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FORMULAE FOR TWO-VARIABLE GREEN FUNCTIONS

FRANCÔIS DIGNE AND JEAN MICHEL

Abstract. Based on results of Digne-Michel-Lehrer (2003) we give two formulae for two-variable Green functions attached to Lusztig induction in a finite reductive group. We present applications to explicit computation of these Green functions, to conjectures of Malle and Rotilio, and to scalar products between Lusztig inductions of Gelfand-Graev characters.

Let $G$ be a connected reductive group with Frobenius root $F$; that is, some power $F^q$ is a Frobenius endomorphism attached to an $F_q$-structure on $G$, where $q^e$ is a power of a prime $p$. Let $L$ be an $F$-stable Levi subgroup of a (non-necessarily $F$-stable) parabolic subgroup $P$ of $G$. Let $U$ be the unipotent radical of $P$ and let $X_U = \{gU \in G/U \mid g^{-1}Fg \in U \cdot F U\}$ be the variety used to define the Lusztig induction and restriction functors $R^G_L$ and $R^G_L$. For $u \in G^F, v \in L^F$ unipotent elements, the two-variable Green function is defined as

$$Q^G_L(u,v) = \text{Trace}((u,v) | \sum_i (-1)^i H_i^e(X_U)).$$

In this paper, using the results of [5], we give two different formulae for two-variable Green functions, and some consequences of these, including proving some conjectures of [15].

The two-variables Green functions occur in the character formulae for Lusztig induction and restriction. In particular, for unipotent elements these formulae read

\begin{itemize}
  \item If $u$ is a unipotent element of $G^F$, and $\psi$ a class function on $L^F$, we have
    \[ R^G_L(\psi)(u) = |L^F|\langle \psi, Q^G_L(u, -) \rangle_{L^F}. \]
  \item If $v$ is a unipotent element of $L^F$, and $\chi$ a class function on $G^F$, we have
    \[ \ast R^G_L(\chi)(v) = |L^F|\langle \chi, Q^G_L(-, v^{-1}) \rangle_{G^F}. \]
\end{itemize}

**Two formulae for two-variable Green functions**

For an element $u$ in a group $G$ we denote by $u^G$ the $G$-conjugacy class of $u$.

**Proposition 2.** Assume either the centre $ZG$ of $G$ is connected and $q > 2$, or $q$ is large enough (depending just on the Dynkin diagram of $G$). Then for $u$ regular, $Q^G_L(u, -)$ vanishes outside a unique regular unipotent class of $L^F$. For $v$ in that class, we have $Q^G_L(u, v) = |v^L|^e$.

\textit{Proof (Rotilio).} Let $\gamma^u_G$ be the normalized characteristic function of the $G^F$-conjugacy class of $u$; that is, the function equal to 0 outside the class of $u$ and to $|C_G(u)^F|$ on that class. For $v' \in L^F$ unipotent, Proposition 1 gives

$$\ast R^G_L(\gamma^u_G)(v') = |L^F|\langle \gamma^u_G, Q^G_L(-, v'^{-1}) \rangle_{G^F} = |L^F|Q^G_L(u, v'^{-1}).$$
Now, by [1, Theorem 15.2], there exists \( v \in L^F \) such that the left-hand side is equal to \( q_{\psi}(v') \). In [1, Theorem 15.2] \( q \) is assumed large enough so that the Mackey formula holds and that \( p \) and \( q \) are such that the results of [10] hold (that is \( p \) almost good and \( q \) larger than some number depending on the Dynkin diagram of \( G \)). The Mackey formula is known to hold if \( q > 2 \) by [2] and the results of [10] hold without condition on \( q \) if \( ZG \) is connected by [17] or [18, Theorem 4.2] (the assumption that \( p \) is almost good can be removed, see [13, section 89] and [12]). The proposition follows.

