A self-dual poset on objects counted by the Catalan numbers

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Abstract

We examine the poset \( P \) of 132-avoiding \( n \)-permutations ordered by descents. We show that this poset is the "coarsening" of the well-studied poset \( Q \) of noncrossing partitions. In other words, if \( x \prec y \) in \( Q \), then \( f(y) \prec f(x) \) in \( P \), where \( f \) is the canonical bijection from the set of noncrossing partitions onto that of 132-avoiding permutations. This enables us to prove many properties of \( P \).

1 Introduction

There are more than 150 different objects enumerated by Catalan numbers. Two of the most carefully studied ones are noncrossing partitions and 132-avoiding permutations. A partition \( \pi = (\pi_1, \pi_2, \ldots, \pi_t) \) of the set \([n] = \{1, 2, \ldots, n\}\) is called noncrossing \cite{2} if it has no four elements \( a < b < c < d \) so that \( a, c \in \pi_i \) and \( b, d \in \pi_j \) for some distinct \( i \) and \( j \). A permutation of \([n]\), or, in what follows, an \( n \)-permutation, is called 132-avoiding \cite{4} if it does not have three entries \( a < b < c \) so that \( a \) is the leftmost of them and \( b \) is the rightmost of them.

Noncrossing partitions of \([n]\) have a natural and well studied partial order: the refinement order \( Q_n \). In this order \( \pi_1 \prec \pi_2 \) if each block of \( \pi_2 \) is the union of some blocks of \( \pi_1 \). The poset \( Q_n \) is known to be a lattice, and it is graded, rank-symmetric, rank-unimodal, and \( k \)-Sperner \cite{6}. The poset \( Q_n \) has been proved to be self-dual in two steps \cite{2}, \cite{5}.

In this paper we introduce a new partial order of 132-avoiding \( n \)-permutations which will naturally translate into one of noncrossing partitions. In this poset, for two 132-avoiding \( n \)-permutations \( x \) and \( y \), we define \( x \prec y \) if the descent set of \( x \) is contained in that of \( y \). (We will provide a natural equivalent, definition, too.) We will see that this new partial order \( P_n \) is a coarsening of the dual of \( Q_n \). In other words, if for two noncrossing partitions \( \pi_1 \) and \( \pi_2 \) we have \( \pi_1 \prec \pi_2 \) in \( Q_n \), then we also

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have \( f(\pi_2) < f(\pi_1) \) in \( P_n \), where \( f \) is a natural bijection from the set of noncrossing partitions onto that of 132-avoiding permutations. This will enable us to prove that \( P_n \) has the same rank-generating function as \( Q_n \), and so \( P_n \) is rank-unimodal, rank-symmetric and \( k \)-Sperner. Furthermore, we will also prove that \( P_n \) is self-dual in a somewhat more direct way than it is proved for \( Q_n \).

2 Our main results

2.1 A bijection and its properties

It is not difficult to find a bijection from the set of noncrossing partitions of \([n]\) onto that of 132-avoiding \( n \)-permutations. However, we will exhibit such a bijection here and analyze its structure as it will be our major tool in proving our theorems. To avoid confusion, integers belonging to a partition will be called \emph{elements}, while integers belonging to a permutation will be called \emph{entries}. An \( n \)-permutation \( x = x_1x_2\cdots x_n \) will always be written in the one-line notation, with \( x_i \) denoting its \( i \)th entry.

Let \( \pi \) be a noncrossing partition of \([n]\). We construct the 132-avoiding permutation \( p = f(\pi) \) corresponding to it. Let \( k \) be the largest element of \( \pi \) which is in the same block of \( \pi \) as 1. Put the entry \( n \) of \( p \) to the \( k \)th position, so \( p_k = n \). As \( p \) is to be 132-avoiding, this implies that entries larger than \( n - k \) are on the left of \( n \) and entries less than or equal to \( n - k \) are on the right of \( n \) in \( q \).

Then we continue this procedure recursively. As \( \pi \) is noncrossing, blocks which contain elements larger than \( k \) cannot contain elements smaller than \( k \). Therefore, the restriction of \( \pi \) to \( \{k + 1, k + 2, \cdots, n\} \) is a noncrossing partition, and it corresponds to the 132-avoiding permutation of \( \{1, 2, \cdots n - k\} \) which is on the left of \( n \) in \( \pi \) by this same recursive procedure.

We still need to say what to do with blocks of \( \pi \) containing elements smaller than or equal to \( k \). Delete \( k \), and apply this same procedure for the resulting noncrossing partition on \( k - 1 \) elements. This way we obtain a 132-avoiding permutation of \( k - 1 \) elements, and this is what we needed for the part of \( p \) on the left of \( n \), that is, for \( \{n - k + 1, n - k + 2, \cdots, n - 1\} \).

So in other words, if \( \pi_1 \) is the restriction of \( \pi \) into \([k - 1]\) and \( \pi_2 \) is the restriction of \( \pi \) into \( \{k + 1, k + 2, \cdots, n\} \), then \( f(\pi) \) is the concatenation of \( f(\pi_1) \), \( n \) and \( f(\pi_2) \), where \( f(\pi_1) \) permutes the set \( \{n - k + 1, n - k + 2, \cdots, n - 1\} \) and \( f(\pi_2) \) permutes the set \([n - k] \).

