Pushing the Efficiency-Regret Pareto Frontier for Online Learning of Portfolios and Quantum States

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Abstract

We revisit the classical online portfolio selection problem. It is widely assumed that a trade-off between computational complexity and regret is unavoidable, with Cover’s Universal Portfolios algorithm, SOFT-BAYES and ADA-BARRONS currently constituting its state-of-the-art Pareto frontier. In this paper, we present the first efficient algorithm, BISONS, that obtains polylogarithmic regret with memory and per-step running time requirements that are polynomial in the dimension, displacing ADA-BARRONS from the Pareto frontier. Additionally, we resolve a COLT 2020 open problem by showing that a certain Follow-The-Regularized-Leader algorithm with log-barrier regularization suffers an exponentially larger dependence on the dimension than previously conjectured. Thus, we rule out this algorithm as a candidate for the Pareto frontier. We also extend our algorithm and analysis to a more general problem than online portfolio selection, viz. online learning of quantum states with log loss. This algorithm, called SCHRÖDINGER’S-BISONS, is the first efficient algorithm with polylogarithmic regret for this more general problem.

Keywords: Portfolio Management, Online Learning, Quantum Learning

1. Introduction

We study the classical online portfolio selection problem (Cover, 1991). In this problem, there are $d$ assets (e.g. stocks) that an investor can invest money in on any given day. On each day, indexed by $t = 1, 2, \ldots, T$, the investor can choose a portfolio over the $d$ assets, which is a distribution of their wealth on the assets, after observing the returns (i.e. ratio of closing price to opening price) of the assets on the previous day. The goal is to compete with the best constant-rebalanced portfolio (CRP) in hindsight, which redistributes wealth on each day to maintain a fixed proportion in each asset. Importantly, we study the case without assumptions on the quality of the returns, i.e. any individual asset might suffer a total loss at any time. On any day, the wealth of the investor increases by a factor equal to the inner product between the portfolio chosen by the investor and the vector of returns for the $d$ assets. The goal is to develop algorithms that minimize the investor’s regret, which is the difference between the logarithm of the total wealth earned by the investor after the $T$ days (starting with an initial wealth of $1$), and the logarithm of the total wealth earned by the best CRP in hindsight. Equivalently, the online portfolio selection problem can be seen as an instance of online convex optimization (OCO), where the loss is the negative logarithm of the inner product between the portfolio and the returns vector.

The online portfolio selection problem can be seen as a special case of a more general problem, viz. online learning of quantum states with log loss. In this problem, the goal is to learn to predict the outcome of a sequence of two-outcome measurements of an unknown quantum state on $\log_2(d)$ qubits. Without going into quantum computing jargon (we refer the reader to (Aaronson et al.,

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and Appendix A.1 for a more detailed discussion of the setting), this online learning problem can be specified as follows. In each time step the learner constructs a quantum state, which is a $d \times d$ positive semidefinite Hermitian matrix of trace 1, and in response, receives a two-outcome measurement, which is a $d \times d$ Hermitian matrix with eigenvalues in $[0, 1]$. The loss of the learner is the negative logarithm of the trace product between the quantum state generated by the learner and the measurement. The trace product can be interpreted as a probabilistic prediction of observing one of two outcomes in the measurement, and hence it is natural to use the log loss for measuring the quality of the prediction. The goal is to minimize regret with respect to the best quantum state in hindsight. It is easy to see that the online portfolio selection problem is exactly the special case of this problem where both the quantum state and loss matrices are restricted to be diagonal matrices. Aaronson et al. (2018) developed regret minimizing algorithms for Lipschitz loss functions of the trace product – in particular, the natural log loss setting was not handled by their algorithms.

Our first main contribution is the development of new algorithms, BISONS for the online portfolios problem and SCHRÖDINGER’S-BISONS for the quantum learning problem, with regret bounds of $O(d^2 \log^2(T))$ and $O(d^3 \log^2(T))$ respectively, and $\tilde{O}(\text{poly}(d))$ \footnote{The $\tilde{O}(\cdot)$ notation suppresses polylogarithmic dependence on $T$ and $d$.} per-iteration running time. This result is noteworthy for two reasons. BISONS is the first algorithm that enjoys polylogarithmic regret with $\tilde{O}(\text{poly}(d))$ memory and running time per-iteration, and we show that the quantum learning problem is only slightly harder than the online portfolios problem. Technically, the BISONS algorithm operates in epochs (inspired by the ADA-BARRONS algorithm of Luo et al. (2018)), with each epoch running a Follow-The-Regularized Leader (FTRL) algorithm with quadratic surrogate losses using the log-barrier regularizer, with an additional linear bias term added to the surrogate loss. The linear bias term is crucial to the analysis and ensures that the regret within any epoch is non-positive, while the final epoch incurs polylogarithmic regret.

Extending the algorithm and its analysis to the quantum learning problem presents several technical challenges. First, the non-commutativity of the matrices involved makes the construction of the linear bias term non-trivial; we use semidefinite programming duality to design the linear term. Second, since the matrices are complex and Hermitian, standard convex analysis machinery such as gradients, Hessians and the intermediate value theorem need to be custom developed for the analysis. As observed earlier, the portfolios problem is a special case of the quantum learning problem when the matrices are all diagonal, and in this case SCHRÖDINGER’S-BISONS collapses to BISONS. Hence, we only give a regret bound analysis for SCHRÖDINGER’S-BISONS using the machinery developed; the bound for BISONS follows automatically.

Our second main contribution is that we provide novel insights about a certain natural FTRL algorithm for the online portfolios problem. Van Erven et al. (2020) conjectured, in a COLT 2020 open problem, that FTRL with log-barrier regularization (denoted LB-FTRL) obtains the optimal $O(d \log(T))$ regret bound. If this is true, this would provide the first (semi-)efficient algorithm with optimal regret. We resolve the COLT 2020 open problem by disproving this conjecture with a lower bound of $\Omega(2^d \log(T) \log \log(T))$ on the regret of the LB-FTRL algorithm. This result effectively removes the LB-FTRL algorithm as a candidate for an optimal trade-off between complexity and regret, since our algorithm obtains superior regret (when $T \leq \exp \exp(d)$) at a significantly better run-time and memory complexity.

Related work. The classical online portfolios has a rich literature starting with Cover (1991), who presented the Universal Portfolios algorithm with optimal regret. However, its fastest known imple-
mentation (Kalai and Vempala, 2000) requires $O(T^2(T + d)^2)$ average per-step computation. Motivated by this inefficiency, early work (Agarwal et al., 2006; Hazan et al., 2007; Hazan and Kale, 2015) developed very efficient second order algorithms – the primary one being Online Newton Step (ONS) – for this problem, under the assumption that the returns of any stock are bounded away from 0 on any day. This assumption translates to a bound $G$ on the gradient of the loss function. ONS obtains $O(Gd\log(T))$ regret at a per-step computational complexity of $\tilde{O}(d^3)$. Simpler first order methods based on online gradient descent (Zinkevich, 2003) or multiplicative weights update (Helmbold et al., 1998) can also be applied to the problem, obtaining regret bounds of $O(G\sqrt{T}\log(d))$ and $O(G\sqrt{T})$ respectively, at a per step complexity of $\tilde{O}(d)$.

Since Cover’s original work did not have a dependence on $G$, recent work has focused on overcoming the dependency on $G$ via both first and second order methods. The SOFT-BAYES algorithm (Orseau et al., 2017) is a first order method that obtains $O(\sqrt{dT\log(d)})$ regret, while preserving linear run-time in $d$. ADA-BARRONS (Luo et al., 2018) is a second order method based on ONS and achieves $O(d^2\log^4(T))$ regret. However, it requires computing the solution of log-barrier FTRL at any point, which increases its per-step complexity to $\tilde{O}(d^{2.5}T)$.

The tradeoff between regret and computational complexity described above is plotted schematically in Figure 1. Characterizing the Pareto frontier of this tradeoff has been a subject of study over two decades. In particular, special attention has been given to the log-barrier FTRL algorithm (Agarwal and Hazan, 2005), which obtains a regret of $O(\min\{G^2d\log(T), d\log^4(T)\})$, but has been conjectured to obtain the optimal $O(d\log(T))$ regret by Van Erven et al. (2020).

The online learning of quantum states problem has a shorter history, being introduced by Aaronson et al. (2018). While the log loss version of the problem hasn’t been studied before, it is easy to see that the log loss is 1-mixable (Vovk, 1995), and hence Vovk’s Aggregating Algorithm can be applied.
to the problem to obtain an algorithm with $O(d^2 \log(T))$ regret – in fact, this algorithm exactly coincides with Cover’s Universal Portfolios algorithm in the online portfolio setting. Implementing this algorithm however is computationally rather inefficient.

**Notation.** For a natural number $d$ we define $[d] := \{1, 2, \ldots, n\}$, and $\Delta([d])$ to be the set of distributions over $[d]$, seen as vectors in $\mathbb{R}^d$. We denote the set of $d \times d$ Hermitian matrices by $\mathcal{H}^d$. Through the paper $\| \cdot \|_p$ denotes the $\ell_p$ norm. Given a vector $v$ and a positive semi-definite matrix $M$, we define the semi-norm $\|v\|_M := \sqrt{\text{Tr}(v^* M v)}$. Given two Hermitian matrices $X, Y$ we define the standard inner product (which is always a real number) between them as $\langle X, Y \rangle := \text{Tr}(X^* Y) = \text{Tr}(XY)$.

We define additional notation required for the analysis of the quantum learning problem in the Appendix C. We use the acronyms PSD for positive semi-definite Hermitian matrices and PD for positive definite Hermitian matrices. In general, throughout the paper we denote matrices with capital letters and vectors by small letters. When denoting functions, capital letters are reserved for functions that are defined as sums of functions.

## 2. Problem setting

**Online Optimal Portfolio:** The agent interacts with the environment in finite time-steps $t = 1, \ldots, T$. At any time-step, the agent picks a portfolio distribution $x_t \in A = \Delta([d])$, observes a non-negative returns vector $r_t \in \mathbb{R}^d_+$ and suffers the log loss

$$f_t(x_t) = f(x_t; r_t) := -\log(\langle x_t, r_t \rangle).$$

Since multiplicative scaling of $r_t$ shifts the loss by a constant independent of $x_t$, the regret is unchanged if we scale $r_t$ so that it lies in $A$. The goal of the agent is to minimize its regret, defined as the cumulative loss compared to the best static action in hindsight.

$$\text{Reg} = \max_{u \in A} \text{Reg}(u) = \max_{u \in A} \sum_{t=1}^T (f_t(x_t) - f_t(u)).$$

(1)

**Quantum Learning with Log Loss:** This problem generalizes the online optimal portfolios problem as follows. The agent’s action set is $\mathcal{A} := \{X | X \in \mathcal{H}^d_+, \text{Tr}(X) = 1\}$. The agent at every round picks a PSD Hermitian matrix $X_t \in \mathcal{A}$, observes a PSD loss matrix $R_t$, which is assumed to be in $\mathcal{A}$ as in the portfolios case, and suffers the log loss

$$f_t(X_t) = f(X_t; R_t) := -\log(\langle X_t, R_t \rangle).$$

The task of the agent is to minimize regret defined analogously to (1). In the Appendix A.1, we show that the above problem formulation captures problem of online learning of quantum states with log loss as described in Aaronson et al. (2018).

## 3. Algorithm

In this section, we present our main algorithm BISONS (Algorithm 1). The algorithm is inspired by the algorithm ADA-BARRONS proposed by Luo et al. (2018), but improves the regret bound.
Algorithm 1: BISONS

input: $T$, $B$, $\eta$, $\beta$.
initialize: $\forall e \in \mathbb{N}: p^e_0 = d1, G^e_0(\cdot) = \hat{F}^e_0(\cdot) = \eta^{-1}R(\cdot), x^e_\tau = u^e_1 = \arg \min_{x \in A} G^e_0(x)$.

$e \leftarrow 1, \tau \leftarrow 1$

for $t = 1, \ldots$ do

$\hat{f}_t \leftarrow$ receive from playing $x_t \leftarrow x^e_\tau$.

$\hat{F}_\tau = \hat{f}_t \leftarrow$ construct according to (2).

$\hat{F}^e_\tau = \hat{F}^e_{\tau-1} + \hat{f}_\tau$

$G^e_\tau = G^e_{\tau-1} + g^e_\tau$, where $g^e_\tau(x) := \hat{f}^e_\tau(x) - \langle x, p^e_\tau - p^e_{\tau-1} \rangle B$

$x^e_{\tau+1} \leftarrow \arg \min_{x \in A} G^e_\tau(x), u^e_{\tau+1} \leftarrow \arg \min_{x \in A} \hat{F}^e_\tau(x)$

$\forall i \in [d]: p^e_{\tau+1,i} = \max\{p^e_{\tau,i}, x^e_{\tau+1,i}^{-1}\}$

if $\exists i: (2(1 + 6\eta)\beta)u^e_{\tau+1,i} \geq (p^e_{\tau+1,i})^{-1}$ then

$e \leftarrow e + 1, \tau \leftarrow 1$ // Reset the algorithm

else

$\tau \leftarrow \tau + 1$

end

end

obtained by Luo et al. (2018) by a factor of $\log^2(T)$, while simultaneously and more importantly improving the run-time by factors polynomial in $T$. BISONS is the first algorithm with constant per-step computational complexity that obtains polylogarithmic regret in the portfolio problem.

The algorithm operates in epochs, where each epoch ends when either the global time reaches $T$ or when a certain reset condition (detailed below) is met. We call an epoch completed if it ends by reset, which sets the internal time $\tau$ of the algorithm back to 1 and lets the algorithm forget all history. Thus, we keep only one copy of all parameters in memory and reset them to the initial values when the epoch is completed.

Let $T_1 \ldots T_E \in [1, T]$ denote the timesteps following a restart trigger event. By convention we set $T_0 = 1$ and $T_{E+1} := T + 1$. We define an epoch $\{E_i\}$ of the algorithm as the period between successive resets of the algorithm, i.e. $E_i := [T_i, T_{i+1} - 1]$. Note that by definition there is no restriction over the length of these epochs and they can be of variable lengths.

On a high level, BISONS works by approximating at every step, the true loss function $f_t(x)$ by a quadratic surrogate loss

$$\hat{f}_t(x) := f_t(x_t) + \langle x - x_t, \nabla f_t(x_t) \rangle + \frac{\beta}{2} \langle x - x_t, \nabla f_t(x_t) \rangle^2,$$

(2)

where $\beta \leq 1$ is an input parameter to the algorithm. Let $e, \tau$ be the epoch and internal time of the algorithm at time $t$, then we define $x_t = x^e_\tau$ and $\hat{f}^e_\tau = \hat{f}_t$. For reasons that become clear in section 4, BISONS further augments the above surrogate loss with a linear bias term, defined at every internal step $\tau$ as

$$g^e_\tau(x) := \hat{f}^e_\tau(x) - \langle x, p^e_\tau - p^e_{\tau-1} \rangle B,$$

(3)
where \( \{p^e_\tau \in \mathbb{R}^d\} \) is an auxiliary sequence maintained by the algorithm and \( B \) is a bias scaling factor which is a parameter input to the algorithm. To produce the output \( x^e_\tau \) BISONS runs FTRL over the biased surrogate losses, i.e.

\[
x^e_\tau := \arg \min_{x \in A} \sum_{s=1}^{\tau-1} g^e_s(x) + \eta^{-1} R(x),
\]

where \( \eta \) is a learning rate parameter and \( R(x) := -\sum_{i=1}^d \log(x_i) \) is the log-barrier regularization. The algorithm further maintains a reference solution \( u^e_\tau \) by running FTRL over the surrogate losses without bias,

\[
u^e_\tau := \arg \min_{x \in A} \sum_{s=1}^{\tau-1} \hat{f}^e_s(x) + \eta^{-1} R(x).
\]

Further, the asset dependent bias \( p \) is updated according to

\[
\forall i \in [d] : p^e_{\tau,i} = \max\{p^e_{\tau-1,i}, x^e_{\tau,i} - 1\}.
\]

Finally, the algorithm is reset (i.e. the bias vector \( p \) is reset and all previous losses are discarded) whenever

\[
\exists i \in [d] : u^e_{\tau+1,i} > \frac{1}{2(1+6\eta)\beta^2} (p^e_{\tau+1,i})^{-1}.
\]

The following theorem and corollary capture our main regret bound for BISONS. We show that the total regret in any completed epoch is always non-positive and the total regret in the last uncompleted epoch is bounded. Summing the regrets over individual epochs (which is only an over-estimation of the true regret) gives the final result.

**Theorem 1** Assuming \( T \geq 110d^2 \), setting the input parameters as \( B = \frac{264}{B} d \log(T), \eta = \frac{1}{4B}, \beta = \frac{11}{18B} \), we have that the regret of BISONS over a completed (i.e. end triggered by the reset condition) epoch against any comparator \( u : \min_i u_i \geq T^{-1} \) is non-positive. Further, for the epoch that runs until the end of time \( T \), the regret is bounded by \( \mathcal{O}(d^2 \log^2(T)) \).

The proof is given in Appendix D, a sketch is provided at the end of Section 4. The following corollary is immediate:

**Corollary 2** Assuming \( T \geq 110d^2 \), the total regret of BISONS with parameters from Theorem 1 is bounded by \( \mathcal{O}(d^2 \log^2(T)) \).

**Runtime:** Note that BISONS only uses quadratic functions (\( \hat{f}_t \) and \( g_t \)) and therefore a succinct representation of these functions can be maintained in time \( \tilde{O}(d^2) \) in each iteration. Further it can be seen that the constrained minimization up to a sufficient accuracy can also be carried out in \( \tilde{O}(\text{poly}(d)) \) time (see Appendix E for details from the more general quantum learning perspective).
3.1. Extension to Quantum Learning

In this section, we describe the Schrödinger’s-BISONS algorithm (formally defined in the appendix as Algorithm 2) for the quantum learning problem. Schrödinger’s-BISONS follows the same structure as BISONS, and uses the same choice of surrogate function \( f_t \), point played \( X_t \), and comparator \( U_t \) as in online optimal portfolio which are still well defined by (2), (4) and (5) respectively.

