Splitting of the chiral critical point and realization of solitonic pion condensate driven by isospin density

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We study the influence of the isospin asymmetry on the phase structure of strongly interacting quark matter near the tricritical point (TCP) using a generalized Ginzburg-Landau approach. The effect has proven to be so drastic, not only bringing about the shift of the location of TCP, but resulting in a rich fine structure at the vicinity of TCP. In particular, we find that an arbitrary small perturbation due to isospin density lifts the degeneracy of TCP making it split into four independent multicritical points. Accordingly, the homogeneous pion condensate and its solitonic counterpart come to occupy large domains in the Ginzburg-Landau coupling space.

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Introduction.—Quantum chromodynamics (QCD) at finite temperature and/or finite density is expected to exhibit a rich phase structure, and has been the subject of extensive theoretical and experimental studies. In particular, several approaches to QCD with two light flavors suggest the existence of a critical end point (CEP) where the first order chiral phase transition turns into a crossover [1]; the CEP continues to a tricritical point (TCP) in the chiral limit where three lines of second-order phase transition meet. Despite many efforts based on the first principle calculations [2], the precise location of CEP in the phase diagram is still controversial.

Recently based on a Ginzburg-Landau (GL) approach [3], and also in other effective models [4], it was shown that such TCP, if any, is actually replaced by a Lifshitz point where three different forms of matter meet: i.e., a phase with broken chiral symmetry, a symmetric (Wigner) phase, and an inhomogeneous phase characterised by an additional translational symmetry breaking. Such inhomogeneous chiral condensates can be viewed as a microscale ordered phase separation [3, 5, 6].

The effect of a finite quark mass on TCP is rather simple: just to turn it into a CEP. Our focus here is the other important ingredient in realistic systems, i.e., the effect of an isospin asymmetry on TCP. Such a flavor symmetry breaking can be caused by a neutrality constraint which should be imposed in any bulk system to prevent the energy density from diverging. The isospin asymmetry effect on the thermodynamics is also important for the physics of heavy ion collisions with neutron rich nuclei at the Fermi energies, as it affects the isospin distillation in the nuclear liquid-gas phase transition [7].

The isospin asymmetry effects were well studied in the physics of color superconductivity at high density, and proved to lead to a rich variety of phases [8]. On the other hand, while there are a few model-based studies in the context of TCP [3], there has been to present, to our knowledge, no systematic analysis based on the GL approach. Here we present for the first time GL analyses of the effects on phases at the vicinity of TCP.

By comparison with other approaches, our GL framework has an advantage that it can give model independent predictions near TCP. Based on it, we take into consideration the charged pion condensate which may become favored by the inclusion of isospin chemical potential [10]. Moreover we incorporate the possibility of inhomogeneous condensate such as a chiral spiral [11] in which the chiral condensate is entangled with the pion condensate. Since we are interested in the response of TCP and its neighborhood against the isospin chemical potential (μI), our strategy is to take μI as a perturbative parameter and expand the GL functional with respect to it. As a result we have many GL couplings, but it is possible to derive universal relations among them; this will be performed based on the assumption that the quark loops are dominant near the TCP. This is justified if it is located at large fugacity region eμI/T ≫ 1. The analyses for the opposite case will be reported elsewhere.

Our findings are: i) an arbitrary small perturbation due to isospin density not only brings about the shift of TCP but makes it split into four independent critical points, and ii) both the homogeneous and inhomogeneous pion condensates come to occupy large domains in the GL coupling space. Although it is beyond the scope of this Letter, these pion condensates near TCP may smoothly continue to pion condensates in nuclear matter at zero temperature [12], or to those with some spatial structures [13]; such meson condensates are of a renewed interest since their analogs may have a chance to be realized in ultracold atomic gases as was recently reported [14]. The fine phase structure near TCP may also have some phenomenological impacts on the physics of heavy ion collisions, and on the proto-neutron star cooling via the neutrino diffusion.

