The aim of this paper is to draw attention to an interesting semilinear parabolic equation that arose when describing the chaotic dynamics of a polymer molecule in a liquid. This equation is nonlocal in time and contains a term, called the interaction potential, that depends on the time-integral of the solution over the entire interval of solving the problem. In fact, one needs to know the “future” in order to determine the coefficient in this term, that is, the causality principle is violated. The existence of a weak solution of the initial boundary value problem is proven. The interaction potential satisfies fairly general conditions and can have arbitrary growth at infinity. The uniqueness of this solution is established with restrictions on the length of the considered time interval.

**KEYWORDS**
Initial boundary value problem, nonlocal-in-time parabolic equation, solvability, uniqueness

**MSC CLASSIFICATION**
35K58; 35Q92

1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary $\partial \Omega$. In the space-time cylinder $\Omega_T = \Omega \times (0, T)$, $T \in (0, \infty)$, we consider the following differential equation:

$$\partial_t u - \Delta u + \varphi \left( \int_0^T u(\cdot, s) \, ds \right) u = 0,$$

where $u = u(x,t)$ is an unknown scalar function, $x = (x_1, \ldots, x_n)$ is the vector of the spatial variables in $\mathbb{R}^n$, $t$ is the time variable in the interval $[0, T]$, and $\varphi$ is a scalar function that will be specified below. We assume that the following boundary and initial conditions are satisfied:

$$u(x, t) = 0 \text{ for } x \in \partial \Omega, \; t \in [0, T],$$

$$u(x, 0) = u_0(x) \text{ for } x \in \Omega,$$

where the function $u_0 : \Omega \to \mathbb{R}$ is prescribed.

An interesting feature of this problem is that Equation (1) contains a nonlocal-in-time term that depends on the integral over the whole interval $(0, T)$ on which the problem is being solved. For this reason, Equation (1) is called global-in-time in the title of the paper. There are many works that study problems for parabolic equations with a memory term. This term includes the integral of the solution from the initial to the current time, and it is not difficult to find appropriate works on this subject. The problems with the memory term differ from ours. In fact, we need to know the “future” in order to determine the coefficient in Equation (1). It should be noted that the problem cannot be reduced to known ones by any
transitions. There are papers that study problems, where the future stands in the data (see, e.g.,1-4). The paper5 is devoted to the investigation of a system of equations that contain an integral of the solution over the entire time interval, but this nonlocality is easily eliminated and the equation is reduced to a parabolic equation with a prescribed combination of initial and final data, as in.2 A problem with a global term is encountered also in the population dynamics.5,7 The nonlocality can be in the data of the problem and in the equation as well. Notice that the equation is ultra-parabolic in this case and the role of the second time plays the age of individuals. The global terms contain the integral with respect to the age. In the previous studies,6,7 independent-of-time problems are considered, so the equation becomes parabolic. Besides, the nonlocality is left only in the data.

Our problem appeared when describing the chaotic dynamics of a single polymer molecule or, as it is also called, a polymer chain in an aqueous solution (see8). The time $t$ in Equation (1) is, in fact, the arc length parameter along the chain. The unknown function $u = u(x,t)$ is the density of probability that the $t$-th segment of the chain is at the point $x$. Since each segment of the chain interacts with all other segments through the surrounding fluid, the equation contains an interaction term which includes an integral of $u$ over the entire chain. Equation (1) is simpler than that obtained in8; however, it looks similar and also contains the term with the integral of the solution from 0 to $T$.

In,9 the weak solvability of the problem is proved for the case where $u$ is a positive bounded function and $\varphi$ is the so called Flory-Huggins potential. The positiveness is a natural requirement since $u$ is the density of probability. The Flory-Huggins potential is a convex increasing function that tends to infinity as its argument approaches a certain positive value. Such an equation can appear in other problems as well. It is worthwhile to investigate it with another potential $\varphi$. In this paper, we consider the potential $\varphi$ which is, in general, not convex and not everywhere increasing. Besides that, we do not require that the solution is positive and bounded.

