Heat conduction and Wiedemann-Franz Law in disordered Luttinger Liquids

Mei-Rong Li\textsuperscript{1} and E. Orignac\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1
\textsuperscript{2}LPTENS, CNRS UMR8549, 24, Rue Lhomond - 75231 Paris Cedex 05, France

We consider heat transport in a Luttinger liquid (LL) with weak disorder and study the Lorenz number for this system. We start at a high-$T$ regime, and calculate both the electrical and thermal conductivities using a memory function approach. The resulting Lorenz number $L$ is independent of $T$ but depends explicitly on the LL exponents. Lowering $T$, however, allows for a renormalization of the LL exponents from their bare values by disorder, causing a violation of the Wiedemann-Franz law. Finally, we extend the discussion to quantum wire systems and study the wire size dependence of the Lorenz number.

PACS Numbers: 71.10.Pm, 72.15.Eb, 72.15.Nj

In this letter, we first examine the WF law for an infinite LL with a nonzero concentration of impurities. Unlike KF, we calculate $\kappa(T)$ and $\sigma_e(T)$ in the high-$T$ regime where impurity scattering can be treated by the memory function method \cite{19,16}. The resulting Lorenz number is independent of $T$ but depends on the LL exponents. It differs from the one obtained by KF. As $T$ decreases, we show that the Lorenz number acquires $T$ dependence as the consequence of instability towards Anderson localized state. Such a violation of the WF law is in sharp contrast to the higher-d FL situation. Finally, we examine the effects of contact in a realistic quantum wire measurement and discuss the wire-size dependence of the Lorenz number.

Conductivities from Memory function method. A LL of spin-1/2 fermions in the presence of impurities is described by the following Hamiltonian \cite{11}

\begin{equation}
H = H_{LL} + H_{\text{imp}} = \int dx \, \mathcal{H}(x),
\end{equation}

\begin{equation}
H_{LL} = \sum_{i=\rho,\sigma} \int dx \left\{ u_i K_i [\pi \Pi_i(x)]^2 + \frac{v_i}{K_i} [\partial_x \phi_i(x)]^2 \right\},
\end{equation}

\begin{equation}
H_{\text{imp}} = -\frac{\sqrt{2}}{\pi} \int dx \, \eta(x) \partial_x \phi_{\rho} + \int dx \left\{ \frac{\xi(x)}{\pi a} e^{i\sqrt{2} \phi_{\rho}} \cos[\sqrt{2} \phi_{\rho}(x)] + \text{H.c.} \right\},
\end{equation}

where $K_i$ are the LL exponents, $u_i$ the velocities of the excitations, $a$ the lattice constant, and $\Pi_i(x)$ and $\phi_i(x)$ are canonical momenta and coordinates, respectively. $\xi(x)$ in Eq. 3 is the $2k_F$ component of the impurity potential, assumed to have a Gaussian distribution with zero average, $\xi(x)\xi^*(x') = D_L \delta(x-x')$. $\eta(x)$ is the forward scattering component. The charge and heat currents $J_c = \int dx \, J_c(x)$ and $J_Q = \int dx \, J_Q(x)$ are obtained from the continuity equations $\partial_t J_c(x) + \partial_x n(x) = 0$ and $\partial_t J_Q(x) + \partial_x \mathcal{H}(x) = 0$, where $n(x) = -\frac{g}{2a} \partial_x \phi_{\rho}(x)$ is the electron density operator. We find

\begin{equation}
J_c(x) = \sqrt{2} u_{\rho} K_{\rho} \Pi_\rho(x),
\end{equation}

\begin{equation}
J_Q(x) = -u_{\rho}^2 \Pi_\rho \partial_x \phi_{\rho} - u_{\sigma}^2 \Pi_\sigma \partial_x \phi_{\sigma} + J_Q^0(x),
\end{equation}

In the following, we examine the effects of contact in a realistic quantum wire measurement and discuss the wire-size dependence of the Lorenz number.
where $J_p^p(x) = \sqrt{2} u_p K_p \eta(x) \Pi_p(x)$. We note that the transformation $\tilde{\phi}_p(x) = \phi_p(x) - \sqrt{2 u_p K_p} \int^x \eta(x') dx'$ eliminates $\eta$ from both $H_{imp}$ and $J_Q$, implying a null effect of $\eta(x)$ on $\sigma_c$ and $\kappa$. Thus we set $\eta(x) \equiv 0$ heaftter. Eqs. (1) and (2) also show that the heat and charge current operators are quadratic and linear in boson operators, respectively. This is crucial in resulting in deviation of the Lorenz number from the universal number at high $T$ even for spinless fermion case as shown below.

