A Generalized-Polymatroid Approach to Disjoint Common Independent Sets in Two Matroids

Kenjiro Takazawa* and Yu Yokoi†
June 2018

Abstract

In this paper, we investigate the classes of matroid intersection admitting a solution for the problem of partitioning the ground set $E$ into $k$ common independent sets, where $E$ can be partitioned into $k$ independent sets in each of the two matroids. For this problem, we present a new approach building upon the generalized-polymatroid intersection theorem. We exhibit that this approach offers alternative proofs and unified understandings of previous results showing that the problem has a solution for the intersection of two laminar matroids and that of two matroids without $(k+1)$-spanned elements. Moreover, we newly show that the intersection of a laminar matroid and a matroid without $(k+1)$-spanned elements admits a solution. We also construct an example of a transversal matroid which is incompatible with the generalized-polymatroid approach.

1 Introduction

For two matroids with a common ground set, the problem of partitioning the ground set into common independent sets is a classical topic in discrete mathematics. That is, extending the celebrated König’s bipartite edge-coloring theorem [15], described below, into general matroid intersection has been well discussed.

Theorem 1.1 (König [15]). For a bipartite graph $G$ and a positive integer $k$, the edge set of $G$ can be partitioned into $k$ matchings if and only if the maximum degree of the vertices of $G$ is at most $k$.

Let $G = (U, V; E)$ be a bipartite graph. A subset $X$ of $E$ is a matching if and only if it is a common independent set of two partition matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$, where $I_1$ (resp., $I_2$) is the family of edge sets in which no two edges are adjacent at $U$ (resp., at $V$). The maximum degree of $G$ coincides with the minimum number $k$ such that $E$ can be partitioned into $k$ independent sets of $M_1$ and also into $k$ independent sets of $M_2$. We then naturally conceive the following problem for a general matroid pair on the common ground set.

Problem 1.2. Given two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ and a positive integer $k$ such that $E$ can be partitioned into $k$ independent sets of $M_1$ and also into $k$ independent sets of $M_2$, find a partition of $E$ into $k$ common independent sets of $M_1$ and $M_2$.

*Hosei University, Tokyo 184-8584, Japan. E-mail: takazawa@hosei.ac.jp
†National Institute of Informatics, Tokyo 101-8430, Japan. E-mail: yokoi@nii.ac.jp
Solving Problem 1.2 amounts to extending Theorem 1.1 into matroid intersection. Such an extension is proved for arborescences in digraphs [5] and the intersection of two strongly base orderable matroids [3], while such an extension is impossible for a simple example of the intersection of a graphic matroid and a partition matroid on the edge set of $K_4$ [21, Section 42.6c]. Indeed, Problem 1.2 is known to be a challenging problem: we only have partial answers in the literature [1, 3, 5, 12, 16], and the famous conjecture of Rota (in [14]) is equivalent to a sufficient condition for Problem 1.2 to have a solution.

An interesting class of matroid intersection for which Problem 1.2 admits a solution is introduced by Kotlar and Ziv [16]. For a matroid $M$ on ground set $E$ and a positive integer $k$, an element $e$ of $E$ is called $k$-spanned if there exist $k$ disjoint sets spanning $e$ (see Section 2.1 for definition). Kotlar and Ziv [16] presented two sufficient conditions (Theorems 3.9 and 3.10) for the common ground set $E$ of two matroids $M_1$ and $M_2$ to be partitioned into $k$ common independent sets, under the assumption that no element of $E$ is $(k+1)$-spanned in either $M_1$ or $M_2$. Since the ground set of a matroid in which no element is $(k+1)$-spanned can be partitioned into $k$ independent sets (Lemma 3.11), these two cases offer classes of matroid intersection for which Problem 1.2 is solvable.

In this paper, we present a new approach to Problem 1.2 building upon the integrality of generalized polymatroids [3, 13], a comprehensive class of polyhedra associated with a number of tractable combinatorial structures. This generalized-polymatroid approach is regarded as an extension of the polyhedral approach to bipartite edge-coloring [21, Section 20.3], and is indeed successful in supermodular coloring [24], which is another matroidal generalization of bipartite edge-coloring (see [20, 21]). Utilizing generalized polymatroids, we offer alternative proofs and unified understandings for some special cases for which Problem 1.2 admits solutions. To be more precise, we first prove the extension of Theorem 1.1 for the intersection of two laminar matroids. Laminar matroids recently attract particular attention based on its relation to the matroid secretary problem (see [7] and references therein), and form a special case of strongly base orderable matroids. Thus, the generalized-polymatroid approach yields another proof for a special case of [3]. We then show alternative proofs for the two cases of Kotlar and Ziv [16], which offer a new understanding of the tractability of the two cases. Moreover, we newly prove that Problem 1.2 admits a solution for the intersection of a laminar matroid and a matroid in the two classes of Kotlar and Ziv [16]. Finally, we show a limit of the generalized-polymatroid approach by constructing an instance of a transversal matroid, another special class of a strongly base orderable matroid, which is incompatible with the generalized-polymatroid approach.

The rest of the paper is organized as follows. In Section 2, we review the definitions and some fundamental properties of matroids and generalized polymatroids. Section 3 is devoted to solving Problem 1.2 for some classes of matroid intersection by our generalized-polymatroid approach. In contrast, in Section 4, we construct a transversal matroid which is incompatible with the generalized-polymatroid approach. Section 5 concludes the paper.

2 Preliminaries

In this section, we review the definition and fundamental properties of matroids and generalized polymatroids. For more details, the readers are referred to [10, 11, 17, 19, 21, 25].
2.1 Matroid

Let $E$ be a finite set. For a subset $X \subseteq E$ and elements $e \in E \setminus X$, $e' \in X$, we denote $X + e = X \cup \{e\}$ and $X - e' = X \setminus \{e'\}$. For a set family $\mathcal{I} \subseteq 2^E$, a pair $(E, \mathcal{I})$ is called a matroid if $\mathcal{I}$ satisfies

(I0) $\emptyset \in \mathcal{I}$,
(I1) $X \subseteq Y \in \mathcal{I}$ implies $X \in \mathcal{I}$, and
(I2) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then $\exists e \in Y \setminus X : X + e \in \mathcal{I}$.

Each member $X$ of $\mathcal{I}$ is called an independent set. In particular, an independent set $B \in \mathcal{I}$ is called a base if it is maximal in $\mathcal{I}$ with respect to inclusion. It is known that all bases have the same size.

For a matroid $M = (E, \mathcal{I})$ and a subset $S \subseteq E$, the restriction $M|S$ of $M$ to $S$ is a pair $(S, \mathcal{I}|S)$, where $\mathcal{I}|S = \{ X \mid X \in \mathcal{I}, X \subseteq S \}$. For any $S \subseteq E$, the restriction $M|S$ is again a matroid.

