TOPOLOGICAL RIGIDITY FOR HOLOMORPHIC
FOLIATIONS

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Abstract. We study analytic deformations and unfoldings of holomorphic foliations in complex projective plane $\mathbb{CP}(2)$. Let $\{F_t\}_{t \in \mathbb{D}_\epsilon}$ be topological trivial (in $\mathbb{C}^2$) analytic deformation of a foliation $F_0$ on $\mathbb{C}^2$. We show that under some dynamical restriction on $F_0$, we have two possibilities: $F_0$ is a Darboux (logarithmic) foliation, or $\{F_t\}_{t \in \mathbb{D}_\epsilon}$ is an unfolding. We obtain in this way a link between the analytical classification of the unfolding and the one of its germs at the singularities on the infinity line. Also we prove that a finitely generated subgroup of $\text{Diff}(\mathbb{C}^n,0)$ with polynomial growth is solvable.

1. Introduction

Let $\text{Fol}(M)$ denote the set of holomorphic foliations on a complex manifold $M$. An analytic deformation of $F \in \text{Fol}(M)$ is an analytic family $\{F_t\}_{t \in Y}$ of foliations on $M$, with parameters on an analytic space $Y$, such that there exists a point "0" $\in Y$ with $F_0 = F$. Here we will only consider deformations where $Y = \mathbb{D} \subset \mathbb{C}$ is a unitary disk. A topological equivalence (resp. analytical equivalence) between two foliations $F_1$ and $F_2$ is a homeomorphism (resp. biholomorphism) $\phi : M \rightarrow M$, which takes leaves of $F_1$ onto leaves of $F_2$, and such that $\phi(\text{Sing}(F_1)) = \text{Sing}(F_2)$. The deformation $\{F_t\}_{t \in \mathbb{D}}$ is topologically trivial (resp. analytically trivial) if there exists a continuous map (resp. holomorphic map) $\phi : M \times \mathbb{D} \rightarrow M$, such that each map $\phi_t = \phi(.,t) : M \rightarrow M$ is a topological equivalence (resp. analytical equivalence) between $F_t$ and $F_0$.

Let $\mathcal{C} \subset \text{Fol}(M)$ be a class of foliations. A foliation $F_0 \in \mathcal{C}$ is topologically rigid in the class if any topologically trivial deformation $\{F_t\}_{t \in \mathbb{D}}$ of $F_0$ with $F_t \in \mathcal{C}$ is analytically trivial.

We also say that $F_0 \in \mathcal{C}$ is $U$-topological rigid in the class $\mathcal{C}$, where $U \subset M$ is an open subset, if any analytic deformation $\{F_t\}_{t \in \mathbb{D}}$ of $F_0$ with $F_t \in \mathcal{C}$, $\forall t$; which is topologically trivial in $U$, is in fact analytically trivial in $M$.

In this part we will be concerned with holomorphic foliations in $\mathbb{CP}(2)$. These foliations are motivated by Hilbert Sixteenth Problem on the number and position of limit cycles of polynomial differential equations.

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in the real plane \((x, y) \in \mathbb{R}^2\) where \(P\) and \(Q\) are relatively prime polynomials. A major attempt in this line was started in 1956 by a seminal work of I. Petrovski and E. Landis \([16]\). They consider \((\ast)\) as a differential equations in the complex plane \((x, y) \in \mathbb{C}^2\), with \(t\) now being a complex time parameter. The integral curves of the vector field are now either singular points which correspond to the common zeros of \(P\) and \(Q\), or complex curves tangent to the vector field which are holomorphically immersed in \(\mathbb{C}^2\). This gives rise to a holomorphic foliation by complex curves with a finite number of singular points. One can easily see that this foliation extends to the complex projective plane \(\mathbb{C}P(2)\), which is obtained by adding a line at infinity to the plane \(\mathbb{C}^2\). Conversely any holomorphic foliation by curves on \(\mathbb{C}P(2)\) is given in an affine space \(\mathbb{C}^2 \hookrightarrow \mathbb{C}P(2)\) by a polynomial vector field \(X = (P, Q) \in \mathfrak{X}(\mathbb{C}^2)\) with \(\gcd(P, Q) = 1\).

Although they didn’t solve this problem, however they introduced a truly novel method in geometric theory of ordinary differential equations. In 1978, Il’yashenko made a fundamental contribution to the problem. Following the general idea of Petrovski and Landis, he studied equations \((\ast)\) with complex polynomials \(P\) and \(Q\) from a topological standpoint without particular attention to Hilbert’s question.

We fix the line at infinity \(L_\infty = \mathbb{C}P(2) \setminus \mathbb{C}^2\) and denote by \(\mathcal{X}(n)\) the space of foliations of degree \(n \in \mathbb{N}\) which leave invariant \(L_\infty\). Let us denote by \(\mathcal{F}(n)\) the space of degree \(n\) foliations on \(\mathbb{C}P(2)\) as introduced in \([10]\). We are interested in the following question:

Under which conditions topologically trivial deformations of a foliation \(\mathcal{F} \in \mathcal{F}(n)\) are analytically trivial?

A remarkable result of Y. Ilyashenko states topological rigidity for a residual set of foliations on \(\mathcal{X}(n)\) if \(n \geq 2\).

More precisely we have:

**Theorem 1.1.** \([8]\) For any \(n \geq 2\) there exists a residual subset \(\mathcal{I}(n) \subset \mathcal{X}(n)\) whose foliations are topologically rigid in the class \(\mathcal{X}(n)\).

This result has been later improved by A. Lins Neto, P. Sad and B. Scardua as follows:

**Theorem 1.2.** \([11]\) For each \(n \geq 2\), \(\mathcal{X}(n)\) contains an open dense subset \(\mathcal{R} \subset \mathcal{X}(n)\) whose foliations are topologically rigid in the class \(\mathcal{X}(n)\).

We stress the fact that in both theorems above we consider deformations \(\{\mathcal{F}_t\}_{t \in \mathbb{D}}\) in the class \(\mathcal{X}(n)\), that is, \(\mathcal{F}_t\) leaves invariant \(L_\infty, \forall t \in \mathbb{D}\); and we assume topological triviality in \(\mathbb{C}P(2)\). This last hypothesis is slightly relaxed by requiring topological triviality for the set of separatrices through the singularities at \(L_\infty\):
The trivial unfolding of 
embeds into an analytic foliation.

if there exists an analytic foliation \( \tilde{F} \) then
\( \tilde{F} \) denotes the set of separatrices of \( F \).

\( \phi \) map and \( \phi \) exists a continuous family of maps
respectively analytically equivalent if there exists a continuous respectively
analytic map \( \phi \).

We also state results for deformations which are topological trivial in
the class \( C \) and obtain in this way a link between the analytical classific ation of the
exceptional divisor is invariant and all of singularities are of saddle-type)
and obtain in this way a link between the analytical classification of the

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\[ \text{3} \]

\[ \text{Theorem 1.3.} \ [11] \text{ For any } n \geq 2, X(n) \text{ contains an open dense subset } S \text{ whose foliations are } s\text{-rigid in the class } X(n). \]

According to [11] a foliation \( F_0 \in X(n) \) is \( s\)-rigid if for any deformation
\( \{F_t\}_{t \in \mathbb{D}} \subset X(n) \) of \( F_0 \) with the \( s\)-triviality property that is: If \( S_t \subset C^2 \)
denotes the set of separatrices of \( F_t \) which are transverse to \( L_\infty \) then there
exists a continuous family of maps \( \phi_t : S_0 \to C^2 \) such that \( \phi_0 \) is the inclusion map and \( \phi_t \) is a continuous injection map from \( S_0 \to C^2 \) with \( \phi_t(S_0) = S_t \);
then \( \{F_t\} \) is analytically trivial.

\[ \text{Remark 1.4.} \text{ Topological triviality in } C^2 \text{ implies } s\text{-triviality.} \]

Let us change now our point of view.

A deformation \( \{F_t\}_{t \in \mathbb{D}} \) of a foliation \( F_0 \) on a manifold \( M \), is an unfolding
if there exists an analytic foliation \( \mathcal{F} \) on \( M \times \mathbb{D} \) with the property that:
\( \mathcal{F}|_{M \times \{t\}} = F_t, \forall t \in \mathbb{D} \). In other words, an unfolding is a deformation which
embeds into an analytic foliation. The trivial unfolding of \( \mathcal{F} \) is given by the
\( \mathcal{F}_t := \mathcal{F}, \forall t \in \mathbb{D} \) and \( \mathcal{F} \) is the product foliation \( \mathcal{F} \times \mathbb{D} \) in \( M \times \mathbb{D} \).

Two unfoldings \( \{F_t\}_{t \in \mathbb{D}} \) and \( \{\tilde{F}_t\}_{t \in \mathbb{D}} \) of \( \mathcal{F} \) are topologically equivalent
respectively analytically equivalent if there exists a continuous respectively
analytic map \( \phi : M \times \mathbb{D} \to M \) such that each map \( \phi_t : M \to M, \phi_t(p) = \phi(p,t) \), is a topological respectively analytical equivalence between \( F_t \) and
\( \tilde{F}_t \).

