Abstract  In this note, we show that some $F$-harmonic maps into spheres are global maxima of the variations of their energy functional on the conformal group of the sphere. Our result extends partially those obtained in El Soufi and Lejune [C.R.A.S. 315(Serie I):1189–1192, 1992] and El Soufi [Compositio Math 95:343–362, 1995] for harmonic and $p$-harmonic maps.

Keywords  $F$-harmonic maps · Stress-energy tensor · $F$-tension

Mathematics Subject Classification  Primary 35B53 · 58Z05 · 53C43

1 Introduction

Harmonic maps have been studied first by Eells and Sampson in the sixties and since then many articles have appeared (see [6,13,17,20,21,25]) to cite a few of them. Extensions to the notions of $p$-harmonic, biharmonic, $F$-harmonic and $f$-harmonic maps were introduced and similar research has been carried out (see [1–3,7,16,19,22,24]). Harmonic maps were applied to broad areas in sciences and engineering including the robot mechanics (see [5,8,9]).

In this paper for a $C^2$-function $F : [0, +\infty[ \rightarrow [0, +\infty[$ such that $F'(t) > 0$ on $t \in ]0, +\infty[$, we look for sufficient conditions which present $F$-harmonic maps into spheres as global maxima of the energy functional. Our result extends similar results obtained in [18,19] for harmonic and $p$-harmonic maps.
Let \((M, g)\) and \(S^n\) be, respectively, a compact Riemannian manifold of dimension \(m \geq 2\) and the unit \(n\)-dimensional Euclidean sphere with \(n \geq 2\) endowed with the canonical metric \(can\) induced by the inner product of \(R^{n+1}\).

For a \(C^1\)-application \(\phi: (M, g) \rightarrow (S^n, can)\), we define the \(F\)-energy functional by

\[
E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) dv_g,
\]

where \(\frac{|d\phi|^2}{2}\) denotes the energy density given by

\[
\frac{|d\phi|^2}{2} = \frac{1}{2} \sum_{i=1}^{m} |d\phi(e_i)|^2
\]

and where \(\{e_i\}\) is an orthonormal basis on the tangent space \(T_xM\) and \(dv_g\) is the Riemannian measure associated to \(g\) on \(M\).

Let \(\phi^{-1}TS^n\) and \(\Gamma(\phi^{-1}TS^n)\) be, respectively, the pullback vector fiber bundle of \(TS^n\) and the space of sections on \(\phi^{-1}TS^n\). Denote by \(\nabla^M\), \(\nabla^{S^n}\) and \(\nabla\), respectively, the Levi-Civita connections on: \(TM, TS^n\) and \(\phi^{-1}TS^n\). Recall that \(\nabla\) is defined by

\[
\nabla_X Y = \nabla_{\phi^{-1}X} Y
\]

where \(X \in TM\) and \(Y \in \Gamma(\phi^{-1}TS^n)\).

Let \(v\) be a vector field on \(S^n\) and denote by \((\gamma_i^v)\) the flow of diffeomorphisms induced by \(v\) on \(S^n\) i.e.

\[
\gamma_0^v = \text{id}, \quad \frac{d}{dt}\gamma_t^v|_{t=0} = v(\gamma_t^v).
\]

Denote by \(\phi_t = \gamma_t^v o \phi\) the flow generated by \(v\) along the map \(\phi\). The first variation formula of \(E_F(\phi)\) is given by

\[
\frac{d}{dt}E_F(\phi_t)|_{t=0} = \int_M F'\left(\frac{|d\phi_t|^2}{2}\right) \langle \nabla_{\phi_t} d\phi_t, d\phi_t \rangle|_{t=0} dv_g
\]

\[
= - \int_M \langle v, \tau_F(\phi) \rangle dv_g
\]

where \(\tau_F(\phi) = \text{trace}_g \nabla F'\left(\frac{|d\phi|^2}{2}\right) d\phi\) is the \(F\)-tension.

