Hamiltonian Relativistic Two-Body Problem: Center of Mass and Orbit Reconstruction

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Abstract
After a short review of the history and problems of relativistic Hamiltonian mechanics with action-at-a-distance inter-particle potentials, we study isolated two-body systems in the rest-frame instant form of dynamics. We give explicit expressions of the relevant relativistic notions of center of mass, we determine the generators of the Poincare’ group in presence of interactions and we show how to do the reconstruction of particles’ orbits from the relative motion and the canonical non-covariant center of mass. In the case of a simple Coulomb-like potential model, it is possible to integrate explicitly the relative motion and show the two dynamical trajectories.

1 Introduction
In Newtonian mechanics the two-body problem is completely understood both in configuration and phase space. The notions of absolute time and absolute
space allow us to describe the two particles of mass $m_i$, $i = 1, 2$, with Euclidean position 3-vectors $\vec{x}_i$ and momenta $\vec{p}_i$ in an inertial frame. For an isolated two-body system the Hamiltonian $H = \sum_{i=1}^{2} \frac{\vec{p}_i^2}{2m_i} + V(|\vec{x}_1 - \vec{x}_2|)$ is the energy generator of the kinematical Galilei group, whose other generators are all interaction independent. With the point (both in coordinate and momenta) canonical transformation $\vec{x} = \frac{m_1}{m_1 + m_2} \vec{x}_1 + \frac{m_2}{m_1 + m_2} \vec{x}_2$, $\vec{p} = \vec{p}_1 + \vec{p}_2$, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\vec{q} = \frac{1}{2} (\vec{p}_1 - \vec{p}_2)$ we can separate the decoupled center of mass from the relative motion: the new Hamiltonian is $H = H_{\text{com}} + H_{\text{rel}}$ with $H_{\text{com}} = \frac{\vec{p}^2}{2m} (m = m_1 + m_2)$ and $H_{\text{rel}} = \frac{\vec{q}^2}{2\mu} + V(|\vec{r}|)$ ($\mu = \frac{m_1 m_2}{m_1 + m_2}$). The relative Hamiltonian $H_{\text{rel}}$ governs the relative motion and, when its Hamilton equations have been solved, the trajectories of the particles are obtained with the inverse canonical transformation $\vec{x}_1 = \vec{x} + \frac{m_1}{m_1 + m_2} \vec{r}$, $\vec{x}_2 = \vec{x} - \frac{m_2}{m_1 + m_2} \vec{r}$, $\vec{p}_1 = \frac{1}{2} \vec{p} + \vec{q}$, $\vec{p}_2 = \frac{1}{2} \vec{p} - \vec{q}$. As a consequence the non-relativistic theory of orbits, for either 2 or N particles, is well understood and developed (see for instance Ref. [1]).

By contrast, in special relativity, where only Minkowski space-time is absolute, where there is no absolute notion of simultaneity and where inertial frames are connected by the transformations generated by the kinematical Poincare’ group, the situation is extremely more complicated and till now there is no completely self-consistent theory of orbits even for the two-body case. This is due to the facts that

i) the particles locations and momenta are now 4-vectors $x^\mu_i$, $p^\mu_i$;
ii) the momenta are not independent but must satisfy mass-shell conditions [since a relativistic particle is an irreducible representation of the Poincare’ group with mass $m_i$ and a value of the spin (only scalar particles will be studied in this paper)];
iii) a simultaneity convention (for instance Einstein’s one identifying inertial frames) for the synchronization of distant clocks has to be introduced, so that the time components $x^0_i$ are no more independent;
iv) the inter-particle interaction potentials appear in the boosts as well as in the energy generator in the instant form of dynamics;
v) the structure of the Poincare’ group implies that there is no definition of relativistic 4-center of mass sharing all the properties of the non-relativistic 3-center of mass.

Since a clarification of all these problems has recently been obtained [2], in this paper we want to illustrate these developments by using the simplest two-body system with a scalar action-at-a-distance (a-a-a-d) interaction, for which a closed Poincare’ algebra can be found in the rest-frame instant form of dynamics, as an example. By using the relativistic generalization [2] of the above quoted non-relativistic canonical basis, we will show that the potential appearing in the energy Hamiltonian (as with $H_{\text{rel}}$) determines the relative motion, while the potentials appearing in the Lorentz boosts (which disappear in the non-relativistic limit), together with the notion of the canonical non-covariant 4-center of mass, contribute to the reconstruction of the actual orbits.
of the two particles.

As a consequence for the first time we have full control on the relativistic theory of orbits and we can start to reformulate at the relativistic level the properties of the Newtonian theory of orbits.

In Section II we give a brief history of the problems that have arisen in past attempts to develop Hamiltonian relativistic mechanics. Then in Section III there is a review of the instant form of dynamics, referred to above, with its two (external and internal) realizations of the Poincaré algebra, while in Section IV there is a review of the three intrinsic notions of relativistic collective center-of-mass-like variables in both the realizations. In Section V there is the relativistic extension of the non-relativistic canonical transformation implementing the separation of the center of mass from the relative variables and how this can be used in general to do the reconstruction of the particle orbits from the relative motion and the vanishing of the internal canonical non-covariant 3-center of mass. In Section VI there is the study of a simple two-body model with a-a-a-distance interaction which correctly reproduces the Poincaré algebra including potential-dependent boosts and energy generators, while in Section VII there is the determination of its orbits with an explicit integration of its equations of motion. A final discussion on the relativistic theory of orbits with its avoidance of the no-interaction theorem is given in the Conclusions. Finally in Appendix A there is a review of the two-body models with first class constraints.
2 Brief History of Hamiltonian Relativistic Mechanics

Relativistic classical particle mechanics with a-a-a-d interactions and its Hamiltonian counterpart arose as an approximation to interactions with a finite time delay (like the electro-magnetic one) and have been quite useful in the treatment of relativistic bound states with an instantaneous approximation of the kernels of field-theoretic equations like the Bethe-Salpeter equation. The starting points for Hamiltonian relativistic particle mechanics were the instant, front and point forms of relativistic Hamiltonian dynamics proposed by Dirac [3]. This approach was an attempt to find canonical realizations of the Poincare’ algebra such that some of the generators, called Hamiltonians (the energy and the boosts in the instant form), are not the direct sum of the corresponding ones for free particles. After the pioneering works of Thomas, Bakamjian and Foldy [4], employing $1/c$ expansions of the Hamiltonians in the instant form (the only one analyzed in this paper), many researches were done as can be seen from the bibliography of Refs. [5].

The main obstacle in the development of models was the Currie - Jordan - Sudarshan no-interaction theorem [6], whose implication was the impossibility, in models with interaction, for the canonical particle 4-positions to be 4-vectors when their time components are put equal to the time of the reference inertial frame. See Refs. [7, 8, 9, 10, 11, 12] for the localization problem and Ref. [13] for a review of the problem of the world-line conditions and for the definition of the covariant particle predictive 4-positions, coinciding with the canonical ones only in absence of interactions.

From these studies it has become clear that relativistic particle mechanics has to be formulated by using Dirac’s theory of constraints [14]. There must be as many mass-shell first class constraints (containing the potentials of the mutual interactions among the particles) as particles. The first consistent two-body model with two first class constraints depending upon a suitable potential was found by Droz-Vincent [15], Todorov [16] and Komar [17] simultaneously and independently (see Appendix A; for $N \geq 3$ a closed form of the $N$ first class constraints is not known). These studies led to the following problems:

a) The study of two- (and N-) particle configurations with a one-to-one correlation among the world-lines. This can be done by adding gauge fixing constraints, so that only the combination of the original constraints describing the mass spectrum of the global system of particles remains first class (moreover there are $N - 1$ pairs of second class constraints). Van Alstine [18] and the authors of Refs. [19] developed consistent two-body models with second class constraints. The avoidance of the relative times in these models has been recently re-interpreted in Ref. [20] as the problem of the synchronization of the clocks associated to the individual particles.

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1 The symbol $\approx 0$ (weakly equal to zero) means that the constraint have been used to get the equality. Let us remember that the constraints can be imposed only after Poisson Brackets are evaluated.
b) The reformulation of the N-body problem with N first class constraints as a form of N-times dynamics \[15, 21\]. As shown in Ref.\[13\], in this way one can write the equations defining Droz-Vincent covariant non-canonical predictive 4-coordinates \[15\]: each of them depends only on the proper time of the associated particle, while the canonical coordinates depend simultaneously on all the proper times when interactions are present (or equivalently on the proper time of the center of mass of the isolated system in the one-time theory). A by-product of these studies was the reformulation of the Newtonian N-body problem as a N-times theory \[22\]: as a consequence a form of no-interaction theorem appears also at the non-relativistic level.

c) The identification of canonical bases containing a relativistic 4-center of mass and relativistic relative variables starting from the original canonical 4-vectors \(x^\mu_i, p^\mu_i\). This was a highly non-trivial problem due to the lack of a unique notion of relativistic center of mass. If we use only the Poincare' generators of the N-particle system, it is possible to define only three such notions: a canonical non-covariant Newton-Wigner-like 3-center of mass \[7, 10, 11\], a non-canonical non-covariant Møller 3-center of energy \[8\] and a non-canonical covariant Fokker-Pryce 3-center of inertia \[9, 10\]. Each then have to be extended to 4-centers (\(\tilde{x}^\mu, R^\mu\) and \(Y^\mu\), respectively). The two non-covariant 4-centers \(\tilde{x}^\mu\) and \(R^\mu\) describe frame-dependent pseudo-world-lines filling the so-called Møller world-tube \[8, 21, 23\] around the world-line of the Fokker-Pryce 4-center of inertia \(Y^\mu\) (in the rest frame the 3 centers coincide). The invariant Møller radius of this world-tube is determined by the Poincare' Casimirs of the particle configuration, \(\rho = \frac{S}{Mc}\).

In Ref.\[2\] there is a full classification of these centers and of their properties and a methodology to find canonical bases of center-of-mass and relative variables, also in the presence of interactions, in the framework of the Wigner-covariant rest-frame instant form of dynamics developed in Refs.\[21, 23\]. This instant form is a special case of parametrized Minkowski theories \[21, 23\] in which the leaves of the 3+1 splitting of Minkowski space-time are inertial hyper-surfaces (simultaneity 3-surfaces called Wigner hyper-planes) orthogonal to a 4-vector \(P^\mu\), coinciding with the conserved 4-momentum \(P^\mu_{sys}\) of the N-particle system in the rest frame: if we define the invariant mass \(M = \sqrt{P_{sys}^2}\), then we have \(P^\mu = M w^\mu(P), w^2(P) = 1\). The 4-vector \(P^\mu\) is canonically conjugate to the canonical non-covariant 4-center of mass \(\tilde{x}^\mu\). Therefore the Wigner hyperplane at time \(\tau\) is the intrinsic rest frame of the isolated system at time \(\tau\).

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\(^2\) This approach was developed to give a formulation of the N-body problem on arbitrary simultaneity 3-surfaces (corresponding to a convention for the synchronization of distant clocks \[20\]), which could be also used as Cauchy surfaces for the electro-magnetic field if present. In this way both the unknown closed form of first class constraints and the special choices of gauge fixings leading to second class ones are avoided. Moreover, the change of clock synchronization convention may be formulated as a gauge transformation not altering the physics and there is no problem in introducing the electro-magnetic field when the particles are charged. The rest-frame instant form corresponds to the gauge choice of the 3+1 splitting whose simultaneity 3-surfaces are the intrinsic rest frame of the given configuration of the isolated system.
In this approach all the particles depend on the scalar rest-frame time \( \tau = Y^\mu u_\mu(P) = \tilde{x}^\mu u_\mu(P) = R^\mu u_\mu(P) \), the Fokker-Pryce 4-center of inertia \( Y^\mu(\tau) \) is the inertial observer origin of the inertial rest frame, and all the first class constraints have been solved to determine the single particle energies. As a consequence we have:

i) \( x_i^\mu(\tau) = Y^\mu(\tau) + \epsilon_i^\mu(P) \eta_i^\tau(\tau), \) where \( \epsilon_i^\mu(P) \) are suitable momentum-dependent space-like 4-vectors orthogonal to \( u^\mu(P) \);

ii) \( p_i^\mu(\tau) \) are suitable solutions of the mass shell constraints. Therefore the particles must have a definite sign of the energy (assumed positive in this paper) and their independent canonical variables are the (Wigner spin-1) position 3-vectors \( \eta_i^\mu(\tau) \) and their conjugate 3-momenta \( \kappa_i^\mu(\tau) \).

iii) There is an external realization of the Poincare’ algebra with generators \( P^\mu, J^{\mu\nu} \) (describing the properties of the Wigner hyper-planes in every inertial frame) and an unfaithful internal (called unfaithful since some generators are weakly vanishing) realization of the Poincare’ algebra, with generators \( M \) (the invariant mass of the system), \( \vec{p}, \vec{j}, \vec{k} \) describing the covariance properties inside the Wigner hyper-planes. The Hamiltonian for the relative motion on the Wigner hyper-plane (replacing the non-relativistic \( H_{\text{rel}} \)) is the the internal energy generator \( M \).

iv) It is now clear that the avoidance of the no-interaction theorem implies the non-covariance of the canonical external Newton-Wigner-like 4-center of mass \( \tilde{x}^\mu \) (a pseudo-world-line intersecting each Wigner hyper-plane in every inertial frame and coinciding with the external Fokker-Pryce center of inertia only when the reference inertial frame coincides with the rest frame): this non-covariance is universally (i.e. independently from the relativistic isolated system under investigation) concentrated on the point-particle-like degrees of freedom describing the decoupled external center of mass of the system.

d) Another problem is the identification of special models suited to the relativistic bound state problem. The model building \([13, 21, 24, 25, 26]\) initially concentrated on the potentials in the energy Hamiltonian, which governs the relative motion (the canonical non-covariant 4-center of mass has free motion). The much harder problem to find the suitable potentials in the Lorentz boosts \([5]\), so that the global Poincare’ algebra is satisfied, was finally solved in Ref.\([27]\) for charged scalar particles interacting with a dynamical electro-magnetic field (with Grassmann-valued electric charges to regularize the self-energies): in the sector of configurations without an independent radiation field the Darwin potential appeared in the energy Hamiltonian (till now it had been obtained only coming down from quantum field theory through instantaneous approximations

\[ \text{3 The internal total 3-momentum is weakly vanishing, } \vec{p} \approx 0, \text{ since these three first class constraints define the rest frame. If the internal Lorentz boosts } \vec{k} \text{ are put weakly equal to zero, } \vec{k} \approx 0, \text{ as gauge fixing constraints to the rest-frame conditions, then it can be shown } [2, 23] \text{ that the 3-position describing the collective center variable inside the Wigner hyper-planes (the 3 internal centers coincide due to the rest-frame conditions as shown in Section IV) coincides with the origin, i.e. is located at the position of the external Fokker-Pryce 4-center of inertia at each instant. Therefore only the external canonical non-covariant 4-center of mass remains as a decoupled point particle, without any double counting.} \]
to the Bethe-Salpeter equation) and suitable related potentials in the boost Hamiltonians. In Ref. [28] analogous results (involving the Salpeter potential) were obtained for charged spinning particles (with Grassmann-valued spins implying Dirac spin 1/2 fermions after quantization).
3 The Inertial Rest-Frame Instant Form of Dynamics

In this Section we will describe the main properties of the inertial rest-frame instant form of dynamics in the case of free particles by anticipating the various notions of relativistic center of mass, which will be clarified in Section IV. In Appendix B there is the definition of non-inertial rest frames, obtained as other special gauges of parametrized Minkowski theories.