For a matroid $M = (E, \mathcal{I})$ and a positive integer $k \in \mathbb{Z}$, we define a set family $\mathcal{I}^k \subseteq 2^E$ by

$$\mathcal{I}^k = \{ X \subseteq E \mid X \text{ can be partitioned into } k \text{ independent sets in } \mathcal{I} \}.$$

The following theorem is a special case of the famous matroid union theorem of Edmonds and Fulkerson [6].

**Theorem 2.1** (Edmonds and Fulkerson [6]). For a matroid $M = (E, \mathcal{I})$ and a positive integer $k \in \mathbb{Z}$, the pair $M^k = (E, \mathcal{I}^k)$ is a matroid.

For a matroid $M = (E, \mathcal{I})$, its rank function $r : 2^E \to \mathbb{Z}_{\geq 0}$ is defined by $r(A) = \max\{|X| \mid X \subseteq A, X \in \mathcal{I}\}$ for any $A \subseteq E$. Then, it is known that $\mathcal{I} = \{ X \mid \forall A \subseteq E : |X \cap A| \leq r(A) \}$ holds. A subset $X \subseteq E$ spans an element $e \in E$ if $r(X + e) = r(X)$.

For a matroid $M = (E, \mathcal{I})$ with rank function $r : 2^E \to \mathbb{Z}_{\geq 0}$ and a positive integer $k \in \mathbb{Z}$, we denote by $r^k : 2^E \to \mathbb{Z}_{\geq 0}$ the rank function of $M^k = (E, \mathcal{I}^k)$. That is, for any $A \subseteq E$ we have

$$r^k(A) = \max\{|X| \mid X \subseteq A, X \text{ can be partitioned into } k \text{ independent sets in } \mathcal{I} \}.$$

2.2 Generalized Polymatroid

Let $E$ be a finite set. A function $b : 2^E \to \mathbb{R} \cup \{\infty\}$ is called submodular if it satisfies the submodular inequality

$$b(A) + b(B) \geq b(A \cup B) + b(B \cap A)$$

for any $A, B \subseteq E$, where the inequality is seen to hold if the left-hand side is infinite. A function $p : 2^E \to \mathbb{R} \cup \{-\infty\}$ is called supermodular if $-p$ is submodular. A pair $(p, b)$ is called paramodular if we have

(i) $p(\emptyset) = b(\emptyset) = 0$,
(ii) $p$ is supermodular, $b$ is submodular, and
(iii) $p$ and $b$ satisfy the cross inequality

$$b(A) - p(B) \geq b(A \setminus B) - p(B \setminus A)$$

for any $A, B \subseteq E$, where the inequality is seen to hold if the left-hand side is infinite.
For a pair of set functions \( p : 2^E \to \mathbb{R} \cup \{-\infty\} \) and \( b : 2^E \to \mathbb{R} \cup \{\infty\} \), we associate a polyhedron \( Q(p, b) \) defined by
\[
Q(p, b) = \{ x \in \mathbb{R}^E \mid \forall A \subseteq E : p(A) \leq x(A) \leq b(A) \},
\]
where \( x(A) = \sum \{ x(e) \mid e \in A \} \). Here, \( p \) serves as a lower bound while \( b \) serves as an upper bound of the polyhedron \( Q(p, b) \). A polyhedron \( P \subseteq \mathbb{R}^E \) is called a generalized polymatroid (for short, a \( g \)-polymatroid) if \( P = Q(p, b) \) holds for some paramodular pair \((p, b)\). It is known \([8, 10]\) that such a paramodular pair is uniquely defined for any \( g \)-polymatroid.

We next introduce the concept of intersecting paramodularity, which is weaker than paramodularity but still yields \( g \)-polymatroids. We say that subsets \( A, B \subseteq E \) are intersecting if none of \( A \cap B \), \( A \setminus B \) and \( B \setminus A \) is empty. A function \( b : 2^E \to \mathbb{R} \cup \{\infty\} \) is called intersecting submodular if it satisfies the submodular inequality for any intersecting subsets \( A, B \subseteq E \). A function \( p : 2^E \to \mathbb{R} \cup \{-\infty\} \) is called intersecting supermodular if \( -p \) is intersecting submodular. A pair \((p, b)\) is called intersecting paramodular if \( p \) and \( b \) are intersecting super- and submodular functions, respectively, and the cross inequality \([1]\) holds for any intersecting subsets \( A, B \subseteq E \).

The following theorem states that an intersecting-paramodular pair \((p, b)\) defines a \( g \)-polymatroid.

**Theorem 2.2 (Frank \([8]\)).** For an intersecting-paramodular pair \((p, b)\) such that \( Q(p, b) \neq \emptyset \), the polyhedron \( Q(p, b) \) is a \( g \)-polymatroid, which is, in addition, integral whenever \((p, b)\) is integral.

In general, the intersection of two integral polyhedra \( P_1 \) and \( P_2 \) is not necessarily integral. For two integral \( g \)-polymatroids, however, the intersection preserves integrality as stated below. This fact plays a key role in our \( g \)-polymatroid approach to Problem 1.2.

**Theorem 2.3 (Integrality of \( g \)-polymatroid intersection \([8]\)).** For two integral \( g \)-polymatroids \( P_1 \) and \( P_2 \), the intersection \( P_1 \cap P_2 \) is an integral polyhedron if it is nonempty.

As this paper studies partitions of finite sets, we are especially interested in vectors in the intersection of a \( g \)-polymatroid and the unit hypercube \([0, 1]^E = \{ x \in \mathbb{R}^E \mid \forall e \in E : 0 \leq x(e) \leq 1 \} \). It is known that the intersection is again a \( g \)-polymatroid.

**Theorem 2.4 (Frank \([8]\)).** For a \( g \)-polymatroid \( P \), if \( P \cap [0, 1]^E \) is nonempty, then the intersection \( P \cap [0, 1]^E \) is again a \( g \)-polymatroid, which is, in addition, integral whenever \( P \) is integral.

Similarly to the definition of \( Q(p, b) \), for a pair of set functions \( p : 2^E \to \mathbb{R} \cup \{-\infty\} \) and \( b : 2^E \to \mathbb{R} \cup \{\infty\} \), we associate the following set family:
\[
\mathcal{F}(p, b) = \{ X \subseteq E \mid \forall A \subseteq E : p(A) \leq |X \cap A| \leq b(A) \}.
\]

The following observation is derived from Theorem 2.4. For a subset \( Y \subseteq E \), its characteristic vector \( \chi_Y \in \{0, 1\}^E \) is defined by \( \chi_Y(e) = 1 \) for \( e \in Y \) and \( \chi_Y(e) = 0 \) for \( e \in E \setminus Y \).

