SILTING COMPLEXES AND GORENSTEIN PROJECTIVE MODULES

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Dedicated to Pu Zhang on the occasion of his 60th birthday

ABSTRACT. We introduce Gorenstein silting modules (resp. complexes), and give the relation with the usual silting modules (resp. complexes). We show that Gorenstein silting modules are the module-theoretic counterpart of 2-term Gorenstein silting complexes; and partial Gorenstein silting modules are in bijection with \(\tau\)-rigid modules for finite dimensional algebras of finite CM-type. We also give the relation between 2-term Gorenstein silting complexes, t-structures and torsion pair in module categories; and generalise the classical Brenner-Butler theorem to this setting; and characterise the global dimension of endomorphism algebras of 2-term Gorenstein silting complexes over an algebra \(A\) by terms of the Gorenstein global dimension of \(A\).

1. Introduction and Preliminaries

Silting complexes were first introduced by Keller and Vossieck [KV] to study t-structures in the bounded derived category of representations of Dynkin quivers. They generalize tilting complexes and, thus, finitely generated tilting modules introduced by Auslander and Solberg [AS1]. Hoshino-Kato-Miyachi [HKM, MM] gave the relation between 2-term silting complexes and torsion pair in module categories. Then Buan-Zhou [BZ1] studied 2-term silting complexes in the bounded homotopy categories, and they generalized the classical tilting theorem to the silting situation. After that, they considered the global dimension of endomorphism algebras of 2-term silting complexes in [BZ2].

Support \(\tau\)-tilting modules are the module-theoretic counterpart of 2-term silting complexes. They were introduced over finite dimensional algebras by Adachi-Iyama-Reiten [AIR], who showed that these modules admit mutation and that there is a mutation-preserving bijection with 2-term silting complexes. Silting modules introduced by Angeleri Hügel-Marks-Vitória [AMV1] over an arbitrary ring who are intended to generalize tilting modules in a similar fashion as 2-term silting complexes generalize 2-term tilting complexes and also, coincide with support \(\tau\)-tilting modules for finite dimensional algebras. Adachi-Iyama-Reiten [AIR] showed how 2-term silting complexes relate with silting modules, t-structure, and co-t-structure. In [MS], they showed that silting modules are in bijection with universal localisations for finite dimensional algebras of finite representation type. Then Angeleri Hügel-Marks-Vitória [AMV2] constructed the bireflective subcategory associated with a partial silting module and studied silting modules over hereditary
rings. Moreover, Aihara [A] gave a necessary condition for silting to be tilting under the self-injective case. Recently, Li-Zhang [LZ] showed the existence of nontrivial Gorenstein projective (support) $\tau$-tilting modules except support $\tau$-tilting modules over selfinjective algebras.

The main idea of Gorenstein homological algebra is to replace projective modules by Gorenstein-projective modules. These modules were introduced by Enochs and Jenda [EJ1] as a generalization of finitely generated module of $G$-dimension zero over a two-sided Noetherian ring, in the sense of Auslander and Bridger [AB]. The subject has been developed to an advanced level, see for example [ABu, AR, Hap, EJ2, Ch, AM, Hol, BI, CFH, BR, J, Chen, GZ, GK, RZ1, RZ2, RZ3]. As the version of tilting modules in Gorenstein homological algebra, the definition of Gorenstein tilting module was introduced (see [AS1, G1, YLO]). Later Zhang [Z] introduced the notion of Gorenstein star modules, and gave a close relation between Gorenstein star modules and Gorenstein tilting modules.

As the correspondence of algebras of finite representation type in Gorenstein homological algebra, Beligiannis [B2] introduced and studied the algebras of finite Cohen-Macaulay type (resp. finite CM-type for simply). For this class of algebras, Gao [G2] introduced the relative transpose $\text{Tr}_G$ by the term of Gorenstein-projective modules and the corresponding Auslander-Reiten formula.

Based on these work, there are the following natural questions:

**Question A:** Which class of modules coincides with the above $\tau_G$-rigid module?

**Question B:** What’s the correspondence in the bounded homotopy category with respect to the above modules?

The answers are given in the paper. We organize the paper as follows. In Section 2, we introduce the Gorenstein silting module and show it is in bijection with the relative
rigid module over an algebra of CM-type. We also show the relations among Gorenstein tilting modules, Gorenstein star modules and Gorenstein silting modules. In Section 3, we introduce the 2-term Gorenstein silting complex and show that the Gorenstein silting module is its module-theoretic counterpart. We also characterise it by the connection with the t-structure and torsion pair, and show the corresponding Brenner-Butler theorem, and characterise the global dimension of endomorphism algebras of 2-term Gorenstein silting complexes over an algebra $A$ by terms of the Gorenstein dimension of $A$.

First we give the main definitions in the paper.

**Definition 2.2** Let $R$ be a Noetherian ring. We say that an $R$-module $T$ is

- partial Gorenstein silting if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that
  
  $(G_{s1})$ $D_{\theta}$ is a relative torsion class (i.e. closed for $G$-epimorphic images, $G$-extensions and coproducts);
  
  $(G_{s2})$ $T$ lies in $D_{\theta}$.

- Gorenstein silting if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that $\text{Gen}_G(T) = D_{\theta}$.

**Definition 2.9** Let $A$ be a finite dimensional $k$-algebra of finite CM-type with the Gorenstein-projective generator $E$. We say that an $A$-module $M$ is $\tau_G$-rigid if $\text{Hom}_{A(Gproj)}((E, M), \tau_G M) = 0$ for the left $A(Gproj)$-module $\text{Hom}_A(E, M)$.

**Definition 3.1** Let $G^\bullet : G_1 \xrightarrow{d_1} G_0$ be a complex with $G_i \in \text{Gproj}_A$ for $i = 0, 1$. We say that $G^\bullet$ is

- 2-term partial Gorenstein silting in $K^b(\text{Gproj}_A)$ if it satisfies the following two conditions:
  
  (i) $G_1 \rightarrow \text{Im} d_1$ and $G_0 \rightarrow \text{Coker} d_1$ are right $\text{Gproj}_A$-approximations;
  
  (ii) $\text{Hom}_{D_{\text{gp}}(A)}(G^\bullet, G^\bullet[1]) = 0$.

- 2-term Gorenstein silting in $K^b(\text{Gproj}_A)$ if it is a 2-term partial Gorenstein silting complex and $\text{thick} G^\bullet = D_{\text{gp}}^b(A)$.

Our main theorems are as follows:

**Theorem A** (Theorem 2.10 and 2.11) Let $A$ be a finite dimensional algebra of finite CM-type with the Gorenstein-projective generator $E$. Let $M$ be an $A$-module in $\text{mod} A$. Then the following statements hold.

1. $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(E, M)$ is a $\tau$-rigid module, where $\tau$ is the Auslander-Reiten translation over $A(Gproj)$.
2. $\tau_G M \cong \tau \text{Hom}_A(E, M)$.
3. $M$ is $\tau_G$-rigid if and only if $M$ is a partial Gorenstein silting $A$-module.
Let \( B = \text{End}_{D_{bgp}^b(A)}(G^\bullet)^{op} \). Consider the subcategories of \( \text{mod} B \)

\[
\mathcal{X}(G^\bullet) = \text{Hom}_{D_{bgp}^b(A)}(G^\bullet, \mathcal{F}(G^\bullet)[1]) \quad \text{and} \quad \mathcal{Y}(G^\bullet) = \text{Hom}_{D_{bgp}^b(A)}(G^\bullet, \mathcal{T}(G^\bullet)).
\]

**Theorem B** (Theorem 3.8, Theorem 3.12 and Theorem 3.16) Let \( G^\bullet : G_1 \longrightarrow G_0 \) be a 2-term Gorenstein silting complex in \( D_{bgp}^b(A) \), where \( G_i \in \text{Gproj} A \). Then the following statements hold.

1. \( (\mathcal{T}(G^\bullet), \mathcal{F}(G^\bullet)) \) is a torsion pair for \( \text{mod} A \).
2. \( (\mathcal{X}(G^\bullet), \mathcal{Y}(G^\bullet)) \) is a torsion pair in \( \text{mod} B \) and there are equivalences
   \[
   \text{Hom}_{D_{bgp}^b(A)}(G^\bullet, -) : \mathcal{T}(G^\bullet) \longrightarrow \mathcal{Y}(G^\bullet),
   \]
   and
   \[
   \text{Hom}_{D_{bgp}^b(A)}(G^\bullet, [1]) : \mathcal{F}(G^\bullet) \longrightarrow \mathcal{X}(G^\bullet).
   \]
3. \( \text{gldim} B \leq \text{Gdim} A + 1 \).

