REFINED STRICHARTZ INEQUALITIES FOR THE WAVE EQUATION

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Abstract. Some analogues of the Schrödinger refined Strichartz inequalities (Du, Guth, Li and Zhang) are obtained for the wave equation. These are used to improve the best known $L^2$ fractal Strichartz inequalities for the wave equation in dimensions $d \geq 4$.

1. Introduction

A special case of the Strichartz inequality for the wave equation asserts that

\[
\|u\|_q \lesssim \|u(\cdot, 0)\|_{H^s(\mathbb{R}^d)} + \|u_t (\cdot, 0)\|_{H^{s-1}(\mathbb{R}^d)} \quad \text{for} \quad \Delta u - u_{tt} = 0,
\]

where $d \geq 2$, $s = 1/2$ and $q = \frac{2(d+1)}{d-1}$ [23, 13]. The purpose of this work is to prove a refinement of this inequality in certain cases, analogous to the refined Strichartz inequalities for the Schrödinger equation in [4, 6, 5]. The refined inequalities for the Schrödinger equation were used in [4, 7] to prove sharp results involving the almost everywhere pointwise convergence of Schrödinger solutions to the initial data. They were also used in [6, 7] to improve the known results on Falconer’s distance set problem in dimensions $n \geq 3$. This was done through Matilla’s approach [16] via the spherical averages of Fourier transforms of measures. Here the refined Strichartz inequality for the wave equation will be used to improve results on the averages over the cone, rather than the sphere. As is well known [24, 8, 3], the optimal decay of these conical averages is related to best possible $s$ in an inequality of the form (1.1), with the $L^q$ norm on the left replaced by an $L^2(\mu)$ norm for a fractal measure $\mu$.

The Fourier transform of a compactly supported Borel measure $\mu$ on $\mathbb{R}^n$ is given by

\[
\hat{\mu}(\xi) := \int e^{-2\pi i \langle \xi, x \rangle} \, d\mu(x).
\]

The measure $\mu$ is called $\alpha$-dimensional if

\[
c_\alpha(\mu) := \sup_{x \in \mathbb{R}^n, r > 0} \frac{\mu(B(x, r))}{r^\alpha} < \infty,
\]

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Given a smooth compact surface $\Gamma \subseteq \mathbb{R}^n$, let $\beta(\alpha, \Gamma)$ be the supremum over all $\beta \geq 0$ which satisfy
\[
\int |\hat{\mu}(R\xi)|^2 \, d\sigma_\Gamma(\xi) \lesssim \|\mu\|c_\alpha(\mu)R^{-\beta} \quad \text{for all } R > 0,
\]
uniformly over all Borel measures $\mu$ with support in the unit ball of $\mathbb{R}^n$. Here $\|\mu\|$ is the total variation norm, the implicit constant is allowed to depend on $\beta$, and $\sigma_\Gamma$ is the surface measure on $\Gamma$. The distance set problem is related to the case where $\Gamma = S^{n-1}$, see also [15] for the connection between $\beta_n(\alpha, S^{n-1})$ and the pointwise convergence of solutions to the wave and Schrödinger equations. Henceforth $\Gamma$ will be the truncated cone
\[
\Gamma := \{(\xi, |\xi|) \in \mathbb{R}^{d+1} : 1 \leq |\xi| \leq 2\}, \quad n = d + 1,
\]
and it may be assumed that $\sigma_\Gamma$ is actually the pushforward of the $d$-dimensional Lebesgue measure under the map $\xi \mapsto (\xi, |\xi|)$.

For $d \geq 3$, the best known lower bounds are
\[
\beta_d(\alpha, \Gamma) \geq \begin{cases} 
\alpha & \text{if } 0 < \alpha \leq \frac{d}{2} - 1 \quad \text{(Matilla [16] and Rogers [20])} \\
\frac{d}{2} + \frac{\alpha}{2} - \frac{1}{4} & \text{if } \frac{d}{2} - 1 < \alpha \leq \frac{d+1}{2} \quad \text{(Cho, Ham and Lee [3])} \\
\alpha - 1 & \text{if } \frac{d+1}{2} < \alpha \leq d + 1 \quad \text{(Sjölin [22]),}
\end{cases}
\]
and the best known upper bounds are
\[
\beta_d(\alpha, \Gamma) \leq \begin{cases} 
\alpha & \text{if } 0 < \alpha \leq d - 2 \\
\frac{d}{2} + \frac{\alpha}{2} - 1 & \text{if } d - 2 < \alpha \leq d \quad \text{(Cho, Ham and Lee [3]).}
\end{cases}
\]
These bounds are equal and give the exact value of $\beta_d(\alpha, \Gamma)$ when $d = 3$. The exact value of $\beta_2(\alpha, \Gamma)$ was determined by Erdoğan in [8]. In Section 4 it is shown that
\[
\beta_d(\alpha, \Gamma) \geq \alpha - 1 + \frac{d - \alpha}{d + 1}, \quad \alpha \in (0, d + 1],
\]
improving (1.3) in the range $\frac{d+1}{2} + \frac{\alpha}{2} < \alpha < d$, which is nonempty only for $d \geq 4$. As in [6], the inequality (1.5) will be shown through the linear refined Strichartz inequality; the proof of which is given in Section 3. In Section 3 a multilinear refined Strichartz inequality will be deduced from the linear one, as in [5]. By using the multilinear version, the bound (1.3) could possibly be improved further through the methods in [7]; the known values of $\beta_d(\alpha, \Gamma)$ for the $d$-dimensional cone tend to mimic the known values of $\beta_d(\alpha, S^{d-1})$ for the sphere of one dimension less.

The upper bound
\[
\beta_d(\alpha, \Gamma) \leq \alpha - 1 + \frac{2(d + 1 - \alpha)}{d + 1}, \quad \alpha \in (0, d + 1), \quad d \geq 5.
\]
will also be proved in Section 5; this is the analogy to the upper bound for the spherical averages, due to Luca and Rogers [15]. Improvement holds in the range $\frac{d+1}{2} < \alpha < d - \frac{1}{d+1}$, and this interval is nonempty only for $d \geq 6$.

For $q \in [1, \infty]$, let $s_q(\alpha, q)$ be the infimum over all $s$ satisfying the inequality
\[
\|u\|_{L_q(\mu)} \lesssim C(\mu) \left(\|u(\cdot, 0)\|_{H^{s}} + \|u_t(\cdot, 0)\|_{H^{s-1}}\right) \quad \text{for } \Delta u - u_{tt} = 0,
\]
where
\[
\|f\|_{H^{s}} := \left(\int |\hat{f}(\xi)|^2 \left(1 + |\xi|^2\right)^s \, d\xi\right)^{1/2},
\]
and it may be assumed that $\mu$ is actually the pushforward of the $d$-dimensional Lebesgue measure under the map $\xi \mapsto (\xi, |\xi|)$. The distance set problem is related to the case where $\Gamma = S^{n-1}$, see also [15] for the connection between $\beta_n(\alpha, S^{n-1})$ and the pointwise convergence of solutions to the wave and Schrödinger equations. Henceforth $\Gamma$ will be the truncated cone
\[
\Gamma := \{(\xi, |\xi|) \in \mathbb{R}^{d+1} : 1 \leq |\xi| \leq 2\}, \quad n = d + 1,
\]
and
\[
C(\mu) := \begin{cases} 
\|\mu\|^{\frac{d}{q} - \frac{1}{2}} c_\alpha(\mu) \frac{1}{2} & \text{if } 1 \leq q \leq 2 \\
c_\alpha(\mu) \frac{1}{2} & \text{if } 2 < q \leq \infty.
\end{cases}
\]

The quantity \(s_d(\alpha, 2)\) is related to \(\beta_d(\alpha, \Gamma)\) through the equation
\[
\beta_d(\alpha, \Gamma) = d - 2s_d(\alpha, 2),
\]
see [24, 8, 3, 20]; a proof is given here in Proposition 5.3. For \(d \geq 3\), the corresponding best known bounds for \(s_d(\alpha, q)\) are
\[
s(\alpha, 2, d) \leq s_d(\alpha, q) \leq \bar{s}(\alpha, 2, d) \quad \text{for } q \in [1, 2],
\]
and
\[
s(\alpha, q, d) \leq s_d(\alpha, q) \leq \bar{s}(\alpha, q, d) \quad \text{for } q \in [2, \infty],
\]
where
\[
s(\alpha, q, d) := \begin{cases} 
\max \left\{ \frac{d}{2} - \frac{\alpha}{q}, \frac{d + 1}{4} \right\} & \text{if } 0 < \alpha \leq 1 \\
\max \left\{ \frac{d}{2} - \frac{\alpha}{q}, \frac{d + 1 + \frac{1}{2} - \alpha}{2q}, \frac{d + 2 - \alpha}{4} \right\} & \text{if } 1 < \alpha \leq d \\
\max \left\{ \frac{d}{2} - \frac{\alpha}{q}, \frac{d + 1 + \frac{1}{2} - \alpha}{2q}, \frac{d + 1 - \alpha}{2q} \right\} & \text{if } d < \alpha \leq d + 1,
\end{cases}
\]
and
\[
\bar{s}(\alpha, 2, d) := \begin{cases} 
\frac{d - \alpha}{2} & \text{if } 0 < \alpha \leq \frac{d - 1}{2} \\
\frac{3d + 1 - \alpha}{8} & \text{if } \frac{d - 1}{2} < \alpha \leq \frac{d + 3}{2} \\
\frac{d + 1 - \alpha}{2} & \text{if } \frac{d + 3}{2} < \alpha \leq d + 1.
\end{cases}
\]

The piecewise intervals in \(\bar{s}(\alpha, q, d)\) are only subdivided this way for simplicity, and could be made slightly more optimal as in \(\bar{s}(\alpha, 2, d)\). The lower bound in (1.9) is from [3], and the lower bound in (1.10) is from [20] (and works even if the constant \(\|\mu\|^{\frac{d}{q} - \frac{1}{2}} c_\alpha(\mu) \frac{1}{2}\) is relaxed to \(c_\alpha(\mu) \frac{1}{2}\)). The upper bound in (1.9) and (1.10) is from [3]. Earlier results were obtained by Oberlin in [17], see also [18]. The inequalities (1.9) and (1.10) determine \(s_3(\alpha, q) = s(\alpha, q, 3)\) for all \(q\). The value of \(s_2(\alpha, q) = s(\alpha, q, 2)\) was determined for all \(q\) in [9], see also [25] for the case \(\alpha \in (0, 1]\).

