Reducing stabilizer circuits without the symplectic group

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Abstract

We start by studying the subgroup structures underlying stabilizer circuits. Then we apply our results to provide two normal forms for stabilizer circuits. These forms are computed by induction using simple conjugation rules in the Clifford group and our algorithms do not rely on a special decomposition in the symplectic group. The first normal form has shape CX-CZ-P-Z-X-H-CZ-P-H, where CX (resp. CZ) denotes a layer of CNOT (resp. CZ) gates, P a layer of phase gates, X (resp. Z) a layer of Pauli-X (resp. Pauli-Z ) gates. Then we replace most of the CZ gates by CNOT gates to obtain a second normal form of type P-CX-CZ-CX-Z-X-H-CZ-CX-P-H. In this second form, both CZ layers have depth 1 and together contain therefore at most $n$ CZ gates. We also consider normal forms for stabilizer states and graph states. Finally we carry out a few tests on classical and quantum computers in order to show experimentally the utility of these normal forms to reduce the gate count of a stabilizer circuit.

1 Introduction

In Quantum Computation, any unitary operation can be approximated to arbitrary accuracy using CNOT gates together with Hadamard, Phase, and $\pi/8$ gates (see Figure 2 for a definition of these gates and [9, Section 4.5.3] for a proof of this result). Therefore, this set of gates is often called the standard set of universal gates. When we restrict this set to Hadamard, Phase and CNOT gates, we obtain the set of Clifford gates. The Pauli group $E_n$ is the group generated by the Pauli gates acting on $n$ qubits (see Figure 1) and the normalizer of the Pauli Group in the unitary group $U_{2^n}$ is called the Clifford group. In his PhD thesis [6, Section 5.8], Gottesman gave a constructive proof of the fact that the Clifford gates generate the Clifford group, up to a global phase. He also introduced the stabilizer formalism [9, Section 685 Avenue de l’Université, 76800 Saint-Étienne-du-Rouvray. France.]

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10.5.1], which turned out to be is a very efficient tool to analyze quantum error-
correction codes [6] and, more generally, to describe unitary dynamics [9, Section
10.5.2]. Indeed, the Gottesman-Knill theorem asserts that a stabilizer circuit (i.e. a
quantum circuit consisting only of Clifford gates) can be simulated efficiently on a
classical computer (see [9, Section 10.5.4] and [6, p. 52]).
Due to their importance in many fields of Quantum Computation, several normal
forms for stabilizer circuits were proposed over the last two decades, with the aim
of reducing the gate count in these circuits. The first normal form proposed by
Aaronson and Gottesman [1] was successively improved by Maslov and Roetteler
[8], Bravyi and Maslov [4] and Duncan et al. [5]. These authors use decomposition
methods in the symplectic group over $\mathbb{F}_2$ in dimension $2n$ [1, 8, 4] or ZX-calculus
[5] in order to compute a normal form. In this paper we provide two normal forms
for stabilizer circuits. The first form is similar to the most recent ones [4, 5] but
is computed with a different method that does not rely on the properties of the
symplectic matrices. We use and induction process based on conjugation rules in
the Clifford group. The second form is obtained from the first one by replacing most
of the $CZ$ gates by $CNOT$ gates. This allows to reduce the 2-qubit gate count by using
an algorithm proposed in 2004 by Patel et al. [10].
This article is structured as follows. Section 2 is a background section on quantum
circuits and Clifford gates that will guide the non-specialist reader through the rest
of the paper. In Section 3 we investigate some remarkable subgroups of the Clifford
Group. The precise description of these group structures allows us to propose, in
Section 4 a polynomial-time algorithm to compute two types of normal forms. In
Section 5 we apply our results to stabilizer states and graph states. Finally, in
section 6 we use a C implementation of our algorithms to provide a few statistics
which empirically show the interest of these normal forms to reduce the gate count
of stabilizer circuits. We also propose an efficient implementation of graph states in
the publicly available IBM quantum computers.

2 Quantum circuits and Clifford gates

In this background section we recall classical notions about quantum circuits and
Clifford gates and we also introduce the main notations used in the paper. In
Quantum Information Theory, a qubit is a quantum state that represents the basic
information storage unit. This state is described by a ket vector in the Dirac notation
$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$ where $a_0$ and $a_1$ are complex numbers such that $|a_0|^2 + |a_1|^2 = 1$.
The value of $|a_i|^2$ represents the probability that measurement produces the value $i$.
The states $|0\rangle$ and $|1\rangle$ form a basis of the Hilbert space $\mathcal{H} \simeq \mathbb{C}^2$ where a one qubit
quantum system evolves. Operations on qubits must preserve the norm and are
therefore described by unitary operators $U$ in the unitary group $U_{2^n}$. In quantum
computation, these operations are represented by quantum gates and a quantum
circuit is a conventional representation of the sequence of quantum gates applied to
the qubit register over time. In Figure 1 we recall the definition of the Pauli gates
mentioned in the introduction. Notice that the states $|0\rangle$ and $|1\rangle$ are eigenvectors
of the Pauli-Z operator respectively associated to the eigenvalues 1 and -1, so the
standard computational basis (|0⟩, |1⟩) is also called the Z-basis. Notice also that X |0⟩ = 1 and X |1⟩ = 0, hence the Pauli-X gate is called the NOT gate. The phase gate P (see Figure 2) is defined by P |0⟩ = |0⟩ and P |1⟩ = i |1⟩. The Hadamard gate H creates superposition since H |0⟩ = \( \frac{1}{\sqrt{2}} (|0⟩ + |1⟩) \). The following useful identities are obtained by direct computation.

\[
\begin{align*}
H^2 &= X^2 = Y^2 = Z^2 = I \\
XZ &= -ZX \\
Y &= iZX \\
HZH &= X \\
P^2 &= Z \\
PXP^{-1} &= Y
\end{align*}
\]

The Pauli group for one qubit is the group generated by the set \{X, Y, Z\}. Any element of this group can be written uniquely in the form \( i^\lambda X^a Z^b \), where \( \lambda \in \mathbb{Z}_4 \) and \( a, b \in \mathbb{F}_2 \).

![Pauli gates](image)

**Figure 1:** The Pauli gates

A quantum system of two qubits A and B (also called a two-qubit register) lives in a 4-dimensional Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \) and the computational basis of this space is \( \{|00⟩, |01⟩, |10⟩, |11⟩\} \). If \( U \) is any unitary operator acting on one qubit, a controlled-\( U \) gate acts on the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \) as follows. One of the two qubits (say qubit A) is the control qubit whereas the other qubit is the target qubit. If the control qubit A is in the state |1⟩ then \( U \) is applied on the target qubit B but when qubit A is in the state |0⟩ nothing is done on qubit B. The CNOT gate (or \( CX \) gate) is the controlled-X gate with control on qubit A and target on qubit B, so the action of CNOT on a two-qubit register is described by : CNOT |00⟩ = |00⟩, CNOT |01⟩ = |01⟩, CNOT |10⟩ = |11⟩, CNOT |11⟩ = |10⟩ (the corresponding matrix is given in Figure 2). Observe that this action can be sum up by the simple formula CNOT \( |xy⟩ \) = \( |x, x \oplus y⟩ \), where \( \oplus \) denotes the XOR operator between two bits \( x \) and \( y \), which is also the addition in \( \mathbb{F}_2 \). In the same way, the reader can check that the controlled-Z operator acts on a a basis vector as CZ \( |xy⟩ = (-1)^{xy} |xy⟩ \). Note that this action is invariant by switching the control and the target. The last 2-qubit gate we need is the SWAP gate defined by SWAP \( |xy⟩ = |yx⟩ \).

On a system of \( n \) qubits, we label each qubit from 0 to \( n-1 \) thus following the usual convention. For coherence we also number the lines and columns of a \( n \times n \) matrix from 0 to \( n-1 \) and we consider that a permutation of the symmetric group \( \mathfrak{S}_n \) is a bijection of \( \{0, \ldots, n-1\} \). The \( n \)-qubit system evolves over time in the
Hilbert space $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{n-1}$ where $\mathcal{H}_i$ is the Hilbert space of qubit $i$. In this space, a state vector of the standard computational basis is classically denoted by $|x\rangle = |x_0x_1 \cdots x_{n-1}\rangle$ where $x_i$ is in $\{0,1\}$. Let $(e_i)_{0 \leq i < n}$ be the canonical basis of $\mathbb{F}_2^n$. It is convenient to identify the binary label $x = x_0x_1 \cdots x_{n-1}$ with the column vector $[x_0, \ldots, x_{n-1}]^T = \sum_i x_i e_i$ of $\mathbb{F}_2^n$, which is also denoted by $x$. So the standard basis is $\{|x\rangle\}_{x \in \mathbb{F}_2^n}$. When we apply locally a single qubit gate $U$ to qubit $i$ of a $n$-qubit register, the corresponding action on the $n$-qubit system is that of the operator $U_i = I \otimes \cdots \otimes I \otimes U \otimes I \otimes \cdots \otimes I = I^\otimes (n-i) \otimes U \otimes I^\otimes (n-i-1)$, where $\otimes$ is the Kronecker product of matrices and $I$ the identity matrix in dimension 2. As an example, if $n = 4$, $H_3 = I \otimes H \otimes I \otimes I$ and $H_3 H_3 = H \otimes I \otimes I \otimes H$. We also use a $n$-bit column vector notation, e.g. $H_0 H_3 = H_{[1,0,0,1]}^\otimes 4$. Observe that, with this notation, one has $U_i = U_{e_i}$. When $U$ is an involution (i.e. $U^2 = I$), the group generated by the $U_i$’s is isomorphic to $\mathbb{F}_2^n$, since it is an abelian 2-group. This is the case for the Pauli and Hadamard gates but not for the phase gates. For instance $H_{[1,0,0,1]}^\otimes 4 H_{[0,0,1,1]}^\otimes 4 = H_{[1,0,0,1]}^\otimes 4 \oplus [0,0,1,1]^\otimes 4 = H_{[1,0,1,0]}^\otimes 4 = H_0 H_2$. Note that the action of $Z_i$ on $|x\rangle = |x_0 \cdots x_{n-1}\rangle$ is described by $Z_i |x\rangle = (-1)^{x_i} |x\rangle$. Hence, if $v = [v_0, \ldots, v_{n-1}]^T \in \mathbb{F}_2^n$, one has
\begin{equation}
Z_v |x\rangle = (-1)^{v^T x} |x\rangle ,
\end{equation}
where $v \cdot x = \sum_i v_i x_i$. In the same way, $P_i |x\rangle = i^{v_i} |x\rangle$, hence
\begin{equation}
P_v |x\rangle = i^{v^T x} |x\rangle .
\end{equation}