Throughout \( A \) is a finite dimensional \( k \)-algebra over a field \( k \), and \( \text{mod} A \) is the category of finitely generated left \( A \)-modules. A module \( G \) of \( \text{mod} A \) is Gorenstein-projective if there is an exact sequence

\[
\cdots \longrightarrow P^{-1} \longrightarrow P^0 \overset{d^0}{\longrightarrow} P^1 \longrightarrow \cdots
\]

of projective modules of \( \text{mod} A \), which stays exact after applying \( \text{Hom}_A(-, P) \) for each projective module \( P \), such that \( G \cong \text{Ker} d^0 \) (see [EJ2]). Denote by \( \text{Gproj} A \) and \( \text{proj} A \) the full subcategories of \( \text{mod} A \) consisting of Gorenstein-projective modules and projective modules, respectively.

Let \( A \) be an abelian category and \( \mathcal{X}, \mathcal{Y} \) the full additive subcategories of \( A \). Let \( M \) be an object of \( A \). Following [AS2], a morphism \( f : X \longrightarrow M \) with \( X \in \mathcal{X} \) is called a right \( \mathcal{X} \)-approximation of \( M \) if any morphism from an object \( \mathcal{X} \) to \( M \) factors through \( f \). \( \mathcal{X} \) is called contravariantly finite if any object in \( A \) admits a right \( \mathcal{X} \)-approximation. A morphism \( g : M \longrightarrow Y \) with \( Y \in \mathcal{Y} \) is called a left \( \mathcal{Y} \)-approximation of \( M \) if any morphism from \( M \) to an object \( \mathcal{Y} \) factors through \( g \). \( \mathcal{Y} \) is called covariantly finite if any object in \( A \) admits a left \( \mathcal{Y} \)-approximation.

Following [EJ2], an exact sequence \( G_1 \overset{d_1}{\longrightarrow} G_0 \longrightarrow M \longrightarrow 0 \) (\( * \)) is called a proper Gorenstein-projective presentation of \( M \) if each \( G_i \) is Gorenstein-projective and \( \text{Hom}_A(G, G_1) \longrightarrow \text{Hom}_A(G, G_0) \longrightarrow \text{Hom}_A(G, T) \longrightarrow 0 \) is exact for any Gorenstein-projective module \( G \). (\( * \)) is minimal if \( G_1 \longrightarrow \text{Im} d_1 \) and \( G_0 \longrightarrow M \) are right \( \text{Gproj} A \)-approximation. Moreover, the exact sequence \( 0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0 \) is called a proper Gorenstein-projective resolution of \( M \) of length \( n \) for some non-negative integer \( n \), if each \( G_i \) is all Gorenstein-projective and \( 0 \longrightarrow \text{Hom}_A(G, G_n) \longrightarrow \cdots \longrightarrow \text{Hom}_A(G, G_0) \longrightarrow \text{Hom}_A(G, M) \longrightarrow 0 \) is exact for any Gorenstein-projective module \( G \). We say that \( M \) has Gorenstein-projective dimension \( d \), denoted by \( \text{Gpd} M \), if \( d \) is the least, and the Gorenstein dimension of \( A \) is defined as follows:

\[
\text{Gdim} A = \sup\{ \text{Gpd} M | M \in \text{mod} A \}.
\]
Now we write $C^b(A)$, $K^b(A)$ and $D^b(A)$ for the bounded complex category, bounded homotopy category and bounded derived category of mod$A$, respectively. Denote by $K^b($Gproj$A)$ (resp. $K^b($proj$A)$) the corresponding bounded homotopy category of Gorenstein-projective modules (resp. projective modules).

1.1. Gorenstein derived category. A complex $C^\bullet \in C^b(A)$ is $GP$-acyclic, if $\text{Hom}_A(G, C^\bullet)$ is acyclic for each $G \in $ Gproj$A$. A chain map $f^\bullet : X^\bullet \to Y^\bullet$ is a $GP$-quasi-isomorphism, if $\text{Hom}_A(G, f^\bullet)$ is a quasi-isomorphism for each $G \in $ Gproj$A$, i.e., there are isomorphisms of abelian groups for any $n \in \mathbb{Z}$,

$$H^n\text{Hom}_A(G, f^\bullet) : H^n\text{Hom}_A(G, X^\bullet) \cong H^n\text{Hom}_A(G, Y^\bullet).$$

Put $K^b_{gpac}(A) := \{ X^\bullet \in K^b(A) \mid X^\bullet \text{ is } GP\text{-acyclic} \}.$ Then $K^b_{gpac}(A)$ is a thick triangulated subcategory of $K^b(A)$. Following [GZ], we have the following definition:

$$D^b_{gp}(A) := K^b(A)/K^b_{gpac}(A),$$

which is called the bounded Gorenstein derived category.

Following [AS1], an exact sequence $0 \to X^\bullet \overset{f}{\to} Y^\bullet \overset{g}{\to} Z^\bullet \to 0$ in $C^b(A)$ is $G$-exact if and only if $0 \to \text{Hom}_A(G, X^\bullet) \to \text{Hom}_A(G, Y^\bullet) \to \text{Hom}_A(G, Z^\bullet) \to 0$ is exact for all $G \in $ Gproj$A$. In the module case, $g$ is called a $G$-epimorphism and $X$ is called a $G$-submodule of $Y$. From [GZ], if the exact sequence $0 \to X \to Y \to Z \to X[1]$ is a triangle in $D^b_{gp}(A)$.

1.2. CM-Auslander algebra. Recall from [B2] that $A$ is of finite Cohen-Macaulay type (resp. finite CM-type for simply), if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein-projective $A$-modules. Let $\{ E_i \}_{i=1}^n$ be all non-isomorphic finitely generated Gorenstein-projective $A$-modules and $E = \bigoplus_{i=1}^n E_i$, and $A(Gproj) = \text{End}_A(E)^{op}$. Then $E$ is an $A$-$A(Gproj)$-bimodule, and $A(Gproj)$ is called the the Cohen-Macaulay Auslander (resp. CM-Auslander for simply) algebra of $A$.

1.3. torsion pair.

Definition 1.1. ([D]) A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of mod$A$ is called a torsion pair provided that:

1. $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$;
2. $\text{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$;
3. $\text{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

1.4. t-structure.

Definition 1.2. ([BBD]) A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of the triangulated category $D^b_{gp}(A)$ is called a t-structure provided that:
(1) $\mathcal{X}[1] \subset \mathcal{X}$ and $\mathcal{Y}[-1] \subset \mathcal{Y}$;
(2) $\text{Hom}_{D_{\text{gp}}(A)}(X, Y[-1]) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
(3) for any $C \in D_{\text{gp}}(A)$, there exists a distinguished triangle

$$X \rightarrow C \rightarrow Y[-1] \rightarrow X[1]$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

2. Gorenstein silting modules

In this section, we introduce and study a class of modules in bijection with the relative rigid module over an algebra of CM-type introduced by Gao, which we call the Gorenstein silting module. We also show the relations among Gorenstein tilting modules, Gorenstein star modules and Gorenstein silting modules.

2.1. Gorenstein silting modules. In this subsection, we introduce the Gorenstein silting module. Before this, we make some preparation.

Throughout this subsection, let $R$ be a Noetherian ring, and $\text{Mod} R$ the category of all left $R$-modules. We denote by $Gp(R)$ (resp. $Gi(R)$) the full subcategory of $\text{Mod} R$ consisting of Gorenstein-projective (resp. Gorenstein-injective) modules.

Now we assume that $Gp(R)$ is contravariantly finite in $\text{Mod} R$. For $R$-modules $M$ and $N$, we compute right derived functors of $\text{Hom}_R(M, N)$ using a Gorenstein-projective resolution of $M$ ([E.2], [Hol]). We will denote these derived functors by $\text{Gext}^i_R(M, N)$.

A short exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is $G$-exact if and only if it is in $\text{Gext}^1_R(L, M)$.

Let $T$ be an $R$-module. Put

$$\text{Pres}_G(T) := \{ M \in \text{Mod} R | \exists \text{ a } G\text{-exact sequence } T_1 \rightarrow T_0 \rightarrow M \rightarrow 0 \}
$$

$$\text{with } T_i \in \text{Add} T \},$$

$$\text{Gen}_G(T) := \{ M \in \text{Mod} R | \exists \text{ a } G\text{-exact sequence } T_0 \rightarrow M \rightarrow 0 \text{ with } T_0 \in \text{Add} T \},$$

and

$$T^{G\perp} := \{ M \in \text{Mod} R | \text{Gext}^1_R(T, M) = 0 \}, \quad T^G \perp := \{ M \in \text{Mod} R | \text{Hom}_R(T, M) = 0 \},$$

where $\text{Add} T$ denotes the subcategory of modules consisting of direct summands of direct sums of $T$.

