Data Consistency in Transactional Storage Systems: A Centralised Approach

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Abstract—Modern distributed databases weaken data consistency guarantees to allow for faster transaction processing. It poses several challenges: formalising user-observable behaviour; then verifying protocols of databases and reasoning about client applications. We abstract such databases to centralised multi-version key-value stores and client views which provide partial information about such stores. We propose an operational semantics that is parametric in the notion of execution test, which determines if a client with a given view is allowed to commit a transaction. Different execution tests give rise to different consistency models, which are equivalent to the declarative consistency models defined in the literature. Using our semantics, we prove the correctness of distributed protocols and analyse the robustness of simple client applications.

1. Introduction

Transactions are the de facto synchronisation mechanism in modern distributed databases. To achieve scalability and performance, distributed databases often use weak transactional consistency guarantees. Much work has been done to formalise the semantics of such consistency guarantees, both declaratively and operationally. On the declarative side, several general formalisms have been proposed, such as dependency graphs\cite{3} and abstract executions\cite{14}, to provide a unified semantics for formulating different consistency models. On the operational side, the semantics of specific consistency models have been captured using reference implementations\cite{23,29,32}. However, unlike declarative approaches, there has been little work on general operational semantics for describing the user-observable behaviour of distributed atomic transactions. Our model comprises a global, centralised key-value store (kv-store) with multi-versioning, and client views inspired by the C11 operational semantics\cite{23}; using these mechanisms, we record all the versions written for each key, and let clients see only a subset of such versions. Our operational semantics is parametric in the notion of execution test\cite{2}, determining if a client with a given view is allowed to commit a transaction. Different execution tests give rise to different consistency models in our semantics. We are able to capture most of the well-known consistency models in a uniform way (§4): e.g., causal consistency (CC), PSI, snapshot isolation (SI) and serialisability (SER). We define these models using kv-stores and views, provide a correspondence between our kv-stores and dependency graphs, and introduce novel proof techniques for demonstrating that our definitions of consistency models are equivalent to existing declarative definitions (§5).

We showcase our semantics by verifying database protocols and analysing client programs (§6). For the former, we show that the COPS protocol of a replicated database satisfies our definition of CC and that the Clock-SI protocol of a partitioned database satisfies our definition of SI. For the later, we show the robustness of applications against our consistency models: we prove that a transactional library comprising a single counter is robust against PSI; and that the library with multiple counters is robust against SI, but not PSI. To our knowledge, our robustness results are the first to take into account client sessions. Without sessions, multiple counters can be proved to be robust against PSI using the static analysis check from\cite{8}. We verify protocols and analyse clients in the same operational semantics. By contrast, in existing declarative literature these two tasks are carried out in different semantics: protocols are verified using abstract executions; clients are analysed using dependency graphs; and equivalence results are used to move between the two.
2. Overview

Consider a simple counter object, Counter($k$), defined over a distributed kv-store. Clients can manipulate the value of key $k$ via two operations, inc($k$) and read($k$):

$$inc(k) \triangleq [x := [k]; \ [k] := x + 1] \quad read(k) \triangleq [x := [k]]$$

Command $x := [k]$ reads the value of key $k$ to local variable $x$ and $[k] := x + 1$ writes the value of $x + 1$ to key $k$. The code of each operation is wrapped in square brackets, denoting that it must be executed \textit{atomically} as a transaction. Correctness of atomic transactions is subtle, depending heavily on implementation details of the distributed kv-store and, in particular, on its \textit{consistency model}.

\textbf{Consistency Models.} A well-known consistency model is that of \textit{serialisability}, where transactions appear to execute in a sequential (serial) order, one after another. However, implementing serialisability for distributed kv-stores comes at a significant performance cost. Instead, implementors are content with \textit{weaker} consistency models, that have been implemented both in replicated and partitioned databases [5] [6] [9] [19] [24] [25] [26] [30] [32] [33].

We motivate these weak consistency models using replicated kv-stores. In such stores, clients run each operation on an arbitrary replica, and propagate the effects of the operations (if any) to other replicas. In this setting, concurrent calls to operations can lead to weak behaviours not present with serialisability. For instance, consider the program below where clients $cl_1$ and $cl_2$ run inc($k$) concurrently:

$$cl_1 : inc(k) \parallel cl_2 : inc(k)$$

\textit{(LU)}

Let us assume that $k$ initially holds value 0. Intuitively, since transactions are executed atomically, after both calls to inc($k$) have terminated, the counter should hold the value 2. Indeed, this is the only outcome allowed under serialisability. However, when clients execute inc($k$) at different replicas, if the kv-store provides no synchronisation mechanism for transactions, then it is possible for both clients to read the same initial value 0 for $k$ at their distinct replicas, update them to 1, and propagate their updates. Consequently, both replicas are unchanged with value 1 for $k$. This weak behaviour is known as the \textit{lost update anomaly}, which is allowed under causal consistency [24] [25] [33].

\textbf{Centralised KV-Stores and Views.} Reasoning about programs in a distributed database by capturing the whole system state may be cumbersome and error-prone, due to the large amount of information that needs to keep tracked. Declarative formalisms overcome this issue by abstracting from the system state, and relying on the history of operations performed by transactions instead.

Here we take a different approach, and abstract from distribution by projecting the local state of each component of the kv-store into a global, \textit{multi-versioned} centralised state: we record all versions of each key written, together with the meta-data of the transactions that access it. Clients use a mechanism called \textit{client views} to determine the subset of versions in the kv-store that are made available to them: this lets clients observe different states of the kv-store, allowing for weak behaviours such as the \textit{lost update anomaly}.

Let us illustrate how we can produce the lost update anomaly in (LU) using our kv-stores. The initial kv-store comprises a single key $k$, with only one version, $\langle 0, t_0, \emptyset \rangle$, stating that $k$ holds value 0, that the version \textit{writer} is the initialising transaction $t_0$ (this version was written by $t_0$), and that the version \textit{reader set} is empty (no transaction has read this version as of yet). Fig. 1a depicts this initial kv-store, with the version represented as a box sub-divided to three sections; proceeding clockwise from the left, they represent the version value, writer and reader set, respectively.

Suppose that $cl_1$ first invokes inc on Fig. 1a. In order to mark the versions that $cl_1$ reads and writes while executing the underlying transaction of inc, $cl_1$ first obtains a unique transaction identifier, $t$, and then proceeds with inc($k$). It then reads the (only) version of $k$ (with value 0), and writes a new value 1 for $k$. The resulting kv-store is depicted in Fig. 1b. Note that the initial version of $k$ is updated to reflect that it has been read by $t$; moreover, versions are ordered (left to right), from the oldest to the newest.

Next, client $cl_2$ invokes inc on Fig. 1b. As there are now two versions available for $k$, we need to determine the version from which $cl_2$ fetches its value, before running inc($k$). This is where \textit{client views} come into play. Intuitively, a view of client $cl_2$ comprises those versions in the kv-store that are \textit{visible} to $cl_2$, i.e. those whose values can be read by $cl_2$. If more than one version is visible, then the newest (right-most) such version is selected, modelling the \textit{last writer wins} resolution policy used by several kv-stores [35]. In our example, there are two possible view candidates for $cl_2$ when running inc($k$) on Fig. 1b: one containing only the initial version of $k$, another containing both versions of $k$. In the former case, the view is depicted in Fig. 1c: running inc($k$) on this view reads 0 and writes a new version with value 1, as depicted in Fig. 1c. Such a kv-store does not contain a version with value 2, despite two increments on $k$, producing the lost update anomaly. In the latter case, running inc($k$) reads the latest value (1) and writes a new version with value 2.

\textbf{Execution Tests.} To avoid the lost update anomaly, distributed kv-stores introduce a check at commit time to ensure that at most one of concurrent transactions writing to the same key commits. This property is known as \textit{write conflict freedom}. In our framework, we can simulate this behaviour by introducing the notion of \textit{execution tests}. Intuitively, an execution test determines whether a client with a given view can execute a transaction. For example, write-conflict freedom can be enforced by requiring that a client commit a transaction writing to $k$ only if its view contains all versions available in the global state for such a view. In our (LU) example, this prevents $cl_2$ from running inc($k$) on Fig. 1b if its view only contains the initial version of $k$. Instead, the $cl_2$ view must contain both versions of $k$, thus enforcing $cl_2$ to write a version with value 2 after running inc($k$).
inc(k) In §6.1, we prove that if the kv-store ensures write-conflict freedom as well as few other properties, then clients can increment and read from a single counter as if the kv-store were (strictly) serialisable.

However, the situation becomes more complicated if the kv-store contains multiple counters. In this case, as each client has its own view on the kv-store, and since the views of clients are independent from each other, it is possible for two clients to observe the increments on two distinct counters, Counter(k₁) and Counter(k₂), in different orders. For instance, consider the following program:

\[
\begin{align*}
cl_0 &: \text{inc}(k_1) \\
cl_1 &: \text{inc}(k_2) \\
cl_2 &: \text{read}(k_1) \\
cl_3 &: \text{read}(k_2)
\end{align*}
\]

Suppose that cl₀ executes first and increments k₁, k₂. Both k₁ and k₂ then have two versions with values 0 and 1. Let us assume that the cl₁ view contains both versions of k₁, but only the initial version of k₂ (value 0). When next client cl₁ executes, it thus reads 1 for k₁ and 0 for k₂; that is, from the point of view of cl₁, the increment of k₁ happens before the increment of k₂. Conversely, let us assume that the cl₂ view contains both versions for k₂, but only the initial version of k₁ (value 0). As such, when cl₂ executes, it reads 0 for k₁ and 1 for k₂; that is, from the point of view of cl₂, the increment of k₂ happens before the increment of k₁. This behaviour is known as the long fork anomaly (Fig. 2).

The long fork anomaly is disallowed under strong models, e.g. SER and SI [20], but allowed under weak models e.g. PSI [32] and CC. To capture such consistency models and rule out the long form anomaly as a possible result of the program (LF), we must strengthen the execution test associated with the kv-store. For SER, we strengthen the execution test by requiring that a client can execute a transaction only if its view contains all the versions available in the global state. For SI, the candidate execution test recovers the order in which updates of versions have been observed by different clients (e.g. cl₁), and allows a transaction to commit only if the observations made by the committing client (e.g. cl₂) are consistent with previous clients (i.e. cl₁); we give the formal definition of this execution test in §6. Under such strengthened execution tests, e.g. the one for SI, in the (LF) example cl₂ cannot observe 1 for k₂ after observing 0 for k₁; this is because cl₁ has already established that the increment on k₂ happens after the one of k₁. In §6.1, we prove that if the kv-store consists of multiple counter objects, and the execution test employed by transactions guarantees SI, then the kv-store behaves as it were (strictly) serialisable. As we demonstrate in §6.2, using execution tests on kv-stores, we can define all well-known consistency models (weak or strong) subject to a few basic conditions. Moreover, in §6.2 we encode two different distributed protocols using kv-stores and views: COPS [26], a geo-replicated protocol for causal consistency, and Clock-SI [19], a protocol for SI under partitioned key-value stores. Each of these protocols is verified against our definition using execution tests.

3. Operational Model

We give the technical definitions of our operational model: the global, centralised key-value stores; the partial client views; and the operational semantics.

Notation. Given a set A, we write \( A \ni a \) to denote that elements of A are ranged over by a and its variants (e.g. \( a', a_1, \ldots \)). Given a relation \( r, r' \subseteq A \times A \), we write \( r', r^+ \) and \( r^* \) for its reflexive, transitive and reflexive-transitive closures, respectively; write \( a_1 \leadsto a_2 \) for \( (a_1, a_2) \in r \); and write \( r_1; r_2 \) for \( \{ (a_1, a_2) \mid \exists a. (a_1, a) \in r_1 \land (a, a_2) \in r_2 \} \).

3.1. Key-Value Stores and Views

Kv-stores are defined using client and transaction identifiers. We assume a countably infinite set of client identifiers \( \text{CLIENT} \ni c \). The set of transaction identifiers is \( \text{TRANSID} \ni \{ t_0 \} \equiv \{ t_{\nu}^n \mid c \in \text{CLIENT} \land n \geq 0 \} \), where \( t_0 \) denotes an initialisation transaction, and \( t_{\nu}^n \) identifies a transaction committed by client \( \nu \). Elements of \( \text{TRANSID} \) are ranged over by \( t, t', \ldots \), and subsets by \( T, T', \ldots \). Let \( \text{TRANSID}_0 \equiv \text{TRANSID} \setminus \{ t_0 \} \). For each \( n \in \mathbb{N} \), \( t_{\nu}^n \) identifies the \( n \)th transaction committed by \( \nu \). We assume each clients is associated with a session, and the order of transactions within a single session is determined by the superscript \( n \) in transaction identifiers of the form \( t_{\nu}^n \): we define the session order as \( \text{SO} \equiv \{ (t, t') \mid \exists c. n, m. \ t = t_{c}^n \land t' = t_{c}^m \land n < m \} \).

Definition 3.1 (Kv-stores). Assume a countably infinite set of keys \( \text{KEY} \ni k \), and a countably infinite set of values \( \text{VAL} \ni v \), including an initialisation value \( v_0 \). The set of versions, \( \text{VERSION} \ni \nu \), is: \( \text{VERSION} \equiv \text{VAL} \times \text{TRANSID} \times \mathbb{T} \). A kv-store, is a function \( K : \text{KEY} \rightarrow \text{VERSION}^* \), where \( \text{VERSION}^* \) is the free monoid generated by \( \text{VERSION} \). The initial key-value store is given by \( K_0 \), where \( K_0(k) \equiv (v_0, t_0, \emptyset) \) for all \( k \in \text{KEY} \).

A version \( \nu = (v, t, \mathbb{T}) \) comprises a value \( v \) and meta-data about the transactions that accessed it: the writer \( t \) identifies the transaction that wrote \( v \); and the set of readers \( \mathbb{T} \) identifies the transactions that read from \( v \). We define \( \text{val}(\nu) \equiv v \), \( \text{w}(\nu) \equiv t \) and \( \text{rs}(\nu) \equiv \mathbb{T} \). Lists of versions (elements of \( \text{VERSION}^* \)) are ranged over by \( \nu, \nu', \ldots \). Given a kv-store \( K \) and a transaction \( t \), we write \( t \in K \) when \( t \) appears as either the writer or amongst the readers of a version in the range of \( K \). We write \( |K(k)| \) for the length of
of \( K(k) \), and write \( K(k, i) \) for the \( i \)th version (indexed from 0) of \( k \) when defined, with \( i \geq 0 \).

We focus on kv-stores whose consistency model enforces the atomic visibility of transactions \([4]\). This ensures that 1) a transaction reads and writes at most one version for each key. We also assume that 2) the list of versions for each key has an initial version carrying the initialisation value \( v_0 \), written by the initialisation transaction \( t_0 \) with an initial empty set of readers. Finally, we assume that 3) the state of a kv-store is consistent with the session order of clients: a client cannot read a version of a key that has been written by a future transaction within the same session; and the order in which versions are written by a client must agree with its session order. A kv-store is well-formed if it satisfies these three conditions (defined formally in Def. 3.1). Henceforth, we assume kv-stores are well-formed, and write KVS to denote the set of well-formed kv-stores.

Clients often have partial views of kv-stores, with different clients seeing different versions of the same key. We define \( \text{views} \) to track the versions seen by clients.

**Definition 3.2 (Views).** A view of a kv-store \( K \) is a function \( u \in \text{views}(K) \triangleq \text{KEY} \rightarrow \mathcal{P}(\mathbb{N}) \) such that for all \( i, i', k, k' \):

\[
0 \in u(k) \land (i \in u(k) \Rightarrow i < |K(k)|) \quad \text{(wf)}
\]

\[
i \in u(k) \land w(K(k, i)) = w(K'(k', i')) \Rightarrow i' \in u(k') \quad \text{(atomic)}
\]

The initial view is denoted by \( u_0 \), where \( u_0(k) = \{ 0 \} \) for all \( k \in \text{KEY} \). A configuration, \( \Gamma \in \text{CONF} \), is a pair \((K, U)\) with \( K \in \text{KVS} \) and \( U \in \text{CLIENT} \stackrel{\text{def}}{=} \text{views}(K) \). The set of initial configurations, \( \text{CONF}_0 \subseteq \text{CONF} \), contains configurations of the form \((K_0, U)\), where \( K_0 \) is the initial kv-store.

Given a configuration \((K, U)\), when the view of client \( cl \in \text{CLIENT} \) is defined, i.e. there exists \( u = U(cl) \), then for each key \( k \in \text{KEY} \), \( u \) determines the sublist of versions in \( K \) that client \( cl \) sees. If \( i, j \in u(k) \) and \( i < j \), then \( cl \) knows about the values carried by versions \( K(k, i) \) and \( K(k, j) \), and knows that these versions are contained in \( K \) with \( K(k, j) \) more up-to-date than \( K(k, i) \). Let \( \text{views} \triangleq \bigcup_{K \in \text{KVS}} \text{views}(K) \) be the set of all views. Given a kv-store \( K \) and two views \( u, u' \in \text{views}(K) \), we define:

\[
u \sqsubseteq u' \iff \forall k \in \text{dom}(K), u(k) \subseteq u'(k).
\]

Given a configuration \((K, U)\) and a client \( cl \) such that \( U(cl) \) is defined, it is possible to define a snapshot of the view \( U(cl) \), which identifies the last write of a client view. This definition assumes that the database satisfies the last write wins resolution policy \([4]\), although our theory can be re-adapted to capture other resolution policies.

**Definition 3.3 (Snapshots).** A snapshot, \( ss \in \text{SNAPSHOT} \triangleq \text{KEY} \rightarrow \text{VAL} \), is a function from keys to values. Given \( K \in \text{KVS} \) and \( u \in \text{views}(K) \), the snapshot of \( u \) in \( K \) is defined by snapshot\((K, u) \triangleq \lambda k. \text{val}(K(k, \text{max}_<(u(k)))) \), where \( \text{max}_<(u(k)) \) is the maximum element in \( u(k) \) with respect to the natural order \( < \) over \( \mathbb{N} \).

### 3.2. Operational Semantics

**Programming Language.** A program \( P \) comprises a finite number of clients, where each client is associated with a unique identifier \( cl \in \text{CLIENT} \), and executes a sequential command \( C \), given by the following grammar:

\[
C_p := x ::= \text{E} | \text{assume}(E) \quad C := \text{skip} | \{ C \} ; C ; C + C + C \quad T_p := C_p | x ::= \text{E} | \{ E \} ; E \quad T := \text{skip} | \{ T_p \} ; T ; T + T \quad T^*
\]

Sequential commands comprise \text{skip}, primitive commands \( C_p \), atomic transactions \( T \), and standard compound commands. Primitive commands (the variable assignment \( x ::= \text{E} \) and the assume statement \text{assume}(E) \) used to encode conditionals) are used for computations based on client-local variables and can hence be invoked without restriction. Transactional commands, \( T \), comprise \text{skip}, primitive commands, primitive transactional commands \( T_p \), and standard compound commands. Primitive transactional commands include lookup \( (x ::= \text{E}) \) and mutation \( (\{ E \} ; E) \) used for reading and writing to kv-stores, respectively, and can be invoked only as part of an atomic transaction \( T \).

A program is a finite partial function from client identifiers to sequential commands. For clarity, we often write \( C_1 ; \ldots ; C_n \) as syntactic sugar for a program \( P \) with \( n \) clients associated with identifiers \( cl_1 ; \ldots ; cl_n \), where each client \( cl_i \) executes \( C_i \). Each client \( cl_i \) is associated with its own client-local stack, \( s_i \in \text{STACK} \triangleq \text{VARS} \rightarrow \text{VAL} \), mapping program variables (ranged over by \( x, y, \ldots \)) to values. We assume a language of expressions built from values and program variables: \( E ::= v | x | E + E \quad \cdots \). The evaluation \( [E]_x \) of expression \( E \) is parametric in the client-local stack, where:

\[
[v]_x \triangleq v \quad [x]_x \triangleq s(x) \quad [E_1 + E_2]_x \triangleq [E_1]_x + [E_2]_x \quad \cdots
\]

**Transactional Semantics.** In our framework, transactions are executed atomically. Intuitively, given a configuration \( \Gamma = (K, U) \), when a client \( cl \) executes a transaction \( T \), it performs the following steps: 1) it constructs a snapshot \( ss \) of \( K \) using its view \( U(cl) \) as defined in Def. 3.3; 2) it executes \( T \) in isolation over the snapshot \( ss \), accumulating the effects (i.e. the reads and writes) of executing \( T \); and 3) it commits \( T \) by incorporating these effects into \( K \).

To capture the effects of executing a transaction \( T \) on a snapshot \( ss \) of kv-store \( K \), we identify the fingerprint of \( T \) on \( ss \) and \( K \). A fingerprint of \( T \) on \( ss \) and \( K \) is a set of read and write operations, where the read operations capture the values \( T \) reads from \( ss \) (prior to overwriting them), and the write operations capture the values \( T \) writes to \( ss \) and aims to commit to \( K \), so long as certain consistency conditions are met. Execution of a transaction in a given configuration may result in more than one fingerprint due to non-determinism.

**Definition 3.4 (Fingerprints).** The set of operations is \( \text{OPS} \triangleq \{(l, k, v) \mid l \in \{ \text{w, x} \} \land k \in \text{KEY} \land v \in \text{VAL} \} \). A fingerprint \( F \) is a subset of operations, \( F \subseteq \text{OPS} \), such that:

\[
\forall k \in \text{KEY}, l \in \{ \text{w, x} \}, (l, k, v_1), (l, k, v_2) \in F \Rightarrow v_1 = v_2.
\]

Note the last constraint ensure that for each key, a fingerprint contains at most one read operation and at most one write operation. This reflects the fact that we work with transactions that are atomically visible \([9]\) that is, reads are taken from a single snapshot of the kv-store, and since clients see either none or all the writes of a transaction, the last write to each key is committed.
We provide an operational description of the behaviour of a transactional command, \( T \), starting from an initial client stack, a snapshot and the empty fingerprint \( \emptyset \). First, we define a transition system describing how client stack and snapshot are updated via primitive transactional commands.

**Definition 3.5.** The transition system, \( \xrightarrow{T} \subseteq (\text{STACK} \times \text{SNAPSHOT}) \times (\text{STACK} \times \text{SNAPSHOT}) \), is defined by:

\[
(s, ss) \xrightarrow{\text{assume}(E)} (s[x \mapsto E], ss) \quad (s, ss) \xrightarrow{x = E} (s[ss \mapsto s([E]_s)], ss) \quad (s, ss) \xrightarrow{E} (s, [ss[[E]_s] \mapsto [E]_s])
\]

Second, we define a *fingerprint function*, \( \text{op} \), computing the fingerprint of primitive transactional commands:

\[
\text{op}(s, ss, x := E) \triangleq \text{op}(s, ss, \text{assume}(E)) \triangleq \epsilon
\]

The empty operation \( \epsilon \) is used for those primitive commands that do not contribute to the fingerprint.

Third, we define a *combination operator*, \( \llcorner : \mathcal{P}(\text{OPS}) \times \text{OPS} \cup \{\epsilon\} \to \mathcal{P}(\text{OPS}) \), adding a read/write operation to a fingerprint, and ignoring the empty operation \( \epsilon \):

\[
F \llcorner (x, k, v) \triangleq \begin{cases} F \cup \{(x, k, v)\} & \text{if } \forall i. (k, v_i) \notin F \\ F & \text{otherwise} \end{cases}
\]

\[
F \llcorner (s, k, v) \triangleq \begin{cases} F \setminus \{(s, k, v)\} & \text{if } \forall i. (s, k_i) \notin F \\ F & \text{otherwise} \end{cases}
\]

Note that a read from \( k \) is added to a fingerprint \( F \) only if \( F \) does not contain an entry for \( k \), thus only recording the first value read for \( k \) (before a subsequent write). Analogously, a write to \( k \) is always added to \( F \) by removing the existing writes, thus only recording the last write to \( k \).

Finally, we have all the ingredients to describe the behaviour of a transactional command. Fig. 2 provides the one-step transactional semantics; the interesting rule is the *TPRIMITIVE* rule which describes how a primitive transactional command updates the client stack, the snapshot and the fingerprint; the remaining rules are standard.

**Definition 3.6 (Fingerprint Set).** Given a client stack \( s \) and a snapshot \( ss \), the *fingerprint set of a transactional command* is:

\[
F \triangleq \{ F : (s, ss, \emptyset), T \rightarrow^{*} (s', ss', F), \text{skip} \}
\]

where \( \rightarrow^{*} \) is the transitive closure of \( \rightarrow \) given in Fig. 2. A set \( F \) in \( F \) is called a *final fingerprint* of \( T \).

It is immediate to see that final fingerprints of \( T \) contain at most one read (resp. one write) operation per key.

**Operational Semantics.** We give the operational semantics of commands and programs. The command semantics describes transitions of the form \( cl \vdash (K, u, s, C) \xrightarrow{\text{cl}} (K', u', s') \), stating that given the kv-store \( K \), view \( u \) and stack \( s \), a client \( cl \) may execute command \( C \) for one step, updating the kv-store to \( K' \), the stack to \( s' \), and the command to its continuation \( C' \). The label \( \lambda \) is either of the form \( (cl, t) \) denoting that \( cl \) executed a primitive command that required no access to \( K \), or \( (el, u', F) \) denoting that \( cl \) committed an atomic transaction with final fingerprint \( F \) under the view \( u' \). Transitions are parametric in the choice of execution test \( ET \), defined shortly in §4. Intuitively, an execution test captures the consistency model under which a transaction executes. In §4 we present the execution tests associated with several well-known consistency models.

Fig. 3 contains the rules for primitive commands and atomic transactions. The rules for the compound commands are straightforward and are given in §A. The rule for primitive commands, \( C_{\text{PRIMITIVE}} \), lifts the transition system \( \xrightarrow{T} \) (Def. 3.5) to those primitive commands that only affect the client stack. The rule for atomic transactions, \( C_{\text{ATOMIC-TRANS}} \), describes the execution of an atomic transaction under execution test \( ET \). The first premise states that the current view \( u \) of the executing command maybe advanced to a newer atomic view \( u'' \) (see Def. 3.2). This corresponds to the client receiving new updates. Given the new view \( u'' \), the transaction obtains a snapshot \( ss \) of the kv-store \( K \), and executes \( T \) locally to completion (skipping), updating the stack to \( s' \), while accumulating the fingerprint \( F \). Note that the resulting snapshot is ignored (denoted by \( - \)) as the effect of the transaction is recorded in the fingerprint \( F \).

The transaction is now ready to commit. The rule picks a fresh transaction identifier \( t \in \text{nextTxId}(cl, K) \), and updates the kv-store via \( K' = \text{update}(K, u, F, t) \) if the commit is permitted by \( ET \) using the judgement \( ET \vdash (K, u') \rightarrow F : (K', u') \). The set \( \text{nextTxId}(cl, K) \) provides the transactions identifiers associated with \( cl \) that are fresh for \( K \); \( \text{nextTxId}(cl, K) \triangleq \{ t_{cl}^m, \forall m. t_{cl}^m \in K \Rightarrow m < n \} \). By construction, all elements of \( \text{nextTxId}(cl, K) \) are greater (with respect to session order SO) than transaction identifiers previously used by \( cl \). The judgement \( ET \vdash (K, u') \rightarrow F : (K', u') \) states that the fingerprint \( F \) is compatible with kv-store \( K \) and view \( u'' \), and the resulting view \( u' \) is compatible with view \( u'' \) and final kv-store \( K' \). In §4 we give several examples of such judgements. Finally, having selected a suitable transaction identifier \( t \), the update \( (K, u, F, t) \) defined below, describes how the final fingerprint \( F \) of \( t \) executed under view \( u \) updates \( K \), update \( (K, u, F, t) \) updates \( K \) with \( F \) as follows: for each read operation \( (x, k, v) \in F \), it adds to the reader set of the last version of \( k \) that is included in the view \( u \) of the client; for each write operation \( (w, k, v) \), it appends a new version \( (v, t, \emptyset) \in K(k) \), where \( :: \) denotes list concatenation. When \( V = v_0 :: \cdots :: v_n \) and \( i = 0 \cdots n \), then \( V[i \mapsto v] \) denotes \( V'' = v_0 :: \cdots :: v_{i-1} :: v' :: v_{i+1} :: v_n \).
as: \(\{\text{update}(K, u, F, t) \mid t \in \text{nextTxd}(K, cl)\}\).

Note that, under the assumption that \(F\) contains at most one read and one write operation per key and the identifier is fresh for \(K\), the transaction update function and the transaction update set for \(cl\) are well-defined.

The last rule of Fig. 3 captures the execution of a program step, given a client environment \(E \in \text{CENV}\). A client environment \(E\) is a function from client identifiers to stacks, associating each client with its stack. We assume that the domain of client environments is the same as the the domain of the program throughout the execution: \(\text{dom}(E) = \text{dom}(P)\). Program transitions are simply defined in terms of the transitions of their constituent client commands. This yields a standard interleaving semantics for concurrent programs; that is, a client performs a reduction in an atomic step without affecting other clients.

### 4. Consistency Guarantees

Consistency guarantees of distributed databases describe what it means for distributed data to be consistent. They have been formally described axiomatically via dependency graphs \([3, 4]\) and abstract execution graphs \([11, 14]\). We formalise the consistency guarantees of our centralised kv-stores by defining a consistency model. A consistency model is a set of kv-stores capturing the possible outcomes obtained when multiple clients commit several transactions each, provided that the effects of such transactions comply with the consistency guarantees of the underlying consistency model. To this end, we define consistency models induced by an execution test. An execution test is a relation which determines whether a client may commit a transaction into a kv-store. We formulate several well-known consistency models over our centralised kv-stores by defining their corresponding execution tests. Later in §5 we demonstrate that our definitions over centralised kv-stores are equivalent to their existing definitions over distributed databases.

An execution test is a set \(ET\) of tuples of the form \((K, u, F, K', u')\), denoting that a client with view \(u\) on kv-store \(K\) may commit an atomic transaction with fingerprint \(F\) and obtain an updated kv-store \(K'\) and an updated view \(u'\). We often write \(ET \vdash (K, u, F) : (K', u')\) for \((K, u, F, K', u') \in ET\).

**Definition 4.1.** An execution test is a set of tuples \(ET \subseteq \text{KVS} \times \text{VIEWS} \times P \times \text{KVS} \times \text{VIEWS}\) such that for all \((K, u, F, K', u') \in ET\) and all \(k, v:\)

\[
\begin{align*}
(v, k, v) &\in F \Rightarrow K(k, \max \{u(k)\}) = v \quad \text{(Ext)} \\
u(k) \neq u'(k) &\Rightarrow \exists \ell. (k, \ell) \in F \quad \text{(ValidViewUpd)}
\end{align*}
\]

The first condition enforces the last-write-wins policy [35]: a transaction always reads the most recent writes from the initial view. The second condition states that a transaction is only allowed to update the view for those keys that have been recorded in the fingerprint.

Note that at this initial stage \(\text{Ext}\) and \(\text{ValidViewUpd}\) are the only required conditions and execution tests are otherwise unrestricted. Further restrictions on execution tests are determined by the underlying consistency model, thus prescribing the consistency guarantees of the model.

Given an execution test \(ET\), we define the ET-trace as a sequence of ET-reductions on configurations that either 1) advances the client view to a more up-to-date view; or 2) commits a fingerprint of a transaction.

**Definition 4.2 (ET-trace).** An action \(\alpha \in \text{ACT}\) is either of the form \((cl, \Xi)\) or \((cl, F)\), where \(cl\) is a client and \(F\) is a fingerprint. Given an execution test \(ET\), the ET-reduction relation, \(\rightsquigarrow_{ET} \subseteq \text{CONF} \times \text{ACT} \times \text{CONF}\), is the smallest relation such that for all \(cl, K, K', \ell, u, u'\) and \(u = U(cl)\):

\[
\begin{align*}
1) \ u \subseteq u' &\Rightarrow (K, U) \rightsquigarrow_{ET} (K, U[cl \mapsto u']) \\
2) \ K' \in \text{update}(K, u, F, cl) \land (K', u, F, K', u') \in ET &\Rightarrow (K', U) \rightsquigarrow_{ET} (K', U[cl \mapsto u'])
\end{align*}
\]

Given an execution test \(ET\), an ET-trace is a sequence of ET-reductions of the form \(\Gamma_0 \rightsquigarrow_{ET} \cdots \rightsquigarrow_{ET} \Gamma_n\).
ET-traces are ranged over by $\tau, \tau', \cdots$; given a ET-trace $\tau$, $|\tau|$ denotes the number of ET-reductions in $\tau$, and for $i = 1, \cdots, n$, $\tau(i)$ denotes the $i$-th reduction of $\tau$.

A consistency model induced by ET is a set of kv-stores resulting from ET-traces starting in an initial configuration.

**Definition 4.3** (Consistency Model). Given an execution test $ET$, the set of configurations induced by ET, $CONF(ET)$, is given by:

$$CONF(ET) \triangleq \{ \Gamma \mid \exists \Gamma_0 \in CONF_0, \Gamma_0 \xrightarrow{ET} \Gamma \}.$$  

The consistency model induced by $ET$ is defined as:

$$CM(ET) \triangleq \{ \Gamma \mid \langle \mathcal{K}, - \rangle \in CONF(ET) \}.$$  

In §B.3 we prove that consistency models are monotonic: if $ET_1 \subseteq ET_2$ then $CM(ET_1) \subseteq CM(ET_2)$.

**Compositionality.** We examine the compositionality of the consistency models induced by execution tests: i.e. given two execution tests $ET_1, ET_2$, does $CM(ET_1 \cap ET_2) = CM(ET_1) \cap CM(ET_2)$ hold? The monotonicity of execution tests guarantees that for all $ET_1, ET_2$, $CM(ET_1 \cap ET_2) \subseteq CM(ET_1) \cap CM(ET_2)$. However, the other direction $CM(ET_1) \cap CM(ET_2) \subseteq CM(ET_1 \cap ET_2)$ does not hold for arbitrary consistency models. Consider the following:

$$ET_1 \cap ET_2 = \{ (k, u_0) \triangleright \{(w, k, 1)\} : (k, u_0) \triangleright \{(w, k', 1)\} : (k', u_0) \}$$

with $K_k = K_0 \triangleright \{(0, t_0, 0)\}$ and $K_{k'} = K_0 \triangleright \{(0, t_0, 0)\}$

As both $ET_1$ and $ET_2$ allow a version with value 1 to be written for $k, k'$, we have $\mathcal{K}' \in CM(ET_1) \cap CM(ET_2)$. However, $ET_1$ and $ET_2$ enforce a different order in which the writes on $k, k'$ must happen; thus $\mathcal{K}' \notin CM(ET_1 \cap ET_2)$.

In this example, compositionality fails because execution tests enforced a particular order in which the updates must be committed, even though such updates are non-conflicting: the kv-store obtained after committing such updates is independent of the commit order. This observation is captured in the following definition:

**Definition 4.4.** Two fingerprints $F_1$ and $F_2$ are conflicting if there exists $k$ such that $(w, k, -) \in F_1 \land (w, k, -) \in F_2$. An execution test $ET$ is commutative, written $com(ET)$, if for all $\mathcal{K}, \mathcal{K}', \mathcal{U}, \mathcal{U}'$, distinct clients $c_1, c_2$, non-conflicting fingerprints $F_1, F_2$, and $u_1, u_2 \in \text{views}(\mathcal{K})$:

$$\text{if } (\mathcal{K}, \mathcal{U}) \xrightarrow{(c_1, F_1)}_{ET} (c_2, F_2) \text{ then } (\mathcal{K}, \mathcal{U}) \xrightarrow{(c_2, F_2)}_{ET} (c_1, F_1)$$

To guarantee the compositionality of two execution tests $ET_1, ET_2$, we require at least one of those to be commutative, say $ET_1$. The main idea is the following: fix a $ET_1$-trace $\tau_1$ and a $ET_2$-trace $\tau_2$, both terminating in a configuration of the form $\langle \mathcal{K}, - \rangle$; then we construct a $(ET_1 \cap ET_2)$-trace terminating in a configuration of the same form by re-ordering the sequence of reductions of $\tau_1$ and the sequence of reductions of $\tau_2$. In §B.3, we show that if $ET_1$ is commutative, we can indeed re-order the sequence of reductions in the $ET_1$-trace leading to a trace $\tau_1'$ such that $|\tau_1'| = |\tau_2|$ and for any $i = 1, \cdots, |\tau_1'|$, the pre and post kv-store of $\tau_1(i)$, coincide with the action, pre and post kv-store of $\tau_2(i)$.

Commutativity alone does not ensure that, for $i = 1, \cdots, |\tau_2|$, the pre-views and post-views of $\tau_1(i)$ match the pre-views and post-views of $\tau_2(i)$, which is necessary to show that $\tau_1$ and $\tau_2$ can be recast as a $(ET_1 \cap ET_2)$-trace. In §B.3 we present three other basic requirements to be satisfied by $ET_1$ and $ET_2$, that guarantee that $\tau_1(i)$ and $\tau_2(i)$ agree on the pre-views and post-views for $i = 1, \cdots, |\tau_2|$. The first two of these requirements, no blind writes and minimum footprint, ensure that the pre-views of the reductions $\tau_1(i)$ and $\tau_2(i)$ match, while the third requirement, which we call monotonic post-views, ensures that the post-views of the reductions $\tau_1(i)$ and $\tau_2(i)$ match.

**Theorem 4.1** (Compositionality). For all $ET_1, ET_2$ with no blind writes, minimum footprints and monotonic post-views: if $com(ET_1)$, then $CM(ET_1 \cap ET_2) = CM(ET_1) \cap CM(ET_2)$; if $com(ET_1) \land com(ET_2)$, then $com(ET_1 \cap ET_2)$.

Most of the execution tests associated with well-known consistency models (introduced shortly) can be tweaked to satisfy no-blind writes, minimum footprints and monotonic post-views without altering their semantics. However, some of these execution tests are inherently non-commutative.

### 4.1. Examples

We now give examples of execution tests in Fig. 4 where the associated consistency models for kv-stores correspond to widely adopted consistency guarantees for distributed databases. Following [34], we distinguish between client- and data-centric consistency models: the former constrain the client views; the latter impose conditions on the structure of the kv-store. In Fig. 5 we give illustrative examples of kv-stores allowed/disallowed by our consistency models.

**Monotonic Reads (MR).** It states that a client cannot lose information from the view and hence can only see increasingly more up-to-date versions. This prevents e.g. the kv-store of Fig. 5a since client $cl$ first reads the latest version of $k$ in $t_2$, and then reads the older, initial version of $k$ in $t_2$. The execution test $ET_{MR}$ ensures that clients can only extend their views.

**Monotonic Writes (MW).** It states that whenever a transaction sees a version written by a client $cl$, then it sees all previous versions written by $cl$. This prevents e.g. the kv-store of Fig. 5b since transaction $t'$ reads the second version of $k_2$, with value $v_2$ written by client $cl$, but it does not read, hence does not see, the second version of $k_1$ with value $v_1$ and previously written by the same client. The execution test $ET_{MW}$ ensures that, prior to executing a transaction, the set of versions included in the view of the client are write prefix-closed with respect to the relation $SO$.

2. In fact, to ensure that $|\tau_1'| = |\tau_2|$ we require to further manipulate $\tau_1$ prior to re-ordering its sequence of $ET_1$-reductions.
| Model | Execution Test: \((K, u) \triangleright \mathcal{F} : (K', u')\) | Model | Execution Test: \((K, u) \triangleright \mathcal{F} : (K', u')\) |
|-------|-------------------------------------------------|-------|-------------------------------------------------|
| MR    | \(j \in u(k) \land w(K(k', i)) \xrightarrow{SO^\prime} w(K(k, j)) \Rightarrow i \in u(k')\) | UA    | \((w, k, -) \in \mathcal{F} \land 0 \leq i < |K(k)| \Rightarrow i \in u(k)\) |
| MW    | \(t \in K' \land t \notin K \land w(K(k', i)) \xrightarrow{SO^\prime} t \Rightarrow i \in u'(k)\) | PSI   | \(ET_{PSI} = ET_{CC} \cap ET_{UA}\) |
| RYW   | \(j \in u(k) \land t \in rs(K(k', i)) \land \neg SO^\prime \Rightarrow w(K(k, j)) \Rightarrow i \in u(k')\) | CP    | \(\{(K, u, F, K', u') \mid \}\} \cap ET_{MR} \cap ET_{RYW}\) |
| WFR   | \(ET_{CC} = ET_{MR} \cap ET_{MW} \cap ET_{RYW} \cap ET_{WFR}\) | SER   | \(0 \leq i < |K(k)| \Rightarrow i \in u(k)\) |

\[
\begin{align*}
\uparrow \triangleq & \forall k', i, j. i \in u(k) \land w(K(k', j)) \xrightarrow{(SO^\prime降温), (WFR\text{-}K\text{-}RW)} w(K(k, i)) \Rightarrow j \in u(k') \\
\downarrow \triangleq & \forall k', i, j. i \in u(k) \land w(K(k', j)) \xrightarrow{(SO^\prime降温), (WW\text{-}K\text{-}RW)} w(K(k, i)) \Rightarrow j \in u(k') \\
WR_K \triangleq & \{(t, t') \mid \exists k, i, t = w(K(k, i)) \land t' \in rs(K(k, i))\} \\
WW_K \triangleq & \{(t, t') \mid \exists k, i, j. t = w(K(k, i)) \land t' = w(K(k, j)) \land i < j\} \\
RW_K \triangleq & \{(t, t') \mid \exists k, i, j. t \in rs(K(k, i)) \land t' = w(K(k, j)) \land i < j \land t \neq t'\} \\
\end{align*}
\]

Figure 4: Execution tests of client-centric (left) and data-centric (right) consistency models, with SO as defined in §3.1. All free variables are universally quantified.

Read Your Writes (RYW). It states that a client must always see the versions previously written by the client itself. This prevents the kv-store in Fig. 5a since the initial version of \(k\) holds value 0 and client \(cl\) tries to increment the value of \(k\) by 1 twice. For its first transaction, it reads the initial value 0 and then writes a new version with value 1. For its second transaction, since the client need not see its own writes, it might read the initial value 0 again and write a new version with value 1. The execution test RYW ensures that, after committing a transaction, the client view includes all the versions it wrote.

Write Follows Reads (WFR). It states that, if a transaction sees a version written by a client \(cl\), then it must also see the versions previously read by \(cl\) (in SO relation). This prevents the kv-store of Fig. 5c since transaction \(t\) reads a version written by \(cl\) but not a version previously read by \(cl\). The execution test ET_{WFR} ensures that a view includes all previously read versions by a client if the view already includes a write from that client.

Causal Consistency (CC). The causal consistency is defined in the literature [10] as the conjunction (composition) of the four session guarantees MR, MW, RYW and WFR: \(CM_{CC} \triangleq CM_{MR} \cap CM_{MW} \cap CM_{RYW} \cap CM_{WFR}\). Analogously, we have defined \(ET_{CC}\) as the conjunction of the execution tests corresponding to the four session guarantees. As we discuss before, we can do this thanks to the compositionality of our execution tests: the composition of several consistency models is equivalent to the consistency model induced by the conjunction of the corresponding execution tests. That is, \(CM(ET_{CC}) = CM(ET_{MR}) \cap CM(ET_{MW}) \cap CM(ET_{RYW}) \cap CM(ET_{WFR})\). As we discuss in §6 and prove in [1], the COPS implementation [20] satisfies \(ET_{CC}\).

Update Atomic (UA). This model has been proposed in [14] and implemented in [25]. UA disallows concurrent transactions writing to the same key, a property known as write-write conflict freedom, that is, when two transactions want to write to the same key, one must see another. This prevents the kv-store of Fig. 1d as \(t, t'\) concurrently increment the initial version of \(k\) by 1. Note that UA generalises RYW: unlike RYW, UA does not require \(t, t'\) to be executed by the same client. The \(ET_{UA}\) ensures write-write conflict freedom by allowing a client to write to key \(k\) only if its view includes all versions of \(k\).

Parallel Snapshot Isolation (PSI). The guarantees of PSI have been defined as the conjunction of the guarantees provided by CC and UA [14]. Analogously, we have defined \(ET_{PSI} = ET_{CC} \cap ET_{UA}\). This definition exploits the compositionality of our execution tests (Theorem 4.1).

Strict serialisability (SER). Serialisability is the strongest consistency model, requiring that there exists a sequential schedule of transactions. The execution test \(ET_{SER}\) thus allows clients to execute transactions only when their view of the kv-store store is complete. This prevents the kv-store in Fig. 5e under serialisability either \(t_1\) or \(t_2\) commits first. In the former case when \(t_1\) commits first, then its write to \(k_2\) must be included in the view of \(t_2\), and thus \(t_2\) should not read the outdated write to \(k_2\) by \(t_0\). The latter case is analogously prohibited. This example is allowed by all the other execution tests in Fig. 4.

Consistent Prefix (CP). When the total order in which transactions commit is known, then CP is described by the following property: if a client sees a transaction \(t\), then it must also see any transaction that commits before \(t\). Our configurations of kv-stores and views only provides partial information about the order in which transactions commit, but this information is enough to define the consistency models we seek. Consider the transaction relations \(WR_K, WW_K\) and \(RW_K\) defined in §3. adapted from well-known transaction relations associated with dependency graphs [3, 4]. In [10] an alternative definition of CP is given: if a client sees a transaction \(t\), then it must also see the subset of transactions that committed before \(t\), that can be computed from SO, \(WR_K, WW_K, RW_K\). This property is captured by \(\uparrow\) in Fig. 4. The pair \((t, t') \in WR_K\) means that \(t\) reads the version of some key \(k\) written by \(t'\), therefore it must be the case that \(t\) commits before \(t'\) starts, and therefore before \(t'\) commits, and similarly for SO. The pair \((t, t') \in RW_K\) means that \(t'\) reads one version for some key that is later overwritten by \(t'\); then \(t\) is prevented.
from seeing the write of \( t' \), and therefore it must be the case that \( t \) commits before \( t' \) commits. As a consequence, if \((t, t') \in \text{WR}_K \) (resp. SO) and \((t'', t') \in \text{WR}_K \), then it must also be the case that \( t \) commits before the commit of \( t' \). Last, if \((t, t') \in \text{WW}_K \) means that \( t' \) overwrites a version written by \( t \) for some key, then it must be the case that \( t \) commits before the commit of \( t' \). The relation \(((\text{SO}; \text{WR}_K) \cup (\text{WR}_K; \text{WW}_K) \cup \text{WW}_K)^+ \ni (t, t') \) captures that, all the transactions \( t \) that must have already committed to the kv-store before commit of \( t' \). The execution test \( \text{ET}_{CP} \) is the intersection of \( \uparrow \) with \( \text{ET}_{MR} \cap \text{ET}_{RYW} \), where the latter enforces a client sees its own commits. For the long fork anomaly Fig. 5 if \( t_{cl_2} \) is the last transaction, it reads and thus sees \( t' \). Given the kv-store we have: \( t \xrightarrow{\text{WR}_K} t' \xrightarrow{\text{SO}} t_{cl_1} \xrightarrow{\text{WR}_K} t', \) which means, by \( \uparrow \), transactions \( t_{cl_2} \) must see \( t \). However, \( t_{cl_2} \) reads a older version of \( k_1 \) than the one written by \( t \). Symmetrically, if \( t_{cl_1} \) is the last transaction, it sees \( t \) and so must see \( t' \).

**Snapshot Isolation (SI).** When the total order in which transactions commit is known then SI can be defined compositionally from CP and UA When we can rely only on a partial order of transaction execution, inferred from kv-stores and views, then this compositional result does not hold. For example, the kv-store of Fig. 5 is included in both CM(ET\(_{CP}\)) and CM(ET\(_{UA}\)), but is disallowed by the execution test ET\(_{SI}\), introduced presently. In our definition, ET\(_{SI}\) by replacing property dagger in ET\(_{CP}\) with \( \uparrow \) (Fig. 4) and by intersecting the result with ET\(_{UA}\). Similarly as for CP, the \( \uparrow \) property captures the fact that if transaction \( t \) sees the writes of another transaction \( t' \), then it must see the subset of transactions committing before \( t' \) that can be computed from SO, \( \text{WR}_K, \text{WW}_K, \text{RW}_K \). However, because snapshot isolation enforces write-conflict freedom, the computation of this subset differs from the one for CP. Under UA consequently SI, the pair \((t, t'') \in \text{WW}_K \) means not only the case that \( t \) commits before \( t'' \), but \( t \) commits before \( t'' \) start. Because \((t'', t') \in \text{RW}_K \) implies that \( t'' \) starts before \( t' \) commits, then it must be the case that when \((t, t') \in \text{WW}_K; \text{RW}_K \) then \( t \) commits before \( t' \) does. In Fig. 5, transaction \( t_4 \) reads the last version of \( k_2 \) written by \( t_3 \) so does not violate UA. From the \( t_3 \) backwards we have edges: \( t_1 \xrightarrow{\text{WW}, \text{WR}_K} t_2 \xrightarrow{\text{RW}_K} t_3 \). Snapshot isolation SI requires that if a transaction sees \( t_3 \) it must see \( t_1 \) by the \( \uparrow \) (it is not the case in \( \uparrow \)). However in Fig. 5, transaction \( t_4 \) only see the initial version of \( k_0 \). As we discuss in §6 and prove in the [2] the Clock-SI protocol [19] satisfies ET\(_{SI}\).

5. KV-Stores and Other Formalisms

In this Section, we discuss how kv-stores and execution tests relate to existing declarative formalisms for specifying consistency models, based on dependency graphs [3] and abstract executions [14]. We give an overview of our results here, and refer the reader to the appendix for more details.

**Relating KV-Stores and Dependency Graphs.** Dependency graphs [3][4] are perhaps the most popular formalism used for specifying transactional consistency models. A dependency graph \( G \) is a directed, labelled graph where its nodes denote transactions and its edges denote certain dependencies between transactions. An example dependency graph is given in Fig. 6a. Each node is labelled with a transaction identifier and a fingerprint. Each edges is labelled with metadata describing the information flow in a run of the database: 1) a session order edge, \( t_1 \xrightarrow{\text{SO}} t_2 \), 2) a read dependency edge, \( t_1 \xrightarrow{\text{WR}} t_2 \), denotes that transaction \( t_2 \) reads a version written by \( t_1 \); 3) a write dependency edge, \( t_1 \xrightarrow{\text{WW}, \text{WR}} t_2 \), denotes that \( t_2 \) overwrites a version written by \( t_1 \); and 4) an anti-dependency edge, \( t_1 \xrightarrow{\text{WW}, \text{WR}} t_2 \), denotes that \( t_2 \) overwrites a version read by \( t_2 \). We give the formal definition of dependency graphs in §3. Observe that we can always extract a dependency graph \( G \) from a kv-store \( K \): we choose the transaction identifiers appearing in \( K \) as the nodes of \( G \), and let SO as defined in §3.1 and WR, WW, RW be as defined in Fig. 4. For example, Fig. 4d corresponds to the dependency graph extracted from the kv-
store in Fig. 1d. In §C we show that this construction can be reversed, thus giving rise to the following result:

**Theorem 5.1.** Dependency graphs are isomorphic to kv-stores.

Using dependency graphs, consistency models are specified by constraining the shape of the graph; typically, such constraints mandate the absence of certain cycles. For example, strict serialisability is defined as those dependency graphs where \((SO \cup VWU \cup WRU RW)^+\) is acyclic. We can always convert a dependency graph-based specification into an execution test-based one by simply checking that, when committing a transaction to kv-store \(K\) and obtaining \(K'\), the dependency graph extracted from \(K'\) contains no cycles prohibited by the dependency-based specification.

**Relating KV-Stores and Abstract Executions.** Abstract executions [11][14] are an alternative formalism for defining consistency models. As with dependency graphs, an abstract execution graph \(X\) is a directed graph with its nodes representing transactions (with each node labelled with a transaction identifier and a set of (read/write) operations), and its edges representing certain relations between transactions. An example abstract execution graph is depicted in Fig. 1b. Each edge is labelled by either the visibility (VIS) or arbitration (AR) relation. The VIS is an irreflexive order on transactions such that \(t_1 \xrightarrow{\text{VIS}} t_2\) denotes that the effects (updates) of \(t_1\) are visible to \(t_2\). The AR is a strict total order on transactions such that \(t_1 \xrightarrow{\text{AR}} t_2\) denotes that the updates performed by \(t_2\) are newer than those of \(t_1\). Moreover, AR contains VIS (VIS \(\subseteq\) AR) and agrees with the session order. Lastly, abstract executions observe the last-write-wins policy: a transaction reading \(k\) always fetches the latest visible write (VIS predecessor) on \(k\). We refer the reader to §D for full details.

Following [16], we can always extract a dependency graph \(G_X\) from an abstract execution \(X\), and thus a kv-store \(K_X\) via Theorem 5.1—see §E for the formal details. We write \(K_X\) for the kv-store extracted from \(X\) using this construction. Moreover, we show that there is a Galois connection between ET \(\uparrow\) traces, the weakest possible execution test and abstract executions (§E).

With dependency graphs, consistency models using abstract executions are defined by constraining the shape of abstract execution graphs via a set of axioms \(A\), e.g. imposing certain conditions on VIS. All consistency models presented in this paper have an equivalent axiomatic definition based on abstract executions [14][16]. Proving the equivalence of execution test-based and abstract execution-based definitions is non-trivial; however, we have observed that all proofs follow the same structure, so long as certain conditions hold. We develop the meta-theory to capture this proof structure. Our meta-theory is non-trivial; we refer the reader to §F.3 for the full details. We give an intuitive account of the two conditions required by our meta-theory, and then state our equivalence theorem.

Given a set of axioms \(A\), we define \(\text{CM}(A) \triangleq \{K_X \mid X \text{ sats. } A\}\). Our first condition is the soundness of an execution test against an axiomatic definition. An execution test ET is sound against an axiomatic definition \(A\) if: for all \(n\) and for all ET-traces \(\tau\) with \(n\) steps, we can construct \(X_0, \ldots, X_n\) and \(K_0 = K_X\) such that for each step \((K_i, u_i) \xrightarrow{\tau} (K_{i+1}, u_{i+1})\) in \(\tau\), the new VIS edges in \(X_{i+1}\) (those not in \(X_i\)) satisfy \(A\). The formal definition of execution test soundness is given in §F.3.

Our second condition is the completeness of an execution test against an axiomatic definition. Let \(t_i\) denote the \(i\)th transaction of \(X\) in its AR order, and \(X'\) denote the restriction of \(X\) to \(t_1 \cdots t_i\). An execution test ET is complete against an axiomatic definition \(A\) if: for all abstract executions \(X\) that satisfy \(A\) containing \(n\) transactions, all \(i \in \{1 \cdots n\}\), views \(u_i, u'_i\), transactions \(t',\) and fingerprints \(F_i\), whenever 1) \(t'\) is the immediate SO-successor of \(t_i\); 2) \(u_i\) includes all visible transactions of \(t_i\); 3) \(u'_i\) includes all visible transactions of \(t'\); and 4) \(F_i\) is the fingerprint of \(t_i\), then ET \(\vdash (K_i, u_i) \xrightarrow{\tau} (K_{i+1}, u_{i+1})\). The formal definition of execution test completeness is given in §F.3.

Finally, we state our equivalence theorem below (Theorem 5.2), with its full proof in §F.3. This theorem ensures that if an execution test is sound and complete against a set of axioms \(A\), then the consistency model induced by ET corresponds to the kv-stores extracted from abstract executions satisfying \(A\).

**Theorem 5.2.** For all ET, \(A\), if ET is sound against \(A\), then: \(\text{CM}(ET) \subseteq \{K_X \mid X \in \text{CM}(A)\}\). For all ET, \(A\), if ET is complete against \(A\), then: \(\{K_X \mid X \in \text{CM}(A)\} \subseteq \text{CM}(ET)\).

In §G we apply Theorem 5.2 and show all our definitions in Fig. 4 are sound and complete against (equivalent to) existing axiomatic definitions on abstract executions.

### 6. Applications

We showcase the applications of our formalism by showing how we can prove the robustness of transactional libraries (§6.1), and verify database iprotocols (§6.2).

#### 6.1. Program Analysis

A transactional library, \(L = \{T_i\}_{i \in I}\), provides a set of operations through which the library clients can access the kv-store. For instance, the set of operations of the counter library on key \(k\) in §2 is \(\text{Counter}(k) = \{\text{inc}(k), \text{read}(k)\}\). A program \(P\) is a client of library \(L\) if the only transactional calls in \(P\) are those to library \(L\) operations. A transactional library \(L\) is robust against an execution test ET if: for all client programs \(P\) of library \(L\), the set of kv-stores by running \(P\) are included in the set of kv-stores obtained by running \(P\) under ET\(\text{SER}\). As an application of our theory, we show how we can prove the robustness of simple transactional libraries. In the following, we write \(\text{CM}(ET, L)\) for the set of kv-stores obtained from running a client program \(P\) of \(L\).

We prove the robustness of single and multiple counters ([2] against ET\(\text{PSI}\) and ET\(\text{SRI}\), respectively. Previous techniques for checking robustness [1][13][16][28] are based on static analysis: since the session order of clients cannot be determined at compile time, these techniques abstract from
sessions. To our knowledge, we give the first robustness proofs that take sessions into account.

Our proof technique for robustness uses the following result, which is the kv-store counterpart of another well-known result on dependency graphs [3].

**Theorem 6.1.** For all kv-stores $K$: $K \in \text{CM}(\text{ET}_{\text{SER}})$ iff $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$ is irreflexive.

Given this theorem, to prove the robustness of $L$ against ET, for an arbitrary a kv-store $K$, we 1) identify a property of $K$ that remains invariant when running any client program of $L$ on $K$; and 2) show that the invariant implies the irreflexivity of $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$.

**Robustness of a Single Counter against ET$_{PSI}$.** Pick an arbitrary key $k$ and $K \in \text{CM}(\text{ET}_{PSI}, \text{Counter}(k))$, with $\text{Counter}(k) = \{\text{inc}(k), \text{read}(k)\}$. Note that clients can only write to $k$ by calling inc$(k)$. As such, since ET$_{PSI}$ enforces write conflict freedom, we have: \forall t > 0. t = w(K(k, i)) \Rightarrow t \in rs(K(k, i−1)). Furthermore, because ET$_{PSI}$ satisfies monotonic reads ($ET_{PSI} \subseteq ET_{MR}$), the order in which clients observe the versions of $k$ (by calling read$(k)$), is consistent with the order of the index of such versions in $K(k)$:

$$\mathcal{K}(k) = (0, t_0, T_0^r \cup \{t_0^w\});(1, t_1^w = t_1^r \cup \{t_1^w\}); \cdots; \cdots; (n, t_n^r \cup \{t_n^w\})$$

where $\{t_n^w\}_{i=1}^n$ is the set of transactions calling inc$(k)$, and $\bigcup_{i=0}^n T_i^r$ is the set of transactions calling read$(k)$.

Next, we define a relation $\rightarrow$ between the transactions in $K$ as the smallest transitive relation that satisfies the following: 1) $t_0^w \rightarrow t$ if $t \in T_0^r$ or $t = t_0^w + i$; 2) $t \rightarrow t'$ for some $t \in T_i^r$, if $t' = t_{i+1}^w$ or $t' \in T_i^r$ and $t \rightarrow_{SO} t'$. Note that $\rightarrow$ yields a total order over the transactions in $K$, and includes $\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K$. As such, $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$ is irreflexive, ensuring the robustness of Counter$(k)$ against ET$_{PSI}$.

