UDC 512.722+512.723; MSC 14J60, 14D20, 14D06

On degeneration of surface in Fitting compactification of moduli of stable vector bundles

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Abstract

The new compactification of moduli scheme of Gieseker-stable vector bundles with the given Hilbert polynomial on a smooth projective polarized surface $(S, H)$, over the field $k = \bar{k}$ of zero characteristic, is constructed in previous papers of the author. Families of locally free sheaves on the surface $S$ are completed by the locally free sheaves on the schemes which are certain modifications of $S$. We describe the class of modified surfaces to appear in the construction.

Keywords: moduli space, semistable coherent sheaves, blowup algebra, algebraic surface.

INTRODUCTION, NOTATION AND CONVENTION

Let $S$ be a nonsingular irreducible projective algebraic surface over an algebraically closed field $k$ of characteristic zero. Fix an ample divisor class $H \in \text{Pic} S$. The symbol $\chi(\cdot)$ will denote, as usually, the Euler-Poincaré characteristic. We work with the notion of (semi)stability of a coherent torsion-free sheaf $E$ on a surface $S$ due to D. Gieseker [1].

Definition 0.1. Coherent torsion-free $\mathcal{O}_S$-sheaf $E$ is stable (respectively, semistable), if for any proper subsheaf $F \subset E$ for $m \gg 0$

$$\frac{\chi(F(mH))}{\text{rk}(F)} < \frac{\chi(E(mH))}{\text{rk}(E)}$$

(respectively, $\frac{\chi(F(mH))}{\text{rk}(F)} \leq \frac{\chi(E(mH))}{\text{rk}(E)}$).

It is well-known [2] that the structure of moduli space for semistable sheaves depends strongly on the choice of polarization $H$. The Gieseker and Maruyama moduli scheme for semistable torsion-free sheaves on the surface $S$, with Hilbert polynomial $P(m) = \chi(E(mH))$ with respect to the class $H$, is denoted by $\overline{M}$. It is known [3] that it is a projective scheme of finite type over $k$. The points corresponding to locally free sheaves (vector bundles) constitute Zariski-open subscheme $M_0$ of $\overline{M}$. Assume that $\overline{M}$ is a fine moduli space. Then there is a trivial family of surfaces $\Sigma := \overline{M} \times S \longrightarrow \overline{M}$ carrying the universal family of stable sheaves $E$. In [4, 5] a projective scheme $\widetilde{M}$ and a nontrivial flat family of (possibly reducible) schemes $\widetilde{\Sigma} \stackrel{\phi}{\longrightarrow} \overline{M}$ endowed with the family of locally free sheaves $\widetilde{E}$, are constructed. In [6]...
the analogous construction (flat families of schemes $\tilde{\Sigma}_i \overset{\tilde{\pi}_i}{\longrightarrow} \tilde{B}_i$ over étale neighborhoods $\tilde{B}_i \overset{\text{étale}}{\longrightarrow} \tilde{M}$, endowed with locally free sheaves $\tilde{E}_i$) is done for the coarse case. The scheme $\tilde{M}$ contains Zariski-open subscheme $\tilde{M}_0$ isomorphic to $M_0$. Moreover, in both (fine and coarse) cases there is a birational morphism of compactifications $\varphi : \tilde{M} \to \overline{M}$ such that $\varphi|_{\tilde{M}_0}$ is isomorphism.

**Convention 0.2.** We work in the notations for the fine case. Although, all the considerations will be valid for the coarse case as well.

The birational morphism $\varphi : \tilde{M} \to \overline{M}$ constructed in papers [4] [5], establishes the correspondence among pairs $(\tilde{S}, \tilde{E}) \in \tilde{M}$ and $\varphi(\tilde{S}, \tilde{E}) = (S, E) \in \overline{M}$. Let $\tilde{y} \in \tilde{M}$ and $\varphi(\tilde{y}) = y$. Hence we mean that the fibre $\pi^{-1}(y) = S$ of family $\Sigma$ is the image of the fibre $\tilde{\pi}^{-1}(\tilde{y}) = \tilde{S}$, and the coherent sheaf $E$ on a fibre $S$ is the image of the vector bundle $\tilde{E}$ on the fibre $\tilde{S}$.

Recall the following definition.

