The Warm DM halo mass function below the cut-off scale.

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ABSTRACT
Warm Dark Matter (WDM) cosmologies are a viable alternative to the Cold Dark Matter (CDM) scenario. Unfortunately, an accurate scrutiny of the WDM predictions with N-body simulations has proven difficult due to numerical artefacts. Here, we report on cosmological simulations that, for the first time, are devoid of those problems, and thus, are able to accurately resolve the WDM halo mass function well below the cut-off. We discover a complex picture, with perturbations at different evolutionary stages populating different ranges in the halo mass function. On the smallest mass scales we can resolve, identified objects are typically centres of filaments that are starting to collapse. On intermediate mass scales, objects typically correspond to fluctuations that have collapsed and are in the process of relaxation, whereas the high mass end is dominated by objects similar to haloes identified in CDM simulations. We then explicitly show how the formation of low-mass haloes is suppressed, which translates into a strong cut-off in the halo mass function. This disfavours some analytic formulations that predict a halo mass function that would extend well below the free streaming mass. We argue for a more detailed exploration of the formation of the smallest structures expected to form in a given cosmology, which, we foresee, will advance our overall understanding of structure formation.

Key words: cosmology:theory - large-scale structure of Universe.

1 INTRODUCTION
Since the 1980s, Warm Dark Matter (WDM) has been an attractive alternative to Cold Dark Matter (CDM) as the main gravitating component in the Universe. For a long time, however, WDM was disfavoured compared to CDM due to, in part, the additional free parameter required (the mass of the DM particle). This has changed recently, and WDM has again attracted the attention of the cosmological community as a viable and competitive cosmological model. Traditionally, the gravitino was favoured as the hypothetical particle that could serve as WDM (e.g. Moroi et al. 1993, and references therein), while, more recently, the interest has focused on the sterile neutrino (see e.g. Boyarsky et al. 2009b, for a recent review).

The reason for the recent revitalisation of WDM is that a suitable AWDM model could reproduce all the successes of \(\Lambda\)CDM on large scales, and, in addition, it may alleviate the alleged tension between the CDM model and some observations. Such measurements concern mainly mass scales of \(\sim 10^{10}M_\odot\) and include: the dynamics of Milky-Way satellites (Boylan-Kolchin et al. 2011, Lovell et al. 2012), the velocity function of HI selected galaxies (Zavala et al. 2009), and the abundance of low mass galaxies at low and high redshifts (Menci et al. 2012).

The key feature of WDM cosmologies that distinguishes them from CDM, is the lack of initial small-scale density fluctuations. Before recombination, WDM particles have relatively high velocities (set when these particles become non-relativistic), which allows them to travel further than a “free-streaming” distance \(R_{fs}\). Thus, particles move out of over-dense regions of size \(R_{fs}\) and smaller, and therefore inflation-generated small-scale density and potential perturbations are washed out. This dissipation is captured as a strong suppression of the mass transfer function below a “cut-off” scale (cf. Bond & Szalay 1983). This difference relative to CDM is expected to affect the abundance of collapsed objects (DM haloes) below the cut-off mass scale \(\sim \) \(10^{10}M_\odot\) and include: the dynamics of Milky-Way satellites (Boylan-Kolchin et al. 2011, Lovell et al. 2012), the velocity function of HI selected galaxies (Zavala et al. 2009), and the abundance of low mass galaxies at low and high redshifts (Menci et al. 2012). The reason for the recent revitalisation of WDM is that a suitable AWDM model could reproduce all the successes of \(\Lambda\)CDM on large scales, and, in addition, it may alleviate the alleged tension between the CDM model and some observations. Such measurements concern mainly mass scales of \(\sim 10^{10}M_\odot\) and include: the dynamics of Milky-Way satellites (Boylan-Kolchin et al. 2011, Lovell et al. 2012), the velocity function of HI selected galaxies (Zavala et al. 2009), and the abundance of low mass galaxies at low and high redshifts (Menci et al. 2012).

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\(^1\) Note that the free-streaming scale today is much smaller than that at early times. Thus, scales below the cut-off scale in the transfer function are, in principle, able to collapse gravitationally, but not those below today’s value of the free-streaming scale.
centrated. In addition, the formation time, as well as the actual halo formation mechanism is modified, as a larger fraction of haloes form directly from the collapse of filaments. Additionally, phase correlations of the overall density field are expected to be related to the mass of the DM particle \cite{Obreschkow et al. 2013}.

Besides the obvious cosmological interest in WDM scenarios, this class of models also offers an interesting test case from a theoretical point of view, especially for cosmological simulations. Free-streaming provides a small-scale limit to the structure formation problem, thus, in principle, it is possible to capture the full hierarchy of objects expected to be relevant during the formation of a halo. This can provide clues as to how hierarchical structure formation proceeds in general, and its connection to cosmological parameters and the spectrum of density fluctuations. In contrast, this is less direct in CDM since there are always structures and interactions not resolved by any given mass and force resolution at late times.

Unfortunately, an accurate numerical scrutiny of the predictions of a WDM model has proven difficult. Despite a multitude of attempts, N-body simulations, and analytical frameworks, have not been able to decisively quantify even one of the simplest properties of the nonlinear field: the abundance of DM haloes. Cosmological WDM simulations show the dominant presence of an artificial population of low-mass DM haloes. This phenomenon originates in the numerical (unphysical) fragmentation of filaments, and it exceeds by far the population of real WDM haloes on scales near the cut-off scale (see e.g. \cite{Avila-Reese et al. 2001, Bode et al. 2001, Wang & White 2007, Melott 2007}. Although higher mass resolution helps to somewhat reduce this problem (e.g. \cite{Schneider et al. 2013}), the mass scale dominated by artificial fragmentation changes only weakly with an increase of the mass resolution. Besides the obvious impact of artificial fragmentation in the halo mass function, it is possible that they also affect the internal properties of DM haloes, which grow by accreting these objects. Finally, it is currently unknown what type of structures, if any, inhabit below the cut-off mass scale. The latter is relevant when modelling galaxy formation in WDM scenarios.

