Strong Universality in Forced and Decaying Turbulence

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The weak version of universality in turbulence refers to the independence of the scaling exponents of the nth order structure functions from the statistics of the forcing. The strong version includes universality of the coefficients of the structure functions in the isotropic sector, once normalized by the mean energy flux. We demonstrate that shell models of turbulence exhibit strong universality for both forced and decaying turbulence. The exponents and the normalized coefficients are time independent in decaying turbulence, forcing independent in forced turbulence, and equal for decaying and forced turbulence. We conjecture that this is also the case for Navier-Stokes turbulence.

I. INTRODUCTION

The statistical theory of fluid turbulence is concerned with correlation functions of the turbulent velocity vector field $u(r, t)$ where $r$ is the spatial position and $t$ the time $t$. Since the velocity field is a vector, multi-point and multi-time correlation functions are in general tensor functions of the vector positions and the scalar times. Naturally such functions have rather complicated forms which are difficult to measure and to compute. Consequently, almost from its very beginning, the statistical theory of turbulence had been discussed in the context of an isotropic and homogeneous model. The notion of isotropic turbulence was first introduced by G. I. Taylor in 1935. It refers to a turbulent flow, in which the statistical averages of every function of the velocity field and its derivatives with respect to a particular frame of axes is invariant to any rotation in the axes. This is a very effective mathematical simplification which, if properly used, can drastically reduce the mathematical complexity of the theory. For this reason, it was very soon adopted by others, such as T. D. Kármán and L. Howarth who derived the Kármán-Howarth equation, and A. N. Kolmogorov who derived the 4/5 law. In fact, most of the theoretical work in turbulence in the past sixty years had been limited to the isotropic model.

Within the homogeneous and isotropic model there developed the notion of universality of turbulence. By universality, we mean the tendency of different turbulent systems to show, for very large Reynolds numbers $Re$, the same small-scales statistical behavior when the measurements are done far away from the boundaries. The statistical objects (see below for definitions) exhibit approximately the same scaling exponents whether they are measured in the atmospheric boundary layer, in a wind tunnel or in a computer simulation, provided they are measured far from the boundaries. Moreover, the accumulated experimental knowledge over the years indicated that not only in forced, stationary turbulence, but also in decaying turbulence, there is a regime of time where the statistical objects exhibit the same scaling properties. This phenomenon was explained by the widely separated time scales (“eddy turn over times”) that characterize large and small length scales in turbulence. While turbulence was decaying on the time scale of the large eddies, the small one had ample time to reach an “energy-flux equilibrium” that in terms of scaling behavior was indistinguishable from forced turbulence. Thus there exists a wide-spread belief that at least from the point of view of scaling exponents, forced and decaying turbulence are in the same universality class, sharing the same scaling exponents of the corresponding correlation functions.

To actually prove this type of universality in experiments and simulations is however far from straightforward. To achieve reasonable precision in the measurement of scaling exponents one needs large ranges of scales where scaling prevails, and this entails large Reynolds numbers. Unfortunately large Reynolds numbers are available usually when anisotropic effects are large, like in the atmospheric boundary layer or in large wind tunnels. Direct Numerical Simulations (DNS) can be used to eliminate anisotropy almost completely (up to lattice anisotropy which are unavoidable in simulations), but they are limited to relatively low $Re$, notwithstanding the very recent simulations of size $10^6$. Decaying turbulence is even harder to characterize precisely, since the effective Reynolds number decreases in time. Thus actual measurements of scaling properties are fraught with difficulties, corrections to scaling, effects of anisotropy and what not. As a result, over the years and also very recently, it was proposed that structure functions in forced and decaying turbulence have different exponents. In this paper we take a strong stand, proposing that the universality that actually exists in turbulence is even stronger than what has been anticipated so far.

To make our point clear, recall that the statistical description of fully developed turbulence employs correlation functions and structure functions. These are ensemble average of velocity differences across a length-scale $R$. In the theoretical studies of turbulence the two most common ensemble averages are over realizations of the forcing when one studies forced turbulence, or over initial conditions when one studies decaying turbulence. The longitudinal structure functions are the simplest such objects, being moments of the longitudinal components of the ve-
locity difference between two points. We will denote the longitudinal structure functions in forced and decaying turbulence by $S_p$ and $F_p$ respectively, with the precise definitions

\[ S_p(R, t) \equiv \langle [(u(r + R, t) - u(r, t)] \cdot \frac{R}{R} \rangle_t \], \quad (1) \]

\[ F_p(R, t) \equiv \langle [(u(r + R, t) - u(r, t)] \cdot \frac{R}{R} \rangle_t \]. \quad (2) \]

