Gravitational waves in magnetized relativistic plasmas

J. Moortgat† and J. Kuijpers†
Department of Astrophysics, University of Nijmegen,
PO Box 9010, 6500 GL Nijmegen, The Netherlands
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We study the propagation of gravitational waves (GW) in a uniformly magnetized plasma at arbitrary angles to the magnetic field. No a priori assumptions are made about the temperature, and we consider both a plasma at rest and a plasma flowing out at ultra-relativistic velocities. In the 3+1 orthonormal tetrad description, we find that all three fundamental low-frequency plasma wave modes are excited by the GW. Alfvén waves are excited by a × polarized GW, whereas the slow and fast magneto-acoustic modes couple to the + polarization. The slow mode, however, doesn’t interact coherently with the GW. The most relevant wave mode is the fast magneto-acoustic mode which in a strongly magnetized plasma has a vanishingly small phase lag with respect to the GW allowing for coherent interaction over large length scales. When the background magnetic field is almost, but not entirely, parallel to the GW’s direction of propagation even the Alfvén waves grow to first order in the GW amplitude. Finally, we calculate the growth of the magneto-acoustic waves and the damping of the GW.

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I. INTRODUCTION

The interaction of gravitational waves (GW) with a magnetized plasma relies on the non-isotropy of the magnetic field and the fact that both GW and electromagnetic waves propagate at the speed of light in vacuum. In vacuum, the GW excites an electromagnetic wave propagating in the same direction as the GW at the speed of light as was first realized by [1]. Propagation through a perfect collisionless fluid does not affect a GW [2], but in an astrophysical plasma the electromagnetic field is coupled to the matter and, indirectly, the GW interacts with both the electromagnetic fields and quantities such as density, pressure and currents [4].

For a summary of previous work on the interaction of a GW with plasma waves we refer to the introduction of [3], referred to from hereon as Paper I, and references therein and [5] for more recent work.

In this paper we study the full problem of a plane fronted, monochromatic GW of either + or × polarization propagating obliquely through a magnetized collisionless plasma. Furthermore, we don’t specify whether the plasma is Poynting flux or matter dominated. Also, we allow for relativistic velocities as we want to apply our results to an ultra-relativistic force-free wind or jet.

The outline of this paper is as follows. We start in Sect. I with a discussion of the Einstein field equations (EFE) describing both the background curvature due to the static energy content and the dynamical interaction of space-time with a time dependent energy-momentum density. In the geometric optics limit (Sect. II A) the GW can be treated just like photons traveling on null geodesics. In Sects. II B – II E we recapitulate how the proper reference frame of an observer (the 3 + 1 orthonormal tetrad frame) is defined by taking snapshots of four-dimensional space-time and using ordinary vector calculus on these spatial hypersurfaces. Also, the covariant derivatives and connection coefficients for such a non-coordinate frame are summarized. A closed set of linearized general relativistic magneto-hydrodynamic (MHD) equations is derived in Sect. III and solved algebraically in Sect. IV. We find that all the fundamental plasma waves are excited: a + polarized GW excites fast magneto-acoustic waves, which was expected from previous idealized calculations in [7] and Paper I. In this paper we extend this result to a realistic fast mode with both electromagnetic and gas properties and find that also the slow magneto-acoustic mode is excited (Sect. IV B). A completely new result is that a × polarized GW propagating at an angle to an ambient magnetic field excites Alfvén waves in the plasma (Sect. IV D). The space-time solutions in the comoving frame are derived in Sect. V and boosted to an ultra-relativistic wind in the observer frame in Sect. VI. In the limit where the phase velocities of the Alfvén and fast mode approach the speed of light the waves can interact coherently, which results in linear growth of the amplitudes. For the Alfvén mode this also requires that the angle with the magnetic field is very small (Sect. VI C).

