TASEP on a ring with internal degrees of freedom

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A totally asymmetric exclusion process on a ring with $\nu$ non-conserved internal degrees of freedom, where particles hop forward with a rate that depends on their internal state, has been studied. We show, using a mapping of the model to a zero range process with $\nu$ different kinds of boxes, that steady state weights can be written in a matrix product form and calculate the spatial correlations exactly. A comparison of the model with an equivalent conserved system reveals that unequal hopping rates of particles belonging to different internal states is responsible for the non-trivial correlations.

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I. INTRODUCTION

Driven diffusive systems have been studied extensively in recent past for their unorthodox non-equilibrium properties. Totally asymmetric simple exclusion process (TASEP) is one such system, initially introduced as a basic model for transport, which has found applications in wide areas of physics and biology. The steady state weights of TASEP, which has been calculated exactly, show collective behaviour like boundary driven phase transitions, non-trivial density and shock-profiles. Several variations of this exclusion model including non-conservation of particles, multiple species have been studied in different contexts.

Recently TASEP with non-conservation of particles having internal degrees of freedom has also been studied. In these models particles are allowed to enter or exit the system from both boundaries and their hop rate in the bulk depend on the internal degree they possess. The explicit conditions on the rates for the steady state of a parity-time invariant system to have a factorized form has been derived. Analytical results for the steady state of these models for generic rates (where non-trivial spatial correlations are expected) are not known.

In this article we introduce a totally asymmetric exclusion process on a ring with non conserved internal degrees of freedom. In this model, apart from the usual forward hopping, each particle in one of the $\nu$ possible internal states can change to any other $\nu$ states with different rates. The steady state of the model could be calculated exactly. Unlike TASEP on a ring, which has an uncorrelated steady state, this model show non-zero spatial correlations when hop rate of particles depend on their internal states.

II. THE MODEL

The model is defined on a one dimensional lattice with periodic boundary conditions. The sites, labelled by $i = 1, 2 \ldots L$, can either be empty or occupied by at most one particle. Each particle can be in one of the $\nu$ possible internal states $I = 1, 2 \ldots \nu$. Correspondingly, at the $i^{th}$ site we take the site variable $s_i = 0$ for vacancy and $s_i = 1, 2 \ldots \nu$ for the different internal states.

As in TASEP, the particles hop towards the right neighbouring site if it is vacant. However, the hop rate depends on the internal state of the particle. Corresponding dynamics is

$$I 0 \xrightarrow{\alpha_I} I,$$

where $I = 1, 2 \ldots \nu$ represents the different internal states of the particles and $\alpha_I$ are the hop rates of the particles in the $I^{th}$ state. In addition to this hopping dynamics, the particles are also allowed to change their internal states when their right neighbouring sites are occupied. In other words

$$I K \xrightarrow{p_{IJ}} J K,$$

where the rates $p_{IJ}$ and $p_{JI}$ do not depend on the state $K$ of the neighbouring particle. Note, that $K$ need not necessarily differ from $I$ or $J$. This dynamics is identical to that of TASEP.

For simplicity we first study the case $\nu = 2$ in details; generalization to arbitrary $\nu$ is straightforward. Particles in the two possible internal states here are denoted by $+$ and $-$ for notational convenience; correspondingly the site variable $s_i = 0, \pm$. Both $+$ and $-$ particles can move only to the rightward vacant site with rates $\alpha_{\pm}$. Otherwise, if the rightward neighbour is occupied, the particles can flip their state ($\pm \rightarrow \mp$) with rates $p_{\pm}$. Explicitly,

$$\pm 0 \xrightarrow{\alpha_{\pm}} 0 \pm \quad \pm \xrightarrow{p_{\pm}} \pm \mp.$$  

Clearly, the total number of particles $N_+ + N_-$ and vacancies $N_0$ are conserved by this dynamics, whereas...
The mapping is described schematically in Fig. 1. Thus, this ZRP with $N_0$ particles distributed in $M = L - N_0$ boxes obey the following dynamics:

(a) particles from a $+(-)$ box moves to the left box with constant rate $\alpha_+ (\alpha_-)$, (b) an empty $+(-)$ box can alter its state with rate $p_+ (p_-)$.

