ON BLOCH–KATO SELMER GROUPS AND IWASAWA THEORY OF $p$-ADIC GALOIS REPRESENTATIONS

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Abstract. A result due to R. Greenberg gives a relation between the cardinality of Selmer groups of elliptic curves over number fields and the characteristic power series of Pontryagin duals of Selmer groups over cyclotomic $\mathbb{Z}_p$-extensions at good ordinary primes $p$. We extend Greenberg’s result to more general $p$-adic Galois representations, including a large subclass of those attached to $p$-ordinary modular forms of level $\Gamma_0(N)$ with $p \nmid N$.

1. Introduction

A classical result of R. Greenberg establishes a relation between the cardinality of Selmer groups of elliptic curves over number fields and the characteristic power series of Pontryagin duals of Selmer groups over cyclotomic $\mathbb{Z}_p$-extensions at good ordinary primes $p$. Our goal in this paper is to extend Greenberg’s result to more general $p$-adic Galois representations, including a large subclass of those coming from $p$-ordinary modular forms of even weight and level $\Gamma_0(N)$ with $p \nmid N$. This generalization of Greenberg’s theorem will play a role in our forthcoming paper on Kolyvagin’s conjecture and a Bloch–Kato formula for higher (even) weight modular forms (\cite{15}).

Let us begin by recalling Greenberg’s result. Let $E$ be an elliptic curve defined over a number field $F$, let $p$ be a prime number and suppose that $E$ has good ordinary reduction at all primes of $F$ above $p$. Moreover, assume that the $p$-Selmer group $\text{Sel}_p(E/F)$ of $E$ over $F$ is finite. Let $F_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$, set $\Gamma := \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p$ and write $\Lambda := \mathbb{Z}_p[\Gamma]$ for the corresponding Iwasawa algebra with $\mathbb{Z}_p$-coefficients, which we identify with the power series $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[[T]]$. Since $\text{Sel}_p(E/F)$ is finite, the $p$-primary Selmer group $\text{Sel}_p(E/F_\infty)$ of $E$ over $F_\infty$ is $\Lambda$-cotorsion, i.e., the Pontryagin dual of $\text{Sel}_p(E/F_\infty)$ is a torsion $\Lambda$-module (\cite[Theorem 1.4]{8}). Greenberg’s result that is the starting point for the present paper (\cite[Theorem 4.1]{8}) states that if $f_E \in \Lambda$ is the characteristic power series of the Pontryagin dual of $\text{Sel}_p(E/F_\infty)$, then

$$f_E(0) \sim \frac{\# \text{Sel}_p(E/F) \cdot \prod_{v \text{ bad}} c_v(E) \cdot \prod_{v \ell p} \#(\tilde{E}_v(F_v))^2}{\#(E(F)_p)^2},$$

where the symbol $\sim$ means that the two quantities differ by a $p$-adic unit, $c_v(E)$ is the Tamagawa number of $E$ at a prime $v$ of bad reduction, $F_v$ is the residue field of $F$ at $v$, $\tilde{E}_v(F_v)$ is the group of $\mathbb{F}_v$-rational points of the reduction $\tilde{E}_v$ of $E$ at $v$ and $E(F)_p$ is the $p$-torsion subgroup of the Mordell–Weil group $E(F)$.

To the best of our knowledge, no generalization of this result is currently available to other settings of arithmetic interest, most notably that of modular forms of level $\Gamma_0(N)$ that are ordinary at a prime number $p \nmid N$ (see, however, \cite[Theorem 3.3.1]{11} for a result for Galois representations in an anticyclotomic imaginary quadratic context). In this article we offer a generalization of this kind. Now let us describe our main result more in detail.
Given a number field $F$ with absolute Galois group $G_F := \text{Gal}(\bar{F}/F)$ and an odd prime number $p$, we consider a $p$-ordinary (in the sense of Greenberg, cf. [7]) representation

$$\rho_F : G_F \longrightarrow \text{Aut}_K(V) \simeq \text{GL}_r(K)$$

where $V$ is a vector space of dimension $r$ over a finite extension $K$ of $Q_p$, equipped with a continuous action of $G_F$. We assume that $\rho_F$ is crystalline at all primes of $F$ above $p$, self-dual and unramified outside a finite set $\Sigma$ of primes of $F$ including those that either lie above $p$ or are archimedean. Writing $\mathcal{O}$ for the valuation ring of $K$, we also fix a $G_F$-stable $\mathcal{O}$-lattice $T$ inside $V$, set $A := T \otimes_{\mathcal{O}} (K/\mathcal{O})$ and assume that there exists a non-degenerate, skew-symmetric, Galois-equivariant pairing

$$T \times A \longrightarrow \mu_{p^\infty},$$

where $\mu_{p^\infty}$ is the group of $p$-power roots of unity, so that $A$ and the Tate twist of the Pontryagin dual of $T$ become isomorphic. Finally, we impose on $V$ a number of technical conditions on invariant subspaces and quotients for the ordinary filtration at primes above $p$; the reader is referred to Assumption 2.1 for details. In particular, in §2.3 we show that, choosing the prime number $p$ judiciously, these properties are enjoyed by the $p$-adic Galois representation attached by Deligne to a modular form of level $\Gamma$. As before, let $F_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$ and set $\Gamma := \text{Gal}(F_\infty/F)$. Let $\Lambda := \mathcal{O}[\Gamma]$ be the Iwasawa algebra of $\Gamma$ with coefficients in $\mathcal{O}$, which we identify with the power series $\mathcal{O}$-algebra $\mathcal{O}[T]$. Finally, let $\text{Sel}_{\text{Gr}}(A/F_\infty)$ be the Greenberg Selmer group of $A$ over $F_\infty$ and let $\text{Sel}_{\text{BK}}(A/F)$ be the Bloch–Kato Selmer group of $A$ over $F$. General control theorems due to Ochiai ([18]) relate $\text{Sel}_{\text{BK}}(A/F)$ and $\text{Sel}_{\text{Gr}}(A/F_\infty)^\vee$, and one can think of our paper as a refinement of [18] in which we describe in some cases the (finite) kernel and cokernel of the natural restriction map $\text{Sel}_{\text{BK}}(A/F) \rightarrow \text{Sel}_{\text{Gr}}(A/F_\infty)^\vee$. If $\text{Sel}_{\text{BK}}(A/F)$ is finite, then $\text{Sel}_{\text{Gr}}(A/F_\infty)$ is a $\Lambda$-cotorsion module, i.e., the Pontryagin dual of $\text{Sel}_{\text{Gr}}(A/F_\infty)^\vee$ of $\text{Sel}_{\text{Gr}}(A/F_\infty)$ is a torsion $\Lambda$-module. Thus, when $\text{Sel}_{\text{BK}}(A/F)$ is finite we can consider the characteristic power series $F \in \Lambda$ of $\text{Sel}_{\text{Gr}}(A/F_\infty)^\vee$.

Our main result (Theorem 6.1) is

**Theorem 1.1.** If $\text{Sel}_{\text{BK}}(A/F)$ is finite, then $F(0) \neq 0$ and

$$\#(\mathcal{O}/F(0) \cdot \mathcal{O}) = \# \text{Sel}_{\text{BK}}(A/F) \cdot \prod_{v \in \Sigma} c_v(A),$$

where $c_v(A)$ is the $p$-part of the Tamagawa number of $A$ at $v$.

The Tamagawa numbers $c_v(A)$ are defined in §3.2. It is worth pointing out that our local assumptions at primes of $F$ above $p$ ensure that the term corresponding to $E(F)_p$ in (1.1) is trivial. Moreover, our conditions on the ordinary filtrations at primes $v$ of $F$ above $p$ force all the terms corresponding to $E_v(F_v)$ in (1.1) to be trivial as well. As will be apparent, our strategy for proving Theorem 6.1 is inspired by the arguments of Greenberg in [3] §4.

We conclude this introduction with a couple of remarks of a general nature. First of all, several of our arguments can be adapted to other $\mathbb{Z}_p$-extensions $F_\infty/F$. However, in general this would require modifying the definition of Selmer groups at primes in $\Sigma$ that are not finitely decomposed in $F_\infty$. For example, suppose that $F$ is an imaginary quadratic field, $T$ is the $p$-adic Tate module of an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at $p$ and $F_\infty$ is the anticyclotomic $\mathbb{Z}_p$-extension of $F$. Since prime numbers that are inert in $F$ split completely in $F_\infty$, some of the arguments described in this paper (e.g., the proof of Lemma 5.13) fail and one needs to work with subgroups (or variants) of $\text{Sel}_{\text{BK}}(A/F)$ that are defined by imposing different conditions at primes in $\Sigma$ that are inert in $F$. For instance, the definition of Selmer groups in [11] §2.2.3 in the imaginary quadratic case requires that all the local conditions at inert primes be trivial, while in [2] one assumes ordinary-type conditions.
at those primes. Since the precise local conditions needed depend on the arithmetic situation being investigated, in this paper we chose to work with the cyclotomic $\mathbb{Z}_p$-extension of $F$ exclusively, thus considering only Bloch–Kato Selmer groups as defined below.

Finally, we remark that our interest in $\text{Sel}_{\text{Gr}}(A/F_\infty)$ instead of the Bloch–Kato Selmer group $\text{Sel}_{\text{BK}}(A/F_\infty)$ of $A$ over $F_\infty$ is essentially motivated by the applications to [15] of the results in this paper. Actually, the results of Skinner–Urban on the Iwasawa main conjecture for the cyclotomic setting ([25]), which play a crucial role in [15], are formulated in terms of $\text{Sel}_{\text{Gr}}(A/F_\infty)$ rather than $\text{Sel}_{\text{BK}}(A/F_\infty)$, which explains the focus of our article. However, alternative settings can certainly be considered; for example, [18, Theorem 2.4] establishes a relation between $\text{Sel}_{\text{BK}}(A/F)$ and $\text{Sel}_{\text{BK}}(A/F_\infty)^G$, and it would be desirable to prove a formula for the value at 0 of the characteristic power series of $\text{Sel}_{\text{BK}}(A/F_\infty)$ analogous to that in Theorem 1.1.

2. Galois representations

We fix the Galois representations that we consider in this paper, specifying our working assumptions. We will then show that these conditions are satisfied by a large class of $p$-ordinary crystalline representations attached to modular forms.

2.1. Notation and terminology. To begin with, we introduce some general notation and terminology. If $p$ is a prime number and $M$ is a $\mathbb{Z}_p$-module $M$, then we write

$$M^\vee := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$$

for the Pontryagin dual of $M$. If $M$ is a module over the Galois group $\text{Gal}(E/L)$ of some (Galois) field extension $E/L$, where $L$ is an extension of $Q$ or $Q_l$ for some prime number $\ell$, then we denote by $M(1)$ the Tate twist of $M$. Moreover, let $\hat{L}$ be a fixed algebraic closure of $L$ and let $G_L := \text{Gal}(\hat{L}/L)$ be the absolute Galois group of $L$. If $\mathfrak{p}_L \subset \hat{L}$ is the group of $p$-power roots of unity in $\hat{L}$, then local Tate duality

$$\langle \cdot, \cdot \rangle : H^i(G_L, M) \times H^{2-i}(G_L, M^\vee(1)) \to H^2(G_L, \mathfrak{p}_L) \simeq Q_p/\mathbb{Z}_p$$

identifies $H^i(G_L, M)$ with $H^{2-i}(G_L, M^\vee(1))$ for $i = 0, 1$.

Let $F$ be a number field, let $\hat{F}$ be a fixed algebraic closure of $F$ and let $G_F := \text{Gal}(\hat{F}/F)$ be its absolute Galois group. For every prime $v$ of $F$ let $F_v$ be the completion of $F$ at $v$ and let $\mathcal{O}_v$ be the valuation ring of $F_v$. Moreover, let $G_v := \text{Gal}(\hat{F}_v/F_v)$ be the absolute Galois group of $F_v$ and let $I_v \subset G_v$ be the inertia subgroup.

