TWISTED LOGARITHMIC MODULES OF FREE FIELD ALGEBRAS

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Abstract. Given a non-semisimple automorphism \( \varphi \) of a vertex algebra \( V \), the fields in a \( \varphi \)-twisted \( V \)-module involve the logarithm of the formal variable, and the action of the Virasoro operator \( L_0 \) on such module is not semisimple. We construct examples of such modules and realize them explicitly as Fock spaces when \( V \) is generated by free fields. Specifically, we consider the cases of symplectic fermions (odd superbosons), free fermions, and \( \beta \gamma \)-system (even superfermions). In each case, we determine the action of the Virasoro algebra.

1. Introduction

The notion of a vertex algebra introduced by Borcherds [9] provides a rigorous algebraic description of two-dimensional chiral conformal field theory [8,12,26], and is a powerful tool for studying representations of infinite-dimensional Lie algebras [31,35]. We will assume that the reader is familiar with the theory of vertex algebras as presented in [32] (see also [23,24,35,44] for other sources). Given an automorphism \( \varphi \) of a vertex algebra \( V \), one considers the so-called \( \varphi \)-twisted \( V \)-modules [5,14,21,24], which are useful for constructing modules of the orbifold subalgebra consisting of elements fixed by \( \varphi \) (see e.g. [18,15,37]). More recently, the notion of a \( \varphi \)-twisted module was generalized to the case when \( \varphi \) is not semisimple [4,28]. This generalization was motivated by logarithmic conformal field theory (see e.g. [3,11,41]) and applications to Gromov–Witten theory (cf. [6,7,16,46]). The main feature of such modules is that the twisted fields involve the logarithm of the formal variable; for this reason we also call them twisted logarithmic modules.

More precisely, let \( z \) and \( \zeta \) be two independent formal variables. If we think of \( \zeta \) as \( \log z \), then the derivatives with respect to \( z \) and \( \zeta \) will become

\[
D_z = \partial_z + z^{-1} \partial_\zeta, \quad D_\zeta = z \partial_z + \partial_\zeta.
\]
For a vector space $W$ over $\mathbb{C}$, a \textit{logarithmic (quantum) field} on $W$ is a linear map from $W$ to the space of formal series of the form
\[
\sum_{m \in S} \sum_{i=0}^{\infty} w_{i,m}(\zeta) z^{i+m}, \quad w_{i,m}(\zeta) \in W[\zeta],
\]
for some finite subset $S$ of $\mathbb{C}$. The space of all logarithmic fields is denoted $\text{LFie}(W)$. Given a vertex algebra $V$ and an automorphism $\varphi$ of $V$, a $\varphi$-\textit{twisted} $V$-\textit{module} is a vector space $W$, equipped with a linear map $Y: V \rightarrow \text{LFie}(W)$ satisfying certain axioms (see [4] for full details).

The fields $Y(a)$ are usually written as $Y(a, z)$ for $a \in V$ (even though they also depend on $\zeta$). One of the axioms is the $\varphi$-\textit{equivariance}
\[
Y(\varphi a, z) = e^{2\pi i D(\zeta)} Y(a, z), \quad a \in V.
\]
As in [4], we will assume that $\varphi = \sigma e^{-2\pi i N}$, where $\sigma \in \text{Aut}(V)$, $N \in \text{Der}(V)$, $\sigma$ and $N$ commute, $\sigma$ is semisimple, and $N$ is locally nilpotent (i.e., nilpotent on every $a \in V$). We denote the eigenspaces of $\sigma$ by
\[
V_{\alpha} = \{ a \in V | \sigma a = e^{-2\pi i \alpha} a \}, \quad \alpha \in \mathbb{C}/\mathbb{Z}.
\]
The $\varphi$-equivariance implies that
\[
X(a, z) = Y(e^{\zeta N} a, z) = Y(a, z)|_{\zeta=0}
\]
is independent of $\zeta$, and the exponents of $z$ in $X(a, z)$ belong to $-\alpha$ for $a \in V_{\alpha}$. For $m \in \alpha$, the $(m+N)$-\textit{th mode} of $a \in V_{\alpha}$ is defined as
\[
a_{(m+N)} = \text{Res}_z z^m X(a, z),
\]
where, as usual, $\text{Res}_z$ denotes the coefficient of $z^{-1}$. Then
\[
Y(a, z) = X(e^{-\zeta N} a, z) = \sum_{m \in \alpha} (z^{-m-N-1} a)_{(m+N)},
\]
where we use the notation $z^{-N} = e^{-\zeta N}$. Notice that $Y(a, z)$ is a polynomial of $\zeta$, since $N$ is nilpotent on $a$. The paper [4] develops the theory of twisted logarithmic modules and, in particular, proves a \textit{Borcherds identity} and \textit{commutator formula} for the modes (1.3). It contains examples of modules in the cases when $V$ is an affine vertex algebra or a Heisenberg vertex algebra. The latter is also known as the \textit{free boson} algebra (see e.g. [32]).

In the present paper, we extend the results of [4] to the case of \textit{symplectic fermions}, which are odd super-analogs of the free bosons. Historically, the symplectic fermions provided the first example of a logarithmic conformal field theory, due to Kausch [41,42]. An orbifold of the symplectic fermion algebra $SF$ under an automorphism of order 2 has the important properties of being $C_2$-cofinite but not rational [1]. More recently, the subalgebras of $SF$ known as the triplet and singlet algebras have generated considerable interest (see [3,43]). Other orbifolds of $SF$ give rise to interesting $W$-algebras [10].

We also consider the examples of \textit{free superfermions} (cf. [32]), which include in particular the \textit{free fermions} and the symplectic bosons (also known
as the bosonic ghost system or \(\beta\gamma\)-system). The \(\beta\gamma\)-system provides another interesting model of logarithmic conformal field theory \[47\]. Many important algebras have free-field realizations by superfermions. In particular, these include: affine Kac–Moody algebras \[17,20,22,33,48\], affine Lie superalgebras \[35\], toroidal Lie algebras \[29,30\], \(W\)-algebras \[18,19,36\], superconformal algebras \[36,39\], the \(W_{1+\infty}\)-algebra and its subalgebras \[34,40,45\]. If \(V\) is one of the free-field algebras, \(\varphi \in \text{Aut}(V)\), and \(A \subset V\) is a subalgebra such that \(\varphi(A) \subset A\), then any \(\varphi\)-twisted \(V\)-module gives rise to a \(\varphi\)-twisted \(A\)-module by restriction. Moreover, such \(A\)-modules are untwisted if \(\varphi\) acts as the identity on \(A\) (i.e., if \(A \subset V^\varphi\)). Thus, we expect the twisted modules constructed in this paper to be useful for studying modules over subalgebras of free-field algebras.

The paper is organized as follows. In Section 2, following \[32\], we review the definition of free superbosons, which include the free bosons in the even case and the symplectic fermions \(SF\) in the odd case. Then we outline the construction of \(\varphi\)-twisted \(SF\)-modules as modules over \(\varphi\)-twisted affine Lie superalgebras, similarly to the even case considered in \[4\]. In order to make the construction explicit, we need to solve a linear algebra problem, which we do in Section 3. Using that, in Section 4 we realize explicitly all highest-weight \(\varphi\)-twisted \(SF\)-modules as certain Fock spaces and we determine the action of the Virasoro algebra on them. In particular, we confirm that the action of the Virasoro operator \(L_0\) is not semisimple (cf. \[3,11,41\]). In Section 5 we study twisted logarithmic modules of free superfermions and determine the action of the Virasoro algebra. In the final Sections 6 and 7 we realize these modules explicitly as Fock spaces in the odd case (free fermions) and the even case (\(\beta\gamma\)-system), respectively.

### 2. Free superbosons

In this section, we review the definition of free superbosons, following \[32\]. Let \(\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1\) be an abelian Lie superalgebra with \(\text{dim } \mathfrak{h} = d < \infty\). Let \((\cdot|\cdot)\) be a non-degenerate even supersymmetric bilinear form on \(\mathfrak{h}\). Thus, 
\[
(b|a) = (-1)^{p(a)p(b)}(a|b)\quad\text{and}\quad(\mathfrak{h}_0|\mathfrak{h}_1) = 0,
\]
where \(p(a)\) denotes the parity of \(a\).

Consider the Lie superalgebra given by the affinization \(\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus C\mathbf{K}\) with commutation relations
\[
[at^m, bt^n] = m\delta_{m,-n}(a|b)K, \quad [\hat{\mathfrak{h}}, K] = 0 \quad (m, n \in \mathbb{Z}),
\]
and \(p(at^m) = p(a)\), \(p(K) = 0\). We will use the notation \(a_{(m)} = at^m\). The free superbosons
\[
a(z) = \sum_{m \in \mathbb{Z}} a_{(m)}z^{-m-1}, \quad a \in \mathfrak{h},
\]
have OPEs given by
\[
a(z)b(w) \sim \frac{(a|b)K}{(z-w)^2}.
\]
The (generalized) Verma module

\[ V = \text{Ind}_{\mathfrak{h}[t] \oplus \mathbb{C}K}^{\hat{\mathfrak{h}}[t]} \mathbb{C} \]

is constructed by letting \( \mathfrak{h}[t] \) act trivially on \( \mathbb{C} \) and \( K \) act as 1. Then \( V \) has the structure of a vertex algebra called the free superboson algebra and denoted \( B^1(\mathfrak{h}) \). The commutator (2.1) is equivalent to the following \( n \)-th products:

\[ a_{(0)} b = 0, \quad a_{(1)} b = (a|b)1, \quad a_{(j)} b = 0 \quad (j \geq 2) \]

for \( a, b \in \mathfrak{h} \), where 1 is the vacuum vector in \( B^1(\mathfrak{h}) \).

