Abstract. Quantitative separation logic (QSL) is an extension of separation logic (SL) for the verification of probabilistic pointer programs. In QSL, formulae evaluate to real numbers instead of truth values, e.g., the probability of memory-safe termination in a given symbolic heap. As with SL, one of the key problems when reasoning with QSL is entailment: does a formula $f$ entail another formula $g$?

We give a generic reduction from entailment checking in QSL to entailment checking in SL. This allows to leverage the large body of SL research for the automated verification of probabilistic pointer programs. We analyze the complexity of our approach and demonstrate its applicability. In particular, we obtain the first decidability results for the verification of such programs by applying our reduction to a quantitative extension of the well-known symbolic-heap fragment of separation logic.

1 Introduction

Separation logic [28] (SL) is a popular formalism for Hoare-style verification of imperative, heap-manipulating and, possibly, concurrent programs. Its assertion language extends first-order logic with two connectives—the separating conjunction $\star$ and the magic wand $\rightsquigarrow$—that enable concise specifications of how program memory, or other resources, can be split-up and combined. SL builds upon these connectives to champion local reasoning about the resources employed by programs. Consequently, program parts can be verified by considering only those resources they actually access—a crucial property for building scalable tools including automated verifiers [45,11,15,43,30], static analyzers [9,23,13], and interactive theorem provers [31]. At the foundation of almost any automated approach based on SL, lies the entailment problem $\varphi \models \psi$: are all models of SL formula $\varphi$ also models of SL formula $\psi$? For example, Hoare-style verifiers need to solve entailments whenever they invoke the rule of consequence, and static

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analyzers ultimately solve entailments to perform abstraction. While undecidable in general [1], the wide adoption of SL and the central role of the entailment problem have triggered a massive research effort to identify SL fragments with a decidable entailment problem [10,16,20,21,26,27,34,39,46,17,19], and to build practical entailment solvers [45,11,15,49].

Probabilistic programs, that is, programs with the ability to sample from probability distributions, are an increasingly popular formalism for, amongst others, designing efficient randomized algorithms [41] and describing uncertainty in systems [22,14]. While formal reasoning techniques for probabilistic programs exist since the 80s (cf., [36,37,48]), they are rarely automated and typically target only simplistic programming languages. For example, verification techniques that support reasoning about both randomization and data structures are, with notable exceptions [50,8], rare—a surprising situation given that randomized algorithms typically rely on dynamic data structures.

Quantitative separation logic (QSL) is a weakest-precondition-style verification technique that targets randomized algorithms manipulating complex data structures; it marries SL and weakest preexpectations [42]—a well-established calculus for reasoning about probabilistic programs. In contrast to classical SL, QSL’s assertion language does not consist of predicates, which evaluate to Boolean values, but expectations (or: random variables), which evaluate to real numbers. QSL has been successfully applied to the verification of randomized algorithms, and QSL expectations have been formalized in Isabelle/HOL [25]. However, reasoning is far from automated—mainly due to the lack of decision procedures or solvers for entailments between expectations in QSL.

This paper presents, to the best of our knowledge, the first technique for automatically deciding QSL entailments. More precisely, we reduce QSL quantitative entailments to classical entailments between SL formulas. Hence, we can leverage two decades of separation logic research to advance QSL entailment checking, and thus also automated reasoning about probabilistic programs.

Contributions. We make the following technical contributions:

- We present a generic construction that reduces the entailment problem for quantitative separation logic to solving multiple entailments in fragments of SL; if we reduce to an SL fragment where entailment is decidable, our construction yields a QSL fragment with a decidable entailment problem.
- We provide simple criteria for whether one can leverage a decision procedure or a practical entailment solver for SL to build an entailment solver for QSL.
- We analyze the complexity of our approach parameterized in the complexity of solving entailments in a given SL fragment; whenever we identify a decidable QSL fragment, it is thus accompanied by upper complexity bounds.
- We use our construction to derive the QSL fragment of quantitative symbolic heaps for which entailment is decidable via a reduction to the Bernays-Schönfinkel-Ramsey fragment of SL [19].

Outline. Section 2 introduces (quantitative) separation logic. Section 3 motivates our approach by providing the foundations for probabilistic pointer program
verification with QSL together with several examples. We present the key ideas and our main contribution of reducing QSL entailment checking to SL entailment checking in Section 4. We analyse the complexity of our approach in Section 5. In Section 6, we apply our approach to obtain the first decidability results for probabilistic pointer verification. Finally, Section 7 discusses related work and Section 8 concludes.

Table 1. Metavariables used throughout this paper.

| Entities            | Metavariables | Domain          |
|---------------------|---------------|-----------------|
| Natural numbers     | $n, i, j, k$  | $\mathbb{N}$    |
| Rational probabilities | $p, q, \alpha, \beta, \gamma, \delta$ | $\mathbb{P}$    |
| Programs            | $C$           | hpGCL           |
| Stacks              | $s$           | Stacks          |
| Heaps               | $h$           | Heaps$_k$       |
| Variables           | $x, y, z$     | Vars            |
| Values              | $v, w$        | Vals            |
| Locations           | $\ell$        | Locs            |
| Predicates          | $\Phi$        | $\mathcal{P}$ (States) |
| one-bounded expectations | $X$      | $\mathbb{E}_{\leq 1}$ |
| SL formulæ          | $\varphi, \psi, \vartheta$ | SL $[\cdot]$ |
| Pure formulæ        | $\pi$         |                 |
| QSL formulæ         | $f, g, u, I$  | QSL $[\cdot]$  |

2 (Quantitative) Separation Logic

2.1 Program States

Let $\text{Vals}$ be a countably infinite set of values, and let $\text{Vars}$ be a countably infinite set of variables with domain $\text{Vals}$. The set of stacks is given by

$$\text{Stacks} = \{ s \mid s : \text{Vars} \rightarrow \text{Vals} \} .$$

Let $\text{Locs} \subseteq \text{Vals}$ be an infinite set of locations. We denote locations by $\ell$ and variations thereof. We fix a natural number $k \geq 1$ and a heap model where finite sets of locations are mapped to fixed-size records over $\text{Vals}$ of size $k$. Put more
| Formula          | Meaning                                                                 |
|------------------|-------------------------------------------------------------------------|
| \( \varphi \)    | \( (s, h) \models \varphi \) if \( \varphi \) is true \( (s, h) \in [\varphi] \) |
| \( \top \)       | \( (s, h) \not\models \top \)                                          |
| \( \neg \psi \)  | \( (s, h) \models \neg \psi \) if \( \neg \psi \) is true \( (s, h) \not\models \psi \) |
| \( \psi \wedge \top \) | \( (s, h) \models \psi \) and \( (s, h) \models \top \)              |
| \( \psi \vee \top \) | \( (s, h) \models \psi \) or \( (s, h) \models \top \)               |
| \( \exists x : \psi \) | \( (s \ [x := v], h) \models \psi \) for some \( v \in \text{Vals} \) |
| \( \forall x : \psi \) | \( (s \ [x := v], h) \models \psi \) for all \( v \in \text{Vals} \) |
| \( \psi \star \top \) | \( (s, h_1) \models \psi \) and \( (s, h_2) \models \top \) for some \( h_1 \star h_2 = h \) |
| \( \psi \rightarrow \top \) | \( (s, h \star h') \models \psi \) for all \( h' \perp h \) with \( (s, h') \models \psi \) |

Formally, the set of heaps is given by

\[
\text{Heaps}_k = \left\{ h \mid h : L \rightarrow \text{Vals}_k, L \subseteq \text{Locs}, |L| < \infty \right\}.
\]

The set of program states is then given by

\[
\text{States} = \{ (s, h) \mid s \in \text{Stacks}, h \in \text{Heaps}_k \}.
\]

Given a program state \((s, h)\) and an expression \(t\) over \(\text{Vars}\), we denote by \(t(s)\) the evaluation of expression \(t\) in \(s\), i.e., the value that is obtained by evaluating \(t\) after replacing any occurrence of any variable \(x \in \text{Vars}\) in \(t\) by the value \(s(x)\). We write \(s \ [x := v]\) to indicate that we set variable \(x\) to value \(v \in \text{Vals}\) in \(s\), i.e.\(^4\),

\[
s \ [x := v] = \lambda y \cdot \begin{cases} v, & \text{if } y = x \\ s(y), & \text{if } y \neq x. \end{cases}
\]

For heap \(h\), \(h \ [\ell := (v_1, \ldots, v_k)]\) is defined analogously. For a given heap \(h : L \rightarrow \text{Vals}_k\), we denote by \(\text{dom}(h)\) its domain \(L\). Two heaps \(h_1, h_2\) are disjoint, denoted \(h_1 \perp h_2\), if their domains do not overlap, i.e., \(\text{dom}(h_1) \cap \text{dom}(h_2) = \emptyset\). The disjoint union of two disjoint heaps \(h_1 : L_1 \rightarrow \text{Vals}_k\) and \(h_2 : L_2 \rightarrow \text{Vals}_k\) is

\[
h_1 \star h_2 : \text{dom}(h_1) \cup \text{dom}(h_2) \rightarrow \text{Vals}_k, \quad (h_1 \star h_2)(\ell) = \begin{cases} h_1(\ell), & \text{if } \ell \in \text{dom}(h_1) \\ h_2(\ell), & \text{if } \ell \in \text{dom}(h_2). \end{cases}
\]

### 2.2 Separation Logic

A predicate \(\Phi \in \mathcal{P}(\text{States})\) is a set of states. A predicate \(\Phi\) is called pure if it does not depend on the heap, i.e., for every stack \(s\) and heaps \(h, h'\), we have \((s, h) \in \Phi\) iff \((s, h') \in \Phi\).

\(^4\) We use \(\lambda\)-expressions to denote functions: Function \(\lambda X \cdot f\) applied to an argument \(v\) evaluates to \(f\) in which every occurrence of \(X\) is replaced by \(v\).
We consider a separation logic SL [A] with standard semantics [47]. A distinguishing aspect is that SL [A] is parametrized by a set A of predicate symbols ψ with given semantics [[ψ]] ∈ P (States). We often identify predicate symbols ψ with their predicates [[ψ]]. Elements of A build the atoms of SL [A]. Our reduction from quantitative entailments to qualitative entailments does not depend on the choice of these predicate symbols. We therefore take a generic approach that allows for user-defined atoms, e.g., list or tree predicates.

**Definition 1.** Let A be a countable set of predicate symbols. Formulae in separation logic SL [A] with atoms in A adhere to the grammar

φ → ϑ | ¬φ | φ ∧ φ | φ ∨ φ | ∃x: φ | ∀x: φ | φ ∗ φ | φ − − φ,

where ϑ ∈ A, and where x ∈ Vars. △

The Boolean connectives ¬, ∧, and ∨ as well as the quantifiers ∃ and ∀ are standard. ∗ is the separation conjunction and − − is the magic wand.

The semantics [[φ]] ∈ P (States) of a formula φ ∈ SL [A] is defined by induction on the structure of φ as shown in Table 2. Recall that we assume the semantics [[ψ]] of predicate symbols ψ ∈ A to be given. We often write (s, h) |= φ instead of (s, h) ∈ [[φ]]. For φ, ψ ∈ SL [A], we say that φ entails ψ, denoted φ |= ψ, if whenever (s, h) |= φ, also (s, h) |= ψ.

**Example 1.** Let Vars = Z, Locs = N > 0, and k = 1. A term t is either a variable x ∈ Vars or the constant 0 ∈ Vars. The set A of predicate symbols is

A = {true, emp, x → t, t = t′, t ̸= t′, ls (t, t′) | x ∈ Vars, t, t′ terms}

Here, apart from standard predicates for true, equalities, and disequalities,

1. emp is the empty-heap predicate, i.e.,

(s, h) |= emp iff dom (h) = ∅,

2. x → t is the points-to predicate, i.e.,

(s, h) |= x → t iff dom (h) = {s(x)} and h(s(x)) = t(s),

3. the list predicate ls (t, t′) asserts that the heap models a singly-linked list segment from t to t′:

(s, h) |= ls (t, t′)

iff dom (h) = ∅ and t(s) = t′(s) or there exist n ≥ 1 and terms t₁, ..., tₙ with tₙ = t′ such that

(s, h) |= t → t₁ ⋆ ⋯ ⋆ tₙ₋₁ → tₙ.

In this setting, SL [A] contains, e.g., the well-known symbolic heap fragment of separation logic with lists. For instance, the SL [A] formula

∃y: ∃z: x → y ⋆ y → z ⋆ ls (z, 0).

asserts that the heap consists of a list with head x of length at least 2. △
Table 3. Semantics of QSL [A] formulae.

| $f$               | $[f] (s, h)$                                      |
|-------------------|--------------------------------------------------|
| $[\psi]$          | $[\psi] (s, h)$                                  |
| $[\pi] \cdot g + [-\pi] \cdot u$ | $[\pi] (s, h) \cdot [g] (s, h) + [-\pi] (s, h) \cdot [u] (s, h)$ |
| $q \cdot g + (1 - q) \cdot u$ | $q \cdot [g] (s, h) + (1 - q) \cdot [u] (s, h)$ |
| $g \cdot u$       | $[g] (s, h) \cdot [u] (s, h)$                   |
| $1 - g$           | $1 - [g] (s, h)$                                |
| $g \max u$        | $\max\{[g] (s, h), [u] (s, h)\}$               |
| $g \min u$        | $\min\{[g] (s, h), [u] (s, h)\}$               |
| $\exists x : g$   | $\max \{[g] (s \mid x := v), h) \mid v \in \text{Vals}\}$ |
| $\mathcal{L} x : g$ | $\min \{[g] (s \mid x := v), h) \mid v \in \text{Vals}\}$ |
| $g \ast u$        | $\max \{[g] (s, h_1) \cdot [u] (s, h_2) \mid h = h_1 \ast h_2\}$ |
| $[\psi] \longrightarrow g$ | $\inf \{[g] (s, h \ast h') \mid h' \perp h \text{ and } [\psi] (s, h) = 1\}$ |

2.3 Quantitative Separation Logic

In quantitative separation logic [8,38], formulae evaluate to non-negative real numbers or infinity instead of truth values. By conservatively extending the weakest preexpectation calculus by McIver & Morgan [40], this enables the compositional verification of probabilistic pointer programs by reasoning about expected list-sizes, probabilities of terminating with an empty heap, and alike.

We consider here a fragment of quantitative separation logic suitable for reasoning about the likelihood of events in probabilistic pointer programs such as, e.g., the probability of terminating in a given symbolic heap. The formulae we consider evaluate to rational probabilities rather than arbitrary reals or infinity. We denote the set $[0, 1] \cap \mathbb{Q}_{>0}$ of rational probabilities by $\mathbb{P}$. Like SL [A], quantitative separation logic is parameterized by a set $\mathfrak{A}$ of predicate symbols $\psi$ with given semantics $[\psi] \in \mathcal{P} (\text{States})$, building the atoms of QSL [A].

Definition 2. Let $\mathfrak{A}$ be a countable set of predicate symbols. Formulae in quantitative separation logic QSL [A] with atoms in $\mathfrak{A}$ adhere to the grammar

\[
\begin{align*}
  f & \rightarrow \ [\psi] \ | \ [\pi] \cdot f + [-\pi] \cdot f \ | \ q \cdot f + (1 - q) \cdot f \ | \ f \cdot f \\
  & \ | \ 1 - f \ | \ f \max f \ | \ f \min f \ | \ \exists x : f \ | \ \mathcal{L} x : f \\
  & \ | \ f \ast f \ | \ [\psi] \longrightarrow f ,
\end{align*}
\]

where $\psi, \pi \in \mathfrak{A}$ with $\pi$ pure, $q \in \mathbb{P}$, and where $x \in \text{Vars}$. \triangle

The semantics of a formula $f \in$ QSL [A] is a (one-bounded) expectation. The set $\mathbb{E}_{\leq 1}$ of one-bounded expectations is defined as

\[
\mathbb{E}_{\leq 1} = \{ X \mid X : \text{States} \rightarrow [0, 1] \} .
\]
We use the Iverson bracket \cite{Iverson} notation \([\Phi]\) to associate with predicate \(\Phi\) its indicator function. Formally,

\[
[\Phi] : \text{States} \to \{0, 1\}, \quad [\Phi](s, h) = \begin{cases} 1, & \text{if } (s, h) \in \Phi \\ 0, & \text{if } (s, h) \notin \Phi \end{cases}.
\]

Given a predicate symbol \(\psi\), we often write \([\psi]\) instead of \([(\psi)]\). The semantics \([f] \in E_{\leq 1}\) of \(f \in \text{QSL } \mathcal{A}\) is defined by induction on the structure of \(f\) in Table 3. We write \(f \equiv g\) if \(f\) and \(g\) are equivalent, i.e. if \([f] = [g]\). Infima and suprema are taken over the complete lattice \(([0, 1], \leq)\). In particular, \(\inf \emptyset = 1\) and \(\sup \emptyset = 0\).