A CNOT gate with target on qubit $i$ and control on qubit $j$ will be denoted $X_{ij}$. The reader will pay attention to the fact that our convention is the opposite of that generally used, where $\text{CNOT}_{ij}$ denotes a CNOT gate with control on qubit $i$ and target on qubit $j$. The reason for this change will appear clearly in the proof of Theorem 2. So, if $i < j$, the action of $X_{ij}$ and $X_{ji}$ on a basis vector $|x\rangle$ is given by
\begin{align}
X_{ij} |x\rangle &= X_{ij} |x_0 \cdots x_i \cdots x_j \cdots x_{n-1}\rangle = |x_0 \cdots x_i \oplus x_j \cdots x_{n-1}\rangle , \quad (9) \\
X_{ji} |x\rangle &= X_{ji} |x_0 \cdots x_i \cdots x_j \cdots x_{n-1}\rangle = |x_0 \cdots x_i \cdots x_j \oplus x_i \cdots x_{n-1}\rangle . \quad (10)
\end{align}

The CZ (resp. SWAP) gate between qubits $i$ and $j$ will be denoted by $Z_{ij}$ (resp. $S_{ij}$). Notice that $Z_{ij} = Z_{ji}$ and $S_{ij} = S_{ji}$. These gates are defined by
\begin{align}
Z_{ij} |x_0 \cdots x_{n-1}\rangle &= (-1)^{x_i x_j} |x\rangle , \quad (11) \\
S_{ij} |x_0 \cdots x_i \cdots x_j \cdots x_{n-1}\rangle &= |x_0 \cdots x_j \cdots x_i \cdots x_{n-1}\rangle . \quad (12)
\end{align}

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|}
\hline
CNOT & $A$ \hspace{1cm} CNOT = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} \hspace{1cm} \text{Phase : } \begin{bmatrix}
1 \\
0
\end{bmatrix} \hspace{1cm} \text{P} = \begin{bmatrix}
1 & 0 \\
0 & i
\end{bmatrix} \\
\hline
Hadamard & $H$ \hspace{1cm} H = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \hspace{1cm} \pi/8 : \begin{bmatrix}
1 \\
0
\end{bmatrix} \hspace{1cm} T = \begin{bmatrix}
1 & 0 \\
0 & e^{i\pi/4}
\end{bmatrix}
\hline
\end{tabular}
\caption{The standard set of universal gates : names, circuit symbols and matrices.}
\end{figure}
Observe that the $X_{ij}$, $S_{ij}$ and $X_i$ gates are $2^n \times 2^n$ permutation matrices while the $Z_{ij}$ and $Z_i$ gates are diagonal matrices with all diagonal entries equal to 1 or $-1$. All these matrices are involutions. The $P_i$ gates are also diagonal matrices but are not involutions since $P_i^2 = Z_i$. We end this section by recalling 3 classical identities. They correspond to the circuit equivalences in Figure 3. Each identity can be proved by checking that the action of its left hand side and of its right hand side on a basis vector $|x\rangle$ is the same.

$$X_{ij} = H_i H_j X_{ji} H_i H_j$$ (13)

$$Z_{ij} = H_i X_{ij} H_j = H_j X_{ji} H_i$$ (14)

$$S_{ij} = X_{ij} X_{ji} X_{ij} = X_{ji} X_{ij} X_{ji}$$ (15)

The Pauli group for $n$ qubits is the group generated by the set $\{X_i, Y_i, Z_i \mid i = 0, \ldots, n-1\}$. Since Identities (1), (2) and (3), any element of this group can be uniquely written in the form $i^\lambda X_u Z_v$, where $\lambda \in \mathbb{Z}_4$ and $u, v \in \mathbb{F}_2^n$. So, using (2), the multiplication rule in the Pauli group is given by

$$i^\lambda X_u Z_v i^{\lambda'} X_{u'} Z_{v'} = i^{\lambda + \lambda'} (1-1) u \cdot v X_u \oplus u \oplus v' Z_v \oplus v'$$ (16)

A stabilizer circuit for $n$ qubit is an element of the group generated by the set $\{P_i, H_i, X_{ij} \mid 0 \leq i, j \leq n-1\}$. This group contains the $S_{ij}$ and $Z_{ij}$ gates since (14) and (15). It also contains the Pauli group, since $Z_i = P_i^2$, $X_i = H_i P_i H_i$ and $Y_i = P_i X_i P_i^{-1} = P_i H_i P_i^2 H_i P_i^3$. In a stabilizer circuit, changes of the overall phase by a multiple of $\pi/4$ are possible since

$$(H_i P_i)^3 = (P_i H_i)^3 = e^{i\pi/4} I.$$ (17)

This last equation can be proved by a direct computation.

| CNOT | CZ | SWAP |
|------|----|------|
| A    | H  | H    |
| B    | H  | H    |
|      | H  | H    |
|      | H  | H    |

**Figure 3:** Classical equivalences of circuits involving CNOT and Hadamard gates.

### 3 Subgroup structures underlying stabilizer circuits

We start by describing the group $(\text{CZ})_n$ which is the group generated by the $Z_{ij}$ gates. The set of parts of $\{(i, j) \mid 0 \leq i < j \leq n-1\}$ is denoted by $B_n$. As noticed
in Section 2, the matrices $Z_{ij}$ are involutions and commute with each other since they are diagonal matrices. So $\langle CZ \rangle_n$ is isomorphic to the abelian 2-group $(B_n, \oplus)$, where $\oplus$ denotes the symmetric difference between two parts of a set (i.e. their union minus their intersection). As a consequence, the order of $\langle CZ \rangle_n$ is $2^{\frac{n(n-1)}{2}}$. For any $B$ in $B_n$, we denote by $Z_B$ the unitary operator in $\langle CZ \rangle_n$ associated to $B$, i.e. $Z_B = \prod_{\{i,j\} \in B}Z_{ij}$. With this notation, Identity [11] can be generalized as

$$Z_B \ket{x} = (-1)^{\sum_{\{i,j\} \in B} x_ix_j} \ket{b}.$$  \hspace{1cm} (18)

To any $B$ in $B_n$ we associate a $n \times n \mathbb{F}_2$-matrix whose entry $(i,j)$ is 1 if $\{i,j\}$ is in $B$. This matrix is symmetric with only zeros on the diagonal and, for convenience, we also denote it by $B$. For example $\{\{i,j\}\}$ also denotes the matrix whose entries are all 0 but entries $(i,j)$ and $(j,i)$ that are 1. Depending on the context, it will be clear if we consider the set or the matrix. Let $q_B$ be the quadratic form defined on $\mathbb{F}_2^n$ by

$$q_B(x) = \sum_{\{i,j\} \in B} x_ix_j = \sum_{i<j} B_{ij}x_ix_j,$$  \hspace{1cm} (19)

where $B_{ij}$ is the entry $(i,j)$ of the matrix $B$. Then Identity [18] can be rewritten as

$$Z_B \ket{x} = (-1)^{q_B(x)} \ket{x}.$$  \hspace{1cm} (20)

Note that $B$ can be viewed as the matrix of the alternating (and symmetric) bilinear form associated to the quadratic form $q_B$.

In a previous work [2], we described the group $\langle \text{CNOT} \rangle_n$ generated by the $X_{ij}$ gates. We recall now some results of [2]. The special linear group on any field $K$ is generated by the set of transvection matrices. In the special case of $K = \mathbb{F}_2$, this set is reduced to the $n(n-1)$ matrices $T_{ij} = I_n + E_{ij}$, where $E_{ij}$ is the matrix with all entries 0 except the entry $(i,j)$ that is 1. So $\text{GL}_n(\mathbb{F}_2) = \text{SL}_n(\mathbb{F}_2)$ is generated by the matrices $T_{ij}$. The following simple property of the $T_{ij}$ matrices will be of great use.

Proposition 1. Multiplying to the left (resp. the right) any matrix $M$ by a transvection matrix $T_{ij}$ is equivalent to add the row $j$ (resp. column $i$) to the row $i$ (resp. column $j$) in $M$.

Applying Proposition 1 to vector $b \in \mathbb{F}_2^n$, we can rewrite Relation [9] as

$$X_{ij} \ket{x} = \ket{T_{ij}x}.$$  \hspace{1cm} (21)

The above considerations lead quite naturally to the following theorem.

