Multifractal analysis of three-dimensional grayscale images: Estimation of generalized fractal dimension and singularity spectrum

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Abstract

A multifractal analysis is performed on a three-dimensional grayscale image associated with a complex system. First, a procedure for generating 3D synthetic images (2D image stacks) of a complex structure exhibiting multifractal behaviour is described. Then, in order to characterize the 3D system, the theoretical calculation of the generalized fractal dimension $D_q$ and two different approaches for evaluating the singularity spectrum $f(\alpha)$ are presented.

1. Introduction

Geometric (mono)fractals are self-similar sets of points. Multifractals (or multifractal measures) are self-similar measures defined on specific set of points [Baveye et al. 2008; Evertsz et al. 1992; Falconer 2003]. In general, the former can be generated by additive processes and the latter by multiplicative cascade of random processes.

The \textit{generalized fractal dimension} $D_q$, which is closely related to the \textit{Rényi entropy} [Rényi 1961], provides a direct measurement of the fractal properties of an object – several values of the \textit{momentum order} $q$ correspond to well-known generalized dimensions, such as the \textit{capacity dimension} (box-counting dimension) $D_0$, the \textit{information dimension} $D_1$, and the \textit{correlation dimension} $D_2$. The \textit{singularity spectrum} $f(\alpha)$ provides information about the scaling properties of the structure [Halsey et al. 1986, Hentschel et al. 1983; Lowen et al. 2005; Theiler 1990].

Nowadays, several techniques can be used to obtain representations of complex structures through 2D (3D) images. Typically, a 3D image is constituted by a stack of 2D images – examples of 3D images are the computed tomography (CT) images.

The aim of the present paper is to \textit{a)} extend a procedure for generating 2D multifractal lattice (Meakin method) to the 3D case; \textit{b)} present the theoretical calculation of the generalized fractal dimension $D_q$ for a 3D multifractal structure generated by a random multiplicative process; and \textit{c)} compare two different approaches for evaluating the singularity spectrum $f(\alpha)$ in the case of 3D images of a complex structure exhibiting multifractal behaviour.
2. Generation of 3D multifractal lattices: Meakin method

In order to investigate the properties of random walks on multifractal lattices, in 1987 P. Meakin devised a procedure for generating two-dimensional multifractal lattices by using a random multiplicative process [Meakin 1987; Stanley et al. 1988]. Recently, new variants of the generator have been proposed [H. Zhou et al. 2011]. For this study, we refer to the original algorithm.

This method is based on an iterative algorithm. Starting with a 2D lattice containing $2 \times 2$ sites, a 2D lattice with $2^k \times 2^k$ sites is generated after $k$ iterations. In the limit $k \to \infty$, the procedure defines a multifractal measure on the 2D space which can be described in terms of a countinous spectrum of singularities of type $\alpha$, each supported on a fractal subset with a fractal dimensionality $f(\alpha)$.

To extend the Meakin method to the 3D case, let us consider a finite volume of linear size $L = 1$ and, associated with it, a 3D lattice containing $2 \times 2 \times 2 = 8$ boxes of linear size $l = 1/2$. In the first stage of construction, eight numbers $p_i$ are chosen, and randomly associated with the boxes of the lattice: $p_1 = p_2 = p_3 = p_4 = \hat{p}$, $p_5 = p_6 = p_7 = p_8 = \tilde{p}$. Each of these numbers may be regarded as a probability – for example, the probability that a specific box is filled. Then, an iterative algorithm is implemented (e.g. by using a recursive function). At each iteration $k$, the lattice is divided in $N(l) = 2^k \times 2^k \times 2^k = 2^{3k}$ boxes of linear size $l = 2^{-k}$.

