q SYK models from deformed Almheiri-Polchinski gravity

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ABSTRACT: In this paper, based on the notion of gauge/gravity duality, we explore q SYK models in the light of Yang-Baxter (YB) deformations. The gravitational counterpart of this duality turns out to be the YB deformed Almheiri-Polchinski (AP) model embodied with hyperbolic potential for the dilaton. The dual geometry is constructed with certain conditions on the classical $\mathcal{R}$-operator which satisfies homogeneous Classical Yang-Baxter Equations (CYBE). The field theory dual to such deformations turns out to be the q SYK model with “appropriate” quantum corrections to the classical IR solution. We identify these quantum effects as being the reminiscent of some UV physics (that comes into play at next to leading order in the large N expansion) over the classical (large N) saddle point solution. With an appropriate tuning of the YB parameter these UV effects get completely decoupled from the rest of the spectrum which thereby reproduces the original q SYK spectrum at strong coupling ($|Jt| \gg 1$).

KEYWORDS: AdS-CFT Correspondence, Gauge-Gravity correspondence, 2D Gravity, 1/$N$ Expansion
1 Overview and motivation

Recently, the Sachdev-Ye-Kitaev (SYK) model [1]-[6] has been proposed as being one of the most promising laboratories in order to understand the origin of holographic duality. In the strict IR limit, the large N version of the theory is believed to be dual to the Lorentzian \(AdS_2\). As a remarkable feature, the strong coupling (\(|Jt| \gg 1\)) version of the model turns out to be \textit{exactly} solvable at the IR fixed point [7]-[31]. However, due to the presence of an infinite tower of zero (Goldstone) modes, the corresponding two point correlations diverge and therefore one needs to move slightly away from the deep infrared in order for the theory to make sense. This in turn is related to the fact that quantum gravity in pure \(AdS_2\) is pathological in the sense that the gravitational back-reaction is infinite. One way to get rid of this instability is to couple the theory with dilaton that eventually breaks some of the IR conformal symmetries of \(AdS_2\).

The quest for a sensible gravity dual [32]-[39] for the SYK model has been a non trivial issue until very recently [40]-[43]. The zero temperature version of the model turns out to be dual to a 3D spacetime \((AdS_2 \times S^1/Z_2)\) that could be viewed as near horizon limit of a near extremal black brane\(^1\). In this model, the dilaton plays the role of the compact third direction \((S^1/Z_2)\) in the dual gravity picture. It turns out that the spectrum associated with the Kaluza-Klein (KK) modes of the scalar fluctuations (propagating in the bulk 3D spacetime) precisely matches to that with corresponding spectrum of bi-local propagators in the \(q\) SYK model at strong coupling. The purpose of the present article is to understand as well as extend the notion of above SYK/AdS duality [40]-[41] in a broader context which we elaborate below.

The motivation behind the present analysis stems from a recent observation where based on the notion of the so called Yang-Baxter (YB) deformations\(^2\) [44]-[50], the authors in [51]-[53] had proposed an interesting modification of the 2D dilaton gravity model [32]-[34] where the quadratic dilaton potential is replaced by a hyperbolic function. The resulting (deformed \((AdS_2)_\eta\)) metric turns out to be a solution for the (YB) deformed version of the Almheiri-Polchinski (AP) model [34]. If the holographic correspondence is believed to hold true for \textit{generic} spacetimes those are solutions to the classical Einstein gravity (with certain amount of matter excitation) then YB deformations of the above form [51]-[53] should have a natural interpretation in terms of bi-local excitation [13]-[14] associated with the large N version of the \(q\) SYK model at strong coupling. However, for the sake of our present analysis, we would restrict our attention towards YB deformations with classical \(\mathcal{R}\)-operator satisfying classical

\(^1\)For generic \(q\) point vertices this interpretation is not quite obvious [41].

\(^2\)The interested reader is encouraged to go through the section 2.1 and references therein for a detailed discussion on YB deformations and its relevance in the context of holographic correspondence.
YB equation (CYBE). We leave the issue corresponding to the modified CYBE for the purpose of future investigations.

With this vision in mind, the goal of the present calculation is therefore to take *holographic* lessons on the implications of YB deformations towards the strong coupling dynamics associated with $q$ SYK model in the large $N$ limit$^3$. Our objective is twofold namely - (I) To explore the *holographic* consequences of YB deformations on the spectrum of $q$ SYK and (II) provide an interpretation in terms of bi-local and/or collective field excitation $^{[13]}$-$^{[14]}$ within the framework of the $q$ SYK model. Our analysis reveals that YB deformation (in the dual gravitational counterpart $^{[51]}$) triggers a non trivial *rescaling$^4*$ associated with collective excitation and/or bi-local fields $^{[13]}$ in the $q$ SYK model. These YB effects could be interpreted as being that of the *modified* “quantum fluctuations” (around the *classical* IR saddle point solution $^{[13]}$) which thereby affects the corresponding two point correlations at strong coupling. As we elaborate in section 2.3 that quantum modifications triggered due to YB deformations (introduced in the dual gravity picture) should be regarded as being that of the *reminiscent* of certain UV effects (over the *classical* large N saddle point solution in the $q$ SYK model) in the same sense as that of the holographic RG flow $^{[54]}$-$^{[57]}$.

The organization of the paper is the following: The entire analysis could be divided mainly into two parts. In the first part of the analysis (Section 2 and 3) we propose a dual gravity picture for the $q$ SYK model in the presence of YB deformations and compute two point correlations $^{[40]}$-$^{[41]}$ in the large $q(\gg 1)$ limit. The strong coupling propagator corresponding to zero modes receives an expected divergent $\mathcal{O}(J)$ contribution as in the undeformed scenario $^{[40]}$-$^{[41]}$. However, the subleading corrections due to YB deformations turn out to be suppressed compared to that with the leading (large $J$) contribution. Next, in section 4, we systematically build up the corresponding $q$ SYK model and identify the YB contributions as being some form of high energy (UV) quantum fluctuations around the IR saddle point solution. Finally, we conclude in section 5.

### 2 YB deformations and $\text{AdS/CFT}$

For the consistency of the analysis as well as the familiarity of the reader, we start our analysis with a preliminary discussion on Yang-Baxter (YB) sigma models and their relevance in gauge/string duality. This eventually helps one to understand the broader perspective behind the present analysis. Having said that, next we move on to the YB

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$^3$For the sake of technical simplicity, in the present analysis we focus only on *bound* states $^{[8]}$ with discrete energy eigen values.

$^4$This could also be interpreted as a field redefinition associated with the collective modes in SYK.
deformed version of the 2D dilaton gravity [51]-[53] and discuss the consequences of
the holographic RG flow (in terms of the YB deformation parameter) on the dynamics
associated to the $q$ SYK model at strong coupling.

2.1 Preliminaries

Yang-Baxter (YB) sigma models [44]-[46] are defined as one parameter integrable
deformations of the principal chiral model (PCM) over some compact Lie group $G$,

$$S_{\text{YB}} = -\frac{1}{2} \int_{W} d\tau d\sigma \gamma^{\alpha\beta} \text{tr} \left( \partial_{\alpha} g g^{-1} \left( \mathbb{I} + \eta \frac{2}{\mathcal{R}(\mathcal{R})} \partial_{\beta} g g^{-1} \right) \right)$$

$$\equiv -\frac{1}{2} \int_{W} d\tau d\sigma \gamma^{\alpha\beta} \text{tr} \left( J_{\alpha} \left( \frac{1 + \eta^{2}}{\mathbb{I} - \eta \mathcal{R}} \right) J_{\beta} \right), \quad J_{\rho} = \partial_{\rho} g g^{-1} \tag{2.1}$$

where, the trace is taken over some representation of the Lie group ($G$) with Lie algebra,
$\mathfrak{g} = \text{Lie}(G)$ and $|\eta| \geq 0$ is the YB deformation parameter. Here, $g(\tau, \sigma)$ is a group valued field that defines a smooth map from the 2D worldsheet ($W$) to the group $G$,

$$g : W \rightarrow G. \tag{2.2}$$

The linear operator $\mathcal{R}$ defines a smooth map,

$$\mathcal{R} : \mathfrak{g} \rightarrow \mathfrak{g} \tag{2.3}$$

that satisfies the (modified) classical YB equation (mCYBE) of the following form,

$$[\mathcal{R}(X), \mathcal{R}(Y)] - \mathcal{R}([\mathcal{R}(X), Y] + [X, \mathcal{R}(Y)]) = \mathcal{C} [X, Y], \quad X, Y \in \mathfrak{g}. \tag{2.4}$$

