6 + 1 Vacua in Supersymmetric QCD with $G_2$ Gauge Group

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Abstract

We consider $\mathcal{N} = 1$ supersymmetric QCD based on the $G_2$ gauge group and involving 3 chiral matter 7–plets $S^i_\alpha$. In that case, the gauge symmetry is broken completely and the instanton–generated superpotential on the classical moduli space is present. If the theory involves the Yukawa term $\lambda f^{\alpha\beta\gamma} S^1_\alpha S^2_\beta S^3_\gamma$, there are six chirally asymmetric vacua. In the limit $\lambda \to 0$, two of the vacua run away to infinity and only 4 asymmetric vacuum states are left. Besides, a chirally symmetric state is always present. We consider also an $O(7)$ model with 4 chiral multiplets in spinor representation. In that case, there are 4 extra “virtual vacua” dwelling at infinity of the moduli space. In a non-renormalizable theory with a quartic term in the superpotential, they show up at finite moduli values.
1 Introduction

The dynamics of the $\mathcal{N} = 1$ supersymmetric QCD has been studied by theorists since the beginning of the eighties [1]. The theories with unitary gauge groups attracted a special attention. In the most simple from ideological viewpoint case when the number of flavours is $N_f = N_c - 1$, the gauge symmetry is broken completely and the theory involves a discrete set of vacuum states. The existence of $N_c \equiv N$ such states with nonvanishing value of the gluino condensate $< \text{Tr} \, \lambda^2 >$ associated with the spontaneous breaking of a discrete symmetry $Z_{2N} \to Z_2$ has been known for a long time. It was noted recently [2] that on top of $N$ chirally asymmetric vacua also a chirally symmetric vacuum with the zero value of the condensate exists.

The vacuum structure of the theory displays itself in a straightforward way in the framework of the effective lagrangian due to Taylor, Veneziano, and Yankielowicz (TVY) [3]. The lagrangian is written for the colorless composite fields (moduli)

$$\Phi^3 = \frac{1}{16\pi^2} \text{Tr} \, W^2, \quad M_{ij} = 2 \tilde{S}_i S_j, \quad i, j = 1, \ldots, N - 1$$

(1.1)

where $W$ is the chiral gauge superfield and $\tilde{S}_i, S_j$ are the matter chiral multiplets in the antifundamental and fundamental representations of the gauge group, respectively. The TVY effective lagrangian presents a Wess–Zumino model with the superpotential

$$W = \Phi^3 \left[ \ln \frac{\Phi^3 \det M}{\Lambda_{\text{SQCD}}^{2N+1}} - 1 \right] - \frac{m}{2} \text{Tr} \, M$$

(1.2)

$(m$ is the common mass for all matter fields). The corresponding potential for the lowest scalar components $\phi, \mu_{ij}$ of the superfields $\Phi, \mathcal{M}_{ij}$ has $N + 1$ degenerate minima. One of them is chirally symmetric $\phi = \mu_{ij} = 0$, and there are also $N$ chirally asymmetric vacuum states.

The latter are clearly seen in the weak–coupling limit $m \ll \Lambda_{\text{SQCD}}$ where we can integrate out the heavy field $\Phi$ associated with the gauge degrees of freedom (i.e. to freeze down $\Phi^3 \det M \equiv \Lambda_{\text{SQCD}}^{2N+1}$) to obtain

$$W = -\frac{\Lambda_{\text{SQCD}}^{2N+1}}{\det \mathcal{M}} - \frac{m}{2} \text{Tr} \, \mathcal{M}$$

(1.3)

The effective lagrangian with the superpotential (1.3) is a true Born–Oppenheimer lagrangian for the light fields and has a better status than the TVY lagrangian (1.2) where light and heavy degrees of freedom are not so nicely separated. The existence of $N$ chirally asymmetric vacua as follows from Eq.(1.3) is a theorem

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\[1\] It is the step where the chirally symmetric state is lost. The latter appears if freezing down $\Phi = 0$. The valleys $\Phi^3 \det \mathcal{M} = \Lambda_{\text{SQCD}}^{2N+1}$ and $\Phi = 0$ in the TVY lagrangian are separated by an ostensibly high but still penetrable energy barrier [3] [3] [3].
of the supersymmetric QCD while the presence of the chirally symmetric vacuum following from Eq. (1.2) is a conjecture. We believe it is true, but the question is still under discussion [4].

The existence of the discrete set of vacua implies the presence of the domain walls interpolating between them [3, 1, 7, 10]. The extensive numerical study of these walls [4, 3, 9, 10] displayed their rich non-trivial structure. A certain kind of the walls (those connecting different chirally asymmetric vacua) exists only for small enough masses. Sometimes the walls are BPS saturated and sometimes they are not, there are “wallsome sphalerons”, etc.

