Theory of thin shells as a spatial two-dimensional continuum in an oblique system of coordinates

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Abstract. In contrast to the known approaches in the theory of shells, based on certain assumptions, and the traditional reduction of a three-dimensional problem to a two-dimensional one, in this paper, static and geometric relations are obtained from the position of the interaction of linear forces and moments in a spatially curved two-dimensional continuum. This approach was applied by E. Reissner, but in an orthogonal grid and for physical components. Relations are presented in tensor, vector, and scalar forms.

1. Introduction

Various methods for constructing versions of the theory of shells based on certain assumptions are known. In this paper, in contrast to the traditional approach of reducing a three-dimensional problem to a two-dimensional one, static and geometric dependences are obtained from the point of view of the interaction of linear forces and moments in a spatially curved two-dimensional continuum. The basic relations of the theory of shells have considerable generality and a compact form. With the help of some transformations and simplifications, these relations are reduced to the equations of the classical theory. The construction of the theory of shells as a two-dimensional continuum was carried out for the general case in a spatial oblique coordinate system, which entailed the use of the apparatus of tensor analysis.

2. Derivation formulas

Let us present some dependencies related to the differential geometry of surfaces, which are necessary for further constructing the proposed theory of shells. These relations are described in detail in the corresponding courses, as well as in the introductory chapters of fundamental monographs [1–4, 6, 9] on the theory of shells with some differences in notation. Therefore, the formulas in this paragraph are given as a reference without detailed derivation.

So, let some parametrised surface be given by the equation \( \vec{r} = \vec{r}(\alpha^1, \alpha^2) \), where \( \alpha^1, \alpha^2 \) are oblique Gaussian coordinates. In this case, the projections of the vector function \( \vec{r} \) of the surface points in the Cartesian coordinate system are given by the continuous single-valued functions

\[
x = x(\alpha^1, \alpha^2), \quad y = y(\alpha^1, \alpha^2), \quad z = z(\alpha^1, \alpha^2).
\]
As is known, the coordinate vectors \( \vec{r} = \partial r / \partial \alpha^i \) \((i = 1,2)\) together with the unit vector of the normal \( \vec{n} = \vec{r}_1 \times \vec{r}_2 / \sqrt{a} \) form the basic basis at any point of the surface; \( \sqrt{a} = A_1 A_2 \sin \chi \), \( A_1, A_2 \) are the Lame parameters, \( \chi \) is the angle between \( \vec{r}_1 \) and \( \vec{r}_2 \).

The first quadratic form of the surface is determined by the equality

\[
I = d\vec{r} \cdot d\vec{r} = (ds)^2 = A_1^2 (d\alpha^1)^2 + 2a_{12} d\alpha^1 d\alpha^2 + A_2^2 (d\alpha^2)^2,
\]

where

\[
A_1^2 = \vec{n} \cdot \vec{n} = \left( \frac{\partial x}{\partial \alpha^i} \right)^2 + \left( \frac{\partial y}{\partial \alpha^i} \right)^2 + \left( \frac{\partial z}{\partial \alpha^i} \right)^2;
\]

\[
a_{12} = \vec{n} \cdot \vec{n}_2 = A_1 A_2 \cos \chi = \frac{\partial x}{\partial \alpha^1} \frac{\partial x}{\partial \alpha^2} + \frac{\partial y}{\partial \alpha^1} \frac{\partial y}{\partial \alpha^2} + \frac{\partial z}{\partial \alpha^1} \frac{\partial z}{\partial \alpha^2};
\]

\[
A_2^2 = \vec{n}_2 \cdot \vec{n}_2 = \left( \frac{\partial x}{\partial \alpha^2} \right)^2 + \left( \frac{\partial y}{\partial \alpha^2} \right)^2 + \left( \frac{\partial z}{\partial \alpha^2} \right)^2.
\]

The physical significance of \( A_1 \) and \( A_2 \) is that they are proportionality coefficients in the formulas connecting the differentials of the arc lengths of the coordinate lines with the differentials of the curvilinear coordinates themselves

\[
ds_1 = A_1 d\alpha^1, \quad ds_2 = A_2 d\alpha^2.
\]

The area of an infinitesimal quadrangular surface element bounded by the coordinate lines \( \alpha^1 = \text{const} \) and \( \alpha^1 + d\alpha^1 = \text{const} \), \( \alpha^2 = \text{const} \) and \( \alpha^2 + d\alpha^2 = \text{const} \), is defined by the equality

\[
d\sigma = \sqrt{a} d\alpha^1 d\alpha^2.
\]

