Interpolation and Extrapolation of Toeplitz Matrices via Optimal Mass Transport

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Abstract—In this work, we propose a novel method for quantifying distances between Toeplitz structured covariance matrices. By exploiting the spectral representation of Toeplitz matrices, the proposed distance measure is defined based on an optimal mass transport problem in the spectral domain. This may then be interpreted in the covariance domain, suggesting a natural way of interpolating and extrapolating Toeplitz matrices, such that the positive semi-definiteness and the Toeplitz structure of these matrices are preserved. The proposed distance measure is also shown to be contractive with respect to both additive and multiplicative noise, and thereby allows for a quantification of the decreased distance between signals when these are corrupted by noise. Finally, we illustrate how this approach can be used for several applications in signal processing. In particular, we consider interpolation and extrapolation of Toeplitz matrices, as well as clustering problems and tracking of slowly varying stochastic processes.

Keywords Covariance interpolation, Optimal mass transport, Toeplitz matrices, Spectral estimation

I. INTRODUCTION

Statistical modeling is a key methodology for estimation and identification and is used throughout the signal processing field. An intrinsic component of such models is covariance estimates, which are extensively used in application areas such as spectral estimation, radar, and sonar [1], [2], wireless channel estimation, medical imaging, and identification of systems and network structures [3], [4]. Although being a classical subject (see, e.g., [5]), covariance estimation has recently received considerable attention. Such contributions include works on finding robust covariance estimates with respect to outliers, as well as methods suitable for handling different distribution assumptions, including families of non-Gaussian distributions [6]–[10]. Another important class of problems is covariance estimation with an inherent geometry that gives rise to a structured covariance matrix. Such structures could arise from stationarity assumptions of the underlying object [11]–[15] or be due to assumptions in, e.g., the underlying network structures in graphical models [16], [17]. In this work, we focus on Toeplitz structures which naturally arise when modeling stationary signals and processes.

Although many methods rely on stationarity for modeling signals, such assumptions are typically not valid over longer time horizons. Therefore, tools for interpolation and morphing of covariance matrices are important for modeling and fusing information. Several such tools for interpolating covariances have recently been proposed in the literature, for example methods based on g-convexity [9], optimal mass transport [18], and information geometry [19]. An alternative approach for such interpolation is to relax, or "lift", the covariances and instead consider interpolation between the lifted objects. For example, in [20] (see also [21]), interpolation between covariance matrices is induced from the optimal mass transport geodesics between the Gaussian density functions with the corresponding covariances.

The topic of optimal mass transport (see, e.g., [22], [23]) was originally introduced in order to address the problem of, in a cost efficient way, supplying construction sites with building material and has been used in many contexts, such as, e.g., economics and resources allocation. Lately, it has also gained interest in application fields such as image processing [24], [25] signal processing [26]–[28], computer vision and machine learning [29]–[33]. In this work, we will utilize optimal mass transport to model changes in the covariance structure of stochastic processes, or signals. To this end, we propose a new lifting approach, where the lifting is made from the covariance domain to the frequency domain, using the fact that any positive semi-definite Toeplitz matrix has a spectral representation. We combine this approach with the frequency domain metric based on optimal mass transport [27], in order to define pairwise distances between Toeplitz matrices. This is done by considering the minimum distance, in the optimal mass transport sense, between the sets of power spectra consistent with each of the Toeplitz matrices. The proposed distance measure is shown to be contractive with respect to additive and multiplicative noise, i.e., it reflects the increased difficulty of discriminating between two stochastic processes if these are corrupted by two realizations of a noise process. Also, we show that the proposed distance measure gives rise to a natural way of interpolating and extrapolating Toeplitz matrices. The interpolation method preserves the Toeplitz structure, the positive semi-definiteness, as well as the zeroth moment of the interpolating/extrapolating Toeplitz matrices. The interpolation method preserves the Toeplitz structure, the positive semi-definiteness, as well as the zeroth moment of the interpolating/extrapolating Toeplitz matrices.

The proposed optimal mass transport problem is in its original form an infinite-dimensional problem. As an alternative to finding solutions using approximations based on discretizations of the underlying space, we show that certain formulations of the problem allows for approximations by a...
semi-definite program using a sum-of-squares representation. Also, we illustrate how the method can be used for interpolation, extrapolation, tracking, and clustering.

This paper is organized as follows. In Section II we provide a brief background on the moment problem, i.e., determining the power spectrum from a partial covariance sequence or finite covariance matrix, as well as introduce the problem of optimal mass transport. Section III introduces the proposed distance notion for positive semi-definite Toeplitz covariance matrices. Here, the dual problem is derived, and properties of the proposed distance notion are described. In Section IV, we formulate a sum-of-squares relaxation of the dual problem. Section V formulates a sum-of-squares relaxation of the dual problem. Section VI provides numerical illustrations of the proposed distance notion, as well as the described applications. Finally, Section VII concludes upon the work.

\section*{Notation}
Let \(M^n\) denote the set of Hermitian \(n \times n\) matrices and let \((\cdot)^T\) denote the transpose, \((\cdot)^H\) the Hermitian transpose, and \((\cdot)^*\) the complex conjugate. Let \(T = (-\pi, \pi]\) and let \(C_{\text{perio}}(T)\) denote the set of continuous and 2-periodic functions on \(T\). The set of linear bounded functionals on \(C_{\text{perio}}(T)\), or equivalently, the dual space of \(C_{\text{perio}}(T)\), is the set of generalized integrable functions on the set \(T\), here denoted by \(\mathcal{M}(T)\). Thus, \(\mathcal{M}(T)\) includes, e.g., functions containing singular parts such as Dirac delta functions [34]. Further, we let \(M_+(T)\) denote the subset of such functions that are non-negative. We use \(\langle \Phi, f \rangle\) to denote the application of the functional \(\Phi\) on \(f\), e.g.,

\[\langle \Phi, f \rangle = \int_T f(\theta)\Phi(\theta)d\theta\]

if \(f \in C_{\text{perio}}(T)\) and \(\Phi \in \mathcal{M}(T)\). For Hilbert spaces, \((\cdot, \cdot)\) is the standard inner product, e.g., when \(X\) and \(Y\) are vectors or matrices, then \(\langle X, Y \rangle = \text{trace}(XY^H)\). We denote matrices by boldface upper-case letters, such as \(X\), whereas vectors are denoted by boldface lower-case letters, such as \(x\).

\section{Background}

\subsection{Stochastic processes and spectral representations}
We will in this work consider complex-valued discrete time stochastic processes, or signals, \(y(t)\) for \(t \in \mathbb{Z}\). These will be assumed to be zero mean and wide sense stationary (WSS), i.e., \(\mathbb{E}(y(t)) = 0\) for all \(t \in \mathbb{Z}\), and the covariance

\[r_k \triangleq \mathbb{E}(y(t)y(t-k))\]

being independent of \(t\). Here, \(\mathbb{E}(\cdot)\) denotes the expectation operator. The frequency content of the process \(y(t)\) may then be represented by the power spectrum, \(\Phi\), i.e., the non-negative function on \(T\) whose Fourier coefficients coincide with the covariances:

\[r_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(\theta)e^{-ik\theta}d\theta\]