As said in the previous Section, in the rest-frame instant form of dynamics Minkowski space-time is foliated with inertial hyper-planes (named Wigner hyper-planes) orthogonal to a 4-vector \( P^\mu \) coinciding with the conserved 4-momentum \( P^\mu_{\text{sys}} \) of the given isolated system in the rest frame. With respect to an arbitrary inertial frame the Wigner hyper-planes are described by the following embedding

\[
z^\mu(\tau, \vec{\sigma}) = x^\mu_s(\tau) + \epsilon^\mu_r(u(P)) \sigma^r,
\]

with \( x^\mu_s(\tau) \) being the world-line of an arbitrary inertial observer. The (so-called radar \[20\]) observer-dependent 4-coordinates \((\tau, \vec{\sigma})\) are the proper time \(\tau\) of this observer and 3-coordinates on the Wigner hyper-planes \(\Sigma_\tau\) having the observer as origin \(\vec{\sigma} = 0\) for every \(\tau\). The space-like 4-vectors \(\epsilon^\mu_r(u(P))\) together with the time-like one \(\epsilon^\mu_o(u(P))\) are the columns of the standard Wigner boost for time-like Poincare’ orbits

\[
\epsilon^\mu_o(u(P)) = u^\mu(P) = P^\mu/\sqrt{P^2}, \quad \epsilon^\mu_r(u(P)) = (-u_r(P); \delta^i_r - \frac{u^i(P) u_r(P)}{1 + u_o(P)}),
\]

\[
\epsilon^\mu_o(u(P)) = \eta^\alpha \eta_{\mu\nu} \epsilon^{\nu}_r(u(P)) = u_\mu(P), \quad \epsilon^r_\mu(u(P)) = \eta^{\alpha} \eta_{\mu\nu} \epsilon^{\nu}_o(u(P)).
\]

(2)

Since we are in the rest frame, we have \(\tau \equiv T_s = u(P) \cdot x_s\). Here \(T_s\) is the scalar rest time of the inertial observer, whose world-line is given by

\[
x^\mu_s(\tau) = x^\mu(0) + u^\mu(P) \tau.
\]

(3)

In the rest-frame instant form the particles’ 4-coordinates, describing their world-lines, and the associated momenta (see Refs.\[2, 21\]) are

\[4\] We use the metric \(\eta_{\mu\nu} = (+ - - -)\).

\[5\] It sends the time-like four-vector \(P^\mu\) to its rest-frame form \(\hat{P}^\mu = \eta \sqrt{P^2}(1; 0)\), where \(\eta = \text{sign} P^0\); see Refs.\[2, 21\]. From now on we restrict ourselves to positive energies, i.e. \(\eta = 1\). While \(\epsilon^\mu_o(u(P))\) and \(\epsilon^\mu_r(u(P))\) are 4-vectors, \(\epsilon^\mu_t(u(P))\) have more complex transformation properties under Lorentz transformations, given in the Conclusions.
\[
x^{\mu}(\tau) = z^{\mu}(\tau, \bar{\eta}_i(\tau)) = x^{\mu}_0(\tau) + \epsilon^{\mu}_i(u(P)) \eta^\mu_i(\tau),
\]
\[
p^{i\mu}(\tau) = \eta_i \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)} \ u^{\mu}(P) + \epsilon^{\mu}_i(u(P)) \kappa_{ir}(\tau) \Rightarrow p^2_i = m_i^2,
\]
\[
\Rightarrow \quad P^{\mu}_{sys} = \sum_{i=1}^{N} p^{\mu}_i = u^{\mu}(P) \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)} + \epsilon^{\mu}_i(u(P)) \sum_{i=1}^{N} \kappa_{ir}(\tau)
\]

This shows that inside the Wigner hyper-planes the N particles (all assumed to have positive energy, i.e. \( \eta_i = 1 \)) are described by the 6N Wigner spin-1 3-vectors \( \bar{\eta}_i(\tau), \bar{\kappa}_i(\tau) \) as independent canonical variables \( \{\eta^\mu_i(\tau), \kappa_{ir}(\tau)\} = \delta_{ij} \delta_{rs} \), \( \{\eta^\mu_i(\tau), \eta^\mu_j(\tau)\} = \{\kappa_{ir}(\tau), \kappa_{js}(\tau)\} = 0 \). To them we must add \( P^{\mu} = \sum_i p^{\mu}_i \) and a canonically conjugate collective variable \( \bar{z}^{\mu} \) \( \{\bar{z}^\mu, P^{\nu}\} = -\eta^{\mu\nu} \).

It turns out \(^6\) (see also Section IV) that the relevant collective variable to be added is the external canonical non-covariant 4-center of mass \( (M = \sqrt{P^2}) \).

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\(^6\) Under Lorentz transformations \( A \) these 3-vectors rotate with Wigner rotations \( P = M(1; 0); \ P^{\mu} = L(P, P)^{\mu}_{\nu} \ P = e^{\mu}_{AB}(u(P)) \ P \) with \( L \) the standard Wigner boost

\[
R^{\nu}_{\ \nu}(\Lambda, P) = [L(\Lambda, P) \Lambda^{-1} L(\Lambda P, P)]^{\nu}_{\ \nu} = \begin{pmatrix} 1 & 0 \\
0 & R^{\nu}_{\ \nu}(\Lambda, P) \end{pmatrix},
\]
\[
R^{\nu}_{\ \nu}(\Lambda, P) = (\Lambda^{-1})^i_{\ \nu} \ P_\alpha (\Lambda^{-1})^\beta_{\ \nu} \frac{P_\beta (\Lambda^{-1})^\alpha_{\ \nu} + \sqrt{P^2}}{P_\beta (\Lambda^{-1})^\alpha_{\ \nu} + \sqrt{P^2}} - \frac{P^\mu}{P^\mu + \sqrt{P^2}} [(\Lambda^{-1})^o_{\ \nu} - (\Lambda^{-1})^o_{\ \nu} (\Lambda^{-1})^m_{\ \nu} P^\mu (\Lambda^{-1})^\mu_{\ \nu}] + \frac{P^\mu}{P^\mu + \sqrt{P^2}} [(\Lambda^{-1})^o_{\ \nu} - (\Lambda^{-1})^o_{\ \nu} (\Lambda^{-1})^m_{\ \nu} P^\mu (\Lambda^{-1})^\mu_{\ \nu}].
\]

As a consequence the scalar product of two of these 3-vectors is a Lorentz scalar.

\(^7\) As shown in Ref.\( ^{[2]} \), the three spin tensors \( S^{\mu\nu} = e^{\mu}_{A}(u(P)) e^{\nu}_{B}(u(P)) S^{AB}, \ S^{\mu\nu} \) and \( \bar{S}^{AB} \) satisfy the Lorentz algebra: \( \{S^{\mu\nu}, S^{\alpha\beta}\} = e^{e}_{\alpha\beta}(u(P)) S^{AB} \), \( \{S^{\mu\nu}, \bar{S}^{\alpha\beta}\} = e^{e}_{\alpha\beta}(u(P)) \bar{S}^{AB} \), \( \{S^{AB}, S^{CD}\} = C_{EBCD}^{EF} S^{EF} \), where \( C^{\mu\nu}_{\alpha\beta} = n^\mu_{\eta} n^\mu_{\eta} \eta^{\nu\alpha} + n^\mu_{\eta} n^\mu_{\eta} \eta^{\nu\alpha} - n^\mu_{\eta} n^\mu_{\eta} \eta^{\nu\alpha} \), \( C_{EBCD}^{EF} = n^E_{\eta} n^E_{\eta} \eta^{\nu\alpha} + n^E_{\eta} n^E_{\eta} \eta^{\nu\alpha} - n^E_{\eta} n^E_{\eta} \eta^{\nu\alpha} \) are the Lorentz structure constants.
\[
\dot{x}^\mu(\tau) = (\dot{x}^\mu(\tau); \ddot{x}(\tau)) = z^\mu(\tau, \tilde{\sigma}) = x_s^\mu(\tau) - \frac{1}{M (P^\sigma + M)} \left[ P_o S^{\mu\sigma} + M (S^{\sigma\nu} - S^{\alpha\beta} \frac{P^\alpha P^\beta}{M^2}) \right],
\]

\[S^{\mu\nu} = J^{\mu\nu} - (x_s^\mu P^\nu - x_s^\nu P^\mu), \quad \bar{S}^{rs} = \sum_{i=1}^{N} (\eta_i^r \kappa_i^s - \eta_i^s \kappa_i^r), \quad \bar{S}^{or} = -\sum_{i=1}^{N} \eta_i^r \sqrt{m_i^2 + \vec{r}_i^2},\]

\[\dot{S}^{uv} = J^{uv} - (\ddot{x}_s^\mu P^\nu - \ddot{x}_s^\nu P^\mu), \quad \bar{S}^{ij} = \delta^{ir} \delta^{js} \bar{S}^{rs}, \quad \bar{S}^{oi} = -\frac{\delta^{ir} \bar{S}^{rs} P^s}{P^o + M}.
\]

(5)

The point with coordinates \(\tilde{x}^\mu(\tau)\) is the decoupled canonical external 4-center of mass, playing the role of a kinematical external 4-center of mass and of a decoupled observer with his parametrized clock (point particle clock). As shown in Refs. [21, 2], when we restrict the parametrized Minkowski theory to the rest-frame instant form and we add the gauge fixing \(\vec{k} \approx 0\) (see footnote 3), we get the result \(\dot{x}_s^\mu(\tau) = \ddot{x}_s^\mu(\tau) = u^\mu(P)\). Therefore the gauge \(\vec{k} \approx 0\) is the natural one, because only in it are both the velocities \(\dot{x}_s^\mu(\tau)\) and \(\ddot{x}_s^\mu(\tau)\) parallel to \(P^\mu\), so that there is no classical zitterbewegung in the associated world-lines.

In conclusion, as a consequence of Eqs. (5), in the rest-frame instant form the standard \(8N\) variables \(x_s^\mu\), \(p_i^\mu\) are re-expressed in terms of the 8 variables \(x_s^\mu\) (or \(\tilde{x}^\mu\)), \(P^\mu = M u^\mu(P)\) and the 6\(N\) variables \(\eta_i\), \(\kappa_i\) restricted by the rest-frame condition \(\vec{p} \approx 0\) and by the associated gauge-fixings (see footnote 3). Therefore we have \(8 + 6N - 6 = 2(3N + 1)\) variables: the lacking \(2(N - 1)\) to arrive to \(8N\) are the \(N - 1\) relative times (they disappeared due to the clock synchronization) and \(N - 1\) relative momenta (they disappeared because, as shown in Eqs. (4), the particles are on the mass shell).

Let us stress that the rest-frame instant form succeeds in separating the relativistic center of mass from the relative motion by means of a splitting of the description of the isolated system into an external one and an internal one.

A) The external description is concerned with the embedding of the Wigner hyper-planes into Minkowski space-time from the point of view of a generic inertial observer. Each Wigner hyper-plane, orthogonal to the conserved 4-momentum \(P^\mu\) of the isolated system, is parametrized by means of canonical coordinate \(\tilde{x}_s^\mu\) (conjugate to \(P^\mu\)), which describes the decoupled collective degrees of freedom of the isolated system (at the non-relativistic level it is the free center of mass \(\tilde{x}\) with Hamiltonian \(H_{\text{com}} = \frac{\vec{p}^2}{2m}\)). There is an external realization of the Poincare’ algebra, which governs the covariance properties of Wigner hyper-planes under Poincare’ transformations (\(\Lambda, \alpha\)).
It can be shown [2] that its generators have the following form \( M = \sqrt{P^2} \); while \( i, j \) are Euclidean indices, \( r, s \) are Wigner spin-1 indices; \( \tilde{S}^{\mu\nu} \) is given in Eq.(5); footnote 7 implies \( \{ P^\mu, P^\nu \} = 0 \), \( \{ P^\mu, J^{\alpha\beta} \} = \eta^{\mu\alpha} P^\beta - \eta^{\mu\beta} P^\alpha \), \( \{ J^{\mu\nu}, J^{\alpha\beta} \} = C^{\mu\nu\alpha\beta} J^{\gamma\delta} \)

\[
P^\mu = \sqrt{M^2 + \vec{P}^2},
\]

\[
K^i = J^{oi} = \tilde{\omega}^o P^i - \tilde{\omega}^i \sqrt{M^2 + \vec{P}^2} - \frac{1}{M + \sqrt{M^2 + \vec{P}^2}} \delta^{ir} P^s \sum_{i=1}^N (\eta^i_\alpha \kappa^s_\alpha - \eta^i_\alpha \kappa^s_\alpha) = \tilde{\omega}^o P^i - \tilde{\omega}^i \sqrt{M^2 + \vec{P}^2} - \frac{\delta^{ir} P^s \epsilon^{rsu} \bar{S}^u}{M + \sqrt{M^2 + \vec{P}^2}}.
\]

Note that both \( \tilde{L}^{\mu\nu} = \tilde{\omega}^\mu P^\nu - \tilde{\omega}^\nu P^\mu \) and \( \tilde{S}^{\mu\nu} \) are conserved.