Let \( \varphi \) be an even automorphism of \( \mathfrak{h} \) such that \( \langle \cdot | \cdot \rangle \) is \( \varphi \)-invariant. As before, we write \( \varphi = \sigma e^{-2\pi i N} \), and denote the eigenspaces of \( \sigma \) by

\[ \mathfrak{h}_\alpha = \{ a \in \mathfrak{h} | \sigma a = e^{-2\pi i \alpha} a \}, \quad \alpha \in \mathbb{C}/\mathbb{Z}. \]

**Definition 2.1** (cf. [4]). The \( \varphi \)-twisted affinization \( \hat{\mathfrak{h}}_\varphi \) is the Lie superalgebra spanned by an even central element \( K \) and elements \( a_{(m+N)} = at^m a_{(m+N)} \) (\( a \in \mathfrak{h}_\alpha, m \in \alpha \)) with parity \( p(a_{(m+N)}) = p(a) \), and the Lie superbracket

\[ [a_{(m+N)}, b_{(n+N)}] = \delta_{m,-n} ((m+N)a|b)K \]

for \( a \in \mathfrak{h}_\alpha, b \in \mathfrak{h}_\beta, m \in \alpha, n \in \beta \).

An \( \mathfrak{h}_\varphi \)-module \( W \) is called restricted if for every \( a \in \mathfrak{h}_\alpha, m \in \alpha, v \in W \), there is an integer \( L \) such that \( (at^{m+i})v = 0 \) for all \( i \in \mathbb{Z}, i \geq L \). We note that every highest weight \( \mathfrak{h}_\varphi \)-module is restricted (see [31]). The automorphism \( \varphi \) naturally induces automorphisms of \( \hat{\mathfrak{h}} \) and \( B^1(\mathfrak{h}) \), which we will denote again by \( \varphi \). Then every \( \varphi \)-twisted \( B^1(\mathfrak{h}) \)-module is a restricted \( \hat{\mathfrak{h}}_\varphi \)-module and, conversely, every restricted \( \hat{\mathfrak{h}}_\varphi \)-module uniquely extends to a \( \varphi \)-twisted \( B^1(\mathfrak{h}) \)-module [4, Theorem 6.3].

We split \( \mathbb{C} \) as a disjoint union of subsets \( \mathbb{C}^+, \mathbb{C}^- = -\mathbb{C}^+ \) and \( \{0\} \) where

\[ \mathbb{C}^+ = \{ \gamma \in \mathbb{C} | \text{Re} \, \gamma > 0 \} \cup \{ \gamma \in \mathbb{C} | \text{Re} \, \gamma = 0, \text{Im} \, \gamma > 0 \}. \]

Then the \( \varphi \)-twisted affinization \( \hat{\mathfrak{h}}_\varphi \) has a triangular decomposition

\[ \hat{\mathfrak{h}}_\varphi = \hat{\mathfrak{h}}^-_\varphi \oplus \hat{\mathfrak{h}}^0_\varphi \oplus \hat{\mathfrak{h}}^+_\varphi, \]

where

\[ \hat{\mathfrak{h}}^\pm_\varphi = \text{span}\{ at^m | a \in \mathfrak{h}_\alpha, \alpha \in \mathbb{C}/\mathbb{Z}, m \in \alpha \cap \mathbb{C}^\pm \} \]

and

\[ \hat{\mathfrak{h}}^0_\varphi = \text{span}\{ at^0 | a \in \mathfrak{h}_0 \} \oplus \mathbb{C}K. \]

Starting from an \( \hat{\mathfrak{h}}^\varphi_\varphi \)-module \( R \) with \( K = I \), the (generalized) Verma module is defined by

\[ M_\varphi(R) = \text{Ind}_{\hat{\mathfrak{h}}^-_\varphi \oplus \hat{\mathfrak{h}}^0_\varphi}^{\hat{\mathfrak{h}}^+_\varphi} R, \]

where \( \hat{\mathfrak{h}}^+_\varphi \) acts trivially on \( R \). These are \( \varphi \)-twisted \( B^1(\mathfrak{h}) \)-modules. In order to describe them more explicitly, first we will obtain canonical forms for all automorphisms \( \varphi \) of \( \mathfrak{h} \) preserving \( \langle \cdot | \cdot \rangle \).
3. Automorphisms preserving a bilinear form

In this section, we study automorphisms $\varphi$ of a finite-dimensional vector space $V$ preserving a nondegenerate bilinear form $(\cdot|\cdot)$. Recall that $\varphi$ can be written uniquely as $\varphi = \sigma e^{-2\pi i N}$ where $\sigma$ is invertible and semisimple, $N$ is nilpotent, and $\sigma$ and $N$ commute. The $\varphi$-invariance of $(\cdot|\cdot)$ is equivalent to:

$$ (\sigma a | \sigma b) = (a | b), \quad (N a | b) + (a | N b) = 0 $$

for all $a, b \in V$. We will assume that the bilinear form $(\cdot|\cdot)$ is either symmetric or skew-symmetric, and will consider these cases separately.

3.1. The symmetric case. The case when $(\cdot|\cdot)$ is symmetric was investigated previously in [4] Section 6. The classification of all $\sigma$ and $N$ satisfying (3.1) can be deduced from the well-known description of the canonical Jordan forms of orthogonal and skew-symmetric matrices over $\mathbb{C}$ (see [25,27]). We include it here for completeness. In the following examples, $V$ is a vector space with a basis $\{v_1, \ldots, v_d\}$ such that $(v_i | v_j) = \delta_{i+j, d+1}$ for all $i, j$, and $\lambda = e^{-2\pi i \alpha_0}$ for some $\alpha_0 \in \mathbb{C}$ such that $-1 < \text{Re} \alpha_0 \leq 0$.

Example 3.1 ($d = 2\ell$).

$$ \sigma v_i = \begin{cases} \lambda v_i, & 1 \leq i \leq \ell, \\ \lambda^{-1} v_i, & \ell + 1 \leq i \leq 2\ell, \end{cases} \quad N v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell - 1, \\ -v_{i+1}, & \ell + 1 \leq i \leq 2\ell - 1, \\ 0, & i = \ell, 2\ell. \end{cases} $$

The symmetry $v_i \mapsto (-1)^i v_{\ell+i}$, $v_{\ell+i} \mapsto (-1)^{i+\ell+1} v_i$ ($1 \leq i \leq \ell$) allows us to switch $\lambda$ with $\lambda^{-1}$ and assume that $\alpha_0 \in \mathbb{C}^+ \cup \{0\}$. If we write $\lambda^{-1} = e^{-2\pi i \beta_0}$ where $-1 < \text{Re} \beta_0 \leq 0$, then $\beta_0 = -\alpha_0$ or $-\alpha_0 - 1$. Thus, after switching the roles of $\alpha_0$ and $\beta_0$ if necessary, we may always assume that $-\frac{1}{2} \leq \text{Re} \alpha_0 \leq 0$ and $\text{Im} \alpha_0 \geq 0$ when $\text{Re} \alpha_0 = -\frac{1}{2}$.

Example 3.2 ($d = 2\ell - 1$ and $\lambda = \pm 1$).

$$ \sigma v_i = \lambda v_i, \quad 1 \leq i \leq 2\ell - 1, \quad N v_i = \begin{cases} (-1)^{i+1} v_{i+1}, & 1 \leq i \leq 2\ell - 2, \\ 0, & i = 2\ell - 1. \end{cases} $$

Since $\lambda = \pm 1$, it follows that $\alpha_0 = 0$ or $-1/2$.

Remark 3.3. After rescaling the basis vectors in Example 3.2, the operator $N$ can be written alternatively in the form

$$ N v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell - 1, \\ -v_{i+1}, & \ell \leq i \leq 2\ell - 2, \\ 0, & i = 2\ell - 1, \end{cases} $$

which more clearly shows the strong relationship between the symmetric and skew-symmetric cases (cf. Example 3.6 below).
Proposition 3.4. Let $V$ be a finite-dimensional vector space, equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and with commuting linear operators $\sigma, \mathcal{N}$ satisfying (3.1), such that $\sigma$ is invertible and semisimple, and $\mathcal{N}$ is nilpotent. Then $V$ is an orthogonal direct sum of subspaces that are as in Examples 3.5 and 3.6.

3.2. The skew-symmetric case. In the case when $\langle \cdot, \cdot \rangle$ is a nondegenerate skew-symmetric bilinear form, we were unable to locate in the literature an analogous classification of symplectic matrices. Below we present two explicit examples of linear operators $\sigma$ and $\mathcal{N}$ satisfying (3.1). Pick a basis $\{v_1, \ldots, v_{2\ell}\}$ for $V$ such that

$$\langle v_i, v_j \rangle = \delta_{i+j,2\ell+1} = -\langle v_j, v_i \rangle, \quad 1 \leq i \leq j \leq 2\ell,$$

and let $\lambda = e^{-2\pi i \alpha}$ as before.

Example 3.5 ($\ell$ is odd, or $\ell$ is even and $\lambda \neq \pm 1$).

$$\sigma v_i = \begin{cases} \lambda v_i, & 1 \leq i \leq \ell, \\ \lambda^{-1} v_i, & \ell + 1 \leq i \leq 2\ell, \end{cases} \quad \mathcal{N} v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell - 1, \\ -v_{i+1}, & \ell + 1 \leq i \leq 2\ell - 1, \\ 0, & i = \ell, 2\ell. \end{cases}$$

As in Example 3.1 the symmetry $v_i \mapsto (-1)^i v_{\ell+i}$, $v_{\ell+i} \mapsto (-1)^{\ell+i} v_i$ ($1 \leq i \leq \ell$) allows us to assume that $\alpha_0 \in \mathbb{C}^- \cup \{0\}$, $-\frac{1}{2} \leq \Re \alpha_0 \leq 0$, and $\Im \alpha_0 \geq 0$ when $\Re \alpha_0 = -\frac{1}{2}$.

