Nonequilibrium Critical Relaxation in the 2D Random Field Ising Model

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Abstract. A particular type of nonequilibrium critical dynamics is discussed in this paper, in which the initial state is generated in the square lattice Ising model with random fields of zero mean and variance $h$. Thus we interpolate between the ordered ($h = 0$) and disordered ($h \to \infty$) initial states. In particular we consider the square lattice Ising model in which for weak enough random fields the clusters of parallel spins are percolating in the initial state. We aim to study the effect of correlations in the initial state on the nonequilibrium dynamics, so we measure the nonequilibrium relaxation of the magnetization and the autocorrelation function. The relaxation of the magnetization, characterized by the initial slip exponent, $\theta$, is not altered by the percolating nature of the initial state. However, the starting part of the magnetization curve is affected by a cluster dissolution effect leading to a reentrance in time.

1. Introduction
Nonequilibrium relaxation has been thoroughly investigated in several papers [1, 2, 3, 4], mainly choosing the high temperature initial phase ($T_{pre} \to \infty$), and quenching the system to the critical temperature $T_c$. Provided a small initial magnetization $m_0$ is introduced in the system, local fluctuations will initiate coarsening, which leads to increasing magnetization. As growing clusters will touch each other, the competition of these will reduce order on the terms of an equilibrium process. For an Ising model with $\sigma = \pm 1$ one measures magnetization $m(t) = \langle \sigma(t) \rangle$ and autocorrelation $G(t, s) = \langle \sigma(t)\sigma(s) \rangle$. A dynamic scaling theory has been derived with an $\epsilon$-expansion up to two loop order by Janssen, Schaub and Schmittman [5]. According to this the $k$th moment of the magnetization has the scaling relation,

$$M^{(k)}(t, \tau, L, m_0) = b^{-k\beta/\nu} M^{(k)}(t/b^{1/\nu} \tau, b^{-1} L, b^{-x_0} m_0).$$

(1)

when lengths, $L$, are rescaled as $L/b$. Here $t$ is the time and $\tau$ denotes the reduced temperature. The scaling form involves the usual static ($\beta$ and $\nu$), and dynamic ($z$) critical exponents, as well as the nonequilibrium exponent $x_0$, which is the scaling dimension of the initial magnetization.

In the initial time regime, for a small $m_0$ the evolution of the magnetization follows

$$m(t) \simeq m_0 \cdot t^\theta,$$  

(2)

where, $\theta = (x_0 - \beta/\nu)/z$, is the so called initial slip exponent. This behaviour is valid up to a time, $t_0 \sim m_0^{-z/x_0}$, after that the relaxation crosses over to an equilibrium decay.
This type of decay is observable also when the initial state is a ferromagnetically ordered one, thus the preparation temperature is ($T_{\text{prep}} = 0$).

Also the autocorrelation function, $G(t, s)$, depends on the initial state. For the fully ordered state we have:

$$G(t, s) \sim (t - s)^{-2\beta/\nu z} g(t/s),$$  

(4)

where for $t/s \gg 1$ $g(t/s) \sim (t/s)^{\beta/\nu z}$, so $G(t, s) \sim t^{-\beta/\nu z}$ for $t \gg s$. If we prepare a high temperature initial state the decay is given by[6]

$$G(t, s) \sim t^{-\lambda/z},$$  

(5)

where the exponent, $\lambda$, can be expressed with the previously introduced exponents as: [7] $\lambda = d - \theta z = d - x_0 + \beta/\nu$.

At this point we mention a formal analogy between nonequilibrium critical dynamics and static critical behaviour in semi-infinite systems [8, 9, 10, 11]. The translational invariance is broken in time in the first, and in space in the second problem. The initial state in dynamics with $T_{\text{prep}} = 0$ ($T_{\text{prep}} = \infty$) corresponds to a semi-infinite system with fixed (open) boundary conditions. In this respect the exponent $x_0$ is analogous to the surface magnetization exponent $x_1$ at an ordinary surface transition.

The nonequilibrium critical behavior of systems might depend on the initial state, in particular on the form of its correlations. Although there are systems where the long-ranged properties do not influence long-time behaviour [12], there are a few examples, where power-law correlations in initial configurations, that determine long-time nonequilibrium dynamics of the system. For example in the 2D XY-model [13, 14, 15], when initial states are prepared at $T_1 > 0$ and the quench is made to $T_2 > T_1$ with both temperatures in the critical phase ($T_2 < T_{KT}$, $T_{KT}$ is the Kosterlitz-Thouless temperature). According to spin-wave theory for $T_2 \ll T_{KT}$ in the autocorrelation function in Eq. (4) the anomalous dimension $x = \beta/\nu$ is replaced by the difference, $x(T_2) - x(T_1)$, since here the decay exponents are temperature dependent. This theoretical prediction has been checked numerically[16].