For the $i$th box, the probability to be filled is:

\[ p_i(k) = \prod_{j=0}^{k} p_j \]  \hspace{1cm} (1)

where $p_j = \hat{p}$ or $p_j = \tilde{p}$. As a result of this multiplicative process, there are $(k + 1)$ sets of boxes sharing the same probability value. For each $j$ between 0 and $k$, $2^{2k\binom{k}{j}}$ boxes of linear size $l = 2^{-k}$ and volume $v = 2^{-3k}$ have the same probability to be filled:

\[ p_i = \hat{p}^j \tilde{p}^{k-j} \]  \hspace{1cm} (2)

each box having the mass equal to:

\[ m_i = p_i v = \hat{p}^j \tilde{p}^{k-j} 2^{-3k} \]  \hspace{1cm} (3)

The total mass of the system at each iteration $k$ is:

\[ m_T = \sum_{i=1}^{N} m_i = \sum_{j=0}^{k} \binom{k}{j} \hat{p}^j \tilde{p}^{k-j} 2^{-3k} = 2^{-k} \sum_{j=0}^{k} \binom{k}{j} \hat{p}^j \tilde{p}^{k-j} = 2^{-k}(\hat{p} + \tilde{p})^k \]  \hspace{1cm} (4)

Hence, for each box $i$ belonging to the set $j$ made of similar boxes, the mass probability (or mass fraction) is:

\[ p_i(l) = \frac{m_i}{m_T} = \frac{\hat{p}^j \tilde{p}^{k-j}}{2^{2k}(\hat{p} + \tilde{p})^k} \]  \hspace{1cm} (5)

*Generation of synthetic images* – The Meakin method has been used to generate synthetic images of 2D complex structures exhibiting multifractal behaviour [Perrier et al. 2006; H. Zhou et al. 2011].

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In this context, the numbers $p_i$ generated by using a random multiplicative process represent pixel values. If the images are in grayscale format, the pixel values lay in the range $[0,255]$ and can be considered as measures of mass ($0 = \text{black} = \text{no mass}; 255 = \text{white} = \text{mass}$). In order to generate a 2D (3D) synthetic image in grayscale format, the following initial values for the iterative algorithm associated with Eq. (1) are chosen: $\dot{p} = 255$ and $\ddot{p} = 255/2$. Note that, at the end of calculations, a rounding procedure has to be applied to ensure that all pixel values are integer numbers.

3. Multifractal Analysis: Theory

The generalized fractal dimension is defined by:

$$D_q = \lim_{l \to 0} \frac{1}{q-1} \frac{\ln \sum_{i=1}^{N(l)} p_i^q(l)}{\ln l}$$

(6)

where $p_i(l)$ is the integrated measure (mass probability) associated with the $i$-th box, $q$ is the momentum order, and $N(l)$ is the number of boxes of linear size $l$. For $q = 1$ (information dimension), the Eq. (6) can be approximated by:

$$D_1 = \lim_{l \to 0} \frac{\sum_{i=1}^{N(l)} p_i(l) \ln p_i(l)}{\ln l}$$

(7)

In the case of 3D multifractal structures generated by a random multiplicative process, the generalized fractal dimension has the following expressions for $q \neq 1$ and $q = 1$ (see Appendix A for detailed calculations):

$$D_q = \frac{\ln \left( \frac{\dot{p}^q + \ddot{p}^q}{2^{2(q-1)}(\dot{p} + \ddot{p})^q} \right)}{(1-q)\ln 2}$$

(8)

$$D_1 = \frac{\ln (2^2(\dot{p} + \ddot{p})) - (\dot{p} \ln (\dot{p}) + \ddot{p} \ln (\ddot{p})) (\dot{p} + \ddot{p})^{-1}}{\ln 2}$$

(9)

Once $D_q$ is known, the singularity spectrum $f(\alpha)$ can be evaluated via a Legendre transformation:

$$f(\alpha(q)) = q\alpha(q) - \tau(q), \quad \alpha(q) = \frac{d\tau(q)}{dq}$$

(10)

where $\tau(q) = (q-1)D_q$ [Halsey et al. 1986].