**Robustness of Multiple Counters against ET$_{SI}$.** Recall from 8[2] that for the library comprising two or more counters we can observe the long fork anomaly disallowed under serialisability (ET$_{SER}$). As such, multiple counters are not robust against ET$_{PSI}$. The long fork anomaly arises because different clients can observe increments over two different counters in a different order. As this anomaly does not arise under SI, we show that if we strengthen ET$_{PSI}$ to ET$_{SI}$, we can analogously recover the robustness of multiple counters. For simplicity, here we assume that the library comprises only two counters: Counters = Counter$(k_1) \cup \text{Counter}(k_2)$.

We use the following result, which is the kv-store counterpart of a well-known result for SI [13, 20]:

**Theorem 6.2.** For all $K \in \text{CM}(\text{ET}_{SI})$, any cycle in $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$ has two adjacent RW$_K$-edges.

Consider a kv-store $K \in \text{CM}(\text{ET}_{SI}, \text{Counters})$. As ET$_{SI} \subseteq ET_{PSI}$, the list of versions of $K(k_1)$ and $K(k_2)$ must both follow the same structure as that described above for a single counter under ET$_{PSI}$. Recall that this structure embeds a total order $\rightarrow$ over transactions in $K(k_1)$ and $K(k_2)$. However, since a client can perform operations over two different counters, the $\rightarrow$ no longer comprises SO$_K$.

We thus define a second relation $\rightarrow$, where $t \rightarrow t'$ if $t \in \{w(K(k_1, i)) \cup rs(K(k_1, i)) \}$ and $t' \in \{w(K(k_2, j)) \cup rs(K(k_2, j)) \}$ with $i, j \in \{1, 2\} \land i \neq j$, and $t \rightarrow_{SO} t'$. Relation $\rightarrow$ tracks the session order of transactions performing operations on different keys. Let $\rightarrow_{\text{ET}_{SI}} (\rightarrow_{ET_{PSI}} \cup \rightarrow)$; we then know that $\rightarrow$ embeds $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$. Furthermore, because Counters contains no operation that accesses both $k_1$ and $k_2$, and since $\rightarrow_{ET_{PSI}}$ is acyclic, any cycle in $\rightarrow$ contains at least two edges from $\rightarrow_{SI}$. In §6[4] we show that the existence of such a cycle implies the existence of a cycle in $\rightarrow$ of the form $t \rightarrow_{et} t_a \rightarrow_{et} t_b \rightarrow_{et} t_c \rightarrow_t$. However, this cannot happen in $K$, as otherwise we would have a cycle in $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$ with no adjacent RW$_K$-edges (a $\rightarrow_{SI}$ edge represents two SO$_K$-related transactions, thus contradicting Theorem 6.2). It then follows that $\rightarrow_{SI}$ is irreflexive, and thus so is $(\text{SO} \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+$.

6.2. Verifying Database Proocols

Our representation of database state as kv-stores and client views closely matches the representation of database state in many implementations of well-known protocols [5 6 9 19 24 25 26 30 32 33]. As such, when verifying such protocols, our state-based formalism is better-suited than the existing graph-based formalisms of dependency graphs and abstract executions. In centralised databases, the state of the centralised server corresponds to a kv-store, and the local snapshots that transactions operate on are extracted from our views. In distributed databases, each replica state corresponds to a view. Moreover, since most distributed databases are eventual consistent, i.e. all replicas eventually agree on the database state, replica states can be collectively encoded as a kv-store. We verify the correctness of two database protocols: COPSi [26] and Clock-SI [19].

COPS. COPS is a fully replicated database: each replica contains all keys, but their associated values may be out of
date. Database clients operate on a single replica; synchronisation among replicas ensures that they are in a consistent (albeit out-of-date) state. The COPS API follows the multiple-readers-single-writer paradigm: at any given time, the database can be accessed by either multiple read-only concurrent transactions, or a single writing transaction. As such, all versions (on all keys) are totally ordered. Each transaction is assigned to a designated partition, known as the coordinator, and the commit timestamp. Each transaction is assigned to a designated partition, known as the coordinator, and the commit timestamp. Each version $\nu$ records a dependency set, written $\text{dep}(\nu)$, tracking the versions on which $\nu$ depends. A client maintains a context tracking the versions that have been either fetched from, or committed to, other replicas. We encode the COPS replicas as a kv-store, and encode client contexts as views. We show that the COPS protocol is causally consistent in that it is sound with respect to the CM($\text{ET}_{\text{CC}}$), where $\text{ET}_{\text{CC}}$ is the causal consistency execution test in Fig. 4. The COPS protocol and our soundness proof are given in § I.1.

**Clock-SI.** Clock-SI is a partitioned database protocol for snapshot isolation (SI), where different partition host disjoint fragments of keys. As with the original defpartition of SI, Clock-SI uses timestamps to maintain different versions. The clocks on different partitions may not agree; however, the difference is assumed to be bounded. Each key keeps a history of versions, where each version comprises a value and the commit timestamp. Each transaction $t$ is assigned to a designated partition, known as the $t$ coordinator. When $t$ starts, it obtains a snapshot and records the snapshot timestamp at its coordinator partition. Transaction $t$ may then read from partitions other than its coordinator, so long as the current timestamp at those partitions are greater than the recorded snapshot time. Once $t$ completes, it may commit if no transaction with a conflicting write has committed since the snapshot time. We encode the partitions collectively as a kv-store. Each snapshot timestamp is encoded as a view, contains all versions committed before that timestamp. We show that Clock-SI is sound with respect to the CM($\text{ET}_{\text{SI}}$), where $\text{ET}_{\text{SI}}$ is the snapshot isolation execution test in Fig. 4. More details and our soundness proof are given in § I.1.

**7. Conclusions and Related Work**

We have introduced an operational semantics for transactional distributed databases, based on a global, centralised kv-store and partial client views. Our semantics is parametric in the definition of an execution test. We capture a large variety of well-known consistency models for replicated and distributed databases by simple changes the execution tests. We have proved the correctness of two real-world protocols employed by distributed databases: COPS [26], a protocol for replicated databases that satisfies our definition of causal consistency; and Clock-SI [19], a protocol for partitioned databases that satisfies our definition of snapshot isolation. We have also demonstrated the usefulness of our framework for program analysis, by proving the robustness of simple transactional libraries against different consistency models. We believe that we are the first to show such results for databases which allow the grouping of transactions into sessions.

In future, we plan to validate further the usefulness of our framework by verifying the correctness of other well-known protocols employed by distributed databases, such as Eiger [27], Wren [33] and Red-Blue [24]; by exploring robustness results for OLTP workloads such as TPC-C [2] and RUBiS [1]; and by exploring other program-analysis techniques such as transactional chopping [15, 31], invariant checking [21, 36] and program logics [22].

**Related Work.** Kaki et al. proposed an operational semantics of programs under different transactional consistency models [22], corresponding to the ANSI/SQL isolation levels [7]. In their framework, transactions work on a local copy of the global state of the system, and the local effects of a transaction are pushed as system state changes upon commit. Because state changes are made immediately available to all clients of a system, this model is not suitable to capture weak consistency models such as PSI or CC. Also, the definition given are not validated against previously known formal definitions.

Nagar and Jagannathan proposed an operational semantics for weak consistency based on abstract executions [28]. Their semantics is parametric in the declarative definition of a consistency model. Because in their model abstract executions are not equipped with sessions, they cannot specify a large variety of consistency models as we do, such as the session guarantees. The authors also present a static analysis tool for determining the robustness of transactional libraries; however, their robustness results rely on the assumption that the underlying database is not equipped with sessions. Because the session order cannot be determined at compile time, their tool cannot be easily extended to a setting where sessions are provided by the database.

Crooks et al. [17] proposed a state-based formal framework for weak consistency models, that also employs notions similar to execution tests and views: commit tests and read states. On one side, the authors capture a greater range of consistency models that we do (in particular, the Read Committed isolation level), and they also exploit their formalism to verify the correctness of protocols. However, they do not consider program analysis. Because their notion of commit tests and read states requires the knowledge of information that cannot be determined at compile time, i.e. the total order of system changes that happened in the database we believe that their framework is not suitable for the development of program analysis techniques.

Several other works have focused on program analysis for transactional systems. Dias et al. [13] developed a separation logic for the robustness of applications under SI. Fekete et al. [20] derived a static analysis check for SI based on dependency graph. Bernardi and Gotsman [8] developed a static analysis check for several consistency models with atomic visibility. Cerone et al. [16] investigated the relationship between abstract executions and dependency graph from an algebraic perspective, and applied it to infer robustness checks for several consistency models.
References

[1] The rubis benchmark. https://rubis.osw2.org/index.html. Accessed: 2019-01-11.

[2] The tpcc-c benchmark, http://www.tpc.org/tpcc/. Accessed: 2019-01-11.

[3] Atul Adya. Weak consistency: A generalized theory and optimistic implementations for distributed transactions. PhD thesis, MIT, 1999.

[4] Atul Adya, Barbara Liskov, and Patrick E. O’Neil. Generalized isolation level definitions. In ICDE, 2000.

[5] Masoud Saeida Ardekani, Pierre Sutra, and Marc Shapiro. G-dur: A middleware for assembling, analyzing, and improving transactional protocols. In Proceedings of the 15th International Middleware Conference, Middleware ’14, pages 13–24, NY, USA, 2014. ACM. ISBN 978-1-4503-2785-5. doi: 10.1145/2663165.2663336. URL: http://doi.acm.org/10.1145/2663165.2663336.

[6] Peter Bailis, Alan Fekete, Ali Ghodsi, Joseph M. Hellerstein, and Ion Stoica. Scalable atomic visibility with RAMP transactions. In 2014 ACM SIGMOD International Conference on Management of Data (SIGMOD), pages 27–38, 2014.

[7] Hal Berenson, Phil Bernstein, Jim Gray, Jim Melton, Elizabeth O’Neil, and Patrick O’Neil. A critique of ANSI SQL isolation levels. In 1995 ACM SIGMOD international conference on Management of data (SIGMOD), pages 1–10, 1995.

[8] Giovanni Bernardi and Alexey Gotsman. Robustness against consistency models with atomic visibility. In 27th International Conference on Concurrency Theory (CONCUR), pages 7:1–7:15, 2016. doi: 10.4230/LIPIcs.CONCUR.2016.7. URL: http://dx.doi.org/10.4230/LIPIcs.CONCUR.2016.7.

[9] Carsten Birgining, Stefan Hildenbrand, Franz Färber, Donald Kossmann, Juchang Lee, and Norman May. Distributed snapshot isolation: Global transactions pay globally, local transactions pay locally. The VLDB Journal, 23(6):987–1011, December 2014. ISSN 1066-8888. doi: 10.1007/s00778-014-0359-9. URL: http://dx.doi.org/10.1007/s00778-014-0359-9.

[10] J. Brzezinski, C. Sobaniec, and D. Wawrzyniak. From session causality to causal consistency. In 12th Euromicro Conference on Parallel, Distributed and Network-Based Processing, 2004. Proceedings., pages 152–158, Feb 2004. doi: 10.1109/EMPDP.2004.1271440.

[11] Sebastian Burckhardt, Michael Fahndrich, Daan Leijen, and Mooyi Sagiv. Eventually consistent transactions. In 22nd European Symposium on Programming (ESOP), page 6786, 2012. URL: https://www.microsoft.com/en-us/research/publication/eventually-consistent-transactions/.

[12] Sebastian Burckhardt, Alexey Gotsman, Hongseok Yang, and Marek Zawirski. Replicated data types: specification, verification, optimality. In 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL), pages 271–284, 2014.

[13] Andrea Cerone and Alexey Gotsman. Analysing snapshot isolation. In 2016 ACM Symposium on Principles of Distributed Computing (PODC), pages 55–64, 2016.

[14] Andrea Cerone, Giovanni Bernardi, and Alexey Gotsman. A framework for transactional consistency models with atomic visibility. In 26th International Conference on Concurrency Theory (CONCUR), pages 58–71, Dagstuhl, 2015.

[15] Andrea Cerone, Alexey Gotsman, and Hongseok Yang. Transaction chopping for parallel snapshot isolation. In 29th International Symposium on Distributed Computing (DISC), pages 388–404, 2015.

[16] Andrea Cerone, Alexey Gotsman, and Hongseok Yang. Algebraic laws for weak consistency. In 27th International Conference on Concurrency Theory (CONCUR), pages 26:1–22:16, 2017.

[17] Natacha Crooks, Youer Pu, Lorenzo Alvisi, and Allen Clement. Seeing is believing: A client-centric specification of database isolation. In Proceedings of the ACM Symposium on Principles of Distributed Computing, PODC ’17, pages 73–82, New York, NY, USA, 2017. ACM. ISBN 978-1-4503-4992-5. doi: 10.1145/3087801.3087802. URL: http://doi.acm.org/10.1145/3087801.3087802.

[18] Ricardo J. Dias, João M. Lourenço, and Nuno Preguiça. Efficient and correct transactional memory programs combining snapshot isolation and static analysis. In HotPar, 2011.

[19] Jaquing Du, Sameh Elkinkey, and Willy Zwaenepoel. Clock-SI: Snapshot isolation for partitioned data stores using loosely synchronized clocks. In SRDS, 2013.

[20] Alen Fekete, Dimitrios Liarokapis, Elizabeth O’Neil, Patrick O’Neil, and Dennis Shasha. Making snapshot isolation serializable. ACM Transactions on Database Systems, 30(2):492–528, 2005.

[21] Alexey Gotsman, Hongseok Yang, Carla Ferreira, Mahsa Najafzadeh, and Marc Shapiro. ‘cause i’m strong enough: Reasoning about consistency choices in distributed systems. In Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL ’16, pages 371–384, New York, NY, USA, 2016. ACM. ISBN 978-1-4503-3549-2. doi: 10.1145/2837614.2837625. URL: http://doi.acm.org/10.1145/2837614.2837625.

[22] Gowtham Khami, Kartik Nagar, Mahsa Najafzadeh, and Suresh Jaganathan. Alone together: Compositional reasoning and inference for weak isolation. Proc. ACM Program. Lang., 2(PPL):27:1–27:34, December 2017. ISSN 2475-1421. doi: 10.1145/3158115. URL: http://doi.acm.org/10.1145/3158115.

[23] Jeehoon Kang, Chung-Kil Hur, Ori Lahav, Viktor Vafeiadis, and Derek Dreyer. A promising semantics for relaxed-memory concurrency. In Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages, POPL 2017, pages 175–189, New York, NY, USA, 2017. ACM. ISBN 978-1-4503-4660-3. doi: 10.1145/3009837.3009850. URL: http://doi.acm.org/10.1145/3009837.3009850.

[24] Cheng Li, Daniel Porto, Allen Clement, Johannes Gehrke, Nuno Preguiça, and Rodrigo Rodrigues. Making geo-replicated systems fast as possible, consistent when necessary. In 10th USENIX Symposium on Operating Systems Design and Implementation (OSDI), pages 265–278, 2012.

[25] Si Liu, Peter Csaba Övéczky, Keshav Santhanam, Qi Wang, Indranil Gupta, and José Meseguer. Rola: A new distributed transaction protocol and its formal analysis. In Alessandra Russo and Andy Schürr, editors, Fundamental Approaches to Software Engineering, pages 77–93, Cham, 2018. Springer International Publishing.

[26] Wyatt Lloyd, Michael J. Freedman, Michael Kaminsky, and David G. Andersen. Don’t settle for eventual: scalable causal consistency for wide-area storage with COPs. In 23rd ACM Symposium on Operating Systems Principles (SOSP), pages 401–416, 2011.

[27] Wyatt Lloyd, Michael J. Freedman, Michael Kaminsky, and David G. Andersen. Stronger semantics for low-latency geo-replicated storage. In Presented as part of the 10th USENIX Symposium on Networked Systems Design and Implementation (NSDI 13), pages 313–328, Lombard, IL, 2013. USENIX. ISBN 978-1-931971-00-3. URL: https://www.usenix.org/conference/nsdi13/technical-sessions/presentation/lloyd.

[28] Kartik Nagar and Suresh Jaganathan. Automated detection of serializability violations under weak consistency. In 29th International Conference on Concurrency Theory, CONCUR 2018, September 4-7, 2018, Beijing, China, pages 41:1–41:18, 2018. doi: 10.4230/LIPIcs.CONCUR.2018.41. URL: https://doi.org/10.4230/LIPIcs.CONCUR.2018.41.

[29] Azalea Raad, Ori Lahav, and Viktor Vafeiadis. On parallel snapshot isolation and release/acquire consistency. In Amal Ahmed, editor, Programming Languages and Systems, pages 940–967, Cham, 2018. Springer International Publishing. ISBN 978-3-319-89884-1.

[30] M. Saeida Ardekani, P. Sutra, and M. Shapiro. Non-monotonic snapshot isolation: Scalable and strong consistency for geo-replicated transactional systems. In 32nd International Symposium on Reliable Distributed Systems (SRDS), pages 163–172, 2013.

[31] D. Shasha, F. Llibret, E. Simon, and P. Valduriez. Transaction chopping: Algorithms and performance studies. ACM Trans. Database Syst., 20(3):325–365, 1995.

[32] Y. Sovran, R. Power, M. K. Aguilera, and J. Li. Transactional storage with COPS. In Second International Symposium on Dependable Systems and Networks, DSN 2018, Luxembourg City, Luxembourg, June 25-28, 2018, pages 1–12, 2018. doi: 10.1109/DSN.2018.00014. URL: https://doi.org/10.1109/DSN.2018.00014.
[34] Andrew S. Tanenbaum and Maarten van Steen. *Distributed Systems: Principles and Paradigms (2Nd Edition)*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 2006. ISBN 0132392275.

[35] Werner Vogels. Eventually Consistent. *Commun. ACM*, 52(1):40–44, January 2009.

[36] Peter Zeller. Testing properties of weakly consistent programs with repliss. In *Proceedings of the 3rd International Workshop on Principles and Practice of Consistency for Distributed Data*, PaPoC ’17, pages 3:1–3:5, New York, NY, USA, 2017. ACM. ISBN 978-1-4503-4933-8. doi: 10.1145/3064889.3064893. URL http://doi.acm.org/10.1145/3064889.3064893
Appendix A.
Operational Semantics on KV-Stores

Definition A.1 (Multi-version Key-value Stores). Assume a countably infinite set of keys $\text{KEY} \ni k$, and a countably infinite set of values $\text{VAL} \ni v$, including an initialisation value $v_0$. The set of versions, $\text{VERSION} \ni \nu$, is: $\text{VERSION} \triangleq \text{VAL} \times \text{TRANSID} \times \mathcal{P}(\text{TRANSID})$. A key-value store, abbreviated to kv-store, is a function $\mathcal{K} : \text{KEY} \rightarrow \text{VERSION}^*$, where $\text{VERSION}^*$ is the free monoid on $\text{VERSION}$. The initial key-value store is given by $\mathcal{K}_0$, where $\mathcal{K}_0(k) \triangleq (v_0, t_0, \emptyset)$ for all $k \in \text{KEY}$. Well-formed key-values store satisfy:

\begin{align}
\forall k, i, j. \ & rs(\mathcal{K}(k, i)) \cap rs(\mathcal{K}(k, j)) \neq \emptyset \lor w(\mathcal{K}(k, i)) = w(\mathcal{K}(k, j)) \Rightarrow i = j \quad (1.1) \\
\forall k. \ & \mathcal{K}(k, 0) = (v_0, t_0, -) \quad (1.2) \\
\forall k, cl, i, j, n, m. \ & w(\mathcal{K}(k, i)) \cup \{w(\mathcal{K}(k, j))\} \cup rs(\mathcal{K}(k, i)) \Rightarrow n < m \quad (1.3)
\end{align}

Given the Def. [3.3], a kv-store $\mathcal{K}$, a key $k$ and a view $u$, we write $\mathcal{K}(k, u)$ as a shorthand for $\mathcal{K}(k, \max_<(u(k)))$. Thus, $\text{snapshot}(\mathcal{K}, u) = \lambda k. \text{val}(\mathcal{K}(k, u))$. The full operational semantics is given in Fig. 7.

Appendix B.
Execution Test and Compositionality

B.1. Sanity Check for ET

Proposition B.1. If $\text{ET}_1 \subseteq \text{ET}_2$ then $\text{CM}(\text{ET}_1) \subseteq \text{CM}(\text{ET}_2)$.

Proof. It is sufficient to prove that $\text{ET}_1 \subseteq \text{ET}_2 \Rightarrow \text{CONF}(\text{ET}_1) \subseteq \text{CONF}(\text{ET}_2)$. We prove it by induction on the length of the traces, $i$.

Base case: $i = 0$. We have $\Gamma_0 \in \text{CONF}(\text{ET}_1)$ and $\Gamma_0 \in \text{CONF}(\text{ET}_2)$. Inductive case: $i + 1$. Suppose identical traces of $\text{ET}_1$ and $\text{ET}_2$ respectively with length $i$. Let the final configuration be $\Gamma_i = (K_i, U_i)$. If the next step is a view shift or a step with empty fingerprint, it trivially holds. If the next step is a step by a client $cl$ with fingerprint $F$, we have $\text{ET}_1 \vdash (K_i, U_i(cl)) \triangleright F : (K_{i+1}, U')$, where $K_{i+1} \in \text{update}(K_i, U_i(cl), F, cl)$. The next configuration from $\text{ET}_1$ is $\Gamma_{i+1} = (K_i, U_i(cl \rightarrow u'))$. Since $\text{ET}_1 \subseteq \text{ET}_2$, so $\text{ET}_2 \vdash (K_i, U_i(cl)) \triangleright F : (K_{i+1}, u')$ holds. It is possible for $\text{ET}_2$ to have the exactly same next configuration $\Gamma_{n+1}$.

B.2. Normal ET Traces

For technical reasons, it will be convenient to adopt a reduction strategy for inferring kv-stores induced by an execution test: such an execution strategy require that clients only commit transactions with non-empty fingerprints, and a client updates its view only immediately before committing a transaction. The next proposition states that all kv-stores induced by an execution test ET can be obtained via a sequence of reductions that adhere to the reduction strategy outlined above. Note that throughout this section, we assume that the execution test ET is fixed.

Definition B.1. Let ET be an execution test. The ET-trace

\[
\Gamma_0 \xrightarrow{\alpha_0} \text{ET} \Gamma_1 \xrightarrow{\alpha_1} \text{ET} \cdots \xrightarrow{\alpha_n} \text{ET} \Gamma_{n+1}
\]

is in normal form if (i) $\Gamma_0$ is initial, and (ii) $\forall i = 0, \ldots, n$ there exists a client $cl_i$ and set of operations $F_i$ such that $\alpha_2 = (cl_i, \varepsilon)$, and $\alpha_{2+i}$ is defined and equal to $(cl_i, F_i)$ where $F_i \neq \emptyset$.

For any trace satisfying $\text{CM}(\text{ET})$, there exists an equivalent normal trace ends up with the same state (Prop. B.2).

Proposition B.2 (Normal ET Traces). Let ET be an execution test, and suppose that $\mathcal{K} \in \text{CM}(\text{ET})$. Then there exists a ET-trace

\[
(K_0, U_0) \xrightarrow{\varepsilon_{\text{ET}}} \cdots \xrightarrow{\varepsilon_{\text{ET}}} (K_n, U_n)
\]

that is in normal form, and such that $K_n = \mathcal{K}$.

Proof. Let $\mathcal{K} \in \text{CM}(\text{ET})$. By definition, there exists a sequence of reductions

\[
(K_0, U_0) \xrightarrow{(cl_0, \mu_0)} \cdots \xrightarrow{(cl_{n-1}, \mu_{n-1})} \xrightarrow{\varepsilon_{\text{ET}}} (K_n, U_n) 
\]  
(2.1)

such that $K_n = \mathcal{K}$. Given an index $i = 1, \ldots, n - 1$, we say that the action $(cl_i, \mu_i)$ is in place if $\mu_i = F_i$ for some $F_i$, $cl_{i-1} = cl_i$, $\mu_{i-1} = \varepsilon$, and if $(cl_j, \mu_j) = (cl_i, \varepsilon)$, for some $j = 0, \ldots, i - 2$, then there exists $j' : j < j' < i$ such that $(cl_{j'}, \mu_{j'}) = (cl_i, F_{j'})$. An action of the form $(cl_i, \mu_i)$ is out of place if it is not in place.
Given the sequence of reductions in Eq. (2.1), we show the following:

1) if the sequence has no action out of place, then there exists a sequence

\[
(K_0', U_0') \xrightarrow{(c_{i_0}', \mu_{i_0}')_E} \cdots \xrightarrow{(c_{i_{m-1}}', \mu_{i_{m-1}}')_E} (K_m', U_m')
\]

that is in normal form, and such that \( K_m' = K_n \), and

2) if the sequence has \( h \) actions out of place, for some \( h > 0 \), then there exists a sequence

\[
(K_0', U_0') \xrightarrow{(c_{i_0}', \mu_{i_0}')_E} \cdots \xrightarrow{(c_{i_{m-1}}', \mu_{i_{m-1}}')_E} (K_m', U_m')
\]

that has \( h - 1 \) actions out of place, and such that \( K_m' = K_n \).

Combining the two facts above, we obtain that if \( K \in CM(ET) \), then there exists a sequence of reductions in formal form whose final configuration is \((K, \_\)\), as we wanted to prove.
1) Suppose that the sequence of reductions from Eq. (2.1) has no action out of place. Let \( i = 0, \ldots, n - 1 \), and consider the greatest index \( i = 0, \ldots, n - 1 \) such that \( \mu_i = \varepsilon \), and either \( i = n - 1 \), or \( \forall F. (cl_{i+1}, \mu_{i+1}) \neq (cl_i, F) \). If such an index does not exist, then the sequence of transitions from Eq. (2.1) is in normal form, and there is nothing to prove. Otherwise, note that for any \( j = i + 1, \ldots, n - 1, \forall F. (cl_j, \mu_j) \neq (cl_i, F) \).

Suppose in fact that there existed an index \( j = i + 1, \ldots, n - 1 \) such that \( (cl_j, \mu_j) = (cl_i, F_j) \) for some \( F_j \), and without loss of generality assume that \( j \) is the smallest such index. This implies that there exists no index \( j' : i < j' < j \) such that \( (cl_{j'}, \mu_{j'}) = (cl_i, F_{j'}) \) for some \( F_{j'} \). Also, it cannot be \( j = i + 1 \), because we are assuming that \( \forall F. (cl_{i+1}, \mu_{i+1}) \neq (cl_i, F) \).

We have that \( j \geq i + 2 \); we also have that \( (cl_j, \mu_j) = (cl_i, F_j), (cl_i, \mu_i) = (cl_i, \varepsilon), \forall f' : i < j < j', \forall F. (cl_{j'}, \mu_{j'}) \neq (cl_i, F) \). By definition, the action \( (cl_j, \mu_j) \) is out of place, contradicting the assumption that the sequence of reduction of Eq. (2.1) has no actions out of place.

We have proved that \( \forall j = i+1, \ldots, n-1, \forall F. (cl_j, \mu_j) \neq (cl_i, F) \). Also, because we are assuming that \( \mu_i = \varepsilon \), and either \( i = n - 1 \) or \( \forall F. (cl_{i+1}, \mu_{i+1}) \neq (cl_i, F) \), then \( \forall j = i+1, \ldots, n-1 \forall \mu. (cl_j, \mu_j) \neq (cl_i, \mu) \). Consider the transition \( (K_i, U_i) \xrightarrow{(cl_i, \mu_i)} \Phi_{ET} (K_{i+1}, U_{i+1}) \).

Let \( u = U_i(cl) \). Because \( \mu_i = \varepsilon \), then it must be the case that \( K_i = K_{i+1}, U_{i+1} = U_i[cl \mapsto u'] \) for some \( u' : u \subseteq u' \).

For any \( j \geq i \), we have that \( cl_j \neq cl_i \). We can replace the transition

\[
(K_j, U_j) \xrightarrow{(cl_j, \mu_j)} \Phi_{ET} (K_{j+1}, U_{j+1})
\]

with

\[
(K_j, U_j(cl_i \mapsto u)) \xrightarrow{(cl_j, \mu_j)} \Phi_{ET} (K_{j+1}, U_{j+1}[cl_i \mapsto u]).
\]

It follows that the sequence of transitions

\[
(K_0, U_0) \xrightarrow{(cl_0, \mu_0)} \Phi_{ET} (K_1, U_1) \xrightarrow{(cl_1, \mu_1)} \Phi_{ET} (K_2, U_2) \xrightarrow{(cl_2, \mu_2)} \cdots \xrightarrow{(cl_{i-1}, \mu_{i-1})} \Phi_{ET} (K_i, U_i) = (K_{i+1}, U_{i+1}[cl_i \mapsto u])
\]

Note that this sequence has one reduction less than the original sequence from (2.1) (specifically, the reduction \( (K_i, U_i) \xrightarrow{(cl_i, \mu_i)} (K_{i+1}, U_{i+1}) \) has been removed). We can repeat this procedure until the resulting sequence of reductions has no index \( i = 0, \ldots, n - 1 \) such that \( \mu_i = \varepsilon \), and either \( i = n - 1 \), or \( \forall F. (cl_{i+1}, \mu_{i+1}) \neq (cl_i, F) \). That is, the resulting sequence of reductions is in normal form, and its final configuration is \( (K_n, U_n) \).

2) Suppose that the sequence from Eq. (2.1) has \( h \) actions out of place, where \( h > 0 \). Let \( i \) be the smallest index such that \( (cl_i, \mu_i) \) is out of place. This means that either \( i = 0 \), or \( (cl_{i-1}, \mu_{i-1}) \neq (cl_i, \varepsilon) \), or there exists an index \( j < i - 1 \) such that \( (cl_j, \mu_j) = (cl_i, \varepsilon) \) and \( \forall F. (cl_j, \mu_j) \neq (cl_i, F) \). Without loss of generality, we can assume that \( i \neq 0 \) and \( (cl_{i-1}, \mu_{i-1}) = (cl_i, \varepsilon) \). This is because we can always transform the sequence of reductions of Eq. (2.1) by introducing a transition of the form \( (K_i, U_i) \xrightarrow{(cl_i, \varepsilon)} \Phi_{ET} (K_i, U_i) \), leading to the sequence of reductions

\[
(K_0, U_0) \xrightarrow{(cl_0, \mu_0)} \Phi_{ET} (K_1, U_1) \xrightarrow{(cl_1, \varepsilon)} \Phi_{ET} (K_2, U_2) \xrightarrow{(cl_2, \mu_2)} \cdots \xrightarrow{(cl_{i-1}, \mu_{i-1})} \Phi_{ET} (K_i, U_i)
\]

Therefore, it must be the case that there exists an index \( j < i - 1 \) such that \( (cl_j, \mu_j) = (cl_i, \varepsilon) \), and \( \forall F. (cl_j, \mu_j) \neq (cl_i, F) \). Let then \( j \) be the smallest such index. Let \( d = (i - 1) - j \) be the number of reductions that separate the configuration \( (K_i, \mu_i) \) from \( (K_{i-1}, \mu_{i-1}) \) in Eq. (2.1). Note that it must be the case that \( d > 0 \). We show that we can construct a sequence of reductions where the distance between these two configurations is reduced to 0: a consequence of this fact is such a sequence of reductions would have exactly \( h - 1 \) actions out of place. Consider the following fragment in the sequence of reductions from Eq. (2.1):

\[
(K_j, U_j) \xrightarrow{(cl_j, \mu_j)} \Phi_{ET} (K_{j+1}, U_{j+1}) \xrightarrow{(cl_{j+1}, \mu_{j+1})} \Phi_{ET} (K_{j+2}, U_{j+2})
\]

We have two possible cases:

- \( cl_{j+1} \neq cl_j \). In this case we can apply Lemma B.2 and infer the sequence of reductions

\[
(K_j, U_j) \xrightarrow{(cl_{j+1}, \mu_{j+1})} \Phi_{ET} (K'_{j+1}, U'_{j+1}) \xrightarrow{(cl_j, \mu_j)} \Phi_{ET} (K_{j+1}, U_{j+1})
\]
which leads to the whole sequence of reductions
\[
(K_0, U_0) \xrightarrow{(c_{l_0}, \mu_0)}_{ET} \cdots \xrightarrow{(c_{l_{j-1}}, \mu_{j-1})}_{ET} (c_{l_j}, \mu_j) \xrightarrow{(c_{l_{j+1}}, \mu_{j+1})}_{ET} (K'_{j+1}, U'_{j+1}) \xrightarrow{(c_{l_{j+2}}, \mu_{j+2})}_{ET} \cdots \xrightarrow{(c_{l_{n-1}}, \mu_{n-1})}_{ET} K_n, U_n
\]

- \(c_{l_{j+1}} = c_{l_j}\). In this case we can apply Lemma B.1 and infer the reduction
\[
(K_j, U_j) \xrightarrow{(c_{l_j}, \varepsilon)}_{ET} (K_{j+2}, U_{j+2})
\]
which leads to the sequence of reductions
\[
(K_0, U_0) \xrightarrow{(c_{l_0}, \mu_0)}_{ET} \cdots \xrightarrow{(c_{l_{j-1}}, \mu_{j-1})}_{ET} (K_j, U_j) \xrightarrow{(c_{l_j}, \varepsilon)}_{ET} (K_{j+2}, U_{j+2}) \xrightarrow{(c_{l_{j+2}}, \mu_{j+2})}_{ET} \cdots \xrightarrow{(c_{l_{n-1}}, \mu_{n-1})}_{ET} K_n, U_n
\]

In both cases, in the resulting sequence of reductions the number of reductions that separate the configuration \((K_j, U_j)\) from \((K_{j-1}, U_{j-1})\) is strictly less than \(d\). We can repeating applying the procedure outlined above until there are no reductions that separate the configuration \((K_j, U_j)\) from \((K_i, U_i)\).

\[\square\]

**Lemma B.1 (Absorption).** If \(\Gamma \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma' \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma''\), then \(\Gamma \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma''\).

**Proof.** Let \(\Gamma = (K, U, \Gamma' = (K', U'))\). By the reduction it must be the case that \(K = K'\), and \(U' = U[c_{l'} \mapsto u']\) for some \(u' : u \subseteq u'\). It must also be the case that \(K' = K''\), and \(U'' = U'[c_{l'} \mapsto u'']\) for some \(u'' : u' \subseteq u''\). Therefore we have that \(K'' = K' = K\), and \(U'' = U'[c_{l'} \mapsto u''] = (U'[c_{l'} \mapsto u'])(c_{l'} \mapsto u'') = U[c_{l'} \mapsto u'']\), and \(u \subseteq u''\). It follows that \(\Gamma = (K, U) \xrightarrow{(c_{l'}, \mu')}_{ET} (K'', U'') = \Gamma''\).

\[\square\]

**Lemma B.2 (Independence of commit).** Let \(\Gamma \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma_1 \xrightarrow{(c_{l''}, \mu'')}_{ET} \Gamma_2 \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma'\) for some \(\Gamma, \Gamma_1, \Gamma_2\) and \(c_{l'} \neq c_{l''}\). Then \(\Gamma \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma_1 \xrightarrow{(c_{l''}, \mu'')}_{ET} \Gamma_2 \xrightarrow{(c_{l'}, \mu')}_{ET} \Gamma'\).

**Proof.** We only consider the case where \(\mu = F\) for some fingerprint \(F\). The case where \(\mu = \varepsilon\) is simpler to prove. Let \(\Gamma = (K, U)\), \(\Gamma_1 = (K_1, U_1)\), \(\Gamma_2 = (K_2, U_2)\). Let also \(u = U(c)\). By the reduction rule we have that \(K_1 = K, U_1 = U[c \mapsto u_1]\) for some \(u_1 : u \subseteq u_1\). Let \(u' = U(c')\); then we have that \(U_1(c') = u'\). Because \((K_1, U_1) \xrightarrow{(c', \mu')}_{ET} (K', U', \mu')\), we have that \(ET \vdash (K_1, u') \triangleright F : (K', u'')\), where \(u'' = U'(c'')\). Because \(K_1 = K\), that means that \(ET \vdash (K, u') \triangleright F : (K', u'')\), then it follows that \((K, U) \xrightarrow{(c', \mu')}_{ET} (K', U'[c \mapsto u'])\) \xrightarrow{(c_{l'}, \mu')}_{ET} (K', U'[c \mapsto u'][c \mapsto u_1]) = (K', U'[c \mapsto u'[c \mapsto u_1]]) = (K', U'[c \mapsto u'[c \mapsto u_1]]) = (K', U'[c \mapsto u'[c \mapsto u_1]]) = (K', U'[c \mapsto u'[c \mapsto u_1]]) = (K, U)\), as we wanted to prove.

\[\square\]

**B.3. Commutativity update**

A desirable property that one would request from execution test is compositionality: the consistency model induced by a composite execution test can be recovered from the consistency models generated by each execution test: that is,

\[\forall ET_1, ET_2, CM(ET_1 \cap ET_2) = CM(ET_1) \cap CM(ET_2)\]

However, execution tests do not always satisfy this property. We find that if \(ET_1\) and \(ET_2\) satisfies certain constraints, the above holds (Theorem B.1). Before showing the Theorem B.1, we first define conflict (Def. B.2) and composition of update.

**Definition B.2.** Two triples \((c_{l_1}, F_1)\) and \((c_{l_2}, F_2)\) are conflicting if either \(c_{l_1} = c_{l_2}\), or there exists a key \(k\) such that \((w, k, -) \in F_1, (w, k, -) \in F_2\).

An execution test is ET is commutative if, whenever \((c_{l_1}, u_1, F_1)\), \((c_{l_2}, u_2, F_2)\) are non-conflicting, and \(u_1, u_2 \in VIEWS(K_0)\), then for any \(K_0, K', U', U''\) we have that

\[
(K_0, U) \xrightarrow{(c_{l_1}, F_1)}_{ET} (K', U') \Rightarrow (K_0, U) \xrightarrow{(c_{l_2}, F_2)}_{ET} (K', U'') \Rightarrow (K_0, U) \xrightarrow{(c_{l_1}, F_1)}_{ET} (K', U')
\]

Given two non-conflict commits, it is possible to swap the commit order as shown in Prop. B.3. The Lemma B.3 and then Cor. B.1 shows that swapping the operations of one key yields the same result. Given Cor. B.1, then Prop. B.3 holds.

**Lemma B.3 (Swapping Operation).** Let \(K\) be a kv-store, \(u \in VIEWS(K)\), \(t \in TRANSID\) and \(F \in P(Ops)\). Let also \(k \in KEY\). Then

1) \(\forall v, (x, k, v) \notin F \land (w, k, v) \notin F \Rightarrow update(K, u, F, t)(k) = K(k)\)
2) \(\forall v. (x, k, -) \in F \land (w, k, v) \notin F \Rightarrow \text{update}(K, u, F, t)(k) = \text{let } (v', t', T') = K(k, \max < (u(k)))\) in \(K(k)[\max < (u(k)) \mapsto (v', t', T' \cup \{t\})]\)

3) \(\forall v, v'. (x, k, v) \notin F \land (w, k, v) \in F \Rightarrow \text{update}(K, u, F, t)(k) = K(k)\)

4) \(\forall v, (x, k, -) \in F \land (w, k, v) \in F \Rightarrow \text{update}(K, u, F, t)(k) = \text{let } (v', t', T') = K(k, \max(u(k)))\) in \(K(k)[\max(u(k)) \mapsto (v', t', T' \cup \{t\})]\)

**Proof.** All the four statements are proved by induction on \(F\), by keeping the variable \(K\) universally quantified in the inductive hypothesis. Statement item 2 and item 3 requires proving item 1 first, while Statement item 4 requires proving all the other statements. Fix then an arbitrary \(k \in K\).

1) Suppose that for any \(v, (x, k, v) \notin F\) and \((w, k, v) \notin F\). We prove that \(\text{update}(K, u, F, t)(k) = K(k)\).
   - **Base case:** \(F = \emptyset\). In this case we have that
     \[
     \text{update}(K, u, \emptyset, t)(k) = \text{let } (v', t', T') = K(k)\ \ \text{in } K(k)[\max < (u(k)) \mapsto (v', t', T' \cup \{t\})] = K(k).
     \]
   - **Suppose that** \(F = F' \cup \{(x, k', v')\}\) for some \(k', v'). Because we are assuming that \((x, k, v) \notin F\) for any \(v \in \text{VAL}\), then it must be the case that
     \[
     k \neq k'.
     \]

   Also, we have that \((x, k, v) \notin F'\) and \((w, k, v) \notin F\) for any \(v \in \text{VAL}\). By inductive hypothesis we can assume
   \[
   \forall k'. \text{ update}(K', u, F', t)(k) = K'(k) \tag{2.3}
   \]

   Therefore we have
   \[
   \text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(x, k', v')\}, t)(k) = \text{let } (v', t', T') = K(k', \max < (u(k')))\ \text{in } K(k')[\max < (u(k')) \mapsto (v', t', T' \cup \{t\})] = K(k') \tag{2.3}
   \]

   **Suppose that** \(F = F' \cup \{(w, k', v')\}\) for some \(v' \in \text{VAL}\). Then it must be the case that
   \[
   k \neq k'.
   \]

   Also, we have that \((x, k, v) \notin F'\) and \((w, k, v) \notin F\) for any \(v \in \text{VAL}\). By inductive hypothesis we can assume
   \[
   \forall k'. \text{ update}(K', u, F', t)(k) = K'(k) \tag{2.5}
   \]

   Therefore we have
   \[
   \text{update}(K, k, F, t)(k) = \text{update}(K, k, F' \cup \{(w, k', v')\}, t)(k) = \text{let } (v', t', T') = K(k)\ \text{in } K[k' \mapsto K(k')][\max < (u(k')) \mapsto (v', t', T' \cup \{t\})] = K(k').
   \]

2) Suppose \((x, k, -) \in F\), and \((w, k, v) \notin F\) for all \(v \in \text{VAL}\). Let \((v, t', T) = K(k, \max < (u(k)))\). We prove that
   \[
   \text{update}(K, u, F, t)(k) = K(k)[u \mapsto (v', t', T \cup \{t\})] = K(k)[\max(u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)[\max(u(k)) \mapsto (v', t', T \cup \{t\})] = K(k) \tag{2.6}
   \]

   **Suppose that** \(F = F' \cup \{(x, k', -)\}\) for some \(k'\). We have two possible cases:
   a) \(k = k'\), in which case we know that \((x, k', v') \notin F'\) for all \(v' \in \text{VAL}\) because of the assumptions that we make on the structure of \(F\). By Lemma 3.3 item 4 we have that
   \[
   \forall k'. \text{ update}(K', u, F', t)(k) = K'(k) \tag{2.6}
   \]

   In this case we have that
   \[
   \text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(x, k', v')\}, t)(k) = \text{let } (v', t', T') = K(k)\ \text{in } K[k \mapsto K(k)][\max < (u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)[\max(u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)[\max(u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)\]

   b) \(k \neq k'\), in which case we know that \((x, k', v') \notin F'\) for all \(v' \in \text{VAL}\) because of the assumptions that we make on the structure of \(F\). By Lemma 3.3 item 4 we have that
   \[
   \forall k'. \text{ update}(K', u, F', t)(k) = K'(k) \tag{2.6}
   \]

   In this case we have that
   \[
   \text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(x, k', v')\}, t)(k) = \text{let } (v', t', T') = K(k)\ \text{in } K[k \mapsto K(k)][\max < (u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)[\max(u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)[\max(u(k)) \mapsto (v', t', T \cup \{t\})] = K(k)\]
b) $k \neq k'$. In this case we know that because $(v, k, -) \in F$, then it must be $(v, k, -) \in F'$. We also know that $\forall v. (w, k, v) \notin F$. By the inductive hypothesis, we have that
\[
\forall K'. \text{ update}(K', u, F', t)(k) = K'(k)[\max_<(u(k)) \mapsto (v, t', T \cup \{t\})]
\] (2.7)

In this case we have
\[
\text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(v, k', -)\}, t)(k)
\]

\[= \text{update}(K, u, F' \cup \{(w, k', v')\}, t)(k)
\]

\[\overset{\text{Def. 3.7}}{=} \text{update}(K[k' \mapsto -], u, F', t)(k)
\]

\[\overset{\text{Eq. 2.9}}{=}
\]

\[K(k)[\max_<(u(k)) \mapsto (v, t', T \cup \{t\})]
\]

\[k \neq k'
\]

\[
\text{F = F'} \cup \{(w, k', v')\}
\]

for some $v' \in \text{VAL}$. Because $(w, k, v) \notin F$ for any $v \in \text{VAL}$, it must be the case that
\[
k \neq k'
\] (2.8)

Because $(v, k, -) \in F$, it must also be the case that $(v, k, -) \in F'$. By the inductive hypothesis, we have that
\[
\forall K'. \text{ update}(K', u, F', t)(k) = K'(k)[\max_<(u(k)) \mapsto (v, t', T \cup \{t\})]
\] (2.9)

It follows that
\[
\text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(v, k', -)\}, t)(k)
\]

\[\overset{\text{Def. 3.7}}{=} \text{update}(K[k' \mapsto -], u, F', t)(k)
\]

\[\overset{\text{Eq. 2.9}}{=}
\]

\[K(k)[\max_<(u(k)) \mapsto (v, t', T \cup \{t\})]
\]

3) Suppose that $(w, k, v) \in F$ for some $v \in \text{VAL}$, and $(v, k, v') \notin F$ for any $v' \in \text{VAL}$. We prove that update$(K, u, F, t)(k) = K(k)[:(v, t, 0)]$.

- Base case: $F = \emptyset$. This case is vacuous, as $(w, k, v) \in F$.
- Suppose that $F = F' \cup \{(v, k', -)\}$ for some $k'$. Note that, because we are assuming that $\{(v, k', v')\} \notin F$ for all $v' \in \text{VAL}$, then it must be the case that

\[
k \neq k'
\] (2.10)

We also have that $\{(v, k', v')\} \notin F'$ for all $v' \in \text{VAL}$, and $(w, k, v) \in F'$. By the inductive hypothesis we have that
\[
\forall K'. \text{ update}(K', u, F', t)(k) = K'(k)[:(v, t, 0)]
\] (2.11)

Therefore, we have that
\[
\text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(v, k', -)\}, t)(k)
\]

\[\overset{\text{Def. 3.7}}{=} \text{update}(K[k' \mapsto -], u, F', t)(k)
\]

\[\overset{\text{Eq. 2.9}}{=}
\]

\[K(k)[:(v, t, 0)]
\]

- Suppose that $F = F' \cup \{(w, k', v')\}$ for some $k'$. We distinguish two possible cases:
  
a) $k = k'$. In this case the structure of $F$ also imposes that $v = v'$, and $(w, k, v'') \notin F$ for any $v'' \in \text{VAL}$. Furthermore, we have that $(v, k, v'') \notin F'$ for any $v'' \in \text{VAL}$. By Lemma B.3 item 1 we have that

\[
\forall K'. \text{ update}(K', u, F', t)(k) = K(k)
\] (2.12)

from which follows
\[
\text{update}(K, u, F, t)(k) = \text{update}(K, u, F' \cup \{(w, k', v')\}, t)(k)
\]

\[= \text{update}(K, u, F' \cup \{(w, k, v)\}, t)(k)
\]

\[\overset{\text{Def. 3.4}}{=}
\]

\[\text{update}(K[k \mapsto K(k)\mapsto [:(v, t, 0)]]), u, F', t)(k)
\]

\[\overset{\text{Eq. 2.12}}{=}
\]

\[K(k)[:(v, t, 0)]
\]

b) $k \neq k'$. In this case we have that, because $(w, k, v) \in F$, then it must be $(w, k, v) \in F'$. Furthermore, we also have that $(v, k, v'') \notin F'$ for any $v'' \in \text{VAL}$. By the inductive hypothesis, we have that
\[
\forall K'. \text{ update}(K', u, F', t)(k) = K(k)[:(v, t, 0)]
\] (2.13)
from which it follows
\[
\text{update}(K, u, \mathcal{F}, t)(k) = \text{update}(K, u, \mathcal{F}' \cup \{(w,k',v')\} , t) \\
\begin{array}{l}
\text{Def. 2.7} \quad \text{update}(K \cup [\mathcal{F} \rightarrow -], u, \mathcal{F}, t)(k) \\
\text{Eq. 2.13} \quad K[k' \rightarrow -](k):=[[v, t, \emptyset]] \\
\end{array}
\]
\[
k \neq k' \Rightarrow K(k) :=[[v, t, \emptyset]]
\]

4) Suppose that \((w, k, v) \in \mathcal{F}\) for some \(v \in \text{Val}\), and \((x, k, -) \in \mathcal{F}\). Let \((K, k, u) = (v', t', \mathcal{T}')\). We prove that
\[
\text{update}(K, u, \mathcal{F}, t)(k) = K(k)[u(k) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]]
\]
by induction on \(\mathcal{F}\):
- \(\mathcal{F} = \emptyset\); this case is vacuous.
- \(\mathcal{F} = \mathcal{F}' \cup \{(x, k', -)\}\). We distinguish two cases, according to whether \(k = k'\) or \(k \neq k'\). If \(k = k'\), then we know that \((w, k, v) \in \mathcal{F}'\) and \((x, k, v') \notin \mathcal{F}\) for any \(v'\) \in \text{Val}\). By Lemma Lemma B.3 item 5 we have that
\[
\forall K'. \text{update}(K, u, \mathcal{F}', t)(k) = K(k)[u(k) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]]
\]
from which it follows that
\[
\text{update}(K, u, \mathcal{F}, t)(k) = \text{update}(K, u, \mathcal{F}' \cup \{(x, k', -)\}, t)(k) \\
\begin{array}{l}
\text{Def. 2.7} \quad \text{update}(K \cup [\mathcal{F} \rightarrow -], u, \mathcal{F}, t)(k) \\
\text{Eq. 2.13} \quad K[k \mapsto K(k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})]] :=[[v, t, \emptyset]] \\
\end{array}
\]
\[
K[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]] \\
\]
If \(k \neq k'\), then we have that both \((x, k, -) \in \mathcal{F}'\) and \((w, k, v) \in \mathcal{F}'\). In this case, by the inductive hypothesis we have that
\[
\forall K'. \text{update}(K', u, \mathcal{F}', t)(k) = K'(k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]]
\]
from which it follows that
\[
\text{update}(K, u, \mathcal{F}, t)(k) = \text{update}(K, u, \mathcal{F}' \cup \{(x, k', -)\}, t)(k) \\
\begin{array}{l}
\text{Def. 2.7} \quad \text{update}(K \cup [\mathcal{F} \rightarrow -], u, \mathcal{F}, t)(k) \\
\text{Eq. 2.13} \quad K[k \mapsto K(k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})]] :=[[v, t, \emptyset]] \\
\end{array}
\]
\[
K[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]] \\
\]
- \(\mathcal{F} = \mathcal{F}' \cup \{(w, k'', v'')\}\) for some \(k'', v''\). Again, there are two possible cases to consider. If \(k = k''\), then \(v = v''\) because of the structure imposed on \(\mathcal{F}\). Furthermore, we have that \((x, k, -) \in \mathcal{F}'\) and \((w, k, v'') \notin \mathcal{F}\) for all \(v''\) \in \text{Val}\). By Lemma B.3 item 2 we have that
\[
\forall K'. \text{update}(K', u, \mathcal{F}', t)(k) = K'(k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]]
\]
We have that
\[
\text{update}(K, u, \mathcal{F}, t)(k) = \text{update}(K, u, \mathcal{F}' \cup \{(w, k'', v'')\}, t)(k) \\
\begin{array}{l}
\text{Def. 2.7} \quad \text{update}(K \cup [\mathcal{F} \rightarrow -], u, \mathcal{F}, t)(k) \\
\text{Eq. 2.13} \quad K[k \mapsto K(k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})]] :=[[v, t, \emptyset]] \\
\end{array}
\]
\[
K[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]] \\
\]
Finally, if \(k \neq k'\), then we have that \((x, k, -) \in \mathcal{F}'\) and \((w, k, v) \in \mathcal{F}'\). By the inductive hypothesis, we obtain
\[
\forall K'. \text{update}(K', u, \mathcal{F}', t)(k) = K'(k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]].
\]
It follows that
\[
\text{update}(K, u, \mathcal{F}, t)(k) = \text{update}(K, u, \mathcal{F}' \cup \{(w, k', -)\}, t) \\
\begin{array}{l}
\text{Def. 2.7} \quad \text{update}(K \cup [\mathcal{F} \rightarrow -], u, \mathcal{F}', t)(k) \\
\text{Eq. 2.13} \quad K[k' \mapsto -](k)[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]] \\
\end{array}
\]
\[
K[\text{max}_{<} (u(k)) \mapsto (v', t', \mathcal{T}' \cup \{t\})] :=[[v, t, \emptyset]] \\
\]
In the following, given a version \( v = (v, t', \mathcal{T}) \) and a set of transaction identifiers \( \mathcal{T}' \), we let \( v \oplus \mathcal{T}' = (v, t', \mathcal{T} \cup \mathcal{T}') \).

Clearly the operator \( \oplus \) is commutative over sets of transactions: \( \forall v, \mathcal{T}, \mathcal{T}' \in \mathcal{P} (\text{Ops}) \). Let also \( k \in \text{Key} \). Then

1) \( \forall i. 0 \leq i < |K(k)| - 1 \land i \neq \max_<(u(k)) \Rightarrow \text{update}(K, u, \mathcal{T}, t)(k, i) = K(k, i) \)
2) \( \forall v. (x, k, -) \in \mathcal{F} \Rightarrow \text{update}(K, u, \mathcal{F}, t)(k, u) = K(k, \max_<(u(k))) \oplus \{ t \} \)
3) \( \forall v. (x, k, v) \notin \mathcal{F} \Rightarrow \text{update}(K, u, \mathcal{F}, t)(k, u) = K(k, \max_<(u(k))) \)
4) \( \forall v. (w, k, v) \in \mathcal{F} \Rightarrow |\text{update}(K, u, \mathcal{F}, t)(k, v)| = |K(k)| + 1 \land \text{update}(K, u, \mathcal{F}, t)(k, |K(k)|) = (v, t, 0) \)
5) \( \forall v. (w, k, v) \notin \mathcal{F} \Rightarrow |\text{update}(K, u, \mathcal{F}, t)(k, v)| = |K(k)| \)

**Proof.** A simple consequence of Lemma B.3.

**Proposition B.3.** Let \( K \in \text{KVS} \), \( u_1, u_2 \in \text{Views}(K) \) and let \( cl_1, cl_2 \in \text{Client} \) be such that \( cl_1 \neq cl_2 \). Let also \( \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{P} (\text{Ops}) \) be such that whenever \( (w, k, -) \in \mathcal{F}_1 \) for some key \( k \), then \( (w, k, v) \notin \mathcal{F}_2 \) for all \( v \in \text{Val} \).

\[
\{ \text{update}(K_1, u_2, \mathcal{F}_2, cl_2) \mid K_1 \in \text{update}(K, u_1, \mathcal{F}_1, cl_1) \} = \{ \text{update}(K_2, u_1, \mathcal{F}_1, cl_1) \mid K_2 \in \text{update}(K, u_2, \mathcal{F}_2, cl_2) \}
\]

**Proof.** Assume \( K_1 = \text{update}(K, u_1, \mathcal{F}_1, t_1) \), \( K_2 = \text{update}(K, u_2, \mathcal{F}_2, t_2) \). It suffices to show that for any key \( k \):

\[
|\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k)| = |\text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k)|
\]

and for any index \( i \) such that \( 0 \leq i < |\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k)| \):

\[
\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k, i) = \text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k, i)
\]

First, fix a key \( k \in \text{Key} \). Note that if \( (w, k, -) \in \mathcal{F}_1 \), then by Corollary B.1, we have that \( |\text{update}(K, u_1, \mathcal{F}_1, t_1)(k)| = |K(k)| \). Because \( \mathcal{F}_1 \) is not conflicting with \( \mathcal{F}_2 \), it must be the case that \( \forall v. (w, k, v) \notin \mathcal{F}_2 \), and therefore by Cor. B.1, we have that

\[
|\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k)| = |K_1(k)| = |K(k)| + 1.
\]

Similarly, because \( \forall v. (w, k, v) \notin \mathcal{F}_2 \) and \( (w, k, -) \in \mathcal{F}_1 \), then

\[
|\text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k)| = |K_2(k)| + 1 = |\text{update}(K, u_2, \mathcal{F}_2, t_2)(k)| = |K(k)| + 1.
\]

Therefore, if \( (w, k, -) \in \mathcal{F}_1 \), we have that

\[
|\text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k)| = |\text{update}(K_1(k), u_2, \mathcal{F}_2, t_2)(k)|
\]

Similarly, we can prove that this claim holds also when \( (w, k, -) \in \mathcal{F}_2 \). Finally, if \( \forall v. (w, k, -) \notin \mathcal{F}_1 \land \forall v. (w, k, v) \notin \mathcal{F}_2 \), then by Cor. B.1, we have that

\[
|\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k)| = |K_1(k)| = |\text{update}(K, u_1, \mathcal{F}_1, t_1)(k)| = |K(k)|
\]

and

\[
|\text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k)| = |\text{update}(K, u_2, \mathcal{F}_2, t_2)(k)| = |K(k)|
\]

This concludes the proof that, for any key \( k \in \text{Key} \),

\[
|\text{update}(K_1, u_2, \mathcal{F}_2, t_2)| = |\text{update}(K_2, u_1, \mathcal{F}_1, t_1)|
\]

Next, fix a key \( k \) and a index \( i \) such that \( 0 \leq i < |K(k)| - 1 \). We show that:

\[
\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k, i) = \text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k, i)
\]

by performing a case analysis on \( u_1 \):

1) \( i \neq \max_<(u_1(k)) \). In this case, by Cor. [B.1][item 1] we have that

\[
K_1(k, i) = \text{update}(K, u_1, \mathcal{F}_1, t_1)(k, i) = K(k, i) \quad (2.18)
\]

and

\[
\text{update}(K_2, u_1, \mathcal{F}_1, t_1)(k, i) = K_2(k, i) \quad (2.19)
\]

Then, we have three possible sub-cases:

a) \( i \neq \max_<(u_2(k)) \): in this case, by Cor. [B.1][item 1] we have that

\[
\text{update}(K_1, u_2, \mathcal{F}_2, t_2)(k, i) = K_1(k, i) \quad \text{Eq. 2.18} \quad K(k, i)
\]
and
\[ \text{update}(K_2, u_1, F_1, t_1) \text{ Eq. 2.19} \]

b) \( i = \max_c(u_2(k)) \), and \((x, k, -) \notin F_2\). In this case the proof is analogous to the previous case, only Cor. B.1 item 3 needs to be applied in place of Cor. B.1 item 2.

c) \( i = \max_c(u_2(k)) \), and \((x, k, -) \in F_2\). In this case we can apply Cor. B.1 item 3 and deduce that
\[ \text{update}(K_1, u_2, F_2, t_2)(k, i) = K_1(k, i) \oplus \{ t_2 \} \] (2.20)

It follows that
\[ \text{update}(K_1, u_2, F_2, t_2)(k, i) = K(k, i) \oplus \{ t_2 \} \] (2.21)

2) \( i = \max_c(u_1(k)) \), \((x, k, -) \notin F_1\). This case is similar to the previous one: we can infer that Equations Eq. (2.18) and Eq. (2.19) are valid in this case using Cor. B.1 item 3, then we can proceed by performing a case analysis on \( u_2 \) and \( F_2 \) as in the previous case.

