Finite thin disc models of four galaxies in the Ursa Major cluster: NGC3877, NGC3917, NGC3949 and NGC4010

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ABSTRACT
Finite thin disc models of four galaxies in the Ursa Major cluster are presented. The models are obtained by means of the Hunter method and the particular solutions are chosen in such a way that the circular velocities are adjusted very accurately to the observed rotation curves of some specific spiral galaxies. We present particular models for the four galaxies NGC3877, NGC3917, NGC3949 and NGC4010 with data taken from the recent paper by Verheijen & Sancici (2001). By integrating the corresponding surface mass densities, we obtain the total mass \( M \) of these four galaxies, all of them being of the order of \( 10^{10} M_\odot \). These obtained values for \( M \) may be taken as a quite accurately estimative of the mass upper bound of these galaxies, since in the model was considered that all their mass was concentrated at the galactic disc. The models can be consider as a first approximation to the obtaining of quite realistic models of spiral galaxies.

Key words: stellar dynamics – galaxies: kinematics and dynamics.

1 INTRODUCTION
Currently, the most accepted description of the composition of spiral galaxies is that a main part of its mass is concentrated in a thin disc, being the other constituents a spheroidal halo, a central bulge and, perhaps, a central black hole (Binney & Tremaine 2008). Now, as all of these components contribute to the gravitational field of a galaxy, obtaining proper models that include the effects of all parts is a rather difficult problem. However, the contribution of all parts is limited to certain distance scales, so that not all components have to be included in a reasonably realistic model. Therefore, it is commonly accepted that many of the main aspects of the galactic dynamics can be described, in a quite approximate way, with models that only consider the thin galactic disc.

Accordingly, the study of the gravitational potential generated by an idealized thin disc is a problem of great astrophysical relevance and so, through the years, different approaches have been used to obtain such kind of thin disc models (see Binney & Tremaine 2008 and references therein). So, once an expression for the gravitational potential has been derived, corresponding expressions for the surface mass density of the disc and for the circular velocity of the disc particles can be obtained. Then, if the expression for the circular velocity can be adjusted to fit the observational data of the rotation curve of a particular galaxy, the total mass can be obtained by integrating the corresponding surface mass density.

However, although most of these thin disc models have surface densities and rotation curves with remarkable properties, many of them mainly represent discs of infinite extension and thus they are rather poor flat galaxy models. Therefore, in order to obtain more realistic models of flat galaxies, it is better to consider methods that permit the obtention of finite thin disc models. Now, a simple method to obtain the gravitational potential, the surface density and the rotation curve of thin discs of finite radius was developed by Hunter (1963), the simplest example of a disc obtained by this method being the well known Kalnajs (1972) disc.

In a previous paper (González & Reina 2006) we use the Hunter method in order to obtain an infinite family of thin discs of finite radius with a well behaved surface mass density, an infinite family of generalized Kalnajs discs. Also, the motion of test particles in the gravitational fields generated by the first four members of this family was studied (Ramos-Caro, López-Suspe & González 2008) and a new infinite family of self-consistent models was obtained as a superposition of members belonging to the generalized Kalnajs family (Pedraza, Ramos-Caro & González 2008).

In González & Reina (2006), the family of disc models was derived by requiring that the surface density behaves as a monotonously decreasing function of the radius, with a maximum at the center of the disc and vanishing at the edge. So, although the mass distribution of this family of discs present a satisfactory behavior in such a way that they could be considered adequate as flat galaxy models, their corresponding rotation curves do not present a so good behavior as they do not reproduce the flat region of the observed rotation curve.
On the other hand, in Pedraza, Ramos-Caro & González (2008) the new family of discs was obtained by superposing the members of the generalized Kalnajs family in order that the resulting surface density can be expressed as a well behaved function of the gravitational potential, in such a way that the corresponding distribution functions can be easily obtained. Furthermore, besides present a well-behaved surface density, the models also present rotation curves with a better behavior than the generalized Kalnajs discs and are radially stable, whereas vertically unstable. Then, apart of the stability problems, these discs can be considered as quite adequate models in order to describe satisfactorily a great variety of galaxies.

