Mottness induced phase decoherence suggests Bose-Einstein condensation in overdoped cuprate high-temperature superconductors

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(Dated: April 5, 2019)

Recent observations of diminishing superfluid phase stiffness in overdoped cuprate high-temperature superconductors challenges the conventional picture of superconductivity. Here, through analytic estimation and verified via variational Monte Carlo calculation of an emergent Bose liquid, we point out that Mottness of the underlying doped holes dictates a strong phase fluctuation of the superfluid at moderate carrier density. This effect turns the expected doping-increased phase stiffness into a dome shape, in good agreement with the recent observation. Specifically, the effective mass divergence due to “jamming” of the low-energy bosons reproduces the observed nonlinear relation between phase stiffness and transition temperature. Our results suggest a new paradigm, in which the high-temperature superconductivity in the cuprates is dominated by physics of Bose-Einstein condensation, as opposed to pairing-strength limited Cooper pairing.

PACS numbers: 74.72.-h, 74.20.Mn, 74.40.Kb, 74.20.Rp

Discovered almost thirty years ago, the exotic phenomenon of high-temperature superconductivity (HT-SC) in the cuprates still remains puzzling to researchers. Conventional superconductivity, a state of the matter that shows no resistance in conducting current, is well described by the standard “BCS” theory via weakly bound “Cooper pairs” that fluctuate in amplitude. On the other hand, the HT-SC in cuprates shows qualitatively different behavior. For example, in the weakly hole doped (“underdoped”) region, the isotope effect increases dramatically and yet the corresponding superconducting transition temperature $T_c$ decreases instead. The observed superconducting gap $\Delta_0$ in the cuprates is typically significantly larger than the canonical value of twice the transition temperature $\sim 2k_B T_c$ in BCS theory. During the phase transition, the measured specific heat shows two clear kinks $10K$ apart, qualitatively different from a standard second-order phase transition from the BCS theory. Furthermore, in the underdoped region, the low-temperature specific heat shows no $T^2$ contribution expected from the observed $d$-wave quasiparticles, but only a dominant $T^3$ instead. In addition, the observed upper critical field $H_{c2}$ does not saturate at low temperature and sometimes even exceeds the Pauli limit. These qualitative distinct features indicates clearly that the HT-SC in the cuprates is of a different nature.

A significant step forward came with the realization of a different physical regime where the fluctuation is predominately in the superconducting phase rather than its amplitude. In that case the superconducting transition temperature $T_c$ would be determined by the phase stiffness, as opposed to the strength of the pairing. This regime is arguably unavoidable in the low doping $\delta$ (low carrier density $n$) regime of cuprates, given the canonical conjugation between carrier density and phase $\phi$, $\Delta n \Delta \phi \sim \hbar$. This crucial realization seems to provide a very natural explanation of the generic “dome” shape of superconducting transition temperature in the cuprates: As shown schematically in Fig. 1(a), in the underdoped side, $T_c$ increases as the phase stiffness grows at higher carrier density (grey dotted line), while in the overdoped side $T_c$ decreases due to reducing pairing interaction (grey dashed line) as in the BCS theory.

However, recent measurement of penetration depth $\lambda$ challenges this traditional picture. Surprisingly, the low-temperature superfluid phase stiffness ($\propto \lambda^{-2}$) also forms a dome shape against doping, qualitatively different from the simple proportionality to carrier density expected in the pairing strength-limited BCS theory. Furthermore, the low-temperature phase stiffness even scales with $T_c$ in the overdoped regime, obeying the same universal Uemura-relation well-known in the underdoped regime. Since this relation implies a phase coherence-limited superconductivity, this new data apparently reveals that even in the overdoped side, the dominant physics is still the phase fluctuation!

