NON-VANISHING OF CENTRAL VALUES OF QUADRATIC HECKE L-FUNCTIONS OF PRIME MODULI IN THE GAUSSIAN FIELD

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ABSTRACT. We study the first and second mollified moments of central values of a quadratic family of Hecke L-functions of prime moduli to show that more than nine percent of the members of this family do not vanish at the central values.

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1. Introduction

Central values of L-functions have received considerable attention in the literature as they carry rich arithmetic information. In general, an L-function is expected to vanish at the central value for a reason. Such a reason may simply come from an observation on the sign of the functional equation or arise from ties with deep assertions such as the Birch and Swinnerton-Dyer conjecture.

For the case of Dirichlet L-functions, it is believed that \( L(\frac{1}{2}, \chi) \neq 0 \) for any Dirichlet character \( \chi \). When \( \chi \) is a primitive quadratic character, this is a conjecture of S. Chowla [3]. In [12], M. Jutila initiated the study on the first two moments of the family of quadratic Dirichlet L-functions. His results imply that Chowla’s conjecture is true for infinitely many such L-functions. By further evaluating the mollified moments, K. Soundararajan [16] showed that at least 87.5% of the members of the same family do not vanish at the central value.

Other than the entire family of quadratic Dirichlet L-functions, it is also intriguing to investigate the non-vanishing issue of the family of quadratic Dirichlet L-functions over prime moduli. In this case, it again follows from the work on Jutila in [12], who also obtained the first moment of the family of quadratic Dirichlet L-functions of prime moduli, that there are infinitely many such L-functions having non-vanishing central values. With more efforts, by combining an evaluation on the mollified first moment and an upper bound for the mollified second moment using sieve methods, S. Baluyot and K. Pratt [1] were able to show that more than nine percent of the members of the quadratic family of Dirichlet L-functions of prime moduli do not vanish at the central value.

In [6], the author studied the mollified first and second moments of a family of quadratic Hecke L-functions in the Gaussian field to show that at least 87.5% of the members of that family do not vanish at the central value. This can be regarded as an analogue to the result of Soundararajan in [16]. In this paper, motivated by the above mentioned result of Baluyot and Pratt in [1], it is our goal to investigate the non-vanishing issue of the quadratic family of Hecke L-functions introduced in [6] with a further restriction to prime moduli.

Throughout the paper, we denote \( K = \mathbb{Q}(i) \) for the Gaussian field and \( \mathcal{O}_K = \mathbb{Z}[i] \) for the ring of integers of \( K \). We say an element \( d \in \mathcal{O}_K \) is odd if \( (d, 2) = 1 \). We also denote \( L(s, \chi) \) for the L-function associated to a Hecke character \( \chi \) and \( \zeta_K(s) \) for the Dedekind zeta function of \( K \). We denote \( \varpi \) for a prime element in \( \mathcal{O}_K \), by which we mean that the ideal \( (\varpi) \) is a prime ideal in \( \mathcal{O}_K \). The expression \( \zeta \) is reserved for the quadratic residue symbol \( (\cdot) \) defined in Section 2.1.

A Hecke character \( \chi \) of \( K \) is said to be of trivial infinite type if its component at infinite places of \( K \) is trivial and a Hecke character \( \chi \) is said to be primitive modulo \( q \) if it does not factor through \( (\mathcal{O}_K/(q'))^\times \) for any divisor \( q' \) of \( q \) such that \( N(q') < N(q) \). It is shown in [7, Section 2.1] that \( \chi_{(1+i)\varpi} \) defines a primitive quadratic Hecke character modulo \( (1+i)^3 \varpi \) of trivial infinite type for any odd prime \( \varpi \in \mathcal{O}_K \). Thus, for technical reasons, instead of considering a family of L-functions of \( \{L(s, \chi_{\varpi})\} \) for primes \( \varpi \) satisfying certain congruence conditions (so that the corresponding L-functions become primitive), we considering the following family of L-functions:

\[
\mathcal{F} = \{L(\frac{1}{2}, \chi_{(1+i)^3 \varpi}) : \varpi \text{ odd prime}\}.
\]

We aim to prove the following result in the paper, which shows that more than nine percent of the members of the above family have non-vanishing central values.
Theorem 1.1. For all large $X$, we have

$$\sum_{(\varpi,2)=1}^{1} 1 \geq 0.0964 \sum_{N(\varpi)\leq X} 1.$$ 

$L(\frac{1}{2}, X_{(1+i)^3}) \neq 0$

Notice here the percentage we obtain in Theorem 1.1 is exactly the same as the one given in [1, Theorem 1.1]. This is not surprising since our proof of Theorem 1.1 follows closely the proof of [1, Theorem 1.1] by Baluyot and Pratt. We now briefly outline the approach of the proof. Let $X$ be a large number and for some fixed $\theta, \vartheta \in (0, \frac{1}{2})$, we define

$$M = X^\theta, \quad R = X^\vartheta.$$ 

We fix a smooth function $\Phi(x)$, compactly supported in $[\frac{1}{2}, 1]$, satisfying $\Phi(x) = 1$ for $x \in [\frac{1}{2} + \frac{1}{\log X}, 1 - \frac{1}{\log X}]$ and $\Phi^{(j)}(x) \ll_j (\log X)^j$ for all $j \geq 0$. Let $H(t)$ be another smooth function to be optimized later such that it is compactly supported in $[-1, 1]$. For $m \in \mathcal{O}_K$, we set

$$b_m = \mu_{[i]}(m)H \left( \frac{\log N(m)}{\log M} \right),$$

where we denote $N(m)$ for the norm of $m$ and $\mu_{[i]}$ for the analogue on $\mathcal{O}_K$ of the usual M"obius function on $\mathbb{Z}$. We use $b_m$ to define the mollifier function $M(\varpi)$ for every odd prime $\varpi$ by

$$M(\varpi) = \sum_{\mathbb{Z} / N(m) \leq M} \frac{b_m}{\sqrt{N(m)}} \chi(1+i)^3(\varpi)(m).$$

Here we recall from [1, Section 2.1] that every ideal in $\mathcal{O}_K$ co-prime to 2 has a unique generator congruent to 1 modulo $(1+i)^3$ which is called primary. Hence in (1.4) and in what follows, a sum of the form $\sum_{m \equiv 1 \mod (1+i)^3}$ indicates that we are summing over primary elements in $\mathcal{O}_K$.

Now we introduce the mollified first moment $S_1$ and the mollified second moment $S_2$ of the family $F$ given in (1.1) as follows:

$$S_1 = \sum_{(\varpi,2)=1}^{1} \log N(\varpi)\Phi \left( \frac{N(\varpi)}{X} \right) L \left( \frac{1}{2}, \chi(1+i)^3 \varpi \right) M(\varpi),$$

$$S_2 = \sum_{(\varpi,2)=1}^{1} \log N(\varpi)\Phi \left( \frac{N(\varpi)}{X} \right) L \left( \frac{1}{2}, \chi(1+i)^3 \varpi \right)^2 M(\varpi)^2.$$

Our aim is to evaluate both $S_1$ and $S_2$ asymptotically. The evaluation of $S_1$ is relatively easy, which is performed in Section [3] and our result is summarized in the following proposition.

Proposition 1.2. Let $0 < \theta < \frac{1}{2}$ be fixed. For all large $X$, we have

$$S_1 = 2 \left( H(0) - \frac{1}{2\theta} H'(0) \right) X + O \left( \frac{X}{\log X} \right).$$

It is a challenging task to evaluate $S_2$. However, for our purpose, we need an upper bound of the right order of magnitude for $S_2$. Thus, instead of evaluating $S_2$ asymptotically, we follow the approach in [1] to use sieves to derive an upper bound for $S_2$ in Section [4]. The result is given in the following Proposition.

Proposition 1.3. Let $\delta > 0$ be small and fixed, and let $\theta, \vartheta$ satisfy $\theta + 2\vartheta < \frac{1}{2}$. If $X \geq X_0(\delta, \theta, \vartheta)$, then

$$S_2 \leq (1 + \delta) \frac{3}{\theta} 2X,$$

where

$$3 = -2 \int_0^1 H(x)H'(x)dx + \frac{1}{\theta} \int_0^1 H(x)H''(x)dx + \frac{1}{\theta} \int_0^1 H'(x)^2 dx$$

$$- \frac{1}{2\vartheta^2} \int_0^1 H'(x)H''(x)dx + \frac{1}{24\vartheta^3} \int_0^1 H''(x)^2 dx.$$

In our proof of Proposition 1.3, the use of sieves allows us to reduce the difficulty of estimating certain character sums over primes to an evaluation on a character sum over algebraic integers in $\mathcal{O}_K$. We then adapt the methods developed by Soundararajan in [10] to treat the resulting sum. The most delicate part of the treatment consists of applying a
two-dimensional Poisson summation to convert the desired character sum to its dual sum. A careful analysis on the
dual sum ultimately leads to the bound of $S_2$ given in \( (1.6) \).

With both Proposition \( 1.2 \) and Proposition \( 1.3 \) available, we apply the Cauchy-Schwarz inequality to see that

\[
\sum_{\substack{x, \bar{x} \geq 1 \\text{L}(\frac{1}{2}, \chi_{1+ix}^s) \neq 0}} \log N(x) \Phi \left( \frac{N(x)}{X} \right) \geq \frac{S_1^2}{S_2}. \tag{1.7}
\]

The optimal choice of $H(t)$ in the above estimation has already been determined in \( [1] \) Section 8], which allows us to
deduce the conclusion given in Theorem \( 1.4 \).

2. Preliminaries

As tools needed in the rest of the paper, we gather here some auxiliary results.

2.1. Residue symbol and Gauss sum. It is well-known that the Gaussian field $K = \mathbb{Q}(i)$ has class number one. We denote $U_K = \{ \pm 1, \pm i \}$ for the group of units in $\mathcal{O}_K$ and $D_K = -4$ for the discriminant of $K$. We say an element $d \in \mathcal{O}_K$ is a perfect square if $d = n^2$ for some $n \in \mathcal{O}_K$ and we denote it by writing $d = \square$. We say an element $d \in \mathcal{O}_K$ is square-free if the ideal $(d)$ is not divisible by the square of any prime ideal in $\mathcal{O}_K$.

Recall that every ideal in $\mathcal{O}_K$ co-prime to 2 has a unique generator called primary. As $(1 + i)$ is the only prime ideal in $\mathcal{O}_K$ that lies above the ideal $(2) \in \mathbb{Z}$, we can fix a generator for every prime ideal $(x) \in \mathcal{O}_K$ by taking $x$ to be $1 + i$ for the ideal $(1 + i)$ and by taking $x$ to be primary otherwise. By further taking 1 as the generator for the ring $\mathbb{Z}[i]$ itself, we extend the choice of the generator for any ideal of $\mathcal{O}_K$ multiplicatively. We denote $G$ for this set of generators. For $a, b \in \mathcal{O}_K$, we write $[a, b]$ for their least common multiple such that $[a, b] \in G$. Similarly, write $(a, b)$ for their greatest common divisor such that $(a, b) \in G$.

For an odd $n \in \mathcal{O}_K$, the quadratic residue symbol $(\frac{a}{n})$ modulo $n$ is first defined when $n = x$ is a prime. In this case, for any $a \in \mathcal{O}_K$, we have $(\frac{a}{n}) = 0$ when $x|a$ and $(\frac{a}{n}) \equiv a^\frac{N(x)-1}{2} \mod x$ with $(\frac{a}{n}) \in \{ \pm 1 \}$ when $(a, x) = 1$. Then the definition is extended to any composite $n$ multiplicatively. Moreover, for $n \in U_K$, we define $(\frac{a}{n}) = 1$.

For two co-prime primary elements $m, n \in \mathcal{O}_K$, we have the following quadratic reciprocity law (see \( [9] \) (2.1))

\[
\left( \frac{m}{n} \right) = \frac{\left( \frac{n}{m} \right)}{n} \cdot \left( \frac{m}{n} \right).	ag{2.1}
\]

Recall that $\chi_c$ is reserved for the quadratic residue symbol $(\frac{z}{n})$. For odd $c$, it is shown in \( [5] \) Section 2.1] that $\chi_c$ can be regarded as a Hecke character of trivial infinite type modulo $16c$, provided that we define $(\frac{z}{n}) = 0$ when $1 + i|a$. We shall henceforth write $\chi_c$ as a Hecke character whose conductor dividing $16c$. We make one exception here that we regard $\chi_{\pm 1}$ as a principal character modulo 1 (so we have $\chi_{\pm 1}(a) = 1$ for all $a \in \mathcal{O}_K$). This further implies that we have $L(s, \chi_{\pm 1}) = \zeta_K(s)$.

For any complex number $z$, we write $e(z) = e^{2\pi i z}$ and we denote

\[
ed(z) = \exp \left( 2\pi i \frac{z}{2i} \frac{z}{2i} \right).
\]

For $r, n \in \mathcal{O}_K$ with $(n, 2) = 1$, the quadratic Gauss sum $g(r, n)$ is defined by

\[
g(r, n) = \sum_{x \mod n} \left( \frac{x}{n} \right) \bar{e} \left( \frac{rx}{n} \right).
\]

Let $\varphi_{[i]}(n)$ denote the number of elements in the reduced residue class of $\mathcal{O}_K/(n)$, we recall the following explicitly evaluations of $g(r, n)$ from \( [9] \) Lemma 2.2.

\textbf{Lemma 2.2.} \ (i) We have

\[
g(rs, n) = \left( \frac{r}{n} \right) g(r, n), \quad (s, n) = 1,
\]

\[
g(k, mn) = g(k, m)g(k, n), \quad m, n \text{ primary and } (m, n) = 1.
\]
Lemma 2.5. Suppose \( \varpi^h \) is the largest power of \( \varpi \) dividing \( k \). (If \( k = 0 \) then set \( h = \infty \).) Then for \( l \geq 1 \),
\[
g(k, \varpi^l) = \begin{cases} 
0 & \text{if } l \leq h \text{ is odd}, \\
\varphi_1(\varpi^l) & \text{if } l \leq h \text{ is even}, \\
-N(\varpi)^{l-1} & \text{if } l = h+1 \text{ is even}, \\
\left( \frac{ik\varpi^{-h}}{\varpi} \right) N(\varpi)^{l-1/2} & \text{if } l = h+1 \text{ is odd}, \\
0, & \text{if } l \geq h+2.
\end{cases}
\]

2.3. The approximate functional equation. Let \( \chi \) be a primitive quadratic Hecke character \( \chi \) of \( K \) of trivial infinite type. A well-known result of E. Hecke says that \( L(s, \chi) \) has an analytic continuation to the entire complex plane and satisfies the following functional equation (see [11, Theorem 3.8])
\[
\Lambda(s, \chi) = W(\chi)(N(m))^{-1/2}\Lambda(1 - s, \chi),
\]
where \( m \) is the conductor of \( \chi \), \( |W(\chi)| = (N(m))^{1/2} \) and
\[
\Lambda(s, \chi) = (|D_K|N(m))^{s/2}(2\pi)^{-s}\Gamma(s)L(s, \chi).
\]

In particular, we have the following functional equation for \( \zeta_K(s) \):
\[
\pi^{-s}\Gamma(s)\zeta_K(s) = \pi^{-(1-s)}\Gamma(1-s)\zeta_K(1-s).
\]

By combining [11, Theorem 3.8] and [7, Lemma 2.2], we see that \( W(\chi(1+i)s, \chi) = g(\chi(1+i)s, \chi) = \sqrt{N((1+i)s)} \) when \( \chi = \chi(1+i)s,d \) for an odd, square-free \( d \in \mathcal{O}_K \). In this case (2.2) becomes
\[
\Lambda(s, \chi(1+i)s, \chi) = \Lambda(1 - s, \chi(1+i)s, \chi).
\]

For \( n \in \mathcal{O}_K \) and \( j \in \mathbb{Z}, j \geq 1 \), we denote \( d_{i,j}(n) \) for the analogue on \( \mathcal{O}_K \) of the usual function \( d_k \) on \( \mathbb{Z} \), so that \( d_{i,j}(n) \) equals the coefficient of \( N(n)^{-s} \) in the Dirichlet series expansion of the \( j \)-th power of \( \zeta_K(s) \). It follows that \( d_{i,1}(n) = 1 \) and when \( n \) is primary,
\[
d_{i,j}(n) = \sum_{a_1 \cdots a_j = n, a_i \equiv 1 \mod (1+i)^3} 1.
\]

We further denote for \( j \in \mathbb{Z}, j \geq 1 \) and any real number \( t > 0 \),
\[
V_j(t) = \frac{1}{2\pi i} \int_{(2)} w_j(s) t^{-s} \frac{ds}{s}, \quad w_j(s) = \left( \frac{2^{5/2}}{\pi} \right)^{js} \left( \frac{\Gamma(\frac{5}{2} + s)}{\Gamma(\frac{5}{2})} \right)^{-j}.
\]

We then note the following approximate functional equation for \( L(\frac{1}{2}, \chi(1+i)s,d)^j \) from [7, Lemma 2.2].

**Lemma 2.4 (Approximate functional equation).** For any odd, square-free \( d \in \mathcal{O}_K \), we have for \( j = 1, 2 \),
\[
L(\frac{1}{2}, \chi(1+i)s,d)^j = \sum_{n \equiv 1 \mod (1+i)^3} \frac{\chi(1+i)s,d(n)d_{i,j}(n)}{N(n)^{\frac{5}{2}}} V_j \left( \frac{N(n)}{N(d)^{1/2}} \right).
\]

The next lemma gives the behaviors of \( V_j(t) \) defined in (2.5) for \( t \to 0^+ \) or \( t \to \infty \), which can be established similar to [10, Lemma 2.1].

**Lemma 2.5.** Let \( j = 1, 2 \). The function \( V_j(\xi) \) is real-valued and smooth on \( (0, \infty) \). We have
\[
V_j(\xi) = 1 + O_\varepsilon(\xi^{-\varepsilon}).
\]

For any fixed integer \( \nu \geq 0 \) and large \( \xi \), we have
\[
V_j^{(\nu)}(\xi) \ll \xi^{\nu + \frac{1}{2}} \exp \left( -\frac{j\xi^{\frac{1}{2}}}{2} \right) \ll_{\nu} \exp \left( -\frac{j\xi^{\frac{1}{4}}}{4} \right).
\]
2.6. Poisson summation. A key ingredient needed in our treatment of the paper is the following two dimensional Poisson summation, which follows from \cite[Lemma 2.7, Corollary 2.8]{[16]}. 

**Lemma 2.7.** Let $n \in \mathcal{O}_K$ be primary and $(\frac{\sigma}{n})$ be the quadratic residue symbol modulo $n$. For any smooth function $W : \mathbb{R}^+ \to \mathbb{R}$ of compact support, we have for $X > 0$, 

$$
\sum_{m \in \mathcal{O}_K \atop (m, 1+i)=1} \frac{\mu^2(n) a_n \chi(n)}{N(n) \leq N} \ll \mu(n) a_n \left| \frac{(m, n)}{N} \right| < \varepsilon (QN)^e(Q + N) \sum_{n_1, n_2 \equiv 1 \mod (1+i)^3 \atop N(n_1), N(n_2) \leq N} |a_{n_1} a_{n_2}|.
$$

Let $M$ be a positive integer, and for each $m \in \mathcal{O}_K$ satisfying $N(m) \leq M$, we write $m = m_1 m_2$ with $m_1$ square-free and $m_2 \in G$. Suppose the sequence $a_n$ satisfies $|a_n| \ll (N)^{\varepsilon}$, then 

$$
\sum_{N(n) \leq N} a_n \left| \frac{(m, n)}{N} \right| < (MN)^{\varepsilon} N(M + N).
$$

The second lemma is a combination of \cite[Lemma 2.15]{[17]} and the proof of \cite[Lemma 2.5]{[10]}. 

**Lemma 2.8.** Suppose $\sigma + it$ is a complex number with $\sigma \geq \frac{1}{2}$. Then 

$$
\sum_{(d, 2)=1 \atop N(d) \leq X} |L(\sigma + it, \chi(1+i)^d)|^4 \ll X^{1+\varepsilon} (1 + |t|^2)^{1+\varepsilon}.
$$

Here the “$*$” on the sum over $d$ means that the sum is restricted to square-free elements $d$ in $\mathcal{O}_K$.