3) \( i = \max_c(u_1(k)) \), \((x, k, -) \in F_1\). We can apply Cor. B.1 item 2 to deduce the following:
\[ K_1(k, i) = \text{update}(K, u_1, F_1, t_1)(k, i) = K(k, i) \oplus \{ t_1 \} \] (2.22)

and
\[ \text{update}(K_2, u_1, F_1, t_1)(k, i) = K_2(k, i) \oplus \{ t_1 \} \] (2.23)

We have two different sub-cases to consider:

a) \( i \neq \max_c(u_2(k)) \), \( i = \max_c(u_2(k)) \) with \((x, k, -) \notin F_2\). In this case, we can apply either Cor. B.1 item 1 (if \( i \neq \max_c(u_2(k)) \)), or Cor. B.1 item 3 (if \( i = \max_c(u_2(k)) \) and \((x, k, -) \notin F_2\)), to obtain
\[ \text{update}(K_1, u_2, F_2, t_2)(k, i) = K_1(k, i) \text{ Eq. 2.22} \]

and
\[ \text{update}(K_2, u_1, F_1, t_1)(k, i) = K_2(k, i) \text{ Eq. 2.23} \]

b) if \( i = \max_c(u_2(k)) \) and \((x, k, -) \in F_2\), then by Cor. B.1 item 2 we obtain that
\[ K_2(k, i) = \text{update}(K, u_2, F_2, t_2)(k, i) = K(k, i) \oplus \{ t_2 \} \] (2.24)

\[ \text{update}(K_1, u_2, F_2, t_2)(k, i) = K_1(k, i) \oplus \{ t_2 \} \] (2.25)

From these facts it follows that
\[ \text{update}(K_1, u_2, F_2, t_2)(k, i) \text{ Eq. 2.25} \]
\[ \text{update}(K_2, u_1, F_1, t_1)(k, i) \text{ Eq. 2.26} \]

Next, note that if \( \forall v \in \text{VAL}. (w, k, v) \notin F_1 \land (w, k, v) \notin F_2 \), then
\[ |\text{update}(K_1, u_2, F_2, t_2)(k)| = |K(k)| = |\text{update}(K_2, u_1, F_1, t_1)(k)| \]

Because we have already proved that
\[ \forall i = 0, \ldots, |K(k)|. \text{update}(K_1, u_2, F_2, t_2)(k, i) = \text{update}(K_2, u_1, F_1, t_1)(k, i) \]

It follows that
\[ \text{update}(K_1, u_2, F_2, t_2)(k) \oplus \{ t_2 \} \]

and there is nothing left to prove.

Suppose then that either \((w, k, v) \in F_1 \) or \((w, k, v) \in F_2 \) for some \( v \). Without loss of generality, let \((w, k, v) \in F_1 \) for some \( v \in \text{VAL} \); because we are assuming that \( F_1 \) does not conflict with \( F_2 \), then it must be the case that \( \forall v' \in \text{VAL}. (w, k, v') \notin F_2 \). Using Cor. B.1 item 5 and Cor. B.1 item 3:
\[ \text{update}(K_1, u_2, F_2, t_2)(k, |K(k)|) = K_1(k, |K(k)|) = \text{update}(K, u_1, F_1, t_1)(|K(k)|) = (v, t_1, \emptyset) \]
\[ \land \text{update}(K_2, u_1, F_1, t_1)(k, |K(k)|) = (v, t_1, \emptyset) \]
We have now proved that if \((\omega, k, v) \in \mathcal{F}_1\), then \(|\text{update}(\mathcal{K}_1, u_2, \mathcal{F}_2, t_2)| = |\text{update}(\mathcal{K}_2, u_1, \mathcal{F}_1, t_1)|\), and for all \(i = 0, \ldots, |\text{update}(\mathcal{K}_1, u_2, \mathcal{F}_2, t_2)| - 1\), update\((\mathcal{K}_1, u_2, \mathcal{F}_2, t_2)(k, i) = \text{update}(\mathcal{K}_2, u_1, \mathcal{F}_1, t_1)(k, i)\). This concludes the proof that for any key \(k\), update\((\mathcal{K}_1, u_2, \mathcal{F}_2, t_2)(k) = \text{update}(\mathcal{K}_2, u_1, \mathcal{F}_1, t_1)(k)\), and therefore update\((\mathcal{K}_1, u_2, \mathcal{F}_2, t_2) = \text{update}(\mathcal{K}_2, u_1, \mathcal{F}_1, t_1)\).

\[\square\]

**B.4. Counter examples for compositionality**

For compositionality of two execution tests, it is not enough that one execution test is commutative. The following three counter examples lead us to further restrict the execution tests.

**B.4.1. No blind write.** Let consider a kv-store \(\mathcal{K} = \{k \mapsto (0, t_0, \emptyset, (1, t, \emptyset))\}:

\[k \mapsto \begin{pmatrix} 0 & t_0 & 1 & t & 0 \end{pmatrix}\]

and the following two execution test:

\[\text{ET}_1 \vdash (\mathcal{K}, u_1) \triangleright \mathcal{F}, (\mathcal{K}', u') \quad \text{ET}_2 \vdash (\mathcal{K}, u_2) \triangleright \mathcal{F}, (\mathcal{K}', u')\]

where \(u_1 = \{k \mapsto \{0\}\}, u' = u_2 = \{k \mapsto \{0, 1\}\}\) and \(\mathcal{F} = \{(\omega, k, 2)\}\). This leads to the final kv-store \(\mathcal{K}' = \mathcal{K}|k \mapsto \mathcal{K}(k):(2, t', \emptyset)\) for some \(t'\):

\[k \mapsto \begin{pmatrix} 0 & t_0 & 1 & t' & 0 \end{pmatrix}\]

It is easy to see \(\text{ET}_1 \cap \text{ET}_2 = \emptyset\) therefore \(\text{CM}(\text{ET}_1 \cap \text{ET}_2) = \emptyset\), yet \(\text{CM}(\text{ET}_1) \cap \text{CM}(\text{ET}_2) = \{\mathcal{K}'\}\). If execution tests enforce no blind write, \(\text{ET}_1\) and \(\text{ET}_2\) cannot produce the same kv-store, thus \(\text{CM}(\text{ET}_1) \cap \text{CM}(\text{ET}_2) = \text{CM}(\text{ET}_1 \cap \text{ET}_2) = \emptyset\), because the initial views are different.

**Definition B.3** (No blind writes). An execution test \(\text{ET}\) has no blind writes if, whenever \(\text{ET} \vdash (\mathcal{K}, u) \triangleright \mathcal{F} \cup \{(\omega, k, -)\} : (\mathcal{K}', u')\), then \((z, k, -) \in \mathcal{F}\).

**B.4.2. minimum footprint.** Let consider a kv-store \(\mathcal{K} = \{k_1 \mapsto (0, t_0, 0, (1, t_1, \emptyset)), k_2 \mapsto (2, t_0, 0, (3, t_2, \emptyset))\}:

\[k_1 \mapsto \begin{pmatrix} 0 & t_0 & 1 & t_1 & 0 \end{pmatrix} \quad k_2 \mapsto \begin{pmatrix} 2 & t_0 & 3 & t_2 & 0 \end{pmatrix}\]

and the following two execution test:

\[\text{ET}_1 \vdash (\mathcal{K}, u_1) \triangleright \mathcal{F}, (\mathcal{K}', u') \quad \text{ET}_2 \vdash (\mathcal{K}, u_2) \triangleright \mathcal{F}, (\mathcal{K}', u')\]

where \(u_1 = \{k_1 \mapsto \{0\}, k_2 \mapsto \{0\}\}, u_2 = \{k_1 \mapsto \{0\}, k_2 \mapsto \{0, 1\}\}\) and \(\mathcal{F} = \{(x, k_1, 0)\}\). This leads to the final kv-store \(\mathcal{K}' = \mathcal{K}|k \mapsto \mathcal{K}(k):(t', \emptyset)\) for some \(t'\):

\[k_1 \mapsto \begin{pmatrix} 0 & t_0 & 1 & t & 0 \end{pmatrix} \quad k_2 \mapsto \begin{pmatrix} 2 & t_0 & 3 & t_2 & 0 \end{pmatrix}\]

It is easy to see \(\text{ET}_1 \cap \text{ET}_2 = \emptyset\) therefore \(\text{CM}(\text{ET}_1 \cap \text{ET}_2) = \emptyset\), yet \(\text{CM}(\text{ET}_1) \cap \text{CM}(\text{ET}_2) = \{\mathcal{K}'\}\).

**Definition B.4** (Minimum footprints). An execution test \(\text{ET}\) has minimum footprints if for any key-value store \(\mathcal{K}\) views \(u, u', u''\) and fingerprint \(\mathcal{F}\),

\[\text{ET} \vdash (\mathcal{K}, u) \triangleright \mathcal{F} : (\mathcal{K}', u') \land \forall k. \ ((-, k, -) \in \mathcal{F} \Rightarrow u(k) = u'(k)) \Rightarrow \text{ET} \vdash (\mathcal{K}, u') \triangleright \mathcal{F} : (\mathcal{K}', u'')\]

If execution tests enforce minimum footprint, given that \(k_2\) does not appear in the fingerprint \(\mathcal{F}\), then we have \(\text{ET}_2 \vdash (\mathcal{K}, u_2) \triangleright \mathcal{F}, (\mathcal{K}', u') \Rightarrow \text{ET}_2 \vdash (\mathcal{K}, u_1) \triangleright \mathcal{F}, (\mathcal{K}', u')\). This means \(\text{CM}(\text{ET}_1) \cap \text{CM}(\text{ET}_2) = \text{CM}(\text{ET}_1 \cap \text{ET}_2) = \{\mathcal{K}'\}\).

**B.4.3. monotonic post-views.** Let consider a kv-store \(\mathcal{K} = \{k \mapsto (0, t_0, 0, (1, t, \emptyset))\}:

\[k \mapsto \begin{pmatrix} 0 & t_0 & 1 & t & 0 \end{pmatrix}\]

and the following two execution test:

\[\text{ET}_1 \vdash (\mathcal{K}, u) \triangleright \mathcal{F}, (\mathcal{K}', u_1) \quad \text{ET}_2 \vdash (\mathcal{K}, u) \triangleright \mathcal{F}, (\mathcal{K}', u_2)\]
where $u = u_1 = \{ k \mapsto \{0\} \}$, $u_2 = \{ k \mapsto \{0, 1\} \}$ and $\mathcal{F} = \{ (x, k, 0) \}$. This leads to the final kv-store $K' = \{ k \mapsto (0, t_0, \{t'\}):(1, t, \emptyset) \}$ for some $t'$:

$$k \mapsto \begin{array}{ccc}
0 & t_0 & 1 & t \\
\end{array}$$

It is easy to see $ET_1 \cap ET_2 = \emptyset$ therefore $CM(ET_1 \cap ET_2) = \emptyset$, yet $CM(ET_1) \cap CM(ET_2) = \{ K' \}$.

**Definition B.5.** An execution test $ET$ has **monotonic post-views** if for any key-value store $K$ views $u, u', u''$ and fingerprint $\mathcal{F}$,

$$ET \vdash (K, u) \triangleright \mathcal{F} : (K', u') \wedge u' \subseteq u'' \Rightarrow ET \vdash (K, u) \triangleright \mathcal{F} : (K', u'')$$

If execution tests enforce **monotonic post-views**, we have $ET_1 \vdash (K, u) \triangleright \mathcal{F}, (K', u_1) \Rightarrow ET_1 \vdash (K, u) \triangleright \mathcal{F}, (K', u_2)$ thus $CM(ET_1) \cap CM(ET_2) = CM(ET_1 \cap ET_2) = \{ K' \}$.

**B.5. Compositionality of $ET$**

To make two execution tests $ET_1, ET_2$ compositional with respect to to function $CM$, they need to satisfy Defs. B.2 to B.5. For all the definitions we have in Fig. 4, it is easy to adapt so that they satisfy Defs. B.3 to B.5 but CP and SI cannot be adapted so to satisfy Def. B.2. Now we can prove compositionality of $ET$ (Theorem B.1).

**Theorem B.1.** Let $ET_1, ET_2$ be two execution tests has no blind writes, minimum footprints and monotonic post-views. If $ET_1$ is commutative, then $CM(ET_1 \cap ET_2) = CM(ET_1) \cap CM(ET_2)$. Furthermore, if $ET_1, ET_2$ are commutative, then $ET_1 \cap ET_2$ is commutative.

**Proof.** Given the definition of the $CM(\cdot)$ function (Def. 4.3), it suffices to prove that $CM(ET_1 \cap ET_2) \subseteq CM(ET_1) \cap CM(ET_2)$ and $CM(ET_1) \cap CM(ET_2) \subseteq CM(ET_1 \cap ET_2)$. The former is proven by the Lemma B.4 and the later is proven by Lemma B.6.

**Lemma B.4.** $CM(ET_1 \cap ET_2) \subseteq CM(ET_1) \cap CM(ET_2)$.

**Proof.** It suffices to prove a stronger result that $CONF(ET_1 \cap ET_2) \subseteq CONF(ET_1) \cap CONF(ET_2)$. By the definition of $CONF(\cdot)$, it suffices to prove for configurations $\Gamma_0$ to $\Gamma_n$:

$$\Gamma_0 \in CONF \wedge \Gamma_0 \overset{ET_1 \cap ET_2}{\longrightarrow} \cdots \overset{ET_1 \cap ET_2}{\longrightarrow} \Gamma_n \Rightarrow \Gamma_0 \overset{ET_1}{\longrightarrow} \cdots \overset{ET_2}{\longrightarrow} \Gamma_n$$  (2.26)

We prove the Eq. (2.26) by induction on the number $n$.

- Base case: $n = 0$. The Eq. (2.26) holds when $n = 0$, because all initial configurations $\Gamma_0$ are included in the $CONF(ET_1)$ and $CONF(ET_2)$ by the definition of the $CONF$ function (Def. 4.3).
- Inductive case: $n = i + 1$. Suppose the Eq. (2.26) holds when $n = i$ for some $i$. Let consider $n = i + 1$ and specifically the last step. For any $\Gamma_{i+1} = (K_{i+1}, U_{i+1})$ induced by $ET_1 \cap ET_2$, there exist some client $cl$, views $u, u'$ and fingerprint $\mathcal{F}$ such that:

$$(K_i, U_i) \overset{cl, \mathcal{F}}{\longrightarrow} \overset{ET_1 \cap ET_2}{\longrightarrow} (K_{i+1}, U_{i+1}) \wedge U_{i+1} = U_i[cl \mapsto u'] \wedge (K_i, u, \mathcal{F}, u') \in ET_1 \cap ET_2$$

Thus, it is easy to see that $\Gamma_i \overset{cl, \mathcal{F}}{\longrightarrow} \overset{ET_1}{\longrightarrow} \overset{ET_2}{\longrightarrow} \Gamma_{i+1}$ by the Lemma B.5.

**Lemma B.5.** If $\Gamma \overset{cl, \mathcal{F}}{\longrightarrow} \overset{ET}{\longrightarrow} \Gamma'$ and $ET \subseteq ET'$, then $\Gamma \overset{cl, \mathcal{F}}{\longrightarrow} \overset{ET}{\longrightarrow} \Gamma'$.

**Proof.** Let $(K, U) = \Gamma$, $(K', U') = \Gamma'$ and $u = U(cl)$ By the definition of $\Gamma \overset{cl, \mathcal{F}}{\longrightarrow} \overset{ET}{\longrightarrow} \Gamma'$ (Def. 4.3), we have $K' \in update(K, u, \mathcal{F}, cl)$ and $U' = U[cl \mapsto u']$ for some $u'$ such that $ET \vdash (K, u) \triangleright \mathcal{F} : (K', u')$. Given that $ET \subseteq ET'$, we know $ET' \vdash (K, u) \triangleright \mathcal{F} : (K', u')$ and so $\Gamma \overset{cl, \mathcal{F}}{\longrightarrow} \overset{ET}{\longrightarrow} \Gamma'$.

**Lemma B.6.** $CM(ET_1) \cap CM(ET_2) \subseteq CM(ET_1 \cap ET_2)$.

**Proof.** By the definition of $CM$ and $CONF(\cdot)$, we prove a stronger result that for an initial configuration $\Gamma_0$, configurations $\Gamma_1$ to $\Gamma_n$ from trace $ET_1$, configurations $\Gamma'_1$ to $\Gamma'_m$ from trace $ET_2$,

$$\Gamma_0 \overset{ET_1}{\longrightarrow} \cdots \overset{ET_1}{\longrightarrow} \Gamma_n \wedge \Gamma_0 \overset{ET_2}{\longrightarrow} \cdots \overset{ET_2}{\longrightarrow} \Gamma'_m \wedge \Gamma_n \vdash \Gamma'_m$$
there exists configurations from $\Gamma_i^\prime$ to $\Gamma_k^\prime$ from trace $ET_1 \cap ET_2$:

$$
\Gamma_0 \xrightarrow{\phi_{ET_1 \cap ET_2}} \Gamma_1^\prime \xrightarrow{\phi_{ET_1 \cap ET_2}} \cdots \xrightarrow{\phi_{ET_1 \cap ET_2}} \Gamma_k^\prime \wedge \Gamma_n|_1 = \Gamma_m|_1 = \Gamma_k^\prime|_1 \wedge \forall cl \in \text{dom}(\Gamma_k|_2), k \in (\Gamma_k|_1). \Gamma_k^\prime|_2 (cl)(k) = \max \{ \Gamma_n|_2 (cl)(k), \Gamma_m|_2 (cl)(k) \} 
$$

(2.27)

We prove Eq. (2.27) by induction on the length $m$ of the trace of $ET_2$.

- Base case: $m = 0$. We have the trace of $ET_1$:

$$
\Gamma_0 \in \text{CONF}_0 \wedge \Gamma_0 \xrightarrow{\phi_{ET_1}} \cdots \xrightarrow{\phi_{ET_1}} \Gamma_n
$$

(2.28)

for some number $n$ and configurations from $\Gamma_0$ to $\Gamma_n$ and the trace of $ET_2$ with only one configuration:

$$
\Gamma_0
$$

(2.29)

By the hypothesis we have $\Gamma_0|_1 = \Gamma_n|_1$, which means that all the steps from the trace of $ET_1$ are view shift. We can pick the trace of $ET_1$ (Eq. (2.28)) as the trace of $ET_1 \cap ET_2$:

$$
\Gamma_0 \xrightarrow{\phi_{ET_1}} \Gamma_1 \xrightarrow{\phi_{ET_1}} \cdots \xrightarrow{\phi_{ET_1}} \Gamma_n
$$

(2.30)

It is easy to see:

$$
\forall cl \in \text{dom}(\Gamma_k|_2), k \in \text{dom}(\Gamma_k|_1). \Gamma_0|_2 (cl)(k) = \max \{ \Gamma_0|_2 (cl)(k), \Gamma_n|_2 (cl)(k) \}
$$

(2.31)

Combine Eq. (2.30) and Eq. (2.31), we prove the Eq. (2.27).

- Inductive case: $m = i + 1$. Suppose that Eq. (2.27) holds when $m = i$. Let consider $m = i + 1$. We have the trace for $ET_1$:

$$
\Gamma_0 \xrightarrow{\phi_{ET_1}} \Gamma_1 \xrightarrow{\phi_{ET_1}} \cdots \xrightarrow{\phi_{ET_1}} \Gamma_n
$$

(2.32)

for some number $n$ and the configurations from $\Gamma_0$ to $\Gamma_n$, and the trace of $ET_2$:

$$
\Gamma_0 \xrightarrow{\phi_{ET_2}} \Gamma_1^\prime \xrightarrow{\phi_{ET_2}} \cdots \xrightarrow{\phi_{ET_2}} \Gamma_k^\prime
$$

(2.33)

It is safe to assume these two traces are in normal form by Prop. [B.2]. Assume a client $cl^\prime_i$ views $u_i, u_{i+1}$ and a fingerprint $F_i^\prime$ that commit to the second last configuration $(\mathcal{K}_i^\prime, \mathcal{U}_i^\prime) = \Gamma_i^\prime$ in the trace of $ET_2$ which yields the final configuration $(\mathcal{K}_{i+1}^\prime, \mathcal{U}_{i+1}^\prime) = \Gamma_{i+1}^\prime$:

$$
(\mathcal{K}_i^\prime, \mathcal{U}_i^\prime) \xrightarrow{cl_i^\prime, F_i^\prime} (\mathcal{K}_{i+1}^\prime, \mathcal{U}_{i+1}^\prime) \wedge ET_2 \vdash (\mathcal{K}_i^\prime, u_i') \triangleright F_i^\prime : (\mathcal{K}_{i+1}^\prime, u_{i+1}')
$$

(2.34)

There are three cases: (i) $F_i^\prime = \emptyset$, (ii) $F_i^\prime = \epsilon$, and (iii) $F_i^\prime \neq \emptyset \wedge F_i^\prime \neq \epsilon$.

- If $F_i^\prime = \emptyset$ or $F_i^\prime = \epsilon$, by the Lemma [B.7] we know $\Gamma_i^\prime = \Gamma_{i+1}^\prime$ from the trace of $ET_2$. Since $\Gamma_n|_1 = \Gamma_n|_1$ where $\Gamma_n$ is the final configuration of the trace of $ET_1$, we now have $\Gamma_i^\prime|_1 = \Gamma_i^\prime|_1$. Applying I.H. that Eq. (2.27) holds when $m = i$, so there exist configurations from $\Gamma_i^\prime$ to $\Gamma_{i+1}^\prime$:

$$
\Gamma_0 \xrightarrow{\phi_{ET_1 \cap ET_2}} \Gamma_1^\prime \xrightarrow{\phi_{ET_1 \cap ET_2}} \cdots \xrightarrow{\phi_{ET_1 \cap ET_2}} \Gamma_{i+1}^\prime \wedge \forall cl \in \text{dom}(\Gamma_k|_2), k \in (\Gamma_k|_1). \Gamma_k^\prime|_2 (cl)(k) = \max \{ \Gamma_{n}|_2 (cl)(k), \Gamma_{m}|_2 (cl)(k) \}
$$

(2.35)

Given the definition of the reduction (Def. [4.2]), when $F = \emptyset$ or $F = \emptyset$ we know $\Gamma_i^\prime|_2 (cl_i) \subseteq \Gamma_{i+1}^\prime|_2 (cl_{i+1})$ thus:

$$
\max \{ \Gamma_{n}|_2 (cl_i), \Gamma_{m}|_2 (cl_i) \} \subseteq \max \{ \Gamma_{n}|_2 (cl_{i+1}), \Gamma_{m}|_2 (cl_{i+1}) \}
$$

(2.36)

Therefore Eq. (2.27) holds when $m = i + 1$ by appending a view shift to the end of the trace in Eq. (2.35):

$$
\Gamma_0 \xrightarrow{\phi_{ET_1 \cap ET_2}} \Gamma_1^\prime \xrightarrow{\phi_{ET_1 \cap ET_2}} \cdots \xrightarrow{\phi_{ET_1 \cap ET_2}} \Gamma_{i+1}^\prime \wedge \Gamma_k^\prime|_2 \left[ \phi \rightarrow \max \{ \Gamma_{n}|_2 (cl_{i+1}), \Gamma_{m}|_2 (cl_{i+1}) \} \right]
$$

(2.37)

- If $F_i^\prime \neq \emptyset \wedge F_i^\prime \neq \epsilon$, by Lemma [B.9] there exists a step $(cl_j, F_j)$ from the trace of $ET_1$ such that:

$$
(\mathcal{K}_j, \mathcal{U}_j) \xrightarrow{cl_j, F_j} (\mathcal{K}_{j+1}^\prime, \mathcal{U}_{j+1}^\prime) \wedge ET_1 \vdash (\mathcal{K}_j, u_j) \triangleright F_j : (\mathcal{K}_{j+1}^\prime, \mathcal{U}_{j+1}^\prime(\mathcal{U}_{j+1}^\prime |_1)) \wedge u_j = U_j(\mathcal{U}_{j+1}^\prime)
$$

(2.37)

for some $j, cl_j, u_j$ and $F_j$ such that $0 \leq j < n, cl_j = cl_i, F_j = F_i^\prime$, and

$$
\forall k. (-, k, -) \in F_j \Rightarrow u_j(k) = u_i^\prime(k)
$$
We apply the commutativity of $ET_1$ until the step shown in Eq. (2.37) is at the end or the second end of the trace of $ET_1$. Let consider the next two steps, $(j+1)$-th and $(j+2)$-th step. Since the trace is in normal form, the $(j+1)$-th step is a view shift by a client $cl_{j+2}$ and $(j+2)$-th step is a concrete step issued by the same client $cl_{j+2}$ under the view $u_{j+2}$:

\[
(K_{j+1}, U_{j+1}) \xrightarrow{cl_{j+2}, \epsilon} ET_1 \xrightarrow{K_{j+1}, [cl_{j+2} \mapsto u_{j+2}]} ET_1 \xrightarrow{cl_{j+2}, F_{j+2}} ET_1 \xrightarrow{K_{j+3}, U_{j+3}} (2.38)
\]

It is known that the client $cl_{j+2}$ is different from $cl_j$ (Lemma B.10) and $F_{j+2}$ writes different keys from $F_j$ (Lemma B.12). Because $cl_j \neq cl_{j+2}$ we can swap the view shift step shown in Eq. (2.38) before the $j$-th step shown in Eq. (2.37) which gives the following:

\[
(K_j, U_j) \xrightarrow{cl_{j+2}, \epsilon} ET_1 \xrightarrow{K_j, [cl_{j+2} \mapsto u_{j+2}]} ET_1 \xrightarrow{cl_{j+2}, F_{j+2}} ET_1 \xrightarrow{K_{j+3}, U_{j+3}} (2.39)
\]

Now let discuss the $(j+2)$-th step. Similarly by the Lemma B.9 there is a step $(cl_p, F_p)$ from the trace of $ET_2$ such that $cl_p = cl_{j+2}$ and $F_p = F_{j+2}$ and $p < i$. Note that the last step from $ET_2$, i.e. $(i+1)$-th step, is not a view shift therefore the i-th step must be a view shift so the p-th step must be before i-th step. This means that the fingerprint $F_p$ does not observe any change by $(i+1)$-th step from the trace of $ET_2$. Therefore $u_{j+2}$ does not observe any change by $j$-th step from the trace of $ET_1$, i.e. $u_{j+2} \in \text{VIEWS}(K_j)$. By Prop. B.3 that allows to swap the two adjacent non-conflict steps from Eq. (2.39), i.e. the last two steps. It follows a new kv-stores $K''_{j+2}$ and a new view environment $U''_{j+2}$ such that:

\[
(K_j, U_j) \xrightarrow{cl_{j+2}, \epsilon} ET_1 \xrightarrow{K_j, [cl_{j+2} \mapsto u_{j+2}]} ET_1 \xrightarrow{cl_{j+2}, F_{j+2}} ET_1 \xrightarrow{K_{j+3}, U_{j+3}} (2.40)
\]

In the Eq. (2.40) the j-th step moves to the right of $(j+2)$-th step. We monotonically move the j-th step until it is at the end or the second end of trace of $ET_1$:

\[
\Gamma_0 \xrightarrow{\epsilon} ET_1 \cdots \xrightarrow{\epsilon} ET_1, \Gamma_j \xrightarrow{\epsilon} ET_1, \Gamma_{m} \xrightarrow{\epsilon} ET_1 \cdots \xrightarrow{\epsilon} ET_1, \Gamma_{n-1} \xrightarrow{cl_j, F_j} ET_1, \Gamma_n
\]

Given the hypothesis that $\Gamma_{n|1} = \Gamma_{i|1}$ and the fact that the last step of the new trace of $ET_1$ (Eq. (2.41)) and the last step of the trace of $ET_2$ (Eq. (2.34)) are the same step, the kv-stores of the second last configurations the new trace of $ET_1$ (Eq. (2.41)) and the one from the trace of $ET_2$ (Eq. (2.34)) are the same $\Gamma''_{n-1|1} = \Gamma''_{i|1}$. Then by applying I.H. that Eq. (2.27) holds when $m = i$, there exists a trace of $ET_1 \cap ET_2$:

\[
\Gamma_0 \xrightarrow{\epsilon} ET_1 \cap ET_2 \cdots \xrightarrow{\epsilon} ET_1 \cap ET_2 \Gamma''_{k-1} \wedge \Gamma''_{n-1|1} = \Gamma''_{i|1} \wedge \forall cl \in \text{dom}(\Gamma''_{k-1|2}), k \in (\Gamma''_{k-1|1}). \Gamma''_{k-1|2}(cl)(k) = \max \{\Gamma_n|2 (cl)(k), \Gamma_i|2 (cl)(k)\}
\]

for some number $k$ and configurations from $\Gamma''_{i|1}$ to $\Gamma''_{k-1}$. By Eq. (2.34) and Eq. (2.41), we have:

\[
ET_1 \vdash (\Gamma_{i|1}, \Gamma_{i|2}(cl_i)) > F_1 : (\Gamma_{i+1|1}, \Gamma_{i+1|2}(cl_i)) \wedge ET_2 : (\Gamma_{n|1}, \Gamma_{n|2}(cl_i)) > F_2 : (\Gamma_{n+1|1}, \Gamma_{n+1|2}(cl_i)) (2.43)
\]

First, for any quadruple in $ET_1$ and $ET_2$, it does not constrain the view for keys that are not appear in the fingerprint before update. That is:

\[(\forall \mathcal{K}, \mathcal{K}', u, u', u'', \mathcal{F}, k. \ (-, -, -, \epsilon) \in \mathcal{F} \wedge u(k) = u'(k) \wedge (\mathcal{K}, u, \mathcal{F}, \mathcal{K}', u'') \in \mathcal{ET} \Rightarrow (\mathcal{K}', u', \mathcal{F}, \mathcal{K}', u'') \in \mathcal{ET})
\]

Given above and Eq. (2.42), we can substitute the configurations $\Gamma_i$ and $\Gamma''_{n-1}$ from Eq. (2.43) by $\Gamma''_{k-1}$. Then, because for any $\mathcal{K}, \mathcal{K}', u, u', u''$ and $\mathcal{F}$, if $(\mathcal{K}, u, \mathcal{F}, \mathcal{K}', u') \in ET_1$ and $u' \subseteq u''$ then $(\mathcal{K}, u, \mathcal{F}, \mathcal{K}', u'') \in ET_1$ and similarly for $ET_2$. It means:

\[
ET_1 \cap ET_2 \vdash (\Gamma''_{k-1|1}, \Gamma''_{k-1|2}(cl_i)) > F_1 : (\Gamma_n|1, \max \{\Gamma_{i+1|2}(cl_i), \Gamma_{n|2}(cl_i)\}) (2.44)
\]
Therefore the Eq. \((2.27)\) holds when \(m = i + 1\) by appending the shown in Eq. \((2.44)\) to the end of the trace shown in Eq. \((2.42)\):

\[
\begin{align*}
\Gamma_0 &\rightarrow_{ET_1, ET_2} \cdots \rightarrow_{ET_1, ET_2} \Gamma_{n-1} \rightarrow_{cl_i} \rightarrow_{ET_1, ET_2} (\Gamma_n |_1, \Gamma_{n-1} |_2 [cl_i \mapsto \max \{ \Gamma_{i+1} |_2 (cl_i), \Gamma_n |_2 (cl_i) \}]) \\
\end{align*}
\]

* If the \(j\)-th step is the second last step of the new trace of \(ET_1\), we have the trace:

\[
\begin{align*}
\Gamma_0 &\rightarrow_{ET_1, ET_2} \cdots \rightarrow_{ET_1} \Gamma_{j-1} \rightarrow_{ET_1} \Gamma_{j} \rightarrow_{cl_j} \rightarrow_{ET_1} \Gamma_{j+1} \rightarrow_{cl_j} \rightarrow_{ET_1} \cdots \\
&\rightarrow_{ET_1} \Gamma_{n-2} \rightarrow_{cl_j} \rightarrow_{ET_1} \Gamma_{n-1} \rightarrow_{cl_j} \rightarrow_{ET_1} \cdots \\
\end{align*}
\]

(2.45)

Since the last step is a view shift, we know \(\Gamma_n |_1 = \Gamma''_{n-1} |_1\), and the rest of proof is the same as the case where \(j\)-th is the last step as shown in Eq. \((2.41)\).

\[\square\]

**Lemma B.7** (No effect from empty fingerprint and epsilon reduction).

\[
\forall \Gamma, \Gamma', cl, u. \quad \Gamma \rightarrow_{cl, \emptyset}^{ET} \Gamma' \lor \Gamma \rightarrow_{cl, \epsilon}^{ET} \Gamma' \Rightarrow \Gamma |_1 = \Gamma' |_1
\]

**Proof.** Let \((\mathcal{K}, \mathcal{U}) = \Gamma\) and \((\mathcal{K}', \mathcal{U}') = \Gamma'\). For the case of empty fingerprint, by the definition of \(\Gamma \rightarrow_{cl, \emptyset}^{ET} \Gamma'\) (Def. 4.2), we have \(\mathcal{K}' \in \text{update}(\mathcal{K}, u, \emptyset, cl)\), and therefore \(\mathcal{K}' = \mathcal{K}\). For the case of view shift, by the definition of \(\Gamma \rightarrow_{cl, \epsilon}^{ET} \Gamma'\) (Def. 4.2) it is easy to see \(\mathcal{K}' = \mathcal{K}\).

\[\square\]

**Lemma B.8** (Transactions persistence).

\[
\forall ET, \Gamma, \Gamma', t, F. \quad \Gamma |_1 (t) = F \land \Gamma \rightarrow_{cl}^{ET} \Gamma' \Rightarrow \Gamma' |_1 (t) = F
\]

**Proof.** It is easy to prove this by case analysis on the reduction relation.

\[\square\]

**Lemma B.9** (Same steps). Given a trace of \(ET_1\) and a trace of \(ET_2\), if the have the same final kv-store, the trace contains the same concrete steps (free variables are globally quantified):

\[
\begin{align*}
\Gamma_0 &\rightarrow_{cl_1, F_1}^{ET_1} \cdots \rightarrow_{cl_n, F_n}^{ET_1} \Gamma_n \land \Gamma_0 \rightarrow_{cl'_1, F'_1}^{ET_2} \cdots \rightarrow_{cl'_m, F'_m}^{ET_2} \Gamma'_m \land \Gamma_n |_1 = \Gamma'_m |_1 \\
&\Rightarrow \forall i : 0 < i \leq n. \quad F_i = \emptyset \lor F_i = \epsilon \lor \exists j : 0 < j \leq m. \quad cl_i = cl'_j \land F_i = F'_j \land (\forall k. \ (-k, -) \in F_i \Rightarrow u_i(k) = u'_j(k))
\end{align*}
\]

**Proof.** We prove by contradiction. First because \(\mathcal{K}_n = \mathcal{K}'_m\), we know that:

\[
\forall t, F. \quad \mathcal{K}_n(t) = F \iff \mathcal{K}'_m(t) = F
\]

(2.46)

Let \(\Gamma_n = (\mathcal{K}_n, \mathcal{U}_n)\) and \(\Gamma'_m = (\mathcal{K}'_m, \mathcal{U}'_m)\). Assume a step \(\Gamma_i \rightarrow_{cl_i}^{ET} \Gamma_{i+1}\) from the trace of \(ET_1\), where the transaction identifier is \(t\) and \(F \neq \emptyset\). It must have a step from the trace of \(ET_2\), which commits some fingerprint via the same transaction identifier \(t\). We know \(\mathcal{K}_n(t) = F\) by Lemma B.8, thus \(\mathcal{K}'_m(t) = F\) by Eq. (2.46). Let assume a key \(k\) that \((-k, -) \in F \land u_i(k) \neq u'_j(k)\) where \(u_i\) and \(u'_j\) are the views immediate before the commit of the fingerprint \(F\) in traces of \(ET_1\) and \(ET_2\) respectively. Since the no blind write assumption, it is safe to assume it is a read operation on the key \(k\). By the definition of the reduction (Def. 4.2 and Lemma B.8) we know read \((\mathcal{K}_n(k)(u_i(k))) \neq \text{read}(\mathcal{K}'_m(k)(u_i(k)))\), which contradicts with \(\mathcal{K}_n = \mathcal{K}'_m\).

\[\square\]

We define \(\text{max}_{cl}(\Gamma)\) that returns the most recent transaction identifier for client \(cl\) in the configuration \(\Gamma\)

\[
\text{max}_{cl}(\mathcal{K}, \mathcal{U}) \triangleq \max \{ t'_{i} | t'_{i} \text{ appear in } \mathcal{K} \}
\]

**Lemma B.10** (Transactions from different clients). Given a trace of \(ET_1\) and a trace of \(ET_2\), if the \(i\)-th step from \(ET_1\) issued by the client \(cl_i\) is the same as the last step from \(ET_2\), then in the trace of \(ET_1\) there is no concrete step issued by the client \(cl_i\) after the \(i\)-th step (free variables are globally quantified):

\[
\begin{align*}
\Gamma_0 &\rightarrow_{cl_1, F_1}^{ET_1} \cdots \rightarrow_{cl_n, F_n}^{ET_1} \Gamma_n \land \Gamma_0 \rightarrow_{cl'_1, F'_1}^{ET_2} \cdots \rightarrow_{cl'_m, F'_m}^{ET_2} \Gamma'_m \land \Gamma_n |_1 = \Gamma'_m |_1 \land F'_m \neq \emptyset \\
&\land \exists i. \quad cl_i = cl'_m \land F_i = F'_m \Rightarrow \forall j > i. \quad F_j = \epsilon \lor F_j = \emptyset \lor cl_i \neq cl_j
\end{align*}
\]
Proof. We prove by deriving contradiction. Assume the last step of the trace of $ET_2$ is:

$$
\Gamma_m' \xrightarrow{cl_m', \mathcal{F}_m'} \Gamma'_m
$$

(2.47)

Assume a step of the trace of $ET_1$:

$$
\Gamma_i \xrightarrow{cl_i, \mathcal{F}_i} \Gamma_i'
$$

(2.48)

where $cl_i = cl_m'$ and $\mathcal{F}_i = \mathcal{F}_m'$. Because these two steps (Eq. (2.47) and Eq. (2.48)) are issued by the same transaction identifier, we know $\max_{cl_m}(\Gamma_m') = \max_{cl_m}(\Gamma_i)$. Assume that there exists a step from the trace of $ET_1$, says $j$-th step, such that:

$$
\Gamma_{j-1} \xrightarrow{cl_j, \mathcal{F}_j} \Gamma_j \wedge j > i \wedge \mathcal{F}_j \neq \emptyset \wedge cl_i = cl_j
$$

Therefore we have $\max_{cl_m}(\Gamma_j) > \max_{cl_m}(\Gamma_i)$ by Lemma B.11. That means $\max_{cl_m}(\Gamma_i) > \max_{cl_m}(\Gamma_j) > \max_{cl_m}(\Gamma_i)$, which contradicts to $\Gamma_n |_{1} = \Gamma_m' |_{1}$.

Lemma B.11 (Reduction following session order).

$$
\forall \Gamma, \Gamma', cl, \mathcal{F}, ET. \Gamma \xrightarrow{cl, \mathcal{F}} \Gamma' \wedge (\mathcal{F} \neq \emptyset \Rightarrow \max_{cl}(\Gamma) < \max_{cl}(\Gamma')) \vee (\mathcal{F} = \emptyset \Rightarrow \max_{cl}(\Gamma) = \max_{cl}(\Gamma'))
$$

Proof. Assume a step $(\mathcal{K}, \mathcal{U}) \xrightarrow{cl, \mathcal{F}} (\mathcal{K}', \mathcal{U}')$. By the definition of $\xrightarrow{\mathcal{F}}$ (Def. 4.2), we know $\mathcal{K}' \in \text{update}(\mathcal{K}, u, \mathcal{F}, cl)$. The update $(\mathcal{K}, u, \mathcal{F}, cl)$ picks a fresh transaction identifier $t'_{cl} |_{m}$ that is greater than any transaction identifiers $t_{cl} |_{m}$ in $\mathcal{K}'$ via nextTxId function, i.e. $m > n$. If the fingerprint $\mathcal{F}$ is not empty, the new identifier appears in $\mathcal{K}'$, so $\max_{cl}(\Gamma) < \max_{cl}(\Gamma')$. Otherwise the fingerprint is empty, the new identifier will not appear anywhere in $\mathcal{K}'$, so $\max_{cl}(\Gamma) = \max_{cl}(\Gamma')$.

Lemma B.12 (Writing different keys). Given a trace of $ET_1$ and a trace of $ET_2$, if the $i$-th step from $ET_1$ that writes to key $k$ is the same as the last step from $ET_2$, then in the trace of $ET_1$ there is no concrete step writing to the key $k$ after the $i$-th step (free variables are globally quantified):

$$
\Gamma_0 \xrightarrow{cl_0, \mathcal{F}_0} \Gamma_n \wedge \Gamma_0 \xrightarrow{cl_1, \mathcal{F}_1} \Gamma_n \wedge \Gamma_n \wedge \Gamma_0 \xrightarrow{cl_1, \mathcal{F}_1} \Gamma_n \wedge \Gamma_n \wedge \Gamma_n \wedge \Gamma_n \wedge \Gamma_n |_{1} = \Gamma_m' |_{1} \wedge \mathcal{F}_m' \neq \emptyset
$$

(2.49)

Proof. We prove this by deriving contradiction. Assume the last step from the trace of $ET_2$:

$$
\Gamma_{m-1} \xrightarrow{cl_m', \mathcal{F}_m'} (\mathcal{K}_m', \mathcal{U}_m')
$$

Assume the transaction identifier for the Eq. (2.49) is $t$, and by the definition of $\xrightarrow{\mathcal{F}}$ (Def. 4.2) we know:

$$
\forall k. (w, k, -) \in \mathcal{F}_m' \Rightarrow \mathcal{K}_n(k) \left( |\mathcal{K}_m(k) - 1 \right) = (-, t, -)
$$

(2.50)

Assume a step of the trace of $ET_1$ that is issued by the same transaction identifier with the same fingerprint:

$$
\Gamma_{i-1} \xrightarrow{cl_i, \mathcal{F}_i} (\mathcal{K}_i, \mathcal{U}_i)
$$

(2.51)

where $cl_i = cl_m'$ and $\mathcal{F}_i = \mathcal{F}_m'$. Given Eq. (2.50) and Eq. (2.51), it follows:

$$
\forall k. (w, k, -) \in \mathcal{F}_i \Rightarrow \mathcal{K}_n(k) \left( |\mathcal{K}_n(k) - 1 \right) = (-, t, -)
$$

Assume a step, says $j$-th, after $i$-th step that writes to the same key:

$$
\Gamma_{j-1} \xrightarrow{cl_j, \mathcal{F}_j} (\mathcal{K}_j, \mathcal{U}_j) \wedge j > i \wedge \forall k. (w, k, -) \in \mathcal{F}_i \cap \mathcal{F}_j
$$

Therefore, by Lemma B.13 we have:

$$
\exists k, i. (w, k, -) \in \mathcal{F}_i \cap \mathcal{F}_j \wedge \mathcal{K}_n(k)(i) = t \wedge \mathcal{K}_n(k)(|\mathcal{K}_n(k) - 1 \neq t
$$

Note that $\mathcal{F}_i = \mathcal{F}_m'$. Since the writer of a version cannot be overwritten, for the final configuration of the trace of $ET_1$ $(\mathcal{K}_n, \mathcal{U}_n)$, we know:

$$
\exists k, i. (w, k, -) \in \mathcal{F}_i \cap \mathcal{F}_j \wedge \mathcal{K}_n(k)(i) = t \wedge \mathcal{K}_n(k)(|\mathcal{K}_n(k) - 1 \neq t
$$

Last, by Eq. (2.50) and $\mathcal{F}_i = \mathcal{F}_m'$, it follows:

$$
\exists k. (w, k, -) \in \mathcal{F}_m' \wedge \mathcal{K}_n(k)(\mathcal{K}_n(k) - 1 \neq t \wedge \mathcal{K}_m'(k)(|\mathcal{K}_m'(k) - 1 |)_{2} = t
$$

which contradicts with $\mathcal{K}_m' = \mathcal{K}_n$. 

\end{document}
Lemma B.13 (Version persistence).

\[ \forall \mathcal{K}, \mathcal{K}', \mathcal{U}, \mathcal{U}', cl, u, \mathcal{F}, i. (\mathcal{K}, \mathcal{U}) \xrightarrow{cl, \mathcal{F}} \mathcal{F} \Rightarrow (\mathcal{K}', \mathcal{U}) \wedge (u, k, -) \in \mathcal{F} \Rightarrow 0 \leq i < |\mathcal{K}'(k)| - 1 \]

Proof. By the definition of \( \xrightarrow{-} \) (Def. 4.2), the \( \mathcal{K}' \) is update \( (\mathcal{K}, u, \mathcal{F}, cl) \). Given the definition of update \( (\mathcal{K}, u, \mathcal{F}, cl) \), it picks a fresh transaction identifier \( t \) such that does not appear in \( \mathcal{K} \). For any write fingerprint \( (u, k, -) \in \mathcal{F} \), a new version is appended to the end of the key \( k \) and the writer (the second projection) is assigned to be the fresh identifier \( t \). Thus we have the proof.

Proposition B.4. if \( ET_1, ET_2 \) are commutative, then \( ET_1 \cap ET_2 \) is commutative.

Proof. Let \( ET_{12} = ET_1 \cap ET_2 \). Assume \( \Gamma_1, \Gamma_2, \Gamma_3, cl, cl', u, u', \mathcal{F}, \mathcal{F}' \) such that:

\[ \Gamma_1 \xrightarrow{cl, \mathcal{F}} \Gamma_2 \xrightarrow{cl', \mathcal{F}'} \Gamma_3 \]

Therefore, we have:

\[ \Gamma_1 \xrightarrow{cl, \mathcal{F}} \Gamma_2 \xrightarrow{cl', \mathcal{F}'} \Gamma_3 \]

Because \( ET_1 \) and \( ET_2 \) are commutative, there exists a configuration \( \Gamma'_2 \) such that:

\[ \Gamma_1 \xrightarrow{cl, \mathcal{F}} \Gamma_2' \xrightarrow{cl', \mathcal{F}'} \Gamma_3 \]

so we have the proof that:

\[ \Gamma_1 \xrightarrow{cl, \mathcal{F}} \Gamma_2' \xrightarrow{cl', \mathcal{F}'} \Gamma_3 \]

\[ \square \]

B.6. Example: CP and SI is not commutative

We should the following counter example why CP and SI are not commutative. Let consider an initial kv-store \( \mathcal{K} \) (some writers are omitted):

\[
\begin{array}{ccc}
| & 0 & 1 & t \\
| k_1 | & 0 & 1 & t \\
| k_2 | & 0 & 0 & t \\
\end{array}
\]

Assume two clients, \( cl_1 \) and \( cl_2 \), want to read the two keys \( k_1 \) and \( k_2 \). Let assume the first client \( cl_1 \) initially has view \( u_1 = \{ k_1 \mapsto 0, k_2 \mapsto \{ \} \} \) With the view, we have the fingerprint for the client \( \mathcal{F}_1 = \{ (x, k_1, 0), (x, k_2, 2) \} \), which leads to the final kv-store \( \mathcal{K}_1' \):

\[
\begin{array}{ccc}
| & 0 & 1 & t \\
| k_1 | & 0 & 0 & t \\
| k_2 | & 0 & 0 & t \\
\end{array}
\]

Assume the view remains the same afterwards. It is easy to see \( ET_{CP} \vdash (\mathcal{K}, u_1) \triangleright \mathcal{F}_1 : (\mathcal{K}_1, u_1) \), and also for \( ET_{SI} \).

Now for the second client \( cl_2 \), assume the view \( u_2 = \{ k_1 \mapsto \{ 0 \}, k_2 \mapsto \{ 0 \} \} \), which leads to the fingerprint \( \mathcal{F}_2 = \{ (x, k_1, 0)(x, k_2, 2) \} \), and the kv-store \( \mathcal{K}_2' \):

\[
\begin{array}{ccc}
| & 0 & 1 & t \\
| k_1 | & 0 & 0 & t \\
| k_2 | & 0 & 0 & t \\
\end{array}
\]

It is trivial that \( ET_{CP} \vdash (\mathcal{K}_1, u_2) \triangleright \mathcal{F}_2 : (\mathcal{K}_2, u_2) \), and also for \( ET_{SI} \).

Commulative allows to swap two fingerprints from different clients and still yields the same kv-store. It is not the case here. If we let \( \mathcal{F}_2 \) commits first we have the following kv-store \( \mathcal{K}_2'' \):

\[
\begin{array}{ccc}
| & 0 & 1 & t \\
| k_1 | & 0 & 0 & t \\
| k_2 | & 0 & 0 & t \\
\end{array}
\]

Over the \( \mathcal{K}_2' \), the view \( u_1 \) is no longer valid because in \( ET_{CP} \):

\[
0 \in u_1(k_2) \wedge w(K(k_1, 1)) \xrightarrow{((SO \cup WR) \cup RW) \cup WW} w(K(k_2, 0))
\]

but \( i \notin u_1(k_2) \). It is similar for \( ET_{SI} \).
B.7. \textit{SI is not intersection of CP and UA on dependency graph}

We refer reader to §C and D for the full details of dependency graphs and abstract execution. Here we show why when on abstract executions, the definition of SI is intersection of CP and UA, but it is not the case on the definition on dependency graphs. Let consider the following dependency graph which is allowed by both CP and UA:

\begin{center}
\begin{tikzpicture}
  
  \node (t0) at (0,0) [circle, draw] {t_0 \((w, k_1, 1)\) \(\(w, k_2, 1\)\)};
  \node (t1) at (2,0) [circle, draw] {t_1 \((w, k_1, 2)\) \(\(x, k_4, 0\)\)};
  \node (t2) at (2,2) [circle, draw] {t_2 \((w, k_3, 4)\) \(\(x, k_2, 0\)\)};
  \node (t3) at (0,2) [circle, draw] {t_3 \((w, k_3, 3)\) \(\(w, k_4, 1\)\)};

  \draw [->] (t0) edge node [swap] {WW} (t2);
  \draw [->] (t1) edge node {WW} (t3);
  \draw [->] (t2) edge node [swap] {RW} (t1);
  \draw [->] (t3) edge node {RW} (t0);

\end{tikzpicture}
\end{center}

Now we convert the graph to abstract executions. First we can only choose one of the following arbitration relations:

\begin{center}
\begin{align*}
& t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \quad \text{or} \quad t_3 \rightarrow t_2 \rightarrow t_0 \rightarrow t_1.
\end{align*}
\end{center}

Such two relations guarantee the abstract executions converted from the dependency graph satisfy SI. Then UA requires \(WW \in VIS\), which means we might have the following abstract execution (the another one is symmetric):

\begin{center}
\begin{tikzpicture}
  
  \node (t0) at (0,0) [circle, draw] {t_0 \((w, k_1, 1)\) \(\(w, k_2, 1\)\)};
  \node (t1) at (2,0) [circle, draw] {t_1 \((w, k_1, 2)\) \(\(x, k_4, 0\)\)};
  \node (t2) at (2,2) [circle, draw] {t_2 \((w, k_3, 4)\) \(\(x, k_2, 0\)\)};
  \node (t3) at (0,2) [circle, draw] {t_3 \((w, k_3, 3)\) \(\(w, k_4, 1\)\)};

  \draw [->] (t0) edge node [swap] {AR, VIS} (t2);
  \draw [->] (t1) edge node {AR, VIS} (t3);
  \draw [->] (t2) edge node [swap] {AR} (t1);
  \draw [->] (t3) edge node {AR} (t0);

\end{tikzpicture}
\end{center}

By the constrain \(AR; VIS \subseteq VIS\), which should be satisfies by both CP and especially SI on abstract execution, we have the following abstract execution:

\begin{center}
\begin{tikzpicture}
  
  \node (t0) at (0,0) [circle, draw] {t_0 \((w, k_1, 1)\) \(\(w, k_2, 1\)\)};
  \node (t1) at (2,0) [circle, draw] {t_1 \((w, k_1, 2)\) \(\(x, k_4, 0\)\)};
  \node (t2) at (2,2) [circle, draw] {t_2 \((w, k_3, 4)\) \(\(x, k_2, 0\)\)};
  \node (t3) at (0,2) [circle, draw] {t_3 \((w, k_3, 3)\) \(\(w, k_4, 1\)\)};

  \draw [->] (t0) edge node [swap] {VIS} (t2);
  \draw [->] (t1) edge node {VIS} (t3);
  \draw [->] (t2) edge node [swap] {AR} (t1);
  \draw [->] (t3) edge node {AR} (t0);

\end{tikzpicture}
\end{center}

This leads to contradiction: \(t_2\) should see \(t_0\) but \(t_2\) did not read the value 1 for \(k_2\) written by \(t_0\).

\textbf{Appendix C.}

\textbf{Relations to Dependency Graphs}

\textit{Dependency graphs} are introduced by Adya to define consistency models of transactional databases \[3\]. They are directed graphs consisting of transactions as nodes, each of which is labelled with transaction identifier and a set of read and write operations, and labelled edges between transactions for describing how information flows between nodes. Specifically, a
Definition C.1. A dependency graph is a quadruple $G = (\mathcal{F}, \text{WR}, \text{WW}, \text{RW})$, where

- $\mathcal{F}_0 : \text{TRANSID} \to \mathcal{P}(\text{Ops})$ is a partial finite mapping from transaction identifiers to the set of operations, where there are at most one read operation and one write operation per key;
- $\text{WR} : \text{KEY} \to \mathcal{P}(\text{dom}(\mathcal{F}) \times \text{dom}(\mathcal{F}))$ is a function that maps each key $k$ into a relation between transactions, such that for any $t, t_1, t_2, k, m, n$:
  - if $(x, k, v) \in \mathcal{T}(t)$, either $v = v_0$ and there exists no $t'$ such that $t' \xrightarrow{\text{WR}(k)} t$, or there exists $t'$ such that $(w, k, v) \in \mathcal{T}(t')$, and $t' \xrightarrow{\text{WR}(k)} t$,
  - if $t_1 \xrightarrow{\text{WR}(k)} t$ and $t_2 \xrightarrow{\text{WR}(k)} t$, then $t_2 = t_1$.
- $\text{WW} : \text{KEY} \to \mathcal{P}(\text{dom}(\mathcal{F}) \times \text{dom}(\mathcal{F}))$ is a function that maps each key into an irreflexive relation between transactions, such that for any $t, t', k, m, n$:
  - if $t \xrightarrow{\text{WW}(k)} t'$, then $(w, k, v) \in \mathcal{T}(t)$, $(w, k, v) \in \mathcal{T}(t')$,
  - if $(w, k, v) \in \mathcal{T}(t)$, $(w, k, v) \in \mathcal{T}(t')$, then either $t = t'$, $t \xrightarrow{\text{WW}(k)} t'$, or $t' \xrightarrow{\text{WW}(k)} t$.
- $\text{RW} : \text{KEY} \to \mathcal{P}(\text{dom}(\mathcal{F}) \times \text{dom}(\mathcal{F}))$ is defined by letting $t \xrightarrow{\text{RW}(k)} t'$ if and only if $(x, k, v) \in \mathcal{T}(t)$, $(w, k, v) \in \mathcal{T}(t')$ and either there exists no $t''$ such that $t'' \xrightarrow{\text{RW}(k)} t$, or $t'' \xrightarrow{\text{RW}(k)} t$, $t'' \xrightarrow{\text{RW}(k)} t'$ for some $t''$.

Let $\mathcal{D}_{\mathcal{G}}$ be the set of all dependency graphs.

Given a dependency graph $G = (\mathcal{F}, \text{WR}, \text{WW}, \text{RW})$, we often commit an abuse of notation and use WR to denote the relation $\bigcup_{k \in \text{KEY}} \text{WR}(k)$; a similar notation is adopted for WW, RW. It will always be clear from the context whether the symbol WR refers to a function from keys to relations, or to a relation between transactions.

Definition C.2. Given a kv-store $K$, the dependency graph $G_K = (\mathcal{F}_K, \text{WR}_K, \text{WW}_K, \text{RW}_K)$ is defined as follows:

- for any $t \neq t_0$, $\mathcal{F}_K(t)$ is defined if and only if there exists an index $i$ and a key $k$ such that either $t = w(K(k, i))$, or $t \in \text{rs}(K(k, i))$; furthermore, $(w, k, v) \in \mathcal{T}(t)$ if and only if $t = w(K(k, i))$ for some $i$, and $(x, k, v) \in \mathcal{T}(t)$ if and only if $t \in \text{rs}(K(k, i))$ for some $i$.
- $t \xrightarrow{\text{WR}_K} t'$ if and only if there exists an index $i : 0 < i < |K(k)|$ such that $t = w(K(k, i))$, and $t' \in \text{rs}(K(k, i))$.
- $t \xrightarrow{\text{WW}_K} t'$ if and only if there exist two indexes $i, j : 0 < i < j < |K(k)|$ such that $t = w(K(k, i))$, $t' = w(K(k, j))$.
- $t \xrightarrow{\text{RW}_K} t'$ if and only if there exist two indexes $i, j : 0 < i < j < |K(k)|$ such that $t \in \text{rs}(K(k, i))$ and $t' = w(k, j)$.

Definition C.3. Given a dependency graph $G = (\mathcal{F}, \text{WR}, \text{WW}, \text{RW})$, we define the kv-store $K_G$ as follows:

1. For any transaction $t \in \text{dom}(\mathcal{F})$ such that $(w, k, v) \in \mathcal{T}(t)$, let $T_k = \{t' | t' \xrightarrow{\text{WR}_K} t\}$, and let $\nu(t, k) = (v, t, T)$.
2. For each key $k$, let $\nu_k^0 = (v_0, t_0, \mathcal{T}_k^0)$, where $\mathcal{T}_k^0 = \{t | (x, k, v) \in \mathcal{T}(t) \land \forall t', \neg(t' \xrightarrow{\text{WR}_K} t)\}$. Let also $\nu_k^0 = \nu(t, k)$ for some $t$ such that $(w, k, v) \in \mathcal{T}(t)$, and such that for any $i, j : 1 < i < j \leq n$, $\nu_k^0 \xrightarrow{\text{WW}_K} \nu_k^0$.

We then let $K_G = \lambda k : \prod_{i=0}^n \nu_k^0$.

Theorem C.1. There is a one-to-one map between kv-stores and dependency graphs.

Proof. We prove that given any a well-formed kv-store $K$, then $G_K$ is a well-formed dependency graph in Prop. C.1 and given any $G$, then $K_G$ is a well-formed kv-store in Prop. C.2. Then we prove the bijection that $K_{G_K} = K$ in Prop. C.3. \qed

Proposition C.1. Let $K$ be a well-formed kv-store. Then $G_K$ is a well-formed dependency graph.

Proof. Let $K$ be a (well-formed) kv-store. We need to show that $G_K = (\mathcal{F}_K, \text{WR}_K, \text{WW}_K, \text{RW}_K)$ is a dependency graph. As a first step, we show that $G_K$ is a dependency graph, i.e. it satisfies all the constraints placed by Def. C.1.

