Orientations and $p$-adic analysis

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Abstract. Matthew Ando produced power operations in the Lubin-Tate cohomology theories and was able to classify which complex orientations were compatible with these operations. The methods used by Ando, Hopkins and Rezk to classify orientations of topological modular forms can be applied to complex K-Theory. Using techniques from local analytic number theory, we construct a theory of integration on formal groups of finite height. This calculational device allows us to show the equivalence of the two descriptions for complex K-Theory. As an application we show that the $p$-completion of the Todd genus is an $E_{\infty}$ map.
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1. Introduction

The theory of power operations in cohomology theories arose with Steenrod operations in Eilenberg MacLane spectra over \( \mathbb{F}_p \). Another example is Atiyah’s construction of power operations in complex K-Theory using representation theory techniques \([8]\). In both cases, a novel description of cohomology operations was obtained. Other examples include tom Dieck power operations for bordism theories associated to classical Lie groups \([16]\).

In general, power operations are internal properties of a spectrum. Thus a natural question to ask is when such structure is preserved by maps of spectra. Recall that an orientation is a map of spectra whose source is the Thom spectrum of a space of \( BO \). Recently, Ando constructed power operations for the Lubin-Tate spectra \( E_n \) \([7]\). In addition, he produced a necessary and sufficient condition for a complex orientation of \( E_n \) to be compatible with the power operations in complex cobordism and \( E_n \).

Continuing this program, Ando along with Hopkins and Strickland generalized this result to different types of Thom spectra associated to various spaces over \( BO \) \([6]\). One of the goals of their article was to explain the existence of the Witten genus in a purely homotopical way. In \([5]\) an \( E_\infty \) map from the Thom spectrum of the eight connected cover of \( BO \), \( BO\langle 8 \rangle \), to the spectrum of topological modular forms was produced. The projection of this map to the elliptic spectrum attached to the Tate curve produces the Witten genus. In order to produce the map to topological modular forms, they needed to study congruences among the Bernoulli numbers of a formal group. An outline of this program, along with its overall place in the tapestry that is mathematics, can be found in the inspiring article of Hopkins \([19]\).

In this note we describe how the necessary and sufficient condition of Ando implies the \( E_\infty \) condition for an orientation of \( p \)-adic complex K-Theory. It will turn out that these conditions are equivalent. One of the key ingredients is a theorem of Nicholas Katz involving measures on a formal group. He was interested in producing congruences for sequences attached to elliptic curves over local number fields that arise as the values of \( p \)-adic L-series at negative integers. We use isogenies on formal groups to produce analogous sequences and congruences between them. We believe that this calculation locates the role of isogenies in the description of \([5]\). I would like to thank both Matthew Ando and Charles Rezk for their infinite patience.

1.1. Notation. Recall that a formal group law over a complete local noetherian ring \( R \) is a power series \( f(x, y) \in R[[x, y]] \) that satisfies the formal properties of a group law. If \( F(x, y) \) is such a power series we will often write

\[ x + F y \]

for the formal sum.

For any \( a \in \mathbb{Z} \) we write \( [a]_F(t) \) for the power series obtained by adding \( t \) to itself \( a \) times over the formal group law. If \( a \) is negative, then we use the inverse power series. See the appendix of \([32]\). In the case when \( F \) is a Lubin-Tate formal group law over \( R \) we write \( [a]_F(t) \) for the endomorphism of \( F \) induced by \( a \in R \). For a formal group law we write \( F[p^n] \) for the set of \( p^n \)-torsion points of \( F \). This set is computed by factoring the \( p^n \) series using the Weierstrass preparation theorem.

An (affine commutative dimension one) formal group \( G \) is a functor from complete local rings to abelian groups that is isomorphic to the formal affine
line \( \hat{A}_1 \) after forgetting to sets. Examples include the formal multiplicative group \( \hat{G}_m = \text{Spf} \mathbb{Z}_p[u, u^{-1}] \) and the formal additive group \( \hat{G}_a = \text{Spf} \mathbb{Z}_p[[u, u^{-1}]] \).

The algebraic multiplicative group will be written as \( G_m = \text{Spec} \mathbb{Z}_p[[u, u^{-1}]] \). All formal groups \( G \) are over the complete local ring \( R \) unless otherwise stated.

A coordinate on a formal group \( G \) is a choice of isomorphism of functors between \( G \) and \( \hat{A}_1 \). Such an isomorphism produces a formal group law by considering the sum of the two natural maps \( t_1, t_2 : R[[t]] \to R[[t_1, t_2]] \).

For a prime \( p \), \( \mathbb{Z}_p \) is the \( p \)-adic integers and \( \mathbb{Q}_p \) the \( p \)-adic rationals. The spectrum \( K_p \) is the \( p \)-completion of periodic complex K-Theory. It is a \( K(1) \) local \( E_\infty \) ring spectrum. We write \( MU \) for the Thom spectrum associated to the space \( BU \) over \( BO \).

1.2. Structure. This article has three main parts. In the first we discuss the role of measures in algebraic topology. We build up the language of integration on a formal group and how to calculate in that setting. In the middle of the article we define Bernoulli numbers analogously to modular forms. We then investigate some of their properties and how they arise in \( K(1) \) local homotopy theory. Finally, we give a brief review of the obstruction theory for \( E_\infty \) orientations and verify the conditions we need for our story. We close with a proof of the main result.

2. The Main Result

Suppose \( \alpha : MU \to K_p \) is a homotopy ring map. By Quillen’s theorem \( \alpha \) determines a formal group law \( F \) over the ring \( \mathbb{Z}_p \). Let \( \exp_F x \) be the strict isomorphism from the formal additive group law to \( F \) over \( \mathbb{Q}_p \). Expanding the Hirzebruch series as

\[
\frac{x}{\exp_F x} = \exp \left( \sum_{k \geq 1} t_k \frac{x^k}{k!} \right)
\]

produces a sequence \( \{t_k\} \) attached to the orientation \( \alpha \).

**Theorem 2.1.** For an odd prime \( p \) the map

\[
D : H_\infty(MU, K_p) \to \pi_0 E_\infty(MU, K_p).
\]

that sends \( \alpha \) to the sequence \( \{t_k\} \) is an isomorphism. Moreover, \( D \) is a section of the forgetful map.

Suppose \( E \) is an \( E_\infty \) ring spectrum and \( X \) is a space. Part of the data attached to \( E \) is a collection of functions

\[
E^0(ES_n \times_{\Sigma_n} X^n) \xrightarrow{P} E^0(X)
\]

for each positive integer \( n \) satisfying some formal identities called power operations. More precise definitions along with some applications can be found in [12]. Fix \( E_\infty \) spectra \( E \) and \( F \). The set of homotopy clashe
de

\[
\begin{align*}
E^0(\Sigma_n \times_{\Sigma_n} X) \xrightarrow{P} E^0(\Sigma_n \times_{\Sigma_n} X^n)
\end{align*}
\]

for each positive integer \( n \) satisfying some formal identities called power operations. More precise definitions along with some applications can be found in [12]. Fix \( E_\infty \) spectra \( E \) and \( F \). The set of homotopy clashe
commutes for all spaces $X$ is written $H_\infty(E, F)$. The space of of $E_\infty$ maps from $E$ to $F$ is denoted $E_\infty(E, F)$.

The method for classifying the components of the space of $E_\infty$ maps from complex cobordism to $K_p$ is described in [5]. Each component corresponds to a sequence $\{t_k\}$ satisfying some conditions. We will make these conditions explicit in Theorem 2.4. The source of $\mathbb{D}$ was completely determined by Ando [7]. We will show that the necessary and sufficient condition describing the source is enough to produce the required conditions on the sequence $\{t_k\}$.

### 2.2. Background.

Quillen calculated the effect of the power operations in complex cobordism on the tautological line bundle over $\mathbb{C}P_\infty$ [30]. One of Ando’s insights was the similarity between this calculation of Quillen and an isogeny in the theory of formal groups, which we describe. Given a finite subgroup of a formal group Lubin constructs the quotient formal group [25]. Part of the data is an isogeny from the group to the quotient. This isogeny has the same general form of Quillen’s calculation. Ando’s work lead to the viewpoint that isogenies can be used to construct power operations in complex oriented cohomology theories.

Suppose $p$ is a prime. Let $E_1$ be the Lubin-Tate spectrum associated to a universal deformation of the Honda formal group law over $F_p$ of height one. It is a $K(1)$ local $E_\infty$ ring spectrum and so has a theory of power operations. For odd primes, $K_p$ is a model for $E_1$.

Since complex cobordism is the universal complex oriented spectrum, a map

$$\alpha : MU \to E_1$$

of naive homotopy ring spectra (compatible with smash products) determines a graded formal group law over $\pi_*E_1$. Since $E_1$ is an even two periodic theory, we can consider the ungraded group law over

$$\pi_0E_1 \cong \mathbb{Z}_p.$$  

Call this formal group law $F_\alpha$. We may drop the subscript $\alpha$ when it becomes notationally convenient. The following result is the necessary and sufficient condition for $\alpha$ to be an $H_\infty$ orientation.

**Theorem 2.3 (Ando)**. The orientation $\alpha$ [1] commutes with the power operations in the source and target if and only if

$$ \prod_{c \in F[p]} t + F c = [p] F(t).$$

It is convenient to use the language of measures to describe the target of $\mathbb{D}$. A $\mathbb{Z}_p$ valued measure on $\mathbb{Z}_p$ is a (continuous) linear functional

$$\mu : \text{Cont}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{Z}_p.$$  

We denote the collection of all such by $M(\mathbb{Z}_p, \mathbb{Z}_p)$. The moments of $\mu$ is the sequence obtained by applying $\mu$ to the function $x^k$ for $k \geq 0$. A description of the target in Theorem 2.1 is

**Theorem 2.4 (Ando, Hopkins, Rezk)**. For odd primes $p$, there is a bijection between elements $\alpha \in \pi_0E_\infty(MU, K_p)$ and sequences

$$\{t_k\} \in \prod_{k \geq 1} \mathbb{Q}_p$$

satisfying
(1) For any \( a \in \mathbb{Z}_p^\times - \{\pm 1\} \) there exists a measure \( \mu \) on the \( p \)-adic units, depending on \( a \), such that
\[
\mu(x^k) = \int_{\mathbb{Z}_p^\times} x^k d\mu(x) = t_k(1 - a^k)(1 - p^{k-1})
\]
for all \( k \geq 1 \) and
\[
-\frac{B_k}{k} \equiv t_k \mod \mathbb{Z}_p.
\]

The rational number \( B_k \) is the classical \( k \)-th Bernoulli number with generating series
\[
x = \frac{e^x - 1}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!}.
\]
The congruence in condition two means that the difference between the \( p \)-adic rationals \( t_k \) and \( -\frac{B_k}{k} \) is a \( p \)-adic integer.

3. Cohomology Operations

In this section, we will recall the connection between \( p \)-adic integration and operations in \( K_p \). The relationship between Bernoulli numbers and \( p \)-adic integration has an interesting relationship with topological \( K \)-Theory. One of the first authors to study this phenomenon was Francis Clarke [13]. Additional work involving the Kummer congruences and its role in the homotopy of the space \( BU \) can be found in [9].

