Boson Stars with Self-Interacting Quantum Scalar Fields

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Abstract

The Klein-Gordon-Einstein equations of classical real scalar fields have time-dependent solutions (periodic in time). We show that quantum real scalar fields can form non-oscillating (static) solitonic objects, which are quite similar to the solutions describing boson stars formed with classical and quantum complex scalar fields (the latter will be studied in this paper). We numerically analyze the difference between them concerning the mass of boson stars. On the other hand, we suggest an interesting test (a viable process that the boson star may undergo in the early universe) for the formation of boson stars. That is, it is questioned that after a second-order phase transition (a simple toy model will be considered here), what is the fate of the boson star composed of quantum real scalar field.

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I. INTRODUCTION

The presence of dark matter has been established indirectly in a wide range of scale of the universe, from that of individual galaxies to the entire universe itself [1]. Though direct measurements of the nature of the dark matter have not yielded any result, speculations on its composition vary from baryonic to non-baryonic matter. Particles like axions and neutralinos are the specific targets for direct observation since the indirect measurements like rotation curves of spiral galaxies and others do not depend on the presence of a particular type of particles. One of the most promising candidates for dark matter is the boson star, which was discovered theoretically over thirty years ago [2] [3]. Until now, the reality of boson stars has been successfully applied to various plausible physical situations [4].

The boson star is a self-gravitating compact solitonic object made up of bosonic fields. Non-interacting complex scalar fields [2] [3] were originally considered for the constituents composing boson stars. In this case, the resultant configurations are typically ‘mini’-boson stars, which have small size and mass. This result originates from the following specific feature of boson stars; the boson star is protected from gravitational collapse by the Heisenberg uncertainty principle, instead of the Pauli exclusion principle that applies to fermionic stars, and the characteristic length scale of the former is much smaller than that of the latter. It has been shown that this situation can be dramatically changed by introducing self-interacting complex scalar fields. The self-interaction effectively generates a repulsive force and the maximum mass of stable boson stars can be enhanced up to a size of the order of ordinary fermionic stars [5].

On the other hand, there is another type of gravitationally bound solitonic object known as oscillating soliton star composed of classical real scalar fields [7]. In this case, the space-time geometry and real scalar field satisfying the Klein-Gordon-Einstein (KGE) equations

Analytic evaluation of the maximum mass and higher order self-interaction effect to boson star configuration can be found in [6].
are time-dependent (periodic in time). Such objects are of interest to study dark matter, since the most promising candidates for dark matter are described by real scalar fields such as the axion [8]. However, even though these solutions are stable with respect to some simple perturbations, it is still unclear whether the stability of the oscillating star can be maintained with general perturbations. Moreover, due to its oscillating feature, real scalar fields may not form primordial solitonic objects in the early universe [10] (cf. [9]), which are expected to play important roles in galaxy formation, the microwave background, and formation of protostars.

In this paper, we report some very interesting results for the self-gravitating solitonic object formed with real scalar fields: there exist gravitationally bound non-oscillating (static) objects composed of quantum real scalar fields, instead of classical real ones. Our (zero-node) solitonic solution is quite similar to the solutions describing boson stars formed with classical [5] and quantum complex scalar fields (the latter will be studied in this paper). The only difference between them at the equation of motion level is just the effective coefficient of the self-interaction term (within the validity considered in this paper, i.e. zero-node solutions, the semi-classical and Hartree (mean-field) approximations). Therefore, non-self-interacting quantum/classical complex and quantum real scalar fields coupled to gravity can form identical static mini-boson stars.

On the other hand, as many viable cosmological scenarios indicate, quantum fields present in early stage of the universe may experience phase transitions by temperature changes. Thus, if the boson star forms in the early universe, it would undergo phase transitions. It might be an interesting issue to consider the consequence of such a phase transition for the boson star.\(^2\) What is the fate of the scalar field and what are the final products after the phase transition, which is presumably second order? In this paper, as a first step,

\(^2\)In [11], a boson star in an evolving cosmological background with time varying gravitational constant (i.e., in the context of scalar-tensor theory of gravity) has been considered.
we consider a simple toy model; without speculating on the detailed procedure of the phase transition, we introduce a plausible form of the Lagrangian for real scalar fields coupled to gravity, which would give the physics after phase transition, and examine that there may still exist boson stars. If they continue to exist, what are the differences between the boson stars before and after phase transition.

