.23)(d)] that $\tau(R) = (0 :_R 0^e_{E})$.

(ii) It is conjectured that $(0 :_R 0^e_{E}) = (0 :_R 0^e_{E})$. This conjecture is known to be true

(a) if $R$ is an excellent Gorenstein local ring (see K. E. Smith [12 p. 48]);
(b) if $R$ is the localization of a finitely generated $\mathbb{N}_0$-graded algebra over an $F$-finite field $K$ (of characteristic $p$ and having $K$ as its component of degree 0) at its unique homogeneous maximal ideal (see G. Lyubeznik and K. E. Smith [5 Corollary 3.4]);
(c) if $R$ is a Cohen–Macaulay local ring which is Gorenstein on its punctured spectrum (see Lyubeznik–Smith [6 Theorem 8.8]); or
(d) if $(R, m)$ is local and an isolated singularity (see Lyubeznik–Smith [6 Theorem 8.12]).

(iii) It was shown in [10 Theorem 3.3] that if $R$ has a big test element, then the big test ideal $\tilde{\tau}(R)$ of $R$ is equal to $(0 :_R \Delta^R(\Phi(E)))$, and the set of big test elements for $R$ is $(0 :_R \Delta^R(\Phi(E))) \cap R^\circ$.

2. **Existence of $S$-test elements in $F$-pure local rings**

Theorem 0.3 was proved by means of an ‘embedding theorem’ [9] Theorem 4.10. Similar embedding theorems were established in [8 Theorem 3.5] and [10 Theorem 3.2]. In this paper, the ideas underlying those theorems are going to be pursued further, and so we begin with three remarks and a lemma that can be viewed as addenda to [10, §2].

The notation in this section is as described in [14].

2.1. **Remark.** Let $\tilde{H}$ denote the graded companion of $H$ described in [10 Example 2.1]. It is easy to check that $\Delta^S(\tilde{H}) = \Delta^S(\tilde{H})$, so that $(0 :_R \Delta^S(\tilde{H})) = (0 :_R \Delta^S(H))$.

2.2. **Remark.** Let $(H^{(\lambda)})_{\lambda \in \Lambda}$ be a non-empty family of $\mathbb{Z}$-graded left $R[x, f]$-modules, and consider the graded product $\prod'_{\lambda \in \Lambda} H^{(\lambda)}$ of the $H^{(\lambda)}$, described in [8 Lemma 2.1] and [10 Example 2.2]. It is routine to check that, if there is an ideal $\mathfrak{a}_0$ of $R$ such that $(0 :_R \Delta^S(H^{(\lambda)})) = \mathfrak{a}_0$ for all $\lambda \in \Lambda$, then $(0 :_R \Delta^S(\prod'_{\lambda \in \Lambda} H^{(\lambda)})) = \mathfrak{a}_0$.

2.3. **Remark.** Assume that the left $R[x, f]$-module $H = \bigoplus_{n \in \mathbb{Z}} H_n$ is $\mathbb{Z}$-graded; let $t \in \mathbb{Z}$. Denote by $H(t)$ the result of application of the $t$th shift functor to $H$; this is described in [10 Example 2.3]. It is clear that $\Delta^S(H(t)) = \Delta^S(H)(t)$, so that $(0 :_R \Delta^S(H(t))) = (0 :_R \Delta^S(H))$.

2.4. **Lemma.** Let $b, h \in \mathbb{N}$ and let $W := \bigoplus_{n \geq b} W_n$ be a graded left $R[x, f]$-module. Let $(g_i)_{i \in I}$ be a family of arbitrary elements of $W_b$. Consider the $b$-place extension $\text{ext}(W; (g_i)_{i \in I}; h)$ of $W$ by $(g_i)_{i \in I}$, defined in [9 Definition 4.5] and [10, §2]. Then $(0 :_R \Delta^S(\text{ext}(W; (g_i)_{i \in I}; h))) = (0 :_R \Delta^S(W))$.

**Proof.** This can be proved by making obvious modifications to the proof, presented in [10, Proposition 2.4(i)], that $(0 :_R \Delta^R(\text{ext}(W; (g_i)_{i \in I}; h))) = (0 :_R \Delta^R(W))$. \qed
The above remarks and lemma are helpful for use in the proof of some of the claims in the following Embedding Theorem.

2.5. **Embedding Theorem.** (See [10, Theorem 3.2].) Let $E$ be an injective cogenerator of $R$. Assume that there exists an $\mathbb{N}_0$-graded left $R[x,f]$-module $H = \bigoplus_{n \in \mathbb{N}_0} H_n$ such that $H_0$ is $R$-isomorphic to $E$.

Let $M$ be an $R$-module. Then there is a family $(L^{(n)})_{n \in \mathbb{N}_0}$ of $\mathbb{N}_0$-graded left $R[x,f]$-modules, where $L^{(n)}$ is an $n$-place extension of the $-n$th shift of a graded product of copies of $H$ (for each $n \in \mathbb{N}_0$), for which there exists a homogeneous $R[x,f]$-monomorphism

$$
\nu : \Phi(M) = \bigoplus_{i \in \mathbb{N}_0} (Rx^i \otimes_R M) \to \prod_{n \in \mathbb{N}_0} L^{(n)} =: K.
$$

Consequently, $(0 : _R \Delta^S(H)) = (0 : _R \Delta^S(K)) \subseteq (0 : _R \Delta^S(\Phi(M)))$.