In fact, several previous studies already raised similar doubts against the common lore that at least in the overdoped regime the superconductivity can still be described by pairing strength-limited BCS-like theories. As shown in Fig. 1(b), angular resolved photo-emission spectroscopy (ARPES) observed a nearly doping independent superconducting gap $\Delta$ near momentum $(\pi/2, \pi/2)$ for almost the entire doping range, in great contrast to the expected proportionality to $T_c$ in the BCS theory. Similarly, Fig. 1(d) shows that even earlier, the observed momentum-dependence of superconducting
FIG. 1: Indications of the essential role of phase fluctuation even in the overdoped cuprates. (a) Schematic phase diagram of hole-doped HTSC. Beyond the antiferromagnetic (AFM) phase at very low doping, the superconductivity exists in a “dome” region of the phase diagram, much smaller than the one predicted by the VMC calculation. This suggests that below the classical phase coherent temperature, $T_0$, additional phase fluctuation must exist in the region in grey. (b) Observed superconducting gap size (triangles) near momentum $(\pi/2,\pi/2)$ [15] displays a doping dependence very different from that of superconducting temperature (purple line), but well captured by $\Delta \propto (1 - 2\delta)\sqrt{T_c}$ (red line). Normalized spectral weight transfer $\Delta W$ from condensed EBL (c) and ARPES [17] (d) both show a $d$-wave form distinct from the $\cos(k_x) - \cos(k_y)$ form of the superconducting order parameter expected in BCS-like theories.

end of the dome) $\delta \sim 5\%$ and $25\%$? Finally and most importantly, what essential nature do these puzzles reveal about the unconventional high-temperature superconductivity in the cuprates?

Here we show that all these important questions can be answered naturally in the scenario of the emergent Bose liquid (EBL) [21–23]. In this low-energy effective theory the doped holes are assumed to be tightly bound into local nearest neighboring pairs due to various higher-energy physics, for example, exclusion of double occupancy [24]—two-dimensional short-range antiferromagnetic correlation [26] and bi-polaronic correlation [27]. The resulting pivotal motion of the emergent bosons is then described by a bond-centered checkerboard lattice (a two-orbital Hamiltonian corresponding to the vertical and horizontal bonds) as shown in Fig. 2(a):

$$H = \sum_{\langle l,l' \rangle} \tau_{ll'} b_{l}^\dagger b_{l'}^\vphantom{\dagger} (+ \text{ constraint})$$

where $b_l$ denotes the annihilation of a boson located at bond $l$, and $\tau_{ll'} = \tau'$ or $\tau''$ is the fully dressed kinetic process involving nearest and second nearest neighboring bonds (which was previously extracted from ARPES dispersion data [21].) It is important to note that the emergent bosons have an “extended hard-core” constraint: all nearest and second nearest neighboring bonds [c.f.: empty ellipses in Fig. 2(a)] are excluded from other bosons, due to the double-occupancy constraint of the underlying doped holes in a Mott insulator. This effective short-range repulsion helps provide the phase stiff-
ness of superfluidity and prevents phase separation due to clustering of the emergent bosons.

Previously this model was shown to offer a simple explanation for the disappearance of superfluidity in cuprates at $\delta \sim 5\%$, as the effective mass of the emergent boson diverges [22]. (The same study also suggested a scenario for the lack of superfluidity below 5% doping.) In a related study, a second kind of “superconducting gap” was found in the quasi-particle spectrum, resulting from coherent kinetic scattering against the Bose-Einstein condensation (BEC) of the EBL [21]:

$$\Delta_k(T) \sim f_d(k) \cdot (\epsilon_k - \epsilon_0) \sqrt{n_0(T)} \tag{2}$$

where $f_d(k) = \frac{1}{2} [\cos(k_x) - \cos(k_y)]$ gives the momentum dependent $d$-wave form factor of the order parameter and $n_0(T)$ the temperature-dependent condensation density. $\epsilon_k - \epsilon_0$ denotes the quasi-particle energy measured from the low-energy band center, and signifies the kinetic origin of the gap (as opposed to the pairing potential). Beyond the “Fermi arc”, where $\epsilon_k$ moves from the chemical potential toward the band center, it provided additional momentum dependence [21] that reproduced nicely the ARPES measurements [15, 17] at various doping levels [c.f. Fig. 1(c)].

We first observe that this model actually “predicted” naturally the above-mentioned nearly doping independent superconducting gap in ARPES measurements [14, 15]. Since the quasi-particles on the Fermi arc are at the chemical potential $\mu$, $\epsilon_k - \epsilon_0$ is given by $\mu - \epsilon_0 \propto 1 - 2\delta$ upon hole doping into the quasi-2D singlet band of the cuprates. Away from the ends of the dome, in the absence of strong quantum fluctuation, the condensation density $n_0(T = 0)$ is roughly proportional to the superfluid density, $n_s(T = 0) \propto T_c$. Together, this gives a weakly doping dependent near-node gap scale $V_\Delta \propto (1 - 2\delta)\sqrt{T_c}$, very similar to the experiments shown in Fig. 1(b), but in great contrast to the stronger dome shape of $T_c$. Particularly, notice that the $1 - 2\delta$ factor shifts the maximum of $V_\Delta$ to a significantly lower doping $\delta \sim 0.12$, compared to $T_c$. Similar to the above mentioned deviation from $d$-wave form of the gap, this again reflects the kinetic nature of the scattering gap absent in typical pairing scenario.