Generally speaking, Equation (1) has features unusual for parabolic equations. First of all, the causality principle is violated. The state of the system depends not only on the past but also on the future. Besides, from a mathematical point of view, the solution of a nonlinear parabolic problem is commonly being constructed locally in time and is extended afterwards. In our case, this procedure is impossible. Finally, as a rule, the local-in-time uniqueness of the solution implies the global one. We cannot prove the uniqueness without restrictions on $T$. However, it is possible that a more skilled author will be able to do this.

In the next section, we define the notion of weak solution of Problem (1)–(3) and formulate Theorem 1, the main result of the paper, that states the weak solvability of the problem. The proof of this result (Section 4) is based on the Tikhonov theorem on the existence of a fixed point of a map $\Psi$. The construction of this map is divided into two standard problems which are considered in Section 3. The mapping $\Psi$ must be weakly continuous. Roughly speaking, we have to show that not only a subsequence but the whole sequence converges weakly.

In Section 5, we present one of possible uniqueness results. We managed to prove the uniqueness of the weak solution of Problem (1)–(3) only for sufficiently small $T$. Even in the case where $\varphi$ is the Flory-Huggins potential, a convex increasing function, and the solution of the problem is a nonnegative bounded function, we are forced to impose a restriction on $T$. Generally speaking, this fact can be explained by physical reasons. Recall that the original problem describes the dynamics of a polymer chain and $T$ is its length. If the chain is too long, it can form knots that significantly affect the chaotic motion of the chain. Nevertheless, from a mathematical point of view, such a situation cannot be considered completely satisfactory. In forthcoming studies, we will try to prove the uniqueness of the solution without restrictions on $T$. Notice that the Cauchy problem for the corresponding ordinary differential equation has a unique solution for all values of $T$. Since we intend to establish only a local in time uniqueness result, it makes no sense to prove it under the most general conditions on the data of the problem. We suppose that the initial value of the solution is a bounded function. This condition is natural for the problem of the polymer chain dynamics. Besides that, we impose an additional restriction on the potential $\varphi$.

## 2 | WEAK STATEMENT OF THE PROBLEM AND MAIN RESULT

At first, we formulate conditions on the potential $\varphi$ which will be fulfilled throughout the paper.

**Assumption 1.** The potential $\varphi : \mathbb{R} \to [0, +\infty)$ is a continuous nonnegative function such that $\varphi(0) = 0$ and $s \mapsto \varphi(s)s$ is a nondecreasing differentiable function whose derivative is bounded on every compact subset of $\mathbb{R}$. 
This assumption admits that the function \( \varphi \) is not convex and not increasing as its argument tends to infinity. Besides that, we do not impose any restrictions on the growth of \( \varphi \) at infinity. Figure 1 shows examples of the possible function \( \varphi \). The functions in the figure are even, but this is not necessary.

We will use the standard Lebesgue and Sobolev spaces \( L^p(\Omega), H^1_0(\Omega), L^2(0, T; H^1_0(\Omega)) \), and \( C(0, T; L^2(\Omega)) \) (see, e.g.,\(^{10,11}\)). As usual, \( H^{-1}(\Omega) \) is the dual space of \( H^1_0(\Omega) \) with respect to the pivot space \( L^2(\Omega) \). The norm and the inner product in \( L^2(\Omega) \) will be denoted by \( \| \cdot \| \) and \( (\cdot, \cdot) \), respectively.

**Definition 1.** Let \( \varphi \) satisfy Assumption 1 and \( u_0 \in L^2(\Omega) \). A function \( u : \Omega_T \to \mathbb{R} \) is said to be a weak solution of Problem (1)–(3) if

1. \( u \in L^2(0, T; H^1_0(\Omega)) \) and \( \varphi(v)u \in L^1(\Omega_T) \), where \( v = \int_0^T u \, dt \);
2. the following integral identity

\[
\int_0^T \int_\Omega (u \partial_t h - \nabla u \cdot \nabla h - \varphi(v)uh) \, dx \, dt + \int_\Omega u_0 h_0 \, dx = 0
\]

holds for an arbitrary smooth in the closure of \( \Omega_T \) function \( h \) such that \( h(x,t) = 0 \) for \( x \in \partial \Omega \) and for \( t = T \). Here, \( h_0 = h_{1=0} \).

The main result of the paper is the following theorem.

