INCOMPRESSIBLE LIMIT OF THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. This paper is concerned with the incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. It is rigorously shown that the weak solutions of the compressible magnetohydrodynamic equations converge to the strong solution of the viscous or inviscid incompressible magnetohydrodynamic equations as long as the latter exists both for the well-prepared initial data and general initial data. Furthermore, the convergence rates are also obtained in the case of the well-prepared initial data.

1. Introduction

Magnetohydrodynamics (MHD) studies the dynamics of compressible quasineutrally ionized fluids under the influence of electromagnetic fields. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas. The compressible viscous MHD equations in the isentropic case take the form (see, e.g., [14, 15, 21])

$$\begin{align*}
\partial_t \hat{\rho} + \text{div} (\hat{\rho} \hat{u}) &= 0, \\
\partial_t (\hat{\rho} \hat{u}) + \text{div} (\hat{\rho} \hat{u} \otimes \hat{u}) + \nabla \hat{P} &= (\text{curl} \hat{H}) \times \hat{H} + \hat{\mu} \Delta \hat{u} + (\hat{\mu} + \hat{\lambda}) \nabla (\text{div} \hat{u}), \\
\partial_t \hat{H} - \text{curl} (\hat{u} \times \hat{H}) &= -\text{curl} (\hat{\nu} \text{curl} \hat{H}), \\
\text{div} \hat{H} &= 0.
\end{align*}$$

(1.1)  (1.2)  (1.3)

Here $x \in \mathbb{T}^d$, a torus in $\mathbb{R}^d$, $d = 2$ or $3$, $t > 0$, the unknowns $\hat{\rho}$ denotes the density, $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_d) \in \mathbb{R}^d$ the velocity, and $\hat{H} = (\hat{H}_1, \ldots, \hat{H}_d) \in \mathbb{R}^d$ the magnetic field, respectively. The constants $\hat{\mu}$ and $\hat{\lambda}$ are the shear and bulk viscosity coefficients of the flow, respectively, satisfying $\hat{\mu} > 0$ and $2\hat{\mu} + d\hat{\lambda} > 0$; the constant $\hat{\nu} > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. $\hat{P}(\hat{\rho})$ is the pressure-density function and here we consider the case

$$\hat{P}(\hat{\rho}) = a\hat{\rho}^\gamma,$$

(1.4)

where $a > 0$ and $\gamma > 1$ are constants.

The well-posedness of the Cauchy problem and initial boundary value problems for (1.1)-(1.3) has been investigated recently. The global existence of weak solutions to the compressible MHD
equations with general initial data was obtained by Hu and Wang \[9, 10\] (also see \[6\] on “variational solutions”). From the physical point of view, one can formally derive the incompressible models from the compressible ones when the Mach number goes to zero and the density becomes almost constant. Based on this observation, Hu and Wang \[11\] proved the convergence of the weak solutions of the compressible MHD equations \[13, 14\] to a weak solution of the viscous incompressible MHD equations. Jiang, Ju and Li \[12\] obtained the convergence towards the strong solution of the ideal incompressible MHD equations in the whole space by using the dispersion property of the wave equation if both the shear viscosity and the magnetic diffusion coefficients go to zero.

In this paper, we shall extend the results on the Cauchy problem in \[12\] to the periodic case. First, we consider the well-prepared initial data for which the oscillations will never appear. We will rigorously show the weak solutions of the compressible MHD equations converge to the strong solution of the ideal incompressible MHD equations in the periodic domain if both the shear viscosity and the magnetic diffusion coefficients go to zero, as well as to the strong solution of the viscous incompressible MHD equations. Furthermore, we shall also give the rates of convergence which are not obtained in \[11, 12\]. Secondly, we consider the case of general initial data. For this case the oscillations (acoustic waves) will appear. Comparing with \[12\] where the Cauchy problem was dealt with, the acoustic waves in the current situation will lose the dispersion property and will interact each other. Thus, here we have to impose more regular conditions than $L^2$ on the initial data to control the oscillating parts. In addition, we have to assume that the Sobolev norm of the oscillating parts is comparable to the magnetic diffusion coefficient in order to deal with the general initial data. We will rigorously prove the convergence of the weak solutions of the compressible MHD equations to the strong solution of the incompressible MHD equations, as well as to the strong solution of the partial viscous incompressible MHD equations.

To begin our argument, we first give some formal analysis. Formally, by utilizing the identity

$$
\nabla(|\tilde{\mathbf{H}}|^2) = 2(\mathbf{H} \cdot \nabla)\tilde{\mathbf{H}} + 2\tilde{\mathbf{H}} \times \text{curl} \tilde{\mathbf{H}},
$$

we can rewrite the momentum equation (1.2) as

$$
\partial_t(\tilde{\rho}\tilde{\mathbf{u}}) + \text{div}(\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \nabla \tilde{P} = (\mathbf{H} \cdot \nabla)\tilde{\mathbf{H}} - \frac{1}{2} \nabla(|\tilde{\mathbf{H}}|^2) + \tilde{\mu}\Delta \tilde{\mathbf{u}} + (\tilde{\mu} + \tilde{\lambda})\nabla(\text{div}\tilde{\mathbf{u}}). \tag{1.5}
$$

By the identities

$$
\text{curl} \text{curl} \tilde{\mathbf{H}} = \nabla \text{div} \tilde{\mathbf{H}} - \Delta \tilde{\mathbf{H}}
$$

and

$$
\text{curl} (\tilde{\mathbf{u}} \times \tilde{\mathbf{H}}) = \tilde{\mathbf{u}}(\text{div} \tilde{\mathbf{H}}) - \tilde{\mathbf{H}}(\text{div}\tilde{\mathbf{u}}) + (\mathbf{H} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{H}},
$$

together with the constraint $\text{div} \tilde{\mathbf{H}} = 0$, the magnetic field equation (1.3) can be expressed as

$$
\partial_t \tilde{\mathbf{H}} + (\text{div}\tilde{\mathbf{u}})\tilde{\mathbf{H}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{H}} - (\mathbf{H} \cdot \nabla)\tilde{\mathbf{u}} = \tilde{\nu}\Delta \tilde{\mathbf{H}}. \tag{1.6}
$$

We introduce the scaling

$$
\tilde{\rho}(x,t) = \rho^*(x,ct), \quad \tilde{\mathbf{u}}(x,t) = c\mathbf{u}^*(x,ct), \quad \tilde{\mathbf{H}}(x,t) = c\mathbf{H}^*(x,ct)
$$
and assume that the viscosity coefficients $\tilde{\mu}$, $\tilde{\xi}$, and $\tilde{\nu}$ are small constants and scaled like

$$
\tilde{\mu} = \epsilon \mu', \quad \tilde{\xi} = \epsilon \lambda', \quad \tilde{\nu} = \epsilon \nu',
$$

where $\epsilon \in (0, 1)$ is a small parameter and the normalized coefficients $\mu'$, $\lambda'$, and $\nu'$ satisfy $\mu' > 0$, $2\mu' + d\lambda' > 0$, and $\nu' > 0$.

With the preceding scalings and the pressure function (1.4), the compressible MHD equations (1.1), (1.5), and (1.6) take the form

$$
\begin{align*}
\partial_t \rho' + \text{div}(\rho' \mathbf{u}') &= 0, \\
\partial_t (\rho' \mathbf{u}') + \text{div}(\rho' \mathbf{u}' \otimes \mathbf{u}') + \frac{a \nabla(\rho') \gamma}{\epsilon^2} &= \mathbf{H}' \cdot \nabla \mathbf{H}' - \frac{1}{2} \nabla(|\mathbf{H}'|^2) \\
&
\quad + \mu' \Delta \mathbf{u}' + (\mu' + \lambda') \nabla(\text{div}\mathbf{u}'), \\
\partial_t \mathbf{H}' + (\text{div}\mathbf{u}') \mathbf{H}' + (\mathbf{u}' \cdot \nabla)\mathbf{H}' - (\mathbf{H}' \cdot \nabla)\mathbf{u}' &= \nu' \Delta \mathbf{H}', \quad \text{div}\mathbf{H}' = 0.
\end{align*}
$$

Moreover, by replacing $\epsilon$ by $\sqrt{a \gamma} \epsilon$, we can always assume $a = 1/\gamma$.

Now, we investigate the incompressible limit of the compressible MHD equations (1.8)-(1.10). Formally let $\epsilon \to 0$ in the equations (1.8)-(1.10), then we obtain from the momentum equation (1.9) that $\rho'$ converges to some function $\tilde{\rho}(t) \geq 0$. If we further assume that the initial datum $\rho'_0$ is of order $1 + O(\epsilon)$ (this can be guaranteed by the initial energy bound (2.4) below), then we can expect that $\tilde{\rho} = 1$. Thus, the continuity equation (1.8) gives $\text{div}\mathbf{u} = 0$. Furthermore, using the assumption

$$
\mu' \to 0, \quad \nu' \to 0 \quad \text{as} \quad \epsilon \to 0,
$$

we obtain the following ideal incompressible MHD equations (suppose that the limits $\mathbf{u}' \to \mathbf{u}$ and $\mathbf{H}' \to \mathbf{H}$ exist)

$$
\begin{align*}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{H} \cdot \nabla)\mathbf{H} + \nabla p + \frac{1}{2} \nabla(|\mathbf{H}|^2) &= 0, \\
\partial_t \mathbf{H} + (\mathbf{u} \cdot \nabla)\mathbf{H} - (\mathbf{H} \cdot \nabla)\mathbf{u} &= 0, \\
\text{div}\mathbf{u} &= 0, \quad \text{div}\mathbf{H} = 0.
\end{align*}
$$

In Section 3 we shall rigorously prove that the weak solutions of the compressible MHD equations (1.8)-(1.10) converge to, as $\epsilon \to 0$, the strong solution of the ideal incompressible MHD equations (1.12)-(1.14) for the well-prepared initial data in the time interval where the strong solution of (1.12)-(1.14) exists. Furthermore, the convergence rates are obtained. To show these results, since the viscosity coefficients go to zero, we lose the spatial compactness property of the velocity and the magnetic field, and the arguments in [11] do not work here. To overcome such difficulty, we shall carefully exploit the energy arguments.

Next, if we assume that the shear and the bulk viscosity coefficients and the magnetic diffusivity coefficient satisfy

$$
\mu' \to \mu > 0, \quad \lambda' \to \lambda, \quad \nu' \to \nu > 0 \quad \text{as} \quad \epsilon \to 0,
$$

we obtain that the viscosity coefficients $\tilde{\mu}$, $\tilde{\xi}$, and $\tilde{\nu}$ are small constants and scaled like

$$
\tilde{\mu} = \epsilon \mu', \quad \tilde{\xi} = \epsilon \lambda', \quad \tilde{\nu} = \epsilon \nu',
$$

and the normalized coefficients $\mu', \lambda', \text{and} \nu'$ satisfy $\mu' > 0$, $2\mu' + d\lambda' > 0$, and $\nu' > 0$.
then the compressible MHD equations (1.12)-(1.14) formally converges to the incompressible MHD equations (suppose that the limits \( u^\epsilon \to u \) and \( H^\epsilon \to H \) exist)

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u - \mu \Delta u + \nabla p - (H \cdot \nabla)H + \frac{1}{2} \nabla (|H|^2) &= 0, \\
\partial_t H + (u \cdot \nabla)H - (H \cdot \nabla)u - \nu \Delta H &= 0, \\
\text{div} u &= 0, \quad \text{div} H = 0.
\end{align*}
\]

(1.16) \hspace{1cm} (1.17) \hspace{1cm} (1.18)

In Sections 3 and 4 we shall prove the convergence to the strong solution of the incompressible viscous MHD equations (1.16)-(1.18) for both the well-prepared and the general initial data. Furthermore, the convergence rates are also obtained for the well-prepared initial data. For the general initial data, we shall also show the convergence to the strong solution of the partial viscous incompressible MHD equations (that is, \( \mu = 0 \) and \( \nu > 0 \) in (1.16)-(1.18)).

There are a lot of studies on the compressible MHD equations in the literature. Besides the aforementioned results, the interested reader can see \([13, 17]\) on the global smooth solutions with small initial data and see \([5, 24]\) on the local strong solution with general initial data. We also mention the work \([25]\) where a MHD model describing the screw pinch problem in plasma physics was discussed and the global existence of weak solutions with symmetry was obtained.

Before ending the introduction, we give the notation used throughout the current paper. We denote the space \( L^q_T(\mathbb{T}^d) \) by

\[
L^q_T(\mathbb{T}^d) = \{ f \in L^q_{\text{loc}}(\mathbb{T}^d) : f^1_{\{ |f| \geq 1/2 \}} \in L^2, f^1_{\{ |f| \leq 1/2 \}} \in L^2 \}.
\]

We use the letters \( C \) and \( C_T \) to denote various positive constants independent of \( \epsilon \), but \( C_T \) may depend on \( T \). For convenience, we denote by \( H^r(\mathbb{T}^d) \) the standard Sobolev space. For any vector field \( \mathbf{v} \), we denote by \( P\mathbf{v} \) and \( Q\mathbf{v} \) the divergence-free part and the gradient part of \( \mathbf{v} \), respectively. Namely, \( Q\mathbf{v} = \nabla \Delta^{-1}(\text{div}\mathbf{v}) \) and \( P\mathbf{v} = \mathbf{v} - Q\mathbf{v} \).

We state our main results in Section 2 and present the proofs for the well-prepared case in Section 3 and the ill-prepared case in Section 4, respectively.