For a morphism $\theta : G_1 \rightarrow G_0$ with $G_1$ and $G_0$ are Gorenstein-projective modules. We consider the class of $R$-modules

$$D_\theta := \{ X \in \text{Mod} R | \text{Hom}_R(\theta, X) \text{ is epic} \}.$$

We first collect some useful properties of $D_\theta$. 

Lemma 2.1. Let $\theta : G_1 \to G_0$ with $G_1$ and $G_0$ are Gorenstein-projective modules, and $T$ the cokernel of $\theta$ such that $\theta$ being the proper Gorenstein-projective presentation of $T$.

(i) $D_\theta$ is closed under $G$-epimorphic images, $G$-extensions and direct products.

(ii) The class $D_\theta$ is contained in $T^{G^\perp}$.

Proof. The statement (i) is easy to prove. Now we prove (ii). There exists the following diagram

\[
\begin{array}{cccccc}
G_1 & \xrightarrow{\theta} & G_0 & \xrightarrow{i} & T & \to 0 \\
\downarrow{\pi} & & \downarrow{i} & & \downarrow{\pi} & \\
\text{Im}\theta & & & & & \\
\end{array}
\]

Consider the $G$-exact sequence $0 \to \text{Im}\theta \xrightarrow{i} G_0 \to T \to 0$. Applying the functor $\text{Hom}_R(-, X)$ for any $X \in D_\theta$, then we get the $G$-exact sequence

$$\text{Hom}_R(G_0, X) \xrightarrow{i^*} \text{Hom}_R(\text{Im}\theta, X) \to \text{Gext}^1_R(T, X) \to 0.$$ 

We show that $i^*$ is surjective. Let $f \in \text{Hom}_R(\text{Im}\theta, X)$. Since $X \in D_\theta$, there is a map $g : G_0 \to X$ such that $f\pi = g\theta = gi\pi$. Since $\pi$ is an epimorphism, we have $f = gi$. Hence $\text{Gext}^1_R(T, X) = 0$, and so $X \in T^{G^\perp}$.

Definition 2.2. We say that an $R$-module $T$ is

- **partial Gorenstein silting** if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that
  
  (Gs1) $D_\theta$ is a relative torsion class (i.e. closed for $G$-epimorphic images, $G$-extensions and coproducts);
  
  (Gs2) $T$ lies in $D_\theta$.

- **Gorenstein silting** if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that $\text{Gen}_G(T) = D_\theta$.

Proposition 2.3. Let $T$ be an $R$-module with the proper Gorenstein-projective presentation $\theta : G_1 \to G_0$. If $T$ is a partial Gorenstein silting module with respect to $\theta$, and for each $P \in \text{Gp}(R)$, there exists a $G$-exact sequence $P \xrightarrow{\phi} T_0 \to T_{-1} \to 0$ with $T_0$ and $T_{-1}$ in $\text{Add}T$ such that $\phi$ is the left $D_\theta$-approximation, then $T$ is a Gorenstein silting module.

Proof. Since $T$ is a partial Gorenstein silting module with respect to $\theta$, it is clear that $\text{Gen}_G(T) \subseteq D_\theta$. Let $X$ be an object in $D_\theta$. Since $\text{Gp}(R)$ is contravariantly finite, there is an $R$-module $E \in \text{Gp}(R)$ and a $G$-epimorphism $E \to X$. By assumption this epimorphism factors through the left $D_\theta$-approximation $\phi : E \to T_0$ via a $G$-epimorphism $g : T_0 \to X$. Thus $X$ lies in $\text{Gen}_G(T)$. This means that $T$ is a Gorenstein silting module.

2.2. Connection with Gorenstein tilting (resp. star) modules. In this subsection, we characterise the relations among Gorenstein silting modules, Gorenstein tilting modules and Gorenstein star modules.
Throughout, let $R$ be a Noetherian ring such that $Gp(R)$ the contravariantly finite subcategory of $\text{Mod}R$.

**Definition 2.4.** ([AS1, G1, YLO]) An $R$-module $T$ is called a Gorenstein tilting module if $T^{G\perp} = \text{Pres}_G(T)$, or equivalently, it satisfies the following three conditions:

$(T1)$ $\text{Gpd}_R T \leq 1$.
$(T2)$ $\text{Gext}^1_R (T, T(I)) = 0$ for all sets $I$.
$(T3)$ For any $P \in Gp(R)$, there exists a $G$-exact sequence $0 \to P \to T_0 \to T_{-1} \to 0$ with each $T_i \in \text{Add}T$.

**Definition 2.5.** ([Z]) An $R$-module $T$ is called a Gorenstein star module if

(i) Any $G$-exact sequence $0 \to X \to T_0 \to Y \to 0$ with $T_0 \in \text{Add}T$ and $X \in \text{Gen}_G(T)$ is $\text{Hom}_R(T, -)$-exact.

(ii) $\text{Gen}_G(T) = \text{Pres}_G(T)$.

**Lemma 2.6.** ([Z Proposition 2.9]) The following statements are equivalent for an $R$-module $T$:

(i) $T$ is a Gorenstein star module and $\text{Gen}_G(T)$ is closed under $G$-extension.

(ii) $\text{Gen}_G(T) = \text{Pres}_G(T) \subseteq T^{G\perp}$.

**Proposition 2.7.** The following hold.

(1) Each Gorenstein tilting $R$-module is Gorenstein silting.

(2) Each Gorenstein silting $R$-module is a Gorenstein star module.

*Proof.* (1) Let $T$ be a Gorenstein tilting $R$-module. Then by definition there exists a $G$-exact sequence as follows:

$$0 \to G_1 \xrightarrow{\theta} G_0 \to T \to 0.$$  

Let $X \in \text{Gen}_G(T)$. Then there exists an $G$-epimorphism $g : T(I) \to X$ such that $\text{Hom}_R(G, g)$ is $G$-epic. Since $\text{Gpd}_R T \leq 1$ and $\text{Gext}^1_R(T, T(I)) = 0$, applying the functor $\text{Hom}_R(T, -)$ to $g$, it follows that $\text{Gext}^1_R(T, X) = 0$, and hence $X \in D_\theta$.

Let $X \in D_\theta$. Then by Lemma 2.6, $X \in T^{G\perp}$. It follows from the definition $T^{G\perp} = \text{Pres}_G(T)$ that $X \in \text{Pres}_G(T)$. This implies that $X \in \text{Gen}_G(T)$. Therefore, we prove that $\text{Gen}_G(T) = D_\theta$, and $T$ is a Gorenstein silting module with respect to $\theta$.

(2) Let $T$ be a Gorenstein silting $R$-module with the Gorenstein-projective presentation $\theta : G_1 \to G_0$. Let $X \in \text{Gen}_G(T)$. Then there is the $G$-epimorphic universal map $u : T(I) \to X$ for some index set $I$. We will show that $K := \text{Ker} u$ lies in $D_\theta = \text{Gen}_G(T)$. Pick $f : G_1 \to K$. Since $T(I)$ lies in $D_\theta$, we get the following commutative diagram of
Recall that a ring $R$ is Gorenstein if $R$ is two-sided Noetherian and has finite injective dimension, both as left and right $R$-module.

**Theorem 2.8.** Let $R$ be a 1-Gorenstein ring and $T$ an $R$-module. Then the following are equivalent.

1. $T$ is a Gorenstein tilting module.
2. $T$ is a Gorenstein silting module.
3. $T$ is a Gorenstein star module and $G_i(R) \subset \text{Pres}_G(T)$.

**Proof.** First by [Z, Theorem 3.2] we know that (1) $\iff$ (3). The theorem immediately follows from Proposition 2.7. □

We finish this section with an important class of examples of (partial) Gorenstein silting modules: $\tau_G$-rigid modules over a finite dimensional $k$-algebra, where $\tau_G$ was introduced in [G2] for an algebra of finite CM-type.

