Asymptotics of determinants of Bessel operators

Estelle L. Basor∗
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407, USA

Torsten Ehrhardt†
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz, Germany

Abstract

For \( a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) the truncated Bessel operator \( B_\tau(a) \) is the integral operator acting on \( L^2[0, \tau] \) with the kernel

\[
K(x, y) = \int_0^\infty t \sqrt{xy} J_\nu(x t) J_\nu(y t) a(t) dt,
\]

where \( J_\nu \) stands for the Bessel function with \( \nu > -1 \). In this paper we determine the asymptotics of the determinant \( \det(I + B_\tau(a)) \) as \( \tau \to \infty \) for sufficiently smooth functions \( a \) for which \( a(x) \neq 1 \) for all \( x \in [0, \infty) \). The asymptotic formula is of the form \( \det(I + B_\tau(a)) \sim G^\tau E \) with certain constants \( G \) and \( E \), and thus similar to the well-known Szegő-Akhiezer-Kac formula for truncated Wiener-Hopf determinants.

1 Introduction

For \( a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) the Bessel operator \( B(a) \) is the integral operator acting on \( L^2(\mathbb{R}_+) \) with the kernel

\[
K(x, y) = \int_0^\infty t \sqrt{xy} J_\nu(x t) J_\nu(y t) a(t) dt.
\]
Here $J_\nu$ is the Bessel function with a parameter $\nu > -1$.

For each $\tau > 0$, the truncated Bessel operator $B_\tau(a)$ is the integral operator acting on $L^2[0, \tau]$ with the same kernel (1). Obviously, $B_\tau(a)$ can be considered as the restriction of $B(a)$ onto $L^2[0, \tau]$, i.e.,

$$B_\tau(a) = P_\tau B(a) |_{L^2[0, \tau]},$$

(2)

where $P_\tau$ is the projection

$$P_\tau : f(x) \mapsto g(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq \tau \\ 0 & \text{for } x > \tau. \end{cases}$$

(3)

For $a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, the Bessel operator $B(a)$ is bounded on $L^2(\mathbb{R}_+)$, and the truncated Bessel operator $B_\tau(a)$ is a trace class operator on $L^2[0, \tau]$. Hence the operator determinant $\det(I + B_\tau(a))$ is well defined for each $\tau$. For more information about trace class operators and related notions we refer to [9].

In this paper we compute the asymptotics of the determinants $\det(I + B_\tau(a))$ as $\tau \to \infty$ for certain continuous functions $a$. Given $a \in L^1(\mathbb{R}_+)$, we denote by $\hat{a}$ the cosine transform of the function $a$:

$$\hat{a}(x) = \frac{1}{\pi} \int_0^\infty \cos(x t) a(t) \, dt.$$  

(4)

A function $a$ defined on $[0, \infty)$ is said to be piecewise $C^2$ on $[0, \infty)$ if there exist $0 = t_0 < t_1 < \ldots < t_N < \infty$, $N \geq 0$, such that $a$ is two times continuously differentiable on each of the intervals $[0, t_1], \ldots, [t_{N-1}, t_N], [t_N, \infty)$, where the derivatives at $t_0, \ldots, t_N$ are considered as one-sided derivatives.

The main result of this paper is as follows.

**Theorem 1.1** Let $\nu > -1$ and suppose the function $b \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ satisfies the following conditions:

(i) $b$ is continuous and piecewise $C^2$ on $[0, \infty)$, and $\lim_{t \to \infty} b(t) = 0$;

(ii) $(1 + t)^{-1/2}b'(t) \in L^1(\mathbb{R}_+)$, $b''(t) \in L^1(\mathbb{R}_+)$.  

Denote by $\hat{b}$ the cosine transform of $b$ and put $a = e^b - 1$. Then

$$\det(I + B_\tau(a)) \sim \exp \left( \tau \hat{b}(0) - \frac{\nu}{2} b(0) + \frac{1}{2} \int_0^\infty x(\hat{b}(x))^2 \, dx \right) \text{ as } \tau \to \infty.$$  

(5)
The proof of this theorem will be given in the last section of this paper (Section 6). Note that the assumptions of $b$ ensure that all expressions appearing in formula (5) are well defined (see also the arguments in the proof).

A result of this kind has already been established by one of the authors in [1] under more restrictive assumptions. There the motivation for considering Bessel determinants was to describe certain densities that occur in random matrix theory (see also [5]). In comparison with [1], the proof of the asymptotic formula given here will be more transparent as we also employ a new algebraic method for the proof [7]. In particular, we remove the quite restrictive assumption that $\|a\|_{L^\infty(\mathbb{R}^+)} < 1$, which was imposed [1]. It is replaced by assumption that $1 + a$ is a function that possesses a logarithm, which is a natural requirement for Szegö-Akhiezer-Kac type formulas [6].

It is notable that for particular values of $\nu$ the Bessel operator $B(a)$ can be written in terms of Wiener-Hopf and Hankel operators: 

$$B(a) = W(a) + H(a) \quad \text{if } \nu = -1/2,$$

$$B(a) = W(a) - H(a) \quad \text{if } \nu = 1/2.$$ 

Here we think of $a$ as a function on $\mathbb{R}^+$ which is extended to an even function on $\mathbb{R}$ by stipulating $a(-x) = a(x)$. Hence in these cases, Theorem 1.1 describes the asymptotics of the determinants of Wiener-Hopf + Hankel operators

$$\text{det}(I + P_{\tau}W(a)P_{\tau} \pm P_{\tau}H(a)P_{\tau})$$

where the symbol $a$ is even.

We should also note here that our proof requires that we establish for general $\nu$ sufficient conditions on a function $a$ such that the Bessel operator $B(a)$ differs from the Wiener-Hopf operator $W(a)$ by a Hilbert-Schmidt operator. For $\nu = \pm 1/2$ this reduces to the condition that the Hankel operator $H(a)$ is Hilbert-Schmidt. However the result for general $\nu$ is of independent interest, much more difficult to obtain, and thus the main focus of the next section of the paper.

Finally, the discrete analogue of computing Toeplitz + Hankel determinants, has been recently investigated by the authors and results have been generalized to the case where the symbol is discontinuous [3] (see also [2, 4]).

2 Operator theoretic preliminaries

In this section, we establish all general operator theoretic facts as well particular results about Bessel operators and Wiener-Hopf operators that we will need later on.

First of all, let us mention that Bessel operators can be defined for arbitrary functions $a \in L^\infty(\mathbb{R}^+)$. For $\nu > -1$, let $H_{\nu}$ denote the Hankel transform

$$H_{\nu} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+), \quad f(x) \mapsto g(x) = \int_0^\infty \sqrt{tx} J_{\nu}(tx) f(t) \, dt. \quad (6)$$
It is well known that $H_\nu$ is selfadjoint and unitary on $L^2(\mathbb{R}_+)$, i.e., $H_\nu^* = H_\nu^{-1} = H_\nu \ [14]$. For $a \in L^\infty(\mathbb{R}_+)$ the Bessel operator $B(a) \in \mathcal{L}(L^2(\mathbb{R}_+))$ is defined by

$$B(a) = H_\nu M(a) H_\nu, \quad (7)$$

where $M(a)$ is the multiplication operator on $L^2(\mathbb{R}_+)$. If $a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, then this definition coincides with the one given in the introduction.

From formula (7) it follows immediately that

$$B(ab) = B(a)B(b) \quad \text{(8)}$$

for all $a, b \in L^\infty(\mathbb{R}_+)$. It is also clear that the Bessel operators are bounded and

$$\|B(a)\|_{\mathcal{L}(L^2(\mathbb{R}_+))} = \|a\|_{L^\infty(\mathbb{R}_+)}. \quad \text{(9)}$$

For $a \in L^\infty(\mathbb{R})$, the two-sided Wiener-Hopf operator $W^0(a) \in \mathcal{L}(L^2(\mathbb{R}))$ is defined by

$$W^0(a) = FM(a)F^{-1}, \quad (10)$$

where $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform and $M(a)$ stands here for the multiplication operator on $L^2(\mathbb{R})$. The usual Wiener-Hopf and Hankel operators acting on $L^2(\mathbb{R}^+_\nu)$ are defined by

$$W(a) = PW^0(a)P|_{L^2(\mathbb{R}_+^\nu)}, \quad H(a) = PW^0(a)JP|_{L^2(\mathbb{R}_+^\nu)}, \quad \text{(11)}$$

where $(Pf)(x) = \chi_{\mathbb{R}_+}(x)f(x)$ and $(Jf)(x) = f(-x)$. For $a \in L^\infty(\mathbb{R})$, these operators are bounded and

$$\|W(a)\|_{\mathcal{L}(L^2(\mathbb{R}_+^\nu))} = \|a\|_{L^\infty(\mathbb{R})}, \quad \|H(a)\|_{\mathcal{L}(L^2(\mathbb{R}_+^\nu))} \leq \|a\|_{L^\infty(\mathbb{R})}. \quad \text{(12)}$$

Moreover, for $a, b \in L^\infty(\mathbb{R})$ the well known identities

$$W(ab) = W(a)W(b) + H(a)H(\tilde{b}), \quad H(ab) = W(a)H(b) + W(a)H(\tilde{b}), \quad \text{(13)}$$

hold, where $\tilde{b}(x) = b(-x)$. These identities are a simple consequence of the facts that $W^0(ab) = W^0(a)W^0(b)$, $I = P + JPJ$ and $JW^0(b) = W^0(\tilde{b})J$.

If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then $W(a)$ and $H(a)$ are integral operators on $L^2(\mathbb{R}_+^\nu)$ with kernel $\hat{a}(x-y)$ and $\hat{a}(x+y)$, respectively, where

$$\hat{a}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt}a(t) \, dt \quad \text{(15)}$$

is the Fourier transform of $a$.  


We remark that if we extend \( a \in L^1(\mathbb{R}_+) \) to an even function \( a_0 \in L^1(\mathbb{R}) \) by stipulating \( a_0(x) = a(|x|) \), \( x \in \mathbb{R} \), then the cosine transform of \( a \) coincides with the Fourier transform of \( a_0 \). Therefore we will use the same notation for the cosine transform \( \mathbb{H} \) and the Fourier transform \( \mathbb{F} \).

In addition to the projection \( P_\tau \), we define the following operators acting on \( L^2(\mathbb{R}_+) \):

\[
W_\tau : f(x) \mapsto g(x) = \begin{cases} f(\tau - x) & \text{for } 0 \leq x \leq \tau \\ 0 & \text{for } x > \tau, \end{cases}
\]

(16)

\[
V_\tau : f(x) \mapsto g(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \tau \\ f(x-\tau) & \text{for } x > \tau, \end{cases}
\]

(17)

\[
V_{-\tau} : f(x) \mapsto g(x) = f(x+\tau),
\]

(18)

and \( Q_\tau = I - P_\tau \). It is readily verified, that \( P_\tau^2 = W_\tau^2 = P_\tau \), \( W_\tau P_\tau = W_\tau \), \( V_\tau V_{-\tau} = Q_\tau \). Moreover, the following identities hold:

\[
W_\tau W(a) W_\tau = P_\tau W(\tilde{a}) P_\tau, \quad P_\tau W(a) V_\tau = W_\tau H(\tilde{a}), \quad V_{-\tau} W(a) P_\tau = H(a) W_\tau.
\]

(19)

**Lemma 2.1**

(a) Let \( a \in L^\infty(\mathbb{R}) \) and \( K \) be a compact operator on \( L^2(\mathbb{R}_+) \) such that \( W(a) + K = 0 \). Then \( a = 0 \) and \( K = 0 \).

