Detailed asymptotic expansions for partitions into powers

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Abstract

Here we examine the number of ways to partition an integer \( n \) into \( k \)th powers when \( n \) is large. Simplified proofs of some asymptotic results of Wright are given using the saddle-point method, including exact formulas for the expansion coefficients. The convexity and log-concavity of these partitions is shown for large \( n \), and the stronger conjectures of Ulas are proved. The asymptotics of Wright’s generalized Bessel functions are also treated.

1 Introduction

Let \( k \) be a positive integer and, following Hardy and Ramanujan, write \( p^k(n) \) for the number of partitions of \( n \) into \( k \)th powers. The generating function for \( p^k(n) \) is

\[
G_k(q) := \sum_{n=0}^{\infty} p^k(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^{mk}}.
\]

To describe the size of \( p^k(n) \) as \( n \to \infty \), define the useful quantities

\[
c_k := \frac{1}{k} \Gamma(1 + \frac{1}{k}) \zeta(1 + \frac{1}{k}), \quad a_k := c_k^{k/(k+1)}, \quad b_k := \frac{a_k}{(2\pi)^{(k+1)/2}(1 + \frac{1}{k})^{1/2}}, \quad h_k := \frac{\zeta(-k)}{2} = -\frac{B_{k+1}}{2(k+1)},
\]

in terms of the gamma function, Riemann zeta function and Bernoulli numbers. Then \( a_k, b_k, c_k \) are positive real numbers and the first few values of \( h_k \) for \( k \geq 1 \) are \(-\frac{1}{24}, 0, \frac{1}{240} \) and 0. Set

\[
M_k(n) := b_k \exp\left((k + 1)a_k(n + h_k)^{1/(k+1)}\right) \left(n + h_k\right)^{3/2-1/(k+1)}.
\]

Hardy and Ramanujan in [HR00, p. 111] stated the main term in the asymptotics of \( p^k(n) \) as \( n \to \infty \) and this was extended by Wright [Wri34, Thm. 2]:

**Theorem 1.1** (Wright, 1934). Let \( k \) and \( R \) be positive integers. There exist \( Q_r(k) \) so that as \( n \to \infty \),

\[
p^k(n) = M_k(n) \left(1 + \sum_{r=1}^{R-1} \frac{Q_r(k)}{(k \cdot a_k)^r (n + h_k)^{r/(k+1)}} + O\left(\frac{1}{n^{R/(k+1)}}\right)\right)
\]

for an implied constant depending only on \( k \) and \( R \).
There has been renewed interest in this result, with Vaughan [Vau15] for \( k = 2 \), and Gafni [Gaf16] for \( k \geq 2 \), giving new treatments of Theorem 1.1 using slightly different parameters. Tenenbaum, Wu and Li [TWL19] showed that the proof of Theorem 1.1 (expressed in the form (4.13)), can be obtained more easily by employing the saddle-point method.

Wright expressed the coefficients \( Q_r(k) \) from (1.4) in a concise formula that seems to have been overlooked by these recent authors. His asymptotic expansions use the generalized Bessel function \( \phi(z) = \phi(\rho, \beta; z) \), see (6.1), which in turn has an expansion that is stated in [Wri33] and finally proved in [Wri35, Thm. 1]. In our notation the formula is as follows.

**Theorem 1.2** (Wright, 1935). For all integers \( r \geq 0 \) and \( k \geq 1 \),

\[
Q_r(k) = r! \left( -\frac{1}{r} \right) \left( -\frac{1}{2} \right)^{-r} \left[ x^{2r} \right] (1 + x)^{1/2} \left( \frac{x}{r} \right)^{-1/k} \left( 1 - \frac{x}{k} \right)^{-1/2-r} ,
\]

where \([x^{2r}]\) indicates the coefficient of \( x^{2r} \) in the succeeding series in \( Q[[x]] \).

Note that the series in parentheses on the right of (1.5) has initial terms

\[
1 - \frac{1}{3} x + \frac{(1 + 2)(1 + 3)}{3 \cdot 4} x^2 - \ldots .
\]

In Section 2 and the beginning of Section 3 we prove both Theorems 1.1 and 1.2 together. This proof begins in the same way as [TWL19], with Propositions 2.1 and 2.2 included for completeness. After that, a simpler choice of saddle-point allows a different treatment that is much more explicit and that also includes the asymptotic expansion coefficients \( W_r(\rho, \beta) \) of \( \phi(\rho, \beta; z) \).

Various properties of \( W_r(\rho, \beta) \), and its special case \( Q_r(k) = W_r(k, -\frac{1}{k}) \), are established in Section 3. De Moivre polynomials are a convenient device for manipulating power series, and the De Moivre polynomial version of (1.5) in (3.2) is easy to work with. For example, \( Q_0(k) = 1 \),

\[
Q_1(k) = -\frac{11k^2 + 11k + 2}{24(k + 1)} , \quad Q_2(k) = -\frac{(k - 1)(k + 2)(23k^2 + 23k + 2)}{1152(k + 1)^2} ,
\]

\[
Q_3(k) = -\frac{(k - 1)(k + 2)(1183k^4 + 10646k^3 + 11139k^2 + 2396k + 556)}{414720(k + 1)^3} ,
\]

and we prove some of the patterns that are already appearing.

The polynomials introduced by De Moivre in 1697 are described briefly as follows and [O’S22] has more information. Let \( n \) and \( k \) be integers with \( k \geq 0 \). For a power series \( a_1 x + a_2 x^2 + a_3 x^3 + \ldots \), without a constant term and with coefficients in \( \mathbb{C} \), we may define the De Moivre polynomial \( A_{n,k}(a_1, a_2, a_3, \ldots) \) by means of the generating function

\[
(a_1 x + a_2 x^2 + a_3 x^3 + \ldots)^k = \sum_{n \in \mathbb{Z}} A_{n,k}(a_1, a_2, a_3, \ldots) x^n .
\]

If \( n \geq k \) then \( A_{n,k}(a_1, a_2, a_3, \ldots) \) is a polynomial in \( a_1, a_2, \ldots, a_{n-k+1} \) of homogeneous degree \( k \) with positive integer coefficients. Some simple properties we will need are

\[
A_{n,k}(0, a_1, a_2, a_3, \ldots) = A_{n-k,k}(a_1, a_2, a_3, \ldots) ,
\]

\[
A_{n,k}(ca_1, ca_2, ca_3, \ldots) = c^k A_{n,k}(a_1, a_2, a_3, \ldots) ,
\]

\[
A_{n,k}(c a_1, c^2 a_2, c^3 a_3, \ldots) = c^n A_{n,k}(a_1, a_2, a_3, \ldots) .
\]

Section 4 gives further results on the asymptotics of \( p^k(n) \), including some interesting explicit expansions for the usual partition function \( p(n) = p^1(n) \). Proposition 4.4 shows for instance that as \( n \to \infty \),

\[
p(n) = \exp \left( \frac{\pi \sqrt{2n/3}}{4\sqrt{3n}} \right) \left( 1 + \sum_{r=1}^{R-1} \frac{\omega_r}{n^{r/2}} + O \left( \frac{1}{n^{R/2}} \right) \right) .
\]
with an explicit determination of the coefficients that seems to be new:

\[
\omega_r = \frac{1}{(4\sqrt{6})^r} \sum_{k=0}^{(r+1)/2} \binom{r+1}{k} \frac{r+1-k}{(r+1-2k)!} \left( \frac{\pi}{6} \right)^{r-2k}.
\]

The convexity and log-concavity of the sequence \( p^k(n) \) for large \( n \) is shown in Section 5. Recall that a sequence \( a(n) \) is convex at \( n \) if \( 2a(n) \leq a(n+1) + a(n-1) \) and log-concave there if \( a(n)^2 \geq a(n+1) \cdot a(n-1) \). Ulas conjectured further inequalities in [Ula21] and we prove them here:

**Theorem 1.3.** For each positive integer \( k \) there exist \( C_k \) and \( D_k \) so that

\[
2p^k(n) < \left( p^k(n+1) + p^k(n-1) \right) \left( 1 - n^{-2} \right) \quad (n \geq C_k),
\]

\[
p^k(n)^2 > p^k(n+1) \cdot p^k(n-1) \cdot \left( 1 + n^{-2} \right) \quad (n \geq D_k).
\]

In fact Theorem 1.3 is stronger than Ulas’s conjectures and follows from the sharper results of Corollaries 5.6 and 5.7. These in turn follow from a detailed study of Theorem 1.1.

Theorem 6.3 generalizes Theorem 1.1 to give the asymptotics of the generalized Bessel function \( G_k(q) \) he developed.

## 2 The asymptotic expansion of \( p^k(n) \)

We give the proofs of Theorem 1.1 and most of Theorem 1.2 in this section. In the usual starting point, by Cauchy’s theorem, and for any \( \sigma > 0 \),

\[
p^k(n) = \frac{1}{2\pi i} \int G_k(q)q^{-n-1} dq
\]

\[
= \frac{1}{2\pi} \int_{\sigma-i\pi}^{\sigma+i\pi} G_k(e^{-s})e^{ns} ds
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} G_k(e^{-\sigma-it})e^{n(\sigma+it)} dt.
\]

### 2.1 Estimates when \( t \) is large

The next result will be used to show that small values of \( t \) in (2.2) make the main contribution.

**Proposition 2.1.** Let \( n, \lambda, \sigma \) and \( M \) be real numbers with \( \lambda, \sigma, M > 0 \). As \( \sigma \to 0^+ \),

\[
\int_{\lambda, \sigma \to 0^+} e^{nit} G_k(e^{-\sigma-it}) dt \ll \sigma^M G_k(e^{-\sigma}),
\]

for an implied constant depending only on \( k \in \mathbb{Z}_{\geq 1}, \lambda \) and \( M \).

**Proof.** This is a special case of Debruyne and Tenenbaum’s result [DT20, Lemma 3.1], with the minor addition of including an extra parameter \( \lambda \). We give their intricate self-contained proof here, for completeness, showing directly how it applies to \( G_k(q) \). Another approach is shown in [TWL19, Lemma 2.3], (with a correction in the final arXiv version).

First let \( \Lambda := \{1^k, 2^k, 3^k, \ldots \} \). It is easy to see that the number of elements of \( \Lambda \) of size at most \( R \geq 0 \) is \( [R^{1/k}] \geq R^{1/k} - 1 \) and we have

\[
|\Lambda \cap [1, R]| \geq \frac{1}{2} R^{1/k} \quad (R \geq 2^k),
\]

\[
|\Lambda \cap [R, 2R]| \geq (2^{1/k} - 1) R^{1/k} \quad (R \geq 0).
\]
For later use in (2.3), choose a real number $R_0$ large enough so that $(2^{1/k} - 1)R_0^{1/k} \geq 2M$, thinking of $k$ and $M$ as fixed.

Next let $\|\vartheta\|$ denote the distance from $\vartheta$ to the nearest integer. For every $d > 0$ the methods in [DT20, p. 732] establish the inequality

$$\frac{|G_k(e^{-\sigma-it})|}{G_k(e^{-\sigma})} \leq \prod_{m \in \Lambda, m \leq d/\sigma} \left( 1 + \frac{16}{e^d} \frac{\|mt/(2\pi)\|^2}{m^2\sigma^2} \right)^{-1/2}$$

(2.6)

and this allows us to give detailed bounds for the integrand in (2.3).

**First case**, when $\lambda \cdot \sigma^{1+1/(3k)} \leq t \leq 2\pi \sigma$. Then $0 \leq mt/(2\pi) \leq 1/2$ when $m \leq 1/(2\sigma)$ and so with $d = 1/2$ in (2.6),

$$\frac{|G_k(e^{-\sigma-it})|}{G_k(e^{-\sigma})} \leq \prod_{m \in \Lambda, m \leq 1/(2\sigma)} \left( 1 + \frac{\lambda^2\sigma^2/(3k)}{5} \right)^{-1/2}.$$  (2.7)

If $\sigma$ satisfies $\sigma \leq 2^{-k-1}$ then the number of factors on the right side of (2.7) is at least $(2\sigma)^{-1/k}/2$ by (2.4). Therefore (2.7) is bounded by

$$\exp \left( -\frac{1}{4} \log \left( 1 + \frac{\lambda^2\sigma^2/(3k)}{5} \right) \right) \leq e^{-C\sigma^{-1/(3k)}} \ll \sigma^M,$$

where $C$ is a positive constant depending only on $\lambda$. This completes the first case.