\footnote{Strictly speaking \(\mathcal{M}(T)\) is the set of signed bounded measures on \(T\) [35].}

for \(k \in \mathbb{Z}\) (see, e.g., [36] Chapter 2). Typically in spectral estimation, one considers the inverse problem of recovering the power spectrum \(\Phi\) from a given set of covariances \(r_k\), for \(k \in \mathbb{Z}\), with \(|k| \leq n - 1\). The condition for any such reconstruction to be valid is that \(\Phi\) should be consistent with the covariance sequence \(\{r_k\}_{|k| \leq n-1}\), i.e., \(\Phi\) should hold for \(|k| \leq n - 1\). The corresponding \(n \times n\) covariance matrix is

\[R = \begin{pmatrix} r_0 & r_{-1} & r_{-2} & \cdots & r_{-n+1} \\ r_{1} & r_0 & r_{-1} & \cdots & r_{-n+2} \\ r_{2} & r_{1} & r_0 & \cdots & r_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n-1} & r_{n-2} & r_{n-3} & \cdots & r_0 \end{pmatrix} \]

which is a Hermitian Toeplitz matrix, since \(y(t)\) is WSS. Thus, expressed in the form of matrices, a spectrum is consistent with an observed partial covariance sequence, or, equivalently, a finite covariance matrix, if \(\Gamma(\Phi) = R\), where \(\Gamma : \mathcal{M}(T) \to M^n\) is the linear operator

\[\Gamma(\Phi) = \frac{1}{2\pi} \int_T a(\theta)\Phi(\theta)a(\theta)^Hd\theta\]

and

\[a(\theta) = \left[1, e^{i\theta}, \ldots, e^{i(n-1)\theta}\right]^T / \sqrt{n}\]

is the Fourier vector. Note that \(\Gamma(\Phi)\) is a Toeplitz matrix, since \(a(\theta)a(\theta)^H\) is Toeplitz for any \(\theta\). It may be noted that for any positive semi-definite Toeplitz matrix, \(R\), there always exists at least one consistent power spectrum; in fact, if \(R\) is positive definite, there is an infinite family of consistent power spectra [37]. It may be noted that for singular Toeplitz covariance matrices, the spectral representation is unique. This fact has recently been successfully utilized in atomic norm minimization problems for grid-less compressed sensing of sinusoidal signals [38]. In this work, we are mainly interested in the non-singular case, where several power spectra are consistent with given covariance matrices. In Section III we will utilize such spectral representations in order to define a notion of distance between pairs of Toeplitz matrices. This distance will be defined in terms of the minimum optimal mass transport cost between the sets of power spectra consistent with the matrices.

\subsection{Optimal mass transport}
The Monge-Kantorovich problem of optimal mass transport is the problem of finding an optimal transport plan between two given mass distributions [22], [23]. The cost of moving a unit mass is defined on the underlying space, and the optimal transport plan is defined as the plan minimizing the total cost. The resulting minimal cost, associated with the optimal transport plan is defined as the plan minimizing the total cost. The idea of utilizing the optimal mass transport cost as a distance measure has been used, e.g., for defining metrics on the space of power spectra [27], whereas the optimal transport plan has been used for tracking stochastic processes with smoothly varying spectral content and for spectral morphing for speech signals.
Recently, the interpretation of the optimal transport plan as providing an optimal association between elements in two mass distributions has been used as a means of clustering in fundamental frequency estimation algorithms [39].

As in [27], we consider the following distance between two spectra $\Phi_0$ and $\Phi_1$:

$$ S(\Phi_0, \Phi_1) \triangleq \min_{M \in \mathcal{M}_+^+(\mathbb{T}^2)} \int_{\mathbb{T}^2} c(\theta, \varphi) M(\theta, \varphi) d\theta d\varphi \quad (5a) $$

subject to

$$ \Phi_0(\theta) = \int_{\mathbb{T}} M(\theta, \varphi) d\varphi \quad (5b) $$

$$ \Phi_1(\varphi) = \int_{\mathbb{T}} M(\theta, \varphi) d\theta \quad (5c) $$

where $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ denotes the 2-D frequency space. Here, the cost function, $c(\theta, \varphi)$, details the cost of moving one unit of mass between the frequencies $\theta$ and $\varphi$. The transport plan, $M(\theta, \varphi)$, specifies the amount of mass moved from frequency $\theta$ to frequency $\varphi$. The objective in (5a) is the total cost associated with the transport plan $M$ and the constraints (5b) and (5c) ensure that $M$ is a valid transport plan from $\Phi_0$ to $\Phi_1$, i.e., the integration marginals of $M$ coincide with the spectra $\Phi_0$ and $\Phi_1$. It may be noted that, due to these marginal constraints, the distance measure $S$ is only defined for spectra of the same mass, or total power. However, $S$ may be generalized in order to allow for mass differences by including a cost for adding and subtracting mass by postulating that the spectra $\Phi_0$ and $\Phi_1$ are perturbations of functions $\Psi_0$ and $\Psi_1$ that have equal mass. As in [27], this may be formulated as

$$ S_c(\Phi_0, \Phi_1) = \min_{\Psi_j \in \mathcal{M}_+^+(\mathbb{T})} S(\Psi_0, \Psi_1, \kappa) + \kappa \sum_{j=0}^1 \| \Psi_j - \Psi_j \|_1 $$

(6)

where $\kappa > 0$ is a used-defined parameter detailing the cost of adding or subtracting mass. Also, if the cost function in the optimal mass transport problem in (5) is chosen as $c(\theta, \varphi) = |\theta - \varphi|^p$, $p \geq 1$ for any metric $d(\theta, \varphi)$ on $\mathbb{T}$, then $W(\Phi_0, \Phi_1) = S(\Phi_0, \Phi_1)^{1/p}$ is the so-called Wasserstein metric on $\mathcal{M}_+^+(\mathbb{T})$. Similarly, for $S_c$, the following theorem holds.

**Theorem 1** ([27]). Let $p \geq 1$, $\kappa > 0$, and let the cost function be $c(\theta, \varphi) = |\theta - \varphi|^p$. Then,

$$ W_c(\Phi_0, \Phi_1) = S_c(\Phi_0, \Phi_1)^{1/p} \quad (7) $$

is a metric on $\mathcal{M}_+^+(\mathbb{T})$.

Consider a situation where we need to discriminate between two signals on the basis of their statistics or of their power spectra. In such cases, additive or multiplicative noise typically impede our ability to differentiate between the two. This was considered in [27] and it was shown that the transportation distance respects this property in the sense that corrupting two signals with additive and (normalized) multiplicative noise decreases their transportation distance. Specifically, the following theorem holds.

**Theorem 2** ([27]). Let $p \geq 1$, $\kappa > 0$, and let the cost function be $c(\theta, \varphi) = |\theta - \varphi|^p$, and let $W_c(\Phi_0, \Phi_1)$ be defined by (7).

Then, $W_c(\Phi_0, \Phi_1)$ is contractive with respect to the additive and normalized multiplicative noise, i.e., with respect to

- $\Phi \mapsto \Phi + \Phi_n$
- $\Phi \mapsto \Phi * \Phi_m$

for any $\Phi_n, \Phi_m \in \mathcal{M}_+^+(\mathbb{T})$, with $\int_0^1 \Phi_m(\theta) d\theta = 1$.

As we shall see, it is possible to construct notions of distance on the space of positive semi-definite Toeplitz matrices that have properties similar to those stated in Theorem 2.

### III. A NOTION OF DISTANCE FOR TOEPLITZ MATRICES

As noted above, any positive semi-definite Toeplitz matrix $R$ has at least one spectral representation, i.e., there exists at least one spectrum $\Phi$ that is consistent with it. Thus, we define the distance, $T$, between two positive semi-definite Toeplitz matrices, $R_0$ and $R_1$, as the minimum transportation cost, as measured by $S$, between spectra consistent with the respective matrices, i.e.,

$$ T(R_0, R_1) \triangleq \min_{\Phi_0, \Phi_1 \in \mathcal{M}_+^+(\mathbb{T})} S(\Phi_0, \Phi_1) \quad (8) $$

subject to

$$ \Gamma(\Phi_j) = R_j, \quad j = 0, 1 $$

Considering the definition of $S$ in (5), this can equivalently be formulated as the convex optimization problem

$$ T(R_0, R_1) = \min_{M \in \mathcal{M}_+^+(\mathbb{T}^2)} \int_{\mathbb{T}^2} c(\theta, \varphi) M(\theta, \varphi) d\theta d\varphi \quad (9) $$