An unfolding \( \{F_t\}_{t \in \mathbb{D}} \) of a foliation \( F_0 \) on \( M \) is said to be topologically
rigid in the class \( \mathcal{C} \subset \mathcal{F}(n) \) if any analytic unfolding \( \{\tilde{F}_t\}_{t \in \mathbb{D}} \) of \( \mathcal{F} \) (\( \tilde{F}_t \in \mathcal{C}, \forall t \)), which is topologically equivalent to \( \{F_t\}_{t \in \mathbb{D}} \), is necessarily analytically equivalent.

These notions rewrite theorems (1.1) and (1.2) as follows:

\[ \text{Theorem 1.5.} \ [8] \text{ For any } n \geq 2 \text{ there exists a residual subset } I(n) \subset X(n) \text{ whose foliations are topologically rigid trivial unfolding in the class } X(n). \]

\[ \text{Theorem 1.6.} \ [11] \text{ For each } n \geq 2, X(n) \text{ contains an open dense subset } \text{Rig}(n) \subset X(n) \text{ whose foliations are topologically rigid trivial unfolding in the class } X(n). \]

We are now in conditions of stating our main results concerning topolog-
ical rigidity. We stress the fact that a priori, our deformations are allowed to
move the line \( L_\infty \) (Theorems A, B and C) which is \( F_0 \)-invariant by hypoth-
esis. We also state results for deformations which are topological trivial in
\( C^2 \) not necessarily in \( CP(2) \). Finally, we may relax the hypothesis of hyper-
bolicity for \( \text{Sing}(F_0) \cap L_\infty \) by allowing quasi-hyperbolic singularities (i.e the
exceptional divisor is invariant and all of singularities are of saddle-type)
and obtain in this way a link between the analytical classification of the
unfolding and the one of its germs at the singularities \( p \in \text{Sing}(F_0) \cap L_\infty \).

Our main results are the following:

\[ \text{Theorem A.} \text{ Given } n \geq 2 \text{ there exists an open dense subset } \text{Rig}(n) \subset X(n) \]
such that any foliation in $\text{Rig}(n)$ is $C^2$-topological rigid: any deformation $\{F_t\}_{t \in \mathbb{D}}$ of $F = F_0$ which is topologically trivial in $\mathbb{C}^2$ must be analytically trivial in $\mathbb{C}P(2)$ for $t \approx 0$.

**Theorem B.** Let $\{F_t\}_{t \in \mathbb{D}}$ be topological trivial (in $\mathbb{C}^2$) analytic deformation of a foliation $F_0$ on $\mathbb{C}^2$ such that:

1. $F_0$ leaves $L_\infty$ invariant,
2. $\forall p \in \text{Sing}(F_0) \cap L_\infty$, $p$ is a quasi-hyperbolic singularity,
3. $F_0$ has degree $n \geq 2$ and exhibits at least two reduced singularities in $L_\infty$.

Then we have two possibilities:

- $F$ is a Darboux (logarithmic) foliation, or
- $\{F_t\}_{t \in \mathbb{D}}$ is an unfolding.

In this last case the unfolding is analytically trivial if and only if given a singularity $p \in \text{Sing}(F_0) \cap L_\infty$ the germ of the unfolding $\{F_t\}_{t \in \mathbb{D}}$ at $p$ is analytically trivial for $t \approx 0$.

We can rewrite Theorem (B) as follows:

**Theorem C.** Let $F_0$ be a foliation on $\mathbb{C}P(2)$ with the following properties:

1. $F_0$ leaves $L_\infty$ invariant,
2. $\forall p \in \text{Sing}(F_0) \cap L_\infty$, $p$ is a quasi-hyperbolic singularity,
3. $\text{Sing}(F_0) \cap L_\infty$ has at least two reduced singularities.

Given two topologically equivalent unfoldings $\{F_t\}_{t \in \mathbb{D}}$ and $\{F^1_t\}_{t \in \mathbb{D}}$ of $F_0$ we have that they are analytically equivalent if and only if the germs of the unfoldings are analytically equivalent at the singular points $p \in \text{Sing}(F_0) \cap L_\infty$.

One of the main ingredients of the proof of above theorems is the Theorem of Nakai and its consequences [1] and [17]. This motivates characterization of solvability of finitely generated subgroups of complex diffeomorphisms. In this direction we have the following theorem which generalizes Theorem (C) of [5]:

**Theorem D.** Let $G$ be a finitely generated subgroup of $\text{Diff}(\mathbb{C}^n, 0)$. If $G$ has a polynomial growth then $G$ is solvable.

2. **Preliminaries**

Let $\mathcal{F}$ be a (singular) foliation on $\mathbb{C}P(2)$ and $L \subset \mathbb{C}P(2)$ be a projective line, which is not an algebraic solution of $\mathcal{F}$ ($L \setminus \text{Sing}(\mathcal{F})$ is not a leaf of $\mathcal{F}$). We say that $p \in L$ is a tangency point of $\mathcal{F}$ with $L$, if either $p \in \text{Sing}(\mathcal{F})$ or $p \not\in \text{Sing}(\mathcal{F})$ and the tangent spaces of $L$ and of the leaf of $\mathcal{F}$ through
$p$, at $p$, coincide. We say that $L$ is invariant by $\mathcal{F}$ if $\forall p \in L \setminus \text{Sing}(\mathcal{F})$, $p$ is a tangency point of $\mathcal{F}$ with $L$. Denote by $T(\mathcal{F}, L)$, the set of tangency points of $\mathcal{F}$ with $L$. According to [10], if $\text{Sing}(\mathcal{F})$ has codimension $\geq 2$ or equivalently the singularities of $\mathcal{F}$ are finitely many points in $\mathbb{CP}(2)$, then there exists an open, dense and connected subset $NI(\mathcal{F})$ of the set of lines in $\mathbb{CP}(2)$, such that every $L \in NI(\mathcal{F})$ satisfies the following properties:

- $L$ is not invariant by $\mathcal{F}$,
- $T(\mathcal{F}, L)$ is an algebraic subset of $L$ defined by a polynomial of degree $k = k(\mathcal{F})$ in $L$ and this number is independent of $L$.

The integer $k(\mathcal{F})$ is called the degree of foliation $\mathcal{F}$. According to [10], a foliation of degree $n$ in $\mathbb{CP}(2)$ can be expressed in an affine coordinate system by a differential equation of the form

$$(P(x, y) + xg(x, y))dy - (Q(x, y) + yg(x, y))dx = 0,$$

where $P$, $Q$ and $g$ are polynomials such that:

1. $P + xg$ and $Q + yg$ are relatively prime,
2. $g$ is homogeneous of degree $n$,
3. $\max\{\deg(P), \deg(Q)\} \leq n$,
4. $\max\{\deg(P), \deg(Q)\} = n$ if $g \equiv 0$.

Let $B_{n+1}$ be space of polynomials of degree $\leq n + 1$ in two variables. Let $V \subset B_{n+1} \times B_{n+1}$ be the subspace of pairs of polynomials of the form $(p + xg, q + yg)$, where $P$, $Q$ and $g$ are as in (2) and (3) above. Clearly $V$ is a vector subspace of $B_{n+1} \times B_{n+1}$. Let $\mathbb{P}(V)$ be the projective space of lines through $0 \in V$. Since the differential equations $(P + xg)dy - (Q + yg)dx = 0$ and $\lambda(P + xg)dy - \lambda(Q + yg)dx = 0$ define the same foliation in $\mathbb{C}^2$, we can identify the set of all foliations of degree $n$ in $\mathbb{CP}(2)$ with a subset $\mathcal{F}(n) \subset \mathbb{P}(V)$. We consider $\mathcal{F}(n)$ with the topology induced by the topology of $\mathbb{P}(V)$. $\mathcal{F}(n)$ is called the space of foliations of degree $n$ in $\mathbb{CP}(2)$.

We consider the following subsets:

- $\mathcal{S}(n) := \{\mathcal{F} \in \mathcal{F}(n) \mid$ the singularities of $\mathcal{F}$ are non-degenerated\}$
- $\mathcal{T}(n) := \{\mathcal{F} \in \mathcal{S}(n) \mid$ any characteristic number $\lambda$ of $\mathcal{F}$ satisfies $\lambda \in \mathbb{C} \setminus \mathbb{Q}_+$\}$ = $\{\mathcal{F} \in \mathcal{S}(n) \mid$ $\mathcal{F}$ has reduced singularities\}$
- $\mathcal{A}(n) := \mathcal{T}(n) \cap \mathcal{X}(n)$
- $\mathcal{H}(n) := \{\mathcal{F} \in \mathcal{A}(n) \mid$ all singularities of $\mathcal{F}$ in $L_{\infty}$ are hyperbolic $\}$

**Proposition 2.1.** [10][11] $\mathcal{X}(n)$ is an analytic subvariety of $\mathcal{F}(n)$ and also if $n \geq 2$ then:

1. $\mathcal{T}(n)$ contains an open dense subset of $\mathcal{F}(n)$.
2. $\mathcal{H}(n)$ contains an open dense subset $\mathcal{M}_1(n)$ such that if $\mathcal{F} \in \mathcal{M}_1(n)$, $n \geq 2$ then:
   - $L_{\infty}$ is the only algebraic solution of $\mathcal{F}$
   - The holonomy group of the leaf $L_{\infty} \setminus \text{Sing}(\mathcal{F})$ is nonsolvable.
3. $\mathcal{T}(n) \subset \mathcal{H}(n) \subset \mathcal{X}(n)$ are open subsets.
Lemma 2.2. Let $\mathcal{F} \in \mathcal{M}_1(n), n \geq 2$; then each leaf $F \neq L_\infty$ is dense in $\mathbb{C}P(2)$.