The second quadratic form of the surface is given by the expression

\[
II = b_{11} (d\alpha^1)^2 + 2b_{12} d\alpha^1 d\alpha^2 + b_{22} (d\alpha^2)^2,
\]

where the coefficients \( b_{ij} \) are determined by the equalities

\[
b_{11} = \vec{r}_{11} \cdot \vec{n} = \frac{\vec{r}_{11} \cdot (\vec{r}_1 \times \vec{r}_2)}{\sqrt{a}} = 1 \frac{\begin{vmatrix}
\partial^2 x & \partial^2 y & \partial^2 z \\
\partial (\partial \alpha^1)^2 & \partial (\partial \alpha^1)^2 & \partial (\partial \alpha^1)^2 \\
\partial (\partial \alpha^2)^2 & \partial (\partial \alpha^2)^2 & \partial (\partial \alpha^2)^2 
\end{vmatrix}}{
\begin{vmatrix}
\partial \alpha^1 & \partial \alpha^1 & \partial \alpha^1 \\
\partial \alpha^1 & \partial \alpha^1 & \partial \alpha^1 \\
\partial \alpha^1 & \partial \alpha^1 & \partial \alpha^1 
\end{vmatrix}}
\]

\[
b_{12} = b_{21} = \vec{r}_{12} \cdot \vec{n} = \frac{\vec{r}_{12} \cdot (\vec{r}_1 \times \vec{r}_2)}{\sqrt{a}} = 1 \frac{\begin{vmatrix}
\partial^2 x & \partial^2 y & \partial^2 z \\
\partial (\partial \alpha^1)^2 & \partial (\partial \alpha^1)^2 & \partial (\partial \alpha^1)^2 \\
\partial (\partial \alpha^2)^2 & \partial (\partial \alpha^2)^2 & \partial (\partial \alpha^2)^2 
\end{vmatrix}}{
\begin{vmatrix}
\partial \alpha^1 & \partial \alpha^1 & \partial \alpha^1 \\
\partial \alpha^1 & \partial \alpha^1 & \partial \alpha^1 \\
\partial \alpha^1 & \partial \alpha^1 & \partial \alpha^1 
\end{vmatrix}}
\]
The third quadratic form of the surface is the square of the differential of the unit normal vector to the surface

$$III = d\bar{n} \cdot d\bar{n} = \bar{n}_1^2 (d\alpha^1)^2 + 2\bar{n}_1 \cdot \bar{n}_2 d\alpha^1 d\alpha^2 + \bar{n}_2^2 (d\alpha^2)^2.$$ 

The last form is not independent, it is expressed through the first two \( III = I \cdot K - II \cdot K_{cp}, \) where \( K, K_{cp} \) are Gaussian (total) and mean curvature of the surface, respectively.

When deriving the basic relations, we need the Gauss-Weingarten formulas for differentiating the vectors of the reciprocal local basis. In surface theory these derivation equalities are analogous to the Serre-Frenet formulas in the theory of spatial curves. Let us represent them in the form

$$\bar{\alpha}_j = \Gamma_{ij}^k \bar{\alpha}_k + b_{ij} \bar{\alpha}; \quad \bar{\alpha}_i = -b_{ik} \bar{\alpha}_k.$$  

(7)

Here \( \Gamma_{ij}^k \) are the Christoffel symbols defined by the equalities

$$
\begin{align*}
\Gamma_{11}^1 &= \frac{1}{a} \left[ A_1 A_2^2 \frac{\partial A_1}{\partial \alpha^1} - a_{12} \left( \frac{\partial a_{12}}{\partial \alpha^1} - A_1 \frac{\partial A_2}{\partial \alpha^1} \right) \right]; \\
\Gamma_{12}^1 &= \Gamma_{21}^1 = \frac{1}{a} \left[ A_1 A_2^2 \frac{\partial A_2}{\partial \alpha^2} - A_1 a_{12} \frac{\partial A_2}{\partial \alpha^2} \right]; \\
\Gamma_{22}^1 &= \frac{1}{a} \left[ A_1^2 \frac{\partial a_{12}}{\partial \alpha^2} - A_2 a_{12} \frac{\partial A_1}{\partial \alpha^2} \right]; \\
\Gamma_{11}^2 &= \frac{1}{a} \left[ A_1^2 \frac{\partial a_{12}}{\partial \alpha^1} - A_1 a_{12} \frac{\partial A_2}{\partial \alpha^1} \right]; \\
\Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{a} \left[ A_1^2 \frac{\partial A_2}{\partial \alpha^1} - A_2 a_{12} \frac{\partial A_1}{\partial \alpha^1} \right]; \\
\Gamma_{22}^2 &= \frac{1}{a} \left[ A_2^2 \frac{\partial A_2}{\partial \alpha^2} - a_{12} \left( \frac{\partial a_{12}}{\partial \alpha^2} - A_1 \frac{\partial A_2}{\partial \alpha^2} \right) \right].
\end{align*}
$$ 

