A twisted conformal field theory description of dissipative quantum mechanics

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Abstract

We show how the recently proposed CFT for a bilayer Quantum Hall system at filling $\nu = \frac{m}{pm+2}$ [1]–[3], the Twisted Model (TM), is equivalent to the system of two massless scalar bosons with a magnetic boundary interaction as introduced in [4], at the so called “magic” points. We are then able to describe, within such a framework, the dissipative quantum mechanics of a particle confined to a plane and subject to an external magnetic field normal to it. Such an analogy is further developed in terms of the TM boundary states, by describing the interaction between an impurity with a Hall system.

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1 Introduction

In ref. [5] the effect of the dissipative term ($\eta \dot{q}$) on the motion of an electron confined in a plane and in the presence of an external magnetic field $B$, normal to the plane, was analyzed. By using the correspondence principle it was possible to quantize the system and to study its time evolution on large time scales ($t \gg \frac{1}{\eta}$) employing coherent states techniques. It was found that the effect of dissipation would simply be accounted for by a rotation combined with a scale transformation on the coordinate $z$ of the electron: $z \rightarrow (\rho e^{i\gamma})z$, $\rho = \left(\frac{\eta^2 + \omega^2}{\omega^2}\right)^{\frac{1}{2}}$, $\gamma = -\arctan\left(\frac{\eta}{\omega}\right)$. As a result, the gaussian width describing the electron in the lowest Landau level (LLL) state would get reduced. Furthermore the current operator, lying in the Hall direction in the absence of dissipation, for $\eta \neq 0$ would acquire a longitudinal component, with a resulting metallic conductance $\sigma_L \neq 0$ for the multielectron system.

It is remarkable that such an effect was soon after proposed in ref. [4] in the context of boundary conformal field theories (BCFT). The authors consider a system of two massless free scalar fields which have a boundary interaction with a periodic potential and furthermore are coupled to each other through a boundary magnetic term. By using a string analogy, this last interaction allows for exchange of momentum of the open string moving in an external magnetic field. The magnetic interaction enhances one chirality with respect to the other producing the same effect of a rotation together with a scale transformation on the fields as for the dissipative system of ref. [5] where the string parameter places the role of dissipation.

It is crucial to observe that conformal invariance of the theory is preserved only at special values of the parameters entering the action, the so called “magic” points.

The aim of this letter is to show that the effective conformal field theory, the twisted model (TM), recently proposed in refs. [1]–[3] is well suited to describe a dissipative system precisely at the “magic” points. In fact the presence of a $Z_m$ twist accounts, in the open string picture, for a mismatch of momentum exchange at its two endpoints. For the special $Z_2$ case the TM has central charge $c = 2$ and describes a system of two layers coupled through a topological defect. It is interesting to notice that the fields which diagonalize such an interaction can be expressed in terms of the original layers fields through a rotation and a scale transformation. The amount of the rotation and scale transformation is fixed by the order of the twist. This is the content of the $m$-reduction procedure introduced in ref. [6]. The paper is organized as follows:

In sec. 2 we recall some results of the dissipative quantum mechanics obtained in refs. [5][4].
In sec. 3 we review the properties of the TM which are relevant to the description of the dissipative effects and make a correspondence with the BCFT approach of ref. [4].

In sec. 4 we analyze the effect of the magnetic field on the TM boundary states (BS) recently introduced in ref. [3] and make a correspondence with the bulk degrees of freedom of ref. [2].

In sec. 5 we give explicitly the duality transformations which relate the UV properties with the IR ones in the TM context.

2 Dissipative quantum mechanics

In the presence of dissipation, the equations of motion of an electron moving in a plane with a magnetic field $B$ transverse to it are given by:

$$\frac{d^2 z}{dt^2} = -(\eta + i\omega) \frac{dz}{dt}, \quad \frac{d^2 \bar{z}}{dt^2} = -(\eta - i\omega) \frac{d\bar{z}}{dt},$$

(1)

where $z(t) = x(t) + iy(t)$, $\omega = B$ (in the units $m = c = e = 1$) and $\eta$ is the viscosity coefficient.