**Definition 0.3.** (sheaf analogue of given in [7 20.2]) Let $X$ be an algebraic scheme, $F_0, F_1$ locally free $\mathcal{O}_X$-sheaves, $\psi : F_1 \to F_0 - \mathcal{O}_X$-module homomorphism. Denote $\mathcal{F} = \text{coker } \psi, r_0 = \text{rk } F_0$. The sheaf of zeroth Fitting ideals of $\mathcal{O}_X$-module $\mathcal{F}$ is defined as

$$
\mathcal{F} \text{fit}^0(\mathcal{F}) = \text{im } (\psi') : \bigwedge^{r_0} F_1 \otimes \bigwedge^{r_0} F_0^\vee \to \mathcal{O}_X),
$$

where $\psi'$ is the associate morphism for $\psi$.

The aim of the present paper is to investigate the structure of fibres of the morphism $\tilde{\pi} : \tilde{\Sigma} \to \tilde{M}$ in general case. In [5] it is proven that the fibre at a general point $\tilde{y} \in \tilde{M}_0$ is isomorphic to $S$ and one component of the fibre at a special point $\tilde{y} \in \tilde{M} \setminus \tilde{M}_0$ is isomorphic to the blowup of $S$ in the sheaf of ideals $\mathcal{F} \text{fit}^0 \mathcal{O}_X^1(\mathcal{E}, \mathcal{O}_S)$. Now we give a description for the whole of the scheme $\tilde{S}$.

As proven in [5] Proposition 3.1, the scheme $\tilde{\Sigma}$ is given by the blowup $\sigma : \tilde{\Sigma} \to \tilde{M} \times \Sigma$ of the trivial family $\tilde{M} \times S$ in the sheaf of ideals $\mathcal{F} \text{fit}^0 \mathcal{O}_X^1(\mathcal{E}, \mathcal{O}_S)$ with $\Delta$ being the scheme-theoretic closure of the image of diagonal immersion $\tilde{M}_0 \times S \hookrightarrow \tilde{M} \times \tilde{M}_0 \times S$ in the product $\tilde{M} \times \Sigma$. The scheme $\Delta$ is isomorphic to $\tilde{M} \times S$. This means that to describe fibres of the projection $\tilde{\pi}$ it is enough to investigate fibres of the composite morphism $\pi \circ \sigma : \tilde{\Sigma} \overset{\sigma}{\longrightarrow} \tilde{M} \times S \overset{\pi}{\longrightarrow} \tilde{M}$.

The paper is organized as follows. In the first section we give some observations about the structure of singularities of semistable torsion-free coherent sheaves on a nonsingular surface. In the second section we construct a flat 1-dimensional family of coherent sheaves such that it contains $\mathcal{E}$ and its general sheaf is locally free. At last, in the third section we derive the scheme structure of $\tilde{S}$.

**Convention 0.4.** As usually we assume that $\overline{M}$ is irreducible. If not, consider each of its irreducible components containing locally free sheaves. We restrict ourselves to the case when $\overline{M}$ contains at least one locally free sheaf.
Convention 0.5. In the whole of the text we omit the subscripts in Ext’s whenever no ambiguity occur: for example, we replace Ext^1_S(E, O_S) by Ext^1(E, O_S). When working with Artinian sheaves the length l(\pi) of a sheaf \pi denotes its Euler-Poincaré characteristic: l(\pi) = \chi(\pi). The same is for zero dimensional subscheme Z \subset S: l(Z) = l(O_Z) = \chi(O_Z). As usually, we denote the Grothendieck scheme of length l zero dimensional quotients of O_S-sheaf F on S as Quot^lF. The point in the Grothendieck scheme, corresponding to the quotient q : F \rightarrow \pi, is denoted as q.

Let r be the rank of coherent sheaves E with Hilbert polynomial P(m). The final result of this paper is given by the following theorem.

**Theorem 0.6.** The fibre of the family \tilde{\pi} : \tilde{\Sigma} \rightarrow \tilde{M} at the point \tilde{y} \in \tilde{M}

i) is isomorphic to S if \tilde{y} \in \tilde{M}_0, or

ii) contains in the class of all Proj \bigoplus_{s \geq 0}(\mathcal{I}_S[t] + (t))^s/(t^{s+1}), with \mathcal{I}_S = \Fitt^0 \Ext^2(\pi, O_S) where \pi denotes the length l Artinian sheaf which is a quotient of the direct sum O_S^r, l \leq c_2, if \tilde{y} \in \tilde{M} \setminus \tilde{M}_0.