(8)

The coefficients \( b_{ij} \) are represented by formulas (6), and \( b_{ij}^k \) are expressed through \( b_{ij} \) and the coefficients of the first quadratic form

$$
\begin{align*}
b_{11}^1 &= \frac{1}{a} \left( A_2^2 b_{11} - a_{12} b_{12} \right); \\
b_{12}^1 &= \frac{1}{a} \left( A_2^2 b_{12} - a_{12} b_{11} \right); \\
b_{22}^1 &= \frac{1}{a} \left( A_2^2 b_{21} - a_{12} b_{22} \right); \\
b_{12}^2 &= \frac{1}{a} \left( A_2^2 b_{22} - a_{12} b_{21} \right).
\end{align*}
$$ 

(9)

For the vectors of the reciprocal local basis
formulas similar to (7) can be written in the form

$$\vec{r}_j = -\Gamma_{jk} \vec{r}^k + b_j \vec{n}.$$  \hspace{1cm} (10)

3. Equilibrium equations. Boundary conditions

Let us select an infinitesimal element of the surface, which will be the element of a certain shell, loaded with the distributed load $\vec{q}$ and moment $\vec{m}$ per unit of undeformed area. Let the vectors of forces $\vec{N}^i$ and moments $\vec{M}^i$ be applied to the four sides of this element in accordance with figure 1 (hereinafter $(i,j) = \partial / \partial \alpha^i$).

$$\sqrt{aM^i} d\alpha^2 \quad \sqrt{aN^i} d\alpha^2 \quad \sqrt{a\vec{q}} da^1 d\alpha^2 \quad \sqrt{a\vec{n}} da^1 \quad \sqrt{aM^i} d\alpha^2 \quad \sqrt{aN^i} d\alpha^2$$

Figure 1. Loading conditions of a shell element.

In this case, vectors of forces and moments can be represented in the form ($i,j = 1,2$)

$$\vec{q} = q^i \vec{r}_j + q_n \vec{n}; \quad \vec{m} = m^i \vec{n} \times \vec{r}_j + m_n \vec{n}. \hspace{1cm} (11)$$

$$\vec{N}^i = N^i \vec{r}_j + Q^i \vec{n}; \quad \vec{M}^i = M^i \vec{n} \times \vec{r}_j + P^i \vec{n}. \hspace{1cm} (12)$$

In contrast to the known theories of shells, here for the moment vectors we introduce the moment components $P^i$ normal to the surface.

The force $\sqrt{aN_1} da^2$ acts on the side $\alpha^1 = const$, whose length is $A_2 da^2$ and the force $\left[\sqrt{aN^1} \right]_1 da^1 da^2$ acts on the side $\alpha^1 + da^1 = const$. Considering the increment of force factors during the transition from the side $\alpha^1 = const$ to the side $\alpha^1 + da^1 = const$, subject to the
principle of equality of action and reaction, we obtain the total force \( \int \sqrt{a N_1^1} d\alpha^1 d\alpha^2 \). Similarly, the total force acting on the sides \( \alpha^2 = \text{const} \) and \( \alpha^2 + d\alpha^2 = \text{const} \) is equal to \( \int \sqrt{a N_2^2} d\alpha^1 d\alpha^2 \).

Taking into account the external forces \( \tilde{q}\sqrt{ad\alpha^1}d\alpha^2 \) acting on the element under consideration, the conditions of the equality of the principal vector to zero are reduced to the vector equation of the equilibrium of forces

\[
\left( \sqrt{a N_1^1} \right)_1 + \sqrt{\tilde{q}} = 0. \tag{13}
\]

Consideration of the increments of the moments when passing from the side \( \alpha^i = \text{const} \) to the side \( \alpha^i + d\alpha^i = \text{const} \) determines the principal moment

\[
\left( \sqrt{a M_1^1} \right)_1 + \left( \sqrt{a M_2^2} \right)_2.
\]