Let \( K(1) \) be Morava \( K \)-Theory of height one at the odd prime \( p \) and \( L_{K(1)} \) Bousfield localization with respect to the homology theory \( K(1) \). Suppose \( f : S \to L_{K(1)} (K_p \wedge K_p) \), i.e. \( f \in (K_p)^0 K_p \) and \( a \in \mathbb{Z}_p^\times \). If \( \psi^a \) is the Adams operation at \( a \), then the composition
\[
S \xrightarrow{f} L_{K(1)} (K_p \wedge K_p) \xrightarrow{\psi^a \wedge \text{id}} L_{K(1)} (K_p \wedge K_p) \to L_{K(1)} K_p \cong K_p
\]
is an element of \( \pi_0 K_p \cong \mathbb{Z}_p \). Letting \( a \) vary shows that elements of \( \pi_0 L_{K(1)} (K_p \wedge K_p) \) are continuous functions on \( \mathbb{Z}_p^\times \). Dualizing this observations yields

**Theorem 3.1 (Adams, Harris, Switzer).** There is an isomorphism
\[
\text{Hom}_{\text{Cont}}(\text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p) = M(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong [K_p, K_p] = K_p^0 K_p.
\]

**Proof.** This statement follows from the three authors’ work on operations in integral \( K \)-Theory [1]. They describe the operations in terms of the generalized binomial coefficients \( \binom{x}{y} \). Using \( K(1) \) localization techniques, we can produce this statement from the integral result. Some details are worked out in [5] and in [18]. A result of this type for the Lubin-Tate spectra \( E_n \) for any \( n \geq 1 \) can be found in [20]. \( \square \)

It is shown in [5] that the effect in \( \pi_{2k} \) is given by multiplication by the \( k \)-th moment, \( \mu(x^k) \). Later in the article (Theorem 7.1) we will describe the effect of an operation on the \( K \)-Theory of infinite complex projective space \( CP^\infty \).

4. Measure Theory

In order to construct the map in Theorem 2.1 we will use the theory of integration on a formal group of finite height.
4.1. p-divisible groups. A $p$-divisible group of height $h$ is a system

$$(G_n, i_n)$$

for $n \geq 1$ where

1. Each $G_n$ is a locally free commutative group scheme of rank $p^{nh}$.
2. $i_n: G_n \to G_{n+1}$ is a homomorphism of commutative group schemes.
3. The sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1}$$

is exact.

Given a $p$-divisible group, $(G_n, i_n)$ the colimit

$$\lim_{\leftarrow} G_n$$

is a functor from rings to abelian groups. This construction does not always yield a formal group.

Suppose $G$ is a (affine commutative dimension one) formal group of finite height $h$ over a complete local ring $R$. Suppose further that the residue field is perfect and has characteristic $p$. These hypothesis imply that multiplication by $p$ on $G$ is an isogeny, that is a finite free map of group schemes. If $\Gamma_n$ denotes the kernel of the isogeny $p^n$ on $G$ and $i_n: \Gamma_n \to \Gamma_{n+1}$ is the induced map on the kernels, the system

$$(\Gamma_n, i_n)$$

is a $p$-divisible group of height $h$ over $R$. The current hypothesis on $R$ forces the colimit of $(\Gamma_n, i_n)$ to be the formal group $G$. See [34] for more details on $p$-divisible groups and a proof of this fact.

The Cartier dual of $\Gamma_n$ is represented by the functor

$$\text{Hom}_{\text{Groups}}(\Gamma_n, \mathbb{G}_m)$$

from complete local rings to abelian groups and is denoted by $\Gamma_n^\vee$. The homomorphism $p: \Gamma_{n+1} \to \Gamma_n$ induces a homomorphism $i_n^\vee: \Gamma_n^\vee \to \Gamma_{n+1}^\vee$ on the duals and the data

$$(\Gamma_n^\vee, i_n^\vee)$$

is a $p$-divisible group. As before, the colimit of $(\Gamma_n^\vee, i_n^\vee)$ is a formal group of height equal to the height of $G$.

Notation 1. The $p$-divisible dual of the formal group $G$ is denoted by $G^\vee$.

4.2. Tate Modules. Suppose $G$ is a formal group. Recall that $G[p^n]$ is the set of $p^n$ torsion points of $G$.

Definition 4.3. The Tate module associated to a formal group $G$ is

$$T_p G := \lim_{\leftarrow} G[p^n].$$

The Tate module is a $\mathbb{Z}_p$ module of rank equal to the height of $G$. The Tate module of the $p$-divisible dual of $G$ is the inverse limit of its $p^i$ torsion points of the dual:

$$T_p G^\vee = \lim_{\leftarrow} G^\vee[p^i].$$
4. MEASURE THEORY

Lemma 4.4 (Tate). If $R_C$ is the ring of integers in the completion of an algebraic closure of $R$ then there is an isomorphism of $Z_p$ modules
$$T_pG^\vee \to \text{Hom}_{R_C-\text{Groups}}(G \times R_C, \mathbb{G}_m \times R_C).$$

It is well known that when $G$ is a Lubin-Tate group of height one we can replace $R_C$ with the ground ring $R$.

4.5. Measures on $G$. For now on, we insist that the height of $G$ is one. In this case, the Tate module of the dual of $G$, $T_pG^\vee$ is isomorphic to $Z_p$ as a $Z_p$-module. The Tate module possesses a topology induced by the inverse limit and thus we can consider the $R$ module
$$\text{Cont}(T_pG^\vee, R)$$
of continuous functions to $R$.

Definition 4.6 ([22]). If $S$ is a complete local algebra over $R$ then we define the $S$-valued measures on $G$ as
$$M(G, S) = \text{Hom}_R^{\text{Cont}}(\text{Cont}(T_pG^\vee, R), S)$$

Notation 2. If $\mu \in M(G, S)$ and $f : T_pG^\vee \to R$ we write
$$\int_G f(x) d\mu(x)$$
for the effect of $\mu$ on the function $f$.

If $l \in Z_p$ we can consider $l \in T_pG^\vee$ using the isomorphism of Tate (Theorem 4.4).

Example 4.7 (Dirac Measure). If $l \in Z_p$ define a measure $\psi_l \in M(G, S)$ by
$$\int_G f(x) d\psi_l(x) = f(l)$$

The measure $d\psi_l$ corresponds to the Adams operation $\psi^l$ under the identification in Theorem 3.1. In light of Theorem 2.4 we are interested in integration on the units.

Definition 4.8. Let
$$(T_pG^\vee)^\times$$
be the complement of $pT_pG^\vee$ inside $T_pG^\vee$.

The units of the Tate module is isomorphic to $Z_p^\times \times G$ since $G$ has height one. It inherits a topology from $T_pG^\vee$ and as above write
$$M(G^\times, S)$$
for the $S$-valued measures on $(T_pG^\vee)^\times$.

We will be interested in the image of the natural map
$$\text{res} : M(G^\times, S) \to M(G, S)$$
defined by restricting the source of a continuous function and then integrating.

Definition 4.9 (N. Katz). A measure $\mu$ is supported on the units if it is in the image of the restriction map.
5. Derivations and Measures

In this section we develop the tools to calculate the moments of a measure on a formal group.

5.1. Mahler’s Theorem. Suppose $S$ is a complete local torsion free $\mathbb{Z}_p$ algebra. Write $M(\mathbb{Z}_p, S)$ for the module of linear functionals $\mu : \text{Cont}(\mathbb{Z}_p, \mathbb{Z}_p) \to S$.

Theorem 5.2 (Mahler). There is an isomorphism

$$M(\mathbb{Z}_p, S) \to S[[t]]$$

Proof. The generalized binomial coefficients

$$\binom{x}{n} = \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!}$$

are continuous functions from $\mathbb{Z}_p$ to itself. Mahler showed that these functions generate all continuous functions from $\mathbb{Z}_p$ to itself [27]. In particular, if $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is continuous then

$$f(x) = \sum_n a_n(f) \binom{x}{n}$$

and the $a_n(f)$ tend to zero $p$-adically. To finish the proof, identify the integral of the function $\binom{x}{n}$ with the $n$-th coefficient of the associated power series. \hfill \Box

Mahler’s theorem does not easily produce the moments, $\mu(x^k)$ of a measure. Work of Katz produces a computational method using derivations on a formal group $G$. We will review this work and produce a slight generalization of Mahler’s Theorem.

5.3. Derivations. Suppose that $G$ is a formal group over $R$ with multiplication given by $m : G \times G \to G$, and ring of functions $O_G$. Recall that $O_G \times O_G$ is a commutative ring with multiplication given by

$$(a,m)(b,n) = (ab, an + bm)$$

Definition 5.4. A derivation on $G$ is a map of commutative rings

$$\omega' : O_G \to O_G \times O_G$$

such that the diagram

$$\begin{CD}
O_G @>{\omega'}>> O_G \times O_G \\
@. @VV{p_1}V \\
O_G @<<< O_G \times O_G
\end{CD}$$

commutes.

We write

$$\omega : O_G \xrightarrow{\omega'} O_G \times O_G \xrightarrow{p_2} O_G$$

for the composition of $\omega'$ with projection onto the second factor.
5. DERIVATIONS AND MEASURES

**Definition 5.5.** A derivation $\omega$ is invariant if the diagram of commutative rings

$$
\begin{array}{ccc}
O_G & \xrightarrow{\omega} & O_G \\
m^* \downarrow & & \downarrow m^* \\
O_G \otimes O_G & \xrightarrow{\omega \otimes 1} & O_G \otimes O_G
\end{array}
$$

commutes.

**Example 5.6.** Suppose $t$ is a coordinate on $G$ and $F(x, y)$ is the induced formal group. Write $F_2(x, y)$ for the partial derivative of $F(x, y)$ with respect to the second variable. The map

$$
D_F = F_2(t, 0) \frac{d}{dt} : O_G \rightarrow O_G
$$

is an invariant derivation.

**Definition 5.7.** If $\omega$ is an invariant derivation on $G$ define

$$
\beta^k : O_G \xrightarrow{\omega^k} O_G \xrightarrow{0^*} R.
$$

where the exponent $k$ means compose $\omega$ with itself $k$ times.

5.8. **Computing the Moments of a Measure.** If $t$ is a coordinate on $G$, elements of the Tate module of the $p$-divisible dual, $T_pG^\vee$, are power series $g(t)$ over $R$ such that $g(x + y) = g(x)g(y)$ and $g(0) = 1$. Given an invariant derivation $\omega$ we can construct a continuous function

$$
\langle \omega, - \rangle : T_pG^\vee \rightarrow R
$$

defined by

$$
\langle \omega, g(t) \rangle = \omega g(t)|_{g=0}
$$

If $\text{Diff}(G)$ is the $p$-completed module of invariant differential operators on $G$ then this pairing shows that $O_G$ is the continuous $R$-linear dual of $\text{Diff}(G)$ [22].

Let $S$ be a complete local torsion free algebra over $R$. Hitting the map

$$
\langle - , - \rangle : \text{Diff}(G) \rightarrow \text{Cont}(T_pG^\vee, R)
$$

with $\text{Hom}_R^\text{Cont}(-, S)$ produces a homomorphism

$$
\mathcal{D}_G : M(G, S) \rightarrow O_G \hat{\otimes} S
$$

**Proposition 5.9.** If $G$ is a height one Lubin-Tate formal group over $\mathbb{Z}_p$ with coordinate $t$, then the natural map

$$
\mathcal{D}_G : M(G, S) \rightarrow O_G \hat{\otimes} S \cong S[[t]]
$$

is an isomorphism.