As in [5], we consider irreducible \( G \)-equivariant local systems on unipotent classes. These local systems are partitioned into “blocks” parametrised by cuspidal pairs formed by a Levi subgroup and a cuspidal local system supported on a unipotent class of that Levi subgroup. Let us call \( Q_{wF}^{I} \) the function defined in [5, 3.1(iii)] relative to an \( F \)-stable block \( I \) of unipotently supported local systems on \( L \) and to \( wF \in W_L(L^F) \) where \((L^F,\iota)\) is the cuspidal datum of \( I \) (for a Levi \( L \) of a reductive group \( G \), we set \( W_G(L) = N_G(L)/L \)).

**Proposition 3.** Assume either \( ZG \) is connected or \( q \) is large enough (depending just on the Dynkin diagram of \( G \)). For \( u \) a unipotent element of \( G^F \) and \( v \) a unipotent element of \( L^F \), we have

\[
Q^G_L(u,v) = |L^F|^{-1} \sum_{I} \sum_{w \in W_L(L^F)} \frac{|Z^0(L^F)wF|}{|W_L(L^F)|} \hat{Q}^{G,F}_w(u)Q_{wF}^{I}(v),
\]

where \( I \) runs over the \( F \)-stable blocks of \( L \) and where \( I_G \) is the block of \( G \) with same cuspidal data as \( I \).

The part of the above sum for \( I \) the principal block is the same formula as [7, Corollaire 4.4].

**Proof.** Proposition 1 applied with \( \psi = Q_{wF}^{L} \) gives, if we write \( R_{L,F}^{G,F} \) instead of \( R_L^G \) to keep track of the Frobenius,

\[
\langle Q_{wF}^{L}, Q^G_L(u,-) \rangle_{L^F} = |L^F|^{-1} R_{L,F}^{G,F}(Q^L_{wF})(u)
\]

Now we have by [5, Proposition 3.2] \( Q_{wF}^{L,F} = R_{L,F}^{G,F} \hat{\chi}_{L^F}(wF) \) where \( \hat{\chi}_{L^F}(wF) = q^{c_{L^F}} \) times the characteristic function of \((L^F, wF)\), a class function on \( L^F \). Here, as in [5, above Remark 2.1], for an irreducible \( G \)-equivariant local system \( \iota \), we denote by \( C_\iota \) the unipotent \( G \)-conjugacy class which is the support of \( \iota \), and if \((L^F, \iota)\) is the cuspidal datum of \( \iota \) we set \( c_\iota = \frac{1}{2} \text{codim} C_{\iota} - \text{dim} Z(L^F) \). In [5, Proposition 3.2] the assumptions on \( p \) and \( q \) come from [10] but by the same considerations than at the end of the proof of Proposition 2 it is sufficient to assume \( ZG \) connected or \( q \) large enough.

By the transitivity of Lusztig induction we get

\[
\langle Q_{wF}^{L,F}, Q^G_L(u,-) \rangle_{L^F} = |L^F|^{-1} R_{L,F}^{G,F}(\hat{\chi}_{L^F}(wF))(u) = |L^F|^{-1} Q^G_{wF}(u),
\]

where \( I_G \) is the block of \( G \) with same cuspidal data as \( I \). Using the orthogonality of the Green functions \( Q_{wF}^{L} \), see [5, Corollary 3.5] (where the assumption that \( p \) is almost good which comes from [9], so can be removed now by [12]) and the fact they form a basis of unipotently supported class functions on \( L^F \), indexed by the \( W_L(L^F) \)-conjugacy classes of \( W_L(L^F)F \), we get the proposition. □
Proposition 3 gives a convenient formula to compute automatically two-variable Green functions. Table 1 gives an example, computed with the package Chevie (see [16]).

We denote by \( Y_i \) the characteristic function of the \( F \)-stable local system \( i \), and by \( A(u) \) the group of components of the centralizer of a unipotent element \( u \).