\section{Main Results}

We first recall the local existence of strong solutions to the ideal incompressible MHD equations (1.12)-(1.14) in the torus \( \mathbb{T}^d \). The proof can be found in \([1,23]\).

\begin{proposition} \([1,23]\) \label{prop:ideal}
Assume that the initial data \( (u,H)|_{t=0} = (u_0,H_0) \) satisfy \( u_0, H_0 \in H^s \) \((s > d/2 + 1)\), and \( \text{div} u_0 = 0, \text{div} H_0 = 0 \). Then, there exist a \( T^* \in (0,\infty) \) and a unique solution \( (u,H) \in L^\infty([0,T^*),H^s) \) to the ideal incompressible MHD equations (1.12)-(1.14) satisfying, for any \( 0 < T < T^* \),

\[
\sup_{0 \leq t \leq T} \{ ||(u,H)(t)||_{H^s} + ||(\partial_t u, \partial_t H)(t)||_{H^{s-1}} + ||\nabla p(t)||_{H^{s-1}} \} \leq C_T.
\]

(2.1)

\end{proposition}

\begin{remark} \label{remark:ideal}
The local existence of strong solutions to the incompressible viscous MHD equations (1.16)-(1.18) was also established in \([1,23]\).
\end{remark}
We prescribe the initial conditions to the compressible MHD equations (1.8)-(1.10) as
\[
\rho^\epsilon|_{t=0} = \rho_0^\epsilon(x), \quad \rho^\epsilon u^\epsilon|_{t=0} = \rho_0^\epsilon(x)u_0^\epsilon(x) \equiv m_0^\epsilon(x), \quad H^\epsilon|_{t=0} = H_0^\epsilon(x),
\]
and assume that
\[
\rho_0^\epsilon \geq 0, \quad \rho_0^\epsilon \in L^\gamma, \quad \rho_0^\epsilon |u_0^\epsilon|^2 \in L^1, \quad H_0^\epsilon \in L^2, \quad \text{div}\,H_0^\epsilon = 0, \quad m_0^\epsilon = 0 \quad \text{for a.e.} \quad \rho_0^\epsilon = 0.
\]
Moreover, we assume that the initial data also satisfy the following uniform bound
\[
\int_{T^d} \left[ \frac{1}{2} \rho_0^\epsilon |u_0^\epsilon|^2 + \frac{1}{2} |H_0^\epsilon|^2 + \frac{a}{c^2(\gamma-1)} \left( (\rho_0^\epsilon)^\gamma - 1 - \gamma(\rho_0^\epsilon - 1) \right) \right] dx \leq C.
\]
The initial energy inequality (2.4) implies that \(\rho_0^\epsilon\) is of order \(1 + O(\epsilon)\).

Under the above assumptions, it was proved in [9] that the compressible MHD equations (1.8)-(1.10) with initial data (2.2)-(2.4) has a global weak solution. More precisely, we have

**Proposition 2.2** ([9]). Let \(\gamma > d/2\). Suppose that the initial data \((\rho_0^\epsilon, u_0^\epsilon, H_0^\epsilon)\) satisfy the assumptions (2.3) and (2.4). Then the compressible MHD equations (1.8)-(1.10) with the initial data (2.2) enjoy at least one global weak solution \((\rho^\epsilon, u^\epsilon, H^\epsilon)\) satisfying

1. \(\rho^\epsilon \in L^\infty(0, \infty; L^\gamma) \cap C([0, \infty), L^r)\) for all \(1 \leq r < \gamma\), \(\rho^\epsilon |u^\epsilon|^2 \in L^\infty(0, \infty; L^1)\), \(H^\epsilon \in L^\infty(0, \infty; L^2)\), and \(u^\epsilon \in L^2(0, T; H^1)\), \(\rho^\epsilon u^\epsilon \in C([0, T], L^\text{weak})\), \(H^\epsilon \in L^2(0, T; H^1)\) and \(C([0, T], L^\text{weak})\) for all \(T \in (0, \infty)\);
2. the energy inequality
\[
\mathcal{E}^\epsilon(t) + \int_0^t \mathcal{D}^\epsilon(s) ds \leq \mathcal{E}^\epsilon(0)
\]
holds with the finite total energy
\[
\mathcal{E}^\epsilon(t) \equiv \int_{T^d} \left[ \frac{1}{2} \rho^\epsilon |u^\epsilon|^2 + \frac{1}{2} |H^\epsilon|^2 + \frac{a}{c^2(\gamma-1)} \left( (\rho^\epsilon)^\gamma - 1 - \gamma(\rho^\epsilon - 1) \right) \right] dx(t)
\]
and the dissipation energy
\[
\mathcal{D}^\epsilon(t) \equiv \int_{T^d} \left[ \mu^\epsilon |\nabla u^\epsilon|^2 + (\mu^\epsilon + \lambda^\epsilon) |\text{div}\,u^\epsilon|^2 + \nu^\epsilon |\nabla H^\epsilon|^2 \right] dx(t);
\]
3. the continuity equation is satisfied in the sense of renormalized solutions, i.e.,
\[
\partial_t b(\rho^\epsilon) + \text{div}(b(\rho^\epsilon)u^\epsilon) + (b'(\rho^\epsilon)\rho^\epsilon - b(\rho^\epsilon))\text{div}u^\epsilon = 0
\]
for any \(b \in C^1(\mathbb{T})\) such that \(b'(z)\) is constant for \(z\) large enough;
4. the equations (1.8)-(1.10) hold in \(\mathcal{D}'(\mathbb{T}^d \times (0, \infty))\).

The main results of this paper can be stated as follows.

**Theorem 2.3.** Let \(s > d/2 + 2\) and \(\mu^\epsilon + \lambda^\epsilon > 0\). Suppose that the initial data \((\rho_0^\epsilon, u_0^\epsilon, H_0^\epsilon)\) satisfy the conditions presented in Proposition 2.2. Assume further that
\[
\int_{T^d} |\rho_0^\epsilon - 1|_1^2 1_{(|\rho_0^\epsilon - 1| \leq \delta)} dx + \int_{T^d} |\rho_0^\epsilon - 1|_1^2 1_{(|\rho_0^\epsilon - 1| > \delta)} dx \leq C\epsilon^2,
\]
\[
||\sqrt{\rho_0^\epsilon} u_0^\epsilon - u_0||_{L^2(T^d)} \leq C\epsilon, \quad ||H_0^\epsilon - H_0||_{L^2(T^d)} \leq C\epsilon
\]
for any $\delta \in (0,1)$, where $\mathbf{u}_0$ and $\mathbf{H}_0$ are defined in Proposition 2.7. We assume that the shear viscosity $\mu'$ and the magnetic diffusion coefficient $\nu'$ satisfy

$$\mu' = \epsilon^\alpha, \quad \nu' = \epsilon^\beta$$  \hspace{1cm} (2.11)

for some constants $\alpha, \beta > 0$ satisfying $0 < \alpha + \beta < 2$. Let $(\mathbf{u}, \mathbf{H})$ be the smooth solution to the ideal incompressible MHD equations (1.12)-(1.14) defined on $[0, T^*)$ with $(\mathbf{u}, \mathbf{H})|_{t=0} = (\mathbf{u}_0, \mathbf{H}_0)$. Then, for any $0 < T < T^*$, the global weak solution $(\rho', \mathbf{u}', \mathbf{H}')$ of the compressible MHD equations (1.8)-(1.10) established in Proposition 2.2 satisfies

$$\int_{\mathbb{T}^d} |\rho' - 1|^2 1_{(|\rho' - 1| \leq \delta)} dx + \int_{\mathbb{T}^d} |\rho' - 1|^1 1_{(|\rho' - 1| > \delta)} dx \leq C_T \epsilon^2,$$

$$||\sqrt{\rho'} \mathbf{u}' - \mathbf{u}||_{L^2(\mathbb{T}^d)}^2 \leq C_T \epsilon, \quad ||\mathbf{H}' - \mathbf{H}||_{L^2(\mathbb{T}^d)}^2 \leq C_T \epsilon$$  \hspace{1cm} (2.12)

for any $t \in [0, T]$, where $\sigma = \min\{\alpha, \beta, 1 - (\alpha + \beta)/2\}$.

The proof of Theorem 2.3 is based on the combination of the modulated energy method, motivated by Brenier [1], the weak convergence method and the refined energy analysis. Masmoudi [19] made use of such idea to study the incompressible, inviscid limit of the compressible Navier-Stokes equations in both the whole space and the torus. Comparing with the proof in [19], here we have to overcome the difficulties caused by the strong coupling of the hydrodynamic motion and the magnetic field.

Furthermore, we can use an idea similar to that described above to obtain the convergence of the compressible MHD equations (1.8)-(1.10) to the incompressible viscous MHD equations (1.16)-(1.18). In fact, we have the following result.

**Theorem 2.4.** Let $s > d/2 + 2$ and $\mu' + \lambda' > 0$. Suppose that the initial data $(\rho'_0, \mathbf{u}'_0, \mathbf{H}'_0)$ satisfy the conditions presented in Proposition 2.2. Assume further that

$$\int_{\mathbb{T}^d} |\rho'_0 - 1|^2 1_{(|\rho'_0 - 1| \leq \delta)} dx + \int_{\mathbb{T}^d} |\rho'_0 - 1|^1 1_{(|\rho'_0 - 1| > \delta)} dx \leq C \epsilon^2,$$

$$||\sqrt{\rho'_0} \mathbf{u}'_0 - \mathbf{u}_0||_{L^2(\mathbb{T}^d)}^2 \leq C \epsilon, \quad ||\mathbf{H}'_0 - \mathbf{H}_0||_{L^2(\mathbb{T}^d)}^2 \leq C \epsilon$$  \hspace{1cm} (2.13)

for any $\delta \in (0,1)$ and for some $\mathbf{u}_0, \mathbf{H}_0 \in H^s(\mathbb{T}^d)$, satisfying $\text{div}\mathbf{u}_0 = 0, \text{div}\mathbf{H}_0 = 0$. We also assume that the shear viscosity $\mu'$ and the magnetic diffusion coefficient $\nu'$ satisfy (1.15). Let $(\mathbf{u}, \mathbf{H})$ be the smooth solution to the incompressible MHD equations (1.16)-(1.18) with $(\mathbf{u}, \mathbf{H})|_{t=0} = (\mathbf{u}_0, \mathbf{H}_0)$. Then, for any $0 < T < T^*$ ( $T^*$ is the maximal time of existence for (1.16)-(1.18)), the global weak solution $(\rho', \mathbf{u}', \mathbf{H}')$ of the compressible MHD equations (1.8)-(1.10) established in Proposition 2.2 satisfies that $\nabla \mathbf{u}'$ and $\nabla \mathbf{H}'$ converge strongly to $\nabla \mathbf{u}$ and $\nabla \mathbf{H}$ in $L^2(0,T; L^2(\mathbb{T}^d))$, respectively. Moreover, for any $t \in [0, T]$, we have

$$\int_{\mathbb{T}^d} |\rho' - 1|^2 1_{(|\rho' - 1| \leq \delta)} dx + \int_{\mathbb{T}^d} |\rho' - 1|^1 1_{(|\rho' - 1| > \delta)} dx \leq C_T \epsilon^2,$$

$$||\sqrt{\rho'} \mathbf{u}' - \mathbf{u}||_{L^2(\mathbb{T}^d)}^2 + ||\mathbf{H}' - \mathbf{H}||_{L^2(\mathbb{T}^d)}^2 \leq C_T \frac{\epsilon}{\sqrt{\mu'}} + C_T \left(\frac{|\mu' - \mu|}{\sqrt{\mu}} + \frac{|\nu' - \nu|}{\sqrt{\nu}}\right).$$  \hspace{1cm} (2.14)
To show Theorem 2.3 besides the techniques mentioned above, we have to employ a new technique, that is, to modulate both the total energy and the partial dissipative energy simultaneously. Moreover, the dissipative effect of the viscous terms is also carefully exploited to obtain the desired results.

Remark 2.2. Comparing with Theorem 2.3, we have gotten the better convergence rates than (2.13) when the shear viscosity and the magnetic diffusion coefficient tend to some positive constants.

Some results in Theorem 2.3 can be extended to the case of general initial data. More precisely, we shall obtain the convergence of the compressible MHD equations (1.16)-(1.18) to the incompressible MHD equations (1.16)-(1.18) for the general initial data under the conditions that the oscillating parts of the initial data have higher regularity and the Sobolev norm of the oscillation parts is comparable to the magnetic diffusion coefficient. This implies that the influence of oscillations on the magnetic field can be balanced by the diffusive effect of the magnetic field, which is one of the new ingredients in our paper.

To describe the result, we write \( \rho^\ell = 1 + \epsilon \varphi^\ell \) and denote
\[
\Pi^\ell(x,t) = \frac{1}{c} \sqrt{\frac{2a}{\gamma - 1} ((\rho^\ell)^\gamma - 1 - \gamma(\rho^\ell - 1))}.
\]
We will use the above approximation \( \Pi^\ell(x,t) \) instead of \( \varphi^\ell \), since we can not obtain any bound for \( \varphi^\ell \) in \( L^\infty(0,T;L^2) \) directly if \( \gamma < 2 \).

Theorem 2.5. Let \( s > 2 + d/2 \) and \( 2\mu + d\lambda > 0 \). Suppose that the initial data \( (\rho_0^0, u_0^0, H_0^0) \) satisfy the conditions presented in Proposition 2.2. Moreover, we assume that \( \sqrt{\rho_0^0} u_0^0 \) converges strongly in \( L^2 \) to some \( \tilde{u}_0 \) satisfying \( Q \tilde{u}_0 \in H^{s-1} \), \( H_0^0 \) converges strongly in \( L^2 \) to some \( H_0 \) with
\[
\int_{\mathbb{T}^d} H_0(x)dx = 0, \quad \Pi^\ell|_{t=0} = \Pi_0^\ell \text{ converges strongly in } L^2 \text{ to some } \varphi_0 \in H^{s-1}, \text{ and}
\]
\[
\|\varphi_0\|_{H^2} + \|Q \tilde{u}_0\|_{H^s} \leq c_0 \nu \tag{2.18}
\]
for some constant \( c_0 > 0 \). Let \( (u, H) \) be the smooth solution to the incompressible MHD equations (1.16)-(1.18) with \( (u, H)|_{t=0} = (u_0, H_0) \in H^{s}({\mathbb{T}^d}) \) satisfying \( u_0 = P\tilde{u}_0 \) and \( \text{div} H_0 = 0 \). Then, for any \( 0 < T < T^\ast \) ( \( T^\ast \) is the maximal time of existence for (1.16)-(1.18)), the global weak solution \( (\rho^\ell, u^\ell, H^\ell) \) of the compressible MHD equations (1.8)-(1.10) established in Proposition 2.2 satisfies

1. \( \rho^\ell \) converges strongly to \( 1 \) in \( C([0,T], L^2_t(\mathbb{T}^d)) \);
2. \( \nabla H^\ell \) converges strongly to \( \nabla H \) in \( L^2(0,T;L^2(\mathbb{T}^d)) \);
3. \( H^\ell \) converges strongly to \( H \) in \( L^\infty(0,T;L^2(\mathbb{T}^d)) \);
4. \( P(\sqrt{\rho^\ell} u^\ell) \) converges strongly to \( u \) in \( L^\infty(0,T;L^2(\mathbb{T}^d)) \);
5. \( \sqrt{\rho^\ell} u^\ell \) converges weakly to \( u \) in \( H^{-1}(0,T;L^2(\mathbb{T}^d)) \).

