Long $\mathcal{N} = 2, 4$ multiplets in supersymmetric mechanics

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Abstract. We define $SU(2|1)$ supermultiplets described by chiral superfields having non-zero external spins with respect to $SU(2) \subset SU(2|1)$ and show that their splitting into $\mathcal{N} = 2, d = 1$ multiplets contains the so-called “long” indecomposable $\mathcal{N} = 2, d = 1$ multiplets ($2,4,2$). We give superfield formulation for this type of $\mathcal{N} = 2$ long multiplets and construct their most general superfield action. A simple example of long $\mathcal{N} = 4, d = 1$ multiplet is also considered, both in the superfield and the component formulations.

1. Introduction

In [1], $SU(2|1)$ supersymmetric mechanics was proposed as a deformation of the standard $\mathcal{N} = 4$ mechanics by a mass parameter $m$. Superfield approach based on the deformed $SU(2|1)$ superspaces allowed to reproduce many previously known models [2, 3, 4, 5] and to construct new ones [6, 7, 8]. In the paper [9], $SU(2|1)$ supersymmetric quantum mechanics was obtained via dimensional reduction from the superconformal model on the four-dimensional curved spacetime $S^3 \times \mathbb{R}$ and applied to compute vacuum energy of the model. For simplicity, the authors considered supersymmetric mechanics in the framework of $\mathcal{N} = 2, d = 1$ supersymmetry and revealed a new type of supermultiplets, the so-called “long multiplets”. As was shown in [13], the long $\mathcal{N} = 2$ multiplet can be embedded into a generalized $SU(2|1)$ chiral multiplet described by a chiral superfield $\Phi_A$ carrying some external index $A$ with respect to the subgroup $SU(2)$ of the supergroup $SU(2|1)$.

Generalizations to $\mathcal{N} = 4$ supersymmetry with various extended sets of component fields were considered in [10, 11, 12]. The main distinguishing feature of long (non-minimal) multiplets is that they accommodate extended sets of component fields. The long $\mathcal{N} = 2$ multiplet [9] can be interpreted as a deformation of the pair of chiral multiplets ($2, 2, 0$) and ($0, 2, 2$) by a mass-dimension parameter, i.e. it has an extended set of component fields ($2, 4, 2$). The long multiplet ($4, 8, 4$) considered in [12] joins two $\mathcal{N} = 4$ chiral multiplets ($2, 4, 2$) through a dimensionless parameter.

In this contribution we give a brief account of the long $\mathcal{N} = 2$ multiplet, as it was discussed in [13], and present some new results for the long $\mathcal{N} = 4$ multiplet suggested in [12]. To be more precise, we give the superfield description for the long $\mathcal{N} = 2, 4$ multiplets which were studied at the component level in [9, 12].
2. $SU(2|1)$ supersymmetric mechanics

We proceed from the centrally-extended superalgebra $\hat{su}(2|1)$ with the following non-vanishing (anti)commutators:

$$\{Q^i, \bar{Q}_j\} = 2m \left( I_j^i - \delta^i_j F \right) + 2\delta^i_j H, \quad [I_j^i, I_k^j] = \delta_j^k I_j^i - \delta_j^i I_k^j,$$

$$[I_j^i, \bar{Q}_l] = \frac{1}{2} \delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j, \quad [I_j^i, Q^k] = \delta^k_j Q^i - \frac{1}{2} \delta^i_j Q^k,$$

$$[F, \bar{Q}_l] = -\frac{1}{2} \bar{Q}_l, \quad [F, Q^k] = \frac{1}{2} Q^k. \quad (1)$$

Its bosonic sector contains the central charge generator $H$ (commuting with all other generators) and the $U(2)_{\text{int}}$ generators $I_j^i$ and $F$. In the limit $m = 0$, this superalgebra becomes the standard $\mathcal{N} = 4, d = 1$ Poincaré superalgebra.

The supersymmetric $SU(2|1)$ transformations of the superspace coordinates $\zeta := \{t, \theta^i, \bar{\theta}^i\}$, $\bar{\theta}^i = (\bar{\theta}_i)$, are given by

$$\delta \theta_i = \epsilon_i + 2m \bar{\theta}_k \theta_i, \quad \delta \bar{\theta}^i = \bar{\epsilon}^i - 2m \epsilon_k \bar{\theta}^k \bar{\theta}^i, \quad \delta t = i \left( \bar{\epsilon}^k \theta_k + \epsilon_k \bar{\theta}^k \right). \quad (2)$$

The $SU(2|1)$ measure invariant under these transformations is

$$d\zeta = dt d^2 \theta d^2 \bar{\theta} \left( 1 + 2m \bar{\theta}^k \theta_k \right), \quad \delta (d\zeta) = 0. \quad (3)$$

The left chiral subspace $\zeta_L = \{t_L, \theta_i\}$, where $t_L$ is defined as

$$t_L = t + i \bar{\theta}^k \theta_k - \frac{i}{2} m (\theta)^2 \left( \bar{\theta} \right)^2, \quad (4)$$

is closed under the $SU(2|1)$ transformations

$$\delta \theta_i = \epsilon_i + 2m \bar{\theta}_k \theta_i, \quad \delta t_L = 2i \bar{\epsilon}^k \theta_k. \quad (5)$$

Conjugating the coordinates of the subspace $\zeta_L$, one obtains the right-chiral subspace $\zeta_R$.

