Solving multi-loop Feynman diagrams using light-front coordinates

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We determine the numerical values of scalar multi-loop two-vertex Feynman diagrams, the generalized sunset diagrams, by integrating all but the longitudinal momenta analytically. For the longitudinal momenta we introduce one collective coordinate, which allows us to determine the numerical value of the diagram efficiently and to an arbitrary accuracy. The imaginary part and the threshold behavior of the diagram are also handled within this framework.

I. INTRODUCTION

Multi-loop Feynman diagrams pose a serious challenge, especially in the massive case where multiple scales arise. Apart from the successes of high-order expansions in $\alpha$ in QED, which provide a stringent test of quantum field theory, Feynman diagrams are also the way in which we understand the most of field theory, since it allows us to quantify it. They form the basis for our understanding of many phenomena, such as asymptotic freedom and gauge invariance. Therefore, we should strive to refine our handling of Feynman diagrams and to extend its context and its applications.

In the late 40’s, the move from time-ordered perturbation theory towards covariant perturbation theory brought about a revolution in field theory. It created a handle on the calculations and people were able the control the divergences. However, despite the successes for scattering experiments, the method failed to deliver for bound-states calculations. Therefore, there has been a constant move ”backwards” to quasi-potential, and time-ordered, formulations, as we can formulate a bound state only in a single time frame, not with the relative time as it exists in the covariant formulation. However, often it is hard to follow the route back from covariant to time-ordered perturbation theory, and then to apply it to an intrinsic non-perturbative problem, such as a bound-state problem.

One distinct method for describing bound-states in field theory is discrete light-cone quantization, where as the time direction a light-like direction is chosen. This has certain advantages, extensively discussed in the literature. However, also here renormalization forms a serious problem, often dealt with rather callously. Although the renormalization for perturbative expansions is reasonably under control, its extension to non-perturbative calculations is far from desired.

In this paper I will tackle a class of simple $n$-loop Feynman diagrams, determine the finite part and show that the light-front approach is particularly useful for that. I will also discuss how to look upon the large set of counterterms in this highly divergent case. These diagrams have been studied extensively in the recent years. However, the simplicity of this approach in Minkowski space is striking. There are no special functions needed, and eventually it depends on the introduction of the collective coordinate $\beta$, analogous to the radial coordinate $r^2$ in Euclidean space. In this case, specifically, the coordinate interpolates smoothly from the threshold value at the center of the kinematical to the edges of the kinematical domain. Eventually, such simple collective coordinates for many-particle systems might extend the applicability of Hamiltonian light-front field theory, as, generally, the problem of the Hamiltonian approach is the control on number of the variables.

II. THE SUNSET DIAGRAM

We consider the $n$-loop Feynman diagram, which consists of $n+1$ lines between two vertices:

$$ I_n = \frac{1}{(2\pi)^n} \int \frac{d^4 k_1 \cdots d^4 k_n}{(k_1^2 - m_1^2)(k_2^2 - m_2^2) \cdots (k_n^2 - m_n^2)((p - k_1 - k_2 - \cdots - k_n)^2 - m_{n+1}^2)}. $$

(2.1)
This diagram the generalized sunset diagram. Lines can be added or removed (see Fig. [I]). The Feynman diagram is covariant and therefore it is only a function of $p^2$ and the masses $m_1, \ldots, m_{n+1}$. We solve it in the frame where $p_\perp = (p^1, p^2) = 0$. We introduce light-front coordinates: $k^\pm = \frac{1}{\sqrt{2}}(k^0 \pm k^3)$, and $x_i = k^+_i / p^+$. The transverse momenta $k^1$ and $k^2$ are unaltered. After the residue integration over the light-front energies $k^-_i$, we find:

$$I_n = \int_\Delta dx_1 \cdots dx_n \int d^2k_1 \cdots d^2k_n \frac{1}{2^n \cdot 1 \cdot \ldots \cdot 1_n} \frac{1}{(1 - \sum x_i)(2p^+ p^- - \beta^{-1} - \alpha)} , \quad (2.2)$$

