Instanton Transition in Thermal and Moduli deformed de Sitter Cosmology

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Abstract

We consider the de Sitter cosmology deformed by the presence of a thermal bath of radiation and/or time-dependent moduli fields. Depending on the parameters, either a first or second order phase transition can occur.

In the first case, an instanton allows a double analytic continuation. It induces a probability to enter the inflationary evolution by tunnel effect from another cosmological solution. The latter starts with a big bang and, in the case the transition does not occur, ends with a big crunch. A temperature duality exchanges the two cosmological branches. In the limit where the pure de Sitter universe is recovered, the tunnel effect reduces to a “creation from nothing”, due to the vanishing of the big bang branch. However, the latter may be viable in some range of the deformation parameter. In the second case, there is a smooth evolution from a big bang to the inflationary phase.

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1 Introduction

The recent astrophysical observations indicate that our universe is in a phase of classical expansion, with a small positive cosmological term, $\Lambda$, representing $60 - 70\%$ of the total energy density. However, at early time, quantum corrections to this trajectory are expected to be of first significance. For instance, drastic non-perturbative effects could occur as topology changes, as can be seen in the context of field theory in semi-classical approximation. As an example, a topology change is provided by the instability of the five dimensional Kaluza-Klein (KK) space-time [1]. In that case, one observes the nucleation of a finite size "bubble of nothing" that is then growing up at the speed of light. The transition is described by tunnel effect in terms of an instanton configuration allowing a "double analytic continuation". This means that at two different Euclidean times $\tau_i$ and $\tau_f$, analytic continuations to real times $\tau = \tau_i + i t$ and $\tau = \tau_f + i t$ are allowed. To be specific, the configuration admits a time independent asymptotic behavior allowing a continuation at $\tau_i = +\infty$ to the KK universe, while another continuation at $\tau_f = 0$ describes the evolution of a bubble.

Actually, the KK universe is suffering from a first order phase transition where bubbles appear instantaneously, grow and coalesce, so that the space “evaporates” into nothing (after an infinite time, since the volume is itself infinite). In some sense, some reversed ideas can be invoked in another example of topology change, namely in Vilenkin’s scenario of the de Sitter space “creation from nothing” [2, 3]. In that case, a finite radius $S^3$ space appears instantaneously by tunnel effect from a space-time state that amounts to the empty set. This $S^3$ “bubble” is then following a de Sitter growing up evolution. The transition involves an instanton, whose shape is an $S^4$ hemisphere. An analytic continuation on its boundary amounts to “gluing” a de Sitter universe, while the fact that the instanton is compact with no other boundary to analytically continue allows an interpretation in terms of a transition from an empty set.

Concerning the birth of the de Sitter space, the instanton method provides an estimate of the transition probability equivalent to the one derived from the Hartle-Hawking wave function [4] approach. The transition amplitude and wave function $\Psi$ being proportional to $e^{-S_E}$, where $S_E$ is the Euclidean action, the probability of the event is

$$p \propto e^{-2S_E}.$$  \hspace{1cm} (1.1)
A selection principle of the cosmological constant $\Lambda$ based on the extremum of $p$ leads to a favored value $\Lambda \to 0_+$, [5]. However, a present too small cosmological term cannot account for the 60% to 70% of the total energy density, which is necessary to explain the recent dark energy data. To remedy to this problem, it as been stressed [6, 7] that since the de Sitter space has an horizon, it can be associated a Hawking temperature. Thus, quantum fluctuations of the metric (or any massless mode introduced in the model) induce a space filling thermal bath. The latter implies an additional radiation term in the action\(^1\) and thus a back reaction on the metric background that deforms the de Sitter solution (see also [9]). In that case, the modified probability transition derived from the Euclidean action is maximal for a non vanishing finite value of $\Lambda$. This computation has also been addressed by the authors of [10] and refined in [11], who considered the Wheeler-de Witt equation in the WKB approximation. They found that the tunnel effect is not connecting the deformed de Sitter expansion to “nothing”, but to what they called a thermally excited era. However, the latter is $\Lambda$-dependent. This point makes the difference with the pure de Sitter space created from a $\Lambda$-independent state, namely the empty set, so that extremizing the transition amplitude with respect to $\Lambda$ was making sense. In the case of the deformed solution, applying this selection rule for $\Lambda$ is thus questionable.

Our first aim is to reconsider the thermally deformed de Sitter space-time. The previous analysis that involves an instanton transition occurs when the radiation energy density is below some critical value. This case corresponds to a first order phase transition. A double analytic continuation on the Euclidean solution implies a probability to connect two cosmological behaviors in real time. The first one is the de Sitter like evolution, while the second one is an era that begins with a big bang and ends with a big crunch, when the transition does not occur. If instead the radiation energy density is above the critical value, the phase transition becomes second order. It describes a smooth evolution from a big bang to a de Sitter like behavior. In all these solutions, the temperature is formally infinite when the radius of the universe vanishes. This means that these evolutions should be trusted as soon as the temperature passes below some upper bound such as the string or Hagedorn temperature $T_H$ of an underlying fundamental string theory.

We then generalize the model by including time dependent moduli-like fields that im-

\(^1\)See [8] for a cosmological scenario based on a cascade of transitions between vacuum and radiation energy.
ply a further deformation of the de Sitter solution. Since, these deformations respect the Robertson-Walker (RW) isometries, they can be expressed in terms of the scale factor $a$ of the universe. The Hubble parameter is taking the form

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 3\lambda - \frac{3k}{a^2} + \frac{C_R}{a^4} + \frac{C_M}{a^6},$$

(1.2)

where the constant and $1/a^2$ terms are associated to the cosmological constant and uniform space curvature. The $1/a^4$ monomial accounts for the effective radiation term discussed before, while the on shell moduli contribution reaches the $1/a^6$ contribution. What we find is that the moduli deformation can be fully absorbed in a redefinition of time. We then describe different cosmological scenarios that depend on $C_R$ and $C_M$. These parameters can be expressed in terms of the cosmological constant, the number of effective “massless” degrees of freedom, the temperature at the transition and the slope of the moduli dependence. If this two parameters model is interesting by itself from a formal point of view, its study is also motivated by the fact that it describes more complicated systems once they are considered on shell. For instance, cases involving a scalar field with a non trivial potential and coupled to moduli with non trivial kinetic terms are treated in [12].

## 2 Effective cosmological models in minisuperspace

We focus on models where space is 3-dimensional\(^2\) and closed. The number of degrees of freedom is important, and actually infinite in the context of string or M-theory. However, a subset of them to be considered in an effective field theory is determined by the ultraviolet cut off set by the fundamental theory. Some of the simplest class of models that can be constructed are considering the so called “minisuperspace” (MSS) approximation [4, 13]. They involve isotropic and homogeneous spaces only, so that the only gravitationnal degree of freedom is the scale factor of the universe, to which some matter fields can be coupled to. The space-time metric is thus

$$ds^2 = \sigma^2 \left( -N^2(t)dt^2 + a^2(t)d\Omega_3^2 \right), \quad \text{where} \quad \sigma^2 = \frac{2}{3\pi M_P^2},$$

(2.1)

\(^2\)By this, we mean that if the fundamental theory is for instance string (or M-)theory, there are compact dimensions whose sizes are dynamically constrained to remain of order $l_s$, the string length.
$M_p$ is the Planck mass and $\sigma a(t)$ is the radius of a 3-sphere. $N$ is a gauge dependent function that can be arbitrarily chosen by a redefinition of time\(^3\). (In our conventions, $N$ and $t$ are dimensionless.) In order to include in the effective action the quantum fluctuations of the full metric and matter degrees of freedom of the theory, we switch on a thermal bath which is created simultaneously with the space-time. The thermal bath involves the degrees of freedom below the temperature scale $T$ and is consisting in an extra radiation term in the effective MSS action. This amounts to take into account the back reaction of the thermal bath on the space-metric i.e. deform the de Sitter solution [6, 7].

The starting point is a gravity action coupled to some matter fields, whose Lagrangian density is $L_m$:

$$S \equiv S_g + S_m = \frac{M_p^2}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda) + \int d^4x \sqrt{-g} L_m. \quad (2.2)$$

Assuming the metric (2.1) in the gravity part and freezing all other degrees of freedom, we obtain the MSS-action,

$$S_g = \frac{1}{2} \int_{t_i}^{t_f} dt \, N a^3 \left( -\frac{1}{N^2} \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{a^2} - \lambda \right) + \frac{1}{2} \left[ \frac{a^2 \dot{a}^2}{N} \right]_{t_i}^{t_f}, \quad \text{where} \quad \lambda = \frac{\Lambda}{3} \sigma^2. \quad (2.3)$$

In this expression, $a$ and $N$ have been chosen positive, without loss of generality. Furthermore, the boundary terms arise from the integration by parts of the second derivative of $a$ in the scalar curvature $R$.

The non-vanishing components of the energy-momentum tensor which are induced by the matter fields (and which are invariant under the isometries of the RW space-time), can be expressed in terms of the energy density $\rho$ and pressure $P$:

$$T_{tt} = \rho \sigma^2 N^2, \quad T_{ij} = P \sigma^2 a^2 \tilde{g}_{ij}, \quad (2.4)$$

where $d\Omega^2_3 = \tilde{g}_{ij} dx^i dx^j$ is the line element on $S^3$. These quantities appear in the equations of motion of $N$ and $a$ i.e. the time-time and space-space Einstein equations:

$$3 \left( \frac{\dot{a}}{a} \right)^2 + 3 \frac{N^2}{a^2} = 12\pi^2 \sigma^4 N^2 \rho + 3\lambda N^2; \quad (2.5)$$

\(^3\)We shall use in the following the time variable $t$ either when the function $N(t)$ remains unspecified, or when it is fixed to $N(t) \equiv 1$. We shall instead use another variable, $\tilde{t}$, when we choose to fix $N(\tilde{t}) = 1/ \left( 1 + \lambda^{-1} \kappa/a^2(\tilde{t}) \right)$, where $\kappa$ is a constant.
\[-2\frac{\ddot{a}}{a} + 2\frac{\dot{a}\dot{N}}{a\dot{N}} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{N^2}{a^2} = 12\pi^2\sigma^4N^2P - 3\lambda N^2. \tag{2.6}\]

Since \(N\) can be fixed to any arbitrary function by a reparameterization of \(t\) in the metric (2.1), this differential system of two unknown functions must be degenerate so that we only have to solve any of the two equations\(^4\), e.g. the Friedman one (2.5). Let us consider some cosmological solutions associated to various matter contents and where phase transitions can occur.

