General analytic formulae for attractor solutions of scalar-field dark energy models and their multi-field generalizations

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We study general properties of attractors for scalar-field dark energy scenarios which possess cosmological scaling solutions. In all such models there exists a scalar-field dominant solution with an energy fraction $\Omega_\phi = 1$ together with a scaling solution. A general analytic formula is given to derive fixed points relevant to dark energy coupled to dark matter. We investigate the stability of fixed points without specifying the models of dark energy in the presence of non-relativistic dark matter and provide a general proof that a non-phantom scalar-field dominant solution is unstable when a stable scaling solution exists in the region $\Omega_\phi < 1$. A phantom scalar-field dominant fixed point is found to be classically stable. We also generalize the analysis to the case of multiple scalar fields and show that for a non-phantom scalar field assisted acceleration always occurs for all scalar-field models which have scaling solutions. For a phantom field the equation of state approaches that of cosmological constant as we add more scalar fields.

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I. INTRODUCTION

Dark energy is one of the most serious mysteries in modern cosmology. The observations of supernova (SN) Ia \cite{1} have continuously confirmed that about 70\% of the total energy density of universe consists of unknown energy which leads to an accelerated expansion of the universe. The existence of dark energy has been also supported from the observations of cosmic microwave background (CMB) \cite{2}, large-scale structure (LSS) \cite{3} and baryon oscillation experiments \cite{4} (see also Refs. \cite{5}).

While cosmological constant can be a natural candidate of dark energy, this suffers from a severe fine-tuning problem if it originates from particle physics (see Refs. \cite{6,7,8,9} for review). Instead of sticking to cosmological constant, a host of alternative scenarios have been proposed – ranging from scalar-field models, Chaplygin gas \cite{10}, braneworld \cite{11} and modified gravity scenarios in General Relativity \cite{12}. Since scalar fields are natural ingredients in particle physics, it is an interesting challenge for theoreticians to construct viable dark energy models using them. A number of scalar-field models have been studied as candidates of dark energy – an incomplete list includes quintessence \cite{13}, k-essence \cite{14}, tachyons \cite{15}, phantoms \cite{16} and (dilatonic) ghost condensates \cite{17,18}. These can at least alleviate the fine-tuning problem of cosmological constant because of their dynamical nature.

A general guidance to obtain viable dark energy models is that the energy density of it is “hidden” during radiation and matter dominant eras and comes out near to the present epoch. If one uses the property of cosmological scaling solutions \cite{19,20,21,22}, the energy density of scalar fields decreases proportionally to that of the background fluid (radiation or matter) independently of the initial conditions \cite{23}. The system can exit from the scaling regime to give rise to the present accelerated expansion if the potential of the field becomes shallow \cite{24,25} or if the coupling $Q$ between dark energy and dark matter becomes important \cite{26}. Especially in the latter case there is an intriguing possibility that the present universe is a scaling attractor with a constant dark energy fraction $\Omega_\phi \simeq 0.7$ \cite{27}. Thus the presence of scaling solutions plays an important role for the model-building of dark energy.

It is well known that exponential potentials give rise to scaling solutions for an ordinary scalar field \cite{28}. The form of the potentials which generate scaling solutions depends on the models of dark energy. For example in the case of a tachyon field the corresponding potential is given by $V(\phi) = V_0 \phi^{-2}$ \cite{29,30,31}. A general algorithm to obtain potentials corresponding to scaling solutions is given in Refs. \cite{15,32}. Interestingly the scalar-field Lagrangian density for the existence of scaling solutions is restricted to be in a compact form $p = X g(X e^{\lambda \phi})$ \cite{18}, where $X$ is the kinematic term of a scalar field $\phi$ and $g$ is an arbitrary function. This analysis can be extended to the more general case in which the Hubble parameter $H$ has a dependence $H^2 \propto \rho^n$ with $\rho$ being the total energy density \cite{33}. By using the formula in Ref. \cite{32} we can easily obtain potentials which generate scaling solutions once we specify dark energy models.

In addition to scaling solutions there exists another important fixed point which corresponds to the scalar-field dominant solution characterized by $\Omega_\phi = 1$. In fact, in the case of an ordinary scalar field with an exponential potential, this solution was used in Ref. \cite{24} to obtain a late-time attractor with an accelerated expansion after the system exits from a scaling regime during radiation and matter dominant eras. In Ref. \cite{34} it was shown that the scalar-field dominant solution also exists for tachyons and dilatonic ghost condensates. In this paper we will

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show that this fixed point always exists for all scalar-field models which possess scaling solutions. We shall also provide a general analytic formula to derive critical points for both the scalar-field dominant solution and the scaling solution.

The stability of critical points against perturbations was analyzed in Ref. [31] by specifying three models of dark energy. We shall generally prove that the non-phantom scalar-field dominant solution is unstable when the stable scaling solution exists in the region $\Omega_\phi < 1$. It will be also shown that the phantom scalar-field dominant solution is classically stable. This analysis is useful when we construct viable dark energy models.

There is another interesting possibility to give rise to an inflationary solution by using multiple scalar fields. In the case of an ordinary scalar field with an exponential potential, many fields can cooperate to sustain inflation even if none is able to do individually [32]. This assisted inflation scenario was applied to dark energy in Refs. [33, 34]. The field energy density mimics that of the background fluid during radiation and matter dominant eras because of the scaling property, which is followed by the scalar-field dominant solution as more fields join to sustain an accelerated expansion. In this paper we shall show that for a non-phantom field this assisted behavior always happens for all scalar-field dark energy models which have scaling solutions.

II. CONDITIONS FOR THE EXISTENCE OF SCALING SOLUTIONS

In this section we shall review the Lagrangian for the existence of scaling solutions presented in Refs. [13, 30]. Let us start with the following 4-dimensional action

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + p(X, \phi) \right] + S_m(\phi),$$

where $R$ is a scalar curvature and $X = -g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi/2$ is a kinematic term of a scalar field $\phi$. $S_m$ is an action for a barotropic fluid which is generally dependent on $\phi$. The reduced Planck mass $M_p$ is set to be unity. The above action covers most of scalar-field dark energy models.