Let $T$ be a continuous $G_F$-module, which we assume to be free of finite rank $r$ over the valuation ring $\mathcal{O}$ of a finite extension $K$ of $Q_p$, where $p$ is an odd prime number. Fix a uniformizer $\pi$ of $\mathcal{O}$ and set $E := \mathcal{O}/(\pi)$ for the residue field of $K$, which is a finite field of characteristic $p$. Define $V := T \otimes \mathcal{O} K$ and $A := V/T = T \otimes \mathcal{O} (K/\mathcal{O})$, so that $T$ is an $\mathcal{O}$-lattice inside $V$. Then $A \simeq (K/\mathcal{O})^r$ and $V \simeq K^r$ as groups and vectors spaces, respectively. Moreover, for every integer $n \geq 0$ there is an isomorphism of $G_F$-modules $T/\pi^n T \simeq A[\pi^n]$, where $A[\pi^n]$ is the $\pi^n$-torsion $\mathcal{O}$-submodule of $A$. Set $T^* := \text{Hom}_{\mathbb{Z}_p}(A, \mathfrak{p}_L) = A^\vee(1)$, $V^* := T^* \otimes K$ and $A^* := V^*/T^* = T^* \otimes K/\mathcal{O}$, equipped with the induced actions of $G_F$. The representation $V$ is self-dual if there is a $G_F$-equivariant isomorphism $V \simeq V^*$. Let $V$ be self-dual and let us fix an isomorphism $\nu : V \to V^*$ as above. Suppose that $\nu(T)$ is homothetic to $T^*$, which means that there exists $\lambda \in K^\times$ such that $\lambda \cdot \nu(T) = T^*$ in $V^*$; this is an identification of $G_F$-modules, as the action of $G_F$ is $\mathcal{O}$-linear. The composition of $\nu$ with the multiplication-by-$\lambda$ map on $V^*$ induces an isomorphism of $G_F$-modules between the quotients $A = V/T$ and $A^* = V^*/T^*$. 


The representation $V$ is said to be ordinary (in the sense of Greenberg) at a prime $v | p$ if there is a short exact sequence of representations
\[0 \rightarrow V^+ \rightarrow V \rightarrow V^- \rightarrow 0\]
such that $I_v$ acts on $V^-$ via the cyclotomic character (see [7]). Set $r^+ := \dim_K(V^+)$ and $r^- := \dim_K(V^-)$. Define $T^+ := V^+ \cap T$ and $T^- := V^- \cap T$, which are free $\mathcal{O}$-submodules of $T$ of ranks $r^+$ and $r^-$, respectively. We also define $A^+ := V^+/T^+ \simeq (K/\mathcal{O})^{r^+}$ and $A^- := V^-/T^- \simeq (K/\mathcal{O})^{r^-}$.

2.2. Assumptions. Notation being as in §2.1 write
\[\rho_V : G_F \rightarrow \text{Aut}_K(V) \simeq \text{GL}_r(K)\]
and
\[\rho_T : G_F \rightarrow \text{Aut}_\mathcal{O}(T) \simeq \text{GL}_r(\mathcal{O})\]
for the Galois representations associated with $V$ and $T$, respectively.

We work under the following assumption, which is slightly more restrictive than the one appearing in [15].

**Assumption 2.1.**

1. $\rho_V$ is unramified outside a finite set of primes of $F$;
2. $\rho_V$ is crystalline at all primes $v | p$;
3. $\rho_V$ is self-dual;
4. $\rho_V$ is ordinary at all primes $v | p$;
5. there is a skew-symmetric, $G_F$-equivariant and non-degenerate pairing
   \[T \times A \longrightarrow \mu_{p^\infty}\]
   that induces a non-degenerate pairing
   \[T/p^mT \times A[p^m] \longrightarrow \mu_{p^m}\]
   for every integer $m \geq 1$, where $\mu_{p^m}$ is the group of $p^m$-th roots of unity (in particular, this gives an isomorphism $T^{\psi}(1) \simeq A$);
6. for each prime $v | p$, one has:
   (a) $H^0(F_v, A) = 0$,
   (b) the largest cotorsion quotient of $H^0(I_v, A^-)$ is trivial,
   (c) $H^0(F_v, (T^-)^\psi(1)) = 0$,
   (d) there exists a suitable basis such that $G_{F_v}$ acts on $A^+$ diagonally via non-trivial characters $\eta_1, \ldots, \eta_r$ that do not coincide with the cyclotomic character.

Let us define
\[\Sigma := \{\text{primes of } F \text{ at which } V \text{ is ramified}\} \cup \{\text{primes of } F \text{ above } p\}\]

(2.1)
\[\cup \{\text{archimedean primes of } F\},\]

which is a finite set.

Remark 2.2. We list some consequences of Assumption 2.1

1. If the inertia group $I_v$ acts on $V^-$ via a non-trivial power of the cyclotomic character, then $H^0(I_v, V^-) = 0$, so $H^0(I_v, T^-) = 0$. Furthermore, $H^0(I_v, A^-) = 0$ because $A^- = T^- \otimes_K K/\mathcal{O}$.
2. If $H^0(F_v, A) = 0$, then the Galois-equivariant isomorphism $A \simeq T^{\psi}(1)$ ensures that $H^0(F_v, (T^-)^\psi(1)) = 0$ as well.
3. If the isomorphism $V \simeq V^+$ induces an isomorphism $(T^+)^\psi(1) \simeq A^+$ (which is always true in the case of ordinary modular forms considered below), then the condition $H^0(F_v, (T^-)^\psi(1)) = 0$ is equivalent to the condition $H^0(F_v, A^+) = 0$, which is implied by (6).
(4) Since \( V \) is ordinary at all \( v \mid p \), it satisfies the so-called \textit{Panchishkin condition} at all such \( v \) (\cite{13} Definition 2.2, \cite{19} §5.4).

2.3. The case of modular forms. We want to check that if the prime number \( p \) is chosen carefully, then Assumption 2.1 is satisfied by the \( p \)-adic Galois representation attached to a newform. Let \( f(q) = \sum_{n \geq 1} a_n(f)q^n \) be a newform of weight \( k \geq 2 \) and level \( \Gamma_0(N) \). Let \( Q_f := Q(a_n(f) \mid n \geq 1) \) be the Hecke field of \( f \), which is a totally real number field. It is well known that the Fourier coefficients \( a_n(f) \) are algebraic integers.

Let \( p \) be a prime number such that \( p \nmid 2N \) and fix a prime \( \mathfrak{p} \) of \( Q_f \) above \( p \). We assume that

\[
\text{(2.2)} \quad \mathfrak{p} \text{ is ordinary for } f.
\]

In other words, we require \( a_p(f) \) to be a \( p \)-adic unit, \( i.e., a_p(f) \in \mathcal{O}^\times \).

Remark 2.3. Let us say that a prime number \( p \) is \textit{ordinary} for \( f \) if \( p \nmid a_p(f) \). Thanks to results of Serre on eigenvalues of Hecke operators (\cite{24} §7.2), one can prove that if \( k = 2 \), then the set of primes that are ordinary for \( f \) has density 1, so it is infinite (see, \( \text{e.g.,} \) \cite{5} Proposition 2.2). On the other hand, it is immediate to check that if \( p \) is an ordinary prime for \( f \) that is unramified in \( Q_f \), then there exists a prime \( \wp \) of \( Q_f \) above \( p \) such that \( f \) is \( \wp \)-ordinary. As a consequence, a weight 2 newform satisfies (2.2) with a suitable \( p \) for infinitely many primes \( p \) (in fact, the set of such primes has density 1). On the contrary, the existence of infinitely many ordinary primes for a modular form of weight larger than 2 is, as far as we are aware of, still an open question.

We also assume that

\[
\text{(2.3)} \quad a_p(f) \not\equiv 1 \pmod{p}.
\]

Write \( K \) for the completion of \( Q_f \) at \( \mathfrak{p} \) and \( \mathcal{O} \) for the valuation ring of \( K \). Let \( V \) be the self-dual twist of the representation \( V_{f,p} \) of \( G_{Q_f} \) attached by Deligne to \( f \) and \( p \) (\cite{4}), so that \( V = V_{f,p}(k/2) \). Choose a \( G_Q \)-stable \( \mathcal{O} \)-lattice \( T \subset V \). The \( p \)-adic representations

\[
\rho_{f,p} : G_Q \rightarrow \text{Aut}_K(V) \simeq \text{GL}_2(K)
\]

and

\[
\rho_{f,p,T} : G_Q \rightarrow \text{Aut}_\mathcal{O}(T) \simeq \text{GL}_2(\mathcal{O})
\]

will play the roles of \( \rho_V \) and \( \rho_T \), respectively. In particular, \( F = Q \) in the notation of (2.2).

Now we show that Assumption 2.1 is satisfied by the representation \( V \). First of all, it is well known that \( V \) is unramified at all primes \( \ell \nmid Np \) and crystalline at \( p \); these properties correspond to conditions \( \text{(1)} \) and \( \text{(2)} \) in Assumption 2.1. Furthermore, \( V \) is the self-dual twist of \( V_{f,p} \), so \( \text{(3)} \) in Assumption 2.1 is satisfied. On the other hand, \( \text{(5)} \) in Assumption 2.1 corresponds to \( \text{[16] Proposition 3.1, (2)} \).

Next, we show that \( \text{(4)} \) and \( \text{(5)} \) in Assumption 2.1 are satisfied. If \( \ell \nmid Np \) and \( \text{Frob}_\ell \) is a geometric Frobenius at \( \ell \), then the characteristic polynomial of \( \rho_{V}(\text{Frob}_\ell) \) is the Hecke polynomial \( X^2 - a_\ell(f)X + p^{k-1} \). Let \( \alpha \in \mathcal{O}^\times \) be the unit root of \( X^2 - a_p(f)X + p^{k-1} \), which exists because \( f \) satisfies (2.2), and let \( \delta \) be the unramified character of the decomposition group \( G_p := \text{Gal}(\overline{Q}_p/Q_p) \) given by \( \delta(\text{Frob}_p) := \alpha \), where \( \text{Frob}_p \in G_p/I_p \) is the arithmetic Frobenius. It is a classical result of Deligne and of Mazur–Wiles (see, \( \text{e.g.,} \) \cite{13} Proposition 3.2]) that the restriction of \( V_{f,p} \) to \( G_p \) is equivalent to a representation of the form \(
\begin{pmatrix}
\delta^{-1} & 0 \\
0 & \chi_{\text{cycl}}
\end{pmatrix}
\)

where \( c \) is a 1-cocycle with values in \( \mathcal{O} \) and \( \chi_{\text{cycl}} : G_p \rightarrow \mathbb{Z}_p^\times \) is the \( p \)-adic cyclotomic character. It follows that the restriction of \( V = V_{f,p}(k/2) \) to \( G_p \) is equivalent to a representation of the form \(
\begin{pmatrix}
\delta^{-1} & 0 \\
0 & \chi_{\text{cycl}}^{-1}
\end{pmatrix}
\)

Thus, \( \text{(4)} \) in Assumption 2.1 is satisfied and there is an exact sequence of \( G_{Q_p} \)-modules

\[
0 \rightarrow V^+ \rightarrow V \rightarrow V^- \rightarrow 0
\]
such that $V^+$ and $V^-$ are 1-dimensional $K$-vector spaces on which $G_{Q_p}$ acts via $\delta^{-1} \cdot \chi_{\text{cyc}}$ and $\delta \cdot \chi_{\text{cyc}}^{-k/2}$, respectively. It follows that (6d) in Assumption 2.1 is satisfied. Now, $I_p$ acts on $A^- \simeq K/O$ via the $(-k/2)$-th power of the cyclotomic character, which is non-trivial, so $H^0(I_p, A^-) = 0$, which shows that (6b) in Assumption 2.1 holds. As for (6c), see part (3) of Remark 2.2.