**Theorem 1.** If \( u_0 \in L^2(\Omega) \), \( T \) is an arbitrary positive number, and \( \varphi \) satisfies Assumption 1, then Problem (1)-(3) has a weak solution \( u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \) such that

\[
\varphi(v) \in L^2(\Omega), \quad \varphi(v)v \in L^2(\Omega), \quad \varphi(v)u^2 \in L^1(\Omega_T), \quad u \in C(0, T; L^2(\Omega)) \text{,}
\]

where \( v = \int_0^T u \, dt \).

The proof of this result will be given in Sections 3 and 4. It is not difficult to see that the solution of the problem is more smooth than stated in Theorem 1. This fact is a consequence of the embedding theorems and the classical theory of partial differential equations. Really, as it follows from Equation (4) below, \( v \in H^1(\Omega) \cap H^1_0(\Omega) \). This implies that \( \varphi(v) \) is a bounded function in the physically reasonable case \( n = 3 \). Due to the continuity of the potential \( \varphi \), the function \( \varphi(v) \) is also bounded. Thus, \( u \) satisfies the parabolic Equation (1) with a bounded coefficient \( \varphi(v) \). Therefore, \( \partial_t u \) and \( \Delta u \) are in \( L^2(\Omega_{\delta,T}) \), where \( \Omega_{\delta,T} = \Omega \times [\delta, T], \delta > 0 \). If \( u_0 \in H^1_0(\Omega) \), then \( \partial_t u, \Delta u \in L^2(\Omega_T) \). The smoothness of the solution of Problem (1)–(3) could be investigated in more detail, but this question is beyond the goal of the present paper. It is possible also to consider various generalizations of the problem considered. In particular, in the case of a nonhomogeneous Equation (1) with a right-hand side in \( L^2(\Omega_T) \), the assertion of Theorem 1 holds true without changes.

## 3 | AUXILIARY RESULTS

In this section, we consider two intermediate problems. The first problem is elliptic and the second one is parabolic.

### 3.1 | Elliptic problem

For every function \( f : \Omega \to \mathbb{R} \), define a function \( v : \Omega \to \mathbb{R} \) as a solution of the following problem:

\[
-\Delta v + \varphi(v)v + f = 0 \text{ in } \Omega, \quad v_{|\partial\Omega} = 0.
\] (4)

**FIGURE 1** Examples of the potential: \( \varphi(s) = \frac{2|s| \tanh |s|}{\cosh |s|} \) and \( \varphi(s) = 2|s| + 3 \sin |s| \)**
This problem is a result of the integration of Equation (1) with respect to \( t \) from 0 to \( T \). The functions \( v \) and \( f \) correspond to \( \int_0^T u(\cdot, t) \, dt \) and \( u(\cdot, T) - u_0 \), respectively. Problem (4) was already considered in a more general situation, namely, with a nonlinear elliptic operator instead of the Laplace operator (see other studies\textsuperscript{12-14}). For every function \( f \in \mathcal{H}^{-1}(\Omega) \), the existence of the solution \( v \in H^1_0(\Omega) \), such that \( \varphi(v) w \) and \( \varphi(v) v^2 \) are in \( L^1(\Omega) \), was proven. The techniques used were different. In Gossez\textsuperscript{12}, the Orlicz spaces were employed under an additional assumption that the function \( \varphi \) is even. In Hess\textsuperscript{13}, the problem was regularized by a problem of a higher order. The order of the approximate problem depended on the dimension \( n \) and was such that its solution was a bounded function. So, there were no difficulties with the integrability of the term \( \varphi(v) v \). Notice that this is the main difficulty of the problem. In Brézis and Browder\textsuperscript{14}, the function \( \varphi(v) v \) was truncated by constants \( \pm k \) and then \( k \) tended to infinity.