The theories with orthogonal and exceptional groups also attracted a considerable attention. They are interesting, in particular, because simple arguments leading to the estimate $I_W = < \text{rank of the group} > + 1$ for the number of vacuum states [11] which work well for the unitary and simplectic groups (the chirally symmetric vacuum appears only in the infinite volume limit [1, 5] and its presence does not invalidate the finite volume calculation of Ref. [11]) fail in this case. Also, orthogonal and exceptional groups do not involve a sufficiently rich center subgroup and the so called toron field configurations [2] are absent.

The previous studies of the theories with exotic groups displayed the following dynamical picture:

The number of (chirally asymmetric) vacuum states in the pure supersymmetric gluodynamics is equal to the Dynkin index of the group $T(G)$ defined as $\text{Tr}\{T^aT^b\} = T(G)\delta^{ab}$ where $T^a$ are the generators in the adjoint representation. This is best seen by noting that, for a general group, instantons involve $2T(G)$ gluino zero modes. Supersymmetric Ward identities + an explicit instanton calculation require that the chiral correlator $< T\{\lambda^\alpha_1\lambda^{\alpha_2}(x_1) \ldots \lambda^\alpha_T(x_{T(G)})\} >$ (here $\alpha = 1, 2$ is the Weyl spinor index) is a non-zero $x_i$-independent constant. By cluster decomposition, that implies the existence of the states where the vacuum expectation value $< \lambda^\alpha_1\lambda^{\alpha_2} >$ is nonzero. $T(G)$ values for the phase of the gluino condensate are allowed [1, 11].

For unitary and symplectic groups, $T(G) = r + 1$ in accordance with the original Witten’s counting [11]. For higher orthogonal groups [starting from $O(7)$] and for exceptional groups, $T(G) > r + 1$. A “traditional” explanation for this mismatch was that the method how the estimate $r + 1$ for the number of states was obtained is not quite rigorous. In was based on the Born–Oppenheimer treatment of the theory on a small spatial torus with topologically trivial boundary conditions. But however small the torus is, the parameter of the Born–Oppenheimer expansion is not small everywhere. An explicit calculation for supersymmetric QED displayed large corrections to the lowest order Born–Oppenheimer hamiltonian in some region of the moduli space [14]. Such corrections could in principle invalidate the vacua counting. Quite recently, however, Witten found a bug in his original reasoning [15]:

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2Such configurations appear formally when a theory with a unitary gauge group is defined on a large 4-dimensional torus. They provide an essential contribution in the Euclidean path integral when the size of the box is small. The question of whether such configurations play an essential role in the limit of large boxes is not yet totally clear. See [18, 1, 15] for a recent discussion.
it turned out that, for complicated groups, the moduli space has two disconnected components. The second component is responsible for some extra vacuum states so that their total number is just $T(G)$! This was checked explicitly for orthogonal groups.

$T(G)$ supersymmetric vacua are seen also in the supersymmetric QCD involving light chiral matter multiplets [16]. However, up to now, only the theories involving the tree–level mass term were considered.

The purpose of this note is to demonstrate that, in the general case when the tree–level lagrangian involves also Yukawa couplings, the number of vacua may be larger. In particular, the $G_2$ theory with 3 chiral 7–plets involves generally six chirally asymmetric vacua.

## 2 Vacuum Structure in $G_2$ theory

$G_2$ theory is defined as a subgroup of $O(7)$ leaving invariant the combination $f^{\alpha\beta\gamma} p_\alpha q_\beta r_\gamma$ where $p_\alpha$, $q_\beta$, and $r_\gamma$ are three arbitrary 7-vectors and $f^{\alpha\beta\gamma}$ is a certain antisymmetric tensor. One particular choice for $f^{\alpha\beta\gamma}$ is

$$
\begin{align*}
 f^{165} &= f^{341} = f^{523} = f^{271} = f^{673} = f^{475} = f^{246} = 1 
\end{align*}
$$

and all other nonzero components are restored by antisymmetry. The form (2.1) can be mnemonicized by drawing the triangle diagram as in Fig. 1. $f^{\alpha\beta\gamma}$ is nonzero only for the indices lying on the same line, with the arrows indicating the order of indices when $f^{\alpha\beta\gamma}$ is positive.
Another way to describe the $G_2$ group is to think of it as of a subgroup of $O(7)$ leaving invariant a real 8–component 7D spinor $\eta$. 14 generators out of 21 generators of $O(7)$ are left. 7 others act on the spinor $\eta$ non–trivially. The rank of $G_2$ is $r = 2$, one unit less than the rank of $O(7)$. In this construction, the tensor $f^{\alpha\beta\gamma}$ is defined as

$$f^{\alpha\beta\gamma} = \eta^T \Gamma^\alpha \Gamma^\beta \Gamma^\gamma \eta$$

(2.2)

where $\eta^T \eta = 1$ and $\Gamma^\alpha$ are 7D Euclidean (real and antisymmetric) $\Gamma$–matrices. The particular form (2.1) is obtained from (2.2) if choosing

$$\Gamma^1 = i\sigma^2 \otimes \sigma^2 \otimes \sigma^2, \Gamma^2 = -i \otimes \sigma^1 \otimes \sigma^2, \Gamma^3 = -i \otimes \sigma^3 \otimes \sigma^2, \Gamma^4 = i\sigma^1 \otimes \sigma^2 \otimes 1,$$

$$\Gamma^5 = -i\sigma^3 \otimes \sigma^2 \otimes 1, \Gamma^6 = -i\sigma^2 \otimes 1 \otimes \sigma^1, \Gamma^7 = -i\sigma^2 \otimes 1 \otimes \sigma^3,$$