For, $\mathcal{C} \neq 0$ one ends up being referring at mCYBE. On the other hand, with $\mathcal{C} = 0$ the resulting equation is termed as CYBE. Besides preserving the integrability of the PCM, the YB sigma models (2.1) also exhibit a classical $q$ deformed $U_{q}(\mathfrak{g}) \times G_{R}$ symmetry where, $q = q(\eta)$ is a function of the YB deformation parameter [46] so that in the limit of the vanishing deformation one recovers the original $G_{L} \times G_{R}$ symmetry of the PCM. When applied to $AdS_{5} \times S^{5}$ string sigma models, YB deformations give rise to interesting consequences in the context of gauge/string duality. It eventually allows us to incorporate a broader class of stringy geometries within the unified framework of

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$^{5}$YB string sigma models corresponding to deformed $AdS_{5} \times S^{5}$ possess a dual interpretation in terms of non commutative (NC) gauge theories living on the boundary. In fact, it has been shown [58]-[62] that the NC parameter corresponding to dual gauge theory is directly related to the classical $\mathcal{R}$- operator of homogeneous CYBE. However, YB deformations corresponding to the classical $\mathcal{R}$-operator satisfying mCYBE lead to geometries that satisfy generalized type IIB SUGRA [63]-[64] in the presence of a Killing vector field.
gauge/string duality. In particular, referring back to the original conjectured duality between type IIB super-strings propagating in $AdS_5 \times S^5$ and that of the $\mathcal{N} = 4$ SYM in 4D, the YB $AdS_5 \times S^5$ string sigma models provide a non trivial generalization of the duality. QFTs those are dual to these geometries could be realized as some form of $q$ deformed dynamical spin chain models [65] that preserves integrability like in the original $\mathcal{N} = 4$ SYM example [66]. Depending on whether the classical $\mathfrak{R}$-operator satisfies the mCYBE [47] or CYBE\textsuperscript{6} [49] one could in principle generate a broader class of dual geometries on the gravitational counterpart. Motivated from the above examples, the purpose of the present article is to generalize the notion of $AdS_2/SYK$ duality [40] in the presence of YB deformations. As mentioned previously in the introduction, the YB deformation of the Almheiri-Polchinski (AP) model has been recently constructed by authors in [51]. This construction follows from a similar spirit of deforming the $AdS_5 \times S^5$ (super)coset with classical $\mathfrak{R}$-operator. However, the dual interpretation of such deformations in terms of SYK d.o.f. is yet to be understood. The purpose of the present analysis therefore is to fill up this gap and provide a systematic understanding of the $q$ SYK model in the presence of YB deformations. However, for the purpose of our present calculation, we would restrict ourselves only to the sector where the classical $\mathfrak{R}$ operator satisfies homogeneous CYBE.

2.2 AP model with YB deformations

We now move towards the most generic solution corresponding to the 2D Einstein-Maxwell-dilaton theory in the presence of Yang-Baxter (YB) corrections [44]-[45] and explore the possibilities regarding a 3D uplift [41] that would essentially stands for the dual gravitational counterpart for the SYK model with (YB rescaled [43]) $q$ fermion interactions at strong coupling [7]. We start our analysis with a systematic review of the $q = 4$ case and then generalizing the solution corresponding to the arbitrary $q$ point interactions that is dual to deformed Almheiri-Polchinski (AP) gravity [51] in (2+1)D.

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\textsuperscript{6}YB $AdS_5 \times S^5$ sigma models corresponding to CYBE exhibit a much richer structure in the sense that they allow us to incorporate a much larger class of stringy geometries within the realm of gauge/string duality. For example, the Lunin Maldacena background [67] has been identified to that with the YB $AdS_5 \times S^5$ sigma models with $\mathfrak{R}$ operator satisfying CYBE [68]. However, the analysis [68] is quite restricted to that with the bosonic subsector of the full supercoset model and a generalization in the presence of fermionic d.o.f. is yet to be accomplished.
2.2.1 The $q = 4$ case

Let us first focus on the background dual to SYK model with four fermion interactions\(^\text{7}\)[40],

\[
    ds^2 = \frac{1}{z^2} (-dt^2 + dz^2) + \left( 1 + \frac{\alpha}{z} \right)^2 dy^2
\]

which by means of the following substitution,

\[
y = \sin^{-1}(\sqrt{\theta})
\]

reverts back the metric of the following form\(^\text{8}\)[41],

\[
    ds^2 = \frac{1}{z^2} (-dt^2 + dz^2) + \frac{(1 + \frac{\alpha}{z})^2}{4|\theta|(1 - |\theta|)} d\theta^2.
\]

The above solution (2.12) could be easily generalized in the presence of YB deformations as\(^\text{9}\),

\[
    ds^2 = \frac{F(z, \alpha)}{z^2} (-dt^2 + dz^2) + \frac{G(z, \alpha)}{4|\theta|(1 - |\theta|)} d\theta^2.
\]

where we have introduced the function\(^\text{10}\),

\[
    F(z, \alpha) = \frac{1}{1 - \frac{\eta^2 \alpha^2}{z^2}}.
\]

Using (2.6), this trivially generalizes to,

\[
    ds^2 = \frac{F(z, \alpha)}{z^2} (-dt^2 + dz^2) + \frac{G(z, \alpha)}{4|\theta|(1 - |\theta|)} d\theta^2.
\]

where we have used a short hand notation for the function,

\[
    G(z, \alpha) = 1 + \frac{1}{2\eta} \log \left( \frac{1 + \eta \alpha/z}{1 - \eta \alpha/z} \right).
\]

\(^7\)We set $L = 1$ throughout our analysis.

\(^8\)Here, the compact direction is constrained by $0 < |\theta| < 1$. Also notice that unlike the previous analysis [41] here one needs to retain terms associated with the SYK coupling $\alpha(\sim 1/J)$ as YB effects manifest at quadratic order in the coupling [43].

\(^9\)It has been recently argued that (2.8) could be systematically uplifted to a 3D geometry [43] which acts as a solution for the (YB) deformed AP model [51] that replaces the standard dilaton potential with a hyperbolic function.

\(^10\)Notice that here we have considered the special case with homogeneous CYBE which amounts of seting, $X.P = \frac{\alpha}{z}$, $X^2 = -1$ and $P^2 = 0$ [51].
2.2.2 The generic $q$ solution

We now intend to construct the deformed AP model corresponding to generic $q$ point vertex. To start with, we note down the $q$ fermion generalization of (2.12),

$$ds^2 = (\sin y)^{2(4/q-1)} \left[ \frac{1}{z^2} (-dt^2 + dz^2) + \left(1 + \frac{\alpha}{z}\right)^2 dy^2 \right]. \quad (2.12)$$

Using (2.6), this precisely reduces to [41],

$$ds^2 = |\theta|^{4/q-1} \left[ \frac{1}{z^2} (-dt^2 + dz^2) + \frac{(1 + \frac{\alpha}{z})^2}{4|\theta|(1-|\theta|)} d\theta^2 \right]. \quad (2.13)$$

Like in the previous example, a straightforward YB generalization of (2.13) is indeed quite trivial,

$$ds^2 = |\theta|^{4/q-1} \left[ \frac{F(z, \alpha)}{z^2} (-dt^2 + dz^2) + \frac{G^2(z, \alpha)}{4|\theta|(1-|\theta|)} d\theta^2 \right] \quad (2.14)$$

which serves as the starting point for our subsequent analysis.

2.3 Remarks on RG flow

Before we conclude this section, several important remarks are in order. In the very first place, one should notice that the above background (2.14) suffers from an unusual metric singularity associated with the function $F(z, \alpha)$. This constraints our bulk calculations within a finite radial cutoff,

$$|z_B| = \eta \alpha \equiv \frac{\eta}{J}. \quad (2.15)$$

The existence of such radial cutoffs turns out to be a very special feature for spacetimes embodied with YB deformations [69]-[70]. One could therefore imagine constraining the bulk spacetime within a singularity surface (also known as the holographic screen [69]-[70]) whose radial location is fixed at a finite distance $z = z_B$ that eventually could be treated as the boundary for the bulk spacetime under consideration. The dual QFT is therefore considered to be living on this holographic screen. As far as the Gauge/Gravity duality is concerned, the above entity (2.15) could be thought of as being that of the energy scale associated with the holographic RG flow. In other words, depending on the (radial) location of the singularity surface one is supposed to probe the dynamics associated with the dual QFT under the RG flow. The UV fixed point associated with this RG flow corresponds to the criteria, $|z| \sim 0$ which yields the metric corresponding to $AdS_2$ with dilaton being completely trivial. On the other
hand, the end point of this RG flow is determined by the criteria, \( |\eta| = \Lambda_{IR} \gg 1 \) where \( \Lambda_{IR} \) stands for that of the deep IR cutoff in the bulk. Imposing this condition on the metric (2.14), the effective 2D background turns out to be\(^{11}\),

\[
ds^2 \bigg|_{2D} \approx \frac{1}{|\varepsilon|^2} (-dT^2 + d\varepsilon^2) \tag{2.16}
\]

where, we have included new variables like,

\[
\varepsilon = 1 - \frac{|\Lambda_{IR}|}{z}, \quad T = \frac{t}{|\Lambda_{IR}|}, \quad |\varepsilon| \ll 1. \tag{2.17}
\]

Here, \( \zeta = \frac{1}{2} \) is the dynamical critical exponent in the deep IR. Therefore, to summarize, the theory flows from a \( \zeta = 1 \) UV conformal fixed point to a \( \zeta = \frac{1}{2} \) IR Lifshitz fixed point as far as YB deformations are concerned.

