Relation of the Bell inequalities with quantum logic, hidden variables and information theory

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Abstract

I review the relation of the Bell inequalities - characteristic of (classical) probabilities defined on Boolean logics - with noncontextual and local hidden variables theories of quantum mechanics and with quantum information.
Quantum mechanics looks radically different from all classical theories of physics. Is that difference fundamental or is it due to our present lack of understanding of quantum mechanics? (That our understanding is not good enough is widely recognized.) Several approaches have been followed in the attempt to answer the question. We comment here briefly on three of them, namely quantum logics, hidden variables and information theory, showing the relation of each one of these with the Bell inequalities.

I. Quantum logic and quantum probability

According to Birkhoff and von Neumann, the difference between quantum and classical theories is radical because it appears at the most fundamental level, the logic. The elements of a logic are the propositions which, using the language of physics, are observables having the possible values 1 (the proposition is true) or 0 (false). Some pairs of propositions are related by the implication ($A$ implies $B$ if $B$ is true whenever $A$ is true). This binary relation endows the logic with the mathematical structure of a partially ordered set (“poset”). Another binary relation associates every proposition with its negation (for each proposition $A$ there exist another one, $A'$, which is true if and only if the first is false). This makes the poset orthocomplemented. The internal operations ”meet” and ”join” endows the poset with a richer structure making it an orthocomplemented lattice. Finally it is assumed that there exist the sure proposition $I$, always true, and the absurd proposition, always false, which makes the lattice complete. From now on any complete and orthocomplemented lattice will be called a logic. Classical logic is a distributive lattice and it is called a Boolean algebra.

In the view of Birkhoff and von Neumann the structure of quantum logic may be derived from the correspondence between propositions and projection operators (which we shall call projectors in the following.) Accordingly these authors postulated that the proposition associated to the projector $\hat{P}$ is true (or false), for a physical system in a given state, if the state-vector $|\Psi\rangle$ is an eigenvector of $\hat{P}$ (or $\hat{I} - \hat{P}$). This assumption gives rise to a trivalent logic where propositions may be, in addition to true or false, also undefined (which happens if $|\Psi\rangle$ is neither an eigenvector of $\hat{P}$ nor an eigenvector of $\hat{I} - \hat{P}$.) As projectors are associated to closed subspaces of the Hilbert space, quantum logic has the mathematical structure of the set of closed subspaces.
From these assumptions it is straightforward to define the fundamental relation of order (or implication) of propositions. We say that, for two propositions $A$ and $B$ we have $A \leq B$ (or $A \Rightarrow B$) if the subspace associated to $B$ contains that associated to $A$. Hence the binary operations ”meet”, $\land$, and ”join”, $\lor$, may be defined in a natural form and it follows that the propositions form a lattice. The lattice is orthocomplemented (the subspaces associated to the proposition $A$ and its negation $A'$ are orthogonal) and complete (there exist the sure proposition, $I$, corresponding to the whole Hilbert space and its negation, $\Phi$, corresponding to the null vector). Up to here everything is similar to what happens in classical logic. But the quantum lattice is not distributive (Boolean) at a difference with the classical one. As a conclusion the authors claimed that the non-Boolean character of the lattice of propositions is the essential characteristic of quantum theory. The details may be seen in the original article.

In the 66 years elapsed since the work of Birkhoff and von Neumann many articles and several books have been devoted to the subject of quantum logic (see e.g. the book of Hooker), in many cases starting from different definitions of quantum propositions. Also some criticisms have aroused in the sense that ”quantum logic” is not a true logic, but just a propositional calculus. Indeed in an ”actual” logic the relations amongst proposition like $A \Rightarrow B$ or $A \land B = C$ should be also considered propositions, which is not necessarily the case in a propositional calculus. But the commented approach to the logic of quantum mechanics is still widely used.