It is this external realization which implements the Wigner rotations of footnote 6.

Let us remark that this realization is universal in the sense that it depends on the nature of the isolated system only through the invariant mass \( M \) (which in turn depends on the relative variables and on the type of interaction).

B) The internal description concerns the relative degrees of freedom of the isolated system inside the Wigner hyper-plane (replacing the absolute Newtonian Euclidean 3-space containing the isolated system). In order to avoid a double counting of the center-of-mass degrees of freedom there is the rest-frame condition, which implies the existence of the following 3 first class constraints on the internal 3-momentum \( \vec{p} = \sum_{i=1}^N \vec{\kappa}_i \approx 0 \). This implies that a collective 3-variable (the internal 3-center of mass) inside each Wigner hyper-plane can be eliminated, so that only \( 3N - 3 \) internal relative canonical variables are independent.

Since the sin tensor \( \tilde{S}^{AB} \) satisfies a Lorentz algebra (see footnote 7), we can build an unfaithful internal realization of the Poincaré algebra, acting inside the Wigner hyperplane, by adding the internal 3-momentum \( \vec{p}^r = \epsilon^r_\mu (u(P)) P^\mu_{sys} \approx \)

\[
\text{In the non-relativistic limit we get the standard description with } \vec{\eta}_i = \vec{\xi}_i, \vec{\kappa}_i = \vec{\nu}_i \text{ in the Newtonian rest frame } \vec{p} \approx 0.
\]
0 and the invariant mass $M = \sqrt{P_\text{sys}^2}$ as the internal energy. The internal Poincare' generators $p^\tau = M$, $p^r$, $j^r = \tilde{S}^r$, $k^r = \tilde{S}^{\tau r}$ can also be found from the energy-momentum tensor of the isolated system evaluated in the associated parametrized Minkowski theory and then restricted to the rest-frame instant form.

The internal Poincare' generators for $N$ free particles are

\begin{align*}
M &= p^\tau = \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2}, \\
\vec{p} &= \sum_{i=1}^{N} \vec{k}_i \approx 0, \\
\vec{j} &= \sum_{i=1}^{N} \vec{\eta}_i \times \vec{k}_i, \\
\vec{k} &= \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2} \vec{\eta}_i, \\
\tilde{S}^r &= \frac{1}{2} \epsilon^r_{\mu \nu} \tilde{S}^{\mu \nu},
\end{align*}

They satisfy the Poincare' algebra (like the external ones)

\begin{align*}
\{p^\tau, p_i\} &= \{p_i, p_j\} = 0, \\
\{p_i, k_j\} &= \delta_{ij} p^\tau = \delta_{ij} M, \\
\{p^\tau, k_j\} &= \{p^r, j_i\} = 0, \\
\{j_i, j_j\} &= \epsilon_{ijk} j_k, \\
\{j_i, k_j\} &= \epsilon_{ijk} k_k, \\
\{k_i, k_j\} &= -\epsilon_{ijk} j_k.
\end{align*}

The Poisson brackets $\{p_i, k_j\} = \delta_{ij} p^\tau$ shows clearly that the presence of interaction potentials in the invariant mass $p^\tau = M$ requires the presence of potentials also in the boost generators $k_i$.

Let us remark that, since we have $\tilde{S}^{AB} = \epsilon^A_{\mu}(u(P)) \epsilon^B_{\nu}(u(P)) \left( J^{\mu \nu} - x^{\mu}_s P^{\nu} + \ddot{x}^{\mu} P^{\mu} \right)$, then Eq.(4) implies $\tilde{S}^{\tau r} = \epsilon^r_{\mu}(u(P)) \epsilon^\tau_{\nu}(u(P)) J^{\mu \nu}$ and $\tilde{S}^{\tau \tau} = u_{\mu}(P) \epsilon^\tau_{\nu}(u(P)) J^{\mu \nu}$ only with the choice $x^{\mu}_s(0) = 0$, i.e. when $x^{\mu}_{s}(\tau) = u^{\mu}(P) \tau$. Moreover, as said in footnote 3, $k^r = \tilde{S}^{\tau r}$ is a gauge variables, whose natural gauge fixing is $\vec{k} \approx 0$.

In Ref.[27] we found the generators for a system of $N$-interacting Grassmann charged particles and electromagnetic fields. We started from the standard classical action reexpressed as a parametrized Minkowski theory. Then
with a suitable canonical transformation we decoupled the scalar and longitudinal gauge degrees of freedom of the electromagnetic field: in this way we obtained a canonical formulation of the radiation gauge (only the transverse vector potentials and electric fields are present) with the emergence of the action-at-a-distance Coulomb potentials among the charged particles. For \( N = 2 \) the internal Hamiltonian and boosts have the following form \([-1/4\pi |\vec{\sigma}|]\)

\[
M = \sqrt{m_1^2 + (\vec{k}_1(\tau) - Q_1 \vec{A}_1(\tau, \vec{\eta}_1(\tau)))^2} + \sqrt{m_2^2 + (\vec{k}_2(\tau) - Q_2 \vec{A}_2(\tau, \vec{\eta}_2(\tau)))^2} + \\
\frac{Q_1 Q_2}{4\pi |\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|} + \int d^3\sigma \frac{1}{2} |\vec{E}_\perp^2 + \vec{B}_\perp^2(\tau, \vec{\sigma})|
\]

\[
k^r = -\sum_{i=1}^{2} \eta_i^r(\tau) \sqrt{m_i^2 + (\vec{k}_i(\tau) - Q_i \vec{A}_i(\tau, \vec{\eta}_i(\tau)))^2} + \\
\sum_{i=1}^{2} \left[ Q_1 Q_2 \sum_{i \neq j} \left( \frac{1}{\sqrt{\eta_j^r}} \frac{\partial}{\partial \eta_j^r} c(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)) - \eta_j^r(\tau) c(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)) \right) \right] + \\
Q_i \int d^3\sigma E_i^r(\tau, \vec{\sigma}) c(\vec{\sigma} - \vec{\eta}_i(\tau)) - \frac{1}{2} \int d^3\sigma \sigma^r |\vec{E}_\perp^2 + \vec{B}_\perp^2(\tau, \vec{\sigma})|,
\] (9)

In the sector without an independent radiation field it can be shown [27] that the Coulomb potential is replaced by the classical Darwin potential. The final form of the internal generators \( M \) and \( \vec{k} \) in this sector is given in Eqs. (6.19) and (6.46) of Ref. [27], respectively. Let us remark that starting from classical electrodynamics we arrive at a Coulomb potential additive to the square roots, and not living inside them like in the toy model at the end of Appendix A, whose rest-frame instant form will be studied in Section VI.
4 The Problem of the Relativistic Center of Mass.

As shown in Refs. [2], given an isolated system with an associated realization of the Poincare' algebra, only three notions of collective 3-variables (coinciding only in the rest frame) can be built in term of them (namely without introducing external variables). This is done by using the group theoretical methods of Refs. [5]. Then, these 3-variables have to be extended to suitable collective 4-variables. They are

i) a non-canonical non-covariant Møller center of energy \[\vec{R}^+\], defining a frame-dependent pseudo-world-line (it is the non-relativistic prescription with the particle energies replacing their masses);

ii) a canonical non-covariant center of mass (or center of spin); it is the classical analogue [10, 11] of the Newton-Wigner position operator [7] and defines a frame-dependent pseudo-world-line;

iii) a non-canonical covariant Fokker-Pryce center of inertia \[\vec{y}^+\], leading to a 4-vector defining a frame-independent world-line.

However, no-one of these candidates to represent the relativistic center of mass has all the properties of the non-relativistic center of mass.

Since in the rest-frame instant form of dynamics we have both an internal and an external realization of the Poincare' algebra, in this Section we shall review the definition of the three collective variables in both cases starting from the internal one.

The internal realization of the Poincare' algebra leads to the following internal (Wigner spin-1) collective 3-variables inside the Wigner hyper-planes (therefore they need not to be extended to 4-variables):

i) a non-canonical internal Møller center of energy \[\vec{R}^+_+\];

ii) a canonical internal center of mass (or center of spin) \[\vec{q}^+_+\];

iii) a non-canonical internal Fokker-Pryce center of inertia \[\vec{y}^+_+\].

We shall see that on the Wigner hyper-planes, due to the rest frame condition \[\vec{p} \approx 0\], all of them coincide: \[\vec{q}^+_+ \approx \vec{R}^+_+ \approx \vec{y}^+_+\].

The 3-variables \[\vec{R}^+_+\], \[\vec{q}^+_+\], \[\vec{y}^+_+\] have the following definitions:

i) The internal Møller 3-center of energy and the associated spin vector are
\[ \vec{R}_+ = -\frac{1}{M} \vec{k} = \frac{\sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2} \vec{n}_i}{\sum_{k=1}^{N} \sqrt{m_k^2 + \vec{k}_k^2}} \]

\[ \vec{S}_R = \vec{j} - \vec{R}_+ \times \vec{p}, \]

\[ \{ R^r_+, p^s \} = \delta^{rs}, \quad \{ R^r_+, M \} = \frac{p^r}{M}, \quad \{ R^r_+, R^s_+ \} = -\frac{1}{M^2} \epsilon^{rus} S^u_R, \]

\[ \{ S^r_R, S^s_R \} = \epsilon^{rus} (S^u_R - \frac{1}{M^2} \vec{S}_R \cdot \vec{p} p^u), \quad \{ S^r_R, M \} = 0. \] (10)

Therefore, the \textit{internal} boost generator may be rewritten as \( \vec{k} = -M \vec{R}_+ \), so that \( \vec{R}_+ \approx 0 \) implies \( \vec{k} \approx 0 \) (see footnote 3).

Note that in the non-relativistic limit \( \vec{R}_+ \) tends the the non-relativistic center of mass \( \vec{q}_{nr} = \frac{\sum_{i=1}^{N} m_i \vec{n}_i}{\sum_{i=1}^{N} m_i} \).

ii) The canonical \textit{internal} 3-center of mass and the associated spin vector are

\[ \vec{q}_+ = -\frac{\vec{k}}{\sqrt{M^2 - \vec{p}^2}} + \frac{\vec{j} \times \vec{p}}{\sqrt{M^2 - \vec{p}^2}} + \frac{\vec{k} \cdot \vec{p} \vec{p}}{M \sqrt{M^2 - \vec{p}^2}} (M + \sqrt{M^2 - \vec{p}^2}), \]

\[ \approx \vec{R}_+ \quad \text{for} \quad \vec{p} \approx 0; \quad \{ \vec{q}_+, M \} = \frac{\vec{p}}{M} \]

\[ \vec{S}_q = \vec{j} - \vec{q}_+ \times \vec{p} = \frac{M \vec{j}}{\sqrt{M^2 - \vec{p}^2}} + \frac{\vec{k} \times \vec{p}}{\sqrt{M^2 - \vec{p}^2} (M + \sqrt{M^2 - \vec{p}^2})} \approx \vec{S} = \vec{j}, \]

\[ \{ \vec{S}_q, \vec{p} \} = \{ \vec{S}_q, \vec{q}_+ \} = 0, \quad \{ S^r_q, S^s_q \} = \epsilon^{rus} S^u_q. \] (11)

iii) The \textit{internal} non-canonical Fokker-Pryce center of inertia \( \vec{y}_+ \) is

\[ \vec{y}_+ = \vec{q}_+ + \frac{\vec{S}_q \times \vec{p}}{\sqrt{M^2 - \vec{p}^2} (M + \sqrt{M^2 - \vec{p}^2})} = \vec{R}_+ + \frac{\vec{S}_q \times \vec{p}}{M \sqrt{M^2 - \vec{p}^2}}, \]

\[ \{ y^r_+, y^s_+ \} = \frac{1}{M \sqrt{M^2 - \vec{p}^2}} \epsilon^{rus} \left[ S^u_q + \frac{\vec{S}_q \cdot \vec{p} p^u}{\sqrt{M^2 - \vec{p}^2} (M + \sqrt{M^2 - \vec{p}^2})} \right]. \] (12)
We have
\[
\vec{q}_+ = \vec{R}_+ + \frac{\vec{S}_q \times \vec{p}}{M (M + \sqrt{M^2 - \vec{p}^2})} = \frac{M \vec{R}_+ + \sqrt{M^2 - \vec{p}^2} \vec{y}_+}{M + \sqrt{M^2 - \vec{p}^2}}.
\]
\[
\vec{p} \approx 0 \Rightarrow \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+.
\]
(13)

We see that the gauge fixings \(\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0\) force the three internal collective 3-variables to coincide with the location of the inertial observer \(x^\mu_s(\tau)\), origin of the 3-coordinates (see Refs.\[2\] for other properties of \(x^\mu_s(\tau)\) in this gauge).