Note that here the hop rates $\alpha_{\pm}$ of ZRP are independent of the number of particles in the departure box. The box dynamics (b) here can produce correlations among boxes (absent in ordinary ZRP) as the non-empty $\pm$ boxes can *not* alter their sign.

III. STEADY STATE FOR $\nu = 2$

A generic configuration can be written as \{ $n_1 \tau_1, n_2 \tau_2, \ldots, n_M \tau_M$ \} where $n_k$ is the number of particles in $k^{th}$ box of type $\tau_k$. It can be shown that steady state of this ZRP has a product measure:

$$P(n_1 \tau_1, n_2 \tau_2, \ldots, n_M \tau_M) \sim f_{\tau_1}(n_1)f_{\tau_2}(n_2) \cdots f_{\tau_M}(n_M) \tag{4}$$

where weight of an individual $\pm$ box containing $n$ particles is $f_{\pm}(n)$.

A. Proof of Product measure

The proof can be constructed using pairwise balance where we find a unique configuration $C''$ for every transition $C \rightarrow C'$ such that

$$P(C)W(C \rightarrow C') = P(C'')W(C'' \rightarrow C). \tag{5}$$

It is sufficient to consider all possible transitions that changes the state $n_k \tau_k$ of the box $k$. Thus $C = \{ n_{k-1}\tau_{k-1}, n_k \tau_k, n_{k+1}\tau_{k+1}, \ldots \}$. Let us construct $C''$ for all possible $C'$s.

**Case-I** $n_k = 0$ : In this case the $k^{th}$ box can change sign. Thus $C' = \{ \ldots n_{k-1}\tau_{k-1}, 0 \tau_k, n_{k+1}\tau_{k+1}, \ldots \}$, where $\tau_k = -\tau_k$. A choice $C'' = C'$ along with Eq. (4) and (5) gives

$$\frac{f_{-}(0)}{f_{+}(0)} = \frac{p_{+}}{p_{-}}. \tag{6}$$

**Case II** $n_k > 0$ : In this case a particle from box $k$ can move to $k - 1$ with rate $\alpha_{\tau_k}$. Thus $C' = \{ \ldots (n_{k-1}+1)\tau_{k-1}, (n_k - 1)\tau_k, n_{k+1}\tau_{k+1}, \ldots \}$. We choose $C'' = \{ \ldots (n_{k-1}+1)\tau_{k-1}, (n_k - 1)\tau_k, (n_{k+1}+1)\tau_{k+1}, \ldots \}$. Then the condition of pairwise balance along with (6) demands

$$\alpha_{\tau_k} \frac{f_{\tau_k}(n_k)}{f_{\tau_k}(n_k-1)} = \alpha_{\tau_{k+1}} \frac{f_{\tau_{k+1}}(n_{k+1}+1)}{f_{\tau_{k+1}}(n_{k+1})} = c \tag{7}$$

The constant $c$ is independent of $n$ and $\tau$. We set $c = \frac{\alpha_+}{\alpha_-}$ without any loss of generality. Since the steady state weights are yet to be normalized, we can take $f_{+}(0) = 1$ without loss of generality. Then, Eqs. (6) and (7) result in

$$f_{+}(n) = 1; \quad f_{-}(n) = p_{+}^{n} \tag{8}$$

where $\alpha = \frac{\alpha_+}{\alpha_-}$ and $p = \frac{p_{+}}{p_{-}}$ are the ratios of hop rates and flip rates respectively.