2.11. Analytical behaviors of certain functions. In this section, we discuss the analytical behaviors of certain functions that are needed in the paper. First, we have the following result that can be established similar to \cite[Lemma 5.3]{[16]}.

**Lemma 2.12.** Let $\alpha, d, \ell \in \mathcal{O}_K$ be primary and let 

$$
(2.7) \quad d_1 = \frac{d}{(d, \alpha)}.
$$

For each $k \in \mathcal{O}_K$, $k \neq 0$, we write $kd_1$ uniquely by 

$$
(2.8) \quad kd_1 = k_1 k_2^2,
$$

with $k_1$ square-free and $k_2 \in G$. For $\Re(s) > 1$, we have 

$$
\sum_{\nu \equiv 1 \mod (1+i)^3 \atop \nu, \alpha d = 1} \frac{d_1(\nu)}{N(\nu)^{3/2}} \left( \frac{d_1}{\nu} \right)^{g(k, \ell, \nu)} = L(s, \chi_{ik_1}) \prod_{\omega \in G} G_0(\omega) G_0(s; k, \ell, \alpha, d) =: L(s, \chi_{ik_1})^2 G_0(s; k, \ell, \alpha, d),
$$

where $G_0(\omega)$ is defined by 

$$
G_0(\omega)(s; k, \ell, \alpha, d) = \left( 1 - \frac{1}{N(\omega)^{s}} \left( \frac{ik_1}{\omega} \right) \right)^2 \text{ if } \omega | 2\alpha d,
$$

where $G_0(\omega)$ is defined by
The function $G_0(s; k, \ell, \alpha, d)$ is holomorphic for $\Re(s) > \frac{1}{2}$. Furthermore, on writing

\[ k = k_3k_4^2, \]

with $k_3$ square-free and $k_4 \in G$, we have uniformly for $\Re(s) \geq \frac{1}{2} + \varepsilon$,

\[ G_0(s; k, \ell, \alpha, d) \ll \varepsilon N(\alpha d k_4 \ell) \frac{\varepsilon N(\ell, k_4^2)}{N(\ell, k_4^2)^{1/2}}. \]

Next, let $\Phi(t)$ be the smooth function appearing in the definition of $S_1$ and $S_2$ given in (1.5) and $V_2(t)$ be given in (2.5), we define

\[ F_y(t) = \Phi(t)V_2 \left( \frac{y}{tX} \right). \]

We further define for $\xi > 0$ and $\Re(w) > 0$,

\[ h(\xi, w) = \int_0^\infty \tilde{F}_{\xi} \left( \frac{\xi}{t} \right)^{1/2} t^{w-1} dt. \]

Before we state our next lemma, we would like to recall that the Mellin transform $\hat{g}(s)$ of a function $g$ is given by

\[ \hat{g}(s) = \int_0^\infty g(t)t^{s-1} dt. \]

Now, we are ready to present a result concerning some analytic properties of $h(\xi, w)$.

**Lemma 2.13.** Let $F_{\xi}$ be defined by (2.9) and let $\xi > 0$. The function $h(\xi, w)$ is an entire function of $w$ in $\Re(w) > -1$ such that

\[ h(\xi, w) = \Phi(1 + w)\xi^w \pi \frac{1}{2\pi i} \int_{(c)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma\left( \frac{3}{4} + s \right)}{\Gamma\left( \frac{7}{4} \right)} \right)^2 \left( \frac{X}{\xi} \right)^s (\pi)^{-2s+2w} \frac{\Gamma(s-w)}{\Gamma(1-s+w)} \frac{ds}{s} \]

for any $c$ with $c > \max\{0, \Re(w)\}$. Moreover, in the region $1 \geq \Re(w) > -1$, it satisfies the bound

\[ h(\xi, w) \ll (1 + |w|)^{3-2\Re(w)} \exp \left( -\frac{\xi^{1/4}}{10X^{1/4}|w| + 1} \right) \xi^{\Re(w)}|\Phi(1+w)|. \]

**Proof.** We recall that for any smooth function $W$, the function $\tilde{W}(t)$ defined in (2.6) can be evaluated in polar coordinates as

\[ \tilde{W}(t) = 4 \int_0^{\pi/2} \int_0^\infty \cos(2\pi tr \sin \theta)W(r^2) \ r dr d\theta. \]
It follows from this and the definition of \( V_2(t) \) in (2.10) that we have, for \( c_s > 2 \),

\[
h(\xi, w) = 4 \int_0^{\pi/2} \int_0^\infty \cos(2\pi \left( \frac{\xi}{t} \right)^{1/2} \right) r \sin \theta \Phi(r^2) V_2 \left( \frac{t}{r^2 X} \right) r \, dr \, d\theta \, dt
\]

\[
= 2 \int_0^{\pi/2} \int_0^\infty \cos(r) \Phi((\frac{rt^{1/2}}{2\pi \xi^{1/2} \sin \theta} )^2) V_2 \left( \frac{t}{X} (\frac{rt^{1/2}}{2\pi \xi^{1/2} \sin \theta} )^{-2} \right) \frac{r}{2\pi \xi^{1/2} \sin \theta} ^{2} \, dr \, d\theta \, dt
\]

\[
= 2 \int_0^{\pi/2} \int_0^\infty \cos(r) V_2 \left( \frac{1}{X} \frac{r}{2\pi \xi^{1/2} \sin \theta} ^{2} \right) \frac{r}{2\pi \xi^{1/2} \sin \theta} ^{2} \, dr \int_0^{\pi/2} \Phi((\frac{rt^{1/2}}{2\pi \xi^{1/2} \sin \theta} )^2) t^{w+1} \, dt
\]

\[
= 2 \Phi(1 + w) \int_0^{\pi/2} \int_0^\infty \cos(r) V_2 \left( \frac{1}{X} \frac{r}{2\pi \xi^{1/2} \sin \theta} ^{2} \right) \frac{r}{2\pi \xi^{1/2} \sin \theta} ^{2} \, dr \int_0^{\pi/2} \Phi((\frac{rt^{1/2}}{2\pi \xi^{1/2} \sin \theta} )^2) t^{w-2} \, dt
\]

\[
= 2 \Phi(1 + w) \xi^w \int_0^{\pi/2} \int_0^\infty \cos(r) \frac{1}{2\pi^i} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right) \frac{1}{X} \frac{r}{2\pi \xi^{1/2} \sin \theta} ^{2} \, dr \int_0^{\pi/2} \Phi((\frac{rt^{1/2}}{2\pi \xi^{1/2} \sin \theta} )^2) \frac{2s}{2\pi \sin \theta} ^{2} \, dt
\]

Now, applying the relation (see [6] Section 2.4)

\[
\int_0^{\pi/2} \cos(r) r^u \, dr = \pi \frac{2^{u-1} \Gamma(\frac{u}{2})}{\Gamma(\frac{2-u}{2})}
\]

we see that

\[
\int_0^{\pi/2} \cos(r) r^{2s-2w} \, dr = \pi \frac{2^{2s-2w-1} \Gamma(s-w)}{\Gamma(1 - s + w)}
\]

Substituting this into the above expression for \( h(\xi, w) \), we immediately obtain (2.11) by noticing that we can now take \( c_s > \max\{0, \Re(w)\} \). This also implies that \( h(\xi, w) \) is an entire function of \( w \) in \( \Re(w) > -1 \).

It remains to establish (2.12). For this, we let \( c = \Re(s) \) and we may assume that \( c \geq 2 \) here. By apply Stirling’s formula (see [11 (5.112)]), we deduce that

\[
\ll \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 (\pi)^{-2s+2w} \frac{\Gamma(s-w)}{\Gamma(1 - s + w)} \frac{1}{s}
\]

\[
\ll \frac{|s|^{2c-1} e^{-\frac{c}{2}|s|^2|\Im(s)|} (1 + |s - w|)^{2c-2\Re(w)-1}}{2^{c-1} e^{-\frac{c}{2}|s|^2|\Im(s)|} (1 + |s|)^{2c-2\Re(w)-1} (1 + |w|)^{2c-2\Re(w)-1}}.
\]

It follows that

\[
\int_{(c)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 (\pi)^{-2s+2w} \frac{\Gamma(s-w)}{\Gamma(1 - s + w)} \frac{ds}{s} \ll e^{-c|s|^{4c-2\Re(w)} - 2(1 + |w|)^{2c-2\Re(w)-1}}.
\]

We now take

\[
c = \max\{2, \frac{\xi^{1/4}}{X^{1/4}(1 + |w|)^{1/2}}\}
\]

to see that the bound given (2.12) follows.

Our next two lemmas provide bounds for certain dyadic sums involving \( G_0 \) and \( h(\xi, w) \).
Lemma 2.14. Let $K, J \geq 1$ be two integers and let $k_2 \geq 2$ be defined in (2.8). Then for $\Re(w) = -\tfrac{1}{2} + \varepsilon$ and any sequence of complex numbers $\delta_{t}$ satisfying $\delta_{t} \ll N(t)^{\varepsilon}$, we have

$$
\sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \left| \sum_{\substack{N(t) = J \\ (t, 2ad) = 1}}^{2J-1} \delta_{t} \sqrt{N(t)} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) \right|^2 \ll_{\varepsilon} (N(ad)JK)^{\varepsilon} J(J + K).
$$

Proof. We write any $k \neq 0, k \in \mathcal{O}_K$ as $k = u_k \prod_{\omega \in G, \, \alpha_i \geq 1} \omega_i^{a_{i}}$ with $u_k \in U_K$. For those $\omega_i$ appearing in this product, we define

$$(2.14) \quad a(k) = \prod_{\omega_i} \omega_i^{n_i+1} \quad \text{and} \quad b(k) = \prod_{\alpha_i \geq 2} \omega_i \prod_{\alpha_i} \omega_i^{n_i-1}.$$  

It follows from part (ii) of Lemma 2.22 and the definition of $\mathcal{G}_0$ in Lemma 2.12 that $\mathcal{G}_0(1 + w; k, \ell, \alpha, d) = 0$ unless $\ell = gm$ with $g|a(k), (m, k) = 1$ and $m$ square-free. We then deduce from this and Lemma 2.2 that when $(\ell, 2ad) = 1$,

$$
\mathcal{G}_0(1 + w; k, \ell, \alpha, d) = \sqrt{N(m)} \frac{ik}{m} \prod_{\omega \in G} \prod_{m} \left( 1 + \frac{2}{N(\omega)^{1+w}} \left( \frac{ik_1}{\omega} \right) \right)^{-1} \mathcal{G}_0(1 + w; k, g, \alpha, d).
$$

We apply the above and the Cauchy-Schwarz inequality to see that

$$
\sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \left| \sum_{\substack{N(t) = J \\ (t, 2ad) = 1}}^{2J-1} \delta_{t} \sqrt{N(t)} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) \right|^2 \ll_{\varepsilon} K^{\varepsilon} \sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \sum_{\substack{g|a(k) \\ N(g) < 2J}} \Psi(k, g),
$$

where

$$
\Psi(k, g) = \left| \sum_{N(t) < N(m) < \frac{2J}{g}} \mu_{[i]}(m) \delta_{gm} \sqrt{N(g)} \mathcal{G}_0(1 + w; k, g, \alpha, d) \frac{ik}{m} \prod_{\omega \in G} \prod_{m} \left( 1 + \frac{2}{N(\omega)^{1+w}} \left( \frac{ik_1}{\omega} \right) \right)^{-1} \mathcal{G}_0(1 + w; k, g, \alpha, d) \right|^2.
$$

Applying the bound for $\mathcal{G}_0$ in Lemma 2.12 in the above expression, we obtain that

$$(2.15) \quad \Psi(k, g) \ll_{\varepsilon} N(adK)^{\varepsilon} N(g)^{1+\varepsilon} \left| \sum_{N(t) < N(m) < \frac{2J}{g}} \mu_{[i]}(m) \delta_{gm} \frac{ik}{m} \prod_{\omega \in G} \prod_{m} \left( 1 + \frac{2}{N(\omega)^{1+w}} \left( \frac{ik_1}{\omega} \right) \right)^{-1} \mathcal{G}_0(1 + w; k, g, \alpha, d) \right|^2.
$$

Note that as $(\frac{ik}{m}) \neq 0$,

$$
\prod_{\omega \in G} \prod_{m} \left( 1 + \frac{2}{N(\omega)^{1+w}} \left( \frac{ik_1}{\omega} \right) \right)^{-1} = \prod_{\omega \in G} \prod_{m} \left( 1 - \frac{4}{N(\omega)^{2+2w}} \left( \frac{ik_1}{\omega} \right) \right) \prod_{\omega \in G} \prod_{m} \left( 1 - \frac{4}{N(\omega)^{2+2w}} \left( \frac{ik_1}{\omega} \right) \right) = \prod_{\omega \in G} \prod_{m} \left( 1 - \frac{4}{N(\omega)^{2+2w}} \right) \sum_{j \equiv 1 \bmod (1+i)^3} \mu_{[i]}(j) d_{[i]}(j) \left( \frac{ik_1}{j} \right).
$$

Using this in (2.15), we see via another application of Cauchy-Schwarz that

$$
\Psi(k, g) \ll_{\varepsilon} N(adK)^{\varepsilon} N(g)^{1+\varepsilon} \sum_{\nu < \frac{2J}{g}} \sum_{\nu < \frac{2J}{g}} \mu_{[i]}(m) \delta_{gm} \frac{ik}{m} \prod_{\omega \in G} \prod_{m} \left( 1 - \frac{4}{N(\omega)^{2+2w}} \right)^{-1} \mathcal{G}_0(1 + w; k, g, \alpha, d).
$$

Relabelling $m$ by $jm$ while noting that $N(\omega) > 3$ for all $\omega | m$, we deduce that for $\Re(w) \geq -\frac{1}{2} + \varepsilon$,

$$
\mu_{[i]}(j) \prod_{\omega \equiv j} \prod_{m} \left( 1 - \frac{4}{N(\omega)^{2+2w}} \right)^{-1} \ll N(j)^{\varepsilon}.
$$
Using this, we see that
\[(2.16) \quad \Psi(k, g) \ll \epsilon N(\alpha d JK)^\varepsilon N(g)^{1+\varepsilon} \sum_{N(j) < \frac{2j}{\varepsilon}} \left| \sum_{\substack{\ell, \alpha \in \mathbb{P} \backslash \{0\} \colon \ell | \alpha d}} \mu_{\ell}^2(m) \Phi_{\gamma}(\frac{i}{m} k) \prod_{\ell \in G \backslash \mathbb{Z}} \left(1 - \frac{4}{N(\varepsilon^{2+2\ell})}\right)^{-1}ight|^2.
\]

Observe that $g|a(k)$ implies $b(g)|k$ by (2.14). For such $k$, we relabel it as $fb(g)$ to obtain from (2.16) that
\[(2.17) \quad \Psi(k, gb) = \sum_{N(g) < 2J} \frac{1}{N(k_2)} \sum_{N(j) < \frac{2j}{\varepsilon}} \mu_{\ell}^2(m) \Phi_{\gamma}(\frac{i}{m} k) \prod_{\ell \in G \backslash \mathbb{Z}} \left(1 - \frac{4}{N(\varepsilon^{2+2\ell})}\right)^{-1}
\]

We write $f = f_1 f_2^2$ with $f_1$ square-free and $f_2 \in G$ to see that the relation $fb(g) d_1 = k_1 k_2^2$ implies that $f_2 k_2$, so that $N(k_2)^{-1} \ll N(f_2)^{-1}$. Applying this in (2.17), we see that the assertion of the lemma follows from Lemma 2.9.

**Lemma 2.15.** Let $N(\alpha) \leq Y$ and let $K, J \geq 1$ be two integers. Then for $\Re(w) = -\frac{1}{2} + \varepsilon$ and any $\gamma \in \mathbb{C}$ satisfying $|\gamma| \leq 1$,
\[
\sum_{K \leq N(k) < 2K} \frac{1}{N(k_2)} \sum_{N(j) < \frac{2j}{\varepsilon}} \frac{\gamma \ell}{N(\ell)} G_0(1 + w; k, \ell, h) h(\frac{N(k)X}{2N(\alpha^2d_1\ell)}; w)\]

is bounded by
\[
\ll \epsilon (1 + |w|^2) \left(1 + w\right)^{8+\varepsilon} \frac{N(d_1)^2 N(\alpha)^{2+\varepsilon} J^2 K^2 N(d)^\varepsilon}{X^{1-\varepsilon}} \exp\left(-\frac{1}{20 \sqrt[5]{N(\alpha)^2 N(d_1) J (1 + |w|^2)}}\right),
\]

and also by
\[
\ll (1 + |w|) N(\alpha) JKX \left|\hat{\Phi}(1 + w)\right|^2 \frac{N(\alpha)^2 N(d_1)(JK + J^2)}{KX}.
\]

**Proof.** We apply Lemma 2.12 and Lemma 2.13 to bound respectively $G_0$ and $h(\xi, w)$ to see that the expression in (2.18) is
\[
\ll \left|\hat{\Phi}(1 + w)\right|^2 (1 + |w|)^{8+\varepsilon} \frac{N(d_1)^2 N(\alpha)^{2+\varepsilon} J^2 K^2 N(d)^\varepsilon}{X^{1-\varepsilon}} \exp\left(-\frac{1}{20 \sqrt[5]{N(\alpha)^2 N(d_1) J (1 + |w|^2)}}\right)
\]
\[
\times \sum_{K \leq N(k) < 2K} \frac{1}{N(k) N(k_2)} \left( \sum_{\substack{\ell, \alpha \in \mathbb{P} \backslash \{0\} \colon \ell | \alpha d}} N(|\ell, k_2^2|) \right)^2
\]
\[
\ll \left|\hat{\Phi}(1 + w)\right|^2 (1 + |w|)^{8+\varepsilon} \frac{N(d_1)^2 N(\alpha)^{2+\varepsilon} J^2 K^2 N(d)^\varepsilon}{X^{1-\varepsilon}} \exp\left(-\frac{1}{20 \sqrt[5]{N(\alpha)^2 N(d_1) J (1 + |w|^2)}}\right)
\]
\[
\times \sum_{K \leq N(k) < 2K} \frac{1}{N(kd_1) N(k_2)} \left( \sum_{\substack{\ell, \alpha \in \mathbb{P} \backslash \{0\} \colon \ell | \alpha d}} N(k_2) \right)^2,
\]

where the last estimation above follows from the observation that $N(|\ell, k_2^2|) \leq N(k_4)^2 \leq N(k_2)^2$. By further writing $N(kd_1) = N(k_1 k_2^2)$, we obtain the first bound of the lemma from the above estimation.
We now derive the second bound by setting \( c = \varepsilon \) to write the integral \((2.11)\) as
\[
\frac{1}{2\pi \iota} \int g(s, w) \left( \frac{X}{\xi} \right)^s ds.
\]
This allows us to see that
\[
\left| \sum_{N(\ell) = J} \gamma_{\ell} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) h(\frac{N(k)X}{2N(\alpha^2 \ell)}, w) \right| \lesssim |\hat{\Phi}(1 + w)| \left( \frac{N(\alpha)^{1+\varepsilon} N(d_1)^{\frac{1}{2} + \varepsilon}}{N(k)^{1-\varepsilon} X^{\frac{1}{2} - \varepsilon}} \right) \int g(s, w) \sum_{N(\ell) = J} \gamma_{\ell} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) |ds|.
\]
Note that \((2.13)\) is still valid when \( c = \varepsilon \) and \( \Re(w) = -\frac{1}{2} + \varepsilon \) so that it implies that \( g(s, w; k) \ll \varepsilon (1 + |w|)^\varepsilon \exp(-\left(\frac{1}{2} - \varepsilon\right)|\Im(s)|) \). We then deduce via the Cauchy-Schwarz inequality that
\[
\left| \sum_{N(\ell) = J} \gamma_{\ell} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) h(\frac{N(k)X}{2N(\alpha^2 \ell)}, w) \right|^2 \lesssim (1 + |w|)^\varepsilon |\hat{\Phi}(1 + w)|^2 \left( \frac{N(\alpha)^{2+\varepsilon} N(d_1)^{1+\varepsilon}}{N(k)^{1-\varepsilon} X^{1-\varepsilon}} \right) \int g(s, w) \sum_{N(\ell) = J} \gamma_{\ell} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) |ds|^2.
\]
Inserting this into \((2.13)\) and applying Lemma \((2.14)\) we readily deduce the second bound of the lemma.