As we shall see in the subsequent section 4 (also see the Appendix A) that these YB deformations are supposed to be modifying the quantum fluctuations around the fixed classical IR background (4.9) of the q SYK model. Therefore in the limit, \( |\eta| \ll 1 \) (which is the limit considered in this paper) quantum \( \mathcal{O} (1/\sqrt{N}) \) modifications due to YB deformations should be regarded as being that of the reminiscent of some high energy (UV) physics over the critical (saddle point) IR background and therefore is a purely large but finite \( J \) effect. However, as expected, in the strict limit of \( \eta = 0 \) these UV effects completely decouple from the rest of the spectrum and one recovers the original “IR physics” corresponding to the \( q \) SYK model \([13]\).

### 3 Correlators in the deformed AP model

#### 3.1 The quadratic action

We start considering a single scalar field over the background (2.14),

\[
S_\varphi = \frac{1}{2} \int dt dz d\theta \sqrt{-g} \mathcal{L}_\varphi \tag{3.1}
\]

where, we define the corresponding scalar Lagrangian as,

\[
\mathcal{L}_\varphi = - (\partial_a \varphi)^2 - \mathcal{V}(\theta) \varphi^2 \tag{3.2}
\]

with the scalar potential \([41]\) introduced as\(^{12}\),

\[
\mathcal{V}(\theta) = \frac{1}{|\theta|^{4/q-1}} \left[ 4 \left( \frac{1}{q} - \frac{1}{4} \right)^2 + m_0^2 + \frac{2V}{J(\theta)} \left( 1 - \frac{2}{q} \right) \delta(\theta) \right]. \tag{3.3}
\]

\(^{11}\)We ignore the trivial dilaton profile near both the fixed points of the RG flow.

\(^{12}\)Here, \( V \) is some unknown parameter of the model that would be eventually fixed under certain specific physical considerations.
Notice that here $V$ is a constant together with,

$$J(\theta) = \frac{|\theta|^{2/q-1}}{2\sqrt{1 - |\theta|}}. \quad (3.4)$$

A straightforward calculation reveals,

$$\sqrt{-g} \mathcal{L}_\varphi = J(\theta) G(z, \alpha) \left[ (\partial_t \varphi)^2 - (\partial_z \varphi)^2 - \frac{4F(z, \alpha)}{z^2 G^2(z, \alpha)} |\theta|(1 - |\theta|)(\partial_\theta \varphi)^2 \right]$$

$$- \frac{J(\theta)}{z^2} F(z, \alpha) G(z, \alpha) \left[ 4 \left( \frac{1}{q} - \frac{1}{4} \right)^2 + m_0^2 + \frac{2V}{J(\theta)} \left( 1 - \frac{2}{q} \right) \delta(\theta) \right] \varphi^2. \quad (3.5)$$

The range for $\theta$ integral ranges between -1 to +1 which clearly hits the delta function discontinuity near $\theta \sim \varepsilon \sim 0$. One way to get rid of this delta function discontinuity is to impose constraints on the scalar field near this singularity [41] which is subjected to the Dirichlet boundary conditions of the following form,

$$\varphi(t, z, \pm 1) = 0. \quad (3.6)$$

A careful analysis on the $\theta$ integral yields the following,

$$S_\varphi = S_\varphi^{(B)} - \int_{\partial \Sigma} dt dz \frac{F(z, \alpha)}{z^2 G(z, \alpha)} \varphi(t, z, 0) B(t, z, \varepsilon) \quad (3.7)$$

where, we set the constraint,

$$B(t, z, \varepsilon) = \left[ 2\theta^2 q_\theta \varphi \right]_{\theta=\varepsilon} + VG^2(z, \alpha) \varphi(t, z, 0) = 0 \quad (3.8)$$

subjected to the fact that, $\varphi(t, z, \varepsilon) = \varphi(t, z, -\varepsilon)$ [41]. This finally leads to the bulk quadratic action of the following form,

$$S_\varphi^{(B)} = \frac{1}{2} \int d^2 x \int_0^1 d\theta J(\theta) \varphi(x, \theta) \hat{D} \varphi(x, \theta); \quad (x = t, z) \quad (3.9)$$

where, we have introduced the operator,

$$\hat{D} = G(z, \alpha) (-\partial_t^2 + \partial_z^2) + G'(z, \alpha) \partial_z + \frac{4F(z, \alpha)}{z^2 G(z, \alpha)} \left[ \theta(1 - \theta) \partial_\theta^2 + \left( \frac{2}{q} - \theta \left( \frac{1}{2} + \frac{2}{q} \right) \right) \partial_\theta \right]$$

$$- \frac{F(z, \alpha)}{z^2} G(z, \alpha) \left[ 4 \left( \frac{1}{q} - \frac{1}{4} \right)^2 + m_0^2 \right]. \quad (3.10)$$
Considering a strong coupling limit ($|J| \gg 1$) it is quite natural to expand (3.10) perturbatively in the coupling $\alpha(\sim 1/J)$ [40] which yields the following expansion of the operator,

$$\hat{D} = \hat{D}^{(0)} + \delta \hat{D} + \mathcal{O}(\alpha^3)$$  \hspace{1cm} (3.11)

where, the individual terms in the expansion (3.11) could be formally expressed as,

$$\hat{D}^{(0)} = -\partial_t^2 + \partial_z^2 + \frac{4}{z^2} \left[ \theta(1-\theta) \partial_\theta^2 + \left( \frac{2}{q} - \theta \left( \frac{1}{2} + \frac{2}{q} \right) \right) \partial_\theta \right]$$

$$- \frac{1}{z^2} \left[ 4 \left( \frac{1}{q} - \frac{1}{4} \right)^2 + m_0^2 \right]$$  \hspace{1cm} (3.12)

$$\delta \hat{D} = \frac{\alpha}{z} \left[ (-\partial_t^2 + \partial_z^2) - \frac{1}{z} \partial_z \right] - \frac{\alpha}{z^3} \left( 1 + \frac{\eta^2 \alpha}{z} \right) \left[ 4 \left( \frac{1}{q} - \frac{1}{4} \right)^2 + m_0^2 \right]$$

$$- \frac{4\alpha}{z^3} \left( 1 - \frac{\alpha (\eta^2 + 1)}{z} \right) \left[ \theta(1-\theta) \partial_\theta^2 + \left( \frac{2}{q} - \theta \left( \frac{1}{2} + \frac{2}{q} \right) \right) \partial_\theta \right].$$  \hspace{1cm} (3.13)

Notice that (3.12) is identical to that with the earlier results in [41]. However on the other hand, we identify (3.13) as the new addition to the SYK spectrum that has its root in YB deformations associated with the $AdS_2$ sigma model [51]-[53]. Going into the Fourier space, one might express the scalar modes as:

$$\varphi(x, \theta) = \int \frac{d\nu dk}{\mathcal{N}_\nu} e^{-iwz} \sqrt{z} \mathcal{Z}_\nu(|wz|) \sigma_k(\theta) \zeta_w(\nu, k)$$  \hspace{1cm} (3.14)

which finally yields the quadratic action of the following form,

$$S_\varphi^{(B)} = \frac{1}{2} \int dt dz d\theta J(\theta) \int \frac{dw' d\nu' dk'}{\mathcal{N}_{\nu'} \mathcal{N}_\nu} e^{-iw'z} \sqrt{z} \mathcal{Z}_\nu^*(|w'z|) \sigma_k'(\theta) \zeta_{w'}(\nu', k') \times \int dw d\nu dk \hat{D} e^{-iwz} \sqrt{z} \mathcal{Z}_\nu(|wz|) \sigma_k(\theta) \zeta_w(\nu, k).$$  \hspace{1cm} (3.15)