Theorem 2. The group $\langle \text{CNOT} \rangle_n$ generated by the CNOT gates acting on $n$ qubits is isomorphic to $\text{GL}_n(\mathbb{F}_2)$. The morphism $\Phi$ sending each gate $X_{ij}$ to the transvection matrix $T_{ij}$ is an explicit isomorphism. The order of $\langle \text{CNOT} \rangle_n$ is $2^{\frac{n(n-1)}{2}} \prod_{i=1}^n(2^i - 1)$.
Proof. Since the matrices $T_{ij}$ generate $GL_n(\mathbb{F}_2)$, it is clear that $\Phi$ is surjective. For Identity (21), a preimage $V$ under $\Phi$ of any matrix $A$ in $GL_n(\mathbb{F}_2)$ must satisfy the relations $V_x = |Ax|$ for any basis vector $|x\rangle$. Since these relations define a unique matrix $V$, $\Phi$ is injective. The order of $GL_n(\mathbb{F}_2)$ is classically obtained by counting the number of basis of the vector space $\mathbb{F}_2^n$. \hfill \Box

The following conjugation rule between the CNOT gates can be proved by checking, thanks to Identity (9), that the action of its left hand side and of its right hand side on any basis vector $|b\rangle$ is the same.

$$X_{ij}X_{jk}X_{ij} = X_{jk}X_{ik} \quad (i, j, k \text{ distinct}) \quad (22)$$

Let $\langle P, CZ \rangle_n$ be the group generated by the set $\{P, Z_{ij}\}$. Any element of the group generated by the $P$, gates can be written uniquely in the form $Z_v P_b$ where $v, b \in \mathbb{F}_2^n$. This group is isomorphic to $(\mathbb{Z}_4^n, +)$, one possible isomorphism associating $Z_v P_b$ to $2v + b$. Hence the group $\langle P, CZ \rangle_n$ is isomorphic to $\mathbb{Z}_4^n \times \mathcal{B}_n$. Any element in $\langle P, CZ \rangle_n$ can be written uniquely in the form $Z_v P_b Z_B$ and

$$Z_v P_b Z_B Z_{ji} P_{b'} Z_{B'} = Z_{v+e' \oplus b} P_{b \oplus b'} Z_{B \oplus B'}, \quad (23)$$

where $bb'$ denotes the $\mathbb{F}_2^n$-vector $\sum_i b_i' e_i$. The conjugation by the $X_{ij}$ gates in $\langle P, CZ \rangle_n$ obey to the 7 rules below. Each equality can be proved by checking, thanks to Identities (7) to (11), that the action of its left hand side and of its right hand side on any basis vector $|x\rangle$ is the same.

$$X_{ij}Z_{ij}X_{ij} = Z_{ij}Z_{ij} \quad (24)$$

$$X_{ij}Z_{ik}X_{ij} = Z_{ik}Z_{jk} \quad (i, j, k \text{ distinct}) \quad (25)$$

$$X_{ij}Z_{pq}X_{ij} = Z_{pq} \quad (p, q \neq i) \quad (26)$$

$$X_{ij}Z_i X_{ij} = Z_i Z_j \quad (27)$$

$$X_{ij}Z_j X_{ij} = Z_j \quad (28)$$

$$X_{ij}P_i X_{ij} = P_i P_j Z_{ij} \quad (29)$$

$$X_{ij}P_j X_{ij} = P_j \quad (30)$$

For any $A$ in $GL_n(\mathbb{F}_2)$, let $X_A = \Phi^{-1}(A)$, where $\Phi$ is the morphism defined in Theorem 2. So, if $[ij]$ denotes the transvection matrix $T_{ij}$, one has $X_{ij} = X_{[ij]}$. Let us denote by $\langle P, CZ, \text{CNOT} \rangle_n$ the group generated by the set $\{P, Z_{ij}, X_{ij}\}$. As described in the following proposition, we can extend relations (24) to (30) to the unitary matrices $Z_v, P_b$ or $Z_B$. 

**Proposition 3.** The group $\langle P, CZ \rangle_n$ is a normal subgroup of $\langle P, CZ, \text{CNOT} \rangle_n$. The conjugation of any element of $\langle P, CZ \rangle_n$ by a CNOT gate is described by the relations

$$X_{[ij]} Z_v X_{[ij]} = Z_{[ij]} v, \quad (31)$$

$$X_{[ij]} P_b X_{[ij]} = Z_{b_{ij}} e_{ij} P_{[ij]} , B_{b_{ij}} \{i, j\} ; \quad (32)$$

$$X_{[ij]} Z_B X_{[ij]} = Z_{B_{ij}} e_{ij} Z_{[ij]} B_{[ij]} ; \quad (33)$$

$$X_A Z_B X_A^{-1} = Z_{q(A^{-1})} Z_{A^{-1}, B^{-1}} ; \quad (34)$$

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where $A^{-T}$ is a shorthand for $(A^T)^{-1}$, $q_B$ is the quadratic form defined by $B$ and $q_B(A)$ is a shorthand for the vector $[q_B(C_0), \ldots, q_B(C_{n-1})]^T$ with $C_i$ being the column $i$ of $A$.

Proof. Identities (31) and (32) are direct consequences of the conjugation relations (27), (28), (29), (30) and Proposition (1) applied to vectors $v$ and $b$. Let us prove Identity (33). Let $B_i = \{ \{p, q\} \in B \mid i \in \{p, q\} \}$, $B_C^i = B \oplus B_i$ and $B_1' = B_i \oplus B_1 \{\{i, j\}\}$. Then $B = B_1 \{\{i, j\}\} \oplus B_1' \oplus B_C^i$. On one hand, $[ji]B[ij] = B_1[B_1][\{\{i, j\}\}] = \{\{i, j\}\}$, $[ji]\{\{i, k\}\}[ij] = \{\{i, k\}, \{j, k\}\}$ when $k \neq j$ and $[ji]\{\{p, q\}\}[ij] = \{\{p, q\}\}$ when $p, q \neq i$. Hence

$$Z_{[ij]B[ij]} = Z_{ij}B_{ij}Z_{ij} \prod_{k \in K_i} Z_{ik} \prod_{j \in K_i} Z_{jk}, \quad (35)$$

where $K_i = \{ k \mid \{i, k\} \in B_1' \}$. On the other hand, $X_{[ij]}Z_{B[ij]} = X_{[ij]}Z_{ij}B_{ij}Z_{ij} \prod_{k \in K_i} Z_{ik} \prod_{j \in K_i} Z_{jk}. \quad (36)$

As $Z_{ij}B_{ij} = Z_{B_{ij}e_{ij}}$, we conclude by comparing (35) and (36). Let us prove Identity (34). Since (33) and (31), it is clear that $X_AZ_{B}X_A^{-1}$ can be written in the form $Z_vZ_{A^{-T}BA^{-1}}$ for some $v$ in $F^{n_2}_2$, so we have to prove that $v = q_B(A^{-1})$. We start from $Z_v = X_AZ_{B}X_A^{-1}Z_{B'}$, where $B' = A^{-T}BA^{-1}$. Let $|\psi\rangle = Z_v|e_i\rangle$, then $|\psi\rangle = (-1)^v|e_i\rangle$. On the other hand, $|\psi\rangle = X_AZ_{B}X_A^{-1}|e_i\rangle$ since $q_B(e_i) = 0$ for any $B' \in B_n$. Besides, $X_A^{-1}|e_i\rangle = |A^{-1}e_i\rangle = |C_i\rangle$ where $C_i$ is the column $i$ of $A^{-1}$, hence $|\psi\rangle = X_AZ_{B}|C_i\rangle = (-1)^{q_B(C_i)}X_A|C_i\rangle = (-1)^{q_B(C_i)}|e_i\rangle$. Finally we see that $v_i = q_B(C_i)$, thus $v = q_B(A^{-1})$. □

From Identity (15), the gate $S_{ij}$ is in $\langle \text{CNOT} \rangle_n$ and is therefore a $X_A$ gate. Let $i(j)$ be the permutation matrix of $GL_n(F_2)$ associated to the transposition $\tau$ of $S_i$ that swaps $i$ and $j$, then $(ij) = [i][j][i][j] = [ij][ij][ij]$ and $S_{ij} = X_{(ij)}$. The group generated by the $S_{ij}$ gates is a subgroup of $\langle \text{CNOT} \rangle_n$ that is isomorphic to $S_n$. The conjugation by $S_{ij}$ is given by $S_{ij}Z_{pq}S_{ij} = Z_{(p)(\tau)(q)}$ (see [3] for further development on CZ and SWAP gates). As in Proposition 3 we prove that

$$X_{(ij)}Z_vX_{(ij)} = Z_{(ij)v}, \quad (37)$$

$$X_{(ij)}P_bX_{(ij)} = P_{(ij)b}, \quad (38)$$

$$X_{(ij)}Z_{B}X_{(ij)} = Z_{(ij)B(ij)}. \quad (39)$$

The main results of this section are summarized by the theorem that follows.

**Theorem 4.** Any element of $\langle P, CZ, CNOT \rangle_n$ admits a unique decomposition in the normal form $Z_vP_bZ_{B}X_{A}$ where $v, b \in F_2^n$, $B \in B_n$, $A \in GL_n(F_2)$. The group $\langle P, CZ, CNOT \rangle_n$ is the semidirect product of its normal subgroup $\langle P, CZ \rangle_n$ with $\langle \text{CNOT} \rangle_n$, i.e. $\langle P, CZ, CNOT \rangle_n = \langle P, CZ \rangle_n \times \langle \text{CNOT} \rangle_n$. The order of $\langle P, CZ, CNOT \rangle_n$ is therefore $2^{n(n+1)} \prod_{i=1}^{n}(2^l - 1)$. 8
Proof. The existence of the decomposition can be proved by induction using Proposition 3 and Identity (23). The algorithm in Figure 4 gives the details of the induction step: let \( C = \prod_{k=1}^{\ell} M_k \), where \( M_k \in \{ P_i, Z_{ij}, X_{ij} \} \) be an element of \( \langle P, CZ, CNOT \rangle_n \), then the form \( Z_v P_b Z_B X_A \) for \( C \) is the result of Algorithm C-to-PZX applied to \( C \) and \( I \), i.e. C-to-PZX\((C, I)\). Observe that the time complexity of this algorithm is only \( O(n^4) \) since we use row and column additions instead of matrix multiplication thanks to Proposition 3. Let us prove the unicity of such a decomposition. Suppose that \( Z_v P_b Z_B X_A = Z_v' P_b' Z_B' X_A' \). If \( A \neq A' \), there exists \( u \in \mathbb{F}_2^n \) such that \( Au \neq A'u \). But this leads to a contradiction because \( Z_v P_b Z_B X_A |u\rangle = Z_v' P_b' Z_B' X_A |u\rangle \), so \( Z_v P_b Z_B |Au\rangle = Z_v' P_b' Z_B' |A'u\rangle \), hence \( |Au\rangle \) and \( |A'u\rangle \) are two different basis vectors that are collinear. So \( A = A' \) and \( Z_v P_b Z_B = Z_v' P_b' Z_B' \). If there exists \( i \) such that \( b_i = 1 \) and \( b_i' = 0 \), then \( P_b |e_i\rangle = i |e_i\rangle \) and \( P_b' |e_i\rangle = |e_i\rangle \), so \( i Z_v Z_B |e_i\rangle = Z_v' Z_B' |e_i\rangle \), hence \( i |e_i\rangle = \pm |e_i\rangle \) which is not possible. Thus \( b = b' \). Finally, if \( Z_v Z_B = Z_v' Z_B' \), we show that \( v = v' \) and \( B = B' \) by comparing their action on \( |e_i\rangle \) and \( |e_i \oplus e_j\rangle \) for any \( i, j \). Since \( \langle P, CZ \rangle_n \) is a normal subgroup of \( \langle P, CZ, CNOT \rangle_n \), the semidirect product structure is a consequence of the existence and uniqueness of the decomposition. The order of \( \langle P, CZ, CNOT \rangle_n \) is computed using Theorem 2. \( \square \)

**Algorithm** : Normal form for a stabilizer circuit of \( P, CZ \) and \( CNOT \) gates.

**Input** : \( (C, C') \), where
- \( C \) is a circuit of \( \ell \) gates in the form \( \prod_{k=1}^{\ell} M_k \), where \( M_k \in \{ P_i, Z_{ij}, X_{ij} \} \),
- \( C' \) is a circuit in the normal form \( Z_v P_b Z_B X_A \).