Finally, we investigate the back-reaction on the GW in Sect. VII and find that as the plasma waves grow, the GW is damped with a group velocity that decreases linearly with distance. An intuitive interpretation of some of these results is presented in Sect. VIII and we end with conclusions in Sect. IX.

Throughout this paper Gaussian geometrized units are adopted (\( G = c = 1 \)). Latin indices \( a \ldots e \) stand for 0, 1, 2, 3, and \( i, j, k \) for spatial components 1, 2, 3.
II. EINSTEIN-MAXWELL EQUATIONS

In Lorentz gauge, the linearized Einstein field equations (EFE) for weak gravitational waves interacting with a magneto-fluid read [8]:

\[ G^{ab} \simeq -\frac{1}{2} \Box h^{ab} = 8\pi \delta T^{ab}, \]  

(1)

where \( \delta T^{ab} \) is the oscillatory part of the energy-momentum tensor. If we further specify to the transverse tensor and (2) take the form [1]:

\[ \Box h^+ = -8\pi (\delta T^{xx} - \delta T^{yy}), \]  

(2a)

\[ \Box h_\times = -8\pi (\delta T^{xy} + \delta T^{yx}). \]  

(2b)

As an example, a GW propagating perpendicularly to a background magnetic field \( B^z \) excites perturbations \( B^z \) that produce an oscillating cross term in the stress-energy tensor and (2) take the form [11]:

\[ \Box h^+(z, t) = 4B^0_B B^1(z, t), \quad \Box h_\times = 0. \]  

(3)

From Maxwell’s equations (coupled to the EFE via the twice contracted Bianchi identities and consequently conservation of energy-momentum \( \nabla_\mu T^{\mu\nu} = 0 \)) one finds wave equations for the plasma quantities in which the GW appears as a source term:

\[ \Box B^1(z, t) \propto h^+(z, t)B^0, \]  

(4)

where \( \Box \) is some wave propagator. (2) together with (1) self-consistently determine the interaction between the GW and the plasma waves (as we will work out in more detail in Sect. VII).

A. Background curvature

The exact (non-linear) Einstein field equations describe the curvature of space-time due to the presence of matter and energy. This curvature is described by the contracted Riemann tensor with an associated characteristic length scale, \( \mathcal{R} \), given by the magnitude of it’s largest components as:

\[ R^{ab} = 8\pi \left( T^{ab} - \frac{1}{2} g^{ab} T \right), \]  

(5a)

\[ \frac{1}{\mathcal{R}^2} \sim (B^0)^2. \]  

(5b)

When the interaction with a magnetic field excites electromagnetic waves growing linearly with distance, viz \( B^1 \propto B^0 h^+ k_z \) we can see from our rough estimate (5b) that the fraction of GW energy that is converted into electromagnetic waves is proportional to \( B^1 / h^+ \propto z / \mathcal{R} \). As an example, a background magnetic field comparable to the surface field of a neutron star (\( B^0 \sim 10^9 \) T) curves space on a scale of \( \sim 10^{10} \) m. If this field would remain constant with distance, all the GW energy would be converted to EM energy on a length scale of order \( \mathcal{R} \). In reality, we will be studying a force-free plasma wind in which the magnetic field falls off linearly with distance. In this case the background curvature also decreases and the interaction length scale is much smaller than the radius of curvature.

The opposite limit of a weak primordial magnetic field with an extremely large spatial extent was discussed by [8] who argued that it is very difficult to keep the interaction coherent on such a scale in a universe that is not a perfect vacuum.

B. Geometric optics

The gravitational waves are assumed to be of the form: \( h^+ \propto H(z)e^{i\omega(z-t)} \) with a slowly varying amplitude such that \( \omega H(z) \gg \frac{\pi}{\mathcal{R}} H(z) \) and \( \frac{\pi}{\mathcal{R}} \ll 1 \). This is the short wavelength, geometric optics limit where GWs behave as rays moving on null geodesics : \( k^\alpha k_\alpha = 0 \) that experience dispersion, refraction, lensing etc. The GW move in an essentially flat Minkowski background, \( g^{ab} \), so the full metric is \( g^{ab} = \eta^{ab} + h^{ab} \) with \( |h^{ab}| \ll 1 \) and self-interactions (of order \( h^2 \)) are negligible.