## 3 Pure de Sitter space

With neither matter nor thermal bath, we have \(\rho = P = 0\). Choosing \(N(t) \equiv 1\), eq. (2.5) is easily solved,

\[a(t) = \frac{\cosh(\sqrt{\lambda}t)}{\sqrt{\lambda}}. \tag{3.1}\]

This corresponds to an usual de Sitter evolution where a phase of contraction from an infinitely past time is followed by an expansion phase when the radius \(a\) has reached its minimum.

However, a different cosmological scenario can give rise to an identical de Sitter expansion. Actually, one can consider the real time evolution for \(t \geq 0\) as the result of a tunnel effect, since an analytic continuation at \(t_i = -i\tau_f = 0\) is allowed. The instanton solution in question is obtained by substituting \(t = -i\tau\) in (3.1)\(^5\),

\[a_E(\tau) = \frac{|\cos(\sqrt{\lambda}\tau)|}{\sqrt{\lambda}}, \tag{3.2}\]

where we find convenient to insist on a conventional positivity of \(a_E\) for future purpose. The Euclidean metric takes the form \(ds_E^2 = (\sigma^2/\lambda)\lambda d\tau^2 + \cos^2(\sqrt{\lambda}\tau)d\Omega_3^2\), so that the instanton is a 4-sphere of radius \(\sigma/\sqrt{\lambda}\), if \(\sqrt{\lambda}\tau \in [-\pi/2, \pi/2]\).

Vilenkin, assuming that the range of the Euclidean time \([\tau_i, \tau_f] = [-\pi/2\sqrt{\lambda}, 0]\), considers this instantonic hemisphere as giving rise to a transition amplitude from nothing (a state of the space-time where it is the empty set) to an \(S^4\) space with de Sitter cosmological evolution [2,3]. As we shall see later, this interpretation is very natural in the deformed de

\(^4\)This is not true for static solutions.

\(^5\)For any field \(f(t)\), we define in general its Euclidean counterpart \(f_E(\tau) \equiv f(-i\tau)\) and its derivative \(\dot{f}_E(\tau) = -i\dot{f}(-i\tau)\).
Sitter solution which is induced by the thermal bath. The Euclidean action,
\[ S_{Eg} = \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau N_E a_E^3 \left( -\frac{1}{N_E^2} \left( \frac{\dot{a}_E}{a_E} \right)^2 + \frac{1}{a_E^2} + \lambda \right) + \frac{1}{2} \left[ \frac{a_E^3 \dot{a}_E}{N_E} \right]_{\tau_i}^{\tau_f}, \] (3.3)
evaluated on shell, has vanishing boundary terms and takes the value
\[ S_{Eg} = -\frac{1}{3\lambda}, \] (3.4)
so that the probability transition
\[ p \propto e^{2/(3\lambda)} \] (3.5)
becomes maximal for a vanishing cosmological constant. Considering the second hemisphere of the \( S^4 \)-instanton, \( \sqrt{\lambda}\tau_i = 0 \) and \( \sqrt{\lambda}\tau_f = \pi/2 \), we note that the previous probability also evaluates the chances for a contracting de Sitter space to simply vanish instantaneously when it reaches its minimal radius at \( t = 0 \).

Another derivation

Before closing this section, we would like to mention an alternative method to solve the Friedman equation (2.5) that will appear useful to express the general deformed de Sitter solution once the thermal and moduli effects are simultaneously present. Consider
\[ N^2(\tilde{t}) = \frac{1}{1 + \lambda^{-1}\kappa/a^2(\tilde{t})}, \] (3.6)
and choose \( \kappa = -1 \) so that the \( 1/a^2 \) term of eq. (2.5) is cancelled out\(^6\),
\[ \left( \frac{\dot{a}}{a} \right)^2 = \lambda. \] (3.7)

There are two solutions to this equation, whose corresponding metrics are
\[ ds^2_{\pm} = a^2 \left( \frac{-d\tilde{t}^2}{1 - \lambda^{-1}e^{\mp2\sqrt{\lambda}\tilde{t}}} + e^{\pm2\sqrt{\lambda}\tilde{t}} d\Omega_3^2 \right), \] (3.8)
defined for \( \tilde{t} > -\log(\lambda)/2\sqrt{\lambda} \) and \( \tilde{t} < \log(\lambda)/2\sqrt{\lambda} \), respectively. These parameterizations do not admit analytic continuations to Euclidean time, and actually correspond to “half solutions”, as can be seen as follows. Integrating in each case
\[ dt = N(\tilde{t}) d\tilde{t}, \] (3.9)
\(^6\)As signaled in footnote 3, remind that the “dot” derivative is here with respect to \( \tilde{t} \), the time variable in the gauge (3.6).
one can choose the time variable \( t \) (corresponding to the gauge \( N \equiv 1 \)) such that

\[
\lambda e^{\pm 2\sqrt{\lambda} \tilde{t}} = \cosh^2(\sqrt{\lambda} t),
\]

(3.10)

to glue the two branches in \( \tilde{t} \) in a smooth way, \( t \in \mathbb{R} \) (see fig. 1). In that case, the metrics \( ds^2_\pm \) are taking a RW form in the \( N \equiv 1 \) gauge (see eq. (2.1)), with \( a \) defined in eq. (3.1).

\[ \text{Figure 1: As a function of } \tilde{t}, \text{ the pure de Sitter scale factor } a \text{ has two branches. As a function of } t, \text{ these branches are glued together in a smooth way.} \]

4 Thermally deformed de Sitter universe

The quantum fluctuations of the previous background metric have been shown to induce an additional effective “radiation term” in the action. This implies a back reaction of the massless gravitationnal degrees of freedom on the metric [6,7]. In other words, the solution of the new equations of motion describes the evolution of a RW closed universe in presence of thermalized radiation. An explicit derivation of the effective thermal contribution can be found in [6, 7], while the consistency of the semi-classical approach has given a similar result [10]. We would like here to reconsider these effects by using thermodynamical and precise dynamical constraints to reveal new features. The results can be applied to any
model involving massless bosonic or fermionic modes (beside the gravitational ones), as long as their vacuum expectation values do not show up in the effective action.

For bosonic (or fermionic) fluctuating states of masses $m_b$ (or $m_f$) in thermal equilibrium at temperature $T$, the general expressions of the energy density $\rho_T$ and pressure $P_T$ are

$$\rho_T = \sum_{\text{boson}} I^B_{\rho}(m_b) + \sum_{\text{fermion}} I^F_{\rho}(m_f), \quad P_T = \sum_{\text{boson}} I^B_P(m_b) + \sum_{\text{fermion}} I^F_P(m_f), \quad (4.1)$$

where

$$I^{B(F)}_{\rho}(m_{b(f)}) = \int_0^\infty dq \frac{q^2 E(m_{b(f)})}{e^{\beta E(m_{b(f)})} - 1}, \quad I^{B(F)}_P = \frac{1}{3} \int_0^\infty dq \frac{q^4 / E(m_{b(f)})}{e^{\beta E(m_{b(f)})} - 1} \quad (4.2)$$

and $E(m_{b(f)}) = \sqrt{q^2 + m^2_{b(f)}}$. If the system is consisting in $n_{B(F)}$ massless degrees of freedom, it is easy to see that the state equation

$$\rho_T = 3P_T = \frac{\pi^4}{15} \left( n_B + \frac{7}{8} n_F \right) T^4 \quad (4.3)$$

is satisfied. These relations combined with the expression of the energy-momentum conservation

$$\dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) (\rho + P) = 0 \quad (4.4)$$

is then implying that

$$aT = \text{constant} \quad (4.5)$$

and that

$$\rho_T = \frac{\delta_T^2}{16\pi^2 \sigma^4 \lambda} \frac{1}{a^4}, \quad (4.6)$$

where $\delta_T^2/(16\pi^2 \sigma^4 \lambda)$ is a positive integration constant. The factor of $\lambda$ in its denominator is chosen for later convenience.

To proceed, we solve the Einstein equations in the case where the energy density and pressure are the thermal ones. This has been done for eq. (2.6) in [6, 7, 10] and [14] in the $N(t) \equiv 1$ gauge. However, we prefer to deal with eq. (2.5) as in [9] but in a way similar to what we did at the end of the last section for the pure de Sitter case, since this will be easier to generalize when we take into account moduli fields. For this purpose, we choose again $N(\tilde{t})$ with a form

$$N^2(\tilde{t}) = \frac{1}{1 + \lambda^{-1} \kappa/a^2(\tilde{t})}, \quad (4.7)$$
and look for $\kappa$ so that the $1/a^4$ thermal contribution is cancelled out. This implies that

$$\kappa^2 + \kappa + \frac{\delta_T^2}{4} = 0 ,$$

(4.8)
a condition that admits real solutions as long as the constraint $\delta_T^2 \leq 1$ is satisfied.

**The case $\delta_T^2 < 1$**

The Friedman equation (2.5) becomes

$$\left(\frac{a}{\dot{a}}\right)^2 + \kappa + \frac{1}{a^2} = \lambda,$$

(4.9)

showing that the solution of our present problem can be related to the one of section 3, the de Sitter case with no thermal effect, up to a shift in the $1/a^2$ curvature term. The hyperbolic cosine solutions of eq. (4.9) for the two roots of (4.8) are giving rise to the metrics

$$ds_{\pm}^2 = \sigma^2 \left( \frac{-dt^2}{u \cosh^2(\sqrt{\lambda} t)} + \frac{1 \pm \sqrt{1 - \delta_T^2}}{2\lambda} \cosh^2(\sqrt{\lambda} t) d\Omega_3^2 \right) ,$$

(4.10)

where

$$u = \frac{1 - \sqrt{1 - \delta_T^2}}{1 + \sqrt{1 - \delta_T^2}} \in [0, 1] ,$$

(4.11)

if $\delta_T^2 > 0$. Since they simply correspond to different gauge choices of functions $N$, we already know that they are equal up to a redifinition of time. This means that the potential singularity of $ds_+^2$ when the denominator of $N(\tilde{t})$ vanishes should be fake. In any case, let us make contact with the more intuitive $N \equiv 1$ gauge by choosing a new time variable $t$ satisfying eq. (3.9). This will in particular allow us to study in an explicit way various analytic continuations.