By assuming the following relations between the canonical momenta $p_z$, $p_{\bar{z}}$ and the velocities $v_z$, $v_{\bar{z}}$ for $\eta \neq 0$:

$$p_z = v_z + \frac{\eta + i\omega}{2} z, \quad p_{\bar{z}} = v_{\bar{z}} + \frac{\eta - i\omega}{2} \bar{z},$$

(2)

it is possible to quantize the system using the correspondence principle and to define the creation and annihilation operators $\hat{b}$ and $\hat{b}^+$

$$\hat{b} = \frac{\omega + i\eta}{\sqrt{2\omega}} \hat{z}_\infty, \quad \hat{b}^+ = \frac{\omega - i\eta}{\sqrt{2\omega}} \hat{z}_\infty,$$

(3)

with commutation relation $[\hat{b}, \hat{b}^+] = 1$. The coordinates $z_\infty$, $\bar{z}_\infty$ describe the center of the Larmor orbit of the electron and $\hat{z}_\infty$, $\hat{\bar{z}}_\infty$ are the corresponding operators. It is now interesting to observe that the time evolution on a large time scale ($t \gg \frac{1}{\eta}$) is simply defined as:

$$\psi_{j, t \to \infty}^\eta (z) = \lim_{t \to \infty} \langle z | U(t, 0) | \psi_j \rangle = \langle z_\infty | \psi_j \rangle,$$

(4)
We are interested in the case in which $\psi_j(z)$ is the wave function of an electron in the lowest Landau level (LLL):

$$
\psi_j(z) = \sqrt{\frac{\omega}{\pi}} \frac{1}{\sqrt{j!}} \left( z \sqrt{\frac{\omega}{2}} \right)^j e^{-\frac{\omega z^2}{4}}.
$$

(5)

In order to build the state $|z(t)\rangle_{t\to\infty}$ it is possible to define the coherent state $|\xi\rangle$ such that $\hat{b}|\xi\rangle = \xi|\xi\rangle$ where $\xi = \frac{\omega-i\eta}{\sqrt{2\omega}}$ and $|\xi\rangle$ is given by $|\xi\rangle = e^{-\frac{|\xi|^2}{4}} e^{\hat{b}^+} |0\rangle$ with the vacuum state $|0\rangle = \sqrt{\frac{\omega}{\pi}} e^{-\frac{\omega z^2}{4}}$ annihilated by $\hat{b}$: $\hat{b}|0\rangle = 0$.

If we require that the zero angular momentum state $\psi_0(z)$ is annihilated by $\hat{b}$ (that is we require unitarity in our description), then we immediately get

$$
\psi_\eta_{j,t\to\infty}(\xi) = e^{-\frac{|\xi|^2}{4}} \left< 0 \left| \sqrt{\frac{\omega^2 + \eta^2}{\pi \omega}} \frac{1}{\sqrt{j!}} \left( \frac{\omega-i\eta}{\sqrt{2\omega}} z \right)^j e^{-\frac{\omega z^2 + \eta^2}{4\omega} |z|^2} \right| 0 \right>
$$

(6)

having expressed $\xi$, $\bar{\xi}$ in terms of the original $z$, $\bar{z}$ variables. By comparing eq. (6) with eq. (5) we can infer that the effect of dissipation can be simply accounted for by making the following transformations on the variable $z$: $z \to (\rho e^{i\gamma})z$, where $\rho = \left( \frac{\eta^2 + \omega^2}{\omega^2} \right)^{\frac{1}{2}}$ and $\gamma = -\arctan \left( \frac{\eta}{\omega} \right)$, that is a rotation plus a scale transformation.

Furthermore for a vector operator $O$ one has the transformation properties:

$$
O_{x,t\to\infty}(z) = \frac{1}{\rho} (O_{x,0} \cos \gamma - O_{y,0} \sin \gamma),
$$

$$
O_{y,t\to\infty}(z) = \frac{1}{\rho} (O_{x,0} \sin \gamma + O_{y,0} \cos \gamma).
$$

(7)