(2.3)

and

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(2.4)

The quantities $v_\alpha = \Gamma_\alpha \eta$ form a real 7–plet which is the fundamental representation of $G_2$.

The theory we want to discuss involves the gauge supermultiplet $V^a$ and 3 chiral 7–plets $S^i_\alpha, i = 1, 2, 3$. Note that though the representation 7 of $G_2$ is real, the chiral fields $S^i_\alpha$, and in particular their lowest scalar components $s^i_\alpha$ are complex.

As we will shortly see, the ground state of the theory corresponds to nonzero values of $< s^i_\alpha >$ which break down completely the $G_2$ gauge group. Indeed, a nonzero vacuum expectation value of one of the fields $< s^1_\alpha > = v_0 \delta^\alpha_7$ breaks $G_2$ down to $SU(3)$. After such a breaking, two other 7–plets are decomposed as $7 = 1 + 3 + 3$. Explicitly: $s^{2,3}_7$ form the singlets, and the triplets $r_\alpha, t_\alpha$ and antitriplets $\bar{r}^\alpha, \bar{t}^\alpha$ are

$$r_\alpha, t_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} s^{2,3}_1 + is^{2,3}_2 \\ s^{2,3}_3 + is^{2,3}_6 \\ s^{2,3}_5 + is^{2,3}_4 \end{pmatrix}, \bar{r}^\alpha, \bar{t}^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} s^{2,3}_1 - is^{2,3}_2, s^{2,3}_3 - is^{2,3}_6, s^{2,3}_5 - is^{2,3}_4 \end{pmatrix}$$

(2.5)

As is well known [1], two triplets and two antitriplets which are left is a minimal necessary set of matter fields to break the remaining $SU(3)$ group completely.

The moduli space for this theory is constructed in the same way as for unitary groups. If the tree–level superpotential is zero (i.e. if the mass of the matter fields and their Yukawa couplings are zero), the classical vacuum energy vanishes whenever the $D$–term vanishes:

$$D^a = \sum_i \bar{s}_i T^a s_i = 0$$

(2.6)

This imposes 14 conditions on $3 \times 7 = 21$ complex or 42 real parameters describing a general scalar field configuration. Further 14 parameters are absorbed by group
rotations. Thus, the classical moduli space involves 14 real or 7 complex bosonic parameters promoted by supersymmetry up to 7 chiral superfields. They can be conveniently chosen as

\[ M_{ij} = S_i \alpha S_j \beta, \quad B = \frac{1}{6} \epsilon^{ijk} f^{\alpha \beta \gamma} S_i^\alpha S_j^\beta S_k^\gamma \]  

(2.7)

Quantum effects result, however, in this theory in a nontrivial instanton–induced superpotential on the moduli space. Holomorphy and symmetry considerations dictate

\[ W_{\text{inst}}(M_{ij}, B) = -\frac{\Lambda_{G_2}^9}{\det M} f \left( \frac{B^2}{\det M} \right) \]  

(2.8)

where \( \Lambda_{G_2} \) is the characteristic scale for the glueball (and gluino-ball) states in the \( G_2 \) supersymmetric gauge theory. The particular form of the function \( f(x) \) is \( f(x) = 1/(1-x) \) [17]. The easiest way to see it is to require that, after the breaking \( G_2 \to SU(3) \) induced by the vacuum expectation value of one of the fields \( < s_i^\alpha > = v_0 \delta_{\alpha 7} \), the instanton–induced superpotential (2.8) would coincide with the one in the \( SU(3) \) theory with 2 chiral triplets and 2 antitriplets:

\[ W_{G_2}^{\text{inst}} \to W_{SU(3)}^{\text{inst}} = -\frac{\Lambda_{SU(3)}^7}{(R^\alpha R_\alpha)(T^\alpha T_\alpha) - (\bar{R}^\alpha R_\alpha)(\bar{T}^\alpha T_\alpha)} \]  

(2.9)

\[ [\Lambda_{SU(3)}^7 \equiv \Lambda_{G_2}^9/(4v_0^2) ] \]

The theory with the superpotential

\[ W^{\text{inst}} = -\frac{\Lambda_{G_2}^9}{\det M - B^2} \]  