In our case, the function \( f \) is not only in \( \mathcal{H}^{-1}(\Omega) \) but in \( L^2(\Omega) \); therefore, we can expect more from the solution of the problem. We need, in particular, that \( \varphi(v) \in L^2(\Omega) \). For brevity, we introduce the following notation: \( \eta(s) = \varphi(s)s \). A function \( v \in H^1_0(\Omega) \) is said to be a weak solution of Problem (4) if \( \eta(v) \in L^1(\Omega) \) and

\[
(\nabla v, \nabla \psi) + (\eta(v), \psi) + (f, \psi) = 0 \quad \text{for all} \quad \psi \in H^1_0(\Omega) \cap L^\infty(\Omega).
\]

**Lemma 1.** Let Assumption 1 be satisfied. For every \( f \in L^2(\Omega) \), Problem (4) has a unique weak solution \( v \in H^1_0(\Omega) \) such that

1. \( \|\nabla v\| \leq d(\Omega) \|f\| \), where \( d(\Omega) \) is the diameter of the domain \( \Omega \);
2. \( \|\varphi(v) v\| \leq \|f\| \);
3. \( \|\varphi(v)\| \leq C \), where the constant \( C \) depends on \( \|f\| \) and \( \Omega \).

**Proof.** As noted above, the weak solvability of Problem (4) is already known (see, e.g., papers\textsuperscript{12-14}). We present here a simple proof of this fact that is based on the Galerkin method. Let \( \{\psi_k\} \) be the orthonormal basis in \( L^2(\Omega) \) that consists of the eigenfunctions of the Laplace operator \(( - \Delta )\) with homogeneous boundary conditions. If we define the inner product in \( H^1_0(\Omega) \) as \((w_1, w_2)_{H^1_0(\Omega)} = (\nabla w_1, \nabla w_2)\) for \( w_1, w_2 \in H^1_0(\Omega) \), then the set \( \{\psi_k\} \) is an orthogonal basis in this space. It is well known that \( \psi_k \in L^\infty(\Omega) \) for every \( k \in \mathbb{N} \) (see, e.g., Brézis\textsuperscript{11}, Sec. 9.8). Let \( H_k \) be the subspace of \( L^2(\Omega) \) spanned by the basis functions \( \{\psi_1, \ldots, \psi_k\} \). For every \( k \in \mathbb{N} \), denote by \( v_k \) a function from \( H_k \) such that

\[
(\nabla v_k, \nabla \psi) + (\eta(v_k), \psi) + (f, \psi) = 0 \quad \text{for all} \quad \psi \in H_k.
\]

By employing the Brouwer fixed-point theorem, it is not difficult to prove the existence of \( v_k \). Notice that the function \( \eta(v_k) \) is bounded and \( (\eta(v_k), \psi) \) is well-defined. If we take \( \psi = v_k \), then we easily find that

\[
\|\nabla v_k\|^2 + \int_\Omega \varphi(v_k)v_k^2 \, dx \leq \|f\| \|v_k\|, \quad k \in \mathbb{N}.
\]

The positiveness of \( \varphi \) and the Poincaré inequality imply that

\[
\|\nabla v_k\| \leq d(\Omega) \|f\|, \quad k \in \mathbb{N},
\]

where \( d(\Omega) \) is the diameter of the domain \( \Omega \).

Therefore, the sequence \( \{v_k\} \) has a subsequence which converges weakly in \( H^1_0(\Omega) \) and almost everywhere in \( \Omega \) to a function \( v \). We denote this subsequence again by \( \{v_k\} \). Due to the continuity of the function \( \eta \), we obtain that

\[
\eta(v_k) \to \eta(v) \quad \text{almost everywhere in} \ \Omega.
\]

Let us prove that the functions \( \eta(v_k) \) are uniformly integrable. Estimates (7) and (8) imply that \( \int_\Omega \eta(v_k) v_k \, dx \leq C_0 \), where the constant \( C_0 \) is independent of \( k \) and depends only on \( \|f\| \). For every measurable set \( A \subset \Omega \) and every positive
number $M$, we introduce the set $A^k_M = \{x \in A \mid |v_k(x)| \geq M\}$. Then,

$$\int_{A^k_M} \eta(v_k) \, dx \leq \frac{1}{M} \int \eta(v_k) \, v_k \, dx \leq \frac{C_0}{M}.$$  