**Proposition 4.** Assume either \( ZG \) is connected or \( q \) is large enough (depending just on the Dynkin diagram of \( G \)). Let \( R_{i, \gamma} \) be the polynomials which appear in [5, Lemma 6.9]. Then

\[
Q^G_{\ell}(u, v) = |v_\ell^{-1}|A(v)|^{-1} \sum_{I} \sum_{i \in I_G, \gamma \in \mathcal{I}_F} \overline{Y_i(u)} Y_{\gamma}(v) R_{i, \gamma} q^{c_i - c_u},
\]

where \( c_i = \frac{1}{2} \text{codim} C_i - \text{dim} Z(L_T). \)

**Proof.** For a block \( I \) of \( L \) and \( v \in I_F \), let \( \hat{Q}_v \) be the function of [5, (4.1)]. Then by [5, (4.4)] applied respectively in \( G \) and \( L \) we have

\[
Q^G_{wF}(u) = \sum_{v \in I_F} \hat{Q}_v(wF) \hat{Y}_u(v) \quad \text{and} \quad Q^L_{wF}(v) = \sum_{\kappa \in I_F} \hat{Q}_\kappa(wF) \hat{Y}_v(\kappa),
\]

where \( \hat{Y}_u = q^{c_u} Y_u \). Thus, using the notation \( Z_{L_T} \) as in [5, 3.3] to denote the function \( wF \mapsto |Z^0(L_T)^wF| \) on \( W_G(L_T)F \), the term relative to a block \( I \) in the formula of Proposition 3 can be written

\[
|L|^{|-1}(Z_{L_T} \sum_{\kappa \in I_F} \hat{Q}_\kappa \hat{Y}_\kappa(v) \sum_{\iota \in I_G, \gamma \in \mathcal{I}_F} \hat{Y}_\iota(u) R_{\iota, \gamma} \hat{Q}_\gamma \hat{Y}_\kappa(wL_T)F).
\]

Applying now [5, Lemma 6.9] this is equal to

\[
|L|^{|-1}(Z_{L_T} \sum_{\kappa \in I_F} \hat{Q}_\kappa \hat{Y}_\kappa(v) \sum_{\iota \in I_G, \gamma \in \mathcal{I}_F} \hat{Y}_\iota(u) R_{\iota, \gamma} \hat{Q}_\gamma \hat{Y}_\kappa(wL_T)F),
\]

We use now [5, Corollary 5.2] which says that, \( \langle \hat{Q}_\gamma, Z_{L_T} \hat{Q}_\kappa \rangle_{wL_T}(L_T)F = 0 \) unless \( C_\gamma = C_\kappa \) and in this last case is equal to

\[
|A(v)|^{-1} \sum_{a \in A(v)} |C^0_{L}(v_a)^F| q^{2c_\gamma} Y_\gamma(v_a) \overline{Y_\kappa(v_a)}.
\]

Thus the previous sum becomes

\[
|L|^{|-1} \sum_{\iota \in I_G, \gamma \in \mathcal{I}_F} \hat{Y}_\iota(u) R_{\iota, \gamma} |A(v)|^{-1} \sum_{a \in A(v)} |C^0_{L}(v_a)^F| q^{2c_\gamma} Y_\gamma(v_a) \sum_{\kappa \in I_F} \overline{Y_\kappa(v_a)Y_\kappa(v)}.
\]

But by [5, (4.5)] we have \( \sum_{\kappa} \overline{Y_\kappa(v_a)Y_\kappa(v)} = \begin{cases} q^{c_\kappa} |A(v)|^F & \text{if } v_a = v \\ 0 & \text{otherwise} \end{cases} \) where \( \kappa \) runs over all local systems. Thus, summing over all the blocks, we get the formula in the statement. \( \square \)

**Corollary 5.** Assume either \( ZG \) is connected or \( q \) is large enough (depending just on the Dynkin diagram of \( G \)). Then for any unipotent elements \( u \in G_F \) and \( v \in L_F \) we have:

(i) \( Q^G_{\ell}(u, v) \) vanishes unless \( v^G \subseteq u^G \subseteq \text{Ind}^G_L(v^L) \), where \( \text{Ind}^G_L(v^L) \) is the induced class in the sense of [14].

