Effect of control procedures on the evolution of entanglement in open quantum systems

Sandeep K Goyal,1 Subhashish Banerjee,2,3 and Sibasish Ghosh1

1Center for Quantum Sciences, The Institute of Mathematical Sciences, CIT Campus, Chennai 600 113, India
2Chennai Mathematical Institute, Padur PO, Siruseri, 603103, India
3Indian Institute of Technology, Jodhpur, Rajasthan

The effect of a number of mechanisms designed to suppress decoherence in open quantum systems is studied with respect to their effectiveness at slowing down the loss of entanglement. The effect of photonic band-gap materials and frequency modulation of the system-bath coupling are along expected lines in this regard. However, other control schemes, like resonance fluorescence, achieve quite the contrary: increasing the strength of the control kills entanglement off faster. The effect of dynamic decoupling schemes on two qualitatively different system-bath interactions are studied in depth. Dynamic decoupling control has the expected effect of slowing down the decay of entanglement in a two-qubit system coupled to a harmonic oscillator bath under non-demolition interaction. However, non-trivial phenomena are observed when a Josephson charge qubit, strongly coupled to a random telegraph noise bath, is subject to decoupling pulses. The most striking of these reflects the resonance fluorescence scenario in that an increase in the pulse strength decreases decoherence but also speeds up the sudden death of entanglement. This demonstrates that the behaviour of decoherence and entanglement in time can be qualitatively different in the strong-coupling non-Markovian regime.

I. INTRODUCTION

Entanglement is one of the basic features that distinguish quantum systems from their classical counterparts, and has its origins in the inherent non-locality of quantum mechanics [1]. It is the most useful resource in quantum information theory [2], and is indispensable for diverse quantum information tasks such as quantum communication, teleportation, quantum error correction, superdense coding, one-way communication etc. In closed systems — that is, systems which are completely isolated from their surroundings — entanglement remains conserved under a local unitary evolution, and decays slowly under a non-local evolution. This makes these systems ideal for quantum information tasks. Closed systems are, however, a rarity in the natural world. More often than not, quantum systems are open, that is, they are in contact with the surrounding environment — a thermodynamic reservoir, for example [3, 4]. Quantum systems are extremely fragile, and the dissipative effects of the environment gives rise to the phenomenon of quantum decoherence [5]. As a result, the system undergoes an asymptotic transition to classicality and hence loses all its entanglement, which is a purely quantum phenomenon. This happens even if the evolution is a local unitary one. Nevertheless, this in itself is not a bad scenario, for if the decoherence rate is low, then entanglement takes a long time to completely disappear and such systems can function as useful quantum devices for sufficient periods of time. However, recent studies [6, 7] have uncovered systems where the rate of loss of entanglement is exponentially higher than the decoherence rate. This results in a finite time to classicality, and consequently, a finite time to the total loss of entanglement — a phenomenon given the name entanglement sudden death (ESD). Systems that suffer from ESD are rendered unusable for quantum tasks. Naturally then, ESD has dire implications for the success of quantum tasks, and has become one of the premier branches of quantum information study in recent times. Some of us have recently investigated this phenomenon for the case of n-qubit states at finite temperature [8], as well as for spatially separated n-mode Gaussian states coupled to local squeezed thermal baths [9].

Given the obvious importance of ESD regarding the success of quantum tasks, it is thus a worthwhile exercise to investigate ways and means of controlling the rate of loss of entanglement. Error-correcting codes [11, 12] and error-avoiding codes [13] (which are also known as decoherence-free subspaces) are such attempts. Open loop decoherence control strategies [14–18] are another class of widely used strategies used to this effect, where the system of interest is subjected to external, suitably designed, time-dependent drivings that are independent of the system dynamics. The aim is to cause an effective dynamic decoupling of the system from the ambient environment. A comparative analysis of some of these methods has been made in [19]. Another mechanism known to slow down the process of decoherence is through manipulation of the density of states. This has been put to use in photonic band-gap materials, which is used to address questions related to the phenomenon of localization of

PACS numbers: 03.65.Yz, 03.67.Bg, 03.67.Pp

*Electronic address: goyal@imsc.res.in
†Electronic address: subhashish@cmi.ac.in
‡Electronic address: sibasish@imsc.res.in
light \[23, 24\].

In this paper, we analyze the evolution of entanglement in two-qubit systems connected to local baths (or reservoirs). A number of studies of entanglement in open quantum systems have been made \[25–27\]. Here we address the need to have a control on the resulting nonunitary evolution, as motivated by the above discussion, and study several methods of doing so. These include manipulation of the density of states in photonic crystals, modulation of the frequency of the system-bath coupling and modulation of external driving on two-qubit systems as examples of systems undergoing Markovian evolution. We also study control methods in systems undergoing non-Markovian evolution. The first of these is dynamic decoupling — which is an open-loop strategy — on a two-qubit system that is in contact with a harmonic oscillator bath. This system undergoes a quantum non-demolition interaction, where dephasing occurs without the system getting damped. The second is a Josephson-junction charge qubit subject to random telegraph noise due to charge impurities. The plan of the paper is as follows. In Section II, we introduce the basic techniques and formalism used in this paper, including the formal way of solving the Lindblad master equation. We also introduce the phenomenon of channel-state duality and the law of entanglement decay, both of which will be used subsequently. In Section III, we study the evolution of entanglement in photonic band gap materials and the law of entanglement decay, both of which will be used in unitary evolution due to this interaction term, and, depending on the type of the system-reservoir (SR) interaction, can be broadly divided into two categories — dissipative and non-dissipative. In the former, the system Hamiltonian does not commute with the interaction Hamiltonian, \([H_S, H_{SR}] \neq 0\), and dephasing occurs along with dissipation and decoherence. In the latter however, these two do commute — \([H_S, H_{SR}] = 0\) — and hence the SR interaction is characterized by a class of energy-preserving measurements where dephasing occurs without damping the system \[28, 29\]. Such a non-dissipative system, as well as the corresponding interaction, is called a Quantum Non-Demolition (QND) system.

We are interested in the time evolution of OQS, i.e., of the system state \(\rho_S\). Let the initial state of the system-bath combination be \(\rho(0)\), and let the state at time \(t\) be \(\rho(t) = U\rho(0)U^\dagger\), where \(U = e^{-iHt}\) is the time evolution operator. The state of the system alone is obtained from \(\rho(t)\) by simply tracing out the bath degrees of freedom: \(\rho_S(t) = Tr_R[\rho(t)]\), where \(Tr_R\) implies a partial trace over the bath. The evolution of the system-bath combination in unitary, and

\[
\dot{\rho}(t) = -i[H, \rho(t)]
\]

is the equation of motion. However, the evolution of the system itself is nonunitary, and thus requires a more general equation of motion which, after the application of the Born, Markov and rotating wave approximations, can be written as

\[
\dot{\rho}_S(t) = -i[H_S, \rho_S(t)] + \sum_j \gamma_j \left( F_j \rho_S(t) F_j^\dagger - \frac{1}{2} \{ \rho_S(t), F_j^\dagger F_j \} \right).
\]

This is a master equation in its Lindblad form. It can be written in super operator form as

\[
\dot{\rho}_{Sij}(t) = \mathcal{L}[\rho_S(t)]
\]

where \(\mathcal{L}\) is the super operator acting on the system state \(\rho_S(t)\) and is effectively a time-derivative, and where \(L\) is the matrix representation of \(\mathcal{L}\). In general, \(\mathcal{L}\) is time independent and the solution of the above equation can be written formally as

\[
\rho_S(t) = A[\rho_S(0)]
\]

where \(V = \exp(\mathcal{L}t)\) is the matrix representation of the time evolution map \(A\). If the system is evolving under unitary evolution \(w\) then the matrix \(V\) is simply \(w \otimes w^*\), where \(w^*\) represents the complex conjugate of \(w\) in a fixed basis.