Here $u(r, t)$ is the velocity field measured at point $r$ at time $t$. $\langle \ldots \rangle_t$ and $\langle \ldots \rangle_i$ stand for ensemble averaging over the forcing and the initial conditions respectively. In writing Eq. 1 we assumed that the forcing is stationary in time, homogeneous and isotropic, and thus $S_p$ is a function of the scalar $R$ only. In writing Eq. 2 we assumed that the initial condition are homogeneous and isotropic. Of course the decaying structure functions are by definition time dependent. The widely spread belief \[ 1, 4, 10 \] is for $R$ values in the inertial range of turbulence (much smaller than the forcing scale but much larger than the dissipation scale), the scaling exponents $\zeta_p$ that characterize $S_p(R)$, i.e. $S_p(\lambda R) = \lambda^{\zeta_p} S_p(R)$, are the same as the scaling exponents that characterize $F_p(R, t)$ for a given value of $t$. Of course also here $R$ should be well in the (time dependent) inertial range and $t$ should be neither too small nor too large \[ 11 \]. In the sequel we refer to the identity of the only scaling exponents of these two sets of objects (if it exists) as “the weak version of universality”. As mentioned above the existence of the weak version of universality is by no means accepted by everybody in the field of turbulence. Since there is no proof of this universality, doubts of its existence linger, and for example in \[ 8 \] it was concluded that the scaling exponents of the two families of statistical objects are not the same. We note however that in the same paper it was stated that the scaling exponents of the longitudinal and transverse structure functions are also not the same. It was shown recently however that such statements stem from incomplete treatments of the effects of anisotropy \[ 12, 13 \], leaving hope that the weak version of universality is still correct.

In fact, in this paper we will propose that not only the weak version of universality is applicable, but in fact also a “strong version of universality” is applicable. By the latter we mean that once properly normalized, the structure functions $F_p$ and $S_p$ agree not only in exponents but also in amplitudes. In the context of the 2nd order structure function this is not a new statement. The universality of ($C_2$ and $C_2$) was already stated in the 80’s by Yaglom \[ 14 \] and Kader \[ 15 \]. Analyzing hundreds of experiments made in different flows under different conditions, Sreenivasan in 1995 came to the conclusion that “the Kolmogorov constant $C_2$ is more or less universal, essentially independent of the flow as well as the Reynolds number (for $R_\lambda > 50$ or so)... with the average value of $C_2 \approx 0.53$ with a standard deviation of about 0.055” \[ 16 \]. Nevertheless the universality of $C_2$ and $C_p$ for $p \leq 4$ is still under debate. For example, very recently in \[ 17 \] the authors argued on the basis of a 256$^3$ DNS “in favor of an “exponents only” universality scenario for forced turbulence”.

We believe that this strong version of universality was never stated before, and the common thinking is that amplitudes depend in a non-universal way on details of the forcing or the preparation of the decaying turbulence. While true, we will argue that the freedom afforded by such details is very limited, amounting at the end to just one free number, which, once taken into account, the strong version of universality applies.

Besides being an issue of fundamental importance to turbulence, there is another reason for returning at this time to the correspondence between force and decaying turbulence. The reason is that the riddle of anomalous scaling of correlation and structure functions in forced turbulent advection (passive and active) had been solved recently. First in the context of the non-generic Kraichnan model of passive scalar advection \[ 18 \], and then, in steps, for passive vectors \[ 18, 20 \], generic passive scalars and vectors \[ 21, 22, 23 \] and finally for generic active scalar and vectors \[ 24, 25, 26 \]. The common thread of this advance is that anomalous scaling is discussed in the context of the decaying (unforced problem), in which one shows that there exist Statistically Preserved Structures (eigenfunctions of eigenvalue 1 of the appropriate propagator of the decaying correlation functions). The decaying problem is independent of forcing, and one shows that the statistics of the forced problem is dominated by the same Statistically Preserved Structures that are identified in the decaying problem. The calculation of the anomalous exponents boils down then to calculating eigenfunctions of linear operators. In these problems the correspondence between the decaying and forced statistics is proven mathematically or demonstrated beyond reasonable doubt by careful numerics. A crucial ingredient in all this progress is that turbulent advection is described by linear pde’s. There is therefore an urgent question how to translate (if it is possible) the newly acquired insights to the non-linear turbulent problem itself, be it the Navier-Stokes equations or any of the shell models that were frequently discussed recently in the context of anomalous scaling. In this paper we make a step in this direction, analyzing the decay of the Sabra shell model \[ 27 \] and showing numerically that the statistics of the decaying state and the forced turbulent state are the same in exponents and in amplitudes up to one freedom (the time dependent mean energy). We opt to work with the shell model rather than the Navier-Stokes equations simply because the accuracy required for our aims exceeds the available scaling ranges and decay times for the latter. We express a strong belief that very similar results can be demonstrated also for Navier-Stokes turbulence. Indeed, in a future publication we will present the theory that stands behind the present numerical findings and demonstrate that the basic structure of that theory is the same for shell models and the Navier-Stokes equations.