We highlight the main differences from the online optimal portfolio in this section. The main differences between the two cases firstly is that the regularizer \( R \) used is the log-det-barrier, which reduces to the log-barrier for diagonal matrices: \( R(X) = -\log \det(X) \). Secondly, and the primary non-trivial step in the generalization, is the appropriate definition of the biases \( P_t \) and the reset condition. Analogous to Algorithm 1, the reset condition is generalised to \( U_t^{e} \preceq \frac{1}{2t(1+\log t)}[P_t^e]^{-1} \), for some biases \( P_t^e \) ensuring \( P_t^e \succeq [X_t^e]^{-1} \) for all \( s,t \) in the same epoch with \( s \leq t \). This ensures that within any epoch \( f_t^e \) stays a valid lower bound for the comparator \( U_t^e \) for that epoch. This property is summarized as Lemma 28 in the appendix.

The main hurdle for extending our results to the quantum setting is to find a suitable bias rule \( P_t^e \) that generalises (6). The goal is to construct \( P_t^e \) that satisfies \( P_{t+1}^e \succeq P_t^e \) and \( P_t^e \succeq [X_t^e]^{-1} \). Unlike in the online optimal portfolio case, there is no canonical “smallest” \( P_t^e \) with that property in general. Instead we choose to look for a choice satisfying these constraints that suffers a small cost of bias.

\[
P_{t+1}^e = P_t^e + [X_{t+1}^e]^{-\frac{1}{2}} \left( I_d - [X_{t+1}^e]^\frac{1}{2} P_t^e [X_{t+1}^e]^\frac{1}{2} \right)_+ \cdot [X_{t+1}^e]^{-\frac{1}{2}}, \tag{7}
\]

where \((\cdot)_+\) is the operator that sets all negative eigenvalues to 0, i.e. if \( M \) is a Hermitian matrix with eigendecomposition \( M = U^*PU + V^*NV \), where \( P \) and \( N \) are diagonal matrices with the non-negative and negative eigenvalues respectively, then \( M_+ = U^*PU \).

**Remark 3** For diagonal matrices, (7) picks \( P_{t}^e(i,i) = \max\{[X_{t}^e]^{-1}(i,i), P_{t-1}^e(i,i)\} \) and is hence a strict generalization of (6).

Surprisingly, we show in the appendix that the cost of bias remains \( O(d \log(T)B) \), so we do not pay anything for this generalization. We note that relying on the “negative regret by linear bias” technique used here is crucial towards obtaining this generalization. It is not clear how to use the “negative regret by increasing learning rate” approach used in ADA-BARRONS here. We now state the theorem governing the regret for Schrödinger’s-BISONS.

**Theorem 4** Assuming \( T \geq 110d^2 \), setting \( B = \frac{2d^4}{\alpha} \log(T), \eta = \frac{1}{\alpha^2}, \beta = \frac{11d}{T} \), the regret of Schrödinger’s-BISONS over a single epoch against any comparator \( U \succeq T^{-\frac{1}{2}}I_d \) is non-positive if the end is triggered by the reset condition. Otherwise, if the algorithm runs until the end of time \( T \), then the regret is bounded by \( O(d^3 \log^2(T)) \).

Theorem 4 can be used to prove the following regret bound for Schrödinger’s-BISONS yields the following corollary analogous to Corollary 2. Missing proofs are in Appendix D.

**Corollary 5** For \( T \geq 110d^2 \), the regret of Schrödinger’s-BISONS is bounded by \( O(d^3 \log^2(T)) \).

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3. See Section 4 for an explanation of what cost of bias means and how it shows up in the analysis.
4. Overview of the Analysis

Intuition for the regret bound. Using quadratic surrogate losses instead of the true losses is a standard technique for improving computation complexity while preserving logarithmic regret (see the Online Newton Step (ONS) method from Hazan et al. (2007)). We use the same quadratic surrogate \( \hat{f}_t \) as the ONS method (with a different choice of \( \beta \)). Such analyses including ONS often require that the surrogate is a lower bound for the function value of the comparator \( u \), i.e. \( \hat{f}_t(u) \leq f_t(u) \) at all time-steps. Since \( u \) is unknown, this is typically enforced by ensuring lower boundedness over the entire domain. However in the case of optimal portfolio, a uniform lower bound requires \( f \) quadratic surrogate losses \( \hat{f} \) as a function over the surrogate losses as comparator for the reset condition. This computation is as costly as \( x_t \), which can be done in \( O(d^2 \log(T)) \) arithmetic operations, in contrast to \( O(dT) \) required by previous algorithms with optimal regret, e.g. ADA-BARRONS (Luo et al., 2018). We first setup some

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\[ \sum_{t=1}^{\tau} \left( \hat{f}_t(x_t) - \hat{f}_t(u) \right) = \sum_{t=1}^{\tau} \left( g_t(x_t) - g_t(u) + \langle x_t - u, p_t - p_{t-1} \rangle B \right) \]

\[ = \text{Reg}_g(u) + \sum_{t=1}^{\tau} \left( \langle x_t, p_t - p_{t-1} \rangle B - \langle u, p_{\tau} - p_0 \rangle B \right). \]

The FTRL regret over the sequence of functions \( g_t \) is bounded via ONS analysis. Further recall that the bias parameters \( p_t \) satisfy for all \( i, p_{ti} = \max_{s \leq t} x^{-1}_{ti} \). Therefore for all \( t, i, p_{ti} - p_{t-1,i} \neq 0 \) implies \( x_{ti} = p^{-1}_{ti} \). We can now bound the cost of bias is any epoch by

\[ \sum_{t=1}^{\tau} \langle x_t, p_t - p_{t-1} \rangle B = \sum_{i=1}^{d} \sum_{t=1}^{\tau} p^{-1}_{ti}(p_{ti} - p_{t-1,i})B \leq \sum_{i=1}^{d} \log(p_{\tau i}/d)B. \]

We show in our analysis that \( p_{ti} \leq T^2 \) at all time-steps, so this term is bounded by \( O(d \log(T)B) \). If a reset is triggered at timestep \( \tau \), then by the reset condition we have for the comparator \( u_{\tau} \) (maintained by the algorithm), \( \exists i \in [d] : u_{\tau i}p_{\tau i} = \Omega(\beta^{-1}). \) Hence the negative regret is of order \( \Omega(d^2) \), which is, given the right tuning, significantly larger than the cost of bias. We argued the above for the comparator \( u_{\tau} \) maintained by the algorithm, which is the FTRL solution of the quadratic surrogate losses \( \hat{f}_t \). This choice of comparator is the core reason behind our runtime improvement. We now explain why this works.
auxiliary notation to simplify our argument. Let $\ell_t(x) = \langle x, r_t \rangle$ be the linear reward at time $t$, then we can rewrite $f_t = h \circ \ell_t$ and $\hat{f}_t = \hat{h}_t \circ \ell_t$, where $y_t := \ell_t(x_t)$ and $h(x), \hat{h}_t(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ are functions defined as

$$h(x) := -\log(x), \quad \hat{h}_t(x) := h(y_t) + (x - y_t)h'(y_t) + \frac{\beta}{2}(x - y_t)^2h'(y_t)^2.$$  

Note that both $h, \hat{h}_t$ are convex functions. We now define an additional function $\hat{f}_t = \hat{h}_t \circ \ell_t$, with $\hat{h}_t(x) = \hat{h}_t(x) x \leq \beta^{-1}y_t$, and $\hat{h}_t(\beta^{-1}y_t) + (x - \beta^{-1}y_t)\hat{h}'(\beta^{-1}y_t)$ otherwise. Geometrically $\hat{h}_t$ coincides with $\hat{h}_t$ for $x$ up to $\beta^{-1}y_t$ and follows its linear extension at $x = \beta^{-1}y_t$ afterwards (see Figure 2). From the convexity of $\hat{h}_t$, it follows that both $\hat{h}_t$ and $\hat{f}_t$ are convex. Furthermore as shown by the following lemma, it holds that $\hat{h}_t$ is a proper lower approximation of $h$ and therefore $\hat{f}_t$ is a proper lower approximation of $f$.

**Lemma 6** For all $x \in (0, \infty) : \hat{h}_t(x) \leq h(x)$, where equality holds for $x = y_t$.

The proof can be found in Appendix D. We have introduced the function $\hat{f}_t$ merely as a tool for the analysis. An important invariant of our algorithm that our reset condition ensures is:

**Lemma 7** Let $\eta \leq \min\{\frac{1}{\beta r^2} \frac{\beta}{1+\beta}, \frac{1}{\beta r}\}$. Consider any epoch $e$ with the reset points $T_{e-1} < T_e \leq T$. Let $L$ represent the length of the epoch, i.e. $L = T_e - T_{e-1}$, we have that, it holds that

$$\min_{x \in \mathcal{A}} \sum_{\tau = 1}^{L} \hat{f}_\tau^e(x) + \eta^{-1}R(x) = \sum_{\tau = 1}^{L} \hat{f}_\tau^e(u_{\tau+1}) + \eta^{-1}R(u_{\tau+1}).$$

While we defer the proof to Appendix D, the high level idea is that the reset condition ensures that $\ell_t(u_{\tau+1}) \leq \beta^{-1}y_s$ for all $s \leq \tau$. That means that the LHS is equal to the RHS around $u_{\tau+1}$. Since $u_{\tau+1}$ by definition is the minimizer of the RHS (which is a strictly convex function), hence it is a local and thereby due to convexity, a global minimizer of the LHS. We are now ready to provide a full proof sketch for Theorem 1.

**Proof sketch of Theorem 1.** Let $\tau$ denote the last time-step of any particular epoch. Then

$$\text{Reg}(u) = \sum_{t=1}^{\tau} (f_t(x_t) - f_t(u)) \leq \sum_{t=1}^{\tau} (\hat{f}_t(x_t) - \hat{f}_t(u)) \quad \text{(by Lemma 6)}$$

$$\leq \max_{u' \in \mathcal{A}} \left( \sum_{t=1}^{\tau} (\hat{f}_t(x_t) - \hat{f}_t(u')) - \eta^{-1}R(u') + \eta^{-1}R(u) \right)$$

$$= \sum_{t=1}^{\tau} (\hat{f}_t(x_t) - \hat{f}_t(u_{\tau+1})) - \eta^{-1}R(u_{\tau+1}) + \eta^{-1}R(u) \quad \text{(by Lemma 7)}$$

$$= \text{Reg}_g(u_{\tau+1}) - \eta^{-1}R(u_{\tau+1}) + \sum_{t=1}^{\tau} \langle x_t - u_{\tau+1}, p_t - p_{t-1} \rangle B + \eta^{-1}R(u).$$
We show in the detailed proof that the FTRL regret over $g$ is bounded by $O\left(\frac{4}{\eta} \log(T)\right)$ and the regularizer is bounded by $O\left(\frac{4}{\eta} \log(T)\right)$ due to the constraint on $u$. As discussed before, the cost of bias is bounded by $O(d \log(T)B)$ and the negative regret in case a reset is triggered is of order $\Omega\left(\frac{B^2}{\eta}\right)$. Set $\beta = \Theta(\eta) = \Theta\left(\frac{1}{B}\right)$, then the regret is bounded by

$$\text{Reg}(u) = O(d \log(T)B) - \Omega(B^2)\mathbb{I}\{\text{reset triggered}\}.$$ 

Finally tuning $B = \Theta(d \log(T))$ completes the proof. 

**Comparison with ADA-BARRONS (Luo et al., 2018)** ADA-BARRONS uses the same surrogate loss as us, but computes $x_t$ via online mirror descent (OMD) updates with increasing learning rate. This technique is closely related to using linear biases (see Foster et al. (2020) for a detailed discussion), however as we show via our application to the quantum learning problem (See Section 3.1), the latter is more flexible and additionally saves a $\log(T)$ factor in the regret. ADA-BARRONS does not use a fixed $\beta$ but instead doubles the parameter $\beta_e$ with every reset. They ensure bounded regret by tuning the negative regret of phase $e$, such that it cancels the $\text{Reg}_g$ term of the next phase $e+1$. Additionally, they show that the total number of epochs is bounded by $\log(T)$. We go a step further and not only cancel the $\text{Reg}_g$ term, but all positive regret contributions. This allows us to use a fixed $\beta$ and saves another $\log(T)$ factor in the regret. Finally, our algorithm uses the FTRL solution over surrogate losses instead of the FTRL solution over the true losses for the comparator $u_t$ as run by ADA-BARRONS. This is made possible via the introduction of the auxiliary functions $f_t$ combined with Lemma 7 and yields the improvement in computational complexity.

5. **Lower bound for FTRL**

In this section, we disprove a COLT 2020 conjecture (Van Erven et al., 2020) regarding FTRL for the online portfolio selection problem. Throughout this section, we consider FTRL with regularizer $R(x) = -\sum_{i=1}^{d} \log(x_i)$, simply referred to as LB-FTRL. In round $t$, this algorithm plays $x_t := \arg\min_{x \in A} F_t(x)$, where $F_t(t) := \eta^{-1} R(x) + \sum_{t'=1}^{t-1} f_t(x)$ and $\eta > 0$ is a constant hyperparameter. This is in some sense a natural choice, since the adversary can “force” the player to operate with this regularization by picking $r_i = e_i$ for $i \in [d]$. Indeed Van Erven et al. (2020) conjectured that FTRL obtains the optimal bound of $O(d \log(T))$, while we prove an exponentially worse lower bound of $\Omega(2^d \log(T) \log \log(T))$. Our main theorem, stated in a slightly abstract fashion for notational convenience, is the following (all missing proofs appear in Appendix F):

**Theorem 8** Let $T > 0$ and let $t_1, \ldots, t_T$ and $o_1, \ldots, o_T$, be sequences of target vectors and associated returns vectors in $\Delta([d])$, which satisfy $\forall j < i : \langle t_i, o_j \rangle = \Omega(1/\text{Poly}(d))$, and $\forall i : \langle t_i, o_i \rangle = 0$, then there exists $T_0 = \text{Poly}(T, d)$, such that for any $T > T_0$ the regret of LB-FTRL against the sequence of reward vectors $r_t$ generated by Algorithm 3 (Appendix F) is lower bounded by

$$\text{Reg} = \Omega(T \log(T) \log \log(T)).$$

**Remark.** This lower bound extends easily to the quantum version of LB-FTRL which uses the log-det regularizer via the observation that when all the loss matrices $R_t$ are diagonal, log-det regularized LB-FTRL reduces to vanilla (log barrier regularized) LB-FTRL.
Lower bound proof sketch. First, we note that the action set $\Delta([d])$ lies in a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. For technical reasons, it will be convenient to work with a full dimensional action set with non-zero volume. Hence, we define the projection operator $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ with kernel $c = \frac{1}{d}1_d$ and $\Pi^{-1}$ its inverse mapping into $A$. Thus $A$ gets mapped to $\Pi A$, which has non-zero volume in $\mathbb{R}^{d-1}$. In a slight overload of notation, we consider $f^\Pi(\tilde{x}; y)$ as a function with argument $\tilde{x} \in \Pi A$ by the identity

$$f^\Pi(\tilde{x}; r) = f(\Pi^{-1}\tilde{x}; r) = -\log(\langle \Pi^{-1}\tilde{x}, r \rangle) = -\log(1/d + \langle \tilde{x}, \Pi r \rangle),$$

and use $\nabla f_t(x) = \nabla f^\Pi(\Pi x; r_t)$, $\nabla^2 f_t(x) = \nabla^2 f^\Pi(\Pi x; r_t)$ as shorthand notation for the gradient and Hessians with respect to the above definition of $f^\Pi$. We define $\nabla f_t(x)$ and $\nabla^2 f_t(x)$ analogously.

The lower bound rests on the following key lemma, which shows that the regret of LB-FTRL is lower bounded by a certain quantity which also appears in the upper bound for FTRL in the standard analysis; so this quantity controls the regret tightly. This is, to the best of our knowledge, a novel idea and is crucial in showing that LB-FTRL does not obtain $\tilde{O}(d \log(T))$ regret in the portfolio problem.

**Lemma 9** The regret of LB-FTRL is lower bounded as follows:

$$\text{Reg} = \Omega \left( \sum_{t=1}^{T} \|\nabla f_t(x_t)\|^2_{(\nabla^2 f_t(x_t))^{-1}} \right).$$

We now give a high level intuition of why a lower bound of $T \log(T)$ is possible. The extra log $\log(T)$ factor requires a careful layering construction that is deferred to the appendix. The main idea of algorithm 3 is to let the agent sequentially visit each of the points $(t_i)_{i=1}^{T}$ for $T^\alpha$ steps ($0 < \alpha < 1$ is some fixed parameter), and the agent receives the return $o_i$ at point $t_i$. Since $t_i$ are on the boundary of the domain, which the agent cannot reach exactly, we refer to visiting $t_i$ if the agent plays $t'_i = (1 - T^{-\alpha})t_i + T^{-\alpha}c$, which is the target pulled towards the center by $T^{-\alpha}$.