Finally, we prove that (6a) is satisfied. Suppose that $H^0(Q_p, A) \neq 0$. Then one can find an element $a \in A[p]$ such that $\sigma(a) = a$ for all $\sigma \in G_p$. Since $T/pT \simeq A[p]$, we may regard $a$ as an element of $T/pT$. Choose a basis $\{e_1\}$ of the 1-dimensional $\mathbb{F}$-vector space $T^+/pT^+$ and complete it to a $\mathbb{F}$-basis $\{e_1, e_2\}$ of $T/pT$. The action of $\sigma \in G_p$ on $T/pT$ is then given by the matrix

$$
\hat{\rho}_{f, p}(\sigma) = \begin{pmatrix}
\delta^{-1} \cdot \chi_{\text{cyc}}^{-k/2} & c \cdot \chi_{\text{cyc}}^{-k/2} \\
0 & \delta \cdot \chi_{\text{cyc}}^{-k/2}
\end{pmatrix} \pmod{p}.
$$

Write $a = x_1e_1 + x_2e_2$ with $x_1, x_2 \in \mathbb{F}$. Let $v|p$ be a prime of $F$, set $\bar{F}_v := \bar{Q}_p$ and let $G_v := \text{Gal}(\bar{F}_v/F_v) \subset G_p$. The action of $\sigma \in G_v$ on $a$ is given by

$$\sigma(a) = (x_1(\delta^{-1} \cdot \chi_{\text{cyc}}^{-k/2})(\sigma) + x_2(c \cdot \chi_{\text{cyc}}^{-k/2}(\sigma))e_1 + x_2(\delta \cdot \chi_{\text{cyc}}^{-k/2}(\sigma))e_2 \pmod{p}.
$$

Then, in light of (2.3), one easily shows from (2.4) that $\sigma(a) \neq a$. First suppose that $x_1 = 0$. Then $x_1 \neq 0$ because $a \neq 0$, so it is enough to find $\sigma \in G_p$ such that $x_1(\delta^{-1} \cdot \chi_{\text{cyc}}^{-k/2})(\sigma) \neq 1$. Let Frobp $\in \text{Gal}(\bar{Q}_p/Q_p)$ be an arithmetic Frobenius. If $k = 2$, then $\text{Frob}_p(a) = x_1\alpha^{-j}e_1$ for all integers $j$, where $\alpha = \alpha (\text{mod } p)$. Since $\alpha \neq 1$, one may find $j$ such that $x_1\alpha^{-j} \neq 1$, and so $\sigma(a) \neq a$. If $k > 2$, then a similar argument works. Namely, choose an integer $j$ such that $x_1\alpha^{-j} \neq 1$, pick a lift $F \in G_p$ of Frobp, and let $\bar{F}$ be the image of $F$ in $\text{Gal}(\bar{Q}_p(\sqrt[k]{\mu_p})/Q_p)$, so that $\chi_{\text{cyc}}(F) = \chi_{\text{cyc}}(\bar{F})$. If $x_1\alpha^{-j} \cdot \chi_{\text{cyc}}^{-k/2}(F) \neq 1$, then we are done; otherwise, since $x_1\alpha^{-j} \neq 1$, we see that $\chi_{\text{cyc}}^{-k/2}(\bar{F} \neq 1$, so the map $\sigma \mapsto \chi_{\text{cyc}}^{-k/2}$ is not the trivial character. It follows that $\chi_{\text{cyc}}^{-k/2} : \text{Gal}(\bar{Q}_p(\sqrt[k]{\mu_p})/Q_p) \to \mathbb{F}_p^\times$ is surjective, and therefore we can find $\sigma \in I_p$ such that $\chi_{\text{cyc}}^{-k/2}(\sigma) \neq 1$. Then

$$
x_1(\delta^{-1} \cdot \chi_{\text{cyc}}^{-k/2})(\sigma) x_1\alpha^{-j} \cdot \chi_{\text{cyc}}^{-k/2}(F) = x_1\alpha^{-j} \cdot \chi_{\text{cyc}}^{-k/2}(F) \sigma
$$

and we are done. The case $x_2 = 0$ is similar; in fact, it suffices to show that there exists $\sigma$ such that $x_2(\delta \cdot \chi_{\text{cyc}}^{-k/2})(\sigma) \neq 1$. Choose $j$ such that $x_2\alpha^{-j} \neq 1$ and fix a lift $F \in G_p$ of Frobp. If $x_2\alpha^{-j} \cdot \chi_{\text{cyc}}^{-k/2}(F) \neq 1$, then we are done, otherwise $\chi_{\text{cyc}}^{-k/2} : \text{Gal}(Q_p(\sqrt[k]{\mu_p})/Q_p) \to \mathbb{F}_p^\times$ is surjective, so we can find $\sigma \in I_p$ such that $\chi_{\text{cyc}}^{-k/2}(\sigma) \neq 1$. It follows that

$$
x_2(\delta \cdot \chi_{\text{cyc}}^{-k/2}(F) \sigma) = x_2\alpha^{-j} \cdot \chi_{\text{cyc}}^{-k/2}(F) \sigma
$$

and the proof of (6a) in Assumption 2.1 is complete.

### 3. Selmer Groups

Notation from 2.1 is in force; moreover, we work under Assumption 2.1. As before, let $F_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. Set $\Gamma := \text{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$, choose a topological generator $\gamma$ of $\Gamma$ and let $\Lambda := \mathcal{O}[[\Gamma]]$ be the Iwasawa algebra of $\Gamma$ with coefficients in $\mathcal{O}$, which can be identified with the formal power series $\mathcal{O}$-algebra $\mathcal{O}[T]$ by sending $\gamma$ to $T + 1$. For every integer $n \geq 0$ write $F_n$ for the $n$-th layer of $F_\infty/F$, i.e., the unique extension $F_n$ of $F$ such that $F_n \subset F_\infty$ and $\text{Gal}(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$ (in particular, $F_0 = F$). For every prime $v$ of $F_n$ denote by $F_{n,v}$ the completion of $F_n$ at $v$, let $G_{n,v} := \text{Gal}(F_{n,v}/F_{n,v})$ be the absolute
Galois group of $F_{n,v}$ and let $I_{n,v} \subset G_{n,v}$ be the inertia subgroup. If $n = 0$, in the previous notation we have $G_n = G_{0,v}$ and $I_v = I_{0,v}$. We also set $G_{\infty,v} := \text{Gal}(F_v/F_{\infty,v})$ and denote by $I_{\infty,v}$ its inertia subgroup.

3.1. Local conditions at $\ell \neq p$. Fix an integer $n \geq 0$. Fix also a prime number $\ell \neq p$ and a prime $v | \ell$ of $F_n$. For $\star \in \{V, A\}$, define

$$H_{\text{ur}}^1(F_{n,v}, \star) := \text{ker} \left( H^1(F_{n,v}, \star) \to H^1(I_{n,v}, \star) \right).$$

Here, as customary, $H^1(F_{n,v}, \star)$ stands for $H^1(G_{n,v}, \star)$. By functoriality, there is a map $H^1(F_{n,v}, V) \to H^1(F_{n,v}, A)$; set

$$H^1_f(F_{n,v}, V) := H_{\text{ur}}^1(F_{n,v}, V), \quad H^1_f(F_{n,v}, A) := \text{im} \left( H^1_f(F_{n,v}, V) \to H^1(F_{n,v}, A) \right).$$

The commutative diagram

$$
\begin{array}{ccc}
0 & \to & H^1_{\text{ur}}(F_{n,v}, V) \to H^1(F_{n,v}, V) \to H^1(I_{n,v}, V) \\
& & \downarrow \quad \downarrow \\
0 & \to & H^1_{\text{ur}}(F_{n,v}, A) \to H^1(F_{n,v}, A) \to H^1(I_{n,v}, A)
\end{array}
$$

shows that $H^1_f(F_{n,v}, A) \subset H^1_{\text{ur}}(F_{n,v}, A)$.

3.2. Local Tamagawa numbers. For every prime $v$ of $F$ we introduce the $p$-part of the Tamagawa number of $A$ at $v$. We shall see, the product of these integers will appear in our main result.

**Lemma 3.1.** The index $[H^1_{\text{ur}}(F_v, A) : H^1_f(F_v, A)]$ is finite.

**Proof.** See [21, Lemma 3.5].

The following notion is well defined thanks to Lemma 3.1.

**Definition 3.2.** Let $v$ be a prime of $F$. The integer

$$c_v(A) := [H^1_{\text{ur}}(F_v, A) : H^1_f(F_v, A)]$$

is the $p$-part of the Tamagawa number of $A$ at $v$.

Recall the finite set $\Sigma$ from (2.1).

**Lemma 3.3.** If $v \notin \Sigma$, then $c_v(A) = 1$.

**Proof.** Since $T$ is unramified outside $Np$, this is [21, Lemma I.3.5, (iv)].

3.3. Local conditions at $p$. Fix an integer $n \geq 0$ and let $v | p$ be a prime of $F_n$.

3.3.1. The Bloch–Kato condition. Let $B_{\text{cris}}$ be Fontaine’s crystalline ring of periods. Define

$$H^1_f(F_{n,v}, V) := \text{ker} \left( H^1(F_{n,v}, V) \to H^1(F_{n,v}, V \otimes_{\mathbb{Q}_p} B_{\text{cris}}) \right)$$

and

$$H^1_f(F_{n,v}, A) := \text{im} \left( H^1_f(F_{n,v}, V) \to H^1(F_{n,v}, A) \right),$$

where the second arrow is induced by the canonical map $H^1(F_{n,v}, V) \to H^1(F_{n,v}, A)$.

3.3.2. The Greenberg condition. As in [21] let $T^+ := T \cap V^+$, $A^+ := V^+/T^+$, $A^- := A/A^+$. For $\star \in \{V, A\}$, define

$$H^1_{\text{ord}}(F_{n,v}, \star) := \text{ker} \left( H^1(F_{n,v}, \star) \to H^1(I_{n,v}, *, \star^-) \right),$$

the map being induced by restriction and the canonical projection $* \to *^-$.
3.3.3. \textit{Comparison between local conditions.} Let $\mathcal{B}_{\text{dR}}$ be Fontaine’s de Rham ring of periods. Define

$$H^1_g(F_v, V) := \ker \left( H^1(F_v, V) \to H^1(F_v, V \otimes_{\mathbb{Q}_p} \mathcal{B}_{\text{dR}}) \right).$$

Since $\mathcal{B}_{\text{cris}}$ is a subring of $\mathcal{B}_{\text{dR}}$, there is an inclusion $H^1_f(F_v, V) \subset H^1_g(F_v, V)$. Since $V$ is crystalline at $p$, by [17, Proposition 12.5.8], one has $D_{\text{cris}, F_v}(V^-) = 0$ (here, as usual, $D_{\text{cris}, F_v}(W) := (W \otimes_{\mathbb{Q}_p} \mathcal{B}_{\text{cris}})^{G_{F_v}}$ for a $G_{F_v}$-representation $W$). Then it follows from [17, Proposition 12.5.7, (2), (ii)] that

$$H^1_f(F_v, V) = H^1_g(F_v, V). \quad (3.1)$$

Moreover, by a result of Flach ([6, Lemma 2]; see also [18, Proposition 4.2]), there is an equality

$$H^1_g(F_v, V) = H^1_{\text{ord}}(F_v, V). \quad (3.2)$$

Combining (3.1) and (3.2) then yields

$$H^1_f(F_v, V) = H^1_{\text{ord}}(F_v, V). \quad (3.3)$$

Finally, in light of (3.3), the commutativity of the square

$$\begin{array}{ccc}
H^1(F_v, V) & \longrightarrow & H^1(F_v, A) \\
\downarrow & & \downarrow \\
H^1(I_v, V^-) & \longrightarrow & H^1(I_v, A^-)
\end{array}$$

shows that $H^1_f(F_v, A) \subset H^1_{\text{ord}}(F_v, A)$.

3.4. \textit{Selmer groups.} Now we introduce Selmer groups in the sense of Bloch–Kato and of Greenberg.