**Theorem 1.** The semantics of QSL \([\mathcal{A}]\) formulae is well-defined, i.e., for all \(f \in \text{QSL } \mathcal{A}\), we have \([f] \in E_{\leq 1}\).

**Proof.** By induction on the structure of \(f\). For details see Appendix A.

Let us go over the individual constructs. Formulae of the form \([\psi]\) are the atomic formulae. \([\pi] \cdot g + [\neg \pi] \cdot u\) is a Boolean choice between \(g\) and \(u\) that does not depend upon the heap since \([\pi]\) is pure. \(q \cdot g + (1 - q) \cdot u\) is a convex combination of \(g\) and \(u\). \(g \cdot u\) is the pointwise multiplication of \(g\) and \(u\). \(1 - g\) is the quantitative (or probabilistic) negation of \(g\). \(g\max u\) and \(g\min u\) is the pointwise maximum and minimum of \(g\) and \(u\), respectively.

\(\exists x : g\) is the supremum quantification that, given a state \((s, h)\), evaluates to the supremum of the set obtained from evaluating \(g\) in \((s[x := v], h)\) for every value \(v \in \text{Vals}\). In our setting, this supremum is actually a maximum. Dually, \(\forall x : g\) is the infimum quantification.

\(\star\) and \(\rightarrow\) are the quantitative analogous of the separating conjunction and the magic wand from separation logic as defined in \cite{DBLP:journals/tcs/BrookeS96}. \(g \star u\) is the quantitative separating conjunction of \(g\) and \(u\). Intuitively speaking, whereas the qualitative separating conjunction maximizes a truth value under all appropriate partitionings of the heap, the quantitative separating conjunction maximizes a probability. \([\psi]\) \(\rightarrow\) \(u\) is the quantitative magic wand. Whereas the qualitative magic wand minimizes a truth value under all appropriate extensions of the heap, the quantitative magic wand minimizes a probability. For an in-depth treatment of these connectives, we refer to \cite{DBLP:journals/tcs/BrookeS96}.

**Example 2.** Let \(\text{Vals}, \text{Locs}, k, \) and \(\mathcal{A}\) be as in Example 1. Then QSL \([\mathcal{A}]\) contains, e.g., a quantitative extension of the symbolic heap fragment of separation logic with lists. For instance, the QSL \([\mathcal{A}]\) formula

\[0.7 \cdot (\exists y : \exists z : [x \mapsto y] \star [y \mapsto z] \star [\text{ls}(z, 0)]) + 0.3 \cdot [\text{emp}]\]

expresses that with probability 0.7 the heap consists of a list with head \(x\) of length at least 2 and that with probability 0.3 the heap is empty. \(\triangle\)

Finally, given \(f, g \in \text{QSL } \mathcal{A}\), we say that \(f\) entails \(g\), denoted \(f \models g\), if

\[\text{for all } (s,h) \in \text{States}: \quad [f](s,h) \leq [g](s,h)\]
Quantitative entailments $f \models g$ generalize classical entailments in the sense that $f$ (pointwise) lower-bounds the quantity $g$. For example, if $g$ assigns to each state the probability that some program $C$ terminates without a memory error, then the entailment $\text{true} \models g$ means that $C$ terminates almost-surely, i.e., with probability one. Our problem statement now reads as follows: Reduce entailment checking in QSL [20] to checking finitely many entailments in SL [31].

3 Entailments in Probabilistic Program Verification

Our primary motivation for studying the entailment problem for quantitative separation logic is to provide foundations for the automated verification of probabilistic pointer programs. In this section, we consider examples of such programs written in hpGCL—an extension of McIver & Morgan’s probabilistic guarded command language (cf., [40]) by heap-manipulating instructions—and the entailments that arise from their verification. We briefly formalize reasoning about hpGCL programs with weakest liberal preexpectations; for a thorough introduction of hpGCL programs and techniques for their verification, we refer to [8,38].

3.1 Heap-manipulating pGCL

Recall from Section 2.1 that heaps map memory locations to fixed-size records (or tuples) of length $k \geq 1$. The set of programs in heap-manipulating probabilistic guarded command language for $k = 1$, $\text{Vars} = \mathbb{Z}$ and $\text{Locs} = \mathbb{N}_{>0}$, denoted hpGCL, is given by the grammar

$$C \rightarrow \text{skip} \quad \text{(effectless program)}$$

$$| \quad x := E \quad \text{(assignment)}$$

$$| \quad \{ C \} \{ p \} \{ C \} \quad \text{(prob. choice)}$$

$$| \quad C ; C \quad \text{(seq. composition)}$$

$$| \quad \text{if} (B) \{ C \} \text{else} \{ C \} \quad \text{(conditional choice)}$$

$$| \quad \text{while} (B) \{ C \} \quad \text{(loop)}$$

$$| \quad x := \text{new} (E) \quad \text{(allocation)}$$

$$| \quad \text{free}(E), \quad \text{(disposal)}$$

$$| \quad x := < E > \quad \text{(lookup)}$$

$$| \quad < E > := E' \quad \text{(mutation)}$$

where $x \in \text{Vars}$, $p \in \mathbb{P}$, $E, E'$ are arithmetic expressions and $B$ is a Boolean expression. We assume that expressions do not depend on the heap. For now, we do not fix a specific syntax for expressions but assume evaluation mappings

$$E : \text{Stacks} \rightarrow \mathbb{Z} \quad \text{and} \quad B : \text{Stacks} \rightarrow \{ \text{true}, \text{false} \} .$$

In addition to the usual control flow structures for sequential composition, conditionals, and loops, skip does nothing, $x := E$ assigns the value $E(s)$ obtained
Table 4. Rules for compositionally computing weakest liberal preexpectations. Here, \( f \) is a QSL \([A]\) formula representing the postexpectation. \( f \left[ x := E \right] \) denotes the substitution of every free occurrence of \( x \) by \( E \) in \( f \). \( E \mapsto \cdot \) desugars to \( S_z : [E \mapsto \cdot \] → \( S \).

| \( C \) | \( \text{wlp}[C](f) \) |
|---|---|
| \( \text{skip} \) | \( f \) |
| \( x := E \) | \( f \left[ x := E \right] \) |
| \( \{ C_1 \} [p] \{ C_2 \} \) | \( p \cdot \text{wlp}[C_1](f) + (1 - p) \cdot \text{wlp}[C_2](f) \) |
| \( C_1 \); \( C_2 \) | \( \text{wlp}[C_1](\text{wlp}[C_2](f)) \) |
| \( \text{if} (B) \{ C_1 \} \text{ else } \{ C_2 \} \) | \( [B] \cdot \text{wlp}[C_1](f) + \neg[B] \cdot \text{wlp}[C_2](f) \) |
| \( x := \text{new}(E) \) | \( \forall y : [y \mapsto E] \rightarrow f \left[ x := y \right] \) |
| \( \text{free}(E) \) | \( [E \mapsto \cdot \] * \( f \) |
| \( x := <E> \) | \( \forall y : [E \mapsto y] \star ([E \mapsto y] \rightarrow f \left[ x := y \right]) \) |
| \( <E> := E' \) | \( [E \mapsto \cdot \] * \( ([E \mapsto E'] \rightarrow f) \) |

from evaluating expression \( E \) in the current program state \((s,h)\) to \( x \), and the probabilistic choice \( \{ C_1 \} [p] \{ C_2 \} \) flips a coin with bias \( p \)—it executes \( C_1 \) if the coin flip yields heads, and \( C_2 \) otherwise. The allocation \( x := \text{new}(E) \) non-deterministically selects a fresh location, stores it in \( x \), and puts a record with value \( E \) on the heap at that location. Since we assume an infinite address space, allocation never fails. Conversely, \( \text{free}(E) \) disposes the record at location \( E \) from the heap; it fails if no such location exists. The mutation \( <E> := E' \) and the lookup \( x := <E> \) update to \( E' \) resp. assign to \( x \) the value stored at location \( E \); both statements fail if the heap contains no such location.

3.2 Weakest Liberal Preexpectations

We formalize reasoning about hpGCL programs in terms of the weakest liberal preexpectation transformer \( \text{wlp} : \text{hpGCL} \rightarrow (\text{QSL}[\mathcal{A}] \rightarrow \text{QSL}[\mathcal{A}]) \), where \( \mathcal{A} \) at least contains formulae of the form \([E \mapsto E']\); Table 4 summarizes the rules for computing \( \text{wlp} \) of loop-free programs on the program structure.

Conceptually, the weakest liberal preexpectation \([\text{wlp}[C](f)](s,h)\) of program \( C \) with respect to postexpectation \( f \in \text{QSL}[\mathcal{A}] \) on \((s,h)\) is the least expected value of \([f]\) (measured in the final states) after successful\(^{5}\) termination of \( C \) on initial state \((s,h)\), plus the probability that \( C \) does not terminate on \((s,h)\). Adding the non-termination probability can be thought of as a partial correctness view: we include the non-termination probability of \( C \) on state \((s,h)\) in the \( \text{wlp} \) of \( C \) just as we include the state \((s,h)\) in the weakest liberal precondition of \( C \) in case \( C \) does not terminate on \((s,h)\).

\(^{5}\) i.e., without encountering a memory error.
A reader familiar with separation logic will realize the close similarity between the rules in Table 4 and the weakest preconditions for SL by Ishtiaq and O’Hearn [28]. The main differences are (1) the use of the quantitative connectives $\star, \rightarrow\star, \cdot$, and $+$, and (2) the additional rule for probabilistic choice, \( \text{wlp}[[C_1] \mid p \{ C_2 \}] (f) \), which is a convex sum that weights \( \text{wlp}[C_1] (f) \) and \( \text{wlp}[C_2] (f) \) by \( p \) and \( (1 - p) \), respectively.

The transformer \( \text{wlp} \) is well-defined in the sense that, for every loop-free \( \text{hpGCL} \)-program and every \( \text{QSL} [\mathfrak{A}] \) formula, we obtain—under mild conditions—again a \( \text{QSL} [\mathfrak{A}] \) formula:

**Theorem 2.** Let \( C \in \text{hpGCL} \) be loop-free and \( \mathfrak{A} \) be a set of predicate symbols. If

1. \( \mathfrak{A} \) contains the points-to predicate for all variables and all expressions occurring in allocation, disposal, lookup and mutation in \( C \),
2. \( \mathfrak{A} \) contains all guards and their negations occurring in \( C \), and
3. all predicates in \( \mathfrak{A} \) are closed under substitution of variables by variables and arithmetic expressions occurring on right-hand sides of assignments in \( C \),

then, for every \( \text{QSL} [\mathfrak{A}] \) formula \( f \), \( \text{wlp}[C] (f) \in \text{QSL} [\mathfrak{A}] \).

**Proof.** By induction on loop-free \( C \). For details see Appendix B.1.

For loops, \( \text{wlp}[\text{while} (B) \{ C \}] (f) \) is typically characterized as the greatest fixed point of loop unrollings. However, we fixed an explicit syntax of formulae instead of allowing arbitrary expectations; the above fixed point is in general not expressible in our syntax.\(^6\) To deal with loops, we thus require a user-supplied invariant \( I \) and apply the following proof rule (cf., [33]) to approximate \( \text{wlp} \):

\[
I \models [\neg B] \cdot f + [B] \cdot \text{wlp}[C'] (I) \quad \text{implied} \quad I \models \text{wlp}[\text{while} (B) \{ C' \}] (f)
\]

Notice that verifying that \( I \) is indeed an invariant via the above rule requires proving an entailment between \( \text{QSL} [\mathfrak{A}] \) formulae.

### 3.3 Interfered Swap

Our first example concerns a program \( C_{\text{swap}} \), implemented in \( \text{hpGCL} \) below, that attempts to swap the contents of two memory locations \( x \) and \( y \). However, since variable \( x \) is shared with a concurrently running process, writing to \( x \) can be unreliable, that is, instead of the intended value, the concurrently running process may write a corrupted value \( \text{err} \) into memory with some probability, say 0.001.

A similar situation occurs, e.g., when using the protocol described in [2].

\[
C_{\text{swap}} :\quad \text{tmp1} := < x > ; \\
\text{tmp2} := < y > ; \\
\{ < x > := \text{tmp2} \} [0.999] \{ < x > := \text{err} \} ; \\
< y > := \text{tmp1} .
\]

\(^6\) It is noteworthy that a sufficiently expressive syntax for weakest preexpectation reasoning without heaps has been developed only recently [7].
We can use \( \text{wlp} \) to verify an upper bound on the probability that an erroneous write operation happened by solving the QSL entailment

\[
\text{wlp}[C_{\text{swap}}] ([x \mapsto z_2] \star [y \mapsto z_1]) \\
\models [z_2 = \text{err}] \cdot ([x \mapsto z_1] \star [y \mapsto z_2]) + [z_2 \neq \text{err}] \cdot (0.999 \cdot ([x \mapsto z_1] \star [y \mapsto z_2])) .
\]

That is, the probability that \( C_{\text{swap}} \) successfully swaps the contents of \( x \) and \( y \) is at most 0.999 if \( y \) does initially not point to the corrupt value \( \text{err} \).

As we will see in Section 6.1, our approach for solving QSL entailments is capable of deciding the above entailment, where \( \text{wlp}[C_{\text{swap}}] ([x \mapsto z_2] \star [y \mapsto z_1]) \) is computed according to the rules in Table 4.

### 3.4 Avoiding Magic Wands

Recall from Table 4 that computing \( \text{wlp} \) introduces a magic wand (\( \mapsto \)) for almost every statement that accesses the heap. This is unfortunate because many decidable separation logic fragments as well as practical entailment solvers do not support magic wands.

In particular, in Section 6.1 we present a QSL fragment with a decidable entailment problem that supports magic wands only on the left-hand side of entailments. Hence, proving a lower bound on the probability that the program \( C_{\text{swap}} \) from above successfully swapped the contents of two memory cells, e.g.,

\[
0.98 \cdot ([x \mapsto z_2] \star [y \mapsto z_1]) \models \text{wlp}[C_{\text{swap}}] ([x \mapsto z_1] \star [y \mapsto z_2]) ,
\]

might still be possible with our technique but requires a different separation logic fragment to reduce to.

Fortunately, we can often avoid introducing magic wands by employing local reasoning and rules for computing \( \text{wlp} \) for specific pre- and postexpectations. In particular, the \( \text{wlp} \) calculus features (1) the frame rule from separation logic, i.e., if no free variable in \( g \) is modified by \( C \), then \( \text{wlp}[C] (f \cdot g) \models \text{wlp}[C] (f) \cdot \text{wlp}[C] (g) \), (2) super-distributivity for convex combinations and maximum, i.e., \( q \cdot \text{wlp}[C] (f) + (1 - q) \cdot \text{wlp}[C] (g) \models \text{wlp}[C] (q \cdot f + (1 - q) \cdot g) \) and \( \text{wlp}[C] (f) \max \text{wlp}[C] (g) \models \text{wlp}[C] (f \max g) \), and (3) monotonicity, i.e., \( f \models g \) implies \( \text{wlp}[C] (f) \models \text{wlp}[C] (g) \). Moreover, we give four examples of specialized rules that avoid magic wands but require specific postexpectations: if \( x \) is not a free variable of \( E \) or \( f \), and \( x \) and \( y \) are distinct variables, then

(i) \( \text{wlp}[x := \langle E \rangle] ([E \mapsto y] : [x = y]) = [E \mapsto y] * f [x := y] ; \)

(ii) \( \text{wlp}[x := E'] ([E \mapsto E'] * f) = [E \mapsto -] * f ; \)

(iii) \( \text{wlp}[x := \text{new}(x)] (\exists y : [x \mapsto y] * f) = f [y := x] ; \) and

(iv) \( \text{wlp}[x := \text{new}(y)] ([x \mapsto -] * f) = f \).