**Definition 1** \(\phi\) is said \(F\)-harmonic if and only if \(\tau_F(\phi) = 0\) i.e. \(\phi\) is a critical point of the \(F\)-energy functional \(E_F\).
Let $v \in \mathbb{R}^{n+1}$ and set $\tilde{v}(y) = v - \langle v, y \rangle y$ for any $y \in S^n$. It is known that $\tilde{v}$ is a conformal vector field on $S^n$ i.e. $(\gamma^v_t)^* can = \alpha^2 can$ where $(\gamma^v_t)_t$ denotes the flow induced by the vector field $\tilde{v}$. The expression of $\alpha_t$ is given in [17] by

$$\alpha_t = \frac{|v|}{|v| ch t + \phi_v sht}.$$  

where $\phi_v (x) = \langle v, \phi (x) \rangle$ and $\langle \ldots \rangle$ the inner product on the Euclidean space $\mathbb{R}^{n+1}$. Denote by $\mathcal{E}(\phi)$ the subspace of $\Gamma(\phi^{-1} TS^n)$ given by

$$\mathcal{E}(\phi) = \{ \tilde{v} \circ \phi, v \in \mathbb{R}^{n+1} \}.$$  

Obviously, if $\phi$ is not constant, $\mathcal{E}(\phi)$ is of dimension $n + 1$.

2 $F$-harmonic maps as global maxima

For any $v \in \mathcal{E}(\phi)$, we denote by $(\gamma^v_t)_{t \in \mathbb{R}}$ the one parameter group of conformal diffeomorphisms on $S^n$ induced by the vector $\tilde{v}$. For a $C^2$-function $F : [0, +\infty[ \to [0, +\infty[$ such that $F'(t) > 0$ in $]0, +\infty[$.

Now we introduce the following tensor field

$$S^F_g(\phi) = F' \left( \frac{|d\phi|^2}{2} \right) \frac{|d\phi|^2}{2} g - \left[ F' \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right] \phi^* can.$$  

For $x \in M$, we set

$$S^F_{g,v}(\phi)(X, X) = \inf \left\{ S^F_{g,v}(\phi)(X, X), X \in T_x M \text{ such that } g(X, X) = 1 \right\}.$$  

The tensor $S^F_g(\phi)(x)$ will be said positive (resp. positive defined) at $x$ if $S^F_{g,v}(\phi)(x) \geq 0$ (resp. $S^F_{g,v}(\phi)(x) > 0$).

Example 1 For $F(t) = \frac{1}{p} (2t)^{\frac{p}{2}}$, with $p = 2$ or $p \geq 4$, $S^F_g(\phi)$ is the stress-energy tensor introduced, respectively, by Eells and Lemaire [12] for $p = 2$ and modulo a multiplied positive constant by El Soufi [16] for $p \geq 4$, so we may call $S^F_g(\phi)$ the stress-energy tensor of $\phi$.

Indeed if $F(t) = t$ then $F'(t) = 1$, $F''(t) = 0$ and

$$S^F_g(\phi) = \frac{|d\phi|^2}{2} g - \phi^* can.$$  

In the case $F(t) = \frac{1}{p} (2t)^{\frac{p}{2}}$, with $p \geq 4$, $F'(t) = (2t)^{\frac{p}{2} - 1}$, $F''(t) = (p - 2) (2t)^{\frac{p}{2} - 2}$ and

$$S^F_g(\phi) = \frac{1}{2} |d\phi|^p g - \frac{p}{2} |d\phi|^{p-2} \phi^* can = \frac{p}{2} \left( \frac{1}{p} |d\phi|^p g - |d\phi|^{p-2} \phi^* can \right).$$
The function $F$ is called admissible if $F$ satisfies

$$
B = \left( \frac{F''(\alpha_1^2 \phi \cdot \frac{|d \phi|^2}{2}) - \alpha_1^2 \phi}{F'(\alpha_1^2 \phi \cdot \frac{|d \phi|^2}{2})} \right) \phi_v \geq 0
$$

and the stress-energy tensor $S^F_g (\phi)$ of $\phi$ fulfills

$$
S^F_g (\gamma_t \phi) \geq \alpha_t^2 \phi. S^F_g (\phi)
$$

where $\gamma_t$ is the one parameter group of conformal transformations induced by the vector field $\nu$ (defined above) on the euclidean sphere $S^n$ and $\alpha_t$ is given by (1).

The function $F(t) = \frac{1}{p} (2t)^{\frac{p}{2}}$ for $p = 2$ and $p \geq 4$ and $t \geq 0$ is admissible.