**Channel-State Duality**— A quantum channel is a conduit for the transmission of quantum as well as classical information, and is essentially a completely positive map between spaces of operators. Any such physical quantum channel acting on a \(d\)-dimensional quantum state

II. PRELIMINARIES

An open quantum system, as defined in the introduction, is exposed to its environment, which is usually a thermal reservoir. The dynamics of such a system is naturally dictated by its interaction with its environment. If \(H\) be the total Hamiltonian of an open system, then \(H = H_S + H_R + H_{SR}\), where \(H_S\) and \(H_R\) are the system and reservoir Hamiltonians respectively and \(H_{SR}\) is the interaction Hamiltonian. Open systems undergo nonunitary evolution due to this interaction term, and, depending on the type of the system-reservoir (SR) interaction, can be broadly divided into two categories — dissipative and non-dissipative. In the former, the system Hamiltonian does not commute with the interaction Hamiltonian, \([H_S, H_{SR}] \neq 0\), and dephasing occurs along with dissipation and decoherence. In the latter however, these two do commute — \([H_S, H_{SR}] = 0\) — and hence the SR interaction is characterized by a class of energy-preserving measurements where dephasing occurs without damping the system \[28, 29\]. Such a non-dissipative system, as well as the corresponding interaction, is called a Quantum Non-Demolition (QND) system.
can be mapped to a positive operator in \( d^2 \) dimensions, and, if the channel is trace-preserving, then the corresponding positive operator will have unit trace. Similarly, a valid density matrix in \( d^2 \)-dimensions can be mapped to a trace preserving physical channel in \( d \) dimensions. Such a two-way mapping between a quantum state and its corresponding higher dimensional operator is called channel-state duality.

The time evolution operator \( \Lambda \) is a physical quantum channel represented by the matrix \( V \). If \( M \) be a valid density matrix corresponding to the map \( \Lambda \), it is given by

\[
M = (I \otimes \Lambda)[|\phi^+\rangle\langle\phi^+|],
\]

where \( |\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle \) is a maximally entangled state in \( d \otimes d \). Here the channel is applied to one side of a maximally entangled state in \( d \otimes d \). We shall use the symbols \( V \) and \( M \) to represent the matrix representation of the time evolution map \( \Lambda \) and a valid density matrix corresponding to it, respectively, throughout the paper.

**Law of entanglement decay**

This law says that the evolution of entanglement in a bipartite entangled state under a local one-sided channel can be fully characterized by its action on a maximally entangled state. The amount of entanglement at any time \( t \) in a given initially entangled two-qubit state \( |\chi\rangle \), under the action of a one-sided quantum channel, is equal to the product of the initial entanglement in the given state and the entanglement in the state which we get by applying the channel on one side of a two-qubit maximally entangled state. Mathematically this can be written as:

\[
\mathcal{C}((I \otimes \Lambda)[|\chi\rangle\langle\chi|]) = \mathcal{C}(|\chi\rangle\langle\chi|) \mathcal{C}((I \otimes \Lambda)[|\phi^+\rangle\langle\phi^+|]),
\]

\( \mathcal{C}() \) being the concurrence. Therefore, it is enough to study the evolution of \( |\phi^+\rangle \) state.

### III. EVOLUTION OF ENTANGLEMENT IN THE PRESENCE OF PHOTONIC CRYSTALS

In this section we consider a system of two level atoms interacting with a periodic dielectric crystal. The periodic dielectric structure gives rise to the photonic band gap [20, 23, 34]. The effect of this particular structure on electromagnetic waves is analogous to the effect semiconductor crystals have on the propagation of electrons, and leads to interesting phenomena like strong localization of light [22], inhibition of spontaneous emission [23] and atom-photon bound states [24, 31, 36]. The origin of such phenomena can ultimately be traced to the photon density of states changing at a rate comparable to the spontaneous emission rates. The photon density of states are of course estimated from the local photon mode density which constitutes the reservoir.

The advantage of studying them is that the presence of the photonic gap suppresses decoherence [37]. Let us consider a two-qubit system. One of these is locally coupled to a photonic crystal reservoir initially kept at temperature \( T = 0 \). In this case the entanglement dynamics can be obtained by studying any one of the qubits individually. We start with the following Hamiltonian:

\[
H = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k a_k^\dagger \sigma_- + g_k^\ast a_k \sigma_+),
\]

where \( \omega_0 \) is the natural frequency of the two level atom, \( \omega_k \) is the energy of the \( k \)-th mode and \( g_k \) is the frequency dependent coupling between the qubit and the photonic crystal, which is acting as the reservoir here. If we restrict the total atom-reservoir system to the case of a single excitation [35], the evolution of a given state of the qubit is then given by [37]:

\[
\rho(t) = \left( \begin{array}{ccc} \rho_{00}(0) & |c(t)|^2 & \rho_{01}(0) e^{i\delta t} \\ c(t)^* & \rho_{10}(0) e^{i\delta t} & 1 - |c(t)|^2 \\ \rho_{11}(0) & c(t)^* & 0 \end{array} \right) + \rho_{00}(0) + \rho_{11}(0),
\]

where

\[
c(t) = \varepsilon (\lambda_+ e^{i\delta t} (1 + \Phi(\lambda_+ e^{i\pi/4})) - \lambda_- e^{-i\delta t} (1 + \Phi(\lambda_- e^{-i\pi/4}))),
\]

\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2k}{(2k+1)^{3/2}} e^{-tk^2},
\]

\[
\varepsilon = \frac{(\alpha^2 - 4\delta)}{2},
\]

\[
\lambda_{\pm} = -\alpha \pm \sqrt{\alpha^2 - 4\delta},
\]

\[
\alpha = \frac{\omega_0^2 d^2}{8\omega_0 c_0 (\pi A)^{3/2}}.
\]

Here \( \delta = \omega_0 - \omega_c \) is the detuning of the atomic frequency and \( \omega_c \) is the upper band-edge frequency. We have made use of the following photon-dispersion relation near the band edge: \( \omega_k \approx \omega_c + A(k-k_0)^2 \), where \( A \approx \omega_c/k_0^2 \), \( d \) is the atomic dipole moment and \( c_0 \) is the vacuum dielectric constant.

The density matrix \( \rho(t) \) in (10) is related to the initial density matrix \( \rho(0) \) by a map \( \Lambda \), given by \( \rho(t) = \Lambda[\rho(0)] \), whose matrix representation is

\[
V = \left( \begin{array}{ccc} |c(t)|^2 & 0 & 0 \\ 0 & c(t) & 0 \\ 0 & 0 & c^*(t) \end{array} \right).
\]

Channel-state duality, explained earlier in Section II, ensures that there exists a two-qubit density matrix \( M \) for every single-qubit channel \( V \). This matrix \( M \) can be written as

\[
M = \Lambda_V(|\Phi^+\rangle\langle\Phi^+|)
\]

\[
= \frac{1}{2} \left( \begin{array}{ccc} |c(t)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - |c(t)|^2 \end{array} \right).
\]
where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a two-qubit maximally entangled state. The concurrence of $M$ is $|c(t)|^2$. The complex function $c$ is a function of the detuning parameter $\delta$ and time $t$. Therefore, we need to see the effect of $\alpha$ on entanglement. If we assume that $\delta = \Delta \alpha^2$, $c$ can then be written in the following simplified form:

$$c(t) = \frac{e^{i\Delta \tau}}{\sqrt{1 - 4\Delta^2}} \left( d_+ e^{i\pi/4} (1 + \Phi(d_+ e^{i\pi/4} \sqrt{t})) - d_- e^{i\pi/4} (1 + \Phi(d_- e^{i\pi/4} \sqrt{t})) \right),$$

where $d_\pm = -1 \pm \sqrt{1 - 4\Delta}$ and $\tau = \alpha^2 t$. Since the entanglement in $M$ is $|c(t)|^2$, it is now a function of $\delta$ and $\tau$.

