Convergence to diffusion waves for solutions of 1D Keller–Segel model

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In this paper, we are concerned with the asymptotic behavior of solutions to the Cauchy problem (or initial-boundary value problem) of one-dimensional Keller–Segel model. For the Cauchy problem, we prove that the solutions time-asymptotically converge to the nonlinear diffusion wave whose profile is self-similar solution to the corresponding parabolic equation, which is derived by Darcy's law. For the initial-boundary value problem, we consider two cases: Dirichlet boundary condition and null-Neumann boundary condition on \((u, \rho)\).

In the case of Dirichlet boundary condition, similar to the Cauchy problem, the asymptotic profile is still the self-similar solution of the corresponding parabolic equation, which is derived by Darcy's law; thus, we only need to deal with boundary effect. In the case of null-Neumann boundary condition, the global existence and asymptotic behavior of solutions near constant steady states are established. The proof is based on the elementary energy method and some delicate analysis of the corresponding asymptotic profiles.

KEYWORDS
asymptotic behavior, Darcy's law, Keller–Segel model, nonlinear diffusion waves

MSC CLASSIFICATION
85A25, 35L65, 35B40

1 | INTRODUCTION

In 1970, E.F. Keller and L.A. Segel proposed a model to describe the aggregation process of cellular slime mold by the chemical attraction in their celebrated work.\(^1\) The model is now known as the Keller–Segel model, which can be written into the following form:

\[
\begin{aligned}
\partial_t u &= a \Delta u - \kappa \nabla \cdot (u \nabla \rho), \\
\partial_t \rho &= b \Delta \rho + \mu u - \lambda \rho.
\end{aligned}
\] (1.1)

Here, \(u = u(x, t)\) denotes the density of bacteria, and \(\rho = \rho(x, t)\) denotes the concentration of chemical substance that mediates the aggregation. \(a, b, \lambda, \mu, \) and \(\kappa\) are positive constants. \(a\) and \(b\) are, respectively, the diffusion coefficients of bacteria and of chemical substance. \(\lambda\) is a constant rate of decrease of the chemical substance. \(\mu\) is a constant rate of the chemical substance production by the bacteria, and \(\kappa\) denotes the intensity of chemotaxis.

As an important biological model, the Keller–Segel model (1.1) has attracted great interest among many scholars, and there have been many important developments. In one-dimension case, for the Neumann initial-boundary value problem of model (1.1), Osaki and Yagi\(^2\) proved that the solutions of the model are global and uniformly bounded if the initial
data are smooth sufficiently. Afterwards Iwasaki et al. showed that every solution must converge to a stationary solution by using the Lojasiewicz–Simon gradient inequality of the Lyapunov function. For the Cauchy problem of model (1.1), Nagai and Yamada gave the large time behavior of bounded solutions. In two-dimensional model, Nagai et al. proved that every bounded solution to the Cauchy problem decays to zero as $t$ goes infinity with small data $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\nabla \rho_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and that the solution behaves like the heat kernel as the self-similar profile. Calvez and Corrias showed that under additional assumptions $u_0 \log(1 + |x|^2) \in L^1(\mathbb{R}^2)$ and $u_0 \log u_0 \in L^1(\mathbb{R}^2)$, any solution with $m(u_0; \mathbb{R}^2) < 8\pi$ to the Cauchy problem exists globally in time. Mizoguchi proved the global existence of solutions without the additional assumptions when $m(u_0; \mathbb{R}^2) < 8\pi$. Later, the result in Biler et al. meant that any critical mass may lead to a global-in-time solution. In $n$-dimensional case ($n \geq 3$), Corrias and Perthame proved existence of weak positive solutions to the Cauchy problem by taking small data $u_0 \in L^r(\mathbb{R}^n)$ and $\nabla \rho_0 \in L^r(\mathbb{R}^n)$ with $\frac{n}{2} < r \leq n$. If the norms $\|u_0\|_{L^2(\Omega)}$ and $\|\nabla \rho_0\|_{L^r(\Omega)}$ are suitably small, it was proved that the global bounded solutions exist; see Cao and Winkler as well as references cited therein. In the case of large initial data, Winkler constructed a Lyapunov functional to prove that the solutions blow up in finite time. For other results related to (1.1), we refer to the interesting works and the references therein, cf. other works.

However, we can observe that in the unbounded region, the above results require that the initial data be the same constant at infinity, specifically zero at infinity. The different states of the initial data at infinity, as we know, have not been studied so far. In this paper, we will consider this problem by the methods introduced by Hsiao and Liu. And it turns out that the solution to (1.1) will converge to the nonlinear diffusion waves.

Precisely, we shall restrict ourselves to the one-dimensional Keller–Segel model (cf. Osaki and Yagi and Iwasaki et al.):

$$
\begin{align*}
\frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x} + k |u| \rho_x = 0, \\
\frac{\partial \rho}{\partial t} - b \frac{\partial \rho}{\partial x} + \lambda \rho - \mu u = 0, \quad x \in \Omega, \quad t > 0,
\end{align*}
$$

where $\Omega = \mathbb{R}$ or $\mathbb{R}^+$. And we will consider the Cauchy problem on $\mathbb{R}$ and the initial-boundary value problem on $\mathbb{R}^+$ of (1.2).

For the Cauchy problem, the initial data are given by

$$
(u, \rho)(x, 0) = (u_0, \rho_0)(x) \rightarrow (u_\pm, \rho_\pm), \quad as \ x \rightarrow \pm \infty.
$$

where $u_+ \neq u_-$, $\rho_+ \neq \rho_-$ and $u_\pm = \frac{\lambda}{\mu} \rho_\pm$.

For the initial-boundary value problem, we consider the initial data

$$
(u, \rho)|_{t=0} = (u_0, \rho_0)(x) \rightarrow (u_+, \rho_+), \quad u_+ > 0, \quad u_+ = \frac{\lambda}{\mu} \rho_+, \quad as \ x \rightarrow +\infty,
$$

and one of the following boundary conditions:

1. Dirichlet boundary condition

$$
 u|_{x=0} = \beta, \quad \rho|_{x=0} = \frac{\mu \beta}{\lambda},
$$

where the constant $\beta$ takes a value on $[u_-, u_+]$ if $u_- < u_+$ (or $[u_+, u_-]$ if $u_- > u_+$);

2. Null-Neumann boundary condition

$$
 u_\pm|_{x=0} = 0, \quad \rho_\pm|_{x=0} = 0.
$$

We expect to prove that the solutions of the Cauchy problem (1.2), (1.3) and the initial-boundary value problem (1.2), (1.4), and (1.5) (or 1.2,1.4,1.6) converge to the nonlinear diffusion waves as in Hsiao and Liu. Regarding the convergence theory on the nonlinear diffusion waves, it is necessary for us to briefly review its development history. For the Cauchy problem, it was studied by Hsiao and Liu for the first time. They proved that the solutions of the Cauchy problem for $p$-system with damping converge time-asymptotically to the nonlinear diffusion waves whose profile is self-similar solution to the corresponding parabolic equation, which is derived by Darcy’s law. Then, by more detailed and accurate energy estimates, Nishihara generalized the result of Hsiao and Liu and obtained a more precise convergence rates. Furthermore, by constructing an appropriate approximate Green function with the energy method together, Nishihara et al. further improved the convergence rates, which is optimal in the sense comparing with the heat equation. For other results, we refer to other studies and the references therein. For the initial-boundary value
problem on a half line $\mathbb{R}^+$, Nishihara and Yang\textsuperscript{26} considered the asymptotic behavior of solutions of the initial-boundary value problem on $\mathbb{R}^+$ to the equations of $p$-system with linear damping and obtained the $L^2$ and $L^\infty$ convergence rates. Later, Marcati et al\textsuperscript{27} improved the $L^2$ convergence rates of Nishihara and Yang\textsuperscript{26}. For other results, see previous works\textsuperscript{28–31} and references cited therein.

Motivated by these preceding results, we shall prove, for the Cauchy problem (1.2)-(1.3), that the solutions globally exist and time-asymptotically converge to the nonlinear diffusion waves, which is self-similar solution to the corresponding parabolic equation given by Darcy’s law. For the initial-boundary value problem, we consider two cases: Dirichlet boundary condition (1.5) or null-Neumann boundary condition (1.6). In the case of Dirichlet boundary condition (1.5), similar to the Cauchy problem, the asymptotic profile is still the self-similar solution of the corresponding parabolic equation; thus, we only need to deal with boundary effect. In the case of null-Neumann boundary condition (1.6), the global existence and asymptotic behavior of solutions near constant steady states are established.

Finally, we briefly give some remarks on our problem and review some key analytical techniques. Firstly, for the Cauchy problem, the main difficult step lies in obtaining the zero-order energy estimates. On the one hand, we find that it is difficult to handle the bad term $\int_0^t \int_{\mathbb{R}} w^2 \, dx \, dr$ since we cannot obtain the uniform-in-time estimates for $\int_0^t \int_{\mathbb{R}} w^2 \, dx \, dr$ and $\int_0^t \int_{\mathbb{R}} z^2 \, dx \, dr$. In order to overcome such a difficulty, we try to use the structure of the reformulated equations to produce a time-space integrable good term $\int_0^t \int_{\mathbb{R}} (\lambda w - \mu z)^2 \, dx \, dr$ (see 2.3.5–2.3.7 in Lemma 2.4). On the other hand, we need to treat the term $\int_0^t \int_{\mathbb{R}} z^2 \, dx \, dr$. By a heuristic analysis, we realize that $\int_0^t \int_{\mathbb{R}} z^2 \, dx \, dr$ can be transformed into $\int_0^t \int_{\mathbb{R}} z \cdot \omega^2 \, dx \, dr$ (see 2.3.15 in Lemma 2.4), and then we can just control it by using new estimate based on the argument in Huang et al.\textsuperscript{32} See Lemma 2.3 and Corollary 2.1 for details. These two aspects are very vital for obtaining the zero-order energy estimates and are conducive to obtaining the higher-order energy estimates. One can see the details of the zero-order energy estimates in Lemma 2.4.

Secondly, in the case of Dirichlet boundary condition, we construct still the self-similar diffusion waves. Therefore, the zero-order and first-order energy estimates are similar to Cauchy problem, and we need only to consider the treatment of the boundary effect. However, for the second-order energy estimates, due to the difficulties in dealing with some boundary terms, we no longer use the methods applied in the Cauchy problem but mainly use the structure of the reformulated Equation (3.1.2) to close the a priori assumption. See Lemmas 3.7 and 3.8 for more details. In the case of Neumann boundary condition, we consider the global existence and asymptotic behavior of solutions near constant steady states $(u_+, \rho_+)$. The analysis is also quite similar to Cauchy problem.

The rest of the paper is organized as follows. In Section 2, we consider the Cauchy problem for the system (1.2). In Section 2.1, the Cauchy problem is reformulated, and main result will be stated. In Section 2.2, we prepare some preliminaries, which will be useful in the proof of our theorem. Section 2.3 is devoted to the proof of our theorem. In Section 3, we show that the corresponding initial-boundary value problem admits a unique global smooth solution. In Section 3.1, we will obtain the convergence in the case of Dirichlet boundary condition; in Section 3.2, we will study the null-Neumann boundary problem.

**Notations:** Hereafter, $C$ denotes some generic positive constants which are only dependent of the initial data and may vary from line to line. $C_\eta$ denotes the generally large positive constant depending on $\eta$. $L^p = L^p(\Omega)$ $(1 \leq p \leq \infty)$ denotes the Lebesgue space with the norm

$$\|f\|_{L^p} = \left( \int_{\Omega} |f(\xi)|^p \, d\xi \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty} = \sup_{\Omega} |f(\xi)|.$$ 

For any integer $m \geq 0$, $H^m(\Omega)$ denotes the usual Sobolev space with the norm

$$\|f\|_m = \left( \sum_{k=0}^{m} \|\partial_\xi^k f\|^2 \right)^{\frac{1}{2}},$$

where $\| \cdot \| = \| \cdot \|_0 = \| \cdot \|_{L^2(\Omega)}$, $\Omega = \mathbb{R}$ or $\mathbb{R}^+$. 


2 | CAUCHY PROBLEM

2.1 | Reformulation of the Cauchy problem and main result

We first reformulate the Cauchy problem (1.2)-(1.3). From Darcy’s law and asymptotic analysis, we notice that the first term $\rho_t$ and the second term $-b\rho_{xx}$ have a faster time-decay with respect to the term $\lambda \rho$. Therefore, we expect the solutions of (1.2) time-asymptotically behave as those of the following system

\[
\begin{align*}
\tilde{u}_t &= a\tilde{u}_{xx} - \kappa [\tilde{u}\tilde{\rho}_x]_x, \\
\lambda \tilde{\rho} - \mu \tilde{u} &= 0,
\end{align*}
\tag{2.1.1}
\]

or

\[
\begin{align*}
\tilde{u}_t &= \left( a - \frac{\kappa \mu}{\lambda} \right) \tilde{u}_x, \\
\tilde{\rho} &= \frac{\mu}{\lambda} \tilde{u}.
\end{align*}
\tag{2.1.2}
\]

Motivated by Hsiao and Liu and Nishihara,\textsuperscript{18,19} we denote $\bar{u}$ by any solutions of (2.1.1) with the same end states as $u(x, 0)$:

\[
\bar{u}(\pm \infty, t) = u_\pm,
\tag{2.1.3}
\]

and set

\[
\tilde{\rho}(\pm \infty, t) = \frac{\mu}{\lambda} u_\pm = \rho_\pm,
\tag{2.1.4}
\]

due to the Darcy’s law.