Proof. First we notice that $F$ must accumulate $L_\infty$. Since $F$ is a non-algebraic leaf it must accumulate at some regular point $p \in L_\infty \setminus \text{Sing}(\mathcal{F})$. Choose a small transverse disk $\Sigma \ni L_\infty$ with $\Sigma \subset V, V$ is a flow-box neighborhood of $p$. We consider the holonomy group $\text{Hol}(\mathcal{F}, L_\infty, \Sigma)$. Then $F$ accumulates the origin $p \in \Sigma$ and since by [15] (see also theorem (3.1)) $G$ has dense pseudo-orbits in a neighborhood the origin, it follows that $F$ is dense in a neighborhood of $p$ in $\Sigma$. Any other leaf $L'$ of $\mathcal{F}$, $L' \neq L_\infty$ must have the same property. Using the continuous dependence of the solutions with respect to the initial conditions we may conclude that $F$ accumulates any point $q \in F'$, $\forall F' \neq L_\infty$. Thus $F$ is dense in $\mathbb{C}^2$ and since $L_\infty$ is $\mathcal{F}$-invariant, $F$ is dense in $\mathbb{C}P(2)$.

□

Proposition 2.3. Let $\{\mathcal{F}_t\}_{t \in \mathbb{D}}, \mathcal{F}_0 = \mathcal{F} \in \mathcal{M}_1(n)$ is an unfolding then it is analytically equivalent to the trivial unfolding of $\mathcal{F}$ for $t \approx 0$.

Proof. Denote by $\tilde{\mathcal{F}}$ the foliation on $\mathbb{C}P(2) \times \mathbb{D}$ such that $\forall t \in \mathbb{D}, \tilde{\mathcal{F}}|_{\mathbb{C}P(2) \times \{t\}} = \mathcal{F}_t$,

$$\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P(2)$$

the canonical projection and

$$\Pi : (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{D} \to \mathbb{C}P(2) \times \mathbb{D}$$

the map

$$\Pi(p, t) := (\pi(p), t).$$

Denote by $\mathcal{F}^* := \Pi^*(\tilde{\mathcal{F}})$, pull-back foliation on $(\mathbb{C}^3 \setminus \{0\}) \times \mathbb{D}$. Then $\mathcal{F}^*$ extends to a foliation on $\mathbb{C}^3 \times \mathbb{D}$ by a Hartogs type argument.

Claim. We may choose an integrable holomorphic 1-form $\Omega$ which defines $\mathcal{F}^*$ on $\mathbb{C}^3 \times \mathbb{D}$ such that

$$\Omega = A(x, t)dt + \sum_{i=1}^{3} B_j(x, t)dx_j,$$

where $B_j$ is a homogeneous polynomial of degree $n + 1$ in $x$, $A$ is a homogeneous polynomial of degree $n + 2$ in $x$, $\sum_{i=1}^{3} x_j B_j(x, t) \equiv 0$ and $\Omega_t := \sum_{i=1}^{3} B_j(x, t)dx_j$ defines $\pi^*(\mathcal{F}_t)$ on $\mathbb{C}^3$.

Proof of the claim. First we remark that by triviality of Dolbeault and Cech cohomology groups of $\mathbb{C}^3 \times \mathbb{D}$, $\mathcal{F}^*$ is given by an integrable holomorphic 1-form, say, $\omega$ in $\mathbb{C}^3 \times \mathbb{D}$.

The restriction $\omega_t := \omega|_{\mathbb{C}^3 \times \{t\}}$ defines $\mathcal{F}^*_t := \pi^*(\mathcal{F}_t)$ in $\mathbb{C}^3$. Thus we may write

$$\omega = \alpha(x, t)dt + \sum_{k=1}^{3} \beta_j(x, t)dx_k = \alpha(x, t)dt + \omega_t$$

Since the radial vector field $R$ is tangent to the leaves of $\mathcal{F}^*$ we have $\omega \circ R = 0$ so that $\omega_t \circ R = 0$, i.e. $\sum_{k=1}^{3} x_k \beta_j(x, t) = 0$. Now we use the Taylor expansion in the variable $x = (x_1, x_2, x_3)$ of $\omega$ around a point $(0, t)$ so that $\omega = \sum_{j=0}^{+\infty} \omega_j$ where $\omega_j(x, t) := \alpha_j(x, t)dt + \sum_{k=1}^{3} \beta_j^k(x, t)dx_k = \alpha_j(x, t)dt + \omega_j^t$ and $\alpha_j, \beta_j^k$ are holomorphic in $(x, t)$, polynomial of degree $j$ in $x$, $\omega_j^t \equiv 0$. Now the main argument is the following:

Lemma 2.4. $\Omega = \alpha_{t+1}dt + \omega_{t+1}^t$ defines $\mathcal{F}^*$ in $\mathbb{C}^3 \times \mathbb{D}$. 
Proof. Indeed, \( \omega \wedge d\omega = 0 \Rightarrow i_R(\omega \wedge d\omega) = i_R(\omega).d\omega - \omega \wedge i_R(d\omega) = 0 \)
\( \omega \wedge i_R(d\omega) = 0 \) (since \( i_R(\omega) = 0 \) \( \Rightarrow i_R(d\omega) = f\omega \) for some holomorphic function \( f \) (Divisor lemma of Saito). Therefore the Lie derivative of \( \omega \) with respect to \( R \) is

\[
L_R(\omega) = i_R(d\omega) + d(i_R(\omega)) = f\omega.
\]

On the other hand since \( \omega = \sum_{j=\nu}^{+\infty} \omega_j = \sum_{j=\nu}^{+\infty}(\alpha_j(x,t)dt + \omega^j_t) \) we obtain

\[
L_R(\omega) = \sum_{j=\nu}^{+\infty} L_R(\alpha_j(x,t)dt + \omega^j_t)
= \sum_{j=\nu}^{+\infty} \frac{d}{dz}[\alpha_j(e^z x,t)dt + \sum_{k=1}^{3} \beta^k_j(e^z x,t)e^z dx_k]|_{z=0}
= \sum_{j=\nu}^{+\infty} [j\alpha_j(x,t)dt + (j+1)\omega^j_t].
\]

Now we write the Taylor expansion also for \( f \) in the variable \( x \). \( f(x,t) = \sum_{j=0}^{+\infty} f_j(x,t) \) where \( f_j(x,t) \) is holomorphic in \( (x,t) \) homogeneous polynomial of degree \( j \) in \( x \). We obtain from (1) and (2)

\[
\sum_{j=\nu}^{+\infty} [j\alpha_j(x,t)dt + (j+1)\omega^j_t] = \sum_{k=0}^{+\infty} f_k(\sum_{l=\nu}^{+\infty} \omega_l) = \sum_{j\geq\nu} \sum_{l+k=j} f_k\omega_l j\alpha_jdt + (j+1)\omega^j_t
= \sum_{l+k=j} f_k\omega_l [f_k\alpha_l dt + f_k\omega^l_t] \quad l \geq \nu \quad \text{and} \quad \forall j \geq \nu
\]

Then

\[
j\alpha_j = \sum_{l+k=j} f_k\alpha_l
\]

(3)

\[
(j+1)\omega^j_t = \sum_{l+k=j} f_k\omega^l_t \quad \forall j \geq \nu \quad \text{and} \quad l \geq \nu
\]

(4)

In particular (3) and (4) imply \( f_0\alpha_\nu = \nu\alpha_\nu \) and \( f_0\omega^\nu_t = (\nu + 1)\omega^\nu_t \) then \( f_0 = \nu + 1, \alpha_\nu = 0 \).

An induction argument shows that:

\[
j \geq \nu \Rightarrow (\alpha_{j+1}dt + \omega^j_t) \wedge \Omega = 0, (\Omega := \alpha_{\nu+1}dt + \omega^\nu_t)
\]
Finally since the degree of the foliation $\mathcal{F} = \mathcal{F}_0$ is $n$ we have $\nu = n + 1$. This proves the lemma (2.4). \hfill \Box

**Lemma 2.5.** There exists a complete holomorphic vector field $X$ on $\mathbb{C}^3 \times \mathbb{D}_\epsilon$, $\mathbb{D}_\epsilon \subset \mathbb{D}$ small subdisk, such that $X(x,t) = \frac{\partial}{\partial t} + \sum_{j=1}^{3} F_j(x,t) \frac{\partial}{\partial x_j}$, $\Omega \circ X = 0$ and $F_j(x,t)$ is linear on $x$.

**Proof.** We may present $\Omega = A(x,t)dt + \sum_{j=1}^{3} B_j(x,t) dx_j = A(x,t)dt + \omega_t$ where $i_R(\omega_t) = 0$, $B_j$ is a homogeneous polynomial of degree $n + 1$ in $x$, $A$ is a homogeneous polynomial of degree $n + 2$ in $x$.