(b) Let \( a \in L^\infty(\mathbb{R}_+) \), \( K \) be a compact operator on \( L^2(\mathbb{R}_+) \) and \( \{C_\tau\}_{\tau \in (0, \infty)} \) be a sequence of bounded operators on \( L^2(\mathbb{R}_+) \) tending to zero in the operator norm as \( \tau \to \infty \) such that \( B_\tau(a) + W_\tau KW_\tau + C_\tau = 0 \). Then \( a = 0 \), \( K = 0 \) and \( C_\tau = 0 \) for all \( \tau \in (0, \infty) \).

Proof. (a): From the equation \( W(a) + K = 0 \) it follows that \( V_{-\tau} W(a) V_\tau + V_{-\tau} K V_\tau = 0 \).
Since \( V_{-\tau} W(a) V_\tau = W(a) \), we obtain that \( W(a) = -V_{-\tau} K V_\tau \). Observing that \( V_{-\tau} \to 0 \) strongly as \( \tau \to \infty \), we take the strong limit of the previous equality and it follows that \( W(a) = 0 \). Hence \( a = 0 \) and \( K = 0 \).

(b): Since \( W_\tau \to 0 \) weakly as \( \tau \to \infty \), the operators \( W_\tau KW_\tau \) converge strongly to zero. Taking the strong limit of \( B_\tau(a) + W_\tau KW_\tau + C_\tau = 0 \) we obtain \( B(a) = 0 \) because \( B_\tau(a) = P_\tau B(a) P_\tau \to B(a) \) strongly. Hence \( a = 0 \) and \( W_\tau KW_\tau + C_\tau = 0 \). Multiplying with \( W_\tau \) from both sides and taking again the strong limit, we conclude that \( 0 = P_\tau K P_\tau + W_\tau C_\tau W_\tau \to K \). Thus \( K = 0 \) and \( C_\tau = 0 \). \( \square \)

### 2.1 Hilbert-Schmidt and trace class conditions

In what follows we establish sufficient conditions for the operators \( H(a) \) and \( B(a) - W(a) \) to be Hilbert-Schmidt. Moreover, we state sufficient conditions such that \( B_\tau(a) \) is a trace class operator for each \( \tau \in (0, \infty) \).
Proposition 2.2 Let $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. If
\[
\int_0^\infty x|\hat{a}(x)|^2 \, dx < \infty,
\] (20)
where $\hat{a}$ is given by (13), then $H(a)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$. 

Proof. As pointed out above, $H(a)$ is an integral operator with kernel $\hat{a}(x + y)$. This operator is Hilbert-Schmidt if and only if the following integral is finite, which is the square of the Hilbert-Schmidt norm of $H(a)$:
\[
\int_0^\infty \int_0^\infty |\hat{a}(x + y)|^2 \, dx \, dy.
\]
This integral coincides with (20). $\square$

Proposition 2.3 If $a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, then $B_{\tau}(a)$ is a trace class operator on $L^2[0, \tau]$ for each $\tau \in (0, \infty)$.

Proof. Here we make use of Mercer’s Theorem [9, Ch.III], which reads as follows: If $m$ is a continuous function on $[0, \tau] \times [0, \tau]$ such that $m(s, t) = m(t, s)$ and
\[
\int_0^\tau \int_0^\tau m(t, s)f(t)f(s) \, dt \, ds \geq 0,
\] (21)
then the (positive semi-definite) operator given by
\[
L^2[0, \tau] \to L^2[0, \tau], \, f(t) \mapsto \int_0^\tau m(t, s)f(s) \, ds
\]
is a trace class operator. This theorem shows that if $a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ is a nonnegative function, then $B_{\tau}(a)$ is trace class on $L^2[0, \tau]$. Indeed, let $m(s, t) = K(s, t)$ be the kernel (1) and note that $K(s, t)$ is continuous since $\nu > -1$. Moreover, the above integral (21) equals
\[
\int_0^\infty a(t)(H_{\nu}f)(t)(H_{\nu}f)(t) \, dt,
\]
which is obviously nonnegative if $a$ is also. Since each function in $L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ can be represented as the linear combination of four nonnegative functions, the assertion follows for the general case. $\square$

The rest of this section is devoted to establishing sufficient conditions under which the difference $B(a) - W(a)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$, where $a(-x) = a(x)$. This result will be crucial for the considerations in the subsequent sections.

As mentioned in the introduction, it is motivated, to some extent, by the fact that $B(a) - W(a) = \pm H(a)$ for $\nu = \mp 1/2$. In this case the desired assertion is clear from Proposition 2.2.
Lemma 2.4 Let \( a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \). Then the operators \( W(a)P_1 \) and \( P_1W(a) \) are Hilbert-Schmidt operators on \( L^2(\mathbb{R}_+) \).

Proof. The operator \( W(a)P_1 \) is Hilbert-Schmidt if and only if
\[
\int_0^1 \int_0^\infty |\hat{a}(x-y)|^2 \, dx \, dy < \infty, \tag{22}
\]
where \( \hat{a}(x) \) is given by (15). Obviously,
\[
\int_0^1 \int_0^\infty |\hat{a}(x-y)|^2 \, dx \, dy \leq \int_{-\infty}^\infty |\hat{a}(x)|^2 \, dx = \int_{-\infty}^\infty |a(t)|^2 \, dt \tag{23}
\]
since \( \hat{a} \) is the Fourier transform of \( a \). This integral is finite because \( L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R}) \). Hence \( W(a)P_1 \) is Hilbert-Schmidt. It follows analogously that \( P_1W(a) \) is a Hilbert-Schmidt operator. \( \square \)

Let us at this point recall the following indefinite integrals for Bessel functions:
\[
\int t J^2_\nu(tx) \, dt = \frac{t^2}{2} \left( J^2_\nu(tx) - J_{\nu+1}(tx)J_{\nu-1}(tx) \right), \tag{24}
\]
\[
\int t J_\nu(tx)J_\nu(ty) \, dt = \frac{txJ_{\nu+1}(tx)J_{\nu}(ty) - tyJ_{\nu}(tx)J_{\nu+1}(ty)}{x^2 - y^2}. \tag{25}
\]
They can be proved in a straightforward manner by using the recursion formulas \( J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \) and \( J_\nu(x) - J_{\nu+1}(x) = 2J'_\nu(x) \). The asymptotic behavior of \( J_\nu(t) \) at zero and at infinity is as follows:
\[
J_\nu(t) = \left( \frac{t}{2} \right)^{\nu} \left( \frac{1}{\Gamma(\nu + 1)} + O(t^2) \right), \quad t \to 0, \tag{26}
\]
\[
J_\nu(t) = \sqrt{\frac{2}{\pi t}} \left( \cos(t - \alpha) - \sin(t - \alpha) \frac{\nu^2 - \frac{1}{4}}{t} + O(t^{-2}) \right), \quad t \to \infty, \tag{27}
\]
where \( \alpha = \frac{\pi}{2} \nu + \frac{\pi}{4} \).

Lemma 2.5 Let \( a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \). Then the operators \( B(a)P_1 \) and \( P_1B(a) \) are Hilbert-Schmidt operators on \( L^2(\mathbb{R}_+) \).

Proof. The operator \( B(a)P_1 \) is Hilbert-Schmidt if and only if
\[
\int_0^1 \int_0^\infty |K(x,y)|^2 \, dx \, dy < \infty, \tag{28}
\]
where \( K(x, y) \) is given by (1). Another interpretation of formula (1) is that for fixed \( y \) the function \( K(x, y) \) is the Hankel transform (3) of the function \( a_y(t) = \sqrt{yt}J_\nu(yt)a(t) \). In other words, \( K(x, y) = (H_\nu a_y)(x) \). Since \( H_\nu \) is an isometry on \( L^2(\mathbb{R}_+) \), it follows that

\[
\int_0^\infty |K(x, y)|^2 \, dx = \int_0^\infty |a_y(t)|^2 \, dt.
\]

Hence (28) is equal to

\[
\int_0^1 \int_0^\infty y J_\nu^2(yt) |a(t)|^2 \, dt \, dy = \int_0^\infty \left( \int_0^1 y J_\nu^2(yt) \, dy \right) |a(t)|^2 \, dt = \frac{1}{2} \int_0^\infty |a(t)|^2 \left( J_\nu^2(t) - J_{\nu+1}(t)J_{\nu-1}(t) \right) \, dt.
\]

Here we have used (24). The term involving the Bessel functions is continuous for \( t \in (0, \infty) \) and behaves at zero like \( O(t^{2\nu}) \), \( \nu > -1 \), and at infinity like \( O(t^{-1}) \). We can split the integral into an integral from zero to one and an integral from one to infinity. Using the fact that \( a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \), it follows that (28) is finite. Hence \( B(a)P_1 \) is a Hilbert-Schmidt operator.

It can be shown analogously that \( P_1 B(a) \) is a Hilbert-Schmidt operator.

\[\square\]

**Lemma 2.6** For each \( t \in [0, \infty) \) and \( x, y \in [1, \infty) \), the following integral exists:

\[
K_t(x, y) = \int_t^\infty \left( s \sqrt{xy} J_\nu(xs) J_\nu(ys) - \frac{2 \cos(xs - \alpha) \cos(ys - \alpha)}{\pi} \right) \, ds. \tag{29}
\]

In particular,

\[
K_0(x, y) = -\frac{\sin(2\alpha)}{\pi(x + y)}, \tag{30}
\]

and with a certain constant \( C_\nu \) depending only on \( \nu \) we have

\[
\left( \int_1^\infty \int_1^\infty |K_t(x, y)|^2 \, dx \, dy \right)^{1/2} \leq \frac{C_\nu}{t} \quad \text{for all } t \in (0, \infty). \tag{31}
\]

**Proof.** From (25), it follows that

\[
\int \left( s \sqrt{xy} J_\nu(xs) J_\nu(ys) - \frac{2 \cos(xs - \alpha) \cos(ys - \alpha)}{\pi} \right) \, ds
\]

\[
= \sqrt{xy} \frac{s x J_{\nu+1}(sx) J_\nu(sy) - s y J_\nu(xs) J_{\nu+1}(sy)}{x^2 - y^2} - \frac{\sin((x + y)s - 2\alpha)}{\pi(x + y)} \frac{\sin((x - y)s)}{\pi(x - y)}
\]

\[
= \sqrt{xy} \frac{s x J_{\nu+1}(sx) J_\nu(sy) + s y J_\nu(xs) J_{\nu+1}(sy)}{2(x + y)} - \frac{\sin((x + y)s - 2\alpha)}{\pi(x + y)} \frac{\sin((x - y)s)}{\pi(x - y)}
\]

\[
+ \sqrt{xy} \frac{s x J_{\nu+1}(sx) J_\nu(sy) - s y J_\nu(xs) J_{\nu+1}(sy)}{2(x - y)} - \frac{\sin((x + y)s - 2\alpha)}{\pi(x + y)} \frac{\sin((x - y)s)}{\pi(x - y)}.
\]
Using the leading term in the asymptotics (27), it is easily seen that the previous expression tends to zero as \( s \to \infty \) for fixed \( x, y \). Hence the integral \( K_t(x, y) \) exists for \( t \in (0, \infty) \).