Now, weights of every configuration in the lattice can be written in terms of the steady state weights of ZRP.

$$P(\{ s_i \} | i = 1, \ldots, L) = P(\{ n_k \tau_k \} | k = 1, \ldots, M) = \prod_{k=1}^{M} f_{\tau_k}(n_k) \tag{9}$$

Spatial indices of the model on a lattice are not carried over to their corresponding ZRP version, which makes calculation of the spatial correlations unreasonably difficult. We follow a method discussed in [12]. First, let us rewrite the steady state weights as a matrix product state, by replacing $s_i$ by the corresponding matrices $X_{s_i}$.

$$P(\{ s_i \}) = Tr[\prod_{i=1}^{L} X_{s_i}] \tag{10}$$

Since we already know $P(\{ s_i \})$ from Eq. (10) in terms of $f_{\tau}(n)s$, $X_{s}s$ must be chosen such that

$$Tr[\prod_{i=1}^{L} X_{s_i}] = \prod_{k=1}^{M} f_{\tau_k}(n_k). \tag{11}$$

We proceed with a simpler notation $X_+ = D, X_- = E$ and $X_0 = A$ and the following choice,

$$D = |d_1\rangle|d_2\rangle; \quad E = |e_1\rangle|e_2\rangle. \tag{12}$$
Now, Eq. (11) and (12) together impose the following conditions on matrix \( A \),
\[
\langle d_2 | A^n | d_1 \rangle = \langle d_2 | A^n | e_1 \rangle = f_+(n) = 1 \\
\langle e_2 | A^n | e_1 \rangle = \langle e_2 | A^n | d_1 \rangle = f_-(n) = p \alpha^n 
\]
(13)
Note, that unlike the usual Matrix Product Ansatz [6, 7], these conditions on \( D, E \) and \( A \) do not depend on the dynamics of the model explicitly. This will be discussed in some length later in III B.

An infinite dimensional representation of these matrices that satisfy Eq. (13), similar to those obtained for the usual ZRP with one kind of boxes [12], can always be constructed. Fortunately for this model, as the usual ZRP with one kind of boxes [12], can always be made to satisfy the required equations with a suitable choice of the auxiliary matrices. An explicit representation of these auxiliary matrices \( \tilde{D}, \tilde{E} \) and \( \tilde{A} \) is also given.

The complete set of equations for the matrices \( D, E \) and \( A \), as required by the MPA, are
\[
\alpha_+ DA = - \tilde{A} D + \tilde{D} A - D \tilde{A} \\
\alpha_- EA = - \tilde{E} A + \tilde{A} E - E \tilde{A} \\
p_+ DD - p_- ED = - \tilde{E} D + \tilde{D} E - D \tilde{E} \\
p_+ DE - p_- EE = - \tilde{E} E + \tilde{E} E - E \tilde{E} 
\]
(16)
where the auxiliaries \( \tilde{D}, \tilde{E} \) and \( \tilde{A} \) also need to be determined. These set of equations do not necessarily have a unique solution. For simplicity we choose \( \tilde{D} = D, \tilde{E} = E \), which reduces the above set of equations to,
\[
p_+ DD = p_+ ED ; \\
p_+ DE = p_- EE \\
(1 - \alpha_+) DA = D \tilde{A} ; \\
(1 - \alpha_-) EA = E \tilde{A} \\
\alpha_+ DA - AD = - \tilde{A} D \\
\alpha_- EA - AE = - \tilde{A} E
\]
(17)

Now we need to check, if \( D, E \) and \( A \) obtained earlier in Eq. (15) satisfy above equation along with some auxiliary matrix \( \tilde{A} \).

It turns out that \( \tilde{A} = \left( \begin{array}{cc} 1 - \alpha_+ & 0 \\ 0 & \alpha(1 - \alpha_-) \end{array} \right) \) consistently solves the above equation along with Eq. (15).