Similarly, recent analytic approaches, based on modifications of the Extended Press-Schechter theory \cite{Press & Schechter 1974, Lacey & Cole 1993}, produce predictions that differ by orders of magnitude from each other in the way in which the halo mass function is suppressed in WDM \cite[compare e.g.][]{Smith & Markovic 2011, Benson et al. 2013}. The reason for this is mainly uncertainties in the formulation of traditional excursion sets, together with possible modifications to the shape of the barrier for collapse near the cut-off scale.

In this paper, we revisit the issue of the halo mass function in WDM cosmologies and propose an answer to the questions posed above. In order to overcome previous limitations, we carry out a suite of high-resolution cosmological N-body simulations that feature a recently developed method to compute gravitational forces \cite{Hahn et al. 2012}. This new technique allows us to alleviate some of the long-known problems originating from employing excessive force resolution compared to the mass resolution in the simulations \cite{Efstathiou & Eastwood 1981, Centrella et al. 1988, Melott & Shandarir 1989, Splinter et al. 1998}, and also to reduce discreteness noise in the large-scale density and tidal fields, which we find are key players in causing artificial fragmentation.

With these tools at hand, we are able to robustly compute for the first time the WDM halo mass function at, and below, the cut-off mass scale. Our simulations unveil systems of different characteristics, and at different stages of formation, populating different mass ranges in the halo mass function. Explicitly, well below the cut-off scale we find dense filaments and sheets that have started to collapse into 3D systems. At larger mass scales we find proto-haloes, systems that are collapsed but still in the process of virialisation. Only above the cut-off scale do we observe systems traditionally regarded as DM haloes. When we consider only haloes, we observe a strong suppression of the halo mass function, together with a cut-off on small masses, however. However, due to the wealth of different types of structures, the position of the cut-off depends on the actual halo definition one wishes to adopt.

The structure of the paper is as follows: In §2 we provide details of the simulation techniques and the construction of the halo catalogues. In §3 we present our results, focusing on the abundance of collapsed objects, the WDM halo mass function, and exploring the characteristics of objects located at different mass ranges in the halo mass function. We then discuss our findings and possible implications. Finally, in §4 we provide a summary of our work along with directions for future work.

2 NUMERICAL TOOLS

In this section, we describe the numerical simulations we use to study dark matter haloes in a WDM scenario. We also describe our halo identification procedure and the construction of the group catalogues.

2.1 Initial conditions

We start by computing the power spectrum of density fluctuations using the fitting formulae of \cite{Eisenstein & Hu 1999}. We adopt a set of cosmological parameters consistent with the published measurements of the WMAP7 data release \cite{Komatsu et al. 2010}. Explicitly: $\Omega_m = 0.276$, $\Omega_{\Lambda} = 0.724$, $\Omega_b = 0.045$, $h = 0.703$, $\sigma_8 = 0.811$ and spectral index $n_s = 0.96$. Note we set the normalisation of the power spectrum according to $\sigma_8$ via a CDM spectrum, so that the amplitude of fluctuations on large-scales is independent of the WDM particle mass.

We then incorporate the effects of a thermally produced warm dark matter particle on the transfer function following the fitting formula of \cite{Bode et al. 2001}. Explicitly, a fit to the WDM transfer function of density perturbations is given by:

$$T_{\text{WDM}}(k) = T_{\text{CDM}}(k) \left[1 + (\alpha k)^2\right]^{-0.5},$$

with

\footnote{See also \cite{Viel et al. 2003} who adopt a very similar fitting function but indicate a simple way in which (non-thermal) sterile neutrinos can be accounted for as well by a change to an effective thermal mass \cite[cf. also][]{Colombi et al. 1996}.}
\(\alpha = 0.05 \left( \Omega_m / 0.4 \right)^{0.15} \times \left( \frac{h}{0.65} \right)^{1.3} \left( \frac{\Omega_{dm}}{1 \text{ keV}} \right)^{-1.15} \left( \frac{g_X}{9} \right)^{0.29} h^{-1}\text{Mpc}, \)

where \(\Omega_{dm}\) is the DM particle mass (or the effective sterile neutrino mass), in units of keV, and \(g_X\) is the number of degrees of freedom that the particle contributes to the number density which is 3/2 in our case.

In this paper we will explore the case where \(\Omega_{dm} = 250\) eV. Our choice is inconsistent with current constraints placed by observations of the Ly-\(\alpha\) forest power spectrum (which set a lower limit at the order of keV [Viel et al. 2005, Boyarsky et al. 2009a]). However, a low \(\Omega_{dm}\) has the advantage of allowing us to resolve structures at redshift zero much below the cutoff mass scale using only relatively modest mass resolution and computational resources. In addition, our results can be readily extended and generalised to other DM particle masses.

For our choice of cosmological parameters and WDM particle mass: \(\alpha = 0.26 h^{-1}\text{Mpc}\), which translates into a free-streaming mass-scale

\[ M_\alpha = \frac{4\pi}{3} \rho (\alpha/2)^3 \approx 7 \times 10^8 h^{-1}\text{M}_\odot \quad \text{(3)} \]

and a half-mode mass-scale (cf. Colin et al. 2008, Schneider et al. 2012)

\[ M_{hm} \approx 4.3 \times 10^3 M_\alpha \approx 3.0 \times 10^{12} h^{-1}\text{M}_\odot. \quad \text{(4)} \]

This mass-scale is where a deficit in the mass-function compared to CDM is expected to be a factor of two.

Using the WDM primordial power spectrum discussed above and the MUSIC code [Hahn & Abel 2011], we then create the initial particle configuration at \(z = 63\) for our numerical experiments. We do this by perturbing a particle distribution, initially arranged in a regular cubic lattice, according to the Zel'dovich approximation.