The evolution of entanglement for different values of $\Delta$ can be seen in the FIGS. 1(a) and 1(b). The inset of the figures depict the evolution of entanglement, computed using concurrence (see appendix). For the usual case of zero band gap, while in FIGS. 1(a) and 1(b), evolution of entanglement is shown for increasing influence of the band gap. In FIG. 1(a), the system is within a gap in the photonic spectrum, indicated by the negative value of $\Delta$ and hence also $\delta$, as a result of which coherence is preserved and the decay of entanglement is arrested. This feature is further highlighted in FIG. 1(b), which is also for the case of negative $\Delta$ of lower order of magnitude than that in FIG. 1(a) and as a result there is a greater persistence of entanglement. Thus we find that with the increase in the influence of the photonic band gap on the evolution, entanglement evolution is preserved longer. From Eq. (12), it can be seen that, following the arguments of the previous section, there is no ESD in this case, a feature corroborated by the FIGS. II.

### IV. FREQUENCY MODULATION

Agarwal and coworkers [18] introduced an open-loop control strategy which involved modulation of the system-bath coupling, with the proviso that the frequency modulation should be carried out at a time scale which is faster than the correlation time scale of the heat bath. The technique of frequency modulation has been used earlier to demonstrate the existence of population trapping states in a two-level system [22]. Raghavan et al. [10] showed the connection between trapping in a two-level system with frequency-modulated fields in quantum optics and dynamic localization of charges moving in a crystal under the action of a time-periodic electric field.

Consider the Hamiltonian given in Eq. (9). Frequency modulation essentially involves a modification of the coupling $g$ — the modulated coupling is $g \exp \{ -i m \sin \nu t \}$, where $m$ is the amplitude and $\nu$ is the frequency of the modulation. Choosing $m$ such that $J_0(m) = 0$, where $J_0$ are the Bessel functions of order zero, the decay of the excited state population can be shown to be significantly arrested. The resulting master equation in the interaction picture, when applied to the evolution of a standard two-level system, is [18]:

$$\frac{\partial \rho}{\partial t} = \frac{-2(\kappa - i \Delta)J_0^2(m)}{(\kappa - i \Delta)^2 + \nu^2} \left\{ C_0^+(S^+ S^- \rho - S^- S^+ \rho) + C_0^-(S^- S^+ \rho - S^+ S^- \rho) \right\} + h.c, \quad \Delta = (\omega_0 - \omega).$$

(13)

where the Bessel function expansion $e^{-i m \sin (\nu t)} = \sum_{l=-\infty}^{1} J_l(m)e^{-i\omega_l t}$ has been used. Additionally, the modified bath correlation functions are assumed to have the forms $C^{--}(t) = C_0^+e^{-\kappa t}e^{i\omega t}$ and $C^{++}(-t) = C_0^-e^{-\kappa t}e^{i\omega t}$, where $\kappa$ is the bath correlation frequency. Now, we have

$$\frac{\partial \rho}{\partial t} = \Lambda[\rho],$$

(14)

$$\Rightarrow \rho(t) = \exp(\Lambda t)\rho(0),$$

(15)

$$\Rightarrow \rho(t)_{ij} = \sum_{kl} \exp(\Lambda t)_{ij,kl}\rho(0)_{kl} = \sum_{kl} V(t)_{ij,kl}\rho(0)_{kl},$$

(16)

where $V = \exp(\Lambda t)$ and $L$ is the matrix representation of $\Lambda$. We obtain the matrices $L$ and $V$ using Eq. (13):

$$L = \begin{pmatrix} -2 \text{Re}(\alpha)C_0^+ & 0 & 0 & 2 \text{Re}(\alpha)C_0^- \\ 0 & -\alpha(C_0^+ + C_0^-) & 0 & 0 \\ 0 & 0 & -\alpha^*(C_0^+ + C_0^-) & 0 \\ 2 \text{Re}(\alpha)C_0^- & 0 & 0 & -2 \text{Re}(\alpha)C_0^+ \end{pmatrix},$$

(17)

$$V = \exp(\Lambda t) = \begin{pmatrix} \frac{1}{T}(C_0^+ - 2 \text{Re}(\alpha)Tt + C_0^-) \\ \frac{1}{T}(C_0^+ + (1 - e^{-2 \text{Re}(\alpha)Tt})) \\ \frac{1}{T}(C_0^- - 2 \text{Re}(\alpha)Tt) \\ e^{-\alpha Tt} \end{pmatrix},$$

(18)

where $\alpha = \frac{2(\kappa - i \Delta)J_0^2(m)}{(\kappa - i \Delta)^2 + \nu^2}$ and $T = C_0^- + C_0^+$. If $M = (I \otimes V)(|\phi^+\rangle \langle \phi^+|)$, then we have

$$M = \begin{pmatrix} M_{11} & 0 & 0 & e^{-\alpha Tt} \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ e^{-\alpha^* Tt} & 0 & 0 & M_{44} \end{pmatrix},$$

(19)
where

\[
M_{11} = \frac{1}{T} \left( C_0^{++} e^{-2\text{Re} \alpha T t} + C_0^{+-} \right),
\]

(20)

\[
M_{22} = \frac{1}{T} \left( C_0^{++} (1 - e^{-2\text{Re} \alpha T t}) \right),
\]

(21)

\[
M_{33} = \frac{1}{T} \left( C_0^{++} (1 - e^{-2\text{Re} \alpha T t}) \right),
\]

(22)

\[
M_{44} = \frac{1}{T} \left( C_0^{++} e^{-2\text{Re} \alpha T t} + C_0^{--} \right).
\]

(23)

If this state \(M\) is separable, the factorization law for entanglement decay \cite{32} allows us to assert that all states will show ESD. The state \(M\) will be separable only if the condition

\[
1 + X^2 - 2X - \frac{T^2}{C_0^{++}C_0^{--}} X = 0,
\]

(24)

holds, where \(X = \exp(-2\text{Re} \alpha T t)\). The roots of the above equation are

\[
X_{\pm} = \frac{1}{2} \left[ 2 + \frac{T^2}{C_0^{++}C_0^{--}} \right] \pm \sqrt{\left(2 + \frac{T^2}{C_0^{++}C_0^{--}} \right)^2 - 4}.
\]

(25)

The negative root is less than unity \((X_- \leq 1)\), implying that there exists, always, a finite and positive time \(t_{ESD}\) at which the system loses all its entanglement. This is given by

\[
t_{ESD} = -\frac{1}{2\text{Re} \alpha T} \log(X_-).
\]

(26)

The modulation factor \(\nu\) appears in the numerator and therefore it can be expected that a higher frequency of modulation should sustain entanglement longer. This is confirmed in the plot of \(t_{ESD}\) against \(\nu\) (FIG. 2). This result is not altogether surprising, for a higher degree of modulation is naturally expected to filter out the influence of the bath and increase the coherence which ultimately results in entanglement sustaining for a longer period of time.

V. RESONANCE FLUORESCENCE

In the previous section, we focused on the decrease in the time to ESD through an increase in the degree of frequency modulation of the system-bath coupling. In this section, we study a system where a two-level atomic transition is driven by an external coherent single-mode field which is in resonance with the transition itself. We shall show that, in this situation, an increase in the Rabi frequency — which plays the role of the modulator — produces the opposite effect by speeding up ESD. The behavior of such driven systems has been well studied in the literature and has found many applications. In contrast to the situation here, Lam and Savage \cite{41} have investigated a two-level atom driven by polychromatic light. The phenomenon of tunneling in a symmetric double-well potential perturbed by a monochromatic driving force was analyzed by Grossmann et al. \cite{42}, while photon-assisted tunneling in a strongly driven double-barrier tunneling diode has been studied by Wagner \cite{43}.