**2.3. Relative rigid modules over algebras of finite CM-type.** From now on, let $A$ be a finite dimensional $k$-algebra of finite CM-type over a field $k$ and $\text{mod} A$ the category of finitely generated left $A$-modules. Use the notation in the introduction. Denote by $A(\text{Gproj})$-mod the category of finitely generated right $A(\text{Gproj})$-modules and $A(\text{Gproj})$-mod the stable category of $A(\text{Gproj})$-mod modulo projective $A(\text{Gproj})$-modules. Let $D : A(\text{Gproj})$-mod $\to \text{mod} A(\text{Gproj})$ be the duality. For Simplicity, we denote the functors $\text{Hom}_A(-,-)$ by $(-,-)$ and $\text{Hom}_{A(\text{Gproj})}(-,-)$ by $A(\text{Gproj})(-,-)$.

Let $M$ be an $A$-module in $\text{mod} A$. Then there is a minimal Gorenstein-projective presentation $G_1 \to G_0 \to M \to 0$ of $M$. This induces the following exact sequences

$0 \to (M,E) \to (G_0,E) \to (G_1,E) \to \text{Tr}_G M \to 0$

and

$0 \to \tau_G M \to D(G_1,E) \to D(G_0,E)$,

with $\text{Tr}_G M \in A(\text{Gproj})$-mod and $\tau_G M \in \text{mod} A(\text{Gproj})$. Recall from [G2] that

$\text{Tr}_G : \text{mod} A/Gp(A) \to A(\text{Gproj})$-mod

is called the relative transpose of $A$, and moreover, $\text{Tr}_G$ is a faithful functor.
Definition 2.9. We say that an $A$-module $M$ is $\tau_G$-rigid if $\text{Hom}_{A\text{-proj}}((E, M), \tau_G M) = 0$ for the left $A(\text{proj})$-module $\text{Hom}_A(E, M)$.

Next we provide some properties for $\tau_G$-rigid module. For an $A$-module $M$, put

$$\perp^0(\tau_G M) := \{ X \in \text{mod}A(\text{proj}) \mid \text{Hom}_{A(\text{proj})}(X, \tau_G M) = 0 \}.$$

Proposition 2.10. Let $M$ be a finitely generated $A$-module and

$$G_1 \xrightarrow{f} G_0 \rightarrow M \rightarrow 0 \quad (\ast)$$

the minimal proper Gorenstein-projective presentation of $M$. Then the following statements hold.

(1) $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(f, M)$ is an epimorphism.

(2) $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(E, M)$ is a $\tau$-rigid module, where $\tau$ is the Auslander-Reiten translation of $A(\text{proj})$.

(3) $\tau_G M \cong \tau(E, M)$.

(4) Suppose that $M$ is $\tau_G$-rigid. Then $\text{Hom}_A(E, \text{Gen}_G M) \subseteq \perp^0(\tau_G M)$.

Proof. There is an exact sequence

$$0 \rightarrow \tau_G M \rightarrow D(G_1, E) \rightarrow D(G_0, E). \quad \text{(**)}$$

Applying $\text{Hom}_{A(\text{proj})}(N, -)$ for any $A(\text{proj})$-module $N$, we have the following commutative diagram of exact sequences:

$$0 \rightarrow A(\text{proj})(N, \tau_G M) \rightarrow A(\text{proj})(N, D(G_1, E)) \rightarrow A(\text{proj})(N, D(G_0, E))$$

$$\cong \quad \cong$$

$$D_A(\text{proj})((E, G_1), N) \rightarrow D_A(\text{proj})((E, G_0), N) \rightarrow D_A(\text{proj})((E, M), N) \rightarrow 0.$$

Then we get that $\text{Hom}_{A(\text{proj})}(N, \tau_G M) = 0$ if and only if the map

$$\text{Hom}_{A(\text{proj})}((E, G_0), N) \rightarrow \text{Hom}_{A(\text{proj})}((E, G_1), N)$$

is an epimorphism.

(1) and (2) Applying $\text{Hom}_A(E, -)$ the $(\ast)$, we can get the projective resolution of $(E, M)$:

$$(E, G_1) \xrightarrow{\sigma} (E, G_0) \rightarrow (E, M) \rightarrow 0.$$ Then we have from above arguments that $\text{Hom}_{A(\text{proj})}((E, M), \tau_G M) = 0$ if and only if $\text{Hom}_{A(\text{proj})}((\sigma, (E, M)))$ is epic if and only if $\text{Hom}_A(f, M)$ is epic. Notice that the statement that $\text{Hom}_{A(\text{proj})}((\sigma, (E, M)))$ is epic means that $\text{Hom}_A(E, M)$ is a partial silting $A(\text{proj})$-module, equivalently, $\text{Hom}_A(E, M)$ is $\tau$-rigid.

(3) Consider the exact sequence

$$0 \rightarrow \tau(E, M) \rightarrow D_A(\text{proj})((E, G_1), A(\text{proj})) \rightarrow D_A(\text{proj})((E, G_0), A(\text{proj})).$$

Then we get from above arguments that $\tau_G M \cong \text{Hom}_{A(\text{proj})}(A(\text{proj}), \tau_G M) \cong \tau(E, M)$.
(4) We have proved in (2) that \( \text{Hom}_A(E, M) \) is a \( \tau \)-rigid module. If \( Y \in \text{Gen}_GM \), then there is the \( G \)-exact sequence \( M^{(I)} \to Y \to 0 \) for some index set \( I \). Therefore we have that \( (E, \text{Gen}_GM) \subseteq \text{Gen}(E, M) \subseteq (E, M)^{\perp_1} \). Since there is the following isomorphisms

\[
\text{Ext}_A^{1}(\text{Gen}_G\text{proj})(E, M), X) \cong D \text{Hom}_A(\text{Gen}_G\text{proj})(\tau^{-1}X, (E, M))
\]

\[
\cong D \text{Hom}_A(\text{Gen}_G\text{proj})(X, \tau(E, M))
\]

\[
\cong D \text{Hom}_A(\text{Gen}_G\text{proj})(X, \tau GM),
\]

it follows that \( X \in (E, M)^{\perp_1} \) if and only if \( X \in \perp_0(\tau GM) \). Hence \( \text{Hom}_A(E, \text{Gen}_GM) \subseteq \perp_0(\tau GM) \).

\[\square\]

**Theorem 2.11.** Let \( A \) be a finite dimensional algebra of finite CM-type and \( M \) an \( A \)-module in \( \text{mod}A \). Then \( M \) is \( \tau_G \)-rigid if and only if \( M \) is a partial Gorenstein silting \( A \)-module.

**Proof.** Let \( G_1 \to \theta \to G_0 \to 0 \) be the minimal proper Gorenstein-projective presentation of \( M \). Since \( G_0 \) and \( G_1 \) are finitely generated, it follows that \( D\theta \) is closed under coproducts. Then we get from Lemma 2.1 that \( D\theta \) is a torsion class. On the other hand, we get from Proposition 2.10(1) that \( M \) is \( \tau_G \)-rigid if and only if \( \text{Hom}_A(\theta, M) \) is surjective. The latter implies that \( M \in D\theta \). Thus \( M \) is \( \tau_G \)-rigid if and only if \( M \) is partial Gorenstein silting. \[\square\]

Recall from [DIJ] that an algebra \( A \) is called \( \tau \)-tilting finite if it admits finite number of isomorphism classes of indecomposable \( \tau \)-rigid modules.

**Definition 2.12.** An algebra \( A \) of finite CM-type is called \( \tau_G \)-tilting finite if it admits finite number of isomorphism classes of indecomposable \( \tau_G \)-rigid modules.

**Corollary 2.13.** Let \( A \) be an Artin algebra of finite CM-type and \( A(\text{Gproj}) \) the CM-Auslander algebra. If \( A(\text{Gproj}) \) is \( \tau \)-tilting finite, then \( A \) is \( \tau_G \)-tilting finite, and further has finite number of isomorphism classes of indecomposable partial Gorenstein silting modules.

**Proof.** By Proposition 2.10(1), we know that an \( A \)-module \( M \) is \( \tau_G \)-rigid if and only if \( \text{Hom}_A(E, M) \) is a \( \tau \)-rigid module. This implies the result by Theorem 2.11 \[\square\]

### 3. 2-term Gorenstein silting complexes

In this section, we introduce the notion of the 2-term Gorenstein silting complex, which is in bijection with the Gorenstein silting module. Then we characterise it by the connection with the t-structure and torsion pair. We also characterise the global dimension of endomorphism algebras of 2-term Gorenstein silting complexes over an algebra \( A \) by terms of the Gorenstein dimension of \( A \). Throughout, we denote by \( A \) a finite dimensional \( k \)-algebra over a field \( k \).
3.1. 2-term Gorenstein silting complexes. In this subsection, we introduce the definition of 2-term Gorenstein silting complexes, and show the links with t-structures and torsion pairs. The Brenner-Butler theorem is given.