By slightly modifying the proof of Theorem 2.5, we can obtain the convergence of compressible MHD equations to the partial viscous incompressible MHD equations when the shear viscosity goes to zero and the magnetic diffusion coefficient goes to a positive constant. The partial viscous
incompressible MHD equations correspond to the case of turbulent flow with very high Reynolds number (where the viscosity of flow can be ignored, see [16]).

**Theorem 2.6.** Let \( s > 2 + d/2 \). Suppose that the conditions in Theorem 2.5 hold. Moreover, we assume that

\[
\nu^\epsilon \to \nu > 0, \quad 2\mu^\epsilon + \lambda^\epsilon \to 2\theta > 0 \quad \text{as} \quad \epsilon \to 0,
\]

and \( \mu^\epsilon = \epsilon^\alpha \) for some constant \( 0 < \alpha < 1 \). Let \((u, H)\) be the smooth solution to the following partially viscous incompressible MHD equations

\[
\partial_t u + (u \cdot \nabla) u + \nabla p - (H \cdot \nabla) H + \frac{1}{2} \nabla(|H|^2) = 0,
\]

\[
\partial_t H + (u \cdot \nabla) H - (H \cdot \nabla) u - \nu \Delta H = 0,
\]

\[
\text{div} u = 0, \quad \text{div} H = 0,
\]

with \((u, H)|_{t=0} = (u_0, H_0) \in H^s(T^3)\) satisfying \(u_0 = P\tilde{u}_0\) and \(\text{div}H_0 = 0\). Then, for any \(0 < T < T^{**}\) (\(T^{**}\) is the maximal time of existence for (1.16)–(1.18)), the global weak solution \((\rho^\epsilon, u^\epsilon, H^\epsilon)\) of the compressible MHD equations 1.3–1.10 established in Proposition 2.2 satisfies

1. \(\rho^\epsilon\) converges strongly to 1 in \(C([0, T], L^2(T^d))\);
2. \(\nabla H^\epsilon\) converges strongly to \(\nabla H\) in \(L^2(0, T; L^2(T^d))\);
3. \(H^\epsilon\) converges strongly to \(H\) in \(L^\infty(0, T; L^2(T^d))\);
4. \(P(\sqrt{\rho^\epsilon} u^\epsilon)\) converges strongly to \(u\) in \(L^\infty(0, T; L^2(T^d))\);
5. \(\sqrt{\rho^\epsilon} u^\epsilon\) converges weakly to \(u\) in \(H^{-1}(0, T; L^2(T^d))\).

**Remark 2.3.** The assumption that \(\Pi^\epsilon_0\) converges strongly in \(L^2\) to some \(\varphi_0\) in fact implies that \(\varphi^\epsilon_0\) converges strongly to \(\varphi_0\) in \(L^2\).

**Remark 2.4.** When taking \(H^\epsilon \equiv 0\) in (1.1)–(1.3), the MHD equations reduce to the classical compressible Navier-Stokes equations. The low Mach number limit problem of the compressible Navier-Stokes equations has been investigated extensively, for instance, see [2, 3, 7, 8, 18]. The interested reader can refer to the survey article [20] for more related results.

**Remark 2.5.** We point out that our arguments in the present paper can be applied to the case of \(H^\epsilon \equiv 0\). In this case, we obtain the convergence of the compressible Navier-Stokes equations to the incompressible Euler or Navier-Stokes equations with general initial data, extending thus the results in [18, 19].

### 3. Proof of Theorems 2.3 and 2.4

In this section, we shall prove our convergence results for the case of well-prepared initial data by combining the modulated energy method, the weak convergence method, and the refined energy analysis.
Proof of Theorem 2.3 We divide the proof into several steps.

Step 1: Basic energy estimates and compact arguments.

By the assumptions on the initial data we obtain, from the energy inequality (2.5), that the total energy $E^{\epsilon}(t)$ has a uniform upper bound for a.e. $t \in [0, T]$, $T > 0$. This uniform bound implies that $\rho^{\epsilon}|u^{\epsilon}|^2$ and $(|\rho^{\epsilon}|^2 - 1)/(\rho^{\epsilon} - 1)$ are bounded in $L^\infty(0, T; L^1)$ and $H^{\epsilon}$ is bounded in $L^\infty(0, T; L^2)$. Using the analysis in [18], we obtain

$$
\int_{T_d} \frac{1}{\epsilon^2} |\rho^{\epsilon} - 1|^2 1_{(|\rho^{\epsilon} - 1| \leq \frac{1}{2})} + \int_{T_d} \frac{1}{\epsilon^2} |\rho^{\epsilon} - 1|^2 1_{(|\rho^{\epsilon} - 1| \geq \frac{1}{2})} \leq C, \tag{3.1}
$$

which implies (2.12) and

$$\rho^{\epsilon} \to 1 \text{ strongly in } C([0, T], L^2(\mathbb{T}^d)). \tag{3.2}$$

From the results in [18], we know that $||u^{\epsilon}||^2_{L^2 L^2} \leq C + C||\nabla u^{\epsilon}||_{L^2 L^2}^2$. Furthermore, the fact that $\rho^{\epsilon}|u^{\epsilon}|^2$ and $|H^{\epsilon}|^2$ are bounded in $L^\infty(0, T; L^1)$ implies the following convergence (up to the extraction of a subsequence $\epsilon_n$):

$$\sqrt{\rho^{\epsilon}} u^{\epsilon} \text{ converges weakly-}^* \text{ to some } J \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)),
$$

$$H^{\epsilon} \text{ converges weakly-}^* \text{ to some } K \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)).$$

Thus, to finish our proof, we need to show that $J = u$ and $K = H$ in some sense and the inequalities (2.13) hold, where $(u, H)$ is the strong solution to the ideal incompressible MHD equations (1.12)-(1.14).

Step 2: The modulated energy functional and the uniform estimates.

We first recall the energy inequality of the compressible MHD equations (1.8)-(1.10), i.e., for almost all $t$, there holds

$$
\frac{1}{2} \int_{T_d} \left[ \rho^{\epsilon}(t)|u^{\epsilon}|^2(t) + |H^{\epsilon}|^2(t) + (\Pi^{\epsilon}(t))^2 \right] + \mu^{\epsilon} \int_0^t \int_{T_d} |\nabla u^{\epsilon}|^2
$$

$$+ (\mu^{\epsilon} + \lambda^{\epsilon}) \int_0^t \int_{T_d} |\nabla u^{\epsilon}|^2 + \nu^{\epsilon} \int_0^t \int_{T_d} |\nabla H^{\epsilon}|^2
$$

$$\leq \frac{1}{2} \int_{T_d} \left[ \rho_0^0 |u_0^0|^2 + |H_0^0|^2 + (\Pi_0^0)^2 \right]. \tag{3.3}
$$

The conservation of energy for the ideal incompressible MHD equations (1.12)-(1.14) reads

$$
\frac{1}{2} \int_{T_d} |u|^2(t) + |H|^2(t) = \frac{1}{2} \int_{T_d} |u_0|^2 + |H_0|^2. \tag{3.4}
$$

Using $u$ to test the momentum equation (1.9), we obtain

$$
\frac{1}{2} \int_{T_d} (\rho^{\epsilon} u^{\epsilon} \cdot u)(t) + \frac{1}{2} \int_0^t \int_{T_d} \rho^{\epsilon} u^{\epsilon} \cdot [(u \cdot \nabla)u - (H \cdot \nabla)H + \nabla p + \frac{1}{2} \nabla(|H|^2)]
$$

$$- \int_0^t \int_{T_d} \left[ (\rho^{\epsilon} u^{\epsilon} \otimes u^{\epsilon}) \cdot \nabla u + (H^{\epsilon} \cdot \nabla)H^{\epsilon} \cdot u - \mu^{\epsilon} \nabla u^{\epsilon} \cdot \nabla u \right] = \int_{T_d} \rho_0^0 u_0^0 \cdot u_0. \tag{3.5}
$$
Similarly, using $\mathbf{H}$ to test the magnetic field equation (1.10), one gets

$$
\int_{T^d} (\mathbf{H} \cdot \mathbf{H})(t) + \int_0^t \int_{T^d} \mathbf{H} \cdot \left( (\mathbf{u} \cdot \nabla)\mathbf{H} - (\mathbf{H} \cdot \nabla)\mathbf{u} \right) + \nu^r \int_0^t \int_{T^d} \nabla \mathbf{H} \cdot \nabla \mathbf{H}
+ \int_0^t \int_{T^d} \left[ (\text{div}\mathbf{u}^r)\mathbf{H} + (\mathbf{u}^r \cdot \nabla)\mathbf{H} - (\mathbf{H}^r \cdot \nabla)\mathbf{u} \right] \cdot \mathbf{H} = \int_{T^d} \mathbf{H}_0^r \cdot \mathbf{H}_0. 
$$

(3.6)

Summing up (3.3) and (3.4), and inserting (3.5) and (3.6) into the resulting inequality, we can deduce the following inequality by a straightforward computation

$$
\frac{1}{2} \int_{T^d} \left\{ |\sqrt{\rho^r}\mathbf{u} - \mathbf{u}|^2(t) + |\mathbf{H}^r - \mathbf{H}|^2(t) + (\Pi^r)^2(t) \right\}
+ \mu^r \int_0^t \int_{T^d} |\nabla \mathbf{u}^r|^2 + (\mu^r + \lambda^r) \int_0^t \int_{T^d} |\text{div}\mathbf{u}^r|^2 + \nu^r \int_0^t \int_{T^d} |\nabla \mathbf{H}^r|^2
\leq \mu^r \int_0^t \int_{T^d} \nabla \mathbf{u}^r \cdot \nabla \mathbf{u} + \nu^r \int_0^t \int_{T^d} \nabla \mathbf{H}^r \cdot \nabla \mathbf{H} - \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot [(\mathbf{H} \cdot \nabla)\mathbf{H}]
- \int_0^t \int_{T^d} (\mathbf{H}^r \cdot \nabla)\mathbf{H} \cdot \mathbf{u} + \int_0^t \int_{T^d} \mathbf{H} \cdot [(\mathbf{u} \cdot \nabla)\mathbf{H} - (\mathbf{H} \cdot \nabla)\mathbf{u}]
+ \int_0^t \int_{T^d} \left[ (\text{div}\mathbf{u}^r)\mathbf{H} + (\mathbf{u}^r \cdot \nabla)\mathbf{H} - (\mathbf{H}^r \cdot \nabla)\mathbf{u} \right] \cdot \mathbf{H} + \frac{1}{2} \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot \nabla (|\mathbf{H}|^2)
+ \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p] - \int_0^t \int_{T^d} (\rho^r \mathbf{u}^r \otimes \mathbf{u}^r) \cdot \nabla \mathbf{u}
+ \int_{T^d} \left[ (\sqrt{\rho^r} - 1)\sqrt{\rho^r}\mathbf{u}^r \cdot \mathbf{u} \right] (t) - \int_{T^d} \left[ (\sqrt{\rho^r} - 1)\sqrt{\rho^r}\mathbf{u}^r \cdot \mathbf{u} \right] (0)
+ \frac{1}{2} \int_{T^d} \left\{ |\sqrt{\rho^r}\mathbf{u} - \mathbf{u}|^2(0) + |\mathbf{H}^r - \mathbf{H}|^2(0) + (\Pi_0^r)^2 \right\}. 
$$

(3.7)

We first deal with the right-hand side of the inequality (3.7). Denoting $\mathbf{w}^r = \sqrt{\rho^r}\mathbf{u}^r - \mathbf{u}$ and $\mathbf{Z}^r = \mathbf{H}^r - \mathbf{H}$, integrating by parts, and using the fact that $\text{div} \mathbf{H}^r = 0$, $\text{div} \mathbf{u} = 0$ and $\text{div} \mathbf{H} = 0$, we find that

$$
- \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot [(\mathbf{H} \cdot \nabla)\mathbf{H}] - \int_0^t \int_{T^d} (\mathbf{H}^r \cdot \nabla)\mathbf{H} \cdot \mathbf{u}
+ \int_0^t \int_{T^d} \mathbf{H}^r \cdot [(\mathbf{u} \cdot \nabla)\mathbf{H} - (\mathbf{H} \cdot \nabla)\mathbf{u}] + \frac{1}{2} \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot \nabla (|\mathbf{H}|^2)
+ \int_0^t \int_{T^d} \left[ (\text{div}\mathbf{u}^r)\mathbf{H} + (\mathbf{u}^r \cdot \nabla)\mathbf{H} - (\mathbf{H}^r \cdot \nabla)\mathbf{u} \right] \cdot \mathbf{H}
= - \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot [(\mathbf{H} \cdot \nabla)\mathbf{H}] + \int_0^t \int_{T^d} (\mathbf{H}^r \cdot \nabla)\mathbf{u} \cdot \mathbf{H}^r
+ \int_0^t \int_{T^d} (\mathbf{u} \cdot \nabla)\mathbf{H} \cdot \mathbf{H}^r - \int_0^t \int_{T^d} (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H}^r
- \int_0^t \int_{T^d} (\mathbf{u}^r \cdot \nabla)\mathbf{H} \cdot \mathbf{H}^r + \int_0^t \int_{T^d} (\mathbf{H}^r \cdot \nabla)\mathbf{H} \cdot \mathbf{u}^r + \frac{1}{2} \int_0^t \int_{T^d} \rho^r \mathbf{u}^r \cdot \nabla (|\mathbf{H}|^2)
= \int_0^t \int_{T^d} (1 - \rho^r)\mathbf{u}^r \cdot [(\mathbf{H} \cdot \nabla)\mathbf{H}] + \int_0^t \int_{T^d} (\mathbf{H}^r - \mathbf{H}) \cdot \nabla \mathbf{u} \cdot (\mathbf{H}^r - \mathbf{H})
$$
\[ + \int_0^t \int_{\Omega} [(H' - H) \cdot \nabla] H \cdot (u' - u) - \int_0^t \int_{\Omega} (u' - u) \cdot \nabla] H \cdot (H' - H) \\
+ \frac{1}{2} \int_0^t \int_{\Omega} (\rho' - 1)u' \nabla(\|H\|^2) \leq \int_0^t \int_{\Omega} (1 - \rho' - 1)u' \nabla(\|H\|^2) \]

