Quantum stress tensor fluctuation effects in inflationary cosmology

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Abstract. We review several related investigations of the effects of the quantum stress tensor of a conformal field in inflationary cosmology. Particular attention will be paid to the effects of quantum stress tensor fluctuations as a source of density and tensor perturbations in inflationary models. These effects can possibly depend upon the total expansion factor during inflation, and hence be much larger than one might otherwise expect. They have the potential to contribute a non-scale invariant and non-Gaussian component to the primordial spectrum of perturbations, and might be observable.

1. Introduction
The gravitational effects of quantum matter fields are often described by a semiclassical theory whereby a suitable renormalized expectation value of the quantum stress tensor operator becomes the source of gravity. The semiclassical Einstein equation may be written as

\[ G_{\mu\nu} = 8\pi G \ell_p^2 \langle T_{\mu\nu} \rangle, \tag{1} \]

where \( \ell_p \) is the Planck length and \( \langle T_{\mu\nu} \rangle \) is the renormalized expectation value. This theory clearly has a wide domain of applicability, as it includes classical general relativity theory as a special case. Furthermore, it is expected to be useful in describing non-classical effects, such as quantum violation of the classical energy conditions.

However, the semiclassical theory is limited in that it does not describe quantum fluctuations of gravity. These fluctuations can arise directly from the dynamical degrees of freedom of the gravitational field itself, the “active” fluctuations. They can also be driven by quantum fluctuations of the quantum stress tensor, the “passive” fluctuations. The latter will be the primary focus of this paper.

The fluctuations of quantum stress tensors and their physical effects have been discussed by several authors in recent years [1, 2, 3, 4, 5, 6, 7]. For a recent review with further references, see Ref. [8]. Quantum stress tensor fluctuations necessarily have a skewed, highly non-Gaussian, probability distribution, although the explicit form of this distribution has only been found in
two-dimensional spacetime models [9]. The basic features of the probability distribution are a lower bound and an infinite, positive tail. The lower bound is at the quantum inequality bound for expectation values in an arbitrary state. For the present purposes, the most important result is the non-Gaussian character, which implies that cosmological perturbations driven by stress tensor fluctuations will be non-Gaussian. Here we will summarize several related pieces of work and try to illustrate their interconnections, but for more details, the reader is referred to the original articles.

2. The spacetime geometry of inflationary cosmology

Here we briefly sketch the assumptions about the spacetime in the version of inflation which we consider. We take the metric to be that of a spatially flat Friedmann-Robertson-Walker model

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) = a^2(\eta) (-d\eta^2 + dx^2 + dy^2 + dz^2).$$

(2)

Here \(t\) is the comoving time, and \(\eta\) the conformal time. The inflationary period will be taken to be de Sitter spacetime with approximately constant curvature, so that

$$a = e^{Ht} = -\frac{1}{H\eta}.$$  

(3)

We may set \(a = 1\) at the end of inflation, which occurs at \(t = 0\), or \(\eta = -1/H\). Most inflationary models are insensitive to initial conditions, but that will not be the case for the effects which we discuss. It will be necessary to impose an initial condition that the effects of quantum fluctuations vanish at some initial time, which will be taken to be \(\eta = \eta_0\). The total expansion factor of the universe from this time to the end of inflation is

$$S = \frac{1}{H|\eta_0|}. $$

(4)

This factor will play a key role in our subsequent discussion.

We assume that inflation ends quickly, with efficient reheating to a subsequent radiation dominated epoch. This assumption is not crucial, but simplifies the discussion. In this case, the radiation immediately after inflation is described by a reheating energy of \(E_R\), where

$$H^2 = \frac{8\pi}{3} \ell_p^2 E_R^4.$$  

(5)

3. Semiclassical effects on gravity waves

In this section, we will discuss an effect in the semiclassical theory, dealing with the expectation value of a stress tensor rather than its fluctuations. However the result has features in common with the stress tensor fluctuation effects to be discussed later. The effects of \(\langle T_{\mu \nu} \rangle\) for a conformal field on the propagation of gravity waves in de Sitter spacetime was recently treated in Ref. [10]. The calculations were based on a formalism developed by Horowitz and Wald [11]. It is assumed that at early times, \(\eta < \eta_0\), there is a linearly polarized plane gravitational wave of the form

$$h_{\mu}^{\nu} = c_0 e_{\mu}^{\nu} (1 + ik\eta) e^{i(k \cdot x - k\eta)},$$

(6)