Using the asymptotics (26) we obtain that the terms involving the Bessel functions tend to zero as \( s \to 0 \) since \( \nu > -1 \). Hence \( K_0(x, y) \) exists and equals (30).

To be precise, the just stated assertions hold for \( x \neq y \). However, if \( x = y \), we can proceed similarly by using (24):

\[
\int \left( sx J_2^2(\nu, s) - \frac{2 \cos^2(\nu s - \alpha)}{\pi} \right) ds = \frac{x s^2}{2} \left( J_{\nu+1}^2(s x) - J_{\nu-1}^2(s x) \right) - \frac{\sin(2 \nu s - 2 \alpha)}{2 \pi x} - \frac{s}{\pi}.
\]

Using the asymptotics (26) with the first and second term it follows that the previous expression tends to zero as \( s \to \infty \). Due to the asymptotics (27) the term containing the Bessel functions tends to zero as \( s \to 0 \). Hence \( K_t(x, x) \) exists for all \( t \in [0, \infty) \) and \( K_0(x, x) \) equals (30).

In order to prove (31) we divide the integral into three parts:

\[
K_t(x, y) = \int_t^\infty k_1(x, y; s) \, ds + \int_t^\infty k_2(x, y; s) \, ds + \int_t^\infty k_3(x, y; s) \, ds,
\]

where

\[
k_1(x, y; s) = \left( \sqrt{x s} J_{\nu}(x s) - \sqrt{\frac{2}{\pi}} \cos(x s - \alpha) \right) \left( \sqrt{y s} J_{\nu}(y s) - \sqrt{\frac{2}{\pi}} \cos(y s - \alpha) \right),
\]

\[
k_2(x, y; s) = \left( \sqrt{x s} J_{\nu}(x s) - \sqrt{\frac{2}{\pi}} \cos(x s - \alpha) \right) \sqrt{\frac{2}{\pi}} \cos(y s - \alpha),
\]

\[
k_3(x, y; s) = \sqrt{\frac{2}{\pi}} \cos(x s - \alpha) \left( \sqrt{y s} J_{\nu}(y s) - \sqrt{\frac{2}{\pi}} \cos(y s - \alpha) \right).
\]

Next we remark that from (26) and (27) it follows that for \( \nu > -1 \),

\[
\left| \sqrt{z} J_{\nu}(z) - \sqrt{\frac{2}{\pi}} \cos(z - \alpha) \right| \leq \frac{\text{const}}{z} \quad \text{for all } z \in (0, \infty),
\]

with a constant that depends only on \( \nu \). Hence \( |k_1(x, y; s)| \leq \text{const} (x y s^2)^{-1} \), whence

\[
\left| \int_t^\infty k_1(x, y; s) \, ds \right| \leq \frac{\text{const}}{x y t}
\]
follows. Partial integration of the second integral in (32) gives
\[
\int_t^\infty k_2(x, y; s) \, ds = \left[ \left( \sqrt{xs} J_\nu(xs) - \sqrt{2 \pi \cos(xs - \alpha)} \right) \frac{\sqrt{2 \sin(y s - \alpha)}}{y} \right]_t^\infty 
- \int_t^\infty x \left( \frac{1}{2 \sqrt{x s}} J_\nu(xs) + \sqrt{x s} J'_\nu(xs) + \sqrt{\frac{2}{\pi \sin(xs - \alpha)}} \left( \frac{2 \sin(y s - \alpha)}{y} \right) ds. \right.
\]

By (33) the first expression is bounded by a constant times \((xyt)^{-1}\). Next observe that for fixed \(\nu\) and arbitrary \(z \in (0, \infty)\) the following identity holds:
\[
\frac{1}{2 \sqrt{z}} J_\nu(z) + \sqrt{z} J'_\nu(z) = \frac{\nu + \frac{1}{2}}{\sqrt{z}} J_\nu(z) - \sqrt{z} J_{\nu+1}(z) 
= \sqrt{\frac{2}{\pi}} \left( \frac{(\nu + \frac{1}{2}) \cos(z - \alpha)}{z} - \sin(z - \alpha) - \left( (\nu + 1)^2 + \frac{1}{4} \right) \frac{\cos(z - \alpha)}{z} + O \left( \frac{1}{z^2} \right) \right). 
\]

Like (33) this follows from (24) and (27). We emphasize that estimate holds not just for \(z \to \infty\) but also for \(z \to 0\), thus uniformly for all \(z \in (0, \infty)\). Thus
\[
\int_t^\infty k_2(x, y; s) \, ds = O \left( \frac{1}{xyt} \right) + \int_t^\infty O \left( \frac{1}{ysz^2} \right) ds + A_\nu \int_t^\infty \frac{\cos(xs - \alpha) \sin(ys - \alpha)}{ys} ds 
\]
with \(A_\nu = \frac{2}{\pi}(\nu^2 + \nu + 1)\). A similar expression can be obtained for the integral involving \(k_3(x, y; s)\). It follows that
\[
\int_t^\infty (k_2(x, y; s) + k_3(x, y; s)) \, ds 
= O \left( \frac{1}{xyt} \right) + A_\nu \frac{xy}{xy} \int_t^\infty \frac{x \cos(xs - \alpha) \sin(ys - \alpha) + y \sin(xs - \alpha) \cos(ys - \alpha)}{s} ds.
\]
The last integral equals
\[
\int_t^\infty \frac{d}{ds} \left( \sin(xs - \alpha) \sin(ys - \alpha) \right) ds.
\]

Another partial integration shows that this equals \(O(t^{-1})\). Summarizing the previous results we can conclude that
\[
K_1(x, y) = O \left( \frac{1}{xyt} \right),
\]
from which the desired assertion (31) follows. \( \square \)

For \(a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)\), we introduce two operators acting on \(L^2[1, \infty)\). Firstly, let
\[
\mathcal{K}_a = Q_1(B(a) - W(a))Q_1|_{L^2[1, \infty)}, \quad (34)
\]
where we stipulate \( a(-x) = a(x), x < 0 \), for the symbol of \( W(a) \). Secondly define the Hankel operator \( H_a \) as the integral operator on \( L^2[1, \infty) \) with kernel

\[
H_a(x, y) = -\frac{\sin(2\alpha)a(0)}{\pi(x+y)} + \frac{1}{\pi} \int_0^\infty \cos((x+y)t - 2\alpha)a(t) \, dt
\]  

(35)

where \( \alpha = \frac{5}{2} \nu + \frac{\pi}{4} \).

Lemma 2.7  Let \( a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) and assume that

(i) \( a \) is continuous on \([0, \infty)\);

(ii) there exist a finite number of points \( 0 < t_1 < \ldots < t_N < \infty, N \geq 1, \) such that \( a \) is two times continuously differentiable on the interval \([0, t_1]\) and one times continuously differentiable on each of the intervals \([t_1, t_2], \ldots, [t_{N-1}, t_N], [t_N, \infty)\);

(iii) \( (1 + t)^{-1/2}a'(t) \in L^1(\mathbb{R}_+) \).

Then \( K_a - H_a \) is a Hilbert-Schmidt operator on \( L^2[1, \infty) \).

Proof. The kernel of the operator \( K_a = Q_1(B(a) - W(a))Q_1 \) is given by

\[
K_a(x, y) = \int_0^\infty \left( t \sqrt{xy} J_\nu(xt) J_\nu(yt) - \frac{\cos(xt - yt)}{\pi} \right) a(t) \, dt.
\]

This combined with (35) yields that \( K_a(x, y) - H_a(x, y) \) equals

\[
\frac{\sin(2\alpha)a(0)}{\pi(x+y)} + \int_0^\infty \left( t \sqrt{xy} J_\nu(xt) J_\nu(yt) - \frac{2\cos(xt - \alpha)\cos(yt - \alpha)}{\pi} \right) a(t) \, dt.
\]

In other words,

\[
K_a(x, y) - H_a(x, y) = \frac{\sin(2\alpha)a(0)}{\pi(x+y)} - \int_0^\infty a(t) \frac{d}{dt} K_t(x, y) \, dt,
\]

(36)

where \( K_t(x, y) \) is given by (29).

We first consider functions \( a(t) \) which are two times continuously differentiable on \([0, \infty)\), have compact support and satisfy \( a'(0) = 0 \). Notice that then \( t^{-1}a'(t) \in L^1(\mathbb{R}_+) \). From formula (30) and partial integration of (36) we obtain

\[
K_a(x, y) - H_a(x, y) = \int_0^\infty a'(t)K_t(x, y) \, dt.
\]

(37)

Notice that \( \lim_{t \to \infty} K_t(x, y) = 0 \) and \( a \in L^\infty(\mathbb{R}_+) \). Equation (34) says that the integral operators on \( L^2[1, \infty) \) with kernel \( tK_t(x, y) \) are Hilbert-Schmidt and their Hilbert-Schmidt norm is
uniformly bounded for all \( t \in (0, \infty) \). Since \( t^{-1}a'(t) \in L^1(\mathbb{R}_+) \), it follows that the operator \( K_a - H_a \) is Hilbert-Schmidt.

Next, we assume that the function \( a(t) \) satisfies the above conditions and in addition \( a(0) = 0 \). In this case, it is easily verified that the following integrals

\[
\int_0^\infty \frac{|a(t)|}{t} dt, \quad \int_0^\infty \frac{|a(t)|}{t^{3/2}(1 + t^{1/2})} dt, \quad \int_0^\infty \frac{|a'(t)|}{\sqrt{t}} dt, \quad \int_0^\infty \frac{d}{dt} \left( \frac{a(t)}{t} \right) dt
\]

are finite.

Since \( a(0) = 0 \), equation (36) becomes

\[
K_a(x, y) - H_a(x, y) = -\int_0^\infty a(t) \frac{d}{dt} K_t(x, y) dt.
\]

Also recall, using the notation from the proof of Lemma 2.6, that we can write the function \(-\frac{d}{dt} K_t(x, y)\) as the sum of three terms \( k_1(x, y; t) + k_2(x, y; t) + k_3(x, y; t)\). We will now write the sum of these three operators as another sum of operators each of which is trace class or Hilbert-Schmidt. To do this we recall that if an integral operator on \( L^2[1, \infty) \) has a kernel given by

\[
K(x, y) = \int_0^\infty h_1(x, t) h_2(y, t) a(t) dt,
\]

then the trace class norm of this operator is at most

\[
\int_0^\infty |a(t)| \left( \int_1^\infty |h_1(x, t)|^2 dx \right)^{1/2} \left( \int_1^\infty |h_2(y, t)|^2 dy \right)^{1/2} dt.
\]

Let us begin with the term \( k_1(x, y; t) \). From (26) and (27) along with the assumption \( \nu > -1 \), we obtain

\[
\int_0^\infty \left| \sqrt{z} J_\nu(z) - \sqrt{2 \pi} \cos(z - \alpha) \right|^2 dz < \infty.
\]

This immediately yields that the trace class norm of the operator given by

\[
\int_0^\infty k_1(x, y; t) a(t) dt
\]

is bounded by a constant times \( \int_0^\infty t^{-1} |a(t)| dt \).