- Let $t \in \text{dom}(\mathcal{F}_K)$, and suppose that $(x, k, v) \in \mathcal{F}_K(t)$. We need to prove that either $v = v_0$, and there exists no $t' \in \text{dom}(\mathcal{F}_K)$ such that $t' \xrightarrow{\text{WR}_K} t$, or $t' \xrightarrow{\text{WR}_K} t$ for some $t' \in \text{dom}(\mathcal{F}_K)$ such that $(w, k, v) \in \mathcal{F}_K(t')$. Because $t \in \text{dom}(\mathcal{F}_K)$, the definition of $G_K$ (and in particular the fact that $\mathcal{F}_K : \text{TRANSID}_0 \to \mathcal{P}(\text{Ops})$) ensures that $t \neq t_0$, transaction $t$ reads a version for a key $k$ that has been written by another transaction $t'$ (write-read dependency WR), overwrites a version of $k$ written by $t'$ (write-write dependency WW), or reads a version of $k$ that is later overwritten by $t'$ (read-write anti-dependency RW). Note that here we purposely use the same names WR, WW, RW as those used in kv-store, since there is one-to-one map between kv-stores and dependency graphs.
We prove that each of the four constraints required by well-formed kv-stores are satisfied by $K_G$, where $G = (\mathcal{T}, \mathcal{W}, \mathcal{W}_R, \mathcal{R}_W)$, $\mathcal{K}_G$ is a well-formed kv-store.

(i) For each key $k$, $\mathcal{K}_G(k, 0) = (v_0, t_0, t_0)$. By construction, we have that $\mathcal{K}_G(k, 0) = \nu_k^0 = (v_0, t_0, t_0)$.

(ii) For all $k \in \text{Key}$. For all $i, j \leq |\mathcal{K}_G(k)|$, $\mathcal{K}_G(k, i) = \nu_k^i = \nu(t, k)$, we have that $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, j)$ for all $i, j \leq |\mathcal{K}_G(k)|$. Without loss of generality, we can assume that $i < j$. First, note that if $i = 0$, then $\mathcal{K}_G(k, i) = t_0$, hence it must be the case that $\mathcal{K}_G(k, j) = t_0$. By construction, it is also the case that $\mathcal{K}_G(k, j) = \nu_k^j$, hence either one of the following is true:

a) If $j = 0$, in which case there is nothing to prove, or
b) If $j > 0$, and $\mathcal{K}_G(k, j) = \nu_k^j$, then for some $t \in \text{dom}(\mathcal{T})$. We have that $\mathcal{K}_G(k, j) = \mathcal{K}_G(k, j) = \nu(t, k)$, and because $t \in \text{dom}(\mathcal{T})$, it must be $t \neq t_0$. Contradiction.

Assume that $i > 0$. Therefore, it must be the case that $\mathcal{K}_G(k, i) = \nu_k^i = \nu(t, k)$ for some $t \in \text{dom}(\mathcal{T})$ such that $(w, k, v) \in \mathcal{T}_G(t_i)$. Finally, note that if it were $i < j$, then by construction we should have that $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, j)$, contradicting the requirement that $\mathcal{W}_R(k)$ is irreflexive. Therefore, it must be $i = j$.

(iii) For all $k \in \text{Key}$. For all $i, j \leq |\mathcal{K}_G(k)|$, $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, j)$ for all $i, j \leq |\mathcal{K}_G(k)|$. Without loss of generality, suppose that $i < j$. We distinguish between two cases:

a) If $i = 0$, then there is nothing to prove, or
b) If $i > 0$, and $\mathcal{K}_G(k, i) = \nu_k^i$, then for some $t \in \text{dom}(\mathcal{T})$. We have that $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, i) = \nu(t, k)$, and because $t \in \text{dom}(\mathcal{T})$, it must be $t \neq t_0$. Contradiction.

Proof. We prove that each of the four constraints required by well-formed kv-stores are satisfied by $\mathcal{K}_G$.

(i) For each key $k$, $\mathcal{K}_G(k, 0) = (v_0, t_0, t_0)$. By construction, we have that $\mathcal{K}_G(k, 0) = \nu_k^0 = (v_0, t_0, t_0)$.

(ii) For all $k \in \text{Key}$. For all $i, j \leq |\mathcal{K}_G(k)|$, $\mathcal{K}_G(k, i) = \nu_k^i = \nu(t, k)$, we have that $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, j)$ for all $i, j \leq |\mathcal{K}_G(k)|$. Without loss of generality, we can assume that $i < j$. First, note that if $i = 0$, then $\mathcal{K}_G(k, i) = t_0$, hence it must be the case that $\mathcal{K}_G(k, j) = t_0$. By construction, it is also the case that $\mathcal{K}_G(k, j) = \nu_k^j$, hence either one of the following is true:

a) If $j = 0$, in which case there is nothing to prove, or
b) If $j > 0$, and $\mathcal{K}_G(k, j) = \nu_k^j$, then for some $t \in \text{dom}(\mathcal{T})$. We have that $\mathcal{K}_G(k, j) = \mathcal{K}_G(k, j) = \nu(t, k)$, and because $t \in \text{dom}(\mathcal{T})$, it must be $t \neq t_0$. Contradiction.

Assume that $i > 0$. Therefore, it must be the case that $\mathcal{K}_G(k, i) = \nu_k^i = \nu(t, k)$ for some $t \in \text{dom}(\mathcal{T})$ such that $(w, k, v) \in \mathcal{T}_G(t_i)$. Finally, note that if it were $i < j$, then by construction we should have that $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, j)$, contradicting the requirement that $\mathcal{W}_R(k)$ is irreflexive. Therefore, it must be $i = j$.

(iii) For all $k \in \text{Key}$. For all $i, j \leq |\mathcal{K}_G(k)|$, $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, j)$ for all $i, j \leq |\mathcal{K}_G(k)|$. Without loss of generality, suppose that $i < j$. We distinguish between two cases:

a) If $i = 0$, then there is nothing to prove, or
b) If $i > 0$, and $\mathcal{K}_G(k, i) = \nu_k^i$, then for some $t \in \text{dom}(\mathcal{T})$. We have that $\mathcal{K}_G(k, i) = \mathcal{K}_G(k, i) = \nu(t, k)$, and because $t \in \text{dom}(\mathcal{T})$, it must be $t \neq t_0$. Contradiction.
b) $i > 0$; by construction, it must be the case that $K_G(k, i) = \nu(t', k)$ for some $t'$ such that $t' \xrightarrow{WR(k)} t$. Furthermore, because we are assuming that $i \leq j$, we also have that $j > 0$, and therefore $K_G(k, j) = \nu(t'', k)$ for some $t''$ such that $t'' \xrightarrow{WR(k)} t$. We have that $t' \xrightarrow{WR(k)} t$, and $t'' \xrightarrow{WR(k)} t$. By definition of dependency graph, this implies that $t' = t''$. We have that $w(K_G(k, i)) = t'$, $w(K_G(k, j)) = t''$, and $t' = t''$; if it were $i < j$, then by construction we would have that $t' \xrightarrow{WW(k)} t'$, contradicting the requirement of dependency graphs that $WW(k)$ is irreflexive. Therefore, it must be the case that $i = j$.

(iv) Suppose the following holds:

\[
\forall k \in \text{dom}(K), cl \in \text{Client}. \forall i, j : 0 \leq i < j < |K_G(k)|. \forall n, m \geq 0.
\]

\[
(t_{cl}^m = w(K_G(k, i)) \land t_{cl}^n \in \{w(K_G(k, j))\}) \cup \text{rs}(K_G(k, i)) \Rightarrow n < m
\]

Let $k \in \text{KEY}$, $cl \in \text{CLIENT}$, $i, j : 0 \leq i < j < |K_G(k)|$. Let also $n, m \geq 0$. First, suppose that $t_{cl}^n = w(K_G(k, i))$. Note that it cannot be $i = 0$, because by construction $w(K_G(k, i)) = t_0 \neq t_{cl}^n$. Therefore, it must be $i > 0$. We prove the following facts:

a) if $t_{cl}^m = w(K_G(k, j))$, then $n < m$. By construction, $K_G(k, i) = \nu(t_{cl}^m, k)$, and $(w, k, _) \in \mathcal{I}(t_{cl}^m)$. Similarly, $K_G(k, j) = \nu(t_{cl}^n, k)$, and $(w, k, _) \in \mathcal{I}(t_{cl}^n)$. Because $i < j$, it must be the case that $t_{cl}^n = w(\nu(t_{cl}^m, k) \xrightarrow{WW(k)} w(\nu(t_{cl}^n, k)) = t_{cl}^m, k)$, and by definition of dependency graph it follows that $n < m$.

b) if $t_{cl}^m \in \text{rs}(K_G(k, i))$, then $n < m$. In this case we have that $t_{cl}^m \xrightarrow{WR(k)} t_{cl}^n$ by construction, hence the definition of dependency graph ensures that $n < m$.

\[\square\]

**Proposition C.3.** For any kv-store $K$, $K_G = K$.

**Proof.** The conversions from kv-store to dependency graph (Def. C.2) and vice versa (Def. C.3) are based per key. Those conversions are well-formed by Prop. C.1 and Prop. C.2. It is sufficient to fix a key $k$ and prove $K_G(k) = K(k)$. We prove $K_G(k) = K(k)$ by induction on the length of $K(k)$.

- Base case: $|K(k)| = 1$. Let $K(k) = (v_0, t_0, T_0)$ for some $T_0$ that read the initial value $v_0$. Given the definition of $G_K$ (Def. C.2), we know that $(v, k, v_0) \in \mathcal{I}(t)$ for all $t \in T_0$ and $WW(k) = WR(k) = RW(k) = \emptyset$.

  Given the definition of $K_G(k)$ (Def. C.3), it is easy to see $K_G(k) = K(k)$.

- Inductive case: $|K(k)| = m + 1$. Suppose $K_G(k) = K(k)$ when $|K(k)| = m$ and let consider $|K(k)| = m + 1$. Let $K(k) = [(v_0, t_0, T_0), \ldots, (v_m, t_m, T_m), (v_{m+1}, t_{m+1}, T_{m+1})]$.

  We now discuss the $WW(k)$, $WR(k)$ and $RW(k)$ relations in $G_K(k)$ and the corresponding versions in $K_G(k)$.

  - For any $(t, t') \in WW(k)$, there are two cases: $t' \neq t_{m+1}$ and $t' = t_{m+1}$. If $t' \neq t_{m+1}$, then $t = t_i$ and $t = t_j$ for some $i$ and $j$ such that $0 < i < j < m + 1$ by the definition of $G_K$ (Def. C.2). By the I.H., we have $w(K_G(k, i)) = t_i$ and $w(K_G(k, j)) = t_j$. If $t' = t_{m+1}$, then $t = t_i$ for some $i$ such that $0 \leq i < m + 1$. By the definition of $K_G(k)$ (Def. C.3), the order of versions is the same as the order of $RW(k)$. That means the version $(v_{m+1}, t_{m+1}, -)$ is the last one, i.e. $(m + 1)$-th, in the $K_G(k)$.

    Combine the two cases above, we know:

    \[
    \forall i : 0 \leq i \leq m + 1. \exists T. K_G(k, i) = (v_i, t_i, T)
    \]  

    (3.1)

  - For any $(t, t') \in WR(k)$. Assume $t = t_i$ for some $i$ that $0 < i \leq m + 1$. Given the definition of $G_K$ (Def. C.2), it must be that $t' \in T_i$. By the definition of $K_G(k)$ and Eq. (3.1), it follows:

    \[
    \forall i : 0 \leq i \leq m + 1. K_G(k, i) = (v_i, t_i, T_i)
    \]  

    (3.2)

  The Eq. (3.2) implies $K(k) = K_G(k)$ and then $K = K_G$. 

\[\square\]

**Appendix D.**

**Operational Semantics of Abstract Executions**

**D.1. Operational Semantics of Abstract Executions**

Abstract executions are a framework originally introduced in [11] to capture the run-time behaviour of clients interacting with a database. In abstract execution, two relations between transactions are introduced: the visibility relation establishes when a transaction observes the effects of another transaction; and the arbitration relation helps to determine the value of a key $k$ read by a transaction, in the case that the transaction observes multiple updates to $k$ performed by different transactions.
Definition D.1. An abstract execution is a triple $\mathcal{X} = (\mathcal{T}, \text{VIS}, \text{AR})$, where

- $\mathcal{T} : \text{TransID}_0 \xrightarrow{f} \mathcal{P}(\text{Ops})$ is a partial, finite function mapping transaction identifiers to the set of operations that they perform,
- $\text{VIS} \subseteq \text{dom}(\mathcal{T}) \times \text{dom}(\mathcal{T})$ is an irreflexive relation, called visibility,
- $\text{AR} \subseteq \text{dom}(\mathcal{T}) \times \text{dom}(\mathcal{T})$ is a strict, total order such that $\text{VIS} \subseteq \text{AR}$, and whenever $t_{cl}^{n} \xrightarrow{\text{AR}} t_{cl}^{m}$, then $n < m$.

The set of abstract executions is denoted by $\text{absExec}$.

Given an abstract execution $\mathcal{X} = (\mathcal{T}, \text{VIS}, \text{AR})$, the notation $\mathcal{T}_X = \mathcal{T}$, $\mathcal{T}_X = \text{dom}(\mathcal{T})$, $\text{VIS}_X = \text{VIS}$ and $\text{AR}_X = \text{AR}$.

The session order for a client $\text{SO}_X(cI) = \{(t_{cl}^{n}, t_{cl}^{m}) \mid cI \in \text{CLIENT} \wedge t_{cl}^{n} \in \mathcal{T}_X \wedge t_{cl}^{m} \in \mathcal{T}_X \wedge n < m\}$

Definition D.2. A resolution policy $\text{RP}$ is a function $\text{RP} : \text{absExec} \times \mathcal{P}(\text{TransID}) \rightarrow \mathcal{P}(\text{Snapshot})$ such that, for any $X_1, X_2$ that agree on a subset of transactions $T$, then $\text{RP}(|X_1, T|) = \text{RP}(|X_2, T|)$. An abstract execution $\mathcal{X}$ satisfies the execution policy $\text{RP}$ if,

$$\forall t \in \mathcal{T}_X, \exists \text{ss} \in \text{RP}(\mathcal{X}, \text{VIS}^{-1}(t)), \forall k, v. (x, k, v) \in \mathcal{X} t \Rightarrow \text{ss}(k) = v$$

Definition D.3. An axiom $\mathcal{A}$ is a function from abstract executions to relations between transactions, $\mathcal{A} : \text{absExec} \rightarrow \mathcal{P}(\text{TransID} \times \text{TransID})$, such that whenever $X_1, X_2$ agree on a subset of transactions $T$, then $\mathcal{A}(X_1) \cap (T \times T) \subseteq \mathcal{A}(X_2)$.

Axioms of a consistency model are constraints of the form $\mathcal{A}(\mathcal{X}) \subseteq \text{VIS}_X$. For example, if we require $\mathcal{A}(\mathcal{X}) = \text{AR}_X$, then the corresponding axiom is given by $\text{AR}_X \subseteq \text{VIS}_X$, thus capturing the serialisability of transactions, i.e. this axiom is equivalent to require that $\text{VIS}_X$ is a total order. The requirement on subsets of transactions on which abstract executions agree will be needed later, when we define an operational semantics of transactions where clients can append a new transaction $t$ at the tail of an abstract execution $\mathcal{X}$, which satisfies an axiom $\mathcal{A}$. This requirement ensures that we only need to check that the axiom is satisfied by the pre-visibility and pre-arbitration relation of the transaction $t$ in $\mathcal{X}'$. In fact, the resulting abstract execution $\mathcal{X}'$ agrees with $\mathcal{X}$ on the set $\mathcal{T}_X$; in this case we'll note that we can rewrite $\mathcal{A}(\mathcal{X}') = \mathcal{A}(\mathcal{X}) \cap (\mathcal{T}_X \times \mathcal{T}_X) \cup (\mathcal{T}_X \times \{t\})$. Then $\mathcal{A}(\mathcal{X}') \cap (\mathcal{T}_X \times \mathcal{T}_X) \subseteq \mathcal{A}(\mathcal{X}) \cap (\mathcal{T}_X \times \mathcal{T}_X) \subseteq \text{VIS}_X \cap (\mathcal{T}_X \times \mathcal{T}_X) \subseteq \text{VIS}_X$, hence we only need to check that $\mathcal{A}(\mathcal{X}') \cap (\mathcal{T}_X \times \{t\}) \subseteq \text{VIS}_X$.

We say that an abstract execution $\mathcal{X}$ satisfies an axiom $\mathcal{A}$, if $\mathcal{A}(\mathcal{X}) \subseteq \text{VIS}_X$. An abstract execution $\mathcal{X}$ satisfies $\text{RP}(\mathcal{A})$, written $\mathcal{X} \models (\text{RP}, \mathcal{A})$, if the abstract execution $\mathcal{X}$ satisfies $\text{RP}$ and $\mathcal{A}$.

Definition D.4 (Abstract executions induced by axiomatic definition). The set of all abstract executions induced by an axiomatic definition, $\text{CM}(\text{RP}, \mathcal{A})$ is defined as $\text{CM}(\text{RP}, \mathcal{A}) \triangleq \{ \mathcal{X} \mid \mathcal{X} \models (\text{RP}, \mathcal{A}) \}$.

The Fig. 8 presents all rules of the operational semantics of programs based on abstract executions. TheACOMMIT rule is the abstract execution counterpart of rule PCOMMIT for kv-stores. TheACOMMIT models how an abstract execution $\mathcal{X}$ evolves when a client wants to execute a transaction whose code is $[T]$. In the rule, $\mathcal{T}$ is the set of transactions of $\mathcal{X}$ that are visible to the client $cl$ that wishes to execute $[T]$. Such a set of transactions is used to determine a snapshot $\text{ss} \in \text{RP}(\mathcal{X}, \mathcal{T})$ that the client $cl$ uses to execute the code $[T]$, and obtain a fingerprint $\mathcal{F}$. This fingerprint is then used to extend abstract execution $\mathcal{X}$ with a transaction from the set $\text{nextTxId}(\mathcal{T}_X, cl)$. Similar PPROG and APROG rule in Fig. 8 models multi-clients concurrency in an interleaving fashion. All the rest rules of the abstract operational semantics in Fig. 8 have a similar counterpart in the kv-store semantics.
\[ \rightarrow : \text{CLIENT} \times ((\text{AbsExecs} \times \text{STACK}) \times \text{CMD}) \times \text{ET} \times \text{LABEL} \times ((\text{AbsExecs} \times \text{STACK}) \times \text{CMD}) \]

**AAtomicTrans**

\[ T \subseteq T_{\text{ET}} \quad \text{s}_s \in \text{RP}(\mathcal{X}, T) \quad (s, s, \emptyset), T \rightarrow^* (s', \emptyset, F), \text{skip} \]

\[ t \in \text{nextTxd}(T_{\text{ET}}, cl) \quad \mathcal{X}' = \text{extend}(\mathcal{X}, t, T, F) \quad \forall A \in A. \{ t' | (t', t) \in A(\mathcal{X}') \} \subseteq T \]

\[ cl \vdash (\mathcal{X}, s), [T] \xrightarrow{(cl, T, F)}_{(\text{RP}, A)} (\mathcal{X}', s'), \text{skip} \]

**APrimitive**

\[ s \xrightarrow{c_p} s' \]

\[ cl \vdash (\mathcal{X}, s), c_p \xrightarrow{(cl, c_p)}_{\text{ET}} (\mathcal{X}, s'), \text{skip} \]

**AChoice**

\[ i \in \{ 1, 2 \} \]

\[ cl \vdash (\mathcal{X}, s), c_1 + c_2 \xrightarrow{(cl, c_1)}_{\text{ET}} (\mathcal{X}, s), c_i \]

**AIter**

\[ cl \vdash (\mathcal{X}, s), c_p \xrightarrow{(cl, c_p)}_{\text{ET}} (\mathcal{X}, s), \text{skip} + (c; c^*) \]

**ASeqSkip**

\[ cl \vdash (\mathcal{X}, s), \text{skip} : c \xrightarrow{(cl, c)}_{\text{ET}} (\mathcal{X}, s), c \]

\[ cl \vdash (\mathcal{X}, s), c_1 : c_2 \xrightarrow{(cl, c_1)}_{\text{ET}} (\mathcal{X}, s'), c_1 ' : c_2 ' \]

Figure 8: Operational Semantics on Abstract Executions

Note that **AAtomicTrans** is more general than Rule **PATomicTrans** in the kv-store semantics. In the latter, the snapshot of a transaction is uniquely determined from a view of the client, in a way that roughly corresponds to the last write wins policy in the abstract execution framework. In contrast, the snapshot of a transaction used in **AAtomicTrans** is chosen non-deterministically from those made available to the client by the resolution policy **RP**, which may not necessarily be last-write-win.

Throughout this report we will work mainly with the **Last Write Wins** resolution policy (Def. D.5). When discussing the operational semantics of transactional programs, we will also introduce the **Anarchic** resolution policy.

**Definition D.5.** The Last Write Wins resolution policy \( \text{RP}_{\text{LWW}} \) is defined as \( \text{RP}_{\text{LWW}}(\mathcal{X}, T) \equiv \{ ss \} \) where

\[ ss = \lambda k. \text{let } \mathcal{T}_k = (T \cap \{ t | (w, k, \_ ) \in \mathcal{X} t \}) \text{ in } \begin{cases} v_0 & \text{if } T_k = \emptyset \\ v & \text{if } (w, k, v) \in \mathcal{X} \text{ max}_{\text{AR}}(\mathcal{T}_k) \end{cases} \]

**D.2. Anarchic Model**

To justify this semantics capture all the possible abstract executions under certain consistency model, we introduce anarchic model. Then by Theorem D.1 the operational semantics captures all possible behaviours of a program \( P \) under certain consistency model.

**Definition D.6 (Anarchic Model).** The anarchic resolution policy \( \text{RP}_{\emptyset} \) is defined as \( \text{RP}_{\emptyset}(\_, \_) = \text{SNAPSHOT} \). The anarchic consistency model is defined axiomatically by the pair \( \emptyset = (\text{RP}_{\emptyset}, \emptyset) \).

Under the anarchic model, the initial snapshot of a transaction \( t \) is non-deterministic and does not depend on its observable transactions \( T \). When committing, the transaction \( t \) is always allowed to do so no matter the state of visibility relation.
Suppose a client cl has the transactional code below:
\[
\begin{align*}
\text{a} & := [k]; \\
\text{if} (\text{a} = v_1) \{ [k'] := v_1 \}
\end{align*}
\]

Let consider executing the transaction under initial state with all keys set to the same initial value \(v_0\). Under \(\otimes\), because the snapshot is chosen non-deterministically, the transaction might start with an initial snapshot where all keys have the value \(v_1\), which are not necessary equal to \(v_0\). Since there is no constraint on the visibility relation, such transaction is always allowed to commit. Consequently, the abstract execution after is the following:

\[(x, k, v_1) \to (w, k', v_1)\]

It is important to note, however, that the set of abstract executions generated by \(P\) is still bound to the structure of the program. For example, executing \(P\) under the anarchic execution model will never lead to an abstract execution with multiple transactions, or to an abstract execution where a transaction writes a key other than \(k'\) is written.

The Def. [D.7] defines all the possible abstract executions induced by a axiomatic definition \((\text{RP}, \Lambda)\).

**Definition D.7** (Programs under axiomatic definition). The semantics of a program \(P\) under a consistency model with axiomatic definition \((\text{RP}, \Lambda)\) is given by

\[\llbracket P \rrbracket_{(\text{RP}, \Lambda)} = \{ \mathcal{X} \mid (\mathcal{X}_0, \mathcal{E}_0, P) \xrightarrow{(\text{RP}, \Lambda)} (\mathcal{X}, \lambda, P_f) \}\]

where \(\mathcal{E}_0 = \lambda cl \in \text{dom}(P). \lambda x.0\) and \(P_f = \lambda cl \in \text{dom}(P). \text{skip}\).

We define the set of all the possible behaviours of a program \(P\) to be \(\llbracket P \rrbracket_{\otimes}\). One may argue that the axiomatic definition \(\otimes\) does not truly represent an anarchic consistency model. Consider the following code:

\[
\begin{align*}
\text{a} & := [k]; \\
\text{b} & := [k]; \\
\text{if} (\text{a} \neq \text{b}) \{ [k'] := v_1 \}
\end{align*}
\]

Under a even weaker anarchic consistency model, it would be possible for program \(P'\) to write the value \(v_1\) for key \(k'\). However, this never happens if \(P'\) is executed under our \(\otimes\). Because we embedded into abstract execution the assumption that transactions only read at most one value for each key.

We can weaken the anarchic behaviours further by lifting this limitation and still retain the validity of all the results contained in this report. However, the constraint that an object is never read twice in transactions is enforced by atomic visibility, so it is unnecessary to lift such limitation of our \(\otimes\). The set of all possible behaviours exhibited by a program \(P\) under a consistency model \((\text{RP}, \Lambda)\) can be defined by intersecting the set of executions that \(P\) exhibits under the anarchic consistency model, with the set of all executions allowed by the axiomatic definition \((\text{RP}, \Lambda)\) (Theorem D.1).

**Theorem D.1.** For any program \(P\) and axiomatic definition \((\text{RP}, \Lambda)\):

\[\llbracket P \rrbracket_{(\text{RP}, \Lambda)} = \llbracket P \rrbracket_{\otimes} \cap \text{CM}(\text{RP}, \Lambda)\]

**Proof.** It is easy to see \(\llbracket P \rrbracket_{(\text{RP}, \Lambda)} \subseteq \text{CM}(\text{RP}, \Lambda)\) by Def. [D.7] so \(\llbracket P \rrbracket_{(\text{RP}, \Lambda)} \subseteq \llbracket P \rrbracket_{\otimes} \cap \text{CM}(\text{RP}, \Lambda).\) For another way, by Prop. [D.1] we know \(\llbracket P \rrbracket_{\otimes} \subseteq \llbracket P \rrbracket_{(\text{RP}, \Lambda)}.\) Then, by the definition of \(\llbracket P \rrbracket_{\otimes}\), it follows \(\llbracket P \rrbracket_{\otimes} \cap \text{CM}(\text{RP}, \Lambda) \subseteq \llbracket P \rrbracket_{(\text{RP}, \Lambda)}.\)  

**Proposition D.1** (Monotonic axiomatic definition). Let define \((\text{RP}_1, \Lambda_1) \sqsubseteq (\text{RP}_2, \Lambda_2)\) as the following:

\[\forall \mathcal{X}, \mathcal{T}. \text{ RP}_2(\mathcal{X}, \mathcal{T}) \subseteq \text{RP}_1(\mathcal{X}, \mathcal{T})\]

and,

\[\forall \mathcal{X}. \bigcup_{A_1 \in A_1} A_1(\mathcal{X}) \subseteq \bigcup_{A_2 \in A_2} A_2(\mathcal{X})\]

then

\[\llbracket P \rrbracket_{(\text{RP}_1, \Lambda_1)} \subseteq \llbracket P \rrbracket_{\otimes}\]

**Proof.** Since \(\otimes \subseteq (\text{RP}_1, \Lambda_1)\), we prove stronger result that for \((\text{RP}_1, \Lambda_1) \sqsubseteq (\text{RP}_2, \Lambda_2)\), the following hold:

\[\forall cl, \mathcal{X}, \mathcal{X}', P, P', s, s'. cl \vdash (\mathcal{X}, s), P \rightarrow_{(\text{RP}_1, \Lambda_1)} (\mathcal{X}', s'), P' \Rightarrow cl \vdash (\mathcal{X}', s'), P \rightarrow_{(\text{RP}_2, \Lambda_2)} (\mathcal{X}', s'), P'\]

We prove it by induction on the derivations. The only interesting case is the \text{PATOMICTRANS} rule. Given an initial runtime abstract execution \(\mathcal{X}\), a set of observable transactions \(\mathcal{T}\), a new transaction identifier \(t\), by the \text{AATOMICTRANS} rule, it is
sufficient to prove, first, all the snapshot under the stronger consistency model is also a valid snapshot under the weaker
one:

\[ \forall \mathcal{X}, \mathcal{T}. \ RP_2(\mathcal{X}, \mathcal{T}) \subseteq RP_1(\mathcal{X}, \mathcal{T}) \]

(4.1)

and second if it is valid to commit a new transition with the observable set \( \mathcal{T} \) under stronger consistency model, it is able
to do so under weaker consistency model:

\[ \bigcup_{A_1 \in A_1} A_1(\mathcal{X'})^{-1}(t) \subseteq \bigcup_{A_2 \in A_2} A_2(\mathcal{X'})^{-1}(t) \]

(4.2)

The Eqs. (4.1) and (4.2) can be proven by \( (RP_1, A_1) \subseteq (RP_2, A_2) \).

Base case: \textsc{Passign}, \textsc{Passume}, \textsc{Pchoice}, \textsc{Ploop}, \textsc{Pseqskip}, \textsc{Ppar}, \textsc{Pwait}. These base cases do not depend on the
consistency model, so they trivial hold because of the hypothesis. Inductive case: \textsc{Pseq}. It is proved directly by applying
the I.H.

\( \square \)

Appendix E.

Relationship between kv-stores and abstract execution

E.1. KV-Store to Abstract Executions

We introduce the definition of the dependency graph induced an abstract execution:

**Definition E.1.** Given an abstract execution \( \mathcal{X} \) that satisfies the last write wins policy, the dependency graph \( \text{GraphOf}(\mathcal{X}) \) is defined by letting

- \( t \xrightarrow{\text{WR}_{\mathcal{X}}(k)} t' \) if and only if \( t = \max_{\text{AR}_{\mathcal{X}}}(\text{visibleWrites}_{\mathcal{X}}(k, t')) \),
- \( t \xrightarrow{\text{WW}_{\mathcal{X}}(k)} t' \) if and only if \( t, t' \in \mathcal{X} ~(w, k, -) \) and \( t \xrightarrow{\text{AR}_{\mathcal{X}}} t' \),
- \( t \xrightarrow{\text{RW}_{\mathcal{X}}(k)} t' \) if and only if either \( (x, k, -) \in \mathcal{X} ~ t, (w, k, -) \in \mathcal{X} ~ t' \) and whenever \( t'' \xrightarrow{\text{WR}_{\mathcal{X}}(k)} t \), then \( t'' \xrightarrow{\text{WW}_{\mathcal{X}}(k)} t' \).

Note that each abstract execution \( \mathcal{X} \) determines a kv-store \( \mathcal{K}_\mathcal{X} \), as a result of Def. E.1 and Theorem C.1. Let \( \mathcal{K} \) be the unique kv-store such that \( \mathcal{G}_\mathcal{K} = \text{GraphOf}(\mathcal{X}) \), then \( \mathcal{K}_\mathcal{X} = \mathcal{K} \). As we will discuss later in this Section, this mapping \( \mathcal{K}(\cdot) \) is NOT a bijection, in that several abstract executions may be encoded in the same kv-store. Because kv-stores abstract away the total arbitration order of transactions.

Upon the relation \( \mathcal{K}_\mathcal{X} = \mathcal{K} \), there is a deeper link between kv-store plus views and abstract exeptions. This notion, named \textit{compatibility}, bases on the intuition that clients can make observations over kv-stores and abstract executions, in
terms of snapshots.

In kv-stores, observations are snapshots induced by views. While in abstract executions, observations correspond to the
snapshots induced by the visible transactions. Note that it is under the condition that the abstract execution adopts \textsc{RPWW}
resolution policy. This approach is analogous to the one used by operation contexts in \cite{12}. Thus, a kv-store \( \mathcal{K} \) is \textit{compatible}
with an abstract execution \( \mathcal{X} \), written \( \mathcal{K} \simeq \mathcal{X} \) if any observation made on \( \mathcal{K} \) can be replicated by an observation made on
\( \mathcal{X} \), and vice-versa.

**Definition E.2.** Given a kv-store \( \mathcal{K} \), an abstract execution \( \mathcal{X} \) is compatible with \( \mathcal{K} \), written \( \mathcal{X} \simeq \mathcal{K} \), if and only if there exists a mapping \( f : \mathcal{P}(\mathcal{T}_\mathcal{X}) \rightarrow \mathcal{VIEWS}(\mathcal{K}) \) such that

- for any subset \( \mathcal{T} \subseteq \mathcal{T}_\mathcal{X} \), then \( \text{RP}_{\text{LWW}}(\mathcal{X}, \mathcal{T}) = \{ \text{snapshot}(\mathcal{K}, f(\mathcal{T})) \} \);
- for any view \( \mathcal{U} \in \mathcal{VIEWS}(\mathcal{K}) \), there exists a subset \( \mathcal{T} \subseteq \mathcal{T}_\mathcal{X} \) such that \( f(\mathcal{T}) = \mathcal{U} \) and \( \text{RP}_{\text{LWW}}(\mathcal{X}, \mathcal{T}) = \{ \text{snapshot}(\mathcal{K}_\mathcal{X}, \mathcal{U}) \} \).

The function \( \text{getView}(\mathcal{X}, \mathcal{T}) \) defines the view on \( \mathcal{K}_\mathcal{X} \) that corresponds to \( \mathcal{T} \) as the following:

\[ \text{getView}(\mathcal{X}, \mathcal{T}) \triangleq \lambda k.\{0\} \cup \{ i \mid w(\mathcal{K}_\mathcal{X}(k, i)) \in \mathcal{T} \} \]

Inversely, the function \( \text{Tx}(\mathcal{K}, \mathcal{U}) \) converts a view to a set of observable transactions:

\[ \text{Tx}(\mathcal{K}, \mathcal{U}) \triangleq \{ w(\mathcal{K}(k, i)) \mid k \in \text{KEY} \land i \in \mathcal{U}(k) \} \]

Given \text{getView}, \text{Tx}, Def. E.2 it follows \( \mathcal{X} \simeq \mathcal{K}_\mathcal{X} \) shown in Theorem E.1.

**Theorem E.1.** For any abstract execution \( \mathcal{X} \) that satisfies the last write wins policy, \( \mathcal{X} \simeq \mathcal{K}_\mathcal{X} \).
Proof. Given the function $\text{getView}(\mathcal{X}, \cdot)$ from $\mathcal{P}(\mathcal{T}_X)$ to $\text{views} (K_X)$, we prove it satisfies the constraint of Def. E.2. Fix a set of transitions $\mathcal{T}$. By the Prop. E.1, the view $\text{getView}(\mathcal{X}, \mathcal{T})$ on $K_X$ is a valid view, that is, $\text{getView}(\mathcal{X}, \mathcal{T}) \in \text{views}(K_X)$. Given that it is a valid view, the Prop. E.2 proves:

$$\text{RP}_{LWW}(\mathcal{X}, \mathcal{T}) = \{\text{snapshot}(K_X, \text{getView}(\mathcal{X}, \mathcal{T}))\}$$  \hspace{1cm} (5.1)

The another way round is more subtle, because $\mathcal{T}$ contains any read only transaction. By Prop. E.3 it is safe to erase read only transactions from $\mathcal{T}$, when calculating the view $\text{getView}(\mathcal{X}, \mathcal{T})$. Last, by Prop. E.4 we prove the following:

$$\text{RP}_{LWW}(\mathcal{X}, \mathcal{T}) = \text{snapshot}(K_X, \mathcal{U})$$  \hspace{1cm} (5.2)

By Eq. (5.1) and Eq. (5.2), it follows $\mathcal{X} \simeq K_X$.

Proposition E.1 (Valid views). For any abstract execution $\mathcal{X}$, and $\mathcal{T} \subseteq \mathcal{T}_X$, $\text{getView}(\mathcal{X}, \mathcal{T}) \in \text{views}(K_X)$.

Proof. Assume an abstract execution $\mathcal{X}$, a set of transactions $\mathcal{T} \subseteq \mathcal{T}_X$, and a key $k$. By the definition of $\text{getView}(\mathcal{X}, \mathcal{T})$, then $0 \in \text{getView}(\mathcal{X}, \mathcal{T})(k)$, and $0 \leq i < |K_X(k)|$ for any index $i$ such that $i \in \text{getView}(\mathcal{X}, \mathcal{T})(k)$. Therefore we only need to prove that $\text{getView}(\mathcal{X}, \mathcal{T})$ satisfies (atomic). Let $j \in \text{getView}(\mathcal{X}, \mathcal{T})(k)$ for some key $k$, and let $t = w(K_X(k, j))$. Let also $w, i$ be such that $w(K_X(k', i')) = t$. We need to show that $i \in \text{getView}(\mathcal{X}, \mathcal{T})(k')$. Note that it $t = t_0$ then $w(K_X(k', i')) = t$ only if $i = 0$, and $0 \in \text{getView}(\mathcal{X}, \mathcal{T})(k')$ by definition. Let then $t \neq t_0$. Because $w(K_X(k, j)) = t$ and $j \in \text{getView}(\mathcal{X}, \mathcal{T})$, then it must be the case that $t \in \mathcal{T}$. Also, because $w(K_X(k', i')) = t$, then $(w, k, \cdot) \notin \mathcal{T}$, it follows that there exists an index $i' \in \text{getView}(\mathcal{X}, t)(k')$ such that $w(K_X(k', i')) = t$. By definition of $K_X$, if $w(K_X(k', i')) = t$, then it must be $i' = i$, and therefore $i \in \text{getView}(\mathcal{X}, t)(k')$.

Proposition E.2 (Visible transactions to views). For any subset $\mathcal{T} \subseteq \mathcal{T}_X$, $\text{RP}_{LWW}(\mathcal{X}, \mathcal{T}) = \{\text{snapshot}(K_X, \text{getView}(\mathcal{X}, \mathcal{T}))\}$.

Proof. Fix $\mathcal{T} \subseteq \mathcal{X}$, and let $\{K\} = \text{RP}_{LWW}(\mathcal{X}, \mathcal{T})$. We prove that, for any $k \in \text{key}$, $K(k) = \text{snapshot}(\text{getView}(\mathcal{X}, \mathcal{T}))(k)$. There are two different cases:

1) $\mathcal{T} \cap \{t \mid (w, k, \cdot) \in \mathcal{X} t\} = \emptyset$. In this case $K(k) = \emptyset$. We know that GraphOf($\mathcal{X}$) satisfies all the constraints required by the definition of dependency graph (I.6). Together with Theorem C.1, it follows that $\text{getView}(\mathcal{X}, \mathcal{T})(k) = \{0\}$, hence

$$\text{snapshot}(K_X, \text{getView}(\mathcal{X}, \mathcal{T}))(k) = \text{val}(K_X(k, 0)) = v_0$$

Note that whenever $(w, k, \cdot) \in \mathcal{X} t$ for some $t$, then $t \notin \mathcal{T}$. Therefore, whenever $(w, k, \cdot) = \text{getView}(\mathcal{X}, \mathcal{T})(k)$ for some $i \geq 0$, then $t \notin \mathcal{T}$.

2) Suppose now that $\mathcal{T} \cap \{t \mid (w, k, \cdot) \in \mathcal{X} t\} \neq \emptyset$. Let then $t = \text{max}_{\text{AR}}(\mathcal{T} \cap \{t \mid (w, k, \cdot) \in \mathcal{X} t\})$. Then $(w, k, v) \in \mathcal{X} t$ for some $v \in \text{val}$. Furthermore, $\text{RP}_{LWW}(\mathcal{X}, \mathcal{T})(k) = v$. By definition, $t' \in \mathcal{T} \cap \{t \mid (w, k, \cdot) \in \mathcal{X} t\}$, then either $t' = t$ or $t' \xrightarrow{\text{AR}} t$. The definition of GraphOf($\mathcal{X}$) gives that $t' \xrightarrow{\text{AR}} t$. Because $(w, k, v) \in \mathcal{X} t$, then there exists an index $i \geq 0$ such that $\text{getView}(\mathcal{X}(k, i)) = (v, t, \cdot)$. Furthermore, whenever $w(k, j) = t'$ for some $t'$ and $j > i$, then it must be the case that $t' \xrightarrow{\text{AR}} t$, and because $\text{getView}(\mathcal{X}(k, i)) = t'$, it follows that $max(\text{getView}(\mathcal{X}, \mathcal{T})(k)) = i$, hence $\text{snapshot}(K_X, \text{getView}(\mathcal{X}, \mathcal{T})) = \text{val}(K_X(k, i)) = v$.

Proposition E.3 (Read-only transactions erasing). Let $u \in \text{views}(K_X)$, and let $\mathcal{T} \subseteq \mathcal{T}_X$ be a set of read-only transactions in $\mathcal{X}$. Then $\text{getView}(\mathcal{X}, \mathcal{T} \cup \text{Tx}(K_X, u)) = u$.

Proof. Fix a key $k$. Suppose that $i \in \text{getView}(\mathcal{X}, \mathcal{T} \cup \text{Tx}(K_X, u))(k)$. By definition, $K_X(k, j) = (-, t, \cdot)$ for some $t \in \mathcal{T} \cup \text{Tx}(K_X, u)$. Because $\mathcal{T}$ only contains read-only transactions, by definition of $K_X$ there exists no index $j$ such that $K_X(k, j) = (-, t', \cdot)$ for some $t' \notin \mathcal{T}$, hence it must be the case that $t \in \text{Tx}(K_X, u)$. By definition of $\text{Tx}$, this is possible only if there exist a key $k'$ and an index $j$ such that $K_X(k', j) = (-, t, \cdot)$. Because $u$ is atomic by definition, and because $K_X(k, i) = (-, t, \cdot)$, we have that $i \in u(k)$. Now suppose that $i \in u(k)$, and let $K_X(k, i) = (-, t, \cdot)$ for some $t$. This implies that $(w, k, \cdot) \in \mathcal{X} t$ by definition. Let $t \in \text{Tx}(K_X, u)$, hence $t \in \mathcal{T} \cup \text{Tx}(K_X, u)$, then for any key $k'$ such that $(w, k', \cdot) \in \mathcal{X} t$, there exists an index $j \in \text{getView}(\mathcal{X}, \mathcal{T} \cup \text{Tx}(K_X, u))$ such that $K_X(k', j) = (-, t, \cdot)$; because kv-stores only allow a transaction to write at most one version per key, then the index $j$ is uniquely determined. In particular, we know that $(w, k, \cdot) \in \mathcal{X} t$, and $K_X(k, i) = (-, t, \cdot)$, from which it follows that $i \in \text{getView}(\mathcal{X}, \mathcal{T} \cup \text{Tx}(K_X, u))(k)$. 

\[\square\]
Proposition E.4 (Views to visible transactions). Given a view \( u \in \text{Views}(K_X) \), there exists \( T \subseteq T_X \) such that getView(\( X', T \)) = \( u \), and \( \text{RP}_{\text{LWW}}(X', T) = \text{snapshot}(K_X, u) \).

Proof. We only need to prove that for any \( u \in \text{Views}(K_X) \), there exists \( T \subseteq T_X \) such that getView(\( X', T \)) = \( u \). Then it follows from Prop. E.2 that \( \text{RP}_{\text{LWW}}(X', T) = \text{snapshot}(K_X, u) \). It suffices to choose \( T = \bigcup_{k \in \text{KEY}} \{ (w(K_X(k, i)) \mid i > 0 \land i \in u(k)) \} \). Fix a key \( k \), and let \( i \in u(k) \). We prove that \( i \in \text{getView}(X', T) \). If \( i = 0 \), then \( i \) \in \text{getView}(X', T) \) by definition. Therefore, assume that \( i > 0 \). Let \( t = w(K_X(k, i)) \). It must be the case that \( t \in T \) and \( i \in \text{getView}(X', T)(k) \).

Next, suppose that \( i \) \in \text{getView}(X', T)(k) \). We prove that \( i \in u(k) \). Note that if \( i = 0 \), then \( i \) \in \text{getView}(X', T)(k) \) because of the definition of views. Let then \( i > 0 \). Because \( i \) \in \text{getView}(X', T)(k) \), we have that \( w(K_X(k, i)) \in T \). Let \( t = w(K_X(k, i)) \). Because \( i > 0 \), it must be the case that \( t \neq t_0 \). By definition, \( t \in T \) only if there exists an index \( j \) and key \( k' \), possibly different from \( k \), such that \( w(K_X(k', j)) = t \) and \( j \in u(k') \). Because \( t \neq t_0 \) we have that \( j > 0 \). Finally, because \( u \) is atomic by definition, \( j \in u(k') \) \( w(K_X(k', j)) = t = w(K_X(k, i)) \), then it must be the case that \( i \in u(k) \), which concludes the proof. □

E.2. KV-Store Traces to Abstract Execution Traces

To prove our definitions using execution test on kv-stores is sound and complete with respect with the axiomatic definitions on abstract executions (§B.4), we need to prove trace equivalence between these two models.

In this section, we only consider the trace that does not involve \( k \)-set operation and only committing fingerprint and view shift. In §B.4, we will go further and discuss the trace installed with \( P \).

Similar to \( \varnothing \), let ET be the most permissive execution test. That is ET \( \vdash (K, u) \triangleright F : (K', u') \) such that whenever \( u(k) \neq u'(k) \) then either \( (u, k, \_ \setminus ) \in F \) or \((\_ \setminus , k, \_ \setminus ) \in F\). We will relate ET-traces to abstract executions that satisfy the last write wins resolution policy, i.e. \( \text{RP}_{\text{LWW}}(\emptyset) \).

To bridge ET-traces to abstract executions, The absExec(\( \tau \)) function converts the trace of ET to a set of possible abstract executions (Def. E.3). In fact, for any trace \( \tau \) and abstract execution \( X' \in \text{absExec}(\tau) \), the last configuration of \( \tau \) is \((K_X, \_ \setminus ) \) (Prop. E.5). We often use \( X' \) for \( X' \in \text{absExec}(\tau) \).

Definition E.3. Given a kv-store \( K \), a view \( \emptyset \), an initial abstract execution \( X_0 = ([], \emptyset, \emptyset) \), an abstract execution \( X' \), a set of transactions \( T \subseteq T_X \), a transaction identifier \( t \) and a set of operations \( F \), the extend function is defined as the follows:

\[
\text{extend}(X', t, T, F) \triangleq \begin{cases} 
\text{undefined} & \text{if } t = t_0 \\
\begin{cases} 
F_X \uplus \{ \langle t \mapsto F \rangle \}, \text{VIS'}', \text{AR}' & \text{if } \dagger \\
\vdash t = t_0' \land \text{VIS'} \vdash \text{VIS}_X \uplus \{(t', t) \mid t \in T \} & \land \text{AR}' = \text{AR}_X \uplus \{(t', t) \mid t' \in T_X \}
\end{cases}
\end{cases}
\]

Given a ET trace \( \tau \), let lastConf(\( \tau \)) be the last configuration appearing in \( \tau \). The set of abstract executions absExec(\( \tau \)) is defined as the smallest set such that:

- \( X_0 \in \text{absExec}((K_0, U_0)) \),
- if \( X' \in \text{absExec}(\tau) \), then \( X' \in \text{absExec}(\tau \rightarrow ET \_ (K, U)) \),
- if \( X' \in \text{absExec}(\tau) \), then \( X' \in \text{absExec}(\tau \rightarrow ET \_ (K, U)) \),
- let \((K', U') = \text{lastConf}(\tau) \); if \( X' \in \text{absExec}(\tau) \), \( F \neq \emptyset \), and \( T = T_X(K, U'(cl)) \cup T_{rd} \) where \( T_{rd} \) is a set of read-only transactions such that \((u, k, v) \notin X' \) for all keys \( k \) and values \( v \) and transactions \( t' \in T_{rd} \), and if the transaction \( t \) is the transaction appearing in lastConf(\( \tau \)) but not in \( K \), then \( \text{extend}(X', t, T, F) \in \text{absExec}(\tau \rightarrow ET \_ (K, U)) \).

Proposition E.5 (Trace of ET to abstract executions). For any ET trace \( \tau \), the abstract execution \( X' \in \text{absExec}(\tau) \) satisfies the last write wins policy, and \((K_X, \_ \setminus ) = \text{lastConf}(\tau) \).

Proof. Fix a ET trace \( \tau \). We prove by induction on the number of transitions \( n \) in \( \tau \).

- Base case: \( n = 0 \). It means \( \tau = (K_0, \_ \setminus ) \). This triple satisfies the constraints of Def. D.1 as well as the resolution policy \( \text{RP}_{\text{LWW}} \). It is also immediate to see that GraphOf(\( X' \)) = ([], \emptyset, \emptyset, \emptyset). In particular, \( T_{\text{GraphOf}(X')} = \emptyset \), and the only kv-store \( K \) such that \( T_{GK} = \emptyset \) is given by \( K = K_0 \). By definition, \( K_X = K_0 \), as we wanted to prove.

- Inductive case: \( n > 0 \). In this case, we have that \( \tau \rightarrow (cl, u) \rightarrow (K, U) \) for some \( cl, \mu, K, U \). The ET trace \( \tau' \) contains exactly \( n-1 \) transitions, so that by induction we can assume that \( X' \) is a valid abstract execution that satisfies \( \text{RP}_{\text{LWW}} \) and \( \text{lastConf}(\tau') = (K_{X'}, U') \) for some \( U' \).
We perform a case analysis on $\mu$. If $\mu = \varepsilon$, then it follows that $K = K_{X_{\mu}}$, and $X_{\mu} = X_{\tau}$ by Def. [5]. Then by the inductive hypothesis $X_{\tau}$ is an abstract execution that satisfies $RPLW$, $lastConf(\tau) = (K, -)$, and $K_{X_{\tau}} = K_{X_{\tau}} = K_{X_{\tau}}$ and there is nothing left to prove.

Suppose now that $\mu = F$, for some $F$. In this case we have that $K = update(K_{X_{\tau}}, U'(cl), F, cl)$. Note that if $F = \emptyset$, then $K = K_{X_{\tau}}$, and $X_{\tau} = X_{\tau}$. By the inductive hypothesis, $X_{\tau}$ is an abstract execution that satisfies $RPLW$, and $K = K_{X_{\tau}} = K_{X_{\tau}}$. Assume then that $F \neq \emptyset$. By definition, $K = update(K_{X_{\tau}}, U'(cl), F, t)$ for some $t \in nextTx(t(cl, K_{X_{\tau}}))$. It follows that $t$ is the unique transaction such that $t \notin K_{X_{\tau}}$, and $t \in K_{X_{\tau}}$ (the fact that $t \in K_{X_{\tau}}$ follows from the assumption that $F \neq \emptyset$). Let $T = Tx(K_{X_{\tau}}, U'(cl))$; then $X_{\tau} = extend(K_{X_{\tau}}, t, T)$. Note that $X_{\tau}$ satisfies the constraints of abstract execution required by Def. [5].

- Because $t \in nextTx(t(cl, K_{X_{\tau}}))$, it must be the case that $t = t_{cl}^{m}$ for some $m \geq 1$; we have that $T_{X_{\mu}} = T_{X_{\tau}, } [t_{cl}^{m} \rightarrow F]$, from which it follows that

$$T_{X_{\tau}} = dom(T_{X_{\tau}}) \cup \{t_{cl}^{m}\} \rightarrow T_{X_{\tau}} \cup \{t_{cl}^{m}\}$$

By inductive hypothesis, $t_{0} \notin T_{X_{\tau}}$, and therefore $t_{0} \notin T_{X_{\tau}} \cup \{t_{cl}^{m}\} = T_{X_{\tau}}$.

- $VIS_{X_{\tau}} \subseteq AR_{X_{\tau}}$. Let $(t', t'') \in VIS_{X_{\tau}}$. Then either $t' = t_{cl}^{m}$ and $t'' \in T_{X_{\tau}}$, or $(t', t'') \in VIS_{X_{\tau}}$. In the former case, we have that $(t', t_{cl}^{m}) \in AR_{X_{\tau}}$ by definition; in the latter case, we have that $(t', t'') \in AR_{X_{\tau}}$, because $X_{\tau}$ is a valid abstract execution by inductive hypothesis, and therefore $(t'', t_{cl}^{m}) \in AR_{X_{\tau}}$. This concludes the proof that $VIS_{X_{\tau}} \subseteq AR_{X_{\tau}}$.

- $VIS_{X_{\tau}}$ is irreflexive. Assume $(t', t'') \in VIS_{X_{\tau}}$, then either $(t', t'') \in VIS_{X_{\tau}}$, and because $VIS_{X_{\tau}}$ is irreflexive by the inductive hypothesis, then $t' \neq t''$; or $t' = t_{cl}^{m}$, $t'' \in T_{X_{\tau}}$, then $t' \neq t_{cl}^{m}$.

- $AR_{X_{\tau}}$ is total. Let $(t', t'') \in T_{X_{\tau}}$. Suppose that $t' \neq t''$. (1) $t' \neq t_{cl}^{m}$, $t'' \neq t_{cl}^{m}$, then it must be the case that $(t', t'') \in T_{X_{\tau}}$: this is because we have already argued that $T_{X_{\tau}} = T_{X_{\tau}} \cup \{t_{cl}^{m}\}$. By the inductive hypothesis, we have that either $(t', t'') \in AR_{X_{\tau}}$, or $(t'', t') \in AR_{X_{\tau}}$. Because $AR_{X_{\tau}} \subseteq AR_{X_{\tau}}$, then either $(t', t'') \in AR_{X_{\tau}}$, or $(t'', t') \in AR_{X_{\tau}}$.

- $AR_{X_{\tau}}$ is irreflexive. It follows the same as the one of $VIS_{X_{\tau}}$.

- $AR_{X_{\tau}}$ is transitive. Assume $(t', t''') \in AR_{X_{\tau}}$, and $(t'', t''') \in AR_{X_{\tau}}$. Note that it must be the case that $(t', t'') \in T_{X_{\tau}}$, by the definition of $AR_{X_{\tau}}$, and in particular $(t', t'') \in AR_{X_{\tau}}$. For $t''''$, we have two possible cases.

1. Either $t''' \in T_{X_{\tau}}$, from which it follows that $(t', t''') \in AR_{X_{\tau}}$, because of $AR_{X_{\tau}}$, is transitive by the inductive hypothesis, then $(t', t''') \in AR_{X_{\tau}}$, and therefore $(t', t'''') \in AR_{X_{\tau}}$.

2. Or $t''' \notin T_{X_{\tau}}$, and because $(t', t'') \in T_{X_{\tau}}$, then $(t', t''') \in AR_{X_{\tau}}$ by definition.

- $SO_{X_{\tau}} \subseteq AR_{X_{\tau}}$. Let $cl'$ be a client such that $(t_{cl}^{m}, t_{cl}^{m}) \in AR_{X_{\tau}}$. If $cl' \neq cl$, then it must be the case that $t_{cl}^{m} \in T_{X_{\tau}}$, and therefore $(t_{cl}^{m}, t_{cl}^{m}) \in AR_{X_{\tau}}$. By the inductive hypothesis, it follows that $i < j$. If $cl' = cl$, then by definition of $AR_{X_{\tau}}$, it must be $i \neq m$. If $j \neq m$, we can proceed as in the previous case to prove that $i < j$. If $j = m$, then note that $t_{cl}^{m} \in T_{X_{\tau}}$ only if $t_{cl}^{m} \in K_{X_{\tau}}$. Because $t_{cl}^{m} \in nextTx(t(cl, K_{X_{\tau}}))$, then we have that $i < m$, as we wanted to prove.

Next, we prove that $X_{\tau}$ satisfies the last write wins policy. Let $t' \in T_{X_{\tau}}$, and suppose that $(i, k, v) \in X_{\tau}, t'$.

- If $t' \neq t$, then we have that $t \in T_{X_{\tau}}$. We also have that $VIS_{X_{\tau}}^{-1}(t') = VIS_{X_{\tau}}^{-1}(t')$, $AR_{X_{\tau}}^{-1}(t') = AR_{X_{\tau}}^{-1}(t')$; finally, for any $t'' \in T_{X_{\tau}}$, $(i, k, v) \in X_{\tau}, t''$ if and only if $(i, k, v) \in X_{\tau}, t''$. Therefore, let $t_{r} = max_{AR_{X_{\tau}}} (VIS_{X_{\tau}}^{-1}(t') \cap \{t' | (i, k, -) \in X_{\tau}, t''\})$. We have that $t_{r} = max_{AR_{X_{\tau}}} (VIS_{X_{\tau}}^{-1}(t') \cap \{t' | (i, k, -) \in X_{\tau}, t''\})$, and because $X_{\tau}$ satisfies the last write wins resolution policy, then $(i, k, v) \in X_{\tau}, t_{r}$. This also implies that $(i, k, v) \in X_{\tau}, t_{r}$.

- Now, suppose that $t' = t$. Suppose that $(i, k, v) \in X_{\tau}, t'$. By definition, we have that $(i, k, v) \in F$. Recall that $\tau = t' \xrightarrow{(cl, F)} \tau_{ET'} (K, U')$, and $lastConf(\tau') = (K_{X_{\tau}}, U')$ for some $U'$. That is,

$$(K_{X_{\tau}}, U') \xrightarrow{(cl, F)} (K, U')$$

which in turn implies that $ET' \vdash (K_{X_{\tau}}, U'(cl)) \vdash F : (K, U(cl))$. Let then $r = max \{ i \mid i \in U'(cl)(k) \}$. By definition of execution test, and because $(i, k, v) \in F$, then it must be the case that $K_{X_{\tau}}(i, k, v) = (v, t'')$, for some $t''$.

We now prove that $t'' = max_{AR_{X_{\tau}}} (VIS_{X_{\tau}}^{-1}(t') \cap \{t' | (i, k, -) \in X_{\tau}, t''\})$. First we have

$$VIS_{X_{\tau}}^{-1}(t') = Tx(K_{X_{\tau}}, U'(cl)) = \{w(K_{X_{\tau}},{i'}) \mid k' \in KEY \land i \in U'(cl)(k')\}$$

Note that $r \in U'(cl)(k)$, and $t'' = w(K_{X_{\tau}}, (k, r))$. Therefore, $t'' \in VIS_{X_{\tau}}^{-1}(t')$. Because $K = update(K_{X_{\tau}}, U'(cl), F, t)$, it must be the case that $w(K_{X_{\tau}}(k, r)) = t''$. Also, because $w(K_{X_{\tau}}(k, r)) = t''$, then
\[(w, k, -) \in X', t'\], or equivalently \((w, k, -) \in T_X(t')\). We have already proved that VIS\[X\] is irreflexive, hence it must be the case that \(t' \neq t\). In particular, because \(X' = \text{extend}(X, t, -,-)\), then we have that \(T_X(t') = T_X[t \mapsto F](t') = T_X(t'')\), hence \((w, k, -) \in T_X(t'').\) Equivalently, \((w, k, -) \in X', t''\). We have proved that \(t' \in \text{VIS}_{X'}(t)\), and \((w, k, -) \in X, t''\).

Now let \(t''\) be such that \(t''' \in \text{VIS}_{X'}(t)\), and \((w, k, -) \in X, t''\). Note that \(t''' \neq t\) because VIS\[X\] is irreflexive.

We show that either \(t''' = t''\), or \(t''' \notin X, t''\). Because \(t''' \in \text{VIS}_{X'}(t)\), then there exists a key \(k'\) and an index \(i \in U'(cl)\) such that \(w(K_{X'}, (k', i)) = t'''\). Because \((w, k, -) \in X, t''\), and because \(t''' \neq t\), then \((w, k, -) \in X, t''\), and therefore there exists an index \(j\) such that \(w(K_{X}, (k, j)) = t'''\). We have that \(w(K_{X}, (k, j)) = w(K_{X'}, (k', i))\), and \(i \in U'(cl)\). By Eq. (atomic), it must be \(j \in U'(cl)\). Note that \(r = \max\{i \mid i \in U'(cl)\}\), hence we have that \(j \leq r\). If \(j = r\), then \(t''' = t''\) and there is nothing left to prove. If \(j < r\), then we have that \((t''', t'') \in \text{AR}_X\), and therefore \((t'''', t''') \in \text{AR}_X\).