The relation (1.8) combined with the bound for \(\beta_d(\alpha, \Gamma)\) in (1.5) will give
\[
s_d(\alpha, 2) \leq \frac{d}{2} - \frac{1}{2} \left( \alpha - 1 + \frac{d - \alpha}{d + 1} \right).
\]

The range of improvement is the same as for \(\beta_d(\alpha, \Gamma)\), given by \(\frac{d + 1 + \frac{2}{d - 1}}{d + 1} < \alpha < d\). There would also be some small improvement for slightly larger \(q\) by interpolation of \(H^s\) spaces. For \(q \geq 4\) the known bounds are already optimal.

The upper bound in (1.6) gives
\[
s_d(\alpha, 2) \geq \frac{d}{2} - \frac{1}{2} \left( \alpha - 1 + \frac{2(d + 1 - \alpha)}{d + 1} \right), \quad \alpha \in (0, d + 1), \quad d \geq 5,
\]
which improves the lower bound in [19] for $q = 2$, $d \geq 6$ in the range $\frac{d+1}{2} < \alpha < d - \frac{1}{2}$. This shows that $s(\alpha, q, d)$ is not the optimal value for all $d$, which disproves a conjecture from [3].

The Strichartz inequality (1.7) was also considered in [20] for a restricted class of measures of the form $\nu = \mu \otimes \lambda$, where $\mu$ is a compactly supported $\alpha$-dimensional measure in the unit ball of $\mathbb{R}^d$ and $d\lambda = \chi_{[0,1]} \, dm$, where $m$ is the Lebesgue measure on $\mathbb{R}$. For this restricted class, the infimum over $s$ in (1.7) was shown in [20] to be related to Falconer’s distance set problem, however the optimal value of $s$ has no relationship to the optimal decay of conical averages in (1.2), since for this restricted class of measures the optimal decay

$$\int |\hat{\nu}(R\xi)|^2 \, d\sigma_\Gamma(\xi) \lesssim I_\alpha(\mu) R^{-(\alpha+2)} \quad \text{for all } R > 0,$$

follows directly from a straightforward computation (see Proposition 5.4) involving the characterisation of the energy $I_\alpha(\mu)$ as

$$I_\alpha(\mu) := \int \int \frac{1}{|x-y|^\alpha} \, d\mu(x) \, d\mu(y) = c_{\alpha,d} \int |\xi|^{\alpha-d} |\hat{\mu}(\xi)|^2 \, d\xi.$$

1.1. Notation. Throughout, let $A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be the unitary defined through the standard basis by

$$e_i \mapsto e_i \quad \text{for} \quad 1 \leq i \leq d-1, \quad \frac{e_{d+1} + e_\ell}{\sqrt{2}} \mapsto e_\ell, \quad \frac{e_{d+1} - e_{d}}{\sqrt{2}} \mapsto e_{d+1},$$

and let $E$ be the extension operator for the subset $\Gamma_+ := \Gamma \cap \{\xi_d \geq 0\}$ of the cone, defined by

$$Ef(x,t) = \int_{\{\xi_d \geq 0\} \cap B(0,2) \setminus B(0,1)} e^{2\pi i (\langle \xi, x \rangle + |\xi| t)} f(\xi) \, d\xi.$$

The restriction to $\xi_d \geq 0$ is a minor technicality to ensure that the normal to the cone at any point $(\xi, |\xi|) \in \Gamma_+$ makes an angle $\geq 1$ with the plane $A^\perp (\mathbb{R}^d \times \{0\})$.

When used as a function the symbol $\pi$ will mean the natural projection from $\mathbb{R}^{d+1}$ to $\mathbb{R}^d$.

The statement that $f$ is essentially supported in $U$ will mean that $|f(x)| \lesssim_N R^{-N} \|f\|_2$ for $x$ outside $U$, where $N$ is arbitrarily large.

For non-negative $X$ and $Y$ the notation $X \lesssim Y$ will mean $X \leq C \epsilon Y$ where $\epsilon$ is arbitrarily small. Similarly $X \approx Y$ means $X \lesssim Y$ and $Y \lesssim X$. For functions the statement $Ef(x) \approx g(x)$ will mean that $|Ef(x) - g(x)| \lesssim R^{-N} \|f\|_2$ for arbitrarily large $N$, where $E$ is an operator (usually the extension operator). For any box $X$ and constant $C \geq 0$, the set $CX$ will the box with the same centre, but with side lengths scaled by $C$.

2. Wave packet decomposition for the cone

The wave packet decomposition used here is based on those in [11, 3, 3, 21, 19]. In the decomposition, the truncated cone is partitioned into caps; to construct these, partition the $(d-1)$-dimensional sphere $S^{d-1}$ into spherical caps $C$ of diameter $\delta^{1/2}$, where $\delta \in (0,1)$, and define each $\tau = \tau_C$ by

$$\tau = \left\{ (\xi, |\xi|) \in \Gamma : \frac{\xi}{|\xi|} \in C \right\}.$$
The set $\tau$ is called a cap at scale $\delta^{1/2}$. It is contained in a box of dimensions
$$
\sim \delta^{1/2} \times \cdots \times \delta^{1/2} \times 1 \times \delta,
$$
where the second last coordinate refers to the flat direction in $\tau$, and the last direction is normal to the cone at $\tau$.

**Proposition 2.1.** Fix $\delta > 0$. Let $B$ be a box in $\mathbb{R}^d$ of dimensions
$$
R^{-1/4} \times \cdots \times R^{-1/4} \times 1,
$$
with sides parallel to the coordinate axes, and let $D$ be a collection of boxes $D \subseteq \mathbb{R}^d$ of dimensions
$$
R^{3/4+\delta} \times \cdots \times R^{3/4+\delta} \times R^{1/2+\delta},
$$
also with sides parallel to the coordinate axes, such that the boxes $(1/2)D$ form a finitely overlapping cover of $\mathbb{R}^d$. Suppose that $\Phi \in C^\infty((4/3)B, \mathbb{R})$ satisfies
$$
|\partial_i^k \Phi| \lesssim_k R^{k/4} \quad \text{for} \quad 1 \leq i \leq d-1 \quad \text{and} \quad |\partial_d^k \Phi| \lesssim_k R^{k/2} \quad \text{for} \quad k \geq 1.
$$
Then for sufficiently large $R$, any $f \in L^2(\mathbb{R}^d)$ supported in $B$ can be decomposed as
$$
f = \sum_D f_D, \quad \text{such that each } f_D \text{ is supported in } (4/3)B, \quad \Phi f_D \text{ is essentially supported in } D,
$$
and
$$
\sum_{D: z \notin D} \Phi f_D(z) \lesssim R^{-N} \|f\|_2, \quad \text{for any } z \in \mathbb{R}^d.
$$

**Proof.** Let $\psi$ be smooth with $0 \leq \psi \leq 1$, such that $\psi$ is equal to 1 on $B$, vanishes outside $(5/4)B$ and satisfies
$$
|\partial_i^k \psi| \lesssim_k R^{k/4} \quad \text{for} \quad 1 \leq i \leq d-1 \quad \text{and} \quad |\partial_d^k \psi| \lesssim_k 1 \quad \text{for} \quad k \geq 1.
$$
Let $\{\phi_D\}_D$ be a smooth partition of unity subordinate to the $(3/4)D$ so that
$$
f = \psi f = \sum_D \psi \left( \widehat{\phi_D} * f \right) =: \sum_D f_D.
$$
The convergence holds in $L^2(\mathbb{R}^d)$ by Plancherel, and it is clear that each $f_D$ is supported in $(4/3)B$. By the finite overlapping property and Plancherel, the $L^2$ norm of $f$ satisfies
$$
\|f\|_2^2 = \sum_{i,j} \langle \phi_D, \tilde{f}, \phi_D, \tilde{f} \rangle \sim \sum_D \|\phi_D \tilde{f}\|_2^2 = \sum_D \|\widehat{\phi_D} * f\|_2^2 \lesssim \sum_D \|f_D\|_2^2,
$$
which proves (2.2).

