Geodesic Nucleation and Evolution of a de-Sitter Brane

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(Dated: March 15, 2005)

Within the framework of Geodesic Brane Gravity, the deviation from General Relativity is parameterized by the conserved bulk energy. The corresponding geodesic evolution/nucleation of a de-Sitter brane is shown to be exclusively driven by a double-well Higgs potential, rather than by a plain cosmological constant. The (hairy) horizon serves then as the locus of unbroken $Z_2$ symmetry. The quartic structure of the scalar potential, singled out on finiteness grounds of the total (including the dark component) energy density, chooses the Hartle-Hawking no-boundary proposal.

The Randall-Sundrum model[1] has in fact re-ignited the interest in brane gravity. A completely different approach, to be referred to as Geodesic Brane Gravity (GBG), has been advocated long ago by Regge-Teitelboim[2]. Originally aiming towards Quantum Gravity, the RT theory has been proposed with the motivation that the first principles which govern the evolution of the entire Universe need not be too different from those which determine the world-line (world-sheet) behavior of a point particle (elementary string). Following RG, the brane Universe can be regarded as a 4-dim extended object, parameterized by means of $x^\mu$ ($\mu = 0, 1, 2, 3$), geodesically floating in some fixed higher dimensional background spanned by $y^A$ ($A = 0, 1, \ldots, N - 1$). The idea was criticized[3] in the past on gauge dependence grounds, but has been reconsidered (in a flat background) by several authors[4][5] since. Being accustomed to General relativity (GR), one may find the RT action deceptively conventional

$$ S = \int \left( \frac{1}{16\pi G} R + L_m \right) \sqrt{-g} \, d^4 x . $$

However, in the spirit of classical particle/string theory, it is now the embedding vector $y^A(x^\mu)$, rather than the induced metric tensor $g_{\mu\nu} = \eta_{AB} g(y^B[g^B_{\mu
u}]$, which is unconditionally elevated to the level of the canonical gravitational field. Whereas Einstein equations are supposed to get drastically modified, it is crucial to emphasize that all other equations of motion (variations with respect to the matter fields) remain absolutely intact. The corresponding RT gravitational field equations, a weaker system (only six independent equations) in comparison with Einstein equations, take the compact form

$$ E^{\mu\nu} \left( g^{\mu\nu} + \Gamma^B_{\mu\nu} y^C + y^C_{\mu\nu} \right) = 0 , $$

where he Einstein tensor

$$ E^{\mu\nu} = \frac{1}{8\pi G} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) - T^{\mu\nu} $$

keeps track of the underlying Einstein-Hilbert Lagrangian. Invoking the extrinsic curvatures $K^{\mu\nu}_{\rho}$, the above equations can be alternatively written in a more geometrically oriented form, namely $E^{\mu\nu} K_{\mu\nu} = 0$. The RT formalism has two important features:

- Every solution of GR equations, namely $E^{\mu\nu} = 0$, is necessarily a solution of the corresponding RT equations. In other words, RT-gravity exhibits a built in GR-limit.

- Owing to the powerful identity of geometrical origin $\eta_{AB} y^{\mu}_{\nu} \left( \eta_{BC} + \Gamma^B_{\nu\lambda} y^C_{\lambda} \right) = 0$, one still recovers energy-momentum conservation $T_{\mu\nu} = 0$.

Within the framework of geodesic brane cosmology, formulated by virtue of 5-dimensional local isometric embedding, only a single independent RT-equation survives, namely

$$ \frac{d}{dt} \left( \sqrt{-g} E^{\mu\nu} y^0 \right) = 0 . $$

A trivial integration gives then rise to

$$ \rho a^3 (a^2 + k)^{1/2} - 3 a (\dot{a}^2 + k)^{3/2} = -\frac{\dot{a}}{\sqrt{3}} . $$

accompanying by $\dot{\rho} + \frac{3}{2} \rho \dot{a} (\rho + P) = 0$. The constant of integration $\omega$, recognized as the conserved bulk energy conjugate to the cyclic embedding time coordinate $y^0(t)$, parameterizes the deviation from the Einstein limit (where $\dot{a}^2 + k \rightarrow \frac{1}{2} \rho a^2$). A physicist equipped with the traditional Einstein formalism, presumably unaware of the underlying RT physics, would naturally re-organize the latter equation into