For $ds_+^2$, one can fix the integration constant such that

$$\cosh^2(\sqrt{\lambda} \tilde{t}) = u + (1 - u) \cosh^2(\sqrt{\lambda} t) ,$$

(4.12)

so that the metric is taking the form $ds_+^2 = \sigma^2 (-dt^2 + a^2 (\tilde{t}(t)) d\Omega_3^2)$, with

$$a = N \sqrt{\varepsilon + \cosh^2(\sqrt{\lambda} \tilde{t})} ,$$

(4.13)

where

$$N = \frac{(1 - \delta_T^2)^{1/4}}{\sqrt{\lambda}} , \quad \varepsilon = \frac{1}{2} \left( \frac{1}{\sqrt{1 - \delta_T^2}} - 1 \right) .$$

(4.14)
For $ds^2_-$, one can have instead
\[ \cosh^2(\sqrt{\lambda}t) = 1 + (u^{-1} - 1)\cosh^2(\sqrt{\lambda}t), \tag{4.15} \]
which has the effect to glue the two branches $\sqrt{\lambda}t > \arccosh(u^{-1/2})$ and $\sqrt{\lambda}t < -\arccosh(u^{-1/2})$ in a smooth way, with $t \in \mathbb{R}$ (see fig. 2). This time variable, has the property to explicitly show that $ds^2_- \equiv ds^2_+$.

Figure 2: When one is considering the time variable $\tilde{t}$ involved in $ds^2_-$ in the pure thermal case, the scale factor $a$ has two branches. In terms of $t$, these branches are glued together in a smooth way. At the minimum, one has $a = a_+$ given in eq. (4.18).

This solution is a deformation of the de Sitter solution (3.1) i.e. a contraction phase followed by an expansion one. As before, the latter can also arise by tunnel effect since the analytic continuation at $t_i = -i\tau_f = 0$ is still valid. Let us consider for a moment the relevant instanton solution obtained under the substitution $t = -i\tau$,
\[ a_E = N\sqrt{\varepsilon + \cos^2(\sqrt{\lambda}\tau)}, \tag{4.16} \]
for the range of Euclidean time $-\pi/2\sqrt{\lambda} \leq \tau \leq 0$. An important point is that at $\sqrt{\lambda}\tau_i = -\pi/2$, $a_E$ is no longer vanishing as in the birth of the de Sitter space (ref. [15] presents
another example of this phenomenon). Instead, a second analytic continuation is allowed, \( \sqrt{\lambda \tau} = -\pi/2 + i\sqrt{\lambda}t \), and another cosmological solution

\[
a = N\sqrt{\varepsilon - \sinh^2(\sqrt{\lambda}t)},
\]

(4.17)
satisfies (2.5) (in the \( N \equiv 1 \) gauge), for \(-\arcsinh\sqrt{\varepsilon} \leq \sqrt{\lambda}t \leq 0\). Actually, this sinh-solution is connected to the cosh-solution (4.13) via the instantonic one (4.16). The cosmological evolution makes use of all above mention solutions.

- The universe starts from a big-bang era at \( t \sim -\lambda^{-1/2}\arcsinh\sqrt{\varepsilon} \), with an evolution following the sinh-solution till \( t = 0 \).
- At \( t = 0 \), either i) the space starts to contract till a big crunch occurs at \( t = \lambda^{-1/2}\arcsinh\sqrt{\varepsilon} \), or ii) there is a first order phase transition that changes instantaneously its radius.
- In case ii), the cosmological evolution for later times, \( t \geq 0 \), follows the inflationary cosh-solution.

Actually, the very early part of these scenarios (and late time in case i)) are not trustable, assuming that the fundamental theory is string like. The degrees of freedom grow exponentially at high temperature, and blow up at the Hagedorn temperature. This effect gives a cut in time before which the sinh-solution is no more valid (and another cut for late time in case i)). A stringy approach is then necessary to cover such ultra hot eras of the universe.

At the transition, the scale factor is jumping,

\[
a_- = \sqrt{\frac{1 - \sqrt{1 - \delta_T^2}}{2\lambda}} \quad \rightarrow \quad a_+ = \sqrt{\frac{1 + \sqrt{1 - \delta_T^2}}{2\lambda}},
\]

(4.18)

(see fig. 3). The transition probability at \( t = 0 \) (see section 5.2) depends on \( \lambda \) and on the deformation parameter \( \varepsilon \) (or equivalently on \( \delta_T^2 \)). The maximal value of this probability is for finite \( \lambda \) and depends on \( \varepsilon \). This is fundamentally different from the pure de Sitter case where the maximal probability is for \( \lambda = 0 \).

It is interesting to make a comparison between the pure de Sitter case and the thermally deformed one. When \( \delta_T^2 \to 0 \), the big bang occurs at \( t_i \sim -\delta_T/2\sqrt{\lambda} \), while we also have \( a_- \sim \delta_T/2\sqrt{\lambda} \) at \( t = 0_- \). This means that Vilenkin’s interpretation of the spontaneous creation of the pure de Sitter space can be seen as the limit when the big bang sinh-cosmological branch has shrunk to nothing (see fig. 4). Actually, the instantonic jump \( a_- \to a_+ \) becomes more drastic as the thermal effects become more negligible, \( \delta_T \ll 1 \), since \( a_+ \sim 1/\sqrt{\lambda} \) in
Figure 3: Cosmological scenario of an expanding RW universe in presence of thermal and time-dependent moduli effects below some critical value. A phase of expansion starts from a big bang and is connected to an inflationary deformed de Sitter one by an instanton. This describes a first order transition, where the scale factor jumps from $a_-$ to $a_+$. 

this limit. On the contrary, the transition from the big bang branch to the cosh-inflationary deformed de Sitter one is smoother and smoother as we approach the critical magnitude of the thermal effects, $\delta_T \to 1$. Actually the amplitude of the instanton solution decreases as we approach this value of the parameter. When $\delta_T = 1$, the instanton solution is constant, implying that its analytic continuations are static, $a(t) \equiv 1/\sqrt{2\lambda}$.

An estimate of the transition probability from the big bang branch to the deformed de Sitter one is $p \propto e^{-2S_{E_{eff}}}$, where

$$S_{E_{eff}} = \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau N_E a_E^3 \left( -\frac{1}{N_E^2} \left( \frac{\dot{a}_E}{a_E} \right)^2 - \frac{1}{a_E^2} + \lambda + \frac{\delta_T^2}{4\lambda a_E^4} \right) + \frac{1}{2} \left[ \frac{a_E^{2}\dot{a}_E}{N_E} \right]_{\tau_i}^{\tau_f} \right) \quad (4.19)$$

is the Euclidean effective action that is giving rise to the desired equations of motion. In this expression, $S_{E_{eff}}$ is evaluated with the solution (4.16) and has vanishing boundary terms for $-\pi/2 \leq \sqrt{\lambda} \tau \leq 0$, (see eq. (5.57)). However, one could sum in principle the instanton contributions for the series of ranges $-(2n+1)\pi/2 \leq \sqrt{\lambda} \tau \leq 0$, $n \in \mathbb{N}$. The total probability would take the form $p_{tot} = \sum_{n \geq 0} c_{2n+1} e^{-(2n+1)2S_{E_{eff}}}$, where $c_{2n+1}$ are positive constants. Sim-
Figure 4: Thermal instanton (in bold) compared to the pure de Sitter $S^4$ one. In the limit where thermal effects are negligible, $\delta_T \ll 1$, the instantonic jump of $a$ is drastic, passing from $a_- \sim \delta_T/2\sqrt{\lambda}$ to $a_+ \sim 1/\sqrt{\lambda}$. When thermal effects are getting closer and closer to their critical magnitude, $\delta_T = 1$, the amplitude of the transition from the big bang branch to the inflationary deformed de Sitter one is decreasing. At the very precise value $\delta_T = 1$, the instanton solution is a constant, $a_E(\tau) \equiv 1/\sqrt{2\lambda}$.

Similarly, the probability to pass from the contracting phase of the deformed de Sitter space to the expanding one would take the form $p'_{tot} = \sum_{n \geq 0} c_{2n} e^{-4nS_{eff}}$, for positive constants $c_{2n}$, (see fig. 5 and [16]).

Introducing the temperature at the transition

It is useful to express the deformation parameter $\delta_T^2$ (or $N$ and $\varepsilon$ in eq. (4.14)) in terms of more intuitive quantities, such as temperatures. First, observe that before and after the transition, the quantity $aT$ remains constant (see eq. (4.5)). Since these evolutions are related by a double analytic continuation, the constants must be equal. This allows us to express $aT$ in terms of either the temperatures $T_+$ at $t = 0_+$ or $T_-$ at $t = 0_-$:

$$T = \frac{T_\pm}{a} \frac{1}{\sqrt{2\lambda}} \sqrt{1 \pm \sqrt{1 - \delta_T^2}}. \quad (4.20)$$

Plugging either of these relations in the expression of $\rho_T$ in eq. (4.3) and comparing the result with eq. (4.6) imposes a consistency condition to be satisfied by $\delta_T^2$. Its solution involves a mean temperature $T_m$ satisfying $T_+ \leq T_m \leq T_-$,

$$\delta_T^2 = \frac{4}{(T_m^2/T^2_\pm + T^2_\pm/T_m^2)^2} \quad \text{where} \quad T_m = \left(\frac{15}{4\pi^6} \frac{\lambda}{(n_B + \frac{2}{3}n_F)}\right)^{1/4} \frac{1}{\sigma}. \quad (4.21)$$
Figure 5: The inflationary phase of the deformed de Sitter space for \( t \geq 0 \) can arise from any contracting phase or big bang phase for \( t \leq 0 \), after a multi-instantonic transition.