We will study the propagation of transverse traceless gravitational waves in a magnetized plasma at arbitrary angle \( \theta \) to a background magnetic field \( B_0 \). Also we don’t specify the temperature (or equivalently the pressure) regime except that the temperature equilibration time between the different particle species is short as compared to other characteristic timescales to comply with a hydrodynamic description. The only other non-vanishing plasma quantities in the equilibrium state are then the energy density \( \mu_0 \) and pressure \( p^0 \). The passing GW will excite small (first-order) perturbations in all plasma quantities, denoted as for instance: \( \mu = \mu_0 + \mu^1 \).

C. No coupling to unmagnetized plasma

The stress-energy tensor for a homogeneous perfect fluid in the rest frame of an observer \( (u^a = (1, 0, 0, 0)) \) is:

\[ T^{ab} = (\rho + p)u^a u^b + pg^{ab}. \]  

(6)

A linearly polarized GW will produce perturbations in \( \rho, p \) and \( u^i \) of order \( h^+ \). However, the \( \delta T^{ii} \) components are all higher order except for the trace \( \delta T^{ii} \), which is purely gauge. Explicitly, in (2) \( T^{xy} = T^{yx} = 0 \) and \( T^{xx} - T^{yy} = 0 \).

Therefore a gravitational wave cannot couple to an unmagnetized perfect fluid in linearized theory [8]. Only relativistically gyrating particles can interact with a \( \times \) polarized GW through non-linear \( (\rho p)\nu_i \nu_j \) terms [10, 11].
D. Space-time split

To simplify equations, we will use the so-called 3+1 split of space-time \[12, 13\]. A time-like observer moving with 4-velocity \(u^\alpha\) perceives space as the 3 dimensional hypersurface \[19\] orthogonal to \(u^\alpha\), and \(u^\alpha\) itself as the time axis. We can define \(u^\alpha(x^\beta)\) at each point in space-time as the direction of time, and define space as the snapshots of space at constant time. Subsequently, we can split equations into their space and time components by using the parallel and orthogonal projection operators \(U^\alpha_\beta \equiv -u^\alpha u_\beta\) and \(H_{ab} \equiv g_{ab} + u_a u_b\), respectively, with \(U^\alpha_\beta + H^a_b = \delta^a_b\).

As an example, the covariant electromagnetic Faraday tensor \(F^{ab}\) and its dual \(\tilde{F}^{ab} \equiv \frac{1}{2}\epsilon^{abcd} F_{cd}\) can be simplified by splitting it into its time and space components:

\[
F^{ab} = (U^a_\gamma U^b_\delta + H^a_\gamma H^b_\delta) F^{cd} = u^a E^b - E^a u^b + \epsilon^{abc} B_c = \epsilon^{abc} B_c,
\]

(7)

where we have defined \(B_a \equiv \frac{1}{2}\epsilon^{abc} F^{bc}\) and \(E^a \equiv F^{ab} u_b\) that reduce to the magnetic and electric field, respectively, in the rest frame of an observer. The last equality in (7) is only valid in the ideal MHD approximation \(E^a = 0\). Since \(E^a = \epsilon^{abc} u_b B_c\) or \(E = -\mathbf{v} \times \mathbf{B}\) the requirement that the comoving electric field vanishes replaces Ohm’s law. Similarly, the covariant Maxwell equations can be split into their space and time components. For a derivation we refer to \[12\].