**Output** : \( C'' \), an equivalent circuit to the product \( CC' \) in normal form.

1. \( C'' \leftarrow C' \);
2. for \( k = \ell \) to 1 do
3. if \( M_k = Z_{ij} \) then
4. \( B \leftarrow B \oplus \{i, j\} \);
5. else if \( M_k = P_i \) then
6. \( v \leftarrow v \oplus b_i e_i \); \( b \leftarrow b \oplus e_i \);
7. else
8. \( v \leftarrow [ji] v \oplus b_i e_i \oplus B_{ij} e_j \); \( B \leftarrow [ji] B_{ij} \oplus b_i \{i, j\} \); \( b \leftarrow [ji] b \); \( A \leftarrow [ij] A \);
9. return \( C'' \);

**Figure 4** : Algorithm C-to-PZX

The C-to-PZX algorithm computes a normal form for particular stabilizer circuits. In [3] we defined the C-to-ZS algorithm that rewrites any circuit of \( SWAP \) and \( CZ \) gates in the form \( Z_B S_{\sigma} \), where \( S_{\sigma} \) is a circuit of \( SWAP \) gates corresponding to the permutation \( \sigma \) in \( S_n \). So the C-to-PZX algorithm can be viewed as an extension of the algorithm C-to-ZS to the set of input gates \( \{ P_i, Z_{ij}, X_{ij} \} \). The C-to-PZX algorithm is of great use because it is a subroutine called by the main algorithm that computes a normal form for any stabilizer circuits (see next section). The conjugation rules below will also be very helpful. Let \( h = \prod_{i=0}^{n-1} H_i \), let \( U^h = h U h^{-1} = h U h \) for any \( U \in U_{2^n} \). We
use $U^{-h}$ as a shorthand notation for $(U^h)^{-1}$.

\begin{align}
    hX_A h^{-1} &= X_A - \tau \\
    p_i^h p_i^{-1} &= e^{i\frac{\pi}{2} H_i} X_i \\
    p_i^h Z_{ik} p_i^{-h} &= Z_{ik} X_{[ik]} p_k \\
    Z_{ij} Z_{ij}^h &= Z_{ij} Z_{ij}^h = H_i H_j X_{(ij)} = X_{(ij)} H_i H_j \\
    Z_{ik} Z_{ij}^h &= Z_{ik} X_{[ik]} (i, j, k \text{ distinct}) \\
    Z_{ij}^h p_j Z_{ij}^h &= p_i^h X_{[ij]} p_j
\end{align}

Identity (40) is a straightforward consequence of (13). Identity (41) comes from (6) and (3). We prove Identity (42) using (29), (17) and (14). We prove Identity (43) using (14) and (15). Identity (44) is proved using (14) and (22). Identity (45) is proved using (14) and (29).

The last identities we need to write a stabilizer circuit in normal form are the conjugation rules of a Pauli product

\begin{align}
    P_i X_u Z_v P_i^{-1} &= i^{u_i} X_u Z_v \oplus u_i e_i \\
    X_{[ij]} X_u Z_v X_{[ij]} &= X_{[ij]} X_u Z_v \oplus u_i e_i e_j = (-1)^{u_i u_j} X_u Z_v \oplus \{i, j\} u \\
    h X_u Z_v h &= X_u Z_v \\
\end{align}

Identity (46) comes from (6) and (3). Identity (47) from (31), (4) and (13). Identity (48) from (47), (2) and (4). Identity (49) from (4). Finally we generalize below Identities (46), (47) and (48).

\begin{align}
    p_b^h X_u Z_v p_b^{-1} &= i^{\sum_i u_i X_u Z_v \oplus \sum_i b_i u_i e_i} = i^{\sum_i u_i X_u Z_v \oplus b u} \\
    X_A X_u Z_v X_A^{-1} &= X_A Z_v X_A - \tau_v \\
    Z_B X_u Z_v Z_B &= (-1)^{q_B(u)} X_u Z_v \oplus B u
\end{align}

## 4 Computing normal forms

In this section we provide algorithms to compute two kinds of normal forms for stabilizer circuits. The first normal form is a generalization of the form $Z_v P_B Z_B X_A$.

**Theorem 5. Normal form for stabilizer circuits**

Any stabilizer circuit $C = \prod_{k=1}^n M_k$, where $M_k \in \{P_i, H_i, X_{[ij]}\}$ can be written in polynomial time $O(\ell n^2)$ in the form

\begin{align}
    H_u P_B Z_B H_u e^{i\varphi} X_u Z_v P_B Z_B X_A ,
\end{align}

where $\alpha, d, \omega, u, v, b \in \mathbb{F}_2^n$, $D, B \in \mathcal{B}_n$, $A \in \text{GL}_n(\mathbb{F}_2)$, and $\varphi \in \{k \pi, k \in \mathbb{Z}\}$.

**Proof.** The proof consists in the description of an algorithm that builds the normal form by induction. This algorithm is called the C-to-NF algorithm and its skeleton
is given in Figure 5. More precisely, we prove by induction on the length ℓ of the input circuit that any stabilizer circuit \( C = \prod_{k=1}^{\ell} M_k \) can be written in the normal form \( H_a P_d Z D e^{i\phi} X_u Z_u P_d Z B X_A \). Then the result comes from a straightforward simplification of \( H_a \) with \( h \). We define \( \alpha \) and \( \omega \) as follows. Let \( Q \) be the set of qubits involved in the subcircuit \( P_d Z D \), i.e. \( Q = \{ i \mid d_i = 1 \} \cup \{ i \mid \exists j, D_{ij} = 1 \} \). If \( a_i = 1 \) and \( i \notin Q \) then \( \alpha_i = \omega_i = 0 \), otherwise \( \alpha_i = a_i \) and \( \omega_i = 1 \). This simplification corresponds to the subroutine simplify mentioned in Figure 5.

The base case of the induction is clear: if \( \ell = 0 \) then \( C = 1 \) and a normal form for \( C \) is \( H_a h \), where \( a = [1, \ldots, 1]^T h \). To realize the induction step, we prove that, for any \( M \in \{ H_a, P_i, X_{[ij]} \} \) and any \( C \) in the form \( H_a P_d Z D e^{i\phi} X_u Z_u P_d Z B X_A \), the product \( MC \) can be written in the same form. The induction step is divided in three cases and a few subcases.

The first case corresponds to the subroutine merge-Hadamard mentioned in Figure 5.

**Case 1** : \( M = H_i \) and \( H_i C = H_u P_i H_a \cdot P_d Z D e^{i\phi} X_u Z_u P_d Z B X_A \), so \( H_i C \) is in normal form.

The second case corresponds to the subroutine merge-phase mentioned in Figure 5.

**Case 2** : \( M = P_i \) and \( P_i C = H_u P_i H_a \cdot P_d Z D e^{i\phi} X_u Z_u P_d Z B X_A \) (we use the dot \cdot just to make reading easier).

**Case 2.1** : \( a_i = 0 \). Then \( H_u P_i H_a = P_i \), so

\[
P_i C = H_u P_i Z D e^{i\phi} X_u Z_u P_d Z B X_A.
\]

**Case 2.2** : \( a_i = 1 \). Then \( H_u P_i H_a = P_i \), so

\[
P_i C = P_i Z D e^{i\phi} X_u Z_u P_d Z B X_A.
\]

**Case 2.2.1** : \( d_i = 0 \). Then \( P_i Z D e^{i\phi} X_u Z_u P_d Z B X_A \).

Let \( D_i = \{ \{ p, q \} \mid i \in \{ \{ p, q \} \} \} \) and \( K_i = \{ k \mid \{ k, i \} \in D \} \), then

\[
P_i Z D e^{i\phi} X_u Z_u P_d Z B X_A = Z D e^{i\phi} X_u Z_u P_d Z B X_A.
\]

**Case 2.2.2** : \( d_i = 1 \). Then \( P_i Z D e^{i\phi} X_u Z_u P_d Z B X_A \), so

\[
P_i C = H_u P_i Z D e^{i\phi} X_u Z_u P_d Z B X_A.
\]
\[P_iC = H_{a \otimes e_i} P_d \otimes e_i X_i \cdot P_i h Z_D P_i^{-1} \cdot h \cdot e^{i(\varphi' + \frac{\pi}{2})} X_u Z_{u'}, P_B Z_B X_A.\]

\[P_iC = H_{a \otimes e_i} P_d \otimes e_i P_i h Z_D P_i^{-1} \cdot (P_i h Z_D P_i^{-1})^{-1} X_i (P_i h Z_D P_i^{-1}) \cdot h \cdot e^{i(\varphi' + \frac{\pi}{2})} X_u Z_{u'}, P_B Z_B X_A.\]

Using Identity 16, 19, 52 and 16, we compute \(\varphi'', u'', v''\) such that

\[P_iC = H_{a \otimes e_i} P_d \otimes e_i P_i h Z_D P_i^{-1} h e^{i\varphi''} X_u Z_{u''} P_B Z_B X_A.\]

Then we proceed as in Case 2.2.1. and we obtain a normal form for \(P_iC\).