We have omitted the case when $\ell$ is even and $\lambda = \pm 1$ in Example 3.5 because it can be rewritten as an orthogonal direct sum of two copies of the following example.

Example 3.6 ($\lambda = \pm 1$).

$$\sigma v_i = \lambda v_i, \quad 1 \leq i \leq 2\ell, \quad \mathcal{N} v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell, \\ -v_{i+1}, & \ell + 1 \leq i \leq 2\ell - 1, \\ 0, & i = 2\ell. \end{cases}$$

Since $\lambda = \pm 1$, we have $\alpha_0 = 0$ or $-1/2$.

Theorem 3.7. Consider a finite-dimensional vector space $V$ with a nondegenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $\sigma$ and $\mathcal{N}$ be commuting linear operators satisfying (3.1), such that $\sigma$ is invertible and semisimple and $\mathcal{N}$ is nilpotent. Then $V$ is an orthogonal direct sum of subspaces as in Examples 3.5 and 3.6.

Proof. Denote by $V_{\lambda}$ the eigenspaces of $\sigma$. Since the form $\langle \cdot, \cdot \rangle$ is nondegenerate and $\sigma$-invariant, it gives isomorphisms $V_{\lambda} \cong (V_{\lambda^{-1}})^*$, while $V_{\lambda} \perp V_{\mu}$ for $\lambda \neq \mu^{-1}$. Hence, we can assume that $V = V_{\lambda} \oplus V_{\lambda^{-1}}$ ($\lambda \neq \pm 1$) or $V = V_{\pm 1}$.

In the first case, pick a basis $\{w_1, \ldots, w_d\}$ for $V_{\lambda}$ in which $\mathcal{N}$ is in lower Jordan form. Then $V_{\lambda^{-1}}$ has a basis $\{w_{d+1}, \ldots, w_{2d}\}$ such that $\langle w_i, w_j \rangle = \delta_{i+j,2d+1}$ for all $i < j$, and $V$ becomes an orthogonal direct sum of subspaces as in Example 3.5.
A Gram–Schmidt process allows us to construct a new basis \( (3.4) \) that \((3.3)\) and \((3.4)\) still hold and 
\[
\langle w_i \mid w_{d+j}\rangle = \delta_{i+j,d+1} \quad \text{for all } 1 \leq i, j \leq d. \quad \text{However, } V' \text{ and } V'' \text{ may not be distinct, and we must consider two cases.}
\]
First, suppose \( V' \) and \( V'' \) are distinct, and hence \( V = V' \oplus V'' \). When \( d \) is odd, \( \varphi \) acts on \( V' \oplus V'' \) as in Example \((3.5)\) When \( d \) is even, we make the following change of basis:

\[
\begin{align*}
  u'_i &= \begin{cases} 
    \frac{1}{\sqrt{2}}(w_i + (-1)^{i+1}w_{d+i}) & \text{if } 1 \leq i \leq \frac{d}{2}, \\
    \frac{1}{\sqrt{2}}((-1)^{i+1}w_i + w_{d+i}) & \text{if } \frac{d}{2} + 1 \leq i \leq d,
  \end{cases} \\
  u''_i &= \begin{cases} 
    \frac{1}{\sqrt{2}}(w_i + (-1)^i w_{d+i}) & \text{if } 1 \leq i \leq \frac{d}{2}, \\
    \frac{1}{\sqrt{2}}((-1)^i w_i + w_{d+i}) & \text{if } \frac{d}{2} + 1 \leq i \leq d.
  \end{cases}
\end{align*}
\]

Then \( U' = \text{span}\{u'_1, \ldots, u'_d\} \) and \( U'' = \text{span}\{u''_1, \ldots, u''_d\} \) are as in Example \(3.6\), and \( V = U' \oplus U'' \) is an orthogonal direct sum.

Finally, consider the case when \( V = V' = V'' \). Then \( d = 2\ell \) is even, and \( V \) has a basis \( \{w_1, \ldots, w_d\} \) such that \((3.3)\) holds. Note that \((3.3)\) and \((3.4)\) imply that \( \langle w_i \mid w_j \rangle = 0 \) whenever \( i + j > 2\ell + 1 \). With the appropriate rescaling we may assume \( \langle w_1 \mid w_{2\ell} \rangle = 1 \). Then

\[
(3.4) \quad \langle w_i \mid w_{2\ell-i+1} \rangle = (-1)^{i+1}, \quad 1 \leq i \leq 2\ell.
\]

A Gram–Schmidt process allows us to construct a new basis \( w'_1, \ldots, w'_{2\ell} \) such that \((3.3)\) and \((3.4)\) still hold and 
\[
\langle w'_i \mid w'_j \rangle = 0 \quad \text{when } i + j > 2\ell + 1.
\]

Thus 
\[
\langle w'_i \mid w'_j \rangle = (-1)^{i+1}\delta_{i+j,2\ell+1} \quad \text{for all } i, j.
\]

Rescaling the basis vectors, we see that \( V \) is as in Example \(3.6\) \( \square \)

Note that Proposition \(3.4\) can be proved similarly to Theorem \(3.7\).

4. Symplectic fermions

In this section, we will continue to use the notation from Section \(2\). In the case when \( \mathfrak{h} \) is even \( (\mathfrak{h} = \mathfrak{h}_0) \), the free superbosons are known simply as free bosons and \( B^1(\mathfrak{h}) \) is called the Heisenberg vertex algebra. Its twisted logarithmic modules were described in \[1\] Section \(6\). Now we will assume that \( \mathfrak{h} \) is odd, i.e., \( \mathfrak{h} = \mathfrak{h}_1 \). In this case, \( B^1(\mathfrak{h}) \) is called the \textit{symplectic fermion algebra} and denoted \( SF \) (see \[1,11,12\]). Then the bilinear form \( \langle \cdot \mid \cdot \rangle \) on \( \mathfrak{h} \) is skew-symmetric. For \( \varphi \) and \( \varphi \) as in Examples \(3.5\) and \(3.6\), we will describe explicitly the \( \varphi \)-twisted affinization \( \mathfrak{h}_\varphi \) and its irreducible highest-weight modules \( M_\varphi(R) \), together with the action of the Virasoro algebra on them.
4.1. Action of the Virasoro algebra. Choose a basis \( \{ v_i \} \) for \( \mathfrak{h} \) satisfying (4.2), where \( \varphi \) acts either as in Example 3.5 or 3.6. Let \( v^j = v_{2\ell - i + 1} \) and \( v^{\ell + i} = -v_{\ell - i + 1} \) (1 \( \leq i \leq \ell \)). The basis \( \{ v^j \} \) is dual to \( \{ v_i \} \) with respect to \( \langle \cdot | \cdot \rangle \), so that \( \langle v_i | v^j \rangle = \delta_{i,j} \). Then

\[
\omega = \frac{1}{2} \sum_{i=1}^{2\ell} v^i_{(-1)} = \sum_{i=1}^{\ell} v^i_{(-1)} v_i \in B^1(\mathfrak{h})
\]

is a conformal vector with central charge \( c = \text{sdim} \mathfrak{h} = \dim \mathfrak{h}_0 - \dim \mathfrak{h}_1 \). Since \( \varphi \omega = \omega \), the modes of \( Y(\omega, z) \) give a (untwisted) representation of the Virasoro Lie algebra on every \( \varphi \)-twisted \( B^1(\mathfrak{h}) \)-module (cf. [4, Lemma 6.8]).

The triangular decomposition (2.5) induces the following normal ordering on the modes of \( \mathfrak{h}_\varphi \):

\[
\varphi (at^m)(bt^n) = \begin{cases} (at^m)(bt^n) & \text{if } m \in \mathbb{C}^- \\ (-1)^{p(a)p(b)}(bt^n)(at^m) & \text{if } m \in \mathbb{C}^+ \cup \{0\}. \end{cases}
\]

On the other hand, the normally ordered product \( :Y(a, z)Y(b, z): \) of two logarithmic fields is defined by placing the part of \( Y(a, z) \) corresponding to powers \( z^\gamma \) with \( \text{Re} \gamma < 0 \) to the right of \( Y(b, z) \) (see [4, Section 3.3]). The two normal orderings of the modes are different in general, as we will see in the proof of the next proposition.

**Proposition 4.1.** Assume \( \mathfrak{h} \) and \( \varphi \) are as in Example 3.5 or 3.6. Then in every \( \varphi \)-twisted module of \( SF = B^1(\mathfrak{h}) \), we have

\[
L_k = \sum_{i=1}^{\ell} \sum_{m \in \mathfrak{h}_0 + \mathbb{Z}} \varphi (v^i t^{-m})(v_i t^{k+m}) \varphi + \delta_{k,0} \frac{\ell}{2} a_0 (a_0 + 1) I.
\]

**Proof.** Assume \( v_1, \ldots, v_\ell \in \mathfrak{h}_0 \) and \( v_{\ell+1}, \ldots, v_{2\ell} \in \mathfrak{h}_\beta \), where \( \beta = -\alpha \). Using [4, Lemma 5.8], (2.5), the skew-symmetry of \( \langle \cdot | \cdot \rangle \), and the fact that \( \langle N v^i | v_i \rangle = 0 \) for \( 1 \leq i \leq \ell \), we obtain

\[
Y(\omega, z) = \sum_{i=1}^{\ell} :X(v^i, z)X(v_i, z): + z^{-2} \ell \left( \frac{\beta_0}{2} \right) I.
\]

If \( \text{Re} a_0 = 0 \), then \( \beta_0 = -\alpha_0 \in \mathbb{C}^+ \cup \{0\} \). So the normal ordering in (4.4) coincides with (4.2), and \( \left( \frac{\beta_0}{2} \right) = \frac{1}{2} a_0 (a_0 + 1) \). Thus \( L_k \) is given by (4.3).