In the next section, we start by studying boson stars composed of quantum complex scalar fields, and compare them with the boson stars formed with the classical complex scalar fields [5]. The boson stars composed of quantum real scalar fields are studied in Sect.3. In Sect.4, we consider boson stars composed of quantum real scalar fields that undergo a second-order phase transition. Discussions and some remarks are given in Sect.5.

**II. COMPLEX QUANTUM SCALAR FIELDS**

Consider a self-interacting complex scalar field Lagrangian given by

\[
L_{\text{matt}} = -\frac{1}{2} \partial^\mu \phi^* \partial_\mu \phi - \frac{1}{2} m^2 |\phi|^2 - \frac{\lambda}{4} |\phi|^4,
\]

(1)

where we assume that the coupling to gauge fields is negligible. Then, the field equations of the scalar field are given by

\[
\nabla^\mu \nabla_\mu \phi - m^2 \phi - \lambda |\phi|^2 \phi = 0,
\]

(2)

and its complex conjugate. For convenience, we introduce two real scalar fields, defined by

\[
\Phi_1 = \frac{1}{\sqrt{2}} (\phi + \phi^*), \quad \Phi_2 = \frac{i}{\sqrt{2}} (\phi - \phi^*),
\]

(3)

and rewrite the field equations as

\[
\nabla^\mu \nabla_\mu \Phi_1 - m^2 \Phi_1 - \lambda \left( \Phi_1^2 + \Phi_2^2 \right) \Phi_1 = 0,
\]

(4)

\[
\nabla^\mu \nabla_\mu \Phi_2 - m^2 \Phi_2 - \lambda \left( \Phi_1^2 + \Phi_2^2 \right) \Phi_2 = 0.
\]

General expressions for solutions to Eqs. (4) can be written as
\[
\Phi_1 = \sum_{nlm} \left( a_{nlm}^* \zeta(r)_nY^l_m e^{-i\omega^l_{nl} t} + a_{nlm}^* \zeta(r)_{nl}^*(Y^l_m)^* e^{i\omega^l_{nl} t} \right),
\]
\[
\Phi_2 = \sum_{nlm} \left( b_{nlm} \eta(r)_nY^l_m e^{-i\omega^l_{nl} t} + b_{nlm}^* \eta(r)_{nl}^*(Y^l_m)^* e^{i\omega^l_{nl} t} \right).
\]

As quantum scalar fields, \( \Phi_1 \) and \( \Phi_2 \) become operators
\[
\hat{\Phi}_1 = \sum_{nlm} \left( \hat{a}_{nlm} \zeta(r)_nY^l_m e^{-i\omega^l_{nl} t} + \hat{a}_{nlm}^\dagger \zeta(r)_{nl}^*(Y^l_m)^* e^{i\omega^l_{nl} t} \right),
\]
\[
\hat{\Phi}_2 = \sum_{nlm} \left( \hat{b}_{nlm} \eta(r)_nY^l_m e^{-i\omega^l_{nl} t} + \hat{b}_{nlm}^\dagger \eta(r)_{nl}^*(Y^l_m)^* e^{i\omega^l_{nl} t} \right),
\]

where \( \hat{a}_{nlm}^\dagger, \hat{b}_{nlm}^\dagger \) and \( \hat{a}_{nlm}, \hat{b}_{nlm} \) are the creation and annihilation operators, respectively, and they satisfy the commutation relations
\[
[a_{\alpha}, a_{\alpha'}] = [b_{\alpha}, b_{\alpha'}] = \delta_{\alpha\alpha'}.
\]

In the ground state of the system, \( n = 1, l = m = 0 \), will be considered and the condition \( \omega^0_{01} = \omega^0_{10} = \omega \) is assumed. In addition, we assume that in the ground state, which is denoted by \( |N > = |N_{\Phi_1} > \otimes |N_{\Phi_2} > \), the number of particles created by \( \hat{a}^\dagger \) is the same as that associated with \( \hat{b}^\dagger \), i.e. \( N_{\Phi_1} = N_{\Phi_2} = N \). In general, \( \zeta(r) \) and \( \eta(r) \) are complex functions, but here we restrict them to real ones. To linearize the field equations (4), we adopt the Hartree approximation such that \( (\hat{\Phi}_1^2 + \hat{\Phi}_2^2) \hat{\Phi}_1 \sim < N | (\hat{\Phi}_1^2 + \hat{\Phi}_2^2) | N > \hat{\Phi}_1 \).