**Definition 3.1.** Let $G^\bullet : G_1 \xrightarrow{d^1} G_0$ be a complex in $D^b_{gp}(A)$ with $G_i \in \text{Gproj}A$ for $i = 0, 1$. We say that $G^\bullet$ is

- **2-term partial Gorenstein silting** if it satisfies the following two conditions:
  (i) $G_1 \xrightarrow{d^1}$ and $G_0 \xrightarrow{\text{Coker}d^1}$ are right $\text{Gproj}A$-approximations;
  (ii) $\text{Hom}_{D^b_{gp}(A)}(G^\bullet, G^\bullet[1]) = 0$;

- **2-term Gorenstein silting** in $K^b(\text{Gproj}A)$ if it is a 2-term partial Gorenstein silting complex and $\text{thick}G^\bullet = K^b(\text{Gproj}A)$.

Let $G^\bullet : G_1 \xrightarrow{d^1} G_0$ be a 2-term Gorenstein silting complex in $K^b(\text{Gproj}A)$. Consider the subcategories of $D^b_{gp}(A)$:

$$D^0_{gp}(G^\bullet) = \{ X^\bullet \in D^b_{gp}(A) \mid \text{Hom}_{D^b_{gp}(A)}(G^\bullet, X^\bullet[i]) = 0, \text{ for } i > 0 \}$$

and

$$D^0_{gp}(G^\bullet) = \{ X^\bullet \in D^b_{gp}(A) \mid \text{Hom}_{D^b_{gp}(A)}(G^\bullet, X^\bullet[i]) = 0, \text{ for } i < 0 \},$$

and the subcategories of $\text{mod}A$:

$$\mathcal{T}(G^\bullet) = \{ X \in \text{mod}A \mid \text{Hom}_{D^b_{gp}(A)}(G^\bullet, X[1]) = 0 \}$$

and

$$\mathcal{F}(G^\bullet) = \{ Y \in \text{mod}A \mid \text{Hom}_{D^b_{gp}(A)}(G^\bullet, Y) = 0 \}.$$ 

Then we have the following facts.

**Lemma 3.2.**

(1) $(D^0_{gp}(G^\bullet), D^0_{gp}(G^\bullet))$ is a t-structure in $D^b_{gp}(A)$.

(2) $\mathcal{T}(G^\bullet) = D^0_{gp}(G^\bullet) \cap \text{mod}A$ and $\mathcal{F}(G^\bullet) = D^0_{gp}(G^\bullet) \cap \text{mod}A$.

**Proof.** The proof of (1) can be found in [HKM, Theorem 1.3]. (2) can be obtained by definitions. □

Next we consider the relation between 2-term Gorenstein silting complexes, t-structures and torsion pairs in module categories. A key lemma is given below.

**Lemma 3.3.** For any $X^\bullet \in D^b_{gp}(A)$ and $n \in \mathbb{Z}$, we have a functorial exact sequence

$$0 \rightarrow \text{Hom}_{D^b_{gp}(A)}(G^\bullet, H^{n-1}(X^\bullet)[1]) \rightarrow \text{Hom}_{D^b_{gp}(A)}(G^\bullet, X^\bullet[n])$$

$$\rightarrow \text{Hom}_{D^b_{gp}(A)}(G^\bullet, H^n(X^\bullet)) \rightarrow 0.$$ 

**Proof.** For $X^\bullet[n] \in D^b_{gp}(A)$, applying $\text{Hom}_{D^b_{gp}(A)}(-, X^\bullet[n])$ to a distinguished triangle

$$G_1 \xrightarrow{d^1} G_0 \rightarrow G^\bullet \rightarrow G_1[1],$$
we have a short exact sequence
\[
0 \to \text{Coker}(\text{Hom}_{D^b_{\text{gp}}(A)}(d^1, X^*[n-1])) \to \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, X^*[n]) \\
\to \text{Ker}(\text{Hom}_{D^b_{\text{gp}}(A)}(d^1, X^*[n])) \to 0.
\]

Since
\[
\text{Ker}(\text{Hom}_{D^b_{\text{gp}}(A)}(d^1, X^*[n])) \cong \text{Ker}(\text{Hom}_A(d^1, H^n(X^*)))
\cong \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, H^n(X^*)),
\]
and
\[
\text{Coker}(\text{Hom}_{D^b_{\text{gp}}(A)}(d^1, X^*[n-1])) \cong \text{Coker}(\text{Hom}_A(d^1, H^{n-1}(X^*)))
\cong \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, H^{n-1}(X^*)[1]),
\]
we get the desired exact sequence. \qed

**Proposition 3.4.** Let $G^*$ be a 2-term Gorenstein silting complex in $K^b(\text{Gproj} A)$, and $C_{\text{gp}}(G^*) := D^{<0}_{\text{gp}}(G^*) \cap D^0_{\text{gp}}(G^*)$ the heart of the induced t-structure $(D^{\leq 0}_{\text{gp}}(G^*), D^\geq 0(G^*))$. Let $B = \text{End}_{D^b_{\text{gp}}(A)}(G^*)^{\text{op}}$.

1. $C_{\text{gp}}(G^*)$ is an abelian category and the short exact sequences in $C_{\text{gp}}(G^*)$ are precisely the triangles in $D^0_{\text{gp}}(A)$ all of whose vertices are objects in $C_{\text{gp}}(G^*)$.

2. For a complex $X^*$ in $D^b_{\text{gp}}(A)$, we have that $X^*$ is in $C_{\text{gp}}(G^*)$ if and only if $H^0(X^*)$ is in $\mathcal{T}(G^*)$, $H^{-1}(X^*)$ is in $\mathcal{F}(G^*)$ and $H^i(X^*) = 0$ for $i \neq -1, 0$.

3. The functor $\text{Hom}_{D^b_{\text{gp}}(A)}(G^*, \cdot) : C_{\text{gp}}(G^*) \to \text{mod} B$ is an equivalence of abelian categories.

4. For any $E \in \text{Gproj} A$, there is a triangle in $D^0_{\text{gp}}(A)$
\[
E \xrightarrow{e} G^* \xrightarrow{f} G'^* \xrightarrow{g} E[1] \quad (\triangle_{G^*})
\]
with $G'^*, G'^* \in \text{add} G^*$.

5. Suppose that $A$ is of finite CM-type with the Gorenstein-projective generator $E$. Then
\[
Q^* : \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, G^*) \xrightarrow{\text{Hom}_{D^0_{\text{gp}}(A)}(G^*, f)} \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, G'^*)
\]
is a 2-term partial silting complex in $K^b(\text{proj} B)$, where $f$ is the map from the triangle $\triangle_{G^*}$ of the Gorenstein-projective generator $E$.

**Proof.** (1) The pair $(D^{\leq 0}_{\text{gp}}(G^*), D^0_{\text{gp}}(G^*))$ is a t-structure in $D^b_{\text{gp}}(A)$. Then we come to the conclusion.

(2) From Lemma 3.3, we have that
\[
D^{\leq 0}_{\text{gp}}(G^*)
= \{X^* \in D^b_{\text{gp}}(A) | \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, H^i(X^*)) = 0 \text{ for } i > 0, \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, H^j(X^*)[1]) = 0 \text{ for } j \geq 0\}
= \{X^* \in D^b_{\text{gp}}(A) | H^i(X^*) = 0 \text{ for } i > 0 \text{ and } \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, H^0(X^*)[1]) = 0\}
= \{X^* \in D^b_{\text{gp}}(A) | H^i(X^*) = 0 \text{ for } i > 0 \text{ and } H^0(X^*) \in \mathcal{T}(G^*)\},
\]
and
\[ D_{\geq 0}^{\operatorname{gp}}(G^\bullet) \]
= \{ X^\bullet \in D_{\operatorname{gp}}^b(A) | \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, H^i(X^\bullet)) = 0 \text{ for } i < 0, \text{ and } \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, H^j(X^\bullet)[1]) = 0 \text{ for } j < -1 \}
= \{ X^\bullet \in D_{\operatorname{gp}}^b(A) | H^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, H^{-1}(X^\bullet)) = 0 \}
= \{ X^\bullet \in D_{\operatorname{gp}}^b(A) | H^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } H^{-1}(X^\bullet) \in \mathcal{F}(G^\bullet) \}.