In agreement with the above considerations, in this paper we explore the possibility of obtain some thin disc models in which the circular velocities can be adjusted very accurately to fit the observed rotation curves. In order to do this, we will consider a differential approach in order to solve the boundary value problem defining the circular velocities can be adjusted very accurately to fit the observed rotation curves. In order to do this, we will consider a differential approach in order to solve the boundary value problem defining

\[ v_c^2(R) = R \frac{\partial \Phi}{\partial R} \bigg|_{z=0}. \]  

(4)

Also, given \( \Phi(R, z) \), the density \( \Sigma(R) \) of the surface distribution of matter can be obtained using the Gauss law and, by using the equation (4), we obtain

\[ \Sigma(R) = \frac{1}{2\pi G} \frac{\partial \Phi}{\partial z} \bigg|_{z=0^+}. \]  

(5)

Thus, in order to have a surface density corresponding to a finite disclike distribution of matter, we impose boundary conditions in the form

\[ \frac{\partial \Phi}{\partial z} \bigg|_{z=0^+} \neq 0; \quad R \leq a, \]  

(6a)

\[ \frac{\partial \Phi}{\partial z} \bigg|_{z=0^+} = 0; \quad R > a, \]  

(6b)

in such a way that the matter distribution is restricted to the disc \( z = 0, 0 \leq R \leq a \).

In order to properly pose the boundary value problem, we introduce oblate spheroidal coordinates, whose symmetry adapts in a natural way to the geometry of the model. These coordinates are related to the usual cylindrical coordinates by the relation \( \text{(Morse & Fesbach 1953)} \)

\[ R = a \sqrt{1 + \xi^2 (1 - \eta^2)}, \]  

(7a)

\[ z = a \xi \eta, \]  

(7b)

where \( 0 \leq \xi < \infty \) and \(-1 \leq \eta < 1 \). The disc has the coordinates \( \xi = 0, 0 \leq \eta^2 < 1 \). On crossing the disc, the \( \eta \) coordinate changes sign but does not change in absolute value. The singular behaviour of this coordinate implies that an even function of \( \eta \) is a continuous function everywhere but has a discontinuous \( \eta \) derivative at the disc.

In terms of the oblate spheroidal coordinates, the Laplace equation can be written as

\[ \frac{\partial}{\partial \xi} \left( (1 + \xi^2) \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( (1 - \eta^2) \frac{\partial \Phi}{\partial \eta} \right) = 0, \]  

(8)

and we need to find solutions that are even functions of \( \eta \) and with the boundary conditions

\[ \frac{\partial \Phi}{\partial \xi} \bigg|_{\xi=0} \neq 0, \]  

(9a)

\[ \frac{\partial \Phi}{\partial \eta} \bigg|_{\eta=0} = 0. \]  

(9b)

According to this, the Newtonian gravitational potential for the exterior of a finite thin disc with an axially symmetric matter density can be written as \( \text{Bateman 1944)} \)

\[ \Phi(\xi, \eta) = -\sum_{n=0}^{\infty} C_{2n} q_{2n}(\xi) P_{2n}(\eta), \]  

(10)

where \( C_{2n} \) are arbitrary constants, \( P_{2n}(\eta) \) and \( q_{2n}(\xi) = \zeta^{2n+1} Q_{2n}(i \xi) \) are the usual Legendre polynomials and the Legendre functions of second kind, respectively.