Combining (1.2) and (2.1.1), we get

\[
\begin{align*}
(u - \bar{u})_t - a(u - \bar{u})_{xx} + \kappa [u\rho_x]_x - \kappa [\bar{u}\bar{\rho}_x]_x &= 0, \\
\rho_t - b\rho_{xx} + \lambda (\rho - \bar{\rho}) - \mu (u - \bar{u}) &= 0.
\end{align*}
\tag{2.1.5}
\]

Setting the perturbation

\[
(w, z)(x, t) = (\rho - \bar{\rho}, u - \bar{u})(x, t),
\tag{2.1.6}
\]

we have the reformulated problem

\[
\begin{align*}
(z_t - a z_{xx} + \kappa [z + \bar{u}] w_x + z \bar{\rho}_x) &= 0, \\
(w_t - b w_{xx} + \lambda w - \mu z + \bar{\rho}_t - b \bar{\rho}_{xx}) &= 0.
\end{align*}
\tag{2.1.7}
\]

with initial data

\[
(w, z)|_{t=0} = (w_0, z_0)(x) \to 0 \text{ as } x \to \pm \infty.
\tag{2.1.8}
\]

**Theorem 2.1** (Cauchy problem). Suppose that both $\delta := |\rho_+ - \rho_-| + |u_+| + |u_-|$ and $\|w_0\|_2 + \|z_0\|_2$ are sufficiently small. Then, the Cauchy problem (2.1.7)-(2.1.8) exists a unique time-global solutions $(w, z)(x, t)$, which satisfies

\[
w \in W^{i, \infty}([0, \infty); H^{2-i}), \quad i = 0, 1, 2, \quad z \in W^{i, \infty}([0, \infty); H^{2-i}), \quad i = 0, 1, 2,
\]

and

\[
\begin{align*}
\sum_{k=0}^{2} (1 + t)^{k} \left( \| \partial_x^k w(t) \|^2 + \| \partial_x^k z(t) \|^2 \right) &+ (1 + t)^{2} \left( \| w(t) \|^2 + \| z(t) \|^2 \right) \\
+ \int_{0}^{t} \left[ \sum_{j=0}^{2} (1 + \tau)^{j} \left( \| \partial_x^{j+1} w(\tau) \|^2 + \| \partial_x^{j+1} z(\tau) \|^2 \right) + \| \partial_x^{j} (\lambda w - \mu z)(\tau) \|^2 \right] d\tau &
\lesssim C \left( \| w_0 \|_2^2 + \| z_0 \|_2^2 + \delta \right).
\tag{2.1.9}
\end{align*}
\]
2.2 Preliminaries

In this subsection, we are going to introduce some fundamental properties of the nonlinear diffusion waves \((\bar{u}, \bar{\rho})\) and some elementary inequalities, which will play an important role later.

Firstly, combining (2.1.2)\(_1\) with (2.1.3), we have

\[
\begin{cases}
\bar{u}_t = (f(\bar{u})\bar{\rho})_x, \\
\bar{u}(\pm \infty, t) = u_\pm,
\end{cases}
\] (2.2.1)

where \(f(\bar{u}) = a - \frac{d \bar{u}}{\bar{\rho}}\). From the previous works in Atkinson and Peletier and van Duyn and Peletier\(^{33,34}\) we can know that (2.2.1) has a unique self-similar solution called nonlinear diffusion wave in the form

\[
\left\{ \begin{array}{l}
\bar{u}(x, t) = \phi \left( \frac{x}{\sqrt{4t}} \right) := \phi(\xi), \quad \xi \in \mathbb{R}, \\
\phi(\pm \infty) = u_\pm.
\end{array} \right.
\] (2.2.2)

Plug (2.2.2)\(_1\) into (2.2.1)\(_1\) and integrate to obtain, for any \(\zeta_0\),

\[
\phi'(\xi) = \frac{\phi'(\zeta_0) f(\phi(\zeta_0))}{f(\phi(\xi))} e^{-\int_{\zeta_0}^{\xi} \frac{s}{\sqrt{4t}}} d\eta,
\] (2.2.3)

\[
\phi(\xi) = \phi(\zeta_0) + \int_{\zeta_0}^{\xi} \frac{\phi'(\eta) f(\phi(\zeta_0))}{f(\phi(\eta))} e^{-\int_{\zeta_0}^{\xi} \frac{s}{\sqrt{4t}}} d\eta
\]

\[
= \phi(\zeta_0) + \int_{\zeta_0}^{\xi} \phi'(\eta) d\eta.
\] (2.2.4)

It has been shown that (2.2.3) with boundary condition (2.2.2)\(_2\) has a unique solution and that is strictly monotone increasing if \(u_+ > u_-\) and decreasing if \(u_+ < u_-\) in Atkinson and Peletier and van Duyn and Peletier\(^{33,34}\) According to (2.2.4) and \(f(\bar{u}) > 0\) (due to \(|u_\pm| < \frac{\alpha d}{\beta \mu}\), we have

\[
|\phi'(\xi)| \leq Ce^{-C_0 \xi^2},
\] (2.2.5)

for some \(C_0, C > 0\) depending on \(u_\pm\). Moreover, \(\phi'(\zeta_0)\) has the following property that

\[
C_1 |u_+ - u_-| \leq |\phi'(\zeta_0)| \leq C_2 |u_+ - u_-|,
\]

where \(C_1\) and \(C_2\) are positive constants depending on \(u_\pm\). Therefore, we can obtain

\[
|\phi'(\xi)| \leq C |u_+ - u_-| e^{-C_0 \xi^2}.
\] (2.2.6)

As one can see in Hsiao and Liu\(^{18}\), it is easy to prove that the self-similar solution \(\phi(\xi)\) satisfies

\[
\sum_{k=1}^{4} \left| \frac{d^k}{d\xi^k} \phi(\xi) \right| + |\phi(\xi) - u_+|_{(\xi > 0)} + |\phi(\xi) - u_-|_{(\xi < 0)} \leq C |u_+ - u_-| e^{-C_0 \xi^2},
\] (2.2.7)

and \(\bar{u}(x, t)\) satisfies the following dissipative properties:

\[
\begin{align*}
\bar{u}_x & = \frac{\phi'(\xi)}{\sqrt{4(1 + t)}}, \\
\bar{u}_t & = -\frac{\xi \phi'(\xi)}{2(1 + t)}, \\
\bar{u}_{xx} & = \frac{\phi''(\xi)}{1 + t}, \\
\bar{u}_{xt} & = -\frac{\phi'(\xi) + \xi \phi''(\xi)}{2(1 + t)^{3/2}}, \\
\bar{u}_{xxx} & = \frac{\phi'''(\xi)}{(1 + t)^{3/2}}, \\
\bar{u}_{xxt} & = -\frac{\xi \phi'''(\xi) + 2 \phi''(\xi)}{2(1 + t)^{3/2}}, \\
\bar{u}_{xxxx} & = \frac{\phi''''(\xi)(1 + t)^{3/2}}{1 + t^2}.
\end{align*}
\] (2.2.8)
From (2.2.7) and (2.2.8), we can prove that \( \bar{u}(x,t) \) satisfies the following decay estimates.

**Theorem 2.2.** For each \( p \in [1, \infty] \) is an integer, the self-similar solution of (2.2.1) holds that

\[
\min \{ u_+, u_- \} \leq \bar{u}(x,t) \leq \max \{ u_+, u_- \},
\]

\[
\left\| \partial_x^k \partial_t^j \bar{u}(t) \right\|_{L^p(\mathbb{R})} \leq C |u_+ - u_-| (1 + t)^{-\frac{k}{2} - j + \frac{1}{2p}}, \quad k, j \geq 0, \quad k + j \geq 1.
\]

Due to \( \bar{\rho} = \frac{\rho}{\bar{u}} \), the dissipative properties of \( \bar{\rho}(x,t) \) are the same as \( \bar{u}(x,t) \).

Next, we introduce an elementary inequality concerning the time-space integrable estimates with the square of the heat kernel as a weight function. For \( \alpha > 0 \), we define

\[
\tilde{\omega}(x,t) = (1 + t)^{-\frac{1}{2}} \exp \left\{-\frac{ax^2}{1 + t}\right\}, \quad g(x,t) = \int_{-\infty}^{x} \tilde{\omega}(y,t)dy.
\]

(2.2.9)

It is easy to check that

\[
\tilde{\omega}_t = \frac{1}{4a} \tilde{\omega}_{xx}, \quad 4ag_t = \tilde{\omega}_x, \quad \|g(\cdot,t)\|_{L^\infty} = \sqrt{\pi} \alpha^{-\frac{1}{2}}.
\]

(2.2.10)

**Theorem 2.3** (see Huang et al.32). For \( 0 < T \leq +\infty \), assume that \( h(x,t) \) satisfies

\[
h_x \in L^2(0,T;L^2(\mathbb{R})), \quad h_t \in L^2(0,T;H^{-1}(\mathbb{R})).
\]

Then the following estimate holds:

\[
\int_0^T \int_{\mathbb{R}} h^2 \tilde{\omega}^2 dx dt \leq 4\pi \|h(0)\|^2 + 4\pi \alpha^{-1} \int_0^T \|h_x(t)\|^2 dt + 8\alpha \int_0^T \langle h_t, h g^2 \rangle_{H^{-1} \times H^1} dt,
\]

(2.2.11)

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( H^{-1}(\mathbb{R}) \times H^1(\mathbb{R}) \).

**Corollary 2.1.** In addition to the condition of Lemma 2.3, we assume further that \( |u_+| < \delta \ll 1, \|h\|_{L^\infty(\mathbb{R})} \leq \epsilon \ll 1 \) and \( h \) satisfies

\[
h_t = a h_{xx} - \kappa [(h + \bar{u})w_x + h \bar{\rho}_x]_x, \quad h(x,0) = h_0(x) \in L^2(\mathbb{R}), \quad h_x(+\infty, t) = 0,
\]

(2.2.12)

where \( a \) and \( \kappa \) are given positive constant and \( \bar{u} \) and \( \bar{\rho} \) are the self-similar solutions of (2.1.1). Then there exists some positive constant \( C > 0 \) such that

\[
\int_0^T \int_{\mathbb{R}} h^2 \tilde{\omega}^2 dx dt \leq C \int_0^T (\|h_x(t)\|^2 + \|w_x(t)\|^2) dt + C \|h_0\|^2.
\]

(2.2.13)
Proof. Using the integration by parts and (2.2.12), we deduce

\[
\langle h_t, h g^2 \rangle_{H^{-1} \times H^1} = \int_R h_t h g^2 \, dx \\
= \int_R \{ a h_{xx} - \kappa [(h + \bar{u}) w_x + h \bar{\rho}_x] \} h g^2 \, dx \\
= a \int_R h_{xx} h g^2 + \kappa \int_R [(h + \bar{u}) w_x + h \bar{\rho}_x] (h g^2 + 2 h g \bar{\omega}) \, dx \\
= -a \int_R h_{xx} h g^2 - 2a \int_R h g \bar{\omega} d x + a \int_R [(h + \bar{u}) w_x + h \bar{\rho}_x] (h g^2 + 2 h g \bar{\omega}) \, dx \\
\leq -2a \int_R h g \bar{\omega} d x + \kappa \int_R [(h + \bar{u}) w_x + h \bar{\rho}_x] (h g^2 + 2 h g \bar{\omega}) \, dx \\
+ \kappa \int_R h \bar{\rho}_x h g^2 \, dx + 2 \kappa \int_R h g \bar{\omega} d x \\
= \sum_{i=1}^5 I_i,
\]

where \( I_i (1 \leq i \leq 5) \) corresponds to the terms on the right-hand side of the above inequality.

Now we estimate \( I_i (1 \leq i \leq 5) \) term by term. By using (2.2.10), Young inequality, and the following inequality,

\[
\int_R h_{xx} h g^2 \, dx \leq C \int_R (1 + t)^{-1/2} |u_+ - u_-| e^{-C_{\kappa} t} \, dx
\]

we can infer that

\[
I_1 \leq \eta \int_R h^2 \bar{\omega}^2 \, dx + C \eta \int_R h_x^2 \, dx, \\
I_2 \leq C \int_R (\| h \|_{L^\infty} + \| \bar{u} \|_{L^\infty}) (w_x^2 g^4 + h_x^2) \, dx \\
\leq C (\epsilon + \delta) \int_R w_x^2 \, dx + C (\epsilon + \delta) \int_R h_x^2 \, dx, \\
I_3 \leq C \int_R (\| h \|_{L^\infty} + \| \bar{u} \|_{L^\infty}) (w_x^2 g^2 + h^2 \bar{\omega}^2) \, dx \\
\leq C (\epsilon + \delta) \int_R h^2 \bar{\omega}^2 \, dx + C (\epsilon + \delta) \int_R w_x^2 \, dx,
\]

\[
I_4 \leq C \int_R h^2 \bar{\rho}_x^2 \, dx + C \int_R h_x^2 g^4 \, dx \\
\leq C \delta^2 \int_R h^2 \bar{\omega}^2 \, dx + C \int_R h_x^2 \, dx, \\
I_5 \leq \eta \int_R h^2 \bar{\omega}^2 \, dx + C \eta \int_R h_x^2 \bar{\rho}_x^2 \, dx \\
\leq \eta \int_R h^2 \bar{\omega}^2 \, dx + C \eta \delta^2 \int_R h_x^2 \bar{\omega}^2 \, dx.
\]

Here, we have used (2.1.1)_2, (2.2.2)_1, and (2.2.7)–(2.2.9).
Inserting the above inequalities into (2.2.14), then integrating the resulting inequality with respect to \( t \), using (2.2.11) and choosing \( \delta, \eta \) sufficiently small, we can prove (2.2.13) easily. The proof of Corollary 2.1 is completed. \(\square\)

### 2.3 Proof of Theorem 2.1

It is generally known that the global existence can be obtained by the classical continuation argument based on the local existence of solutions and a priori estimates. The local existence of solutions to the reformulated Cauchy problem (2.1.7) and (2.1.8) can be established by the standard iteration argument. The details are omitted. In order to prove Theorem 2.1 for brevity, we only devote ourselves to obtaining the a priori estimates under the a priori assumption

\[
N(T) := \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^{2} (1 + t)^k \left( \| \partial_x^k w(t) \|^2 + \| \partial_x^k z(t) \|^2 \right) \right\} \leq \varepsilon_0^2, \tag{2.3.1}
\]

for some \( 0 < \varepsilon_0 \ll 1 \).