Let us first assume that this is possible and that we only need to care about these returns in the Hessian. By Lemma 9, the regret is lower bounded by

$$\text{Reg} = \Omega \left( \sum_{t=1}^{T} \|\nabla f_t(x_t)\|^2_{(\nabla^2 f_t(x_t))^{-1}} \right) = \Omega \left( \sum_{t=1}^{T} \frac{\|\nabla f_t(x_t)\|^2}{\text{Tr}(\nabla^2 f_t(x_t))} \right).$$

During the $T^\alpha$ times we visit $t_i$ and receive $o_i$, the term $\|\nabla f_t(t'_i; o_i)\|^2$ is of order $T^{2\alpha}$ (ignoring dimension dependence), since it scales with $\langle (1 - T^{-\alpha})t_i + T^{-\alpha}c, o_i \rangle^{-2} = T^{2\alpha} \langle c, o_i \rangle^{-2}$. The trace in the denominator (ignoring the regularizer) after the $m$-th visit of $t_i$, is

$$\sum_{j < i} T^\alpha \|\nabla f(t'_i; o_j)\|^2 + m \|\nabla f_t(t'_i; o_i)\|^2 = \mathcal{O}(T \text{Poly}(d) T^\alpha) + m \|\nabla f(t'_i; o_i)\|^2,$$

which uses $\langle t'_i, o_j \rangle = \Omega(1/\text{Poly}(d))$ for $i < j$. We can assume that $T$ is large enough that $T^{\alpha/2} = \Omega(\text{Poly}(d))$, so for any $m > T^{\alpha/2}$, the denominator is of order $\mathcal{O}(m \|\nabla f(t'_i; o_i)\|^2)$. Hence the

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4. Let $U$ be a $(d-1) \times d$ matrix whose columns form an orthonormal basis for the subspace orthogonal to $c$. Then $\Pi$ can be defined as $\Pi x = U^* x$, and $\Pi^{-1}$ as $\Pi^{-1} v = U v + c$. 


stability is approximated by
\[
\sum_{i=1}^{T} \sum_{m=\lceil T^a \rangle}^{T^a} \frac{\| \nabla \Pi f(t'_i; o_i) \|^2}{m \| \nabla \Pi f(t'_i; o_i) \|^2} = \Omega(T \log(T)).
\]
This shows that the stability is large if the agent’s trajectory can be controlled. In fact, this is possible without increasing the trace of the Hessian significantly. To ensure that the agent visits the points \(t'_i\), we interleave the \(o_i\) returns by additional movement-returns \(r_i\), which satisfy \(\| \nabla \Pi r_i \| = O(T^{-\frac{1}{2}})\). Since the contribution to the Hessian is quadratic, the cumulative contribution to the Hessian trace of all movement steps does not exceed \(O(\text{Poly}(d))\), which is negligible in the argument above. Finally, one needs to show that the required number of movement-returns is small enough such that the sequence does not exceed \(T\) time steps. In our detailed proof, we show that this always holds for \(\alpha = \frac{1}{8}\) and sufficiently large \(T\).

**Exponential lower bound for LB-FTRL.** Equipped with Theorem 8, we are ready to derive an exponential lower bound for LB-FTRL. We define the following sequence of target point sets for any \(k \in [d-1]\): \(\mathcal{T}_k := \left\{ \frac{1}{d} x \mid x \in \{0, 1\}^d, \|x\|_1 = k \right\}\), i.e. the sets where exactly \(k\) components of the vector are non-zero, and these are of equal size. Define the combined sequence by adding the sets in increasing order of \(k\), with arbitrary ordering within a set \(t_1, \ldots, t_T = (t \in \mathcal{T}_1), \ldots, (t \in \mathcal{T}_{d-1})\). For each \(i \in [T]\), define the associated returns vector by \(o_i := \frac{1}{d-\|t_i\|_0}(1_d - \|t_i\|_0 t_i)\), i.e. the complement vector that is non-zero iff \(t_i\) is zero, normalized so that it lies in \(\Delta([d])\).

**Lemma 10** For all \(i < j\), it holds \(\langle t_i, o_j \rangle = \Omega(1/d^2)\), as well as \(\langle t_i, o_i \rangle = 0\).

**Proof** The second equality follows trivially by construction. For the first observe that for any \(j < i\), the number of non-zero components in \(t_j\) does not exceed the number of non-zero components in \(t_i\). That means that if \(t_i\) has \(k\) non-zero entries, then \(o_j\) has at least \(d - k\) non-zero entries. Since \(t_j \neq t_i\), \(o_j \neq o_i\), there is at least one component of non-zero values overlapping. Finally all non-zero components are least of size \(\frac{1}{d}\), which completes the proof.

**Corollary 11** The worst-case regret of LB-FTRL for any \(T > \text{Poly}(d)\) is \(\Omega(2^d \log(T) \log \log(T))\).

**Proof** Combine Lemma 10 with Theorem 8 and observe that the constructed sequence is of length \(2^d - 2\).

6. Conclusion

We have presented BISONS, the first algorithm with \(\hat{O}(\text{poly}(d))\) memory and per-step running time that obtains near optimal regret in the optimal portfolio problem without any assumptions on the gradient. Further, we have shown that key techniques in our algorithm BISONS can be adapted to work with the more general setting of quantum learning with log loss as well, at an additional factor of \(d\) in the regret.

Further, we showed that previous conjectures about LB-FTRL are wrong and that the worst-case regret of LB-FTRL is at least of order \(2^d \log(T) \log \log(T)\). In the natural regime of \(T \ll \exp(\exp(d))\) the regret of BISONS outperforms LB-FTRL at a significantly lower run-time and memory complexity. Therefore we practically eliminate LB-FTRL as a candidate for optimal trade-off between regret and computational complexity.
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Appendix A. Quantum Learning: Preliminaries

A.1. Reductions for Online Learning of Quantum States with Log Loss

In this section we describe the online learning of quantum states problem as described in (Aaronson et al., 2018), and show that when the loss function is the log loss (and more generally, the KL-divergence), the problem can be cast in the form in Section 2. Recall that a quantum state on $\log_2 d$ qubits is a $d \times d$ Hermitian PSD matrix of trace 1. A two-outcome measurement is a $d \times d$ Hermitian matrix with eigenvalues in $[0, 1]$. When a quantum state $X$ is measured using a two-outcome measurement $E$, the result is a Bernoulli random variable with probability of 1 being $\langle X, E \rangle$.

Aaronson et al. (2018) formulated the problem of online learning of quantum states as follows. In each round $t$, the learner constructs a quantum state $X_t$. In response, nature provides a two-outcome measurement $E_t$ and a value $b_t \in [0, 1]$. The value $b_t$ may be considered to be an approximation of $\langle X, E_t \rangle$ for some unknown quantum state $X$ that we’re trying to learn, or it can be thought of as the outcome in $\{0, 1\}$ of measuring the state $X$ using $E_t$. However, as is standard in online learning, the pair $(E_t, b_t)$ doesn’t have to be consistent with any quantum state. The quality of the learner’s prediction is given by a loss function $\ell : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$, and the loss in round $t$ is computed as $\ell((X_t, E_t), b_t)$. The goal is to minimize the regret, defined in the usual way as

$$\text{Reg} = \sum_{t=1}^{T} \ell((X_t, E_t), b_t) - \min_{\text{quantum state } X} \sum_{t=1}^{T} \ell((X, E_t), b_t).$$

We now show that in either of the following two settings, the problem can be recast in the form given in Section 2.

Setting 1: $b_t \in \{0, 1\}$, and $\ell$ is the log loss, i.e. $\ell(p, b) = -\log(bp + (1-b)(1-p))$.

In this case, note that by setting the loss matrix to be $R_t := b_tE_t + (1-b_t)(I_d - E_t)$, we have $-\log(\langle X, R_t \rangle) = \ell(\langle X, E_t \rangle, b_t)$ for any quantum state $X$. This completes the reduction to the form in Section 2.

Setting 2: $b_t \in [0, 1]$, and $\ell$ is the KL-divergence, i.e. $\ell(p, b) = b \log\left(\frac{b}{p}\right) + (1-b) \log\left(\frac{1-b}{1-p}\right)$.

In each round $t$, sample a Bernoulli random variable $y_t$ with probability of 1 being $b_t$. Then, setting $R_t = \frac{b_t}{1-b_t} E_t + \frac{1-b_t}{1-b_t} (I_d - E_t)$, it is easy to check that $\mathbb{E}_{y_t}[-\log(\langle X, R_t \rangle)] = \ell(\langle X, E_t \rangle, b_t)$ for any quantum state $X$. This completes a randomized reduction to the form in Section 2. Note that setting 1 is the special case of this setting when $b_t \in \{0, 1\}$, and in this case the randomized reduction becomes deterministic and coincides with the reduction described for setting 1.

A.2. Preliminary Notation, Definitions and Useful Properties

In this section, we collect some basic notation, definitions and useful properties which allow for the extension of the usual concepts in online convex optimization to the case when the domain is Hermitian matrices.

Notation: Recall that, we denote the set of $d \times d$ Hermitian matrices by $\mathcal{H}_d^d$, the set of $d \times d$ positive semi-definite Hermitian matrices by $\mathcal{H}_d^+$. Further, given two Hermitian matrices $X, Y$ we define the standard inner product between them as $\langle X, Y \rangle := \text{Tr}(X^*Y) = \text{Tr}(XY)$. Given a $d \times d$ matrix $M$, we use $M_{\text{flat}}$ denotes its canonical flattening which is $d^2$ dimensional vector obtained by serializing the columns of $M$. We further define $\overline{\mathcal{H}}_d^d_{\text{flat}} = \{ \overline{M} | M_{\text{flat}} \in \mathcal{H}_d^d \}$. Note that $\mathcal{H}_d^d_{\text{flat}}$ forms a subspace
in the vector space $\mathbb{C}^{d^2}$. Let $\dim\mathcal{H}_d$ be the dimension of the subspace and let $\Pi_{\mathcal{H}_d} \in \mathbb{C}^{d^2 \times \dim\mathcal{H}_d}$ be a projection operator from $\mathbb{C}^{d^2}$ on to this subspace. For convenience due to repeated usage through the paper given a $d \times d$ Hermitian matrix $M$, we define its vectorization as the projection of its canonical flattening, i.e.

$$\overrightarrow{M} = M_{\text{flat}} \Pi_{\mathcal{H}_d}$$

Further define the subset of matrices $\mathcal{K} := \mathcal{H}^{\dim\mathcal{H}_d}$. This set of matrices will be used to define the Hessian matrices in the complex case.

**Definitions and Useful Properties:** Note that as such the gradient of a real-valued function over the set of complex numbers does not exist (unless the function is a constant function). However, since we are dealing with the set of Hermitian matrices, we can define appropriate notions of a gradient and Hessian that allows for the same treatment of the proof as in the case of real matrices.

**Definition 12** We say a function $f : \mathcal{H}^d \rightarrow \mathbb{R}$ admits the gradient function $\nabla f : \mathcal{H}^d \rightarrow \mathcal{H}^d$, if for all PD $X$ it satisfies

$$\forall Y \in \mathcal{H}^d : \lim_{h \rightarrow 0} \frac{f(X + h(Y - X)) - f(X)}{h} = \langle (Y - X), \nabla f(X) \rangle .$$

Additionally we that say the function $f : \mathcal{H}^d \rightarrow \mathbb{R}$ also admits the Hessian function $\nabla^2 f : \mathcal{H}^d \rightarrow \mathcal{K}$ if for PD $X$ it satisfies

$$\forall Y \in \mathcal{H}^d : \lim_{h \rightarrow 0} \langle Y - X, \nabla f(X + h(Y - X)) - \nabla f(X) \rangle = (\overrightarrow{Y} - \overrightarrow{X})^* \nabla^2 f(X)(\overrightarrow{Y} - \overrightarrow{X}) .$$

A simple property to note is that the admissibility of gradient and Hessian function is additive, i.e. if $f$ admits gradient function $\nabla f$ and $g$ admits $\nabla g$, then $f + g$ admits $\nabla f + \nabla g$. The same property holds for the Hessian.

**Lemma 13 (Chain rule.)** Let $X$ be a PD matrix and $Y$ be a PSD matrix. Further define the function $g(t) = tY + (1 - t)X$. If $f$ admits the gradient function $\nabla f$, we have that for any $\alpha \in [0, 1)$

$$(f \circ g)'(\alpha) = \langle Y - X, \nabla f(g(\alpha)) \rangle .$$

If $f$ further admits Hessians, then

$$(f \circ g)''(\alpha) = (\overrightarrow{Y} - \overrightarrow{X})^* \nabla^2 f(g(\alpha))(\overrightarrow{Y} - \overrightarrow{X}) .$$

**Proof**

$$\begin{align*}
(f \circ g)'(\alpha) &= \lim_{h \rightarrow 0} \frac{(f \circ g)(\alpha + h) - (f \circ g)(\alpha)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(g(\alpha) + (Y - X)h) - f(g(\alpha))}{h} \\
&= \langle Y - X, \nabla f(g(\alpha)) \rangle .
\end{align*}$$

The proof for the case of the second derivative is analogous.

We now extend the notion of Bregman divergence in the following.
Definition 14  For any function $f$ that admits the gradient function $\nabla f$ at $X$, we define the Bregman divergence for any PD matrix $X$, and PSD matrix $Y$.

$$D_f(Y, X) = f(Y) - f(X) - \langle Y - X, \nabla f(X) \rangle.$$  

The following lemma establishes the extension of the intermediate value theorem which holds over Hermitian matrices.

Lemma 15 (Intermediate value theorem) For any $f$ that admits the gradient function $\nabla f$ and Hessian function $\nabla^2 f$, and for all PD matrices $X$ and PSD matrices $Y$, there exists an $\alpha \in [0, 1]$ and a matrix $\Xi := \alpha X + (1 - \alpha)Y$, such that

$$D_f(Y, X) = \frac{1}{2}(\mathbf{Y} - \mathbf{X})^* \nabla^2 f(\Xi)(\mathbf{Y} - \mathbf{X}).$$

Proof Define the function $g(t) = tY + (1-t)X$. Then consider the scalar function $f \circ g : [0, 1] \to \mathbb{R}$. Then using the derivation in Lemma 13 we have that

$$D_f(Y, X) = D_{f \circ g}(1, 0),$$

where for the real function $f \circ g$ we use the standard definition of Bregman divergence which coincides with Definition 14. Now by the intermediate value theorem for real valued functions, there exists $\alpha \in [0, 1]$ such that

$$D_{f \circ g}(1, 0) = \frac{1}{2}(1 - 0)^2 (f \circ g)''(\alpha).$$

Using Lemma 13 and combining the above we get the requisite conclusion.

The following lemmas establish that the natural (derived via an extension of the real case) definition of gradient and Hessians are admissible for the loss functions of interest as well as the regularizer.

Lemma 16 For any PSD matrix $R$, and PD matrix $X$, the function $f(X) = -\log(\langle X, R \rangle)$ admits the gradient the function $\nabla f(X) := -\frac{R}{\langle X, R \rangle}$ and the Hessian function $\nabla^2 f(X) := \frac{\mathbf{R}^*}{\langle X, R \rangle^2}$.

Proof The requisite property for the gradient can be verified by the following calculations,

$$\lim_{h \to 0} \frac{-\log(\langle X + h(Y - X), R \rangle) + \log(\langle X, R \rangle)}{h} = \lim_{h \to 0} \frac{-\log(1 + h \frac{\langle Y - X, R \rangle}{\langle X, R \rangle})}{h} = \lim_{h \to 0} \frac{-h \frac{\langle Y - X, R \rangle}{\langle X, R \rangle} + o(h^2)}{h} = -\frac{\langle Y - X, R \rangle}{\langle X, R \rangle}. $$

Similarly for the Hessian, consider the following,

$$\lim_{h \to 0} h^{-1} \left( \frac{-\langle Y - X, R \rangle}{\langle X + h(Y - X), R \rangle} + \frac{\langle Y - X, R \rangle}{\langle X, R \rangle} \right) = \lim_{h \to 0} h^{-1} \left( \frac{h \langle Y - X, R \rangle^2}{\langle X + h(Y - X), R \langle X, R \rangle \langle X, R \rangle} \right) = \frac{\langle Y - X, R \rangle^2}{\langle X, R \rangle^2} = \frac{(\mathbf{Y} - \mathbf{X})^* \mathbf{R}^* \mathbf{R} \mathbf{R}^*(\mathbf{Y} - \mathbf{X})}{\langle X, R \rangle^2}. $$


Lemma 17  For any PSD matrix $R$, and PD matrix $X$, the function $f(X) = -\log \det(X)$ admits the gradient function $\nabla f(X) := -X^{-1}$. Further given a PD matrix $X$, define $\nabla^2 f(X) : \mathcal{H}^d_+ \in \mathcal{H}^d$ as the matrix satisfying the following for all Hermitian matrices $M$

$$\overrightarrow{M} \nabla^2 f(X) \overrightarrow{M} = \text{Tr}(MX^{-1}MX^{-1}).$$

We have that the function $f(x)$ additionally admits the Hessian function $\nabla^2 f$.

**Proof**  We use the following results from [Hjorungnes and Gesbert (2007)](Hjorungnes and Gesbert (2007)) (Table 2), which show that the differential along the Hermitian matrices are equal to real symmetric ones. The differential of $\log \det(Z)$ is $\text{Tr}(Z^{-1} dZ)$ and the differential of $Z^{-1}$ is $-Z^{-1} dZZ^{-1}$. In our case, the differential is $dZ = (Y - X)$ and $Z$ is evaluated at $X$, hence

$$\lim_{h \to 0} h^{-1}(-\log \det(X + h(Y - X)) + \log \det(X)) = -\text{Tr}((Y - X)X^{-1}),$$

and

$$\lim_{h \to 0} -\text{Tr}((Y - X)(X + h(Y - X))^{-1}) + \text{Tr}((Y - X)X^{-1}) = \text{Tr}((Y - X)X^{-1}(Y - X)X^{-1}).$$

Finally, we show that there exists a Hermitian matrix $P \in \mathcal{K}$ such that for all $M \in \mathcal{H}$ we have that,

$$\text{Tr}(MX^{-1}MX^{-1}) = \overrightarrow{M}^* P \overrightarrow{M}.$$

To this end note that

$$\text{Tr}(MX^{-1}MX^{-1}) = \text{Tr}(X^{-\frac{1}{2}}MX^{-\frac{1}{2}}) = \text{vec}(X^{-\frac{1}{2}}MX^{-\frac{1}{2}})^* \text{vec}(X^{-\frac{1}{2}}MX^{-\frac{1}{2}}).$$

Since $X^{-\frac{1}{2}}MX^{-\frac{1}{2}}$ is linear in $M$, there exists a linear operator $N \in \mathcal{K}$ such that

$$\text{vec}(X^{-\frac{1}{2}}MX^{-\frac{1}{2}}) = N \overrightarrow{M}.$$

Therefore we have that

$$\text{vec}(X^{-\frac{1}{2}}MX^{-\frac{1}{2}})^* \text{vec}(X^{-\frac{1}{2}}MX^{-\frac{1}{2}}) = \overrightarrow{M}^* N^* N \overrightarrow{M},$$

which completes the argument defining $P = N^* N$. ■

In the following lemma proves the Hessian of the $-\log \det()$ function is PD and lower bounded over the Hermitian subspace.