**Acknowledgements.** The author expresses her deep and sincere gratitude to M. Reid in Mathematics Research Centre of the University of Warwick, Great Britain, V. V. Shokurov and D. B. Kaledin in V. A. Steklov Institute of RAS, Moscow, Russia, and M. E. Sorokina in Yaroslavl State Pedagogical University, Yaroslavl, Russia, for discussions and comments.

1 SINGULARITY OF E

In [5] we proved that the main (dominating S) component \tilde{S}_0 of \tilde{S} is the blowup of S in the sheaf of ideals \Fitt^0 \Ext^1(E, O_S). In this section we investigate the class of sheaves who appear as \Ext^1(E, O_S) for various semistable E.

For any torsion-free sheaf E there is exact triple

\[ 0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow \pi \rightarrow 0. \] (1.1)

Since S is nonsingular surface, the double dual sheaf E^{\vee\vee} is locally free and the cokernel \pi is Artinian sheaf.

**Definition 1.1.** In (1.1) the cokernel \pi is said to be a **singularity sheaf** of E. When necessary, we reflect this fact in the notation \pi_E := E^{\vee\vee}/E.

The form of Hilbert polynomial of E is determined by the geometry of the surface S, choice of polarization H, rank r and Chern classes c_1, c_2 of the sheaf E. In any case, all possible l’s are bounded by the inequality 0 \leq l \leq c_2. But for c_2 fixed, and various S, H, r, c_1, the collections of possible l’s can be different. However, there is no explicit description of such l’s up to now.

Now recall the notion of slope-(semi)stability ascending to D. Mumford and F. Takemoto. We use the following definition.
Definition 1.2. Coherent torsion-free sheaf $E$ is \textit{slope-(semi)stable} with respect to the polarization $H$ if for any proper subsheaf $F \subset E$ the following holds:

$$
\frac{c_1(F) \cdot H}{\text{rk}(F)} < \frac{c_1(E) \cdot H}{\text{rk}(E)}, \quad \text{respectively,} \quad \frac{c_1(F) \cdot H}{\text{rk}(F)} \leq \frac{c_1(E) \cdot H}{\text{rk}(E)}.
$$

Semistability implies slope-semistability, slope-stability implies stability.

The following simple remark shows that the possible $l$’s may cover the interval $[0, c_2] \subset \mathbb{Z}$ not completely.

Remark 1.3. For $r, c_1, H$ such that $r$ and $(c_1 \cdot H)$ coprime, the slope-semistability implies slope-stability. In this case all semistable sheaves are slope-stable. Let the invariants $r, c_1, c_2, l$, $l \leq c_2$, be such that there are no slope-stable sheaves $E$ with $\text{rk}(E) = r, c_1(E) = c_1, c_2(E) = c_2$ but there exist at least one slope-stable vector bundle $F$ of the same rank and first Chern class $c_2(F) = c_2 - l$. Then for any Artinian sheaf $\mathcal{N}$ of length $l$ the slope-stable kernel $E$ of the morphism $F \rightarrow \mathcal{N}$ must not exist. This means that length-$l$-sheaves do not appear as cokernels of exact triples (1.1).

Example 1.4. These effects can be observed even in the classical case $S = \mathbb{P}^2, r = 2, c_1 = -1, c_2 = 2$. The corresponding moduli variety of semistable (stable=slope-stable) coherent sheaves contains nonempty locus $M_0$ of locally free sheaves [8, Ch. II, 4]. Assume that there is a stable sheaf $E$ with $l = 2$. Then for $E^\vee\vee$ compute Bogomolov’s discriminant: $\Delta(E^\vee\vee) = 2rc_2(E^\vee\vee) - (r - 1)c_1^2(E^\vee\vee) = -1$. This contradicts the slope-stability of $E^\vee\vee$ and shows that for nonlocally free $E$ there is only one possibility $E^\vee\vee / E \cong k$.

Definition 1.5. The Artinian sheaf $\mathcal{N}$ is said to be $(S, H, r, P(m))$-\textit{admissible} if there is an exact $\mathcal{O}_S$-triple (1.1) where coherent sheaf $E$ of rank $r$ with Hilbert polynomial $P(m)$ is semistable with respect to the polarization $H$.

