An Approach Based on Active Constraint Strategy for Solving The Portfolio Optimization Problem

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Abstract. The mathematical model of portfolio optimization has been largely written in terms of minimizing the risk, given the return. The difficulty of this model is to deal with the quadratic programming model due to Markowitz. This situation has been overcome by the recent progress in algorithmic research, and the introduction of linear risk function. This paper deals with the portfolio selection problem with minimum transaction lots. A neighbourhood search algorithm based on active constrained strategy is proposed to solve the mixed integer programming model. The algorithm starts from the solution of the relaxed problem to find a solution which is close to the continuous solution.

1. Introduction

It is well known that the paper in [1] regarding to the portfolio selection problem has addressed a first mathematical formulation. However, there are some drawbacks recognized to the Mean-Variance (MV) model i.e., its high computational complexity and the input problem of estimating \(2n + n(n - 1)/2\) parameters (expected returns, variances and covariances), which made the model a milestone in finance theory, but a scarcely used tool in practice. Even though there are several attempts in literature to linearize the quadratic objective function see [2], [3] and [4].

Nowadays MV model consisting of more than a few thousand assets have been solved changing dramatically the practical role portfolios. Real time solutions are attainable through the use of interior point algorithm for quadratic programming problem [5], or by using compact factorizations and piecewise linear approximations [6] and [7].

The first linear model for portfolio selection is due to Konno and Yamazaki [8]. The linear form of the model is made possible by the use of a risk function different from the classical portfolio variance, namely the portfolio absolute deviation. A relevant feature of the model is that no probabilistic assumptions are made on the securities rates of return, while in the case the rates of return are multivariate normally distributed the model is shown to be equivalent to Markowitz’s one. The Konno and Yamazaki’s model, the so-called Mean Absolute Deviation (MAD), has been applied by Zenios and Kang [9] to a mortgage-backed securities portfolio optimization. The author showed how a suitable choice for the coefficients in the linear combination gives rise to a model equivalent to Konno and Yamazaki’s but halving its number of constraints. A similar result has been independently obtained by Feinstein and Thapa [11].

The largest part of the portfolio selection models which have been proposed in the literature are based on the assumption of a perfect fractionability of the investments in such a way that the portfolio...
fraction for each security could be represented by a real variable. In the real world, securities are negotiated as multiples of a minimum transaction lot (the so called rounds). As a consequence of considering rounds, solving a portfolio selection problem requires finding the solution of a mixed integer programming model. When applied to real problems, the tractability of the integer model is subject to the availability of algorithms able to find a good, even if not optimal, integer solution in a reasonable amount of time. A general mixed integer model including real characteristics of the problem has been presented in Speranza [12], where a simple heuristic is proposed and tested for the case when minimum transaction lost are considered. The problem with fixed transaction costs with and without minimum transaction lots has been studied in [13].

In this paper we show that, when rounds are taken into account, the problem of finding a feasible solution is, independently of the risk function, NP-complete. Moreover, new algorithm, based on integrating search in [16] is proposed for the solution of the model with rounds. As the number of securities selected by a standard (quadratic or linear) portfolio optimization model is observed to be almost always smaller than 20, the heuristics proposed herein are based upon the idea of constructing and solving mixed integer subproblems which consider subsets of the investment choices available. The subsets are generated by exploiting the information obtained from the relaxed problem (selected securities and reduced costs). The heuristics have the relevant advantage of being general. Different mixed integer models can be of interest in portfolio selection if, for instance, transaction costs are considered. The presented algorithms can be applied or easily generalized to such models.

2. The mean Semi-absolute Deviation Model with Minimum Lot Constraints

Markowitz’s original work was based on the rule that the investor does consider expected return as a desirable thing and variance as an undesirable one. Analytically, this implies that given \( |S| \) securities, where \( S \) is the set of investment alternatives (securities), and a level \( \rho \) of expected return the model turns out to be a quadratic programming problem as follows:

\[
\min \sum_{i \in S} \sum_{j \in S} \sigma_{ij} x_i x_j, \\
\sum_{i \in S} r_i x_i = \rho, \\
\sum_{i \in S} x_i = 1, \\
x_i \geq 0, \ i \in S, 
\]

Where \( x_i \) represents the percentage of money invested in security \( i \), \( r_i = E(R_i) \) with \( R_i \) the random variable representing the return of security \( i \) and \( \sigma_{ij} \) is the covariance between returns of security \( i \) and of security \( j \). The most commonly adopted assumption for this model is multivariate normally distributed rates of return.