Claim. $\forall t \in \mathbb{D}_\epsilon$ ($\epsilon \geq 0$ small enough) we have $\text{Sing}(\mathcal{F}_t) \subset \{ A(.,t) = 0 \}$.

**Proof of the claim.** Since $\Omega \wedge d\Omega = 0$ we have the coefficients of $dt \wedge dx_i \wedge dx_j$ equal to zero, that is:

\begin{equation}
A(\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j}) + B_j \frac{\partial B_i}{\partial t} - B_i \frac{\partial B_j}{\partial t} + B_i \frac{\partial A}{\partial x_j} - B_j \frac{\partial A}{\partial x_i} = 0
\end{equation}

Now given $p_0 \in \text{Sing}(\mathcal{F}_{t_0})$, $(t_0 \approx 0$, so that $\mathcal{F}_{t_0} \in \mathcal{M}_1(n)$) we have from (5) that $(B_i(p_0,t_0) = B_i(p_0,t_0) = 0) : A(p_0,t_0)\{\frac{\partial B_i}{\partial x_i}(p_0,t_0) - \frac{\partial B_j}{\partial x_j}(p_0,t_0)\}$. Since $\mathcal{F}_{t_0} \in T(n)$ we have $\frac{\partial B_i}{\partial x_i}(p_0,t_0) \neq \frac{\partial B_j}{\partial x_j}(p_0,t_0)(i \neq j)$ and $A(p_0,t_0) = 0$.

Using now Noether’s lemma for foliations we conclude that there exist $F_j(x,t)$ holomorphic in $(x,t)$, homogeneous polynomial of degree $1 = (n + 2) - (n + 1)$ in $x$, such that $A(x,t) = \sum_{j=1}^{3} F_j(x,t)B_j(x,t)$. Now we define $X(x,t) := \frac{\partial}{\partial t} + \sum_{j=1}^{3} F_j(x,t) \frac{\partial}{\partial x_j}$ so that $\Omega \circ X = A - \sum_{j=1}^{3} F_jB_j = 0$.

In addition $X$ is complete because each $F_j$ is of degree one in $x$. The flow of $X$ writes $X_{\psi}(x,t) = (\psi(x,t),t + \tau)$. Clearly the $\psi : \mathbb{C}^3 \{0\} \rightarrow \mathbb{C}^3 \{0\}$ defines an analytic equivalence between $\mathcal{F}$ and $\mathcal{F}_x$. The proposition (1.9) is now proved. \hfill \Box

Another important remark is the following:

**Proposition 2.6.** Let $\mathcal{F}$, $\mathcal{G}$ be foliations with hyperbolic singularities on $\mathbb{C}P(2)$. Assume that $L_\infty$ is the only algebraic leaf of $\mathcal{F}$ and that $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{G}|_{\mathbb{C}^2}$ are topologically equivalent. Then $L_\infty$ is also $\mathcal{G}$-invariant.

**Proof.** Let $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a topological equivalence between $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{C}^2$. We notice that given a singularity $p \in \text{Sing}(\mathcal{F}) \cap L_\infty$, there exist local coordinates $(x,y) \in U$, $x(p) = y(p) = 0$, $L_\infty \cap U = \{y = 0\}$ such that $\mathcal{F}|_U : xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $U \cap \text{Sing}(\mathcal{F}) = \{p\}$. Let $U^* = U \setminus (L_\infty \cap U)$, $V^* = \phi(U^*) \subset \mathbb{C}^2$, $\Gamma := (x = 0)$, $\Gamma^* := \Gamma \cap U^* = \Gamma \setminus \{p\}$, $\Gamma$ is the local separatrix of $\mathcal{F}$ at $p$, transverse to $L_\infty$. We put $\Gamma^*_1 = \phi(\Gamma^*) \subset V^*$. We remark that $\Gamma^*_1$ is contained in a leaf of $\mathcal{G}$ and it is closed in $V^*$. On the other hand if we take any local leaf $L$ of $\mathcal{F}|_{U^*}$, $L \neq \Gamma$; then by the hyperbolicity of $p \in \text{Sing}(\mathcal{F})$ we have that $L$ accumulates $\Gamma$. Thus the image $L_1 = \phi(L)$ is a leaf of $\mathcal{G}|_{V^*}$ that accumulates $\Gamma^*_1 \neq L_1$. 

Assume by contradiction that $L_\infty$ is not $G$-invariant. The curve $\Gamma^*_1 \subset \mathbb{C}^2$ accumulates $L_\infty$. By the Flow Box Theorem, a point of accumulation $q \in L_\infty \cap \bar{\Gamma}^*_1$ which is not a singularity of $G$, must be a point near to which the closure (in $\mathbb{C}P(2)$) $\bar{\Gamma}^*_1$ is analytic.

Thus if there are no singularities of $G$ in $\Gamma^*_1 \cap L_\infty$ then $\bar{\Gamma}^*_1$ is an algebraic $G$-invariant curve in $\mathbb{C}P(2)$. This implies that if $L_0$ is the leaf of $F$ on $\mathbb{C}P(2)$ that contains $\Gamma^*$ then $\bar{L}_0$ is an algebraic invariant curve and $F$-invariant. Since $\bar{L}_0 \neq L_\infty$ we have a contradiction to our hypothesis. Therefore $\bar{\Gamma}^*_1$ must accumulate to some singularity $r$ of $G$ in $L_\infty$. Once again by the local behavior of the leaves close to $\Gamma^*_1$ and due to the fact that $r$ is hyperbolic, it follows that $\bar{\Gamma}^*_1$ is locally a separatrix of $G$ at $r$. Since $L_\infty$ is not $G$-invariant, we have two local separatrices $\Lambda_1$, $\Lambda_2$ for $G$ at $r$ with $\Lambda_j \not\subset L_\infty$, $j = 1, 2$. Thus $\bar{\Gamma}^*_1$ is locally contained in $\Lambda_1 \cup \Lambda_2$ and in particular $\bar{\Gamma}^*_1$ is analytic around $r$. Since (as we have seen) $\bar{\Gamma}^*_1$ is also analytic around the points $q \in \text{Sing}(G)$, it follows that $\bar{\Gamma}^*_1$ is analytic in $\mathbb{C}P(2)$ and once again it is an algebraic curve. Again we conclude that $\Gamma$ is contained in an algebraic leaf of $F$, other than $L_\infty$. Contradiction! \hfill \Box

The proof given above also shows us:

**Proposition 2.7.** Let $F$, $G$ be foliations on $\mathbb{C}P(2)$ both leaving invariant the line $L_\infty$. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a topological invariant equivalence for $F|_{\mathbb{C}^2}$ and $F|_{\mathbb{C}^2}$. Then $\phi$ takes the separatrix set $S_F$ onto the separatrix set $S_G$.

Here $S_F$ and $S_G$ are respectively the set of separatrices of $F$ and $G$ in $\mathbb{C}^2$ that are transverse to $L_\infty$ at some singular point $p \in \text{Sing}(F)$.

**Corollary 2.8.** Let $F_0 \in \mathcal{H}(n)$, $n \geq 2$. Then any $\mathbb{C}^2$-topologically trivial deformation $\{F_t\}_{t \in \mathbb{C}}$ of $F_0$, is a deformation in the class $\mathcal{H}(n)$ and it is also $s$-trivial if we consider $t \approx 0$.

*Proof.* First we recall that $\mathcal{H}(n)$ is open in $\mathcal{X}(n)$. Thus it remains to use proposition (2.6) to conclude that $F_t \in \mathcal{H}(n)$, $\forall t \approx 0$ and then we use proposition (2.7) to conclude that $\{F_t\}_{t \approx 0}$ is $s$-trivial. \hfill \Box

3. Fixed points and one-parameter pseudogroup

$\text{Diff}(\mathbb{C},0)$ denotes the group of germs of complex diffeomorphisms fixing $0 \in \mathbb{C}$, $f(z) = \lambda z + \sum_{n \geq 2} a_n z^n$; $\lambda \neq 0$.

Let $G \subset \text{Diff}(\mathbb{C},0)$ be a finitely generated subgroup with a set of generators $g_1, \cdots, g_r \in G$ defined in a compact disk $\bar{D}_r$.

**Theorem 3.1.** [1], [15] Suppose $G$ is nonsolvable. Then:

1. The basin of attraction of (the pseudo-orbits of) $G$ is an open neighborhood of the origin $\Omega, (0 \in \Omega)$

2. Either $G$ has dense pseudo-orbits in some neighborhood $0 \in V \subset \Omega$ or there exists an invariant germ of analytic curve $\Gamma$ (equivalent to $\text{Im} z^k = 0$ for some $k \in \mathbb{N}$) where $G$ has dense pseudo-orbits and such that $G$ has also dense pseudo-orbits in each component of $V \setminus \Gamma$. 


(3) $G$ is topologically rigid: Given another nonsolvable subgroup $G' \subset \text{Diff}(\mathbb{C},0)$ and a topological conjugation $\phi : \Omega \to \Omega'$ between $G$ and $G'$, then $\phi$ is holomorphic in a neighborhood of 0.

(4) There exists a neighborhood $0 \in W \subset V \subset \Omega$ where $G$ has a dense set of hyperbolic fixed points.