E. Proper reference frame

In describing the interaction of a GW with a plasma, one has two choices for the reference frame. One is the transverse-traceless coordinate frame, discussed in the previous sections, which is tuned to a GW with metric:

\[
g^{ab}_{TT} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 + h_+(z, t) & h_x(z, t) & 0 \\
0 & h_x(z, t) & 1 - h_+(z, t) & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

(8)

The natural reference frame of an observer, however, is an orthonormal tetrad frame (ONF). The basis vectors that remain orthogonal in the presence of a TT plane polarized GW are:

\[
e_0 = \left(\frac{\partial}{\partial \gamma}, 0, 0, 0\right), \quad e_1 = \left(0, (1 - \frac{h_+}{2}) \frac{\partial}{\partial \sigma}, -\frac{h_x}{2} \frac{\partial}{\partial \sigma}, 0\right),
\]

\[
e_2 = \left(0, -\frac{h_x}{2} \frac{\partial}{\partial \sigma}, (1 + \frac{h_+}{2}) \frac{\partial}{\partial \sigma}, 0\right), \quad e_3 = \left(0, 0, 0, \frac{\partial}{\partial \gamma}\right),
\]

(9)

where the partial derivatives reflect the notion that in curved space-time, tangent vectors and partial derivatives are equivalent. With respect to these basis vectors the metric is \(\eta^{ab}\).

Covariant derivatives are defined as: \(\nabla_a T_{bc} = e_a T_{bc} - \Gamma^d_{ab} T_{dc} - \Gamma^d_{ac} T_{bd}\), where the connection coefficients are linear combinations of the commutation functions \(\gamma^{bc}_{ab}\) and not derivatives of the metric as in a coordinate frame \[8, 20\]:

\[
\Gamma_{abc} = \frac{1}{2} (\eta_{ad}\gamma^{d}_{cb} - \eta_{bd}\gamma^{d}_{ca} + \eta_{cd}\gamma^{d}_{ab}),
\]

(10)

and are skew in the first two indices, \(\Gamma_{(abc)} = 0\).

There seems to be some inconsistency in the literature on the explicit form of the connection coefficients. In a coordinate frame, the connection coefficients are given by the Christoffel symbols. For the metric \(\eta\) the Christoffel symbols are \(\frac{\partial}{\partial t}\) and \(\frac{\partial}{\partial \gamma}\) and have 12 non-vanishing components for each polarization. The explicit form of these components are given in \[11\] and \[7\], but it should be noted that in \[14\] one of course also has the symmetric components, and in \[10\] \(\Gamma^0_{10}, \Gamma^0_{01}\) should be \(\Gamma^1_{10}, \Gamma^1_{01}\).

In the orthonormal tetrad frame the connection coefficients only have 8 non-vanishing components for each polarization:

\[
\Gamma^{[02]}_{1} = -\Gamma^{[01]}_{1} = \frac{1}{2} \frac{\partial h_+}{\partial \gamma}, \quad \Gamma^{[32]}_{1} = -\Gamma^{[31]}_{1} = \frac{1}{2} \frac{\partial h_+}{\partial z},
\]

\[
\Gamma^{[20]}_{1} = -\Gamma^{[02]}_{1} = \frac{1}{2} \frac{\partial h_\times}{\partial \gamma}, \quad \Gamma^{[13]}_{2} = -\Gamma^{[32]}_{1} = \frac{1}{2} \frac{\partial h_\times}{\partial z}
\]

where \(\Gamma^{[01]}_{1}\) stands for \(\Gamma^{[01]}_{1}\) etc.

The Einstein field equations and the equations of geodesic deviation are derived from the Riemann curvature tensor, that in the ONF is given to first order in \(h_+, h_\times\) by:

\[
R^{a}_{bcd} = e_c \Gamma^{a}_{bd} - e_d \Gamma^{a}_{bc} + O(h^2).
\]

(11)

The Ricci tensor \(R_{ab}\) is just a contraction of (11) and to first order reduces to the same form as in the TT coordinate frame. This means that the Einstein field equations in the proper reference frame also have the same form as in \[11\] (we will use this in Sect. \[VII\] when discussing the damping of the GW).