\textsuperscript{13}In principle the integral over $\nu$ could be thought of as being that of the sum over both discrete (bound states) modes as well as the continuous modes/scattering states [8]. However, for the purpose of our present analysis we would like to restrict our computations only for those of the discrete modes with $\nu = 3/2 + 2n$ ($n = 0, 1, 2, \ldots$) that also sets, $\mathcal{N}_\nu = (2\nu)^{-1}$ [13]. Therefore one might thought of replacing the integral as a sum over discrete modes and/ or the bound states,

$$\int d\nu \rightarrow \sum_{\nu = 3/2 + 2n}.$$
Our next task would be to diagonalize the operator (3.11), \( \hat{D} = \hat{D}^{(0)} + \hat{D}^{(1)} \) and obtain the corresponding spectra. This eventually translates into a problem of solving the corresponding eigenvalue equation associated with this operator. Before we proceed further, it is customary to express the operator (3.11) in the following form,

\[
\hat{D} = \hat{D}_z(\alpha, \eta, z) + \frac{4}{z^2} \mathcal{F}(\alpha, \eta, z) \hat{D}_\theta(\theta) \tag{3.16}
\]

where, the individual components could be formally expressed as,

\[
\hat{D}_z = \left( 1 + \frac{\alpha}{z} \right) \left( \partial_z^2 - \partial^2 \right) - \frac{\alpha}{z^2} \partial_z - \frac{\mu^2}{z^2} \tag{3.17}
\]

\[
\hat{D}_\theta(\theta) = \theta(1 - \theta) \partial^2_\theta + \left( \frac{2}{q} - \theta \left( \frac{1}{2} + \frac{2}{q} \right) \right) \partial_\theta \tag{3.18}
\]

together with the function,

\[
\mathcal{F} = 1 - \frac{\alpha}{z} \left( 1 - \frac{\alpha(\eta^2 + 1)}{z} \right)
\]

\[
\mu^2 = \mu^2_0(q) \left( 1 + \frac{\alpha}{z} \left( 1 + \frac{\eta^2 \alpha}{z} \right) \right) ; \quad \mu^2_0(q) = m^2_0 + 4 \left( \frac{1}{q} - \frac{1}{4} \right)^2. \tag{3.19}
\]

### 3.2 Diagonalization of \( \hat{D}_\theta \)

We start with proposing the eigenvalue equation corresponding to the angular operator,

\[
\hat{D}_\theta(\theta) \sigma_k(\theta) = -\frac{k^2}{4} \sigma_k(\theta) \tag{3.20}
\]

which has a solution of the following form,

\[
\sigma_k(\theta) = C_1(-1)^{\frac{n}{2}} \theta^{\frac{n}{2}} \theta^{\frac{n}{2}} F_1 \left( a_1, a_2; 2 - \frac{2}{q}; \theta \right) + C_2 F_1 \left( b_1, b_2; 2; \theta \right) \tag{3.21}
\]

where, the individual entities could be formally expressed as,

\[
a_n = \frac{(-1)^n}{4q} \left( \sqrt{q} \left( 4qk^2 + q - 8 \right) + 16 - (-1)^n (4 - 3q) \right),
\]

\[
b_n = \frac{(-1)^n}{4q} \left( \sqrt{q} (4k^2q + q - 8) + 16 + (-1)^n (q - 4) \right). \tag{3.22}
\]

Here, \( C_{1,2} \) are integration constants that needs to be fixed using boundary conditions. Substituting the above solution (3.21) into (3.8) and subsequently expanding in the
small variable $\theta(\sim 0)$ one finds\textsuperscript{14},
\[ C_1 + \frac{k^2 q}{8} C_2 \approx \frac{V}{2} C_2. \quad (3.23) \]

On the other hand, imposing the other boundary condition (3.6) one finds,
\[ C_1 \approx \frac{C_2}{32} (q + \pi + 6 \log 2). \quad (3.24) \]

Combining (3.23) and (3.24) we find,
\[ V = \frac{k^2 q}{4} + \frac{1}{16} (q + \pi + 6 \log 2) + O(1/q). \quad (3.25) \]

Setting\textsuperscript{15},
\[ k^2 \approx 2 - \frac{4}{q^2} \quad (3.26) \]

for zero modes in the large $q$ limit we finally fix the potential,
\[ V(q) = \frac{9q}{16} + \pi + 6 \log 2 + O(1/q). \quad (3.27) \]

### 3.3 The Green’s function

The action (3.15) could be expressed as a perturbation in the coupling $\alpha$ namely,
\[ S^{(B)}_\varphi = S^{(0)}_\varphi + \alpha S^{(1)}_\varphi + \alpha^2 S^{(2)}_\varphi. \quad (3.28) \]

In the following, we evaluate each of these above entities separately.

#### 3.3.1 Zeroth order computation

We first note down the action at zeroth order in $\alpha$,
\[ S^{(0)}_\varphi = \frac{1}{2} \int dt dz d\theta J(\theta) \int \frac{dw' d\nu' dk'}{N_\nu N_{\nu'}} e^{-iw't} \sqrt{z} Z_{\nu'}(|w'z|) \sigma_k(\theta) \zeta_{w'}(\nu', k') \times \int dw d\nu dk \hat{D}^{(0)}(\nu, k) e^{-iwt} \sqrt{z} Z_{\nu}(|wz|) \sigma_\nu(\theta) \zeta_w(\nu, k) \quad (3.29) \]

\textsuperscript{14}In order to arrive at the above condition (3.23) we have made use of the fact that the parameter $q(\gg 1)$ (together with the fact $|q\theta| < 1$) of the theory is large enough so that one can consider an expansion in $1/q$ and keep terms only up to leading order in the expansion.

\textsuperscript{15}This solution is obtained (in the large $q(\gg 1)$ limit) by demanding that the leading order propagator (3.31) associated to zero modes has a pole.
where, the zeroth order operator could be expressed as usual,
\[
\hat{D}^{(0)} = -\partial_t^2 + \partial_z^2 - \frac{\mu_0^2(q)}{z^2} + \frac{4}{z^2} \hat{D}_\theta(\theta).
\] (3.30)

Using the completeness relation for the Bessel function together with the orthogonality condition [41] for \(\sigma_k(\theta)\) it is indeed trivial to show,
\[
S^{(0)}_{\varphi} = \frac{1}{2} \int d\nu dk C(k) \frac{C(k)}{N_{\nu} N_{\nu'}} \zeta_{\nu}(\nu, k) (\nu^2 - \nu_0^2(q)) \zeta_{\nu'}(\nu, k)
\] (3.31)
where, we have introduced,
\[
\nu_0^2(q) = \mu_0^2(q) + k^2 + \frac{1}{4}.
\] (3.32)

It is noteworthy to mention that \(Z_{\nu}(|wz|)\) satisfies the standard Bessel equation,
\[
(z^2 \partial_z^2 + z \partial_z + w^2 z^2) Z_{\nu}(|wz|) = \nu^2 Z_{\nu}(|wz|).
\] (3.33)

### 3.3.2 Yang-Baxter shift

The first order shift in the action (3.28) could be formally expressed as,
\[
S^{(1)}_{\varphi} = \frac{1}{2} \int dt dz d\theta J(\theta) \int \frac{dw' dk'}{N_{\nu} N_{\nu'}} e^{-iwt} \sqrt{z} Z_{\nu'}(|w' z|) \sigma_{\nu'}(\theta) \zeta_{\nu'}(\nu', k')
\times \int d\nu d\nu' d\nu'' e^{-iwt} \sqrt{z} Z_{\nu}(|wz|) \sigma_{\nu}(\theta) \zeta_{\nu}(\nu, k)
\] (3.34)
where, the operator above in (3.34) is given by,
\[
\hat{D}^{(1)} = \frac{1}{z} (-\partial_t^2 + \partial_z^2) - \frac{1}{z^2} \partial_z - \frac{\mu_0^2(q)}{z^3} - \frac{4}{z^2} \hat{D}_\theta(\theta).
\] (3.35)

Using (3.33), it is quite straightforward to show,
\[
\hat{D}^{(1)} e^{-iwt} \sqrt{z} Z_{\nu}(|wz|) \sigma_{\nu}(\theta) = \frac{1}{z^{5/2}} (-z \partial_z + G(\nu, q, k)) e^{-iwt} Z_{\nu}(|wz|) \sigma_{\nu}(\theta)
\] (3.36)
where, the new entity above could be expressed as,
\[
G(\nu, q, k) = \nu^2 - \frac{3}{4} - \mu_0^2(q) + k^2.
\] (3.37)