Now assume \( \text{Re} a_0 < 0 \). Then \( \beta_0 = -\alpha_0 - 1 \in \mathbb{C}^- \), and the ordering of the modes in (4.4) differs from (4.2) when \( k = 0 \) for

\[
-(v_i t^{-\beta_0})(v^i t^{\beta_0}) \overset{\varphi}{=} (v^i t^{\beta_0})(v_i t^{-\beta_0}) + \beta_0 I.
\]

Finally, we note that \( \beta_0 + \left( \frac{\beta_0}{2} \right) = \frac{1}{2} a_0 (a_0 + 1) \). Thus after reordering to match (4.2), \( L_k \) is given by (4.3). \( \square \)
Remark 4.2. Similarly, when $\mathfrak{h}$ is even and $\varphi$ is as in Example 3.1 we have

$$L_k = \sum_{i=1}^{\ell} \sum_{m \in \alpha_0 + \mathbb{Z}} \varphi^i (v^i t^{-m}) (v^i t^{k+m}) \varphi^i - \delta_{k,0} \frac{\ell}{2} \alpha_0 (\alpha_0 + 1) I$$

in any $\varphi$-twisted $B^1(\mathfrak{h})$-module. In the case of Example 3.2 we have

$$L_k = \frac{1}{2} \sum_{i=1}^{d} \sum_{m \in \alpha_0 + \mathbb{Z}} \varphi^i (v^i t^{-m}) (v^i t^{k+m}) \varphi^i - \delta_{k,0} \frac{d}{4} \alpha_0 (\alpha_0 + 1) I.$$ 

These results agree with \cite[Section 6]{4}.

Note that the normal orderings in (4.3), (4.5), (4.6) can be omitted for $k \neq 0$. In the following subsections, we will compute explicitly the actions of $\hat{\mathfrak{h}}_{\varphi}$ and $L_0$ on $M_{\varphi}(R)$ when $\mathfrak{h}$ is odd as in Examples 3.4, 3.6.

4.2. The case of Example 3.5. Recall that for any $\varphi$-twisted $SF$-module, the logarithmic fields are given by (1.4). Assume that $N$ acts on $\mathfrak{h}$ as in Example 3.5 Since this action is the same as in Example 3.1 the logarithmic fields $Y(v_j, z)$ are the same as in \cite[Section 6.4]{3}:

$$Y(v_j, z) = \sum_{i=j}^{\ell} \sum_{m \in \alpha_0 + \mathbb{Z}} \frac{(-1)^{i-j}}{(i-j)!} \varphi^i (v^i t^{m}) z^{-m-1},$$

$$Y(v_{\ell+j}, z) = \sum_{i=j}^{\ell} \sum_{m \in -\alpha_0 + \mathbb{Z}} \frac{1}{(i-j)!} \varphi^i (v^i t^{m}) z^{-m-1},$$

for $1 \leq j \leq \ell$.

The Lie superalgebra $\hat{\mathfrak{h}}_{\varphi}$ is spanned by an even central element $K$ and odd elements $v^i t^{m+\alpha_0}, v^i t^{m-\alpha_0}$ ($1 \leq i \leq \ell$, $m \in \mathbb{Z}$), where $\alpha_0 \in \mathbb{C}^- \cup \{0\}$ and $-1 < \text{Re} \alpha_0 \leq 0$. By (2.3), the only nonzero brackets in $\hat{\mathfrak{h}}_{\varphi}$ are given by:

$$[v^i t^{m+\alpha_0}, v^j t^{n-\alpha_0}] = (m + \alpha_0) \delta_{m,-n} \delta_{i+j,2\ell+1} K + \delta_{m,-n} \delta_{i+j,2\ell} K,$$

for $1 \leq i \leq \ell$, $\ell+1 \leq j \leq 2\ell$, $m, n \in \mathbb{Z}$. Notice that the elements of $\hat{\mathfrak{h}}_{\varphi}^-$ act as creation operators on $M_{\varphi}(R)$. Throughout the rest of the section, we will represent them as anti-commuting variables as follows:

$$v^i t^{m+\alpha_0} = \xi_{i,m}, \quad v^j t^{n-\alpha_0} = \xi_{j,n},$$

for $1 \leq i \leq \ell$, $\ell+1 \leq j \leq 2\ell$, and $m \geq 0$, $n \geq 1$.

The precise triangular decomposition (2.5) depends on whether $\alpha_0 \in \mathbb{C}^-$ or $\alpha_0 = 0$. Suppose first that $\alpha_0 \in \mathbb{C}^-$. Then $\hat{\mathfrak{h}}_{\varphi}^0 = \mathbb{C} K$ and $R = \mathbb{C}$. Equations (4.8) and (4.9) imply that

$$M_{\varphi}(R) \cong \bigwedge \left( \xi_{i,m}, \xi_{\ell+i,m+1} \right)_{1 \leq i \leq \ell, m=0,1,2,...}.$$
Using the commutation relations (4.8) and the fact that \( h_{\varphi}^+ R = 0 \), we obtain the action of \( h_{\varphi}^+ \) on \( M_{\varphi}(R) \):

\[
v_i t^{m+\alpha_0} = (m + \alpha_0) \partial_{\xi_{2\ell-i+1,m}} + (1 - \delta_{i,\ell}) \partial_{\xi_{2\ell-i,m}},
\]

\[
v_{\ell+i} t^{m-\alpha_0} = -(n - \alpha_0) \partial_{\xi_{\ell-i+1,n}} + (1 - \delta_{i,\ell}) \partial_{\xi_{\ell-i,n}},
\]

where \( 1 \leq i \leq \ell \), \( m \geq 1 \), \( n \geq 0 \). By Proposition 4.1, the action of \( L_0 \) is

\[
L_0 = \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i,m} \left( (m - \alpha_0) \partial_{\xi_{i,m}} - (1 - \delta_{i,1}) \partial_{\xi_{i-1,m}} \right)
\]

\[
+ \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} \xi_{\ell+i,m} \left( (m + \alpha_0) \partial_{\xi_{\ell+i,m}} + (1 - \delta_{i,1}) \partial_{\xi_{\ell+i-1,m}} \right)
\]

\[
+ \frac{\ell}{2} \alpha_0 (\alpha_0 + 1) I.
\]

Now we consider the case when \( \alpha_0 = 0 \). Then \( h_{\varphi}^0 R = \text{span} \{ v_i t^0 \}_{1 \leq i \leq 2\ell} \otimes \mathbb{C}K \). We let

\[
R = \bigwedge (\xi_{i,0}, \xi_{2\ell,0})_{1 \leq i \leq \ell},
\]

where the action of \( h_{\varphi}^0 \) on \( R \) is given by

\[
v_i t^0 = \xi_{i,0}, \quad 1 \leq i \leq \ell \text{ or } i = 2\ell,
\]

\[
v_{\ell+i} t^0 = \partial_{\xi_{2\ell-j,0}}, \quad \ell + 1 \leq j \leq 2\ell - 1.
\]

Therefore, by (4.12),

\[
M_{\varphi}(R) \cong \bigwedge (\xi_{i,m}, \xi_{2\ell,m}, \xi_{j,m+1})_{1 \leq i \leq \ell, \ell+1 \leq j \leq 2\ell-1, m=0,1,2,...},
\]

where the action of \( h_{\varphi}^+ \) is given by

\[
v_i t^m = m \partial_{\xi_{2\ell-i+1,m}} + (1 - \delta_{i,\ell}) \partial_{\xi_{2\ell-i,m}},
\]

\[
v_{\ell+i} t^m = -m \partial_{\xi_{\ell-i+1,m}} + (1 - \delta_{i,\ell}) \partial_{\xi_{\ell-i,m}},
\]

for \( 1 \leq i \leq \ell \) and \( m \geq 1 \). The action of \( L_0 \) is

\[
L_0 = \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i,m} \left( m \partial_{\xi_{i,m}} - (1 - \delta_{i,1}) \partial_{\xi_{i-1,m}} \right)
\]

\[
+ \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} \xi_{\ell+i,m} \left( m \partial_{\xi_{\ell+i,m}} + (1 - \delta_{i,1}) \partial_{\xi_{\ell+i-1,m}} \right) - \xi_{1,0} \xi_{2\ell,0}.
\]
4.3. The case of Example 3.6. Let \( \mathfrak{h} \) be as in Example 3.6. Then, by (1.4), in any \( \varphi \)-twisted \( SF \)-module,

\[
Y(v_j, z) = \sum_{i=j}^{\ell} \sum_{m \in \mathbb{N} + \mathbb{Z}} \frac{(-1)^{i-j}}{(i-j)!} \zeta^{i-j} (v_i t^m) z^{-m-1}
\]

\[
+ (-1)^{j} \sum_{i=j+1}^{2\ell} \sum_{m \in \mathbb{N} + \mathbb{Z}} \frac{1}{(i-j)!} \zeta^{i-j} (v_i t^m) z^{-m-1},
\]

(4.14)

\[
Y(v_{\ell+j}, z) = \sum_{i=j}^{\ell} \sum_{m \in \mathbb{N} + \mathbb{Z}} \frac{1}{(i-j)!} \zeta^{i-j} (v_{\ell+i} t^m) z^{-m-1},
\]

for \( 1 \leq j \leq \ell \) and \( \alpha_0 = 0 \) or \(-1/2\).