The remaining part of the integral has $\sigma < t/(2\pi) \leq 1/2$. Dirichlet’s approximation theorem lets us find a close rational number $a/q$ to $t/(2\pi)$:

$$\frac{t}{2\pi} = a/q \pm r \quad \text{for} \quad 1 \leq q \leq 3R_0, \quad (a, q) = 1, \quad 0 \leq r \leq \frac{1}{3R_0q}. \quad (2.8)$$

Following [DT20, Lemma 3.1], we consider two subcases separately.

**“Minor arcs”** when $\sigma < t/(2\pi) \leq 1/2$ and $2\sigma/(3q) < r \leq 1/(3R_0q)$. Start with (2.6) for $d = 1$. Since $2/(3rq) < 1/\sigma$, it follows that

$$\frac{|G_k(e^{-\sigma-it})|}{G_k(e^{-\sigma})} \leq \prod_{1/(3rq) \leq m \leq 2/(3rq)} \left( 1 + \frac{\|mt/(2\pi)\|^2}{m^2\sigma^2} \right)^{-1/2} \leq \prod_{1/(3rq) \leq m \leq 2/(3rq)} \left( 1 + \frac{5r^2}{4\sigma^2} \right)^{-1/2}, \quad (2.9)$$

as $\|mt/(2\pi)\| \geq 1/(3q)$ here. By (2.5) the number of factors in the product is at least $(2^{1/k} - 1)(3rq)^{-1/k}$.

If $2\sigma/(3q) < r \leq \sqrt{\sigma}$ then $3rq \leq 9R_0\sqrt{\sigma}$ and (2.9) implies

$$\frac{|G_k(e^{-\sigma-it})|}{G_k(e^{-\sigma})} \leq \left( 1 + \frac{5}{9q^2} \right)^{-C_0\sigma^{-1/(2k)}} \leq e^{-C_1\sigma^{-1/(2k)}} \ll \sigma^M,$$

where $C_0$ and $C_1$ are positive constants depending only on $k$ and $R_0$.

If $\sqrt{\sigma} < r \leq 1/(3R_0q)$ then the number of factors in the product on the right of (2.9) is at least $(2^{1/k} - 1)R_0^{1/k}$, and this more than $2M$ by our original choice of $R_0$. Hence we again find

$$\frac{|G_k(e^{-\sigma-it})|}{G_k(e^{-\sigma})} \leq \left( 1 + \frac{5}{4\sigma} \right)^{-M} \ll \sigma^M.$$

**“Major arcs”** when $\sigma < t/(2\pi) \leq 1/2$ and $0 \leq r \leq 2\sigma/(3q)$. Note first that if $q = 1$ in (2.8) then we must have $a = 0$ and $t/(2\pi) = r$. This implies $\sigma < r$ and $r \leq 2\sigma/3$, an impossibility. Therefore, every $t$ in the major arcs has a rational approximation with denominator $q$ in the range $2 \leq q \leq 3R_0$. For each of these
we may choose distinct $m_{j,q}$ in $\Lambda$ for $1 \leq j \leq M$ that are not multiples of $q$. With this choice, let $\sigma$ be small enough that $\sigma \leq 1/m_{j,q}$ is always true. By (2.6) with $d = 1$,

$$\frac{|G_k(e^{-\sigma-it})|}{G_k(e^{-\sigma})} \leq \prod_{j=1}^{M} \left(1 + \frac{5}{9q^2 m_{j,q}^2} \right)^{-1/2} \leq \prod_{j=1}^{M} \left(1 + \frac{5}{9q^2 m_{j,q}^2} \right)^{-1/2} \ll \sigma^M,$$

since $m_{j,q} t/(2\pi) = \ell/q \pm m_{j,q} r$ with $q \not\mid \ell$ and $0 \leq m_{j,q} r \leq 2/(3q)$ implies $\| m_{j,q} t/(2\pi) \| \geq 1/(3q)$. This last case completes the proof of Proposition 2.1.

### 2.2 Further setting up

Put $\Phi_k(s) := \log G_k(e^{-s})$.

**Proposition 2.2.** Let $k$ and $L$ be positive integers. Suppose $s \in \mathbb{C}$ has $\text{Re}(s) > 0$ and $|\arg s| \leq \pi/4$. Then

$$\Phi_k(s) = \frac{k c_k}{s^{1/k}} + \frac{1}{2} \log \left( \frac{s}{(2\pi)^k} \right) + h_k s + \Phi^*_k(s) \quad \text{where} \quad \Phi^*_k(s) = O(|s|^L),$$

(2.10)

for $c_k$ and $h_k$ in (1.1), (1.2) and an implied constant depending only on $k$ and $L$.

**Proof.** Now we are following [TWL19, Lemma 2.1]. The Mellin transform of $\Phi_k(s)$ is computed there as $\zeta(z + 1)\zeta(kz)\Gamma(z)$, so that

$$\Phi_k(s) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z + 1)\zeta(kz)\Gamma(z) \frac{dz}{sz} \quad (\text{Re}(s) > 0).$$

The integrand is meromorphic and for $z = x + i y$ with $|y|$ large and $-L \leq x \leq 2$ we have from the usual estimates

$$\zeta(z + 1)\zeta(kz)\Gamma(z)s^{-z} \ll |y|^{A - \pi|y|/2} |s^{-x-iy}|$$

for $A$ and the implied constant depending only on $k$ and $L$. Also

$$|s^{-x-iy}| = \exp(-\text{Re}(x + iy) \log s)) = \exp(-x \log |s| + y \arg s)) \ll |s|^{-x}e^{|y|/4}.$$

Therefore the integrand has exponential decay as $|y| \to \infty$ and moving the contour of integration from the line with real part 2 to the line with real part $-L$ is justified. The only poles crossed are at $z = 1/k$ and $z = -1$ along with a double pole at $z = 0$; the poles of $\Gamma(z)$ at $z = -2, -3, \ldots$ cancel with the zeros of $\zeta(z + 1)\zeta(kz)$ since $k \in \mathbb{Z}_{\geq 1}$ and $\zeta(z)$ has zeros at the negative even integers. The residues of $\zeta(z + 1)\zeta(kz)\Gamma(z)s^{-z}$ at $z = 1/k, 0$ and $-1$ are

$$\frac{\Gamma(\frac{1}{k})\zeta(1 + \frac{1}{k})}{ks^{1/k}} = \frac{k c_k}{s^{1/k}}, \quad \frac{1}{2} \log \left( \frac{s}{(2\pi)^k} \right), \quad \frac{\zeta(-k)s}{2} = h_k s,$$

respectively, and (2.10) follows.

Set

$$n_k := n + h_k, \quad F_k(s) := \frac{k c_k}{s^{1/k}} + n_k s,$$

and the integrand in (2.1) satisfies

$$G_k(e^{-s})e^{ns} = e^{\Phi_k(s) + ns} = e^{F_k(s)} s^{1/2} (2\pi)^{-k/2} e^{\Phi^*_k(s)},$$

(2.11)

For the saddle-point method, the path of integration in (2.1) is usually chosen to pass through a point where $\frac{d}{ds}$ of the integrand is zero. This is the point $\sigma_n$ used in [TWL19]. However, there is some leeway, and here we use the more convenient point $s_n$ which is the saddle-point of the factor $e^{F_k(s)}$. In other words,

$$s_n = s_{n,k} := (c_k/n_k)^{k/(k+1)} \implies F'_k(s_n) = 0.$$

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\(^1\)In one of the earliest applications of the saddle-point method, Riemann used this flexibility in obtaining asymptotics for $\zeta(s)$ on the critical line; see for example [O'S23, Sect. 2.1].

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For large $n$ we see that $s_n$ is a positive real number that approaches 0 with increasing $n$. 

Taking $\sigma = s_n$ and $\lambda = c_k^{-1/3}$ in (2.2) and Proposition 2.1 write

$$p^k(n) = E_1(n) + \frac{1}{2\pi} \int_{c_k^{-1/3}s_n}^{c_k^{-1/3}s_n+1/(3k)} \exp\left(F_k\left(s_n + it\right)\right) \frac{(s_n + it)^{1/2}}{(2\pi)^{k/2}} e^{\Phi^*(s_n+it)} dt$$

(2.12)

Therefore, in this notation,

$$E_1(n) := \frac{1}{2\pi} \int_{c_k^{-1/3}s_n+1/(3k)}^{c_k^{-1/3}s_n} G_k\left(e^{-s_n-it}\right) e^{\pi i(s_n+it)} dt.$$  

(2.13)

To make the coming computations more manageable, apply the change of variables $w = n_k(s_n + it)$ in (2.12) to find

$$p^k(n) - E_1(n) = \left(2\pi \right)^{-k/2} n_k^{-3/2} \frac{1}{2\pi i} \int_{U-iU^{2/3}}^{U+iU^{2/3}} \left( w^{1/2} \exp\left( w + \frac{k c_k n_k^{1/k}}{w^{1/k}} \right) e^{\Phi^*(w/n_k)} \right) dw,$$

where $U = c_k^{k/(k+1)} n_k^{1/(k+1)}$. This integral naturally breaks into the large and small parts

$$I_N := \frac{1}{2\pi i} \int_{U-iU^{2/3}}^{U+iU^{2/3}} w^{-\beta} \exp\left( w + \frac{N}{w^\rho} \right) dw,$$

(2.14)

$$I'_N := \frac{1}{2\pi i} \int_{U-iU^{2/3}}^{U+iU^{2/3}} w^{-\beta} \exp\left( w + \frac{N}{w^\rho} \right) e^{\Phi^*(w/n_k)} dw,$$

(2.15)

respectively, for $\beta = -1/2$ and the simpler variables

$$N = k c_k n_k^{1/k}, \quad \rho = 1/k.$$

Therefore, in this notation,

$$p^k(n) = E_1(n) + \left(2\pi \right)^{-k/2} n_k^{-3/2} \left( I_N + I'_N \right).$$

(2.16)

### 2.3 The asymptotics of $I_N$

Now we focus on $I_N$ where we may fix $\rho \in \mathbb{R}_{\geq 0}, \beta \in \mathbb{C}$ and let real $N$ tend to infinity. In terms of these variables,

$$U = \left(\rho N\right)^{1/(\rho+1)}$$

so that $NU^{-\rho} = U/\rho$.  