We note that we have not attempted to explicitly include thermal velocities (on top of gravitationally induced velocities) in our initial conditions, since it is both negligible for the results of this paper as well as unclear how a proper implementation would proceed. The RMS velocity of the microscopic WDM particles following the Fermi-Dirac distribution (for thermally produced warm dark matter) is given at redshift \(z\) by [Bode et al. 2001]

\[ \sigma_v \approx 0.043 (1 + z) \times \left( \frac{\Omega_m}{0.3} \right)^{1/3} \left( \frac{h}{0.65} \right)^{2/3} \left( \frac{1.5}{g_X} \right)^{1/3} \left( \frac{1 \text{ keV}}{m_{\text{dm}}} \right)^{4/3} \text{km s}^{-1}.\]

For the \(\Omega_{dm} = 0.25\) keV chosen in this paper, we find \(\sigma_v = 0.28\) km s\(^{-1}\) at \(z = 0\). Clearly, a value considerably smaller than those arising due to nonlinear structure formation. For instance, the virial velocity of a typical DM halo in our simulations is 100-1000 km s\(^{-1}\). Of course, this RMS velocity can be of relevance in the most central parts of the haloes, determining details of the phase-space density there (e.g. Dalcanton & Hoger 2001), as well as the thickness of caustics (e.g. White & Vogelsberger 2009).

Although the RMS velocity corresponds to a microscopic value, it is sometimes regarded as a macroscopic one, and implemented in N-body simulations as random kicks of simulation particles (e.g. Colin et al. 2008, Macciò et al. 2013). We emphasise, however, that this is simply an Ansatz and that simulation particles represent a coarse-grained phase-space distribution, thus each of them averages over a statistical ensemble with a negligibly small dispersion around this mean. On the contrary, a kick to a single simulation particle is equivalent to a locally coherent motion of a large ensemble of actual WDM particles. This leads to a velocity spectrum inconsistent with the results of linear perturbation theory [Colin et al. 2008], and it is clear that the evolution of this numerical setup is not equivalent to the evolution of the fine-grained distribution function. Therefore, neglecting this dispersion is a very good, as well as convenient, approximation when one is only concerned with the mass of DM haloes, as it is in our case.

### 2.2 Gravitational Evolution

We perform a series of cosmological N-body simulations evolving 1024\(^3\) particles inside a cubical region of \(L = 80 h^{-1}\text{Mpc}\) a side. For our choice of cosmological parameters, each of these simulation particles has a mass equal to \(3.65 \times 10^7 h^{-1}\text{M}_\odot\). This mass resolution and volume is sufficient to have a fair sample of haloes located at the half-mode mass, which is resolved with almost 100,000 particles.

We evolve simulation particles using a memory-efficient version of the P-Gadget3 code [Springel 2005], which was originally developed and optimised for the Millennium-XXL project [Angulo et al. 2012]. In this code we have implemented three different methods to compute gravitational forces. In the remainder of the paper we refer to them as **Tree-PM**, **PM** and **T4PM**, and are described in the following:

1. **Tree-PM**: This method corresponds to the standard numerical configuration followed in state-of-the-art calculations. Long-range interactions are calculated using a PM method ([Hockney & Eastwood 1981]), whereas short-range forces are calculated using a multipole expansion of the force field together with a Tree algorithm ([Barnes & Hut 1986]). In order to reduce two-body scattering and binary particle systems (among other artefacts), forces need to be softened on small scales. We do this in our runs using a Plummer-equivalent softening length equal to \(5h^{-1}\text{kpc}\).

2. **PM**: Here, gravitational forces are only given by the PM method, i.e. we compute the gravitational potential field on a grid by solving the Poisson equation using Fourier methods. In our runs we use a grid of 2048\(^3\) points and forces are Gaussian smoothed on scales roughly equal to twice the grid size, \(2\Delta x = 80 h^{-1}\text{kpc}\). This length scale matches the mean inter-particle separation. As argued by [Angulo et al. 2011], this numerical configuration suppresses undesired collisionality of the N-body system, and it is particularly successful (compared to a **Tree-PM** run) in following accurately the gravitational interaction of baryons and DM.

3. **T4PM**: This method is an implementation of the algorithm proposed by [Hahn et al. 2012]. In short, a Delaunay triangulation of the Lagrangian particle distribution defines a phase-space element (a tetrahedron) that can be reconstructed at any desired later time to reconstruct the respective density field. At all times, the density of each tetrahedron is assumed to be uniform. In practice, we represent the contribution of each tetrahedron to the total mass field using 4 virtual particles (they carry mass, but do not interact directly with the fluid) whose spatial distribution matches the monopole and the quadrupole of the parent tetrahedra. We deposit these particles onto a 4096\(^3\) mesh with CIC interpolation and com-
compute forces using the PM method, as described above. The spatial resolution of these runs is $40\ h^{-1}\text{kpc}$, twice as high as in the PM case.

In Fig. 1 we show the density projection of a subregion of our WDM simulations, as computed by the three methods described above. We will analyse this figure in detail in the next section, but it is readily apparent that all three runs display the same large-scale structure, but they differ on small scales.

Additionally, we have carried out a set of CDM N-body simulations, with identical force and mass resolution as the WDM runs. The initial conditions of all simulation boxes were generated using identical white-noise fields, which simplifies their comparison.

2.2.1 Computational Performance

The computational resources required for the three methods are similar but differ systematically. For our WDM runs, the Tree-PM, PM and T4PM runs require about 3000, 1000 and 3500 CPU-hours, respectively. For the Tree-PM run, $\sim 50\%$ of the time was employed in the construction and walking of the tree, while the overhead of the T4PM respect to the PM method is caused by our on-the-fly calculation of the initial tessellation together with the position calculation and depositing of the mass-carriers particles. The peak memory consumption of the T4PM run is $\sim 600\text{Gb}$, to be compared with $\sim 150\text{Gb}$ employed by the PM run. The difference is dominated by an extra set of pointers needed to recover the initial connectivity of the phase-space tessellation. We remark that the extra factor of $\sim 4$ is small considering that we effectively have 24 times more particles representing the density and force field. For a dramatic increase in force accuracy it hence seems very worthwhile to afford this additional cost in memory and run time. In addition, our implementation is suboptimal in terms of memory consumption: it is possible to carry out the T4PM run with a memory footprint identical to that of PM, at the cost of slightly more CPU time.