In 1991 Konno and Yamazaki [8] developed a new approach having an important implication in portfolio analysis especially when the previous assumptions are not satisfied. In their original formulation of the \( L_1 \) risk model, Konno and Yamazaki proposed the following risk function:

\[
\min \omega(x) = E \left[ \sum_{j \in S} R_j x_j - E \left[ \sum_{j \in S} R_j x_j \right] \right].
\]

The random variable \( R_j \) still represents the rate of return, while \( x_j \) is the amount of money invested in security \( j \).

According to Konno and Yamazaki, \( r_{ij} \) is the realization of the random variable \( R_i \) during the period \( t \) and is obtainable through historical (or forecasted) data. Alternative models in which different scenarios for the rates of returns are taken into account are described in [10]. In particular, they assume that the mean of \( R_i \) can be estimated as
\[ r_j = E[R_j] = \frac{\sum_{t=1}^{T} r_{jt}}{T} \]  

where \( T \) is the length of the time horizon, and that \( w(x) \) can be reformulated as follows:

\[ w(x) = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j \in S} (r_{jt} - r_j) x_j \right| \]  

This objective function is equivalent to the following linear program:

\[
\begin{align*}
\min & \quad \frac{\sum_{t=1}^{T} y_t}{T} \\
y_t + \sum_{j \in S} (r_{jt} - r_j) x_j & \geq 0, \quad t = 1, \ldots, T \\
y_t - \sum_{j \in S} (r_{jt} - r_j) x_j & \geq 0, \quad t = 1, \ldots, T
\end{align*}
\]

If the risk measured by means of the mean semi-absolute deviation instead of the absolute deviation, as in Speranza [10], the objective function is

\[
\Sigma_{t=1}^{T} \left[ \min \{0, \Sigma_{j \in S} (r_{jt} - r_j) x_j \} \right] \]

and can be rewritten as

\[
\min \frac{\sum_{t=1}^{T} y_t}{T}
\]

that is with a smaller number of constraints. It has been shown in [10] that (11) is equivalent to (7) and thus to the variance under the assumption of multivariate normally distributed returns.

Since the model based on a mean semi-absolute deviation risk function is linear, it becomes natural to introduce new specifications deriving from market structure as well as from operative constraints.

We briefly describe the required notation for the mixed integer model with minimum transaction lot constraints. We denote by \( c_j \) the purchasing price for the minimum lot of security \( j \). In this way, for each security, the minimum lot is expressed in terms of money and is equivalent to \( c_j = N_j p_j \), where \( p_j \) is the market price for security \( j \) at the date of the purchase of the portfolio and \( N_j \) is the number of units of security \( j \) required as minimum quantity. Trivially, it is \( c_j = p_j \) when asset \( j \) is traded without minimum lot.

The integer variable \( x_j \), \( \forall j \in S \), represents the number of minimum lots, for each security \( j \), which will make part of the total available amount of money that the investor decides to put in security \( j \).

The constant \( d_j \), which may vary according to market conditions and agreement types, expresses the transaction cost proportional to the value of the purchase. Since the proportional transaction costs can be directly incorporated in the price, from now on we assume that the price \( c_j \) includes all possible proportional transaction costs.

The mixed integer linear program for the portfolio selection problem with minimum lot constraints is

\[
\begin{align*}
\min & \quad \frac{\sum_{t=1}^{T} y_t}{T} \\
y_t + \sum_{j \in S} (r_{jt} - r_j) c_j x_j & \geq 0, \quad t = 1, \ldots, T \\
C &= \sum_{j \in S} c_j x_j
\end{align*}
\]
\sum_{j \in S} r_j c_j x_j \geq \rho C, 
C_0 \leq C \leq C_1, 
0 \leq x_j \leq u_j, \text{ integer } j \in S, 
v_t \geq 0, \ t = 1, ..., T. 

Constrain (16) defines as C the total portfolio expenditure. The constraint on the expected return (17) implies that the selected portfolio has a combined rate of return greater than \( \rho \). With constraint (18), the unknown investment \( C \) is fixed to range between \( C_0 \) and \( C_1 \), i.e. between the minimum and the maximum amount of money available for the investment. Finally, constraints (19) define limitations on the value each \( x_j \) can take, being \( c_j u_j \) an upper bound on the investment in security \( j \).