The singularity spectrum can also be directly (without knowing $D_q$) evaluated by using the method proposed by Chhabra et al. [1989]. The first step of this approach consists of defining a family of normalized measures $\mu(q)$:

$$\mu_i(q,l) = \frac{[p_i(l)]^q}{\sum_{j=1}^{N(l)} [p_j(l)]^q}$$

(11)

For each box $i$, the normalized measure $\mu_i(q,l)$ depends on the order of the statistical moment and on the box size and it takes values in the range $[0,1]$ for any value of $q$. Then, the two functions $f(q)$ and $\alpha(q)$ are evaluated:
\[ f(q) = \lim_{l \to 0} \frac{\sum_{i=1}^{N(l)} \mu_i(q, l) \ln \mu_i(q, l)}{\ln l} \]  
\[ \alpha(q) = \lim_{l \to 0} \frac{\sum_{i=1}^{N(l)} \mu_i(q, l) \ln p_i(l)}{\ln l} \]

For each \( q \), values of \( f(q) \) and \( \alpha(q) \) are obtained from the slope of plots of \( \sum_{i=1}^{N(l)} \mu_i(q, l) \ln \mu_i(q, l) \) versus \( \ln l \) and \( \sum_{i=1}^{N(l)} \mu_i(q, l) \ln p_i(l) \) versus \( \ln l \) over the entire range of box size values under consideration. Finally, the singularity spectrum \( f(\alpha) \) is constructed from these two data sets.

4. Results and Discussion

For this study, a stack of 128 images with a resolution of \((128 \times 128)\) pixels is generated by using the Meakin method (Eq. (1), number of iterations \( k = 7 \)). The image stack represents a 3D image of a finite volume of linear size \( L = 128 \) [pixels]. For this complex structure, we evaluate the singularity spectrum \( f(\alpha) \) by using the two different methods previously introduced; in order to compare the indirect and direct approaches, we adopt the same set of 21 values of moment orders: \( q = \{-5.0, -4.5, -4.0, \ldots, 4.0, 4.5, 5.0\} \).

To indirectly evaluate \( f(\alpha) \), we calculate \( D_q \) first. Since the synthetic images have been generated by using a random multiplicative process, the generalized fractal dimension can be derived from Eq. (8) and (9) with \( \dot{\rho} = 255 \) and \( \ddot{\rho} = 255/2 \). Fig. 1 shows \( D_q \) as function of the order moment. Then, we derive the singularity strength and the Hausdorff dimension of normalized measures as function of \( q \) (Fig. 2) and, finally, the singularity spectrum (Fig. 3). As expected, the capacity dimension \((q = 0)\) is \( D_0 = 3 \) and the curve \( f(\alpha) \) is convex with a single maximum at \( q = 0 \) \((\alpha = 3.085)\) and with infinite slope at \( q = \mp \infty \) [Halsey et al. 1986].

Note that, for a multifractal system, the variation of generalized fractal dimension with the order moment quantifies the nonuniformity. The maximum value of the \( D_q \) is associated with the least-dense points on the fractal and the minimum value corresponds to the most-dense points [Theiler 1990].

In order to perform multifractal analysis of 2D (3D) grayscale images and, in particular, to directly evaluate the singularity spectrum \( f(\alpha) \) by using the Chhabra method, a program – Munari – has been developed [Milazzo 2010]. The application is written in C++ and, at this stage, it processes grayscale images in ASCII format.

During the initial processing stage, the mass probabilities (Eq. (5)) associated with the box-shaped parts of the images need to be evaluated. A partitioning procedure could be used to carry out this calculation – at each step, a cell (a square box in 2D; a cubic box in 3D) is divided into sub-cells (\( 2^2 \) in 2D, \( 2^3 \) in 3D) and \( p_i(l) \) is calculated for each of them. However, a different approach is adopted by the Munari program: an aggregation procedure consisting of grouping together \( 2^2 \) adjoining cells in the 2D case and \( 2^3 \) in the 3D case. This choice is determined by the fact that it is less computationally intensive – the higher the number of the initial measures (i.e. the number of pixels) to be processed, the more efficient the aggregation is in comparison to the partitioning procedure. Note that the algorithms within Munari have been designed to be general and, as a result, they are independent from the resolution of the images and from the values of box sizes and moment orders used by the Chhabra method.

