A UNIVERSAL GAP FOR THE DIAMETER OF ORBIT SPACES OF COMPACT LIE GROUP ACTIONS ON SPHERES, AND CONSEQUENCES IN QUANTUM CONTROL.

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ABSTRACT. In this paper, we prove the existence of a universal gap for controllability (minimum time) of finite dimensional quantum systems. This is equivalent to the existence of a gap in the diameter of orbit spaces of compact connected Lie group unitary actions on the Hermitian spheres.

It is known that such a gap does not exist for finite group orthogonal actions. The existence of this gap for connected compact Lie groups has already been conjectured in the literature.

1. INTRODUCTION

We consider finite dimensional quantum control systems (Σ), i.e.:

\[ \dot{x} = Ax + \sum_{i=1}^{p} B_i x u_i \quad (\Sigma) \]

\[ x \in \mathbb{C}^n, \quad u_i \in \mathbb{R}, \]

\[ A, B_i \text{ skew-adjoint matrices.} \]

In practice, the drift \( A \) represents the \((n\text{-dimensional})\) Schrödinger dynamics of the system and the \( B_i \)'s are the laser controls.

The unit of time is the “time of the drift”, i.e. we assume that \(||A|| = 1\), where the norm is the standard matrix norm associated with the Hermitian norm over \( \mathbb{C}^n \).

In quantum dynamics, the state \( x \) evolves on the unit (real) sphere \( S^{2n-1} \subset \mathbb{C}^n \). The minimum time \( T(\Sigma) \) of the system (Σ) is defined as the supremum of the minimum times necessary to connect two points of the unit sphere \( S^{2n-1} \) with trajectories of (Σ), corresponding to arbitrary \( L^\infty \) controls \( u_i(t) \). In other terms,

\[ T(\Sigma) = \sup_{X,Y\in S^{2n-1}} \inf\{T \geq 0 \text{ s.t. } \gamma(0) = X, \gamma(T) = Y, \gamma \text{ trajectory of } \Sigma\}. \]

A conjecture, due to Andrei Agrachev, is the following:

Conjecture 1. There is a universal gap \( \delta > 0 \) for the minimum time, i.e. whatever the dimension \( n \), and whatever the quantum system (Σ), either \( T(\Sigma) = 0 \), or \( T(\Sigma) \geq \delta \).
We will prove this conjecture, as a direct consequence of the following Theorem 1.

Let $G$ be any connected Lie subgroup of $U(n)$, the unitary group over $\mathbb{C}^n$. The diameter $D(G)$ is the maximum distance in $\mathbb{C}^n$ of two $G$-orbits in $S^{2n-1}$. Clearly,

$$D(G)^2 = 2 \sup_{X,Y \in S^{2n-1}} \inf_{g \in G} (1 - \text{Re}(<g.X,Y>)).$$

Here, $<.,.>$ is the usual Hermitian scalar product over $\mathbb{C}^n$, i.e. $<X,Y> = X'\overline{Y}$ ($X'$ being the transpose of $X$, and $\overline{Y}$ being the conjugate of $Y$).

**Theorem 1.** There is a universal gap $\delta' > 0$ for the diameter of unitary actions, i.e. for nontransitive actions over the sphere, it holds $D(G) \geq \delta'$ for some universal constant $\delta' > 0$.

It is clear that Agrachev’s conjecture follows from Theorem 1. Actually, given a system $(\Sigma)$ one applies the theorem to the group $G$ (or its closure) whose Lie Algebra is generated by the $B_i$, $i = 1, \ldots, p$, as a subalgebra of $u(n)$.

A similar conjecture relative to isometric group actions on spheres is proposed in [7]. Also, in [8], it is shown that, for finite orthogonal actions, this similar conjecture is false. Therefore the connectedness assumption here is crucial. Other related references are [3, 5, 6, 9].

Our proof consists of several successive reductions of the problem. First, in Section 2 we reduce the problem to irreducible actions of $G$, where $G$ is simple, simply connected. Second, in Section 3 we reduce to basic irreducible representations of such groups. This allows also to evacuate the case of the exceptional groups, since their basic representations are in finite number. In Section 4, we reduce to the case of the spin representations, which we treat in a quite different way in Section 5.