The $SU(2|1)$ covariant derivatives are defined as

$$D^i = \left[ 1 + m \bar{\theta}^k \theta_k - \frac{3m^2}{8} (\theta)^2 \left( \bar{\theta} \right)^2 \right] \frac{\partial}{\partial \theta_i} - m \bar{\theta}^j \theta_j \frac{\partial}{\partial \theta_j} - i \bar{\theta}^i \partial_t,$$

$$+ m \bar{\theta} \bar{F} - m \bar{\theta}^i \left( 1 - m \bar{\theta}^k \theta_k \right) \bar{I}_j^i,$$

$$D_j = - \left[ 1 + m \theta^k \theta_k - \frac{3m^2}{8} (\theta)^2 \left( \bar{\theta} \right)^2 \right] \frac{\partial}{\partial \bar{\theta}^j} + m \theta^j \theta_j \frac{\partial}{\partial \theta_j} + i \theta_j \partial_t,$$

$$- m \theta \bar{F} + m \theta_i \left( 1 - m \bar{\theta}^k \theta_k \right) \bar{I}_j^i, \quad (6)$$

where $\bar{F}$ and $\bar{I}_j^i$ are the “matrix” parts of the generators $F$ and $I^i_j$. The latter non-trivially act on the covariant derivatives:

$$\bar{I}_j^i D_l = \delta_i^l \bar{D}_j - \frac{1}{2} \delta^l_j \bar{D}_i, \quad \bar{I}_j^i D^k = \frac{1}{2} \delta^k_j D^i - \delta^i_j D^k,$$

$$\bar{F} \bar{D}_l = \frac{1}{2} \bar{D}_l, \quad \bar{F} D^k = - \frac{1}{2} D^k. \quad (7)$$

An $SU(2|1)$ superfield $\Phi_A$ can carry an external $U(2)$ representation corresponding to these matrix parts and it transforms according to this representation as

$$\delta \Phi_A = \left( i \delta h \bar{F} - i \delta h_{ij} \bar{I}^{ij} \right) \Phi_A,$$

$$\delta h = - im \left( \epsilon_k \bar{\theta}^k + \bar{\epsilon}^k \theta_k \right), \quad \delta h_{ij} = im \left( \epsilon_i \bar{\theta}^j + \bar{\epsilon}^j \theta_i \right) \left( 1 - m \bar{\theta}^k \theta_k \right). \quad (8)$$
2.1. Chiral superfields

Chiral $SU(2|1)$ superfields can carry non-zero external spins $s$ with respect to $SU(2) \subset SU(2|1)$. The simplest chiral superfield with $s = 0$ has the field contents $(2, 4, 2)$. As compared to the $SU(2)$ singlet chiral superfields, the number of component fields in the superfield $\Phi_A$ carrying non-zero external spins $s = 1/2, 1, \ldots$ increases according to

\[
(2|2s + 1), (4|2s + 1), (2|2s + 1)).
\]

The decomposition of the $s = 0$ chiral supermultiplet $(2, 4, 2)$ into $N = 2$ multiplets is given by a direct sum of two chiral multiplets, $(2, 2, 0)$ and $(0, 2, 2)$. Below, we will consider the analogous decompositions of the $s \neq 0$ chiral multiplets.

The singlet $(s = 0)$ chiral superfield $\Phi$ in the $U(2)$ representation $(0, 2\kappa)$ satisfies the chirality condition

\[
\mathcal{D}_I \Phi = 0, \quad \tilde{F} \Phi = 2\kappa \Phi, \quad \tilde{F}^I \Phi = 0.
\]

In the case of $s = 1/2, 1, 3/2, \ldots$, the chiral superfield $\Phi_{(i_1 \ldots i_{2s})}$ belongs to the $U(2)$ representation $(s, 2\kappa)$ and is defined by the constraints

\[
\mathcal{D}_I \Phi_{(i_1 \ldots i_{2s})} = 0, \quad \tilde{F} \Phi_{(i_1 \ldots i_{2s})} = 2\kappa \Phi_{(i_1 \ldots i_{2s})}, \quad \tilde{F}^I \Phi_{(i_1 \ldots i_{2s})} = 2\kappa \Phi_{(i_1 \ldots i_{2s})}.
\]

$SU(2|1)$ transformations of chiral superfields can be found from [5].