With this general solution for the gravitational potential, the circular velocity can be written as

\[ v_c^2(\tilde{R}) = \frac{\tilde{R}^2}{\eta} \sum_{n=1}^{\infty} C_{2n} q_{2n}(0) P_{2n}'(\eta), \]  

(11)

where \( \tilde{R} \) is the effective radius of the disc, \( q_{2n}(0) = 2n+1 \) and the prime denotes the derivative with respect to \( \eta \).
while the surface matter density is given by

$$\Sigma(\bar{R}) = \frac{1}{2\pi a_G} \sum_{n=0}^{\infty} C_{2n}(2n+1)q_{2n+1}(0)P_{2n}(\eta),$$  

which allows to compute the value of the total mass $M$. Accordingly, all the quantities characterizing the thin disc model are determined in terms of the set of constants $C_{2n}$, which can be determined from the observational data corresponding to rotation curves for some particular galaxy.

### 3 A FAMILY OF PARTICULAR MODELS

We now explore the possibility of obtain particular thin disc models in which the expression for the circular velocity can be very accurately adjusted with the observed data from the rotation curve of a given galaxy. However, in order to do this, first the sum must be limited to a finite number of terms. This correspond to take $C_{2n} = 0$ for $n > m$, with $m$ a positive integer. So, after replace the derivatives of the Legendre polynomials, the expression (11) can be cast as

$$v_c^2(\bar{R}) = \sum_{n=1}^{m} A_{2n} \bar{R}^{2n},$$  

where the $A_{2n}$ constants are related with the previous constants $C_{2n}$, through the relation

$$C_{2n} = \frac{4n+1}{4n(2n+1)} \sum_{k=0}^{m} A_{2k} q_{2k}(0) \int_{-1}^{1} \eta(1-\eta^2)^{k} P'_n(\eta) d\eta,$$  

for $n \neq 0$, which is obtained by equaling expressions (11) and (14) and by using properties of the Legendre polynomials [Arken & Weber 2005]. Then, if the constants $A_{2n}$ are determined by fitting the observational data of the corresponding rotation curve, the relation (15) gives the values of the constants $C_{2n}$, that defines the particular thin disc model through (10).

As we can see, the value of $C_0$ it is not determined by expression (15). However, it is clear from (12) that the surface mass density diverges at the disc edge, when $\eta = 0$, unless that we impose the condition [Hunt 1963]

$$\sum_{n=0}^{m} C_{2n}(2n+1)q_{2n+1}(0)P_{2n}(0) = 0,$$  

that, after use the properties of the Legendre functions, leads to the expression

$$C_0 = \sum_{n=1}^{m} (-1)^{n+1} C_{2n},$$  

which gives the value of $C_0$, and then of the total mass $M$, in terms of the $A_{2n}$.

The previous expressions imply then that any particular thin disc model will be completely determined by a set of constants $A_{2n}$, which must be chosen in such a way that the circular velocities can be adjusted very accurately to fit the observed rotation curves. Furthermore, as the powers $\bar{R}^{2n}$ are a set of linearly independents functions, the expression (14) is quite adequate to be numerically adjusted to any set of data. Accordingly, expression (14) can be considered as a kind of “universal rotation curve” for flat galaxies, which can easily be adjusted to the observed data of the rotation curve of any particular spiral galaxy.

Now then, besides the circular velocity, there are two other important quantities concerning the kinematics of the models, which describe the stability against radial and vertical perturbations of particles in quasi-circular orbits [Binney & Tremaine 2008]. These two quantities, which must be positive in order to have stable circular orbits, are the epicyclic or radial frequency, defined as

$$\kappa^2(R) = \frac{\partial^2 \Phi_{\text{eff}}}{\partial \bar{R}^2} \bigg|_{\eta = 0},$$  

and the vertical frequency, defined as

$$\nu^2(R) = \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \bigg|_{\eta = 0},$$  

where

$$\Phi_{\text{eff}} = \Phi(R, z) + \frac{\ell^2}{2R^2},$$  

is the effective potential and $\ell = R \nu_c$ is the specific axial angular momentum.