The paper is organized as follows: in Sec. \[ 11 \] we intro-
duce the shell model and the numerical simulations that we perform. We present the data for the energy decay and explain what is the time domain for which we should compare the decaying and the forced statistics. In Sec. III we present the results for forced structure functions for different types of forcing. In determining the exponents and the amplitudes of these functions one has to be extra careful - we explain that one needs to find fits to functions throughout their range of existence. It is not enough to plot log-log plots for the inertial range. In Sect. IV we present the data for the decaying correlation functions, and explain how to find their exponents and amplitudes once the time dependence is taken into account. We explain theoretically that the decaying structure functions contain sub-leading contributions that decay fast toward small scales and do not affect the leading scaling exponents. The results of our calculations are summarized in Table 1 which is the central result of this paper, giving strong support to the conjecture of strong universality. In Sec. V we present a summary and some concluding remarks.

II. MODEL AND ENERGY DECAY

A. Model and objectives

The Sabra shell model \[27\], like all shell models, is a reduced dynamical description of turbulence in terms of complex variables \(u_n\) which represent velocity amplitudes associated with wavenumber \(k_n = k_0 \lambda^n\). The equations of motion are

\[
\frac{du_n}{dt} = i(a k_{n+1} u_{n+2} u_n^* + b k_n u_{n+1} u_n^* - c k_{n-1} u_{n-1} u_{n-2}) - \nu k_n^2 u_n + f_n ,
\]

where the symbol * stands for complex conjugation and \(\nu\) is the “viscosity”. The coefficients \(a, b\) and \(c\) are chosen such that \(a + b + c = 0\). This guarantees the conservation of the “energy”

\[
E = \sum_n |u_n|^2 ,
\]

in the inviscid (\(\nu = 0\)) forceless limit. As it is well known, the Sabra model has a second quadratic invariant, analogous to the helicity in fluid mechanics, of the form

\[
H = \sum_n (a/c)^n |u_n|^2 .
\]

In this paper we will compare the statistics of the forced solution (with the forcing \(f_n\) restricted to the first and second shells, \(n = 1, 2\)) to the statistics of the decaying problem with \(f_n = 0\) for all \(n\). The comparison will be presented in terms of the time-independent forced structure functions \(S_n\) and time dependent decaying structure functions \(F_n\) defined as follows:

\[
S_2(k_n) = \langle |u_n|^2 \rangle , \quad F_2(k_n, t) = \langle |u_n|^2 \rangle ,
\]

\[
S_3(k_n) = \text{Im}(u_{n-1} u_n u_{n+1}^*) , \quad F_3(k_n, t) = \text{Im}(u_{n-1} u_n u_{n+1}^*) ,
\]

\[
S_4(k_n) = \langle |u_n|^4 \rangle , \quad F_4(k_n, t) = \langle |u_n|^4 \rangle ,
\]

\[
S_6(k_n) = \langle |u_n|^6 \rangle , \quad F_6(k_n, t) = \langle |u_n|^6 \rangle .
\]

Here \(\langle \ldots \rangle_t\) and \(\langle \ldots \rangle_i\) represent averaging with respects to realizations of the forcing and the initial condition respectively for the forced and decaying problem.