3.4.1. \textit{The Bloch–Kato Selmer group.} Fix an integer $n \geq 0$. The \textit{Bloch–Kato Selmer group of $A$ over $F_n$} is

$$\text{Sel}_{\text{BK}}(A/F_n) := \ker \left( H^1(F_n, A) \to \prod_v H^1(F_{n,v}, A) \right),$$

where $v$ varies over all primes of $F_n$ and the arrow is induced by the localization maps. Moreover, define the \textit{Bloch–Kato Selmer group of $A$ over $F_{\infty}$} as

$$\text{Sel}_{\text{BK}}(A/F_{\infty}) := \varprojlim_n \text{Sel}_{\text{BK}}(A/F_n),$$

the direct limit being taken with respect to the usual restriction maps in Galois cohomology.

3.4.2. \textit{The Greenberg Selmer group.} Fix an integer $n \geq 0$. The \textit{Greenberg Selmer group of $A$ over $F_n$} is

$$\text{Sel}_{\text{Gr}}(A/F_n) := \ker \left( H^1(F_n, A) \to \prod_{v \not\mid p} H^1(F_{n,v}, A) \times \prod_{v \mid p} H^1(I_{n,v}, A) \right),$$

where $v$ varies over all primes of $F_n$ and the arrow is induced by the localization maps. Moreover, define the \textit{Greenberg Selmer group of $A$ over $F_{\infty}$} as

$$\text{Sel}_{\text{Gr}}(A/F_{\infty}) := \varprojlim_n \text{Sel}_{\text{Gr}}(A/F_n),$$

the direct limit being taken again with respect to the restriction maps.
Remark 3.4. For Galois representations associated with modular forms, the Selmer group considered in [25] is the Greenberg Selmer group. In many cases, the strict Selmer group, which is defined as the Greenberg Selmer group with the difference that the local condition at a prime $v$ above $p$ is taken to be the kernel of the map

$$H^1(F_n, A) \rightarrow H^1(F_{n,v}, A)$$

is equal to the Bloch–Kato Selmer group (see, e.g., [10, (23)]).

4. Characteristic power series

As before, let $F_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$ and put $\Gamma := \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p$.

With notation as in (3.4), set

$$S := \text{Sel}_{Gr}(A/F_\infty).$$

Furthermore, let

$$X := S^\vee = \text{Hom}(S, \mathbb{Q}_p/\mathbb{Z}_p)$$

be the Pontryagin dual of $S$. By the topological version of Nakayama’s lemma (see, e.g., [1, p. 226, Corollary]), the $\Lambda$-module $X$ is finitely generated.

4.1. Invariants and coinvariants of Selmer groups. In what follows, $S^\Gamma$ (respectively, $S_\Gamma$) denotes the $O$-module of $\Gamma$-invariants (respectively, $\Gamma$-coinvariants) of $S$.

Proposition 4.1. If $\text{Sel}_{BK}(A/F)$ is finite, then $S^\Gamma$ is finite and $S$ is a cotorsion $\Lambda$-module.

Recall that, by definition, $S$ is $\Lambda$-cotorsion if $X$ is $\Lambda$-torsion.

Proof. By [18, Theorem 2.4], which can be applied in our setting using [17, Lemma 12.5.7], $\text{Sel}_{BK}(A/F)$ is finite if and only if $\text{Sel}_{BK}(A/F_\infty)^\Gamma$ is. On the other hand, by [18, Proposition 4.2] combined with (3.3), $\text{Sel}_{BK}(A/F_\infty)^\Gamma$ is finite if and only if $S^\Gamma$ is. Now fix a topological generator $\gamma$ of $\Gamma$. By Pontryagin duality, the finiteness of $S^\Gamma$ is equivalent to the finiteness of $X/(\gamma - 1)X$. By [11, p. 229, Theorem], it follows that $X$ is $\Lambda$-torsion, which means that $S$ is a cotorsion $\Lambda$-module. □

Remark 4.2. The proof of Proposition 4.1 actually shows that the finiteness of $\text{Sel}_{BK}(A/F)$ is equivalent to the finiteness of $S^\Gamma$. Moreover, by [6, Theorem 3] and [18, Proposition 4.1, (1)], for every $n \geq 0$ the finiteness of $\text{Sel}_{BK}(A/F_n)$ is equivalent to the finiteness of $\text{Sel}_{Gr}(A/F_n)$.

In particular, when $\text{Sel}_{BK}(A/F)$ is finite we can consider the characteristic power series $F \in O[[T]]$ of $X$.

Proposition 4.3. If $S^\Gamma$ is finite, then $S_\Gamma$ is finite, $F(0) \neq 0$ and $\#(O/F(0)O) = \#S^\Gamma/\#S_\Gamma$.

Proof. Proceed as in the proof of [8, Lemma 4.2], which only deals with the $O = \mathbb{Z}_p$ case but carries over to our more general setting in a straightforward way. □

Remark 4.4. Since $\Gamma$ has cohomological dimension 1 and $S$ is a direct limit of torsion groups, $H^2(\Gamma, S) = 0$; moreover, since $\Gamma \cong \mathbb{Z}_p$, we have $H^1(\Gamma, S) = S_\Gamma$. It follows that

$$\#S^\Gamma/\#S_\Gamma = \#H^0(\Gamma, S)/\#H^1(\Gamma, S)$$

is the Euler characteristic of $M$.

From now on we work under

Assumption 4.5. The group $\text{Sel}_{BK}(A/F)$ is finite.
In light of Assumption 4.5, it follows from Propositions 4.1 and 4.3 that

\[(4.1) \quad S \text{ is } \Lambda\text{-cotorsion, } S^\Gamma \text{ and } S_\Gamma \text{ are finite, } \mathcal{F}(0) \neq 0\]

and

\[(4.2) \quad \#(\mathcal{O}/\mathcal{F}(0) \cdot \mathcal{O}) = \frac{\#S^\Gamma}{\#S_\Gamma}.\]

In the following sections we shall study the quotient on the right hand side of (4.2).

5. Relating \(\text{Sel}_{\text{BK}}(A/F)\) and \(S^\Gamma\)

Recall that Assumption 4.5 is in force. As remarked in (4.1), \(S_\Gamma\) is finite as well. There is a natural map of finite groups

\[(5.1) \quad s : \text{Sel}_{\text{BK}}(A/F) \rightarrow S^\Gamma.\]

Then

\[(5.2) \quad \#S^\Gamma = \frac{\#\text{Sel}_{\text{BK}}(A/F) \cdot \# \text{coker}(s)}{\# \text{ker}(s)}.\]

Our next goal is to study the orders of the kernel and of the cokernel of \(s\).

5.1. The map \(r\). Set

\[P_{\text{BK}}(A/F) := \prod_v \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)}\]

and for every integer \(n \geq 0\) set also

\[P_{\text{Gr}}(A/F_n) := \prod_{v \mid p} \frac{H^1(F_{n,v}, A)}{H^1_\text{ur}(F_{n,v}, A)} \times \prod_{v \nmid p} \frac{H^1(F_{n,v}, A)}{H^1_\text{ord}(F_{n,v}, A)},\]

where products are taken over all primes of \(F\) and of \(F_n\), respectively. With this notation available, we can write

\[\text{Sel}_{\text{BK}}(A/F) = \ker\left(H^1(F, A) \rightarrow P_{\text{BK}}(A/F)\right)\]

and

\[\text{Sel}_{\text{Gr}}(A/F_n) = \ker\left(H^1(F_n, A) \rightarrow P_{\text{Gr}}(A/F_n)\right)\]

Finally, we define

\[(5.3) \quad P_{\text{Gr}}(A/F_\infty) := \lim_{\rightarrow n} P_{\text{Gr}}(A/F_n) = \prod_{v \mid p} \frac{H^1(F_{\infty,v}, A)}{H^1_\text{ur}(F_{\infty,v}, A)} \times \prod_{v \nmid p} \frac{H^1(F_{\infty,v}, A)}{H^1_\text{ord}(F_{\infty,v}, A)}\]

where the direct limit is taken with respect to the restriction maps; we also note that if a prime of \(F_n\) splits completely in \(F_m\) for \(m \geq n\), then the corresponding map is the diagonal embedding. By definition, there is an equality

\[\text{Sel}_{\text{Gr}}(A/F_\infty) = \ker\left(H^1(F_\infty, A) \rightarrow P_{\text{Gr}}(A/F_\infty)\right),\]

By construction (see §3.4 and (5.3)), there are natural maps \(P_{\text{BK}}(A/F) \rightarrow P_{\text{Gr}}(A/F)\) and \(P_{\text{Gr}}(A/F) \rightarrow P_{\text{Gr}}(A/F_\infty)\), which produce a map

\[r : P_{\text{BK}}(A/F) \rightarrow P_{\text{Gr}}(A/F_\infty).\]

For every prime \(v\) of \(F\) let \(w\) be a prime of \(F_\infty\) above \(v\). The map \(r\) is given by a product

\[r = \prod_{v,w} r_{v,w},\]

where

\[r_{v,w} : \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \rightarrow \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ur}(F_{\infty,w}, A)}\]
if \( v \nmid p \), while
\[
\rho_{v,w} : \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \to \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ord}(F_{\infty,w}, A)}
\]
if \( v \mid p \). Our next goal is to study the kernel of \( \rho \) prime by prime.

5.1.1. The map \( \rho_{v,w} \) for \( v \nmid p \). Assume that \( v \nmid p \). Let \( \Sigma \) be the finite set of primes of \( F \) introduced in \([\text{2.1}]\). We will distinguish two cases: \( v \notin \Sigma \) and \( v \in \Sigma \).

**Lemma 5.1.** If \( v \notin \Sigma \), then \( \rho_{v,w} \) is injective.

**Proof.** By Lemma \([\text{3.3}]\), the kernel of \( \rho_{v,w} \) is the kernel of the restriction map
\[
(5.4) \quad \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \to \frac{H^1(I_v, A)}{H^1_\text{ur}(I_v, A)} = \text{Hom}(I_v, A)
\]
With self-explaining notation, there are injections
\[
\frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \to \frac{H^1(I_v, A)}{H^1_\text{ur}(I_v, A)} = \text{Hom}(I_v, A)
\]
and
\[
\frac{H^1(F_{\infty,w}, A)}{H^1_\text{ur}(F_{\infty,w}, A)} \to \frac{H^1(I_{\infty,w}, A)}{H^1_\text{ur}(I_{\infty,w}, A)} = \text{Hom}(I_{\infty,w}, A),
\]
where the equalities are a consequence of the fact that \( A \) is unramified at \( v \). Therefore, the kernel of \([\text{3.3}]\) is contained in the kernel of the natural map
\[
(5.5) \quad \text{Hom}(I_v, A) \to \text{Hom}(I_{\infty,w}, A).
\]
Since \( v \) is unramified, \( F_{\infty,w} \) is a \( \mathbb{Z}_p \)-subextension of the unramified \( \mathbb{Z}_p \)-extension of \( F_v \). Thus, \( I_v = I_{\infty,w} \) and \([\text{5.5}]\) is injective, which concludes the proof.