where

$$\beta^{-1} = \frac{m_1^2}{x_1} + \cdots + \frac{m_n^2}{x_n} + \frac{m_{n+1}^2}{1 - \sum x_i} , \quad (2.3)$$
$$\alpha = \frac{k_1^2}{x_1} + \cdots + \frac{k_n^2}{x_n} + \frac{(\sum k^2_i)^2}{1 - \sum x_i} . \quad (2.4)$$

The domain $\Delta$ is given by $x_i \geq 0$ and $\sum x_i \leq 1$. This integral is the corresponding light-front diagram, equivalent to the Feynman diagram. If we translate the transverse momenta successively starting from $k_{\perp n}$:

$$k_{\perp i} = l_{\perp i} - \left( \sum_{j=1}^{i} k^\perp_j \right) \left( \frac{1}{x_i} + \frac{1}{1 - \sum_{j=1}^{i} x_j} \right)^{-1} \left( \frac{1}{1 - \sum_{j=1}^{i} x_j} \right) , \quad (2.5)$$

we find that the $\alpha$ reduces to a pure quadratic form in $l_{\perp i}$:

$$\alpha = \sum_{i=1}^{n} l^2_{\perp i} \left( \frac{1}{x_i} + \frac{1}{1 - \sum_{j=1}^{i} x_j} \right) . \quad (2.6)$$

The integral is divergent, and requires counterterms $c_0, c_1 p^2, \ldots, c_{n-1} (p^2)^{n-1}$. We do so by subtracting the $(n-1)$-th order Taylor expansion in $p^-$ around $p^2 = 0$, implemented through the multiplication of the integral with the proper moments:

$$I_n = \int_\Delta I \to \mathcal{I}^\text{reg}_n = \int_\Delta I J^n , \quad (2.7)$$

where

$$J = \frac{p^2}{\beta^{-1} + \alpha} . \quad (2.8)$$

which will lead to

$$\mathcal{I}^\text{reg}_n = \int_\Delta dx_1 \cdots dx_n d^2l_{\perp 1} \cdots d^2l_{\perp n} (p^2)^n \quad (2.9)$$

If we scale the transverse momenta accordingly and perform the angular integrations we obtain the integral:

$$\mathcal{I}^\text{reg}_m = \pi^n \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \int_\Delta dx_1 \cdots dx_n \frac{d^2l_{\perp 1} \cdots d^2l_{\perp n} (p^2)^n}{2^{n+1}(p^2 - \sum z_i - \beta^{-1} - \alpha)(\beta^{-1} + \alpha)^n} , \quad (2.10)$$

where

$$z_i = l^2_{\perp i} \left( \frac{1}{x_i} + \frac{1}{1 - \sum_{j=1}^{i} x_j} \right) . \quad (2.11)$$

In order to integrate over $z_i$ we express the integrand as a series in $p^2$. For each separate term we can integrate over all $z_i$, which yields:

$$\mathcal{I}^\text{reg}_m = -\frac{\pi^n}{2^{n+1}} \int_\Delta d^n x (p^2)^{n-1} \sum_{i=0}^{\infty} \frac{1}{(n + i) \cdots (i + 1)} (p^2 \beta)^{i+1} . \quad (2.12)$$
We can write the series as an analytical function:

\[
\mathcal{T}_{n}^{\text{reg}} = \int_{\Delta} d^{n}x \frac{(-\pi)^{n}}{2^{n+1}(n-1)!\beta^{n-1}}(1 - \mathcal{T}_{n-1})(1 - p^{2}\beta)^{n-1}\ln \left[1 - p^{2}\beta - i\epsilon\right],
\]

where \(\mathcal{T}_{n-1}\) stands for the \((n-1)\)-th order Taylor expansion around \(p^{2} = 0\). The Taylor expansion yields the following polynomial:

\[
\mathcal{T}_{z(n-1)}(1 - z)^{n-1}\ln[1 - z] = \sum_{k=1}^{n-1} \left( \sum_{j=0}^{k} (-1)^{j}n! \frac{(-1)^{j}n!}{(n-j)!j!(k-j+1)} \right) z^{k}.
\]