### 3. The mollified first moment

In this section, we prove Proposition \(1.2\). By applying Lemma \(1.2\) and the definition of \( M(\varpi) \) in \((1.4)\) in the expression of \( S_1 \) in \((1.8)\), we see that
\[
S_1 = 2 \sum_{N(m) \leq M} \frac{1}{\sqrt{N(m)}} \sum_{n \equiv 1 \mod (1+i)^3} \frac{1}{\sqrt{N(n)}} \sum_{\varpi = 2, 1} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) V_1 \left( \frac{N(n)}{\sqrt{N(\varpi)}} \right) \left( 1 + i \right)^5 \varpi \frac{m}{mn},
\]
\[
= S_1^\square + S_1^\neq,
\]
where
\[
S_1^\square = 2 \sum_{N(m) \leq M} \frac{1}{\sqrt{N(m)}} \sum_{n \equiv 1 \mod (1+i)^3} \frac{1}{\sqrt{N(n)}} \sum_{\varpi = 2, 1} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) V_1 \left( \frac{N(n)}{\sqrt{N(\varpi)}} \right) \left( 1 + i \right)^5 \varpi \frac{m}{mn},
\]
\[
S_1^\neq = 2 \sum_{N(m) \leq M} \frac{1}{\sqrt{N(m)}} \sum_{n \equiv 1 \mod (1+i)^3} \frac{1}{\sqrt{N(n)}} \sum_{\varpi = 2, 1} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) V_1 \left( \frac{N(n)}{\sqrt{N(\varpi)}} \right) \left( 1 + i \right)^5 \varpi \frac{m}{mn}.
\]

#### 3.1. Evaluation of \( S_1^\square \)
As \( b_m \) is supported on square-free elements in \( \mathcal{O}_K \), we see that \( mn = \square \) if and only if \( n = mk^2 \) with \( k \in \mathcal{O}_K \). By making a change of variable \( n = mk^2 \) with \( k \) being primary, we deduce that
\[
S_1^\square = 2 \sum_{\varpi = 2, 1} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) \sum_{N(m) \leq M} \frac{1}{\sqrt{N(m)}} \sum_{k \equiv 1 \mod (1+i)^3} \frac{1}{N(k)} V_1 \left( \frac{N(mk^2)}{\sqrt{N(\varpi)}} \right).
\]

The rapid decay of \( V_1 \) in Lemma \((2.2)\) implies that the contribution from those \( k \) with \( (k, \varpi) \neq 1 \) is \( O_A(X^{-A}) \) for any large number \( A \). Moreover, the condition \( (m, \varpi) = 1 \) is automatically satisfied as \( N(m) \leq M < N(\varpi) \). We may thus ignore these two conditions and apply the definition \((2.3)\) of \( V_1(\xi) \) to see that
\[
\sum_{k \equiv 1 \mod (1+i)^3} \frac{1}{N(k)} V_1 \left( \frac{N(mk^2)}{\sqrt{N(\varpi)}} \right) = \frac{1}{2\pi} \int w_1(s) \left( 1 - \frac{1}{2i+2s} \right) \zeta_K(1+2s)N(\varpi)^{s/2}N(m)^{-s} \frac{ds}{s}.
\]
Now we note the following convexity bounds for $\zeta_K(s)$ and $\zeta'_K(s)$ for $0 < \Re(s) < 1$:

\begin{equation}
\zeta_K(s), \zeta'_K(s) \ll (1 + |s|^2)^{-\frac{1}{2} + \varepsilon}.
\end{equation}

Here the bound for $\zeta_K(s)$ follows from \textbf{[11] Exercise 3, p. 100} and the bound for $\zeta'_K(s)$ can be obtained via a similar convexity principle.

By moving the line of integration to $\Re(s) = -\frac{1}{2} + \varepsilon$, we apply \textbf{[12]} along with the rapid decay of the gamma function in vertical strips to see that the integral on this line is $\ll_{\frac{1}{2} + \varepsilon} \left( \frac{N(m)}{X} \right)^{\frac{1}{2} - \varepsilon} N(m)^{\frac{1}{2} - \varepsilon}$ and this contributes an error term of size $\ll X^{\frac{1}{2} + \varepsilon} M^{1/2} = O(X^{1-\varepsilon})$ to $S_1^{\square}$ by noticing the definition of $M$ in \textbf{[12]} and the estimation that $b_m \ll 1$.

We also obtain a contribution from the residue of a pole at $s = 0$, which we write as an integral along a small circle around 0 to see that

\begin{equation}
S_1^{\square} = 2 \sum_{(m,2) = 1} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) \sum_{N(m) \leq M} \frac{b_m}{N(m)}
\times \frac{1}{2\pi i} \int \frac{w_1(s)}{|s| = \frac{1}{2\pi i}} \left( 1 - \frac{1}{2^{1+2s}} \right) \zeta_K(1+2s) N(\varpi)^{s/2} N(m)^{-s} \frac{ds}{s} + O(X^{1-\varepsilon}).
\end{equation}

Notice that $b_m = \mu_i(m) H \left( \frac{\log N(m)}{\log M} \right)$ and we have by the Fourier inversion formula that

\begin{equation}
H(t) = \int_{-\infty}^\infty h(z) e^{-t(1+iz)} \, dz,
\end{equation}

where

\begin{equation}
h(z) = \frac{1}{2\pi} \int_{-\infty}^\infty e^s H(t) e^{itz} \, dt.
\end{equation}

Applying the above, we see that

\begin{equation}
\sum_{N(m) \leq M} \frac{b_m}{N(m)^{1+s}} = \int_{-\infty}^\infty h(z) \sum_{m \equiv 1 \mod (1+i)^3} \frac{\mu_i(m)}{N(m)^{1+s+\frac{3+iz}{2\log M}}} \, dz
= \int_{-\infty}^\infty h(z) \left( 1 - \frac{1}{2^{1+s+\frac{3+iz}{2\log M}}} \right)^{-1} \zeta_K^{-1} \left( 1 + s + \frac{1+iz}{\log M} \right) \, dz.
\end{equation}

Integration by parts shows that

\begin{equation}
h(z) \ll_{s} \frac{1}{(1 + |z|)^2},
\end{equation}

which implies that we have for any large number $A$,

\begin{equation}
\sum_{N(m) \leq M} \frac{b_m}{N(m)^{1+s}} = \int_{|z| \leq \sqrt{\log M}} h(z) \left( 1 - \frac{1}{2^{1+s+\frac{3+iz}{2\log M}}} \right)^{-1} \zeta_K^{-1} \left( 1 + s + \frac{1+iz}{\log M} \right) \, dz
+ O \left( \frac{1}{(\log X)^A} \right).
\end{equation}

For $|s| = \frac{1}{2\log X}$ and $|z| \leq \sqrt{\log M}$, we expand $\left( 1 - \frac{1}{2^{1+s+\frac{3+iz}{2\log M}}} \right)^{-1} \zeta_K^{-1} \left( 1 + s + \frac{1+iz}{\log M} \right)$ into a power series (note that $\zeta_K(s)^{-1} \sim \frac{1}{s} (s - 1)$ when $s$ is near 1) to obtain that

\begin{equation}
\sum_{N(m) \leq M} \frac{b_m}{N(m)^{1+s}} = \frac{8}{\pi} \int_{|z| \leq \sqrt{\log M}} h(z) \left( s + \frac{1+iz}{\log M} \right) \, dz
+ O \left( \frac{1}{(\log X)^2} \right).
\end{equation}

Again by (3.5), we may extend back the integration above to all $z$ with an negligible error. Then we have

\begin{equation}
\int_{-\infty}^\infty h(z) \left( s + \frac{1+iz}{\log M} \right) \, dz = sH(0) - \frac{1}{\log M} H'(0),
\end{equation}

since by the expression for $H(t)$ given in \textbf{[13]}, we have that

\begin{equation}
H'(t) = -(1+iz) \int_{-\infty}^\infty h(z) e^{-t(1+iz)} \, dz.
\end{equation}
Thus we conclude that
\begin{equation}
\sum_{N(m) \leq M, \, m \equiv 1 \mod (1+i)^3} \frac{b_m}{N(m)^{1+s}} = 8 \pi \left( sH(0) - \frac{1}{\log M H'(0)} \right) + O \left( \frac{1}{(\log X)^2} \right). \tag{3.6}
\end{equation}

We apply (3.6) to (3.3) and use the definition of $\omega_1(s)$ given in (2.5) to see that
\begin{align*}
S_1^\square &= \frac{16}{\pi} \sum_{(\omega,2)=1} \log N(\omega) \Phi \left( \frac{N(\omega)}{X} \right) \\
&\times \frac{1}{2\pi i} \int_{|s| = \frac{1}{2}+2\pi} \omega_1(s) \left( 1 - \frac{1}{2^{1+2s}} \right) \zeta_K(1+2s) N(\omega)^{s/2} \left( sH(0) - \frac{1}{\log M H'(0)} \right) ds + O \left( \frac{X}{\log X} \right) \\
&= \frac{16}{\pi} \sum_{(\omega,2)=1} \log N(\omega) \Phi \left( \frac{N(\omega)}{X} \right) \\
&\times \frac{1}{2\pi i} \int_{|s| = \frac{1}{2}+2\pi} \left( \frac{2^{s/2}}{\pi} \right)^s \frac{\Gamma \left( \frac{1}{2} + s \right)}{\Gamma (s)} \left( 1 - \frac{1}{2^{1+2s}} \right) \zeta_K(1+2s) N(\omega)^{s/2} \left( sH(0) - \frac{1}{\log M H'(0)} \right) ds + O \left( \frac{X}{\log X} \right).
\end{align*}

Now, the integral above can be evaluated according to the formula
\begin{equation}
\text{Res} g(s) = \frac{1}{(n-1)!} \left. \frac{d^{n-1}}{ds^{n-1}} g(s) \right|_{s=0}
\end{equation}
for a function $g(s)$ having a pole of order at most $n$ at $s=0$. We then arrive at
\begin{equation}
S_1^\square = \sum_{(\omega,2)=1} \log N(\omega) \Phi \left( \frac{N(\omega)}{X} \right) \left( H(0) - \frac{\log N(\omega)}{2\log M H'(0)} \right) + O \left( \frac{X}{\log X} \right). \tag{3.7}
\end{equation}

We may replace $\log N(\omega)/\log M$ in the sum above by $\log X/\log M$ in view of the support of $\Phi$. Then applying the prime ideal theorem and partial summation, we see that
\begin{equation}
S_1^\square = 4 \left( H(0) - \frac{\log X}{2\log M H'(0)} \right) \hat{\Phi}(1) X + O \left( \frac{X}{\log X} \right). \tag{3.8}
\end{equation}

3.2. Decomposition of $S_1^{\square}$. Our remaining task is to bound $S_1^{\square}$. By writing $n = rk^2$ with $r$ being primary, square-free and $k$ primary, we see that the condition $mn \neq \square$ in (6.1) is equivalent to $m \neq r$, as both $m$ and $r$ are primary and square-free. This allows us to recast $S_1^{\square}$ as
\begin{align*}
S_1^{\square} &= 2 \sum_{N(m) \leq M, \, m \equiv 1 \mod (1+i)^3} \frac{b_m}{N(m)} \sum_{r \equiv 1 \mod (1+i)^3} \sum_{k \equiv 1 \mod (1+i)^3} \frac{\mu_2^2(r)}{N(k) \sqrt{N(r)}} \\
&\times \sum_{(\omega,2)=1} \log N(\omega) \Phi \left( \frac{N(\omega)}{X} \right) V_1 \left( \frac{N(rk^2)}{\sqrt{N(\omega)}} \right) \left( 1 + i \right)^5 \frac{2}{r We make changes of variables $m \to gm, r \to gr$ with $g = (m,r)$ to further recast $S_1^{\square}$ as
\begin{align*}
S_1^{\square} &= 2 \sum_{g \equiv 1 \mod (1+i)^3} \frac{\mu_2^2(g)}{N(g)} \sum_{N(m) \leq M/N(g), \, m \equiv 1 \mod (1+i)^3} \frac{b_{mg}}{N(m)} \sum_{r \equiv 1 \mod (1+i)^3} \sum_{k \equiv 1 \mod (1+i)^3} \frac{\mu_2^2(r)}{N(r)} \sum_{N(mr) \leq M/N(g), \, m \equiv 1 \mod (1+i)^3} \frac{1}{N(mr)} \\
&\times \sum_{(\omega,2)=1} \log N(\omega) \Phi \left( \frac{N(\omega)}{X} \right) V_1 \left( \frac{N(grk^2)}{\sqrt{N(\omega)}} \right) \left( 1 + i \right)^5 \frac{2}{mr \sqrt{gk^2}} \right)
\end{align*}

We deduce from Lemma 2.34 that we may truncate the sums over $k, r$ to $N(k) \leq X^{1+\epsilon}$ and $N(r) \leq X^{1+\epsilon}$ with negligible errors. Notice that this also implies that $N(k) < N(\omega)$. As we also have $N(g) \leq M < N(\omega)$, we conclude
that we have \( \frac{\log X}{\sqrt{X}} = 1 \). We then extend the sum on \( k \) to infinity again by Lemma 2.5 to see that

\[
S_1^* = \sum_{g \equiv 1 \mod (1+i)^3} \frac{\mu_2^2(g)}{N(g)} \sum_{\substack{N(m) \leq M/N(g) \\ m \equiv 1 \mod (1+i)^3 \atop (m,2g) = 1 \atop r \equiv 1 \mod (1+i)^3 \atop N(r) \leq X^{1/2+s} \atop (r,mg) = 1 \atop N(mr) > 1}} \frac{b_{mg}}{N(m)} \left( 1 + \frac{i}{m} \right) \sum_{\substack{r \equiv 1 \mod (1+i)^3 \\ N(r) \leq X^{1/2+s} \atop (r,mr) = 1 \atop N(mr) > 1}} \frac{\mu_2^2(r)}{N(r)} \left( 1 + \frac{i}{r} \right) \sum_{k \equiv 1 \mod (1+i)^3} \frac{1}{N(k)}
\]

\[
\times \sum_{\varpi \equiv 1 \mod (1+i)^3} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) V_1 \left( \frac{N(grk^2)}{\sqrt{N(\varpi)}} \right) \chi_{mr}(\varpi) + O(X^{-1}).
\]

We may now write \( \varpi = u_{\varpi} \varpi' \) with \( u_{\varpi} \in U_K \). As the treatments are similar, we may further assume that \( \varpi \) is primary, so that we can apply the quadratic reciprocity law \( (2.1) \) with \( (2.5) \) to obtain that

\[
S_1^* = 2 \sum_{g \equiv 1 \mod (1+i)^3} \frac{\mu_2^2(g)}{N(g)} \sum_{\substack{N(m) \leq M/N(g) \\ m \equiv 1 \mod (1+i)^3 \atop (m,2g) = 1 \atop r \equiv 1 \mod (1+i)^3 \atop N(r) \leq X^{1/2+s} \atop (r,mr) = 1 \atop N(mr) > 1}} \frac{b_{mg}}{N(m)} \left( 1 + \frac{i}{m} \right) \sum_{\substack{r \equiv 1 \mod (1+i)^3 \\ N(r) \leq X^{1/2+s} \atop (r,mr) = 1 \atop N(mr) > 1}} \frac{\mu_2^2(r)}{N(r)} \left( 1 + \frac{i}{r} \right) \sum_{k \equiv 1 \mod (1+i)^3} \frac{1}{N(k)}
\]

\[
\times \int \left( \frac{25/2}{\pi} \right)^s \left( \frac{\Gamma(\frac{g}{2} + s)}{\Gamma(\frac{|g|}{2})} \right) \left( 1 - \frac{1}{2^{1+2s}} \right) \zeta_K(1+2s)N(gr)^{-s}
\]

\[
\times \sum_{\varpi \equiv 1 \mod (1+i)^3} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) N(\varpi)^{s/2} \chi_{mr}(\varpi) \frac{ds}{s} + O(X^{-1}).
\]

We denote \( \Lambda_1(n) \) for the von Mangoldt function on \( K \), so that \( \Lambda_1(n) \) equals the coefficient of \( N(n)^{-s} \) in the Dirichlet series expansion of \( \zeta_K(s) / \zeta_K(1) \). Then we have

\[
\sum_{\varpi \in \mathcal{O_G}} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) \chi_{mr}(\varpi) N(\varpi)^{s/2} = \sum_{n \in \mathcal{O}} \Lambda_1(n) \Phi \left( \frac{N(n)}{X} \right) \chi_{mr}(n) N(n)^{s/2} + O(X^{1/2}).
\]

It is easy to check that the contribution of the error term above to \( S_1^* \) is \( O(X^{1-\varepsilon}) \) for sufficiently small \( \varepsilon = \varepsilon(\theta) > 0 \). Now, we define for any function \( g(t) \) and any complex number \( s \),

\[
g_s(t) = g(t)t^{s/2}.
\]

Using this notation, we have

\[
S_1^* = 2 \sum_{g \equiv 1 \mod (1+i)^3} \frac{\mu_2^2(g)}{N(g)} \sum_{\substack{N(m) \leq M/N(g) \\ m \equiv 1 \mod (1+i)^3 \atop (m,2g) = 1 \atop r \equiv 1 \mod (1+i)^3 \atop N(r) \leq X^{1/2+s} \atop (r,mr) = 1 \atop N(mr) > 1}} \frac{b_{mg}}{N(m)} \left( 1 + \frac{i}{m} \right) \sum_{\substack{r \equiv 1 \mod (1+i)^3 \\ N(r) \leq X^{1/2+s} \atop (r,mr) = 1 \atop N(mr) > 1}} \frac{\mu_2^2(r)}{N(r)} \left( 1 + \frac{i}{r} \right)
\]

\[
\times \frac{1}{2\pi i} \int_{\frac{1}{2} - i \log X}^{\frac{1}{2} + i \log X} \left( \frac{25/2}{\pi} \right)^s \left( \frac{\Gamma\left(\frac{g}{2} + s\right)}{\Gamma\left(\frac{|g|}{2}\right)} \right) \left( 1 - \frac{1}{2^{1+2s}} \right) \zeta_K(1+2s)X^{s/2}N(gr)^{-s}
\]

\[
\times \sum_{n \in \mathcal{O}} \Lambda_1(n) \Phi_s \left( \frac{N(n)}{X} \right) \chi_{mr}(n) \frac{ds}{s} + O(X^{1-\varepsilon}),
\]
where we truncate the integral to $|\Im(s)| \leq (\log X)^2$ due to the rapid decay of the gamma function in vertical strips. We further split the first expression on the right-hand side above into a sum of two terms:

$$S^\delta = E_1 + E_2 + O(X^{1-\varepsilon}),$$

with $E_1$ restricting the sums over $m, r$ to $N(mr) \ll \exp(w\sqrt{\log X})$ and $E_2$ the opposite, where $w > 0$ is a fixed sufficiently small constant.

### 3.3. Evaluation of $E_1$.

In this section we estimate $E_1$, which is given by

$$E_1 = 2 \sum_{g \equiv 1 \mod (1+i)^3} \frac{\mu_0^2(g)}{N(g)} \sum_{m \equiv 1 \mod (1+i)^3} h_{mg} \left( \frac{1+i}{m} \right) \sum_{r \equiv 1 \mod (1+i)^3} \frac{\mu_r^2(r)}{N(r)} \left( \frac{1+i}{r} \right) \times \frac{1}{2\pi i} \int_{\frac{1}{2}+it-\frac{1}{2}+it} \left( \frac{\zeta(s)}{\zeta(1)} \right)^s \left( \prod_{n \leq N} \frac{n}{X} \right) \chi_{mr}(n) \frac{ds}{s}.$$

By partial summation, we have that

$$\sum_{n \in G} \Lambda_{[i]}(n) \phi_s \left( \frac{N(n)}{X} \right) \chi_{mr}(n) = -\int_0^\infty \frac{1}{X} \Phi'_s \left( \frac{u}{X} \right) \left( \sum_{n \in G, \chi \leq u} \Lambda_{[i]}(n) \chi_{mr}(n) \right) du.$$

Combining [11] Theorem 5.13 and [11] Theorem 5.35 together with [11] (5.52)], by noting that the conductor of the primitive character $\chi_{mr}$ is $\ll \exp(w\sqrt{\log X}) \leq \exp(2w\sqrt{\log X})$, we see that

$$\sum_{n \in G, \chi \leq u} \Lambda_{[i]}(n) \chi_{mr}(n) = -\frac{u^{\beta_1}}{\beta_1} + O \left( u \exp(w\sqrt{\log X}) \exp(-c_1\sqrt{\log u}) \right),$$

for an absolute constant $c_1 > 0$. Here the term $-\frac{u^{\beta_1}}{\beta_1}$ appears only when $L(s, \chi_{mr})$ has a real zero $\beta_1 > 1 - \frac{2}{\log N(mr)}$ for a positive constant $c_2$.