The Lie superalgebra \( \hat{\mathfrak{h}}_\varphi \) is spanned by an even central element \( K \) and odd elements \( v_i t^m \) \( (1 \leq i \leq 2\ell, m \in \mathbb{N} + \mathbb{Z}) \). The brackets in \( \hat{\mathfrak{h}}_\varphi \) are:

\[
[v_i t^m, v_j t^n] = m \delta_{m, -n} \delta_{i, j, 2\ell+1} K + \delta_{m, -n} (1 - 2\delta_{i, \ell}) \delta_{i + j, 2\ell} K,
\]

\[
[v_{\ell+i} t^m, v_j t^n] = -m \delta_{m, -n} \delta_{i, j, \ell+1} K + \delta_{m, -n} \delta_{i, j, \ell} K,
\]

for \( 1 \leq i \leq \ell, 1 \leq j \leq 2\ell, m, n \in \mathbb{N} + \mathbb{Z} \). To determine explicitly \( \hat{\mathfrak{h}}_\varphi^0 \), we need to consider separately the cases \( \alpha_0 = 0 \) or \(-1/2\).

First, we assume \( \alpha_0 = 0 \). Then \( \mathfrak{h}_0^\varphi = \text{span}\{v_i t^0\}_{1 \leq i \leq 2\ell} \oplus K \). We let

\[
R = \bigwedge (\xi_{i,0}, \xi_{2\ell,0})_{1 \leq i \leq \ell} \quad \text{(where} \xi_{\ell,0}^2 = -\frac{1}{2})
\]

with the action of \( \hat{\mathfrak{h}}_\varphi^0 \) given by (4.12). Again, we will let the creation operators from \( \hat{\mathfrak{h}}_-^\varphi \) act by (4.9). Thus \( \hat{\mathfrak{h}}_\varphi(R) \) is again as in (4.13) but with \( \xi_{\ell,0}^2 = -1/2 \). The action of \( \hat{\mathfrak{h}}_\varphi^+ \) on \( \hat{\mathfrak{h}}_\varphi(R) \) is given by

\[
v_i t^m = m \partial \xi_{2\ell-i+1, m} + (1 - 2\delta_{i, \ell}) \partial \xi_{2\ell-i, m},
\]

\[
v_{\ell+i} t^m = -m \partial \xi_{\ell-i+1, m} + (1 - \delta_{i, \ell}) \partial \xi_{\ell-i, m},
\]

for \( 1 \leq i \leq \ell, m \geq 1 \). The action of \( L_0 \) is

\[
L_0 = \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i,m} \left( m \partial \xi_{i, m} - (1 - \delta_{i,1}) \partial \xi_{i-1, m} \right)
\]

\[
+ \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} \xi_{i,m} \left( m \partial \xi_{i+1,m} + (1 - 2\delta_{i,1}) \partial \xi_{i+1, m} \right) - \xi_{0,0} \xi_{2\ell,0}.
\]

Second, we consider the case when \( \alpha_0 = -1/2 \). Then \( \hat{\mathfrak{h}}_\varphi^0 = \mathbb{C} K \). We represent the elements of \( \hat{\mathfrak{h}}_\varphi \) on \( \hat{\mathfrak{h}}_\varphi(R) \) as \( v_i t^{-m-1/2} = \xi_{i,m} \) for \( 1 \leq i \leq 2\ell \) and \( m \geq 0 \). Then

\[
\hat{\mathfrak{h}}_\varphi(R) \cong \bigwedge (\xi_{i,m})_{1 \leq i \leq 2\ell, m=0,1,2,...},
\]
The action of $\hat{h}_\varphi^+$ on $M_\varphi(R)$ is

$$v_\ell t^{m+1/2} = \left( m + \frac{1}{2} \right) \partial \xi_{\ell-i+1,m} + \left( 1 - 2 \delta_{i,\ell} \right) \partial \xi_{\ell-i,m},$$

$$v_{\ell+i} t^{m+1/2} = - \left( m + \frac{1}{2} \right) \partial \xi_{\ell-i+1,m} + \left( 1 - \delta_{i,\ell} \right) \partial \xi_{\ell-i,m},$$

for $1 \leq i \leq \ell$, $m \geq 0$. The action of $L_0$ is

$$L_0 = \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i,m} \left( \left( m + \frac{1}{2} \right) \partial \xi_{i,m} - (1 - \delta_{i,1}) \partial \xi_{i-1,m} \right) + \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{\ell+i,m} \left( \left( m + \frac{1}{2} \right) \partial \xi_{\ell+i,m} + (1 - 2 \delta_{i,1}) \partial \xi_{\ell+i-1,m} \right) - \frac{\ell}{8} J.$$

**Remark 4.3.** Let $\dim \mathfrak{h} = 2$ in Example [3.6]. The triplet algebra $\mathfrak{W}(1,2) \subset SF$ is generated by the elements (see [11, 43]):

$$W^+ = -v_1(-2)v_1, \quad W^0 = -v_1(-2)v_2 - v_2(-2)v_1, \quad W^- = -v_2(-2)v_2.$$

Then $\sigma = I$ on $\mathfrak{W}(1,2)$ and $\mathcal{N}$: $W^+ \mapsto W^0 \mapsto 2W^- \mapsto 0$. Hence, the restriction of any $\varphi$-twisted module of $SF$ to $\mathfrak{W}(1,2)$ is a $\varphi$-twisted module of $\mathfrak{W}(1,2)$, in which $Y(W^-, z)$ is independent of $\zeta$ while the fields $Y(W^+, z)$ and $Y(W^0, z)$ are logarithmic.

### 5. Free Superfermions

In this section, we study twisted logarithmic modules of the free superfermion algebras. First, let us review the definition of free superfermions given in [32]. Let $\mathfrak{a}$ be an abelian Lie superalgebra with $\dim \mathfrak{a} = d < \infty$, and $(\cdot | \cdot)$ be a nondegenerate even anti-supersymmetric bilinear form on $\mathfrak{a}$. Thus $(b|a) = -(1) p(a)p(b)$ and $(\mathfrak{h}_0|\mathfrak{h}_1) = 0$. The Clifford affinization of $\mathfrak{a}$ is the Lie superalgebra

$$C_\mathfrak{a} = \mathfrak{a}[t, t^{-1}] \oplus CK$$

with commutation relations

$$(5.1) \quad [at^m, bt^n] = (a|b) \delta_{m,-n-1} K, \quad [C_\mathfrak{a}, K] = 0$$

for $m, n \in \mathbb{Z}$, where $p(at^m) = p(a)$ and $p(K) = 0$. The free superfermions

$$a(z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}, \quad a_m = at^m,$$

have OPEs given by

$$a(z)b(w) \sim \frac{(a|b)K}{z-w}.$$

The (generalized) Verma module

$$V = \text{Ind}_{\mathfrak{a}[t][\oplus CK]}^{C_\mathfrak{a}} \mathbb{C}$$
is constructed by letting $a[t]$ act trivially on $\mathbb{C}$ and $K$ act as 1. Then $V$ has the structure of a vertex algebra called the *free superfermion algebra* and denoted $F^1(a)$. The brackets (5.1) are equivalent to the following $n$-th products in $F^1(a)$:

\[(5.2) \quad a_{(0)}b = (a|b)1, \quad a_{(j)}b = 0 \quad (j \geq 1),\]

where 1 is the vacuum vector. In the even case ($a = a_0$), the free superfermions are also known as symplectic bosons or as the bosonic ghost system ($\beta\gamma$-system). In the odd case ($a = a_1$), they are just called *free fermions*.

5.1. **Twisted logarithmic modules of $F^1(a)$**. Letting $\varphi$ be an automorphism of $a$ such that $(\cdot | \cdot)$ is $\varphi$-invariant, we write as before $\varphi = \sigma e^{-2\pi i \lambda}$, and denote the eigenspaces of $\sigma$ by

$$a_\alpha = \{a \in a | \sigma a = e^{-2\pi i \lambda} a\}, \quad \alpha \in \mathbb{C}/\mathbb{Z}.$$  

**Definition 5.1.** The *$\varphi$-twisted Clifford affinization* $(C_a)_\varphi$ is the Lie superalgebra spanned by elements $at^m$ ($a \in a_\alpha, m \in \alpha$) with $p(at^m) = p(a)$ and an even central element $K$. The Lie superbracket in $(C_a)_\varphi$ is given by

\[(5.3) \quad [at^m, bt^n] = \delta_{m-n-1}(a|b)K, \quad [K, at^m] = 0,\]

for $a \in a_\alpha, b \in a_\beta, m \in \alpha, n \in \beta$.

**Remark 5.2.** Since the brackets (5.3) do not depend on $\lambda$, we have $(C_a)_\varphi = (C_a)_\sigma$. In particular, $(C_a)_\varphi = C_a$ if $\varphi = e^{-2\pi i \lambda}$.

As in the case of superbosons, $\varphi$ naturally induces automorphisms of $C_a$ and $F^1(a)$. As before, a $(C_a)_\varphi$-module $W$ will be called *restricted* if for every $a \in a_\alpha, m \in \alpha, \nu \in W$, there is an integer $L$ such that $(at^{m+i})\nu = 0$ for all $i \in \mathbb{Z}, i \geq L$.

**Theorem 5.3.** Every $\varphi$-twisted $F^1(a)$-module $W$ has the structure of a restricted $(C_a)_\varphi$-module with $(at^m)\nu = a_{(m+N)}(a|\nu)$ for $a \in a_\alpha, m \in \alpha, \nu \in W$. Conversely, every restricted $(C_a)_\varphi$-module uniquely extends to a $\varphi$-twisted $F^1(a)$-module.