$$ \dot{a}^2 + k = \frac{1}{3} (\rho + \Delta \rho) a^2 , $$

squeezing all 'anomalous' pieces into $\Delta \rho$. Our physicist may rightly conclude[6] that the FRW evolution of the Universe is governed by the effective energy density $\rho + \Delta \rho$ rather than by the primitive $\rho$, and thus may further identify (or just use the language of) $\Delta \rho \equiv \rho_{\text{dark}}$. A simple algebra reveals the cubic consistency relation

$$ (\rho + \rho_{\text{dark}}) \rho_{\text{dark}}^2 = \frac{\omega^2}{a^8} . $$

In this paper, we focus attention on the prototype case involving a minimally coupled scalar field $\phi(t)$, subject
to the standard equation of motion
\[ \ddot{\phi} + \frac{3}{a} \dot{\phi} + \frac{dW(\phi)}{d\phi} = 0 , \] (8)
involving some (yet unspecified) scalar potential \( W(\phi) \).

Two exclusive features of Geodesic Brane Cosmology, relevant for our discussion, are worth noting, namely

- **Positive definite total energy density:** To be specific, Eq. (4) tells us that \( \rho_{\text{total}} \equiv \rho + \rho_{\text{dark}} \geq 0 \). Curiously, this conclusion holds even for \( \rho < 0 \).

- **Cosmic duality:** FRW cosmological evolution cannot tell the configuration \( \{ \rho, \rho_{\text{dark}} \} \) from its dual \( \{ \rho + 2\rho_{\text{dark}}, -\rho_{\text{dark}} \} \), both sharing a common \( \rho_{\text{total}} \). A pedagogical example can be provided by the 'empty' \( \rho = 0 \) case, whose dual happens to constitute a scalar field theory governed by a quintessence-type potential.

To uncover the mysteries of the so-called dark component \( \rho_{\text{dark}} \), we start by asking a simple minded question: Under what conditions can we obtain eternal deSitter evolution? It is well known that Einstein’s GR requires the introduction of a positive cosmological constant \( \rho = \Lambda > 0 \). Counter intuitively, however, within the framework of RT-gravity, the presence of a scalar field appears mandatory for deriving the exact deSitter solution. Adopting (say) the spatially-closed \( k > 0 \) case, it remains to find the tenable scalar potential \( W(\phi) \), if any, capable of supporting \( \rho_{\text{total}} = \Lambda > 0 \).

Guided by eq. (4), the thing to notice now is the split
\[ \rho = \Lambda + \frac{\omega}{\Lambda^{1/2} a^4} , \] \[ \rho_{\text{dark}} = -\frac{\omega}{\Lambda^{1/2} a^4} . \] (9)
Differentiating \( \rho = \frac{1}{2} \dot{\phi}^2 + W(\phi) \), we substitute \( \frac{dW}{d\phi} + \ddot{\phi} \) by \( -\frac{3}{a} \dot{\phi} \), to learn that
\[ \dot{\phi}^2 = \frac{4\omega}{3\Lambda^{1/2} a^4} , \] (10)
and appreciate the fact that associated with our \( \omega > 0 \) is a negative dark energy component \( \rho_{\text{dark}} < 0 \) (we recall in passing the existence of a yet unspecified \( \rho_{\text{dark}} > 0 \) dual theory with identical brane evolution). We also find that
\[ W = \Lambda + \frac{\omega}{3\Lambda^{1/2} a^4} , \] (11)
and would like, in search of a differential equation for \( W(\phi) \), to also express \( \frac{dW}{d\phi} \) as a parametric function of \( a \). To do so, we calculate \( \ddot{\phi} \) and plug the result into the scalar field equation. We find
\[ \frac{dW}{d\phi} = \pm \frac{1}{a^3} \sqrt{\frac{4\omega}{3\Lambda^{1/2}} \left( \frac{1}{3} \Lambda a^2 - k \right)} . \] (12)
The next step is to define a function \( f(\phi) \),
\[ f(\phi) = \frac{1}{3} \Lambda - \frac{3k^2 \Lambda^{1/2}}{\omega} (W(\phi) - \Lambda) , \] (13)
which turns out to satisfy the differential equation
\[ \left( \frac{df}{d\phi} \right)^2 = \frac{3k^2 \Lambda^{1/2}}{\omega} f . \] (14)
The solution of this differential equation is rather serendipitous: The unique scalar potential capable of supporting an inflationary deSitter brane is a double-well Higgs potential, given explicitly by
\[ W(\phi) = \Lambda + \frac{3\Lambda^{1/2} k^2}{16\omega} \left( \phi^2 - \frac{4 \omega \Lambda^{1/2}}{9k^2} \right)^2 \] (15)
Associated with this double-well potential, but relying on certain creation initial conditions (to be specified soon) is the full \( k > 0 \) solution
\[ a(t) = \sqrt{\frac{3k}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3} t} , \] (16)
\[ \phi(t) = \sqrt{\frac{4 \omega \Lambda^{1/2}}{9k^2}} \tanh \sqrt{\frac{\Lambda}{3} t} . \] (17)
On symmetry (and forth coming Euclidean) grounds, we find it rewarding to follow Hartle and Hawking\(^2\) and define the proper scalar field \( b(t) \propto a(t)\phi(t) \). In this language, the cosmological evolution of the system is described by the hyperbola
\[ a(t)^2 - b(t)^2 = \frac{3k}{\Lambda} . \] (18)
The emerging deSitter inflationary scheme, accompanied by the auxiliary scalar field, deviates conceivably from the conventional GR prescription. Created with a finite radius of \( a_0 = \sqrt{\frac{3k}{\Lambda}} \), while sitting at the top of the potential hill \( W_0 = \Lambda \left( 1 + \frac{\omega \Lambda^{1/2}}{27k^2} \right) \), the exponentially growing brane slides down the hill towards the absolute minimum of the theory. The latter is conveniently located at the Einstein limit \( W_{\infty} = \Lambda \). The scalar field, at the meantime, recovering from the non-conventional creation initial conditions
\[ \phi_0 = 0 , \quad \dot{\phi}_0 = \sqrt{\frac{4 \omega \Lambda^{1/2}}{9k^2}} , \] (19)
grows monotonically on the way to eventually picking up its vacuum expectation value