Notice that for $\Re(s)$ bounded, we have uniformly in $s$ that

$$\int_0^\infty \frac{1}{X} |\Phi'_s \left( \frac{u}{X} \right)| \, du = \int_0^\infty |\Phi'_s(u)| \, du \ll |s| + 1.$$

Applying the above estimation, we see from (3.11) that the contribution of the error term to $E_1$ is

$$\ll X \exp(c_3(w - c_1)\sqrt{\log X})$$

for some absolute constant $c_3 > 0$.

By an analogue of Page’s theorem for the family of quadratic Hecke $L$-functions (which can be established by using combining the arguments in [11] §14, page 95, [11] Theorem 5.28 (1) and [11] Lemma 5.9), there exists a fixed absolute constant $c_4 > 0$ such that we have at most one character $\chi_{mr}$ (notice as shown above, the conductor of $\chi_{mr}$ is $\leq \exp(2w\sqrt{\log X})$) for which the $L$-function $L(s, \chi_{mr})$ has a real zero $\beta_1$ satisfying

$$\beta_1 > 1 - \frac{c_4}{2w\sqrt{\log X}}.$$
Now, by choosing \( w > 0 \) sufficiently small in terms of \( c_1 \) in \( 3.13 \) and applying the above to \( 3.11 \), we see that for some positive constant \( c_5 \) and some bounded power of two denoted by \( \gamma^* \),

\[
E_1 = -\frac{2}{2\pi i} \sqrt{N(\gamma^*)} X^{\beta_1} \int_{-\infty-i(\log X)^2}^{\infty+i(\log X)^2} \left( \frac{25/2}{\pi} \right)^s \left( \frac{\Gamma(1+s)}{\Gamma(1/2)} \right) \left( 1 - \frac{1}{2^{1+s}} \right) \times X^{s/2} \zeta_K(1+2s) \tilde{\Phi} \left( \frac{s}{2} + \beta_1 \right) \sum_{1 < N(mr) \leq \exp(w \sqrt{\log X})} \mu^2_{[i]}(r) \frac{1 + i}{N(r)^s} \sum_{g \equiv 1 \pmod{(1+i^3)} \gamma^* = q^*} \frac{\mu^2_{[i]}(g)b_{mg}}{N(g)^{1+s}} \frac{ds}{s} + \mathcal{O} \left( X \exp(-c_5 \sqrt{\log X}) \right).
\]

Using \( b_{mg} = \mu_{[i]}(mg)H\left( \frac{\log N(mg)}{\log M} \right) \) and applying Fourier inversion given in \( 3.3 \), we obtain

\[
\sum_{g \equiv 1 \pmod{(1+i^3)} \gamma^* = q^*} \frac{\mu^2_{[i]}(g)b_{mg}}{N(g)^{1+s}} \frac{ds}{s} = \mu_{[i]}(m) \int_{-\infty}^{\infty} \frac{1}{N(m)^{1+s}} h(z) \prod_{\varpi \in G} \left( 1 - \frac{1}{\gamma^* = q^*} \right) \zeta^{-1}_K \left( 1 + s + \frac{1 + iz}{\log M} \right) dz.
\]

We may truncate the above integral in \( 3.14 \) to \( |z| \leq \sqrt{\log M} \) with a negligible error using \( 3.5 \). This leads to

\[
E_1 = -\frac{2}{2\pi i} \sqrt{N(\gamma^*)} X^{\beta_1} \sum_{N(m) \leq M, N(r) \leq X^{1+s}} \mu_{[i]}(m) \mu^2_{[i]}(r) \left( 1 + i \right) \frac{1}{2\pi i} \int_{-\infty-i(\log X)^2}^{\infty+i(\log X)^2} \left( \frac{25/2}{\pi} \right)^s \left( \frac{\Gamma(1+s)}{\Gamma(1/2)} \right) N(r)^{-s} \left( \frac{1 + i}{rm} \right) \times \zeta_K(1+2s) X^{s/2} \tilde{\Phi} \left( \frac{s}{2} + \beta_1 \right) \int_{|z| \leq \sqrt{\log M}} \frac{1}{N(m)^{1+s}} h(z) \prod_{\varpi \in G} \left( 1 - \frac{1}{\gamma^* = q^*} \right) \zeta^{-1}_K \left( 1 + s + \frac{1 + iz}{\log M} \right) dz \frac{ds}{s} + \mathcal{O} \left( \frac{X}{\log X} \right).
\]

We move line of integration over \( s \) in \( 3.15 \) to \( \Re(s) = -\frac{c_6}{\log \log X} \), for some small \( c_6 > 0 \) such that there is no zero of \( \zeta_K(1+s+\frac{1+iz}{\log M}) \) in the region \( \Re(s) \geq -\frac{c_6}{\log \log X} \), \( \Im(s) \leq (\log X)^2 \). The contribution to \( E_1 \) of the integration over \( s \) on the new line of integration is \( O(X/\log X) \). There is also a contribution from the residue at \( s = 0 \), which we write as an integral along a small circle around 0 to see that

\[
E_1 = -\frac{2}{2\pi i} \sqrt{N(\gamma^*)} X^{\beta_1} \sum_{N(m) \leq M, N(r) \leq X^{1+s}} \mu_{[i]}(m) \mu^2_{[i]}(r) \left( 1 + i \right) \frac{1}{2\pi i} \int_{|s| = \frac{1}{\log X}} \left( \frac{25/2}{\pi} \right)^s \left( \frac{\Gamma(1+s)}{\Gamma(1/2)} \right) N(r)^{-s} \times \zeta_K(1+2s) X^{s/2} \tilde{\Phi} \left( \frac{s}{2} + \beta_1 \right) \int_{|z| \leq \sqrt{\log M}} \frac{1}{N(m)^{1+s}} h(z) \prod_{\varpi \in G} \left( 1 - \frac{1}{\gamma^* = q^*} \right) \zeta^{-1}_K \left( 1 + s + \frac{1 + iz}{\log M} \right) dz \frac{ds}{s} + \mathcal{O} \left( \frac{X}{\log X} \right).
\]

(3.16)

Notice that when \( |s| = \frac{1}{\log X} \), we have

\[
\zeta_K(1+2s) \ll X, \quad \zeta^{-1}_K \left( 1 + s + \frac{1 + iz}{\log M} \right) \ll \frac{1}{\log X}.
\]
Applying the above bounds in (3.10) and estimating things trivially, we deduce that

\[(3.17) \quad E_1 \ll \frac{X^{\beta_1}}{N(q^s)^{1/2-\varepsilon}} + O\left(\frac{X}{\log X}\right).\]

Now we need an upper bound for \(\beta_1\). For this, we recall a result of Landau [13] says that for an algebraic number field \(F\) of degree \(n = 2\) and any primitive ideal character \(\chi\) of \(F\) with conductor \(q\), we have for \(X > 1\),

\[\sum_{\chi(I) \leq X} \chi(I) \ll |N_F(q) \cdot D_F|^{1/3} \log^2(|N_F(q) \cdot D_F|X^{1/3}),\]

where \(N_F(q)\), \(N_F(I)\) denotes the norm of \(q\) and \(I\) respectively, \(D_F\) denotes the discriminant of \(F\) and \(I\) runs over integral ideas of \(F\).

It follows from this that we have, for any Hecke character \(\chi\) modulo \(q\) of trivial infinite type in \(K\),

\[(3.18) \quad L(1, \chi) = \sum_{I \neq 1 \atop N(I) \leq N(q)} \frac{\chi(I)}{N(I)} + \sum_{N(I) > N(q)} \frac{\chi(I)}{N(I)} \ll \sum_{I \neq 1 \atop N(I) \leq N(q)} \frac{1}{N(I)} + \int_{N(q)}^{\infty} \frac{d}{u} \left(\sum_{N(I) \leq u} \chi(I)\right) \ll \log N(q).\]

Similarly, we have that

\[(3.19) \quad L'(1, \chi) \ll \log^2 N(q).\]

We then deduce from (3.18), (3.19) and the proof of [11] Theorem 5.28 (2) that we have the following analogue in \(K\) of Siegel’s theorem, i.e. for any primitive quadratic Hecke character \(\chi\) modulo \(q\) of trivial infinite type, we have that

\[(3.20) \quad \beta_1 < 1 - \frac{c_2(\varepsilon)}{N(q)^{\varepsilon}},\]

where \(c_2(\varepsilon) > 0\) is an ineffective constant depending only on \(\varepsilon\).

Applying (3.20) in (3.17), then treating the resulting upper bound of \(E_1\) according to whether \(N(q^s)\) is \(\leq (\log X)^3\) or not, we see that

\[(3.21) \quad E_1 \ll \frac{X^{1 - \frac{c_2(\varepsilon)}{N(q)}}}{N(q^s)^{1/2-\varepsilon}} + O\left(\frac{X}{\log X}\right) \ll \frac{X}{\log X}.\]

### 3.4. Evaluation of \(E_2\).

In this section we estimate \(E_2\). We let \(q = mr\) in (3.11) and we break \(N(q)\) into dyadic segments to see that

\[(3.22) \quad E_2 \ll (\log X)^{(1)} \sum_{Q = 2^j \atop \exp(w \sqrt{\log X}) \ll Q \ll MX^{1/2+\varepsilon}} \mathcal{E}(Q),\]

where

\[\mathcal{E}(Q) = Q^{-\frac{1}{2} + \varepsilon} \sum_{n \in S(Q)} \left|\sum_{n \in G} \Lambda_{ij}(n) \Phi_{s_0} \left(\frac{n}{X}\right) \chi(n)\right|.\]

Here \(S(Q)\) is defined as in Lemma 2.11, \(s_0 \in \mathbb{C}\) such that \(\Re(s_0) = \frac{1}{\log X}\) and \(|\Im(s_0)| \leq (\log X)^2\).

We now employ zero-density estimates to estimate \(\mathcal{E}(Q)\). For this, we write \(\Phi_{s_0}\) using inverse Mellin transform to obtain that

\[(3.23) \quad \sum_{n \in G} \Lambda_{ij}(n) \Phi_{s_0} \left(\frac{N(n)}{X}\right) \chi(n) = \frac{1}{2\pi i} \int_{(2)} X^w \Phi \left(w + \frac{s_0}{2}\right) \left(-\frac{L'}{L}(w, \chi)\right) dw.\]

Note that integration by parts implies that for every non-negative integer \(j\), we have

\[(3.24) \quad \left|\Phi(\sigma + it + \frac{s_0}{2})\right| \ll_{\sigma, j} (\log X)^j \left(1 + \left|\frac{\log(s_0)}{2}\right|\right)^{-j}.\]

On the other hand, it follows from the functional equation (2.3) that for any primitive Hecke character \(\chi\) modulo \(m\) of trivial infinite type in \(K\), we have

\[L(s, \chi) = W(\chi)(N(m))^{-1/2}(|D_K|N(m))^{(1-2s)/2}(2\pi)^{1-2s} \frac{\Gamma(s)}{\Gamma(1-s)} L(1-s, \chi).\]

Upon taking logarithmic derivative on both sides above, we see that

\[
\frac{L'(s, \chi)}{L(s, \chi)} = -\log(|D_K|N(m)) - \log(2\pi) + \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(1-s)}{\Gamma(1-s)} - \frac{L'(1-s, \chi)}{L(1-s, \chi)}.
\]
Applying the estimation (see Theorem C.1) that for \( |s| > \delta, |\arg s| < \pi - \delta, \)
\[
\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(\frac{1}{|s|}\right),
\]
we deduce that on the line \( \Re(w) = -\frac{1}{2} \) we have
\[
\left| \frac{L'}{L}(w, \chi) \right| \ll \log(N(q)|w|).
\]

We now shift the line of integration in (3.23) to \( \Re(w) = -\frac{1}{2} \) and we apply the above bound of \( L'/L \) and (3.24) to bound the integral on this line. Along with contributions from residues of poles at all zeros \( \rho = \beta + i\gamma \) of \( L(s, \chi) \) in the critical strip, we obtain that
\[
\sum_{n \in G} \Lambda_{[\nu]}(n) \Phi_n \left( \frac{N(n)}{X} \right) \chi(n) = \sum_{L(\rho, \chi) = 0} X^\rho \hat{\Phi} \left( \rho + \frac{s_0}{2} \right) + O \left( \frac{(\log X)^{O(1)}}{X^{1/2}} \right).
\]

To bound the right side of (3.28), we introduce some notations here. For a primitive Hecke character \( \chi \) modulo \( q \) of trivial infinite type, let \( N(T, \chi) \) denote the number of zeros of \( L(s, \chi) \) in the rectangle
\[
0 \leq \beta \leq 1, \quad |\gamma| \leq T.
\]

Then we have Theorem 5.8 for \( T \geq 2, \)
\[
N(T, \chi) \ll T \log(N(q)T).
\]

We further define for \( \frac{1}{2} \leq \alpha \leq 1, \)
\[
N(\alpha, Q, T) = \sum_{\chi \in \mathcal{S}(Q)} N(\alpha, T, \chi),
\]
where \( N(\alpha, T, \chi) \) denotes the number of zeros \( \rho = \beta + i\gamma \) of \( L(s, \chi) \) in the rectangle
\[
\alpha \leq \beta \leq 1, \quad |\gamma| \leq T.
\]

We note the following analogue of Heath-Brown’s zero-density estimate for \( L \)-functions of quadratic Dirichlet characters Theorem 3 to see that for \( \frac{1}{2} < \alpha \leq 1, \)
\[
N(\alpha, Q, T) \ll (QT)^{\varepsilon}(T^{3/2}Q)^{(2-2\alpha)/(3/2-\alpha)}.
\]

In fact, the above estimation follows by modifying the proof of Corollary 1.6 using the earlier mentioned large sieve result of K. Onodera on quadratic residue symbols in the Gaussian field. More specifically, we replace the bound given in (5.4)] by
\[
(Q^2X)^{\varepsilon}T^{1/2-\alpha}(QT)^{1/2}(X^{1/2} + Q^{1/2}).
\]

Also, we replace the bound given in (5.5)] by
\[
(QY)^{\varepsilon}(QX^{1-2\alpha} + Y^{2-2\alpha}),
\]
where \( 1 \leq X \leq Y. \) Then by setting \( X = Q, Y = (T^{3/2}Q)^{1/(3/2-\alpha)}, \) we deduce (3.21).

Now, combining (3.24) and (3.26), we see that the contribution in (3.25) to \( E(Q) \) from those \( \rho \) with \( |\gamma| > Q^\varepsilon \) is
\[
\ll X(\log X)^A Q^{1+\varepsilon} Q^{-A\varepsilon},
\]
for any large number \( A. \) We compute the contribution of this bound and the error term in (3.25) to \( E(Q) \) by noting that \( Q \gg \exp(w\sqrt{\log X}) \) to obtain that
\[
E(Q) \ll X \exp(-w\sqrt{\log X}) + Q^{-\frac{1}{2} + \varepsilon} \sum_{\chi \in \mathcal{S}(Q)} \sum_{L(\rho, \chi) = 0} X^\beta.
\]

In (3.28), we separate the zeros \( \rho \) according to whether \( \beta < \frac{1}{2} + \varepsilon_0 \) or \( \beta \geq \frac{1}{2} + \varepsilon_0 \) for a suitable small \( \varepsilon_0 > 0 \) such that when \( Q < X^{1-\varepsilon} \), we have by (3.26),
\[
Q^{-\frac{1}{2} + \varepsilon} \sum_{\chi \in \mathcal{S}(Q)} \sum_{L(\rho, \chi) = 0} X^\beta \ll X^{\frac{1}{2} + \varepsilon_0} Q^{1/2+\varepsilon} \ll X^{1-\varepsilon}.
\]
For those zeros with $\beta \geq \beta_0$ we apply partial summations to see that
\[ Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho, \chi) = 0 \\beta_0 \leq \beta \leq 1 \\epsilon \leq \sqrt{Q}}} X^\beta = -Q^{-\frac{1}{2}+\varepsilon} \int_{\beta_0}^{1} X^\alpha dN(\alpha, Q, Q^\varepsilon) \]
\[ \ll Q^{-\frac{1}{2}+\varepsilon} X^{\beta_0} N(\beta_0, Q, Q^\varepsilon) + \log X Q^{-\frac{1}{2}+\varepsilon} \int_{\beta_0}^{1} X^\alpha N(\alpha, Q, Q^\varepsilon) d\alpha. \]
(3.30)

Note that when $\alpha \leq 1$, we have $1/(3/2 - \alpha) \leq 2$, so that the bound (3.27) implies that
\[ N(\alpha, Q, Q^\varepsilon) \ll Q^{4(1-\alpha)+\varepsilon}. \]

Replacing this bound for $N(\alpha, Q, Q^\varepsilon)$ in the last integral in (3.30), we see that when $Q \geq X^{1/4}$, we see that the integrand in the right-hand side expression of (3.30) is maximized when $\alpha = \beta_0$, it follows from this and (3.30) that
\[ Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho, \chi) = 0 \\beta_0 \leq \beta \leq 1 \\epsilon \leq \sqrt{Q}}} X^\beta \ll X^{\beta_0} Q^{4(1-\beta_0)-1/2+\varepsilon} \ll X^{\beta_0} X^{4(1-\beta_0)-1/2+\varepsilon} \ll X^{1-\varepsilon}, \]
when $\beta_0 > \beta_1 + \varepsilon$ with $\beta_1 = 5/6$. When $Q < X^{1/4}$, the integrand in the right-hand side expression of (3.30) is maximized when $\alpha = 1$, in which case we have
\[ Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho, \chi) = 0 \\beta_0 \leq \beta \leq 1 \\epsilon \leq \sqrt{Q}}} X^\beta \ll X^{\beta_0} Q^{4(1-\beta_0)-1/2+\varepsilon} + XQ^{4(1-1)-1/2+\varepsilon} \ll X^{1-\varepsilon} + XQ^{-1/2+\varepsilon}, \]
(3.31)
since we have that $Q \ll \exp(w\sqrt{\log X})$.

We now argue inductively by setting
\[ \beta_{n+1} = \frac{3\beta_n - 1}{2} + \beta_n \]
(3.33)
to see that when $1/2 < \beta_0 < \beta_n + \epsilon$, we have $1/(3/2 - \alpha) \leq 1/(3/2 - \beta_n) + \epsilon$ so that when $Q \geq X^{(3/2 - \beta_n)/2}$, the estimation (3.31) is valid when $\beta_0 > \beta_{n+1} + \varepsilon$. On the other hand, the estimation (3.32) is valid when $Q < X^{(3/2 - \beta_n)/2}$ and $\beta_0 > \beta_{n+1} + \varepsilon$.

One checks that the sequence $\{\beta_n\}_{n \geq 1}$ defined in (3.33) is decreasing and satisfies $1/2 < \beta_n < 1$ for all $n \geq 1$ (recall that $\beta_1 = 5/6$). It follows that $\lim_{n \to \infty} \beta_n$ exists and a little computation shows that
\[ \lim_{n \to \infty} \beta_n = \frac{1}{2}. \]

It follows that, for our choice of $\varepsilon > 0$ above, we can achieve, after finitely number of iterations, that
\[ Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho, \chi) = 0 \\beta_0 \leq \beta \leq 1 \\epsilon \leq \sqrt{Q}}} X^\beta \ll X^{1-\varepsilon} + XQ^{-1/2+\varepsilon}. \]
(3.34)

We now combine (3.25), (3.29) and the above estimation to see that for some positive constant $c_8$,
\[ E(Q) \ll X Q^{-1/2+\varepsilon} + X^{1-\varepsilon} \ll X \exp(-c_8w\sqrt{\log X}). \]

Applying the above bound to (3.22), we see that for some absolute constant $c > 0$,
\[ (3.34) \quad E_2 \ll X \exp(-cw\sqrt{\log x}) . \]

3.5. **Conclusion.** We now combine (3.8), (3.10), (3.21) and (3.34) to see that Proposition 1.2 follows.