In any logic (orthocomplemented and complete lattice) it is straightforward to define a probability distribution (or ”state”):

**Definition 1** If $L$ is a logic, a probability distribution is a mapping $p: L \rightarrow [0, 1]$ with the axioms

1) $p(\Phi) = 0, p(I) = 1$, where $\Phi$ (I) is the absurd (sure) proposition,
2) If $\{A_i\}$ is a sequence such that $A_i \leq A'_j$, $A'$ being the negation of $A$, for all pairs $i \neq j$, then $\sum_i p(A_i) = p(\lor A_i)$,
3) For any sequence $\{A_i\}$, $p(A_i) = 1 \forall i \Rightarrow p(\land A_i) = 1$,

Thus from quantum logic, as defined by Birkhoff and von Neumann, we get quantum probability, whilst the classical, Boolean, logic provides the standard probability theory. Indeed the above axioms are simply a generalization of the axioms of probability as stated by Kolmogorov.
II. The Bell inequalities

A discrimination between classical and quantum probability is provided by the Bell inequalities, derived as follows. For any two proposition $A, B \in \mathcal{L}$ we may define a function, $d(A,B)$, by

$$d(A,B) = p(A \vee B) - p(A \wedge B).$$

That function has the properties

$$0 \leq d(A,B) \leq 1, d(A,A) = 0, d(A,A') = 1, \tag{2}$$

and provides some measure of the "distance" between two propositions in a given state (probability distribution). The function is called a metric (pseudometric) if the following additional property holds (does not hold) true

$$d(A,B) = 0 \Rightarrow A = B, \tag{3}$$

but this property is not very relevant for our purposes. More important are the following triangle inequalities, which are (are not generally) fulfilled if the lattice is (is not) Boolean

$$|d(A,B) - d(A,C)| \leq d(B,C) \leq d(A,B) + d(A,C). \tag{4}$$

As the Boolean character provides the essential difference between classical and quantum theories, according to Birkhoff and von Neumann, we see that the triangle inequalities (4) give a criterion to distinguish both theories. These inequalities are closely related to the Bell inequalities as shown in the following, although in mathematical theory of probability the inequalities (4) were known well before Bell’s work.

In quantum mechanics, if we consider three compatible propositions, \{A, B, C\}, (associated to pairwise commuting projectors) the inequalities (4) hold true because the lattice of commuting projectors is distributive. On the other hand, if two of the propositions, say A and B, are not compatible then their distance is not defined because quantum mechanics does not provide a joint probability of two incompatible observables (and it is assumed that they cannot be measured simultaneously). However there are quadrilateral inequalities, derived from the triangular ones (4), which may be violated by quantum mechanics and tested empirically. In fact, if we consider four projectors \{A, B, C, D\} it is easy to see that the inequalities (4) lead to

$$d(A,D) \leq d(A,B) + d(B,C) + d(C,D). \tag{5}$$
At a difference with the inequalities (4), now all four distances are defined in quantum mechanics if every pair involve commuting projectors (that is if \([A, D] = [A, B] = [B, C] = [C, D] = 0\)). We see that the inequality (5) and the other three obtained by permutations involving the four projectors are necessary conditions for the existence of a \textit{classical} joint probability distribution defined on the set of projectors. The are cases where quantum mechanics predicts violations of one of the inequalities, which leads to Bell’s theorem (see next section.)

Inequality (5) is equivalent to the following one

\[
p_B + p_C \geq p_{AB} + p_{BC} + p_{CD} - p_{DA},
\]

where \(p_A\) (or \(p_{AB}\)) is the probability that \(A\) (or \(AB\)) is true. This is called a \textit{Bell inequality} and, in this form it was derived by Clauser and Horne. Instead of projectors, taking the values 0 or 1, we might use observables taking the values -1 or +1. They are trivially related to the projectors by

\[
a = 2A - 1, b = 2B - 1, etc.
\]

and the inequality (5) takes the form of Clauser-Horne-Shimony-Holt (CHSH):

\[
|\langle ab \rangle + \langle bc \rangle + \langle cd \rangle - \langle ad \rangle| \leq 2,
\]

where \(\langle ab \rangle\) means the expectation value of the product of \(a\) and \(b\). Therefore these, CHSH, and the Clauser-Horne inequalities (5) are equivalent.