Similar rules have been used successfully for symbolic execution with separation logic in non-probabilistic settings [12]. Combining the above rules with framing, distributivity, and monotonicity often allows avoiding magic wands. In such cases, we have a richer set of decidable SL fragments upon which to build solvers.
for QSL entailments at our disposal. Coming back to the entailment (†) from above and writing \( C_{\text{swap}} = C_1; C_2; C_3; C_4 \), we calculate

\[
\begin{align*}
\text{wlp}[C_{\text{swap}}](\{x \mapsto z_1\} \ast \{y \mapsto z_2\}) & \models \text{wlp}[C_{\text{swap}}](\{y \mapsto \text{tmp1}\} \ast \{x \mapsto \text{tmp2}\} \cdot \{\text{tmp1} = z_2\} \cdot \{\text{tmp2} = z_1\}) \\
& \models \text{wlp}[C_1; C_2; C_3](\{y \mapsto \text{tmp1}\} \ast \{x \mapsto \text{tmp2}\} \cdot \{\text{tmp1} = z_2\} \cdot \{\text{tmp2} = z_1\})) \\
& \models \text{wlp}[C_1; C_2; C_3](\{y \mapsto \text{tmp1}\} \ast \{x \mapsto \text{tmp2}\} \cdot \{\text{tmp1} = z_2\} \cdot \{\text{tmp2} = z_1\})) \\
& \models \text{wlp}[C_1](0.999 \cdot (\{y \mapsto z_1\} \ast \{\text{tmp1} = z_2\} \cdot \{x \mapsto -\})) + 0.001 \cdot [\text{false}] \\
& \models 0.999 \cdot \text{wlp}[C_1](\{x \mapsto z_2\} \ast \{y \mapsto z_1\}) + 0.001 \cdot [\text{false}] \\
& \models 0.999 \cdot (\{x \mapsto z_2\} \ast \{y \mapsto z_1\}) + 0.001 \cdot [\text{false}] \\
\end{align*}
\]

which yields a preexpectation without magic wand. Hence, we obtain a magic wand-free entailment in (†). We have used our technique to transform this quantitative entailment into several qualitative entailments and checked them successfully using the separation logic extension of CVC4 [46]. Detailed calculations, the resulting qualitative entailments, and the input for CVC4 in SMT-LIB 2 format are found in Appendix B.2.

### 3.5 Randomized List Population

Our second example populates a singly-linked list by flipping coins and adding a list element until the coin flip yields heads, i.e., we consider the program

\[
C_{\text{populate}}: \quad \text{while}(c \neq 0) \{ \\
\quad \{c := 0\} [0.5] \{x := \text{new}(x)\} \\
\},
\]

where \( x \) is the head of a linked list. Assume we would like to determine a lower bound on the probability that the above program does not crash and produces a list of length at least two\(^7\). For that, recall from Example 1 the separation logic formula \( \text{ls}(x, y) \) for singly-linked list segments. The aforementioned probability is then given by \( \text{wlp}[C_{\text{populate}}](f) \) for postexpectation

\[
f = \exists y: \exists z: \{x \mapsto y\} \ast \{y \mapsto z\} \ast [\text{ls}(z, 0)].
\]

\(^7\) plus the probability of nontermination, which is 0.
We propose the loop invariant $I$ below to show that $I \models \text{wl}p[C_{\text{populate}}](f)$, i.e., $I$ is a lower bound on the sought-after probability.

$$I = \exists y: \ [x \mapsto y] \ast ( [c = 0] \cdot \exists z: \ [y \mapsto z] \ast \text{ls}(z, 0)]$$

$$+ [c \neq 0] \cdot 1/2 \cdot ( \exists z: \ [y \mapsto z] \ast \text{ls}(z, 0) + 1/2 \cdot \text{ls}(z, 0)))$$

To verify that $I$ is indeed a loop invariant (hint: it is), we need to prove that

$$I \models [c = 0] \cdot f + [c \neq 0] \cdot \text{wl}p[[\{ c := 0 \} \{ 0.5 \} \{ x := \text{new} (x) \}]](I)$$

As described in Section 3.4, we can compute $\text{wl}p$ in a way such that the resulting formula contains no magic wands. Our reduction from QSL entailments to standard SL entailments then allows us to discharge the above invariant check using existing separation logic solvers with support for fixed list predicates, e.g., [45].

4 Quantitative Entailment Checking

We present our main contribution of reducing entailment checking in QSL $[\mathcal{A}]$ to entailment checking in SL $[\mathcal{A}]$. We consider the key observations leading to our reduction in Section 4.1. We then deal with the formalization and more technical considerations of our approach in Sections 4.2 and 4.3.

4.1 Idea and Key Observations

We reduce entailment checking in QSL $[\mathcal{A}]$ to entailment checking in SL $[\mathcal{A}]$, i.e.,

Given $f, g \in \text{QSL} [\mathcal{A}]$, we reduce checking $f \models g$ to checking finitely many entailments of the form $\varphi \models \psi$ with $\varphi, \psi \in \text{SL} [\mathcal{A}]$.

We instantiate QSL $[\mathcal{A}]$ and SL $[\mathcal{A}]$, respectively, for the sake of concreteness. For that, we fix the set $\mathcal{A}$ of predicate symbols given by

$$\mathcal{A} = \{ \text{true, emp, } x = y, x \neq y, x \mapsto y \mid x, y \in \text{Vars} \}$$

Now, consider the following entailment $u_1 \models u_2$ as a running example:

$$u_1 = 0.4 \cdot ([x \mapsto y] \ast [y \mapsto z]) + 0.6 \cdot [x \mapsto y] \models 0.6 \cdot ([x \mapsto y] \ast \text{true}) = u_2.$$  

Intuitively speaking, $u_1$ expresses that with probability 0.4 the heap consists of two cells where $x$ points to $y$ and separately $y$ points to $z$, and that with probability 0.6 the heap consists of a single cell where $x$ points to $y$. Formula $u_2$ expresses that with probability 0.6 the heap contains a cell where $x$ points to $y$. How can we reduce the problem of checking whether $u_1 \models u_2$ holds to checking finitely many entailments in SL $[\mathcal{A}]$? We rely on two key observations:

Observation 1. For every $f \in \text{QSL} [\mathcal{A}]$, the set

$$\text{Eval}(f) = \{ [[f]](s, h) \mid (s, h) \in \text{States} \} \subset \mathbb{P}$$

is finite. Moreover, there is an effectively constructible finite and sound overapproximation $\text{Val}[f]$ of $\text{Eval}(f)$, i.e., $\text{Eval}(f) \subseteq \text{Val}[f]$. 


Example 3. Consider the expectation $u_1$ from our running example: We have $\text{Eval}(u_1) = \{0, 0.4, 0.6\}$. We construct a finite overapproximation of $\text{Eval}(u_1)$ as follows: First, we observe that both subformulae $g_1$ and $g_2$ evaluate to a value in $\{0,1\}$, i.e., $\text{Val}[g_1] = \text{Val}[g_2] = \{0,1\}$. From $\text{Val}[g_1]$ and $\text{Val}[g_2]$, we obtain a finite overapproximation $\text{Val}[u_1]$ of $\text{Eval}(u_1)$ given by

$$\text{Val}[u_1] = \{0.4 \cdot \alpha + 0.6 \cdot \beta \mid \alpha \in \text{Val}[g_1], \beta \in \text{Val}[g_2]\} = \{0, 0.4, 0.6, 1\}.$$ 

Notice that $\text{Val}[u_1]$ is a proper superset of $\text{Eval}(u_1)$ since $1 \notin \text{Eval}(u_1)$. △

We consider the construction of $\text{Val}[f]$ for arbitrary $f \in \text{QSL}[A]$ in Section 4.2.

Observation 2. Given $f \in \text{QSL}[A]$ and a probability $\alpha \in \mathbb{P}$, there is an effectively constructible $\text{SL}[A]$ formula, which we denote by $[\alpha \leq f]$, such that $(s,h)$ is a model of $[\alpha \leq f]$ if and only if $f$ evaluates at least to $\alpha$ on state $(s,h)$, i.e.,

$$\frac{(s,h) \models [\alpha \leq f]}{\text{in } \text{SL}[A]} \iff \frac{\alpha \leq [f](s,h)}{\text{in } \text{QSL}[A]}.$$ 

We can thus lower bound $\text{QSL}[A]$ formulae in terms of $\text{SL}[A]$ formulae.

Example 4. Continuing our running example, we construct $[0.5 \leq u_1]$, i.e., an $\text{SL}[A]$ formula evaluating to true on state $(s,h)$ if and only if $u_1$ evaluates at least to 0.5. We start by considering the subformulae of $u_1$. Since both $g_1$ and $g_2$ embed $\text{SL}[A]$ predicates, we have for every $\alpha \in \mathbb{P}$

$$[\alpha \leq g_1] = \text{true} \text{ if } \alpha = 0 \text{ else } x \mapsto y \star y \mapsto z$$

and

$$[\alpha \leq g_2] = \text{true} \text{ if } \alpha = 0 \text{ else } x \mapsto y.$$ 

The intuition is as follows: $\alpha = 0$ lower bounds every probability. Conversely, if $\alpha > 0$ then $\alpha$ lower bounds $g_1$ (resp. $g_2$) on state $(s,h)$ if and only if $(s,h)$ satisfies the predicate $g_1$ (resp. $g_2$). Now, when does $u_1$ evaluate at least to 0.5? Given $\text{Val}[g_1]$ and $\text{Val}[g_2]$ and the fact that the valuation of $u_1$ is a convex combination of the valuations of $g_1$ and $g_2$, there are (at most) two cases: Either both $g_1$ and $g_2$ evaluate to (at least) 1, or $g_2$ (but not necessarily $g_1$) evaluates to (at least) 1. Given $[1 \leq g_1]$ and $[1 \leq g_2]$, the aforementioned informal disjunction translates to a formal disjunction in $\text{SL}[A]$:

$$[0.5 \leq u_1] = ([1 \leq g_1] \land [1 \leq g_2]) \lor [1 \leq g_2]$$

$$= ((x \mapsto y \star y \mapsto z) \land x \mapsto y) \lor x \mapsto y.$$ 

Notice that—as it is the case for $\text{Val}[u_1]$—we construct $[0.5 \leq u_1]$ syntactically. In particular, we disregard that the disjunct $(x \mapsto y \star y \mapsto z) \land x \mapsto y$ is unsatisfiable and therefore equivalent to false. △

We provide the construction of $[\alpha \leq f]$ for arbitrary $\text{QSL}[A]$ formulae $f$—including quantitative quantifiers and the magic wand—in Section 4.3.

Finally, Observations 1 and 2 together yield our reduction from $f \models g$ to finitely many entailments in $\text{SL}[A]$. Intuitively speaking, we formalize that
Table 5. Inductive definition of $\text{Val}[f]$.

| $f \in \text{QSL}[\mathfrak{A}]$ | $\text{Val}[f] \subseteq \mathbb{P}$ |
|---------------------------------|-----------------------------------|
| $[\psi]$                        | $\{0, 1\}$                       |
| $[\pi] \cdot g + [\neg\pi] \cdot u$ | $\text{Val}[g] \cup \text{Val}[u]$ |
| $q \cdot g + (1 - q) \cdot u$    | $\rho \cdot \text{Val}[g] + (1 - \rho) \cdot \text{Val}[u]$ |
| $g \cdot u$                     | $\text{Val}[g] \cdot \text{Val}[u]$ |
| $1 - g$                         | $1 - \text{Val}[g]$               |
| $g \max u$                      | $\text{Val}[g] \max \text{Val}[u]$ |
| $g \min u$                      | $\text{Val}[g] \min \text{Val}[u]$ |
| $\exists x : g$                 | $\text{Val}[g]$                   |
| $\exists x : g$                 | $\text{Val}[g]$                   |
| $g \star u$                     | $\text{Val}[g] \cdot \text{Val}[u]$ |
| $[\psi] \rightarrow g$         | $\text{Val}[g]$                   |

whenever $f$ evaluates at least to $\alpha$, then $g$ too evaluates at least to $\alpha$
equivalently in terms of finitely many $\text{SL}[\mathfrak{A}]$ entailments. Put more formally, since $\text{Val}[f]$ is finite, we have

$$f \models g$$

iff for all $(s, h)$: $[f](s, h) \leq [g](s, h)$ \hspace{1cm} (by definition)

iff for all $(s, h)$ and all $\alpha \in \text{Val}[f]$: $\alpha \leq [f](s, h)$ implies $\alpha \leq [g](s, h)$ \hspace{1cm} (by Observation 1)

iff for all $(s, h)$ and all $\alpha \in \text{Val}[f]$: $(s, h) \models [\alpha \leq f]$ implies $(s, h) \models [\alpha \leq g]$ \hspace{1cm} (by Observation 2)

iff for all $\alpha \in \text{Val}[f]$: $[\alpha \leq f] \models [\alpha \leq g]$ \hspace{1cm} (by definition)

Example 5. Reconsider our running example. Since $|\text{Val}[u_1]| = 4$, the $\text{QSL}[\mathfrak{A}]$ entailment $u_1 \models u_2$ is equivalent to the four entailments

$$[\alpha \leq u_1] \models [\alpha \leq u_2] \quad \text{for} \quad \alpha \in \{0, 0.4, 0.6, 1\}$$

in $\text{SL}[\mathfrak{A}]$, each of which actually holds. \(\triangle\)

4.2 Constructing Finite Overapproximations of $\text{Eval}(f)$

We consider the formal construction underlying Observation 1 from the previous section, i.e., given $f \in \text{QSL}[\mathfrak{A}]$, we provide a syntactic construction of a finite
overapproximation $\Val[f]$ of $\Eval(f)$. This construction is by induction on the structure of $f$ as shown in Table 5. For that, we define some shorthands. Given $\alpha \in \mathbb{P}$, $V,W \subseteq \mathbb{P}$, and a binary operation $\circ : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$, we define

$$\alpha \cdot V = \{ \alpha \cdot \beta \mid \beta \in V \} \quad \text{and} \quad V \circ W = \{ \beta \circ \gamma \mid \beta \in V, \gamma \in W \} .$$

Let us now go over the individual cases.

The case $f = \llbracket \psi \rrbracket$. We have $\llbracket \psi \rrbracket (s,h) \in \{0,1\}$ by definition.

The case $f = [\pi] \cdot g + [\neg \pi] \cdot u$. For every $(s,h)$, the formula $f$ either evaluates to $\llbracket g \rrbracket (s,h)$ or to $\llbracket u \rrbracket (s,h)$, depending on whether $(s,h) \models \pi$ holds.

The case $f = p \cdot g + (1-p) \cdot u$. The formula $f$ evaluates to $p \cdot \alpha + (1-p) \cdot \beta$ for some $\alpha \in \Val[g]$ and $\beta \in \Val[u]$.

The case $f = g \cdot u$ or $f = g \circ u$. The formula $f$ evaluates to $\alpha \cdot \beta$ for some $\alpha \in \Val[g]$ and $\beta \in \Val[u]$.

The case $f = 1 - g$. The formula $f$ evaluates to $1 - \alpha$ for some $\alpha \in \Val[g]$.

The case $f = g \circ u$ for $\circ \in \{\max, \min\}$. Since $\max$ and $\min$ are defined point-wise, the formula $f$ evaluates to some value $\alpha \circ \beta$ for $\alpha \in \Val[g]$ and $\beta \in \Val[u]$.

The case $f = \exists x : g$ or $f = \forall x : g$. Since $\Val[g]$ overapproximates the set of all valuations of $g$, quantitative quantifiers do not add any valuation.

The case $f = [\psi] \longrightarrow g$. Recall that

$$\llbracket f \rrbracket (s,h) = \inf \{ \llbracket g \rrbracket (s,h \star h') \mid h' \perp h \text{ and } \llbracket \psi \rrbracket (s,h') = 1 \} .$$

If the above set is non-empty, the infimum is actually a minimum and therefore $f$ evaluates to some value in $\Val[g]$. If the above set is empty, then $\llbracket f \rrbracket (s,h) = 1$. It is easy to verify that 1 is necessarily an element of $\Val[g]$ (cf., Lemma 4).