It follows from Lemma \([\text{5.1}]\) that \( \ker(\rho) \) is the subgroup of \( \mathcal{P}_\text{BK}(A/F) \) consisting of elements \( s \) such that \( \rho_{v,w}(s) = 0 \) for all \( w \mid v \) with \( v \in \Sigma \). Thus, upon setting
\[
\mathcal{P}_\text{BK}^\Sigma(A/F) := \prod_{v \in \Sigma} \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)}
\]
and
\[
\mathcal{P}_\text{BK}^\Sigma(A/\infty) := \prod_{\substack{w \mid p \\ v \in \Sigma, v \mid p}} \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ur}(F_{\infty,w}, A)} \times \prod_{\substack{w \mid p \\ v \notin \Sigma}} \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ord}(F_{\infty,w}, A)},
\]
it follows that \( \ker(\rho) \subset \mathcal{P}_\text{BK}^\Sigma(A/F) \); more precisely, \( \ker(\rho) \) coincides with the kernel of the restriction map
\[
(5.6) \quad g : \mathcal{P}_\text{BK}^\Sigma(A/F) \to \mathcal{P}_\text{GI}^\Sigma(A/\infty)^\Gamma.
\]
We conclude that
\[
(5.7) \quad \# \ker(\rho) = \prod_{\substack{w \mid p \\ v \in \Sigma}} \# \ker(\rho_{v,w}).
\]

**Lemma 5.2.** If \( v \in \Sigma \) and \( v \nmid p \), then \( \# \ker(\rho_{v,w}) = c_v(A) \).

**Proof.** The map \( \rho_{v,w} \) splits as a composition
\[
\rho_{v,w} : \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \to \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \to \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ur}(F_{\infty,w}, A)} \to \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ur}(F_{\infty,w}, A)}.
\]
Thus, since \( \#(H^1_\text{ur}(F_v, A)/H^1_\text{ord}(F_v, A)) = c_v(A) \), it suffices to show that the map
\[
(5.8) \quad \frac{H^1(F_v, A)}{H^1_\text{ur}(F_v, A)} \to \frac{H^1(F_{\infty,w}, A)}{H^1_\text{ur}(F_{\infty,w}, A)}
\]
is injective. Let \( F_v^{\text{ur}} \) be the maximal unramified extension of \( F_v \) and set \( G_v^{\text{ur}} := \text{Gal}(F_v^{\text{ur}}/F_v) \). There is a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1_{\text{ur}}(F_v, A) & \rightarrow & H^1(F_v, A) & \rightarrow H^1(I_v, A)^{G_v^{\text{ur}}} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^1_{\text{ur}}(F_{\infty,w}, A) & \rightarrow & H^1(F_{\infty,w}, A) & \rightarrow H^1(I_{\infty,w}, A) & 
\end{array}
\]

with exact rows. Notice that, since \( G_v^{\text{ur}} \simeq \prod_{\ell} \mathbb{Z}_\ell \), the surjectivity of the right non-trivial map in the top row of (5.9) stems from the vanishing of \( H^2(G_v^{\text{ur}}, A) \) ([22] Proposition 1.4.10, (2)). It follows that the kernel of the map in (5.3) can be identified with the kernel of the rightmost vertical map in (5.9), which is isomorphic (by the inflation-restriction exact sequence) to \( H^1(I_v/I_{\infty,w}, A^{1_{\infty,w}})^{G_v^{\text{ur}}} \). Since \( F_{\infty,w}/F_v \) is unramified if \( v \nmid p \), we have \( I_v = I_{\infty,w} \), so \( H^1(I_v/I_{\infty,w}, A^{1_{\infty,w}}) = 0 \). It follows that (5.3) is injective, which completes the proof. □

5.1.2. The map \( r_{v,w} \) for \( v \mid p \). Now we study the local conditions at a prime \( v \mid p \). Recall that, by (3.3), we have \( H^1_{\text{ord}}(F_v, V) = H^1(F_v, V) \). Moreover, as explained in (3.3.3) there is an inclusion \( H^1_{\text{ord}}(F_v, A) \subset H^1_{\text{ord}}(F_v, A) \).

The lemma below, whose proof uses in a crucial way the triviality of local invariants from part (6a) in Assumption 2.1, forces the terms corresponding to \( \tilde{E}_v(F_v) \) in (1.1) to be trivial for all primes \( v \) of \( F \) above \( p \).

Lemma 5.3. If \( v \mid p \), then \( r_{v,w} \) is injective.

Proof. To begin with, note that the map \( r_{v,w} \) can be written as the composition

\[
r_{v,w} : H^1(F_v, A)_{\text{ord}} \rightarrow H^1(F_v, A)_{\text{ord}} \rightarrow H^1(F_{\infty,w}, A)_{\text{ord}} 
\]

where the first arrow is induced by the identity map of \( H^1(F_v, A) \) and the second by the obvious (restriction) map \( H^1(F_v, A) \rightarrow H^1(F_{\infty,w}, A) \). Our strategy is to prove that both maps appearing in (5.10) are injective. As we shall see, the proof of the injectivity of the second map is, thanks to Assumption 2.1, straightforward, while dealing with the first map is much more delicate.

We first take care of the second map in (5.10). Let us consider the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1_{\text{ord}}(F_v, A) & \rightarrow & H^1(F_v, A) & \rightarrow H^1(I_v, A^-) & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^1_{\text{ord}}(F_{\infty,w}, A) & \rightarrow & H^1(F_{\infty,w}, A) & \rightarrow H^1(I_{\infty,w}, A^-). & 
\end{array}
\]

The kernel of the rightmost vertical arrow is isomorphic (by the inflation-restriction exact sequence) to \( H^1(I_v/I_{\infty,w}, (A^-)^{1_{\infty,w}}) \). By Assumption 2.1 we know that \( (A^-)^{1_{\infty,w}} = 0 \) (cf. part (1) of Remark 2.2), so the second arrow in (5.10) is injective. It follows that \( \ker(r_{v,w}) \) is just the kernel of the first map in (5.10). We tackle the study of this map by adapting arguments from [3] and [18, Proposition 4.2]. Of course, proving that the above-mentioned map is injective amounts to showing that \( H^1_{\text{ord}}(F_v, A) = H^1_{\text{ord}}(F_v, A) \).
Let us consider the commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \rightarrow & H^1(F_v, V) & \rightarrow & H^1(F_v, V) & \rightarrow & H^1(I_v, V)_{\text{tors}} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1(I_v, T)_{\text{tors}} & \rightarrow & H^1(I_v, T)_{\text{tors}} & \rightarrow & H^1(I_v, T)_{\text{tors}} \\
\end{array}
\]

The map \( b \) splits as a composition

\[
b : H^1(F_v, V) \xrightarrow{b'} H^1(F_v, V^-) \xrightarrow{b''} H^1(I_v, V^-)_{G^ur_v}.
\]

The cokernel of \( b' \) injects into \( H^2(F_v, V^+) \), which is isomorphic to \( H^0(F_v, V^-)^{(1)} \) by local Tate duality. Since the \( H^0(F_v, T^-)^{(1)} \) is trivial due to Assumption 2.1, the latter group is trivial. On the other hand, the cokernel of \( b'' \) injects into \( H^2(G^ur_v, (V^-)^I_v) \), which is trivial because \( V^- \) has trivial \( I_v \)-invariants by Assumption 2.2. Therefore, the map \( b \) is surjective. Moreover, local Tate duality identifies \( H^2(F_v, T) \) with \( H^0(F_v, T^\vee(1)) \), which is trivial by Assumption 2.1 (cf. part (2) of Remark 2.2), hence \( H^2(F_v, T) = 0 \). The snake lemma then gives an isomorphism \( \text{coker}(a) \cong \text{coker}(g) \). Now we study the cokernel of \( a \). Let us consider the commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \rightarrow & H^1(F_v, T)_{\text{tors}} & \rightarrow & H^1(F_v, T) & \rightarrow & H^1(F_v, T)_{\text{tors}} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1(I_v, T^-)_{G^ur_v} & \rightarrow & H^1(I_v, T^-)_{G^ur_v} & \rightarrow & H^1(I_v, T^-)_{G^ur_v} & \rightarrow & 0 \\
\end{array}
\]

The cokernel of \( e \) is a subgroup of \( H^1(G^ur_v, H^1(I_v, T^-)_{G^ur_v}) \). The group \( H^1(I_v, T^-)_{G^ur_v} \) is (isomorphic to) the largest cotorsion quotient of \( H^0(I_v, A^-) \), which is trivial by part (6b) of Assumption 2.1. Thus, \( H^1(G^ur_v, H^1(I_v, T^-)_{G^ur_v}) \) is trivial too, and so is \( \text{coker}(e) \). It follows that the natural map \( \text{coker}(h) \rightarrow \text{coker}(a) \) is surjective. On the other hand, the map \( h \) can be written as

\[
h : H^1(F_v, T) \xrightarrow{h'} H^1(F_v, T^-) \xrightarrow{h''} H^1(I_v, T^-)_{G^ur_v}.
\]

The cokernel of \( h' \) injects into \( H^2(F_v, T^+) \), whose dual \( H^0(F_v, (T^-)^I_v) \) is trivial thanks to part (6b) of Assumption 2.1. Moreover, the cokernel of \( h'' \) injects into \( H^2(G^ur_v, (T^-)^I_v) \), which is trivial because \( (T^-)^I_v = 0 \) by Assumption 2.2 (cf. part (1) of Remark 2.2). Thus, the map \( h \) is surjective, and we conclude that the cokernel of \( a \) is trivial. Since \( \text{coker}(a) \cong \text{coker}(g) \), it follows that the cokernel of \( g \) is trivial as well. This means that \( g \) is surjective, i.e., \( H^1(F_v, A) = H^1(I_v, A) \), as was to be shown. \( \square \)
If follows from a combination of equality (5.7) and Lemmas 5.2 and 5.3 that
\[
\# \ker(r) = \prod_{v \in \Sigma} c_v(A).
\] (5.11)

5.2. Surjectivity of restriction. Denote by \(F_\Sigma\) the maximal extension of \(F\) unramified outside \(\Sigma\), and for any \(\text{Gal}(F_\Sigma/F)\)-module \(M\) set \(H^1(F_\Sigma/F, M) := H^1(\text{Gal}(F_\Sigma/F), M)\). The symbol \(H^1(F_\Sigma/F, M)\) will have an analogous meaning. It follows from Lemma 5.1 that
\[
\text{Sel}(A/F) = \ker \left( H^1(F_\Sigma/F, A) \xrightarrow{\delta_\Sigma} P_{\text{BK}}^\Sigma(A/F) \right)
\]
and
\[
\text{Sel}(G,F/F_\infty) = \ker \left( H^1(F_\Sigma/F_\infty, A) \xrightarrow{\delta_{\infty,\Sigma}} P_G^\Sigma(A/F_\infty) \right),
\]
where \(\delta_\Sigma\) and \(\delta_{\infty,\Sigma}\) are the restriction maps. For notational convenience, if \(M\) is a \(G_F\)-module and \(F'\) is an algebraic extension of \(F\), then we set \(M(F') := H^0(F', M)\).

The following lemma, whose proof uses (a global consequence of) part (6a) of Assumption 2.1, implies that the term corresponding to \(E(F)_{p}\) in (1.1) is trivial.

**Lemma 5.4.** \(A(F_\infty) = 0\).

**Proof.** Since \(A = \varprojlim_p A[\pi^n]\), it suffices to show that \(A[\pi](F_\infty) = 0\). Recall that \(A[\pi]\) is a finite-dimensional vector space over the finite field \(E\) of characteristic \(p\); in particular, \(A[\pi]\) is finite. It follows that if \(x \in A[\pi]\), then the stabilizer of \(x\) in \(G_F\) is a closed and finite index subgroup of \(G_F\), so it is equal to \(G_{F'}\) for a suitable finite extension \(F'\) of \(F\). If, moreover, \(x \in A[\pi](F_\infty)\), then \(F' \subset F_\infty\), hence \(F'' = F_n\) for a suitable \(n \geq 0\) and \(x \in A[\pi](F_n)\). Thus, we are reduced to showing that \(A[\pi](F_n) = 0\) for all \(n \geq 0\). Since \(A[\pi](F) = 0\) by part (6a) of Assumption 2.1 and \([F_{m+1}: F_m] = p\) for all \(m \geq 0\), the triviality of \(A[\pi](F_n)\) for all \(n \geq 0\) follows from (23) Proposition 26) by induction on \(n\).

**Remark 5.5.** If we imposed additional assumptions on the Galois representation \(\rho_V\), then we could avoid using part (6a) of Assumption 2.1 to deduce that \(A(F_\infty) = 0\). For example, if we required the reduction of \(\rho_F\) modulo \(p\) to be irreducible with non-solvable image (which in the case of non-CM modular forms is true for all but finitely many \(p\), cf. [13, Lemma 3.9], [20, §2]), then we could show that \(A(F_\infty)\) is trivial by proceeding as in the proof of [14, Lemma 2.4] (see also [13, Lemma 3.10, (2)]). However, the local vanishing from part (6a) of Assumption 2.1 plays a much more delicate role in the proof of Lemma 5.3 which is the reason why we decided to assume this condition right from the outset.