where \(c_0\) is a constant, \(e_{\mu}^{\nu}\) is the polarization tensor, and \(k\) is the wavenumber. Next the coupling to the expectation value of the quantized matter field is assumed to be switched on at \(\eta = \eta_0\). Its effect is to add a correction term \(h''_{\mu}^{\nu}(x)\), which has the same functional form as \(h_{\mu}^{\nu}(x)\), but differs
in amplitude and phase. Most importantly, its amplitude at the end of inflation depends upon the expansion factor $S$ and upon $k$. In the case that the conformal field is the electromagnetic field, the fractional correction is

$$\Gamma = \left| \frac{h'_\mu}{h^\nu_\mu} \right| = \frac{1}{5\pi} s^2 H S k.$$  \hspace{1cm} (7)

This effect grows with increasing $S$ and $k$, but its total magnitude is limited by the requirement that $\Gamma \lesssim 1$ for the one-loop approximation to hold.

Nonetheless, this effect could have observational consequences in the form of a modification of the tensor perturbations predicted by inflation. Inflationary models predict a nearly scale invariant spectrum of both scalar and tensor perturbations, both of which arise from the quantum fluctuations of nearly free fields and are Gaussian in character. The tensor perturbations result from the fluctuations of quantized linear perturbations of de Sitter spacetime [12, 13, 14]. They have not yet been observed, but are expected to leave signatures in the cosmic microwave background that could be seen in the future. The effect described by Eq. (7) modifies the normalization of the vacuum graviton modes in a way which breaks the scale invariance, and increases the power on shorter wavelengths.

4. Negative power spectra

One of the remarkable properties of quantum stress tensor fluctuations is that they can introduce negative power spectra of fluctuations. In most statistical processes, this is not possible. The well-known Wiener-Khinchin [15, 16] theorem states that the Fourier transform of a correlation function is a power spectrum. A corollary of this theorem is that the power spectrum can normally be written as the expectation value of a squared quantity, and hence must be positive. However, the latter result can fail in quantum field theory, and negative power spectra are possible [17]. The basic reason is that quantum correlation functions can be highly singular at coincident points, and the expectation value in the Wiener-Khinchin theorem may not exist.

A simple example is the spectrum of vacuum energy density fluctuations of the quantized electromagnetic field in flat spacetime. The energy density correlation function is

$$C_0(\tau, r) = \langle \rho(t, \mathbf{x}) \rho(t', \mathbf{x'}) \rangle = \frac{(\tau^2 + 3r^2)(r^2 + 3\tau^2)}{\pi^4 (r^2 - \tau^2)^6},$$  \hspace{1cm} (8)

where $\tau = t - t'$ and $r = |\mathbf{x} - \mathbf{x'}|$. Its spatial Fourier transform is

$$\hat{C}_0(\tau, k) = -\frac{k^4 \sin(k \tau)}{960\pi^5 \tau},$$  \hspace{1cm} (9)

and the power spectrum is

$$P_0(k) = \hat{C}_0(0, k) = -\frac{k^5}{960\pi^5}.$$  \hspace{1cm} (10)

The negative sign in the power spectrum essentially interchanges correlations and anticorrelations, compared to what one would have with a positive spectrum of the same functional form.

5. Density perturbations in inflation

Just as fluctuations of the free graviton field in de Sitter spacetime can lead to observable tensor perturbations, fluctuations of the inflaton field can produce density perturbations [18, 19, 20, 21,
The resulting spectrum of nearly scale invariant, Gaussian fluctuations has apparently been observed in the temperature fluctuations of the cosmic microwave background [23]. However, quantum stress tensor fluctuations can produce an additional, non-scale invariant, non-Gaussian contribution, which was studied in Refs. [24, 25]. This effect arises from the quantum fluctuations of the comoving energy density of a conformal field in its vacuum state. In a dynamic model treated in Ref. [25], these fluctuations couple to the dynamics of the inflaton field and produce a spectrum of density perturbations proportional to \( S^2 \),

\[
\mathcal{P}(k) = \frac{8\ell^4 p^2 S^2}{75}.
\]  

Note that

\[
\mathcal{P}(k) = 4\pi k^3 P(k),
\]

is the quantity usually called the power spectrum in cosmology, and which is independent of \( k \) for a scale invariant spectrum. Thus the effect of the quantum stress tensor fluctuations is a blue-tilted, non-Gaussian contribution.