(2.17)

Also write

$$f_N(w) := w + \frac{N}{w^\rho}$$

with

$$f_N^{(j)}(w) = \begin{cases} -\rho & \text{for } j = 0, \\ \frac{N}{(j+1)!} & \text{for } j \geq 1. \end{cases}$$

(2.21)

$$I_N = \frac{1}{2\pi} \int_{-U^{2/3}}^{U^{2/3}} \exp(f_N(U + it))(U + it)^{-\beta} dt.$$  

Assume from here on that $N$ is large enough to ensure that $U > 1$. Then $f_N(z)$ is holomorphic in a domain containing the disk of radius $U^{2/3}$ about $U$. Expanding $f_N(z)$ in its Taylor series there shows

$$I_N = \frac{e f_N(U)}{2\pi U^\beta} \int_{-U^{2/3}}^{U^{2/3}} \exp\left( \frac{f_N''(U)}{2} (it)^2 \right) g_N(t) dt,$$

(2.19)

where $g_N(z)$ is holomorphic on the same disk and given by

$$g_N(z) = \exp\left( \sum_{j=3}^{\infty} \frac{f_N^{(j)}(U)}{j!} (iz)^j \right) \cdot \left( 1 + \frac{iz}{U} \right)^{-\beta}.$$
Lemma 2.3. Suppose $z \in \mathbb{C}$ with $|z| \leq U^{2/3}$. Then for these $z$ values and $N$ large enough we have the bound $g_N(z) \ll 1$ and the development

$$g_N(z) = \sum_{j=0}^{\infty} d_j \left( \frac{iz}{U} \right)^j \quad (2.20)$$

for

$$d_j = \sum_{\ell=0}^{j/3} \frac{U^\ell}{\rho^\ell \ell!} \sum_{m=3\ell}^{j} \left( -\beta \right) A_{m-2\ell,\ell} \left( \frac{-\rho}{3}, \frac{-\rho}{4}, \ldots \right). \quad (2.21)$$

Proof. To bound $g_N(z)$ when $|z| \leq U^{2/3}$, first note with (2.18) that

$$\sum_{j=3}^{\infty} f_N^{(j)}(U) (iz)^j \ll \sum_{j=3}^{\infty} \left( -\frac{\rho}{j} \right) \frac{N}{U^{\rho+j}} U^{2j/3} = \frac{1}{\rho} \sum_{j=3}^{\infty} \left( -\frac{\rho}{j} \right) (-1)^j U^{1-j/3}.$$

As $1 - j/3 \leq -j/6$ for $j \geq 6$, we have

$$\sum_{j=3}^{\infty} f_N^{(j)}(U) (iz)^j \ll 1 + \sum_{j=6}^{\infty} \left( -\frac{\rho}{j} \right) (-1)^j U^{-j/6} \leq \left( 1 - U^{-1/6} \right)^{-\rho} \ll 1,$$

and hence $g_N(z) \ll 1$ as we claimed. To find $d_j$ write

$$\exp \left( \sum_{j=3}^{\infty} \frac{f_N^{(j)}(U)}{j!} (iz)^j \right) = \sum_{m=0}^{\infty} (iz)^m \sum_{\ell=0}^{m/3} \frac{1}{\ell!} A_{m,\ell} \left( 0, 0, \frac{f_N^{(3)}(U)}{3!}, \frac{f_N^{(4)}(U)}{4!}, \ldots \right)$$

$$= \sum_{m=0}^{\infty} \left( \frac{iz}{U} \right)^m \sum_{\ell=0}^{m/3} \frac{N^\ell}{\ell!} U^{-\rho \ell} A_{m-2\ell,\ell} \left( \frac{-\rho}{3}, \frac{-\rho}{4}, \ldots \right),$$

where we used (1.10), (1.11) and (1.12). Developing $(1 + iz/U)^{-\beta}$ with the binomial theorem and interchanging summations completes the proof of (2.21). \square

Lemma 2.4. Suppose $J$ is a positive integer and $z \in \mathbb{C}$ with $|z| \leq U^{2/3}$. Then for $N$ large enough

$$g_N(z) = \sum_{j=0}^{J-1} d_j \left( \frac{iz}{U} \right)^j + g_{N,J}(z) \quad \text{with} \quad g_{n,J}(z) = O \left( \frac{|z|^J}{U^{2J/3}} \right) \quad (2.22)$$

and an implied constant independent of $N$.

Proof. The same proof as for Lemma 2.3 demonstrates that $g_N(z) \ll 1$ for $|z| \leq R := 2U^{2/3}$. By Taylor's theorem, [Ahl78] pp. 125-126,

$$g_N(z) - \sum_{j=0}^{J-1} d_j \left( \frac{iz}{U} \right)^j = \frac{z^J}{2\pi i} \int_{|w|=R} \frac{g_N(w)}{w^J (w-z)} dw. \quad (2.23)$$

Since $|w-z| \geq R/2$, it follows that the right side of (2.23) is $\ll |z|^J/R^J$ as we wanted. \square

With (2.17) and (2.18), set

$$Y := \frac{f''_N(U)}{2} = \left( \frac{-\rho}{2} \right) \frac{N}{U^{\rho+2}} = \left( \frac{-\rho}{2} \right) \frac{1}{\rho U}. \quad (2.24)$$

For any positive integer $J$ write $I_N = I^*_N - E_2(N) + E_3(N)$ with

$$I^*_N := \frac{e^{f_N(U)}}{2\pi U^\beta} \int_{-\infty}^{\infty} \exp\left(-Y \cdot t^2\right) \frac{2J-1}{2} \sum_{j=0}^{2J-1} \frac{d_j}{U} \frac{t^j}{j!} \, dt,$$  \hspace{1cm} (2.25)

$$E_2(N) := \frac{e^{f_N(U)}}{2\pi U^\beta} \int_{|t|>U^{2/3}} \exp\left(-Y \cdot t^2\right) \frac{2J-1}{2} \sum_{j=0}^{2J-1} \frac{d_j}{U} \frac{t^j}{j!} \, dt,$$  \hspace{1cm} (2.26)

$$E_3(N) := \frac{e^{f_N(U)}}{2\pi U^\beta} \int_{-U^{2/3}}^{U^{2/3}} \exp\left(-Y \cdot t^2\right) g_{N,2J}(t) \, dt.$$  \hspace{1cm} (2.27)

Only the even values of $j$ give nonzero contributions in (2.25) and (2.26), so $j$ may be replaced by $2j$ in the summands. The well-known identity

$$\int_{-\infty}^{\infty} e^{-Y \cdot t^2} t^{2j} \, dt = \sqrt{\pi} j! \frac{(-1)^j}{Y^{j+1/2}} \quad (Y > 0, j \in \mathbb{Z}_{\geq 0}),$$  \hspace{1cm} (2.28)

implies

$$I^*_N = \frac{e^{f_N(U)}}{2\pi U^\beta} \sum_{j=0}^{J-1} \sqrt{\pi} j! \frac{(-1)^j}{U^{2j} Y^{j+1/2}} \frac{d_{2j}}{j!}.$$  \hspace{1cm} (2.29)

Then the relations

$$f_N(U) = (1 + \frac{1}{\rho})U, \quad \frac{1}{U^{2j} Y^{j+1/2}} = \left(\frac{2U}{\rho+1}\right)^{1/2} \left(-\rho\right)^{-j} \frac{\rho^j}{U^j},$$  \hspace{1cm} (2.30)

show

$$I^*_N = \frac{U^{1/2-\beta} e^{(1+\frac{1}{\rho})U} J-1}{\sqrt{2\pi(\rho+1)}} \sum_{j=0}^{J-1} j! \frac{(-\rho)}{2} \left(-\rho\right)^{-j} \frac{\rho^j}{U^j} \frac{d_{2j}}{j!}.$$  \hspace{1cm} (2.31)

With (2.21) the sum in (2.30) is

$$\sum_{j=0}^{J-1} j! \frac{(-\rho)}{2} \left(-\rho\right)^{-j} \sum_{\ell=0}^{2j/3} \frac{\rho^{J}\ell}{U^{j-\ell}!} \sum_{m=3\ell}^{2j} \left(-\beta\right) A_{m-2\ell,\ell},$$

and writing $r = j - \ell$, this equals

$$\sum_{r=0}^{J-1} \frac{\rho^r}{U^r} \sum_{\ell=0}^{2r} \left(\frac{r+\ell}{r}\right) \left(-\frac{1}{2}\right) \left(-\frac{\rho}{2}\right)^{-r-\ell} \sum_{m=3\ell}^{2r+2\ell} \left(-\beta\right) A_{m-2\ell,\ell}.$$  \hspace{1cm} (2.32)

Using

$$\left(\frac{r+\ell}{r}\right) \left(-\frac{1}{2}\right) \left(-\frac{\rho}{2}\right)^{-r-\ell} = \left(\frac{-\frac{1}{2}}{r}\right) \left(-\frac{1}{2}\right)^{-r},$$

along with the substitution $v = 2r + 2\ell - m$ makes the inner sums in (2.31) into

$$\mathcal{W}_r(\rho, \beta) := r! \left(\frac{-\frac{1}{2}}{r}\right) \sum_{\ell=0}^{2r} \left(-\frac{1}{2}\right) \left(-\frac{\rho}{2}\right)^{-r-\ell} \sum_{v=0}^{2r} \left(-\beta\right) A_{2r-v,\ell} \left(\frac{-\rho}{3}, \frac{-\rho}{4}\right),$$  \hspace{1cm} (2.33)

so that finally,

$$I^*_N = \frac{U^{1/2-\beta}}{\sqrt{2\pi(\rho+1)}} \exp\left((1 + \frac{1}{\rho})U\right) \left(1 + \sum_{r=1}^{J-1} \frac{\rho^r \mathcal{W}_r(\rho, \beta)}{U^r}\right).$$  \hspace{1cm} (2.34)

This is our main term and the next result is demonstrated by showing that $E_2$ and $E_3$ are smaller.
Proposition 2.5. Fix $\rho > 0$ and $\beta \in \mathbb{C}$. Set $U := (\rho N)^{1/(\rho+1)}$ and

$$I_N := \frac{1}{2\pi i} \int_{U-iU^{2/3}}^{U+iU^{2/3}} w^{-\beta} \exp\left( w + \frac{N}{w^r} \right) dw$$

as in (2.14). Then as real $N \to \infty$,

$$I_N = \frac{U^{1/2-\beta}}{\sqrt{2\pi}(\rho+1)} \exp\left( (1 + \frac{1}{\rho})U \right) \left( 1 + \sum_{r=1}^{R-1} \frac{\rho^r \mathcal{W}_r(\rho, \beta)}{U^r} + O\left( \frac{1}{U^R} \right) \right),$$

(2.34)

for an implied constant depending only on $R \in \mathbb{Z}_{\geq 1}$, $\rho$ and $\beta$.

Proof. We begin by bounding $E_2(N)$ in (2.26); the parameter $J$ will be chosen later. For all $a, r \geq 0$ and $c > 0$

$$\int_a^\infty e^{-cx} x^r \, dx \ll c^{-r-1}((ac)^r + 1)e^{-ac}$$

(2.35)

for an implied constant depending only on $r$. This elementary bound is shown in [O'S21, Eq. (2.8)] for example. So the integral in (2.26) satisfies

$$\int_{U^{2/3}}^\infty \exp(-Y \cdot t^2)t^{2j} \, dt = \frac{1}{2} \int_{U^{4/3}}^\infty \exp(-Y \cdot x)x^{j-1/2} \, dx$$

$$\ll Y^{-j-1/2} \left( (U^{4/3}Y)^{j-1/2} + 1 \right) \exp\left( -U^{4/3}Y \right)$$

$$\ll U^{4j/3+1/2} \exp\left( -\frac{4+1}{2}U^{1/3} \right).$$

Since $d_{2j} \ll U^{2j/3}$ is easily seen from (2.21), we obtain

$$E_2(N) \ll \frac{1}{|U^\beta|} \exp\left( (1 + \frac{1}{\rho})U \right) \sum_{j=0}^{J-1} \frac{d_{2j}}{U^{2j}} U^{4j/3+1/2} \exp\left( -\frac{4+1}{2}U^{1/3} \right)$$

$$\ll \frac{U^{1/2}}{|U^\beta|} \exp\left( (1 + \frac{1}{\rho})U \right) \exp\left( -\frac{4+1}{2}U^{1/3} \right).$$

(2.36)

Next we bound $E_3(N)$ in (2.27). With Lemma 2.4 and (2.28),

$$E_3(N) \ll \frac{1}{|U^\beta|} \exp\left( (1 + \frac{1}{\rho})U \right) \int_{-\infty}^\infty e^{-Y t^2} \frac{t^{2j}}{U^{4j/3}} \, dt$$

$$\ll \frac{1}{|U^\beta|} \exp\left( (1 + \frac{1}{\rho})U \right) \frac{1}{Y^{J+1/2}} \ll \frac{U^{1/2}}{|U^\beta|} \exp\left( (1 + \frac{1}{\rho})U \right) \frac{1}{U^{J/3}}.$$ (2.37)

As $I_N = I_N' - E_2(N) + E_3(N)$, it follows from (2.35), (2.36) and (2.37) that

$$I_N = \frac{U^{1/2-\beta}}{\sqrt{2\pi}(\rho+1)} \exp\left( (1 + \frac{1}{\rho})U \right) \left( 1 + \sum_{r=1}^{J-1} \frac{\rho^r \mathcal{W}_r(\rho, \beta)}{U^r} + O\left( \exp\left( -\frac{4+1}{2}U^{1/3} \right) + \frac{1}{U^{J/3}} \right) \right).$$

Choosing $J = 3R$ then completes the proof.

\[\square\]

2.4 Final steps

The companion integral $I_N'$ in (2.15) to $I_N$ in (2.14) is supposed to be smaller. This is proved next.
Proposition 2.6. Fix $\rho, x > 0$ and $\beta \in \mathbb{C}$. With $N > 0$ set $U := (\rho N)^{1/(\rho+1)}$ and

$$I'_N := \frac{1}{2\pi} \int_{-U^{2/3}}^{U^{2/3}} \exp(f_N(U + it))(U + it)^{-\beta}\left(e^{\Phi^*((U+it)/x)} - 1\right) dt$$

as in (2.15), (with $x$ replacing $n_k$). Assume $x > U$. Then for any $L > 0$,

$$I'_N \ll \frac{U^{1/2}}{|U^\beta|} \exp\left(1 + \frac{1}{\rho}\right) U^L \frac{L}{x^L},$$

(2.38)

as real $N \to \infty$, for an implied constant depending only on $L$, $\rho$ and $\beta$.