On the other hand from the external realization of the Poincaré algebra we get the following three external collective 3-variables (the canonical \(\vec{q}_s\), the Møller \(\vec{R}_s\) and the Fokker-Pryce \(\vec{Y}_s\))

\[
\vec{R}_s = -\frac{1}{P^0} \vec{K} = (\vec{x} - \frac{\vec{p}}{P^0} \vec{x}^0) - \frac{\vec{S} \times \vec{P}}{P^0 (P^0 + M)},
\]

\[
\vec{q}_s = \vec{x} - \frac{\vec{p}}{P^0} \vec{x}^0 = \vec{R}_s + \frac{\vec{S} \times \vec{P}}{P^0 (P^0 + M)} = \frac{P^0 \vec{R}_s + M \vec{Y}_s}{P^0 + M},
\]

\[
\vec{Y}_s = \vec{q}_s + \frac{\vec{S} \times \vec{P}}{M (P^0 + M)} = \vec{R}_s + \frac{\vec{S} \times \vec{P}}{P^0 M},
\]

\[
\{R^r_s, R^s_s\} = -\frac{1}{(P^0)^2} \epsilon^{r su} \Omega^u_s, \quad \vec{\Omega}_s = \vec{J} - \vec{R}_s \times \vec{P},
\]

\[
\{q^r_s, q^s_s\} = 0, \quad \{Y^r_s, Y^s_s\} = \frac{1}{M P^0} \epsilon^{r su} \left[ \vec{S}^u + \frac{\vec{S} \cdot \vec{P} P^u}{M (P^0 + M)} \right],
\]

\[
\vec{P} \cdot \vec{q}_s = \vec{P} \cdot \vec{R}_s = \vec{P} \cdot \vec{Y}_s,
\]

\[
\vec{P} = 0 \Rightarrow \vec{q}_s = \vec{Y}_s = \vec{R}_s.
\]
(14)

All of them have the same velocity and coincide in the Lorentz rest frame where \(P^0 = M (1; \vec{0})\)

As shown in Refs.\[2\], the requirement that the relations \(\tau \equiv T_s = u(P) \cdot x_s = u(P) \cdot \tilde{x} = u(P) \cdot Y_s = u(P) \cdot R_s\) holds on Wigner hyper-planes allows us to extend the external collective 3-variables to the following external 4-variables:

i) the external non-canonical and non-covariant Møller 4-center of energy \(R^\mu_s\) (a frame-dependent pseudo-world-line, whose intersection with the Wigner hyper-plane has 3-coordinate \(\sigma^r_R\));

ii) the external canonical non-covariant 4-center of mass \(\tilde{x}^\mu\) (a frame-dependent
pseudo-world-line, whose intersection with the Wigner hyper-plane has 3-coordinate \( \tilde{\sigma}^r \));

iii) the external covariant non-canonical Fokker-Pryce 4-center of inertia \( Y^\mu_\sigma \) (a frame-independent world-line, whose intersection with the Wigner hyper-plane has 3-coordinate \( \sigma^r \)).

They have the following definitions\(^3\)

\[
\tilde{x}^\mu = (\tilde{x}^0; \tilde{x}) = (\tilde{x}^0; \tilde{q}_s + \frac{\tilde{\rho}^s}{\rho^s} \tilde{x}^0) \equiv x^\mu_s + \epsilon^\mu_s(u(P)) \tilde{\sigma}^u,
\]

\[
Y^\mu_s = (\tilde{x}^0; \tilde{Y}_s) =
\]

\[
= \tilde{x}^\mu + \epsilon^\mu_s(P) \frac{(\tilde{S} \times \tilde{P})^r}{M [1 + u^\alpha(P)]} \equiv x^\mu_s + \epsilon^\mu_s(u(P)) \sigma^r_Y,
\]

\[
R^\mu_s = (\tilde{x}^0; \tilde{R}_s) =
\]

\[
= \tilde{x}^\mu - \epsilon^\mu_s(P) \frac{(\tilde{S} \times \tilde{P})^r}{M u^\alpha(P)[1 + u^\alpha(P)]} \equiv x^\mu_s - \epsilon^\mu_s(u(P)) \sigma^r_R, \quad (15)
\]

and we have

\[
\sigma^r_Y = \epsilon_{r \mu}(u(P)) [x^\mu_s - Y^\mu_s] = \tilde{\sigma}^r + \frac{\tilde{S}^r_s u^s(P)}{1 + u^\alpha(P)} =
\]

\[
= M R^r_s \approx M q^r_s \approx 0,
\]

\[
\Rightarrow \quad x^\mu_s(\tau) = x^\mu(0) + Y^\mu_s(\tau), \text{ when } \tilde{q}_s \approx 0, \quad (16)
\]

As a consequence the gauge fixings \( \tilde{q}_s \approx \tilde{R}_s \approx \tilde{Y}_s \approx 0 \) implies that we can choose the inertial observer \( x^\mu_s \) to coincide with the covariant (non-canonical) external Fokker-Pryce center of inertia by choosing \( x^\mu(0) = 0 \).

Let us remark that in the inertial rest-frame instant form with the gauge fixings \( \tilde{h} \approx 0 \), i.e. \( \tilde{q}_s \approx \tilde{R}_s \approx \tilde{Y}_s \approx 0 \), the only non-zero generators of the

\[ ^9 \text{As shown in Refs.}\ [2, 21] \text{the canonical variables } \tilde{x}^\mu, P^\mu, \text{ can be replaced by the new canonical basis } T_s = u(P) \cdot \tilde{x}, \epsilon_s = \sqrt{P^2}, \tilde{z}^i = \sqrt{P^2} (\tilde{x}^i - \frac{\rho^i}{\rho} \tilde{z}^0), h^i = P^i/\sqrt{P^2} \text{ restricted by the first class constraint } \epsilon_s - M \approx 0. \text{ The gauge fixing } T_s \approx \tau, \epsilon_s \equiv M, \text{ at the level of Dirac brackets. The 3-vector } \tilde{z}/M \text{ is the non-covariant canonical 3-center of mass (the classical counterpart of the ordinary Newton-Wigner position operator), which has 3-velocity } \tilde{h} \text{ (or 3-momentum } M \tilde{h}). \text{ Note that } \tilde{z} \text{ and } \tilde{h} \text{ are non-evolving Jacobi data. In the text we used } \tilde{x}^\mu \text{ and } P^\mu \text{ instead of their expressions in terms of } \tilde{z} \text{ and } \tilde{h} \text{ (} \tilde{x}^\mu = \sqrt{1 + \tilde{h}^2} (\tau + \frac{\tilde{h}^2}{2} \tilde{z}^0), \tilde{z} = \sqrt{1 + \tilde{h}^2} \tilde{h}, P^\mu = M \sqrt{1 + \tilde{h}^2}, \tilde{P} = M \tilde{h} \text{ to simplify the notation. The external Poincare' algebra } \{, \} \text{ is satisfied with the following form of the generators: } P^\mu = M \sqrt{1 + \tilde{h}^2}, \tilde{P} = M \tilde{h}, J^{ij} = z^i \tilde{h}^j - z^j \tilde{h}^i + \tilde{S}^{ij}, \}

K^i = -\sqrt{1 + \tilde{h}^2} z^i + \frac{\tilde{S}^{ij} \tilde{z}_j}{1 + \sqrt{1 + \tilde{h}^2}} \]
internal Poincare’ algebra are $M$ and $\vec{j} = \vec{S}$; they contain all the information about the isolated system and generate the dynamical U(2) algebra of Ref. [29]. As is evident from Eqs. (6) also the external Poincare’ generators depend only on the generators of this U(2) algebra.

Since we are in an instant form of dynamics, in the presence of interactions among the constituents of the isolated system only the internal generators $M$ and $\vec{k}$ will contain the interaction potentials, $M \mapsto M_{(\text{int})}$, $\vec{k} \mapsto \vec{k}_{(\text{int})}$, but only the ones inside $M_{(\text{int})}$ contribute to the U(2) algebra. As shown in the next Section, the potentials inside $\vec{k}_{(\text{int})}$ contribute to the elimination of the internal 3-centers by means of the gauge fixings $\vec{k}_{(\text{int})} \approx 0$.

As anticipated in Section II, in each Lorentz frame one has different pseudo-world-lines describing $R^\mu_s$ and $\tilde{x}^\mu_s$: the canonical 4-center of mass $\tilde{x}^\mu_s$ lies nearer to $Y^\mu_s$ than $R^\mu_s$ in every (non rest-) frame. In an arbitrary Lorentz frame, the pseudo-world-lines associated with $\tilde{x}^\mu_s$ and $R^\mu_s$ fill a Møller world-tube around the world-line $Y^\mu_s$ of the covariant non-canonical Fokker-Pryce 4-center of inertia $Y^\mu_s$. 

18
5 The Canonical Transformation to Relative Variables.

From the previous Sections it is clear that in the rest-frame instant form the N-body problem in the free case is described by the $8 + 6N$ canonical variables $\tilde{x}, P^\mu, \tilde{\eta}_i, \tilde{\kappa}_i, i = 1, \ldots, N$, restricted by the 6 conditions $\tilde{p} \approx 0, \tilde{q}_+ \approx 0$ and with $P^\mu = M u^\mu(P), M = \sum_{i=1}^N \sqrt{m_i^2 + \tilde{\kappa}_i^2}$, and $\tau \equiv T_s = u(P) \cdot \tilde{x}$, so that there are only $6 + 6(N-1)$ independent canonical variables $[\tilde{z}, \tilde{h}, \tilde{\rho}_{qa}, \tilde{\pi}_{qa}, a = 1, \ldots, N-1]$ like in the non-relativistic case (see footnote 9 for the definition of $\tilde{z}$ and $\tilde{h}$).

We have now to find the canonical transformation

$$
\begin{array}{ccc}
\tilde{x}^\mu & \tilde{\eta}_i & \tilde{\kappa}_i \\
\tilde{p} & \tilde{q}_+ & \tilde{\rho}_{qa} & \tilde{\pi}_{qa}
\end{array}
$$

$$
\tilde{p} \approx 0, \quad \tilde{q}_+ \approx 0, \quad (17)
$$

defining the $6(N-1)$ relativistic relative variables $\tilde{\rho}_{qa}, \tilde{\pi}_{qa}, a = 1, \ldots, N-1$, so that the spin (barycentric angular momentum) becomes $\tilde{S}_{q} = \sum_{a=1}^{N-1} \tilde{\rho}_{qa} \times \tilde{\pi}_{qa}$.

Let us stress that this cannot be a point transformation, because of the momentum dependence of the relativistic internal center of mass $\tilde{q}_+$.

Since $\tilde{q}_+$ and $\tilde{p}$ are known from Eqs. (11) and (7) respectively, we have only to find the internal conjugate variables appearing in the canonical transformation (17). They have been determined in Ref. [2] by using the technique (the Gartenhaus-Schwarz transformation) of Ref. [30] and starting from a set of canonical variables defined in Ref. [21]. Then, starting from the naive internal center-of-mass variable $\tilde{\eta}_+ = \frac{1}{N} \sum_{i=1}^N \tilde{\eta}_i$, we defined relative variables $\tilde{\rho}_a, \tilde{\pi}_a$ based on the following family of point canonical transformations

$$
\begin{array}{ccc}
\tilde{\eta}_i & \tilde{\kappa}_i \\
\tilde{\rho}_a & \tilde{\pi}_a
\end{array}
$$

$$
\begin{array}{ccc}
\tilde{q}_+ & \tilde{p} & \tilde{\rho}_{qa} & \tilde{\pi}_{qa}
\end{array}
$$

, \quad a = 1, \ldots, N-1,
\[ \bar{\eta}_+ = \frac{1}{N} \sum_{i=1}^{N} \bar{\eta}_i, \quad \bar{p} = \sum_{i=1}^{N} \bar{\kappa}_i \approx 0, \]

\[ \bar{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \bar{\eta}_i, \quad \bar{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \bar{\kappa}_i, \]

\[ \{ \eta^+_i, \kappa^+_j \} = \delta_{ij} \delta^{rs}, \quad \{ \eta^+_i, p^+_s \} = \delta^{rs}, \quad \{ \rho^+_a, \pi^+_b \} = \delta_{ab} \delta^{rs}, \]

\[ \bar{\eta}_i = \bar{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \gamma_{ai} \bar{\rho}_a, \quad \bar{\kappa}_i = \frac{1}{N} \bar{p} + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \bar{\pi}_a, \]

(18)

In order that the above brackets be satisfied, the numerical parameters \( \gamma_{ai} \) must satisfy the relations \( \sum_{i=1}^{N} \gamma_{ai} = 0, \sum_{i=1}^{N} \gamma_{ai} \gamma_{bi} = \delta_{ab}, \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} = \delta_{ij} - \frac{1}{N} \), which depend on \( \frac{1}{2} (N - 1)(N - 2) \) free parameters.

Then, (in Appendix B of Ref. [31]), we gave the closed form of the canonical transformation (17) for arbitrary \( N \), which turned out to be point in the momenta but, unlike the non-relativistic case, non-point in the configurational variables.