**IV. CORRELATION FUNCTIONS** \((\nu = 2)\)

To calculate the partition function one needs to take care of the conservation of total number of 0s. Instead, we work in grand canonical ensemble (GCE), where the fugacity \( z \) associated with ‘\( A \)’s fixes the average density of 0s. Thus the partition function in GCE is
\[
Z = Tr[(D + E + zA)^{L}] = Tr[T^L] 
\]
where we have used \( T = D + E + zA = \left( \begin{array}{cc} 1 + z & p \\ 1 & p + z\alpha \end{array} \right) \) for convenience. The eigenvalues of \( T \) are
\[
\lambda_{\pm} = \frac{1}{2} \left[ 1 + p + z + \alpha z \pm \sqrt{(1 + p + z + \alpha z)^2 - 4z(\alpha + p + \alpha z)} \right] 
\]
(19)

**FIG. 2:** The main figure compares Eqs. (21) and (22) (solid lines) with \((++)\) (circle) and \(<++>\) (cross) obtained from Monte-carlo simulation of a system of size \( L = 1000, \alpha = 0.3 \) and \( p = 0.6 \). The correlation function \( C_{+++} \) is shown in the inset, where solid line corresponds to Eq. (25).

\[
T_{11}^n \simeq \frac{\lambda_+^n (\lambda_+ - p - \alpha z)}{\lambda_+ - \lambda_-}, \quad T_{12}^n \simeq \frac{p \lambda_+^n}{\lambda_+ - \lambda_-}
\]
straightforward in this formulation. For example, 

\[ C_{00} = \langle 00 \rangle - (0)^2 = \frac{p z^2(1-\alpha)^2}{\lambda_+^2(\lambda_+ - \lambda_-)^2} \]  

We have compared these results with those obtained from the Monte-carlo simulations. In Fig. 2 we have shown variation of \( \langle + \rangle \) and \( \langle ++ \rangle \) with \( \rho_0 \). The parameters \( \alpha = 0.3 \), \( p = 0.6 \) and the system size \( L = 1000 \) were fixed. Solid lines correspond to Eqs. (21) and (24) respectively, where \( \rho_0 \) was obtained from Eq. (20). Inset of Fig. 2 compares Eq. (25) with \( C_{++} \) obtained from simulations.

Similarly, in Fig. 3 we have compared the average velocity of 0s with Eq. (23). The inset therein shows variation of \( C_{00} = \langle 00 \rangle - (0)^2 \).

V. General Results for \( \nu > 2 \)

In this section we will extend the results of section III A and obtain the exact steady state weights of the model with generic \( \nu > 2 \).

Here, it is evident that the dynamics (1) and (2) can also be mapped to that of a ZRP with \( \nu \) kinds of boxes, where a particle from a randomly chosen box of kind \( I \) moves to the left box with a constant rate \( \alpha_I \) and an empty box can change its internal state. As before we look for a factorized steady state satisfying pairwise balance condition. Accordingly, for non-empty boxes, we obtain

\[ \alpha_I f_I(n+1) = c, \forall I = 1, 2 \ldots \nu \]  

similar to Eq. (7). Here again \( c \) is an arbitrary constant, independent of the number of particles \( n \) and the state of the box \( I \), can be taken to be 1 without any loss of generality. Thus \( f_I(n) = f_I(0)/\alpha_I^n \). The relative weights of the empty boxes, again, satisfy

\[ \frac{f_J(0)}{f_I(0)} = \frac{p_{IJ}}{p_{IJ}} \]  

for all possible pairs \( (I, J) \). This condition, which is a generalized version of the Eq. (3), demands that the steady state cannot have a product measure form if all the \( \nu(\nu - 1) \) rates \( p_{IJS} \) are independent; they must be related in the following way

\[ \frac{p_{IJK}}{p_{IJK}} = \frac{p_{IK}}{p_{IK}} \]  

This set of equations imposes \( (\nu - 1)(\nu - 2)/2 \) constraints leaving \( (\nu - 1)(\nu/2 + 1) \) independent \( p_{IJS} \). Further, we take \( f_I(0) = 1 \) as the weights are not normalized yet, which results in

\[ f_I(n) = \frac{1}{\alpha_I^n} \]  

and

\[ f_I(n) = \frac{p_{IJ}}{p_{IJ}} \frac{1}{\alpha_I^n} \]  

\( \forall I > 1 \).