The proof of the theorem is identical to that of [1, Theorem 6.3] and is omitted. It follows from $(C_a)_\varphi = (C_a)_\sigma$ that every $\varphi$-twisted $F^1(a)$-module $W$ has the structure of a $\sigma$-twisted $F^1(a)$-module, and vice versa. More precisely, if $Y: F^1(a) \rightarrow \text{LFe}(W)$ is the state-field correspondence as a $\varphi$-twisted module, then the state-field correspondence as a $\sigma$-twisted module is given by the map $X$ from (1.2) for $a \in a$. Conversely, given $X$, we can determine $Y$ from (1.2) for $a \in a$. However, the relationship is more complicated for elements $a \in F^1(a)$ that are not in the generating set $a$. In particular, we will see below that the action of the Virasoro algebra is different, so that $L_0$ is not semisimple in a $\varphi$-twisted module while it is semisimple in a $\sigma$-twisted module.
We will split $\mathbb{C}$ as a disjoint union of subsets $\mathbb{C}^+_\frac{1}{2}$, $\mathbb{C}^-_{-\frac{1}{2}}$ and $\{-\frac{1}{2}\}$ where

$$C^+_\frac{1}{2} = -\frac{1}{2} + \mathbb{C}^+, \quad C^-_{-\frac{1}{2}} = \frac{1}{2} - \mathbb{C}^+, \quad \text{and } \mathbb{C}^+$$

is given by (2.3). The $\varphi$-twisted Clifford affinization $(C_a)_\varphi$ has a triangular decomposition

$$(5.5) \quad (C_a)_\varphi = (C_a)^-_{\varphi} \oplus (C_a)^0_{\varphi} \oplus (C_a)^{+}_{\varphi},$$

where

$$(C_a)^{\pm}_{\varphi} = \text{span}\{at^m \mid a \in a_\alpha, \alpha \in \mathbb{C}/\mathbb{Z}, m \in \alpha \cap \mathbb{C}^\pm_{\frac{1}{2}}\}$$

and

$$(C_a)^0_{\varphi} = \text{span}\{at^{-\frac{z}{2}} \mid a \in a_{-\frac{1}{2}}\} \oplus \mathbb{C}K.$$

Starting from a $(C_a)^0_{\varphi}$-module $R$ with $K = I$, the (generalized) Verma module is defined by

$$M_{\varphi}(R) = \text{Ind}_{(C_a)^0_{\varphi}}^{(C_a)^+_{\varphi}} R,$$

where $(C_a)^+_{\varphi}$ acts trivially on $R$. These are $\varphi$-twisted $F^1(a)$-modules, and in the following sections we will realize them explicitly as Fock spaces and will determine the action of the Virasoro algebra on them.

### 5.2. Action of the Virasoro algebra.

Pick bases $\{v_i\}$ and $\{v^i\}$ of $a$ such that $p(v_i) = p(v^i)$ and $(v_i|v^j) = \delta_{i,j}$. Then

$$(5.6) \quad \omega = \frac{1}{2} \sum_{i=1}^{d} v^i(-2)v_i \in F^1(a), \quad d = \dim a,$$

is a conformal vector with central charge $c = -\frac{1}{2}\text{sdim } a$.

Let $S: a \to a$ be the linear operator given by $Sa = a_0 a$ for $a \in a_\alpha$, where, as before, $a_0 \in \alpha$ is such that $-1 < \text{Re } a_0 \leq 0$. In the next result, we use the normally ordered product from [4, Section 3.3] (cf. Section 4.1).

**Proposition 5.4.** In every $\varphi$-twisted $F^1(a)$-module, we have

$$2Y(\omega, z) = \sum_{i=1}^{d} :\partial_z X(v^i, z)X(v_i, z): - z^{-1} \sum_{i=1}^{d} :X(Nv^i, z)X(v_i, z): - z^{-2} \text{str } \left(\begin{array}{c} S \\ 2 \end{array}\right) I,$$

where $\text{str}$ denotes the supertrace.

**Proof.** Recall that in any vertex algebra, $(Ta)_{(j)b} = -ja_{(j-1)b}$, where $T$ is the translation operator (see e.g. [32]). By replacing $a$ with $Ta$ in [4] Lemma 5.8 and using [4] (4.3), we obtain

$$:(D_z Y(a, z)) Y(b, z): = - \sum_{j=-1}^{N-1} jz^{-j-1}Y\left(l\left(\begin{array}{c} S + N \\ j \end{array}\right) a_{(j-1)} b, z\right)$$
for sufficiently large \( N \) (depending on \( a, b \)). Due to (5.2), when \( a, b \in a \), the right-hand side reduces to

\[
Y(a(-2)b, z) - z^{-2}\left(\frac{S + N}{2}\right)a|b) I.
\]

Now using (1.1) and (1.4), we observe that

\[
D_z Y(a, z) |_{\zeta = 0} = \partial_z X(a, z) - z^{-1} X(Na, z).
\]

Finally, we note that

\[
\sum_{i=1}^{d} \left(\frac{S}{2}\right) v_i = -\sum_{i=1}^{d} (-1)^{p(v_i)} v_i \left(\frac{S}{2}\right) v_i = -\text{str}\left(\frac{S}{2}\right).
\]

Then the rest of the proof is as in [4, Lemma 6.4].

Now let us assume that \( a \) can be written as the direct sum of two isotropic subspaces \( a^- = \text{span}\{v_i\} \) and \( a^+ = \text{span}\{v^j\} \) (1 \( \leq i \leq \ell \)), where, as before, \( (v_i|v^j) = \delta_{i,j} \) and \( d = \dim a = 2\ell \). Following [32, Section 3.6], we let

\[
(5.8) \quad \omega^\lambda = (1 - \lambda)\omega^+ + \lambda\omega^- \quad (\lambda \in \mathbb{C}),
\]

where

\[
\omega^+ = \sum_{i=1}^{\ell} v_i(-2)v_i, \quad \omega^- = -\sum_{i=1}^{\ell} (-1)^{p(v_i)} v_i(-2)v_i.
\]

Then \( \omega^\lambda \) is a conformal vector in \( F^1(a) \) with central charge

\[
c_\lambda = (6\lambda^2 - 6\lambda + 1) \text{sdim} a.
\]

In particular, \( \omega^{1/2} \) coincides with (5.6). We denote the corresponding family of Virasoro fields as

\[
L^\lambda(z) = Y(\omega^\lambda, z) = (1 - \lambda)L^+(z) + \lambda L^-(z).
\]

Their action can be derived from the proof of Proposition 5.4 as follows.

**Corollary 5.5.** If \( \varphi(\omega^+) = \omega^+ \), then in every \( \varphi \)-twisted \( F^1(a) \)-module

\[
L^+(z) = \sum_{i=1}^{\ell} :\partial_z X(v^i, z)\colon X(v_i, z) - z^{-1} \sum_{i=1}^{\ell} :X(Nv^i, z)X(v_i, z) - z^{-2}\text{str}\left(\frac{S^+}{2}\right)I,
\]

where \( S^+ \) is the restriction of \( S \) to \( a^+ \).

If the automorphism \( \varphi \) is as in Example 3.6, then a short calculation gives \( N(\omega^\lambda) = (2\lambda - 1)v_{\ell+1}(-2)v_{\ell+1} \). This implies that only the modes of \( Y(\omega^{1/2}, z) = L^{1/2}(z) \) yield an untwisted representation of the Virasoro algebra on a \( \varphi \)-twisted \( F^1(a) \)-module. If \( \varphi \) is as in Examples 3.1 or 3.5, then \( N(\omega^\lambda) = 0 \) and \( L^\lambda(z) \) yields an untwisted representation of the Virasoro algebra for any \( \lambda \in \mathbb{C} \).
Remark 5.6. In the special case when $a$ is even with $\dim a = 2$ (i.e., when we have a $\beta\gamma$-system of rank 1), the above Virasoro fields resemble but are different from those of [2 (3.15)].

5.3. Subalgebras of free superfermions. Suppose that $\dim a = 2\ell$ as in Examples [3.1, 3.5]. It is well-known that the elements

$$u_i = v_i \in F^1(a), \quad u_{\ell+i} = Tv_{\ell+i} \in F^1(a) \quad (1 \leq i \leq \ell)$$

(where $T$ is the translation operator) are generators of the free superboson algebra $B^1(a) \subset F^1(a)$. This is the Heisenberg vertex algebra in the case of Example [3.1] and the symplectic fermion algebra $SF$ in the case of Example [3.5] (see e.g. [11]). Here $u_i$ plays the role of $v_i$ from Section [3] and from [4, Section 6.3].

When $\phi$ is the automorphism of $F^1(a)$ from Examples [3.1, 3.5] then $\phi$ restricts to an automorphism of $B^1(a)$ of the same type, since $\phi$ commutes with $T$. Thus, any $\phi$-twisted $F^1(a)$-module restricts to a $\phi$-twisted $B^1(a)$-module. In such a module, the logarithmic fields corresponding to the generators are given by (cf. [4 (4.3)]):

$$Y(u_i, z) = Y(v_i, z), \quad Y(u_{\ell+i}, z) = D_2 Y(u_{\ell+i}, z) \quad (1 \leq i \leq \ell).$$

Then $u_i(m+N) = v_i(m+N)$ for $m \in \alpha_0 + \mathbb{Z}$, and using (5.7) we obtain

$$u_{\ell+i}(m+N) = -mv_{\ell+i}(m-N) + (1 - \delta_{i,\ell})v_{\ell+i+1}(m-N)$$

for $m \in -\alpha_0 + \mathbb{Z}$. The action of these modes on $M_{\phi}(R)$ is related to the $\phi$-twisted modules constructed in Section [4] and [4, Section 6.3] by a linear change of variables.