(3.8)

Substituting (3.8) and (3.9) into the inequality (3.7), we conclude that

\[ \int_0^t \int_{\Omega} \left( (1 - \sqrt{\rho'}) u' \cdot \nabla] H \cdot (u' - u) - \int_0^t \int_{\Omega} (u' - u) \cdot \nabla] H \cdot (H' - H) \\
+ \frac{1}{2} \int_0^t \int_{\Omega} (\rho' - 1)u' \nabla(\|H\|^2) \leq \int_0^t \int_{\Omega} (1 - \sqrt{\rho'}) u' \nabla(\|H\|^2) \]

(3.9)

and

\[ \int_0^t \int_{\Omega} \left[ (\rho' u' \cdot \nabla] u + \nabla p) - \int_0^t \int_{\Omega} (\rho' u' \otimes u') \cdot \nabla u \\
- \int_0^t \int_{\Omega} (\rho' - \sqrt{\rho'}) u' \cdot ((u \cdot \nabla) u) + \int_0^t \int_{\Omega} (\nabla' - u) \cdot \nabla] u \cdot w' + \int_0^t \int_{\Omega} \rho' u' \cdot \nabla p \\
- \int_0^t \int_{\Omega} (\rho' - u) \cdot \nabla] \frac{|u|^2}{2}, \right) \]

(3.10)

Substituting (3.8) and (3.9) into the inequality (3.7), we conclude that

\[ \|w'(t)\|_{L^2}^2 + ||Z'(t)||_{L^2}^2 + ||\Pi'(t)||_{L^2}^2 + 2\mu' \int_0^t \int_{\Omega} |\nabla u'|^2 \\
+ 2(\mu' + \lambda') \int_0^t \int_{\Omega} |\text{div} u'|^2 + 2\nu' \int_0^t \int_{\Omega} |\nabla H'|^2 \leq 2C \int_0^t (\|w'(\tau)\|_{L^2}^2 + ||Z'(\tau)||_{L^2}^2)(||\nabla u(\tau)||_{L^\infty} + ||\nabla H(\tau)||_{L^\infty}) d\tau \\
+ ||w'(0)||_{L^2}^2 + ||Z'(0)||_{L^2}^2 + ||\Pi'(0)||_{L^2}^2 + 2 \sum_{i=1}^{6} R_i^2(t), \]

where

\[ R_1(t) = \mu' \int_0^t \int_{\Omega} \nabla u' \cdot \nabla u + \nu' \int_0^t \int_{\Omega} \nabla H' \cdot \nabla H, \]

\[ R_2(t) = \int_0^t \int_{\Omega} \left( [(\sqrt{\rho'} - 1)\sqrt{\rho'} u' \cdot u] (t) - \int_0^t \int_{\Omega} [(\sqrt{\rho'} - 1)\sqrt{\rho'} u' \cdot u] (0) \\
+ \int_0^t \int_{\Omega} (\rho' - \sqrt{\rho'}) u' \cdot ((u \cdot \nabla) u), \right) \]

\[ R_3(t) = \int_0^t \int_{\Omega} (Z' \cdot \nabla) H \cdot [(1 - \sqrt{\rho'}) u'] - \int_0^t \int_{\Omega} \{[(1 - \sqrt{\rho'}) u'] \cdot \nabla] H \cdot Z', \]

\[ R_4(t) = \int_0^t \int_{\Omega} (1 - \rho') u' \cdot [(H \cdot \nabla] H] + \frac{1}{2} \int_0^t \int_{\Omega} (\rho' - 1)u' \cdot \nabla(\|H\|^2), \]
\[ R_5^\epsilon(t) = \int_0^t \int_{\mathbb{T}^d} \rho^\epsilon \mathbf{u}^\epsilon \cdot \nabla \rho, \]
\[ R_6^\epsilon(t) = -\int_0^t \int_{\mathbb{T}^d} (\sqrt{\rho^\epsilon} \mathbf{u}^\epsilon - \mathbf{u}) \cdot \nabla (|\mathbf{u}|^2 / 2). \]

**Step 3: Convergence of the modulated energy functional.**

To show the convergence of the modulated energy functional (3.10) and to finish our proof, we have to estimate the reminders \( R_i^\epsilon(t), \) \( i = 1, \ldots, 6. \)

First, in view of (3.1) and the following two elementary inequalities
\[ |\sqrt{x} - 1|^2 \leq M|x - 1|, \quad |x - 1| \geq \delta, \quad \gamma \geq 1, \]  \[ |\sqrt{x} - 1|^2 \leq M|x - 1|^2, \quad x \geq 0 \]  for some positive constants \( M \) and \( \delta, \) \( \gamma, \) we obtain
\[ \int_{\mathbb{T}^d} |\sqrt{\rho^\epsilon} - 1|^2 = \int_{|\rho^\epsilon| - 1| \leq \frac{1}{2}} |\sqrt{\rho^\epsilon} - 1|^2 + \int_{|\rho^\epsilon| - 1| \geq \frac{1}{2}} |\sqrt{\rho^\epsilon} - 1|^2 \leq M \int_{|\rho^\epsilon| - 1| \leq \frac{1}{2}} |\rho^\epsilon - 1|^2 + M \int_{|\rho^\epsilon| - 1| \geq \frac{1}{2}} |\rho^\epsilon - 1|^\gamma \leq M \epsilon^2. \]  (3.13)

Now, we begin to estimate the terms \( R_i^\epsilon(t), \) \( i = 1, \ldots, 6. \) For the term \( R_1^\epsilon(t), \) by Young’s inequality and the regularity of \( \mathbf{u} \) and \( \mathbf{H}, \) we have
\[ |R_1^\epsilon(t)| \leq \frac{\mu^\epsilon}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla \mathbf{u}^\epsilon|^2 + \frac{\nu^\epsilon}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla \mathbf{H}|^2 + C_T \mu^\epsilon + C_T \nu^\epsilon. \]  (3.14)

For the term \( R_2^\epsilon(t), \) by Hölder’s inequality, the estimate (3.13), the assumption on the initial data, the estimate on \( \sqrt{\rho^\epsilon} \mathbf{u}^\epsilon, \) and the regularity of \( \mathbf{u}, \) we infer that
\[ |R_2^\epsilon(t)| \leq C \epsilon + ||(\mathbf{u} \cdot \nabla)\mathbf{u}||_{L^\infty} \left( \int_{\mathbb{T}^d} |\sqrt{\rho^\epsilon} - 1|^2 \right)^\frac{1}{2} \left( \int_{\mathbb{T}^d} \rho^\epsilon |\mathbf{u}^\epsilon|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{T}^d} |\nabla \mathbf{H}|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{T}^d} |\nabla \mathbf{H}|^2 \right)^\frac{1}{2} \leq C_T \epsilon. \]  (3.15)

For the term \( R_3^\epsilon(t), \) making use of the inequality (3.13), the basic inequality (2.10), the estimates on \( \mathbf{u}^\epsilon \) and \( \mathbf{H}^\epsilon, \) the regularity of \( \mathbf{H}, \) the assumption (2.11), and Sobolev’s imbedding theorem, we get
\[ |R_3^\epsilon(t)| \leq \left( ||(\mathbf{H} \cdot \nabla)\mathbf{H}(t)||_{L^\infty} + ||\nabla \mathbf{H}(t)||_{L^\infty} \cdot ||\mathbf{H}(t)||_{L^\infty} \right) \times \left( \int_0^t \int_{\mathbb{T}^d} |\sqrt{\rho^\epsilon} - 1|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{T}^d} |\mathbf{u}^\epsilon|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{T}^d} |\mathbf{u}^\epsilon|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{T}^d} |\nabla \mathbf{H}(t)|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{T}^d} |\nabla \mathbf{H}(t)|^2 \right)^\frac{1}{2} \leq C_T \epsilon \left( 1 + (\mu^\epsilon)^{-\frac{1}{2}} + ||\sqrt{\rho^\epsilon} - 1||_{L^\infty(0, T; L^2)} \left( \int_0^t ||\mathbf{u}^\epsilon(\tau)||_{L^4}^2 d\tau \right)^\frac{1}{2} \left( \int_0^t ||\mathbf{H}^\epsilon(\tau)||_{L^4}^2 d\tau \right)^\frac{1}{2} \right). \]
\[ \leq C_T \epsilon (1 + (\mu')^{-\frac{1}{2}}) + C_T \epsilon [\|u'|\|_{L^2(0,T;L^2)} + \|\nabla u'\|_{L^2(0,T;L^2)}] \]
\[ \times (\|H'\|_{L^2(0,T;L^2)} + \|\nabla H'\|_{L^2(0,T;L^2)}) \]
\[ \leq C_T \epsilon (1 + (\mu')^{-\frac{1}{2}}) + C_T [1 + (\mu')^{-\frac{1}{2}}] \cdot [1 + (\mu')^{-\frac{1}{2}}] \]
\[ \leq C_T \epsilon^{1-\alpha/2} + C_T \epsilon^\sigma \leq C_T \epsilon^\sigma, \quad (3.16) \]

where \( \sigma = 1 - (\alpha + \beta)/2. \)

For the term \( R_5(t) \), one can utilize the inequality (3.13), the estimates on \( u' \) and \( \sqrt{\rho'} u' \), the regularity of \( H \), and \( \rho' - 1 = \rho' - \sqrt{\rho'} + \sqrt{\rho'} - 1 \) to deduce
\[ |R_5(t)| \leq (\|H \cdot \nabla H\|_{L^\infty} + \|\nabla (\|H\|^2)\|_{L^\infty}) \left( \int_0^t \int_{T^d} |\sqrt{\rho'} - 1|^2 \right)^{\frac{1}{2}} \]
\[ \times \left[ \left( \int_0^t \int_{T^d} |u'|^2 \right)^{\frac{1}{2}} + \left( \int_0^t \int_{T^d} \rho' |u'|^2 \right)^{\frac{1}{2}} \right] \]
\[ \leq C_T \epsilon (1 + (\mu')^{-\frac{1}{2}}) \leq C_T \epsilon^{1-\alpha/2}. \quad (3.17) \]

Using (2.1), (3.11) and (3.12), the term \( R_5(t) \) can be bounded as follows.
\[ |R_5(t)| = \left| \int_0^t \int_{T^d} \rho' u' \cdot \nabla p \right| \]
\[ = \int_{T^d} \left\{ \left( (\rho' - 1)p(t) - ((\rho' - 1)p)(0) \right) - \int_0^t \int_{T^d} (\rho' - 1) \partial_t p \right\} \]
\[ \leq \left( \int_{|\rho' - 1| \leq \frac{1}{2}} |\rho' - 1|^2 \right)^{\frac{1}{2}} \left[ \left( \int_{T^d} |p(t)|^2 \right)^{\frac{1}{2}} + \left( \int_{T^d} |p(0)|^2 \right)^{\frac{1}{2}} \right] \]
\[ + \left( \int_{|\rho' - 1| \geq \frac{1}{2}} |\rho' - 1|^2 \right)^{\frac{1}{2}} \left[ \left( \int_{T^d} |p(t)||\nabla p| \right)^{\frac{2}{3}} + \left( \int_{T^d} |p(0)||\nabla p| \right)^{\frac{2}{3}} \right] \]
\[ + \int_0^t \left( \int_{|\rho' - 1| \leq \frac{1}{2}} |\rho' - 1|^2 \right)^{\frac{1}{2}} \left( \int_{T^d} |\partial_t p(t)|^2 \right)^{\frac{1}{2}} \]
\[ + \int_0^t \left( \int_{|\rho' - 1| \geq \frac{1}{2}} |\rho' - 1|^2 \right)^{\frac{1}{2}} \left( \int_{T^d} |\partial_t p(t)||\nabla p| \right)^{\frac{2}{3}} \]
\[ \leq C_T (\epsilon + \epsilon^{2/\kappa}) \leq C_T \epsilon, \quad (3.18) \]

where \( \kappa = \min\{2, \gamma\} \) and we have used the conditions \( s > 2 + d/2 \) and \( \gamma > 1. \)

Finally, to estimate the term \( R_6(t) \), we rewrite it as
\[ R_6(t) = -\int_0^t \int_{T^d} \sqrt{\rho'} \cdot \nabla u' \cdot \nabla (\frac{|u'|^2}{2}) \]
\[ = \int_0^t \int_{T^d} \sqrt{\rho'} (\sqrt{\rho'} - 1) u' \cdot \nabla (\frac{|u'|^2}{2}) - \int_0^t \int_{T^d} \rho' u' \cdot \nabla (\frac{|u'|^2}{2}) \]
\[ = R_{61}(t) + R_{62}(t), \quad (3.19) \]

where
\[ R_{61}(t) = \int_0^t \int_{T^d} \sqrt{\rho'} (\sqrt{\rho'} - 1) u' \cdot \nabla (\frac{|u'|^2}{2}), \]
\[ R'_{62}(t) = \int_0^t \int_{\mathbb{T}^d} (\rho' - 1) \partial_t \left( \frac{|u|^2}{2} \right) - \int_{\mathbb{T}^d} \left[ \left( (\rho' - 1) \left( \frac{|u|^2}{2} \right) \right)(t) - \left( (\rho' - 1) \left( \frac{|u|^2}{2} \right) \right)(0) \right]. \]

Applying arguments similar to those used for \( R'_{61}(t) \) and \( R'_{62}(t) \), we arrive at the following boundedness

\[ |R'_6(t)| \leq |R'_{61}(t)| + |R'_{62}(t)| \leq C_T \epsilon. \tag{3.20} \]

Inserting the estimates (3.14)-(3.20) into (3.10) and applying Gronwall’s inequality, we conclude

\[
||w^\epsilon(t)||^2_{L^2} + ||Z^\epsilon(t)||^2_{L^2} + ||\Pi^\epsilon(t)||^2_{L^2} \\
\leq \bar{C} \left[ ||w^\epsilon(0)||^2_{L^2} + ||Z^\epsilon(0)||^2_{L^2} + ||\Pi_0^\epsilon||^2_{L^2} + C_T \epsilon^\sigma \right], \quad \text{for a.e. } t \in [0, T], \tag{3.21}
\]

where

\[ \bar{C} = \exp \left\{ C \int_0^T \left[ ||\nabla u(\tau)||_{L^\infty} + ||\nabla H(\tau)||_{L^\infty} \right] d\tau \right\} < +\infty. \tag{3.22} \]

Now, letting \( \epsilon \) go to 0, we obtain \( K = H \) in \( L^\infty(0, T; L^2) \) and \( J = u \) in \( L^\infty(0, T; L^2) \). The inequality (2.13) follows from (2.10) and (3.21) directly. Thus, we complete the proof. \( \Box \)

**Proof of Theorem 2.4** For simplicity we assume here that \( \mu^\epsilon \equiv \mu \), \( \lambda^\epsilon \equiv \lambda \), and \( \nu^\epsilon \equiv \nu \) are constants, independent of \( \epsilon \), satisfying \( \mu > 0 \), \( \mu + \lambda > 0 \), and \( \nu > 0 \). The case (4.7) can be treated similarly. The proof of Theorem 2.4 is similar to that of Theorem 2.3. Since the viscosity is involved here, we have to modulate the part of dissipation energy in the energy inequality (2.5). We state the main different points in the proof here.