The main result of the present work is that in this model there exists strong universality. This means that in the bulk of the inertial interval the “decaying” structure functions \(F_p(k_n, t)\) take on the form

\[
F_p(k_n, t) = C_p \left( \frac{\bar{\varepsilon}(t)}{\kappa_0} \right)^{p/3} \lambda^{-n} \zeta_p ,
\]

with the same anomalous scaling exponents \(\zeta_p\) and the same dimensionless constants \(C_p\), as in the scaling laws of the “forced” structure functions

\[
S_p(k_n, t) = C_p \left( \frac{\varepsilon(t)}{\kappa_0} \right)^{p/3} \lambda^{-n} \zeta_p .
\]

Here

\[
\bar{\varepsilon}(t) = \langle \varepsilon(t) \rangle_t , \quad \bar{\varepsilon} = \langle \varepsilon(t) \rangle_i .
\]

In these equations the instantaneous value of the energy flux going through \(n\)th shell (in a given realization) is

\[
\varepsilon_n(t) = 2 k_n \left[ -a \lambda \text{Im}(u_n(t) u_{n+1}(t) u_{n+2}^*) + c \lambda \text{Im}(u_{n-1}(t) u_n(t) u_{n+1}^*) \right]
\]

(for more details, see \[27\]). The only difference between Eqs. 7 and 8 is that the energy flux \(\bar{\varepsilon}(t)\), averaged over the statistics of the initial conditions, is decaying in time, while the energy flux \(\varepsilon(t)\), averaged over the statistics of the forcing, is time independent. Eqs. 7 and 8 imply that the probability distribution function (PDF) of the velocity fluctuations \(u_n\) on scales \(k_n\) within the inertial range in decaying turbulence can be obtained from the corresponding PDF in stationary forced turbulence (and vice versa). This is achieved simply by the interchange \(\bar{\varepsilon} \leftrightarrow \varepsilon\). Moreover, strong universality means that the PDF dependence on \(\bar{\varepsilon}\) for the fine scales in decaying turbulence is independent of the initial conditions, providing that the Reynolds number \(Re \gg 1\) and all the initial energy is concentrated in the region of large scales (first shells). Similarly, the PDF dependence on \(\varepsilon\) for the fine scales in forced turbulence is independent of the statistics of forcing for \(Re \gg 1\), provided that the forcing is concentrated in the region of large scales.

The rest of this paper is devoted to substantiating these (strong) propositions, which can be summarized...
in the terms of the existence of a probability distribution function
\[ P(u_1, \ldots, u_N, t) = [v(t)]^{-N} P(x_1, \ldots, x_N), \quad (11) \]
in which \( x_i = u_i / v(t) \) and \( v(t) \propto [\epsilon_n(t)]^{1/3} \) is the corresponding velocity scale.

B. Simulations

The calculations presented below were carried out for the Sabra model with 28 shells, \( \lambda = 2, a = 1, b = c = -0.5, k_0 = 2^{-4} \) and \( \nu = 10^{-7} \). In our simulations we employed two different types of forcing. We denote them as Forced 1 and Forced 2. Forced 1 has white noise added to the equation of the first shell. Forced 2 is forced by a Gaussian force on the first shell which is correlated exponentially in time. In both cases the amplitude of the force in the first shell was chosen \( f_1 = 0.01 \), while the forcing amplitude in the second shell was adjusted to reduce the helicity input. The calculations presented below were carried out for the first two shell velocities were defined as \( |u_1(0)|^2 = \alpha E_0 \) and \( |u_2(0)|^2 = (1 - \alpha) E_0 \) with \( \alpha \) random, uniformly distributed in the interval \([0,1]\). The phases in both shells were random, uniformly distributed in the interval \([0, 2\pi]\). A 4th order Runge-Kutta scheme with adaptive time step was applied. The total energy decay was followed for 9 decades in time. The statistical objects were accumulated during 5 decades in time.

Two recording schemes for the decaying turbulence were applied. In one case (denoted as Decay 1), the data were recorded starting after a short transient time with \( E_0 = 10 \). For this case, the data was averaged over 13200 initial conditions. In another case (Decay 2), the data were recorded when the energy in each realization have reached the value \( E = 0.1 \) with \( E_0 = 5 \). This data was averaged over 37000 initial conditions. The decay of the total energy for two cases, plotted with an appropriate time shift, are shown in Fig. 1. It is clear that the two schemes are equivalent for the study of the advanced stages of the decay.