Moreover, forces also create moments. When considering the sides \( \alpha^i = \text{const} \) and \( \alpha^i + d\alpha^i = \text{const} \), the forces applied to them determine the moment (figure 2)

\[
\tilde{r} \times \left[ \sqrt{a N_1^1} + \left( \sqrt{a N_1^1} \right)_i d\alpha^2 + \tilde{r} \times \left( -\sqrt{a N_1^1} \right) d\alpha^2 \right] \approx \left( \tilde{r}_+ - \tilde{r}_- \right) \times \sqrt{a N} \sqrt{a d\alpha^1} d\alpha^2 = \tilde{r} \times \sqrt{a N} \sqrt{a d\alpha^1} d\alpha^2.
\]

![Diagram](image-url)

**Figure 2.** Moments created by the action of forces.

Similarly, considering the sides \( \alpha^2 = \text{const} \) and \( \alpha^2 + d\alpha^2 = \text{const} \), we get

\[
\tilde{r}_2 \times \sqrt{a d\alpha^1} d\alpha^2.
\]
The condition of equality of the principal moment of all forces and moments to zero, taking into account the external surface $m$, can finally be reduced to the vector equation of the moment equilibrium

$$\left(\sqrt{a}M^i\right)_i + \vec{r}_i \times \left(\sqrt{a}N^i\right)_i + \sqrt{a}\vec{m} = 0.$$  \hspace{1cm} (14)

Each of equations (13), (14) is equivalent to three scalar equations. Expanding equation (13) with the help of (7) and sequentially multiplying by the coordinate vectors of the auxiliary basis $\vec{r}^1, \vec{r}^2, \vec{n}$ ($\vec{r}_i \cdot \vec{r}_j = \delta^i_j$ is the Kronnecker symbol, $\vec{r}_i \cdot \vec{n} = 0$), we obtain the known equilibrium equations in scalar form

$$\nabla_i N^{ik} - b_i^k Q^j + q_i^k = 0, \quad k = 1, 2$$

$$\nabla_i Q^j + b_i^k N^{ik} + q_n = 0,$$

where $\nabla_i N^{ik} = \frac{1}{\sqrt{a}} \left(\sqrt{a}N^{ik}\right)_i + N^{ij} \Gamma^k_j$; $\nabla_i Q^j = \frac{1}{\sqrt{a}} \left(\sqrt{a}Q^j\right)_i$.

Equations of moment equilibrium (14), considering the relation $\Gamma^j_i = \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial \alpha^i}$, can be transformed to the form

$$\nabla_i \tilde{M}^i + \vec{r}_i \times \tilde{N}^i + \tilde{m} = 0,$$  \hspace{1cm} (16)

where $\nabla_i \tilde{M}^i = \frac{\partial \tilde{M}^i}{\partial \alpha^i} + \Gamma^j_i \tilde{M}^j$.

Expanding (16) with the help of (11), (12) and sequentially multiplying scalar by $(\vec{n} \times \vec{r}^1)$, $(\vec{n} \times \vec{r}^2)$ and $\vec{n}$, we obtain three equations of equilibrium of moments

$$\nabla_i M^{ik} - b_{ij} c^{ij} P^j - Q^k + m^k = 0, \quad k = 1, 2$$

$$\nabla_i P^j + c_{ik} \left(N^{ik} + b_i^k M^{ij}\right) + m_n = 0,$$

where

$$\nabla_i M^{ik} = \frac{\partial M^{ik}}{\partial \alpha^i} + \Gamma^j_i M^{kj} + \Gamma^j_i M^{kj} = \frac{1}{\sqrt{a}} \left(\sqrt{a}M^{ik}\right)_j + \Gamma^j_i M^{kj}; \quad \nabla_i P^j = \frac{\partial P^j}{\partial \alpha^i} + \Gamma^j_i P^j = \frac{1}{\sqrt{a}} \left(\sqrt{a}P^j\right)_j.$$

Without taking into account $P^j$ and $m_n$, equations (17) are the same as those known from the literature sources [3, 9]. Taking these quantities into account, the third moment equation is not algebraic and not satisfied identically. It also implies that for the moment components $P^j$, the component $m_n$ of the surface moment is determining.