From the basic energy inequality (2.5), we obtain that, for a.e. \( t \in [0, T] \), \( \rho' |u'|^2 \) and \( (\rho')^\gamma - 1 - \gamma (\rho' - 1) / \epsilon^2 \) are bounded in \( L^\infty(0, T; L^1) \), \( H^\epsilon \) is bounded in \( L^\infty(0, T; L^2) \), \( \nabla u^\epsilon \) is bounded in \( L^2(0, T; L^2) \), and \( \nabla H^\epsilon \) is bounded in \( L^2(0, T; L^2) \). Therefore, we have

\[ \rho' \rightarrow 1 \text{ strongly in } C([0, T], L^2(\mathbb{T}^d)), \]

and \( u^\epsilon \) is bounded in \( L^2(0, T; L^2) \). The boundedness of \( \rho' |u'|^2 \) and \( |H^\epsilon|^2 \) in \( L^\infty(0, T; L^1) \) implies the following convergence (up to the extraction of a subsequence \( \epsilon_n \)):

\[ \sqrt{\rho' u^\epsilon} \text{ converges weakly-}* \text{ to some } \tilde{J} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)), \]

\[ H^\epsilon \text{ converges weakly-}* \text{ to some } \tilde{K} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)). \]

Our main task in this section is to show that \( \tilde{J} = u \) and \( \tilde{K} = H \) in some sense, where \( (u, H) \) is the strong solution to the viscous incompressible MHD equations (1.16)-(1.18).

Next, we shall also modulate the energy inequality (2.5). The conservation of energy for the viscous incompressible MHD equations (1.16)-(1.18) reads

\[ \frac{1}{2} \int_{\mathbb{T}^d} |u|^2 + |H|^2(t) + \int_0^t \int_{\mathbb{T}^d} \left[ \mu |\nabla u|^2 + \nu |\nabla H|^2 \right] = \frac{1}{2} \int_{\mathbb{T}^d} |u_0|^2 + |H_0|^2. \tag{3.23} \]

Similarly to Step 2, we use \( u \) to test the momentum equation (3.9) to deduce

\[ \int_{\mathbb{T}^d} (\rho' u^\epsilon \cdot u)(t) + \int_0^t \int_{\mathbb{T}^d} \rho' u^\epsilon \cdot \left[ (u \cdot \nabla) u - (H \cdot \nabla) H - \mu \Delta u + \nabla p \right] + \frac{1}{2} \nabla (|H|^2) \]
Summing up (3.3) and (3.23), and inserting (3.5), (3.6) with $H$

By virtue of (3.8) and (3.9), we can rewrite the inequality (3.26) as follows

$$ -\int_0^t \int_{\mathbb{T}^d} [(\rho' u' \otimes u') \cdot \nabla u + (H' \cdot \nabla)H' \cdot u - \mu \nabla u' \cdot \nabla u] = \int_{\mathbb{T}^d} \rho' u'_0 \cdot u_0. \quad (3.24) $$

Then, we test (3.10) by $H$ to infer that

$$ \int_{\mathbb{T}^d} (H' \cdot H)(t) + \int_0^t \int_{\mathbb{T}^d} H' \cdot [(u \cdot \nabla) H - (H \cdot \nabla) u - \nu \Delta H] + \nu \int_0^t \int_{\mathbb{T}^d} \nabla H' \cdot \nabla H $$

$$ + \int_0^t \int_{\mathbb{T}^d} [(\text{div} u') H' + (u' \cdot \nabla)H' - (H' \cdot \nabla)u'] \cdot H = \int_{\mathbb{T}^d} H'_0 \cdot H_0. \quad (3.25) $$

Summing up (3.3) and (3.23), and inserting (3.5), (3.6) with $\mu' \equiv \mu$ and $\lambda' \equiv \lambda$, (3.24), and (3.26) into the resulting inequality, we deduce the following inequality by a straightforward calculation

$$ \frac{1}{2} \int_{\mathbb{T}^d} \left\{ \sqrt{\rho'} u' - u \right\}^2(t) + |H' - H|^2(t) + (\Pi')^2(t) \right\} $$

$$ + \mu \int_0^t \int_{\mathbb{T}^d} |\nabla u' - \nabla u|^2 + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla H' - \nabla H|^2 + (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} |\text{div} u'|^2 $$

$$ + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla H'|^2 + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla H|^2 $$

$$ \leq -\int_0^t \int_{\mathbb{T}^d} \rho' u' \cdot [(H \cdot \nabla)H] - \int_0^t \int_{\mathbb{T}^d} (H' \cdot \nabla)H' \cdot u $$

$$ + \int_0^t \int_{\mathbb{T}^d} (H' \cdot [(u \cdot \nabla) H - (H \cdot \nabla) H] + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \rho' u' \cdot \nabla(|H|^2) $$

$$ + \int_0^t \int_{\mathbb{T}^d} [(\text{div} u') H' + (u' \cdot \nabla)H' - (H' \cdot \nabla)u'] \cdot H + \mu \int_0^t \int_{\mathbb{T}^d} (1 - \rho') u' \Delta u $$

$$ + \int_0^t \int_{\mathbb{T}^d} \rho' u' \cdot [(u \cdot \nabla) u + \nabla p] - \int_0^t \int_{\mathbb{T}^d} (\rho' u' \otimes u') \cdot \nabla u $$

$$ + \int_{\mathbb{T}^d} [(\sqrt{\rho'} - 1) \sqrt{\rho'} u' \cdot u](t) - \int_{\mathbb{T}^d} [(\sqrt{\rho'} - 1) \sqrt{\rho'} u' \cdot u](0) $$

$$ + \frac{1}{2} \int_{\mathbb{T}^d} \left\{ \sqrt{\rho'} u' - u \right\}^2(0) + |H' - H|^2(0) + (\Pi_0')^2 \right\}. \quad (3.26) $$

By virtue of (3.8) and (3.9), we can rewrite the inequality (3.26) as follows

$$ \frac{1}{2} \int_{\mathbb{T}^d} \left\{ \sqrt{\rho'} u' - u \right\}^2(t) + |H' - H|^2(t) + (\Pi')^2(t) \right\} $$

$$ + \mu \int_0^t \int_{\mathbb{T}^d} |\nabla u' - \nabla u|^2 + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla H' - \nabla H|^2 $$

$$ + (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} |\text{div} u'|^2 + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla H'|^2 + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} |\nabla H|^2 $$

$$ \leq \frac{1}{2} \int_{\mathbb{T}^d} \left\{ \sqrt{\rho'} u' - u \right\}^2(0) + |H' - H|^2(0) + (\Pi_0')^2 \right\} $$

$$ + R'_2(t) + R'_4(t) + R'_5(t) + R'_6(t) + R'_7(t) + R'_8(t), \quad (3.27) $$
where $R^e_2(t)$ and $R^e_i(t)$, $i = 4, 5, 6$, are the same as before, and

$$R^e_2(t) = \mu \int_0^t \int_{\mathbb{T}^d} (1 - \rho^e) u^e \Delta u,$$

$$R^e_i(t) = \int_0^t \int_{\mathbb{T}^d} (\nabla \cdot \nabla) H \cdot [(1 - \sqrt{\rho^e}) u^e] - \int_0^t \int_{\mathbb{T}^d} \{(1 - \sqrt{\rho^e}) u^e \cdot \nabla\} H \cdot \nabla u^e.$$ 

Form the previous arguments on $R^e_2(t)$ and $R^e_i(t)$, $i = 4, 5, 6$, we get

$$|R^e_2(t)| + \sum_{i=4}^6 |R^e_i(t)| \leq C T \epsilon. \quad (3.28)$$

Now, we estimate $R^e_2(t)$ and $R^e_5(t)$. Using the inequality $\frac{\mu}{2}$, Holder’s inequality, the estimates on $u^e$ and $\sqrt{\rho^e} u^e$, the regularity of $u$, and $\rho^e - 1 = \rho^e - \sqrt{\rho^e} + \sqrt{\rho^e} - 1$, we obtain

$$|R^e_2(t)| \leq \mu \|\Delta u(t)\|_{L^\infty} \left( \int_0^t \int_{\mathbb{T}^d} |\nabla - 1|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{T}^d} |u^e|^2 \right)^{\frac{1}{2}} + \left( \int_0^t \int_{\mathbb{T}^d} \rho^e \|u^e\|^2 \right)^{\frac{1}{2}}$$

$$\leq C T \epsilon \left( 1 + \frac{1}{\sqrt{\mu}} \right) \leq C T \epsilon \frac{\mu}{\sqrt{\mu}} \quad (3.29)$$

For the term $R^e_5(t)$, we can make use of $\frac{\mu}{2}$, and the estimates on $u^e$ and $H^e$, the regularity of $H$, the assumption $\|\cdot\|_{L^\infty}$, and Sobolev’s imbedding theorem to deduce

$$|R^e_5(t)| \leq \left( \frac{\|H \cdot \nabla H\|_{L^\infty}}{L^\infty} + \|\nabla H(t)\|_{L^\infty} \cdot \|H(t)\|_{L^\infty} \right)$$

$$\left( \int_0^t \int_{\mathbb{T}^d} |\nabla - 1|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{T}^d} |u^e|^2 \right)^{\frac{1}{2}} + \left( \int_0^t \int_{\mathbb{T}^d} \rho^e \|u^e\|^2 \right)^{\frac{1}{2}}$$

$$\leq C T \epsilon \left( 1 + \frac{1}{\sqrt{\mu}} \right) + \|\nabla H(t)\|_{L^\infty} \left( \int_0^t \int_{\mathbb{T}^d} |u^e|^2 \right)^{\frac{1}{2}} \left( \int_0^t \|H^e\|_{L^2} d\tau \right)^{\frac{1}{2}}$$

$$\leq C T \epsilon \left( 1 + \frac{1}{\sqrt{\mu}} \right) + C T \epsilon \left( \|u^e\|_{L^2(0,T;L^2)} + \|\nabla H^e\|_{L^2(0,T;L^2)} \right)$$

$$\leq C T \epsilon \left( 1 + \frac{1}{\sqrt{\mu}} \right) + C T \epsilon \left( 1 + \mu^{-\frac{1}{2}} \cdot \left( 1 + \nu^{-\frac{1}{2}} \right) \right)$$

$$\leq C T \epsilon + C T \epsilon / \sqrt{\mu \nu} \leq C T \epsilon / \sqrt{\mu \nu} \quad (3.30)$$

Now, substituting (3.28), (3.29) into (3.27) and applying Gronwall’s inequality, we conclude

$$\|w^e(t)\|_{L^2}^2 + \|Z^e(t)\|_{L^2}^2 + \|\Pi^e(t)\|_{L^2}^2$$

$$\leq \tilde{C} \left[ \|w^e(0)\|_{L^2}^2 + \|Z^e(0)\|_{L^2}^2 + \|\Pi^e(0)\|_{L^2}^2 + C T \epsilon / \sqrt{\mu \nu} \right], \text{ for a.e. } t \in [0, T], \quad (3.31)$$

where $\tilde{C}$ is defined by (3.22). Combining (2.15) with (3.31) we obtain (2.17). Substituting (3.31) into (3.27), we conclude that $\nabla u^e$ converges to $\nabla u$ strongly in $L^2(0, T; L^2(T^d))$ and $\nabla H^e$ to $\nabla H$ strongly in $L^2(0, T; L^2(T^d))$. This completes the proof of Theorem 2.4. \( \square \)
4. Proof of Theorem 2.5

In this section we shall study the incompressible limit of the compressible MHD equations (1.8)-(1.10) with general initial data. Compared with the case of the well-prepared initial data, the main difficulty here is to control the oscillations caused by the initial data. For simplicity, we assume here that $\mu^* \equiv \mu$, $\lambda^* \equiv \lambda$, and $\nu^* \equiv \nu$ are constants, independent of $\epsilon$, satisfying $\mu > 0$, $2\mu + d\lambda > 0$, and $\nu > 0$.