2.2. The case $s = 1/2$

The $SU(2|1)$ chiral superfield $\Phi_i$ ($i = 1, 2$) in the $U(2)$ representation $(1/2, 2\kappa)$ is defined by the constraints

\[
\mathcal{D}_I \Phi_i = 0, \quad \tilde{F}^I \Phi_i = \delta^I_i \Phi_i - \frac{1}{2} \delta^I_i \Phi_i, \quad \tilde{F} \Phi_i = 2\kappa \Phi_i.
\]

The chirality condition is solved by

\[
\Phi_i (t_L, \theta_i, \bar{\theta}_k) = (1 + 2m \theta^I \theta_i)^{-\kappa} \left[ 1 - \frac{3m^2}{16} (\theta^I)^2 \right] \phi_i (t_L, \theta_i) - m \left( \frac{1}{2} \delta^I_i \bar{\theta}^k \theta_i - \theta^I \theta_i \right) \phi_j (t_L, \theta_i),
\]

\[
\phi_i (t_L, \theta_i) = z_i + \theta_i \psi - \sqrt{2} \theta^k \psi_{(ik)} + \theta_k \bar{\theta}^k B_i.
\]

The superfields $\Phi_i$ and $\phi_i$ transform as

\[
\delta \Phi_i = m \left( 1 - m \theta^I \theta_i \right) \left[ \frac{1}{2} \delta^I_i \left( \epsilon_i \bar{\theta}^k + \bar{\epsilon}^k \theta_i \right) - \left( \epsilon_i \bar{\theta}^j + \bar{\epsilon}^j \theta_i \right) \right] \Phi_j + 2\kappa m \left( \epsilon_i \bar{\theta}^k + \bar{\epsilon}^k \theta_i \right) \Phi_i,
\]

\[
\delta \phi_i = 4\kappa m \left( \epsilon^k \bar{\theta}_k \right) \phi_i + 2m \left( \frac{1}{2} \delta^I_i \epsilon^k \bar{\theta}^k - \bar{\epsilon}^k \theta_i \right) \phi_j.
\]

The relevant $SU(2|1)$ transformations of the component fields read

\[
\delta z^i = - \epsilon^i \psi - \sqrt{2} \epsilon_k \psi_{(ik)}^{(ik)}, \quad \delta B^i = - \epsilon^i \left( i \nabla t \psi - \frac{m}{2} \psi \right) - \sqrt{2} \epsilon_k \left( i \nabla t \psi_{(ik)}^{(ik)} + \frac{3m}{2} \psi_{(ik)} \right),
\]

\[
\delta \psi = \epsilon^k \left( i \nabla t z^k + \frac{3m}{2} z_k \right) - \epsilon^k B_k, \quad \delta \psi_{(ik)} = \sqrt{2} \epsilon^k \left[ i \nabla t z^i + \frac{m}{2} z^i \right] - \sqrt{2} \epsilon (i B^k),
\]

where

\[
\nabla t = \partial_t + 2im \kappa, \quad \nabla t = \partial_t - 2im \kappa.
\]
2.3. Decomposition into $\mathcal{N} = 2$ multiplets

Singling out the subset of $\mathcal{N} = 2$ transformations associated with the parameter $\epsilon_1 \equiv \epsilon$ in (15), we can identify the component fields $\left(z_i, \psi^{(ik)}, \psi, B^i\right)$ with the system of three $\mathcal{N} = 2$ multiplets:

- one long multiplet $(2, 4, 2)_l \oplus$ one short multiplet $(2, 2, 0) \oplus$ one short multiplet $(0, 2, 2)$.

The $\mathcal{N} = 2$-irreducible multiplets $(2, 2, 0)$ and $(0, 2, 2)$ are composed of the fields $(z_2, \psi^{(11)})$ and $(\psi^{(22)}, B^2)$, while the rest of component fields $(z_1, \psi^{(12)}, \psi, B^1)$ forms a multiplet with the field contents $(2, 2, 0)$ that was called “long” multiplet [9]. In the limit $m = 0$, the indecomposable long $\mathcal{N} = 2$ multiplet $(2, 4, 2)_l$ splits into the direct sum of two “short” irreducible $\mathcal{N} = 2$ multiplets $(2, 2, 0)$ and $(0, 2, 2)$. At $m \neq 0$, such a splitting cannot be accomplished by any field redefinition.

In the general case $s > 0$, we have the following sum of $\mathcal{N} = 2$ multiplets:

- $2s$ long multiplets $\oplus$ one short multiplet $(2, 2, 0) \oplus$ one short multiplet $(0, 2, 2)$.

From this decomposition one can figure out that all long multiplets have mass-dimension parameters proportional to $m$.