Equilibrium equations (13), (14), which are valid in the interior of the shell, must be supplemented with static boundary conditions, which can be represented in the form

$$\tilde{N} = A_j \tilde{N}^j \cos(\nu, \alpha^j); \quad \tilde{M} = A_j \tilde{M}^j \cos(\nu, \alpha^j),$$  \hspace{1cm} (18)

where $\tilde{N}, \tilde{M}$ are the specified edge forces and moments; $\nu$ – tangential normal to the surface boundary line.
4. Geometric relationships

Let us define possible displacements and possible deformations as a system of infinitely small kinematically possible displacements and deformations, which allow us to express possible work in such a form that equations (13), (14), (18) will be equivalent to the following

$$
\iint_{(\sigma)} (\tilde{N}^i \delta \vec{e}_i + \tilde{M}^i \delta \vec{k}_i) \sqrt{\alpha_1 \alpha_2} d\alpha^2 = \iint_{(\sigma)} (\tilde{q} \delta \vec{u} + \tilde{m} \delta \vec{\phi}) \sqrt{\alpha_1 \alpha_2} d\alpha^2 + \oint (\tilde{N} \delta \vec{u} + \tilde{M} \delta \vec{\phi}) ds,
$$

(19)

$\delta \vec{e}_i, \delta \vec{k}_i$ – vectors of possible deformations and rotations (bending and torsion), respectively; $\delta \vec{u}, \delta \vec{\phi}$ – vectors of possible translational and rotational displacements.

In the classical theory of shells, the necessary and sufficient condition for the fulfillment of (19) is equations (13), (14), (18). Let us use equation (19) in the inverse form: we shall assume that (13), (14), (18) are given, and with their help for arbitrary $\tilde{N}^i$ and $\tilde{M}^i$ from (19) we shall exclude the quantities $\tilde{q}, \tilde{m}, \tilde{N}, \tilde{M}$:

$$
\iint_{(\sigma)} (\tilde{N}^i \delta \vec{e}_i + \tilde{M}^i \delta \vec{k}_i) \sqrt{\alpha_1 \alpha_2} d\alpha^2 = \oint \left( A_i \tilde{N}^i \delta \vec{u} + A_i \tilde{M}^i \delta \vec{\phi} \right) \cos(\alpha, \alpha') ds -
\iint_{(\sigma)} \left( \sqrt{\alpha} \tilde{N}^i \right)_i \delta \vec{u} + \left( \sqrt{\alpha} \tilde{M}^i \right)_i + \vec{r} \times \left( \sqrt{\alpha} \tilde{N}^i \right) \delta \vec{\phi} \right) d\alpha^1 \alpha^2.
$$

(20)

Using the formula for integration by parts, we obtain the equalities

$$
- \iint_{(\sigma)} \left( \sqrt{\alpha} \tilde{N}^i \right)_i \delta \vec{u} d\alpha^1 \alpha^2 = \iint_{(\sigma)} \tilde{N}^i \left( \delta \vec{u} \right)_i \sqrt{\alpha_1 \alpha_2} d\alpha^2 - \oint \sqrt{\alpha} \delta \vec{u} \left( \tilde{N}^1 \alpha^2 + \tilde{N}^2 \alpha^1 \right)
$$

$$
- \iint_{(\sigma)} \left( \sqrt{\alpha} \tilde{M}^i \right)_i \delta \vec{\phi} d\alpha^1 \alpha^2 = \iint_{(\sigma)} \tilde{M}^i \left( \delta \vec{\phi} \right)_i \sqrt{\alpha_1 \alpha_2} d\alpha^2 - \oint \sqrt{\alpha} \delta \vec{\phi} \left( \tilde{M}^1 \alpha^2 + \tilde{M}^2 \alpha^1 \right).
$$

(21)

The contour integral in (20) can be represented in the form

$$
\oint \left( A_i \tilde{N}^i \delta \vec{u} + A_i \tilde{M}^i \delta \vec{\phi} \right) \cos(\alpha, \alpha') ds = \oint \tilde{N}^1 \delta \vec{u} A_1 + \tilde{M}^1 \delta \vec{\phi} A_2 \cos(\alpha, \alpha') d\alpha^2 + \oint \tilde{N}^2 \delta \vec{u} A_1 + \tilde{M}^2 \delta \vec{\phi} A_2 \cos(\alpha, \alpha') d\alpha^2 + \oint \tilde{N}^2 \delta \vec{u} A_1 + \tilde{M}^2 \delta \vec{\phi} A_2 \cos(\alpha, \alpha') d\alpha^2 = \oint \sqrt{\alpha} \delta \vec{u} \left( \tilde{N}^1 \alpha^2 + \tilde{N}^2 \alpha^1 \right) + \oint \sqrt{\alpha} \delta \vec{\phi} \left( \tilde{M}^1 \alpha^2 + \tilde{M}^2 \alpha^1 \right)
$$

(22)

In this expression $\cos(\alpha, \alpha') = \cos(90 - \chi) = \sin \chi$ (figure 3); and similarly, $\cos(\alpha, \alpha') = \sin \chi$.