This implies that \( T_+ T_- = T_m^2 \) and that \( \delta_f^2 \) is invariant under the duality transformation

\[
\frac{T_+}{T_m} \leftrightarrow \frac{T_m}{T_+} \equiv \frac{T_-}{T_m}.
\] (4.22)

Actually, this operation exchanges the scale factor of the thermally deformed de Sitter evolution,

\[
a = \frac{1}{\sqrt{2\lambda}} \sqrt{1 + \frac{T_m^4 - T_+^4}{T_m^4 + T_+^4} \cosh(2\sqrt{\lambda} t)} , \quad t > 0 ,
\] (4.23)

with the big bang / big crunch one,

\[
a = \frac{1}{\sqrt{2\lambda}} \sqrt{1 + \frac{T_m^4 - T_-^4}{T_m^4 + T_-^4} \cosh(2\sqrt{\lambda} t)} , \quad t < 0 .
\] (4.24)

In some sense, the instantonic transition between them is “inversing” the temperature, since \( T(t) \geq T_- \) for \( t \leq 0 \), while \( T(t) \leq T_+ = T_m^2/T_- \) for \( t \geq 0 \). If one wishes, one can also rewrite
the constants $N$ and $\varepsilon$ of eq. (4.14) in terms of either $T_+$ or $T_-,$

$$\mathcal{N} = \frac{1}{\sqrt{\lambda}} \sqrt{\frac{1 - (T_+/T_m)^{\pm4}}{1 + (T_+/T_m)^{\pm4}}}, \quad \varepsilon = \frac{1}{(T_+/T_m)^{\pm4} - 1},$$

(4.25)

and note that the function $N^2$ appearing in eq. (4.7) satisfies $N^2(\tilde{t}) = 1/(1 - T^2(\tilde{t})/T^2_{\pm})$ for $ds^2_{\pm},$ respectively.

**Focussing on the $\lambda$-dependance of $\delta_T^2$**

We would like here to consider the dependance on $\lambda$ of $\delta_T^2.$ This is relevant, for instance, if one wants to minimize the Euclidean action with respect to $\lambda$ (keeping fixed the number of degrees of freedom and temperatures $T_{\pm}.$) Doing so was considered as a selection principle for the cosmological constant in [6, 7, 10, 11, 17]. The reason for this was based on the fact that this action (see eq. (5.57)) is controlling the probability transition between the two cosmological evolutions. However, there are two hypothesis for this argument to be considered. The first one is that the viable universes are the deformed de Sitter ones, i.e. that the big bang / big crunch evolution is too hot or too short in time. However, this condition implies that $\delta_T^2$ is small enough, which in turn can be translated into a condition on $\lambda$ itself (see eq. (4.21)). The second is that the probability to start the $\lambda$-dependent big bang evolution is almost constant as a function of $\lambda.$

Eq. (4.21) can be rewritten as

$$\delta_T^2 = \frac{\nu_+}{\lambda} \frac{4}{(1 + \nu_+ / \lambda)^2} = \frac{\nu_-}{\nu_- (1 + \lambda / \nu_-)^2},$$

(4.26)

where

$$\nu_\pm = \frac{4\pi^6}{15} \left( n_B + \frac{7}{8} n_F \right) T_{\pm}^4 \sigma^4,$$

(4.27)

and reproduces the result of [11]. Actually, $\lambda$ is the geometric mean of $\nu_+$ and $\nu_-,$

$$\left( \frac{T_+}{T_m} \right)^4 = \frac{\nu_+}{\lambda} = \frac{\lambda}{\nu_-} = \left( \frac{T_m}{T_-} \right)^4 \leq 1,$$

(4.28)

so that eq. (4.26) is relevant to derive an expansion of $\delta_T^2$ for small $\nu_+ / \lambda,$ whose leading term reproduces the result of [6, 7, 10]. Eq. (4.26) goes beyond the small $\nu_+ / \lambda$ approximation by taking consistently into account the back reaction term $\delta_T^2$ that appears in the r.h.s. of eq. (4.20).
A conceptual difference of our approach compared to that of [6, 7, 10] is that we insist on the difference between the two temperatures $T_\pm$, where $T_- \geq T_+$. Thus, the only stringy constraint to not reach the Hagedorn temperature during the cosmological evolution is $T_- \ll T_H$. In that case, the temperature $T(t)$ remains small, $T(t) \ll T_H$, for all $t > t_H$ (and $t < -t_H$ in the big crunch branch), where $t_H$ is the Hagedorn time.

The case $\delta_T^2 > 1$

We cannot use any more a function $N$ of the form (4.7) to map the Friedman equation of the present case to the pure de Sitter one. We thus actually solve eq. (2.5) in a more straightforward way. In the gauge $N \equiv 1$, one can immediately express a monotonically increasing time $t$ as a function of $a$:

$$t(a) = \int_0^a \frac{u \, du}{\sqrt{\lambda u^4 - u^2 + \delta_T^2/4\lambda}} + t_i \equiv \frac{1}{2\sqrt{\lambda}} \text{arctanh} \left( \frac{2\lambda a^2 - 1}{2\sqrt{\lambda} \sqrt{\lambda a^4 - a^2 + \delta_T^2/4\lambda}} \right), \quad a \geq 0,$$

(4.29)

where the argument of the square root is never vanishing and we have chosen the integration constant $t_i = -(4\lambda)^{-1/2}\text{artanh}\delta_T^{-1}$. Inverting the function $t(a)$, one finds

$$a(t) = \frac{1}{\sqrt{2\lambda}} \sqrt{1 + \delta_T^2 - 1 \sinh(2\sqrt{\lambda} t)}, \quad t \geq t_i \equiv -\frac{1}{2\sqrt{\lambda}} \text{arsinh} \left( \frac{1}{\sqrt{\delta_T^2 - 1}} \right).$$

(4.30)

The cosmological evolution described by this solution starts with a big bang at $t = t_i$, while for large positive time, it is inflationary. It thus looks similar to the case $\delta_T^2 < 1$, both at the beginning of time and for large $t$, when the first order transition has occurred. However, in the present case, the evolution from the small to very large scale factors is smooth, (see fig. 6). Actually, the solution (4.30) has an inflection point arising at $t = t_{\text{inf}}$, where $a(t_{\text{inf}}) = a_{\text{inf}}$,

$$t_{\text{inf}} = \frac{1}{2\sqrt{\lambda}} \text{arsinh} \left( \frac{\delta_T - 1}{\delta_T + 1} \right), \quad a_{\text{inf}} = \sqrt{\frac{\delta_T}{2\lambda}},$$

(4.31)

associated to a second order phase transition. As in the case $\delta_T^2 < 1$, the solution should be trusted as soon as the temperature is below $T_H$.

For completeness, we signal that another solution to eq. (2.5) is found under the time reversal $t \rightarrow -t$ and is thus monotonically decreasing. A de Sitter like universe is contracting since an infinitely past time and evolves in a smooth way around $t = t_{\text{inf}}$ toward a branch that ends with a big crunch.

The case $\delta_T^2 = 1$
Figure 6: Cosmological scenario of an expanding RW universe in presence of thermal and non trivial time-dependent moduli effects above some critical value. A phase of expansion starts from a big bang and evolves smoothly toward an inflationary one. This describes a second order transition.

Taking the limit $\delta_T^2 \to 1$ in the solutions found in the cases $\delta_T^2 < 1$ and $\delta_T^2 > 1$ reaches the same static solution,

$$a(t) \equiv a_0 \quad \text{where} \quad a_0 = \frac{1}{\sqrt{2\lambda}}.$$  \hfill (4.32)

It corresponds to a universe with an $S^3$-space of constant radius. Actually, the $\delta_T^2 \to 1_-$ limit amounts to send the big bang to an infinitely past time (see eqs. (4.14), (4.17)), while keeping the instant the transition occurs at $t = 0$. Similarly, the $\delta_T^2 \to 1_+$ limit sends $t_i \to -\infty$ (see eq. (4.30)), while keeping $t_{\inf}$ finite. In both cases, the solutions are thus “stretched” and converge toward a constant. In the two cases, it is however possible to redefine the big bang time at a conventional $t = 0$ by a $\delta_T^2$-dependent shift in the definition of time. Taking only then the limit $\delta_T^2 \to 1$ reaches solutions where the dynamical part of the solutions are not sent to infinity any more.

To find these new solutions, one can write the Friedman equation (2.5) in the form

$$(a\dot{a})^2 = \lambda (a^2 - a_0)^2,$$  \hfill (4.33)
and derive two monotonically increasing solutions for the time as a function of $a$,

$$t(a) = \frac{1}{\sqrt{\lambda}} \int_{0}^{a} \frac{u \, du}{a_0^2 - u^2}, \quad 0 \leq a < a_0,$$

(4.34)

and

$$t(a) = \frac{1}{\sqrt{\lambda}} \int_{a_0}^{a} \frac{u \, du}{u^2 - a_0^2}, \quad a > a_0.$$

(4.35)

Integrating explicitly these expressions and inverting the functions $t(a)$ reaches then the two cosmological evolutions:

$$a(t) = a_0 \sqrt{1 - e^{-2\sqrt{\lambda} t}}, \quad t \geq 0,$$

(4.36)

and

$$a(t) = a_0 \sqrt{1 + e^{2\sqrt{\lambda} t}}, \quad t \in \mathbb{R}.$$

(4.37)

The first one starts with a big bang arising at $t = 0$, converges quickly toward the static solution, (and should be trusted as soon as $T(t) < T_H$). The second is a deviation from the static solution toward the inflationary phase. Two other monotonically decreasing solutions are found under the time reversal $t \rightarrow -t$ in eqs. (4.36) and (4.37). The four solutions are drawn on fig. 7.