The relevant $\mathcal{N} = 2$ subalgebra of (11) can be identified with

$$\{Q, \bar{Q}\} = 2(H - \Sigma), \quad Q^2 = \bar{Q}^2 = 0,$$

where

$$Q \equiv Q^1, \quad \bar{Q} \equiv \bar{Q}^1, \quad \Sigma \equiv m \left( F - I_1 \right).$$

(17)

(18)

If we forget about the $su(2|1)$ origin of this superalgebra, the presence of the central $\Sigma$ is not necessary since it can always be removed by a field redefinition, and $H - \Sigma$ can be chosen as the Hamiltonian:

$$H - \Sigma \rightarrow H \equiv i\partial_t.$$

(19)

For the long multiplet $(2, 4, 2)_l$, this shift can be performed through the redefinitions

$$z = z_1 e^{i(2\kappa - 1/2) mt}, \quad \xi = \left( \frac{\psi}{\sqrt{2}} - \psi^{(12)} \right) e^{i(2\kappa - 1/2) mt},$$

$$B = -B^1 e^{i(2\kappa - 1/2) mt}, \quad \pi = \left( \frac{\psi}{\sqrt{2}} + \psi^{(12)} \right) e^{i(2\kappa - 1/2) mt}.$$  

(20)

Then, the corresponding $\mathcal{N} = 2$ supersymmetry transformations,

$$\delta z = -\sqrt{2} \epsilon \xi, \quad \delta \xi = \sqrt{2} i\tilde{\epsilon} \tilde{z},$$

$$\delta \pi = -\sqrt{2} \epsilon B + \sqrt{2} m \tilde{\epsilon} \tilde{z}, \quad \delta B = \sqrt{2} i\tilde{\epsilon} \tilde{\pi} - \sqrt{2} m \tilde{\epsilon} \tilde{\xi},$$

(21)

close on the standard $\mathcal{N} = 2$ supersymmetry algebra

$$\{Q, \bar{Q}\} = 2H, \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0, \quad H = i\partial_t.$$

(22)

The free $SU(2|1)$ Lagrangian is given by

$$\mathcal{L}_{\text{free}} = \frac{1}{4} \int d^2 \theta d^2 \bar{\theta} \left( 1 + 2m \theta^k \bar{\theta}^k \right) \Phi \bar{\Phi}.$$  

(23)
Rewriting it in terms of the component fields,

\[
L_{\text{free}} = \nabla_t z^i \nabla_t \bar{z}_i + i \left( \bar{\psi} \nabla_t \psi + \psi \nabla_t \bar{\psi} \right) + \frac{i}{2} \left( \bar{\psi}_{(ik)} \nabla_t \psi^{(ik)} + \psi^{(ik)} \nabla_t \bar{\psi}_{(ik)} \right) + B_i \bar{B}_i \\
- \frac{i}{2} m \left( z_i \nabla_t \bar{z}_i - \bar{z}_i \nabla_t z_i \right) - \frac{3m^2}{4} z^i \bar{z}_i + \frac{m}{2} \left( \bar{\psi} \psi - 3\psi^{(ik)} \bar{\psi}^{(ik)} \right),
\]

one can check that it splits into a sum of the three free $\mathcal{N} = 2$ Lagrangians:

\[
L_{\text{free}} = L_{(2,4,2)}^{\text{free}} + L_{(0,2,2)}^{\text{free}} + L_{(2,2,0)}^{\text{free}}.
\]

After the redefinition (20), the component Lagrangian of the long multiplet in (25) reads

\[
L_{(2,4,2)}^{\text{free}} = \dot{z} \dot{\bar{z}} + \frac{i}{2} \left( \bar{\xi} \dot{\xi} - \dot{\bar{\xi}} \xi \right) + \frac{i}{2} \left( \bar{\pi} \dot{\pi} - \dot{\bar{\pi}} \pi \right) + BB - m \left( \bar{\xi} \pi + \pi \bar{\xi} \right) - m^2 \bar{z} z.
\]

3. The long $\mathcal{N} = 2$ multiplet

Now we consider a superfield description for the long $\mathcal{N} = 2$ supermultiplet defined by the transformations (21). First we define $\mathcal{N} = 2$ covariant derivatives $D, \bar{D},$

\[
D = \frac{\partial}{\partial \theta} - i \partial \partial_t, \quad \bar{D} = - \frac{\partial}{\partial \bar{\theta}} + i \partial \partial_t, \quad \{D, \bar{D}\} = 2i \partial_t.
\]

The $\mathcal{N} = 2, d = 1$ superspace coordinates $\{t, \theta, \bar{\theta}\}$ transform in the familiar way:

\[
\delta \theta = \epsilon, \quad \delta \bar{\theta} = \bar{\epsilon}, \quad \delta t = i \left( \epsilon \bar{\theta} + \bar{\epsilon} \theta \right).
\]

The chiral superfields are defined by the standard conditions

\[
\bar{D} Z = 0, \quad \bar{D} \Pi = 0.
\]

The bosonic superfield $Z$ describes an irreducible multiplet $(2, 2, 0)$, while the fermionic superfield $\Pi$ has the field contents $(0, 2, 2)$. The component expansion of $\Pi$ and $Z$ reads

\[
Z = z + \sqrt{2} \theta \xi - i \theta \bar{\theta} \bar{z}, \quad \Pi = \pi + \sqrt{2} \theta B - i \theta \bar{\theta} \bar{\pi}.
\]