**Proof.** If $t$ is a coordinate on $G$ then $a \in \mathbb{Z}_p$ can be identified with $F_2(t, 0)^a \in T_pG^\vee$. Thus the continuous function

$$
\langle D_F^k, - \rangle : T_pG^\vee \rightarrow \mathbb{Z}_p
$$

corresponds to the function

$$
x^k : \mathbb{Z}_p \rightarrow \mathbb{Z}_p.
$$

The result now follows from Theorem [5.2].
Corollary 5.10. Under the correspondence
\[ \mu \mapsto f_\mu \]
of Proposition 5.9 we have the formula
\[ \int_G x^k d\mu(x) = \beta^k f_\mu \]
where the derivation we are using to define \( \beta \) is \( D_F \).

**Proof.** We can calculate. Define \( c_{m,k} \) by
\[ x^k = \sum_{m=0}^{k} c_{m,k} \binom{x}{m} \]
If \( s \) is the coordinate on \( \hat{G}_m = \text{Spf} \mathbb{Z}_p[[u, u^{-1}]] \) given by \( u - 1 \) then \( D = (1 + s) \frac{d}{ds} \). It is easy to show that
\[ D^k x^m = c_{m,k} \]
for \( k \leq m \). We can obtain the result for any height one Lubin-Tate group by pulling back the derivation \( D \) along an isomorphism to \( \hat{G}_m \). Explicitly, if \( \Theta : G \to \hat{G}_m \) then pulling back induces the formula
\[ \beta^k f(t) = \beta^k_{\hat{G}_m} f(\Theta(s)). \]
\[ \square \]

Remark 5.11. If \( G \) produces a Landweber exact theory \( E \) with \( \pi_0 E = R \), this statement can be realized in topology by writing down the paring between \( E^0 \mathbb{C}P^\infty \) and \( E^\infty \mathbb{C}P^\infty \). The later is the ring of invariant derivations on \( G \) generated by \( D_F^k \).

Definition 5.12. If \( \mu \) is a \( S \)-valued measure on \( G \) then the sequence \( \beta^k(f_\mu) \) for \( k \geq 0 \) is the moments of the measure \( \mu \).

The calculation in [5.10] depends on a coordinate, however the isomorphism is natural in the following sense.

Lemma 5.13. Suppose \( s \) and \( t \) are coordinates on the formal group \( G \). Then the diagram
\[ M(G, S) \xrightarrow{=} M(G, S) \xrightarrow{\Theta} S[[t]] \]
commutes.

**Proof.** The coordinates \( s \) and \( t \) determine a pair of isomorphic formal group laws over \( R \). The associated strict isomorphism, say \( \Theta \), is the right hand vertical map in the above diagram. \[ \square \]
6. A Theorem of Katz

We are interested in the image of the restriction map
\[ \text{res} : M(G^\times, S) \to M(G, S). \]

**Theorem 6.1 (N. Katz).** Let \( R \) be a complete local noetherian domain with residue characteristic \( p \). Suppose \( G \) is a formal group of height one over \( R \) and \( t \) is a coordinate on \( G \). Let \( S \) be a complete local \( R \)-algebra. An \( S \)-valued measure \( \mu \in M(G, S) \) is supported on the units if and only if \( f_\mu \in S[[t]] \) satisfies
\[ \text{Trace}_G f = T_G f := \sum_{c \in F_p} f_\mu(t + F_p c) = 0. \]

This statement, without proof, can be found in [22] where it is also claimed to be true when \( G \) is the formal group associated to a supersingular elliptic curve. We will verify only the part of the theorem we need for our purposes. Namely when \( G \) is a height one Lubin-Tate group over \( \mathbb{Z}_p \). As a warm up, we check the result for \( G = \hat{G}_m \) with a specified coordinate. We obtain the general statement using deformation theory.

In Proposition 7.1 we show that the effect of a cohomology operation on the tautological line bundle \( L \) over \( \mathbb{C}P^\infty \) is given by \( f_\mu \). This calculation allows us to show that Theorem 6.1 is equivalent to the Madsen, Snaith and Tornehave result describing the image of
\[ \Omega^\infty : [K_p, K_p] \to [\Omega^\infty K_p, \Omega^\infty K_p]. \]

In other words,

**Corollary 6.2.** There is a commutative diagram
\[
\begin{array}{ccc}
M(\hat{G}_m, \mathbb{Z}_p) & \xrightarrow{\text{res}} & M(\hat{G}_m, \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
[K_p, K_p] & \xrightarrow{\Omega^\infty} & [\Omega^\infty K_p, \Omega^\infty K_p].
\end{array}
\]

In order to prove Theorem 6.1 we will describe a measure on \( G \) in terms of Riemann sums.

**6.3. Measures on \( \Gamma_n \).** Recall that \( \Gamma_n \) is the kernel of multiplication by \( p \) on \( G \). Define \( M(\Gamma_n, S) \) as the \( S \)-module of linear functionals
\[ \text{Hom}(\text{Maps}(T_p \Gamma_n^\vee, R), S) = \text{Hom}(\text{Maps}(\Gamma_n^\vee, R), S). \]

If \( \mu \in M(\Gamma_n, S) \) we will often write
\[ \int_{\Gamma_n} f(x) d\mu(x) \]
for the effect of \( \mu \) on the function \( f \).

A coordinate \( t \) on \( G \) determines an isomorphism
\[ M(\Gamma_n, S) \cong \text{Hom}(\text{Maps}(\mathbb{Z}/p^n, R), S). \]

and there is an isomorphism
\[ M(\Gamma_n, S) \cong S[\mathbb{Z}/p^k] \cong S[\gamma]/\gamma^{p^n} - 1. \]
If \( \chi_i \) is the characteristic function at \( i \) then
\[
\mu \mapsto \sum_{i=0}^{p^n-1} \left( \int \chi_i(x) d\mu(x) \right) \gamma^i
\]

Hitting the inverse system
\[\ldots \to \Gamma_n \to \Gamma_{n-1} \to \ldots\]
with the functor \( M(-,S) \) produces the inverse system
\[\ldots \to M(\Gamma_n,S) \to M(\Gamma_{n-1},S) \to \ldots\]
Each map sends a measure \( \mu \) on \( \Gamma_n \) to the measure \( \bar{\mu} \) defined by
\[
\int_{\Gamma_{n-1}} f d\bar{\mu} := \int_{\Gamma_n} \tilde{f} d\mu.
\]
Here \( \tilde{f} \) is the composite of \( f \) with the natural projection:
\[
\begin{array}{ccc}
\mathbb{Z}/p^n & \to & \mathbb{Z}/p \\
\gamma & \mapsto & \bar{\gamma} \\
\end{array}
\]

For example, if \( a \in \mathbb{Z}/p^n \) the Dirac measure \( d\psi_a \) is mapped to \( d\bar{\psi}_{\bar{a}} \) where \( \bar{a} \) is the class represented by the reduction of \( a \) modulo \( p^{n-1} \). In general we have
\[
\sum_{i=0}^{p^n-1} a_i(n) d\psi_i \mapsto \sum_{i=0}^{p^n-1} a_i(n) d\bar{\psi}_i.
\]
However, each class modulo \( p^{n-1} \) gets hit precisely \( p \) times. Collecting terms, we can rewrite this sum as
\[
\sum_{j=0}^{p^{n-1}-1} \sum_{l=0}^{p-1} a_{i+l p^{n-1}}(n) d\psi_j.
\]
In other words, we have the formula
\[
(2) \quad a_i(n-1) = \sum_{l=0}^{p-1} a_{i+l p^{n-1}}(n).
\]

6.4. Density of Dirac Measures. Recall from our discussion on Tate modules that \( G \) is isomorphic to its associated \( p \)-divisible group. Putting this together with the discussion of the last section we have

**Lemma 6.5.** There is an isomorphism
\[
M(G,S) \cong \lim_{\to} M(\Gamma_n,S).
\]

**Proof.** Observe
\[
M(G,S) = M(\lim\Gamma_n,S) = \lim_{\to} M(\Gamma_n,S).
\]
We are now in a position to describe an analytic version of integration on $G$. Suppose $\mu$ is in $M(G,S)$. The previous lemma permits us to write $\mu$ uniquely as a sequence

$$ s_n = \sum_{i=0}^{p^n-1} a_i(k) d\psi_i $$

using the isomorphism in [23]. A statement of the following form can be found in [23].

**Proposition 6.6.** With the above notation, if $f: T_p G' \to R$ is any continuous function then

$$ \int_G f(x) d\mu(x) = \lim_{n \to \infty} \sum_{i=0}^{p^n-1} a_i(n) f(\gamma_i) $$

**Proof.** We begin by showing that the limit converges. To see this recall that the $a_i(n)$ and $a_j(n+1)$ are related; we have the formula

$$ a_i(n) = \sum_{l=0}^{p-1} a_{i+lp^n}(n+1) $$

from line (2).

If we rewrite the $n$-th term in the limit using this formula we obtain

$$ \sum_{i=0}^{p^n-1} \sum_{l=0}^{p-1} a_{i+lp^n}(n+1) f(i) $$

We need to compare this sum with

$$ \sum_{j=0}^{p^{n+1}-1} a_j(n+1) f(j). $$

We can rewrite line (5) as

$$ \sum_{i=0}^{p^n-1} \sum_{l=0}^{p-1} a_{i+lp^n}(n+1) f(i + lp^n). $$

There are the same number of terms in this new expression and in (4). Comparing the coefficient of $a_{i+lp^n}(n+1)$ shows that we need to compare $f(i)$ and $f(i + lp^n)$. Since $f$ was uniformly continuous ($R$ is compact) these two function values are uniformly bounded, thus the difference of the sums (4) and (5) is small uniformly and so the limit must exist.

To finish we need to show that this limit computes the value of the linear functional $\mu$. To this end, we investigate the map, call it $\rho$, that sends the sequence

$$ \left\{ s_n = \sum_{i=0}^{p^n-1} a_i(n)(t+1)^i \right\} $$

associated to the measure $\mu$ to the linear functional $\mu'$ defined by

$$ \int_G f(t) d\mu' = \lim_{k \to \infty} \sum_{i=0}^{p^n-1} a_i(n) f(i). $$
We will show that $\rho$ is the identity.

The function $\rho$ is a $S$ module map and is the identity on the Dirac measures. The restriction of $\rho$ to $\mathbb{Z}/p^n$ is also the identity since everything can be computed via the Dirac measures there. Thus the diagram

$$
\begin{array}{ccc}
M(\mathbb{Z}/p^{n+1}) & \xrightarrow{\rho} & M(\mathbb{Z}/p^{n+1}) \\
\downarrow & & \downarrow \\
M(\mathbb{Z}/p^n) & \xrightarrow{\rho} & M(\mathbb{Z}/p^n)
\end{array}
$$

commutes and has identity morphisms for horizontal arrows. Taking limits we see that

$$
\rho: M(G, S) \rightarrow M(G, S)
$$

must be the identity. $\square$

### 6.7. A Special Case of Katz’s Theorem.

In this section we will verify Theorem 6.1 in the case $G = \widehat{\mathbb{G}}_m$. To help the flow of the proof we check the result for the Dirac measures. Let $G = \widehat{\mathbb{G}}_m$ and $t$ the coordinate given by $u - 1$. The power series associated to the Dirac measure $\psi_l$ in this case is $(1 + t)^l$.