C. The law of energy decay

We first discuss the total energy decay, where the total energy \( E(t) \) is defined as
\[ E(t) = \sum_n |u_n|^2. \quad (12) \]

In the Navier-Stokes case the law of energy decay had been intensively studied following the influential works of Taylor [2], Kolmogorov [4, 5], Batchelor and Townsend [10, 28]. For recent development, see, e.g. Ref. [8] and references therein. It was found that \( E(t) \propto t^{-n} \) with the decay exponent, \( n \), ranging between 1 and 5/2. This large degree of uncertainty stems from difficulties in pinpointing the energy spectrum at scales larger than the energy containing scale \( L \). It is also not easy to determine how \( L \) depends on time. Take for example the case of grid turbulence in a wind tunnel. Immediately behind the grid \( L \) is of the order of the mesh size. It increases however with the distance from the grid. Downstream \( L \) may saturate at the wind tunnel diameter. In this regime the phenomenological analysis [8] predicts \( n = 2 \), which is a number that is not in contradiction with experiments [11]. The same prediction \( (n = 2) \) was reached in DNS of the Navier Stokes equation [7] and for the GOY shell model [29]. This prediction was shown to be in agreement with numerical simulation in which \( L \) is time independent due to the special choice of initial conditions.

For the sake of completeness we review the theoretical analysis of [29] and show that our simulations of the Sabra shell model are in excellent agreement with its predictions. Consider a decaying solution with the energy initially concentrated, say, in the first two shells, see Fig. 2, upper panel. Time is measured in natural time units \( T \) which are determined by the characteristic time of the first shell, \( T = 1/5 k_0 \sqrt{E_0} \) (\( E_0 \) is the total initial energy). One sees that during one \( T \) the energy cascaded down to the 6th shell, and during 2\( T \) down to the 12th shell. At later times the cascade process accelerates, and the energy goes from the 13th shell to “infinite” shells during a time that is roughly between 2.75\( T \) and 3\( T \). As expected, the completion of the cascade process requires a finite time \( T_\ast \) of a few units \( T \). In our case \( T_\ast \approx 3T \).

For \( t > T_\ast \) the total energy of the system \( E(t) \) begins
FIG. 2: The behavior of $F_2$. Development of the energy cascade (upper panel). The decay phase (lower panel).

to decay with

$$\frac{dE(t)}{dt} = -\bar{\varepsilon}_i(t). \quad (13)$$

Accordingly, the time dependent Reynolds number $Re(t)$, which is proportional to $\sqrt{E(t)}$, decreases, and the viscous cutoff $k_d(t)$ is moving toward smaller shell numbers, as is shown in Fig. 2 lower panel. For example, $k_d(10^5T) \approx 2^{12}$, $k_d(10^8T) \approx 2^3$ and the inertial interval almost disappears. For larger times all the energy is contained in the first shell, and it decays exponentially,

$$E(t) \propto \exp[-2\nu k_0^2 t], \quad (14)$$

following the linear part of the equation of motion for the first shell.

For intermediate times, which in our simulations span the eight orders of magnitude for $3T < t < 10^8T$, the slope of plots of $\log_2 F_2(k_n)$ vs $n$ remains more or less constant. This is a manifestations of the time independence of the scaling exponents. Taking this as a fact, one immediately see from Eq. (12) for $p=2$ that $E(t) \propto [\bar{\varepsilon}_i(t)]^{2/3}$ and hence $\bar{\varepsilon}_i(t) \propto [E(t)]^{3/2}$. Note that this result is independent of the precise value of the scaling exponent $\zeta_2$, anomalous or not. Thus Eq. (13) can be presented as

$$\frac{dE(t)}{dt} = -\kappa [E(t)]^{3/2}, \quad (15)$$

with a pre-factor $\kappa$, which may be expressed via the parameters of the shell model, $k_0$, $\zeta_2$ and $C_2$. Approximately, $\kappa \approx k_0$. In our calculations, both Decay 1 and Decay 2, $\kappa = 0.0687$, while $k_0 = 0.0625$. The solution of Eq. (15) is

$$E(t) = E_\ast \frac{t^2}{(t + t_\ast)^2}, \quad (16)$$

where $t_\ast = 2/\kappa \sqrt{E_\ast}$ and $E_\ast$ is the integration constant.

The results of the numerical simulations for the total energy, $E(t)$, (cf. Fig. 3 solid line), are in excellent agreement with Eq. (16), which is shown in the figure as a dashed line. The total energy decay was followed for 9 decades in time. After 1.5 decades of transient behavior, the decay of total energy follows very closely the $t^{-2}$ law, until at about 6 decades the viscous scale reaches the first shells and the decay become exponential in agreement with Eq. (14).

III. FORCED STRUCTURE FUNCTIONS

In this section we present results for the forced structure functions. As far as the scaling exponents are concerned, there is not much novelty in this section, the exponents are basically the same as those reported in
a number of previous publications. The new aspect stressed here is that of the coefficients $C_p$ of the structure functions, cf. Eq. (5). We demonstrate that these coefficients are independent of the forcing, and that the scaling form proposed in Eq. (8) is indeed universal.