Applying functor $\mathcal{E}xt^\bullet(-, \mathcal{O}_S)$ to (1.1) one gets immediately

$$
\mathcal{E}xt^1(E, \mathcal{O}_S) = \mathcal{E}xt^2(\mathcal{N}, \mathcal{O}_S)
$$

and we have

Proposition 1.6. Class of all sheaves of ideals $\mathcal{F}itt^0 \mathcal{E}xt^1(E, \mathcal{O}_S)$ for all semistable $E$ of fixed rank $r$ and Hilbert polynomial $P(m)$, contains in the class of sheaves as follows

$$
\mathcal{F}itt^0 \mathcal{E}xt^2(\mathcal{N}, \mathcal{O}_S)
$$

for all $q : \mathcal{O}_S^{\oplus r} \rightarrow \mathcal{N}, q \in \text{Quot}^l \mathcal{O}_S^{\oplus r}, l \leq c_2$.

2 REMOVABILITY OF SINGULARITY

We prove (with respect to convention 0.3) that any semistable coherent sheaf can be include into some 1-dimensional flat family of sheaves such that the general sheaf of the family is locally free.
Take $m \gg 0$ such that $E(mH)$ is globally generated. Consider the vector space $V \cong H^0(S, E(mH))$ and Grothendieck’s Quot-scheme $\text{Quot} = \text{Quot}^{P(m)}(V \otimes \mathcal{O}_S(-mH))$ parameterizing quotient sheaves

$$V \otimes \mathcal{O}_S(-mH) \rightarrow E$$

with Hilbert polynomial equal to $P(m) = \chi(E(mH))$. We work as usually with the quasiprojective subscheme $Q \subset \text{Quot}$ constituted by all quotients (2.1) with $E$ semistable and with isomorphism $H^0(S, E(mH)) \cong V$. Grothendieck’s scheme $\text{Quot}$ carries the universal quotient sheaf $E_{\text{Quot}}$, let $E_Q$ be its restriction onto $Q$. Let $Q_0$ be the open subscheme of $Q$ whose points correspond to locally free quotient sheaves $E$.

The further consideration contains the usage of Bertini’s theorem and one needs the smoothness of $Q$. If it is not so, replace it by any smooth resolution, for example due to H. Hironaka [9].

Since $Q$ is quasiprojective, there is a projective space $\mathbb{P}^N$ together with locally closed immersion $i: Q \hookrightarrow \mathbb{P}^N$. Fix any point $q$ in the image $i(Q)$ such that $q$ corresponds to $E$. Consider the set of all hyperplanes in $\mathbb{P}^N$ passing through $q$. Choose hyperplane $H_1$ such that the intersection scheme $Q_{(1)} = i(Q) \cap H_1$ containing $q$ and meeting $i(Q_0)$ is irreducible and nonsingular. It is possible by Bertini’s theorem [10, Ch. III, Corollary 7.9]. Clearly, $i(Q_0) \cap Q_{(1)}$ forms an open subset in $Q_{(1)}$. Now repeat the procedure replacing $Q$ by $Q_{(1)}$. Iterating the process one comes to $Q_{(d)}$ being irreducible curve for some $d > 0$.

### 3 STRUCTURE OF MODIFIED SURFACES

Here we derive the scheme structure of surfaces $\widetilde{S}$ as projective spectra of appropriate algebras. As usually $\mathcal{O}_{\tilde{S}}$ denotes the structure sheaf of the surface $S$ and let $\mathcal{I}_S \subset \mathcal{O}_S$ be the sheaf of ideals to be blown up in $S$.

As direct computation with blowup equations shows, the scheme $\widetilde{S}$ can carry quite sophisticated structure. The main component $\widetilde{S}_0$ admits singularities and each of other components can carry nonreduced scheme structure.