Proof of Theorem 2.5. As stated in the proof of Theorem 2.4, we obtain from the basic energy inequality (2.3) that, for a.e. $t \in [0, T]$, $\rho^* |\mathbf{u}^*|^2$ and $((\rho^*)^\gamma - 1 - \gamma(\rho^* - 1)) / \epsilon^2$ are bounded in $L^\infty(0, T; L^1)$, $\mathbf{H}^*$ is bounded in $L^\infty(0, T; L^2)$, $\nabla \mathbf{u}^*$ is bounded in $L^2(0, T; L^2)$, and $\nabla \mathbf{H}^*$ is bounded in $L^2(0, T; L^2)$. Therefore, we have

$$\rho^* \to 1 \quad \text{strongly in} \quad C([0, T], L^2_2(\mathbb{T}^d)), \quad (4.1)$$

and $\mathbf{u}^*$ is bounded in $L^2(0, T; L^2)$. The fact that $\rho^* |\mathbf{u}^*|^2$ and $|\mathbf{H}^*|^2$ are bounded in $L^\infty(0, T; L^1)$ gives the following convergence (up to the extraction of a subsequence $\epsilon_n$):

$$\sqrt{\rho^*} \mathbf{u}^* \text{ converges weakly-}\ast \text{ to some } \mathbf{J} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)), \quad (4.2)$$

$$\mathbf{H}^* \text{ converges weakly-}\ast \text{ to some } \mathbf{K} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)). \quad (4.3)$$

Our main task in this section is to show that $\mathbf{J} = \mathbf{u}$ and $\mathbf{K} = \mathbf{H}$ in some sense, where $(\mathbf{u}, \mathbf{H})$ is the strong solution to the incompressible viscous MHD equations (1.16)-(1.18). The key point is to control the oscillations caused by the initial data. This can be done as follows.

Step 1: Description and cancelation of the oscillations.

In order to describe the oscillations caused by the initial data, we employ the “filtering” method which has been used previously by several authors, see [2.7,18,19].

We project the momentum equation (1.9) on the “gradient vector-fields” to find

$$\partial_t Q(\rho^* \mathbf{u}^*) + Q[\text{div}(\rho^* \mathbf{u}^* \otimes \mathbf{u}^*)] - (2\mu + \lambda) \nabla \text{div} \mathbf{u}^* + \frac{1}{2} \nabla(|\mathbf{H}^*|^2)$$

$$- Q[(\mathbf{H}^* \cdot \nabla) \mathbf{H}^*] + \frac{a}{\epsilon^2} \nabla \left( (\rho^*)^\gamma - 1 - \gamma(\rho^* - 1) \right) + \frac{1}{\epsilon^2} \nabla (\rho^* - 1) = 0. \quad (4.2)$$

Noticing $\rho^* = 1 + \epsilon \phi^*$, we can write (1.8) and (1.12) as

$$\epsilon \partial_t \phi^* + \text{div} Q(\rho^* \mathbf{u}^*) = 0, \quad (4.3)$$

$$\epsilon \partial_t Q(\rho^* \mathbf{u}^*) + \nabla \phi^* = \epsilon \mathbf{F}^*, \quad (4.4)$$

where $\mathbf{F}^*$ is given by

$$\mathbf{F}^* = - Q[\text{div}(\rho^* \mathbf{u}^* \otimes \mathbf{u}^*)] + (2\mu + \lambda) \nabla \text{div} \mathbf{u}^* - \frac{1}{2} \nabla(|\mathbf{H}^*|^2) + Q[(\mathbf{H}^* \cdot \nabla) \mathbf{H}^*] - \frac{a}{\epsilon^2} \nabla \left( (\rho^*)^\gamma - 1 - \gamma(\rho^* - 1) \right). \quad (4.5)$$
Therefore, we introduce the following group defined by
\[ L(\tau) = e^{\tau L}, \quad \tau \in \mathbb{R}, \]
where \( L \) is the operator defined on \( \mathcal{D}'(\mathbb{R}^d) \) with \( \mathcal{D}'(\mathbb{R}^d) = \{ \phi \in \mathcal{D}', \int_{\mathbb{R}^d} \phi(x) dx = 0 \} \), by
\[ L(\phi, \psi) = \left( \begin{array}{c} -\text{div} \psi \\ -\nabla \phi \end{array} \right). \]
Then, it is easy to check that \( e^{\tau L} \) is an isometry on each \( H^r \times (H^r)^d \) for all \( r \in \mathbb{R} \) and for all \( \tau \in \mathbb{R} \). Denoting
\[ \bar{\phi}(\tau), \bar{\psi}(\tau) = e^{\tau L}(\phi, \psi), \]
we have
\[ \frac{\partial \bar{\phi}}{\partial \tau} = -\text{div} \bar{\psi}, \quad \frac{\partial \bar{\psi}}{\partial \tau} = -\nabla \bar{\phi}. \]
Thus, \( \frac{\partial^2 \bar{\phi}}{\partial \tau^2} - \Delta \bar{\phi} = 0. \)

In the sequel, we shall denote
\[ U^\varepsilon = \begin{pmatrix} \varphi^\varepsilon \\ Q(\rho^\varepsilon u^\varepsilon) \end{pmatrix}, \quad V^\varepsilon = L\left( -\frac{t}{\varepsilon} \right) \begin{pmatrix} \varphi^\varepsilon \\ Q(\rho^\varepsilon u^\varepsilon) \end{pmatrix}, \]
and use the following approximations
\[ \bar{U}^\varepsilon = \begin{pmatrix} \Phi^\varepsilon \\ Q(\sqrt{\rho^\varepsilon} u^\varepsilon) \end{pmatrix}, \quad \bar{V}^\varepsilon = L\left( -\frac{t}{\varepsilon} \right) \begin{pmatrix} \Phi^\varepsilon \\ Q(\sqrt{\rho^\varepsilon} u^\varepsilon) \end{pmatrix}, \]
which satisfy
\[ \| U^\varepsilon - \bar{U}^\varepsilon \|_{L^\infty(0, T; L^{\frac{2s}{s-1}}(\mathbb{T}^d))} \to 0 \quad \text{as} \quad \varepsilon \to 0. \] (4.6)

With this notation, we can rewrite the equations (4.3)-(4.4) as
\[ \partial_t U^\varepsilon = \frac{1}{\varepsilon} L U^\varepsilon + \bar{F}^\varepsilon, \]
or equivalently
\[ \partial_t V^\varepsilon = L\left( -\frac{t}{\varepsilon} \right) \bar{F}^\varepsilon, \] (4.7)
where (and in what follows) \( \bar{V} \) denotes \((0, \psi)^T\).

It is easy to check that \( F^\varepsilon \), given by (4.5), is bounded in \( L^2(0, T; H^{-s_0}(\mathbb{T}^d)) \) for some \( s_0 \) \((s_0 \in \mathbb{R})\). Hence, \( V^\varepsilon \) is compact in time. Moreover, by virtue of the energy inequality (2.5) and the boundedness of the linear projector \( P \), \( V^\varepsilon \in L^\infty(0, T; L^{\frac{2s}{s-1}}(\mathbb{T}^d)) \) uniformly in \( \varepsilon \). Thus,
\[ V^\varepsilon \text{ converges strongly to some } \bar{V} \text{ in } L^r(0, T; H^{-s'}(\mathbb{T}^d)) \] (4.8)
for all \( s' > s_0 \) and \( 1 < r < \infty \).

Denote \( \theta = 2\mu + \lambda, \quad \mathcal{L}_1(\tau) \) the first component of \( L(\tau) \), and \( \mathcal{L}_2(\tau) \) the last \( d \) components of \( L(\tau) \). If we had sufficient compactness in space, then we could pass the limit in (4.7) and obtain the following limit system for the oscillating parts
\[ \partial_t \bar{V} + Q_1(u, \bar{V}) + Q_2(\bar{V}, \bar{V}) - \frac{\theta}{2} \Delta \bar{V} = 0, \] (4.9)
where $u$ is the strong solution of the viscous incompressible MHD equations \([1.16]-[1.18]\). $Q_1$ is a linear form of $V$ defined by

$$Q_1(v, V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \left( \begin{array}{c} 0 \\
\text{div}(v \otimes \mathcal{L}_2(s)V + \mathcal{L}_2(s)V \otimes v) \end{array} \right) ds,$$  \quad (4.10)

and $Q_2$ is a bilinear form of $V$ defined by

$$Q_2(V, V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \left( \begin{array}{c} 0 \\
\text{div}(\mathcal{L}_2(s)V \otimes \mathcal{L}_2(s)V) + \frac{2-\epsilon}{2} \nabla(\mathcal{L}_1(s)V)^2 \end{array} \right) ds$$  \quad (4.11)

for any divergence-free vector field $v \in L^2(\mathbb{T}^d)^	ext{d}$ and any $V = (\phi, \nabla q)^T \in L^2(\mathbb{T}^d)^{d+1}$. Actually, the convergence in \(4.10\) and \(4.11\) can be guaranteed by the following Proposition.

**Proposition 4.1** \([19]\). For all $v \in L^{r_1}(0, T; L^2)$ and $V \in L^{r_2}(0, T; L^2)$, we have the following weak convergences ($r_1$ and $r_2$ are such that the products are well defined)

$$w - \lim_{\epsilon \to 0} \mathcal{L} \left( \begin{array}{c} 0 \\
\text{div}(v \otimes \mathcal{L}_2(\frac{t}{\epsilon})V + \mathcal{L}_2(\frac{t}{\epsilon})V \otimes v) \end{array} \right) = Q_1(v, V),$$  \quad (4.12)

$$w - \lim_{\epsilon \to 0} \mathcal{L} \left( \begin{array}{c} 0 \\
\text{div}(\mathcal{L}_2(\frac{t}{\epsilon})V \otimes \mathcal{L}_2(\frac{t}{\epsilon})V) + \frac{2-\epsilon}{2} \nabla(\mathcal{L}_1(\frac{t}{\epsilon})V)^2 \end{array} \right) = Q_2(V, V).$$  \quad (4.13)

The viscosity term in the oscillation equations \(4.9\) is obtained by the following Proposition.

**Proposition 4.2** \([19]\). Suppose that the same hypothesis as in Proposition 4.1 on $V$ holds. Then, we have

$$\frac{\theta}{2} \nabla V = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \left( \begin{array}{c} 0 \\
\theta \mathcal{L}_2(s)V \end{array} \right) ds.$$  \quad (4.14)

The following propositions, the proof of which can be found in \([19]\), play an important role in our subsequent analysis.

**Proposition 4.3** \([19]\). For all $v, V, V_1$ and $V_2$ (regular enough to define all the products), we have

$$\int_{\mathbb{T}^d} Q_1(v, V) V = 0, \quad \int_{\mathbb{T}^d} Q_2(V, V) V = 0,$$  \quad (4.15)

$$\int_{\mathbb{T}^d} [Q_1(v, V_1)V_2 + Q_1(v, V_2)V_1] = 0,$$  \quad (4.16)

$$\int_{\mathbb{T}^d} [Q_2(V_1, V_1)V_2 + 2 Q_2(V_1, V_2)V_1] = 0.$$  \quad (4.17)

**Proposition 4.4** \([19]\). Using the symmetry of $Q_2$, we can extend the equality \(4.13\) in Proposition 4.1 to the case:

$$w - \lim_{\epsilon \to 0} \mathcal{L} \left( \begin{array}{c} 0 \\
\text{div}[\mathcal{L}_2(\frac{t}{\epsilon})V_1 \otimes \mathcal{L}_2(\frac{t}{\epsilon})V_2 + \mathcal{L}_2(\frac{t}{\epsilon})V_2 \otimes \mathcal{L}_2(\frac{t}{\epsilon})V_1] \end{array} \right)$$
Moreover, the above identity holds for $V_1 \in L^q(0, T; H^r)$ and $V_2 \in L^p(0, T; H^{-r})$ with $r \in \mathbb{R}$ and $1/p + 1/q = 1$. Also, (4.18) can be extended to the case where we replace $V_2$ in the left-hand side by a sequence $V_2^\varepsilon$ such that $V_2^\varepsilon$ converges strongly to $V_2$ in $L^p(0, T; H^{-r})$.

**Step 2: The modulated energy functional and uniform estimates.**
Let $V^0$ be the solution of the following system

$$
\partial_t V^0 + Q_1(u, V^0) + Q_2(V^0, V^0) - \frac{\theta}{2} \Delta V^0 = 0
$$

with initial data

$$
V^0|_{t=0} = (\varphi_0, Q u_0)^T,
$$

where $u$ is the strong solution of the viscous incompressible MHD equations [(1.16)-(1.18)] with initial velocity $u_0$. From [19], we know that the Cauchy problem [(4.19)-(4.20)] has a unique global strong solution.