2.3 Dark Matter Haloes

For each of our simulations, we produce on-the-fly friends-of-friends (FoF; Davis et al. 1985) halo catalogues. We use a non-standard linking length parameter of $b = 0.05$ times the mean inter-particle separation, keeping objects with 20 or more particles. This unusual choice of $b$ (as compared to $b = 0.2$) is required to avoid large FoF haloes percolating the cosmic web. We will return to this point below.

For each FoF halo, we compute a spherical-overdensity (SO) mass, taking the centre of mass of the parent FoF group as the SO centre. We define the halo boundary as the sphere of radius $R_{200}$, which contains a mean density of 200 times the critical density, $\rho_{\text{crit}}$. Therefore, the mass of the halo is $M_{200} = 4\pi R_{200}^3 200 \rho_{\text{crit}}$.

We discard substructures from our catalogues whose $R_{200}$ spheres overlap with that of a more massive halo. At $z = 0$ this procedure finds 8359, 3422 and 2916 objects with mass $M_{200} > 10^{15}h^{-1}M_\odot$ in the WDM Tree-PM, PM, and T4PM runs, respectively. These are a factor of 15 – 40 smaller than in a CDM Tree-PM run, where we detect 127 133 structures. In Fig. 1 we overplot this halo catalogue on top of the dark matter density field.

In passing, we note that even though the FoF algorithm with the standard choice, $b = 0.2$, works satisfactorily for the Tree-PM run, it fails to deliver a reasonable halo catalogue for the other two cases. While in the Tree-PM case the filaments are broken into small haloes, in the T4PM and PM runs a strong artificial fragmentation is absent (as can be seen in Fig. 1), thus filaments and sheets are more homogeneous with sharp dense cores: genuine two and one dimensional dense structures exist. The FoF algorithm links these filaments to dark matter haloes located at their ends, and with other nearby haloes. We show examples of this problem in Fig. 2, where we display a projection of the particles associated with the two most massive FoF haloes at $z = 1$ (top row) and $z = 0$ (bottom row),

\footnote{Each particle contributes to six distinct tetrahedra, and each tetrahedra is represented by four particles.}
Figure 2. The 'standard' linking length $b = 0.2$ selects large parts of a WDM simulation once the forces are captured accurately enough that filaments do not artificially fragment. Density projections of the particles belonging to the two most massive FOF-“haloes” in our WDM T4PM simulation of a 250eV dark matter model. Objects at $z = 1$ (top row) and $z = 0$ (bottom row) are shown. These haloes have a mass of $2.7 \times 10^{14}h^{-1}M_{\odot}$ and $1.6 \times 10^{14}h^{-1}M_{\odot}$ ($z = 1$), and of $6.4 \times 10^{14}h^{-1}M_{\odot}$ and $2.8 \times 10^{14}h^{-1}M_{\odot}$ ($z = 0$), as identified in the T4PM runs. The largest $b = 0.2$ FoF structure at $z = 0$ has a mass of $6 \times 10^{14}h^{-1}M_{\odot}$ and spans almost $10h^{-1}$Mpc. At $z = 1$, the failure of the FoF algorithm is even worse: the biggest FoF halo spans almost one quarter of the simulation box size!

In order to avoid such problems, we employed a small linking length that ensures that only local high density peaks are selected as the starting point for our SO halo catalogues. We tested that the resulting SO halo mass function was insensitive to small changes in $b$ about our preferred value of 0.05. However, for values approaching $b = 0.2$, the mass function agrees only with the cases with smaller linking lengths at the highest mass end. On any other mass scale, it shows a notorious deficit of structures. This is because a large fraction of small haloes are artificially linked to form a single larger FoF structure, and thus are not present in our list of SO candidate haloes.

3 RESULTS

The main goal of this paper is to quantify the abundance of haloes expected in WDM cosmologies, especially below the cut-off scale. An accurate account of this is important, firstly, to establish robustly the predictions of WDM which can then be tested against observational data, and secondly, to understand more generally the collapse and assembly of DM haloes in the presence of a resolved cut-off scale in the perturbation spectrum. This in turn can help to understand the formation and properties of micro-haloes expected for some CDM particle candidates (e.g. the neutralino). In addition, these cosmologies offer a test of the methods and implementations of N-body simulations.

3.1 Halo abundance – dependence on the numerical method

Previous numerical simulations have not been able to explore the cut-off mass scale because it is dominated by a population of low-mass haloes aligned within filaments. This phenomenon has been reported in numerical simulation for decades: including early works (e.g. Melott & Shandarin 1989; Avila-Reese et al. 2001; Bode et al. 2001; Knebe et al. 2003), and also recent state-of-the-art runs (Wang & White 2003; Lovell et al. 2012; Schneider et al. 2012).

Initially, it was not clear whether a real and physical fragmentation of filaments could be in place, or if it has its origin in numerical inaccuracies. This has settled recently, and there is a consensus that these haloes are numerical artefacts. Evidence for this is that their spatial distribution is closely related to the initial unperturbed particle load, and that their abundance changes (albeit slowly) with mass resolution (in fact $\propto m_p^{1/3}$). In particular, Wang & White (2003) have analysed this problem in detail and concluded that their presence is caused by non-zero small-scale fluctuations of the 1D-projected density field. This is related to warnings of Melott & Shandarin (1989) about using excessive force resolution compared to the mass resolution as it leads to fragmentation also in 2D cosmological simulations. This is indeed the regime in which state-of-the-art simulations are carried out: the typical force resolution is set to a value $10 - 100$ times smaller than the mean interparticle separation.