4. **The mollified second moment**

We now begin our proof of Proposition 1.3. As a preparation, we first include some results from the sieve methods.
4.1. Tools from sieve methods. We denote $1_{\mathcal{A}}(n)$ for the indicator function of a set $\mathcal{A}$ of algebraic integers in $\mathcal{O}_K$, so that $1_{\mathcal{A}}(n) = 1$ when $n \in \mathcal{A}$ and $1_{\mathcal{A}}(n) = 0$ otherwise. Then we have
\[(4.1) \quad 1_{\{n: n \text{ prime}\}} \leq 1_{\{n: n P(z_0) = 1\}} 1_{\{n: n P(R)/P(z_0) = 1\}},\]
where $R$ is defined as in $[1.2]$ and
\[(4.2) \quad z_0 = \exp((\log X)^{1/3}), \quad P(y) = \prod_{\omega \in G, N(\omega) \leq y} \omega, \quad y > 2.
\]

We write $\omega_i(n)$ for the number of distinct prime ideal factors of $(n)$. We apply Brun's upper bound sieve condition (see $[5.6.1]$) to see that
\[(4.3) \quad 1_{\{n: n P(z_0) = 1\}}(n) \leq \sum_{b \in G, b | n P(z_0)} \mu_{i}(b), \quad \omega_{i}(b) \leq 2r_0
\]
where
\[r_0 = \lfloor (\log X)^{1/3} \rfloor.
\]

Let $G(t)$ be a non-negative smooth function, compactly supported on $[-1, 1]$ satisfying $|G(t)| \ll 1$, $|G^{(j)}(t)| \ll_j (\log X)^{-1}$ for $j \geq 1$ and $G(t) = 1 - t$ for $0 \leq t \leq 1 - (\log X)^{-1}$. Then we have
\[(4.4) \quad 1_{\{n: n P(R)/P(z_0) = 1\}}(n) \leq \left( \sum_{d \in G, \frac{d}{|d| n}} \mu_{i}(d) G(\frac{\log N(d)}{\log R}) \right)^2
\]
\[= \sum_{j, k \in G, N(j), N(k) \leq R, (j, k) | n} \mu_{i}(j) \mu_{i}(k) G(\frac{\log N(j)}{\log R}) G(\frac{\log N(k)}{\log R}).\]

Now $[4.2]$, $[4.3]$ and $[4.4]$ implies that
\[(4.5) \quad 1_{\{n: n \text{ prime}\}}(n) \leq \sum_{d \in G, \frac{d}{|d| n}} \lambda_d,
\]
where the coefficients $\lambda_d$ are defined by
\[(4.6) \quad \lambda_d = \sum_{b \in G} \sum_{b \in G, \omega_i(b) \leq 2r_0} \mu_{i}(b) \mu_{i}(m) \mu_{i}(n) G\left(\frac{\log N(m)}{\log R}\right) G\left(\frac{\log N(n)}{\log R}\right) G\left(\frac{\log N(k)}{\log R}\right).
\]

Similar to what is pointed out in the paragraph above $[1.5]$, we have $\lambda_d \neq 0$ for $N(d) \leq D$, where
\[(4.7) \quad D = R^2 \exp(2(\log X)^{2/3}) \ll_\varepsilon R^2 X^{\varepsilon}.
\]

We end this section by listing a few lemmas needed in the paper, which are analogous to $[1.5]$, Lemma 5.1-5.4.

**Lemma 4.2.** Let $0 < \delta < 1$ be a fixed constant, $r$ a positive integer with $r \asymp (\log X)^\delta$, and $z_0$ as in $[4.2]$. Let $G$ be the set of generators of ideals in $\mathcal{O}_K$ chosen in Section $[2.7]$. Suppose that $g$ is a multiplicative function on $G$ such that uniformly for all primes $\omega \in G$, we have $|g(\omega)| < 1$. Then uniformly for all $t \in \mathcal{O}_K$,
\[
\sum_{b \in G} \frac{\mu_{i}(b)}{N(b)} g(b) = \prod_{\omega \in G, N(\omega) \leq z_0, \omega \not| t} \left( 1 - \frac{g(\omega)}{N(\omega)} \right) + O\left( \exp(-r \log \log r) \right).
\]

**Lemma 4.3.** Let $z_0 = \exp((\log X)^{1/3})$. Let $G(t)$ be as above and $G$ be the set of generators of ideals in $\mathcal{O}_K$ chosen in Section $[2.7]$. Suppose $h$ is a function on $G$ such that uniformly for all primes $\omega \in G$, $|h(\omega)| \ll_\varepsilon N(\omega)^{-\varepsilon}$. For a fixed
real number $A > 0$, there exists a function $E_0(X)$ depending only on $X, G$ and $\vartheta$ with $E_0(X) \to 0$ as $X \to \infty$, such that uniformly for $N(\ell) \ll X^{O(1)}$,

$$\sum_{m, n \in G} \frac{\mu(\phi(m)) \mu(\phi(n))}{N(\phi(m), \phi(n))} \cdot G \left( \frac{\log N(m)}{\log R} \right) \cdot G \left( \frac{\log N(n)}{\log R} \right) \prod_{\varpi \in G} \left( 1 + h(\varpi) \right)$$

(4.8)

$$- \frac{4}{\pi} \cdot \frac{1 + E_0(X)}{\log R} \cdot \prod_{\varpi \in G} \left( 1 - \frac{1}{N(\varpi)} \right)^{-1} + O_{\epsilon, A} \left( \frac{1}{(\log R)^A} \right),$$

**Lemma 4.4.** Let $\lambda_d$ and $D$ be as defined in [4.4] and (4.4), respectively. Let $G$ be the set of generators of ideals in $\mathcal{O}_K$ chosen in Section 2.1. Suppose that $g$ is a multiplicative function on $G$ such that $g(\varpi) = 1 + O(N(\varpi)^{-\varepsilon})$ for all primes $\varpi \in G$. Then with $E_0(X)$ as in Lemma 4.4 we have uniformly in $N(\ell) \ll X^{O(1)}$,

$$\sum_{d \in G} \frac{\lambda_d}{N(d)} g(d) \sum_{\varpi \in G} h(\varpi) = - \frac{4}{\pi} \cdot \frac{1 + E_0(X)}{\log R} \cdot \prod_{\varpi \in G} \left( 1 - \frac{1}{N(\varpi)} \right)^{-1}$$

$$\times \sum_{\varpi \in G} \frac{g(\varpi) h(\varpi)}{N(\varpi)} \prod_{\varpi' \in G \atop \varpi' \not\mid \varpi \varpi, \varpi' \text{ prime}} \left( 1 - \frac{g(\varpi')}{N(\varpi')} \right) + O_{\epsilon} \left( \frac{1}{(\log R)^{2020}} \right),$$

uniformly for all $\ell \in \mathcal{O}_K$ such that $\log N(\ell) \ll \log X$. (Here, the index $\varpi'$ runs over primes $\varpi'$.)

As the above lemmas can be established similarly to [2.1], Lemma 5.1-5.4, we omit the proofs by only pointing out that constant $4/\pi$ in (4.8) (and hence in Lemma 4.4 and 4.5) comes from expanding

$$\frac{\zeta_K \left( 1 + \frac{2+i\zeta_1 + i\zeta_2}{\log R} \right)}{\zeta_K \left( 1 + \frac{1+i\zeta_1}{\log R} \right) \zeta_K \left( 1 + \frac{1+i\zeta_2}{\log R} \right)}$$

into Laurent series and noting that the residue of $\zeta_K(s)$ at $s = 1$ equals $\pi/4$.

### 4.6 Initial treatment

Now we are ready to estimate $S_2$. As $\Phi$ is supported on $[\frac{1}{2}, 1]$, we have $\log N(\varpi) \leq \log X$, so that by positivity we may apply the sieve given [4.5] to see that

$$S_2 \leq (\log X)S^+,$$

where

$$S^+ = \sum_{(n, 2) = 1} \mu_2^2(n) \left( \sum_{d \in G} \frac{\lambda_d}{d \phi(d)} \right) \Phi \left( \frac{N(n)}{X} \right) L(\frac{1}{2}, \chi_{(1+i)^{s_n}})^2 M(n)^2,$$

As $d | n$ and $n$ is odd, we know that $d$ is also odd. Thus, $d \in G$ implies that $d$ is primary. Also, we may write $\lambda_d = \mu_2^2(d) \lambda_d$ since $\lambda_d \neq 0$ only for square-free $d$ by [4.5]. We further write

$$\mu_2^2(n) = N_Y(n) + R_Y(n),$$

where

$$N_Y(n) = \sum_{\ell \in \ell | n \atop N(\ell) \leq Y} \mu(|\ell|), \quad R_Y(n) = \sum_{\ell \in \ell | n \atop N(\ell) > Y} \mu(|\ell|),$$

and

$$L(\frac{1}{2}, \chi_{(1+i)^{s_n}})^2 = \prod_{p} \left( 1 - \frac{1}{p^{1+i}} \right)^2.$$
for some parameter $Y$ to be determined later. Now, we apply Lemma 2.4 and 2.5 to write $L(\frac{1}{2}, \chi_{(1+i)^5n})^2 = D_2(n)$, where

$$D_2(n) = 2 \sum_{\nu \equiv 1 \text{ mod } (1+i)^3} \frac{d_{[i],2}(\nu)}{N(\nu)^{\frac{s}{2}}} \frac{1}{2} \left( N(\nu) \frac{(1+i)^5n}{\nu} \right)$$

(4.12)

$$= \frac{2}{2\pi i} \int (\frac{\Phi(s)}{\pi}) \left( \frac{1}{\Gamma(\frac{1}{2})} \right)^2 N(n)^s \left( \frac{1}{2} + s, \chi_{(1+i)^5} \right) \mathcal{E}(s, 2) \frac{ds}{s}.$$  

(4.13)

Here $c > 1/2$ and

$$\mathcal{E}(s, k) = \prod_{\substack{\varpi \in G \backslash \mathbb{H} \atop \varpi \equiv k}} \left( 1 - \frac{\chi_{(1+i)^5}(\varpi)}{N(\varpi)^{1/2+s}} \right)^2.$$  

Applying (4.10) and (4.12) in (4.9), we see that

$$S^+ = S_N^+ + S_R^+,$$

where

$$S_N^+ = \sum_{(n,2) = 1} N_Y(n) \left( \sum_{d|n, N(d) \leq D} \mu_{[i]}(d) \lambda_d \right) \Phi \left( \frac{N(n)}{X} \right) D_2(n)M(n)^2,$$

(4.14)

$$S_R^+ = \sum_{(n,2) = 1} R_Y(n) \left( \sum_{d|n, N(d) \leq D} \mu_{[i]}(d) \lambda_d \right) \Phi \left( \frac{N(n)}{X} \right) D_2(n)M(n)^2.$$  

4.7. Evaluation of $S_R^+$. In this section we evaluate $S_R^+$. We apply the divisor bound and the observation that $|\lambda_d| \ll N(d)^{c}$ by (4.9) to deduce that

$$|R_Y(n)| \ll N(n)^c, \quad \left| \sum_{\substack{d|n \atop N(d) \leq D}} \mu_{[i]}(d) \lambda_d \right| \ll N(n)^c,$$

(4.15)

Further note that $R_Y(n) = 0$ unless $n = \ell^2 h$ with $N(\ell) > Y$ and $h$ square-free. This together with the above bounds implies that, via Cauchy-Schwarz,

$$S_R^+ \ll X^c \sum_{Y < N(\ell) \leq \sqrt{X}} \sum_{X/2N(\ell)^2 < N(h) \leq X/N(\ell)^2} \mu_{[i]}^2(h)|M(\ell^2 h)^2|D_2(\ell^2 h)^2|$$

$$\ll X^c \sum_{Y < N(\ell) \leq \sqrt{X}} \left( \sum_{X/2N(\ell)^2 < N(h) \leq X/N(\ell)^2} \mu_{[i]}^2(h)|M(\ell^2 h)^2|^2 \right)^{1/2} \left( \sum_{X/2N(\ell)^2 < N(h) \leq X/N(\ell)^2} \mu_{[i]}^2(h)|D_2(\ell^2 h)^2|^2 \right)^{1/2}.$$  

Now, we write $M(\ell^2 h)^2$ as

$$M(\ell^2 h)^2 = \sum_{N(m) \leq M^2} \frac{\alpha(m)}{\sqrt{N(m)}} \left( \frac{(1+i)^5}{\ell} \right) \left( \frac{1+i}{m} \right),$$

where $|\alpha(m)| \ll N(m)^c$. Then Lemma 2.9 allows us to deduce that

$$\sum_{X/2N(\ell)^2 < N(h) \leq X/N(\ell)^2} \mu_{[i]}^2(h)|M(\ell^2 h)^2|^2 \ll X^{-c} \left( \frac{X}{N(\ell)^2} + M^2 \right).$$

(4.16)
Next, we deduce from (4.12) that

\[ D_2(\ell^2 h) = \frac{2}{2\pi i} \int (\frac{2^{5/2}}{\pi})^{2s} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 N(\ell^2 h)^s L^2 \left( \frac{1}{2} + s, \chi_{(1+i)^5 h} \right) \mathcal{E}(s, 2\ell) \frac{ds}{s}. \]

We evaluate \( D_2(\ell^2 h) \) by moving the line of integration to \( c = \frac{1}{\log X} \) without encountering any poles. We then apply Cauchy-Schwarz to see that

\[ |D_2(\ell^2 h)|^2 \ll X^\varepsilon \int \left| \frac{1}{2} + s \right|^2 \left| L \left( \frac{1}{2} + s, \chi_{(1+i)^5 h} \right) \right|^4 |ds|. \]

It then follows from Lemma 2.10 that we have

\[ \sum_{X/2N(\ell^2) < N(h) \leq X/N(\ell)^2} \mu_i^2(h)|D_2(\ell^2 h)|^2 \ll \frac{X^{1+\varepsilon}}{N(\ell)^2}. \]

We thus deduce from (4.10), (4.10) and (4.17) that

\[ S_R \ll X^{\varepsilon} \left( \frac{X}{Y} + X^{1/2}M \right). \]

4.8. Decomposition of \( S_N^+ \). In this section, we begin to evaluate \( S_N^+ \). We apply (1.4) and (4.12) in (4.14) to see that

\[ S_N^+ = 2 \sum_{d \equiv 1 \mod (1+i)^3} \mu_i^2(d) \lambda_d \sum_{N(m_1), N(m_2) \leq M} \frac{b_{m_1} b_{m_2}}{\sqrt{N(m_1 m_2)}} \sum_{\nu \equiv 1 \mod (1+i)^3} \frac{d_{i,2}(\nu)}{N(\nu)^2} \left( \frac{1+i}{} N_1 m_2 \right) Z(d, \nu, m_1 m_2; X, Y), \]

where

\[ Z = Z(d, \nu, m_1 m_2; X, Y) = \sum_{(n, 2)=1} \frac{N_Y(n) \Phi \left( \frac{N(n)}{X} \right)}{N(n)} V_2 \left( \frac{N(\nu)}{N(n)} \right) \left( \frac{n}{m_1 m_2 \nu} \right). \]

Here \( F_y(t) \) is defined in (2.29) and the last equality above follows from (4.11) by noting that when \( n \) is odd, \( \ell | n \) and \( \ell \in G \) implies that \( \ell \) is primary.

As both \( \alpha \) and \( d \) are primary and square-free, we have \( |\alpha^2, d| = \alpha^2 d_1 \), where \( d_1 \) is defined in (2.37). Therefore, we can rewrite \( n \) as \( \alpha^2 d_1 m \) in (4.19) to recast \( Z \) as

\[ Z = \sum_{\alpha \equiv 1 \mod (1+i)^3} \mu_i(\alpha) \left( \frac{d_1}{m_1 m_2 \nu} \right) \sum_{(m, 2)=1} F_N(\nu) \left( \frac{N(\alpha^2 d_1 m)}{X} \right) \left( \frac{m}{m_1 m_2 \nu} \right) = \frac{X}{2N(m_1 m_2 \nu)} \sum_{\alpha \equiv 1 \mod (1+i)^3} \mu_i(\alpha) \left( \frac{1+i}{m_1 m_2 \nu} \right) \sum_{k \in G_K} (-1)^{N(k)} F_N(\nu) \left( \frac{N(k)X}{2N(\alpha^2 d_1 m_1 m_2 \nu)} \right) g(k, m_1 m_2 \nu), \]

where the last equality above follows from Lemma 2.7.
Applying the above expression of $Z$ in (4.19), we deduce that

$$S_N = \sum_{N(d) \leq D \atop d \equiv 1 \mod (1+i)^3} \mu_3^2(d) \lambda_d \sum_{N(m_1), N(m_2) \leq M \atop m_1, m_2 \equiv 1 \mod (1+i)^3} b_{m_1} b_{m_2} N^{-3/2}(m_1 m_2) \sum_{\nu \equiv 1 \mod (1+i)^3 \atop (\nu, d) = 1} \frac{d_{[x]}(\nu)}{N(\nu)^{1/2}}$$

(4.20)

$$\times \sum_{N(\alpha) \leq Y \atop \alpha \equiv 1 \mod (1+i)^3} \frac{\mu_3(\alpha)}{N(\alpha^2 d_1)} \sum_{i \in \mathcal{O}_K} (-1)^{N(k)} \bar{F}_{\nu}(0) \left( \frac{N(k)X}{\sqrt{2N(\alpha^2 d_1 m_1 m_2)}} \right) g(k, m_1 m_2)$$

$$= \mathcal{T}_0 + \mathcal{B},$$

where $\mathcal{T}_0$ singles out the term $k = 0$ of the first expression on the right-hand side of (4.20) while $\mathcal{B}$ being the rest.

4.9. Evaluation of $\mathcal{T}_0$. It follows from Lemma 2.2 that $g(0, n) = \varphi[y](n)$ if $n = \square$ and $g(0, n) = 0$ otherwise. This implies that

$$\mathcal{T}_0 = \sum_{N(d) \leq D \atop d \equiv 1 \mod (1+i)^3} \mu_3^2(d) \lambda_d \sum_{N(m_1), N(m_2) \leq M \atop m_1, m_2 \equiv 1 \mod (1+i)^3} b_{m_1} b_{m_2} N^{-3/2}(m_1 m_2) \sum_{\nu \equiv 1 \mod (1+i)^3 \atop (\nu, d) = 1} \frac{d_{[x]}(\nu)}{N(\nu)^{1/2}}$$

(4.21)

$$\times \sum_{N(\alpha) \leq Y \atop \alpha \equiv 1 \mod (1+i)^3} \frac{\mu_3(\alpha)}{N(\alpha^2 d_1)} \bar{F}_{\nu}(0) \varphi[y](m_1 m_2).$$

We now extend the sum over $\alpha$ to all elements in $\mathcal{O}_K$. As $\varphi[y](n) \leq N(n)$, the error term introduced is

$$\ll \sum_{N(d) \leq D \atop d \equiv 1 \mod (1+i)^3} |\lambda_d| \sum_{N(m_1), N(m_2) \leq M \atop m_1, m_2 \equiv 1 \mod (1+i)^3} \frac{|b_{m_1} b_{m_2}|}{\sqrt{N(m_1 m_2)}} \sum_{\nu \equiv 1 \mod (1+i)^3 \atop (\nu, d) = 1} \frac{d_{[x]}(\nu)}{N(\nu)^{1/2}} \sum_{N(\alpha) > Y \atop \alpha \equiv 1 \mod (1+i)^3} \frac{1}{N(\alpha^2 d_1)} \ll X^{1+\varepsilon},$$

(4.22)

where the last estimation above follows from the observation that $\bar{F}_{\nu}(0) \ll 1$ for all $N(\nu) > X^{1+\varepsilon}$, together with the bounds that $|\lambda_d| \ll N(d)^\varepsilon$ by (4.6) and $|b_n| \ll 1$ by (1.3).