The third and last case corresponds to the subroutine merge–CNOT mentioned in Figure 5.

**Case 3**: \(M = X_{[ij]}\).

\[X_{[ij]}C = H_a \cdot H_a X_{[ij]} H_a \cdot P_d Z_D h e^{i\varphi} X_u Z_{u'} P_B Z_B X_A.\]

**Case 3.1**: \(a_i = a_j = 0\).

Then \(H_a X_{[ij]} H_a = X_{[ij]}\) and \(X_{[ij]}C = H_a X_{[ij]} P_d Z_D h e^{i\varphi} X_u Z_{u'} P_B Z_B X_A.\)

Let \(Z_{w'} P_d Z_D' X_{[ij]} = C\text{-to-PZX}(X_{[ij]}, P_d Z_D)\), then

\[X_{[ij]}C = H_a Z_{w'} P_d Z_D' X_{[ij]} h e^{i\varphi} X_u Z_{u'}, P_B Z_B X_A,\]

\[X_{[ij]}C = H_a Z_{w'} P_d Z_D' \cdot h \cdot e^{i\varphi} X_{[ij]} u' Z_{[ij]} u \cdot X_{[ij]} P_B Z_B X_A,\]

\[X_{[ij]}C = H_a Z_{w'} P_d Z_D' \cdot h \cdot e^{i\varphi} X_{[ij][u''w']} Z_{[ij]} u' \cdot X_{[ij]} P_B Z_B X_A.\]

Let \(Z_{w''} P_{B'} Z_{B''} X_{A''} = C\text{-to-PZX}(X_{[ij]}, P_B Z_B X_A)\), then

\[X_{[ij]}C = H_a P_d Z_D h e^{i\varphi} X_{[ij][u''w']} Z_{[ij]} u'' \cdot P_B Z_{B''} X_{A''}\]

and \(X_{[ij]}C\) is in normal form.

**Case 3.2**: \(a_i = a_j = 1\).

Then \(H_a X_{[ij]} H_a = X_{[ij]}\) and we proceed as in case 3.1, swapping \(i\) and \(j\).

**Case 3.3**: \(a_i = 1\) and \(a_j = 0\).

Then \(H_a X_{[ij]} H_a = Z_{ij}\) and a normal form for \(X_{[ij]}C\) is

\[H_a P_d Z_{D\oplus[i,j]} h e^{i\varphi} X_u Z_{u'} P_B Z_B X_A.\]

**Case 3.4**: \(a_i = 0\) and \(a_j = 1\).

Then \(H_a X_{[ij]} H_a = Z_{ij}^h\) and \(X_{[ij]}C = H_a Z_{ij}^h P_d Z_D h e^{i\varphi} X_u Z_{u'} P_B Z_B X_A.\)

**Case 3.4.1**: \(D_i = 0\).

\[X_{[ij]}C = H_a \cdot Z_{ij}^h P_d Z_{ij}^h \cdot Z_{ij}^h Z_D Z_{ij}^h \cdot h \cdot Z_{ij} e^{i\varphi} X_u Z_{u'}, Z_{ij} P_B Z_B X_A\]

Let \(\varphi' = \varphi + u_i u_j \pi, u' = u, u'' = v \oplus u_i u_j \oplus u_j e_i\) and \(B' = B \oplus \{i, j\}\), then

\[X_{[ij]}C = H_a \cdot Z_{ij}^h P_d Z_{ij}^h \cdot Z_{ij}^h Z_D Z_{ij}^h \cdot h \cdot e^{i\varphi'} X_u Z_{u''} \cdot P_B Z_B X_A.\]

Let \(D_i = \{p, q\} \in D \mid i \in \{p, q\}\) and \(K_i = \{k \mid \{i, k\} \in D\}\), then \(D_i \cap D_j = \emptyset\) since \(D_i = 0\) and \(Z_{ij}^h Z_D Z_{ij} = Z_{D\oplus D_i \oplus D_j} \cdot Z_{ij} \cdot \prod_{k \in K_i} Z_{ik} \cdot \prod_{k \in K_j} Z_{jk} Z_{ij}\).

**Case 3.4.2**: \(D_i \neq 0\).

\[Z_{ij}^h Z_D Z_{ij} = Z_{D\oplus D_i \oplus D_j} \cdot Z_{ij} \cdot \prod_{k \in K_i} Z_{ik} \cdot \prod_{k \in K_j} Z_{jk} \cdot Z_{ij} X_{[ij]}.\]

Let \(Z_{w''} Z_D X_{A'} = C\text{-to-PZX}(Z_{ij} Z_D Z_{ij}^h, 1)\), then \(A' = \prod_{k \in K_i}[j k] \prod_{k \in K_j}[i k]\) and

\[X_{[ij]}C = H_a \cdot Z_{ij}^h P_d Z_{ij}^h \cdot Z_{w''} Z_D X_{A'} \cdot h \cdot e^{i\varphi'} X_u Z_{u''} \cdot P_B Z_B X_A.\]

So \(X_{[ij]}C\) is \(H_a \cdot Z_{ij}^h P_d Z_{ij}^h \cdot Z_{w''} Z_D X_{A'} \cdot h \cdot e^{i\varphi'} X_u Z_{u''} \cdot P_B Z_B X_A, \)

and \(A'' = \prod_{k \in K_i}[j k] \prod_{k \in K_j}[i k]\).

Let \(e^{i\varphi''} X_{u''} Z_{u''} X_{A''} = C\text{-to-PZX}(X_{A''}, P_B Z_B X_A).\)
Then $X_{[ij]} C = H_a \cdot Z_{ij}^b P_d Z_{ij}^h \cdot Z_{w^r} Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}$, so

$$X_{[ij]} C = H_a \cdot Z_{ij}^b Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}.$$ 

**Case 3.4.1.1** : $d_i = d_j = 0$.

Then $Z_{ij}^h P_d Z_{ij}^h = P_d$ and $X_{[ij]} C = H_a P_d Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}$, so $X_{ij} C$ is in normal form.

**Case 3.4.1.2** : $d_i = 0$ and $d_j = 1$.

Then $Z_{ij}^h P_d Z_{ij}^h = Z_{ij}^h P_d Z_{ij}^h P_{d \oplus e_j} = P_{d \oplus e_j} X_{[ij]} P_d$

Hence $X_{[ij]} C = H_a \cdot P_{d \oplus e_j} X_{[ij]} P_d \cdot Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}$.

Observe that $P_d Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}$ is already in normal form, so we merge $X_{[ij]} C$ with this form using case 3.1. As no Hadamard gate is created in case 3.1, we can use case 2.2 to merge $P_{d \oplus e_j}$. Finally we merge $H_a$ using case 1 and obtain thereby a normal form for $X_{[ij]} C$.

**Case 3.4.1.3** : $d_i = 1$ and $d_j = 0$.

We proceed as in case 3.4.1.2, swapping $i$ and $j$.

**Case 3.4.1.4** : $d_i = d_j = 1$.

Then $Z_{ij}^h P_d Z_{ij}^h = Z_{ij}^h P_d Z_{ij}^h P_{d \oplus e_i \oplus e_j} = P_{d \oplus e_i \oplus e_j} X_{[ij]} P_{d \oplus e_i \oplus e_j}$.

Since Identity $[30]$, $P_{d \oplus e_i \oplus e_j}$ and $P_{d \oplus e_i \oplus e_j}$ commutes, so $Z_{ij}^h P_d Z_{ij}^h = P_{d \oplus e_i \oplus e_j} X_{[ij]} P_{d \oplus e_i \oplus e_j}$.

Hence $X_{[ij]} C = H_a \cdot P_{d \oplus e_i \oplus e_j} X_{[ij]} P_{d \oplus e_i \oplus e_j} \cdot Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}$.

Observe that $Z_{D'}^e a v^w Z_{w^r} Z_{P_d} Z_{B'} Z_{A'}$ is already in normal form, so we merge $X_{[ij]} C$ using this form case 3.1. As no Hadamard gate is created in case 3.1, we can use cases 2.2 and 2.2.1 to merge $P_{d \oplus e_i \oplus e_j}$. As no Hadamard gate is created in case 2.2.1, we can use case 2.1 to merge $P_d$ and case 3.1 to merge $X_{[ij]}$. Again, no Hadamard gate is created in cases 2.1 and 3.1, so we finally merge $P_{d \oplus e_i \oplus e_j}$ (case 2.2) and $H_a$ (case 1) and obtain a normal form for $X_{[ij]} C$.

**Case 3.4.2** : $D_{ij} = 1$.

$X_{[ij]} C = H_a \cdot Z_{ij}^h Z_{ij}^d \cdot P_d Z_{D'}^e a v^w Z_{D}^e a v^w Z_{B} Z_{A'}$, where $D' = D \oplus \{i, j\}$, hence $D'_{ij} = 0$.

$X_{[ij]} C = H_a \cdot H_{ij} X_{[ij]} Z_{ij}^h \cdot P_d Z_{D'}^e a v^w Z_{D}^e a v^w Z_{B} Z_{A'}$.

We use the conjugation rules (37), (38) and (39) by the SWAP gate $X_{[ij]}$ and we merge the Hadamard gates to obtain

$X_{[ij]} C = H_{i,j} X_{[ij]} Z_{ij}^h \cdot P_d Z_{D'}^e a v^w Z_{D}^e a v^w Z_{B} Z_{A'}$.

Finally we proceed as in case 3.4.1, since $\{i, j\} \notin \{ij\}_{D'(ij)}$.

