A hypercyclic finite rank perturbation of a unitary operator

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Abstract

A unitary operator $V$ and a rank 2 operator $R$ acting on a Hilbert space $\mathcal{H}$ are constructed such that $V + R$ is hypercyclic. This answers affirmatively a question of Salas whether a finite rank perturbation of a hyponormal operator can be supercyclic.

MSC: 47A16, 37A25
Keywords: Hypercyclic operators, Hyponormal operators, Unitary operators, Finite rank operators

1 Introduction

All vector spaces in this article are assumed to be over the field $\mathbb{C}$ of complex numbers. Symbol $\mathbb{R}$ stands for the field of real numbers, $\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{N}$ is the set of positive integers and $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$. For a subset $A$ of a Banach space $X$, $\text{span}(A)$ stands for the linear span of $A$ and $\overline{\text{span}}(A)$ denotes the closure of $\text{span}(A)$. For a Banach space $X$, $L(X)$ is the Banach algebra of continuous linear operators on $X$. In what follows symbol $\mu$ stands for the normalized Lebesgue measure on $\mathbb{T}$. Instead of $L_1(\mathbb{T}, \mu)$ and $L_2(\mathbb{T}, \mu)$ we simply write $L_1(\mathbb{T})$ and $L_2(\mathbb{T})$. We use the $\langle x, y \rangle$ notation for the scalar product of vectors $x$ and $y$ in a Hilbert space $\mathcal{H}$. A compact subset of a metric space is called perfect if it is non-empty and has no isolated points.

Recall that a continuous linear operator $T$ on a topological vector space $X$ is called hypercyclic if there exists $x \in X$ such that the orbit $\{ T^n x : n \in \mathbb{Z}_+ \}$ is dense in $X$ and $T$ is called supercyclic if there is $x \in X$ for which the projective orbit $\{ \lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{Z}_+ \}$ is dense in $X$. We refer to [2] for additional information on hypercyclicity and supercyclicity. In particular, it is well known that there are no hypercyclic operators on finite dimensional Hausdorff topological vector spaces and there are no supercyclic operators on Hausdorff topological vector spaces of finite dimension $\geq 2$. Thus when speaking of hypercyclicity or supercyclicity of operators, we always assume that the underlying space is infinite dimensional.

Recall also that a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$ is called hyponormal if $\|Tx\| \geq \|T^* x\|$ for any $x \in \mathcal{H}$, where $T^*$ is the Hilbert space adjoint of $T$. Equivalently, $T$ is hyponormal if and only if $T^* T - TT^* \succeq 0$.

Hilden and Wallen [7] observed that there are no supercyclic normal operators. Kitai [8] proved that there are no hypercyclic hyponormal operators. A result simultaneously stronger than those of Hilden and Wallen and of Kitai was obtained by Bourdon [5], who demonstrated that a hyponormal operator can not be supercyclic. This motivated Salas [3] to raise the following question.

**Question S.** Can a finite rank perturbation of a hyponormal operator be supercyclic?

The above question is also reproduced in [6]. It is worth noting that Bayart and Matheron [3] constructed a unitary operator on a Hilbert space $\mathcal{H}$, which is supercyclic on $\mathcal{H}$ endowed with the weak topology (=weakly supercyclic). In the present paper we answer Question S affirmatively.

**Theorem 1.1.** There exist a unitary operator $V$ and a bounded linear operator $R$ of rank at most 2 acting on a Hilbert space $\mathcal{H}$ such that $T = V + R$ is hypercyclic.

The idea of the proof is the following. We consider the unitary multiplication operator $U$ on $L_2(\mathbb{T})$, $Uf(z) = zf(z)$, and construct $h, g \in L_2(\mathbb{T})$ and a closed linear subspace $\mathcal{K}$ of $L_2(\mathbb{T})$ such that $\mathcal{K}$ is invariant for $U + S$, where $Sf = \langle f, g \rangle h$, the restriction $T \in L(\mathcal{K})$ of $U + S$ to $\mathcal{K}$ is hypercyclic.
and $T$ admits the decomposition $T = V + R$, with $V \in L(K)$ being unitary and $R \in L(K)$ having rank at most 2. We prove the hypercyclicity of $T$ by means of applying a criterion of Bayart and Grivaux [1] in terms of unimodular point spectrum. We construct $g, h$ and $K$ with the required properties using a result of Belov [4] on the distribution of values of functions defined by lacunary trigonometric series. Note also that the described scheme immediately produces a hypercyclic rank 1 perturbation of a Hilbert space contraction. Indeed, if $P$ is the orthoprojection of $L_2(\mathbb{T})$ onto $K$, then $T = (PU)|_K + (PS)|_K$, $(PU)|_K$ is a contraction on $K$ and $(PS)|_K$ is a rank 1 operator on $K$. Thus we have the following corollary, which is of independent interest.

**Corollary 1.2.** There exist a contraction $A$ and a bounded rank 1 linear operator $S$ acting on a Hilbert space $\mathcal{H}$ such that $T = A + S$ is hypercyclic.

The following lemma summarizes the properties of $h$ and $g$ we need in order to run the described procedure. This is the key ingredient in the proof of Theorem 1.1.

**Lemma 1.3.** There exist $h, g \in L_2(\mathbb{T})$ and a perfect compact set $K \subset \mathbb{T}$ such that

- $\lambda \mapsto h_\lambda$ is a continuous map from $K$ to $L_2(\mathbb{T})$, where $h_\lambda(z) = \frac{h(z)}{\lambda - z}$;  
  \begin{equation}
  (1.1)
  \end{equation}

- $\langle h, g_1 \rangle = 0$, $\langle h_\lambda, g \rangle = 1$ and $\langle h_\lambda, g_1 \rangle = \lambda^{-1}$ for each $\lambda \in K$, where $g_1(z) = zg(z)$.  
  \begin{equation}
  (1.2)
  \end{equation}

In Section 2 Theorem 1.1 is reduced to Lemma 1.3. The latter is proved in Section 3. We discuss further possibilities in Section 4.

## 2 Reduction of Theorem 1.1 to Lemma 1.3

In this section we assume Lemma 1.3 to be true and prove Theorem 1.1. We start by deriving the following lemma from Lemma 1.3.