The next term involving \( k_2 \) is a bit more complicated, but still follows the computation of Lemma 2.6. We write

\[
\int_0^\infty k_2(x, y; t) a(t) dt = \int_0^\infty \left( \sqrt{xt} J_\nu(xt) - \sqrt{2 \pi} \cos(xt - \alpha) \right) a(t) \sqrt{2 \pi} \cos(yt - \alpha) dt.
\]
We use integration by parts to write this as two terms. The first term is given by
\[
\left( \sqrt{xt} J_\nu(xt) - \sqrt{2 \pi} \cos(xt - \alpha) \right) a(t) \sqrt{\frac{2 \sin(yt - \alpha)}{y}} \bigg|_{t=0}^\infty.
\]
Since \( a(t) \) is bounded at infinity, \( a(t) = O(\sqrt{t}) \) as \( t \to 0 \) and \( \nu > -1 \), this expression is zero.

The next term yields, from differentiating the first factor, two terms one of which is
\[
\int_0^\infty \left( \sqrt{xt} J_\nu(xt) - \sqrt{2 \pi} \cos(xt - \alpha) \right) a'(t) \sqrt{\frac{2 \sin(yt - \alpha)}{y}} dt.
\]

The trace norm of this operator using the same argument as above is at most a constant times
\[
\int_0^\infty t^{-1/2} |a'(t)| dt
\]
and is thus finite. So we are left with one last term
\[
\int_0^\infty x \left( \frac{1}{2 \sqrt{xt}} J_\nu(xt) + \sqrt{xt} J_\nu'(xt) + \sqrt{2 \pi} \sin(xt - \alpha) \right) a(t) \sqrt{\frac{2 \sin(yt - \alpha)}{y}} dt.
\]

Following Lemma 2.6 and equation (27) we rewrite the Bessel functions to obtain an error term of the form
\[
\int_0^\infty x h(xt) a(t) \sqrt{\frac{2 \sin(yt - \alpha)}{y}} dt,
\]
where \( h(z) \) is \( O(z^p) \) with \( p = \min\{\nu - 1/2, -1\} \) for small \( z \) and \( O(1/z^2) \) for large \( z \). The norm of the term
\[
\frac{\sin(yt - \alpha)}{y}
\]
in \( L^2[1, \infty) \) is uniformly bounded for \( t \in [0, \infty) \) and the estimate of the norm of \( x h(xt) \) in \( L^2[1, \infty) \) is \( O(t^{-3/2}) \) as \( t \to 0 \) and \( O(t^{-2}) \) as \( t \to \infty \). Thus we have a trace norm estimate of a constant times
\[
\int_0^\infty \frac{|a(t)|}{t^{3/2}(1 + t^{1/2})} dt,
\]
which is finite. Hence (38) has bounded trace norm. Thus we have one remaining term
\[
A_\nu \int_0^\infty \frac{\cos(xt - \alpha) \sin(yt - \alpha)}{yt} a(t) dt.
\]

Now the computation for the last term \( k_3 \) is exactly the same as above with the variables reversed, so once again we combine these terms into one final integral,
\[
\frac{A_\nu}{xy} \int_0^\infty \left( x \cos(xt - \alpha) \sin(yt - \alpha) + y \sin(xt - \alpha) \cos(yt - \alpha) \right) \frac{a(t)}{t} dt.
\]
or
\[
\frac{A_\nu}{xy} \int_0^\infty \frac{d}{dt} \left( \sin(xt - \alpha) \sin(yt - \alpha) \right) \frac{a(t)}{t} \, dt.
\]
We integrate by parts one more time so that the above is
\[
\left[ \frac{A_\nu}{xy} \sin(xt - \alpha) \sin(yt - \alpha) \right]_{t=0}^\infty - \frac{A_\nu}{xy} \int_0^\infty \sin(xt - \alpha) \sin(yt - \alpha) \left( \frac{a(t)}{t} \right)' \, dt.
\]
From our assumptions on \(a(t)\), these expressions are easily seen to be \(O\left(\frac{1}{xy}\right)\) and are hence Hilbert-Schmidt.

To complete the proof for arbitrary functions \(a\) satisfying the assumptions of the lemma, let \(f\) be a two times continuously differentiable function on \(\mathbb{R}_+\) with compact support and \(f(0) = 1, f'(0) = 0\). We decompose \(a = a_1 + a_2\), where \(a_1(t) = a(0) f(t)\) and \(a_2(t) = a(t) - a(0) f(t)\). The function \(a_1\) fulfills the first assumed conditions, and the function \(a_2\) satisfies the second assumptions. This completes the proof.

Lemma 2.8 Let \(a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)\) and assume that \(a\) is continuous and piecewise \(C^2\) on \([0, \infty)\). Assume also that \(\lim_{x \to \infty} a(x) = 0\) and \(a'' \in L^1(\mathbb{R}_+)\). Then \(\mathcal{H}_a\) is Hilbert-Schmidt.

Proof. We integrate the following integral twice by parts where \(x \geq 2\):
\[
\frac{1}{\pi} \int_0^\infty \cos(xt - 2\alpha) a(t) \, dt = \left[ \frac{\sin(xt - 2\alpha)}{\pi x} a(t) \right]_0^\infty - \int_0^\infty \frac{\sin(xt - 2\alpha)}{\pi x} a'(t) \, dt
\]
\[
= \frac{\sin(2\alpha) a(0)}{\pi x} + \sum_{i=0}^{n-1} \left( \left[ \frac{\cos(xt - 2\alpha)}{\pi x^2} a'(t) \right]_t^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \frac{\cos(xt - 2\alpha)}{\pi x^2} a''(t) \, dt \right)
\]
where \(0 = t_0 < t_1 < \ldots t_{n-1} < t_n = \infty\) are the points where the derivatives are discontinuous. From this it follows that
\[
|\mathcal{H}_a(x, y)| \leq \frac{C}{(x+y)^2}.
\]
Hence this operator is Hilbert-Schmidt.

Proposition 2.9 Let \(a \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)\) and assume that
\[
(i) \ a \text{ is continuous and piecewise } C^2 \text{ on } [0, \infty), \text{ and } \lim_{t \to \infty} a(t) = 0;
\]
\[
(ii) \ (1+t)^{-1/2}a' \in L^1(\mathbb{R}_+) \text{ and } a'' \in L^1(\mathbb{R}_+).
\]
Then the operator $B(a) - W(a)$ is Hilbert-Schmidt on $L^2(\mathbb{R}_+)$. 

Proof. We write the operators $B(a)$ and $W(a)$ as follows:

$$B(a) = P_1B(a) + Q_1B(a)P_1 + Q_1B(a)Q_1$$
$$W(a) = P_1W(a) + Q_1W(a)P_1 + Q_1W(a)Q_1$$

Hence by Lemma 2.4 and Lemma 2.5, $B(a) - W(a) = \mathcal{K}_a + \text{Hilbert-Schmidt}$. 

Now it remains to apply Lemma 2.7 and Lemma 2.8. 

\[\Box\]

3 The algebraic approach

Here we follow, essentially, the method developed in [7]. It is useful to point out that the next sections use Banach algebra techniques to compute determinants. This is done without the knowledge that the operator $B(a) - W(a)$ is trace class, but merely Hilbert-Schmidt.

3.1 A Banach algebra of functions

Let $S$ stand for the set of all functions $a \in L^\infty(\mathbb{R}_+)$ such that the following properties are fulfilled:

(i) $H(a)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ where $a(-x) := a(x)$;

(ii) $B(a) - W(a)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$. 

We introduce a norm in $S$ by stipulating

$$\|a\|_S = \|a\|_{L^\infty(\mathbb{R}_+)} + \|H(a)\|_{C_2(L^2(\mathbb{R}_+))} + \|B(a) - W(a)\|_{C_2(L^2(\mathbb{R}_+))}. \quad (39)$$

**Proposition 3.1** $S$ is a Banach algebra.

Proof. The linearity and the completeness are easy to verify. It remains to show that $a, b \in S$ implies $ab \in S$ and that $\|a\|_S \|b\|_S \leq \text{const} \|ab\|_S$.

Indeed, let $a, b \in S$. Obviously $ab \in L^\infty(\mathbb{R}_+)$. Moreover,

$$H(ab) = H(a)W(b) + W(a)H(b),$$
$$B(ab) - W(ab) = B(a)B(b) - W(a)W(b) - H(a)H(b)$$
$$= B(a)\left(B(b) - W(b)\right) + \left(B(a) - W(b)\right)W(b) - H(a)H(b).$$
It follows that both $H(ab)$ and $B(ab) - W(ab)$ are Hilbert Schmidt. Moreover,
\[
\|H(ab)\|_2 \leq \|H(a)\|_2 \|b\|_\infty + \|a\|_\infty \|H(b)\|_2, \\
\|B(ab) - W(ab)\|_2 \leq \|a\|_\infty \|B(b) - W(b)\|_2 + \|B(a) - W(b)\|_2 \|b\|_\infty + \|H(a)\|_2 \|H(b)\|_2.
\]
From this the norm estimate is easy to obtain.

3.2 A Banach algebra of Wiener-Hopf operators

Let $\mathcal{B}$ be the set of all operators of the form
\[
A = W(a) + K
\]
where $a \in \mathcal{S}$ and $K$ is a trace class operator on $L^2(\mathbb{R}_+)$. We define in $\mathcal{B}$ a norm by
\[
\|A\|_B = \|a\|_S + \|K\|_{\mathcal{C}_1(L^2(\mathbb{R}_+))}.
\]
This definition is correct since $a$ and $K$ are uniquely determined by the operator $A$. In fact, this is a consequence of Lemma 2.1(a).

**Proposition 3.2** $\mathcal{B}$ is a Banach algebra.

Proof. The linearity and the completeness can be shown straightforwardly. As above, we prove that $A, B \in \mathcal{B}$ implies $AB \in \mathcal{B}$ and a corresponding norm estimate. In fact, let
\[
A = W(a) + K, \quad B = W(b) + L,
\]
where $a, b \in \mathcal{S}$ and $K, L \in \mathcal{C}_1(L^2(\mathbb{R}_+))$. Then
\[
AB = \left(W(a) + K\right)\left(W(b) + L\right)
= W(ab) - H(a)H(b) + KW(b) + W(a)L + KL.
\]
Noting that $ab \in \mathcal{S}$ and $H(a)H(b)$ is trace class, it follows that $AB \in \mathcal{B}$. Moreover,
\[
\|AB\|_B \leq \|ab\|_S + \|H(a)\|_2 \|H(b)\|_2 + \|K\|_1 \|b\|_\infty + \|a\|_\infty \|L\|_1 + \|K\|_1 \|L\|_1
\leq \text{const} \|a\|_S \|b\|_S + \|K\|_1 \|b\|_S + \|a\|_S \|L\|_1 + \|K\|_1 \|L\|_1.
\]
Thus $\|AB\|_B \leq \text{const} \|A\|_B \|B\|_B$. 

\[\square\]
3.3 A Banach algebra of sequences of Bessel operators

We are going to introduce a Banach algebra of sequences $\{A_\tau\}$, $\tau \in (0, \infty)$, which contains the sequences $\{B_\tau(a)\}$ of Bessel operators. First of all, let $\mathcal{N}$ be the set of all sequences $\{C_\tau\}$, $\tau \in (0, \infty)$, where

- $C_\tau$ is a trace class operator on $L^2(\mathbb{R}_+)$ for all $\tau \in (0, \infty)$;
- $\sup_{\tau \in (0, \infty)} \|C_\tau\|_{C_1(L^2(\mathbb{R}_+))} < \infty$;
- $\lim_{\tau \to \infty} \|C_\tau\|_{C_1(L^2(\mathbb{R}_+))} = 0$.