**Lemma 5.6.** The map \(\delta_\Sigma\) is surjective.

**Proof.** We apply some results from [8, Section 4], so we first recall the setting of [8]. Let \(M\) be a \(G_F\)-module isomorphic to \((Q_p/Z_p)^d\) for some integer \(d \geq 1\). As in [8], define \(T^* := \text{Hom}_{Z_p}(M, \mu_{p^\infty}) = M^*(1), V^* := T^* \otimes_{Z_p} Q_p\) and \(M^* := V^*/T^* \otimes_{Z_p} Q_p/Z_p\). Let
\[
H^1(F_v, M) \times H^1(F_v, T^*) \longrightarrow Q_p/Z_p
\]
be the local Tate pairing, which is perfect. Suppose that for each \(v \in \Sigma\) we have a divisible subgroup \(L_v\) of \(H^1(F_v, M)\), then consider the Selmer group
\[
\text{Sel}(M/F) := \ker \left( H^1(F_\Sigma/F, M) \xrightarrow{\gamma} \prod_{v \in \Sigma} \frac{H^1(F_v, M)}{L_v} \right).
\]
Furthermore, denote by \(U_v^*\) the orthogonal complement of \(L_v\) under the local Tate pairing. Write \(L_v^*\) for the image in \(H^1(F_v, M^*)\) of the \(Q_p\)-subspace of \(H^1(F_v, V^*)\) generated by the
image of $U_v^*$ under the natural map $H^1(F_v, T^*) \to H^1(F_v, V^*)$. Set

$$\text{Sel}(M^*/F) := \ker \left( H^1(F_{\Sigma}/F, M^*) \to \prod_{v \in \Sigma} \frac{H^1(F_v, M^*)}{L_v^*} \right).$$

The arguments in [8, pp. 100–101] show that if $\text{Sel}(M^*/F)$ is finite and $H^0(F_{\Sigma}/F, M^*)$ is trivial, then $\gamma$ is surjective.

Now we return to our setting, with $M = A$, $T$ and $V$ as in Assumption 2.1. To apply the results explained above, we recall that, by Assumption 2.1, the image of $T$ in $V^* = V^\Sigma(1)$ under the isomorphism $V \simeq V^*$ is homothetic to $T^*$. Furthermore, the Bloch–Kato conditions $H^1_{\text{gr}}(F_v, A)$ and $H^1_{\text{gr}}(F_v, A^*)$ are orthogonal under local Tate duality ([3, Proposition 3.8]). Thus, $\text{Sel}_{BK}(A/F) \simeq \text{Sel}_{BK}(A^*/F)$, so the finiteness of $\text{Sel}_{BK}(A/F)$ is equivalent to that of $\text{Sel}_{BK}(A^*/F)$. To conclude the proof we only need to check that $H^0(F_{\Sigma}/F, A^*)$ is trivial. Since $T$ and $T^*$ are homothetic, and $A = V/T$, $A^* = V^*/T^*$, it is enough to show that $H^0(F_{\Sigma}/F, A)$ is trivial. Since $A$ is unramified outside $\Sigma$, we have $A = H^0(F_{\Sigma}/F, A)$, so $H^0(F, A) = H^0(F_{\Sigma}/F, A)$. It follows from Lemma 5.4 that $H^0(F_{\Sigma}/F_{\infty}, A) = 0$ and, a fortiori, $H^0(F_{\Sigma}/F, A) = 0$. \hfill \Box

5.3. An application of the snake lemma. It follows from Lemma 5.6 that there is a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \text{Sel}_{BK}(A/F) & \to & H^1(F_{\Sigma}/F, A) & \to \mathcal{P}_{BK}^{\Sigma}(A/F) & 0 \\
& & s & \downarrow h & \delta_{\Sigma} & \downarrow g & \\
0 & \to & S^\Gamma & \to & H^1(F_{\Sigma}/F_{\infty}, A^\Gamma) & \to \mathcal{P}_{G_{\text{gr}}}^{\Sigma}(A/F_{\infty})^\Gamma & 0
\end{array}
$$

with exact rows, where $s$ is as in (5.1), $h$ is restriction, $g$ is as in (5.6) and $\delta_{\infty, \Sigma}^\Gamma$ is the map induced by $\delta_{\infty, \Sigma}$ between $\Gamma$-invariants.

Lemma 5.7. The map $h$ is an isomorphism and the map $s$ is injective.

Proof. By inflation-restriction, there is an exact sequence

$$0 \to H^1(\Gamma, A(F_{\infty})) \to H^1(F_{\Sigma}/F, A) \xrightarrow{h} H^1(F_{\Sigma}/F_{\infty}, A^\Gamma) \to H^2(\Gamma, A(F_{\infty})), $$

and the lemma follows from Lemma 5.4 and the injection $\ker(s) \hookrightarrow \ker(h)$. \hfill \Box

Recall from (4.11) that $S^\Gamma$ is finite, so $\ker(s)$ is finite.

Lemma 5.8. $\#S^\Gamma = \# \text{Sel}_{BK}(A/F) \cdot \# \ker(g)$.

Proof. From (5.12), the snake lemma gives an exact sequence

$$\ker(h) \to \ker(g) \to \ker(s) \to \ker(h).$$

On the other hand, by Lemma 5.7 both $\ker(h)$ and $\ker(g)$ are trivial, so (5.13) yields an isomorphism $\ker(g) \simeq \ker(s)$. In particular, $\# \ker(s) = \# \ker(g)$, and the searched-for equality follows immediately from (5.2). \hfill \Box

The proposition below provides a crucial step towards our main result.

Proposition 5.9. $\#S^\Gamma = \# \text{Sel}_{BK}(A/F) \cdot \prod_{\nu \in \Sigma} c_v(A)$.

Proof. Combine Lemma 5.8 the equality $\ker(r) = \ker(g)$ and (5.11). \hfill \Box
5.4. Finite index \(\Lambda\)-submodules. Our present goal is to generalize [8 Proposition 4.9], which shows that if \(E/F\) is an elliptic curve, then \(H^1(F_2/F,E[p^\infty])\) does not have proper \(\Lambda\)-submodules of finite index. As a consequence, we will see that \(S_1 = 0\).

In order to prove this non-existence result, we need four lemmas.

**Lemma 5.10.** The map \(g\) is surjective.

*Proof.* Using the fact that \(\Gamma\) has cohomological dimension 1, one can check that each of the local components defining \(g\) is surjective. \(\square\)

The second lemma we are interested in is a generalization of [8 Lemma 2.3].

**Lemma 5.11.** Let \(v_n\) be the prime of \(F_n\) above \(p\), denote by \(F_{n,v_n}\) the completion of \(F_n\) at \(v_n\) and set \(G_{F_{n,v_n}} := \text{Gal}(\hat{Q}_p/F_{n,v_n})\). Let \(\psi : G_{F_n} \to \mathcal{O}^\times\) be a character and write \((K/\mathcal{O})(\psi)\) for the group \(K/\mathcal{O}\) with \(G_{F_n}\)-action given by \(\psi\). If \(\psi|_{G_{F_{n,v_n}}}\) is non-trivial and does not coincide with the cyclotomic character, then the \(\mathcal{O}\)-corank of \(H^1(F_{n,v_n},(K/\mathcal{O})(\psi))\) is \([F_{n,v_n} : \hat{Q}_p]\).

*Proof.* To simplify the notation, put \(C := (K/\mathcal{O})(\psi)\), \(M := F_{n,v_n}\) and \(G_M := \text{Gal}(\hat{Q}_p/M)\). Let \(T(C)\) be the \(p\)-adic Tate module of \(C\), set \(V(C) := T(C) \otimes_{\mathcal{O}} K\) and define

\[
\begin{align*}
h_i := \dim_{\hat{Q}_p}(H^i(M,V(C)))
\end{align*}
\]

for \(i = 0,1,2\). Since \(\dim_{\hat{Q}_p}(V(C)) = [K : \hat{Q}_p]\), the Euler characteristic of \(V(C)\) over \(M\) is given by the formula

\[(5.14) \quad h_0 - h_1 + h_2 = -[M : \hat{Q}_p] \cdot [K : \hat{Q}_p].\]

Let \(K(\psi)\) denote \(K\) (viewed as a \(K\)-vector space over itself) equipped with the \(G_{F_n}\)-action induced by \(\psi\). There is an isomorphism \(V(C) \simeq K(\psi)\), so \(H^0(M,V(C)) = 0\), where the restriction of \(\psi\) to \(G_M\) is non-trivial by assumption. Let \(Q_p(1)\) be the Tate twist of \(Q_p\) and define

\[
V(C)^* := \text{Hom}_{\hat{Q}_p}(V(C),Q_p(1)).
\]

The \(H^0(M,V(C)^*) = 0\) because \(V(C) \simeq K(\psi)\) and \(\psi\) does not coincide with the cyclotomic character. Poitou–Tate duality implies that \(H^2(M,V(C)^*) = (K/\mathcal{O})(\psi)^{-1}\). Since \(H^0(M,V(C)^*) = 0\), we deduce that \(H^2(M,V(C)) = 0\). Therefore, \(h_0 = h_2 = 0\) and it follows from (5.14) that \(h_1 = [M : \hat{Q}_p] : [K : \hat{Q}_p]\).

We want to prove that the \(\mathbb{Z}_p\)-corank of \(H^1(M,C)\) is equal to \(h_1\). This completes the proof of the lemma because, by what we have just shown, the \(\mathbb{Z}_p\)-corank of \(H^1(M,C)\) is then equal to \([M : \hat{Q}_p] : [K : \hat{Q}_p]\) and so its \(\mathcal{O}\)-corank must be equal to \([M : \hat{Q}_p]\). To prove this claim, note that the short exact sequence

\[
0 \to T(C) \to V(C) \to C \to 0
\]

induces an exact sequence

\[
H^1(M,T(C)) \to H^1(M,V(C)) \to H^1(M,C) \to H^2(M,T(C)).
\]

Set \((T(C))^* := \text{Hom}(T(C),\mathbb{Z}_p)\). By Poitou–Tate duality, \(H^2(M,T(C))^*\) is identified with \(H^0(M,T(C)^*)\). Since the action of \(G_M\) on \(T(C)\) is via \(\psi\), we get an isomorphism

\[
H^0(M,T(C)^*) \simeq H^0(M,(K/\mathcal{O})(\chi_{\text{cyc}}\psi^{-1})),
\]

where \((K/\mathcal{O})(\chi_{\text{cyc}}\psi^{-1}) = K/\mathcal{O}\) with Galois action twisted by \(\chi_{\text{cyc}}\psi^{-1}\). On the other hand, \(\psi \neq \chi_{\text{cyc}}\) by assumption, so \(H^0(M,T(C)^*) = 0\). Thus, \(H^2(M,V(C))\) is also trivial. Now, by [26 Proposition 2.3], the image of \(H^1(M,T(C))\) in \(H^1(M,V(C))\) is a lattice, say \(P\), so the quotient of \((M,V(C))\) by \(P\), which is just \(H^1(M,C)\) because \(H^2(M,T(C)) = 0\), is isomorphic to \((Q_p/\mathbb{Z}_p)^{h_1}\). Taking \(\mathbb{Z}_p\)-duals shows that the \(\mathbb{Z}_p\)-corank of \(H^1(M,C)\) is \(h_1\), as was to be proved. \(\square\)
Lemma 5.12. If \( v \nmid p \), then \( H^0(F_v, A) \) is finite.