On the other hand, we also use the semi-classical approximation in Einstein’s equations
\[
G_\mu^\nu = 8\pi G < N | \hat{T}_\mu^\nu | N >,
\]
where \( G_\mu^\nu \) is the Einstein tensor and \( \hat{T}_\mu^\nu \) is the energy-momentum operator that is obtained by quantizing the energy-momentum tensor
\[
T_\mu^\nu = \frac{1}{2} (\partial^\mu \phi^* \partial_\nu \phi + \partial^\mu \phi^* \partial_\nu \phi^*) - \frac{1}{2} g_\nu^\mu \left( \partial^\alpha \phi^* \partial_\alpha \phi + m^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4 \right)
\]
\[
= \frac{1}{2} (\partial^\mu \Phi_1 \partial_\nu \Phi_1 + \partial^\mu \Phi_2 \partial_\nu \Phi_2) - \frac{1}{4} g_\nu^\mu \left( \partial^\alpha \Phi_1 \partial_\alpha \Phi_1 + \partial^\alpha \Phi_2 \partial_\alpha \Phi_2 + m^2 (\Phi_1^2 + \Phi_2^2) + \frac{\lambda}{4} (\Phi_1^2 + \Phi_2^2)^2 \right).
\]

The background spacetime is restricted to be static and spherically symmetric. Thus, in Schwarzschild coordinates, it can be written as
\[
ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega_2^2,
\]
where \( d\Omega_2^2 \) denotes the metric of a two-dimensional unit sphere.
After performing some algebraic calculations and reparametrizations in Eqs. (7) and (4), we obtain the equations

\[
\frac{A'}{A^2 x} + \frac{1}{x^2} \left(1 - \frac{1}{A}\right) = \left(\frac{\Omega^2}{B} + 1\right) \left(\sigma_1^2 + \sigma_2^2\right) + \frac{A}{4} \left(3 (\sigma_1^4 + \sigma_2^4) + 4\sigma_1^2\sigma_2^2\right) + \frac{1}{A} \left((\sigma_1')^2 + (\sigma_2')^2\right),
\]

\[
\frac{B'}{AB x} - \frac{1}{x^2} \left(1 - \frac{1}{A}\right) = \left(\frac{\Omega^2}{B} - 1\right) \left(\sigma_1^2 + \sigma_2^2\right) - \frac{A}{4} \left(3 (\sigma_1^4 + \sigma_2^4) + 4\sigma_1^2\sigma_2^2\right) + \frac{1}{A} \left((\sigma_1')^2 + (\sigma_2')^2\right),
\]

\[
\sigma_1'' + \left(\frac{2}{x} + \frac{B'}{2B} - \frac{A'}{2A}\right) \sigma_1' + A \left[\left(\frac{\Omega^2}{B} - 1\right) \sigma_1 - \Lambda (\sigma_1^2 + \sigma_2^2) \sigma_1\right] = 0,
\]

\[
\sigma_2'' + \left(\frac{2}{x} + \frac{B'}{2B} - \frac{A'}{2A}\right) \sigma_2' + A \left[\left(\frac{\Omega^2}{B} - 1\right) \sigma_2 - \Lambda (\sigma_1^2 + \sigma_2^2) \sigma_2\right] = 0,
\]

(10)

where \(x \equiv mr, \sigma_1 \equiv (4\pi GN)^{1/2} \zeta(r), \sigma_2 \equiv (4\pi GN)^{1/2} \eta(r), \Omega \equiv \omega/m, \Lambda \equiv \lambda/4\pi Gm^2,\) and the prime denotes the derivative with respect to \(x.\) In Eqs. (10), we have used the approximation, \(N \gg 1.\) In this paper, we consider only a special type of solution given by \(\sigma_1 = \sigma_2 = \sigma/\sqrt{2}.\) Then, the equations become

\[
\frac{A'}{A^2 x} + \frac{1}{x^2} \left(1 - \frac{1}{A}\right) = \left(\frac{\Omega^2}{B} + 1\right) \sigma^2 + \frac{5\Lambda}{8} \sigma^4 + \frac{(\sigma')^2}{A},
\]