In [17] the nonequilibrium dynamics of the d-dimensional spherical model is studied starting from an initial state which has long-range correlations. Depending on $d$ and on the correlation exponent $\alpha$, different types of critical ageing behaviour can be observed. In both examples described above there is quasi-long-range order in the initial state which is characterized by a fractal dimension, $d_f$, which is larger than the fractal dimension $d_f = d - x$ of the critical state.

Here we consider a new type of preparation technique, in which the initial state is due to the effect of random fields rather than the temperature. To be specific we consider here the ground states of the Random Field Ising Model (RFIM) at $T = 0$ as the initial configurations. The RFIM is defined by the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i,$$  

(6)

where $\langle i, j \rangle$ are nearest neighbor sites of a square lattice. The random fields, $h_i$, have a Gaussian distribution with zero mean and standard deviation $h$, so that:

$$P(h_i) = \frac{1}{\sqrt{2\pi h^2}} \exp \left[ -\frac{h_i^2}{2h^2} \right].$$  

(7)

In two dimensions magnetic long-range order is destroyed by any value of the random field even at $T = 0$[18]. In this case it is of interest to study the structure of domains of the parallel
spins. The wall between differently magnetized domains has a characteristic length-scale, $L_b$, which is called the breaking up length and given by:[20]:

$$L_b \sim \exp \left( \frac{A}{\Delta^2} \right), \quad (8)$$

where $\Delta = h/J$ is the relative field strength, and $A$ is a constant. By decreasing the strength of the random field the typical size of parallel domains is increasing, and as noticed by Seppälä and Alava [19] there is a percolation transition for $\Delta_c \approx 1.65$ [19]. According to numerical calculations[19, 21] the fractal dimension of the percolating cluster is the same as for standard percolation$^{22}$ $d_f = 91/48$.

It is furthermore to be noted that the initial states prepared with small and large $\Delta$, correspond to the initial states prepared in the low and high temperature limits, respectively. As $\Delta \to \infty$ the spins follow the direction of the random field, which is equivalent to a high-temperature initial state. On the contrary for $\Delta = 0$ the initial state is ordered as for $T = 0$.

In the following section we present our numerical results, which are discussed in the final section. We note that some aspects of our study has been published in Ref.[23].

2. Numerical results.

Ground states of the Random Field Ising Model can be computed exactly using a combinatorial optimization method$^{24}$. Here we have generated ground states of the square lattice model having field strengths $1.10 < \Delta < 4$, and the system sizes are $L = 128 \ldots 256$. For each cases averages are made over 10000 realization of the disorder. In the samples we have a small initial magnetization $m_i = 0.0390625$, and after the quench the relaxation is made by the standard heat-bath algorithm. For a given sample the thermal average is made by using $\sim 20$ different set of random numbers.

Investigating the relaxation process for large enough values of $\Delta$ the magnetization shows similar effects as if the initial state is prepared with $T_{prep} = \infty$, as described in Eq. 2. On the contrary for small enough $\Delta$ the magnetization has a unique reentrance in time as shown in Figure 1. The preset initial magnetization $m_i$ decreases for the first Monte Carlo steps until a minimum is reached at $t_{min}$ (regime #1). This timescale depends on the strength of the preparing random field as well as on the system size, as will be discussed later. This decrease is followed by the increase of the magnetization due to the known nonequilibrium coarsening process in the period $t_{min} < t < t_0$ (regime #2), and finally there is the equilibrium decay for $t > t_0$ (regime #3).

The unexpected short-time reentrance behaviour can be explained by looking at the snapshots of the configurations at different times as given in the insets of Figure 1. At the initial state at $t = 0$ there are more or less homogenous clusters of parallel spins of typical size of the breaking-up length, $L_b$. Immediately after the quench these compact clusters start to dissolve as seen in the snapshot at $t = 5$. This process is locally the same as the equilibrium relaxation starting at $T_{prep} = 0$, so basically Eq. 3 should describe the decay of the magnetization. During this decay the magnetization is reduced a factor of $L_b^d$ with $d = 2$, which yields a prediction for the position of the minimum, $t_{min}$, as $t_{min}^{x/z} \sim L_b^d$, so

$$\ln t_{min} \sim \ln L_b \sim \frac{1}{\Delta^2} \quad (9)$$

After the clusters have been dissolved, the standard nonequilibrium relaxation process will start.

During the time regime #2 the increase of the magnetization has the same type of asymptotic behavior, characterized by the same initial slip exponent, $\theta$. This is illustrated in Figure 2 in which one can see some cross-over effects in particular for smaller values of $\Delta$. For example for
Figure 1. Typical evolution of magnetization in time in a double logarithmic scale. The three regimes are: 1 cluster dissolution, 2 nonequilibrium growth through coarsening and 3 equilibrium relaxation. Typical configurations (up spins: white, down spins: black) are presented in the snapshots for a system with $\Delta = 1.4$ and $L = 128$.

