SYMBOLIC MODELS FOR NETWORKED CONTROL SYSTEMS

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ABSTRACT. Networked control systems (NCS) are spatially-distributed systems in which communication between sensors, controllers, and actuators is supported by a shared communication network that is subject to variable communication delays, quantization errors, packet losses, limited bandwidth, and other practical non-idealities. This work investigates the problem of constructively deriving symbolic models of NCS by simultaneously considering the mentioned network non-idealities. One can employ the obtained abstracted models to synthesize symbolic controllers enforcing rich specifications over NCS. Examples of such specifications include properties expressed as formulae in linear temporal logic (LTL) or as automata on infinite strings.

1. INTRODUCTION

Networked control systems (NCS) are spatially distributed systems in which sensors, controllers, and actuators communicate through shared communication channels. The analysis and synthesis of NCS have received significant attention in the last few years because they offer many advantages such as increasing architecture flexibility and reduced installation and maintenance costs. However, non-idealities of the network introduce new challenges for the analysis of the behavior of the plant (such as its stability) and the synthesis of control schemes. One can categorize the non-idealities of the network into: (i) quantization errors, (ii) packet dropouts, (iii) time-varying sampling/transmission intervals, (iv) time-varying communication delays, and (v) communication constraints (scheduling protocols).

Recently, there have been many studies focused mostly on the stability properties of NCS: in \cite{7}, (iii)-(v) are simultaneously considered; in \cite{11} (i), (ii), and (iv) are taken into account; \cite{1} studies (ii) and (v); \cite{5} focuses on (ii) and (iii); in \cite{10, 22} (ii)-(iv) are considered; and in \cite{18} (i), (iii), and (v). Despite all the progress on stability analysis of NCS, as reported in \cite{7, 11, 1, 5, 10, 22, 18}, there are no notable results in the literature dealing with more complex objectives, such as verification or (controller) synthesis for logical specifications. Examples of those specifications include linear temporal logic (LTL) formulae, or automata on infinite strings, which are not amenable to be dealt with existing approaches for NCS.

A promising direction to investigate these complex properties is the use of symbolic models \cite{21}. Symbolic models are abstract descriptions of the original dynamics, where each abstract state (or symbol) corresponds to an aggregate of states in the concrete system. When a finite symbolic model is obtained and is formally put in relationship with the original system, one can leverage algorithmic machinery for controller synthesis of symbolic systems \cite{21} to automatically synthesize hybrid controllers for the original model \cite{21}. To the best of our knowledge, the only results available in the literature on the construction of symbolic models for NCS are the ones in \cite{9, 8}. The work in \cite{9, 8} considers the network non-idealities (i), (ii), and (iv) simultaneously. However, the results in \cite{9, 8} exhibit several possible shortcomings: they are limited to grid-based symbolic models, which severely suffer from the curse of dimensionality; they only consider static symbolic controllers (i.e. memoryless) whereas for general temporal logic specifications the symbolic controllers are often dynamic (i.e. with memory) \cite{6}; the possibility of out-of-order packet arrivals is not considered; only specifications expressed in terms of some types of nondeterministic automata can be addressed; and, furthermore, the given specification needs to be reformulated in an extended state-space, in order to construct a more complex specification that is applicable to the obtained symbolic model.
In this paper, we provide a construction of symbolic models for NCS using available symbolic models obtained exclusively for the plant. One can thus use existing results to provide symbolic models for the plant, such as the grid-based approaches in [14, 23] or formula-guided (non-grid-based) approaches [21] and construct the symbolic models for the NCS from those. As long as there exists some type of symbolic abstraction of the plant, one can always use the results provided in this paper to construct symbolic models for the NCS. We explicitly consider the network non-idealities (i), (ii), and (iv) simultaneously. Furthermore, relying on symbolic abstractions, one can easily incorporate scheduling constraints (v) as well. We also consider explicitly possible out-of-order packet arrivals and message rejection, i.e. the effect of older data being neglected because more recent data is available. Our work is not limited to problems where the controller needs to be static. As a result, we enable the study of larger classes of logic specifications such as those expressed as general LTL formulae or as automata on infinite strings, without requiring any additional reformulation.

2. Control Systems & (In)Stability Notions

2.1. Notation. The identity map on a set $A$ is denoted by $1_A$. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_+^*$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. Given a set $A$, define $A^{n+1} = A \times A^n$ for any $n \in \mathbb{N}$. Given a vector $x \in \mathbb{R}^n$, we denote by $x_i$ the $i$-th element of $x$, and by $\|x\|$ the infinity norm of $x$, namely, $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, where $|x_i|$ denotes the absolute value of $x_i$. Given an interval $[a, b] \subseteq \mathbb{R}$ with $a \leq b$, we denote by $[a; b]$ the set $[a, b] \cap \mathbb{N}$.

Given a measurable function $f : \mathbb{R}_+^* \to \mathbb{R}^n$, the (essential) supremum of $f$ is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (\text{ess sup}) \{\|f(t)\|, t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_+^* \to \mathbb{R}_+^*$, is said to belong to class $K$ if it is strictly increasing and $\gamma(0) = 0$; $\gamma$ is said to belong to class $K_\infty$ if $\gamma \in K$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ is said to belong to class $K\mathcal{L}$ if, for each fixed $s$, the map $\beta(r, s)$ belongs to class $K_\infty$ with respect to $r$ and, for each fixed nonzero $r$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, $R^{-1}$ denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$. When $R$ is an equivalence relation on a set $A$, we denote by $[a]$ the equivalence class of $a$, by $A/R$ the set of all equivalence classes, and by $\pi_R : A \to A/R$ the natural projection map taking a point $a \in A$ to its equivalence class $\pi(a) = [a] \in A/R$.

2.2. Control systems. The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1. A control system is a tuple $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $U \subseteq \mathbb{R}^m$ is the compact input set;
- $\mathcal{U}$ is a subset of the set of all measurable functions of time from intervals of the form $[a, b] \subseteq \mathbb{R}$ to $U$ with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $Q \subseteq \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that $\|f(x, u) - f(y, u)\| \leq Z\|x - y\|$ for all $x, y \in Q$ and all $u \in \mathcal{U}$.

A curve $\xi : [a, b] \to \mathbb{R}^n$ is said to be a trajectory of $\Sigma$ if there exists $v \in \mathcal{U}$ satisfying:

$$\xi(t) = f(\xi(t), v(t)),$$

for almost all $t \in [a, b]$. Although we have defined trajectories over open domains, we shall refer to trajectories $\xi : [0, t] \to \mathbb{R}^n$ defined on closed domains $[0, t]$, $t \in \mathbb{R}^+$, with the understanding of the existence of a trajectory $\xi' : [a, b] \to \mathbb{R}^n$ such that $\xi = \xi'|_{[0, t]}$ with $a < 0$ and $b > t$. We also write $\xi_{x_0}(t)$ to denote the point reached at time $t$ under the input $v$ from the initial condition $x = \xi_{x_0}(0)$; the point $\xi_{x_0}(t)$ is uniquely determined, since the assumptions on $f$ ensure existence and uniqueness of trajectories [20].
A control system $\Sigma$ is said to be forward complete if every trajectory is defined on an interval of the form $[\alpha, \infty]$. Sufficient and necessary conditions for a control system to be forward complete can be found in [4].

2.3. (In)Stability notion. Some of the existing results, recalled in this paper, require certain (in)stability properties on $\Sigma$. First, we recall the stability property, introduced in [3], as defined next.

**Definition 2.2.** A control system $\Sigma$ is incrementally input-to-state stable (\(\delta\)-ISS) if it is forward complete and there exist a $KL$ function $\beta$ and a $K_{\infty}$ function $\gamma$ such that for any $t \in \mathbb{R}^+_0$, any $x, x' \in \mathbb{R}^n$, and any $v, v' \in U$, the following condition is satisfied:

\[
\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq \beta (\|x - x'\|, t) + \gamma (\|v - v'\|_{\infty}),
\]

Now we recall the instability property, introduced in [23], as defined next.