Since the function $\eta$ is continuous, there exists a constant $\gamma(M)$ such that $|\eta(s)| \leq \gamma(M)$ for $s \in [-M, M]$. In fact, as $\eta$ is nondecreasing, $\gamma(M) = \max\{-\eta(-M), \eta(M)\}$. Thus,

$$\int_{A \setminus A^k_M} |\eta(v_k)| \, dx \leq \gamma(M) \mu(A),$$

where $\mu(A)$ is the Lebesgue measure of the set $A$. These inequalities imply that

$$\int_{A} |\eta(v_k)| \, dx \leq \frac{C_0}{M} + \gamma(M) \mu(A).$$

For arbitrary $\epsilon > 0$, we take $M = \epsilon/(2C_0)$ and $\delta = \epsilon/(2\gamma(M))$. Then, we find that $\int_{A} |\eta(v_k)| \, dx < \epsilon$ for an arbitrary measurable set $A \subset \Omega$ such that $\mu(A) < \delta$. Thus, the uniform integrability of $\eta(v_k)$ is proven.

This fact together with (9) and the Vitali convergence theorem (see, e.g., Makarov and Podkorytov, Sec. 4.8.7) enable us to conclude that $\eta(v) \in L^1(\Omega)$ and $\eta(v_k) \rightarrow \eta(v)$ in $L^1(\Omega)$ as $k \rightarrow \infty$. Now we are able to pass to the limit in (6) as $k \rightarrow \infty$. As a result, we find that $v$ satisfies (5), which means that $v$ is a weak solution of Problem (4). Since the function $\eta$ is nondecreasing, this solution is unique.

Let us prove that $v$ satisfies the estimates stated in the lemma. The first estimate is a direct consequence of (8). In order to prove the second one, for every $m \in \mathbb{N}$, we introduce the truncated function

$$v_m(x) = \begin{cases} m, & v(x) \geq m, \\ v(x), & -m < v(x) < m, \\ -m, & v(x) \leq -m. \end{cases}$$

Let us take $\psi = \eta(v_m)$ in (5). Since $\eta(v) \eta(v_m) \geq \eta^2(v_m)$, we easily find that

$$\int_{\Omega} \eta'(v_m) |\nabla v_m|^2 \, dx + \|\eta(v_m)\| \leq \|f\| \|\eta(v_m)\|.$$  

The sequence $\{\eta^2(v_m)\}$ converges to $\eta^2(v)$ almost everywhere in $\Omega$; therefore, the second estimate of the lemma follows from the Fatou lemma.

Finally, let us prove the third estimate. If $A_1 = \{x \in \Omega \mid |v(x)| \geq 1\}$, then

$$\int_{A_1} \varphi^2(v) \, dx \leq \int_{\Omega} \varphi^2(v) \, v^2 \, dx \leq \|f\|^2.$$  

Since the function $\varphi$ is continuous, there exists a constant $\gamma$ such that $\varphi^2(s) \leq \gamma$ for $s \in [-1, 1]$. Thus,

$$\int_{\Omega \setminus A_1} \varphi^2(v) \, dx \leq \gamma \mu(\Omega).$$

These inequalities imply the required estimate with $C^2 = \|f\|^2 + \gamma \mu(\Omega)$. \qed

We denote by $V$ the mapping from $L^2(\Omega)$ into $H^1_0(\Omega)$ such that $v = V(f)$ is the unique weak solution of Problem (4).

**Lemma 2.** Let $\{f_k\}$ be a sequence in $L^2(\Omega)$ that converges to $f$ weakly in this space. If $v_k = V(f_k)$ and $v = V(f)$, then
1. \( v_k \to v \) in \( H^1_0(\Omega) \) as \( k \to \infty \);
2. \( \varphi(v_k) \to \varphi(v) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \).

Proof. It is not difficult to see that \( v_k \to v \) is the weak solution of the following problem:

\[-\Delta(v_k - v) + \varphi(v_k)v_k - \varphi(v)v + f_k - f = 0 \text{ in } \Omega, \quad (v_k - v)|_{\partial\Omega} = 0.\]

Since the function \( s \mapsto \eta(s) = \varphi(s)s \) is non-decreasing, we obtain that

\[
\int_{\Omega} |\nabla(v_k - v)|^2 \, dx \leq \int_{\Omega} |(f_k - f)(v_k - v)| \, dx \leq \|f_k - f\|_{H^{-1}(\Omega)} \|v_k - v\|_{H^1_0(\Omega)}.
\]

The first assertion of the lemma follows from the fact that \( \|f_k - f\|_{H^{-1}(\Omega)} \to 0 \) as \( k \to \infty \).