To find the spatial correlation functions we rewrite the steady state in a matrix product form which is a direct
generalization of the two species case; each particle of species \( I \) is replaced by \( D_I \) and vacancies by \( A \). The non-commuting set of matrices \( D_I \) and \( A \) must satisfy Eq. (11) with the above \( f_I \). This can be achieved by choosing \( D_I = |d_i⟩⟨d_i| \), similar to Eq. (12), resulting in

\[
⟨d_i|A^n|d_i⟩ = f_I(n)
\]

which is a generalized form of Eq. (13).

It is straightforward to find a \( \nu \) dimensional representation of these matrices; \( |d| = \sum_i |i⟩, ⟨d| = \frac{1}{p_{i1}} |i⟩ \) and \( A = Diag(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_\nu}) \). Here, \( \{|i⟩\} \) are the standard basis set for the \( \nu \)-dimensional vector space.

Let us discuss the \( \nu = 3 \) case in some details. Explicitly, the dynamics is

\[
\begin{align*}
10 \xrightarrow{\alpha_{11}} 01; & \quad 20 \xrightarrow{\alpha_{21}} 02; \quad 30 \xrightarrow{\alpha_{31}} 03 \\
1I \xrightarrow{p_{12}} 2I; & \quad 2I \xleftarrow{p_{21}} 3I; \quad 1I \xrightarrow{p_{13}} 3I, \quad \forall I \neq 0.
\end{align*}
\]

The model is mapped to a ZRP with three different kinds of boxes. Its steady state has product measure only when the rates \( p_{IJ} \) follow Eq. (30): \( \frac{p_{I2} p_{23}}{p_{21} p_{32}} = \frac{p_{13}}{p_{31}} \).

Thus, any five out of these six \( p_{IJ} \)'s can be chosen independently and the sixth one is fixed by the above equation.

The spatial correlations of this system on the lattice can be calculated in a straightforward manner using the prescription described for \( \nu = 2 \). All one needs is the explicit representation of the matrices, which are,

\[
D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad D_2 = \frac{p_{12}}{p_{21}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix};
\]

\[
D_3 = \frac{p_{13}}{p_{31}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad A = \begin{pmatrix} \frac{1}{\alpha_1} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & \frac{1}{\alpha_3} \end{pmatrix} \quad (32)
\]

The correlation functions here are found to be qualitatively same as that of \( \nu = 2 \) case.

**VI. DISCUSSION AND CONCLUSION**

Several multiple species exclusion processes with or without conservation have been studied earlier, mostly numerically, in different contexts. In this article we study a multispecies exclusion model where exact steady state weights and spatial correlations are calculated analytically.

This model with \( \nu = 1 \) is identical to TASEP, which, on a ring, has spatially uncorrelated steady state. When internal degrees of freedom are introduced (i. e. \( \nu > 1 \)), two new features appear in the dynamics: i) particles in different internal states hop with different rates, ii) the particles can change their internal states resulting in non-conservation of number of particles belonging to each state. To investigate which of these features generates spatial correlations we revisit \( \nu = 2 \) case of the model with the nonconserving part of the dynamics (2) replaced by a conserving one,

\[
\pm 0 \xrightarrow{\alpha} 0 \pm - \xrightarrow{\frac{1}{\alpha}} - + . \quad (33)
\]

Symmetric exchange of + and − particles ensures that particle current gets contribution only from the hoping dynamics, as was the case in the non-conserved system.