The numerical nature of the fragmentation is also illustrated in Fig. 4 where we show a density projection of a $20h^{-1}$Mpc thick slab through our three simulation boxes at $z = 0$. The visualization technique is identical for all three panels, and corresponds to a CIC density (for the T4PM run, we project the flow tracers, not the 24 times more abundant mass carriers); thus any difference is a result of real discrepancies in the spatial distribution of particles. In these images, we overplot the halo distribution over the underlying dark matter field. We display only haloes with SO mass $M_{200} > 2 \times 10^{10}h^{-1}M_{\odot}$, which is the resolution limit of our simulations, as we will discuss below. It is straightforward to see that all three runs, which use different methods to compute gravitational forces, display the same large-scale structure while, however, differences exist on small scales.

In the top panel, we display results obtained with the most commonly used method to compute gravitational forces (c.f. Section 2.1), labelled as Tree-PM. Fragmentation of filaments into small clumps is clearly visible in several places, for instance, in the two filaments located in the lower half of the image. These clumps are indeed very dense and are identified as haloes by our FoF-SO algorithm, and are thus highlighted by red circles. In the middle panel, which shows the PM simulation, forces are effectively softened below the mean inter-particle separation and low-mass haloes aligned with the filaments are considerably less abundant. In the bottom image, displaying the T4PM run, artificial fragmentation virtually does not exist! Even though this run has a force resolution twice as high as in the PM case.

Therefore, we see that the fragmentation of filaments is closely related to the force calculation, or more precisely, to the combination of force and mass resolution. We note that,
in these simulations, the crucial difference is not the actual method to compute forces (e.g. a PM versus a Tree+PM), but the chosen force resolution for a given mass resolution. We have explicitly tested this assertion by varying the size of the mesh in the PM run. An excessive force resolution causes local minima in the global potential around simulation particles, which eventually grow and accrete neighbouring particles. In this sense the T4PM method has the advantage of smoothing these minima since it provides a smoother representation of the mass field (see e.g. Kaehler et al. 2012) and thus more accurately captures the smooth but dense structure of the density field in regions of strong anisotropic compression. This was already qualitatively seen by Hahn et al. (2012), who found that the T4PM method suppresses artificial fragmentation in WDM scenarios. The advantage of T4PM has the price that the density in the inner regions of haloes is overestimated. This is because the evolution of highly distorted Lagrangian phase-space elements in regions of strong mixing cannot be represented correctly by the piecewise linear approximation to the distribution function that is only tracked by the Lagrangian motion of the particles. In principle this limitation can be overcome by an adaptive mass refinement (Hahn, Angulo & Abel, 2013, in prep.). Nevertheless, this limitation has a very minor effect on the halo masses, and thus, on our results regarding the halo mass function.

In Fig. 3 we can quantitatively see the differences in the predicted number of DM haloes at $z = 0$, as a function of their SO mass, $M_{200}$, for our three methods. For comparison, we also display the halo mass function analogously constructed for a CDM simulation with matching volume and mass resolution, and where forces are computed using the Tree-PM and PM method. For the latter, we use a mesh of 2048$^3$ cells, matching the force resolution of the T4PM run. The vertical dashed line indicates a mass of $2 \times 10^{10} h^{-1} M_{\odot}$, or, equivalently, $\sim 700$ particles. This is an estimate for the mass limit above which we expect our results to be numerically robust.

We choose this limit by comparing the resulting mass function in the CDM case for the PM and Tree-PM force methods. Below $M_{\text{min}} = 2 \times 10^{10} h^{-1} M_{\odot}$, the PM mass function shows a strong deficit of haloes, this is (1) because the force resolution of the PM run ($80 h^{-1} \text{kpc}$) is simply too low to resolve (and keep bound) some low-mass density peaks, which in the Tree-PM run collapse as haloes; and (2) since it is plausible that even in CDM the lowest masses are a mixture of artificial fragments and true haloes, as suggested by the strong increase of the number of “peakless” haloes at low particle counts (cf. Ludlow & Porciani 2011). Above $M_{\text{min}}$ this does not seem to be important and the mass functions are consistent with each other, showing only a small offset caused by a systematic underestimation of masses in the PM case. This is an indication that our results above $M_{\text{min}}$ are numerically robust, and that the differences regarding artificial fragmentation are a result of our improved estimation of the force field and the respective reduction of discreteness effects, rather than due to a product of a somewhat low force resolution. Another aspect supporting this is the fact that the amount of low-mass haloes in the T4PM run is lower than in the PM case, despite the former having higher force resolution. This is the opposite to what is expected if the suppresion were caused by a lack of force resolution.

As can be seen in the lower panel of Fig. 3 above $M \sim 3 \times 10^{11} h^{-1} M_{\odot}$ the halo mass function of our WDM three runs agree well with each other. The suppression with respect to CDM reaches a factor or two at $M = 4 \times 10^{12} h^{-1} M_{\odot}$, but it can be as large as a factor of 20. These values are consistent with previous studies, though slightly stronger than those reported by Schneider et al. (2013), however, the discrepancy is most likely due to the different halo definitions. The qualitative agreement with other works also is not surprising, given that the Tree-PM method has been the choice of those studies and it agrees with our other two methods. It is when we consider smaller masses, were this is no longer true, that we can enter into a regime hidden to previous simulations.

Below $M \sim 3 \times 10^{11} h^{-1} M_{\odot}$, and for over an order of magnitude in halo mass, the T4PM and PM methods deviate systematically from the Tree-PM run. The characteristic upturn in the halo mass function produced by artificial fragmentation is not present in either the PM or the T4PM run. The differences reach a maximum factor of $\sim 10$ and 7, respectively at our mass resolution limit $M \sim 2 \times 10^{10} h^{-1} M_{\odot}$. All of this is consistent with the qualitative picture provided in Fig. 1.

On the other hand, despite the lack of spurious fragmentation, there is no sign of a sharp cut-off, even in the T4PM run, as expectations raised from previous works suggest (e.g. Benson et al. 2013; Schneider et al. 2013). The abundance of low-mass objects only decreases slowly and shows a mild upturn at $\sim 5 \times 10^{10} h^{-1} M_{\odot}$. We explore this issue in more detail next.

