Size Scaling of Velocity Field in Granular Flows through Apertures

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For vertical velocity field \(v_z(r, z; R)\) of granular flow through an aperture of radius \(R\), we propose a size scaling form \(v_z(r, z; R) = v_z(0, 0; R)f(r/R, z/R)\) in the region above the aperture. The length scales \(R_e = R - 0.5d\) and \(R_a = R + k_2d\), where \(k_2\) is a parameter to be determined and \(d\) is the diameter of granule. The effective acceleration, which is derived from \(v_z\), follows also a size scaling form \(a_{\text{eff}}(0, 0; R)R^{-1}f(r/R, z/R)\). For granular flow under gravity \(g\), there is a boundary condition \(a_{\text{eff}}(0, 0; R) = -g\) which gives rise to \(v_z(0, 0; R) = \sqrt{gR}\) with \(\lambda = -1/\theta(0, 0)\). Using the size scaling form of vertical velocity field and its boundary condition, we can obtain the flow rate \(W = C_2\rho\sqrt{g}R^{D-1}R^{-1/2}\), which agrees with the Beverloo law when \(R \gg d\). The vertical velocity fields \(v_z(r, z; R)\) in three-dimensional (3D) and two-dimensional (2D) hoppers have been simulated using the discrete element method (DEM) and GPU program. Simulation data confirm the size scaling form of \(v_z(r, z; R)\) and the \(R\)-dependence of \(v_z(0, 0; R)\).

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Granular materials flowing through apertures show many unusual physical properties and have been studied extensively for decades\cite{1}. Depending on the size of apertures, granular flows can have three patterns which are continuous, intermittent, and jammed respectively. Contrary to fluids, flow rate of granular materials in the continuous pattern does not depend on the height of the granular layer above the aperture. Beverloo et al.\cite{2} proposed an empirical expression of granular material flow rate \(W\) driven by gravity as

\[
W = C_\rho \sqrt{g}(R - kd)^{D-1/2},  
\]

(1)

where \(g\) is the gravitational acceleration, \(\rho\) is the bulk density, \(R\) is the radius of aperture, \(d\) is the granule diameter, and \(D\) is the dimensionality of hopper. \(C\) and \(k\) are two fitted parameters. The Beverloo law was related to a hypothesis of free-fall arch, which was introduced by Hagen\cite{3} and developed lately by Brown and Richards\cite{4}. Velocities of the granules above the arch are considered to be negligible. Below the arch, granules fall freely\cite{1, 3-6}. However, the acceleration profiles obtained experimentally by Rubio-Largo et al.\cite{7} are against the existence of free-fall arch described by the Heaviside function\cite{7}.

For granular flow on conveyor belt, Bao et al.\cite{8} found that 2D flow rate \(W\) is proportional to \(R\) when the velocity of conveyor belt \(u < u_c\) and becomes proportional to \((R - kd)^{3/2}\) when \(u > u_c\). Further experiments of granular flow on conveyor belt were performed by Aguirrue et al.\cite{9, 10}. Using the DEM\cite{11}, microdynamic variable distributions of the granular flow in 3D cylindrical hoppers have been investigated by Zhu et al.\cite{12}. Under general gravity \(g^*\), Dorbolo et al.\cite{13} measured the mass flow rate which depends on the square root of the gravity \(g^*\)\cite{13}.

In this Letter, we provide a mechanism to understand the dependences of flow rate \(W\) on the radius \(R\) of apertures in hoppers. A schematic diagram of hopper is shown in Fig.1. At any position of a hopper, an average velocity \(v\) of granular flow can be defined. Because of the symmetry of hopper, vertical velocity \(v_z\) depends on \(r\) and \(z\). Furthermore, \(v_z\) should be related to \(R\) and the radius of hopper \(H\). For hoppers with \(H \gg R\), the dependency of \(v_z\) on \(H\) can be neglected and we have \(v_z = v_z(r, z; R)\).

Using DEM and GPU program, granular flows in hopper have been simulated\cite{14, 15}. The correlation of velocities in different positions of hopper can be calculated from the simulation data. It was found that the velocities in the region above the aperture are correlated strongly. For a finite system near its critical point, the system is

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FIG. 1. Schematic diagram of granular flow through an aperture.
correlated strongly and there is the finite-size scaling [16]. Similar to the finite-size scaling of critical phenomena, we propose a size scaling form of the vertical velocity field as

\[
v_z(r, z; R) = v_z(0, 0; R) f(r/R_r, z/R_z),
\]

where \( R_r \) and \( R_z \) are the radial and vertical sizes of the region. Because of the granule diameter \( d \), we anticipate that \( R_r = R - 0.5d [17] \). The scale length \( R_z = R + k_2d \), where \( k_2 \) is the only parameter to be determined.