Now let $\mathcal{F}$ stand for the set of all sequences $\{A_\tau\}$, $\tau \in (0, \infty)$, which are of the form

$$A_\tau = B_\tau(a) + W_\tau K W_\tau + C_\tau$$  \hspace{1cm} (43)

where $a \in \mathcal{S}$, $K$ is a trace class operator on $L^2(\mathbb{R}_+)$ and $\{C_\tau\} \in \mathcal{N}$. For such sequences we introduce a norm by

$$\|\{A_\tau\}\|_\mathcal{F} = \|a\|_S + \|K\|_{C_1(L^2(\mathbb{R}_+))} + \sup_{\tau \in (0, \infty)} \|C_\tau\|_{C_1(L^2(\mathbb{R}_+))}. \hspace{1cm} (44)$$

This definition is correct because Lemma 2.1(b) implies that for a given sequence $\{A_\tau\}$ the function $a$ and the operators $K$ and $C_\tau$ are determined uniquely.

Moreover, for sequences $\{A_\tau^{(1)}\}, \{A_\tau^{(2)}\} \in \mathcal{F}$ and $\lambda^{(1)}, \lambda^{(2)} \in \mathbb{C}$ we define algebraic operations by

$$\lambda^{(1)} \{A_\tau^{(1)}\} + \lambda^{(2)} \{A_\tau^{(2)}\} = \{\lambda^{(1)} A_\tau^{(1)} + \lambda^{(2)} A_\tau^{(2)}\}, \quad \{A_\tau^{(1)}\} \{A_\tau^{(2)}\} = \{A_\tau^{(1)} A_\tau^{(2)}\}. \hspace{1cm} (45)$$

We will provide $\mathcal{F}$ with these algebraic operations and the above norm.

**Proposition 3.3** $\mathcal{F}$ is a Banach algebra, and $\mathcal{N}$ is a closed two-sided ideal of $\mathcal{F}$.

Proof. The only non-trivial statement to prove is that $\{A_\tau^{(1)}\} \in \mathcal{F}$ and $\{A_\tau^{(2)}\}$ implies that $\{A_\tau^{(1)}\} \{A_\tau^{(2)}\} \in \mathcal{F}$ and the corresponding norm estimate. Let

$$A_\tau^{(j)} = B_\tau(a_j) + W_\tau K_j W_\tau + C_\tau^{(j)}, \quad j = 1, 2,$$  \hspace{1cm} (46)

where $a_j \in \mathcal{S}$, $K_j$ is trace class and $\{C_\tau^{(j)}\} \in \mathcal{N}$. Using for brevity the notation $R(a) = B(a) - W(a)$, consider first

$$P_\tau B(a_1)Q_\tau B(a_2)P_\tau = P_\tau W(a_1)Q_\tau W(a_2)P_\tau + P_\tau W(a_1)Q_\tau R(a_2)P_\tau + P_\tau R(a_1)Q_\tau W(a_2)P_\tau + P_\tau R(a_1)Q_\tau R(a_2)P_\tau$$

$$= W_\tau H(a_1)H(a_2)W_\tau + W_\tau H(a_1)W_\tau R(a_2)P_\tau + P_\tau R(a_1)H(a_2)W_\tau + P_\tau R(a_1)Q_\tau R(a_2)P_\tau.$$
Since $H(a_j)$ and $R(a_j)$ are Hilbert-Schmidt operators and $V^*_r = V_{-r}$ and $Q_r$ converge strongly to zero on $L^2(\mathbb{R}^+)$ as $\tau \to \infty$, it follows that the last three terms belong to $\mathcal{N}$. Thus, it can be seen that $\{P_r B(a_1) Q_r B(a_2) P_r\} \in \mathcal{F}$, and the norm can be estimated by $\|a_1\|_S \|a_2\|_S$.

From this it follows that

$$B_r(a_1) B_r(a_2) = P_r B(a_1) B(a_2) P_r - P_r B(a_1) Q_r B(a_2) P_r - P_r B(a_1) P_r - \tau H(a_1) H(a_2) W_\tau + D_r^{(1)}$$

where $\{D_r^{(1)}\} \in \mathcal{N}$ with $\|\{D_r^{(1)}\}\|_F \leq \|a_1\|_S \|a_2\|_S$. In particular, that $\{B_r(a_1)\} \{B_r(a_2)\} \in \mathcal{F}$.

Furthermore, observe that

$$B_r(a_1) W_\tau K_2 W_\tau = P_r W(a_1) W_\tau K_2 W_\tau + P_r R(a_1) W_\tau K_2 W_\tau = \tau W_\tau W(a_1) P_r K_2 W_\tau + P_r R(a_1) W_\tau K_2 W_\tau = \tau W_\tau W(a_1) K_2 W_\tau - W_\tau W(a_1) Q_\tau K_2 W_\tau + P_r R(a_1) W_\tau K_2 W_\tau$$

where the last two terms belong to $\mathcal{N}$ since $Q_\tau \to 0$ strongly and $W_\tau \to 0$ weakly. Hence

$$B_r(a_1) W_\tau K_2 W_\tau = \tau W_\tau W(a_1) K_2 W_\tau + D_r^{(2)},$$

where $\{D_r^{(2)}\} \in \mathcal{N}$ with $\|\{D_r^{(2)}\}\|_F \leq \|a_1\|_S \|K_2\|_1$. By a similar argument we obtain that

$$\tau W_\tau K_1 W_\tau B_r(a_2) = \tau W_\tau K_2 W_\tau + D_r^{(3)},$$

where $\{D_r^{(3)}\} \in \mathcal{N}$ with $\|\{D_r^{(3)}\}\|_F \leq \|K_1\|_1 \|a_2\|_S$. Finally,

$$\tau W_\tau K_1 W_\tau K_2 W_\tau = \tau W_\tau K_2 W_\tau + D_r^{(4)},$$

where $\{D_r^{(4)}\} = \{-W_\tau K_1 Q_\tau K_2 W_\tau\} \in \mathcal{N}$ and $\|\{D_r^{(4)}\}\|_F \leq \|K_1\|_1 \|K_2\|_1$.

Summarizing the above we can conclude that

$$A_r^{(1)} A_r^{(2)} = B_r(a_1 a_2) + W_\tau K W_\tau + D_r,$$  \hspace{1cm} (47)

where

$$K = W(a_1) K_2 + K_1 W(a_2) + K_1 K_2 - H(a_1) H(a_2)$$  \hspace{1cm} (48)

and $\{D_r\} \in \mathcal{N}$ with $\|\{D_r\}\|_F \leq \text{const} \|\{A_r^{(1)}\}\|_F \|\{A_r^{(2)}\}\|_F$. Hence $\{A_r^{(1)}\} \{A_r^{(2)}\} \in \mathcal{F}$.

Noting that

$$\|K\|_1 \leq \|a_1\|_\infty \|K_2\|_1 + \|K_1\|_1 \|a_2\|_\infty + \|K_1\|_1 \|K_2\|_1 + \|H(a_1)\|_2 \|H(a_2)\|_2, \leq (\|a_1\|_S + \|K_1\|_1) (\|a_2\|_S + \|K_2\|_1),$$

we obtain that

$$\|\{A_r^{(1)}\} \{A_r^{(2)}\}\|_F = \|a_1 a_2\|_S + \|K\|_1 + \|D_r\|_F \leq \text{const} \|\{A_r^{(1)}\}\|_F \|\{A_r^{(2)}\}\|_F.$$  

Finally, the fact that $\mathcal{N}$ is an ideal of $\mathcal{F}$ follows from formulas (47) and (48) with $a_1 = 0$ and $K_1 = 0$ or $a_2 = 0$ and $K_2 = 0$, respectively. \hfill \Box
3.4 Banach algebra ideals and homomorphisms

In the following proposition we introduce a Banach algebra homomorphism that links the Banach algebras $\mathcal{F}$ and $\mathcal{B}$, which have been introduced in the previous sections. This result also shows that the quotient Banach algebra $\mathcal{F}/\mathcal{N}$ is isomorphic $\mathcal{B}$.

Proposition 3.4 The mapping $\Phi : \mathcal{F} \rightarrow \mathcal{B}$ defined by

$$\Phi : \{A_\tau\} = \{B_\tau(a) + W_\tau KW_\tau + C_\tau\} \mapsto A = W(a) + K,$$

is a surjective Banach algebra homomorphism with kernel $\mathcal{N}$.

Proof. The linearity and surjectivity are obvious. The continuity follows from the definition of the norms:

$$\|\{A_\tau\}\|_\mathcal{F} = \|a\|_S + \|K\|_1 + \|\{C_\tau\}\|_\mathcal{F} \quad \text{and} \quad \|A\|_\mathcal{B} = \|a\|_S + \|K\|_1.$$  \hspace{1cm} (49)

It is also clear that the kernel equals $\mathcal{N}$. The only point that needs an explanation is the multiplicativity. Assume that sequences $\{A^{(1)}_\tau\}$ and $\{A^{(2)}_\tau\}$ are given by (46). Then $\{A_\tau\} = \{A^{(1)}_\tau\}\{A^{(2)}_\tau\}$ is given by formulas (47) and (48). Hence

$$\Phi(\{A_\tau\}) = W(a_1a_2) + W(a_1)K_2 + K_1W(a_2) + K_1K_2 - H(a_1)H(a_2).$$

On the other hand,

$$\Phi(\{A^{(j)}_\tau\}) = W(a_j) + K_j.$$  \hspace{1cm} \Box

This implies that $\Phi(\{A_\tau\}) = \Phi(\{A^{(1)}_\tau\})\Phi(\{A^{(2)}_\tau\})$.

Let $\mathcal{J} \subseteq \mathcal{F}$ be the set

$$\mathcal{J} = \{\{W_\tau KW_\tau + C_\tau\} : K \in C_1(L^2(\mathbb{R}_+)) \text{ and } \{C_\tau\} \in \mathcal{N}\}.$$  \hspace{1cm} (50)

Formulas (46), (47) and (48) and the definition of the norm in $\mathcal{F}$ show that $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{F}$. One can form the quotient Banach algebra $\mathcal{F}/\mathcal{J}$. By

$$\pi : \mathcal{F} \rightarrow \mathcal{F}/\mathcal{J}, \{A_\tau\} \mapsto \{A_\tau\} + \mathcal{J},$$  \hspace{1cm} (51)

we denote the canonical homomorphism.

Furthermore, we introduce the linear and continuous mapping

$$\Lambda : \mathcal{S} \rightarrow \mathcal{F}, \ a \mapsto \{B_\tau(a)\}.$$  \hspace{1cm} (52)

Proposition 3.5 The mapping $\pi \circ \Lambda : \mathcal{S} \rightarrow \mathcal{F}/\mathcal{J}$ is a Banach algebra isomorphism.
Proof. It is obvious that $\pi \circ \Lambda$ is linear, continuous and surjective. In regard the multiplicativity we may refer to formulas (14) and (17) again, which can apply in order to conclude that

$$B_\tau(a_1)B_\tau(a_2) = B_\tau(a_1a_2) + K + C_\tau$$

with a certain $K \in C_1(L^2(\mathbb{R}_+))$ and $\{C_\tau\} \in \mathcal{N}$. Finally, the mapping is in fact an isomorphism, since for $\{A_\tau\} = \{B_\tau(a) + K + C_\tau\}$ the following norm equality holds,

$$\|\{A_\tau\} + \mathcal{J}\|_{\mathcal{F}/\mathcal{J}} = \|a\|_S,$$

which in turn follows immediately from the definition of the norm in $\mathcal{F}$.

\[\square\]

4 Asymptotic analysis

We start with an auxiliary result concerning the exponential of an element of $\mathcal{F}$. Recall that the exponential of an element of a Banach algebra is defined by the (absolutely convergent) sum

$$e^b = \sum_{n=0}^{\infty} \frac{b^n}{n!}. \quad (53)$$

For any Banach algebra homomorphism $\xi$ we have $\xi(e^b) = e^{\xi(b)}$.

