Proof of the equivalence of the symplectic forms derived from the canonical and the covariant phase space formalisms

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We prove that, for any theory defined over a space-time with boundary, the symplectic form derived in the covariant phase space is equivalent to the one derived from the canonical formalism.

I. INTRODUCTION

In physics, we can identify, roughly speaking, two somewhat disjoint frameworks to deal with a field theory: the canonical and the covariant. The former breaks all the objects of the theory into spatial and normal components. The main advantages are that we gain a dynamical perspective and some universal structures like the symplectic form of a cotangent bundle. This provides a way to approach the problem numerically \cite{1} (essential in the study of gravitational waves and LIGO observations) and a starting point for the Hamiltonian quantization \cite{2–5}. A price to pay is the apparent loss of some symmetries.

The covariant approach, on the other hand, considers fields over the whole space-time \cite{6}. Some noteworthy advantages are: symmetries are explicit, the study of null infinity is easier, there are methods to compute conserved quantities, and higher-derivative theories are treated on equal footing as 1st-order ones \cite{7–9}. All this has important consequences in effective and perturbation theory, both of which are relevant to the study of string theories, edge modes, corner and BMS algebras, or the analysis of consistent deformations \cite{10–19}. However, one drawback is that there are no known canonical structures on the spaces involved. In particular, to have a symplectic structure, one has to rely on the covariant phase space (CPS) formalism and fix a local action.

The question that arises naturally is if both approaches are equivalent. It is possible to prove that in many aspects they are. However, the equivalence of the symplectic forms was unknown in general. The problem has been studied in concrete relevant examples, like GR without boundaries \cite{20, 21}. Also in \cite{22}, an analytical prove of the symplectic equivalence is provided over the reduced phase space (using Poisson brackets) for 1st-order theories with no boundaries. It is important to mention that adding boundaries complicates the matter a great deal even in concrete examples. In that sense, our paper \cite{23} was a breakthrough because it provided a way to map any theory with boundary to another one without boundary in the CPS formalism. Although this approach would simplify the upcoming computations, we will not use it here as it requires the introduction of a lot of notation and definitions. However, in \cite{23} we also provided a geometric language that bridges between the mathematical formalism (\textit{\textgamma}–jets framework) and the standard physics notation. We will see in this paper that this language turns out to be essential to prove, in full generality, the equivalence of both symplectic structures. As a byproduct, the equivalence shows that our proposal of a CPS symplectic structure in manifolds with boundaries (introduced also in \cite{23}) is the most natural one. Not only for its cohomological nature, as explained in that paper, but also because it is equivalent to the one coming from the canonical formalism.

II. THE GEOMETRIC ARENA

A. The spacetime

Consider a globally hyperbolic \textit{n}-manifold \(M\) (up to diffeomorphism, \(M = \mathbb{R} \times \Sigma\)) with boundary \(\partial M = \mathbb{R} \times \partial \Sigma\). We have the inclusions \(\overline{\partial} \Sigma : \partial \Sigma \hookrightarrow \Sigma, \: \overline{\partial} \Sigma : \partial \Sigma \to M, \) and \(\iota_t : \Sigma \hookrightarrow \{t\} \times \Sigma \subset M.\) As usual, we have the exterior derivative \(\text{d}\), the wedge product \(\wedge\), the interior derivative \(\iota\), and the Lie derivative \(\mathcal{L}\).

In this setting it is crucial to keep track of the orientations. We orient \(\mathbb{R}\) with the standard volume form \(dt\) and \(\Sigma\) with some volume form \(\text{vol}_\Sigma\) (which, up to easy to handle technicalities, can be equally understood as an \((n-1)\)-form on \(\Sigma\) or on \(M\)). We then orient \(M\) with

\[
\text{vol}_M := dt \wedge \text{vol}_\Sigma
\]

(1)

Boundary values are oriented so that Stokes’ theorem holds

\[
\int_M \text{d} \alpha = \int_{\partial M} \iota^* \alpha \quad \int_{\Sigma} \text{d} \beta = -\int_{\partial \Sigma} \mathcal{L} \beta
\]

(2)