(2.10)

does not have a ground state at all — the energy is positive at any finite value of the moduli. When adding to Eq.(2.10) the tree–level superpotential, a discrete number of vacuum states appear. To find how many, we have to study the full effective theory with the superpotential

\[ W = -\frac{1}{2(\det M - B^2)} - \frac{m}{2} \text{Tr } M - \lambda B \]  

(2.11)

(from now on, everything will be measured in the units of \( \Lambda_{G_2} \), and the factor 2 is put downstairs in the first term for convenience). Basically, we have to find the points in the moduli space where all the F–terms vanish:

\[ \partial W/\partial(\text{moduli}) = 0 \]  

(2.12)

Let us see first what happens when \( \lambda = 0 \). Assuming \( M_{ij} = V_0^2 \delta_{ij}, B = 0 \) and substituting it in the superpotential (2.11), the condition (2.12) is reduced to

\[ m(v_0^2)^4 = 1 \]  

(2.13)
The equation (2.13) for the moduli \( v_0^2 \) has 4 roots corresponding to 4 chirally asymmetric vacuum states. This result is well known \([16]\).

Let us now study the general case \( \lambda \neq 0 \). The previous Ansatz with \( B = 0 \) does not go through the equations (2.12) anymore. We can still assume \( M_{ij} = V_0^2 \delta_{ij} \), but have to allow for a nonzero \( B \).

Note first that the equations (2.12) are written somewhat symbolically and should be handled with some care. Effectively, one can write them down in our case as

\[
\frac{\partial W}{\partial \mu_{ij}} = 0, \quad \frac{\partial W}{\partial b} = 0 \tag{2.14}
\]

All the solutions of the equation system (2.14) describe true supersymmetric vacua. However, some other choice of variables (defining e.g. the superfield of canonical dimension 1 \( C = B^{1/3} \) and solving the equation \( \partial W / \partial c = 0 \)) could lead to extra fake solutions. To write the equations quite correctly, we have to take accurately into account the metrics on the moduli space, in other words to choose the variables so that the kinetic term in the effective lagrangian would have a standard quadratic form.

The latter is induced by the kinetic term of the matter fields in the fundamental theory:

\[
\mathcal{L}_{\text{matt}}^{\text{kin}} = \int d^4 \theta \sum_i \bar{S}^i S^i \tag{2.15}
\]

To proceed quite generally, one should resolve the constraint (2.6), choose a group orientation, express \( S_\alpha^i \) via 7 complex moduli and substitute these expressions in Eq. (2.15). This is a complicated technical problem which has not yet been solved. Fortunately, we do not need this. It suffices to act in the framework of the Ansatz

\[
M_{ij} = V_0^2 \delta_{ij}, \quad B \neq 0 \tag{2.16}
\]

In that case, a simple convenient parametrization for the matter superfields can be chosen:

\[
\begin{align*}
S_\alpha^1 &= V_0 \delta_{\alpha \gamma} \\
S_7^2 &= 0, \quad R_\alpha = \frac{1}{2} \begin{pmatrix} iV_1 \\ V_2 \\ 0 \end{pmatrix}, \quad \bar{R}^\alpha = \frac{1}{2} (-iV_1, V_2, 0) \\
S_7^3 &= 0, \quad T_\alpha = \frac{1}{2} \begin{pmatrix} V_1 \\ iV_2 \\ 0 \end{pmatrix}, \quad \bar{T}^\alpha = \frac{1}{2} (V_1, -iV_2, 0), \tag{2.17}
\end{align*}
\]

where \( R_\alpha, T_\alpha \) and \( \bar{R}^\alpha, \bar{T}^\alpha \) are defined in Eq. (2.5). The parametrization (2.17) corresponds to the vanishing D–terms as it should and implies the following values of the moduli: \( M_{11} = V_0^2, \quad M_{22} = M_{33} = (V_1^2 + V_2^2)/2, \quad M_{i \neq j} = 0 \). Many other
parametrizations with $D^a = 0$ and the same values of the moduli are possible, but they all are obtained from Eq. (2.17) by colour and flavour rotations.

If we want to keep the condition (2.16), the constraint $2V_0^2 = V_1^2 + V_2^2$ should be imposed. But it is not really necessary: one can as well keep three free parameters $V_0, V_1, V_2$, and the condition $2V_0^2 = V_1^2 + V_2^2$ will be automatically satisfied at the vacuum points where all the F–terms (2.12) vanish. With the parametrization (2.17), the kinetic term (2.15) has a simple form

$$L_{\text{kin}} = \int d^4 \theta (\bar{V}_0 V_0 + \bar{V}_1 V_1 + \bar{V}_2 V_2)$$

(2.18)

That means that the correct form of the equation (2.12) for the vacuum states is just

$$\frac{\partial W}{\partial v_0} = \frac{\partial W}{\partial v_1} = \frac{\partial W}{\partial v_2} = 0.$$ 