(ii) \( |v_\ell L_F| \cdot |A(v)| Q^G_{\ell}(u, v) \) is an integer and is a polynomial in \( q \) with integral coefficients.
Proof. For (i), we use [5, Lemma 6.9(i)] which states that \( R_{i, \gamma} = 0 \) unless \( C_\gamma \subseteq \overline{C_i} \subseteq \text{Ind}^G_L(C_\gamma) \). Since \( \tilde{\mathcal{Y}}_\nu(v) \) vanishes unless \( C_\nu \ni v \), the only non-zero terms in the formula of proposition 4 have \( C_\gamma \ni v \), whence the result since \( \tilde{\mathcal{Y}}_\nu(u) \) vanishes unless \( C_i \ni u \).

For (ii), we start with

**Lemma 6.** \( q^{c_{i, \gamma}} R_{i, \gamma} \) is a polynomial in \( q \) with integral coefficients.

**Proof.** The defining equation of the matrix \( \tilde{\mathcal{R}} = \{ q^{c_{i, \gamma}} R_{i, \gamma} \}_{i, \gamma} \) reads (see the proof of [5, Lemma 6.9(i)]):

\[
\tilde{\mathcal{R}} = P_G C_G I C_L^{-1} P_L^{-1}
\]

where \( C_G \) is the diagonal matrix with diagonal coefficients \( q^{c_{i}} \) for \( i \in I_G \), and \( C_L \) is the similar matrix for \( L \) and \( I \), where \( P_G \) is the matrix with coefficients \( \{ P_{i', i} \}_{i', i} \in I_G \) where these polynomials are those defined in [11, 6.5], and \( P_L \) is the similar matrix for \( L \) and \( I \), and finally \( I \) is the matrix with coefficients

\[
I_{i, \gamma} = \langle \text{Ind}^{W_G(L_x)}_{W_L(L_z)} \tilde{\varphi}_\gamma, \tilde{\varphi}_i \rangle_{W_G(L_z)}
\]

where \( \tilde{\varphi}_i \) is the character of \( W_L(L_z) \) which corresponds to the generalised Springer correspondence to \( \gamma \) (and similarly for \( \tilde{\varphi}_\gamma \)). Since \( P_L \) and \( P_G \) are unitriangular matrices with coefficients integral polynomials in \( q \), thus \( P_L^{-1} \) also, it suffices to prove that \( C_G I C_L^{-1} \) has coefficients polynomial in \( q \), or equivalently that

\[
\text{if } \langle \text{Ind}^{W_G(L_x)}_{W_L(L_z)} \tilde{\varphi}_\gamma, \tilde{\varphi}_i \rangle_{W_G(L_z)} \neq 0, \text{ then } c_i - c_\gamma \geq 0.
\]

We now use [5, Proposition 2.3(ii)] which says that the non-vanishing above implies \( C_\gamma \subseteq \overline{C_i} \subseteq \text{Ind}^G_L(C_\gamma) \). We now use that, according to the definitions, \( c_i - c_\gamma = \dim B_u^G - \dim B_v^L \) where \( B_u^G \) is the variety of Borel subgroups of \( G \) containing an element \( u \) of the support of \( \gamma \), and where \( B_v^L \) is the variety of Borel subgroups of \( L \) containing an element \( v \) of the support of \( \gamma \). Now the lemma follows from the fact that by [14, Theorem 1.3 (b)] we have \( \dim B_u^G = \dim B_v^L \) if \( u \) is an element of \( \text{Ind}^G_L(C_\gamma) \), and that \( \dim B_u^G \) is greater for \( u \in \text{Ind}^G_L(C_\gamma) - \text{Ind}^G_L(C_\gamma) \).

Now (ii) results from the lemma: since the \( \tilde{\mathcal{Y}} \) have values algebraic integers, by Proposition 4 the expression in (ii) is a polynomial in \( q \) with coefficients algebraic integers. But, since \( L^F \) is a Lefschetz number (see for example [8, 8.1.3]), the expression in (ii) is a rational number; since this is true for an infinite number of integral values of \( q \) the expression in (ii) is a polynomial with integral coefficients.