3. The Basic Approach
Consider a MILP problem with the following form
Minimize \( P = c^T x \) 
Subject to \( Ax \leq b \) 
\( x \geq 0 \) 
\( x_j \) integer for some \( j \in J \)
A component of the optimal basic feasible vector \( (x_B)_k \), to MILP solved as continuous can be written as
\( (x_B)_k = \beta_k - \alpha_{kj} (x_N) - \cdots - \alpha_{kj} (x_N) - \cdots - \alpha_{kj} (x_N) - m(x_N) n - m \)

Note that, this expression can be found in the final tableau of Simplex procedure. If \( (x_B)_k \) is an integer variable and we assume that \( \beta_k \) is not an integer, the partitioning of \( \beta_k \) into the integer and fractional components is that given
\( \beta_k = [\beta_k] + f_k, 0 \leq f_k \leq 1 \)
suppose we wish to increase \( (x_B)_k \) to its nearest integer, \( ([\beta]+1) \). Based on the idea of suboptimal solutions we may elevate a particular nonbasic variable, say \( (x_N)_j^* \), as one of the element of the vector \( \alpha_{kj^*} \), is negative. Let \( \Delta_j^* \) be amount of movement of the non variable \( (x_N)_j^* \), such that the numerical value of scalar \( (x_N)_k \) is integer. Referring to Eqn.(25), \( \Delta_j^* \) can then be expressed as
\( \Delta_j^* = 1 - \frac{f_k}{-\alpha_{kj^*}} \)
while the remaining nonbasic stay at zero. It can be seen that after substituting (27) into (25) for \( (x_N)_j^* \), and taking into account the partitioning of \( \beta_k \) given in (26), we obtain
\( (x_B)_k = [\beta] + 1 \)
Thus, \( (x_B)_k \) is now an integer.

3.1. The Role Of Nonbasic Variables
It is now clear that a nonbasic variable plays an important role to integerize the corresponding basic variable. Therefore, the following result is necessary in order to confirm that there must be a non-integer variable to work with in integerizing process.

Theorem 3. Suppose the MILP problem (21)-(24) has an optimal solution, then some of the nonbasic variables \( (x_N)_j, j = 1, \ldots, n \), must be non-integer variables.

Proof.
Solving problem as a continuous of slack variables (which are non-integer, except in the case of equality constraint). If we assume that the vector of basic variables \( x_B \) consists of all the slack variables then all integer variables would be in the nonbasic vector \( x_N \) and therefore integer valued.
It is clear that the other components, \((x_\text{B})_{i \neq k}\), of vector \(x_\text{B}\) will also be affected as the numerical value of the scalar \((x_\text{N})_j^*\) increases to \(\Delta_j^*\). Consequently, if some element of vector \(\alpha_\text{N}^*\), i.e., \(\alpha_i\) for \(i \neq k\), are positive, then the corresponding element of \(x_\text{B}\) will decrease, and eventually may pass through zero. However, any component of vector \(x\) must not go below zero due to the non-negativity restriction. Therefore, a formula, called the minimum ratio test is needed in order to see what is the maximum movement of the nonbasic \((x_\text{N})_j^*\) such that all components of \(x\) remain feasible. This ratio test would include two cases.

a. A basic variable, \((x_\text{B})_{i \neq k}\) decreases to zero (lower bound) first.

b. The basic variable, \((x_\text{B})_k\) increases to an integer.

Specifically, corresponding to each of these two cases above, one would compute

\[
\theta_1 = \min_{i \neq k; \alpha_i > 0} \left( \beta_i / \alpha_i \right) \quad (28)
\]

\[
\theta_2 = \Delta_j^* \quad (29)
\]

How far one can release the nonbasic \((x_\text{N})_j^*\) from its bound of zero, such that vector \(x\) remains feasible, will depend on the ratio test \(\theta^*\) given below

\[
\theta^* = \min(\theta_1, \theta_2) \quad (30)
\]

Obviously, if \(\theta^* = \theta_1\), one of the basic variable \((x_\text{B})_{i \neq k}\) will hit the lower bound before \((x_\text{B})_k\) becomes integer. If \(\theta^* = \theta_2\), the numerical value of the basic variable \((x_\text{B})_k\) will be integer and feasibility is still maintained. Analogously, we would be able to reduce the numerical value of the basic variable \((x_\text{B})_k\) to its closest integer \([b_k]\). In this case the amount of movement of a particular nonbasic variable, \((x_\text{N})_j^*\), corresponding to any positive element of vector \(\alpha_j^*\), is given by

\[
\Delta_j = f_k / \alpha_{kj}^* \quad (31)
\]

In order to maintain the feasibility, the ratio test \(\theta^*\) is still needed. Consider the movement of a particular nonbasic variable, \(\Delta\), as expressed in Eqns.(31) and (25). The only factor that one needs to calculate is the corresponding element of vector \(\alpha\). A vector \(\alpha_j\) can be expressed as

\[
\alpha_j = B^{-1} a_j, j = 1, \ldots, n - m \quad (32)
\]