Suppose $l$ is prime to $p$, then $\psi_l$ is clearly supported on the units. We will write $\zeta$ for a primitive $p$ root of unity over $\mathbb{Z}_p$ and $\zeta_j = \zeta^j - 1$. We compute:

$$
\begin{align*}
\sum_{j=0}^{p-1} (1 + (t + \zeta^j))l &= \sum_{j=0}^{p-1} (1 + t + \zeta^j - 1 + t\zeta^j - t)^j \\
&= \sum_{j=0}^{p-1} (\zeta^j(t + 1))^j \\
&= (t + 1)^j \sum_{j=0}^{p-1} \zeta^{jl} = 0.
\end{align*}
$$

The last sum vanishes because $p$ is prime to $l$ and we are summing over the $p$-th roots of unity.

**Lemma 6.8 (N. Katz).** Suppose $t$ is the coordinate on $\widehat{\mathbb{G}}_m$ given by $u - 1$ with associated formal group law $F$. A measure $\mu$ is supported on the units if and only if the corresponding power series $f_\mu(t) \in S[[t]]$ satisfies

$$
\sum_{j=0}^{p-1} f_\mu(x + F \zeta_j) = 0
$$

**Proof.** First, suppose we have a measure $\mu$ that is supported on the units and the corresponding power series

$$
f_\mu(t) = \sum_{i} \left( \int_{G} \left( \frac{x}{i} \right) d\mu(x) \right) t^i
$$

given by Mahler’s theorem.
The coefficient of $t^l$ in the trace of $f_\mu(t)$ is
\[
\int \sum_{j=0}^{p-1} \zeta^{jx} \left( \frac{x}{l} \right) d\mu(x).
\]
However, the integrand vanishes on the units and so by hypothesis the integral must be zero.

For the converse, suppose the trace of $f_\mu(t)$ is zero. This time we know

\[
\int \sum_{j=0}^{p-1} \zeta^{jx} \left( \frac{x}{l} \right) d\mu(x) = 0
\]

since it computes the coefficient of $t^l$ in the trace of $f_\mu(t)$. Using the Riemann sum representation (Proposition 6.6) we can compute (6) via
\[
\lim_{n} \sum_{i=0}^{p^n-1} a_i(n) \sum_{j=0}^{p-1} \zeta^{ji} \left( \frac{i}{l} \right) = 0.
\]
We can rewrite this expression as

\[
\lim_{n} \sum_{i=0}^{p^n-1} a_i(n) \left( \frac{i}{l} \right) = 0.
\]
We have dropped the factor of $p$ since $R$ is torsion free.

Now suppose $f: T_p G^\vee \to \mathbb{Z}_p$ is a continuous that vanishes on $T_p G^\vee$. By Mahler’s theorem we can represent $f$ as
\[
f(x) = \sum_l b_l \left( \frac{x}{l} \right).
\]
To compute its integral we need to calculate
\[
\lim_{n} \sum_{i=0}^{p^n-1} a_i(n) f(i) = \lim_{n} \sum_{i=0}^{p^n-1} a_i(n) \sum_{l=0}^{p^n-1} b_l \left( \frac{i}{l} \right) = \sum_{l=0}^{p^n-1} b_l \lim_{n} \sum_{i=0}^{p^n-1} a_i(n) \left( \frac{i}{l} \right) = 0.
\]
The second equality follows since $T_p G^\vee$ is compact; by uniform continuity
\[
\int \sum_{G} b_n \left( \frac{x}{n} \right) d\mu = \sum_{n} \int_{G} b_n \left( \frac{x}{n} \right) d\mu.
\]
The third equality follows from the hypothesis $f((T_p G^\vee)^\times) = 0$. The last equality follows from the calculation in (7). $\square$
6.9. Proof of the General Case. Let $\phi$ be a lift of the Frobenius over $\mathbb{F}_p$ to $\mathbb{Z}_p$. The power series $\phi$ has coefficients in $\mathbb{Z}_p$ and satisfies $\phi = pt + O(t)$, $\phi \equiv t^p \mod p$.

Let $G = G_\phi$ be the associated Lubin-Tate formal group law over $\mathbb{Z}_p$ with special endomorphism $\phi$. Since $\hat{G}_m$ is the universal formal group law of height one there exists a strict isomorphism $\Theta: \hat{G}_m \rightarrow G$.

**Theorem 6.10.** Suppose $t$ is a coordinate on $G$. Under the identification $D_G: M(G, S) \rightarrow S[[t]]$ a measure $\mu$ is supported on the units if and only if $T_G f_\mu(t) = 0$.

**Proof.** The diagram

$$
\begin{array}{ccc}
M(\hat{G}_m, S) & \rightarrow & M(\hat{G}_m, S) \\
\downarrow \Theta & & \downarrow \Theta \\
M(G^\times, S) & \rightarrow & M(G, S)
\end{array}
$$

commutes and the vertical arrows are isomorphism.

Suppose the trace of $f(t) \in S[[t]]$ with respect to $G$ vanishes. Under the change of variables $t \mapsto \Theta(s)$, $T_G f(t) = 0$ maps to $T_{\hat{G}_m} f(\Theta(s)) = 0$. Observe that the set of $\Theta(v)$ for $v \in G[p]$ is the same as the set of $u \in \hat{G}_m[p]$ since $\Theta$ is an isomorphism.

Pulling back along $\Theta$ produces the change of variables formula $D_F^k f(t)|_{s=0} = \Theta^*(D_F^k f(\Theta(s)))|_{s=0} = D^k f(\Theta(s))|_{s=0}$.

It follows from Lemma 5.13 that the measure associated to $f(t)$ via the isomorphism $D_G: M(G, S) \rightarrow S[[t]]$ coincides with the measure determined by $f(\Theta(s))$ under $D_{\hat{G}_m}: M(G, S) \rightarrow S[[s]]$ as they produce the same moments. The claim now follows from Lemma 6.8. One can use a similar argument to prove the “only if” part of the theorem. \[\square\]

7. Measures and $\mathbb{C}P^\infty$

A measure on $\hat{G}_m$ determines a cohomology operation on $K_p$, recall Theorem 3.1. In this section we describe the effect of these operations on $\mathbb{C}P^\infty$.

**Theorem 7.1.** The effect of an operation $\mu$ on the tautological line bundle over $\mathbb{C}P^\infty$ is the series $f_\mu$.

Once we understand the proof of this fact, the equivalence of Theorem 6.1 and the result of Madsen, Snaith and Tornehave will be an easy consequence. The argument is based on section 2.3 of [26].
7.2. Collection of Statements. Recall that for $p$-adic K-Theory, the cohomology of $\mathbb{C}P^\infty$ can be computed via the cohomology of the cyclic groups of $p$-powers. Explicitly we have

**Lemma 7.3.** There is an isomorphism

\[ K_p^0 \mathbb{C}P^\infty \cong \lim_{\to} K_p^0 (B\mathbb{Z}/p^i). \]  

We produced a similar statement for $S$-valued measures on $G$

\[ M(G, S) \cong \lim_{\to} M(\Gamma_n, S). \]

**Definition 7.4.** Let $\hat{A}_p$ be the set of $H$-maps from $\Omega^\infty K_p$ to itself.

We quote a lemma of Adams describing $\hat{A}_p$. It is Lemma 6 in [2]

**Lemma 7.5 (Adams).** There is an isomorphism

\[ \epsilon: \hat{A}_p \cong K_p^0 \mathbb{C}P^\infty \]

which sends an operation $\mu$ to its effect on the tautological line bundle $L$.

The above lemma together with line (8) allows us to write any element of $\hat{A}_p$ uniquely as a sequence

\[ \left\{ s_n = \sum_{i=0}^{p^n-1} a_i(n) \psi^i \right\} \]

where $\psi^i$ is the Adams operation at $i$ and $a_i(n) \in \mathbb{Z}_p$.

Our goal is the following

**Theorem 7.6.** Suppose we have chosen a coordinate for $\mathbb{G}_m$ with induced formal group law $F$. Under these conditions, the diagram

\[ M(\mathbb{G}_m, \mathbb{Z}_p) \xrightarrow{\text{res}} M(\mathbb{G}_m, \mathbb{Z}_p) \xrightarrow{D_{\mathbb{G}_m}} \mathbb{Z}_p[[t]] \]

\[ K_p^0 K_p \xrightarrow{\Omega^\infty} \hat{A}_p \xrightarrow{\epsilon} K_p^0 \mathbb{C}P^\infty \]

commutes.

Observe that an unstable operation $\phi \in \hat{A}_p$ is in the image of $\Omega^\infty$ if and only if the corresponding measure is in the image of the restriction map.

The proof of Theorem 7.6 will take a few steps.

7.7. The K-Theory of $\mathbb{C}P^\infty$. We exploit the similarity of Line (8) and Line (9) to calculate the effect of a measure on the cohomology of $\mathbb{C}P^\infty$.

The direct system

\[ \ldots \mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^{n-1} \to \ldots \]

leads to the inverse system

\[ \ldots KB(\mathbb{Z}/p^n) \to KB(\mathbb{Z}/p^{n-1}) \to \ldots \]

It is easier to describe the maps in the system if we use Atiyah’s calculation to replace these gadgets with the representation rings

\[ \ldots R(\mathbb{Z}/p^n) \otimes \mathbb{Z}_p \to R(\mathbb{Z}/p^{n-1}) \otimes \mathbb{Z}_p \to \ldots \]
These are easily computed as

\[ \ldots Z_p[s]/(s^{p^n} - 1) \rightarrow Z_p[s]/(s^{p^{n-1}} - 1) \rightarrow \ldots \]

Where \( s \) is the one dimensional irreducible representation

\[ \mathbb{Z}/p^n \rightarrow GL_1(\mathbb{C}) \cong \mathbb{C}^\times \]

that sends the generator \( \gamma \) to the appropriate primitive \( p \)-th root of unity. These one dimensional representations are compatible under the structure maps. Thus we have no notation to describe the source of the representation.

We can now describe the maps involved. It is very similar to what we saw in the measure theory situation involving \( \Gamma_n \). The map induced on the representation rings by multiplication by \( p \) sends the generator \( s \) to \( s^p \). Thus we have for a general element

\[ \sum_{i=0}^{p^n-1} a_i(n)s^i \rightarrow \sum_{i=0}^{p^{n-1}} a_i(n)s^{ip}. \]

Collecting terms once more, we can rewrite the image as

\[ \sum_{j=0}^{p^{n-1}-1} \sum_{l=0}^{n-1} a_j+lps^{j} \cdot \sum_{i=0}^{l}. \]

We can connect this to unstable operations via an isomorphism

\[ \hat{A}_p \cong \lim_n K^n_p B(\mathbb{Z}/p^n) \cong \lim_k R(B\mathbb{Z}/p^n) \otimes \mathbb{Z}_p \]

such that any operation corresponds to a sequence

\[ \left\{ \frac{s^n}{s^{p^n} - 1} \right\} \]

where \( \psi_i \) matches with \( s^i \) inside

\[ K^0_p B(\mathbb{Z}/p^n) \cong \mathbb{Z}_p[\{s\}]/(s^{p^n} - 1). \]

This is the approach of Madsen, Snaith and Tornehave.