Summarizing our considerations on $\Val[f]$, we get:

**Theorem 3.** For every $f \in \text{QSL } [\mathbb{A}]$, the effectively constructible set $\Val[f] \subseteq \mathbb{P}$ given in Table 5 satisfies

$$|\Val[f]| < \infty \quad \text{and} \quad \Eval(f) \subseteq \Val[f] .$$

**Proof.** Straightforward by induction on $f$.

### 4.3 Lower Bounding $\text{QSL } [\mathbb{A}]$ by $\text{SL } [\mathbb{A}]$ Formulae

We now consider the formal construction underlying Observation 2 from Section 4.1. That is, given $f \in \text{QSL } [\mathbb{A}]$ and $\alpha \in \mathbb{P}$, we provide the syntactic construction of an $\text{SL } [\mathbb{A}]$ formula $\lceil \alpha \preceq f \rceil$ evaluating to true on state $(s,h)$ if and
only if \( f \) evaluates at least to \( \alpha \) on \((s,h)\). This construction relies on \( \text{Val}[f] \) from the previous section and is given by induction on the structure of \( f \) as shown in Table 6. We consider the individual constructs. For that, we fix some state \((s,h)\).

The case \( f = [\psi] \). There are two cases. If \( \alpha = 0 \), then \( \alpha \) trivially lower bounds the value of \([\psi]\). Conversely, if \( \alpha > 0 \), then \( \alpha \) lower bounds \([\psi]\) on state \((s,h)\) if and only if \((s,h)\) satisfies \( \psi \).

For the composite cases, recall that by Theorem 3 there are effectively constructible finite sets \( \text{Val}[g], \text{Val}[u] \) covering all values \( g \) and \( u \) evaluate to.

The case \( f = [\pi] \cdot g + [\neg \pi] \cdot u \). The formula \( f \) represents a Boolean choice between the formulae \( g \) and \( u \), depending on the truth value of \( \pi \). Hence, there are two cases: If \((s,h)\) does satisfy \( \pi \), then \( \alpha \) lower bounds \( f \) iff \( \alpha \) lower bounds \( g \). Conversely, if \((s,h)\) does not satisfy \( \pi \), then \( \alpha \) lower bounds \( f \) iff \( \alpha \) lower bounds \( u \).

The case \( f = p \cdot g + (1-p) \cdot u \). Since \( \text{Val}[g] \) and \( \text{Val}[u] \) cover every possible valuation of \( g \) and \( u \), respectively, it follows that \( \alpha \) lower bounds the valuation of \( f \) if and only if there are \( \beta \in \text{Val}[g] \) and \( \gamma \in \text{Val}[u] \) such that (1) \( \beta \) lower bounds \( g \), (2) \( \gamma \) lower bounds \( u \), and (3) \( \alpha \) lower bounds the convex sum \( p \cdot \beta + (1-p) \cdot \gamma \).

The case \( f = g \cdot u \). The reasoning is analogous to the previous case.
The case $f = 1 - g$. We write $\alpha \leq \llbracket 1 - g \rrbracket(s, h)$ equivalently as $-(1 - \alpha < \llbracket g \rrbracket(s, h))$. In order to turn the strict inequality into a non-strict one, we consider the successor $\delta$ of $1 - \alpha$ in $\text{Val} [g]$, i.e., the smallest $\delta$ in $\text{Val} [g]$ greater than $1 - \alpha$. Since $\text{Val} [g]$ is finite, such a $\delta$ always exists if $1 - \alpha \neq 1$. We illustrate the idea in the following picture, where all elements in $\text{Val} [g]$ are marked by $\bullet$.

![Diagram showing the successor $\delta$ of $1 - \alpha$ in $\text{Val} [g]$]

For the successor $\delta$, checking if $\delta$ is a lower bound of $\llbracket g \rrbracket(s, h)$ is equivalent to checking if $1 - \alpha$ is a strict lower bound - if $\delta$ is not a lower bound, then we ran out of possible valuations that are strictly lower bounded by $1 - \alpha$.

The case $f = g \circ u$ for $\circ \in \{\max, \min\}$. The probability $\alpha$ lower bounds the maximum of $g$ and $u$ on state $(s, h)$ if and only if $\alpha$ lower bounds $g$ or $\alpha$ lower bounds $u$. For $\circ = \min$, the reasoning is dual.

The case $f = 3x: g$. Recall that

$$\llbracket f \rrbracket(s, h) = \max \{ \llbracket g \rrbracket(s[x:=v], h) \mid v \in \text{Vals} \}.$$  

Now observe that $\alpha$ lower bounds the above maximum if and only if $\alpha$ lower bounds some element of the above set, i.e., if and only if there is some $v$ with

$$\alpha \leq \llbracket g \rrbracket(s[x:=v], h)$$  

which is equivalent to  

$$(s, h) \models \exists x : [\alpha \leq g].$$

The case $f = L x: g$. Recall that

$$\llbracket f \rrbracket(s, h) = \min \{ \llbracket g \rrbracket(s[x:=v], h) \mid v \in \text{Vals} \}.$$  

Since $\alpha$ lower bounds the above minimum if and only if $\alpha$ lower bounds all elements of the above set, the reasoning is dual to the previous case.

The case $f = g \ast u$. Recall that

$$\llbracket f \rrbracket(s, h) = \max \{ \llbracket g \rrbracket(s, h_1) \cdot \llbracket u \rrbracket(s, h_2) \mid h = h_1 \ast h_2 \}.$$  

Since $\text{Val}[g]$ and $\text{Val}[u]$ cover every possible valuation of $g$ and $u$, respectively, $\alpha$ lower bounds the evaluation of $f$ on $(s, h)$ iff there are $\beta \in \text{Val}[g], \gamma \in \text{Val}[u]$ and $h_1, h_2$ with $h_1 \ast h_2 = h$ such that (1) $\beta$ lower bounds $g$ on $(s, h_1)$, (2) $\gamma$ lower bounds $u$ on $(s, h_2)$, and (3) $\alpha$ lower bounds $\beta \cdot \gamma$. Given such $\beta$ and $\gamma$, we can phrase this equivalently in $\text{SL}[2]$ as

$$(s, h) \models [\beta \leq g] \ast [\gamma \leq u].$$

The case $f = \llbracket \psi \rrbracket \rightarrow g$. Recall that

$$\llbracket f \rrbracket(s, h) = \inf \{ \llbracket g \rrbracket(s, h \ast h') \mid h' \perp h \text{ and } \llbracket \psi \rrbracket(s, h') = 1 \}.$$
Probability $\alpha$ lower bounds the above infimum if and only if for every extension $h'$ of the heap $h$ such that the stack $s$ together with $h'$ satisfy $\psi$, probability $\alpha$ is a lower bound on $\llbracket g \rrbracket (s, h \ast h')$. Put more formally, the latter statement reads

$$\forall h' \perp h \text{ with } (s, h') \models \psi : (s, h \ast h') \models [\alpha \leq g] \,,$$

which is equivalent to $(s, h) \models \psi \rightarrow [\alpha \leq g]$.

Our construction thus applies to arbitrary QSL $[A]$ formulae and we get:

**Theorem 4.** For every $f \in \text{QSL} [A]$ and all $\alpha \in \mathbb{P}$ there is an effectively constructible $\text{SL} [A]$ formula $[\alpha \leq f]$ such that for all $(s, h) \in \text{States}$, we have

$$(s, h) \models [\alpha \leq f] \iff \alpha \leq \llbracket f \rrbracket (s, h) \,.$$

**Proof.** By induction on $f$. See Appendix C for details.

Finally, we obtain our main theorem.

**Theorem 5.** Entailment checking in $\text{QSL} [A]$ reduces to entailment checking in $\text{SL} [A]$, i.e., for all $f, g \in \text{QSL} [A]$, we have

$$f \models g \iff \text{for all } \alpha \in \text{Val}[f] : \ [\alpha \leq f] \models [\alpha \leq g] \,.$$

**Proof.** Follows from Theorems 3 and 4 and the reasoning at the end of Section 4.1.

**Remark 1 (Avoiding true in $\text{SL} [A]$ entailments).** Formulae of the form $[\alpha \leq f] \in \text{SL} [A]$ may introduce the atom $\text{true}$, which is not admitted by some decidable separation logic fragments, such as [26]. Fortunately, we can avoid $\text{true}$ in $[\alpha \leq f]$ formulae. $\text{true}$ is only required in formulae of the form $[0 \leq f]$, which arise in two situations when applying Theorem 5: (1) in entailment checks of the form $[0 \leq f] \models [0 \leq g]$, which always hold and can thus be omitted, and (2) if $f = p \cdot g + (1 - p) \cdot u$. In the latter case, if we have $\alpha \neq 0$ in

$$[\alpha \leq f] = \bigvee_{\beta \in \text{Val}[g], \gamma \in \text{Val}[u], p \cdot \beta + (1 - p) \cdot \gamma \geq \alpha} [\beta \leq g] \land [\gamma \leq u] \,,$$

then either $\beta \neq 0$ or $\gamma \neq 0$ holds for every disjunct. Hence, subformulae of the form $[0 \leq g]$ or $[0 \leq u]$ can be omitted, as well.

## 5 Complexity

We now analyze the complexity of our approach. Recall that Theorem 5 reduces checking $f \models g$ in QSL $[A]$ to checking

$$\text{for all } \alpha \in \text{Val}[f] : \ [\alpha \leq f] \models [\alpha \leq g]$$

in $\text{SL} [A]$. We consider two aspects: (1) the number of $\text{SL} [A]$ entailments and (2) the size of the resulting $\text{SL} [A]$ formulae occurring in each entailment. We
express these quantities in terms of the size of a QSL $[\mathfrak{A}]$ formula $f$ and a SL $[\mathfrak{A}]$ formula $\varphi$ and denote them as $|f|$ and $|\varphi|$ respectively. In these sizes, we count every construct in the formula and require that the size of atoms are defined at instantiation. Moreover, we assume that every atom in $\mathfrak{A}$ is at least of size 1 and especially the atom true is of size 1. Additionally we count in a QSL $[\mathfrak{A}]$ formula $f$ the constructs that increase the number of possible evaluation results of $f$, namely $q \cdot g + (1 - q) \cdot u$, $g \cdot u$ and $g \ast u$, and denote it as $|f|_p$.\footnote{For a formal definition see Table 9 on page 37.}

We will see that for an entailment $f \models g$ in QSL $[\mathfrak{A}]$, (1) the number of SL $[\mathfrak{A}]$ entailments is in $2^{O(|f|_p)}$ in the worst case (see Theorem 6) and (2) the size of the resulting SL $[\mathfrak{A}]$ formulae are in $O(|f|) \cdot 2^{O(|f|_p^2)}$ and $O(|g|) \cdot 2^{O(|g|_p^2)}$ respectively in the worst case (see Theorem 7). Now let us assume we have an entailment checker for SL $[\mathfrak{A}]$ formulae that can solve entailments of the form $[\alpha \preceq f] \models [\alpha \preceq g]$ and which has a runtime complexity of SL-Time($n, m$) where $n$ and $m$ are the size of SL $[\mathfrak{A}]$ formulae on the left and right side of an entailment respectively. Putting the above together, checking the entailment $f \models g$ in QSL $[\mathfrak{A}]$ then has a runtime complexity of

$$2^{O(|f|_p)} \cdot \text{SL-Time} \left( O(|f|) \cdot 2^{O(|f|_p^2)}, O(|g|) \cdot 2^{O(|g|_p^2)} \right)$$

$$+ O(|f|) \cdot 2^{O(|f|_p^2)} + O(|g|) \cdot 2^{O(|g|_p^2)}.$$  

If we furthermore reasonably assume that SL-Time($n, m$) is at least linear in both arguments (otherwise the entailment checker can only check trivial entailments anyway), the runtime complexity simplifies to

$$2^{O(|f|_p)} \cdot \text{SL-Time} \left( O(|f|) \cdot 2^{O(|f|_p^2)}, O(|g|) \cdot 2^{O(|g|_p^2)} \right).$$

As for aspect (1), we first observe that checking $f \models g$ by means of Theorem 5 requires checking $|\text{Val}[f]|$ entailments in SL $[\mathfrak{A}]$. However, only the constructs we count with $|f|_p$ increase the number of possible evaluations, which in turn will also increase the size of the overapproximation $\text{Val}[f]$. Every time any of these constructs occur, the number of possible evaluations $\text{Eval}(f)$ may double. Consequently, also the overapproximation $\text{Val}[f]$ doubles in size when any of these constructs occur. Other constructs do not increase the number of evaluations, but instead inherit the evaluations from their subformulae.

**Theorem 6.** We have $|\text{Val}[f]| \leq 2^{|f|_p + 1}$. Hence, checking $f \models g$ by means of Theorem 5 requires checking $2^{O(|f|_p)}$ entailments in SL $[\mathfrak{A}]$.

**Proof.** By induction on $f$. For details see Appendix D.

For the size of the resulting SL $[\mathfrak{A}]$ formulae, i.e., aspect (2), recall that we construct entailments of the form

$$[\alpha \preceq f] \models [\alpha \preceq g].$$
We thus determine an upper bound on the size of any SL\(\mathcal{A}\) formula \([\alpha \preceq f]\). Here we make a similar observation as in aspect (1): whenever one of the constructs we count with \(|f|_p\) appears, the size of the formula increases by the exponential factor \(|\text{Val}[f]|\). Such a multiplication of increasing exponential expressions then results asymptotically in a squared exponent. The other constructs increase the size by only a constant per construct. By combining both observations we can finally conclude an upper bound on the size of the formula \([\alpha \preceq f]\).

**Theorem 7.** For any formula \(f \in \text{QSL}\mathcal{A}\) and all probabilities \(\alpha \in \mathbb{P}\), the SL\(\mathcal{A}\) formula \([\alpha \preceq f]\) has at most size \(3 \cdot |f| \cdot 2^{(|f|_p+1)^2}\). Hence the size of the formula \([\alpha \preceq f]\) is in \(O(|f|) \cdot 2^{O(|f|_p^2)}\).

**Proof.** By induction on \(f\). For details see Appendix D.

**Remark 2 (Complexity of SL\(\mathcal{A}\) Entailments in QSL\(\mathcal{A}\)).** By Theorem 6 and Theorem 7, the number of entailments and the size of formulae \([\alpha \preceq f]\) is only exponential if \(|f|_p\) is not constant. However, we would assume that an entailment \(f \models g\) in QSL\(\mathcal{A}\), where neither in \(f\) nor in \(g\) the probabilistic choice \(p \cdot g + (1-p) \cdot u\) appears, should have a similar runtime complexity as SL\(\mathcal{A}\) entailment. While it is easy to see that \(\text{Val}[f] = \{0,1\}\) has constant size in this setting, the size of the formula is still exponential. In the case where no probabilistic choice is present, we generate multiple exponentially-sized tautologies of the form \([0 \preceq f]\). However, due to Remark 1 we can eliminate all occurrences of \([0 \preceq f]\). That means, if \(f\) does not contain \(p \cdot g + (1-p) \cdot u\), then for \(\alpha \neq 0\), we can construct an equivalent formula to \([\alpha \preceq f]\) in such a way that its size is in \(O(|f|)\) and \(|\text{Val}[f]| = 2\).

\[\triangle\]

6 Application: Decidable hpGCL Verification

Since entailment in full separation logic is undecidable, it is common to consider fragments of separation logic with a (semi-)decidable entailment problem. Given a QSL\(\mathcal{A}\) fragment \(Q\), we provide sufficient and easy-to-check characterizations on SL\(\mathcal{A}\) fragments \(S\) ensuring that entailment checking in \(Q\) reduces to entailment checking in \(S\). This simplifies the search for decidable fragments of quantitative separation logic.

We then apply our results in Section 6.1 to show the decidability of entailment checking for quantitative symbolic heaps—a quantitative extension of the well-known symbolic heap fragment of separation logic—and demonstrate the applicability to the verification of probabilistic pointer programs.

Our reduction from entailments in QSL\(\mathcal{A}\) to entailments in SL\(\mathcal{A}\) relies on the construction of the \([\alpha \preceq f]\) formulae from Section 4.3. This suggests to define:

**Definition 3.** Let \(Q\) be a QSL\(\mathcal{A}\) fragment. We say that an SL\(\mathcal{A}\) fragment \(S\) is \(Q\)-admissible if \([\alpha \preceq f]\) \(\in S\) holds for all \(f \in Q\) and all \(\alpha \in \mathbb{P}\). \[\triangle\]
Table 7. SL [A] requirements for entailment checking in QSL [A].

| Q fragment contains | S contains/is closed under         |
|---------------------|----------------------------------|
| [ψ]                | ψ, true                          |
| [π] · f + [−π] · g | π, −π, ∧, ∨                      |
| p · f + (1 − p) · g | ∧, ∨                             |
| f · g              | ∧, ∨                             |
| 1 − f              | ¬, true                          |
| f max g            | ∨                                |
| f min g            | ∧                                |
| ∃ x : f            | ⊤                                |
| ∀ x : f            | ⊤                                |
| f ⋆ g              | *, ∨                             |
| [ψ] → f            | ψ → ·                            |

The syntactic nature of our construction of the S formulae \([\alpha \preceq f]\) allows for a syntactic criterion on SL [A] fragments to be Q-admissible.