Let us compute the worst case time complexity of the C-to-NF algorithm. The complexity of the C-to-PZX algorithm is $O(\ell n)$, where $\ell$ is the gate count of the input circuit. As we apply the C-to-PZX algorithm to circuits of $O(n)$ gates (cases 2.2 and 3.4.1), the cost of each call to C-to-PZX is $O(n^2)$ operations. Conjugating a Pauli gate $X_{u^r} Z_{u}$ by $X_A$ requires only $O(n)$ operations because we use a decomposition in less than $n$ (resp. $2n$) transvections of matrix $A$ in case 2.2.1 (resp. case 3.4.1) and each conjugation by a transvection costs $O(1)$ operations (17). Conjugating a Pauli gate $X_{u}$ by $(P_{d} Z_{D}' P_{d}^{-1})^{-1}$ (case 2.2.2) requires $O(n^2)$ operations. Other rewrites require $O(n)$ or $O(1)$ operations. Therefore the cost of merging a $M_k$ gate in the normal form at each induction step is $O(n^2)$ operations. As we need $\ell$ steps to write
Remark 6. The global phase $\varphi$ of a quantum circuit is generally considered as unimportant since it is physically unobservable. However, we decided not to neglect it during the computation process of the normal form since knowing its exact value is, at least, of purely mathematical interest. Moreover, calculating the exact value of $\varphi$ does not require much additional work.

Remark 7. The C-to-NF algorithm can also take CZ, SWAP, Z, X, or Y gates as input since $Z_{ij} = H_i X_{ij} H_i$, $S_{ij} = X_{[ij]} X_{[ji]} X_{[ij]}$, $Z_i = P_i^2$, $X_i = H_i P_i^2 H_i$ and $Y_i = P_i X_i P_i^{-1} = P_i H_i P_i^2 H_i P_i^3$.

Remark 8. Notice that the form $Z_{ij} P_i Z_B X_A$ defined in Theorem 4 is just a particular case of the normal form $H_n P_d Z_D H_e e^{i\varphi} X_u Z_v P_b Z_B X_A$. The C-to-NF algorithm can be applied to any stabilizer circuit of type $\prod_{k=1}^{\ell} M_k$, where $M_k \in \{P_i, Z_{ij}, X_{[ij]}\}$ and yields in this case a circuit in the normal form $Z_v P_b Z_B X_A$. In this sense, the C-to-NF algorithm is an extension of the C-to-PZX algorithm to any stabilizer circuit.

Remark 9. Unlike the particular normal form $Z_{ij} P_b Z_B X_A$ computed by the C-to-PZX algorithm, the general normal form $H_n P_d Z_D H_e e^{i\varphi} X_u Z_v P_b Z_B X_A$ is not unique. For instance, since Identity 43 one has $H_i H_j Z_{ij} H_j Z_{ij} = Z_{ij} H_i H_j X_{(ij)}$.

---

**Algorithm**: Normal form for a stabilizer circuit.

**INPUT**: $C$, a circuit of length $\ell$ in the form $\prod_{k=1}^{\ell} M_k$, where $M_k \in \{H_i, P_i, X_{[ij]}\}$.

**OUTPUT**: $\text{NF}$, an equivalent circuit to $C$ in the normal form $H_n P_d Z_D H_e e^{i\varphi} X_u Z_v P_b Z_B X_A$.

1. $n v \leftarrow$ null vector; $n m \leftarrow$ null matrix; $I \leftarrow$ identity matrix;
2. $\alpha \leftarrow [1, \ldots, 1]^T$; $\omega \leftarrow [1, \ldots, 1]^T$;
3. $d \leftarrow n v$; $D \leftarrow n m$; $\varphi \leftarrow 0$; $u \leftarrow n v$; $v \leftarrow n v$; $b \leftarrow n v$; $B \leftarrow n m$; $A \leftarrow I$;
4. $\text{NF} \leftarrow H_n P_d Z_D H_e e^{i\varphi} X_u Z_v P_b Z_B X_A$;
5. for $k = \ell$ to 1 do
   6. if $M_k = H_i$ then
   7. merge-Hadamard($\text{NF}, i$);
   8. else if $M_k = P_i$ then
   9. merge-phase($\text{NF}, i$);
   10. else
   11. merge-CNOT($\text{NF}, i, j$);
   12. simplify($\text{NF}$);
   13. return $\text{NF}$;

---

**Figure 5**: Algorithm C-to-NF

Note that the form \[ Z_{ij} \] is given as a matrix product and corresponds actually to a quantum circuit in the form $\text{CX-CZ-P-Z-X-H-CZ-P-H-P}$\(^1\). To obtain a quantum circuit

\(^1\)The reader who is not used to quantum circuits must pay attention to the following fact: the circuits act to the right of the state $\ket{\psi}$ presented to their left but the associated operators act to the left of $\ket{\psi}$, so the order of the gates in the circuit is inverted comparing to the form $\ket{\psi}$.
corresponding to the normal form, one can use the algorithm proposed in 2004 by Patel et al. [10] in order to write the matrix $A$ as a product of transvections. Our normal form is similar to the forms proposed by Duncan et al. [5] (H-Z-P-CX-H-CZ-P-Z-H) or by Bravyi and Maslov [4] (X-Z-P-CX-CZ-H-CZ-H-P). In these 3 normal forms, the 2-qubit gate count is asymptotically dominated by the two CZ gate layers which contain together up to $n(n - 1)$ gates. Indeed, the CNOT gate layer can be decomposed in $O(n^2/\log n)$ CNOT gates thanks to the algorithm by Patel et al. [10]. Therefore it would be interesting to compute another normal form where the gate count is dominated by the CNOT gate layers. To this end we need the following definition and lemma.

**ALGORITHM**: Reduction of a matrix in $\mathcal{B}_n$.

**INPUT**: $B$, a matrix in $\mathcal{B}_n$.

**OUTPUT**: $(B', A)$, where
- $B' \in \mathcal{B}_n$ is a reduced matrix congruent to $B$,
- $A \in \text{GL}_n(\mathbb{F}_2)$ satisfies the congruence relation $B' = A^TBA$.

1. $B' \leftarrow B$;
2. $A \leftarrow \text{Identity}$;
3. /* pivot[j] = true, if j has already been chosen as a pivot */
4. for $j = 0$ to $n - 1$ do
5.   pivot[j] $\leftarrow$ false;
6. for $j = 0$ to $n - 2$ do //nothing to do on column n - 1
7.   if pivot[j] or card$\{i \mid B'_{ij} = 1\} = 0$ then
8.     continue;
9. /* choosing pivot */
10.  $p \leftarrow \min\{i \mid B'_{ij} = 1\}$;
11.  pivot[p] $\leftarrow$ true;
12. /* Step a : eliminating the remaining 1’s on column j and line j */
13.  for $r = p + 1$ to $n - 1$ do
14.    if $B'_{rj} = 1$ then
15.      $B' \leftarrow [rp]B'[pr]$;
16.      $A \leftarrow A[pr]$;
17. /* Step b : eliminating the remaining 1’s on line p and column p */
18.  for $c = j + 1$ to $n - 1$ do
19.    if $B'_{pc} = 1$ then
20.      $B' \leftarrow [cj]B'[jc]$;
21.      $A \leftarrow A[jc]$;
22. return($B', A$);

**Figure 6**: Algorithm : reduction of a matrix in $\mathcal{B}_n$

**Definition 10.** We say that a matrix $B \in \mathcal{B}_n$ is reduced when each column and each line of $B$ contains at most one non-zero entry, i.e. $Z_B$ corresponds to a CZ circuit of depth 1.

**Lemma 11.** For any $B \in \mathcal{B}_n$, there exists an upper triangular matrix $A \in \text{GL}_n(\mathbb{F}_2)$ and a reduced matrix $B_{\text{red}} \in \mathcal{B}_n$ such that $B_{\text{red}} = A^TBA$. 

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Proof. $B$ is the matrix of an alternating bilinear form with respect to the canonical basis $(e_i)_{i=0...n-1}$. The equality $B_{\text{red}} = A^TBA$ is just the classical change of basis formula, where $A$ is the matrix of the new basis. A construction of $A$ and $B_{\text{red}}$ is given by the algorithm in Figure 6. We use Gaussian elimination (i.e. multiplication by transvection matrices, cf. Proposition 1) on columns and rows of the matrix $B$ to construct step by step the matrices $A$ and $B_{\text{red}}$ (see Example 12).

Example 12. Computing a reduced matrix in $\mathcal{B}_7$.

**INPUT**: $B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$j = 0$

Choosing pivot : $p \leftarrow 3; \text{pivot}[3] \leftarrow \text{true}';

Step a : $[53]B[35] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Step b : $[40][10][53]B[35][01][04] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$j = 1$

Choosing pivot : $p \leftarrow 2; \text{pivot}[2] \leftarrow \text{true}';

Step a : $[62][52][40][10][53]B[35][01][04][25][26] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$
After merging the Cation (subroutine $X_{\alpha,\omega,d,b,u,v}$)

Let $Z_{GL}$ form

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The proof consists in the description of an algorithm that takes a stabilizer

Step $b$: $[51][41][62][52][40][10][53][01][04][25][26][14][15] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

\section*{Theorem 13. CZ-reduced normal form}

Any stabilizer circuit $C = \prod_{k=1}^{\ell} M_k$, where $M_k \in \{P_i, H, X_{ij}\}$ can be written in the form

$$H_{\alpha}P_d X_{A_1} Z_{D_{\text{red}}} H_{\omega} e^{i\phi} X_u Z_v X_{A_2} Z_{B_{\text{red}}} X_{A_3} P_b,$$  \hspace{1cm} (54)

where $\alpha, \omega, d, b, u, v \in \mathbb{F}_2$, $D_{\text{red}}$ and $B_{\text{red}}$ are reduced matrices in $B_n$, $A_1, A_2, A_3 \in \text{GL}_n(\mathbb{F}_2)$, $A_1$ and $A_3$ are upper triangular matrices and $\varphi \in \{k_{\frac{\pi}{2}}, k \in \mathbb{Z}\}$.