Using the following identity [40],
\[
\partial_z Z_{\nu}(|wz|) = \frac{\nu}{z} Z_{\nu}(|wz|) - |w|(J_{\nu+1}(|wz|) - \xi_{\nu} J_{\nu-1}(|wz|))
\] (3.38)
one might further simplify (3.36) as\textsuperscript{16},

\[
\mathcal{D}(\nu) e^{-iwt} \sqrt{z} \mathcal{Z}_\nu(|wz|) \sigma_k(\theta) = \left[ \frac{1}{2^{3/2}} (G(\nu, q, k) - \nu) \mathcal{Z}_\nu(|wz|) + \frac{|w|}{2^{3/2}} \mathcal{Z}_{\nu+1}(|wz|) \right] e^{-iwt} \sigma_k(\theta).
\]

(3.39)

Substituting (3.39) into (3.34) we find,

\[
S^{(1)}_\nu = \Delta_1 + \Delta_2
\]

(3.40)

where, the individual entities could be formally expressed as,

\[
\Delta_1 = -\frac{1}{2} \int dw \int dk \nu' \mathcal{C}(k) \left| \frac{\zeta_w(\nu', k)}{N_\nu N_{\nu'}} \right| \mathcal{H}_1(\nu, \nu', q) \zeta_w(\nu, k)
\]

\[
\mathcal{H}_1 = \frac{4 \cos \left( \frac{1}{2} \pi (\nu' - \nu) \right)}{\pi \left( \nu'^4 - 2\nu'^2 (\nu'^2 + 1) + (\nu'^2 - 1)^2 \right)}
\]

and,

\[
\Delta_2 = \frac{1}{2} \int dw \int dk \nu' \mathcal{C}(k) \left| \frac{\zeta_w(\nu', k)}{N_\nu N_{\nu'}} \right| \mathcal{H}_2(\nu, \nu', q) \zeta_w(\nu, k)
\]

\[
\mathcal{H}_2 = \frac{2 \cos \left( \frac{1}{2} \pi (\nu - \nu') \right)}{\pi ((\nu + 1)^2 - \nu'^2)}.
\]

(3.41)

(3.42)

Finally, we note down the correction at quadratic order in the coupling,

\[
S^{(2)}_\nu = \frac{1}{2} \int dw \int dk \nu' \mathcal{C}(k) \left| \frac{\zeta_w(\nu', k)}{N_\nu N_{\nu'}} \right| \Sigma(\nu, \nu', q) \zeta_w(\nu, k)
\]

\[
\Sigma = 16 \sin \left( \frac{1}{2} \pi (\nu' - \nu) \right) \left( \mu_0^2(q) + k^2 \eta^2 + k^2 \right) \left( (\nu' - \nu)^2 - 4(\nu'^2 - \nu^2)((\nu' + \nu)^2 - 4) \right).
\]

(3.43)

We focus on the zero modes \cite{40} with \( \nu = \nu' = 3/2 \) and set the scalar mass at the BF bound, \( m_0^2 = -\frac{1}{4} \) which finally yields the two point correlation in the momentum space,

\[
\langle \zeta_{-w}(3/2, k \sim \sqrt{2}) \zeta_{w}(3/2, k \sim \sqrt{2}) \rangle_{q \gg 1} \approx -\frac{9\pi}{4C(\sqrt{2})|w|\Gamma(\alpha)}
\]

(3.44)

where we have defined,

\[
\Gamma(\alpha) = \alpha + \frac{8\pi}{15} \alpha^2(\eta^2 + 1) + \mathcal{O}(\alpha^3).
\]

(3.45)

\textsuperscript{16}Here, we have summed over only real modes with discrete energy eigen values with, \( \nu = 3/2 + 2n \).
The second term on the R.H.S. of (3.45) is precisely the YB contribution to the spectrum of zero modes associated with the $q$ SYK model in the limit of large $q$. Using (3.44) we finally express the space time propagator as,

$$\langle \Phi(x, \theta) \Phi(x', \theta') \rangle_{q \gg 1} \approx -\frac{9\pi}{4\Gamma(\alpha)}|zz'|^{1/2}\Theta(\theta, \theta') \int \frac{dw}{|w|} e^{-iw(t-t')} Z_{3/2}(|wz|) Z_{3/2}(|wz'|)$$

$$\Theta(\theta, \theta') = \frac{\sigma_k(\theta) \sigma_k(\theta')}{C(k)} \bigg|_{k^2 \sim 2+O(1/q^2)}.$$  

(3.46)

A straightforward computation reveals,

$$\sigma_k(0)^2 \bigg|_{k^2 \sim 2+O(1/q^2)} \approx C_1^2$$

(3.47)

which thereby yields the space time propagator,

$$\langle \Phi(x, 0) \Phi(x', 0) \rangle_{q \gg 1} \approx -\frac{9\pi C_1^2}{4\alpha C(k)} \left(1 - \frac{8\pi}{15} \alpha(\eta^2 + 1)\right) |zz'|^{1/2}$$

$$\times \int \frac{dw}{|w|} e^{-iw(t-t')} Z_{3/2}(|wz|) Z_{3/2}(|wz'|)$$

(3.48)

together with,

$$C(k) = \int_0^1 d\theta J(\theta) \sigma_k^2(\theta).$$

(3.49)

Notice that (3.48) is precisely of the form [7] and is sort of expected from the earlier analysis [40]. The leading order contribution goes as $O(J)$ [40] in the limit of large $J$ which thereby diverges. However, the sub-leading (YB) contributions turn out to be suppressed (compared with the leading term) in the limit of strong coupling.

4 $q$ SYK model with YB shift

The purpose of this section is to find an appropriate interpretation for the YB deformations (that acts on $AdS_2$ coset model) in the $q$ SYK model and explore its consequences on the two point function. In particular, we would be interested in the limit of large $N(\gg 1)$ and strong coupling $|Jt|(\gg 1)$ that should be identified with the dual gravitational counterpart described in the previous section.

4.1 Basics

The Hamiltonian corresponding to $q$ SYK model could be formally expressed as [7],

$$\mathcal{H} \sim \sum J_{i_1i_2..i_q} \chi_{i_1} \chi_{i_2} .. \chi_{i_q}$$

(4.1)
where, \( q \) is even integer and \( \chi_i \)s are so called Majorana fermions satisfying,
\[
\{\chi_i, \chi_j\} = \delta_{ij}. \tag{4.2}
\]

Here, \( J_{ijkl} \)s are random couplings satisfying a Gaussian distribution,
\[
\langle J_{ijkl}^2 \rangle = \frac{(q - 1)! J^2}{N^{q-1}}. \tag{4.3}
\]

In the large \( N \) limit, the model is solved by introducing collective excitations \([13]\),
\[
\Psi(t_1, t_2) = \frac{1}{N} \sum_{i=1}^{N} \chi_i(t_1) \chi_i(t_2) \tag{4.4}
\]
which yields the collective action of the following form,
\[
S_{col}[\Psi] = \frac{N}{2} \int dt \partial_t \Psi(t, t') \bigg|_{t'=t} + \frac{N}{2} \text{Tr} \log \Psi - \frac{J^2 N}{2q} \int dt_1 dt_2 \Psi^q(t_1, t_2). \tag{4.5}
\]

Clearly, in the strict IR limit one might ignore the kinetic term which leads to the effective action of the following form,
\[
S^{(IR)}_{col}[\Psi] \approx \frac{N}{2} \text{Tr} \log \Psi - \frac{J^2 N}{2q} \int dt_1 dt_2 \Psi^q(t_1, t_2) \tag{4.6}
\]
which clearly possesses a reparametrization invariance \([14]\),
\[
\Psi(t_1, t_2) \to \Psi_f(t_1, t_2) = |f'(t_1)f'(t_2)|^{1/q} \Psi(f(t_1), f(t_2)). \tag{4.7}
\]
This reparametrisation invariance is an artifact of the existence of an one parameter family of infinite vacua associated with the low energy effective theory at the IR critical point. Expansion around one of these vacua thereby breaks the global symmetry spontaneously generating an infinite tower of Goldstone modes (the so called zero modes) that is responsible for an infinite correlation in the strict IR limit of the theory.