The free superboson algebra $B^1(a)$ has a conformal vector (cf. (4.1))

$$\omega' = \sum_{i=1}^{\ell} u_{2\ell-i+1}(-1)u_i.$$

Since

$$u_{2\ell+1-i}(-1) = (Tv_{2\ell+1-i})(-1)v_i = v_{2\ell+1-i}(-2)v_i = v_i(-2)v_i,$$

we have $\omega' = \omega^+ \in F^1(a)$ (see (5.8)). It follows that the action of $Y(\omega', z)$ on $M_{\phi}(R)$ is equivalent to the action of $L^+(z)$. The actions of $L(z) = L^{1/2}(z)$ and $L^+(z)$ will be computed explicitly in the following two sections.

Another important subalgebra of $F^1(a)$ is the $W_{1+\infty}$-algebra [34]. It is generated by the following elements similar to $\omega^+$:

$$\nu^n = \sum_{i=1}^{\ell} v_{(-n)}^i v_i, \quad n = 1, 2, 3, \ldots$$

so that $\nu^2 = \omega^+$ and $\nu^1$ generates the Heisenberg algebra. The automorphism $\phi$ of $F^1(a)$ from Examples [3.1, 3.5] satisfies $\phi(\nu^n) = \nu^n$ for all $n$. Therefore, any $\phi$-twisted $F^1(a)$-module restricts to an (untwisted) module
of $\mathcal{W}_{1+\infty}$. The fields $Y(p^n, z)$ in such a module can be computed as in Proposition 5.4 and Corollary 5.5.

$$Y(p^n, z) = \frac{1}{(n-1)!} \sum_{i=1}^{\ell} iX((\partial_z - z^{-1}\mathcal{N})^{n-1}v^i, z)X(v_i, z):$$

$$+ (-1)^{n-1}z^{-n}\text{str}\left(\begin{pmatrix} S^+ \\ n \end{pmatrix} I\right).$$

(5.10)

Other important realizations by free superfermions are those of classical affine Lie (super)algebras [20, 22, 33, 38]. Here we discuss just one example. Let us assume, as before, that $\mathfrak{a}$ can be written as the direct sum of two isotropic subspaces $\mathfrak{a}^- = \text{span}\{v_i\}$ and $\mathfrak{a}^+ = \text{span}\{v^i\}$ ($1 \leq i \leq \ell$), where $(v_i|v^j) = \delta_{ij}$. We label the basis vectors so that the odd part $(\mathfrak{a}^-)_1$ is spanned by $\{v_i\}_{i=1,\ldots,m}$, and the even part $(\mathfrak{a}^-)_0$ is spanned by $\{v_i\}_{i=m+1,\ldots,m+n}$ where $\ell = m + n$. Then, by [38, Proposition 3.1], the elements $c_{ij} = v_i v_j (1 \leq i, j \leq \ell)$ provide a realization of the affine Lie superalgebra $\mathfrak{gl}(m|n)$ inside the free superfermion algebra $F^1(\mathfrak{a})$. It is easy to see that if $\varphi$ is the automorphism of $F^1(\mathfrak{a})$ from Examples 3.1, 3.2, then $\varphi$ preserves $\mathfrak{gl}(m|n)$. In fact, $\varphi$ acts as an inner automorphism of $\mathfrak{gl}(m|n)$; hence, the corresponding $\varphi$-twisted modules are as in [4, Section 6.1].

6. Free fermions

In this section, we will compute explicitly the actions of $(C_\varphi)_{\varphi}$ and $L_0$ on $M_{\varphi}(R)$ when $\mathfrak{a}$ is odd as in Examples 3.1, 3.2. Let $\{v_i\}$ be a basis for $\mathfrak{a}$ such that $(v_i|v_j) = \delta_{i+j,d+1}$ ($1 \leq i, j \leq d$), and $\varphi$ acts as in Example 3.1 or 3.2. Then the basis defined by $v^i = v_{d-i+1}$ is dual to $\{v_i\}$ with respect to $(|)$, and a conformal vector $\omega$ is given by (5.6).

6.1. The case of Example 3.1. Assume that $\dim \mathfrak{a} = 2\ell$, and $\sigma$ and $\mathcal{N}$ act as in Example 3.1. The logarithmic fields $Y(v_j, z)$ are given by (5.7). The Lie superalgebra $(C_\varphi)_{\varphi}$ is spanned by an even central element $K$ and odd elements $v_i t^{m+\alpha_0}$, $v_{\ell+i} t^{m-\alpha_0}$ ($1 \leq i \leq \ell, m \in \mathbb{Z}$). The nonzero brackets in $(C_\varphi)_{\varphi}$ are given by:

$$[v_i t^{m+\alpha_0}, v_j t^{n-\alpha_0}] = \delta_{m,-n-1}\delta_{i+j,2\ell+1}K, $$

for $1 \leq i \leq \ell$, $\ell + 1 \leq j \leq 2\ell$, $m, n \in \mathbb{Z}$. The elements of $(C_\varphi)_{\varphi}$ act as creation operators on $M_{\varphi}(R)$. Throughout the rest of this section, we will represent them as anti-commuting variables as follows:

(6.1) $$v_i t^{m+\alpha_0} = \xi_{i,m}, \quad v_i t^{n-\alpha_0-1} = \xi_{j,n}, $$

for $v_i \in \mathfrak{a}_+$, $v_j \in \mathfrak{a}_{-\alpha}$, and $m \geq 1$, $n \geq 0$. The precise triangular decomposition (5.5) depends on whether $\alpha_0 = -1/2$ or $\alpha_0 \in \mathbb{C}_+^{1/2}$ (cf. (5.4)).

Suppose first that $\alpha_0 \in \mathbb{C}_{-\frac{1}{2}}$. Then $(C_\varphi)_{\varphi}^0 = CK$ and $R = \mathbb{C}$. Thus

(6.2) $$M_{\varphi}(R) \cong \bigwedge\left((\xi_{i,m+1}, \xi_{\ell+i,m})\right)_{1 \leq i \leq \ell, m = 0,1,2,\ldots}. $$
The action of \((C_a)_\varphi^+\) on \(M_\varphi(R)\) is given explicitly by
\[
v_{t}^{m+\alpha_0} = \partial_{\xi_{2i-l+1,m}}, \quad v_{t+i}^{m-\alpha_0-1} = \partial_{\xi_{i-1,m}},
\]
where \(1 \leq i \leq \ell, \ m \geq 0, \ n \geq 1\). By Proposition 5.4, the action of \(L_0\) is
\[
L_0 = \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} \xi_{i,m} \left( (m - \alpha_0 - \frac{1}{2}) \partial_{\xi_{i,m}} - (1 - \delta_{i,1}) \partial_{\xi_{i-1,m}} \right)
+ \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i+1,m} \left( (m + \alpha_0 + \frac{1}{2}) \partial_{\xi_{i+1,m}} + (1 - \delta_{i,1}) \partial_{\xi_{i-1,m}} \right)
+ \frac{\ell}{2} \alpha_0 (\alpha_0 - 1) I,
\]
which corresponds to (4.11) after relabeling of the variables.

Now we consider the case when \(\alpha_0 = -1/2\). Then
\[
(C_a)^0_\varphi = \text{span}\{v_i t^{-1/2}\}_{1 \leq i \leq d} \oplus C K,
\]
where \(d = 2\ell\). We let
\[
R = \bigwedge (\xi_{\ell+i,0})_{1 \leq i \leq \ell}
\]
with
\[
v_{t+1} t^{-1/2} = \xi_{i+1,0}, \quad v_{i} t^{-1/2} = \partial_{\xi_{2i-l+1,0}} \quad (1 \leq i \leq \ell).
\]
Therefore, by (6.1), \(M_\varphi(R)\) is again given by (6.2). The action of \((C_a)_\varphi^+\) is given by
\[
v_{t+i} t^{-m/2} = \partial_{\xi_{2i-l+1,m}}, \quad v_{t+i} t^{n-1/2} = \partial_{\xi_{i-1,m}},
\]
for \(1 \leq i \leq \ell, \ m \geq 0, \ n \geq 1\). The actions of \(L_0\) and \(L_0^+\) are given by (6.3) and (6.4) respectively, each with \(\alpha_0 = -1/2\).

6.2. **The case of Example 3.2** Let \(a\) be as in Example 3.2. The logarithmic fields \(Y(v_j, z)\) are the same as in [4, Section 6.5]:
\[
Y(v_j, z) = \sum_{i=j}^{2\ell-1} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{(-1)^{(i-j)(i-j-1)/2}}{(i-j)!} \zeta^{i-j} (v_{i} t^m) z^{m-1}
\]
for \(1 \leq j \leq 2\ell - 1\). The Lie superalgebra \((C_a)_\varphi\) is spanned by an even central element \(K\) and odd elements \(v_i t^{m+\alpha_0}\) (\(1 \leq i \leq 2\ell - 1, m \in \mathbb{Z}\)) where \(\alpha_0 = -\frac{1}{2} or 0\). The brackets in \((C_a)_\varphi\) are given by

\[
[v_i t^{m+\alpha_0}, v_j t^{n-\alpha_0}] = \delta_{m,-n-1}\delta_{i+j,2\ell}K,
\]

for \(1 \leq i, j \leq 2\ell - 1, m, n \in \mathbb{Z}\).

We let the creation operators from \((C_a)_\varphi^+\) act by the first equation of (6.1). The triangular decomposition \((5.5)\) depends on whether \(\alpha_0 = -\frac{1}{2}\) or \(\alpha_0 = 0\).