By using then (4) and (20) in (13), we can easily obtain the relation

$$\kappa^2(R) = \frac{1}{R} \frac{dv_c^2}{dR} + \frac{2\nu_c^2}{R^2},$$  

in such a way that, by using (14), the epicyclic frequency can be cast as

$$\kappa^2(\bar{R}) = \sum_{n=0}^{m} 2(n+1)A_{2n} \bar{R}^{2n-2},$$  

where $\bar{\nu} = a \nu$. Also, from the expression for the Laplace operator in cylindrical coordinates and using (4), (19) and (20), is easy to see that

$$\nu^2(R) = \frac{\nabla^2 \Phi}{z = 0} - \frac{1}{R} \frac{dv_c^2}{dR},$$  

so, as the potential is a solution of the Laplace equation, by using (4) the vertical frequency can be written as

$$\nu^2(\bar{R}) = - \sum_{n=0}^{m} 2nA_{2n} \bar{R}^{2n-2},$$  

where $\bar{\nu} = a \nu$.

### 4 FITTING OF DATA TO THE MODELS

In order to adjust the previous model to real observed data, we choose four spiral galaxies of the Ursa Major cluster, the galaxies NGC3877, NGC3917, NGC3949 and NGC4010. The corresponding data are taken from the recent paper by Verheijen & Sancici (2001), which presents the results of an extensive 21 cm-line synthesis imaging survey of 43 galaxies in the nearby Ursa Major cluster using the Westerbork Synthesis Radio Telescope. For each rotation curve data, we take the value of $\alpha$ as given by the last tabulated value of the radius. Accordingly, we are assuming that the radii of the galaxies are defined by the last observed data. Then, by taking the radii normalized in units of $\alpha$, we make a non-linear least square fit of the data with the general relation (14), by considering in each case a value of $m$ less than the number of available data points.

In Figure 1 we show the adjusted rotation curves for the four
Table 1. Constants $C_{2n} \,[\text{km}^2\text{s}^{-2}]$ and values of $m$.

|       | NGC3877 | NGC3917 | NGC3949 | NGC4010 |
|-------|---------|---------|---------|---------|
| $m$   | 6       | 7       | 5       | 7       |
| $C_0$ | 17452.78| 11258.92| 15686.77| 8801.12 |
| $C_2$ | 26564.93| 17210.85| 24051.95| 12875.61|
| $C_4$ | 13926.79| 8998.92 | 13011.66| 4944.66 |
| $C_6$ | 7478.82 | 4573.34 | 7127.38 | 1117.08 |
| $C_8$ | 4053.13 | 2213.64 | 4577.68 | 866.64  |
| $C_{10}$ | 1887.23 | 1063.01 | 2096.77 | 1019.52 |
| $C_{12}$ | 498.29  | 640.40  | 818.98  |         |
| $C_{14}$ | 264.69  |         | 419.19  |         |

Table 2. Morphological type, radius $a$ and total mass $M$.

|       | Type | $a \,[\text{kpc}]$ | $M \,[\text{kg}]$ | $M \,[\text{M}_\odot]$ |
|-------|------|---------------------|------------------|-------------------------|
| NGC3877 | Sc   | 11.74               | $9.47 \times 10^{40}$ | $4.76 \times 10^{10}$          |
| NGC3917 | SSc  | 15.28               | $7.95 \times 10^{40}$ | $3.95 \times 10^{10}$          |
| NGC3949 | Sbc  | 8.72                | $6.32 \times 10^{40}$ | $3.18 \times 10^{10}$          |
| NGC4010 | SBd  | 10.84               | $4.41 \times 10^{40}$ | $2.22 \times 10^{10}$          |

galaxies considered. The points with error bars are the observations, as reported in Verheijen & Sancici (2001), while the solid line is the rotation curve determined from (14) with the values for the $A_{2n}$, given by the numerical fit. As we can see, the relation (14) fit quite accurately to the observed data of the four galaxies considered. Then, from the obtained values for $A_{2n}$, the corresponding values of the $C_{2n}$ are determined by using (15) and (17). In Table 1 we present the values of $C_{2n}$ for the four galaxies as well as the corresponding value of $m$ used in (14). In Table 2 we indicate, for each galaxy, the morphological type according to the Hubble’s classification of galaxies, the radius $a$ in kpc and the total mass $M$, both in kg and in solar mass units ($M_\odot$).