Explicitly, for \( N = 2 \) we have \( [\gamma_{11} = -\gamma_{12} = \frac{1}{\sqrt{2}} \), so that \( \bar{\rho} = \bar{\eta}_1 - \bar{\eta}_2, \) \( \bar{\pi} = \frac{1}{2} (\bar{\kappa}_1 - \bar{\kappa}_2) \)]
The inverse canonical transformation is

\[ M = \sqrt{m_1^2 + \kappa_1^2 + m_2^2 + \kappa_2^2}, \quad \vec{S}_q = \vec{\rho}_q \times \vec{\pi}_q, \]

\[ \vec{q}_+ = \frac{\sqrt{m_1^2 + \kappa_1^2} \vec{\eta}_1 + \sqrt{m_2^2 + \kappa_2^2} \vec{\eta}_2}{\sqrt{M^2 - \vec{p}^2}} + \frac{(\vec{\eta}_1 \times \kappa_1, \vec{\eta}_1 \times \kappa_2) \times \vec{p}}{\sqrt{M^2 - \vec{p}^2} (M + \sqrt{M^2 - \vec{p}^2})} - \frac{\vec{\rho} \cdot \vec{p}}{M \sqrt{M^2 - \vec{p}^2}} (M + \sqrt{M^2 - \vec{p}^2}), \]

\[ \vec{p} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0, \]

\[ \vec{\pi}_q = \vec{\pi} - \frac{\vec{p}}{\sqrt{M^2 - \vec{p}^2}} \left[ \frac{1}{2} \left( \sqrt{m_1^2 + \kappa_1^2} - \sqrt{m_2^2 + \kappa_2^2} \right) - \frac{\vec{\rho} \cdot \vec{p}}{\sqrt{M^2 - \vec{p}^2}} (M - \sqrt{M^2 - \vec{p}^2}) \right] \approx \vec{\pi} = \frac{1}{2} (\vec{\kappa}_1 - \vec{\kappa}_2), \]

\[ \vec{\rho}_q = \vec{\rho} + \left( \frac{\sqrt{m_1^2 + \kappa_1^2}}{\sqrt{m_1^2 + \kappa_1^2}} \right) \frac{\vec{\rho} \cdot \vec{p}}{M \sqrt{M^2 - \vec{p}^2}} \approx \vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2, \]

\[ \Rightarrow M = \sqrt{M^2 + \vec{p}^2} \approx M = \sqrt{m_1^2 + \vec{\pi}_1^2 + \sqrt{m_2^2 + \vec{\pi}_2^2}}, \]

\[ \vec{q}_+ \approx \vec{R}_+ \approx \vec{\eta}_1 \sqrt{m_1^2 + \vec{\pi}_1^2} + \vec{\eta}_2 \sqrt{m_2^2 + \vec{\pi}_2^2}. \]
In Eqs. (20) we used explicitly the gauge fixing \( \vec{q}_+ \approx 0 \).

As shown in Refs. [32], [21] and their bibliography, a-a-a-d interactions inside the Wigner hyperplane may be introduced either under (scalar and vector potentials) or outside (scalar potential like the Coulomb one) the square roots appearing in the free Hamiltonian. Since a Lagrangian density in presence of action-at-a-distance mutual interactions is not known and since we are working in an instant form of dynamics, the potentials in the constraints restricted to hyper-planes must be introduced \textit{by hand} [see, however, Ref. [27] for their evaluation starting from the Lagrangian density for the electro-magnetic interaction]. The only restriction is that the Poisson brackets of the modified constraints must generate the same algebra of the free ones.

In the rest-frame instant form the most general Hamiltonian with action-at-a-distance interactions is

\[
M_{(int)} = \sum_{i=1}^{N} \sqrt{m_i^2 + U_i + [\vec{k}_i - \vec{V}]^2} + V, \tag{21}
\]

where

\[
U = U(\vec{k}, \vec{\eta} - \vec{\eta}_j \neq k), \quad V = V_o(|\vec{\eta} - \vec{\eta}_j|) + V' (\vec{k}, \vec{\eta} - \vec{\eta}_j).
\]

If we use the canonical transformation [17] defining the relativistic canonical internal 3-center of mass (now it is interaction-dependent, \( \vec{q}_{(int)} \)) and relative variables on the Wigner hyperplane, with the rest-frame conditions \( \vec{p} \approx 0 \), the rest frame Hamiltonian for the relative motion becomes

\[
M_{(int)} \approx \sum_{i=1}^{N} \sqrt{m_i^2 + \tilde{U}_i + [\sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_{qa} - \tilde{V}_i]^2 + \tilde{V}}, \tag{22}
\]

where

\[
\begin{align*}
\tilde{U}_i &= U(\sqrt{N} \sum_{a=1}^{N-1} \gamma_{ak} \vec{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ah} - \gamma_{ah}) \vec{\rho}_{qa}), \\
\tilde{V}_i &= \tilde{V}_i(\sqrt{N} \sum_{a=1}^{N-1} \gamma_{aj} \vec{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \vec{\rho}_{qa}), \\
\tilde{V} &= V_o(\frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \vec{\rho}_{qa}) + V' (\sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \vec{\rho}_{qa}).
\end{align*}
\]

(23)

In order to build a realization of the internal Poincare’ group, besides \( M_{(int)} \) we need to know the potentials appearing in the internal boosts \( \vec{k}_{(int)} \) (being an instant form, \( \vec{p} \approx 0 \) and \( j \) are the free ones).
Since the 3-centers $\vec{R}_+\approx \vec{q}_+$ become interaction dependent, the final canonical basis $\vec{q}_+, \vec{p}, \vec{\rho}_{qa}, \vec{\pi}_{qa}$ is not explicitly known in the interacting case. For an isolated system, however, we have $M = \sqrt{\mathcal{M}^2 + \vec{p}\,^2} \approx \mathcal{M}$ with $\mathcal{M}$ independent of $\vec{q}_+$ (or $\vec{p}$ in the internal Poincare' algebra). This suggests that the same result should hold true even in the interacting case. Indeed, by its definition, the Gartenhaus-Schwartz transformation [30], [2] gives $\vec{\rho}_{qa} \approx \vec{\rho}_a$, $\vec{\pi}_{qa} \approx \vec{\pi}_a$ also in presence of interactions, so that we get

$$M_{(int)}|\vec{p}=0 = \left( \sum_i \sqrt{m_i^2 + U_i + (\vec{\kappa}_i - \vec{V}_i)^2 + V_i} \right)|\vec{p}=0 = \sqrt{\mathcal{M}_{(int)}^2 + \vec{p}\,^2}|\vec{p}=0 =$$

$$= \mathcal{M}_{(int)}|\vec{p}=0 = \sum_i \sqrt{m_i^2 + \tilde{U}_i + (\tilde{\kappa}_i - \tilde{V}_i)^2 + \tilde{V}},$$

where the potentials $\tilde{U}_i, \tilde{V}_i, \tilde{V}$ are now functions of $\tilde{\pi}_{qa} \cdot \tilde{\pi}_{qb}, \tilde{\pi}_{qa} \cdot \tilde{\rho}_{qb}, \tilde{\rho}_{qa} \cdot \tilde{\rho}_{qb}$.

Unlike in the non-relativistic case, the canonical transformation [19] is now interaction dependent (not even a point transformation in the momenta), since $\vec{q}_+$ is determined by a set of Poincare' generators depending on the interactions. The only thing to do in the generic situation is therefore to use the free relative variables [19] even in the interacting case. We cannot impose anymore, however, the natural gauge fixings $\vec{q}_+ \approx 0 (k \approx 0)$ of the free case, since it is replaced by $\vec{q}_{+int} \approx 0$ (namely by $\vec{k}_{(int)} \approx 0$), the only gauge fixing identifying the centroid with the external Fokker-Pryce 4-center of inertia also in the interacting case. Once written in terms of the canonical variables [19] of the free case, the equations $\vec{k}_{(int)} \approx 0$ can be solved for $\vec{q}_{+}$, which takes a form $\vec{q}_{+} \approx \vec{f}(\vec{\rho}_{aq}, \vec{\pi}_{aq})$ as a consequence of the potentials appearing in the boosts. Therefore, for $N = 2$, the reconstruction of the relativistic orbit by means of Eqs.(20) in terms of the relative motion is given by (similar equations hold for arbitrary $N; \vec{\rho}_{aq} \approx \vec{\rho}_a$, $\vec{\pi}_{aq} \approx \vec{\pi}_a$)

$$\vec{n}_i(\tau) \approx \vec{q}_+(\vec{\rho}_q, \vec{\pi}_q) + \frac{1}{2} \left[ (-)^{i+1} - \frac{m_1^2 - m_2^2}{M^2} \right] \vec{\rho}_q \rightarrow c \rightarrow \infty \frac{1}{2} \left[ (-)^{i+1} - \frac{m_1 - m_2}{m} \right] \vec{\rho}_q,$$

$$\vec{\kappa}_i(\tau) \approx (-)^{i+1} \vec{\pi}_q(\tau),$$

$$\downarrow$$

$$x_i^\mu(\tau) = z_{wigner}^\mu(\tau, \vec{n}_i(\tau)) = u^\mu(P) \tau + e^\mu(u(P)) \eta_i^\mu(\tau),$$

$$p_i^\mu(\tau) = \sqrt{m_i^2 + \vec{n}_i^2(\tau) u^\mu(P) + e^\mu(u(P)) \kappa_i(\tau)}.$$  

While the potentials in $M_{(int)}$ determine $\vec{\rho}_q(\tau)$ and $\vec{\pi}_q(\tau)$ through Hamilton equations, the potentials in $\vec{k}_{(int)}$ determine $\vec{q}_+(\vec{\rho}_q, \vec{\pi}_q)$. It is seen, therefore -
as it should be expected - that the relativistic theory of orbits is much more complicated than in the non-relativistic case, where the absolute orbits $\vec{\eta}_i(t)$ are proportional to the relative orbit $\vec{\rho}_q(t)$ in the rest frame.
A Simple 2-Particle Model with a-a-a-d Interaction.

Instead of the physically more relevant but complicated system of Ref. [27], whose internal Hamiltonian and boosts in the rest-frame instant form are given in Eq. (9), let us study a simple two-body system with an a-a-a-d interaction, defined at the end of Appendix A in terms of two first class constraints. As we shall see its treatment in the constraint formalism leads to a realization of the Poincaré algebra only in the rest frame. Therefore, let us look at its reformulation in the rest-frame instant form, where the rest-frame conditions are automatically contained.

In the rest-frame instant form we may define the model by making the ansatz that the generators of the internal realization of the Poincaré algebra have the form [we use \( \vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2 \) of Eqs. (19)]

\[
M_{(\text{int})} = \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{\rho}^2)} + \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{\rho}^2)}, \\
\vec{\rho} = \kappa_1 + \kappa_2, \\
\vec{j} = \vec{\eta}_1 \times \kappa_1 + \vec{\eta}_2 \times \kappa_2, \\
\vec{k}_{(\text{int})} = -\vec{\eta}_1 \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{\rho}^2)} - \vec{\eta}_2 \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{\rho}^2)}. 
\] (26)

Let us verify this ansatz by checking whether these generators satisfy the Poincaré algebra.

It is self evident that one has

\[
[j_i, j_j] = \varepsilon_{ijk} j_k, \quad [j_i, k_{(\text{int})j}] = \varepsilon_{ijk} k_{(\text{int})k}. 
\] (27)

We examine the other Poincaré brackets. First note that

\[
[p_i, k_{(\text{int})j}] = [\kappa_{1i} + \kappa_{2j}, -\eta_{1j} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{\rho}^2)} - \eta_{2j} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{\rho}^2)}] = \\
= \delta_{ij} [\sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{\rho}^2)} + \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{\rho}^2)}] = \\
= \delta_{ij} M_{(\text{int})}. 
\] (28)

Next, examine \([\Phi'(x) = \frac{d\Phi(x)}{dx}]\)
Again the rest-frame condition implies
\[ \vec{p} \approx 0 \]
implies
\[ [M_{(int)}, k_{(int)i}] \approx 0 \approx \vec{p}. \quad (30) \]

The remaining crucial bracket is
\[
[k_{(int)i}, k_{(int)j}] = \frac{[-\eta_{1i} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)} - \eta_{2i} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)} - \eta_{1j} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)} - \eta_{2j} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}]}{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2) - m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)} =
\]
\[
= [\eta_{1i} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)}, \eta_{1j} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)}] +
+ [\eta_{2i} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}, \eta_{2j} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}] +
+ [\eta_{1i} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)}, \eta_{2j} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}] +
+ [\eta_{2i} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}, \eta_{1j} \sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)}] =
\]
\[
\eta_{1j} \kappa_{1i} - \eta_{1i} \kappa_{1j} + \eta_{2j} \kappa_{2i} - \eta_{2i} \kappa_{2j} -
- \eta_{1i} \eta_{2j} \frac{2 \Phi'(\vec{p}^2) \vec{\rho} \cdot (\kappa_1 + \kappa_2)}{\sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}} +
+ \eta_{2i} \eta_{1j} \frac{2 \Phi'(\vec{p}^2) \vec{\rho} \cdot (\kappa_1 + \kappa_2)}{\sqrt{m_1^2 + \kappa_1^2 + \Phi(\vec{p}^2)} \sqrt{m_2^2 + \kappa_2^2 + \Phi(\vec{p}^2)}}. \quad (31)
\]

Again the rest-frame condition implies
\[
[k_{(int)i}, k_{(int)j}] \approx \eta_{1j} \kappa_{1i} - \eta_{1i} \kappa_{1j} + \eta_{2j} \kappa_{2i} - \eta_{2i} \kappa_{2j} =
\]
\[
-\varepsilon_{ijk} \varepsilon_{klm} (\eta_{1l} \kappa_{1m} + \eta_{2l} \kappa_{2m}) = -\varepsilon_{ijk} j_k. \quad (32)
\]
From Eq. (10) the interacting form of the canonical internal 3-center of mass is weakly equal to the 3-center of energy due to the rest frame condition $\vec{p} \approx 0$

$$\vec{q}_+ \approx \vec{R}_+ = -\frac{\vec{k}_{(int)}}{M_{(int)}}, \quad (33)$$

Eqs. (19), (20) (as well as Eq. (30) in Appendix A) imply the following forms of $\vec{k}_{(int)}$ and $M_{(int)}$

$$\vec{k}_{(int)} \approx -\vec{\eta}_1 \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\vec{\rho}^2)} - \vec{\eta}_2 \sqrt{m_2^2 + \vec{\pi}^2 + \Phi(\vec{\rho}^2)},$$

$$M_{(int)} \approx \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\vec{\rho}^2)} + \sqrt{m_2^2 + \vec{\pi}^2 + \Phi(\vec{\rho}^2)} = M_{(int)}. \quad (34)$$

In the free case, with $M = \sqrt{m_1^2 + \vec{\pi}^2} + \sqrt{m_2^2 + \vec{\pi}^2}$, Eqs. (20) imply

$$\vec{\eta}_1 \approx \vec{q}_+ + \frac{1}{2} (1 - \frac{m_1^2 - m_2^2}{M^2}) \vec{\rho} \approx \frac{1}{2} (1 - \frac{m_1^2 - m_2^2}{M^2}) \vec{\rho},$$

$$\vec{\eta}_2 \approx \vec{q}_+ - \frac{1}{2} (1 + \frac{m_1^2 - m_2^2}{M^2}) \vec{\rho} \approx -\frac{1}{2} (1 + \frac{m_1^2 - m_2^2}{M^2}) \vec{\rho}. \quad (35)$$

and also

$$\sqrt{m_1^2 + \vec{\pi}^2} = \frac{1}{2} (M + \Delta) = \frac{M}{2} (1 + \frac{m_1^2 - m_2^2}{M^2}),$$

$$\sqrt{m_2^2 + \vec{\pi}^2} = \frac{1}{2} (M - \Delta) = \frac{M}{2} (1 - \frac{m_1^2 - m_2^2}{M^2}), \quad (36)$$

where $\Delta = \sqrt{m_1^2 + \vec{\pi}^2} - \sqrt{m_2^2 - \vec{\pi}^2}, \quad M \Delta = m_1^2 - m_2^2$.