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3.2 The nature of collapsed structures

In order to explore the nature of the objects below the mass cut-off, and the origin of the mild upturn at \( \sim 5 \times 10^{10} h^{-1} M_\odot \), we have visually inspected all haloes found above our resolution limit in the T4PM run. In Fig. 4 we provide density projections of six randomly chosen objects found in six disjoint mass bins, which serve as examples of the type of objects that populate different regions of the halo mass function. We display mass carriers in a region of twice the size of the virial radius around each target halo. It is readily apparent that truly different structures are identified at the different mass scales. We now enumerate the most common features found in different mass bins:

1) In the leftmost column (smallest mass bin), we find density peaks just undergoing collapse, with highly disturbed morphologies, irregular boundaries and that usually show no clear center. In addition, we also find locally overdense regions that are however incompatible with the concept of a virialized dark matter halo. Most of these correspond to caustics – usually located at the radius where the particle orbits first turn back after crossing a potential minimum. Another, somewhat less common, occurrence of these are the centres of very dense filaments, and also the caustics of filaments.

2) In the next mass bin, we typically find objects where there is clearly one of the three axes that has collapsed recently. These objects usually show a roughly round external iso-density contour, and a bar-like feature at their centre, which is the remnant of the filament whose folding produced the collapse of the objects.

3) Objects in the third mass bin show less strong disturbances. They correspond to roughly spherically symmetric objects, but they clearly show many caustics, resulting from the continuous folding of the phase-space sheet. Commonly, they also show a bar, as those in the previous mass bin.

4-5-6) Finally, in the three most massive bins, we find systems similar to those we usually find in CDM simulations and that can be unequivocally categorised as fully collapsed DM structures, with a well defined centre and approximate spherical...
symmetry. The objects have much more clearly undergone an isotropic virialization than objects on lower mass scales.

It is interesting to note that the evolutionary state of structures of increasing mass resembles different stages of halo formation in Hot dark matter cosmologies. Firstly, the local tidal field halts the expansion of a density perturbation along one axis, which eventually collapses, creating a ‘pancake’. Then, collapse along a second orthogonal axes produces a filament. Material is accreted along that filament and the third and final axis collapses. Then, a relaxation process occurs, which finally gives rise to a DM halo in the sense of an approximately spherical object showing clear density enhancements. Here all three axes have collapsed, but the dense core survives. The third case correspond to dense sheets and filaments where sometimes the collapse of a further axis has started.

This picture is supported by Fig. 5 where we show the descendant halo mass at z = 0 of haloes detected at z = 0.5. We link haloes at these two redshifts by finding the object at z = 0 that contains the majority of the particles associated with a FoF halo at z = 0.5. We highlight as filled orange symbols the z = 0.5 haloes that are the most massive progenitor of a z = 0 halo.

From this figure we can see that almost all haloes increase their mass consistent with a hierarchical picture of structure formation. However, haloes of different initial masses grow by significantly different amounts. Those of $10^{13} \ h^{-1} M_{\odot}$ increase their mass by 30%, on average. At the expected half-mode mass scale, $2 \times 10^{12} \ h^{-1} M_{\odot}$, the increment is typically a factor of 2. Whereas, at $10^{11} \ h^{-1} M_{\odot}$ it is a factor of 15! This very rapid mass increase at low masses has the consequence that most of the haloes below $10^{11} \ h^{-1} M_{\odot}$ (e.g. those in the first two columns of Fig. 4) are well above this limit by $z = 0$. This is consistent with the picture given above in which the objects found below the mass scale are simply a transient stage of halo formation, thus they quickly increase their mass and sit above the cut-off scale, where the growth proceeds at a slower pace. This is to be contrasted with the CDM case, where low-mass haloes have the earliest formation redshifts.

Out of the 1413 points we display, there are 47 which are located below $1 \times 10^{11} \ h^{-1} M_{\odot}$ at both $z = 0.5$ and $z = 0$. These haloes are not compatible with our interpretation, and they could be rare occasion where our mass resolution is not sufficient to avoid absolutely all fragmentation. Despite this, this is a very small population, which will not affect our results.

3.3 The abundance of virialized structures

With the ideas discussed above in hand, we now return to the issue of the halo mass function in WDM cosmologies. Upon visual inspection of the members of our FoF-SO catalogue, it became obvious that many of the entries did not comply with the features usually found in halo catalogues built from CDM simulations. Consequently, we visually inspected and classified all haloes in our 74PM run into one of three groups.

1. “Not Halos”: In the first category we include all objects that appear as clear failures of our halo finder algorithm. These enclose mainly three cases: one corresponds to outer caustics of large haloes (which sometimes are located further than the virial radius), another to descendants of haloes that have flown trough a more massive system (these are stripped of most of their mass, but their core survives). The third case correspond to dense sheets and filaments where sometimes the collapse of a further axis has started.

2. “Proto-Halos” Our second category contains haloes that are not fully formed yet, but show clear isolated 3D density enhancements. Here all three axes have collapsed, but the density peak has not fully virialized: we include here all objects from highly anisotropic systems, that appear just after a violent collapse, to much more quiet haloes, where only minor departures from a smooth mass distribution exist.

3. “Halos” The third and final category contains systems which can be unambiguously defined as a halo in the traditional sense of approximately spherical objects showing clear three dimensional virialization, and that resemble those seen in CDM simulations.

We note that we attempted to perform automatic classifications using several different halo properties. Unfortunately,

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4 We find that the small number of haloes that see a reduction of their mass are systems that were accreted by a larger halo, their outskirts removed by tidal stripping, but the denser core survives in a orbit that yield them to outside the virial radius of the host halo, and thus are identified as separate haloes but with a reduced mass.
The Warm DM halo mass function below the cut-off scale

none of them could satisfactorily separate the categories mentioned above. Some of the measures introduced in Abel et al. (2012) or methods inspecting the shape of the Lagrangian patch of the identified structures (such as that proposed by e.g. Lovell et al. 2012) may help in automating such a procedure. On the other hand, since proto-haloes, as well as some “Not Halo” objects, are likely simply early stages of halo formation, they are also likely to correspond to peaks in the initial conditions with only their collapse time differing from those corresponding to “Halos”. Thus, Lagrangian approaches might not clearly separate our three classes of objects. An additional complication for automatic classifications is that the haloes in the critical regime, $< 10^{11} h^{-1} M_\odot$, are resolved with only a few thousand particles which is not enough to perform a detailed analysis of their internal structure. We will defer further exploration of these issues to future work.