For the proof, we use the classification of representations of compact simply connected simple Lie groups, which is the same as the one of (representations of) complex simple Lie algebras, or equivalently of their real compact forms. In fact, at all steps our proofs are very simple, and we always get much more than needed. For that reason, we think that may be there is a much simpler proof. However, we were not able to get it.

2. **Reduction to irreducible representations of compact simple Lie groups**

First let us go for a while to Formula (1.2). It is easy to see that finding the universal lower bound $\delta'$ to $D(G)^2$ is equivalent to finding the upper bound $\varepsilon = 1 - \frac{\delta'}{2}$ to the quantity $R(G)$,

$$R(G) = \inf_{X,Y \in S^{2n-1}} \sup_{g \in G} \text{Re}(<g.X,Y>) \leq \varepsilon < 1$$

This last condition is implied by

$$M(G) = \inf_{X,Y \in S^{2n-1}} \sup_{g \in G} |<g.X,Y>| \leq \varepsilon < 1$$
Facing a system $\Sigma$, we are given a unitary representation $\Phi$ of the compact connected group $G$. Such a group is automatically covered by the direct product of a compact connected, simply connected Lie group by a torus. Therefore, without restriction, $G$ may be assumed to be a direct product of a torus and several simple compact, connected, simply connected Lie groups. We will assume this from now on.

What is also clear is that $\Phi$ has to be irreducible: if not, the diameter of $\Phi(G)$ is at least $\sqrt{2}$ by the invariance of two orthogonal subspaces.

In fact, we will prove that the (sufficient) condition (2.2) is always satisfied when $G$ is the direct product of compact connected, simply connected simple Lie groups. Hence we can forget about the toric component of $G$: coefficients $\langle \Phi(g), X, Y \rangle$ will just be multiplied by a character of the toric component, which does not change the quantity $M(G)$.

At this point, we may assume that $\Phi$ is a tensor product of unitary irreducible representations of several simple, compact, connected, simply connected Lie groups.

**Remark 1.** This shows already that there is a lower bound $\delta(n)$ depending on the dimension $n$, since we are reduced to a finite number of cases.

The very simple following lemma shows that we may restrict to the case where $G$ is a simple factor:

**Lemma 1.** A tensor product $\Phi \otimes \Phi'$ of two unitary irreducible representations of connected simple Lie groups $G, G'$ over Hermitian spaces $V, V'$, meets $M(\Phi(G) \otimes \Phi'(G')) \leq \frac{1}{\sqrt{2}}$.

**Proof.** Let $e_1, e_2$ be orthonormal vectors in $V$ and $e'_1, e'_2$ be orthonormal vectors in $V'$ (all real). Let $S(V), S(V'), S(V \otimes V')$ denote the unit spheres in $V, V', V \otimes V'$ respectively. We have $S(V) \otimes S(V') \subset S(V \otimes V')$. Choose $X = e_1 \otimes e'_1 \in S(V) \otimes S(V')$ and $Y = \frac{1}{\sqrt{2}}(e_2 \otimes e'_2 + e_1 \otimes e'_1) \in S(V \otimes V')$.

Let us estimate $m(g, g') = |\langle \Phi \otimes \Phi'(g, g')X, Y \rangle|$. Of course $m(g, g')$ is of the form $|\langle Z \otimes W, Y \rangle|$, where $Z \in S(V), W \in S(V')$, and

$$m(g, g') = |\langle Z \otimes W, Y \rangle| = \frac{1}{\sqrt{2}}|z_1w_1 + z_2w_2|$$

and

$$\sup_{g, g' \in G} m(g, g') = \sup_{|z_1|^2 + |z_2|^2 \leq 1, |w_1|^2 + |w_2|^2 \leq 1} \frac{1}{\sqrt{2}}|z_1w_1 + z_2w_2| = \frac{1}{\sqrt{2}}$$

from what $M(\Phi(G) \otimes \Phi'(G')) \leq \frac{1}{\sqrt{2}}$. \hfill \qed

3. Reduction to basic representations and evacuation of exceptional groups

From now on, $G$ is Lie, compact, simple, connected, simply connected, and $\Phi$ is a unitary irreducible representation of $G$. 
Basic representations are those which do not come from a Cartan product of irreducible representations of $G$. By the following Lemma 2, we may evacuate the non-basic representations.