**Example 3.1.** By locality one can replace the original nonsingular surface by the affine subset $U \cong k^2 = \text{Spec} \{x, y\}$. Take $I_S = \Gamma(U, \mathcal{I}_S) = (x^2, y)$. Consider the trivial family $T_U = U \times \text{Spec} k[t] = \text{Spec} k[x, y, t]$ with natural projection $\pi: T_U \rightarrow \text{Spec} k[t]$. We blow up $T_U$ in the nonreduced point with ideal $I_S$ in the fibre over $b_0 = \{t = 0\}$. This is equivalent to the blowup $\sigma: \widetilde{T_U} \rightarrow T_U$ of the point with ideal $I = (x^2, y, t)$ on $T_U$. The scheme $\widetilde{T_U}$ is given in the direct product $T_U \times \mathbb{P}^2 \cong k^3 \times \mathbb{P}^2$ for $\mathbb{P}^2 = \text{Proj} k[u, v, w]$, by usual blowup relations: $x^2: y: t = u: v: w$. The exceptional divisor $\mathcal{E}$ of the blowup morphism $\sigma: \widetilde{T_U} \rightarrow T_U$ carries a nonreduced (“double”) structure; its equations are $x^2 = y = t = 0$. The fibre $\widetilde{S}$ of composite morphism $\pi \circ \sigma$ over $b_0$ consists of two components: $\widetilde{S} = \widetilde{S}_0 \cup \widetilde{S}_1$. In our case $\widetilde{S}_0$ has a quadratic singularity; in the affine chart $v \neq 0$, $z := u/v$ its equation in the neighborhood of singular point is $x^2 = yz$. The component $\widetilde{S}_1 = \mathcal{E}$. 
For the general consideration form a polynomial extension $\mathcal{O}_S[t]$ for $t$ transcendental over $\mathcal{O}_S$, let $(t) \subset \mathcal{O}_S[t]$ be principal ideal sheaf. Set $\mathcal{I} := \mathcal{I}_S[t] + (t) \subset \mathcal{O}_S[t]$. Set as well $T := \text{Proj} \mathcal{O}_S[t]$, and $\mathcal{O}_Z := \mathcal{O}_T/\mathcal{I}$. Clearly, $T = \text{Spec} \mathcal{k}[t] \times S$. Denote $B := \text{Spec} \mathcal{k}[t]$, and the zero point on the base $b_0 = \{t = 0\}$. Let $\pi : T \to B$ be the projection induced by the $\mathcal{O}_S[t]$ - algebra morphism $\mathcal{k}[t] \to \mathcal{O}_S[t]$. The latter morphism is obtained by the extension of the structure morphism $\mathcal{k} \to \mathcal{O}_S$.

Form a graded sheaf algebra $\mathcal{A} := \bigoplus_{s \geq 0} I_s$, and $\tilde{T} := \text{Proj} \mathcal{A}$. There is a projective (blowup) morphism $\sigma : \tilde{T} \to T$ induced by the natural $\mathcal{O}_S[t]$-algebra morphism $\mathcal{k}[t] \to \mathcal{O}_S[t]$ onto the zero graded component.

**Proposition 3.2.** The fibre of composite morphism $\sigma \pi$ at $b_0 \in B$ equals

$$\tilde{S} = (\text{Proj} \mathcal{A}) \times_T \pi^{-1}(b_0) = \text{Proj} \bigoplus_{s \geq 0} \mathcal{I}^s/(t^{s+1}).$$

**Example 3.3.** Take $\mathcal{I}_S = \mathcal{m}_p$ – the sheaf of maximal ideals corresponding to a reduced point $p \in S = \pi^{-1}(b_0)$. After the restriction to appropriate affine neighborhood $U$ one has $I = (x, y, t)$, henceforth $A = \Gamma(U, \mathcal{A}) = \bigoplus_{s \geq 0} (x, y, t)^s$, $T = \text{Spec} \mathcal{k}[x, y, t]$, $\tilde{T} = \text{Proj} \bigoplus_{s \geq 0} (x, y, t)^s$. For the exceptional divisor $E = \sigma^{-1}(p, b_0)$ one has

$$E = (\text{Proj} \bigoplus_{s \geq 0} (x, y, t)^s) \times_T (p, b_0) = \text{Proj} \bigoplus_{s \geq 0} (x, y, t)^s/(x, y, t)^{s+1}.$$