Finally, we need to prove that \(K = K_{X'}\). Recall \(K = \text{update}(K_{X'}, U'(cl), F, t), X'\) = extend\((X, t, U'(cl), F)\), and \(X_{t'} = \text{update}(X', t, X_{t'} X', U'(cl), F)\). The result follows then from Prop. E.6

**Proposition E.6 (extend matching update).** Given an abstract execution \(X\), a set of transactions \(T \subseteq T_X\), a transaction \(t \notin T_X\), and a fingerprint \(F \subseteq P\) (Ops), if the new abstract execution \(X' = \text{extend}(X, T, t, F)\), and the view \(u = \text{getView}(K_{X'}, T)\), then \(\text{update}(K_{X'}, u, F, t) = K_{X'}\).

**Proof.** Let \(G = G_{\text{update}(K_{X'}, u, F, t)}\), \(G' = \text{GraphOf}(X')\). Note that \(K_{X'}\) is the unique kv-store such that \(G_{K_{X'}} = \text{GraphOf}(X')\). It suffices to prove that \(G = G'\). Because the function \(G\) is injective, it follows that update\((K_{X'}, u, F, t) = K_{X'}\), as we wanted to prove.

The proof is a consequence of Lemma E.1 and Lemma E.2 Consider the dependency graph \(G_{K_{X}}\). Recall that \(K_{X}\) is the unique kv-store such that \(G_{K_{X}} = \text{GraphOf}(X)\). We prove that \(G_{K_{X}} = G'_{K_{X}}\). Assume that \(G_{K_{X}} = G'_{K_{X}}\) from the last two it follows that \(\text{RW}_{G_{K_{X}}} = \text{RW}_{G_{K_{X}}}\).

- It is easy to see \(G_{K_{X}} = G'_{K_{X}}\).
- \(\text{WR}_{G_{K_{X}}} = \text{WR}_{G_{K_{X}}}\). Let \(K = K_{X}\). Suppose that \(t' = \text{WR}_{G_{K}}(k)\) \(= \text{WR}_{G_{K}}(k)\) \(\to t''\) for some \(t''\). By Lemma E.2, we have that either \(t' = \text{WR}_{G_{K}}(k)\) \(\to t''\), or \(t'' = t, (x, k, -) \in F, t' = \max_{\text{WW}_{G_{K}}(k)}\{w(k, i) \mid i \in u(k)\}\).
- If \(t' = \text{WR}_{G_{K}}(k)\) \(\to t''\), then because \(G_{K_{X}} = \text{GraphOf}(X)\), we have that \(t' = \text{max}_{\text{WW}_{G_{K}}(k)}(T) \cap \{t'' \mid (x, k, -) \in X, t''\}\).
- If \(t'' = t, (x, k, -) \in F, t' = \max_{\text{WW}_{G_{K}}(k)}\{w(K_{X}, (x, k, -)) \mid i \in u(k)\}\), then we also have that \(t' = \max_{\text{WW}_{G_{K}}(k)}(T) \cap \{t'' \mid (x, k, -) \in X, t''\}\).

Again, it follows from Lemma E.1 that \(t' = \text{WR}_{G_{K}}(k)\) \(\to t''\).
- \(\text{WW}_{G_{K}} = \text{WW}_{G_{K}}\). The \(\text{WW}_{G_{K}} = \text{WW}_{G_{K}}\) follows the similar reasons as \(\text{WR}_{G_{K}} = \text{WR}_{G_{K}}\).

**Lemma E.1 (Graph to abstract execution extension).** Let \(X\) be an abstract execution, \(t \notin T_X \cup \{t_0\}\) be a transaction identifier \(T_X\), and \(F \subseteq T_X\). Let \(T \subseteq T_X\) be a set of transaction identifiers. Let \(G = \text{GraphOf}(X)\), \(G' = \text{GraphOf}(\text{extend}(X, t, T, F))\). We have the following:

1. For any \(t' \in T_{G_{K'}}\), either \(t' \in T_{G}\) and \(T_{G}(t') = T_{G}(t')\), or \(t' = t\) and \(T_{G}(t') = F\).
2. \(t' = \text{WR}_{G_{K}}(k)\) \(\to t''\) if and only if \(t' = \text{max}_{\text{WW}_{G_{K}}(k)}(T) \cap \{t'' \mid (x, k, -) \in X, t''\}\).
3. \(t' = \text{WW}_{G_{K}}(k)\) \(\to t''\) if and only if \(t' = \text{max}_{\text{WW}_{G_{K}}(k)}(T) \cap \{t'' \mid (x, k, -) \in X, t''\}\).

**Proof.** Fix a key \(k\). Let \(X' = \text{extend}(X, t, T, F)\). Recall that \(G' = \text{GraphOf}(X')\).

1. By definition of extend, and because \(t \notin T_X\), we have that \(T_X = T_X \cup \{t\}\). Furthermore, \(T_X(t') = F\), from which it follows that \(T_{G}(t') = F\). For all \(t' \in T_X\), we have that \(T_{G}(t') = \text{GraphOf}(t')\).
2. There are two cases that either the \(t''\) already exists in the dependency graph before, or it is the newly commited transaction.
- Suppose that \(t' = \text{WR}_{G_{K}}(k)\) \(\to t''\) for some \(t', t'' \in T_{G_{K'}}\). By definition, \((x, k, -) \in X, t''\), and \(t' = \text{max}_{\text{WW}_{G_{K}}(k)}(T) \cap \{t'' \mid (x, k, -) \in X, t''\}\). Because \(t'' \in T_{G_{K'}}\), it follows that \(t'' \neq t\). By definition, \(\text{VIS}_{X'}^{-1}(t'') = \text{VIS}_{X'}^{-1}(t'')\). also,
whenever $t_a, t_b \in \text{VIS}^{-1}_X(t)$ we have that $t_a, t_b \in T_X$, and therefore if $t_a \xrightarrow{\text{AR}x_a} t_b$, then it must be the case that $t_a \xrightarrow{\text{AR}x_a} t_b$; also, $T_X(t_a) = T_X(t_b)$. As a consequence, we have that

$$\max_{\text{AR}x_a} (\text{VIS}^{-1}_X(t) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \}) = \max_{\text{AR}x_a} (\text{VIS}^{-1}_X(t) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \}) = t'$$

and therefore $t' \xrightarrow{\text{WR}G} t$. 

- Suppose now that $(x, k, -) \in F$, and $t' = \max_{\text{WW}(k)G} (T)$. By Definition, $t' = \max_{\text{AR}x_a} (T) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \}$, and $T = \text{VIS}_X^{-1}(t)$. Because $T \subseteq T_X$, we have that for any $t_a, t_b$, if $t_a \xrightarrow{\text{AR}x_a} t_b$, then $t_a \xrightarrow{\text{AR}x_a} t_b$; and $T_X(t_a) = T_X(t_b)$. Therefore,

$$t' = \max_{\text{AR}x_a} (\text{VIS}^{-1}_X(t) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \})$$

from which it follows that $t' \xrightarrow{\text{WR}G} t$.

Now, suppose that $t' \xrightarrow{\text{WR}G} t''$ for some $t, t'' \in T_T = T_X$. We have that $(x, k, -) \in X \cdot t', (w, k, -) \in X \cdot t''$, and $t'' = \max_{\text{AR}x_a} (\text{VIS}^{-1}_X(t) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \})$. We also have that $T_X = T_X \cup \{ t \}$. We perform a case analysis on $t''$.

- If $t'' = t$, then by definition of extend we have that $\text{VIS}_X^{-1}(t) = T$. Note that $T \subseteq T_X$, so that for any $t_a, t_b \in T_X$, we have that $t_a \xrightarrow{\text{AR}x_a} t_b$ if and only if $t_a \xrightarrow{\text{AR}x_a} t_b$, and $(w, k, v) \in X \cdot t_a$ if and only if $(w, k, v) \in X \cdot t_a$. Thus, $t' = \max_{\text{AR}x_a} (T \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \}) = \max_{\text{WW}(k)G} (T)$.

- If $t'' \in T_X$, then it is the case that $t' = \max_{\text{AR}x_a} (\text{VIS}_X^{-1}(t'')) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \}$. Similarly to the case above, we can prove that $\text{VIS}_X^{-1}(t'') = \text{VIS}_X^{-1}(t)$, for any $t_a, t_b \in T_X$, $(w, k, v) \in X \cdot t_a$ implies $(w, k, v) \in X \cdot t_a$, and $t_a \xrightarrow{\text{AR}x_a} t_b$ implies $t_a \xrightarrow{\text{AR}x_a} t_b$, from which it follows that $t' = \max_{\text{AR}x_a} (\text{VIS}_X^{-1}(t'')) \cap \{ t'' \mid (w, k, -) \in X \cdot t'' \}$, and therefore $t' \xrightarrow{\text{WR}G} t''$.

3) Similar to $\text{WR}(k)G$, there are two cases that either the $t''$ already exists in the dependency graph before, or it is the newly committed transaction.

- Suppose that $t' \xrightarrow{\text{WW}G} t''$ for some $t', t'' \in T_X$. Then $(w, k, -) \in X \cdot t', (w, k, -) \in X \cdot t''$, and $t' \xrightarrow{\text{AR}x_a} t''$. By definition of extend, it follows that $t' \xrightarrow{\text{AR}x_a} t''$, and because $t', t'' \in T_X$, hence $t', t'' \neq t$, then $(w, k, -) \in X \cdot t', (w, k, -) \in X \cdot t''$. By definition, we have that $t' \xrightarrow{\text{WW}G} t''$.

- Suppose that $(w, k, -) \in X \cdot t', (w, k, -) \in F$. Because $t' \in T_X$, we have that $t' \neq t$, hence $(w, k, -) \in X \cdot t'$. By definition, $T_X(t') = F$, hence $(w, k, -) \in X \cdot t'$. Finally, the definition of extend ensures that $t' \xrightarrow{\text{AR}x_a} t$. Combining these three facts together, we obtain that $t' \xrightarrow{\text{WW}G} t''$.

Now, suppose that $t' \xrightarrow{\text{WW}G} t''$ for some $t', t'' \in T_X$. Then $t' \xrightarrow{\text{AR}x_a} t''$, $(w, k, -) \in X \cdot t', (w, k, -) \in X \cdot t''$. Recall that $T_G = T_X = T_X \cup \{ t \}$. We perform a case analysis on $t''$.

- If $t'' = t$, then the definition of extend ensures that $t' \xrightarrow{\text{AR}x_a} t$ implies that $t \in T_X$, hence $t' \neq t$. Together with $(w, k, -) \in X \cdot t'$, this leads to $(w, k, -) \in X \cdot t'$.

- If $t'' \in T_X$, then $t'' \neq t$. The definition of extend ensures that $t' \xrightarrow{\text{AR}x_a} t$. This implies that $t', t'' \in T_X$, hence $t', t'' \neq t$, and $T_X(t') = T_X(t')$. This implies that $(w, k, -) \in X \cdot t', (w, k, -) \in X \cdot t''$, and therefore $t' \xrightarrow{\text{WW}G} (k)t''$. 

\[ \square \]

**Lemma E.2** (Graph to kv-store update). Let $K$ be a kv-store, and $u \in \text{VIEW}(K)$. Let $t \notin K$, and $F \subseteq P(\text{Ops})$, and let $K' = \text{update}(K, u, F, t)$. Let $G = G_K$, $G' = G_K'$; then for all $t', t'' \in T_G$ and keys $k$,

- $G \xrightarrow{\text{WR}(k)} t \rightarrow F$,

- $t' \xrightarrow{\text{WR}(k)} t''$ if and only if either $t' \xrightarrow{\text{WR}(k)} t''$, or $(x, k, -) \in F$ and $t' = \max_{\text{WW}(k)G} \{ w(K(k, i)) \mid i \in u(k) \}$,

- $t' \xrightarrow{\text{WW}(k)G} t''$ if and only if either $t' \xrightarrow{\text{WW}(k)G} t''$, or $(w, k, -) \in F$ and $t' = w(K(k, -))$.

**Proof.** Fix $k \in \text{KEY}$. Because $t \notin K$, then $t \notin T_G$, and by definition of update we obtain that $\{ t' \mid t' \in K' \} = \{ t' \mid t' \in K \} \cup \{ t \}$. It follows that $T_G = T_G \cup \{ t \}$.

1) Suppose that $(x, k, v) \in G' t$. By definition, there exists an index $i$ such that $K'(k, i) = (v, - \cup \{ t' \} \cup \{ - \})$. Because $K' = \text{update}(K, u, F, t)$, it is immediate to observe that $K'(k, i) = (v, - \cup \{ t' \} \cup \{ - \})$, and therefore $(x, k, v) \in G' t$. Conversely, note that if $(x, k, v) \in G' t$, then there exists an index $i = 0, \cdots, |K'(k)| - 1$ such that $K'(k, i) = (v, - \cup \{ t' \} \cup \{ - \})$. As
Similarly, if \((w, k, v, t') \in \mathcal{G}\), then there exists an index \(i = 0, \ldots, |K(k)| - 1\) such that \(K(k, i) = (v, t', v)\). It follows that \(K'(k, i) = (v, t', -)\), hence \((w, k, v, t') \in \mathcal{G'}\). Conversely, if \((w, k, v, t') \in \mathcal{G'}\), then there exists an index \(i = 0, \ldots, |K'(k)| - 1\) such that \(K(k, i) = (v, t', -)\). We have two possible cases: either \(i < |K'(k)| - 1\), leading to \(t' \neq t\) and \(K(k, i) = (v, t', -)\), or equivalently \((w, k, v) \notin \mathcal{G}\); or \(i = |K'(k)| - 1\), leading to \(t' = t\), and \(K(k, i) = (v, t, \emptyset)\) for some \(v\) such that \((w, k, v) \in \mathcal{F}\).

Putting together the facts above, we obtain that \(\mathcal{T}_G = \mathcal{T}_G [t \to \mathcal{F}]\), as we wanted to prove.

2) There are two cases that either the \(t''\) already exists in the dependency graph before, or it is the newly committed transaction.

- Suppose that \(t' \xrightarrow{\text{WR}_{G}(k)} t''\) By definition, there exists an index \(i = 0, \ldots, |K(k)| - 1\) such that \(K(k, i) = (-, t', \{t''\} \cup -)\). It is immediate to observe, from the definition of update, that \(K'(k, i) = (-, t', \{t''\} \cup -)\), and therefore \(t' \xrightarrow{\text{WR}_{G'}(k)} t''\).

- Next, suppose that \((w, k, ) \in \mathcal{F}\) and \(t' = \max_{\text{WR}_{G}(k)} \{w(K(k, i)) \mid i \in u(k)\}\). By definition, \(K(k, i) = (-, t', -)\), where \(i = \max(u(k))\). This is because \(t' \xrightarrow{\text{WR}_{G}(k)} t''\) if and only if \(t' = w(K(k, i))\), \(t'' = w(K(k, j))\) for some \(i \neq j\) such that \(j < j\). The definition of update now ensures that \(K'(k, i) = (-, t', \{t''\} \cup -)\), from which it follows that \(t' \xrightarrow{\text{WR}_{G'}(k)} t''\).

Conversely, suppose that \(t' \xrightarrow{\text{WR}_{G'}(k)} t''\). Recall that \(\mathcal{T}_G = \mathcal{T}_G \cup \{t\}\), hence either \(t'' \in \mathcal{T}_G\) or \(t'' = t\).

- If \(t'' = t\), then it must be the case that \(t' = t\). The definition of update ensures that \(K'(k, i) = (-, t', -)\), \(K'(k, j) = (-, t'', -)\), and \(i < j\). The fact that \(i < |K'(k)| - 1\) leads to \((w, k, ) \notin \mathcal{F}\), and consequently \(t') \xrightarrow{\text{WR}_{G'}(k)} t'\).

3) Similar to \(\text{WR}(k)_G\), there are two cases that either the \(t''\) already exists in the dependency graph before, or it is the newly committed transaction.

- Suppose that \(t' \xrightarrow{\text{WW}_{G}(k)} t''\). By definition, there exist two indexes \(i, j\) such that \(K(k, i) = (-, t', -)\), \(K(k, j) = (-, t'', -)\), and \(i < j\). The definition of update ensures that \(K'(k, i) = (-, t', -)\), \(K'(k, j) = (-, t'', -)\), and because \(i < j\) we obtain that \(t' \xrightarrow{\text{WW}_{G'}(k)} t''\).

- Suppose that \((w, k, ) \in \mathcal{F}\). Then \(K'(k, |K(k)|) = (-, t, -)\). Let \(t' \in \mathcal{T}_G\); by definition there exists an index \(i = 0, \ldots, |K(k)|\) such that \(K(k, i) = (-, t', -)\). It follows that \(K'(k, i) = (-, t', -)\), and because \(i < |K'(k)|\), then we have that \(t' \xrightarrow{\text{WW}_{G'}(k)} t\).

Conversely, suppose that \(t' \xrightarrow{\text{WW}_{G'}(k)} t''\). Because \(\mathcal{T}_G = \mathcal{T}_G \cup \{t\}\), we have two possibilities. Either \(t'' = t\), or \(t'' \in \mathcal{T}_G\).

- If \(t'' = t\), then it must be the case that \((w, k, ) \notin \mathcal{G}\), or equivalently there exists an index \(i = 0, \ldots, |K'(k)| - 1\) such that \(K'(k, i) = (-, t, -)\). Because \(t \notin \mathcal{K}\), and for every index \(i = 0, \ldots, |K'(k)| - 1\), \(K(k, i) = (-, t, -)\) \(K(k, i) = (-, t, -)\), then it necessarily has to be \(i = |K'(k)| - 1\). According to the definition of update, this is possible only if \((w, k, ) \notin \mathcal{F}\). Finally, note that because \(t' \xrightarrow{\text{WW}_{G'}(k)} t\), then there exists an index \(j < |K'(k)| - 1\) such that \(K'(k, j) = (-, t', -)\). The fact that \(j < |K'(k, i)| - 1\) From Cor. [B.1] we obtain that \(K(k, j) = (-, t', -)\), or equivalently \(t'' = w(K(k, j))\).

- If \(t'' \in \mathcal{T}_G\), then there exist two indexes \(i, j\) such that \(j < |K'(k, j)| - 1\), \(K(k, j) = (-, t'', -)\), \(i < j\), and \(K(k, i) = (-, t' -)\). It is immediate to observe that \(K(k, i) = (-, t', -)\), \(K(k, j) = (-, t'', -)\), from which \(t' \xrightarrow{\text{WW}_{G'}(k)} t''\) follows.

E.3. Abstract Execution Traces to KV-Store Traces

We show to construct, given an abstract execution \(X\), a set of ET\_traces KVtrace(ET\_\(X\)) in normal form such that for any \(\tau \in \text{KVtrace}(ET_{\tau}, X)\), the trace \(\tau\) satisfies lastConf(\(\tau\)) = (\(K_X, \emptyset\)). We first define the cut(\(X, n\)) function in
Def. E.4 which gives the prefix of the first \( n \) transactions of the abstract execution \( \mathcal{X} \). The \( \text{cut}(\mathcal{X}, n) \) function is very useful for later discussion.

**Definition E.4.** Let \( \mathcal{X} \) be an abstract execution, let \( n = |T_\mathcal{X}| \), and let \( \{t_i\}_{i=1}^n \subseteq T_\mathcal{X} \) be such that \( t_i \xrightarrow{\text{AR}_\mathcal{X}} t_{i+1} \). The \textit{cut} of the first \( n \) transactions from an abstract execution \( \mathcal{X} \) is defined as the follows:

\[
\begin{align*}
\text{cut}(\mathcal{X}, 0) & \triangleq ([], \emptyset, \emptyset) \\
\text{cut}(\mathcal{X}, i + 1) & \triangleq \text{extend} \left( \text{cut}(\mathcal{X}, i), t_{i+1}, \text{VIS}_\mathcal{X}^{-1}(t_{i+1}), \mathcal{F}_X(t_{i+1}) \right)
\end{align*}
\]

**Proposition E.7** (Well-defined cut). For any abstract execution \( \mathcal{X}, \mathcal{X} = \text{cut}(\mathcal{X}, |T_\mathcal{X}|) \).

**Proof.** This is an instantiation of Lemma E.3 by choosing \( i = |T_\mathcal{X}| \).

**Lemma E.3** (Prefix). For any abstract execution \( \mathcal{X} \), and index \( i : i \leq j \leq |T_\mathcal{X}| \), if \( T_\mathcal{X} = \{t_i\}_{i=1}^n \) be such that \( t_i \xrightarrow{\text{AR}_\mathcal{X}} t_{i+1} \), then \( \text{cut}(\mathcal{X}, i) = \mathcal{X}_i \) where

\[
\begin{align*}
\mathcal{F}_X(t) & = \begin{cases} 
\mathcal{F}_X(t) & \text{if } \exists j \leq i, t = t_j \\
\text{undefined} & \text{otherwise}
\end{cases} \\
\text{VIS}_\mathcal{X} = \{ (t, t') \in T_\mathcal{X} : t \xrightarrow{\text{VIS}_\mathcal{X}} t' \} \\
\text{AR}_\mathcal{X} = \{ (t, t') \in T_\mathcal{X} : t \xrightarrow{\text{AR}_\mathcal{X}} t' \}
\end{align*}
\]

**Proof.** Fix an abstract execution \( \mathcal{X} \). We prove by induction on \( i = |T_\mathcal{X}| \).

- Base case: \( i = 0 \). Then \( \mathcal{F}_X = [], \text{VIS}_\mathcal{X} = \emptyset, \text{AR}_\mathcal{X} = \emptyset, \) which leads to \( \mathcal{X}' = \text{cut}(\mathcal{X}, 0) \).
- Inductive case: \( i = i' + 1 \). Assume that \( \text{cut}(\mathcal{X}, i') = \mathcal{X}_{i'} \). We prove the following:
  - \( \mathcal{F}_{\text{cut}(\mathcal{X}, i)} = \mathcal{F}_\mathcal{X} \). By definition,
    \[
    \mathcal{F}_{\text{cut}(\mathcal{X}, i)}(t_i) = \mathcal{F}_\mathcal{X}(t_i) \quad \text{if } \exists j \leq i, t = t_j
    \]
    \[
    \text{VIS}_{\text{cut}(\mathcal{X}, i)} = \text{VIS}_\mathcal{X}_{i'} \cup \{ (t_j, t_{j'}) \in \text{VIS}_\mathcal{X} : j = 1, \ldots, i' \}
    \]
    \[
    \text{AR}_{\text{cut}(\mathcal{X}, i)} = \{ (t_j, t_{j'}) \in \text{AR}_\mathcal{X} : j = 1, \ldots, i' \}
    \]
  - \( \text{VIS}_{\text{cut}(\mathcal{X}, i)} = \text{VIS}_{\mathcal{X}_i} \). Note that, by inductive hypothesis, \( \mathcal{F}_{\text{cut}(\mathcal{X}, i')} = \mathcal{F}_{\mathcal{X}_{i'}} = \{ t_j \}_{j=1}^{i'} \). We have that
    \[
    \text{VIS}_{\text{cut}(\mathcal{X}, i)} = \text{VIS}_{\mathcal{X}_{i'}} \cup \{ (t_j, t_{j'}) \in \text{VIS}_{\mathcal{X}} : j = 1, \ldots, i' \}
    \]
    \[
    = \{ (t_j, t_{j'}) \in \text{VIS}_{\mathcal{X}} : j = 1, \ldots, i' \}
    \]
    \[
    = \text{VIS}_{\mathcal{X}_{i}}
    \]
  - \( \text{AR}_{\text{cut}(\mathcal{X}, i)} = \text{AR}_{\mathcal{X}_{i}} \). It follows the same way as the above.

Let \( \text{CLIENT}(\mathcal{X}) \triangleq \{ cl \mid \exists n, t_{cl}^n \in T_\mathcal{X} \} \). Given an abstract execution \( \mathcal{X} \), a client \( cl \) and an index \( i : 0 \leq i < |T_\mathcal{X}| \), the function \( \text{nextTx}(\mathcal{X}, cl, i) \triangleq \text{min}_{\text{AR}_\mathcal{X}} \{ t_{cl}^i \mid t_{cl}^i \notin \text{cut}(\mathcal{X}, i) \} \). Note that \( \text{nextTx}(\mathcal{X}, cl, i) \) could be undefined. The conversion from abstract execution tests to \( ET \) traces is in Def. E.3.

**Definition E.5.** Given an abstract execution \( \mathcal{X} \) and an index \( i : 0 \leq i < |T_\mathcal{X}| \), the function \( \text{KVtrace}(\text{ET}_\mathcal{T}, \mathcal{X}, i) \) is defined as the smallest set such that:

- \( (K_0, \text{cl} \in \text{CLIENT}(\mathcal{X}), \lambda k. \{ 0 \}) \in \text{KVtrace}(\text{ET}_\mathcal{T}, \mathcal{X}, 0) \),
- suppose that \( \tau \in \text{KVtrace}(\text{ET}_\mathcal{T}, \mathcal{X}, i) \) for some \( i \). Let
  - \( t = \text{min}_{\text{AR}_\mathcal{X}} (T_\mathcal{X} \setminus \text{cut}(\mathcal{X}, i)) \),
  - \( cl, n \) be such that \( t = t_{cl}^n \),
  - \( u = \text{getView}('A', \text{VIS}_\mathcal{X}^{-1}(t_{cl}^n)) \),
  - \( u' = \text{getView}('A', T) \), where \( T \) is an arbitrary subset of \( T_\mathcal{X} \) if \( \text{nextTx}(\mathcal{X}, cl, i + 1) \) is undefined, or is such that \( T \subseteq \{ (\text{AR}_\mathcal{X})^{-1}(t) \cap \text{VIS}_\mathcal{X}^{-1}(\text{nextTx}(cl, i + 1)) \} \),
  - \( \mathcal{F} = \mathcal{F}_X(t) \),
  - \( (K, \mathcal{U}_r) = \text{lastConf}(\tau) \), and
  - \( K = \text{update}(K, u, \mathcal{F}, t) \).

Then

\[
(\tau \xrightarrow{(cl, e)^{\mathcal{F}_{\text{ET}_\mathcal{T}}}} (K, \mathcal{U}_r[cl \mapsto u])) \xrightarrow{(cl, e)^{\mathcal{F}_{\text{ET}_\mathcal{T}}}} (K, \mathcal{U}_r[cl \mapsto u']) \in \text{KVtrace}(\text{ET}_\mathcal{T}, \mathcal{X}, i + 1)
\]

Last, the function \( \text{KVtrace}(\text{ET}_\mathcal{T}, \mathcal{X}) \triangleq \text{KVtrace}(\text{ET}_\mathcal{T}, \mathcal{X}, |T_\mathcal{X}|) \).

Proposition E.8 (Abstract executions to trace ET). Given an abstract execution \( \mathcal{X} \) satisfying RP\_WW, and a trace \( \tau \in KV\text{trace}(\mathcal{E}_\tau, \mathcal{X}) \), then lastConf(\( \tau \)) = (K_{\mathcal{X}}, X, \tau) \). Then the result follows from Prop. E.7.

\[ \mathcal{X} \]
\[ \text{Lemma E.4} \]
\[ \text{Proposition E.8} \]

**Proof.** Let \( \mathcal{X} \) be an abstract execution that satisfies the last write wins policy. Let \( n = |T_\mathcal{X}| \). Fix \( i = 0, \ldots, n \), and let \( \tau \in KV\text{trace}(\mathcal{E}_\tau, \mathcal{X}, i) \). We prove, by induction on \( i \), that \( \tau \in CM(\mathcal{E}_\tau) \), and lastConf(\( \tau \)) = (K_{\mathcal{cut}(\mathcal{X}, i)}, \tau). Then the result follows from Prop. E.7.

- **Base case:** \( i = 0 \). By definition, \( \tau = (K_0, U_0) \), where \( U_0 = \lambda cl \in Client(\mathcal{X}).k. \{0\} \). Clearly, we have that \( \tau \in CM(\mathcal{E}_\tau) \).
- **Inductive case:** \( i = i + 1 \). Let \( t_i = min_{\mathcal{AR}_X}(T_\mathcal{X} \setminus T_{\mathcal{cut}(\mathcal{X}, i)}) \), and suppose that \( t_i = t_i^m \) for some client \( \mathcal{c} \) and index \( m \). Fix \( u = getView(\mathcal{X}, VIS_X^{-1}(t_i)) \), and \( F = T_{\mathcal{X}}(t_i) \). We prove that there exists a trace \( \tau' \in KV\text{trace}(\mathcal{E}_\tau, \mathcal{X}, i) \) and a set \( T \) such that:

1) if nextTx\((cl, \mathcal{X}, i) \) is undefined then \( T \subseteq VIS_X^{-1}(nextTx\((cl, \mathcal{X}, i) \)) \cap (AR_X^{-1})?\((t_i)\)

2) the new trace \( \tau \) such that

\[ \tau = \tau' \frac{(cl, c)}{(K_{\mathcal{X}}, U_{\tau'}[cl \mapsto u])} \frac{(cl, F)}{(K, U_{\tau'}[cl \mapsto u'])} \]

where \((K_{\tau'}, U_{\tau'}) = lastConf(\( \tau' \)) \) and \((K, u, F) = update(K_{\mathcal{cut}(\mathcal{X}, i) \}, u, F, t_i) \) and \( u' = getView(\mathcal{X}, T) \). By inductive hypothesis, we may assume that \( \tau' \in CM(\mathcal{E}_\tau) \), and \( K_{\tau'} = K_{\mathcal{cut}(\mathcal{X}, i)} \). We prove the following facts:

1) \( K = K_{\mathcal{extend}(\mathcal{cut}(\mathcal{X}, i))} \). Because of Prop. E.6 and Prop. E.7 we obtain

\[ K = update(K_{\mathcal{X}, u, F, t_i}) \]
\[ = update(K_{\mathcal{cut}(\mathcal{X}, i)}, getView(\mathcal{X}, VIS_X^{-1}(t_i), T_{\mathcal{X}}(t_i), t_i)) \]
\[ = K_{\mathcal{extend}(\mathcal{cut}(\mathcal{X}, i))} \]

2) \((K_{\tau'}, U_{\tau'}) \frac{(cl, c)}{(K_{\mathcal{X}}, U_{\tau'}[cl \mapsto u])} \frac{(cl, F)}{(K, U_{\tau'}[cl \mapsto u'])} \]

It suffices to prove that \( U_{\tau'}(cl) \subseteq u \) for any key \( k \). By Lemma E.3 we have that \( T_{\mathcal{cut}(\mathcal{X}, i)} = \{ t_j \}_{j=1}^j \), for some \( t_1, \ldots, t_i \) such that whenever \( 1 \leq j < j' \leq i \), then \( t_j \frac{AR_X^{-1}}{T} t_j' \). We consider two possible cases:

- For all \( j : 1 \leq j \leq i \), and \( h \in N \), then \( t_j \neq t_h \). In this case we have that no transition contained in \( \tau' \) has the form \((-, -, \tau) \), from which it is possible to infer that \( U_{\tau'}(cl) = \lambda k. \{0\} \). Because \( u = getView(\mathcal{X}, VIS_X^{-1}(t_i)) \), then by definition we have that \( 0 \in u(k) \) for all keys \( k \in Y \). It follows that \( U_{\tau'}(cl) \subseteq u \).

- There exists an index \( j : 1 \leq j \leq i \) and an integer \( h \in N \) such that \( t_j = t_h \). Without loss of generality, let \( j \) be the largest such index. It follows that the last transition in \( \tau' \) of the form \((-, -, \tau) \), from which it is possible to infer that \( U_{\tau'}(cl) = \lambda k. \{0\} \). Because \( u = getView(\mathcal{X}, VIS_X^{-1}(t_i)) \), then by definition we have that \( 0 \in u(k) \) for all keys \( k \in Y \). It follows that \( U_{\tau'}(cl) \subseteq u \).

3) \((K_{\mathcal{X}}, U_{\tau'}[cl \mapsto u]) \frac{(cl, F)}{(K, U_{\tau'}[cl \mapsto u'])} \]

It suffices to show that \( ET_{\tau} \vdash (K_{\mathcal{X}}, u) \vdash F : (K, u') \). That is, it suffices to show that \( u \in Views(K_{\mathcal{X}}), u' \in Views(K) \), and whenever \( (z, k, v) \in F \), then \( max_{(u(k))}(v, -) \).

The first two facts are a consequence of Lemma E.5 \( K_{\mathcal{X}} = K_{\mathcal{cut}(\mathcal{X}, i)} \) and \( K_{\mathcal{cut}(\mathcal{X}, i)} \). The last one that if \( (x, k, v) \in F \) then \( max_{(u(k))}(v, -) \) follows the fact that \( X \) satisfies the last write wins policy and the fact that \( u = getView(\mathcal{X}, T_1) \).

\[ \square \]

**Lemma E.4 (Monotonic getView).** Let \( \mathcal{X} \) be an abstract execution, and let \( T_1 \subseteq T_2 \subseteq T_{\mathcal{X}} \). Then getView(\( \mathcal{X}, T_1 \)) \subseteq getView(\( \mathcal{X}, T_2 \)).
Proof. Fix \( k \in \text{Key} \). By definition

\[
\text{getView}(\mathcal{X}, T_1)(k) = \{0\} \cup \{ i \mid w(\mathcal{K}_\mathcal{X}(k, i)) \in T_1 \}
\]

then it follows that \( \text{getView}(\mathcal{X}, T_1) \subseteq \text{getView}(\mathcal{X}, T_2) \).

Lemma E.5 (Valid view on cut of abstract execution). Let \( \mathcal{X} \) be an abstract execution, with \( T_\mathcal{X} = \{ t_i \}_{i=1}^n \) for \( n = |T_\mathcal{X}| \), and \( i : 0 \leq i < n \) such that \( t_i \xrightarrow{\text{AR}_\mathcal{X}} t_{i+1} \). Assuming \( T_0 = \emptyset \), and \( T_i \subseteq \text{AR}^{-1}(i) \) for \( i : 0 \leq i \leq n \), then \( \text{getView}(\mathcal{X}, T_i) \in \text{VIEWS}(\mathcal{K}_{\text{cut}(\mathcal{X}, i)}) \).

Proof. We prove by induction on the index \( i \).

- Base case: \( i = 0 \). It follows \( T_0 = \emptyset \), and \( \text{getView}(\mathcal{X}, T_0) = \lambda k \cdot \{0\} \). We also have that \( \mathcal{K}_{\text{cut}(\mathcal{X}, 0)} = \lambda k \cdot \{(v_0, t_0, 0)\} \), hence it is immediate to see that \( \text{getView}(\mathcal{X}, T_0) \in \text{VIEWS}(\mathcal{K}_{\text{cut}(\mathcal{X}, 0)}) \).
- Inductive case: \( i = i' + 1 \). Suppose that for any \( T \subseteq (\text{AR}_\mathcal{X}^{-1})(i') \), then \( \text{getView}(\mathcal{X}, T) \in \text{VIEWS}(\mathcal{K}_{\text{cut}(\mathcal{X}, i)}) \). Let consider the set \( T_i \). Note that, because of Prop. E.6, we have that

\[
\mathcal{K}_{\text{cut}(\mathcal{X}, i)} = \mathcal{K}_{\text{extend}(\text{cut}(\mathcal{X}, i), T_i, \text{VIS}_\mathcal{X}^{-1}(T_i))), T_i} = \text{update}(\mathcal{K}_{\text{cut}(\mathcal{X}, i')}, \text{getView}(\text{VIS}_\mathcal{X}^{-1}(T_i)), T_i)
\]

There are two possibilities:

- \( t_i \notin T_i \), where case \( T_i \subseteq (\text{AR}_\mathcal{X}^{-1})(i') \). From the inductive hypothesis we get \( \text{getView}(\mathcal{X}, T_i) \in \text{VIEWS}(\mathcal{K}_{\text{cut}(\mathcal{X}, i')}) \). Note that \( \mathcal{K}_{\text{cut}(\mathcal{X}, i')} \) only contains the transactions identifiers from \( t_1 \) to \( t_i \); in particular, it does not contain \( t_i \). Because \( \mathcal{K}_{\text{cut}(\mathcal{X}, i)} = \text{update}(\mathcal{K}_{\text{cut}(\mathcal{X}, i')}, T_i, T_i) \), then by Lemma E.5 we have that \( \text{getView}(\mathcal{X}, t_i) \in \text{VIEWS}(\mathcal{K}_{\text{cut}(\mathcal{X}, i)}) \).

- \( t_i \in T_i \). Note that for any key \( k \) such that \( (w, k, -) \notin \mathcal{F}_\mathcal{X}(T_i) \), then \( \text{getView}(\mathcal{X}, T_i)(k) = \text{getView}(\mathcal{X}, T_i \setminus \{ t_i \})(k) \); and for any key \( k \) such that \( (w, k, -) \in \mathcal{F}_\mathcal{X}(T_i) \), then \( \text{getView}(\mathcal{X}, T_i)(k) = \text{getView}(\mathcal{X}, T_i \setminus \{ t_i \})(k) \cap \{ j \mid w(\mathcal{K}_\mathcal{X}(k, i')) = t_j \} \). In the last case, the index \( j \) must be such that \( j < |\mathcal{K}_{\text{cut}(\mathcal{X}, i)}| - 1 \), because we know that \( t_i \in \mathcal{K}_{\text{cut}(\mathcal{X}, i)} \). It follows from this fact and the inductive hypothesis, that \( \text{getView}(\mathcal{X}, T_i) \in \text{VIEWS}(\mathcal{K}_{\text{cut}(\mathcal{X}, i)}) \).

Lemma E.6 (update preserving views). Given a kv-store \( \mathcal{K} \), a transactions \( t \notin \mathcal{K} \), views \( u, u' \in \text{VIEWS}(\mathcal{K}) \), and set of operations \( \mathcal{F} \), then \( u \in \text{update}(\mathcal{K}, u', \mathcal{F}, t) \).

Proof. Immediate from the definition of update. Note that \( t \notin \mathcal{K} \) ensures that \( u \) still satisfies atomic with respect to the new kv-store update \( \text{update}(\mathcal{K}, u', \mathcal{F}, t) \).

E.4. Galois connection

Now we can prove Theorem E.2. The last statement in Theorem E.2 implies that there is a Galois connection between the set of ET-\( \tau \)-traces, and the set of abstract executions that satisfy the last write wins policy. The lower and upper adjoints of this connection are the lifting of the functions absExec(\( \cdot \)) and KVtrace(\( \cdot \)) to sets of ET-\( \tau \)-traces and abstract executions, respectively. However, these two sets are not isomorphic: when converting a set of abstract executions into kv-traces, we abstract away the pairs \( \text{VIS}_\mathcal{K}, t' \) in the visibility relation where \( t \) is a read-only transaction. When converting a ET-\( \tau \)-trace into a set of abstract executions, we (partially) lose the information about the views of clients immediately after it executed a transaction.

Theorem E.2 (Galois connection between kv-store trace and abstract execution). Given a ET-\( \tau \)-trace \( \tau \), there exists a set of abstract executions absExec(\( \sigma \)) such that lastConf(\( \tau \) = \( \{ \mathcal{K} \mid \mathcal{X} \) for any \( \mathcal{K} \in \text{absExec}(\tau) \). Given an abstract execution \( \mathcal{X} \) satisfying the last write wins resolution policy, there exists a set of ET-\( \tau \)-traces KVtrace(\( \mathcal{X} \)) in normal form such that lastConf(\( \tau \) = \( \{ \mathcal{K} \mid \mathcal{X} \) for any \( \tau \in \text{KVtrace}(\mathcal{X}) \).

Proof. It can be derived from Prop. E.5 and Prop. E.8.

Corollary E.1. CM(ET-\( \tau \)) = \{ \mathcal{K} \mid \mathcal{X} \) satisfies RP_{LWW} \}.

Proof. It follows by Theorem E.2.
Appendix F.
The Sound and Complete Constructors of the KV-Store Semantics with Respect to Abstract Executions

In this Section we first define the set of ET-traces generated by a program P. Then we prove correctness our semantics on kv-stores, meaning that if a program P executing under the execution test ET terminates in a state (K, _), then K ∈ CM(ET).

F.1. Traces of Programs under KV-Stores

The P traces(ET, P) is the set of all possible traces generated by the program P starting from the initial configuration (K0, U0).

Definition F.1. Given an execution test ET a program P and a state (K, U, E), the P traces(ET, P, K, U, E) function is defined as the smallest set such that:
- (K, U) ∈ P traces(ET, P, K, U, E)
- if τ ∈ P traces(ET, P′, K′, U′, E′) and ((K, U, E), P) \(\xrightarrow{\text{cl}}\) ET (K′, U′, E′), then τ ∈ P traces(ET, P, K, U, E)
- if τ ∈ P traces(ET, P′, K′, U′, E′) and (K, U, E), P \(\xrightarrow{(\text{cl}, \text{u}, \text{F})}\) (K′, U′, E′, P′), then
  \[
  \left( (K, U) \underset{\text{cl}}{\rightarrow} ET \left( (K, U[cl \mapsto u]) \right) \underset{\text{cl}}{\rightarrow} ET \tau \right) \in P traces(ET, P, K, U, E)
  \]

The set of traces generated by a program P under the execution test ET is then defined as P traces(ET, P) \(\triangleq\) P traces(ET, P, K0, U0, E0), where U0 = \(\lambda cl \in \text{dom}(P).\lambda k.\{0\}\), and E0 = \(\lambda cl \in \text{dom}(P).\lambda a.0\).

Proposition F.1. For any program P and execution test ET, P traces(ET, P) ⊆ CONF(ET) and \(\tau \in P traces(ET, P)\) is in normal form.

Proof. First, by the definition of P traces, it only constructs trace in normal form. It is easy to prove that for any trace \(\tau\) in P traces(ET, P), by induction on the trace length, the trace is also in CONF(ET).

Corollary F.1. If a trace in the following form
\[
(K_0, U_0, E_0, P) \rightarrow_{E T} \cdots \rightarrow_{E T} (K, U, E, \lambda cl \in \text{dom}(P).\text{skip})
\]
then K ∈ CM(ET).

Proof. By the definition of P traces, there exists a corresponding trace \(\tau \in P traces(ET, P)\). By Prop. F.1, such trace \(\tau \in CONF(ET)\), therefore K ∈ CM(ET) by definition of CM(ET).

Similar to \([P]_{(R_P)}\) (Def. D.7), the function \([P]_{E T}\) is defined as the following:
\[
[P]_{E T} = \{ K \mid (K_0, U_0, E_0, P) \xrightarrow{\lambda cl \in \text{dom}(P).\text{skip}} K \}
\]
where E0 = \(\lambda cl \in \text{dom}(P).\lambda x.0\) and Pf = \(\lambda cl \in \text{dom}(P).\text{skip}\).

Proposition F.2. For any program P and execution test ET: \([P]_{E T} = [P]_{E T \cap CM(ET)}\).

Proof. We prove a stronger result that for any program P and execution test ET, P traces(ET, P) = P traces(ET \(\cap\) CONF(ET)). It is easy to see P traces(ET, P) \(\subseteq\) P traces(ET \(\cap\) P). By Prop. F.1 we know P traces(ET, P) \(\subseteq\) CONF(ET). Therefore P traces(ET, P) \(\subseteq\) P traces(ET \(\cap\) CONF(ET)).

Let consider a trace \(\tau\) in P traces(ET \(\cap\) CONF(ET)). By inductions on the length of trace, every step that commits a new transaction must satisfy ET as \(\tau \in CONF(ET)\). It also reduce the program P since \(\tau \in P traces(ET \cap CONF(ET))\). By the definition P traces(ET, P), we can construct the same trace \(\tau\) so that \(\tau \in P traces(ET, P)\).
F.2. Adequate of KV-Store Semantic

Our main aim is to prove that for any program \( P \), the set of kv-stores generated by \( P \) under ET corresponds to all the possible abstract executions that can be obtained by running \( P \) on a database that satisfies the axiomatic definition \( A \). In this sense, we aim to establish that our operational semantics is adequate.

More precisely, suppose that a given execution test ET captures precisely a consistency model defined in the axiomatic style, using a set of axioms \( A \) and a resolution policy RP over abstract executions. That is, for any abstract execution \( X \) that satisfies the axioms \( A \) and the resolution policy RP, then KVtrace(ET,\( X \)) \( \cap \) CM(ET) \( \neq \emptyset \); and for any \( \tau \in \text{CM}(ET) \), there exists an abstract execution \( X' \in \text{absExec}(\tau) \) that satisfies the axioms \( A \) and the resolution policy RP.

To recall, we have already defined what is the set of all possible behaviours that can be produced by a program \( P \) under a given consistency model \( CM \), for which an axiomatic definition (\( RP, A \)) is known in thm:consistency-intersect-anarchic. It in turn requires: (i) the set of all possible behaviours that may be exhibited by a program \( P \), independently of the consistency model, i.e. \([P]_\oplus \) that is defined in Defs. D.6 and D.7, and (ii) defining the set of all possible behaviours that are allowed by a given consistency model \( CM \), i.e. \( CM(RP, A) \) that is defined in Def. D.4. Then the set of all possible behaviours of \( P \) under \( CM \) is obtained by intersecting the two sets above (Theorem D.1). The kv-store semantics is intrinsically not expressive enough to tackle (i). By Cor. E.1, only those kv-store arising from abstract executions satisfying the last write wins resolution policy can be captured in the kv-store framework. Further than Cor. E.1 we now consider the program \( P \). The Prop. F.3 and Prop. F.4 show the connection between reduction steps between the last write win resolution policy (\( \text{RP}_{\text{LWW}}, \emptyset \)) and the most permissive execution test \( ET_\top \).

**Proposition F.3** (Permissive execution test to last write win). Suppose that \((K, U, \mathcal{E}), \mathcal{P} \overset{(cl,u,F)}{\rightarrow}_{ET_\top} \left(K', U', \mathcal{E}', \mathcal{P}'\right)\). Assuming an abstract execution \( \mathcal{X} \) such that \( K_X = K \), and a set of read-only transactions \( T \subseteq T_X \), then there exists an abstract execution \( \mathcal{X}' \) such that \( K_{X'} = K' \), and

\[
\left(\mathcal{X}, \mathcal{E}\right), \mathcal{P} \overset{(cl,t,U(k,u,F)}{\rightarrow}_{ET_\top} \left(\mathcal{X}', \mathcal{E}', \mathcal{P}'\right).
\]

**Proof.** Suppose that \((K, U, \mathcal{E}), \mathcal{P} \overset{(cl,u,F)}{\rightarrow}_{ET_\top} \left(K', U', \mathcal{E}', \mathcal{P}'\right)\). This transition can only be inferred by applying Rule PSINGLETHREAD, meaning that

- \( P(cl) \rightarrow C \) for some command \( C \),
- \( cl \vdash (K, U(cl), \mathcal{E}(cl)), C \overset{(cl,u,F)}{\rightarrow}_{ET} (K', u', s'), C' \) for some \( u', s' \), and
- \( U' = U[cl \rightarrow u'] \), \( \mathcal{E}' = \mathcal{E}[cl \rightarrow s'] \) and \( \mathcal{P}' = \mathcal{P}[cl \rightarrow C] \).

Let \( \mathcal{X} \) be such that \( K_{\mathcal{X}} = K \), and let \( T \subseteq T_X \) be a set of read-only transactions in \( \mathcal{X} \). It suffices to show that there exists an abstract execution \( \mathcal{X}' \) such that \( K_{\mathcal{X}'} = K' \), and

\[
cl \vdash (\mathcal{X}, \mathcal{E}(cl)), C \overset{(cl,t,U(k,u,F)}{\rightarrow}_{ET_\top} (\mathcal{X}', s'), C'.
\]

By the ASINGLETHREAD rule, we obtain

\[
(\mathcal{X}, \mathcal{E}), \mathcal{P} \overset{(cl,t,U(k,u,F)}{\rightarrow}_{ET_\top} (\mathcal{X}', \mathcal{E}', \mathcal{P}').
\]

Now we perform a rule induction on the derivation of the transition

\[
cl \vdash (K, U(cl), \mathcal{E}(cl)), C \overset{(cl,u,F)}{\rightarrow}_{ET_\top} (K', u', ss'), C'.
\]

Base case: PCOMMIT. This implies that

- \( C = \left[U\right] \) for some \( U \), and \( C' = \text{skip} \),
- \( U(cl) \subseteq u \),
- let \( ss = \text{snapshot}(K, u) \); then \( (\mathcal{E}(cl), ss, \emptyset) \rightarrow^* (s', \_ , F) \),
- \( hh' = \text{update}(K, u, t, F) \) for some \( t \in \text{nextTxId}(K, cl) \), and
- \( ET_\top = K, u \triangleright F : u' \).

Choose an arbitrary set of of read-only transactions \( T \subseteq T_X \). We observe that \text{getView}(\mathcal{X}, T \cup T(K, u)) = u \) since \( K_X = K \) and Prop. E.3. We can now apply Prop. E.2 and ensure that \( \text{RP}_{\text{LWW}}(\mathcal{X}, T \cup T(K, u)) = \{ss\} \). Let \( \mathcal{X}' = \text{extend}(\mathcal{X}, t, T \cup T(K, u), F) \). Because \text{getView}(\mathcal{X}, T \cup T(K, u)) = u, \mathcal{K}_X = K, \) then by Prop. E.6 we have that \( K_{\mathcal{X}'} = \text{update}(K, u, t, F) = K' \). To summarise, we have that \( T \cup T(K, u) \subseteq T_X, ss \in \text{RP}_{\text{LWW}}(\mathcal{X}, T \cup T(K, u)) \), \( (\mathcal{E}(cl), ss, \emptyset) \rightarrow^* (s', \_ , F) \) and \( t \in \text{nextTxId}(T_X, cl) \). Now we can apply ACOMMIT and infer

\[
cl \vdash (\mathcal{X}, \mathcal{E}(cl)), \left[U\right] \overset{(cl,t,U(k,u,F)}{\rightarrow}_{ET_\top} (\mathcal{X}', s'), \text{skip}
\]

which is exactly what we wanted to prove.
Base case: PPrimitive,PChoice,PIter,PSeqSkip. These cases are trivial since they do not alter the state of $\mathcal{K}$. Inductive case: PSeq. It is derived by the I.H.

**Proposition F.4** (Last write win to permissive execution test). Suppose that $(\mathcal{X}, \mathcal{E}), P \xrightarrow{(cl,T,F)}_{(RP_{UWW},@)} (\mathcal{X}', \mathcal{E}')$. Then for any $\mathcal{U}$ and $u \in \text{Views}(\mathcal{K})$ such that $u \subseteq \text{getView}(\mathcal{X}, T)$, the following holds:

$$
(K_X, U[cl \mapsto u], \mathcal{E}), P \xrightarrow{(cl, \text{getView}(\mathcal{X}, T), F)}_{ET+} (K_X', U, \mathcal{E}', P').
$$

**Proof.** Suppose that $(\mathcal{X}, \mathcal{E}), P \xrightarrow{(cl,T,F)}_{(RP_{UWW},@)} (\mathcal{X}', \mathcal{E}')$. Fix a function $\mathcal{U}$ from clients in $\text{dom}(P)$ to views in $\text{Views}(K)$, and a view $u \subseteq \text{getView}(\mathcal{X}, T)$. We show that $(K_X, U[cl \mapsto u], \mathcal{E}) \xrightarrow{(cl, \text{getView}(\mathcal{X}, T), F)}_{ET+} (K_X', U, \mathcal{E}', P')$.

Note that the transition $\mathcal{X}, \mathcal{E}, P \xrightarrow{(cl,T,F)}_{(RP_{UWW},@)} (\mathcal{X}', \mathcal{E}'), P'$ can only be inferred using ASINGLETHREAD rule, from which it follows that

$$
cl \vdash (\mathcal{X}, \mathcal{E}(cl)), P(cl) \xrightarrow{(cl,T,F)}_{(RP_{UWW},@)} (\mathcal{X}', s'), \mathcal{C}'
$$

for some $s'$ such that $\mathcal{E}' = \mathcal{E}[cl \mapsto s']$ and $\mathcal{C}'$ such that $P' = P[cl \mapsto \mathcal{C}]$. It suffices to show that

$$
cl \vdash (K_X, U, \mathcal{E}(cl)), P(cl) \xrightarrow{(cl, \text{getView}(K_X, T), F)}_{ET+} (K_X', U(cl), s'), \mathcal{C}'
$$

Then by applying PSINGLETHREAD we obtain

$$
(K_X, U[cl \mapsto u], \mathcal{E}), P \xrightarrow{(cl, \text{getView}(K_X, T), F)}_{ET+} (K_X', U, \mathcal{E}', P').
$$

The rest of the proof is performed by a rule induction on the derivation to infer

$$
cl \vdash (\mathcal{X}, \mathcal{E}(cl)), P(cl) \xrightarrow{(cl,T,F)}_{(RP_{UWW},@)} (\mathcal{X}', s'), \mathcal{C}'
$$

Base case: ACommit. In this case we have that

- $P = [T]$,
- $P' = \text{skip}$,
- $(E(cl), ss, \emptyset), T \rightarrow^* (s', \_ , F), \text{skip}$ for an index $ss \in \text{RP}_{UWW}(\mathcal{X}, T)$, and
- $\mathcal{X}' = \text{extend}(\mathcal{X}, t, T, F)$ for some $t \in \text{nextTId}(\mathcal{X}, cl)$.

Furthermore, it is easy to see by induction on the length of the derivation $(E(cl), ss, \emptyset), T \rightarrow^* (s', \_ , F), \text{skip}$, that whenever $(x, k, v) \in F$ then $ss(k) = v$. Note that snapshot($K_X, \text{getView}(\mathcal{X}, T)) = ss$ by Prop. E.2. Also, if $(x, k, v) \in F$ then $ss(k) = v$, which is possible only if $K_X(k, \text{max}_{<}(\text{getView}(\mathcal{X}, T)(k))) = (v, \_ , \_ )$. This ensures that $ET+ \vdash (K_X, \text{getView}(\mathcal{X}, T)) \triangleright F : U(cl)$. We can now combine all the facts above to apply rule PCommit

$$
cl \vdash (K_X, u, \mathcal{E}(cl)), [T] \xrightarrow{(cl, \text{getView}(K_X, T), F)}_{ET+} (K'_X, U(cl), s'), \text{skip},
$$

where $K'_X = \text{update}(K_X, t, \text{getView}(\mathcal{X}, T), F)$. Recall that $\mathcal{X}' = \text{extend}(\mathcal{X}, T, t, F)$. Therefore by Prop. E.6 we have that $K' = K_X$, which concludes the proof of this case.

Base case: APrimitive,AChoice,AIter,ASeqSkip. These cases are trivial since they do not alter the state of $\mathcal{X}$. Inductive case: ASeq. It is derived by the I.H. 

**Corollary F.2.** For any program $P$,

$$
[P]_{ET+} = \{ \gamma \in [P]_{(\text{RP}_{UWW},@)} \}
$$

**Proof.** It can be derived by Prop. F.4 and Prop. F.3.

**F.3. Soundness and Completeness Constructor**

We now show how all the results illustrated so far can be put together to show that the kv-store operational semantics is sound and complete with respect to abstract execution operational semantics.
F.3.1. Soundness. Recall that in the abstract execution operational semantics, a client \( cl \) loses information of the visible transactions immediately after it commits a transaction. Yet such information is indirectly presented when the next transaction from the same client is committed. To define the soundness judgement (Def. F.3), we introduce a notion of invariant (def:invariant-for-clients) to encode constraints on the visible transactions after each commit.

Definition F2 (Invariant for clients). A client-based invariant condition, or simply invariant, is a function \( I : \text{AbsExecs} \times \text{Client} \rightarrow \mathcal{P}(\text{TRANSID}) \) such that for any \( cl \) we have that \( I(X, cl) \subseteq T_X \), and for any \( cl' \) such that \( cl' \neq cl \) we have that \( I(\text{extend}(X, \ell_{cl'}, -, -), cl) = I(X, cl) \).

Definition F3 (Soundness judgement). An execution test \( ET \) is sound with respect to an axiomatic definition \( (\text{RP}_{\text{LWW}}, A) \) if and only if there exists an invariant condition \( I \) such that if assuming that

- a client \( cl \) having an initial view \( u \), commits a transaction \( t \) with a fingerprint \( F \) and updates the view to \( u' \), which is allowed by \( ET \) i.e. \( ET \vdash (K, u) \triangleright F : (K', u') \) where \( K' = \text{update}(K, u, F, t) \),
- a \( X \) such that \( K_X = K \) and \( I(X, cl) \subseteq \text{Tx}(K, u) \),

then there exist a set of read-only transactions \( T_{rd} \) such that

- the view \( u \) satisfies \( A \), i.e. \( \forall A \in A \cdot \{ t' \mid (t', t) \in A(X') \} \subseteq \text{Tx}(K, u) \cup T_{rd} \),
- the invariant is preserved, i.e. \( I(X', cl) = \text{Tx}(K', u') \) for some \( X' \) that \( K' = K_{X'} \).

Theorem F1 (Soundness). If \( ET \) is sound with respect to \( (\text{RP}_{\text{LWW}}, A) \), then

\[ \text{CM}(ET) \subseteq \{ K \mid \exists X \in \text{CM}(\text{RP}_{\text{LWW}}, A) : K_X = K \} \]

Proof. Let \( ET \) be an execution test that is sound with respect to an axiomatic definition \( (\text{RP}_{\text{LWW}}, A) \). Let \( I \) be the invariant that satisfies Def. F.3. Let consider an ET-trace \( \tau \). Because of Prop. E.2 we can assume that \( \tau \) is in normal form. Without lose generality, we can also assume that the trace does not have transitions labelled as \((-\), \emptyset\). Thus we have that the following trace \( \tau \):

\[ \tau = (K_0, U_0) \cdot (c_{d_0}, e) \cdot (K_0, U_0) \cdot (c_{d_1}, F_0) \cdot (K_1, U_1) \cdot (c_{d_2}, e) \cdot \ldots \cdot (c_{d_n}, F_{n-1}) \cdot (K_n, U_n) \]

For any \( i : 0 \leq i \leq n \), let \( \tau_i \) be the prefix of \( \tau \) that contains only the first \( 2i \) transitions. Clearly \( \tau_i \) is a valid ET-trace, and it is also a ET\( \tau \)-trace. By Prop. E.5 any abstract execution \( X_i \in \text{absExec}(\tau_i) \) satisfies the last write wins policy. We show by induction on \( i \) that we can always find an abstract execution \( X_i \in \text{absExec}(\tau_i) \) such that \( X_i \models A \) and \( I(X_i, cl) \subseteq T_{cl,i} \) for any client \( cl \) and set of transactions \( T_{cl,i} = \text{Tx}(X_i, U_i(cl)) \cup T_{rd,i} \), and read-only transactions \( T_{rd} \) in \( X_i \). If so, because \( X_i \) satisfies the last write wins policy, then it must be the case that \( X_i \models (\text{RP}_{\text{LWW}}, A) \). Then by choosing \( i = n \), we will obtain that \( X_n \models (\text{RP}_{\text{LWW}}, A) \). Last, by Prop. E.5 \( K_{X_n} = K_n \), and there is nothing left to prove. Now let prove such \( X_i \in \text{absExec}(\tau_i) \) always exists.

Base case: \( i = 0 \). Let \( X_0 \) be the only abstract execution included in \( \text{absExec}(\tau_0) \), that is \( X_0 = ([], \emptyset, \emptyset) \). For any \( A \in A \), it must be the case that \( A(X_0) \subseteq T_{X_0} = \emptyset \), hence the inequation \( A(X_0) \subseteq \text{VIS}_X \) trivially satisfies. Furthermore, for the client invariant \( I \) we also require that \( I(X_0, -) \subseteq T_{X_0} = \emptyset \); for any client \( cl \) we can choose \( T_{cl,0} = \text{Tx}(U_0(cl)) \cup \emptyset = \emptyset \). Therefore \( I(X_0, cl) = \emptyset \cup \emptyset = T_{cl,0} \).