III. Hidden variables theories

The question of hidden variables in quantum mechanics aroused soon after the formulation of the theory during the years 1925-26. It was explicitly mentioned in the book by von Neumann in 1932, where he derived a celebrated no hidden variables theorem. From that time many books and articles have been devoted to the subject. Nevertheless there is no sharp definition of hidden variables (HV) theory which is widely accepted. I propose the following:

\textbf{Definition 2} HV is a theory physically equivalent to quantum mechanics (that is giving the same predictions for all experiments) which has the formal structure of classical statistical mechanics.
The definition may be illustrated in the following table giving the correspondence of concepts in experiments, standard quantum theory and a possible HV theory:

**Table I. Correspondence of concepts**

| EMPIRICAL | QUANTUM THEORY | HV THEORY |
|-----------|----------------|-----------|
| physical system | Hilbert space $H$ | phase space $\Lambda$ |
| state | vector $|\Psi\rangle \in H$ | probability density $\rho(\lambda)$ |
| observable $A$ | self-adjoint operator $\hat{A}$ | function $A(\lambda)$ |
| expectation value | $\langle \Psi | \hat{A} | \Psi \rangle$ | $\int A(\lambda) \rho(\lambda) d\lambda$ |
| correlation | if $\hat{A} \hat{B} = \hat{B} \hat{A}$, $\langle \Psi | \hat{A} \hat{B} | \Psi \rangle$ | $\int A(\lambda) B(\lambda) \rho(\lambda) d\lambda$ |

The parameter (or parameters) $\lambda$ is usually called the *hidden variable*. Two observables, $A$ and $B$, which are associated to commuting operators, $\hat{A}$ and $\hat{B}$, are said *compatible*. The correlation may be extended to more than two compatible observables. It is easy to see that the latter equality implies the equality of the joint probability distributions of compatible observables. In fact, it is enough to substitute $\exp\left(i\xi\hat{A}\right)$ for $\hat{A}$ and $\exp[i\xi A(\lambda)]$ for $A(\lambda)$ in the equality, and similarly for $B$, in order to show the equality of the characteristic function of the joint probability distribution. On the other hand, it is well known that quantum mechanics does not provide joint distributions of observables not compatible (the associated operators noncommuting). For the sake of clarity, in the Table we have considered only quantum pure states. The most general states are associated to density operators, $\hat{\rho}$, whence the quantum expectation value and correlation should be written, respectively

$$Tr\left(\hat{\rho}\hat{A}\right), Tr\left(\hat{\rho}\hat{A}\hat{B}\right).$$

In order to make clear what is the content of the theorems against HV theories, discussed later, I propose the following

**Definition 3** A simple experiment consists of the preparation of a state of a physical system, followed by the evolution of the system and finishing with the measurement of a set of compatible observables.

**Definition 4** A composite experiment consists of several simple experiments with the same preparation and the same subsequent evolution, but measuring different sets of compatible observables in each simple experiment.
With these definitions we may state the following theorem:

**Theorem 5** For any simple experiment there exists a HV theory.

**Proof:** The essential part of the proof is to show that for any state $|\Psi\rangle$ and two compatible observables $A$, $B$ the expectation may be obtained in the form

$$\langle \Psi | \hat{A} \hat{B} | \Psi \rangle = \int A(\lambda) B(\lambda) \rho(\lambda) \, d\lambda. \quad (9)$$

For simplicity we consider just two observables, but the generalization to any finite number is trivial. In order to proceed with the proof we recall that there exists a complete set of orthonormal vectors which are simultaneous eigenvectors of two commuting self-adjoint operators. Let us label $|\lambda\rangle$ one of the common eigenvectors of $\hat{A}$ and $\hat{B}$. Complete means that

$$\int |\lambda\rangle \langle \lambda | \, d\lambda = 1, \quad (10)$$

which leads to

$$\langle \Psi | \hat{A} \hat{B} | \Psi \rangle = \int d\lambda d\lambda' d\lambda'' \langle \Psi | \lambda \rangle \langle \lambda | \hat{A} | \lambda' \rangle \langle \lambda' | \hat{B} | \lambda'' \rangle \langle \lambda'' | \Psi \rangle$$

$$= \int d\lambda \langle \Psi | \lambda \rangle \langle \lambda | \hat{A} | \lambda \rangle \langle \lambda | \hat{B} | \lambda \rangle \langle \lambda | \Psi \rangle \quad (11)$$

$$= \int d\lambda |\langle \Psi | \lambda \rangle|^2 \langle \lambda | \hat{A} \rangle \langle \lambda | \hat{B} \rangle \langle \lambda | \Psi \rangle. \quad (12)$$