In Fig. 6 we show the relative contribution of each of these three categories to the WDM halo mass function. It is very interesting to see that the groups are clearly localized at different mass scales, although some overlap exists. High masses are dominated by standard dark matter haloes. Right below the cut-off, recently collapsed systems dominate. And the lowest masses receive a similar contribution from Proto-Haloes and from failures of our FoF-SO algorithm.

Before we continue, we would like to note that, as in most classifications, the division between these three groups is somewhat arbitrary. This is accentuated by the subjective nature of our visual inspection. For these reasons, we emphasise that the distinction between different categories just provides a qualitative assessment of the nature of objects at different mass scales, and of how they affect the WDM halo mass function.

Another point to note is that the fine division between the categories does depend on the force resolution employed.

We have explored this by carrying out our T4PM run using a PM mesh a factor of eight smaller, thus degrading our spatial resolution by a factor of two. There are three aspects worth noting.

1) The amount of filaments/sheets in our catalogues, as well as the sum of Haloes plus Proto-Haloes remains roughly the same when the force resolution is varied. This is because the time of collapse of a filament depends mostly on large-scale tidal and density fields, which are less sensitive to the force resolution.

2) The distinction between Haloes and Proto-Haloes is very different at different force resolutions. With higher force resolution, caustics are created more rapidly, there is more mixing, and haloes appear more relaxed. Note that due to computational limitations, it is not possible for us to increase the mass resolution of our runs needed to increase further the force resolution.

3) The frequency of some type of FoF failures changes considerably with force resolution. In this case, we find that the number of Not-Haloes at the low mass end increased substantially, mainly due to an increase in the number of caustics – a lower force resolution allows the turn-around radius to move outwards.

Figure 6. Relative contribution of different types of objects to the total WDM halo mass function. The black histogram shows the fraction of “Halos”. The green histogram shows the abundance of “Proto-Haloes”, whereas the green line indicates the fraction of objects that our SO-FOF algorithm wrongly identified as haloes. See the text for more details about our classification method.

Figure 7. Contribution of different types of objects to the WDM halo mass function. The red line show the abundance of standard dark matter haloes. The green line represents haloes in final stages of formation, while the blue line displays the abundance of objects in initial stages of formation. Finally, the magenta line shows objects incorrectly identified as haloes by our algorithm. See the text for more details on the classification, and Fig. 4 for examples of structures in the various categories. Black lines of different styles show the mass function expected in different analytical formulations, as specified in the legend. Vertical dashed line indicate a limit where the abundance of haloes is not affected by finite force resolution. The bottom panel shows the ratio of our results to the expectations in a CDM scenario.
From this, we can conclude that the mass functions for the Haloes and the Haloes plus Proto-Haloes samples should provide the range in which we expect the mass function of virialised haloes to lie.

We are now in the position to provide the most important result of our paper. In Fig. 1 we show the WDM halo mass function for two catalogues. The first one, denoted by a green dotted line, shows the abundance of systems categorised as Haloes, the second case, denoted by a blue dashed line, adds in the contribution of Proto-Haloes. Thus, the blue and green lines provide upper and lower limits to the abundance of collapsed objects in WDM cosmologies. These data are well fitted by the following functional form:

\[
\frac{n}{n_{\text{cdm}}} (M) = \frac{1}{2} \left( 1 + \frac{M_1}{M} \right)^{-1} \left[ 1 + \text{erf} \left( \log \frac{M}{M_2} \right) \right]
\]

with \(M_1 = 3.9 \times 10^{12} h^{-1} M_\odot\) set by the half-mode mass scale, and \(M_2\) corresponds to the location of the small-scale cut-off, which we find to be \(5.2 \times 10^{10}\) for the Haloes catalogue, and \(2.1 \times 10^{11} h^{-1} M_\odot\) for the Haloes plus Proto-Haloes sample. The best fits are displayed as dashed lines in Fig. 1.

We clearly see that once we neglect the contribution of failures of our halo finder, the WDM halo mass function shows a strong cut-off at small masses, and the upturn seen in Fig. 3 essentially disappears. The strong cut-off implies that there are no collapsed peaks below some scale, in agreement with what one would naively expect from the cut-off in the transfer function which predicts no small-scale density perturbations. This rules out the scenario in which a substantial population of low-mass haloes are created as a result of nonlinear self-interactions of the density field. However, a quantitative comparison between the cut-off in the transfer function and that in the mass function is not straightforward, as the substantial differences between our two samples indicate.

To illustrate this point, in Fig. 7 we also show three predictions for the shape of the WDM halo mass function. Dotted line shows the recent results of Schneider et al. (2013) that is based on a sharp-k filter calibrated with a Tree-PM simulation, whereas the solid line is the prediction of Schneider et al. (2013) based on the EPS formalism and extrapolating the results of N-body simulations. Dash-dotted line is the model of Benson et al. (2013), who attempt to incorporate the effects of the thermal velocity of the WDM particle. All predictions differ largely. The Schneider et al. (2013) prediction, for instance, does not predict any cut-off in the halo mass function, whereas the other two models do so and roughly match the Haloes sample. However, as we discussed below, our data only provides a lower limit for the halo mass function, and the differences in the halo definition become important for a quantitative comparison.

The lack of an unique answer is a natural consequence of the complexity of structure formation and of the need for a common definition of a "halo" when numerical simulations and analytical formalism are compared. This is also true in CDM, but accentuated in WDM since low-mass scales are dominated by the collapse of filaments and by systems in the process of relaxation or collapse. Despite the added complexity, examining the cut-off scale can be extremely useful to isolate successful as well as problematic aspects of analytic formulations, which in turn could lead to improvements in their foundations and ultimately to a better understanding of structure formation and assembly in all cosmological models of structure formation.