**Scalar products of induced Gelfand-Graev characters**

The pretext for this section is as follows: in [2, Remark 3.10] is pointed the problem of computing \( (R^F_G \Gamma_\iota, R^G_L \Gamma_\iota)_G \) when \( (G, F) \) is simply connected of type \( \tilde{E}_6 \), when \( L \) is of type \( A_2 \times A_2 \), and when \( \iota \) corresponds to a faithful character of \( Z(L)/Z^0(L) \), and checking that the value is the same as given by the Mackey formula. We show now various ways to do this computation, where in this section we assume \( p \) and \( q \) large enough for all the results of [5] to hold (in particular, we assume \( p \) good for \( G \), thus not solving the problem of loc. cit. where we need \( q = 2 \)).
Let $Z = Z(G)$, and let $\Gamma_z$ be the Gelfand-Graev character parameterized by $z \in H^1(F, Z)$, see for instance [3, Definition 2.7]. Let $u_z$ be a representative of the regular unipotent class parametrized by $z$. As in [11, 7.5 (a)] for $\iota$ an $F$-stable local system on the regular unipotent class we define $\Gamma_\iota = c \sum_{\gamma \in H^1(F, Z)} \chi_\iota(u_z) \Gamma_z$ where $c = \frac{|Z/Z^0|}{|H^1(F, Z)|}$.

Note that the cardinality $|C_G(u_z)\mathcal{F}|$ is independent of $z$; actually it is equal to $|Z(G)^F| q^{\kappa_{kG}}$ (see [1, 15.5]). Thus we will denote this cardinality $|C_{G^F}(u)|$ where $u \in G^F$ is any regular unipotent element. There exists a character $\zeta$ of $H^1(F, Z)$ and a root of unity $b_\iota$ (see [4, above 1.5]) such that $\chi_\iota(u_z) = b_\iota \zeta(z)$. With these notations, we have

**Proposition 7.** We have $\Gamma_\iota = \eta_G \sigma_\zeta^{-1} c |C_{G^F}(u)| D\chi_\iota$ where $\eta_G$ and $\sigma_\zeta$ are defined as in [4, 2.5].

**Proof.** This proposition could be deduced from [6, Theorem 2.8] using [4, Theorem 2.7]. We give here a more elementary proof.

With the notations of [3, (3.5)] we have $D\Gamma_\iota = \sum_{z \in H^1(F, Z)} c_{z, z'} \gamma_{z'}$. By [4, lemma 2.3] we have $c_{z, z'} = c_{z, z'-1}$ and $\sum_{z \in H^1(F, Z)} \chi(z) c_{z, z'} = \eta_G \sigma_\zeta^{-1}$. It follows that

$$c^{-1}_\iota b^{-1}_\iota D\Gamma_\iota = \sum_{z \in H^1(F, Z)} \chi(z) D\Gamma_\iota = \sum_{z, z' \in H^1(F, Z)} \chi(z) c_{z, z'} \gamma_{z'} = \sum_{z, z' \in H^1(F, Z)} \chi(z) c_{z, z'-1} \gamma_{z'} = \sum_{z' \in H^1(F, Z)} \chi(z') \gamma_{z'} \sum_{z'' \in H^1(F, Z)} c_{z, z'-1} \chi(z'') = \eta_G \sigma_\zeta^{-1} \sum_{z' \in H^1(F, Z)} \chi(z') \gamma_{z'} = \eta_G \sigma_\zeta^{-1} b^{-1}_\iota |C_{G^F}(u)| \chi_\iota$$

$\Box$

**Proposition 8.** If $\iota$ is a local system supported on the regular unipotent class of $L$ and $\mathcal{I}$ denotes its block, we have

$$< R_G^L \Gamma_\iota, R_G^L \Gamma_\iota >_{G^F} = \frac{|Z(L)|^2}{Z^0(L)} \sum_{w \in W_L(L_z)} \frac{|Z^0(L_z)^w F| |W_G(L_z)| (|w F|)^{W_G(L_z)} \cap W_L(L_z) |}{(w F)^{W_G(L_z)} |}$$

Note that in a given block $\mathcal{I}$ there is at most one local system supported by the regular unipotent class (see [4, Corollary 1.10]).