**Lemma 1.** Let Q be a QSL [A] fragment. If an SL [A] fragment S satisfies the requirements provided in Table 7, then S is Q-admissible.

**Proof.** By induction on f. For details see Appendix E.

Finally, we provide a sufficient criterion for the decidability of entailment in QSL [A] fragments given SL [A] fragments with a decidable entailment problem. Since entailment checks \(\varphi \models \psi\) in SL [A] can often (but not always) be reduced to unsatisfiability checks \(\varphi \land \neg \psi\), we take a more fine-grained perspective and distinguish between fragments for the left- and the right-hand side of entailments, respectively. This distinction matters when, e.g., SL [A] fragments with a decidable satisfiability problem impose restrictions on quantifiers (cf., [19]).

**Theorem 8.** Let \(Q_1, Q_2\) be QSL [A] fragments, and let \(S_1, S_2\) be SL [A] fragments. If \(S_1\) is Q\(_1\)-admissible and \(S_2\) is Q\(_2\)-admissible, then

\[\varphi \models \psi \text{ for } \varphi \in S_1, \psi \in S_2 \text{ is decidable}\]

implies

\[g \models f \text{ for } g \in Q_1, f \in Q_2 \text{ is decidable} .\]

**Proof.** This is a consequence of Theorem 5. 

6.1 Quantitative Symbolic Heaps

We now demonstrate that our approach can facilitate the automated verification of probabilistic pointer programs by providing a sample QSL fragment with a decidable entailment problem.

Recall that QSL [A] is parameterized by a set \(A\) of predicate symbols. We obtain the quantitative symbolic heap fragment of QSL by instantiating \(A\).
Definition 4. Let \( \mathfrak{A} \) be the set of predicate symbols given by
\[
\mathfrak{A} = \{ \text{true}, \text{emp} \} \cup \{ x \mapsto (y_1, \ldots, y_k) \mid x, y_1, \ldots, y_k \in \text{Vars} \} \cup \{ x = y, x \neq y, x = y \land \text{emp}, x \neq y \land \text{emp} \mid x, y \in \text{Vars} \}.
\]
Then the set \( \text{QSH} \) of quantitative symbolic heaps is given by the grammar
\[
f \rightarrow [\psi] \mid [\pi] \cdot f + [\neg \pi] \cdot f \mid q \cdot f + (1 - q) \cdot f \mid \exists x : f \mid f \star f.
\]
Quantitative symbolic heaps naturally extend the symbolic heap fragment of separation logic. Intuitively speaking, a quantitative symbolic heap \( f \) specifies probability (sub-)distributions over (symbolic) heaps. By applying Theorem 5, we obtain the following decidability result.

Theorem 9. For loop- and allocation-free \( \text{hpGCL} \) programs \( C \) (that only perform pointer operations, no arithmetic, and guards from the pure fragment of \( \mathfrak{A} \)) and \( f_1, f_2 \in \text{QSH} \), it is decidable whether the entailment \( \text{wlp}[\llbracket C \rrbracket](f_1) \models f_2 \) holds.

Hence, for loop- and allocation-free programs \( C \) as above, upper bounds (in terms of quantitative symbolic heaps \( f_2 \)) on the probability \( \text{wlp}[\llbracket C \rrbracket](f_1) \) of terminating in a given quantitative symbolic heap \( f_1 \) are decidable. We refer to Section 3.3 for an example entailment involving quantitative symbolic heaps. In the sequel, we show how to prove the above result.

Proof of Theorem 9. The proof relies on extended quantitative symbolic heaps \( \text{eQSH} \), which include magic wands with points-to formulae on their left-hand side.

Definition 5. The set \( \text{eQSH} \) of extended quantitative symbolic heaps is given by the grammar
\[
g \rightarrow [\psi] \mid [\pi] \cdot g + [\neg \pi] \cdot g \mid q \cdot g + (1 - q) \cdot g \mid g \star g
\]
\[
\mid \exists x : g \mid [x \mapsto (y_1, \ldots, y_k)] \longrightarrow g.
\]
Notice that indeed \( \text{QSH} \subseteq \text{eQSH} \).

Lemma 2. For every loop- and allocation-free program \( C \in \text{hpGCL} \) without arithmetic and only with guards of the pure fragment of \( \mathfrak{A} \), extended quantitative symbolic heaps are closed under \( \text{wlp}[\llbracket C \rrbracket] \), i.e.,
\[
\text{for all } g \in \text{eQSH} : \quad \text{wlp}[\llbracket C \rrbracket](g) \in \text{eQSH}.
\]
In particular, since \( \text{QSH} \subseteq \text{eQSH} \), we have
\[
\text{for all } f \in \text{QSH} : \quad \text{wlp}[\llbracket C \rrbracket](f) \in \text{eQSH}.
\]

Proof. By induction on the structure of loop- and allocation-free program \( C \). See Appendix F for details.
Hence, if \( g \models f \) is decidable for \( g \in \text{eQSH} \) and \( f \in \text{QSH} \), Theorem 9 follows.

**Lemma 3.** For \( g \in \text{eQSH} \) and \( f \in \text{QSH} \), it is decidable whether \( g \models f \) holds.

**Proof.** We employ Lemma 1 to determine two SL\( [\exists] \) fragments \( S_1, S_2 \) such that \( S_1 \) is eQSH-admissible and \( S_2 \) is QSH-admissible. Then, by Theorem 8, decidability of \( g \models f \) follows from decidability of \( \varphi \models \psi \) for \( \varphi \in S_1 \) and \( \psi \in S_2 \). For that, we exploit the equivalence

\[
\varphi \models \psi \quad \text{iff} \quad \varphi \land \neg \psi \text{ is unsatisfiable}.
\]

The latter is decidable by \cite{19}, Theorem 3.3 since \( \varphi \land \neg \psi \) is equivalent to a formula of the form \( \exists^{*} \forall^{*} \vartheta \) with \( \vartheta \) quantifier-free and no formula \( \vartheta_1 \rightarrow \vartheta_2 \) occurring in \( \vartheta \) contains a universally quantified variable. See Appendix F for details.

## 7 Related Work

**Weakest preexpectations.** Weakest precondition reasoning was established in a classical setting by Dijkstra \cite{18} and has been extended to provide semantic foundations for probabilistic programs by Kozen \cite{37,36} and McIver & Morgan \cite{40}, who also coined the term weakest preexpectations. Their relation to operational models is studied in \cite{24}. Moreover, weakest preexpectation reasoning has been shown to be useful for obtaining bounds on the expected resource consumption \cite{44} and, in particular, the expected run-time \cite{32} of probabilistic programs.

**Logics for probabilistic pointer programs.** Although many algorithms rely on randomized dynamic data structures, formal reasoning about programs that are both probabilistic and heap manipulating has received scarce attention. A notable exception is the work by Tassarotti and Harper \cite{50}, who introduce a concurrent separation logic with support for probabilistic reasoning, called Polaris. Their focus is on program refinement, employing a semantic model that is based on the idea of coupling, which underlies recent work on probabilistic relational Hoare logics \cite{4}. However, no other decision procedures targeting entailments for QSL or other logics targeting probabilistic pointer programs exist.

**Leveraging SL research.** As shown in Table 7, building QSL entailment checkers by employing our reduction technique requires the availability of SL fragments that support certain logical operations, and whose entailment problem is decidable. Since the inception of separation logic \cite{28}, the latter has been extensively studied. In particular, the symbolic heap fragment of SL has received a lot of attention. Table 8 gives an overview of related approaches.

---

\( \ast \) is always covered. Supported (Boolean or separating) connectives are marked with “+”, unsupported ones with “−”. “∗” means that the restrictions on the connective are more involved. “Pure” means that the connective can only appear in pure formulae and “flat” means that the quantifier needs to be on the outermost level.
Table 8. SL fragments with decidable entailment problem.

| Paper | ¬ | ∧ | ∨ | → | ∃ | ∀ | Ind. predicates | Complexity          |
|-------|---|---|---|----|---|---|-----------------|---------------------|
| [1]   | pure | pure | pure | flat | – | – | user defined    | EXPTime-hard        |
| [10]  | – | – | – | – | – | – | Lists           | Polynomial          |
| [20]  | – | – | – | – | – | – | user defined    | 2-EXPTime-complete  |
| [21]  | – | – | – | – | – | – | user defined    | 2-EXPTime-complete  |
| [26]  | – | + | – | – | flat | – | user defined    | ?                   |
| [27]  | – | – | – | – | flat | – | user defined    | EXPTime-complete    |
| [34]  | – | – | – | – | – | – | user defined    | 2-EXPTime          |
| [39]  | * | + | – | – | – | – | user defined    | 2-EXPTime          |
| [46]  | + | + | + | – | – | – | ?               |                     |
| [17]  | – | – | – | – | List | – | Polynomial     | PSPACE-complete     |
| [19]  | + | + | + | * | * | – | user defined    | PSPACE-complete     |

8 Discussion and Conclusion

We studied entailment checking in QSL by means of a reduction to entailment checking in SL. We analyzed the complexity of our approach and demonstrated its applicability by means of several examples. In particular, our reduction yields the first decidability result for probabilistic pointer program verification.

Our primary goal was to investigate the entailment problem for QSL to pave the way for automated verification of probabilistic pointer programs. Theorem 8 provides a generic result that enables building upon the large body of work dealing with classical SL entailments to obtain both theoretical and practical insights. Theoretically, Theorem 8 gives sufficient criteria to derive QSL fragments with a decidable entailment problem from a classical SL fragment. We derived a QSL fragment such that reasoning about a simple probabilistic heap-manipulating language becomes decidable. More practically, Theorem 8 allows reusing existing (possibly incomplete) SL solvers to solve the entailments derived by our construction—an empirical evaluation of how well existing solvers can deal with these entailments is an interesting direction for future work.

We believe that our fine-grained complexity analysis demonstrates that our approach can be practically feasible: the exponential blow-up in Theorem 7 stems from the number of probabilistic constructs in the given QSL formulae. We expect the number of such constructs to be small for many randomized algorithms. We remark that existing approaches on checking quantitative entailments between heap-independent expectations encounter similar exponential blow-ups (cf., [35,6]). There is thus some evidence that such exponential blow-ups do not prohibit one from automatically verifying non-trivial properties. We are not aware of work on checking quantitative entailments between expectations that avoids such exponential blow-ups.

Future work includes considering richer classes of QSL and applications of entailment checking such as $k$-induction [6]. Another interesting direction is the applicability of our reduction to other approaches that aim for local reasoning about the resources employed by probabilistic programs, such as [50,3,5].
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Appendix

A Proof of Theorem 1

Theorem 1. The semantics of QSL [$\mathcal{A}$] formulae is well-defined, i.e., for all $f \in \text{QSL} [\mathcal{A}]$, we have $[f] \in E_{\leq 1}$.

Proof. By induction on $f$. For the base case $f = [\psi]$, we have $[[\psi]] (s, h) \in \{0, 1\}$. For all other cases, it is straightforward to prove that $[f] (s, h) \in [0, 1]$, since $[0, 1]$ is closed under all operations used in Table 3 if they are defined. It is only left to prove that $[\exists x : g]$, $[\forall x : g]$ and $[g \star u]$ are well-defined. For that the max and min need to be defined on the given sets, i.e. that the sets

$$\{ [g] (s [x:=v], h) \mid v \in \text{Vals} \}$$

$$\{ [g] (s, h_1) \cdot [u] (s, h_2) \mid h = h_1 \star h_2 \}$$

are non-empty and finite. Since $\text{Vals}$ is non-empty and $h = h_1 \star h_2$ where $h_\emptyset$ is the heap with $\text{dom} (h_\emptyset) = \emptyset$, both sets are non-empty. Lastly, from Theorem 3 it follows directly, that both sets are also finite.

B Appendix to Section 3

B.1 Proof of Theorem 2

Theorem 2. Let $C \in \text{hpGCL}$ be loop-free and $\mathcal{A}$ be a set of predicate symbols. If

1. $\mathcal{A}$ contains the points-to predicate for all variables and all expressions occurring in allocation, disposal, lookup and mutation in $C$,
2. $\mathcal{A}$ contains all guards and their negations occurring in $C$, and
3. all predicates in $\mathcal{A}$ are closed under substitution of variables by other variables and arithmetic expressions occurring on right-hand sides of assignments in $C$,

then, for every QSL [$\mathcal{A}$] formula $f$, $\text{wp}[C] (f) \in \text{QSL} [\mathcal{A}]$.

Proof. First we remark that since $\mathcal{A}$ is closed under substitution of variables by arithmetic expressions occurring on right-hand sides of assignments in $C$, so is QSL [$\mathcal{A}$]. For a formula $f \in \text{QSL} [\mathcal{A}]$, substitutions are always only handed to the atoms in $\mathcal{A}$. Since $\mathcal{A}$ is closed under substitution of variables by arithmetic expressions occurring on right-hand sides of assignments, we have for every such expression $E$ on a right-hand side of an assignment and $\psi \in \mathcal{A}$ that also $\psi [x:=E] \in \mathcal{A}$, where the semantics $[[\psi [x:=E]]]$ of $\psi [x:=E]$ is defined as

$$[[\psi [x:=E]]] = \{ (s [x:=E(s)], h) \in \psi \mid (s, h) \in \text{States} \} .$$

Now we prove the theorem by induction on $C$. 

For the base case $C = \text{skip}$, we have $\text{wlp}[\text{skip}](f) = f \in \text{QSL}[\mathfrak{A}]$ by assumption.

For the base case $C = x := E$, we have $\text{wlp}[x := E](f) = f[x:=E] \in \text{QSL}[\mathfrak{A}]$ since $\text{QSL}[\mathfrak{A}]$ is closed under the substitution of $x$ by $E$.

For the base case $C = x := \text{new}(E)$, we have

$$\text{wlp}[x := \text{new}(E)](f) = \Lambda y: [y \mapsto E] \rightarrow f[x:=y] \in \text{QSL}[\mathfrak{A}]$$

because $\text{QSL}[\mathfrak{A}]$ is closed under substitution of $x$ by $y$ and $\mathfrak{A}$ contains the points-to formula $[y \mapsto E]$ by assumption.

For the base case $C = \text{free}(E)$, we have

$$\text{wlp}[\text{free}(E)](f) = [E \mapsto -] \star f = \exists x: [E \mapsto x] \star f \in \text{QSL}[\mathfrak{A}]$$

because $\mathfrak{A}$ contains $[E \mapsto x]$ by assumption.

For the base case $C = x := <E>$, we have

$$\text{wlp}[x := <E>](f) = \exists x: [E \mapsto y] \star ([E \mapsto xy] \rightarrow f[x:=y]) \in \text{QSL}[\mathfrak{A}]$$

because $\text{QSL}[\mathfrak{A}]$ is closed under substitution of $x$ by $y$ and $\mathfrak{A}$ contains the points-to formulae $[E \mapsto y]$ by assumption.

For the base case $C = <E> := E'$, we have

$$\text{wlp}[<E> := E'](f) = [E \mapsto -] \star ([E \mapsto E'] \rightarrow f)$$

$$= (\exists x: [E \mapsto x] \star ([E \mapsto E'] \rightarrow f) \in \text{QSL}[\mathfrak{A}]$$

because $\mathfrak{A}$ contains the points-to formulae $[E \mapsto x]$ and $[E \mapsto E']$ by assumption.

For all other composite cases we assume for some fixed, but arbitrary loop-free programs $C_1, C_2 \in \text{hpGCL}$ that for all $g \in \text{QSL}[\mathfrak{A}]$ we have $\text{wlp}[C_1](g) \in \text{QSL}[\mathfrak{A}]$ and for all $u \in \text{QSL}[\mathfrak{A}]$ we have $\text{wlp}[C_2](u) \in \text{QSL}[\mathfrak{A}]$.