To see that this is a bijection note that we can recover the largest element of the block containing the entry 1 from the position of \( n \) in \( p \) and then proceed recursively.

Example 1 If \( \pi = (\{1, 4, 6\}, \{2, 3\}, \{5\}, \{7, 8\}) \), then \( f(\pi) = 64573812 \).

Example 2 If \( p = (\{1, 2, \cdots, n\}) \), then \( f(p) = 12 \cdots n \).
Example 3 If \( p = (\{1\}, \{2\}, \cdots, \{n\}) \), then \( f(p) = n \cdots 21 \).

The following definition is widely used in the literature.

**Definition 1** Let \( p = p_1 p_2 \cdots p_n \) be a permutation. We say the \( i \) is a descent of \( p \) if \( p_i > p_{i+1} \). The set of all descents of \( p \) is called the descent set of \( p \) and is denoted \( D(p) \).

Now we are in a position to define the poset \( P_n \) of 132-avoiding permutations we want to study.

**Definition 2** Let \( x \) and \( y \) be two 132-avoiding \( n \)-permutations. We say that \( x \prec_P y \) (or \( x < y \) in \( P_n \)) if \( D(x) \subset D(y) \).

Clearly, \( P_n \) is a poset as inclusion is transitive. It is easy to see that in 132-avoiding permutations, \( i \leq 1 \) is a descent if and only if \( p_{i+1} \) is smaller than every entry on its left, (such an element is called a left-to-right minimum). So \( x \prec_P y \) if and only if the set of positions in which \( x \) has a left-to-right minimum is a proper subset of that of those positions in which \( y \) has a left-to-right minimum. The Hasse diagram of \( P_4 \) is shown on the Figure below.

![Figure 1: The Hasse diagram of \( P_4 \).](image)

The following proposition describes the relation between the blocks of \( x \) and the descent set of \( f(x) \).

**Proposition 1** The bijection \( f \) has the following property: \( i \in D(f(x)) \) if and only if \( i + 1 \) is the smallest element of its block.
Proof: By induction on $n$. For $n = 1$ and $n = 2$ the statement is true. Now suppose we know the statement for all positive integers smaller than $n$. Then we distinguish two cases:

1. If 1 and $n$ are in the same block of $x$, then the construction of $f(x)$ simply starts by putting the entry $n$ to the last slot of $f(x)$, then deleting the element $n$ from $x$. Neither of these steps alters the set of minimal elements of blocks or that of descents in any way. Therefore, the algorithm is reduced to one of size $n - 1$, and the proof follows by induction.

2. If the largest element $k$ of the block containing 1 is smaller than $n$, then as we have seen above, $f$ constructs the images of $x_1$ and $x_2$ which will be separated by the entry $n$. Therefore, by the induction hypothesis, the descents of $f(x)$ are given by the minimal elements of the blocks of $x_1$ and $x_2$, and these are exactly the blocks of $x$. There will also be a descent at $k$ (as the entry $n$ goes to the $k$th slot), and that is in accordance with our statement as $k + 1$ is certainly the smallest element of its block.

We point out that this implies that $P_n$ is equivalent to a poset of noncrossing partitions in which $\pi_1 < \pi_2$ if the set of elements which are minimal in their block in $\pi_1$ is contained in that of elements which are minimal in their block in $\pi_2$.

2.2 Properties of $P_n$

Now we can prove the main result of this paper.

Theorem 1 The poset $P_n$ is coarser than the dual of the poset $Q_n$ of noncrossing partitions ordered by refinement. That is, if $x < y$ in $Q_n$, then $f(y) < f(x)$ in $P_n$.

Proof: If $x < y$, then each block of $x$ is a subset of a block of $y$. Therefore, if $z$ is the minimal element of a block $B$ of $y$, then it is also the minimal element of the block $E$ of $x$ containing it as $E \subseteq B$. Therefore, the set of elements which are minimal in their respective blocks in $x$ contains that of elements which are minimal in their respective blocks in $y$. By Proposition 1 this implies $D(f(y)) \subset D(f(x))$. ◊

Now we apply this result to prove some properties of $P_n$. For definitions, see [7].

Theorem 2 The rank generating function of $P_n$ is equal to that of $Q_n$. In particular, $P_n$ is rank-symmetric, rank-unimodal and $k$-Sperner.

Proof: By proposition 1, the number of 132-avoiding permutations having $k$ descents equals that of noncrossing partitions having $k$ blocks, and this is known to be the $(n,k)$ Narayana-number $\frac{1}{n} \cdot \binom{n}{k} \frac{n}{k-1}$. 