**Definition 3.2.** A control system $\Sigma$ is incrementedally forward complete (\(\delta\)-FC) if it is forward complete and there exist continuous functions $\beta : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0$ and $\gamma : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0$ such that for each fixed $s$, the functions $\beta(r, s)$ and $\gamma(r, s)$ belong to class $K_{\infty}$ with respect to $r$, and for any $t \in \mathbb{R}^+_0$, any $x, x' \in \mathbb{R}^n$, and any $v, v' \in U$, the following condition is satisfied:

\[
\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq \beta (\|x - x'\|, t) + \gamma (\|v - v'\|_{\infty}, t).
\]

We refer the interested readers to the results in [3] (resp. [23]) providing a characterization (resp. description) of $\delta$-ISS (resp. $\delta$-FC) in terms of the existence of so-called incremental Lyapunov functions.

3. Systems & Approximate Equivalence Notions

3.1. Systems. We now recall the notion of systems, introduced in [21], that we use later to describe NCS as well as their symbolic abstractions.

**Definition 3.1.** A system $S$ is a tuple $S = (X, X_0, U, \rightarrow, Y, H)$ consisting of: a (possibly infinite) set of states $X$; a (possibly infinite) set of initial states $X_0 \subseteq X$; a (possibly infinite) set of inputs $U$; a transition relation $\rightarrow \subseteq X \times U \times X$; a set of outputs $Y$; and an output map $H : X \rightarrow Y$.

A transition $(x, u, x') \in \rightarrow$ is also denoted by $x \xrightarrow{u} x'$. If $x \xrightarrow{u} x'$, state $x'$ is called a $u$-successor of state $x$. We denote by $Post_u(x)$ the set of all $u$-successors of a state $x$ and by $U(x)$ the set of inputs $u \in U$ for which $Post_u(x)$ is nonempty.

System $S$ is said to be:

- *metric*, if the output set $Y$ is equipped with a metric $d : Y \times Y \rightarrow \mathbb{R}^+_0$;
- *finite* (or *symbolic*), if $X$ and $U$ are finite sets;
- *countable*, if $X$ and $U$ are countable sets;
- *deterministic*, if for any state $x \in X$ and any input $u \in U$, $|Post_u(x)| \leq 1$;
- *nondeterministic*, if there exist a state $x \in X$ and an input $u \in U$ such that $|Post_u(x)| > 1$;

Given a system $S = (X, X_0, U, \rightarrow, Y, H)$, we denote by $|S|$ the size of $S$, defined as $|S| := |\rightarrow|$, which is equal to the total number of transitions in $S$. Note that it is more reasonable to consider $|\rightarrow|$ as the size of $S$ rather than $|X|$, as it is the transitions of $S$ that are required to be stored rather than just the states of $S$.

3.2. System relations. We recall the notions of (alternating) approximate (bi)simulation relation, introduced in [13] [19], which are useful to relate properties of NCS to those of their symbolic models. First we recall the notions of approximate (bi)simulation relation, introduced in [13].

**Definition 3.2.** Let $S_a = (X_a, X_{a0}, U_a, \xrightarrow{a}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \xrightarrow{b}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $d$. For $\varepsilon \in \mathbb{R}^+_0$, a relation $R \subseteq X_a \times X_b$ is said to be an $\varepsilon$-approximate simulation relation from $S_a$ to $S_b$ if the following three conditions are satisfied:
(i) for every \( x_{a0} \in X_{a0} \), there exists \( x_{b0} \in X_{b0} \) with \((x_{a0}, x_{b0}) \in R\);
(ii) for every \((a, x_b) \in R\) we have \( d(H_a(x_a), H_b(x_b)) \leq \varepsilon\);
(iii) for every \((a, x_b) \in R\), the existence of \( x_a \xrightarrow{u_a} x'_a \) in \( S_a \) implies the existence of \( x_b \xrightarrow{u_b} x'_b \) in \( S_b \) satisfying \((x'_a, x'_b) \in R\).

A relation \( R \subseteq X_a \times X_b \) is said to be an \( \varepsilon \)-approximate bisimulation relation between \( S_a \) and \( S_b \) if \( R \) is an \( \varepsilon \)-approximate simulation relation from \( S_a \) to \( S_b \) and \( R^{-1} \) is an \( \varepsilon \)-approximate simulation relation from \( S_b \) to \( S_a \).

System \( S_a \) is \( \varepsilon \)-approximately simulated by \( S_b \), or \( S_b \) \( \varepsilon \)-approximately simulates \( S_a \), denoted by \( S_a \preceq_S S_b \), if there exists an \( \varepsilon \)-approximate simulation relation from \( S_a \) to \( S_b \). System \( S_a \) is \( \varepsilon \)-approximate bisimilar to \( S_b \), denoted by \( S_a \cong_S S_b \), if there exists an \( \varepsilon \)-approximate bisimulation relation between \( S_a \) and \( S_b \).

Note that when \( \varepsilon = 0 \), condition (ii) in the above definition is modified as \((x_a, x_b) \in R\) if and only if \( H_a(x_a) = H_b(x_b) \), and \( R \) becomes an exact (bi)simulation relation, as introduced in [17].

As explained in [19], for nondeterministic systems we need to consider relationships that explicitly capture the adversarial nature of nondeterminism. Furthermore, these types of relations become crucial to enable the refinement of symbolic controllers [21].

**Definition 3.3.** Let \( S_a = (X_a, X_{a0}, U_a, \xrightarrow{a} Y_a, H_a) \) and \( S_b = (X_b, X_{b0}, U_b, \xrightarrow{b} Y_b, H_b) \) be metric systems with the same output sets \( Y_a = Y_b \) and metric \( d \). For \( \varepsilon \in \mathbb{R}_0^+ \), a relation \( R \subseteq X_a \times X_b \) is said to be an alternating \( \varepsilon \)-approximate simulation relation from \( S_a \) to \( S_b \) if conditions (i) and (ii) in Definition 3.2, as well as the following condition, are satisfied:

(iii) for every \((a, x_b) \in R\) and for every \( u_a \in U_a(a) \) there exists some \( u_b \in U_b(x_b) \) such that for every \( x'_b \in \text{Post}_{u_b}(x_b) \) there exists \( x'_a \in \text{Post}_{u_a}(a) \) satisfying \((x'_a, x'_b) \in R\).

A relation \( R \subseteq X_a \times X_b \) is said to be an alternating \( \varepsilon \)-approximate bisimulation relation between \( S_a \) and \( S_b \) if \( R \) is an alternating \( \varepsilon \)-approximate simulation relation from \( S_a \) to \( S_b \) and \( R^{-1} \) is an alternating \( \varepsilon \)-approximate simulation relation from \( S_b \) to \( S_a \).

System \( S_a \) is alternately \( \varepsilon \)-approximately simulated by \( S_b \), or \( S_b \) alternately \( \varepsilon \)-approximately simulates \( S_a \), denoted by \( S_a \preceq_{AS} S_b \), if there exists an alternating \( \varepsilon \)-approximate simulation relation from \( S_a \) to \( S_b \). System \( S_a \) is alternately \( \varepsilon \)-approximately bisimilar to \( S_b \), denoted by \( S_a \cong_{AS} S_b \), if there exists an alternating \( \varepsilon \)-approximate bisimulation relation between \( S_a \) and \( S_b \).

Note that when \( \varepsilon = 0 \), \( R \) becomes an exact alternating (bi)simulation relation, as introduced in [2].

It can be readily seen that the notions of approximate (bi)simulation relation and of alternating approximate (bi)simulation relation coincide when the systems involved are deterministic as in Definition 3.1.