To get an accurate determination of the scaling exponents and of the coefficients, and to be able to demonstrate strong universality, it is mandatory to fit the measured data to model functions that contain the presumed dissipative behavior. Failing to do so results in inaccuracies that may lead to infinite confusions. As a first step in analysis of the data we fit the normalized structure functions $S_{p}/E^{n/2}$ with the fit formula Eq. (44) from [27]:

$$P_p(k_n) = A_p k_n^{\zeta_p} \left( 1 + \alpha_p \frac{k_n}{k_{d,p}} \right)^{\mu_p} \exp \left[ - \left( \frac{k_n}{k_{d,p}} \right)^x \right],$$

(17)

where $A_p, \zeta_p, \alpha_p, \mu_p, k_{d,p},$ or $n_{d,p} \equiv \log_2 k_{d,p}$ are the fit parameters and $x = \log_3(1 + \sqrt{5})/2$ is the exponent of the viscous range. The parameters $A_p, \zeta_p$ determine the behavior of $P_p(k_n)$ in the inertial interval, $k_{d,p}$ determines the viscous cutoff. The “auxiliary parameters” $\alpha_p, \mu_p$ correct the behavior in the transient inertial-viscous region. To obtain the best fit we minimize the error function:

$$E = \sqrt{\sum_n \left( 1 - \frac{\log_{10} P_p(k_n)}{\log_{10} S_p(k_n)} \right)^2},$$

(18)

where $S_p$ refers to the numerically obtained data. Both sets of forced structure functions data were fit with all the shells taken into account except the first two and the last three shells, to minimize the boundary effects.

The quality of the fit may be seen in Fig. 4. The forced data are shown normalized by the respective total energy, but not compensated by $k_n^{\zeta_p}$. Then the different structure functions are separated and data for Force 1 and Force 2 cases may be distinguished.

The fit procedure allows us to express the structure function in the inertial range as $S_p = A_p E^{n/2} k_n^{\zeta_p}$. To calculate the coefficients $C_p$ we have now estimate the value of the energy flux $\tilde{\epsilon}_I$ [see Eq. (5)]. We use the exact result for $S_3$ to express $\tilde{\epsilon}_I$ via $A_3$ and parameters of the model Eq. (20). The coefficient $C_p$ of the structure functions, other than $S_3$, may be therefore written as:

$$C_p = \frac{\alpha_p}{(2a_3(a-c)^{n/3}}.$$  

The results are summarized in the Table [2]. Before we discuss this Table, which is the central result of this paper, we turn to the analysis of the decaying structure functions and add their analogous results to the Table as well.

IV. DECAYING STRUCTURE FUNCTIONS

In this section we present results for the decaying structure functions, including the numerical support for the strong universality proclaimed in Eq. (7). We caution the reader (and whoever wants to repeat these calculations in other systems, including Navier-Stokes decaying turbulence), that the issue is fraught with sub-leading contributions, even in the isotropic sector. One obvious sub-leading term is provided by the rate of change of $F_p(k_n, t)$ which is coupled, via the infinite hierarchy of equations, to terms involving $F_{p+1}(k_m, t)$, with $m$ of the order of $n$. To see this phenomenon clearly and to learn how to take it into account we discuss first the case of the 3rd order structure function which can be dealt with analytically.

A. Sub-leading corrections to the scaling of decaying turbulence

The easiest case for theoretical analysis of the scaling behavior of the decaying structure functions $F_p(k_n, t)$ is the case $p = 3$, for which in the forced case $S_3(k_n)$ is known exactly [27]. For simplicity we will discuss here the helicity-free case, for which

$$S_3(k_n) = \tilde{\epsilon}_I / k_n(c-a).$$

(20)

We will show now that in the decaying case strong universality is realized, but only well within the inertial range. In the vicinity of the energy containing scales there are significant sub-leading corrections caused by the time dependence of $\tilde{\epsilon}_I(t)$.
dependence, parameters in Eq. (17) are found to be in the intervals \( \alpha_p \sim 0.5 \div 2 \) and \( \mu_p \sim 0.6 \div 2.3 \). The error bars for each parameter correspond to the error function \( E \), Eq. (13) equal \( \approx \sqrt{2E_{\text{min}}} \) with all other parameters set to their optimal values.