**Lemma 2.1.** There exist a Hilbert space $\mathcal{H}$, a unitary operator $U \in L(\mathcal{H})$, $h \in \mathcal{H}$, $S \in L(\mathcal{H})$ with $S(\mathcal{H}) = \text{span} \{ h \}$, a perfect compact set $K \subseteq \mathbb{T}$ and a continuous map $\lambda \mapsto h_\lambda$ from $K$ to $\mathcal{H} \setminus \{ 0 \}$ such that

- $(U + S)h_\lambda = \lambda h_\lambda$ for each $\lambda \in K$;  
  \begin{equation}
  (2.1)
  \end{equation}

- either $h, U^{-1}h \in K$ or $h, U^{-1}h \notin K$, where $K = \text{span} \{ h_\lambda : \lambda \in K \}$.
  \begin{equation}
  (2.2)
  \end{equation}

**Proof.** Let $\mathcal{H} = L_2(\mathbb{T})$ and $U \in L(\mathcal{H})$, $Uf(z) = zf(z)$. Obviously, $U$ is unitary. Let also $K \subset \mathbb{T}$ and $h, g \in \mathcal{H}$ be the perfect compact set and the functions provided by Lemma 1.3. For $\lambda \in K$, let $h_\lambda(z) = \frac{h(z)}{\lambda - z}$. According to (1.1), $h_\lambda \in \mathcal{H}$ for each $\lambda \in K$ and the map $\lambda \mapsto h_\lambda$ from $K$ to $\mathcal{H}$ is continuous. By (1.2), $h_\lambda \neq 0$ for every $\lambda \in K$. Define $S \in L(\mathcal{H})$ by the formula $Sf = (f, g)h$. By (1.2), $g \neq 0$ and therefore $S(\mathcal{H}) = \text{span} \{ h \}$. It remains to verify (2.1) and (2.2).

Taking into account the specific shape of $h_\lambda$ and $U$, one can easily see that

$$U h_\lambda = \lambda h_\lambda - h \quad \text{and} \quad U^{-1} h_\lambda = \lambda^{-1} h_\lambda + \lambda^{-1} h \quad \text{for each} \ \lambda \in K.$$  
  \begin{equation}
  (2.3)
  \end{equation}

By (1.2), $\langle h_\lambda, g \rangle = 1$ and therefore $Sh_\lambda = h$ for every $\lambda \in K$. Thus the first equality in (2.3) implies that $(U + S)h_\lambda = \lambda h_\lambda$ for each $\lambda \in K$. That is, (2.1) is satisfied. In order to prove (2.2) it suffices to verify that $h \in K$ if and only if $U^{-1}h \in K$.

First, assume that $h \in K$. Then there exists a sequence $\left\{ \sum_{j=1}^{k_n} c_{j,n} h_{\lambda_{j,n}} \right\}_{n \in \mathbb{N}}$ with $\lambda_{j,n} \in K$ and $c_{j,n} \in \mathbb{C}$ such that

$$\sum_{j=1}^{k_n} c_{j,n} h_{\lambda_{j,n}} \to h \quad \text{in} \ \mathcal{H} \quad \text{as} \quad n \to \infty.$$  
  \begin{equation}
  (2.4)
  \end{equation}

By (1.2), $(h_\lambda, U g) = \lambda^{-1}$ for any $\lambda \in K$ and $(h, U g) = 0$. Using these equalities and taking the scalar product with $U g$ in (2.4), we obtain

$$\sum_{j=1}^{k_n} \frac{c_{j,n}}{\lambda_{j,n}} \to 0 \quad \text{as} \quad n \to \infty.$$  
  \begin{equation}
  (2.5)
  \end{equation}
Applying $U^{-1}$ to (2.4), we get
\[ \sum_{j=1}^{k_n} c_{j,n} U^{-1} h_{\lambda_j,n} \rightarrow U^{-1} h \text{ in } H \text{ as } n \rightarrow \infty. \]
Using the second equality in (2.3), we obtain
\[ \left( \sum_{j=1}^{k_n} \frac{c_{j,n}}{\lambda_j,n} \right) U^{-1} h + \sum_{j=1}^{k_n} \frac{c_{j,n}}{\lambda_j,n} h_{\lambda_j,n} \rightarrow U^{-1} h \text{ in } H \text{ as } n \rightarrow \infty. \]
By (2.5), \( \sum_{j=1}^{k_n} \frac{c_{j,n}}{\lambda_j,n} h_{\lambda_j,n} \rightarrow U^{-1} h \text{ as } n \rightarrow \infty. \) Hence $U^{-1} h \in K$. Thus $h \in K$ implies $U^{-1} h \in K$.
Now we assume that $U^{-1} h \in K$. Then there exists a sequence \( \left\{ \sum_{j=1}^{k_n} c_{j,n} h_{\lambda_j,n} \right\}_{n \in \mathbb{N}} \) with $\lambda_j,n \in K$ and $c_{j,n} \in \mathbb{C}$ such that
\[ \sum_{j=1}^{k_n} c_{j,n} h_{\lambda_j,n} \rightarrow U^{-1} h \text{ in } H \text{ as } n \rightarrow \infty. \] (2.6)
By (1.2), $\langle h, \lambda \rangle = 1$ for any $\lambda \in K$ and $\langle U^{-1} h, g \rangle = \langle h, U g \rangle = 0$. Using these equalities and taking the scalar product with $g$ in (2.3), we obtain
\[ \sum_{j=1}^{k_n} c_{j,n} \rightarrow 0 \text{ as } n \rightarrow \infty. \] (2.7)
Applying $U$ to (2.6), we get \( \sum_{j=1}^{k_n} c_{j,n} U h_{\lambda_j,n} \rightarrow h \text{ in } H \text{ as } n \rightarrow \infty. \) Using the first equality in (2.3), we see that
\[ -\left( \sum_{j=1}^{k_n} c_{j,n} \right) h + \sum_{j=1}^{k_n} \lambda_{j,n} c_{j,n} h_{\lambda_j,n} \rightarrow h \text{ in } H \text{ as } n \rightarrow \infty. \]
By (2.7), \( \sum_{j=1}^{k_n} \lambda_{j,n} c_{j,n} h_{\lambda_j,n} \rightarrow h \text{ as } n \rightarrow \infty. \) Hence $h \in K$. Thus $U^{-1} h \in K$ implies that $h \in K$.

We also need the following criterion of hypercyclicity by Bayart and Grivaux [1].