(11)

\[
\frac{B'}{AB x} - \frac{1}{x^2} \left(1 - \frac{1}{A}\right) = \left(\frac{\Omega^2}{B} - 1\right) \sigma^2 - \frac{5\Lambda}{8} \sigma^4 + \frac{(\sigma')^2}{A}.
\]

(12)

\[
\sigma'' + \left(\frac{2}{x} + \frac{B'}{2B} - \frac{A'}{2A}\right) \sigma' + A \left[\left(\frac{\Omega^2}{B} - 1\right) \sigma - \Lambda \sigma^3\right] = 0
\]

(13)

Note that comparing (11)-(13) with the KGE equations for classical complex scalar fields [5], only the coefficients of self-interaction terms in (11) and (12) are changed as \(\Lambda/2 \rightarrow 5\Lambda/8,\) while the field equation (13) remains without any change. Such a change results in modifications of some properties of the boson stars, especially the maximum mass of stable boson stars. This can be seen in the following numerical analysis.

\[\text{Note that while Eqs.}(4)\) and (8) are invariant under a rotation in the internal space given by \(\Phi_1 \rightarrow \Phi_1 \cos \Theta + \Phi_2 \sin \Theta, \Phi_2 \rightarrow -\Phi_1 \sin \Theta + \Phi_2 \cos \Theta,\) where \(\Theta\) is a constant, in Eq.(10) such a symmetry does not exist. This is simply because \(\sigma_1, \sigma_2\) in Eq.(10) are not field variables but (redefined) radial functions of the mode solutions. So, there is no reason maintaining the internal symmetry in the level of Eq.(10).\]
Substituting an ansatz of the metric function $A$ by

$$A(x) = \left[1 - \frac{2M(x)}{x}\right]^{-1},$$

(14)

into the above equations and using the boundary conditions

$$M(0) = 0, \quad \sigma(0) = \sigma_c, \quad \sigma'(0) = 0, \quad \text{and} \quad B(\infty) = 1,$$

(15)

we obtain the mass of boson stars as a function of $\sigma_c$ for $\Lambda = 100$ in Fig.1. In Fig.1, it is shown that the maximum mass of boson stars composed of quantum complex scalar fields is greater than that in the case of classical complex scalar fields. This is true for other values of $\Lambda$ as can be seen in Fig.2. The lines in Fig.2 are chosen in a way that they hold for $\Lambda \gg 1$; in the case of quantum fields $(5\Lambda/8) M_{\max} \approx 0.223\Lambda^{1/2}M_p^2/m$, while $M_{\max} \approx 0.22\Lambda^{1/2}M_p^2/m$ for the classical fields $(\Lambda/2)$ [5]. It has also to be mentioned that in Fig.1, there is a crossing (at $\sigma_c \approx 0.132$) of the two mass-curves, and when $\sigma_c > 0.132$, the mass of the boson stars formed by the classical complex scalar field is greater than that of the boson stars formed by a quantum complex scalar field.

As mentioned above, the difference between the two cases of classical and quantum fields is in the coefficient of the self-interaction term in the Einstein equations (11) and (12), i.e. $\Lambda/2$ for the classical field and $5\Lambda/8$ for the quantum one. Why does the difference appear? In fact, the KGE equations for the classical complex scalar field considered in Ref. [5] become the equations for a “real” field after eliminating the time-dependent part. This is because the authors of Ref. [5] have taken the solution of the complex field which has the form of $\phi(r, t) = \Phi(r) \exp(-i\omega t)$ with real $\Phi(r)$. In our case, the KGE equations given in (10) are for two “real” fields. Even though we chose a special form of the solution given by $\sigma_1 = \sigma_2 = \sigma/\sqrt{2}$ to compare with the classical case studied in [5], the resulting equations given by (11)-(13) already include the interaction between the two “real” fields as well as

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4The quoted term of “real” field will be used for the spatial part of a mode function, not for an ordinary field.
self-interactions of each fields, and this is the reason why the coefficients of self-interaction term are effectively different from each other in the quantum and classical cases, even though we have started with the same Lagrangian (1).