Furthermore, if $H$ is $x$-torsion-free, then so too is $\Phi(M)$, and $\Delta(\Phi(M)) \subseteq \Delta(H)$ and $(0 : _R \Delta^S(\Phi(M))) \subseteq \Delta(H)$.

**Proof.** The existence of $K$ and $\nu$ with the stated properties were proved in [10, Theorem 3.2].

The existence of the $R[x,f]$-monomorphism $\nu$ shows that

$$(0 : _R \Delta^S(K)) \subseteq (0 : _R \Delta^S(\Phi(M))),$$

while Remarks 2.2 and 2.3 and Lemma 2.4 show that $(0 : _R \Delta^S(K)) = (0 : _R \Delta^S(H))$.

Now suppose $H$ is $x$-torsion-free. It follows from [8, Lemmas 2.3 and 2.8] that $K$ is $x$-torsion-free and that $\Delta(K) = \Delta(H)$. The existence of the $R[x,f]$-monomorphism $\nu$ shows that $\Phi(M)$ is $x$-torsion-free and $R[x,f]$-isomorphic to an $R[x,f]$-submodule of $K$; therefore $\Delta(\Phi(M)) \subseteq \Delta(K) = \Delta(H)$. Finally, $(0 : _R \Delta^S(\Phi(M))) \subseteq \Delta(H)$. $\square$

2.6. **Theorem.** Suppose that the local ring $(R, m)$ is $F$-pure. Then $R$ has an $S$-test element.

In more detail, set $E := E_R(R/m)$. Recall that $b^{S,\Phi(E)}$ denotes the intersection of all the minimal members of the set $\{p \in \text{Spec}(R) \cap \Delta(\Phi(E)) : p \cap S \neq \emptyset\}$ (see [1.4]). Then $S \cap b^{S,\Phi(E)}$ is (non-empty and) equal to the set of $S$-test elements for $R$. Furthermore, $b^{S,\Phi(E)} = (0 : _R \Delta^S(\Phi(E)))$.

**Proof.** By Theorem 0.3, the set $\Delta(\Phi(E))$ of $\Phi(E)$-special $R$-ideals is finite. Consequently, $\Delta^S(\Phi(E))$ is finite, and Proposition 1.5 shows that $(0 : _R \Delta^S(\Phi(E))) \cap S \neq \emptyset$.

We use the Embedding Theorem 2.5 with $\Phi(E)$ playing the rôle of $H$. We conclude that, for every $R$-module $M$, we have

$$(0 : _R \Delta^S(\Phi(E))) \subseteq (0 : _R \Delta^S(\Phi(M))).$$

Thus $(0 : _R \Delta^S(\Phi(E)))$ is precisely the set of elements of $R$ that annihilate $\Delta^S(\Phi(M))$ for every $R$-module $M$. Since $\Delta^S(\Phi(M))$ is an $R[x,f]$-submodule of $\Phi(M)$ (for each $R$-module $M$), it follows that $(0 : _R \Delta^S(\Phi(E))) \cap S$ (which is non-empty) is the set of $S$-test elements for $R$.

It also follows from Proposition 1.5 that $\Delta^S(\Phi(E)) = \text{ann}_{\Phi(E)}(b^{S,\Phi(E)}R[x,f])$ and $(0 : _R \Delta^S(\Phi(E))) = b^{S,\Phi(E)}$. $\square$
In our applications of these ideas, we shall frequently take $S$ to be the complement in $R$ of the union of finitely many prime ideals. In that case, a little more can be said.

2.7. Lemma. Let $A$ be a commutative ring and let $p_1, \ldots, p_n \in \text{Spec}(A)$. Set $T := A \setminus \bigcup_{i=1}^n p_i$, and let $a$ be an ideal of $A$ such that $a \cap T \neq \emptyset$. Then $a$ can be generated by elements of $a \cap T$.

Proof. Let $a'$ be the ideal of $A$ generated by $a \cap T$. Then

$$a \subseteq (a \cap T) \cup (a \cap (A \setminus T)) \subseteq a' \cup p_1 \cup \cdots \cup p_n.$$  

Since $a \cap T \neq \emptyset$, we have, for every $i \in \{1, \ldots, n\}$, that $a \nsubseteq p_i$. Therefore, by the Prime Avoidance Theorem (in the form given in [4, Theorem 81]), we must have $a \subseteq a'$.

2.8. Corollary. Suppose that the local ring $(R, m)$ is $F$-pure, and $S = R \setminus \bigcup_{i=1}^n p_i$, where $p_1, \ldots, p_n \in \text{Spec}(R)$. It was shown in Theorem 2.6 that $R$ has an $S$-test element. Set $E := E_R(R/m)$. The $S$-test ideal of $R$, that is, the ideal of $R$ generated by all $S$-test elements for $R$, is $b^{S, \Phi(E)}$, the smallest member of $I(\Phi(E))$ that meets $S$. In symbols, $\tau^S(R) = b^{S, \Phi(E)}$.