For that, take any metric \(g_\Sigma\) on \(\Sigma\) and denote \(\nu^i\) the unit vector field \(\gamma\)-normal to \(\partial \Sigma\). Consider the adapted metric \(g_\nu\) on \(\mathbb{R} \times \Sigma\) such that \(g_\nu^{a\beta} = -\partial^a \partial^b + (\nu^a) (\nu^b) \gamma^j\). Then the unit vector field \(\gamma\)-normal to \(\partial M\) is (up to push-forward) \(\nu^i\), i.e., \(V^a := (\nu^a) \nu^i\). Finally, we define

\[
\text{vol}_{\partial \Sigma} := \iota_\nu \text{vol}_\Sigma \quad \text{vol}_{\partial M} := \iota_\nu \text{vol}_M
\]

(3)

It is important to notice that

\[
\text{vol}_{\partial M} = \iota_\nu (dt \wedge \text{vol}_\Sigma) = -dt \wedge \text{vol}_{\partial \Sigma}
\]

(4)
B. The space of fields

Let \( \mathcal{F} \) be a space of tensor fields (of any tensorial character) over \( M \). This space is \( \infty \)-dimensional and non-linear in general. Although it can rigorously be described with the \( \infty \)-jets formalism, for our purposes it is enough to think of \( \mathcal{F} \) as a standard smooth manifold with the usual operators such as the exterior derivative \( d \), the wedge product \( \wedge \), the interior derivative \( \iota \), or the Lie derivative \( \mathcal{L} \). Here, \( \nabla \) is a vector field of \( \mathcal{F} \) (see [23] for a careful discussion). Of course, \( \mathcal{F} \) can consist of different types of tensor fields, hence \( \mathcal{F} = \mathcal{F}^r \times \cdots \times \mathcal{F}^n \) with the fields labeled as \( (\phi^I)_{I=1,\ldots, N} \in \mathcal{F} \).

III. THE COVARIANT PHASE SPACE FORMALISM IN A NUTSHELL

Roughly speaking, the CPS method studies the space of solutions of a theory over the whole space-time. The equations of motion are not dynamic but, rather, give conditions for the fields to be solutions. We devote this section to summarizing how to define a presymplectic form canonically associated with a local action. For a detailed discussion and some applications see [23–26].

Consider a local action \( \mathcal{S} : \mathcal{F} \rightarrow \mathbb{R} \) given by

\[
\mathcal{S} = \int_M L - \int_{\partial M} \mathcal{T}
\]

where \( (L, T) \in \Omega^{(n, 0)}(M \times \mathcal{F}) \times \Omega^{(n, -1, 0)}(\partial M \times \mathcal{F}) \), known as CPS bulk and boundary Lagrangians, are top-forms on \( (M, \partial M) \) and 0-forms on \( \mathcal{F} \) (they are bigraded forms). We assume that they are locally constructed, i.e., when evaluating \( (L(\phi), T(\phi)) \in \Omega^n(M) \times \Omega^{n-1}(\partial M) \) at \( p \in M \), they only depend on \( \phi(p) \), \( \phi(p) \), and finitely many of its derivatives at \( p \).

It is a standard result [23, 27] that the \((n, 1)\)-form \( d\mathcal{S} \) can be split as

\[
d\mathcal{S} = E_1 \wedge \mathcal{S}_1^1 + d\Theta
\]

for some \((n, 0)\)-forms \( E_1 \) (Euler-Lagrange forms) and some \((n, 1)\)-form \( \Theta \) (bulk symplectic potential, uniquely defined up to an exact form). With these ingredients, we define the space of solutions

\[
\text{Sol}(\mathcal{S}) = \{ \phi \in \mathcal{F} / (E_1(\phi), \mathcal{T}(\phi)) = (0, 0) \} \rightarrow \mathcal{F}
\]

and the symplectic structure associated with an embedding \( \mathcal{S} : \Sigma \rightarrow M \)

\[
\Omega_{\mathcal{S}} = \int_{\Sigma} d\mathcal{H} - \int_{\partial \Sigma} d\mathcal{H}
\]

\( \Omega_{\mathcal{S}} \) is independent of the choice of Lagrangians (as long as they define the same \( \mathcal{S} \)) and of the symplectic potentials chosen in (6) and (8). Moreover, if we denote \( \Omega_{\mathcal{S}} := \int_{\Sigma} \mathcal{H} \) the pullback of the symplectic form to \( \text{Sol}(\mathcal{S}) \), then it can be proved that \( \Omega_{\mathcal{S}} \) does not depend on the embedding either. Thus, we have constructed a presymplectic form on \( \text{Sol}(\mathcal{S}) \) canonically associated with \( \mathcal{S} \).