subject to

$$ \Gamma \left( \int_{\mathbb{T}} M(\theta, \varphi) d\varphi \right) = R_0 $$

$$ \Gamma \left( \int_{\mathbb{T}} M(\theta, \varphi) d\theta \right) = R_1 $$

Note that the formulation in (9) is only defined for covariance matrices with the same zeroth covariances, or, equivalently, covariance matrices whose consistent spectra have the same mass. However, in order to allow for mass differences, $T(R_0, R_1)$ can be generalized in analog with (6) as

$$ T_c(R_0, R_1) \triangleq \min_{\Phi_j \in \mathcal{M}_+^+(\mathbb{T})} S_c(\Phi_0, \Phi_1) \quad (10) $$

subject to

$$ \Gamma(\Phi_j) = R_j \quad j = 0, 1 $$

or, equivalently,

$$ T_c(R_0, R_1) = \min_{M \in \mathcal{M}_+^+(\mathbb{T}^2)} \int_{\mathbb{T}^2} c(\theta, \varphi) M(\theta, \varphi) d\theta d\varphi + \kappa \left( \left\| \int_{\mathbb{T}} M(\theta, \varphi) d\varphi - \Psi_0 \right\|_1 \right) \quad (11) $$

subject to

$$ \Gamma(\Psi_j) = R_j, \quad j = 0, 1 $$

Typically, the cost function $c$ would be selected to be symmetric in its arguments, in which case $T_c$ would be symmetric as well, i.e., $T_c(R_0, R_1) = T_c(R_1, R_0)$ for any Toeplitz matrices $R_0, R_1 \in \mathcal{M}_+^+$. Although many possible choices of such cost functions exist, we will in most examples presented herein consider $c(\theta, \varphi) = |e^{i\theta} - e^{i\varphi}|^2$, i.e., the cost function quantifies distances as the square of the distance between the
corresponding points on the unit circle. As we show in Section \[\text{[V]}\] this particular choice of cost function allows for a sum-of-squares relaxation of the dual formulation of \([\text{[II]}]\). This dual formulation is presented next.

**A. The dual formulation**

In order to study properties of the distance notion \(T_c\), we consider the dual formulation of \([\text{[II]}]\), where we assume that the cost function, \(c\), is non-negative on \(\mathbb{T}^2\). In order to address this problem, we first note that the adjoint operator \(\Gamma^*: M^n \to C_{\text{perio}}(\mathbb{T})\) of the operator \(\Gamma\) is

\[
\Gamma^*(R)(\theta) = (a(\theta))^H R a(\theta)
\]

since

\[
\langle (\Gamma(\Phi), R) \rangle = \int_\mathbb{T} a(\theta) \Phi(\theta) (a(\theta))^H d\theta \cdot R
\]

where, in the first line, the inner product is the one associated with \(M^n\), and the second line is the bilinear form with arguments \(\Gamma^*(R) \in C_{\text{perio}}(\mathbb{T})\) and \(\Phi \in M_n(\mathbb{T})\). With this result, we can derive an expression of the dual problem by considering the Lagrangian relaxation of \([\text{[II]}]\). The Lagrangian is given by

\[
\mathcal{L}_\kappa(M, \Psi_0, \Psi_1, A_0, A_1) = \int_{\mathbb{T}} c(\theta, \varphi) M(\theta, \varphi)d\theta d\varphi
\]

\[
+ \langle R_0 - \Gamma(\Psi_0), A_0 \rangle + \langle R_1 - \Gamma(\Psi_1), A_1 \rangle
\]

\[
+ \kappa \left( \left\| \int_{\mathbb{T}} M(\theta, \varphi)d\varphi - \Psi_0 \right\|_1^2 + \kappa \left\| \int_{\mathbb{T}} M(\theta, \varphi)d\theta - \Psi_1 \right\|_1^2 \right)
\]

\[
= \langle A_0, R_0 \rangle + \langle A_1, R_1 \rangle + \int_{\mathbb{T}} c(\theta, \varphi) M(\theta, \varphi)d\theta d\varphi
\]

\[
- \langle \Psi_0, \Gamma^*(A_0) \rangle - \langle \Psi_1, \Gamma^*(A_1) \rangle
\]

\[
+ \kappa \left( \left\| \int_{\mathbb{T}} M(\theta, \varphi)d\varphi - \Psi_0 \right\|_1^2 + \kappa \left\| \int_{\mathbb{T}} M(\theta, \varphi)d\theta - \Psi_1 \right\|_1^2 \right)
\]

Note that the Lagrange multiplier matrices \(A_0\) and \(A_1\) may be taken as Hermitian matrices, as \(R_0 - \Gamma(\Psi_0)\) and \(R_1 - \Gamma(\Psi_1)\) are Hermitian, and thus all inner products are real. Considering the infimum of \(\mathcal{L}_\kappa\) with respect to \(\Psi_0\) and \(\Psi_1\), it may be noted that this is only finite if \(\Gamma^*(A_0)(\theta) \leq \kappa\) and \(\Gamma^*(A_1)(\varphi) \leq \kappa\), for all \(\theta, \varphi \in \mathbb{T}\). If this is satisfied, the Lagrangian is, for any fixed non-negative \(M\), minimized by \(\Psi_0\) and \(\Psi_1\) given by

\[
\Psi_0(\theta) = \mathbb{1}_{\{\Gamma^*(A_0)(\theta) \in [-\kappa, \kappa]\}} \int_{\mathbb{T}} M(\theta, \varphi)d\varphi
\]

\[
\Psi_1(\varphi) = \mathbb{1}_{\{\Gamma^*(A_1)(\varphi) \in [-\kappa, \kappa]\}} \int_{\mathbb{T}} M(\theta, \varphi)d\theta
\]

where \(\mathbb{1}_{\{\cdot\}}\) is the indicator function. Using this, and considering the infimum with respect to \(M\), we arrive at

\[
\inf_{\substack{M \in M_n(\mathbb{T}^2) \\
\Psi_0, \Psi_1 \in M_n(\mathbb{T})}} \mathcal{L}_\kappa = \left\{ \begin{array}{ll}
\langle A_0, R_0 \rangle + \langle A_1, R_1 \rangle & \text{if } (A_0, A_1) \in \Omega_c^c \\
-\infty & \text{otherwise}
\end{array} \right.
\]

3Strictly speaking, this is the pre-adjoint operator of \(\Gamma\).

where

\[
\Omega_c^c = \{A_0, A_1 \in M^n | \Gamma^*(A_0)(\theta) + \Gamma^*(A_1)(\varphi) \leq c(\theta, \varphi), \Gamma^*(A_0)(\theta) \leq \kappa, \Gamma^*(A_1)(\varphi) \leq \kappa \text{ for all } \theta, \varphi \in \mathbb{T} \}
\]

where we have used that the cost function \(c\) is non-negative. This yields the dual problem.

**Proposition 1.** For any cost function \(c(\theta, \varphi)\), such that \(c(\theta, \varphi) \geq 0\), for all \(\theta, \varphi \in \mathbb{T}\), the dual problem of \([\text{[II]}]\) is

\[
\text{maximize } \langle A_0, R_0 \rangle + \langle A_1, R_1 \rangle
\]

where

\[
\Omega_c^c = \{A_0, A_1 \in M^n | \Gamma^*(A_0)(\theta) + \Gamma^*(A_1)(\varphi) \leq c(\theta, \varphi), \Gamma^*(A_0)(\theta) \leq \kappa, \Gamma^*(A_1)(\varphi) \leq \kappa \text{ for all } \theta, \varphi \in \mathbb{T} \}
\]

Since the primal and dual problems are convex and the set of feasible points \(\Omega_c^c\) has non-empty interior for any \(\kappa \geq 0\), Slater’s condition (see, e.g., [40]) gives that strong duality holds and hence the duality gap between \([\text{[II]}]\) and \([\text{[12]}]\) is zero. Also, note the strong resemblance in form compared to the dual of \([\text{[9]}]\), which is given by \([27]\).

\[
\text{maximize } \int_{\mathbb{T}} \lambda_0(\theta) \psi_0(\theta)d\theta + \int_{\mathbb{T}} \lambda_1(\varphi) \psi_1(\varphi)d\varphi
\]

subject to

\[
\lambda_0(\theta) + \lambda_1(\varphi) \leq c(\theta, \varphi) \forall \theta, \varphi \in \mathbb{T}
\]

\[
\lambda_0(\theta) \leq \kappa \forall \theta \in \mathbb{T}
\]

\[
\lambda_1(\varphi) \leq \kappa \forall \varphi \in \mathbb{T}
\]

Similarly, for the case when the zeroth covariances of \(R_0\) and \(R_1\) are required to be equal, the dual problem is

\[
\text{maximize } \langle A_0, R_0 \rangle + \langle A_1, R_1 \rangle
\]

where

\[
\Omega_c = \{A_0, A_1 \in M^n | \Gamma^*(A_0)(\theta) + \Gamma^*(A_1)(\varphi) \leq c(\theta, \varphi) \forall \theta, \varphi \in \mathbb{T} \}
\]

This is the dual problem of \([\text{[9]}]\), where adding and subtracting mass is not allowed in the transport problem.