In particular, the big test ideal $\tau(R)$ is the smallest member of $I(\Phi(E))$ of positive height. (We interpret the height of the improper ideal $R$ as $\infty$.)

Proof. We saw in Theorem 2.6 that the set $S \cap b^{S, \Phi(E)}$ is non-empty and equal to the set of $S$-test elements for $R$. By Lemma 2.7, the ideal $b^{S, \Phi(E)}$ can be generated by elements of $S \cap b^{S, \Phi(E)}$.

For the final claim, take $S = R^\circ$ and note that a proper ideal of $R$ has positive height if and only if it meets $R^\circ$.

We shall actually use variations of the Embedding Theorem 2.5 in Proposition 2.10 below.

2.9. Definitions. Suppose that $(R, m)$ is local and $F$-pure; set $E := E_R(R/m)$. Let $M$ be an $R$-module.

(i) We define the finitistic $S$-tight closure $0^*_{E}^{S\Phi} 0$ in $M$ to be $\bigcup_N 0^*_N$, where the union is taken over all finitely generated submodules $N$ of $M$.

(ii) We define the finitistic $S$-test ideal $\tau_{E}^{S\Phi}(R)$ of $R$ to be $\bigcap_L (0 : R 0^*_L)$, where the intersection is taken over all finitely generated $R$-modules $L$.

2.10. Proposition. Suppose that $(R, m)$ is local and $F$-pure; set $E := E_R(R/m)$.

(i) For every $R$-module $M$, we have $I(\Phi(M)) \subseteq I(\Phi(E))$ and $(0 :_R \Delta^S(\Phi(E))) \subseteq (0 :_R \Delta^S(\Phi(M))) \subseteq I(\Phi(E))$.

(ii) The ideal $(0 :_R 0^*_{E}^{S\Phi})$ annihilates $(0 :_R 0^*_{L})$ for every finitely generated $R$-module $L$.

(iii) We have $\tau_{E}^{S\Phi}(R) = (0 :_R 0^*_{E}^{S\Phi})$, and this ideal belongs to $I(\Phi(E))$.

(iv) For every $R$-module $M$, we have $(0 :_R 0^*_E) \subseteq (0 :_R 0^*_M)$.

(v) We have $b^{S, \Phi(E)} = (0 :_R 0^*_E)$. Consequently, when $S$ is the complement in $R$ of the union of finitely many prime ideals, then the $S$-test ideal $\tau^S(R)$ of $R$ is equal to $(0 :_R 0^*_E)$.
Proof. (i) This follows from the Embedding Theorem 2.3 with $H$ taken as $\Phi(E)$.

(ii) By Krull’s Intersection Theorem, $\bigcap_{n \in \mathbb{N}} n^a L = 0$. We can therefore express the zero submodule of $L$ as the intersection of a countable family $(Q_i)_{i \in \mathbb{N}}$ of irreducible submodules of finite colength. (A submodule of $L$ is said to be irreducible if it is proper and cannot be expressed as the intersection of two strictly larger submodules.) Note that $E_R(L/Q_i) = E$ for all $i \in \mathbb{N}$.

The $R$-monomorphism $\Lambda_0 : L \to \prod_{i \in \mathbb{N}} L/Q_i$ for which $\lambda_0(g) = (g + Q_i)_{i \in \mathbb{N}}$ for all $g \in L$ can be extended to a homogeneous $R[x, f]$-homomorphism

$$
\lambda : \Phi(L) = \bigoplus_{n \in \mathbb{N}_0} R x^n \otimes_R L \to \prod_{i \in \mathbb{N}} \Phi(L/Q_i) = \prod_{i \in \mathbb{N}} \left( \bigoplus_{n \in \mathbb{N}_0} R x^n \otimes_R (L/Q_i) \right)
$$

whose restriction to the $n$th component of $\Phi(L)$, for $n \in \mathbb{N}_0$, satisfies $\lambda(rx^n \otimes g) = (rx^n \otimes (g + Q_i))_{i \in \mathbb{N}}$ for all $r \in R$ and $g \in L$. It is clear that the $R$-monomorphism $\lambda_0$ (that is, the component of degree 0 of $\lambda$) maps $0^*_{L^g}$ into $\prod_{i \in \mathbb{N}_0} 0^*_{L/Q_i}$. But $L/Q_i$ is $R$-isomorphic to a finitely generated submodule of $E$, and so $0^*_{L/Q_i}$ is annihilated by $(0 :_R 0^*_{E^{fg,S}})$ (for all $i \in \mathbb{N}$). It follows that $0^*_{L^g}$ is annihilated by $(0 :_R 0^*_{E^{fg,S}})$.