## 5 De Sitter deformation by moduli fields

In this section, we generalize the thermally deformed de Sitter solution by including the effect of the non trivial time dependence of some moduli fields. To be specific, our choice of matter action to be added to (2.3) is

$$S_m = \frac{M_p^2}{16\pi} \int d^4x \sqrt{-g} \left( -g^{\mu\nu} \partial_\mu \chi_i \partial_\nu \chi_i \right),$$

(5.1)

where the $\chi_i$’s are the moduli fields with neither potential, nor interactions with the thermal system considered before. Assuming the RW metric (2.1) and the presence of the thermal corrections discussed before, the modified MSS action becomes

$$S_{\text{eff}} = \frac{1}{2} \int_{t_i}^{t_f} dt \, N a^3 \left( -\frac{1}{N^2} \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{a^2} - \lambda - \frac{\delta_T^2}{4\lambda a^4} + \frac{1}{6N^2} \dot{\chi}_i^2 \right) + \frac{1}{2} \left[ \frac{a^2 \dot{a}}{N} \right]_{t_i}^{t_f},$$

(5.2)

where $\delta_T^2$ is a new positive constant. The energy-momentum tensor can be expressed in terms of

$$\rho = \frac{1}{12\pi^2 \sigma^4 N^2} \left( 3\delta_T^2 N^2 \frac{1}{a^4} + 1 \right),$$

(5.3)
Figure 7: Cosmological scenario of an expanding RW universe in presence of thermal and non trivial time-dependent moduli effects at a critical value. A phase of expansion starts from a big bang and converges quickly toward a static universe. After a very long time, the universe can either die in a big crunch or enter into an inflationary phase.

\[ P = \frac{1}{12\pi^2\sigma^2N^2} \left( \frac{\delta^2_P}{4\lambda} \frac{N^2}{a^4} + \frac{1}{2} \chi_i^2 \right). \]  
(5.4)

The equation of motion of \( \chi_i \) is trivially solved,

\[ \dot{\chi}_i = \frac{2\sqrt{2}\delta\chi_i}{3\lambda} \frac{N}{a^3}, \]  
(5.5)

where \( 2\sqrt{2}\delta\chi_i/3\lambda \) is an integration constant, whose form is chosen for later convenience.

Defining

\[ \delta^2 = \sum_i \delta^2_{\chi_i}, \]  
(5.6)

the moduli contributions of \( \rho \) and \( P \) become

\[ \rho_\chi = P_\chi = \frac{\delta^2}{27\pi^2\sigma^4\lambda^2} \frac{1}{a^6}. \]  
(5.7)

To solve the Friedman equation (2.5) in presence of the extra \( 1/a^6 \)-term induced by the motion of the \( \chi_i \) fields, we follow the method implemented in the previous sections. We look for a real constant \( \kappa' \) in the definition of \( N(\tilde{t}) \),

\[ N^2(\tilde{t}) = \frac{1}{1 + \lambda^{-1}\kappa'/a^2(\tilde{t})}, \]  
(5.8)
so that the extra $1/a^6$ contribution is cancelled. This can be achieved by choosing $\kappa'$ such that
\[ P(\kappa') = \kappa'^3 + \kappa'^2 + \frac{\delta_T^2}{4} \kappa' - \frac{4}{27} \delta^2 \chi = 0. \quad (5.9) \]
This polynomial always has a real positive root (see the appendix). Using it, eq. (2.5) is taking the form
\[ (\frac{\dot{a}}{a})^2 + \frac{1 + \kappa'}{a^2} = \lambda + \frac{\kappa'^2 + \kappa' + \delta_T^2 / 4}{\lambda a^4}. \quad (5.10) \]
The above equation has the same form with that of the $1/a^4$-thermally deformed de Sitter one discussed previously, up to a shift in both coefficients of the $1/a^2$-curvature term and the $1/a^4$-thermal term. Using a simple rescaling of the radius $a$ (by a factor $\sqrt{1 + \kappa'}$), one can immediately deduce the solutions from the previous section in terms of the time variable $\tilde{t}$. This implies in particular that the analogue of the order parameter $\delta_T^2$ of the pure thermal case is now the positive quantity
\[ \Delta = \frac{4 \kappa'^2 + 4 \kappa' + \delta_T^2}{(1 + \kappa')^2} = \frac{16 \delta^2 \chi}{27 \kappa' (1 + \kappa')^2}, \quad (5.11) \]
where these two definitions of $\Delta$ are equivalent as a consequence of eq. (5.9). In the appendix, we shown that the condition $\Delta < 1$ is equivalent to the statement that the parameters $\delta_T^2$ and $\delta^2 \chi$ satisfy the inequalities,
\[ \delta_T^2 < 1 \quad \text{and} \quad \delta^2 \chi < h(\delta_T^2), \quad (5.12) \]
where
\[ h(\delta_T^2) = \frac{1}{2} \left( \sqrt{1 - \frac{3}{4} \delta_T^2} + 1 \right)^2 \left( \sqrt{1 - \frac{3}{4} \delta_T^2} - \frac{1}{2} \right). \quad (5.13) \]
This defines a domain in the $(\delta^2 \chi, \delta_T^2)$-plane shown in fig. 8. At the origin of this domain, one has $\Delta = 0$, while $\Delta = 1$ on its boundary. Outside, this region, one has $\Delta > 1$.

**The case $\Delta < 1$**

Eq. (5.10) admits the cosmological evolution
\[ a(\tilde{t}) = \mathcal{N}' \sqrt{\varepsilon'} + \cosh(\sqrt{\lambda} \tilde{t}), \quad (5.14) \]
where
\[ \mathcal{N}' = \sqrt{\frac{1 + \kappa'}{\lambda} (1 - \Delta)^{1/4}} \quad \text{and} \quad \varepsilon' = \frac{1}{2} \left( \frac{1}{\sqrt{1 - \Delta}} - 1 \right). \quad (5.15) \]
Figure 8: Phase diagram in the $(\delta^2_\chi, \delta^2_c)$-plane. In the domain $\delta^2_\chi \leq h(\delta^2_c)$ defined in eq. (5.12), a first order phase transition between two cosmological branches can occur. Outside the domain, there is a solution describing a second order phase transition. On the critical curve $\delta^2_\chi = h(\delta^2_c)$, there are cosmological evolutions asymptotic to a static solution. Furthermore, one can introduce the temperatures $T_+ < T_-$ in the interior of the domain to replace the two parameters $\delta^2_\chi$ and $\delta^2_c$. On the boundary curve, these temperatures converge and their common value can be considered as a function $T_{cT}(\delta^2_c)$ or $T_{c\chi}(\delta^2_\chi)$.

To proceed, we switch to the more intuitive $N \equiv 1$ gauge. This requests a redefinition of the time variable dictated by eq. (3.9):

$$t = \int_0^\bar{t} dv \sqrt{\frac{\cosh^2(\sqrt{\lambda}v) + \varepsilon'}{\cosh^2(\sqrt{\lambda}v) + \varepsilon' + \varepsilon''}} \quad \text{where} \quad \varepsilon'' = \frac{\kappa'}{(1 + \kappa')\sqrt{1 - \Delta}}. \quad (5.16)$$

An explicit form of $t$ as a function of $\bar{t}$ can actually be determined, however we find more convenient to work with the above integral representation. The scale factor $a(\bar{t}(t))$ takes a simple form in terms of the function $\bar{t}(t)$ found by inverting the definition (5.16),

$$a(\bar{t}(t)) = N' \sqrt{\varepsilon' + \cosh^2(\sqrt{\lambda} \bar{t}(t))}. \quad (5.17)$$

Some comments are in order:

- The slope $d\bar{t}/dt$ is positive everywhere, and for large $\bar{t}$ one has $t \sim \bar{t}$. This means that the moduli deformation of the thermal de Sitter solution appears as a mild redefinition of time and normalization factor.

- An Euclidean solution is defined via the analytic continuation, $\bar{t} = -i\bar{\tau}$ i.e. $t = -i\tau$,

$$a_E(\bar{\tau}(\tau)) = N' \sqrt{\varepsilon' + \cos^2(\sqrt{\lambda} \bar{\tau}(\tau))} \quad \text{with} \quad \tau = -\int_\bar{\tau}^\theta dv \sqrt{\frac{\cos^2(\sqrt{\lambda}v) + \varepsilon'}{\cos^2(\sqrt{\lambda}v) + \varepsilon' + \varepsilon''}}. \quad (5.18)$$
The range of Euclidean time is
\[-\pi/2 \leq \sqrt{\lambda} \tilde{\tau} \leq 0 \quad \text{i.e.} \quad \tau_i \leq \tau \leq 0, \]
(5.19)
where \( \tau_i \) is given by
\[
\tau_i = - \int_{\tau_i}^{0} dv \sqrt{\frac{\cos^2(\sqrt{\lambda}v) + \varepsilon'}{\cos^2(\sqrt{\lambda}v) + \varepsilon' + \varepsilon''}}.
(5.20)
The boundary terms in eq. (3.3) vanish.

- The Euclidean solution admits, (as in the pure thermal case), two analytic continuations, one for each boundary. The one associated to \( \tau_i, \sqrt{\lambda} \tilde{\tau} = -\pi/2 + i\sqrt{\lambda}t \) i.e. \( \sqrt{\lambda} \tilde{\tau} = \sqrt{\lambda} \tau_i + i\sqrt{\lambda}t \), gives rise to a sinh-type cosmological solution,
\[
a(t(t)) = N' \sqrt{\varepsilon' - \sinh^2(\sqrt{\lambda} \tilde{t}(t))} \quad \text{with} \quad t = - \int_{\tilde{t}}^{0} dv \sqrt{\frac{\varepsilon' - \sinh^2(\sqrt{\lambda}v)}{\varepsilon' + \varepsilon'' - \sinh^2(\sqrt{\lambda}v)}}.
(5.21)
The range of time is
\[-\frac{1}{\sqrt{\lambda}} \arcsinh \sqrt{\varepsilon'} \leq \tilde{t} \leq 0 \quad \text{i.e.} \quad t_i = - \int_{\arcsinh \sqrt{\varepsilon'}}^{0} dv \sqrt{\frac{\varepsilon' - \sinh^2(\sqrt{\lambda}v)}{\varepsilon' + \varepsilon'' - \sinh^2(\sqrt{\lambda}v)}} \leq t \leq 0. \]
(5.22)
The full scenario is thus similar to the one presented in the previous section. There is a big bang at \( t = t_i \) and the universe expands till \( t = 0 \), before either contracting or experimenting a first order transition to an inflationary phase. At the transition, the scale factor is jumping,
\[
a_+ = \sqrt{(1 + \kappa') \frac{1 + \sqrt{1 - \Delta}}{2\lambda}} \quad \longrightarrow \quad a_- = \sqrt{(1 + \kappa') \frac{1 + \sqrt{1 - \Delta}}{2\lambda}}.
(5.23)

Thus, the present moduli deformation is generalizing the pure thermal one discussed previously, (see fig. 3).