For the case $C = \{ C_1 \} [p] \{ C_2 \}$, we have

$$\text{wlp}[C](f) = p \cdot \text{wlp}[C_1](f) + (1 - p) \cdot \text{wlp}[C_2](f) \in \text{QSL}[\mathfrak{A}]$$

by the induction hypothesis.

For the case $C = C_1 ; C_2$, we have $\text{wlp}[C_2](f) \in \text{QSL}[\mathfrak{A}]$ by the induction hypothesis, thus we also have $\text{wlp}[C_1](\text{wlp}[C_2](f)) \in \text{QSL}[\mathfrak{A}]$ by the induction hypothesis.

For the case $C = \text{if} \ (B) \ { C_1 } \ \text{else} \ { C_2 }$, we have

$$\text{wlp}[C](f) = [B] \cdot \text{wlp}[C_1](f) + [-B] \cdot \text{wlp}[C_2](f) \in \text{QSL}[\mathfrak{A}]$$

by the induction hypothesis and since $\mathfrak{A}$ contains $B$ and $\lnot B$ by assumption.

This concludes the proof.
B.2 \(C_{\text{swap}}\) Example

Full Computation

\[
\begin{align*}
\text{wlp}[C_{\text{swap}}]( [x \mapsto z_1] \star [y \mapsto z_2]) \\
= & \quad \text{wlp}[C_{\text{swap}}]( [y \mapsto z_2] \star ([x \mapsto z_1] \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad \text{(monotonicity)} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [y \mapsto \text{tmp}1] \star ([x \mapsto \text{tmp}2] \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad \text{(variable substitution)} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) ( [y \mapsto -] \star ([x \mapsto \text{tmp}2] \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad \text{(Rule (ii))} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) (0.999 \cdot \text{wlp}[< x > := \text{tmp}2]( [y \mapsto -] \\
& \quad \star ([x \mapsto \text{tmp}2] \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad + 0.001 \cdot \text{wlp}[< x > := \text{err}( [y \mapsto -] \\
& \quad \star ([x \mapsto \text{tmp}2] \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad \text{(wlp application)} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) (0.999 \cdot \text{wlp}[< x > := \text{tmp}2]( [y \mapsto -] \\
& \quad \star ([x \mapsto \text{tmp}2] \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad + 0.001 \cdot [\text{false}] \quad \text{(monotonicity)} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) (0.999 \cdot ([x \mapsto -] \star ([y \mapsto -] \\
& \quad \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad + 0.001 \cdot [\text{false}] \quad \text{(Rule (ii))} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) (0.999 \cdot \text{wlp}[C_{\bigstar} C_{\bigstar} C_{\bigstar}]( [x \mapsto -] \star ([y \mapsto -] \\
& \quad \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad + 0.001 \cdot [\text{false}] \quad \text{(wlp application)} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) (0.999 \cdot \text{wlp}[C_{\bigstar} C_{\bigstar} C_{\bigstar}]( [y \mapsto -] \star ([y \mapsto -] \\
& \quad \cdot ([\text{tmp}1 = z_2] \cdot [\text{tmp}2 = z_1]))) \\
& \quad + 0.001 \cdot [\text{false}] \quad \text{(super-distributivity)} \\
= & \quad \text{wlp}[C_{\text{swap}}]( [C_{\bigstar} C_{\bigstar} C_{\bigstar}]) (0.999 \cdot \text{wlp}[C_{\bigstar} C_{\bigstar} C_{\bigstar}]( [y \mapsto z_1] \cdot [\text{tmp}2 = z_1]) \\
& \quad \star ([\text{tmp}1 = z_2] \cdot [x \mapsto -]) \\
& \quad + 0.001 \cdot [\text{false}] \quad \text{(monotonicity)}
\end{align*}
\]
\[ wlp[C_1] (0.999 \cdot ([y \mapsto z_1] \ast ([\text{tmp1} = z_2] \cdot [x \mapsto -])) + 0.001 \cdot [\text{false}]) \]

(Rule (i))

\[ 0.999 \cdot wlp[C_1] ([y \mapsto z_1] \ast ([\text{tmp1} = z_2] \cdot [x \mapsto -])) + 0.001 \cdot wlp[C_1] ([\text{false}]) \]

(super-distributivity)

\[ 0.999 \cdot wlp[C_1] ([y \mapsto z_1] \ast ([\text{tmp1} = z_2] \cdot [x \mapsto z_2])) + 0.001 \cdot [\text{false}] \]

(monotonicity)

\[ 0.999 \cdot wlp[C_1] (([x \mapsto z_2] \cdot [\text{tmp1} = z_2]) \ast [y \mapsto z_1]) + 0.001 \cdot [\text{false}] \]

(commutativity)

\[ 0.999 \cdot ([x \mapsto z_2] \ast [y \mapsto z_1]) + 0.001 \cdot [\text{false}] \]

(Rule (i))

**Necessary Separation Logic Entailments**

\[ ((([x \mapsto z_2] \ast [y \mapsto z_1]) \land \text{false}) \lor ([x \mapsto z_2] \ast [y \mapsto z_1])) \lor \text{false} \]

\[ ((([x \mapsto z_2] \ast [y \mapsto z_1]) \land \text{false}) \lor ([x \mapsto z_2] \ast [y \mapsto z_1])) \]

\[ ((([x \mapsto z_1] \ast [y \mapsto z_2]) \land \text{false}) \lor ([x \mapsto z_2] \ast [y \mapsto z_1])) \]

\[ ((([x \mapsto z_1] \ast [y \mapsto z_2]) \land \text{false}) \lor ([x \mapsto z_2] \ast [y \mapsto z_1])) \]

\[ ([x \mapsto z_2] \ast [y \mapsto z_1]) \land \text{false} \]

\[ ([x \mapsto z_2] \ast [y \mapsto z_1]) \land \text{false} \]

**SMT-LIB 2 File**

(set-logic QF_ALL_SUPPORTED)
(declare-sort Loc 0)
(declare-heap (Loc Int))
(declare-const x Loc)
(declare-const y Loc)
(declare-const z1 Int)
(declare-const z2 Int)
(assert (or (and (or (and (sep (pto x z2) (pto y z1)) false)
(sep (pto x z2) (pto y z1))) false)
(not (or (and (sep (pto x z2) (pto y z1)) false)
(sep (pto x z2) (pto y z1)))))
(or (and (or (and (sep (pto x z2) (pto y z1)) false)
(sep (pto x z2) (pto y z1)))
(not (or (and (sep (pto x z2) (pto y z1)) false)
(sep (pto x z2) (pto y z1))))
(and (and (sep (pto x z2) (pto y z1)) false)
(not (and (sep (pto x z2) (pto y z1)) false)))))))
(check-sat)
Lemma 4. For all formulae \( f \in QSL[A] \) we have \( 0, 1 \in \text{Val}[f] \).

Proof. By induction on \( f \).

For the base case \( f = [\psi] \), we have \( 0, 1 \in \{0, 1\} = \text{Val}[f] \).

For all other composite cases we assume for some fixed, but arbitrary \( g, u \in QSL[A] \) that \( 0, 1 \in \text{Val}[g] \) and \( 0, 1 \in \text{Val}[u] \).

For the induction step \( f = [\pi] \cdot g + [\neg \pi] \cdot u \), we have by the induction hypothesis \( 0, 1 \in \text{Val}[g] \subseteq \text{Val}[f] \).

For the induction step \( f = p \cdot g + (1 - p) \cdot u \), we have by the induction hypothesis \( 0 = 0 \cdot 0 \) and \( 1 = 1 \cdot 1 \) that \( 0, 1 \in \text{Val}[f] \).

For the induction step \( f = g \cdot u \), we have by the induction hypothesis \( 0, 1 \in \text{Val}[g] \) and \( 0, 1 \in \text{Val}[u] \) thus with \( 0 = \max(0, 0) \) and \( 1 = \max(1, 1) \) we also have \( 0, 1 \in \text{Val}[f] \).

For the induction step \( f = 1 - g \), we have by the induction hypothesis \( 0, 1 \in \text{Val}[g] \), thus with \( 0 = 1 - 1 \) and \( 1 = 1 - 0 \) we also have \( 0, 1 \in \text{Val}[f] \).

The induction step \( f = g \min u \) is analogous to the previous case.

For the induction steps \( f = \mathcal{L} x : g \), we have by the induction hypothesis \( 0, 1 \in \text{Val}[g] = \text{Val}[f] \).

The induction step \( f = \mathcal{E} x : g \) is analogous to the previous case.

The induction step \( f = g \star u \) is analogous to the case \( g \cdot u \).

For the induction step \( f = [\psi] \rightarrow g \), we have by the induction hypothesis \( 0, 1 \in \text{Val}[g] = \text{Val}[f] \).

This concludes the proof.

Theorem 4. For every \( f \in QSL[A] \) and all \( \alpha \in \mathbb{P} \) there is an effectively constructible \( \text{SL}[A] \) formula \( [\alpha \leq f] \) such that for all \((s, h) \in \text{States}\), we have

\[
(s, h) \models [\alpha \leq f] \iff \alpha \leq \llbracket f \rrbracket(s, h).
\]

Proof. By induction on \( f \).

The case \([\psi]\). If \( \alpha = 0 \) then \( \alpha \leq \llbracket [\psi] \rrbracket(s, h) \) trivially. If \( 0 < \alpha \), then

\[
\alpha \leq \llbracket [\psi] \rrbracket(s, h) \iff \llbracket [\psi] \rrbracket(s, h) = 1 \iff (s, h) \models \psi.
\]
For the composite cases, now assume that for some arbitrary, but fixed formulae \( g, u \in \text{QSL} \{ \mathfrak{A} \} \) and all probabilities \( \beta, \gamma \in \mathbb{P} \) there are effectively constructible \( \text{SL} \{ \mathfrak{A} \} \) formulae \( \lceil \beta \leq g \rceil \) and \( \lceil \gamma \leq u \rceil \) such that for all \((s, h) \in \text{States} \),

\[
\beta \leq \llbracket g \rrbracket (s, h) \text{ iff } (s, h) \models \lceil \beta \leq g \rceil \quad \text{and} \quad \gamma \leq \llbracket u \rrbracket (s, h) \text{ iff } (s, h) \models \lceil \gamma \leq u \rceil .
\]

The case \( f = [\pi] \cdot g + [\neg \pi] \cdot u \).

\[
\alpha \leq \llbracket [\pi] \cdot g + [\neg \pi] \cdot u \rrbracket (s, h) \quad \text{iff} \quad \alpha \leq [\pi] (s, h) \cdot \llbracket g \rrbracket (s, h) + [\neg \pi] (s, h) \cdot \llbracket u \rrbracket (s, h).
\]

\[
(s, h) \models \pi \text{ and } \alpha \leq \llbracket g \rrbracket (s, h) \quad \text{or} \quad (s, h) \models \neg \pi \text{ and } \alpha \leq \llbracket u \rrbracket (s, h).
\]

\[
(s, h) \models \pi \text{ and } (s, h) \models [\alpha \leq g] \quad \text{or} \quad (s, h) \models \neg \pi \text{ and } (s, h) \models [\alpha \leq u].
\]

\[
(s, h) \models (\pi \land [\alpha \leq g]) \lor (\neg \pi \land [\alpha \leq u]).
\]

\( (s, h) \models (\pi \land [\alpha \leq g]) \lor (\neg \pi \land [\alpha \leq u]) \) (IH)

The case \( f = p \cdot g + (1 - p) \cdot u \).

\[
\alpha \leq \llbracket p \cdot g + (1 - p) \cdot u \rrbracket (s, h) \quad \text{iff} \quad \alpha \leq p \cdot \llbracket g \rrbracket (s, h) + (1 - p) \cdot \llbracket u \rrbracket (s, h).
\]

\[
\text{there are } \beta \in \text{Val} [g], \gamma \in \text{Val} [u] \text{ with } p \cdot \beta + (1 - p) \cdot \gamma \geq \alpha \text{ with}
\]

\[
\beta \leq \llbracket g \rrbracket (s, h) \text{ and } \gamma \leq \llbracket u \rrbracket (s, h) \quad \text{(monotonicity)}
\]

\[
\text{there are } \beta \in \text{Val} [g], \gamma \in \text{Val} [u] \text{ with } p \cdot \beta + (1 - p) \cdot \gamma \geq \alpha \text{ with}
\]

\[
(s, h) \models [\beta \leq g] \text{ and } (s, h) \models [\gamma \leq u].
\]

\[
(s, h) \models \bigvee_{\beta \in \text{Val} [g], \gamma \in \text{Val} [u], p \cdot \beta + (1 - p) \cdot \gamma \geq \alpha} [\beta \leq g] \land [\gamma \leq u].
\]

\( (\text{Val} [g] \text{ and Val} [u] \text{ are finite}) \)

The case \( f = g \cdot u \) is analogous to the previous case.
The case $f = 1 - g$.

\[
\alpha \leq \llbracket 1 - g \rrbracket (s, h)
\]

iff

\[
\alpha \leq 1 - \llbracket g \rrbracket (s, h)
\]

iff

\[
0 = \alpha \text{ or } 0 < \alpha \text{ and } \alpha \leq 1 - \llbracket g \rrbracket (s, h)
\]

iff

\[
0 = \alpha \text{ or } 1 > 1 - \alpha \text{ and } 1 - \alpha \geq \llbracket g \rrbracket (s, h)
\]

iff

\[
0 = \alpha \text{ or } 1 > 1 - \alpha \text{ and } \text{not } 1 - \alpha < \llbracket g \rrbracket (s, h)
\]

iff

\[
0 = \alpha \text{ or } 1 > 1 - \alpha \text{ and } \text{not: there exists } \delta \in \text{Val}[g] \text{ such that } 1 - \alpha < \delta \leq \llbracket g \rrbracket (s, h) \quad (\delta \in \text{Val}[g])
\]

iff

\[
0 = \alpha \text{ or } 1 > 1 - \alpha \text{ and } \text{not } \min \{ \beta \in \text{Val}[g] \mid \beta > 1 - \alpha \} \leq \llbracket g \rrbracket (s, h)
\]

(†, see below)

iff

\[
0 = \alpha \text{ or } 0 < \alpha \text{ and } (s, h) \models \lnot [\delta \preceq g] \text{ for } \delta = \min \{ \beta \in \text{Val}[g] \mid \beta > 1 - \alpha \}
\]

iff

\[
0 = \alpha \text{ or } 0 < \alpha \text{ and } (s, h) \models [\delta \preceq g] \text{ for } \delta = \min \{ \beta \in \text{Val}[g] \mid \beta > 1 - \alpha \}
\]

iff

\[
0 = \alpha \text{ or } 0 < \alpha \text{ and } (s, h) \models \lnot [\delta \preceq g] \text{ for } \delta = \min \{ \beta \in \text{Val}[g] \mid \beta > 1 - \alpha \}
\]

Regarding †: If there exists a $\delta \in \text{Val}[g]$ with $\delta \leq \llbracket g \rrbracket (s, h)$, then we also have for all $\beta \leq \delta$ that $\beta \leq \llbracket g \rrbracket (s, h)$ by transitivity. Since furthermore $\text{Val}[g]$ is finite, $1 - \alpha < 1$ and $1 \in \text{Val}[g]$ we can also pick the smallest $\beta \in \text{Val}[g]$ satisfying $1 - \alpha < \beta$. For the other direction we have that since $\text{Val}[g]$ is finite, $1 \in \text{Val}[g]$ and $1 > 1 - \alpha$, the set $\{ \beta \in \text{Val}[g] \mid \beta > 1 - \alpha \}$ is finite and non-empty. Thus there also exists an $\delta$ such that $1 - \alpha < \delta \leq \llbracket g \rrbracket (s, h)$.