**Lemma 2.** Let $\Phi$ be the Cartan product of two unitary irreducible representations $\Phi_1$ and $\Phi_2$. Then $M(\Phi(G)) \leq \sqrt{2}$.

**Proof.** Let $l_1, l_2$ be two unit lowest weight vectors of $\Phi_1$ and $\Phi_2$ respectively. Let $h_1, h_2$ be two unit highest weight vectors of $\Phi_1$ and $\Phi_2$ respectively.

It is known that:
1. $l_1 \otimes l_2$ is a (unit) lowest weight vector of $\Phi$,
2. $h_1 \otimes h_2$ is a (unit) highest weight vector of $\Phi$,
3. $h_1 \otimes h_2$ is in the irreducible component of $\Phi_1 \otimes \Phi_2$ corresponding to $\Phi$ (by definition) but also $l_1 \otimes l_2$ is in the same irreducible component.

Now, we chose again $X = h_1 \otimes h_2$ and $Y = \frac{1}{\sqrt{2}}(h_1 \otimes h_2 + l_1 \otimes l_2)$ and evaluate

$$m(g) = \sup_{g \in G} | \langle \Phi(g)X, Y \rangle |$$

Again, $m(g)$ is of the form

$$m(g) = \sup_{g \in G} | \langle \Phi_1(g)h_1 \otimes \Phi_2(g)h_2, Y \rangle | \leq \sup_{||V||=1} | \langle V \otimes W, Y \rangle | \leq \frac{1}{\sqrt{2}}$$

with the same reasoning as in the proof of Lemma 1. The lemma follows. □

At this step, it remains only to examine the case where $\Phi$ is a basic representation.

It is important to notice that we can immediately evacuate the cases where $G$ is one of the exceptional groups $E_6, E_7, E_8, F_4, G_2$ : actually, basic representations of these exceptional groups are in finite number, and hence they do not count in our problem.

### 4. Case of basic representations except spin

We may now assume that $\Phi$ is a basic, unitary irreducible representation of a connected simply connected compact form group $G$ of type $A_n, B_n, C_n$ or $D_n$.

Let us first deal with the case of $A_n$.

#### 4.1. Case of $A_n$. Let $\Phi$ be the natural representation of $G = SU(n+1)$. It acts transitively over $S^{2n+1}$. The basic representations of $G$ are the representations $\Phi^{(k)}$ acting (unitarily) on the $k^{th}$ exterior power $V^k = \Lambda^k \mathbb{C}^{n+1} = \Lambda^k V$ in the natural way, $k = 1, \ldots, n$. Let us chose $X = e_1 \wedge e_2 \wedge \ldots \wedge e_k$ and $Y = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \wedge \ldots \wedge e_k + e_{k+1} \wedge e_{k+2} \wedge \ldots \wedge e_{2k})$ for $k \leq \frac{n+1}{2}$.

Again, it is not too hard to see that $| \langle \Phi^{(k)}(g)X, Y \rangle | \leq \frac{1}{\sqrt{2}}$ for all $g \in G$ and $M(\Phi^{(k)}(G)) \leq \frac{1}{\sqrt{2}}$.

For the other values of $k$ a similar reasoning holds using the contragredient (=conjugate) representation to $\Phi$ and the duality $\Lambda^k V \rightleftharpoons \Lambda^{n-k+1} V$.

Therefore, we can exclude $A_n$.

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1. Along the paper, we mostly adopt the terminology and notations of Dynkin, from his paper [4].
4.2. Case of $C_n$. The elementary representation $\Phi$ is now the standard representation over $V = \mathbb{C}^{2n}$ of the real compact form $G = Sp(n)$ of $Sp(2n, \mathbb{C})$. It acts transitively on $S(V)$. All basic representations $\Phi^{(k)}$, $k = 1, \ldots, n$ act over $\Lambda^k V$ again in natural way, but the $\Phi^{(k)}$ being not irreducible, $\Phi^{(k)}$ is the highest weight component of $\Phi^{(k)}$. Lemma 0.3 page 360, paragraph 6, point 32, of [11] claims that inside the highest weight component $\Phi^{(k)}$ of $\Phi^{(k)}$, there are two vectors of the form $e_{11} \wedge e_{12} \wedge \ldots \wedge e_{1k}$ and $e_{j1} \wedge e_{j2} \wedge \ldots \wedge e_{jk}$, with $i l \neq j r$ for all $r, l$. Therefore, exactly the same reasoning as for $A_n$ allows to exclude $C_n$.