The fibre $\tilde{S} = (\sigma \pi)^{-1}(b_0)$ is given by the fibered product

$$\begin{array}{ccc}
\tilde{T} & \xrightarrow{\sigma} & T \\
\downarrow & & \uparrow \\
\tilde{S} & \xrightarrow{\sigma} & S
\end{array}$$

with $\sigma$ be the restriction of $\sigma$ onto the fibre $\tilde{S}$ over $b_0$. Passing to algebras one has that the algebra $R = \bigoplus_{s \geq 0} R_s$ for $\tilde{S}$ is the coproduct of graded algebras given by

$$R_s = (x, y, t)^s \prod_{k[x, y, t]} \mathcal{k}[x, y].$$
The push-out diagram of \( k[x, y, t] \)-modules

\[
\begin{array}{ccc}
0 & \quad & 0 \\
\downarrow & & \downarrow \\
(t) & \quad & (t) \\
\downarrow & \quad & \downarrow \\
k[x, y, t] & \xrightarrow{-t^s} & (x, y, t)^s \\
\downarrow & & \downarrow \\
k[x, y] & \rightarrow & R_s \\
\downarrow & & \downarrow \\
0 & \quad & 0
\end{array}
\]  \quad (3.2)

\[
\begin{array}{ccc}
0 & \quad & 0 \\
\downarrow & & \downarrow \\
(t) & \quad & (t) \\
\downarrow & \quad & \downarrow \\
0 & \quad & 0
\end{array}
\]

\[
0 \rightarrow k[x, t] \rightarrow (x, t)^s \rightarrow C \rightarrow 0
\]

\[
0 \rightarrow k[x, y] \rightarrow R_s \rightarrow C \rightarrow 0
\]

\[
0 \rightarrow 0
\]

\[
r_s = (x, y, t)^s/(t^{s+1}) \quad (3.3)
\]

The universal property of \( R \) as a coproduct is checked immediately.

The inclusion of the exceptional divisor \( E \) into \( \widetilde{S} \) is defined by the epimorphism of algebras

\[
\bigoplus_{s \geq 0} (x, y, t)^s/(t^{s+1}) \twoheadrightarrow \bigoplus_{s \geq 0} (x, y, t)^s/(x, y, t)^{s+1}. \quad (3.4)
\]

As well the inclusion of the main component \( \widetilde{S}_0 \hookrightarrow \widetilde{S} \) is defined by the epimorphism

\[
\bigoplus_{s \geq 0} (x, y, t)^s/(t^{s+1}) \twoheadrightarrow \bigoplus_{s \geq 0} (x, y)^s. \quad (3.5)
\]

**Proof of proposition 3.2** Let \( Z \subset T \) be zero dimensional subscheme defined by the sheaf of ideals \( \mathcal{I} \). The exceptional divisor \( E \) of the blowup \( \sigma \) is given by

\[
E := (\text{Proj} \mathcal{I}) \times_T Z = \text{Proj} \bigoplus_{s \geq 0} \mathcal{I}^s/\mathcal{I}^{s+1}.
\]

The fibered square (3.1) relates to the (sheaf) algebra

\[
\mathcal{R} = \bigoplus_{s \geq 0} \mathcal{R}_s \quad (3.6)
\]

as a coproduct

\[
\mathcal{R}_s = \mathcal{I}^s \coprod_{\mathcal{O}_S} \mathcal{O}_S, \quad s \geq 0.
\]

For (3.2) and (3.3) one has straightforward generalizations and

\[
\mathcal{R}_s = \mathcal{I}^s/(t^{s+1}). \quad (3.7)
\]
Analogously to (3.4), (3.5) there are the inclusions of exceptional divisor $E$ and of the main component $\tilde{S}_0$ defined by the sheaf algebra epimorphisms
\[
\bigoplus_{s \geq 0} \mathcal{I}^s / (t^{s+1}) \twoheadrightarrow \bigoplus_{s \geq 0} \mathcal{I}^s / \mathcal{I}^{s+1},
\]
\[
\bigoplus_{s \geq 0} \mathcal{I}^s / (t^{s+1}) \twoheadrightarrow \bigoplus_{s \geq 0} \mathcal{I}_S^s
\]
respectively. Hence, $\tilde{S} = \text{Proj } \mathcal{R}$ for $\mathcal{R}$ given by (3.6), (3.7).

Remark 3.4. Let $E_0$ be the exceptional divisor of the main component $\tilde{S}_0$, i.e. exceptional divisor for the blowup morphism $\sigma_0 = \sigma|_{\tilde{S}_0} : \tilde{S}_0 \to S$. It is easily seen that the main component $\tilde{S}_0 = \text{Proj } \bigoplus_{s \geq 0} \mathcal{I}_S^s$ of special fibre $\tilde{S}$ meets $E$ precisely at $E_0 = \text{Proj } \bigoplus_{s \geq 0} \mathcal{I}_S^s / \mathcal{I}_S^{s+1}$.

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