Indeed, for $F(t) = \frac{1}{p} (2t)^{\frac{p}{2}}$ we have $B = 0$ and for any conformal diffeomorphism $\gamma$ on the euclidean sphere, we have

$$
S^F_g (\gamma \phi) = \frac{1}{2} |d (\gamma \phi)|^p \ g - \frac{p}{2} |d (\gamma \phi)|^{p-2} \phi^* can
$$

so if we let $|d (\gamma \phi)|^2 = \alpha_t^2 \phi. |d \phi|^2$, we get

$$
S^F_g (\gamma \phi) = \alpha_t^2 \phi. \left( \frac{1}{2} |d \phi|^p \ g - \frac{p}{2} |d \phi|^{p-2} \phi^* can \right)
$$

$$
= \alpha_t^2 \phi. S^F_g (\phi).
$$

The $F(t) = 1 + at - e^{-t}$, for $t \in [0, +\infty[$ where $a = \max_{x \in M} \frac{|d \phi|^2}{2}$ is admissible provided that the conformal diffeomorphism on the euclidean sphere $S^n$ is contracting that means that the function $\phi_v$ given in the expression of (1) is nonnegative.

Indeed, we have

$$
B = \left( -\alpha_1^2 \phi \frac{e^{-\alpha_1^2 \phi \cdot \frac{|d \phi|^2}{2}}}{a + e^{-\alpha_1^2 \phi \cdot \frac{|d \phi|^2}{2}}} + \frac{e^{-|d \phi|^2}}{2} \phi_v \right)
$$

Putting $u = \alpha_1^2 \phi \in [0, 1]$, we consider the function $\varphi (u) = -u \frac{e^{-|d \phi|^2}}{a + e^{-|d \phi|^2}} + \frac{e^{-|d \phi|^2}}{2}$, we get

$$
\varphi' (u) = \left( \frac{|d \phi|^2}{2} u - a - e^{-|d \phi|^2} u \right) \frac{e^{-|d \phi|^2}}{2} \left( a + e^{-|d \phi|^2} u \right) - u \frac{e^{-|d \phi|^2}}{a + e^{-|d \phi|^2}} + \frac{e^{-|d \phi|^2}}{2}
$$
and it is obvious that \( \varphi'(u) \leq 0 \), hence \( \varphi \) is a decreasing function on \([0, 1]\) i.e. \( \varphi(u) \geq \varphi(1) = 0 \). Consequently \( B \geq 0 \).

Now

\[
S^F_g (\gamma \circ \phi) (X; X) = \left( a + e^{-\frac{|d(\gamma \circ \phi)|^2}{2}} \right) \frac{|d (\gamma \circ \phi)|^2}{2} g (X, X)
- \left[ \left( a + e^{-\frac{|d(\gamma \circ \phi)|^2}{2}} \right) - \frac{|d (\gamma \circ \phi)|^2}{2} e^{-\frac{|d(\gamma \circ \phi)|^2}{2}} \right] (\gamma \circ \phi)^* \text{can} (X, X).
= \alpha^2 \circ \phi \left( a + e^{-\frac{|d\phi|^2}{2}} \right) \frac{|d\phi|^2}{2} g (X, X)
- \alpha^2 \circ \phi \left[ a + e^{-\frac{|d\phi|^2}{2}} \right] \alpha^2 \circ \phi \frac{|d\phi|^2}{2} e^{-\frac{|d\phi|^2}{2}} \phi^* \text{can} (X, X)
= \alpha^2 \circ \phi \left( a + e^{-\frac{|d\phi|^2}{2}} \right) \left[ \frac{1}{2} |d\phi|^2 g (X, X) - \phi^* \text{can} (X, X) \right]
+ \alpha^2 \circ \phi \frac{|d\phi|^2}{2} e^{-\frac{|d\phi|^2}{2}} \phi^* \text{can} (X, X).
\]

An other example is the following function \( F(t) = (1 + 2t)^\alpha \) where \( 0 < \alpha < 1 \), the \( F\)-energy is the \( \alpha\)-energy of Sacks-Uhlenbeck (see [23]). In fact

\[
B = (\alpha - 1) \left( \frac{1}{1 + \alpha^2 \circ \phi \circ |d\phi|^2} - \frac{1}{1 + |d\phi|^2} \right) \frac{1}{\alpha}
\]

provided that \( \phi_v \geq 0 \).