In the absence of impurities, both $J_c$ and $J_Q$ are conserved currents. In fact, $J_Q$ is (up to a prefactor) the total momentum of the system, and its conservation results from translational invariance. With impurities, neither $J_c$ and $J_Q$ are conserved and finite conductivities can be expected [13]. At sufficiently high $T$, quantum effects are cut off by inelastic thermal processes, and disorder scattering can be treated within the Born approximation. This cuts conductivities to leading order in $D_\xi$. A convenient formalism to do this is the memory function method [13,19], in which the finite-frequency electrical and thermal conductivities are expressed in terms of the memory functions $M_j(\omega) = \chi_{j}^{-1} \omega^{-1} \{ \langle\langle F_j; F_j\rangle\rangle - \langle\langle F_j; F_j\rangle\rangle_{\omega=0} \} (j = c, Q)$ as

$$\sigma_c(\omega) = i e^2 \chi_c[\omega + M_c(\omega)]^{-1}, \quad \kappa(\omega) = i \chi_Q T^{-1} \omega + M_Q(\omega)]^{-1}, \quad (6)$$

with $\chi_c = T 0^{1/2} d\lambda (J_c(0)^l J_c(i\lambda))$ the static current susceptibilities, $F_j(x) = \langle F_j(x) H \rangle$, and $\langle\langle F_j; F_j\rangle\rangle = - \int dxdx' \int_0^\infty dt e^{i\omega t} \langle F_j(x, t) F_j(x', 0) \rangle H$. To leading order in $D_\xi$, both $\chi_c$ and $\chi_Q$ take their pure LL values, $\chi_c \simeq 2 \pi u_p K_p \chi_Q \simeq 2 \pi u_p / (2\pi T)^2$, and

$$\langle\langle F_j; F_j\rangle\rangle_{\omega} \simeq D_\xi 2 K_i + 2(\pi a) / K_i - 2 u_p K_p - u_s K_s + K_s A_j(\omega) \times \sin \left( \frac{\pi K_i}{2} B \left( \frac{K_i}{2} - \frac{\omega}{2\pi T} - 1 - K_i \right) \right), \quad (7)$$

where $K_i = K_p + K_s$, $B(x, y)$ is the Euler Beta function, $\alpha_c = 1 - K_i$, $\alpha_Q = \alpha_c - 2$, $A_c(\omega) = K_p^2 / 2 \pi^2$, and $A_Q(\omega) = \pi \left[ K_i^2 / 2 + (\omega/2\pi T)^2 \right] (K_i^2 + K_s^2) / 2 (K_i + 1)$. Inserting Eqs. (3) into Eqs. (6,7), and after some straightforward algebra, we find that the dc conductivities become

$$\sigma_c(T) \simeq \frac{e^2 G(K_i)}{\pi T^2 (K_i/2)} \frac{a}{\pi D} \left( \frac{2\pi a T}{u_s} \right)^{2-K_s} \quad = 2 e_0(T), \quad (9)$$

$$\kappa(T) \simeq \frac{2\pi^2 T}{9} \frac{K_i}{K_i^2/2} \frac{u_p + u_s}{u_p^2 K_p + u_s^2 K_s} \frac{\Gamma(K_i)}{2} \times \frac{a}{\pi D} \left( \frac{2\pi a T}{u_s} \right)^{2-K_s} \propto C_v(T) \epsilon_0(T), \quad (10)$$

where $D = (2D_i a / u_s^2 u_p) (u_s / u_p')^{K_s}$ is a dimensionless disorder parameter [11], $\Gamma(x)$ the Gamma function, $C_v(T) \propto T$ the specific heat of a pure LL, and $\epsilon_0(T)$ the elastic mean free path [8]. Both $\sigma_c$ and $\kappa$ exhibit power-law behavior in $T$ with nonuniversal exponents.