Proof. If \( H^0(F_v, A) \) is not finite, then there exists a \( p \)-divisible subgroup \( B \subset A \) that is fixed by \( G_{F_v} \). Let us choose an element \( b \in B[p^M] \setminus \{0\} \) for some integer \( M \geq 1 \). By part (4) of Assumption 2.1, for every integer \( m \geq 1 \) there is a skew-symmetric, \( G_p \)-equivariant and non-degenerate pairing

\[
\langle \cdot, \cdot \rangle_m : A[p^m] \times A[p^m] \rightarrow \mathbb{Z}_p.
\]

Choose \( c \in A[p^M] \) such that \( (c, b)_M = \zeta_{p^r} \) for a non-trivial, primitive \( p^r \)-th root of unity \( \zeta_{p^r} \) (so \( r \leq M \)). Using the fact that the representation \( \rho_T \) is continuous and \( T/p^mT \approx A[p^m] \) for all \( m \geq 1 \), there is a finite extension \( H/F_v \) with \( b, c \in H^0(H, A) \). Since \( B \) is divisible, for every \( N \geq M \) we can pick an element \( b_N \in B[p^N] \) with the property that \( p^{N-M}b_N = b \). Then \( (c, b_N)_N = \zeta_{p^r} \) for some \( p^N \)-th root of unity \( \zeta_{p^r} \) such that \( \zeta_{p^N} = \zeta_{p^r} \); in particular, \( \zeta_{p^N} \) is a primitive \( p^{N-M+r} \)-th root of unity. By assumption, \( b_N \in H^0(H, A) \), and then the Galois-equi-variance of \( (\cdot, \cdot)_N \) ensures that

\[
(c, b_N)_N = \sigma(\zeta_{p^r})
\]

for all \( \sigma \in G_H \). Since \( N \) is arbitrary and \( \zeta_{p^N} \) is a primitive \( p^{N-M+r} \)-th root of unity, this implies that \( H \) contains the cyclotomic \( \mathbb{Z}_p \)-extension of \( F_v \), which is impossible because the extension \( H/F_v \) is finite. \( \square \)

The proof of the next lemma, which follows from [7] Propositions 1 and 2, proceeds along the lines of the arguments in [8], p. 94

Lemma 5.13. The \( \Lambda \)-corank of \( \mathcal{P}^\Sigma_{Gr}(A/F_\infty) \) is \( (r - r^+, [F : Q]) \).

Proof. Let \( \mathcal{P}^\Sigma_{Gr}(A/F_\infty) \) denote the factor in \( \mathcal{P}_{Gr}^\Sigma(A/F_\infty) \) corresponding to the place \( v \). For \( v \nmid p \), \( \mathcal{P}^\Sigma_{Gr}(A/F) \) is \( \Lambda \)-cotorrison by [2] Proposition 1. This result is clear if the prime \( v \) is archimedean, as in this case this module has exponent 2. For finite primes \( v \nmid p \), \( \mathcal{P}^\Sigma_{Gr}(A/F) \) is finite; this is a consequence of [7] Proposition 2 combined with Lemma 5.12 and the isomorphism \( A \approx A^{\ell}(1) \). Since the map \( \mathcal{P}^\Sigma_{Gr}(A/F) \rightarrow \mathcal{P}^\Sigma_{Gr}(A/F_\infty) \) is surjective by Lemma 5.10, \( \mathcal{P}^\Sigma_{Gr}(A/F_\infty) \) is finite as well, which implies that \( \mathcal{P}^\Sigma_{Gr}(A/F_\infty) \) is \( \Lambda \)-cotorrison; here we are using [8] (2), p. 79 and the fact that all the primes in \( \Sigma \) are finitely decomposed in \( F_\infty \).

Now let \( v \mid p \) be a place of \( F_\infty \) and let \( \Gamma_v \) be the corresponding decomposition group. By [7] Proposition 1, the \( \mathbb{Z}_p[\Gamma_v] \)-corank of \( H^1(F_{\infty,v}, A) \) is \( [F_v : Q_p] \cdot [K : Q_p] \) (to apply this result, note that \( K/O \simeq (Q_p/\mathbb{Z}_p)^{[K:Q_p]} \) as groups). Thus, the \( \mathbb{O}[\Gamma_v] \)-corank of \( H^1(F_{\infty,v}, \mathbb{A}) \) is \( r \cdot [F_v : Q_p] \cdot [K : Q_p] \). There is an isomorphism \( \mathbb{A}^+ \simeq (K/O)^{r^+} \) of \( \mathbb{O} \)-modules and the action of \( G_{F_v} \) on \( \mathbb{A}^+ \) is via the unramified characters \( \eta_1, \ldots, \eta_{r^+} \), which are non-trivial thanks to condition (6) in Assumption 2.1. If \( v_p \) denotes the prime of \( F_n \) above \( p \), then Lemma 5.11 guarantees that the \( \mathcal{O} \)-corank of \( H^1(F_{n,v_n}, A^+) \) is \( r^+ \cdot [F_{n,v_n} : Q_p] \). This implies that \( H^1(F_{\infty,v}, A^+) \) has \( \mathcal{O}[\Gamma_v] \)-corank \( r^+ \cdot [F_v : Q_p] \). Consequently, the \( \mathcal{O}[\Gamma_v] \)-corank of the quotient \( H^1(F_{\infty,v}, A)/H^1_{\text{tor}}(F_{\infty,v}, A) \) is \( (r - r^+) \cdot [F_v : Q_p] \). Finally, the well-known formula

\[
\sum_{v \mid p} [F_v : Q_p] = [F : Q]
\]

(see, e.g., [12] p. 39, Corollary 1) concludes the proof. \( \square \)

Now we can prove a result establishing, in particular, the non-existence of proper \( \Lambda \)-submodules of finite index of \( H^1(F_\infty/F_\infty, A) \).

Proposition 5.14. The \( \Lambda \)-module \( H^1(F_\Sigma/F_\infty, A) \) is cofinitely generated of rank \( r^- \cdot [F : Q] \) and has no proper \( \Lambda \)-submodules of finite index.
and before, a topological generator \(H\).

Proof. Since \(\text{Sel}_{G_1}(A/F_\infty)\) is \(\Lambda\)-cotorsion by Proposition 4.1 it follows from Lemma 5.13 that the \(\Lambda\)-corank of \(H^1(F_\Sigma/F_\infty, A)\) is at most \(r^- \cdot [F : Q]\). By [7, Proposition 3 and (34)], the \(\Lambda\)-corank of \(H^1(F_\Sigma/F_\infty, A)\) is \(r^- \cdot [F : Q]\) and the \(\Lambda\)-corank of \(H^2(F_\Sigma/F_\infty, A)\) is 0. Since \(H^2(F_\Sigma/F_\infty, A)\) is \(\Lambda\)-cocoercive by [7, Proposition 4], we deduce that \(H^2(F_\Sigma/F_\infty, A) = 0\). The lemma is a consequence of [7, Proposition 5].

5.5. \textbf{Triviality of coinvariants.} Recall that \(S = \text{Sel}_{G_1}(A/F_\infty)\). Our goal here is to prove that \(S^\Gamma = 0\). First of all, we need a lemma on the interaction between Pontryagin duals and torsion submodules, which is valid in a slightly more general context.

Lemma 5.15. Let \(N\) be a compact \(\Lambda\)-algebra, let \(I\) be an ideal of \(\Lambda\) and write \(N[I]\) for the \(I\)-torsion submodule of \(N\). There is an isomorphism of \(\Lambda\)-modules

\[
N^\vee/I N^\vee \cong N[I]^\vee.
\]

Proof. Write \(I = (x_1, \ldots, x_n)\) and consider the map

\[
\xi : N \rightarrow \prod_{i=1}^n x_i N, \quad n \mapsto (x_i n)_{i=1,\ldots,n},
\]

whose kernel is equal to \(N[I]\). If \(i : N[I] \hookrightarrow N\) denotes inclusion then Pontryagin duality gives an exact sequence of \(\Lambda\)-modules

\[
\left(\prod_{i=1}^n x_i N\right)^\vee \xrightarrow{\xi^\vee} N^\vee \xrightarrow{i^\vee} N[I]^\vee \rightarrow 0;
\]

here the surjectivity of \(i^\vee\) is a consequence of \(N[I]\) being closed in \(N\), hence compact. On the other hand, sending \((\varphi_1, \ldots, \varphi_n)\) to \(\prod_i \varphi_i\) gives an isomorphism between \(\prod_i (x_i N)^\vee\) and \(\left(\prod_{i=1}^n x_i N\right)^\vee\), so we can rewrite (5.15) as

\[
\prod_{i=1}^n (x_i N)^\vee \xrightarrow{\xi^\vee} N^\vee \xrightarrow{i^\vee} N[I]^\vee \rightarrow 0.
\]

In light of (5.16), we want to check that \(\text{im}(\xi^\vee) = IN^\vee\). First of all, let \(\varphi = \sum_{i=1}^n x_i \varphi_i \in IN^\vee\), with \(\varphi_i \in N^\vee\) for all \(i = 1, \ldots, n\). Then \(\varphi = \xi^\vee((\varphi_1|x_1 N, \ldots, \varphi_n|x_n N))\), which shows that \(\varphi \in \text{im}(\xi^\vee)\). Conversely, let \(\varphi \in \text{im}(\xi^\vee)\); by definition, for every \(i = 1, \ldots, n\) there exists \(\varphi_i \in (x_i N)^\vee\) such that \(\varphi = \xi^\vee((\varphi_1, \ldots, \varphi_n))\). For every \(i\), the \(\Lambda\)-module \(x_i N\) is compact because \(N\) is, hence the inclusion \(x_i N \hookrightarrow N\) gives a surjection \(N^\vee \rightarrow (x_i N)^\vee\). Now for every \(i = 1, \ldots, n\) choose a lift \(\psi_i \in N^\vee\) of \(\varphi_i\). It follows that \(\varphi = \sum_{i=1}^n x_i \psi_i \in IN^\vee\), and the proof is complete.

The vanishing of \(S^\Gamma\) will be a consequence of the following result.

Proposition 5.16. \(H^1(F_\Sigma/F_\infty, A)^{\Gamma} = 0\).

Proof. By Proposition 5.14 it suffices to show that \(H^1(F_\Sigma/F_\infty, A)^{\Gamma}\) is finite. Let us denote by \(H^1(F_\Sigma/F_\infty, A)_{\Lambda\text{-div}}\) the maximal \(\Lambda\)-divisible submodule of \(H^1(F_\Sigma/F_\infty, A)\) and define the \(\Lambda\)-module \(Q\) via the short exact sequence

\[
0 \rightarrow H^1(F_\Sigma/F_\infty, A)_{\Lambda\text{-div}} \rightarrow H^1(F_\Sigma/F_\infty, A) \rightarrow Q \rightarrow 0.
\]

The Pontryagin dual \(Q^\vee\) of \(Q\) is the torsion \(\Lambda\)-submodule of the Pontryagin dual \(Y\) of \(H^1(F_\Sigma/F_\infty, A)\); it follows from Proposition 5.14 that there is a pseudoisomorphism

\[
Y \sim \Lambda^{-\cdot [F : Q]} \oplus Q^\vee,
\]

and \(Q\) is a cofinitely generated cotorsion \(\Lambda\)-module. Set \(M := H^1(F_\Sigma/F_\infty, A)\) and fix, as before, a topological generator \(\gamma\) of \(\Gamma\). Combining the vanishing of \(H^2(\Gamma, N)\) for all torsion
discrete \( \Gamma \)-modules \( N \) (see, e.g., [9, Corollary 4.27]) with the identifications \( H^1(\Gamma, N) = N/(\gamma - 1)N = N_{\Gamma} \) for every \( \Gamma \)-module \( N \), exact sequence (5.17) yields an exact sequence
\[
0 \rightarrow M_{A, \text{div}}^\Gamma \rightarrow M^\Gamma \rightarrow Q^\Gamma \rightarrow (M_{A, \text{div}})^\Gamma \rightarrow M_{\Gamma} \rightarrow Q_{\Gamma} \rightarrow 0
\]
in Galois cohomology. Since \((M_{A, \text{div}})^\Gamma = 0\) because \(M_{A, \text{div}}\) is \( \Lambda \)-divisible, it follows from (5.19) that \( M_{\Gamma} \cong Q_{\Gamma} \). Therefore, it is enough to show that \( Q_{\Gamma} \) is finite. Furthermore, since \( Q \) is a finitely generated cotorsion \( \Lambda \)-module, the exact sequence
\[
0 \rightarrow Q^\Gamma \rightarrow Q \xrightarrow{(\gamma - 1)} Q \rightarrow Q_{\Gamma} \rightarrow 0
\]
shows that \( Q^\Gamma \) and \( Q_{\Gamma} \) have the same \( \mathcal{O} \)-corank. Thus, we are reduced to proving that the \( \mathcal{O} \)-corank of \( Q^\Gamma \) is 0.