Therefore, we get that \( X^\bullet \in C_{\operatorname{gp}}(G^\bullet) \) if and only if \( H^0(X^\bullet) \in \mathcal{T}(G^\bullet), H^{-1}(X^\bullet) \in \mathcal{F}(G^\bullet) \) and \( H^i(X^\bullet) = 0 \text{ for } i \neq -1,0. \)

(3) The proof can be found in [HKM, Theorem 1.3].

(4) Let \( G''\bullet \rightarrow E[1] \) be a right add-\( G^\bullet \)-approximation of \( E[1] \). Extend it to a triangle
\[
E \rightarrow H^\bullet \rightarrow G''\bullet \rightarrow E[1],
\]
where \( H^\bullet \) is a 2-term complex in \( K^b(\operatorname{Gproj}A) \). By applying the functors \( \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, -) \) and \( \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(-, G^\bullet) \) to the triangle (*), we have that
\[
\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, H^1[1]) = 0 \text{ and } \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(H^\bullet, G^1[1]) = 0.
\]
Applying \( \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(-, H^\bullet) \) yields \( \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(H^\bullet, H^1[1]) = 0. \) Hence \( G^\bullet \oplus H^\bullet \) is a 2-term partial Gorenstein silting complex in \( K^b(\operatorname{Gproj}A) \). The triangle (*) shows that \( E \in \operatorname{thick}(G^\bullet \oplus H^\bullet) \) and so \( G^\bullet \oplus H^\bullet \) is a 2-term Gorenstein silting complex. Therefore, we get the desired triangle \( \triangle_{G^\bullet}. \)

(5) Let \( \alpha \) be a morphism in \( \operatorname{Hom}_{K^b(\operatorname{proj}B)}(Q^\bullet, Q^1[1]) \). Then it has the following form
\[
\begin{array}{ccc}
\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, G^\bullet) & \xrightarrow{\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, f)} & \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, G''\bullet) \\
\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, G^\bullet) & \xrightarrow{\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, f)} & \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, G''\bullet) \\
\end{array}
\]
and there is a morphism \( h : G^\bullet \rightarrow G''\bullet \) such that \( \alpha = \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, h). \) Since \( \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(E, E[1]) = 0 \), there are unique morphisms \( h_1, h_2 \) such that the following diagram
\[
\begin{array}{c}
E \xrightarrow{e} G^\bullet \xrightarrow{f} G''\bullet \xrightarrow{g} E[1] \\
\downarrow h_1 \quad \downarrow h_3 \quad \downarrow h_2 \quad \downarrow h_1[1] \\
G^\bullet \xrightarrow{f} G''\bullet \xrightarrow{g} E[1] \xrightarrow{-e[1]} G^\bullet[1].
\end{array}
\]
is commutative. So there is a morphism \( h_3 \) such that \( h_2 = gh_3 \), and also,
\[ g(h - h_3 f) = gh - gh_3 f = gh - h_2 f = 0. \]
Hence there is a morphism \( h_4 \) such that \( h - h_3 f = f h_4. \) Applying \( \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, -) \) to \( h - h_3 f \) yields
\[
\alpha = \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, h_3)\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, f) + \operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, f)\operatorname{Hom}_{D_{\operatorname{gp}}^b(A)}(G^\bullet, h_4),
\]
which implies that \( \alpha \) regarded as a map in \( \operatorname{Hom}_{K^b(\operatorname{proj}B)}(Q^\bullet, Q^1[1]) \) is null-homotopic. Thus, \( Q^\bullet \) is a 2-term partial silting complex in \( K^b(\operatorname{proj}B). \)
Let $X \in \text{mod}A$. Consider the canonical sequence of $X$:

$$0 \rightarrow tX \xrightarrow{i_X} X \rightarrow X/tX \rightarrow 0,$$

where $tX = \sum \text{Im}f$ with $f \in \text{Hom}_A(H^0(G^\bullet), X)$ such that any $g : E \rightarrow X$ factors through $f$ for any $E \in \text{Gproj}A$. We collect some properties of $T(G^\bullet)$ and $\mathcal{F}(G^\bullet)$.

**Lemma 3.5.** The following hold:

1. $T(G^\bullet)$ is closed under $G$-epimorphic images.
2. $\mathcal{F}(G^\bullet)$ is closed under $G$-submodules.
3. For any $X \in \text{mod}A$, $\text{Hom}_A(H^0(G^\bullet), i_X)$ is an isomorphism.

**Proof.** (1) Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be a $G$-exact sequence in $\text{mod}A$. Applying $\text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, -)$, we get the following exact sequence

$$\text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, Y[1]) \rightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, Z[1]) \rightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, X[2]).$$

Since $\text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, X[2]) = 0$, then we have that $Y \in T(G^\bullet)$ implies $Z \in T(G^\bullet)$.

(2) Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be a $G$-exact sequence in $\text{mod}A$. Applying $\text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, -)$, we get the following exact sequence

$$0 \rightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, X) \rightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, Y) \rightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, Z).$$

Hence $Y \in \mathcal{F}(G^\bullet)$ implies $X \in \mathcal{F}(G^\bullet)$.

(3) By the definition of $tX$, we immediately obtain the desired isomorphism. \qed

**Lemma 3.6.** There exists a triangle in $D^b_{\text{gp}}(A)$ for a 2-term Gorenstein silting complex $G^\bullet$ of the form

$$H^{-1}(G^\bullet)[1] \rightarrow G^\bullet \rightarrow H^0(G^\bullet) \rightarrow H^{-1}(G^\bullet)[2].$$

**Proof.** Let $G^\bullet := 0 \rightarrow \text{Im}d^1 \rightarrow G_0 \rightarrow 0$. We have a $G$-exact sequence

$$0 \rightarrow \text{Ker}d^1[1] \rightarrow G^\bullet \rightarrow G^\bullet^\ast \rightarrow 0$$

in $C^b(A)$. Since $G^\bullet$ is $\mathcal{GP}$-quasi-isomorphic to $H^0(G^\bullet)$, then $G^\bullet \cong H^0(G^\bullet)$ in $D^b_{\text{gp}}(A)$, we get the desired triangle in $D^b_{\text{gp}}(A)$ of the form

$$H^{-1}(G^\bullet)[1] \rightarrow G^\bullet \rightarrow H^0(G^\bullet) \rightarrow H^{-1}(G^\bullet)[2].$$

\qed

**Lemma 3.7.** For any $X \in \text{mod}A$, we have a functorial isomorphism

$$\text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, X) \cong \text{Hom}_A(H^0(G^\bullet), X)$$

and a monomorphism

$$\text{Hom}_{D^b_{\text{gp}}(A)}(H^0(G^\bullet), X[1]) \rightarrow \text{Hom}_A(G^\bullet, X[1]).$$
Proof. Applying $\text{Hom}_{D^b_{\text{gp}}(A)}(-, X)$ to the triangle in Lemma 3.3
\[ H^{-1}(G^*)[1] \longrightarrow G^* \longrightarrow H^0(G^*) \longrightarrow H^{-1}(G^*)[2] \]
and using that there is no non-zero negative extensions between modules, we get the required isomorphism and monomorphism. \hfill \Box

Theorem 3.8. The following are equivalent for a complex $G^* : G_1 \longrightarrow G_0$ with $G_i \in \text{GprojA}.$

\begin{enumerate}
\item $(1)$ $T(G^*)$ is a 2-term Gorenstein silting complex in $D^b_{\text{gp}}(A).$
\item $(2)$ $T(G^*) \cap F(G^*) = 0$ and $H^0(G^*) \in T(G^*).$
\item $(3)$ $T(G^*) \cap F(G^*) = 0$ and $t(X) \in T(G^*), \ X/tX \in F(G^*)$ for all $X \in \text{modA}.$
\item $(4)$ $(T(G^*), F(G^*))$ is a torsion pair for modA.
\end{enumerate}

Proof. (1) $\Leftrightarrow$ (2) $\text{Hom}_{D^b_{\text{gp}}(A)}(G^*, G^*[i]) = 0$ for all $i > 0$ if and only if $H^0(G^*) \in T(G^*)$ by Lemma 3.3. For any $X \in T(G^*) \cap F(G^*), \ \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, X[n]) = 0$ for all $n \in \mathbb{Z}$ and hence $X = 0.$ Conversely, let $X^* \in D^b_{\text{gp}}(A)$ with $\text{Hom}_{D^b_{\text{gp}}(A)}(G^*, X[n]) = 0$ for all $n \in \mathbb{Z}.$ Then by Lemma 3.3, $H^n(X^*) \in T(G^*) \cap F(G^*) = 0.$

(2) $\Rightarrow$ (3) Let $X \in \text{modA}.$ Since $H^0(G^*) \in T(G^*), \ \text{it follows that } tX \in T(G^*).$ Next, since there is an isomorphism by Lemma 3.7
\[ \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, X/tX) \cong \text{Hom}_A(H^0(G^*), X/tX), \]
and $\text{Hom}_A(H^0(G^*), i_X)$ is an isomorphism, it follows that $\text{Hom}_{D^b_{\text{gp}}(A)}(G^*, X/tX) = 0$ and hence $X/tX \in F(G^*).$

(3) $\Rightarrow$ (4) It can be obtained by the definition.