The fact that this contribution has not yet been observed can be interpreted as placing an upper bound on \( S \), which is found to be

\[
S \lesssim 10^{42} \left( \frac{10^{12} \text{ GeV}}{E_R} \right)^3. 
\]  

This bound is consistent with adequate inflation to solve the horizon and flatness problems, which requires \( S > 10^{23} \).

### 6. Gravity waves from stress tensor fluctuations in inflation

In addition to the scalar (density) perturbations, stress tensor fluctuations can also create tensor perturbations, which are gravity waves [26]. Here the fluctuations of spatial components of the stress tensor can couple to the transverse, tracefree components of the gravitational field. The result at the end of inflation is a spectrum of gravity wave fluctuations with a power spectrum given by

\[
\mathcal{P}(k) = \frac{4\ell^4}{3\pi} k^2 H^2 S^2 \left(1 + k^2 H^{-2} \right). 
\]  

This is an example of a negative power spectrum, as well as one which is strongly blue-tilted. This result assumes that the coupling between the quantum stress tensor and the gravitational field is switched on suddenly at \( \eta = \eta_0 \). However, more gradual switching leads to qualitatively similar results. If \( S \) were sufficiently large, then the resulting tensor perturbations in the CMB spectrum should have already been observed. This leads to the constraint

\[
S \lesssim 10^{46} \left( \frac{10^{12} \text{ GeV}}{E_R} \right)^3, 
\]  

which is somewhat weaker than the bound, Eq. (13), which comes from density perturbations.

However, gravity wave fluctuations are potentially detectable at much smaller scales than are the density perturbations. The shorter wavelengths of the primordial density perturbation spectrum have presumably been erased by nonlinear classical effects by now, but gravity waves interact very weakly and should still exist. This raises the possibility of detecting the primordial gravity waves from stress tensor fluctuations in gravity wave detectors, in the form of background...
noise corresponding to the spectrum given in Eq. (14). LIGO has placed limits \([27]\) of \(h \lesssim 10^{-24}\) on scales of the order of \(10^2\) km, leading to the constraint

\[
S < 10^{23} \left( \frac{10^{10} \text{GeV}}{E_R} \right)^3.
\]

This result is compatible with adequate inflation to solve the horizon and flatness problems only if

\[
E_R \lesssim 10^{10} \text{GeV}.
\]

This would put a non-trivial constraint on inflationary models. An even more exciting possibility is that Earth or space-based gravity wave detectors might have a chance of detecting the effects of quantum stress tensor fluctuations during inflation.

### 7. The transplanckian issue

There is an important qualification to all of the results which depend upon \(S\), the expansion factor during inflation. This is that quantum field modes above the Planck scale in energy are required. Consider, for example, the example in the previous section of a gravity wave with a present proper wavelength of \(100\) km. If we assume \(E_R \approx 10^{10} \text{GeV}\), then there has been an expansion by a factor of about \(10^{23}\) since reheating to redshift to the current CMB temperature. This, combined with an additional expansion of at least \(10^{23}\) during inflation, implies that this mode would have had a proper wavelength of about \(10^{-3}\) \(\ell_p\) at the beginning of inflation. The quantum field would have to have fluctuations on this scale to create such a gravity wave mode.

This raises questions as to whether the framework of quantum field theory on a fixed or weakly fluctuating background can be trusted at this scale. However, an even more extreme version of the transplanckian issue arises in black hole thermodynamics. Hawking’s \([28]\) derivation of of black hole radiance relies upon modes which begin far above the Planck energy. The fact that the Hawking effect gives a beautiful unification of gravity, thermodynamics, and quantum theory can be considered to be a powerful argument to take transplanckian modes seriously. It is true that it is possible to derive the Hawking effect without transplanckian modes \([29, 30]\), but only at the price of introducing modified dispersion relations which break local Lorentz symmetry and hence postulate new physics. An analogous prescription in cosmology is to consider modes only after they redshift below the Planck energy in the comoving frame \([25]\). There has been an extensive discussion of the possible role of transplanckian modes in inflationary cosmology. (See Ref. \([25]\) for a lengthy list of references.)

The dependence of the gravity wave spectrum upon a positive power of \(S\) might seems to contradict a theorem due to Weinberg \([31]\), which was generalized by Chaicherdsakul \([32]\). This theorem states that radiative corrections during inflation should not grow faster than a logarithm of the scale factor. However, as was discussed in more detail in Ref. \([25]\), the effect of density perturbations growing as a power of \(S\) is really due to high frequency modes at the initial time, and is hence always large rather than growing. This applies to all the effects discussed in the present paper.