It remains to check the support conditions on $\Phi f_D$. Let $w = \Phi \psi$, so that by (2.1) and (2.3),
$$
|\partial_i^k w| \lesssim_k R^{k/4} \quad \text{for} \quad 1 \leq i \leq d-1 \quad \text{and} \quad |\partial_d^k w| \lesssim_k R^{k/2} \quad \text{for} \quad k \geq 1.
$$
After integrating by parts $L$ times, this implies that
$$
\tilde{w}(z) \lesssim \frac{m(B)}{|z|^{R-3/4}} \quad \text{for} \quad 1 \leq i \leq d-1 \quad \text{and} \quad \tilde{w}(z) \lesssim \frac{m(B)}{|z|^{R-1/2}} \quad \text{for} \quad z \in \mathbb{R}^d.
$$
Hence for $z \notin D$,
$$
\left| \Phi f_D(z) \right| = \left| \tilde{w} \left( \phi_D \tilde{f} \right)(z) \right|$$
This finishes the proof. □

Let $\tau$ be a cap in the cone at scale $R^{-1/4}$. Then any $f \in L^2(\pi(\tau))$ can be decomposed in $L^2$ as $f = \sum_T f_T$ with

\begin{equation}
\sum_T \|f_T\|_2^2 \lesssim \|f\|_2^2,
\end{equation}

such that each $f_T$ has support in $(4/3)\pi(\tau)$, where the $T$’s are boxes with dimensions

\[ \sim R^{3/4+\delta} \times \ldots \times R^{3/4+\delta} \times R^{1/2+\delta} \times R, \]

with long axis normal to $\tau$, short axis in the flat direction of $\tau$, and such that the restriction of $E f_T$ to $B(0, R)$ is essentially supported in $T$, with

\begin{equation}
\sum_{T: (x,t) \notin T} |E f_T(x,t)| \lesssim R^{-N} \|f\|_2 \quad \text{for } |(x,t)| \leq R.
\end{equation}

Proof. Since the condition $|(x,t)| \leq R$ is rotation invariant, after applying a rotation of $\mathbb{R}^{d+1}$ which fixes the cone, it may be assumed that the flat direction of $\tau$ is $e_{d+1}$. After then applying the unitary $A$, it may be assumed that the normal to $\tau$ is completely in the $e_{d+1}$-direction, and that $(4/3)\tau$ is contained in the graph of the function

\[ h(\omega_1, \ldots, \omega_d) := \frac{\sum_{i=1}^{d-1} \omega_i^2}{2\omega_d}, \]

restricted to $(4/3)B$ where

\[ B := \left[ -R^{-1/4}, R^{-1/4} \right] \times \ldots \times \left[ -R^{-1/4}, R^{-1/4} \right] \times \left[ \sqrt{2}, 2\sqrt{2} \right] \subseteq \mathbb{R}^d. \]

Let $f_T = f_D$ from Proposition 2.1 where each $T = D \times x$. The inequality \((2.4)\) and the support condition on the $f_T$’s both follow from Proposition 2.1. Moreover,

\[ E f_T(x,t) = \int e^{2\pi i \langle \omega, x \rangle + h(\omega)t} f_T(\omega) \, d\omega = \Phi_t f_D(x), \]

where $\Phi_t(\omega) := e^{2\pi i h(\omega)t}$ restricted to $(4/3)B$, and from the definition of $h$,

\[ |\partial_i^k \Phi_t|_i(\omega) \lesssim_k R^{ik/4} \quad \text{for } 1 \leq i \leq d-1 \quad \text{and} \quad |\partial_d^k \Phi_t|_d(\omega) \lesssim_k R^{k/2}, \]

for all $k \geq 1$, uniformly for $|t| \leq R$ and $\omega \in (4/3)B$. By Proposition 2.1 this means that for each fixed $t$ with $|t| \leq R$, $E f_T(\cdot, t)$ is essentially supported in $D$. Therefore the restriction of $E f_T$ to $B(0, R)$ is essentially supported in $T$, and the inequality \((2.5)\) follows from Proposition 2.1. This finishes the proof. □
3. Linear refined Strichartz inequality

The proof of the linear refined Strichartz inequality will use the sharp decoupling theorem for the truncated cone from [2].

**Theorem 3.1.** Let $\delta \in (0, 1)$, partition the truncated cone into caps $\tau$ at scale $\delta^{1/2}$, and let $f = \sum_{\tau} f_{\tau}$ be a function on $\mathbb{R}^{d+1}$ such that the support of $f_{\tau}$ is contained in the $\delta$-neighbourhood of $\tau$. Then for any $\epsilon > 0$,

$$\|f\|_q \leq C_{\epsilon} \delta^{-\epsilon} \left( \sum_{\tau} \|f_{\tau}\|^q \right)^{2/q}, \quad \text{where } q = \frac{2(d+1)}{d-1}. $$

Most of the notation used throughout is similar to that in [4], to emphasise the analogy with the Schrödinger case (see also Subsection 1.1).

**Theorem 3.2.** Suppose that $f \in L^2(\mathbb{R}^d)$ is supported in $B(0, 2) \setminus B(0, 1)$, and let $Y = \bigcup Q$ be a collection of lattice $R^{1/2}A^{-1}\mathbb{Z}^{d+1}$-cubes inside a ball of radius $R$. If $\|Ef\|_{L^q(\mathbb{R}^d)}$ is essentially constant as $Q$ varies over $Y$, and if the cubes are arranged in slabs of the form $A^* (\mathbb{R}^d \times [R^{1/2}j, R^{1/2}j + R^{1/2}])$ with $j \in \mathbb{Z}$, such that each slab intersecting $Y$ contains $\sigma$ cubes in $Y$, then for any $\epsilon > 0$,

$$\|Ef\|_{L^q(Y)} \leq C_{\epsilon} R^\epsilon \sigma^{-\gamma} \|f\|_2, $$

where $\gamma = \frac{1}{2} - \frac{\epsilon}{\delta}$ and $q = \frac{2(d+1)}{d-1}$.

**Remark 3.3.** To apply induction, it will be easier to prove a slightly more general statement, where each $Q$ in the theorem is replaced by $CQ$, for a constant $C \in [1, 2]$, but the slabs are unchanged. The inductive assumption will be that this slightly more general statement of the theorem holds whenever $R$ is replaced everywhere in the theorem by $\tilde{R}$, for any $\tilde{R} \leq R^{3/4}$ ($3/4$ is not important, any exponent in $(1/2, 1)$ would work).

Given $\epsilon > 0$, the inductive assumption is that

$$\|Ef\|_{L^q(Y)} \leq C_{\epsilon} \tilde{R}^{\epsilon} \sigma^{-\gamma} \|f\|_2, $$

for any $f$ satisfying the (generalised) assumptions of the theorem, and any $\tilde{R} \leq R^{3/4}$. Using this, it will be shown that for any $\delta > 0$, the inequality

$$\|Ef\|_{L^q(Y)} \leq C_{\epsilon} C_{\delta} R^{O(\delta)} R^{\epsilon/2} \sigma^{-\gamma} \|f\|_2, $$

holds for any $f$ satisfying the (generalised) assumptions of the theorem. By choosing $\delta > 0$ small enough, depending only on $\epsilon$, and then taking $R$ large enough (depending only on $\epsilon$), this will give

$$\|Ef\|_{L^q(Y)} \leq C_{\epsilon} R^\epsilon \sigma^{-\gamma} \|f\|_2, $$

with the same constant $C_{\epsilon}$, which will close the induction and prove the theorem. To simplify notation the $R^\epsilon$ and $R^\delta$ factors will be absorbed in the symbols $\approx$ and $\ll$, and therefore this argument will not be carried out explicitly.

**Proof of Theorem 3.2.** After replacing $f$ by $e^{-2\pi i (\langle x_0, \xi \rangle + t_0 |\xi|)} f$ it may be assumed that the ball of radius $R$ containing $Y$ is centred at the origin. Fix $\delta > 0$, and use Proposition 2.2 to decompose

$$f = \sum_{\tau, \Box} f_{\tau, \Box}, \quad Ef = \sum_{\tau, \Box} Ef_{\tau, \Box},$$

where
where the caps $\tau$ partitioning the truncated half cone $\Gamma_+$ are at scale $\sim R^{-1/4}$, each contained in a corresponding box with dimensions
\[ \sim R^{-1/4} \times \cdots \times R^{-1/4} \times 1 \times R^{-1/2}, \]
and each $E_f\Box$ has distributional Fourier transform supported in the cap $\tau$ corresponding to $\Box$. When restricted to $B(0, R)$, each $E_f\Box$ is essentially supported in a set $\Box$ of dimensions
\[ \sim R^{3/4+\delta} \times \cdots \times R^{3/4+\delta} \times R^{1/2+\delta} \times R, \]
with long axis normal to $\tau$, and short axis in the flat direction of $\tau$. Although each set $\Box$ depends on a cap $\tau$, this will be suppressed in the notation. Each $E_f\Box$ can be partitioned further
\[ (3.1) \quad E_f\Box \approx \sum_{2^S \cap \Box \neq \emptyset} \eta_S E_f\Box \quad \text{in } B(0, R), \]
where for each $\tau$, the sets $S$ partition physical space and have the same axis orientations as $\Box$, with dimensions
\[ R^{1/2+\delta} \times \cdots \times R^{1/2+\delta} \times R^{1/4+\delta} \times R^{3/4+\delta}. \]
The functions $\eta_S$ form a smooth partition of unity with each $\eta_S \lesssim 1$ on $S$, non-negative, essentially supported on $2S \cap N_{R^{1/2+\delta}}(S)$, such that $\hat{\eta_S}$ is supported in a box of dimensions
\[ (3.2) \quad \sim R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1/4} \times R^{-1/2}, \]
around the origin, with long axis in the flat direction of $\tau$. Such a partition can be constructed by using the Poisson summation formula at scale one on the integer lattice to get a smooth non-negative function $\eta$ with
\[ \sum_{k \in \mathbb{Z}^{d+1}} \eta(x - k) = 1 \quad \text{for all } x \in \mathbb{R}^{d+1}, \quad \text{and } \text{supp } \hat{\eta} \subseteq B(0, c_d). \]
Rescaling by the dimensions in (3.2) and then grouping the functions together corresponding to scaled lattice points in $S$ gives the required functions $\eta_S$.