The most generic particle conserving two species model \( \tau \tau' \rightarrow \tau' \tau \) with rate \( w_{\tau \tau'} \), where \( \tau \neq \tau' \in \{0, 1, 2\} \) has been discussed in Section 7 of the review article [7]. It has been shown there that the steady state of this generic model can be obtained using Matrix Product Ansatz, when the rates satisfy a set of conditions, Eqs.(7.8)-(7.13) therein. The particle conserving model discussed here is a special case of the above with \( w_{10} = \alpha_+, w_{20} = \alpha_-, w_{12} = 1 = w_{21} \), and all other rates are zero. Here, MPA provides an exact solution only when \( \alpha_+ = \alpha_- \); corresponding steady state turns out to be spatially uncorrelated. This indicates that non-conservation is irrelevant, at least for \( \alpha_+ = \alpha_- \), as the non-conserving model (3) also provides an uncorrelated steady state in this case. To understand, if non-conservation plays any role in generating spatial correlations, we study model (33) numerically for generic rates \( \alpha_\pm \). These Monte-Carlo simulations reveal that non-trivial spatial correlations appear for \( \alpha_+ \neq \alpha_- \). In Fig. 4 we compare some of the two-point correlations obtained from Monte-Carlo simulations of the conserved system with the known exact results of the corresponding (33) non-conserved model. Clearly the spatial correlations of the conserved model are qualitatively similar to those of the non-conserved case. Thus, conservation of internal...
degrees is an irrelevant criterion in developing spatial correlations in exclusion processes, rather the unequal hop rates are responsible.

In conclusion, we have studied an exclusion process on a one dimensional system with periodic boundary, where particles carry $\nu$ internal degrees of freedom. Along with a directional hopping dynamics that depends on the internal state of the particle non-conservation is introduced by allowing a particle to change its internal state when the target site is occupied. We show that this model can be mapped to a zero range process where particles distributed in $\nu$ different kinds of boxes hop with different rates and empty boxes can change their state. The steady state weights of the exclusion process could be written in matrix product form using the exact single box weights obtained from ZRP. Exact spatial correlations have been calculated exploiting the matrix product form. From the comparison of these correlations with those of a corresponding conserved model we conclude that the unequal hop rates of particles belonging to different internal states is the possible cause for the non-trivial spatial correlations observed on a ring.

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[1] B. Schmittmann and R. K. P. Zia, Phase Transitions and Critical Phenomena, Vol. 17, eds. C. Domb and J.L. Lebowitz, (Academic Press, N.Y., 1995).
[2] F. Spitzer, Adv. Math. 5, 246 (1970).
[3] K. Nishinari, D. Chowdhury, and A. Schadschneider, Phys. Rev. E 67, 036120 (2003); K. Nishinari, Y. Okada, A. Schadschneider, and D. Chowdhury, Phys. Rev. Lett. 95, 118101 (2005);
[4] T. M. Liggett, Stochastic Interacting Systems: Voter, Contact and Exclusion Processes, (Springer, Newyork, 1999); G. M. Schütz, Phase transitions and critical phenomena, Vol. 19, eds. C. Domb and J.L. Lebowitz, (Academic Press, London, 2000).
[5] B. Derrida, Phys. Rep. 301, 65 (1998).
[6] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, J. Phys. A: Math. Gen. 26, 1493 (1993).
[7] R. A. Blythe and M. R. Evans, J. Phys. A: Math. Theor. 40, R333 (2007).
[8] M. R. Evans, P. A. Ferrari and K. Mallick, J. Stat. Phys. 135, 217(2009).
[9] A. Ayyer, J. L. Lebowitz, E. R. Speer, J. Stat Phys. 135, 1009 (2009).
[10] F. Tabatabaei and G. M. Schütz, Phys. Rev. E 74, 051108 (2006).
[11] M. R. Evans and T. Hanney, J. Phys. A: Math. Gen. 38, R195 (2005).
[12] U. Basu and P. K. Mohanty, J. Stat. Mech. L03006 (2010).
[13] The density of particles in any internal state in the conserved model, which is fixed, corresponds to the average density of particles in the same internal state for the non-conserved model.