The imaginary part follows directly from the natural logarithm of Eq. (2.13):

\[
\lim_{\epsilon \to 0} \ln(-x - i\epsilon) = \ln|x| - i\pi \theta(x),
\]

which lead to a finite amplitude, unaffected by the renormalization procedure which reminiscence appears in Eq. (2.13) in the form of the subtracted Taylor expansion.

### III. LONGITUDINAL INTEGRATION

After the integration over the \(k^{-}\)'s and the \(k_{\perp}^{-}\)'s, we are left with an integration of the longitudinal momentum fractions \(x_{i}\), which cannot be performed analytically. However, note that the integrand of Eq. (2.13) depends only on one particular combination of the longitudinal momenta, namely \(\beta\). This \(\beta\) is a smooth function of the longitudinal momenta \(x_{i}\), and ranges between:

\[
0 \leq \beta \leq \left( \sum_{i=1}^{n+1} m_{i} \right)^{-2} = b,
\]

where for \(\beta = b\) the longitudinal momentum fractions \(x_{i}\) equal \(m_{i}\sqrt{b}\). Therefore the \(n\)-dimensional integral over \(\Delta\) reduces to the determination of the integration measure \(\mu\) for the integration over \(\beta\):

\[
\int_{\Delta} d^{n}xf(\beta) = \int_{0}^{b} \mu(\beta) d\beta f(\beta).
\]

The volume of the domain \(\Delta\) is \(\Gamma(n+1)^{-1}\). Once this measure is determined, it can be used for all values of \(p^{2}\). The measure \(\mu\) can be determined via several means, for example, with Monte Carlo integration. The threshold behavior of the diagram is dominated by the the values of \(\beta\) close to \(b\). For this purpose we can make an analytical expansion of \(\mu\) around \(b\). We find that:

\[
\mu(\beta) \sim \frac{1}{2} \Omega_{n} b^{-\frac{n+1}{2}} \left[ \prod_{i=1}^{n+1} \sqrt{m_{i}} \right] (b - \beta)^{-\frac{n+2}{2}},
\]

where \(\Omega_{n}\) is the surface area of the unit sphere in \(n\) dimensions. Note that as some of the masses tend to zero, the exponent in the measure will be larger than \((n-2)/2\). The addition of a zero mass particle, \(m_{i} = 0\), leads to a flat direction in \(\beta\) with respect to the longitudinal momentum fraction \(x_{i}\) at the threshold. Therefore the harmonic approximation breaks down. However, \(\beta\) and the measure \(\mu\) are well-defined as long as at least one particle is massive.

#### A. The integration measure

Apart from series expansion and Monte-Carlo integration mentioned above, we can determine the measure iteratively. Given the integration measure \(\mu_{n}(\beta_{n})\) for \(n\) momenta, the integration measure \(\mu_{n+1}(\beta_{n+1})\) for \(n + 1\) variables can be expressed as

\[
\int_{0}^{b_{n+1}} \mu_{n+1}(\beta_{n+1}) d\beta_{n+1} = \int_{0}^{b_{n}} \int_{0}^{1} \mu_{n}(\beta_{n}) d\beta_{n} y^{n} dy,
\]
where
\[
\beta_{n+1} = \left( \frac{m^2}{1 - y} + \frac{1}{\beta_n y} \right)^{-1},
\]
with \(m\) the mass of the added particle, with longitudinal momentum fraction \(1 - y\). The other longitudinal momenta are scaled by a factor \(y\) such that the total longitudinal momentum remains 1.