**Lemma 4.1** Let $\{A_\tau\} \in \mathcal{F}$, then $\{e^{A_\tau}\} = e^{\{A_\tau\}} \in \mathcal{F}$.

Proof. Obviously, $e^{\{A_\tau\}} \in \mathcal{F}$. Hence $e^{\{A_\tau\}} = \{\hat{A}_\tau\}$, where $\{\hat{A}_\tau\}$ is a sequence of operator in $\mathcal{F}$. For each fixed $\tau_0 \in (0, \infty)$, the mapping $\xi_{\tau_0} : \{A_\tau\} \in \mathcal{F} \mapsto A_{\tau_0} \in \mathcal{L}(L^2(\mathbb{R}_+))$ is a Banach algebra homomorphism. We apply these homomorphisms to the equation $e^{\{A_\tau\}} = \{\hat{A}_\tau\}$ and obtain that $e^{A_{\tau_0}} = \hat{A}_{\tau_0}$. Hence $\{A_\tau\} = \{e^{A_\tau}\}$, and the assertion is proved. \[\square\]

The following result can be considered as a very crucial point in our argument.

**Proposition 4.2** Let $b \in \mathcal{S}$ and $\{A_\tau\} = \{B_\tau(e^b)e^{-B_\tau(b)}\}$. Then there exist an operator $K \in C_1(L^2(\mathbb{R}_+))$ and a sequence $\{C_\tau\} \in \mathcal{N}$ such that

$$\{A_\tau\} = \{P_\tau + W_\tau KW_\tau + C_\tau\}. \quad (54)$$

Moreover, the operator $K$ is determined by the identity

$$I + K = W(e^b)e^{-W(b)}. \quad (55)$$
Proof. First of all, the previous lemma implies that the sequence \( \{A_\tau\} \) is contained in \( \mathcal{F} \). Moreover, the following identity holds:

\[
\{A_\tau\} = \{B_\tau(e^b)e^{-B_\tau(b)}\} = \{B_\tau(e^b)\}e^{-\{B_\tau(b)\}} = \Lambda(e^b)e^{-\Lambda(b)}.
\]

Applying the canonical homomorphism \( \pi \) to this identity, we obtain

\[
\pi(\{A_\tau\}) = \pi(\Lambda(e^b)e^{-\Lambda(b)}) = \pi(\Lambda(e^b))e^{-\pi(\Lambda(b))} = ((\pi \circ \Lambda)(e^b))e^{-(\pi \circ \Lambda)(b)}.
\]

Using the fact that \( \pi \circ \Lambda \) is a homomorphism, it follows that

\[
\pi(\{A_\tau\}) = (\pi \circ \Lambda)(e^b e^{-b}) = \{P_\tau\} + \mathcal{J}.
\]

Hence (54) is proved. Finally we apply the homomorphism \( \Phi \) to this identity. From the definition of this homomorphism, identity (55) follows immediately.

An operator \( A \) is said to be of determinant class if \( A = I + K \) where \( K \) is a trace class operator. In this case, the operator determinant of \( A \) is well defined.

A conclusion of the previous proposition is that the operator \( W(e^b)e^{-W(b)} \) is of determinant class for each \( b \in \mathcal{S} \). Hence the following operator determinant

\[
E[b] = \det W(e^b)e^{-W(b)}
\]

is well defined for each \( b \in \mathcal{S} \).

**Lemma 4.3** Let \( A_\tau = P_\tau + W_\tau KW_\tau + C_\tau \), where \( K \in \mathcal{C}_1(L^2(\mathbb{R}_+)) \) and \( \{C_\tau\} \in \mathcal{N} \). Then

\[
\lim_{\tau \to \infty} \det A_\tau = \det(I + K).
\]

Proof. We remark that

\[
\det A_\tau = \det(I + W_\tau KW_\tau + C_\tau) = \det(I + P_\tau KP_\tau + W_\tau C_\tau W_\tau).
\]

Since \( Q^*_\tau = Q_\tau \to 0 \) strongly on \( L^2(\mathbb{R}_+) \), it follows that this equals \( \det(I + K + \tilde{C}_\tau) \) with \( \tilde{C}_\tau \) tending to zero in the trace class norm.

In order to prepare the theorem below, let us recall Proposition 2.3, which says that for each \( b \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) the operator \( B_\tau(b) \) is a trace class operator on \( L^2[0, \tau] \) for each \( \tau \in (0, \infty) \). Consequently, the operator trace of \( B_\tau(b) \) is well defined. Moreover, the following result holds.
Proposition 4.4  For each \( b \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) the operator \( B_\tau(e^b) \) is of determinant class for each \( \tau \in (0, \infty) \).

Proof. The set \( L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) is a Banach algebra without unit element. Upon adjoining the constant functions on \( \mathbb{R}_+ \) to \( L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \), one obtains a Banach algebra with unit element. From the series expansion of \( e^b \) it follows that \( e^b - 1 \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) whenever \( b \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \). Now Proposition 2.3 implies that \( B_\tau(e^b - 1) \) is of determinant class for each \( \tau \in (0, \infty) \).

These considerations show that all the expressions appearing in equation (59) below are well defined. This equation is the asymptotic result in which the justifications of this and the previous section culminate.

Theorem 4.5  Let \( b \in S \cap L^1(\mathbb{R}_+) \). Then

\[
\lim_{\tau \to \infty} \frac{\det B_\tau(e^b)}{e^{\text{trace } B_\tau(b)}} = E[b]
\]

where \( E[b] \) is given by (57).

Proof. We first use Proposition 4.2 in connection with Lemma 4.3 and obtain that

\[
\lim_{\tau \to \infty} \det \left( B_\tau(e^b)e^{-B_\tau(b)} \right) = \det(I + K)
\]

where \( I + K \) is given by (55). Thus \( \det(I + K) \) equals \( E[b] \). Now we observe that

\[
\det \left( B_\tau(e^b)e^{-B_\tau(b)} \right) = \left( \det B_\tau(e^b) \right) e^{-\text{trace } B_\tau(b)},
\]

which is a consequence of the just stated facts that \( B_\tau(b) \) is a trace class operator and \( B_\tau(e^b) \) is of determinant class. Notice that \( S \subseteq L^\infty(\mathbb{R}_+) \),

In order to prove the main result stated in the introduction, it is now clear that we have to find an explicit expression for the operator determinant \( E[b] \) and to determine the asymptotics of trace \( B_\tau(b) \) as \( \tau \to \infty \). This will be done next.

5  Further calculations

5.1  Trace determination

In what follows we evaluate the asymptotics of the trace of \( B_\tau(b) \) as \( \tau \to \infty \). The computation goes along the lines of [1, Sect. 3].
Proposition 5.1 Let $b \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, and assume that $b$ is continuous and piecewise $C^1$ on $[0, \infty)$ such that $(1 + t)^{-1}b' \in L^1(\mathbb{R}_+)$. Then

$$\text{trace } B_\tau(b) = \frac{\tau}{\pi} \int_0^\infty b(t) \, dt - \frac{\nu}{2} b(0) + o(1), \quad \tau \to \infty. \tag{60}$$

Proof. From formula (1) for the kernel of Proposition 5.1 

\begin{align*}
\text{trace } B_\tau(b) &= \int_0^\tau \int_0^\infty x \cdot J_\nu^2(xt)b(t) \, dtdx \\
&= \int_0^\tau b(t) t \int_0^\tau x \cdot J_\nu^2(xt) \, dxdt \\
&= \frac{\tau^2}{2} \int_0^\tau b(t) \left( J_\nu^2(\tau t) - J_{\nu+1}(\tau t)J_{\nu-1}(\tau t) \right) dt \\
&= \int_0^\tau b\left(\frac{t}{\tau}\right) \frac{t}{2} \left( J_\nu^2(t) - J_{\nu+1}(t)J_{\nu-1}(t) \right) dt.
\end{align*}

Here (24) and (26) has been used. Due to the asymptotics (27) we obtain

$$\frac{t}{2} \left( J_\nu^2(t) - J_{\nu+1}(t)J_{\nu-1}(t) \right) = \frac{1}{\pi} + \frac{\sin(2(t - \alpha))}{t} + O(t^{-2}), \quad \text{as } t \to \infty. \tag{61}$$

Hence the following integral exists:

$$F(\xi) = -\int_\xi^\infty \left( \frac{t}{2} \left( J_\nu^2(t) - J_{\nu+1}(t)J_{\nu-1}(t) \right) - \frac{1}{\pi} \right) dt.$$

From integration by parts it follows that

$$\text{trace } B_\tau(b) - \frac{1}{\pi} \int_0^\infty b\left(\frac{t}{\tau}\right) dt = \left[ b\left(\frac{t}{\tau}\right) F(t) \right]_{t=0}^\infty - \frac{1}{\tau} \int_0^\infty b'\left(\frac{t}{\tau}\right) F(t) \, dt.$$

Hence

$$\text{trace } B_\tau(b) = \frac{\tau}{\pi} \int_0^\infty b(t) \, dt - b(0) F(0) - \int_0^\infty b'(t) F(t \tau) \, dt.$$

The asymptotics (61) implies that $F(t) = O(t^{-1})$ as $t \to \infty$. Obviously, $F$ is continuous on $[0, \infty)$. Hence there exists a constant $C > 0$ such that $F(t) \leq C(1 + t)^{-1}$ for all $t \in [0, \infty)$. We estimate the last integral of the last equation as follows:

$$\left| \int_0^\infty b'(t) F(t \tau) \, dt \right| \leq \int_0^\infty |b'(t)| \frac{C}{1 + \tau t} \, dt \leq \int_0^{1/\tau} |b'(t)| \frac{C}{1 + \tau t} \, dt + \int_0^\infty |b'(t)| \frac{C}{1 + \tau t} \, dt \leq C \int_0^{1/\tau} |b'(t)| \, dt + C \left( \sup_{t \geq 1/\tau} \frac{1 + t}{1 + \tau t} \right) \int_0^\infty |b'(t)| \frac{1}{1 + t} \, dt.$$
This expression tends to zero as $\tau \to \infty$. Notice that

$$- F(0) = \int_0^\infty \left( \frac{t}{2} \left( J^2_\nu(t) - J_{\nu+1}(t) J_{\nu-1}(t) \right) - \frac{1}{\pi} \right) dt$$

$$= \int_0^\infty \left( \frac{t}{2} \left( J^2_\nu(t) + J^2_{\nu+1}(t) \right) - \frac{1}{\pi} \right) dt - \nu \int_0^\infty J_{\nu+1}(t) J_\nu(t) dt = - \frac{\nu}{2}.$$ 

Here we have used $J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{t} J_\nu$. The first integral equals zero, which can be seen from a direct calculation based on (24) and the asymptotics (26) and (27). The last integral equals $\frac{1}{2}$, see, for instance, [10, Sect. 6.512-3]. \hfill \Box

5.2 Operator determinant calculation

In nearly the last step of our program we calculate the operator determinant (57) for a certain class of sufficiently smooth functions $b$. We remark that (only) in this section we are dealing with functions $b$ defined on $\mathbb{R}$ that are not necessarily even.