(4) $\Rightarrow$ (2) We just need to prove that $H^0(G^*) \in T(G^*).$ By Lemma 3.7
\[ 0 = \text{Hom}_{D^b_{\text{gp}}(A)}(G^*, F(G^*)) \cong \text{Hom}_A(H^0(G^*), F(G^*)), \]
it follows from $(T(G^*), F(G^*))$ is a torsion pair that $H^0(G^*) \in T(G^*).$ \hfill \Box

Remark 3.9. Note that the torsion pair $(T(G^*), F(G^*))$ coincides with $(D_\theta, T_{\perp 0})$ defined in the subsection 2.1.

Proof. Let $G^* : G_1 \xrightarrow{\theta} G_0$ be a 2-term Gorenstein silting complex in $D^b_{\text{gp}}(A),$ and $T = H^0(G^*) = \text{Coker} \theta.$ On one hand, consider the distinguished triangle in $D^b_{\text{gp}}(A)$
\[ G_1 \xrightarrow{\theta} G_0 \longrightarrow G^* \longrightarrow G_1[1]. \]
Applying the functor $\text{Hom}_{D^b_{\text{gp}}(A)}(-, X)$ for any $A$-module $X,$ there is the induced exact sequence
\[ \text{Hom}_{D^b_{\text{gp}}(A)}(G_0, X) \longrightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G_1, X) \longrightarrow \text{Hom}_{D^b_{\text{gp}}(A)}(G^*[1], X) \longrightarrow 0. \]
Since $\text{Hom}_{D^b_{\text{gp}}(A)}(G_1, X) \cong \text{Hom}_A(G_1, X)$ with $i = 0, 1,$ we get that $X \in D_\theta$ if and only if $\text{Hom}_{D^b_{\text{gp}}(A)}(G^*, X[1]) = 0$ if and only if $X \in T(G^*).$
On the other hand, since
\[ \text{Hom}_A(T, X) \cong \text{Hom}_{D^b_{\text{gp}}(A)}(T, X) \cong \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, X), \]
we get that \( X \in T^{\perp 0} \) if and only if \( X \in \mathcal{F}(G^\bullet) \). \( \square \)

**Proposition 3.10.** Let \( G^\bullet \) be a 2-term Gorenstein silting complex in \( D^b_{\text{gp}}(A) \) and \( (T(G^\bullet), \mathcal{F}(G^\bullet)) \) the torsion pair induced by \( G^\bullet \).

1. For any \( X \in \text{mod}A \), \( X \in \text{add}H^0(G^\bullet) \) if and only if \( X \) is Ext-projective in \( T(G^\bullet) \).
2. For any \( X \in T(G^\bullet) \), there is a \( G \)-exact sequence \( 0 \to L \to T_0 \to X \to 0 \) with \( T_0 \in \text{add}H^0(G^\bullet) \) and \( L \in T(G^\bullet) \).

**Proof.** Assume that \( X \) is Ext-projective in \( T(G^\bullet) \). Since \( T(G^\bullet) = \text{Fac}H^0(G^\bullet) \), there is a \( G \)-exact sequence
\[ 0 \to L \to T_0 \xrightarrow{\alpha} X \to 0, \]
where \( T_0 \xrightarrow{\alpha} X \) is a right \( \text{add}H^0(G^\bullet) \)-approximation. Since \( \text{Hom}_A(H^0(G^\bullet), \alpha) \) is an epimorphism, we have that \( \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, \alpha) \) is an epimorphism. Applying \( \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, -) \) to (**) we have an exact sequence
\[ \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, T_0) \xrightarrow{\text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, \alpha)} \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, X) \to \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, L[1]) \to 0. \]
Then \( \text{Hom}_{D^b_{\text{gp}}(A)}(G^\bullet, L[1]) = 0 \) which implies that \( L \) is in \( T(G^\bullet) \). Thus, by assumption, the sequence (**) splits, and hence \( X \) is in \( \text{add}H^0(G^\bullet) \).

By the monomorphism in Lemma 3.7 we have that \( \text{add}H^0(G^\bullet) \) is Ext-projective in \( T(G^\bullet) \). \( \square \)

**Theorem 3.11.** Suppose that \( A \) is a Gorenstein algebra of finite CM-type with the Gorenstein projective generator \( G \), and \( A(\text{Gproj}) = \text{End}_A(E)^{\text{op}} \). Let \( G^\bullet : G_1 \xrightarrow{\theta} G_0 \) be a 2-term complex in \( D^b_{\text{gp}}(A) \), and \( T = H^0(G^\bullet) = \text{Coker}\theta \). Then the following statements are equivalent.

1. \( T \) is a Gorenstein silting module with respect to \( \theta \) in \( \text{mod}A \).
2. \( G^\bullet : G_1 \xrightarrow{\theta} G_0 \) is a 2-term Gorenstein silting complex in \( D^b_{\text{gp}}(A) \).

**Proof.** First, we claim that \( T \) is a partial Gorenstein silting module with respect to \( \theta \) if and only if \( G^\bullet \) is a 2-term partial Gorenstein silting complex.

Assume that \( T \) is a partial Gorenstein silting module. By definition we know that \( \text{Hom}_A(\theta, T) \) is an epimorphism. Then we get that \( \text{Hom}_{A(\text{Gproj})}((E, \theta), (E, T)) \) is an epimorphism from the proof of Proposition 2.10]. Let \( \sigma := \text{Hom}_A(E, \theta) \). By the following diagram,

\[ \text{Hom}_A(E, G_1) \xrightarrow{\sigma} \text{Hom}_A(E, G_0) \]
\[ \text{Hom}_A(E, G_1) \xrightarrow{\sigma} \text{Hom}_A(E, G_0) \xrightarrow{\pi} \text{Hom}_A(E, T) \]
there exists a morphism $g : \text{Hom}_A(E,G_0) \to \text{Hom}_A(E,T)$, such that $f = g\sigma$. Since
\[ \text{Hom}_A(G,G_1) \] is projective, there exists a morphism
\[ h : \text{Hom}_A(E,G_1) \to \text{Hom}_A(E,G_0), \]
such that $f = \pi h$. Similarly since $\text{Hom}_A(E,G_0)$ is projective, there exists a morphism
\[ s_0 : \text{Hom}_A(E,G_0) \to \text{Hom}_A(E,G_0), \]
such that $g = \pi s_0$. Therefore $\pi h = \pi s_0 \sigma$, i.e., $\pi(h - s_0 \sigma) = 0$. It follows that there is
\[ s_1 : \text{Hom}_A(E,G_1) \to \text{Hom}_A(E,G_1), \]
such that $h - s_0 \sigma = s_1$, which shows that $h$ is null-homotopic. This implies that
\[ \text{Hom}_A(G_{\text{proj}})((E,G^\bullet), (E,G^\bullet)[1]) = 0. \]
Therefore, we get that $\text{Hom}_{D_{\text{gr}}(A)}(G^\bullet, G^\bullet[1]) = 0$. This implies that $G^\bullet$ is a 2-term partial Gorenstein silting complex.

Conversely, if $G^\bullet$ is a 2-term partial Gorenstein silting complex, then by the diagram above, we have that $h = s_0 \sigma + s_1$. Then $f = \pi h = \pi s_0 \sigma + \pi s_1 = (\pi s_0) \sigma$, which means that $\text{Hom}_A(G_{\text{proj}})((E,\theta), (E,T))$ is an epimorphism. Therefore $\text{Hom}_A(\theta,T)$ is an epimorphism, and so $T \in D_{\theta}$. This implies that $T$ is a partial Gorenstein silting module with respect to $\theta$.