(iii) By part (ii), we have $(0 :_R 0^*_{E^{fg,S}}) \subseteq \bigcap_{i \in \mathbb{N}_0} (0 :_R 0^*_{L^g})$, where the intersection is taken over all finitely generated $R$-modules $L$. Thus $(0 :_R 0^*_{E^{fg,S}}) \subseteq \tau^{fg,S}(R)$. On the other hand, by definition, $\tau^{fg,S}(R)$ annihilates $\bigcup_{N \in \mathbb{N}_0} 0^*_{N^g}$, where the union is taken over all finitely generated submodules $N$ of $E$; therefore $\tau^{fg,S}(R) \subseteq (0 :_R 0^*_{E^{fg,S}})$.

For each finitely generated $R$-module $L$,

$$
\Delta^S(\Phi(L)) = 0^*_{L^g} \oplus 0^*_{R_x \otimes_R L} \oplus \cdots \oplus 0^*_{R_x^n \otimes_R L} \oplus \cdots,
$$

by Example [3](ii), so that $(0 :_R \Delta^S(\Phi(L))) = \bigcap_{n \in \mathbb{N}_0} (0 :_R 0^*_{R_x \otimes_R L})$. Note that $R x^n \otimes_R L$ is a finitely generated $R$-module, for each $n \in \mathbb{N}_0$. It therefore follows that

$$
\bigcap_{n \in \mathbb{N}_0} (0 :_R \Delta^S(\Phi(L))) = \bigcap_{n \in \mathbb{N}_0} (0 :_R 0^*_{L^g}),
$$

where in both cases the intersection is taken over all finitely generated $R$-modules $L$. Therefore $\tau^{fg,S}(R)$ is equal to the above ideal $\bigcap_{n \in \mathbb{N}_0} (0 :_R \Delta^S(\Phi(L)))$, and the latter is in $\mathcal{I}(\Phi(E))$ because each $(0 :_R \Delta^S(\Phi(L)))$ is (by part (ii)) and $\mathcal{I}(\Phi(E))$ is closed under taking arbitrary intersections (by [7 Corollary 1.12]).

(iv) By [10] Lemma 3.1, there is a family of graded left $R[x, f]$-modules $(H^{(\lambda)})_{\lambda \in \Lambda}$, with each $H^{(\lambda)}$ equal to $\Phi(E)$, and a homogeneous $R[x, f]$-homomorphism

$$
\mu : \Phi(M) \to \prod_{\lambda \in \Lambda} H^{(\lambda)}
$$

such that its component $\mu_0$ of degree 0 is a monomorphism. Since $\mu_n(sx^n \otimes m) = sx^n \mu_0(m)$ for all $m \in M$, $s \in S$ and $n \in \mathbb{N}_0$, it follows that the $R$-monomorphism $\mu_0$ maps $0^*_{M^g}$ into a direct product of copies of $0^*_{E^S}$. Therefore $(0 :_R 0^*_{E^{fg,S}})$ annihilates $0^*_{M^g}$. Note that this is true for each $R$-module $M$.

(v) It now follows from part (iv) and the fact (see Example [3](ii)) that

$$
\Delta^S(\Phi(E)) = 0^*_{E^g} \oplus 0^*_{R_x \otimes_R E} \oplus \cdots \oplus 0^*_{R_x^n \otimes_R E} \oplus \cdots
$$

that $(0 :_R 0^*_{E^{fg,S}}) = (0 :_R \Delta^S(\Phi(E)))$. But $(0 :_R \Delta^S(\Phi(E))) = b^{S,\Phi(E)}$, by Theorem 2.6.
When $S$ is the complement in $R$ of the union of finitely many prime ideals, it follows from Corollary 2.8 that $\tau^S(R) = b^{S,\Phi(E)}$. \hfill \qed

2.11. Remarks. Suppose that $(R, m)$ is local and $F$-pure; set $E := E_R(R/m)$.

(i) In the special case in which $S$ is taken to be $R^c$, Proposition 2.10(v) reduces to the (probably well-known) result that the big test ideal $\tau(R) = \tau^{R^c}(R)$ of $R$ is equal to $(0 :_R 0^E_{R^c})$.

(ii) In the special case in which $S$ is taken to be $R^c$, the first part of Proposition 2.10(iii) reduces to (a special case of) a result of Hochster–Huneke [2, Proposition (8.23)(d)] about the test ideal $\tau(R)$:

$$
\tau(R) = \tau^{fg, R^c}(R) = (0 :_R 0^E_{R^c}) = (0 :_R 0^{E\bar{}}).
$$

We have seen that, over an $F$-pure local ring $(R, m)$, the set $\mathcal{I}(\Phi(E))$ (where $E := E_R(R/m)$) of radical ideals includes the test ideal $\tau(R)$, the big test ideal $\tau(R) = b^{R^c, \Phi(E)} = \tau^{R^c}(R)$ of $R$ and the $S$-test ideal, for each multiplicatively closed subset $S$ of $R$ which is the complement in $R$ of the union of finitely many prime ideals. It is natural to ask whether every member of the finite set $\mathcal{I}(\Phi(E))$ occurs as the $S'$-test ideal for some multiplicatively closed subset $S'$ of $R$. In Theorem 2.12 below, we shall answer this question in the affirmative.