Lorenz number at high $T$. From Eqs. (6) and (10), one has

$$L = \frac{\kappa}{\sigma_c T} = \frac{\pi^2}{9\kappa(T)} \frac{u_p + u_s}{u_p^2 K_p + u_s^2 K_s} \frac{1 + K_i}{K_i}, \quad (11)$$

$K_p, K_s, u_p, u_s$ are bare intrinsic parameters, so $L$ is independent of $T$, and a generalized WF law is obeyed. In the noninteracting case, $K_p = K_s = 1$, $u_p = u_s = v_F$, the universal Lorenz number $L = L_0$ is recovered from Eq. (1).

The result (11) can be reduced to the spinless LL case, by making $u_p = u_s = u, K_p = K_s = K$. We obtain

$$L' = \left( \frac{\pi^2}{9\kappa(T)} \right) (K^2 - 2K)^{-1} \quad (12)$$

It follows that $L' < L_0$ for an attractive interaction case $K > 1$, as a result of tendency towards a superconducting state with high electrical conduction but poor thermal conduction: While for a repulsion case $K < 1$, $L' > L_0$, indicating tendency towards a weakly pinned charge density wave (CDW) state with better heat transport than charge transport. It is instructive to compare $L'$ with $L(KF) = (\pi^2/9)(K^2 + 2K)^{-1}$ obtained by KF for a single strong impurity at $K < 1$ [20]. We see that $L' - L(KF) = 2(K - 1)^2/9K^2(K + 2) \geq 0$. From physical point of view, $L(KF)$ and $L'$ can be roughly understood as the results for a strongly pinned and a weakly pinned CDW phase, respectively, so that we should expect a larger $L(KF)$ than $L'$. This puzzle can be resolved by noting that some Hamiltonian terms causing heat but no charge conduction are neglected in the derivation of the tunneling Hamiltonian used to calculate $L(KF)$ [24]. Including these terms explicitly for $K = 1/2$ indeed leads to $L(KF) = \pi^2/9L(KF) > L'$.

We would also like to remark that the single strong impurity case considered by KF corresponds to a low energy fixed point which cannot be reached from the limit of weak impurity scatterers at high concentration $\epsilon$ we consider [21]. The KF fixed point is realized when the renormalized impurity strength $t_B(\ell) = e^2(2-2K)/l_B(0)$ becomes of the order of the high energy cutoff $W \sim u/a$. This requires the renormalized length $a t_B' = a(\ell_B(0)/\ell_B(\ell))^{1/(2-2K)}$ to remain still much smaller than the inter-impurity distance $1/c$. In the high impurity concentration limit, $c \to \infty$, $t_B \to 0$ with $D = ct_B^2$ fixed, obviously such regime cannot be observed.

Lorenz number with decreasing $T$. The enhanced quantum interference effects, which are responsible for occurrence of the Anderson localization at $T < T_{loc} = u_s/\ell_{loc} (\ell_{loc} = a D^{-1/(3-K_s)} [11]$), lead to renormalization of $K_p, K_s, \kappa, \sigma_c$ and $D$. Now we study the influence of such renormalization on $L$ for $T$ still much larger than $T_{loc}$, so that a perturbative renormalization group (RG) theory can be used. We first neglect renormalization of $K_p$ and $K_s$ by $D$. The RG flow equation determining the renormalization of $D$ reads [11] $dD(l)/dl = (3 - K_s) D(l)$, which leads to
$$D(l)/D = e^{(3-K_f)l}.$$  \tag{13}$$

It is clear that $K_f = 3$ defines a metal-insulator transition (MIT) line: for $K_f < 3$, $D(l)$ grows exponentially with renormalized length, and the Anderson localization will take place eventually; Whereas for $K_f > 3$, $D(l)$ flows to zero, corresponding to a delocalized phase. By defining a $T$-dependent scale $l^*(T)$ from $ae^{l^*(T)} = u/a/(2\pi T) = l_{th}(T)(l_{th}$, the thermal length), we see that Eq. \[(3)\] yields $(a/\pi D)(2\pi aT/u_a)^{2-K_f} = ae^{l^*(T)}/[\pi D(l^*)]$, which allows for a simple RG interpretation for $\sigma$ and $\kappa$ in Eqs. \[(1)-(10)\]: They can be obtained by applying the RG flow up to the scale $l^*(T)$ at which $T$ becomes order of the energy cutoff $\pi u_a/a$. At this scale, thermal effects suppress quantum effects, and the use of the memory function method is justified. One can then perform a memory function calculation in which the bare parameters $D$ and $a$ are replaced by $D(l^*)$ and $ae^{l^*}$, respectively.