Remark 3.2. In the case $G$ is nonsolvable and contains some $f \in G$ with $f'(0)^n \neq 1$, $\forall n \in \mathbb{Z} \setminus \{0\}$ (i.e., $f'(0) = e^{2\pi i \lambda}, \lambda \notin \mathbb{Q}$) we have the following:

Dense Orbits Property: There exists a neighborhood $0 \in V \subset \Omega$ where the pseudo-orbits of $G$ are dense.

**Holomorphic deformations in $\text{Diff}(\mathbb{C},0)$:**
Let $g \in \text{Diff}(\mathbb{C},0)$ defined in some open neighborhood $0 \in \Omega$. A *holomorphic (one-parameter) deformation* of $g$ is a map $G : \mathbb{D}_\epsilon \to \text{Diff}(\mathbb{C},0)$, $(\epsilon > 0)$ which verifies the four properties:

(1) $G(0) = g$ as germs

(2) The Taylor expansion coefficients of $G(t)$ depend holomorphically on $t$

(3) The radii of convergence of $G(t)$ and $G(t)^{-1}$ are both uniformly minorated by some constant $R \geq 0$ $(\forall t \in \mathbb{D}_\epsilon)$

(4) The modules of the linear coefficient of $G(t)$ is uniformly minorated by some constant $C \geq 0$. In particular $|(G(t)^{-1})'(0)|$ is uniformly majored by $0 < t < \infty$.

Given a finitely generated pseudo-group $G \subset \text{Diff}(\mathbb{C},0)$ with a set of generators $g_1, \ldots, g_r \in G$; a holomorphic (one parameter) deformation of $G$ is given by holomorphic deformation of $g_j$, $j = 1, \ldots, r$. We may restrict ourselves to the following situation:

$G_t$ is an one-parameter analytic deformation of $G$ with $t \in \mathbb{D}$, $G_0 = G$.

We have $g_{1,t}, \ldots, g_{r,t}$ as a set of generators for $G_t$, all of them defined in a disk $\mathbb{D}_\delta$ (uniformly on $t$). We will consider dynamical and analytical properties of such deformations. The results we state below have their proofs reduced to the following case which is studied in [23].

$g_{1,t}(z) = g_1(z) + tz^{D+1}$ where $D \in \mathbb{N}$ is fixed,

$g_{2,t}(z) = g_2(z), \ldots, g_{r,t}(z) = g_r(z)$.

For such deformations we have:

**Theorem 3.3.** [23] Given a hyperbolic fixed point $p \approx 0$ for a word $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ in $G$, we consider the corresponding word $f_t = f = f_n \circ \cdots \circ f_{1,t}$ in $G_t$. Then $f_t$ has a hyperbolic fixed point $p(t)$ given by the implicit differential equation with initial conditions:

$$\frac{dp(t)}{p(t)^{D+1} dt} = \frac{f'_t(p(t))}{f'_1(p(t)) - 1} f'_{1,t}(p(t)), \quad p(0) = p.$$

In particular $p(t)$ depends analytically in $t$ as well as its multiplicator $f'_t(p(t))$. This holds for $|t| < \epsilon$ if $\epsilon > 0$ is small enough.
We also have:

**Theorem 3.4.** [22] Let $f$ and $g$ be two non-commuting complex diffeomorphisms defined in some neighborhood of the origin $0 \in \mathbb{C}$, fixed by $f$ and $g$. Assume that $f'(0) = e^{2\pi i \lambda}$, $g'(0) = e^{2\pi i \mu}$ with $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\text{Re} \lambda, \text{Re} \mu \notin \mathbb{Q}$. Then there exist some bound $K > 0$ and some radius $r_0 > 0$ such that if $r \in (0, r_0)$ and $|t| \leq Kr$ then the orbits of the pseudo-group generated by $g$ and $f_t(z) = t + f(z - t)$ are dense in $\mathbb{D}_r$.

**Corollary 3.5.** [21] Let $f$ and $g$ be as above. Any holomorphic deformation of the subgroup $\langle f, g \rangle \subset \text{Diff}(\mathbb{C}, 0)$ preserves locally at the origin the dense orbits property.

4. **Growth of finitely generated subgroups of $\text{Diff}(\mathbb{C}^n, 0)$**

Consider a group $G$ generated by $S = \{g_1, \ldots, g_k\}$. Each element $g \in G$ can be represented by a word $g_1^{p_1}g_2^{p_2} \cdots g_k^{p_k}$ and $|p_1| + |p_2| + \cdots + |p_k|$ is called the length of the word. The norm $\|\gamma\|$ (relative to $S$) is defined as the minimal length of the words representing $g$. Let $B(n)$ be the set of elements $g \in G$ with $\|g\| \leq n$. The growth function of $g$ with respect to the set of generators $S$ is defined as $\gamma := \gamma_G(n) := |B(n)|$.

We say that a function $f : \mathbb{N} \to \mathbb{R}$ is dominated by a function $g : \mathbb{N} \to \mathbb{R}$, denoted by $f \preceq g$, if there is a constant $C > 0$ such that $f(n) \leq g(Cn)$ for all $n \in \mathbb{N}$. Two functions $f, g : \mathbb{N} \to \mathbb{R}$ are called equivalent, denoted by $f \sim g$, if $f \preceq g$ and $g \preceq f$. It is known that for any two finite sets of generators $S_1$ and $S_2$ of a group, the corresponding two growth functions are equivalent. Note also that if $|S| = k$, then $\gamma(n) \leq k^n$.

Growth of group $G$ is called exponential if $\gamma(n) \sim e^n$. Otherwise the growth is said to be subexponential. Growth of group $G$ is called polynomial if $\gamma(n) \sim n^c$ for some $c > 0$. If $\gamma(n) \succeq n^c$ for all $c$, the growth of $G$ is said to be superpolynomial. If the growth is subexponential and superpolynomial, it is called intermediate.

**Examples.** The finitely generated abelian groups have polynomial growth. More precisely if $S = \{g_1, \ldots, g_k\}$ is a minimal generating set for a free abelian group $G$ then the growth function is

$$\gamma(n) = \sum_{l=0}^{m} 2^l \binom{m}{l} \binom{n}{l}.$$ 

Also the finitely generated nilpotent groups are of polynomial growth [24].

The free groups with $k \geq 2$ generators have exponential growth. Milnor and Wolf in [14] and [24] showed that a finitely generated solvable group $G$ has exponential growth unless $G$ contains a nilpotent subgroup of finite index.

If $G$ is a finite extension of a group of polynomial growth, then $G$ itself has polynomial growth. So we conclude if a finitely generated group $G$ has a nilpotent subgroup of finite index then $G$ has polynomial growth. Conversely
the finitely generated linear groups (Tits [20]), the finitely generated polycyclic groups (Wolf [24]) and the finitely generated subgroups of Diff(\(\mathbb{R}^n,0\)) (Plante-Thurston [17]) with polynomial growth have nilpotent subgroups with finite index. Finally in an extraordinary work [7], Gromov settled the problem and proved if a finitely generated group \(G\) has polynomial growth then \(G\) contains a nilpotent subgroup of finite index.

In [6], Grigorchuk constructed a family of groups of intermediate growth which are the only known examples of such groups.

Denote by \(\text{Diff}(\mathbb{C}^n,0)\), the group of germs of complex diffeomorphisms fixing the origin. Let \(G \subset \text{Diff}(\mathbb{C}^n,0)\) be a finitely generated subgroup. Nonsolvable finitely generated subgroups of complex diffeomorphism in dimension \(n = 1\) play a fundamental role in dynamical study of holomorphic vector fields in \((\mathbb{C}^2,0)\). In fact the holonomy groups of irreducible components of desingularization of the germ of a nondicritical foliation at a singularity in \(\mathbb{C}^2\) are finitely generated subgroups of \(\text{Diff}(\mathbb{C}^n,0)\). Theorem of Nakai and its consequences [1], [15] provide a new dynamical information for the “generic” foliation whose holonomy groups are nonsolvable as we will see later. This motivates characterization of solvability of finitely generated subgroups of complex diffeomorphisms. Our objective is to prove a finitely generated subgroup of \(\text{Diff}(\mathbb{C}^n,0)\) with polynomial growth is solvable. Moreover if we know there is no finitely generated subgroups with intermediate growth then it implies a nonsolvable subgroup of \(\text{Diff}(\mathbb{C}^n,0)\) has exponential growth.

We recall a useful lemma due to Gromov [7]:

**Lemma 4.1.** Let \(G\) be a finitely generated (abstract) group of polynomial growth. Then the commutator subgroup \([G,G]\) is also finitely generated.

The following lemma is proved in [17]:

**Lemma 4.2.** Suppose that \(G\) has polynomial growth of degree \(k\) and that

\[
H_0 \subset H_1 \subset \cdots \subset H_n \subset G
\]

is a finite sequence of subgroups such that for each \(i \ (1 \leq i \leq n)\) there is a non trivial homomorphism \(f_i : H_i \to \mathbb{R}^l\) such that \(H_i \subset \text{Ker} f_i\). Then \(n \leq k\).

**Remark 4.3.** A finitely generated solvable group of polynomial growth is polycyclic.

**Theorem D.** Let \(G\) be a finitely generated subgroup of \(\text{Diff}(\mathbb{C}^n,0)\). If \(G\) has a polynomial growth then \(G\) is solvable.