$$\langle \phi \rangle = \sqrt{\frac{4\omega\Lambda^{1/2}}{9k^2}} \equiv V.$$  \hspace{1cm} (20)

Altogether, accompanied by a remarkable seesaw-type $\rho \leftrightarrow \rho_{\text{dark}}$ interplay, deSitter inflation is described, within the framework of geodesic brane cosmology, as a spontaneous symmetry breaking process, with GR eventually recovered at the absolute minimum. On the practical side, there is no need to artificially engineer the shape of a slow-rolling scalar potential in order to maximize the inflation period; in RT-gravity, an ordinary Higgs potential can produce eternal inflation.

Two important remarks are in order:
(i) For $k < 0$, the situation is very much alike. Truly, this time one faces

$$a(t) \sim \sinh \sqrt{\frac{\Lambda}{3}} t, \quad \phi(t) \sim \coth \sqrt{\frac{\Lambda}{3}} t, \hspace{1cm} (21)$$

but the Higgs potential stays invariant under $k \to -k$. Nucleated with size zero, accompanied by a monotonically decreasing scalar field, our exponentially growing open brane slides again towards the $W_\infty = \Lambda$ Einstein limit. However, contrary to the closed $k > 0$ case where only the inner section ($0 \leq \phi \leq V$) of the potential was involved, it is the outer section ($V \leq \phi < \infty$) which participates in the $k < 0$ game. For $k = 0$, the situation is less complicated, with the Higgs potential reducing to a simple mass term.

(ii) The deSitter metric can also take the static radially symmetric form

$$ds^2 = -(1 - \frac{1}{3}\Lambda R^2) dT^2 + \frac{dR^2}{(1 - \frac{1}{3}\Lambda R^2)} + R^2 d\Omega^2,$$  \hspace{1cm} (22)

exhibiting an event horizon at $R = \sqrt{\frac{3}{\Lambda}}$. Reflecting the $\rho \leftrightarrow \rho_{\text{dark}}$ seesaw interplay between the primitive and the dark energy densities, the auxiliary scalar field plays in this coordinate system an apparently paradoxical non-static role. To see the point, consider (say) the patch $R \leq \sqrt{\frac{3}{\Lambda}}$ covered by

$$\sqrt{\frac{\Lambda}{3k}} R = r \cosh \sqrt{\frac{\Lambda}{3}} t,$$  \hspace{1cm} (23)

$$\coth \sqrt{\frac{\Lambda}{3}} t = \sqrt{1 - kr^2} \coth \sqrt{\frac{\Lambda}{3}} t.$$  \hspace{1cm} (24)