Proof. By Proposition 2.2 we know $\Phi^*((U+it)/x) \ll U^L/x^L$ for $|t| \leq U$. Hence, for $U/x < 1$,

$$e^{\Phi^*((U+it)/x)} - 1 \ll U^L/x^L.$$ 

As in (2.19), using $g_N(z) \ll 1$ from Lemma 2.3,

$$I'_N = \frac{e^{f_N(U)}}{2\pi U^\beta} \int_{-U^{2/3}}^{U^{2/3}} \exp\left(\frac{f'_N(U)}{2}(it)^2\right) g_N(t)\left(e^{\Phi^*((U+it)/x)} - 1\right) dt$$

$$\ll \frac{e^{f_N(U)}}{|U^\beta|} \int_{-U^{2/3}}^{U^{2/3}} \exp(-Yt^2) \frac{U^L}{x^L} dt$$

$$\ll \frac{e^{f_N(U)}}{|U^\beta|} \frac{U^L}{x^L} \int_{-\infty}^{\infty} \exp(-Yt^2) dt,$$

and (2.38) follows by employing (2.24), (2.28) and the left identity in (2.29). \qed

Proof of Theorem 1.1. Recall (1.1), (1.2), (1.3), and make the substitutions, $(n_k := n + h_k)$,

$$N = kc_k n_k^{1/k}, \quad \rho = 1/k, \quad \beta = -1/2, \quad U = a_k n_k^{1/(k+1)} , \quad x = n_k,$$

(2.39)

in Propositions 2.5 and 2.6, going back to our original variables. Then, after choosing $L$ large enough in (2.38) that $Lk \geq R$, (2.16) implies

$$p^k(n) - E_1(n) = \mathcal{M}_k(n) \left(1 + \sum_{r=1}^{R-1} \frac{W_r(1/k, \rho, x)}{(k \cdot a_k)^r n_k^{r/(k+1)}} + O\left(\frac{1}{nR/(k+1)}\right)\right),$$

(2.40)

as $n \to \infty$, with $Q_r(k) := W_r(1/k, -\frac{1}{2})$ given in (2.32).

Finally, we claim that $E_1(n)$ may be moved inside the error term in (2.40). Applying Proposition 2.1 to (2.13) shows

$$E_1(n) \ll e^{ns_n} s_n^M P(e^{-s_n}) \ll s_n^M e^{F_k(s_n)} s_n^{1/2} e^{\Phi^*(s_n)} \ll s_n^M e^{F_k(s_n)},$$

where we also needed (2.11) and the bound in (2.10). Use the equality $F_k(s_n) = (k+1)a_k n_k^{1/(k+1)}$ to find

$$E_1(n) \ll \mathcal{M}_k(n) \cdot n_k^{-(M-2)k/(k+1)},$$

and choosing any $M$ satisfying $(M-2)k \geq R$ proves the claim. This completes the proof of Theorem 1.1 while also giving the explicit formula $W_r(1/k, -\frac{1}{2})$ from (2.32) for the expansion coefficients $Q_r(k)$. \qed
3 Properties of $Q_r(k)$ and $W_r(\rho, \beta)$

In this section we develop some properties of $W_r(\rho, \beta)$ for $\rho > 0$ and $\beta \in \mathbb{C}$. These quantities will be required for the asymptotic expansion of Wright’s generalized Bessel function in Section 6. As seen in (2.40), the special case $W_r(\frac{1}{2}, -\frac{1}{2})$ equals $Q_r(k)$ which is needed in the asymptotic expansion of $p^k(n)$.

Define the series in $\mathbb{Q}[\rho][[x]]$

$$S(\rho, x) := \left(1 + x\right)^{-\rho} - 1 - \left(\frac{-\rho}{1}\right)x = 1 + \left(\frac{-\rho}{2}\right)x + \left(\frac{-\rho}{4}\right)x^2 + \ldots$$

$$= 1 - \frac{\rho + 2}{3}x + \frac{(\rho + 2)(\rho + 3)}{3 \cdot 4}x^2 - \ldots.$$  

**Theorem 3.1.** For all integers $r \geq 0$, the coefficients $W_r(\rho, \beta)$ defined in (2.32) satisfy

$$W_r(\rho, \beta) = r! \left(\frac{-\frac{1}{2}}{r}\right) \left(\frac{-\rho}{2}\right)^{-r} [x^{2r}] (1 + x)^{-\beta} S(\rho, x)^{-1/2-r},$$  

(3.1)

where again $[x^{2r}]$ indicates the coefficient of $x^{2r}$ in the succeeding series.

**Proof.** With $\sum_{v=0}^{\infty} (-\beta)_v x^v = (1 + x)^{-\beta}$ and

$$\sum_{v=0}^{\infty} A_{v, \ell} \left(\frac{-\rho}{3}, \frac{-\rho}{4}, \ldots\right) x^v = \left(\frac{-\rho}{3}\right)x + \left(\frac{-\rho}{4}\right)x^2 + \ldots \right) \ell,$$

we see that the inner sum in (2.32) equals

$$[x^{2r}] (1 + x)^{-\beta} \left(\left(1 + x\right)^{-\rho} - 1 - \left(\frac{-\rho}{1}\right)x - \left(\frac{-\rho}{2}\right)x^2 \right) / x^2 \right)^\ell$$

$$= [x^{2r}] (1 + x)^{-\beta} (S(\rho, x) - 1)^\ell \left(\frac{-\rho}{2}\right)^\ell.$$  

Hence,

$$W_r(\rho, \beta) = r! \left(\frac{-\frac{1}{2}}{r}\right)^{2r} \sum_{\ell=0}^{\frac{1}{2}-r} \left(\frac{-\rho}{\ell}\right)^{-r-\ell} [x^{2r}] (1 + x)^{-\beta} (S(\rho, x) - 1)^\ell \left(\frac{-\rho}{2}\right)^\ell$$

$$= r! \left(\frac{-\frac{1}{2}}{r}\right) \left(\frac{-\rho}{2}\right)^{-r} [x^{2r}] \sum_{\ell=0}^{\frac{1}{2}-r} \left(\frac{-\rho}{\ell}\right)^{-r} (1 + x)^{-\beta} (S(\rho, x) - 1)^\ell$$

$$= r! \left(\frac{-\frac{1}{2}}{r}\right) \left(\frac{-\rho}{2}\right)^{-r} \left[ x^{2r} \right] (1 + x)^{-\beta} S(\rho, x)^{-1/2-r},$$

as we wanted to show.  

Then Theorem 1.2 is the special case of Theorem 3.1 with $\rho = 1/k$ and $\beta = -1/2$. Also by (2.32),

$$Q_r(k) = r! \left(\frac{-\frac{1}{2}}{r}\right)^{2r} \sum_{\ell=0}^{\frac{1}{2}-r} \left(\frac{1}{k} - \frac{1}{\ell}\right)^{-r-\ell} \sum_{v=0}^{2r} \left(\frac{v}{2}\right) A_{2r-v, \ell} \left(\frac{v}{3}, \frac{1}{4}, \ldots\right).$$  

(3.2)

**Lemma 3.2.** We have

$$W_r(\rho, \beta) = \frac{V_r(\rho, \beta)}{\rho^r(\rho + 1)^r},$$

for $V_r(\rho, \beta)$ a polynomial of degree at most $2r$ in $\mathbb{Q}[\rho, \beta]$.  

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Proof. Rewrite the binomial coefficients in (2.32) as
\[
\binom{-\rho}{j+2} = \rho (\rho + 1) (-1)^j w_j \quad \text{for} \quad w_j := \frac{(\rho + 2)(\rho + 3) \cdots (\rho + j + 1)}{(j + 2)!}.
\]
when \(j \geq 0\), (with \(w_0 = 1/2\)). Then by (1.11), (1.12),
\[
A_{v, \ell} \left( \binom{-\rho}{3}, \binom{-\rho}{4}, \ldots \right) = (-1)^v (\rho + 1)^\ell A_{v, \ell}(w_1, w_2, w_3, \ldots),
\]
and inserting this into (2.32) and simplifying shows
\[
W_r(\rho, \beta) = \binom{-\frac{1}{2}}{r} \frac{r!}{\rho^r (\rho + 1)^r} \sum_{\ell=0}^{2r} \left( -\frac{1}{2} - r \right) 2^{r+\ell} \sum_{v=0}^{2r} \binom{-\beta}{v} (-1)^v A_{2r-v, \ell}(w_1, w_2, w_3, \ldots). \tag{3.3}
\]
Since \(w_j\) is a degree \(j\) polynomial in \(\rho\), it follows from (1.12) that \(A_{2r-v, \ell}(w_1, w_2, w_3, \ldots)\) is a degree \(2r-v\) polynomial in \(\rho\), while \((-\beta\choose v)\) is a degree \(v\) polynomial in \(\beta\).

\[\begin{align*}
\text{Corollary 3.3.} \quad & \text{There exist } P_r(x) \text{ in } \mathbb{Q}[x], \text{ of degree at most } 2r, \text{ so that} \\
& Q_r(k) = \frac{P_r(k)}{(k + 1)^r},
\end{align*}\]

With Corollary 3.3, we see that the definition of \(Q_r(k)\) extends from \(k \in \mathbb{Z}_{\geq 1}\) to \(k\) being any number except \(-1\). Examples of \(P_r(x)\) for \(r = 1, 2, 3\) may be seen in (1.7), (1.8). Numerically examining the zeros of \(P_r(x)\) for \(r \leq 20\) we see they are mostly real, with a small number of complex roots slightly to the left of the imaginary axis. By the rational roots test, the only rational roots for \(r \leq 20\) are \(x = 1\) and \(x = -2\).

\[\begin{align*}
\text{Proposition 3.4.} \quad & \text{For } r \geq 2 \text{ the polynomial } P_r(x) \text{ has roots } x = 1 \text{ and } x = -2. \\
& \text{Proof.} \quad \text{When } k \text{ equals } 1 \text{ we have} \\
& Q_r(1) = r! \left( -\frac{1}{r} \right) \left( -\frac{1}{2} \right)^r \left( x^{2r} \right) \left( 1 + x \right)^{1/2} \left( \frac{(1 + x)^{-1} - (1 - x)}{(-1)^x} \right)^{-1/2-r} \\
& = r! \left( -\frac{1}{r} \right) \left( x^{2r} \right) \left( 1 + x \right)^{r+1}
\end{align*}\]

by (1.5) and so \(Q_r(1)\) and \(P_r(1)\) equal 0 for \(r \geq 2\).

When \(k\) equals \(-2\) we have
\[
Q_r(-2) = r! \left( -\frac{1}{r} \right) \left( -\frac{1}{8} \right)^r \left( x^{2r} \right) \left( 1 + x \right)^{1/2} \left( \frac{(1 + x)^{1/2} - 1 - x/2}{-1/8x^2} \right)^{-1/2-r}. \tag{3.4}
\]

Let \(y := (1 + x)^{1/2}\), a series in \(x\) with coefficients \(\binom{1/2}{j}\). Then \(x = y^2 - 1\) and we find that the series to the right of \(\left[ x^{2r} \right]\) in (3.4) equals
\[
y(-8y + 8 + 4x)^{-1/2-r}x^{2r+1} = y(1 + y)^{2r+1}/2^{2r+1}.
\]

Now we claim, for positive integers \(m\), that \((1 + y)^m\), when considered as a series in \(x\), has zero coefficients for \(x^j\) when \(\lfloor m/2 \rfloor + 1 \leq j \leq m - 1\). If we notice that
\[
(1 + y)^m + (1 - y)^m = 2 \sum_{j=0}^{m/2} \binom{m}{2j} y^{2j} = 2 \sum_{j=0}^{m/2} \binom{m}{2j} (1 + x)^j
\]
Proposition 3.6. Let \( m \) be a positive integer.