Lemma 18  For the function $f = -\log \det()$, the Hessian function $\nabla^2 f$ satisfies the following properties.

- For any PD $X$, and any Hermitian $M \neq 0$, we have that

$$\overrightarrow{M}^* \nabla^2 f(X) \overrightarrow{M} > 0$$
• For any PD \( X \), s.t. \( \text{Tr}(X) \leq 1 \), and any Hermitian \( M \), we have that
\[
\overrightarrow{M} \nabla^2 f(X) \overrightarrow{M} \geq \|M\|^2
\]

**Proof** By definition in Lemma 17 we have that
\[
\overrightarrow{M} \nabla^2 f(X) \overrightarrow{M} = \text{Tr}(MX^{-1}MX^{-1}) = \text{Tr}((X^{-1/2}MX^{-1/2})^2) = \sum_i \lambda_i^2 (X^{-1/2}MX^{-1/2}) > 0,
\]
where the last inequality follows because we know that \( M \) is Hermitian and not identically 0. Next we show that for any Hermitian matrix \( M \) and any PD matrix \( X \) such that \( \text{Tr}(X) \leq 1 \), we have that
\[
\overrightarrow{M} \nabla^2 f(X) \overrightarrow{M} \geq \|\overrightarrow{M}\|^2.
\]
This is proved as follows. By the spectral theorem, we can write \( X^{-1} \) as \( U^* \text{diag}(\Lambda) U \), where \( U \) is a unitary matrix, and \( \Lambda \) is the vector of eigenvalues of \( X^{-1} \), which are all at least 1 since \( \text{Tr}(X) \leq 1 \). Now by Lemma 17 we have
\[
\overrightarrow{M} \nabla^2 f(X) \overrightarrow{M} = \text{Tr}(MU^* \text{diag}(\Lambda) U) = \text{Tr}(\overrightarrow{M} \text{diag}(\Lambda) \overrightarrow{M} \text{diag}(\Lambda)),
\]
where \( \overrightarrow{M} := UMU^* \). Now consider the function \( f : \mathbb{R}^d_+ \rightarrow \mathbb{R} \) defined as
\[
f(\lambda) = \text{Tr}(\overrightarrow{M} \text{diag}(\Lambda) \overrightarrow{M} \text{diag}(\Lambda)).
\]
An easy calculation using the fact that \( \overrightarrow{M} \) is Hermitian yields, for any \( i \in [d] \),
\[
\frac{\partial f(\lambda)}{\partial \lambda_i} = \sum_{k \neq i} |\tilde{M}_{ik}|^2 \lambda_k + 2|\tilde{M}_{ii}|^2 \lambda_i \geq 0.
\]
Since \( \Lambda \geq 1 \) entrywise, the above inequality implies that
\[
\overrightarrow{M} \nabla^2 f(X) \overrightarrow{M} = f(\Lambda) \geq f(1) = \text{Tr}(\overrightarrow{M}^2) = \|\overrightarrow{M}\|^2.
\]

Further the following easy to verify lemma establishes the admissibility of gradients of the intermediate functions maintained by the algorithm. The definition of surrogate functions and the biased surrogate functions are naturally extended to the quantum learning case from the definitions provided in (2), (3).

**Lemma 19** The surrogate function \( \hat{f}_t \) defined in (2) admits the following gradient and Hessian
\[
\nabla \hat{f}_t(X) := (1 + \beta \langle X - X_t, \nabla f_t(X_t) \rangle) \nabla f_t(X_t),
\]
\[
\nabla^2 \hat{f}_t(X) := \beta \nabla f_t(X_t) \nabla f_t(X_t)^*.
\]
Further the biased surrogate function \( g_c^e \) defined in (3) admits the following gradient and Hessian
\[
\nabla g_c^e(X) := \nabla \hat{f}_t^e(X) + B
\]
\[
\nabla^2 g_c^e(X) := \nabla^2 \hat{f}_t^e(X).
\]
We now recall the definitions provided in the algorithm,

\[
G^e_\tau(X) := \sum_{s=1}^{\tau} g^e_s(X) + \eta^{-1} R(X) \quad \text{and} \quad X^e_{\tau+1} := \arg\min_{X \in A} G^e_\tau(X).
\]

\[
\hat{F}^e_\tau(X) := \sum_{s=1}^{\tau} \hat{f}^e_s(X) + \eta^{-1} R(X) \quad \text{and} \quad U^e_{\tau+1} := \arg\min_{X \in A} \hat{F}^e_\tau(X).
\]

Using Lemmas 17, 19 we can analogously define admissible gradients and Hessian for both \(G^e_\tau, \hat{F}^e_\tau\). We now have the following analogue for the minimality condition.

**Lemma 20** We have that the following statements hold for all \(\tau, c\),

- \(X^e_{\tau+1} \succ 0, U^e_{\tau+1} \succ 0\), i.e. lie in the interior of the action set \(A\)
- Given any Hermitian matrices \(X, U\) such that \(\text{Tr}(X) = \text{Tr}(U) = 0\), we have that
  \[
  \langle \nabla G^e_\tau(X^e_{\tau+1}), X \rangle = 0 \quad \text{and} \quad \langle \nabla \hat{F}^e_\tau(U^e_{\tau+1}), U \rangle = 0
  \]

**Proof** The first statement is immediate by noting that for any \(X \succeq 0\) with at least one eigenvalue approaching 0, we have that \(R(X)\) and thus \(G^e_\tau\) approaches \(\infty\) and for all PD matrices \(\in A\), \(G^e_\tau\) is finite.

For the second statement we will prove the first inequality. The proof for the second inequality is analogous. We assume \(X \neq 0\), otherwise the statement is immediate. Since \(X_{\tau+1} \succ 0\), there exists a \(\delta > 0\) and a matrix \(X^+ = X_{\tau+1}^e + \delta X\) such that \(X^+ \in A\). Consider the function \(X(\alpha) := \alpha X^e_{\tau+1} + (1 - \alpha)X^+\) over \(\alpha \in [0, 1]\). We have that for all \(\alpha\) there exists some \(\alpha' \in [0, \alpha]\) such that the following holds

\[
G^e_\tau(X(\alpha)) - G^e_\tau(X^e_{\tau+1}) = \alpha \langle \nabla G^e_\tau(X^e_{\tau+1}), X^+ - X^e_{\tau+1} \rangle + D G^e_\tau(X(\alpha), X^e_{\tau+1})
\]

\[
= \alpha \langle \nabla G^e_\tau(X^e_{\tau+1}), X^+ - X^e_{\tau+1} \rangle + \frac{\alpha^2}{2} \langle \overrightarrow{X^+ - X^e_{\tau+1}}^*, \nabla^2 G^e_\tau(X(\alpha')) \langle \overrightarrow{X^+ - X^e_{\tau+1}}^* \rangle \rangle \geq 0
\]

Since \(X^e_{\tau+1}\) is the minimizer we have that for all \(\alpha\),

\[
\alpha \langle \nabla G^e_\tau(X^e_{\tau+1}), X^+ - X^e_{\tau+1} \rangle + \frac{\alpha^2}{2} \langle \overrightarrow{X^+ - X^e_{\tau+1}}^*, \nabla^2 G^e_\tau(X(\alpha')) \langle \overrightarrow{X^+ - X^e_{\tau+1}}^* \rangle \rangle \geq 0
\]

Using very coarse bounds obtained through combining Lemmas 19, 16 and 17 it is easy to see that there exists a finite number \(L\) independent of \(\alpha\) (but potentially dependent on other problem parameters like \(T\)), such that \(\nabla^2 G^e_\tau(X(\alpha')) \preceq L \cdot I_{d^2}\). This further implies that for all \(\alpha \in [0, 1]\) we have that

\[
\alpha \langle \nabla G^e_\tau(X^e_{\tau+1}), X^+ - X^e_{\tau+1} \rangle + \frac{Ld^2\alpha^2}{2} \geq 0.
\]

Now in case \(\langle \nabla G^e_\tau(X^e_{\tau+1}), X^+ - X^e_{\tau+1} \rangle < 0\), then we can set \(\alpha\) to be appropriately small such that the above expression is strictly negative which is a contradiction. Hence we have that

\[
\langle \nabla G^e_\tau(X^e_{\tau+1}), X^+ - X^e_{\tau+1} \rangle \geq 0 \implies \langle \nabla G^e_\tau(X^e_{\tau+1}), X \rangle \geq 0.
\]
Repeating the argument by replacing $X$ with $-X$ gives that $\langle \nabla G^e_T(X^e_{T+1}), X \rangle \leq 0$ and thus $\langle \nabla G^e_T(X^e_{T+1}), X \rangle = 0$.

We provide the proof of the following lemma whose restriction over the reals is well-known and is used repeatedly in the proofs of Online Newton Step like algorithms.

**Lemma 21** Given a sequence of PD matrices $X_1 \preceq X_2 \preceq X_3 \ldots X_T$, we have that

$$
\sum_{t=1}^{T-1} \langle X_t^{-1}, X_{t+1} - X_t \rangle \leq \log(\det(X_T)) - \log(\det(X_1))
$$

**Proof** We first begin by providing the proof of a simpler statement which implies the above statement via a simple summation. Given two PD matrices $X \preceq Y$, we have that

$$\langle Y^{-1}, Y - X \rangle \leq \log(\det(Y)) - \log(\det(X))$$

To prove the above we consider the following function $\phi(\alpha)$ defined as

$$\phi(\alpha) := \log \det(\alpha Y + (1 - \alpha)X)$$

Using Lemma 13 and the calculations in Lemma 17 we see that $\phi$ is a concave function over $\alpha$ and that

$$\phi'(\alpha) = \langle Y - X, (\alpha Y + (1 - \alpha)X)^{-1} \rangle.$$ 

Therefore using concavity we have that $\phi'(1) \leq \phi(1) - \phi(0)$ which implies the requisite statement by substitution.

\[\square\]
Appendix B. Algorithm for Quantum Learning with Log Loss

**Algorithm 2:** SCHRODINGER’S-BIONS

**input:** $T, \beta, \eta, \beta$.

**initialize:** $\forall e \in \mathbb{N} : P_0^e = dI_e, G_0^e(\cdot) = \hat{F}_0^e(\cdot) = \eta^{-1}R(\cdot), X_1^e = U_1^e = \text{arg} \min_{X \in A} G_0^e(X)$.

$e \leftarrow 1, \tau \leftarrow 1$

for $t = 1, \ldots$ do

$f_t \leftarrow$ receive from playing $X_t \leftarrow X_\tau^e$.

$\hat{f}_\tau^e = f_t \leftarrow$ construct according to (2).

$F_\tau^e = \hat{F}_\tau^e + f_t$

$G_\tau^e \leftarrow G_{\tau-1}^e + g_t^e$, where $g_t^e(X) := \hat{f}_\tau^e(X) - (X, P_\tau^e - P_{\tau-1}^e)B$.

$X_{\tau+1}^e \leftarrow \text{arg} \min_{X \in A} G_\tau^e(X), U_{\tau+1}^e \leftarrow \text{arg} \min_{X \in A} F_\tau^e(X)$

$P_{\tau+1}^e = P_{\tau}^e + [X_{\tau+1}^e]^\frac{1}{2} \left( I_d - [X_{\tau+1}^e]^\frac{1}{2} P_{\tau}^e [X_{\tau+1}^e]^\frac{1}{2} \right) [X_{\tau+1}^e]^{-\frac{1}{2}}$

if $U_{\tau} \neq \frac{1}{2(1+6\eta)(\beta)}P_t^{-1}$ then

$e \leftarrow e + 1, \tau \leftarrow 1$ // Reset the algorithm

else

$\tau \leftarrow \tau + 1$

end

end

Appendix C. Preliminary definitions and properties

In this section we provide some general definitions and other properties necessary for the analysis of the BISONS algorithm. Given a PSD matrix $A \in \mathcal{H}_+^d$, we associate a norm over $\mathcal{H}_+^d$, defined for any $W \in \mathcal{H}_+^d$ as

$$
\|W\|_A = \sqrt{\text{Tr}(WA^2W)} = \sqrt{\text{Tr}((A^{1/2}W^{1/2})^2)}.
$$

**Lemma 22** For any positive semi-definite $A$, $\|\cdot\|_A$ is a pseudo-norm.

**Proof** The only non-trivial property is the triangle inequality. We have

$$
\|V + W\|_A^2 = \text{Tr}((A^{1/2}(V + W)A^{1/2})^2) = \|V\|_A^2 + \|W\|_A^2 + 2\text{Tr}((A^{1/2}V A^{1/2})(A^{1/2}W A^{1/2}))
$$

$$
\leq \|V\|_A^2 + \|W\|_A^2 + 2\sqrt{\text{Tr}((A^{1/2}V A^{1/2})^2)\text{Tr}((A^{1/2}W A^{1/2})^2) = (\|V\|_A + \|W\|_A)^2}
$$

where the inequality is due to the fact that for PSD matrices $A, B$, we have that $\text{Tr}(AB) \leq \sqrt{\text{Tr}(A^2)\text{Tr}(B^2)}$, which follows from the Cauchy-Schwarz inequality.

**Lemma 23** For any PD matrices $A, B$ such that

$$
\|A - B\|_{B^{-1}} \leq \lambda
$$

for some $\lambda \geq 0$. Then, it holds that the eigenvalues of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ lie within the interval $[1 - \lambda, 1 + \lambda]$. 

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Proof We have
\[ \|A - B\|^2_{B^{-1}} = \text{Tr}((A - B)B^{-1}(A - B)B^{-1}) \]
\[ = \text{Tr}((B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - I)^2) \]
\[ = \sum_{i=1}^{d} \left( \text{ev}_i(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) - 1 \right)^2 \leq \lambda^2, \]
where \( \text{ev}_i \) represents the \( i^{th} \) eigenvalue. Therefore every eigenvalue satisfies
\[ |\text{ev}_i(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) - 1| \leq \lambda. \]

For any PSD matrix \( A, B \) define
\[ [A, B] := \{ \alpha A + (1 - \alpha)B \mid \alpha \in [0, 1] \}. \] (8)

**Lemma 24** For any PD matrix \( A, B \) such that
\[ \max_{C \in [A, B]} \|A - B\|_{C^{-1}} \leq \lambda, \]
for some \( \lambda \geq 0 \). Then it holds for all \( D, E \in [A, B] \):
\[ D \preceq (1 + \lambda)E, \quad D^{-1} \preceq (1 + \lambda)E^{-1}. \]

**Proof** Since \( D, E \in [A, B] \), there exists \( c : |c| \leq 1 \) such that \( D - E = c(A - B) \). Hence
\[ \|D - E\|_{D^{-1}} \leq \lambda. \]
Applying Lemma 23 completes the first part. Repeating the same argument, but now starting with \( \|D - E\|_{E^{-1}} \leq \lambda \), yields the second claim. \( \blacksquare \)

**Appendix D. BISONS detailed analysis**

In this section we provide the details for the analysis of our algorithms 1, 2, eventually proving Theorems 1 and 4. Before delving into the analysis we request the reader to familiarize themselves with the requisite notation, definition and properties listed out in Sections C, B. Since BISONS is a special case of SCHRODINGER’s-BISONS, we will provide the analysis focused on the quantum learning case, i.e. the domain will be PSD Hermitian matrices, however all the statements will hold when these matrices are real and diagonal as will be the case for the online optimal portfolio.

We first provide a proof of Lemma 6. We further begin the core analysis by providing some useful auxiliary lemmas and the lemmas governing the stability of the output of the algorithm in the next two subsections. We will restrict attention in the next two subsections to any fixed epoch and there for brevity we will remove the epoch superscript \( e \), from the lemma statements as well as proofs. All the statements should be understood to hold for any particular epoch.
D.1. Proof of Lemma 6

Proof Equality at $x = y_t$ holds by construction. We have $h'(x) = -x^{-1}$, which is concave and $\hat{h}'(x) = \min\{-\left(1 + \beta\right)y_t^{-1} + \beta xy_t^{-2}, -\beta y_t^{-1}\}$, which is piece-wise linear. A quick calculation shows $h'(y_t) = \hat{h}'(y_t)$ and $h'(\beta y_t) = \hat{h}'(\beta y_t)$. Hence for $x < y_t$, we have $h'(x) < \hat{h}'(x)$ and for $\beta y_t \geq x > y_t$, we have $h'(x) > \hat{h}'(x)$. Finally, the derivative of $h'(x)$ is monotonically increasing which implies $h'(x) > \hat{h}'(x)$ for $x > \beta^{-1} y_t$, which completes the proof.

D.2. Auxiliary Lemmas

In this section we collect some basic lemmas regarding the matrices $X_\tau, P_\tau$ generated by the algorithm. We recall the definition of $P_\tau$ defined in (7) as

$$P_\tau := P_{\tau-1} + X_{\tau}^{-\frac{1}{2}} \left( I_d - X_{\tau}^{\frac{1}{2}} P_{\tau-1} X_{\tau}^{\frac{1}{2}} \right) + X_{\tau}^{-\frac{1}{2}} X_{\tau+1}^{\frac{1}{2}} (I_d - X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}}) + X_{\tau+1}^{-\frac{1}{2}},$$

which in particular implies that for all $\tau$,

$$X_{\tau+1}^{\frac{1}{2}} (P_{\tau+1} - P_\tau) X_{\tau+1}^{\frac{1}{2}} = \left( I_d - X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}} \right).$$

The next two lemmas state the main properties satisfied by our choice of $P_\tau$. These properties prompt the choice of the definition for $P_\tau$.

**Lemma 25** We have that for all $\tau$,

$$P_\tau \succeq P_{\tau-1} \quad \text{and} \quad P_\tau \succeq X_{\tau}^{-1}.$$

Proof The first statement is immediate from the definition of $P_\tau$. For the second inequality note that

$$X_{\tau}^{\frac{1}{2}} P_\tau X_{\tau}^{\frac{1}{2}} = X_{\tau}^{\frac{1}{2}} P_{\tau-1} X_{\tau}^{\frac{1}{2}} + \left( I_d - X_{\tau}^{\frac{1}{2}} P_{\tau-1} X_{\tau}^{\frac{1}{2}} \right) + \geq I_d,$$

which implies that $P_\tau \succeq X_{\tau}^{-1}$.