**Lemma 7.8.** The diagram

\[ \begin{array}{ccc}
M(\mathbb{Z}/p^n) & \rightarrow & M(\mathbb{Z}/p^{k-1}) \\
\downarrow & & \downarrow \\
\mathbb{Z}_p[t]/(s^{pk} - 1) & \rightarrow & \mathbb{Z}_p[t]/(s^{p^{k-1}} - 1)
\end{array} \]

where the vertical maps send \( d\psi_i \) to \( s^i \) commutes.

**Proof.** Obvious. \( \square \)

Indeed, the vertical arrows are isomorphisms and so the limits are isomorphic:

\[ \lim_k M(\mathbb{Z}/p^k, \mathbb{Z}_p) \cong \lim_k \mathbb{Z}_p[s]/(s^{p^k} - 1). \]
We are now in the situation to compute over $\mathbb{Z}_p$. Suppose $\mu \in M(\mathbb{Z}_p^\times)$. Let $\bar{\mu}$ be the corresponding element of $M(\mathbb{Z}/p^k)$. It is represented by a formal sum

$$
\sum_{i=0}^{p^n-1} a_i(n) d\psi_i.
$$

Likewise, if we consider $\mu$ as an element of $\hat{A}_p$ and then consider its image inside $K_p(B\mathbb{Z}/p^n)$ it is described by

$$
\mu(s) = \sum_{i=0}^{p^n-1} b_i(n) s^i.
$$

We have described the map between these two gadgets. It sends $s^i$ to $d\psi_i$. It remains to check if $a_i(n) = b_i(n)$ for each $i$ if we consider $\mu$ as a measure on $\mathbb{Z}_p^\times$ or as an operation.

It’s clear that if $\mu$ is a sum of Adams operations, these two coefficients agree since they are both counting the number of copies of an Adams operation in the original operation. This remains true when passing to limits and the result of the discussion is

**Proposition 7.9.** The diagram

$$
\begin{array}{ccc}
M(\mathbb{Z}_p^\times) & \longrightarrow & \lim_k M(\mathbb{Z}/p^n, \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
K_p^0 K_p & \longrightarrow & \lim_k K_p^0 (B\mathbb{Z}/p^n)
\end{array}
$$

commutes.

**7.10. Proof of Theorem [7.6]** In order to prove Theorem [7.6] we will first verify it for a specific coordinate on $\mathbb{G}_m$. We will then use formal techniques to produce the result for any other coordinate.

We are studying the diagram

$$
\begin{array}{ccc}
M(\mathbb{Z}_p^\times) & \longrightarrow & \lim_k M(\mathbb{Z}/p^n) \\
\downarrow & & \downarrow \\
K_p^0 K_p & \longrightarrow & \lim_k K_p^0 (B/\mathbb{Z}/p^n)
\end{array}
\longrightarrow
\begin{array}{ccc}
M(\mathbb{Z}_p) & \longrightarrow & \mathbb{Z}_p[[t]] \\
\downarrow & & \downarrow \\
\hat{A}_p & \longrightarrow & K(\mathbb{C} P^\infty)
\end{array}
$$

Proposition [7.9] shows that the left-hand rectangle commutes. Two complete our proof, we have to compute the the two paths around the right-hand rectangle.

The top horizontal arrow is computed via

$$
\mu \mapsto \sum_{j=0}^{\infty} \left( \int_{\mathbb{C}_p} \left( \frac{x}{x_j} \right) d\mu \right) t^j
$$

In other words we are computing $f_\mu(t)$, the power series associated to the measure by Mahler’s theorem. We can use the Riemann sums description of integration in
Proposition 6.6 to compute the right hand side of (12). If $\mu$ is represented by the sequence 
\[ s_n = \sum_i a_i(n)d\psi_i \]
then the $j$-th coefficient of (12) can be computed via
\[ \lim_n \sum_{i=0}^{p^n-1} a_i(n) \binom{i}{j} \]

The bottom row is similar. The isomorphisms between $\hat{A}_p$ and $K^0_p(\mathbb{C}P^\infty)$ sends an operation to its effect on the tautological line bundle. The element $\mu(L)$ inside the K-Theory of $\mathbb{Z}/p^n$ is represented by the element
\[ \mu(s) = \sum_{i=0}^{p^n-1} a_i(n)s^i. \]
with the same $a_i(n)$ by Proposition 7.9. The sequence
\[ s_n = \sum_{i=0}^{p^n-1} a_i(n)s^i \]
is mapped to the sequence
\[ s_n = \sum_{i=0}^{p^n-1} a_i(n)\psi^i \]
inside $\hat{A}_p$. The isomorphism of Adams sends this sequence to
\[ s_n = \sum_{i=0}^{p^n-1} a_i(n)L^i \]
inside $K^0_p(\mathbb{C}P^\infty)$. Finally, under the isomorphism of this cohomology group with a power series ring the sequence is sent to
\[ s_n = \sum_{i=0}^{p^n-1} a_i(n)(1+t)^i \]
We can compute the power series representing this sequence in a natural way using the fact that $\mathbb{Z}_p[[t]]$ is complete with respect to the $(p,t)$ topology.

Putting all of this together, the bottom row of our diagram sends $\mu$ to
\[ \lim_k \sum_{i=0}^{p^n-1} a_i(n)L^i = \lim_k \sum_{i=0}^{p^n-1} a_i(n)(1+t)^i = \lim_k \sum_{i=0}^{p^n-1} a_i(n) \sum_{l=0}^i \binom{i}{l} t^l \]
Pulling off the coefficient of $t^j$ and comparing with the calculation of (12) we obtain the same element in the power series ring as claimed.
7.11. Changing Coordinates. We have asked for a specific coordinate to be chosen in order to make the calculation of the previous section. To finish the theorem we have to understand how the theory changes as we change the coordinate.

Suppose for the moment that we have another coordinate on the formal multiplicative group and thus a new group law for K-Theory called \( F \).

Let \( \Theta : F \to (x + y + xy) \) be a strict isomorphism whose existence follows from deformation theory. We write \( s \) for the coordinate on \( \widehat{\mathbb{G}}_m \) and \( t \) for that of \( F \). We have the change of variables \( t \mapsto \Theta(s) \).

Recall the invariant derivation \( D = (1 + s) \frac{d}{ds} \) and

\[
D_F = \frac{1}{\log_F(t)} \frac{d}{dt} = F_2(t,0) \frac{d}{dt}.
\]

We can now state our main result for this section:

**Proposition 7.12.** Under the bijection in Proposition 5.9 the effect of \( \mu \) on \( L \) is given by \( f_\mu(t) \).

**Proof.** The point is that pulling back \( D \) along \( \Theta \) induces \( D_F \) and this is compatible with our identification of \( M(G,S) \) with \( S[[t]] \), recall Lemma 5.13. Write

\[
\Theta : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p[[s]]
\]

for the map that sends \( t \) to \( \Theta(s) \). It is classical that the diagram

\[
\begin{array}{ccc}
[K_p,K_p] & \xrightarrow{F} & \mathbb{Z}_p[[t]] \\
\downarrow_{G} & & \downarrow_{\Theta} \\
\mathbb{Z}_p[[s]] & \\
\end{array}
\]

commutes where the formal group law labels the isomorphism. It follows that if \( L = (1 + s) \in \mathbb{Z}_p[[s]] \) then \( L = 1 + \Theta^{-1}(t) \in \mathbb{Z}_p[[t]] \).

Pushing symbols around produces

\[
\int_G (1 + \Theta^{-1}(t))^x d\mu(x) = f_\mu(t)
\]

and the claim is verified. \( \square \)

8. Bernoulli Numbers

Let \( R \) be a complete local ring that is torsion free with perfect residue field of characteristic \( p \). Consider the collection of triples \( (G,S,t) \) where \( S \) is a torsion free complete local algebra over \( R \), \( G \) is a finite height formal group over \( S \) and \( t \) is a coordinate on \( G \).

**Definition 8.1.** A Bernoulli number of weight \( k \geq 0 \) over \( R \) is a function \( B \) that sends a triple \( (G,S,t) \) to an element of \( S \otimes \mathbb{Q} \) such that

1. \( B \) is invariant under isomorphic data
2. For any \( a \in S^\times \) we have \( B(G,S,at) = a^kB(G,S,t) \).
3. \( B \) behaves well under pull backs.

The coordinate \( at \) is the composition

\[
G \xrightarrow{t} \widehat{\mathbb{A}}_1 \xrightarrow{a} \widehat{\mathbb{A}}_1.
\]
The third condition requires some explanation. Suppose $G$ lives over $S$ and we have a map

$$g: S \rightarrow S'$$

of complete local $R$ algebras. The pull back $g^*G$ is a formal group over $S'$. Condition three says there must be an equation in $S'$ of the form

$$B(g^*G, S', g^*t) = g(B(G, S, t)).$$

In other words, the Bernoulli numbers of a pull back are the push forward of the Bernoulli numbers. We will often simplify the notation $B(G, R, t)$ to $B(G, t)$ in the case $S = R$. Definition 8.1 is formal version of a Katz $p$-adic modular form [21].

8.2. The Fundamental Example. In this section we will construct the fundamental example of a Bernoulli number of weight $k$. It is unclear at the moment whether or not we construct the unique Bernoulli number of weight $k$ up to multiplication by a constant. In addition, we are unable to produce a coordinate free version of the fundamental example. The canonical derivation

$$D: O_G \rightarrow \text{Lie}^\vee \otimes O_G$$

may permit the removal of the coordinate from our test data. However, it is not clear what the target for the Bernoulli numbers should be in that situation.

Recall that any formal group over a torsion free ring is isomorphic to the additive group after tensoring with the rationals. Let $\log_G$ be a formal group logarithm for $G$. It is an isomorphism

$$\log_G: G \rightarrow \hat{G}_a$$

of formal groups over $S \otimes \mathbb{Q}$ between the additive group and $G$.

Let $u$ be the coordinate on the formal additive group

$$\hat{G}_a \xrightarrow{u} \hat{A}_1$$

where $G_a = \text{Spf} \mathbb{Z}_p[u]$.

The function $\log_G, t$ corresponds to the appropriate composition in the diagram

$$G \xrightarrow{t} \hat{A}_1$$

$$\log_G \downarrow \hat{G}_a \xrightarrow{u} \hat{A}_1.$$  

It is an element of $O_G \otimes \mathbb{Q}$ and we can consider the element

$$\frac{\log_G t}{t} \in O_G^* \otimes \mathbb{Q}.$$

Denote the maximal ideal of $O_G$ by $m_{O_G}$. The Iwasawa logarithm is a homomorphism

$$\log: 1 + m_{O_G} \rightarrow O_G \otimes \mathbb{Q}$$

of abelian groups. It is constructed by considering the power series expansion of $\log(1 - x)$ about zero and extending to a $p$-adically complete ring.

Recall the invariant derivation

$$D_F = F_2(t, 0) \frac{d}{dt}$$

and the associated map

$$\beta^k: O_G \rightarrow S.$$
Our fundamental example of a Bernoulli number is

**Definition 8.3.** Define

\[ B^k(G, S, t) := \beta^k \left( \log \log G t \right) \]

It is straightforward to verify the three properties of Definition 8.1.

**Example 8.4.** Consider the triple \((\mathbb{Z}_p, \widehat{G}_m, t)\) where \(t\) is the coordinate \(1 - u\). In this situation, the function \(B^k\) returns the classical Bernoulli numbers \(-B_k^k\). One way to see this is to use the following approach.