The striking consequence of the above relations (see ref.[5]) is that, if one starts with the current density operator $\hat{J}$ which accounts for a Hall conductance $\sigma_H = \frac{1}{\omega}$ (being $\left< \hat{J} \right> = (-\frac{E}{\omega},0)$ only, by applying the transformations given in eq. (7) one obtains

$$
\left< \hat{J}_\eta \right> = \left( -\frac{\omega}{\omega^2 + \eta^2} E, \frac{\eta}{\omega^2 + \eta^2} E \right)
$$

(8)

with a resulting metallic conductance $\sigma^L = \frac{\eta}{\omega^2 + \eta^2}$ different from zero! The brackets above indicate an expectation value.
It is remarkable that such an effect was soon after proposed in ref. [4] in the context of BCFT. A system of two massless scalar fields in 1 + 1 dimensions is considered, which are free in the bulk except for boundary interactions, which couple them. Its action is given by

\[ S = S_{\text{bulk}} + S_{\text{pot}} + S_{\text{mag}} \]

where:

\[ S_{\text{bulk}} = \frac{\alpha}{4\pi} \int_0^T dt \int_0^l d\sigma \left( (\partial_\mu X)^2 + (\partial_\mu Y)^2 \right), \]

(9)

\[ S_{\text{pot}} = \frac{V}{\pi} \int_0^T dt \left( \cos X(t, 0) + \cos Y(t, 0) \right), \]

(10)

\[ S_{\text{mag}} = \frac{i}{4\pi} \int_0^T dt \left( X \partial_t Y - Y \partial_t X \right)_{\sigma=0}. \]

(11)

In the equations above \( \alpha \) determines the strength of dissipation and is related to the potential \( V \), as it can be seen, by rescaling the fields; \( \beta \) is related to the strength of the magnetic field \( B \) orthogonal to the \( X - Y \) plane, as \( \beta = 2\pi B \).

The magnetic term introduces a coupling between \( X \) and \( Y \) at the boundary keeping conformal invariance. Such a symmetry gets spoiled by the presence of the interaction potential term except for the magic points \((\alpha, \beta) = \left( \frac{1}{n^2 + 1}, \frac{n}{n^2 + 1} \right), n \in \mathbb{Z} \). For such parameters values the theory is conformal invariant for any potential strength \( V \). It is possible to express all the degrees of freedom of such a system in terms of the boundary states, which can be easily constructed in two steps:

- By considering first the magnetic interaction term it is easy to see that the net effect of the magnetic field is a chiral \( o(2) \) rotation of the Neumann boundary state \( |N> \) as:

\[ |B_0> = \sec \left( \frac{\delta}{2} \right) e^{i\delta R_M} |N> . \]

(12)

Above the rotation operator \( R_M \) is given by

\[ R_M = (y_0 X - x_0 Y) + \sum_{n>0} \frac{i}{n} (\alpha_n^Y \alpha_n^X - \alpha_n^X \alpha_n^Y) \]

(13)

and the rotation parameter \( \delta \) is defined in terms of the parameters \( \alpha, \beta \) as \( \tan \left( \frac{\delta}{2} \right) = \frac{\beta}{\alpha} \).

- By considering in addition the effect of the potential term one obtains the boundary state \( |B_V> \) as:

\[ |B_V> = \sec \left( \frac{\delta}{2} \right) e^{i\delta R_M} e^{-H_{\text{pot}}(2X_L') - H_{\text{pot}}(2Y_L')} |N > | X' > | Y' > \]

(14)
where the rotated and rescaled coordinates $X', Y'$ have been introduced as:

$$
X' = \cos \frac{\delta}{2} \left( \cos \frac{\delta}{2} X - \sin \frac{\delta}{2} Y \right),
$$

$$
Y' = \cos \frac{\delta}{2} \left( \sin \frac{\delta}{2} X + \cos \frac{\delta}{2} Y \right).
$$

(15)

Since the $o(2)$ rotation commutes with the Virasoro generators $L_n$, the state $|B_V\rangle$ satisfies the Ishibashi condition $\left( L_n - \tilde{L}_{-n} \right) |B_V\rangle = 0$. Notice the strong resemblance of the above relations with the transformation properties given in eq. (7), describing the effect of dissipation on the vector operator $O$. We will come back later to further comment on this analogy, after introducing the TM in section 3. Let us only observe here that it is possible to get further insight into this analogy by considering the partition function $Z_{NB_V}$ defined as:

$$
Z_{NB_V} = \sec \left( \frac{\delta}{2} \right) < N | q^{L_0+\tilde{L}_0} | B_V >
$$

(16)

because, in the open string language, the rotation $\mathcal{R}_M$ introduces now twisted boundary conditions in the $\sigma$ direction.