The “passive” superfield transformations $\delta \Pi = \delta Z = 0$ amount to the two independent sets of transformations for the component fields:

\[
\delta z = -\sqrt{2} \epsilon \xi, \quad \delta \xi = \sqrt{2} i \epsilon \bar{z}, \quad \delta \pi = -\sqrt{2} \epsilon B, \quad \delta B = \sqrt{2} i \epsilon \bar{\pi}.
\]

The long multiplet is described by the pair of fermionic and bosonic $\mathcal{N} = 2$ superfields $\Psi$ and $Z$ (of the same dimension as $\Pi$ and $Z$ above) which are subjected to the following conditions, with $m$ being a deformation parameter:

\[
\bar{D} \Psi = \sqrt{2} m Z, \quad \bar{D} Z = 0.
\]

As a solution of (32) (non-singular at $m = 0$), the superfield $\Psi$ can be represented as

\[
\Psi = \Pi - \sqrt{2} m \theta Z, \quad \bar{D} \Pi = 0.
\]

The first condition in (32) expresses some components of $\Psi$ through the components of $Z$ and so forces the superfunction $\Pi$ to transform through $Z$. 
The transformations $\delta \Psi = \delta Z = 0$ give rise to the following transformation law for $\Pi$:

$$\delta \Pi = \sqrt{2} m \epsilon Z.$$  \hfill (34)

It generates deformed $\mathcal{N} = 2$ supersymmetry transformations which coincide with the transformations (21). Thus the considered multiplet involves an irreducible chiral multiplet $(2, 2, 0)$ and a set of fields $(0, 2, 2)$ which are described by the chiral $d = 1$ superfunctions $Z$ and $\Pi$, respectively. The quantity $Z$ is a chiral superfield, while $\Pi$ has the non-standard transformation law (34) $\sim m$. Thus the parameter $m$ is a deformation parameter responsible for unifying the two former chiral “short” multiplets into a single “long” multiplet.

3.1. Invariant Lagrangians

The most general Lagrangian of the long multiplet can be written down as

$$L_{(\Psi, Z)} = \int d\bar{\theta} d\theta \left[ \bar{D} \bar{D} Z \frac{\partial}{\partial \theta} h_0 \left( Z, \bar{Z} \right) + \Psi \bar{\Psi} h_1 \left( Z, \bar{Z} \right) + \mu h_2 \left( \Psi, Z \right) \right],$$  \hfill (35)

where $h_0$, $h_1$, $h_2$ are arbitrary functions and $\mu$ is a mass-dimension parameter. The free Lagrangian of the long multiplet is given by a sum of the three superfield invariants:

$$L_{(2, 4, 2)} = \frac{1}{4} \int d\bar{\theta} d\theta \left( \bar{D} \bar{D} Z \frac{\partial}{\partial \theta} Z + 2 \bar{\Psi} \bar{\Psi} - 2 \mu ZZ \right).$$  \hfill (36)

In the component form it reads

$$L_{(2, 4, 2)} = \bar{\xi} \ddot{z} + \frac{i}{2} (\ddot{\xi} - \ddot{\bar{\xi}}) + \frac{i}{2} (\ddot{\bar{\pi}} - \ddot{\pi}) + \ddot{B} \bar{B} - \mu \left[ i \frac{1}{2} (z \bar{z} - \bar{z} z) + \bar{\xi} \xi \right].$$  \hfill (37)

The free Lagrangian (26) corresponds to the choice $\mu = 0$.

4. The long $\mathcal{N} = 4$ multiplet

In this section, we give a superfield description of the indecomposable long $\mathcal{N} = 4$ supermultiplet suggested in [12]. The standard $\mathcal{N} = 4$ supersymmetry superalgebra is formed by the (anti)commutators

$$\{Q_i, \bar{Q}_j\} = 2 \delta^i_j H, \quad \{Q^i, Q^j\} = \{\bar{Q}_i, \bar{Q}_j\} = 0, \quad H = i \partial_t.$$  \hfill (38)

The covariant $\mathcal{N} = 4, d = 1$ derivatives are defined in the standard way as

$$D^i = \frac{\partial}{\partial \theta^i} - i \bar{\theta}^i \partial_t, \quad \bar{D}_j = - \frac{\partial}{\partial \bar{\theta}^j} + i \theta_j \partial_t, \quad \{D^i, \bar{D}_j\} = 2 \delta^i_j H.$$  \hfill (39)

The $\mathcal{N} = 4, d = 1$ superspace coordinates $\{t, \theta_i, \bar{\theta}^i\}$ undergo the transformations:

$$\delta \theta_i = \epsilon_i, \quad \delta \bar{\theta}^i = \bar{\epsilon}^i, \quad \delta t = i \left( \epsilon_i \bar{\theta}^i + \bar{\epsilon}^i \theta_i \right).$$  \hfill (40)