An easy application of Sobolev inequality for \( L^\infty \), we can obtain inequalities

\[
\| \partial_x^k w(\cdot, t) \|_{L^\infty} \leq \sqrt{2} \varepsilon_0 (1 + t)^{-\frac{3}{2} - \frac{k}{2}}, \quad k = 0, 1, \tag{2.3.2}
\]

\[
\| \partial_x^k z(\cdot, t) \|_{L^\infty} \leq \sqrt{2} \varepsilon_0 (1 + t)^{-\frac{3}{2} - \frac{k}{2}}, \quad k = 0, 1, \tag{2.3.3}
\]

which will be used later.

Now we turn to establish (2.1.9), which will be given by a series of lemmas.

**Theorem 2.4.** If \( \varepsilon_0 \) and \( \delta \) are small enough, it holds that

\[
\| w(t) \|^2 + \| z(t) \|^2 + \int_0^t \left( \| w_\lambda(\tau) \|^2 + \| z_\lambda(\tau) \|^2 + \| (\lambda w - \mu z)(\tau) \|^2 \right) \, d\tau \leq C \left( \| w_0 \|^2 + \| z_0 \|^2 + \delta \right), \tag{2.3.4}
\]

for \( 0 \leq t \leq T \).

**Proof.** Firstly, multiplying (2.1.7) by \( \lambda w \), integrating the resulting equality with respect to \( x \) over \( \mathbb{R} \), we obtain

\[
\frac{d}{dt} \int_\mathbb{R} \frac{\lambda w^2}{2} \, dx + b \lambda \int_\mathbb{R} w_x^2 \, dx + \lambda^2 \int_\mathbb{R} w^2 \, dx - \lambda \mu \int_\mathbb{R} wz \, dx + \int_\mathbb{R} \lambda w(\tilde{\rho}_t - b \tilde{\rho}_x) \, dx = 0. \tag{2.3.5}
\]

Multiplying (2.1.7) by \( -\mu z \) and integrating it with respect to \( x \) over \( \mathbb{R} \), we have

\[
-\mu \int_\mathbb{R} w_z \, dx - b \mu \int_\mathbb{R} w_z z_\lambda \, dx - \lambda \mu \int_\mathbb{R} wz \, dx + \mu^2 \int_\mathbb{R} z^2 \, dx - \int_\mathbb{R} \mu z(\tilde{\rho}_t - b \tilde{\rho}_x) \, dx = 0. \tag{2.3.6}
\]

Next, we try to use the structure of Keller–Segel model to produce the good term: \( \int_\mathbb{R} (\lambda w - \mu z)^2 \, dx \). By summing (2.3.5) and (2.3.6), it follows that

\[
\frac{d}{dt} \int_\mathbb{R} \frac{\lambda w^2}{2} \, dx + b \lambda \int_\mathbb{R} w_x^2 \, dx + \int_\mathbb{R} (\lambda w - \mu z)^2 \, dx + \int_\mathbb{R} (\lambda w - \mu z)(\tilde{\rho}_t - b \tilde{\rho}_x) \, dx = \mu \int_\mathbb{R} w_z \, dx + b \mu \int_\mathbb{R} w_z z_\lambda \, dx. \tag{2.3.7}
\]
By applying integration by parts and using (2.1.7)1, one can obtain

\[
\begin{align*}
\mu \int_R w_tz \, dx &= \frac{d}{dt} \int_R \mu wz \, dx - \mu \int_R wz \, dx \\
&= \frac{d}{dt} \int_R \mu wz \, dx - \mu \int_R w \left( azt \, dx - \kappa \left[ (z + \bar{u})w_x + z\bar{\rho}_x \right] \right) \, dx \\
&= \frac{d}{dt} \int_R \mu wz \, dx + a \mu \int_R wz_\alpha \, dx - \kappa \mu \int_R (z + \bar{u})w_x + z\bar{\rho}_x \, dx \\
&= \frac{d}{dt} \int_R \mu wz \, dx + a \mu \int_R wz_\alpha \, dx - \kappa \mu \int_R (z + \bar{u})w_x + z\bar{\rho}_x \, dx.
\end{align*}
\] (2.3.8)

Taking (2.3.8) into (2.3.7), we get

\[
\begin{align*}
\frac{d}{dt} \int_R \left( \frac{\lambda w^2}{2} - \mu wz \right) \, dx + b\lambda \int_R w^2 \, dx + \int_R (\lambda w - \mu z)(\bar{\rho}_t - b\bar{\rho}_\alpha) \, dx \\
= (a + b)\mu \int_R w_xz_\alpha \, dx - \kappa \mu \int_R (z + \bar{u})w^2 \, dx - \kappa \mu \int_R w_z\bar{\rho}_x \, dx.
\end{align*}
\] (2.3.9)

Now we need to estimate the last two terms on the right-hand side of (2.3.9). By using Lemma 2.2 and (2.3.3), we can derive

\[
-\kappa \mu \int_R (z + \bar{u})w_x^2 \, dx \leq C \int_R (||z||_{L^\infty} + ||\bar{u}||_{L^\infty})w_x^2 \, dx \leq C(\epsilon_0 + \delta) \int_R w_x^2 \, dx,
\] (2.3.10)

and

\[
-\kappa \mu \int_R w_z\bar{\rho}_x \, dx \leq \eta \int_R w_x^2 \, dx + C\eta \int_R z^2\bar{\delta}^2 \, dx \leq \eta \int_R w_x^2 \, dx + C\delta^2 \int_R z^2\bar{\delta}^2 \, dx,
\] (2.3.11)

where in the last inequality we have taken \( h = z \) in (2.2.15). Hence, putting (2.3.10)–(2.3.11) into (2.3.9), and using Young inequality, then choosing \( \eta \) suitably small to arrive at

\[
\frac{d}{dt} \int_R \left( \frac{\lambda w^2}{2} - \mu wz \right) \, dx + b\lambda \int_R w^2 \, dx + \frac{1}{4} \int_R (\lambda w - \mu z)^2 \, dx \leq \frac{1}{2} \int_R (\bar{\rho}_t - b\bar{\rho}_\alpha)^2 \, dx + C\delta \int_R z^2\bar{\delta}^2 \, dx + C \int_R z^2 \, dx.
\] (2.3.12)

Now we only need to estimate the last term on the right-hand side of (2.3.12).

Multiplying (2.1.7)1 by \( z \) and integrating it with respect to \( x \), we obtain

\[
\frac{d}{dt} \int_R \frac{z^2}{2} \, dx + a \int_R z_x^2 \, dx = \kappa \int_R (z + \bar{u})w_xz_\alpha \, dx + \kappa \int_R z\bar{\rho}_xz_\alpha \, dx.
\] (2.3.13)

Similar to the treatment of (2.3.10) and (2.3.11), one has

\[
\frac{d}{dt} \int_R \frac{z^2}{2} \, dx + \frac{a}{2} \int_R z_x^2 \, dx \leq C(\epsilon_0 + \delta) \int_R w_x^2 \, dx + C\delta \int_R z^2\bar{\delta}^2 \, dx.
\] (2.3.14)

Multiplying (2.3.14) by a big positive constant \( K \) and summing it to (2.3.12), then using Lemma 2.2, we have

\[
\begin{align*}
\frac{d}{dt} \int_R \left( \frac{\lambda w^2}{2} + \frac{Kz^2}{2} - \mu wz \right) \, dx + b\lambda \int_R w^2 \, dx + \frac{K}{4} \int_R z_x^2 \, dx + \frac{1}{2} \int_R (\lambda w - \mu z)^2 \, dx \\
\leq C\delta(1 + t)^{-\frac{1}{2}} + C\delta \int_R z^2\bar{\delta}^2 \, dx.
\end{align*}
\] (2.3.15)
Integrating the above inequality with respect to $t$, and taking $h = z$ in (2.2.13), we reach (2.3.4). The proof of Lemma 2.4 is completed.

**Theorem 2.5.** If $\epsilon_0$ and $\delta$ are small enough, it holds that

$$
(1+\epsilon)(\|w_x(t)\|^2 + \|z_{xx}(t)\|^2) + \int_0^t (1+\tau) \left( \|w_{xx}(\tau)\|^2 + \|z_{xx}(\tau)\|^2 + \|(\lambda w_x - \mu z_{xx})(\tau)\|^2 \right) \, d\tau \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right),
$$

for $0 \leq t \leq T$.

**Proof.** Differentiating (2.1.7) in $x$ to obtain

$$
\begin{cases}
  z_{xx} - a z_{xx} + \kappa [(z + \bar{u})w_x + z\bar{\rho}_x]_{xx} = 0, \\
  w_{xx} - bw_{xx} + \lambda w_x - \mu z_x + \bar{\rho}_x - b\bar{\rho}_{xx} = 0.
\end{cases}
$$

(2.3.17)

Firstly, multiplying (2.3.17)$_2$ by $\lambda w_x$ and integrating it with respect to $x$ over $\mathbb{R}$, we have

$$
\frac{d}{dt} \int_{\mathbb{R}} \frac{\lambda w_x^2}{2} \, dx + b\lambda \int_{\mathbb{R}} w_{xx}^2 \, dx + \lambda^2 \int_{\mathbb{R}} w_x^2 \, dx - \lambda \mu \int_{\mathbb{R}} w_xz_x \, dx + \int_{\mathbb{R}} \lambda w_x (\bar{\rho}_x - b\bar{\rho}_{xx}) \, dx = 0.
$$

(2.3.18)

Multiplying (2.3.17)$_2$ by $(-\mu z_x)$ and integrating it with respect to $x$ over $\mathbb{R}$, we have

$$
-\mu \int_{\mathbb{R}} w_{xx}z_x \, dx - b\mu \int_{\mathbb{R}} w_{xx}z_{xx} \, dx - \lambda \mu \int_{\mathbb{R}} w_xz_x \, dx + \mu^2 \int_{\mathbb{R}} z_{xx}^2 \, dx - \int_{\mathbb{R}} \mu z_x (\bar{\rho}_x - b\bar{\rho}_{xx}) \, dx = 0.
$$

(2.3.19)

Then summing (2.3.18) and (2.3.19) to obtain the good term $\int_{\mathbb{R}} (\lambda w_x - \mu z_{xx})^2 \, dx$, as follows:

$$
\frac{d}{dt} \int_{\mathbb{R}} \frac{\lambda w_x^2}{2} \, dx + b\lambda \int_{\mathbb{R}} w_{xx}^2 \, dx + \int_{\mathbb{R}} (\lambda w_x - \mu z_{xx})^2 \, dx + \int_{\mathbb{R}} (\lambda w_x - \mu z_{xx}) (\bar{\rho}_x - b\bar{\rho}_{xx}) \, dx = \mu \int_{\mathbb{R}} w_xz_x \, dx + b\mu \int_{\mathbb{R}} w_{xx}z_{xx} \, dx.
$$

(2.3.20)

By applying integration by parts and using (2.3.17)$_1$, one can obtain

$$
\mu \int_{\mathbb{R}} w_{xx}z_x \, dx = \frac{d}{dt} \int_{\mathbb{R}} \mu w_xz_x \, dx - \mu \int_{\mathbb{R}} w_xz_{xx} \, dx
= \frac{d}{dt} \int_{\mathbb{R}} \mu w_xz_x \, dx - \int_{\mathbb{R}} w_x \left( az_{xx} - \kappa [(z + \bar{u})w_x + z\bar{\rho}_x]_{xx} \right) \, dx
= \frac{d}{dt} \int_{\mathbb{R}} \mu w_xz_x \, dx + a\mu \int_{\mathbb{R}} w_{xx}z_{xx} \, dx - \kappa \mu \int_{\mathbb{R}} w_{xx} \left( (z + \bar{u})w_x + z\bar{\rho}_x \right) \, dx
= \frac{d}{dt} \int_{\mathbb{R}} \mu w_xz_x \, dx + a\mu \int_{\mathbb{R}} w_{xx}z_{xx} \, dx - \kappa \mu \int_{\mathbb{R}} (z + \bar{u})w_x^2 \, dx
- \kappa \mu \int_{\mathbb{R}} (z + \bar{u})w_{xx} \, dx - \kappa \mu \int_{\mathbb{R}} z\bar{\rho}_x \, dx.
$$

(2.3.21)
Putting (2.3.21) into (2.3.20), we get

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\lambda w_x^2}{2} - \mu w_{xx} \right) dx + b \lambda \int_{\mathbb{R}} w_{xx}^2 dx + \int_{\mathbb{R}} (\lambda w_x - \mu z_x)^2 dx \\
+ \int_{\mathbb{R}} (\lambda w_x - \mu z_x) (\beta_{xt} - b \beta_{xx}) dx = (a + b) \mu \int_{\mathbb{R}} w_{xx} z_{xx} dx - \kappa \mu \int_{\mathbb{R}} (z + \bar{u}) w_{xx}^2 dx - \kappa \mu \int_{\mathbb{R}} (z_x + \bar{u}_x) w_x w_{xx} dx \\
- \kappa \mu \int_{\mathbb{R}} z_x \beta_x w_{xx} dx - \kappa \mu \int_{\mathbb{R}} z \beta_{xx} w_{xx} dx \\
= (a + b) \mu \int_{\mathbb{R}} w_{xx} z_{xx} dx + \sum_{i=6}^{9} I_i.
\]

(2.3.22)

Similar to the treatment of (2.3.10), we have

\[
I_6 \leq C \int_{\mathbb{R}} (\|z\|_{L^\infty} + \|\bar{u}\|_{L^\infty}) w_{xx}^2 dx \leq C (\epsilon_0 + \delta) \int_{\mathbb{R}} w_{xx}^2 dx.
\]

(2.3.23)

Recall that \(|\bar{u}_x| \leq C \delta (1 + t)^{-1}\), from (2.1.1)2, (2.3.3), and Young inequality, it is easy to derive that

\[
I_7 + I_8 = -\kappa \mu \int_{\mathbb{R}} (z_x + \bar{u}_x) w_x w_{xx} dx - \kappa \mu \int_{\mathbb{R}} z_x \beta_x w_{xx} dx \\
\leq \eta \int_{\mathbb{R}} w_x^2 dx + C_\eta \int_{\mathbb{R}} (\|z_x\|_{L^\infty}^2 + \|\bar{u}_x\|_{L^\infty}^2) w_{xx}^2 dx + C_\eta \int_{\mathbb{R}} \|\beta_x\|_{L^\infty}^2 z_{xx}^2 dx \\
\leq \eta \int_{\mathbb{R}} w_x^2 dx + C_\eta \left[ \epsilon_0^2 (1 + t)^{-2} + \delta^2 (1 + t)^{-1} \right] \int_{\mathbb{R}} w_{xx}^2 dx + C_\eta \delta^2 (1 + t)^{-1} \int_{\mathbb{R}} z_{xx}^2 dx \\
\leq \eta \int_{\mathbb{R}} w_x^2 dx + C_\eta (\epsilon_0^2 + \delta^2) (1 + t)^{-1} \int_{\mathbb{R}} (w_{xx}^2 + z_{xx}^2) dx.
\]

(2.3.24)

Similar to the calculation of (2.3.11), we obtain

\[
I_9 \leq \eta \int_{\mathbb{R}} w_{xx}^2 dx + C_\eta \delta^2 (1 + t)^{-1} \int_{\mathbb{R}} z_{xx}^2 dx.
\]

(2.3.25)

Putting (2.3.23)–(2.3.25) into (2.3.22), then choosing \(\eta\) suitably small, we can conclude

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\lambda w_x^2}{2} - \mu w_{xx} \right) dx + \frac{b \lambda}{4} \int_{\mathbb{R}} w_{xx}^2 dx + \frac{1}{2} \int_{\mathbb{R}} (\lambda w_x - \mu z_x)^2 dx \\
\leq \frac{1}{2} \int_{\mathbb{R}} (\beta_{xt} - b \beta_{xx})^2 dx + C(\epsilon_0 + \delta) (1 + t)^{-1} \int_{\mathbb{R}} (w_{xx}^2 + z_{xx}^2) dx \\
+ C \delta (1 + t)^{-1} \int_{\mathbb{R}} z_{xx}^2 dx + C \int_{\mathbb{R}} z_{xx}^2 dx.
\]

(2.3.26)

Now we only need to estimate the last term on the right-hand side of (2.3.26).