Roughly speaking, since larger self-interaction energy exerts greater repulsive force in the formation of boson stars, the ‘quantum’ effect, i.e. using the quantum field rather than the classical one to form a boson star, enhances the maximum mass of the boson stars. (In fact, when the self-interaction is introduced, density of the boson star is approximately proportional to $\Lambda^{-1}$ so that the boson stars become dilute. However, since its radius is approximately proportional to $\Lambda^{1/2}$, the maximum mass of the boson star is to be proportional to $\Lambda^{1/2}$. This argument is available for the case that the self-interaction energy is comparable to the mass energy of the field, i.e. $\Lambda >> 0$. See Ref. [6] for a detailed analysis.)

Of course, the validity of our argument has to be restricted to solutions that are considered in this paper and [5]. However, it is reasonable to study the difference between the roles of classical and quantum fields in the formation of boson stars. Another point that has to be mentioned is that the mass curves in Fig.1 cross each other at $\sigma_c \approx 0.132$, and the boson star masses for the classical and quantum cases reverse in magnitude. This phenomenon can not be described by the above argument. To understand such a behavior, a detailed analytical approach would be required.

**III. REAL QUANTUM SCALAR FIELDS**

Now, let us consider a real scalar field for the formation of boson stars. The real massive scalar field Lagrangian including self-interaction is written as

$$L_{\text{matt}} = -\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4} \Phi^4.$$  \hspace{1cm} (16)

And the energy-momentum tensor and field equation are obtained as

$$T^\nu_\mu = \partial^\mu \Phi \partial_\nu \Phi - \frac{1}{2} g^\mu_\nu \left( \partial^\alpha \Phi \partial_\alpha \Phi + m^2 \Phi^2 + \frac{\lambda}{2} \Phi^4 \right),$$ \hspace{1cm} (17)

$$\nabla^\mu \nabla_\mu \Phi - m^2 \Phi - \lambda \Phi^3 = 0.$$ \hspace{1cm} (18)
From the general form of solutions of the field equation, the field operator is given by

$$\hat{\Phi} = \sum_{nlm} \left( \hat{d}_{nlm} \Phi(r)_{nl} Y^l_m e^{-i\omega_n t} + \hat{d}^\dagger_{nlm} \Phi^*(r)_{nl} (Y^l_m)^* e^{i\omega_n t} \right),$$

(19)

where the creation and annihilation operators, $\hat{d}_{nlm}$ and $\hat{d}^\dagger_{nlm}$, satisfy the commutation relations $[\hat{d}_{\alpha'}, \hat{d}_{\alpha}] = \delta_{\alpha\alpha'}$. Again we only consider the ground state. Note that we take the function $\Phi(r)$ as a complex function. Then, defining new real functions

$$\xi(r) \equiv \frac{1}{\sqrt{2}} (\Phi(r) + \Phi^*(r)), \quad \chi(r) \equiv \frac{i}{\sqrt{2}} (\Phi(r) - \Phi^*(r)),$$

(20)

and taking the Hartree and semi-classical approximations, we obtain the KGE equations given by

$$\frac{A'}{A^2} x + \frac{1}{x^2} (1 - \frac{1}{A}) = \left( \frac{\Omega^2}{B} + 1 \right) \left( \rho_1^2 + \rho_2^2 \right) + \frac{3A}{4} \left( \rho_1^2 + \rho_2^2 \right)^2 + \frac{1}{\Lambda} \left( (\rho_1')^2 + (\rho_2')^2 \right),$$

$$\frac{A'}{AB^2} x - \frac{1}{x^2} (1 - \frac{1}{A}) = \left( \frac{\Omega^2}{B} - 1 \right) \left( \rho_1^2 + \rho_2^2 \right) - \frac{3A}{4} \left( \rho_1^2 + \rho_2^2 \right)^2 + \frac{1}{\Lambda} \left( (\rho_1')^2 + (\rho_2')^2 \right),$$

$$\rho_1' + \left( \frac{\mu}{x} + \frac{A'}{2B} - \frac{A'}{2A} \right) \rho_1 + A \left[ \left( \frac{\Omega^2}{B} - 1 \right) \rho_1 - \Lambda (\rho_1^2 + \rho_2^2) \rho_1 \right] = 0,$$

$$\rho_2' + \left( \frac{2}{x} + \frac{A'}{2B} - \frac{A'}{2A} \right) \rho_2 + A \left[ \left( \frac{\Omega^2}{B} - 1 \right) \rho_2 - \Lambda (\rho_1^2 + \rho_2^2) \rho_2 \right] = 0,$$