**The case \( \Delta > 1 \)**

Eq. (5.10) admits the \( \tilde{t} \)-dependent solution
\[
a(\tilde{t}) = \sqrt{\frac{1 + \kappa'}{2\lambda}} \sqrt{1 + \sqrt{\Delta - 1} \sinh(2\sqrt{\lambda} \tilde{t})}, \quad \tilde{t} \geq \tilde{t}_i \equiv - \frac{1}{2\sqrt{\lambda}} \arcsinh \frac{1}{\sqrt{\Delta - 1}},
(5.24)
we want to express in terms of a time variable associated to the natural \( N \equiv 1 \) gauge. Integrating eq. (3.9), one can define \( t \) as a function of \( \tilde{t} \) by
\[
t = \int_{0}^{\tilde{t}} dv \sqrt{\frac{\sqrt{\Delta - 1} \sinh(2\sqrt{\lambda}v) + 1}{\sqrt{\Delta - 1} \sinh(\sqrt{2}\lambda v) + 1 + \frac{2\kappa'}{1 + \kappa'}}},
(5.25)
where

\[ t \geq t_i \equiv -\int_{t_i}^{0} dv \sqrt{\frac{\sqrt{\Delta - 1} \sinh(2\sqrt{\lambda}v) + 1}{\sqrt{\Delta - 1} \sinh(\sqrt{2\lambda}v) + 1 + \frac{2\kappa'}{1+\kappa}}}, \]

(5.26)

and rewrite

\[ a(\tilde{t}(t)) = \sqrt{\frac{1 + \kappa}{2\lambda}} \sqrt{1 + \frac{\sqrt{\Delta - 1} \sinh(2\sqrt{\lambda}\tilde{t}(t))}{\sqrt{\Delta - 1} \sinh(\sqrt{2\lambda}\tilde{t}(t))}}. \]

(5.27)

As in the previous case, the slope \( d\tilde{t}/dt \) is always positive and \( t \sim \tilde{t} \) for large \( \tilde{t} \). The solution is thus a mild deformation of the pure thermal case. It starts with a big bang at \( t = t_i \) and is monotonically increasing. For large time, the behavior is inflationary. The evolution can again be interpreted as a second order phase transition since there is a unique inflection point, (see fig. 6). To show this, one can express \( dt/da \) as a function of \( a \) from the Friedman equation (in the \( N \equiv 1 \) gauge) and take a derivative. One finds that the inflection point arises at the radius \( a_{\text{inf}} \), where

\[ \lambda a_{\text{inf}}^2 > 0 \quad \text{satisfies} \quad x^3 - \frac{\delta^2}{4} x - \frac{8}{27} \delta^2 = 0. \]

(5.28)

Finally, a monotonically decreasing cosmological solution is found under the time reversal \( t \rightarrow -t \).

**The case \( \Delta = 1 \)**

The static solution obtained in the limit \( \Delta \rightarrow 1 \) in the previous cases is

\[ a(t) \equiv a_0 \quad \text{where} \quad a_0 = \sqrt{\frac{1 + \kappa'}{2\lambda}}. \]

(5.29)

Beside this constant radius universe, we again have the two \( \tilde{t} \)-dependent evolutions

\[ a(\tilde{t}) = a_0 \sqrt{1 \mp e^{\mp 2\sqrt{\lambda}\tilde{t}}}. \]

(5.30)

To consider them in the \( N \equiv 1 \) gauge, we integrate eq. (3.9) in each case and find

\[ a(\tilde{t}(t)) = a_0 \sqrt{1 - e^{-2\sqrt{\lambda}\tilde{t}(t)}} \quad \text{where} \quad t = \int_0^{\tilde{t}} dv \sqrt{\frac{1 - e^{-2\sqrt{\lambda}v}}{1 + \frac{2\kappa'}{1+\kappa} - e^{-2\sqrt{\lambda}v}}}, \]

(5.31)

for \( \tilde{t} \geq 0 \), i.e. \( t \geq 0 \), and

\[ a(\tilde{t}(t)) = a_0 \sqrt{1 + e^{2\sqrt{\lambda}\tilde{t}(t)}} \quad \text{where} \quad t = \int_0^{\tilde{t}} dv \sqrt{\frac{1 + e^{2\sqrt{\lambda}v}}{1 + \frac{2\kappa'}{1+\kappa} + e^{2\sqrt{\lambda}v}}}, \]

(5.32)
for arbitrary \( \tilde{t} \) and \( t \). As in the cases \( \Delta \neq 1 \), one has \( d\tilde{t}/dt > 0 \) and this redefinition of time does not change the qualitative behavior of the solutions, (see fig. 7). They are monotonically increasing as in the pure thermal case, while two other solutions obtained under \( t \to -t \) are decreasing.

5.1 Other parameterizations of the solutions when \( \Delta \leq 1 \)

The temperatures at the 1st order transition

We would like to replace the parameters \( \delta_2^2 \) and \( \delta_T^2 \) that appear in the scale factor solution by the temperatures \( T_\pm \) at \( t = 0_\pm \). From the definition (4.6) (with \( \delta_T^2 \) replacing \( \delta_T^2 \)) and the relation (4.3), using the fact that \( aT \equiv a_+T_+ = a_-T_- \), one has

\[
T_\pm(\delta_\chi^2, \delta_T^2) = T_m \frac{\sqrt{\delta_T^2}}{\sqrt{(1 + \kappa') (1 \pm \sqrt{1-\Delta})}},
\]

where the temperature \( T_m \) has been defined in eq. (4.21). On the critical curve that is delimiting the domain (5.12), one has \( \delta_\chi^2 = h(\delta_T^2) \). This implies \( \Delta = 1 \) and thus \( T_+ = T_- \). It is shown in the appendix that in this case, \( \kappa' \) equals \( \kappa'_{\Delta_+} \) defined in eq. (A.4). One can then deduce the critical temperature \( T_{cT} \) on the boundary of the domain (5.12) by

\[
T_{cT}(\delta_T^2) = T_m \sqrt{\frac{2 \delta_T^2}{\sqrt{\delta_T^2}}} \sqrt{1 + \sqrt{1 - \frac{3}{4} \delta_T^2}},
\]

plotted on fig. 9. The temperature on the critical curve can also be considered as a function of \( \delta_\chi^2 \). This can be done by expressing \( \delta_T^2 = h^{-1}(\delta_\chi^2) \),

\[
\delta_T^2 = \frac{4}{3} - \frac{4}{3} \left( (A + B)^{1/3} + (A - B)^{1/3} - \frac{1}{2} \right)^2,
\]

where

\[
A = \frac{1}{8} + \delta_\chi^2, \quad B = \sqrt{\delta_\chi^2 (\delta_\chi^2 + \frac{1}{4})}, \quad \text{for} \quad \delta_\chi^2 \leq 1.
\]

Defining \( T_{c\chi}(\delta_\chi^2) \equiv T_\pm(\delta_\chi^{-1}(\delta_T^2), \delta_\chi^2) \) (see fig. 8), one obtains

\[
T_{c\chi}(\delta_\chi^2) = T_m 3^{1/4} \left( 1 - \frac{(A + B)^{1/3} + (A - B)^{1/3} - \frac{1}{2}}{(A + B)^{1/3} + (A - B)^{1/3} + \frac{1}{2}} \right)^{1/4}.
\]
Figure 9: Critical temperature as a function of $\delta_T^2$. It is defined on the curve $\delta_X^2 = h(\delta_T^2)$.

Since $T_+ a_+ = T_- a_-$, one has

$$\Delta = \frac{4}{(T_+/T_- + T_-/T_+)^2},$$

a relation that generalizes eq. (4.21) and the duality $T_+ \leftrightarrow T_-$. Actually, for eqs. (5.38) and (4.21) to have precisely the same form, one can introduce a mean temperature $T_m' = T_+ T_-,$

$$T_m' = T_m \left( \frac{\delta_T^2}{4\kappa^2 + 4\kappa' + \delta_T^2} \right)^{1/4},$$

where $T_m$ has been defined in eq. (4.21). However, this definition is less useful than its counterpart in the pure thermal case. This is due to the fact that $T_m'$ depends on the point $(\delta_T^2, \delta_X^2)$ of the parameter space, while in the pure thermal case, $\delta_X^2 = \kappa' = 0$ so that the mean temperature $T_m$ is independent of the remaining parameter $\delta_T^2$.

We may express the evolution of the scale factor $a(\tilde{t}(t))$ in each branch with $T_\pm$,

$$a(\tilde{t}(t)) = \frac{1}{\sqrt{2\lambda(1-A)}} \sqrt{1 + \frac{T_\pm^2 - T_\mp^2}{T_\pm^2 + T_\mp^2} \cosh(2\sqrt{\lambda} \tilde{t}(t))}, \quad t > 0$$

and

$$a(\tilde{t}(t)) = \frac{1}{\sqrt{2\lambda(1-A)}} \sqrt{1 + \frac{T_\pm^2 - T_\mp^2}{T_\pm^4 + T_\mp^4} \cosh(2\sqrt{\lambda} \tilde{t}(t))}, \quad t < 0,$$
where $\kappa'$ has been expressed in terms of $A$ that involves $T_\pm$ and $T_m$

$$1 + \kappa' = \frac{1}{1 - A} \quad \text{where} \quad A = \frac{1 - T_+^2 T_-^2 / T_m^4}{(T_+ / T_+ + T_- / T_+)^2}. \quad (5.42)$$

To obtain the above expression for $A$, we have used eq. (5.11) to express $\delta_\gamma^2$ in terms of $\Delta$ and $\kappa'$. Then we combine the result with eqs. (5.33) and (5.38).