During a time period \( \delta t \), granules at \( (r, z) \) move in average to \( (r + \delta r, z + \delta z; R\delta t) \) and have a velocity change \( \delta v_z = v_z(r, z + \delta z; R) - v_z(r, z; R) \). We can calculate an effective acceleration as

\[
a_{\text{eff}}(r, z; R) = \lim_{\delta t \to 0} \frac{\delta v_z}{\delta t} = \frac{\partial v_z(r, z; R)}{\partial z} v_z(r, z; R). \tag{3}
\]

According to Eq.2, we get

\[
a_{\text{eff}}(r, z; R) = \frac{v_z^2(0, 0; R)}{R_z} \theta(r/R_r, z/R_z), \tag{4}
\]

where

\[
\theta(r, z) = f(r, z) f_{z}^{(1)}(r, z), \tag{5}
\]

\[
f_{z}^{(1)}(r, z) = \frac{\partial f(r, z)}{\partial z}. \tag{6}
\]

At the center of aperture, the scaling function \( f(0, 0) = 1 \) and the effective acceleration

\[
a_{\text{eff}}(0, 0; R) = \frac{v_z^2(0, 0; R)}{R_z} f_{z}^{(1)}(0, 0). \tag{7}
\]

When hopper radius \( R \gg d \), granules can fall freely at aperture center and there is the boundary condition

\[
a_{\text{eff}}(0, 0; R) = -g. \tag{8}
\]

We can then get the velocity

\[
v_z(0, 0; R) = \sqrt{g\lambda R_z^{1/2}}, \tag{9}
\]

where \( \lambda = -1/f_{z}^{(1)}(0, 0) \).

For a \( D \)-dimensional hopper, mass flow rate \( W \) can be calculated as

\[
W = \int_{0}^{R_r} v_z(r, 0; R) \rho S_D (r - r)^{D-2} dr, \tag{10}
\]

where \( S_1 = 2 \) and \( S_2 = 2\pi \). Using Eqs.(2) and (9), we obtain

\[
W = C_1 \rho \sqrt{g\lambda R_z^{1/2}} R^{D-1} R_z^{1/2}, \tag{11}
\]

with \( C_1 = S_D [\lambda f_{z}^{(1)}(0, 0)]^{-1/2} f_0^1 f(x, 0) x^{D-2} dx \). In asymptotical case \( R \gg d \), our Eq.(11) becomes

\[
W \approx C_1 \rho \sqrt{g\lambda R_z R^{D-2}} \]

and is in agreement with the Beverloo law.

To test the size scaling of velocity field above, we have made a series of simulations about granular flow through an aperture with the model of Ref.[14]. The granule has the diameter \( d = 1 \) \( \text{mm} \) and density \( \rho = 2500 \) \( \text{kg/m}^3 \). Further parameters are listed in Table 1.

To check the size scaling in general dimensionality, granular flows both in 3D and 2D hoppers have been simulated. In 3D hoppers with radius \( R = 60d \), \( N = 4400000 \) granules are studied for apertures with radius \( R = 5d, 10d, 15d, 20d, 25d \). The snapshots of positions and velocities of granules are taken every 10000 steps with step \( \tau = 5.0 \times 10^{-7} \) seconds. In 2D hoppers with the radius \( R = 100d, N = 200000 \) granules have been simulated through apertures with radius \( R = 8d, 10d, 15d, 20d, 25d \). The snapshots of 2D hoppers are taken every 20000 steps.

The vertical velocity field \( v_z(r, h; R) \) can be obtained by two averages of simulation data. From a snapshot, vertical velocities of the granules inside a region with radius from \( r - 0.5d \) to \( r + 0.5d \) and height from \( h - 0.5\delta_h \) to \( h + 0.5\delta_h \) are known and their averages can be calculated. After the second average over snapshot, vertical velocity field \( v_z(r, h; R) \) can be obtained. We chose \( \delta_h = d, 1.5d \) for 3D and 2D hopper, respectively.

| TABLE I. Parameters of granular model. |
|--------------------------------------|
| Quantity                             | Symbol | Value       |
| Poisson’s ratio                      | \( \nu \) | 0.2         |
| Friction of spheres                  | \( \mu \) | 0.5         |
| Coefficient of restitution           | \( \varepsilon \) | 0.8         |
| Shear modulus (Pa)                   | \( G \) | \( 3.0 \times 10^{10} \) |
| Elastic modulus (Pa)                 | \( E \) | \( 7.2 \times 10^{10} \) |
| Density of spheres (kg/m\(^3\))     | \( \rho \) | \( 2.5 \times 10^{3} \) |
| Diameter of spheres (m)              | \( d \) | \( 1.0 \times 10^{-3} \) |
The different curves on the left side collapse together. As shown in Fig.4(b), the solid line is drawn with the slope \( k_2 \) of 2D hoppers with aperture radius \( R \). According to Eq.2, the scaled velocity field \( v_2(r, z; R)/v_2(0, 0; R) \) is plotted in Fig.8 as a function of \( r/R \) and \( z/R \) with \( R_t = R - 0.5d \) and \( R_e = R + 0.1d \). The different curves of \( v_2(r, z; R) \) in Fig.6 collapse together in Fig.8. In Fig.8, the size scaling form of Eq.2 is also confirmed by our simulation data of 2D hoppers.