Multiplying (2.3.17)1 by \(z_{xx}\), integrating it with respect to \(x\), we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}} \frac{z_{xx}^2}{2} dx + a \int_{\mathbb{R}} z_{xx}^2 dx = \kappa \int_{\mathbb{R}} (z + \bar{u}) w_{xx} z_{xx} dx + \kappa \int_{\mathbb{R}} (z_x + \bar{u}_x) w_x z_{xx} dx + \kappa \int_{\mathbb{R}} z_x \beta_x z_{xx} dx + \kappa \int_{\mathbb{R}} z \beta_{xx} z_{xx} dx.
\]

(2.3.27)
Similar to the treatment of (2.3.23), (2.3.24), and (2.3.25), one has

\[
\frac{d}{dt} \int_{\mathbb{R}} z^2_0 \, dx + \frac{a}{2} \int_{\mathbb{R}} z^2_2 \, dx \leq C(\varepsilon_0 + \delta) \int_{\mathbb{R}} w^2_0 \, dx + C(\varepsilon_0 + \delta)(1 + t)^{-1} \int_{\mathbb{R}} (w^2_2 + z^2_0) \, dx + C(1 + t)^{-1} \int_{\mathbb{R}} z^2 \alpha \, dx.
\] (2.3.28)

Multiplying (2.3.28) by a big positive constant \(K\) and summing it to (2.3.26), then using Lemma 2.2, we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\lambda w^2_2}{2} + \frac{Kz^2_0}{2} - \mu w_z z_a \right) \, dx + \frac{b \lambda}{8} \int_{\mathbb{R}} w^2_0 \, dx + \frac{K a}{4} \int_{\mathbb{R}} z^2_2 \, dx + \frac{1}{2} \int_{\mathbb{R}} (\lambda w_x - \mu z_a)^2 \, dx \\
\leq C(1 + t)^{-\frac{1}{2}} + C(\varepsilon_0 + \delta)(1 + t)^{-1} \int_{\mathbb{R}} (w^2_2 + z^2_0) \, dx + C(1 + t)^{-1} \int_{\mathbb{R}} z^2 \alpha \, dx.
\] (2.3.29)

Integrating (2.3.29) over \((0, t)\) and taking \(h = z\) in (2.2.13), together with (2.3.4), we get

\[
\|w_x(t)\|^2 + \|z_a(t)\|^2 + \int_0^t \left( \|w_{xx}(r)\|^2 + \|z_{ax}(r)\|^2 + \|(\lambda w_x - \mu z_a)(r)\|^2 \right) \, dr \leq C \left( \|w_0\|^2_2 + \|z_0\|^2_2 + \delta \right).
\] (2.3.30)

Multiplying (2.3.29) by \((1 + t)\), and integrating it with respect to \(t\), then by applying (2.2.13) and (2.3.4), one can immediately obtain (2.3.16). The proof of Lemma 2.5 is completed. \(\square\)

**Theorem 2.6.** If \(\varepsilon_0\) and \(\delta\) are small enough, it holds that

\[
(1 + t)^2 (\|w_{xx}(t)\|^2 + \|z_{ax}(t)\|^2) + \int_0^t (1 + r)^2 (\|w_{xxx}(r)\|^2 + \|z_{axx}(r)\|^2 + \|(\lambda w_x - \mu z_a)(r)\|^2) \, dr \\
\leq C \left( \|w_0\|^2_2 + \|z_0\|^2_2 + \delta \right),
\] (2.3.31)

for \(0 \leq t \leq T\).

**Proof.** Similar to Lemmas 2.4 and 2.5, from \(\int_{\mathbb{R}} \lambda w_{xx} \times \partial_z^2 (2.1.7)_2 \, dx - \int_{\mathbb{R}} \mu z_{xx} \times \partial_z^2 (2.1.7)_2 \, dx\), then applying integration by parts and the equation \(\partial_t^2 (2.1.7)_1\), we can get

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\lambda w^2_2}{2} - \mu w_z z_a \right) \, dx + b \lambda \int_{\mathbb{R}} w^2_0 \, dx + \int_{\mathbb{R}} (\lambda w_x - \mu z_a)^2 \, dx \\
+ \int_{\mathbb{R}} (\lambda w_x - \mu z_a)(\tilde{p}_{xx} - b \tilde{p}_{axx}) \, dx \\
= (a + b) \mu \int_{\mathbb{R}} w_{xxx} z_{ax} \, dx - \kappa \mu \int_{\mathbb{R}} w_{xx}(z + \tilde{u}) w_x + z \tilde{p}_{ax} \, dx \\
= (a + b) \mu \int_{\mathbb{R}} w_{xxx} z_{ax} \, dx - \kappa \mu \int_{\mathbb{R}} w_{xxx} z_{ax} w_x + z \tilde{p}_{ax} \, dx - 2 \kappa \mu \int_{\mathbb{R}} w_{xxx} z_{ax} w_x dx \\
- \kappa \mu \int_{\mathbb{R}} w_{xxx} \tilde{u}_x w_x \, dx \\
= (a + b) \mu \int_{\mathbb{R}} w_{xxx} z_{ax} \, dx + \sum_{i=10}^{17} I_i.
\] (2.3.32)

By applying (2.3.2)-(2.3.3) and Young inequality, we get

\[
I_{10} + I_{11} \leq \eta \int_{\mathbb{R}} w^2_{xxx} \, dx + C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}} z^2_{xxx} \, dx + C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}} w^2_{xx} \, dx.
\] (2.3.33)
Noting that $|\bar{u}_x| \leq C\delta(1+t)^{-\frac{1}{2}}$ and $|\bar{u}_{xx}| \leq C\delta(1+t)^{-1}$, using (2.3.1) and Young inequality, one yields that

$$I_{12} + I_{13} + I_{14} + I_{15} \leq \eta \int_R w_{xx}^2 \, dx + C_\eta \int_R (|\bar{u}_{xx}|^2 + |\bar{u}_x|^2 + |\bar{u}_{xx}|^2 + |\bar{u}_{xxx}|^2 + |\bar{u}_{xxxx}|^2 + |\bar{u}_{xxxxx}|^2) \, dx$$
$$\leq \eta \int_R w_{xx}^2 \, dx + C_\eta \delta^2(1 + t)^{-2} \int_R w_{xx}^2 \, dx + C_\eta \delta^2(1 + t)^{-1} \int_R w_{xx}^2 \, dx$$
$$+ C_\eta \delta^2(1 + t)^{-1} \int_R z_{xx}^2 \, dx + C_\eta \delta^2(1 + t)^{-2} \int_R z_{xxx}^2 \, dx.$$  \hspace{1cm} (2.3.34)

Similar to the calculation of (2.3.10) and (2.3.11), we obtain

$$I_{16} \leq C(\epsilon_0 + \delta) \int_R w_{xx}^2 \, dx.$$  \hspace{1cm} (2.3.35)

and

$$I_{17} \leq \eta \int_R w_{xx}^2 \, dx + C_\eta \delta^2(1 + t)^{-2} \int_R z_{xx}^2 \omega^2 \, dx.$$  \hspace{1cm} (2.3.36)

Putting (2.3.33)–(2.3.36) into (2.3.32), by applying Young inequality, and choosing $\eta$ suitably small, we derive that

$$\frac{d}{dt} \int_R \left( \lambda w_{xx}^2 \right) \, dx + \frac{b \lambda}{2} \int_R w_{xx}^2 \, dx + \frac{1}{2} \int_R \left( \lambda w_{xx} \mu w_{xx} \right) \, dx$$
$$\leq \frac{1}{2} \int_R \left( \rho_{xx} - b \rho_{xxx} \right) \, dx + C(\epsilon_0 + \delta)(1 + t)^{-1} \int_R (w_{xx}^2 + z_{xx}^2) \, dx$$
$$+ C\delta(1 + t)^{-2} \int_R (w_{xx}^2 + z_{xx}^2) \, dx + C\delta(1 + t)^{-2} \int_R z_{xx}^2 \omega^2 \, dx.$$  \hspace{1cm} (2.3.37)

Next, multiplying $\delta_x^2(2.1.7)_1$ by $Kz_{xx}$ ($K$ is sufficiently large) and integrating it with respect to $x$ over $\mathbb{R}$, then similar to the treatment of (2.3.33)–(2.3.36), we can reach

$$\frac{d}{dt} \int_R \frac{Kz_{xx}^2}{2} \, dx + \frac{Ka}{2} \int_R z_{xx}^2 \omega^2 \, dx \leq C(\epsilon_0 + \delta)(1 + t)^{-1} \int_R (w_{xx}^2 + z_{xx}^2) \, dx$$
$$+ C\delta(1 + t)^{-2} \int_R (w_{xx}^2 + z_{xx}^2) \, dx + C\delta(1 + t)^{-2} \int_R z_{xx}^2 \omega^2 \, dx.$$  \hspace{1cm} (2.3.38)

Summing (2.3.38) with (2.3.37), and using Lemma 2.2, we can obtain

$$\frac{d}{dt} \int_R \left( \lambda w_{xx}^2 \right) \, dx + \frac{Ka}{4} \int_R z_{xx}^2 \omega^2 \, dx$$
$$\leq C\delta(1 + t)^{-2} \int_R (w_{xx}^2 + z_{xx}^2) \, dx + C(\epsilon_0 + \delta)(1 + t)^{-1} \int_R (w_{xx}^2 + z_{xx}^2) \, dx$$
$$+ C\delta(1 + t)^{-2} \int_R z_{xx}^2 \omega^2 \, dx.$$  \hspace{1cm} (2.3.39)

Integrating (2.3.39) over $(0, t)$, then taking $h = z$ in (2.2.13), together with using (2.3.4) and (2.3.16), we reach

$$\|w_{xx}(t)\|^2 + \|z_{xx}(t)\|^2 + \int_0^t \left( \|w_{xxx}(r)\|^2 + \|z_{xxx}(r)\|^2 + \|\lambda w_{xx} - \mu z_{xx}\|^2 \right) \, dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right).$$  \hspace{1cm} (2.3.40)

Multiplying (2.3.39) by $(1 + t)^2$, integrating with respect to $t$, then using (2.2.13), (2.3.4), and (2.3.16), we can immediately obtain (2.3.31). The proof of Lemma 2.6 is completed.
Theorem 2.7. If $\varepsilon_0$ and $\delta$ are small enough, it holds that

$$
(1 + t)^2 \left( \|w(t)\|^2 + \|z(t)\|^2 \right) + \int_0^t (1 + r)^2 \left( \|w_k(r)\|^2 + \|z(t)\|^2 + \|\lambda w - \mu z\|^2 \right) dr \\
\leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right),
$$

(2.3.41)

for $0 \leq t \leq T$.