The analysis of the said driven system begins with its Hamiltonian which, when written in the interaction picture, is \(H_{SR} = -E(t) \cdot D(t)\). Here \(E(t) = e^{-i\omega_0 t} + e^{i\omega_0 t}\) is the electric field strength of the driving mode (treated classically), \(\omega_0\) is the atomic transition frequency and \(D(t)\) is the dipole moment operator in the interaction picture. The driven two-level system is coupled to a thermal reservoir of radiation modes. If \(\gamma_0\) be the spontaneous rate due to coupling with the thermal reservoir
and $N = N(\omega_0)$ be the Planck distribution at the atomic transition frequency $\omega_0$, then the evolution of this composite system is given by the following master equation \[\frac{d}{dt}\rho(t) = \frac{i\Omega}{2}[\sigma_+ + \sigma_-, \rho(t)] + \frac{\gamma_0(N+1)}{2}[2\sigma_- \rho(t)\sigma_+ - \sigma_+ \sigma_- \rho(t) - \rho(t)\sigma_+ \sigma_-] + \frac{\gamma_0 N}{2}[2\sigma_+ \rho(t)\sigma_- - \sigma_- \sigma_+ \rho(t) - \rho(t)\sigma_- \sigma_+],\] (27)

where $\Omega = 2\epsilon \cdot d^*$ is the Rabi frequency, $d$ being the transition matrix element of the dipole operator. The term $-(\Omega/2)[\sigma_+ + \sigma_-]$ characterizes the interaction between the atom and the external driving field in the rotating wave approximation. As usual, $\sigma_\pm$ are the atomic raising and lowering operators, respectively.

Let us take two identical qubits and, as before, assume that one of them interacts locally with a thermal bath and is subject to monochromatic driving by an external coherent field. The master equation (Eq. 27) yields the corresponding matrices $V_{rf}$ and $M_{rf}$ (where the subscript $rf$ stands for resonance fluorescence):

$$V_{rf} = \begin{pmatrix}
a_1 & a_2 & a_2^* & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
a_1^* & b_2^* & b_3^* & b_4^* \\
d_1 & -a_2 & -a_2^* & d_4
\end{pmatrix},$$

(28)

$$M_{rf} = \frac{1}{2} \begin{pmatrix}
a_1 & a_2 & b_1 & b_2 \\
a_2 & a_4 & b_3 & b_4 \\
a_1^* & b_2^* & d_1 & -a_2 \\
b_2^* & b_4^* & -a_2^* & d_4
\end{pmatrix},$$

(29)

where

$$a_1 + a_4 = 1 + \left(1 - X^3 \left(\cos(\mu t) - \frac{\gamma}{4\mu} \sin(\mu t)\right)\right) S_3 + \frac{i\Omega}{\mu} X^3 \sin(\mu t) (S_- + S_+),$$

$$a_1 - a_4 = X^3 \left[\cos(\mu t) - \frac{\gamma}{4\mu} \sin(\mu t)\right],$$

$$a_2 = \frac{i\Omega}{\mu} X^3 \sin(\mu t),$$

$$b_1 + b_4 = -X^2 (S_+ + S_-) - \frac{i\Omega}{\mu} X^3 \sin(\mu t) S_3 + X^3 \left(\cos(\mu t) + \frac{\gamma}{4\mu} \sin(\mu t)\right) (S_- - S_+),$$

$$b_1 - b_4 = \frac{i\Omega}{\mu} X^3 \sin(\mu t),$$

$$b_{2,3} = \frac{1}{2} X^2 \pm X^3 \left(\cos(\mu t) + \frac{\gamma}{4\mu} \sin(\mu t)\right),$$

$$d_1 + d_4 = 2 - (a_1 + a_4),$$

$$d_1 - d_4 = -(a_1 - a_4),$$

$$X = e^{-\frac{\gamma t}{2}},$$

$$S_+ = -\frac{i\Omega \gamma}{\gamma^2 + 2\Omega^2},$$

$$S_- = S_+^*,$$

$$S_3 = -\frac{\gamma \Omega \gamma}{\gamma^2 + 2\Omega^2},$$

$$\gamma = \gamma_0 (2N + 1),$$

$$\mu = \sqrt{\Omega^2 - (\gamma/4)^2}.$$
fields on the system of interest, with the aim of achieving dynamic decoupling of the system from the environment \[14, 16–18\]. Bang-Bang control is a particular form of such decoupling where the decoupling interactions are switched on and off at a rate faster than that set by the environment. The application of suitable radio frequency (RF) pulses, applied fast enough, averages out unwanted effects of the environment and suppresses decoherence. In this section, we compare the effect of Bang-Bang decoupling on the evolution of entanglement in systems connected to two different types of baths. One bath type is composed of infinitely many harmonic oscillators at a finite non-zero temperature \(T\) and couples locally to a two-level atom acting as the qubit, while the other adds random telegraph noise to a Josephson-junction charge qubit. It has been shown for the former case that all two-qubit states shows ESD at finite \(T\) \[9\].

A. Bang-Bang decoupling when the bath consists of harmonic oscillators

1. Quantum Non-Demolition Interaction

Let us consider the interaction of a qubit with a bath of harmonic oscillators where the system Hamiltonian commutes with the interaction Hamiltonian so that there is no exchange of energy between the system and the bath — this is quantum non-demolition dynamics \[28, 29\]. The only effect of the bath will be on the coherence elements of the qubit evolution, which will decay in time at the rate \(\gamma\). The total Hamiltonian for the system plus bath is:

\[
H_0 = H_q + H_B + H_I; \quad (30)
\]

\[
H_q = \omega_0 \sigma_z; \quad H_B = \sum_k \omega_k b_k^\dagger b_k; \quad H_I = \sum_k \sigma_z (g_k b_k^\dagger + g_k^* b_k).
\]

Here the system Hamiltonian \(H_q\) commutes with the interaction Hamiltonian \(H_I\) and the evolution of such a system is called pure dephasing. For simplicity we will work in the interaction picture where the density matrix of the system plus bath and the interaction Hamiltonian transform as:

\[
\dot{\rho}(t) = e^{i(H_q + H_B)t} \rho(t) e^{-i(H_q + H_B)t} \quad (31)
\]

\[
\dot{\tilde{H}}(t) = \sigma_z \sum_k (g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}). \quad (32)
\]
From here we can write the total time evolution operator for the system plus bath as

$$\hat{U}(t_0, t) = T \exp \left\{ -i \int_{t_0}^{t} ds \hat{H}(s) \right\}$$

$$= \exp \left\{ \frac{\sigma_z}{2} \sum_k |b_k|^2 e^{i \omega_k t_0} \xi_k(t - t_0) - b_k e^{-i \omega_k t_0} \xi_k^*(t - t_0) \right\},$$

(33)

where $\xi_k(t) = \frac{2q_a}{\omega_k}(1 - \exp(i \omega_k t))$. We are interested in calculating

$$\hat{\rho}_0(t) = (0|\text{Tr}_B \left\{ \hat{U}(t_0, t) \hat{\rho}(t_0) \hat{U}^\dagger(t_0, t) \right\}|1).$$

(34)

Assuming that in the beginning the bath and the qubit were uncorrelated and that the bath is in a thermal state, we have [14]:

$$\hat{\rho}_0(t) = \hat{\rho}_0(0)e^{-\gamma(t_0, t)},$$

(35)

where

$$\gamma(t_0, t) = \sum_k \frac{|\xi_k(t - t_0)|^2}{2} \coth \left( \frac{\omega_k}{2T} \right).$$

(36)

The matrix representation of the evolution operator $V$ can be written from here as:

$$V = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-\gamma(t_0, t)} & 0 & 0 \\
0 & 0 & e^{-\gamma(t_0, t)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

(37)

To see the evolution of entanglement, it is enough to evolve the maximally entangled state $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and study the evolution of entanglement. Evolving one subsystem in state $|\phi^+\rangle$ gives rise to the density matrix:

$$M = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & e^{-\gamma(t_0, t)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
e^{-\gamma(t_0, t)} & 0 & 0 & 1
\end{pmatrix}.$$  

(38)

The concurrence in the state $M$ is directly proportional to $e^{-\gamma(t_0, t)}$.