The pressure density and the energy density of the field are given, respectively, by

$$p_\phi = p, \quad \rho_\phi = 2X \frac{\partial p}{\partial X} - p.$$  

We introduce a scalar charge $\sigma$ defined by

$$\sigma = -\frac{1}{\sqrt{-g}} \frac{\delta S_m(\phi)}{\delta \phi},$$

which corresponds to the coupling between the field $\phi$ and the barotropic fluid. This coupling can lead to an accelerated expansion even when scaling solutions can not do so in the absence of the interaction [25]. See Refs. [37] for recent works about the interacting dark energy. A similar coupling arises in scalar-tensor theories [22, 26] and in neutrino models of dark energy in which the mass of neutrinos depends on a scalar field $\phi$ [37].

In a flat Friedmann-Robertson-Walker (FRW) background with a scale factor $a$, the continuity equation for the field $\phi$ is given by

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = -Q \rho_m \dot{\phi},$$

where $\dot{a}/a$ is the Hubble rate. Meanwhile the fluid energy density, $\rho_m$, satisfies

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = Q \rho_m \dot{\phi}.$$  

In what follows the equation of state of a barotropic fluid, $w_m = p_m/\rho_m$, is assumed to be constant.

The Hubble parameter $H$ obeys the Friedmann equation

$$3H^2 = \rho_\phi + \rho_m.$$  

The fractional densities of $\rho_\phi$ and $\rho_m$ are defined by

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3H^2}.$$  

These satisfy the relation $\Omega_\phi + \Omega_m = 1$ from Eq. [4]. We note that reconstruction equations suitable for the comparison to SN Ia observations were obtained in Ref. [38] for the action [11].

Cosmological scaling solutions are characterized by the condition

$$\frac{\rho_\phi}{\rho_m} = \text{const} \neq 0.$$  

In this case $\Omega_\phi$ and $\Omega_m$ are constant from Eq. [4]. We also assume that $Q$ and $w_\phi$ do not vary in the scaling regime. Note that the constant coupling $Q$ appears in theories in which the mass of dark matter or neutrinos depends exponentially on the scalar field, see e.g., Refs. [37, 38].

From the condition [8] we find $(\log \rho_\phi)^* = (\log \rho_m)^*$. Then we obtain the following relation from Eqs. [4] and [9]:

$$\frac{d\phi}{dN} = \frac{3\Omega_\phi}{Q}(w_m - w_\phi) = \text{const},$$

where $N \equiv \int H dt$ is the number of $e$-foldings. This gives the following scaling behavior:

$$\frac{d \ln \rho_\phi}{dN} = \frac{d \ln \rho_m}{dN} = -3(1 + w_{eff}),$$

where the effective equation of state is

$$w_{eff} \equiv \frac{w_\phi \rho_\phi + w_m \rho_m}{\rho_\phi + \rho_m} = w_m + \Omega_\phi(w_\phi - w_m).$$
This expression of \( w_{\text{eff}} \) can be used irrespective of the fact that scaling solutions exist or not.

The definition of \( X \) gives the relation

\[
2X = H^2 \left( \frac{d\phi}{dN} \right)^2 \propto H^2 \propto (\rho_\phi + \rho_m),
\]

(12)

where we used Eq. (9). Hence the scaling property of \( X \) is the same as \( \rho_\phi \) and \( \rho_m \), which leads to the relation

\[
\frac{d\ln X}{dN} = -3(1 + w_{\text{eff}}).
\]

(13)

The pressure density \( p_\phi = w_\phi \rho_\phi \) scales in the same way as \( \rho_\phi \), i.e., \( \frac{d\ln p_\phi}{dN} = -3(1 + w_{\text{eff}}) \). Using this relation together with Eqs. (9) and (12) we find

\[
\frac{\partial \ln p_\phi}{\partial \ln X} - 1 \frac{\partial \ln p_\phi}{\partial \phi} = 1,
\]

(14)

where

\[
\lambda \equiv \frac{Q + w_m - \Omega_\phi(w_m - w_\phi)}{\Omega_\phi(w_m - w_\phi)}.
\]

(15)

From Eq. (14) the existence of scaling solutions restricts the Lagrangian density in the form:

\[
p(X, \phi) = X g (X e^{\lambda \phi}),
\]

(16)

where \( g \) is an arbitrary function in terms of \( Y = X e^{\lambda \phi} \).

This expression was first derived in Ref. 18 and was extended to the more general case in which the Friedmann equation is given by \( H^2 \propto (\rho_\phi + \rho_m)^n \). The quantity \( Y \) is conserved along the scaling solution, i.e., \( Y = X e^{\lambda \phi} = \text{constant} \).

Although we derived the expression (16) under the assumption that \( Q \) is a constant, this can be generalized to the case in which \( Q \) depends on the field \( \phi \). This case will be presented in elsewhere 19. We also note that the r.h.s. of Eq. (9) is kept to be a constant for \( Q \to 0 \), since \( w_\phi \to w_m \) for scaling solutions in this limit. Thus the above discussion for the derivation of Eq. (16) is valid even for \( Q = 0 \) by taking the limit \( Q \to 0 \).

Equations (9), (12) and (15) give the relation

\[
3H^2 = \frac{2(\lambda + Q)^2}{3(1 + w_m)^2} X.
\]

(17)

From Eqs. (11) and (15) we find

\[
w_{\text{eff}} = \frac{w_m \lambda - Q}{Q + \lambda}.
\]

(18)

When \( Q = 0 \) this reduces to \( w_{\text{eff}} = w_m \). Hence an accelerated expansion is not possible for a non-relativistic fluid \((w_m = 0)\). However the presence of the coupling \( Q \) leads to an accelerated expansion \((w_{\text{eff}} < -1/3)\) for \( Q > (3w_m + 1)\lambda/2 \).