**Lemma 26** For any $\tau$, we have

$$\langle X_{\tau+1}, P_{\tau+1} - P_\tau \rangle = \langle P_{\tau+1}^{-1}, P_{\tau+1} - P_\tau \rangle$$

Proof Recall, by definition

$$P_{\tau+1} = P_\tau + X_{\tau+1}^{-\frac{1}{2}} \left( I_d - X_{\tau+1}^{\frac{1}{2}} P_\tau X_{\tau+1}^{\frac{1}{2}} \right) + X_{\tau+1}^{-\frac{1}{2}} X_{\tau+2}^{\frac{1}{2}} (I_d - X_{\tau+2}^{\frac{1}{2}} P_{\tau+1} X_{\tau+2}^{\frac{1}{2}}) +.$$

Hence

$$\langle X_{\tau+1}, P_{\tau+1} - P_\tau \rangle = \Tr \left( \left( I_d - X_{\tau+1}^{\frac{1}{2}} P_\tau X_{\tau+1}^{\frac{1}{2}} \right) + \right) = \sum_{i=1}^{d} \max\{1 - \lambda_i, 0\},$$

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where \(\lambda_i\) are the eigenvalues of \(X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}}\). For the RHS, we have

\[
\langle P_{\tau+1}^{-1}, P_{\tau+1} - P_{\tau} \rangle = \langle (X_{\tau+1}^{\frac{1}{2}} P_{\tau+1} X_{\tau+1}^{\frac{1}{2}})^{-1}, X_{\tau+1}^{\frac{1}{2}} (P_{\tau+1} - P_{\tau}) X_{\tau+1}^{\frac{1}{2}} \rangle.
\]

Note that \((X_{\tau+1}^{\frac{1}{2}} P_{\tau+1} X_{\tau+1}^{\frac{1}{2}}) = X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}} + \left( I - X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}} \right)\) modifies the eigenvalues of \(X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}}\) such that they are lower bounded by 1. Therefore

\[
\langle (X_{\tau+1}^{\frac{1}{2}} P_{\tau+1} X_{\tau+1}^{\frac{1}{2}})^{-1}, X_{\tau+1}^{\frac{1}{2}} (P_{\tau+1} - P_{\tau}) X_{\tau+1}^{\frac{1}{2}} \rangle = \sum_{i=1}^{d} \frac{\max\{1 - \lambda_i, 0\}}{\max\{1, \lambda_i\}} = \sum_{i=1}^{d} \max\{1 - \lambda_i, 0\},
\]

where the last equality follows from the nominator being non-zero only if the denominator is 1. 

The following is a useful lemma we collect here.

**Lemma 27** For any \(\tau\), it holds that

\[
\|P_{\tau+1} - P_{\tau}\|_{X_{\tau+1}} \leq \|X_{\tau+1} - X_{\tau}\|_{X_{\tau}^{-1}}.
\]

**Proof** Denote \(\tilde{D} = X_{\tau+1}^{\frac{1}{2}} (P_{\tau+1} - P_{\tau}) X_{\tau+1}^{\frac{1}{2}}\), therefore we have that

\[
\tilde{D} = \left( I - X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}} \right)_{+}.
\]

We have

\[
\|P_{\tau+1} - P_{\tau}\|_{X_{\tau+1}}^2 = \text{Tr}(\tilde{D}^2) = \text{Tr} \left( (I - X_{\tau+1}^{\frac{1}{2}} P_{\tau} X_{\tau+1}^{\frac{1}{2}})_{+} \right)^2 \\
\leq \text{Tr} \left( (I - X_{\tau+1}^{\frac{1}{2}} X_{\tau}^{-1} X_{\tau+1}^{\frac{1}{2}})_{+} \right)^2 \\
\leq \text{Tr} \left( (I - X_{\tau+1}^{\frac{1}{2}} X_{\tau}^{-1} X_{\tau+1}^{\frac{1}{2}}) \right)^2 \\
= \|X_{\tau+1} - X_{\tau}\|_{X_{\tau}^{-1}}^2,
\]

where the first inequality uses the fact \(P_{\tau} \succeq X_{\tau}^{-1}\) from Lemma 25.

Finally as a result of our reset condition we have the following lemma.

**Lemma 28** Let \(s \leq \tau\) be two time indices belonging to the same epoch, such that the reset condition was not triggered upto time index \(\tau - 1\). Then we have that

\[
U_{\tau} \preceq \beta^{-1} X_s
\]

**Proof** Due to the reset condition, we know that \(U_{\tau} \preceq (2(1 + 6\eta))^{-1} \preceq \beta^{-1} P_{\tau}^{-1}\). Further since by Lemma 25, \(P_{\tau}^{-1} \preceq X_s\), the lemma follows.
D.3. Stability Lemmas

In this section we show that successive iterates $X_\tau, U_\tau, P_\tau$ and $X_{\tau+1}, U_{\tau+1}, P_{\tau+1}$ do not move too far away from each other due to the log barrier, establishing the requisite stability of our method. These results are summarized in lemmas 29 and 30. Our stability lemmas hold under the following constraints over the algorithm parameters $\eta, \beta$.

\[
\eta \leq \min \{ \frac{1}{4B}, \frac{\beta}{4}, \frac{1}{63} \} \quad (10)
\]

\[
\beta \leq \sqrt{2} - 1 \quad (11)
\]

\[
T \geq \max \{ 2d, \beta^{-1} \} \quad (12)
\]

Lemma 29  If $\eta$ satisfies constraint (10), then for any $t \in [T]$:

\[
\frac{1}{1 + 6\eta} X_{\tau+1} \preceq X_\tau \preceq (1 + 6\eta) X_{\tau+1},
\]

as well as

\[
P_{\tau+1} \preceq (1 + 6\eta) P_\tau.
\]

Lemma 30  If $\eta$ and $\beta$ satisfy constraints (10) and (11) and no reset is triggered at time $\tau - 1$, then

\[
U_{\tau+1} \preceq 2U_\tau.
\]

We first present some auxiliary lemmas and then prove the above stability lemmas.

Lemma 31  For any $\tau$, it holds that

\[
X_{\tau+1}^{-1} \preceq (1 + \lambda) X_\tau^{-1} \quad \Rightarrow \quad P_{\tau+1} \preceq (1 + \lambda) P_\tau.
\]

Proof  We assume LHS above is true. By Lemma 25 we have that,

\[
X_{\tau+1}^{-1} \preceq (1 + \lambda) X_\tau^{-1} \preceq (1 + \lambda) P_\tau.
\]

Therefore we have that,

\[
I_d \preceq (1 + \lambda) X_{\tau+1}^{\frac{1}{2}} P_\tau X_{\tau+1}^{\frac{1}{2}}.
\]

Hence by definition of $P_{\tau+1}$,

\[
X_{\tau+1}^{\frac{1}{2}} (P_{\tau+1} - (1 + \lambda) P_\tau) X_{\tau+1}^{\frac{1}{2}} = (I_d - X_{\tau+1}^{\frac{1}{2}} P_\tau X_{\tau+1}^{\frac{1}{2}}) + - \lambda X_{\tau+1}^{\frac{1}{2}} P_\tau X_{\tau+1}^{\frac{1}{2}}
\]

\[
\preceq (I_d - (1 + \lambda) X_{\tau+1}^{\frac{1}{2}} P_\tau X_{\tau+1}^{\frac{1}{2}}) + = 0.
\]

Finally, this implies

\[
(P_{\tau+1} - (1 + \lambda) P_\tau) \preceq 0 \quad \Leftrightarrow \quad P_{\tau+1} \preceq (1 + \lambda) P_\tau
\]

as claimed. \qed

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Lemma 32  If $\eta$ satisfies constraint (10), then for all $\tau$ we have that,

$$\max_{\xi \in [X_{\tau}, X_{\tau+1}]} \|X_{\tau+1} - X_{\tau}\|_{\Xi^{-1}} \leq 6\eta$$

Proof  The proof follows by induction. Set by convention $X_0 = X_1$, then the condition holds for $\tau = 0$. Now assuming that the condition holds for all time-steps including $\tau - 1$, we prove in the following that this holds for $\tau$. We will show this by contradiction. To this end suppose that $\max_{\xi \in [X_{\tau}, X_{\tau+1}]} \|X_{\tau+1} - X_{\tau}\|_{\Xi^{-1}} > 6\eta$. By continuity there exists a point $X \in (X_{\tau}, X_{\tau+1})$ such that $\max_{\xi \in [X_{\tau}, X]} \|X - X_{\tau}\|_{\Xi^{-1}} = 6\eta$. Now recall the definitions,

$$G_{\tau}(X) := \sum_{s=1}^{\tau} g_s(X) + \eta^{-1}R(X) \quad \text{and} \quad X_{\tau+1} := \arg\min_{X \in A} G_{\tau}(X).$$

In the latter half of the proof will show that the condition on $X$ implies that $G_{\tau}(X) \geq G_{\tau}(X_{\tau})$. We will first show why establishing the above leads to a contradiction. So we assume $G_{\tau}(X) \geq G_{\tau}(X_{\tau})$. To this end consider the scalar function $\phi(\alpha)$ for $\alpha \in [0, 1]$ defined as

$$\phi(\alpha) = G_{\tau}(\alpha X_{\tau+1} + (1 - \alpha)X_{\tau}).$$

Let $\alpha_X \in (0, 1)$ correspond to the unique $\alpha$ such that

$$X = \alpha X_{\tau+1} + (1 - \alpha)X_{\tau}.$$  

The assumption $G_{\tau}(X) \geq G_{\tau}(X_{\tau})$ implies that $\phi(\alpha_X) \geq \phi(0)$. Further since $X_{\tau+1}$ is the minimizer, we have that $\phi(1) \leq \phi(0)$. We will now show that $\phi$ is a strictly convex function and that will contradict the above derived statements. To show that $\phi$ is strictly convex we use Lemma 13 to establish that for any $\alpha$

$$\phi''(\alpha) = (\overrightarrow{X}_{\tau+1} - \overrightarrow{X}_{\tau})^{\ast} \nabla^2 G_{\tau}(\alpha X_{\tau+1} + (1 - \alpha)X_{\tau})((\overrightarrow{X}_{\tau+1} - \overrightarrow{X}_{\tau})).$$

Further Lemma 19 establishes that for any $Y$, $\nabla^2 G_{\tau}(Y) \succeq \eta^{-1}R(Y)$. Using the calculation of $\nabla^2 R$ in Lemma 17 and noting that $X_{\tau}, X_{\tau+1}$ are positive-definite, we get that

$$\phi''(\alpha) = (\overrightarrow{X}_{\tau+1} - \overrightarrow{X}_{\tau})^{\ast} \nabla^2 G_{\tau}(\alpha X_{\tau+1} + (1 - \alpha)X_{\tau})((\overrightarrow{X}_{\tau+1} - \overrightarrow{X}_{\tau}) > 0,$n

which establishes the strict convexity of $\phi$ and therefore the contradiction.

All that is left to show now is that $G_{\tau}(X) \geq G_{\tau}(X_{\tau})$. To this end consider the following. Using gradient and Hessian via Lemma 19 and via the intermediate value lemma (Lemma 15), we have that there exists $\Psi \in [X_{\tau}, X]$ such that

$$G_{\tau}(X) = G_{\tau}(X_{\tau}) + \langle \nabla G_{\tau}(X_{\tau}), X - X_{\tau} \rangle + \frac{1}{2}(\overrightarrow{X}_{\tau} - \overrightarrow{X}_{\tau})^{\ast} \nabla^2 G_{\tau}(\Psi)((\overrightarrow{X}_{\tau} - \overrightarrow{X}_{\tau})).$$

Further noting that fact that $\nabla G_{\tau}(\cdot) - \nabla g_{\tau}(\cdot) = \nabla G_{\tau-1}(\cdot)$, $X_{\tau} := \arg\min_{X \in A} G_{\tau-1}(X)$ and using Lemma 20 we get that

$$G_{\tau}(X) \geq G_{\tau}(X_{\tau}) + \langle \nabla g_{\tau}(X_{\tau}), X - X_{\tau} \rangle + \frac{1}{2}(\overrightarrow{X}_{\tau} - \overrightarrow{X}_{\tau})^{\ast} \nabla^2 G_{\tau}(\Psi)((\overrightarrow{X}_{\tau} - \overrightarrow{X}_{\tau})).$$
Further using $\nabla^2 G_\tau(\cdot) \succeq \eta^{-1} \nabla^2 R(\cdot)$ and considering the computation of $\nabla^2 R$ presented in Lemma 17 and that for any $\Xi, \Phi \in [X_\tau, X_{\tau+1}]$ we have that $\Xi^{-1} \preceq (1 + 6\eta)\Phi^{-1}$ according to Lemma 24, we get that

$$G_\tau(X) \geq G_\tau(X_\tau) + \langle \nabla g_\tau(X_\tau), X - X_\tau \rangle + \frac{1}{2}(X - \overline{X}_\tau)^* \nabla^2 G_\tau(\Psi)(X - \overline{X}_\tau) \geq G_\tau(X_\tau) - \|\nabla g_\tau(X_\tau)\|_{X_\tau} \|X - X_\tau\|_{X_{\tau}} + \frac{1}{2\eta(1 + 6\eta)^2} \max_{\Xi \in [X_\tau, X_{\tau+1}]} \|X - X_\tau\|_{\Xi} - 6\eta \|\nabla g_\tau(X_\tau)\|_{X_\tau} + \frac{18\eta}{(1 + 6\eta)^2}.$$ 

We now show that $\|\nabla g_\tau(X_\tau)\|_{X_\tau} \leq \frac{3}{(1 + 6\eta)^2}$ which completes the inductive step. We have

$$\|\nabla g_\tau(X_\tau)\|_{X_\tau} \leq \left\| \nabla \hat{f}_\tau(X_\tau) \right\|_{X_\tau} + B \|P_\tau - P_{\tau-1}\|_{X_\tau}$$

$$= \sqrt{\text{Tr}((X_\tau^2 R_\tau X_\tau^2)^2)} + B \|P_\tau - P_{\tau-1}\|_{X_\tau} \leq 1 + \|X_\tau - X_{\tau-1}\|_{X_{\tau-1}} B$$

$$\leq 1 + 6\eta B \leq \frac{5}{2} = \frac{3}{(1 + (\sqrt{6/5} - 1))^2} \leq \frac{3}{(1 + 6\eta)^2},$$

where the last set of inequalities follow from the constraints on $\eta$ defined in (10) and induction assumption.

We now prove the stability lemmas, Lemma 29 and Lemma 30.

**Proof** [Proof of Lemma 29] Combining Lemma 32 and 23 yields the claim for $X$. By Lemma 31 this follows for $P$ as well.

**Proof** [Proof of Lemma 30] We will show that

$$\|U_{\tau+1} - U_\tau\|_{U^{-1}_\tau} \leq 1,$$

which by Lemma 23 implies the statement of the lemma. To show the above we use a similar proof structure as in the case of Lemma 32 and assume for contradiction that $\|U_{\tau+1} - U_\tau\|_{U^{-1}_\tau} > 1$. Once again by continuity we have that there exists a point $U \in [U_\tau, U_{\tau+1}]$ such that $\|U - U_\tau\|_{U^{-1}_\tau} = 1$. Now recall the definitions,

$$\hat{F}_\tau(X) := \sum_{s=1}^{\tau} \hat{f}_s(X) + \eta^{-1} R(X) \quad \text{and} \quad U_{\tau+1} := \arg\min_{X \in \mathcal{A}} \hat{F}_\tau(X).$$

In the latter half of the proof will show that the condition on $U$ implies that $\hat{F}_\tau(X) \geq \hat{F}_\tau(X_\tau)$. We will first show why establishing the above leads to a contradiction. So we assume $\hat{F}_\tau(X) \geq \hat{F}_\tau(X_\tau)$. To this end consider the scalar function $\phi(\alpha)$ for $\alpha \in [0, 1]$ defined as

$$\phi(\alpha) = \hat{F}_\tau(\alpha U_{\tau+1} + (1 - \alpha)U_\tau).$$
Let $\alpha_U \in (0, 1)$ correspond to the unique $\alpha$ such that

$$U = \alpha U_{\tau+1} + (1 - \alpha)U_\tau.$$ 

The assumption $\hat{F}_\tau(U) \geq \hat{F}_\tau(U_\tau)$ implies that $\phi(\alpha_U) \geq \phi(0)$. Further since $U_{\tau+1}$ is the minimizer, we have that $\phi(1) \leq \phi(0)$. We will now show that $\phi$ is a strictly convex function and that will contradict the above derived statements. To show that $\phi$ is strictly convex we use Lemma 13 to establish that for any $\alpha$

$$\phi''(\alpha) = (\hat{U}_{\tau+1} - \hat{U}_\tau)\nabla^2 \hat{F}_\tau(\alpha U_{\tau+1} + (1 - \alpha)U_\tau))(\hat{U}_{\tau+1} - \hat{U}_\tau).$$

Further Lemma 19 establishes that for any $Y$, $\nabla^2 \hat{F}_\tau(Y) \succeq \eta^{-1}R(Y)$. Using the calculation of $\nabla^2 R$ in Lemma 17 and noting that $U_\tau, U_{\tau+1}$ are positive-definite, we get that

$$\phi''(\alpha) = (\hat{U}_{\tau+1} - \hat{U}_\tau)\nabla^2 \hat{F}_\tau(\alpha U_{\tau+1} + (1 - \alpha)U_\tau))(\hat{U}_{\tau+1} - \hat{U}_\tau) > 0,$$

which establishes the strict convexity of $\phi$ and therefore the contradiction.