For each fixed set $\Box$, sort the boxes $S$ close to $\Box$ into sets $S_{\lambda}$ according to the dyadic value $\lambda$ of $\|E_f\Box\|_{L^q(2S)}$. By ignoring the very small values of $\lambda$ which do not contribute significantly, there are $\lesssim \log R$ relevant values of $\lambda$, which will be the only ones considered from now on. For each fixed $\lambda$ and dyadic number $\sigma_{\Box}$, define $S_{\lambda, \sigma_{\Box}}$ by $S \in S_{\lambda, \sigma_{\Box}}$ if and only if there are $\sim \sigma_{\Box}$ boxes in $S_{\lambda}$ inside the slab of width $R^{3/4+\delta}$ parallel to the tangent plane at $\tau$. These slabs are different to those in the theorem statement, but will be rotated and scaled to use the inductive assumption. There are $\lesssim \log R$ values of $\sigma_{\Box}$, so there are $\lesssim (\log R)^2$ total relevant pairs $\lambda$ and $\sigma_{\Box}$. For any set $\Box$ let $Y_{\Box, \lambda, \sigma_{\Box}}$ be the union of boxes $S$ in $S_{\lambda, \sigma_{\Box}}$. By Fubini (and the condition (2.5) in Proposition 2.2),
\[ E_f \approx_{L^q} \sum_{\lambda, \sigma_{\Box}} \sum_{\Box} \eta_{Y_{\Box, \lambda, \sigma_{\Box}}} E_f\Box \quad \text{in } B(0, R), \quad \text{where } \eta_{Y_{\Box, \lambda, \sigma_{\Box}}} := \sum_{S \subseteq Y_{\Box, \lambda, \sigma_{\Box}}} \eta_S, \]
where the symbol $\approx_{L^q}$ here means the difference has $L^q(B(0, R)) \|f\|_{L^q}$ norm $\lesssim R^{-N}\|f\|_2$. By the triangle inequality and the standard pigeonhole principle, there is a fixed
pair \((\lambda_0, \sigma_{\Box})\) such that
\[
\|Ef\|_{L^q(\mathbb{R})} \lesssim \left\| \sum_{\Box \in \mathcal{B}} \eta_{\Box, \lambda_0, \sigma_{\Box}} Ef_{\Box} \right\|_{L^q(\mathbb{R})} + R^{-N}\|f\|_2,
\]
for a fraction \(\approx 1\) of the cubes \(Q\). The quantity \(R^{-N}\|f\|_2\) is a remainder term which absorbs the very small values of \(\lambda\) and the error terms in the wave packet decomposition, and it may essentially be ignored in most of the inequalities that follow. Henceforth write \(Y_{\Box} = Y_{\Box, \sigma_{\Box}}\). The sets \(\Box\) can be sorted according to the dyadic value of \(\|f_{\Box}\|_2\), and since there are only \(\sim \log R\) relevant dyadic values, the triangle inequality and the standard pigeonhole principle applied again to the remaining cubes satisfying (3.3) yield a subset \(B\) of sets \(\Box\) such that \(\|f_{\Box}\|_2\) is essentially constant over \(B\) and
\[
\|Ef\|_{L^q(\mathbb{R})} \lesssim \left\| \sum_{\Box \in \mathbb{B}} \eta_{\Box, \lambda_0, \sigma_{\Box}} Ef_{\Box} \right\|_{L^q(\mathbb{R})} + R^{-N}\|f\|_2,
\]
again for a fraction \(\approx 1\) of the cubes \(Q\). By pigeonholing the remaining cubes satisfying (3.3) again, there is a dyadic number \(\mu\) and a fraction \(\approx 1\) of the cubes \(CQ\), with union \(Y'\), such that the bound in (3.4) still holds and for each \(CQ \subseteq Y'\) the cube \(R^{1/2}Q\) intersects \(\sim \mu\) of the sets \(Y_{\Box}\) with \(\Box \in B\).
Let \(\eta_Q\) be smooth non-negative functions such that \(\eta_Q \sim 1\) on \(2Q\), with each \(\eta_Q\) compactly supported in a ball of radius \(\sim R^{-1/2}\), and such that \(\sum_Q \eta_Q \lesssim 1\) (this can be done, for example, by using Poisson summation again). To apply decoupling, the functions \(\eta_{\Box, \lambda_0, \sigma_{\Box}} Ef_{\Box}\) have Fourier transform supported in an \(\sim R^{-1/2}\) neighbourhood of \(\tau\), since the short directions in the sets \(S \subseteq Y_{\Box}\) are in the flat direction of \(\tau\). Moreover, for each fixed \(\tau\), the sets \(\Box\) corresponding to \(\tau\) form a finitely overlapping cover of physical space and have side lengths at least as large as those of \(Q\), which means that each cube \(R^{1/2}Q\) intersects \(\lesssim 1\) set \(\Box\) associated to \(\tau\), for each \(\tau\). Hence the disjointness assumption in the decoupling theorem applies and so for each \(CQ \subseteq Y'\),
\[
\|Ef\|_{L^q(\mathbb{R})} \lesssim \left\| \sum_{\Box \in \mathcal{B}} \eta_{\Box, \lambda_0, \sigma_{\Box}} Ef_{\Box} \right\|_{L^q(\mathbb{R})} + R^{-N}\|f\|_2
\] by (3.4),
\[
\lesssim \left( \sum_{\Box \in \mathcal{B}} \|\eta_{\Box, \lambda_0, \sigma_{\Box}} Ef_{\Box}\|_q^2 \right)^{1/2} + R^{-N}\|f\|_2
\] by Theorem 3.1
\[
\lesssim \mu^{\frac{1}{2} - \frac{1}{q}} \left( \sum_{\Box \in \mathcal{B}} \|\eta_{\Box, \lambda_0, \sigma_{\Box}} Ef_{\Box}\|_q^q \right)^{1/q} + R^{-N}\|f\|_2.
\]
Raising both sides to the power \( q \) and summing over \( CQ \subseteq Y' \) gives, since 
\[ \sum_q \eta_q \lesssim 1 \] and since \( \eta_Q \) is essentially supported on \( 2Y := \bigcup_{S \leq Y} 2S \),
\[ \|Ef\|_{L^q(Y')}^q \lesssim \mu^q\left( \frac{1}{2} - \frac{1}{q} \right) \sum_{\square \in \mathcal{B}} \|Ef\|_{L^q(2Y')}^q + R^{-Nq}\|f\|_2^q. \]

Since \( Y' \) contains a fraction \( \approx 1 \) of the cubes in \( Y \), and since the cubes contribute equally, this gives
\[ \|Ef\|_{L^q(Y')} \lesssim \mu^q\left( \frac{1}{2} - \frac{1}{q} \right) \sum_{\square \in \mathcal{B}} \|Ef\|_{L^q(2Y')} + R^{-Nq}\|f\|_2 \]  
(3.5)
\[ \lesssim \mu^q\left( \frac{1}{2} - \frac{1}{q} \right) \sigma_\square \sum_{\square \in \mathcal{B}} \|f\|_2^q \]

(by Lorentz rescaling, see below)
\[ \lesssim \mu^q\left( \frac{1}{2} - \frac{1}{q} \right) \sigma_\square \|B\|_{1, \frac{1}{q}}^q \|f\|_2^q, \]  
(3.6)

since \( \|f\|_2 \) is essentially constant in \( \square \) and \( \sum_{\square \in \mathcal{B}} \|f\|_2^2 \lesssim \|f\|_2^2 \). To justify the Lorentz rescaling step in (3.5), after a rotation it may be assumed that the flat direction of the cap \( \tau \) corresponding to \( \square \) is \( \frac{\sqrt{2} \xi}{\sqrt{\tau}} \). Let
\[ B_E = \left( R^{1/4} x_1, \ldots, R^{1/4} x_{d-1}, x_d, R^{1/2} x_{d+1} \right), \]
and define \( \eta \) and \((\bar{x}, \bar{t})\) by
\[ (\eta, \eta) = A^* BA(\xi, \xi) \quad \text{for} \quad \xi \in \tau, \quad (\bar{x}, \bar{t}) = A^* B^{-1} A(x, t). \]