**Lemma 2.5.** For an integral intersecting-paramodular pair \((p, b)\), the polyhedron \( Q(p, b) \cap [0, 1]^E \) is a convex hull of the characteristic vectors of the members of \( \mathcal{F}(p, b) \).
Proof. By Theorem 2.4, \( Q(p, b) \cap [0, 1]^E \) is integral, and hence all its vertices are \((0,1)\)-vectors. Also, by the definition of \( Q(p, b) \) and \( F(p, b) \), we have \( y \in Q(p, b) \cap [0, 1]^E \) if and only if \( y = \chi_Y \) for some \( Y \in F(p, b) \). Thus, the vertices of \( Q(p, b) \cap [0, 1]^E \) coincides with the characteristic vectors of the members of \( F(p, b) \).

3 Generalized-Polymatroid Approach

In this section, we exhibit some cases of matroid intersection for which a solution of Problem 1.2 can be constructed by utilizing the g-polymatroid intersection theorem (Theorem 2.3). In Section 3.1 we describe a general method to apply Theorem 2.3 for solving Problem 1.2. In Section 3.2 we use this method to prove an extension of Theorem 1.1 to the intersection of two laminar matroids, a special case of intersection of two strongly base orderable matroids [3]. In Section 3.3, we utilize this method for alternative proofs for two classes of matroid intersection due to Kotlar and Ziv [16]. Finally, in Section 3.4 we present a new class of matroid intersection for which Problem 1.2 admits a solution: intersection of a laminar matroid and a matroid in Kotlar and Ziv’s classes.

3.1 General Method

With the notations introduced in Section 2.1 now Problem 1.2 is reformulated as follows.

Problem 1.2 (reformulated). Given matroids \( M_1 = (E, \mathcal{I}_1) \) and \( M_2 = (E, \mathcal{I}_2) \) and a positive integer \( k \) such that \( E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k \), find a partition \( \{X_1, X_2, \ldots, X_k\} \) of \( E \) such that \( X_j \in \mathcal{I}_1 \cap \mathcal{I}_2 \) for \( j = 1, 2, \ldots, k \).

Our general method to solve Problem 1.2 is to find \( X \in \mathcal{I}_1 \cap \mathcal{I}_2 \) such that \( E \setminus X \in \mathcal{I}_1^k \cap \mathcal{I}_2^k \) with the aid of g-polymatroid intersection, replace \( E \) and \( k \) with \( E \setminus X \) with \( k - 1 \), respectively, and iterate. The following proposition, which can be proved by combining Theorems 2.2, 2.3 and Lemma 2.5, shows a necessary condition that this method can be applied.

Proposition 3.1. Let \( M_1 = (E, \mathcal{I}_1) \), \( M_2 = (E, \mathcal{I}_2) \) be matroids and \( k \in \mathbb{Z} \) be a positive integer with \( E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k \). If there exists an integral intersecting-paramodular pairs \((p_i, b_i)\) such that

\[
\mathcal{F}(p_i, b_i) = \{ X \subseteq E \mid X \in \mathcal{I}_i, \ E \setminus X \in \mathcal{I}_i^{k-1} \}
\]

for each \( i = 1, 2 \), then there exists a subset \( X \subseteq E \) such that \( X \in \mathcal{I}_1 \cap \mathcal{I}_2 \) and \( E \setminus X \in \mathcal{I}_1^{k-1} \cap \mathcal{I}_2^{k-1} \).

Proof. We first show that \( Q(p_i, b_i) \cap [0, 1]^E \) is nonempty for each \( i = 1, 2 \). Because \( E \in \mathcal{I}_i^k \), there is a partition \( \{X_1, X_2, \ldots, X_k\} \) of \( E \) such that \( X_j \in \mathcal{I}_i \) for each \( j = 1, 2, \ldots, k \). For each \( j \), we have \( X_j \in \mathcal{I}_i \) and \( E \setminus X_j = \bigcup_{\ell \neq j} X_\ell \in \mathcal{I}_i^{k-1} \), and hence \( X_j \in \mathcal{F}(p_i, b_i) \). As \( \{X_1, X_2, \ldots, X_k\} \) is a partition of \( E \), the vector \( x := \left( \frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k} \right)^\top \in \mathbb{R}^E \) coincides with \( \sum_{j=1}^k \frac{1}{k} \chi_{X_j} \), which is a convex combination of the characteristic vectors of \( X_j \in \mathcal{F}(p_i, b_i) \) \( (j = 1, 2, \ldots, k) \). Then, Lemma 2.5 implies \( x \in Q(p_i, b_i) \cap [0, 1]^E \).

Now \( Q(p_1, b_1) \cap Q(p_2, b_2) \cap [0, 1]^E \) includes the vector \( \left( \frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k} \right)^\top \), and hence is nonempty. Then, by combining Theorems 2.2, 2.3, 2.4, we obtain that \( Q(p_1, b_1) \cap Q(p_2, b_2) \cap [0, 1]^E \) is an integral nonempty polyhedron, and hence it contains a \((0,1)\)-vector \( y \). Let \( Y \subseteq E \) be the set satisfying \( \chi_Y = y \). Then \( y \in Q(p_1, b_1) \cap Q(p_2, b_2) \cap [0, 1]^E \) implies \( Y \in \mathcal{F}(p_1, b_1) \cap \mathcal{F}(p_2, b_2) \), which means \( Y \in \mathcal{I}_1 \cap \mathcal{I}_2 \) and \( E \setminus Y \in \mathcal{I}_1^{k-1} \cap \mathcal{I}_2^{k-1} \).

\( \square \)
In order to use our method, $M_1$ and $M_2$ should belong to a class of matroids in which each member $M = (E, \mathcal{I})$ with $E \in \mathcal{I}^k$ admits an integral intersecting-paramodular pair $(p, b)$ satisfying \([2]\) and the restriction $M|(E \setminus X)$ with any $X \in \mathcal{F}(p, b)$ belongs to this class again with $k$ replaced by $k - 1$. In the subsequent subsections, we show that the class of laminar matroids and the two matroid classes in \([16]\) have this property.

**Remark 3.2.** An advantage of our approach is that there is no constraint straddling the two matroids $M_1$ and $M_2$. In other words, our approach can deal with any pair of matroids such that each of them admits an intersecting-paramodular pair required in Proposition 3.1. This makes a contrast with some previous works \([3, 16]\), which assume that the two matroids are in the same matroid class. Indeed, utilizing this fact, we provide a result (Theorems 3.15 and 3.16) that is not included in previous works.