2. Dephasing under Bang-Bang dynamics

The aim of Bang-Bang decoupling, acting on the system of interest, is to hit it with a sequence of fast radio-frequency pulses with the aim of slowing down decoherence (see FIG. 3). Adding the radio frequency term to the system-plus-bath Hamiltonian $H_0$ (Eq. 38), we get

$$H(t) = H_0 + H_{RF}(\omega_0, t),$$

$$H_{RF}(t) = \sum_{n=1}^{N_p} U^{(n)} \{ \cos[\omega_0(t - t^{(n)})] \sigma_x + \sin[\omega_0(t - t^{(n)})] \sigma_y \},$$

(39)

(40)

The term $H_{RF}$ acts only on the system of interest — here the qubit. It represents a sequence of $n_p$ identical pulses, each of duration $\tau_p$, applied at instants $t = t^{(n)}$. The separation between the pulses is $\tau = \Delta t$. The decay rate for this pulsed sequence evolution is [14]:

$$\gamma_p(N, \Delta t) = \sum_k \frac{|\eta_k(N, \Delta t)|^2}{2} \coth \left( \frac{\omega_k}{2T} \right),$$

(41)

where

$$|\eta_k(N, \omega_k \Delta t)|^2 = 4(1 - \cos(\omega_k \Delta t))^2 \times \left(N + \sum_{n=0}^{N-1} 2n \cos[2(N - n)\omega_k \Delta t] \right).$$

(42)

In [14] it has also been shown that $|\eta_k|^2 \leq |\xi_k|^2$ which implies that decoherence is suppressed. Also, it is evident that a lower value of $\omega$ implies a lower value of $\gamma$. Consequently, we can conclude that that Bang-Bang decoupling slows down entanglement decay.

B. Josephson Junction qubit

Although solid state nanodevices satisfy the requirements of large scale integrability and flexibility in design, they are subject to various kinds of low-energy excitations in the environment and suffer from decoherence problems. There have been a number of proposals in this context about the implementation of quantum computers using superconducting nanocircuits [44-47]. Experiments highlighting the quantum properties of such devices have already been performed [48-49]. Here the concept of a Josephson-junction qubit comes into prominence. A charge-Josephson qubit is a superconducting island connected to a circuit via a Josephson junction and a capacitor. The computational states are associated with charge $Q$ in the island and are mixed by Josephson tunneling. For temperatures much lower than the Josephson energy, $k_B T \ll E_j$, the Hamiltonian

$$H_Q = \frac{e}{2} \sigma_z - \frac{E_j}{2} \sigma_z,$$  

(43)
with the charging energy $E_C$ dominating the Josephson energy. Here, $\epsilon \equiv \epsilon(V) = 4E_C(1 - C_2V/e)$, $C_2$ is the capacitance of the capacitor connected to the island and $V$ is the external gate voltage.

An important source of decoherence in the operation of Josephson charge qubits is due to fluctuating background charges (BCs) (charge impurities). These are believed to originate in random traps for single electrons in dielectric materials surrounding the superconducting island. These fluctuations cause the $1/f$ noise, also known as random telegraph noise, at low frequencies, and are directly observed in single electron tunneling devices [52, 54]. This has also been studied in the context of fractional statistics in the Quantum Hall Effect [55]. This noise, arising out of decoherence, is modeled [51] by considering each of the BCs by a localized impurity level connected to a fermionic band, i.e., the quantum impurity is described by the Fano Anderson model. This is the quantum analogue of the classical model of $N$ independent, randomly activated bistable processes. For a single impurity, the total Hamiltonian is:

$$H = H_Q - \frac{v}{2}b^\dagger b\sigma_z + H^I,$$  \hspace{1cm} (45)

where

$$H^I = \epsilon_c b^\dagger b + \sum_k \left[ T_k c_k^\dagger b + h.c. \right] + \sum_k \epsilon_k c_k^\dagger c_k.$$  \hspace{1cm} (46)

Here $H^I$ describes the BC, $b$ represents the impurity charge in the localized level $\epsilon_c$, $c_k$ the electron in the band with energy $\epsilon_k$, and $H_Q$ is as in Eq. (41). The impurity electron may tunnel to the band with amplitude $T_k$. The BC produces an extra bias $v$ for the qubit via the coupling term $(v/2)b^\dagger b\sigma_z$. An important scale is the switching rate $\gamma = 2\pi\rho(\epsilon_c)|T|^2$, where $\rho(\epsilon_c)$ is the density of states of the band and it is assumed that we are working in the the relaxation regime of the BC where the tunneling rate to each fermionic band is approximately equal. The fraction $v/\gamma$ determines whether the operational regime of the qubit is weak ($v/\gamma \ll 1$) or strong ($v/\gamma > 1$). Studying the single BC case is important, since it has been shown [51] that the effect of multiple BCs can be trivially extended from that of a single BC. For multiple strongly coupled BCs producing $1/f$ noise, the effect of a large number of slow fluctuators is minimal and pronounced features of discrete dynamics such as saturation and transient behavior are seen. There are two special operational points for the qubit [41]: (a) $\epsilon = 0$, corresponding to charge degeneracy and (b) $E_j = 0$, for the case of pure dephasing [56, 57], where tunneling can be neglected. We will consider this case later in detail and make a comparison of ESD, for the case of pure dephasing, between the harmonic oscillator and $1/f$ baths.

In order to study the effect of the BC on the dynamics of the qubit, the general procedure is to calculate the unitary evolution of the entire system plus bath and trace out the bath degree of freedom, i.e., $\rho_Q(t) = \text{tr}_B(W(t))$, $W(t)$ being the full density matrix. In the weak coupling limit a master equation for $\rho_Q(t)$ can be written [58]. The results in the standard weak coupling approach are obtained at lowest order in the coupling $v$, but it has been pointed out that higher orders are important for a $1/f$ noise [50, 51].

The failure of the standard weak coupling approach is due to the fact that the $1/f$ environment includes fluctuators which are very slow on the time scale of the reduced dynamics. To circumvent this problem one considers another approach in which a part of the bath is treated on the same footing as the system [52]. We study the evolution of this new system and later trace out the extra part which belongs to the bath, i.e., $\rho(t) = \text{Tr}_B\{W(t)\}$. We then obtain $\rho_Q(t)$ from $\rho(t)$ as $\rho_Q(t) = \text{Tr}_B\{\rho(t)\}$, where $fb$ stands for fermionic band. In that regard we split the Hamiltonian (45) into a system Hamiltonian

$$H_0 = H_Q - \frac{v}{2}b^\dagger b\sigma_z + v^\dagger b^\dagger b^\dagger b$$  \hspace{1cm} (47)

and environment Hamiltonian

$$H_E = \sum_k \epsilon_k c_k^\dagger c_k$$  \hspace{1cm} (48)

coupled by $\mathcal{V} = \sum_k \left[T_k c_k^\dagger b + h.c.\right]$. The eigenstates of $H_0$ are product states of the form $|\theta\rangle|n\rangle$, e.g.,

$$|a\rangle = |\theta_+\rangle|0\rangle,$$

$$|b\rangle = |\theta_-\rangle|0\rangle,$$

$$|c\rangle = |\theta'_+\rangle|1\rangle,$$

$$|d\rangle = |\theta'_-\rangle|1\rangle,$$

with corresponding energies

$$\frac{-\Omega}{2} < \frac{\Omega}{2}, -\frac{\Omega'}{2} + \epsilon_c, \frac{\Omega'}{2} + \epsilon_c.$$  \hspace{1cm} (49)

The two level splittings are $\Omega$ which belongs to the bath, i.e., $\left\{H_B\right\} = \text{Tr}_B\{W(t)\}$ from

$$\frac{-\Omega}{2} < \frac{\Omega}{2}, -\frac{\Omega'}{2} + \epsilon_c, \frac{\Omega'}{2} + \epsilon_c.$$  \hspace{1cm} (49)

\hspace{1cm} (49)
$G^>(-t)$. This problem has a very interesting symmetry: the diagonal and off diagonal elements do not mix if the initial state of the charge particle is a diagonal density matrix in the BC. Therefore, we can divide the Redfield tensor elements in two parts, one which corresponds to population (diagonal elements) and other which corresponds to coherence (off diagonal elements).