**Theorem BG.** Let $X$ be a separable infinite dimensional Banach space, $T \in L(X)$ and assume that there exists a continuous Borel probability measure $\nu$ on the unit circle $\mathbb{T}$ such that for each Borel set $A \subseteq \mathbb{T}$ with $\nu(A) = 1$, the space
\[ X_A = \text{span} \left( \bigcup_{z \in A} \ker (T - zI) \right) \] (2.8)
is dense in $X$. Then $T$ is hypercyclic.

**Corollary 2.2.** Let $X$ be a separable infinite dimensional Banach space, $T \in L(X)$ and assume that there exists a perfect compact set $K \subseteq \mathbb{T}$ and a continuous map $\lambda \mapsto x_\lambda$ from $K$ to $X$ such that $Tx_\lambda = \lambda x_\lambda$ for each $\lambda \in K$ and $\text{span} \{ x_\lambda : \lambda \in K \}$ is dense in $X$. Then $T$ is hypercyclic.

**Proof.** Since $K$ is a perfect compact subset of $\mathbb{T}$, we can pick a continuous Borel probability measure $\nu$ on the unit circle $\mathbb{T}$ such that $K$ is exactly the support of $\nu$. Let now $A \subseteq \mathbb{T}$ be a Borel measurable set such that $\nu(A) = 1$. Since $K$ is the support of $\nu$, $B = A \cap K$ is dense in $K$. Clearly $x_\lambda \in X_A$ for each $\lambda \in K$ and $\text{span} \{ x_\lambda : \lambda \in K \}$ is dense in $X$. Hence $X_A$ is dense in $X$. By Theorem BG, $T$ is hypercyclic.

**Lemma 2.3.** Let $U$ be a unitary operator acting on a Hilbert space $H$ and $K$, $K_+$ and $K_-$ be closed linear subspaces of $H$ such that $K \subseteq K_+ \cap K_-$, dim $K_+/K$ = dim $K_-/K$ = 1, $U(K) \subseteq K_+$, $U^{-1}(K) \subseteq K_-$, $U(K) \not\subseteq K$ and $U^{-1}(K) \not\subseteq K$. Then there exist a unitary operator $V \in L(K)$ and a bounded linear operator $A : K \rightarrow H$ of rank at most 1 such that $U|_K = V + A$.

**Proof.** Let $X = U^{-1}(K) \cap K$ and $Y = U(K) \cap K$. Clearly $X$ and $Y$ are closed linear subspaces of $K$. Moreover, $U(X) = K \cap U(K) = Y$.
Since $U(K) \subseteq K_+$ and dim $K_+/K$ = 1, we have dim $K/X \leq 1$. Similarly, since $U^{-1}(K) \subseteq K_-$ and dim $K_-/K$ = 1, we see that dim $K/Y \leq 1$. On the other hand, the relations $U(K) \not\subseteq K$
and $U^{-1}(K) \not\subseteq K$ imply that $X \neq K$ and $Y \neq K$. Thus $\dim K/X = \dim K/Y = 1$. Now we can pick $x, y \in K$ such that $\|x\| = \|y\| = 1$, $x$ is orthogonal to $X$, $y$ is orthogonal to $Y$ and $K = X \oplus \text{span}\{x\} = Y \oplus \text{span}\{y\}$. Define the operator $V : K \to H$ be the formula

$$V u = U u + \langle u, x \rangle (y - U x), \quad u \in K.$$ 

It is easy to see that $U|_X = V|_X$ and $V x = y$. Since $U(X) = Y$, $V$ maps $X$ isometrically onto $Y$. Thus $V x = y$, $x$ spans the orthocomplement of $X$ and $y$ spans the orthocomplement of $Y$, we see that $V$ maps $K$ onto itself isometrically. Thus $V \in L(K)$ is a unitary operator. It remains to notice that according to the last display, $U|_K = V + A$, where the bounded linear operator $A : K \to H$ is given by the formula $A u = \langle u, x \rangle (y - U x)$ and therefore has rank at most $1$.

**Lemma 2.4.** Let $U$ be a unitary operator acting on a Hilbert space $H$, $h \in H$, $S \in L(H)$ with $S(H) \subseteq \text{span}\{h\}$ and $K$ be a closed linear subspace of $H$ invariant for the operator $U + S$. Assume also that $(U + S)(K)$ is dense in $K$ and either $h, U^{-1} h \in K$ or $h, U^{-1} h \not\in K$. Then the restriction $T \in L(K)$ of $U + S$ to $K$ can be expressed as $T = V + R$, where $V \in L(K)$ is a unitary operator on $K$ and $R \in L(K)$ has rank at most $2$.

**Proof.** If $x \in K$, then $U x = T x - S x \in K_+ = \text{span}(K \cup \{h\})$. Thus $U(K) \subseteq K_+$. Applying $U^{-1}$ to the equality $U x = T x - S x$, we obtain $U^{-1} T x = x + U^{-1} S x \in K_- = \text{span}(K \cup \{U^{-1} h\})$ for each $x \in K$. Since $T$ has dense range, $U^{-1}(K) \subseteq K_-$. If $h \in K$ and $U^{-1} h \in K$, then $K_+ = K_- = K$ and therefore $K$ is an invariant subspace for $U$ and $U^{-1}$. Hence the restriction $V \in L(K)$ of $U$ to $K$ is unitary and $T = V + R$ with $R = S|_K$ being of rank at most $1$. If $S|_K = 0$, then $T$ is the restriction of $U$ to $K$ and therefore $T$ is an isometry. Since $T$ also has dense range, $T$ is unitary. Thus $T$ has the required shape with $V = T$ and $R = 0$.

It remains to consider the case $h \notin K$, $U^{-1} h \notin K$ and $S|_K \neq 0$. Since $h \notin K$ and $U^{-1} h \notin K$, $K$ is a closed hyperplane in $K_+$ and in $K_-$. Since $S|_K \neq 0$, $S(H) \subseteq \text{span}\{h\}$ and $h, U^{-1} h \notin K$, the equalities $U x = T x - S x$ and $U^{-1} T x = x + U^{-1} S x$ for $x \in K$ imply that $U(K) \not\subseteq K$ and $U^{-1}(K) \not\subseteq K$. Thus all conditions of Lemma 2.3 are satisfied. By Lemma 2.3 there is a unitary operator $V \in L(K)$ and a bounded linear operator $A : K \to H$ of rank at most $1$ such that $U|_K = V + A$. Thus $T = V + R$, where $R = A + S|_K$. Clearly $R = T - V$ takes values in $K$ and has rank at most $2$ as a sum of two operators $A$ and $S|_K$ from $K$ to $H$ of rank at most $1$.