If we choose the function \( g(Y) = 1 - c/Y \) in Eq. (10), we obtain the Lagrangian density for an ordinary scalar field with an exponential potential: \( p = X - ce^{-\lambda \phi} \). The function \( g(Y) = 1 + cY \) gives the dilatonic ghost condensate model with Lagrangian density \( p = -X + cX^2 \). When we choose the function \( g(Y) = c\sqrt{1 - 2Y/Y} \) together with the introduction of a new field \( \varphi = (2/\lambda)e^{\lambda \phi/2} \), we obtain the tachyon Lagrangian \( p = -V(\varphi)i \sqrt{1 - \varphi^2} \) with an inverse square potential \( V(\varphi) = 4e/(\lambda^2 \varphi^2) \).

### III. FIXED POINTS

In the previous section we showed that the Lagrangian (16) possesses scaling solutions. There exist other fixed points for the system (17). We shall derive fixed points relevant to dark energy and analyze the stability of them against perturbations. Note that critical points were derived for three dark energy models in Ref. [31] by specifying the functional form of \( g(Y) \). In this paper we show that this is possible for all scalar-field models that possess scaling solutions. In what follows we shall consider the case of positive values of \( Q \) and \( \lambda \).

From Eqs. (2) and (4) together with the Lagrangian (16), we find that the scalar field obeys the equation of motion:

\[
\ddot{\phi} + 3HA p_X \phi + \lambda X [1 - A(y + 2Y g')^r] = -AQ \rho_m.
\]

(19)

where a prime represents a derivative with respect to \( Y \) and

\[
A(Y) \equiv \left[ g(Y) + \frac{5Y g'(Y) + 2Y^2 g''(Y)}{1 - (g(Y) + 2Y g')^r} \right]^{-1},
\]

(20)

\[
p_X(Y) \equiv \frac{d\rho}{dX} = g(Y) + Y g'(Y).
\]

(21)

Combining Eqs. (18) and (19) with (10), we find

\[
\dot{H} = -X p_X - \frac{1}{2}(1 + w_m) \rho_m.
\]

(22)

We introduce the following dimensionless quantities which are useful to study the dynamical system:

\[
x \equiv \frac{\dot{\phi}}{\sqrt{6H}}, \quad y \equiv \frac{e^{-\lambda \phi/2}}{\sqrt{3H}}.
\]

(23)

The variables \( x \) and \( y \) can be regarded as a “kinematic” term and a “potential” term, respectively. Since we do not consider a contracting universe, \( y \) is positive from its definition. The variable \( Y = X e^{\lambda \phi} \) is expressed in terms of \( x \) and \( y \):

\[
Y = x^2/y^2.
\]

(24)

Equation (10) gives the constraint equation \( \Omega_\phi + \Omega_m = 1 \) with

\[
\Omega_\phi \equiv x^2(g + 2Y g').
\]

(25)

The equation of state for the scalar field \( \phi \) is given by

\[
w_\phi = \frac{g}{g + 2Y g'}.
\]

(26)
It may be useful to notice the relation
\[ \Omega_\phi w_\phi = g x^2, \quad w_\phi = -1 + \frac{2x^2}{\Omega_\phi} p_X. \] (27)

This shows that the field behaves as a phantom \((w_\phi < -1)\) for \(p_X < 0\) provided that \(\Omega_\phi > 0\).

From Eqs. 11 and 29, we get the following autonomous equations for \(x\) and \(y\):
\[
\frac{dx}{dN} = \frac{3x}{2} \left[ 1 + gx^2 - w_m(\Omega_\phi - 1) - \sqrt{\frac{6}{3}} \lambda x \right]
+ \sqrt{\frac{6A}{2}} [(Q + \lambda) \Omega_\phi - Q - \sqrt{6}(g + Y g')x],
\] (28)
\[
\frac{dy}{dN} = \frac{3y}{2} \left[ 1 + gx^2 - w_m(\Omega_\phi - 1) - \sqrt{\frac{6}{3}} \lambda x \right].
\] (29)

We can obtain fixed points of the system by setting \(dx/dN = 0\) and \(dy/dN = 0\). From Eq. 29, we find that \(y = 0\) corresponds to one of the fixed points. However, this is irrelevant to dark energy, since we need the contribution of a “potential” term to give rise to an accelerated expansion. In addition, this fixed point is unstable against perturbations 31.

Then the fixed points we are interested in satisfy the following equations:
\[ \sqrt{6} \lambda x = 3 \left[ 1 + gx^2 - w_m(\Omega_\phi - 1) \right], \] (30)
\[ \sqrt{6}(g + Y g')x = (Q + \lambda) \Omega_\phi - Q. \] (31)

Using Eq. 27, we find that \(g + Y g' = \Omega_\phi(1 + w_\phi)/2x^2\). Then Eqs. 30 and 31 can be written in the form:
\[ x = \frac{\sqrt{6}[1 + (w_\phi - w_m) \Omega_\phi + w_m]}{2\lambda}, \] (32)
\[ = \frac{\sqrt{6}(1 + w_\phi) \Omega_\phi}{2[(Q + \lambda) \Omega_\phi - Q]}, \] (33)
which leads to
\[(\Omega_\phi - 1) [(w_\phi - w_m)(Q + \lambda) \Omega_\phi + Q(1 + w_m)] = 0. \] (34)

Hence we find two cases:

- (i) A scalar-field dominant solution with
  \[ \Omega_\phi = 1. \] (35)
- (ii) A scaling solution with
  \[ \Omega_\phi = \frac{(1 + w_m)Q}{(w_m - w_\phi)(Q + \lambda)}. \] (36)

In the case of an ordinary scalar field, for example, the solution (i) corresponds to the 4-th fixed point in Table I of Ref. 20, whereas the solution (ii) corresponds to the 5-th fixed point. The above discussion shows that similar fixed points exist for all scalar-field dark models which possess scaling solutions. In what follows we shall study the properties of the critical points in more details.