Therefore all we need to establish is that $\hat{F}_\tau(U) \geq \hat{F}_\tau(U_\tau)$. To this end consider the following. Using gradient and Hessian via Lemma 19 and via the intermediate value lemma (Lemma 15), we have that there exists $\Xi \in [U, U_\tau]$ such that

$$\hat{F}_\tau(U) = \hat{F}_\tau(U_\tau) + \langle \nabla \hat{F}_\tau(U_\tau), U - U_\tau \rangle + \frac{1}{2}(\hat{U} - \hat{U}_\tau)\nabla^2 \hat{F}_\tau(\Xi)(\hat{U} - \hat{U}_\tau).$$

Further noting that fact that $\nabla \hat{F}_\tau(\cdot) - \nabla \hat{f}_\tau(\cdot) = \nabla \hat{F}_{\tau-1}(\cdot)$, $U_\tau := \arg \min_{U \in A} \hat{F}_{\tau-1}(X)$ and using Lemma 20 we get that

$$\hat{F}_\tau(U) \geq \hat{F}_\tau(U_\tau) + \langle \nabla \hat{f}_\tau(U_\tau), U - U_\tau \rangle + \frac{1}{2}(\hat{U} - \hat{U}_\tau)\nabla^2 \hat{F}_\tau(\Xi)(\hat{U} - \hat{U}_\tau).$$

Further using $\nabla^2 \hat{F}_\tau(\cdot) \succeq \eta^{-1} \nabla^2 R(\cdot)$ and considering the computation of $\nabla^2 R$ presented in Lemma 17 we have that,

$$\hat{F}_\tau(U) \geq \hat{F}_\tau(U_\tau) - \left\| \nabla \hat{f}_\tau(U_\tau) \right\|_{U_\tau} \| U - U_\tau \|_{U_{\tau+1}} + \frac{1}{2\eta}(\hat{U} - \hat{U}_\tau)\nabla^2 R(\Xi)(\hat{U} - \hat{U}_\tau)$$

$$= \hat{F}_\tau(U_\tau) - \left\| \nabla \hat{f}_\tau(U_\tau) \right\|_{U_\tau} \| U - U_\tau \|_{U_{\tau+1}} + \frac{1}{2\eta} \| U - U_\tau \|^2_{\Xi_{\tau+1}}$$

$$\geq \hat{F}_\tau(U_\tau) - \left\| \nabla \hat{f}_\tau(U_\tau) \right\|_{U_\tau} \| U - U_\tau \|_{U_{\tau+1}} + \frac{1}{8\eta} \| U - U_\tau \|^2_{U_{\tau+1}}$$

$$\geq \hat{F}_\tau(U_\tau) - \left\| \nabla \hat{f}_\tau(U_\tau) \right\|_{U_\tau} + \frac{1}{2\beta}.$$ 

The first inequality follows from Cauchy-Schwartz, the second to last inequality follows from Lemma 23 and the last inequality follows from the definition of $U$ and the constraint on $\eta$ given by (10).

Finally, note using Lemma 19 that $\nabla \hat{f}_\tau(U_\tau) = -\left(1 + \beta - \beta \frac{\langle U_\tau, R_\tau \rangle}{\langle X_\tau, R_\tau \rangle} \right) \frac{R_\tau}{\langle X_\tau, R_\tau \rangle}$. Since no reset is triggered at time $\tau - 1$, we have using Lemma 28 that $0 \leq \frac{\langle U_\tau, R_\tau \rangle}{\langle X_\tau, R_\tau \rangle} \leq \frac{1}{\beta}$. Therefore we have that

$$\left\| \nabla \hat{f}_\tau(U_\tau) \right\|_{U_\tau} = \left(1 + \beta - \beta \frac{\langle U_\tau, R_\tau \rangle}{\langle X_\tau, R_\tau \rangle} \right) \frac{\sqrt{\text{Tr}(U_\tau R_\tau U_\tau R_\tau)}}{\langle X_\tau, R_\tau \rangle} \leq \left(1 + \beta - \beta \frac{\langle U_\tau, R_\tau \rangle}{\langle X_\tau, R_\tau \rangle} \right) \langle U_\tau, R_\tau \rangle.$$
Maximizing the above expression over all choice of \( \frac{U_{\tau}, R_{\tau}}{\langle X_{\tau}, R_{\tau} \rangle} \in [0, 1/\beta] \) we get that
\[
\left\| \nabla \hat{F}_{\tau}(U_{\tau}) \right\|_{U_{\tau}} \leq \frac{(1 + \beta)^2}{4\beta} \leq \frac{1}{2\beta},
\]
which follows by the constraint on \( \beta \) in (11). Using this and plugging it into the bound for \( \hat{F}_{\tau}(U) \) completes the proof.

Finally we provide some loose upper bounds on the inverses of the iterates.

**Lemma 33** If \( \eta \) and \( \beta \) satisfy constraints (10) and (11), we have that for any \( \tau \) such that the reset condition is not triggered up to index \( \tau - 1 \),
\[
U_{\tau}^{-1} \preceq \left( \frac{(1 + \beta)^2 T}{4\beta} \eta + d \right) I_d
\]
and
\[
X_{\tau}^{-1} \preceq P_{\tau}^{-1} \preceq \left( \frac{(1 + \beta)^2 T}{4\beta^2} - \eta \right) I_d
\]
If further \( T \) satisfies constraint (12), then the last term is upper bounded by \( T^2 I_d \).

**Proof** Lemma 20 shows that for all Hermitian matrices \( H \), such that \( \text{Tr}(H) = 0 \) we have that \( \langle \nabla \hat{F}_{\tau-1}(U_{\tau}), H \rangle \). Further by definition \( \hat{F}_{\tau-1}(U_{\tau}) \) is Hermitian. These facts imply that \( \nabla \hat{F}_{\tau-1}(U_{\tau}) = \gamma I_d \) for some \( \gamma \in \mathbb{R} \). Substituting the definition of \( \nabla \hat{F}_{\tau-1} \) we get that
\[
\gamma I_d = \sum_{s=1}^{\tau-1} \left( -\frac{(1 + \beta) R_s}{\langle X_s, R_s \rangle} + \beta \frac{\langle U_{\tau}, R_s \rangle R_s}{\langle X_s, R_s \rangle^2} \right) - \eta^{-1} U_{\tau}^{-1}.
\]
Using Lemma 28 we get that \( \langle U_{\tau}, R_s \rangle \leq \frac{\langle X_s, R_s \rangle}{\beta} \) and therefore the above equality implies that
\[
\eta^{-1} U_{\tau}^{-1} \preceq -\gamma I_d
\]
Further using (13) we have that
\[
\gamma = \langle U_{\tau}, \nabla \hat{F}_{\tau-1}(U_{\tau}) \rangle = \langle U_{\tau}, \left( \sum_{s=1}^{\tau-1} \left( -\frac{(1 + \beta) R_s}{\langle X_s, R_s \rangle} + \beta \frac{\langle U_{\tau}, R_s \rangle R_s}{\langle X_s, R_s \rangle^2} \right) - \eta^{-1} U_{\tau}^{-1} \right) \rangle
\]
\[
= \sum_{s=\tau-1}^{\tau-1} \left( -\frac{(1 + \beta) \langle U_{\tau}, R_s \rangle}{\langle X_s, R_s \rangle} + \beta \frac{\langle U_{\tau}, R_s \rangle^2}{\langle X_s, R_s \rangle^2} \right) - \frac{d}{\eta}
\]
\[
\geq -\frac{(1 + \beta)^2 T}{4\beta} - \frac{d}{\eta}
\]
Combining these leads to
\[
U_{\tau}^{-1} \preceq \left( \frac{(1 + \beta)^2 T}{4\beta} \eta + d \right) I_d
\]
By the reset condition we have \( P_{\tau} \preceq \frac{1}{\beta} U_{\tau}^{-1} \), which completes the first part of the lemma. Finally, since \( \eta \leq \beta/4, (1 + \beta)^2 \leq 3 \) and \( T \geq \max(2d, \beta^{-1}) \), we have
\[
\frac{(1 + \beta)^2 T}{4\beta^2} \eta + \frac{d}{\beta} \leq T^2.
\]
D.4. Main proofs

For the next three lemmas, once again for brevity we drop the epoch superscript. Further we define the inherent dimension of the problem $\tilde{d}$ as $d$ for the standard optimal portfolio case and $d^2$ for the quantum case. The next lemma bounds the cost of bias in our algorithm.

**Lemma 34** Let $\eta, \beta, T$ satisfy constraints (10)-(12). Consider any epoch $e$ with the reset points $\mathcal{T}_{e-1} < \mathcal{T}_e \leq T$. Let $L$ represent the length of the epoch, i.e. $L = \mathcal{T}_e - \mathcal{T}_{e-1}$. Then the cost of bias within the epoch is bounded as follows

$$\sum_{\tau=1}^{L} \langle X_\tau, P_\tau - P_{\tau-1} \rangle = \sum_{\tau=1}^{L} \langle P_{\tau}^{-1}, P_\tau - P_{\tau-1} \rangle \leq 2d \log(T).$$

**Proof** By using Lemma 21 and Lemma 26, we have

$$\sum_{\tau=1}^{L} \langle X_\tau, P_\tau - P_{\tau-1} \rangle = \sum_{\tau=1}^{L} \langle P_{\tau}^{-1}, P_\tau - P_{\tau-1} \rangle \leq \log \det(P_L/d).$$

Finally by Lemma 33, we have $P_L \leq T^2 \mathbf{I}_d$, which completes the proof.

The following lemma bounds the regret with respect to biased surrogate functions $g_\tau$ within an epoch.

**Lemma 35** Let $\eta, \beta, T$ satisfy constraints (10)-(12). Consider any epoch $e$ with the reset points $\mathcal{T}_{e-1} < \mathcal{T}_e \leq T$. Let $L$ represent the length of the epoch, i.e. $L = \mathcal{T}_e - \mathcal{T}_{e-1}$. The FTRL-regret with respect to any comparator $U$ over the functions $(g_\tau)_{\tau=1}^{L}$ is bounded by

$$\sum_{\tau=1}^{L} (g_\tau(X_\tau) - g_\tau(U)) \leq \frac{11 \tilde{d}}{\beta} \log(T) + \eta^{-1} R(U),$$

where $\tilde{d}$ is the inherent dimension of the problem: $d^2$ for the PSD case and $d$ for the simplex case.

**Proof** It can be verified that the conditions for Lemma 44 are satisfied with factor $(1 + 6\eta)^2 \leq \frac{6}{5}$ due to combining Lemma 32 and 23.

Recall that we denote the canonical vectorization of a matrix $X$ as $\overrightarrow{X}$. Further denote $\nabla \overrightarrow{X}$ as the gradient with respect to this vectorization. In an overload of notation, we define for a vector $\overrightarrow{X} \in \mathbb{R}^{d^2}$ and PSD matrix $M \in \mathbb{R}^{d^2 \times d^2}$, the semi-norm $\| \overrightarrow{X} \|_M = \sqrt{\langle \overrightarrow{X}, M \overrightarrow{X} \rangle}$ (recall for matrices $X, M \in \mathbb{R}^{d \times d}$, we defined $\| X \|_M = \sqrt{\text{Tr}(X^T MX)}$) then

$$\sum_{\tau=1}^{L} (g_\tau(X_\tau) - g_\tau(U)) - \eta^{-1} R(U) \leq \frac{3}{5} \sum_{\tau=1}^{L} \| \nabla g_\tau(X_\tau) \|^2_{(\nabla^2 G_\tau(X_\tau))^{-1}}$$

$$\leq \sum_{\tau} \left( \| \nabla \hat{f}_\tau(X_\tau) \|^2_{(\nabla^2 G_\tau(X_\tau))^{-1}} + \frac{3}{2} B^2 \| \hat{P}_\tau - \hat{P}_{\tau-1} \|^2_{(\nabla^2 G_\tau(X_\tau))^{-1}} \right),$$
where we used \((a + b)^2 \leq \lambda a^2 + \frac{\lambda}{\lambda - 1} b^2\) for any \(\lambda > 1\), generalizing it appropriately to vectors. We deal with the above two terms separately. To control the first term we note using Lemma 19 that

\[
\nabla^2 G_\tau(X_\tau) = \eta^{-1} \nabla^2 R(X_\tau) + \sum_{s=1}^{\tau} \frac{\beta}{2} \hat{\nabla} f_s(X_s) \hat{\nabla} f_s(X_s)^*.
\]

Using Lemmas 17, 18 we get that for any \(\tau\)

\[
\left\| \hat{\nabla} f_\tau(X_\tau) \right\|^2_{(\nabla^2 G_\tau(X_\tau))^{-1}} \leq \left( \hat{\nabla} f_\tau(X_\tau)[\hat{\nabla} f_\tau(X_\tau)]^* \right) \left( \eta^{-1} I_d + \sum_{s=1}^{\tau} \frac{\beta}{2} \hat{\nabla} f_s(X_s) \hat{\nabla} f_s(X_s)^* \right)^{-1}.
\]

Using Lemma 21, the following computation follows.

\[
\sum_{\tau=1}^{L} \left\| \hat{\nabla} f_\tau(X_\tau) \right\|^2_{(\nabla^2 G_\tau(X_\tau))^{-1}} \leq \frac{2}{\beta} \text{log det}(I_d + \sum_{\tau=1}^{L} \frac{\eta \beta}{2} \hat{\nabla} f_\tau(X_\tau) \hat{\nabla} f_\tau(X_\tau)^*)
\]

\[
\leq \frac{2}{\beta} \tilde{d} \text{log} \left( \frac{(\tilde{d} + \frac{\eta \beta}{2} T \max_{t \in [\tau]} \left\| \hat{\nabla} f_t(X_t) \right\|^2) / \tilde{d} \right).
\]

By Lemma 33, we have \(X_\tau \succeq T^{-2} I_d\) for all \(\tau \in [L]\), hence

\[
\left\| \hat{\nabla} f_\tau(X_\tau) \right\|^2 = \frac{(R_\tau, R_\tau)}{(X_\tau, R_\tau)^2} \leq T^4.
\]

Further, since \(T > 2d\) and \(\eta \leq \frac{1}{2}\), we have

\[
\sum_{\tau=1}^{L} \left\| \hat{\nabla} f_\tau(X_\tau) \right\|^2_{(\nabla^2 G_\tau(X_\tau))^{-1}} \leq \frac{10}{\beta} \tilde{d} \text{log}(T).
\]
For the second norm, we have

\[
\sum_{t=1}^{\tau} \left\| \bar{P}_t - \bar{P}_{t-1} \right\|_{(\nabla^2G_r(X_t))^{-1}}^2 \leq \eta \sum_{t=1}^{\tau} \left\| P_t - P_{t-1} \right\|_{X_r}^2 \quad \text{(by Lemma 17)}
\]

\[
= \eta \sum_{t=1}^{\tau} \text{Tr} \left( (I_d - X_r^{1/2}P_{t-1}X_r^{1/2})^2 + (I_d - X_r^{1/2}P_{t-1}X_r^{1/2})_+ \right)
\]

\[
\leq \eta \sum_{t=1}^{\tau} \text{Tr} \left( (I_d - X_r^{1/2}P_{t-1}X_r^{1/2})_+ \right)
\]

\[
= 6\eta^2 \sum_{t=1}^{\tau} \langle X_r, X_r^{-1/2}(I_d - X_r^{1/2}P_{t-1}X_r^{1/2})_+X_r^{-1/2} \rangle
\]

\[
= 6\eta^2 \sum_{t=1}^{\tau} \langle P_{t-1}^{-1}X_r - P_{t-1} \rangle \leq 12\eta^2 d \log(T),
\]

where we use \( X_r^{1/2}X_{t-1}X_r^{1/2} \geq \frac{1}{1+6\eta}I_d \) by Lemma 29 for the third inequality and Lemma 34 for the last equality. By \( \eta \leq \frac{1}{4B}, \beta \leq \sqrt{2} - 1 \) by constraint (10),(11), we have

\[
\frac{3}{2}B^2 \sum_{t=1}^{\tau} \left\| \bar{P}_t - \bar{P}_{t-1} \right\|_{(\nabla^2G_r(X_t))^{-1}}^2 \leq \frac{36d}{32} \log(T) \leq \frac{\tilde{d}}{\beta} \log(T).
\]

Combining both bounds completes the proof. 

The following lemma lower bounds the negative regret contribution we get.