### 4.2 Correlation function

The critical large \( N(\gg 1) \) saddle point solutions could be formally expressed as \([14]\),
\[
\Psi_0(t_1, t_2) = \text{sgn}(t_1 - t_2) \frac{s}{|t_1 - t_2|^{2/q}}, \quad s^q = \frac{\tan\left(\frac{\pi}{q}\right)}{J^2 \pi} \left(\frac{1}{2} - \frac{1}{q}\right). \tag{4.8}
\]
Next, we consider fluctuations around this IR critical point\textsuperscript{17},

$$\Psi(t_1, t_2) \approx \Psi_0(t_1, t_2) + \sqrt{\frac{2}{N}} \zeta_{YB}. \quad (4.9)$$

Based on our analysis in Appendix A, we propose the following rescaling (or field re-definition) associated with the collective field excitation in the $q$ SYK model,

$$\zeta_{YB}(t_1, t_2) = \frac{\zeta(t_1, t_2)}{F_\eta(t_1, t_2)}, \quad F_\eta(t_1, t_2) = \frac{1}{1 - \frac{4\nu^2}{J^2}(t_1 - t_2)^2}. \quad (4.10)$$

Following the notion of Gauge/Gravity duality, one should be able to identify these (4.10) as the (rescaled) scalar fluctuations in the bulk $(AdS_2)_\eta$ and therefore the corresponding correlation functions should also agree. In order to make the above identification, it is useful to introduce the following coordinates,

$$t = \frac{1}{2}(t_1 + t_2), \quad z = \frac{1}{2}(t_1 - t_2) \quad (4.11)$$

and expand the fluctuations in a complete basis\textsuperscript{18} [40],

$$\zeta_{YB}(t, z) = \sum_{\nu=3/2+2n} \int dw \tilde{\psi}_{\nu w}(z) b_{\nu w}(t, z), \quad \tilde{\psi}_{\nu w}(z) = \frac{\psi_{\nu w}}{F_\eta(z)}$$

$$b_{\nu w}(t, z) = \phi(\nu) e^{\pm|w|t} sgn(z) Z_{\nu}(|wz|). \quad (4.12)$$

Substituting (4.12) into (4.6), the action at the quadratic order in the fluctuations could be formally expressed as,

$$S^{(2)} = \frac{1}{2} \int \zeta_{YB}(t_1, t_2) K(t_1, t_2; t_3, t_4) \zeta_{YB}(t_3, t_4) \quad (4.13)$$

where, the kernel [14] is given by,

$$K(t_1, t_2; t_3, t_4) = \Psi_0^{-1}(t_1, t_3) \Psi_0^{-1}(t_2, t_4) + (q - 1) J^2 \delta(t_1 - t_3) \delta(t_2 - t_4) \Psi_0^{-2}(t_1, t_2). \quad (4.14)$$

Using (4.14), one might express the full contribution (4.13) into following two parts,

$$S^{(2)} = S^{(2)}_I + (q - 1) J^2 S^{(2)}_{II} \quad (4.15)$$

which we evaluate one by one.

\textsuperscript{17}As mentioned previously, one should think of YB effects (from the perspective of collective excitation in $q$ SYK model) as being that of the modified “quantum mechanical” fluctuations (in the sense of $1/N$ expansion) around the classical IR saddle point solution ($\Psi_0$).

\textsuperscript{18}We shall fix the pre-factor $\phi(\nu)$ in the Appendix B using consistency criteria.
4.2.1 Evaluation of \( S^{(2)}_I \)

We first note down,

\[
S^{(2)}_I = \frac{1}{2} \int dt_1 dt_2 dt_3 dt_4 \Psi_0^{-1}(t_1, t_3) \zeta_{YB}(t_3, t_4) \Psi_0^{-1}(t_4, t_2) \zeta_{YB}(t_2, t_1) \tag{4.16}
\]

where, we have used the fact that the collective fields are antisymmetric in their arguments [13]. Using (4.12) and the following identity\(^{19}\),

\[
b_{\nu w}(t_1, t_2) = g(\nu, q) \int \frac{dt_a dt_b}{| t_a - t_b |} \Psi_0(t_1, t_a) \Psi_0(t_2, t_b) b_{\nu w}(t_a, t_b) \tag{4.17}
\]

one could further rearrange (4.16) as,

\[
S^{(2)}_I = \frac{1}{2} \sum_{\nu} \int dw \psi_{\nu - w} \frac{N_{\nu} \gamma(q)}{| b(\nu) |} \left( 1 - \frac{\eta^2 \Pi^{(2)}_I(w, \nu)}{2 J^2 N_{\nu}^{'\gamma}} \right) \phi^2(\nu) \psi_{\nu w} + O(\eta^4 / J^4) \tag{4.18}
\]

where we have introduced the function,

\[
\Pi^{(2)}_I(w, \nu) = \frac{w^2}{\nu(\nu^2 - 1)} \tag{4.19}
\]

together with the normalization for the Bessel function [41],

\[
\int_{0}^{\infty} \frac{dz}{z} Z_{\nu'}(|z|) Z_{\nu}(|z|) = N_{\nu} \delta_{\nu \nu'}. \tag{4.20}
\]

4.2.2 Evaluation of \( S^{(2)}_I \)

Finally, we note down the action,

\[
S^{(2)}_I = \frac{1}{2} \int dt_1 dt_2 \zeta_{YB}(t_1, t_2) \Psi_0^{-2}(t_1, t_2) \zeta_{YB}(t_1, t_2). \tag{4.21}
\]

Using (4.8) one could further simplify (4.21) as,

\[
S^{(2)}_I = -\frac{1}{2} \sum_{\nu} \int dw \psi_{\nu - w} \Pi^{(2)}_I(w, \nu, q) \phi^2(\nu) \psi_{\nu w} + O(\eta^4 / J^4) \tag{4.22}
\]

where, we have repackaged the above expression (4.22) in terms of a new function,

\[
\Pi^{(2)}_I(w, \nu, q) = \left( \frac{s^q}{4} \right)^{1-2/q} \left| w \right|^{1-4/q} \Gamma \left( 1 - \frac{2}{q} \right) \Gamma \left( \nu - \frac{1}{2} + \frac{2}{q} \right) \sqrt{\pi} \Gamma \left( \frac{3}{2} - \frac{2}{q} \right) \Gamma \left( \nu + \frac{3}{2} - \frac{2}{q} \right) \left[ 1 - \frac{2 \eta^2 w^2}{J^2} \left( \frac{1 - \frac{2}{q}}{\frac{3}{2} - \frac{2}{q}} \right) \left( \frac{| - \nu + \frac{1}{2} | + \frac{2}{q}}{\nu + \frac{3}{2} - \frac{2}{q}} \right) \right]. \tag{4.23}
\]

\[^{19}\text{See Appendix B for the details of the derivation.}\]
4.2.3 The $q = 4$ case

We first consider the special case with $q = 4$. With this choice the action (4.15) turns out to be,

$$
S^{(2)} = \frac{3J}{32\sqrt{\pi}} \sum_{\nu=3/2+2n} \mathcal{N}_\nu \psi^2(\nu) \int dw \tilde{\Psi}^{*}_{\nu-w}(|\tilde{h}(\nu)| - 1)\tilde{\psi}_{\nu w} 
$$

where, we have introduced,

$$
\tilde{\psi}_{\nu w} \approx \psi_{\nu w} \left( 1 - \eta^2 w^2 \frac{1}{J^2 \nu^2 - 1} \right) 
$$

(4.25)

together with the fact, $|\tilde{h}(\nu)| = |h(\nu)|^{-1}$. Notice that (4.24) is identical to (A.11) once expressed in appropriate coordinates [13]. This finally yields the propagator,

$$
\langle \Psi(t, z)\Psi(t', z') \rangle_{q=4} \approx \text{sgn}(zz') \sum_{\nu=3/2+2n} 16\sqrt{\pi} 3J\mathcal{N}_\nu \times \int dw e^{-iw(t-t')} \frac{Z_\nu(|wz|)Z_\nu(|wz'|)}{(|\tilde{h}(\nu)| - 1) \left( 1 - \frac{\eta^2 w^2}{J^2 \nu^2 - 1} \right)} 
$$

which precisely matches to that with [13] in the limit of the vanishing ($\eta/J \to 0$) deformations. Furthermore, one also encounters a pole for those of the zero ($\nu = 3/2$) modes that recives nontrivial corrections due to finite coupling ($J$)[13].