A. Scalar-field dominant solutions

When \(\Omega_\phi = 1\), Eqs. 27 and 30 give the equation of state:
\[ w_\phi = -1 + \frac{\sqrt{6}\lambda}{3} x. \] (37)

In this case the effective equation of state of the system is given by \(w_{\text{eff}} = w_\phi\) by Eq. 11. The condition for an accelerated expansion, \(w_{\text{eff}} < -1/3\), corresponds to
\[ \lambda x < \sqrt{\frac{6}{3}}. \] (38)

From Eq. 31 we find that \(x\) is given by \(x = \lambda/\sqrt{6p_X}\). Then \(w_\phi\) is written in the form
\[ w_\phi = -1 + \frac{\lambda^2}{3p_X}. \] (39)

Hence one has \(w_\phi > -1\) for \(p_X > 0\) and \(w_\phi < -1\) for \(p_X < 0\). For an ordinary scalar field with an exponential potential, the Lagrangian density is given by \(p = X - c \lambda \phi^2\). Since \(p_X = 1\) in this case the equation of state is \(w_\phi = -1 + \lambda^2/3\). This agrees with the result obtained in Ref. 20. In the case of a phantom field with an exponential potential \((p = -X - c \lambda \phi^2)\), we obtain \(w_\phi = -1 - \lambda^2/3\).

From Eqs. 30 and 31 we find
\[ g(Y) = \frac{\sqrt{6}\lambda x - 3}{3x^2}, \quad Y g'(Y) = \frac{6 - \sqrt{6}\lambda x}{6x^2}. \] (40)

Once we specify a scalar-field dark energy model, i.e., the function \(g(Y)\), we get \(Y\) and \(x\) from Eq. 11. Using the relation 29, we obtain the scalar-field dominant fixed point \((x, y)\). The equation of state is known by using Eq. 37.

Let us consider the ordinary (phantom) scalar field with \(g(Y)\) given by \(g(Y) = e - cY / (e + 1)\) for the ordinary field and \(e = -1\) for the phantom). From Eq. 11, we find \(Y = c\lambda^2/(6 - c\lambda^2)\) and
\[ x = \frac{\lambda}{\sqrt{6}e}, \quad y = \sqrt{\frac{e}{e - 1}} \left[ 1 - \frac{\lambda^2}{6} \right]. \] (41)

This agrees with what was obtained in Refs. 20, 31.

In the case of dilatonic ghost condensate with \(g(Y) = -1 + cY\), Eq. 11 leads to
\[ x_{1,2} = \frac{-\sqrt{6}f_\pm(\lambda)}{4}, \quad cY_{1,2} = \frac{1}{2} + \frac{\lambda^2 f_\pm(\lambda)}{16}, \] (42)
where
\[ f_\pm(\lambda) = 1 \pm \sqrt{1 + 16/(3\lambda^2)}. \] (43)

This result agrees with what was obtained in Ref. 13, 31. We have two fixed points in this model. Since \(f_+(\lambda) > 0\)
and \( f_-(\lambda) < 0 \), the critical point \((x_1, y_1)\) corresponds to a phantom one with \( w_0 < -1 \) from Eq. (37), whereas the point \((x_2, y_2)\) satisfies \( w_0 > -1 \). In the latter case the condition \( \Delta_1 \) for the accelerated expansion corresponds to \( \lambda < \sqrt{6}/3 \). The phantom divide is characterized by the condition \( p_X = 0 \), i.e., \( cY = 1/2 \). From Eq. (42) we find \( 1/3 < cY_1 < 1/2 \) and \( cY_2 > 1/2 \). In the limit \( \lambda \to 0 \) both fixed points approach the phantom divide at \( cY = 1/2 \) (i.e., \( w_0 = -1 \)).

**B. Scaling solutions**

When \( \Omega_{\phi} \) is given by \( \Omega_{\phi} \), Eq. (36) leads to

\[
x = \frac{\sqrt{6}(1 + w_m)}{2(Q + \lambda)}.
\]

This is equivalent to Eq. (37). From Eqs. (11) and (36) we obtain the effective equation of state given in (18). Equations (37) and (38) give

\[
\Omega_{\phi} = \frac{Q(Q + \lambda) + 3(1 + w_m)p_X}{(Q + \lambda)^2}.
\]

From Eqs. (11) and (18) we find that \( x \) and \( w_{\text{eff}} \) are independent of the form of \( g(Y) \). Meanwhile \( Y \) and \( y \) depend upon the models of dark energy. From Eqs. (30) and (31) we find

\[
(1 - w_m)g(Y) - 2w_M g'(Y) = \frac{-2Q(Q + \lambda)}{3(1 + w_m)}.
\]

Once the function \( g(Y) \) is specified, one gets \( Y = x/\sqrt{Y} \) by using this equation. Then we obtain \( \Omega_{\phi} \) and \( w_{\phi} \) from Eqs. (18) and (27).

For example, in the case of an ordinary scalar field with \( g(Y) = 1 - c/Y \), we obtain

\[
Y = \frac{3c(1 + w_m)^2}{2Q(Q + \lambda) + 3(1 - w_m^2)},
\]

\[
y = \frac{\left[2Q(Q + \lambda) + 3(1 - w_m)^2\right]^{1/2}}{2c(Q + \lambda)^2},
\]

which agrees with the result obtained in Ref. 31.