**Lemma 36**  \( \eta, \beta \) satisfy constraints (10) and (11), then for any \( \tau \), the negative regret is bounded by

\[
-\langle U_{\tau+1}, P_\tau - P_0 \rangle B \leq \mathbb{I}\{ \text{reset happened at } \tau \} \left( -\frac{5B}{12\beta} + dB \right)
\]

**Proof**  If no reset happened at \( \tau \), we have \( P_\tau \succeq P_0 \) and the term is bounded by 0. Otherwise by Lemma 29 and the reset condition, we have

\[
\langle U_{\tau+1}, P_\tau \rangle \geq \frac{1}{1+6\eta} \langle U_{\tau+1}, P_{\tau+1} \rangle \geq \frac{1}{2(1+6\eta)^2\beta}.
\]

By the constraint (10), we have \((1+6\eta)^2 \leq \frac{6}{\beta}\). Using \( P_0 = dI_d \) completes the proof.
Proof of Theorem 1 and Theorem 4. We use \( \tilde{d} \) to denote \( d^2 \) in the full PSD case and \( d \) in the regular portfolio case (i.e. all matrices are diagonal matrices). With

\[
B = \frac{264}{5} \tilde{d} \log(T)
\]

\[
\beta = \frac{11 \tilde{d}}{7 B d}
\]

\[
\eta = \frac{1}{4 B},
\]

the constraints can be seen to (10)-(12) be satisfied. Consider any epoch \( e \) with the reset points \( T_{e-1} < T_e \leq T \). Let \( L \) represent the length of the epoch, i.e. \( L = T_e - T_{e-1} \). We drop the superscript \( e \) below for brevity. Then for any comparator \( U \succeq T^{-1}1_d \), we have that

\[
\text{Reg}_e(U) = \sum_{t = T_{e-1}}^{T_e-1} (f_t(X_t) - f_t(U)) \leq \sum_{\tau = 1}^{L} (\hat{f}_\tau(X_\tau) - \hat{f}_\tau(U)) \quad \text{(by Lemma 6)}
\]

\[
\leq \max_{U' \in A} \sum_{\tau = 1}^{L} (\hat{f}_\tau(X_\tau) - \hat{f}_\tau(U')) - \eta^{-1}R(U') + \eta^{-1}R(U)
\]

\[
= \sum_{\tau = 1}^{L} (\hat{f}_\tau(X_\tau) - \hat{f}_\tau(U_{\tau+1})) - \eta^{-1}R(U_{\tau+1}) + \eta^{-1}R(U) \quad \text{(by Lemma 7)}
\]

\[
= \sum_{t = 1}^{\tau} (g_t(X_\tau) - g_t(U_{\tau+1})) - \eta^{-1}R(U_{\tau+1}) + \sum_{t = 1}^{\tau} (X_t - U_{\tau+1}, B(P_t - P_{t-1})) + \underbrace{\eta^{-1}R(U)}_{\leq \eta^{-1}d \log(T)}
\]

\[
\leq \frac{11}{\beta} \tilde{d} \log(T) + 2d \log(T)B + \frac{d \log(T)}{\eta} - \frac{5B}{12\beta} - dB \| \text{reset happened at } \tau \| \quad \text{(by Lemma 34-36)}
\]

\[
\leq \frac{11}{\beta} \tilde{d} \log(T) + 7d \log(T)B - \frac{5B}{12\beta} \| \text{reset happened at } \tau \|
\]

\[
= \frac{3696}{5} \tilde{d} \log^2(T) - \frac{3696}{5} \| \text{reset happened at } \tau \|.
\]

Proof of Corollary 2 and Corollary 5. We use \( \tilde{d} \) to denote \( d^2 \) in the full PSD case and \( d \) in the regular portfolio case (i.e. all matrices are diagonal matrices). Define \( U^0 = \text{arg } \min_{U \in A} \sum_{t = 1}^{T} f_t(X) \) and \( U = (1 - \frac{d}{T})U^0 + \frac{d}{T}(\frac{1}{2}1_d) \). By construction \( U \succeq T^{-1}1_d \) is satisfied. As denoted earlier \( T_1, \ldots, T_E \) are the reset points of Algorithm 2 over the game with \( T_0 \) steps, and \( T_0 = 1 \) and \( T_{E+1} = T + 1 \) by convention. We now derive the following succession of inequalities

\[
\text{Reg} \leq \text{Reg}(U) - T \log \left( 1 - \frac{d}{T} \right) \leq \text{Reg}(U) + O(d)
\]

\[
\leq \sum_{e = 0}^{E} \sum_{t \in E_e} (f_t(X_t) - f_t(U)) + O(d) \leq O(d \tilde{d} \log^2(T)) ,
\]
where the first inequality follows via a simple bound on the optimality gap between $U^0$ and $U$ and the last step uses the epoch-wise regret bounds established in Theorem 1 and Theorem 4.

Proof of Lemma 7 In the proof we omit the superscript $e$ for brevity. Define for any $s$, the set $\mathcal{D}_s := \{ X \in \mathcal{A} \mid \langle X, R_s \rangle \leq \beta^{-1} \langle X_s, R_s \rangle \}$. As we have shown in Section 4 we have that for all $s$, $\hat{f}_t|\mathcal{D}_s = \hat{f}_s|\mathcal{D}_s$ where $l|s$ for a function $l$ and a set $S$ denotes the restriction of the function $l$ on the set $S$. The first step is to show the following for any step $\tau$

$$D_\tau := \{ U \in \mathcal{A} \mid U \preceq \beta^{-1} P^{-1} \} \subset \bigcap_{s=1}^{\tau} \mathcal{D}_s.$$ 

To derive the above, note that due to $U \in D_\tau$, considering any $s \leq \tau$ and noting that $R_s \succeq 0$, we have

$$\frac{\langle U, R_s \rangle}{\langle X_s, R_s \rangle} \leq \sup_{R' \in \mathcal{H}_+^d} \frac{\langle U, R \rangle}{\langle X_s, R' \rangle} = \sup_{R' \in \mathcal{H}_+^d} \frac{\langle U, X^{-\frac{1}{2}} R X^{-\frac{1}{2}} \rangle}{\langle X_s, X^{-\frac{1}{2}} R X^{-\frac{1}{2}} \rangle} = \sup_{R' \in \mathcal{H}_+^d} \frac{\langle X^{-\frac{1}{2}} U X^{-\frac{1}{2}}, R' \rangle}{\text{Tr}(R')} \leq \max_i \text{ev}_i(X^{-\frac{1}{2}} U X^{-\frac{1}{2}}) \leq \max_i \text{ev}_i(P^\frac{1}{2} U P^\frac{1}{2}) \leq \beta^{-1},$$

which concludes that claim. Next we show that $U_{\tau+1} \in \text{int}(D_L)$. Since $L - 1$ did not trigger a reset, we know that $U_L \prec \frac{1}{2(1+6\eta)} L^{-1}$. By Lemma 29 and 30, we have $U_{\tau+1} \preceq 2U_L$ and $P^{-1} \preceq (1+6\eta) P^{-1}_{L+1}$. Hence $U_{\tau+1} \prec \beta^{-1} P^{-1}_{\tau+1}$. Finally since $\hat{f}_t|\mathcal{D}_s = \hat{f}_s|\mathcal{D}_s$ and $U_{\tau+1}$ is by definition the minimizer

$$U_{\tau+1} = \arg \min_{X \in \mathcal{A}} \sum_{s=1}^{t} \hat{f}_s(X) + \eta^{-1} R(X),$$

this implies that $U_{\tau+1}$ is a local minimum and by convexity a global minimum of the LHS in Lemma 7.

Appendix E. Solving the SCHRÖDINGER’S-BIONS optimization problem

In each iteration of SCHRÖDINGER’S-BIONS (Algorithm 2), the main computational effort is in solving the optimization problems

$$X_{\tau+1}^e \leftarrow \arg \min_{X \in \mathcal{A}} G^e_{\tau}(X) \quad \text{and} \quad U_{\tau+1}^e \leftarrow \arg \min_{X \in \mathcal{A}} \hat{f}^e_{\tau}(X).$$

We now show that these can be rewritten as convex minimization problems over a bounded convex subset of $\mathbb{R}^{d^2}$, such that the gradient for the objective can be computed in $O(\text{poly}(d))$ time. Also, it suffices to solve these optimization problems to an accuracy of $\frac{1}{\text{poly}(T)}$ with negligible impact on the regret. Hence, the optimization can be done via a method like ellipsoid or Vaidya’s algorithm in $O(\text{poly}(d))$ time per iteration.

Towards the above goal, we first identify $\mathcal{H}^d$ with the real space $\mathbb{R}^{d^2}$ simply by enumerating the real and imaginary parts of the $\frac{d^2-d}{2}$ lower triangular entries excluding the diagonal entries, and then the $d$ real diagonal entries. Let $\phi : \mathcal{H}^d \rightarrow \mathbb{R}^{d^2}$ denote this mapping. It is obvious that $\phi$ is linear.
and a bijection. Thus, if \( f : \mathcal{H}^d \rightarrow \mathbb{R} \) is a convex function, then \( f \circ \phi^{-1} \) is also convex. Furthermore, \( \phi(A) \) is a bounded convex set. So it suffices to show that \( G_r^e \) and \( \hat{F}_r^e \) are convex functions on \( A \).

We show this for \( G_r^e \), the reasoning for \( \hat{F}_r^e \) is analogous.

\[
G_r^e(X) = \eta^{-1} R(X) + \sum_{s=1}^{\tau} \hat{f}_s^e(X) - \langle X, P_r^e - P_0^e \rangle B.
\]

It is well-known that the log-det regularizer \( R \) is convex over \( \mathcal{H}^d \) (one way to see that is to use the fact \( -\log \det(X) = -\text{Tr}(\log(X)) \), and then use the operator concavity of \( \log(X) \), which follows from Löwner’s theorem (Löwner, 1934)). The last term \( -\langle X, P_r^e - P_0^e \rangle B \) is linear and therefore convex, so it remains to show that \( \hat{f}_s^e(X) \) is convex for any \( s \). From the definition of \( \hat{f}_s^e \) in (2) we see that we only need to show that \( \langle X, \nabla f_t(X_i) \rangle^2 \) is convex. But this follows because \( X \mapsto \langle X, \nabla f_t(X_i) \rangle \) is a linear function of \( X \) mapping \( X \) to a real number since \( \nabla f_t(X_i) \) is Hermitian, and \( u \mapsto u^2 \) is convex over real numbers.

Finally, turning to gradient computation for \( G_r^e \), note that \( \nabla R(X) = -X^{-1} \) which can be computed in \( \tilde{O}(d^3) \) time. Then, it is easy to see that we can combine all the quadratic surrogate functions \( \hat{f}_s^e \) into a single quadratic function that we can maintain in \( \tilde{O}(\text{poly}(d)) \) memory over the iterations, and thus we can compute the gradient of \( \sum_{s=1}^{\tau} \hat{f}_s^e(X) \) in \( \tilde{O}(\text{poly}(d)) \) time as well. The gradient of \( -\langle X, P_r^e - P_0^e \rangle B \) is just \( B(P_r^e - P_0^e) \). Thus, we can compute gradients of \( G_r^e \) in \( \tilde{O}(\text{poly}(d)) \) time.

We note that the running time can be further improved by using Newton’s method since the functions \( G_r^e \) and \( \hat{F}_r^e \) are actually self-concordant, since all the component functions (log-det, linear, and quadratic) are self-concordant.

Appendix F. FTRL lower bound omitted proofs.

First, we prove Lemma 9. This lemma follows from Lemma 45, since \( \Pi A \) has non-zero volume, and the fact that Assumption 3 holds, as shown by the following lemma:

Lemma 37  For any \( \eta > 0 \), LB-FTRL satisfies Assumption 3 with \( c_2 = \frac{1}{(1+\eta)^2} \).

Proof First note that for any \( x, y \in \text{int}(\Delta([d])) \), we have

\[
\nabla^2_{\Pi} F_t(x) = \sum_{s=1}^{t} \frac{(\Pi r_s)(\Pi r_s)^\top}{\langle x, r_s \rangle^2} + \sum_{i=1}^{d} \frac{(\Pi e_i)(\Pi e_i)^\top}{\langle x, e_i \rangle^2} \geq \min_{i \in [d]} \frac{y_{t,i}^2}{x_{t,i}^2} \nabla^2_{\Pi} F_t(y).
\]

Hence we need to prove that for \( c_2 = \frac{1}{(1+\eta)^2} \), we have \( \min_{i \in [d]} x_{t,i} / x_{t,i}^\lambda \geq \sqrt{c_2} \).

We have

\[
D_{G_1^\lambda}(x_t^\lambda, x_t) + D_{G_2^\lambda}(x_t, x_t^\lambda) = \langle x_t^\lambda - x_t, \nabla G_1^\lambda(x_t^\lambda) - \nabla G_1^\lambda(x_t) \rangle = \langle x_t^\lambda - x_t, -\lambda \nabla f_t(x_t) \rangle = \lambda \left( \frac{\langle x_t^\lambda, r_t \rangle}{\langle x_t, r_t \rangle} - 1 \right).
\]

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Let $H^\lambda_t(x) = G^\lambda_t - \sum_{s=1}^{l-1} f_s(x)$, then
\[D_H^\lambda_t(x^\lambda_t, x_t) + D_H^\lambda_t(x_t, x^\lambda_t) = \langle x^\lambda_t - x_t, \nabla H^\lambda_t(x^\lambda_t) - \nabla H^\lambda_t(x_t) \rangle = \eta^{-1} \sum_{i=1}^{d} \left( \frac{x^\lambda_{t,i}}{x_{t,i}} + \frac{x_{t,i}}{x^\lambda_{t,i}} - 2 \right) + \lambda \left( \frac{\langle x^\lambda_t, r_{t,i} \rangle}{\langle x_t, r_{t,i} \rangle} + \frac{\langle x_t, r_{t,i} \rangle}{\langle x^\lambda_t, r_{t,i} \rangle} - 2 \right).

Since by construction $\nabla^2 G^\lambda_t \geq \nabla^2 H^\lambda_t$, we have
\[D_G^\lambda_t(x^\lambda_t, x_t) + D_G^\lambda_t(x_t, x^\lambda_t) \geq D_H^\lambda_t(x^\lambda_t, x_t) + D_H^\lambda_t(x_t, x^\lambda_t),
\]
which implies
\[\lambda \left( 1 - \frac{\langle x_t, r_{t,i} \rangle}{\langle x^\lambda_t, r_{t,i} \rangle} \right) \geq \eta^{-1} \sum_{i=1}^{d} \left( \frac{x^\lambda_{t,i}}{x_{t,i}} + \frac{x_{t,i}}{x^\lambda_{t,i}} - 2 \right).
\]

Let $z = \arg \min_{x \in [d]} \frac{x^\lambda_{t,i}}{x_{t,i}}$, then this results in
\[(1 - z) \geq \eta^{-1}(z^{-1} - z - 2) \iff z \geq \frac{1}{1 + \eta},
\]
as required.

In the remainder of this section, we use $f(d, T) = O(g(d, T))$, $f(d, T) = \Omega(g(d, T))$ to mean that there exists universal constants $C > c > 0$ and $T_0 = \operatorname{Poly}(d)$ such that for all $T > T_0$, it holds $f(d, T) \leq C g(d, T)$ and $f(d, T) \geq c g(d, T)$ respectively. $\operatorname{Poly}(x)$ hereby means that there exists some fixed exponent $a \in [0, \infty)$ such that the statement holds for $x^a$. Finally $f(d, T) = \Theta(g(d, T))$ means $f(d, T) = O(g(d, T))$ and $f(d, T) = \Omega(g(d, T))$ hold simultaneously. Also recall that we assume $T > T_0 = \operatorname{Poly}(T, d)$, specifically we will use $T \leq T^\alpha$ throughout this section.

Define the scaling factors $(c_{i,j})_{i,j=0} = 1 - 2^i T^{-\alpha}$, where $I = \lceil \frac{n}{\alpha} \log_2(T^\alpha) \rceil$. For $x \in \Delta([d])$, we define the “pulling to the center” operator $(s)$, by $x^{(s)} = \Pi^{-1} c_s \Pi x = c_s x + (1 - c_s) c$.

**Basic calculations:** By definition $c_{i,j} = \Theta(1)$ for all $i \in [I] \cup \{0\}$. Further we have for any $s, s' \in [I] \cup \{0\}$:
\[1 - c_sc_{s'} = (2^s + 2^{s'}) T^{-\alpha} - 2^{s + s'} T^{-2\alpha} = \Theta(2^{\max(s, s')} T^{-\alpha}).
\]

For any $x, y \in \Delta([d])$, we have
\[\langle x^{(s)}, y^{(s')} \rangle = \frac{1}{d} + \langle \Pi x^{(s)}, \Pi y^{(s')} \rangle = \frac{1}{d} \cdot c_sc_{s'} \langle \Pi x, \Pi y \rangle = \frac{1 - c_sc_{s'}}{d} + c_sc_{s'} \langle x, y \rangle.
\]

By the assumption on the sequence, we have for any $j < i$:
\[\langle t_i^{(s)}, o_j^{(s')} \rangle = \Omega(\langle t_i, o \rangle) = \Omega \left( \frac{1}{\operatorname{Poly}(d)} \right) \tag{14}
\]
\[\langle t_i^{(s)}, o_i^{(s')} \rangle = \frac{1 - c_sc_{s'}}{d} = \Theta \left( \frac{2^{\max(s, s')}}{dT^\alpha} \right) \tag{15}
\]
Algorithm 3: Sequence for large regret.

**Input:** \( (t_i, o_i)_{i=1}^T, \alpha = \frac{1}{8}, T \)

for \( i = 1, 2, \ldots, T \) do
  for \( k = 1, \ldots, T^\alpha \) do
    for \( s = 1, \ldots, \lceil \frac{1}{8} \alpha \log_2(T) \rceil \) do
      while \( x_t \neq t_i^{(s)} \) do
        \( r_t \leftarrow \text{move-to-x}(t_i^{(s)}; F_{t-1}) \)
        \( t \leftarrow t + 1 \)
      end
      \( r_t \leftarrow o^{(s)}(t_i) \)
      \( t \leftarrow t + 1 \)
    end
  end
end

**Function move-to-x** \( (x; F) \):
  \( g \leftarrow \Pi \nabla F(x) \)
  \( g \leftarrow \min \{ T^{-\frac{1}{2}} / \|g\|_2, d \max \{1 - \langle \Pi t, \Pi r \rangle, 0\} \} g \)
  return \( \Pi^{-1} g \)

Bounding the movement steps. The main result of this section is the following Lemma.

**Lemma 38** The number of movement steps up to time \( T \) is bounded by \( O \left( \text{Poly}(d)T^{3\alpha + \frac{3}{2}} \log(T)^2 \right) \).

In order to prove this Lemma, we first require the following.

**Lemma 39** The while routine over move-to-x for a target \( t \) up from time \( t \) requires \( \tau \leq \frac{2^{1/2}}{d} \| \nabla \Pi F_t(t) \|_2 + 1 \) steps.

**Proof** We have reached the target, if at time \( t + \tau \) holds \( \nabla \Pi F_{t+\tau}(t) = 0 \). We select the movement returns \( r_s \) for \( s \in \{t, \ldots, t + \tau - 1\} \) such that

\[
\| \nabla \Pi F_{s+1}(t) \|_2 = \max \{0, \| \nabla \Pi F_s(t) \|_2 - \| \nabla \Pi f_{s+1}(t) \|_2 \}.
\]

When we cannot reach the target in one step, the norm of the gradient is

\[
\| \nabla \Pi f_{s+1}(t) \|_2 = \frac{\| \Pi r_s \|}{d + \langle \Pi t, \Pi r_s \rangle} \geq \frac{T^{-1/2}}{d + T^{-1/2}} \geq \frac{d}{2} T^{-1/2}.
\]

Hence the number of steps \( \tau \) until the norm is 0 is bounded by

\[
\tau \leq \frac{2^{1/2}}{d} \| \nabla \Pi F_t(t) \|_2 + 1.
\]
Lemma 40  For any movement-return \( r \) and any \( x, y \in \Delta([d]) \), it holds
\[
\| \nabla_\Pi f(x; r) - \nabla_\Pi f(y; r) \| = O(d^2 T^{-1}) .
\]

Proof
\[
\| \nabla_\Pi f(x; r) - \nabla_\Pi f(y; r) \|
= \left| \frac{1}{1/d + \langle \Pi x, \Pi r \rangle} - \frac{1}{1/d + \langle \Pi y, \Pi r \rangle} \right| \| \Pi r \|
\leq \left( \frac{1}{1/d - T^{-\frac{1}{2}}} - \frac{1}{1/d + T^{-\frac{1}{2}}} \right) T^{-\frac{1}{2}} = \frac{2T^{-1}}{1/d^2 - T^{-1}} = O(d^2 T^{-1}) ,
\]
where we use that movement returns by construction satisfy \( \| \Pi r \| \leq T^{-\frac{1}{2}} \) and \( \| \Pi x \| \leq \| x \| \leq 1 \) for any \( x \in \Delta([d]) \).