**B. Properties of distance notion \(T_c\)**

For the distance notion \(T_c\) in \([\text{[10]}]\), the following proposition holds.

**Proposition 2.** Let the cost function \(c\) be non-negative and such that \(c(\theta, \theta) = 0\), for all \(\theta \in \mathbb{T}\). Then, the distance measure \(T_c\) defined in \([\text{[11]}]\) is contractive with respect to additive noise. If \(c\) is also shift-invariant, i.e., \(c(\theta - \phi, \varphi - \phi) = c(\theta, \varphi)\) for all \(\theta, \varphi, \phi \in \mathbb{T}\), then \(T_c\) is also contractive with respect to multiplicative noise whose zeroth covariance is smaller than or equal to unity.

The statement of this proposition is that when two stochastic processes become contaminated by noise, the distance notion \(T_c\) decreases and hence the processes become harder to
distinguish. Intuitively, this is a desirable property of \( T_c \), as additive or multiplicative noise should indeed impede ones ability to discriminate between two processes. Proposition 2 may be proven by utilizing results in [27]. However, in the interest of making the exposition self-contained, we will here provide a direct proof based on the dual formulation in (12). Here, we prove the first part of the proposition, i.e., the case of additive noise. For the case of multiplicative noise, the proof is deferred to Appendix A.

**Proof.** Consider two processes with covariance matrices \( \mathbf{R}_0 \) and \( \mathbf{R}_1 \) and assume that they are both additively corrupted by an independent noise process with covariance \( \mathbf{R}_w \). This results in processes with covariances \( \mathbf{R}_0' = \mathbf{R}_0 + \mathbf{R}_w \) and \( \mathbf{R}_1' = \mathbf{R}_1 + \mathbf{R}_w \), respectively. From the dual formulation in (12), the distance between these covariance matrices is given by

\[
T_c(\mathbf{R}_0', \mathbf{R}_1') = \max_{(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_c} \langle \mathbf{A}_0, \mathbf{R}_0 \rangle + \langle \mathbf{A}_1, \mathbf{R}_1 \rangle + \langle \mathbf{A}_0 + \mathbf{A}_1, \mathbf{R}_w \rangle
\]

Let \( \Phi_w \) be any spectrum consistent with the noise covariance \( \mathbf{R}_w \), i.e., with \( \Gamma(\Phi_w) = \mathbf{R}_w \). Then we have that

\[
\langle \mathbf{A}_0 + \mathbf{A}_1, \mathbf{R}_w \rangle = \langle \mathbf{A}_0 + \mathbf{A}_1, \Gamma(\Phi_w) \rangle = \langle \Gamma^*(\mathbf{A}_0) + \Gamma^*(\mathbf{A}_1), \Phi_w \rangle = \int (\Gamma^*(\mathbf{A}_0)(\theta) + \Gamma^*(\mathbf{A}_1)(\theta)) \Phi_w(\theta) d\theta
\]

As \( (\mathbf{A}_0, \mathbf{A}_1) \in \Omega_c \), it holds that

\[
\Gamma^*(\mathbf{A}_0)(\theta) + \Gamma^*(\mathbf{A}_1)(\theta) \leq c(\theta, \theta) = 0
\]

and since \( \Phi_w \geq 0 \), we get

\[
\langle \mathbf{A}_0 + \mathbf{A}_1, \mathbf{R}_w \rangle \leq 0
\]

Hence, it follows that

\[
T_c(\mathbf{R}_0', \mathbf{R}_1') \leq \max_{(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_c} \langle \mathbf{A}_0, \mathbf{R}_0 \rangle + \langle \mathbf{A}_1, \mathbf{R}_1 \rangle = T_c(\mathbf{R}_0, \mathbf{R}_1)
\]

It may be noted that the assumptions made in Proposition 2 regarding the cost function \( c \) are quite mild, allowing for a large class of potential cost functions. In particular, \( c(\theta, \varphi) = |e^{i\theta} - e^{i\varphi}|^2 \) satisfies the conditions of Proposition 2 for both additive and multiplicative noise.

**IV. INTERPOLATION, EXTRAPOLATION, AND TRACKING**

The formulation in (8) does not only define a notion of distance between two Toeplitz covariance matrices, \( \mathbf{R}_0 \) and \( \mathbf{R}_1 \); it also provides a means of forming interpolating matrices, i.e., defining intermediate covariance matrices \( \mathbf{R}_\tau \), for \( \tau \in (0,1) \). In order to define the interpolating matrices, we again utilize the spectral representation of positive semi-definite Toeplitz matrices. To this end, we note that, given an optimal transport plan, \( M \), found as the functional minimizing (9), one may define spectra intermediate to the marginals \( \int \tau M(\theta, \varphi) d\varphi \) and \( \int M(\theta, \varphi) d\theta \) by linearly shifting the frequency locations of the spectral mass, as dictated by \( M \). That is, any mass transferred from \( \phi \) to \( \phi + \varphi \) is defined to, at \( \tau \in [0,1] \), be located at frequency \( \phi + \tau \varphi \). Using this, the intermediate spectrum is given by

\[
\Phi^M_\tau(\theta) \triangleq \int T_\tau \delta_0(\{\phi + \tau \varphi\}_{mod T}) M(\phi, \phi + \varphi) d\phi d\varphi = \int T M(\theta - \tau \varphi, \theta + (1 - \tau) \varphi) d\varphi
\]

Here, \( \delta_0 \) is the Dirac delta function localized at \( \theta \), i.e., \( \delta_0(\phi) = \delta(\phi - \theta) \) and the integrands are extended periodically with period \( 2\pi \) outside the domain of integration. Also, we denote by \( \{x\}_{mod T} \) the value in \( T \) that is congruent with \( x \) modulo \( 2\pi \). Based on this definition of the interpolating spectrum \( \Phi^M_\tau \), the corresponding interpolating covariance matrix, \( \mathbf{R}_\tau \) is defined as the unique Toeplitz matrix consistent with this spectrum, i.e.,

\[
\mathbf{R}_\tau \triangleq \Gamma(\Phi^M_\tau)
\]

\[
= \int T \mathbf{a}(\theta) \left( \int T M(\theta - \tau \varphi, \theta + (1 - \tau) \varphi) d\varphi \right) \mathbf{a}(\theta)^T d\theta
\]

\[
= \int T \mathbf{a}(\{\phi + \tau \varphi\}_{mod T}) M(\phi, \phi + \phi) \mathbf{a}(\{\phi + \tau \varphi\}_{mod T})^T d\phi d\varphi
\]

for \( \tau \in [0,1] \). To simplify the following exposition, let \( \mathcal{I}_\tau(M) = \Gamma(\Phi^M_\tau) \) denote the linear operator in (15) that maps a transport plan to an interpolating covariance matrix, i.e., \( \mathbf{R}_\tau = \mathcal{I}_\tau(M) \). It may be noted from (15) that \( \mathcal{I}_\tau(M) \) is well-defined also for \( \tau \neq [0,1] \), i.e., the formulation allows also for extrapolation. The following proposition follows directly from the definition in (15).