Inductive case: \( i + 1 \) where \( i < n \). By the inductive hypothesis, there exists an abstract execution \( X_i \) such that

- \( X_i \models A \) for all \( A \in A \), and
- \( I(X_i, cl) \subseteq T_{cl,i} \) for any client \( cl \) and set of transactions \( T_{cl,i} = \text{Tx}(X_i, U_i(cl)) \).

We have two transitions to check, the view shift and committing a transaction.

- the view shift transition \( (K_i, U_i) \cdot (c_{d_i}, e) \cdot (K_i, U_i') \). By definition, it must be the case that \( U_i'(cl) = U_i(cl) \cup u_i' \) for some \( u_i' \) such that \( U_i(cl) \subseteq u_i' \). Let \( (T_{cl,i}') = \text{Tx}(K_i, U_i'(cl)) \); then we have \( T_{cl,i} = \text{Tx}(K_i, U_i(cl)) \subseteq \text{Tx}(K_i, U_i'(cl)) = (T_{cl,i}') \). As a consequence, \( I(X_i, cl) \subseteq T_{cl,i} \subseteq (T_{cl,i}') \).
- the commit transition \( (K_i, U_i') \cdot (c_{d,i}, F_i) \cdot (K_{i+1}, U_{i+1}) \). A necessary condition for this transition to appear in \( \tau \) is that \( ET \vdash (K_i, U_i(cl)) \triangleright F_i : (K_{i+1}, U_{i+1}(cl)) \). Because \( I \) is the invariant to derive that \( ET \) is sound with respect to \( A \), and because \( I(X_i, cl) \subseteq (T_{cl,i}') \), then by Def. F.3 we have the following:
  - there exists a set of read-only transactions \( T_{rd} \) such that
    \[ \{ t' \mid (t', t_{(cl,i)}) \in A(K_{i+1}) \} \subseteq (T_{cl,i}') \cup T_{rd} \]
  where \( t_{(cl,i)} \in \text{nextTxId}(K_i(cl)) \) and \( X_{i+1} = \text{extend}(X_i, t_{(cl,i)}, (T_{cl,i}') \cup T_{rd}, F_i) \).

Because \( X_i \in \text{absExec}(\tau_i) \), by definition of \( \text{absExec}(\tau) \) we have that \( X_{i+1} \in \text{absExec}(\tau) \) (under the assumption that \( F_i \neq \emptyset \)), and because lastConf\( (\tau_{i+1}) = (K_{i+1}, -) \), then \( K_{X_{i+1}} = K_{i+1} \).

Now we need to check if \( X_{i+1} \models A \) and the invariant \( I \) is preserved.
− \( A(X_{i+1}) \subseteq VIS^{X_{i+1}} \) for any \( A \in A \). Fix \( A \in A \) and \((t', t) \in A(X_{i+1})\). Because \( X_{i+1} = extend(X_i, t_{cl(i)}, (T_t^i)' \cup T_{rd}, F_i) \), we distinguish between two cases.
  * If \( t = t_{cl(i)} \), then it must be the case that \( t' \in (T_t^i)' \cup T_{rd} \), and by definition of extend(−) we have that \((t', t_{cl(i)}) \in VIS_{X_i+1}\).
  * If \( t \neq t_{cl(i)} \), then we have that \( t', t' \in T_{X_i} \). Because \( X_i \) and \( X_{i+1} \) agree on \( T_{X_i} \), then \((t', t) \in A(X_i)\). Because \( X_i \models A \), then \((t', t) \in VIS_{X_i}\). By definition of extend, it follows that \((t', t) \in VIS_{X_{i+1}}\).

− Finally, we show the invariant is preserved. Fix a client \( cl' \).
  * If \( cl' = cl \), then we have already proved that \( I(X_{i+1}, cl) \subseteq T_{cl'+1} \).
  * if \( cl' \neq cl \), then note that \( U_i(cl') = U_i(cl') = U_i+1(cl') \), and in particular \((T_t^{cl'}) = Tx(X_i, U_i(cl')) = Tx(X_i, U_i+1(cl'))\). By the inductive hypothesis we know that \( I(X_i, cl) \subseteq T_{cl'+1} \), and by the definition of invariant, we have \( I(X_{i+1}, cl) \subseteq T_{cl'+1} \). □

**Corollary F.3.** If \( ET \) is sound with respect to \((RP_{LWW}, A)\), then for any program \( P \), \( \{ P \}_{ET} \subseteq \{ K_\chi \mid \chi \in \{ P \}_{RP_{LWW}, A} \} \).

**Proof.**

\[
\begin{align*}
\{ P \}_{ET} & \xrightarrow{\text{Prop. F.3}} \{ P \}_{ET} \cap \text{CM(ET)} \\
\text{Corollary F.3} & \subseteq \{ K_\chi \mid \chi \text{ satisfies } RP_{LWW} \} \cap \text{CM(ET)} \\
\text{Theorem F.3} & \subseteq \{ K_\chi \mid \chi \text{ satisfies } RP_{LWW} \land \chi \in \text{CM(RP_{LWW}, A)} \} \\
\text{Theorem F.3} & = \{ K_\chi \mid \chi \in \{ P \}_{RP_{LWW}, A} \} 
\end{align*}
\]

□

**F.3.2. Completeness.** The Completeness judgement is in Def. F.4. Given a transaction \( t_i \) from client \( cl \), it converts the visible transactions \( VIS^{-1}_{X_i}(t_i) \) into view and such view should satisfy the ET. Note that \( \chi \) does not contain precise information about final view after update, yet the visible transactions of the immediate next transaction from the same client \( cl \) include those information.

**Definition F.4.** An execution test \( ET \) is **complete** with respect to an axiomatic definition \((RP_{LWW}, A)\) if, for any abstract execution \( \chi \in \text{CM(RP_{LWW}, A)} \) and index \( i : 1 \leq i < |T_X| \) such that \( t_i \xrightarrow{AR} t_{i+1} \), there exist an initial view \( u_i \) and a final view \( u_i' \) where

- \( u_i = \text{getView}(\chi, VIS^{-1}_{X_i}(t_i)) \),
- let \( t_i = t_{cl} \) for some \( cl, n \),
  - if the transaction \( t'_i = \text{min}_{SO_X} \left\{ t' \mid t_i \xrightarrow{SO} t' \right\} \) is defined, then \( u' = \text{getView}(\chi, T_i) \) where \( T_i \subseteq (AR_{X_i}^{-1})?t_i \cap VIS^{-1}_{X_i}(t_i) \),
  - otherwise \( u' = \text{getView}(\chi, T_i) \) where \( T_i \subseteq (AR_{X_i}^{-1})?t_i \),
- \( ET \vdash (K_{cut(X,i-1)}, u_i) \triangleright T_{X}(t_i) : (K_{cut(X,i)}, u_i') \).

**Theorem F.2.** Let \( ET \) be an execution test that is complete with respect to an axiomatic definition \((RP_{LWW}, A)\). Then \( \text{CM(RP_{LWW}, A)} \subseteq \text{CM(ET)} \).

**Proof.** Fix an abstract execution \( \chi \in \text{CM(RP_{LWW}, A)} \). For any \( i : 1 \leq i < |T_X| \), suppose that \( t_i \) is the \( i \)-th transaction follows the arbitrary order, i.e. \( t_i \xrightarrow{AR_X} t_{i+1} \) and let \( cl \) be the client of the \( i \)-th step, i.e. \( t_i = t_{cl} \). Because \( ET \) is complete with respect to \((RP_{LWW}, A)\), for any step \( i \) we can find an initial views \( u_i \) and a final view \( u_i' \) such that

- \( u_i = \text{getView}(\chi, VIS^{-1}_{X_i}(t_i)) \),
- there exists a set of transactions \( T_i \) such that \( \text{getView}(\chi, T_i) = u_i \), and either \( \text{min}_{SO_X} \left\{ t' \mid t_i \xrightarrow{SO} t' \right\} \) is defined and \( T_i \subseteq (AR_{X_i}^{-1})?t_i \cap VIS^{-1}_{X_i}(t_i) \), or \( T_i \subseteq (AR_{X_i}^{-1})?t_i \),
- \( ET \vdash (K_{cut(X,i-1)}, u_i) \triangleright T_{X}(t_i) : (K_{cut(X,i)}, u_i') \).

Given above, let \( K_i = \text{cut}(X,i) \) and \( F_i = T_{X}(t_i) \). Define the views for clients as

\[
U_0 = \lambda k.(v_0, t_0, \emptyset) \quad K_i = \text{update}(K_{i-1}, u_i, F_i, t_i)
\]

and the ke-stores as

\[
K_0 = \lambda k.(v_0, t_0, \emptyset) \quad K_i = \text{update}(K_{i-1}, u_i, F_i, t_i)
\]
Now by Prop. E.8 we have that the following sequence of $ET_T$-reductions
\[
(K_0, U_0) \xrightarrow{(cl_1, \epsilon)} (K_0, U_0') \xrightarrow{(cl_1, F_1)} (K_1, U_1) \xrightarrow{(cl_2, \epsilon)} \cdots \xrightarrow{(cl_n, F_n)} (K_n, U_n)
\]
Note that $K_i = K_{\text{cut}(x_i)}$. Because $ET \vdash (K_{\text{cut}(x_{i-1}), u_i') \cup F_i : (K_i, u_i))$, or equivalently $ET \vdash (K_{\text{cut}(x_{i-1}), U_{i-1}(cl_i)} \cup F_i : (K_{\text{cut}(x_{i-1}), U_i(cl_i)))$, therefore
\[
(K_0, U_0) \xrightarrow{(cl_1, \epsilon)} (K_0, U_0') \xrightarrow{(cl_1, F_1)} (K_1, U_1) \xrightarrow{(cl_2, \epsilon)} \cdots \xrightarrow{(cl_n, F_n)} (K_n, U_n)
\]
It follows that $K_n \in \text{CM}(ET)$ then $K_n = K_{\text{cut}(x, n)} = K_x$, and there is nothing left to prove. $\square$

**Corollary F.4.** If $ET$ is complete with respect to $(\text{RP}_{\text{LWW}}, \mathcal{A})$, then for any program $\mathcal{P}$, $\{K_\mathcal{A} \mid \mathcal{A} \in [\mathcal{P}](\text{RP}_{\text{LWW}}, \mathcal{A})\} \subseteq [\mathcal{P}]_{ET}$.

**Proof.**
- \{K_\mathcal{A} \mid \mathcal{A} \text{ satisfies } \text{RP}_{\text{LWW}} \wedge \mathcal{A} \in \text{CM}(\text{RP}_{\text{LWW}}, \mathcal{A})\} \subseteq \{K_\mathcal{A} \mid \mathcal{A} \text{ satisfies } \text{RP}_{\text{LWW}} \} \cap \text{CM}(ET)$
- \{K_\mathcal{A} \mid \mathcal{A} \text{ satisfies } \text{RP}_{\text{LWW}} \} \subseteq [\mathcal{P}]_{ET} \cap \text{CM}(ET)
- \text{Cor. E.8}
- \text{Prop. F.2}

$\square$

**Appendix G.**
The Soundness and Completeness of Execution Tests

We now show using Defs. F.3 and F.4 to prove the soundness and completeness of execution tests with respect to axiomatic definitions. It is sufficient to match these two definition, then by Corrs. F.3 and F.4 we have $\text{CM}(ET) = \{K_\mathcal{A} \mid \mathcal{A} \in \text{CM}(\text{RP}_{\text{LWW}}, \mathcal{A})\}$.

**G.1. Monotonic Read MR**

The execution test $ET_\text{MR}$ is sound with respect to the axiomatic definition $(\text{RP}_{\text{LWW}}, \{\lambda \mathcal{A}. \text{VIS}_\mathcal{A} ; \text{SO}_\mathcal{A}\})$. We choose an invariant as the following,
\[
I(\mathcal{A}, cl) = \left( \bigcup_{t_n \in T_X \mid n \in \mathbb{N}} \text{VIS}^{-1}_X(t_n) \right) \setminus T_{rd}
\]
where $T_{rd}$ is all the read-only transactions in $\bigcup_{t_n \in T_X \mid n \in \mathbb{N}} \text{VIS}^{-1}_X(t_n)$. Assume a kv-store $K$, an initial and a final view $u, u'$ a fingerprint $F$ such that $ET_{MR} \vdash (K, u) \triangleright F : (K', u')$. Also choose an arbitrary $cl$, a transaction identifier $t \in \text{nextTxD}(K, cl)$, and an abstract execution $\mathcal{A}$ such that $K_\mathcal{A} = K$ and
\[
I(\mathcal{A}, cl) \subseteq \text{Tx}(K, u)
\]
(7.1)

Let $\mathcal{A}' = \text{extend}(\mathcal{A}, t, F, \text{Tx}(K, u) \cup T_{rd})$. We now check if $\mathcal{A}'$ satisfies the axiomatic definition and the invariant is preserved:
- $\{t' \mid (t', t) \in \text{VIS}_\mathcal{A} ; \text{SO}_\mathcal{A}\} \subseteq \text{Tx}(K, u) \cup T_{rd}$. Suppose that $t' \xrightarrow{\text{VIS}_\mathcal{A}} t'' \xrightarrow{\text{SO}_\mathcal{A}} t$ for some $t', t''$. We show that $t' \in I(\mathcal{A}, cl)$ and then Eq. (7.1) ensures that $t' \in \text{Tx}(K, u) \cup T_{rd}$. Suppose $t'' \xrightarrow{\text{SO}_\mathcal{A}} t$, then $t'' = t_n^0$ for some $n \in \mathbb{N}$. Because $t_n^0 \neq t$ and $T_{X'} \setminus T_X = \{t\}$, we also have that $t'' \in X$. By the invariant of $I(\mathcal{A}, cl)$, we have that $\text{VIS}^{-1}_X(t_n^0) \subseteq I(\mathcal{A}, cl)$; because $t' \xrightarrow{\text{VIS}_\mathcal{A}} t''$ and $t'' \neq t$ we have that $t' \xrightarrow{\text{VIS}_\mathcal{A}} t''$ and therefore $t' \in I(\mathcal{A}, cl)$.
- $I(\mathcal{A}', cl) \subseteq \text{Tx}(\mathcal{A}', u') = \text{Tx}(K', u')$. In this case, because $ET_{MR} \vdash (K, u) \triangleright F : (K', u')$, then it must be the case that $u \subseteq u'$. A trivial consequence of this fact is that $\text{Tx}(K, u) \subseteq \text{Tx}(K, u')$. Also, because $\mathcal{A}' = \text{extend}(\mathcal{A}, t, \text{Tx}(K, u) \cup T_{rd})$,
Let \( \mathcal{C} \) be an arbitrary \( K \), and assume a kv-store \( T \) as given by the view after update:

\[
\mathcal{C}(\mathcal{K}, u) = T(\mathcal{K}, u) \cup T_{rd}.
\]

Using all these facts, we obtain

\[
I(X', cl) = \left( \bigcup \{ t_n \in X' \mid n \in \mathbb{N} \} \right) \setminus T_{rd}
= \left( \left( \bigcup \{ t_n \in X' \mid n \in \mathbb{N} \} \right) \setminus T_{rd} \right) \cup \left( \bigg( X' \setminus \cup T_{rd} \right)
= I(X, cl) \cup T(\mathcal{K}, u)
\]

We show that the execution test \( ET_{MR} \) is complete with respect to the axiomatic definition (\( RP_{LWW}, \{ \lambda \mathcal{X}. (VIS_X; SO_X) \} \)). Let \( \mathcal{X} \) be an abstract execution that satisfies the definition \( CM(RP_{LWW}, \{ \lambda \mathcal{X}. (VIS_X; SO_X) \}) \), and consider a transaction \( t \in T_{X} \). Assume \( i \)th transaction \( t_i \) in the arbitrary order, and let \( u_i = getView(\mathcal{X}, VIS_X^{-1}(t_i)) \).

We have two possible cases:

- the transaction \( t_i \) is defined. In this case let \( u'_i = getView(\mathcal{X}, (AR_X^{-1})?((t_i) \cap VIS_X^{-1}(t_i))) \).

Note that \( t_i \in SO_X \), and because \( \mathcal{X} \vdash VIS_X; SO_X \), it follows that \( VIS_X^{-1}(t_i) \subseteq VIS_X^{-1}(t'_i) \). We also have that \( VIS_X^{-1}(t_i) \subseteq (AR_X^{-1})?((t_i)) \) because of the definition of abstract execution. It follows that

\[
VIS_X^{-1}(t_i) \subseteq (AR_X^{-1})?((t_i)) \cap VIS_X^{-1}(t'_i).
\]

Recall that \( u_i = getView(\mathcal{X}, VIS_X^{-1}(t_i)) \), and \( u'_i = getView(\mathcal{X}, (AR_X^{-1})?((t_i) \cap VIS_X^{-1}(t_i))) \). Thus we have that \( u_i \subseteq u'_i \), and therefore \( ET_{MR} \vdash (K_{cut}(X,i), u_i) \vdash X(t_i): (K_{cut}(X,i+1), u'_i) \).

- the transaction \( t_i \) is not defined. In this case, let \( u'_i = getView(\mathcal{X}, (AR_X^{-1})?((t_i))) \). As for the case above, we have that \( u_i \subseteq u'_i \), and therefore \( ET_{MR} \vdash (K_{cut}(X,i), u_i) \vdash X(t_i): (K_{cut}(X,i+1), u'_i) \).

**G.2. Monotonic Write MW**

The execution test \( ET_{MW} \) is sound with respect to the axiomatic definition (\( RP_{LWW}, \{ \lambda \mathcal{X}. (VIS_X; SO_X) \} \)). Pick the invariant as empty set given the fact of no constraint on the view after update:

\[
I(X, cl) = \emptyset.
\]

Assume \( \mathcal{K}, u, u' \) a fingerprint \( F \) such that \( ET_{MW} \vdash (\mathcal{K}, u) \vdash F : (\mathcal{K}', u') \). Also choose an arbitrary \( cl \), a transaction identifier \( t \in nextTk(\mathcal{K}, cl) \), and an abstract execution \( \mathcal{X} \) such that \( \mathcal{K}_X = \mathcal{K} \) and \( I(\mathcal{X}, cl) = \emptyset \subseteq T(\mathcal{K}, u) \). Let \( \mathcal{X}' = extend(X, k, Tx(\mathcal{K}, u) \cup T_{rd}, F) \). Note that since the invariant is empty set, it remains to prove that there exists a set of read-only transactions \( T_{rd} \) such that:

\[
\forall t'. (t', t) \in SO_{X'} \quad VIS_{X'} \Rightarrow t' \in T(\mathcal{K}, u) \cup T_{rd}
\]

Initially we take \( T_{rd} = \emptyset \), and by closing the \( T(\mathcal{K}, u) \) with respect to the relation \( SO_{X'}; VIS_{X'} \), we will add more read-only transactions into the set \( T_{rd} \). Suppose \( (t', t) \in SO_{X'}; VIS_{X'} \), that is \( t' \xrightarrow{SO_{X'}} t'' \xrightarrow{VIS_{X'}} t \). We perform a case analysis on if \( t'' \) has write:

- If the transaction \( t'' \) writes a key. For the new abstract execution \( \mathcal{X}' \), the visible transactions for \( t \) must come from \( T(X(\mathcal{K}, u) \cup T_{rd}) \). It means \( t'' \in T(\mathcal{K}, u) \cup T_{rd} \). Then given that \( t'' \) is not a read-only transaction, we have \( t'' \in T(\mathcal{K}, u) \).

Now there are two cases:

- if \( t'' \) is a read-only transaction, we include \( t'' \in T_{rd} \).
- if \( t'' \) has at least one write, it is easy to see \( t' \in T(X(\mathcal{K}, u) \cup T_{rd}) \) since \( j \in u(k) \land w(\mathcal{K}(k', j)) \Rightarrow i \in u(k') \).

- If the transaction \( t'' \in T_{rd} \) is a read-only transaction, since \( T_{rd} \) is initial empty, there must exist a later transaction \( t''' \) from the same client that writes a key, and such transaction \( t''' \) is included in \( T(\mathcal{K}, u) \):

\[
t' \xrightarrow{SO_{X'}} t'' \xrightarrow{VIS_{X'}} t''' \xrightarrow{VIS_{X'}} t \in T(\mathcal{K}, u)
\]

Since \( SO \) is transitive, therefore \( t' \xrightarrow{SO_{X'}} t'' \xrightarrow{VIS_{X'}} t \), which we have already proven \( t' \in T(X(\mathcal{K}, u) \cup T_{rd}) \) we will include \( t' \) in \( T_{rd} \). Since there are finite transactions from a client in a trace, there must exist a \( T_{rd} \) in the end.
The execution test $ET_{MW}$ is complete with respect to the axiomatic definition $(RP_{LWW}, \{\lambda X : \text{SO}_X \cap \text{VIS}_X\})$. Let $X$ be an abstract execution that satisfies the definition $CM(RP_{LWW}, \{\lambda X : \text{SO}_X \cap \text{VIS}_X\})$, and consider a transaction $t \in T_X$. Assume $i$-th transaction $t_i$ in the arbitrary order, and let view $u_i = \text{getView}(X, \text{VIS}_X^{-1}(t_i))$. We also pick any final view such that $u'_i \subseteq \text{getView}(X, \text{AR}_X^{-1}(t_i))$. It suffices to prove $ET_{MW} \vdash (K_{\text{cut}(X,i-1)}, u'_i) \triangleright F_X(t_i) : (K_{\text{cut}(X,i-1)}, u'_i)$. It means to prove the follows:

$$
\forall j, m, k, k'. j \in u(k) \land \text{w}(K_{\text{cut}(X,i-1)}(k', m)) \xrightarrow{\text{SO}_X} \text{w}(K_{\text{cut}(X,i-1)}(k, j)) \Rightarrow m \in u(k')
$$  \hspace{1cm} (7.2)

Assume $j$ and $k'$ such that $j \in u(k')$, which means $\text{w}(K_{\text{cut}(X,i-1)}(k, j)) \in \text{VIS}_X^{-1}(t_i)$. Now let consider transaction $t$ that commits before $t$ from the same session, i.e. $t \xrightarrow{\text{SO}_X} \text{w}(K_{\text{cut}(X,i-1)}(k, j))$. By the constraint $\lambda X : \text{SO}_X \cap \text{VIS}_X$, the transaction $t \in \text{VIS}_X^{-1}(t_i)$. It means that in the kv-store $K_{\text{cut}(X,i-1)}$ every version written by $t = \text{w}(K_{\text{cut}(X,i-1)}(k', m))$ should be included in the view $m \in u(k')$. Thus we have the proof of Eq. (7.2).

G.3. Read Your Write RYW

The execution test $ET_{RW}$ is sound with respect to the axiomatic definition $(RP_{LWW}, \{\lambda X : \text{SO}_X\})$. We pick an invariant for the $ET_{RW}$ as the following:

$$I(X, cl) = \left(\bigcup \left\{t_{cl}^m \in T_X \mid n \in \mathbb{N}\right\} \setminus \text{T}_{rd}\right)
$$

where $\text{T}_{rd}$ is all the read-only transactions in $\bigcup \left\{t_{cl}^m \in T_X \mid n \in \mathbb{N}\right\}(\text{SO}_X^{-1}(t_{cl}^m))$. Assume a kv-store $K$, an initial and a final view $u, u'$ a fingerprint $F$ such that $ET_{RW} \vdash (K, u) \triangleright F : (K', u')$. Also choose an arbitrary $cl$, a transaction identifier $t_{cl}^m \in \text{nextTxId}(K, cl)$, and an abstract execution $X$ such that $K_X = K$ and $I(X, cl) \subseteq \text{Tx}(K, u)$. Let a new abstract execution $X' = \text{extend}(X, t_{cl}^m, F, \text{Tx}(K, u) \cup \text{T}_{rd})$. We need to prove that $X'$ satisfies the constraint and the invariant is preserved:

- $t \in \text{Tx}(K, u) \cup \text{T}_{rd}$ for all $t$ such that $t \xrightarrow{\text{SO}_X} t_{cl}^m$. Assume a transaction $t$ such that $t \xrightarrow{\text{SO}_X} t_{cl}^m$. It immediately implies that $t = t_{cl}^m$ where $m < n$ and $t_{cl}^m \in X$. Thus we prove that

$$t \in \left(\bigcup \left\{t_{cl}^m \in T_X \mid n \in \mathbb{N}\right\} \setminus \text{T}_{rd}\right)
$$

- $I(X', cl) \subseteq \text{Tx}(K_X', u')$. Let $\text{T}_{rd}' = \text{T}_{rd}$ if the new transaction $t_{cl}^m$ has writes, otherwise $\text{T}_{rd}' = \text{T}_{rd} \cup \left\{t_{cl}^m\right\}$. First we have

$$I(X', cl) = \left(\bigcup \left\{t_{cl}^m \in T_{X'} \mid m \in \mathbb{N}\right\} \setminus \text{T}_{rd}'\right)
$$

Note that $t_{cl}^m$ is the latest transaction committed by the client $cl$. For any transaction $t \in (\text{SO}_X^{-1}(t_{cl}^m)) \setminus \text{T}_{rd}'$ that has write, because execution test requires $z \in u'(k)$ for any key $k$ and index $z$ such that $w(K_X(k, z)) \xrightarrow{\text{SO}_X} t$, then $t \in \text{Tx}(K_X', u')$ as we wanted.

The execution test $ET_{RW}$ is complete with respect to the axiomatic definition $(RP_{LWW}, \{\lambda X : \text{SO}_X\})$. Let $X$ be an abstract execution that satisfies the definition $CM(RP_{LWW}, \{\lambda X : \text{SO}_X\})$. Assume $i$-th transaction $t_i$ in the arbitrary order, and let view $u_i = \text{getView}(X, \text{VIS}_X^{-1}(t_i))$. We construct the final view $u'_i$ depending on whether $t_i$ is the last transaction from the client.

- If the transaction $t'_i = \text{minSO}_X \left\{t' \mid t_i \xrightarrow{\text{SO}_X} t'\right\}$ is defined, then $u'_i = \text{getView}(X, T_i)$ where $T_i \subseteq (\text{AR}_X^{-1}(t_i)) \cap \text{VIS}_X^{-1}(t_i')$. For given the definition $\lambda X : \text{SO}_X$, we know $\text{SO}_X^{-1}(t_i') \subseteq \text{VIS}_X^{-1}(t'_i)$. We pick $T_i = (\text{AR}_X^{-1}(t_i)) \cap \text{SO}_X^{-1}(t'_i) = (\text{SO}_X^{-1}(t'_i))$. To recall $u'_i = \text{getView}(X, T_i)$, therefore $ET_{RW} \vdash (K_{\text{cut}(X,i-1), u_i}) \triangleright F_X(t_i) : (K_{\text{cut}(X,i-1), u'_i})$.

- If there is no other transaction after $t_i$ from the same client, we pick $u'_i = \text{getView}(X, T_i)$ where $T_i = (\text{SO}_X^{-1}(t_i))$ and $T_{cut} \subseteq \text{VIS}_X^{-1}(t_i')$, so $ET_{RW} \vdash (K_{\text{cut}(X,i-1), u_i}) \triangleright F_X(t_i) : (K_{\text{cut}(X,i), u'_i})$. 


G.4. Write Following Read WFR

The write-read relation on $X$ is defined as the following:

$$\text{WR}(X,k) \triangleq \{ (t,t') \mid \exists v. (u,w,v) \in X \land (x,k,v) \in X \land t' \land t = \max_{\text{AR}}(\text{VIS}^{-1}(t')) \}$$

The notation $\text{WR}_X$ is defined as $\text{WR}_X \triangleq \bigcup_{k \in \text{KEY}} \text{WR}(X,k)$. Note that for a kv-store $K$ such that $K = K_X$, by the definition of $K = K_X$, the following holds:

$$\text{WR}_X = \{ (t,t') \mid \exists k,i. K(k,i) = (-,t,t' \cup -) \}$$

Note that such $\text{WR}_X$ coincides with $\text{WR}_G$ and $\text{WR}_K$.

The execution test $\text{ET}_{\text{WFR}}$ is sound with respect to the axiomatic definition ($\lambda X.\text{WR}_X; (SO_X)?; \text{VIS}_X$). We pick the invariant as $I(X,cl) = \emptyset$, given the fact of no constraint on the view after update. Assume a kv-store $K$, an initial and a final view $v,u$ a fingerprint $F$ such that $\text{ET}_{\text{WFR}}(K,u) \triangleright F : (K',u')$. Also choose an arbitrary $cl$, a transaction identifier $t \in \text{nextTxId}(K,cl)$, and an abstract execution $X$ such that $K_X = K$ and $I(X,cl) = \emptyset \subseteq \text{Tx}(K,u)$. Let $X' = \text{extend}(X,\text{Tx}(K(u),F))$. Note that since the invariant is empty set, it remains to prove the following (the read-only transactions set is empty):

$$\forall t'. (t',t) \in \text{WR}(X',k); (SO_X)\Rightarrow t' \in \text{Tx}(K,u)$$

Suppose $(t',t) \in \text{WR}(X',k); (SO_X)$; $\text{VIS}_X$ for some key $k$, that is, $t' \xrightarrow{\text{WR}(X',k)} t'' \xrightarrow{\text{SO}_X} t''' \xrightarrow{\text{VIS}_X} t$ for some transaction $t'''$. It immediately implies that $t''' \in \text{Tx}(K,u)$ by $X' = \text{extend}(X,\text{Tx}(K(u),F))$. Because $t' \xrightarrow{\text{WR}(X',k)} t''$, there exists an index $i$ such that $K(k,i) = (-,t',t'' \cup -)$. By the execution test $\text{ET}_{\text{WFR}}$, we have $i \in u(k)$ then $t' \in \text{Tx}(K(u))$.

The execution test $\text{ET}_{\text{WFR}}$ is complete with respect to the axiomatic definition ($\lambda X.\text{WR}(X',k); (SO_X)?; \text{VIS}_X$). Assume $i$-th transaction $t_i$ in the arbitrary order, and let view $u_i = \text{getView}(X',\text{VIS}_X^{-1}(t_i))$. We also pick any final view such that $u_i' \subseteq \text{getView}(X,\text{AR}_X^{-1}(t_i))$. Note that there is nothing to prove for $u_i'$, so it is sufficient to prove the following:

$$\forall k,k',m,j,t'. j \in u(k) \land t' \in \text{rs}(\text{cut}(X,i-1)(k',m)) \land t \xrightarrow{\text{SO}_X} w(\text{cut}(X,i-1)(k,j)) \Rightarrow m \in u(k')$$

Given a key $k$ and an index $j$ such that $j \in u(k)$, it means that the writer $t$ of the version $K(k,i-1)(k,j)$ is visible, i.e. $t \in \text{VIS}_X^{-1}(t_i)$. Assume some $t'$ such that $(t',t) \in (SO_X)$ and reads a version of some key $K(k',m)$. Therefore, we know the writer of the key $t''' = w(K(k',i))$ has a write-read edge to $t'$, i.e. $t''' \xrightarrow{\text{WR}_X} t'$. By the constraint on abstract execution $X$, we know $t''' \in \text{VIS}_X^{-1}(t_i)$, which means $m \in u(k')$ by the definition of $\text{getView}$.

G.5. Causal Consistency CC

The wildly used definition on abstract executions for causal consistency is that $\text{VIS}$ is transitive. Yet it is for the sake of elegant definition, while there is a minimum visibility relation given by $(\text{WR}_X \cup \text{SO}_X)^+; \text{VIS}_X \subseteq \text{VIS}_X$ (Lemma G.1).

**Lemma G.1.** For any abstract execution $X$ under last-write-win, if it satisfies the following:

$$(\text{WR}_X \cup \text{SO}_X)^+; \text{VIS}_X \subseteq \text{VIS}_X \quad \text{SO}_X \subseteq \text{VIS}_X$$

There exists a new abstract execution $X'$ where $T_X = T_X', \text{AR}_X = T_X'; \text{VIS}_X \subseteq \text{VIS}_X'$ and under last-write-win $\mathcal{T}_X(t) = \mathcal{T}_X'(t)$ for all transactions $t$.

**Proof.** To recall, the write-read relation under a key $\text{WR}(X,k)$ is defined as $\text{WR}(X,k) \triangleq \{ (t,t') \mid \exists v. (u,w,v) \in X \land (x,k,v) \in X \land t' \land t = \max_{\text{AR}}(\text{VIS}^{-1}(t')) \}$. Given an $X$ that satisfies the following

$$(\text{WR}_X \cup \text{SO}_X)^+; \text{VIS}_X \subseteq \text{VIS}_X \quad \text{SO}_X \subseteq \text{VIS}_X$$

we erase some visibility relation for each transaction following the order of arbitration AR until the visibility is transitive. Assume the $i$-th transaction $t_i$ with respect to the arbitration order. Let $R_i$ denote a new visibility for transaction $t_i$ such that $R_{i,2} = \{ t_i \}$ and the visibility relation before (including) $t_i$ is transitive. Let $X_i = \text{cut}(X,i)$ and $\text{VIS}_i = \bigcup_{0 \leq k \leq i} R_i$. For each step, says $i$-th step, we preserve the following:

$$\text{VIS}_i; \text{VIS}_i \subseteq \text{VIS}_i \tag{7.3}$$

$$\forall t. (t,t_i) \in R_i \Rightarrow (t,t_i) \in (\text{WR}_i \cup \text{SO}_i)^+ \tag{7.4}$$

- **Base case:** $i = 1$ and $R_1 = \emptyset$. Assume it is from client $cl$. There is no transaction committed before, so $\text{VIS}_1 = \emptyset$ and $\text{VIS}_1 \subseteq \text{VIS}_1$ as Eq. (7.3).
• Inductive case: \(i\)-th step. Suppose the \((i-1)\)-th step satisfies Eq. (7.3) and Eq. (7.4). Let consider \(i\)-th step and the transaction \(t_i\). Initially we take \(R_i\) as empty set. We first extend \(R_i\) by closing with respect to \(WR_i\) and prove that it does not affect any read from the transaction \(t_i\). Then we will do the same for \(SO_i\).

- \(WR_i\). For any read \((r, k, v) \in t_i\), there must be a transaction \(t_j\) that \(t_j \xrightarrow{WR(X, \lambda, k)} t_i\) and \(j < i\). We include \((t_j, t_i) \in R_i\). Let consider all the visible transactions of \(t_j\). Assume a transaction \(t' \in VIS_{i-1}^{-1}(t_j)\), thus \(t' \in VIS_j^{-1}(t_j)\). It is safe to include \((t', t_i) \in R_i\) without affecting the read result, because those transaction \(t'\) is already visible for \(t_i\) in the abstract execution \(X\): by Eq. (7.4) we know \(R_i \subseteq (WR_j \cup SO_j)^+ \subseteq (WR_X \cup SO_X)^+\), and by the definition of \(WR(X, \lambda, k)\) we know \(WR(X, \lambda, k) \subseteq VIS_X\).

- Given \(SO_X \subseteq VIS_X\), we include \((t_j, t_i)\) for some \(t_j\) such that \(t_j \xrightarrow{SO_X} t_i\). For the similar reason as \(WR\), it is safe to includes all the visible transactions \(t'\) for \(t_j\), i.e. \(t' \in R_j^{-1}\).

By the construction, both Eq. (7.3) and Eq. (7.4) are preserved. Thus we have the proof.

By Lemma 4.1, the execution test \(ET_{\lambda}x_{\lambda}X\) is sound with respect to the axiomatic definition \((RP_{LWW}, \lambda X, (SO_X \cup WR_X)^+, VIS_X, \lambda X, SO_X)\). We pick an invariant for the \(ET_{\lambda}x_{\lambda}X\) as the union of those for \(MR\) and \(RYW\) shown in the following:

\[
I_1(X, cl) = \left( \bigcup \{ t_{cl}^n \in \mathcal{T}_X | n \in \mathbb{N} \} VIS_X^{-1}(t_{cl}^n) \right) \setminus \mathcal{T}_rd
\]

\[
I_2(X, cl) = \left( \bigcup \{ t_{cl}^n \in \mathcal{T}_X | n \in \mathbb{N} \} VIS_X^{-2}(t_{cl}^n) \right) \setminus \mathcal{T}_rd
\]

where \(\mathcal{T}_rd\) is all the read-only transactions included in both \(\left( \bigcup \{ t_{cl}^n \in \mathcal{T}_X | n \in \mathbb{N} \} VIS_X^{-1}(t_{cl}^n) \right)\) and \(\left( \bigcup \{ t_{cl}^n \in \mathcal{T}_X | n \in \mathbb{N} \} VIS_X^{-2}(t_{cl}^n) \right)\). Assume a kv-store \(\mathcal{K}\), an initial and a final view \(u, u'\) a fingerprint \(F\) such that \(ET_{\lambda}x_{\lambda}X = (\mathcal{K}, u) \supset F : u'\). Also choose an arbitrary \(cl\), a transaction identifier \(t_{cl}^n \in nextTxId(\mathcal{K}, cl)\), and an abstract execution \(X\) such that \(\mathcal{K}_X = \mathcal{K}\) and \(I_1(X, cl) \cup I_2(X, cl) \subseteq \mathcal{T}(\mathcal{K}, u)\). Let a new abstract execution \(X' = extend(X, t_{cl}^n, F, Tx(\mathcal{K}, u) \cup \mathcal{T}_rd)\). We are about to prove there exists an extra set of read-only transactions \(\mathcal{T}_rd\) such that:

\[
\forall t. (t, t_{cl}^n) \in SO_X \Rightarrow t \in Tx(\mathcal{K}, u) \cup \mathcal{T}_rd \cup \mathcal{T}_rd' (7.5)
\]

\[
\forall t. (t, t_{cl}^n) \in (SO_X \cup WR_X)^+ \cup VIS_X \Rightarrow t \in Tx(\mathcal{K}, u) \cup \mathcal{T}_rd \cup \mathcal{T}_rd' (7.6)
\]

\[
I_1(X', cl) \cup I_2(X', cl) \subseteq \mathcal{T}(\mathcal{K}(X'), u') (7.7)
\]

The invariant \(I_2\) implies Eq. (7.5) as the same as \(RYW\) in §G.3.

To prove Eq. (7.6), let \(\mathcal{T}_rd' = \emptyset\) initially, and more read-only transactions will be added in \(\mathcal{T}_rd'\) until the Eq. (7.6) holds. Assume transaction \(t\) such that \((t, t_{cl}^n) \in (SO_X \cup WR_X)^+ \cup VIS_X\). That is, there exists some transaction \(t'\) such that \(t \xrightarrow{VIS_X} t' \xrightarrow{WR_X} t_{cl}^n\). We consider two cases for \(t'\): \(t'\) is also visible by previous transactions from the same client; or \(t'\) is a newly visible transaction for the client.

- If \(t'\) is also visible by previous transactions from the same client, it means \(t' \xrightarrow{VIS_X} t_{cl}^m\) for some \(m < n\). The edge already exists before, therefore \(t' \xrightarrow{VIS_X} t_{cl}^m, c \subseteq \mathcal{T}_rd\). Since \(t \xrightarrow{(SO_X \cup WR_X)^+ \cup VIS_X} t'\) and \(SO_X \subseteq VIS_X\), we know \(t \xrightarrow{VIS_X} t_{cl}^m\). Because of \(I_1\) and \(t_{cl}^m \xrightarrow{SO_X} t_{cl}^n\), then \(t \in I_1 \cup \mathcal{T}_rd \subseteq Tx(\mathcal{K}, u) \cup \mathcal{T}_rd\).

- If \(t'\) is a newly visible transaction for the client \(cl\), it suffices to prove \(t \in Tx(\mathcal{K}, u) \cup \mathcal{T}_rd \cup \mathcal{T}_rd'\) for some \(t \xrightarrow{(SO_X \cup WR_X)^+ \cup VIS_X} t_{cl}^n\). Since \(t' \xrightarrow{VIS_X} t_{cl}^m\) so \(t \in Tx(\mathcal{K}, u) \cup \mathcal{T}_rd \cup \mathcal{T}_rd'\). More specifically, \(t'\) is not visible for the client before, we know \(t' \in Tx(\mathcal{K}, u) \cup \mathcal{T}_rd\). We perform case analysis if \(t'\) has write.

* If \(t'\) writes to some keys, i.e. \(t' \in Tx(\mathcal{K}, u)\), because CC satisfies MW and WFR, and by the execution tests for MW and WFR (the proofs follow §G.2 and §G.4), the \(t'\) is either already in \(Tx(\mathcal{K}, u)\), or \(t'\) is a read-only and we include it in \(\mathcal{T}_rd\).

* If \(t'\) is a read-only transaction, given that \(\mathcal{T}_rd'\) initially is empty set, we know there exists a third transaction \(t''\) that writes to some keys and it satisfies \(t \xrightarrow{(SO_X \cup WR_X)^+ \cup VIS_X} t' \xrightarrow{(SO_X \cup WR_X)^+ \cup VIS_X} t'' \xrightarrow{VIS_X} t_{cl}^n\). Since \(t''\) has a write, it means \(t \xrightarrow{(SO_X \cup WR_X)^+ \cup VIS_X} t' \xrightarrow{(SO_X \cup WR_X)^+ \cup VIS_X} t'' \xrightarrow{VIS_X} t_{cl}^n\).

4. For two relation \(R_1, R_2, R_1; R_2 \subseteq R_2 \iff R_1 \cup R_2 \subseteq R_2\)
\[
\text{if } t \xrightarrow{WR,x'} t' \xrightarrow{SO,x'} t'' \xrightarrow{VIS,x'} t''' \xrightarrow{cl}, \text{ this is exactly WFR. Therefore, the } t \text{ is either already in } Tx(K, u), \text{ or } t \text{ is a read-only and we include it in } T_{rd}.
\]

\[
\text{if } t \xrightarrow{SO,x'} t' \xrightarrow{SO,x'} t'' \xrightarrow{VIS,x'} t''' \xrightarrow{cl}, \text{ because SO is transitive, we have } t \xrightarrow{SO,x'} t'' \xrightarrow{VIS,x'} t''' \xrightarrow{cl}. \text{ By previous case we already know } t \in Tx(K, u) \cup T_{rd}.
\]

- Finally the new abstract execution preserves the invariant \( I_1 \) and \( I_2 \) because CC satisfies MW and RW. The proofs are the same as those in \( \S G.1 \) and \( \S G.3 \).

The execution test \( ET_{CC} \) is complete with respect to the axiomatic definition \( (RP_{LWW}, \{ \lambda X. \text{VIS}_X \}; \lambda Y. \text{VIS}_Y, \lambda Y. \text{SO}_Y) \). For MR, since \( \text{VIS}_X; \text{SO}_X \subseteq \text{VIS}_Y; \text{VIS}_Y \subseteq \text{VIS}_X \), the proof is as the same proof as in \( \S G.1 \). For MW, since \( \text{SO}_X; \text{VIS}_X \subseteq \text{VIS}_Y; \text{VIS}_X \subseteq \text{VIS}_X \), the proof is as the same proof as in \( \S G.2 \). For RW, since \( \text{WR}_X; \text{VIS}_X \subseteq \text{VIS}_X; \text{VIS}_Y \subseteq \text{VIS}_X \), the proof is as the same proof as in \( \S G.3 \).

G.6. Update Atomic UA

Given abstract execution \( \mathcal{X} \), we define write-write relation for a key \( k \) as follows:

\[ \text{WW}(\mathcal{X}, k) \triangleq \{(t, t') \mid t \xrightarrow{AR_X} t' \wedge (w, k, -) \in t \wedge (w, k, -) \in t'\} \]

Then, the notation \( \text{WW}_K \triangleq \bigcup_{k \in \text{KEY}} \text{WW}(\mathcal{X}, k) \). Note that for a kv-store \( K \) such that \( K = K_{\mathcal{X}} \), by the definition of \( K = K_{\mathcal{X}} \), the following holds:

\[ \text{WW}_K = \{(t, t') \mid \exists k, i, j. \ t = w(K(k, i)) \wedge t' = w(K(k, j)) \wedge i < j\} \]

Also the \( \text{WW}_K \) coincides with \( \text{WW}_G \) and \( \text{WW}_K \).

The execution test \( ET_{UA} \) is complete with respect to the axiomatic definition \( (RP_{LWW}, \{ \lambda X. \text{WW}_X \}) \). We pick the invariant as \( I(\mathcal{X}, cl) = \emptyset \), given the fact of no constraint on the final view. Assume a kv-store \( K \), an initial and a final view \( u, u' \) a fingerprint \( F \) such that \( ET_{UA} \vdash (K, u) \models F = (K', u') \). Also choose an arbitrary \( cl \), a transaction identifier \( t \in \text{nextTxId}(K, cl) \), and an abstract execution \( \mathcal{X} \) such that \( K_{\mathcal{X}} = K \) and \( I(\mathcal{X}, cl) = \emptyset \subseteq Tx(K, u) \). Let \( \mathcal{X}' = \text{extend}(\mathcal{X}, t, Tx(K, u), F) \). Note that since the invariant is empty set, it remains to prove the following:

\[ \forall t', t \xrightarrow{WW_X} t \Rightarrow t' \in Tx(K, u) \]

Assume a transaction \( t' \) that writes to a key \( k \) as \( t \) such that \( t \xrightarrow{WW_X} t \). Since that \( t' \) is a transaction already existing in \( K \), we have \( w(K(k, i)) = t' \) for some index \( i \). By the execution test of UA, we know \( i \in u(k) \) therefore \( t' \in Tx(K, u) \).

The execution test \( ET_{UA} \) is complete with respect to the axiomatic definition \( (RP_{LWW}, \{ \lambda X. \text{WW}_X \}) \). Assume \( i \)-th transaction \( t_i \) in the arbitrary order, and let view \( u_i = \text{getView}(\mathcal{X}, \text{VIS}_{\mathcal{X}}^{-1}(t_i)) \). We also pick any final view such that \( u_i \subseteq \text{getView}(\mathcal{X}, (\text{AR}_{\mathcal{X}}^{-1})'(t_i)) \). Note that there is nothing to prove for \( u_i \), so it is sufficient to prove the following:

\[ \forall k. (w, k, -) \in \mathcal{F}(t_i) \Rightarrow \forall j : 0 < j < |K_{\text{cut}(\mathcal{X}, i-1)}(k)|, j \in u_i(k) \]

Let consider a key \( k \) that have been overwritten by the transaction \( t_i \). By the constraint of \( X \) that \( \text{WW}_X \subseteq \text{VIS}_X \), for any transaction \( t \) that writes to the same key \( k \) and committed before \( t_i \), they are included in the visible set \( t \in \text{VIS}_{\mathcal{X}}^{-1}(t_i) \). Note that \( t \xrightarrow{WW_X}, t_i \Rightarrow t \xrightarrow{AR_X}, t_i \Rightarrow t \in K_{\text{cut}(\mathcal{X}, i-1)} \). Since that the transaction \( t \) write to the key \( k \), it means \( w(K_{\text{cut}(\mathcal{X}, i-1)}(k, j)) = t \) for some index \( j \). Then by the definition of getView, we have \( j \in u_i(k) \).

G.7. Consistency Prefix CP

Given abstract execution \( \mathcal{X} \), we define read-write read-write relation:

\[ \text{RW}(\mathcal{X}, k) \triangleq \{(t, t') \mid t \xrightarrow{AR_X} t' \wedge (x, k, -) \in t \wedge (w, k, -) \in t'\} \]

It is easy to see \( \text{RW}(\mathcal{X}, k) \) can be derived from \( \text{WW}(\mathcal{X}, k) \) and \( \text{WR}(\mathcal{X}, k) \) as the following:

\[ \text{RW}(\mathcal{X}, k) = \{(t, t') \mid \exists t''. (t'', t) \in \text{WR}(\mathcal{X}, k) \wedge (t'', t') \in \text{WW}(\mathcal{X}, k)\} \]

Then, the notation \( \text{RW}_X \triangleq \bigcup_{k \in \text{KEY}} \text{RW}(\mathcal{X}, k) \). Note that for a kv-store \( K \) such that \( K = K_{\mathcal{X}} \), by the definition of \( K = K_{\mathcal{X}} \), the following holds:

\[ \text{RW}_X = \{(t, t') \mid \exists k, i, j. \ t \in \text{rs}(K(k, i)) \wedge t' = w(K(k, j)) \wedge i < j\} \]

The \( \text{RW}_X \) also coincides with \( \text{RW}_G \) and \( \text{RW}_K \).
An abstract execution $\mathcal{X}$ satisfies consistency prefix (CP), if it satisfies $AR_{\mathcal{X}}; VIS_{\mathcal{X}} \subseteq VIS_{\mathcal{X}}$ and $SO_{\mathcal{X}} \subseteq VIS_{\mathcal{X}}$. Given the definition, there is a corresponding definition in dependency graph by solve the following inequalities:

$$WR \subseteq VIS$$
$$WW \subseteq AR$$
$$VIS \subseteq AR$$
$$VIS: RW \subseteq AR$$
$$AR: AR \subseteq AR$$
$$SO \subseteq VIS$$
$$AR: VIS \subseteq VIS$$

By solving the inequalities the visibility and arbitration relations are:

$$AR \triangleq ((SO \cup WR); RW^? \cup WW \cup R)^+$$
$$VIS \triangleq ((SO \cup WR); RW^? \cup WW \cup R)^+ \ni (SO \cup WR)$$

for some relation $R \subseteq AR$. When $R = \emptyset$, it is the smallest solution therefore the minimum visibility required.

**Lemma G.2.** For any abstract execution $\mathcal{X}$, if it satisfies

$$(SO \cup WR); RW^? \cup WW \cup R)^+ \ni VIS_{\mathcal{X}} \subseteq VIS_{\mathcal{X}}$$
$$SO_{\mathcal{X}} \subseteq VIS_{\mathcal{X}}$$

then there exists a new $\mathcal{X}'$ such that $T_{\mathcal{X}} = T_{\mathcal{X}'}$, under last-write-win $\mathcal{X}'(t) = \mathcal{X}'(t)$ for all transactions $t$, and the relations satisfy the following:

$$AR_{\mathcal{X}'}; VIS_{\mathcal{X}'} \subseteq VIS_{\mathcal{X}'}$$
$$SO_{\mathcal{X}'} \subseteq VIS_{\mathcal{X}'}$$

and vice versa.

**Proof.** Assume abstract execution $\mathcal{X}'$ that satisfies $AR_{\mathcal{X}'}; VIS_{\mathcal{X}'} \subseteq VIS_{\mathcal{X}'}$ and $SO_{\mathcal{X}'} \subseteq VIS_{\mathcal{X}'}$. We already show that:

$$AR_{\mathcal{X}'} = ((SO \cup WR); RW^? \cup WW \cup R)^+$$
$$VIS_{\mathcal{X}'} = ((SO \cup WR); RW^? \cup WW \cup R)^+ \ni (SO \cup WR)$$

for some relation $R \subseteq AR_{\mathcal{X}'}$. If we take $R = \emptyset$, we have the proof for:

$$SO \subseteq VIS_{\mathcal{X}}$$

For another way, we pick the $R$ that extends $((SO \cup WR); RW^? \cup WW \cup R)^+$ to a total order. \hfill $\square$

By Lemma G.2 to prove soundness and completeness of $ET_{CP}$, it is sufficient to use the definition:

$$\{RP_{WWW}, \{\lambda \mathcal{X}. ((SO \cup WR); RW^? \cup WW \cup R)^+ \ni VIS_{\mathcal{X}}, \lambda \mathcal{X}. SO_{\mathcal{X}}\}\}$$

For the soundness, we pick the invariant as the following:

$$I_1(\mathcal{X}, cl) = \left(\bigcup_{\{t^i_{cl} \in T_X \mid i \in \mathbb{N}\}} VIS_{\mathcal{X}}^{-1}(t^i_{cl})\right) \setminus T_{rd}$$
$$I_2(\mathcal{X}, cl) = \left(\bigcup_{\{t^i_{cl} \in T_X \mid i \in \mathbb{N}\}} (SO_{\mathcal{X}}^{-1})(t^i_{cl})\right) \setminus T_{rd}$$

where $T_{rd}$ is all the read-only transactions included in both $\left(\bigcup_{\{t^i_{cl} \in T_X \mid i \in \mathbb{N}\}} VIS_{\mathcal{X}}^{-1}(t^i_{cl})\right)$ and $\left(\bigcup_{\{t^i_{cl} \in T_X \mid i \in \mathbb{N}\}} (SO_{\mathcal{X}}^{-1})(t^i_{cl})\right)$. Assume a key-value store $\mathcal{K}$, an initial and a final view $u, u'$ a fingerprint $\mathcal{F}$ such that $ET_{CP} \vdash (\mathcal{K}, u) \models \mathcal{F} \models (\mathcal{K}', u')$. Also choose an arbitrary $cl$, a transaction identifier $t^i_{cl} \in \text{nextTxId}(\mathcal{K}, cl)$, and an abstract execution $\mathcal{X}'$ such that $\mathcal{K}_X = \mathcal{K}$ and $I_1(\mathcal{X}, cl) \cup I_2(\mathcal{X}, cl) \subseteq T_X(\mathcal{K}, u)$. Let a new abstract execution $\mathcal{X}' = \text{extend}(\mathcal{X}, t^i_{cl}, \mathcal{F}, T_X(\mathcal{K}, u) \cup T_{rd})$. We are about to prove that there exists an extra set of read-only transaction $T_{rd}'$ such that:

$$\forall t. (t, t^n_{cl}) \in SO_{\mathcal{X}'} \Rightarrow t \in T_X(\mathcal{K}, u) \cup T_{rd} \cup T_{rd}' \quad (7.8)$$
$$\forall t. (t, t^n_{cl}) \in ((SO_{\mathcal{X}'} \cup WR_{\mathcal{X}'}) \cup SO_{\mathcal{X}'})^+ \ni VIS_{\mathcal{X}'}$$
$$\Rightarrow t \in T_X(\mathcal{K}, u) \cup T_{rd} \cup T_{rd}'$$
$$I_1(\mathcal{X}', cl) \cup I_2(\mathcal{X}', cl) \subseteq T_X(\mathcal{K}, u') \quad (7.9)$$

- the invariant $I_2$ implies the Eq. (7.8) where the proof is the same as $RYW$ in §G.3
For Eq. (7.9), it is sufficient to prove one step inclusion, i.e.

\[ \forall t \in (\mathcal{K}, u) \setminus \mathcal{T}_d \cup T \cap \mathcal{T}'_d \]

To prove above, let \( \mathcal{T}'_d \) initially be empty set. We will add more read-only transactions until it satisfies Eq. (7.9). Assume a transaction \( t \) such that \( (t, t'_d) \in ((\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{RW}_x)'; \mathcal{WW}_x); \mathcal{V}_x \). There exists a transaction \( t' \) such that \( t (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{RW}_x)'; \mathcal{WW}_x) \mathcal{T}_d \rightarrow t' \mathcal{V}_x \mathcal{T}_d \). It follows \( t' \in (\mathcal{K}, u) \cup \mathcal{T}_d \cup \mathcal{T}'_d \). Note that \( t \) and \( t' \) must exist in the abstract execution \( \mathcal{X} \) before update. There are two cases: \( t' \) writes to at least a key; or \( t' \) is a read-only transaction.

- If \( t' \) writes to at least a key, then \( t' \in (\mathcal{K}, u) \). Now we perform case analysis if \( t \) is a read-only transaction.
  * If \( t \) has write, we prove \( t \in (\mathcal{K}, u) \). Recall the \( \dagger \) is defined as the following:

\[ \dagger \equiv \forall k, k', i, j. \quad i \in u(k) \wedge w(k', j) \rightarrow (\mathcal{SO}_x, \mathcal{WR}_x)'; (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{WW}_x); \mathcal{V}_x \rightarrow w(k, j) \Rightarrow j \in u(k') \]

Since \( \mathcal{WR}_x, \mathcal{WW}_x \) and \( \mathcal{SO}_x \) coincide with \( \mathcal{WR}_x, \mathcal{WW}_x \) and \( \mathcal{X} \) respectively. Also because \( t \) write to at least one key, it is easy to see there exists some version \( k'' \), \( m \) such that \( t = \mathcal{W}(k', m) \) and \( m \in u(k'') \). By definition of \( \mathcal{T}_d \), it follows \( t \in (\mathcal{K}, u) \).

- If \( t \) is a read-only transaction, we add it into \( \mathcal{T}'_d \).

Now assume \( t' \) is a read-only transaction, then either \( t' \in \mathcal{T}'_d \) or \( t' \in \mathcal{T}'_d \). More specifically we have three cases: (i) \( t' \in (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{RW}_x)' \); (ii) \( t' \in (\mathcal{SO}_x, \mathcal{WR}_x)'; (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{WW}_x) \); (iii) \( t' \in \mathcal{T}'_d \).

Assume \( t' \in (\mathcal{SO}_x, \mathcal{WR}_x)'; (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{WW}_x) \). It means \( t' \) is visible for some previous transaction \( t_m \) from the same client \( cl \), i.e.

\[ t (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{RW}_x)' \rightarrow t_m \mathcal{V}_x \mathcal{T}_d \]

Note that all the edges before \( t_m \) must exist in \( \mathcal{X} \). Since \( \mathcal{X} \) satisfies the \( (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{V}_x \), we have \( t \mathcal{V}_x \rightarrow t_m \mathcal{V}_x \) and then \( t \in (\mathcal{SO}_x, \mathcal{WR}_x)' \); (\( \mathcal{SO}_x, \mathcal{WR}_x)' \); \mathcal{WW}_x); \mathcal{V}_x \). By the invariant \( I_1 \), it means \( t \in (\mathcal{SO}_x, \mathcal{WR}_x)' \); \mathcal{WW}_x); \mathcal{V}_x \).

- If \( t' \in (\mathcal{SO}_x, \mathcal{WR}_x)' \); (\( \mathcal{SO}_x, \mathcal{WR}_x)' \); \mathcal{WW}_x); \mathcal{V}_x \), given that \( \mathcal{SO}_x \) is transitive, then either \( t \mathcal{WR}_x \rightarrow t_m \mathcal{WR}_x \) or \( t \mathcal{WR}_x \rightarrow t_m \mathcal{WR}_x \).

- If \( t \mathcal{WR}_x \rightarrow t_m \mathcal{WR}_x \). The WR edge must exists in \( \mathcal{X} \). Because \( \mathcal{WR}_x \subseteq \mathcal{V}_x \), then \( t \mathcal{V}_x \rightarrow t_m \mathcal{V}_x \mathcal{T}_d \).

Note that all the edges before \( t_m \) must exist in \( \mathcal{X} \). Since \( \mathcal{X} \) satisfies the \( (\mathcal{SO}_x, \mathcal{WR}_x)'; \mathcal{V}_x \), we have \( t \mathcal{V}_x \rightarrow t_m \mathcal{V}_x \) and then \( t \in (\mathcal{SO}_x, \mathcal{WR}_x)' \); (\( \mathcal{SO}_x, \mathcal{WR}_x)' \); \mathcal{WW}_x); \mathcal{V}_x \). By the invariant \( I_1 \), it means \( t \in (\mathcal{SO}_x, \mathcal{WR}_x)' \); (\( \mathcal{SO}_x, \mathcal{WR}_x)' \); \mathcal{WW}_x); \mathcal{V}_x \).