IV. THE CANONICAL FORMALISM IN A NUTSHELL

The canonical (CAN) formalism deals, roughly speaking, with “instant fields” that evolve according to some dynamical equations. By evolving specific initial data, we obtain a curve in the space of “instantaneous fields” which corresponds (up to integrability issues) to a solution over the whole space-time. We devote this section to summarizing the results necessary for the present work. For now, we will focus on 1st-order theories and delay the generalization to higher order ones to section VI.

Consider the space \( \mathcal{Q} = \mathcal{Q}^1 \times \cdots \times \mathcal{Q}^M \) of fields over \( \Sigma \) and its tangent bundle \( T\mathcal{Q} \) (its standard geometric operators like \( d\mathcal{L} \) or \( \wedge \) are denoted as the ones of \( \mathcal{F} \)). It consists of elements of the form \( (q^1, \ldots, q^M; v^1, \ldots, v^M) \). Consider a local CAN Lagrangian-action \( \mathcal{L} : T\mathcal{Q} \rightarrow \mathbb{R} \) given by

\[
\mathcal{L} = \int_{\Sigma} \mathcal{L} - \int_{\partial \Sigma} \mathcal{L}
\]

where \( (L, l) \in \Omega^{(n, 0)}(\Sigma \times T\mathcal{Q}) \times \Omega^{(n, 0)}(\Sigma \times T\mathcal{Q}) \) are some (locally constructed and possibly time-dependent) CAN Lagrangians. \( \mathcal{L} \) is historically called Lagrangian but it plays a role similar to the action (5) although not entirely equal since the time integration is missing. So we define the CAN action \( \mathcal{S} : C^\infty(\Sigma, \mathcal{Q}) \rightarrow \mathbb{R} \)

\[
\mathcal{S}(q) = \int_{\Sigma} \mathcal{L}(q(t), q(t)) dt
\]

The analogues to equations (6) and (8) are

\[
d\mathcal{S} = A_1^{(0)} \wedge \mathcal{Q}_1^1 + A_1^{(1)} \wedge d\mathcal{Q}_1^1 + d\mathcal{\Theta}
\]

\[
d\mathcal{S} - j^* \mathcal{\Theta} = B_1^{(0)} \wedge \mathcal{Q}_1^1 + B_1^{(1)} \wedge d\mathcal{Q}_1^1 - d\mathcal{\Theta}
\]

Taking the \(d\)-exterior derivative of (11), using Stokes’ theorem, and integrating by parts with respect to time (notice that in \( \mathcal{S} \) every \( v^i \) is replaced by \( \dot{q}^i \)), we obtain

\[
d\mathcal{S} = \int_{\Sigma} \mathcal{L}(A_1^{(0)} \wedge \mathcal{Q}_1^1 - \int_{\partial \Sigma} (B_1^{(0)} - B_1^{(1)} \wedge d\mathcal{Q}_1^1)
$(A^0_I - A^{(1)}_I, B^0_I - B^{(1)}_I)$ are the bulk and boundary dynamical equations (if some of them do not involve time derivatives, we obtain bulk or boundary constraints).