**Proof.** Put \(G_1 = [G,G]\) and \(G_{i+1} = [G_i,G_i]\) for \(i \in \mathbb{N}\). By the lemma (4.2) it is enough to show that for each \(G_i\) there is a non trivial homomorphism \(f_i \in \text{Hom}(G_i,\mathbb{R}^l)\). Notice that \(G_i \subset \text{Ker} f_{i+1}\) if there are such \(f_i\)’s. For simplicity denote by \(H := G_i\). By the lemma (4.1) \(H\) is finitely generated. Take a symmetric system of generators \(S := \{h_1, \cdots, h_k\}\). We may write
\[ h_i(z) = z + \tilde{h}_i(z) \] for \( i = 1, \cdots, k \) since elements of \( H \) are tangent to the identity. If \( u, v \in H \) then
\[ \widetilde{(v \circ u)}(z) = \tilde{u}(z) + \tilde{v}(z) + [\tilde{v}(u(z)) - \tilde{v}(z)]. \]

Write \( \tilde{u}(z) = (\tilde{u}_1(z), \cdots, \tilde{u}_n(z)) \) as a real function and \( z = (x_1, \cdots, x_{2n}) \in \mathbb{R}^{2n} \). We have
\[
(*) \quad \tilde{v}(u(z)) = \tilde{v}(z) + \sum_{i=1}^{2n} \tilde{u}_i(z) \int_0^1 \frac{\partial \tilde{v}}{\partial x_i}(x_1 + t\tilde{u}_1, \cdots, x_{2n} + t\tilde{u}_{2n}) dt.
\]

We choose a sequence \( \{z_m\}_{m=1}^{\infty} \) in \( \mathbb{C}^n \) converges to the origin such that for at least an index \( j \in \{1, \cdots, k\} \), all terms of the sequence \( \{\tilde{h}_j(z_m)\} \) are different from zero. put \( M_m = \max\{|\tilde{h}_1(z_m)|, \cdots, |\tilde{h}_k(z_m)|\}. \) \( \forall m \in \mathbb{N}, M_m > 0 \) and \( \forall i \in \{1, \cdots, k\} \) the sequence \( \{\tilde{h}_i(z_m)/M_m\} \) converges to, say, \( b_i \). If \( h \) is an arbitrary element of \( H \) such that \( \tilde{h}(z_m)/M_m \) converges to \( b \), then for each generator \( h_i \) the sequence \( \{(h_i \circ \tilde{h})(z_m)/M_m\} \) converges to \( b + b_i \). In fact from \( (*) \)
\[
\lim_{m \to \infty} M_m^{-1}\left(\tilde{h}_i(h(z_m)) - \tilde{h}(z_m)\right) = 0.
\]

Now define \( f : H \to \mathbb{R}^{2n} \) as following:
\[
h \mapsto \lim_{m \to \infty} \tilde{h}(z_m)/M_m.
\]

\( f \) is well defined and non trivial homomorphism.

\[ \square \]

5. Proof of theorem A

We use the terminology of [11] and some of the original ideas of [8]. Let therefore \( \{\mathcal{F}_t\}_{t \in \mathbb{D}} \) be a \( C^2 \)-topological trivial deformation of \( \mathcal{F}_0 \in \mathcal{H}(n) \), \( n \geq 2 \). As we have proved in corollary (2.8) there exists \( \epsilon > 0 \) such that \( \{\mathcal{F}_t\}_{t \in \mathbb{D}_\epsilon} \) is a s-trivial deformation of \( \mathcal{F}_0 \) in the class \( \mathcal{H}(n) \). Now we consider the continuous foliation \( \tilde{\mathcal{F}} \) on \( \mathbb{C}P(2) \times \mathbb{D}_\epsilon \) defined as follows:

- \( \text{Sing}(\tilde{\mathcal{F}}) = \bigcup_{|t|<\epsilon} \text{Sing}(\mathcal{F}_t) \times \{t\} \)
- The leaves of \( \mathcal{F}_t \) are the intersections of the leaves of \( \tilde{\mathcal{F}} \) with \( \mathbb{C}P(2) \times \{t\}, \forall |t| < \epsilon \).

Because of the topological triviality, \( \tilde{\mathcal{F}} \) is a continuous foliation on \( \mathbb{C}^2 \times \mathbb{D}_\epsilon \). This foliation extends to a continuous foliation on \( \mathbb{C}P(2) \times \mathbb{D}_\epsilon \) by adding the leaf with singularities \( L_\infty \times \mathbb{D}_\epsilon \). In order to prove that \( \tilde{\mathcal{F}} \) is holomorphic we begin by proving that it has holomorphic leaves and then it is transversely holomorphic. This is basically done by the following lemma:

**Lemma 5.1.** Let \( p_1, \cdots, p_{n+1} \in L_\infty \) be the singularity of \( \mathcal{F}_0 \) in \( L_\infty \). Then

1. There exist analytic functions \( p_j(t), t \in \mathbb{D}_\epsilon \) such that \( \{p_1(t), \cdots, p_{n+1}(t)\} = \text{Sing}(\mathcal{F}_t) \cap L_\infty \), \( p_j(0) = p_j, j = 1, \cdots, n+1 \).
Fix $q \in L_\infty \setminus \Sing(F_0)$ and take small simple loops $\alpha_j \in \pi_1(L_\infty \setminus \Sing(F_0), q)$ and a small transverse disk $\Sigma \pitchfork L_\infty$. Then for $\epsilon > 0$ small we have:

(2) The holonomy group $G_t := \text{Hol}(F_t, L_\infty, \Sigma_t) \subset \text{Diff}(\Sigma, q)$ is generated by the holonomy maps $f_{j,t}$ associated to the loops $\alpha_j$ ($\alpha_j$ is also simple loop around $p_j(t)$).

In particular we obtain

(3) $\{G_t\}_{t \in \mathbb{D}_\epsilon}$ is an one-parameter holomorphic deformation of $G_0 = \text{Hol}(F_0, L_\infty, \Sigma)$.

(4) The group $G_t$ is nonsolvable with the density orbits property, a dense set $\eta_t \subset \Sigma \times \{t\}$ of hyperbolic fixed points around the origin $(q,t)$. Moreover, given any $p_0 \in \eta_0$, $p_0 = f_0(p_0)$, there exists an analytic curve $p_t \in \eta_t$ such that $p(0) = p_0$, $f_t(p_t) = p_t$ where $f_t \in G_t$ is the corresponding deformation of $f_0$.

Using above lemma we prove that $\hat{F}$ is holomorphic close to $L_\infty \times \mathbb{D}_\epsilon$:

Given a point $p_0 \in \eta_0$ and $f_0 \in G_0$ as above, the curve $p(t)$ and $f_t \in G_t$ given by (iv) above we have $\{p(t), |t| < \epsilon\} \subset \hat{L}_{p_0} \cap (\Sigma \times \mathbb{D}_\epsilon)$ where $\hat{L}_{p_0}$ is the $\hat{F}$-leaf through $p_0$. On the other hand $\hat{L}_{p_0}$ is already holomorphic along the cuts $\hat{L}_{p_0} \cap (CP(2) \times \{t\})$ for $L_{p_0,t}$ for $p_0 = (p_0', 0)$. This implies that $\hat{L}_{p_0}$ is analytic.

Since the curves $\{p(t), |t| < \epsilon\}$ with $p_0 \in \eta_0$ are analytic and locally dense around $\{q\} \times \mathbb{D}_\epsilon \subset \Sigma \times \mathbb{D}_\epsilon$ it follows that any leaf $\hat{L}$ of $\hat{F}$ is a uniform limit of holomorphic leaves $\hat{L}_{p_0}$ and it is therefore holomorphic. Thus $\hat{F}$ has holomorphic leaves. We proceed to prove that it is transversely holomorphic. This is in fact a consequence of topological rigidity theorem [15] for nonsolvable groups of $\text{Diff}(\mathbb{C}, 0)$.

Fix transverse section $\Sigma \pitchfork L_\infty$ as above. We may assume that $\Sigma \subset V$ where $V$ is a flow-box neighborhood for $F_0$ with $q \in V$. The homeomorphisms $\phi_t : \mathbb{C}^2 \to \mathbb{C}^2$ take the separatrices $S_0$ of $F_0$ onto the set of separatrices $S_t$ of $F_t$. Now we use the following proposition:

**Proposition 5.2.** Given $F \in \mathcal{H}(n)$, $n \geq 2$, the set of separatrices $S_F$ of $F$ is dense in $\mathbb{C}P(2)$ and it accumulates densely a neighborhood of the origin for any transverse disk $\Sigma \pitchfork L_\infty$, $q \notin \Sing F$.

**Proof.** Indeed, given a separatrix $\Gamma \subset S_F$ the leaf $L \supset \Gamma$ is nonalgebraic for $F \in \mathcal{H}(n)$. This implies that $L \setminus \Gamma$ accumulates $L_\infty$ and therefore any transverse disk $\Sigma$ as above is cut by $L$. Now it remains to use the density of the pseudo-orbits of $\text{Hol}(F, L_\infty)$ stated in theorem (1.15).