**Proposition 3.** For any \( \tau \in \mathbb{R} \), the following basic properties hold for \( \mathbf{R}_\tau = \mathcal{I}_\tau(M) \):

a) If \( \mathbf{R}_0 \) and \( \mathbf{R}_1 \) have the same zeroth covariance, then it is also the zeroth covariance of \( \mathbf{R}_\tau \).

b) The matrix \( \mathbf{R}_\tau \) is a Toeplitz matrix.

c) The matrix \( \mathbf{R}_\tau \) is positive semi-definite.

Due to these properties, the proposed method offers a way of interpolating the covariances of, e.g., slowly varying time series as the interpolant \( \mathbf{R}_\tau \) allows for modeling linear changes in the spectrum of the process.

**Remark 1.** The interpolation approach generalizes trivially to the formulation in (11) between the covariances \( \Gamma(\mathbf{I}_0^M) \) and \( \Gamma(\mathbf{I}_1^M) \) under the assumption that \( \mathbf{I}_0 \) and \( \mathbf{I}_1 \) are perturbations of \( \mathbf{I}_0^M \) and \( \mathbf{I}_1^M \), respectively. In order to define an interpolation and extrapolation procedure from \( \mathbf{R}_0 \) to \( \mathbf{R}_1 \) where there is a cost \( \kappa \) for adding and subtracting mass, one may, along with the interpolation path \( \mathcal{I}_\tau(M) \), linearly add the part corresponding to the added and subtracted mass, i.e.,

\[
\mathbf{R}_\tau = \mathcal{I}_\tau(M) + (1 - \tau) \Gamma(\mathbf{I}_0 - \mathbf{I}_1^M) + \tau \Gamma(\mathbf{I}_1 - \mathbf{I}_1^M)
\]

Note that in this scenario, positive semi-definiteness cannot be guaranteed for the extrapolation case.

**A. Comparison with other methods**

The properties in Proposition 3 distinguish the proposed interpolant \( \mathbf{R}_\tau \) from other proposed matrix geodesics. As an
example, consider the basic method of forming interpolants using convex combinations of $\mathbf{R}_0$ and $\mathbf{R}_1$, i.e., $\mathbf{R}_\text{conv} = \tau \mathbf{R}_0 + (1 - \tau) \mathbf{R}_1$, for $\tau \in [0, 1]$. This preserves the Toeplitz structure, as well as the zeroth covariance and the positive semi-definiteness. However, from a spectral representation point of view, the convex combination gives rise to fade-in fade-out effects, i.e., only spectral modes directly related to $\mathbf{R}_0$ and $\mathbf{R}_1$ can be represented, and there can be no shift in the location of these modes (see also Example [7]) and Figure [3]. Other more sophisticated options include, e.g., the geodesic with respect to $g$-convexity [9]

$$\mathbf{R}_\tau = \mathbf{R}_0^{1/2} \left( \mathbf{R}_0^{-1/2} \mathbf{R}_1 \mathbf{R}_0^{-1/2} \right)^T \mathbf{R}_0^{1/2} \quad (17)$$

and the geodesic in [20, 21], which builds on optimal mass transport of Gaussian distributions and can be expressed as

$$\tilde{\mathbf{R}}_\tau = \left( (1 - \tau) \mathbf{R}_0^{1/2} + \tau \mathbf{R}_1^{1/2} \right) \left( (1 - \tau) \mathbf{R}_0^{1/2} + \tau \mathbf{R}_1^{1/2} \right)^H$$

$$U = \mathbf{R}_1^{-1/2} \mathbf{R}_0^{-1/2} \left( \mathbf{R}_0^{-1/2} \mathbf{R}_1 \mathbf{R}_0^{-1/2} \right)^{1/2} \quad (18)$$

Although both of these geodesics preserve positive semi-definiteness, they neither preserve the Toeplitz structure nor the zeroth covariance. As noted above, the three properties in Proposition [3] hold for any $\tau \in \mathbb{R}$ for the proposed approach, and thus directly allows for extrapolating using (15). In contrast, it may be noted that for the linear combination $\mathbf{R}_\text{conv}$ there are no guarantees that the resulting matrix is positive semi-definite if $\tau \notin [0, 1]$. Also, note that the alternative geodesics, $\tilde{\mathbf{R}}_\tau$ and $\tilde{\mathbf{R}}_\tau$, do not naturally generalize to extrapolation.

B. Tracking of slowly varying processes

The proposed interpolant $\mathbf{R}_\tau = \mathcal{I}_\tau(M)$ may also be readily used for tracking slowly varying stochastic process. As noted above, $\mathcal{I}_\tau(M)$ allows for the modeling of slow, i.e., locally linear, shifts in the location of spectral power. Building on this property, we can extend the optimal transport problem in [8], in order to fit a covariance path $\mathbf{R}_\tau$ to a sequence of $J$ covariance matrix estimates, $\mathbf{R}_j$, for $j = 1, \ldots, J$. As $\mathbf{R}_\tau$ is unambiguously determined from a transport plan $M$ via $\mathcal{I}_\tau(M)$, this tracking problem may be formulated as the convex optimization problem

$$\min_{M \in M_+^T(\mathbb{D}^2)} \int_{\mathbb{T}^2} c(\theta, \varphi) M(\theta, \varphi) d\theta d\varphi + \lambda \sum_{j=1}^J \left\| \mathcal{I}_\tau(M) - \mathbf{R}_j \right\|_F^2 \quad (19)$$

where $\lambda > 0$ is a user-specified regularization parameter. As may be noted from (10), the optimal transport plan $M$ is here determined as the one that minimizes not only the transport cost, but also takes into account the deviations of the interpolant $\mathbf{R}_j = \mathcal{I}_\tau(M)$ from the available covariance matrix estimates $\mathbf{R}_j$. The behavior of this construction is illustrated in the numerical section.

![Figure 1. Path of the off-diagonal element of the covariance matrix $\mathbf{R}_\tau$ for the case $n = 2$, when choosing $\mathbf{R}_0$ and $\mathbf{R}_1$ according to (26).](image)

C. Clustering: Barycenter computation

As a further example, we will see that the barycenter with respect to the distance notion $T_\kappa$ may be formulated as a convex optimization problem. This might be desirable in clustering or classification applications, where one is interested in either identifying classes of signals or processes based on their covariance matrices or associate a given covariance matrix with such a signal class. Considering the case of clustering, assume that $L$ covariance matrices, $\mathbf{R}_\ell$, $\ell = 1, \ldots, L$, are available. Then, we may define their barycenter via $T_\kappa$ according to

$$\mathbf{R}_\text{bary} = \arg \min_{\mathbf{R} \in M_+^T} \sum_{\ell=1}^L T_\kappa(\mathbf{R}, \mathbf{R}_\ell) \quad (20)$$

i.e., as the covariance matrix that minimizes the sum of $T_\kappa$ for the set of covariance matrices $\mathbf{R}_\ell$. Explicitly, $\mathbf{R}_\text{bary}$ solves the convex optimization problem

$$\min_{\mathbf{R} \in M_+^T, \Psi_\ell \in M_+^T} \sum_{\ell=1}^L \int_{\mathbb{T}^2} c(\theta, \varphi) M_\ell(\theta, \varphi) d\theta d\varphi + \kappa \sum_{\ell=1}^L \left\| \int_{\mathbb{T}} M_\ell(\theta, \phi) d\phi - \Psi_\ell \right\|_1$$

$$+ \kappa \sum_{\ell=1}^L \left\| \int_{\mathbb{T}} M_\ell(\theta, \phi) d\theta - \Phi_\ell \right\|_1$$

subject to $\Gamma(\Phi_\ell) = \mathbf{R}, \Gamma(\Psi_\ell) = \mathbf{R}_\ell, \ell = 1, \ldots, L$.