### 2.1 Proof of Theorem 1.1 modulo Lemma 1.3

Lemma 2.1 guarantees the existence of a unitary operator $U$ acting on a Hilbert space $H$, $h \in H$, $S \in L(H)$ with $S(H) \subseteq \text{span}\{h\}$, a perfect compact subset $K$ of $T$ and a continuous map $\lambda \mapsto h_\lambda$ from $K$ to $H \setminus \{0\}$ such that (2.1) and (2.2) are satisfied.

Let $K$ be the space defined in (2.2). According to (2.1), $K$ is invariant for $U + S$. Let $T \in L(K)$ be the restriction of $U + S$ to $K$. By (2.1), $T h_\lambda = \lambda h_\lambda$ and therefore $h_\lambda$ are linearly independent for $\lambda \in K$. By definition of $K$, $\text{span}\{h_\lambda : \lambda \in K\}$ is a dense subspace of $K$. Thus $K$ is separable and infinite dimensional. Corollary 2.2 implies that $T$ is hypercyclic.

On the other hand, the equalities $T h_\lambda = \lambda h_\lambda$ imply that $\text{span}\{h_\lambda : \lambda \in K\}$ is contained in $T(K)$ and therefore $T(K)$ is dense in $K$. Then (2.2) and Lemma 2.4 imply that $T$ is a sum of a unitary operator and an operator of rank at most $2$ as required in Theorem 1.1.

### 3 Lemma 1.3: preparation and proof

To make the idea of the proof of Lemma 1.3 more transparent, we note that the scalar product of the functions $f_1, f_2 \in L_2(T)$ can be written in terms of a contour integral: $\langle f_1, f_2 \rangle = \frac{1}{2\pi i} \oint_T \frac{f_1(z) \overline{f_2(z)}}{z} dz$. Thus condition (1.2) reads as

$$\oint_T \frac{h(w) \overline{g(w)}}{w^2} dw = 0, \quad \oint_T \frac{h(w) g(w)}{z - w} dw = 2\pi i \quad \text{and} \quad \oint_T \frac{h(w) \overline{g(w)}}{z - w} dw = \frac{2\pi i}{z} \quad \text{for} \quad z \in K.$$
Assuming that the function \( \psi(z) = \frac{h(z)g(z)}{2\pi iz} \) is continuous and vanishes on \( K \), the above display can be rewritten as
\[
\int_T \frac{\psi(w)}{w} \, dw = 0, \quad \int_T \frac{\psi(w) - \psi(z)}{z - w} \, dw = 1 \quad \text{and} \quad \int_T \frac{\psi(w) - \psi(z)}{(z - w)^2} \, dw = z^{-1} \quad \text{for } z \in K.
\]

We prove Lemma 3.3 by constructing \( K \) and an appropriate function \( \psi \) and then splitting it into a product to recover \( h \) and \( g \).

3.1 Auxiliary results

The next few lemmas certainly represent known facts. We state them in a convenient for our purposes form, different from the one usually found in the literature. For the sake of completeness we sketch their proofs. For a subset \( A \) of a metric space \( (M, d) \), the symbol \( \text{dist}(x, A) \) stands for the distance from \( x \in M \) to \( A \): \( \text{dist}(x, A) = \inf_{y \in A} d(x, y) \). Speaking of \( \mathbb{T} \), we always assume that it carries the metric inherited from \( \mathbb{C} \).

Lemma 3.1. Let \( F \) be an uncountable closed subset of \( \mathbb{T} \). Then there exists a perfect compact set \( K \subset F \) such that \( f_\alpha \in L_2(\mathbb{T}) \) for any \( \alpha \in (0, 1/2) \), where \( f_\alpha(z) = (\text{dist}(z, K))^{-\alpha} \).

The above lemma immediately follows from the next result.

Lemma 3.2. Let \([a, b]\) be a bounded closed interval in \( \mathbb{R} \) and \( F \) be an uncountable closed subset of \([a, b]\). Then there exists a perfect compact set \( K \subset F \) such that
\[
\int_a^b (\text{dist}(x, K))^{-\alpha} \lambda(dx) < \infty, \quad \text{for each } \alpha < 1, \tag{3.1}
\]
where \( \lambda \) is the Lebesgue measure on the real line.

Proof. For a subset \( A \) of the real line, we say that \( x \in \mathbb{R} \) is a left accumulation point for \( A \) if \((x - \varepsilon, x) \cap A \) is uncountable for any \( \varepsilon > 0 \). Similarly \( x \) is a right accumulation point for \( A \) if \((x, x + \varepsilon) \cap A \) is uncountable for any \( \varepsilon > 0 \). It is a well-known fact and an easy exercise that for any uncountable subset \( A \) of \( \mathbb{R} \), all points of \( A \) except for countably many are left and right accumulation points of \( A \). We construct \( K \) by means of a procedure similar to the one used to construct the standard Cantor set. For each \( n \in \mathbb{N} \), let \( \Omega_n = \{0, 1\}^n \) be endowed with the lexicographical ordering: 
\( \varepsilon < \varepsilon' \) if and only if \( \sum_{j=1}^{n} \varepsilon_j 2^{n-j} < \sum_{j=1}^{n} \varepsilon'_j 2^{n-j} \). Using the fact that all points of \( F \), except for countably many, are left and right accumulation points for \( F \), we can easily construct (inductively with respect to \( n \)) elements \( a^n_\varepsilon, b^n_\varepsilon \in F \) for \( n \in \mathbb{N} \) and \( \varepsilon \in \Omega_n \) such that:
\[
\begin{align*}
 a^{n+1}_{\varepsilon,0} &= a^n_\varepsilon & \text{and} & & b^{n+1}_{\varepsilon,1} = b^n_\varepsilon \quad \text{for any } n \in \mathbb{N} \text{ and } \varepsilon \in \Omega_n; \tag{3.2} \\
 a^n_\varepsilon &< b^n_\varepsilon < a^n_{\varepsilon'} < b^n_{\varepsilon'} \quad \text{for } n \in \mathbb{N}, \varepsilon, \varepsilon' \in \Omega_n, \varepsilon < \varepsilon'; \tag{3.3} \\
 b^n_\varepsilon - a^n_\varepsilon &< \frac{1}{n!} \quad \text{for any } n \in \mathbb{N} \text{ and } \varepsilon \in \Omega_n; \tag{3.4} \\
 a^n_\varepsilon &\text{ is a right accumulation point for } F \text{ for any } n \in \mathbb{N} \text{ and } \varepsilon \in \Omega_n; \tag{3.5} \\
 b^n_\varepsilon &\text{ is a left accumulation point for } F \text{ for any } n \in \mathbb{N} \text{ and } \varepsilon \in \Omega_n. \tag{3.6}
\end{align*}
\]