**IV. STABILITIES OF FIXED POINTS**

In this section we shall study the stability of critical points obtained in the previous section. Remarkably this can be carried out without specifying the functional form of \( g(Y) \). Let us consider small perturbations \( \delta x \) and \( \delta y \) about the critical point \((x_c, y_c)\), i.e.,

\[
x = x_c + \delta x, \quad y = y_c + \delta y, \quad Y = Y_c + \delta Y.
\]

At linear level \( \delta Y \) is related to \( \delta x \) and \( \delta y \) via

\[
\delta Y = 2 \left( \frac{x_c}{y_c} \delta x - \frac{x_c}{y_c} \delta y \right).
\]

The function \( g(Y) \) is expanded as

\[
g(Y) = g_c + g'_c(Y - Y_c) + \frac{g''_c}{2}(Y - Y_c)^2 + \cdots,
\]

where \( g_c \equiv g(Y_c) \). When we consider linear perturbations of \( g(Y) \), it is sufficient to neglect the terms higher than the first order. However when we evaluate linear perturbations of the \( g + Y g' \) term in Eq. (25), we need to take into account the second-order term in Eq. (51), i.e.,

\[
\delta(g + Y g') = (2g'_c + Y g''_c) \delta y.
\]

Similarly the perturbation of the fractional energy density, \( \Omega_{\phi} = x^2(g + 2Y g') \), is given by

\[
\delta \Omega_{\phi} = 2x_c A_c \delta x - 2x_c (3Y_c g'_c + 2Y_c^2 g''_c) \delta y,
\]

where \( A_c = [g_c + 5Y_c g'_c + 2Y_c^2 g''_c]^{-1} \).

Let us study linear perturbations about a scalar-field dominant solution characterized by Eq. (36) and a scaling solution characterized by Eq. (30). We recall that these critical points satisfy the equations \( \Omega_{\phi} = x^2(g + 2Y g') \) and \( \Omega_{\phi} \), and from Eqs. (26) and (27) we get the following perturbation equations:

\[
\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix},
\]

where the elements of the matrix \( \mathcal{M} \) are

\[
a_{11} = -3 + \frac{\sqrt{6}}{2}(2Q + \lambda) x_c + 3x_c^2 (g_c + Y_c g'_c)
\]

\[-3w_m A_c^{-1} x_c^2,
\]

\[
a_{12} = y_c \left[ -3g'_c x_c Y_c^2 + \frac{3x_c}{y_c} - \sqrt{6}(Q + \lambda) Y_c
\right.
\]

\[+ 3w_m x_c Y_c (A_c^{-1} - g_c - 2Y_c g'_c)
\]

\[+ \sqrt{6} A_c \frac{(Q + \lambda) \Omega_{\phi} + Q}{2y_c^2} \right],
\]

\[
a_{21} = \frac{y_c}{2} \left[ -\sqrt{6} \lambda + 6x_c \left((1 - w_m)g_c + (1 - 5w_m)Y_c g'_c \right)
\right.
\]

\[+ 12 w_m x_c Y_c^2 g''_c \],
\]

\[
a_{22} = 3g'_c x_c^2 Y_c (3w_m - 1) + 6w_m x_c^2 Y_c^2 g''_c.
\]

We checked that these agree what was obtained in Ref. 31 for three dark energy models. The eigenvalues of the matrix \( \mathcal{M} \) are

\[
\mu_\pm = \frac{a_{11} + a_{22}}{2} \left[ 1 \pm \sqrt{1 - \frac{4(a_{11} a_{22} - a_{12} a_{21})}{(a_{11} + a_{22})^2}} \right].
\]

The fixed point is stable against perturbations when both \( \mu_+ \) and \( \mu_- \) are negative or when \( \mu_\pm \) have negative real
parts. Meanwhile it is unstable when either \( \mu_+ \) or \( \mu_- \) is positive.

In what follows we study the case of a non-relativistic barotropic fluid \((w_m = 0)\). We shall consider the stability of fixed points for (i) the scalar-field dominant solution and (ii) the scaling solution, separately.

### A. Scalar-field dominant solutions

When \( \Omega_\phi = 1 \) we have \((g_\phi + V_\phi g_\phi^2)X_c = \lambda/\sqrt{6} \) from Eq. [31]. Then we find \( a_{11} = 0 \) from Eq. [37]. This means that the eigenvalues of the matrix \( \mathbf{M} \) are

\[
\begin{align*}
\mu_+ &= a_{11} = -3 + \sqrt{6}(Q + \lambda)x_c, \\
\mu_- &= a_{22} = -3 + \sqrt{6}2\lambda x_c. 
\end{align*}
\]

Making use of Eq. [37], the second eigenvalue is expressed as \( \mu_- = -(3/2)(1 - w_\phi) \). Hence one has \( \mu_- < 0 \) for \( w_\phi < 1 \). The first eigenvalue \( \mu_+ \) is negative for

\[
x_c < \frac{\sqrt{6}}{2(Q + \lambda)}. \tag{62}
\]

Since we are considering the case of positive values of \( Q \) and \( \lambda, \mu_- \) becomes automatically negative when the condition \([63]\) is satisfied. Hence the scalar-field dominant fixed point \((\Omega_\phi = 1)\) is stable for \( x_c < \sqrt{6}/2(Q + \lambda) \), whereas it is unstable for \( x_c > \sqrt{6}/2(Q + \lambda) \).

In the case of a non-phantom scalar field characterized by \( p_X > 0 \), the stability condition \([64]\) can be written as

\[
p_X > \frac{\lambda(Q + \lambda)}{3}, \tag{63}
\]

where we used \( p_X X_c = \lambda/\sqrt{6} \). Meanwhile for a phantom field this stability condition corresponds to \( p_X < \lambda(Q + \lambda)/3 \). Since \( p_X < 0 \) this is automatically satisfied for positive values of \( Q \) and \( \lambda \) and also for \( Q = 0 \) with any \( \lambda \). Hence the scalar-field dominant solution is stable for phantom fields. This was found in Ref. \([31]\) for several models of dark energy, but we showed that this property persists for any form of \( g(Y) \).