Concerning the main puzzles of the strong phase fluctuation and diminishing superfluidity at overdoped regime, we will now show that the EBL also provides a simple resolution through its “extended hard-core constraint”. Focusing on the $d$-wave bosons relevant to the BEC in Fig. 2(a), we first integrate out the $s$-wave bosonic degrees of freedom. The resulting one-orbital Hamiltonian

$$\hat{H} = \sum_{<i,i'>} \tilde{\tau}_{ii'} d_i^\dagger d_{i'} \quad (\text{+ interaction & constraint}) \tag{3}$$

corresponds to larger local $d$-wave Wannier functions [cf: Fig. 2(b)] that can hop to the first, second and third neighbors with $\tilde{\tau}_{ii'}$ with a correspondingly more extended hard-core constraint over 16 bonds. Here $d_i^\dagger$ denotes creation of a $d$-wave boson at site $i$. As highlighted in Fig. 2(c), this constraint forbids occupation of 12 surrounding sites by other low-energy $d$-wave bosons.

Such a large extended hard-core implies that the essential kinetic processes will be easily renormalized at moderate density due to blocking and jamming between the low-energy $d$-wave bosons. Figure 2(d) shows a simple estimation of the special case, in which each of the bosons reach minimum non-zero mobility on average, being able to hop back and forth between two sites. The corresponding density, 2 holes (1 boson) / 8 atomic sites $= 25\%$, marks the approximate maximum doping level of superfluidity, above which the $d$-wave bosons become nearly impossible to move and thus unable to maintain phase coherence. Interestingly, this 25% coincides very well with the experimentally observed end of the superconducting dome in overdoped cuprates in general [28]. Therefore, this mechanism offers a simple and natural explanation of the key puzzle of strong phase fluctuation in the overdoped cuprates.

We employ the well-known Gutzwiller $g$-factor approximation [21, 30] $\tilde{\tau}_{ii'} \rightarrow g_{ii'} \tilde{\tau}_{ii'}$ as the simplest way to capture this renormalization of the kinetic process. The $g$-factors are calculated via two numerical approaches [31]. First, we count statistically the probability of each hopping under the constraint at various doping level. Figure 3(a) shows that the resulting $g$-factors are approximately $g \sim 1 - \delta/0.3$. Second, we calculate the $g$-factor using VMC based on a noninteracting BEC wavefunction under the constraint. The resulting $g$-factors in Fig. 3(b) resemble Fig. 3(a) very well. As expected, both results...
We now demonstrate the “dome” shape of $T_c$ with our simple picture. Estimated from the standard BEC of a uniform boson gas, the condensation temperature $T_c \propto n^{2/3}/m^*$ is a simple function of density $n$ and effective mass $m^*$. Obviously, following $\tilde{r}$, $1/m^*$ and $T_c$ are also renormalized by $g$. Figure 3(c) plots $T_c \propto \delta^{2/3}$ (in black) and the renormalized $T_c \rightarrow gT_c$ (in red). Indeed $gT_c$ is strongly suppressed at higher doping and eventually vanishes around 30%, where $m^*$ diverges in an average sense. (The Gutzwiller approximation is expected to underestimate the short-range fluctuation when the jamming becomes severe at high density, resulting in a slight overestimation of the persistence of the superfluidity beyond 25% doping.) If we further incorporate the previously proposed doping dependent mass of the boson [22], which contains an additional mass divergence at around 5% due to level crossing, the $gT_c$ in Fig. 3(d) reproduces the experimentally observed dome shape quite well. In short, the phase fluctuation can indeed grow in the overdoped regime when the suppression of kinetic processes overcomes the increasing density.