**Proof.** When $\iota$ is supported by the regular unipotent class we have $\mathcal{Q}_\iota = 1$, see the beginning of section 7, bottom of page 130 in [5]. Using this in the last formula of the proof of [5, Proposition 6.1], we get that $\Gamma_\iota^L$ is up to a root of unity equal to $|A(C_l)| |W_L(L_z)|^{-1} \sum_{w \in W_L(L_z)} |Z^0(L_z)^w F| Q_{w F}$. Since $R_G^L Q_{w F} = Q_{w F}$, we get

$$< R_G^L \Gamma_\iota, R_G^L \Gamma_\iota >_{G^F} = |A(C_l)|^2 |W_L(L_z)|^{-2} \sum_{w, w' \in W_L(L_z)} |Z^0(L_z)^w F| |Z^0(L_z)^w F| < Q_{w F} Q_{w' F}>_{G^F}.$$
By [5, 3.5] the last scalar product is zero unless $wF$ and $w'F$ are conjugate in $W_G(L_I)$, and is equal to $|C_{W_G(L_I)}(wF)|/|Z^0(L_I)^{wF}|$ otherwise. We get
\[
\langle R_L^G \Gamma_1, R_L^G \Gamma_1 \rangle_{G^F} = |A(C_j)|^2 |W_L(L_I)|^{-2} \sum_{w \in W_L(L_I)} |Z^0(L_I)^{wF}| |C_{W_G(L_I)}(wF)||C_{W_G(L_I)}(wF)^{wF}| \cap W_L(L_I)|,
\]
which gives the formula of the proposition since $A(C_j) = Z(L)/Z(L)^0$. \hfill \Box

**Corollary 9.** Let $\iota$ and $\iota'$ be local systems supported on the regular unipotent class of $G$, and $I, I'$ be their respective blocks: then

\begin{enumerate}[(i)]
\item \[
\langle \Gamma^G_\iota, \Gamma^G_\iota' \rangle_{G^F} = \begin{cases} 0 & \text{if } \iota \neq \iota', \\ \frac{|Z(G)|^2}{Z^0(G)^F} |q^{\dim Z(L_I) - \dim Z(G)}| & \text{if } \iota = \iota'. \end{cases}
\]
\item \[
\langle \iota, \iota' \rangle_{G^F} = \begin{cases} 0 & \text{if } \iota \neq \iota', \\ q^{\rkss G} |Z^0(G)|^{F-1} & \text{if } \iota = \iota'. \end{cases}
\]
\end{enumerate}

**Proof.** The functions $Q^G_{wF}$ and $Q^G_{w'F}$ are orthogonal to each other when $I_G \neq I'_G$ (see [9, V, 24.3.6] where the orthogonality is stated for the functions $X_\iota$). Since there is a unique $\iota$ in a given block supported on the regular unipotent class, we get the orthogonality in (i). In the case $\iota = \iota'$ in (i), the specialization $L = G$ in Proposition 8 is $\langle \Gamma^G_\iota, \Gamma^G_\iota \rangle_{G^F} = \frac{|Z(G)|^2}{Z^0(G)^F} \sum_{w \in W_G(L_I)} |Z^0(L_I)^{wF}|/|W_G(L_I)|$. By [5, Corollary 5.2], where we use that $Q_\iota = 1$ when $\iota$ has regular support, we have $\sum_{w \in W_G(L_I)} |Z^0(L_I)^{wF}|/|W_G(L_I)| = q^{-2c} |C_G(u)_0^{F}|$. Whence $\langle \Gamma^G_\iota, \Gamma^G_\iota \rangle_{G^F} = |Z(G)|^2 q^{-\rkss G + \dim Z(L_I)} |C_G(u)_0^{F}|$. Using $|C_G(u)_0^{F}| = q^{\rkss L_I} |Z(G)^F|$, we get (i).