### 8. Discussion

Here we have reviewed three distinct, but related, quantum effects in inflation which depend upon the expansion factor \(S\) during inflation. The first is the effect of the expectation value of a conformal field upon the amplitude of gravity waves in de Sitter spacetime \([10]\). The second is the effect of energy density fluctuations of such a field upon the spectrum of primordial density
perturbations [24, 25]. The third effect is the role of quantum stress tensor fluctuations in generating a spectrum of primordial gravity waves [26]. The latter two effects are associated with non-Gaussian fluctuations, and all three effects grow as $S$ increases and produce a blue-tilted spectrum, in which there is more power on shorter wavelengths. The gravity waves produced by stress tensor fluctuations are an example of a negative power spectrum, which is usually forbidden, but is possible in quantum field theory [17].

All three effects have the potential to produce observable effects, either in the cosmic microwave background radiation, or in gravity wave detectors. However, all three effects also rely upon the existence of transplanckian modes. As such, they have the possibility to open an observational window on transplanckian physics.

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References

[1] Wu C-H and Ford L H 2001 Phys. Rev. D 64 045010 quant-ph/0012144.
[2] Borgman J and Ford L H 2004 Phys. Rev. D 70 064032 gr-qc/0307043.
[3] Hu B L and Verdaguer E 2004 Living Rev. Rel. 7 3 gr-qc/0307032.
[4] Ford L H and Woodard R P 2005 Class. Quant. Grav. 22 1637 gr-qc/0411003.
[5] Ford L H and Roman T A 2005 Phys. Rev. D 72 105010 gr-qc/0506026.
[6] Thompson R T and Ford L H 2006 Phys. Rev. D 74 024012 gr-qc/0601137.
[7] Perez-Nadal G, Roura A and Verdaguer E 2010 JCAP 1005:036 arXiv:0911.4870.
[8] Ford L H and Wu C H 2008 AIP Conf. Proc. 977 145 arXiv:0710.3787.
[9] Fewster C J, Ford L H, and Roman T A 2010 Phys. Rev. D 81 121901(R) arXiv:1004.0179.
[10] Hsiang J-T, L H Ford, Lee D-S, and Yu H-L 2011 Phys. Rev. D 83 084027 arXiv:1012.1582.
[11] Horowitz G T and Wald R M 1980 Phys. Rev. D 21 1462.
[12] Starobinsky A A 1979 JETP Lett. 30 682.
[13] Abbott L F and Wise M B 1984 Nucl. Phys. B 244 541.
[14] Allen B 1988 Phys. Rev. D 37 2078.
[15] Wiener N 1930 Acta. Math Stockholm 55 117.
[16] Khinchin A 1934 Math. Ann. 109 604.
[17] Hsiang J-T, Wu C-H, and Ford L H 2011 Phys. Lett. A 375 2296 arXiv:1012.3226.
[18] Mukhanov V and Chibisov G 1981 JETP Lett. 33 532.
[19] Guth A H and Pi S-Y 1982 Phys. Rev. Lett. 49 1110.
[20] Hawking S W 1982 Phys. Lett. B 115 295.
[21] Starobinsky A A 1982 Phys. Lett. B 117 175.
[22] Bardeen J M, Steinhardt P J, and Turner M S 1983 Phys. Rev. D 28 679.
[23] Komatsu E 2011 et al., Astrophys. J. Suppl. 192 18 arXiv:1001.4538.
[24] Wu C-H, Ng K-W, and Ford L H 2007 Phys. Rev. D 75 103502 gr-qc/0608002.
[25] Ford L H, Miao S-P, Ng K-W, Woodard R, and Wu C-H 2010 Phys. Rev. D 82 043501 arXiv:1005.4530.
[26] Wu C-H, Hsiang J-T, Ford L H, and Ng K-W, arXiv:1105.1155.
[27] Sutton P J 2008 J. Phys. Conf. Ser. 110 062024.
[28] Hawking S W 1975 Commun. Math. Phys. 43 199.
[29] Unruh W G 1995 Phys. Rev. D 51 2827 gr-qc/9409008.
[30] Corley S and Jacobson T 1996 Phys. Rev. D 54 1568 hep-th/9601073.
[31] Weinberg S 2005 Phys. Rev. D 72 043514.
[32] Chaichereaksul K 2007 Phys. Rev. D 75 063522.