**Remark 3.3.** Each iteration step (and hence the total computation) in the above argument can be executed in polynomial time in the following manner. Suppose that $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ satisfy all the assumptions in Proposition 3.1 and the membership oracles of $\mathcal{I}_1$ and $\mathcal{I}_2$ are provided. Since each $(p_i, b_i)$ is intersecting paramodular and $\mathcal{F}(p_i, b_i) = \{ X \subseteq E \mid X \in \mathcal{I}_i, \ E \setminus X \in \mathcal{I}_i^{k-1} \}$ holds, the problem of finding $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $E \setminus X \in \mathcal{I}_1^{k-1} \cap \mathcal{I}_2^{k-1}$ is reduced to generalized-matroid intersection \([10, 24]\). To solve this, we need a membership oracle and some member of $\mathcal{F}(p_i, b_i)$ for each $i = 1, 2$. As $\mathcal{F}(p_i, b_i) = \{ X \subseteq E \mid X \in \mathcal{I}_i, \ E \setminus X \in \mathcal{I}_i^{k-1} \}$, we can find some member of $\mathcal{F}(p_i, b_i)$ by Edmonds’ matroid partition algorithm \([4]\). By the same algorithm, we can also simulate a membership oracle of $\mathcal{I}_i^{k-1}$, and hence of $\mathcal{F}(p_i, b_i)$ efficiently.

### 3.2 Intersection of Two Laminar Matroids

In this section, we prove that Problem 1.2 is solvable for laminar matroids by our generalized-polymatroid approach. Since a laminar matroid is a generalization of a partition matroid, this extends the bipartite edge-coloring theorem of König \([15]\). On the other hand, since a laminar matroid is strongly base orderable, this proof amounts to another proof for a special case of strongly base orderable matroids by Davies and McDiarmid \([3]\).

We first define the concept of laminar matroids. A subset family $\mathcal{A}$ of a finite set $E$ is called *laminar* if $A_1, A_2 \in \mathcal{A}$ implies $A_1 \subseteq A_2, A_2 \subseteq A_1$, or $A_1 \cap A_2 = \emptyset$. Let $\mathcal{A} \subseteq 2^E$ be a laminar family and $q : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ be a capacity function. Let $\mathcal{I}$ be a family of subsets $X$ satisfying all capacity constraints, i.e.,

$$\mathcal{I} = \{ X \subseteq E \mid \forall A \in \mathcal{A} : |X \cap A| \leq q(A) \} .$$

Then it is known that $(E, \mathcal{I})$ is a matroid, which we call the *laminar matroid* induced from $\mathcal{A}$ and $q$.

It is known that a laminar matroid is a special case of a strongly base orderable matroid \([2]\).

**Definition 3.4** (Strongly base orderable matroid \([2]\)). A matroid is *strongly base orderable* if for each pair of bases $B_1, B_2$ there exists a bijection $\pi : B_1 \to B_2$ such that for each subset $X$ of $B_1$ the set $\pi(X) \cup (B_1 \setminus X)$ is a base again.

Thus, it follows from the result of Davies and McDiarmid \([3]\) that Theorem 1.1 can be extended to the intersection of laminar matroids.
Theorem 3.5 (Davies and McDiarmid [3]). For laminar matroids \( M_1 = (E, \mathcal{I}_1) \) and \( M_2 = (E, \mathcal{I}_2) \) and a positive integer \( k \) such that \( E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k \), there exists a partition \( \{X_1, X_2, \ldots, X_k\} \) of \( E \) such that \( X_j \in \mathcal{I}_1 \cap \mathcal{I}_2 \) for each \( j = 1, 2, \ldots, k \).

In the rest of this subsection, we present an alternative proof for this theorem via the generalized-polymatroid approach. We first observe some properties of laminar matroids. It is known and can be easily observed that the class of laminar matroids is closed under taking restrictions.

Lemma 3.6. Let \( M = (E, \mathcal{I}) \) be a laminar matroid induced from a laminar family \( \mathcal{A} \) and a capacity function \( q : \mathcal{A} \to \mathbb{Z}_{\geq 0} \). Then for any subset \( S \subseteq E \), the restriction \( M|_S \) of \( M \) to \( S \) is a laminar matroid induced from a laminar family \( \mathcal{A}_S := \{ A \cap S \mid A \in \mathcal{A} \} \) and a capacity function \( q_S : \mathcal{A}_S \to \mathbb{Z}_{\geq 0} \) defined by \( q_S(A \cap S) = q(A) \) for \( S \cap A \in \mathcal{A}_S \).

The next lemma states that, if \( M = (E, \mathcal{I}) \) is a laminar matroid, then \( M^k = (E, \mathcal{I}^k) \) is also a laminar matroid.

Lemma 3.7. Let \( M = (E, \mathcal{I}) \) be a laminar matroid induced from a laminar family \( \mathcal{A} \) and a capacity function \( q : \mathcal{A} \to \mathbb{Z}_{\geq 0} \). Then for a positive integer \( k \), the matroid \( M^k = (E, \mathcal{I}^k) \) is a laminar matroid defined by

\[
\mathcal{I}^k = \{ X \subseteq E \mid \forall A \in \mathcal{A} : |X \cap A| \leq k \cdot q(A) \}.
\]

Proof. We show that, for any \( X \subseteq E \), there exists a partition \( \{Y_1, Y_2, \ldots, Y_k\} \) of \( X \) with \( Y_j \in \mathcal{I} \) \((j = 1, \ldots, k) \) if and only if \( |X \cap A| \leq k \cdot q(A) \) for any \( A \in \mathcal{A} \). The necessity is clear, because each \( Y_j \) satisfies \( |Y_j \cap A| \leq q(A) \) for any \( A \in \mathcal{A} \). For the sufficiency, suppose \( |X \cap A| \leq k \cdot q(A) \) for any \( A \in \mathcal{A} \). Let \( X = \{e_1, e_2, \ldots, e_{|X|}\} \) (i.e., give indices for the elements in \( X \)), so that for all \( A \in \mathcal{A} \) the elements in \( X \cap A \) have consecutive indices. This can be done easily because \( \mathcal{A} \) is a laminar family.\( ^4 \) For each \( j \in \{1, 2, \ldots, k\} \), let \( Y_j = \{ e_\ell \in X \mid \ell = j \mod k \} \). Then, \( \{Y_1, Y_2, \ldots, Y_k\} \) is a partition of \( X \), and for each \( Y_j \) and \( A \in \mathcal{A} \), we have \( |Y_j \cap A| \leq \lceil |X \cap A|/k \rceil \) by the definition of the indices. Because \( |X \cap A| \leq k \cdot q(A) \), this implies \( |Y_j \cap A| \leq q(A) \) for all \( A \in \mathcal{A} \). Thus, we have \( Y_j \in \mathcal{I} \) for each \( j \in \{1, 2, \ldots, k\} \).