### B. Scaling solutions

The scaling solution satisfies Eq. [44]. When \( w_m = 0 \) we have the relation \( g_c x_c^2 = -Q/(Q + \lambda) \) and also \( \Omega_\phi = -Q/(Q + \lambda) + 2Y_c g_\phi^2 x_c^2 \). Hence we find that the components of the matrix \( \mathbf{M} \) are

\[
\begin{align*}
a_{11} &= -\frac{3}{2}(1 - \Omega_\phi), \quad a_{12} = \frac{x_c^2 - A_c}{x_c y_c}a_{22}, \\
a_{21} &= \frac{y_c}{x_c}a_{11}, \quad a_{22} = -3Y_c g_\phi^2 x_c^2. \tag{64}
\end{align*}
\]

From Eq. [60] we get the eigenvalues of the matrix \( \mathbf{M} \), as

\[
\mu_\pm = \xi_1 \left[ 1 \pm \sqrt{1 - \xi_2^2} \right], \tag{65}
\]

where

\[
\begin{align*}
\xi_1 &= \frac{3(2Q + \lambda)}{4(Q + \lambda)}, \\
\xi_2 &= \frac{8(1 - \Omega_\phi)(Q + \lambda)[\Omega_\phi(Q + \lambda) + Q]}{3(2Q + \lambda)^2} A_c. \tag{66}
\end{align*}
\]

This shows that \( \xi_1 < 0 \) for positive \( Q \) and \( \lambda \). When \( Q = 0, \xi_1 \) is automatically negative (\( \xi_1 = -3/4 \)). \( \mu_+ \) is negative or has a negative real part. Meanwhile the sign of \( \mu_- \) depends upon that of \( \xi_2 \). When \( \xi_2 > 0 \) the scaling solution is stable, whereas it is unstable for \( \xi_2 < 0 \).

In order to get a viable scaling solution we require the condition \( \Omega_\phi < 1 \). Hence the stability of the scaling solution is dependent on the sign of \( A_c \), which can be expressed as

\[
A_c = (p_{X_c} + 2X_c p_{X_c, X_c})^{-1}. \tag{68}
\]

This quantity is related to the speed of sound:

\[
\xi_s^2 = \frac{p_X}{\rho_X} = \frac{p_X}{p_X + 2X p_{XX}}, \tag{69}
\]

which appears as a coefficient of the \( k^2/a^2 \) term in perturbation equations \([41, 42]\) \((k \text{ is a comoving wavenumber})\).

While the classical fluctuations may be regarded to be stable when \( \xi_s^2 > 0 \), it was shown in Ref. \([18]\) that the stability of quantum fluctuations requires the conditions \( p_X > 0 \) and \( p_X + 2X p_{XX} > 0 \). Hence if we impose the quantum stability, \( A_c \) is positive. Then the scaling solution with \( \Omega_\phi < 1 \) is a stable attractor. In the case of an ordinary scalar field with \( p = X - V(\phi) \) one has \( A_c = 1 \), which shows that the scaling solution is stable. Meanwhile for an ordinary phantom field with \( p = -X - V(\phi) \), it is unstable since \( A_c = -1 \).

From Eq. [53] with \( w_m = 0 \) we find that the condition, \( \Omega_\phi < 1 \), corresponds to

\[
p_X < \frac{\lambda(Q + \lambda)}{3}. \tag{70}
\]

When this is satisfied, the stability condition \([60]\) for the scalar-field dominant fixed point is violated. Hence when we have a stable scaling solution with \( \Omega_\phi < 1 \) and \( A_c > 0 \), the scalar-field dominant fixed point with \( p_X > 0 \) is always unstable. Hence the system chooses either a scaling solution with \( \Omega_\phi < 1 \) or a scalar-field dominant solution with \( \Omega_\phi = 1 \) and \( p_X > 0 \) as an attractor. In Ref. \([31]\) this property was found by specifying the functional form of \( g(Y) \), but we have shown that this holds for all scalar-field models which possess scaling solutions.

We note that the scaling solution can be an attractor even for a phantom \((p_X < 0)\) provided that the condition \( p_X + 2X p_{XX} > 0 \) is satisfied. In the case of the
dilatonic ghost condensate model \((p = -X + ce^{\lambda_Y}X^2)\), for example, this is realized for \(1/3 < cY < 1/2\). The scalar-field dominant solution \(cY_1 = 1/2 + \lambda^2 f_-(\lambda)/16\) in Eq. (42) also exists in the phantom region characterized by \(1/3 < cY_1 < 1/2\). In the subsection A we showed that this case is also a stable attractor. We recall that \(x_i \equiv 1/\sqrt{6}A\) also corresponds to a negative sound of speed, thus suffering from the instability of perturbations even at the classical level [18]. See Refs. [17, 43] for related works.

V. MULTI-FIELD DARK ENERGY

We can generalize the analysis to the case in which many fields are present with a same Lagrangian density for the each field. In the case of an ordinary scalar field with an exponential potential several fields can cooperate to drive an accelerated expansion even if none is able to do individually [52, 53, 34]. We will show that this generally happens for the Lagrangian density which possesses cosmological scaling solutions. We shall study the case without a coupling between dark energy and dark matter \((Q = 0)\) throughout this section. This is because an accelerated expansion is realized even for \(Q = 0\) in the presence of multiple scalar fields.

Let us consider \(n\) scalar fields \((\phi_1, \phi_2, \cdots, \phi_n)\) with the Lagrangian density:

\[
p = \sum_{i=1}^{n} X_i g(X_i e^{\lambda_Y \phi_i}),
\]

where \(X_i = -g_{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i /2\) and \(g\) is an arbitrary function. In the presence of a barotropic fluid with an equation of state \(w_m = p_m/\rho_m\), the constraint equation is given by \(\Omega_m \cdot \Omega_\phi = 1\) with a fractional density

\[
\Omega_\phi = \sum_{i=1}^{n} \Omega_{\phi_i} = \sum_{i=1}^{n} x_i^2 \left[ g(Y_i) + 2Y_i g'(Y_i) \right].
\]

We also obtain the equation for the Hubble rate:

\[
\frac{1}{H} \frac{dH}{dN} = -\frac{3}{2} \left[ 1 + \sum_{i=1}^{n} g(Y_i) x_i^2 - w_m (\Omega_\phi - 1) \right].
\]