Once we have the Lagrangian formalism, we proceed to introduce the Hamiltonian formalism which, instead of living in the tangent bundle $T\mathcal{Q}$, lives on the cotangent bundle $T^*\mathcal{Q}$. The advantage of the latter is its canonical symplectic form, which plays an essential role on the Hamiltonian formulation. Indeed, denoting the elements of $T^*\mathcal{Q}$ as $(q;p) = (q^1, \ldots, q^M; p_1, \ldots, p_M)$, the canonical symplectic structure is given by

$$\Omega_{T^*\mathcal{Q}} := dq^I \wedge dp_I$$

The usual pairing between position and momenta applies when evaluated over fields. In order to go from the Lagrangian (tangent bundle) to the Hamiltonian (cotangent bundle), we use the fiber derivative $FL: T\mathcal{Q} \rightarrow T^*\mathcal{Q}$. For each $(q;v) \in T_q\mathcal{Q}$ we define $FL_{(q,v)} \in T^*_q\mathcal{Q}$ as

$$FL_{(q,v)}(w):= \frac{d}{dt} \bigg|_{t=0} L(q;v_t)$$

where $v_t$ is a curve in $T_q\mathcal{Q}$ with $v_0 = v$ and $\frac{d}{dt}|_{t=0} v_t = w$. In general, this map is not surjective and the relevant symplectic structure is actually the one induced on its image (the primary constraint submanifold), i.e., induced by the inclusion $\gamma_{FL}: F\mathcal{L}(T\mathcal{Q}) \rightarrow T^*\mathcal{Q}$. Since over $F\mathcal{L}(T\mathcal{Q})$ we have $p = F\mathcal{L}(\mathcal{Q})$, we schematically obtain

$$\Omega_{\mathcal{L}} := J_{FL}^*\Omega_{T^*\mathcal{Q}} := dq^I \wedge dp_I \mathcal{L} = dq^I \wedge dp_I (FL_{(q,v^I)})$$

where by $v^I$ we mean $(0, \ldots, 0, v^I, 0, \ldots, 0)$. In order to give a concrete sense to (15), we rely on the geometric language introduced before. The key observation is that the momenta can be rewritten as

$$p_I(q;w^I) := \mathcal{L}_{(0,w^I)} = \epsilon_{(0,w^I)} \mathcal{L} = \int_\Sigma \epsilon_{(0,w^I)} \mathcal{L} = \int_\Sigma \mathcal{L}_{(0,w^I)}$$

where $\epsilon_{(0,w^I)} \mathcal{L}$ denotes the function that relates the top-form $\mathcal{L}$ with the volume form $\epsilon_\mathcal{L}$. Plugging these expressions of $(\mathcal{L}, \tilde{\mathcal{L}})$ into (5) and using Fubini’s theorem we get

$$S = \int_\Sigma \int_{L^n} (\int_{\partial \Sigma} (-\tilde{\mathcal{L}})) dt$$

A minus sign appears in the boundary because, from (4), we have $(\mathcal{L}, \tilde{\mathcal{L}})$ in (5) and using Fubini’s theorem we get

$$\mathcal{L} = \int_\Sigma \mathcal{L}_{(0,w^I)} \mathcal{L} = \int_\partial \mathcal{L}_{(0,w^I)} \mathcal{L}$$

Taking its exterior derivative and using the $p-q$ pairing finally allows us to rewrite (15) as

$$\Omega_{\mathcal{L}} = \int_\Sigma dq^I \wedge dp_I - \int_{\partial \Sigma} dB^{(1)}_I \wedge dq^I$$

V. PROVING THE EQUIVALENCE FOR FIRST-ORDER THEORIES

In the previous sections we have seen that, on the one hand, we can derive a symplectic structure $\Omega_{\mathcal{L}}$ canonically associated with a CPS action $S$ (defined over the space of fields of $\mathcal{M}$). On the other hand, from a CAN Lagrangian-action $\mathcal{L}$ defined over some $T\mathcal{Q}$, we get the symplectic structure $\Omega_{\mathcal{L}}$ canonically associated with $\mathcal{L}$. It is well known that we can go from the CPS formalism to the CAN one by performing a $(1, n-1)$-decomposition (there are many equivalent ways of doing it). Therefore, for a given theory we end up with two symplectic structures canonically associated with it. The issue of understanding the relation between them has been a long-lasting open question that we answer now.

A. CANonicalizing the Lagrangian

In order to go from the CPS Lagrangians $(\mathcal{L}, \tilde{\mathcal{L}})$ to the CAN Lagrangians $(L, l)$, we need Fubini’s theorem.