### 3.3. Existence of symbolic models for control systems.

Let us define a metric system \( S_r(\Sigma) := (X_r, X_{r0}, U_r, \xrightarrow{r} Y_r, H_r) \), capturing all the information contained in the plant \( \Sigma \), assumed to be forward complete, at the sampling times:

- \( X_r = \mathbb{R}^n \);
- \( X_{r0} = \mathbb{R}^n \);
- \( U_r = U \);
- \( x_r \xrightarrow{u_r} x'_r \) if there exists a trajectory \( \xi_{x_r, u_r} : [0, \tau] \rightarrow \mathbb{R}^n \) of \( \Sigma \) satisfying \( \xi_{x_r, u_r}(\tau) = x'_r \);
- \( Y_r = \mathbb{R}^n/Q \) for some given equivalence relation \( Q \subseteq X_r \times X_r \);
- \( H_r = \pi_Q \).
We recall now some of the existing results on the existence of symbolic abstractions. Assume that the output set $Y_\tau$ is equipped with a metric $d_{Y_\tau} : Y_\tau \times Y_\tau \to \mathbb{R}^d_+$. We also notice that the set of states and inputs of $S_\tau(\Sigma)$ exists a countable deterministic abstraction $S_\tau(\Sigma)$ is a deterministic system in the sense of Definition 3.1 since (cf. Subsection 2.2) the trajectory of $\Sigma$ is uniquely determined. We also assume that the metric $d_{Y_\tau}$ is the natural infinity norm metric. We recall the following theorem from [14].

**Theorem 3.4.** Consider a $\delta$-ISS control system $\Sigma$. For any $\varepsilon \in \mathbb{R}^+_0$ and any sampling time $\tau \in \mathbb{R}^+$, there exists a countable deterministic abstraction $S_\tau(\Sigma)$ such that $S_\tau(\Sigma) \cong_s S_\tau(\Sigma)$ (equivalently $S_\tau(\Sigma) \cong_{\delta AS} S_\tau(\Sigma)$).

Now we recall a result, borrowed from [23], on the existence of an abstraction $S_\tau(\Sigma)$ for $\Sigma$ without requiring any stability assumption on $\Sigma$.

**Theorem 3.5.** Consider a $\delta$-FC control system $\Sigma$. For any $\varepsilon \in \mathbb{R}^+_0$ and any sampling time $\tau \in \mathbb{R}^+$, there exists a countable nondeterministic abstraction $S_\tau(\Sigma)$ such that $S_\tau(\Sigma) \cong_{\delta AS} S_\tau(\Sigma)$.

Although the abstractions $S_\tau(\Sigma)$ in Theorems 3.4 and 3.5 are countable, if one is interested in the dynamics of $\Sigma$ over a compact set $D \subset \mathbb{R}^n$, then they are also finite.

The relationships established in Theorem 3.5 are weaker than the relationships established in Theorem 3.4 in the sense that failing to find a controller for the desired specifications on $S_\tau(\Sigma)$ does not prevent the existence of a controller for $\Sigma$ satisfying the same specifications.

Note that the specific abstractions $S_\tau(\Sigma)$ in Theorems 3.4 and 3.5 provided in [14, 23] are grid-based abstractions that suffer from the curse of dimensionality. For some specific classes of control systems $\Sigma$, one can also construct abstractions that are exactly bisimilar to $\Sigma$ using some equivalence relation $Q \subset \mathbb{R}^n \times \mathbb{R}^n$. We refer the interested readers to [21, Theorem 8.10] to consult the results on abstractions not based on grids.

**Remark 3.6.** Consider the metric system $S_\tau(\Sigma)$ admitting an abstraction $S_\tau(\Sigma)$. Since the plant $\Sigma$ is forward complete, one can readily verify that given any state $x_\tau \in X_\tau$ there always exists a $v_\tau$-successor of $x_\tau$ for any $v_\tau \in U_\tau$. Hence, $U_\tau(x_\tau) = U_\tau$ for any $x_\tau \in X_\tau$. Therefore, without loss of generality, one can also assume that $U_\tau(x_\tau) = U_\tau$ for any $x_\tau \in X_\tau$.

4. **Networked Control Systems**

Consider a NCS as depicted schematically in Figure 1. The NCS consists of a forward complete plant $\Sigma = (\mathbb{R}^n, U, \mathcal{U}, f)$ which is connected to a symbolic controller, explained in more detail in the next subsection, over a communication network that induces delays ($\Delta^{sc}$ and $\Delta^{ca}$). The state measurements of the plant are sampled by a time-driven sampler at times $s_k := k\tau$, $k \in \mathbb{N}_0$, and we denote $x_k := \xi(s_k)$. The discrete-time control values computed by the symbolic controller at times $s_k$ are denoted by $u_k$. Time-varying network-induced delays, i.e. the sensor-to-controller delay ($\Delta_k^{sc}$) and the controller-to-actuator delay ($\Delta_k^{ca}$), are included in the model. Moreover, packet dropouts in both channels of the network can be incorporated in the delays $\Delta_k^{sc}$ and $\Delta_k^{ca}$ as long as the maximum number of subsequent dropouts over the network is bounded [13]. Finally, the varying computation time, needed to evaluate the symbolic controller, is incorporated into $\Delta_k^{sc}$. We assume that the time-varying delays are bounded and are integer multiples of the sampling time $\tau$, i.e. $\Delta_k^{sc} := N_k^{sc}\tau$, where $N_k^{sc} \in [N_{k_{\min}}^{sc}, N_{k_{\max}}^{sc}]$, and $\Delta_k^{ca} := N_k^{ca}\tau$, where $N_k^{ca} \in [N_{k_{\min}}^{ca}, N_{k_{\max}}^{ca}]$, for some $N_{k_{\min}}^{sc}, N_{k_{\max}}^{sc}, N_{k_{\min}}^{ca}, N_{k_{\max}}^{ca} \in \mathbb{N}_0$. Under these assumptions, there is no difference in assuming that both the controller and the actuator act in an event-driven fashion (i.e. they respond instantaneously to newly arrived data) or time-driven fashion.

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1Let us recall that the notions of alternating approximate (bi)simulation and of approximate (bi)simulation relation coincide when the systems involved are deterministic as per Definition 3.1.
(i.e. they respond to newly arrived data at the sampling instants $s_k$). Furthermore, we model the occurrence of message rejection, i.e. the effect of older data being neglected because more recent data is available before the older data arrival, similarly to the work in [10, 22]. The zero-order-hold (ZOH) function (see Figure 1) is placed before the plant $\Sigma$ to transform the discrete-time control inputs $u_k$, $k \in \mathbb{N}_0$, to a continuous-time control input $\upsilon(t) = u_k \ast (t)$, where $k \ast (t) := \max \{ k \in \mathbb{N}_0 | s_k + \Delta^c_{k} \leq t \}$. As argued in [10, 22], in the sampling interval $[s_k, s_{k+1}]$, $\upsilon(t)$ can be explicitly described by

$$v(t) = u_{k^\ast(t)} - N^c_{\max} - j$$

for $t \in [s_k, s_{k+1}]$, where $j^\ast \in [0; N^c_{\max} - N^c_{\min}]$ is defined as:

$$j^\ast = \hat{f} \left( \hat{N}^c_{N^c_{\min}}, \ldots, \hat{N}^c_{N^c_{\max}} \right),$$

where $\hat{N}_k$, for $k \in [N^c_{\min}; N^c_{\max}]$, is the delay suffered by the control packet sent $k$ samples ago, namely $\hat{N}^c_{N^c_{\max} - i} = N^c_{k-N^c_{\max} + i}$ for any $i \in [0; N^c_{\max} - N^c_{\min}]$ and