Let $L^2_{1/2}(\mathbb{R})$ stand for the Banach space of all Lebesgue measurable function $f$ defined on $\mathbb{R}$ for which

$$\|f\|_{L^2_{1/2}(\mathbb{R})} = \left( \int_{-\infty}^{\infty} (1 + |x|)|f(x)|^2 \, dx \right)^{1/2} < \infty. \quad (62)$$

By $C^\infty_0(\mathbb{R})$ we denote the Banach algebra of all continuous functions $f$ on $\mathbb{R}$ for which $f(x) \to 0$ as $x \to \pm \infty$, where the multiplication is defined pointwise and the norm is the supremum norm on $\mathbb{R}$.

Let $\mathcal{W}_0$ stand for the set of all functions $a$ defined on $\mathbb{R}$ which are the inverse Fourier transform of a function $\hat{a} \in L^1(\mathbb{R}) \cap L^2_{1/2}(\mathbb{R})$, i.e.,

$$a(x) = \int_{-\infty}^{\infty} e^{ixt} \hat{a}(t) \, dt \quad (63)$$

(see also (14)). The norm in $\mathcal{W}_0$ is defined as

$$\|a\|_{\mathcal{W}_0} := \|\hat{a}\|_{L^1(\mathbb{R})} + \|\hat{a}\|_{L^2_{1/2}(\mathbb{R})}. \quad (64)$$

A routine computation shows that the set $L^1(\mathbb{R}) \cap L^2_{1/2}(\mathbb{R})$ is a Banach algebra with the multiplication defined as the convolution on $\mathbb{R}$, i.e.,

$$(\hat{a} * \hat{b})(x) = \int_{-\infty}^{\infty} \hat{a}(x-y) \hat{b}(y) \, dy. \quad (65)$$

Since $\mathbf{F}^{-1}(\hat{a} * \hat{b}) = (\mathbf{F}^{-1}\hat{a}) \cdot (\mathbf{F}^{-1}\hat{b})$, the set $\mathcal{W}_0$ is also a Banach algebra with the multiplication defined pointwise. Moreover, $\mathcal{W}_0$ is continuously embedded into $C^\infty_0(\mathbb{R})$. 24
Let \( W_0^+ \) and \( W_0^- \), respectively, be the sets of all functions \( a \in W_0 \), where \( a = F^{-1} \hat{a} \), for which \( \hat{a} \) is supported on \([0, \infty)\) and \((-\infty, 0]\), respectively. It is easily seen that \( W_0^+ \) and \( W_0^- \) are Banach subalgebras of \( W_0 \), for which the decomposition \( W_0 = W_0^+ + W_0^- \) holds.

The Banach algebras \( W_0 \) and \( W_0^\pm \) do not contain a unit element. The corresponding Banach algebras with unit element are obtained by adjoining the constant functions on \( \mathbb{R} \), i.e., \( W = \mathbb{C} \oplus W_0 \) and \( W^\pm = \mathbb{C} \oplus W_0^\pm \).

Due to the fact that \( W \subset L^\infty(\mathbb{R}) \), for each \( a \in W \) the two-sided Wiener-Hopf operator \( W^0(a) \) is well defined by (11).

**Proposition 5.2** Let \( a \in W \), where \( a = \alpha + F^{-1} \hat{a} \) with \( \alpha \in \mathbb{C} \) and \( \hat{a} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Then \( W^0(a) = \alpha I + K \), where \( K \) is the integral operator on \( L^2(\mathbb{R}) \) with the kernel \( k(x, y) = \hat{a}(x-y) \).

Proof. Obviously, \( W^0(a) = \alpha I + W^0(F^{-1} \hat{a}) \). Let \( f \in L^1(\mathbb{R}) \) and denote by \( \hat{a} \ast f \) the convolution of \( \hat{a} \) and \( f \). Then \( F^{-1}(\hat{a} \ast f) = (F^{-1} \hat{a}) \cdot (F^{-1} f) \). Hence \( \hat{a} \ast f = W^0(F^{-1} \hat{a})f \). By an approximation argument the same equality holds also for all functions \( f \in L^2(\mathbb{R}) \). This proves the assertion. \( \square \)

For each \( a \in W \) the Wiener-Hopf operator \( W(a) \) and the Hankel operator \( H(a) \) are well defined in the sense of (11). Moreover, the previous proposition implies that under the same hypotheses, \( W(a) \) equals \( \alpha I \) plus an integral operator with the kernel \( \hat{a}(x-y) \) on \( L^2(\mathbb{R}_+) \) and \( H(a) \) is an integral operator with the kernel \( \hat{a}(x+y) \) on \( L^2(\mathbb{R}_+) \).

The following result is taken from [13], where a formula for the evaluation of a certain type of operator determinant has been established. This formula represents a generalization of Pincus’ formula. The classical Pincus’ formula has been obtained in [13, 11] and employed in [15] in the calculation of the asymptotics of Toeplitz determinants.

**Proposition 5.3** Let \( H \) be a Hilbert space and let \( A, B \in \mathcal{L}(H) \) be such that the commutator \( AB - BA \) is a trace class operator. Then \( e^Ae^Be^{-A-B} \) is of determinant class and

\[
\det e^Ae^Be^{-A-B} = \exp \left( \frac{1}{2} \text{trace} (AB - BA) \right). \tag{66}
\]

Next we establish an explicit formula for the operator determinant \( \det W(e^b)e^{-W(b)} \) for functions \( b \in W \).

**Theorem 5.4** Let \( b \in W \), where \( b = \beta + F^{-1} \hat{b} \) with \( \beta \in \mathbb{C} \) and \( \hat{b} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Then \( W(e^b)e^{-W(b)} \) is of determinant class and

\[
\det W(e^b)e^{-W(b)} = \exp \left( \frac{1}{2} \text{trace} H(b)H(\hat{b}) \right) = \exp \left( \frac{1}{2} \int_0^\infty x \hat{b}(x)\hat{b}(-x) \, dx \right). \tag{67}
\]
Proof. According to the remarks made after Proposition 5.2, for \( a \in \mathcal{W} \) the Hankel operators \( H(a) \) and \( H(\hat{a}) \) (where \( \hat{a}(t) = a(t^{-1}) \)) are integral operators with the kernels \( \hat{a}(x + y) \) and \( \hat{a}(-x - y) \), respectively. Since \( \hat{a} \in L^2_{1/2}(\mathbb{R}) \), these Hankel operators are Hilbert-Schmidt operators on \( L^2(\mathbb{R}_+) \) and the estimate
\[
\|H(a)\|_{C_2(L^2(\mathbb{R}_+))} \leq \|a\|_W \tag{68}
\]
holds (see also the argument given in the proof of Proposition 2.2).

Moreover, from formula (13) it follows that \( W(ac) = W(a)W(c) \) whenever \( a \in \mathcal{W}_- \) or \( c \in \mathcal{W}_+ \). In fact, \( H(a) = H(\hat{c}) = 0 \). From this it is not difficult to conclude that
\[
W(e^{b_+}) = e^{W(b_+)} \quad \text{and} \quad W(e^{b_-}) = e^{W(b_-)} \tag{70}
\]
for all \( b_+ \in \mathcal{W}_+ \) and all \( b_- \in \mathcal{W}_- \).

Each function \( b \in \mathcal{W} \) can be decomposed into \( b = b_+ + b_- \) with \( b_\pm \in \mathcal{W}_\pm \). We define \( A = W(b_-) \) and \( B = W(b_+) \). Using formulas (70) and (13) it follows that
\[
e^A e^B = e^{W(b_-)} e^{W(b_+)} = W(e^{b_-}) W(e^{b_+}) = W(e^{b_-} e^{b_+}) = W(e^b).
\]
Obviously, \( e^{-A-B} = e^{-W(b_-)-W(b_+)} = e^{-W(b)} \). Again from formula (13) it follows that \( AB - BA = W(b_-)W(b_+) - W(b_+)W(b_-) = W(b_- b_+) - W(b_+ b_-) = H(b_+)H(b_-) - H(b)H(b) \).

Since both these Hankel operators are Hilbert-Schmidt (see (68)), \( AB - BA \) is a trace class operator. Thus we can apply Proposition 5.3 and obtain the first part of the desired formula.

The second part of the formula follows from the fact that the kernels of the operators \( H(b) \) and \( H(\hat{b}) \) are given by \( \hat{b}(x + y) \) and \( \hat{b}(-x - y) \), respectively. \( Q.E.D. \)

For our purposes it is not enough to have formula (57) proved for functions \( b \in \mathcal{W} \). In what follows we establish this formula also for a slightly different class of functions. The proof is based on an approximation argument.

Let \( \mathcal{S}_0 \) stand for the set of all functions \( a \in L^\infty(\mathbb{R}) \) such that both \( H(a) \) and \( H(\hat{a}) \) are Hilbert-Schmidt operators on \( L^2(\mathbb{R}_+) \). We introduce a norm in \( \mathcal{S}_0 \) as follows:
\[
\|a\|_{\mathcal{S}_0} = \|a\|_{L^\infty(\mathbb{R})} + \|H(a)\|_{C_2(L^2(\mathbb{R}_+))} + \|H(\hat{a})\|_{C_2(L^2(\mathbb{R}_+))}.
\tag{71}
\]

**Proposition 5.5** \( \mathcal{S}_0 \) is a Banach algebra.

Proof. We restrict our considerations to showing that \( a, b \in \mathcal{S}_0 \) implies that \( ab \in \mathcal{S}_0 \) and that \( \|ab\|_{\mathcal{S}_0} \leq \text{const} \|a\|_{\mathcal{S}_0} \|b\|_{\mathcal{S}_0} \).

\[26\]
In fact, if \( a, b \in S_0 \), then it follows from formula (14) that also \( ab \in S_0 \). Moreover,
\[
\|H(ab)\|_2 = \|a\|_\infty \|H(b)\|_2 + \|H(a)\|_2 \|b\|_\infty,
\]
\[
\|H(\tilde{a}b)\|_2 = \|a\|_\infty \|H(\tilde{b})\|_2 + \|H(\tilde{a})\|_2 \|b\|_\infty.
\]
This implies the desired norm estimate. \( \square \)

Let \( B_0 \) stand for the set of all operators of the form
\[
A = W(a) + K
\]
where \( a \in S_0 \) and \( K \) is a trace class operator on \( L^2(\mathbb{R}_+) \). We define in \( B_0 \) a norm by
\[
\|A\|_{B_0} = \|a\|_{S_0} + \|K\|_{C_1(L^2(\mathbb{R}_+))}.
\]
By Lemma 2.1(a), this definition is correct.

In the same way as in the proof of Proposition 3.2 one can show that \( B_0 \) is a Banach algebra. This Banach algebra possesses the ideal \( C_1(L^2(\mathbb{R}_+)) \), and the canonical homomorphism is denoted by \( \pi_0 : B_0 \to B_0/C_1(L^2(\mathbb{R}_+)) \). The mapping \( \Lambda_0 : a \in S_0 \mapsto W(a) \in B_0 \) is a continuous linear mapping. Moreover, the mapping \( \pi_0 \circ \Lambda_0 : S_0 \to B_0/C_1(L^2(\mathbb{R}_+)) \) is a Banach algebra isomorphism.