As $m_1 m_2 \neq \square$, the sum over $m_1, m_2, \nu$ in (4.22) is $\ll X^{1+\varepsilon}$. Also, it follows from the definition of $d_1$ in 2.7 that

$$\sum_{N(\alpha) > Y \atop \alpha \equiv 1 \mod (1+i)^3} \frac{1}{N(\alpha^2 d_1)} = \frac{1}{N(d)} \sum_{j \mid d} \varphi[j] \sum_{N(\alpha) > Y \atop \alpha \equiv 1 \mod (1+i)^3} \frac{\alpha}{N(\alpha)^{1/2}} \ll \frac{1}{N(d)^{1-\varepsilon}}.$$

(4.23)

We thus conclude that the expressions in (4.22) are further bounded by $O(X^{1+\varepsilon}/Y)$. Hence by (4.21), we have

$$\mathcal{T}_0 = \sum_{N(d) \leq D \atop d \equiv 1 \mod (1+i)^3} \mu_3^2(d) \lambda_d \sum_{N(m_1), N(m_2) \leq M \atop m_1, m_2 \equiv 1 \mod (1+i)^3} b_{m_1} b_{m_2} N^{-3/2}(m_1 m_2) \sum_{\nu \equiv 1 \mod (1+i)^3 \atop (\nu, d) = 1} \frac{d_{[x]}(\nu)}{N(\nu)^{1/2}}$$

(4.24)

$$\times \sum_{\alpha \equiv 1 \mod (1+i)^3 \atop \alpha \equiv 1 \mod (1+i)^3} \frac{\mu_3(\alpha)}{N(\alpha^2 d_1)} \bar{F}_{\nu}(0) \varphi[y](m_1 m_2) + O \left( \frac{X^{1+\varepsilon}}{Y} \right).$$
Now we express the sum on $\alpha$ in terms of an Euler product to arrive that

$$
\mathcal{T}_0 = \frac{4X}{3\zeta_K(2)} \sum_{N(d) \leq D \atop d \equiv 1 \mod (1+i)^3} \frac{\mu^2_{[i]}(d) \lambda_d}{N(d)} \prod_{\nu \in \mathbb{Z}/d} \left( \frac{N(\nu)}{N(\nu) + 1} \right) \sum_{\nu \equiv 1 \mod (1+i)^3} \frac{d_{[i],2}(\nu)}{N(\nu)^{d/2}} F_N(\nu)(0) \prod_{\nu \mid m_{1, m_{2 \nu}}} \left( \frac{N(\nu)}{N(\nu) + 1} \right) + O \left( \frac{X^{1+\varepsilon}}{Y} \right).
$$

(4.24)

We apply Lemma 4.3 to see that

$$
\sum_{N(d) \leq D \atop d \equiv 1 \mod (1+i)^3} \frac{\mu^2_{[i]}(d) \lambda_d}{N(d)} \prod_{\nu \in \mathbb{Z}/d} \left( \frac{N(\nu)}{N(\nu) + 1} \right) = \frac{4}{\pi} \cdot \frac{1}{\log R} \prod_{\nu \in \mathbb{Z}/d} \left( \frac{N(\nu)}{N(\nu) + 1} \right) \prod_{\nu \equiv 1 \mod (1+i)^3} \frac{d_{[i],2}(\nu)}{N(\nu)^{d/2}} F_N(\nu)(0).
$$

(4.25)

We now write $o(1)$ for $E_0(X)$ as $E_0(X) \to 0$ when $X \to \infty$. It also follows from trivial estimation that we can omit the condition $N(\nu) \leq z_0$ in (4.25). Applying these in (4.25) and then to (4.24), we see that

$$
\mathcal{T}_0 = \frac{8}{\pi} X \frac{1 + o(1)}{\log R} \mathcal{T}_0 + O \left( \frac{X}{(\log R)^{2020}} + \frac{X^{1+\varepsilon}}{Y} \right),
$$

(4.26)

where

$$
\mathcal{T}_0 = \sum_{N(m_1), N(m_2) \leq M \atop m_1, m_2 \equiv 1 \mod (1+i)^3} \frac{b_{m_1} b_{m_2}}{N(m_1 m_2)^{1/2}} \sum_{\nu \equiv 1 \mod (1+i)^3} \frac{d_{[i],2}(\nu)}{N(\nu)^{d/2}} F_N(\nu)(0).
$$

(4.27)

We apply Lemma 4.3 and Lemma 4.4 to see that

$$
\bar{F}_{N(\nu)}(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(N(x + yi)) V_2 \left( \frac{N(\nu)}{X \Re(N(x + yi))} \right) \, dx \, dy
$$

$$
= \frac{1}{2\pi i} \int_{(2)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma\left( \frac{1}{2} + s \right)}{\Gamma\left( \frac{1}{2} \right)} \right)^2 \left( \frac{X}{N(\nu)} \right)^s \left( \int_{-\infty}^{\infty} \Phi(N(x + yi)) \Re(N(x + yi)) \, dx \right) ds
$$

$$
= \frac{\pi}{2\pi i} \int_{(2)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma\left( \frac{1}{2} + s \right)}{\Gamma\left( \frac{1}{2} \right)} \right)^2 \left( \frac{X}{N(\nu)} \right)^s \Phi(1 + s) \, ds,
$$

(4.28)

since

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(N(x + yi)) \Re(N(x + yi)) \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} \Phi(r^2) \rho^{2s} r \, dr \, d\theta = \pi \Phi(1 + s).
$$

Applying (4.3), (4.6) and (4.28) in (4.27), we obtain for $c = \frac{1}{\log X}$,

$$
\mathcal{T}_0 = \frac{\pi}{2\pi i} \int_{(c)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma\left( \frac{1}{2} + s \right)}{\Gamma\left( \frac{1}{2} \right)} \right)^2 \left( \frac{X}{\Re(\nu)} \right)^s \Phi(1 + s) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1) h(z_2)
$$

$$
\times \sum_{m_1, m_2 \equiv 1 \mod (1+i)^3 \atop m_1 \equiv 1 \mod (1+i)^3} \frac{\mu_{[i]}(m_1) \mu_{[i]}(m_2) d_{[i],2}(\nu)}{N(m_1 m_2)^s N(m_1)^{1-\delta_{m_2,0}} N(m_2)^{1-\delta_{m_1,0}} N(\nu)^s} \, dz_1 dz_2 ds.
$$

(4.29)
We write the sum over $m_1, m_2, \nu$ as an Euler product to see that

\[
\sum_{m_1, m_2 \nu \equiv 1 \pmod{m_1 m_2 \nu} \neq 0} \frac{\mu_\nu(m_1) \mu_\nu(m_2) d_\nu(\nu)}{N(m_1 m_2 \nu)}^\frac{1}{2} \frac{N(m_1)}{1 + i \frac{\log M}{2}} \frac{N(m_2)}{1 + i \frac{\log M}{2}} \frac{N(\nu)}{N(\nu)^s} = \zeta_K(1 + 2s) \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) Q \left( 1 + i \frac{\log M}{2}, 1 + i \frac{\log M}{2}, s \right),
\]

where $Q(w_1, w_2, s)$ is holomorphic and uniformly bounded in the region $\Re(w_1), \Re(w_2), \Re(s) \geq -\varepsilon$, which satisfies

\[
Q(0, 0, 0) = 1.
\]

Applying (4.30) in (1.29), we deduce that

\[
\Upsilon_0 = \frac{\pi}{2\pi i} \int_{c} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(1 + s)}{\Gamma(\frac{1}{2})} \right)^2 X^s \Phi(1 + s) \zeta_K^3 \left( 1 + 2s \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1) h(z_2) \times \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) Q \left( 1 + i \frac{\log M}{2}, 1 + i \frac{\log M}{2}, s \right) \frac{dz_1 dz_2 ds}{s}.
\]

We may truncate the integrals above to $|z_1|, |z_2| \leq \sqrt{\log M}$ and $|\Im(s)| \leq (\log X)^2$ due to the rapid decay of the gamma function in vertical strips and (3.3). Notice further that similar to [14] Theorem 6.7, we can show that there exists a constant $c'$ such that when $\Re(z) \geq c'/|\Im(z)|$ and $|\Im(z)| \geq 1$, we have

\[
\zeta_K(1 + z) \ll |\Im(z)| \quad \text{and} \quad \frac{1}{\zeta_K(1 + z)} \ll |\Im(z)|.
\]

We then change the contour of integration over $s$ to the path consisting of the line segment $L_1$ from $\frac{1}{\log M} - i(\log X)^2$ to $-\frac{c'}{\log M} - i(\log X)^2$, the line segment $L_2$ from $-\frac{c'}{\log M} - i(\log X)^2$ to $-\frac{c'}{\log M} + i(\log X)^2$, and the line segment $L_3$ from $-\frac{c'}{\log M} + i(\log X)^2$ to $\frac{1}{\log M} + i(\log X)^2$. The contributions of the integrals on the new lines are negligible due to the rapid decay of the gamma function on $L_1$ and $L_3$ and the estimation $X^s \ll \exp \left( -c'/|\Im(z)| \right)$ on $L_2$. We are then left with the contribution from a residue of the pole at $s = 0$, which we present as an integral along a circle centered at 0 to obtain

\[
\Upsilon_0 = \frac{\pi}{2\pi i} \int_{c} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(1 + s)}{\Gamma(\frac{1}{2})} \right)^2 X^s \Phi(1 + s) \zeta_K^3 \left( 1 + 2s \right) \int_{|z| \leq \sqrt{\log M}} h(z_1) h(z_2) \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) \zeta_K \left( 1 + \frac{1 + i \frac{\log M}{2}}{s} \right) Q \left( 1 + i \frac{\log M}{2}, 1 + i \frac{\log M}{2}, s \right) \frac{dz_1 dz_2 ds}{s} + O \left( \frac{1}{(\log X)^{2020}} \right).
\]

The main contribution to $\Upsilon_0$ comes from the first terms of the Laurent expansions of the zeta functions and $Q$. We then deduce via (3.31) that

\[
\Upsilon_0 = \frac{\pi}{16\pi i} \int_{c} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(1 + s)}{\Gamma(\frac{1}{2})} \right)^2 X^s \Phi(1 + s) \int_{|z_1| \leq \sqrt{\log M}} h(z_1) h(z_2) \times \left( \frac{\log M}{2 + i z_1 + i z_2} \right)^2 \left( \frac{1 + i z_1}{\log M} + s \right)^2 \left( \frac{1 + i z_2}{\log M} + s \right)^2 dz_1 dz_2 ds + O \left( \frac{1}{(\log X)^{1-\varepsilon}} \right).
\]

In the above expression, we extend the integrals over $z_1, z_2$ to $\mathbb{R}^2$ with a negligible error by (3.3). Then applying the relation (see [4] (7.3.12))

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1) h(z_2) \frac{(1 + i z_1)^j (1 + i z_2)^k}{2 + i z_1 + i z_2} dz_1 dz_2 = (-1)^{j+k} \int_0^{\infty} H^{(j)}(t) H^{(k)}(t) dt,
\]

we can simplify the integral further.
we deduce that
\[
\mathcal{T}_0 = \frac{\pi}{16\pi i} \int_{|s|=\frac{1}{2}} \left( \frac{2^{5/2}}{\pi} + \frac{2s}{\Gamma \left( \frac{1}{2} + s \right)} \right)^2 X \hat{\Phi}(1 + s) \left\{ \frac{1}{(\log M)^3} \int_0^1 H''(t)^2 \, dt 
\right.
\]
\[
- \frac{4s}{(\log M)^2} \int_0^1 H'(t)H''(t) \, dt + \frac{2s^2}{\log M} \int_0^1 H(t)H''(t) \, dt + \frac{4s^2}{\log M} \int_0^1 H'(t)^2 \, dt 
\]
\[
- 4s^3 \int_0^1 \left( H(t)H'(t) + s^4 \log M \int_0^1 H(t)^2 \, dt \right) \frac{ds}{s^4} + O \left( \frac{1}{(\log X)^{1-\varepsilon}} \right).
\]

We now apply (3.7) to evaluate the above integral as a residue to arrive at
\[
\mathcal{T}_0 = \frac{\pi \hat{\Phi}(1)}{8} \left\{ \frac{1}{6} \left( \frac{\log X}{\log M} \right)^3 \int_0^1 H''(t)^2 \, dt - 2 \left( \frac{\log X}{\log M} \right)^2 \int_0^1 H'(t)H''(t) \, dt 
\right.
\]
\[
+ 2 \frac{\log X}{\log M} \int_0^1 H(t)H''(t) \, dt + \frac{\log X}{\log M} \int_0^1 H'(t)^2 \, dt - 4 \int_0^1 H(t)H'(t) \, dt \right\} + O \left( \frac{1}{(\log X)^{1-\varepsilon}} \right).
\]

Substituting the above into (1.20), and noting that
\[
\hat{\Phi}(1) = \frac{1}{2} + O \left( \frac{1}{\log X} \right),
\]
we obtain that
\[
\hat{\Phi}(1) = \frac{1}{2} + O \left( \frac{1}{\log X} \right),
\]

4.10. **Evaluation of $\mathcal{B}$: the principal terms.** In this section, we begin to evaluate $\mathcal{B}$. We recall from (1.20) that we have
\[
\mathcal{B} = X \sum_{N(d) \leq D} \frac{\mu^2 \mu [d]}{d \equiv 1 \mod (1+i)^3} \lambda_d \sum_{\substack{N(m_1), N(m_2) \leq M \\mod (1+i)^3 \\mod (1+i)^3 \\mod (1+i)^3}} \frac{b_{m_1} b_{m_2}}{N(m_1 m_2)^{3/2}} Q,
\]

where
\[
Q = \sum_{\nu \equiv 1 \mod (1+i)^3} \frac{d_{\nu} \mu [\nu]}{(\nu, d) = 1} \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu [\alpha]}{N(\alpha^2 d_1)} \left( \frac{d_1}{m_1 m_2} \right) \sum_{k \in \mathcal{O}_K} \frac{(-1)^k}{N(k)} \left( \frac{N(k) X}{2N(\alpha^2 d_1 m_1 m_2)} \right) g(k, m_1 m_2 \nu).
\]

We apply Mellin inversion to see that for any $c > 1$,
\[
Q = \sum_{\alpha \equiv 1 \mod (1+i)^3} \frac{\mu [\alpha]}{N(\alpha^2 d_1)} \left( \frac{d_1}{m_1 m_2} \right) \sum_{k \in \mathcal{O}_K \setminus \mathbb{Z}^n_0} (-1)^k \left( \frac{N(k) X}{2N(\alpha^2 d_1 m_1 m_2)} \right) g(k, m_1 m_2 \nu) \, dw,
\]

\[
\times \frac{1}{2\pi i} \int_{(c)} \frac{N(k) X}{2N(\alpha^2 d_1 m_1 m_2)} \frac{d_{\nu} \mu [\nu]}{(\nu, d) = 1} \sum_{\nu \equiv 1 \mod (1+i)^3} \frac{d_{\nu} \mu [\nu]}{N(\nu)^{1/2+w}} \left( \frac{d_1}{\nu} \right) g(k, m_1 m_2 \nu) \, dw,
\]
where \( h \) is defined in (2.10). We further apply Lemma 2.12 to recast \( Q \) as

\[
Q = \sum_{N(\alpha) \leq Y} \sum_{\alpha \equiv 1 (\mod (1+i)^3)} \frac{\mu_{ij}(\alpha)}{N(\alpha^2d_1)} \left( \frac{d_1}{m_1m_2} \right) \sum_{k \in \mathcal{O}_K, k \neq 0} (-1)^{N(k)}
\times \frac{1}{2\pi i} \int \frac{N(k)X}{2N(\alpha^2d_1m_1m_2)} w) L(1 + w, \chi_{ik_1})^2 \mathcal{G}_0(1 + w; k, m_1m_2, \alpha, d) \, dw.
\]

Observe that integration by parts shows that when \( \Re(w) \geq -\frac{1}{2} + \varepsilon \), we have for any integer \( j \geq 1 \),

\[
(4.35) \quad \Phi(1 + w) \ll \frac{1}{(1 + |w|)^j}.
\]

It follows Lemma 2.12, Lemma 2.13 and the above estimation that we can move the line of integration of the \( w \)-integral in (4.34) to \( c = -\frac{1}{2} + \varepsilon \). We encounter a pole at \( w = 0 \) only when \( \chi_{ik_1} \) is a principal character, which holds if and only if \( k_1 = \pm i \) and this is further equivalent to \( kd_1 = \pm ij^2 \) for some \( j \in \mathbb{G} \). We thus deduce that

\[
(4.36) \quad Q = P_+ + P_- + R,
\]

where

\[
P_\pm = \text{Res}_{w = 0} \sum_{N(\alpha) \leq Y} \sum_{\alpha \equiv 1 (\mod (1+i)^3)} \frac{\mu_{ij}(\alpha)}{N(\alpha^2d_1)} \left( \frac{d_1}{m_1m_2} \right) \sum_{k \in \mathcal{O}_K, k \neq 0} (-1)^{N(k)}
\times h\left( \frac{N(k)X}{2N(\alpha^2d_1m_1m_2)} , w \right) \zeta_K(1 + w)^2 \mathcal{G}_0(1 + w; k, m_1m_2, \alpha, d),
\]

\[
R = \sum_{N(\alpha) \leq Y} \sum_{\alpha \equiv 1 (\mod (1+i)^3)} \frac{\mu_{ij}(\alpha)}{N(\alpha^2d_1)} \left( \frac{d_1}{m_1m_2} \right) \sum_{k \in \mathcal{O}_K, k \neq 0} (-1)^{N(k)}
\times \frac{1}{2\pi i} \int_{(-\frac{1}{2} + \varepsilon)} h\left( \frac{N(k)X}{2N(\alpha^2d_1m_1m_2)} , w \right) L(1 + w, \chi_{ik_1})^2 \mathcal{G}_0(1 + w; k, m_1m_2, \alpha, d) \, dw.
\]

We treat \( P_\pm \) first. Note that \( d_1 \) is square-free by (2.7) since \( d \) is square-free. It follows that \( kd_1 = \pm ij^2 \) for some \( j \in \mathbb{G} \) if and only if \( k = \pm id_1 j^2 \) for some \( j' \in \mathbb{G} \). Thus we relabel \( k \) as \( \pm id_1 j^2 \) with \( j \) running through all elements in \( \mathbb{G} \) in (4.37) and apply Lemma 2.13 to see that for \( c > \frac{1}{2} \), we have

\[
(4.38) \quad P_\pm = \text{Res}_{w = 0} \sum_{N(\alpha) \leq Y} \sum_{\alpha \equiv 1 (\mod (1+i)^3)} \frac{\mu_{ij}(\alpha)}{N(\alpha^2d_1)} \zeta_K(1 + w)^2 \Phi(1 + w) X^w \pi \frac{2^5/2}{\pi} \int \left( \frac{2^5/2}{\pi} \right)^{2s} \frac{\Gamma((1/2) + s)}{\Gamma((1/2))} \left( \frac{\Gamma((1/2) + 2w)}{\Gamma(1 - s + w)} \right)^2 \frac{\Gamma(s - w)}{\Gamma(1 - s + w)} \frac{\Gamma(s - w)}{\Gamma(1 - s + w)}
\times (2N(\alpha^2m_1m_2))^{-w} \sum_{j \in \mathbb{G}} (-1)^{N(j)} N(j)^{-2s+2w} \left( \frac{d_1}{m_1m_2} \right) \mathcal{G}_0(1 + w; \pm id_1 j^2, m_1m_2, \alpha, d) \, ds.
\]

Note that part (ii) of Lemma 2.22 implies that for \( j \in \mathbb{G}, \varpi \nmid 2\alpha d \) and \( \beta \geq 1 \),

\[
\left( \frac{d_1}{m_1m_2} \right) g(\pm id_1 j^2, \varpi^\beta) = g(\pm ij^2, \varpi^\beta).
\]

This allows us to deduce from the definition of \( \mathcal{G}_0 \) given in Lemma 2.12 that

\[
(4.39) \quad \left( \frac{d_1}{m_1m_2} \right) \mathcal{G}_0(1 + w; \pm id_1 j^2, m_1m_2, \alpha, d) = \mathcal{G}(1 + w; \pm ij^2, m_1m_2, \alpha d),
\]
where for any $k, \ell, \alpha \in \mathcal{O}_K$ and $s \in \mathbb{C}$, we define $\mathcal{G}(s; k, \ell, \alpha) = \prod_{\varpi \in G} \mathcal{G}_\varpi(s; k, \ell, \alpha)$ similar to that given in [10 (5.8)] by

$$
\mathcal{G}_\varpi(s; k, \ell, \alpha) = \left(1 - \frac{1}{N(\varpi)^s} \left(\frac{i k_1}{\varpi}\right)\right)^2 \text{ if } \varpi \mid 2\alpha,
$$

(4.40)

$$
\mathcal{G}_\varpi(s; k, \ell, \alpha) = \left(1 - \frac{1}{N(\varpi)^s} \left(\frac{i k_1}{\varpi}\right)\right)^2 \sum_{r=0}^\infty \frac{r+1}{N(\varpi)^s} \frac{g(k; \varpi^{r+\ord_{\varpi}(\ell)})}{N(\varpi)^{r/2}} \text{ if } \varpi \nmid 2\alpha.
$$

Here $k_1$ is the unique element in $\mathcal{O}_K$ that we have $k = k_1k_2^2$ with $k_1$ being square-free and $k_2 \in G$.