$$\hat{f} \left( \hat{N}^c_{N^c_{\min}}, \ldots, \hat{N}^c_{N^c_{\max}} \right) = \max \left\{ \arg \min_j \hat{g} \left( j, \hat{N}^c_{N^c_{\min}}, \ldots, \hat{N}^c_{N^c_{\max}} \right) \right\},$$

where

$$\hat{g} \left( j, \hat{N}^c_{N^c_{\min}}, \ldots, \hat{N}^c_{N^c_{\max}} \right) = \min \left\{ \max \left\{ 0, \hat{N}^c_{N^c_{\max} - j} + j - N^c_{\max} \right\}, \max \left\{ 0, \hat{N}^c_{N^c_{\max} - 1 - j} + j - N^c_{\max} + 1 \right\}, \ldots, \max \left\{ 0, \hat{N}^c_{N^c_{\min} - N^c_{\min}} \right\}, 1 \right\},$$

with $j \in [0; N^c_{\max} - N^c_{\min}]$. Note that the expression for the continuous-time control input in (4.1) and (4.2) takes into account the possible out-of-order packet arrivals and message rejection. For example, in Figure 3, the time-delays in the controller-to-actuator branch of the network are allowed to take values in $\{\tau, 2\tau, 3\tau\}$, resulting in a message rejection at time $s_{k+2}$.
4.1. Symbolic controller. A symbolic controller is a mechanism that determines which inputs \( u_k \in U \) should be fed into the system \( \Sigma \) based on the observed states \( x_k \in \mathbb{R}^n \). We refer the interested readers to \cite{21} to consult the formal definition of symbolic controllers. Although for some LTL specifications such as safety or reachability it is sufficient to consider only static controllers (i.e., without memory) \cite{22}, we do not limit our work by this assumption. Hence the approach presented in what follows is applicable to dynamic controllers (i.e., the controller has a memory), which are required to address general LTL specifications \cite{6}. Due to the presence of a ZOH, from now on we assume that the set \( U \) contains only curves that are constant over intervals of length \( \tau \in \mathbb{R}^+ \) and take values in \( U \), i.e.:

\[
U = \left\{ v: \mathbb{R}_0^+ \to U \mid v(t) = v((s-1)\tau), t \in [(s-1)\tau, s\tau[, s \in \mathbb{N} \right\}.
\]

Similarly to what was assumed in the connection between controller and plant, we also consider the possible occurrence of message rejection for the measurement data sent from the sensor to the symbolic controller. The symbolic controller uses \( \hat{x}_k \) as an input at the sampling times \( s_k := k\tau \), where

\[
(4.3) \quad \hat{x}_k = x_{k+\ell_\star - N_{sc}^{max}},
\]

where \( \ell_\star \in [0; N_{sc}^{max} - N_{sc}^{min}] \) is defined as:

\[
(4.4) \quad \ell_\star = \tilde{f} \left( \tilde{N}_{N_{sc}^{min}}, \ldots, \tilde{N}_{N_{sc}^{max}} \right),
\]

where \( \tilde{N}_k \), for \( k \in [N_{sc}^{min}; N_{sc}^{max}] \), is the delay suffered by the measurement packet sent \( k \) samples ago, namely \( \tilde{N}_{N_{sc}^{min} - i} = N_{sc}^{max} - i \) for any \( i \in [0; N_{sc}^{max} - N_{sc}^{min}] \) and

\[
\tilde{f} \left( \tilde{N}_{N_{sc}^{min}}, \ldots, \tilde{N}_{N_{sc}^{max}} \right) = \operatorname{arg\,min}_{\ell} \tilde{g} \left( \ell, \tilde{N}_{N_{sc}^{min}}, \ldots, \tilde{N}_{N_{sc}^{max}} \right),
\]

where

\[
\tilde{g} \left( \ell, \tilde{N}_{N_{sc}^{min}}, \ldots, \tilde{N}_{N_{sc}^{max}} \right) = \min \left\{ \max \left\{ 0, \tilde{N}_{N_{sc}^{max}} - \ell - N_{sc}^{max} \right\}, \max \left\{ 0, \tilde{N}_{N_{sc}^{max}} - 1 - \ell + N_{sc}^{max} - 1 \right\}, \ldots, \max \left\{ 0, \tilde{N}_{N_{sc}^{min}} - N_{sc}^{min} \right\}, 1 \right\},
\]

with \( \ell \in [0; N_{sc}^{max} - N_{sc}^{min}] \). Note that the expression for the input of the controller in (4.3) and (4.4) takes into account the possible out-of-order packet arrivals and message rejection.

4.2. Describing NCS as metric systems. Given \( S_\tau(\Sigma) \) and the NCS \( \Sigma \), now consider the metric system \( S(\hat{\Sigma}) := (X, X_0, U, \longrightarrow, Y, H) \), capturing all the information contained in NCS \( \hat{\Sigma} \), where:

- \( X = \{ X_\tau \cup q \}^{N_{sc}^{max}} \times U_\tau^{N_{ca}^{max}} \times [N_{sc}^{min}, N_{sc}^{max}]^{N_{sc}^{max}} \times [N_{ca}^{min}, N_{ca}^{max}]^{N_{ca}^{max}} \), where \( q \) is a dummy symbol;
- \( X_0 = \{ (x_0, q, \ldots, q, v_0, \ldots, v_0, N_{sc}^{max}, \ldots, N_{sc}^{max}, N_{ca}^{max}, \ldots, N_{ca}^{max}) \mid x_0 \in X_\tau, v_0 \in U_\tau \}; \)
- \( U = U_\tau; \)
- \( (x_1, \ldots, x_{N_{sc}^{max}}, v_1, \ldots, v_{N_{ca}^{max}}, \tilde{N}_1, \ldots, \tilde{N}_{N_{sc}^{max}}, \tilde{N}_1, \ldots, \tilde{N}_{N_{ca}^{max}}) \longrightarrow (x', x_1, \ldots, x_{N_{sc}^{max}-1}, v_1, \ldots, v_{N_{ca}^{max}-1}, \tilde{N}_1, \ldots, \tilde{N}_{N_{ca}^{max}-1}) \) for all \( \tilde{N} \in [N_{sc}^{min}, N_{sc}^{max}] \) and all \( \tilde{N} \in [N_{ca}^{min}, N_{ca}^{max}] \) if there exists transition \( x_j \longleftarrow x_{N_{sc}^{max}-j} \tau \) in \( S_\tau(\Sigma) \) where \( j_\star = \tilde{f} \left( \tilde{N}_{N_{sc}^{min}}, \ldots, \tilde{N}_{N_{sc}^{max}} \right) \) in (4.2);
- \( Y = Y_\tau \times Y_\tau; \)
- \( H \left( x_1, \ldots, x_{N_{sc}^{max}}, v_1, \ldots, v_{N_{ca}^{max}}, \tilde{N}_1, \ldots, \tilde{N}_{N_{sc}^{max}}, \tilde{N}_1, \ldots, \tilde{N}_{N_{ca}^{max}} \right) = (H_\tau(x_1), H_\tau(x_{N_{sc}^{max}} - \ell_\star)) \) where \( \ell_\star = \tilde{f} \left( \tilde{N}_{N_{sc}^{min}}, \ldots, \tilde{N}_{N_{sc}^{max}} \right) \) in (4.4). With a slight abuse of notation, we assume that \( H_\tau(q) := q. \)
Let us remark that the set of states and inputs of $S(\Sigma)$ are uncountable and that $S(\Sigma)$ is a nondeterministic system in the sense of Definition 3.1 since depending on the values of $\bar{N}$ and $\tilde{N}$, more than one $v$-successor of any state of $S(\Sigma)$ may exist.

**Remark 4.1.** Note that the output value of any state of $S(\Sigma)$ is a pair: the first entry is the output of the plant available at the sensors at times $s_k := k\tau$, and the second one is the output of the plant available at the controller at the same times $s_k$ taking into consideration the occurrence of message rejection.