Therefore in the free case we have the following expression for the 3-coordinates $\vec{\eta}_i$

$$\vec{\eta}_1 \approx \vec{q}_+ + \frac{1}{2} \left( 1 - \frac{m_1^2 - m_2^2}{M^2} \right) \vec{\rho} = \vec{q}_+ + \frac{\sqrt{m_1^2 + \vec{\pi}^2}}{M} \vec{\rho},$$

$$\vec{\eta}_2 \approx \vec{q}_+ - \frac{1}{2} \left( 1 + \frac{m_1^2 - m_2^2}{M^2} \right) \vec{\rho} = \vec{q}_+ - \frac{\sqrt{m_1^2 + \vec{\pi}^2}}{M} \vec{\rho}. \quad (37)$$

If we use the canonical basis $\vec{q}_+, \vec{p}, \vec{\rho}, \vec{\pi}$ of the free case also in our simple interacting case, Eqs. (37) must be replaced by Eqs. (25). To this end we must find the functions $\vec{q}_+(\vec{\rho}, \vec{\pi})$ from the gauge conditions $\vec{q}_+^{(int)} \approx \vec{R}_+^{(int)} \approx 0$, namely from $\vec{k}_{(int)} \approx 0$.

In our simple interacting case, from Eq. (19), (36) and (37) we get
− \mathcal{M}_{(int)} \tilde{q}_{+}^{(int)} \approx \tilde{k}_{(int)} \approx
\approx -\bar{q}_+ \left[ \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} + \sqrt{m_2^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} \right] - 2 \bar{\rho} \sqrt{m_2^2 + \vec{\pi}^2} \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} - \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} \sqrt{m_2^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)}.

(38)

If, in analogy to the free case, we define \( \Delta_{(int)} = \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} - \sqrt{m_2^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} \), we have \( \mathcal{M}_{(int)} \Delta_{(int)} = m_1^2 - m_2^2 \) and we get

\[
\sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} = \frac{1}{2} (\mathcal{M}_{(int)} + \Delta_{(int)}) = \frac{\mathcal{M}_{(int)}}{2} (1 + \frac{m_1^2 - m_2^2}{M_{(int)}^2}),
\]

\[
\sqrt{m_2^2 + \vec{\pi}^2 + \Phi(\bar{\rho}^2)} = \frac{1}{2} (\mathcal{M}_{(int)} - \Delta_{(int)}) = \frac{\mathcal{M}_{(int)}}{2} (1 - \frac{m_1^2 - m_2^2}{M_{(int)}^2}).
\]

\[
\Rightarrow \mathcal{M} = \sum_{i=1}^{2} \sqrt{m_i^2 + \vec{\pi}^2} = \sqrt{\left[ \mathcal{M}_{(int)} \frac{1}{2} (1 + \frac{m_1^2 - m_2^2}{M_{(int)}^2}) \right]^2 - \Phi(\bar{\rho}^2) + \sqrt{\left[ \mathcal{M}_{(int)} \frac{1}{2} (1 - \frac{m_1^2 - m_2^2}{M_{(int)}^2}) \right]^2 - \Phi(\bar{\rho}^2)}.
\]

(39)

As a consequence, Eq. (38) may be written in the form

\[
\tilde{k}_{(int)} \approx -\mathcal{M}_{(int)} \left( \tilde{q}_+ + \frac{2 \mathcal{M}_{(int)} \frac{m_1^2 - m_2^2}{2 M^2} - \bar{\rho}^2 \frac{m_1^2 - m_2^2}{2 \mathcal{M}_{(int)}^2}}{2 \mathcal{M}} \right) =
\approx -\mathcal{M}_{(int)} \tilde{q}_+ + \bar{\rho} \mathcal{M}_{(int)} \frac{m_1^2 - m_2^2}{2 M^2} - \bar{\rho}^2 \frac{m_1^2 - m_2^2}{2 \mathcal{M}_{(int)}^2} \bar{\rho} =
\issue{40}
\]

Therefore the gauge fixing condition \( \tilde{q}_+^{(int)} \approx 0 \), or \( \tilde{k}_{(int)} \approx 0 \), gives

\[
\tilde{q}_+ \approx \tilde{q}_+ (\bar{\rho}, \vec{\pi}) = \frac{m_1^2 - m_2^2}{2} \left( \frac{1}{M^2} - \frac{1}{M_{(int)}^2} \right) \bar{\rho}.
\]

(41)
so that, by using the inverse canonical transformation (20), in our simple interacting case we get the following reconstruction of the 3-coordinates \( \vec{\eta}_i \) and of the 4-coordinates \( x^\mu_i \) for our simple interacting relativistic model we get that the rest-frame 3-coordinates \( \vec{\eta}_i \) are still proportional to the relative variable \( \vec{\rho} \) [for more complex models there could be a component along \( \vec{\pi} \) coming from the function \( q_+ (\vec{\rho}, \vec{\pi}) \)]. However, instead of the numerical proportionality constants of the non-relativistic case, we have a non-trivial dependence on the total constant fixed c.m. energy (\( \mathcal{M}_{(int)} = \sum_{i=1}^{2} \sqrt{m_1^2 + \vec{\pi}^2 + \Phi(\vec{\rho}^2)} \)).
7 Evaluation of the Orbits in the Simple Relativistic Two-Body Problem with a Coulomb-like Potential.

For illustrative purposes, we make the following choice for the a-a-a-d potential \( \Phi \)

\[
\Phi(\vec{\rho}) = -2 \mu \frac{e^2}{\rho}, \quad \rho = \sqrt{\vec{\rho}^2}.
\]

(44)

This Coulomb-like potential is not to be confused with the real Coulomb potential between charged particles, which is outside the square roots as shown in Eq.(9) and which produces completely different relativistic orbits. However both models may have the same non-relativistic limit for suitable choices of the parameters.

The invariant mass \( M_{(int)} \) of the two-body model (the Hamiltonian of its relative motion) in the rest-frame instant form is

\[
M_{(int)} \approx M_{(int)} = \sqrt{m_1^2 + \pi^2 - 2 \mu \frac{e^2}{\rho}} + \sqrt{m_2^2 + \pi^2 - 2 \mu \frac{e^2}{\rho}}.
\]

(45)

Instead of studying the Hamilton equations for \( \vec{\rho}, \pi \) with \( M_{(int)} \) as Hamiltonian, we will find the orbits using Hamilton-Jacobi methods. Since the potential is a central one, our orbit is confined to a plane with

\[
\pi^2 = \pi_\rho^2 + \pi_\phi^2.
\]

(46)

Since both the time and the angle are cyclic, the generating function is

\[
S = W_1(\rho) + \alpha_\phi \phi - w t \equiv W_1(\rho, \phi) - w t,
\]

(47)

with \( w \) the invariant total c.m. energy. The Hamilton-Jacobi equation is

\[
\sqrt{(\frac{\partial W_1}{\partial \rho})^2 + \frac{\alpha_\phi^2}{\rho^2} + m_1^2 - 2 \mu \frac{e^2}{\rho}} + \sqrt{(\frac{\partial W_1}{\partial \rho})^2 + \frac{\alpha_\phi^2}{\rho^2} + m_2^2 - 2 \mu \frac{e^2}{\rho}} = w,
\]

(48)

This leads to [the function \( b^2(w) \) is defined in Eq. (83) of Appendix A]

\[
(\frac{\partial W_1}{\partial \rho})^2 + \frac{\alpha_\phi^2}{\rho^2} - 2 \mu \frac{e^2}{\rho} = b^2(w),
\]

(49)

\[\text{See Refs.} [33] \text{ for the relativistic Kepler or Coulomb problem with respect to a fixed center and Refs.} [34] \text{ for its use. Let us remark that the techniques of Refs.} [33] \text{ could be applied to the results of Ref.} [27] \text{ to describe the relativistic Darwin two-body problem. Instead it is not known the internal boost} \ \vec{k} \text{ satisfying} \ \{p_i, k_j\} = \delta_{ij} M, \text{ in the case of the pure Coulomb interaction:} \ M = \sum_{i=1}^{2} \sqrt{m_i^2 + \vec{p}_i^2 + \frac{Q_1 Q_2}{4 \pi \epsilon_0 |\vec{r}_1 - \vec{r}_2|}}.\]
and so we get
\[ W_1(\rho, \phi) = \int d\rho \sqrt{b^2(w) - \frac{\alpha_\phi^2}{\rho^2} + 2 \mu \frac{e^2}{\rho} + \alpha_\phi \phi}. \] (50)

The new coordinate canonically conjugate to the new momentum \(\alpha_\phi\) is the constant
\[ \beta_2 = \frac{\partial W}{\partial \alpha_\phi} = -\int \frac{\alpha_\phi d\rho}{\rho^2 \sqrt{b^2(w) - \frac{\alpha_\phi^2}{\rho^2} + 2 \mu \frac{e^2}{\rho}}} + \phi. \] (51)

If we define
\[ u = \frac{1}{\rho}, \] (52)
we get
\[ \beta_2 = \frac{\partial W}{\partial \alpha_\phi} = \int \frac{\alpha_\phi du}{\sqrt{b^2(w) - \alpha_\phi^2 u^2 + 2 \mu e^2 u}} + \phi, \] (53)
or
\[ \phi = \beta_2 = \int \frac{du}{\sqrt{\frac{b^2(w) - \alpha_\phi^2 u^2}{\alpha_\phi^2} + 2 \mu e^2 u - u^2}}. \] (54)

This result leads to the ellipse (we consider only bounded orbits)
\[ \frac{1}{\rho} = \frac{\mu e^2}{\alpha_\phi^2} [1 + \sqrt{1 + \frac{b^2(w) \alpha_\phi^2}{\mu^2 e^2} \cos(\phi - \beta_2)}]. \] (55)

Eqs. (54) and (55) allow us to determine the orbit of the relative motion
\[ \vec{\rho} = \rho (\cos \phi \hat{i} + \sin \phi \hat{j}). \] (56)

Let us compare these results with the non-relativistic limit. In the non-relativistic Kepler or Coulomb case [1], Eq. (55) is replaced by the following expression (in the non-relativistic limit we have \(b^2(w) \to 2\mu E\) with the energy \(E = w - m_1 - m_2\))
\[ \frac{1}{\rho} = \frac{\mu e^2}{\alpha_\phi^2} [1 + \sqrt{1 + \frac{2E \alpha_\phi^2}{\mu^2 e^2} \cos(\phi - \beta_2)}]. \] (57)

If we use the non-relativistic limit into Eqs. (42) for the relation among \(\vec{\eta}_i\) and \(\vec{\rho}\), we get the non-relativistic expressions
\[ \vec{\eta}_1 = \frac{\alpha_2}{m_1 e^2} \frac{1}{[1 + \sqrt{1 + \frac{2 E}{\mu e^2}} \cos(\phi - \beta_2)]} (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}), \]
\[ \vec{\eta}_2 = -\frac{\alpha_2}{m_2 e^2} \frac{1}{[1 + \sqrt{1 + \frac{2 E}{\mu e^2}} \cos(\phi - \beta_2)]} (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}). \]  

For the relativistic counterparts, given in Eq. (42), we have from Eqs. (39)

\[ w = \mathcal{M}(\text{int}) = \sum_{i=1}^{2} \sqrt{m_i^2 + \vec{p}^2 - 2\mu \frac{e^2}{\rho}}, \]
\[ \mathcal{M} = \sqrt{\left[ \frac{w}{2} \left( 1 + \frac{m_1^2 - m_2^2}{w^2} \right) \right]^2 + 2\mu \frac{e^2}{\rho} + \sqrt{\left[ \frac{w}{2} \left( 1 - \frac{m_1^2 - m_2^2}{w^2} \right) \right]^2 + 2\mu \frac{e^2}{\rho}}} = \mathcal{M}(w, \rho). \]  

In this case, from Eqs. (42) we have

\[ \vec{\eta}_1 \approx \frac{1}{2} \rho (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}) [1 - \frac{m_1^2 - m_2^2}{w^2}], \]
\[ \vec{\eta}_2 \approx -\frac{1}{2} \rho (\cos \phi \mathbf{i} + \sin \phi \mathbf{j}) [1 + \frac{m_1^2 - m_2^2}{w^2}], \]  

where for \( \rho \) we have to use the solution given in Eq. (55).