We do not really need conditions (3.3) and (3.4) in what follows. They are included in order to enable us to run the inductive procedure. Now we can define
\[
K = \bigcap_{n=1}^{\infty} \bigcup_{\varepsilon \in \Omega_n} [a^n_\varepsilon, b^n_\varepsilon]. \tag{3.7}
\]

Compactness and non-emptiness of \( K \) are obvious. Actually, \( K \) is homeomorphic to \( \{0, 1\}^\mathbb{N} \) with the 2-element space \( \{0, 1\} \) carrying the discrete topology (=homeomorphic to the standard Cantor set). Indeed, the map from \( \{0, 1\}^\mathbb{N} \) to \( K \), which sends a 0–1 sequence \( \{\varepsilon_1, \varepsilon_2, \ldots\} \) to the unique common
point of the nested sequence \([a^n_{\varepsilon_1, \ldots, \varepsilon_n}, b^n_{\varepsilon_1, \ldots, \varepsilon_n}]\) of closed intervals is a homeomorphism. Thus \(K\) is perfect. The above observations show also that the set \(A = \{a^n_\varepsilon : n \in \mathbb{N}, \varepsilon \in \Omega_n\}\) is dense in \(K\). Since \(A \subset F\) and \(F\) is closed, \(K \subset F\). It remains to show that (3.1) is satisfied. According to (3.4),

\[
\lambda \left( \bigcup_{\varepsilon \in \Omega_n} [a^n_\varepsilon, b^n_\varepsilon] \right) < \frac{2^n}{n!} \to 0 \quad \text{as} \quad n \to \infty.
\]

By (3.7), \(\lambda(K) = 0\). Clearly \([a, b] \setminus K\) is the union of disjoint open intervals \(I_0 = (a, a_1^0), I_1 = (b_1^1, b)\) and \(J^n_j = (b^n_j, a^n_{j+1})\) for \(n \in \mathbb{N}\) and \(1 \leq j \leq 2^n - 1\), where \(\Omega_n = \{\varepsilon^1, \ldots, \varepsilon^{2^n}\}, \varepsilon^1 < \ldots < \varepsilon^{2^n}\). Condition (3.4) and the fact that each \(J^n_j\) is contained in one of the intervals of the shape \([a^n_{\varepsilon-1}, b^n_{\varepsilon+1}]\) implies that the length \(\lambda(J^n_j)\) satisfies \(\lambda(J^n_j) < \frac{1}{(n-1)!}\) for \(n \geq 2\). Fix \(\alpha < 1\). Direct calculation shows that the function \(\dist(x, K)^{-\alpha}\) is integrable on \(I_0, I_1\) and each of \(J^n_j\) and

\[
\int_{J^n_j} (\dist(x, K))^{-\alpha} \lambda(dx) = 2 \int_0^{\lambda(J^n_j)/2} t^{-\alpha} dt = \frac{2^n(\lambda(J^n_j))^{1-\alpha}}{1-\alpha}.
\]

(3.8)

Since \(\lambda(K) = 0\), we see that (3.1) is equivalent to

\[
\sum_{n=2}^{\infty} \sum_{j=1}^{2^n-1} \int_{J^n_j} (\dist(x, K))^{-\alpha} \lambda(dx) < \infty.
\]

Since \(\lambda(J^n_j) < \frac{1}{(n-1)!}\), from (3.8) it follows that convergence of the above series reduces to convergence of \(\sum_{n=2}^{\infty} \frac{2^n}{((n-1)!)^{1-\alpha}}\). Thus (3.1) is satisfied.

For \(0 < \alpha \leq 1\), the symbol \(H_\alpha(T)\) stands for the space of functions \(f : T \to \mathbb{C}\) satisfying the Hölder condition with the exponent \(\alpha\). That is, \(f \in H_\alpha(T)\) if and only if there is \(C > 0\) such that \(|f(z) - f(w)| \leq C|z - w|^\alpha\) for all \(z, w \in T\). The next lemma provides a formula, which is a variant of the integral formula for the conjugate function. It is, of course, true under much weaker restrictions on the function involved.

**Lemma 3.3.** Let \(\{a_n\}_{n \in \mathbb{Z}}\) be a sequence of complex numbers such that \(\sum_{n=-\infty}^{\infty} |a_n| < \infty\) and the function \(f : T \to \mathbb{C}, f(z) = \sum_{n=-\infty}^{\infty} a_n z^n\) belongs to \(H_\alpha(T)\) for some \(\alpha \in (0, 1]\). Then for each \(z \in T\), the function \(w \mapsto \frac{w f(w) - f(z)}{z - w}\) is Lebesgue integrable and

\[
\int_T \frac{w f(w) - f(z)}{z - w} \mu(dw) = f_-(z), \quad \text{where} \quad f_-(z) = \sum_{n=-\infty}^{-1} a_n z^n.
\]

(3.9)

**Proof.** Since \(f \in H_\alpha(T)\), there exists \(C > 0\) such that \(|f(z) - f(w)| \leq C|z - w|^\alpha\) for any \(z, w \in T\). For fixed \(z \in T\), let \(f_z(w) = \frac{w f(w) - f(z)}{z - w}\). Then \(|f_z(w)| \leq C|z - w|^{\alpha-1}\). It immediately follows that \(f_z \in L_1(T)\). It remains to verify (3.9). First, observe that (3.9) is equivalent to

\[
\frac{1}{2\pi i} \int_T \frac{f(w) - f(z)}{z - w} dw = f_-(z),
\]

(3.10)

where the contour \(T\) is encircled counterclockwise. Using the Cauchy formula, one easily show that for any \(n \in \mathbb{Z}\),

\[
\frac{1}{2\pi i} \int_T \frac{w^n - z^n}{z - w} dw = \begin{cases} 0 & \text{if } n \geq 0, \\ z^n & \text{if } n < 0. \end{cases}
\]