Generalized Ginzburg-Landau approach at finite μI.—Let us start with writing the most general GL potential for the chiral four vector φ(x) = (σ(x), π(x)) with σ(x) ∼ −⟨ψψ⟩, π(x) ∼ −⟨ψΓ τψ⟩. We retain up to the sixth order in the order parameter and its spatial derivative so as to allow for a minimal description of TCP. Then
the GL functional can be decomposed into three parts:
\[ \Omega_{\text{GL}}[\sigma(x), \pi(x)] = \omega_0 + \delta \omega_M + \delta \omega_l, \]
with \( \omega_0 \) being the chiral SU(2)_R \times SU(2)_L invariant part, and \( \delta \omega_M (\delta \omega_l) \) being the feedback from current quark mass (isospin density). The form of each part is rather stringently constrained by symmetry as we describe below. First, \( \omega_0 \) can be set as
\[ \omega_0 = \frac{\alpha_2}{2} \sigma^2 + \frac{\alpha_4}{4} (\phi^2)^2 + \frac{\alpha_{6b}}{4} (\nabla \phi)^2 + \frac{\alpha_6}{6} (\sigma^2)^3 + \frac{\alpha_{6b}}{6} (\phi, \nabla \phi)^2 + \frac{\alpha_{6c}}{6} (\phi^2 (\nabla \phi)^2 - (\phi, \nabla \phi)^2), \]

where \((\phi, \nabla \phi)\) denotes the inner product: \( \sigma \nabla \sigma + \pi \cdot \nabla \pi \).

Second, \( \delta \omega_M = -h \sigma \) breaks the chiral symmetry explicitly: \( SU(2)_L \times SU(2)_R \to SU(2)_V \). The GL coupling \( h \) is proportional to the quark mass \( m \) for light flavors. Lastly, the \( \delta \omega_l \) represents the response to isospin density, which is our main focus here,
\[ \delta \omega_l = \frac{\beta_2}{2} \pi_c + \frac{\beta_4}{4} \pi_c^2 + \frac{\beta_{4b}}{4} (\phi^2 - \pi_c^2) \pi_c^2 + \frac{\beta_{4c}}{4} (\nabla \pi_c)^2, \]
where \( \pi_c = (\pi_1, \pi_2) \) is the charged pion doublet. This term breaks \( SU(2)_L \times SU(2)_R \) down to \( U(1)_{\text{flavor}} \times U(1)_{\text{isospin}} \). Negative \( \beta_2, \beta_4, \beta_{4b} \) favor the homogeneous pion condensate \( [\pi_c] \neq 0 \) while negative \( \beta_{4c} \) does inhomogeneous one. It is safe to neglect the quartic terms \( \beta_4, \beta_{4b}, \beta_{4c} \) as long as we are only interested in the extreme vicinity of TCP where both \( |\pi_c|, |\phi| \) are much smaller than the order of \( \mu_1 \).

We here retain them because we are interested in the structure of phases where the order parameters become comparable with \( \mu_1 \). To summarise the symmetry structure, \( SU(2)_L \times SU(2)_R \) is broken to \( SU(2)_V \) due to \( \delta \omega_M \), and then further down to \( U(1)_{\text{flavor}} \) via \( \delta \omega_l \). The residual \( U(1)_{\text{isospin}} \) may be broken spontaneously via the formation of charged pion condensate.