This formulation allows for using, e.g., K-means clustering (see, e.g., [41]) in order to identify classes of covariance matrices, as well as classify a given covariance matrix according to these classes. Classification of a covariance matrix $\mathbf{R}$ according to classes defined by a set of barycenters $\mathbf{R}^{(j)}$, $j = 1, \ldots, J$, may then be formulated as

$$\arg \min_{j \in \{1, \ldots, J\}} T_\kappa(\mathbf{R}, \mathbf{R}^{(j)}_{\text{bary}}) \quad (22)$$
In Section VI-D we present a simple illustration of this potential application, considering unsupervised clustering of phonemes.

V. SUM-OF-SQUARES RELAXATION

In order to solve the dual problem in (12) in practice, it has to be implemented as a finite dimensional problem, e.g., by gridding the space $\mathbb{T}^2$ and thereby approximating the set $\Omega_c$ using a finite number of constraints. However, for the special case of $c(\theta, \varphi) = |e^{i\theta} - e^{i\varphi}|^2$, this can also be done using a sum-of-squares (SOS) relaxation (see, e.g., [42]) of (12). For simplicity of notation, we present only the case without $\kappa (\kappa = \infty)$, i.e., the dual of (9) as formulated in (13). To this end, identify

$$z = e^{i\theta}, \ w = e^{i\varphi}$$

which allows us to write

$$c(\theta, \varphi) = 2 - zw^{-1} - z^{-1}w$$

$$\Gamma^*(A_0)(\theta) = \sum_{k=1}^{n} \sum_{k=1}^{n} |A_0|_{k, \ell} z^{\ell-k}$$

$$\Gamma^*(A_1)(\varphi) = \sum_{k=1}^{n} \sum_{k=1}^{n} |A_1|_{k, \ell} w^{\ell-k}$$

Thus, the set of constraints defining the feasible set $\Omega_c$ is given by $\Gamma^*(A_0)(\theta) + \Gamma^*(A_1)(\varphi) \leq c(\theta, \varphi)$, for all $\theta, \varphi \in \mathbb{T}$, or, equivalently,

$$2 - zw^{-1} - z^{-1}w - \sum_{k=1}^{n} |A_0|_{k, \ell} z^{\ell-k} - \sum_{k=1}^{n} |A_1|_{k, \ell} w^{\ell-k} \geq 0$$

(23)

Note that in the two-dimensional trigonometric polynomial (23), the coefficient for $z^{-k_1}w^{-k_2}$ is equal to

$$2 - \text{diag}(A_0 + A_1)^T 1 \quad \text{for} \quad k_1 = k_2 = 0$$

$$-\text{diag}(A_0, k_1)^T 1 \quad \text{for} \quad k_1 \in \mathbb{Z}_{\neq 0} \text{ and } k_2 = 0$$

$$-\text{diag}(A_1, k_2)^T 1 \quad \text{for} \quad k_1 = 0 \text{ and } k_2 \in \mathbb{Z}_{\neq 0}$$

$$-1 \quad \text{for} \quad k_1 = 1 \text{ and } k_2 = -1$$

$$-1 \quad \text{for} \quad k_1 = -1 \text{ and } k_2 = 1$$

$$0 \quad \text{otherwise}$$

where $\mathbb{Z}_{\neq 0} = \{ k \in \mathbb{Z} | |k| \leq n, k \neq 0 \}$. Here, $\text{diag}(X, k)$ denotes the column vector containing the elements on the $k$th super-diagonal of the matrix $X$, if $k > 0$, and the elements on the $k$th sub-diagonal, if $k < 0$, with $1$ denoting a column vector of ones of appropriate dimension.

In order to formulate a computationally feasible optimization problem, we remove the non-negativity constraint for the two-dimensional polynomial in (23) and instead impose that it should have a sum-of-squares representation [42], [43]. In particular, we impose that the polynomial in (23) should be of the form

$$P(z, w) = (z^{-1})^T Qz$$

(24)

where $Q \in \mathbb{M}^{m^2}$ is positive semi-definite,

$$z = \begin{bmatrix} 1 & w & \ldots & w^{m-1} \end{bmatrix}^T \otimes \begin{bmatrix} 1 & z & \ldots & z^{m-1} \end{bmatrix}^T$$

$$z^{-1} = \begin{bmatrix} 1 & w^{-1} & \ldots & w^{-m+1} \end{bmatrix}^T \otimes \begin{bmatrix} 1 & z^{-1} & \ldots & z^{-m+1} \end{bmatrix}^T$$

and $\otimes$ is the Kronecker product. Note that any polynomial on this form is non-negative by definition, and, furthermore, for any non-negative polynomial $P^*$, there is a sequence of polynomials, $P_m$, on the form (24) such that $\|P_m - P^*\|_{\infty} \to 0$ as $m \to \infty$.

Next, note that the coefficients of $P$ are associated to the elements of $Q$ in (24) according to

$$p_{k_1, k_2} = \text{trace}(T_{k_1, k_2} Q), \quad \text{for} \quad -m + 1 \leq k_1, k_2 \leq m - 1$$

where

$$T_{k_1, k_2} = T_{k_2} \otimes T_{k_1}$$

and $T_k$ is the matrix with ones on the $k$th diagonal and zeros elsewhere. Putting these facts together by requiring that the polynomial (23) can be written as a sum-of-square (24), we
approximate \( (13) \) by the semi-definite program (SDP)

\[
\begin{align*}
\text{maximize} & \quad \langle A_0, R_0 \rangle + \langle A_1, R_1 \rangle \\
\text{subject to} & \quad Q \succeq 0 \\
& \quad \text{trace} (Q) = 2 - \text{diag} (A_0 + A_1)^T 1 \\
& \quad \text{trace} (T_{-1,1} Q) = -1 \\
& \quad \text{trace} (T_{k_1,0} Q) = -\text{diag} (A_0, k_1)^T 1, \; k_1 \in \mathbb{Z}_{n \setminus 0} \\
& \quad \text{trace} (T_{0,k_2} Q) = -\text{diag} (A_1, k_2)^T 1, \; k_2 \in \mathbb{Z}_{n \setminus 0} \\
& \quad \text{trace} (T_{k_1,k_2} Q) = 0, \; \text{if } k_1 k_2 = 1 \\
& \quad \text{or } |k_1 k_2| \leq m^2
\end{align*}
\]

It is worth noting that the \( P \) defined by the optimal \( Q \) in (24) is non-negative (actually, strictly positive) on the unit torus. Thus, for the solution of the SDP problem in (25), there will be a corresponding feasible solution of (13). Therefore, any solution to (25) will give a lower bound for the optimal objective value of (13). However, any non-negative polynomial may be arbitrarily well approximated by \( P \) of the form (24) by a suitable choice the degree \( m \), and thus the maximal objective value of (25) will converge to the maximal objective value of (13) as \( m \) grows. By comparison, directly discretizing \( \mathbb{T} \), thereby approximating \( \Omega \) using a finite number of constraints, yields an upper bound for the maximal objective value of (13). In this case, the maximal objective value will also converge to that of (13), this time from above, as the spacing of the discretization of the grid becomes finer.

VI. NUMERICAL EXAMPLES

In this section, we present some numerical examples illustrating different aspects and application areas of the proposed distance notion \( T_{\nu} \), as well as the interpolant \( R_{\tau} = I_{\tau} (M) \).

A. Trajectory example

In order to illustrate the behavior of the proposed interpolation method, we consider a simple scenario with covariance matrices of dimension \( n = 2 \). Consider covariance matrices of the form

\[
R_0 = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1 & re^{i\varphi} \\ re^{-i\varphi} & 1 \end{bmatrix}
\]

(26)

where \( r \in [-1, 1] \). Thus, considering interpolating paths \( R_{\tau} \), these will be on the form

\[
R_{\tau} = \begin{bmatrix} 1 & r_{1,\tau} \\ r_{1,\tau} & 1 \end{bmatrix}
\]

(27)

Figure 1 displays the real and imaginary part of \( r_{1,\tau} \) for \( \tau \in [0, 1] \) when varying the magnitude \( r \) of the off-diagonal element of \( R_0 \) and \( R_1 \) between 0 and 1. As can be seen, the trajectories of \( r_{1,\tau} \) approximately correspond to convex combinations \((1 - \tau)R_0 + \tau R_1 \). At the other extreme, the interpolant corresponding to covariance matrices \( R_0 \) and \( R_1 \) that have consistent spectra that are close to being singular will also have almost singular consistent spectra.