The driving force of a GW on a test particle is also described through the Riemann tensor in the form of the equations of geodesic deviation:

\[
\frac{d^2 x^i}{d\tau^2} = -R_{0ij0} x^j = \frac{1}{2} \begin{pmatrix}
\frac{\partial}{\partial \gamma} \hat{h}_+ & \hat{h}_\times & 0 \\
\hat{h}_\times & -\hat{h}_+ & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix},
\]

(12)

where \(\hat{h}_+, \hat{h}_\times = \frac{\partial^2}{\partial \tau^2} h_+, \hat{h}_\times\). This equation will be important for our interpretation of the interaction with a magnetic field in Sect. \[VIII\].

III. GENERAL RELATIVISTIC MHD

Throughout this paper we assume that the MHD approximation is valid, viz. that the plasma is a collisionless one-fluid with negligible viscosity, resistivity and heat
flow. In contrast to some definitions of the MHD approximation, however, we do allow for relativistic velocities which means that displacement currents cannot be neglected and a generalized definition of the Alfvén velocity is needed (Sect. IV). We don’t restrict the angle between the GW propagation and the background magnetic field, but without loss of generality choose it to lie in the $x$-$z$ plane.

A. Coupling to the electromagnetic field

A gravitational wave propagating through a uniform magnetic field $B^0$ produces a Lorentz force $F_L$ given by the projection of Ampère’s law, $\nabla B^a = j^a$, perpendicular to $B^0$:

$$F_L = j^1 \times B^0 - \frac{(B^0 \cdot \nabla) B^1}{4\pi} - \frac{\nabla (B^0 \cdot B^1)}{4\pi} - \frac{|B^0|^2 \partial v^1}{4\pi} + \frac{B^0}{4\pi} \frac{\partial}{\partial t} (v^1 \cdot B^0) - \frac{j_e B^0}{4\pi}.$$  \hspace{1cm} (13)

Faraday’s law, $\nabla B^a = 0$, is given in the 3 + 1 split by:

$$\frac{\partial B^1}{\partial t} = (B^0 \cdot \nabla) v^1 - B^0 (\nabla \cdot v^1) - j_B.$$  \hspace{1cm} (14)

The projection of (14) onto $B_0$ governs the evolution of the magnetic energy density:

$$\frac{\partial}{\partial t} \frac{B^0 B^1}{4\pi} = B^0 \cdot \nabla v^1 \frac{B^0}{4\pi} - \frac{|B^0|^2}{4\pi} \nabla \cdot v^1 - j_B \frac{B^0}{4\pi}.$$  \hspace{1cm} (15)

The GW source terms in (13) are given by:

$$j_E = \frac{B^0}{2} \frac{\partial}{\partial z} \begin{pmatrix} -h_\times \\ h_+ \\ 0 \end{pmatrix}, \quad j_B = -\frac{B^0}{2} \frac{\partial}{\partial t} \begin{pmatrix} h_+ \\ h_\times \\ 0 \end{pmatrix}. $$  \hspace{1cm} (16)

B. Eq. of state & Conservation of number density

From the first law of thermodynamics, $dU = dQ - pdV$, we can find the internal energy per unit mass, $U$, as a function of the pressure, $p$, the specific volume per unit mass, $V = 1/\rho$, and the heat flow that, however, vanishes under the MHD condition: $dQ = 0$. We assume an adiabatic equation of state $p = K \rho^\gamma$, where $\gamma$ is the adiabatic index, which lies in $4/3 \leq \gamma \leq 5/3$. The total relativistic matter energy density with respect to the 4-velocity of an ideal fluid (with the velocity of light included explicitly) is given by:

$$\mu = \rho (c^2 + U) = \rho c^2 + \frac{p}{\gamma - 1}.$$  \hspace{1cm} (17)

and the relativistic enthalpy is defined by $w^0 = \mu^0 + p^0$. From these expressions we can derive the proper relativistic sound velocity by considering the change in pressure with $\mu$ at constant entropy:

$$c_s^2 = \left. \frac{\partial p}{\partial \mu} \right|_{ad} = \frac{\gamma p^0}{w^0}. $$  \hspace{1cm} (18)