(1)⇒(2) By Theorem 3.8 we prove that $T(G^\bullet) \cap F(G^\bullet) = 0$ and $H^0(G^\bullet) \in T(G^\bullet)$. From Remark 3.9 we have $T = H^0(G^\bullet) \in D_{\theta} = T(G^\bullet)$. Let $X \in T(G^\bullet) \cap F(G^\bullet) = D_{\theta} \cap T_{\leq 0} = \text{Gen}_C(T) \cap T_{\leq 0}$. Then there is a $G$-epimorphism $T_0 \to X \to 0$ with $T_0 \in \text{Add}T$, and $\text{Hom}_A(T,X) = 0$. Therefore we can get from the induced exact sequence
\[ 0 \to (X,X) \to (T_0,X) \text{ that } X = 0. \]
Thus $G^\bullet : G_1 \to G_0$ is a 2-term Gorenstein silting complex in $D_{\text{gr}}(A)$.

(2)⇒(1) By the above claim, we see that $T$ is a partial Gorenstein silting module, and so $\text{Gen}_C(T) \subseteq D_{\theta}$. From Proposition 3.9, for any $X \in D_{\theta} = T(G^\bullet)$, there is a $G$-exact sequence
\[ 0 \to L \to T_0 \to X \to 0 \]
with $T_0 \in \text{add}H^0(G^\bullet) = \text{Add}T$ and $L \in T(G^\bullet)$. Then we get that $X \in \text{Gen}_C(T)$. Therefore $T$ is a Gorenstein silting module with respect to $\theta$ in mod$A$. \hfill \Box

Let $B = \text{End}_{D_{\text{gr}}(A)}(G^\bullet)^{\text{op}}$. Consider the subcategories of mod$B$
\[ \mathcal{X}(G^\bullet) = \text{Hom}_{D_{\text{gr}}(A)}(G^\bullet, F(G^\bullet)[1]) \text{ and } \mathcal{Y}(G^\bullet) = \text{Hom}_{D_{\text{gr}}(A)}(G^\bullet, T(G^\bullet)). \]
Then we can draw the Brenner-Butler theorem in this setting.

**Theorem 3.12.** Let $G^\bullet$ be a 2-term Gorenstein silting complex in $D_{\text{gr}}(A)$. Then
\[ (\mathcal{X}(G^\bullet), \mathcal{Y}(G^\bullet)) \]
is a torsion pair in mod$B$ and there are equivalences
\[ \text{Hom}_{D_{\text{gr}}(A)}(G^\bullet, -) : T(G^\bullet) \to \mathcal{Y}(G^\bullet), \]
and
\[ \text{Hom}_{D_{\text{gr}}(A)}(G^\bullet, [-1]) : F(G^\bullet) \to \mathcal{X}(G^\bullet). \]
The equivalences send $G$-exact sequences with terms in $T(G^\bullet)$ (resp. $F(G^\bullet)$) to short exact sequences in mod$B$. 
Proof. This follows from Proposition 3.4 (1) and (3), using that \( T(G^\bullet) \cup F(G^\bullet) \subseteq C_{gp}(G^\bullet) \).

We finish this section with an interesting property of the 2-term Gorenstein silting complex over a finite dimensional Gorenstein algebra \( A \) of finite CM-type with the Gorenstein-projective generator \( E \).

Let \( G^\bullet : G_1 \overset{d_1}{\longrightarrow} G_0 \) be the 2-term complex over \( \text{Gproj}A \), and set

\[
P^\bullet : \text{Hom}_A(E, G_1) \longrightarrow \text{Hom}_A(E, G_0).
\]

**Proposition 3.13.** Suppose that \( A \) is a Gorenstein algebra. Then \( G^\bullet \) is a 2-term Gorenstein silting complex in \( D^{b}_{gp}(A) \) if and only if \( P^\bullet \) is a 2-term silting complex in \( D^{b}(A(\text{Gproj})) \).

Proof. Since \( \text{Hom}_A(E, -) \) is a fully faithful functor, we have that

\[
\text{Hom}_{D^{b}(A(\text{Gproj}))}(P^\bullet, P^\bullet[1]) \cong \text{Hom}_{D^{b}_{gp}(A)}(G^\bullet, G^\bullet[1]).
\]

On the other hand, from [GZ], there is a triangle-equivalence \( D^{b}_{gp}(A) \cong D^{b}(A(\text{Gproj})) \) induced by \( \text{Hom}_A(E, -) \). This completes the proof.

3.2. **On global dimension.** In this subsection, we compare the global dimension between \( A \) and \( B \), where \( G^\bullet \) is a 2-term Gorenstein silting complex in \( D^{b}_{gp}(A) \), and \( B = \text{End}_{D^{b}_{gp}(A)}(G^\bullet)^{op} \).

Recall from [IY], for full subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( D^{b}_{gp}(A) \), denote

\[
\mathcal{X} \ast \mathcal{Y} := \{ Z \in D^{b}_{gp}(A) \mid \text{there exists a triangle } X \longrightarrow Z \longrightarrow Y \longrightarrow X[1] \text{ in } D^{b}_{gp}(A) \text{ with } X \in \mathcal{X} \text{ and } Y \in \mathcal{Y} \}.
\]

By the octahedral axiom, we have that \((\mathcal{X} \ast \mathcal{Y}) \ast Z = \mathcal{X} \ast (\mathcal{Y} \ast Z).\) Call \( \mathcal{X} \) extension closed if \( \mathcal{X} \ast \mathcal{X} = \mathcal{X} \). Now fix \( GP = \text{add} G^\bullet \) and \( GP_c = GP \cap C_{gp}(G^\bullet) \).

**Lemma 3.14.** \( C_{gp}(G^\bullet) \subseteq GP \ast GP[1] \ast \cdots \ast GP[d + 1] \) for some non-negative integer \( d \).

Proof. Note that

\[
C_{gp}(G^\bullet) \subseteq D^{\leq 0}_{gp}(G^\bullet) \subseteq GP \ast GP[1] \ast \cdots \ast GP[l - 1] \ast GP[l]
\]

for some \( l > 0 \). For any \( X^\bullet \in C_{gp}(G^\bullet) \), we have \( H^i(X^\bullet) = 0 \) for \( i \neq -1, 0 \). Taking a projective resolution \( P^\bullet \) of \( X^\bullet \), then there exists some non-negative integer \( d \) such that \( H^i(P^\bullet) = 0 \) for \( i > 0 \) or \( i < -d - 1 \). Therefore

\[
\text{Hom}_{D^{b}_{gp}(A)}(X^\bullet, G^\bullet[i]) = 0, \quad i \geq d + 2,
\]

which implies that \( X^\bullet \in GP \ast GP[1] \ast \cdots \ast GP[d + 1] \).

**Lemma 3.15.** For the complex \( X^\bullet \in C_{gp}(G^\bullet) \cap (GP_c \ast GP_c[1] \ast \cdots \ast GP_c[m]) \) for some \( m \geq 0 \), we have \( \text{pd}\text{Hom}_{D^{b}_{gp}(A)}(G^\bullet, X^\bullet)_B \leq m \).
Proof. Let $X^*_0 = X^*$. There are triangles

$$X^*_{i+1} \to O^*_i \overset{g_i}{\to} X^*_i \to X^*_i[1], \quad 0 \leq i \leq m - 1$$

where $O^*_i \in \text{GP}_c$ and $X^*_i \in \text{GP}_c \ast \text{GP}_c[1] \ast \cdots \ast \text{GP}_c[m - i]$. Since $\text{Hom}_{D^b(A)}(G^*, G^*[i]) = 0$ for all $i > 0$, we have that $g_i$ is a right GP-approximation of $X^*_i$. Then we get the following induced exact sequence

$$\text{Hom}_{D^b(A)}(G^*, X^*_{i+1}) \to \text{Hom}_{D^b(A)}(G^*, O^*_i) \to \text{Hom}_{D^b(A)}(G^*, X^*_i) \to 0.$$

Then we get that

$$\text{pdHom}_{D^b(A)}(G^*, X^*_i)_B \leq \text{pdHom}_{D^b(A)}(G^*, X^*_{i+1})_B + 1.$$

Therefore $\text{pdHom}_{D^b(A)}(G^*, X^*)_B \leq \text{pdHom}_{D^b(A)}(G^*, X^*_m)_B + m = m$. □

**Theorem 3.16.** Assume that $A$ has Gorenstein dimension $d$ for some positive integer $d$. Then $\text{gldim} B \leq d + 1$.

Proof. If $G^*$ is Gorenstein tilting, then $G^* \in \mathcal{C}_{gp}(G^*)$. Therefore we have that $\text{GP} = \text{GP}_c$.

It follows from Lemmas 3.14 and 3.15 that $\text{gldim} B \leq \text{Gdim} A + 1$. □

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