2.12. Theorem. Suppose that the local ring $(R, m)$ is $F$-pure, and set $E := E_R(R/m)$. Let $a \in \mathcal{I}(\Phi(E))$. Then there exists a multiplicatively closed subset $S$ of $R$ such that $a$ is the $S$-test ideal of $R$. Moreover, $S$ can be taken to be the complement in $R$ of the union of finitely many prime ideals.

Proof. If $a = R$, then we can take $S = \{1\}$ or $S = R \setminus m$. We therefore assume henceforth in this proof that $a$ is proper.

Let $p_1, \ldots, p_t$ be the (distinct) associated prime ideals of $a$; recall from [7, Theorem 3.6 and Corollary 3.7] that they all belong to $\mathcal{I}(\Phi(E))$. Let $\mathcal{T}$ be the set of all prime members of $\mathcal{I}(\Phi(E))$ which neither contain, nor are contained in, any of $p_1, \ldots, p_t$. Let $q_1, \ldots, q_u$ be the maximal members of the set of prime ideals in $\mathcal{I}(\Phi(E))$ that are properly contained in one of $p_1, \ldots, p_t$, and let $\mathcal{U} := \{q_1, \ldots, q_u\}$. (Observe that $\mathcal{T}$ and/or $\mathcal{U}$ could be empty; for example, both are empty if $a = 0$.)

Set $S := R \setminus \bigcup_{q \in \mathcal{T} \cup \mathcal{U}} q$. Our aim is to show that $a$ is the $S$-test ideal $\tau^S(R)$ of $R$.

It follows from Corollary 2.8 that $\tau^S(R) = b^{S, \Phi(E)}$, the intersection of the minimal members of the set of prime ideals in $\mathcal{I}(\Phi(E))$ that meet $S$.

Let $p \in \mathcal{I}(\Phi(E)) \cap \text{Spec}(R)$. Then, since $\mathcal{I}(\Phi(E))$ and therefore $\mathcal{T}$ and $\mathcal{U}$ are finite, $p \cap S = \emptyset$ if and only if $p$ is contained in some $q \in \mathcal{T} \cup \mathcal{U}$.

Let $i \in \{1, \ldots, t\}$. Then $p_i$ meets $S$, or else $p_i \subseteq q_j$ for some $j \in \{1, \ldots, u\}$, and as $q_j$ is properly contained in one of $p_1, \ldots, p_t$, this would lead to a contradiction to the minimality of the primary decomposition $a = p_1 \cap \cdots \cap p_t$. Furthermore, $p_i$ must be a minimal member of the set $\mathcal{J} := \{p \in \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) : p \cap S \neq \emptyset\}$, for otherwise there would exist $p \in \mathcal{J}$ with $p \subseteq p_i$, so that $p$ would be contained in one of $q_1, \ldots, q_u$ and therefore disjoint from $S$. This shows that $p_1, \ldots, p_t$ are all associated primes of $b^{S, \Phi(E)}$. To complete the proof, it is enough for us to show that there is no other associated prime of this ideal.

So suppose that $p \in \text{ass} b^{S, \Phi(E)} \setminus \{p_1, \ldots, p_t\}$ and seek a contradiction. Then $p$ must contain, or be contained in, $p_i$ for some $i \in \{1, \ldots, t\}$ (or else it would be in $\mathcal{T}$ and
disjoint from \(S\); if \(p \supset p_i\), then \(p\) would be contained in \(q_j\) for some \(j \in \{1, \ldots, t\}\) and so would be disjoint from \(S\); if \(p \supseteq p_i\), then \(p\) could not be a primary component of the radical ideal \(b^{S, \Phi(E)}\). Thus each possibility leads to a contradiction. Therefore \(\text{ass } b^{S, \Phi(E)} = \{p_1, \ldots, p_t\}\) and \(b^{S, \Phi(E)} = a\).

\[\square\]

3. The complete case

In this section we shall concentrate on the case where \((R, \mathfrak{m})\) is local, \(F\)-pure and complete.

3.1. Theorem. Suppose \((R, \mathfrak{m})\) is local, \(F\)-pure and complete. Set \(E := E_R(R/\mathfrak{m})\). Let \(c \in \mathcal{I}(\Phi(E))\) with \(c \neq R\). In the light of Theorem \(2.12\) let \(p_1, \ldots, p_w\) be prime ideals of \(R\) for which the multiplicatively closed subset \(S = R \setminus \bigcup_{i=1}^w p_i\) of \(R\) satisfies \(c = \tau^S(R)\). Set \(J := \Delta^S(\Phi(E))\), a graded left \(R[x, f]\)-module.

(i) We have \(J = 0_E^* \oplus 0_{R[x] \otimes_R E}^* \oplus \cdots \oplus 0_{R[x^n] \otimes_R E}^* \oplus \cdots\).

(ii) When we regard \(J\) as a graded left \((R/c)[x, f]\)-module in the natural way, it is \(x\)-torsion-free and has \(\mathcal{I}_{R/c}(J) = \{g/c : g \in \mathcal{I}(\Phi(E)) : g \supseteq c\}\).

(iii) The 0th component \(J_0\) of \(J\) is \((0 : E)\); as \(R/c\)-module, this is isomorphic to \(E_{R/c}((R/c)/(m/c))\).

(iv) The ring \(R/c\) is \(F\)-pure.