Consider the expansion

\[ \frac{t \exp'_G t}{\exp_G t} = \sum_{k \geq 0} b_k \frac{t^k}{k!}. \]

Using logarithmic differentiation, we can show that

\[ \frac{t}{\exp_G t} = \exp \left( \sum_{k \geq 1} -b_k \frac{t^k}{k!} \right). \]

To compute \(\beta^k \log \log G t\), we pull back along the homomorphism \(\exp_G t\) and obtain

\[ \frac{d^k}{dx^k} \log \frac{\log(\exp_G x)}{\exp_G x} \bigg|_{x=0}. \]

This can be simplified to

\[ \frac{d^k}{dx^k} \log \frac{x}{\exp_G x} \bigg|_{x=0} \]

and we can read off the value as \(-\frac{b_k}{k}\).

In this example, \(\exp_G x = 1 - e^{-x}\) so

\[ \frac{x \exp'_G x}{\exp_G x} = \frac{xe^{-x}}{1 - e^{-x}} = \frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!} \]

and we see that \(B^k(\mathbb{Z}_p, \widehat{G}_m, t) = -\frac{B_k}{k}\) as claimed.

The construction of \(B^k\) is natural with respect to change of coordinates. For example, if \(s\) is another coordinate than the diagram

\[ \begin{array}{ccc}
O_G \otimes \mathbb{Q} & \xrightarrow{\log} & O_G \otimes \mathbb{Q} \\
| & & | \\
O_G \otimes \mathbb{Q} & \xrightarrow{\beta^k} & S \otimes \mathbb{Q} \\
| & & | \\
O_G \otimes \mathbb{Q} & \xrightarrow{\log} & O_G \otimes \mathbb{Q} \\
| & & | \\
O_G \otimes \mathbb{Q} & \xrightarrow{\beta^k} & S \otimes \mathbb{Q}
\end{array} \]

communes where the vertical maps are induced by the change of coordinate \(t \mapsto s\).
9. The $V$ Operator

The quotient of a formal group by a finite subgroup has played an important role in the study of elliptic curves and more recently in algebraic topology. We begin by recalling a fundamental result due to Lubin [24].

**Theorem 9.1 (Lubin).** If $F$ is a formal group of finite height over $R$ and $H$ is a finite subgroup scheme of $F$ then $F/H$ is a formal group.

A proof of this statement can be found in Lubin’s original article, the work of Demazure and Gabriel or in [6]. The quotient formal group comes with an isogeny $f_H: F \to F/H$ with kernel the group scheme $H$. When the subgroup $H$ is invariant under the appropriate Galois action, both $f_H$ and the quotient $G/H$ can be defined over $R$.

We are interested in the case when $H$ is the kernel of the isogeny induced by multiplication by $p$. We use the construction of the quotient to define an operator on Bernoulli numbers analogous to that in the theory of Katz modular forms.

If $t$ is a coordinate on $G$ then $f_H$ induces a map of formal group laws. The associated power series

$$f_H^t = \prod_{c \in G[p]} t + G c$$

is Lubin’s isogoney. We denote by $t_H$ the coordinate induced on the quotient $G/H$.

It is determined by insisting that the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{f_H} & G/H \\
\downarrow t & & \downarrow t_H \\
\widehat{A}_1 & \xrightarrow{f_H} & \widehat{A}_1
\end{array}
$$

commute, see [7].

**Definition 9.2.** If $B$ is a weight $k \geq 0$ generalized Bernoulli number let $B|V$ be the function on triples $(G, S, t)$ whose values are

$$B|V(G, S, t) := B(G/H, S, t_H)$$

where $H$ is the kernel of multiplication by $p$ on $G$.

**Lemma 9.3.** The function $B|V$ is a Bernoulli number.

**Proof.** This is a formal exercise involving the naturalality of the quotient. □

9.4. The $K(1)$ Local Logarithm. Recall that for an $E_\infty$ ring spectrum $R$, there is an associated ring spectrum $gl_1 R$ known as the unit spectrum. This spectrum was first considered in [28]. Some of the formal properties of this spectrum, along with its construction can be found in [33].

**Theorem 9.5 (Bousfield, Kuhn).** There is a functor $\Phi$ from spaces to spectra such that for any $E_\infty$ ring spectrum $R$, there is a weak equivalence

$$\Phi \Omega^\infty(R) \sim L_{K(1)} R.$$
The result as stated is due to Bousfield and for higher chromatic level follows from work of Kuhn [11], [24]. Since $\Omega_\infty R$ and $GL_1 R$ have weakly equivalent base point components, Theorem 9.5 produces a natural transformation of cohomology theories.

$\ell_1 : gl_1 R \to L_{K(1)} g\ell_1 R \cong L_{K(1)} R$.

Thus if $X$ is a space we obtain a group homomorphism $\ell_1 : (R^0 X)^\times \to (L_{K(1)} R)^0(X)$ that is natural in $X$. Rezk calculates the effect of this map at all chromatic levels and for any commutative $K(n)$ local $S$-algebra $R$ [33]. The spectrum $K_p$ is a $K(1)$ local $E_\infty$ spectrum; the above discussion yields

$\ell_1 : gl_1 K_p \to L_{K(1)} gl_1 K_p \cong L_{K(1)} K_p \cong K_p$

**Theorem 9.6 (Rezk).** Suppose $X$ is a finite complex and $\psi^p$ is the Adams operations at $p$ in $K_p$. If $x \in (K_p^0(X))^\times$
then

$\ell_1 x = \left(1 - \frac{\psi^p}{p}\right) \log x = \frac{1}{p} \log \frac{x^p}{\psi^p x}$

We can use the fact that $K_p CP^\infty = \varprojlim K_p B\mathbb{Z}/p^n$ to compute the effect of $\ell_1$ on the cohomology of $CP^\infty$.

**Proposition 9.7.** Suppose $a \in \mathbb{Z}_p^\times - \{\pm 1\}$. If $\langle a \rangle(t) = \frac{[a](t)}{t}$ then

$\beta^k \ell_1 \langle a \rangle(t) = (1 - a^k) \left(B^k(\hat{G}_m, \mathbb{Z}_p, t) - \frac{B^k(V(\hat{G}_m, \mathbb{Z}_p, t))}{p}\right)$

**Proof.** Observe that in Rezk’s formula for $\ell_1$, the $V$ operator plays the role of the Adams operation. These are identical since both are defined by the quotient with respect to the kernel of the multiplication by $p$ map on $G$. On the other hand, the homomorphism $\exp_F(t)$ induces the formula

$\beta^k f(t) = D^k_x f(\exp_F(x)) \big|_{x=0}$
as $D_F$ pulls back to $\frac{d}{dx}$ along $\exp_F(t)$. Using this formula and Definition 8.3, a quick calculation produces the claim. \qed

10. Coleman Norm Operator and the Ando Condition

Our fundamental observation is the similarity between the Ando condition (Theorem 2.3) and the Coleman norm operator [15].

Let $H$ be the kernel of multiplication by $p$ on $G$. It is a finite subgroup scheme since multiplication by $p$ is an isogeny.

**Theorem 10.1 (R. Coleman, Ando-Hopkins-Strickland, Demazure-Gabriel).** There is a multiplicative operator

$N_G : O_G \to O_G$
such that if $t$ is any coordinate inducing the group law $F$ we have

$N_G g \circ [p]_F(t) = \prod_{c \in F[p^n]} g(t +_F c)$
A number theoretic proof can be found R. Coleman’s article. See [6] for a version involving level structures on $G$. As an example we have

**Lemma 10.2.** If $a \in \mathbb{Z}_p$ then the following conditions are equivalent:

1. $N_G t = t$
2. $N_G[a]_F(t) = [a]_F(t)$

**Proof.** For ease of reading, we drop the subscript $G$ on the norm operator. We have the chain of equalities.

\[
N t \circ [a]_F(t) \circ [p]_F(t) = N t \circ [p]_F(t) \circ [a]_F(t) = \prod_{v \in F[p]} [a]_F(t) + [p]_F(v) = N [a]_F(t) \circ [p]_F(t).
\]

However, since $[p]_F(0) = 0$, it is an invertible power series under composition and we obtain the identity

\[
N t \circ [a]_F(t) = N [a]_F(t).
\]

We can now read off the claim. \hfill $\Box$

Recall from the discussion following Lubin’s result (Theorem 9.1) that there is a quotient isogeny over $R$ associated to the finite subgroup $H$:

\[
f_H^* : O_{G/H} \to O_G
\]

**Definition 10.3.** A coordinate satisfies the Ando condition if the two elements of $O_G$ determined by $f_H^* H$ and $[p]_G(t)$ coincide.

Since Lubin’s isogeny $f_H$ is given by $\prod_{c \in H} t + F c$, it is easy to show that the Ando condition is equivalent to the coordinate being fixed by the Coleman norm operator.

**Proposition 10.4.** If $t$ is a coordinate on $G$ that satisfying the Ando condition, then

\[
B^k(G,t)|V = p^k B^k(G,t) \in R \otimes \mathbb{Q}
\]

**Proof.** We use a change of variables to compute $B_k(G,t)|V$. Let $x_H$ be the coordinate induced on the quotient and

\[
f_H : G \to G/H
\]

Lubin’s isogeny. By definition

\[
B^k(G,t)|V = B^k(G/H,t_H).
\]

To compute the value we must pair $D_{G/H}$, the invariant derivation induced on the quotient, with

\[
\log \frac{\log_{G/H} t_H}{t_H}.
\]

However, under base change along $f_H$ the derivation $D_{G/H}$ maps to $D_G$. Thus

\[
\beta_{G/H}^k \log \frac{\log_{G/H} t_H}{t_H} = \beta_G^k \log \frac{\log_{G/H} f_H t}{f_H t}.
\]
There are two observations to make at this point. First, the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{f_H} & G/H \\
\downarrow & & \downarrow \\
\hat{G}_a & \xrightarrow{\log G/H} & \hat{G}\end{array}
\]

is commutative. The second, and more important, is that the Ando condition implies

\[f_H t_H = [p]_G t\]

as elements of \(O_G\). Since the \(V\) operator is induced by the cokernel of multiplication by \(p\) we have

\[B^k(G/H, t_H) = \beta_G^k \log \frac{[p]_G(t)}{[p]_G t}.\]

A change of variables argument similar to the one used in the proof of Proposition 9.7 shows

\[\beta_G^k \log \frac{[p]_G(t)}{[p]_G t} = p^k B^k(G, t).\]

Since \(\langle a \rangle(t)\) is an element of \((K_p \mathbb{C}P^\infty)^\times\), \(l_1 \langle a \rangle(t)\) is a power series with integral coefficients and so determines a measure on \(\hat{G}_m\). We have already computed the moments of this measure in Proposition 9.7. However, we are interested in producing moments on the \(p\)-adic units i.e. \(l_1 \langle a \rangle(t)\) in the image of the restriction map

\[\text{res} : M(\mathbb{Z}_p^\times, \mathbb{Z}_p) \to M(\mathbb{Z}_p, \mathbb{Z}_p)?\]

Recall the key result of Katz, Theorem 6.10.