Ignoring the quark mass term \( \delta \omega_M \), we have eleven couplings \( \{\alpha_2, \alpha_4, \alpha_{6b}, \alpha_6, \alpha_{6b}, \alpha_6, \alpha_d, \beta_2, \beta_4, \beta_{4b}, \beta_{4c}\} \). We can reduce the number of couplings significantly using the expansion about \( \mu_1 = 0 \) as we demonstrate below. In order to make this, we assume that at the vicinity of TCP quark loops are dominant to gluonic ones, which may be justified provided that TCP is located at large fugacity region. The feedback of quark loops to the energy is
\[ \Delta \Omega = - \frac{TN_c}{V} \sum_{n=2}^{\infty} \frac{1}{n!} (S_0 \Sigma(x))^n, \]
where \( V \) denotes the volume of the unit cell of periodic condensate, and \( \text{Tr} \) should be taken over the Dirac, flavor and functional indices. \( S_0 = \text{diag} (S_u, S_d) \) is the bare quark propagator, and \( \Sigma(x) = \sigma(x) I + i \gamma^5 \pi(x)\tau^1 \) is the self-energy. Since the potential is symmetric under \( U(1)_{\text{flavor}} \times U(1)_{\text{isospin}} \) that leaves \( \sigma^2 + \pi_0^2 \) and \( \pi_c^2 \) independently invariant, we set \( \pi_2 = \pi_0 = 0 \) without any loss of generality to derive the GL couplings. The propagator in the momentum space is \( S(p) = \text{diag} (p_u^{-1}, p_d^{-1}) \) with \( p_p^2 = (i \omega_m + \mu_f \cdot p) \) where the subscript \( f \) refers to the flavor index for \( u \) and \( d \) quarks, and \( \omega_m \) is the fermionic Matsubara frequency. We do not show all the results, but for example \( \beta_2 \) and \( \alpha_4 \) have the expressions:
\[ \beta_2 = 4TN_c \sum_{\mu,p} \frac{(p_u - p_d)^2}{p_p^2}, \alpha_4 = 4TN_c \sum_{\mu,p} \frac{p_p^2 + p_d^2}{p_p^2 p_d^2}. \]
Let us start with the GL couplings \( \alpha \)'s which enter in \( \omega_0 \). From the expressions for \( \alpha \)'s, we can derive the universal relations \( \alpha_{6b} = \frac{\alpha_4}{2}, (\alpha_{6b}, \alpha_{6c}, \alpha_{6d}) = (5, 3, 1/2) / \alpha_6 \). Although \( \omega_0 \) is the chirally symmetric part, the couplings themselves can have dependence on \( \mu_1^2 \), whose expansion should have the following general structure:
\[ \left( \begin{array}{c} \alpha_2 \\ \alpha_4 \\ \alpha_6 \end{array} \right) = \left( \begin{array}{ccc} 1 / a \mu_1^2 & 0 & (\mu_1^2) \\ 0 & 1 & b \mu_1^2 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} \alpha_2^{(0)} \\ \alpha_4^{(0)} \\ \alpha_6^{(0)} \end{array} \right), \]
where \( a, b \) are just some constants and the subscript \( (0) \) denotes the quantity in the absence of isospin density. Explicit computation leads \( a = 0, b = 1 \). Similarly the expansion of \( \delta \omega_l \) is cast into the form:
\[ \left( \begin{array}{c} \beta_2 \\ \beta_4 \end{array} \right) = \left( \begin{array}{c} c \mu_1^2 \\ 0 \end{array} \right) \left( \begin{array}{c} \mu_1^2 \mu_1^4 \end{array} \right) \left( \begin{array}{c} \alpha_2^{(0)} \\ \alpha_4^{(0)} \end{array} \right), \]
and we found \( c = -1/2, c = d = b(c) = -1. \) When \( \alpha_6^{(0)} > 0 \), \( \beta_2 = -\mu_1^2 \alpha_4^{(0)}/2 \) acts as the main driving force to the formation of homogeneous pion condensate, whose diagrammatic expression is shown in Fig. 1. Since we can use \( \alpha_6^{(0)} > 0 \) to set the energy scale, we are left with only three parameters \( \{\alpha_2^{(0)}, \alpha_4^{(0)}, \mu_1^2\} \). We study the phase diagram in the space of \( \{\alpha_2^{(0)}, \alpha_4^{(0)}\} \) and how it is affected by \( \mu_1^2 \). It has in principle a unique map onto the QCD phase diagram at the vicinity of the critical point where the chiral condensate has a size less than or of order \( \mu_1 \).

Just to avoid notational confusion, we suppress the subscript \( (0) \) for \( \alpha \)'s in the following. Moreover we scale every quantity with energy dimension in the unit \( \alpha_6^{-1/2} \). In particular we have \( \alpha_6 = 1 \). Using the scaling argument as in \[ 2 \], it turns out that any critical condition is of the form \( f(\alpha_2^{(0)}/\alpha_4^{(0)}, \mu_1^2/\alpha_4^{(0)}) = 0 \) where \( f \) is some function having two arguments. From this, we see that even when \( \mu_1 \)
is magnified as $\lambda \mu_1$ with $\lambda$ being an arbitrary real number, the phase diagram stays the same once we redefine $(\alpha_2, \alpha_4)$ by $(\lambda^4 \alpha_2, \lambda^2 \alpha_4)$. In particular, we conclude that the coordinate of any critical point in $(\alpha_2, \alpha_4)$-plane, if any, should scale as $\alpha_2 \propto \mu_1^4$, $\alpha_4 \propto \mu_1^2$.