(v) We have \(\mathcal{I}(\Phi_{R/c}(J_0)) \subseteq \mathcal{I}_{R/c}(J)\), so that

\(\{d : d\) is an ideal of \(R\) with \(d \supseteq c\) and \(d/c \in \mathcal{I}(\Phi_{R/c}(J_0))\} \subseteq \mathcal{I}(\Phi_R(E))\).

Proof. Set \(\overline{R} := R/c\).

(i) It follows from Example \(1.3\)(ii) that

\(\Delta^S(\Phi(E)) = 0_E^* \oplus 0_{R[x] \otimes_R E}^* \oplus \cdots \oplus 0_{R[x^n] \otimes_R E}^* \oplus \cdots\).

(ii) Since \(\text{gr-ann}_{R[x, f]} \Delta^S(\Phi(E)) = cR[x, f]\), we see that \(c\) annihilates \(\Delta^S(\Phi(E))\), and so the latter inherits a structure as an \(x\)-torsion-free left \(\overline{R}[x, f]\)-module. As the \(\overline{R}[x, f]\)-submodules of \(J\) are exactly the \(R[x, f]\)-submodules of \(\Phi(E)\) contained in \(J\), the claim about \(\mathcal{I}_{\overline{R}}(J)\) is clear.

(iii) Note that \(c = b^{S, \Phi(E)} = \tau^S(R)\), and that, by Proposition \(2.10\)(v), this is the \(R\)-annihilator of \(0_E^*\). Since \(R\) is complete, we can conclude that \(0_E^* = (0 : E)\), by Matlis duality (see, for example, \(\Pi\) p. 154).

(iv),(v) Let \(N\) be an \(\overline{R}\)-module. Use the Embedding Theorem \(2.5\) over the ring \(\overline{R}\) (with \(J\) playing the rôles of \(H\)) to deduce that \(\Phi_{\overline{R}}(N)\) is \(x\)-torsion-free and \(\mathcal{I}(\Phi_{\overline{R}}(N)) \subseteq \mathcal{I}_{\overline{R}}(J)\). These are true for each \(\overline{R}\)-module \(N\), and, in particular, for \(J_0\). It follows that \(\overline{R}\) is \(F\)-pure.

The final claim follows from the description of \(\mathcal{I}_{\overline{R}}(J)\) given in part (ii).

\[\square\]

3.2. Corollary. Suppose that \((R, \mathfrak{m})\) is local, \(F\)-pure and complete. Denote \(E_R(R/\mathfrak{m})\) by \(E\). Let \(c \in \mathcal{I}(\Phi(E))\) with \(c \neq R\). Denote \(R/c\) by \(\overline{R}\), and note that \(\overline{R}\) is \(F\)-pure, by Theorem \(3.1\)(iv). Let \(T\) be a multiplicatively closed subset of \(\overline{R}\) which is the complement in \(\overline{R}\) of the union of finitely many prime ideals.

(i) If \(h\) denotes the unique ideal of \(R\) that contains \(c\) and is such that \(h/c = \tau_{fg,T}(\overline{R})\), the finitistic \(T\)-test ideal of \(\overline{R}\), then \(h \in \mathcal{I}(\Phi(E))\).
(ii) In particular, if \( \mathfrak{b} \) denotes the unique ideal of \( R \) that contains \( \mathfrak{c} \) and is such that \( \mathfrak{b}/\mathfrak{c} = \tau(R) \), the test ideal of \( \overline{R} \), then \( \mathfrak{b} \in \mathcal{I}(\Phi(E)) \).

(iii) If \( \mathfrak{g} \) denotes the unique ideal of \( R \) that contains \( \mathfrak{c} \) and is such that \( \mathfrak{g}/\mathfrak{c} = \tau^T(\overline{R}) \), the \( T \)-test ideal of \( \overline{R} \), then \( \mathfrak{g} \in \mathcal{I}(\Phi(E)) \).

(iv) In particular, if \( \mathfrak{g}' \) denotes the unique ideal of \( R \) that contains \( \mathfrak{c} \) and is such that \( \mathfrak{g}'/\mathfrak{c} = \overline{\tau}(R) \), the big test ideal of \( \overline{R} \), then \( \mathfrak{g}' \in \mathcal{I}(\Phi(E)) \).

Proof. Use the notation of Theorem 3.1. Note that, as \( \mathfrak{R} \)-module, \( J_0 \) is the injective envelope of the simple \( \mathfrak{R} \)-module.

(i) By Proposition 2.10(iii), we have \( \tau_{\mathfrak{g}, T}(\overline{R}) \in \mathcal{I}(\Phi_{\mathfrak{R}}(J_0)) \). The result therefore follows from Theorem 3.1(v).

(ii) This is a special case of part (i): take \( T = \overline{R} \).

(iii) By Proposition 2.10(v), we have \( \tau^T(\overline{R}) \in \mathcal{I}(\Phi_{\mathfrak{R}}(J_0)) \). The result therefore follows from Theorem 3.1(v).

(iv) This is a special case of part (iii): take \( T = \overline{R}^0 \). \( \square \)

The remainder of the paper is devoted to the provision of some examples of the above ideas.