We first consider the case when \(\alpha_0 = -1/2\). Then \((C_a)^0_\varphi\) is given by (6.5) with \(d = 2\ell - 1\). We let

\[
R = \bigwedge (\xi_{i,0})_{\ell \leq j \leq 2\ell - 1} \quad \text{(where } \xi_{i,0}^2 = \frac{1}{2}),
\]

with

\[
v_j t^{-1/2} = \xi_{j,0}, \quad v_i t^{-1/2} = \partial_{\xi_{2\ell-i,0}}
\]

for \(1 \leq i \leq \ell - 1\) and \(\ell \leq j \leq 2\ell - 1\). Therefore,

\[
M_\varphi(R) \cong \bigvee (\xi_{i,m+1}, \xi_{j,m})_{1 \leq i \leq \ell - 1, \ell \leq j \leq 2\ell - 1, m = 0, 1, 2, \ldots},
\]

where the action of \((C_a)_\varphi^+\) on \(M_\varphi(R)\) is given by

\[
v_i t^{m-1/2} = \partial_{\xi_{2\ell-i,m}}, \quad 1 \leq i \leq 2\ell - 1, m \geq 1.
\]

The action of \(L_0\) is

\[
L_0 = \sum_{i=1}^{2\ell-1} \sum_{m=1}^{\infty} \xi_{i,m} \left( m\partial_{\xi_{i,m}} - (-1)^i(1 - \delta_{i,1})\partial_{\xi_{i-1,m}} \right)
\]

\[
+ \sum_{i=\ell+2}^{2\ell-1} \left( -1 \right)^{i+1} \xi_{i,0}\partial_{\xi_{i-1,0}} + (-1)^\ell \xi_{\ell+1,0}\xi_{\ell,0} + \frac{2\ell - 1}{16} t.
\]

7. Bosonic ghost system

Now we will compute explicitly the actions of \((C_a)_\varphi\) and \(L_0\) on \(M_\varphi(R)\) when \(a\) is even as in Examples 3.5 and 3.6. Let \(\{v_i\}_{1 \leq i \leq 2\ell}\) be a basis for \(a\) such that \(\langle v_i | v_j \rangle = \delta_{i+j,2\ell+1}\) (\(1 \leq i \leq j \leq 2\ell\)), and \(\varphi\) acts as in Example 3.5 or 3.6. Then the basis defined by \(v^i = v_{2\ell-i+1}\), \(v^{\ell+i} = -v_{\ell-i+1}\) (\(1 \leq i \leq \ell\)) is dual to \(\{v_i\}\) with respect to \(\langle \cdot | \cdot \rangle\), and a conformal vector is given by (5.6).
7.1. The case of Example 3.5. Assume that $\sigma$ and $N$ act as in Example 3.5. The logarithmic fields $Y(v_j, z)$ are given by (4.7). The Lie algebra $(C_0)_\sigma$ is spanned by a central element $K$ and elements $v_i t^m + \alpha_0, v_{\ell+i} t^m - \alpha_0$ ($1 \leq i \leq \ell, m \in \mathbb{Z}$). The nonzero brackets in $(C_0)_\sigma$ are given by
\[
[v_i t^m + \alpha_0, v_j t^m - \alpha_0] = \delta_{m, -n-1} \delta_{i+j, 2\ell+1} K,
\]
for $1 \leq i \leq \ell, \ell + 1 \leq j \leq 2\ell, m, n \in \mathbb{Z}$. The elements of $(C_0)_\sigma$ act as creation operators on $M_\phi(R)$. Throughout the rest of this section, we will represent them as commuting variables as follows
\[
(7.1) \quad v_i t^m + \alpha_0 = x_{i,m}, \quad v_j t^m - \alpha_0 = x_{j,n},
\]
for $1 \leq i \leq \ell, \ell + 1 \leq j \leq 2\ell$ and $m \geq 1, n \geq 0$. Again, the precise triangular decomposition (5.5) depends on whether $\alpha_0 \in \mathbb{C}^+_\sigma$ or $\alpha_0 = -\frac{1}{2}$.

Consider first the case when $\alpha_0 \in \mathbb{C}^+_\sigma$. Then $(C_0)_\sigma^0 = CK, R = \mathbb{C}$, and
\[
(7.3) \quad M_\phi(R) = \mathbb{C}[x_{i,m+1}, x_{\ell+i,m}]_{1 \leq i \leq \ell, m=0,1,2,...}.
\]
The action of $(C_0)_\sigma^0$ on $M_\phi(R)$ is given explicitly by
\[
(7.4) \quad v_i t^m + \alpha_0 = \partial x_{2\ell-i+1, m}, \quad v_{\ell+i} t^m - \alpha_0 = -\partial x_{\ell-i+1, n},
\]
for $1 \leq i \leq \ell, m \geq 0, n \geq 1$. The action of $L_0$ is
\[
L_0 = \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{i,m} \left( (m - \alpha_0 - \frac{1}{2}) \partial x_{i,m} - (1 - \delta_{i,1}) \partial x_{i-1,m} \right) + \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} x_{\ell+i,m} \left( (m + \alpha_0 + \frac{1}{2}) \partial x_{\ell+i,m} + (1 - \delta_{i,1}) \partial x_{\ell+i-1,m} \right) - \frac{\ell}{2} \alpha_0^2 I.
\]
By Corollary 7.5, the action of $L_0^+$ is given by:
\[
L_0^+ = \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{i,m} \left( (m - \alpha_0) \partial x_{i,m} - (1 - \delta_{i,1}) \partial x_{i-1,m} \right) + \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} x_{\ell+i,m} \left( (m + \alpha_0) \partial x_{\ell+i,m} + (1 - \delta_{i,1}) \partial x_{\ell+i-1,m} \right) - \frac{\ell}{2} \alpha_0 (\alpha_0 - 1) I,
\]
which agrees with [4] Lemma 6.8 up to relabeling of the variables.

Now consider the case when $\alpha_0 = -\frac{1}{2}$. Then $(C_0)_\sigma^0$ is given by (6.5) with $d = 2\ell$. We let
\[
(7.7) \quad R = \mathbb{C}[x_{i,0}]_{1 \leq i \leq \ell},
\]
with
\[
v_{\ell+i} t^{1/2} = x_{\ell+i,0}, \quad v_{\ell} t^{-1/2} = \partial x_{2\ell-i+1, 0} \quad (1 \leq i \leq \ell).
\]
Then $M_{\varphi}(R)$ is given by (7.3), where the action of $(C_{\varphi})^+_{\varphi}$ is given by
\begin{equation}
\begin{aligned}
    v_t \ell^{m-1/2} &= \partial_{\ell^{2\ell-i+1},m}, \\
    v_{\ell+1} \ell^{m+1/2} &= -\partial_{\ell^{2\ell+i+1},m+1},
\end{aligned}
\end{equation}
for $1 \leq i \leq \ell$ and $m \geq 1$. The actions of $L_0$ and $L_0^+$ are given by (7.5) and (7.6) respectively, each with $\alpha_0 = -\frac{1}{2}$.

7.2. The case of Example 3.6. Let $a$ be as in Example 3.6. The logarithmic fields are given by (4.1). The Lie algebra $(C_{\varphi})_{\varphi}$ is spanned by a central element $K$ and elements $v_i \ell^m + \alpha_0$ ($1 \leq i \leq 2\ell$) where $\alpha_0 = 0$ or $-1/2$. The brackets in $(C_{\varphi})_{\varphi}$ are given by (7.1). We let the creation operators from $(C_{\varphi})_{\varphi}$ act on $M_{\varphi}(R)$ by (7.2). As before, the triangular decomposition depends on whether $\alpha_0 = 0$ or $-1/2$.

We first consider the case when $\alpha_0 = 0$. Then $(C_{\varphi})^0_{\varphi} = \mathbb{C} K$, $R = \mathbb{C}$, $M_{\varphi}(R)$ is given by (7.3), and the action of $(C_{\varphi})^+_{\varphi}$ is given by (7.4) with $\alpha_0 = 0$. The action of $L_0$ is
\begin{equation}
    L_0 = \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{i,m} \left( m - \frac{1}{2} \right) \partial_{x_i,m} - (1 - \delta_{i,1}) \partial_{x_{i-1},m} \\
    + \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{\ell+i,m} \left( m + \frac{1}{2} \right) \partial_{x_{\ell+i,m}} + (1 - 2\delta_{i,1}) \partial_{x_{\ell+i-1,m}} \\
    + \frac{1}{2} \sum_{i=1}^{\ell} x_{\ell+i,0} \partial_{x_{\ell+i,0}} + \sum_{i=2}^{\ell} x_{\ell+i,0} \partial_{x_{\ell+i-1,0}}.
\end{equation}

Now assume $\alpha_0 = -1/2$. Then $(C_{\varphi})^0_{\varphi}$, $R$, and $M_{\varphi}(R)$ are the same as in the case when $\alpha_0 = -1/2$ in Section 7.1. The action of $L_0$ is
\begin{equation}
    L_0 = \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{i,m} \left( m \partial_{x_i,m} - (1 - \delta_{i,1}) \partial_{x_{i-1},m} \right) \\
    + \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{\ell+i,m} \left( m \partial_{x_{\ell+i,m}} + (1 - 2\delta_{i,1}) \partial_{x_{\ell+i-1,m}} \right) \\
    + \sum_{i=2}^{\ell} x_{\ell+i,0} \partial_{x_{\ell+i-1,0}} + \frac{1}{2} x_{\ell+1,0}^2 - \frac{\ell}{8} I.
\end{equation}

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