It follows that (3.10) and therefore (3.9) is satisfied for \(f\) being a trigonometric polynomial (= a Laurent polynomial). Now consider the sequence \(\{p_n\}_{n \in \mathbb{N}}\) of Fejér sums for \(f\):

\[
p_n(z) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) a_j z^j.
\]
Clearly \( \{p_n\} \) converges to \( f \) uniformly on \( \mathbb{T} \) as \( n \to \infty \). On the other hand, since \( p_n \) is the convolution of \( f \) with the \( n^{th} \) Fejér kernel \( \Pi_n \) and the latter is positive and has integral 1, we immediately have \(|p_n(z) - p_n(w)| \leq C|z - w|^{\alpha} \) for any \( z, w \in \mathbb{T} \) and any \( n \in \mathbb{N} \) (the continuity modulus of any Fejér sum of a continuous function on \( \mathbb{T} \) does not exceed the continuity modulus of the function itself). Hence for each \( w \in \mathbb{T} \), \( w \neq z \), we have

\[
\frac{w(p_n(w) - p_n(z))}{z-w} \to \frac{w(f(w) - f(z))}{z-w} \quad \text{and} \quad \left| \frac{w(p_n(w) - p_n(z))}{z-w} \right| \leq C|z - w|^{\alpha - 1} \quad \text{for any } n \in \mathbb{N}.
\]

Applying the Lebesgue dominated convergence theorem and the fact that \( (3.9) \) is true for trigonometric polynomials, we obtain

\[
\int_{\mathbb{T}} \frac{w(f(w) - f(z))}{z-w} \mu(dw) = \lim_{n \to \infty} \int_{\mathbb{T}} \frac{w(p_n(w) - p_n(z))}{z-w} \mu(dw) = \lim_{n \to \infty} (p_n)_-(z) = f_-(z). \quad \square
\]

**Lemma 3.4.** Let \( a \in \mathbb{R} \) and \( b \in \mathbb{N} \) be such that \( b > a > 1 \) and \( f : \mathbb{T} \to \mathbb{C} \) be defined by the formula

\[
f(z) = \sum_{n=1}^{\infty} a^{-n} z^b^n.
\]

Then \( f \in H_\alpha(\mathbb{T}) \), where \( \alpha = \log_b a \).

**Proof.** Let \( z, w \in \mathbb{T} \), \( z \neq w \). Pick \( m \in \mathbb{N} \) such that \( b^{-m} \leq \frac{|z-w|}{2} \leq b^{1-m} \). Clearly \(|f(z) - f(w)| \leq \sum_{j=1}^{\infty} a^{-j} |z^b^j - w^b^j| \). Using the estimate \(|z^b^j - w^b^j| \leq 2 \) for \( j \geq m \) and the estimate \(|z^b^j - w^b^j| \leq b^j |z - w| \) for \( j < m \), we obtain

\[
|f(z) - f(w)| \leq |z-w| \sum_{j=1}^{m-1} a^{-j} b^j + 2 \sum_{j=m}^{\infty} a^{-j} \leq \frac{a|z-w|}{b-a} (b/a)^m + \frac{2a}{a-1} a^{-m}.
\]

Since \( b^{-m} \leq \frac{|z-w|}{2} \leq b^{1-m} \), we have \( a^{-m} = (b^{-m})^\alpha \leq 2^{-\alpha} |z - w|^\alpha \) and \( (b/a)^m = (b^{-m})^{\alpha-1} \leq (2b)^{1-\alpha} |z - w|^{\alpha-1} \). Hence, according to the above display, \(|f(z) - f(w)| \leq C|z - w|^\alpha \) with \( C = \frac{a^{2-\alpha}}{b-a} + \frac{2a(2b)^{1-\alpha}}{a-1} \). Thus \( f \in H_\alpha(\mathbb{T}) \), as required. \( \square \)

**Lemma 3.5.** Let \( \alpha \in (1/4, 1] \), \( f \in H_\alpha(\mathbb{T}) \), \( K \subset \mathbb{T} \) be a non-empty compact set and \( f(z) = 0 \) for every \( z \in K \). Then for each \( z \in K \), the function \( f_z(w) = \frac{f(w)(\text{dist}(w,K))^\alpha}{z-w} \) belongs to \( L_2(\mathbb{T}) \). Moreover the map \( z \mapsto f_z \) from \( K \) to \( L_2(\mathbb{T}) \) is continuous.

**Proof.** Since \( f \in H_\alpha(\mathbb{T}) \), there is \( C > 0 \) such that \(|f(z) - f(w)| \leq C|z - w|^\alpha \) for any \( z, w \in \mathbb{T} \).

Let \( z \in K \). Since \( f(z) = 0 \), we have \(|f(w)| \leq C|z - w|^\alpha \) for each \( w \in \mathbb{T} \). Moreover, \( \text{dist}(w,K) \leq |w - z| \) for each \( w \in \mathbb{T} \). Thus \(|f_z(w)| \leq C|z - w|^{2\alpha-1} \) for any \( w \in \mathbb{T} \). Since \( \alpha > \frac{1}{4} \) it follows that \( f_z \in L_2(\mathbb{T}) \) for any \( z \in K \).

It remains to show that the map \( z \mapsto f_z \) from \( K \) to \( L_2(\mathbb{T}) \) is continuous. Clearly it is enough to show that there is \( c(\alpha) > 0 \) such that \(|f_z - f_s|^2 \leq c(\alpha)|z-s|^{4\alpha-1} \) for any \( z, s \in K \). In order to get rid of the dead weight of constants, we temporarily assume the following notation. We write \( A \ll B \) if there is a constant \( c \) depending on \( \alpha \) only such that \( A \leq cB \). Thus we are going to show that \(|f_z - f_s|^2 \ll |z-s|^{4\alpha-1} \). Let \( z, s \in K \), \( z \neq s \). Since

\[
|f_z(w) - f_s(w)| = \frac{|z-s||f(w)|}{|z-w||s-w|} (\text{dist}(w,K))^\alpha,
\]

\( \text{dist}(w,K) \leq \min\{|w - z|, |w - s|\} \) and \(|f(w)| \leq C(\min\{|w - z|, |w - s|\})^\alpha \),

we see that

\[
|f_z(w) - f_s(w)| \ll |z-s| (\min\{|w - z|, |w - s|\})^{2\alpha}.
\]