Proof. Firstly, having $\int_\mathbb{R} \lambda w \times \partial_t (2.1.7)_2 dx - \int_\mathbb{R} \mu z \times \partial_t (2.1.7)_2 dx$, then applying integration by parts and the equation $\partial_t (2.1.7)_1$, we can get

$$
\frac{d}{dt} \int_\mathbb{R} \left( \frac{\lambda w^2}{2} - \mu w z_0 \right) dx + b \int_\mathbb{R} w^2 \gamma dx + \int_\mathbb{R} (\lambda w - \mu z) dx \\
+ \int_\mathbb{R} (\lambda w - \mu z_0)(\bar{\rho} - b \bar{w}) dx \\
= (a + b) \mu \int_\mathbb{R} w_0 z_0 dx - \kappa \mu \int_\mathbb{R} w_k (z + \bar{u}) w_x + z \bar{p}_x dx \\
= (a + b) \mu \int_\mathbb{R} w_0 z_0 dx - \kappa \mu \int_\mathbb{R} w_0 w_x z_0 dx - \kappa \mu \int_\mathbb{R} w_k \bar{u} w_x dx - \kappa \mu \int_\mathbb{R} w_k \bar{p}_x dx \\
- \kappa \mu \int_\mathbb{R} w_k \bar{p}_x dx - \kappa \mu \int_\mathbb{R} (z + \bar{u}) w^2 dx \\
= (a + b) \mu \int_\mathbb{R} w_0 z_0 dx + \sum_{i=18}^{22} I_i.
$$

(2.3.42)

Noting that $|\bar{u}_k| \leq C \delta (1 + t)^{-1}$, using (2.1.1)_2 and (2.3.2), we have

$$
I_{18} + I_{19} + I_{20} \\
\leq \eta \int_\mathbb{R} w^2 dx + C \eta \int_\mathbb{R} \|w_x\|^2 \gamma dx + C \eta \int_\mathbb{R} \|w_t\|^2 \gamma dx + C \eta \int_\mathbb{R} \|z\|^2 \gamma dx \\
\leq \eta \int_\mathbb{R} w^2 dx + C \eta (1 + t)^{-1} \int_\mathbb{R} z_t dx + C \eta \delta^2 (1 + t)^{-2} \int_\mathbb{R} w^2 dx + C \eta \delta^2 (1 + t)^{-1} \int_\mathbb{R} z^2 dx \\
\leq \eta \int_\mathbb{R} w^2 dx + C \eta (\varepsilon_0^2 + \delta^2) (1 + t)^{-1} \int_\mathbb{R} z^2 dx + C \eta \delta^2 (1 + t)^{-2} \int_\mathbb{R} w^2 dx.
$$

(2.3.43)

Similar to the calculation of (2.3.10) and (2.3.11), we obtain

$$
I_{21} \leq \eta \int_\mathbb{R} w^2 dx + C \eta \delta^2 (1 + t)^{-2} \int_\mathbb{R} z^2 \delta^2 dx,
$$

(2.3.44)

and

$$
I_{22} \leq C (\varepsilon_0 + \delta) \int_\mathbb{R} w^2 dx.
$$

(2.3.45)

Putting (2.3.43)–(2.3.45) into (2.3.42), and choosing $\eta$ suitably small, we can conclude that

$$
\frac{d}{dt} \int_\mathbb{R} \left( \frac{\lambda w^2}{2} - \mu w z_0 \right) dx + b \int_\mathbb{R} w^2 \gamma dx + \frac{b \lambda}{4} \int_\mathbb{R} w^2 dx + \frac{1}{2} \int_\mathbb{R} (\lambda w - \mu z_0)^2 dx \\
\leq \frac{1}{2} \int_\mathbb{R} (\bar{w}_t - b \bar{w})^2 dx + C (\varepsilon_0 + \delta) (1 + t)^{-1} \int_\mathbb{R} z^2 dx + C \delta (1 + t)^{-2} \int_\mathbb{R} w^2 dx \\
+ C \delta (1 + t)^{-1} \int_\mathbb{R} z^2 \delta^2 dx + C \int_\mathbb{R} z^2 dx.
$$

(2.3.46)
Similarly, we can get from $\int_R Kz_t \times \partial_t (2.1.7)_1 dx$ ($K$ is sufficiently large) that

$$\frac{d}{dt} \int_R \frac{Kz_t^2}{2} dx + \frac{Ka}{2} \int_R z_{xxt}^2 dx \leq C(\epsilon_0 + \delta) \int_R w_{x}^2 dx + C(\epsilon_0 + \delta)(1 + t)^{-1} \int_R z_t^2 dx + C\delta(1 + t)^{-2} \int_R w_t^2 dx + C\delta(1 + t)^{-2} \int_R z_t^2 \omega_t^2 dx.$$  

(2.3.47)

Summing (2.3.47)–(2.3.46), and using Lemma 2.2, we can get

$$\frac{d}{dt} \int_R \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) dx + \frac{b\lambda}{8} \int_R w_{xt}^2 dx + \frac{Ka}{4} \int_R z_{xxt}^2 dx + \frac{1}{2} \int_R (\lambda w_t - \mu z_t)^2 dx \leq C\delta(1 + t)^{-2} + C\delta(1 + t)^{-2} \int_R w_t^2 dx + C\delta(1 + t)^{-2} \int_R z_t^2 \omega_t^2 dx$$

$$+ C(\epsilon_0 + \delta)(1 + t)^{-1} \int_R z_t^2 dx.$$  

(2.3.48)

Now we only need to estimate the last term on the right-hand side of (2.3.48). By using the equation (2.1.7)_1 and Lemma 2.2, together with (2.3.3), it is direct to derive that

$$\int_R z_t^2 dx \leq C \int_R (w_{x}^2 + z_{x}^2) dx + C(1 + t)^{-1} \int_R (w_t^2 + z_t^2) dx + C\delta(1 + t)^{-1} \int_R z_t^2 \omega_t^2 dx.$$  

(2.3.49)

Noting that $(\lambda w_t)^2 \leq 2(\lambda w_t - \mu z_t)^2 + 2(\mu z_t)^2$, it follows from (2.3.48) and (2.3.49) that

$$\frac{d}{dt} \int_R \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) dx + \frac{b\lambda}{8} \int_R w_{xt}^2 dx + \frac{Ka}{4} \int_R z_{xxt}^2 dx + \frac{1}{2} \int_R z_t^2 dx \leq C\delta(1 + t)^{-2} + C \int_R (w_{x}^2 + z_{x}^2) dx + C(1 + t)^{-1} \int_R (w_t^2 + z_t^2) dx + C\delta(1 + t)^{-1} \int_R z_t^2 \omega_t^2 dx.$$  

(2.3.50)

Integrating (2.3.50) with respect to $t$, then employing (2.2.13), (2.3.4), and (2.3.16), we obtain

$$\|w_t(t)\|^2 + \|z_t(t)\|^2 + \int_0^t \left( \|w_{xt}(r)\|^2 + \|z_{xt}(r)\|^2 + \|w_t(r)\|^2 + \|z_t(r)\|^2 \right) dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right).$$  

(2.3.51)

Multiplying (2.3.50) by $(1 + t)$, we integrate it to obtain

$$(1 + t)\|w_t(t)\|^2 + \|z_t(t)\|^2 + \int_0^t (1 + r) \left( \|w_{xt}(r)\|^2 + \|z_{xt}(r)\|^2 + \|w_t(r)\|^2 + \|z_t(r)\|^2 \right) dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right).$$  

(2.3.52)

At last, multiplying (2.3.48) by $(1 + t)^{\delta}$, then using (2.2.13), (2.3.4), and (2.3.52), we can conclude (2.3.41). The proof of Lemma 2.7 is completed.

Combining Lemmas 2.4-2.7, one can verify that a priori assumption $N(T) \leq \epsilon_0^2 \ll 1$ is closed. In fact, under the a priori assumption, it is easy to infer that (2.1.9) holds, the assumption $N(T) \leq \epsilon_0^2 \ll 1$ always holds provided $\delta$ and $\epsilon_0$ are sufficiently small. The global existence of the solutions to the Cauchy problem (2.1.7)-(2.1.8) follows from the standard continuation argument based on the local existence and the a priori estimates. The proof of Theorem 2.1 is completed.
3 | INITIAL-Boundary VALUE PROBLEM

3.1 | The case of Dirichlet boundary condition

3.1.1 | Reformulation of the problem and theorem

In this subsection, we consider the problems (1.2) and (1.4) with the Dirichlet boundary condition (1.5). Motivated by Nishihara and Yang, in the case of \( u_+ \neq u_- \), without loss of generality, we assume \( u_- < u_+ \), letting \( \rho_t = 0 \) and \( -b \rho_x x = 0 \) in (1.2), we have \( u_t - au_{xx} + \kappa [u \rho_x]_x = 0 \) and \( \lambda \rho - \mu u = 0 \). To construct the diffusion waves \( (\tilde{u}, \tilde{\rho})(x, t) \), it is known that we have a self-similar solution \( \tau = \phi(x/\sqrt{1 + t}) \) satisfying

\[
\begin{align*}
\rho_t = \left( a - \frac{\kappa \mu}{\lambda} \right) \rho, & \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
\rho|_{x=\pm\infty} = u_\pm.
\end{align*}
\]

for any constant \( u_- > 0 \). Therefore, for \( u_- < \beta < u_+ \), there exists a unique \( \tilde{u}(x, t) \) in the form of \( \phi(x/\sqrt{1 + t}) \) satisfying

\[
\begin{align*}
\tilde{u}_t - \left( a - \frac{\kappa \mu}{\lambda} \tilde{\rho} \right) \tilde{u}_x = 0, \\
\tilde{u}|_{x=0} = \beta, \quad \tilde{u}|_{x=\infty} = u_+.
\end{align*}

(3.1.1)

Defining the perturbation as (2.1.6), we have the reformulated problem

\[
\begin{align*}
z_t - z_{xx} + \kappa [z + \tilde{u}] w_x + z \tilde{\rho}_x = 0, \\
w_t - bw_{xx} + \lambda w - \mu z + \tilde{\rho}_t - b \tilde{\rho}_{xx} = 0,
\end{align*}

(3.1.2)

with the initial-boundary data

\[
\begin{align*}
(w, z)|_{t=0} = (w_0, z_0)(x) \to 0 & \quad \text{as } x \to \infty, \\
(w, z)|_{x=\infty} = (0, 0).
\end{align*}

(3.1.3)

**Theorem 3.1.** (Dirichlet boundary). Suppose that \( u_- < u_+ \) and \( \delta := |u_+| + |u_-| \). There exists a positive constant \( \varepsilon_0 \) such that if \( \delta + \|w_0\|_2 + \|z_0\|_2 < \varepsilon_0 \), then the initial-boundary value problem (3.1.2)-(3.1.3) admits a unique time-global solution \( (w, z)(x, t) \), which satisfies

\[
w \in C^{1,\infty}([0, \infty); H^{3-i}), \quad i = 0, 1, 2, \quad z \in C^{1,\infty}([0, \infty); H^{3-i}), \quad i = 0, 1, 2,
\]

and

\[
\|w(t)\|_2^2 + \|z(t)\|_2^2 + \|w_t(t)\|_2^2 + \|z_t(t)\|_2^2 + \int_0^t \left( \|w_x(r)\|_1^2 + \|z_x(r)\|_1^2 + \|w_x(r)\|_1^2 + \|z_x(r)\|_1^2 + \|w_{xx}(r)\|_2^2 + \|z_{xx}(r)\|_2^2 \right) \, dr
\]

\[
\leq C \left( \|w_0\|_2^2 + \|z_0\|_2^2 + \delta \right).
\]

(3.1.4)

Moreover, we have

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^+} \|(w, z)(x, t)\|_1 = 0,
\]

(3.1.5)

or

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^+} \|((\rho - \tilde{\rho}, u - \tilde{u})(x, t))\|_1 = 0.
\]

(3.1.6)

that is to say that the solution \( (u, \rho)(x, t) \) to the initial-boundary value problem (1.2), (1.4), and (1.5) tends time-asymptotically to the diffusion waves.
Remark 3.1. As one can see from Theorem 3.1, the solution \((w, z)(x, t)\) has no decay rate. The main reason is that we encounter difficulties in dealing with some boundary terms and finally use the structure of the reformulated equations to close the a priori assumption.

3.1.2 Preliminaries

In this subsection, we will give some fundamental dissipative properties of nonlinear diffusion waves \((\bar{u}, \bar{\rho})(x, t)\) on the half line \(\mathbb{R}^+\), which is similar to Lemma 2.2, and some inequalities concerning the heat kernel on the half line \(\mathbb{R}^+\).

**Theorem 3.2.** For each \(p \in [1, \infty)\) is an integer and \(u_- \leq u_+\), it is easy to verify that

\[
\begin{align*}
  u_- &\leq \bar{u}(x, t) \leq u_+ , \\
  \left\| \partial_t^k \partial_x^l \bar{u}(x, t) \right\|_{L^p} &\leq C |u_+ - u_-|(1 + t)^{-\frac{k}{2} - \frac{l}{p} + \frac{1}{2}}, \quad k, l \geq 0, \ k + l \geq 1 , \\
  \left\| \partial_t^k \partial_x^l \bar{\rho}(x, t) \right\|_{L^p} &\leq C |u_+ - u_-|(1 + t)^{-\frac{k}{2} - \frac{l}{p} + \frac{1}{2}}, \quad k, l \geq 0, \ k + l \geq 1.
\end{align*}
\]

(3.1.7)

For the inequalities concerning the heat kernel on the half line \(\mathbb{R}^+\), we only need to define

\[
\tilde{\omega}(x, t) = (1 + t)^{-\frac{1}{2}} \exp \left\{ - \frac{\alpha x^2}{1 + t} \right\}, \quad g(x, t) = \int_0^x \tilde{\omega}(y, t) dy.
\]

(3.1.8)

It is easy to check that

\[
4ag_t = \tilde{\omega}_x, \quad \|g(\cdot, t)\|_{L^\infty} = \frac{1}{2} \sqrt{\pi} \alpha^{-\frac{1}{2}}.
\]

Similar to the calculation of (2.2.11) and (2.2.13), we can get the inequalities in Lemma 3.3 and Corollary 3.1, and the details are omitted.

**Theorem 3.3.** For \(0 < T \leq +\infty\), assume that \(h(x, t)\) satisfies

\[
h_x \in L^2(0, T ; H^2(\mathbb{R}^+)) , \quad h_t \in L^2(0, T : H^{-1}(\mathbb{R}^+)).
\]

Then the following estimate holds:

\[
\int_0^T \int_{\mathbb{R}^+} h^2 \tilde{\omega}^2 dx dt \leq \pi \|h(0)\|^2 + \pi \alpha^{-1} \int_0^T \|h_x(t)\|^2 dt + 8\alpha \int_0^T \langle h_t, h g^2 \rangle_{H^{-1} \times H^1} dt,
\]

(3.1.9)

where \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(H^{-1}(\mathbb{R}^+) \times H^1(\mathbb{R}^+)\).