The matter density $\rho$ can be solved from the covariant conservation of proper number density, $n = \rho/\sqrt{\epsilon}$, which is given by $\nabla_a (nu^a) = 0$. In the orthonormal comoving frame to first order one finds:

$$\frac{\partial}{\partial t} \rho^1 + \rho_0 \nabla \cdot v^1 = 0. $$  \hspace{1cm} (19)

C. Energy conservation

The conservation of energy and momentum follows from the divergence of the EFE (11) as:

$$\nabla_b T^{ab} = \nabla_b \cdot [(\mu + p)u^a u^b + pg^{ab}] - F^{ab} j_b = 0. $$  \hspace{1cm} (20)

Contracting (20) with $U^c_a$ leads to the energy conservation equation:

$$\frac{\partial}{\partial t} [\Gamma^2 (\mu + p) - p] + \nabla \cdot [\Gamma^2 (\mu + p) v] = 0. $$  \hspace{1cm} (21)

In the comoving frame ($\Gamma = 1$) we can eliminate $\mu$ in favor of $p$ from (17) and (19):

$$\frac{\partial}{\partial t} p^1 + \gamma p^0 \nabla \cdot v^1 = 0. $$  \hspace{1cm} (22)

Note that from hereon, (19) is redundant and can be dropped from our set of equations altogether. It can be solved separately to find $\rho^1$ from $v^1$.

In the cold non-relativistic limit when the internal energy is negligible with respect to the rest-mass energy ($p \ll \mu$), (21) reduces to (19) as in Paper I.

D. Conservation of momentum

The equation of motion or momentum conservation in a $3 + 1$ split is projected out by $H^c_a$:

$$\frac{\partial}{\partial t} \left[ \frac{\Gamma^2}{2} (\mu + p) v + \nabla \cdot \left[ \frac{\Gamma^2}{2} (\mu + p) vv + pl \right] \right] = j \times B,$$

$$(\mu^0 + p^0) \frac{\partial v^1}{\partial t} + \nabla p^1 = j^1 \times B^0. $$  \hspace{1cm} (23)

The Lorentz force couples the matter to the electromagnetic fields and the GW source terms, which is apparent from combining (13) and (23):

$$w_{tot} \frac{\partial v^1}{\partial t} = -\nabla p^1 + \frac{(B^0 \cdot \nabla) B^1}{4\pi} - \nabla \left( \frac{B^0 \cdot B^1}{4\pi} \right) + \frac{B^0}{4\pi} \frac{\partial (v^1 \cdot B^0)}{\partial t} - \frac{j_E B^0}{4\pi},$$  \hspace{1cm} (24)
where we have defined:

$$w_{\text{tot}} = u^0 + \frac{|B^0|^2}{4\pi}.$$  \hspace{1cm} (25)

This form of the equations of motion is also found by explicitly evaluating the divergence of the electromagnetic part of the stress-energy tensor, $\nabla_b T^{b\mu} = \frac{1}{4\pi} \nabla_b (F^a_{\mu} F^{bc} - \frac{1}{4} g^{ab} F^{cd} F_{cd})$, instead of using Ampère’s law. The time projection, $\nabla_b T^{b\mu}$, then results in a conservation equation for the total energy density, e.g. (15) and (22) combined.

To summarize, in this section we have obtained a closed set of partial differential equations of $z, t$ for the 16 variables $\mathbf{B}, \mathbf{E}, \mathbf{j}, \tau, \mathbf{v}, \mu, p, \rho$ that constitute the general relativistic MHD description of a GW propagating through a magnetized relativistic plasma. We will first solve these equations algebraically in Fourier and Laplace space in the next section and subsequently derive the space-time solutions in Sect. V.