4.2.4 Large $q$ limit

Finally, we take a specific limit of our model and focus on the large $q(\gg 1)$ sector. A straightforward computation reveals the following shift,

$$
\Delta S^{(2)} = \frac{1}{2} \sum_{\Delta \nu} \int dw \psi_{\nu-w} \mathcal{N}_\nu \Lambda(w, \Delta \nu, q) \psi^{2}(\Delta \nu) \psi_{\nu w} 
$$

(4.27)

where, the dispersion relation corresponding to those of the bound states\(^{20}\) could be obtained by studying the pole structure associated with the two point function,

$$
\Lambda(w, \nu, q) = a(\nu, q)w^3 + b(\nu, q)w^2 + c(\nu, q)w + d(\nu, q) 
$$

(4.28)

\(^{20}\)Remember that the bound states are obtained from those of the simple poles which are (distributed along the real $\nu$ axis [8]) defined by the condition $|\tilde{h}(\nu)| = 1$. This condition is equivalent to that of finding discrete roots ($\nu = 3/2 + 2n$) associated with the transcendental equation,

$$
\tan(\frac{\pi \nu}{2}) = \frac{2\nu}{3} 
$$

- 19 -
where the individual coefficients could be formally expressed as,
\[
\begin{align*}
\alpha(\nu, q) &= \frac{2\eta^2}{3\pi J^2 (\nu^2 - \frac{9}{4})(\nu^2 - \frac{1}{4})}, \\
b(\nu, q) &= -\frac{3\sqrt{\pi} \eta^2 q}{64 J^2 (\nu^2 - 1)}, \\
\delta(\nu, q) &= -\frac{1}{2\pi (\nu^2 - \frac{1}{4})}, \\
f(\nu, q) &= \frac{3\sqrt{\pi} q}{64}.
\end{align*}
\] (4.29)

In the deep IR, one could ignore terms \(O(w^3)\) being very small compared to that with the other terms in (4.28) and solve for the zeros of (4.28) perturbatively in the (inverse) coupling \((J^{-1})\). This finally leads to the dispersion relation of the following form,
\[
|\Delta \nu| \approx \frac{32}{3\pi^{3/2} q} \left| w \right| \left(1 - \frac{9\pi^3 q^2}{4096} \left(\frac{\eta}{J}\right)^2\right).
\] (4.30)

From (4.30) it is worthwhile to mention that in the presence of vanishing deformations, the energy eigen values corresponding to those of the discrete modes receive corrections which go as \((\sim q^{-1})\) \([41]\). However, this value is further reduced and/or corrected due to the presence of YB deformations by a finite amount which is subjected to the fact that the entity, \(q|\eta| < 1\) in the limit when \(q \gg 1\).

5 Concluding remarks

We now summarize our paper with a brief conclusion. The goal of the present analysis was to explore the effects of Yang-Baxter (YB) deformations on the \(q\) SYK spectrum at strong coupling. Our observation is twofold-(I) The YB effects enter as a quantum fluctuation (at next to leading order in the large \(N\) expansion) over the classical saddle point solution in the deep IR, and (II) the ratio \(\frac{\eta}{J}\) introduces a RG scale in the problem which follows from the notion of holographic RG flow in the bulk. The UV (\(CFT_1\)) fixed point of this RG flow is characterized by the ratio, \(\frac{\eta}{J} = 0\) whereas, on the other hand, the IR (Lifshitz) fixed point is subjected to the constraint, \(|\frac{\eta}{J}| \gg 1\). Therefore, in the limit, \(|\frac{\eta}{J}| \ll 1\) (which has been considered in the present analysis), the YB effects manifest themselves as highly “UV” (compared to that of the classical IR solution and fluctuations around it) quantum corrections and therefore is not of much significance from the perspective of the original SYK spectrum. In other words, these YB effects could be regarded as being that of the reminiscent of some short distance physics that lefts its imprint in the SYK spectrum at strong coupling. With an appropriate tuning of both the YB parameter (\(\eta\)) as well as the SYK coupling (\(J\)) one might therefore completely get rid of these modes (by setting, \(\frac{\eta}{J} = 0\)) leaving behind the original SYK spectrum in the deep IR \([13]\).
Before we conclude, a few important observations are of worth emphasizing. First of all, the above duality seems to be a special feature associated with classical $\mathcal{R}$ operator satisfying CYBE. This duality certainly does not hold for $\mathcal{R}$ operator satisfying mCYBE. Finally, and most importantly, due to the presence of naked singularity in the bulk spacetime, the finite temperature realization of the above correspondence seems to be far from triviality. This turns out to be a special feature for YB deformations corresponding to mCYBE [51]. Interestingly, the zero temperature version of the SYK model corresponds to the case with CYBE where the duality seems to be well understood.

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A Effective scalar action in $(AdS_2)_\eta$

The purpose of this short analysis is to provide a justification for the YB scaling introduced in (4.10). Consider an effective action for scalar fields on $(AdS_2)_\eta$ [13],

$$S_\varphi = \frac{1}{2} \int d^2 x \sqrt{|g|} \left( -g^{ab} \partial_a \varphi_m \partial_b \varphi_m - \left( \frac{p_m^2}{4} - \frac{1}{4} \right) \varphi_m^2 \right)$$

$$= -\frac{1}{2} \int d^2 x \sqrt{|g|} \mathcal{L}_\varphi$$

(A.1)

where the background metric is given by the $\eta$ deformed $AdS_2$ supercoset (2.14),

$$g_{ab} = diag(-F(z, \alpha)/z^2, F(z, \alpha)/z^2)$$

$$F = \frac{1}{1 - \eta^2 \alpha^2 z^2}.$$  \hspace{1cm} (A.2)

A straightforward computation yields,

$$\sqrt{|g|} \mathcal{L}_\varphi = -\varphi_m \left( \Box^{(2)} - \frac{F(z, \alpha)}{z^2} \left( \frac{p_m^2}{4} - \frac{1}{4} \right) \right) \varphi_m.$$  \hspace{1cm} (A.3)

On the other hand, it is trivial to compute the D’Alembertian\footnote{Here, $D_B = z^2 \partial^2_z + z \partial_z - z^2 \partial^2_t$ is the Bessel differential operator.} acting on the scalar fields and repackaging everything according to [40],

$$\nabla^a \nabla_a \varphi_m = \frac{z^2}{F(z, \alpha)} \Box^{(2)} \varphi_m = \sqrt{z} D_B \left( \frac{\tilde{\varphi}_m}{\sqrt{z}} \right) - \frac{\tilde{\varphi}_m}{4} F(z, \alpha).$$  \hspace{1cm} (A.4)

\hspace{1cm}
Here we have rescaled the scalar fluctuations as\textsuperscript{22},
\[ \tilde{\varphi}_m = \frac{\varphi_m}{F(z, \alpha)} \tag{A.5} \]
and introduced a new function,
\[ \tilde{F}(z, \alpha) = F(z, \alpha) \left( 1 - \frac{17\alpha^2\eta^2}{z^2} \right) \tag{A.6} \]
that in the appropriate limit reproduces the original results of \cite{40}. Next, we propose
the following \textit{non local} field redefinition,
\[ \varphi_m(t, z) = \left( \frac{3J}{8\sqrt{\pi}} \right)^{1/2} z^{1/2} \sqrt{f(\sqrt{D_B})} \tilde{\varphi}_m(t, z). \tag{A.7} \]
which upon substitution into (A.1) yields,
\[ S_\varphi = \frac{1}{2} \frac{3J}{8\sqrt{\pi}} \int dt \int_0^\infty dz \frac{dz}{z} \varphi_m(D_B - \tilde{p}_m^2)f(\sqrt{D_B})\tilde{\varphi}_m. \tag{A.8} \]
The pole has been found to be shifted due to YB deformations,
\[ \tilde{p}_m^2 = F(z, \alpha) \left( \frac{p_m^2 - \eta^2\alpha^2}{4z^2} \right). \tag{A.9} \]
Finally, implementing the original definition \cite{13},
\[ \tilde{h}(\nu) - 1 = (\nu^2 - \tilde{p}_m^2)f(\nu) \tag{A.10} \]
one could further re-express (A.8) as,
\[ S_\varphi = \frac{1}{2} \frac{3J}{8\sqrt{\pi}} \int dt \int_0^\infty dz \frac{dz}{z} \varphi_m \left( \tilde{h}(\sqrt{D_B}) - 1 \right) \tilde{\varphi}_m. \tag{A.11} \]

\section*{B Derivation of the identity (4.17)}

The purpose of this short section is to provide the mathematical detail for the identity
(4.17) used in the main text of this paper. We start with the following entity,
\[ \mathcal{I}(t_1, t_2) = \int dt_a dt_b \frac{\Psi_0(t_1, t_a)\Psi_0(t_2, t_b)\mathcal{B}_{\nu a}(t_a, t_b)}{|t_a - t_b|}. \tag{B.1} \]
\textsuperscript{22}This further confirms that the bi-local excitation (4.10) in the dual $q$ SYK model should also get rescaled accordingly.
At the IR critical point the collective excitation could be formally expressed as \([14]\),

\[
\Psi_0(t) = \text{sgn}(t) \frac{s}{t^{2/q}} \tag{B.2}
\]

which has a Fourier transform,

\[
\Psi_0(w) \bigg|_{w \sim 0} = \int_{-\infty}^{\infty} dt e^{iwt} \Psi_0(t) \approx -is \text{sgn}(w) \frac{\Gamma \left( \frac{q-2}{q} \right)}{|w|^{1-2/q}}. \tag{B.3}
\]