The $R_{ii,nn}$ elements which affect the population are:

$$R_{ii,nn} = \int_0^\infty \{ \chi_{in} G^>(\tau)e^{i\omega_{nn}\tau} + \chi_{in} G^<(\tau)e^{-i\omega_{nn}\tau} \}$$

$$= \chi_{in} [iG^>(\omega_{nn})] .$$

(50)

Here $n \neq i$ and $\chi_{in} = \langle |n\rangle \langle b|/i|^2 | + \langle |n\rangle \langle b|/i|^2 |$, and

$$R_{ii,ii} = - \sum_k \chi_{ik} [iG^>(\omega_{ik})] .$$

(51)

Now we can calculate the elements which are responsible for the coherence part. In the adiabatic regime we have $\gamma \sim \Omega - \Omega' \ll \Omega \sim \Omega'$, i.e., where the BCs are not static and the mixing of $\rho_{ab}$ and $\rho_{cd}$ as well as their conjugates cannot be neglected. Hence the non-zero elements of $R$ tensor which affect the coherence are the following:

$$R_{ab,ab} = -\frac{\gamma}{2} \left[ 1 - c^2\delta - s^2\delta' + i(c^2w + s^2w') \right] ,$$

$$R_{cd,cd} = -\frac{\gamma}{2} \left[ 1 + c^2\delta + s^2\delta' + i(c^2w - s^2w') \right] ,$$

$$R_{ab,cd} = \frac{c^2\gamma}{2} [1 + \delta - iw] ,$$

$$R_{cd,ab} = \frac{c^2\gamma}{2} [1 - \delta - iw] .$$

Here

$$c = \cos[(\theta - \theta')/2] ,$$

$$s = \sin[(\theta - \theta')/2] ,$$

$$\delta = t_{ca} + t_{db} ,$$

$$\delta' = t_{da} + t_{cb} ,$$

$$w = w_{ca} - w_{cb} ,$$

$$w' = w_{da} - w_{cb} ,$$

$$t_{ij} = \frac{1}{2} \tanh \left( \frac{\beta \omega_{ij}}{2} \right) ,$$

$$w_{ij} = -\frac{1}{\pi} R \left\{ \psi \left( \frac{\pi + i \beta \omega_{ij}}{2\pi} \right) \right\} ,$$

and $\psi(z)$ is the digamma function.

Now we can construct the explicit form of the matrix $R$ which is:

$$R = \begin{pmatrix}
R_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & R_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_{3,1} & 0 & 0 & 0 & 0 & R_{3,2} & 0 & 0 & 0 & R_{3,3} & 0 & 0 & 0 & R_{3,4} \\
0 & y_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_{4,1} & 0 & 0 & 0 & 0 & R_{4,2} & 0 & 0 & 0 & R_{4,3} & 0 & 0 & 0 & R_{4,4}
\end{pmatrix} ,$$

(52)

where

$$z_- = -\frac{\gamma}{2} \left[ 1 - c^2\delta - s^2\delta' + i(c^2w + s^2w') \right] ,$$

$$z_+ = -\frac{\gamma}{2} \left[ 1 + c^2\delta + s^2\delta' + i(c^2w - s^2w') \right] ,$$

and

$$\psi(z) = \log(z).$$
FIG. 5: Contour of the entanglement after time \( t = 5 \) for all values of \( E_j \) and \( \epsilon \) (Josephson junction Hamiltonian parameters). The temperature in this case is equal to \( 0 \), \( \gamma = 1 \), \( \kappa = v/\gamma = 0.45 \).

\[
y_+ = \frac{\gamma^2}{2} [1 + \delta - iw], \quad y_- = \frac{\gamma^2}{2} [1 - \delta - iw], \quad R_{i,j} = R_{ii,jj}.
\]

Exponential of the matrix \( R \) is the matrix representation of the evolution channel. Therefore, \( V = \exp(Rt) \). This gives us the evolution for the qubit plus the charge impurity. From here we can find the evolution map acting on the qubit \( V_a \) (see appendix for more details).

The two parameters \( \epsilon \) and \( E_j \) in the Hamiltonian for the charge Josephson qubit \( H_Q \) play a crucial role in the decoherence properties of the system. For example, if \( E_j = 0 \), the system Hamiltonian \( H_Q \) commutes with the interaction Hamiltonian. This situation is called non-demolition evolution or pure dephasing. In this case there is no energy exchange between system and the bath. On the other hand when we have \( \epsilon = 0 \), the system Hamiltonian does not commute with the interaction Hamiltonian. Therefore, the two situations are qualitatively different.

We present a plot of entanglement in the phase space of \( \epsilon \) and \( E_j \) by evolving a maximally entangled state of two qubits with the bath (of charge impurities) acting only on one qubit. The qubit is evolved for a fixed time \( t \) and entanglement calculated for different values of \( \epsilon \) and \( E_j \), FIG. 6. As seen from the figure, as the value of \( E_j \) increases, for fixed \( \epsilon \), the entanglement in the system increases; on the other hand, for fixed \( E_j \) as \( \epsilon \) increases, entanglement decreases. This is counterintuitive because with the increase in the value of \( E_j \), dissipation increases.

FIG. 7 compares the time-evolution of entanglement for the harmonic oscillator bath and the charge-impurities bath, both under pure dephasing. While entanglement decay is exponential for the case of \( 1/f \) noise, it is slower for a bath of harmonic oscillators. We compare, in FIG. 7, the time-evolution of entanglement for various values of Josephson energy \( (E_j) \) starting with the pure dephasing case given by \( E_j = 0 \). We see that with an increase in the value of Josephson energy, entanglement remaining in the system increases. This is consistent with FIG. 5.

Decoherence produced by background charges depends qualitatively on the ratio \( \kappa = v/\gamma \), where \( \kappa \ll 1 \) denotes the weak-coupling regime and \( \kappa > 1 \) is the strong coupling regime. The latter gives rise to qualitative new properties. We find (see FIG. 8(b)), for \( \kappa > 1 \), that the time-evolution of entanglement does not depend on \( \kappa \). This is in contrast to the weak coupling regime, where the time-evolution of entanglement does depend on \( \kappa \), as seen in FIG. 8(a), where an increase in \( \kappa \) leads to a decrease in entanglement. Naturally, decoherence due to
the bath forces entanglement to decay with time for both cases.

1. Evolution operator with Bang-Bang interaction

The Josephson charge qubit in contact with an $1/f$ bath is now subject to fast pulses, under the Bang-Bang dynamical decoupling scheme. The Hamiltonian for this radio frequency pulse is the same as in Eq. (40). If the time for which a pulse is active is $\pi$, then the evolution operator for the pulse may be written as $V_p = I \otimes i\sigma_x$

where $2U\tau_p = \pm \pi$. The total evolution can therefore be written as

$$V_{total} = (V_p V_S(\tau))^{2N}$$

where $2N\tau = t$. Since the RF pulses act on the system for very short amounts of time, the evolution of the system can safely be assumed to be governed only by the dynamical map $V_p$ for the time period during which the pulse is operating. As can be seen from the FIGS. 10 to 12 with the application of bang-bang pulses, the system exhibits the phenomenon of ESD.

Let us consider the case where $E_j = \epsilon = 1$. Let us also fix the pulse strength to be $U = 50\pi$ and ensure that the pulses act for very short times. As defined earlier, the ratio of the BC bias $v$ and the switching rate $\gamma$ defines the weak and strong coupling regimes, the former designated by $\frac{v}{\gamma} < 1$ and the latter by $\frac{v}{\gamma} > 1$.