**Proposition 10.5.** Suppose \(G\) is a height one formal group over \(\mathbb{Z}_p\) and \(t\) is a coordinate that satisfies the Ando condition. For any \(a \in \mathbb{Z}_p^\times - \{\pm 1\}\)

\[T_G l_1 \langle a \rangle F(t) = \sum_{c \in F[p]} l_1 \langle a \rangle F(t + F c) = 0.\]

**Proof.** It follows from our discussion around Lemma 10.2 that if the coordinate \(t\) satisfies the Ando condition then both \(t\) and \([a]_F(t)\) are fixed by the norm operator \(N_G\). Since \(N_G\) is a multiplicative operator, the divided \(a\) series

\[\langle a \rangle_F(t) = \frac{[a]_F(t)}{t}\]

is also fixed by the norm operator. We can now rewrite the sum as a product using the formula of Rezk (Theorem 9.6). Simplicity the expression inside the finite product will produce the claim. \(\square\)

This proof is formal. If \(f(t) \in (K_p \mathbb{C}P^\infty)^\times\) is any invertible power series, then the trace of \(l_1 N_G f(t)\) vanishes. Moreover, one can show that the \(a\) restricted form of the Bousfield-Kuhn idempotent

\[\phi : \tilde{\mathbb{A}}_p \to [K_p, K_p] \xrightarrow{\Omega^\infty} [\Omega^\infty K_p, \Omega^\infty K_p] \]

is given by the map \(id - \frac{1}{p} T_G\).
Corollary 10.6. If \( t \) satisfies the Ando condition then the sequence
\[
B^k(G, t)(1 - a^k) (1 - p^{k-1})
\]
arises as the moments of a measure on \( G^\times \).

Proof. It follows from Theorem 6.10 and Proposition 10.5 that the measure attached to \( l_1(a)_G(t) \) is supported on the units. The calculation in Proposition 9.7 together with Proposition 10.4 produces the sequence above as the moments of this measure. \( \square \)

11. Obstruction Theory of AHR

Given any space \( B \) over \( BO \), we can produce an \( E_\infty \) ring spectrum \( MB \). The space of \( E_\infty \) orientations is the homotopy pull back in the square
\[
\begin{array}{ccc}
E_\infty(MB, R) & \to & \text{Spectra}(gl_1S/b, gl_1R) \\
\downarrow & & \downarrow \\
\text{Spectra}(S, R) & \to & \text{Spectra}(gl_1S, gl_1R).
\end{array}
\]

This description is due to Ando, Blumberg, Gepner, Hopkins and Rezk and is discussed in generality in [4]. The results that are closer to what we require can be found in the first few sections of [5].

11.1. The Calculation for \( MU \). If \( X \) is an \( E_\infty \) ring spectrum, we write
\[
i: S \to X
\]
for the unit. We abuse notation and write
\[
i: gl_1S \to gl_1X
\]
for the map induced on the associated unit spectra.

We will mimic the notation of [3] for connected covers of the space \( BU \). For example
\[
BU(4) = BSU
\]
is the four connected cover of \( BU \) and
\[
BU(0) = BU \times \mathbb{Z}.
\]
There are associated spectra
\[
\begin{align*}
bu(0) &= bu \\
bu(2) &= bu \\
bu(4) &= bsu
\end{align*}
\]
whose zeroth spaces is \( BU(2k) \).

Let
\[
u(2) = \Sigma^{-1} bu(2).
\]
The stable \( j \)-homomorphism leads to a cofiber sequence
\[
u(2) \xrightarrow{j} gl_1S \to gl_1S//u(2).
\]
**Definition 11.2** (Ando, Blumberg, Gepner, Hopkins, Rezk). Define $\text{MU}$ as the homotopy pushout in the diagram of $\text{E}_\infty$ ring spectra

\[
\begin{array}{ccc}
\Sigma_+^\infty \Omega^\infty \text{gl}_1 S & \rightarrow & S \\
\downarrow & & \downarrow \\
\Sigma_+^\infty \Omega^\infty \text{gl}_1 S/\langle 2 \rangle & \rightarrow & \text{MU}
\end{array}
\]

Applying the functor $\text{E}_\infty(-, R)$ to this diagram yields

\[
\begin{array}{ccc}
\text{E}_\infty(\text{MU}, R) & \rightarrow & \text{E}_\infty(\Sigma_+^\infty \Omega^\infty \text{gl}_1 S/\langle 2 \rangle, R) \\
\downarrow & & \downarrow \\
\{i\} & \rightarrow & \text{E}_\infty(\Sigma_+^\infty \Omega^\infty \text{gl}_1 S, R).
\end{array}
\]

It is shown in [4] that $(\Sigma_+^\infty \Omega^\infty, \text{gl}_1)$ form an adjoint pair up to homotopy and the upper horizontal arrow in Definition 11.2 is the counit of the adjunction. Applying the adjoint twice, diagram 14 becomes

\[
\begin{array}{ccc}
\text{E}_\infty(\text{MU}, R) & \rightarrow & \text{Spectra}(\text{gl}_1 S/\langle 2 \rangle, \text{gl}_1 R) \\
\downarrow & & \downarrow \\
\{i\} & \rightarrow & \text{Spectra}(\text{gl}_1 S, \text{gl}_1 R)
\end{array}
\]

and the space $\text{E}_\infty(\text{MU}, R)$ is weakly equivalent to the homotopy pullback. The long exact sequence in homotopy groups associated to this diagram shows that the components of the space of orientations are in bijection with null homotopies of the sequence

\[u(2) \rightarrow \text{gl}_1 S \rightarrow \text{gl}_1 R\]

In other words, we are interested in producing maps $\alpha : \text{gl}_1 S/\langle 2 \rangle \rightarrow \text{gl}_1 R$ in the homotopy category so that the diagram

\[
\begin{array}{ccc}
u(2) & \rightarrow & \text{gl}_1 S \\
\downarrow & \text{inj} & \downarrow \\
\text{gl}_1 S/\langle 2 \rangle & \rightarrow & \text{gl}_1 R
\end{array}
\]

commutes.

The first step in the AHR approach to the description of $\text{E}_\infty$ orientations of complex K-Theory is to show that there exists an $\text{E}_\infty$ map $\text{MU} \rightarrow K_p$.

**Lemma 11.3.** The set $\pi_0 \text{E}_\infty(\text{MU}, K_p)$ is non empty.

**Proof.** To see that there is such a null homotopy we will show that

\[\left[ u(2), \text{gl}_1 K_p \right] = 0 \]

using the splitting of the units of K-theory due to Adams, Priddy and others. Since $u(2)$ is one connected, we can compute on the 1 connected covers,

\[\left[ u(2), \text{gl}_1 K_p(1) \right]\]

This follows from Proposition 5.19 in [5].
Proposition 11.4 (Adams-Priddy, Madsen-Snaith-Tornehave). There is a splitting,
\[ gl_1 K_p(1) = \Sigma^2 H\mathbb{Z}_p \times \Sigma^4 bu_p^\wedge = \Sigma^2 H\mathbb{Z}_p \times bsu_p^\wedge. \]

The proposition can be found on page 416 of Madsen, Snaith and Tornehave [26]. Thus our calculation comes down to inspecting \([u(2), bu_p^\wedge]\) and \([u(2), \Sigma^2 H\mathbb{Z}_p]\). Both of these groups vanish as the K-Theory of \(u\) and the singular cohomology of \(u\) vanish.

These two observations combined with the splitting of \(gl_1 K_p(1)\) and a Künneth theorem shows that
\[ [u(2), gl_1 K_p(1)] \cong [u(2), gl_1 K_p] = 0. \]
Thus there is an \(E_\infty\) map \(MU \to K_p\) and \(\pi_0 E_\infty(MU, K_p)\) is non empty. \(\square\)

11.5. The Miller Invariant. Haynes Miller showed that congruence condition in Theorem 2.4 is true for any orientation of complex K-Theory [29]. We review this result. Suppose \(E\) is an even two periodic theory that is complex oriented such that \(\pi_0 E = \mathbb{R}\) is torsion free. Recall the Bernoulli number \(B^k\) from Definition 8.3.

If \(\alpha : MU \to E\) is any ring map, write
\[ B^k(G, R, \alpha) \]
for the effect of \(B^k\) on the triple \((G, R, t)\) where \(t\) is the coordinate induced by \(\alpha\) and \(G\) is the formal group of \(E\).

Theorem 11.6 (H. Miller). If \(\alpha, \beta : MU \to E\) are two naive ring maps of spectra, then
\[ B^k(G, R, \alpha) \equiv B^k(G, R, \beta) \mod R. \]
The congruence means that the difference of the two values is integral i.e. is an element of \(R\). Hence Miller’s calculation provides a necessary condition for \(E_\infty\) orientability.

11.7. The Characteristic map of AHR. We require a definition from [3] before stating the main result of this section. Fix a positive integer \(n\). Consider the set \(A_n\) of polynomials with \(p\)-adic rational coefficients that satisfy the following condition. If \(b \in \mathbb{Z}_p^\times\) and \(h(z) = \sum_{i \geq n} a_i z^i \in A_n\) then \(\sum_{i \geq n} a_i b^i\) is a \(p\)-adic integer.

Note that any polynomial in \(A_n\) is a continuous function from \(\mathbb{Z}_p^\times\) to \(\mathbb{Z}_p\).

Definition 11.8 (GKC). A sequence of \(p\)-adic integers \(\{t_k\}\) for \(k \geq n\) satisfy the generalized Kummer congruences if for any \(h(z) = \sum_{i \geq n} a_i z^i \in A_n\) we have
\[ \sum_{i \geq n} a_i t_i \in \mathbb{Z}_p. \]

Example 11.9. Suppose \(G\) is a Lubin-Tate group of height one over \(\mathbb{Z}_p\) with coordinate \(t\). If \(\mu \in M(G^\times, S)\) then set
\[ t_k = \int_{G^\times} \langle D^k_F, - \rangle d\mu. \]
If \( h(z) = \sum_{i \geq n} a_i z^i \in A_n \) then
\[
\int_{G^\times} h(z) d\mu(z) = \sum_{i \geq n} a_i t_i \in \mathbb{Z}_p.
\]
Thus if \( \{t_k\} \) is the moments of a measure on the units, then it satisfies the generalized Kummer congruences. The converse is also true; the set \( A_1 \) is dense in \( \text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \).

To obtain the classical Kummer congruences, observe that if \( m \equiv n \mod (p - 1)p^{k-1} \) then by Fermat’s little theorem
\[
\int_{G^\times} x^m d\mu \equiv \int_{G^\times} x^n d\mu \mod p^k.
\]
This approach to Kummer like congruences is attributed to B. Mazur.

**Lemma 11.10.** There is an isomorphism between
\[
[bu(2), K_p \otimes \mathbb{Q}]
\]
and the set of sequences \( \{b_k\} \in \mathbb{Q}_p \) for \( k \geq 1 \).

**Proof.** This is Definition 5.9 in [5]. Let \( v^k \) be the Bott element in \( \pi_{2k}bu \). The isomorphism sends a map \( f \) to the sequence \( s(f) \) defined by
\[
s(f)_k = f_* v^k.
\]
That is we look at the effect of \( f \) on \( v^k \) in degree \( 2k \) for \( k \geq 1 \). \( \Box \)

**Theorem 11.11 (AHR).** Fix a positive integer \( n \). The natural map
\[
K_0^0 K_p \to [bu(2), K_p \otimes \mathbb{Q}] \cong \prod_{k \geq n} \mathbb{Q}_p
\]
sending an operation \( \mu \) to the sequence \( \pi_{2k} \mu \) is injective. Moreover, the image of \( K_0^0 K_p \) is identified with the set of sequences \( \{t_k\} \) satisfying the generalized Kummer congruences for \( k \geq n \).