In this coordinate system, the auxiliary $T$-dependent scalar field acquires the form

$$\phi(T, R) = \frac{V \sqrt{1 - \frac{1}{3}\Lambda R^2} \sinh \sqrt{\frac{\Lambda}{3}} T}{\sqrt{1 + (1 - \frac{1}{4}\Lambda R^2) \sinh^2 \sqrt{\frac{4}{3}} T}}$$  \hspace{1cm} (25)

giving rise to double-kink configuration (a kink-antikink configuration for $R \geq \sqrt{\frac{3}{\Lambda}}$ scalar hair. In almost every point $R$ in space, elegantly avoiding the no-hair theorems of GR, the scalar field connects $\phi(-\infty, R) \to -V$ with $\phi(\infty, R) \to V$. It is exclusively on the event horizon, where the scalar field, experiencing an infinite gravitational red-shift, gets frozen in its unbroken phase! In other words, the hairy event horizon appears as the locus of unbroken $\mathbb{Z}_2$ symmetry. The generality of this statement may (or maynot) extend beyond the scope of the present work.

To enter the Euclidean regime we perform the Wick rotation $t \to -i (\tau - \frac{\pi i}{2} \sqrt{\frac{3}{\Lambda}})$: The exact $k > 0$ solution Eqs. 16 is transformed into

$$a(t) \to a_E(\tau) = \sqrt{\frac{3k}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3}} \tau,$$  \hspace{1cm} (26)

$$\phi(t) \to i\phi_E(\tau) = i\sqrt{\frac{4\omega\Lambda^{1/2}}{9k^2}} \cosh \sqrt{\frac{\Lambda}{3}} \tau.$$  \hspace{1cm} (27)

The fact that the scalar field turns purely imaginary puts us in a less familiar territory, in some sense reminding us of the Coleman-Lee scheme. The imaginary time evolution is then best described by the circle

$$a_E^2(\tau) + b_E^2(\tau) = \frac{3k}{\Lambda},$$  \hspace{1cm} (28)

recognized as the analytic continuation of eq.18. This makes the familiar deSitter Euclidean time periodicity $\Delta \tau = 2\pi \sqrt{\frac{3}{\Lambda}}$ manifest, and opens the door for a generalized Hawking-Hartle no-boundary proposal.

It is now important to find out which potential actually governs the imaginary time evolution of $\phi_E$? Traditionally, we have been accustomed to the upside-down potential $W_E(\phi_E) = -W(\phi_E)$, but this is definitely not the case here. Euclidizing the time derivatives in the scalar field equation, and simultaneously taking care of $\phi \to i\phi_E$, brings us back to the original form

$$\phi'' + \frac{3}{a_E} a'_E \phi'_E + \frac{\partial W_E}{\partial \phi_E} = 0,$$  \hspace{1cm} (29)

only with $W_E(\phi_E) = +W(i\phi_E)$. The resulting potential in the Euclidean regime is then

$$W_E(\phi_E) = \Lambda + \frac{3\Lambda^{1/2}k^2}{16\omega} \left( \phi_E^2 + \frac{4\omega\Lambda^{1/2}}{9k^2} \right)^2,$$  \hspace{1cm} (30)

although quartic, this potential is strikingly not of the double-well type. Furthermore, as depicted in Fig 1
FIG. 1: Geodesic deSitter brane nucleation: Evolving from Euclidean no-boundary initial conditions, the Euclidean to Lorentzian transition can only occur at $\phi = \phi_E = 0$. GR is asymptotically recovered at the Einstein limit of the Lorentzian regime.

The absolute minimum of $W_E(\phi_E)$ is tangent to the local maximum of $W(\phi)$. This is by no means coincidental. $\phi = \phi_E = 0$ is the only point where the Euclidean to Lorentzian transition, to be referred to as brane nucleation\(^{10}\), can actually occur.

We now attempt to go one step beyond de-Sitter inflation. To do so, we would like to commit ourselves to a certain type of scalar potentials, but soon realize that so far we have not really decoded the principles underlying the tenable potential $W(\phi)$. Two related questions are then in order:

1. Why must $W(\phi)$ exhibit a quartic behavior?
2. Is the quartic potential a mandatory ingredient of geodesic brane cosmology?

The answers to these questions is rooted, quite unexpectedly, within the Hartle-Hawking no-boundary ansatz\(^9\). We are about to prove, by exclusively predicting a finite non-vanishing total energy density at the origin, that the quartic structure of the potential actually chooses the no-boundary initial conditions.