The next lemma provides an integral intersecting-paramodular pair \((p, b)\) satisfying the condition in Proposition 3.1 for a laminar matroid.

Lemma 3.8. Let \( M = (E, \mathcal{I}) \) be a matroid induced from a laminar family \( \mathcal{A} \) and a function \( q : \mathcal{A} \to \mathbb{Z}_{\geq 0} \) and suppose \( E \in \mathcal{I}^k \) for a positive integer \( k \). Define \( p : 2^E \to \mathbb{Z} \cup \{-\infty\} \) and \( b : 2^E \to \mathbb{Z} \cup \{\infty\} \) by

\[
p(A) = |A| - (k - 1) \cdot q(A) \quad (A \in \mathcal{A}),
\]

\[
b(A) = q(A) \quad (A \in \mathcal{A}),
\]

where \( p(B) = -\infty \), \( b(B) = \infty \) for all \( B \in 2^E \setminus \mathcal{A} \). Then \((p, b)\) is an integral intersecting-paramodular pair satisfying \( \mathcal{F}(p, b) = \{ X \subseteq E \mid X \in \mathcal{I}, \ E \setminus X \in \mathcal{I}^{k-1} \} \).

\(^4\) Let \( \mathcal{A}_X = \{ X \} \cup \{ X \cap A \mid A \in \mathcal{A} \} \cup \{ \{ e \} \mid e \in X \} \). Since \( \mathcal{A} \) is laminar, \( \mathcal{A}_X \) is also laminar. Let \( T \) be a tree representation of \( \mathcal{A}_X \), i.e., the node sets of \( T \) is \( \mathcal{A}_X \) and a node \( A \) is a child of \( A' \) if \( A \subseteq A' \) and there is no \( A'' \) with \( A \subseteq A'' \subseteq A' \). Then each leaf is the singleton of an element in \( X \). Let \( X = \{ e_1, e_2, \ldots, e_{|X|} \} \) so that the indices represent the order in which the corresponding leaves are found in depth-first search from the root node \( X \). These indices satisfy the required condition.
Proof. Since $\mathcal{A}$ is laminar and the values of $p$ and $b$ are finite only on $\mathcal{A}$, there is no intersecting pair of subsets of $E$ both of which have finite function values. Thus, $(p, b)$ is trivially intersecting paramodular.

For any $X \subseteq E$, the condition $\forall A \in \mathcal{A} : |X \cap A| \geq p(A) = |A| - (k - 1) \cdot q(A)$ is equivalent to $\forall A \in \mathcal{A} : |(E \setminus X) \cap A| \leq (k - 1) \cdot q(A)$, and hence equivalent to $E \setminus X \in \mathcal{I}^{k-1}$ by Lemma 3.7. Also, $\forall A \in \mathcal{A} : |X \cap A| \leq b(A) = q(A)$ is equivalent to $X \in \mathcal{I}$. Thus we have $X \in \mathcal{F}(p, b)$ if and only if $X \in \mathcal{I}$ and $E \setminus X \in \mathcal{I}^{k-1}$ hold. \hfill $\square$

Now we show that Problem 1.2 can be solved for any pair of laminar matroids using the generalized-polymatroid approach.

Proof of Theorem 3.3. We show the theorem by induction on $k$. The case $k = 1$ is trivial. Let $k \geq 2$ and suppose that the statement holds for $k - 1$. By Lemma 3.8, for each $i = 1, 2$, there exists an integral intersecting-paramodular pair $(p_i, b_i)$ such that $\mathcal{F}(p_i, b_i) = \{X \subseteq E \mid X \in \mathcal{I}_i, E \setminus X \in \mathcal{I}_i^{k-1}\}$. Then, by Proposition 3.1, there exists $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ satisfying $E \setminus X \in \mathcal{I}_1^{k-1} \cap \mathcal{I}_2^{k-1}$. By Lemma 3.6, the restrictions $M'_1 := M_1|(E \setminus X)$ and $M'_2 := M_2|(E \setminus X)$ are laminar. Therefore, by the induction hypothesis, $E \setminus X$ can be partitioned into $k - 1$ common independent sets of $M'_1$ and $M'_2$, and hence of $M_1$ and $M_2$. Thus, $E$ can be partitioned into $k$ common independent sets. \hfill $\square$

3.3 Intersection of Two Matroids without $(k + 1)$-Spanned Elements

Let $M = (E, \mathcal{I})$ be a matroid and $k$ be a positive integer. Recall that an element $e \in E$ is said to be $k$-spanned in $M$ if there exist $k$ disjoint sets spanning $e$ (including the trivial spanning set $\{e\}$).

Consider a class of matroids such that no element is $(k + 1)$-spanned. Kotlar and Ziv [16] provided two cases for which Problem 1.2 admits solutions.

Theorem 3.9 (Kotlar and Ziv [16]). Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids with rank functions $r_1$ and $r_2$ and suppose $r_1(E) = r_2(E) = d$ and $|E| = k \cdot d$. If no element of $E$ is $(k + 1)$-spanned in either $M_1$ or $M_2$, then $E$ can be partitioned into $k$ common bases.

Theorem 3.10 (Kotlar and Ziv [16]). Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids. If no element of $E$ is $3$-spanned in either $M_1$ or $M_2$, then $E$ can be partitioned into two common independent sets.

Note that, in Theorems 3.9 and 3.10, the condition $E \in \mathcal{I}_1^k \cap \mathcal{I}_2^k$ is not explicitly assumed. However, it can be easily proved by induction on $|E|$.

Lemma 3.11 (Kotlar and Ziv [16]). If no element of a matroid $M = (E, \mathcal{I})$ is $(k + 1)$-spanned, then $E \in \mathcal{I}^k$.

We provide unified proofs for Theorems 3.9 and 3.10 via the generalized-polymatroid approach, by constructing integral paramodular pairs satisfying (2) in Proposition 3.1. We first show that the cross-inequality condition, which is required for paramodularity, is equivalent to a seemingly weaker condition.