We assume that the output set $Y$ is equipped with the metric $d_Y$ that is induced by the metric $d_Y$, as the following: given any $x := (x_1, \ldots, x_{N_{sc}^{max}}, v_1, \ldots, v_{N_{sc}^{max}}, \bar{N}_1, \ldots, \tilde{N}_1, \ldots, \tilde{N}_{N_{sc}^{max}})$ and $x' := (x'_1, \ldots, x'_{N_{sc}^{max}}, v'_1, \ldots, v'_{N_{sc}^{max}}, \bar{N}'_1, \ldots, \tilde{N}'_1, \ldots, \tilde{N}'_{N_{sc}^{max}})$ in $X$, we set

$$d_Y(x, x') := \max \{d_Y(H(x_k), H(x'_k)) \mid k \in [N_{sc}^{min}, N_{sc}^{max}]\},$$

for some given $x \in R^n$ and $d_Y = (H(x), H(q)) = +\infty$ for any $x \in R^n$ and $d_Y = (H(q), H(q)) = 0$. Hence, two states of $S(\Sigma)$ are $\varepsilon$-close if not only their first entries are $\varepsilon$-close but also if the second entries are too.

### 5. Symbolic Models for NCS

This section contains the main contributions of the paper. We show the existence and construction of symbolic models for NCS by using existing symbolic models for the plant $\Sigma$, namely,

$$S_q(\Sigma) := (X_q, X_q, 0, q, \tau, Y_q, H_q).$$

Define the following system:

$$S_s(\Sigma) := (X_s, X_s, 0, s, Y_s, H_s),$$

where

- $X_s = \{x_s \cup \{0\}\}^{N_{sc}^{max}} \times U_s^{N_{ca}^{max}} \times [N_{sc}^{min}, N_{sc}^{max}]^{N_{sc}^{max}} \times [N_{ca}^{min}, N_{ca}^{max}]^{N_{ca}^{max}}$.
- $X_s = \{x_s, 0, q, 0, \ldots, 0, u_s, 0, 0, 0, \ldots, 0, 0, \ldots, 0\}.$
- $U_s = U_q$.
- $j_s = f(\tilde{N}_{N_{ca}^{min}}, \ldots, \tilde{N}_{N_{ca}^{max}})$ in (4.2).
- $Y_s = Y_q \times Y_q$.
- $H_s(x_s, \ldots, x_s, 0, u_s, \ldots, u_s, 0, \ldots, 0, \tilde{N}_s, \ldots, \tilde{N}_s, \tilde{N}_s, \ldots, \tilde{N}_s, \tilde{N}_s, \ldots, \tilde{N}_{N_{sc}^{max}}) = (H_q(x_s), H_q(x_s))$ with a slight abuse of notation, we set $H_q(q) := q$.

It can be readily seen that the system $S_s(\Sigma)$ is countable or symbolic if the system $S_q(\Sigma)$ is countable or symbolic, respectively. Although $S_q(\Sigma)$ may be a deterministic system, $S_s(\Sigma)$ is always a nondeterministic system, since depending on the values of $\tilde{N}_s$ and $\bar{N}_s$, more than one $u_s$-successor of any state of $S_s(\Sigma)$ may exist.

**Remark 5.1.** Note that, with the output map defined as we suggest, the synthesis of controllers should be performed using the first entries of the output pairs to define the satisfaction of properties. This is so because usually specifications are expressed in terms of the outputs exhibited by the plant, i.e. what is available at
the sensors before the network. However, the controller refinement (and any interconnection analysis) should make use of the second entry of the output pairs as those are the outputs received by the controllers. In the present paper we do not dive further into these issues, which are left as object of future research.

We can now state the first main results of the paper.

**Theorem 5.2.** Consider a NCS $\tilde{\Sigma}$ and suppose that there exists an abstraction $S_q(\Sigma)$ such that $S_q(\Sigma) \preceq_{AS} S_r(\Sigma) \preceq_{S} S_q(\Sigma)$. Then we have $S_q(\tilde{\Sigma}) \preceq_{AS} S(\tilde{\Sigma}) \preceq_{S} S_q(\tilde{\Sigma})$.

The proof of Theorem 5.2 is provided in the Appendix.

**Corollary 5.3.** Consider a NCS $\tilde{\Sigma}$ and suppose that there exists an abstraction $S_q(\Sigma)$ such that $S_q(\Sigma) \cong_{AS} S_q(\Sigma)$. Then we have $S_q(\tilde{\Sigma}) \cong_{AS} S(\tilde{\Sigma})$.

**Proof.** Using Theorem 5.2 one gets that $S_q(\Sigma) \preceq_{AS} S_r(\Sigma)$ implies $S_q(\tilde{\Sigma}) \preceq_{AS} S(\tilde{\Sigma})$. In a similar way, one can show that $S_r(\Sigma) \preceq_{AS} S_q(\Sigma)$ implies $S(\tilde{\Sigma}) \preceq_{AS} S_q(\tilde{\Sigma})$ which completes the proof. □

**Remark 5.4.** By consulting the formal definition of symbolic controllers in [21], one can readily verify the existence of two static functions $\varphi : X_r \rightarrow X_q$ and $\psi : U_q \rightarrow U$, inside the symbolic controllers, associating to any $x_r \in X_r$, one symbol $x_q \in X_q$ and to any symbol $u_q \in U_q$ one control value $u_r \in U$, respectively, as shown in Figure 3. Since the functions $\varphi$ and $\psi$ are static, without violating the main results one can shift those functions toward sensor and actuator in the NCS as shown in Figure 3. If $S_q(\Sigma)$ is symbolic, then $U_q$ and $X_q$ are finite sets. Hence, one can automatically take care of limited bandwidth constraints without introducing additional quantization errors. Note that for the grid-based symbolic abstractions $S_q(\Sigma)$ proposed in [14, 23], one has: $\psi = 1_{U_q}$ and $\varphi : x \rightarrow \lfloor x \rfloor_\eta$, where $\lfloor x \rfloor_\eta \in \mathbb{R}^n_\eta$ such that $\|x - \lfloor x \rfloor_\eta\| \leq \eta/2$ for a given state space quantization parameter $\eta \in \mathbb{R}^+$. The next subsection provides similar results as the ones in Theorem 5.2 and Corollary 5.3 when the symbolic controller is static.

![Diagram](image-url)
5.1. Results for static symbolic controllers. Assuming that the symbolic controller is static, both delays $\Delta_k^c$ and $\Delta_k^a$ can be captured by a single delay $\Delta_k := \Delta_k^c + \Delta_k^a$ \[10, 22\] and shifted to the controller-to-actuator branch of the network, i.e. denoting by $\Delta_k^{sc/ca}$ the delays in the new model: $\Delta_k^{sc} = 0$ and $\Delta_k^{ca} = \Delta_k$. Therefore, one can also consider the occurrence of message rejection in the control data.