![Figure 3](image-url)
Considering the properties of the triple scalar product
\[
\left( \vec{r} \times \sqrt{\alpha} \vec{N}^i \right) \cdot \delta \vec{\varphi} = -\left( \vec{r} \times \delta \vec{\varphi} \right) \cdot \sqrt{\alpha} \vec{N}^i
\]
and equalities (21), (22), equation (20) is reduced to the form
\[
\int_{\sigma} \left( \vec{N}^i \delta \vec{\varepsilon}_i + \vec{M}^i \partial \delta \vec{\varepsilon}_i \right) \sqrt{\alpha} d\alpha^1 d\alpha^2 = \int_{\sigma} \left( \left( \delta \vec{u} \right)_{ij} + \vec{r} \times \delta \vec{\varphi} \right) + \vec{M}^i \left( \partial \delta \vec{\varphi} \right)_{ij} \sqrt{\alpha} d\alpha^1 d\alpha^2. \tag{23}
\]

Due to the arbitrariness of the vectors \( \vec{N}^i \) and \( \vec{M}^i \), (23) implies the relations between possible deformations and possible displacements
\[
\delta \vec{\varepsilon}_i = \left( \delta \vec{u} \right)_{ij} + \vec{r} \times \delta \vec{\varphi}; \quad \delta \vec{\varepsilon}_i = \left( \delta \vec{\varphi} \right)_{ij}.
\]
From this, passing to actual deformations and displacements, we obtain
\[
\vec{e}_i = \vec{u}_{ij} + \vec{r} \times \vec{\varphi}; \quad \vec{k}_i = \vec{\varphi}_{ij}.	ag{24}
\]

We represent the vectors of deformations and rotations, translational and rotational displacements in the form
\[
\vec{e}_i = \varepsilon_i^j \vec{r}^j + \gamma_i \vec{n}; \quad \vec{k}_i = \kappa_i^j \vec{n} \times \vec{r}^j + \lambda_i \vec{n};
\]
\[
\vec{u} = u_j \vec{r}^j + w \vec{n}; \quad \vec{\varphi} = \phi_j \vec{n} \times \vec{r}^j + \omega \vec{n}.	ag{25}
\]

Here, in contrast to the classical theory of shells, the vectors \( \vec{k}_i \) take into account the components \( \lambda_i \), directed along the normal to the surface and corresponding to \( P^i \) in expansions (12).

Expanding vector equations (24) using Gauss-Weingarten formulas (7), (10), we obtain the relations between deformations and displacements in scalar form
\[
\varepsilon_{ij} = u_{ji} - \Gamma_{ki}^j u_k - b_{ij} w - c_{ij} \omega, \quad \gamma_i = w_j + b_{ij} u_k + \phi_j; \tag{27}
\]
\[
\kappa_{ij} = \phi_{ij} - \Gamma_{ki}^j \phi_k - c_{jk} b_{ij} \omega, \quad \lambda_i = \omega_j + c_{jk} b_{ij} \phi_j. \tag{28}
\]

Comparing equalities for \( \lambda_i \) with expressions for \( \zeta_j \) [9, p.84] or \( \zeta_j \) [4, p.84] formally introduced when obtaining geometric relations, we come to the conclusion that these quantities are completely identical to each other. For these quantities, their specific geometric significance is not indicated, although the assumption that "they represent some parameters of deformation" was made in work [8, p.103]. The proposed version of the theory of shells allows us to give auxiliary geometric quantities \( \zeta_j \) or \( \zeta_j \) a specific physical interpretation corresponding to \( \lambda_i \) in expansions (25). So, these components are the components of the bending deformation vectors \( \vec{k}_i \) (torsional deformation in the middle surface) normal to the middle surface of the shell.

5. Deformation compatibility equations

Expressions (24) from the obvious relations \( \vec{\varphi}_{12} = \vec{\varphi}_{21} \) and \( \vec{u}_{12} = \vec{u}_{21} \) allow us to obtain two vector equations of compatibility of deformations
\[
\vec{k}_{2,1} - \vec{k}_{1,2} = 0; \quad \vec{e}_{2,1} - \vec{e}_{1,2} + \vec{r}_2 \times \vec{k}_2 - \vec{r}_1 \times \vec{k}_1 = 0.	ag{29}
\]
That is, the homogeneous equilibrium equations and (29) express the vector form of the staticogeometric analogy of the theory of thin shells.