Hence

\[
||f_z - f_s||^2 \ll |z-s|^2 \int_{\mathbb{T}} \frac{(\min\{|w - z|, |w - s|\})^{4\alpha}}{|z-w|^2|s-w|^2} \mu(dw).
\]
As for any two distinct points in the unit circle, for \( z \) and \( s \) we can find \( a, b \in \mathbb{R} \) such that \( 0 < b \leq \frac{\pi}{2} \) and \( \{z, s\} = \{e^{i(a+b)}, e^{i(a-b)}\} \). Clearly \( b \ll |z-s| \ll b \). Using this notation, the last display and straightforward symmetry considerations, we get

\[
\|f_z - f_s\|^2 \ll b^2 \int_0^\pi \frac{|e^{it} - e^{ib}|^4|e^{it} - e^{-ib}|^2}{|e^{it} - e^{-ib}|^2} dt.
\]

Since \( |t - b| \ll |e^{it} - e^{ib}| \ll |t - b| \) and \( |t + b| \ll |e^{it} - e^{-ib}| \ll |t + b| \) for \( t \in [0, \pi] \), we have

\[
\|f_z - f_s\|^2 \ll b^2 \int_0^\pi \frac{|t - b|^{4\alpha - 2}}{|t + b|^2} dt.
\]

We split the integration interval \([0, \pi]\) into the union of \([0, 2b]\) and \([2b, \pi]\). Since \( |t + b|^2 \ll b^2 \) for \( 0 \leq t \leq 2b \) and \( \frac{|t-b|^{4\alpha-2}}{|t+b|^2} \ll t^{4\alpha-4} \) for \( 2b \leq t \leq \pi \), we get

\[
b^2 \int_0^{2b} \frac{|t-b|^{4\alpha-2}}{|t+b|^2} dt \ll \int_0^{2b} \frac{|t-b|^{4\alpha-2}}{|t+b|^2} dt \ll b^{4\alpha-1} \text{ and } b^2 \int_{2b}^{\pi} \frac{|t-b|^{4\alpha-2}}{|t+b|^2} dt \ll b^2 \int_{2b}^{\pi} t^{4\alpha-4} dt \ll b^{4\alpha-1}.
\]

By the last two displays, \( \|f_z - f_s\|^2 \ll b^{4\alpha-1} \ll |z-s|^{4\alpha-1} \), which completes the proof. \( \square \)

The following Theorem is due to Belov [4, Corollary 3.1].

**Theorem B.** Let \( \alpha, \beta > 0, \lambda > 2, M \geq 0, \{\lambda_n\}_{n \in \mathbb{N}} \) be a sequence of positive numbers, \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of complex numbers and \( g : \mathbb{R} \rightarrow \mathbb{C} \) be such that

\[
|g(x) - g(y)| \leq M|x-y| \text{ for any } x, y \in \mathbb{R}, \quad \sum_{n=1}^{\infty} |a_n| < \infty, \quad \alpha(1+\beta) \leq 1,
\]

\[
\frac{\lambda_{m+1}}{\lambda_m} \geq \lambda, \quad |a_m| \leq \beta \sum_{n=m+1}^{\infty} |a_n| \quad \text{and} \quad 2\pi \frac{\lambda^{-1}}{\lambda^2} \left(M + \sum_{n=1}^{m} |a_n| \lambda_n\right) \leq \alpha |a_{m+1}| \lambda_{m+1} \quad \text{for each } m \in \mathbb{N}.
\]

Assume also that \( x_0 \in \mathbb{R} \), \( \varphi : \mathbb{R} \rightarrow \mathbb{C} \) is defined by the formula

\[
\varphi(x) = g(x) + \sum_{n=1}^{\infty} a_n e^{i\lambda_n x} \quad \text{and} \quad I = [x_0 - \Delta, x_0 + \Delta], \quad \text{where} \quad \Delta = \frac{2\pi \lambda}{(\lambda^2 - 1)\lambda}.
\]

Then \( \varphi^{-1}(w) \cap I \) is uncountable for any \( w \in \mathbb{C} \) satisfying

\[
\frac{\theta}{1+\beta} \sum_{n=1}^{\infty} |a_n| \leq |g(x_0) - w| \leq (1-\alpha) \sum_{n=1}^{\infty} |a_n|.
\]

**Remark 3.6.** The main point in [4] is to find \( \varphi : \mathbb{R} \rightarrow \mathbb{C} \) defined by an absolutely convergent lacunary trigonometric series with the continuity modulus as small as possible and with \( \varphi(\mathbb{R}) \) having non-empty interior in \( \mathbb{C} \). The latter means that \( \varphi \) defines a Peano curve. Belov’s construction allows not only to ensure that certain complex numbers belong to \( \varphi(\mathbb{R}) \) but also that they are attained by \( \varphi \) uncountably many times. We take an advantage of the latter property.

### 3.2 Proof of Lemma 1.3

Consider the functions

\[
\gamma, \psi : \mathbb{T} \rightarrow \mathbb{C}, \quad \gamma(z) = \sum_{n=1}^{\infty} 8^{1-n} z^{2^nz} \quad \text{and} \quad \psi(z) = \gamma(z) + \gamma(z^{-1}) = \sum_{n=1}^{\infty} 8^{1-n} (z^{2^n} + z^{-2^n}). \quad (3.11)
\]

Since \( \log_{2^6} 8 = 1/3 \), Lemma 3.4 implies that \( \gamma \in H_{1/3}(\mathbb{T}) \). Hence \( \psi \in H_{1/3}(\mathbb{T}) \). If \( \varphi : \mathbb{R} \rightarrow \mathbb{C} \) is defined by the formula

\[
\varphi(x) = \gamma(e^{2\pi ix}) = \sum_{n=1}^{\infty} 8^{1-n} e^{2\pi 2^nx},
\]
then \( \varphi \) is 2\( \pi \)-periodic and has the shape exactly as in Theorem B with \( g = 0 \), \( a_n = 8^{1-n} \) and \( \lambda_n = 2\pi 2^{\beta n} \). Now we put \( M = 0 \), \( \lambda = 2^\beta \), \( \beta = 7 \) and \( \alpha = 1/8 \). It is straightforward to verify that all conditions of Theorem B are satisfied. Since \( \sum\limits_{n=1}^{\infty} |a_n| = (1 - \alpha) \sum\limits_{n=1}^{\infty} |a_n| = 1 \), Theorem B implies that \( \varphi^{-1}(w) \) is uncountable if \( |w| = 1 \). Hence \( \gamma^{-1}(w) \) is uncountable for each \( w \in \mathbb{T} \). In particular, the closed set \( F = \{z \in \mathbb{T} : \gamma(z) = i\} \) is uncountable. (3.12)