Given $S_\tau(\Sigma)$ and the NCS $\bar{\Sigma}$, now consider the metric system $\bar{S}(\bar{\Sigma}) := (X, X_0, U, \longrightarrow, Y, H)$, capturing all the information contained in the NCS $\bar{\Sigma}$, where:

- $X = X_\tau \times U^{N_{\max}} \times [N_{\min}, N_{\max}]^{\tau_{\max}}$
- $X_0 = \left\{(x_0, v_0, \ldots, v_0, N_{\max}, \ldots, N_{\max}) \mid x_0 \in X_\tau, v_0 \in U\right\}$
- $U = U_\tau$
- $(x_1, v_1, \ldots, v_{N_{\max}}, N_1, \ldots, N_{N_{\max}}) \overset{\nu}{\rightarrow} (x', v, v_1, \ldots, v_{N_{\max}-1}, N, N_1, \ldots, N_{N_{\max}-1})$ for all $N \in [N_{\min}, N_{\max}]$ if there exists transition $x_1 \overset{\nu^{N_{\max}-1}}{\rightarrow} x$ in $S_\tau(\Sigma)$ where $j_* = \hat{f}(N_{\min}, \ldots, N_{\max} \tau)$ in \([4.2]\).
- $Y = Y_\tau$
- $H(x_1, v_1, \ldots, v_{N_{\max}}, N_1, \ldots, N_{N_{\max}}) = H_\tau(x_1)$

where $N_{\min} = N_{\min}^{sc} + N_{\min}^{ca}$ and $N_{\max} = N_{\max}^{sc} + N_{\max}^{ca}$. Note that the set of states and inputs of $\bar{S}(\bar{\Sigma})$ are uncountable and that $\bar{S}(\bar{\Sigma})$ is a nondeterministic system, since depending on the values of $N$, more than one $\nu$-successor of any state of $\bar{S}(\bar{\Sigma})$ may exist.

We now propose a symbolic model for the NCS $\bar{\Sigma}$ using an existing symbolic model for $\Sigma$, namely,

$$S_q(\Sigma) := \left(X_q, X_0, U, \rightarrow_q, Y_q, H_q\right),$$

Define the following system:

$$\mathcal{S}_s(\bar{\Sigma}) := \left(X_s, X_0, U_s, \rightarrow_s, Y_s, H_s\right),$$

where

- $X_s = X_q \times U_q^{N_{\max}} \times [N_{\min}, N_{\max}]^{\tau_{\max}}$
- $X_0 = \left\{(x_0, u_0, \ldots, u_0, N_{\max}, \ldots, N_{\max}) \mid x_0 \in X_q, u_0 \in U_q\right\}$
- $U_s = U_q$
- $(x_{s1}, u_{s1}, \ldots, u_{sN_{\max}}, N_{s1}, \ldots, N_{sN_{\max}}) \overset{u_{s}}{\longrightarrow} (x'_{s}, u_{s}, u_{s1}, \ldots, u_{s(N_{\max}-1)}, N_s, N_{s1}, \ldots, N_{s(N_{\max}-1)})$ for all $N_s \in [N_{\min}, N_{\max}]$ if there exists transition $x_{s1} \overset{u_{s(N_{\max}-1)}}{\rightarrow q} x'_{s}$ in $S_q(\Sigma)$ where $j_* = \hat{f}(N_{s1}, \ldots, N_{sN_{\max}})$ in \([4.2]\).
- $Y_s = Y_q$
- $H_s(x_{s1}, u_{s1}, \ldots, u_{sN_{\max}}, N_{s1}, \ldots, N_{sN_{\max}}) = H_q(x_{s1})$

It can be readily seen that the system $\mathcal{S}_s(\bar{\Sigma})$ is countable or symbolic if the system $S_q(\Sigma)$ is countable or symbolic, respectively. Although $S_q(\Sigma)$ may be a deterministic system, $\mathcal{S}_s(\bar{\Sigma})$ is always a nondeterministic one, since depending on the values of $N_s$, more than one $u_s$-successor of any state of $\mathcal{S}_s(\bar{\Sigma})$ may exist.

Note that Theorem \[5.2\] and Corollary \[5.3\] are still valid for systems $\bar{S}(\bar{\Sigma})$ and $\mathcal{S}_s(\bar{\Sigma})$, as the following:

**Theorem 5.5.** Consider a NCS $\bar{\Sigma}$ and suppose that the symbolic controller is static and there exists an abstraction $S_q(\Sigma)$ such that $S_q(\Sigma) \preceq_{AS} S_\tau(\Sigma) \preceq_S S_q(\Sigma)$. Then we have $\mathcal{S}_s(\bar{\Sigma}) \preceq_{AS} \bar{S}(\bar{\Sigma}) \preceq_S \mathcal{S}_s(\bar{\Sigma})$.

**Proof.** The proof is analogous to the one of Theorem \[5.2\] \qed
**Corollary 5.6.** Consider a NCS \( \tilde{\Sigma} \) and suppose that the symbolic controller is static and there exists an abstraction \( S_q(\Sigma) \) such that \( S_q(\Sigma) \cong_{AS} S_\ast(\Sigma) \). Then we have \( \mathfrak{S}_\ast(\tilde{\Sigma}) \cong_{AS} \mathfrak{S}(\tilde{\Sigma}) \).

**Proof.** The proof is analogous to the one of Corollary 5.3 \( \square \)

6. **Comparison with the Existing Results in the Literature**

We compare the results provided here with the ones provided in [9, 8] in terms of the size of the proposed symbolic models. For the sake of a fair comparison, assume that we use also a grid-based symbolic abstraction for the plant \( \Sigma \) using the same sampling time and quantization parameters as the ones in [9, 8]. By assuming that we are only interested in the dynamics of \( \Sigma \) on a compact set \( D \subset \mathbb{R}^n \), the size of the set of states of the symbolic models, provided in [9, 8], is:

\[
|X_s| = \sum_{i \in \{1\} \cup \{N_{\min}; N_{\max}\}} |D|_{\eta i}.
\]

Meanwhile, the size of the set of states for the abstractions provided by Theorem 5.2 and Corollary 5.3 is:

\[
|X_s|^\ast = \left( |D|_{\eta i} + 1 \right)^{N_{\sc max}} \cdot |U|_{\mu i}^{N_{\ca max}} \cdot |\gamma|_{\eta i}^{N_{\sc max} + 1} \cdot |\delta|_{\mu i}^{N_{\sc max} + 1} \cdot |\eta|_{\eta i}^{N_{\ca max} + 1} \cdot |\mu|_{\mu i}^{N_{\ca max} + 1},
\]

and for the ones provided by Theorem 5.5 and Corollary 5.6 is:

\[
|X_s| = \left( |D|_{\eta i} \cdot |U|_{\mu i} \right)^{N_{\max} \cdot (N_{\max} - N_{\min} + 1)n_{\max}},
\]

where \( |D|_{\eta i} = D \cap [\mathbb{R}^n]_{\eta i} \) and \( |U|_{\mu i} = U \cap [\mathbb{R}^m]_{\mu i} \) for some quantization parameters \( \eta, \mu \in \mathbb{R}^+ \).

One can easily verify that the size of the symbolic models proposed in [9, 8] is at most:

\[
|\mathfrak{S}_\ast(\Sigma)| = |X_s| \cdot |U|_{\mu i} \cdot (N_{\max} - N_{\min} + 1) \cdot K = \left( \sum_{i \in \{1\} \cup \{N_{\min}; N_{\max}\}} |D|_{\eta i} \right) \cdot |U|_{\mu i} \cdot (N_{\max} - N_{\min} + 1) \cdot K,
\]

where \( K \) is the maximum number of \( u \)-successors of any state of the symbolic model \( S_q(\Sigma) \) for \( u \in |U|_{\mu i} \). Note that with the results in Theorem 5.3 one has \( K = 1 \) because \( S_q(\Sigma) \) is a deterministic system, while with the ones from Theorem 3.5 one has \( K \geq 1 \) because \( S_q(\Sigma) \) is a nondeterministic system and the value of \( K \) depends on the functions \( \beta \) and \( \gamma \) in (2.2), see [23] for more details. The sizes of the symbolic models provided in this paper are at most:

\[
|\mathfrak{S}_\ast(\Sigma)| = |X_s| \cdot |U|_{\mu i} \cdot (N_{\sc max} - N_{\sc min} + 1) \cdot (N_{\ca max} - N_{\ca min} + 1)
\]

\[
= \left( |D|_{\eta i} + 1 \right)^{N_{\sc max}} \cdot |U|_{\mu i}^{N_{\ca max} + 1} \cdot (N_{\sc max} - N_{\sc min} + 1)\cdot (N_{\ca max} - N_{\ca min} + 1) \cdot K,
\]

\[
|\mathfrak{S}_\ast(\Sigma)| = |X_s| \cdot |U|_{\mu i} \cdot (N_{\max} - N_{\min} + 1) = \left( |D|_{\eta i} \cdot |U|_{\mu i} \right)^{N_{\max} + 1} \cdot (N_{\max} - N_{\min} + 1) \cdot K,
\]

with the same \( K \) as in (6.1). For the sake of a fair comparison, one should compare the sizes in (6.1) and (6.3) because in both symbolic models \( S_\ast(\Sigma) \) and \( \mathfrak{S}_\ast(\Sigma) \) it is assumed that the symbolic controllers are static.