**Corollary 1.** In addition to the condition of Lemma 3.3, assume that \(\delta := |u_+| + |u_-| \ll 1, \|h\|_{L^\infty(\mathbb{R}^+)} \leq \varepsilon \ll 1\) and \(h\) satisfies

\[
h_t = a h_{xx} - \kappa [(h + \bar{u}) w_x + h \bar{\rho}_x]_x, \quad h(x, 0) = h_0(x) \in L^2(\mathbb{R}^+), \quad h_x(+\infty, t) = 0,
\]

(3.1.10)

where \(a\) and \(\kappa\) are given positive constant, \(\bar{u}\) is the self-similar solutions of (3.1.1), and \(\bar{\rho} = \frac{\kappa}{a} \bar{u}\). Then there exists some positive constant \(C\) such that

\[
\int_0^T \int_{\mathbb{R}^+} h^2 \tilde{\omega}^2 dx dt \leq C \int_0^T (|h_x(r)|^2 + \|w_x(r)\|^2) dr + C \|h_0\|^2.
\]

(3.1.11)

Finally, we show the following lemma (see Matsumura and Nishihara\(^{35}\)).

**Theorem 3.4.** If \(g(t) \geq 0, g(t) \in L^1(0, \infty)\) and \(g'(t) \in L^1(0, \infty)\), then \(g(t) \to 0\) as \(t \to \infty\).
3.1.3 Proof of Theorem 3.1

In this subsection, we devote ourselves to the proof of Theorem 3.1. It is well-known that the global existence can be obtained by the continuation argument based on the local existence of solutions and a priori estimates. The local existence of (3.1.2) and (3.1.3) can be easily derived by using the standard method, and its proof is omitted for brevity. In the following, our main effort will be to prove the a priori estimates of the solution \((w, z)(x, t)\) under the a priori assumption

\[
N(T) := \sup_{0 \leq t \leq T} \left( ||w||_{L^2}^2 + ||z||_{L^2}^2 \right) \leq \epsilon_0^2,
\]

where \(0 < \epsilon_0 \ll 1\).

Also, from (3.1.2), \(z|_{t=0} = 0\) and \(w|_{t=0} = 0\) give the following boundary conditions:

\[
z(0, t) = z_0(0, t) = w(0, t) = w_t(0, t) = 0,
\]

\[
|w_{xx}(0, t)| = |(\tilde{\rho}_t - b\tilde{\rho}_{xx})(0, t)| = | - b\tilde{\rho}_{xx}(0, t)| \leq \| - b\tilde{\rho}_{xx}\|_{L^\infty} \leq C\delta(1 + t)^{-1}.
\]

With (3.1.13)-(3.1.14) in hand, we now turn to prove Theorem 3.1, which will be given by the following series of lemmas.

**Theorem 3.5.** Under the assumptions of Theorem 3.1, we have

\[
||w(t)||^2 + ||z(t)||^2 + \int_0^t \left( ||w(x)(r)||^2 + ||z(x)(r)||^2 + (|\lambda w - \mu z|(r)|^2) \right) dr \leq C \left( ||w_0||^2 + ||z_0||^2 + \delta \right).
\]

for \(0 \leq t \leq T\).

**Proof.** In a similar method as Lemma 2.4, after \(\int_{\mathbb{R}^+} \lambda w \times (3.1.2)_2 dx - \int_{\mathbb{R}^+} \mu z \times (3.1.2)_2 dx\), applying (3.1.2)_1, the a priori assumption (3.1.12) and Lemma 3.2, we can get

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w^2}{2} - \mu wz \right) dx + \frac{b\lambda}{4} \int_{\mathbb{R}^+} w_x^2 dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w - \mu z)^2 dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^+} (\tilde{\rho}_t - b\tilde{\rho}_{xx})^2 dx + C\delta \int_{\mathbb{R}^+} z^2 \tilde{\omega}^2 dx + C \int_{\mathbb{R}^+} z_x^2 dx.
\]

(3.1.16)

Here, we have used inequality \(\int_{\mathbb{R}^+} z^2 \tilde{\rho}_x^2 dx \leq C\delta^2 \int_{\mathbb{R}^+} z^2 \tilde{\omega}^2 dx\) and (3.1.13).

Next, from \(\int_{\mathbb{R}^+} z \times (3.1.2)_1 dx\), then similar to the treatment of (2.3.10) and (2.3.11), we have from (3.1.13) and the a priori assumption (3.1.12) that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{z^2}{2} dx + \frac{a}{2} \int_{\mathbb{R}^+} z_x^2 dx \leq C(\varepsilon_0 + \delta) \int_{\mathbb{R}^+} w_x^2 dx + C\delta \int_{\mathbb{R}^+} z^2 \tilde{\omega}^2 dx.
\]

(3.1.17)

Multiplying (3.1.17) by a big positive constant \(K\) and summing it to (3.1.16), we derive

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w^2}{2} + \frac{Kz^2}{2} - \mu wz \right) dx + \frac{b\lambda}{8} \int_{\mathbb{R}^+} w_x^2 dx + \frac{Ka}{4} \int_{\mathbb{R}^+} z_x^2 dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w - \mu z)^2 dx \\
\leq C\delta(1 + t)^{-\frac{1}{2}} + C\delta \int_{\mathbb{R}^+} z^2 \tilde{\omega}^2 dx.
\]

(3.1.18)

Integrating the resulting inequality with respect to \(t\), then taking \(h = z\) in (3.1.11) leads to (3.1.15). The proof of Lemma 3.5 is completed. □
Theorem 3.6. Under the assumptions of Theorem 3.1, we have

\[
\|w_x(t)\|^2 + \|z_x(t)\|^2 + \int_0^t \left( \|w_{xx}(r)\|^2 + \|z_{xx}(r)\|^2 + (\lambda w_x - \mu z_x)(r) \right) \, dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right),
\]

(3.1.19)

for \(0 \leq t \leq T\).

Proof. In a similar way as Lemma 2.5, from \(\int_{\mathbb{R}^+} \lambda w_x \times \partial_x (3.1.12)_2 \, dx - \int_{\mathbb{R}^+} \mu w_x \times \partial_x (3.1.12)_2 \, dx\), then applying \(\partial_x (3.1.12)_1\), the a priori assumption (3.1.12) and Lemma 3.2, we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_x^2}{2} - \mu w_x z_x \right) \, dx + \frac{b\lambda}{4} \int_{\mathbb{R}^+} w_{xx}^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w_x - \mu z_x)^2 \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^+} (\tilde{\rho}_x - b \tilde{\rho}_{xx})^2 \, dx + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} (w_x^2 + z_x^2) \, dx + C \delta (1 + t)^{-1} \int_{\mathbb{R}^+} z_x^2 \tilde{\omega}_x^2 \, dx + C \int_{\mathbb{R}^+} z_{xx}^2 \, dx
\]

(3.1.20)

where we have used inequality \(\int_{\mathbb{R}^+} z_x^2 \tilde{\omega}_x^2 \, dx \leq C \delta^2 (1 + t)^{-1} \int_{\mathbb{R}^+} z_x^2 \omega_x^2 \, dx\). Since \(z_x(0, t) = 0\), we only need to estimate the last term on the right-hand side of (3.1.20). Employing the Sobolev inequality and (3.1.14) yields

\[
-bw_{xx}(0, t)(\lambda w_x - \mu z_x)(0, t) \leq C \delta (1 + t)^{-1} \left( \|w_x - \mu z_x\|_{L^\infty} \right) \leq C \delta (1 + t)^{-1} \left( \|w_x - \mu z_x\|_{L^2}^2 \right) \leq C \delta (1 + t)^{-1} + C \delta \|w_x - \mu z_x\|_{L^2}^2 + C \delta \|w_{xx}\|_{L^2}^2 + C \delta \|z_{xx}\|_{L^2}^2.
\]

(3.1.21)

Substituting (3.1.21) into (3.1.20), one obtains that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_x^2}{2} - \mu w_x z_x \right) \, dx + \frac{b\lambda}{8} \int_{\mathbb{R}^+} w_{xx}^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^+} (\lambda w_x - \mu z_x)^2 \, dx \\
\leq C \delta (1 + t)^{-2} + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} (w_x^2 + z_x^2) \, dx + C \delta (1 + t)^{-1} \int_{\mathbb{R}^+} z_x^2 \tilde{\omega}_x^2 \, dx.
\]

(3.1.22)

Next, multiplying \(\partial_x (3.1.12)_1\) by \(Kz_x (K\) is sufficiently large) and integrating it with respect to \(x\) over \(\mathbb{R}^+\), similar to the treatment of (2.3.22), together with the a priori assumption (3.1.12), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{Kz_x^2}{2} \, dx + \frac{K}{2} \int_{\mathbb{R}^+} z_{xx}^2 \, dx \leq C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} w_{xx}^2 \, dx + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} (w_x^2 + z_x^2) \, dx + C \delta (1 + t)^{-1} \int_{\mathbb{R}^+} z_x^2 \tilde{\omega}_x^2 \, dx
\]

(3.1.23)

where the boundary term \(Kz_x(0, t)|-\alpha z_{xx} + \kappa(z + \tilde{u})w_x + z \tilde{\rho}_x|_x(0, t) = -Kz_x(0, t)z_x(0, t) = 0\) due to (3.1.13). Thus, combining (3.1.22) and (3.1.23), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_x^2}{2} + \frac{Kz_x^2}{2} - \mu w_x z_x \right) \, dx + \frac{b\lambda}{16} \int_{\mathbb{R}^+} w_{xx}^2 \, dx + \frac{Ka}{4} \int_{\mathbb{R}^+} z_{xx}^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^+} (\lambda w_x - \mu z_x)^2 \, dx \\
\leq C \delta (1 + t)^{-2} + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} (w_x^2 + z_x^2) \, dx + C \delta (1 + t)^{-1} \int_{\mathbb{R}^+} z_x^2 \tilde{\omega}_x^2 \, dx.
\]

(3.1.24)

Integrating (3.1.24) with respect to \(t\) and taking \(h = z\) in (3.1.11), together with (3.1.15), we get the desired inequality (3.1.19). The proof of Lemma 3.6 is completed. \(\square\)
Theorem 3.7. Under the assumptions of Theorem 3.1, we have

\[
(\|w_t(t)\|^2 + \|z_t(t)\|^2) + \int_0^t (\|w_\alpha(t)\|^2 + \|z_\alpha(t)\|^2 + \|\lambda w_t - \mu z_t\|^2) \, dt \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right),
\]  

(3.1.25)

for \(0 \leq t \leq T\).

Proof. In a similar way as Lemma 2.7, from \(\int_{\mathbb{R}^+} \lambda w_t \times \partial_t (3.1.2) \, dx - \int_{\mathbb{R}^+} \mu z_t \times \partial_t (3.1.2) \, dx\), then applying \(\partial_t (3.1.2)_1\), the \textit{a priori} assumption (3.1.12) and Lemma 3.2, together with (3.1.13), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b}{4} \int_{\mathbb{R}^+} w_t^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^+} (\dot{\rho}_\alpha - b \tilde{\rho}_{\alpha t})^2 \, dx + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} z_t^2 \, dx + C(1 + t)^{-2} \int_{\mathbb{R}^+} w_t \, dx \\
+ C\delta(1 + t)^{-2} \int_{\mathbb{R}^+} \tilde{z}_t^2 \, dx + C \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.1.26)

Next, multiplying \(\partial_t (3.1.2)_1\) by \(Kz_t\) (\(K\) is sufficiently large) and integrating it with respect to \(x\) over \(\mathbb{R}^+\), similar to the treatment of (2.3.42), together with the \textit{a priori} assumption (3.1.12) and (3.1.13), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b}{2} \int_{\mathbb{R}^+} w_t^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^+} (\dot{\rho}_\alpha - b \tilde{\rho}_{\alpha t})^2 \, dx + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} z_t^2 \, dx + C(1 + t)^{-2} \int_{\mathbb{R}^+} w_t \, dx \\
+ C\delta(1 + t)^{-2} \int_{\mathbb{R}^+} \tilde{z}_t^2 \, dx + C(1 + t)^{-2} \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.1.27)

Combining (3.1.26) and (3.1.27), we derive

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b}{8} \int_{\mathbb{R}^+} w_t^2 \, dx + \frac{K}{4} \int_{\mathbb{R}^+} z_t^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 \, dx \\
\leq C\delta(1 + t)^{-2} + C\delta(1 + t)^{-2} \int_{\mathbb{R}^+} w_t \, dx + C\delta(1 + t)^{-2} \int_{\mathbb{R}^+} z_t^2 \, dx + C(\epsilon_0 + \delta) \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.1.28)

Then by applying the equation (3.1.2)_1, Lemma 3.2, and the \textit{a priori} assumption (3.1.12), we get

\[
\int_{\mathbb{R}^+} z_t^2 \, dx \leq C \int_{\mathbb{R}^+} (w_{tx}^2 + z_{tx}^2) \, dx + C \int_{\mathbb{R}^+} (w_t^2 + z_t^2) \, dx + C(1 + t)^{-1} \int_{\mathbb{R}^+} \tilde{z}_t^2 \, dx.
\]  

(3.1.29)

Plugging (3.1.29) into (3.1.28), it is easy to derive that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b}{8} \int_{\mathbb{R}^+} w_t^2 \, dx + \frac{K}{4} \int_{\mathbb{R}^+} z_t^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 \, dx \\
\leq C\delta(1 + t)^{-2} + C \int_{\mathbb{R}^+} (w_{tx}^2 + z_{tx}^2) \, dx + C \int_{\mathbb{R}^+} (w_t^2 + z_t^2) \, dx + C(1 + t)^{-1} \int_{\mathbb{R}^+} \tilde{z}_t^2 \, dx.
\]  

(3.1.30)

Integrating the resulting inequality with respect to \(t\), then taking \(h = z\) in (3.1.11), together with (3.1.15) and (3.1.19), we can immediately obtain (3.1.25). The proof of Lemma 3.7 is completed. \qed

Theorem 3.8. Under the assumptions of Theorem 3.1, we have

\[
\|w_\alpha(t)\|^2 + \|z_\alpha(t)\|^2 \leq C \left( \|w_0\|^2 + \|z_0\|^2 + \delta \right),
\]  

(3.1.31)

for \(0 \leq t \leq T\).
Proof. From the Equation (3.1.2) and Lemma 3.2, it is easy to obtain that
\[
\int_{\mathbb{R}^+} w_{xx}^2 \, dx \leq C \int_{\mathbb{R}^+} (w_t^2 + w^2 + z_t^2) \, dx + C\delta. \tag{3.1.32}
\]
By using (3.1.15) and (3.1.25), we can reach
\[
\| w_{xx} \|^2 \leq C (\| w_0 \|^2 + \| z_0 \|^2 + \delta). \tag{3.1.33}
\]
Similarly, by using the equation (3.1.2), we also have
\[
\int_{\mathbb{R}^+} z_{xx}^2 \, dx \leq C \int_{\mathbb{R}^+} (z_t^2 + z^2 + z_{xx}^2 + w_t^2 + w_{xx}^2) \, dx. \tag{3.1.34}
\]
Applying (3.1.15), (3.1.19), (3.1.25), and (3.1.33), it holds that
\[
\| z_{xx} \|^2 \leq C (\| w_0 \|^2 + \| z_0 \|^2 + \delta). \tag{3.1.35}
\]
Combining (3.1.33) and (3.1.35), we obtain (3.1.31). The proof of Lemma 3.8 is completed.