Each scalar field, \(\phi_i\), satisfies the equation of motion \[40\] with \(Q = 0\). Defining dimensionless quantities

\[
x_i = \frac{\phi_i}{\sqrt{6}H} \quad \text{and} \quad y_i = e^{-\lambda_Y \phi_i}/2,
\]

we get the following differential equations:

\[
\frac{dx_i}{dN} = \frac{3x_i}{2} \left[ 1 + \sum_{i=1}^{n} g(Y_i) x_i^2 - w_m (\Omega_\phi - 1) - \frac{\sqrt{6}}{3} \lambda_i x_i \right]
\]

\[
+ \frac{\sqrt{6}A}{2} \left[ \lambda_i \Omega_\phi - \sqrt{6} \left( g(Y_i) + Y_i g'(Y_i) \right) x_i \right],
\]

(74)

\[
\frac{dy_i}{dN} = \frac{3y_i}{2} \left[ 1 + \sum_{i=1}^{n} g(Y_i) x_i^2 - w_m (\Omega_\phi - 1) - \frac{\sqrt{6}}{3} \lambda_i x_i \right].
\]

(75)

Then the fixed points we are interested in \((y_i \neq 0)\) satisfy

\[
\lambda_i x_i = \sqrt{6} \left( g(Y_i) + Y_i g'(Y_i) \right) / g(Y_i) + 2Y_i g(Y_i)
\]

\[
= \frac{\sqrt{6}}{2} \left[ 1 + \sum_{i=1}^{n} g(Y_i) x_i^2 - w_m (\Omega_\phi - 1) \right].
\]

(77)

Since the equation of state of the each scalar field is given by

\[
\omega_{\phi_i} = g(Y_i)/g(Y_i) + 2Y_i g(Y_i)
\]

\[
= 1/2 \left( 1 + w_{\phi_i} \right).
\]

(78)

Then from Eq. [74] we get

\[
w_{\phi_i} = -1 + \frac{\sqrt{6}}{3} \lambda_i x_i.
\]

(79)

Equation [74] indicates that the quantities \(\lambda_i x_i\) are independent of \(i = 1, 2, \cdots, n\). Hence we set

\[
\lambda_1 x_1 = \cdots = \lambda_i x_i = \cdots = \lambda_n x_n \equiv \lambda x.
\]

(80)

Then from Eq. [74], \(w_{\phi_i}\) are same for all scalar fields, i.e.,

\[
w_{\phi_i} = \cdots = w_{\phi_i} = \cdots = w_{\phi_n} \equiv w_\phi.
\]

(81)

From Eq. [74] the quantities \(Y_i\) are determined by \(\lambda_i x_i\), which means that \(Y_i\) are also same independent of \(i\):

\[
Y_1 = \cdots = Y_i = \cdots = Y_n \equiv Y.
\]

(82)

Taking note the relation

\[
\sum_{i=1}^{n} g(Y_i) x_i^2 = \sum_{i=1}^{n} w_{\phi_i} \Omega_{\phi_i} = w_\phi \Omega_{\phi},
\]

(83)

we find that Eq. [74] yields

\[
(\Omega_\phi - 1)(w_\phi - w_m) = 0.
\]

(84)

Hence we have two fixed points: (i) scalar-field dominant solution with \(\Omega_\phi = 1\) and (ii) scaling solution with \(w_\phi = w_m\). In what follows we shall discuss these cases separately.
A. Scalar-field dominant solutions ($\Omega_\phi = 1$) and
assisted dark energy

From Eqs. [31] and [32], Eq. [70] is written as

$$\lambda x = \frac{\sqrt{6} g(Y) + Y g'(Y)}{g(Y) + 2Y g'(Y)}. \quad (85)$$

Using Eq. [31] with $\Omega_\phi = 1$, we find that Eq. [70] is given by

$$\lambda x = \frac{\sqrt{6}}{2} \left[ 1 + g(Y) x^2 \lambda^2 \sum_{i=1}^{n} \frac{1}{\lambda_i^2} \right]. \quad (86)$$

Hence if we choose

$$\frac{1}{\lambda^2} = \sum_{i=1}^{n} \frac{1}{\lambda_i^2}, \quad (87)$$

Equation [31] yields

$$\lambda x = \frac{\sqrt{6}}{2} \left[ 1 + g(Y) x^2 \right]. \quad (88)$$

Equations [35] and [36] show that the system effectively reduces to that of the single field with $\lambda$ given by Eq. [37]. From Eq. [36] and $\Omega_\phi = 1$ the quantity

$$p_X = g(Y) + Y g'(Y)$$

satisfies the relation $\sqrt{6} \rho_X = \lambda$. Hence from Eq. [38] with Eqs. [31] and [32], we get the equation of state:

$$w_\phi = -1 + \frac{\lambda^2}{3 \rho_X}, \quad (89)$$

which is the same expression as Eq. [39].

Let us consider the case of a non-phantom scalar field ($p_X > 0$). Equation [39] shows that the presence of multiple scalar fields leads to the decrease of $\lambda^2$ relative to the single-field case. The quantity $Y$ is known by Eq. [38] in terms of $\lambda x$ once $g(Y)$ is specified. Then $p_X(Y) = g(Y) + Y g'(Y)$ is determined by the quantity $\lambda x$ which is unaffected by adding scalar fields. This means that the effect of multiple scalar fields to the equation of state only appears for $\lambda^2$ in Eq. [39]. This effect works to reduce the equation of state toward $w_\phi = -1$ as $\lambda$ decreases. Hence even if inflation does not occur in the single-field case because of large $\lambda$, the presence of many scalar fields can lead to an accelerated expansion by reducing $\lambda$. Thus we have shown that assisted acceleration occurs for all dark energy models which possess scaling solutions.

In the case of a phantom field ($p_X < 0$) the presence of many scalar fields leads to the increase of $w_\phi$ toward $-1$, which is different from the assisted acceleration. In any case the equation of state approaches that of cosmological constant ($w_\phi = -1$) as we add more scalar fields.