Expressing the superpotential (2.11) via $V_0, V_1, V_2$:

$$W = -\frac{1}{2} \left( \frac{V_0^2 + V_1^2 + V_2^2}{2} - \frac{\lambda}{2} V_0 (V_2^2 - V_1^2) \right)$$

(2.19)

the equations (2.12) acquire a form

$$mv_0 + \frac{\lambda}{2} (v_2^2 - v_1^2) = \frac{1}{v_0^2 v_1^2 v_2^2}$$

$$mv_1 - \lambda v_0 v_1 = \frac{1}{v_0^2 v_1^2 v_2^2}$$

$$mv_2 + \lambda v_0 v_2 = \frac{1}{v_0^2 v_1^2 v_2^2}$$

(2.20)

This equation system has 12 solutions, the six pairs of them differing only by the sign of $v_0$ (and of $v_2^2 - v_1^2$) and corresponding to the same values of the moduli $v_0^2, b = v_0 (v_2^2 - v_1^2)/2$. The relation $b = -\lambda v_0^2 / m$ holds while $u = v_0^2$ satisfies the equation

$$mu^4 \left( 1 - \frac{\lambda^2}{m^2} u \right)^2 = 1$$

(2.21)

The equation (2.21) has 6 roots as was announced.

Let us see what happens in the limit of small $\lambda$, the smallness being characterized by a dimensionless parameter

$$|\kappa| = \left| \frac{\lambda^2}{m^{9/4}} \right| \equiv \left| \lambda^2 (\Lambda/m)^{9/4} \right| \ll 1$$

(2.22)

(do not forget that $\lambda$ and $m$ are generally complex). Then 4 of the roots are very close to the known solutions of the equation (2.13), but on top of this, there are two extra roots at large values of $u$: $u = m^2/\lambda^2 \pm \lambda^2 / m^{5/2}$. In the limit $\lambda \to 0$, these new vacua run away at infinity of the moduli space and decouple.
The latter statement can be attributed a precise meaning. Different vacua are separated by the domain walls — planar field configurations interpolating between one vacuum on the left and another vacuum on the right. The domain walls have a surface energy density $\epsilon$. The value of $\epsilon$ is the measure of the height of the barrier separating different vacua. It turns out [18],[8],[9] that the wall energy density satisfies a strict lower bound

$$\epsilon \geq 2|W_1 - W_2|$$

(2.23)

where $W_{1,2}$ are the values of the superpotential at the vacua 1,2 between which the wall interpolates. The bound (2.23) has the same nature as the celebrated BPS lower bound for the mass of the magnetic monopole. It often happens that an actual supersymmetric domain wall is BPS saturated, i.e. its energy density is given by the bound (2.23). But sometimes it is not so [5, 6]. Only a detailed numerical study can answer the question what kind of domain walls exist in this model, whether they are BPS saturated or not, and how that depends on the value of the parameter $\kappa$.

Something can be said, however. First, when $\kappa = 0$ and only 4 vacua are left, the domain walls between them are BPS–saturated and have the energy density $\epsilon = 2\sqrt{2}|m|^{3/4}$ for the walls connecting the “adjacent” vacua with, say, $u = m^{-1/4}$ and $u = i m^{-1/4}$ and $\epsilon = 4|m|^{3/4}$ for the walls connecting the “opposite” vacua with $u = \pm m^{-1/4}$ or with $u = \pm i m^{-1/4}$. The point is that the effective Higgs lagrangian is exactly the same here as for the $SU(4)$ theory with 3 chiral quartets and 3 antiquartets whose domain walls were studied in Ref.[6]. When $\kappa$ is nonzero and small so that a couple of new vacua at large values of $u$ appear, the values of the superpotential (2.19) at these vacua are also large $\approx m^3/2\lambda^2$. The bound (2.23) dictates that the energy density of a wall connecting an “old” and a “new” vacua is larger than $|m^3/\lambda^2|$ and tends to infinity in the limit $\lambda \to 0$.

When we increase $|\kappa|$, the new vacua move in from infinity and, at $|\kappa| \sim 1$, the values of $u_{\text{vac}}$ for the vacua of both types are roughly the same. It is interesting that, at $\kappa = \frac{2}{3\sqrt{3}}\sqrt[4]{4} \approx 0.385\sqrt[4]{4}$, two of the vacua (an “old” one and a “new” one) become degenerate. At $|\kappa| \gg 1$ [3], 6 vacua find themselves at the vertices of a perfect hexagon on the complex $u$ – plane. The complex roots of Eq.(2.21) in the units of $m^{-1/4}$ for 3 illustrative values of $\kappa$ are displayed in Fig. 2.