It can be readily verified that if \( |D|_{\eta i} \) is much bigger than \( |U|_{\mu i} \) \( \left( |D|_{\eta i} \gg |U|_{\mu i} \right) \) which is often the case, \( |\mathfrak{S}_\ast(\Sigma)| \) can be much smaller than \( |\mathfrak{S}_\ast(\Sigma)| \) especially for large values of \( N_{\max} \). The symbolic model \( \mathfrak{S}_\ast(\Sigma) \) can also have a smaller size for large values of \( N_{\max} \) and for \( |D|_{\eta i} \gg |U|_{\mu i} \), as shown in the following numerical example.
Example 6.1. Consider a plant $\Sigma$ such that $D = [-1, 1] \times [-1, 1]$, $U = [0, 1]$, $\eta = 0.1$, and $\mu = 1$. Assume that the delays in different parts of the network are as the following: $N_{\text{min}}^{\text{nc}} = 1$, $N_{\text{max}}^{\text{nc}} = 2$, $N_{\text{min}}^{\text{ca}} = 2$, and $N_{\text{max}}^{\text{ca}} = 3$. Using equations (6.1), (6.2), and (6.3), one obtains:

$$|S_*(\Sigma)| = 6.1594 \times 10^{13}K, \quad |S_*(\Sigma)| = 3.2932 \times 10^8K, \quad |S_*(\Sigma)| = 1.8662 \times 10^7K.$$  

It can be readily verified that the sizes of our proposed abstractions $S_*(\Sigma)$ and $S_*(\Sigma)$ are roughly $2 \times 10^5$ and $3 \times 10^6$ times smaller than the one of $S_*(\Sigma)$, proposed in [9, 8], respectively.

7. Discussion

In this paper we have provided a construction of symbolic models for NCS, subject to variable communication delays, quantization errors, packet losses, and limited bandwidth, using available symbolic models for the plant (not limited to grid-based ones). Furthermore, our approach allows us to treat any specification expressed as formulae in LTL or as automata on infinite strings without requiring additional reformulations. Finally, we have shown that the proposed methodology also results, in general, in smaller abstractions than similar approaches in the literature [9, 8].

Future work will concentrate on: 1) providing efficient implementations of the symbolic models, the existence of which has been shown in this work; 2) the construction of symbolic models for NCS with stochastic approaches in the literature [9, 8].

8. Appendix

Proof of Theorem 5.2. We start by proving $S_*(\Sigma) \subseteq_{AS} S(\Sigma)$. Since $S_q(\Sigma) \subseteq_{AS} S_*(\Sigma)$, there exists an alternating $\varepsilon$-approximate simulation relation $R$ from $S_q(\Sigma)$ to $S_*(\Sigma)$. Consider the relation $\bar{R} \subseteq X_s \times X_s$ defined by $(x_s, x) \in \bar{R}$, where $x_s = (x_{s1}, \ldots, x_{sN_{\text{max}}^{\text{sc}}}, u_{s1}, \ldots, u_{sN_{\text{max}}^{\text{ca}}}, \tilde{N}_{s1}, \ldots, \tilde{N}_{sN_{\text{max}}^{\text{ca}}})$ and $x = (x_{11}, \ldots, x_{N_{\text{max}}^{\text{sc}}}, u_{11}, \ldots, u_{N_{\text{max}}^{\text{ca}}}, \tilde{N}_{11}, \ldots, \tilde{N}_{N_{\text{max}}^{\text{ca}}})$, if and only if $\tilde{N}_{si} = N_i$, $\forall i \in [1; N_{\text{max}}^{\text{ca}}]$, $\tilde{N}_{ij} = N_j$, $\forall j \in [1; N_{\text{max}}^{\text{ca}}]$, $(x_k, x_k) \in R$, $\forall k \in [1; N_{\text{max}}^{\text{ca}}]$, and for each $u_{si}$ and the corresponding $v_i$ there exists $x'_i \in \text{Post}_{u_{si}}(x_s)$ such that $(x_s, \xi_{x'_i}((\tau_i))) \in R$ for any $i \in [1; N_{\text{max}}^{\text{ca}}]$ and any $x_s, x \in R$. Note that if $U_r = U_q$ and they are finite then the last condition of the relation $\bar{R}$ is nothing more than requiring $u_{si} = v_i$ for any $i \in [1; N_{\text{max}}^{\text{ca}}].$

Consider $x_0 := (x_{s0}, q, \ldots, q, u_{s0}, \ldots, u_{s0}, N_{\text{max}}^{\text{sc}}, \ldots, N_{\text{max}}^{\text{ca}}, N_{\text{max}}^{\text{sc}}, \ldots, N_{\text{max}}^{\text{ca}}) \in X_s$. Due to the relation $R$, there exist $x_{s0} \in X_r$ such that $(x_{s0}, x_0) \in R$ and $v_0 \in U_r$ such that there exists $x'_{s0} \in \text{Post}_{u_{s0}}(x_s)$ satisfying $(x'_{s0}, \xi_{x'_{s0}}(\tau)) \in R$ for any $(x_s, x) \in R$. Hence, by choosing $x_0 := (x_{01}, q, q, \ldots, q, u_{01}, \ldots, u_{01}, N_{\text{max}}^{\text{sc}}, \ldots, N_{\text{max}}^{\text{ca}}, N_{\text{max}}^{\text{ca}}, \ldots, N_{\text{max}}^{\text{ca}}) \in X_s$, one gets $(x_{s0}, x_0) \in \bar{R}$ and condition (i) in Definition 3.3 is satisfied.

Now consider any $(x_s, x) \in \bar{R}$, where $x_s = (x_{s1}, \ldots, x_{sN_{\text{max}}^{\text{sc}}}, u_{s1}, \ldots, u_{sN_{\text{max}}^{\text{ca}}}, \tilde{N}_{s1}, \ldots, \tilde{N}_{sN_{\text{max}}^{\text{ca}}}, \tilde{N}_{s1}, \ldots, \tilde{N}_{sN_{\text{max}}^{\text{ca}}})$ and $x = (x_{11}, \ldots, x_{N_{\text{max}}^{\text{sc}}}, u_{11}, \ldots, u_{N_{\text{max}}^{\text{ca}}}, \tilde{N}_{11}, \ldots, \tilde{N}_{N_{\text{max}}^{\text{ca}}}, \tilde{N}_{11}, \ldots, \tilde{N}_{N_{\text{max}}^{\text{ca}}})$. Since $\tilde{N}_{si} = N_i$, $\forall i \in [1; N_{\text{max}}^{\text{ca}}]$, and $\tilde{N}_{ij} = N_j$, $\forall j \in [1; N_{\text{max}}^{\text{ca}}]$, and using definitions of $S_*(\Sigma)$ and $S(\Sigma)$, one obtains $H_s(x_s) = (x_{s1}, x_{sk})$ and $H(x) = (x_{11}, x_{1k})$, for some $k \in [N_{\text{min}}^{\text{sc}}, N_{\text{max}}^{\text{sc}}]$ (cf. Definitions 5.1 and 3.1). Since $(x_{si}, x_i) \in R$, $\forall i \in [1; N_{\text{max}}^{\text{sc}}]$, one gets $d_{xy}, (H_q(x_{si}), H_r(x_i)) \leq \varepsilon$, $\forall i \in [1; N_{\text{max}}^{\text{ca}}]$. Therefore,