(21)

where $x \equiv m \rho_1 \equiv (4\pi GN)^{1/2} \xi(r)$, $\rho_2 \equiv (4\pi GN)^{1/2} \chi(r)$, $\Omega \equiv \omega / m$, $\Lambda \equiv \lambda / 4\pi G m^2$, and we use the metric (9) and the approximation $N \gg 1$. Taking a special form of the solution for $\rho_1 = \rho_2 = \sigma / \sqrt{2}$, above equations become

$$\frac{A'}{A^2} x + \frac{1}{x^2} (1 - \frac{1}{A}) = \left( \frac{\Omega^2}{B} + 1 \right) \sigma^2 + \frac{3A}{4} \sigma'^2 + \frac{(\sigma')^2}{A},$$

(22)

$$\frac{A'}{AB^2} x - \frac{1}{x^2} (1 - \frac{1}{A}) = \left( \frac{\Omega^2}{B} - 1 \right) \sigma^2 - \frac{3A}{4} \sigma'^2 + \frac{(\sigma')^2}{A},$$

(23)

$$\sigma'' + \left[ \frac{2}{x} + \frac{A'}{2B} - \frac{A'}{2A} \right] \sigma' + A \left[ \left( \frac{\Omega^2}{B} - 1 \right) \sigma - \Lambda \sigma \right] = 0.$$ 

(24)

Note that the above equations can be obtained from (10) by substituting $\sigma_1 = \sigma$, $\sigma_2 = 0$.

Comparing Eqs. (22)-(24) with (11)-(13), we find that the only difference is the coefficient of the self-interaction term. Thus, the KGE equations (22)-(24) should have solutions that are much like the solutions describing the boson star composed of classical/quantum complex scalar fields. As mentioned above, since the coefficient $3\Lambda/4$ is greater than those in the
classical and quantum complex scalar field cases, which are $\frac{\Lambda}{2}$ and $5\frac{\Lambda}{8}$, respectively, the maximum mass of boson star formed by quantum real scalar fields should be larger than those of other cases. Indeed, using the ansatz (14) and the boundary conditions given by (15), we calculate numerically the mass of boson star as a function of $\sigma_c$ for various values of $\Lambda$ in Fig.3. In Fig.1 and Fig.2, we compare the maximum mass of boson stars composed of quantum real scalar fields with other cases, and confirm that the order of magnitude of maximum masses follows that of the coefficient of self-interaction term, i.e. $3\frac{\Lambda}{4} > 5\frac{\Lambda}{8} > \frac{\Lambda}{2}$. (The solid line in Fig.2 is the relation $M_{\text{max}} \approx 0.225\Lambda^{1/2} M_p^2 / m$, which holds for $\Lambda >> 1$ for the case of quantum real scalar fields.) This result can be interpreted in the same way as given in Sect.2; comparing (21) with (10), the Einstein equations in (21) include the term representing the interaction of “real” fields, $(3\frac{\Lambda}{2})\rho_1^2 \rho_2^2$, which has a larger coefficient than in other cases, e.g. $\Lambda \sigma_1^2 \sigma_2^2$ in the Einstein equations in (10).

IV. PHASE TRANSITION

In this section, we consider a phase transition that the boson star can undergo in early stages of the universe. Our model of the system after the phase transition is a simple one with Lagrangian for real scalar fields given by

$$L_{\text{matt}}^{ph} = -\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi + \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4} \Phi^4. \quad (25)$$

In order to simulate the condition of phase transition the mass $m$ in the Lagrangian is taken to be a function of time $t$. Then, in the process of the phase transition, we consider $m^2 \to -m^2$ after phase transition. Such a procedure is commonly used in describing a second-order phase transition, and compares well to the ideas of spontaneous symmetry breaking which is the theoretical underpinning for such a phase transition.