Similarly, the low-temperature limit of the phase stiffness $\lambda^{-2}$ would also suffer from the jamming reduced kinetic process. Phase stiffness $\lambda^{-2} \propto n_s/m^*$ is proportional to the superfluid density $n_s$. Near the above mentioned mass divergence dictated quantum phase transition points, $n_s \propto (1/m^*)^2 \delta$ must vanish with $1/m^*$ with an exponent $\beta > 0$, leading to $\lambda^{-2} \propto (1/m^*)^{1+\beta} \delta$. Figure 4(b) shows that this phase stiffness indeed forms a dome shape, vanishing at around $\delta \sim 5\%$ and $\sim 30\%$.

Particularly, due to the extended hard-core constraint of the EBL, the phase stiffness reduces in the overdoped regime against the growing density, as observed in recent experiments [10, 11]. (Our results’ deviation from experiment merely reflects the above-mentioned overestimation of phase coherence at high doping in our simple Gutzwiller treatment.)

Furthermore, the effective mass divergence of EBL has an important physical consequence in the relation between $\lambda^{-2}$ and $T_c$ [22]. Estimated from $T_c \propto \delta^{2/3}/m^*$ for a free boson gas, one finds $\lambda^{-2} \propto T_c^{1+\beta}$. Therefore, distinct from the simple linear relation expected from standard phase fluctuation scenario, $\lambda^{-2}$ vs $T_c$ must show a zero slope as $T_c$ approaches zero (since $\beta > 0$). This provides a natural explanation of the super-linear relation in the experimental observations shown in Fig. 4(a) and (c). In fact, with $\beta = 1$ this formula describes very well the observed relation in the entire low-$T_c$ region in both underdoped and overdoped sides.

Such an apparent “symmetry” near the quantum phase transition points between the underdoped and overdoped sides is puzzling within the current lore, given that one is supposedly governed by fluctuation in the phase and the other in the amplitude. Especially, the observed physical properties of the normal-state above $T_c$ in these two regions are quite distinct [28, 33, 34]. In our EBL picture, the contrast in non-superfluid properties is easily understood from the rising importance of the jamming effect discussed above. Concerning the higher-energy features, Fig. 4(d)-(f) show that in the overdoped region the renormalized s-wave (in blue) and d-wave bosons (in red) suffer a significant renormalization, quite distinct from the underdoped and optimally doped region. Yet, the renormalized low-energy coherent d-wave bosonic band near the underdoped (d) and the overdoped transition points becomes heavier (flatter) in a similar fashion (through very different microscopic mechanism though.) In other words, the low-energy phenomenon of superfluidity experiences a similar loss of phase stiffness due to the effective mass divergence despite the distinctly different underlying higher-energy physics.

From this perspective, the non-superconducting “normal state” above 25% doping should have interesting properties. While energetically most favorable locally, the pure d-wave boson would suffer from severe jamming and loss of kinetic energy. It would thus tend to morph into a p-wave boson whose cigar shape allows them to remain mobile at a much larger doping [31]. Therefore, the EBL is expected to behave as an unconventional metal consisting of fluctuating d- and p-wave bosons. Obviously, at very high doping, the electronic correlation will eventually become weaker and the emergent bosons will decompose into weakly bound fermions. However, the observations of robust paramagnon dispersion at 40% doping [20] and linear resistivity at 30% doping [10] both suggest strongly that such decomposition is not immedi-
ate after the disappearance of superconductivity.

It is also interesting to realize that in every energy scale our EBL experiences important influence of the Mottness of the underlying doped holes through the suppression of their double occupation at each Cu site. First, with the help of kinetic energy, it lays the foundation of nearest neighboring antiferromagnetic [35] and bi-polaronic [27] correlations that provides a strong tendency to bind doped holes into pairs [24] [35]. Then, it forces the bound pairs to occupy two atomic sites and thus indirectly establishes $d$-wave form [22], since a single-site local pair would necessarily be of $s$-wave. Finally, it produces the extended hard-core constraint of the EBL that ultimately establishes above that the diminishing superfluid stiffness is a natural outcome of jamming in the EBL at around 25-30% doping due to the Mottness of the doped holes. Our study indicates that, in the entire doping range of the superconducting dome, the unconventional superconductivity can be described naturally by the superfluidity of an EBL, without resorting to a crossover into BCS-like amplitude-fluctuating descriptions with weakened pairing interactions. While various observations [10] [27] [58] are already consistent with this new paradigm in the underdoped region, future investigations on the existence of 2e charge quanta of carriers in the overdoped region, for example via shock noise experiments, would provide decisive evidence to this important long-standing puzzle of modern physics.