\textbf{Proof.} The proof consists in the description of an algorithm that takes a stabilizer circuit $C$ as input and return an equivalent circuit in the CZ-reduced normal form $H_{\alpha}P_d X_{A_1} Z_{D_{\text{red}}} H_{\omega} e^{i\phi} X_u Z_v X_{A_2} Z_{B_{\text{red}}} X_{A_3} P_b$. This algorithm is called the C-to-CZredNF algorithm. We start by applying the C-to-NF algorithm without the final simplification (subroutine simplify) to the input and obtain a circuit $C = H_{\alpha}P_d Z_{D} e^{i\phi} X_u Z_v P_b X_{A}$. Let $Z_{w'} P_{u'} Z_{B'} = X^{-1}_{A'} \cdot P_b Z_{B'} \cdot X_A$, then

$C = H_{\alpha}P_d Z_{D} e^{i\phi} X_u Z_v X_{A} Z_{D_{\text{red}}} H_{\omega} e^{i\phi} X_u Z_v \in A' - u' \cdot X_{A} Z_{B_{\text{red}}} P_b Z_{B'}$. Let $D_{\text{red}}$ (resp. $B_{\text{red}}$) be a reduction of $D$ (resp. $B'$). Let $D_{\text{red}} = A'^T D A'$ and $B_{\text{red}} = A'^T B' A''$, where $A', A'' \in \text{GL}_n(\mathbb{F}_2)$. Since $[34]$ one has $X_{A'} Z_{D_{\text{red}}} X_{A'}^{-1} = Z_{q_{D_{\text{red}}}(A'-1)} Z_{D}$. and $X_{A''} Z_{B_{\text{red}}} X_{A''}^{-1} = Z_{q_{B_{\text{red}}}(A''-1)} Z_{B'}$, hence

$C = H_{\alpha}P_d X_{A'} Z_{D_{\text{red}}} X_{A'}^{-1} Z_{D_{\text{red}}}(A'-1) H_{\omega} e^{i\phi} X_u Z_v \in A' - u' \cdot X_{A} Z_{B_{\text{red}}}(A''-1) X_{A''} Z_{B_{\text{red}}} X_{A''}^{-1} P_b Z_{B'}$.

After merging the $Z$ gates with $X_u Z_{v \in A' - u'}$, we obtain:
Let $H$ be the column $i$.

Finally, we build $H_a$ and $H_u$ after simplifying $H_a$ with $h$ as follows. Let $C_i$ (resp. $L_i$) be the column $i$ (resp. the line $i$) of $A'$ and $(e_i)_{i=0...n-1}$ be the canonical basis of $F_2^n$. Let $Q$ be the set of qubits involved in the subcircuit $P_dX_{A'}Z_{Dred}$, $i.e$ $Q = \{i \mid d_i = 1\} \cup \{i \mid \exists j, D_{red}ij = 1\} \cup \{i \mid L_i \neq e_i\}$ or $C_i \neq e_i$). If $a_i = 1$ and $i \notin Q$ then $\alpha_i = \omega_i = 0$, otherwise $\alpha_i = a_i$ and $\omega_i = 1$.

Neglecting the global phase $\varphi$, the $CZ$-reduced normal form [54] corresponds to a quantum circuit in the form $P\cdotCX\cdot CZ\cdot CX\cdot Z\cdot X\cdot H\cdot CZ\cdot CX\cdot P\cdot H$, where each $CZ$ gate layer contains less than $\frac{n^2}{2}$ $CZ$ gates. Using the algorithm by Patel et al. [10] to decompose the three GL$_n(F_2)$-matrices $A_i$ in transvections yields a circuit that contains $O\left(\frac{n^2}{\log(n)}\right)$ two-qubit gates, which is better, in theory, than the $O(n^2)$ two-qubit gate count of the first normal form. Whether or not this second normal form brings a real practical advantage compared to the first one depends, however, on the value of the constant $c$ such that the two-qubit gate count remains lower than $c\cdot\frac{n^2}{\log(n)}$. In Section[5] we propose a value for $c$. We summarize the algorithms presented in this paper in Table[1]

| Input circuit of... | Algorithm | Normal Form |
|---------------------|-----------|-------------|
| SWAP, CZ            | $\rightarrow CZ$ | $Z_B\sigma$ |
| SWAP, CNOT, CZ, P, Z| $\rightarrow$PZX | $Z_eP_bZ_BX_A$ |
| SWAP, CNOT, CZ, P, Z, X, Y, H | C-to-NF | $H_uP_dZ_Dh_e^{i\varphi}X_uZ_vP_bZ_BX_A$ |
| SWAP, CNOT, CZ, P, Z, X, Y, H | C-to-CZredNF | $H_uP_dX_{A_1}Z_{Dred}h_e^{i\varphi}X_uZ_vX_{A_2}Z_{Bred}X_{A_3}P_b$ |

Table 1: Algorithms and normal forms

5 Application to stabilizer states and graph states

A stabilizer state $|S\rangle$ for a $n$-qubit register can be written in the form $|S\rangle = C|0\rangle^\otimes n$ where $C$ is a stabilizer circuit [11 Theorem 1]. A graph state $|G\rangle$ is a special case of a stabilizer state that can be written in the form $|G\rangle = Z_B^\otimes |+\rangle = Z_B h|0\rangle^\otimes n$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is the eigenvector corresponding to the eigenvalue 1 of the Pauli-$X$ gate and $h = H^\otimes n[7]$. The graph $G$ associated to the graph state $|G\rangle$ is the graph of order $n$ whose vertices are labeled by the $n$ qubits and whose set of edges is $B = \{\{i, j\} \mid B_{ij} = 1\}$. Let $C = \prod_{k=1}^\ell M_k$ be the product of $\ell$ Clifford gates $M_k \in \{P, H, X\}$. Applying the C-to-NF algorithm to $C$ without performing the final simplification (subroutine simplify) yields $C = H_uP_dZ_Dh_e^{i\varphi}X_uZ_vP_bZ_BX_A$ [4] $e^{i\varphi}H_uZ_uP_dZ_DhZ_pP_dZ_BX_A$. Since the subcircuit $Z_uP_dZ_BX_A$ has no effect on the ket $|0\rangle^\otimes n$, one has (neglecting the global phase $\varphi$) $|S\rangle = H_uZ_uP_dZ_Dh|0\rangle^\otimes n = H_uZ_uP_d|G\rangle$. 


where \(|G\rangle\) is the graph state \(Z_B h |0\rangle^{\otimes n}\). So, using the C-to-NF algorithm, we obtain a proof of the well known statement that any stabilizer state \(|S\rangle\) is equivalent to a graph state \(|G\rangle\) under local Clifford operations: there exists a stabilizer circuit \(C'\) composed of local Clifford gates only (i.e. phase and Hadamard gates) such that \(|S\rangle = C' |G\rangle\) (see [11, theorem 1]). Moreover our method provides straightforwardly a possible value for the circuit \(C'\) and the graph \(G\).

**Proposition 14.** For any stabilizer state \(|S\rangle\), there exists a graph state \(|G\rangle\) and 3 vectors \(u, v, w\) in \(\mathbb{F}_2^n\) such that

\[
|S\rangle = H_u Z_v P_w |G\rangle. 
\]

Applying the C-to-CZredNF algorithm to the special case of a CZ-gate circuit \(Z_B (B \in B_n)\) yields

\[
Z_B = Z_v X_A Z_{B_{red}} X_{A^{-1}},
\]

where \(A\) is an upper triangular matrix in \(\text{GL}_n(\mathbb{F}_2)\) and \(B_{red}\) is a reduced matrix in \(B_n\). Using (56), we write the graph state \(|G\rangle = Z_B h |0\rangle^{\otimes n}\) in the form \(|G\rangle = Z_v X_A Z_{B_{red}} h |0\rangle^{\otimes n}\). Since (10), we obtain \(|G\rangle = Z_v X_A Z_{B_{red}} h X_{A^{-1}} |0\rangle^{\otimes n}\). As a CNOT circuit has no effect on the ket \(|0\rangle^{\otimes n}\), one has \(|G\rangle = Z_v X_A Z_{B_{red}} h |0\rangle^{\otimes n}\). Hence the proposition that follows.

**Proposition 15.** Any graph state \(|G\rangle\) can be written in the form

\[
|G\rangle = Z_v X_A Z_{B_{red}} |+\rangle^{\otimes n}\,
\]

where \(v \in \mathbb{F}_2^n\), \(A \in \text{GL}_n(\mathbb{F}_2)\) is an upper triangular matrix, \(Z_{B_{red}}\) is a CZ circuit of depth 1.

**Example 16.** The entangled state of a 5-qubit register \(|\text{GHZ}\rangle_5\) can be easily implemented as a stabilizer state: \(|\text{GHZ}\rangle_5 = X_{[43]} X_{[32]} X_{[21]} X_{[10]} H_0 |0\rangle^{\otimes 5}\). Applying the C-to-NF algorithm on the input \(C = X_{[43]} X_{[32]} X_{[21]} X_{[10]} H_0\) yields the following normal form: \(C = H_1 H_2 H_3 H_4 Z_{01} Z_{02} Z_{03} Z_{04} H_1 H_2 H_3 H_4 X_{[21]} X_{[31]} X_{[41]} X_{[43]} X_{[32]}\) (the reader can use our C implementation of the algorithm, see next section). So \(|\text{GHZ}\rangle_5 = H_1 H_2 H_3 H_4 Z_{01} Z_{02} Z_{03} Z_{04} h |0\rangle^{\otimes 5}\), hence \(|\text{GHZ}\rangle_5\) is Local Clifford equivalent to the star graph state \(|G\rangle = Z_{\left\{(0,1),(0,2),(0,3),(0,4)\right\}} |+\rangle^{\otimes 5}\) (Proposition [12]). Then we use the C-to-CZredNF algorithm on the input \(Z_{\left\{(0,1),(0,2),(0,3),(0,4)\right\}}\) and we obtain \(Z_{\left\{(0,1),(0,2),(0,3),(0,4)\right\}} = X_{[14]} X_{[13]} X_{[12]} Z_{01} X_{[14]} X_{[13]} X_{[12]}\), hence \(|G\rangle = X_{[14]} X_{[13]} X_{[12]} Z_{01} |+\rangle^{\otimes 5}\) (Proposition [13]).