The values of the superpotential $W(u_{\text{vac}}, b_{\text{vac}})$ for the same values of $\kappa$ in the units of $m^{3/4}$ are shown in Fig. 3. We see that, for large $\kappa$, 6 vacua are clustered in 3 pairs (each pair corresponding to the opposite vertices of the hexagon in Fig. 2c). The values of the superpotential for two vacua of the same pair are close and hence the energy barrier between them is small. Indeed, for very small masses and fixed $\lambda$, one can neglect the mass term in the superpotential in which case the vacuum solutions occur at $v_0^2 = 0$ while the value of $b$ is fixed at

$\kappa = \frac{2}{3\sqrt{3}}\sqrt[4]{4} \approx 0.385\sqrt[4]{4}$

To assure that $u_{\text{vac}}$ are still large (compared to $\Lambda$) so that we are still in the Higgs phase and the light and heavy degrees of freedom are separated, we should keep $|\lambda/m^{3/4}| \ll 1$. But if $m$ is small enough, this condition can be fulfilled at arbitrary large $|\kappa|$.
Figure 2: Solutions of the equation $u^4(1 - \kappa u)^2 = 0$ for a) $\kappa = .16$, b) $\kappa = .385$, and c) $\kappa = 100$. The cross marks out the double degenerate root at $\kappa = .385$.

Figure 3: Values of the superpotential $W_{\text{vac}}$. 

a) $\kappa = .16$  

b) $\kappa = .385$  

c) $\kappa = 100$
The splitting of vacua inside the pair is the effect of higher order in $1/\kappa$.

All 6 vacua are chirally asymmetric, i.e. the gluino condensate $< \lambda^a \lambda^a >$ is formed. The latter is fixed by the Konishi identity \[ \sum_i T(R_i) < \lambda^a \lambda^a >= -16\pi^2 \left\langle Q_i \frac{\partial W_{\text{tree}}}{\partial Q_i} \right\rangle \] (2.24)

where $T(R_i)$ are Dynkin indices of the group representations where the matter fields $Q_i$ lie (in our case, $Q_i \equiv S_i^\alpha$, $T(7) = 1$ and $\sum_i T(R_i) = 3$). We have

$$< \lambda^a \lambda^a > = \frac{16\pi^2}{3} [3m v_0^2 + 3\lambda b] = 16\pi^2 m u_i \left( 1 - \frac{\lambda^2}{m^2} u_i \right)$$

(2.25)

where $u_i$ is the $i$-th root of the equation (2.21). For small $|\kappa|$, $< \lambda^a \lambda^a > \sim m^{3/4}$ for old vacua while for the new vacua the gluino condensate is suppressed $< \lambda^a \lambda^a > \sim \lambda^2 m^{-3/2} \sim \kappa m^{3/4}$. When $\lambda \to 0$ it vanishes altogether. However, these new states have nothing to do with the chirally symmetric state of Ref.[2]. The latter is realized at zero values of all moduli (including also the moduli describing gauge degrees of freedom) and there are BPS domain walls with finite energy density connecting the true chirally symmetric vacuum with chirally asymmetric ones. The states we have found have large vacuum expectation values for all moduli, a large superpotential $W(u_{\text{vac}}, b_{\text{vac}})$, and decouple in the limit $\lambda \to 0$.

### 3 0(7) theory

The next in complexity theory where $T(G) \neq r + 1$ is the theory based on the $O(7)$ gauge group. The rank of $O(7)$ is $r = 3$ while $T[O(7)] = 5$. Consider the $\mathcal{N} = 1$ supersymmetric gauge model involving 4 chiral matter multiplets $S_i^\alpha$ in spinor representation. In this theory, the gauge symmetry is completely broken. Indeed, v.e.v. of one of the spinor fields $< S_1^1 > \sim \eta$ break $O(7)$ down to $G_2$ after which 3 other spinors are decomposed as $8 \to 1 + 7$ (explicitly: $S_2^3 = (P_0 + P_\mu \Gamma_\mu) \eta$ etc.). Three 7–plets break down the remaining $G_2$ completely as was discussed in the previous section.

Originally, the theory involved $4 \times 8 = 32$ complex degrees of freedom in the matter sector. Imposing the conditions $D^a = 0$ and fixing the gauge eliminates 21 degrees of freedom so that the classical moduli space involves 11 complex parameters. 10 of them can be presented as $M_{ij} = S_i^T S_j$. To construct the eleventh one, note that the tensor product of two $O(7)$ spinors involves a scalar, a vector, an antisymmetric tensor with two indices and an antisymmetric tensor with 3 indices: $8 \times 8 = 1$.
Multiplying it over by itself, we obtain 4 different scalars in the tensor product $8 \times 8 \times 8 \times 8$. Three of these scalars can be written as $(S^T_1 S_2)(S^T_3 S_4)$, $(S^T_1 S_3)(S^T_2 S_4)$, and $(S^T_1 S_4)(S^T_2 S_3)$ while the fourth one presents a non-trivial quartic group invariant not reducible to the products $\mathcal{M}_{12}\mathcal{M}_{34}$ etc. Its explicit form is

$$H = \frac{1}{24} \epsilon^{ijkl}(S^T_i \Gamma_\mu S_j)(S^T_k \Gamma_\mu S_l)$$  \hspace{1cm} (3.1)$$

The invariant (3.1) is antisymmetric in flavour and colour indices.