In order to prove the convergence results in Theorem 2.5, we have to bound the term

$$
\left\| \sqrt{\rho} u^\varepsilon - u - L_2 \left( \frac{t}{\varepsilon} \right) V^0 \right\|^2_{L^2(\mathbb{T}^d)} + \left\| H^\varepsilon - H \right\|^2_{L^2(\mathbb{T}^d)} + \left\| \Pi^\varepsilon - L_1 \left( \frac{t}{\varepsilon} \right) V^0 \right\|^2_{L^2(\mathbb{T}^d)}.
$$

To this end, we first recall the following energy inequality of the compressible MHD equations [(1.16)-(1.18)]:

$$
\frac{1}{2} \int_{\mathbb{T}^d} \left[ \rho'(t)|u^\varepsilon|^2(t) + |H|^{2}(t) + (\Pi')(t)^2 \right] + \mu \int_0^t \int_{\mathbb{T}^d} |\nabla u|^2 + \nu \int_0^t \int_{\mathbb{T}^d} |\nabla H|^2
\leq \frac{1}{2} \int_{\mathbb{T}^d} \left[ \rho_0 |u_0|^2 + |H_0|^2 + (\Pi_0)^2 \right], \quad \text{for a.e. } t \in [0, T].
$$

On the other hand, the conservation of energy for the incompressible viscous MHD equations [(1.16)-(1.18)] reads

$$
\frac{1}{2} \int_{\mathbb{T}^d} |u|^2(t) + |H|^{2}(t) + \int_0^t \int_{\mathbb{T}^d} [\mu |\nabla u|^2 + \nu |\nabla H|^2] = \frac{1}{2} \int_{\mathbb{T}^d} |u_0|^2 + |H_0|^2.
$$

For the system [(4.19)], Proposition 4.3 implies that

$$
\int_{\mathbb{T}^d} Q_1(u, V^0) V^0 = 0, \quad \int_{\mathbb{T}^d} Q_2(V^0, V^0) V^0 = 0,
$$

from which the following equality follows.

$$
\frac{1}{2} \int_{\mathbb{T}^d} |V^0|^2 + \frac{\theta}{2} \int_{\mathbb{T}^d} |\nabla V^0|^2 = \frac{1}{2} \int_{\mathbb{T}^d} |V^0(t = 0)|^2.
$$
Using \( \mathcal{L}_1(\frac{T}{\epsilon})\mathbf{V}^0 \) as a test function and noticing \( \rho' = 1 + \epsilon \varphi' \), we obtain the following weak formulation of the continuity equation (1.8):

\[
\int_{T^d} \mathcal{L}_1\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \varphi'(t) + \frac{1}{\epsilon} \int_0^t \int_{T^d} \left[ \text{div}\left( \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) \varphi' + \text{div}(\rho' \mathbf{u'} \cdot \mathbf{V}^0) \right] \\
- \int_0^t \int_{T^d} \mathcal{L}_1\left(\frac{T}{\epsilon}\right) \partial_t \mathbf{V}^0 \varphi' = \int_{T^d} \varphi_0 \varphi'_0.
\]

(4.24)

We use \( \mathbf{u} \) and \( \mathcal{L}_2(\frac{T}{\epsilon})\mathbf{V}^0 \) to test the momentum equation (1.9) respectively, to deduce

\[
\int_{T^d} (\rho' \mathbf{u'} \cdot \mathbf{u})(t) + \int_0^t \int_{T^d} \rho' \mathbf{u'} \cdot \left[ (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{H} \cdot \nabla) \mathbf{H} - \mu \Delta \mathbf{u} + \nabla p + \frac{1}{2} \nabla(|\mathbf{H}|^2) \right] \\
- \int_0^t \int_{T^d} \left[ (\rho' \mathbf{u'} \otimes \mathbf{u'}) \cdot \nabla \mathbf{u} + (\mathbf{H'} \cdot \nabla) \mathbf{H} \cdot \mathbf{u} - \mu \nabla \mathbf{u'} \cdot \nabla \mathbf{u} \right] = \int_{T^d} \rho'_0 \mathbf{u}_0' \cdot \mathbf{u}_0
\]

(4.25)

and

\[
\int_{T^d} (\rho' \mathbf{u'} \cdot \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0)(t) + \frac{1}{\epsilon} \int_0^t \int_{T^d} \rho' \mathbf{u'} \cdot \nabla\left( \mathcal{L}_1\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) \\
- \int_0^t \int_{T^d} \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \partial_t \mathbf{V}^0 \varphi' - \int_0^t \int_{T^d} (\rho' \mathbf{u'} \otimes \mathbf{u'}) \cdot \nabla\left( \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) \\
+ \int_0^t \int_{T^d} \left[ \mu \nabla \mathbf{u'} \cdot \nabla\left( \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) + (\mu + \lambda) \text{div} \mathbf{u'} \text{div} \left( \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) \right] \\
- \int_0^t \int_{T^d} (\mathbf{H'} \cdot \nabla) \mathbf{H} \cdot \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 - \int_0^t \int_{T^d} \frac{1}{2} |\mathbf{H'}|^2 \text{div} \left( \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) \\
- \int_0^t \int_{T^d} \left( 1 - \frac{1}{2} \gamma \right) \text{div} \left( \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) = \int_{T^d} \rho'_0 \mathbf{u}_0' \cdot Q \mathbf{u}_0.
\]

(4.26)

Similarly, we test (1.10) by \( \mathbf{H} \) to get

\[
\int_{T^d} (\mathbf{H'} \cdot \mathbf{H})(t) + \int_0^t \int_{T^d} \mathbf{H'} \cdot \left[ (\mathbf{u} \cdot \nabla) \mathbf{H} - (\mathbf{H} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{H} \right] + \nu \int_0^t \int_{T^d} \nabla \mathbf{H'} \cdot \nabla \mathbf{H} \\
+ \int_0^t \int_{T^d} \left[ (\text{div} \mathbf{u'}) \mathbf{H'} + (\mathbf{u'} \cdot \nabla) \mathbf{H'} - (\mathbf{H'} \cdot \nabla) \mathbf{u'} \right] \cdot \mathbf{H} = \int_{T^d} \mathbf{H}_0' \cdot \mathbf{H}_0.
\]

(4.27)

Summing up (4.21), (4.22) and (4.23), inserting (4.24)-(4.27) into the resulting inequality, and using the fact

\[
\int_0^t \int_{T^d} \left( \frac{T}{\epsilon} \right) \partial_t \mathbf{V}^0 \cdot \mathbf{U} = \int_0^t \int_{T^d} \partial_t \mathbf{V}^0 \cdot \mathbf{V}^0,
\]

we deduce, after a straightforward calculation, the following inequality:

\[
\frac{1}{2} \int_{T^d} \left\{ \left| \sqrt{\rho'} \mathbf{u'} - \mathbf{u} - \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right|^2 (t) + |\mathbf{H'} - \mathbf{H}|^2 (t) + \left| \Pi' - \mathcal{L}_1\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right|^2 (t) \right\} \\
+ \mu \int_0^t \int_{T^d} \left| \nabla \left( \mathbf{u'} - \mathbf{u} - \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right) \right|^2 + \nu \int_0^t \int_{T^d} |\nabla \mathbf{H'} - \nabla \mathbf{H}|^2 \\
+ \frac{\mu}{2} \int_0^t \int_{T^d} \left( |\nabla \mathbf{H'}|^2 + |\nabla \mathbf{H}|^2 \right) + (\mu + \lambda) \int_0^t \int_{T^d} \text{div} \left( \mathbf{u'} - \mathbf{u} - \mathcal{L}_2\left(\frac{T}{\epsilon}\right) \mathbf{V}^0 \right)^2
\]
where

\[ A^i_1(t) = \int_{\mathbb{T}^d} \left[ (\sqrt{\rho'} - 1)\sqrt{\rho'} u^r \cdot \left( u + L_2 \left( \frac{T}{\epsilon} \right) \right) \right] (t) - \int_{\mathbb{T}^d} \left[ (\Pi' - \varphi') L_1 \left( \frac{T}{\epsilon} \right) \right] (0) + \int_{\mathbb{T}^d} \left[ (\Pi' - \varphi') L_1 \left( \frac{T}{\epsilon} \right) \right] (0), \tag{4.29} \]

\[ A^i_2(t) = \int_{0}^{t} \int_{\mathbb{T}^d} |\rho' - 1|u^r \cdot \nabla p - \mu \int_{0}^{t} \int_{\mathbb{T}^d} (\rho' - 1)u^r \Delta u, \tag{4.30} \]

\[ A^i_3(t) = -\frac{\theta}{2} \int_{0}^{t} \int_{\mathbb{T}^d} \Delta \psi \cdot \nabla \mu - \mu \int_{0}^{t} \int_{\mathbb{T}^d} \nabla u^r \cdot \nabla \left( L_2 \left( \frac{T}{\epsilon} \right) \right), \tag{4.31} \]

\[ A^i_4(t) = -\frac{\theta}{2} \int_{0}^{t} \int_{\mathbb{T}^d} |\nabla \psi|^2 + \mu \int_{0}^{t} \int_{\mathbb{T}^d} |\nabla L_2 \left( \frac{T}{\epsilon} \right) |^2 + (\mu + \lambda) \int_{0}^{t} \int_{\mathbb{T}^d} \left( \int_{0}^{t} |\nabla L_2 \left( \frac{T}{\epsilon} \right) |^2 \right), \tag{4.32} \]

\[ A^i_5(t) = \int_{0}^{t} \int_{\mathbb{T}^d} \rho' u^r \cdot [(H \cdot \nabla)H] - \int_{0}^{t} \int_{\mathbb{T}^d} (H^r \cdot \nabla)H^r \cdot u \]

\[ + \int_{0}^{t} \int_{\mathbb{T}^d} H^r \cdot [(u \cdot \nabla)H - (H \cdot \nabla)H] + \frac{1}{2} \int_{0}^{t} \int_{\mathbb{T}^d} \rho' u^r \cdot \nabla |H|^2 \]

\[ + \int_{0}^{t} \int_{\mathbb{T}^d} [(\psi u^r) \cdot (H^r \cdot \nabla)H^r - (H^r \cdot \nabla)u^r] \cdot H, \tag{4.33} \]

\[ A^i_6(t) = \int_{0}^{t} \int_{\mathbb{T}^d} \rho' u^r \cdot [(u \cdot \nabla)u] - \int_{0}^{t} \int_{\mathbb{T}^d} \rho' u^r \otimes u^r \cdot \nabla u + L_2 \left( \frac{T}{\epsilon} \right) \]

\[ - \frac{\gamma - 1}{2} \int_{0}^{t} \int_{\mathbb{T}^d} (\Pi')^2 \nabla \cdot \left( L_2 \left( \frac{T}{\epsilon} \right) \right), \tag{4.34} \]

\[ A^i_7(t) = \int_{0}^{t} \int_{\mathbb{T}^d} \left[ \Omega_1 (u, \psi, \nabla) + \Omega_2 (\psi, \nabla) \right] \cdot \nabla 

\[ \leq \frac{1}{2} \int_{\mathbb{T}^d} \left\{ |\sqrt{\rho'} u^r - u - L_2 \left( \frac{T}{\epsilon} \right) \psi_0|^2(0) + |F - H|^2(0) + |F' - L_1 \left( \frac{T}{\epsilon} \right) \psi_0|^2(0) \right\} \]

\[ + \sum_{i=1}^{8} A^i_1(t), \tag{4.28} \]

Step 3: Convergence of the modulated energy functional.

To show the convergence of the modulated energy functional (4.28), we need to estimate the remainders $A^i_1(t), i = 1, \ldots, 8$. In the sequel, we will denote by $\omega^r(t)$ any sequence of time-dependent functions which converges to 0 uniformly in $t$. For convenience, we also denote $w^r = \sqrt{\rho'} u^r - u - L_2 \left( \frac{T}{\epsilon} \right) \psi_0, Z^r = H^r - H,$ and $\Psi^r = \Pi' - L_1 \left( \frac{T}{\epsilon} \right) \psi_0$.

For the term $A^i_1(t)$, we employ (2.5), (3.13), the regularity of $u$ and (4.6), and follow a procedure similar to that for $R^i_2(t)$, to obtain

\[ |A^i_1(t)| \leq C_T \epsilon + \omega^r(t). \tag{4.37} \]
On the other hand, the term $A_2^t(t)$ has the same bound as $R_3^t(t) + R_7^t(t)$, thus

$$|A_2^t(t)| \leq C_T \epsilon.$$  \hfill (4.38)

To bound the term $A_3^t(t)$, we integrate by parts and use the fact $\mathcal{L}_2(\hat{q})V^0 = \nabla \hat{q}^\epsilon$ for some function $\hat{q}^\epsilon$ and Proposition 4.2 to infer

$$\frac{1}{2} \mu \int_0^t \int_{\mathbb{T}^d} \Delta V \cdot V^0 + \omega^\epsilon(t),$$

$$-(\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} \text{div} u^\epsilon \cdot \text{div} \left[ \mathcal{L}_2 \left( \frac{Z}{\epsilon} \right) V^0 \right] = (\mu + \lambda) \int_0^t \int_{\mathbb{T}^d} \Delta u^\epsilon \cdot \mathcal{L}_2 \left( \frac{Z}{\epsilon} \right) V^0$$

$$= \frac{1}{2} \mu \int_0^t \int_{\mathbb{T}^d} \Delta V \cdot V^0 + \omega^\epsilon(t),$$

and

$$-\frac{\theta}{2} \int_0^t \int_{\mathbb{T}^d} \Delta V^0 \cdot V^\epsilon = -\frac{\theta}{2} \int_0^t \int_{\mathbb{T}^d} \Delta V \cdot V^0 + \omega^\epsilon(t).$$

Thus, recalling $\theta = 2\mu + \lambda$, one has

$$A_3^t(t) = \omega^\epsilon(t).$$  \hfill (4.39)

Similarly, it follows from Proposition 4.2 that

$$A_4^t(t) = \omega^\epsilon(t).$$  \hfill (4.40)

Recalling (4.38) and using Hölder’s inequality, the inequalities (2.34) and (6.13), the regularity of $H$, and Sobolev’s imbedding theorem, we conclude

$$A_5^t(t) \leq \int_0^t \int_{\mathbb{T}^d} \left( 1 - \rho^\epsilon \right) u^\epsilon \cdot \left[ (H \cdot \nabla)H \right] + \int_0^t \left\| Z^\epsilon(\tau) \right\|_{L^2}^2 \left\| \nabla u(\tau) \right\|_{L^\infty} d\tau$$

$$+ \int_0^t \left[ \left\| w^\epsilon(\tau) \right\|_{L^2}^2 + \left\| Z^\epsilon(\tau) \right\|_{L^2}^2 \right] \left\| \nabla H(\tau) \right\|_{L^\infty} d\tau$$

$$+ \int_0^t \int_{\mathbb{T}^d} \left[ (1 - \sqrt{\rho^\epsilon}) u^\epsilon + \mathcal{L}_2 \left( \frac{Z}{\epsilon} \right) V^0 \right] \cdot \nabla \left[ (1 - \sqrt{\rho^\epsilon}) u^\epsilon + \mathcal{L}_2 \left( \frac{Z}{\epsilon} \right) V^0 \right]$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \left( \rho^\epsilon - 1 \right) u^\epsilon \nabla \| H \|^2$$

$$\leq \int_0^t \left[ \left\| w^\epsilon(\tau) \right\|_{L^2}^2 + \left\| Z^\epsilon(\tau) \right\|_{L^2}^2 \right] \cdot \left[ \left\| \nabla u(\tau) \right\|_{L^\infty} + \left\| \nabla H(\tau) \right\|_{L^\infty} \right] d\tau$$

$$+ \int_0^t \int_{\mathbb{T}^d} \left[ H^\epsilon \cdot \nabla \right] H \cdot \mathcal{L}_2 \left( \frac{Z}{\epsilon} \right) V^0 - \left( \mathcal{L}_2 \left( \frac{Z}{\epsilon} \right) V^0 \cdot \nabla \right) H \cdot H^\epsilon \right\] d\tau + C_T \epsilon.$$  \hfill (4.41)
The term $A_6^e(t)$ can be rewritten as

$$A_6^e(t) = -\int_0^t \int_{T^d} (\omega^e \otimes \omega^e) \cdot \nabla \left( \mathbf{u} + \mathcal{L} \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) - \frac{\gamma - 1}{2} \int_0^t \int_{T^d} |\Psi|^2 \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
- \int_0^t \int_{T^d} \left\{ \sqrt{\rho} \mathbf{u}^e \otimes \left( \mathbf{u} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + \left( \mathbf{u} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \otimes \sqrt{\rho} \mathbf{u}^e \right\} \cdot \nabla \left( \mathbf{u} + \mathcal{L} \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
+ \int_0^t \int_{T^d} \left( \mathbf{u} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \otimes \left( \mathbf{u} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \cdot \nabla \left( \mathbf{u} + \mathcal{L} \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
+ \int_0^t \int_{T^d} \frac{\gamma - 1}{2} \left| \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right|^2 \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) - (\gamma - 1) \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \Pi^e \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
+ \int_0^t \int_{T^d} (\rho^e - \sqrt{\rho}) \mathbf{u}^e \cdot (\left( \mathbf{u} \cdot \nabla \right) \mathbf{u}) - \int_0^t \int_{T^d} (\sqrt{\rho} \mathbf{u}^e - \mathbf{u}) \cdot \nabla \left( \frac{\mathbf{u}^2}{2} \right). \tag{4.42}$$