In the following section we will introduce the two interacting layers system in the picture of the twisted theory and show that the interlayer interaction is diagonalized by the effective fields $X, \phi$ which are related to the layers fields $Q^{(1)}, Q^{(2)}$ just by the relation given in eq. (15) for $\alpha = \beta$. Also a generalized construction of the partition function $Z_{NB_V}$ will be performed in that context.

### 3 The TM model

In order to introduce the TM model let us consider a system of two interacting parallel layers of $2D$ electrons in a strong perpendicular magnetic field $B$. The filling factor $\nu(\alpha) = \frac{1}{2p+2}$ is the same for the two $\alpha = 1, 2$ layers (balanced system) while the total filling is $\nu = \frac{1}{p+1}$. The CFT description for such a system can be given in terms of two compactified bosons $Q^{(a)}$ (with central charge $c = 2$) defined on the single layer “$a$”.

We review now the construction of the $Q^{(a)}$ fields in the TM description in its key steps, which will turn out to be also relevant for the analogy we are proposing. We show how the $m$-reduction procedure is equivalent to the effect of a magnetic boundary term; in other words the magnetic term can result in a twist on the neutral field.
Starting from the Fubini field
\[ Q(z) = q - i pln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \]  
\tag{17}
compactified on a circle with radius \( R^2 = 1/\nu = 2(p + 1) \) \cite{2}, we perform the transformation \( z \rightarrow e^{i\theta_j z} \) and get \( Q(e^{i\theta_j z}) \), where \( z = e^{-i\frac{2\pi x}{L}} \) and \( \theta_j = \frac{2\pi j}{m} \), \( j = 0, ..., m - 1 \).

For \( m = 2 \) (which regards the two layers system considered in the paper) there are two possible values, \( \theta_0 = 0, \theta_1 = \pi \), with the corresponding fields:

\[ Q(z) = q - i pln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \equiv Q^{(1)}(z) \]  
\tag{18}
\[ Q(-z) = q + \pi p - i pln z + i \sum_{n \neq 0} \frac{a_n}{n} (-1)^n z^{-n} \equiv Q^{(2)}(z). \]  
\tag{19}

Summing and subtracting, we get the charged field \( X(z) \) and the neutral one \( \phi(z) \) which satisfies twisted boundary conditions \( \phi(e^{i\pi z}) = -\phi(z) \). The \( X(z) \) and \( \phi(z) \) fields, which have the profound meaning of diagonalizing the interlayer interaction (see ref. \cite{2} and following section) can be rewritten in a more enlightening form as:

\[ X(z) = \cos(\theta/4) \left( \sin(\theta/4)Q^{(1)}(z) + \cos(\theta/4)Q^{(2)}(z) \right) \]  
\tag{20}
\[ \phi(z) = \cos(\theta/4) \left( \cos(\theta/4)Q^{(1)}(z) - \sin(\theta/4)Q^{(2)}(z) \right). \]  
\tag{21}

Such a transformation consists of a scale transformation plus a rotation; for \( \theta = \pi \) the fields \( X(z) \) and \( \phi(z) \) of the \( Z_2 \) twisted theory introduced in refs. \cite{1}\cite{2} are obtained and the transformations above coincide with the transformations given in eqs. (15) for \( \delta = \frac{\pi}{2} \).

In this context it is possible to express the effect of a boundary magnetic term as:

\[ |B_0(\delta) > = \sec(\theta/4)e^{i\delta YeM} |N(\theta - 2\delta) > \]  
\tag{22}
where

\[ |N(\theta) > = \sqrt{R} \sum_{n > 0} e^{\sum_{n > 0}^{(X)} a_{\alpha_n}^{(X)} e^{\sum_{n > 0}^{(\phi)} a_{\alpha_n}^{(\phi)}}} |w_X, 0 > \otimes |w_\phi, 0 > \]  
\tag{23}
and the rotation $R_M$, defined in eq. (13), is now given in terms of $X$ and $\phi$.