A.I. Lurie and A.L. Goldenweiser were the first to draw attention to the staticogeometric analogy in the theory of thin shells. In the following investigations, the ways of its application turned out to be remarkably diverse [4, 5, 9]. But in the general case, the staticogeometric analogy considered in these sources cannot be called complete, since there are “extra” equalities and quantities, which is indicated in the fundamental monograph [4]. In this sense, the proposed version of the theory of shells gives a complete correspondence between static and geometric relations and quantities.

Expanding the vector relations (29), we obtain six scalar equations, which, without taking into account the components \( \gamma_i \) of transverse deformation, coincide with the known ones [4, p.84; 9, p.86]

\[
\begin{align*}
\nabla \cdot \left( e^{\mu \nu} c^{\lambda \mu} \kappa_{\lambda \mu} \right) & - b_k \left( - e^{\mu \nu} \lambda_j \right) = 0, (k = 1, 2) \\
\nabla \cdot \left( - e^{\mu \nu} \lambda_j \right) + b_k \left( e^{-\mu \nu} c^{\lambda \mu} \kappa_{\lambda \mu} \right) & = 0.
\end{align*}
\]

(30)

The homogeneous equilibrium equations corresponding to (15), (17) and the compatibility equations (30) are identical to each other in structure, which makes it possible to establish a correspondence between scalar static and geometric quantities:

\[
N^{ik} \leftrightarrow e^{ij} c^{kl} \kappa_{kl}, \quad Q^i \leftrightarrow -c^{ij} \lambda_j; \quad M^{ik} \leftrightarrow -c^{ij} c^{kl} \epsilon_{ij}, \quad P^i \leftrightarrow -c^{ij} \gamma_j.
\]

(31)

In the known theories of shells, the components of the transverse shear deformation \( \gamma_i \) are not associated with the components of static quantities (\( P^i \)), and the transverse shear forces \( Q^i \) correspond to the formally introduced quantities \( \zeta_j \) [9, p.84] or \( \xi_i \) [4, p.84] without interpreting their specific physical meaning (in the proposed version, these are the components \( \lambda_i \) of the bending deformation vector \( \tilde{k}_i \) normal to the middle surface).

Based on the staticogeometric analogy, we can introduce the stress functions

\[
\tilde{U} = U^i \tilde{r}_j + W \tilde{n}, \quad \tilde{\Phi} = \Phi^j \tilde{n} \times \tilde{r}_j + \Omega \tilde{n},
\]

(32)
dual to vectors \( \tilde{u} \) and \( \tilde{\phi} : \tilde{U} \leftrightarrow \tilde{\phi}, \quad \tilde{\Phi} \leftrightarrow \tilde{u} \). In this case, \( \tilde{M}^i \) and \( \tilde{N}^i \) are expressed in terms of \( \tilde{U} \) and \( \tilde{\Phi} \) using formulas similar in structure to (27), (28):

\[
\begin{align*}
M^j & = e^{ik} \left( \Phi^k \Omega^{i} - \Gamma^{i} \Omega^k + a_k W \right) + b_k \left( - e^{ik} b_j \Omega^i + c_{ik} U^j \right) \\
P^i & = e^{ik} \left( \Omega^{i} - e^{ik} b_j \Omega^i + c_{ik} U^j \right) \\
N^j & = e^{ik} \left( U^i \Omega^{k} + \Gamma^{i} U^k - b_k W \right) + b_k \left( - c^{ik} b_j \Omega^i + c_{ik} U^j \right)
\end{align*}
\]

(33)

6. Physical relationships

For the proposed version of the theory of shells to have a closed structure, it is necessary to add elasticity relations to the static and geometric equations. Obtained in the traditional way, these equations of the theory of shells, considering the transverse shear, in scalar form are [7, p. 87].

\[
N^{ik} = KE^{ikjs} \epsilon_{js}, \quad Q^i = k^2 K (1 - \nu) \gamma^i, \quad M^{ik} = DE^{ikjs} \kappa_{js},
\]

(34)
where \( E^{ik} = a^{ik} c^{ks} \); \( \nu \) – Poisson’s ratio;
\[
K = \frac{Eh}{1-\nu^2} \quad \text{– stiffness of a strip beam with a width equal to one, working as part of a plate subject to}
\]
uniform extension parallel to the strip axis;
\[
D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{– cylindrical stiffness (stiffness of the same strip but bending as part of a plate along a}
\]
cylindrical surface).