4.3. Case of $B_n, D_n$. Simply connected compact forms are the groups $Spin(2n+1)$, $Spin(2n)$ that are the (universal) double covers of $SO(2n+1)$, $SO(2n)$.

The basic representations fall into two classes: those who come from representations of $SO(2n+1)$, $SO(2n)$ via the covering mapping and the remaining that are the 3 series of basic spinor representations (one for $Spin(2n+1)$ and two for $Spin(2n)$).

In this section, we discuss the first class.

First, for the natural representation $\Phi$ of $G = SO(r)$, it holds

$$R(\Phi(G)) \leq 0, \quad M(\Phi(G)) \leq \frac{1}{\sqrt{2}}$$

Second, for $Spin(2n+1)$, we have $n - 1$ basic representations that come from this natural representation of $SO(2n+1)$ over $V = \mathbb{C}^n$. As previously, they are denoted by $\Phi^{(k)}$ and they act (irreducibly now) on $\Lambda^k V$, $k = 1, \ldots, n - 1$. For $Spin(2n)$ it is the same, for $k = 1, \ldots, n - 2$.

Therefore, we conclude for these basic representations in the same way as for $A_n$.

5. CASE OF SPIN REPRESENTATIONS

At this step, it remains only to consider the 3 series of basic spin representations mentioned above.

The reasoning will be (almost) exactly the same in the 3 cases. Therefore, we give the proof in the case of the basic spin representation of $Spin(2n+1)$ only. The main change in the two other cases is the dimension of the representation that will be $2^{n-1}$ in place of $2^n$, which has no consequence on the main result.

5.1. Preliminaries.

5.1.1. Technical Lemma. Real constants $k, K > 0$ are given, together with a family $\{A_\alpha\}_{\alpha \in A}$ of $2^n \times 2^n$ complex matrices, $\|A_\alpha\| \leq K$.

We assume that all matrices $A_\alpha$ satisfy the following condition: each column has $kn^2$ non-zero elements, at most. We then have the following technical lemma.

Lemma 3. Let $e_i$ with $i = 1, \ldots, 2^n$ be the canonical basis of $\mathbb{C}^{2^n}$. For each $j \in \{1, \ldots, 2^n\}$, $\alpha \in A$, define the set of indexes:

$$I_{\alpha,j} := \{i \in \{1, \ldots, 2^n\} \text{ such that } < A_\alpha e_i, e_j > \neq 0\}.$$ 

Assume that for each $j \in \{1, \ldots, 2^n\}, \alpha \in A$ it holds $|I_{\alpha,j}| \leq kn^2$. Then, for all $\varepsilon > 0$, if $n$ is large enough, it exists a sequence $i_n, j_n$ such that for all $\alpha \in A$ it holds
\[ |< A^n e_i, e_j | \leq \varepsilon. \]

**Proof.** \(< A^k e_i, e_j > \neq 0 \) for at most \((kn^2)^2\) indices \(j\), and \(< A^k e_i, e_j > \neq 0 \) for at most \((kn^2)^r = k' n^{2r}\) values of \(j\).

Set \( \delta_r = \sum_{k=r}^{+\infty} \frac{k^k}{k!} \). Assume that \( r \) is large enough to have \( \delta_r \leq \varepsilon \), and assume that \( n \) is large enough to have \( k^r n^{2r} < 2^n \).

Define \( \Phi_{i,j} \) as follows:

\[ \Phi_{i,j} = \sum_{k=r}^{+\infty} \frac{< A^k e_i, e_j >}{k!}, \]

it then holds,

\[ |\Phi_{i,j}| = \sum_{k=r}^{+\infty} \frac{|A^k|}{k!} \leq \delta_r \leq \varepsilon \]

\( \square \)

5.1.2. **Technical facts.** We recall that \( \text{Spin}(2n + 1) \) is the simply connected double cover of \( \text{SO}(2n + 1) \). The Lie algebra of both groups is \( \text{so}(2n + 1) \), both being compact real forms of \( \text{SO}(2n + 1, \mathbb{C}) \).