Since \( v_k \to v \) in \( H^1_0(\Omega) \) as \( k \to \infty \), the sequence \( \{v_k\} \) converges to \( v \) in measure and, as \( \varphi \) is a continuous function, the sequence \( \{\varphi(v_k)\} \) converges in measure to \( \varphi(v) \). Lemma 1 states that the sequence \( \{\varphi(v_k)\} \) is bounded in \( L^2(\Omega) \); therefore, due to the Vitali convergence theorem (see, e.g., Makarov and Podkorytov\(^{15}\), Sec. 4.8.7), \( \varphi(v_k) \to \varphi(v) \) in \( L^p(\Omega) \) as \( k \to \infty \) for \( p \in [1,2) \). Consequently,

\[
\int_{\Omega} (\varphi(v_k) - \varphi(v)) \, h \, dx \to 0 \text{ as } k \to \infty
\]

for every \( h \in L^\infty(\Omega) \). The density of \( L^\infty(\Omega) \) in \( L^2(\Omega) \) implies the second assertion of the lemma. \( \square \)

The advantage of the lemma just proven is that we have established the convergence results not for a subsequence, but for the entire sequence \( \{V(f_k)\} \). These results will be used for the proof of the weak continuity of the mapping \( \Psi \) in the Tikhonov theorem.

### 3.2 Parabolic problem

We consider the following parabolic problem:

\[
\partial_t u - \Delta u + \zeta u = 0 \text{ in } \Omega_T, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0,
\]

where \( \zeta : \Omega \to \mathbb{R} \) is an independent of \( t \) nonnegative function. We suppose that \( \zeta \in L^2(\Omega) \) and \( u_0 \in L^2(\Omega) \). This problem is standard, and we omit the proof of its unique weak solvability as well as various justifications (see, e.g., Gajewski et al\(^{10}\)). A little trouble is that the function \( \zeta \) is in \( L^2(\Omega) \) only, which can be easily overcome if we consider the solution in \( L^2(0, T; H^1_0(\Omega) \cap L^2(\Omega, \zeta)) \), where \( L^2(\Omega, \zeta) \) is the Hilbert space with the norm \( \|u\|_{L^2(\Omega, \zeta)} = \int_\Omega |u|^2 \, dx \). The weak solution of Problem (10) satisfies the following energy estimate:

\[
\frac{1}{2} \|u(\cdot, s)\|^2 + \int_0^s \|\nabla u(t)\|^2 \, dt + \int_0^s \int_\Omega \zeta(t) u^2 \, dx \, dt \leq \frac{1}{2} \|u_0\|^2
\]

for almost all \( s \in [0, T] \). Besides that, \( \partial_t u \) belongs to the space \( L^2(0, T; (H^1_0(\Omega) \cap L^2(\Omega, \zeta))^*) \), where \( (H^1_0(\Omega) \cap L^2(\Omega, \zeta))^* \) is the conjugate space to \( H^1_0(\Omega) \cap L^2(\Omega, \zeta) \). As a consequence of this fact, we find that \( u \in C(0, T; L^2(\Omega)) \). Thus, the function \( u_T = u|_{t=T} \) is well-defined as an element of \( L^2(\Omega) \) and (11) holds for all \( s \in [0, T] \).

For every nonnegative function \( \zeta \in L^2(\Omega) \), we denote by \( U(\zeta) \) the unique weak solution of problem (10) and by \( U_T(\zeta) \) the function \( U(\zeta)|_{t=T} \). Our goal is to investigate the dependence of \( U_T \) on \( \zeta \).