Then it is clear that \((1 + y)^m\) is the sum of a polynomial in \( x \) of degree at most \( \lfloor m/2 \rfloor \) and a series whose lowest degree term has degree \( m \). This proves our claim.

It follows that \( y(y + 1)^{2r+1} = (y + 1)^{2r+2} - (y + 1)^{2r+1} \) has zero coefficients for \( x^j \) when \( r + 2 \leq j \leq 2r \) and in particular \( Q_r(-2) \) and \( P_r(-2) \) equal 0 for \( r \geq 2 \).

The above claim may also be proved by means of the generalized binomial series \( B_{-1}(x) \) from [GKP94, p. 203]. We have

\[
\left( \frac{1 + y}{2} \right)^m = \left( \frac{1 + (1 + x)^{1/2}}{2} \right)^m = B_{-1}(x/4)^m = 1 + \sum_{j=1}^{\infty} m j (m-j-1) x^j 4^j \tag{3.5}
\]

and the binomial coefficient in (3.5) is zero for \( \lfloor m/2 \rfloor + 1 \leq j \leq m - 1 \). See [O'S, Sects. 8, 9] where (3.5) and generalizations are proved using Lagrange inversion. Note that (3.5) is an identity for formal power series and valid for \( m \) a variable or any complex number.

If \( k = 1 \) then \( p^1(n) \) is the usual partition function \( p(n) \). We have seen with Proposition 3.4 that \( Q_r(1) = 0 \) for \( r \geq 2 \), and therefore Theorem 1.1 implies:

**Theorem 3.5.** Let \( n \) and \( R \) be positive integers. As \( n \to \infty \),

\[
p(n) = \exp \left( \frac{\pi}{4 \sqrt{3}} \frac{2(n-1/24)}{(n-1/24)} \right) \left( 1 - \frac{1}{\pi} \frac{\sqrt{2/3(n-1/24)}}{2(n-1/24)} + O \left( \frac{1}{n^{R/2}} \right) \right), \tag{3.6}
\]

with an implied constant depending only on \( R \).

This of course also follows from the stronger results of Hardy and Ramanujan or Rademacher, taking just the first term in their asymptotic expansions. From [Rad73, p. 278]:

\[
p(n) = \exp \left( \frac{\pi}{4 \sqrt{3}} \frac{2(n-1/24)}{(n-1/24)} \right) \left( 1 - \frac{1}{\pi} \frac{\sqrt{2/3(n-1/24)}}{2(n-1/24)} + O \left( \exp \left( -\frac{\pi}{2} \frac{\sqrt{2/3(n-1/24)}}{2(n-1/24)} \right) \right) \right). \tag{3.7}
\]

In the \( k = -2 \) case, \( Q_r(-2) = 0 \) for \( r \geq 2 \) by Corollary 3.3 and Proposition 3.4. Then it is interesting to speculate that there are similar expansions to (3.6) and (3.7) for a function “\( p^{-2}(n) \)”.

Lastly in this section we give an explicit formula for \( W_r(\rho, \beta) \) that involves only binomial and multinomial coefficients.

**Proposition 3.6.** For \( r \geq 0 \),

\[
W_r(\rho, \beta) = r! \left( \frac{1}{r} \right)^{2r} \sum_{\ell=0}^{2r} \binom{3r + 1/2}{2r - \ell} \left( \frac{2}{r(r+1)} \right)^{r+\ell} \times \sum_{j_1+j_2+j_3=\ell} \binom{-1/2-r}{j_1,j_2,j_3} \binom{-j_3 \rho - \beta}{2r+2\ell-j_2} (-1)^{j_1} \rho^{j_2}.
\]

To find a simpler formula for the De Moivre polynomial in (2.32), we first describe a general technique for dealing with shifted coefficients. Applying the straightforward identity

\[
A_{m,\ell}(a_2, a_3, \ldots) = \sum_{j=0}^{\ell} (-a_1)^{\ell-j} \binom{\ell}{j} A_{m+j, j}(a_1, a_2, \ldots),
\]

\( r \) times shows:
Proposition 3.7. For $r, \ell \geq 0$ we have that $A_{m,\ell}(a_{r+1}, a_{r+2}, \ldots)$ equals
\[
\sum_{j_1 + j_2 + \cdots + j_r + 1 = \ell} \binom{\ell}{j_1, j_2, \ldots, j_r+1} (-a_1)^{j_1} (-a_2)^{j_2} \cdots (-a_r)^{j_r} A_{m+J+rj_r+1,j_r+1}(a_1, a_2, \ldots) \quad (3.8)
\]
where $J$ means $(r-1)j_1 + (r-2)j_2 + \cdots + 1j_{r-1}$ and the summation is over all $j_1, \ldots, j_r+1 \in \mathbb{Z}_{\geq 0}$ with sum $\ell$.

Noting that
\[
A_{m,\ell}\left(\binom{z}{0}, \binom{z}{1}, \binom{z}{2}, \ldots\right) = \binom{\ell z}{m - \ell},
\]
lets us apply Proposition 3.7 with $r = 3$ and $a_j = \left(\frac{-\rho}{j-1}\right)$ to find that $A_{m,\ell}\left(\binom{-\rho}{0}, \binom{-\rho}{1}, \binom{-\rho}{2}, \ldots\right)$ equals
\[
\sum_{j_1 + j_2 + j_3 + j_4 = \ell} \binom{\ell}{j_1, j_2, j_3, j_4} (-1)^{j_1+j_3} \rho^{j_2} \frac{(-\rho)}{2} \binom{-\rho j_4}{m+2j_1+j_2+2j_4}. \quad (3.9)
\]

Proof of Proposition 3.6. Using (3.9) in (2.32) and reordering the summation to use the Chu-Vandermonde identity
\[
\sum_{v=0}^{\infty} \binom{-\beta}{v} \binom{-\rho j_4}{2r - v + 2j_1 + j_2 + 2j_4} = \binom{-\rho j_4 - \beta}{2r + 2j_1 + j_2 + 2j_4},
\]
leads to
\[
\frac{W_r(\rho, \beta)}{r! (-\rho)^{2r}} = \sum_{\ell=0}^{2r} \binom{-\ell - 1}{\ell} \times \sum_{j_1 + j_2 + j_3 + j_4 = \ell} \binom{\ell}{j_1, j_2, j_3, j_4} (-1)^{j_1+j_3} \rho^{j_2} \frac{(-\rho)}{2} \binom{-\rho j_4}{2r + 2j_1 + j_2 + 2j_4}. \quad (3.10)
\]

Note that the upper limit of the sum over $v$ in (2.32) may be increased to $\infty$ because $A_{2r-v,\ell}$ is zero for $v > 2r$. For simplicity, with $z \in \mathbb{C}$ and $j_1 + j_2 + j_3 + j_4 = \ell$, we can write
\[
\binom{z}{j_1, j_2, j_3, j_4} = z(z-1) \cdots (z-\ell+1) = \binom{z}{j_1, j_2, j_3, j_4!}.
\]
Hence (3.10) is
\[
\sum_{j_1 + j_2 + j_3 + j_4 \leq 2r} \binom{-\ell - 1}{j_1, j_2, j_3, j_4} (-1)^{j_1+j_3} \rho^{j_2} \frac{(-\rho)}{2} \binom{-\rho j_4}{2r - j_1 - j_2 + j_4} = \binom{-\rho j_4 - \beta}{2r + 2j_1 + j_2 + 2j_4}.
\]

One further simplification comes from summing over $j_3$:
\[
\sum_{j_3=0}^{2r-j_1-j_2-j_4} \binom{z}{j_1, j_2, j_3, j_4} (-1)^{j_3} = \sum_{j_3=0}^{2r-j_1-j_2-j_4} \binom{z-j_1-j_2-j_4}{j_3} (-1)^{j_3} = \binom{z-2r}{2r-j_1-j_2-j_4}.
\]

This made use of the identity
\[
\sum_{\ell=0}^{m} (-1)^{\ell} \binom{z}{\ell} = (-1)^{m} \binom{z-1}{m} \quad (m \in \mathbb{Z}, z \in \mathbb{C}),
\]
from [GKP94 (5.16)], and the basic relation $\binom{k}{z} = (-1)^{k} \binom{k-z-1}{k}$. Inserting (3.11) with $z = -1/2 - r$ and relabeling $j_4$ as $j_3$ finishes the proof. \qed
4 Further asymptotics

To reduce the notation, set $Q^*_n(k) := Q^*_r(k)/(ka_k)^r$. If we define

$$q_{k,R}(n) := M_k(n) \left(1 + \sum_{r=1}^{R-1} \frac{Q^*_n(k)}{(n + h_k)^{r/(k+1)}}\right),$$

then Theorem 1.1 implies

$$\frac{p^k(n)}{M_k(n)} = \frac{q_{k,R}(n)}{M_k(n)} + O\left(n^{-R/(k+1)}\right),$$

for positive integers $n$ and $R$. Now $q_{k,R}(n)$ makes sense for all values of $n$ except $n = -h_k$ and we can consider $q_{k,R}(n + \delta)$ for real $n$ and $\delta$ with $n \to \infty$ and $\delta$ fixed. Recall the definition of $M_k(n)$ in (1.3).

**Lemma 4.1.** For $n > 0$ large enough we have the expansions

$$\exp\left((\alpha(n + \delta))^{1/(k+1)} - \alpha \cdot n^{1/(k+1)}\right) = \sum_{m=0}^{R-1} \frac{C_1(m, \delta)}{n^{m/(k+1)}} + O\left(\frac{1}{n^{R/(k+1)}}\right),$$

$$\frac{n^{3/2 - 1/(k+1)}}{(n + \delta)^{3/2 - 1/(k+1)}} = \sum_{m=0}^{R-1} \frac{C_2(m, \delta)}{n^{m/(k+1)}} + O\left(\frac{1}{n^{R/(k+1)}}\right),$$

$$\frac{\sum_{r=0}^{R-1} \frac{Q^*_n(r)}{(n + \delta)^{r/(k+1)}}}{(n + \delta)^{1/2 - 1/(k+1)}} = \sum_{m=0}^{R-1} \frac{C_3(m, \delta)}{n^{m/(k+1)}} + O\left(\frac{1}{n^{R/(k+1)}}\right),$$

for implied constants independent of $n$, where

$$C_1(m, \delta) = \sum_{\frac{m}{k+1} \leq \ell \leq \frac{m}{k}} \frac{\alpha^{(k+1)\ell - m} \delta^{\ell}}{(k+1)\ell - m)! \times \frac{A_{\ell, (k+1)\ell - m}\left(\frac{1}{k+1}, \frac{1}{2}, \ldots\right)}{(k+1)\ell - m)! \times \frac{A_{\ell, (k+1)\ell - m}\left(\frac{1}{k+1}, \frac{1}{2}, \ldots\right)}{r}}$$

$$C_2(m, \delta) = \begin{cases} (-\frac{1}{2} + \frac{1}{k+1})\delta^{m/(k+1)} & \text{if } (k+1) | m; \\ 0 & \text{if } (k+1) \nmid m, \end{cases}$$

$$C_3(m, \delta) = \sum_{0 \leq \ell \leq \frac{m}{k+1}} \left(\frac{\ell - m}{k+1}\right) \delta^{\ell} Q^*_m((k+1)\ell)(k).$$

The indices $\ell$ in (4.6) and (4.8) run over all integers satisfying the given inequalities.