Under this change of variables, the boxes \( 2S \) are sent to cubes \( 2Q \) of side length \( 2R^{1/4+\delta} \), whose union is defined to be \( \hat{Y} \). Moreover,
\[ \|Ef\|_{L^q(2Y')} \sim \|E'g\|_{L^q(\hat{Y})} \quad \text{and} \quad \|f\|_2 \sim \|g\|_2, \]
where
\[ g(\eta) := R^{d+1} \left| \frac{d\xi}{d\eta} \right| (f_{\square} \circ \pi)(\xi, |\xi|), \]
and \( E' \) is the extension operator for a compact subset \( \Gamma_1 \) of the cone, which is contained in
\[ \{(\xi, |\xi|) : \xi_d \geq 0 \} \cap B \left( 0, 3\sqrt{2} \right) \setminus B(0, 1), \]
provided \( \tau \) is at scale \( cR^{-1/4} \) for some small constant \( c \) depending only on \( d \) (which may be assumed). The values \( \|Ef\|_{L^q(2S)} \sim \|E'g\|_{L^q(2Q)} \) are essentially constant over \( S, Q \) by definition of the dyadic values \( \lambda \). The slabs are of the form \( A^* \left( \mathbb{R}^d \times [t_0, t_0 + R^{1/4+\delta}] \right) \) and each contains \( \sim \sigma_\square \) cubes, all inside a ball of radius \( R^{1/2+2\delta} \). Even though \( \Gamma_1 \) may be slightly outside the truncated half cone, by scaling \( g \) slightly this only affects the inequality by a constant factor. Hence the generalised statement of the theorem at scale \( R^{1/2+2\delta} \) can be applied to give
\[ \|Ef\|_{L^q(Y')} \lesssim \sigma_\square \|f\|_2 \]
(3.7)
which is (3.5). Taking the \( q \)-th root of (3.5) gives
\[ \|Ef\|_{L^q(Y')} \lesssim \mu^q \|B\|_{1, \frac{1}{q}}^{-1} \|f\|_2, \quad \text{where} \quad \gamma = \frac{1}{2} - \frac{1}{q}. \]
To finish the proof, it suffices to show that \( \mu \lesssim \frac{\sigma m(Y)}{\sigma} \). The multiplicity \( \mu \) satisfies
\[
\mu m(Y) \lesssim \mu m(Y')
\]
\[
\sim \mu \sum_{CQ \subseteq Y'} m(Q)
\]
\[
\sim \sum_{CQ \subseteq Y'} \sum_{\square \in \mathcal{B}} m(Q)
\]
by definition of \( \mu \),
\[
\lesssim \sum_{CQ \subseteq Y'} \sum_{\square \in \mathcal{B}} \frac{R^{1/4} m \left( \square \cap R^{3\delta} CQ \right)}{\# \square \cap R^{2\delta} Q \neq \emptyset}
\]
\[
\lesssim \sum_{\square \in \mathcal{B}} \frac{R^{1/4} m \left( \square \cap R^{3\delta} Y' \right)}{\# \square \cap R^{2\delta} Q \neq \emptyset}
\]
by Fubini,
\[
\leq \sum_{\square \in \mathcal{B}} \frac{R^{1/4} m \left( \square \cap R^{3\delta} Y \right)}{\# \square \cap R^{2\delta} Q \neq \emptyset}.
\]

Therefore it suffices to show that for each \( \square \in \mathcal{B} \),
\[
m \left( \square \cap R^{3\delta} Y \right) \lesssim \frac{\sigma m(Y) R^{-1/4}}{\sigma}.
\]
By breaking both sides of this inequality into slabs \( A^* \left( \mathbb{R}^d \times [t, t + CR^{1/2+3\delta}] \right) \) covering \( R^{3\delta} Y \) and containing \( \approx \sigma \) cubes \( Q \) in each slab, it suffices to show that
\[
(3.8) \quad m \left( \square \cap R^{3\delta} A^* \left( \mathbb{R}^d \times [t, t + CR^{1/2+3\delta}] \right) \right) \lesssim \sigma m(Q) R^{-1/4},
\]
where \( m(Q) \) only depends on \( R \). The long axis of the set \( \square \) makes an acute angle \( \geq 1 \) with the \( CR^{1/2+3\delta} \) slab, and therefore intersects it in a set of diameter \( \lesssim R^{3/4} \).

Hence there are \( \lesssim \sigma_{\square} \) sets \( S \subseteq Y \) in the intersection on the left hand side. Since each such set \( S \) has long axis in the same direction as \( \square \), the intersection of \( S \) with the \( CR^{1/2+3\delta} \) slab is contained in a box of dimensions
\[
\sim R^{1/2+\delta} \times \cdots \times R^{1/2+\delta} \times R^{1/4+\delta} \times R^{1/2+3\delta},
\]
and therefore the intersection has measure \( \lesssim m(Q) R^{-1/4} \). Adding up the contributions of each \( S \) gives \( \Box \Box \), and this yields
\[
\mu \lesssim \frac{\sigma_{\square} |\mathcal{B}|}{\sigma}.
\]
Substituting this bound into \( \Box \Box \) gives
\[
\|Ef\|_{L^\infty(Y)} \lesssim \sigma^{-1} \|f\|_2.
\]
Therefore, by the induction on scales argument explained in Remark \( \Box \Box \), this proves the theorem. \( \square \)

**Remark 3.4.** A similar example to the Schrödinger case \( \Box \Box \) shows that the inequality is sharp: \( Y \) is a collection of \( \sigma \) disjoint tubes of radius \( R^{1/2} \) and length \( R \), \( f \) is an essentially orthogonal sum \( f = \sum_{\nu} f_\nu \) over the different tubes with \( \|f_\nu\|_2 \) constant in \( \nu \), and each \( Ef_\nu \) essentially constant and supported on a thinner tube inside the larger one.

To be more precise, let \( \phi \) be a Schwartz function on \( \mathbb{R}^d \), with \( 0 \leq \hat{\phi} \leq 1 \), and \( \phi \) equal to 1 on \( B((3/2)e_d, \epsilon) \) and vanishing outside \( B((3/2)e_d, 2\epsilon) \) for fixed small \( \epsilon > 0 \). Then \( |E\hat{\phi}| \sim 1 \) on a ball \( B(0, C_\epsilon) \) for some small \( C_\epsilon > 0 \), and \( E\phi \) is essentially
supported on $B(0, R^2)$). Let $\tau$ be a cap in the cone at scale $R^{-1/2}$ with centre line in the direction $e_{d+1} + e_d$. Define $\phi_\tau$ on the Fourier side by $\hat{\phi}_\tau(\xi) = R^{d/2} \hat{\phi} \circ \pi(A^*BA(\xi, |\xi|))$, where $B(\omega_1, \ldots, \omega_d, \omega_{d+1}) = (R^{1/2}\omega_1, \ldots, R^{1/2}\omega_d, \omega_{d+1}, R\omega_{d+1})$.

Then $\|E\phi_\tau\|_1 \sim 1$ on $(A^*BA)B(0, C_\nu)$ by a change of variables, and the restriction of $E\hat{\phi}_\tau$ to $B(0, R)$ is essentially supported on a larger box of dimensions $R^{1/2+\delta} \times \cdots \times R^{1/2+\delta} \times R^d \times 2R$, by an integration by parts as in the wave packet decomposition. Let $\{T_{\nu}\}$ be a finitely overlapping cover of $B(0, R)$ with translates of this box by points in $A^*(Z^d \times \{0\}^d)$, each containing a corresponding translate $S_{\nu}$ of the smaller box. Let $c(\nu)$ be the centre of $S_\nu$. Define $\phi_{\tau, \nu}$ on the Fourier side by $\hat{\phi}_{\tau, \nu}(\xi) = e^{-2\pi i \nu \cdot \langle \xi, \theta \rangle} \hat{\phi}_\tau(\xi)$, and let $f_{\nu} = \phi_{\tau, \nu}$. Let $T$ be a subcollection of boxes $T_{\nu}$ which intersect $B(0, R/2)$ and are $R^{1/2+2\delta}$-separated from each other, such that $|T| = \sigma$. Let $f = \sum_{T_{\nu} \in T} f_{\nu}$. Cover each $S_\nu$ with $R^{1/2-2\delta}$ disjoint lattice $R^{1/2+2\delta} A^* Z^{d+1}$-cubes $Q$, and let $Y$ be the union of all such $Q$, over all $T_{\nu} \in T$. By a change of variables, for any $Q$ intersecting $S_\nu$,

$$\|E f\|_{L^q(Q)}^q \approx \|E f_{\nu}\|_{L^q(Q \cap S_\nu)}^q \approx R^{-1/2} \|E f_{\nu}\|_{L^q(S_\nu)} \approx R^{-1/2}.$$

Similarly $\|E f\|_{L^q(Y)} \approx \sigma$ and $\|f\|_2 \approx \sigma$, which gives $\|E f\|_{L^q(Y)} \gtrsim \sigma^{-\gamma} \|f\|_2$. It is then straightforward to find a subset $Y' \subseteq Y$ with a fraction $\approx 1$ of the cubes in $Y$, and approximately the same number of cubes in each slab, such that $\|E f\|_{L^q(Q)}$ is constant over $Y'$, so this verifies that the theorem is sharp.