By (3.12) and Lemma 3.1 there is a perfect compact set to verify (1.2). First, from (3.14) and (3.13) it follows that for each \( \varphi \) and (1.2) are satisfied for the just specified \( \psi \) and \( h \). Obviously, \( h \in L_2(\mathbb{T}) \), where \( g(z) = -iz^{-1} \text{dist}(z, K)^{-1/3} \). (3.13)

and \( \psi \) is defined in (3.11). In order to prove Lemma 3.4, it suffices to verify that conditions (1.1) and (1.2) are satisfied for the just specified \( K, h \) and \( g \).

First, observe that \( \gamma(z) = i \) and therefore \( \gamma(z^{-1}) = \gamma(Tz) = \gamma(z) = -i \) for \( z \in F \). Thus using (3.11) and the inclusion \( K \subseteq F \), we get

\[
\gamma(z) = i, \quad \gamma(z^{-1}) = -i \quad \text{and} \quad \psi(z) = 0 \quad \text{for each} \ z \in K.
\] (3.15)

Since \( \psi \in H_{1/3}(\mathbb{T}) \), (3.14), (3.15) and Lemma 3.6 imply that \( h_z \in L_2(\mathbb{T}) \) for each \( z \in K \), where \( h_z(w) = \frac{h(w)}{1 - \langle z, w \rangle} \) and the map \( z \mapsto h_z \) from \( K \) to \( L_2(\mathbb{T}) \) is continuous. Thus (1.1) is satisfied. It remains to verify (1.2). First, from (3.14) and (3.13) it follows that for each \( z \in K \),

\[
\langle h_z, g \rangle = \int_{\mathbb{T}} h_z(w)g(w) \mu(dw) = i \int_{\mathbb{T}} \frac{\psi(w) \text{dist}(w, K)^{1/3}}{z-w} \text{dist}(w, K)^{-1/3} \mu(dw) = i \int_{\mathbb{T}} \frac{w \psi(w)}{z-w} \mu(dw).
\]

Applying Lemma 3.3, we see that \( \langle h_z, g \rangle = i \psi_-(z) \). According to (3.11), \( \psi_-(z) = \gamma(z^{-1}) \). By (3.15), \( \gamma(z^{-1}) = -i \) and therefore \( \langle h_z, g \rangle = i \psi_-(z) = i(-i) = 1 \). Thus

\[
\langle h_z, g \rangle = 1 \quad \text{for each} \ z \in K.
\] (3.16)

Next, let \( g_1 \in L_2(\mathbb{T}) \) be defined by the formula

\[
g_1(z) = zg(z) = -idist(z, K)^{-1/3}.
\] (3.17)

Then using (3.14) and (3.17), we obtain

\[
\langle h_z, g_1 \rangle = \int_{\mathbb{T}} h_z(w)g_1(w) \mu(dw) = i \int_{\mathbb{T}} \frac{\psi(w) \text{dist}(w, K)^{1/3}}{z-w} \text{dist}(w, K)^{-1/3} \mu(dw) = i \int_{\mathbb{T}} \frac{\psi(w)}{z-w} \mu(dw).
\]

Applying Lemma 3.3, we see that \( \langle h_z, g_1 \rangle = i(\psi_0)_-(z) \), where \( \psi_0(z) = z^{-1} \psi(z) \). Using (3.11), we have \( (\psi_0)_-(z) = z^{-1} \gamma(z^{-1}) \). By (3.15), \( \gamma(z^{-1}) = -i \) and therefore \( \langle h_z, g_1 \rangle = iz^{-1} \psi_-(z) = i(-i)z^{-1} = z^{-1} \). Thus

\[
\langle h_z, g_1 \rangle = z^{-1} \quad \text{for each} \ z \in K.
\] (3.18)

Finally, from (3.14) and (3.17) it follows that

\[
\langle h, g_1 \rangle = \int_{\mathbb{T}} h(w)g_1(w) \mu(dw) = i \int_{\mathbb{T}} \psi(w) \mu(dw) = i \langle \psi, 1 \rangle,
\]

where \( 1 \) is the constant 1 function. On the other hand, looking at the shape (3.11) of the Fourier series of \( \psi \), we immediately see that \( \langle \psi, 1 \rangle = 0 \). Hence \( \langle h, g_1 \rangle = 0 \), which together with (3.16) and (3.18) implies (1.2). The proof of Lemma 1.3 is complete and so is the proof of Theorem 1.1.
4 Concluding remarks

If we replace finite rank perturbations by compact perturbations, Question S becomes relatively easy. Namely, an operator of the shape $I + K$ can be hypercyclic [10], where $K$ is a compact operator on a separable infinite dimensional Hilbert space. Moreover, $K$ may be chosen to be nuclear. On the other hand, an operator of the shape $I + K$ with $K$ being of finite rank, can not be cyclic.

Theorem [11] naturally gives rise to the following question.

**Question 4.1.** Does there exist a hypercyclic rank 1 perturbation of a unitary operator?

It is worth noting that the above proof of Theorem [11] provides a hypercyclic rank 1 perturbation of a unitary operator if we can construct $K$, $h$ and $g$ as in Lemma [3] with the additional property that $h \in \text{span} \{h_\lambda : \lambda \in K\}$. This additional requirement seems to be difficult to achieve.

Recall that a bounded linear operator $T$ on a Banach space $X$ is called mixing if for any two non-empty open sets $U, V \subseteq X$, $T^n(U) \cap V \neq \emptyset$ for all sufficiently large $n \in \mathbb{N}$. Equivalently $T$ is mixing if and only if for any infinite set $A \subset \mathbb{N}$, there exists $x = x(A) \in X$ such that $\{T^n x : x \in A\}$ is dense in $X$. Thus mixing condition is a strong form of hypercyclicity. The following question seems to be natural and interesting.

**Question 4.2.** Does there exist a mixing finite rank perturbation of a hyponormal operator?

**Acknowledgements.** The author would like to thank the referee for helpful comments.

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