A similar analysis can be made for inclusion of renormalization of $K_\rho$ and $K_\sigma$ by $D$. We start from the bare parameters $K_\rho(0)$, $K_\sigma(0)$ and $D_\xi(0)$ and follow their RG flow, being described by Eqs. (3.4) in Ref. \[11\], up to the scale $l^*(T)$. Since $D_\xi(l^*)$ $\ll$ 1 still holds for $T \gg T_{loc}$, but the quantum phase coherence has started to lose, Eqs. \[(3)-(10)\] are valid again \[24\], but with the bare $K_\rho$ and $K_\sigma$ replaced by the renormalized ones, $K_\rho(l^*)$ and $K_\sigma(l^*)$, respectively. The resulting RG-improved $L$ acquires $T$ dependence through $K_f$ and $K_\sigma$, and the generalized WF law holding at high $T$ breaks down. This is in contrast to a higher-$d$ FL case, in which such a renormalization is always negligible \[3\].

We proceed with a discussion of the spinless fermion case for simplicity. The above analysis suggests that $L$ in Eq. \[(3)\] as a function of the initial parameter $K$ becomes

$$L(T) = (\pi^2/9e^2) \{[K(l^*)]^{-2} + 2[K(l^*)]^{-1}\}, \tag{14}$$

where $K(l^*)$ can be obtained from the RG flow equation \[(11)\], $dK(l)/dl = -K^2(l)D(l)/2$. For a generic initial $K$ away from the MIT line, the RG flow line in the $K$-$D$ plane is almost vertical, and the deviation of $L$ from its high-$T$ value is in order of $D$ which is insignificant unless $T$ becomes close to $T_{loc}$. Alternatively, considering the second order terms in $D$ in $\sigma$, and $\kappa$ one also obtains a $T$-dependent power-law correction term to $L(T)$ with the exponent predicted by the RG method.

However, in the vicinity of the MIT line, the RG flow is no longer vertical, leading to a faster variation of $L$ with $T$. $K(l^*) - K$ is found to be proportional to $\sim \sqrt{D(l^*)} - D$, indicating that $L(T)$ shown in Eq. \[(14)\] is systematic. $L(T)$ for this case is shown in Fig. \ref{fig:lorenz_number}. We emphasize that such a behavior is not expected in a higher-$d$ FL case, in which the Lorenz number is constant even close to the MIT point \[2\].

Lorenz number in dirty quantum wires. An experimental realization for the measurement of the transport properties of a LL is through a finite size wire connected to leads \[25\] or a carbon nanotube \[26\]. The theory presented here for a bulk material needs to be modified, as the boundary condition imposed by the contact is shown \[27\] to drastically influence the transport through the wire. Charge crosses the contact in terms of pieces of fractional charges \[28\], whereas heat is carried by plasmon modes and they cross the contact like waves passing through barriers \[22\]. The Lorenz number will obtain a nontrivial wire size dependence as shown below. For simplicity we only discuss about spinless LL wires, and leave the discussion of the spinful case for a future publication \[29\]. Besides, we follow Ref. \[27\] and model the leads as 1d noninteracting FLs.

We first briefly recall the clean wire case. The electrical conductance $G$ is purely from the contact and is quantized \[27\], $G = G_0 = e^2/2\pi$ (for $h = 1$), which implies a perfect transmission of the total charge after a long time (dc) measurement \[28\]. The mismatch of the plasmon wave velocities at the two sides of the contact leads to $T$-dependent thermal conductance $K_{th}$ normalized by $T$ \[22\], which, at low temperatures $T \ll u/d$ ($d$ the wire size), asymptotically approaches to a constant $K_{th}/T \simeq \pi/6$ \[22\]. The WF law is typically violated for a generic $T$ and is restored at $T \ll u/d$.