And for vector field \( X \) on \( M \), we have

\[
S^F_g (\gamma \circ \phi) (X; X) = 2\alpha \left( 1 + \alpha^2 \circ \phi \circ |d\phi|^2 \right) \frac{\alpha - 1}{\alpha} \alpha^2 \circ \phi \circ \frac{|d\phi|^2}{2} g (X, X)
- \left[ 2\alpha \left( 1 + \alpha^2 \circ \phi \circ |d\phi|^2 \right) ^{\alpha - 1} + 4\alpha (\alpha - 1) \left( 1 + \alpha^2 \circ \phi \circ |d\phi|^2 \right)^{\alpha - 2} \alpha^2 \circ \phi \right]
\times \alpha^2 \circ \phi \phi^* \text{can} (X, X)
= \left[ 2\alpha \left( 1 + \alpha^2 \circ \phi \circ |d\phi|^2 \right) ^{\alpha - 1} \alpha^2 \circ \phi \left( \frac{|d\phi|^2}{2} g (X, X) - \phi^* \text{can} (X, X) \right) \right]
+ 4\alpha (1 - \alpha) \left( 1 + \alpha^2 \circ \phi \circ |d\phi|^2 \right) ^{\alpha - 2} \alpha^2 \circ \phi \phi^* \text{can} (X, X)
\]
and taking account of the positivity of the stress-energy tensor of $\phi$ and the fact that $\phi_v \geq 0$, we infer that

$$S^F_g (\gamma \circ \phi) (X, X) \geq \alpha^4 \circ \phi . S^F_g \phi (X, X).$$

**Remark 1** $\phi_v \geq 0$ occurs for example if $\phi (M)$ is included in the positive half-sphere $S^{n+} = \{ x \in S^n : \langle x, v \rangle \geq 0 \}$.

In this section we state the following result

**Theorem 1** Let $F : [0, +\infty[ \to [0, +\infty[$ be an admissible function and $\phi$ be an $F$-harmonic map from a compact $m$-Riemannian manifold $(M, g)$ ($m \geq 2$) into the Euclidean sphere $S^n$ ($n \geq 2$). Then for any conformal diffeomorphism $\gamma$ on $S^n$, $E_F (\gamma \circ \phi) \leq E_F (\phi)$ (resp. $E_F (\gamma \circ \phi) < E_F (\phi)$) provided that the stress-energy tensor $S^F_g (\phi)$ is positive (respectively positive defined).

**Remark 2** In case $F(t) = \frac{1}{p} \left( \frac{2}{t} \right)^p$, $p = 2$ or $p \geq 4$ the condition $\phi_v \geq 0$ is not needed since $B = 0$, so our result recover the ones by El-Soufi in [16, 18].

To prove Theorem 1, we need the following lemmas

**Lemma 1** Let $\phi : (M, g) \to (N, h)$ be a smooth map and $\gamma$ be a conformal diffeomorphism on $N$, then the $F$-tension of the map $\gamma \circ \phi$, is given by

$$\tau_F (\gamma \circ \phi) = 2 \alpha^{-1} \circ \phi . F' (\alpha^2 \circ \phi \frac{|d \phi|^2}{2}) d \gamma \left( d \phi_v - \frac{|d \phi|^2}{2} \nabla \circ \alpha \circ \phi \right)$$

$$+ f d \gamma \left( \tau_F (\phi) \right) + d \gamma \left( F' \left( \frac{|d \phi|^2}{2} \right) d \phi \left( \nabla f \right) \right).$$

where $f = - \frac{F' (\alpha^2 \circ \phi \frac{|d \phi|^2}{2})}{F' (\frac{|d \phi|^2}{2})}$ and $\gamma^* \circ \alpha = \alpha^2 \circ \alpha$.

**Proof** We follow closely the proof in ([19]).

$$\tau_F (\gamma \circ \phi) = \text{trace}_g \nabla \left( F' \left( \frac{|d (\gamma \circ \phi)|^2}{2} \right) d (\gamma \circ \phi) \right)$$

$$= F' \left( \frac{|d (\gamma \circ \phi)|^2}{2} \right) \text{trace} \left( \nabla d (\gamma \circ \phi) \right)$$

$$+ d (\gamma \circ \phi) \left( \nabla F' \left( \frac{|d (\gamma \circ \phi)|^2}{2} \right) \right)$$

where $\nabla F' \left( \frac{|d (\gamma \circ \phi)|^2}{2} \right)$ is the gradient of $F' \left( \frac{|d (\gamma \circ \phi)|^2}{2} \right)$ in $M$. 