Theorem

Let $A$ and $B$ be manifolds oriented with $\text{vol}_A$ and $\text{vol}_B$ respectively. We orient $A \times B$ with $\text{vol}_A \wedge \text{vol}_B$ and consider an integrable function $f : A \times B \rightarrow \mathbb{R}$, then

$$\int_{A \times B} f \text{vol}_A \wedge \text{vol}_B = \int_A \left( \int_B f \text{vol}_B \right) \text{vol}_A$$

Let us now use the $(1, n-1)$-decomposition induced from $M = \mathbb{R} \times \Sigma$ to decompose the CPS Lagrangians. In general, given $\alpha \in \Omega^k(M)$, we have $\alpha = dt \wedge \alpha_1 + \cdots + \alpha_n$ with $\alpha_1 := \ell_{\partial_1} \alpha$ and $\alpha^n := \ell_{\partial_n} (dt \wedge \alpha)$. Since the CPS Lagrangians are top-forms, we simply have $L = dt \wedge L_1$ and $\tilde{\mathcal{L}} = dt \wedge \tilde{L}_1$. Besides, there exist $F, f$ such that

$$dt \wedge L_\perp = L = F \text{vol}_M(1) \rightarrow F = \frac{L_\perp}{\text{vol}_M}$$

$$dt \wedge \tilde{L}_\perp = \tilde{\mathcal{L}} = f \text{vol}_M(4) \rightarrow f = \frac{-\tilde{L}_\perp}{\text{vol}_M}$$

where $\frac{1}{\text{vol}_M}$ denotes the function that relates the top-form $\omega$ with the volume form $\epsilon_\omega$. Plugging these expressions of $(L, \tilde{\mathcal{L}})$ into (5) and using Fubini’s theorem we get

$$S = \int_\Sigma \left( \int_{L_\perp} - \int_{\partial \Sigma} (-\tilde{L}_\perp) \right) dt$$

All this allows us to identify $L$ with $\ell_{\partial_1} L$ and $l$ with $-\ell_{\partial_n} \tilde{L}$. To formalize this identification, we perform the $(1, n-1)$-decomposition on the fields $(\phi^I, \beta)$ and their derivatives. Since the theory is of 1st order, we end up with some tangential fields $(\tilde{q}^I, \tilde{\beta})$ and their velocities $\tilde{v}^I := \partial_\phi \tilde{q}^I$. We now express $(\ell_{\partial_1} L, -\ell_{\partial_n} \tilde{L})$ in terms of $(\tilde{q}^I, \tilde{v}^I)$ and pull everything back to $\Sigma$ (where the positions and velocities are denoted as $(q^I, v^I)$) to obtain $(L, l)$. Plugging them into equation (10) leads to a CAN Lagrangian-action associated with $S$ that we denote $\mathcal{L}_B$.

B. CANonicalizing the symplectic potentials

Now we want to rewrite the symplectic potentials derived from the CPS formalism in a way suitable for the com-
parison with the CAN formalism. For that, we compute

\[ E_I \wedge d\phi^I + d\Theta = dL = dt \wedge d\phi^I L \]
\[ = dt \wedge (\hat{A}^{(0)}_I \wedge dq^I + \hat{A}^{(1)}_I \wedge d\phi^I + \hat{A}^{(2)}_I \wedge dq^I + d\Theta) = dt \wedge (\hat{A}^{(0)}_I - L_\partial_t \hat{A}^{(1)}_I) \wedge dq^I +
\[ + d\left( \iota_{\partial_t} (dt \wedge \hat{A}^{(1)}_I \wedge dq^I) - dt \wedge \Theta \right) \]

In the \( \uparrow \) equality we have "lifted" (12) from \( \Sigma \) to \( M \) (the corresponding objects are denoted with a hat) taking into account that, although additional terms might seem to appear, they have a \( dt \) in them so they vanish in our previous computation. We know from [23, 28, 29], that if \( r < n \) and \( s > 0 \), then any \( d \)-closed \((r, s)\)-form is also \( d \)-exact. So there exists some \( Z \in \Omega^{(n-2,1)}(M) \) such that

\[ \Theta = \iota_{\partial_t} (dt \wedge \hat{A}^{(1)}_I \wedge dq^I) - dt \wedge \Theta + dZ \] (18)