**Proof.** Writing $w = n^{-1/(k+1)}$, we find on the left of (4.3) the function

$$\psi(w, \delta) := \frac{1}{w} \left(1 + \delta w^{k+1}\right)^{1/(k+1)} - 1 = \sum_{r=1}^{\infty} \left(\frac{1}{k+1}\right)_r \delta^r w^{(k+1)r-1},$$

which is holomorphic for $w \in \mathbb{C}$ with $|w| < \delta^{1/(k+1)}$. Therefore $\exp(\alpha \cdot \psi(w, \delta))$ is holomorphic on the same domain and, by the usual Taylor theorem with remainder,

$$\exp(\alpha \cdot \psi(w, \delta)) = \sum_{m=0}^{R-1} C_1(m, \delta) w^m + O(|w|^R).$$

This shows that the form of (4.3) is correct. To compute the coefficients in (4.9) write

$$\exp(\alpha \cdot \psi(w, \delta)) = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} w^{-j} \sum_{r=1}^{\infty} \left(\frac{1}{k+1}\right)_r \delta^r w^{(k+1)r-1}$$

$$= \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} w^{-j} \sum_{\ell=1}^{\infty} \delta^{(k+1)\ell} \times A_{\ell, j}\left(\frac{1}{k+1}, \frac{1}{2}, \ldots\right)$$

$$= \sum_{m=0}^{\infty} w^m \sum_{0 \leq j \leq \ell \leq \infty} \frac{\alpha^j}{j!} \delta^\ell A_{\ell, j}\left(\frac{1}{k+1}, \frac{1}{2}, \ldots\right)$$
and (4.6) follows. The other parts are shown similarly using the binomial theorem.

For \( m = 0 \) we have \( C_1(0, \delta) = C_2(0, \delta) = C_3(0, \delta) = 1 \). Fix \( \alpha \) in \( C_1(m, \delta) \) as \((k + 1)a_k\) and define

\[
S_k(r, \delta) := \sum_{j_1 + j_2 + j_3 = r} C_1(j_1, \delta) \cdot C_2(j_2, \delta) \cdot C_3(j_3, \delta).
\] (4.10)

Then using Lemma [4.1] with \( n \) replaced by \( n + h_k \) in (4.1) easily gives:

**Corollary 4.2.** For each fixed \( \delta \), as \( n \to \infty \),

\[
q_{k,R}(n + \delta) = M_k(n) \left( 1 + \sum_{r=1}^{R-1} \frac{S_k(r, \delta)}{(n + h_k)^{r/(k+1)}} + O \left( \frac{1}{n^{R/(k+1)}} \right) \right)
\] (4.11)

for an implied constant depending only on \( k, R \) and \( \delta \).

The term \( M_k(n + \delta) \) has a similar expansion to (4.11), using just \( C_1 \) and \( C_2 \), and in particular we need

\[
M_k(n + \delta)/M_k(n) = 1 + O(1/n^{1/(k+1)})
\] (4.12)

as \( n \to \infty \) for an implied constant depending only on \( k \) and \( \delta \). Replacing \( n \) by \( n - h_k \) in Corollary [4.2] and letting \( \delta = h_k \) means with (4.12) and (4.12) that \( p^k(n) \approx q_{k,R}(n) = q_{k,R}(n - h_k + \delta) \) and we obtain the next result.

**Theorem 4.3.** Let \( n, k \) and \( R \) be positive integers. As \( n \to \infty \),

\[
p^k(n) = b_k \frac{\exp\left((k + 1)a_k \cdot n^{1/(k+1)}\right)}{n^{3/2 - 1/(k+1)}} \left( 1 + \sum_{r=1}^{R-1} \frac{S_k(r, h_k)}{n^{r/(k+1)}} + O \left( \frac{1}{n^{R/(k+1)}} \right) \right)
\] (4.13)

for an implied constant depending only on \( k \) and \( R \).

Note that \( C_1(m, 0) = C_2(m, 0) = \delta_{m,0} \) and \( C_3(m, 0) = Q_m^*(k) \). It follows that \( S_k(r, 0) = Q_r^*(k) \). As \( h_k = 0 \) for \( k \) even, we see that (4.13) agrees with (1.4) in this case. When \( k \) is odd, (4.13) gives a simpler asymptotic expansion for \( p^k(n) \), in terms of \( n \) instead of \( n + h_k \), though this comes at the expense of requiring more complicated coefficients \( S_k(r, h_k) \). Theorem 1 of [TWL19] is given in the form of Theorem 4.3 with the notation \( \gamma_k \), for \( S_k(r, h_k) \). We can compute, (see the proof of Theorem 5.4),

\[
S_k(1, h_k) = Q_1^*(k) = Q_1(k)/(ka_k) \quad (k \geq 2),
\]

agreeing with [TWL19], and also

\[
S_k(2, h_k) = Q_2(k)/(ka_k)^2 + \delta_{k,2} \cdot a_k h_k \quad (k \geq 2),
\]

recalling (1.7).

In the \( k = 1 \) case of Theorem 4.3 a simplification of the coefficients \( \omega_r := S_1(r, -1/2) \) in (4.10) is possible. This is shown next and improves on [OS22] Prop. 7.1.

**Proposition 4.4.** Let \( n \) and \( R \) be positive integers. As \( n \to \infty \),

\[
p(n) = \frac{\exp\left(\pi \sqrt{2n/3}\right)}{4\sqrt{3n}} \left( 1 + \sum_{r=1}^{R-1} \frac{\omega_r}{n^{r/2}} + O \left( \frac{1}{n^{R/2}} \right) \right),
\] (4.14)

with an implied constant depending only on \( R \), where

\[
\omega_r = \frac{1}{4(\sqrt{6})^r} \sum_{k=0}^{(r+1)/2} \binom{r+1}{k} \frac{r+1-k}{(r+1-2k)!} \frac{\pi}{6}^{-2k}.
\] (4.15)
Proof. Let \( c := \pi \sqrt{2/3} \). By Theorem 3.3 we can write

\[
p(n) = \frac{\exp \left( \pi \sqrt{2/(3n)} \right)}{4\sqrt{3n}} \left( \sum_{y=1}^\infty \frac{\exp \left( c \sqrt{\frac{1}{24n}} (1 - y) \right)}{1 - \frac{1}{c \sqrt{n} (1 - 1/24n)}} \right) \left( 1 - \frac{1}{c \sqrt{n} (1 - 1/24n)} \right) + O \left( \frac{1}{n^{1/2}} \right).
\]

For \( z := 1/\sqrt{n} \), the above inner component is

\[
\exp \left( \frac{c}{z} \left( \sqrt{1 - \frac{z^2}{24}} - 1 \right) \right) \left( \frac{1}{1 - z} - \frac{1}{\alpha (1 + x)^{3/2}} \right),
\]

and, as a function of \( z \), this is holomorphic in a neighborhood of \( z = 0 \) with Taylor expansion \( 1 + \sum_{r=1}^{R-1} \omega_r z^r + O(|z|^R) \) for some coefficients \( \omega_r \). This proves (4.14).

To find a formula for \( \omega_r \), let \( \alpha := c \sqrt{n} \) and \( x := -1/(24n) \) and rewrite the inner component as

\[
\exp \left( \alpha (\sqrt{1 + x} - 1) \right) \left( \frac{1}{1 + x} - \frac{1}{\alpha (1 + x)^{3/2}} \right) = \sum_{j=0}^{\infty} \xi_j(\alpha) x^j,
\]

which we will treat as a formal series in \( x \). Integrating (4.16) twice with respect to \( x \) shows

\[
\xi_j(\alpha) = (j + 1)(j + 2) [x^{j+2}] \frac{4}{\alpha^2} \exp \left( \sqrt{1 + x} - 1 \right)
\]

\[
= (j + 1)(j + 2) [x^{j+2}] \frac{4}{\alpha^2} \sum_{k=0}^{\infty} \frac{x^k}{k!} (\sqrt{1 + x} - 1)^k.
\]

Let \( y = \sqrt{1 + x} \) and we will use (3.5) in the form

\[
\left( \frac{1 + y}{2} \right)^m = \left( \frac{1 + (1 + x)^{1/2}}{2} \right)^m = B_{-1}(x/4)^m = \sum_{j=0}^{\infty} \frac{m}{m-j} \binom{m-j}{j} x^j 4^j,
\]

which is valid for all \( m \in \mathbb{C} \) when \( m \notin \mathbb{Z}_{\geq 0} \). Then

\[
\xi_j(\alpha) = (j + 1)(j + 2) \sum_{k=0}^{j+2} \binom{\alpha}{2} [x^{j+2-k}] \left( \frac{2(\sqrt{1 + x} - 1)}{x} \right)^k
\]

\[
= (j + 1)(j + 2) \sum_{k=0}^{j+2} \binom{\alpha}{2} \frac{1}{k!} [x^{j+2-k}] \left( \frac{1+y}{2} \right)^{-k}
\]

\[
= (j + 1) \sum_{k=0}^{j+1} \binom{-1}{k} \alpha^{j-k} \binom{j+k+1}{k},
\]

after simplifying. Since \( \xi_j(\alpha) x^j \) in (4.16) contains terms with factors \( \alpha^{j-k} x^j = \frac{e^{j-k}}{(-24)^j} n^{-(j+k)/2} \),

\[
\omega_r = \sum_{j+k=r, k \leq j+1} \frac{(-1)^k (j+1)}{2^{j+k} (j+1-k)!} \binom{j+k+1}{k} \frac{e^{j-k}}{(-24)^j}
\]

and this reduces to (4.15). \( \square \)

A further notable asymptotic expansion of \( p(n) \) is given by Brassesco and Meyroneinc in [BM20] based on probabilistic methods. The natural parameter in this context is

\[
Y_n := 1 + 2 \left( \frac{2\pi^2}{3} \left( n - \frac{1}{24} \right) + \frac{1}{4} \right)^{1/2},
\]

and we may give a short proof of one of their main results, based on Theorem 3.5 (or 3.7).
Proposition 4.5. [BM20 Prop. 2.2] As \( n \to \infty \) we have

\[
p(n) = \frac{2\pi^2}{3\sqrt{3}} \frac{e^{(Y_n-1)/2}}{Y_n^2} \left( 1 + \sum_{r=1}^{R-1} d_r \frac{1}{Y_n^r} + O\left( \frac{1}{Y_n^R} \right) \right),
\]

with an implied constant depending only on \( R \), for

\[
d_r = \frac{r+1}{(-4)^r} \sum_{k=0}^{r+1} \binom{2r}{k} \frac{(-2)^k}{(r+1-k)!}.
\]

Proof. Substituting \( Y_n/2\sqrt{1-2/Y_n} \) for \( \pi \sqrt{2/3(n-1/24)} \) in Theorem 3.5 produces

\[
p(n) = \frac{2\pi^2}{3\sqrt{3}} \frac{\exp\left( \frac{Y_n}{2}\sqrt{1-2/Y_n} \right)}{Y_n^2(1-2/Y_n)} \left( 1 - \frac{2}{Y_n\sqrt{1-2/Y_n}} + O\left( \frac{1}{Y_n^R} \right) \right)
\]

\[
= \frac{2\pi^2}{3\sqrt{3}} \frac{e^{(Y_n-1)/2}}{Y_n^2} \left( \exp\left( \frac{Y_n}{2}\sqrt{1-2/Y_n} \right) - 2 \right) \left( 1 - \frac{2}{Y_n\sqrt{1-2/Y_n}} \right) + O\left( \frac{1}{Y_n^R} \right).
\]

Write \( x = -2/Y_n \) and the above component

\[
\exp\left( \frac{1+x/2 - \sqrt{1+x}}{x} \right) \left( \frac{1}{1+x} + \frac{x}{(1+x)^{3/2}} \right)
\]

is holomorphic in a neighborhood of \( x = 0 \) with a Taylor expansion \( 1 + \sum_{r=1}^{R-1} (-2)^r d_r x^r + O(|x|^R) \) for some numbers \( d_r \). This proves (4.18) and it only remains to find the formula for \( d_r \).