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Since $\gamma$ is a conformal diffeomorphism on $S^n$, we have

$$
\tau_F(\gamma \circ \phi) = F' \left( \alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \tau(\gamma \circ \phi) + d(\gamma \circ \phi) \left( \nabla F' \left( \alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \right)
$$

$$
= F' \left( \alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \left( \text{trace}_g \nabla' d\gamma (d\phi, d\phi) + d\gamma \tau(\phi) \right)
$$

$$
+ d(\gamma \circ \phi) \left( \nabla F' \left( \alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \right)
$$

$$
= F' \left( \alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \left( \text{trace}_g \nabla' d\gamma (d\phi, d\phi) + \frac{1}{F' \left( \frac{|d\phi|^2}{2} \right)} d\gamma (\tau_F(\phi)) \right)
$$

$$
- \frac{F'(\alpha^2 \circ \phi \frac{|d\phi|^2}{2})}{F' \left( \frac{|d\phi|^2}{2} \right)} d(\gamma \circ \phi) \left( \nabla F' \left( \frac{|d\phi|^2}{2} \right) \right)
$$

$$
+ d(\gamma \circ \phi) \left( \nabla F' \left( \alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \right).
$$

Putting $f = \frac{F'(\alpha^2 \circ \phi \frac{|d\phi|^2}{2})}{F' \left( \frac{|d\phi|^2}{2} \right)}$ we get

$$
\tau_F(\gamma \circ \phi) = F'(\alpha^2 \circ \phi \frac{|d\phi|^2}{2}) \text{trace}_g \nabla' d\gamma (d\phi, d\phi) + f d\gamma (\tau_F(\phi))
$$

$$
+ F' \left( \frac{|d\phi|^2}{2} \right) d(\gamma \circ \phi) (\nabla f).
$$

Now since $\gamma : (N, \gamma^*\text{can}) \to (N, \text{can})$ is an isometry then, if $\tilde{\nabla}$ denotes the connection corresponding to $\gamma^*\text{can}$, we have

$$
\nabla' d\gamma(X, Y) = d\gamma (\tilde{\nabla}_X Y - \nabla_X Y)
$$

and since (see [19])

$$
\tilde{\nabla}_X Y - \nabla_X Y = \alpha^{-1} ((X, \nabla) Y + (Y, \nabla) X - (X, Y) \nabla \alpha)
$$

we obtain

$$
\text{trace}_g \nabla' d\gamma (d\phi, d\phi) = 2\alpha^{-1} \circ \phi. d\gamma \left( d\phi (\nabla \alpha \circ \phi) - \frac{|d\phi|^2}{2} \nabla \alpha \circ \phi \right).
$$
Finally we infer that

$$\tau_F (\gamma o \phi) = 2\alpha^{-1} o \phi. F'(\alpha^2 o \phi \frac{|d\phi|^2}{2}) d\gamma \left( d\phi (\nabla \alpha o \phi) - \frac{|d\phi|^2}{2} \nabla \alpha o \phi \right)$$

$$+ f d\gamma (\tau_F (\phi)) + F'(\frac{|d\phi|^2}{2}) d\gamma o d\phi (\nabla f).$$

\[\square\]

**Lemma 2** Let $\phi$ be an $F$-harmonic map from an $m$-dimensional Riemannian manifold $(M, g)$ $(m \geq 2)$ into the Euclidean unit sphere $(S^n, \text{can})$ $(n \geq 2)$.

Then for any $v \in \mathbb{R}^{n+1} - \{0\}$ and any $t_0 \in \mathbb{R}$ we have

$$\frac{d}{dt} E_F (\gamma_t o \phi)_{t=t_0} = -2 sht_{t_0} \frac{1}{|v|} \int_M \alpha_{t_0}^3 o \phi. F' \left( \alpha_{t_0}^2 o \phi \frac{|d\phi|^2}{2} \right) \left( |d\phi|^2 |\bar{\nu} o \phi|^2 - |d\phi|^2 \right) d\nu_g$$

$$- \int_M \alpha_{t_0}^2 o \phi. F' \left( \frac{|d\phi|^2}{2} \right) \left( d\phi (\nabla f_{t_0}) , \bar{\nu} o \phi \right) d\nu_g$$

where $f_{t_0} = \frac{F'(\alpha_{t_0}^2 o \phi \frac{|d\phi|^2}{2})}{F'(|d\phi|^2)}$, $\gamma^* \text{can} = \alpha_{t_0}^2 \text{can}$ and

$$\alpha_{t_0} = \frac{|v|}{\phi_{t_0} sht_{t_0} + |v| cht_{t_0}}.$$ 

**Proof** Recall that the first variation formula of the $F$-energy is given by

$$\frac{d}{dt} E_F (\gamma_t o \phi)_{t=t_0} = - \int_M \left( \tau_F (\gamma_t o \phi) , \bar{\nu} o (\gamma_t o \phi) \right) d\nu_g.$$ 