Now, as the set of constants $C_{2n}$ it defines completely each particular thin disc model, we can easily compute all the physical quantities characterizing each galaxy. However, as explicit expressions for the gravitational potential $\Phi(R, z)$ and the surface mass density $\Sigma(R)$ can be easily obtained by using the values of the $C_{2n}$ at expressions (10) and (12), we will not present them here. Instead, we plot in Figure 2 the surface densities for the four galaxies, as functions of the dimensionless radial coordinate $eR = R/a$. For the four galaxies we obtain a well behaved surface mass density, having a maximum at the disc center and then decreasing until vanish at the disc edge.

In a similar way, we can compute the epiciclic and vertical frequencies by using (22), (24) and the values of the constants $A_{2n}$ obtained from the numerical fit. However, as with the surface mass densities, we will not present the explicit expressions here and, instead, we only show the corresponding plots. So, in Figure 3 we show the plots of the epiciclic frequencies for the four galaxies considered and, in Figure 4, the corresponding plots of the vertical frequencies. From the plots at Figure 4, we can see that only the galaxy NGC4010 presents a small region of radial instability near the disc edge. On the other hand, as it is shown at Figure 4, the four galaxies are instable against vertical perturbations.
Figure 2. Plots of the surface densities $\Sigma \times 10^{-3}$ in kg/m$^2$, as functions of the dimensionless radial coordinate $\tilde{R} = R/a$, for the spiral galaxies NGC3877, NGC3917, NGC3949 and NGC4010.

Figure 3. Plots of the epiciclic frequencies $\kappa^2 \times 10^{-5}$ in (km/s)$^2$, as functions of the dimensionless radial coordinate $\tilde{R} = R/a$, for the spiral galaxies NGC3877, NGC3917, NGC3949 and NGC4010.
5 DISCUSSION

We presented four particular thin disc models adjusted in order to accurately fit the observed data of the rotation curves for the galaxies NGC3877, NGC3917, NGC3949 and NGC4010 of the Ursa Major cluster. These models present well behaved surface densities, that resembles the observed luminosity profile of many spiral galaxies, and the obtained values for the corresponding total mass $M$ it agrees with the expected order of magnitude. Accordingly, the here obtained expression for the circular velocity, equation (14), can be considered as a kind of “universal rotation curve” for flat galaxies, which can easily be adjusted to the observed data of the rotation curve of any particular spiral galaxy.

On the other hand, in one of the models we obtain a small region near the disc edge with instability against radial perturbations. Now, as the models are completely determined by the set of constants $A_{2n}$, which are fixed by the numerical fit of the rotation curve data, there are not free parameters that can be adjusted by requiring radial stability. A possible solution for this problem may be consider a less restrictive numerical fit that leaves some free parameters. This can be done by taking the summation in expression (14) until a value of $m$ greater than the available number of data points.

However, the models present a central region with strong instabilities against vertical perturbations of particles in quasi-circular orbits. This result was expected as a consequence of the fact that the models only consider the thin galactic disc. Indeed, as we can see from expression (23), vertical instability will be always present in models constructed from solutions of Laplace equation and adjusted in such a way that their circular velocities reproduce the central region of the observed rotation curves, where the velocity rises linearly with the radius. Therefore, more realistic models must be considered that include the non-thin character of the galactic disc, or the mass contribution of the spheroidal halo.

In agreement with the above considerations, we can consider the set of models here presented as a first approximation to the obtaining of quite realistic models of spiral galaxies. In particular, we believe that the values of $M$ that were obtained for the four galaxies studied may be taken as a quite accurately estimative of the mass upper bound of these galaxies, since in the model was considered that all their mass was concentrated at the galactic disc. Accordingly, we are working now in a generalization of the model that includes the mass contribution of the spheroidal halo, in such a way that we can overcome the vertical instability problem and obtain some estimative of the relative contributions of the halo and the disc to the total mass of the galaxies.

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