Lemma 3.12. A pair $(p, b)$ of set functions satisfies the cross inequality (1) for any $A, B \subseteq E$ if and only if it satisfies the following inequality for every pair of disjoint subsets $\tilde{A}, \tilde{B} \subseteq E$ and an element $e \in E \setminus (\tilde{A} \cup \tilde{B})$:

$$b(\tilde{A} + e) - b(\tilde{A}) \geq p(\tilde{B} + e) - p(\tilde{B}).$$
Proof. The necessity is obvious, since (3) is obtained by substituting \( A = \tilde{A} + e \) and \( B = \tilde{B} + e \) into (1). For sufficiency, we show (1) for arbitrary \( A, B \subseteq E \) under the assumption of (3). Let 
\[ A \cap B = \{e_1, e_2, \ldots, e_m \} \] 
where \( m = |A \cap B| \) and define \( \hat{A}_\ell = (A \setminus B) \cup \{e_1, e_2, \ldots, e_{\ell-1}\} \) and \( \hat{B}_\ell = (B \setminus A) \cup \{e_{\ell+1}, e_{\ell+2}, \ldots, e_m\} \) for each \( \ell \in \{1, 2, \ldots, m\} \). Then \( \hat{A}_\ell \) and \( \hat{B}_\ell \) are disjoint and \( e_\ell \in E \setminus (\hat{A}_\ell \cup \hat{B}_\ell) \), and hence we have \( b(\hat{A}_\ell + e_\ell) - b(\hat{A}_\ell) \geq p(\hat{B}_\ell + e_\ell) - p(\hat{B}_\ell) \) for \( \ell = 1, 2, \ldots, m \).

As we have \( \hat{A}_1 = A \setminus B \), \( \hat{A}_m + e_m = A \), \( \hat{B}_1 + e_1 = B \), and \( \hat{B}_m = B \setminus A \), it follows that

\[
b(A) - b(A \setminus B) = \sum_{\ell=1}^{m} b(\hat{A}_\ell + e_\ell) - b(\hat{A}_\ell) \geq \sum_{\ell=1}^{m} p(\hat{B}_\ell + e_\ell) - p(\hat{B}_\ell) = p(B) - p(B \setminus A).
\]

Thus, \( A \) and \( B \) satisfy the cross inequality (1). \qed

Lemma 3.12 states that, the range of the subsets \( A, B \subseteq E \) in the cross inequality (1) can be narrowed so that \( |A \cap B| = 1 \). This characterization of paramodularity is of independent interest and somewhat similar to the fact that submodularity is characterized by the local submodularity (see, e.g., [21, Theorem 44.1]) or the diminishing return property (see, e.g., [22, 23]). We also remark that the submodularity of \( p \) and the supermodularity of \( b \) are not assumed in Lemma 3.12.

Now an integral paramodular pair satisfying the condition in Proposition 3.1 is constructed as follows.

**Lemma 3.13.** Let \( M = (E, I) \) be a matroid with rank function \( r : 2^E \rightarrow \mathbb{Z}_{\geq 0} \). For a positive integer \( k \), suppose that no element is \((k+1)\)-spanned in \( M \). Define \( p : 2^E \rightarrow \mathbb{Z} \) and \( b : 2^E \rightarrow \mathbb{Z} \) by

\[
p(A) = |A| - r^{k-1}(A) \quad (A \subseteq E), \]

\[
b(A) = r(A) \quad (A \subseteq E).
\]

Then \((p, b)\) is an integral paramodular pair such that \( F(p, b) = \{X \subseteq E \mid X \in I, E \setminus X \in I^{k-1}\} \).

Proof. It directly follows from the definitions of \( p \) and \( b \) that (i) \( p(\emptyset) = b(\emptyset) = 0 \), (ii) \( p \) is supermodular, \( b \) is submodular. Then, to prove that \((p, b)\) is paramodular, it remains to show the cross inequality (1) for any \( A, B \subseteq E \). By Lemma 3.12, it suffices to show (3) for any disjoint \( \tilde{A}, \tilde{B} \subseteq E \) and an element \( e \in E \setminus (\tilde{A} \cup \tilde{B}) \), where (3) is rephrased as follows by the definitions of \( p \) and \( b \):

\[
\left( r(\tilde{A} + e) - r(\tilde{A}) \right) + \left( r^{k-1}(\tilde{B} + e) - r^{k-1}(\tilde{B}) \right) \geq 1.
\]

Take a maximal independent set \( X \) of \( M \) subject to \( X \subseteq \tilde{A} \) and a maximal independent set \( Y \) of \( M^{k-1} \) subject to \( Y \subseteq \tilde{B} \). It is sufficient to show \( X + e \in I \) or \( Y + e \in I^{k-1} \), because they respectively imply \( r(\tilde{A} + e) \geq r(\tilde{A}) + 1 \) or \( r^{k-1}(\tilde{B} + e) \geq r^{k-1}(\tilde{B}) + 1 \). Note that \( Y \in I^{k-1} \) can be partitioned into \( k-1 \) independent sets \( Y_1, Y_2, \ldots, Y_{k-1} \in I \). Also, \( k+1 \) subsets \( \{e\}, X, Y_1, Y_2, \ldots, Y_{k-1} \) are all disjoint. Because no element is \((k+1)\)-spanned in \( M \), it follows that \( e \) is not spanned by at least one of \( X, Y_1, Y_2, \ldots, Y_{k-1} \). Note that \( Y_j + e \in I \) for some \( j \) implies \( Y + e \in I^{k-1} \) by the definition of \( I^{k-1} \). We then have \( X + e \in I \) or \( Y + e \in I^{k-1} \). Thus, the paramodularity of \((p, b)\) is proved.

We next show \( F(p, b) = \{X \subseteq E \mid X \in I, E \setminus X \in I^{k-1}\} \). For any \( X \subseteq E \), the condition \( \forall A \subseteq E : |X \cap A| \geq p(A) = |A| - r^{k-1}(A) \) is equivalent to \( \forall A \subseteq E : |(E \setminus X) \cap A| \leq r^{k-1}(A) \), and hence equivalent to \( E \setminus X \in I^{k-1} \). Also, \( \forall A \subseteq E : |X \cap A| \leq b(A) = r(A) \) is equivalent to \( X \in I \). Thus we have \( X \in F(p, b) \) if and only if \( X \in I \) and \( E \setminus X \in I^{k-1} \) hold. \qed
Combining Lemmas 3.13 and Proposition 3.14 yields the following proposition.

**Proposition 3.14.** Let \( M_1 = (E, \mathcal{I}_1) \) and \( M_2 = (E, \mathcal{I}_2) \) be two matroids. If no element of \( E \) is \((k+1)\)-spanned in either \( M_1 \) or \( M_2 \), then there exists a subset \( X \subseteq E \) such that \( X \in \mathcal{I}_1 \cap \mathcal{I}_2 \) and \( E \setminus X \in \mathcal{I}_1^{k-1} \cap \mathcal{I}_2^{k-1} \).

Using this proposition, we can provide unified proofs for Theorems 3.9 and 3.10.