Under the duality $T_+ \leftrightarrow T_-$, the two cosmological solutions $a(\tilde{t}(t))$, for $t > 0$ and $t < 0$ are interchanged:

$$(T_+ \leftrightarrow T_-) \quad \iff \quad (a(\tilde{t}(t)) \text{ for } t > 0 \leftrightarrow a(\tilde{t}(t)) \text{ for } t < 0). \quad (5.43)$$

For completeness we display the constants $N', \varepsilon'$ and $\varepsilon''$ appearing in eqs. (5.17), (5.16) and (5.21), in terms of $T_\pm$,

$$N' = \frac{1}{\sqrt{\lambda(1 - A)}} \sqrt{\frac{1 - (T_+ / T_-)^2}{1 + (T_+ / T_-)^2}}, \quad \varepsilon' = \frac{1}{(T_- / T_+)^2 - 1}, \quad \varepsilon'' = \frac{1 - T_+^2 T_-^2 / T_m^4}{(T_- / T_+)^2 - (T_+ / T_-)^2}. \quad (5.44)$$

**Parameterization with $\delta_T^2$ and $\delta_\chi^2$**

Let us determine explicit expressions of the scale factor in terms of $\delta_T^2$ and $\delta_\chi^2$, when $\Delta \leq 1$. This will be useful in the next section, when we study the probability transition. Since the parameters $\delta_T^2$ and $\delta_\chi^2$ are chosen in the domain (5.12), the polynomial $\mathcal{P}$ in eq. (5.45) has three real roots, (see appendix). Noting that $\mathcal{P}(-\lambda a_\pm) = 0$, as can be checked with eq. (5.11), the polynomial $\mathcal{P}$ is actually,

$$\mathcal{P}(x) = (x - \kappa')(x + \lambda a_-^2)(x + \lambda a_+^2). \quad (5.45)$$

Using Cardan’s formulas, one finds

$$\kappa' = \frac{1}{3} \left( \sqrt[3]{4 - 3 \delta_T^2} \cos \left( \frac{\theta}{3} \right) - 1 \right), \quad (5.46)$$

$$a_\pm^2 = \frac{1}{3\lambda} \left( 1 - \sqrt[3]{4 - 3 \delta_T^2} \cos \left( \frac{\theta \pm 2\pi}{3} \right) \right), \quad (5.47)$$

where

$$\theta = \arccos \left( \frac{16 \delta_\chi^2 + 9 \delta_T^2 - 8}{(4 - 3 \delta_T^2)^{3/2}} \right). \quad (5.48)$$
We note that all values of $\theta$ in $[0, \pi]$ can be reached. At the origin $(\delta^T_\chi, \delta^2_\chi) = (0, 0)$ of the domain (5.12), one has $\theta = \pi$, while $\theta$ vanishes on the critical curve $\delta^2_\chi = h(\delta^T_\chi)$. The constant $\mathcal{N}'$ of eqs. (5.17) and (5.21) can be determined by expressing $\sqrt{1 - \Delta}$ from the ratio $a^2_+ / a^2_-$,

$$\mathcal{N}' = \left( \frac{1 - 2\delta^2_\chi}{3} \right)^{1/4} \sqrt{\frac{2}{\lambda} \sin \left( \frac{\theta}{3} \right)}.$$  \hspace{1cm} (5.49)

The relations $a^2_+ = \mathcal{N}'^2 \xi'$ and $\kappa' = \lambda \mathcal{N}'^2 \xi''$ give then

$$\xi' = \frac{1 - \sqrt{4 - 3\delta^2_\chi} \cos \left( \frac{\theta - 2\pi}{3} \right)}{\sqrt{3} \sqrt{4 - 3\delta^2_\chi} \sin \left( \frac{\theta}{3} \right)}, \quad \xi'' = \frac{\sqrt{4 - 3\delta^2_\chi} \cos \left( \frac{\theta}{3} \right) - 1}{\sqrt{3} \sqrt{4 - 3\delta^2_\chi} \sin \left( \frac{\theta}{3} \right)}.$$  \hspace{1cm} (5.50)

For completeness, the temperatures at the transition are also found to be

$$T_\pm = T_m \frac{\sqrt{\frac{2}{3} \delta^T_\chi}}{\sqrt{1 - \sqrt{4 - 3\delta^2_\chi} \cos \left( \frac{\theta \pm 2\pi}{3} \right)}}.$$  \hspace{1cm} (5.51)

### 5.2 First order transition probability

When $\Delta \leq 1$, the transition probability at $t = 0$ between the big bang branch and the deformed de Sitter cosmology can be estimated by computing the Euclidean action. Changing the integration variable $d\tau = da_E / a_E$ in the Euclidean analog of the action (5.2) gives, on shell,

$$S_{E,\text{eff}} = -\frac{1}{\lambda} \int_{a^-}^{a^+} \frac{da_E}{a_E} \left( \sqrt{\mathcal{P}(-\lambda a^2_E)} + \frac{4}{27} \frac{\lambda^2 \delta^2_\chi}{\sqrt{\mathcal{P}(-\lambda a^2_E)}} \right),$$  \hspace{1cm} (5.52)

where $\mathcal{P}$ is given in eq. (5.45) and enters in the Euclidean equation of motion

$$\left( \frac{\dot{a}_E}{a_E} \right)^2 = \frac{N^2_E \mathcal{P}(-\lambda a^2_E)}{\lambda^2}.$$  \hspace{1cm} (5.53)

we have used. Clearly, $S_{E,\text{eff}}$ is negative. To explicitly evaluate it, we prefer considering its integral form on $\tilde{\tau}$ and use eq. (2.5) in Euclidean time to be on shell,

$$S_{E,\text{eff}} = -\int_{\tilde{\tau}_-}^{\tilde{\tau}_+} d\tilde{\tau} \ N_E a^3_E \left( \frac{1}{a^2_E} - \lambda - \frac{\delta^2_T}{4 \lambda a^4_E} \right).$$  \hspace{1cm} (5.54)

In this expression, $a_E(\tilde{\tau})$ and $N_E(\tilde{\tau}) \equiv d\tau / d\tilde{\tau}$ are referring to eq. (5.18). The result can be expressed in terms of complete elliptic integrals of the first kind, $K(k)$, and second kind,
\[ E(k), \]
\[ S_{E,\text{eff}} = -\frac{1}{3^{5/4}} \frac{1}{\lambda} (4 - 3\delta_T^2)^{1/4} \sqrt{\sin \left(\frac{\theta + \pi}{3}\right)} \left( E(k) - \frac{\sqrt{4 - 3\delta_T^2 \cos \left(\frac{\theta}{3}\right)} - 1 + \frac{3}{2} \delta_T^2}{\sqrt{3} \sqrt{4 - 3\delta_T^2} \sin \left(\frac{\theta + \pi}{3}\right)} K(k) \right), \]

where
\[ k = \sqrt{\frac{\sin \left(\frac{\theta}{3}\right)}{\sin \left(\frac{\theta + \pi}{3}\right)}}. \]

To have a better intuition of the behavior of this action as a function of the parameters \( \delta_T^2 \) and \( \delta_\chi^2 \), we concentrate on the boundary of the domain (5.12):

- On the side \( \delta_\chi^2 = 0 \) corresponding to the pure thermal case, the action becomes
\[ S_{E,\text{eff}} = -\frac{1}{3\lambda} \sqrt{\frac{1 + \sqrt{1 - \delta_T^2}}{2}} \left( E(k) - \left(1 - \sqrt{1 - \delta_T^2}\right) K(k) \right), \]

where
\[ k = \frac{2(1 - \delta_T^2)^{1/4}}{\sqrt{1 + \sqrt{1 - \delta_T^2}}}. \]

To derive these expressions, one can use the fact that \( \kappa' \) in eq. (5.46) vanishes and is thus giving rise to an identity satisfied by \( \theta \). Eq. (5.57) reproduces the result of [10]. At the origin \( (\delta_T^2, \delta_\chi^2) = (0, 0) \), one has \( k = 1 \) and \( E(1) = 1 \), so that the result \(-1/3\lambda\) of the pure de Sitter case is recovered. At \( (\delta_T^2, \delta_\chi^2) = (1, 0) \), \( k = 0 \) and \( E(0) = K(0) = \pi/2 \), so that the action vanishes. This is consistent with eq. (5.52) since \( a_+ = a_- \) and \( \delta_\chi^2 \) vanishes.

- On the critical curve \( \delta_\chi^2 = h(\delta_T^2) \), the action is taking the form
\[ S_{E,\text{eff}} = -\frac{\pi}{9\lambda} \left(1 + \sqrt{1 - \frac{3}{4} \delta_T^2}\right) \left(\sqrt{1 - \frac{3}{4} \delta_T^2} - 1/2\right) \frac{(1 - \frac{3}{4} \delta_T^2)^{1/4}}{(1 - \frac{3}{4} \delta_T^2)^{1/4}}. \]

If it is explicitly negative for \( \delta_T^2 \leq 1 \), it vanishes for \( \delta_T^2 = 1 \) only. This shows that even if \( a_+ \equiv a_- \) on the critical curve, the term proportional to \( \delta_\chi^2 \) in the integrand of eq. (5.52) contributes, due to the fact that the denominator vanishes. The probability \( p \) refers in this case to the transition (after an infinite time) from the big bang branch (5.31) to the inflationary one (5.32). The quantity \( (1 - p) \) refers to the transition (after an infinite time) form the big bang branch (5.31) to its time reversal i.e. describing a big crunch. This is the case since considering the solutions (5.17) and (5.21) of the case \( \Delta < 1 \), and performing

\[ \text{Up to minor misprint errors in [10].} \]
shifts on the origins of times, give rise to (5.31) and (5.32) in the limit $\Delta \to 1$.

- The side $\delta_T^2 = 0$ is corresponding to the pure moduli deformation of the de Sitter solution. From eq. (5.59), one finds that at the corner $(\delta_T^2, \delta_\chi^2) = (0, 1)$ of the domain (5.12), the action is taking the value $-\pi/9\lambda$. Since this result is close to the $-1/3\lambda$ at the origin of the $(\delta_T^2, \delta_\chi^2)$-plane, the dependence of the action on $\delta_\chi^2$ seems to be mild along this axis.