B. Interpolation and extrapolation for DOA

Next, we illustrate the proposed methods ability to produce interpolants \( R_{\tau} \) that are consistent with locally linear changes in the frequency location of spectral power. This is illustrated using a direction-of-arrival (DOA) estimation problem. Consider a uniform linear array (ULA) with 15 sensors with half-wavelength sensor spacing and a scenario where two covariance matrices

\[
R_0 = \frac{1}{2} \sum_{\ell=1}^{3} a(\theta_{0,\ell})a^H(\theta_{0,\ell}) + \sigma^2 I
\]

\[
R_1 = \frac{1}{2} a(\theta_{1,1})a^H(\theta_{1,1}) + \frac{1}{4} \sum_{\ell=1}^{3} a(\theta_{1,\ell})a^H(\theta_{1,\ell}) + \sigma^2 I
\]

and

\[
\text{subject to } R_{\tau} = \begin{bmatrix} 1 & r_{1,\tau} \\ r_{1,\tau} & 1 \end{bmatrix}
\]

(27)

Figure 1 displays the real and imaginary part of \( r_{1,\tau} \) for \( \tau \in [0, 1] \) when varying the magnitude \( r \) of the off-diagonal element of \( R_0 \) and \( R_1 \) between 0 and 1. As can be seen, the trajectories of \( r_{1,\tau} \) approximately correspond to convex combinations \((1 - \tau)R_0 + \tau R_1 \). At the other extreme, the interpolant corresponding to covariance matrices \( R_0 \) and \( R_1 \) that have consistent spectra that are close to being singular will also have almost singular consistent spectra.
are available. Here \( \theta_1^{(0)} = \theta_1^{(1)} = -50^\circ, \theta_2^{(0)} = 30^\circ, \theta_2^{(1)} = 20^\circ, \) and \( \theta_3^{(1)} = 40^\circ, \) and \( \sigma^2 = 0.05. \) Such a scenario may be interpreted as a target at \( \theta_2^{(0)} \) splitting up into two targets at \( \theta_2^{(1)} \) and \( \theta_3^{(1)} \) as time progresses, whereas the target at \( \theta_1^{(0)} \) stays put. We use the proposed method in (9), as \( \mathbf{R}_0 \) and \( \mathbf{R}_1 \) have the same zeroth covariance, in order to find the optimal transport map \( \mathbf{M}. \) Then, using (13), we compute covariance matrices \( \mathbf{R}_\tau, \) for \( \tau \in [0, 2], \) i.e., we both interpolate on \( \tau \in [0, 1] \) and extrapolate on \( \tau \in (1, 2]. \) This is then compared to the basic interpolant \( \mathbf{R}_\tau^{\text{conv}} \) based on convex and linear combinations of \( \mathbf{R}_0 \) and \( \mathbf{R}_1, \) as well as the more sophisticated covariance matrix geodesics \( \mathbf{R}_\tau \) and \( \hat{\mathbf{R}}_\tau, \) as defined in (17) and (18), respectively. For these four cases, we then estimate the corresponding inter- and extrapolated spectra using the correlogram, i.e., as

\[
\Phi^{\text{corr}}(\mathbf{X}, \theta) = a(\theta)^H \mathbf{X} a(\theta)
\]

(28)

where \( \mathbf{X} \) is substituted for the four different covariance interpolants. The results for the proposed interpolant \( \mathbf{R}_\tau \) are shown in Figure 2. As can be seen, \( \mathbf{R}_\tau \) indeed models a scenario where one of the targets has a constant location, whereas the second target splits up into two smaller targets. Note also that \( \mathbf{R}_\tau \) implies that the smaller target continue linearly with respect to the look-angle, \( \theta, \) also for the extrapolation case, i.e., for \( \tau > 1. \) In contrast, the convex (linear for \( \tau > 1 \)) combination \( \mathbf{R}_\tau^{\text{conv}}, \) as shown in Figure 3, display undesirable behavior; clear fade-in fade-out effects are visible, and non-negativity is violated as \( \tau \to 2 \) due to \( \mathbf{R}_\tau^{\text{conv}} \) becoming indefinite. Similar objections may be raised against the geodesics \( \hat{\mathbf{R}}_\tau \) and \( \hat{\mathbf{R}}_\tau, \) shown in Figures 4 and 5, both displaying fade-in fade-out effects and thereby fail to model any displacement of the targets. Also, note that the total power of the signal varies greatly as \( \tau \) goes from 0 to 2, especially for \( \hat{\mathbf{R}}_\tau. \)

\footnote{Note that \( \theta \) in this example denotes spatial frequency. For simplicity, we retain the notation \( a(\theta) \) also for this case.}

C. Tracking of an AR-process

Next, we illustrate the approach in (19) for the tracking of signals with slowly varying spectral content. To this end, consider a complex autoregressive (AR) process with one complex, time-varying pole. The pole is placed at a constant radius of 0.9, and moves from the frequency 0.4\( \pi \) to 0.6\( \pi. \) Spectral estimates based directly on covariance matrix estimates \( \mathbf{R} \) are shown in Figure 6. These covariance matrix estimates are obtained as the outer product estimate, based on 100 samples each, where the overlap between each estimate is 80 samples. As can be seen, the spectral estimates are very noisy and vary greatly in power. Using five of these covariance matrix estimates \( \mathbf{R}_\tau \) evenly spaced throughout the signal, we solve (19) in order to obtain an estimated covariance path, \( \hat{\mathbf{R}}_\tau. \) The resulting spectra, estimated using (28), are shown in Figure 7. As can be seen, the path resulting from the proposed method allows for a smooth tracking of the shift in spectral content.

D. K-means clustering

As a simple illustration of how to utilize the barycenter formulation in Section IV-C, we consider the application of unsupervised clustering of phonemes. Specifically, we consider 7 utterances: 3 utterances of the phoneme /ael/, 2 utterances of /loy/, and 2 utterances of /n/. For each utterance, we estimate a covariance matrix of dimension \( n = 10 \) and then run a K-means algorithm that alternates between classifying each estimated covariance matrix according to (22) and computing new barycenters according to (20). We initiate the algorithm by choosing initial barycenters as convex combinations of the available covariance matrix estimates, and demand separation into three clusters. The algorithm is then run until convergence, i.e., until the classification has stabilized. The results are shown in Table 1. As can be seen, the K-means algorithm utilizing (20) and (22) is indeed able to correctly identify three clusters corresponding to the three different phonemes. Table 1 presents the corresponding distance matrix, i.e., the matrix detailing the distance \( T_\kappa \) between each covariance
matrix estimate and each barycenter. As can be seen, the clusters are quite well-separated.

E. Fixed cost for moving mass

In some scenarios, there might be a relatively large noise component present in both \( R_0 \) and \( R_1 \), where the noise power is localized in frequency and this localization is the same in both \( R_0 \) and \( R_1 \). This might be the case in, e.g., DOA estimation scenarios where a source of interest is moving in the presence of a stationary interferer. Then, if the source for example moves past the location of the interferer, the optimal transport problem in (29) may result in a power association such that the source and the interferer are mixed together. Such a scenario is illustrated in Figure 8, showing estimated spectra obtained from the interpolant \( R_1 \) as given by (29) with the cost function \( c(\theta, \varphi) = |e^{i\theta} - e^{i\varphi}|^2 \). Here, the source is moving from look-angle \( \theta = 30^\circ \) to \( \theta = -20^\circ \), whereas there is a fixed interferer located at \( \theta = 0^\circ \) having a third of the power of the source. In order to avoid this type of problem and promoting transport plans that avoid transporting stationary masses, the cost function may be modified according to

\[
c(\theta, \varphi) = \begin{cases} 
1 + |e^{i\theta} - e^{i\varphi}|^2 & \text{if } \theta \neq \varphi \\
0 & \text{if } \theta = \varphi 
\end{cases} \tag{29}
\]

Thus, the cost function is here formulated such that there is a fixed, baseline cost of moving any mass. It may be noted that this cost function satisfies the assumptions of Proposition 2 and thus the proposition holds also for this choice of \( c(\theta, \varphi) \). The resulting estimated spectra obtained from the interpolant \( R_1 \) resulting from solving (9) using this modified cost function is shown in Figure 9. As can be seen, the source and the interferer are now well separated throughout the interpolated path.