**Proof of Theorem 3.9.** By Proposition 3.14, there is \( X \in \mathcal{I}_1 \cap \mathcal{I}_2 \) with \( E \setminus X \in \mathcal{I}_1^{k-1} \cap \mathcal{I}_2^{k-1} \). Because \( r_1(E) = r_2(E) = d \) and \( |E| = k \cdot d \), the subsets \( X \) and \( E \setminus X \) should be common bases of \((M_1, M_2)\) and \((M_1^{k-1}, M_2^{k-1})\), respectively. For each matroid \( M_i \) \((i = 1, 2)\), since every element in \( E \setminus X \) is spanned by \( X \) but not \((k+1)\)-spanned, we see that no element in \( E \setminus X \) is \(k\)-spanned in \( M_i|\!(E \setminus X)\). Thus, \( X \in \mathcal{I}_1 \cap \mathcal{I}_2 \) and restrictions \( M_1|\!(E \setminus X) \) and \( M_2|\!(E \setminus X) \) satisfy the assumption of Theorem 3.9 with \( k \) replaced by \( k - 1 \). By induction, \( E \setminus X \) can be partitioned into \( k - 1 \) common independent sets. Thus, the proof is completed.

**Proof of Theorem 3.10.** By just applying Proposition 3.14 with \( k = 2 \), we obtain a common independent set \( X \in \mathcal{I}_1 \cap \mathcal{I}_2 \) satisfying \( E \setminus X \in \mathcal{I}_1 \cap \mathcal{I}_2 \). Thus, the proof is completed.

The original proofs for Theorems 3.9 and 3.10 [15] have no apparent relation. For these two theorems, we have shown unified proofs by our generalized-polymatroid approach. This offers a new understanding of the conditions in Theorems 3.9 and 3.10; they are nothing other than conditions under which our induction method works.

### 3.4 Intersection of a Laminar Matroid and a Matroid without \((k+1)\)-Spanned Elements

As mentioned in Remark 3.2, our g-polymatroid approach does not require the two matroids to be in the same matroid class, and thus can deal with an arbitrary pair of matroids which have appeared in this section. That is, we can obtain a solution of Problem 1.2 for a new class of matroid intersection, i.e., the intersection of a laminar matroid and a matroid without \((k+1)\)-spanned elements. The following theorems can be immediately derived from combining the proofs of Theorems 3.5, 3.9, and 3.10.

**Theorem 3.15.** Let \( k \) be a positive integer, \( M_1 = (E, \mathcal{I}_1) \) be a laminar matroid such that \( E \in \mathcal{I}_k \), and \( M_2 = (E, \mathcal{I}_2) \) be a matroid with rank function \( r_2 \) such that \(|E| = k \cdot r_2(E)\) and no element is \((k+1)\)-spanned in \( M_2 \). Then, there exists a partition \( \{X_1, X_2, \ldots, X_k\} \) of \( E \) such that \( X_j \in \mathcal{I}_1 \cap \mathcal{I}_2 \) for each \( j = 1, 2, \ldots, k \).

**Theorem 3.16.** Let \( M_1 = (E, \mathcal{I}_1) \) be a laminar matroid such that \( E \in \mathcal{I}_2 \) and \( M_2 = (E, \mathcal{I}_2) \) be a matroid with rank function \( r_2 \) such that \(|E| = 2 \cdot r_2(E)\) and no element is \(3\)-spanned in \( M_2 \). Then, there exists a partition \( \{X_1, X_2\} \) of \( E \) such that \( X_1, X_2 \in \mathcal{I}_1 \cap \mathcal{I}_2 \).

### 4 Example Incompatible with the G-polymatroid Approach

As mentioned before, the class of strongly base orderable matroids admits a solution for Problem 1.2 [9], and our g-polymatroid approach can deal with laminar matroids, a special class of strongly base
orderable matroids. Hence we expect that the g-polymatroid approach can be applied to strongly base orderable matroids. In this subsection, however, we exhibit its difficulty by constructing an example of a transversal matroid, another simple special case of a strongly base orderable matroid, which admits no intersecting-paramodular pair required in Proposition 3.1.

For a bipartite graph $G = (E, F; A)$ with color classes $E, F$ and edge set $A$, let $I$ be a family of subsets $X$ of $E$ such that $G$ has a matching that is incident to all elements in $X$. Then it is known that $(E, I)$ is a matroid [6], which we call the transversal matroid induced from $G$.

Example 4.1. Let $E = \{e_1, e_2, e_3, e_4, e'_1, e'_2, e'_3, e'_4\}$ and $G$ be a bipartite graph with color classes $E$ and $F$. Figure 1 depicts $G$, where the white and black nodes represent $E$ and $F$, respectively.

![Figure 1: A bipartite graph that induces a transversal matroid $M = (E, I)$ in Example 4.1](image)

We show that transversal matroids are incompatible with our g-polymatroid approach by proving that the transversal matroid $M$ in Example 4.1 admits no intersecting-paramodular pair $(p, b)$ satisfying $F(p, b) = \{ X \subseteq E \mid X \in I, E \setminus X \in I^{k-1} \}$, i.e., we cannot use Proposition 3.1 for $M$. For this purpose, we prepare the following fact (see e.g., [9, 17, 18]).

Lemma 4.2. For any integral intersecting-paramodular pair $(p, b)$, if $J := F(p, b) \neq \emptyset$, then $(E, J)$ is a generalized matroid, i.e., $J$ satisfies the following axioms:

1. If $X, Y \in J$ and $e \in Y \setminus X$, then $X + e \in J$ or $\exists e' \in X \setminus Y : X + e - e' \in J$.
2. If $X, Y \in J$ and $e \in Y \setminus X$, then $Y - e \in J$ or $\exists e' \in X \setminus Y : Y - e + e' \in J$.

Now the incompatibility of the transversal matroid $M$ in Example 4.1 is established by the following proposition.

Proposition 4.3. For the transversal matroid $M = (E, I)$ given in Example 4.1, there is no integral intersecting-paramodular pair $(p, b)$ satisfying $F(p, b) = \{ X \subseteq E \mid X \in I, E \setminus X \in I^{k-1} \}$ with $k = 2$, while $E$ can be partitioned into two independent sets in $M$.

Proof. The latter statement is obvious: $E$ can be partitioned into two bases $\{e_1, e_2, e_3, e_4\}$ and $\{e'_1, e'_2, e'_3, e'_4\}$.