4 CONCLUSIONS

In this paper, we were able to overcome the notorious difficulties associated with numerical simulations of the evolution of warm dark matter models. For decades, the numerical fragmentation of filaments created an artificial population of low mass haloes which dominated the halo mass function on small mass scales. Here, we demonstrated that it is possible to avoid such artefacts by employing a force resolution consistent with the mass resolution of the simulations. With this, and for the first time, we could explore the halo mass function below the cut-off scale. We discovered a picture more complex than that present in CDM simulations.

Structure formation in scenarios that have a small scale cut off in the power spectrum proceeds quite differently than in the CDM model. In CDM, haloes form mainly from accreting smaller haloes, which have the earliest formation times and the longest time to relax. On the contrary, in WDM, a large fraction of haloes form from the direct collapse of filaments and small-mass haloes are typically the most recently formed, and thus are still in a process of virialisation.

We find that this formation mechanism is imprinted in the WDM halo mass function: essentially different stages of halo formation dominate the counts at different masses. On the smallest mass scales we can resolve, identified objects are typically centres of filaments that are starting to collapse. On intermediate mass scales, objects typically correspond to fluctuations that have collapsed and are in the process of relaxation, whereas the high mass end is dominated by objects similar to haloes identified in CDM simulations.

In addition, we found that traditional group-finders produce, on small mass scales, catalogues with objects not consistent with the definition of a halo. In a CDM calculation, essentially all dense structures are always part of haloes, and even the simplest approaches, such as the FoF algorithm, succeed in selecting appropriate objects above a certain mass limit (e.g. Warren et al. 2009; More et al. 2011). On the other hand, in WDM these approaches prove to be unreliable. The small-scale cut-off in the primordial transfer function, implies that there are dense structures (e.g. filaments, caustics of large haloes) that can be incorrectly considered as haloes. These misidentifications dominate the WDM mass function below the cut-off scale. For the simulations we considered here, we could bypass this problem by visually inspecting and classifying different systems. This is not prohibitively demanding, thanks to the relatively low number of systems in our runs. However, of course, this is in general not true, and improved halo finders are desirable for future work.

After neglecting these failures of halo identification, we observe a strong cut-off in the halo mass function, with very few objects found below the cut-off scale. These correspond to haloes undergoing rapid collapse and virialisation. Our results indicate that the cutoff scale in the initial density fluctuations does indeed translate into a comparably strong cutoff scale in the halo mass function (Fig. 7). This implies that filaments and sheets that formed in early stages of gravitational collapse are remarkably stable and do not fragment due to the lack of small scale perturbations.

Our work poses several questions. The most natural one is the role of artificial fragmentation in the internal properties of DM haloes. In the absence of spurious haloes, filaments get denser and the accretion of material happens continuously from filaments, not as a sequential accretion of small dense haloes which would likely behave differently dynamically. It is
not clear whether this difference will impact, for instance, the concentration of haloes near the cut-off scale. Thus, we plan to carry out a suite of higher-resolution simulations to address this.

Another question arises from the fact that the exact mass scale at which the cut-off in the halo mass function is located, does depend on the exact definition of what constitutes a halo. Especially, it depends on the separation between a fully formed halo and a halo in the process of formation and/or relaxation. A priori it is thus not clear what one may wish to call a halo. Functional definitions might involve regions of sufficiently high DM density that would allow the associated baryons to collapse and cool, or surface densities that allow for gravitational lensing or more theoretically inspired measures such as a limit on the velocity anisotropy of the dark matter particles or perhaps particular axis ratios inferred from the velocity ellipsoid or inertial tensor. Whatever the exact definition one may choose the abundance of “haloes” fulfilling these choices will likely vary quite dramatically.

Therefore, any key ingredient for a new halo finder will be related directly to the properties that one aims to select for. For instance, the isotropy of the velocity dispersion, or the number of streams, and include the requirement that all three axes have collapsed (Falck et al. 2012) perhaps augmented with local measures of the velocity dispersion in all three directions (Abel et al. 2012). A common definition is also needed for a proper comparison with analytic models for structure formation. It will be desirable for future work to investigate whether there are generic definitions that sensibly define a concept of dark matter halo that is applicable to CDM as well as WDM computational cosmology questions.

Even once the issue of halo definition is decided upon, there is another problem related to the numerical methods. We find that the exact virialisation state as well as the number of caustics depend sensitively on the force resolution employed in our simulations. This is different from standard CDM runs because of the difference in the formation mechanism, and because of a special combination of force and mass resolution. However, this also opens an exciting possibility: because of the free-streaming scale in WDM, there is also a upper limit to the density and also a scale for small-scale gravitational interactions, which could eventually allow us to simulate the full range of length and mass scales relevant for the formation of a halo.

The question of halo definition, and the role of artificial fragmentation, is closely related to the issue of what is the minimum halo mass and/or virialisation stage for a halo to host galaxies. Fragmentation of gas into stars may well still occur in places not regarded as haloes in the traditional sense, perhaps already in regions of filamentary collapse: anywhere where a local gravitational potential well is already aggregating the baryons. On the contrary, the continuous folding of the phase-space sheet might continuously shock-heat the gas, and the cut-off in the respective galaxy stellar-mass function might not trivially relate to that in the halo mass function.

Unfortunately, given the strong limits on the mass of the potential WDM particle from Lyman-α forest constraints (Viel et al. 2005; Boyarsky et al. 2009a) the mass scale on which such a different galaxy formation scenario could be related to the observable Universe is severely limited, and other physics might be more relevant. Nevertheless, WDM scenarios remain of great interest to obtain a much better theoretical understanding of how dark matter haloes assemble and how the collisionless fluid virializes. In particular, they very closely resemble the events that lead to the very first dark matter haloes even in a CDM scenario albeit on radically different mass and length scales (Diemand et al. 2003; Goerdt et al. 2007; Ishiyama et al. 2010). We foresee our results stimulating a more detailed exploration of the formation of the smallest structures expected to form in a given cosmology, which will, hopefully, advance our overall understanding of structure formation.

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