Figure 2. Nonequilibrium increase of the magnetization in the time regime #2 in a double logarithmic scale. The continuous lines belong to different values of $\Delta = 1.4, 1.6, 1.8, 2.0, 2.2, 2.6, 3.0$ from bottom to top. The behaviour of the fully disordered initial state is also shown as reference (dashed line). All values are averaged over 200000 runs.

Figure 3. Autocorrelations as a function of time in a log-log scale for different values of $\Delta = 3.0, 2.6, 2.2, 2.0, 1.8, 1.6, 1.4$ from bottom to top. The dashed line represents the asymptotic behavior starting from the fully disordered initial state.

$\Delta = 1.6$ the effective initial slip exponent has a smaller value for early times and only for long enough time will this exponent reach the asymptotic value which corresponds to the $\Delta = \infty$ curve in Figure 2.

The autocorrelation function, $G(t, s)$, for waiting time, $s = 0$, is shown in Figure 3. Starting from the fully disordered state it obeys the power law form in Eq. (4). If we start now with an initial state prepared by random fields the autocorrelation function also presents a cross-over region, which is due to the existence of ordered domains in the initial state. However, in the large time asymptotic region the autocorrelation function has a power-law decay having the
same exponent $\lambda/z = 0.73$, independent of $\Delta$.

In the rest of the paper we study in more details the reentrance phenomena of the relaxation process, i.e. the time regime #1 in Figure 1. In Figure 4 we show the size dependence of the reentrance for a fixed value of $\Delta = 1.80$. The minima of the curves becomes deeper with increasing $L$ and their position is shifted to the right and the scale of these changes is of the order of $\sim \ln L$. These corrections are probably due to the fact that the size of the largest connected cluster in a system of linear size $L$ varies as $\ln L$. 

Next, we consider the cluster dissolution process and in Figure 5 we show the starting relaxation steps for different values of $\Delta$. In a log-log scale the curves for different $\Delta$-s have approximately the same slope, which is well described by the equilibrium decay exponent, $\beta/\nu z = 0.0576$, which is indicated by a dotted line in Figure 5. This result supports the theoretical argument described in Eq. (9), namely if $L$ is close to the breaking up length the system consists of just a few large clusters, the two phases of up and down spins are clearly separated, while the magnetization is close to zero.

Finally, in Figure 6 we have a closer look on the location of the minima for different values of $\Delta$. Using the asymptotic result in Eq. (9) we plot $\ln t_{\text{min}}$ as a function of $1/\Delta^2$. Indeed, the data points in the range of $\Delta = 1.65 \ldots 1.95$ are well described by a straight line, giving support to Eq. (9). On the contrary the points for $\Delta < 1.65$ have some deviation from the straight line. Recalling that percolation transition takes place at $\Delta_c \approx 1.65$ these deviations can be explained with the influence caused by the transition. We can formulate that the percolation point introduces a finite time and finite length scale of values $t_{\text{min}}$ and $L_b$, respectively, measured at $\Delta_c$. 

**Figure 4.** Reentrance of the magnetization for different system sizes $L = 64, 90, 128, 180, 256$, top to bottom at a field strength of $\Delta = 1.80$.

**Figure 5.** Cluster dissolution and the decay of the magnetization in the starting time steps in a log-log scale for different values of the random field: $\Delta = 1.10 \ldots 1.95$ with a resolution of 0.05 from bottom to top. The dotted line denotes the equilibrium decay rate $\beta/\nu z = 0.0576$.

**Figure 6.** Locations of minima are plotted for each value of $\Delta$ as a function of $1/\Delta^2$. The initial region up to $\Delta \leq 1.65$ is fit on a straight line according to Eq. (9).
3. Discussion
In this work we have described an interesting reentrance in time in nonequilibrium critical relaxation measurements, when the initial state is prepared by the application of random fields of different strength. The competitive behavior of dissolving existing clusters on one hand and the usual domain growth on the other yields this unconventional feature in nonequilibrium critical dynamics. Note by tuning values of the random field $\Delta$ and the initial magnetization $m_i$ one can adjust the borders of different regimes in Figure 1, so that both $t_{\text{min}}$ and $t_0$ can be macroscopic. The percolation point in the initial system - given by a critical strength of random fields - has an impact on the nonequilibrium dynamics by introducing new finite time and length scales.

Eventually, we note that nonequilibrium relaxation based on random fields can be applied to other systems, too. For example for the 3D Ising model the initial state is either ordered (for weak random fields) or disordered (for strong ones)[25]. Most interesting is however the borderline situation when at the critical value of the strength of the random field the initial state is critical and described by a random fixed point.

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