VII. CONCLUSIONS AND FURTHER DIRECTIONS

In this work, we have proposed a notion of distance for positive semi-definite Toeplitz matrices. By considering spectral representations of such matrices, the proposed measure is based on distances, in an optimal mass transport sense, between families of spectra consistent with the Toeplitz matrices. We have shown that the proposed distance measure, under some mild assumptions, is contractive with respect to additive and multiplicative noise. The proposed measure may be used to, for example, define inter- and extrapolation of Toeplitz matrices, being of interest in applications such as tracking of slowly varying signals.

A future direction of this methodology is to generalize the distance measure for structured matrices corresponding to input-to-state covariances in the THREE framework for spectral analysis [12]. This generalizes ideas from beamspace processing, enabling the user to improve resolution and robustness in power spectral estimation over selected frequency bands [44], [45]. This will be the subject of further research.

| Utterance | Class 1 | Class 2 | Class 3 |
|-----------|---------|---------|---------|
| True phoneme | /aɛ/ | /aɛ/ | /oɪ/ |
| Class | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 |
| 3 | 2 | 3 | 3 |

**TABLE I**

Clustering of 7 utterances into three clusters using a K-Means algorithm utilizing the barycenter formulation in Section IV-C. The third row indicates the identified classes as given by the algorithm.

| Utterance | Class 1 | Class 2 | Class 3 |
|-----------|---------|---------|---------|
| 1 | 0.001 | 0.017 | 0.168 |
| 2 | 0.061 | 0.121 | 0.447 |
| 3 | 0.258 | 0.374 | 0.426 |

**TABLE II**

Distance, as measured by \( T_\kappa \), between each utterance and each barycenter for the three identified clusters.
Appendix

A. Proof of Proposition 2 for multiplicative noise

Here, we present the proof for $T_{\kappa}$ being contractive with respect to multiplicative noise.

Proof. Let the noise covariance matrix be $\mathbf{R}_w$, implying that the covariances of the contaminated processes are $\mathbf{R}_0' = \mathbf{R}_0 \oplus \mathbf{R}_w$ and $\mathbf{R}_1' = \mathbf{R}_1 \oplus \mathbf{R}_w$, respectively. Let $\Phi_w$ be any spectrum consistent with $\mathbf{R}_w$, i.e., $\Gamma(\Phi_w) = \mathbf{R}_w$. Also, let the zeroth covariance of the noise process be smaller than or equal to 1, so that

$$\int \Phi_w(\theta) d\theta \leq 1$$

We have

$$\begin{align*}
T_{\kappa}(\mathbf{R}_0', \mathbf{R}_1') &= \max_{(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+} \langle \mathbf{A}_0, \mathbf{R}_0 \oplus \mathbf{R}_w \rangle + \langle \mathbf{A}_1, \mathbf{R}_1 \oplus \mathbf{R}_w \rangle \\
&= \max_{(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+} \langle \mathbf{A}_0 \ominus \mathbf{R}_w, \mathbf{R}_0 \rangle + \langle \mathbf{A}_1 \ominus \mathbf{R}_w, \mathbf{R}_1 \rangle \\
&= \max_{(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+} \langle \mathbf{A}_0, \mathbf{R}_0 \rangle + \langle \mathbf{A}_1, \mathbf{R}_1 \rangle \\
& \quad \text{subject to} \\
& \mathbf{A}_0 = \mathbf{A}_0 \ominus \mathbf{R}_w \\
& \mathbf{A}_1 = \mathbf{A}_1 \ominus \mathbf{R}_w
\end{align*}$$

In order to show that $(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+$ implies $(\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_1) \in \Omega_0^+$, we note the following. Assume that $(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+$. Then,

$$\Gamma^*(\mathbf{A}_0)(\theta - \phi) + \Gamma^*(\mathbf{A}_1)(\varphi - \phi) \leq c(\theta - \phi, \varphi - \phi) = c(\theta, \varphi)$$

for all $\phi, \theta, \varphi \in \mathbb{T}$, where the last equality follows from the assumption that $c$ is shift-invariant. Define

$$\tilde{\Phi}_w(\phi) = \Phi_w(-\phi)$$

which satisfies $\int \tilde{\Phi}_w(\phi) d\phi \leq 1$ and $\tilde{\Phi}_w(\phi) \geq 0$, $\forall \phi \in \mathbb{T}$. Multiplying both sides of (30) with $\tilde{\Phi}_w(\phi)$ and integrating with respect to $\phi$ then yields

$$\left( \Gamma^*(\mathbf{A}_0) \ast \tilde{\Phi}_w \right)(\theta) + \left( \Gamma^*(\mathbf{A}_1) \ast \tilde{\Phi}_w \right)(\varphi) \leq c(\theta, \varphi)$$

for all $\theta, \varphi \in \mathbb{T}$. Similarly,

$$\left( \Gamma^*(\mathbf{A}_0) \ast \tilde{\Phi}_w \right)(\theta) \leq \kappa, \left( \Gamma^*(\mathbf{A}_1) \ast \tilde{\Phi}_w \right)(\varphi) \leq \kappa$$

for all $\theta, \varphi \in \mathbb{T}$. Further, for any $\theta \in \mathbb{T}$, we have

$$\left( \Gamma^*(\mathbf{A}_0) \ast \tilde{\Phi}_w \right)(\theta) = \int_{\mathbb{T}} a(\varphi) H_{\mathbf{A}_0} a(\varphi) \tilde{\Phi}_w(\theta - \varphi) d\varphi$$

where the last equality uses the definition of the operator $\Gamma(\cdot)$. As $\delta_{\theta}(\varphi)$ is a spectrum consistent with the rank-one covariance matrix $\mathbf{R}_{\theta} = a(\theta)a(\theta)^H$, we have, by the properties of the Fourier transform, that $(\Phi_w \ast \delta_{\theta})(\varphi)$ is a spectrum consistent with the covariance matrix $\mathbf{R}_w \ominus \mathbf{R}_{\theta}$, i.e.,

$$\langle \mathbf{A}_0, \Gamma(\Phi_w \ast \delta_{\theta}) \rangle = \langle \mathbf{A}_0, \mathbf{R}_w \ominus \mathbf{R}_{\theta} \rangle$$

and therefore

$$\left( \Gamma^*(\mathbf{A}_0) \ast \tilde{\Phi}_w \right)(\theta) = \langle \mathbf{A}_0, \mathbf{R}_w \ominus \mathbf{R}_{\theta} \rangle$$

where the third equality follows from the definition of $\hat{\mathbf{A}}_0$ and the last from the definition of $\Gamma^*(\cdot)$. Using the same reasoning for $\mathbf{A}_1$, we thus have that $(\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_1) \in \Omega_0^+$ implies $(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+$. Combined, this yields

$$T_{\kappa}(\mathbf{R}_0', \mathbf{R}_1') \leq \max_{(\mathbf{A}_0, \mathbf{A}_1) \in \Omega_0^+} \langle \hat{\mathbf{A}}_0, \mathbf{R}_0 \rangle + \langle \hat{\mathbf{A}}_1, \mathbf{R}_1 \rangle = T_{\kappa}(\mathbf{R}_0, \mathbf{R}_1)$$

$\square$

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