For (ii), we apply Proposition 7 in (i), using that $D$ is an isometry and that $\sigma_G\gamma = q^{\rkss L_I^*}$ by [4, proposition 2.5]. \hfill \Box

A particular case of Proposition 8 is

**Corollary 10.** If $(L, \iota)$ is a cuspidal pair, that is $L = L_I$, then
\[
\langle R_L^G \Gamma_1^L, R_L^G \Gamma_1^L \rangle_{G^F} = \frac{|Z(L)|}{Z^0(L)}^2 |W_G(L)||Z(L)|^{0,F}.
\]

We remark that this coincides with the value predicted by the Mackey formula
\[
\langle R_L^G \Gamma_1^L, R_L^G \Gamma_1^L \rangle_{G^F} = \sum_{x \in L^F \setminus (L_L/L)_L^F} \langle x^R_L \cap L, \Gamma_1^L, x^{-1}_R L \Gamma_1^L \rangle_{(L_L/L)_L^P}
\]
Indeed, since the block $I$ which contains the local system $\iota$ is reduced to the unique cuspidal local system $(C, \iota)$ where $C$ is the regular class of $L$, all terms in the Mackey formula where $L \cap x^R L \neq L$ vanish. Thus the Mackey formula reduces to
\[
\langle R_L^G \Gamma_1^L, R_L^G \Gamma_1^L \rangle_{G^F} = \sum_{x \in W_G(L)} \langle \Gamma_1^L, x^{-1}_R L \Gamma_1^L \rangle_{L^F}
\]
and any $x$ in $W_L(L)$ acts trivially on $H^1(F, Z(L))$ since, the map $h_L$ being surjective, any element of $H^1(F, Z(L))$ is represented by an element of $H^1(F, Z(G))$; thus all the terms in the sum are equal, and we get the same result as Corollary 10 by applying Corollary 10 in the case $G = L$.

Another method for computing $\langle R_L^G DT_1, R_L^G DT_1 \rangle_{G^F}$ would be to use Proposition 1 and the values of the two-variable Green functions.
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We give these values in the following table in the particular case of $2E_6$ for the $F$-stable standard Levi subgroup of type $A_2 \times A_2$, for $q \equiv -1$ (mod 3), so that $F$ acts trivially on $Z(G)/Z^0(G)$. This table has been computed in Chevie using Proposition 3. The method is to compute the one-variable Green functions which appear in the right-hand side sum by the Lusztig-Shoji algorithm; note that even though the characteristic function of cuspidal character sheaves are known only up to a root of unity, this ambiguity disappears when doing the sum, since such a scalar appears multiplied by its complex conjugate. However the Lusztig-Shoji algorithm depends also on the knowledge that when $\iota$, with support the class of the unipotent element $u$, is parameterized by $(u, \chi)$ where $\chi \in \text{Irr}(A(u))$ then for the unipotent element $u_\iota \in G^F$ parameterized by $a \in H^1(F,A(u))$ we have $\chi(u_\iota) = \chi(a)$. We assume that this hold. This is known when $\iota$ is in the principal block, but not for the two blocks with cuspidal datum supported on the Levi subgroup of type $A_2 \times A_2$.

Note that the table shows that the values of $|v^L|Q^G_{\iota}(u,v)$ are not in general polynomials with integral coefficients but may have denominators equal to $|A(v)|$.