**Lemma 3.** Let \( u_0 \in L^2(\Omega) \) and a sequence of nonnegative functions \( \{\zeta_k\} \) converges weakly in \( L^2(\Omega) \) to a function \( \zeta \). Then, \( U_T(\zeta_k) \to U_T(\zeta) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \).
Proof. Notice that the lemma asserts the convergence of the entire sequence \( \{U_T(\zeta_k)\} \). For brevity, we denote by \( u_k \) and \( u \) the functions \( U(\zeta_k) \) and \( U(\zeta) \), respectively. Let \( h : \Omega_T \to \mathbb{R} \) be an arbitrary smooth function such that \( h|_{\partial\Omega} = 0 \). As it follows from (11),

\[
\int_0^T \int_\Omega |\nabla u_k|^2 \, dx \, dt \leq \frac{1}{2} ||u_0||^2, \quad k \in \mathbb{N}.
\]

This estimate implies that

\[
\left\| \nabla \int_0^T u_k h \, dt \right\| \leq C, \quad k \in \mathbb{N},
\]

where the constant \( C \) depends, of course, on \( h \). Therefore, the sequence \( \{u_k\} \) has a subsequence \( \{u_{k'}\} \) such that

\[
u_{k'} \to w \text{ weakly in } L^2(0,T;H^1_0(\Omega)),
\]

\[
\int_0^T u_{k'} h \, dt \to \int_0^T w h \, dt \text{ in } L^2(\Omega),
\]

as \( k' \to \infty \), where \( w \) is some function. As a consequence of the second relation, we have that

\[
\int_0^T \int_\Omega \zeta_{k'} u_{k'} h \, dx \, dt \to \int_0^T \int_\Omega \zeta w h \, dx \, dt \quad \text{as } k' \to \infty.
\]

Here, we have used the fact that the functions \( \zeta_k \) do not depend on \( t \). The passage to the limit as \( k' \to \infty \) in the weak formulation of (10) and the uniqueness of the solution of this problem imply that \( w = U(\zeta) \). Besides that, Equation (10) implies that

\[
\int_\Omega (U_T(\zeta_{k'}) - U_T(\zeta)) h(\cdot, T) \, dx = - \int_0^T (\nabla(u_{k'} - u) \cdot \nabla h + (\zeta_{k'} u_{k'} - \zeta u) h) \, dx dt \to 0 \quad \text{as } k' \to \infty.
\]

Since the set of smooth functions is dense in \( L^2(\Omega) \), we conclude that \( U_T(\zeta_{k'}) \to U_T(\zeta) \) weakly in \( L^2(\Omega) \) as \( k' \to \infty \). Thus, we have proven that every subsequence of the sequence \( \{U_T(\zeta_k)\} \) has a subsequence that converges weakly to \( U_T(\zeta) \) in \( L^2(\Omega) \). The uniqueness of the limit yields the assertion of the lemma. \( \square \)

4 | WEAK SOLVABILITY OF THE PROBLEM

In order to prove the weak solvability of Problem (1) to (3) stated in Theorem 1, we employ the Tikhonov fixed-point theorem which states that, for a reflexive separable Banach space \( X \) and a closed convex bounded set \( E \subset X \), if a mapping \( \Psi : E \to E \) is weakly sequentially continuous, then \( \Psi \) has at least one fixed point in \( E \).

We take \( X = L^2(\Omega), E = \{w \in L^2(\Omega) | \|w\| \leq ||u_0|| \} \) and define the mapping \( \Psi \) as follows: for every \( w \in E, \Psi(w) = U_T(\varphi(v)) \), where \( v = V(w - u_0) \). The operators \( V \) and \( U_T \) were introduced in the previous section. If \( u \) is the weak solution of the original problem, then \( \int_0^T u \, dt = V(u_T - u_0) \) and

\[
u_T = U_T \left( \varphi \left( \int_0^T u \, dt \right) \right) = \Psi(u_T).
\]

Thus, the fixed point of the mapping \( \Psi \) is a function that corresponds to \( u_T \). If we know \( u_T \), we define the function \( v = V(u_T - u_0) \) and finally find the weak solution of the original problem \( u = U(\varphi(v)) \).
As it follows from the properties of the operators \( V \) and \( U \), the mapping \( \Psi \) is well-defined on \( L^2(\Omega) \). Besides that, for every nonnegative function \( \zeta \in L^2(\Omega) \), the function \( U(\zeta) \) satisfies (11), which implies that \( \| U_\tau(\zeta) \| \leq \| u_0 \| \). Thus, \( \Psi(E) \subset E \). It remains to prove the weak sequential continuity of \( \Psi \).