This specific image stack \((128 \times 128 \times 128)\) pixels) is processed by using six values of box sizes: \( l = \)
\{2, 4, 8, 16, 32, 64\}. The singularity spectrum that is obtained after the calculations is identical to the curve \( f(\alpha) \) evaluated by using the indirect method (Fig. 3).

5. Conclusions

A multifractal analysis has been carried out on a three-dimensional grayscale image. The synthetic images were generated by using a random multiplicative process. First, we have extended a procedure for generating multifractal lattice (Meakin method) and the theoretical calculation of the generalized fractal dimension for this kind of structure, to the 3D case. Then, two different methods (indirect and direct) have been used to evaluate the singularity spectrum for the complex structure under analysis. They both have given the same result. Finally, we have presented Munari, a new program for image processing to carry out multifractal analysis on 2D and 3D systems. This study is part of the validation tests that have been and will continue to be implemented to ensure reliability and stability of the application.
Appendix A

Theoretical calculation of the generalized fractal dimension for a 3D multifractal structure generated by a random multiplicative process

The Meakin method is based on an iterative algorithm. Let us consider a finite volume of linear size $L = 1$ and, associated with it, a 3D lattice containing $2 \times 2 \times 2 = 8$ boxes of linear size $l = 1/2$. At each iteration $k$, the lattice is divided in $N(l) = 2^k \times 2^k \times 2^k = 2^{3k}$ boxes of linear size $l = 2^{-k}$. Each box $i$ belonging to the set $j$ made of similar boxes has the integrated measure (mass probability) given by Eq. (5).

Thus, the generalized fractal dimension for $q \neq 1$ and $q = 1$ can be derived as follows:

$$D_q = \lim_{l \to 0} \frac{1}{q - 1} \ln \sum_{j=1}^{N(l)} p_i^q(l) = \lim_{k \to \infty} \frac{1}{q - 1} \ln \frac{\sum_{j=0}^{k} 2^{2k} \left( \frac{p^j \bar{p}^{k-j}}{2^{2k} (\bar{p} + \bar{p})^k} \right)^q}{\ln 2^{-k}}$$

$$= \lim_{k \to \infty} \frac{1}{q - 1} \ln \left( \frac{2^{2k} (\bar{p} + \bar{p})^k \sum_{j=0}^{k} \binom{k}{j} (p^j \bar{p}^{k-j})}{\ln 2^{-k}} \right) =$$

$$= \lim_{k \to \infty} \frac{1}{q - 1} \ln \left( \frac{(p \bar{p} + \bar{p} \bar{p})^k}{2^{2(k-1)} (\bar{p} + \bar{p})^q} \right) =$$

$$= \lim_{k \to \infty} \frac{k \ln \left( \frac{p \bar{p} + \bar{p} \bar{p}}{2^{2(q-1)} (\bar{p} + \bar{p})^q} \right)}{-k \ln 2} =$$

$$= \ln \left( \frac{p \bar{p} + \bar{p} \bar{p}}{2^{2(q-1)} (\bar{p} + \bar{p})^q} \right) / (1 - q) \ln 2$$

\[ (14) \]
\[ D_1 = \lim_{l \to 0} \frac{\sum_{i=1}^{N(l)} p_i(l) \ln p_i(l)}{\ln l} = \lim_{k \to \infty} \frac{\sum_{j=0}^{k} 2^{2k} \binom{k}{j} (\frac{\dot{p}^j \ddot{p}^{k-j}}{2^{2k}(\dot{p} + \ddot{p})^k}) \ln \left( \frac{\dot{p}^j \ddot{p}^{k-j}}{2^{2k}(\dot{p} + \ddot{p})^k} \right)}{\ln 2^{-k}} = \]