Phase diagram with homogeneous states only.—Let us first discuss the consequence of nonzero $\mu_1$ to homogeneous condensates. We show the phase diagram in the lower panel of Fig. 2. Just for comparison, we also show in the upper panel how $\delta \omega_M$ affects TCP and its neighborhood in the absence of $\mu_1$. In the latter case we actually find CEP at $(2.28h^{1/5}, -2.25h^{2/5})[13]$. In contrast, we find a drastic change of phase diagram due to nonzero $\mu_1$ in the former case. As expected, we find that pion condensate replaces a major part of the phase of chiral condensate in particular for $\alpha_4 > 0$. This is because $\beta_2 = -\mu_1^2 \alpha_4/2 < 0$ favors the pion condensation. In fact the second-order phase boundary between the Wigner and pion condensed phases can be derived from the condition of vanishing quadratic term:

$$0 = \frac{\partial^2 \Omega_{GL}(0, \pi)}{\partial \pi^2} \bigg|_{\pi = 0} = \alpha_2 - \mu_1^2 \alpha_4 / 2.$$ 

Another notable point in the phase diagram is the shift of the location of TCP from $(0, 0)$ to $(0, -\mu_1^2)$ labeled by TCP’. The shift itself is consistent with model-based studies [9], and the location can be understood by noting

$$\Omega_{GL}[\sigma, 0] = \frac{\alpha_2}{2} \sigma^2 + \frac{\alpha_4 + \mu_1^2}{4} \sigma^4 + \frac{1}{6} \sigma^6,$$

where we see that $\alpha_2 = 0$ together with $\alpha_4 + \mu_1^2 = 0$ defines a new TCP. For the region of $(\alpha_2 < 0, \alpha_4 < 0)$, there is a competition between the chiral and pion condensates, leading to a first order phase transition between two phases as depicted in the figure. The coordinate of characteristic points is listed in the Table I.

Phase diagram with inhomogeneous states included.—We now address the question what is the impact of inclusion of inhomogeneous phases. We restrict the crystal structure to one dimensional ones [3, 10]. We analyse three cases, (i) $\sigma(z) \neq 0, \pi(z) = 0$, (ii) $\sigma(z) = 0, \pi(z) \neq 0$ and (iii) $\sigma(z) \neq 0, \pi(z) \neq 0$. For the case (i) or (ii), we can solve analytically the Euler-Lagrange (EL) equation [3, 5, 17], leading to a solitonic condensate

$$\sigma(z) [\text{or } \pi(z)] = \sqrt{\nu} \text{sn}(kz; \nu),$$

with $\nu$ denoting the elliptic modulus. In the case (iii), we can not solve the EL equation analytically. We take here the variational method instead. The variational state we consider here is one so called chiral spiral [11] where $\sigma$ and $\pi$ are entangled: $\sigma(z) = m \cos(qz), \pi(z) = m \sin(qz)$. This assumption is motivated by the study of the Gross-Neveu model [18]. We compute the phase diagram by compurer energies for the three cases. Resulting phase diagram is displayed in Fig. 3. We find no window for the chiral spiral, but notice a new fine structure appearing below TCP’. TCP’ changes into a Lifshitz point from which a solitonic chiral condensate $\sigma(\chi)$ expands between the homogeneous $\sigma$ and Wigner phases; this is indeed expected [3]. In this case, however, we notice that $\sigma(\chi)$-phase does not continue to the region of $\alpha_4 \ll -\mu_1^2$; a major part of solitonic $\sigma$ island is taken over by a