Since \( \hat{\phi}^I \) and \( \iota_{\partial_t} \) commute (\( \partial_t \) is tangent to the boundary), if we compute \( \hat{\phi}^I \Theta \), the first term vanishes. Hence

\[ b_I \wedge dq^I - d\Theta = -dt \wedge (\hat{A}^{(0)}_I \hat{q}^I - \hat{\phi}^I \Theta) = -dt \wedge (\hat{B}^{(0)}_I - L_\partial_t \hat{B}^{(1)}_I) \wedge dq^I -
\[ - d\left( \iota_{\partial_t} (dt \wedge \hat{B}^{(1)}_I \wedge dq^I) - dt \wedge \hat{\phi} + \hat{\phi} \hat{Z} - dt \hat{\tau} \right) \]

Thus, there exists some \( \tau \in \Omega^{(n-3,1)}(\partial M) \) such that

\[ \tau = \iota_{\partial_t} (dt \wedge \hat{B}^{(1)}_I \wedge dq^I) - dt \wedge \hat{\phi} + \hat{\phi} \hat{Z} - dt \tau \] (19)

Finally, we rewrite the CPS symplectic potentials in a more suitable way:

\[ \Theta = \hat{A}^{(1)}_I \wedge dq^I + dt \wedge (\cdots) + dZ \]
\[ \Theta = \hat{B}^{(1)}_I \wedge dq^I + dt \wedge (\cdots) - dt \wedge \hat{\phi} + \hat{\phi} \hat{Z} - dt \tau \] (20)

C. CANonicalizing the symplectic form

Since the symplectic structure over \( \text{Sol}(\mathbb{S}) \) does not depend on the embedding, we consider \( t_\partial \) (where \( dt = 0 \), so in particular \( t_\partial \hat{A}_I = A_I \) and \( t_\partial \hat{B}_I = B_I \)) and we finally obtain the desired equivalence

\[ \Omega_\Sigma \binom{9}{20} \int_\Sigma dq^{(1)}_I \wedge d\phi^I - \int_{\partial\Sigma} d\phi^I \wedge dq^I = \Omega_{\Sigma c} \int_\Sigma dq^{(1)}_I \wedge d\phi^I = \Omega_{L \Sigma} \] (21)

VI. PROVING THE EQUVALENCE FOR HIGHER ORDER THEORIES

In section V.A we obtained the 1st-order CAN Lagrangians from the CPS ones by breaking \((\phi^I)_t \) into \((q^I, \hat{q}^I)_t \). However, for general theories higher order “velocities” \( (\hat{L}_\partial \hat{A}^{(k)}_I \hat{q}^I)_{k=1} \) will appear. While the CPS formalism does not change, the CAN one changes drastically. We devote this section to briefly summarizing how to prove the equivalence for higher order theories.

A. The higher order canonical formalism in a nutshell

For a detailed description, see [30]. For our purposes, we only need to generalize some of the equations of section V.A. First, equation (12) has to be changed to

\[ dL = \sum_{\mu} A^{(\mu)} I \wedge dq^I_\mu + d\Theta \]
\[ d\Theta = \sum_{\mu} B^{(\mu)} I \wedge dq^I_\mu - d\Theta \] (22)

where \( q^I_\mu \) are the positions, \( \hat{q}^I_\mu \) their velocities, \( q_\mu \) their accelerations, and so on up to \( \mu = K \). Thus, we are working on the \( K \)-th tangent bundle \( T^K \mathbb{Q} \). The Hamiltonian formalism takes place in \( T^*(T^K \mathbb{Q}) \) where we have \( \{ q_\mu \}_{\mu=0}^{K-1} \) and their momenta \( \{ p^{(\mu)}_I \}_{\mu=0}^{K-1} \). The canonical symplectic structure generalizing (13) is

\[ \Omega_{T^*(T^K \mathbb{Q})} := \sum_{\mu=0}^{K-1} dq^{(\mu)}_I \wedge dp^{(\mu)}_I \] (23)