It is not possible to obtain the missing relation between \( P^i \) and \( \lambda^i \) in the same way. But, relying on
the staticogeometric analogy (31) and expression (34) for \( Q^i \), this relation can be represented in the form
\[
P^i = k^i D \lambda^i ,
\]
where \( k^i \) – coefficient determined empirically (analogue of \( k^2 \) in formula (34) for \( Q^i \)).

In vector form, relations (34), (35) are
\[
\begin{align*}
\vec{N}^i & = K \left[ a^{ik} \tilde{\varepsilon}^i_k + \nu e^{ik} (\tilde{\varepsilon}^i_k \times \tilde{n}) + [k^2 (1-\nu) \gamma_i - a^{ik} \gamma_k] \tilde{n} \right] \\
\vec{M}^i & = D \left[ a^{ik} \tilde{\kappa}^i_k + \nu e^{ik} (\tilde{\kappa}^i_k \times \tilde{n}) + [k^2 \lambda_i - a^{ik} \lambda_k] \tilde{n} \right]
\end{align*}
\]
Dependences (34), and hence (36) also, were obtained using the Kirgoff-Love hypotheses: the
normal to the middle surface before deformation remains normal to it after deformation, while keeping
the length of the normal segment; normal stresses perpendicular to the middle surface are small
compared to other stresses (\( \sigma^{33} = 0 \)). In addition, when deriving them, the quantities \( z b\_k \) are
neglected in comparison with unity (\( z \) is directed along the normal to the middle surface). Because of
this [3, p.95]
\[
g^{ik} \approx a^{ik} , \quad \varepsilon_{zi}^{ik} = g^{is} g^{kj} \varepsilon_{sj}^{z} \approx a^{is} a^{kj} (\varepsilon_{sj} + z \kappa_{sj} ) .
\]
Therefore, dependences (34), (35) or (36) can be called only the formulas of the first approximation.
Considering the minor terms in them makes the general relations of the theory of shells symmetric,
which are more convenient for theoretical studies.

There is no doubt that further refinement and a more rigorous derivation of the physical relations of
the theory of shells is a potentially productive task. Two ways are possible here. The first way is to
develop programmes for the experimental determination of the force-deformation relations without
explicit binding to the spatial nature of the structure. The second way consists in the further
development of analytical methods for deriving these relations, which, in their structure, give an idea
of the shell as a three-dimensional continuum. In this direction, the most widespread are various
asymptotic methods, which were used in the works of K.Z. Galimov, A.L. Goldenweiser, V.V.
Novozhilov, E. Reissner and others.

7. Conclusion
In the proposed version of the theory of thin shells, in contrast to the classical approaches, the
components directed along the normal to the middle surface are taken into account in the vectors of
linear moments, and in the vectors of bending deformation – the corresponding torsion components.
This made it possible to obtain the correspondence of each static quantity to the geometric one and to
each homogeneous equation of equilibrium – the equation of compatibility of deformations. In the
known theories of shells, the components of the transverse shear deformation are not associated with
any components of static quantities, and some formally introduced quantities correspond to the
transverse shear forces. These quantities are also present in the proposed version, but they have a specific physical significance, namely, these are components of the bending deformation vector normal to the middle surface. All this justifies taking into account the introduced components from a theoretical point of view, since from a practical point of view, their retention is impractical: this leads to an increase in the degree of the differential equations of equilibrium (the third scalar moment equation is no longer satisfied identically, as in the classical theory of shells).

References
[1] Ambartsumyan S A 1974 General theory of anisotropic shells Moscow: Nauka 446 p
[2] Vekua I N 1982 Some general methods for constructing variants of the theory of shells Moscow: Nauka 288 p
[3] Galimov K Z 1975 Fundamentals of the nonlinear theory of thin shells Kazan 326 p
[4] Goldenweiser A L 1976 Theory of elastic thin shells Moscow: Nauka 512 p
[5] Novozhilov V V 1962 Theory of thin shells Leningrad: State union publishing house of the shipbuilding industry 431 p
[6] Ogibalov P N, Koltunov M A 1969 Shells and plates Moscow 695 p
[7] Theory of shells considering the transverse shear / Ed. K Z Galimov 1977 Kazan: Publishing house of KSU 211 p
[8] Filin A P 1987 Elements of the theory of shells Leningrad: Stroyizdat 384 p
[9] Chernykh K F Part 1 1962 Part 2 1964 Linear theory of shells: In 2 parts Leningrad Part 1 274 p Part 2 395 p