Like in the previous case, one can show that the theory involves an instanton–generated superpotential. Holomorphy and symmetry considerations dictate

$$\mathcal{W}^{\text{inst}}_{O(7)} = -\frac{\Lambda_{11}^{\text{O}(7)}}{\det \mathcal{M}} f \left( \frac{H^2}{\det \mathcal{M}} \right)$$  \hspace{1cm} (3.2)$$

The function $f$ is fixed by the requirement that after the breaking induced by the vacuum expectation value of one of the spinor fields $< S^1_\alpha > = v_0 \eta$, the superpotential (3.2) is reduced to the superpotential (2.10) in the $G_2$ theory. Noting that $\det \mathcal{M}_{O(7)} \rightarrow -v_0^2 \det \mathcal{M}_{G_2}$ and $H \rightarrow -v_0 B$, we obtain

$$\mathcal{W}^{\text{inst}} = -\frac{\Lambda_{11}^{\text{O}(7)}}{\det \mathcal{M} + H^2}$$  \hspace{1cm} (3.3)$$

The main difference with the $G_2$ theory is that we are not able here to write down a Yukawa term in the superpotential on the tree level. No group invariant can be constructed out of just 3 spinors.

Let us relax, however, for a moment the requirement that the theory is renormalizable. Then one could add the term $\sigma H$ with a dimensionful coupling $\sigma$ in the superpotential. The full superpotential would have the form quite analogous to Eq. (2.11):

$$\mathcal{W} = -\frac{1}{2(\det \mathcal{M} + H^2)} - \frac{m}{2} \text{Tr} \mathcal{M} + \sigma H$$  \hspace{1cm} (3.4)$$

($\Lambda \equiv 1$). Proceeding in the same way as in the $G_2$ case, we search for the solutions of the equation (2.12) in the form $\mu_{ij} = v_0^2 \delta_{ij}; h \neq 0$. One can show that the equation (2.12) written in correct variables implies $h = -\sigma u^3/m$ where $u = v_0^2$ satisfies the equation

$$mu^5 \left( 1 - \frac{\sigma^2}{m^2 u^2} \right)^2 = 1$$  \hspace{1cm} (3.5)$$

The equation (3.5) has nine solutions. For small $|\sigma| (|\sigma^2/m^{12/5}| \ll 1)$, 5 of them are the conventional ones $u = \sqrt[5]{1/m}$ while 4 others are new solutions with large values of $u \sim \pm m/\sigma$. They are separated from the conventional vacua by a high BPS energy.
barrier $\epsilon \sim m^2/\sigma$. If allowing $|\sigma^2/m^{12/5}| \sim 1$, $u_{\text{vac}}$ for all 9 vacua are of the same order. At $\sigma^2/m^{12/5} = 5/9(2/3)^{8/5}\sqrt{T}$ (the analog of the point $\lambda^2/m^2 = 2/(3\sqrt{3})\sqrt{T}$ for the $G_2$ theory), two of the vacua become degenerate. For large $\sigma/m^{6/5}$, $u_{\text{vac}}$ are placed in the vertices of a perfect 9-gon while $\mathcal{W}(u_{\text{vac}}, h_{\text{vac}})$ are grouped in 3 clusters with 3 vacua each.

However, we cannot really consider a non-renormalizable theory. A theory with $\sigma = 0$ and involving only the mass term in the superpotential has just 5 vacua. 4 extra “virtual” vacua dwell at infinity of the moduli space and are separated from the conventional states by an infinitely high barrier.

4 Chirally symmetric phase.

The presence of the chirally symmetric phase is not seen in the framework of the effective Higgs lagrangian (2.11). To see it, we have to study the full TVY effective lagrangian involving also the superfield $\Phi^3 = W^a W^a/(32\pi^2)$ describing the gauge degrees of freedom. The analog of Eq. (1.2) for the $G_2$ theory is

$$\mathcal{W} = \Phi^3 \left[ \ln \frac{\Phi^3 (\det M - B^2)}{\Lambda^9} - 1 \right] - \frac{m}{2} \text{Tr} M - \lambda B \quad (4.1)$$

The superpotential (4.1) is rigidly fixed by the requirements: i) the chiral and conformal anomalies of the underlying gauge theory are reproduced correctly, ii) when $m = \lambda = 0$, the effective theory is symmetric under the transformations $\theta \to \theta e^{i\beta}, M_{ij} \to M_{ij} e^{-2i\beta/3}, B \to B e^{-i\beta}, \Phi \to \Phi e^{2i\beta/3}$ induced by the anomaly–free chiral transformation in the underlying theory $\theta \to \theta e^{i\beta}, W^a \to W^a e^{i\beta}, S^i_\alpha \to S^i_\alpha e^{-i\beta/3}$, and iii) when integrating out the heavy field $\Phi$ by freezing down $\Phi^3 (\det M - B^2) \equiv 1$, the effective Higgs lagrangian (2.11) is reproduced.