In this theory, the vacuum expectation value of the field does not vanish, but $<\Phi>_{\text{vac}} = m\sqrt{\lambda}$. Now, shifting the field, $\Phi \to \Phi + m/\sqrt{\lambda}$, and the potential energy, $V(\Phi) \to V(\Phi) - m^4/(4\lambda)$, we obtain a new Lagrangian written as
\[ L_{\text{matt}}^{\text{new}} = -\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - m^2 \Phi^2 - \frac{\lambda}{4} \Phi^4 - m \sqrt{\lambda} \Phi^3. \]  

(26)

Then, the energy-momentum tensor and field equation are evaluated from the Lagrangian (26);

\[ T^\mu_\nu = \partial^\mu \Phi \partial_\nu \Phi - \frac{1}{2} g^\mu_\nu \left( \partial^\alpha \Phi \partial_\alpha \Phi + 2m^2 \Phi^2 + \frac{\lambda}{2} \Phi^4 + 2m \sqrt{\lambda} \Phi^3 \right), \]  

(27)

\[ \nabla^\mu \nabla_\mu \Phi - 2m^2 \Phi - \lambda \Phi^3 - 3m \sqrt{\lambda} \Phi^2 = 0. \]  

(28)

Comparing (27) and (28) with (17) and (18) for the case before the phase transition, we see that the scalar field becomes effectively more massive, \( m \rightarrow \sqrt{2} m \), through the phase transition, and an additional non-linear interaction term with a coupling constant proportional to \( m \sqrt{\lambda} \) is included in the energy-momentum tensor and field equation. Within the validity of the Hartree and semiclassical approximation, however, the non-linear coupling term proportional to \( m \sqrt{\lambda} \) does not contribute to the KGE equations, i.e. \( \langle N | \hat{\Phi}^3 | N \rangle = 0 \) in the Einstein equations and \( \langle N | \hat{\Phi} | N \rangle = 0 \) in the field equation due to the semiclassical and Hartree approximation, respectively. Thus, effectively the consequence of phase transition is just to enhance the mass of scalar fields.

Explicitly, the KGE equations are written as

\[ \frac{A'}{Ax} + \frac{1}{x^2} \left( 1 - \frac{1}{A} \right) = \left( \frac{\Omega^2}{B} + 2 \right) \sigma^2 + \frac{3\Lambda}{4} \sigma^4 + \frac{(\sigma')^2}{A}, \]  

(29)

\[ \frac{B'}{ABx} + \frac{1}{x^2} \left( 1 - \frac{1}{A} \right) = \left( \frac{\Omega^2}{B} - 2 \right) \sigma^2 - \frac{3\Lambda}{4} \sigma^4 + \frac{(\sigma')^2}{A}, \]  

(30)

\[ \sigma'' + \left( \frac{2}{x} + \frac{B'}{2B} - \frac{A'}{2A} \right) \sigma' + A \left[ \left( \frac{\Omega^2}{B} - 2 \right) \sigma - \Lambda \sigma^3 \right] = 0. \]  

(31)

More conveniently, reparametrizing the above equations by \( y \equiv \sqrt{2} x, \Omega \equiv \Omega / \sqrt{2} \), we obtain

\[ \frac{A'}{Ay} + \frac{1}{y^2} \left( 1 - \frac{1}{A} \right) = \left( \frac{\Omega^2}{B} + 1 \right) \sigma^2 + \frac{3\Lambda}{8} \sigma^4 + \frac{(\sigma')^2}{A}, \]  

(32)

\[ \frac{B'}{ABy} + \frac{1}{y^2} \left( 1 - \frac{1}{A} \right) = \left( \frac{\Omega^2}{B} - 1 \right) \sigma^2 - \frac{3\Lambda}{8} \sigma^4 + \frac{(\sigma')^2}{A}, \]  

(33)

\[ \sigma'' + \left( \frac{2}{y} + \frac{B'}{2B} - \frac{A'}{2A} \right) \sigma' + A \left[ \left( \frac{\Omega^2}{B} - 1 \right) \sigma - \frac{\Lambda}{2} \sigma^3 \right] = 0, \]  

(34)

where prime denotes the derivative with respect to \( y \). Note that the coefficient of the self-interaction term in (34) is not \(-\Lambda\) but \(-\Lambda/2\), which is different from (13) and (24).
The KGE equations given in (32)-(34) are obtained by just replacing $\Lambda \rightarrow \Lambda/2$ in all the equations (22)-(24). In other words, the phase transition makes the scalar field effectively less self-interactive. This result apparently tells us that after phase transition the boson star composed of real scalar fields is to have a smaller mass than in the case before phase transition.