The indecomposable long $\mathcal{N} = 4$ supermultiplet is parametrized by a real dimensionless parameter $\alpha$ and is described by the system of complex $\mathcal{N} = 4$ superfields $V$ and $W$ obeying the constraints

$$\bar{D}_i V = i \alpha D_i W, \quad \bar{D}_i W = 0.$$  \hfill (41)
In the limit $\alpha = 0$, these constraints are reduced to those defining two ordinary $(2, 4, 2)$ chiral multiplets. The constraints (41) are solved by

$$V \left( t, \theta_i, \bar{\theta}^j \right) = V_0 \left( t, \theta_i, \bar{\theta}^j \right) + i\alpha \bar{\theta}_k \frac{\partial}{\partial \theta_k} W \left( t, \theta_i, \bar{\theta}^j \right), \quad \bar{D}_k V_0 = 0,$$

implying the following transformation properties for the involved objects

$$\delta V_0 (t_L, \theta_i) = -i\alpha \bar{\epsilon}_k \frac{\partial}{\partial \theta_k} W (t_L, \theta_i), \quad \delta W = \delta V = 0.$$  

In components, the solution (42) reads

$$V = y + \sqrt{2} \theta_i \xi^i + \theta_i \theta^j A + i \bar{\theta}^i \theta_j \theta_i \xi^i - \frac{1}{4} \bar{\theta}^i \bar{\theta}^j \theta_i \theta^j \bar{\psi}^i - i \alpha \bar{\theta}_i \left( \sqrt{2} \bar{\psi}^i + 2\theta^i C - i \bar{\theta}^i \bar{\theta}^j \bar{\psi}^j \right),$$

$$W = x + \sqrt{2} \theta_i \psi^j + \theta_i \theta^j C + i \bar{\theta}^i \theta_j \theta_i \psi^j - \frac{1}{4} \bar{\theta}^i \bar{\theta}^j \theta_i \theta^j \bar{\psi}^j,$$

with

$$\delta y = -\sqrt{2} \bar{\epsilon}_i \xi^i - \sqrt{2} i \alpha \bar{\epsilon}_i \psi^j, \quad \delta \xi^i = \sqrt{2} i \bar{\epsilon}^i (\bar{\psi}^i - \alpha C) - \sqrt{2} \epsilon^i A, \quad \delta A = -\sqrt{2} i \bar{\epsilon}_i \xi^i,$$

$$\delta x = -\sqrt{2} \bar{\epsilon}_i \psi^j, \quad \delta \psi^j = \sqrt{2} i \bar{\epsilon}^j \bar{\psi}^i - \sqrt{2} \epsilon^j C, \quad \delta C = -\sqrt{2} i \bar{\epsilon}_i \psi^j.$$  

The components of $W$ have the standard transformations inherent to the chiral multiplet $(2, 4, 2)$, while the transformations of the remaining fields $y$, $\xi^i$, $A$ acquire additional pieces involving the components of $W$ (they are proportional to $\alpha$).

4.1. Lagrangian

The general kinetic Lagrangian is written as

$$\mathcal{L}^{\text{kin}} = \frac{1}{4} \int d^2 \theta \, d^2 \bar{\theta} \, f \left( V, \bar{V}, W, \bar{W} \right),$$

where $f$ is just an arbitrary real function of superfields. Like in (42), we can define six bilinear invariant kinetic Lagrangians:

$$V \bar{V}, \quad W \bar{W}, \quad V \bar{W} + W \bar{V}, \quad i \left( V \bar{W} - W \bar{V} \right), \quad V^2 + \bar{V}^2, \quad i \left( V^2 - \bar{V}^2 \right).$$

Possible terms $VW$ and $\bar{W} \bar{V}$ do not contribute. Dependence on $\alpha$ remains only in the superfield $V$, so only five out of six bilinear kinetic Lagrangians involve the parameter $\alpha$.

One can write a superpotential Lagrangian for the chiral superfield $W$ as

$$\mathcal{L}_1^{\text{pot}} = \int d^2 \theta \, \mathcal{F} (W) + \int d^2 \bar{\theta} \, \bar{\mathcal{F}} (\bar{W}).$$

Another option is to write the following superpotential Lagrangian:

$$\mathcal{L}_2^{\text{pot}} = \int d^2 \theta \, h' (W) V_0 + \int d^2 \bar{\theta} \, \bar{h}' (\bar{W}) \bar{V}_0.$$

The transformation property (43) of $V_0$ allows to represent transformations of this term as

$$\delta \mathcal{L}_2^{\text{pot}} = -i\alpha \bar{\epsilon}_k \int d^2 \theta \, \frac{\partial}{\partial \theta_k} h (W) + \text{c.c.} = 0.$$
Above superpotential Lagrangians have no dependence on \( \alpha \) since the chiral superfunction \( V_0 \) corresponds just to the limit \( \alpha = 0 \) of \( V \).