This has the structure of the right side of eq. (9) provided we identify $\langle \lambda | \hat{A} | \lambda \rangle$ with the function $A(\lambda)$ and $|\langle \Psi | \lambda \rangle|^2$ with the density $\rho(\lambda)$. Indeed the density is positive and normalized (the latter because eq. (10)). Eq. (11) follows from the equality

$$\langle \lambda | \hat{A} | \lambda' \rangle = \langle \lambda | \hat{A} | \lambda \rangle \delta(\lambda - \lambda'), \quad (13)$$

$\delta$ being Dirac’s delta, which is a consequence of $|\lambda\rangle$ and $|\lambda'\rangle$ being eigenvectors of $\hat{A}$.

We see that hidden variables are always possible, a fact made clair by J. S. Bell in 1966. However, some families of HV theories are excluded, for instance those in which expectations fulfil linear relations of the form

$$\langle \Psi | \hat{A} + \hat{B} | \Psi \rangle = \int [A(\lambda) + B(\lambda)] \rho(\lambda) \, d\lambda, \quad (14)$$
The impossibility of such HV theories is the content of von Neumann’s theorem mentioned above. Assumption (14) is unphysical, as pointed out by Bell, which shows that von Neumann’s theorem is not very relevant. More physical requirements are noncontextuality and locality which we discuss in the following.

IV. Noncontextual hidden variables

Definition 6 A HV theory is noncontextual if there exists a joint probability distribution for all observables of the system (even if some of them are not compatible.)

In particular this implies that the marginal for the variable A in the joint distribution of the compatible observables A and B is the same as the marginal for A in the joint distribution of the compatible observables A and C, even if B and C are not compatible. For this reason noncontextuality is sometimes stated saying that the result of measuring A does not depend on the context (in particular, the result is the same whether we measure A together with B or we measure A together with C; remember that A, B, C cannot be measured simultaneously, that is with the same experimental set up). The latter property is true in quantum mechanics, but the existence of a joint distribution is a stronger constraint. What is required is the existence of some function of all the observables, p(A,B,C...), which fulfils the mathematical properties of a joint probability distribution and it is such that the marginals for every subset of compatible observables is the same given by quantum mechanics. The said distribution is just a mathematical object (it cannot be measured if some of the observables are not compatible) but their mere existence puts constraints which may be tested empirically.

It is not difficult to see that the existence of a joint distribution for the observables A, B, C,... is equivalent to the existence of a positive normalized function, \( \rho(\lambda) \) of a variable or set of variables, \( \lambda \), and functions A(\( \lambda \)), B(\( \lambda \)), C(\( \lambda \))... However a joint probability distribution cannot be obtained with the construction of eq. (12) if the observables are not compatible. This is because a complete orthonormal set of simultaneous eigenvectors of \( \hat{A}, \hat{B}, \hat{C},... \) may not exist if the operators do not commute pairwise. What may be obtained are several HV theories, one for each simple experiment. For instance, let us consider a composite experiment consisting of two simple ones. In the first,
where we measure $A$ and $B$, a HV theory should provide the functions $\rho_1(\lambda)$, $A_1(\lambda)$, $B_1(\lambda)$. In the second, where we measure $A$ and $C$, a HV theory would give $\rho_2(\lambda)$, $A_2(\lambda)$, $C_2(\lambda)$. The two HV theories together might be called a HV theory for the composite experiment. It would be noncontextual if $\rho_1(\lambda) = \rho_2(\lambda)$ and $A_1(\lambda) = A_2(\lambda)$, if this does not happen it should be contextual.