The boundary state for the (untwisted) twisted sector in the folded theory is obtained from the above rotation on the Neumann boundary state when $\delta = 0$, $\pi$ and can be seen as due to a boundary magnetic term according to [4].

Finally, if we perform the rescaling $z \to z^{\frac{1}{2}}$, $a_{2n+1} \to \sqrt{2}a_{n+\frac{1}{2}}$, $q \to \frac{\sqrt{2}}{2}$, for $\alpha = \beta$ (that is also for $\eta = \omega$), we obtain the $X(z)$ and $\phi(z)$ fields in the standard form; that is the twisted CFT just constructed (see refs. [1][2]) can be conjectured to represent the correct CFT which describes dissipative quantum mechanics (DQM) of ref. [5].

The full set of characters and the partition function for this model was given in [2]. In [3] the effect of the presence of an impurity in the system was analyzed by mapping the impurity into a boundary state by using the well known folding procedure and the boundary partition function $Z_{AB}$ was given. In the rest of the paper we will try to convince the reader that such a conjecture is correct even in the context of boundary CFT. More precisely we will study the effect of the magnetic field on the BS of the TM model [3] and construct the correspondent partition function $Z_{NBV}$ (see eq. (16)). In the following section we will explicitly see how the boundary state so constructed is related, for the different values of $\delta = 0$ or $\delta = \pi$, to the description of the bulk degrees of freedom of the TM.

4 Twisted Model and magnetic boundary interaction

In order to analyze the effect of a magnetic boundary term in the TM we can resort to the fermionized theory and define a pair of left-moving fermions as:

$$\psi_1 = c_1 e^{\frac{i}{2}(Q^{(2)}+Q^{(1)})} = c_1 e^{iX} \quad \psi_2 = c_2 e^{-\frac{i}{2}(Q^{(2)}-Q^{(1)})} = c_2 e^{i\phi} \quad (24)$$

where $c_i$, $i = 1, 2$ are cocycles necessary for the anticommutation.

At the first non-trivial “magic” point $\alpha = \beta = \frac{1}{2}$ of ref. [4] corresponding in our model to the $m = 2$, $p = 0$ case it is very simple to obtain the action of the magnetic boundary term on the Neumann state $|N>$. In fact, separating the two Dirac fermions into real and imaginary parts, $\varphi_1 = \psi_{11} + i\psi_{12}$, $\varphi_2 = \psi_{21} + i\psi_{22}$, we get four left-moving Majorana fermions given by $\psi = (\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}) = (\cos X, \sin X, \cos \phi, \sin \phi)$ and a corresponding set of right-moving ones. In this new
language the magnetic term acts only on the fourth Majorana fermion as $R_M = e^{2i\delta}$ where $\delta = 0 (\delta = \frac{\pi}{2})$ for the untwisted (twisted) sector of our theory, being its action the identity for the other components. Now we add a potential term which acts on the Majorana as:

$$R_P = \begin{pmatrix}
\cos(2V) & -\sin(2V) & 0 & 0 \\
\sin(2V) & \cos(2V) & 0 & 0 \\
0 & 0 & \cos(2V) & -\sin(2V) \\
0 & 0 & \sin(2V) & \cos(2V)
\end{pmatrix}. \quad (25)$$

So the overall rotation of the corresponding fermionic boundary states is $R = R_M R_P$.

In terms of $R$ the partition function $Z_{AB}$, where $|A>$ is the Neumann boundary state $|N>$ and $|B>$ is the magnetic-potential BS $|B_V>$, can be rewritten as:

$$Z_{NB_V}(\delta, V) = \langle N| e^{-L(L_0+L_0)}|B_V(\delta) >= \sqrt{2}(q)^{-2/3} \prod_{n=1}^{\infty} \det \left( I + q^{n-\frac{1}{2}} R \right) \quad (26)$$

where $q = e^{2i\pi \tau}$ and $I$ is the identity matrix.