IV. RENORMALIZATION

In this paper we described a method to find the finite part of any diagram of the type of Fig. 1. The divergences are removed using the Taylor expansion in the external momentum. The counterterms, \(c_0, c_1p^2\) till \(c_n(p^2)^{n-1}\), for the sunset diagram can be expressed as divergent integrals depending on the masses \(m_1, \ldots, m_{n+1}:
\[
c_k = \int dx_1 \cdots dx_n \int \frac{d^2k_{1\perp} \cdots d^2k_{n\perp}}{2^{n+1}x_1 \cdots x_n (1 - \sum x_i) (\alpha + \beta^{-1})^{k+1}},
\]
which defines their relation with other renormalization schemes. The other renormalization schemes will find that the counterterms in Eq. (4.1) equal an infinite constant, to be removed, and a finite part, a function of the masses, which is the finite renormalization. The use of Taylor expansion became disfavor, because of two complications. Firstly, in the case of multiple external momenta, which is not clear which combination of external momenta should serve as variable in the Taylor expansion; different choices will lead to different results, and do not automatically guarantee locality. Secondly, in the case of gauge theories, an extremely consistent scheme, which treats a whole class of integrals in the same way, is required such that the gauge invariance is preserved. Dimensional regularization has for a long time been the only scheme satisfying this consistency, which preserved algebraic relations existing among different integrands of Feynman integrals. For example, the fermion-loop correction to the gauge propagator \(\Pi^{\mu\nu}\) must be transverse, therefore the two parts to \(\Pi^{\mu\nu}\), namely \(g^{\mu\nu}\Pi\) and \(p^\mu p^\nu\Pi\) must be handled in the same way such that \(p^2\Pi = -\Pi\), which is difficult problem for an arbitrary regularization scheme, since the two terms have different degrees of divergence.

However, for an Hamiltonian approach, such as light-front field theory, the renormalization at the level of the integrand is required, if one wants to carry the renormalization procedure over from the covariant renormalization. Note that a straightforward cut-off procedure breaks covariance, since it cannot be applied to the energy part of the covariant integration. The integration of the energies and the regularization should be interchangeable, such that locality is guaranteed. So, although it leads to further complications which requires careful analysis, the Taylor expansion is the way forward for the Hamiltonian approach.

The natural choice of renormalization for a Hamiltonian approach, is to make the self-energy contributions vanish as all the particles are on-shell. However, this is not consistent with the covariant, local and therefore true, renormalization. If the sum energy is the sum of the on-shell energies, it does not mean that the energy is shared out evenly; a large part of the amplitude might arise from the case that both particles are off-shell in different directions. Therefore it is essential to treat the subtractions as pure constants.

Even more, although we can generate finite terms in light-front perturbation theory, for a proper light-front approach we should take the procedure one step further and determine the corresponding finite wave function. However, this is far beyond the scope of this paper.

V. RESULTS

Central to this approach is the actual shape of the measure \(\mu(\beta)\). For particles of equal mass, we find that the measure is most spread over the whole range of \(\beta\). As the masses start to deviate the measure peaks more and more at low values of \(\beta\). However, the measure stays finite, even for massless particles. In Fig. 2 we show two scaled, normalized set of measures, one for equal mass particles, and one for particles with increasing masses. Note that the increasing masses peak more at lower values of \(\beta\), due to the leading contributions from the heavy particles carrying large momentum fractions. In Fig. 3 we compare the measures for two massive particles and a number of particles with equal, but small, or vanishing, masses.