The impossibility of noncontextual theories is established by the following

**Theorem 7** Noncontextual HV theories do not exist for all (composite) experiments.

This is usually called Kochen-Specker theorem after the authors who proved it in 1967. However the theorem had been actually proved one year earlier by Bell and it is a rather direct consequence of a theorem proved in 1957 by Gleason. We shall give here a proof inspired in the celebrated theorem of Bell against local hidden variables.

**Proof:** It is enough to exhibit a particular type of composite experiment where the quantum predictions are incompatible with the existence of a joint probability distribution for all observables. We consider four dichotomic observables, $A$, $B$, $C$ and $D$, each of which may take the values 0 or 1. We assume that $A$ and $C$ are not compatible, and $B$ and $D$ are also not compatible, the remaining pairs being compatible. The corresponding operators will be projectors, i.e. $\hat{A}^2 = \hat{A}$, etc., all pairs commuting except

$$[\hat{A}, \hat{C}] \neq 0, \quad [\hat{B}, \hat{D}] \neq 0.$$  \hspace{1cm} (15)

Let us label $p_A$ the probability of $A = 1$, $p_{AB}$ the probability that $A = B = 1$, etc. The existence of a joint distribution means that there are 15 positive quantities

$$p_A, p_B, p_C, p_D, p_{AB}, p_{AC}, p_{AD}, p_{BC}, p_{BD}, p_{CD}, p_{ABC}, p_{ABD}, p_{ACD}, p_{BCD}, p_{ABCD},$$  \hspace{1cm} (16)

which should fulfil the relations

$$0 \leq p_{ABCD} \leq p_{ABC} \leq p_{AB} \leq p_A \leq 1,$$  \hspace{1cm} (17)

and those obtained by all permutations of the labels. Only 8 of these quantities may be measured (and they are predicted by quantum mechanics), namely

$$p_A, p_B, p_C, p_D, p_{AB}, p_{AD}, p_{BC}, p_{CD}.$$  \hspace{1cm} (18)
The remaining 7 quantities cannot be measured, the corresponding observables not being compatible, and quantum mechanics gives no value for them.

The question is whether there exist 7 quantities fulfilling all constraints of the type (17) which added to the 8 measurable ones provide the desired joint probability distribution (16). Now a necessary condition for the existence of a joint probability distribution is the fulfillement of the Bell inequalities discussed above. A sufficient condition is the fulfillement of the 4 Bell inequalities obtained by suitable permutation of labels in (3) or (8) (that is substituting A for C or D for B or both). The rest of the proof consists of showing that there are states and observables for which quantum mechanics violates the inequalities, which may be seen elsewhere, e. g.

V. Local hidden variables

An important class of HV theories are local HV theories. The concept of local applies to EPR experiments. We call EPR an experiment where we prepare locally a system which is later divided in two subsystems, each of which moves in a different direction. Measurements on each subsystem are later made at space-like separation (in the sense of relativity theory).

Definition 8 A HV theory is local if, for any EPR experiment where we may measure one of several observables, \( A_i \), of the first subsystem and one of several observables, \( B_j \), on the second, there exist a joint probability distribution for all the observables \( \{A_i, B_j; i, j = 1, 2, \ldots\} \).

The impossibility of local HV theories is established by the celebrated Bell’s theorem of 1964.

Theorem 9 Local HV theories do not exist for all (EPR) experiments.

Proof: The proof is the same as for noncontextual HV theories, but considering an EPR experiment. That is, the observables A, C belong to one subsystem and B, D to the other subsystem. In particular, this guarantees that the pairs \( \{A, B\}, \{A, D\}, \{C, B\}, \{C, D\} \) are compatible because they belong to spacelike separated regions (the condition that spacelike separated observables are compatible is called microcausality in quantum field theory).

The class of local theories is wider than that of noncontextual HV theories because the constraints in their definition are weaker. Indeed in local theories
the existence of a joint distribution is only required for EPR experiments, but noncontextual theories assume it for all experiments. Consequently the empirical disproof is easier for noncontextual theories than for local theories. In the former it is enough to perform a composite experiment where the measurements are made locally, the latter requires measurements at spacelike separation.