Suppose to the contrary that $F(p, b) = \{ X \subseteq E \mid X \in I, E \setminus X \in I \}$ holds for some integral intersecting-paramodular pair $(p, b)$. By Lemma 4.2, then $J := F(p, b) = \{ X \subseteq E \mid X \in I, \overline{X} \in I \}$ satisfies (J1) and (J2), where $\overline{X}$ denotes $E \setminus X$. 
Let $X := \{e_1, e_2, e'_1, e'_2\}$ and $Y := \{e_3, e_4, e'_3, e'_4\}$. We can observe $X, X \setminus Y \in I$, and hence $X, Y \in J$. Since $X + e_3 \not\in J$ and $X \setminus Y = \{e_1, e_2\}$, by (J1), $X + e_3 - e_1 \in J$ or $X + e_3 - e_2 \in J$ must hold. However, it holds that $X + e_3 - e_1 = \{e_2, e_3, e'_1, e'_2\} \not\in I$, implying $X + e_3 - e_1 \not\in J$, and it also holds that $X + e_3 - e_2 = \{e_2, e_4, e'_3, e'_4\} \not\in I$, implying $X + e_3 - e_2 \not\in J$, a contradiction. 

5 Conclusion

In this paper, we have presented a new approach based on the generalized-polymatroid intersection theorem \[8\] to the problem of partitioning the common ground set of two matroids into common independent sets. This approach is an extension of the polyhedral proof for König’s bipartite edge-coloring theorem \[15\], and is known to be successful for supermodular coloring \[24\]. We have exhibited that the generalized-polymatroid approach can provide alternative proofs and unified understandings for some extensions of König’s theorem for laminar matroids, which form a special class of strongly base orderable matroids \[3\], and for the two classes of matroids without \((k + 1)\)-spanned elements \[16\]. Then we have shown a new class of matroid intersection for which König’s theorem can be extended: intersection of a laminar matroid and a matroid in the classes of \[16\]. We have also shown that the generalized-polymatroid approach does not work for some transversal matroid, which is also a special case of a strongly base orderable matroid.

A key ingredient of our approach is Proposition 3.1: certain integral intersecting-paramodular pairs enable us to find a desired partition by induction. An advantage of this approach is that it does not require the two matroids to belong to the same class of matroids, which has enabled us to attain a new extension of König’s theorem by combining the previous results. While the problem is still challenging, our approach suggests a new way to obtain new classes of matroid intersection admitting an extension of König’s theorem by finding such integral intersecting-paramodular pairs.

Acknowledgments

The authors would like to thank Kazuo Murota for his valuable comments. The first author is partially supported by JST CREST Grant Number JPMJCR1402, JSPS KAKENHI Grant Numbers JP16K16012, JP26280004, Japan. The second author is supported by JST CREST Grant Number JPMJCR14D2, JSPS KAKENHI Grant Number JP18K18004, Japan.

References

[1] R. Aharoni and E. Berger: The intersection of a matroid and a simplicial complex, Transactions of the American Mathematical Society, 358 (2006), pp. 4895–4917.

[2] R. A. Brualdi: Common transversals and strong exchange systems, Journal of Combinatorial Theory, 8 (1970), pp. 307–329.

[3] J. Davies and C. McDiarmid: Disjoint common transversals and exchange structures, Journal of the London Mathematical Society, 2 (1976), pp. 55–62.

[4] J. Edmonds: Minimum partition of a matroid into independent subsets, Journal of Research National Bureau of Standards Section B, 69 (1965), pp. 67–72.
[5] J. Edmonds: Edge-disjoint branchings, R. Rustin, editor, *Combinatorial Algorithms*, Algorithmics Press, 1973, pp. 285–301.

[6] J. Edmonds and D. R. Fulkerson: Transversals and matroid partition, *Journal of Research National Bureau of Standards Section B*, 69 (1965), pp. 147–157.

[7] T. Fife and J. Oxley: Laminar matroids, *European Journal of Combinatorics*, 62 (2017), pp. 206–216.

[8] A. Frank: Generalized polymatroids, *Finite and Infinite Sets (Proc. Sixth Hungarian Combinatorial Colloquium, 1981)*, Colloquia Mathematica Societatis János Bolyai 37, North-Holland, 1984, pp. 285–294.

[9] A. Frank: *Connections in Combinatorial Optimization*, Oxford Lecture Series in Mathematics and its Applications, 38, Oxford University Press, Oxford, 2011.

[10] A. Frank and É. Tardos: Generalized polymatroids and submodular flows, *Mathematical Programming*, 42 (1988), pp. 489–563.

[11] S. Fujishige: *Submodular Functions and Optimization*, Annals of Discrete Mathematics, 58, Elsevier, Amsterdam, second edition, 2005.

[12] N. J. A. Harvey, T. Király, and L. C. Lau: On disjoint common bases in two matroids, *SIAM Journal on Discrete Mathematics*, 25 (2011), pp. 1792–1803.

[13] R. Hassin: Minimum cost flow with set-constraints, *Networks*, 12 (1982), pp. 1–21.

[14] R. Huang and G.-C. Rota: On the relations of various conjectures on latin squares and straightening coefficients, *Discrete Mathematics*, 128 (1994), pp. 225–236.

[15] D. Kőnig: Graphok és alkalmazásuk a determinánsok és a halmazok elméletére (Hungarian; Graphs and their application to the theory of determinants and sets), *Mathematikai és Természettudományi Értesítő*, 34 (1916), pp. 104–119.

[16] D. Kotlar and R. Ziv: On partitioning two matroids into common independent subsets, *Discrete Mathematics*, 300 (2005), pp. 239–244.

[17] K. Murota: *Discrete Convex Analysis*, SIAM, Philadelphia, 2003.

[18] K. Murota and A. Shioura: M-convex function on generalized polymatroid, *Mathematics of Operations Research*, 24 (1999), pp. 95–105.

[19] J. Oxley: *Matroid Theory*, Oxford University Press, Oxford, second edition, 2011.

[20] A. Schrijver: Supermodular colourings, *Matroid Theory* (L. Lovász and A. Recski, eds.), North-Holland, Amsterdam, 1985, pp. 327–343.

[21] A. Schrijver: *Combinatorial Optimization: Polyhedra and Efficiency*, Springer, Heidelberg, 2003.
[22] T. Soma and Y. Yoshida: A generalization of submodular cover via the diminishing return property on the integer lattice, *Advances in Neural Information Processing Systems 28* (C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama and R. Garnett, eds.), 2015, pp. 847–855.

[23] T. Soma and Y. Yoshida: Maximizing monotone submodular functions over the integer lattice, *Integer Programming and Combinatorial Optimization: Proceedings of the 18th IPCO, LNCS 9682* (Q. Louveaux and M. Skutella, eds.), Springer, 2016, pp. 325–336.

[24] É. Tardos: Generalized matroids and supermodular colourings, *Matroid Theory* (L. Lovász and A. Recski, eds.), North-Holland, Amsterdam, 1985, pp. 359–382.

[25] D. J. A. Welsh: *Matroid Theory*, Academic Press, London, 1976.