The normalized velocity \( v_2(r, 0; R)/v_2(0, 0; R) \) of 2D silo has been investigated experimentally by A. Janda et al. [5]. As a function of \( r/R \), their experimental data of different \( R \) collapse together. For both 2D and 3D silos, S.M. Rubio-Largo et al.[7] studied experimentally the normalized effective acceleration \( a_{eff}(0, z; R)/g \) as a function of \( z/R \) and there is no sign of existence of the free-fall arch.

Under general gravity \( g^* \), the boundary condition becomes \( a_{eff}(0, 0; R) = -g^* \) and \( \sqrt{g} \) of the mass flow rate.
in Eq. 11 is replaced by $\sqrt{g'\lambda}$, which is in agreement with experimental results of Dorbolo et al. [13].

For granular flows on conveyor belt, we have the boundary condition $v(0, 0; R) = u$ when $u$ is small. Then the mass flow rate can be calculated as $W = C_2 \rho u R_f^{D-1}$ with $C_2 = S_{D-1} \cdot \int_0^1 f(x, 0)x^{D-2}dx$. This result is in agreement with the experimental results [8–10]. If the velocity of conveyor belt is larger than a critical value, there is no boundary condition for $v(0, 0; R)$. The boundary condition now becomes $a_{\text{eff}}(0, 0; R) = -a_f$ with $a_f$ related to the friction force between granules and the belt and $W \propto \rho \sqrt{a_f} R_f^{D-1} R_z^{1/2}$. Therefore, the $R$-dependence of flow rate $W$ on conveyor belt with $D = 2$ switches from $R_e$ to $R_f R_z^{1/2}$ with the increase of belt velocity, as found in the experiment [8].

In summary, a size scaling form of vertical velocity field in granular flow through an aperture with radius $R$ is proposed as $v_z(r, z; R) = v_2(0, 0; R)f(r/R_e, z/R_z)$ in the region above the aperture. The length scales $R_e = R - 0.5d$ and $R_z = R + k z_d$, where $k z_d$ is a parameter to be determined. From $v_z(r, z; R)$, we can get an effective acceleration $a_{\text{eff}}(r, z; R)$ and its size scaling form $a_{\text{eff}}(r, z; R) = v_z^2(0, 0; R) R_z^{-1} f(r/R_e, z/R_z) f_z^{(1)}(r/R_e, z/R_z)$, where $f_z^{(1)}$ is the partial derivative of scaling function. For granular flow under gravity $g$, there is a boundary condition $a_{\text{eff}}(0, 0; R) = -g$ which gives rise to $v_z(0, 0; R) = \sqrt{\lambda g R_z}$ with $\lambda = -1/f_z^{(1)}(0, 0)$. Then we get the flow rate $W = C_4 \rho \sqrt{\lambda g} R_e^{D-1} R_z^{1/2}$, which agrees with the Beverloo law when $R \gg d$. For granular flow on conveyor belt with small speed $u$, there is a boundary condition $v_z(0, 0; R) = u$ and the flow rate $W = C_2 \rho u R_f^{D-1}$. When $u$ is larger than a critical value, there will be a fixed friction force acting on granules by conveyor belt and the boundary condition now is related the effective acceleration. This can explain the switch of the $R$-dependence of flow rate $W$ with conveyor belt speed $u$ as observed in the experiment [8].

Using DEM and GPU program, granular flows under gravity have been simulated for 3D and 2D hoppers and different radius of the aperture. Our simulation data confirm the size scaling form of $v_z(r, z; R)$ proposed above. Furthermore, the $R$-dependence of the vertical velocity at the center of aperture $v_z(0, 0; R) = \sqrt{\lambda g R_z}$ is in agreement with our simulation data.

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FIG. 7. Scaled vertical velocity field \( v_z(r, z; R)/v_z(0, 0; R) \) of 2D hoppers is plotted as a function of \( r/R \) and \( z/R \) with \( R_e = R - 0.5d \) and \( R_a = R + 0.1d \).

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FIG. 8. Vertical velocity field $v_z(r, z; R)$ of 2D hopper is plotted versus $r$ at $z = 0$ in (a) and versus $z$ at $r = 0$ in (c). The scaled velocity field $v_z(r, z; R)/v_z(0, 0; R)$ is presented with respect to $r/R$ in (b) and $z/R$ in (d).