The fact that the proofs of both theorems are very similar has been a source of misunderstanding, like the assertion that locality is not needed in order to prove Bell’s theorem. I hope that in our presentation the point is more clear. But in order to stress the distinction between noncontextual and local I give an illustrative example.

Let us assume that we want to perform a test of Bell’s inequality using two spin-1/2 particles prepared in a singlet (zero total spin) state. An example could be the dissociation of a molecule consisting of two sodium atoms. We should measure the spin components along two directions for each atom (in four different simple experiments, see section 3 for the definition of simple experiment). These directions define the four projectors involved in the Bell inequality. If the inequality is violated we would have an empirical disproof of noncontextual hidden variables theories. However if we want to test local theories, the measurements should be performed at space-like separation, which is a rather strong requirement.

For instance, we might use two Stern-Gerlach apparatuses each of length L. If the atoms move at velocity v, in opposite directions, the duration of the measurement would be L/v. The condition that the measurements are space-like separated means that the distance, d, between the Stern-Gerlach apparatuses should fulfil

\[ d > 2L \frac{c}{v}. \]

This inequality involves the velocity of light as it should, locality (in the sense of Bell) being a relativistic concept. An estimate of the minimal distance is obtained if we use a typical energy involved in dissociation, say 1 eV, and L is of the order of a few centimeters. We get for the minimal distance several kilometers. Thus the empirical violation of local hidden variables is far more difficult than the violation of non-contextual ones.
VI. Quantum information

The amount of information is quantified with the concept of entropy. In classical physics, if we have a continuous random variable, \( \lambda \), with a probability distribution \( \rho(\lambda) \), the entropy, \( S^C \), as defined by Shannon is

\[
S^C = -\int \rho(\lambda) \log \rho(\lambda) \, d\lambda. \tag{19}
\]

The quantum entropy was defined by von Neumann in terms of the density operator, \( \hat{\rho} \), with an expression which looks similar to that one, namely

\[
S^Q = -\text{Tr} (\hat{\rho} \log \hat{\rho}). \tag{20}
\]

In both cases \( S \geq 0 \) and the entropy increases with the lack of information, so that the pure states (maximal information) corresponds to \( S = 0 \).

There are two other properties which hold true for both classical and quantum entropy:

**Concavity:** \( \lambda S(\rho_a) + (1 - \lambda) S(\rho_b) \leq S(\lambda \rho_a + (1 - \lambda) \rho_b) \), \( 0 \leq \lambda \leq 1 \),

where \( \rho_a \) stands for either the classical probability density, \( \rho_a(\lambda) \), or the quantum density operator, \( \hat{\rho}_a \), and similarly \( \rho_b \) for a different probability density or density operator of the same system.

**Subadditivity:** \( S(\rho_{12}) \leq S(\rho_1) + S(\rho_2) \),

where \( \rho_{12} \) stands for either the classical probability density, \( \rho_{12}(\lambda_1, \lambda_2) \), or the quantum density operator, \( \hat{\rho}_{12} \), the subindex 1 (2) referring to the first (second) subsystem of a composite system, and we have

\[
\rho_1(\lambda_1) = \int \rho_{12}(\lambda_1, \lambda_2) \, d\lambda_2, \hat{\rho}_1 = \text{Tr}_2 \hat{\rho}_{12}. \tag{21}
\]