The smoothness of the Euclidean manifold at the origin dictates the specific $\tau \to 0$ behavior $a_E(\tau) \simeq \sqrt{k}\tau$, but may in principle allow for $b_E(\tau) \simeq \frac{p\sqrt{k}}{\tau^{j-1}}$. Now, assuming the asymptotic power behavior

$$W(\phi) \simeq \lambda \phi^N \quad (\lambda > 0),$$

the scalar equation of motion eq.\(^{29}\) can be fulfilled (to the leading order) only provided

$$jN = 2(j + 1),$$
$$N\lambda p^{(N-2)} + j(j - 2) = 0.$$  

This in turn implies $\rho \sim \tau^{-2(j+1)}$ but $\rho_{\text{total}} \sim \tau^{4(j-1)}$. Consequently, fully consistent with our expectations, $N$ gets uniquely fixed by insisting on a finite non-vanishing total energy density as $a_E \to 0$. This singles out

$$j = 1 \Rightarrow N = 4.$$  

The special no-boundary initial conditions then read

$$a_E \simeq \sqrt{k}\tau, \quad b_E \simeq \frac{k}{4\lambda},$$

accompanied by the finite total energy density

$$\rho_{\text{total}} \simeq \left(\frac{4\omega\lambda}{3k^2}\right)^2$$

While the no-boundary initial conditions are $\omega$-independent, it is the RT bulk energy $\omega$ (used to parameterize the deviation from GR) that actually fixes the finite value of the total energy density.

It is interesting to note that had we carried out a similar calculation for an $n$-dim brane (we skip the details of the proof), we would have encountered the famous scale invariant behavior

$$W(\phi) \sim \phi^{2n(n+2)},$$

which happens to be quartic for the special $n = 4$ case of interest. This indicates that, within the framework of geodesic brane cosmology, there exists a linkage between the apparently disconnected ideas of Hawking-Hartle no-boundary proposal and global conformal invariance, pointing presumably towards geodesic dilaton cosmology.

Finally, on semi-realistic grounds, while adopting the quartic Higgs potential, it makes sense to exercise the option of setting

$$W_{\text{min}}(\phi) = 0.$$  

The price for eliminating the residual cosmological constant from the Einstein limit is a finite (yet enhanced in comparison with standard cosmology) amount of inflation. On the other hand, subject to the consistent no-boundary initial conditions eq.\(^{25}\), we know that the classical Euclidean evolution is fully determined once the conserved bulk energy $\omega$ gets specified. Naturally, this provokes a new set of questions:

1. Does the global structure of the Euclidean manifold still exhibit, in some cases, imaginary time periodicity?
2. Under what circumstances, if any, does the total energy density evolve free of singularity? If a singularity does occur, what is its nature?
FIG. 2: Imaginary time periodicity and energy density regularity are demonstrated, for $\omega = \omega_{1,3,5}$, by means of closed Lissajous-like trajectories in the $\{a_E, b_E\}$ plane. The $\omega = \omega_{2,4,\ldots}$ case is associate with Coleman-Dellucia\cite{11} Euclidization. If $\omega \neq \omega_n$, a $\cos(\ln c)$-type singularity is developed upon returning to the origin.

3. Can our nucleation conditions $a'_E = \phi_E = 0$, or else Coleman-Dellucia\cite{11} conditions $a'_E = \phi'_E = 0$, be met at some finite Euclidean time $\tau_0$?

We claim, skipping the analytic proof (to be published elsewhere) but based on detailed numerical calculations, that $\tau$-periodicity, $a_E, b_E$-regularity, and the occurrence of spontaneous nucleation, share in fact the one and the same origin. They can all be simultaneously achieved provided

$$\omega = \omega_n$$

(39)

is properly quantized, in agreement with some previous WKB approximation\cite{5}. The integer $n$ counts the total number of times the proper scalar field $b_E$ crosses the absolute minimum of the potential during half a period. For the $n$-odd case of interest ($n$-even is associated with Coleman-Dellucia), reflecting the interplay of two periodicities, we encounter (see Fig.2) $n$-loop closed trajectories in the $\{a_E, b_E\}$ plane which resemble the Lissajous figures. Notice that the Euclidean de-Sitter configuration Eq.\ref{eq:28} clearly belongs to the $n = 1$ category.

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