Such a loss in the boson star mass caused by the phase transition, however, does not appear to be a serious shortcoming for describing the dark matter. Rather, the fact that the boson star can still exist after a phase transition, seems to deserve our attention.

V. DISCUSSIONS

In this paper, we have considered self-gravitating solitonic objects made up of quantum complex/real scalar fields. It has been shown that quantum complex/real scalar fields may compose the boson stars that are similar to that formed by classical complex scalar fields. The difference in the theories considered here appears only in the coefficient of the self-interaction term in the Einstein equations, i.e. they are $\Lambda/2$ for classical complex fields, $5\Lambda/8$ for quantum complex ones, and $3\Lambda/4$ for quantum real ones. Numerically we have verified, Fig.1 and Fig.2, that the maximum mass of the boson stars increases with the magnitude of the coefficient of the self-interaction term. This result can be understood as follows; after eliminating the time-dependent part the KGE equations for the quantum complex/real scalar fields become effectively for two “real” fields as (10) and (21). Then, the equations include interactions between the two “real” fields as well as self-interactions of each fields. On the other hand, the KGE equations for the classical complex fields considered in Ref. [5] become effectively for one “real” field after eliminating the time-dependent part and contain just the self-interaction term of the “real” field. Therefore, the coefficients of the resulting interaction terms in the KGE equations for the quantum complex/real scalar fields, which are given by (11), (12), and (22), (23), are greater than that in the case of the classical complex scalar fields. According to the argument that more interaction energy
generates effectively a larger maximum mass of the boson star, the maximum masses of the boson stars composed of quantum complex/real scalar fields are greater than that in the case of classical complex scalar fields. This argument, however, has a limitation due to the presence of a crossing between the mass-curves in Fig.1 and Fig.2.

In fact, the existence of such solitonic solutions does not guarantee that the bosonic compact objects can be formed in the universe. (cf. [12]) It has been shown that there is a process that is able to describe the formation of the bosonic compact objects in the early universe [10]. The process, which is similar to the way of describing the settling of collisionless star systems, starts with collapsing due to a gravitational instability analogous to the Jeans instability. Then, after undergoing the so called gravitational cooling mechanism ejecting part of the scalar field, bosonic compact objects could be formed from the primordial bosonic cloud. In the case of complex scalar fields, this mechanism works to form mini-boson stars. However, in general, the oscillatons made up of classical real scalar fields can be formed only in a short dynamical time scale, but are unstable in such a state. Thus, without introducing additional proviso such as fragmentation of the primodial bosonic cloud, the classical real scalar fields would be ruled out as a candidate for the dark matter.

In this paper, we have shown that quantum real scalar fields can form the *boson star* rather than the *oscillaton*. Especially, without the self-interaction $\Lambda = 0$, the mini-boson star composed of the quantum real scalar fields becomes exactly identical to that made up of the classical (and quantum) complex scalar fields. Thus, considering the quantum effect, the real scalar field can be saved, and can be the most promising candidate for dark matter.

As a corollary, which would be an interesting issue in the study of bosonic objects, the quantum complex/real scalar fields in excited states may be considered. The solitonic solutions, if they exist, are unlikely to be static in both (complex and real) cases. However, it is not obvious if such quantum fields in excited states can be ruled out completely as candidates for the dark matter.

Another interesting test for the boson star composed of the quantum real scalar fields is the second-order phase transition. In this analysis, we have shown that the phase transition
makes the fields effectively more massive, $m \rightarrow \sqrt{2}m$, or equivalently less self-interactive, $\Lambda \rightarrow \Lambda/2$, and more importantly the boson stars can exist even after the phase transition.

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FIG. 1. Boson star masses as a function of $\sigma_c$ when $\Lambda = 100$. $3\Lambda/4$, $5\Lambda/8$, $\Lambda/2$ denote the cases of real and complex quantum scalar fields, and classical complex ones, respectively.
FIG. 2. Maximum masses of boson stars as a function of $\Lambda$. $3\Lambda/4$, $5\Lambda/8$, $\Lambda/2$ denote the cases of real and complex quantum scalar fields, and classical complex ones, respectively.
FIG. 3. Mass of boson stars composed of real quantum scalar fields as a function of \(\sigma_c\) when \(\Lambda = 0, 10, 50, 100, 250\).