This remark happens to remain true for arbitrary fixed $\delta_T^2$. To understand this, one can note in eq. (5.54) that the $\delta_\chi^2/a^6$ contribution of $\dot{\chi}_i^2$ in the action $S_{E_{\text{eff}}}$ has disappeared on shell. Thus, the only dependence on $\delta_\chi^2$ occurs through the mild deformation $\varepsilon''$ appearing in $N_E$. This is confirmed on the 3-dimensional plot of fig. 10, where $S_{E_{\text{eff}}}$ appears almost constant when $\delta_\chi^2$ varies from 0 to $h(\delta_T^2)$, at fixed $\delta_T^2$.

![Figure 10: Euclidean action $S_{E_{\text{eff}}}$ as a function of the radiation parameter $\delta_T^2$ and the moduli deformation $\delta_\chi^2$. The variable $(\delta_T^2, \delta_\chi^2)$ spans the domain (5.12). The dependence on $\delta_\chi^2$ is weak, while the absolute value of $S_{E_{\text{eff}}}$ decreases when $\delta_T^2$ is switched on. (We have chosen $\lambda = 1$ on this plot.)](image)

From fig. 10, one can also see that the transition amplitude $p \propto e^{-2S_{E_{\text{eff}}}}$ is larger and larger when $\delta_T^2$ decreases from $h^{-1}(\delta_T^2)$ to 0, at fixed $\delta_\chi^2$. However, the magnitude of $p$ highly depends on $\lambda$. For instance, $p$ may not be large even for small $\delta_T^2$, due to the $1/\lambda$-dependance of $S_{E_{\text{eff}}}$, for large $\lambda$. 

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At fixed $\lambda$ and $\delta_\chi^2$, switching on $\delta_T^2$ increases $\varepsilon'$ (see eqs. (5.50) and (5.48)), and thus $2|t_i|$ (see eq. (5.22)), the range of time along the big bang / big crunch evolution. From eq. (5.51), larger and larger $\delta_\chi^2$ imply also lower and lower temperature $T_-$. As a result, increasing $\delta_\chi^2$ makes this cosmological branch more and more viable.

Similarly, at fixed $\lambda$ and $\delta_\chi^2 \neq 0$, switching on $\delta_T^2$ has the effect to increase $2|t_i|$. However, $T_-$ varies from 0 to $T_c\chi$, with the risk for the universe to be too hot. The case $\delta_\chi^2 = 0$ has a different behavior. When $\delta_T^2$ grows up from 0 to 1, $2|t_i|$ increases as well but the temperature $T_-$ varies from $+\infty$ to $T_m$.\(^8\) Along this axis, $\delta_T^2$ is thus making the big bang / big crunch branch more viable.

Finally, when we reach the limit case corresponding to $(\delta_T^2, \delta_\chi^2)$ sitting on the critical curve, after the universe is born with a big bang, it converges quickly toward a static state for a very long time, i.e. an $S^3$ of constant radius

$$a \equiv a_\pm = a_0 = \sqrt{\frac{1 + \sqrt{1 - \frac{3}{4}\delta_T^2}}{3\lambda}},$$

(5.60)

where $a_0$ has already been defined in eq. (5.29). However, it finally enters in an inflationary phase or dies in a big crunch. Also, one can note that in the case $\Delta > 1$, the big bang evolution is longer and longer when $\Delta \rightarrow 1_+\), since $t_i \rightarrow -\infty$, (see eqs. (5.24) and (5.26)), but the final state on the universe is always inflationary.

6 Conclusions

In this work, we analyze some aspects concerning inflationary solutions and possible transitions between different cosmological behaviors. Our starting point is a bare de Sitter background deformed by the presence of a thermal bath and the motion of moduli fields. The thermal bath summarizes the effects of light degrees of freedom in a consistent way and shows up as a $C_R/a^4$ correction in the MSS-action. The moduli contribution gives a $C_M/a^6$ correction to the energy density.

\(^8\) Note that $T_- \rightarrow +\infty$ when $\delta_\chi^2 \equiv 0$ and $\delta_T^2 \rightarrow 0$, while $T_- \equiv 0$ when $\delta_T^2 \equiv 0$ and $\delta_\chi^2 \neq 0$. This ambiguity in the definition of $T_-$ in the pure de Sitter universe is due to the fact that taking $\delta_\chi^2 = 0$ (i.e. $\kappa' = 0$) or $\delta_T^2 = 0$ are operations that do not commute (see eq. (5.39)).
motion parameter space \((\delta_T^2, \delta_\chi^2)\). We solve the gravitational equations in all cases and find that a first order phase transition can occur inside this domain, while a second order one arises outside.

In the second case, a smooth transition always occurs between the big bang cosmological evolution (as soon as \(T(t) < T_H\)) and the inflationary behavior. In the first case, we calculate the transition probability between the big bang cosmological evolution (where \(T_- \leq T(t) < T_H\)) and the inflationary branch (where \(T(t) \leq T_+\)). The two solutions are connected into each other via a gravitational instanton allowing a double analytic continuation. We find, the existence of a temperature duality \(T_+ \leftrightarrow T_-\) that interchanges the two solutions. Inside the domain of the parameter space, the origin \((\delta_T^2, \delta_\chi^2) \simeq (0,0)\) corresponds to the pure de Sitter case where the big-bang branch disappears and the instanton connects the inflationary universe to “nothing”. In the two other extreme cases, i) the pure thermal deformation \((\delta_T^2, \delta_\chi^2) \simeq (1,0)\) and ii) the pure moduli deformation \((\delta_T^2, \delta_\chi^2) \simeq (0,1)\), the big bang branch remains almost static for a very long period of time. However, the probability transition is minimal in case i) and maximal in case ii). Thus, the late future of these evolutions is most probably contracting (till a big crunch occurs) in case i), and inflationary in case ii).

The above analysis and results can be applied to more complex systems where the radiative and temperature corrections are effectively taking the form of cosmological, curvature and radiation terms during the time evolution [12]. These richer models occur in string effective no-scale supergravities i.e. associated to \(N = 1\) string compactifications where \(N = 1\) supersymmetry is spontaneously broken. First and second order phase transitions are again involved [12]. The stringy origin of these models is giving us the possibility to go beyond the field theory approximation. In particular, it is possible to study the big bang cosmological evolution above the Hagedorn temperature [18] as well as “the stringy analog of the wave function of the universe” [19].

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Appendix

We would like to find in this appendix the phase diagram in the \((\delta_T^2, \chi^2)\)-plane that is delimiting when the solutions (5.17) and (5.21) for the scale factor are valid, and when it is instead the evolution (5.24) that is relevant. The two first solutions exist when \(\Delta \leq 1\), while the last one occurs when \(\Delta \geq 1\).

First of all, we note that the polynomial \(P\) defined in eq. (5.9) has three roots, whose product is \(4\delta_\chi^2/27\). If two of them are complex conjugate, the third one is thus real positive. If on the contrary they are all real, one or three of them must be positive. However, since the sum of these real roots is \(-1\), only one is positive. In any case, we always have a single real positive root, \(\kappa'\), and eventually two other real negative ones.

For \(\delta_T^2 > 4/3\), the derivative \(P'\) of the polynomial defined in eq. (5.9) is everywhere positive, so that \(P\) has a single root. When
\[
\delta_T^2 \leq \frac{4}{3},
\]
the derivative \(P'\) admits two roots,
\[
\kappa'_\pm = -\frac{1}{3} \left(1 \mp \sqrt{1 - \frac{3}{4} \delta_T^2}\right).
\]
One always has \(P(\kappa'_\pm) \leq 0\), while \(P(\kappa'_+) \geq 0\) (so that \(P\) has 3 real roots) if and only if \(\delta_\chi^2 \leq h(\delta_T^2)\), where \(h(\delta_T^2)\) is defined in eq. (5.13). Actually, this condition implies in particular that \(h(\delta_T^2) \geq 0\), which is equivalent to \(\delta_T^2 \leq 1\). As a conclusion, the polynomial \(P\) has three real roots, (one positive and two negative), if the inequalities (5.12) are satisfied, while it has only one real root in all other cases.

Let us determine when \(\Delta \leq 1\). This condition can be translated into a degree two inequality for \(\kappa'\) that admits solutions if the condition (A.1) is satisfied. In that case, \(\Delta \leq 1\) is equivalent to having
\[
\kappa'_{\Delta^-} \leq \kappa' \leq \kappa'_{\Delta^+},
\]
where
\[ \kappa'_{\Delta \pm} = \frac{1}{3} \left( \pm \sqrt{4 - 3\delta_T^2} - 1 \right). \] (A.4)

Since having \( \kappa'_{\Delta +} < 0 \) is equivalent to satisfying \( \delta_T^2 > 1 \), the single real root \( \kappa' \) that \( \mathcal{P} \) has in this case is outside the range \( [\kappa'_{\Delta -}, \kappa'_{\Delta +}] \), since it is positive. Thus, \( \delta_T^2 > 1 \) implies \( \Delta > 1 \).

Let us then concentrate on the case \( \delta_T^2 \leq 1 \). We note that when the parameters \( \delta_\chi^2 \) and \( \delta_T^2 \) are chosen on the critical curve defined by \( \delta_\chi^2 = h(\delta_T^2) \), one has \( \mathcal{P}(\kappa'_{\Delta +}) = 0 \). This means that the single real positive root \( \kappa' \) of \( \mathcal{P} \) is then precisely \( \kappa'_{\Delta +} \). In that case, we have \( \Delta = 1 \).

If instead we take \( \delta_\chi^2 > h(\delta_T^2) \), the only real root of \( \mathcal{P} \) is positive and satisfies \( \kappa' > \kappa'_{\Delta +} \), due to the fact that \( \mathcal{P}'(x) > 0 \) for any \( x > 0 \). We thus have \( \Delta > 1 \). On the contrary, if \( \delta_\chi^2 < h(\delta_T^2) \), the real root of \( \mathcal{P} \) which is positive satisfies \( \kappa' < \kappa'_{\Delta +} \) for the same reason, so that we have \( \Delta < 1 \).

As a conclusion, the solutions (5.17) and (5.21) corresponding to the case \( \Delta \leq 1 \) arise when the point \( (\delta_T^2, \delta_\chi^2) \) belongs to the interior of the domain (5.12) of fig. 8. Outside this domain, the evolution (5.24) associated to \( \Delta \geq 1 \) is the correct one. On the critical curve \( \delta_\chi^2 = h(\delta_T^2) \), one has \( \Delta = 1 \).

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