There is, however, a property which dramatically distinguish classical from quantum entropy. In fact in the case of a system consisting of two subsystems, the classical, Shannon’s, entropy fulfills

\[
S^C(\rho_{12}) \geq \max \{ S^C(\rho_1), S^C(\rho_2) \}, \tag{22}
\]

whilst the quantum entropy fulfills the weaker triangle inequality

\[
S^Q(\hat{\rho}_{12}) \geq |S^Q(\hat{\rho}_1) - S^Q(\hat{\rho}_2)|. \tag{23}
\]
In my opinion, the fact that the quantum entropy does not fulfill an inequality similar to (22) is highly paradoxical, I would even say bizarre. In fact, (23) allows for the possibility that both $S_Q(\hat{\rho}_1)$ and $S_Q(\hat{\rho}_2)$ are positive whilst $S_Q(\hat{\rho}_{12})$ is zero. This should be interpreted saying that we have complete information about a composite system whilst we have incomplete information about every subsystem. This contrast with the classical, and intuitive, idea that full information about the whole means that we have complete information about every part. In my view this is indicative that the concept of "complete" information in quantum theory is not the same as in classical physics, and the different meanings of completeness has been the source of misunderstandings about the interpretation of quantum theory, e.g. in the debate between Einstein and Bohr.

The violation of an inequality similar to (22) is closely related to the violation of the Bell inequality. But in order to establish the connection it is necessary to introduce the concept of linear entropy. Actually, although the definitions of entropy (19) and (20) are standard and in some sense an optimum, it is possible to give alternative definitions of entropy which fulfill the essential properties of concavity and subadditivity. The most simple are the so-called linear entropies

$$S^{CL} = 1 - \int \rho(\lambda)^2 d\lambda, S^{QL} = 1 - Tr(\hat{\rho}^2).$$

(24)

The desired connection between linear entropy and the Bell inequalities has been studied by several authors in the last few years. For instance Horodecki et al. proved that the inequality (22) is a sufficient condition for the Bell inequalities. A slightly stronger result may be stated as follows

**Theorem 10** The inequality

$$MNS^{QL}(\hat{\rho}_{12}) + MN - M - N \geq NS^{QL}(\hat{\rho}_1) + MS^{QL}(\hat{\rho}_2),$$

(25)

where $M$ and $N$ are the dimensions of the Hilbert spaces of the two subsystems, is a sufficient condition for all Bell inequalities (1) or (8) which may be got using two dichotomic observables of each subsystem.

**Proof:** We consider observables \{a, b\} for the first particle and \{c, d\} for the second, all of which may take values 1 or -1, and the associated operators, $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$. We define the Bell operator, $\hat{B}$, by

$$\hat{B} = \hat{a} \otimes \hat{b} + \hat{c} \otimes \hat{b} + \hat{c} \otimes \hat{d} - \hat{a} \otimes \hat{d}.$$  

(26)
It is easy to see that

\[ \text{Tr} \hat{B} = 0, \text{Tr} \left( \hat{B}^2 \right) = 4MN, \]  

(27)

and that the Bell inequality (8) is violated if

\[ |\beta| > 2, \beta \equiv \text{Tr} \left( \hat{B} \hat{\rho}_{12} \right), \]  

(28)

(whilst quantum mechanics predicts just \(|\beta| \leq 2\sqrt{2}\)). Now the obvious inequality

\[ \text{Tr} \left( \hat{\rho}_{12} - \frac{1}{N} \hat{\rho}_1 \otimes \hat{I}_2 - \frac{1}{M} \hat{I}_1 \otimes \hat{\rho}_2 + \frac{1}{MN} \hat{I}_1 \otimes \hat{I}_2 + \lambda \hat{B} \right)^2 \geq 0, \lambda \in R, \]  

(29)

where \( \hat{I}_1 (\hat{I}_2) \) is the identity operator for the first (second) particle, gives a quadratic expression in the variable \( \lambda \). We get, after some algebra

\[ M \text{NT} \text{r} \left( \hat{\rho}^2_{12} \right) - N \text{Tr} \left( \hat{\rho}^2_1 \right) - M \text{Tr} \left( \hat{\rho}^2_2 \right) \geq \frac{1}{4} (\beta^2 - 4). \]  

(30)

Hence the inequality (24) implies \(|\beta| \leq 2\), which proves the theorem.

Actually the inequality (24) is rather strong, and therefore not very useful, if either \( M > 2 \) or \( N > 2 \) or both, and it is trivial if either \( M = 1 \) or \( N = 1 \). Consequently its main interest is the case \( M = N = 2 \), where it is a consequence of the inequality (22) characteristic of classical information theory.

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