To make comparison with [12], we can consider two bilinear superpotential terms

\[
\gamma_1 \int d^2 \theta V_0 W + \bar{\gamma}_1 \int d^2 \bar{\theta} \bar{V}_0 \bar{W} \quad \text{and} \quad \gamma_2 \int d^2 \theta W^2 + \bar{\gamma}_2 \int d^2 \bar{\theta} \bar{W}^2.
\]

(51)

Here, \( \gamma_1 \) and \( \gamma_2 \) are complex parameters of mass dimension. These bilinear superpotential terms in components generate the so called “Super-Zeeman” invariant terms of ref.[12], one of them containing expressions proportional to \( \alpha \) and corresponding to a coupling to an external magnetic field. In [12], such terms were also referred to as Wess-Zumino type terms. This Wess-Zumino term can in fact be eliminated by redefining the fields (in the notations of ref. [12]) as

\[
F_4 \rightarrow F_4 - \alpha \phi_6, \quad \psi_1 \rightarrow \psi_1 + \alpha \psi_8, \quad \psi_2 \rightarrow \psi_2 + \alpha \psi_7.
\]

(52)

Then all “Super-Zeeman” invariant terms become independent of \( \alpha \), and the same is true for our general superpotential Lagrangians.

In our notations, Wess-Zumino terms appear only after the elimination of auxiliary fields\(^1\).

An example of such a term is given in the next subsection.

### 4.2. Free model

As an instructive example, let us consider the simple free Lagrangian given by

\[
L^\text{free} = \frac{1}{4} \int d^2 \theta d^2 \bar{\theta} \left( V \bar{V} + (1 - \alpha^2) W \bar{W} \right) + \frac{1}{4} \int d^2 \theta (2\mu_1 V_0 + \mu_2 W) W + \frac{1}{4} \int d^2 \bar{\theta} (2\mu_1 \bar{V}_0 + \mu_3 \bar{W}) \bar{W},
\]

(53)

where \( \mu_1 \) and \( \mu_2 \) are real parameters of the mass dimension. The coefficient \( (1 - \alpha^2) \) in front of \( W \bar{W} \) was chosen to gain the correctly normalized kinetic terms in the off-shell Lagrangian:

\[
L^\text{free} = \frac{\dot{\psi} \dot{\bar{\psi}} + \dot{x} \dot{\bar{x}}}{1 + \alpha^2} + \frac{i}{2} \left( \dot{\bar{\xi}} \xi - \dot{\xi} \bar{\xi} \right) + \frac{i}{2} \left( \dot{\bar{\psi}} \psi - \dot{\psi} \bar{\psi} \right) + \frac{\alpha \mu_2}{1 + \alpha^2} \bar{\phi}_6 - \frac{\mu_1}{1 + \alpha^2} x \bar{x},
\]

\[
\left( C + \bar{C} \right) \left( \bar{C} \psi_1 + \bar{C} \bar{\psi} \right) + \mu_2 \left( \bar{C} \bar{x} + C \bar{x} + \frac{1}{2} \bar{\psi}^i \psi_i + \frac{1}{2} \bar{\psi}^i \bar{\psi} \right).
\]

(54)

After eliminating the auxiliary fields \( A \) and \( C \) by their equations of motion,

\[
\left( 1 + \alpha^2 \right) C = \alpha \dot{y} - \mu_1 \dot{y} - \mu_2 \dot{x}, \quad A = - \mu_1 \bar{x},
\]

(55)

and neglecting a total time-derivative, we obtain the on-shell Lagrangian

\[
L^\text{free} = \frac{\dot{\psi} \dot{\bar{\psi}} + \dot{x} \dot{\bar{x}}}{1 + \alpha^2} + \frac{i}{2} \left( \dot{\bar{\xi}} \xi - \dot{\xi} \bar{\xi} \right) + \frac{i}{2} \left( \dot{\bar{\psi}} \psi - \dot{\psi} \bar{\psi} \right) + \frac{\alpha \mu_2}{1 + \alpha^2} \left( \bar{\phi}_6 - \left( \mu_1 \bar{y} + \mu_2 \bar{x} \right) \right) x \bar{x}
\]

\[
+ \frac{\mu_1}{1 + \alpha^2} \left( \xi^i \psi_i + \bar{\xi}^i \bar{\psi} \right) + \frac{\mu_2}{2} \left( \bar{\psi}^i \psi_i + \bar{\psi}^i \bar{\psi} \right).
\]

(56)