In FIG. 9 we plot the time-evolution of entanglement, with the coupling strength as parameter. For weak coupling, we find that $t_{ESD}$ initially increases with coupling strength. This continues till a turning point is reached at $\frac{v}{\gamma} = 0.38$ when $t_{ESD} \approx 880$. After this, with increase in coupling strength, $t_{ESD}$ starts to decrease. As a result, a kink appears in the corresponding entanglement vs time plot, FIGS. 9 10. The receding of $t_{ESD}$ with increase in coupling strength continues well into the strong coupling regime, i.e. for $5.05 < \frac{v}{\gamma} < 5.5$. It, however, does not go to zero, but rather chooses to saturate at the threshold value of $t_{ESD} \approx 10$, FIG. 11. The “turning” and the “saturation” features are well captured in FIG. 11, where we plot $t_{ESD}$ against $\frac{v}{\gamma}$ and keep the pulse strength and durations fixed. We observe a crossover phenomenon around $\frac{v}{\gamma} \approx 0.38$, where the value of $t_{ESD}$ rises sharply, only to fall back again even quicker.

In FIG. 12 we plot the behavior of $t_{ESD}$ with $E_j$ and find that, as we increase $E_j$ and thus move away from the pure dephasing situation, the time to ESD keeps increasing.

VII. CONCLUSIONS

The importance of the sustenance of entanglement in quantum systems cannot be overstated. In this paper, we have studied a variety of control procedures aimed at doing exactly that. Most of these are designed to suppress decoherence at the level of single qubits. A majority of the systems considered in this paper are qubits coupled with harmonic oscillator baths at finite temperature $T$, the couplings being either of dissipative or of the dephasing type. The time-evolution of entanglement when such a bath acts on one side of the two-qubit maximally entangled state is known. In the commonly occurring dissipative case, entanglement decays asymptotically at
FIG. 9: Effect of bang-bang decoupling on entanglement, in the weak coupling regime. If we compare this plot with FIG. S(a), with the curves corresponding to the same values of $\kappa$, we see that the bang-bang decoupling causes entanglement to disappear faster in time for a fixed value of the coupling strength. Here the parameters are as in FIG. (8(a)) and the pulse strength is $U = 50\pi$ with time for which the pulse was activated is $\tau_p = 0.01$. In the inset we have the evolution of entanglement for very small range ($0.01$ to $0.1$) of coupling $\kappa$. The thickest curve is the one corresponding to $\kappa = 0.38$. This curve is important in the sense that it has the largest $t_{ESD}$.

FIG. 10: Entanglement evolution for the coupling parameter range $0.3 < \kappa < 0.4$, with the uppermost curve corresponding to $\kappa = 0.3$ and the lowest (bottom) curve corresponding to $0.4$. One can see from this plot the formation and disappearance of the kink.

FIG. 11: Effect of bang-bang decoupling on entanglement in the strong coupling region. Here again we can see the effect of bang-bang decoupling on the entanglement if we compare this plot with FIG. S(b). Here the parameters are as in FIG. S(b) and the pulse strength is $U = 50\pi$ with the pulse duration $\tau = 0.01$.

FIG. 12: $t_{ESD}$ is plotted as a function of coupling strength $\kappa$. Here we can see that there is a clear distinction between the strong and the weak coupling region. As we increase $\kappa$ the $t_{ESD}$ tends to freeze and asymptotic value of $t_{ESD}$ is around 10. The parameters used are as in the previous plots.

zero temperature, whereas it dies a sudden death at finite non-zero temperatures. Squeezing the initial bath states increases the time to ESD.

As stated, the aim of most control procedures is to suppress decoherence. For the case of photonic crystals, the design allows the system to conserve coherence when it is within the photonic band gap. Modulating the frequency of the system-bath coupling aims to suppress decoherence by shifting the system out of the spectral influence of the bath. In both these cases it is found that the suppression of decoherence is accompanied by a corresponding increase in $t_{ESD}$.

However, it will be erroneous to naïvely suppose that this is the norm. The exact opposite phenomenon is observed for the case of resonance fluorescence, where the coupling between the bath and a two-level atomic system forced by an external resonant field, is modulated. It is seen that an increase in the external field frequency $\Omega$, the Rabi frequency, results in a faster decay of entanglement (FIG. 3). A further non-trivial effect observed is the saturation in the time to ESD: $t_{ESD}$ does not go below a threshold value no matter what the Rabi frequency. In what could point towards a possible explanation of
FIG. 13: \( t_{ESD} \) is plotted as a function of \( E_i \). This shows that as we go away from pure dephasing the \( t_{ESD} \) increases. The parameters used are as in the previous plots.

this phenomenon, we observe that the sudden death time stops being dependent on the Rabi frequency at \( \Omega = \frac{\gamma_0}{4} \); strikingly, this happens to be the boundary between the overdamped \( \Omega < \frac{\gamma_0}{4} \) and underdamped \( \Omega > \frac{\gamma_0}{4} \) regimes.

The latter and significant portion of the paper, comprising Sec. VI, is entirely devoted to dynamic decoupling schemes where RF pulses, applied at short time-intervals, smoothes out unwanted effects due to environmental interactions. We discuss two qualitatively different system-bath models: the first being the usual qubit and harmonic oscillator bath pair with pure dephasing or QND interaction; and the second being a bath of charge impurities, simulating \( 1/f \) (telegraph) noise, acting on a Josephson-junction charge qubit. In the former, the decay of entanglement is always asymptotic, the only effect of the RF pulses being in a slowing down of the rate of loss. For the later case, we see that though the entanglement decay is exponential in the absence of quick RF pulses, an application of the pulses brings out the phenomenon of ESD. A very interesting phenomenon, observed in the strong coupling regime, was the decrease in the time to ESD with increasing pulse strengths. This is extremely counterintuitive, and brings into perspective the fact the in the non-Markovian strong coupling regime, the dynamics of entanglement can be different than that of decoherence. This calls for the need to have careful exhaustive studies of entanglement in these regimes.

Acknowledgments

We wish to thank Somdeb Ghose for his suggestions to improve the readability of the article. SB thanks T. P. Pareek for a useful discussion.

Appendix (A): Calculation for concurrence

Concurrence is a measure for entanglement of formation for a mixed state of two-qubit system given by Hill et al. [3]. Concurrence is defined as \( C = max(0, \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4) \), where \( \lambda_i \) are the square root of the eigenvalues of the matrix \( R = \rho \tilde{\rho} \). Here \( \tilde{\rho} = \sigma_2 \otimes \sigma_2 \rho \sigma_2 \otimes \sigma_2 \) and the complex conjugate is taken in the standard basis.

For some simple density matrices we can calculate the concurrence very easily. For example, consider the matrix

\[
M = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & e^{-\gamma t} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
e^{-\gamma t} & 0 & 0 & 1
\end{pmatrix}
\]

The \( R \) matrix will be:

\[
R = M\tilde{M}
\]

and

\[
\tilde{M} = \sigma_2 \otimes \sigma_2 \tilde{\sigma}_2 \otimes \sigma_2 \\
= \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & e^{-\gamma t} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
e^{-\gamma t} & 0 & 0 & 1
\end{pmatrix}
\]

\[
R = \frac{1}{4} \begin{pmatrix}
1 + e^{-2\gamma t} & 0 & 0 & 2e^{-\gamma t} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2e^{-\gamma t} & 0 & 0 & 1 + e^{-2\gamma t}
\end{pmatrix}
\]

The set \( \{\lambda_i\} \) is equal to \( \{(1 + e^{-\gamma t})/2, (1 - e^{-\gamma t})/2\} \). The concurrence for this state is \( C = e^{-\gamma t} \).