If choosing the symmetric Ansatz $M_{ij} = V_0^2 \delta_{ij}$, adding to the superpotential (4.1) kinetic terms (As was mentioned before, the kinetic terms for the moduli $M_{ij}, B$ are induced by the kinetic terms of the matter fields in the original theory. The kinetic term for the moduli $\Phi$ is not known and we just choose it in the simplest possible form $\int \Phi \Phi^* d^4\theta$), and solving the equations (2.12), we obtain 6 chirally asymmetric vacua and also a chirally symmetric vacuum with $\phi = \mu_{ij} = b = 0$.

Note that, for $\lambda = 0$, we can forget about the moduli $B$ altogether and the effective lagrangian has exactly the same form as for the supersymmetric QCD with the completely broken $SU(4)$ gauge group. The theory has BPS domain walls connecting the chirally symmetric vacuum with chirally asymmetric ones and also the walls interpolating between different asymmetric vacua whose dynamics depends on the value of mass $m$ in a nontrivial way (3). The dynamics of the domain walls with nonzero $\lambda$ is yet to be studied.
5 Discussion.

Our main result that modifying the tree–level superpotential by adding Yukawa terms can bring about new vacuum states coming from infinity of the moduli space is rather natural. One can remind in this respect that the number of vacua in the supersymmetric Wess–Zumino model is determined by the form of the superpotential. For a polynomial superpotential, the number of vacua is $n - 1$ where $n$ is the highest power in the polynomial. If we add to the superpotential the term $X^{n+1}$ with a small coefficient, a new vacua at large values of $|x_{\text{vac}}|$ appears and the qualitative picture is exactly the same as in Fig. 2.

Bearing this in mind, it is even surprising that we managed to find only one clean example based on the $G_2$ gauge group where extra vacuum states appear. $O(7)$ example was a lame one: one cannot really consider a non-renormalizable theory with quartic term in the superpotential. To keep a theory renormalizable, we are allowed to add only cubic terms with dimensionless couplings. A distinguishing feature of $G_2$ is the presence of a triple group invariant $f^{\alpha\beta\gamma}p_{\alpha}q_{\beta}r_{\gamma}$ made out of fundamental representations. Such an invariant is absent for conventional groups (an exception is $SU(3)$, but to be able to invoke a cubic term there we must have at least 3 flavours in our disposal in which case an instanton-induced superpotential is not generated and the physics of the model is different ℙ). May be one could construct further renormalizable examples of the theories where the number of asymmetric vacua is not given by the Dynkin index counting by considering more complicated representations, playing around with higher exceptional groups where triple invariants are typically present and/or relaxing the condition that the gauge group is broken completely.

Note that our finding does not modify the conclusion that the number of chirally asymmetric vacua in a pure supersymmetric Yang–Mills theory is always $T(G)$. The latter is obtained from supersymmetric QCD by tending the masses of the matter fields to infinity. If doing that in $G_2$ theory with keeping $\lambda$ fixed (we have to keep $\lambda$ small enough not to run in the Landau pole problem), two extra vacua run away to infinity and decouple.

The main lesson to be learned from our result is that the number of vacua in a supersymmetric theory is not necessarily related to any discrete symmetry of the theory which is spontaneously broken: there is no trace of a $Z_6$ – symmetry in our model. Earlier, we constructed a two–dimensional example with several vacuum states whose presence did not follow from symmetry considerations ℙ. This example was not even supersymmetric: it was just $QCD_2$ involving adjoint rather than fundamental fermions and based on the $SU(N)$ gauge group with even $N \geq 4$. The presence of the chirally symmetric vacuum state in $4D$ supersymmetric QCD is also not dictated by any symmetry considerations.

In the 2–dimensional case, we did not manage, however, to construct these degenerate vacuum states explicitly and only presented strong arguments in favour of their existence. Likewise, the chirally symmetric phase in SQCD$_4$ is seen only in
the framework of the TVY lagrangian whose status is not completely clear. On the contrary, the effective lagrangian (2.11) with the kinetic terms (2.18) is a true Born–Oppenheimer lagrangian if the conditions $|m| \ll 1$, $|m/\lambda^{3/4}| \ll 1$ are fulfilled, and the existence of 6 chirally asymmetric states in this theory is a theorem.

An interesting question to be studied is whether these extra vacuum states are still present in the theory at finite volume. Based on the fact that it is not so in QCD$_2$ [13], neither it is the case for the chirally symmetric phase [5], our guess is that they are not. That would imply that, at finite volume (however large it is), the energy of two extra states we have found is not strictly zero like it is not strictly zero for the chirally symmetric state. It should tend to zero exponentially fast in the limit $V \to \infty$.

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