By Lemma 1 and the fact that $\phi$ is $F$-harmonic we get

$$\frac{d}{dt} E_F (\gamma_t o \phi)_{t=t_0} = - \int_M 2\alpha_{t_0} o \phi. F' \left( \alpha_{t_0}^2 o \phi \frac{|d\phi|^2}{2} \right)$$

$$\times \left( (\nabla \alpha_{t_0} o \phi) \nabla \alpha_{t_0} o \phi , \bar{\nu} o \phi \right) d\nu_g$$

$$- \int_M F' \left( \frac{|d\phi|^2}{2} \right) \left( d\phi (\nabla f_{t_0}) , \bar{\nu} o \phi \right) d\nu_g$$

and since (see [19])
we have

\[
\langle \nabla \alpha_{t_o} \phi, \bar{\phi} \rangle = -\left( \frac{\alpha_{t_o}}{|v|} \right)^2 sh(t_o) |\phi|^2.
\]

Now let \((e_1, \ldots, e_m)\) be an orthogonal basis on \(M\)

\[
\left( (\nabla \alpha_{t_o} \phi)^T, \bar{\phi} \right) = \sum_{i=1}^{m} \langle \nabla \alpha_{t_o} \phi, d\phi(e_i) \rangle \langle \bar{\phi}, d\phi(e_i) \rangle
\]

\[
= -sh(t_o) \left( \frac{\alpha_{t_o}}{|v|} \right)^2 \sum_{i=1}^{m} (\bar{\phi}, d\phi(e_i))^2
\]

\[
= -sh(t_o) \left( \frac{\alpha_{t_o}}{|v|} \right)^2 |d\phi|.
\]

Hence

\[
\frac{d}{dt} \left( \frac{\gamma^{G}_{t_o}}{\nabla \phi_{t_o}} \right) = -\frac{sh(t_o)}{|v|} \int_M \alpha_{t_o}^3 \phi \cdot F' \left( \frac{|d\phi|^2}{2} \right) \left( \frac{|d\phi|^2}{2} - |\bar{\phi}|^2 \right) d\nu_g
\]

\[
- \int_M \alpha_{t_o}^2 \phi \cdot F' \left( \frac{|d\phi|^2}{2} \right) \langle \nabla \phi, \bar{\phi} \rangle d\nu_g.
\]

\[\square\]

We set

\[
g(t) = \int_M \alpha_{t_o}^2 \phi \cdot F' \left( \frac{|d\phi|^2}{2} \right) \langle \nabla \phi, \bar{\phi} \rangle d\nu_g.
\]

Lemma 3

\[
g(t) = \int_M \alpha_{t_o}^3 \phi \cdot F'' \left( \frac{|d\phi|^2}{2} \right) |d\phi|^2 \langle \nabla \phi, \bar{\phi} \rangle d\nu_g
\]

\[
+ \int_M \alpha_{t_o}^2 \phi \cdot \left( F'' \left( \frac{|d\phi|^2}{2} \right) \alpha_{t_o}^2 \phi - \frac{F' \alpha_{t_o}^2 \phi \cdot \left| \frac{d\phi}{2} \right|^2}{F' \left( \frac{|d\phi|^2}{2} \right)} F'' \left( \frac{|d\phi|^2}{2} \right) \right) d\nu_g
\]

\[
\times \frac{\phi_v}{|v|} |d\phi|^2 d\nu_g.
\]
Proof First, we compute $\nabla f_t$

$$
\nabla f_t = \frac{F''(\alpha_t^2 \phi \cdot \frac{|d\phi|^2}{2})}{F'} \left( \alpha_t \phi \cdot |d\phi|^2 \nabla (\alpha_t \phi) + \alpha_t^2 \phi \cdot (\nabla d\phi, d\phi) \right)
$$