In order to go from the Lagrangian to the Hamiltonian formulation, we have to generalize the fiber derivative. For our purposes it is enough to generalize (16). The standard way is

\[ p^{(K)}_I (q; w^I) := \mathcal{L}_{(0,w^I)} L \]
\[ p^{(K)}_I (q; w^I) := \mathcal{L}_{(0,w^I)} L \cdot \frac{d}{dt} p^{(\mu+1)}_I (q; w^I) \]

for \( \mu = 1, \ldots, K - 1 \). It is not hard to prove by (backwards) induction that

\[ p^{(\mu)}_I (\cdot) := \sum_{k=\mu}^{K} \left( \frac{d}{dt} \right)^{k-\mu} \left\{ \int_{\Sigma} A^{(k)}_I \wedge d\phi_\Omega - \int_{\partial\Sigma} B^{(k)}_I \wedge d\phi_\Omega \right\} \]

Taking \( K = 1 \) leads, as expected, to (17).

B. Equivalence for higher order theories

Reasoning by induction, using computations similar to those of section V.B, and equation (22) (where \( q^{(\mu)}_I \) has to be replaced by \( (\hat{L}_\partial \hat{A}^{(k)}_I \hat{q}^I) \) since we are in the CPS formalism), we obtain

\[ \frac{dL}{dt} \wedge \sum_{\mu} (-\hat{L}_\partial \hat{A}^{(k)}_I \hat{q}^I) \wedge dq^I_\mu + d\Theta \]
\[ \frac{dL}{dt} \wedge \sum_{\mu} (-\hat{L}_\partial \hat{B}^{(k)}_I \hat{q}^I) \wedge dq^I_\mu - d\Theta \] (25)
where (up to exact forms)
\[
\Theta = \sum_{k=1}^{K} \sum_{\mu=0}^{k-1} (-\mathcal{L}_{\partial_k})^{k-1-\mu} \hat{A}^{(k)}_I \wedge \dd(L_{\partial_k})^\mu \dot{q}^I - d\theta \wedge (\cdot \cdot \cdot)
\]
\[
\bar{\Theta} = \sum_{k=1}^{K} \sum_{\mu=0}^{k-1} (-\mathcal{L}_{\partial_k})^{k-1-\mu} \hat{B}^{(k)}_I \wedge \dd(L_{\partial_k})^\mu \dot{q}^I - d\theta \wedge (\cdot \cdot \cdot)
\]
Plugging them into (9), considering an embedding \( \iota_{\hat{\nu}} \), and taking into account that in CAN \( (\mathcal{L}_{\partial_k})^\mu \dot{q}^I \) is identified with \( q^I(\hat{\nu}) \), we obtain the general equivalence
\[
\Omega_3^{(24)} = \Omega_3^{(24)}
\]

VII. CONCLUSIONS AND COMMENTS

We have proved the equivalence of the symplectic form induced by the Hamiltonian formalism and the one derived in the covariant phase space. The equivalence, which has been an open question for several decades, holds for theories of any order and even when boundaries are present. The proof relies strongly on the geometric formalism introduced in [23], where we also checked the equivalence in some concrete examples. Finally, this work also proves that the symplectic form introduced in [23] for manifolds with boundaries is the natural one.