Let us now turn to the case of a wire with impurities. We consider the case of $l_{loc}$ being much larger than $d$ and/or $l_{th}$ so that we can neglect Anderson localization. The total electrical resistance is the sum of the contact resistance $G_0^{-1}$ and the wire resistance $G_{wire}^{-1}$, resulting in the following total conductance,

$$G = (G_0^{-1} + G_{wire}^{-1})^{-1}. \tag{15}$$

$G_{wire}$ depends on the relative values of the two length scales $d$ and $l_{th}(T)$.

1) At $d \gg l_{th}$ ($T \gg u/d$), a simple Ohm law holds for the electrical conductance, i.e., $G_{wire} = \sigma(T)/d = G_02\pi l_{th}/d$ with $\sigma(T)$ and $l_{th}$ being found in Eq. \[(1)\]. Eq. \[(13)\] reduces, for $d \ll l_{th}(T)$, to Maslov’s perturbation

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Lorenz number as a function of temperature close to the MIT line for spinless fermions. $W \sim u/a$ is a high energy cutoff. The initial conditions are $K(0) = 3/2, D(0) = 10^{-2}$.}
\end{figure}
result \[ G \simeq G_0[1 - d/(2\pi l_d)] \], and, for \( d \gg l_d(T) \), crosses to the Drude formula. In this regime, the thermal conductance of even a clean wire, \( \kappa_{\text{th}} \), from Eq. (9) of Ref. \[24\], depends on the details of the barrier, so that the thermal conductance of the dirty wire becomes a rather non-universal function of disorder, wire size, and temperature \[29\].

2) At \( d \ll \theta_{\text{th}} \) \( (T \ll u/d) \), Eq. (9) of Ref. \[23\] indicates that heat is carried by plasmon modes of wavelength much larger than the distance between the contacts, and thermal conductivity of the clean wire becomes universal. In this regime, we can make use of the \( T \rightleftharpoons u/d \) equivalence \[31\] to obtain an explicit \( d \)-dependent Lorenz number. According to Ogata and Fukuyama, the finite size effect can be taken into account within the memory function formalism by replacing \( \omega \) in Eqs. (16) and (17) by \( i2u/d \). This leads to

\[
G(d) = \frac{ue^2}{\pi} \frac{1}{2u + 2uC_c(u, K)\beta(d)}, \tag{16}
\]

\[
\kappa_{\text{th}}(d) = \frac{\pi uT}{3} \frac{1}{2u + 2uC_{\text{th}}(u, K)\beta(d)}. \tag{17}
\]

with \( \beta(d) = (d/l_{\text{loc}})^{3\gamma - 2K} \), and \( C_c(u, K) \) and \( C_{\text{th}}(u, K) \) being easily found by comparing Eqs. (16) and (17) with Eqs. (6) and (7) respectively. In writing down Eq. (16) we have carefully taken into account the screening of the electric field coming from the leads \[22\]. We recover a universal Lorenz number in this regime in the absence of impurities in agreement with \[22\]. In the presence of impurities, Eqs. (16) and (17) immediately yield

\[
L(d) \simeq L_0 \frac{1 + C_c(u, K) \beta(d)}{1 + C_{\text{th}}(u, K) \beta(d)}, \tag{18}
\]

which is a function of the wire size since \( C_c(u, K) \neq C_{\text{th}}(u, K) \) in the presence of interactions. Thus, deviation from WF law is obtained in finite size systems with impurities even for \( T \ll u/d \).

**Conclusion.** We have investigated the WF law in a disordered LL system. At high \( T \) where thermal effects cut off Anderson localization, the Lorenz number is constant in \( T \), and, in the spinless fermion case, smaller (larger) than the universal \( L_0 \) for attraction (repulsion) between the fermions. Its dependence on the LL exponents is different from the one obtained in a single strong impurity case \[20\]. As \( T \) goes down, Anderson localization effects induce an interplay between disorder and electronic interactions, which is responsible for the violation of the WF law. When a dirty 1d wire is connected to leads we find the Lorenz number becomes a function of the wire length.

**Acknowledgments.** We are indebted to N. Andrei, P. J. Hirschfeld, A. Rosch, and Y.-J. Wang for valuable discussion. We also thank T. Giamarchi, Y. Suzumura, and I. Vekhter for comments on the manuscript.