However, in the most generic case one should consider the shifted (classical) background,

\[
\Psi_{cl}(t_{12}) = \Psi_0(t_{12}) + \frac{1}{J^{1+2/q}} \Psi_1(t_{12}) + \mathcal{O}(1/J^{2+2/q}) \tag{B.4}
\]

where, the first order shift could be formally expressed as \([14]\),

\[
\Psi_1(t) = \text{sgn}(t) \frac{B_1(q)}{t^{1+2/q}}, \quad B_1(q) = \frac{\alpha_G}{\sqrt{q}} \frac{2(q-1/2) \tan \left( \frac{\pi}{q} \right)}{\pi} \left( \frac{1}{2} - \frac{1}{q} \right). \tag{B.5}
\]

The corresponding Fourier transform could be formally expressed as,

\[
\Psi_1(w) \bigg|_{w \sim 0} = \int_{-\infty}^{\infty} dt e^{iwt} \Psi_1(t) \approx -B_1 \text{sgn}(w) \Gamma \left( \frac{2}{q} \right) |w|^{2/q}. \tag{B.6}
\]

Combining both (B.3) and (B.6) it is in fact trivial to see,

\[
\lim_{J \to \infty} \Psi_{cl}(w) \bigg|_{|w| \sim 0} = \Psi_0(w) \left( 1 - i \frac{B_1(q)}{s(q)} \frac{\Gamma \left( \frac{2}{q} \right)}{\Gamma \left( \frac{q-2}{q} \right)} \frac{|w|}{J} \right) \approx \Psi_0(w). \tag{B.7}
\]

Using (B.3), one could re-express (B.1) as,

\[
\mathcal{I}(t_1, t_2) = -|\mathcal{M}(q)| \int \frac{dt_a dt_b}{|t_a - t_b|} \int \frac{dp sgn(p)}{|p|^{1-2/q}} e^{-ip(t_1 - t_a)} \times \int \frac{d\tilde{p} sgn(\tilde{p})}{|\tilde{p}|^{1-2/q}} e^{-i\tilde{p}(t_2 - t_b)} \times sgn(t_a - t_b) e^{i \frac{\pi}{2} (t_a + t_b)} Z_{\nu} \left( \frac{t_a - t_b}{2} \right) \tag{B.8}
\]

where, we have introduced new function,

\[
|\mathcal{M}(q)| = s^2 \left[ \Gamma \left( \frac{q-2}{q} \right) \right]^2. \tag{B.9}
\]
Using the Cauchy product rule \([71]\) formula for two infinite series,

\[
\sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j = \sum_{k=0}^{\infty} c_k, \quad c_k = \sum_{l=0}^{k} a_l b_{k-l}
\]  

(B.10)

this further yields,

\[
\mathcal{I}(t_1, t_2) = -|\mathcal{M}(q)| e^{-iw_\tau_2} \int e^{2i\tau_1 dw} \times \int \frac{dz_1 dp \text{sgn}(p)\text{sgn}(w-p)}{z_1} e^{i(2p-w)(z_1-z_2)} \text{sgn}(z_1) \mathcal{Z}_\nu(|wz_1|)
\]  

(B.11)

where, we have introduced new variables,

\[
\tau_1 = \frac{1}{2}(t_a + t_b), \quad \tau_2 = \frac{1}{2}(t_1 + t_2)\\
\quad z_1 = \frac{1}{2}(t_a - t_b), \quad z_2 = \frac{1}{2}(t_1 - t_2).
\]  

(B.12)

Finally, using the integral representation for the delta function it is obvious to obtain,

\[
\mathcal{I}(t_1, t_2) = -\frac{|\mathcal{M}(q)|}{2} e^{-iw_\tau_2} \int \frac{dz_1 dp \text{sgn}(p)\text{sgn}(w-p)}{z_1} e^{i(2p-w)(z_1-z_2)} \text{sgn}(z_1) \mathcal{Z}_\nu(|wz_1|)
\]  

(B.13)

subjected to the fact that \(|w|\) is a constant. Notice that here we have defined \(\nu' = \frac{1}{2} - \frac{2}{q}\) which satisfies the criteria \(\text{Re}(\nu') \leq 1/2\) for \(q \geq 2\) and thereby allows us to define the following integral representations for Bessel functions,

\[
J_{\nu'}(x) = \frac{2}{\sqrt{\pi}} \frac{(\frac{x}{2})^{-\nu'}}{\Gamma(\frac{1}{2} - \nu')} \int_1^{\infty} \frac{\sin(xt)}{(t^2 - 1)^{\nu'+\frac{1}{2}}} dt \\
Y_{\nu'}(x) = -\frac{2}{\sqrt{\pi}} \frac{(\frac{x}{2})^{-\nu'}}{\Gamma(\frac{1}{2} - \nu')} \int_1^{\infty} \frac{\cos(xt)}{(t^2 - 1)^{\nu'+\frac{1}{2}}} dt.
\]  

(B.14)

Following the change in variable of the form,

\[
p = t + w - 1 \equiv t + t_0
\]  

(B.15)

and thereby finally setting, \(t_0 = 1\) it is indeed quite trivial to show,

\[
\int \frac{dp \text{sgn}(p)\text{sgn}(w-p)}{|p||w-p|^{\nu'+1/2}} e^{i(2p-w)(z_1-z_2)} = \int_1^{\infty} dt \frac{e^{iw(z_1-z_2)t}}{(t^2 - 1)^{\nu'+\frac{1}{2}}}
\]  

(B.16)
which by means of (B.14) finally yields,
\[
\mathcal{I}(t_1, t_2) = \frac{\sqrt{\pi} |M(q)|}{2^{\nu+2}} \Gamma \left( \frac{2}{q} \right) e^{-iw\tau_2} \\
\times \int \frac{du_1}{u_1} (u_1 - u_2)^\nu (Y_\nu(u_1 - u_2) - iJ_\nu'(u_1 - u_2))sgn(u_1)Z_\nu(|u_1|) \quad (B.17)
\]
where, we have rescaled the variables as \( u_i \equiv wz_i \ (i = 1, 2) \). Since in the present analysis we are mostly concerned with bound states [8] with real energy eigen values therefore it is sufficient to consider \( Z_\nu(|x|) \equiv J_\nu(|x|) \) with \( \nu = 3/2 + 2n(n = 0, 1, 2,..) \).

Moreover for the purpose of our analysis (as we are interested in zero modes) it is perfectly consistent to set, \( \nu = \nu' (= 3/2) \). Using the identity,
\[
K_\nu(z) = \frac{\pi}{2} i^{\nu-1} [J_\nu(iz) + iY_\nu(iz)] \quad (B.18)
\]
it is in fact quite straightforward to show,
\[
\mathcal{I}(t_1, t_2) = \frac{|M(q)|}{\sqrt{\pi} 2^{\nu+2}} \Gamma \left( \frac{2}{q} \right) e^{-iw\tau_2} \int \frac{du_1}{u_1} (u_1 - u_2)^\nu sgn(u_1)K_\nu(u_1 - u_2)Z_\nu(|u_1|) \quad (B.19)
\]
Comparing both (B.1) and (B.19) we therefore finally conclude,
\[
b_{\nu w}(t_1, t_2) = g(\nu, q) \int \frac{dt_a dt_b}{|t_a - t_b|} \Psi_0(t_1, t_a) \Psi_0(t_2, t_b) b_{\nu w}(t_a, t_b) \quad (B.20)
\]
where, we have instated the prefactor as,
\[
g(\nu, q)^{-1} = \frac{|M(q)|}{4\sqrt{\pi} \varphi(\nu)} \Gamma(\nu) \Gamma \left( \frac{2}{q} \right). \quad (B.21)
\]
From consistency requirements [13] we demand,
\[
g(\nu, q = 4) = -\frac{3}{16\sqrt{\pi}} \frac{1}{|h(\nu)|}, \quad h(\nu) = -\frac{3}{2\nu} \tan(\pi\nu/2) \quad (B.22)
\]
which thereby sets the constant,
\[
\varphi(\nu) = -\frac{3}{128} \frac{\Gamma(\nu)}{|h(\nu)|}. \quad (B.23)
\]
Using (B.23), we finally note down,
\[
g(\nu, q) = \frac{3\sqrt{\pi}}{32\Gamma \left( \frac{2}{q} \right) |M(q)| |h(\nu)|} = -\frac{\gamma(q)}{|h(\nu)|} \quad (B.24)
\]
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