We have to bound all the terms on the right-hand side of (4.42). Keeping in mind that $\text{div} \mathbf{v} = 0$ and applying Proposition 4.4, one obtains

$$\int_0^t \int_{T^d} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \otimes \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \cdot \nabla \left( \mathbf{u} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + \frac{\gamma - 1}{2} \left| \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right|^2 \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
= -\int_0^t \int_{T^d} \left\{ \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \otimes \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + \frac{\gamma - 1}{2} \nabla \left| \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right|^2 \right\} \cdot (\mathbf{V}^0 + \hat{\mathbf{u}})
= -\int_0^t \int_{T^d} \mathcal{L} \left( - \frac{\tau}{\varepsilon} \right) \left( \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) \otimes \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + \frac{\gamma - 1}{2} \nabla \left| \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right|^2 \right) \cdot (\mathbf{V}^0 + \hat{\mathbf{u}})
= -\int_0^t \int_{T^d} Q_2(\mathbf{V}^0, \mathbf{V}^0) \cdot (\mathbf{V}^0 + \hat{\mathbf{u}}) + \omega^e(t) = \omega^e(t). \tag{4.43}$$

From Proposition 4.4 we get

$$-\int_0^t \int_{T^d} \left\{ \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \otimes \sqrt{\rho} \mathbf{u}^e + \sqrt{\rho} \mathbf{u}^e \otimes \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right\} \cdot \nabla \left( \mathbf{v} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
- \int_0^t \int_{T^d} (\gamma - 1) \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \Pi^e \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
= \int_0^t \int_{T^d} \left\{ \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \otimes \sqrt{\rho} \mathbf{u}^e + \sqrt{\rho} \mathbf{u}^e \otimes \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + (\gamma - 1) \nabla \left( \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \Pi^e \right) \right\}
\cdot \left( \mathbf{u} + \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right)
= \int_0^t \int_{T^d} \mathcal{L} \left( - \frac{\tau}{\varepsilon} \right) \left( \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \otimes \sqrt{\rho} \mathbf{u}^e + \sqrt{\rho} \mathbf{u}^e \otimes \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + (\gamma - 1) \nabla \left( \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \Pi^e \right) \right)
\cdot (\mathbf{V}^0 + \hat{\mathbf{u}})
= \int_0^t \int_{T^d} \mathcal{L} \left( - \frac{\tau}{\varepsilon} \right) \left( \text{div} \left( \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \otimes Q(\sqrt{\rho} \mathbf{u}^e) + Q(\sqrt{\rho} \mathbf{u}^e) \otimes \mathcal{L}_2 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \right) + (\gamma - 1) \nabla \left( \mathcal{L}_1 \left( \frac{\tau}{\varepsilon} \right) \mathbf{V}^0 \Pi^e \right) \right)
\cdot (\mathbf{V}^0 + \hat{\mathbf{u}}) \tag{4.44}$$
Similarly, we have

\[
\int_0^t \int_{T^d} \{ \sqrt{\rho} \mathbf{u}^t \otimes \mathbf{u} + \mathbf{u} \otimes \sqrt{\rho} \mathbf{u}^t \} \cdot \nabla \left( \mathbf{u} + L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \right) = \int_0^t \int_{T^d} \mathcal{Q}_1 \left( \mathbf{u}, \mathbf{V}^0 \right) \mathbf{V}^0 + \mathbf{w}^\epsilon(t) \right),
\]

and

\[
\int_0^t \int_{T^d} \left\{ \mathbf{u} \otimes L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 + L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \otimes \mathbf{u} \right\} \cdot \nabla \left( \mathbf{u} + L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \right) = -\int_0^t \int_{T^d} \mathcal{Q}_1 \left( \mathbf{u}, \mathbf{V}^0 \right) \mathbf{V}^0 + \mathbf{w}^\epsilon(t).
\]

On the other hand, using the basic energy equality (2.5) and the regularity of \( \mathbf{u} \), following the same process as for \( R_5^\epsilon(t) \), we obtain

\[
\left| \int_0^t \int_{T^d} \left( \rho^\epsilon - \sqrt{\rho} \right) \mathbf{u}^t \cdot \left( \mathbf{u} \otimes \nabla \mathbf{u} \right) - \int_0^t \int_{T^d} \left( \sqrt{\rho} \mathbf{u}^t - \mathbf{u} \right) \cdot \nabla \left( \frac{\| \mathbf{u} \|^2}{2} \right) \right| \leq C_T \epsilon.
\]

The term \( A_5^\epsilon(t) \) can be rewritten as

\[
A_5^\epsilon(t) = \int_0^t \int_{T^d} \left[ \mathcal{Q}_1 \left( \mathbf{u}, \mathbf{V}^0 \right) + \mathcal{Q}_2 \left( \mathbf{V}^0, \mathbf{V}^0 \right) \right] \cdot \mathbf{V} + \mathbf{w}^\epsilon(t).
\]

Substituting (4.43)-(4.48) into (4.42), we conclude that

\[
|A_6^\epsilon(t)| + |A_5^\epsilon(t)| \leq C \int_0^t \left( \| w^\epsilon(t) \|^2 + \| \Psi^\epsilon(t) \|^2 \right) \left( \| \nabla \mathbf{u} \|_{L^\infty} + \left\| \nabla L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \right\|_{L^\infty} \right) d\tau + C_T \epsilon + \mathbf{w}^\epsilon(t).
\]

Thus, we insert the estimates on \( A_i^\epsilon(t) \) (\( i = 1, \cdots, 7 \)) into (4.28) to obtain

\[
\int_0^t \int_{T^d} \left\{ |w^\epsilon(t)|^2 + |Z^\epsilon(t)|^2 \right\} (t) + \mu \int_0^t \int_{T^d} \left| \nabla \left( \mathbf{u}^\epsilon(t) - \mathbf{u} - L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \right) \right|^2 + \frac{\nu}{2} \int_0^t \int_{T^d} \left( \| \nabla \mathbf{H}^\epsilon(t) \|_{L^2}^2 + \| \nabla \mathbf{H}^\epsilon(t) \|_{L^2}^2 \right) + \left( \mu + \lambda \right) \int_0^t \int_{T^d} \left| \nabla \left( L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \right) \right|^2 \leq \int_0^t \left( \| w^\epsilon(t) \|_{L^2}^2 + \| Z^\epsilon(t) \|_{L^2}^2 \right) \left[ \| \nabla \mathbf{u}(t) \|_{L^\infty} + \| \mathbf{H}(t) \|_{L^\infty} + \| \nabla L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \|_{L^\infty} \right] d\tau
\]

\[
+ \frac{1}{2} \int_0^t \int_{T^d} \left\{ |w^\epsilon(0)|^2 + |Z^\epsilon(0)|^2 + |\Psi^\epsilon(0)|^2 \right\} + C_T \epsilon + \mathbf{w}^\epsilon(t) + A_6^\epsilon(t) + A_5^\epsilon(t),
\]

where

\[
A_5(t) = \int_0^t \int_{T^d} \left[ \left( \nabla \cdot \mathbf{H} \cdot \nabla \cdot \mathbf{H} \right) L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 - \left( L \left( \frac{T}{\epsilon} \right) \mathbf{V}^0 \cdot \nabla \right) \mathbf{H} \cdot \mathbf{H}^t \right].
\]
Now, to deal with \( A'_\epsilon(t) \) and \( A''_\epsilon(t) \), we denote \( \bar{H}'_0 = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} H'_0(x)dx \) to deduce from the magnetic field equation (1.10) that

\[
\int_{\mathbb{T}^d} H'(x,t)dx = \int_{\mathbb{T}^d} H'_0(x)dx = \bar{H}'_0|\mathbb{T}^d|.
\]  

(4.52)

The assumption that \( H'_0 \) converges strongly in \( L^2 \) to some \( H'_0 \) implies

\[
\left| \int_{\mathbb{T}^d} H'_0(x) - H'_0(x)dx \right| \leq \int_{\mathbb{T}^d} |H'_0(x) - H'_0(x)|dx \\
\leq |\mathbb{T}^d|^{\frac{1}{2}} \left( \int_{\mathbb{T}^d} |H'_0(x) - H'_0(x)|^2dx \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad \epsilon \to 0,
\]

whence

\[
\bar{H}'_0 \to \bar{H}'_0 \quad \text{as} \quad \epsilon \to 0.
\]  

(4.53)

Using Hölder’s inequality, Sobolev’s imbedding theorem, Poincaré’s inequality, the isometry property of \( L \), and (4.52) and (4.53), we find that

\[
|A'_\epsilon(t)| + |A''_\epsilon(t)| \leq \frac{\nu}{4} \int_0^t \left[ ||\nabla H'||^2 + ||\nabla H'||^2 \right] d\tau + \omega(t) \\
+ \frac{1}{\nu} \left( ||\varphi_0||^2_{H^2} + ||Q\varphi_0||^2_{H^2} \right) \int_0^t \left[ ||\nabla H'||^2 + ||\nabla H'||^2 \right] d\tau.
\]  

(4.54)

Thus, substituting (4.53) into (4.50) and using (2.18), we deduce by Gronwall’s inequality that, for almost all \( t \in [0, T] \),

\[
||w'(t)||^2_{L^2} + ||z'(t)||^2_{L^2} + ||\psi'(t)||^2_{L^2} \\
\leq \tilde{C} \left\{ ||w'(0)||^2_{L^2} + ||z'(0)||^2_{L^2} + ||\Pi'_0 - \varphi_0||^2_{L^2} + C\tau \epsilon + \sup_{0 \leq s \leq t} \omega'(s) \right\},
\]  

(4.55)

where

\[
\tilde{C} = \exp \left\{ C \int_0^T \left[ ||\nabla u(\tau)||_L^\infty + ||\nabla H(\tau)||_L^\infty + ||\nabla L_2 \left( \frac{t}{\epsilon} \right) V^0||_L^\infty \right] d\tau \right\} < +\infty.
\]

Step 4: End of the proof of Theorem 2.5

Letting \( \epsilon \) go to zero in (4.55), we see that \( H' \) converges strongly to \( H \) in \( L^\infty(0, T; L^2(\mathbb{T}^d)) \).

Hence, \( \bar{K} = H \). Combining (4.50) with (4.54), we can easily prove that \( \nabla H' \) converges strongly to \( \nabla H \) in \( L^2(0, T; L^2(\mathbb{T}^d)) \).

Next, it suffices to prove (4) and (5) in Theorem 2.5 Noting that \( P(L_2 \left( \frac{t}{\epsilon} \right) V^0) = 0 \) and the fact that the projection operator \( P \) is a bounded linear mapping from \( L^2 \) to \( L^2 \), we obtain, with the help of (4.55), that

\[
\sup_{0 \leq t \leq T} ||P(\sqrt{\rho'} u') - u||_{L^2} = \sup_{0 \leq t \leq T} \left\| P \left( \sqrt{\rho'} u' - u - L_2 \left( \frac{t}{\epsilon} \right) V^0 \right) \right\|_{L^2} \\
\leq \sup_{0 \leq t \leq T} \left\| \sqrt{\rho'} u' - u - L_2 \left( \frac{t}{\epsilon} \right) V^0 \right\|_{L^2} \\
\to 0 \quad \text{as} \quad \epsilon \to 0.
\]  

(4.56)
Therefore, (4) is proved. Utilizing (4.1), we deduce from (1.8) that \( \text{div} \ u^\varepsilon \) converges weakly to 0 in \( H^{-1}((0, T) \times \mathbb{T}^d) \). Thus we obtain easily that \( Q u^\varepsilon \) converges weakly to 0 in \( H^{-1}(0, T; L^2(\mathbb{T}^d)) \). In view of the fact that \( u^\varepsilon \) is bounded in \( L^2(0, T; L^2(\mathbb{T}^d)) \) and \( \sqrt{\rho^\varepsilon} \) converges strongly to 1 in \( C([0, T], L^2(\mathbb{T}^d)) \), we see that \( Q(\sqrt{\rho^\varepsilon} u^\varepsilon) \) converges weakly to 0 in \( H^{-1}(0, T; L^2(\mathbb{T}^d)) \). Obviously, the fact \( \sqrt{\rho^\varepsilon} u^\varepsilon = P(\sqrt{\rho^\varepsilon} u^\varepsilon) + Q(\sqrt{\rho^\varepsilon} u^\varepsilon) \) implies the weak convergence of \( \sqrt{\rho^\varepsilon} u^\varepsilon \) to \( u \) in \( H^{-1}(0, T; L^2(\mathbb{T}^d)) \). The proof of Theorem 2.5 is finished.

Theorem 2.6 can be shown by slightly modifying the proof of Theorem 2.5, and therefore, we omit its proof here.

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