### 6 Empirical validation

We implement the algorithms presented in this paper in the C language with a text-based user interface. The source code is available at: https://github.com/marcbataille/stabilizer-circuits-normal-forms

To decompose the \(\text{GL}_n(\mathbb{F}_2)\)-matrices of the normal forms, we use the algorithm by Patel et al. with a value of \([\log_2(n)/2]\) for the parameter \(m\) (see [10]). Our program
is fast and can write in normal form a 40000 gates random stabilizer circuit for a 200-qubit register in a few seconds using a basic laptop. The manual mode of the program reproduces the induction steps of the C-to-NF algorithm, while the statistics mode provides convenient tools to show empirically the interest (and the limits) of our normal forms to reduce stabilizer circuits. A few significant results are presented below and we invite the reader to use our program to obtain his own statistics. We use samples of 100 random stabilizer circuits and we compute the average length and the average 2-qubit gate count of both normal forms (results are given in percentage of the input). The probability of choosing a CNOT gate is 0.8 while the probability of choosing a phase or a Hadamard gate is 0.1 for each type (these proportions can be easily modified in the program).

| Input circuits of length \(n^2/2\) | All gate count | 2-qubit gate count |
|-----------------------------------|----------------|-------------------|
| \(n\) | Input | N.F. | CZ-red. N.F. | Input | N.F. | CZ-red. N.F. |
| 10 | 100% | 165% | 172% | 100% | 153% | 157% |
| 20 | 100% | 199% | 188% | 100% | 208% | 193% |
| 50 | 100% | 181% | 152% | 100% | 209% | 173% |
| 100 | 100% | 166% | 129% | 100% | 200% | 153% |
| 200 | 100% | 157% | 113% | 100% | 193% | 137% |
| 300 | 100% | 152% | 103% | 100% | 188% | 126% |

| Input circuits of length \(n^2\) | All gate count | 2-qubit gate count |
|-----------------------------------|----------------|-------------------|
| \(n\) | Input | N.F. | CZ-red. N.F. | Input | N.F. | CZ-red. N.F. |
| 10 | 100% | 114% | 113% | 100% | 104% | 101% |
| 20 | 100% | 104% | 96% | 100% | 109% | 98% |
| 50 | 100% | 90% | 76% | 100% | 105% | 87% |
| 100 | 100% | 83% | 64% | 100% | 100% | 76% |
| 200 | 100% | 79% | 56% | 100% | 96% | 68% |
| 300 | 100% | 76% | 52% | 100% | 94% | 63% |

| Input circuits of length \(2n^2\) | All gate count | 2-qubit gate count |
|-----------------------------------|----------------|-------------------|
| \(n\) | Input | N.F. | CZ-red. N.F. | Input | N.F. | CZ-red. N.F. |
| 10 | 100% | 61% | 60% | 100% | 55% | 54% |
| 20 | 100% | 52% | 48% | 100% | 54% | 49% |
| 50 | 100% | 45% | 38% | 100% | 52% | 43% |
| 100 | 100% | 42% | 32% | 100% | 50% | 38% |
| 200 | 100% | 39% | 28% | 100% | 48% | 34% |
| 300 | 100% | 38% | 26% | 100% | 47% | 32% |

The experimental results clearly show that the C-to-CZredNF algorithm is better than the C-to-NF algorithm in terms of circuit reduction. From 20 qubits, we observe that the C-to-CZredNF algorithm is able to reduce the 2-qubit gate count, not only
in average, but for all circuits of length $n^2$ tested in the experiment (second table). In fact, this observation is just a particular case of the conjecture that follows.

**Conjecture 17.** Using the C-to-CZredNF algorithm, any $n$-qubit stabilizer circuit can be transformed into an equivalent circuit that contains less than $\frac{3n^2}{\log(n)}$ two-qubit gates.

We check this conjecture up to 600 qubits on about one thousand random circuits of different lengths.

We focus now on the particular case of CZ circuits and we evaluate to what extent Identity 56 is helpful to reduce the 2-qubit gate count in an implementation of a CZ circuit (corresponding to a matrix $Z_B$) that is based on Pauli-Z, CNOT and CZ gates. Again we use the algorithm by Patel et al. with a value of $\lceil \log_2(n)/2 \rceil$ for the parameter $m$ in order to decompose the matrix $A$ and we obtain an equivalent circuit to $Z_B$. It appears that such an implementation of a CZ circuit is interesting in some cases from about 20 qubits. For instance a 20-qubit CZ circuit corresponding to a matrix $B$ with all non-diagonal entries equal to 1 (complete graph $K_{20}$, 190 CZ gates) is reduced to an equivalent circuit of 164 two-qubit gates (CNOT + CZ). For a high number of qubits, the reduction is more interesting. The table below shows results for 200-qubit CZ circuits. We use samples of 100 random circuits of the same length. All input circuits contain distinct CZ gates and we count only the 2-qubit gates (CNOT + CZ) of the output circuit.

| Input length | Output Max. | Avg. | Min. |
|--------------|-------------|------|------|
| 5000         | 216.2 %     | 214.8 % | 213.5 % |
| 10000        | 108.9 %     | 108.4 % | 107.8 % |
| 12000        | 91.0 %      | 90.4 %  | 89.7 %  |
| 14000        | 77.9 %      | 77.4 %  | 76.9 %  |
| 16000        | 68.1 %      | 67.6 %  | 67.2 %  |
| 18000        | 60.1 %      | 59.4 %  | 58.2 %  |
| 19000        | 55.7 %      | 54.2 %  | 52.7 %  |

We end this section by a simple example which highlights the usefulness that can have Proposition 15, and more generally the C-to-CZredNF algorithm, to reduce the gate count of stabilizer circuits implemented in a real-life quantum machine. We implement in the publicly available 5-qubit ibmq_belem device (https://quantum-computing.ibm.com/) the graph state

$$|K_5\rangle = Z_B |+\rangle ^n,$$

where $B$ is the adjacency matrix of the complete graph $K_5$. This graph-state is Local Clifford equivalent to $|\text{GHZ}_5\rangle$, just like the star graph state $Z_{\{0,1\},\{0,2\},\{0,3\},\{0,4\}} |+\rangle ^n$ mentioned in the previous section (see e.g. [7, Section 4.1]). From Proposition 15, we obtain (using our computer program):

$$|K_5\rangle = Z_2Z_3X_{[34]}X_{[23]}X_{[12]}X_{[02]}X_{[24]}X_{[23]}Z_{01}Z_{23} |+\rangle ^n. \quad (59)$$

Observe that the form 59 contains only 8 two-qubit gates comparing to the 10 CZ gates of the form 58. The reduction is much more impressive when we consider the circuit that is really implemented in the quantum computer using only native gates.
Indeed, the CZ gate is not native in the IBM quantum devices and is simulated thanks to Identity (14). The Hadamard gate is implemented from the $R_z(\pi/2)$ and $\sqrt{X}$ gates, since $H = e^{i\pi/4}R_z(\pi/2)\sqrt{X}R_z(\pi/2)$, where $\sqrt{X} = \frac{1}{2} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix}$ and $R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$. Moreover full connectivity is not achieved and the direct connections allowed between two qubits are given by a graph. The graph of the 5-qubit ibmq_belem device is $\{\{1, 0\}, \{1, 2\}, \{1, 3\}, \{3, 4\}\}$. So, to implement a CNOT gate between qubits without direct connection (e.g. qubits 2 and 3), it is necessary to simulate it from the native CNOT gates using methods we described on a previous work [2, Section 3]. Due to its similarities to the compilation process in classical computing, the rewriting process that transforms an input circuit with measurements into a native gate circuit giving statistically the same measurement results, is called transpilation on the IBM quantum computing website. We remark that the transpiled circuit corresponding to the form 58 contains 43 CNOT gates and 69 single qubit gates which is far more than the 16 CNOT’s and the 17 single gates of the transpiled circuit corresponding to the form 59 (see circuits below).

$$|K_5\rangle = Z_B |+\rangle^\otimes n :$$
\[ |K_5\rangle = Z_2 Z_3 X_{[12]} X_{[13]} X_{[02]} X_{[24]} X_{[23]} Z_{01} Z_{23} |+\rangle \otimes n : \]
Conclusion and future work

Gottesman proved in his Phd thesis that any unitary matrix in the Clifford group is uniquely defined, up to a global phase, by its action by conjugation on the Pauli gates $X_i$ and $Z_i$. This central statement of Gottesman stabilizer formalism can be used to compute normal forms for $n$-qubit stabilizer circuits via the symplectic group over $\mathbb{F}_2$ in dimension $2n$ (e.g. [6, pp.41,42]). In this paper we showed that it is possible to compute normal forms in polynomial time without using this formalism. The reader who is used to work with the symplectic group will notice that our induction process can also be applied inside this group, giving rise to a decomposition of type

$$M_\sigma \begin{bmatrix} I_n & 0 \\ D & I_n \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A^{-T} \end{bmatrix}$$

for the symplectic matrix associated to the form $53$, where $\tilde{B}$ (resp. $\tilde{D}$) is a symmetric matrix corresponding to $P_bZ_B$ (resp. $P_dZ_D$), $A \in \text{GL}_n(\mathbb{F}_2)$ is the invertible matrix corresponding to the CNOT subcircuit $X_A$, and $M_\sigma$ is a permutation matrix corresponding to a circuit of Hadamard gates.

Along with this article we also provided a C implementation of all our algorithms as well as a few basic statistics that help to understand how normal forms can be helpful to reduce the gate count of stabilizer circuits. We applied our results to graph states and we checked experimentally the practical utility of normal forms to implement this type of stabilizer state on real-life quantum computers.

In [2] (resp. [3]), we studied the emergence of entanglement in particular stabilizer circuits, namely CNOT (resp. CZ plus SWAP) circuits. Our goal was to find out what kind of entanglement can be created when those simple circuits act on a fully factorized state. It would be interesting to understand how well normal forms could help to extend these studies to any stabilizer circuit. We leave this topic for future work.

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