It follows from Lemma 2.22 and the above expression of $\mathcal{G}(s; k, \ell, \alpha)$ that we have $\mathcal{G}(s; ij^2, \ell, \alpha) = \mathcal{G}(s; -ij^2, \ell, \alpha)$ for $j \in G$. Thus applying (4.39) to (4.38), we see that

$$
P_{\pm} = \sum_{w=0} \frac{\mu[I](\alpha)}{N(\alpha^2 d_1)} \zeta_K(1+w)^2 \tilde{\Phi}(1+w)X^w
$$

(4.41)

$$
\times \frac{\pi}{2\pi i} \int \left(\frac{2^{5/2}}{\pi}\right)^{2s} N(\alpha)^{2s-2w} \left(\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{1}{2})}\right)^2 \Gamma_1(s-w) \mathcal{H}(s-w, 1+w; m_1 m_2, w) \frac{ds}{s},
$$

where

$$
\Gamma_1(s) = \left(2^{-1/2} \pi \right)^{-2s} \frac{\Gamma(s)}{\Gamma(1-s)},
$$

and where we define for $|v-1| \leq 1/\log X$, and any $u$ with $\Re(u) > 1/2$,

$$
\mathcal{H}(u, v; \ell, \alpha) = N(l)^u \sum_{j \in G} (-1)^{N(j)} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha).
$$

We further note that around $s = 1$,

$$
\zeta_K(s) = \frac{\pi}{4} \cdot \frac{1}{s-1} + \gamma_K + O(s-1),
$$

where $\gamma_K$ is a constant. This allows us to deduce from (4.41) and (4.42) that

$$
P_{\pm} = \frac{\pi^3}{4^2} \sum_{\substack{N(\alpha) \leq Y \\alpha \equiv 1 \mod (1+i)^3 \\alpha \equiv 1 \mod (1+i)^3 \\alpha \equiv 1 \mod (1+i)^3 \\alpha \equiv 1 \mod (1+i)^3}} \frac{\mu[I](\alpha)}{N(\alpha^2 d_1)} \tilde{\Phi}(1)
$$

(4.43)

$$
\times \frac{1}{2\pi i} \int \left(\frac{2^{5/2}}{\pi}\right)^{2s} N(\alpha)^{2s} \left(\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{1}{2})}\right)^2 \Gamma_1(s) \mathcal{H}(s, 1; m_1 m_2, w) \frac{ds}{s},
$$

Note that we have

$$
\sum_{j \in G} (-1)^{N(j)} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha)
$$

$$
= - \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha) + \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha)
$$

$$
= - \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha) + 2 \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha)
$$

$$
= - \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha) + 2(1-2^{-a}) \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; ij^2, \ell, \alpha)
$$

$$
= \left(1 - 2^{-a} \right) \sum_{j \in G} N(j)^{-2a} \mathcal{G}(v; j^2, \ell, \alpha),
$$

respectively.
where the last step above follows by noting that we have \( \mathcal{G}(v; i(1 + i)j^2, \ell, \alpha) = \mathcal{G}(v; ij^2, \ell, \alpha) \). We deduce from this that
\[
(4.44) \quad \mathcal{H}(u, v; \ell, \alpha) = -N(l)^u(1 - 2^{1-2u}) \sum_{j \in \mathbb{G}} N(j)^{-2u} \mathcal{G}(v; ij^2, \ell, \alpha).
\]

We now write
\[
\ell = \ell'_1 \ell'_2, \quad \mu^2_{[\ell]}(\ell'_1) = 1, \quad \ell'_2 \in \mathbb{G},
\]
and apply the above definition of \( \mathcal{G} \) in (4.40) and Lemma 2,2 to obtain from (4.44) that
\[
(4.45) \quad \mathcal{H}(u, v; \ell, \alpha) = -N(l)(1 - 2^{1-2u})N(\ell'_1)^u \zeta_K(2u)\zeta_K(2u + 1)\mathcal{H}(u, v; \ell, \alpha),
\]
where
\[
(4.46) \quad \mathcal{H}_1(u, v; \ell, \alpha) = \prod_{\varpi \in \mathbb{G}} \mathcal{H}_{1, \varpi}(u, v; \ell, \alpha),
\]
with
\[
(4.47) \quad \mathcal{H}_{1, \varpi}(u, v; \ell, \alpha) = \left( \frac{1 - 1/N(\varpi)^u}{(1 - 1/N(\varpi)^u)^3} \right)^2 \left( \frac{1}{N(\varpi)^u} - \frac{2}{N(\varpi)^{2u+1}} - \frac{1}{N(\varpi)^{2u+2}} + \frac{1}{N(\varpi)^{2u+3}} \right)
\]
if \( \varpi \mid 2\alpha \),
\[
= \left( \frac{1 - 1/N(\varpi)^u}{(1 - 1/N(\varpi)^u)^3} \right)^2 \left( \frac{1}{N(\varpi)^u} + \frac{2}{N(\varpi)^{2u+1}} - \frac{1}{N(\varpi)^{2u+2}} - \frac{1}{N(\varpi)^{2u+3}} \right)
\]
if \( \varpi \nmid 2\alpha \),
\[
= \left( \frac{1 - 1/N(\varpi)^u}{(1 - 1/N(\varpi)^u)^3} \right)^2 \left( 1 - \frac{1}{N(\varpi)^u} + \frac{2}{N(\varpi)^{2u+1}} + \frac{1}{N(\varpi)^{2u+2}} - \frac{1}{N(\varpi)^{2u+3}} \right)
\]
if \( \varpi \nmid \ell'_1 \),
\[
= \left( \frac{1 - 1/N(\varpi)^u}{(1 - 1/N(\varpi)^u)^3} \right)^2 \left( \frac{1}{N(\varpi)^u} - \frac{2}{N(\varpi)^{2u+1}} + \frac{1}{N(\varpi)^{2u+2}} + \frac{1}{N(\varpi)^{2u+3}} \right)
\]
if \( \varpi \mid \ell'_1 \).

It follows from (4.45) to (4.47) that
\[
\mathcal{H}(s, 1; \ell, \alpha), \quad \frac{\partial}{\partial s} \mathcal{H}(s, 1; \ell, \alpha), \quad \frac{\partial}{\partial w} \mathcal{H}(s, w; \ell, \alpha) \bigg|_{w=1} \ll N(l)^{1+\varepsilon}N(\ell'_1)^{s-1/2}(N(\alpha)X)^{\varepsilon}(1 + |s|)^{1+\varepsilon}.
\]

This allows us to shift the line of integration in (4.43) to \( \Re(s) = \frac{1}{\log X} \) to see that the integral on the new line is bounded by
\[
\ll N(m_1 m_2)N(\ell'_1)^{-\frac{1}{2}+\varepsilon}N(\alpha)^{\varepsilon}X^\varepsilon \int \left| \Gamma \left( \frac{1}{2} + s \right) \right|^2 \max(|\Gamma'_1(s)|, |\Gamma'_1(s)|)(1 + |s|)^{1+\varepsilon} |ds|
\]
\[
\ll N(m_1 m_2)N(\ell'_1)^{-\frac{1}{2}+\varepsilon}N(\alpha)^{\varepsilon}X^\varepsilon,
\]
where we write
\[
m_1 m_2 = \ell'_1 \ell'_2, \quad \mu^2_{[\ell]}(\ell'_1) = 1, \quad \ell'_2 \in \mathbb{G}.
\]

Note that (4.41) is equivalent to (4.43), so we can now set \( c = 1/ \log X \) there. We then drop the condition \( N(\alpha) \leq Y \) in the sum over \( \alpha \) and apply an argument similar to that in (4.23) to estimate the sum over \( N(\alpha) > Y \) in (4.41) to see that
\[
(4.48) \quad \mathcal{P}_+ = \text{Res}_{w=0} \zeta_K(1 + w)^2 \hat{\Theta}(1 + w)^w X^w
\]
\[
\times \frac{\pi}{2\pi i} \int_{\gamma-\pi i/(\pi X)}^{\gamma+\pi i/(\pi X)} \left( \frac{2^{s/2}}{\pi} \right) \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 \Gamma(1 - s - w)K(s - w, 1 + w; m_1 m_2) \frac{ds}{s} + O \left( \frac{N(m_1 m_2)N(\ell'_1)^{-\frac{1}{2}+\varepsilon}X^{\varepsilon}}{N(d)^{1-\varepsilon}Y^{1-\varepsilon}} \right),
\]
where
\[
K(s, w; \ell, d) = \sum_{\alpha \equiv 1 \mod (1+i)^3 \atop (\alpha, d) = 1} \frac{\mu_{[\ell]}(\alpha)}{N(\alpha)^{2-2s}N(d)^s} \mathcal{H}(s, w; \ell, \alpha d).
\]
We can recast $\mathscr{P}_\pm$ given in (4.48) more explicitly as
\[
\mathscr{P}_\pm = \frac{\pi^3}{4^2} \int \frac{1}{2\pi i} \left( \frac{2^{5/2}}{\pi} \right)^2 \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 \Gamma_1(s) \mathcal{K}(s, 1; m_1 m_2, d) \\
\times \left( \log X + \frac{\Phi'}{\Phi} + \frac{8}{\pi \gamma} - \frac{\partial}{\partial \alpha} \mathcal{K}(s, 1; m_1 m_2, d) + \frac{\partial}{\partial \alpha} \mathcal{K}(s, w; m_1 m_2, d) \right)_{\alpha = 1} / \Gamma_1(s) \frac{ds}{s} \\
+ O \left( \frac{N(m_1 m_2) N(f_1)^{- \frac{1}{2} + \varepsilon} X^\varepsilon}{N(d)^{1 - \varepsilon} Y^{1 - \varepsilon}} \right).
\]

Since $\lambda_d \ll N(d)^\varepsilon$ by (4.6) and $b_m \ll 1$ by (13), it thus follows from (4.55) that
\[
\sum_{N(d)^2 \leq \mathcal{D}} \mu^2_{\alpha}(d) \lambda_d \sum_{N(m_1), N(m_2) \leq M} \frac{b_m b_{m_2}}{N(m_1 m_2)^{3/2}} O \left( \frac{N(m_1 m_2) N(f_1)^{- \frac{1}{2} + \varepsilon} X^\varepsilon}{N(d)^{1 - \varepsilon} Y^{1 - \varepsilon}} \right) \ll X^\varepsilon D^\varepsilon M^\varepsilon.
\]

4.11. Evaluation of $\mathcal{B}$: the remainder terms. In this section, we estimate $\mathcal{R}$ given in (4.57). We denote
\[
\mathcal{R}(\ell, d) = \frac{1}{N(\ell)} \sum_{N(\alpha) \leq Y} \frac{\mu_{\alpha}(\alpha)}{\alpha^2 d_1} \left( d_1 / \ell \right) \sum_{k \in \mathbb{Z}, \ell \neq 0} (-1)^N(k) \\
\times \frac{1}{2\pi i} \int_{(-\frac{1}{2} + \varepsilon)} h \left( \frac{N(k)}{2N(\alpha^2 d_1 \ell)}, w \right) L(1 + w, \chi_{ik_1}) \mathcal{G}_0(1 + w; k, \ell, \alpha, d) dw
\]

to see that $\mathcal{R} = N(m_1 m_2) \mathcal{R}(m_1 m_2, d)$. We let $\beta_{\ell, d} = \frac{\mathcal{R}(\ell, d)}{|\mathcal{R}(\ell, d)|}$ if $\mathcal{R}(\ell, d) \neq 0$, and $\beta_{\ell, d} = 1$ otherwise. Thus $|\mathcal{R}(\ell, d)| = \beta_{\ell, d} \mathcal{R}(\ell, d)$ and we deduce from (4.51) that for integers $J, V \geq 1$,
\[
\sum_{N(d)=V} \sum_{N(\ell)=J} \frac{2V-1}{2J-1} \mathcal{R}(\ell, d) = \sum_{N(d)=V} \sum_{N(\ell)=J} 2V - 1 2J - 1 \beta_{\ell, d} \mathcal{R}(\ell, d) \ll \sum_{N(\alpha) \leq Y} \sum_{k \in \mathbb{Z}, \ell \neq 0} \left( \frac{1}{N(\alpha)^2} \right) \sum \int_{(-\frac{1}{2} + \varepsilon)} U(\alpha, k, w) |dw|,
\]
where
\[
U(\alpha, k, w) = \sum_{N(d)=V} \sum_{N(\ell)=J} \frac{1}{N(1)} \left| L(1 + w, \chi_{ik_1}) \right| \sum_{N(\ell)=J} \left( \frac{1}{N(\ell)} \right) \mathcal{G}_0(1 + w; k, \ell, \alpha, d) h \left( \frac{N(k)}{2N(\alpha^2 d_1 \ell)}, w \right).
\]

We apply the Cauchy-Schwarz inequality inequality to see that for an integer $K \geq 1$,
\[
\sum_{K \leq N(k) < 2K} U(\alpha, k, w) \ll \left( \sum_{N(d)=V} \sum_{K \leq N(k) < 2K} \left( \frac{1}{N(1)} \right) \left| L(1 + w, \chi_{ik_1}) \right| \right)^{1/2} \\
\times \left( \sum_{N(d)=V} \sum_{K \leq N(k) < 2K} \left( \frac{1}{N(1)} \right) \left| \sum_{N(\ell)=J} \left( \frac{1}{N(\ell)} \right) \mathcal{G}_0(1 + w; k, \ell, \alpha, d) h \left( \frac{N(k)}{2N(\alpha^2 d_1 \ell)}, w \right) \right|^2 \right)^{1/2},
\]
where $k_2$ is defined by (2.4). Note that (2.4) implies $N(d_1) \geq N(d)/N(\alpha)$, so that we have

$$\sum_{N(d) = V \atop (d, 2) = 1} \frac{1}{N(d_1)} \sum_{K \leq N(k) < 2K} N(k_2) |L(1 + w, \chi_{k_1})|^4 \ll \frac{N(\alpha)}{V} \sum_{0 \neq N(k_1) \ll KV} |L(1 + w, \chi_{k_1})|^4 \sum_{N(k_2) \ll \frac{N(d)}{d_1 k_1 k_2}} 1.$$

where the last estimation above follows from Lemma 2.10 and partial summation. We then deduce from this and (4.53) that

$$\sum_{K \leq N(k) < 2K} U(\alpha, k, w) \ll \left( N(\alpha)^{1+\varepsilon} K^{1+\varepsilon} V^\varepsilon (1 + |w|^2)^{1+\varepsilon} \right)^1,$$

Applying (4.35) together with the first bound of Lemma 2.15 implies that we may restrict the sum of the right-hand side of (4.54) to $K = 2^j \leq N(\alpha)^2 V J (1 + |w|^2)(\log X)^4$, in which case we apply the second bound in Lemma 2.15 to (1.41) to see that

$$\sum_{K \leq N(k) < 2K} U(\alpha, k, w) \ll \varepsilon (1 + |w|^2)^{\frac{1}{2}+\varepsilon} |\tilde{\Phi}(1 + w)| N(\alpha J K V X)^\varepsilon \left( \frac{N(\alpha)^3 V (J K + J^2)}{X} \right)^{\frac{1}{2}} \ll \varepsilon (1 + |w|^2)^{\frac{1}{2}+\varepsilon} |\tilde{\Phi}(1 + w)| (N(\alpha J K V X)^\varepsilon \frac{\alpha^\frac{3}{2} V J}{X^{\frac{3}{2}}},$$

Substituting this into (1.42) and summing over $K = 2^j, K \leq N(\alpha)^2 V J (1 + |w|^2)(\log X)^4$, we deduce via (4.35) that

$$\sum_{N(d) = V \atop (d, 2) = 1} \sum_{(\ell, 2\alpha d) = 1} |\mathcal{R}(\ell, d)| \ll \frac{V^{1+\varepsilon} J^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon}}{X^{\frac{3}{2}-\varepsilon}}.$$

As $\mathcal{R} = N(m_1 m_2) \mathcal{R}(m_1 m_2, d)$, we apply the bounds that $\lambda_d \ll N(d)^{\varepsilon}$ by (1.7) and $b_m \ll 1$ by (1.3) to derive from (4.55) that

$$\sum_{d \equiv 1 \bmod (1+i)^3} \mu_d^2 \lambda_d \sum_{N(m_1), N(m_2) \leq M \atop m_1, m_2 \equiv 1 \bmod (1+i)^3} \frac{b_{m_1} b_{m_2}}{N(m_1 m_2)^{3/2}} \mathcal{R} \ll \frac{D^{1+\varepsilon} M^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon}}{X^{\frac{3}{2}-\varepsilon}}.$$
4.12. Gathering the terms. We deduce from (4.33), (4.36), (4.49), (4.50) and (4.56) that

\[
\mathcal{B} = 2X \sum_{N(d) \leq D} \mu_{\mathcal{O}}(d) \lambda_d \sum_{m_1, m_2 \equiv 1 \mod (1+i)^3} \frac{b_{m_1} b_{m_2}}{N(m_1 m_2)^{3/2}} \frac{\pi^3}{42} \hat{\Phi}(1) 
\times \frac{1}{2\pi i} \int_{(\log X)} \left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 \Gamma_1(s) \mathcal{K}(s, 1; m_1 m_2, d) 
\times \left( \log X + \frac{\Phi'(1)}{\Phi(1)} + \frac{8}{\pi} \gamma - \frac{\partial}{\partial s} \mathcal{K}(s, 1; m_1 m_2, d) \right) + \frac{\partial}{\partial s} \mathcal{K}(s, w; m_1 m_2, d) \bigg|_{w=1} - \frac{\partial}{\partial s} \frac{\Gamma_1(s)}{\gamma} \right) \frac{ds}{s} 
+ O \left( \frac{X^{1+\varepsilon} D^\varepsilon M^\varepsilon}{Y^{1-\varepsilon}} + X^{\frac{1}{2}+\varepsilon} D^{1+\varepsilon} M^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon} \right).
\]

We now require that the values of \( \theta, \vartheta \) defined in (4.12) to satisfy

\[
\theta + 2\vartheta < \frac{1}{2}.
\]

Also, for small \( \delta = \delta(\theta, \vartheta) \), we set the parameter \( Y \) defined in (4.10) to be

\[
Y = X^{\delta}.
\]

This way, we see that

\[
O \left( \frac{X^{1+\varepsilon} D^\varepsilon M^\varepsilon}{Y^{1-\varepsilon}} + X^{\frac{1}{2}+\varepsilon} D^{1+\varepsilon} M^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon} \right) = O(X^{1-\varepsilon}).
\]