From Lemmas 3.5–3.8, one can get the desired inequality (3.1.4). Therefore, applying the local existence result and the continuity argument, one can extend the local solution for problem (3.1.2)-(3.1.3) globally.

Finally, we have to show the desired large time behavior in Theorem 3.1. Multiplying \( \partial_t (3.1.2)_1 \) by \( w \), then using Lemma 3.2 and (3.1.14), we have
\[
\left| \frac{d}{dt} \| w(t) \|^2 \right| \leq C (\| w_x(t) \|^2 + \| w_{xx}(t) \|^2 + \| z_x(t) \|^2)^2 + C\delta^2 (1 + t)^{-\frac{1}{2}} + C\delta (1 + t)^{-1} \| w_{xx}(t) \|_{L^\infty}
\leq C\delta (1 + t)^{-2} + C (\| w_x(t) \|^2 + \| w_{xx}(t) \|^2 + \| z_x(t) \|^2)
\tag{3.1.36}
\]
Hence, using the estimate (3.1.4) implies that
\[
\int_0^\infty \left( \| w_x(t) \|^2 + \left| \frac{d}{dt} \| w_x(t) \|^2 \right| \right) \, dt < \infty. \tag{3.1.37}
\]
From Lemma 3.4, we can derive
\[
\lim_{t \to +\infty} \| w_x(\cdot, t) \| = 0. \tag{3.1.38}
\]
Similarly, it is easy to obtain that
\[
\int_0^\infty \left( \| z_x(t) \|^2 + \left| \frac{d}{dt} \| z_x(t) \|^2 \right| \right) \, dt < \infty, \tag{3.1.39}
\]
then it follows that
\[
\lim_{t \to +\infty} \| z_x(\cdot, t) \| = 0. \tag{3.1.40}
\]
Applying the Sobolev inequality, (3.1.38) and (3.1.40) easily lead to the large-time behavior (3.1.5) of the solution. This ends the proof of Theorem 3.1.

### 3.2 The case of Neumann boundary condition

#### 3.2.1 Reformulation of the problem and theorem

We now turn to the problem (1.2) and (1.4) with the Neumann boundary condition (1.6). As in the preceding subsection, we first reformulate the problems (1.2) and (1.4). Assume that the steady state of the one-dimensional Keller–Segel model (1.2) is trivial, taking the form of
\[
u = u_+, \quad \rho = \rho_+, \tag{3.2.1}
\]
provided that
\[
\rho_+ = \frac{\mu}{\lambda} u_+. \tag{3.2.2}
\]
Let \( z = u - u_+ \), \( w = \rho - \rho_+ \). Then \((w, z)\) satisfies

\[
\begin{aligned}
  z_t - az_{xx} + \kappa[(z + u_+)w_+ - 1] = 0 , \\
  w_t - bw_{xx} + \lambda w - \mu z = 0,
\end{aligned}
\]

with the initial data

\[
(w, z)|_{t=0} = (w_0, z_0)(x) \to 0 \text{ as } x \to \infty,
\]

and the boundary condition

\[
(w_x, z_x)|_{x=0} = (0,0).
\]

**Theorem 3.9. (Neumann boundary).** Suppose that both \( \delta_0 := |u_+| \) and \( \|w_0\|_2 + \|z_0\|_2 \) are sufficiently small. Then there exists a unique time-global solution \((w, z)(x, t)\) of the initial-boundary value problem (3.2.3)–(3.2.5), which satisfies

\[
w \in C^{1,\infty}([0, \infty); H^{2-i}) \quad i = 0, 1, 2, \quad z \in C^{1,\infty}([0, \infty); H^{2-i}) \quad i = 0, 1, 2,
\]

and

\[
\sum_{k=0}^2 (1 + t)^k \left( \|\partial_x^k w(t)\|^2 + \|\partial_x^k z(t)\|^2 \right) + (1 + t)^2 \left( \|w_0(t)\|^2 + \|z_0(t)\|^2 \right)
\]

\[
+ \int_0^t \sum_{j=0}^2 \left( (1 + r)^j \left( \|\partial_x^{j+1} w(r)\|^2 + \|\partial_x^{j+1} z(r)\|^2 \right) + \|\partial_x^j (\lambda w - \mu z)(r)\|^2 \right)
\]

\[
+ (1 + r)^2 \left( \|w_\alpha(r)\|^2 + \|z_\alpha(r)\|^2 + \|\lambda w_t - \mu z_t\|^2 \right) \right) \, dr 
\]

\[
\leq C \left( \|w_0\|_2^2 + \|z_0\|_2^2 \right).
\]

### 3.2.2 Proof of Theorem 3.9

In this subsection, we devote ourselves to the proof of Theorem 3.9. As long as a priori estimates are proved, Theorem 3.9 follows in the standard method by combining it with the local-in-time existence and uniqueness as well as the continuity argument. Therefore, in what follows, we only estimate the solution \((w, z)(x, t)\), \(0 < t < T < \infty\), to the initial-boundary value problem (3.2.3)–(3.2.5) under the a priori assumption

\[
N(T) := \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^2 (1 + t)^k \left( \|\partial_x^k w(t)\|^2 + \|\partial_x^k z(t)\|^2 \right) \right\} \leq \epsilon_0^2 ,
\]

for some \( 0 < \epsilon_0 \ll 1 \), and other details are omitted for simplicity.

By the Sobolev inequality, it is easy to deduce that

\[
\|\partial_x^k w(\cdot, t)\|_{L^\infty} \leq \sqrt{2\epsilon_0} (1 + t)^{-\frac{1}{2} - \frac{k}{2}}, \quad k = 0, 1,
\]

\[
\|\partial_x^k z(\cdot, t)\|_{L^\infty} \leq \sqrt{2\epsilon_0} (1 + t)^{-\frac{1}{2} - \frac{k}{2}}, \quad k = 0, 1,
\]

which will be frequently used in the sequel.

It can be checked that

\[
z_\alpha(0, t) = z_\alpha(0, t) = w_x(0, t) = w_x(0, t) = w_{xxx}(0, t) = 0,
\]

where we have used the Equation (3.2.3)\_2.

In fact, Theorem 3.9 will be proved by the following series of lemmas, and the estimates obtained below are formally quite similar to those in Section 2.3.

**Theorem 3.10.** Under the assumptions of Theorem 3.9, we have

\[
\|w(t)\|^2 + \|z(t)\|^2 + \int_0^t \left( \|w_\alpha(r)\|^2 + \|z_\alpha(r)\|^2 + \|\lambda w_t - \mu z_t\|^2 \right) \, dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 \right).
\]

for \( 0 \leq t \leq T \).
Similarly, in order to produce good term by using the structure of reformulated equations, we can get from \( \int_{\mathbb{R}} \lambda w \times (3.2.3)_2 \, dt - \int_{\mathbb{R}} (\lambda w - \mu z) \times (3.2.3)_2 \, dt \) that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{\lambda w^2}{2} \, dx + b\lambda \int_{\mathbb{R}^+} w_+^2 \, dx + \int_{\mathbb{R}^+} (\lambda w - \mu z)^2 \, dx = \mu \int_{\mathbb{R}^+} w_+z \, dx + b\mu \int_{\mathbb{R}^+} w_+ z_\alpha \, dx - bw\alpha(0,t)(\lambda w - \mu z)(0,t). \tag{3.2.11}
\]

By applying integration by parts and using the Equation (3.2.3), one can obtain

\[
\mu \int_{\mathbb{R}^+} w_+z \, dx = \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_+z \, dx - \int_{\mathbb{R}^+} w_+ z \, dx
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_+z \, dx - \int_{\mathbb{R}^+} w \{ az_\alpha - \kappa [(z+u_+)w_+]_x \} \, dx
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_+z \, dx + a\mu \int_{\mathbb{R}^+} w_+ z_\alpha \, dx - \kappa \mu \int_{\mathbb{R}^+} (z+u_+)w_+^2 \, dx + \mu w(0,t)\{ az_\alpha - \kappa [(z+u_+)w_+]_x \}(0,t)
\]

\[
\leq \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_+z \, dx + a\mu \int_{\mathbb{R}^+} w_+ z_\alpha \, dx + \kappa \mu \int_{\mathbb{R}^+} (\|z\|_L^+ + |u_+|)w_+^2 \, dx
\]

\[
\leq \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_+z \, dx + a\mu \int_{\mathbb{R}^+} w_+ z_\alpha \, dx + \kappa \mu (\epsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_+^2 \, dx,
\]

where we have used (3.2.8) and (3.2.9).

Putting (3.2.12) into (3.2.11), then applying (3.2.9) and Young inequality, we can get

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w^2}{2} - \mu wz \right) \, dx + \frac{b\lambda}{4} \int_{\mathbb{R}^+} w_+^2 \, dx + \int_{\mathbb{R}^+} (\lambda w - \mu z)^2 \, dx \leq C \int_{\mathbb{R}^+} z_\alpha^2 \, dx. \tag{3.2.13}
\]

Next, multiplying (3.2.3) by \( Kz \) (\( K \) is sufficiently large) and integrating it with respect to \( x \) over \( \mathbb{R}^+ \), we have from (3.2.8)-(3.2.9) that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{Kz^2}{2} \, dx + \frac{Ka}{2} \int_{\mathbb{R}^+} z_\alpha^2 \, dx \leq C \int_{\mathbb{R}^+} (\|z\|_L^+ + |u_+|)w_+^2 \, dx
\]

\[
\leq C(\epsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_+^2 \, dx. \tag{3.2.14}
\]

Hence, combining (3.2.13) and (3.2.14), we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w^2}{2} + \frac{Kz^2}{2} - \mu wz \right) \, dx + \frac{b\lambda}{8} \int_{\mathbb{R}^+} w_+^2 \, dx + \frac{Ka}{4} \int_{\mathbb{R}^+} z_\alpha^2 \, dx + \int_{\mathbb{R}^+} (\lambda w - \mu z)^2 \, dx \leq 0. \tag{3.2.15}
\]

Integrating the above inequality with respect to \( t \), we conclude (3.2.10). The proof of Lemma 3.10 is completed. \( \square \)

**Theorem 3.11.** Under the assumptions of Theorem 3.9, we have

\[
(1 + t)(\|w_+(t)\|^2 + \|z_\alpha(t)\|^2) + \int_0^t (1 + r) \left( \|w_+(r)\|^2 + \|z_\alpha(r)\|^2 + \|\lambda w_+ - \mu z_\alpha\|^2 \right) \, dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 \right),
\]

for \( 0 \leq t \leq T \). \[3.2.16\]
Proof. Similarly, by \( \int_{\mathbb{R}^+} \lambda w_x \times \partial_x (3.2.3)_2 \, dx - \int_{\mathbb{R}^+} \mu z_x \times \partial_x (3.2.3)_2 \, dx \), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{\lambda w_x^2}{2} \, dx + b \lambda \int_{\mathbb{R}^+} w_{xx}^2 \, dx + \int_{\mathbb{R}^+} (\lambda w_x - \mu z_x)^2 \, dx
= \mu \int_{\mathbb{R}^+} w_{xx} z_x \, dx + b \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} \, dx - bw_{xx}(0,t)(\lambda w_x - \mu z_x)(0,t). \tag{3.2.17}
\]

Now we estimate the first term in the right hand of (3.2.17). By applying integration by parts and using \( \partial_x (3.2.3)_1 \), together with (3.2.9), one can obtain

\[
\mu \int_{\mathbb{R}^+} w_{xx} z_x \, dx = \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_{xx} z_x \, dx - \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} \, dx
= \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_{xx} z_x \, dx - \mu \int_{\mathbb{R}^+} w_x \{ \alpha z_{xxx} - \kappa [(z + u_+) w_x]_{xx} \} \, dx
= \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_{xx} z_x \, dx + a \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} \, dx - \kappa \mu \int_{\mathbb{R}^+} w_{xx} [(z + u_+) w_x]_{x} \, dx
+ \mu w_x(0,t) z_{x}(0,t)
= \frac{d}{dt} \int_{\mathbb{R}^+} \mu w_{xx} z_x \, dx + a \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} \, dx - \kappa \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} w_x \, dx
- \kappa \mu \int_{\mathbb{R}^+} (z + u_+) w_{xx}^2 \, dx. \tag{3.2.18}
\]

Firstly, by using Young inequality and (3.2.8), we have

\[
-\kappa \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} w_x \, dx \leq \eta \int_{\mathbb{R}^+} w_{xx}^2 \, dx + C \eta \int_{\mathbb{R}^+} \| z_x \|_{L^\infty}^2 w_x^2 \, dx
\leq \eta \int_{\mathbb{R}^+} w_{xx}^2 \, dx + C \eta \varepsilon_0 (1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_x^2 \, dx. \tag{3.2.19}
\]

Secondly, it is easy to derive that

\[
-\kappa \mu \int_{\mathbb{R}^+} (z + u_+) w_{xx}^2 \, dx \leq C \int_{\mathbb{R}^+} (\| z \|_{L^\infty} + |u_+|) w_{xx}^2 \, dx \leq C (\varepsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xx}^2 \, dx. \tag{3.2.20}
\]

Substituting (3.2.19) and (3.2.20) into (3.2.18), and summing the resulting inequality to (3.2.17), then by using Young inequality and (3.2.9), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_x^2}{2} - \mu w_{xx} z_x \right) \, dx + \frac{b \lambda}{4} \int_{\mathbb{R}^+} w_{xx}^2 \, dx + \int_{\mathbb{R}^+} (\lambda w_x - \mu z_x)^2 \, dx \leq C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_x^2 \, dx + C \int_{\mathbb{R}^+} z_{xx}^2 \, dx. \tag{3.2.21}
\]

Multiplying \( \partial_x (3.2.3)_1 \) by \( K z_x \) (\( K \) is sufficiently large) and integrating it with respect to \( x \) over \( \mathbb{R}^+ \), then similar to the treatment of (3.2.19) and (3.2.20), we can get from (3.2.9) that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{K z_x^2}{2} \, dx + \frac{K a}{2} \int_{\mathbb{R}^+} z_{xx}^2 \, dx \leq C (\varepsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xx}^2 \, dx + C (1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_x^2 \, dx. \tag{3.2.22}
\]
Therefore, combining (3.2.21) and (3.2.22), we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_{xx}^2}{2} - \mu w_x z_x \right) dx + \frac{b \lambda}{8} \int_{\mathbb{R}^+} w_x^2 dx + \frac{K a}{4} \int_{\mathbb{R}^+} z_{xx}^2 dx + \int_{\mathbb{R}^+} \left( \lambda w_x + \frac{K e_0}{4} \right)^2 dx \leq C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_x^2 dx.
\] (3.2.23)

Integrating (3.2.23) over \((0, t)\), we get

\[
\|w_x(t)\|^2 + \|z_x(t)\|^2 + \int_0^t \left( \|w_{xx}(\tau)\|^2 + \|z_{xx}(\tau)\|^2 + \|\lambda w_x - \mu z_x(\tau)\|^2 \right) d\tau \leq C \left( \|w_0\|^2 + \|z_0\|^2 \right).
\] (3.2.24)

Multiplying (3.2.23) by \((1 + t)\), and integrating it with respect to \(t\), by applying (3.2.10), one can immediately obtain (3.2.16). The proof of Lemma 3.11 is completed. 