3.3. Example. Let \( K \) be an algebraically closed field of characteristic \( p \), and assume that \( p \geq 5 \) and that \( p \equiv 1 \pmod{3} \). Let \( R' = K[[X, Y, Z]] \), where \( X, Y, Z \) are independent indeterminates, and \( \mathfrak{a} = (X^3 + Y^3 + Z^3) \in \text{Spec}(R') \). By Huneke [3, Examples 4.7 and 4.8], \( R := R'/\mathfrak{a} \) is \( F \)-pure, and the test ideal \( \tau(R) = \mathfrak{m} \). Because \( R \) is Gorenstein and excellent, the test ideal \( \tau(R) \) is equal to the big test ideal \( \overline{\tau}(R) \), by (1.9)(ii)(a). This means that \( \mathfrak{m} \) must be the smallest ideal in \( \mathcal{I}(\Phi(E)) \) of positive height, so that \( \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) = \{0, \mathfrak{m}\} \).

3.4. Reminder. Suppose that \( (R, \mathfrak{m}) \) is local and \( F \)-pure, and set \( E = E_R(R/\mathfrak{m}) \).

In the case where \( R \) is an \( (F \)-pure \) homomorphic image of an \( F \)-finite regular local ring, Janet Cowden Vassilev showed in [13, §3] that there exists a strictly ascending chain \( 0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t \subset \tau_{t+1} = R \) of radical ideals of \( R \) such that, for each \( i = 0, \ldots, t \), the reduced local ring \( R/\tau_i \) is \( F \)-pure and its test ideal is exactly \( \tau_{i+1}/\tau_i \). If \( R \) is complete, all of \( \tau_0, \tau_1, \ldots, \tau_t \) and all their associated primes belong to \( \mathcal{I}(\Phi(E)) \) (by Corollary 3.2(ii) and [7, Theorem 3.6 and Corollary 3.7]).

3.5. Lemma. Assume that \( (R, \mathfrak{m}) \) is local, \( F \)-pure and complete. Set \( E = E_R(R/\mathfrak{m}) \).

(i) There is a strictly ascending chain \( 0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t \subset \tau_{t+1} = R \) of radical ideals of \( R \) such that, for each \( i = 0, \ldots, t \), the reduced local ring \( R/\tau_i \) is \( F \)-pure and its test ideal is \( \tau_{i+1}/\tau_i \). We call this the test ideal chain of \( R \). All of \( \tau_0 = 0, \tau_1, \ldots, \tau_t \), and all their associated primes, belong to \( \mathcal{I}(\Phi(E)) \).

(ii) There is a strictly ascending chain \( 0 = \tilde{\tau}_0 \subset \tilde{\tau}_1 \subset \cdots \subset \tilde{\tau}_w \subset \tau_{w+1} = R \) of radical ideals in \( \mathcal{I}(\Phi(E)) \) such that, for each \( i = 0, \ldots, w \), the reduced local ring \( R/\tilde{\tau}_i \) is \( F \)-pure and its big test ideal is \( \overline{\tau}_{i+1}/\overline{\tau}_i \). We call this the big test ideal chain of \( R \). All of \( \tilde{\tau}_0 = 0, \tilde{\tau}_1, \ldots, \tilde{\tau}_w \), and all their associated primes, belong to \( \mathcal{I}(\Phi(E)) \).

Note. We have not assumed that \( R \) is \( F \)-finite in part (i). If the conjecture mentioned in [13, ii] turns out to be true, then the big test ideal chain and the test ideal chain of \( R \) would coincide.
Proof. (i) We know from Theorem 0.1 that \( E \) can be given a structure as an \( x \)-torsion-free left \( R[x,f] \)-module (that extends its \( R \)-module structure). It therefore follows from work of the present author in [8, Corollary 3.8] that, in this complete case, the test ideal chain of \( R \) exists. By Corollary 3.2(ii), all the terms in the test ideal chain of \( R \) belong to \( \mathcal{I}(\Phi(E)) \), and all the associated prime ideals of these ideals also belong to \( \mathcal{I}(\Phi(E)) \), by [7, Theorem 3.6 and Corollary 3.7].

(ii) Let \( c \in \mathcal{I}(\Phi(E)) \) with \( c \neq R \). By Theorem 3.1(iv), the ring \( R/c \) is \( F \)-pure. By Corollary 3.2(iv), if \( g' \) denotes the unique ideal of \( R \) that contains \( c \) and is such that \( g'/c = \bar{\tau}(R/c) \), then \( g' \in \mathcal{I}(\Phi(E)) \). One can therefore construct \( \bar{\tau}_1(R), \bar{\tau}_2(R), \ldots \) successively until some \( \bar{\tau}_{t+1}(R) = R \), when the process stops. Use Corollary 3.2(iv) and [7, Theorem 3.6 and Corollary 3.7] again to complete the proof.

We can now use some of Vassilev’s computations in [13, §3] to give some examples.