Finally we get:

$$Z_{NB_V}(\delta, V) = \sqrt{2} \frac{\theta_3(V|\tau)}{\eta(\tau)} \sqrt{\frac{\theta_3(V|\tau)}{\eta(\tau)}} \sqrt{\frac{\theta_3(\delta + V|\tau)}{\eta(\tau)}}, \quad (27)$$

where $\delta = 0 (\delta = \frac{\pi}{2})$ for the untwisted (twisted) sector.

On the other hand, choosing $|A>$ to be the vacuum state we can compute the partition functions $Z_{AB}$ where $|B>$ are all the BS for the TM obtained in ref. [3]. In terms of the characters defined in [2] (for $i = 0, 1$ and $f = 0, 1$)

$$\tilde{\chi}_{(i,0),f}(\tau) = \frac{\theta_3(\tau) \theta_3(\tau) + (-)^f \sqrt{\theta_3(\tau) \theta_4(\tau)}}{\eta(\tau)} + (-)^f \frac{\theta_4(\tau) \theta_4(\tau) + (-)^f \sqrt{\theta_4(\tau) \theta_3(\tau)}}{2 \eta(\tau)};$$

$$\chi_{(0,0)}^+(\tau) = \frac{\sqrt{2} \eta(\tau)}{\theta_3(\tau) \theta_3(\tau)}; \quad \chi_{(1,0)}^+(\tau) = \frac{\theta_2(\tau) \theta_2(\tau)}{2 \eta(\tau)}.$$  \quad (28)

we get, by using eqs.(27,28):

$$Z_{NB_V}(\delta = 0, V = 0) = \tilde{\chi}_{(0,0),0}(\tau) + \tilde{\chi}_{(1,0),0}(\tau) + \tilde{\chi}_{(0,0),1}(\tau) + \tilde{\chi}_{(1,0),1}(\tau);$$

$$Z_{NB_V}(\delta = 0, V = \frac{\pi}{2}) = 2\sqrt{2}\chi_{(0,0)}^-(\frac{1}{2}) \tau); \quad Z_{NB_V}(\delta = \frac{\pi}{2}, V = 0) = 2\chi_{(0,0)}^+(\frac{1}{2}) \tau);$$

$$Z_{NB_V}(\delta = \frac{\pi}{2}, V = \frac{\pi}{2}) = 2\chi_{(1,0)}^+(\frac{1}{2}) \tau). \quad (29)$$
We will comment on such results in the following section, resorting to the duality properties of the TM vacua, which are explicitly evidenced in its analogy with the one impurity Kondo model.

5  Duality properties and comments

Our description adapts very closely to a system of two interacting Luttinger liquids coupled resonantly through an impurity placed in between. Indeed, as stated in ref. [7], the problem of tunneling between two different quantum Hall states at \( \nu = 1 \) and \( \nu = 1/3 \) can be mapped onto that of tunneling through a barrier in a \( g_L = 1/2 \) Luttinger liquid; in particular, if the two layers are equivalent the two tunneling barriers are symmetric. This is the condition for a perfect resonance, where the system flows to the perfectly transmitting fixed point [8]. Equivalently such a model can be mapped to an anisotropic two channel Kondo problem in which the occupation of the impurity state corresponds to the state of the Kondo spin, the two leads are the two channels, the tunneling amplitudes play the role of the transverse couplings and the scaling dimensions of the fields are related to anisotropy. So we get:

\[
\mathcal{L}^{UV} \propto V \left( S^x_j \cos(X(0)) \cos(\phi(0)) + S^y_j \sin(X(0)) \cos(\phi(0)) \right) \tag{30}
\]

where \( X \) and \( \phi \) coincide with the charged and neutral fields introduced in eqs. (20) and (21) and \( S^{x,y}_j \) are \( su(2)_q \) impurity spins in the spin \( j \) representation. As pointed out in ref. [9] there is a duality between the UV and the IR regime in the Kondo problem. Indeed let us consider the anisotropic Kondo model defined by a boundary Lagrangian as in eq. (30) where \( X, \phi \) are massless bosonic fields. Such an interaction induces a flow from Neumann (UV) to Dirichlet (IR) boundary conditions as \( V \) increases. The integrability constraint requires the infrared Lagrangian to contain a single non-trivial term:

\[
\mathcal{L}^{IR} \propto V_D \left( S^x_{j-1/2} \cos\left( \frac{\tilde{X}(0)}{g_L} \right) \cos\left( \frac{\tilde{\phi}(0)}{g_L} \right) + S^y_{j-1/2} \sin\left( \frac{\tilde{X}(0)}{g_L} \right) \cos\left( \frac{\tilde{\phi}(0)}{g_L} \right) \right) \tag{31}
\]

where now \( \tilde{X}(0), \tilde{\phi}(0) \) are the “dual” bosonic fields\(^3\) and the spins are in the representation of spin \( j - 1/2 \). In conclusion the duality results in the following trans-

\(^3\)The fields \( X(0,t), \phi(0,t), \tilde{X}(0,t), \tilde{\phi}(0,t) \) appearing in eqs. (30), (31) are expressed in the folded system description as \( (left) \pm (right) \) components respectively (see ref. [3] for a more detailed description).
formations:

\[ g_L \to \frac{1}{g_L}, \quad V \to V_D, \quad S_j^{x(y)} \to S_{j-1/2}^{x(y)}. \quad (32) \]

If we remember the relation \( R = 2\sqrt{g_L} \) [10], we see that duality maps the compactification radius \( R^2 \) to \( 4/R^2 \), a half-integer spin to an integer one (i.e. no spin) and conversely. At the same time the weak coupling limit goes to the strong one, Neumann boundary conditions to Dirichlet ones and a description in terms of Laughlin quasiholes to one in terms of electrons. It also maps an untwisted sector to a twisted one \( (\delta \to \delta + \pi/2) \) and conversely. In such a context, the theory without spin is dual to our TM model in which the presence of a spin-1/2 impurity gives rise to twisted boundary conditions [3]. In fact the above duality can be explicitly realized as follows. Let us consider the UV vacuum corresponding to the boundary partition function \( Z_{NBV}(\delta = 0, V = \pi/2) \); by performing the transformations: \( \tau \to -\frac{1}{\tau} \) and \( \delta = 0 \to \delta = \pi/2 \) we find

\[ Z_{NBV}(\delta = 0, V = \frac{\pi}{2}) \to Z_{NBV}(\delta_D = \frac{\pi}{2}, V_D = \frac{\pi}{2}) \quad (33) \]

i.e. the partition function \( Z_{NBV}(\delta_D = \frac{\pi}{2}, V_D = \frac{\pi}{2}) \) corresponding to the IR vacuum. The finite renormalized value of \( V_D \) is the consequence of the non-perturbative multi-solitons corrections (see ref.[4] for details). Moreover the effect of the spin-1/2 impurity (a quasihole impurity in our case) is to induce twisted boundary conditions (i.e. \( \delta = 0 \to \pi/2 \)). This is the analog of the non-Fermi liquid behavior of the over-screened Kondo problem. Moreover, the duality between the two fixed points with twisted (untwisted) boundary conditions implies that two equivalent descriptions exist in which the role of the electrons and quasiholes gets exchanged. That extends the duality between Laughlin quasiparticles and electrons to the quasiholes which characterize the paired states. In this case the impurity spin, which is not present in the classic problem of tunneling between edge states in the fractional Quantum Hall Effect, plays a crucial role, as discussed before. The impurity spin leads us to define a larger group with respect to the modular transformations in which it also plays a role: the “duality” group.

It is useful now to give a physical interpretation of our results in terms of dissipation. The present framework adapts very well to the description of an impurity interacting with a thermal bath which is realized in terms of two kinds of light particles (i.e. the two Quantum Hall Fluids). The dissipation is given by the kinetic term for the two fluids and a periodic potential is added. This problem can be solved in the two regimes of strong and weak “corrugation” [11] and flow occurs toward the
stable point. When a magnetic term is added for the impurity, the system drastically changes its properties and a more stable point with $\alpha = \beta = 1/2$ appears. The effect of $\beta \neq 0$ is to rotate the current $\vec{J}$ getting a “metallic” component (see eq. (8)). In the two layers system this implies that a current flows between the layers due to the twisted boundary conditions.

Finally we point out that this description can be also applied to a system of two Branes interacting with strings as it was proposed in [12].

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