This work is supported by National Science Foundation of China (NSFC) #11674220 and 11745006 and Ministry of Science and Technology #2016YFA0300500 and 2016YFA0300501. FY acknowledges support from NSFC # 11674025.

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Supplemental Materials: Mottness induced phase decoherence suggests Bose-Einstein condensation in overdoped cuprate high-temperature superconductors

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STATISTICAL COUNTING OF GUTZWILLER g-FACTOR

We perform the statistical counting of the Gutzwiller, $g$-factor $g_{ii'}$, numerically: In a $M = 10 \times 10$ square lattice of the $d$-wave boson, we randomly places $N \equiv \frac{4}{3}M$ bosons and keep only the configurations that satisfied the 13-site extended hard core constraint described in the main text. Within these valid configurations, we then count the probability $P$ of “legal” hoppings for each first, second and third neighboring hopping $d_i \cdot d_{i'}$ that do not lead to violation of the constraint afterward.

This probability is closely related to the Gutzwiller $g$-factor $[S1]$, except that it ignore details of the lowest energy states. Specifically, the definition of the Gutzwiller $g$-factor involves the ratio of thermal average of the hopping process $g_{ii'} = \langle d_i \cdot d_{i'} \rangle_c / \langle d_i \cdot d_{i'} \rangle_0$, evaluated using low-energy states with and without the constraint (denoted by subscripts $c$ and 0.) In our case, the $d_i$ is the creation operator of local $d$-wave boson. Therefore, this $g$-factors should conceptually be energy and temperature dependent, especially around the scale of the interaction that induces this constraint. On the other hand, below this energy scale where the constraint is enforced, the $g$-factor should be quite energy- and temperature-independent, and thus not very sensitive to the detailed structure of the low-lying states. This makes such statistical counting a rather good approximation of the actual $g$-factor. Indeed, as shown in the main text, the resulting $g$-factor resembles the VMC calculation very well. This energy independence (within the scale of our low-energy Hamiltonian) also allows us to apply on an approximate ground state in our VMC calculation.

VARIATIONAL MONTE CARLO (VMC) CALCULATION OF GUTZWILLER g-FACTOR

Due to the extremely large many-body Hilbert space of bosonic systems, we can only afford to evaluate the Gutzwiller $g$-factor using approximate wave function. Luckily, as argued above, the results should be quite insensitive to this choice of approximate wave function. The calculation detail is shown as following:

We calculate the ratio between the expectation values of the hopping term $\langle d_i \cdot d_{i'} \rangle$ in the “Gutzwiller-projected” Bose-Einstein codensation (BEC) state and that in the mean-field BEC state as the Gutzwiller $g$-factor, $g_{ii'}$,

$$g_{ii'} = \frac{\langle \text{BEC} | P_G d_i \cdot d_{i'} P_G | \text{BEC} \rangle}{\langle \text{BEC} | d_i \cdot d_{i'} | \text{BEC} \rangle}.$$  \hspace{1cm} (S1)

Here, the $P_G$ represents the extended hard-core constraint imposed on the bosonic hole-pairs, and the BEC mean-field state $|\text{BEC}\rangle$ is expressed by the following normalized formula,

$$|\text{BEC}\rangle \equiv \frac{1}{\sqrt{N!}} (d_{k=0}^d)^N |0\rangle = \frac{1}{\sqrt{M!}} \sum_{i=1}^M \langle d_i \rangle^N |0\rangle,$$  \hspace{1cm} (S2)

where $M$ and $N$ represent the total site number and boson numbers respectively, with $N = M\delta/2$.

The denominator of Eq. [S1] can be easily obtained as,

$$\langle \text{BEC} | d_i \cdot d_{i'} | \text{BEC} \rangle = \frac{1}{M} \sum_k e^{-i k \cdot (R_i - R_{i'})} \langle \text{BEC} | d_k d_k | \text{BEC} \rangle = \frac{1}{M} \langle \text{BEC} | d_i d_0 | \text{BEC} \rangle = \frac{N}{M} = \delta/2.$$  \hspace{1cm} (S3)