**Theorem 3.12.** Under the assumptions of Theorem 3.9, we have

\[
(1 + t)^2 \left( \|w_{xx}(t)\|^2 + \|z_{xx}(t)\|^2 \right) + \int_0^t (1 + \tau)^2 \left( \|w_{xxx}(\tau)\|^2 + \|z_{xxx}(\tau)\|^2 + \|\lambda w_{xx} - \mu z_{xx}(\tau)\|^2 \right) d\tau \leq C \left( \|w_0\|^2 + \|z_0\|^2 \right).
\] (3.2.25)

for \(0 \leq t \leq T\).

**Proof.** Similarly, from \(\int_{\mathbb{R}^+} \lambda w_{xx} \times \partial_x^2 (3.2.3) dx - \int_{\mathbb{R}^+} \mu z_{xx} \times \partial_x^2 (3.2.3) dx\), then by applying \(\partial_x^2 (3.2.3)\) and (3.9), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_{xx}^2}{2} - \mu w_{xx} z_x \right) dx + b \lambda \int_{\mathbb{R}^+} w_{xx}^2 dx + \int_{\mathbb{R}^+} \left( \lambda w_{xx} - \mu z_{xx} \right)^2 dx
\]

\[
= (a + b) \mu \int_{\mathbb{R}^+} w_{xxx} z_{xx} dx - \kappa \mu \int_{\mathbb{R}^+} w_{xxx} (z + u_+) w_x dx - bw_{xxx}(0, t) \lambda w_{xx} - \mu z_{xx}(0, t)
\]

\[- \kappa \mu \int_{\mathbb{R}^+} (z + u_+) w_{xxx}^2 dx.
\] (3.2.26)

Firstly, applying Young inequality and (3.2.8), one gets

\[
- \kappa \mu \int_{\mathbb{R}^+} w_{xxx} z_{xx} dx \leq \eta \int_{\mathbb{R}^+} w_{xxx}^2 dx + C \eta \int_{\mathbb{R}^+} \|w_x\|_{L^\infty}^2 z_{xx}^2 dx
\]

\[
\leq \eta \int_{\mathbb{R}^+} w_{xxx}^2 dx + C \eta \epsilon_0^2 (1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} z_{xx}^2 dx,
\] (3.2.27)

and

\[
-2 \kappa \mu \int_{\mathbb{R}^+} w_{xxx} z_{xx} dx \leq \eta \int_{\mathbb{R}^+} w_{xxx}^2 dx + C \eta \int_{\mathbb{R}^+} \|z_x\|_{L^\infty}^2 w_{xx}^2 dx
\]

\[
\leq \eta \int_{\mathbb{R}^+} w_{xxx}^2 dx + C \eta \epsilon_0^2 (1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_{xx}^2 dx.
\] (3.2.28)

Secondly, similar to treatment of (3.2.20), we have

\[
- \kappa \mu \int_{\mathbb{R}^+} (z + u_+) w_{xxx}^2 dx \leq C \int_{\mathbb{R}^+} (\|z\|_{L^\infty} + |u_+|) w_{xxx}^2 dx \leq C(\epsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xxx}^2 dx.
\] (3.2.29)
Putting (3.2.27)–(3.2.29) into (3.2.26), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_{xx}^2}{2} - \mu w_{xx} z_{xx} \right) dx + \frac{b\lambda}{4} \int_{\mathbb{R}^+} w_{xx}^2 dx + \int_{\mathbb{R}^+} (\lambda w_{xx} - \mu z_{xx})^2 dx \\
\leq C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} (w_{xx}^2 + z_{xx}^2) dx + C \int_{\mathbb{R}^+} z_{xx}^2 dx.
\]

(3.2.30)

Next, multiplying \( \partial_t^2 (3.2.31) \) by \( K z_{xx} \) (\( K \) is sufficiently large) and integrating it with respect to \( x \) over \( \mathbb{R}^+ \), then similar to the treatment of (3.2.27), (3.2.28), and (3.2.29), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{K z_{xx}^2}{2} dx + \frac{K a}{2} \int_{\mathbb{R}^+} z_{xx}^2 dx \leq C(\epsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xx}^2 dx + C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} (w_{xx}^2 + z_{xx}^2) dx - K z_{xx}(0, t) z_{xx}(0, t) \\
\leq C(\epsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xx}^2 dx + C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} (w_{xx}^2 + z_{xx}^2) dx,
\]

(3.2.31)

where in the last inequality, we have used (3.2.9).

Combining (3.2.30) and (3.2.31), it follows that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_{xx}^2}{2} + \frac{K z_{xx}^2}{2} - \mu w_{xx} z_{xx} \right) dx + \frac{b\lambda}{8} \int_{\mathbb{R}^+} w_{xx}^2 dx + \frac{K a}{4} \int_{\mathbb{R}^+} z_{xx}^2 dx + \int_{\mathbb{R}^+} (\lambda w_{xx} - \mu z_{xx})^2 dx \\
\leq C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} (w_{xx}^2 + z_{xx}^2) dx.
\]

(3.2.32)

Integrating (3.2.32) over \((0, t)\) and using (3.2.16), we have

\[
\|w_{xx}(t)\|^2 + \|z_{xx}(t)\|^2 + \int_0^t \left( \|w_{xx}(r)\|^2 + \|z_{xx}(r)\|^2 + \|\lambda w_{xx} - \mu z_{xx}\|^2 \right) dr \leq C \left( \|w_0\|_2^2 + \|z_0\|_2^2 \right).
\]

(3.2.33)

Multiplying (3.2.32) by \((1 + t)^2\), integrating with respect to \( t \), then using (3.2.16), we obtain (3.2.25). The proof of Lemma 3.12 is completed. 

**Theorem 3.13.** Under the assumptions of Theorem 3.9, we have

\[
(1 + t)^2(\|w_t(t)\|^2 + \|z_t(t)\|^2) + \int_0^t (1 + \tau)^2 \left( \|w_{xx}(\tau)\|^2 + \|z_{xx}(\tau)\|^2 + \|\lambda w_t - \mu z_t(\tau)\|^2 \right) d\tau \leq C \left( \|w_0\|_2^2 + \|z_0\|_2^2 \right).
\]

(3.2.34)

for \( 0 \leq t \leq T \).

**Proof.** Similarly, from \( \int_{\mathbb{R}^+} \lambda w_t \times \partial_t (3.2.3) dx - \int_{\mathbb{R}^+} \mu z_t \times \partial_t (3.2.3) dx \), from \( \partial_t (3.2.3) \) and \((3.2.9)\), we can obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} - \mu w_t z_t \right) dx + b\lambda \int_{\mathbb{R}^+} w_t^2 dx + \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 dx \\
= (a + b) \mu \int_{\mathbb{R}^+} w_{xx} z_{xx} dx - \kappa \mu \int_{\mathbb{R}^+} w_{xx} [(z + u_+)w_3] dx - b w_{xx}(0, t)(\lambda w_t - \mu z_t)(0, t) \\
+ \mu w_t(0, t) [(z + u_+)w_3]_t (0, t) \\
= (a + b) \mu \int_{\mathbb{R}^+} w_{xxx} z_{xxx} dx - \kappa \mu \int_{\mathbb{R}^+} w_{xx} w_{xx} z_{xx} dx - \kappa \mu \int_{\mathbb{R}^+} (z + u_+)w_{xx}^2 dx.
\]

(3.2.35)
Firstly, using Young inequality and (3.2.8), it follows that

\[
-\kappa \mu \int_{\mathbb{R}^+} w_{xt} w_{zt} \, dx \leq \eta \int_{\mathbb{R}^+} w_{xt}^2 \, dx + C_\eta \int_{\mathbb{R}^+} \|w_{xt}\|_{L^2}^2 \, dx \\
\leq \eta \int_{\mathbb{R}^+} w_{xt}^2 \, dx + C_\eta \varepsilon_0^2 (1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.2.36)

Next, by using the similar method as (3.2.29), we have

\[
-\kappa \mu \int_{\mathbb{R}^+} (z + u_+)w_{xt}^2 \, dx \leq C \int_{\mathbb{R}^+} (\|z\|_{L^\infty} + |u_+|)w_{xt}^2 \, dx \leq C(\varepsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xt}^2 \, dx.
\]  

(3.2.37)

Taking (3.2.36) and (3.2.37) into (3.2.35), and using Young inequality, we can conclude that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b \lambda}{4} \int_{\mathbb{R}^+} w_{xt}^2 \, dx + \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 \, dx \leq C(\varepsilon_0(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} z_t^2 \, dx + C \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.2.38)

Now, we try to treat the second term in the right-hand side of the above inequality. Multiplying $\partial_t (3.2.3)_1$ by $Kz_t$ ($K$ is sufficiently large) and integrating it with respect to $x$ over $\mathbb{R}^+$, we can derive

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{Kz_t^2}{2} + \frac{Ka}{2} \int_{\mathbb{R}^+} z_{tt}^2 \, dx \leq C(\varepsilon_0(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} z_t^2 \, dx \leq C(\varepsilon_0 + \delta_0) \int_{\mathbb{R}^+} w_{xt}^2 \, dx + C \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.2.39)

Combining (3.2.38) and (3.2.39), one can immediately obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b \lambda}{8} \int_{\mathbb{R}^+} w_{xt}^2 \, dx + \frac{Ka}{4} \int_{\mathbb{R}^+} z_{tt}^2 \, dx + \int_{\mathbb{R}^+} (\lambda w_t - \mu z_t)^2 \, dx \leq C(\varepsilon_0(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} z_t^2 \, dx.
\]  

(3.2.40)

By using the Equations (3.2.3)_1 and (3.2.8), it follows that

\[
\int_{\mathbb{R}^+} z_t^2 \, dx \leq C \int_{\mathbb{R}^+} z_{tt}^2 \, dx + C \int_{\mathbb{R}^+} w_{xt}^2 \, dx + C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_t^2 \, dx.
\]  

(3.2.41)

Noting that $(\lambda w_t)^2 \leq 2(\lambda w_t - \mu z_t)^2 + 2(\mu z_t)^2$, combining (3.2.40) and (3.2.41) and choosing $\varepsilon_0$ small enough yields

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \left( \frac{\lambda w_t^2}{2} + \frac{Kz_t^2}{2} - \mu w_t z_t \right) \, dx + \frac{b \lambda}{8} \int_{\mathbb{R}^+} w_{xt}^2 \, dx + \frac{Ka}{4} \int_{\mathbb{R}^+} z_{tt}^2 \, dx + \frac{\lambda^2}{2} \int_{\mathbb{R}^+} w_t^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^+} z_t^2 \, dx \leq C \int_{\mathbb{R}^+} (w_{xx} + z_{xx})^2 \, dx + C(1 + t)^{-\frac{1}{2}} \int_{\mathbb{R}^+} w_t^2 \, dx.
\]  

(3.2.42)

Integrating (3.2.42) with respect to $t$, then employing (3.2.10) and (3.2.16), we obtain

\[
\|w_t(t)\|^2 + \|z_t(t)\|^2 + \int_0^t \left( \|w_{xt}(r)\|^2 + \|z_{tt}(r)\|^2 + \|w_t(r)\|^2 + \|z_t(r)\|^2 \right) \, dr \leq C \left( \|w_0\|^2 + \|z_0\|^2 \right).
\]  

(3.2.43)
Multiplying (3.2.42) by \((1 + t)\), we integrate it to obtain

\[
(1 + t)(\|w(t)\|^2 + \|z(t)\|^2) + \int_0^t (1 + \tau) \left( \|w_\tau(\tau)\|^2 + \|z_\tau(\tau)\|^2 + \|w_\tau(\tau)\|^2 + \|z_\tau(\tau)\|^2 \right) d\tau 
\leq C \left( \|w_0\|^2 + \|z_0\|^2 \right).
\]

(3.2.44)

At last, multiplying (3.2.40) by \((1 + t)^2\), then using (3.2.44), we derive (3.2.34). The proof of Lemma 3.13 is complete. Recalling Lemmas 3.10–3.13, we complete the proof of Theorem 3.9.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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