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[1] E. Gourgoulhon. 3+1 formalism in general relativity: bases of numerical relativity, volume 846. Springer Science & Business Media, 2012.
[2] N.M.J. Woodhouse. Geometric quantization. Oxford university press, 1997.
[3] A. Ashtekar and J. Lewandowski. Background independent quantum gravity: A status report. Class. Quant. Grav., 21(15):R53, 2004. [arXiv:gr-qc/0404018].
[4] J. Margalef-Bentabol. Towards general relativity through parametrized theories. PhD thesis, Universidad Carlos III de Madrid, 2018. [arXiv:1807.05534].
[5] B.A. Juárez-Aubry, J.F. Barbero G., J. Margalfe-Bentabol, and E.J. Villaseñor. Quantization of scalar fields coupled to point masses. Classical and Quantum Gravity, 32(24):245009, 2015.
[6] M.J. Gotay, J. Isenberg, J.E. Marsden, and R. Montgomery. Momentum maps and classical relativistic fields, part I: Covariant field theory. Preprint, 1998. [arXiv:physics/9801019].
[7] K.S. Stelle. Renormalization of higher derivative quantum gravity. Phys. Rev. D, 16:953, 1977.
[8] ———. Classical gravity with higher derivatives. Gen. Rel. Grav., 9:353, 1978.
[9] H. Lu, A. Perkins, C.N. Pope, and K.S. Stelle. Black holes in higher-derivative gravity. Phys. Rev. Lett., 114 (17):171601, 2015. [arXiv:1502.01028].
[10] G. Barnich and M. Henneaux. Consistent couplings between fields with a gauge freedom and deformations of the master equation. Phys. Lett. B, 311:123, 1993. [arXiv:hep-th/9304057].
[11] G. Barnich, F. Brandt, and M. Henneaux. Local BRST cohomology in the antifield formalism: I. General theorems. Commun. Math. Phys., 174:57, 1995. [arXiv:hep-th/9405109].
[12] ———. Local BRST cohomology in the antifield formalism: II. Application to Yang-Mills theory. Commun. Math. Phys., 174:93, 1995. [arXiv:hep-th/9405109].
[13] G. Odak and S. Speziale. Brown-York charges with mixed boundary conditions. JHEP, 11(11):224, 2021. [arXiv:2109.02883].
[14] L. Freidel, R. Oliveri, D. Pranzetti, and S. Speziale. Extended corner symmetry, charge bracket and Einstein’s equations. JHEP, 09(083):083, 2021. [arXiv:2104.12881].
[15] L. Freidel, M. Geiller, and D. Pranzetti. Edge modes of gravity. Part I. Corner potentials and charges. JHEP, 11(11):026, 2020. [arXiv:2006.12527].
[16] W. Wieland. Gravitational SL(2,R) algebra on the light cone. JHEP, 07(057):057, 2021. [arXiv:2104.05803].
[17] ———. Null infinity as an open Hamiltonian system. JHEP, 04(095):095, 2021. [arXiv:2012.01889].
[18] J.F. Barbero G., B. Díaz, J. Margalfe-Bentabol, and E.J.S. Villaseñor. Concise symplectic formulation for tetrad gravity. Phys. Rev. D, 103(2):024051, 2021. [arXiv:2101.00661].
[19] ———. Hamiltonian Gotay-Nester-Hinds analysis of the parametrized unimodular extension of the Holst action. Phys. Rev. D, 103(6):064062, 2021. [arXiv:2101.12311].
[20] J. Frauendiener and G.A.J. Sparling. On the symplectic formalism for general relativity. Proc. R. Soc. Lond. A, 436:141–153, 1992.
[21] A. Ashtekar and A. Magnon-Ashtekar. On the symplectic structure of general relativity. Commun. Math. Phys., 86:55, 1982.
[22] G. Barnich, M. Henneaux, and C. Schomblond. Covariant description of the canonical formalism. Phys. Rev. D, 44(4):R939, 1991.
[23] J. Margalfe-Bentabol and E.J.S. Villaseñor. Geometric formulation of the covariant phase space methods with boundaries. Phys. Rev. D, 103:025011, 2021. [arXiv:2008.01842].
[24] J.F. Barbero G., J. Margalfe-Bentabol, V. Varo, and E.J.S. Villaseñor. Covariant phase space for gravity with boundaries: Metric versus tetrad formulations. Phys. Rev. D, 104:044048, 2021. [arXiv:2103.06362].
[25] ———. Palatini gravity with nonmetricity, torsion, and boundaries in metric and connection variables. Phys. Rev. D, 104:044046, 2021. [arXiv:2105.07053].

[26] ———. On the on-shell equivalence of general relativity and Holst theories with nonmetricity, torsion, and boundaries. Preprint, 2022. [arXiv:2201.12141].

[27] I.M. Anderson. *The variational bicomplex*. Technical report, Utah State Technical Report, 1989.

[28] R.M. Wald. *On identically closed forms locally constructed from a field*. J. Math. Phys., 31(10):2378, 1990.

[29] F. Takens. *A global version of the inverse problem of the calculus of variations*. J. Differ. Geom., 14(4):543, 1979.

[30] M. De León and P.R. Rodrigues. *Generalized Classical Mechanics and Field Theory*. Elsevier, 2011.