- Last, \( t' \in \mathcal{T}'_d \). Since \( \mathcal{T}'_d \) initially is empty set, there exists another write transaction \( t'' \) such that:

\[ t (\mathcal{SO}_x, \mathcal{WR}_x)' \); (\( \mathcal{SO}_x, \mathcal{WR}_x)' \); \mathcal{WW}_x); \mathcal{V}_x \rightarrow t'' \mathcal{V}_x \]

If \( t \) has write, by Eq. (7.11) then \( t \in (\mathcal{K}, u) \). Otherwise if \( t \) is a read only transaction, we add it into \( \mathcal{T}'_d \).

- Since CP satisfies \( \mathcal{RYW} \) and \( \mathcal{MP} \), thus invariants \( I_1 \) and \( I_2 \) are preserved after update.

For completeness, we prove the three parts of the execution test separately.

- Since \( \mathcal{SO}_x \subseteq \mathcal{V}_x \), the proof for \( \mathcal{ET}_\mathcal{RYW} \) is the as in § G.1.

- For any \( \mathcal{V}_x \) satisfies the constraint for CP, by Lemma G.2, it satisfies that

\[ \mathcal{V}_x \equiv (\mathcal{SO}_x, \mathcal{WR}_x); \mathcal{RW}_x \); \mathcal{WW}_x); \mathcal{V}_x \); \mathcal{V}_x \]

for some relation \( R \). It means \( \mathcal{V}_x \); \mathcal{SO}_x \subseteq \mathcal{V}_x \). Therefore it is complete with respect to \( \mathcal{ET}_\mathcal{MR} \).

Let consider the \( \dagger \). Assume \( i \)-th transaction \( t \) in an arbitrary order, and let view \( u_i = \text{getView}(\mathcal{X}, \mathcal{V}_x \mathcal{X}_x(t_i)) \). We also pick any final view such that \( u_i \subseteq \text{getView}(\mathcal{X}, \mathcal{AR}_x \mathcal{X}_x(t_i)) \). Note that there is nothing to prove for \( u_i \) since the \( \dagger \) does not constrain the \( u_i \).

Recall the \( \dagger \):

\[ \dagger \equiv \forall k, k', m, j. \quad m \in u(k) \wedge w(k', j) \rightarrow (\mathcal{SO}_x, \mathcal{WR}_x); (\mathcal{SO}_x, \mathcal{WR}_x); \mathcal{WW}_x) \rightarrow w(k, m) \Rightarrow j \in u(k') \]
Assume $j \in u_i(k)$ for some key $k$ and index $i$. It means the writer of the version is visible by the transaction $t_i$, i.e., $w(K(k, i)) \in VIS^{-1}(t_i)$. Let the $K = K_{\text{ext}(i, i-1)}$. We need to prove the following:

$$\forall k, k', m, j, t, t'. m \in u(k) \land w(K(k, m)) \in VIS^{-1}(t_i) \land w(K(k', j)) \Rightarrow w(K(k', j)) \in VIS^{-1}(t_i)$$

(7.12)

Since $WR_K$, $WW_K$ and $RW_K$ coincide with $WR_X$, $WW_X$ and $RW_X$ respectively, and $((SO \cup WR); RW? \cup WW)^+ ; VIS_X \subseteq VIS_X$. It implies Eq. (7.12).

G.8. Parallel Snapshot Isolation PSI

The axiomatic definition for PSI is

$$\text{(RP}_L \text{WW}, \{\lambda X. VIS; VIS_X, \lambda X. SO_X, \lambda X. WW_X\})$$

First the completeness follows CC in §G.3 and UA in §G.6. Similarly, by Lemma G.1, there exist a minimum visibility such that

$$\text{(RP}_L \text{WW}, \{\lambda X. WR_X; VIS_X, \lambda X. SO_X, \lambda X. WW_X\})$$

Given the minimum visibility, the soundness proof follows CC in §G.5 and UA in §G.6.

G.9. Snapshot Isolation SI

The axiomatic definition for SI is

$$\text{(RP}_L \text{WW}, \{\lambda X. AR_X; VIS_X, \lambda X. SO_X, \lambda X. WW_X\})$$

By a lemma proven in [3], for any $X$ satisfies the SI there exists an equivalent $X'$ with minimum visibility $VIS_{X'} \subseteq VIS_X$ satisfying

$$\text{(RP}_L \text{WW}, \{\lambda X'. ((SO_{X'} \cup WW_{X'} \cup WR_{X'}); RW_{X'}?^+ ; VIS_{X'}, \lambda X'. (WW_{X'} \cup SO_{X'}))\})$$

Under the minimum visibility $VIS$ all the transactions still have the same behaviour as before, meaning they do not violate last-write-win.

To prove the soundness, we pick the invariant as the following:

$$I_1(X, cl) = \left(\bigcup \{t'^i_{cl} \in T_X \mid i \in \mathbb{N}\} VIS_X^{-1}(t_i')\right) \setminus T_{rd}$$

$$I_2(X, cl) = \left(\bigcup \{t'^i_{cl} \in T_X \mid i \in \mathbb{N}\} (SO_X^{-1}(t_i')\right) \setminus T_{rd}$$

where $T_{rd}$ is all the read-only transactions included in both $\left(\bigcup \{t'^i_{cl} \in T_X \mid i \in \mathbb{N}\} VIS_X^{-1}(t_i')\right)$ and $\left(\bigcup \{t'^i_{cl} \in T_X \mid i \in \mathbb{N}\} (SO_X^{-1}(t_i')\right)$. Assume a kv-store $K$, an initial and a final view $u, u'$ a fingerprint $F$ such that $ET_{SI} \vdash (K, u) \triangleright F : (K', u')$. Also choose an arbitrary $cl$, a transaction identifier $t'^i_{cl} \in \text{nextTxId}(K, cl)$, and an abstract execution $X'$ such that $K_K = K$ and $I_1(X, cl) \cup I_2(X, cl) \subseteq T_X(K, u)$. Let a new abstract execution $X = \text{extend}(X, t'^i_{cl}, F, Tx(K, u) \cup T_{rd})$. We are about to prove there exists an extra set of read-only transaction $T'_{rd}$ such that:

$$\forall t. (t, t'^i_{cl}) \in SO_{X'} \Rightarrow t \in Tx(K, u) \cup T_{rd} \cup T'_{rd}$$

(7.13)

$$\forall t. (t, t'^i_{cl}) \in WW_{X'} \Rightarrow t \in Tx(K, u) \cup T_{rd} \cup T'_{rd}$$

(7.14)

$$\forall t. (t, t'^i_{cl}) \in ((SO_{X'} \cup WW_{X'} \cup WR_{X'}); RW_{X'}?^+ ; VIS_{X'}) \Rightarrow t \in Tx(K, u) \cup T_{rd} \cup T'_{rd}$$

(7.15)

$$I_1(X', cl) \cup I_2(X', cl) \subseteq Tx(K_X, u')$$

(7.16)

- The invariant $I_2$ implies Eq. (7.13) as the same as RYW in §G.3.
- Since SI also satisfies UA, the Eq. (7.14) can be proven as the same as UA in §G.6.
- Note that for Eq. (7.15), it is sufficient to prove one step instead of transitive. That is,

$$\forall t. (t, t'^i_{cl}) \in ((SO_{X'} \cup WW_{X'} \cup WR_{X'}); RW_{X'}?^+ ; VIS_{X'}) \Rightarrow t \in Tx(K, u) \cup T_{rd} \cup T'_{rd}$$
Let $T_d'$ initially be empty set. More read-only transactions will be added in until Eq. 7.15 holds. Assume a transaction $t$ such that 

$$(t, t_i') \in (\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; \text{VIS}_X$$

It means $(t \xrightarrow{((\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; \text{RW}_X)} t') \xrightarrow{\text{VIS}_X} t_i'_{cl}$ for some transaction $t'$. There are two cases: $t'$ writes to at least one key; or $t'$ is read-only.

- $t'$ writes to at least one key. Assume that the transaction $t'$ writes to a key $k$. Now we perform case analysis if $t$ is a read-only transaction.

  * if $t$ has write, then $t \in \text{Tx}(\mathcal{K}, u)$. Recall the $\uparrow$ is defined as the following:

$$\uparrow \equiv \forall k', i, j. i \in u(k) \land w(k', j) \rightarrow \text{(SO}_X \cup \text{WW}_X \cup \text{WR}_X)? \rightarrow w(\mathcal{K}(k, i)) \Rightarrow j \in u(k')$$

  (7.17)

Since WR$_X$, WW$_X$ and RW$_X$ coincide with WR$_X$, WW$_X$ and RW$_X$ respectively. Also because $t$ writes to at least one key, it is easy to see there exists some version $k''$, $m$ such that $t = w(\mathcal{K}(k'', m))$ and $m \in u(k'')$. By definition of $\text{Tx}$, it follows $t \in \text{Tx}(\mathcal{K}, u)$.

  * if $t$ is a read-only transaction, we add it into $T_d'$.

- If $t'$ is a read-only transaction, we know $t' \in T_d$ or $t' \in T_d'$. More specifically, we have three cases: (i) $t' \in \bigcup \{t_i \in \text{Tx} \mid i \in \mathbb{N}\}$; (ii) $t' \in \bigcup \{t_i \in T_X \mid i \in \mathbb{N}\}$; (iii) $t' \in T_d'$.

  * Assume $t' \in \bigcup \{t_i \in \text{Tx} \mid i \in \mathbb{N}\}$. There exist a previous transaction from the same client $t_{cl}^m$ such that $m < n$ and 

$$t \xrightarrow{((\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; \text{RW}_X)} t_{cl}^m \xrightarrow{\text{VIS}_X} t_i'$$

Since those edges already exists in the abstract execution $\mathcal{X}$, and by the constraints of $\mathcal{X}$ we have $t \xrightarrow{\text{VIS}_X} t_i'$. Therefore by the invariant $I_1$, either $t \in \text{Tx}(\mathcal{K}, u) \cup T_d$.

  * Assume $t' \in \bigcup \{t_i \in T_X \mid i \in \mathbb{N}\}$. Note that $t'$ is a read-only transaction so we can simplify the edge, 

$$t \xrightarrow{((\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; \text{RW}_X)} t_{cl}^m \xrightarrow{\text{VIS}_X} t_i'$$

for some $m$ such that $m < n$. The path from $t$ to $t_{cl}^m$ must exist in the abstract execution before update and they satisfy the constraint, so $t \xrightarrow{\text{VIS}_X} t_{cl}^m$. Therefore by the invariant $I_2$, we have $t \in \text{Tx}(\mathcal{K}, u) \cup T_d$.

  * $t' \in T_d'$. Given that $T_d'$ initially is empty, there is another transaction $t''$ that has at least a write such that:

$$t \xrightarrow{((\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; \text{RW}_X)} t'' \xrightarrow{\text{VIS}_X} t_{cl}^m \xrightarrow{\text{VIS}_X} t_i'$$

If $t$ has write, then $\uparrow$ then $t \in \text{Tx}(\mathcal{K}, u)$, otherwise we add $t$ into $T_d'$.

- Since $I_1$ satisfies $\text{RYW}$ and $\text{MR}$, thus invariants $I_1$ and $I_2$ are preserved, that is, Eq. 7.16. To prove completeness, we prove four parts of the execution test separately.

  * It is easy to see it is complete with respect to $\text{UA}$ and $\text{RYW}$ as $\text{WW} \cup \text{SO} \subseteq \text{VIS}$. The details are the same as proofs for $\text{UA}$ in §7.6 and $\text{RYW}$ in §7.3.

  * By the 13, we know that for any abstract execution that satisfies $(\text{WW}_X \cup \text{SO}_X) \subseteq \text{VIS}_X$ and $(\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; \text{VIS}_X \subseteq \text{VIS}_X$, the visibility relation will have the following form for some relation $R$:

$$\text{VIS}_X = (((\text{SO}_X \cup \text{WW}_X \cup \text{WR}_X)?; R)^*; (\text{SO}_X \cup \text{WR}_X \cup \text{WW}_X)$$

Given above, it is easy to see $\text{VIS}_X; \text{SO}_X \subseteq \text{VIS}_X$ since $\text{SO}$ is transitive, thus we have the proof for $\text{MR}$ where the detail is the same same $\text{MR}$ in §7.1.

  * Let consider $\uparrow$. Assume $i$th transaction $t_i$ in the arbitrary order, and let view $u_i = \text{getView}(\mathcal{X}, \text{VIS}_X^{-1}(t_i))$. We also pick any final view such that $u_i' \subseteq \text{getView}(\mathcal{X}, \text{AR}_X^{-1}(t_i))$. Note that there is nothing to prove for $u_i'$ since $\uparrow$ does not constrain the $u_i'$s. Let the $\mathcal{K} = \mathcal{K}_{\text{cut}}(\mathcal{X}, i-1)$. Now we need to prove the following:

$$\forall k, k', m, j, t, t', t'' \in \mathcal{K}(k, m) \land w(k', j) \in \text{VIS}_X^{-1}(t_i) \land m \in u(k) \land w(k', j) \rightarrow \text{(SO}_X \cup \text{WW}_X \cup \text{WR}_X)? \rightarrow w(\mathcal{K}(k, m)) \Rightarrow w(k', j) \in \text{VIS}_X^{-1}(t_i)$$

(7.18)
G.10. Serialisability SER

The execution test $ET_{UA}$ is sound with respect to the axiomatic definition

$$ (RP_{LWW}, \{ \lambda X. AR \} ) $$

We pick the invariant as $I(\chi, cl) = \emptyset$, given the fact of no constraint on the view after update. Assume a kv-store $K$, an initial and a final view $u, u'$ a fingerprint $F$ such that $ET_{SER} \vdash (K, u) \models F : (K', u')$. Also choose an arbitrary $cl$, a transaction identifier $t \in nextTxl(K, cl)$, and an abstract execution $\chi$ such that $K\chi = K$ and $I(\chi, cl) = \emptyset \subseteq TX(K, u)$. Let $\chi' = extend(\chi, t, TX(K, u), F)$. Note that since the invariant is empty set, it remains to prove there exists a set of read-only transactions $T_{rd}$ such that:

$$ \forall t', t' \xrightarrow{AR_{\chi'}} t \Rightarrow t' \in TX(K, u) \cup T_{rd} $$

Since the abstract execution satisfies the constraint for SER, i.e. $AR \subseteq VIS$, we know $AR = VIS$. Since $TX(K, u)$ contains all transactions that write at least a key, we can pick a $T_{rd}$ such that $TX(K, u) \cup T_{rd} = TX(K, u)$, which gives us the proof.

The execution test $ET_{UA}$ is complete with respect to the axiomatic definition $(RP_{LWW}, \{ \lambda X. AR_X \} )$. Assume $i$-th transaction $t_i$ in the arbitrary order, and let view $u_i = getView(\chi, VIS^{-1}(t_i))$. We also pick any final view such that $u'_i \subseteq gettimeofday(\chi, (AR^{-1}_{\chi})(t_i))$. Note that there is nothing to prove for $u'_i$. Now we need to prove the following:

$$ \forall k, j, 0 \leq j < |K_{cut(\chi, i-1)}(k)| \Rightarrow j \in u_i(k) $$

Because $VIS^{-1}(t_i) = AR^{-1}(t_i) = \{ t \mid t \text{ appears in } K_{cut(\chi, i-1)} \}$, so for any key $k$ and index $j$ such that $0 \leq j < |K_{cut(\chi, i-1)}(k)|$, the $j$-th version of the key contains in the view, i.e. $j \in u(k)$.

Appendix H.
Program Analysis

We give two applications of our theory aimed at showing the robustness of a transactional library against a given consistency model. The first application considers a single counter library, and proves that it is robust against Parallel Snapshot Isolation. The second application aims at proving the robustness of two counters against Snapshot Isolation.

Code for Counter Objects. We start by reviewing the transactional code for the increment and read operations provided by a counter object over a key $k$, denoted as $inc(k)$ and $read(k)$, respectively.

$$ inc(k) = \begin{cases} &a := [k]; \\ &[k] := a + 1; \end{cases} \quad read(k) = \begin{cases} &a := [k]; \end{cases} $$

Clients can interact with the key-value store only by invoking the $inc(k)$ and $read(k)$ operations. A transactional library is a set of transactional operations. For a single counter over key $k$, we define the transactional library $Counter(k) = \{ inc(k), read(k) \}$, while for multiple counters over a set of keys $K = \{ k_i \}_{i \in I}$, respectively, we define $Counter(K) = \bigcup_{i \in I} Counter(k_i)$.

KV-store semantics of a transactional library. Given the transactional code $[T]$, we define $F(K, u, [T])$ to be the fingerprint that would be produced by a client that has view $u$ over the kv-store $K$, upon executing $[T]$. For the $inc(k)$ and $read(k)$ operations discussed above, we have that $F(K, u, inc(k)) = \{ (\epsilon, k, n), (u, k, n + 1) \mid n = snapshot(K, u) \}$, and $F(K, u, read(k)) = \{ (\epsilon, k, n) \mid n = snapshot(K, u) \}$. Given an execution test $ET$, and a transactional library $L = \{ [T_i] \}_{i \in I}$, we define the set of valid ET-traces for $tL$ as the set $\text{Traces}(ET, \{ [T_i] \}_{i \in I})$ of ET-traces in which only ET-reductions of the form

$$ (K_0, U_0) \xrightarrow{(CL_0, \lambda_0)} ET (K_1, U_1) \xrightarrow{(CL_1, \lambda_1)} ET \cdots \xrightarrow{(CL_{n-1}, \lambda_{n-1})} ET (K_n, U_n), $$

where for any $j = 0, \cdots, n - 1$, either $\lambda_j = \epsilon$ or $\lambda_j = F(K_j, U_j(CL_j), [T_i])$ for some $i \in I$. Henceforth we commit an abuse of notation and write $(K, U) \xrightarrow{[T]} ET (K', U')$ in lieu of $(K, U) \xrightarrow{(CL, \lambda)} ET (K', U')$. We also let $\text{KVStores}(ET, \{ [T_i] \}_{i \in I})$ be the set of kv-stores that can be obtained when clients can only perform operations from $\{ [T_i] \}_{i \in I}$ under the execution test $ET$. Specifically,

$$ \text{KVStores}(ET, \{ [T_i] \}_{i \in I}) = \{ K \mid \bigcup_{(K_0, U_0) \xrightarrow{(CL_0, \lambda_0)} ET \cdots \xrightarrow{(CL_{n-1}, \lambda_{n-1})} ET (K_n, U_n)} \in \text{Traces}(ET, \{ [T_i] \}_{i \in I}) \}. $$
Anomaly of a single counter under Causal Consistency. It is well known that the transactional library consisting of a single counter over a single key, Counter$(k)$, implemented on top of a kv-store guaranteeing Causal Consistency, leads to executions for the kv-store that cannot be simulated by the same transactional library implemented on top of a serialisable kv-store. For simplicity, let us assume that \textup{KEY} = \{ke\}. Let $K_0 = [k \mapsto (0, t_0, \emptyset)]$, $K_1 = [k \mapsto (0, t_0, \{c_{cl}\})): (0, t_{cl_1}, \emptyset): (0, t_{cl_2}, \emptyset)$. Let also $u_0 = [k \mapsto 0]$. Then we have that \[
abla \left( K_0, [c_{cl} \mapsto u_0, c_{cl} \mapsto u_0] \right) \xrightarrow{\text{ET}_{cc}} \left( K_1, [c_{cl} \mapsto \_\_c_{cl} \mapsto u_0] \right) \xrightarrow{\text{ET}_{cc}} \left( K_2, \_\_ \right) \]
By looking at the kv-store $K_2$, we immediately find a cycle in the graph induced by the relations $SO_{K_2}, WR_{K_2}, WW_{K_2}, RW_{K_2}$: $t_{cl_1} \xrightarrow{RW} t_{cl_2} \xrightarrow{RW} t_{cl_1}$. Following from Theorem 6.1 then which proves that $K_2$ is not included in $CM(ET_{SER})$, i.e. it is not serialisable.

Robustness of a Single counter under Parallel Snapshot Isolation. Here we show that the single counter library Counter$(k)$ is robust under any consistency model that guarantees both write conflict detection (formalised by the execution test $ET_{UA}$), monotonic reads (formalised by the execution test $ET_{MR}$) and read your writes (formalised by the execution test $ET_{RW}$). Because $ET_{PSI}$ guarantees all such consistency guarantees, i.e. $CM(ET_{PSI}) \subseteq CM(ET_{MR} \cap ET_{RW} \cap ET_{UA})$, then it also follows that a single counter is robust under Parallel Snapshot Isolation.

**Proposition H.1.** Let $K \in KVStores(ET_{UA} \cap ET_{MR} \cap ET_{RW}, Counter(k))$. Then there exist $\{t_i\}_{i=1}^{n}$ and $\{T_i\}_{i=0}^{n}$ such that \[
K(k) = ((0, t_0, T_0 \uplus \{t_1\}) \cdots ((n-1, t_{n-1}, T_{n-1} \uplus \{t_n\})) :: (n, t_n, T_n) \]
\[
\forall i = 0, \cdots, n. T_i \cap \{t_i\}_{i=0}^{n} = \emptyset \]
\[
\forall t, t', \forall i, j = 0, \cdots, n. t \xrightarrow{SO} t' \wedge t \in \{t_i\} \cup T_i \Rightarrow \left( (t' = t_j \Rightarrow i < j) \wedge (t' \in T_j \Rightarrow i \leq j) \right) \]

**Proof.** It suffices to prove that the properties (8.1)-(8.2), (8.3) given in Prop. H.1 are invariant under $(ET_{MR} \cap ET_{RW} \cap ET_{UA})$-reductions of the form \[
\begin{align*}
(K, \mathcal{U}) \xrightarrow{(cl, inc(k))} & ET_{UA} \cap ET_{MR} \cap ET_{RW} (K', \mathcal{U}') \\
(K, \mathcal{U}) \xrightarrow{(cl, read(k))} & ET_{UA} \cap ET_{MR} \cap ET_{RW} (K', \mathcal{U})
\end{align*}
\]
To this end, we will need the following auxiliary result which holds for any configuration $(K, \mathcal{U})$ that can be obtained under the execution test $ET_{RW} \cap ET_{MR}$:

\[
\forall i, j, n, m, cl, k. t_{cl} \in \{w(K(k, i))\} \cup rs(K(k, i)) \wedge t_{cl} \in \{w(K(k, j))\} \cup rs(K(k, j)) \wedge m < n \wedge i \in UL(cl) \Rightarrow j \in UL(cl)(k).
\]

Suppose that there exist two sets $\{t_i\}_{i=1}^{n}$ and $\{T_i\}_{i=0}^{n}$ such that $(K, \{t_i\}_{i=1}^{n}, \{T_i\}_{i=0}^{n})$ satisfies the properties (8.1)-(8.3). We prove that, for transitions of the form (8.4)-(8.5), there exist an index $m$ and two collections $\{t_i\}_{i=1}^{m}, \{T_i\}_{i=0}^{m}$ such that $(K', \{t_i\}_{i=1}^{m}, \{T_i\}_{i=0}^{m})$ satisfies the properties (8.1)-(8.3). We consider the two transitions separately.

- Assume that \[
(K', \{t_i\}_{i=1}^{n+1}, \{T_i\}_{i=0}^{n+1}) \text{ satisfies Property (8.1). Recall that } (K, \{t_i\}_{i=1}^{n}, \{T_i\}_{i=0}^{n}) \text{ satisfies (8.1), i.e. }
\]
\[
K(k) = ((0, t_0, T_0 \uplus \{t_1\}) \cdots ((n-1, t_{n-1}, T_{n-1} \uplus \{t_n\})) :: (n, t_n, T_n) \]
\[
\text{It follows that } K'(k) = ((0, t_0, T_0 \uplus \{t_1\}) \cdots ((n-1, t_{n-1}, T_{n-1} \uplus \{t_n\})) :: (n+1, t_{n+1}, T_{n+1}) \]
\[
\text{We call that } T_{n+1} = \emptyset \text{ and that } T'_{i} \cap \{t_{n+1}\} = \emptyset.
\]
- Assume that $(K', \{t_i\}_{i=1}^{n+1}, \{T_i\}_{i=0}^{n+1})$ satisfies Property (8.2). Let $i = 0, \cdots, n+1$. If $i = n+1$, then $T_i = \emptyset$, from which $T_i \cap \{t_{n+1}\} = \emptyset$ follows. If $i < n+1$, then because $(K, \{t_i\}_{i=1}^{n}, \{T_i\}_{i=0}^{n})$ satisfies Property (8.2), then $T_i \cap \{t_{n+1}\} = \emptyset$. Finally, because $t_{n+1}$ was chosen to be fresh with respect to the transaction identifiers appearing in $K$, and $T_i \subseteq rs(K(k, i))$, the fact that $T_i \cap \{t_{n+1}\} = \emptyset$. Then, we also have that $T_i \cap \{t_{n+1}\} = \emptyset$.
satisfies Property (8.3), then if $t' = t_j$ it follows that $i < j$, and if $t' \in T_j$ it follows that $i \leq j$, as we wanted to prove. If $t \in \{t_{n+1}\} \cup T_{n+1}$, then it must be $t = t_{n+1}$ because $T_{n+1} = \emptyset$. Recall that $t_{n+1}$ is the transaction identifier that was used to update $K$ to $K'$, i.e. $K' = \text{update}(K, U(cl), t_{n+1})$. By definition of update, it follows that $t_{n+1} \in \text{nextTxDl}(K, cl)$, and because $t_{n+1} \rightarrow t'$, then $t'$ cannot appear in $K'$. In particular, $t' \notin \{t_i\}_{i=0}^{n+1} \cup \bigcup \{T_j\}_{j=0}^{n+1}$, hence in this case there is nothing to prove. Finally, if $t' \in \{t_{n+1}\} \cup T_{n+1}$, then it must be the case that $t' = t_{n+1}$.

If $t = t_j$, because $t \rightarrow t'$ and $t' = t_{n+1}$, it cannot be $t = t_{n+1}$, hence it must be $i \leq n < n+1$.

* Suppose that $(K', U) \xrightarrow{\text{cl, read}(k)} \text{ET}_{UA} \cap \text{ET}_{MR} \cap \text{ET}_{RYY} (K', U')$.

As in the previous case, we have that $K' = \text{update}(K, U(cl), t, \{(x, k, i)\})$, where $m = \text{snapshot}(K, U(cl))(k)$ - in particular, because $(K, \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.1), then it must be the case that $m = \max(K, U(cl))(k)$ - and $t \in \text{nextTxDl}(K, cl)$. For $i = 0, \ldots, n$, let $T_j' := T_i$ if $i \neq m$, $T_j' = T_i \cup \{t\}$ if $i = m$. Then we have that $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies properties (8.1)-(8.3). Putting all these facts together, we obtain the following:

1. $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.1). Without loss of generality, suppose that $m < n$. Because $(K, \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.1), we have that

   $$K(k) = \{(0, t_0, T_0 \cup \{t_1\}) \ddots \{(m, t_m, T_m \cup \{t_{m+1}\}) \ddots \{(n-1, t_{n-1}, T_{n-1} \cup \{t_n\})\}\} = (n, t_n, T_n),$$

   and from the definition of update it follows that

   $$K(k) = \{(0, t_0, T_0 \cup \{t_1\}) \ddots \{(m, t_m, T_m \cup \{t\}) \ddots \{(n-1, t_{n-1}, T_{n-1} \cup \{t\})\}\} = (n, t_n, T_n).$$

2. $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.2). Recall that $m = \max(K, U(cl))(k)$; let $i = 0, \ldots, n$. Let again $i = \max(K, U(cl))(k)$. If $i \neq m$, then $T_i' = T_i$, and because $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.2) we have that $T_i' \cap \{t_i\}_{i=0}^n = \emptyset$. If $i = m$, then we have that $T_i' = T_i = T_m \cup \{t\}$, where we recall that $t \in \text{nextTxDl}(K, cl)$. Because $(K, \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.2), we have that $T_m \cap \{(t_i)_{i=0}^n = \emptyset$. Finally, because $t \in \text{nextTxDl}(K, cl)$, then it must be the case that for any $i = 0, \ldots, n$, $t \notin \{w(K', (i, k))\}_{i=0}^n = \{t_i\}_{i=0}^n$, where the last equality follows because we have already proved that $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.1).

3. $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.3). Let $t', t''$ be such that $t' \rightarrowSO t''$. Suppose also that $t'' \in \{t_j\} \cup T_j$ for some $j = 0, \ldots, n$. We consider two different cases:

   + $t' = t_i$. Suppose then that $t'' = t_j$ for some $j = 0, \ldots, n$. Because $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.3), then it must be the case that $i < j$. Otherwise, suppose that $t'' \in T_j$ for some $j = 0, \ldots, n$. If $j \neq m$, then $T_j' = T_j$, and because $(K', \{t_i\}_{i=0}^n, \{T_i\}_{i=0}^n)$ satisfies Property (8.3), we have that $i \leq j$. Otherwise, $T_j' = T_m \cup \{t\}$. Without loss of generality, in this case we can assume that $t'' = t$ (we have already shown that if $t'' \in T_j$, then it must be $i \leq j$. Recall that $j = m = \max(K, U(cl))(k)$, by the Definition of ETUA it must be the case that $U(cl) = \{k \rightarrow \{0, \ldots, j\}\}$. It also follows that $t = t_{cl}$ for some $p \geq 0$, and because $t' \rightarrowSO t'' = t$, then $t' = t_{cl}$ for some $q < p$. Because of Property (8.6), and because $t' = t_i = w(K, (k, i))$, then it must be the case that $i \in U(cl)(k)$. Hence $i \leq m = j$.

   + $t' \in T_j$. We need to distinguish the cases $i \neq m$, leading to $T_j' = T_j$, or $i = m$, in which case $T_j' = T_m \cup \{t\}$. If either $i \neq m$, or $i = m$ and $t \in T_m$, then we can proceed as in the case $t' = t_i$. Otherwise, suppose that $i = m$ and $t' = t$. Then, because $t' \rightarrowSO t''$, and $t \in \text{nextTxDl}(K, cl)$, it must be the case that $t = t_{cl}$ for some $p \geq 0$, and whenever $t_{cl} \in k$, then $t_{cl} \rightarrowSO t$. In particular we cannot have that $t'' \in k$, because $t' \rightarrowSO t''$, which concludes the proof.

- $(K', U')$ satisfies Property (8.6).

\[ \square \]

**Corollary H.1.** Given $K \in \text{KVStores}(\text{ET}_{UA} \cap \text{ET}_{MR} \cap \text{ET}_{RYY}, \text{Counter})$, then $\text{GraphOf}(K)$ is acyclic.

**Proof.** Let $\{t_i\}_{i=1}^n, \{T_i\}_{i=0}^n$ be such that $(\{t_i\}_{i=1}^n, \{T_i\}_{i=0}^n)$ satisfies properties (8.1)-(8.3). First, we define a partial order between transactions appearing in $K$ as the smallest relation $\rightarrowSO$ such that for any $t, t', t''$ and $i, j, 0, \ldots, n$:

\[
\begin{align*}
  t \in T_j & \implies t_i \rightarrowSO t, \\
  i < j & \implies t_i \rightarrowSO t_j, \\
  t \in T_i \land i < j & \implies t \rightarrowSO t_j, \\
  t, t' \in T_i \land i < j & \implies t \rightarrowSO t', \\
  t \rightarrowSO t' & \implies t \rightarrowSO t''.
\end{align*}
\]
It is immediate that if \( t \rightarrow t' \) then either \( t \in \{t_i\} \cup T_i, t' = \{t_j\} \cup T_j \) for some \( i, j \) such that \( i < j \), or \( t = t_i, t' \in T_i \), or \( t, t' \in T_i \) and \( t \stackrel{\text{SO}}{\rightarrow} t' \). A consequence of this fact, is that \( \rightarrow \) is irreflexive.

Next, observe that we have the following:

- whenever \( t \rightarrow \text{WR}_K \rightarrow t' \), there exists an index \( i = 0, \ldots, n \) such that \( t = t_i \), and either \( i < n \) and \( t' \in T_i \cup \{t_i+1\} \), or \( i = n \) and \( t' \in T_i \); by definition, we have that \( t \rightarrow t' \);
- whenever \( t \rightarrow \text{WW}_K \rightarrow t' \), there exist two indexes \( i, j : 0 \leq i < j \leq n \) such that \( t = t_i, t' = t_j \); again, we have that \( t \rightarrow t' \);
- whenever \( t \rightarrow \text{RW}_K \rightarrow t' \), there exist two indexes \( i, j : 0 \leq i < j \leq n \) such that \( t = t_i, t' = t_j \), or \( t = t_i+1, i + 1 < j \) and \( t' = t_j \); in both cases, we obtain that \( t \rightarrow t' \);
- whenever \( t \rightarrow \text{SO}_K \rightarrow t' \), then \( t \in \{t_i\} \cup T_i \) for some \( i = 0, \ldots, n \), and either \( t' = t_j \) for some \( i < j \), or \( t' \in T_j \) for some \( i \leq j \); it follows that \( t \rightarrow t' \).

We have proved that \( \rightarrow \) is an irreflexive relation, and it contains \( (\text{SO}_K \cup \text{WR}_K \cup \text{WW}_K \cup \text{RW}_K)^+ \); because any subset of an irreflexive relation is itself irreflexive, we obtain that \( \text{Graph}(K) \) is acyclic. \( \square \)

**Corollary H.2.** KVStores(ET_{PSI} (Counter(k))) \( \subseteq \) KVStores(ET_{SER} (Counter(k))).

**Proof.** Let \( K \in \text{KVStores(ET}_{PSI} (Counter(k))). \) Because ET_{PSI} \( \supseteq \) ET_{MR} \( \cap \) ET_{RW} \( \cap \) ET_{UA}, we have that \( K \in \text{KVStores(ET}_{MR} \cap ET_{RW} \cap ET_{UA} (Counter(k))). \) By Cor. H.1, we have that Graph(K) is acyclic. We can now employ the construction outlined in \( \text{[10]} \) to recover an abstract execution \( \mathcal{X} = (\mathcal{E}_K, \mathcal{V}_K, AR) \) such that \( \mathcal{X} \subseteq \mathcal{V}_K \) and \( AR \subseteq \mathcal{V}_K \), and Graph(\( \mathcal{X} \)) = Graph(K). Finally, the results from \( \text{§G.10} \) establish that, from \( \mathcal{X} \) we can recover a ET_{SER}-trace in Traces(ET_{SER} (Counter(k))) whose last configuration is \( (K'_{\text{last}}, cl) \), and Graph(\( K' \)) = Graph(\( \mathcal{X} \)) = Graph(K), leading to \( K' = K \). It follows that \( K \in \text{KVStores(ET}_{SER} (Counter(k))). \) \( \square \)

Multiple counters are not robust against PSI. Suppose that the kv-store contains multiple keys \( k, k', \ldots \), each of which can be accessed and modified by clients using the code of transactional libraries Counter(k), Counter(k'), \ldots. We show that in this case it is possible to have the interactions of two client with the kv-store result in a non-serialisable final configuration.

More formally, suppose that \( \text{KEY} = \{k_1, k_2\} \), and let \( \text{Counter} = \bigcup_{k \in \text{KEY}} \text{Counter}(k) \). Let also

\[
\begin{align*}
K_0 &= \{k_1 \rightarrow (0, t_0, 0), k_2 \rightarrow (0, t_0, 0)\} \\
K_1 &= \{k_1 \rightarrow (0, t_0, \{t_1^1\}), k_2 \rightarrow (0, t_0, \{t_1^1\})\} \\
K_2 &= \{k_1 \rightarrow (0, \{t_2^1\}), k_2 \rightarrow (0, \{t_2^1\})\} \\
K_3 &= \{k_1 \rightarrow (0, \{t_3^1\}), k_2 \rightarrow (0, \{t_3^1\})\} \\
K_4 &= \{k_1 \rightarrow (0, \{t_4^1\}), k_2 \rightarrow (0, \{t_4^1\})\} \\
U_0 &= \{cl_1 \rightarrow \{k_1 \rightarrow 0, k_2 \rightarrow 0\}, cl_2 \rightarrow \{k_1 \rightarrow 0, k_2 \rightarrow 0\}\} \\
U_1 &= \{cl_1 \rightarrow \{k_1 \rightarrow 0, 0, 1\}, cl_2 \rightarrow \{k_1 \rightarrow 0, 0, 1\}\} \\
U_2 &= \{cl_1 \rightarrow \{k_1 \rightarrow 0, 0, 1\}, cl_2 \rightarrow \{k_1 \rightarrow 0, 0, 1\}\} \\
U_3 &= \{cl_1 \rightarrow \{k_1 \rightarrow 0, 0, 1\}, cl_2 \rightarrow \{k_1 \rightarrow 0, 0, 1\}\} \\
U_4 &= \{cl_1 \rightarrow \{k_1 \rightarrow 0, 0, 1\}, cl_2 \rightarrow \{k_1 \rightarrow 0, 0, 1\}\} \\

\end{align*}
\]

Observe that we have the sequence of ET_{PSI}-reductions

\[
(K_0, U_0) \xrightarrow{cl_1, \text{inc}(k_1)} (K_1, U_1) \xrightarrow{cl_2, \text{inc}(k_2)} (K_2, U_2) \xrightarrow{cl_3, \text{read}(k_2)} (K_3, U_3) \xrightarrow{cl_4, \text{read}(k_1)} (K_4, U_4)
\]

and therefore \( K_4 \notin \text{KVStores(ET}_{PSI} (Counter(k))). \) On the other hand, for Graph(K_4) we have the following cycle, which proves that \( K_4 \notin \text{KVStores(ET}_{SER} (Counter(k))). \):

\[
\begin{align*}
\text{SO}_{K_{\downarrow 1}}^{t_1_{cl_1}} &\xrightarrow{t_2_{cl_1}^{SO_{K_{\downarrow 2}}}} t_1_{cl_2}^{RW_{K_{\downarrow 1}}} &\xrightarrow{t_2_{cl_2}^{SO_{K_{\downarrow 2}}}} t_1_{cl_1}^{RW_{K_{\downarrow 2}}} &\xrightarrow{t_1_{cl_1}^{RW_{K_{\downarrow 2}}}} t_2_{cl_1}^{SO_{K_{\downarrow 2}}} &\xrightarrow{t_2_{cl_2}^{SO_{K_{\downarrow 2}}}} t_1_{cl_1}^{RW_{K_{\downarrow 2}}} \\
\end{align*}
\]

Robustness of multiple counters under Snapshot Isolation. The reason why multiple counters fail to be robust under Parallel Snapshot Isolation is that two different clients can observe increments over different counters to have been executed in different order. For example, in the example above we had that client \( cl_1 \) observed the initial value of \( k_2 \) after it incremented \( k_1 \), and vice-versa \( cl_2 \) observed the initial value of \( k_1 \) after it incremented \( k_2 \). This situation does not arise when the execution test employed by clients guarantees Snapshot Isolation: by definition, this consistency model ensures that there is a total order among all the the updates performed over \( k_1 \) and \( k_2 \), and that is consistent with the order in which such updates are observed by clients.

We show that multiple counters are robust against Snapshot Isolation. The proof strategy we employ is the same as for the proof of robustness of a single counter against PSI: we characterise the shape of kv-stores that can be obtained as a
result of multiple clients invoking arbitrary operations on counters, and we show that the corresponding dependency graph has no cycle. To this end, we will need the following, auxiliary result that characterises cycles in Snapshot Isolation:

**Proposition H.2.** Let $\mathcal{K} \in \text{CM}(ET_{SI})$. Then the relation $((SO_{\mathcal{K}} \cup WR_{\mathcal{K}} \cup WW_{\mathcal{K}}); RW_{\mathcal{K}}^+)^+$ is irreflexive. Alternatively, $\mathcal{K}$ only admits cycles with two consecutive $RW_{\mathcal{K}}$-edges.

**Proof.** This is a straightforward consequence of the correspondence between $ET_{SI}$-traces and abstract executions that satisfy the axiomatic definition of SI (§4.9), and the fact that for any such abstract execution $\mathcal{X}$, the relation $((SO_{\mathcal{X}} \cup WR_{\mathcal{X}} \cup WW_{\mathcal{X}}); RW_{\mathcal{X}}^+)^+$ is irreflexive (13][16][20). \hfill \Box

We only consider the case where the transactional library contains two counters, i.e. we consider the transactional library $\text{Counter}((k_1, k_2))$ for some $k_1, k_2 \in \text{KEY}$. However, our line of reasoning can be generalised to an arbitrary number of counters.

Let also $\mathcal{K}$ be in $\text{KVStores}(ET_{SI}, \text{Counter}((k_1, k_2)))$. Because $ET_{SI} \subseteq ET_{PSI}$, $\mathcal{K}(k_1)$ and $\mathcal{K}(k_2)$ satisfy the properties from Proposition H.1. Furthermore, none of the transaction identifiers appearing in $\mathcal{K}$ appears both in $\mathcal{K}_{k_1}$ and $\mathcal{K}_{k_2}$.

**Proposition H.3.** Let $\mathcal{K}$ be in $\text{KVStores}(ET_{SI}, \text{Counter}((k_1, k_2)))$. Let $n_1 = |\mathcal{K}(k_1)|$, $n_2 = |\mathcal{K}(k_2)|$. Then there exist $(t_{i1}^{k_1})_{i=1}^{n_1-1}$, $(t_{i2}^{k_2})_{i=1}^{n_2-1}$, $(T^{k_1}_{i1})_{i=0}^{n_1}$, $(T^{k_2}_{i2})_{i=0}^{n_2}$ such that, for $h = 1, 2$:

$$\mathcal{K}(k_h) = \left(0, t_0, T^{k_h}_{0} \uplus \{t_1^{k_h}\} ; \cdots ; (n - 1, t_{n-1}^{k_h}, T_{n-1}^{k_h}) \right) : (n - 1, t_{n-1}^{k_h}, T_{n-1}^{k_h})$$

$$\forall i = 0, \cdots, n, T^{k_h}_{i} \cap \{t_{i}^{k_h}\} = 0 \quad (8.7)$$

$$\forall t, t', \forall i, j = 0, \cdots, n_h - 1. t \xrightarrow{SO} t' \land t \in \{t_{i}^{k_h}\} \cup T^{k_h}_{i} \quad \Rightarrow \quad \begin{cases} (t' = t_{i}^{k_h} \Rightarrow i < j) \\ (t' \in T^{k_h}_{i} \Rightarrow i \leq j) \end{cases} \quad (8.8)$$

$$\left(\{t_{i}^{k_h}\} \cup \bigcup_{i=0}^{n_h-1} T^{k_h}_{i}\right) \cap \left(\{t_{i}^{k_2}\} \cup \bigcup_{i=0}^{n_2-1} T^{k_2}_{i}\right) = 0 \quad (8.10)$$

**Proof.** Each of the properties $(8.1), (8.8), (8.9)$ can be proved as in Prop. H.1. The fact that $(\mathcal{K}, \{t_{i1}^{k_1}\}_{i=1}^{n_1-1}, \{t_{i2}^{k_2}\}_{i=1}^{n_2-1}, (T^{k_1}_{i1})_{i=0}^{n_1}, (T^{k_2}_{i2})_{i=0}^{n_2})$ also satisfies Property $(8.10)$ follows from the fact that each of the operations in $\text{Counter}((k_1, k_2))$ only access a single key. \hfill \Box

Following the proof pattern we employed for a single counter under $ET_{PSI}$, we now define a strict partial order $\rightarrow$ among transaction identifiers appearing in a store $\mathcal{K} \in \text{KVStores}(ET_{SI}, \text{Counter}((k_1, k_2)))$, and show that such a total order covers the relation $((SO_{\mathcal{K}} \cup WR_{\mathcal{K}} \cup WW_{\mathcal{K}} \cup RW_{\mathcal{K}})^+)^+$. Because clients may invoke counter operations on different keys, and because the order in which a clients observes different versions of different keys is regulated by $ET_{SI}$, the definition of the total order is considerably more complicated than the one obtained for a single-key counter under $ET_{PSI}$.

**Definition H.1.** Let $\mathcal{K} \in \text{KVStores}(ET_{SI}, \text{Counter}((k_1, k_2)))$, and let $n_1 = |\mathcal{K}(k_1)|$, $n_2 = |\mathcal{K}(k_2)|$, and $\{t_{i1}^{k_1}\}_{i=1}^{n_1-1}, \{t_{i2}^{k_2}\}_{i=1}^{n_2-1}, (T^{k_1}_{i1})_{i=0}^{n_1}, (T^{k_2}_{i2})_{i=0}^{n_2}$ the indexes and sets such that $(\mathcal{K}, \{t_{i1}^{k_1}\}_{i=1}^{n_1-1}, \{t_{i2}^{k_2}\}_{i=1}^{n_2-1}, (T^{k_1}_{i1})_{i=0}^{n_1}, (T^{k_2}_{i2})_{i=0}^{n_2})$ satisfies the property from Proposition H.3. Let $h, l \in \{1, 2\}$ with $h \neq l$; we let $(-\rightarrow, \rightarrow \rightarrow)$ be the smallest relation such that

$t \in T^{k_h} \quad \Rightarrow t^{k_h} \rightarrow t$

$i < j \quad \Rightarrow t^{k_h} \rightarrow t^{k_j}$

$t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t' \quad \Rightarrow t^{k_h} \rightarrow t'$

$t \rightarrow t'' \rightarrow \rightarrow t \quad \Rightarrow t^{k_h} \rightarrow t'$

$t \in (T^{k_h} \cup \{t_{i}^{k_h}\}) \land t' \in (T^{k_l} \cup \{t_{j}^{k_l}\}) \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$t \rightarrow t'' \lor t \rightarrow t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

$\forall t \in T^{k_h} \land t \xrightarrow{SO} t' \quad \Rightarrow t \rightarrow t'$

Given $\mathcal{K} \in \text{CM}(ET_{SI}, \text{Counter}((k_1, k_2)))$, It is important to observe that the definition of $\rightarrow$ is a generalisation of the relation with the same notation we defined for a single counter over $ET_{PSI}$, and therefore it is irreflexive. Furthermore, whenever $t \xrightarrow{WR_{\mathcal{K}} \cup WW_{\mathcal{K}} \cup RW_{\mathcal{K}}} t'$, it must be the case that $t \rightarrow t'$ (the same does not hold anymore in the case of $t \rightarrow t'$).

We also observe that whenever $t \rightarrow t'$, then it cannot be $t \xrightarrow{WR_{\mathcal{K}} \cup WW_{\mathcal{K}} \cup RW_{\mathcal{K}}} t'$, because the relation $\rightarrow$ only relates transactions accessing different keys, and the structure of the multiple counter library ensures that transactions always access exactly one of the two keys $k_1, k_2$. Finally, note that whenever $t \xrightarrow{SO_{\mathcal{K}}} t'$, then either $t, t'$ access the same key, and therefore $t \rightarrow t'$, or $t, t'$ access different keys, and therefore $t \rightarrow t'$. It follows that $(SO_{\mathcal{K}} \cup WR_{\mathcal{K}} \cup WW_{\mathcal{K}} \cup RW_{\mathcal{K}})^+)$ is contained in $(\rightarrow \cup \rightarrow \rightarrow)^+ \rightarrow$; therefore, to prove that $\mathcal{K} \in \text{CM}(ET_{SER}, \text{Counter}((k_1, k_2)))$, it suffices to prove that the relation $\rightarrow$ is irreflexive.
Proposition H.4. Let $K \in \text{KVStores}(ET_{SI}, \text{Counter})$: the relation $\rightarrow$ defined by Def. [41] is irreflexive.

Proof. By contradiction. Suppose that there exists a transaction identifier $t \in K$ such that $t \rightarrow t$. Then there exists a cyclic sequence of transaction identifiers $t^0, \ldots, t^n$ such that $t^n = t^0 = t$, and

$$t^0 \rightarrow t^1 \rightarrow \cdots \rightarrow t^n.$$ 

First note that, because $\rightarrow$ is irreflexive, then it must be the case that the cycle above contains at least one edge of the form $t^i \rightarrow t^{i+1}$. Because the relation $\rightarrow$ only relates transactions accessing different object, then it must be the case that the cycle above also contains a second edge $t^j \rightarrow t^{j+1}$, where $j \neq i$: this is because if the cycle above contains exactly one edge labelled as $\rightarrow$, then $t^0 = t^n$ would need to access both keys $k_1$ and $k_2$, contradicting the fact that the operations from $\text{Counter}((k_1, k_2))$ only access a single object. Thus, the cycle above can be rewritten as

$$t \rightarrow t^a (\rightarrow \cup \rightarrow)^+ t^b \rightarrow t^c (\rightarrow \cup \rightarrow)^+ t,$$

where $t = t^i$ for some $i = 0, \ldots, n - 1$. Note that it cannot be $t^a = t^b$, because otherwise we would have $t \rightarrow \rightarrow t$, or equivalently $t \underbrace{\rightarrow \rightarrow \cdots \rightarrow}_{SO_K} t$, contradicting the acyclicity of the relation $SO_K$. Also note that the relation $(\rightarrow \cup \rightarrow)^+ : \rightarrow$ can be rewritten as $(\rightarrow^+ \cup \rightarrow)^+$, and because $\rightarrow$ is transitive, this is equivalent to $(\rightarrow^+ \cup \rightarrow)^+$, where $\rightarrow^+$ is the reflexive closure of $\rightarrow$. We have determined that the presence of a cycle in the relation $(SO_K \cup WR_K \cup WW_K \cup RW_K)^+$ implies the existence of a cycle of the form

$$t \rightarrow \rightarrow \cdots \rightarrow \rightarrow t.$$ 

However, because any edge labelled as $\rightarrow$ cannot correspond to a RWK-edge, it follows that the original cycle cannot have two adjacent RWK-edges. This contradicts Proposition [H.2].

Appendix I.

Verification of implementations

We verify two protocols, COPS and Closk-SI, that the former is a full replicated implementation for causal consistency and the latter is a shard implementation for snapshot isolation.

I.1. COPS

I.1.1. Code.

Structure. COPS is a fully replicated database protocol for causal consistency. There are two versions, that the simplified version only supports either a single read or a single write per transaction, and the full version supports either multiple reads or a single write pe transactions. Here we verify the full version.

The overall database is modelled as a key-value store. Each key, instead of a single value, is associated with a list of versions, consisting of value, version number $\text{VersionNo}$ and dependencies $\text{Dep}$. COPS relies on the version number to resolve conflict, that is, the write with greater version number wins. Version number composite by time (higher bits) and replica identifier (lower bits). Since the replica identifiers are full ordered, therefore version numbers are full ordered. The dependencies is a set of versions (pairs of keys and versions numbers) that the current version depends on.

1 $\text{VersionNo} ::= \text{LocalTime} + \text{ID}$
2 $\text{Dep} ::= \text{Set(Key,VersionNo)}$
3 $\text{KV} ::= \text{Key} \rightarrow \text{List(Val,VersionNo,Dep)}$

Code 1: COPS Structure

Under the hood, there are many replicas, where each replica has a unique identifier and contains the full key-value store yet might be out of data. Replica also tracks its local time.

1 $\text{Replicas} ::= \text{ID} \rightarrow (\text{KV,LocalTime})$

Code 2: COPS Replicas

To track session order, Client interact with a replica via certain API. To call those API, client has the responsibility to provide context $\text{Ctx}$ which contains versions that has been read from or written to the replica from the same client.

1 $\text{Ctx} ::= \{$
2 $\quad \text{readers} : \text{List(Key,VersionNo,Dep)}$
3 $\quad \text{writers} : \text{List(Key,VersionNo,Dep)}$
4 $\}$

Code 3: Client context
Write. The client call **put** to commit a new write to a key $k$ with value $v$ with the context $ctx$. It extracts the dependencies from the context, by unioning all the versions inside the context, then calls the replica API $\text{ver} = \text{put}_\text{after}(k,v,\text{deps},\text{dep}_\text{to}_\text{nearest}(\text{deps}),\text{dep})$, which require to provide the dependencies so that the replica can check whether they are exists. Note $\text{dep}_\text{to}_\text{nearest}(\text{deps})$ is for performance by providing the latest versions. The replica returns a version number allocated for the new version for key $k$, and then the version is added into the context.

```c
Ctx put(k,v,ctx) {
    // add up all the read and write dependency
    deps = ctx_to_dep(ctx);
    // call replica API
    ver = put_after(k,v,deps,dep_to_nearest(deps));
    // update context
    ctx.writers += (k,ver,deps);
    return ctx;
}
```

**Code 4: Client API for write**

The **put_after** waits until all the versions contained in $\text{nearest}$ exists, consequently all the versions contained in $\text{deps}$ exists. The replica increments the local time and insert the new version with version numbers $\text{time} + \text{id}$ (local time concatenating replica identifier) to the key $k$. At this point, replica returns the new version number to client and later on it will broadcast to other replica.

```c
VersionNo put_after(k,v,deps,nearest,vers) {
    for (k,ver) in nearest
        wait until (_,ver,_) \in \text{kv}(k);
    time = inc(local_time);
    // appending local kv with a new version.
    list_isnert(kv[k], (v, (time ++ id), deps ));
    asyn_brordcase(k, v, (time ++ id), deps);
    return (local_time + id);
}
```

**Code 5: Replica API for write**

When a replica receives a update message, it checks the existence of versions included in the dependencies and then adds the new version to the replica. Last, the replica updates the local time if the new version’s time is greater than the local time.

```c
on_receive(k,v,ver,deps) {
    for (k',ver') in deps
        wait until (_,ver',_) \in \text{kv}(k');
    time = inc(local_time);
    // appending local kv with a new version.
    list_isnert(kv[k], (v, (time ++ id), deps ));
    (remote_local_time + id) = ver;
    local_time = max(remote_local_time, local_time);
}
```

**Code 6: Receive update message**

5. It uses message queue for broad-casting
Read. To read multiple keys ks in a transaction, client calls the `get_trans(ks,ctx)`. Note that between two reads for different keys, the replica might be interleave and schedule other transactions. The challenge here is to ensure all the values are consistent, i.e. overall they satisfy the causal consistency. That is, if the transaction fetches a version \( \nu \) for key \( k \), and this version \( \nu \) depends on another version \( \nu' \) for another key \( k' \), then the transaction should at least fetches \( \nu' \) for the key \( k' \), or any later version for the key \( k' \).

The algorithm, in the first phase, optimistically reads the current latest version for each key from the replica via the replica API \( \text{rst}[k] = \text{get_by_version}(k, \text{LATEST}) \). In the second phase, it computes the maximum version \( \text{ccv}[k] \) from any dependencies \( \text{rst}[k].\text{deps} \) read in the first phase. Such \( \text{ccv}[k] \) is the minimum version that should be fetched. Therefore at the end, it only needs to re-fetched the specifically version \( \text{ccv}[k] \), if the version fetched in the first phase is older than \( \text{ccv}[k] \).

```
1 List(Val) get_trans(ks,ctx) {
2   for k in ks {
3     // only guarantee to read up-to-date value
4     // the moment reading the individual key
5     rst[k] = get_by_version(k, LATEST);
6   }
7
8   for k in ks {
9     ccv[k] = max (ccv[k], rst[k].ver);
10    for dep in rst[k].deps
11      if (dep.key \in ks)
12        ccv[k] = max (ccv[dep.key], dep.vers);
13  }
14
15  for k in ks
16    if ccv[k] > rst[k].vers
17      rst[k] = get_by_version(k, ccv[k]);
18
19  // update the ctx
20  for (k,ver,deps) in rst
21    ctx.readers += (k,ver,deps);
22
23  return to_vals(rst);
24 }
```

**Code 7: Reads**

The client API `get_by_version(k, ver)` returns the version `ver` for key `k`.

```
1 (Val,Version,Dep) get_by_version(k,ver) {
2   if (ver = LATEST){
3     ver = max(kv[k].vers);
4   }
5
6   wait until (_,ver,_) \in kv(k);
7
8   let (val,ver,deps) from kv[k];
9   return (val,ver,deps);
10 }
```

**Code 8: Replica API for read**

I.1.2. Verification.

**Semantics the code.** Let \( r \in \text{REPLS} \) denotes the set of totally ordered replicates. Each replicate can have multiple clients, and each clients can commit a sequence of either read-only transitions or single-write transactions. To model these, we annotate the transaction identifier with replicate \( r \), client \( cl \), local time of the replicate \( n \) and read-only transactions counter \( n' \), i.e. \( t_{(r,cl)}^{(n,r,n')} \). Note that the \( (n,r,n') \) can be treated as a single number that \( n \) are the higher bits, \( r \) the middle bits and \( n' \) the lower bits. For a new single-write transaction, it is allocated with a transaction identifier with larger local time, and for a read-only transactions, it is allocated with a transaction identifier with larger read-only counter. There is a total order among transitions from the same replica and from the same client.

To model the dependencies of each version, We extend version from Def. 3.1 with the set of all versions it dependencies on, \( dp \in \mathcal{P}(\text{KEY} \times \text{TRANSID}) \). The function \( \text{deps}(\nu) \) denotes the dependencies set of the version. We use \( \hat{K} \) for key-value store whose versions contain the dependencies.

We use view to model the client context, that is, a version is included in a context if and only if such version is in the view of the client. We also use view to model a replica state, that is, if a replica contains a version if and only if such
version in the view of the replica. For readability, we annotate view with either a replica, "ur", or a client, "ucld". The view environment is extended with replicas and their views, "U : (REPLS × CLIENT) → Views". We give the following semantics to capture the behaviours of the code.

Write. For purpose of verification, we eliminate code for performance, and put the client API and replica API in the same function (Code 9).

```plaintext
1 // mixing the client API and system API
2 put(repl,k,v,ctx) {
3     // Dependency for previous reads and writes
4     deps = ctx_to_depcntx();
5     // increase local time and appending local kv with a new version.
6     list_insert(repl.kv[k], (v, (local_time + id), deps));
7     // update dependency for writes
8     ctx.writers += (k, (local_time + id), deps);
9     // broad case
10    async_broadcase(k, v, (time ++ id), deps);
11    // update dependency for reads
12    ctx.readers += (k, ver, deps);
13    return (local_time + id);
14 }
```

Code 9: put

The following is the rule corresponds to Code 9.

**PUT**

\[
(dp = \{(k', t) \mid \exists i. \ i \in u_{cl}(k') \land t = w(K(k', i)) \lor (k', t) \in deps(K(k', i))\} \implies \text{skip})
\]

\[
t = \min\left\{t_{(r', cl', dp)}^n \mid \forall k', i \in u_{cl}(k'), n. \ t_{(r', cl', dp)}^{n'} = w(K(k', i)) \Rightarrow n' > n\right\}
\]

\[
u_{cl} = u_{r}[k \mapsto u_{r}(k) \uplus \{K'(k) - 1\}]\]

\[
u_{cl}' = u_{r}[K \mapsto K(k) \uplus \{K'(k) - 1\}]
\]

The first premise is to execute the transaction locally (Fig. 2). Since there is only a write, the snapshot ss can be any snapshot. The second line computes the dependency set for the new write operation, by collecting all the writers of versions included in the view ur. The third line simulates the increment of local time. Even though we do not directly track the local time of a replica, yet the local time can compute as the maximum time contained in the replica’s view ur. The forth and fifth simulates the updates of the replica’s key-value store, and the last premise simulates the update of client context.