| Time   | \( C_2 \)    | \( C_4 \)    | \( n_{1,2} \) | \( C_6 \)    | \( n_{4,6} \) |
|--------|--------------|--------------|--------------|--------------|--------------|
| Forced | 0.73 ± 0.07  | 17.0         | 0.60 ± 0.08  | 16.24        | 0.62 ± 0.15  |
| Forced | 0.73 ± 0.07  | 17.0         | 0.60 ± 0.08  | 16.24        | 0.61 ± 0.15  |
| Decay 1 | 0.72 ± 0.05  | 17.2         | 0.62 ± 0.06  | 17.0         | 0.79 ± 0.15  |
| Decay 1 | 0.72 ± 0.06  | 16.2         | 0.62 ± 0.07  | 16.0         | 0.79 ± 0.15  |
| Decay 1 | 10^3        | 13.5         | 0.61 ± 0.08  | 13.2         | 0.77 ± 0.15  |
| Decay 1 | 10^4        | 11.0         | 0.61 ± 0.09  | 10.2         | 0.75 ± 0.15  |
| Decay 2 | 100         | 15.5         | 0.6 ± 0.2    | 15.2         | 0.70 ± 0.20  |
| Decay 2 | 10^3        | 13.3         | 0.6 ± 0.15   | 13.1         | 0.66 ± 0.20  |
| Decay 2 | 10^4        | 10.7         | 0.6 ± 0.2    | 10.5         | 0.72 ± 0.25  |

To find these corrections, consider the equation of motion of the 2’nd order structure in the inertial interval (i.e. for \( k_n \ll k_d \) (Eq. (9) of Ref. [27]):

\[
\frac{dF_2(k_n,t)}{2k_n dt} = a\lambda F_3(k_n+1,t) + bF_3(k_n,t) + \frac{c}{\lambda}F_3(k_n-1,t)
\]  

In the stationary case, the LHS of this equation vanishes and Eq. (20) is a solution. In the decaying case, let

\[
F_3(k_n) = F_3^{(0)}(k_n,t) + \delta F_3(k_n,t),
\]

\[
F_3^{(0)}(k_n) = \frac{\xi(t)}{k_n(c-a)}
\]

In the intermediate time regime \( F_2(k_n,t) \propto (t + t_*)^{-2} \),

\[
\frac{dF_2(k_n,t)}{2k_n dt} \approx -F_2(k_n,t)/k_n (t + t_*)
\]  

Comparing with Eq. (21), we see that the leading solution for \( F_3 \) gains a sub-leading term \( \delta F_3(k_n,t) \propto k_n^{(1+\zeta_2)} \) or more precisely

\[
\delta F_3(k_n,t) \approx \frac{F_2(k_n,t)}{(t + t_*) k_n (a\lambda^{-\zeta_2} + b + c\lambda^{\zeta_2})}.
\]  

Notice, that \( F_3^{(0)}(k_n,t) \) and \( \delta F_3(k_n,t) \) have the same time dependence, \( \propto (t + t_*)^{-3} \), but different scaling. As a result, their ratio is time independent; the sub-leading term does not become relatively smaller in time. On the other hand it decays relatively to the leading term as \( k_n \) increases:

\[
\frac{\delta F_3(k_n,t)}{F_3^{(0)}(k_n,t)} \propto \frac{1}{\lambda^{n\zeta_2}}.
\]  

Although the 3rd order structure function is easiest to handle, it is clear that there will always be a subleading term added to \( F_p \) from the time derivative of \( F_{p-1} \) which appears in the infinite hierarchy of equations. Since these equations always have \( k_n \) on the RHS, one can immediately guess the general form of the correction to scaling for \( F_p \), i.e.

\[
\frac{\delta F_p(k_n,t)}{F_p(k_n,t)} \propto \frac{1}{\lambda^{n(1+\zeta_p-\zeta_p)}}.
\]  

This means that the sub-leading term of the \( p \)-order structure functions decreases toward small scales roughly as \( \lambda^{-2n/3} \) for “normal” K41 scaling, and somewhat slower for anomalous scaling. Strong universality of the turbulent statistics is thus expected only deeply in the inertial interval.