Therefore, in order to get a particular element of vector \(\alpha_j\) we should be able to distinguish the corresponding column of matrix \([B]^{-1}\). Suppose we need the value of element \(\alpha_{kj}\), letting \(v_k^T\) be the k-th column vector of \([B]^{-1}\), we then have

\[
v_k^T = e_k^T B^{-1} \quad (33)
\]

subsequently, the numerical value of \(\alpha_{kj}^*\) can be obtained from

\[
\alpha_{kj}^* = v_k^T a_j^* \quad (34)
\]

in Linear Programming (LP) terminology the operation conducted in Eqns. (33) and (34) is called the pricing operation. The vector of reduced costs \(d_j\) can be used to measure the deterioration of the objective function value caused by releasing a nonbasic variable from its bound. Consequently, in deciding which nonbasic should be released in the integerizing process, the vector \(d_j\) must be taken into account, such that deterioration is minimized. Recall that the minimum continuous solution provides a lower bound to any integer-feasible solution. Nevertheless, the amount of movement of particular nonbasic variable as given in Eqns. (27) or (31), depends in some way on the corresponding element of vector \(\alpha_j\). Therefore it can be observed that the deterioration of the objective function value due to releasing a nonbasic variable \((x_\text{N})_j^*\), so as to integerize a basic variable \((x_\text{B})_k\) may be measured by the ration

\[
\frac{d_k}{\alpha_{kj}^*} \quad (35)
\]

where \(|a|\) means the absolute value of scalar \(a\).
In order to minimize the deterioration of the optimal continuous solution we then use the following strategy for deciding which nonbasic variable may be increased from its bound of zero, that is,

$$\min_j \left[ \frac{d_k}{\alpha_{kn}} \right], j = 1, \ldots, n - m \quad (36)$$

From the “active constraint” strategy and the partitioning of the constraints corresponding to basic (B), superbasic (S) and nonbasic (N) variables we can write

$$\begin{bmatrix} B & S & N \\ \mathbf{I} & N \\ \end{bmatrix} \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \\ \mathbf{x}_S \\ \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b}_N \\ \end{bmatrix}$$

$$B\mathbf{x}_B + S\mathbf{x}_S + N\mathbf{x}_N = \mathbf{b} \quad (37)$$

$$\mathbf{x}_N = \mathbf{b}_N \quad (38)$$

The basis matrix B is assumed to be square and nonsingular, we get

$$\mathbf{x}_B = \beta - W\mathbf{x}_S - \alpha \mathbf{x}_N$$

$$\beta = B^{-1}\mathbf{b} \quad (40)$$

$$W = B^{-1}S \quad (41)$$

$$\alpha = B^{-1}N \quad (42)$$

Expression (39) indicates that the nonbasic variables are being held equal to their bound. It is evident through the “nearly” basic expression of Eqn. (40), the integerizing strategy discussed in the previous section, designed for MILP problem can be implemented. Particularly, we would be able to release a nonbasic variable from its bound, Eqn (39) and exchange it with a corresponding basic variable in the integerizing process, although the solution would be degenerate.

3.2. Pivoting

Currently, we are in a position where particular basic variable, \((\mathbf{x}_B)_k\) is being integerized, thereby a corresponding nonbasic variable, \((\mathbf{c}_N)_j^*\), is being released from its bound of zero. Suppose the maximum movement of \((\mathbf{x}_N)_j^*\) satisfies

$$\theta^* = \Delta_j^*$$

such that \((\mathbf{x}_B)_k\) is integer valued to exploit the manner of changing the basis, we would be able to move \((\mathbf{x}_N)_j^*\) into B (to replace \((\mathbf{x}_B)_k\) and integer-valued \((\mathbf{x}_B)_k\) into S in order to maintain the integer solution. We now have a degenerate solution since a basic variable is at its bound. The integerizing process continues with a new set \([B,S]\). In this case, eventually we may end up with all of the integer variables being superbasic.

4. Conclusions

In this paper the model for portfolio selection with minimum rounds has been applied. Based on the idea of suboptimal solutions we may elevate a particular nonbasic variable, say \((\mathbf{c}_N)_j^*\), above its bound of zero, in such a way that would force the non-integer basic variable to take an integer value while the remaining nonbasic stay at zero. Therefore a nonbasic variable plays an important role to integerize the corresponding basic variable. Among the possible future directions of research, on one side it is of interest to evaluate the performance of the proposed heuristic when applied to the variant of Markowitz’s model which takes the rounds into account, although some problems may arise with the solution of a quadratic model with integer variables. On the other hand, the present model can be used to manage a selection portfolio problem based on derivatives, which implies a higher difficulty of risk management due to their asymmetry.
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