Read. The following is a simplified algorithm by directly taking a list of versions ccv satisfies causal consistency constraint, i.e. the second phase of Code 7, and then read the versions indicated by ccv. The simplified algorithm is easier to understand, yet it is only for verification purpose because the extremely bad performance.

```plaintext
1 // A simplified version by guessing
2 // a ccv satisfying dependency constraints
3 // and then read versions indicated by ccv.
4 // Note that it is a weaker version of the original code,
5 // as the original implementation fetches the latest versions
6 // by for keys by a sequence of atomic get_by_version calls
7 List(Val) get_trans(ks,ctx) {
8     take ccv: \forall k \in ks.
9         (\_, \_, deps) := get_by_version(k,ccv[k]) \land \forall dep \in deps.
10            dep.key \in ks \Rightarrow ccv[dep.key] >= dep.ver
11    for k in ks
12        rst[k] = get_by_version(k,ccv[k]);
13    // update the ctx
14    for (k,verdeps) in rst
15         ctx.readers += (k,ver,deps);
16    return to_vals(ks);
```
The following is the rule for read-only transaction:

**GetTrans**

\[ u_{cl} \subseteq u'_{cl} \subseteq u_r \quad \rightarrow \quad \text{Code 10, lines 16 and 17} \]

\[(\forall k', m, \nu. \ \ And \nu = \text{ccv}(\bar{K}, k, u_{cl}) \land (k', w(\bar{K}(k', m))) \in \text{deps}(\nu) \Rightarrow m \in u'_{cl}(k')) \quad \rightarrow \quad \text{Code 10, lines 8 to 10} \]

\[ T = x_1 := [k_1]; \ldots; x_j := [k_j]; \]

\[ (s, \text{getMax}(\bar{K}, u'_{cl}), \emptyset), T \rightarrow (s', \emptyset, s', \text{skip}) \quad \rightarrow \quad \text{Code 10, lines 12 and 13} \]

\[ t^{(n', r, m)}(r, cl) = \max \left\{ t^{(\bar{r}, \bar{z})}(r, cl) \mid (\bar{r}, \bar{z}) \in \bar{K} \right\}. \quad K' = \text{update} \left( K, u'_{cl}, \mathcal{F}, t^{(n', r, m+1)}(r, cl) \right) \]

\[ r, cl \vdash \bar{K}, u_r, u_{cl}, s, [x_1 := [k_1]; \ldots; x_j := [k_j]] \quad \rightarrow \quad \text{K', ur, u'_{cl}, s', skip} \]

**ClientCommit**

\[ r, cl \vdash \bar{K}, \bar{U}(r), \bar{U}(cl), s, \mathcal{P}(cl) \quad \rightarrow \quad \text{K', u_r, u'_{cl}, s', C'} \]

\[ \bar{K}, \bar{U}, \bar{C}, \mathcal{P} \quad \rightarrow \quad \text{K', U[r \mapsto u'_{cl}[cl \mapsto u_{cl}], \mathcal{E}[cl \mapsto s'], \mathcal{P}[cl \mapsto C'] \}} \]

The first premiss pick a new view \( u'_{cl} \) for client, which later we will see it should be a new that corresponds to the ccv. Note that the new client view cannot see more versions than the replica to which the client connects. The second premiss constrains the new client view \( u'_{cl} \). It says for the version \( \nu \) that is included in the client view and has the maximum writers,

\[ \text{ccv}(\bar{K}, k, u_{cl}) \equiv \max \left\{ (v, t, T, dp) \mid \exists i. \ (v, t, T, dp) = \bar{K}(k, i) \land i \in u_{cl}(k) \right\} \]

\[ (v, t^{(n, r, m')}, T, dp) > (v', t^{(n', r', m''')}, T', dp') \iff (n, r, m') > (n'', r, m''') \]

any dependency of the version \( \nu \) is also included in the view. Note that \( \text{ccv}[k] \) corresponds \( \text{ccv}(\bar{K}, k, u_{cl}) \), where \( k = k \).

The third line simulates local execution of the read-only transaction. The client always fetches the version with the maximum writer it can observed for each key. Which is computed by getMax function. It is different from snapshot because snapshot fetches the latest snapshot with respect to the position in the list, but later one we will prove getMax and snapshot are equivalent.

\[ \text{getMax}(\bar{K}, u_{cl}) \equiv \lambda k. \left( \max_i \left\{ (v, t, T, dp) \mid \exists i. \ (v, t, T, dp) = \bar{K}(k, i) \land i \in u_{cl}(k) \right\} \right) \]

The rest part are trivial be picking a new transaction identifier with larger read counter and committing to key-value store.

The last rule for receiving a update. A replica updates its local state only if all the dependencies has been receive as shown in Code 5.

\[ \text{Sync} \]

\[ u_r = \bar{U}(r)[k \mapsto \bar{U}(r)(k) \cup i] \]

\[ (\forall k', m, \nu. \ \nu = \bar{K}(k, i) \land (k', w(\bar{K}(k', m))) \in \nu \Rightarrow m \in u'_{cl}(k')) \]

\[ \bar{K}, \bar{U}, \bar{C}, \mathcal{P} \rightarrow \bar{K}, \bar{U}[r \mapsto u'_{cl}], \mathcal{E}, \mathcal{P} \]

The premiss says, the first line, the replica \( r \) receive a new version \( i \) for key \( k \), and, the second line, only if all the dependencies of the new update already in the replica’s view.

**COPS Key-value store.** To verify the algorithm, we prove that for any COPS trace produced by the algorithm, there exists a corresponding causal consistency trace. First we prove that the key-value stores from COPS trace satisfy Def. [3.1]

**Theorem 1.1 (Well-formed COPS key-value).** Let ignore the dependencies of versions from \( \bar{K} \). Given the initial key-value store \( \bar{K}_0 \), initial views \( \bar{U}_0 \) and some programs \( \mathcal{P}_0 \), for any \( \bar{K}_i \) and \( \bar{U}_i \) such that:

\[ \bar{K}_0, \bar{U}_0, \mathcal{E}_0, \mathcal{P}_0 \rightarrow \bar{K}_i, \bar{U}_i, \mathcal{E}_i, \mathcal{P}_i \]

The key-value store \( \bar{K}_i \) satisfies Def. [3.1] and any replica or client view \( u \) from \( \bar{U}_i \) is a valid view of the key-value store, i.e. \( u \in \text{Views}(\bar{K}_i) \).

**Proof.** We need to prove the \( \bar{K}_i \) satisfies the well-formed conditions, and any view \( u_i \text{Views}(\bar{K}_i) \). We prove it by introduction on the length \( i \):

- **Base case:** \( i = 0 \). It holds trivially since each key only has the initial version \( (v_0, t_0, \emptyset, \emptyset) \). Since there is only the initial version for each key, it is easy to see that any view \( u_0 \) satisfying the well-formed conditions in Def. [3.2]

- **Inductive case:** \( i > 0 \). Suppose it holds when \( i \), let consider \( i + 1 \). We perform case analysis on the possible next step:
Similarly, a replica observes all its own transactions. Because a transaction can write to at most one key, then by Lemma I.3 and the premiss of P.

Proof. Lemma I.1 (Unique writer). Each version has a unique writer.

\begin{align}
\forall j. 0 \leq j < |\bar{K}_i(k)| \Rightarrow w(\bar{K}_i(k, j)) \neq t \tag{9.1} \\
\forall j, n. t_{(r, cl)}^{(n, r, -)} \in \{w(\bar{K}_i(k, j))\} \cup \text{rs}(\bar{K}_i(k, j)) \Rightarrow n < n' \tag{9.2}
\end{align}

Lemma I.2 implies Eq. (9.1) and Eq. (9.2). Thus the new key-value store $\bar{K}_{i+1}$ satisfies the well-formed conditions. Now let consider the views, especially the views of the replica $u'_r$ and the client $u'_{cl}$. Since that views $u'$ from different replicas or clients remain unchanged, by I.H. they satisfy $u' \in \text{VEWS}(\bar{K}_{i+1})$. The new view for replica $u'_r = u_r[k \mapsto |\bar{K}_{i+1}(k)| - 1]$ where $u_r$ is the replica’s view before updating and the writer of the last version of $k$ is $t$. Because $t$ is a single-write transaction, so the new view $u'_r$ still satisfies the atomic read. For similar reason, the new view for client $u'_{cl}$ still satisfies atomic read. Therefore we have $u'_{r}, u'_{cl} \in \text{VEWS}(\bar{K}_{i+1}).$

- **GetTrans** Assume the client $cl$ of a replica $r$ commits a read-only transaction $t$. Since it is a read-only transaction, it suffice to prove the following:

\begin{align}
\forall k, j. 0 \leq j < |\bar{K}_i(k)| \Rightarrow t \notin \text{rs}(\bar{K}_i(k, j)) \tag{9.3}
\end{align}

Lemma I.2 implies Eq. (9.3). Thus the new key-value store $\bar{K}_{i+1}$ satisfies the well-formed conditions. Now let consider the views. Since only the client view has changed, it is sufficient to prove that $u'_{cl} \in \text{VEWS}(\bar{K}_{i+1})$, where $u'_{cl}$ is the new client. By Lemma I.1 the new view $u'_{cl}$ satisfies the atomic read constraint. Therefore $u'_{cl} \in \text{VEWS}(\bar{K}_{i+1})$.

**Lemma I.3 (Observe its own).** A client observes all its own transactions, that is, for any key-value store $K$ and view $u_{cl}$:

\[\forall i, k, v, dp, t, T, t_{(-, cl)}^{(-, -)} \Rightarrow (v, t, T, dp) = \bar{K}(k, i) \land t = t_{(-, cl)}^{(-, -)} \lor t_{(-, cl)}^{(-, -)} \in T \Rightarrow i \in u_{cl}(k)\]

Similarly, a replica observes all its own transactions.
COPS normal trace. We define COPS semi-normal trace then normal trace. Later, we prove for any COPS trace, there exists an equivalent normal trace.

The dependency of a transaction $\text{deps}(\bar{K}, t)$ is defined as:

- if $t$ is a single-write transaction:
  \[
  \text{deps}(\bar{K}, t) \triangleq dp \text{ where } \exists k, j. \text{lastConf}(\tau)_{1}(k, j) = (-, -, -, dp)
  \]

- if $t$ is a read-only transaction:
  \[
  \text{deps}(\bar{K}, t) \triangleq \bigcup_{\nu} \{ \nu \mid \exists k, j. \nu = \bar{K}(k, j) \land t \in rs(\nu) \} \text{deps}(\nu)
  \]

Given a trace $\tau$, the function $\text{maxVersion}(\tau, t)$ returns the version that the transaction $t$ depends on and is the last one begin reduced:

\[
\text{maxVersion}(\tau, t) \triangleq t_i \quad \text{the } t_i \text{ is the last one appears in } \tau \text{ such that } (k', t_i) \in \text{deps}(\text{lastConf}(\tau)_{1}, t) \land t_i \notin \bar{K}_i \land t_i \notin \bar{K}_{i+1}
\]

Give two COPS’s traces $\tau$ and $\tau'$, $\bar{K}$ being the final state of $\tau$ and $\bar{K}'$ for $\tau'$, if the two traces are equivalent, if and only if,

\[
\forall t, F, dp. \ t \in K, F = \text{RWset}(t, K) \land dp = \text{RWset}(t, K) \Rightarrow t \in K', F = \text{RWset}(t', K) \land dp = \text{RWset}(t', K)
\]

where $\text{RWset}(t, K)$ is the read-write set of $t$. Note that $\text{RWset}(\cdot)$ is well-defined by Theorem I.1.

A COPS’s semi-normal trace is a trace $\tau$ if it is in the form that read-only transactions $t_{rd}$ directly follows its $\text{maxVersion}(\tau, t_{rd})$ or another read-only transaction $t'_{rd}$ such that $\text{maxVersion}(\tau, t_{rd}) = \text{maxVersion}(\tau, t'_{rd})$.

**Corollary I.1.** For any COPS’s trace $\tau$, there exists a equivalent semi-normal trace $\tau'$ such that $\text{lastConf}(\tau) = \text{lastConf}(\tau')$.

**Proof.** It is easy to prove by induction on the numbers of read-only transactions that are not in the wanted position. Let take the first one for those read-only transactions $t_{rd}$ who does not follows its $\text{maxVersion}(\tau, t_{rd})$. It is safe to move the reduction step to the right in the trace, until it directly follows its $\text{maxVersion}(\tau, t_{rd})$ or another read-only transaction $t'_{rd}$ such that $\text{maxVersion}(\tau, t_{rd}) = \text{maxVersion}(\tau, t'_{rd})$.

A COPS’s normal trace is a trace $\tau$ if it is semi-normal trace and the single-write transactions commit in the same order of the transaction identifiers.

**Theorem I.2** (normal trace). For any COPS’s trace $\tau$, there exists a equivalent normal trace $\tau'$ such that

**Proof.** By Cor. I.1 let consider a semi-normal trace $\tau$. It is easy to prove by induction on the single-write transactions that are out of order. Let take the first single-write transaction $t = trcln^l$ that is out of order. Suppose it is the $i$-step. Assume $t$ write to key $k$ with value $v$. Assume the preview write-only transaction $t' = trcln^l u'^m$ and it follows $(n, r, u', v') < (n'', r', v'')$. It means if a view $u$ includes both $t$ and $t'$, and it $t'$ also write to the same key $k$ with value $v'$, the $u$ will fetch the value from $t'$:

\[
\text{getMax}(\bar{K}_i, u)(k) = v'
\]

(9.4)

The intuition here is to swap the reduction steps of these two transactions and those read-only transactions following them. We assume $t$ and $t'$ write to the same key, otherwise it is safe to swap. For any read-only transactions $t_{rd} \in T_{rd}$ following $t$, we knows $t' \notin \text{deps}(() \bar{K}_i, t_{rd})$, otherwise it violate $\text{maxVersion}(\tau, t_{rd}) = t$. Thus it is safe to swap the reduction steps.

**Corollary I.2.** For the PUT and GETTRANs, it is safe to replace the function $\text{getMax}(\cdot)$ with the function $\text{snapshot}(\cdot)$.

**Proof.** By Theorem I.2 each trace $\tau$ has an equivalent normal trace $\tau'$. Assume $i$-th step in the trace $\tau'$. Assume the key-value store before is $\bar{K}_i$. For any versions $m$ and $j$ from the same key in the $\tau'$:

\[
0 < m < j < |\bar{K}_i(k)| \Rightarrow w(\bar{K}_i(k, m)) < w(\bar{K}_i(k, j))
\]

thus it is safe to snapshot $\cdot$.
COPS normal trace to $\text{ET}_{\text{CC}}$ trace. Given Theorem 1.1 and Cor. 1.2 for any COPS’s trace $\tau$, there exists a trace $\tau'$ that is a normal trace and satisfies $\text{ET}_{\tau'}$. Then by the following theorem (Theorem 1.3), we prove the COPS trace satisfies $\text{ET}_{\text{CC}}$.

**Theorem I.3** (COPS satisfying causal consistency). Given a trace starting from the initial key-value store $\bar{K}_0$, initial views $U_0$ and some programs $P_0$, for any $\bar{K}_i$ and $U_i$ such that:

$$\bar{K}_i, U_i, E_i, P_i \xrightarrow{u_i,F} \bar{K}_{i+1}, U_{i+1}, E_{i+1}, P_{i+1}$$

then the $i$-th step satisfies $\text{ET}_{\text{CC}}$, i.e.

$$\text{ET}_{\text{CC}} \vdash (\bar{K}_i, u_i) \triangleright F : (\bar{K}_{i+1}, \bar{U}_{i+1}(c))$$

**Proof.** We introduce an invariant $I$ on the view of client that if the view includes a version, it also includes all the version it depends on, that is,

$$\forall cl, u_cl, k, k', j, m, dp. \quad u_cl = \bar{U}_i(cl) \land j \in u_cl(k') \land (--, --, -) = \bar{K}_i(k, j) \land (k', w(\bar{K}_i(k', m))) \in dp$$

Since $\text{ET}_{\text{CC}} = \text{ET}_{\text{MR}} \cap \text{ET}_{\text{MW}} \cap \text{ET}_{\text{RYW}} \cap \text{ET}_{\text{WFR}}$, we prove the four constraints separately. In each case we need to consider $\text{PUT}$ and $\text{GETTRANS}$.

- $\text{ET}_{\text{MR}}$. It is easy to see $u_cl \subseteq \bar{U}_{i+1}(cl)$.
- $\text{ET}_{\text{MW}}$. By Lemma 1.3 and the invariant, we have to prove. Now we need to re-establish the invariant $I_{i+1}$.
  - For $\text{GETTRANS}$, the client cl commits a read-only transaction. By the premiss of the rule, the new view $\bar{U}_{i+1}(cl)$ satisfies the invariant.
  - For $\text{PUT}$, the new view $\bar{U}_{i+1}(cl) = u_cl[k \rightarrow j]$, where the new version $\bar{K}_{i+1}(k, j)$ is written by the client cl. Let $dp$ be the dependency for the new view, i.e. $dp = \text{deps}((\bar{K}_{i+1}(k, j)))$. By the premiss, the $dp$ includes all version included in $u_cl$, therefore the invariant has been preserved.
- $\text{ET}_{\text{RYW}}$. Let consider two cases:
  - For $\text{GETTRANS}$, the client cl commits a read-only transaction. By Lemma 1.3 and $u_cl \subseteq \bar{U}_{i+1}(cl)$, we have the prove.
  - For $\text{PUT}$, the new view $\bar{U}_{i+1}(cl) = u_cl[k \rightarrow j]$, where the new version $\bar{K}_{i+1}(k, j)$ is written by the client cl. Then by $u_cl \subseteq \bar{U}_{i+1}(cl)$, we have the proof.
- $\text{ET}_{\text{WFR}}$. By Lemma 1.3 and the invariant, we have to prove. Now we need to re-establish the invariant $I_{i+1}$.
  - For $\text{GETTRANS}$, the client cl commits a read-only transaction. By the premiss of the rule, the new view $\bar{U}_{i+1}(cl)$ satisfies the invariant.
  - For $\text{PUT}$, the new view $\bar{U}_{i+1}(cl) = u_cl[k \rightarrow j]$, where the new version $\bar{K}_{i+1}(k, j)$ is written by the client cl. Let $dp$ be the dependency for the new view, i.e. $dp = \text{deps}((\bar{K}_{i+1}(k, j)))$. By the premiss, the $dp$ includes all version included in $u_cl$, therefore the invariant has been preserved.

**Lemma I.4.** Given the initial key-value store $\bar{K}_0$, initial views $U_0$ and some programs $P_0$, for any $\bar{K}_i$ and $U_i$ such that:

$$\bar{K}_0, U_0, E_0, P_0 \xrightarrow{*} \bar{K}_i, U_i, E_i, P_i$$

for any versions $(v, t, \mathcal{E}, dp) = \bar{K}_i(k, j)$ and $(v', t', \mathcal{E}', dp') = \bar{K}_i(k', m)$, if $t$ depends on $t'$, i.e.

$$t' \xrightarrow{S_0'} t \lor \exists t''. \quad t'' \in \mathcal{T'} \land t'' \xrightarrow{S_0'} t$$

then $t'$ and $dp'$ is in the dependency $dp$, i.e.

$$(-, t') \in dp \land dp' \subseteq dp$$

**Proof.** We prove by induction on the length of trace $i$.

- Base case: $i = 0$. It trivially holds.
- Inductive case: $i > 0$. Suppose $\bar{K}_i$ satisfies the property, let consider the next key-value store $\bar{K}_{i+1}$. We perform case analysis.
  - PUT. Assume a client cl commit a transaction $t$ that installs a new version to key $k$. Assume the initial view for the client is $u_cl$. In the new key-value store $\bar{K}_{i+1}$, for versions other than $\bar{K}_{i+1}(k, |\bar{K}_{i+1}| - 1)$, they still satisfy the property by I.H. By Lemma 1.3, it follows that for any preview transaction from the same client $t'$:
    $$t' \xrightarrow{S_0'} t \lor \exists t''. \quad t'' \in \mathcal{T'} \land t'' \xrightarrow{S_0'} t$$
the version written by the transaction $t'$ is included in the view $u_d$. By the premiss of PUT that how the new $dp$ is constructed, we have proof.

- GETTRANS. Since it only commits a read-only transaction, the $\bar{K}_{i+1}$ satisfies the property by I.H.

\[\square\]

1.2. Clock-SI

1.2.1. Code.

**Structure.** Clock-SI is a partitioned distributed NoSQL database, which means each server, also called shard, contains part of keys and does not overlap with any other servers. Clock-SI implements snapshot isolation. To achieve that, each shard tracks the physical time. Note that times between shards do not match, but there is a upper bound of the difference.

```
1 Shard :: ID -> { clockTime }
```

Code 11: Shard

A key maintains a list of values and their versions. A version is the time when such value is committed.

```
1 VersionNo :: Time
2 KV :: Keys -> List{ Val, VersionNo }
3 (each key is associated with a shard)
```

Code 12: Key-value store

The idea behind Clock-SI is that a client starts a transaction in a shard, and the shard is responsible for fetching value from other shards if keys are not stored in the local shard. During execution, a transaction tracks the write set.

```
1 WS :: Key -> Val
```

Code 13: Write set

At the end, the transaction commits all the update in the write set, and the local shard acts as coordinator to update keys either locally or remotely. To commit a transaction, Clock-SI use two-phase commits protocol. A transaction has four states:

- active, the transaction is still running;
- prepared, shards receive the update requests from the coordinator;
- committing, shards receive the update confirmations from the coordinator;
- committed, the transaction commits successfully.

To implement SI, a transaction also tracks its snapshot time so it knows which version should be fetched. Also a transaction tracks the prepared and committing times, which are used to postpone other transactions’ reads if those transactions’ snapshots time are greater.

```
1 State :: { active, prepared, committing, committed }
2 Trans :: { state, snapTime, prepareTime, commitTime, ws }
```

Code 14: Transaction runtime

**Start Transaction.** Clock-SI proposes two versions, with or without session order. Here we verify the one with session order. To start a transaction, the client contacts a shard and provides the previous committing time. The shard will return a snapshot time, which is greater than the committing time provided, for the new transaction. Note that client might connects to a different shard from last time, which means that the shard might have to wait until the shard local time is greater than the committing time.

```
1 startTransaction( Trans t, Time ts )
2   wait until ts < getClockTime();
3   t.snapshotTime = getClockTime();
4   t.state = active;
```

Code 15: Transaction runtime

From this point, such transaction will always interact with the shard and the shard will act as coordinator if necessary.

**Read.** A transaction $t$ might read within the transaction if the key has been updated by the same transaction before, that is, read from the write set $ws$. A transaction $t$ might read from the original shard if the key store in the shard, but it has to wait until any other transactions $t'$ commit successfully who are supposed to commit before the current transaction’s snapshot time, i.e. $t'$ are in prepared or committing stage and the corresponding time is less the $t$ snapshot time.
Read( Trans t, key k )
if ( k in t.ws ) return ws(k);
if ( k is updated by t' and t'.state = committing and t.snapshotTime > t'.committingTime)
  wait until t'.state == committed;
if ( k is updated by t' and t'.state = prepared and t.snapshotTime > t'.preparedTime
  and t.snapshotTime > t'.committingTime )
  wait until t'.state == committed;
return K(k,i), where i is the latest version before t.snapshotTime;

Code 16: Read from original shard

If the key is not stored in the original shard, the original shard sends a read request to the shard containing the key. Because of time difference, the remote shard’s time might before the snapshot time of the transaction. In this case, the shard wait until the time catches up.

On read k request from a remote transaction t
wait until t.snapshotTime < getClockTime()
return read(t,k);

Code 17: Read from original shard

Commit Write Set. If all the keys in the write set are hosted in the original shard that the transaction first connected, the write set only needs to commit local.

localCommit( Trans t )
if noConcurrentWrite(t)
  t.state = committing;
  t.commitTime = getClockTime();
  log t.commitTime;
  log t.ws;
  t.state = committed;

Code 18: Local Commit

To commit local, it first checks, by noConcurrentWrite(t), if there is any transaction t’ that writes to the same key as the transaction new transaction t, and the transaction t’ commit after the snapshot of t. Since writing database needs time, it sets the transaction state to committing and log the commitTime, before the updating really happens. During committing state, other transactions will be pending, if they want to read the keys being updated. Last, the state of transaction is set to committed.

To commit to several shards, Clock-SI uses two-phase protocol.

distributedCommit( Trans t )
  for p in t.updatedPartitions
    send "prepare t" to p;
  wait receiving "t prepared" from all participants, store into prep;
  t.state = committing;
  t.commitTime = max(prep);
  log t.commitTime;
  t.state = committed;
  for p in t.updatedPartitions
    send "commit t" to p;

On receiving "prepare t"
  if noConcurrentWrite(t)
    log t.ws and t.coordinator ID
    t.state = prepared;
    t.prepareTime = getClockTime();
    send "t prepared" to t.coordinator

On receiving "commit t"
  log t.commitTime
  t.state = committed

Code 19: Distributed Commit

The original shard, who acts as the coordinator, sends "prepare t" to shards that will be updated. Any shard receiving "prepare t" checks, similarly, if there is any transaction write to the same key committing after the snapshot time. If the check passes, the shard logs the write set and the coordinator shard ID, set the state to prepared, and sends the local time to the coordinator. Once the coordinator receives all the prepared messages, it starts the second phase by setting the state to committing. Then the coordinator picks the largest time from all the prepared messages as the commit time for the new transaction. Since the write set has been logged in the first phase, so here it can immediately set the state...
to be committed. Last, the coordinator needs to send \texttt{commit} \( t \) to other shards so they will log the commit time and set the state to committed. Note that participants have different view for the new transaction from the coordinator, but it guarantees eventually they all updated to committed with the same commit time.

### 1.2.2. Verification.

\textbf{Structure.} We model the database use key-value store from Def. \ref{def:kv} yet here it is necessary to satisfy the well-formed conditions. Transaction identifier \( t'_n \) are labelled with the committing time \( n \). Sometime we also write \( t'_n \) or omit the client label, i.e. \( t^n \) and \( t' \).

Database is partitioned into several shards. A shard \( r \in R \subseteq \text{SHARDS} \) contains some keys which are disjointed from keys in other shards. The shardOf \( (k) \) denotes the shard where the key \( k \) locates.

Shards and clients are associated with clock times, \( c \in \text{CLOCKTIMES} \equiv \mathbb{N} \), which represent the current times of shards and clients. We use notation \( C \in (\text{SHARDS} \cup \text{CLIENT}) \cap \text{CLOCKTIMES} \).

We will use notation \( \lfloor T \rfloor \) to denote the static code of a transaction, and \( \lfloor T \rfloor C \) for the runtime of a transaction, where \( c \) is the snapshot time and \( F \) is the read-write set. Note that in the model, we only distinguishes \texttt{active} and \texttt{committed} state, since the prepared and committing are only for better performance.

\textbf{Start Transaction.} To start a transaction, it picks a random shard \( r \) as the coordinator, reads the local time \( C(r) \) as the snapshot, and sets the initial read-write set to be an empty set. Also the client time is updated to the snapshot time. For technical reason, we also update the shard time to avoid time collision to other transaction about to commit. Note that in real life, all the operations running in a shard take many time cycles, so it is impossible to have time collision.

\[
\text{STARTTRANS} \quad c < C(r) \quad C' = C[r \mapsto C(r) + 1] \quad \rightarrow \quad \text{simulate time elapses}
\]

\[
cl \vdash C, c, s, \lfloor T \rfloor \quad C(r), C', s, \lfloor T \rfloor C(r)
\]

\textbf{Read.} The clock-SI protocol includes some codes related to performance which does not affect the correctness. Clock-SI distinguishes a local read/commit and a remote read/commit, yet it is sufficient to assume all the read and commit are "remote", while the local read and commit can be treated as self communication. Similarly we assume a transaction always commits to several shards.

\begin{enumerate}
  \item On receive \textquote{\texttt{read}(t,k)'} \{ 
  \item if ( k in t.ws ) return ws(k);
  \item assert( t.snapshotTime < getClockTime() )
  \item for t' that writes to k:
  \hspace{1em} if(t.snapshotTime > t'.preparedTime
  \hspace{2em} || t.snapshotTime > t'.committingTime)
  \hspace{3em} assert( t.state == committed )
  \hspace{1em} return K(k,i), where i is the latest version before t.snapshotTime;
\end{enumerate}

\texttt{Code 20: simplified read}

If the key exists in the write set \( F \), the transaction read from the write set immediately.

\[
\text{READTRANS} \quad k = \llbracket E \rrbracket_s \quad (w,k,v) \in F \quad \rightarrow \quad \text{Code 20, line 2}
\]

\[
cl \vdash K, c, s, \lfloor x : E \rfloor \quad F \quad c, F \llbracket (w,k,v) \rrbracket \quad \rightarrow \quad K, c, s, \lfloor x \mapsto v \rfloor, \llbracket \text{skip} \rrbracket F \llbracket (w,k,v) \rrbracket
\]

Otherwise, the transaction needs to fetch the value from the shard. The first premise says the transaction must wait until the shard local time \( C(\text{shardOf} (k)) \) is greater than the snapshot time \( c \). If so, by the second line, the transaction fetches the latest version for key \( k \) before the snapshot time \( c \).

\[
\text{READREMOTE} \quad k = \llbracket E \rrbracket_s \quad (w,k,v) \notin F \quad c < C(\text{shardOf} (k)) \quad \rightarrow \quad \text{Code 20, line 4}
\]

\[
n = \max \{ n' \mid \exists j, t'' = w(K(k,j)) \land n' < c \} \quad \rightarrow \quad \text{Code 20, line 11}
\]

\[
cl \vdash K, c, s, \lfloor x : E \rfloor \quad F \quad c, F \llbracket (w,k,v) \rrbracket \quad \rightarrow \quad K, c, s, \lfloor x \mapsto v \rfloor, \llbracket \text{skip} \rrbracket F \llbracket (w,k,v) \rrbracket
\]

\textbf{Write.} Write will not go to the shard until committing time. Before it only log it in the write set.

\[
\text{WRITE} \quad k = \llbracket E_1 \rrbracket_s \quad v = \llbracket E_2 \rrbracket_s
\]

\[
cl \vdash K, c, s, \llbracket E_1 \rrbracket \quad F \quad cl, F \llbracket (w,k,v) \rrbracket \quad \rightarrow \quad K, c, s, \llbracket \text{skip} \rrbracket F \llbracket (w,k,v) \rrbracket
\]
Commit. We also assume transaction always commit to several shards and the local commit is treated as self-communication.

```plaintext
commit(Trans t)
  for p in t.updatedPartitions
    send "prepare t" to p;
  wait receiving "t prepared" from all participants, store into prep;
  t.state = committing;
  t.commitTime = max(prep);
  log t.commitTime;
  t.state = committed;
  for p in t.updatedPartitions
    send "commit t" to p;
```

Code 21: simplified commit

Note that Clock-SI uses two phase commit: the coordinator (the shard that the client directly connects to) distinguishes "committing" state and "committed" state, where in between the coordinator pick the committing time and log the write set, and the participants distinguishes "prepared" state and "committed" state. Such operations are for possible network partition or single shard errors, and allowed a more fine-grain implementations which do not affect the correctness, therefore it suffices to assume they are one atomic step.

\[
\begin{align*}
\forall k, i. \ (w, k, -) ∈ F ∧ w(Κ(k,i)) < c & \implies K' = \text{commitKV}(Κ, c, t^n_0, F) \implies \text{Code 21, lines 4 to 6} \\
n = \max\{c' \mid \exists k, i. \ (w, k, -) ∈ F ∧ c' = C(\text{shardOf}(k)) \cup \{c\}\} & \implies \text{Code 21, lines 20 and 21} \\
∀ r. \ (r ∈ \{\text{shardOf}(k) \mid (w, k, -) ∈ F\}) \implies C'(r) = C(r) & \implies \text{simulate time elapses}
\end{align*}
\]

\[
\begin{align*}
r, cl ⊬ Κ, c, C, s, [\text{skip}]_{c} & c, c, F, n \xrightarrow{\text{commitKV}(Κ, c, t^n_0, F),} K', n + 1, C', s, \text{skip}
\end{align*}
\]

To commit the new transaction, it needs to check, by the first premiss, there is no other transactions writing to the same keys after the snapshot time. If it passes, by the second line it picks the maximum time \(n\) among all participants as the commit time. The new key-value store \(K' = \text{commitKV}(Κ, c, t^n_0, F)\), where

\[
\begin{align*}
\text{commitKV}(Κ, c, t, F ∪ \{(w, k, v)\}) & \triangleq \text{let } n = \max\{n' \mid \exists j. \ t^{n'} = w(Κ(k,j)) ∧ n' < c\} \\
& \text{and } Κ(k,i) = (v, t^n, T) \\
& \text{and } K' = \text{commitKV}(Κ, c, t, F) \\
& \text{in } K'[k ↦ K'(k)[i ↦ (v, t^n, T ∪ \{t\})]]
\end{align*}
\]

\[
\begin{align*}
\text{commitKV}(Κ, c, t, F ∪ \{(w, k, v)\}) & \triangleq \text{let } K' = \text{commitKV}(Κ, c, t, F) \\
& \text{in } K'[k ↦ K'(k)[i ↦ (v, t^n, T ∪ \{t\})]]
\end{align*}
\]

Note that commitKV is similar to update by appending the new version to the end of a list. The commitKV also updates versions read by the new transaction using the snapshot time of the transaction. Last, like STARTTRANS we update the client time after the commit time, i.e. \(n + 1\) and simulate time elapses for all shards updated.

**Time Tick.** For technical reasoning, we have non-deterministic time elapses.

\[
\begin{align*}
Κ, C, C', E, P & \xrightarrow{r, c, C[r → C(r) + 1], E, P} Κ, C, C'[r ↦ C(r) + 1], E, P
\end{align*}
\]

**CLIENTSTEP**

\[
\begin{align*}
Κ, C, C', E, P & \xrightarrow{c, c, E[cl ↦ c], C'[r ↦ C'[r ↦ C(r)]}, E[cl ↦ c], C[cl ↦ c]}
\end{align*}
\]
**Verification.** Clock-SI allows interleaving, yet for any clock-si trace $\tau$ there exists an equivalent trace $\tau'$ where transactions do not interleave with others (Theorem I.4). Furthermore, in such trace $\tau'$, transactions are reduced in their commit order.

**Theorem I.4** (Normal clock-SI trace). A clock-SI trace $\tau$ is a clock-SI normal trace if it satisfies the following: there is no interleaving of a transaction,

\[
\forall cl, c, \tau = \cdots \xrightarrow{cl, c, \mathcal{F}, t, i, n} \cdots \xrightarrow{cl, c', \mathcal{F}, t, j, m} \cdots
\]

and transactions in the trace appear in the committing order,

\[
\forall cl, c, \tau = \cdots \xrightarrow{cl, c, \mathcal{F}, t, i, n} \cdots \xrightarrow{cl, c', \mathcal{F}, t, j, m} \cdots
\]

For any clock-SI trace $\tau$, there exists an equivalent normal trace $\tau'$ which has the same final configuration as $\tau$.

**Proof.** Given a trace $\tau$, we first construct a trace $\tau'$ that satisfies Eq. (9.6), by swapping steps. Let take the first two transactions $t_{cl_1}^n$ and $t_{cl_2}^m$ that are out of order, i.e. $n > m$ and

\[
\tau = \cdots \xrightarrow{cl_1, c_1, \mathcal{F}, t, i, n} \cdots \xrightarrow{cl_2, c_2, \mathcal{F}, t, j, m} \cdots
\]

By Lemma I.5, the two clients are different $cl_1 \neq cl_2$ and thus two steps are unique in the trace. We will construct a trace that $t_{cl_1}^n$ commits after $t_{cl_2}^m$.

- First, it is important to prove that $t_{cl_1}^n$ does not read any version written by $t_{cl_2}^m$. By Lemma I.17 the snapshot time $c_j$ of $t_{cl_1}^n$ is less than the commit time, i.e. $c_j < m$, therefore $c_j < n$. By the READ rule, $c_j < n$ implies the transaction $t_{cl_1}^n$ never read any version written by $t_{cl_2}^m$.

- Let consider any possible time tick for those shard $r$ that has been updated by $t_{cl_2}^m$, that is, $r = \text{shardOf}(k)$ for some key $k$ that $(\mathcal{w}, k, -) \in \mathcal{F}$ and

\[
\tau = \cdots \xrightarrow{cl_1, c_1, \mathcal{F}, t, i, n} \cdots \xrightarrow{cl_2, c_2, \mathcal{F}, t, j, m} \cdots
\]

Since $c_j < m < n < c$, therefore such time tick will not affect the transaction $t_{cl_1}^n$, which means it is safe to move the time tick step after the $t_{cl_1}^n$.

Now we can move the commit of $t_{cl_1}^n$ and time tick steps similar to Eq. (9.7) after the commit of $t_{cl_2}^m$,

\[
\tau' = \cdots \xrightarrow{cl_2, c_2, \mathcal{F}, t, j, m} \cdots
\]

We continually swap the out of order transaction until the newly constructed trace $\tau'$ satisfying Eq. (9.6).

Now let consider Eq. (9.5). Let take the first transaction $t$ whose read has been interleaved by other transaction or a time tick.

- If it is a step that the transaction $t$ read from local state,

\[
\tau = \cdots \xrightarrow{cl, c, \mathcal{F}, \mathcal{R}(z, k, v), i, -} \cdots \xrightarrow{cl, c, \mathcal{F}, t, j, m} \cdots
\]

then by READTRANS we know $\mathcal{F} \mathcal{R}(z, k, v) = \mathcal{F}$, and it is safe to swap the two steps as the following

\[
\tau' = \cdots \xrightarrow{cl, c, \mathcal{F}, t, j, m} \cdots
\]

- If it is a step that the transaction $t$ read from remote, the step might be interleaved by a step from other transaction or time tick step.

* if it is interleaved by the commit of other transaction $t' = t_{cl'}^n$, that is

\[
\tau = \cdots \xrightarrow{cl, c, \mathcal{F}, t, j, m} \cdots
\]

where $cl' \neq cl$.

* if the transaction $t'$ does not write to any key $k$ that is read by $t$,

\[
\forall k. (z, k, -) \in \mathcal{F} \Rightarrow (w, k, -) \notin \mathcal{F}
\]
In this case, it is safe to swap the two steps

\[ t' = \cdots \xrightarrow{cl,c,F',m} \xrightarrow{cl,c,F,\bot} \cdots \xrightarrow{cl,c,F''} \cdots \]

* if the transaction \( t' \) write to a key \( k \) that is read by \( t \),

\[ (x,k,-) \in F \land (w,k,-) \in F' \]

Let \( r = \text{shardOf}(k) \). By the ReadRemote, we know the current clock time for the shard \( r \) is greater than \( c \) which is the snapshot time of \( t \), that is, \( C'(r) > c \). Then by Commit, the commit time of \( t' \) is picked as the maximum of the shards it touched, i.e. \( m \geq C'(r) \). Now by the ReadRemote and \( m \geq c \), it is safe to swap the two steps since the new version of \( k \) does not affect the \( t \).

- if it is interleaved by the read of other transaction \( t' \), that is

\[ \tau = \cdots \xrightarrow{cl,c,F,\bot} K, C, C', E, P \xrightarrow{cl,c,F',\bot} \cdots \xrightarrow{cl,c,F''} \cdots \]

Because reads have no side effect to any shard by ReadRemote, it is safe to swap the two steps

\[ \tau' = \cdots \xrightarrow{cl,c,F',\bot} \xrightarrow{cl,c,F,\bot} \cdots \xrightarrow{cl,c,F''} \cdots \]

- if it is interleaved by a time tick step,

\[ \tau = \cdots \xrightarrow{cl,c,F \circ \tau(x,k,v),\bot} K, C, C', E, P \xrightarrow{r,c'} \cdots \xrightarrow{cl,c,F''} \cdots \]

* if the transaction \( t \) does not read from the shard \( r \), it means for any key \( k \),

\[ \text{shardOf}(k) \neq r \]

In this case, it is safe to swap the two steps

\[ \tau' = \cdots \xrightarrow{r,c'} \xrightarrow{cl,c,F \circ \tau(x,k,v),\bot} \cdots \xrightarrow{cl,c,F''} \cdots \]

* if the transaction \( t \) read from the shard \( r \), it means that there exists a key \( k \)

\[ \text{shardOf}(k) = r \]

By the ReadRemote, we know the current clock time for the shard \( r \) is greater than the snapshot time of \( t \), that is, \( C'(r) > c \). Then by TimeTick, we have \( c' > C'(r) \). Now by the ReadRemote and \( c' > c \), it is safe to swap the two steps.

\[ \square \]

**Lemma 1.5 (Monotonic client clock time).** The clock time associated with a client monotonically increases, That is, given a step

\[ K, C, C', E, P \xrightarrow{cl,c,F} K', C', E', P' \]

then for any clients \( cl \),

\[ C(cl) \leq C(cl') \]

**Proof.** It suffices to only check the ClientStep rule which is the only rule updates the client clock time, especially, it is enough to check the client \( cl \) that who starts or commits a new transaction.

* COMMIT. Let \( c \) be the clock time before committing, \( c = C(cl) \). By the premiss of the rule, the new client time \( n + 1 \) satisfies that,

\[ n = \max \left( \{ c' \mid \exists k. \ (-,k,-) \in F \land c' = C(\text{shardOf}(k)) \} \cup \{ c \} \right) \]

It means \( c < (n + 1) \).

* STARTTRANS. Let \( c \) be the clock time before taking snapshot, \( c = C(cl) \). By the premiss of the rule the new client time \( C'(r) \) for a shard \( r \), such that \( c < C'(r) \).

\[ \square \]

**Lemma 1.6 (No side effect local operation).** Any transactional operation has no side effect to the shard and key-value store,

\[ K, C, C', E, P \xrightarrow{cl,c,F,\bot} K', C', E', P' \Rightarrow K = K' \]

**Proof.** It is easy to see that StartTrans, ReadTrans, ReadRemote and Write do not change the state of key-value store.

\[ \square \]
Clock-SI also has a notion view which corresponds the snapshot time. The following definition \( \text{viewOf}(K, c) \) extracts the view from snapshot time.

**Definition I.1.** Given a normal clock-SI trace \( \tau \) and a transaction \( t_{cl} \), such that

\[
\tau = \cdots \text{cl}, c, \emptyset, \downarrow \cdots \text{cl}, c, F, \downarrow \cdots \text{cl}, c, F, c', \downarrow \cdots
\]

the initial view of the transaction is defined as the following:

\[
\text{viewOf}(K, c) \triangleq \lambda \{ i \mid \exists t^n. w(K(k, i)) = t^n \wedge n < c \}
\]

Given the view \( \text{viewOf}(K, c) \) for each transaction, we first prove that clock-si produces a well-formed key-value store (Def. [I.1]).

**Lemma I.7.** Given any key-value store \( K \) and snapshot time \( c \) from a clock-SI trace \( \tau \),

\[
\tau = \cdots \rightarrow K, c, c', \downarrow \cdots \text{cl}, c, F, c' \cdots
\]

\( \text{viewOf}(K, c) \) and \( \text{viewOf}(K, c') \) (Def. [I.1]) produce well-formed views.

**Proof.** It suffices to prove that Eq. (atomic) in Def. [I.2]. Assume a key-value store \( K \) and a snapshot time \( c \). Suppose a version \( i \) in the view \( i \in \text{viewOf}(K, c)(k) \) for some key \( k \). By Def. [I.1] the version is committed before the snapshot time, i.e. \( t^n = w(K(k, i)) \wedge n < c \). Assume another version \( t^n = w(K(k', j)) \) for some key \( k' \) and index \( j \). By Def. [I.1] we have \( j \in \text{viewOf}(K, c)(k') \). Similarly \( \text{viewOf}(K, c') \) is a well-formed view.

Second, given the view \( \text{viewOf}(K, c) \) for each transaction, both \text{commitKV} \ and \text{update} \ produce the same result.

**Lemma I.8.** Given a normal clock-SI trace \( \tau \) and a transaction \( t_{cl} \), such that

\[
\tau = \cdots \text{cl}, c, \emptyset, \downarrow \cdots \text{cl}, c, F, \downarrow \cdots \text{cl}, c, F, c', \downarrow \cdots
\]

the following holds:

\[
\text{commitKV}(K, c, t^c_{cl}, F) = \text{update}(K, \text{viewOf}(K, c), F, t^c_{cl})
\]

**Proof.** We prove by induction on \( F \).

- Base case: \( F = \emptyset \). It is easy to see that \( \text{commitKV}(K, c, t^c_{cl}, \emptyset) = K = \text{update}(K, \text{viewOf}(K, c), F, t^c_{cl}) \).

- Inductive case: \( F \uplus (w, k, v) \). Because in both functions, the new version is installed at the tail of the list associated with \( k \),

\[
\text{commitKV}(K, c, t^c_{cl}, F \uplus (w, k, v)) = \text{commitKV}(K, c, t^c_{cl}, F) [k \mapsto K(k) :: [(v, t, \emptyset)]]
\]

\[
= \text{update}(K, \text{viewOf}(K, c), F, t^c_{cl}) [k \mapsto K(k) :: [(v, t, \emptyset)]]
\]

\[
= \text{update}(K, \text{viewOf}(K, c), F \uplus (w, k, v), t^c_{cl})
\]

- Inductive case: \( F \uplus (x, k, v) \). Let \( K(k, i) \) be the version being read. That is, the writer \( t^n = w(K(k, i)) \) is the latest transaction written to the key \( k \) before the snapshot time \( c \),

\[
n = \max \{ n' \mid \exists j, t^{n'} = w(K(k, j)) \wedge n' < c \}
\]

Let the new version \( \nu = \text{val}(K(k, i)), w(K(k, i)), rs(K(k, i)) \uplus \{ t^c_{cl} \} \). By Lemma I.7 it follows \( i \in \text{viewOf}(K, c)(k) \), then by Lemma I.9 the version is the latest one \( i = \max(\text{viewOf}(K, c)(k)) \). Therefore, we have,

\[
\text{commitKV}(K, c, t^c_{cl}, F \uplus (x, k, v)) = \text{commitKV}(K, c, t^c_{cl}, F) [k \mapsto K(k)[i \mapsto \nu]]
\]

\[
= \text{update}(K, \text{viewOf}(K, c), F, t^c_{cl}) [k \mapsto K(k)[i \mapsto \nu]]
\]

\[
= \text{update}(K, \text{viewOf}(K, c), F \uplus (x, k, v), t^c_{cl})
\]
Lemma 1.9 (Strictly monotonic writers). Each version for a key has a writer with strictly greater clock time than any versions before:

$$\forall K, k, i, j, t^m, t^n. \ w(K(k, i)) = t^n \land w(K(k, j)) = t^m \land i < j \Rightarrow n < m$$

By Theorem 1.4, it is sufficient to only consider normal clock-SI trace. Since transactions do not interleave in a normal clock-SI trace, all transactional execution can be replaced by Fig. 2.

Theorem 1.5 (Simulation). Given a clock-SI normal trace $\tau$, a transaction $t^m_c$ from the trace, and the following transactional internal steps

$$K_0, c_0, C_0, s_0, \{T\} \xrightarrow{cl.c,i,\downarrow} \ldots \xrightarrow{cl.c,F,n} K_i, c_i, C_i, s_i, [\text{skip}]$$

for some $i$, there exists a trace

$$(s_0, \text{snapshot}(K_0, \text{viewOf}(K_0, c)), \emptyset), T \rightarrow^* (s_i, ss_i, F_i), \text{skip}$$

that produces the same final fingerprint in the end.

Proof. Given the internal steps of a transaction

$$K_0, c_0, C_0, s_0, \{T_0\}_c \xrightarrow{F_0} \ldots \xrightarrow{F_i} K_i, c_i, C_i, s_i, \{T_i\}_c$$

We construct the following trace,

$$(s_0, \text{snapshot}(K_0, \text{viewOf}(K_0, c)), F_0), T_0 \rightarrow^* (s_i, ss_i, F_i), T_i$$

Let consider how many transactional internal steps.

- Base case: $i = 0$. In this case,

$$K_0, c_0, C_0, s_0, \{T\}_c$$

It is easy to construct the following

$$(s_0, \text{snapshot}(K, \text{viewOf}(K, c)), F_0), T_0$$

- Inductive case: $i + 1$. Suppose a trace with $i$ steps,

$$K_0, c_0, C_0, s_0, \{T_0\}_c \xrightarrow{F_0} \ldots \xrightarrow{F_i} K_i, c_i, C_i, s_i, \{T_i\}_c$$

and a trace

$$(s_0, \text{snapshot}(K_0, \text{viewOf}(K_0, c)), F_0), T_0 \rightarrow^* (s_i, ss_i, F_i), T_i$$

Now let consider the next step.

- READTRANS. In this case

$$K_i, c_i, C_i, s_i, \{T_i\}_c \xrightarrow{F_i} \ldots \xrightarrow{F_i} K_{i+1}, c_{i+1}, C_{i+1}, s_{i+1}, \{T_{i+1}\}_c$$

such that

$$F_{i+1} = F_i \triangleleft (x, k, v) = F_i \land (w, k, v) \in F_i$$

for some key $k$ and value $v$, and

$$T_i = x := [E] ; T \land [E]_{s_i} = k \land s_{i+1} = s_i[\rightarrow v] \land T_{i+1} \equiv \text{skip}; T$$

for some variable $x$, expression $E$ and continuation $T$. Since $(w, k, v) \in F_i$, it means $ss_i(k) = v$ for the local snapshot. By the TPRIMITIVE, we have

$$(s_i, ss_i, F_i), T_i \rightarrow x := [E] ; T \rightarrow (s_{i+1}, ss_i, F_{i+1}), T_{i+1}$$

- READREMOTE. In this case

$$K_i, c_i, C_i, s_i, \{T_i\}_c \xrightarrow{F_i} \ldots \xrightarrow{F_i} K_{i+1}, c_{i+1}, C_{i+1}, s_{i+1}, \{T_{i+1}\}_c$$

such that

$$F_{i+1} = F_i \triangleleft (x, k, v) = F_i \cup \{(x, k, v)\} \land \forall v. (w, k, v') \notin F_i$$

for some key $k$ and value $v$, and

$$T_i = x := [E] ; T \land [E]_{s_i} = k \land s_{i+1} = s_i[\rightarrow v] \land T_{i+1} \equiv \text{skip}; T$$
for some variable x, expression E and continuation T. By the premiss of the READREMOTE, the value read is from the last version before the snapshot time:
\[
n = \max \{ n' \mid \exists j. t^m = w(K(k, j)) \land n' < c \} \land \text{val}(K(k, n)) = v
\]

By the definition of \( u_0 = \text{viewOf}(K_0, c) \) and snapshot \( (K_0, u_0) \) and the fact that there is no write to the key \( k \), it follows \( ss_i(k) = v \). Thus, by the TPRIMITIVE, we have
\[
(s_i, ss_i, F_i), T_i \rightarrow x := [E]; T \rightarrow (s_{i+1}, ss_i, F_{i+1}), T_{i+1}
\]

- WRITE. In this case
\[
\forall k, t_i \rightarrow K_i, c_i, s_i, [T_i]_c \xrightarrow{F_i} K_{i+1}, c_{i+1}, s_{i+1}, [T_{i+1}]_c
\]
such that
\[
F_{i+1} = F_i \ll (w, k, v) = F_i \setminus \{(w, k, v') \mid v' \in \text{VAL}\} \cup \{(x, k, v)\}
\]

for some key \( k \) and value \( v \), and
\[
T_i \equiv [E_i] := E_2; T \land \lceil E_1 \rceil_{s_i} = k \land \lceil E_2 \rceil_{s_i} = v \land T_{i+1} \equiv \text{skip}; T
\]

for some expressions \( E_1 \) and \( E_2 \), and continuation \( T \). By the TPRIMITIVE, it is easy to see:
\[
(s_i, ss_i, F_i), T_i \rightarrow [E_i] := E_2; T \rightarrow (s_{i+1}, ss_i, F_{i+1}), T_{i+1}
\]

By Def. [L1] Lemma [L7] and Theorem [L5] we know for each clock-SI trace, there exists a trace that satisfies ET \( \subseteq \). Last, we prove such trace also satisfies ET \( \subseteq \).

**Theorem I.6** (Clock-SI satisfying SI). For any normal trace clock-SI trace \( \tau \), and transaction \( t^n \) such that
\[
\tau = \cdots \xrightarrow{\text{cl},c,F,P} K, C, C', E, P \xrightarrow{\text{cl},c,F,P} K', C'', E', P' \xrightarrow{\cdots}
\]
the transaction satisfies ET \( \subseteq \), i.e., ET \( \subseteq \vdash (K, \text{viewOf}(K, c)) \xrightarrow{F} \text{viewOf}(K, C''(cl))

**Proof.** Recall ET \( \subseteq \) = \{ (K, u, F, u') \mid \} \cap ET_{\text{MR}} \cap ET_{\text{RW}} \cap ET_{\text{UA}} \) Note that final view of the client, \( C''(cl) = n + 1 \). We prove the four parts separately.

- \{ (K, \text{viewOf}(K, c), F, \text{viewOf}(K, C''(cl))) \mid \} Assume a version \( i \in \text{viewOf}(K, c)(k) \) for some key \( k \). Suppose a version \( K(k', j) \) such that
\[
w(K(k', j)) \xrightarrow{((\text{SO} \cup \text{WR} \cup \text{WW} \cup \text{K}) + \text{RW} = ?)} w(K(k, i))
\]

Let \( t^n = w(K(k', j)) \) and \( t^m = w(K(k, i)) \). By Lemmas [L10] and [L11] we know \( n < m \) then \( j \in \text{viewOf}(K, c)(k') \).

- **ET_{MR}.** By COMMIT, we know \( c \leq n < C''(cl) \) then \( \text{viewOf}(K, c) \subseteq \text{viewOf}(K, C''(cl)) \).

- **ET_{MR}.** By COMMIT, for any write \((w, k, v) \in F\), there is a new version written by the client \( cl \) in the \( K' \),
\[
w(K'(k, |K''(k)| - 1)) = t^n_{cl}
\]

Since \( n < C''(cl) \), it follows \( |K'(k)| - 1 \in \text{viewOf}(K, C''(cl))(k) \).

- **ET_{UA}.** By the premiss of COMMIT, for any write \((w, k, v) \in F\), any existing versions of the key \( k \) must be installed by some transactions before the snapshot time of \( c \),
\[
\forall k, i. \ (w, k, v) \in F \land w(K(k, i)) < c
\]

It implies that
\[
\forall i. \ i \in \text{dom}(K(k)) \Rightarrow i \in \text{viewOf}(K, c)(k)
\]

**Lemma I.10** (RW\(\subseteq\)). Given a normal clock-SI trace \( \tau \), and two transactions \( t^n_{cl} \) and \( t^m_{cl} \) from the trace
\[
\tau = \cdots \xrightarrow{\text{cl},c,F,P} K, C, C', E, P \xrightarrow{\text{cl},c,F,P} \cdots \land \tau = \cdots \xrightarrow{\text{cl},c,F,P} K', C'', E', P' \xrightarrow{\text{cl},c',F',m} \cdots
\]

Suppose the final state of the trace \( \tau = K''(\cdot) \), if \( t^n_{cl} \xrightarrow{\text{RW} = ?} t^m_{cl} \) then the snapshot time of \( t^n_{cl} \) took snapshot before the commit time of \( t^m_{cl} \), i.e. \( c \leq m \).
Lemma I.11 (WR\(_K\), WW\(_K\) and SO\(_K\)). Given a normal clock-SI trace \(\tau\), and two transactions \(t^m_{cl}\) and \(t^n_{cl}\) from the trace

\[
\tau = \cdots \rightarrow k', c', c'', \mathcal{E}', \mathcal{P} \xrightarrow{c', c', \mathcal{F}, m} \cdots \rightarrow k, c, c', \mathcal{E}, \mathcal{P} \xrightarrow{c', c', \mathcal{F}, m} \cdots
\]

Suppose the final state of the trace \(\tau\) is \(K''\). If \(t^n_{cl} \xrightarrow{\text{WR}_K} t^m_{cl}\), then the transaction \(t^n_{cl}\) commit before the commit time of \(t^m_{cl}\), i.e. \(n < m\). Similarly, \(n < m\) for the relations WW\(_K\) and SO\(_K\).

Proof.

- **WR\(_K\)**. Since \(t^n_{cl} \xrightarrow{\text{WR}_K} t^m_{cl}\), it is only possible that the later commits after the former,

\[
\tau = \cdots \rightarrow k, c, c', \mathcal{E}, \mathcal{P} \xrightarrow{c, c, \mathcal{F}, n} \cdots \rightarrow k', c', c'', \mathcal{E}', \mathcal{P} \xrightarrow{c', c', \mathcal{F}, m} \cdots
\]

By Lemma I.12 we know \(n < m\).

- **WW\(_K\)**. By the definition of WW\(_K\) and Lemma I.9 we know \(n < m\).

- **SO\(_K\)**. By the definition of SO\(_K\) and Lemma I.5 we know \(n < m\).

\[\square\]

Lemma I.12 (Reader greater than writer). Assume a trace \(\tau\) and two transactions \(t^n_{cl}\) and \(t^m_{cl}\),

\[
\tau = \cdots \rightarrow k, c, c', \mathcal{E}, \mathcal{P} \xrightarrow{c, c, \mathcal{F}, n} \cdots \rightarrow k', c', c'', \mathcal{E}', \mathcal{P} \xrightarrow{c', c', \mathcal{F}, m} \cdots
\]

Assume the final state of key-value store of the trace is \(K''\). If \(t^m_{cl}\) reads a version written by \(t^n_{cl}\)

\[
\mathcal{W}(K''(k, i)) = t^n \land t^m \in \mathcal{R}(K''(k, j))
\]

Then, the snapshot times of readers of a version is greater then the commit time of the writer \(n < c'\)

Proof. Trivially, \(\mathcal{W}(K''(k, i)) = t^n\). By the READREMOTE, it follows \(n = \max \{ n' \mid \exists j. t^n = \mathcal{W}(K(k, j)) \land n' < c' \}\) which implies \(n < c'\).

\[\square\]

Lemma I.13 (Commit time after snapshot time). The commit time of a transaction is after the snapshot time. Suppose the following step,

\[
K, c, c', \mathcal{E}, \mathcal{P} \xrightarrow{c, c, \mathcal{F}, n} K', c'', c'', \mathcal{E}', \mathcal{P}'
\]

then \(c < n\).

Proof. It is easy to see by CLIENTSTEP and then COMMIT that

\[
n > n - 1 \max \{ \{ c' \mid \exists k, (\cdot, k, \cdot) \in \mathcal{F} \land c' = \mathcal{C}'(\text{shardOf}(k)) \} \cup \{ c \} \}
\]

so \(c < n\).

\[\square\]
Lemma I.14 (Monotonic shard clock time). The clock time associated with a shard monotonically increases. Suppose the following step,
\[ K, C, C', E, P \xrightarrow{\text{cl, ft, n}} K', C'', E', P' \]
then
\[ \forall r \in \text{dom}(C'). \ C'(r) \leq C'''(r) \]

Proof. We perform case analysis on rules.
- **TIMETick** By the rule there is one shard \( r' \) ticks time \( C'''(r') = C'(r) + 1 > C'(r) \).
- **CLIENTStep**. There are further five cases, yet only **STARTTrans** and **COMMIT** change the shard's clock times.
  - **STARTTrans** By the rule a new transaction starts in a shard \( r' \) and triggering the shard \( r' \) ticks time \( C'''(r') = C'(r) + 1 > C'(r) \).
  - **COMMIT** By the rule the transaction commits their fingerprint \( F \) to those shards \( r' \) it read or write, and triggering the shard \( r' \) ticks time \( C'''(r') = C'(r) + 1 > C'(r) \).