We have the following situation:

1) For equal masses, the relativistic and non-relativistic expressions are identical with \( b^2(\mathcal{M}(\text{int})) \rightarrow 2\mu E \).

2) In the limit in which one of the masses becomes very great (say \( m_2 \)) then, since we have

\[ \frac{m_1^2 - m_2^2}{w^2} \rightarrow_{m_2 \rightarrow \infty} -1, \]  

the relativistic and non-relativistic expressions are identical also.

3) If we introduce the new notation

\[ w = \frac{m_1 + m_2}{\sqrt{\omega}}, \]  

then the relativistic orbits become

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\[ \vec{\eta}_1 = \frac{1}{2} \rho(\phi) (\cos \phi \, \mathbf{i} + \sin \phi \, \mathbf{j}) \left[ 1 - \frac{m_1 - m_2}{m_1 + m_2} \omega \right], \]
\[ \vec{\eta}_2 = -\frac{1}{2} \rho(\phi) (\cos \phi \, \mathbf{i} + \sin \phi \, \mathbf{j}) \left[ 1 + \frac{m_1 - m_2}{m_1 + m_2} \omega \right]. \] (63)

Since \( \omega \) is a constant of motion, the main difference between the relativistic and the non-relativistic orbits is the proportionality constant between individual particle coordinates and the relative coordinate, which, however, is now dependent on the invariant mass of the system.
8 Conclusions

We have given a complete treatment of the Hamiltonian two-body problem in the rest-frame instant form, arising from parametrized Minkowski theories when the dynamics is described with respect to the inertial intrinsic rest frame of the isolated system with its simultaneity 3-surfaces given by the Wigner hyper-planes. The existence of two realizations of the Poincare’ group (the external one and the unfaithful internal one inside the Wigner hyper-planes), together with the clarification of the only three intrinsic notions of a center-of-mass-like collective variable allow us to solve all the kinematical problems and to define canonical transformations for the separation of the center of mass from the relative motion as is possible in Newtonian mechanics.

In the rest-frame instant form of dynamics there is a natural gauge fixing $\vec{k}(\text{int}) \approx 0$ to the rest-frame conditions $\vec{p} \approx 0$, which allows us to clarify completely the determination of the relativistic orbits inside the Wigner hyper-planes. With this gauge fixing it is possible to describe the isolated system from the point of view of an inertial observer, whose world-line $x_{\mu}^{\phi}(\tau) = Y_{\mu}(\tau) = u_{\mu}(P) \tau$ is the (covariant non-canonical) Fokker-Pryce center of inertia and the only non-vanishing generators of the internal Poincare’ algebra are the interaction-dependent invariant mass $M$ and the interaction-independent spin $\vec{S}$. For every isolated system there is the universal realization of the external Poincare’ algebra, whose generators depend upon the external canonical 3-center-of-mass variables $\vec{z}$, $\vec{h}$ and upon the system, but only through its invariant mass $M$ and its spin $\vec{S}$. The simplest model with a-a-a-d interaction is studied in detail.

To reconstruct the actual trajectories in Minkowski space-time in the above inertial frame, we have to use Eqs. (6)

$$
x_{\mu}^{\phi}(\tau) = u_{\mu}(P) \tau + c_{\mu}^{s}(P) \eta_{i}(\tau),
$$

$$
p_{\mu}^{\phi}(\tau) = \sqrt{p_{\mu}^{2}(\tau) + \kappa_{ir}(\tau)} = \sqrt{m_{1}^{2} + R_{\mu}(\tau) u_{\mu}(P) + c_{\mu}^{s}(P) \kappa_{ir}(\tau)}. \tag{64}
$$

To eliminate the momenta and to get a purely configurational description one should invert the first half of the Hamilton equations, $\dot{\vec{p}} = \{\vec{p}, \mathcal{H}_{\text{int}}\}$, to get $\vec{\pi}$ in terms of $\vec{\rho}$ and $\dot{\vec{\rho}} (\dot{f} = \frac{df}{d\tau})$.

Under Lorentz transformations $\Lambda$ generated by the external Poincare’ group, under which we have $c_{\mu}^{s}(u(\Lambda P)) = (R^{-1}(\Lambda, P))_{\sigma}^{\mu} \Lambda_{\mu}^{\nu} c_{\nu}^{s}(u(P))$ and $\eta_{i}^{\tau} = R_{s}(\Lambda, P) \eta_{i}^{\nu}$ (see footnote 6), the derived quantities $x_{\mu}^{\phi}$ and $p_{\mu}^{\phi}$ transform co-

\footnote{See footnote 9. Even if $\vec{z}$ is not covariant due to the no-interaction theorem, there is no problem, because it describes a globally decoupled pseudo-particle: all physical results are described by the Wigner-covariant relative variables inside the Wigner hyper-planes.}

34
variantly as 4-vectors. However the world-lines $x_\mu^i(\tau)$ are no more canonical variables, because they depend on the (non-canonical) Fokker-Pryce center of inertia. This, together with the non-covariance of the canonical center of mass $\tilde{x}$, is the way out from the no-interaction theorem in the rest-frame instant form.

Once the world-lines $x_\mu^i(\tau)$ are known in terms of the rest-frame time $\tau = T_s = u(P) \cdot x_s = u(P) \cdot Y = u(P) \cdot \tilde{x}$, one can geometrically introduce different affine parameters $\tau_i$ on each world-line. Then as shown in Ref.13 one could arrive at the result $x_\mu^i(\tau) = \tilde{x}_\mu(\tau_1, \tau_2) = q_\mu(\tau_i)$, where $q_\mu(\tau_i)$ are the Droz-Vincent predictive covariant coordinates quoted in point b) of Section II and obtained from the equations given in Appendix A by taking $x_\mu^i = q_\mu^i$ as a Cauchy condition on a Wigner hyper-plane. These predictive coordinates implement the geometrical idea that each world-line may be reparametrized independently from the other even in presence of interactions.

Having understood both the kinematical and dynamical problems of relativistic orbit theory, the next step is to try to define a perturbation theory around relativistic orbits as has been done in the non-relativistic case: it could be relevant for the special relativistic approximation of relativistic binaries in general relativity, till now treated only in the post-Newtonian approximation.

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12 Namely the rest-frame instant form satisfies the world-line condition, since its synchronization of clocks (the one-to-one correlation between the world-lines) is a generalization of the gauge fixing $P \cdot (x_1 - x_2) \approx 0$ in models with second class constraints, as shown in Ref.13. As a consequence the world-lines have an objective existence. However, in parametrized Minkowski theories one could choose different 3+1 splittings of Minkowski space-times corresponding to different one-to-one correlations (different conventions for the synchronization of clocks). Since each 3+1 splitting is equivalent to a different choice of non-inertial frame with its inertial forces (see Appendix B for the non-inertial rest frames), the new world-lines will be different (they are obtained from those in the rest-frame instant form by means of a gauge transformation sending an inertial frame into a non-inertial one). This is the interpretation of the so-called frame dependence of the world-lines quoted in Ref.13, where it was connected to a semantic problem.

35
A Two-Body Relativistic Hamiltonian Mechanics with Two First-Class Constraints.

In constraint dynamics for classical relativistic spinless particles one begins by introducing compatible generalized mass shell constraints. We work with constraints that involve potentials that are independent of the relative momenta \((P^\mu = p_1^\mu + p_2^\mu, M = \sqrt{P^2}, r^\mu = x_1^\mu - x_2^\mu, \{x_i^\mu, p_j^\nu\} = -\delta_{ij} \eta^{\mu\nu})\)

\[
\begin{align*}
\mathcal{H}_1 &= p_1^2 - m_1^2 - \Phi_1(r, P) \approx 0, \\
\mathcal{H}_2 &= p_2^2 - m_2^2 - \Phi_2(r, P) \approx 0.
\end{align*}
\] (65)

We call the scalars \(\Phi_i(r, P)\) quasi-potentials (energy dependent potentials that describe deviations from the free mass shell conditions \(p_i^2 - m_i^2 \approx 0\)).

Assuming that the model derives from an unknown reparametrization invariant Lagrangian (so that the canonical Hamiltonian vanishes), the Hamiltonian is defined in terms of the constraints only by

\[
\mathcal{H} = \lambda_1(\tau) \mathcal{H}_1 + \lambda_2(\tau) \mathcal{H}_2.
\] (66)

in which \(\lambda_i\) are arbitrary Lagrange multipliers (called Dirac multipliers). The constraints forming this Hamiltonian must be such that the time rate of change of the constraints vanishes when the constraint are imposed. With the time rate of change of an arbitrary dynamical variable \(f\) given by

\[
\frac{df}{d\tau} = \{f, \mathcal{H}\},
\] (67)

we get \[13\]

\[
\frac{d\mathcal{H}_1}{d\tau} = \{\mathcal{H}_1, \mathcal{H}\} = \lambda_1(\tau) \{\mathcal{H}_1, \mathcal{H}_1\} + \lambda_2(\tau) \{\mathcal{H}_1, \mathcal{H}_2\} \approx \lambda_2(\tau) \{\mathcal{H}_1, \mathcal{H}_2\},
\] (68)

and similarly

\[
\frac{d\mathcal{H}_2}{d\tau} \approx \lambda_1(\tau) \{\mathcal{H}_2, \mathcal{H}_1\}.
\] (69)

Thus, the constraints are constants (their time derivatives weakly vanish) provided that

\[
\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0.
\] (70)

This is called the compatibility condition. Thus, we must have

\[13\] The Dirac multipliers, being functions only of \(\tau\), have zero Poisson bracket with phase space functions.
\[ 2 \mathbf{p}_1 \cdot \{ \mathbf{p}_1, \Phi_2 \} + 2 \mathbf{p}_2 \cdot \{ \Phi_1, \mathbf{p}_2 \} + \{ \Phi_1, \Phi_2 \} \approx 0. \]  

(71)

We assume that the scalar functions \( \Phi_i \) depend on the following variables

\[ \Phi_i = \Phi_i\left(\frac{r^2}{2}, \frac{r^2}{2}, M\right), \]  

(72)

where

\[ r^\mu = \frac{\mathbf{r} \cdot P^\mu}{\mu^2}, \quad r^\mu_\perp = r^\mu - r^\mu_\parallel, \quad P \cdot r_\perp = 0. \]  

(73)

Thus, our compatibility condition becomes

\[ -4 \mathbf{p}_1 \cdot r_\perp \frac{\partial \Phi_2}{\partial r_\perp^2} - 4 \mathbf{p}_1 \cdot r_\parallel \frac{\partial \Phi_2}{\partial r_\parallel^2} - 4 \mathbf{p}_2 \cdot r_\perp \frac{\partial \Phi_1}{\partial r_\perp^2} - 4 \mathbf{p}_2 \cdot r_\parallel \frac{\partial \Phi_1}{\partial r_\parallel^2} + \{ \Phi_1, \Phi_2 \} \approx 0. \]  

(74)

The simplest solution is

\[ \Phi_1 = \Phi_2 = \Phi\left(\frac{r^2}{2}, M\right), \]  

(75)

because it implies the following strong satisfaction of the compatibility condition

\[ \{ \mathcal{H}_1, \mathcal{H}_2 \} = -4 P \cdot r_\perp \frac{\partial \Phi\left(\frac{r^2}{2}, M\right)}{\partial r_\perp^2} = 0. \]  

(76)

This is the original Droz-Vincent, Todorov, Komar model \cite{15, 16, 17}. More general forms of the functions \( \Phi_i \) are possible for which \( \{ \mathcal{H}_1, \mathcal{H}_2 \} \approx 0 \), being proportional to the constraints themselves.

We define the canonical relative momentum by

\[ q^\mu = \frac{\varepsilon_2 p^\mu_1 - \varepsilon_1 p^\mu_2}{M}, \]  

(77)

with

\[ \varepsilon_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \quad \varepsilon_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}. \]  

(78)

These constituent particle rest-energies are defined so that

\[ \varepsilon_1 + \varepsilon_2 = M, \quad \varepsilon_1 - \varepsilon_2 = \frac{m_1^2 - m_2^2}{M}. \]  

(79)

This definition is reinforced by

\[ -\frac{\mathbf{p}_1 \cdot P}{M} = -\frac{-P^2 + p_1^2 - p_1^2}{2M} \approx \varepsilon_1, \]  

\[ -\frac{\mathbf{p}_2 \cdot P}{M} = -\frac{-P^2 + p_1^2 - p_2^2}{2M} \approx \varepsilon_2. \]  

(80)
where Eq. (77) was used. Using $P^\mu = p_1^\mu + p_2^\mu$ and Eq. (77) gives

$$p_1^\mu = \epsilon_1 P^\mu + q^\mu, \quad p_2^\mu = \epsilon_1 P^\mu - q^\mu.$$

(81)

In term of these variables the difference of the constraints depends on the relative energy in the rest frame

$$\mathcal{H}_1 - \mathcal{H}_2 = p_1^2 + m_1^2 - p_2^2 - m_2^2 = 2 P \cdot q \approx 0,$$

(82)

where we have used

$$\epsilon_1^2 - m_1^2 = \epsilon_2^2 - m_2^2 = \frac{1}{4M^2}(M^4 - 2(m_1^2 + m_2^2)M^2 + (m_1^2 - m_2^2)^2) \equiv b^2(M).$$

(83)

On the other hand, the sum of the two constraints, determines the mass spectrum of the two-body system. It can be written as

$$q^2 + \Phi(\frac{r_1^2}{2}, M) - b^2(M) \approx 0,$$

(84)

or

$$q^2 + \Phi(\frac{r_2^2}{2}, M) - b^2(M) \approx 0,$$

(85)

in the rest frame (where $q^\mu \approx 0$ and $r_1^2 \approx r_2^2$). To get the mass spectrum this equation has to be solved for $M = \sqrt{P^2}$.