Integrating (4.20) finds

\[
d_r = (-2)^r (r+1) [x^{r+1}] \exp\left( \frac{1+x/2 - \sqrt{1+x}}{x} \right) 4(1+x/2 + \sqrt{1+x}) \sqrt{1+x}^{r+1} k!
\]

\[
= (-2)^r (r+1) [x^{r+1}] \exp\left( \frac{1+x/2 - \sqrt{1+x}}{x} \right) \frac{4}{k!} \left( \frac{1+x/2 + \sqrt{1+x}}{\sqrt{1+x}} \right)^k.
\]

We used the fact that terms with \( k \leq r+1 \) make the only contributions to the coefficient of \( x^{r+1} \) since the series expansion of the argument of \( \exp \) begins \( x/8 - x^2/16 + \cdots \). Let \( y = \sqrt{1+x} \) and we may again use our techniques from Propositions 3.4 and 3.5. As in (3.4), it is simpler to have an \( x^2 \) denominator and so

\[
d_r = (-2)^r (r+1) \sum_{k=0}^{r+1} \frac{4}{k!} [x^{r+1-k}] \frac{1+x/2 - \sqrt{1+x}}{x^2} \frac{1+x/2 + \sqrt{1+x}}{\sqrt{1+x}} k!
\]

\[
= (-2)^r (r+1) \sum_{k=0}^{r+1} \frac{1}{k!} [x^{r+1-k}] \frac{1+y}{y} \frac{2}{2^{2k}}
\]

\[
= (-2)^r (r+1) \sum_{k=0}^{r+1} \frac{1}{k!} [x^{r+1-k}] \frac{4(1+y/2)}{3^{2k}} \frac{2^{3-2k}}{2^{3-2k}}.
\]

where we integrated (4.21) to get (4.22). Use (3.5) to find the desired coefficient in (4.22) and then simplify to obtain (4.19). \( \square \)

Comparing Propositions 4.4-4.5 and Theorem 3.5 shows that Theorem 3.5 is to be preferred as it is the simpler and more accurate result.
5 Convexity and log-concavity

With the help of Corollary 4.2, we consider in this section the ratio

\[
\frac{p_k(n + \delta)}{p_k(n)} \approx \frac{q_k,R(n + \delta)}{q_k,R(n)} \approx \left( 1 + \sum_{r=1}^{R-1} \frac{S_k(r, \delta)}{(n + h_k)^{r/(k+1)}} \right) \left( 1 + \sum_{r=1}^{R-1} \frac{Q_r^*(k)}{(n + h_k)^{r/(k+1)}} \right)^{-1}.
\]

**Lemma 5.1.** Fix a positive integer \( R \). For \( n > 0 \) large enough we have the expansion

\[
\left( \sum_{r=0}^{R-1} \frac{Q_r^*(k)}{n^{r/(k+1)}} \right)^{-1} = \sum_{m=0}^{R-1} \frac{C_4(m)}{n^{m/(k+1)}} + O\left( \frac{1}{n^{R/(k+1)}} \right),
\]

where

\[
C_4(m) = \sum_{\ell=0}^{m} (-1)^\ell A_m,\ell(Q_1^1(k), Q_2^2(k), Q_3^3(k), \ldots)
\]

and the implied constant in (5.1) depends only on \( k \) and \( R \).

**Proof.** This is similar to the proof of Lemma 4.1. See [O'S22, Prop. 3.2] for the De Moivre polynomial formula for the coefficients of the multiplicative inverse of a power series.

Define

\[
T_k(r, \delta) := \sum_{j_1+j_2+j_3+j_4=r} C_1(j_1, \delta) \cdot C_2(j_2, \delta) \cdot C_3(j_3, \delta) \cdot C_4(j_4).
\]

The next result follows from Corollary 4.2 and Lemma 5.1.

**Corollary 5.2.** Let \( R \) be a positive integer and \( \delta \) a fixed real number. As \( n \to \infty \),

\[
\frac{q_k,R(n + \delta)}{q_k,R(n)} = 1 + \sum_{r=1}^{R-1} \frac{T_k(r, \delta)}{(n + h_k)^{r/(k+1)}} + O\left( \frac{1}{n^{R/(k+1)}} \right)
\]

for an implied constant depending only on \( k \), \( R \) and \( \delta \).

We also have

**Corollary 5.3.** Let \( R \) be a positive integer and \( \delta \) a fixed integer. As \( n \to \infty \),

\[
\frac{p_k(n + \delta)}{p_k(n)} = 1 + \sum_{m=1}^{R-1} \frac{T_k(m, \delta)}{(n + h_k)^{m/(k+1)}} + O\left( \frac{1}{n^{R/(k+1)}} \right),
\]

for an implied constant depending only on \( k \), \( R \) and \( \delta \).

**Proof.** Use (4.2) to show

\[
\frac{p_k(n + \delta)}{p_k(n)} = \frac{M_k(n + \delta)}{M_k(n)} \left( \frac{q_k,R(n + \delta)}{M_k(n + \delta)} + O\left( \frac{1}{n^{R/(k+1)}} \right) \right) \left( \frac{q_k,R(n)}{M_k(n)} + O\left( \frac{1}{n^{R/(k+1)}} \right) \right)^{-1} = \left( \frac{q_k,R(n + \delta)}{M_k(n)} + O\left( \frac{1}{n^{R/(k+1)}} \right) \right) \left( \frac{q_k,R(n)}{M_k(n)} + O\left( \frac{1}{n^{R/(k+1)}} \right) \right)^{-1}.
\]

The error terms simplify to \( O(n^{-R/(k+1)}) \) by (4.1), (4.12) and hence

\[
\frac{p_k(n + \delta)}{p_k(n)} = \left( \frac{q_k,R(n + \delta)}{q_k,R(n)} + O\left( \frac{1}{n^{R/(k+1)}} \right) \right) \left( 1 + O\left( \frac{1}{n^{R/(k+1)}} \right) \right)^{-1} = \frac{q_k,R(n + \delta)}{q_k,R(n)} + O\left( \frac{1}{n^{R/(k+1)}} \right),
\]

with (5.4) now following from Corollary 5.2.
One further definition is required:

\[ F(m) = F_k(m) := -\frac{1}{k+1} \sum_{j=1}^{m-k-1} j Q_j^*(k) C_4(m - k - 1 - j). \]  

(5.5)

**Theorem 5.4.** Let \( k \) and \( \delta \) be integers with \( k \geq 2 \). As \( n \to \infty \)

\[
\frac{p^k(n + \delta)}{p^k(n)} = 1 + \sum_{m=k}^{2k+2} \frac{T_k(m, \delta)}{(n + h_k)m/(k+1)} + O\left(\frac{1}{n(2k+3)/(k+1)}\right)
\]

where the implied constant depends only on \( k \) and \( \delta \), and we have the evaluations

\[
T_k(k, \delta) = a_k \delta,
T_k(k + 1, \delta) = (\frac{1}{k+1} - \frac{3}{2}) \delta,
T_k(m, \delta) = F(m) \delta \quad (k + 2 \leq m \leq 2k - 1),
T_k(2k, \delta) = F(2k) \delta + \frac{1}{2} a_k^2 \delta^2,
T_k(2k + 1, \delta) = F(2k + 1) \delta + a_k(\frac{3}{2(k+1)} - 2) \delta^2,
T_k(2k + 2, \delta) = F(2k + 2) \delta + \left( a_k F(k + 2) + \left(\frac{1}{k+1} - \frac{3}{2}\right)\right) \delta^2,
\]

where \( \frac{1}{6} a_k^3 \delta^3 \) must be added to the expression for \( T_k(2k + 2, \delta) \) if \( k = 2 \).

**Proof:** This proof examines the components of \( T_k(m, \delta) \) in (5.3). Recall (4.6), (with \( \alpha = (k + 1)a_k \)), (4.7), (4.8) and (5.2). We have

\[ C_1(0, \delta) = C_2(0, \delta) = C_3(0, \delta) = C_4(0) = 1. \]

For \( 1 \leq m \leq 2k + 1 \) we have \( C_1(m, \delta) = 0 \) except for

\[ C_1(k, \delta) = a_k \delta, \quad C_1(2k, \delta) = a_k^2 \delta^2 / 2, \quad C_1(2k + 1, \delta) = -k a_k \delta^2 / (2(k + 1)). \]  

(5.6)

Also \( C_1(2k + 2, \delta) = 0 \) except when \( k = 2 \), in which case it is \( a_k^3 \delta^3 / 6 \). For \( 1 \leq m \leq 2k + 2 \) we have \( C_2(m, \delta) = 0 \) except for

\[ C_2(k + 1, \delta) = (\frac{1}{k+1} - \frac{3}{2}) \delta, \quad C_2(2k + 2, \delta) = (\frac{1}{k+1} - \frac{3}{2}) \delta^2. \]  

(5.7)

Next, by (4.8),

\[ C_3(m, \delta) = Q_m^*(k) + \begin{cases} 0 & \text{if } 0 \leq m \leq k + 1, \\ (1 - \frac{m}{k+1}) Q_{m-k-1}^*(k) \delta & \text{if } k + 2 \leq m \leq 2k + 2. \end{cases} \]  

(5.8)

We claim that also

\[
\sum_{j_3 + j_4 = m} C_3(j_3, \delta) \cdot C_4(j_4) = \begin{cases} \delta_{m,0} & \text{if } 0 \leq m \leq k + 1, \\ F(m) \delta & \text{if } k + 2 \leq m \leq 2k + 2. \end{cases}
\]

(5.9)

To see this note that for all \( m \geq 0 \), \( \sum_{j_3 + j_4 = m} Q_j^*(k) \cdot C_4(j_4) = \delta_{m,0} \) since a power series divided by itself is 1. Then (5.8) implies (5.9) when \( m \leq k + 1 \). For \( k + 2 \leq m \leq 2k + 2 \), the left side of (5.9) equals

\[
\sum_{j_3 + j_4 = m} Q_j^*(k) \cdot C_4(j_4) + \sum_{j_3 = k+2}^{m} (1 - \frac{j_3}{k+1}) Q_{j_3-k-1}^*(k) \delta \cdot C_4(m - j_3)
\]

\[
= \delta \sum_{j_3 = k+1}^{m} (1 - \frac{j_3}{k+1}) Q_{j_3-k-1}^*(k) \cdot C_4(m - j_3) = \delta F(m),
\]

and we have established the claim (5.9).

Now \( T_k(m, \delta) \) can be computed for \( 1 \leq m \leq 2k + 2 \) by looking at all possible summands in (5.3) using (5.6), (5.7) and (5.9).

\[ \qed \]
Theorem 5.5. Let $\delta$ be an integer. As $n \to \infty$

$$\frac{p^k(n + \delta)}{p^k(n)} = 1 + \sum_{m=1}^{4} \frac{T_1(m, \delta)}{(n + h_1)^{m/2}} + O\left(\frac{1}{n^{5/2}}\right)$$

where the implied constant depends only on $\delta$ and we have

- $T_1(1, \delta) = a_1 \delta$,
- $T_1(2, \delta) = -\delta + \frac{1}{2} a_1^2 \delta^2$,
- $T_1(3, \delta) = \frac{1}{8a_1^2} \delta - \frac{5}{4} a_1 \delta^2 + \frac{1}{8} a_1^3 \delta^3$,
- $T_1(4, \delta) = \frac{1}{8a_1^2} \delta + \frac{5}{4} \delta^2 - \frac{3}{8} a_1^2 \delta^3 + \frac{1}{8} a_1^4 \delta^4$.

The odd powers of $\delta$ cancel in the next corollary of Theorems 5.4 and 5.5.

Corollary 5.6. Let $k$ and $\delta$ be integers with $k \geq 1$. As $n \to \infty$,

$$\frac{1}{2} \left( \frac{p^k(n + \delta)}{p^k(n)} + \frac{p^k(n - \delta)}{p^k(n)} \right) = 1 + \frac{1}{2} a_k^2 \delta^2 \frac{1}{(n + h_1)^{2-2/(k+1)}} - \frac{(2 - \frac{3}{2(k+1)}) a_k \delta^2}{(n + h_k)^{2-1/(k+1)}} + O\left(\frac{1}{n^2}\right), \quad (5.10)$$

where the implied constant depends only on $k$ and $\delta$.