Returning to our argumentation we fix any $p \in \Sigma$, separatrix $(p_0) \in \Gamma_0 \subset S_0$ of $F_0$ and denote by $P(\Gamma_0, p)$ the local plaque of $F_0|V$ that is contained in $\Gamma_0 \cap V$ and contains the fixed point $p$. Put $\Gamma_t = \phi_t(\Gamma_0)$ and consider the map $t \mapsto p(t) := P(\Gamma_t, p)$. Clearly we may write $p(t) = \phi_t(P(\Gamma_0, p)) \cap \Sigma \times \{t\}$ by choosing $\Sigma$ and $|t|$ small enough. This map $t \mapsto p(t)$ is holomorphic as a consequence of proposition below:
Proposition 5.3. Given any singularity $p^0_j \in \text{Sing}\mathcal{F}$ there exists a connected neighborhood $(p^0_j) \in \mathcal{U}_j$, a neighborhood $\mathcal{U} \ni \mathcal{F}_0$ in $\mathcal{S}(n)$ and a holomorphic map $\psi_j : \mathcal{U} \rightarrow \mathcal{U}_j$ such that $\forall \mathcal{F} \in \mathcal{U}$, $\psi_j(\mathcal{F}) = \text{Sing}\mathcal{F} \cap \mathcal{U}_j$, $\psi_j(\mathcal{F}_0) = p^0_j$. In particular, if $\{\mathcal{F}_t\}_{t \in \mathbb{C}}$ is a deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$; then given $\Gamma_0 \in \mathcal{S}_0 = \mathcal{S}_\mathcal{F}$, $\Sigma \in \mathcal{L}_\infty$, $V$ and $p \in \Gamma_0 \cap \Sigma$ as above, there exist analytic curves $p_j(t)$ and $p(t)$ such that: $p_j(t) = \text{Sing}\mathcal{F}_t \cap \mathcal{U}_j$, $p_j(0) = p^0_j$, $p(t) = P(\Gamma_t, p(t))$, $p(0) = p$ and $p(t) \in \Gamma_t \cap \Sigma$.

Roughly speaking, the proposition says that both the singularities and the separatrices of a foliation with nondegenerate singularities, move analytically under analytic deformations of the foliation.

Finally we define $h_t(p) := p(t)$ obtaining this way an injective map in a dense subset of $\Sigma$ ($\mathcal{F}_0$ has dense separatrices in $(\Sigma, q)$), so that by the $\lambda$-lemma for complex mapping we may extend $h_t$ to a map that $h_t : \Sigma \rightarrow \Sigma$. Moreover, it is clear that if $f_{j,t}$ is a holonomy map as above then we have $h_t(f_{j,0}(p)) = f_{j,t}(h_t(p))$.

Because $f_0$ and $f_t$ fix the separatrices. Therefore, by density we have $h_t \circ f_{j,0} = f_{j,t} \circ h_t$, $\forall j \in \{1, \cdots, n + 1\}$ and the mapping $h_t$ conjugates the holonomy groups $G_t = \text{Hol}(\mathcal{F}_t, \mathcal{L}_\infty, \Sigma)$ and $G_0$. By the topological rigidity theorem $h_t$ is holomorphic which implies that $\mathcal{F}$ is transversely holomorphic close to $\mathcal{L}_\infty \times \mathbb{D}_\epsilon$ [15]. The density of $\mathcal{S}_t$, $\forall t$ assures that $\mathcal{F}$ is in fact holomorphic in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$.

Summarizing the discussion above we have:

Proposition 5.4. Let $\{\mathcal{F}_t\}_{t \in \mathbb{C}}$ be a $\mathbb{C}^2$-topologically trivial deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. Then there exists $\epsilon > 0$ such that $\{\mathcal{F}_t\}_{t \in \mathbb{D}_\epsilon}$ is an unfolding of $\mathcal{F}_0$ in $\mathbb{C}P(2)$.

The end of the proof of Theorem A. The proof is a consequence of propositions (2.3) and (5.4) above.

6. Generalizations

Theorem (A) may be extended to a more general class of foliations on $\mathbb{C}P(2)$ as well as to foliations on other projective spaces. This is the goal of this section. Before going further into generalizations we state a kind of Noether’s lemma for foliations.

Lemma 6.1. Let $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ be a holomorphic unfolding of a foliation $\mathcal{F}_0$ of degree $n$ on $\mathbb{C}P(2)$. Assume that for each singularity $p \in \text{Sing}(\mathcal{F}_0) \cap \mathcal{L}_\infty$ the germ of unfolding at $p$ is analytically trivial. Then there exists $\epsilon > 0$ such that $\{\mathcal{F}_t\}_{t \in \mathbb{D}_\epsilon}$ is analytically trivial.

Proof. Denote by

$\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{C}P(2)$ the canonical projection and by

$\Pi : (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{D} \rightarrow \mathbb{C}P(2) \times \mathbb{D}$ the map $\Pi(p, t) := (\pi(p), t)$. 
Choose a holomorphic integrable 1-form $\Omega$ which defines $\tilde{\mathcal{F}}^*$ extension of $\Pi^*(\mathcal{F})$ to $\mathbb{C}^3 \times \mathbb{D}$, so that we may choose

$$\Omega = A(x,t)dt + \sum_{i=1}^{3} B_j(x,t)dx_j,$$

where $A, B_j$ are holomorphic in $(x,t) \in \mathbb{C}^3 \times \mathbb{D}$, homogeneous polynomial in $x$ of degree $n+2$, $n+1$; $\sum_{i=1}^{3} x_j B_j = 0$. The foliation $\pi^*(\mathcal{F}_t)$ extends to $\mathbb{C}^3$ and this extension $\mathcal{F}_t^*$ is given by $\Omega_t = 0$ for $\Omega_t := \sum_{i=1}^{3} B_j dx_j$.

**Claim.** Given point $q \in \mathbb{C}^3 \times \mathbb{D}$, $q \notin \{0\} \times \mathbb{D}$, there exist a neighborhood $U(q)$ of $q$ in $\mathbb{C}^3 \times \mathbb{D}$ and local holomorphic vector field $X_q \in \mathfrak{X}(U(q))$ such that $A = \Omega \circ X_q$ in $U(q)$, for $\epsilon$ small enough.

*Proof of the claim.* If $q = (x_1, t_1)$ with $x_1 \notin \text{Sing}(\mathcal{F}_0)$ then $x_1 \notin \text{Sing}(\mathcal{F}_t)$ for $|t|$ small enough and in particular $x_1 \notin \text{Sing}(\mathcal{F}_{t_1})$. Thus the existence of $X_q \in \mathfrak{X}(U(q))$ is assured in this case. On the other hand if $x_1 \in \text{Sing}(\mathcal{F}_0)$ then we still have the existence of $X_q \in \mathfrak{X}(U(q))$ because of the local analytical triviality hypothesis for the unfolding at $x_1$.

Using the claim we obtain an open cover $\{U_{\alpha}\}_{\alpha \in \mathbb{Q}}$ of $M := \mathbb{C}^3 \setminus \{0\} \times \mathbb{D}$ with $U_{\alpha}$ connected and $X_{\alpha} \in \mathfrak{X}(U_{\alpha})$ such that $A = \Omega \circ X_{\alpha}$ in $U_{\alpha}$, $\forall \alpha \in \mathbb{Q}$. Let $U_{\alpha} \cap U_{\beta} \neq \emptyset$ then we put $X_{\alpha \beta} := (X_{\alpha} - X_{\beta})|_{U_{\alpha} \cap U_{\beta}}$ to obtain $X_{\alpha \beta} \in \mathfrak{X}(U_{\alpha} \cap U_{\beta})$ such that $\Omega \circ X_{\alpha \beta} = 0$. Take now the rotational vector field

$$Y = \text{rot}(B_1, B_2, B_3)$$

$$= \left( \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} \right) \frac{\partial}{\partial x_1} + \left( \frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial}{\partial x_2} + \left( \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial}{\partial x_3}.$$

$Y \in \mathfrak{X}(\mathbb{C}^3 \times \mathbb{D})$ and for each $t \in \mathbb{D}$ we have $i_Y(\text{Vol}) = d\Omega_t$ where Vol $= dx_1 \wedge dx_2 \wedge dx_3$ is the volume element of $\mathbb{C}^3$ in the $x$-coordinates. Fixed now $q = (x_1, t_1) \notin \text{Sing}(\Omega_{t_1})$ then the leaf of $\mathcal{F}_t^*$ through $q$ is spanned by $Y(q)$ the radial vector field $R(q)$, as a consequence of the remark above: actually, we have $i_{R(q)}(\text{Vol}) = i_R(d\Omega_t) = (n+1)\Omega_t$.