\[ = \lim_{k \to \infty} \frac{\sum_{j=0}^{k} \binom{k}{j} \dot{p}^j \ddot{p}^{k-j} \ln \left( \frac{\dot{p}^j \ddot{p}^{k-j}}{2^{2k}(\dot{p} + \ddot{p})^k} \right)}{-k (\dot{p} + \ddot{p})^k \ln 2} = \]

\[ = \lim_{k \to \infty} \frac{\sum_{j=0}^{k} \binom{k}{j} \dot{p}^j \ddot{p}^{k-j} \ln (\dot{p}^j \ddot{p}^{k-j})}{-k (\dot{p} + \ddot{p})^k \ln 2} - \sum_{j=0}^{k} \binom{k}{j} \dot{p}^j \ddot{p}^{k-j} \ln (2^2 (\dot{p} + \ddot{p})^k) = \]

\[ = \lim_{k \to \infty} \frac{\ln (\dot{p}) \sum_{j=0}^{k} \binom{k}{j} \dot{p}^j \ddot{p}^{k-j} + \ln (\dot{p}) \sum_{j=0}^{k} (k - j) \binom{k}{j} \dot{p}^j \ddot{p}^{k-j}}{-k (\dot{p} + \ddot{p})^k \ln 2} + \]

\[ + \frac{\ln (2^2 (\dot{p} + \ddot{p})^k) \sum_{j=0}^{k} \binom{k}{j} \dot{p}^j \ddot{p}^{k-j}}{k (\dot{p} + \ddot{p})^k \ln 2} = \]

\[ = \lim_{k \to \infty} \frac{\ln (\dot{p}) \dot{p} \sum_{j=0}^{k-1} \binom{k-1}{j} \dot{p}^j \ddot{p}^{k-1-j} + \ln (\dot{p}) k \ddot{p} \sum_{j=0}^{k-1} (k-1) \binom{k-1}{j} \dot{p}^j \ddot{p}^{k-1-j}}{-k (\dot{p} + \ddot{p})^k \ln 2} + \]

\[ + \frac{\ln (2^2 (\dot{p} + \ddot{p})^k) \sum_{j=0}^{k} \binom{k}{j} \dot{p}^j \ddot{p}^{k-j}}{k (\dot{p} + \ddot{p})^k \ln 2} = \]

\[ = \lim_{k \to \infty} \frac{k (\dot{p} \ln (\dot{p}) (\dot{p} + \ddot{p})^{k-1} + \ddot{p} \ln (\dot{p}) (\dot{p} + \ddot{p})^{k-1})}{-k (\dot{p} + \ddot{p})^k \ln 2} + \]

\[ + \frac{k \ln (2^2 (\dot{p} + \ddot{p})) (\dot{p} + \ddot{p})^k}{k (\dot{p} + \ddot{p})^k \ln 2} = \]

\[ = \frac{(\dot{p} \ln (\dot{p}) + \ddot{p} \ln (\dot{p})) (\dot{p} + \ddot{p})^{-1}}{-\ln 2} + \frac{\ln (2^2 (\dot{p} + \ddot{p}))}{\ln 2} = \]

\[ = \frac{\ln (2^2 (\dot{p} + \ddot{p})) - (\dot{p} \ln (\dot{p}) + \ddot{p} \ln (\dot{p})) (\dot{p} + \ddot{p})^{-1}}{\ln 2} \]
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Fig. 1 – The generalized fractal dimension $D_q$ as function of the order moment.

Fig. 2 – The singularity strength $\alpha(q)$ and the Hausdorff dimension of normalized measures $f(q)$ as function of the order moment.
Fig. 3 – The singularity spectrums $f(\alpha)$ obtained by using the *indirect* and *direct* methods. The two curves coincide.