- The root system of \( \text{so}(2n + 1, \mathbb{C}) \) is \( \{ \pm \lambda_p, \pm \lambda_p \pm \lambda_q \}, p, q = 1, ..., n \), where \( \pm \lambda_p \neq \pm \lambda_q \).
- Another compact real form (conjugate to \( \text{so}(2n + 1) \) by an inner automorphism) is obtained, following [2] by

\[
(5.1) \quad \text{so}(2n + 1) \simeq \sum_{i=1}^{n} \gamma_i \alpha_i \sqrt{-1} + \sum_{\alpha} (\mu_\alpha E_\alpha + \bar{\mu}_\alpha E_{-\alpha}),
\]

where the first sum is over the simple roots \( \alpha_i \) and the second sum is over the positive roots \( \alpha \), where \( \gamma_i \) are real, and \( \mu_\alpha \) are complex numbers.

- The system of weights of the spin representation is

\[
1 \over 2 (\pm \lambda_1 \pm \lambda_2 \ldots \pm \lambda_n)
\]

with \( < \lambda_i, \lambda_i > = 1 \), \( < \lambda_i, \lambda_j > = 0 \), \( i \neq j \), where the scalar product is given by the (opposite of) Killing form.

- The maximal weight (in a usual canonical order) is \( 1 \over 2 (\lambda_1 + \lambda_2 + \ldots + \lambda_n) \), the weight spaces are one-dimensional.

5.2. **Proof of the exclusion of the spin representation of** \( B_n \). The first point is that the radius of surjectivity (for the metric associated with the Killing form) of the exponential mapping of \( \text{so}(2n + 1) \to \text{SO}(2n + 1) \) is \( \pi \). Therefore, due to the double cover, the radius of surjectivity of \( \text{Exp} : \text{so}(2n + 1) \to \text{Spin}(n) \) is \( R_\text{s} = 2\pi \).

Let \( \Phi \) be the (basic) spinor representation, \( \Phi : \text{Spin}(n) \to U(\mathbb{C}^{2^n}) \). Let \( \varphi \) denote the associated Lie algebra representation.
For any \( x \in \text{so}(2n + 1) \), we will consider the matrix of \( \varphi(x) \) in a basis of orthonormal weight vectors, relative to the weight decomposition 5.1 above.

If we can prove that the family of matrices \( \varphi(x) \), \( ||x|| \leq R_s = 2\pi \) meets the assumptions of Lemma \( \ref{lemma} \), then we prove that some (orthonormal) coefficient \( \langle \Phi(g)X, Y \rangle > 0 \) is smaller than \( \frac{1}{2} \), for \( n \) large enough. Hence \( M(\text{spin}(2n + 1)) \leq \frac{1}{2} \) for \( n \) large enough. It remains only (for \( \text{Spin}(2n + 1) \)) a finite number of cases. Then, Theorem \( \ref{theorem} \) and the corresponding Conjecture \( \ref{conjecture} \) are proven.

The condition \( \langle \varphi(x)e_i, e_j \rangle \neq 0 \) for at most \( 3n^2 \) values of the index \( j \) is due to the relations between the weight spaces, and the dimension of the Lie algebra \( \text{so}(2n + 1) \).

The condition \( ||\varphi(x)|| \leq K \) is proved as follows:

Any \( x \in \text{so}(2n + 1) \) is diagonalized, \( x = h'\Delta h \), \( h \in \text{SO}(2n + 1) \). Then, \( \varphi(x) = \Phi(h')^*\varphi(\Delta)\Phi(h) \) and \( ||\varphi(x)|| = ||\varphi(\Delta)|| \). \( \Delta \) is in the Cartan subalgebra associated with the decomposition 5.1. \( \varphi(\Delta) \) is diagonal, and if \( \xi \) is a weight vector associated with the weight \( \Lambda \), then \( \varphi(\Delta)\xi = \langle \Delta, \Lambda > \xi \). Then, the norm of \( \varphi(\Delta) \) is the maximum modulus of the values \( \langle \Delta, \Lambda > \) for \( ||\Delta|| \leq R_s \) and \( \Lambda \) a weight of \( \Phi \), \( ||\Lambda|| \leq 1 \).

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