3.6. Examples. Let \( K \) be an algebraically closed field of characteristic \( p \). In these examples, \( X, Y, Z, W \) denote independent indeterminates over \( K \), and \( x, y, z, w \) denote the natural images of \( X, Y, Z, W \) (respectively) in \( R'/a = R \) for appropriate choices of \( R' \) and a proper ideal \( a \) of \( R' \).

(i) As in Vassilev [13, Example 3.12(1)], take 
\[ R' = K[[X, Y, Z]], \quad a = (XY, XZ, YZ) \quad \text{and} \quad R = R'/a. \]

For this \( R \), the test ideal chain is \( 0 \subset m \subset R \). Since \( R \) is an isolated singularity, we have \( \tau(R) = \bar{\tau}(R) \), by [1, §2](d). Therefore \( m \) is the smallest ideal of positive height in \( \mathcal{I}(\Phi(E)) \). Thus in this case,
\[ \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) = \{(x, y), (x, z), (y, z), m\}. \]

(ii) As in Vassilev [13, Example 3.12(2)], take 
\[ R' = K[[X, Y, Z, W]], \quad a = (XYZ, XYW, XZW, YZW) \quad \text{and} \quad R = R'/a. \]

For this \( R \), the test ideal chain is \( 0 \subset (xy, xz, xw, yz, yw, zw) \subset m \subset R \). It therefore follows from Lemma 3.5(i) that
\[ \{ (x, y), (x, z), (x, w), (y, z), (y, w), (z, w), (x, y, z), (x, y, w), (x, z, w), (y, z, w), m \} \subseteq \mathcal{I}(\Phi(E)) \cap \text{Spec}(R). \]

(iii) As in Vassilev [13, Example 3.12(3)], take 
\[ R' = K[[X, Y, Z]], \quad a = (XY, YZ), \quad \text{and} \quad R = R'/a. \]

For this \( R \), the test ideal chain is \( 0 \subset m \subset R \). Because \( R \) is an isolated singularity, \( \tau(R) = \bar{\tau}(R) \), by [1, §2](d). This means that \( m \) must be the smallest ideal in \( \mathcal{I}(\Phi(E)) \) of positive height, so that \( \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) = \{(x, z), (y), m\}. \)

(iv) As in Vassilev [13, Example 3.12(4)], take 
\[ R' = K[[X, Y, Z, W]], \quad a = (XY, ZW) \quad \text{and} \quad R = R'/a. \]

For this \( R \), the test ideal chain is \( 0 \subset (xy, xz, xw, yz, yw, zw) \subset m \subset R \), so that
\[ \{(x, z), (x, w), (y, z), (y, w), (x, y, z), (x, y, w), (x, z, w), (y, z, w), m\} \subseteq \mathcal{I}(\Phi(E)) \cap \text{Spec}(R) \]
by Lemma 3.5(i).
S-TIGHT CLOSURE OVER AN $F$-PURE LOCAL RING

REFERENCES

[1] R. Fedder, ‘$F$-purity and rational singularity’, Transactions Amer. Math. Soc. 278 (1983) 461–480.
[2] M. Hochster and C. Huneke, ‘Tight closure, invariant theory and the Briançon-Skoda Theorem’, J. Amer. Math. Soc. 3 (1990) 31–116.
[3] C. Huneke, ‘Tight closure, parameter ideals and geometry’, Six lectures on commutative algebra, Eds. J. Elias, J. M. Giral, R. M. Miró-Roig and S. Zarzuela, Progress in Mathematics 166 (Birkhäuser, Basel, 1998), pp. 187–239.
[4] I. Kaplansky, Commutative rings (Allyn and Bacon, Boston, 1970).
[5] G. Lyubeznik and K. E. Smith, ‘Strong and weak $F$-regularity are equivalent for graded rings’, American J. Math. 121 (1999) 1279–1290.
[6] G. Lyubeznik and K. E. Smith, ‘On the commutation of the test ideal with localization and completion’, Transactions Amer. Math. Soc. 353 (2001) 3149–3180.
[7] R. Y. Sharp, ‘Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure’, Transactions Amer. Math. Soc. 359 (2007) 4237–4258.
[8] R. Y. Sharp, ‘Graded annihilators and tight closure test ideals’, J. Algebra 322 (2009) 3410–3426.
[9] R. Y. Sharp, ‘An excellent $F$-pure ring of prime characteristic has a big tight closure test element’, Transactions Amer. Math. Soc. 362 (2010) 5455–5481.
[10] R. Y. Sharp, ‘Big tight closure test elements for some non-reduced excellent rings’, J. Algebra 349 (2012) 284-316.
[11] D. W. Sharpe and P. Vámos, Injective modules, Cambridge Tracts in Mathematics and Mathematical Physics 62 (Cambridge University Press, Cambridge, 1972).
[12] K. E. Smith, ‘Tight closure of parameter ideals’, Inventiones math. 115 (1994) 41–60.
[13] J. C. Vassilev, ‘Test ideals in quotients of $F$-finite regular local rings’, Transactions Amer. Math. Soc. 350 (1998) 4041–4051.

School of Mathematics and Statistics, University of Sheffield, Hicks Building, Sheffield S3 7RH, United Kingdom
E-mail address: R.Y.Sharp@sheffield.ac.uk