Appendix (B): Calculation for the evolution map \( V_s \) acting on the qubit from the evolution map of the qubit plus charge-impurity

The matrix representation of the evolution map for qubit plus charge-impurity \( V \) is achieved by \( \exp(Rt) \) where \( R \) is given in Eq. 15 [2]. This map is in the basis \( \{|\theta_\pm\rangle\} \otimes \{|\sigma_\pm\rangle|j\} \). To make it computationally easier we need to write it in the basis \( \{|\theta_\pm\rangle|\theta_\pm\rangle\} \otimes \{|i\rangle|j\} \), since the matrix representation of the map acting on the qubit is in the basis \( \{|\theta_\pm\rangle|\theta_\pm\rangle\} \). To change the basis we need the assistance of a unitary matrix (in this case permutation matrix) \( P \) which is defined as:
\[ P(|a\rangle|b\rangle|c\rangle|d\rangle) = |a\rangle|c\rangle|b\rangle|d\rangle; \]  
\[ |a\rangle|b\rangle|c\rangle|d\rangle = \left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) \otimes \left(\begin{array}{c} b_1 \\ b_2 \end{array}\right) \otimes \left(\begin{array}{c} c_1 \\ c_2 \end{array}\right) \otimes \left(\begin{array}{c} d_1 \\ d_2 \end{array}\right), \]  
\[ |a\rangle|c\rangle|b\rangle|d\rangle = \left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) \otimes \left(\begin{array}{c} c_1 \\ c_2 \end{array}\right) \otimes \left(\begin{array}{c} b_1 \\ b_2 \end{array}\right) \otimes \left(\begin{array}{c} d_1 \\ d_2 \end{array}\right). \]  
\[ \Rightarrow P = I \otimes p \otimes I, \]  
where \[ p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

After conjugating the matrix \( V \) by \( P \) we get:
\[ \hat{V} = PV P^T. \]  

We can write this 16 \times 16 matrix \( \hat{V} \) as a 4 \times 4 matrix, where each of the element itself is a 4 \times 4 matrix \( V_{ij} \) where \( i, j \in \{1, 2, 3, 4\} \). Then the map \( V_{ij} \) acting on qubit is simply \( V_{ij} = \text{Tr}(Z_{ij}) \). From here we can get the corresponding \( M \) matrix. Solving it analytically, in the present case, is not easy. Therefore, we use numerical methods to calculate the evolution operator and entanglement evolution for a system of qubits.

[1] J. S. Bell, Physics 1, 195 (1964).
[2] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[3] W. H. Louisell, *Quantum Statistical Properties of Radiation* (John Wiley and Sons, 1973).
[4] A. O. Caldeira and A. J. Leggett, Physica A 121, 587 (1983).
[5] H. Breuer and F. Petruccione, *Open Quantum Systems* (Oxford University Press, 2002).
[6] W. H. Zurek, Phys. Today 44, 36 (1991); Prog. Theor. Phys. 87, 281 (1993).
[7] T. Yu and J. H. Eberly, Phys. Rev. Lett. 93 140404 (2004).
[8] A. Al-Qasimi and D. V. F. James, Phys. Rev. A 77, 012117 (2008).
[9] S. K. Goyal and S. Ghosh, arXiv:1003.1248.
[10] S. K. Goyal and S. Ghosh, arXiv:1005.4224.
[11] D. Calderbank and P. Shor, Phys. Rev. A 54, 1098 (1996); A. Steane, Proc. Roy. Soc., London, Ser. A 452, 2551 (1996).
[12] P. W. Shor, Phys. Rev. A 52, 2493 (1995); E. Knill and R. Laflamme, Phys. Rev. A 55, 900 (1997).
[13] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997); L. M. Duan and G. C. Guo, *ibid.* 79, 1953 (1997); D. A. Lidar, I. L. Chuang and K. B. Whaley, *ibid.* 81, 2594 (1998).
[14] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998).
[15] M. Ban, J. Mod. Opt. 45, 2513 (1998); L. M. Duan and G. C. Guo, Phys. Lett. A 261, 139 (1999).
[16] D. Vitali and P. Tombesi, Phys. Rev. A 59, 4178 (1999); Phys. Rev. A 65, 012305 (2001).
[17] L. Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999).
[18] G. S. Agarwal, Phys. Rev. A 61, 013809 (1999); G.S. Agarwal, M. O. Scully, and H. Walther, Phys. Rev. Lett. 86, 4271 (2001); Phys. Rev. A 63, 044101 (2001).
[19] P. Facchi, S. Tasaki, S. Pascazio, H. Nakazato, A. Tokuse and D. A. Lidar, Phys. Rev. A 71, 022302 (2005).
[20] S. John, Phys. Rev. Lett. 53, 2169 (1984).
[21] E. Yablonovitch, Phys. Rev. Lett. 59, 2059 (1987).
[22] S. John, Phys. Rev. Lett. 58, 2486 (1987).
[23] E. Yablonovitch, T. J. Gmiter and K. M. Leung, Phys. Rev. Lett. 67, 2295 (1991).
[24] S. John and T. Quang, Phys. Rev. A 50, 1764 (1994).
[25] S. Banerjee, V. Ravishankar and R. Srikanth, Eur. Phys. Jr. D, 121, 587 (2010).
[26] S. Banerjee, V. Ravishankar and R. Srikanth, Ann. of Phys. (N. Y.), 128, 588 (2010).
[27] I. Chakrabarty, S. Banerjee and N. Siddharth, arxiv:1006.1856 (2010).
[28] S. Banerjee and R. Ghosh, J. Phys. A: Math. Theo. 40, 13735 (2007); eprint quant-ph/0703054.
[29] S. Banerjee and R. Srikanth, Eur. Phys. J. D 46, 335 (2008); eprint quant-ph/0611161.
[30] M-D Choi, Linear algebra and its applications 10, 285-290 (1975).
[31] A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972).
[32] T. Konrad, F. Melo, M. Tiersch, C. Kasztelan, A. Aragao and A. Buchleitner, Nature Physics 4 99 (2008).
[33] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997); W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[34] K. M. Ho, C. T. Chan and C. M. Soukoulis, Phys. Rev. Lett. 65, 3152 (1990).
[35] M. Lewenstein, J. Zakrzewski and T. W. Mossberg, Phys. Rev. A 38 808 (1988).
[36] Y. Yang and S. Y. Zhu, Phys. Rev. A 62 013805 (2000).
[37] F. Wang, Z. Zhang and R. Liang, Phys. Rev. A 78 042320 (2008).
[38] B. M. Garraway, Phys. Rev. A 55, 2290 (1997).
[39] G. S. Agarwal and W. Harshawardhan, Phys. Rev. A, 50 R4465 (1994).
[40] S. Raghavan, V. M. Kenkre, D. H. Dunlap, A. R. Bishop and M. I. Salkola, Phys. Rev. A 54 1781 (1996).
[41] P. K. Lam and C. M. Savage, Phys. Rev. A 50 3500 (1994).
[42] F. Grossmann, T. Dittrich, P. Jung and P. Hanggi, Phys. Rev. Lett. 67 516 (1991).
[43] M. Wagner, Phys. Rev. A 51 798 (1995).
[44] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cam-
[45] T. J. Osborne, F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).

[46] Y. Makhlin, G. Schön and A. Shnirman, Nature 386, 305 (1999).

[47] G. Falci, R. Fazio, G. M. Palma, J. Siewert and V. Vedral, Nature 407, 355 (2000).

[48] Y. Nakamura, Yu. A. Pashkin, J. S. Tsai, Nature 398, 786 (1999).

[49] J. R. Friedman, V. Patel, W. Chen, S. K. Tolpygo and J. E. Lukens, Nature 406, 43 (2000).

[50] A. Shnirman, Y. Makhlin and G. Schön, Phys. Scr. T102, 147 (2002); Y. Makhlin, G. Schön and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).

[51] E. Paladino, L. Faoro, G. Falci and R. Fazio, Phys. Rev. Lett. 88, 228304 (2002).

[52] E. Paladino, L. Faoro and G. Falci, Adv. in Solid State Phys. 43, 747-762 (2003).

[53] A. B. Zorin, F.-J. Ahlers, J. Niemeyer, T. Weimann and H. Wolf, Phys. Rev. B 53, 13682 (1996).

[54] Y. Nakamura, Yu. A. Pashkin, T. Yamamoto and J. S. Tsai, Phys. Rev. Lett., 88, 047901 (2002).

[55] C. L. Kane, Phys Rev. Lett 90, 226802 (2003).

[56] J. Bergli and L. Faoro, arXiv:cond-mat/0609073 (2006).

[57] B. Abel and F. Marquardt, arXiv:0805.0962073 (2008).

[58] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, Atom-Photon Interactions, Wiely-Interscience (1993).