4. Multilinear refined Strichartz inequality

As in the Schrödinger case [5], the linear refined Strichartz inequality for the wave equation implies a multilinear version. The proof is similar to the one in [5]; it utilises the $k$-linear multilinear Kakeya inequality in $R^n$ from [10]. To state this, given $\nu > 0$, a collection $T_1, \ldots, T_k$ of $k$ sets consisting of tubes $T_i \in T_i$, of infinite length and equal radius, is called $\nu$-transverse if

$$|v_1 \land \cdots \land v_k| \geq \nu \quad \text{for all } T_i \in T_i, \text{ with } 1 \leq i \leq k,$$

where $v_i$ is the infinite direction in $T_i$.

**Theorem 4.1.** Suppose that $2 \leq k \leq n$ and that $T_1, \ldots, T_k$ are $\nu$-transverse families of tubes in $R^n$ of radius $\delta$. if $\frac{kn}{k-1} < q \leq \infty$ then

$$\left\| \sum_{T_i \in T_i} \chi_{T_i} \right\|_{L^{n/k}(R^n)} \leq C(\nu, q, n) \prod_{i=1}^k \left( \delta^{n/q} |T_i| \right).$$

The preceding theorem in dimension $n$ will be applied with $n = d + 1$. The statement that $f_1, \ldots, f_k \in L^2(R^d)$ are transversely supported in $B(0, 2) \setminus B(0, 1)$ will mean that each supp $f_i \subseteq B(0, 2) \setminus B(0, 1)$, and any $k$-tuple $(\xi_1, \ldots, \xi_k)$ with $\xi_i \in \text{supp } f_i$ satisfies

$$|G(\xi_1) \land \cdots \land G(\xi_k)| \gtrsim 1,$$

where $G(\xi)$ is the unit normal to the cone at $(\xi, |\xi|)$.

**Theorem 4.2.** Suppose that $2 \leq k \leq d + 1$ and that $f_1, \ldots, f_k \in L^2(R^d)$ are transversely supported in $B(0, 2) \setminus B(0, 1)$, and let $Y = \bigcup_{j=1}^N Q_j$ be a collection of
Therefore, to finish the proof it suffices to show that \( \gamma \) is constant in \( j \) for each \( i \), that each cube \( Y \) is \( \approx \) the same for all \( f_i \), and so on. Using (3.7) for each \( f_i \subseteq \mathbb{Z}^1 \), and so on). The sets \( B_i \) as in the proof of Theorem 3.2. Applying Hölder’s inequality gives

\[
\left\| \prod_{i=1}^{k} |E_{f_i}|^{1/k} \right\|_{L^q(Y)} \leq \prod_{i=1}^{k} \|E_{f_i}\|^{1/k}_{L^q(Y)}.
\]

To each \( f_i \), apply the same steps as in the proof of Theorem 3.2 up to (5.7), except without replacing the cubes \( Q \) by \( CQ \), and instead choose the set \( Y' \) of cubes in \( Y \) to be the same for each \( i \) (by doing this first for \( f_1 \), and then \( f_2 \) on the good cubes from \( f_1 \), and so on). The sets \( \Box_i \), \( Y_i \), \( B_i \) and parameters \( \mu_i \), \( \sigma_{\Box_i} \) are all defined as before. Using (5.7) for each \( f_i \), and the generalised version of Theorem 3.2 at (3.5) (instead of the inductive assumption) yields

\[
(4.2) \quad \left\| \prod_{i=1}^{k} |E_{f_i}|^{1/k} \right\|_{L^q(Y)} \approx \left( \prod_{i=1}^{k} \frac{\mu_i}{|B_i| \sigma_{\Box_i}} \right)^{\gamma/k} \prod_{i=1}^{k} \|f_i\|_{L^q(Y)}^{1/k}.
\]

Therefore, to finish the proof it suffices to show that

\[
(4.3) \quad N \prod_{i=1}^{k} \mu_i^{1/k} \leq \prod_{i=1}^{k} \left( \frac{|B_i| \sigma_{\Box_i}}{\mu_i} \right)^{1/k}.
\]

To this end, for each \( Y_i \), let \( Z_{\Box_i} = \bigcup_{S \subseteq Y_i} F(S) \), where for each box \( S \subseteq Y_i \), of dimensions

\[
R^{1/2+\delta} \times \cdots \times R^{1/2+\delta} \times R^{1/4+\delta} \times R^{3/4+\delta},
\]

the set \( F(S) \) is defined to be the box with dimensions

\[
R^{1/2+4\delta} \times \cdots \times R^{1/2+4\delta} \times R^{1/2+4\delta} \times R^{3/4+4\delta},
\]

with the same centre and orientations as \( S \), obtained by expanding \( S \) by \( \approx R^{1/4} \) in the second last direction and \( \approx 1 \) in all the other directions. This adjustment ensures that if the expanded cube \( R^{2^{k_0}}Q \) of side length \( R^{1/2+2\delta} \) intersects \( Y_i \), then \( Q \) is completely contained in \( Z_{\Box_i} \). Recall that each \( \mu_i \) was chosen so that for each \( Q \subseteq Y' \), there are \( \sim \mu_i \) sets \( \Box_i \in B_i \) such that \( Y_{\Box_i} \) intersects \( R^{2^{k_0}}Q \).

Let \( \{B\}_B \) be a finitely overlapping cover of \( B(0, R) \) by balls of radius \( R^{3/4} \) such that each cube \( Q \subseteq Y' \) is completely contained in some \( B \). Define

\[
\mathbb{B}_i = \{ \Box_i \in \mathbb{B}_i : \Box_i \cap R^{2^k}B \neq \emptyset \}.
\]

Invoking the definition of each \( \mu_i \) in the left hand side of (4.3) yields

\[
N \prod_{i=1}^{k} \mu_i^{1/k} \leq \frac{1}{R^{2^{k_0}}} \sum_{Q \subseteq Y'} \int_Q \prod_{i=1}^{k} \mu_i^{1/k} \, dx.
\]
\[ \sum_{B} \sum_{Q \in B \cap Y} \int_{Q} \prod_{i=1}^{k} \left( \sum_{\Box_{i} \in B_{i,B}} \chi_{\Box_{i}} \right) \frac{1}{d} \, dx \lesssim \frac{1}{R^{d+1}} \sum_{B} \int_{B} \prod_{i=1}^{k} \left( \sum_{\Box_{i} \in B_{i,B}} \chi_{\Box_{i}} \right) \frac{1}{d} \, dx. \]

Each \( \chi_{\Box_{i}} \) in the product can be bounded by \( \sum_{T \in \mathcal{T}_{i}} \chi_{T} \) where each \( T \) is a tube of radius \( R^{1/2+5\delta} \) and infinite length in the long direction of \( \Box_{i} \) through the centre line of some \( S \subseteq Y_{\Box_{i}} \) which intersects \( R^{d} B \). Since all the sets \( S \) therefore come from \( \lesssim 1 \) slab of width \( \approx R^{3/4} \) in \( \Box_{i} \), there are \( \lesssim \sigma_{\Box_{i}} \) such tubes \( T \in \mathcal{T}_{i} \), for each \( i \). Therefore, applying the \( k \)-dimensional multilinear Kakeya inequality (4.1) with \( n = d + 1 \) and \( q \) close to \( k \) gives

\[ \left( k \prod_{i=1}^{k} |\mathbb{B}_{i,B}| \right) \approx \frac{1}{R^{d+1}} \sum_{B} \int_{B} \prod_{i=1}^{k} \left( \sum_{\Box_{i} \in B_{i,B}} \chi_{\Box_{i}} \right) \frac{1}{d} \, dx \lesssim \frac{1}{R^{d+1}} \int_{B(0,2R)} \prod_{i=1}^{k} \left( \sum_{\Box_{i} \in B_{i}} \chi_{\Box_{i}} \right) \frac{1}{d} \, dx \lesssim k \prod_{i=1}^{k} |\mathbb{B}_{i}|, \]

by the multilinear Kakeya inequality. Combining this with (4.1) and (4.2) gives

\[ \left\| \prod_{i=1}^{k} |E f_{i}|^{1/k} \right\|_{L^{q}(Y)} \lesssim N^{-\frac{(k-1)}{2}} \prod_{i=1}^{k} \| f_{i} \|_{2}^{1/k}, \]

which finishes the proof. \( \square \)

5. Decay of conical averages

This section contains a lower bound on the decay rate \( \beta_{d}(\alpha, \Gamma) \) of the conical averages, followed by a proof of the relationship between \( \beta_{d}(\alpha, \Gamma) \) and \( s_{d}(\alpha, 2, d) \). The section concludes with an upper bound on \( \beta_{d}(\alpha, \Gamma) \).

Similarly to Corollary 3.3 of [3], the lower bound follows from Theorem 3.2, the proof is included for completeness.