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Therefore $P_n$ is graded, rank-symmetric and rank-unimodal, and its rank generating function is the same as that of $Q_n$, as $Q_n$ too is graded by the number of blocks (and is self-dual). As $P_n$ is coarser than $Q_n$, any antichain of $P_n$ is an antichain of $Q_n$, and the $k$-Sperner property follows. \hfill \diamond

We need more analysis to prove that $P_n$ is self-dual, that is, that $P_n$ is invariant to “being turned upside down”. Denote $\text{Perm}_n(S)$ the number of 132-avoiding $n$-permutations with descent set $S$. The following lemma is the base of our proof of self-duality. For $S \subseteq [n - 1]$, we define $\alpha(S)$ to be the “reverse complement” of $S$, that is, $i \in \alpha(S) \iff n - i \notin S$.

**Lemma 1** For any $S \subseteq [n - 1]$, we have $\text{Perm}_n(S) = \text{Perm}_n(\alpha(S))$.

**Proof:** By induction on $n$. For $n = 1, 2, 3$ the statement is true. Now suppose we know it for all positive integers smaller than $n$. Denote $t$ the smallest element of $S$.

1. Suppose that $t > 1$. This means that $x_1 < x_2 < \cdots < x_t$, and that $x_1, x_2, \ldots, x_t$ are consecutive integers. Indeed, if there were a gap among them, that is, there were an integer $y$ so that $y \neq x_i$ for $1 \leq i \leq t$, while $x_1 < y < x_t$, then $x_1, x_t, y$ would be a 132-pattern. So once we know $x_1$, we have only one choice for $x_2, x_3, \ldots, x_t$. This implies

$$\text{Perm}_n(S) = \text{Perm}_{n-(t-1)}(S - (t-1)), \tag{1}$$

where $S - (t-1)$ is the set obtained from $S$ by subtracting $t-1$ from each of its elements.

On the other hand, we have $n - t + 1, n - t + 2, \ldots, n - 1 \in \alpha(S)$, meaning that $x_{n-t+1} > x_{n-t+2} > \cdots > x_n$, and also, we must have $(x_{n-t+1}, x_{n-t+2}, \ldots, x_n) = (t-1, t-2, \ldots, 1)$, otherwise a 132-pattern is formed. Therefore,

$$\text{Perm}_n(\alpha(S)) = \text{Perm}_{n-(t-1)}(\alpha(S)|n - (t-1)) \tag{2}$$

where $\alpha(S)|n - (t-1)$ is simply $\alpha(S)$ without its last $t-1$ elements. Clearly, $\text{Perm}_{n-(t-1)}(S - (t-1)) = \text{Perm}_{n-(t-1)}(\alpha(S)|n - (t-1))$ by the induction hypothesis, so equations (1) and (2) imply $\text{Perm}_n(S) = \text{Perm}_n(\alpha(S))$.

2. If $t = 1$, but $S \neq [n - 1]$, then let $u$ be the smallest index which is not in $S$. Then again, $x_u$ must be the smallest positive integer $a$ which is larger than $x_{u-1}$ and is not equal to some $x_i$, $1 \leq u - 1$, otherwise $x_{u-1} x_u a$ would be a 132-pattern. So again, we have only one choice for $x_u$. On the other hand, the largest index in $\alpha(S)$ will be $n - (u - 1)$. Then as above, we will only have once choice for $x_{n-u}$. Now we can delete $u$ from $S$ and $n - u$ from $\alpha(S)$ and proceed by the induction hypothesis as in the previous case.

3. Finally, if $S = [n - 1]$, then the statement is trivially true as $\text{Perm}_n(S) = \text{Perm}_n(\alpha(S)) = 1$.

So we have seen that $\text{Perm}_n(S) = \text{Perm}_n(\alpha(S))$ in all cases. \hfill \diamond

Now we are in position to prove our next theorem.
Theorem 3 The poset $P_n$ is self-dual.

Proof: It is clear that in $P_n$ permutations with the same descent set will cover the same elements and they will be covered by the same elements. Therefore, such permutations form orbits of $Aut(P_n)$ and they can be permuted among each other arbitrarily by elements of $Aut(P_n)$. One can think of $P_n$ as a Boolean algebra $B_{n-1}$ in which some elements have several copies. One natural anti-automorphism of a Boolean-algebra is “reverse complement”, that is, for $S \subseteq [n-1]$, $i \in \alpha(S) \iff n - i \notin S$. To show that $P_n$ is self-dual, it is therefore sufficient to show that the corresponding elements appear with the same multiplicities in $P_n$. So in other words we must show that there are as many 132-avoiding permutations with descent set $S$ as there are with descent set $\alpha(S)$. And that has been proved in the Lemma. ◇

2.3 Further directions

It is natural to ask for what related combinatorial objects we could define such a natural partial order which would turn out to be self-dual and possibly, have some other nice properties. *Two-stack sortable permutations* [8] are an obvious candidate. It is known [1] that there are as many of them with $k$ descents as with $n - 1 - k$ descents, however, the poset obtained by the descent ordering is not self-dual, even for $n = 4$, so another ordering is needed. Another candidate could be the poset of the recently introduced noncrossing partitions for classical reflection groups [3], some of which are self-dual in the traditional refinement order.

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