$$
- \frac{F'(\alpha_t^2 \phi \cdot \frac{|d\phi|^2}{2})}{F'} F'' \left( \frac{|d\phi|^2}{2} \right) \langle \nabla d\phi, d\phi \rangle
$$

$$
= \frac{F''(\alpha_t^2 \phi \cdot \frac{|d\phi|^2}{2})}{F'} \alpha_t \phi |d\phi|^2 \nabla (\alpha_t \phi)
$$

$$
+ \left( \frac{F''(\alpha_t^2 \phi \cdot \frac{|d\phi|^2}{2})}{F'} \alpha_t^2 \phi - \frac{F'(\alpha_t^2 \phi \cdot \frac{|d\phi|^2}{2})}{F'} F'' \left( \frac{|d\phi|^2}{2} \right) \right) \langle \nabla d\phi, d\phi \rangle.
$$

Then

$$
g(t) = \int_M \alpha_{t_0}^2 \phi \cdot F' \left( \frac{|d\phi|^2}{2} \right) |d\phi (\nabla f_{t_0}, \bar{\nu} \phi) d\nu_g
$$

$$
= \int_M \alpha_{t_0}^3 \phi \cdot F'' \left( \frac{|d\phi|^2}{2} \right) |d\phi|^2 \langle d\phi (\nabla \alpha_{t_0} \phi), \bar{\nu} \phi \rangle d\nu_g
$$

$$
+ \int_M \alpha_{t_0}^2 \phi \cdot \left( F''(\alpha_{t_0}^2 \phi \cdot \frac{|d\phi|^2}{2}) \alpha_{t_0}^2 \phi - \frac{F'(\alpha_{t_0}^2 \phi \cdot \frac{|d\phi|^2}{2})}{F'} F'' \left( \frac{|d\phi|^2}{2} \right) \right)
$$

$$
\times \langle d\phi (\nabla |d\phi|^2), \bar{\nu} \phi \rangle d\nu_g. \quad (4)
$$

Let $\{e_1, \ldots, e_m\}$ be a basis of $T_xM$ which diagonalizes $\phi^*\gamma$, we have

$$
\left\langle d\phi \left( \frac{|d\phi|^2}{2} \right), \bar{\nu} \phi \right\rangle = \sum_{i,j} \nabla_{e_i} d\phi \cdot \bar{\nu} \phi \cdot d\phi (e_j) \cdot \bar{\nu} \phi \cdot d\phi (e_j)
$$

$$
\left\langle d\phi \left( \frac{|d\phi|^2}{2} \right), \bar{\nu} \phi \right\rangle = -\sum_{i,j} \frac{d}{dt} \mid_{t=0} \gamma_{\phi^*} |d\phi (e_j)|^2
$$

Likewise we get

$$
\left\langle \left[ \bar{\nu} \phi, d\phi (e_j) \right], d\phi (e_j) \right\rangle = -\frac{d}{dt} \mid_{t=0} \alpha_t^2 |d\phi (e_j)|^2
$$

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and taking account of (1) we obtain that

$$\langle [\nu \phi, d \phi (e_j)], d \phi (e_j) \rangle = \frac{\phi v}{|v|} |d \phi (e_j)|^2$$

so we infer that

$$\langle d \phi \left( \nabla \frac{|d \phi|^2}{2} \right), \nu \phi \rangle = \frac{\phi v}{|v|} |d \phi|^2 + \langle \nabla_e \nu \phi, d \phi (e_j) \rangle$$

and

$$\langle \nabla_e \nu \phi, d \phi (e_j) \rangle = \nabla_e \langle \nu \phi, d \phi (e_j) \rangle - \langle \nu \phi, \nabla_e d \phi (e_j) \rangle$$

$$= \nabla_e \langle v, d \phi (e_j) \rangle - \langle \nu \phi, \nabla_e d \phi (e_j) \rangle$$

$$= \langle v, \nabla_e d \phi (e_j) \rangle - \langle \nu \phi, \nabla_e d \phi (e_j) \rangle$$

$$= \langle v - \nu \phi, \nabla_e d \phi (e_j) \rangle = 0.$$ 

Hence

$$\langle d \phi \left( \nabla \frac{|d \phi|^2}{2} \right), \nu \phi \rangle = \frac{\phi v}{|v|} |d \phi|^2.$$ 

(5)