**Corollary 5.1.** For any \( \alpha \in (0, d+1] \),

\[ \beta_{d}(\alpha, \Gamma) \geq \alpha - 1 + \frac{d - \alpha}{d + 1}. \]
Proof. Without loss of generality, it may be assumed that Γ is restricted to ξ ∈ Ω. Let μ : R^d → R^d be the natural projection. If μ is a Borel measure supported in the unit ball with cα(μ) < ∞, then for any R > 0,

\[(5.1)\quad \left( \int |\tilde{\mu}(R\xi)|^2 d\sigma_{\Gamma} \right)^{1/2} = \sup_{\|f\|_2 = 1} \left| \int \tilde{\mu}(R\xi) (f \circ \pi) (\xi) d\sigma_{\Gamma}(\xi) \right| .\]

For any fixed f with \(\|f\|_2 = 1\),

\[\left| \int \tilde{\mu}(R\xi) (f \circ \pi) (\xi) d\sigma_{\Gamma}(\xi) \right| = \left| \int \tilde{\mu}(\xi) (f \circ \pi) (\xi/R) d\sigma_{\Gamma}(\xi) \right| = \left| \int Ef(-Rx, -Rt) d\mu(x,t) \right| \leq R^{-\alpha} \|Ef\|_{L^1(\mu_R)},\]

where \(\mu_R(E) := R^\alpha \mu(-R^{-1}E)\) for any Borel set E. The measure \(\mu_R\) satisfies \(c_\alpha(\mu_R) \leq c_\alpha(\mu)\). If \(\psi\) is a fixed non-negative Schwartz function equal to 1 on the ball \(B(0,3\sqrt{2})\), and vanishing outside a slightly larger ball, then \(\tilde{E}f = \tilde{E}f\psi\), which implies that \(|Ef| \leq |Ef\| \psi\). Therefore using Fubini and the Schwartz decay of \(\psi\) gives

\[\|Ef\|_{L^1(\mu_R)} \leq \int_{B(0,2R)} |Ef| H dx + R^{-N} \|f\|_2,\]

where \(H := |\tilde{\psi}| \mu_R\).

By the uncertainty principle (see e.g. the scaled version of Lemma 2.5 in [9] with \(s = 1\)), the function \(H\) satisfies

\[\|H\|_\infty \leq c_\alpha(\mu), \quad \|H\|_1 \leq \|\mu\| R^\alpha, \quad \text{and} \quad \int_{B((x,t),r)} H dy \leq c_\alpha(\mu)r^\alpha,\]

for every \((x,t) \in \mathbb{R}^{d+1}\) and \(r > 0\) (the third inequality holds for \(r < 1\) by the first one). Partition \(B(0,2R)\) into cubes \(Q\) of side length \(\sqrt{2}R\), and sort each cube \(Q\) into sets \(S_\lambda\) according to the dyadic value \(\lambda\) of \(\|Ef\|_{L^1(Q)}\). For each \(\lambda\), sort the cubes \(Q\) in \(S_\lambda\) further according to the number of cubes \(\sigma\) in \(S_\lambda\) lying in the \((2R)^{1/2}\) slab parallel to \(A^* (\mathbb{R}^d \times \{0\})\) which contains \(Q\). Since there are \(\leq (\log R)^2\) significant pairs \((\lambda, \sigma)\), by standard pigeonholing and the triangle inequality there exists a pair \((\lambda, \sigma)\) with associated cubes \(Y = \bigcup Q\) such that (ignoring the term \(R^{-N}\|f\|_2\))

\[\left\| Ef H^{1/2} \right\|_{L^2(B(0,2R))} \lesssim \left\| Ef H^{1/2} \right\|_{L^2(Y)} .\]

Therefore, with \(\gamma = \frac{1}{2} - \frac{1}{q}\), applying Theorem 5.2 gives

\[\int_{B(0,2R)} |Ef| H dx \lesssim \|\mu\|^{1/2} R^{\alpha/2} \left\| Ef H^{1/2} \right\|_{L^2(B(0,2R))} \text{ by Cauchy-Schwarz,}\]

\[\lesssim \|\mu\|^{1/2} R^{\alpha/2} \left\| Ef H^{1/2} \right\|_{L^2(Y)} \leq \|\mu\|^{1/2} R^{\alpha/2} \left\| Ef \right\|_{L^q(Y)} \left\| H^{1/2} \right\|_{L^{q'}(Y)} \text{ where } \frac{1}{q'} = 1 - \frac{1}{q},\]

\[\lesssim \|\mu\|^{1/2} c_\alpha(\mu)^{1/2} R^{\alpha/2} \sigma^{-\gamma} \|f\|_2 \left( R^{1/2} \sigma R^{\alpha/2} \right)^{\gamma} = \|\mu\|^{1/2} c_\alpha(\mu)^{1/2} R^{\frac{1}{2}(\alpha + \gamma + \alpha \gamma)} \|f\|_2.\]
This proves the inequality $\beta$-characterisation of the Cauchy-Schwarz inequality, so this implies that $s_d(\alpha, 2) \leq \frac{d}{2} - \frac{1}{2} \left( \alpha - 1 + \frac{d - \alpha}{d + 1} \right)$.

Corollary 5.2. For any $\alpha \in (0, d + 1)$,

$$s_d(\alpha, 2) \leq \frac{d}{2} - \frac{1}{2} \left( \alpha - 1 + \frac{d - \alpha}{d + 1} \right).$$

Corollary 5.2 is a consequence of Corollary 5.1 combined with the following relation between $s_d(\alpha, 2)$ and $\beta_d(\alpha, \Gamma)$ from [24], the proof is included here for convenience, but is essentially the same as the one in [20].

Proposition 5.3. For any $\alpha \in (0, d + 1)$,

$$\beta_d(\alpha, \Gamma) = d - 2s_d(\alpha, 2).$$

Proof. Assume now that $E$ denotes the extension operator for the unrestricted cone, and let $\mu$ be a finite Borel measure with support in the unit ball. For any fixed $f$ with $\text{supp } f \subseteq B(0, 2) \setminus B(0, 1)$ and $\|f\|_2 = 1$,

$$\left| \int \tilde{\mu}(R\xi)(f \circ \pi)(\xi) \, d\sigma_\Gamma(\xi) \right| = \left| \int \tilde{\mu}(\xi)(f \circ \pi)(\xi/R) \, dR_\#\sigma_\Gamma(\xi) \right| = R^{-d} \left| \int Ef_R \, d\mu \right| \leq R^{-d} \|Ef_R\|_{L^1(\mu)},$$

(5.3)

where $f_R(\xi) := f(-\xi/R)$. Since $\text{supp } f_R \subseteq B(0, 2R) \setminus B(0, R)$, the definition of $s_d(\alpha, 1)$ implies that

$$R^{-d} \|Ef_R\|_{L^1(\mu)} \lesssim R^{-\left( \frac{d}{2} - s_d(\alpha, 1) \right)} \|\mu\|^{1/2} c_\alpha(\mu)^{1/2}.$$  

Taking the supremum over $\|f\|_2 = 1$ with $\text{supp } f \subseteq B(0, 2) \setminus B(0, 1)$ yields

$$\int |\tilde{\mu}(R\xi)|^2 \, d\sigma_\Gamma(\xi) \lesssim R^{-\left( d - 2s_d(\alpha, 1) \right)} \|\mu\| c_\alpha(\mu).$$

This proves the inequality $\beta_d(\alpha, \Gamma) \geq d - 2s_d(\alpha, 1)$. But $s_d(\alpha, 1) \leq s_d(\alpha, 2)$ by the Cauchy-Schwarz inequality, so this implies that

$$\beta_d(\alpha, \Gamma) \geq d - 2s_d(\alpha, 2).$$

For the other direction, let $f$ be given and let $g = \hat{f}\chi_{B(0, 2R) \setminus B(0, R)}$. The characterisation of the $L^2(\mu)$ norm through the distribution function is

$$\int |Eg|^2 \, d\mu = \int_0^\infty \lambda \mu \{ |Eg| > \lambda \} \, d\lambda.$$

For each $\lambda > 0$, define the probability measure $\mu_\lambda$ by

$$\mu_\lambda(F) = \frac{\mu(F \cap \{ |Eg| > \lambda \})}{\mu \{ |Eg| > \lambda \}},$$

for any Borel set $F$.

For any finite Borel measure $\nu$ supported in the unit ball,

$$\left| \int Eg \, d\nu \right| = \left| \int (g \circ \pi)(\xi) \hat{\nu}(\xi) \, d\sigma_\Gamma(\xi) \right|.$$
\[
\lesssim \|g\|_2 \left( R^{d-\beta_d(\alpha,\Gamma)} \|\nu\| \ c_\alpha(\nu) \right)^{1/2} = \|g\|_2 R^{d-\beta_d(\alpha,\Gamma)} \|\nu\|^{1/2} c_\alpha(\nu)^{1/2}.
\]

By repeating this with \(\nu\) replaced by the restriction of \(\nu\) to a set where \(\arg Eg\) is essentially constant, the absolute value can be moved inside the integral to get

\[
\|Eg\|_{L^1(\nu)} \lesssim \|g\|_2 R^{d-\beta_d(\alpha,\Gamma)} \|\nu\|^{1/2} c_\alpha(\nu)^{1/2}.
\]

Hence, taking \(\nu = \mu_\lambda\) yields

\[
\lambda \leq \|Eg\|_{L^1(\mu_\lambda)} \lesssim (R^{d-\beta_d(\alpha,\Gamma)} \|\mu_\lambda\|^{1/2} \|g\|_2 \leq \left( \frac{c_\alpha(\mu)}{\mu(\{Eg > \lambda\})} \right)^{1/2} R^{d-\beta_d(\alpha,\Gamma)} \|g\|_2.
\]