Taking note the relation $g(Y_i) \lambda_i^2 = w_\phi_i \Omega_{\phi_i}$ and $g(Y) x^2 = w_\phi$, together with [31] and [32], the fractional energy density of each scalar field is

$$\Omega_{\phi_i} = \frac{x_i^2 \lambda_i^2}{\lambda^2} \quad (90)$$

In the two-field case, for example, one has $\Omega_{\phi_1} = \lambda_2^2/(\lambda_1^2 + \lambda_2^2)$ and $\Omega_{\phi_2} = \lambda_1^2/(\lambda_1^2 + \lambda_2^2)$. The energy density is distributed so that the condition [31] is satisfied.

B. Scaling solutions

The scaling solution satisfies $w_{\phi_i} = w_\phi = w_m$ ($i = 1, 2, \cdots , n$). Hence we do not have an accelerated expansion for a non-relativistic dark matter ($w_m = 0$). From Eqs. [77] and [83] we obtain

$$x_i = \frac{\sqrt{6}(1 + w_m)}{2 \lambda_i}. \quad (91)$$

This is similar to the single field case, see Eq. [43] with $Q = 0$. From Eq. [48], one gets $\Omega_{\phi_i} = \sqrt{6} x_i p_X / \lambda_i$. Then by using Eq. [91] we find $\Omega_{\phi_i} = 3(1 + w_m) p_X / \lambda_i^2$. This gives the total energy fraction of the scalar field:

$$\Omega_{\phi} = \sum_{i=1}^{n} \frac{3(1 + w_m) p_X}{\lambda_i^2}, \quad (92)$$

where $\lambda$ is defined in Eq. [39]. In the case of an ordinary scalar field with $p_X = 1$, this result agrees with what was obtained in Ref. [34].

In order for the scaling solution to be physically meaningful, we require the condition $\Omega_{\phi} < 1$, i.e.,

$$p_X < \frac{\lambda^2}{3(1 + w_m)}. \quad (93)$$

In the case of a non-relativistic dark matter this is equivalent to the condition [43] with $Q = 0$. Since the multi-field system is now reduced to an effectively single-field system, the stability analysis of critical points discussed in Sec. I persist in the presence case as well. Thus the scaling solution is stable if a scalar-field dominant solution with $p_X > 0$ is unstable, and vice versa. This is actually what was found in Ref. [34] for an ordinary field with an exponential potential.

As discussed in Ref. [34] we can consider an interesting situation in which the solution is in the scaling regime characterized by $w_{\phi_i} = w_m$ during radiation and matter dominant eras and then enters the regime of an accelerated expansion as more and more fields join the assisted scalar-field dominant attractor. Although this requires a fine-tuning, it is interesting that the presence of many scalar fields leads to the possibility of an accelerated expansion.

VI. CONCLUSIONS

In this paper we have provided a general method to derive critical points relevant to dark energy for the models which possess scaling solutions. The existence of scaling solutions restricts the Lagrangian density to be in the
form $p = X g(x t)$ where $g$ is an arbitrary function. This includes a wide variety of dark energy models such as quintessence, phantoms, tachyons and dilatonic ghost condensates. Since $\lambda$ is constant, we obtain autonomous equations (22) and (24) for two variables $x$ and $y$ defined in Eq. (26). For the models in which scaling solutions do not exist, one has another differential equation for $\lambda$ since $\lambda$ is a dynamically varying quantity. Even in this case we can apply the results obtained in this paper by considering “instantaneous” critical points $(x(N), y(N))$.

The fixed points relevant to dark energy are (i) the scalar-field dominant solution ($\Omega_\phi = 1$) and (ii) the scaling solution with $\rho_\phi / \rho_m = \text{constant} \neq 0$. In the former case the scalar-field equation of state is generally given by Eq. (37). The fixed points can be derived by using Eq. (10) when we specify the function $g(Y)$. Depending upon the models of dark energy the fixed points exist in the region given by $p_X > 0$ or $p_X < 0$. In some of the models like dilatonic ghost condensate there are two fixed points both in the regions $p_X > 0$ and $p_X < 0$. In the case of scaling solutions the variable $x$ and the effective equation of state $w_{	ext{eff}}$ are uniquely determined as Eqs. (14) and (18) independently of the models. The variables $g$ and $Y = x^2 / y^2$ are known by Eq. (16) once we specify the form of $g(Y)$.

The stability of fixed points was discussed in Sec. [19] in the presence of a non-relativistic barotropic fluid without specifying any form of $g(Y)$. The stability condition for the scalar-field dominant solution is given by Eq. (48) for a non-phantom field ($p_X > 0$). When this condition is satisfied, the scaling solution exists in the region which is not physically meaningful ($\Omega_\phi > 1$). When the quantity $\Delta_c$ defined in Eq. (48) is positive, the scaling solution is a stable attractor for $\Omega_\phi < 1$. The positivity of $\Delta_c$ is ensured if we impose the stability of quantum fluctuations. Hence when the stable scaling solution exists in the region $\Omega_\phi < 1$, the scalar-field dominant solution with $p_X > 0$ is unstable.

The above discussion shows that the solution chooses either the scalar-field dominant solution with $p_X > 0$ or the scaling solution with $\Omega_\phi < 1$. When there exists a phantom scalar-field dominant fixed point ($p_X < 0$), we found that this solution is classically stable. If both the scaling solution and this phantom scalar-field dominant solution are stable critical points, the solutions choose either of them depending upon initial conditions.

Finally we generalized our analysis to the case in which more than one field is present with the same form of the Lagrangian density. We showed that the system effectively reduces to that of the single-field with $\lambda$ modified as Eq. (37) for the scalar-field dominant solution. Since the presence of many scalar fields leads to the decrease of $\lambda$ toward 0, the equation of state approaches that of cosmological constant as we add more fields. Thus we have shown that for a non-phantom fluid the assisted acceleration occurs for all scalar-field dark energy models which have scaling solutions.

It is really remarkable that the derivation of critical points, their stability and assisted acceleration can be analyzed in a unified way without specifying any form of $g(Y)$. We hope that the results obtained in this paper will be useful to constrain scalar-field dark energy models from future high-precision observations. It is certainly of interest to place strong observational constraints not only from SN Ia but from CMB and LSS by using the equations of matter perturbations derived in Refs. [42].

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