The case $f = g \max u$.

\[
\alpha \leq \llbracket g \max u \rrbracket (s, h)
\]

iff

\[
\alpha \leq \max(\llbracket g \rrbracket (s, h), \llbracket u \rrbracket (s, h))
\]

iff

\[
\alpha \leq \llbracket g \rrbracket (s, h) \text{ or } \alpha \leq \llbracket u \rrbracket (s, h)
\]

iff

\[
(s, h) \models [\alpha \leq g] \text{ or } (s, h) \models [\alpha \leq u]
\]

(II)

iff

\[
(s, h) \models [\alpha \leq g] \lor [\alpha \leq u]
\]

The case $f = g \min u$ is analogous to the previous case.
The case \( f = 2 \cdot x : g \).

\[
\alpha \leq \left[ [2 \cdot x : g] (s, h) \right]
\]

\[
{\begin{align*}
& \text{iff } \alpha \leq \max \left\{ [[g] (s [x := v], h) \mid v \in \text{Val}_s] \right\} \\
& \text{iff } \text{there is a } v \in \text{Val}_s \text{ with } \alpha \leq [[g] (s [x := v], h)] \quad \text{(Eval} (g) \text{ is finite)} \\
& \text{iff } \text{there is a } v \in \text{Val}_s \text{ with } (s [x := v], h) \models [\alpha \leq g] \quad \text{(IH)} \\
& \text{iff } (s, h) \models \exists x : [\alpha \leq g]
\end{align*}}
\]

The case \( f = 4 \cdot x : g \) is analogous to the previous case.

The case \( f = g \cdot u \).

\[
\alpha \leq \left[ [g \cdot u] (s, h) \right]
\]

\[
\begin{align*}
& \text{iff } \alpha \leq \max \left\{ [[g] (s, h_1) \cdot [[u] (s, h_2) \mid h = h_1 \cdot h_2} \\
& \text{iff } \text{there exists } h_1, h_2 \in \text{Heaps}_g \text{ with } h = h_1 \cdot h_2 \text{ and } \alpha \leq [[g] (s, h_1) \cdot [[u] (s, h_2)] \quad \text{(Eval} (g) \text{ and Eval} (u) \text{ are finite)} \\
& \text{iff } \text{there exists } h_1, h_2 \in \text{Heaps}_g \text{ with } h = h_1 \cdot h_2 \text{ and } \beta \in \text{Val}_g, \gamma \in \text{Val}_u \text{ with } \beta \cdot \gamma \geq \alpha \text{ and } \\
& \quad \beta \leq [[g] (s, h_1) \text{ and } \gamma \leq [[u] (s, h_2)) \quad \text{(monotonicity)} \\
& \text{iff } \text{there exists } h_1, h_2 \in \text{Heaps}_g \text{ with } h = h_1 \cdot h_2 \text{ and } \beta \cdot \gamma \geq \alpha \text{ and } \\
& \quad \text{there exists } h_1, h_2 \in \text{Heaps}_g \text{ with } h = h_1 \cdot h_2 \text{ and } \\
& \quad (s, h_1) \models [\beta \leq g] \text{ and } (s, h_2) \models [\gamma \leq u] \quad \text{(IH)} \\
& \text{iff } \text{there exists } h_1, h_2 \in \text{Heaps}_g \text{ with } h = h_1 \cdot h_2 \text{ and } \\
& \quad (s, h) \models [\beta \leq g] \cdot [\gamma \leq u] \\
& \text{iff } (s, h) \models \bigvee_{\beta \in \text{Val}_g, \gamma \in \text{Val}_u, \beta \cdot \gamma \geq \alpha} [\beta \leq g] \cdot [\gamma \leq u] \\n& \quad \text{(Val}_g \text{ and Val}_u \text{ are finite)}
\end{align*}
\]

The case \( f = [\psi] \rightarrow g \).

\[
\alpha \leq \left[ [\psi] \rightarrow g \right] (s, h)
\]

\[
\begin{align*}
& \text{iff } \alpha \leq \inf \{ [[g] (s, h \cdot h') \mid h' \perp h \text{ and } [\psi] (s, h') = 1\} \\
& \text{iff } \text{for all } h' \in \text{Heaps}_g \text{ with } h' \perp h \text{ and } (s, h') \models \psi \text{ we have } \alpha \leq [[g] (s, h \cdot h') \\
& \quad \Downarrow \text{ see below} \\
& \text{iff } \text{for all } h' \in \text{Heaps}_g \text{ with } h' \perp h \text{ and } (s, h') \models \psi \text{ we have } (s, h \cdot h') \models [\alpha \leq g] \quad \text{(IH)} \\
& \text{iff } (s, h) \models \psi \rightarrow [\alpha \leq g]
\end{align*}
\]
Theorem 3.0 \( \mathcal{A} \) ∈ \( \mathfrak{P} \). Then by the induction hypothesis, we have

\[ |\psi|_{p} = 0. \]

This concludes the proof.

D  Appendix to Section 5

Theorem 6. We have \( |\text{Val} [f]| \leq 2^{|f|_{P} + 1} \). Hence, checking \( f \models g \) by means of Theorem 5 requires checking \( 2^{O(|f|_{P})} \) entailments in SL \( \mathcal{A} \).

Proof. We prove this by induction of \( f \).

For the base case for \( f = [\psi] \) we have \( |\text{Val} ([\psi])| = 2^{0+1} = 2^{[\psi]_{P} + 1} \).

For the composite cases we assume that for some fixed, but arbitrary formulae \( g, u \in \text{QSL} \mathcal{A} \), the inequalities \( |\text{Val} [g]| \leq 2^{|g|_{P} + 1} \) and \( |\text{Val} [u]| \leq 2^{|u|_{P} + 1} \) hold.

For the induction steps \( f = [\pi] \cdot g + [-\pi] \cdot u \), we distinguish three cases.

1. \( |g|_{P} = 0 \). Then by the induction hypothesis, we have \( |\text{Val} [g]| \leq 2. \) By Lemma 4, we have \( \{0, 1\} = \text{Val} [g] \) and \( 0, 1 \in \text{Val} [u] \). However, then the union will not increase the set, i.e. \( \text{Val} [f] = \text{Val} [u] \). Finally we have \( |\text{Val} [f]| = |\text{Val} [u]| \leq 2^{|u|_{P} + 1} = 2^{|f|_{P} + 1} \) by the induction hypothesis.

2. \( |u|_{P} = 0 \) is analogous.

3. \( 0 < |g|_{P}, |u|_{P} \). Then the size of the set is at most the sum of each set \( |\text{Val} [f]| \leq |\text{Val} [g]| + |\text{Val} [u]| \). By the induction hypothesis we then have

\[ |\text{Val} [f]| \leq 2^{|g|_{P} + 1} + 2^{|u|_{P} + 1} \leq 2 \cdot 2^{|g|_{P}} \cdot 2^{|u|_{P}} = 2^{|g|_{P} + |u|_{P} + 1} = 2^{|f|_{P} + 1}. \]

because \( 0 < |g|_{P}, |u|_{P} \).
For the induction steps $f = q \cdot g + (1 - q) \cdot u$, we have that $|\text{Val}[f]| \leq |\text{Val}[g]| \cdot |\text{Val}[u]|$. By the induction hypothesis we deduce the upper bound

$$|\text{Val}[f]| \leq 2^{|g|_{p+1}} \cdot 2^{|u|_{p+1}} + 2^{|g|_{p+1} + |u|_{p+1} + 1} = 2^{|f|_{p+1}}.$$  

The induction step $f = g \cdot u$ is analogous to the previous case.

For the induction step $f = 1 - g$, we have that the set does not change in size $|\text{Val}[f]| = |\text{Val}[g]|$ and $|f|_p = |g|_p$. Thus this follows directly from the induction hypothesis.

The induction steps $f = g \min u$ and $f = g \max u$ are analogous to case $f = [\pi] \cdot g + [-\pi] \cdot u$.

The induction steps $f = \mathcal{S} x: g$ and $f = \mathcal{L} x: g$ are analogous to the case $f = 1 - g$.

The induction step $f = g \star u$ is analogous to the case $f = g \cdot u$.

The induction step $f = [\psi] \longrightarrow g$ is analogous to the case $f = 1 - g$.

**Theorem 7.** For any formulae $f \in \text{QSL}[\mathfrak{A}]$ and all probabilities $\alpha \in \mathbb{P}$, the SL [\mathfrak{A}] formula $[\alpha \leq f]$ has at most size $3 \cdot |f| \cdot 2^{|f|_{p+1}^2}$. Hence the size of the formula $[\alpha \leq f]$ is in $O(|f|) \cdot 2^{O(|f|_{p+1}^2)}$.

**Proof.** We show by induction on $f$ that $|[\alpha \leq f]| \leq 3 \cdot |f| \cdot 2^{|f|_{p+1}^2}$ for all $\alpha \in \mathbb{P}$.

For the base case $f = [\psi]$ we have $|[\alpha \leq f]| \leq |\psi| = |\psi| \cdot 2^1 \leq 3 \cdot |f| \cdot 2^{|f|_{p+1}^2}$.

For the composite cases we assume that for some arbitrary, but fixed formulae $g, u \in \text{QSL}[\mathfrak{A}]$ and all probabilities $\alpha \in \mathbb{P}$ the inequalities for both formulae $|[\alpha \leq g]| \leq 3 \cdot |g| \cdot 2^{|g|_{p+1}^2}$ and $|[\alpha \leq u]| \leq 3 \cdot |u| \cdot 2^{|u|_{p+1}^2}$ hold.

For the induction step $f = [\pi] \cdot g + [-\pi] \cdot u$ we have that

$$|\alpha \leq f| = |([\pi] \land [\alpha \leq u]) \lor ([\neg \pi] \land [\alpha \leq u])| = |[\pi] + 1 + |\alpha \leq u|| + 1 + |[\neg \pi] + 1 + |\alpha \leq u|| = |\pi| + |\alpha \leq g| + |\neg \pi| + |\alpha \leq u|| + 3 \leq |\pi| + 3 \cdot |g| \cdot 2^{|g|_{p+1}+1} + |\neg \pi| + 3 \cdot |u| \cdot 2^{|u|_{p+1}+1} + 3 \leq 3 \cdot (|\pi| + |g| + |\neg \pi| + |u| + 1) \cdot 2^{|g|_{p+1}+|u|_{p+1}+1} \leq 3 \cdot (|\pi| + |g| + |\neg \pi| + |u| + 1) \cdot 2^{|f|_{p+1}^2} = 3 \cdot |f| \cdot 2^{|f|_{p+1}^2}.$$

$$(f|_p = |g|_p + |u|_p)$$

$$f = 3 \cdot |f| \cdot 2^{|f|_{p+1}^2}.$$
For the induction step $f = p \cdot g + (1 - p) \cdot u$, we have for $\beta, \gamma \in \mathbb{P}$ maximizing $|\beta \leq g|$ and $|\gamma \leq u|$ that

$$ |\alpha \leq f| \leq (|\text{Val}[g]| \cdot |\text{Val}[u]|) \cdot (|\beta \leq g| + |\gamma \leq u| + 2) ,$$

thus using the induction hypothesis and Theorem 6

$$ |\alpha \leq f| \leq \left(2^{\langle |g|_{p} + 1 \cdot 2^{|u|_{p} + 1}\rangle} \cdot (3 \cdot |g| \cdot 2^{\langle |g|_{p} + 1\rangle^2} + 3 \cdot |u| \cdot 2^{\langle |u|_{p} + 1\rangle^2} + 2) \right). \quad \text{(†)}$$

Remark that $1 \leq |f|_{p}$ by assumption. We now prove the upper bound on the right side by distinguishing two cases on $|f|_{p}$:

1. $|f|_{p} = 1$. Then $|g|_{p} = |u|_{p} = 0$ and we have:

$$ |\alpha \leq f| \leq 2 \cdot 2 \cdot (3 \cdot |g| \cdot 2 + 3 \cdot |u| \cdot 2 + 2) \quad \text{(† and } |f|_{p} = 1)$$

$$ = 2^3 \cdot 3 \cdot (|g| + |u| + 1/5)$$

$$ \leq 2^3 \cdot 3 \cdot |f| \quad \text{(} |f| = |g| + |u| + 1)$$

$$ \leq 3 \cdot |f| \cdot 2^{\langle |f|_{p} + 1\rangle^2} \quad \text{(} |f|_{p} = 1)$$

2. $|f|_{p} > 1$. Here we require a bit more mathematical tools in form of Lemma 5. Then we have:

$$ |\alpha \leq f| \leq \left(2^{\langle |g|_{p} + 1 \cdot 2^{|u|_{p} + 1}\rangle} \cdot (3 \cdot |g| \cdot 2^{\langle |g|_{p} + 1\rangle^2} + 3 \cdot |u| \cdot 2^{\langle |u|_{p} + 1\rangle^2} + 2) \right). \quad \text{(†)}$$

$$ = 2^{\langle |f|_{p} + 1\rangle} \cdot (3 \cdot |g| \cdot 2^{\langle |g|_{p} + 1\rangle^2} + 3 \cdot |u| \cdot 2^{\langle |u|_{p} + 1\rangle^2} + 2) \quad \text{(|} |f|_{p} = |g| + |u| + 1)$$

$$ \leq 3 \cdot (|g| + |u|) \cdot 2^{\langle |f|_{p} + 1\rangle \cdot 2^{\langle |g|_{p} + |u|_{p} + 1\rangle^2}}$$

$$ + 3 \cdot (|g| + |u|) \cdot 2^{\langle |f|_{p} + 1\rangle \cdot 2^{\langle |g|_{p} + |u|_{p} + 1\rangle^2}}$$

$$ + 2 \cdot 2^{\langle |f|_{p} + 1\rangle^2} \leq 2 \cdot 3 \cdot (|g| + |u|) \cdot 2^{\langle |f|_{p} + 1\rangle \cdot 2^{\langle |g|_{p} + |u|_{p} + 1\rangle^2}} + 2^{\langle |f|_{p} + 2\rangle} \quad \text{(|} |f| = |g| + |u| + 1)$$

$$ \leq 2 \cdot 3 \cdot (|f| \cdot 2^{\langle |f|_{p} + 1\rangle \cdot 2^{\langle |g|_{p} + |u|_{p} + 1\rangle^2}} + 2^{\langle |f|_{p} + 2\rangle} \quad \text{(|} |f|_{p} = |g| + |u| + 1)$$

$$ = 2 \cdot 3 \cdot (|f| \cdot 2^{\langle |f|_{p} + 1\rangle \cdot 2^{\langle |f|_{p} + 2\rangle}} + 2^{\langle |f|_{p} + 2\rangle} \quad \text{(|} |f|_{p} = |g| + |u| + 1)$$

$$ \leq 3 \cdot |f| \cdot (2^{\langle |f|_{p} + 1\rangle^2} + 2^{\langle |f|_{p} + 2\rangle}) \quad \text{(by } 1 < |f|_{p} \text{ and Lemma 5)}$$

The induction steps for $f = g \cdot u$ is analogous to the previous case.
For the induction step \( f = 1 - g \) we have that
\[
\begin{align*}
|\lceil \alpha \leq f \rceil| & \leq |\lceil \alpha \leq g \rceil| + 1 \\
& \leq 3 \cdot |g| \cdot 2^{(|g|_p+1)^2} + 1 \quad \text{(IH)} \\
& \leq 3 \cdot |f| \cdot 2^{(|f|_p+1)^2} + 2^{(|f|_p+1)^2} \quad (0 \leq |g|_p = |f|_p) \\
& = 3 \cdot (|g| + 1/3) \cdot 2^{(|f|_p+1)^2} \\
& \leq 3 \cdot |f| \cdot 2^{(|f|_p+1)^2} \quad (|f| = |g| + 1)
\end{align*}
\]

The induction steps for \( g_{\max u}, g_{\min u}, \mathcal{E}_x: g \) and \( \mathcal{L}_x: g \) are analogous to the previous case.

The induction steps for \( g* u \) is analogous to the case \( f = p \cdot g + (1 - p) \cdot u \).

For the induction step \( \lceil \psi \rceil \rightarrow g \) we have that
\[
\begin{align*}
|\lceil \alpha \leq g \rceil| & = |\lceil \alpha \leq g \rceil| + |\psi| + 1 \\
& \leq 3 \cdot |g| \cdot 2^{(|g|_p+1)^2} + |\psi| + 1 \quad \text{(IH)} \\
& \leq 3 \cdot |g| \cdot 2^{(|f|_p+1)^2} + (|\psi| + 1) \cdot 2^{(|f|_p+1)^2} \quad (0 \leq |g|_p = |f|_p) \\
& = 3 \cdot (|g| + |\psi|/3 + 1/3) \cdot 2^{(|f|_p+1)^2} \\
& \leq 3 \cdot |f| \cdot 2^{(|f|_p+1)^2} \quad (|f| = |g| + |\psi| + 1)
\end{align*}
\]

This concludes the proof.