Assume for the moment that \(\|g\|_2 = 1\). Using the trivial bound \(\|Eg\|_\infty \lesssim R^{d/2}\) and (5.4) gives

\[
\int |Eg|^2 \, d\mu \leq R^{d-\beta_d(\alpha,\Gamma)} \lambda \mu(\{|Eg| > \lambda\}) \, d\lambda + \int_{R^{d-\beta_d(\alpha,\Gamma)}} \lambda \mu(\{|Eg| > \lambda\}) \, d\lambda \lesssim R^{d-\beta_d(\alpha,\Gamma)} c_\alpha(\mu) \|g\|_2^2,
\]

and by scaling this still holds even if \(\|g\|_2 \neq 1\). Fix \(\epsilon > 0\). Applying the triangle inequality results in

\[
\left\| E\hat{f} \right\|_{L^2(\mu)} \lesssim c_\alpha(\mu)^{1/2} \left\| \hat{f} \chi_{B(0,1)} \right\|_2 + \sum_{j=0}^\infty \int_{R^{d-2j}} \lambda R^{d-\beta_d(\alpha,\Gamma) + 2j} \|Eg\|_2^2 \left\| \hat{f} \chi_{B(0,2R) \setminus B(0,R)} \right\|_2.
\]

The \(\ell^2\) Cauchy-Schwarz inequality then yields

\[
\left\| E\hat{f} \right\|_{L^2(\mu)} \lesssim c_\alpha(\mu)^{1/2} \left\| \hat{f} \right\|_{H^{d-\beta_d(\alpha,\Gamma) + 2j}}^2,
\]

which shows that

\[
\beta_d(\alpha,\Gamma) \leq d - 2\sigma_d(\alpha,2),
\]

and therefore proves the inequality. \(\square\)

The following calculation shows that if the class of measures is restricted, there is no relation between the conical averages and fractal Strichartz inequalities.

**Proposition 5.4.** If \(\nu = \mu \otimes \lambda\), where \(\mu\) is a Borel measure with compact support in the unit ball of \(\mathbb{R}^d\) and \(d\lambda = \chi_{[0,1]} \, dm\), where \(m\) is the Lebesgue measure on \(\mathbb{R}\), then

\[
(5.5) \quad \int |\hat{\nu}(R\xi)|^2 \, d\sigma_\Gamma(\xi) \lesssim I_\alpha(\mu) R^{-(\alpha + 2)} \text{ for all } R > 0.
\]

**Proof.** The left hand side of (5.5) is

\[
\int_{\Gamma} |\hat{\nu}(R\xi)|^2 \, d\sigma_\Gamma = \int_{B(0,2) \setminus B(0,1)} |\hat{\nu}(R\xi, R\xi)|^2 \, d\xi.
\]
\[
\begin{align*}
&= R^{-d} \int_{B(0,2R) \setminus B(0,R)} |\hat{\nu}(\xi, |\xi|)|^2 \, d\xi \\
&= R^{-d} \int_{B(0,2R) \setminus B(0,R)} |\hat{\mu}(\xi)|^2 |\chi_{[0,1]}(|\xi|)|^2 \, d\xi \\
&\lesssim R^{-d} \int_{B(0,2R) \setminus B(0,R)} |\hat{\mu}(\xi)|^2 |\xi|^{-2} \, d\xi \\
&\lesssim R^{-(\alpha+2)} \int_{B(0,2R) \setminus B(0,R)} |\xi|^{\alpha-d} |\hat{\mu}(\xi)|^2 \, d\xi \\
&\lesssim R^{-(\alpha+2)} I_\alpha(\mu).
\end{align*}
\]

This proves the proposition. \qed

Returning to the case of a general measure, the following upper bound for the decay of the conical averages is based on the counterexample for the spherical case from [15].

**Proposition 5.5.** Let \( d \geq 5 \). Then
\[
\beta_\alpha(\alpha) \leq \alpha - 1 + \frac{2(d+1-\alpha)}{d+1} \quad \text{for} \quad \alpha \in (0, d+1).
\]

**Proof.** Let \( \alpha \in (0, d+1) \), let \( \epsilon, \kappa \in (0, 1) \) and let \( R > 1 \). Define
\[
\Lambda = (R^{\kappa-1} \mathbb{Z}^{d+1} + B(0, \epsilon R^{-1})) \cap B(0, 1),
\]
and let \( \mu \) be the Lebesgue measure on \( \mathbb{R}^{d+1} \) restricted to \( \Lambda \).

Assume that \( R \) is large enough to ensure that \( R^\kappa > 2 \), which makes the balls disjoint. If \( 0 < r \leq \epsilon R^{-1} \) then
\[
\frac{\mu(B(x, r))}{r^\alpha} \leq r^{d+1-\alpha} \lesssim R^{\alpha-(d+1)}.
\]

If \( \epsilon R^{-1} \leq r \leq R^{\kappa-1} \) then
\[
\frac{\mu(B(x, r))}{r^\alpha} \lesssim (\epsilon R^{-1})^{(d+1)-\alpha} \sim R^{\alpha-(d+1)}.
\]

If \( R^{\kappa-1} \leq r \leq 1 \), then choose a positive integer \( N \) so that \( r \sim NR^{\kappa-1} \), so that \( N \lesssim R^{1-\kappa} \). Then
\[
\frac{\mu(B(x, r))}{r^\alpha} \leq \frac{N^{d+1}(\epsilon R^{-1})^{d+1}}{N^\alpha R^{(\kappa-1)\alpha}} \lesssim R^{-\kappa(d+1)}.
\]

Since \( \mu \) is supported in the unit ball, this gives
\[
c_\alpha(\mu) \lesssim \max \left( R^{-\kappa(d+1)}, R^{\alpha-(d+1)} \right).
\]

Define \( \kappa \in (0, 1) \) by \( \kappa(d+1) = (d+1)-\alpha \). Then
\[
\|\mu\| \lesssim R^{\alpha-(d+1)}, \quad c_\alpha(\mu) \lesssim R^{\alpha-(d+1)}.
\]

Hence for every function \( f \) on the cone with \( \|f\|_{L^2(\sigma_\Gamma)} \leq 1 \),
\[
(5.6) \quad \left| \int f(\sigma_\Gamma(Rx)) \, d\mu(x) \right| \lesssim R^{\alpha-(d+1)} R^{\frac{d(\kappa-1)}{2}}.
\]

Let
\[
E = \{ (\xi, |\xi|) : R^\kappa(\xi, |\xi|) \in \mathbb{Z}^{d+1} \}.
\]
By adding up the lattice points on each of the \( \sim R^\kappa \) relevant spheres of integer radius \( \sim R^\kappa \), and applying the number theoretic estimate which says that for \( d \geq 5 \) there are \( \sim R^{d-2} \) lattice points on each sphere [12, Theorem 20.2], the cardinality of \( E \) is

\[
|E| \sim R^{\kappa(d-1)}.
\]

For a fixed small \( \rho > 0 \), to be chosen later (not depending on \( R \)), let

\[
\Omega = \Omega(\rho) = \{ \omega \in \Gamma : \text{dist}(\omega, E) \leq \rho R^{-1} \}, \quad f = \frac{\chi_{\Omega}}{\| \chi_{\Omega} \|_{L^2(\sigma_\Gamma)}}.
\]

Then for large \( R \),

\[
\sigma_\Gamma(\Omega) \sim |E| R^{-d}.
\]

Let \( F = \mathbb{Z}^{d+1} \cap B(0, R^{1-\kappa}) \), so that

\[
\| \chi_{\Omega} \|_{L^2(\sigma_\Gamma)} \left| \int \hat{f} \sigma_\Gamma(Rx) \, d\mu(x) \right| = \int_{B(0,1)} \int_{\Gamma} e^{-2\pi i \xi \cdot Rx} \chi_{\Omega}(\xi) \, d\sigma_\Gamma(\xi) \, d\mu(x)
\]

\[
= \sum_n \sum_{\omega \in E} \int_{B(\Gamma \cap B(\omega, \rho R^{-1}))} e^{-2\pi i \xi \cdot Rx} \, d\sigma_\Gamma(\xi) \, d\mu(x).
\]

For \( (\xi, x) \) in the domain of integration, and the corresponding \( (\omega, n) \), the Cauchy-Schwarz inequality gives

\[
|\langle \xi, R x \rangle - \langle \omega, R^n n \rangle| \leq |\langle \xi - \omega, R x \rangle| + |\langle \omega, R(x - R^n n) \rangle| \leq 3(\rho + \epsilon).
\]

Hence

\[
\| \chi_{\Omega} \|_{L^2(\sigma_\Gamma)} \left| \int \hat{f} \sigma_\Gamma(Rx) \, d\mu(x) \right|
\]

\[
= \sum_n \sum_{\omega \in E} \int_{B(\Gamma \cap B(\omega, \rho R^{-1}))} 1 + O(\rho + \epsilon) \, d\sigma_\Gamma(\xi) \, d\mu(x)
\]

\[
\sim |E| |F| (\epsilon R^{-1})^{d+1} (\rho R^{-1})^d (1 + O(\rho + \epsilon)).
\]

Therefore if \( \epsilon, \rho \) are taken small enough (not depending on \( R \)), then

\[
\left| \int \hat{f} \sigma_\Gamma(Rx) \, d\mu(x) \right| \sim |E|^{1/2} |F| R^{-\frac{d}{2} - 1}.
\]

Combining this with (5.6) and (5.7) gives

\[
\beta_d(\alpha, \Gamma) \leq \alpha - 1 + \frac{2(d+1-\alpha)}{d+1},
\]
by the definition of \( \kappa \).

\[ \square \]

**References**

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