$$d_Y(H_s(x_s), H(x)) = \max \{d_{xy} | H_q(x_{si}), H_r(x_i) \} \leq \varepsilon,$$

and condition (ii) in Definition 3.3 is satisfied.
Let us now show that condition (iii) in Definition 3.3 holds. Consider any \((x, \hat{x}) \in \hat{R}\), where \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\), \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\). Consider any \(u \in U(x_1) = U_q\). Using the relation \(R\), there exist \(v \in U(x) = U_r\) and \(x \in \text{Post}_{u_1}(x)\) such that \((\hat{x}, \xi_{x, u}(\tau)) \in \mathcal{R}\) for any \((x, \hat{x}) \in \hat{R}\). Now consider any \(x' = (x', 1, \ldots, x'_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N}) \in \text{Post}_{u_1}(x) \subseteq X\) for some \(\hat{N} \in [N_{\min}^{\text{sc}}, N_{\max}^{\text{sc}}]\) and \(\hat{N} \in [N_{\min}^{\text{ca}}, N_{\max}^{\text{ca}}]\) where \(x' = \xi_{x, u}(\tau)\) for some given \(k \in [N_{\min}^{\text{ca}}, N_{\max}^{\text{ca}}]\) (cf. Definition \(S(\Sigma)\)). Because of the relation \(R\), there exists \(x' \in \text{Post}_{u_1}(x) \subseteq X(\hat{\Sigma})\) such that \((x', x') \in \mathcal{R}\). Hence, due to the definition \(S(\Sigma)\), one can choose \(x' = (x'_1, 1, \ldots, x'_{i_1} N_{i_1}, u_{i_1}, \ldots, u_{i_N} N_{i_N} - 1, \hat{N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N} - 1) \in \text{Post}_{u_1}(x) \subseteq X\). Due to the relation \(R\), one can readily verify that \(d_{y_{\tau}}(H_q(x'), H_{\tau}(x')) \leq \varepsilon\). Since \(d_{y_{\tau}}(H_q(x), H_{\tau}(x)) \leq \varepsilon\), one gets

\[
d_y(H(x), H(x')) = \max \{d_{y_{\tau}}(H_q(x), H_{\tau}(x))\} \leq \varepsilon,
\]

for some given \(k \in [N_{\min}^{\text{sc}} - 1, N_{\max}^{\text{sc}} - 1]\) (cf. Definitions \(S_q(\Sigma)\) and \(S(\Sigma)\)). Hence, \((x', x') \in \hat{R}\) implying that condition (iii) in Definition 3.3 holds.

Now we prove \(S(\Sigma) \subseteq S_q(\Sigma)\). Since \(S_q(\Sigma) \subseteq S(\Sigma)\), there exists an \(\varepsilon\)-approximate simulation relation \(R\) from \(S(\Sigma)\) to \(S_q(\Sigma)\). Consider the relation \(R \subseteq X \times X\) defined by \((x, \hat{x}) \in \hat{R}\), where \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\) and \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\), if and only if \(\hat{N}_{i_1} = \hat{N}_{i_1}, \forall i \in [1; N_{\max}^{\text{sc}}]\). Now consider any \((x, \hat{x}) \in \hat{R}\), \(\forall k \in [1; N_{\max}^{\text{sc}}]\), and for each \(v_k\) and the corresponding \(ux_{i_1}\) there exists a \(x' \in \text{Post}_{ux_{i_1}}(x)\) such that \((\xi_{x, u}(\tau), x') \in \mathcal{R}\) for any \(i \in [1; N_{\max}^{\text{ca}}]\) and any \((x, \hat{x}) \in \hat{R}\).

Now consider any \((x, \hat{x}) \in \hat{R}\), where \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\) and \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\). Since \(\hat{N}_{i_1} = \hat{N}_{i_1}, \forall i \in [1; N_{\max}^{\text{ca}}]\), and using definitions of \(S(\Sigma)\) and \(S_q(\Sigma)\), one obtains \(H(x) = (x_1, x_k)\) and \(H_q(x) = (x_1, x_k)\), for some \(k \in [N_{\min}^{\text{ca}}, N_{\max}^{\text{ca}}]\) (cf. Definitions \(S_q(\Sigma)\) and \(S(\Sigma)\)). Now consider any \((x, \hat{x}) \in \hat{R}\), \(\forall k \in [1; N_{\max}^{\text{sc}}]\), one gets \(d_{y_{\tau}}(H(x), H_q(x)) \leq \varepsilon\), \(\forall i \in [1; N_{\max}^{\text{sc}}]\). Therefore, \(d_y(H(x), H_q(x)) \leq \varepsilon\) and condition (ii) in Definition 3.2 is satisfied.

Let us now show that condition (iii) in Definition 3.2 holds. Consider any \((x, \hat{x}) \in \hat{R}\), where \(x = (x_1, 1, \ldots, x_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N})\). Consider any \(v \in U(x) = U_r\). Using the relation \(R\), there exist \(u \in U(x) = U_q\) and \(x \in \text{Post}_{u_1}(x)\) such that \((\xi_{x, u}(\tau), \hat{x}) \in \mathcal{R}\) for any \((x, \hat{x}) \in \hat{R}\). Now consider any \(x' = (x', 1, \ldots, x'_{i_1} N_{i_1}, u_{i_1}, 1, \ldots, u_{i_N}, N_{i_N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N}) \in \text{Post}_{u_1}(x) \subseteq X\) for some \(\hat{N} \in [N_{\min}^{\text{sc}}, N_{\max}^{\text{sc}}]\) and \(\hat{N} \in [N_{\min}^{\text{ca}}, N_{\max}^{\text{ca}}]\) where \(x' = \xi_{x, u}(\tau)\) for some given \(k \in [N_{\min}^{\text{ca}}, N_{\max}^{\text{ca}}]\) (cf. Definition \(S(\Sigma)\)). Because of the relation \(R\), there exists \(x' \in \text{Post}_{u_1}(x) \subseteq S(\Sigma)\) such that \((x', x') \in \mathcal{R}\). Hence, due to the definition \(S_1(\Sigma)\), one can choose \(x' = (x'_1, 1, \ldots, x'_{i_1} N_{i_1}, u_{i_1}, u_{i_1}, \ldots, u_{i_N} N_{i_N} - 1, \hat{N}, \hat{N}_{i_1}, \ldots, \hat{N}_{i_N} - 1) \in \text{Post}_{u_1}(x) \subseteq X\). Due to the relation \(R\), one can readily verify that \(d_{y_{\tau}}(H_{\tau}(x'), H_q(x')) \leq \varepsilon\). Since \(d_{y_{\tau}}(H_{\tau}(x), H_q(x)) \leq \varepsilon\), one gets

\[
d_y(H(x'), H_q(x')) = \max \{d_{y_{\tau}}(H_{\tau}(x'), H_q(x'))\} \leq \varepsilon,
\]

2Note that if \(N_{\max}^{\text{sc}} = 0\), then \(x_{1(-1)} = x_{1}'\) and \(x_{-1} = x'_{0}\).
for some given \(k \in [N_{\min} - 1; N_{\max} - 1]\) (cf. Definitions \(S_{\ast}(\tilde{\Sigma})\) and \(S(\tilde{\Sigma})\)). Hence, \((x', x'_\ast) \in \tilde{R}\) implying that condition (iii) in Definition 3.2 holds.

\[\square\]

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\[\text{\textsuperscript{3}Note that if } N_{\min} = 0, \text{ then } x_{(-1)} = x'_\ast \text{ and } x_{-1} = x'. \]
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