**Lemma 5.** For all natural numbers \( n > 1 \), we have \( 2^{n^2+n+2} + 2^{n+2} \leq 2^{(n+1)^2} \).

**Proof.** By induction over \( n \).

For the base case \( n = 2 \), we have \( 2^{2^2+2+2} + 2^{2+2} = 2^8 + 2^4 < 2^8 + 2^8 = 2^{(2+1)^2} \).

Now we assume that for some fixed, but arbitrary natural number \( n > 1 \) the inequality \( 2^{n^2+n+2} + 2^{n+2} \leq 2^{(n+1)^2} \) holds.

For the induction step \( n \rightarrow n + 1 \), we have
\[
\begin{align*}
2^{(n+1)^2+n+1+2} + 2^{n+1+2}
& = 2^{n^2+2n+1+n+1+2} + 2^{n+1+2} \\
& = 2^{n^2+n+2} \cdot 2^{n+2} + 2^{n+2} \cdot 2 \\
& \leq 2^{2n+2} \cdot \left(2^{n^2+n+2} + 2^{n+2}\right) \quad \text{(IH)} \\
& = 2^{2n+2} \cdot 2^{(n+1)^2} \\
& = 2^{n^2+4n+3} \\
& \leq 2^{n^2+4n+4} \\
& = 2^{(n+1)^2}.
\end{align*}
\]
This concludes the proof.

E Appendix to Section 6

Lemma 1. Let \( Q \) be a QSL [\( S \)] fragment. If an SL [\( S \)] fragment \( S \) satisfies the requirements provided in Table 7, then \( S \) is Q-admissible.

Proof. By induction on \( f \).

For the base case \( f = [\psi] \), we have \( [\alpha \leq [\psi]] = \text{true} \) if \( \alpha = 0 \) otherwise \( [\psi] \), thus \text{true} and \( \psi \) are required.

For all other composite cases we assume for some fixed, but arbitrary \( g,u \in \text{QSL} [\mathcal{A}] \) that for all \( \beta,\gamma \in \mathbb{P} \) the formulae \( [\beta \leq g] \) and \( [\gamma \leq u] \) satisfy the requirements.

For the case \( f = [\pi] \cdot g + [\neg \pi] \cdot u \), we have \( [\alpha \leq f] = (\pi \land [\alpha \leq g]) \lor (\neg \pi \land [\alpha \leq u]) \), since by the induction hypothesis \( [\alpha \leq g] \) and \( [\alpha \leq u] \) already satisfy all requirements, we only require additionally \( \pi,\neg \pi,\land \) and \( \lor \).

For the case \( f = p \cdot g + (1 - p) \cdot u \), we have

\[
[\alpha \leq f] = \bigvee_{\beta \in \text{Val}[g], \gamma \in \text{Val}[u]} [\beta \leq g] \land [\gamma \leq u].
\]

Here we have two observations:

1. There are only finitely many disjunctions since \( \text{Val}[g] \) and \( \text{Val}[u] \) is finite by Theorem 3.
2. The disjunctions is not empty since \( 1 \in \text{Val}[g] \) and \( 1 \in \text{Val}[u] \) by Lemma 4 and \( p \cdot 1 + (1 - p) \cdot 1 = 1 \geq \alpha \) for all \( \alpha \in \mathbb{P} \).

Thus, for any \( \alpha \in \mathbb{P} \), we can construct the big disjunction by only using \( \land,\lor \), \( [\beta \leq g] \) and \( [\gamma \leq u] \) for all \( \beta,\gamma \in \mathbb{P} \). Since \( [\beta \leq g] \) and \( [\gamma \leq u] \) satisfy all requirements by the induction hypothesis, we only require additionally \( \land \) and \( \lor \).

The case \( f = g \cdot u \) is analogous to the previous case.

For the case \( f = 1 - g \), we have \( [\alpha \leq f] = \text{true} \) if \( \alpha = 0 \) otherwise \( \neg[\delta \leq g] \) where \( \delta = \min \{ \beta \in \text{Val}[g] \mid \beta > 1 - \alpha \} \). Remark that we compute \( \delta \) during the construction of \( [\alpha \leq f] \) and not during the checking of \( (s,h) \models [\alpha \leq f] \). Since \( [\delta \leq g] \) already satisfy all requirements by the induction hypothesis, we thus only additionally require \( \neg \) and \text{true}.

The cases \( f = g \max u, f = g \min u, f = \exists x : g \) and \( f = \forall x : g \) are analogous to the case \( f = [\pi] \cdot g + [\neg \pi] \cdot u \).

The case \( f = g \ast u \) is analogous to the case \( f = p \cdot g + (1 - p) \cdot u \).

The case \( f = [\psi] \longrightarrow g \) is analogous to the case \( f = [\pi] \cdot g + [\neg \pi] \cdot u \).

This concludes the proof.
F Appendix to Section 6.1

Theorem 9. For loop- and allocation-free hpGCL programs $C$ (that only perform pointer operations, no arithmetic and guards of the pure fragment of $A$) and $f_1, f_2 \in QSH$, it is decidable whether the entailment $\text{wlp}[C](f_1) \models f_2$ holds.

Proof of Theorem 9 The proof requires extended quantitative symbolic heaps:

Definition 5. The set $eQSH$ of extended quantitative symbolic heaps is given by the grammar

$$g \rightarrow [\Phi] \mid [B] \cdot g + [\neg B] \cdot g \mid q \cdot g + (1 - q) \cdot g \mid g \star g \mid \exists x: g \mid [x \mapsto (y_1, \ldots, y_k)] \star g .$$

△

Notice that indeed $QSH \subseteq eQSH$.

Lemma 2. For every loop- and allocation-free program $C \in \text{hpGCL}$ without arithmetic and only with guards of the pure fragment of $A$, extended quantitative symbolic heaps are closed under $\text{wlp}[C]$, i.e.,

for all $g \in eQSH$: $\text{wlp}[C](g) \in eQSH$.

In particular, since $QSH \subseteq eQSH$, we have

for all $f \in QSH$: $\text{wlp}[C](f) \in eQSH$.

Proof. Since we not allow arithmetic in expressions, we only have expressions of the form $E = y$. Moreover, $eQSH$ is trivially closed under the substitution $g[x:=y]$ for any $x, y \in \text{Vars}$. Now we prove the lemma by induction on the structure of loop- and allocation-free program $C$ with $k = 1$, however the proof is easy to adapt for any $k$.

For the base case $C = \text{skip}$ we have $\text{wlp}[\text{skip}](g) = g \in eQSH$.

For the base case $C = x := y$ we have

$$\text{wlp}[x := y](g) = g[x:=y] \in eQSH .$$

For the base case $C = \text{free}(y)$ we have

$$\text{wlp}[\text{free}(y)](g) = \exists z: [y \mapsto z] \star g \in eQSH$$

where $z$ is fresh.

For the base case $C = x := <y>$ we have

$$\text{wlp}[x := <y>](g) = \exists z: [x \mapsto z] \star ([x \mapsto y] \star f[x:=y]) \in eQSH$$

where $z$ is fresh.
For the base case $C = <x> := y$ we have

$$\text{wlp}[<x> := y] (g) = \exists z: [x \mapsto z] \star ([x \mapsto y] \longrightarrow f) \in \text{eQSH}$$

where $z$ is fresh.

For all other composite cases we assume for some fixed, but arbitrary loop- and allocation-free programs $C_1, C_2 \in \text{hpGCL}$ without arithmetic and only with guards of the pure fragment of $\mathfrak{A}$ such that for all $g \in \text{eQSH}$: $\text{wlp}[C_1] (g) \in \text{eQSH}$ and for all $u \in \text{eQSH}$: $\text{wlp}[C_2] (u) \in \text{eQSH}$.

For the case $C = \{C_1\} [p] \{C_2\}$ we have

$$\text{wlp}[\{C_1\} [p] \{C_2\}] (g) = p \cdot \text{wlp}[C_1] (g) + (1 - p) \cdot \text{wlp}[C_2] (g) \in \text{eQSH}$$

by the induction hypothesis.

For the case $C = C_1 ; C_2$ we have

$$\text{wlp}[C_1 ; C_2] (g) = \text{wlp}[C_1] (\text{wlp}[C_2] (g)) \in \text{eQSH}$$

by the induction hypothesis.

For the case $C = \text{if} (B) \{C_1\} \text{else} \{C_2\}$ we have either $B = (x = y)$, then

$$\text{wlp}[\text{if} (x = y) \{C_1\} \text{else} \{C_2\}] (g) = [x = y] \cdot \text{wlp}[C_1] (g) + [x \neq y] \cdot \text{wlp}[C_2] (g) \in \text{eQSH}$$

by the induction hypothesis; or $B = (x \neq y)$ which is analogous.

This concludes the proof.

Hence, if $g \models f$ is decidable for $g \in \text{eQSH}$ and $f \in \text{QSH}$, Theorem 9 follows.

**Lemma 3.** For $g \in \text{eQSH}$ and $f \in \text{QSH}$, it is decidable whether $g \models f$ holds.

**Proof.** We employ Lemma 1 to determine two $\text{SL} [\mathfrak{A}]$ fragments $S_1, S_2$ such that $S_1$ is $\text{eQSH}$-admissible and $S_2$ is $\text{QSH}$-admissible. Then, by Theorem 8, decidability of $g \models f$ follows from decidability of $\varphi \models \psi$ for $\varphi \in S_1$ and $\psi \in S_2$. For that, we exploit the equivalence

$$\varphi \models \psi \quad \text{iff} \quad \varphi \land \neg \psi \text{ is unsatisfiable}.$$

The latter is decidable by [19, Theorem 3.3] since $\varphi \land \neg \psi$ is equivalent to a formula of the form $\exists^* \forall^* \vartheta$ with $\vartheta$ quantifier-free and such that no formula $\vartheta_1 \longrightarrow \vartheta_2$ occurring in $\vartheta$ contains a universally quantified variable. The $\text{eQSH}$-admissible $\text{SL} [\mathfrak{A}]$ fragment $S_1$ is given by

$$\varphi \longrightarrow \Phi$$

$$\begin{align*}
\varphi \land \varphi \\
\varphi \lor \varphi \\
\exists x: \varphi \\
\varphi \star \varphi \\
x \mapsto (y_1, \ldots, y_k) \longrightarrow \varphi.
\end{align*}$$
The QSH-admissible SL[∀] fragment $S_2$ is given by

$$
\psi \rightarrow \Phi
$$

| $\psi \land \psi$               |
| $\psi \lor \psi$               |
| $\exists x : \psi$            |
| $\psi \star \psi$            |

**Lemma 6.** Every $\varphi \in S_1$ is equivalent to a formula $\psi = \exists x_1, \ldots, x_n : \emptyset$ for some $n \in \mathbb{N}$ with $\emptyset \in S_1$ quantifier-free.

**Proof.** By induction on $\varphi$. For the base case $\varphi = \Phi$ we choose $n = 0$ and have nothing to show. The cases $\land, \lor, \exists$ are standard. For the remaining cases, we reason as follows: As the induction hypothesis assume that for some arbitrary, but fixed, $\varphi_1, \varphi_2 \in S_1$ there are $\psi_1 = \exists x_1, \ldots, x_n : \emptyset_1$ and $\psi_2 = \exists y_1, \ldots, y_m : \emptyset_2$ with $\emptyset_1, \emptyset_2 \in S_1$ quantifier-free and $\psi_1 \equiv \varphi_1$ and $\psi_2 \equiv \varphi_2$. Furthermore, assume without loss of generality that $x_1, \ldots, x_n$ do not occur in $\emptyset_2$ and that $y_1, \ldots, y_m$ do not occur in $\emptyset_1$.

**The case $\varphi = \varphi_1 \star \varphi_2$.** For every $(s, h) \in \text{States}$, we have

$$(s, h) \models \varphi_1 \star \varphi_2$$

iff there are $h_1, h_2$ with $h_1 \star h_2 = h$ such that

$$(s, h_1) \models \varphi_1 \text{ and } (s, h_2) \models \varphi_2$$

iff there are $v_1, \ldots, v_n, w_1, \ldots, w_m$ such that

$$(s[x_1:=v_1]\ldots[x_n:=v_n], h_1) \models \emptyset_1 \text{ and } (s[y_1:=w_1]\ldots[y_m:=w_m], h_2) \models \emptyset_2$$

(by I.H.)

iff there are $v_1, \ldots, v_n, w_1, \ldots, w_m$ such that

$$(s[x_1:=v_1]\ldots[x_n:=v_n], h_1) \models \emptyset_1 \text{ and } (s[y_1:=w_1]\ldots[y_m:=w_m], h_2) \models \emptyset_2$$

iff there are $v_1, \ldots, v_n, w_1, \ldots, w_m$ such that

$$(s[x_1:=v_1]\ldots[x_n:=v_n], h_1) \models \emptyset_1 \text{ and } (s[y_1:=w_1]\ldots[y_m:=w_m], h_2) \models \emptyset_2$$

(variables do not overlap)

iff $$(s, h) \models \exists v_1, \ldots, w_n, w_1, \ldots, w_m : \emptyset_1 \star \emptyset_2.$$
have

\[(s, h) \models x \mapsto (z_1, \ldots, z_k) \longrightarrow \varphi_1\]

iff \(\text{true}\)

iff \((s, h) \models \exists x_1, \ldots, x_n : x \mapsto (z_1, \ldots, z_k) \longrightarrow \vartheta_1\).

(variables do not overlap)

If \(s(x) \not\in \text{dom}(h)\), we have

\[(s, h) \models x \mapsto (z_1, \ldots, z_k) \longrightarrow \varphi_1\]

iff \((s, h \ast s(x) \mapsto (s(z_1), \ldots, s(z_k))) \models \varphi_1\)

iff \((s, h \ast s(x) \mapsto (s(z_1), \ldots, s(z_k))) \models \exists x_1, \ldots, x_n : \vartheta_1\) \hspace{1cm} (by I.H.)

iff there are \(v_1, \ldots, v_n\) such that

\((s [x_1 := v_1] \ldots [x_n := v_n], h \ast s(x) \mapsto (s(z_1), \ldots, s(z_k))) \models \vartheta_1\)

(variables do not overlap)

iff there are \(v_1, \ldots, v_n\) such that

\((s [x_1 := v_1] \ldots [x_n := v_n], h) \models x \mapsto (z_1, \ldots, z_k) \rightarrow \vartheta_1\)

(variables do not overlap)

iff \((s, h) \models \exists x_1, \ldots, x_n : x \mapsto (z_1, \ldots, z_k) \rightarrow \vartheta_1\).

This completes the proof.

**Lemma 7.** Every \(\varphi \in S_2\) is equivalent to a formula \(\psi = \exists x_1, \ldots, x_n : \vartheta\) for some \(n \in \mathbb{N}\) with \(\vartheta \in S_2\) quantifier-free.

**Proof.** Analogous to the proof of Lemma 6.

Now let \(\varphi \in eQSH\) and \(\psi \in QSH\). By Lemma 6 and Lemma 7 there are \(\varphi' = \exists x_1, \ldots, x_n : \vartheta_1\) and \(\psi' = \exists y_1, \ldots, y_m : \vartheta_2\) with \(\vartheta_1 \in S_1, \vartheta_2 \in S_2\) quantifier-free and \(\varphi \equiv \varphi'\) and \(\psi \equiv \psi'\). Notice that \(\vartheta_2\) does not contain \(\longrightarrow\). We may without loss of generality assume that \(x_1, \ldots, x_n\) do not occur in \(\vartheta_2\) and that \(y_1, \ldots, y_m\) do not occur in \(\vartheta_1\). Hence, we get

\(\varphi \models \psi\)

iff \(\varphi \land \neg \psi\) is unsatisfiable

iff \(\exists x_1, \ldots, x_n : \vartheta_1 \land \neg (\exists y_1, \ldots, y_m : \vartheta_2)\) is unsatisfiable (by above reasoning)

iff \((\exists x_1, \ldots, x_n : \vartheta_1) \land (\forall y_1, \ldots, y_m : \neg \vartheta_2)\) is unsatisfiable

iff \((\exists x_1, \ldots, x_n : \forall y_1, \ldots, y_m : \vartheta_1 \land \neg \vartheta_2\) is unsatisfiable , \hspace{1cm} (standard prenexing)

since \(\vartheta_1 \land \neg \vartheta_2\) is quantifier free and since none of \(y_1, \ldots, y_m\) occur in an instance of \(\longrightarrow\) occurring in \(\vartheta_1 \land \neg \vartheta_2\), the claim follows.