Since $\frac{1}{2} a_k^2 \delta^2$ in $\text{(5.10)}$ is $> 0$ when $\delta \neq 0$, it follows that

$$2p^k(n) \leq p^k(n + \delta) + p^k(n - \delta) \quad (5.11)$$

for all $n$ sufficiently large. Hence, with $\delta = 1$, $p^k(n)$ is asymptotically convex for each fixed $k \geq 1$ as $n \to \infty$. Conjecture 3.4 of [Ula21] claims

$$2p^k(n) \leq \left( p^k(n + 1) + p^k(n - 1) \right) \left( 1 - n^{-k} \right) \quad (5.12)$$

for large enough $n$ depending on $k \geq 2$. As the second term on the right of $\text{(5.10)}$ is greater than $n^{-2}$ for $n$ large enough, we obtain the stronger estimate $\text{(1.13)}$.

We have the further easy consequence of Theorems 5.4 and 5.5.

Corollary 5.7. Let $k$ and $\delta$ be integers with $k \geq 1$. As $n \to \infty$,

$$\frac{p^k(n + \delta)p^k(n - \delta)}{p^k(n)^2} = 1 - \frac{(1 - \frac{1}{k+1}) a_k \delta^2}{(n + h_k)^{2-1/(k+1)}} + O\left(\frac{1}{n^{2+1/(k+1)}}\right), \quad (5.13)$$

where the implied constant depends only on $k$ and $\delta$.

Since $-(1 - \frac{1}{k+1}) a_k \delta^2$ in $\text{(5.13)}$ is $< 0$ when $\delta \neq 0$, it follows that

$$p^k(n)^2 \geq p^k(n + \delta) \cdot p^k(n - \delta) \quad (5.14)$$

for all $n$ sufficiently large. Hence, with $\delta = 1$, $p^k(n)$ is asymptotically log-concave for each fixed $k \geq 1$ as $n \to \infty$. Conjecture 3.5 of [Ula21] has

$$p^k(n)^2 \geq p^k(n + 1) \cdot p^k(n - 1) \cdot \left( 1 + n^{-k} \right) \quad (5.15)$$

for large enough $n$ depending on $k \geq 2$. Since $-1$ times the second term on the right of $\text{(5.13)}$ is greater than $n^{-2}$ for $n$ large enough, we obtain the improvement $\text{(1.14)}$. A similar result appears in [Ben20, Eq. (28)], based on the main theorem of [Gaf16], though unfortunately it contains an error.
The case \( k = 1 \) and \( \delta = 1 \) of Corollary 5.7 shows
\[
\frac{p(n+1)p(n-1)}{p(n)^2} = 1 - \frac{\pi}{2\sqrt{6}} \left( \frac{1}{n-\frac{1}{2\pi}} \right)^{3/2} + \frac{1}{(n-\frac{1}{2\pi})^2} + O\left( \frac{1}{n^{5/2}} \right),
\]
and this implies that
\[
\frac{p(n+1)p(n-1)}{p(n)^2} \left( 1 + \frac{\pi}{2\sqrt{6}} \frac{1}{n^{3/2}} \right) > 1,
\]
for large enough \( n \). Chen, Wang and Xie proved in [CWX16] that (5.17) is true for \( n \geq 45 \), establishing a conjecture of DeSalvo and Pak.

We lastly note that the terms \( F(m) \) from (5.5) cancel in the proofs of Corollaries 5.6 and 5.7. Our formulas for \( Q_r(k) \) were not needed and so these two corollaries ultimately follow just from Theorem 1.1.

Extending (5.10) and (5.13) to include more terms does require values of \( Q_r(k) \).

### 6 Further connections with the work of Wright

For \( \rho > 0 \) and \( z \in \mathbb{C} \), Wright defined the function
\[
\phi(z) = \phi(\rho, \beta; z) := \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell! \cdot \Gamma(\ell\rho + \beta)}.
\]
This is an entire function of \( z \), and the case \( \rho = 1 \) corresponds to the \( J \)-Bessel function. See [Wri33] for some further properties of \( \phi(\rho, \beta; z) \), including the following integral representation:
\[
\phi(\rho, \beta; z) = \frac{1}{2\pi i} \int_{\mathcal{H}} w^{-\beta} \exp\left( w + \frac{z}{w^\rho} \right) dw,
\]
where \( \mathcal{H} \) is the usual Hankel contour from \( -\infty \), running below the real line, circling the origin in a positive direction and then running above the real line back to \( -\infty \). Then (6.2) follows very simply by substituting into (6.1) Hankel's formula
\[
\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{\mathcal{H}} e^{w} w^{-s} dw \quad (s \in \mathbb{C}).
\]

Now the connection to
\[
I_N := \frac{1}{2\pi i} \int_{U+iU/2}^{U+iU/3} w^{-\beta} \exp\left( w + \frac{N}{w^\rho} \right) dw,
\]
in Proposition 2.5 is clear. Recall that \( U = (\rho N)^{1/(\rho+1)} \) is the saddle-point of \( \exp(w + N/w^\rho) \). We may take the contour in (6.3) to be part of \( \mathcal{H} \), and define \( \mathcal{H}^* \) to be the remaining part, so that
\[
\phi(\rho, \beta; N) = I_N + \frac{1}{2\pi i} \int_{\mathcal{H}^*} w^{-\beta} \exp\left( w + \frac{N}{w^\rho} \right) dw.
\]
As the integral over \( \mathcal{H}^* \) in (6.4) does not include the saddle-point \( U \), we expect it to be relatively small. Proving this next will allow us to show the asymptotic expansion of \( \phi(\rho, \beta; N) \) as \( N \to \infty \).

**Lemma 6.1.** Let \( y \) and \( \rho \) be real with \( |y| \leq 1/2 \) and \( \rho \geq 0 \). Then
\[
\text{Re}\left((1+iy)^{-\rho}\right) \leq 1 - \psi(\rho)y^2 \quad \text{for} \quad \psi(\rho) = \begin{cases} \frac{2\rho}{3} & \text{if} \quad 0 \leq \rho \leq 1, \\ \frac{2\rho}{5} & \text{if} \quad 1 \leq \rho. \end{cases}
\]

**Proof.** We have
\[
\text{Re}\left((1+iy)^{-\rho}\right) \leq (1+y^2)^{-\rho/2} \leq (1-\frac{4}{3}y^2)^{\rho/2} = \sum_{j=0}^{\infty} \binom{\rho/2}{j} (-\frac{4}{3})^j y^{2j}.
\]
Except for the first, the terms in the above series are all \( \leq 0 \) when \( 0 \leq \rho \leq 1 \). We obtain (6.5) in this case by omitting the terms with \( j \geq 2 \). Since \( (1 - \frac{4}{3} y^2)^{\rho/2} \) is decreasing in \( \rho \),
\[
\text{Re}((1 + iy)^{-\rho}) \leq (1 - \frac{4}{3} y^2)^{1/2} \leq 1 - \frac{2}{3} y^2,
\]
for \( \rho \geq 1 \), where the second inequality may be verified by squaring both sides.

**Proposition 6.2.** Fix \( \rho > 0 \) and \( \beta \in \mathbb{C} \). As real \( N \to \infty \),
\[
\frac{1}{2\pi i} \int_{\mathcal{H}^*} w^{-\beta} \exp\left(w + \frac{N}{w^\rho}\right) \, dw \ll |U^{-\beta}| \exp\left((1 + \frac{1}{\rho})U\right) \cdot \exp\left(-C U^{1/3}\right),
\]
with \( C > 0 \) depending only on \( \rho \), and the implied constant depending only on \( \rho \) and \( \beta \).

**Proof.** We can let the top half of \( \mathcal{H}^* \) follow the vertical line from \( U + i U^{2/3} \) to \( U + i U/2 \), the arc \( \frac{\pi}{2} U e^{it} \) for \( \pi/6 \leq t \leq \pi \), and then the real line from \( -\frac{\pi}{2} U \) to \( -\infty \). We will bound the integral on this contour; the contribution from the symmetric bottom half of \( \mathcal{H}^* \) will be the same. For \( w = U + iy \) with \( U^{2/3} \leq y \leq U/2 \), the integrand is
\[
\ll |w|^{-\beta} \exp\left(U + \frac{N}{U^\rho} \left(1 - \psi(\rho) \frac{y^2}{U^2}\right)\right)
\ll |U^{-\beta}| \exp\left((1 + \frac{1}{\rho})U\right) \exp\left(-\frac{1}{\rho} \psi(\rho) U^{1/3}\right).
\]
The integrand on the arc is
\[
\ll |U^{-\beta}| \exp\left(\frac{\pi}{2} U \cos(t) + \frac{N}{U^\rho} \left(\frac{\pi}{2}\right)^{-\rho} \cos(\rho t)\right).
\]
As this is decreasing with \( t \), the same bound (6.7) applies on the arc. Finally, for the horizontal piece,
\[
\frac{1}{2\pi i} \int_{-\infty}^{-\frac{\pi}{2} U} w^{-\beta} \exp\left(w + \frac{N}{w^\rho}\right) \, dw = \frac{1}{2\pi i} \int_{-\frac{\pi}{2} U}^{\frac{\pi}{2} U} x^{-\beta} e^{-\beta \pi} \exp\left(-x + \frac{N}{x^\rho e^{\rho \pi}}\right) \, dx
\ll \int_{U}^{\infty} x^{-\beta} \exp\left(-x + \frac{N}{x^\rho \cos(\rho \pi)}\right) \, dx
\ll \exp\left(-\frac{1}{2} U + \frac{N}{U^\rho}\right) \int_{U}^{\infty} x^{-\beta} e^{-x/2} \, dx
\ll \exp\left((1 + \frac{1}{\rho})U\right) \exp\left(-\frac{1}{2} U\right),
\]
and this is smaller than we need.

So the integral in Proposition 6.2 fits inside the error term of Proposition 2.5 and with (6.4) we have proved the following result, a special case of Theorems 1 and 2 of [Wri35].

**Theorem 6.3.** Fix \( \rho > 0 \), \( \beta \in \mathbb{C} \) and a positive integer \( R \). Then as real \( N \to \infty \),
\[
\phi(\rho, \beta; N) = \frac{U^{1/2 - \beta}}{\sqrt{2\pi (\rho + 1)}} \exp\left((1 + \frac{1}{\rho})U\right) \left(1 + \sum_{r=1}^{R-1} \frac{\rho^r \gamma_r(\rho, \beta)}{U^r} + O\left(\frac{1}{U^R}\right)\right),
\]
for \( U = (\rho N)^{1/(\rho + 1)} \) and an implied constant depending only on \( \rho \), \( \beta \) and \( R \).

For example, Zagier’s function \( H_2(x) \) in [Zag21, p. 14] is \( 2 x^3 \phi(2, 4; x^2) \) and the asymptotics quoted there follow from (6.3), at least for \( x \) real. Wright in fact found the asymptotics of \( \phi(\rho, \beta; z) \) as \( z \in \mathbb{C} \) goes to infinity in any direction. See also [BKPR17] where the asymptotics of \( \phi(-k/(k + 1), 1; z) \) are required in establishing a more detailed version of a conjecture of Andrews on partitions without \( k \) consecutive part sizes; Wright covered the case \(-1 < \rho < 0\) needed there too.

Comparing the identical asymptotics of Theorem 6.3 in the case (2.39) with Theorem 1.1 makes it evident that our main theorem may be restated succinctly with Wright’s function:
Corollary 6.4. Let $k$ be a positive integer and $T$ any positive real. As $n \to \infty$, 
\[ p^k(n) = \frac{\phi \left( \frac{1}{k}, -\frac{1}{2}; \frac{k}{k+1} \left( n + \frac{1}{n^k} \right) \left( 1 + O \left( \frac{1}{n^T} \right) \right) \right)}{(2\pi)^{k/2} (n + \frac{1}{n^k})^{3/2}} \]  \tag{6.9}
for an implied constant depending only on $k$ and $T$.

Theorem 1 of [Wri34] gives a much stronger version of (6.9) with the error replaced by $O(e^{-\alpha_k n^{1/(k+1)}})$ for some $\alpha_k > 0$. This corresponds to taking the first term in Wright’s theory. Theorem 3 of [Wri34] shows that the absolute error can be made $O(e^{-\alpha_k n^{1/(k+1)}})$, for any $\alpha_k > 0$, by taking enough further terms. The results in [Wri34] are shown by uncovering the elaborate structure of $G_k(q)$ near roots of unity. For this see also Schoenfeld [Sch44] and recent work of Zagier [Zag21]. We hope to return to these topics in a future paper.

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