**Proposition 5.6** Let \( b \in S_0 \). Then \( H(b)H(\tilde{b}) \) is a trace class operator and \( W(e^b)e^{-W(b)} \) is of determinant class. Moreover, the mapping
\[
b \in S_0 \mapsto W(e^b)e^{-W(b)} \in B_0
\]
is continuous.

Proof. The first assertion is obvious since both Hankel operators are Hilbert-Schmidt.

Now let \( A = W(e^b)e^{-W(b)} \). Observe that \( A = \Lambda_0(e^b)e^{-\Lambda_0(b)} \), and thus \( A \) belongs to \( B_0 \). Taking into account that both \( \pi_0 \) and \( \pi_0 \circ \Lambda_0 \) are homomorphisms it follows that
\[
\pi_0(A) = ((\pi_0 \circ \Lambda_0)(e^b)) e^{-(\pi_0 \circ \Lambda_0)(b)} = (\pi_0 \circ \Lambda_0)(e^b e^{-b}) = I + C_1(L^2(\mathbb{R}_+)).
\]
Thus \( A - I \) is a trace class operator.

The last assertion follows essentially from the fact that \( \Lambda_0 \) is continuous and that the exponential function (in \( S_0 \) and \( B_0 \)) is continuous. \( \square \)

**Theorem 5.7** Let \( b \in S_0 \) and assume in addition that \( b \in C_0^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \). Then
\[
\det W(e^b)e^{-W(b)} = \exp \left( \frac{1}{2} \text{trace } H(b)H(\tilde{b}) \right).
\]

27
Proof. Let \( \hat{b} \) stand for the Fourier transform of \( b \) (see (13)). Since \( b \in L^1(\mathbb{R}) \), the Fourier transform \( \hat{b} \) is a function in \( C_0^\infty(\mathbb{R}) \subset L^\infty(\mathbb{R}) \).

For \( \lambda \in (0, \infty) \), let \( \rho_\lambda(x) = \lambda \pi^{-1}(1 + \lambda^2 x^2)^{-1} \) stand for the Poisson kernel, and let \( b_\lambda \) stand for the convolution of \( \rho_\lambda \) with \( b \), i.e.,

\[
    b_\lambda(x) = \int_{-\infty}^{\infty} \frac{\lambda b(t)}{\pi(1 + \lambda^2(x-t)^2)} \, dt.
\]

(76)

Since \( \rho_\lambda \) and \( b \) belong to \( L^1(\mathbb{R}) \), also \( b_\lambda \in L^1(\mathbb{R}) \). Remark that the Fourier transform of \( \rho_\lambda \) is equal to \( (F\rho_\lambda)(x) = (2\pi)^{-1}e^{-|x|/\lambda} \). It follows that the Fourier transform \( \hat{b}_\lambda \) of \( b_\lambda \) is given by

\[
    \hat{b}_\lambda = F(\rho_\lambda * b) = 2\pi(F\rho_\lambda) \cdot (Fb) = e^{-|x|/\lambda}\hat{b}(x).
\]

(77)

We see immediately that \( \hat{b}_\lambda \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Thus \( b_\lambda = F^{-1}\hat{b}_\lambda \), whence it follows that \( b_\lambda \in W_0 \). We can apply Theorem 5.4 and obtain that

\[
    \det W(e^{b_\lambda})e^{-W(b_\lambda)} = \exp\left(\frac{1}{2}\text{trace } H(b_\lambda)H(\hat{b}_\lambda)\right)
\]

(78)

holds for all \( \lambda \in (0, \infty) \).

Another consequence of (77) is that

\[
    H(b_\lambda) = M_\lambda H(b)M_\lambda \quad \text{and} \quad H(\hat{b}_\lambda) = M_\lambda H(\hat{b})M_\lambda,
\]

where \( M_\lambda \) stands for the multiplication operator on \( L^2(\mathbb{R}_+) \) with the function \( e^{-x/\lambda} \), \( x \in \mathbb{R}_+ \). Here we have to observe that the above Hankel operators are integral operators whose kernels are given in terms of by \( \hat{b}_\lambda \) and \( b_\lambda \), respectively.

Since \( (M_\lambda)^* = M_\lambda \) converges strongly on \( L^2(\mathbb{R}_+) \) to the identity operator as \( \lambda \to \infty \), it follows that \( H(b_\lambda) \to H(b) \) and \( H(\hat{b}_\lambda) \to H(\hat{b}) \) in the Hilbert-Schmidt norm as \( \lambda \to \infty \). Hence

\[
    H(b_\lambda)H(\hat{b}_\lambda) \to H(b)H(\hat{b})
\]

in the trace class norm as \( \lambda \to \infty \).

From (73) and the assumption that \( b \in C_0^\infty(\mathbb{R}) \) we obtain that \( b_\lambda \to b \) in the norm of \( L^\infty(\mathbb{R}) \). Hence, together with what has just been said, \( b_\lambda \to b \) in the norm of \( S_0 \). Now we employ the last statement of Proposition 5.4 and obtain that

\[
    W(e^{b_\lambda})e^{-W(b_\lambda)} \to W(e^b)e^{-W(b)}
\]

in the norm of \( \mathcal{B}_0 \). Since both \( W(e^{b_\lambda})e^{-W(b_\lambda)} \) and \( W(e^b)e^{-W(b)} \) are operators of the form identity plus trace class operator, it follows from the particular definition of the Banach algebra \( \mathcal{B}_0 \) that

\[
    W(e^{b_\lambda})e^{-W(b_\lambda)} - I \to W(e^b)e^{-W(b)} - I
\]

in the trace class norm as \( \lambda \to \infty \). By passing to the limit \( \lambda \to \infty \) in (78), the desired identity follows. \( \square \)

The following corollary is an immediate consequence of the previous theorem.
Corollary 5.8 Let \( b \in C_0^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \) be such that \( \hat{b} = Fb \in L^2_{1/2}(\mathbb{R}) \). Then

\[
\det W(e^b)e^{-W(b)} = \exp \left( \frac{1}{2} \int_0^\infty x b(x)\hat{b}(-x) \, dx \right).
\] (79)

Proof. From Proposition 2.3 it follows that the Hankel operators \( H(b) \) and \( H(\hat{b}) \) are integral operators with the kernels \( \hat{b}(x + y) \) and \( \hat{b}(-x - y) \), respectively, and that both integral operators are Hilbert-Schmidt. Hence \( b \in S_0 \) and trace \( H(b)H(\hat{b}) \) can be expressed as the above integral.

\[ \square \]

Notice that \( \mathcal{W} \subseteq S_0 \). However, the example of \( \hat{a}(x) = \text{sign}(x)e^{-|x|} \), \( a(x) = 2ix/(1 + x^2) \) shows that \( a \in \mathcal{W}_0 \) but \( a \notin L^1(\mathbb{R}) \). Thus Corollary 5.8 does not cover all of Theorem 5.4.

6 Proof of the main result

Proof of Theorem 1.1. Suppose that the function \( b \in L^\infty(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \) satisfies the assumptions (i), (ii) of Theorem 1.1. We are going to identify \( b \) with its even continuation, \( \hat{b}(x) = b(-x) \).

Let \( \hat{b} \) stand for the Fourier transform of the even function \( b \), i.e., the cosine transform (see (4)), and integrate twice by parts:

\[
\hat{b}(x) = \frac{1}{\pi} \int_0^\infty \cos(xt)b(t) \, dt = \left[ \frac{\sin(xt)}{\pi x}b(t) \right]_0^\infty - \int_0^\infty \frac{\sin(xt)}{\pi x}b'(t) \, dt
\]

\[
= \sum_{i=0}^{n-1} \left( \left[ \frac{\cos(xt)}{\pi x^2}b'(t) \right]_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \frac{\cos(xt)}{\pi x^2}b''(t) \, dt \right),
\]

where \( 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = \infty \) are the points where the derivatives are discontinuous. From this it follows that \( \hat{b}(x) = O(1/x^2) \) as \( x \to \pm \infty \). Moreover, since \( b \in L^1(\mathbb{R}^+) \), we have \( \hat{b} \in L^\infty(\mathbb{R}) \). Hence we can conclude that \( \hat{b} \in L^2_{1/2}(\mathbb{R}) \).

Firstly, this means that we are in a position to apply Corollary 5.8.

Secondly, the Hankel operator \( H(b) \) is Hilbert-Schmidt (see Proposition 2.2). Moreover, the assumptions (i) and (ii) also imply that \( B(b) - W(b) \) is a Hilbert-Schmidt operator (see Proposition 2.9). Thus from the definition of \( S \) in Section 3.1 it follows that \( b \in S \). This allows us to apply Theorem 4.5.

Finally, the assumptions (i) and (ii) also imply that the assumptions of Proposition 5.1 are fulfilled.

Combining Theorem 4.5 with Proposition 5.1 and Corollary 5.8 yields the desired claim of Theorem 1.1. \[ \square \]
References

[1] E.L. Basor. – Distribution functions for random variables for ensembles of positive hermitian matrices, Comm. Math. Phys. 188 (1997), 327–350.

[2] E.L. Basor, T. Ehrhardt. – On a class of Toeplitz + Hankel operators, New York J. Math. 5 (1999), 1–16.

[3] E.L. Basor, T. Ehrhardt. – Asymptotic formulas for determinants of a sum of finite Toeplitz and Hankel matrices, Math. Nachr. 228 (2001), 5–45.

[4] E.L. Basor, T. Ehrhardt. – Asymptotic formulas for the determinants of symmetric Toeplitz + Hankel matrices, to appear in: Operator Theory: Advances and Applications.

[5] E.L. Basor, C.A. Tracy. – Variance Calculations and the Bessel kernel, J. Statistical Physics 73, no. 2 (1993), 415–421.

[6] A. Böttcher, B. Silbermann. – Analysis of Toeplitz Operators, Springer, Berlin, 1990.

[7] T. Ehrhardt. – A new algebraic approach to the Szegö-Widom Limit Theorem, submitted.

[8] T. Ehrhardt. – A generalization of Pincus’ formula and Toeplitz determinant identities, to appear in: Archiv der Mathematik.

[9] I. Gohberg, I.M. Krein. – Introduction to the theory of linear nonselfadjoint operators in Hilbert space, Amer. Math. Soc. Transl. Math. Monographs 18, Providence, R.I., 1969.

[10] I.S. Gradshteyn, I.M. Ryzhik. – Table of Integrals, Series, and Products, 5th edition, Academic Press, San Diego, 1994.

[11] J.W. Helton, R.E. Howe. – Integral operators: traces, index, and homology, In: Proceedings of the Conference in Operator Theory, Lecture Notes in Math. 345, Springer, Berlin, 1973, 141–209.

[12] M.L. Metha. – Random Matrices, Academic Press, San Diego, 1991.

[13] J.D. Pincus. – On the trace of commutators in the algebra of operators generated by an operator with trace class self-commutator, unpublished, 1972.

[14] E.C. Titchmarsh. – Introduction to the theory of Fourier integrals, Oxford, 1937.

[15] H. Widom. – Asymptotic behavior of block Toeplitz matrices and determinants. II, Adv. Math. 21 (1976), 1–29.