### B. Numerical results

All calculated statistical objects were normalized by the total energy, \( F_p(k_n,t)/E^{p/2}(t) \). All the normalized, compensated decaying structure functions show a plateau. To enrich the statistics the data were first normalized by the total energy and then averaged over one tenth of a temporal decade. For the same reasons that were explained in the case of the forced objects, the fit region for the decaying structure functions was chosen from \( n = 3 \) to \( n = n_{4,p} + 5 \) for the Decay 1 data and from \( n = 5 \) to \( n = n_{4,p} + 5 \) for the Decay 2. For \( t = 10^5 T \) \( n_{4} \approx 7 \) and only very few shells may be considered as the “inertial interval”. Therefore the fit parameters become unreliable. The quality of the fit can be seen in Fig. 4 for \( t \) between \( 20 T \) and \( 10^4 T \). For \( t \leq 20 T \) the flux equilibrium cannot be guaranteed and the coefficients \( C_p \) and the exponents may be not universal. This is definitely the case for \( t \leq 10 T \). The decaying structure functions are plotted normalized by the total energy and compensated to emphasize the fact that main effect is the shift of \( k_{d,p} \).
FIG. 5: The normalized compensated decaying structure functions for Decay 1 data averaged over 13200 initial conditions and over one tenth of the decade in time. The symbols show the calculated data at different times (defined in the legend). The dashed lines denote the fits for the corresponding structure functions. The best fit scaling exponents, used for the compensation ($\zeta_2 = 0.728$, $\zeta_3 = 1.1$, $\zeta_4 = 1.254$ and $\zeta_6 = 1.72$) are the same as in the forced case. The Decay 2 data show similar behavior.

V. SUMMARY AND DISCUSSION

In summary, we analyzed, using the Sabra shell model of turbulence, “forced” structure functions $S_p(k_n)$ for two types of forcing, and “decaying” structure functions $F_p(k_n, t)$ for two types of initial conditions, (called Decay 1 and Decay 2). For Decay 1 we considered four times, which differ in order of magnitudes ($t = 20T, 10^2T, 10^3T, \text{and} 10^4T$, where $T$ is the characteristic time of the 1st shell). For Decay 2 we considered three different times $t = 10^2T, 10^3T, \text{and} 10^4T$. In all these cases we found the scaling exponents, $\zeta_p$, and the dimensionless amplitudes, $C_p$, for the three even orders $p = 2, 4$ and 6. The results are collected in Table 1 together with our estimates of the error bars.

We can state that our results support the conjecture of strong universality within the numerical accuracy. Concerning the fact that before the data analysis presented above the raw results contained objects differing by orders of magnitude, the degree of precision of the identity of the amplitudes $C_p$, and the exponents $\zeta_p$, shown in Table 1 should be taken very seriously. We propose that all the results presented by previous authors with negative indications about universality (even of the weak type) stem from problems in handling the corrections to scaling, either from anisotropy or from dissipative or other boundary effects.

It should be stressed at this point that the strong universality observed here is not expected in the much simpler problem of turbulent advection. The difference stems from the linearity of the advection problem vs. the nonlinearity of the Navier-Stokes problem and its
shell counterpart. In the linear advection problem one finds equations for the statistical objects that decouple for each order $p$. The independent $p$-order equations determine the anomalous scaling exponent $c_p$ from solvability conditions, leaving the amplitude $C_p$ to be found by matching the scale invariant correlation function in the inertial interval with its non-universal “boundary conditions” at energy contained scales. The amplitudes $C_p$ depend therefore on the details of the non-universal forcing, and the statistics of the turbulent advection problem may exhibit weak universality only. In contrast, the nonlinearity of the Navier Stokes equations and their shell-model counterparts leads to coupled, hierarchical equations for all the $p$-order statistical objects that have to be solved for simultaneously. This rigid structure allows much less freedom than the linear advection problem, leading to the possibility of strong universality.

Finally, we comment on a possible theoretical support for the strong universality conjecture. We propose that a necessary condition for strong universality is the locality of interaction, which allows to formulate (see Refs. [30, 31, 32]) the hierarchy of equations in terms of $\zeta$ for each order $p$. The independent $p$-order equations determine the $\zeta$-order statistical objects that have to be solved over scales, which is built in the shell models of turbulence, is an assumption in the Richardson-Kolmogorov cascade picture of turbulence, see, e.g. [14]. The locality of interaction was demonstrated in Ref. [33], using the Belinicher-L’vov transformation of the Navier-Stokes equations [34], which allows to eliminate from the theory the sweeping effect. Once we have a theory in terms of inertial-range objects, it is quite acceptable that amplitudes should be universal as well, up to an overall single parameter which is the energy flux per unit time and mass. An elaboration of these ideas will be presented in a future publication. At this point we finish with the conjecture that strong universality is a property shared also by Navier-Stokes turbulence.

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