![FIG. 2. (color online). Phase structure for homogeneous condensates: Solid (dashed) line represents first (second) order transition, while double dashed line does the pseudo critical line for crossover. (Upper panel): The effect of $h$ in the absence of $\mu_1$. When $h = 0$, TCP is located at the origin; we have a second order critical line $\alpha_2 = 0$ for $\alpha_4 > 0$ and a first order one $\alpha_2 = \frac{1}{16} \alpha_4^2$ for $\alpha_4 < 0$. Nonzero $h$ turns TCP into a CEP whose location is shifted to $(2.28h^{1/5}, -2.25h^{2/5})$. (Lower panel): The effect of $\mu_1$ for $h = 0$. Asymptotic behaviours of the first order critical line separating the $\sigma$ phase from the $\pi$ phase can be derived analytically: $\alpha_2 \rightarrow -\frac{3}{4} \alpha_4 + \frac{1}{2} \alpha_4^2 + O(\mu_1^2)$ as $\alpha_4 \rightarrow -\infty$, while $\alpha_2 \rightarrow -\frac{3}{4} \mu_1^2 \alpha_4 + \frac{1}{8} \alpha_4^2$ as $\alpha_4 \rightarrow 0$.](https://example.com/fig2.png)

| Table I. Location of critical points; CEP, TCP’ and P in Fig. 2. Q, R in Fig. 3. |
|-----------------|-----------------|-----------------|-----------------|
| CEP            | $h = \frac{27}{2} \mu_1^2$ | $\frac{5}{3} \frac{27}{2} \mu_1^2$ | $\frac{5}{3} \frac{27}{2} \mu_1^2$ |
| TCP’           | 0               | $-\mu_1^2$     | Lifshitz t-critical point |
| P              | 0               | 0               | Lifshitz b-critical point |
| Q              | $3 \mu_1^2 / 32$ | $-3 \mu_1^2 / 2$ | Lifshitz b-critical point |
| R              | $0.21 \mu_1^2$  | $-2.22 \mu_1^2$ | critical (end) point |
solitonic pion condensate \( \pi(x) \). This is caused by the term \(-|\alpha_4| - \mu_1^2(\nabla \sigma)^2\) in the potential where \( \mu_1^2 \) disfavors \( \sigma(x) \). Since the quartic term affects thermodynamics via the square of its coefficient, \( \sim \alpha_4^2 - 2\mu_1^2|\alpha_4| \), the effect overwhelms the others at negative large \( \alpha_4 \). This makes \( \pi(x) \)-phase replace \( \sigma(x) \)-phase eventually. As a consequence there appear two additional multicritical points denoted by Q and R in the figure. The precise locations of these points are listed in the Table. \( \Box \)

Let us briefly discuss the effect of quark mass. Since it breaks explicitly the chiral symmetry down to SU(2)_V, the phase boundary between the Wigner and the \( \sigma \neq 0 \) phase would be smoothen out. We expect, however, that pionic phases will not be affected so dramatically since it is characterized by the spontaneous breaking of U(1)_A and so is nothing to do with the explicit breaking of axial symmetry. The detailed analyses are now under investigation and will be reported elsewhere.

In conclusion, we performed a systematic GL analysis on the effect of isospin asymmetry on TCP and phases of its neighborhood. By incorporating the effect of isospin density perturbatively at the leading order, we first derived the GL potential which works at the vicinity of TCP where the order parameters become comparable with \( \mu_1 \). Based on it, we studied how the isospin asymmetry affects the phase structure. We found that it has several remarkable effects; it does not only cause a shift of the location of TCP, but also brings about the development of sizable region for homogeneous and inhomogeneous pion condensates. This in turn leads to the appearance of several new multicritical points.

Let us finally make some speculative remarks about the phases found near TCP. First, although we found a first order phase transition between the solitonic \( \sigma(x) \) and \( \pi(x) \) phases, it may be replaced with a fine structure once we determine a suitable functional form for two condensates solving the coupled EL equation; for instance, a phase with \( \sigma(x) \) entangled with \( \pi(x) \) may show up. Second, the pion condensed phases near TCP may continue smoothly to those discussed in nuclear matter at low temperature \cite{12}. Lastly, multicritical points observed near TCP may have some experimental signatures such as those discussed for CEP \cite{19}. These clearly deserve further investigations.

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