Lemma 41  For any \( x \in \Delta([d]) \) and \( s \in [I] \cup \{0\} \), the largest possible gradient of any regularizer part \( r_i(x^{(s)}) = f(x^{(s)}; e_i), i \in [d] \) is bounded by
\[
\max_{x \in \Delta([d])} \left\| \nabla_\Pi r_i(x^{(s)}) \right\| = O \left( \frac{d T^\alpha}{2^s} \right) .
\]

Proof
\[
\left\| \nabla_\Pi r_i(x^{(s)}) \right\| = \left\| \frac{\Pi e_i}{\langle x^{(s)}, e_i \rangle} \right\| \leq \frac{d}{1 - c_s} = d \frac{T^\alpha}{2^s} ,
\]
where we used
\[
\langle x^{(s)}, e_i \rangle = \frac{1 - c_s}{d} + c_s \langle x, e_i \rangle \geq \frac{1 - c_s}{d} .
\]

Proof [Proof of Lemma 38] For the initial move-to-x, we have \( F_0 (t^{(0)}_1) = R(t^{(0)}_1) \), hence by combining Lemma 41 and 39, we require \( O(dT^{\alpha + \frac{1}{2}}) \) initial steps. Afterwards, we need to bound the steps between any two targets \( t^{(s)}_{k'} \), \( t^{(s')}_{k''} \), where \( k \leq k' \). Assume this switch happens at time \( \tau \leq T \) (since the Lemma statement is concerned with movement steps before time \( T \)), directly after the agent observed a return \( o^{(s)}_k \) at target \( t^{(s)}_k \). Hence
\[
\left\| \nabla_\Pi F_\tau(t^{(s')}_{k''}) \right\| \leq \left\| \nabla_\Pi f(t^{(s')}_{k''}; o^{(s)}_k) \right\| + \left\| \nabla_\Pi F_{\tau-1}(t^{(s')}_{k''}) - \nabla_\Pi F_{\tau-1}(t^{(s)}_k) \right\| ,
\]
where we use that \( \left\| \nabla_\Pi F_{\tau-1}(t^{(s)}_k) \right\| = 0 \) since the agent was in that point when receiving \( r_\tau \).

Splitting the time-steps into movement-returns \( \mathcal{M}_\tau := \{ t \in [\tau] \mid \| \Pi r_t \| \leq T^{-1/2} \} \) and regular
returns yields
\[
\left\| \nabla_{t_1} f(t_{k_i}^{(s)}; o_{k_i}^{(s)}) \right\| + \left\| \nabla_{t_1} F_{t-1}(t_{k_i}^{(s)}; o_{k_i}^{(s)}) - \nabla_{t_1} F_{t-1}(t_{k_i}^{(s)}) \right\|
\]
\[
\leq \left\| \nabla_{t_1} R(t_{k_i}^{(s)}) - \nabla_{t_1} R(t_{k_i}^{(s)}) \right\| + \left\| \sum_{s \in M_r} \nabla_{t_1} f_s(t_{k_i}^{(s)}) - \nabla_{t_1} f_s(t_{k_i}^{(s)}) \right\|
\]
\[
+ T^\alpha \left( \sum_{j=1}^{k-1} \sum_{r=0}^{I} \left( \left\| \nabla_{t_1} f(t_{k_i}^{(s)}; o_{j}^{(r)}) \right\| + \left\| \nabla_{t_1} f(t_{k_i}^{(s)}; o_{j}^{(r)}) \right\| \right) \right)
\]
\[
+ \sum_{r=0}^{I} \left( \left\| \nabla_{t_1} f(t_{k_i}^{(s)}; o_{j}^{(r)}) \right\| + \left\| \nabla_{t_1} f(t_{k_i}^{(s)}; o_{j}^{(r)}) \right\| \right)
\]
\[
\leq O(d^2 T^\alpha) + O(d^2) \quad \text{(Lemma 41 and 40)}
\]
\[
+ O \left( \max_{j \leq \ell, r, r' \in [I] \cup \{0\}} \frac{T^\alpha T \log(T)}{\langle t_{k_i}^{(r)}, o_{j}^{(r')} \rangle} \right) + O \left( \max_{j \leq \ell, r, r' \in [I] \cup \{0\}} \frac{T^\alpha \log(T)}{\langle t_{k_i}^{(r)}, o_{j}^{(r')} \rangle} \right)
\]
\[
= O(\text{Poly}(d)T^{2\alpha} \log(T)) . \quad \text{(Equation (14) and (15))}
\]

The proof is completed by applying Lemma 39, noting that the number of switches is bounded by \( I{T} \leq T^\alpha \log(T) \).

Bounding the Hessian trace. We first bound the Hessian trace of movement-steps.

**Lemma 42** The movement time-steps \( M_r \) for any \( \tau \leq T \) and any \( t \in \Delta([d]) \) satisfy

\[
\sum_{t \in M_r} \left\| \nabla_{t_1} f(t; r_t) \right\|^2 = O(d^2) .
\]

**Proof** By construction \( \|\Pi r_t\| \leq T^{-\frac{1}{2}} \), so

\[
\left\| \nabla_{t_1} f(t; r_t) \right\|^2 = \frac{\|\Pi r_t\|^2}{(\frac{1}{d} + \langle \Pi t, \Pi r_t \rangle)^2} \leq \frac{T^{-1}}{(\frac{1}{d} - T^{-\frac{1}{2}})^2} = O(d^2 T^{-1}) .
\]

Summing over less than \( T \) time-steps completes the proof.

We are ready to bound the total Hessian.

**Lemma 43** Assume \( \tau \leq T \) is the time-step where the \( m \)-th iteration through targets \( (t_i^{(s)})_{s=0}^I \) is completed, then the trace of the Hessian at any target \( t_i^{(s)} \) is bounded by

\[
\text{Tr}(\nabla_{t_1}^2 F_{t}(t_i^{(s)})) = O \left( (\text{Poly}(d) + m(s + 1)) d^2 T^{2\alpha} \right) \left\| \Pi o_{i} \right\|^2 .
\]

**Proof** We split the trace into 4 terms below based on various contributions from (a) the regularizer, (b) the time steps \( M_r \) where the returns are movement-returns selected by the move-to-x subroutine,
(c) the returns $o_j^{(s)}$ selected for $j < i$ and (d) the returns selected for targets $t_i^{(s)}$, $s \in [I] \cup \{0\}$. The first two terms are bounded by Lemma 41 and 42 respectively.

$$\text{Tr}(\nabla^2 F_t(t_i^{(s)})) = \sum_{i=1}^{d} \| \nabla \Pi f_t(t_i^{(s)}; e_i) \|^2 + \sum_{s \in \mathcal{M}_r} \| \nabla f_s(t_i^{(s)}) \|^2 + \sum_{j=1}^{i-1} \sum_{s'=0}^{l} \| \nabla f(t_i^{(s)}; o_j^{(s')}) \|^2 + m \sum_{s'=0}^{l} \| \nabla f(t_i^{(s)}, o_j^{(s')}) \|^2 \leq d^3 \frac{T^{2\alpha}}{2^{2s}} + O(d^2) + \max_{j<i,s' \in [I] \cup \{0\}} \frac{T I}{\langle t_i^{(s)}, o_j^{(s')} \rangle^2} + m \sum_{s'=0}^{l} \| \Pi o_i \|^2 \leq O \left( d^3 \frac{T^{2\alpha}}{2^{2s}} \right) + O(\text{Poly}(d)T^{\alpha} \log(T)) + m \left( \sum_{s'=0}^{s} 2^{-2s} + \sum_{s'=s+1}^{l} 2^{-2s'} \right) d^2 T^{2\alpha} \| \Pi o_i \|^2 = O \left( \text{Poly}(d) \frac{T^{2\alpha}}{2^{2s}} \right) + O \left( (m+1)d \frac{T^{2\alpha}}{2^{2s}} \right) \| \Pi o_i \|^2 ,$$

where we use equations (14) and (15) and the fact that $T \leq T^\alpha$. $O(T^{\alpha} \log(T)) = O(\frac{T^{2\alpha}}{2^{2s}})$ follows from

$$2^{2s} T^{-\alpha} \log(T) \leq T^{-\alpha/3} \log(T) = O(1) .$$

Finally, observe

$$0 = \langle t_i, o_i \rangle = \frac{1}{d} + \langle \Pi t_i, \Pi o_i \rangle \geq \frac{1}{d} - \| \Pi o_i \| .$$

Hence

$$O(\text{Poly}(d) \frac{T^{2\alpha}}{2^{2s}}) = O(\text{Poly}(d) \frac{T^{2\alpha}}{2^{2s}}) \| \Pi o_i \|^2 ,$$

which concludes the proof.

\section*{F.1. Main lower bound proof}

\textbf{Proof} [Proof of Theorem 8] By Lemma 38, there are $O(\text{Poly}(d)T^{3\alpha+\frac{1}{s}} \log^2(T))$ movement-returns before time $T$ and the algorithm walks through $O(T I) = O(T^{\alpha} \log(T))$ regular returns, hence for $\alpha = \frac{1}{8}$, $O(\text{Poly}(d)T^{7/8} \log^3(T)) = O(T^{15/16})$ and there exists a sufficiently large $T_0 = \text{Poly}(d,T)$, such that the algorithm finishes before time $T$.

Next we bound the stability term. We have

$$\| \nabla f_t(x_i) \|^2 \langle \nabla^2 F_t(x_i)^{-1} \rangle \geq \frac{\| \nabla f_t(x_i) \|^2}{\text{Tr}(\nabla^2 F_t(x_i))} .$$
For the \( m \)-th time of visiting \( t_i^{(s)} \), the denominator is by Lemma 43 bounded by \( \mathcal{O}((\text{Poly}(d) + m(s + 1)))d^2 T^{2\alpha} \frac{T}{2s} \|\Pi o\|_2^2 \). For \( m \geq T^{\alpha/2} \), the trace bound simplifies to \( \mathcal{O}(m(s + 1)d^2 2^{-2s} T^{2\alpha} \|\Pi o\|_2^2) \), since we assume \( T^{\alpha/2} = \Omega(\text{Poly}(d)) \). The nominator is
\[
\left\| \nabla \Pi f(t_i^{(s)}; o_i^{(s)}) \right\|^2 = \Theta(d^2 2^{-2s} T^{2\alpha} \|\Pi o_i\|_2^2).
\]
For the total stability, we have
\[
(\text{stab}) \geq \sum_{i=1}^{T} \sum_{m=T^{\alpha/2}}^{T} \sum_{s=0}^{I} \frac{1}{m(s + 1)} = \Omega(T \log(T) \log(I)).
\]
Finally \( \log(I) = \Theta(\log \log(T)) \) completes the proof.

**Appendix G. Follow-The-Regularized-Leader analysis**

Both our main results rely on the standard analysis for FTRL, which we revisit in this section. Vanilla FTRL is used for online learning over a convex action set \( \mathcal{X} \), where the environment picks a sequence convex loss functions \( (g_t)_{t=1}^{T} \) from some function space \( \mathcal{G} \). The input to FTRL is a regularizer \( R : \mathcal{X} \to \mathbb{R} \) and the algorithm plays
\[
x_t = \arg\min_{x \in \mathcal{X}} G_{t-1}(x) := \arg\min_{x \in \mathcal{X}} \sum_{s=1}^{t-1} g_s(x) + \eta^{-1} R(x).
\]
We consider in this paper special cases of FTRL that allow for a simple regret analysis.

**Assumption 1** The action set \( \mathcal{X} \subset \mathbb{R}^d \) is compact and the regularizer \( \nabla R(x) \) is strictly convex, twice continuously differentiable and goes to infinity on the boundary of \( \mathcal{X} \).

This assumption is directly satisfies by the simplex \( \mathcal{A} = \Delta([d]) \) and the log-barrier regularizer. Furthermore the log loss and log-barrier regularization ensure the following.

**Assumption 2** There exists a universal constant \( c_1 \), such that for any sequence of functions \( g_1, \ldots g_T \), any point \( \bar{x}_t \) on the line between \( x_t \) and \( x_{t+1} \), satisfies
\[
\nabla^2 G_t(\bar{x}_t) \preceq c_1 \nabla^2 G_t(x_t).
\]

**Assumption 3** There exists a universal constant \( c_2 \), such that for any sequence of functions \( g_1, \ldots g_T \), the interpolation between \( x_t \) and \( x_{t+1} \) defined by
\[
x_t^\lambda := \arg\min_{x \in \mathcal{X}} G_{t-1}(x) + g_t(x) - (1 - \lambda) \langle x, \nabla g_t(x_t) \rangle,
\]
satisfies for any \( \lambda \in [0, 1] \)
\[
\nabla^2 G_t(x_t^\lambda) \succeq c_2 \nabla^2 G_t(x_t).
\]
For any FTRL algorithm satisfying the assumptions above, the regret is tightly lower and upper bounded as shown in the following lemmas.

The following lemma gives an upper bound on the regret. We will prove this lemma even for the quantum case. We refer the reader to Section C for relevant definitions of gradient, Hessian and Bregman divergences in that setting.

**Lemma 44** Under Assumptions 1 and 2, the regret of FTRL is upper bounded for any comparator $u$ by

$$\sum_{t=1}^{T} (g_t(x_t) - g_t(u)) \leq \frac{C_1}{2} \sum_{t=1}^{T} \left\| \nabla g_t(x_t) \right\|^{2}_{(\nabla^2 G_t(x_t))^{-1}} + \frac{R(u) - R(x_1)}{\eta}.$$

**Proof** We have

$$\sum_{t=1}^{T} (g_t(x_t) - g_t(u)) = \sum_{t=1}^{T} (G_t(x_t) - G_t(x_{t+1})) + G_T(x_{T+1}) - G_T(u) + \eta^{-1}(R(u) - R(x_1)).$$

For the upper bound, since $x_{t+1}$ minimizes $G_t$ we have that $\forall x \in A : \langle x - x_{t+1}, \nabla G_t(x_{t+1}) \rangle = 0$ (For the quantum learning case this is explicitly derived in Lemma 20). By Taylor’s theorem, there exists $\lambda \in [0, 1]$ such that $D_{G_t}(x_{t+1}, x_t) = \frac{1}{2} \left\| x_{t+1} - x_t \right\|^2_{\nabla^2 G_t(x_t)}$ (For the quantum case this statement is explicitly proven in Lemma 15). Therefore we have that

$$G_t(x_t) - G_t(x_{t+1}) = D_{G_t}(x_t, x_{t+1})$$

$$= \langle x_t - x_{t+1}, \nabla G_t(x_t) - \nabla G_t(x_{t+1}) \rangle - D_{G_t}(x_{t+1}, x_t)$$

$$= \langle x_t - x_{t+1}, \nabla g_t(x_t) \rangle - \frac{1}{2} \left\| x_{t+1} - x_t \right\|^2_{\nabla^2 G_t(x_t)}$$

$$\leq \left\| x_t - x_{t+1} \right\| \left\| \nabla g_t(x_t) \right\| \left\| \nabla G_t(x_t) \right\|^{-1} - \frac{1}{2} \left\| x_{t+1} - x_t \right\|^2_{\nabla^2 G_t(x_t)}$$

$$\leq \frac{1}{2} \left\| \nabla g_t(x_t) \right\|^2_{\nabla^2 G_t(x_t)} - \frac{C_1}{2} \left\| \nabla g_t(x_t) \right\|^2_{\nabla^2 G_t(x_t)}.$$

The above statement combined with the decomposition above implies the statement of the lemma.

**Lemma 45** If $A$ has non-zero volume in its embedded space and Assumptions 1 and 3 are satisfied, then the regret is lower bounded by

$$\frac{C_2}{2} \sum_{t=1}^{T} \left\| \nabla g_t(x_t) \right\|^2_{(\nabla^2 G_t(x_t))^{-1}} \leq \max_{u \in A} \sum_{t=1}^{T} (g_t(x_t) - g_t(u)).$$

**Proof** We have

$$\sum_{t=1}^{T} (g_t(x_t) - g_t(u)) = \sum_{t=1}^{T} (G_t(x_t) - G_t(x_{t+1})) + G_T(x_{T+1}) - G_T(u) + \eta^{-1}(R(u) - R(x_1)).$$
For the lower bound, we can simply lower bound $\max_{u'}$ by picking $u' = x_{T+1}$ and omit the last two terms. It remains to analyse the first term. Given that $R(x) \to \infty$ on the boundary of $X$, the points $x_t$ are all strictly in the interior of $A$.

$$G_t(x_t) - G_t(x_{t+1}) = D_{G_t}(x_t, x_{t+1})$$
$$= D_{G_t}(\nabla G_t(x_{t+1}), \nabla G_t(x_t))$$
$$= \frac{1}{2} \|\nabla G_t(x_{t+1}) - \nabla G_t(x_t)\|^2_{\nabla^2 G_t((1-\lambda)\nabla G_t(x_t) + \lambda \nabla G_t(x_{t+1}))}$$
$$= \frac{1}{2} \|g_t(x_t)\|^2_{(\nabla^2 G_t(x_t))^{-1}} \geq \frac{c^2}{2} \|g_t(x_t)\|^2_{(\nabla^2 G_t(x_t))^{-1}}.$$

The above statement using the decomposition implies the lemma.