Table 1. Values of $|v^L|Q^G_{\iota}(u,v)$ for $G = 2E_6(q)$ simply connected and $L = A_2(q^2)/(q - 1)^2$, for $q \equiv -1$ (mod 3).

| \begin{array}{l}
v/u \\ \hline
(111,111) & 0 & 0 & 0 & 0 & 0 & 0 \\
21, 21 & 0 & 0 & 0 & 0 & 0 & 1 \\
3, 3 & 1 & 0 & 0 & (4q + 1)/3 & 2q/3 & 2q/3 \\
3, 3(\zeta_3) & 0 & 1 & 0 & (4q + 1)/3 & 2q/3 & 2q/3 \\
3, 3(\zeta_2) & 0 & 0 & 1 & (4q + 1)/3 & 2q/3 & 2q/3 |
\end{array} |
$$\begin{array}{|c|c|c|c|c|}
\hline
v \setminus u & 2A_2 + A_1 & 2A_2 + A_{1(3)} & 2A_2 + A_{1(2)} & 2A_2 \\
\hline
111, 111 & \Phi_2 \Phi_3 & \Phi_2 \Phi_3 & \Phi_2 \Phi_3 & \Phi_2 \Phi_3 \\
21, 21 & (2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2 & (2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2 & (2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2 & 3q^3 \Phi_2 \Phi_3 \Phi_6 \\
3, 3 & q^6 \Phi_2 \Phi_3 & 0 & 0 & q^6 \Phi_2 \Phi_3 \Phi_6 \\
3, 3_{(3)} & 0 & q^6 \Phi_2 \Phi_3 & 0 & 0 \\
3, 3_{(2)} & 0 & 0 & q^6 \Phi_2 \Phi_3 & 0 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|c|}
\hline
v \setminus u & 2A_{2(3)} & 2A_{2(2)} & A_2 + 2A_1 & A_2 + A_1 \\
\hline
111, 111 & \Phi_2^4 \Phi_3 \Phi_6 & \Phi_2^4 \Phi_3 \Phi_6 & (2q^4 + q^3 + q^2 + q + 1)q^3 \Phi_2 & \Phi_2^4 \Phi_3 \Phi_6 \\
21, 21 & 3q^4 \Phi_2^4 \Phi_3 \Phi_6 & 3q^4 \Phi_2^4 \Phi_3 \Phi_6 & (2q^4 + q^3 + q^2 + q + 1)q^3 \Phi_2 & (3q^4 + 2q^3 + q^2 + 2q^2 + 2q + 2q^2 + 2q + 1)q^3 \Phi_2 \Phi_3 \Phi_6 \\
3, 3 & 0 & 0 & 0 & 0 \\
3, 3_{(3)} & q^6 \Phi_2^4 \Phi_3 \Phi_6 & 0 & 0 & 0 \\
3, 3_{(2)} & 0 & q^6 \Phi_2^4 \Phi_3 \Phi_6 & 0 & 0 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|}
\hline
v \setminus u & 3A_1 & 2A_1 \\
\hline
111, 111 & (3q^6 + q^5 + q^4 + q^3 + q^2 + 1)q^3 \Phi_2 \Phi_3 \Phi_6 & (3q^6 + q^5 + q^4 + q^3 + q^2 + 1)q^3 \Phi_2 \Phi_3 \Phi_6 \\
21, 21 & (-3q^6 - 3q^5 - 3q^4 - 2q^3 - q^2 - q - 1)q^3 \Phi_2 \Phi_3 \Phi_6 & (3q^6 + q^5 + q^4 + q^3 + q^2 + 1)q^3 \Phi_2 \Phi_3 \Phi_6 \\
3, 3 & 0 & 0 \\
3, 3_{(3)} & 0 & 0 \\
3, 3_{(2)} & 0 & 0 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|}
\hline
v \setminus u & A_1 & 1 \\
\hline
111, 111 & (2q^{10} + q^9 + q^8 + q^7 + 2q^6 + q^5 + q^4 + q^3 + q^2 + 1)q^3 \Phi_2 \Phi_3 \Phi_6 \Phi_{10} & \Phi_2 \Phi_3 \Phi_2 \Phi_3 \Phi_6 \Phi_{10} \Phi_{12} \Phi_{18} \\
21, 21 & 0 & 0 \\
3, 3 & 0 & 0 \\
3, 3_{(3)} & 0 & 0 \\
3, 3_{(2)} & 0 & 0 \\
\hline
\end{array}$$

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