Now, a direct calculation shows that

\[
\mathcal{K}(s, 1; \ell, d) = - \frac{N(\ell)}{\sqrt{N(\ell_1)}} \frac{4^s + 4^{-s} - \frac{5}{2}}{4^s} \zeta_K(2s) \zeta_K(2s+1) \frac{\varphi(\ell|d)^2}{N(d)N(d)} \prod_{\mathfrak{c} \equiv \ell \mod \ell_1} \left( 1 + \frac{1}{N(\mathfrak{c})} \right) 
\times \prod_{\mathfrak{c} \equiv \ell \mod \ell_1} \left( N(\mathfrak{c})^s + N(\mathfrak{c})^{-s} \right) \prod_{\mathfrak{c} \equiv 2d \mod \ell_1} \left( 1 - \frac{1}{N(\mathfrak{c})^2} - \frac{1}{N(\mathfrak{c})^{-2}} \right) \left( 1 - \frac{1}{N(\mathfrak{c})^{2s+1}} - \frac{1}{N(\mathfrak{c})^{-2s+1}} \right) 
\times \prod_{\mathfrak{c} \equiv d \mod \ell_1} \left( 1 - \frac{1}{N(\mathfrak{c})^{2s+1}} - \frac{1}{N(\mathfrak{c})^{-2s+1}} \right) 
\times \prod_{\mathfrak{c} \equiv 2d \mod \ell_1} \left( 1 - \frac{1}{N(\mathfrak{c})^{2s+1}} - \frac{1}{N(\mathfrak{c})^{-2s+1}} \right) 
\times \prod_{\mathfrak{c} \equiv d \mod \ell_1} \left( 1 - \frac{1}{N(\mathfrak{c})^{2s+1}} - \frac{1}{N(\mathfrak{c})^{-2s+1}} \right).
\]

Moreover, we use the functional equation (2.4) for \( \zeta_K(s) \) and the relation (see [4, §10, (3)])

\[
\Gamma(s) \Gamma(s + \frac{1}{2}) = 2^{1-2s} \pi^{1/2} \Gamma(2s)
\]

to see that we have

\[
\left( \frac{2^{5/2}}{\pi} \right)^{2s} \left( \frac{\Gamma(\frac{1}{2} + s)}{\Gamma(\frac{1}{2})} \right)^2 \Gamma_1(s) \mathcal{K}(s, 1; m_1 m_2, d) = \left( \frac{2^{5/2}}{\pi} \right)^{-2s} \left( \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(\frac{1}{2})} \right)^2 \Gamma_1(-s) \mathcal{K}(-s, 1; m_1 m_2, d).
\]

Further calculation shows that

\[
\frac{\partial}{\partial s} \frac{\mathcal{K}(s, w; \ell, d)}{\mathcal{K}(s, 1; \ell, d)} \bigg|_{w=1} - \frac{\partial}{\partial s} \frac{\mathcal{K}(s, 1; \ell, d)}{\mathcal{K}(s, 1; \ell, d)} = - \log N(\ell_1) - 2 \frac{\zeta_K(2s)}{\zeta_K(2s+1)} + 2 \frac{\zeta_K'(2s)}{\zeta_K(2s+1)} + \Psi(s),
\]

where

\[
\Psi(s) = \int_1^\infty \frac{\log \zeta_K(t-s)}{t^2} dt.
\]
where
\[
\Psi(s) = \sum_{\varpi \in \mathbb{G}, \varpi|d} \left( \frac{2 \log N(\varpi)}{N(\varpi) - 1} + \frac{2 \log N(\varpi)}{N(\varpi)^{1-2s} - 1} + \frac{2 \log N(\varpi)}{N(\varpi)^{1+2s} - 1} \right) + 2 \log 2 + \frac{6 \log 2}{(1-2s)} (1-2^{1-2s}) + \sum_{\varpi \in \mathbb{G}, \varpi|\ell} \frac{2 \log N(\varpi)}{N(\varpi) - 1} - \sum_{\varpi \in \mathbb{G}, \varpi|\ell} \frac{2 \log N(\varpi)}{N(\varpi) + 1}
\]
(4.63)
\[
+ \sum_{\varpi \in \mathbb{G}, \varpi|\ell} \left( \frac{2 \log N(\varpi)}{N(\varpi) - 1} - \frac{2 \log N(\varpi)}{N(\varpi)} \right) \frac{1 + \frac{2}{N(\varpi)^{2s}} - \frac{1}{N(\varpi)} \left( \frac{1}{N(\varpi)^{2s}} + \frac{N(\varpi)^{-2s}}{N(\varpi)^{2s} + N(\varpi)^{-2s}} \right)}{1 + \frac{2}{N(\varpi)^{2s}} + \frac{1}{N(\varpi)} \left( \frac{1}{N(\varpi)^{2s}} + \frac{N(\varpi)^{-2s}}{N(\varpi)^{2s} + N(\varpi)^{-2s}} \right)} \right)
= \Psi(-s).
\]

This implies that
\[
\left. \frac{\partial}{\partial s} \tilde{K}(s, w; \ell) \right|_{w=1} - \left. \frac{\partial}{\partial s} \tilde{K}(s, 1; \ell) \right|_{s=1} = \Gamma'_1(s) \tilde{K}(s, 1; \ell) - \Gamma'_1(-s) \tilde{K}(s, 1; \ell).
\]
(4.64)

We move the line of integration in (4.57) to \( \Re(s) = -\frac{1}{\log X} \) to encounter a pole at \( s = 0 \). A change of variable \( s \rightarrow -s \) together with (4.61) and (4.64) shows that the integral on the new line is the negative of the original integral in (4.57). It follows that the original integral equals half of the residue of the pole at \( s = 0 \). Representing this residue as an integral along the circle \( |s| = \frac{1}{\log X} \) and applying (4.59), we see that

\[
B = X \sum_{N(d) \leq D, \atop d \equiv 1 \pmod{(1+i)^3}} \mu^2_{1,1}(d) \lambda_d \sum_{N(M_1, M_2) \leq M, \atop m_1, m_2 \equiv 1 \pmod{(1+i)^3}} b_{m_1, m_2} \frac{1}{N(m_1, m_2)^{3/2}} \frac{\pi^3}{32} \frac{\tilde{\Phi}(1)}{\Phi(1)}
\]
(4.65)
\[
\times \left( \log X + \frac{\tilde{\Phi}'(1)}{\Phi(1)} + \frac{8}{\pi} \gamma_K \right) - \left. \frac{\partial}{\partial s} \tilde{K}(s, 1; m_1, m_2, d) \right|_{w=1} - \left. \frac{\partial}{\partial s} \tilde{K}(s, w; m_1, m_2, d) \right|_{w=1} - \Gamma'_1(s) \tilde{K}(s, 1; m_1, m_2, d)
\]
\[
= X \sum_{N(d) \leq D, \atop d \equiv 1 \pmod{(1+i)^3}} \mu^2_{1,1}(d) \lambda_d \tilde{\rho}_{1,1}(d)^2 \sum_{\varpi \in \mathbb{G}, \varpi|d} \left( 1 - \frac{1}{N(\varpi)^{2s+1}} \right) \left( 1 - \frac{1}{N(\varpi)^{-2s+1}} \right)
\]
\[
\times \prod_{\varpi \in \mathbb{G}, \varpi|d} \left( 1 - \frac{1}{N(\varpi)^{2s+1}} \right) \left( 1 - \frac{1}{N(\varpi)^{-2s+1}} \right) \sum_{\varpi \in \mathbb{G}, \varpi|d} J(\varpi, s),
\]
where
\[
J(\varpi, s) = \frac{2 \log N(\varpi)}{N(\varpi)^{1+2s} - 1} + \frac{2 \log N(\varpi)}{N(\varpi)^{1-2s} - 1} + \frac{2 \log N(\varpi)}{N(\varpi)} \left( 1 + \frac{2}{N(\varpi)^{2s}} - \frac{1}{N(\varpi)} \left( N(\varpi)^{2s} + N(\varpi)^{-2s} \right) \right) \frac{1 + \frac{2}{N(\varpi)^{2s}} + \frac{1}{N(\varpi)} \left( N(\varpi)^{2s} + N(\varpi)^{-2s} \right)}{1 + \frac{2}{N(\varpi)^{2s}} + \frac{1}{N(\varpi)} \left( N(\varpi)^{2s} + N(\varpi)^{-2s} \right)}.
\]
Both $\Sigma_1$ and $\Sigma_2$ can be evaluated similar to the sums defined in \[ (7.9.5), (7.9.6) \], using Lemma \[ 4.4 \] and Lemma \[ 4.5 \] respectively. The result is

$$
\Sigma_1 = \frac{4}{\pi} \cdot \frac{2N(m_1 m_2)}{\varphi(m_1 m_2)} \prod_{\omega \in [m_1 m_2]} \left(1 - \frac{1}{N(\omega)^2}\right) \left(1 + o(1)\right) \frac{1}{\log R} + O\left(\frac{1}{(\log R)^{2020}}\right),
$$

\begin{equation}
\Sigma_2 = -\frac{4}{\pi} \cdot \frac{2N(m_1 m_2)}{\varphi(m_1 m_2)} \prod_{\omega \in [m_1 m_2]} \left(1 - \frac{1}{N(\omega)^2}\right) \left(1 + o(1)\right) \frac{1}{\log R} \times \sum_{\omega \in [m_1 m_2]} J(\omega, s) \frac{N(\omega) + 1}{N(\omega) + 1} \left(1 - \frac{1}{N(\omega)^{1+2s}}\right) + O\left(\frac{1}{(\log R)^{2020}}\right).
\end{equation}

Applying \[ (4.66) \] in \[ (4.65) \], together with \[ (4.61) \], \[ (4.62) \], we see that

\begin{equation}
B = \frac{\pi^2}{6z_K(2)} \Phi(1) X \frac{1 + o(1)}{\log R} \sum_{N(m_1), N(m_2) \leq M} b_{m_1} b_{m_2} \prod_{\omega \in G, \varphi(\omega) \equiv \ell_1} \left(1 - \frac{1}{N(\omega)^2}\right) (\log R) \left(1 - \frac{1}{N(\omega)^{1+2s}}\right) + O\left(\frac{1}{(\log R)^{2020}}\right),
\end{equation}

where

$$
\eta_1(\omega, s) = \frac{2 \log N(\omega)}{N(\omega) - 1} - \left(\frac{2 \log N(\omega)}{N(\omega)} - 1 + \frac{2}{N(\omega) + 1} - \frac{1}{N(\omega)^2} \right) \left(\frac{N(\omega)^{2s}}{N(\omega)} + \frac{N(\omega)^{-2s}}{N(\omega)}\right),
$$

$$
\eta_2(\omega, s) = \frac{2 \log N(\omega)}{N(\omega) - 1} - \eta_1(\omega, s),
$$

$$
\eta_3(\omega, s) = \frac{2 \log N(\omega)}{N(\omega) - 1} - \eta_1(\omega, s).
$$

For $|s| = \frac{1}{\log X}$, we further introduce the following sums:

$$
\Upsilon_1 = \sum_{N(m_1), N(m_2) \leq M} b_{m_1} b_{m_2} \prod_{\omega \in G, \varphi(\omega) \equiv \ell_1} \left(1 - \frac{1}{N(\omega)^2}\right) \sum_{\omega \in [m_1 m_2]} (N(\omega)^s + N(\omega)^{-s})
$$

$$
\Upsilon_2 = \sum_{N(m_1), N(m_2) \leq M} b_{m_1} b_{m_2} \prod_{\omega \in G, \varphi(\omega) \equiv \ell_1} \left(1 - \frac{1}{N(\omega)^2}\right) \sum_{\omega \in [m_1 m_2]} (N(\omega)^s + N(\omega)^{-s}) \log N(\ell_1),
$$

$$
\Upsilon_3 = \sum_{N(m_1), N(m_2) \leq M} b_{m_1} b_{m_2} \prod_{\omega \in G, \varphi(\omega) \equiv \ell_1} \left(1 - \frac{1}{N(\omega)^2}\right) \sum_{\omega \in [m_1 m_2]} (N(\omega)^s + N(\omega)^{-s}) \sum_{\omega \in G, \varphi(\omega) \equiv \ell_1} \eta_2(\omega, s),
$$

$$
\Upsilon_4 = \sum_{N(m_1), N(m_2) \leq M} b_{m_1} b_{m_2} \prod_{\omega \in G, \varphi(\omega) \equiv \ell_1} \left(1 - \frac{1}{N(\omega)^2}\right) \sum_{\omega \in [m_1 m_2]} (N(\omega)^s + N(\omega)^{-s}) \sum_{\omega \in G, \varphi(\omega) \equiv \ell_1} \eta_3(\omega, s).
$$
These sums are similar to those defined in \[ (7.9.14)-(7.9.17) \] and can be evaluated accordingly. Again by keeping in mind that the residue of \( \zeta_K(s) \) at \( s = 1 \) equals \( \pi/4 \), we have that

\[
Y_1 = 6 \zeta_K(2) \left( \frac{4}{\pi} \right)^3 \left( \frac{1}{\log M} \int_0^1 H''(t)^2 \, dt - \frac{2s^2}{\log M} \int_0^1 H(t)H''(t) \, dt + s^4 \log M \int_0^1 H(t)^2 \, dt \right) + O \left( \frac{1}{(\log X)^{4-\varepsilon}} \right),
\]

\[
Y_2 = 6 \zeta_K(2) \left( \frac{4}{\pi} \right)^3 \left( -\frac{4}{\log^2 M} \int_0^1 H'(t)H''(t) \, dt + 4s^2 \int_0^1 H(t)H'(t) \, dt \right) + O \left( \frac{1}{(\log X)^{3-\varepsilon}} \right),
\]

\[
Y_3, Y_4 \leq \frac{1}{(\log X)^{3/2-\varepsilon}}.
\]

We now sum over \( m_1, m_2 \) in \[ (4.07) \] and apply the above estimations together with the observation that \( \zeta_K(2s) \) is bounded when \( s \) is near 0 and \( \zeta_K(1+2s) \ll \log X \) when \( |s| = 1/\log X \) to deduce that

\[
\mathcal{B} = \frac{4^3}{\pi} \Phi(1) \frac{1}{\log R} \frac{1}{2\pi} \int_{|s|=\frac{1}{\log X}} \Gamma_1(s) \left( \frac{2^{3/2}}{\pi} \left( \frac{\Gamma \left( \frac{1}{2} + s \right)}{\Gamma(\frac{1}{2})} \right)^2 \zeta_K(2s) \zeta_K(2s + 1) \right.
\]

\[
\left. \times \left( \frac{5}{2} - 4^s - 4^{-s} \right) \left( \log X + \frac{\Phi(1)}{\Phi(1)} + \frac{8}{\pi} \gamma_K - \frac{\Gamma'_1(s)}{\Gamma(s)} - \frac{2\zeta_K(2s)}{\zeta_K(2s + 1)} + \frac{2G_K}{\zeta_K(2s + 1)} \right) \right)
\]

\[
+ 2 \log 2 \left( \frac{6 \log 2}{(1 - 2^{1+2s})(1 - 2^{1-2s})} \sum_{\varpi \in \mathcal{G}} \eta_1(\varpi, s) \right) \left( \frac{1}{\log^3 M} \int_0^1 H''(t)^2 \, dt \right.
\]

\[
- \frac{2s}{\log M} \int_0^1 H(t)H''(t) \, dt + s^4 \log M \int_0^1 H(t)^2 \, dt \right) \left. - \frac{4}{\log^2 M} \int_0^1 H'(t)H''(t) \, dt \right)
\]

\[
+ 4s^2 \int_0^1 H(t)H'(t) \, dt \right) \left. \frac{ds}{s} + O \left( \frac{X}{(\log X)^{3/2-\varepsilon}} \right) \right).
\]

We further use the fact that \( \zeta_K(0) = -\frac{1}{2} \) (see \[ 2 \] p. 27) for an computation of \( \zeta_K(0) \)) and that when \( s \to 0 \)

\[
\Gamma_1(s) \zeta_K(2s + 1) = \frac{\pi}{8} \cdot \frac{1}{s^2} + O(\frac{1}{s})
\]
to evaluate the integral in \[ (4.68) \] to obtain that

\[
\mathcal{B} = 4X \Phi(1) \frac{1}{\log R} \left( \frac{\log X}{2\log M} \int_0^1 H(t)H''(t) \, dt - \int_0^1 H(t)H'(t) \, dt \right) + O \left( X(\log X)^{-3/2+\varepsilon} \right).
\]

We then conclude from \[ (4.12), (4.13), (4.20), (4.32) \] and the above expression that

\[
S^+ = 2X \left( \frac{1}{\log R} \left( \frac{\log X}{\log M} \right)^3 \int_0^1 H''(t)^2 \, dt \right.
\]

\[
- \frac{1}{2} \left( \frac{\log X}{\log M} \right)^2 \int_0^1 H(t)H''(t) \, dt + \frac{\log X}{\log M} \int_0^1 H(t)H''(t) \, dt + \frac{\log X}{\log M} \int_0^1 H'(t)^2 \, dt
\]

\[
- 2 \int_0^1 H(t)H'(t) \, dt \right) \left. + O \left( \frac{X}{(\log X)^{3/2-\varepsilon}} + \frac{X^{1+\varepsilon}}{Y} + X^{1+\varepsilon}M \right) \right).
\]

Recalling the definition of \( M, R \) from \[ 1.2 \] and \( Y \) from \[ 4.58 \], we see that the assertion of Proposition \[ 1.3 \] follows for the above expression of \( S^+ \).

5. PROOF OF THEOREM 1.1

In this section, we complete the proof of Theorem 1.1 by first applying \[ 4.7 \], Proposition 1.2 and Proposition 1.3 to see that, for fixed sufficiently small \( \delta_0 > 0 \),

\[
\frac{2X}{\left( 1 + \delta_0 \right)} \cdot \frac{\phi(H(0) - \frac{1}{\delta_0} H'(0))^2}{3} \leq \sum_{\varpi \neq 2} \log N(\varpi) \Phi \left( \frac{N(\varpi)}{X} \right) \leq \sum_{\varpi \neq 2} \frac{1}{X/2 \leq N(\varpi) \leq X} \frac{1}{L(\frac{1}{2}, \chi(1+i)^s) \neq 0}.
\]
As the left side of (5.1) is an increasing function of $\vartheta$, we take $\vartheta = \frac{1}{2}(\frac{1}{2} - \theta) - \varepsilon$ to be the largest possible value allowed by the condition of Proposition 1.3. This leads to

$$\sum_{(\pi, 2) = 1, X/2 \leq N(\pi) \leq X} \frac{1}{L(\frac{1}{2}, \chi_{1+i})^{\gamma_{\pi}}} \geq \frac{2X}{(1 + 2\delta_0) \log X} \cdot \vartheta,$$

with

$$\vartheta = \frac{1}{2} \left( \frac{1}{2} - \theta \right) \left( H(0) - \frac{1}{2\pi} H'(0) \right)^2.$$

The optimal $H$ that maximizes $\vartheta$ is determined in [1, Section 8] to be a smooth approximation on $[0, 1]$ of the function $H(x)$ given by

$$H(x) = (1 - x)^2 \left( 2 + \frac{3}{2\theta} + x \right).$$

We then proceed as in [1, Section 8] to see that

$$\sum_{(\pi, 2) = 1, X/2 \leq N(\pi) \leq X} \frac{1}{L(\frac{1}{2}, \chi_{1+i})^{\gamma_{\pi}}} \geq \frac{2X}{(1 + O(\delta_0)) \log X} \cdot \rho(\theta),$$

where

$$\rho(\theta) = \frac{1}{2} \left( \frac{1}{2} - \theta \right) \left( 1 - \frac{1}{(1 + 2\theta)^3} \right).$$

By setting $\theta = \theta_0$ to be the unique positive root $\theta_0 = 0.17409\ldots$ of the polynomial $16\theta^4 + 32\theta^3 + 24\theta^2 + 12\theta - 3$ that maximizes $\rho(\theta)$ on $(0, \frac{1}{2})$, we have

$$\rho(\theta_0) = 0.09645\ldots.$$

We substitute this in (5.2) and sum over dyadic intervals. As

$$\sum_{(\pi, 2) = 1, N(\pi) \leq X} 1 = (1 + o(1)) \frac{4X}{(\log X)},$$

the assertion of Theorem 1.1 then follows.

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