The relevant on-shell transformations are given by

\[
\delta y = -\sqrt{2} \epsilon_i \xi^i - \sqrt{2} i \alpha \epsilon_i \psi_i, \quad \delta \xi^i = \frac{\sqrt{2} i \epsilon^i}{1 + \alpha^2} \left( \dot{y} + \alpha \left( \mu_1 \bar{y} + \mu_2 \bar{x} \right) \right) + \sqrt{2} \mu_1 \epsilon \bar{x},
\]

\[
\delta x = -\sqrt{2} \epsilon_i \psi_i, \quad \delta \psi^i = \sqrt{2} i \epsilon^i \dot{x} - \sqrt{2} i \frac{\epsilon^i}{1 + \alpha^2} \left( \alpha \dot{y} - \mu_1 \bar{y} - \mu_2 \bar{x} \right).
\]

(57)

\(^1\) Such a possibility was also mentioned in [12].
The on-shell Lagrangian (56) contains the Wess-Zumino type term describing an interaction between two chiral $\mathcal{N} = 4$ multiplets $(2, 4, 2)$:

$$\sim \frac{\alpha \mu_2 \left( x \dot{y} + \dot{x} \dot{y} \right)}{1 + \alpha^2}. \quad (58)$$

This term matches with the statement of [12] that the elimination of auxiliary fields induces additional terms which can be treated as a coupling to an external magnetic field.

When $\mu_1 = \mu_2 = 0$, we can make rescaling $y \to \sqrt{1 + \alpha^2} y$ in the Lagrangian (56) and obtain the $\alpha$-independent Lagrangian

$$\mathcal{L}^{\text{free}}|_{\mu_1=\mu_2=0} = \dot{y} \dot{\bar{y}} + \dot{x} \dot{\bar{x}} + \frac{i}{2} \left( \xi^i \dot{\xi}^i - \dot{\xi}^i \xi^i \right) + \frac{i}{2} \left( \dot{\psi}^i \psi^i - \psi^i \dot{\psi}^i \right). \quad (59)$$

The transformations (57) become

$$\delta y = \frac{1}{\sqrt{1 + \alpha^2}} \left[ -\sqrt{2} \epsilon_i \xi^i - \sqrt{2} i \alpha \bar{\epsilon}_i \psi^i \right], \quad \delta \xi^i = \frac{\sqrt{2} i \epsilon_i \bar{y}}{\sqrt{1 + \alpha^2}},$$

$$\delta x = -\sqrt{2} \epsilon_i \psi^i, \quad \delta \psi^i = \sqrt{2} i \epsilon_i \bar{x} - \sqrt{2} \alpha \epsilon^i \bar{y}. \quad (60)$$

The Lagrangian (59) is thus invariant under supersymmetry transformations with various parameters $\alpha$, since it has no dependence on $\alpha$. For instance, it is invariant under the undeformed $\alpha = 0$ transformations

$$\delta y = -\sqrt{2} \eta_i \xi^i, \quad \delta \xi^i = \sqrt{2} i \eta^i \dot{y}, \quad \delta x = -\sqrt{2} \eta_i \psi^i, \quad \delta \psi^i = \sqrt{2} i \eta^i \dot{x}. \quad (61)$$

Their closure with (60) yields additional bosonic transformations

$$\delta y = a \dot{x}, \quad \delta x = \bar{a} \dot{y}, \quad (62)$$

which leave the Lagrangian (59) invariant and commute (on-shell) with the supersymmetric transformations (60) for any $\alpha$. It would be interesting to see whether a similar phenomenon takes place in the interaction case too.

5. Summary and outlook

We have shown how long $\mathcal{N} = 2$, $d = 1$ multiplets can be embedded into $SU(2|1)$ chiral multiplets in the framework of $SU(2|1)$ supersymmetric mechanics. They naturally appear in $SU(2|1)$ mechanics of chiral multiplets, when the chiral superfield $\Phi_A$ carries some external index $A$ with respect to the subgroup $SU(2)$ of the supergroup $SU(2|1)$. We studied this multiplet in the framework of $\mathcal{N} = 2$ superspace and constructed its general superfield action.

We considered the long $\mathcal{N} = 4$ multiplet [12] within the standard $\mathcal{N} = 4$ superspace. Defining and solving the constraint (41), we obtained the superfields $V$ and $W$ describing the long multiplet $(4, 8, 4)_1$ and constructed general superfield Lagrangians consisting of the kinetic (sigma-model type) and superpotential Lagrangians. We considered the free Lagrangian (54), where the superpotential term $\sim \mu_2$ is responsible for appearing Wess-Zumino type term in the on-shell Lagrangian (56).

In conclusion, we outline some further possible lines of investigation.

- Quantization of the model (56) and construction of the Hilbert space of wave functions.
- Study of some other generalizations of long multiplets to the standard flat $\mathcal{N} = 4, d = 1$ supersymmetry [10][11].
• Generalizing the constraints (41) to the $SU(2|1)$ covariantized constraints:

$$\bar{D}_k V = i\alpha D_k W, \quad D_k W = 0. \quad (63)$$

Such a generalization is possible for the second type of the $SU(2|1)$ chirality [6], when the spinor derivatives are inert under the induced $U(1)$ transformations.

• Answering the question whether it is possible to find out $d > 1$ analogs of long multiplets.

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