Since we have $\{x_1^\mu, \mathcal{H}_1\} \neq 0$, $\{x_2^\mu, \mathcal{H}_2\} \neq 0$, Droz-Vincent covariant non-canonical positions [13, 15] $q_1^\mu$ are defined as the solutions of the two equations $\{q_1^\mu, \mathcal{H}_2\} = 0$, $\{q_2^\mu, \mathcal{H}_1\} = 0$.

If in Eq. (85) we consider a $M$-independent potential in the rest frame we get that

$$q^2 = \frac{1}{4M^2} \left[ M^4 - 2(m_1^2 + m_2^2)M^2 + (m_1^2 - m_2^2)^2 \right],$$

(86)

is modified to

$$q^2 + \Phi(\frac{1}{2} r^2) = \frac{1}{4M^2} \left[ M^4 - 2(m_1^2 + m_2^2)M^2 + (m_1^2 - m_2^2)^2 \right],$$

(87)

which is the rest-frame form of a covariant two-body constraint dynamics [23] involving two generalized mass shell constraints of the form

$$p_1^2 - m_1^2 - \Phi \approx 0, \quad p_2^2 - m_2^2 - \Phi \approx 0,$$

(88)

with
\[ \Phi = 2 \mu V \left( \frac{1}{2} \vec{r}^2 \right), \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \]  

(89)
i.e. a form suitable for the non-relativistic limit.

Solving Eq. (87) algebraically for \( M \) and choosing all positive square roots leads to:

\[ M = \sqrt{m_1^2 + \vec{q}^2 + \Phi \left( \frac{1}{2} \vec{r}^2 \right)} + \sqrt{m_2^2 + \vec{q}^2 + \Phi \left( \frac{1}{2} \vec{r}^2 \right)}. \]  

(90)

In Section VI the rest-frame instant form of this model is studied in detail. In particular the form of the generators of the internal Poincaré group is given.

**B Inertial and Non-Inertial Rest Frames.**

The inertial rest-frame instant form identifies the 3+1 splitting of Minkowski space-time corresponding to the inertial rest frame of every isolated system centered on the inertial observer associated with the Fokker-Prucovnon-canonical external 4-center of inertia. In this instant form the mutual interactions among the particles of an isolated N-body system are described as acting on the points of the particle world-lines simultaneous in the Wigner hyper-planes, leaves of the inertial rest frame.

Instead in parametrized Minkowski theories one considers 3+1 splittings of Minkowski space-time corresponding to *non-inertial frames* centered on arbitrary time-like observers. If \( x^\mu = z^\mu(\tau, \vec{\sigma}) \) describes the embedding of the associated simultaneity 3-surfaces \( \Sigma_\tau \) into Minkowski space-time, so that the metric induced by the coordinate transformation \( x^\mu \mapsto \sigma^A = (\tau, \vec{\sigma}) \) is \( g_{AB}(\tau, \vec{\sigma}) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} \frac{\partial z^\nu(\sigma)}{\partial \sigma^B} \) [13], the basic restrictions on the 3+1 splitting (i.e. of forming a nice foliation with space-like leaves) are the Møller conditions [20]

\[ g_{\tau\tau}(\sigma) > 0, \]
\[ g_{rr}(\sigma) < 0, \quad \left| \begin{array}{cc} g_{rr}(\sigma) & g_{rs}(\sigma) \\ g_{sr}(\sigma) & g_{ss}(\sigma) \end{array} \right| > 0, \quad \det [g_{rs}(\sigma)] < 0, \]

\[ \Rightarrow \det [g_{AB}(\sigma)] < 0. \]  

(91)

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14 The choice of Eq. (77) for the relative momentum is the relativistic generalization of \( \vec{q} = (m_2 \vec{p}_1 - m_1 \vec{p}_2)/(m_1 + m_2) = \mu d\vec{r}/dt \). The alternative choice of \( q^\mu = (p_1^\mu - p_2^\mu)/2 \) would lead to the constraint \( 2P \cdot q = m_1^2 - m_2^2 \) in place of Eq. (82). However it would lead to the same result, Eq. (87), for \( \vec{q}^2 \) even for unequal mass since the relative energy is not zero for this choice of \( q \) unlike for that given in Eq. (77). Hence the expressions in Eqs. (90) and (34) for the c.m. energy are the same with both choices of the relative momentum.

15 The 4-vectors \( \zeta_\nu(\tau, \vec{\sigma}) = \frac{\partial x^\nu(\sigma)}{\partial \sigma^\nu} \) are tangent to \( \Sigma_\tau \). If \( L^\mu(\tau, \vec{\sigma}) \) is the unit normal to \( \Sigma_\tau \) [proportional to \( e^\mu_{\alpha\beta\gamma} [z_\alpha(\tau, \vec{\sigma}) \zeta_\beta(\tau, \vec{\sigma}) \zeta_\gamma(\tau, \vec{\sigma})] \)], we have \( \zeta_\nu(\tau, \vec{\sigma}) = \frac{\partial x^\nu(\sigma)}{\partial \sigma^\nu} = N(\tau, \vec{\sigma}) L^\mu(\tau, \vec{\sigma}) + N^\nu(\tau, \vec{\sigma}) \zeta_\nu(\tau, \vec{\sigma}) \) where \( N(\tau, \vec{\sigma}) \) and \( N^\nu(\tau, \vec{\sigma}) \) are the lapse and shift functions, respectively.
Moreover, the simultaneity 3-surfaces $\Sigma_\tau$ must tend to space-like hyperplanes at spatial infinity: $z^\mu(\tau, \vec{\sigma}) \rightarrow |\vec{\sigma}| \rightarrow \infty x^\mu_0(\tau) + \epsilon^\mu_\nu \sigma^\nu$ and $g_{AB}(\tau, \vec{\sigma}) \rightarrow |\vec{\sigma}| \rightarrow \infty \eta_{AB}$, with the $\epsilon^\mu_\nu$’s being 3 unit space-like 4-vectors tangent to the asymptotic hyper-plane, whose unit normal is $\epsilon^\mu_\tau$ [the $\epsilon^\mu_A$ form an asymptotic cotetrad, $\epsilon^\mu_A \eta^{AB} \epsilon^\nu_B = \eta^{\mu\nu}$].

As shown in Refs.\cite{20, 36}, Eqs.\cite{91} forbid rigid rotations: only differential rotations are allowed and the simplest example is given by those 3+1 splittings whose simultaneity 3-surfaces are hyper-planes with rotating 3-coordinates described by the embeddings ($\sigma = |\vec{\sigma}|$)

$$z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + \epsilon^\mu_\nu R^\nu_s(\tau, \sigma) \sigma^s,$$

$$R^\nu_s(\tau, \sigma) \rightarrow \sigma \rightarrow \infty \delta^\nu_s, \quad \partial_A R^\nu_s(\tau, \sigma) \rightarrow \sigma \rightarrow \infty 0,$$

$$R(\tau, \sigma) = \tilde{R}(\beta_a(\tau, \sigma)), \quad \beta_a(\tau, \sigma) = F(\sigma) \tilde{\beta}_a(\tau), \quad a = 1, 2, 3,$$

$$\frac{dF(\sigma)}{d\sigma} \neq 0, \quad 0 < F(\sigma) < \frac{1}{A \sigma} \tag{92}$$

Each $F(\sigma)$ satisfying the restrictions of the last line, coming from Eqs.\cite{91}, gives rise to a global differentially rotating non-inertial frame.

As shown in Refs.\cite{2}, given the Lagrangian of every isolated system, one makes the coupling to an external gravitational field and then replaces the external metric with the $g_{AB}(\tau, \vec{\sigma})$ associated to a Møller-admissible 3+1 splitting. The resulting action principle $S = \int d\tau d^3\sigma \mathcal{L}(\text{matter}, g_{AB}(\tau, \vec{\sigma}))$ depends upon the system and the embedding $z^\mu(\tau, \vec{\sigma})$ and is invariant under frame-preserving diffeomorphisms: $\tau \mapsto \tau'(\tau, \vec{\sigma})$, $\sigma^i \mapsto \sigma'^i(\vec{\sigma})$. This special-relativistic general covariance implies the vanishing of the canonical Hamiltonian and the following 4 first class constraints

$$\mathcal{H}_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - \epsilon l_\mu(\tau, \vec{\sigma}) \mathcal{M}(\tau, \vec{\sigma}) - \epsilon z_{\mu\nu}(\tau, \vec{\sigma}) h^{\nu s}(\tau, \vec{\sigma}) \mathcal{M}_s(\tau, \vec{\sigma}) \approx 0,$$

$$\{\mathcal{H}_\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma})\} = 0 \tag{93}$$

where $\rho_\mu(\tau, \vec{\sigma})$ is the momentum conjugate to $z^\mu(\tau, \vec{\sigma})$ and $[\sum_u h^{\nu u} g_{us}](\tau, \vec{\sigma}) = \delta^\nu_s$. $\mathcal{M}(\tau, \vec{\sigma}) = T_{\tau\tau}(\tau, \vec{\sigma})$ and $\mathcal{M}_s(\tau, \vec{\sigma}) = T_{\tau s}(\tau, \vec{\sigma})$ are the energy- and momentum-densities of the isolated system in $\Sigma_\tau$-adapted coordinates.\footnote{For N free particles we have $\mathcal{M}(\tau, \vec{\sigma}) = \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{m_i^2 + h^{\nu\nu}(\tau, \vec{\sigma}) \kappa_{ii}(\tau) \kappa_{ii}(\tau)}$, $\mathcal{M}_s(\tau, \vec{\sigma}) = \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa_{ii}(\tau)$.}

Since the matter variables have only $\Sigma_\tau$-adapted Lorentz-scalar indices, the 10 constant of the motion corresponding to the generators of the external Poincare’ algebra are
\[ P^\mu = \int d^3\sigma \rho^\mu(\tau, \bar{\sigma}), \]
\[ J^{\mu\nu} = \int d^3\sigma \left[ z^\mu \rho^\nu - z^\nu \rho^\mu \right](\tau, \bar{\sigma}). \] (94)

The Hamiltonian gauge transformations generated by the constraints (93) change the form and the coordinatization of the simultaneity 3-surfaces \( \Sigma_\tau \) (on each of these surfaces all the clocks are synchronized): as a consequence the embeddings \( z^\mu(\tau, \bar{\sigma}) \) are gauge variables, so that the choice of the non-inertial frame and in particular of the convention for the synchronization of distant clocks [20] is a gauge choice in this framework. All the inertial and non-inertial frames compatible with the Møller conditions (91) are gauge equivalent for the description of the dynamics of isolated systems.

The inertial rest-frame instant form of Section III is associated to the special gauge \( z^\mu(\tau, \bar{\sigma}) = x^\mu_s(\tau) + \epsilon^\mu_r(u(P)) \sigma^r \), \( x^\mu_s(\tau) = Y^\mu_s(\tau) = u^\mu(P) \tau \), selecting the inertial rest frame of the isolated system centered on the Fokker-Pryce 4-center of inertia and having as instantaneous 3-spaces the Wigner hyper-planes, whose internal 3-vectors transform as Wigner spin-1 3-vectors.

Another particularly interesting family of 3+1 splittings of Minkowski space-time is defined by the embeddings:

\[ z^\mu(\tau, \bar{\sigma}) = Y^\mu_s(\tau) + F^\mu(\tau, \bar{\sigma}) = u^\mu(P) \tau + F^\mu(\tau, \bar{\sigma}), \]
\[ \rightarrow_{\sigma \rightarrow \infty} u^\mu(P) \tau + \epsilon^\mu_r(u(P)) \sigma^r, \] (95)

with \( F^\mu(\tau, \bar{\sigma}) \) satisfying Eqs. (91).

In this family the simultaneity 3-surfaces \( \Sigma_\tau \) tend to Wigner hyper-planes at spatial infinity, where they are orthogonal to a 4-vector \( P^\mu = M w^\mu(P) \), where \( M = \sqrt{P^2_{\text{sys}}} \) is the invariant mass of the isolated system. As a consequence, there are asymptotic inertial observers with the world-lines parallel to that of the Fokker-Pryce 4-center of inertia, namely there are the rest-frame conditions \( p_r = \epsilon^\mu_r(u(P)) P_\mu = 0 \), so that the embeddings (95) define global Møller-admissible non-inertial rest frames [17].

In these non-inertial rest frames the external Poincaré’ generators [18] are given by Eqs. (94) evaluated on the embeddings (95) and taking into account the constraints (93). The internal Poincaré’ generators \( M, p^r \approx 0, j^r, k^r \) can be identified from the energy-momentum tensor of the isolated system restricted

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17 The only ones existing in tetrad gravity, due to the equivalence principle, in globally hyperbolic asymptotically flat space-times without super-translations as shown in Ref. [23] and its bibliography.

18 They are fixed by boundary conditions giving the value of constants of the motion corresponding to asymptotic symmetries of these 3+1 splittings.
to the embedding (95). While $M$ is the effective Hamiltonian for the motion of the relative variables, $k^r \approx 0$ is the natural gauge fixing for the rest-frame conditions $p^r \approx 0$.

Since we are in non-inertial rest frames, inside the internal energy- and boost- densities there are the inertial potential sources of the relativistic inertial forces [36]: they are contained in the spatial components of the metric $g_{rs}(\tau, \vec{\sigma})$ associated with the embeddings (95) and describe the appearances of phenomena in non-inertial frames.

For N-body systems with a-a-a-d interactions each non-inertial rest frame (95) describes the same world-lines with different correlations among them, determined by their intersection with the simultaneity 3-surfaces $\Sigma_{\tau}$, and with the extra inertial forces. The difficulty in obtaining these non-inertial descriptions is the necessity to have an explicit Lagrangian with interactions to be used as a starting point. For instance they could be studied for the systems of Refs. [27,28] containing the electro-magnetic field.
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