The numerator of Eq. [S1] can be evaluated as,

$$\langle \text{BEC} | P_G d_i \cdot d_{i'} P_G | \text{BEC} \rangle = \sum_\alpha |\alpha\rangle |\alpha \rangle \langle \alpha | \langle \alpha | d_i \cdot d_{i'} | \beta \rangle \rangle \cdot \frac{\langle \beta | P_G | \text{BEC} \rangle}{\langle \alpha | P_G | \text{BEC} \rangle} = \sum_\alpha P_\alpha B_\alpha,$$  \hspace{1cm} (S4)

where $P_\alpha \equiv |\alpha\rangle |\alpha \rangle \langle \alpha |$ represents the weight of each configuration $|\alpha\rangle$ in the Gutzwiller-projected wave function $P_G |\text{BEC}\rangle$, and $B_\alpha = \sum_\beta \langle \alpha | d_i \cdot d_{i'} | \beta \rangle \rangle \cdot \frac{\langle \beta | P_G | \text{BEC} \rangle}{\langle \alpha | P_G | \text{BEC} \rangle}$ represents the measurement for the configuration $|\alpha\rangle$. The weight
BEYOND OVERDOPED SYSTEM: BOSE METAL CONSISTING OF AN INCOHERENT \( p \)-WAVE EBL

Since we are interested in the superfluid phase stiffness in this study, our main study focuses only on the jamming of the \( d \)-wave emergent bosons. In this section we briefly discuss the possible phase beyond the overdoped superfluid regime, where the jamming of the \( d \)-wave emergent bosons costs them too much kinetic energy.

Let’s first recall that at low density, the freely propagating \( d \)-wave bosons gain most kinetic energy, \( 2 \epsilon_d = 2 \tau'' - 4 \tau' \), compared to those of other local symmetries. This energy can be approximately split into two contributions: locally forming a \( d \)-wave form (a four-site Wannier function) and propagating in the system. The former contribution can be easily estimated by solving a local four-site problem and turns out to be \( \epsilon_d \), exactly half of the total kinetic energy. Similarly, a \( p \)-wave emergent boson would have a total kinetic energy \( 2 \epsilon_p = -2 \tau'' \), half of which is also gained by forming local \( p \)-wave symmetry. Therefore, when \( \tau' > \tau'' \), \( 2 \epsilon_d < 2 \epsilon_p \) and \( d \)-wave form is dominant.

As discussed in the main text, as the density increases upon higher doping, the large hard core of the \( d \)-wave boson starts to suffer significantly the jamming effect. Represented by the Gutzwiller factor \( g_d \), a crude estimation of the jamming-induced renormalized kinetic energy can be made: \( (1 + g_d) \epsilon_d \). Similarly, for the \( p \)-wave boson, the energy reduces to \( (1 + g_p) \epsilon_p \).

Interestingly, as shown in Fig. S1(a) due to the narrower shape of the \( p \)-wave bosons, they naturally suffer a lot less from the jamming effect than the \( d \)-wave bosons. Consequently, the \( p \)-wave boson can survive a much larger doping, approximately \( \delta = 50\% \) based on the estimation in Fig. S1(b). Indeed, Fig. S1(c) shows a much slower decay in \( g_p \), against \( \delta \) from a simple statistical counting. Therefore, one can expect that at high-enough doping, the \( d \)-wave symmetry should no longer be dominant, while the bosons are forced to adapt to \( p \)-wave more by the kinetic effects. Indeed, Fig. S1(d) shows that for a fixed \( \tau' \) and \( \tau'' \) (from optimal doping), \( (1 + g_p) \epsilon_p \) will eventually become lower than \( (1 + g_d) \epsilon_d \). Therefore, assuming that the emergent Bose liquid remains intact beyond the superconducting dome, the system should become a Bose metal consisting of incoherent \( p \)-wave bosons, similar to the underdoped regime, except with much stronger jamming and scattering effects. Such a strongly fluctuating EBL should present non-Fermi liquid behaviors, as found in Ref. S2, S3. This is also consistent with the similar direction of the charge order found in beyond underdoped and beyond overdoped systems S4.

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FIG. S1: (a) Extended hard-core constraint of site-centered $p$-wave boson. (b) Simple estimation of highest density of $p$-wave boson with minimal non-zero kinetic energy. (c) Gutzwiller $g$-factor of pure $p$- and $d$-wave boson for $\tau'$ calculated by statistical counting. (d) Average kinetic energy of $d$- and $p$-wave boson modified by jamming effect, estimated with a fixed $\tau'$ and $\tau''$. At $\delta \sim 25\%$, the energy of $p$-wave boson becomes lower than $d$-wave boson.