Let \( \{w_k\} \) be an arbitrary sequence in \( E \) that converges to \( w \in E \) weakly in \( L^2(\Omega) \). We need to prove that \( \Psi(w_k) \to \Psi(w) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \). It is a simple consequence of the results obtained in Section 3. Due to Lemma 2, \( \varphi(v_k) \to \varphi(v) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \), where \( v_k = V(w_k - u_0) \) and \( v = V(w - u_0) \). In turn, Lemma 3 implies that \( \Psi(w_k) = U_\tau(\varphi(v_k)) \to U_\tau(\varphi(v)) = \Psi(w) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \). Thus, the weak solvability of Problem (1)-(3) is proven.

5 | UNIQUENESS OF THE SOLUTION

In this section, we prove the following uniqueness theorem:

**Theorem 2.** Let Assumption 1 be satisfied. Assume that there exist constants \( K \) and \( M \) such that \( |u_0| \leq K \) almost everywhere in \( \Omega \) and \( |\varphi'(s)| \leq M \) for \( s \in [-KT,KT] \). If \( MKT^2 < 2 \), then the weak solution of Problem (1) to (3) is unique.

**Proof.** Suppose that this problem has two weak solutions \( u_1 \) and \( u_2 \). Denote by \( u \) its difference \( u_1 - u_2 \). Then,

\[
\partial_t u - \Delta u + \varphi(v_1)u_1 - \varphi(v_2)u_2 = 0, \quad u|_{\partial\Omega} = u|_{t=0} = 0,
\]

where \( v_i(x) = \int_0^T u_i(x,t)dt, \) \( i = 1,2 \). Since Equation (1) admits the maximum principle, \( |u_i| \leq K \) and \( |u_2| \leq K \) almost everywhere in \( \Omega_T \) and \( \int_0^T |u_i|dt \) \( \leq KT, i = 1,2 \), almost everywhere in \( \Omega \). Therefore, the equation for \( u \) holds in \( L^2(0,T;H^{-1}(\Omega)) \). The multiplication of this equation by \( u \) and the integration over \( \Omega \) lead to the following equality:

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \int_{\Omega} \| \nabla u \|^2 + \int_{\Omega} \varphi(v_1)u^2 \mathrm{d}x + \int_{\Omega} (\varphi(v_1) - \varphi(v_2)) u_2 u \mathrm{d}x = 0,
\]

which implies that

\[
\frac{1}{2} \| u(\cdot, t) \|^2 + \int_0^t \| \nabla u(\cdot, s) \|^2 \mathrm{d}s \leq MK \int_0^t \| v(\cdot) \| \| u(\cdot, s) \| \mathrm{d}x \mathrm{d}s
\]

for all \( t \in [0,T] \), where \( v = v_1 - v_2 \). Since

\[
\int_0^t \int_{\Omega} |v(x)| |u(x,s)| \mathrm{d}x \mathrm{d}s = \int_0^t \int_{\Omega} |v(x)| |u(x,s)| \mathrm{d}x \mathrm{d}s \leq \| v \| \| u(\cdot, s) \| \mathrm{d}s \leq \int_0^t \| u(\cdot, s) \| \mathrm{d}s \int_0^t \| u(\cdot, s) \| \mathrm{d}s,
\]

we obtain the following inequality:

\[
\| u(\cdot, t) \|^2 \leq 2MK \int_0^t \| u(\cdot, s) \| \mathrm{d}s \int_0^t \| u(\cdot, s) \| \mathrm{d}s,
\]

which holds for all \( t \in [0,T] \). If \( MKT^2 < 2 \), this inequality implies that \( u = 0 \). Really, let us denote \( \xi(t) = \int_0^t \| u(\cdot, s) \| \mathrm{d}s \). Then (12) can be rewritten as follows:

\[
(\xi'(t))^2 \leq 2MK\xi(T)\xi(t), \quad t \in (0, T].
\]

This inequality yields that \( \xi(t) \leq t^2MK\xi(T)/2 \) for all \( t \in [0,T] \) and, in particular, \( \xi(T) \leq T^2MK\xi(T)/2 \). If \( T^2MK < 2 \), then the only solution of this inequality is \( \xi(T) = 0 \), which implies that \( u = 0 \).

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CONFlict of interest
The author declares no potential conflict of interests.

ORCiD
Victor N. Starovoitov https://orcid.org/0000-0002-0392-3180

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