Given thus $U_{\alpha \beta} := U_{\alpha} \cap U_{\beta} \neq \emptyset$, since $\Omega_t(X_{\alpha \beta})$ we have that $X_{\alpha \beta}$ is tangent to $\mathcal{F}_t^*$ outside the points $(x,t) \in \text{Sing}(\Omega_t)$ so that we can write $X_{\alpha \beta} = g_{\alpha \beta} R + h_{\alpha \beta} Y$ for some holomorphic functions $g_{\alpha \beta}$, $h_{\alpha \beta} \in \mathcal{O}(U_{\alpha \beta} \setminus \text{Sing}(\Omega_t))$. Since $\text{Sing}(\Omega_t)$ is an analytic set of codimension $\geq 2$, Hartogs extension theorem [10] implies that $g_{\alpha \beta}, h_{\alpha \beta}$ extend holomorphically to $U_{\alpha \beta}$. Now if $U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \neq \emptyset$ then

$$0 = X_{\alpha \beta} + X_{\beta \gamma} + X_{\gamma \alpha} = (g_{\alpha \beta} + g_{\beta \gamma} + g_{\gamma \alpha}) R + (h_{\alpha \beta} + h_{\beta \gamma} + h_{\gamma \alpha}) Y$$

and since $R$ and $Y$ are linearly independent outside $\text{Sing}(\Omega_t)$ we obtain: $g_{\alpha \beta} + g_{\beta \gamma} + g_{\gamma \alpha} = 0$, $h_{\alpha \beta} + h_{\beta \gamma} + h_{\gamma \alpha} = 0$.

Thus $(g_{\alpha \beta}), (h_{\alpha \beta})$ are additive cocycles in $\mathcal{M}$ and by Cartan’s theorem (for $\mathbb{C}^{n+1} \setminus \{0\}$, $n \geq 2$) these cocycles are trivial, that is, $\exists g_{\alpha}, h_{\alpha} \in \mathcal{O}(U_{\alpha})$
such that if \( U_\alpha \cap U_\beta \neq \phi \) then \( g_{\alpha\beta} = g_\alpha - g_\beta h_{\alpha\beta} = h_\alpha - h_\beta \) in \( U_\alpha \cap U_\beta \).

This gives \( X_\alpha - X_\beta = g_{\alpha\beta}R + h_{\alpha\beta}Y(g_\alpha R + h_\alpha Y) - (g_\beta R + h_\beta Y) \)
in \( U_\alpha \cap U_\beta \neq \phi \). Thus, in \( U_\alpha \cap U_\beta \neq \phi \) we obtain \( X_\alpha - g_\alpha R - h_\alpha Y = X_\beta - g_\beta R - h_\beta Y \) and this gives a global vector field \( \tilde{X} \in \mathfrak{X}(M) \) such that \( \tilde{X}|_{U_\alpha} := X_\alpha - g_\alpha R - h_\alpha Y \) and this gives a global vector field \( \tilde{X} \in \mathfrak{X}(M) \) such that \( \tilde{X}|_{U_\alpha} := X_\alpha - g_\alpha R - h_\alpha Y \).

It remains to prove that we may choose \( \tilde{X} \) polynomial in the variable \( x \).

Indeed, we write \( \tilde{X} = \sum_{k=0}^{\infty} \tilde{X}_k \) for the Taylor expansion of \( \tilde{X} \) around the origin, in the variable \( x \).

Then \( \tilde{X}_k \) is holomorphic in \((x,t)\) and homogeneous polynomial of degree \( k \) in the variable \( x \). We have \( A = \Omega_t \circ \tilde{X} = \sum_{k=0}^{+\infty} \Omega_t(\tilde{X}_k) \) and since it is polynomial homogeneous of degree \( n + 2 \) in \( x \) it follows that \( k \neq 1 \Rightarrow \Omega_t(\tilde{X}_k) = 0 \) and \( \Omega_t(\tilde{X}_1) = A \). Since \( \tilde{X}_1 \) is linear, the flow of \( \tilde{X}_1 \) gives an analytic trivialization for \( \{F_t\}_{t \in \mathbb{D}_e} \). □

7. Quasi-hyperbolic foliations

Now we recall some of the features coming from [13]. A germ of holomorphic foliation at \( 0 \in \mathbb{C}^2 \), is quasi-hyperbolic if after its reduction of singularities process [18], we obtain an exceptional divisor that is a finite union of invariant projective lines meeting transversely at double points and a foliation with Saddle-type singularities: \( xdy - \lambda ydx = 0, \lambda \in (\mathbb{C} - \mathbb{R}) \cup \mathbb{R}_- \).

In [13] we also find the notion of generic quasi-hyperbolic germs of foliation with some dynamical restrictions on the structure of the foliation after the reduction process.

The outstanding result is:

**Theorem 7.1.** A topological trivial deformation of a generic quasi-hyperbolic germ of foliation is an equisingular unfolding.

Using the concept of singular holonomy [3] we may strengthen this results as follows:

**THEOREM E.** Let \( \{F_t\}_{t \in \mathbb{D}} \) be a topologically trivial analytic deformation of a germ of quasi-hyperbolic foliation \( F_0 \) at \( 0 \in \mathbb{C}^2 \). We have the following possibilities:

(i): \( F_0 \) admits a Liouvillian first integral and all its projective holonomy groups are solvable,

(ii): \( \{F_t\}_{|t|<\varepsilon} \) is an equisingular unfolding.

**Proof.** Assume that all the projective singular holonomy groups of \( F_0 \) are solvable. In this case according to [3], \( F_0 \) has a Liouvillian first integral. (Here we use strongly the fact that \( F_0 \) is quasi-hyperbolic) We may therefore consider the case where some component of the exceptional divisor has non-solvable singular holonomy group. This implies topological rigidity and abundance of hyperbolic fixed points as well as the dense orbits property for
this group as well as for all the projective singular holonomy groups, which are the main ingredients in the proof of theorem (7.1) and \( \{F_t\}_{|t|<\epsilon} \) is an unfolding.

**Proof of Theorem B.** First we remark that by the topological triviality on \( \mathbb{C}P(2) \) we may assume that \( L_\infty \) is an algebraic leaf for \( F_t \) and that \( \phi_t(L_\infty) = L_\infty, \forall t \in \mathbb{D} \). In fact, we take \( S_t = \phi_t(L_\infty) \subset \mathbb{C}P(2) \). Then \( S_t \) is compact \( F_t \)-invariant and of dimension one, so that \( S_t \) is an algebraic leaf of \( F_t \). By a well-known theorem of Zariski \( S_t \) is smooth. Since the self-intersection number is a topological invariant we conclude that \( S_t \) has self-intersection number one and by Bezout’s theorem \( S_t \) has degree one, that is, \( S_t \) is a straight line in \( \mathbb{C}P(2) \).

The problem here is that \( S_t \) may do not depend analytically on \( t \). That is where we use the hypothesis that there exist at least two reduced singularities \( p_1, p_2 \in \text{Sing}(F_0) \cap L_\infty \). Since \( p_j \) is reduced there exists an analytic curve \( p_j(t) \in \text{Sing}(F_t) \) such that \( p_j(t) \) is reduced singularity of \( F_t \) and \( p_j(t) = \phi_t(p_j), p_j(0) = p_j \), since the line \( S_t \) contains \( p_1(t) \neq p_2(t) \) it follows that \( S_t \) depends analytically on \( t \) and there exists a unique automorphism \( T_t : \mathbb{C}P(2) \to \mathbb{C}P(2) \) such that \( T_t(S_t) = S_0 = L_\infty; T_t(p_j(t)) = p_j, j = 1, 2 \).

Thus \( \psi_t = T_t \circ \phi_t : \mathbb{C}P(2) \to \mathbb{C}P(2) \) gives a topological trivialization for the deformation \( \{F_t^1\}_{t \in \mathbb{D}} \) of \( F_0 \), where \( F_t^1 := T_t(F_t) \), and \( L_\infty \) is an algebraic leaf of \( F_t^1, \forall t \in \mathbb{D} \). Thus we may assume that \( L_\infty \) is \( F_t \)-invariant, \( \forall t \in \mathbb{D} \).

Now we proceed after performing the reduction of singularities for \( F_0 \big|_{L_\infty} \) we consider the exceptional divisor \( D = \bigcup_j D_j, D_0 \cong L_\infty, D_j \cong \mathbb{C}P(1), \forall j \in \{1, \ldots, r\} \) and observe that if the singular holonomy groups of the components \( D_j \) are all solvable then according to [3] (using the fact that the singularities \( p \in \text{Sing}(F_0) \cap L_\infty \) are quasi-hyperbolic) we get that \( F \) is a Darboux (logarithmic) foliation. We assume therefore that some singular holonomy group is nonsolvable, then it follows that by definition of singular holonomy group and due to the fact that the divisor \( D \) is invariant and connected and has saddle-singularities at the corners, we can conclude that all components of \( D \) has nonsolvable singular holonomy groups. This implies, that each germ of \( \{F_t\}_{t \in \mathbb{D}} \) at a singular point \( p \in \text{Sing}(F_0) \cap L_\infty \) is an unfolding (these germs are evidently topologically trivial). Using now arguments similar to the ones in proof of (6.1) we conclude that \( \{F_t\}_{t \in \mathbb{D}_\epsilon} \) is an unfolding for \( \epsilon > 0 \) small enough.

If we assume that for any singularity \( p \in \text{Sing}(F_0) \cap L_\infty \) the germs of unfolding is analytically trivial, then as consequence of (6.1) we conclude that \( \{F_t\}_{t \in \mathbb{D}_\epsilon} \) is analytically trivial for \( \epsilon > 0 \) small enough. Theorem B is now proved.

**Remark 7.2.** Above theorem is still true if one replace condition (3) by the following

\[ (3') \phi_t(L_\infty) = L_\infty, \forall t \in \mathbb{D}. \]
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