Now set

$$\varphi(t_o) = 2 sht_o |v| \int_M \alpha^3_{t_o} \phi . F' \left( \alpha^2_{t_o} \phi . \frac{|d \phi|^2}{2} \left( - \frac{|d \phi|^2}{2} |\nu \phi|^2 + |d \phi_v|^2 \right) \right) d v_g$$

$$- \int_M \alpha^3_{t_o} \phi . F'' \left( \alpha^2_{t_o} \phi . \frac{|d \phi|^2}{2} \right) \frac{|d \phi|^2}{2} \langle d \phi \left( \nabla (\alpha_{t_o} \phi) \right), \nu \phi \rangle d v_g$$

and since by (2) we have

$$\langle d \phi \left( \nabla (\alpha_{t_o} \phi) \right), \nu \phi \rangle = - \frac{sht_o}{|v|} \alpha^2_{t_o} \phi |d \phi_v|^2$$

we get

$$\varphi(t_o) = 2 sht_o |v| \int_M \alpha^3_{t_o} \phi . \left[ \left( F' \left( \alpha^2_{t_o} \phi . \frac{|d \phi|^2}{2} \right) + \alpha^2_{t_o} \phi . \frac{|d \phi|^2}{2} F'' \left( \alpha^2_{t_o} \phi . \frac{|d \phi|^2}{2} \right) \right) d v_g \right]$$

$$\times |d \phi_v|^2 - F' \left( \alpha^2_{t_o} \phi . \frac{|d \phi|^2}{2} \right) \frac{|d \phi|^2}{2} |\nu \phi|^2 \right] d v_g.$$ 

(6)
or

\[ \varphi(t_o) = -2 \frac{\text{sh}t_o}{|v|} \int_M \alpha^3_{t_o} \phi \cdot S^F_g (\gamma_{t_o}^v \phi) \, dv_g. \]

**Proof** (of Theorem 1) Recall (see [14]) that for any conformal diffeomorphism \( \gamma \) of the unit sphere \( S^n \) there exist an isometry \( r \in O(n + 1) \), a real number \( t \geq 0 \) and a vector \( v \in R^{n+1} - \{0\} \) such that \( \gamma = ro \gamma_{t}^v \), so it suffices to consider \( \gamma_{t}^v \) with \( t \geq 0 \) and \( v \in R^{n+1} - \{0\} \).

On the other hand

\[ \frac{d}{dt} E_F (\gamma_{t}^v \phi) = \varphi(t) + \chi(t) \]

where

\[ \chi(t) = - \int_M \alpha^2_t \phi \cdot \left( F''(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) \alpha^2_t \phi - \frac{F'(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) F''(\frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} \right) \frac{\phi^v}{|v|} \times |d\phi|^2 \, dv_g \]

and \( \varphi(t) \) is given by (6). Now, since the function \( F \) is admissible we infer that \( \chi(t) \leq 0 \). Since the energy stress-tensor \( S^F_g (\gamma \phi) = S^F_g (\gamma_{t}^v \phi) \) of \( \gamma \phi \) is positive (resp. positive defined) by assumption and

\[ S^F_g (\gamma_{t}^v \phi) = F'(\frac{|d(\gamma_{t}^v \phi)|^2}{2}) \frac{|d(\gamma_{t}^v \phi)|^2}{2} g - \left[ F'(\frac{|d(\gamma_{t}^v \phi)|^2}{2}) + \frac{|d(\gamma_{t}^v \phi)|^2}{2} F''(\frac{|d(\gamma_{t}^v \phi)|^2}{2}) \right] (\gamma_{t}^v \phi)^* \text{can} \]

\[ = \alpha^2_t \phi \left[ F'(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) \frac{|d\phi|^2}{2} g - F'(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) \right] - \alpha^2_t \phi \frac{|d\phi|^2}{2} F''(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) \phi^* \text{can} \]

so the tensor

\[ F'(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) \frac{|d\phi|^2}{2} g - \left( F'(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) + \alpha^2_t \phi \frac{|d\phi|^2}{2} F''(\alpha^2_t \phi \cdot \frac{|d\phi|^2}{2}) \right) \phi^* \text{can} \]

is positive (resp. positive defined). Consequently \( \varphi(t) \leq 0 \) (resp. \( \varphi(t) < 0 \)) for any \( t \geq 0 \) and the proof of Theorem 1 is complete. \( \square \)
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