Lemma 5.13 implies that the \( \mathcal{O} \)-corank of \( \mathcal{P}^\Sigma_{\text{Gr}}(A/F_\infty)^\Gamma \) is \( r^- \cdot [F : Q] \). By Lemma 5.10 and the finiteness of \( \ker(g) \), the \( \mathcal{O} \)-corank of \( \mathcal{P}^\Sigma_{\text{BK}}(A/F) \) is \( r^- \cdot \cdot [F : Q] \) as well. In addition, by Lemma 5.6, the surjection \( \delta_\Sigma : H^1(F_\Sigma/F, A) \rightarrow \mathcal{P}^\Sigma_{\text{BK}}(A/F) \) has the finite group \( \text{Sel}_{\text{BK}}(A/F) \) as its kernel, hence
\[
\text{corank}_\mathcal{O}(H^1(F_\Sigma/F, A)) = r^- \cdot [F : Q].
\]

Now consider the inflation-restriction exact sequence
\[
0 \rightarrow H^1(\Gamma, A(F_\infty)) \rightarrow H^1(F_\Sigma/F, A) \xrightarrow{\theta} H^1(F_\Sigma/F_\infty, A)^\Gamma \rightarrow H^2(\Gamma, A(F_\infty)).
\]
The map \( \theta \) is an isomorphism because, by Lemma 5.4, \( A(F_\infty) = 0 \). Thus, in particular, we get an equality
\[
\text{corank}_\mathcal{O}(H^1(F_\Sigma/F, A)) = \text{corank}_\mathcal{O}(H^1(F_\Sigma/F_\infty, A)^\Gamma).
\]
Taking \( I = (\gamma - 1) \) and \( N = Q \) in Lemma 5.13, we obtain an isomorphism of \( \Lambda \)-modules \((Q^\gamma)^\Gamma \cong (Q^\Gamma)^\gamma \). On the other hand, exact sequence (5.20) with \( Q^\gamma \) in place of \( Q \) shows that \((Q^\gamma)^\Gamma \) and \((Q^\Gamma)^\gamma \) have the same \( \mathcal{O} \)-rank. We surmise that
\[
\text{rank}_\mathcal{O}((Q^\gamma)^\Gamma) = \text{rank}_\mathcal{O}((Q^\gamma)^\Gamma) = \text{rank}_\mathcal{O}((Q^\Gamma)^\gamma) = \text{corank}_\mathcal{O}(Q^\Gamma).
\]
Analogously, since \( Y = H^1(F_\Sigma/F_\infty, A)^\gamma \), we get an equality
\[
\text{rank}_\mathcal{O}(Y^\Gamma) = \text{corank}_\mathcal{O}(H^1(F_\Sigma/F_\infty, A)^\Gamma).
\]
In light of (5.23) and (5.24), pseudoisomorphism (5.13) ensures that
\[
\text{corank}_\mathcal{O}(H^1(F_\Sigma/F_\infty, A)^\Gamma) = r^- \cdot [F : Q] + \text{corank}_\mathcal{O}(Q^\Gamma).
\]
Finally, combining (5.21), (5.22) and (5.25) gives \( \text{corank}_\mathcal{O}(Q^\Gamma) = 0 \), as was to be shown. \( \Box \)

Now we can turn to the vanishing result that will play a crucial role in the proof of our main theorem.

**Corollary 5.17.** \( S_{\Gamma} = 0 \).

**Proof.** With notation as in 5.22, set \( \tilde{P}^\Sigma_{\text{Gr}}(A/F_\infty) := \text{im}(\delta_{\Sigma, \infty}) \subset \mathcal{P}^\Sigma_{\text{Gr}}(A/F_\infty) \), so that there is a short exact sequence
\[
0 \rightarrow S \rightarrow H^1(F_\Sigma/F_\infty, A) \xrightarrow{\delta_{\Sigma, \infty}} \tilde{P}^\Sigma_{\text{Gr}}(A/F_\infty) \rightarrow 0
\]
of \( \Gamma \)-modules. Diagram (5.12) and Lemma 5.10 imply that \( \delta_{\Sigma, \infty} \) is surjective. It follows that \( \delta_{\infty, \Sigma}^\Gamma \) induces a surjection
\[
\delta_{\infty, \Sigma}^\Gamma : H^1(F_\Sigma/F_\infty, A)^\Gamma \rightarrow \tilde{P}^\Sigma_{\text{Gr}}(A/F_\infty)^\Gamma,
\]
which we denote by the same symbol. We can extract from the long exact sequence in cohomology associated with (5.26) an exact sequence
\[ H^1(F_\Sigma/F_\infty, A)^{\Gamma} \xrightarrow{\delta_{\infty, \Sigma}^\Gamma} \hat{P}^\Sigma_{\text{Gr}}(A/F_\infty)^{\Gamma} \rightarrow S_\Gamma \rightarrow H^1(F_\Sigma/F_\infty, A)^{\Gamma}. \]
Since \( \delta_{\infty, \Sigma}^\Gamma \) is surjective, we deduce that \( S_\Gamma \) embeds into \( H^1(F_\Sigma/F_\infty, A)^{\Gamma} \), and the triviality of \( S_\Gamma \) follows from Proposition 5.10. \( \square \)

**Remark 5.18.** A result analogous to Corollary 5.17 in an anticyclotomic imaginary quadratic setting can be found in [11, Lemma 3.3.5].

6. Main result

Putting together the results we have collected so far, we obtain the main theorem of this paper. For the convenience of the reader, we recall our setting.

Let \( F \) be a number field, let \( \mathcal{O} \) be the valuation ring of a finite field extension \( K \) of \( \mathbb{Q}_p \) and let \( T \) be a free \( \mathcal{O} \)-module that is equipped with a continuous action of the absolute Galois group of \( F \) satisfying Assumption 2.2. Let \( F_\infty/F \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \) and let \( \Lambda := \mathcal{O}[\Gamma] \) be the associated Iwasawa algebra, where \( \Gamma := \text{Gal}(F_\infty/F) \simeq \mathbb{Z}_p \). Let \( A := T \otimes_{\mathcal{O}} (K/\mathcal{O}) \), let \( \text{Sel}_{\text{Gr}}(A/F_\infty) \) be the Greenberg Selmer group of \( A \) over \( F_\infty \) and let \( \text{Sel}_{BK}(A/F) \) the Bloch–Kato Selmer group of \( A \) over \( F \). Finally, let \( c_v(A) \) be the Tamagawa number of \( A \) at a prime \( v \) of \( F \) and denote by \( \Sigma \) the (finite) set of primes \( v \) of \( F \) such that either \( v \) is archimedean or \( v \) lies above \( p \) or \( A \) is ramified at \( v \). Notice that the finiteness of \( \text{Sel}_{BK}(A/F) \) in the statement below is Assumption 4.5.

**Theorem 6.1.** Suppose that \( \text{Sel}_{BK}(A/F) \) is finite. Then
1. \( \text{Sel}_{\text{Gr}}(A/F_\infty) \) is \( \Lambda \)-cotorsion;
2. if \( F \) is the characteristic power series of the Pontryagin dual of \( \text{Sel}_{\text{Gr}}(A/F_\infty) \), then \( F(0) \neq 0 \);
3. there is an equality
\[ \#(\mathcal{O}/F(0) \cdot \mathcal{O}) = \# \text{Sel}_{BK}(A/F) \cdot \prod_{v \in \Sigma} c_v(A). \]

**Proof.** Parts (1) and (2) are [4.1], while part (3) follows by combining [11, Proposition 5.9] and Corollary 5.17. \( \square \)

**Remark 6.2.** The equality in part (3) of Theorem 6.1 can be equivalently formulated as
\[ \text{length}_{\mathcal{O}}(\mathcal{O}/F(0) \cdot \mathcal{O}) = \text{length}_{\mathcal{O}}(\text{Sel}_{BK}(A/F)) \cdot \text{ord}_p \left( \prod_{v \in \Sigma} c_v(A) \right), \]
where \( \text{length}_{\mathcal{O}}(\ast) \) denotes the length of the \( \mathcal{O} \)-module \( \ast \) and \( \text{ord}_p \) is the \( p \)-adic valuation.

**Remark 6.3.** An analogue of part (3) of Theorem 6.1 when \( F \) is an imaginary quadratic field and \( F_\infty \) is the anticyclotomic \( \mathbb{Z}_p \)-extension of \( F \) is provided by [11, Theorem 3.3.1].

**References**

1. P. N. Balister and S. Howson, *Note on Nakayama’s lemma for compact \( \Lambda \)-modules*, Asian J. Math. 1 (1997), no. 2, 224–229.
2. M. Bertolini and H. Darmon, *Iwasawa’s Main Conjecture for elliptic curves over anticyclotomic \( \mathbb{Z}_p \)-extensions*, Ann. of Math. (2) 162 (2005), no. 1, 1–64.
3. S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.
4. P. Deligne, *Formes modulaires et représentations \( t \)-adiques*, Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Lecture Notes in Math., vol. 175, Springer, Berlin, 1971, pp. 139–172.
5. A. Fischman, *On the image of Λ-adic Galois representations*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 2, 351–378.
6. M. Flach, *A generalisation of the Cassels–Tate pairing*, J. Reine Angew. Math. **412** (1990), 113–127.
7. R. Greenberg, *Iwasawa theory for p-adic representations*, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 97–137.
8. ______, *Iwasawa theory for elliptic curves*, Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Math., vol. 1716, Springer, Berlin, 1999, pp. 51–144.
9. H. Hida, *Modular forms and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 69, Cambridge University Press, Cambridge, 2000.
10. B. Howard, *Variation of Heegner points in Hida families*, Invent. Math. **167** (2007), no. 1, 91–128.
11. D. Jetchev, C. Skinner, and X. Wan, *The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one*, Camb. J. Math. **5** (2017), no. 3, 369–434.
12. S. Lang, *Algebraic number theory*, second ed., Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994.
13. M. Longo and S. Vigni, *A refined Beilinson–Bloch conjecture for motives of modular forms*, Trans. Amer. Math. Soc. **369** (2017), no. 10, 7301–7342.
14. ______, *Kolyvagin systems and Iwasawa theory of generalized Heegner cycles*, Kyoto J. Math. **59** (2019), no. 3, 717–746.
15. ______, *Kolyvagin’s conjecture and the Bloch–Kato formula for modular forms*, preprint (2020).
16. J. Nekovář, *Kolyvagin’s method for Chow groups of Kuga–Sato varieties*, Invent. Math. **107** (1992), no. 1, 99–125.
17. ______, *Selmer complexes*, Astérisque (2006), no. 310.
18. T. Ochiai, *Control theorem for Bloch–Kato’s Selmer groups of p-adic representations*, J. Number Theory **82** (2000), no. 1, 69–90.
19. A. A. Panchishkin, *Motives over totally real fields and p-adic L-functions*, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 4, 989–1023.
20. K. A. Ribet, *On l-adic representations attached to modular forms II*, Glasgow Math. J. **27** (1985), 185–194.
21. K. Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000.
22. ______, *Euler systems and Kolyvagin systems*, Arithmetic of L-functions, IAS/Park City Math. Ser., vol. 18, Amer. Math. Soc., Providence, RI, 2011, pp. 449–499.
23. J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, vol. 42, Springer-Verlag, New York, 1977.
24. ______, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401.
25. C. Skinner and E. Urban, *The Iwasawa Main Conjectures for GL2*, Invent. Math. **195** (2014), no. 1, 1–277.
26. J. Tate, *Relations between K2 and Galois cohomology*, Invent. Math. **36** (1976), no. 1, 257–274.

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