**Definition 11.12.** Define
\[
b: \pi_0 E_\infty (MU, K_p) \to [bu(2), K_p \otimes \mathbb{Q}] \cong \prod_{k \geq 1} \mathbb{Q}_p.
\]
by
\[
b(\alpha) = \left\{ B^k(\hat{G}_m, \mathbb{Z}_p, \alpha) \right\}.
\]
This is called the characteristic map in [5].

**Theorem 11.13 (AHR).** For odd primes \( p \), the characteristic map
\[
b: \pi_0 E_\infty (MU, K_p) \to [bu(2), K_p \otimes \mathbb{Q}]
\]
identifies the set \( \pi_0 E_\infty (MU, K_p) \) with the set of sequences \( b_k \in \prod_{k \geq 1} \mathbb{Q}_p \) satisfying
\[
(1) \text{ For any } a \in \mathbb{Z}_p^\times - \{\pm\} \text{ the sequence } b_k(1 - a^k)(1 - p^{k-1}) \text{ arises as the moments of a measure on } \mathbb{Z}_p^\times \text{ and}
\]
\( (2) \ b_k \equiv -\frac{B_k}{k} \mod \mathbb{Z}_p. \)
PROOF. Since \( \pi_0 E_\infty(MU, K_p) \) is non empty, we can run the AHR program. Their argument for MO(8) orientations of \( KO_p \) in [5] can be used in this situation. The details are in [35]. We provide a sketch of the proof.

Recall that the complex unstable Adams conjectured was verified in [17]. The conjecture implies that the the top horizontal row in the diagram

\[
\begin{array}{ccc}
J_c & \longrightarrow & BU(p) \\
\downarrow & & \downarrow 1-\psi^a \\
GL_1 S(p) & \longrightarrow & (GL_1 S/U)(p) \longrightarrow BU(p) \longrightarrow BGL_1 S(p)
\end{array}
\]

is null as a map of infinite loop spaces for any \( p \)-adic unit \( a \). We can fill this diagram in

\[
\begin{array}{ccc}
J_c & \longrightarrow & BU(p) \\
\downarrow A & & \downarrow B \\
GL_1 S(p) & \longrightarrow & (GL_1 S/U)(p) \longrightarrow BU(p) \longrightarrow BGL_1 S(p)
\end{array}
\]

using the universal property. Applying the Bousfield-Kuhn functor \( \Phi \) (Theorem [9,5]) yields the top half of

\[
\begin{array}{cccc}
L_{K(1)gl_1 S} & \longrightarrow & K_p & \longrightarrow 1-\psi^a \\
\Phi_A & & \Phi_B & = \\
L_{K(1)gl_1 S/u} & \longrightarrow & K_p & \longrightarrow L_{K(1)gl_1 S} \\
\Sigma^{-1} L_{K(1)gl_1 K_p} \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & L_{K(1)gl_1 K_p} & \longrightarrow L_{K(1)gl_1 K_p} \otimes \mathbb{Q} \longrightarrow L_{K(1)gl_1 K_p} \otimes \mathbb{Q}/\mathbb{Z} \\
\Sigma^{-1} K_p \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & K_p & \longrightarrow K_p \otimes \mathbb{Q} \\
l_1 & \longrightarrow & l_1 & \longrightarrow l_1 & \longrightarrow l_1 & \longrightarrow l_1
\end{array}
\]

We have used the fact that \( K_p \) is a model for \( L_{K(1)bu(2)} \). The map \( \Phi A \) is an expression of the \( K(1) \) local sphere at odd primes and is a weak equivalence by Adams, Baird, Bousfield and Ravenell [10], [31]. It follows that \( \Phi B \) is a weak equivalence.

A map \( \alpha \) making Diagram [15] commute corresponds to a null homotopy of \( u(2) \rightarrow gl_1 S \rightarrow gl_1 K_p \). Since \( \Phi B \) and \( l_1 \) are weak equivalences, to give a map \( \alpha \) up to homotopy is equivalent to producing

\[
l_1 \circ \alpha \circ \Phi B \in [K_p, K_p].
\]

Let \( t_k(\alpha) \) be the sequence of moments attached to this operation. Producing a map \( \alpha \) is equivalent to giving a map \( \beta \) since \( L_{K(1)gl_1 S} \) is torsion. Write \( t_k(\beta) \) for the effect of \( \beta \) : \( K_p \rightarrow L_{K(1)gl_1 K_p} \otimes \mathbb{Q} \) in homotopy. A priori, \( t_k(\beta) \in \mathbb{Q}_p \).

The formula of Rezk (Theorem [9,10]) for the \( K(1) \) local logarithm shows that \( l_1 \) is multiplication by \( (1 - p^{k-1}) \) in degree \( 2k \). Thus, to give a a pair of maps \( \alpha \)}
and $\beta$ making Diagram (15) commute is equivalent to producing a sequence $t_k(\beta)$ such that
\[(1 - a^k) (1 - p^{k-1}) t_k(\beta) = \tau_0 t_k(\alpha) = t_k(\alpha) .\]
The sequence in Equation (16) satisfies the generalized Kummer congruences for $k \geq 1$ by Theorem 11.11. Example 11.9 show $t_k(\alpha)$ must arises as the moments of a measure on the units.

The congruence condition follows from Miller’s theorem 11.6. □

This is the first instance where we have insisted that $p$ be an odd prime. The obstacle at the prime 2 is the cofiber sequence involving the $K(1)$ local sphere. At the even prime, we must use the Adams map on real $K$-Theory $KO_2$. The upshot is that a global result for $MO(8)$ i.e. $MString$ orientations is obtained for real $K$-Theory. A method for classifying complex orientations for complex $K$-Theory at the prime 2 seems mysterious at the current juncture.

12. $H_\infty$ Implies $E_\infty$

In this section we complete our proof of Theorem 2.1.

**Theorem 12.1.** There exists an injective map
\[ \mathbb{D} : H_\infty(MU, K_p) \to \pi_0 E_\infty(MU, K_p). \]

**Proof.** Consider the function which sends the class represented by $\alpha : MU \to K_p$ satisfying the Ando condition to the sequence
\[(17) \quad B^k(\alpha) = B^k(\mathbb{G}_m, \mathbb{Z}_p, \alpha) .\]
Corollary 10.6 shows that the sequence (17) for $k \geq 1$ satisfies the first condition of the AHR classification (Theorem 2.4). Miller’s result (Theorem 11.6) shows that the congruence condition is satisfied. To see that the map is injective, observe that distinct classes, $\alpha$ and $\beta$ in $H_\infty(MU, K_p)$ determine distinct sequences $B^k(\alpha)$ and $B^k(\beta)$ and thus distinct components in $E_\infty(MU, K_p)$.

**Corollary 12.2.** There is an isomorphism
\[ \mathbb{D} : H_\infty(MU, K_p) \to \pi_0 E_\infty(MU, K_p) \]

**Proof.** We check that composition with the forgetful map
\[ C : \pi_0 E_\infty(MU, K_p) \to H_\infty(MU, K_p) \]
is the identity. To begin, suppose $\alpha : MU \to K_p$ is merely $H_\infty$. The map $\mathbb{D}$ identifies $\alpha$ with the $E_\infty$ orientation corresponding to the sequence $B^k(\alpha)$. Under the identification in Theorem 11.13 this component is identified with $\alpha$. The other composition is similar. □

The author feels that this calculation locates the power operations in the classification of $E_\infty$ maps due to AHR. As an application we have
Corollary 12.3. For odd primes $p$, the Todd genus

$$MU \to K_p$$

is an $E_\infty$ map.

Proof. The Todd genus corresponds to the orientation $1 - u$ on $\hat{\mathbb{G}}_m$ over $\mathbb{Z}_p$. The Bernoulli number $B_k$ returns $-\frac{B_k}{k}$ as seen in Example 8.4. One can check that the Ando condition is satisfied for this choice of coordinate and so this the Todd genus is an $E_\infty$ map. □

The Todd genus corresponds to the measure on $\mathbb{Z}_p^\times$ whose moments are

$$-\frac{B_k}{k} \left(1 - a^k\right) \left(1 - p^{k-1}\right).$$

This is the measure considered in Chapter II section 6 of [23] and is known as the Mazur measure.

12.4. Why $\langle a \rangle_F(t)$? We close by producing a conceptual argument for the appearance of the divided $a$ series in our calculations. First recall the cannibalistic class $\rho$. Suppose

$$
\begin{array}{ccc}
V & \downarrow & X \\
\end{array}
$$

is a complex vector bundle (of virtual rank zero) over a connected base space $X$. It is classical that the cohomology of the Thom space, $K_p(X^V)$, is a free rank one module over the cohomology of the base. A generator is known as a Thom class, $U(V)$, of the bundle $V$. A complex orientation determines a Thom class and the Thom isomorphism

$$\tau_V : K_p(X) \to K_p(X^V)$$

is multiplication by this Thom class. The following denition is essentially due to Bott.

Definition 12.5. Suppose $a$ is a unit in $\mathbb{Z}_p$. Given a complex bundle $V$ as above define a class

$$\rho_a U(V) = \frac{\psi^a U(V)}{U(V)} \in (K_p X)^\times$$

A few remarks are necessary. First, since $\psi^a$ is a ring map, $\psi^a U(V)$ is a generator for $K_p(X^V)$ as a $K_p(X)$ module. The ratio in this definition is the unique element of $K_p(X)$ that takes the Thom class $U(V)$ to the class represented by $\psi^a(U(V))$.

The cannibalistic class $\rho_a$ produces an element in the units of the base. As the bundle $V$ was arbitrary we have constructed a map of spectra

$$bu(2) \to gl_1 K_p.$$

Example 12.6. Let $L$ be the tautological line bundle over $\mathbb{C}P^\infty$. Suppose we have a complex orientation $\alpha$ of $K_p$. Let $t$ be the Euler class of $L$ induced by this orientation. In this situation we have the formula

$$\rho_a(t) = \frac{[a]_F(t)}{t} = \langle a \rangle_F(t).$$
Using the obstruction theory described above, an $E_\infty$ orientation is given by a null homotopy of

$$u(2) \to gl_1S \to gl_1K_p.$$ 

We have shown that such a null homotopy exists in Section 11.3. If we precede the above chain of spectra with the stable Adams conjecture and follow with the $K(1)$ local logarithm we obtain the diagram

\[
\begin{array}{ccc}
u(2) & \xrightarrow{\psi^a-1} & gl_1S \\
\downarrow & & \downarrow \\
u(2) & \xrightarrow{gl_1} & gl_1K_p \\
\downarrow & & \downarrow l_1 \\
C & \xrightarrow{\rho} & K_p \\
\downarrow & & \\
\mathbb{C}P^\infty & \xrightarrow{e(L)} & bu(2) \\
\end{array}
\]

The arrow labeled $e(L)$ is Euler class of $L$.

This diagram shows that we are comparing the given orientation with $\psi^a$ of that orientation. As seen from the diagram, this ratio is the cannibalistic class:

$$\rho: bu(2) \to gl_1K_p.$$ 

This class followed by $l_1$ is represented by $l_1(a)\mathcal{F}(t)$ in $(K_p\mathbb{C}P^\infty)^\times$. We would like to contribute this observation to Charles Rezk.
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