For the inspection of the amplitudes, I fitted the measures with a five parameter function, which fits the measure with an accuracy within a few percent. The accuracy for the massless case is higher than for the massive case, in the former case it is below a percent. The function depend on the parameters \(\gamma_1, \gamma_2, \cdots, \gamma_5\):
\[ \tilde{\mu}(\tilde{\beta}) = \gamma_1 (1 - \tilde{\beta})^{\gamma_2} + \gamma_3 (1 - \tilde{\beta})^{\gamma_5} \tilde{\beta}^{\gamma_4}, \]  

(5.1)

where \( \tilde{\beta} = \beta/b \) and \( \tilde{\mu} = \Gamma(n+1)\mu/b \), such that the axis and the measure are normalized to unity. The fitted parameters for the two extreme cases, one case with all masses equal, and one case with the first two masses 1.0, and all the other masses zero are given respectively in Table I and Table II.

VI. CONCLUSIONS

I have derived a straightforward, and largely analytical, way to determine the finite part of the sunset diagram. Both the threshold behavior and the full amplitude can be determined accurately. I removed the divergent parts by subtracting the corresponding Taylor expansion. The mass dependence of the diagram, for the scaled external momentum \( bp^2 \), is only weak over the whole range of masses, and this dependence appears solely in the integration measure \( \mu(\beta) \).

A careful analysis of the Feynman parameterization \( \beta \) could also yield a similar variable \( \beta \). However, it requires one to work in Euclidean space, where the imaginary part does not come for free. Also the poles in dimension space, which are the different subdivergencies of the integral, are transferred to singularities in the parameter space, which renders dimensional regularization invalid. Therefore the removal of the lower-order Taylor expansion is essential. It seems possible to extend the light-front approach to more complicated diagrams, which could contain two and more light-front intermediate states. This requires the introduction of more \( \beta \) variables. The determination of the measures stays essentially the same. Eventually, the light-front approach might be more convenient for the calculation of multi-loop diagrams, as it sees such a diagram as a transition via collection of successive intermediate states, which can all be handled separately, and do not grow as wildly as the number of the subgraphs of a complicated covariant multi-loop Feynman diagram.

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FIG. 1. The generalized sunset diagram

FIG. 2. Two sets of integration measures, all normalized to unity, left with one to six loops, where all the masses are equal, and right, with one to six loops including heavier and heavier particles: \{1.0, 1.0\}, \{1.0, 1.0, 2.0\}, \{1.0, 1.0, 2.0, 3.0\}, \ldots, \{1.0, 1.0, 2.0, \ldots, 6.0\}. The variable \(\beta\) is scaled such that it runs from zero to one.
FIG. 3. Two sets of integration measures, all normalized to unity, left with one to six loops, where the first two masses are 1.0, and the other masses are small (0.01) for the solid line, or zero, for the dashed line. Note the rapid decline at $\beta = 1$.

| $m$          | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ |
|--------------|-------------|-------------|-------------|-------------|-------------|
| 1.0, 1.0, 1.0 | .661422     | 1.01707     | 1.23111     | .815085     | .0064803    |
| (1.0)$^4$    | .746938     | .476909     | 1.66776     | .814929     | .526326     |
| (1.0)$^5$    | .817047     | 1.08804     | 3.45241     | .950327     | .982281     |
| (1.0)$^6$    | .874748     | 1.55609     | 5.90012     | 1.06571     | 1.44554     |
| (1.0)$^7$    | .921155     | 1.94665     | 9.09230     | 1.16310     | 1.88020     |

| $m$          | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ |
|--------------|-------------|-------------|-------------|-------------|-------------|
| 1.0, 1.0, 0.0 | .988223     | .538897     | 1.07238     | .706459     | .490497     |
| (1.0)$^2$, (0.0)$^2$ | 1.48154     | 1.33267     | 2.60100     | .706458     | 1.66423     |
| (1.0)$^2$, (0.0)$^3$ | 1.97032     | 2.19627     | 4.63872     | .702103     | 2.75778     |
| (1.0)$^2$, (0.0)$^4$ | 2.45665     | 3.05772     | 7.09066     | .698004     | 3.84503     |
| (1.0)$^2$, (0.0)$^5$ | 2.94061     | 3.92001     | 9.91789     | .694266     | 4.92679     |