IV. PLASMA WAVES

In a relativistic, magnetized plasma the proper sound velocity of a compressional wave is given in [13] and we define the Alfvén velocity of a non-compressional shear wave in the magnetic field, $u_A$, and the velocity of a mixed magneto-acoustic wave (MSW), $u_m$, by:

$$u^2_A = \frac{|B^0|^2}{4\pi w_{\text{tot}}},$$ \hspace{1cm} (26a)

$$u^2_m = \frac{\gamma p^0}{w_{\text{tot}}} + \frac{|B^0|^2}{4\pi w_{\text{tot}}}.$$ \hspace{1cm} (26b)

To construct a wave equation for the plasma perturbations we take a second time derivative of (24) and eliminate $B^1$ by (14, 15) and $J^1$ by (22). In terms of the characteristic velocities (26a, 26b) we find:

$$\left[ \frac{\partial^2}{\partial t^2} - u^2_m \nabla \nabla \right] v^1 - \left[ u_A \frac{\partial^2}{\partial t^2} - (u_A \cdot \nabla) \nabla \right] v^1 \cdot u_A = \left( u_A \cdot \nabla \right)^2 v^1 - u_A (u_A \cdot \nabla) \nabla \cdot v^1 + \text{GW terms},$$ \hspace{1cm} (27)

where the GW source terms are now given by:

$$\sqrt{w_{\text{tot}}} \left[ \nabla (j_B \cdot u_A) - \frac{\partial}{\partial t} (j_E \times u_A) - (u_A \cdot \nabla) j_B \right].$$

A. Symmetric matrix representation

We are considering GW-propagation along the $z$-axis with the wave vector $k = (0, 0, k)$ at an arbitrary angle $\theta$ to the ambient magnetic field, which is assumed to lie in the $x-z$ plane. To solve the system of differential equations algebraically, we Fourier transform with respect to time and Laplace transform the spatial part to allow for growing amplitudes as in Paper I. The wave equation (27) can be written in a symmetric matrix representation since the non-linear general relativistic MHD equations form a set of symmetric hyperbolic partial differential equations:

$$Dv^1 = J^1_{\text{GW}},$$ \hspace{1cm} (28)

where $D$ is:

$$\begin{pmatrix}
\omega^2 (1-u^2_A) - k^2 u^2_A & 0 & (\omega^2 - k^2) u_A \| u_A \| \\
0 & \omega^2 - k^2 u^2_A & 0 \\
-(\omega^2 - k^2) u_A \| u_A \| & 0 & \omega^2 (1-u^2_A) - k^2 (u_m^2 - u_A^2) \\
\end{pmatrix},$$

and, for $h_{+}, \times \propto \epsilon^{\omega(z-t)}$ as discussed in Sect. VII, the GW source terms $J^1_{\text{GW}}$ are:

$$J^1_{\text{GW}} = -i\omega u_A \| h^+ \left( \begin{array}{c} u_A \| h^+ \\ u_A \| h^x \\ -u_A \| h^+ \end{array} \right).$$ \hspace{1cm} (29)

B. Dispersion relation

A non-trivial solution for the plasma waves in (28) requires the determinant of $D$ to vanish. Solving for $k = k(\omega)$, since $\omega$ is fixed by the driving GW, we find six solutions:

$$\omega = \pm k_A u_A \cos \theta = \pm k_A u_A \|,$$ \hspace{1cm} (30a)

$$\omega = \pm \frac{k_B}{\sqrt{2}} \sqrt{(u_m^2 + c_s^2 u_A^2)} \sqrt{1 \pm \sqrt{1 - \sigma}},$$ \hspace{1cm} (30b)

where we have defined the auxiliary parameter:

$$\sigma(\theta) \equiv \frac{4c_s^2 u_A^2}{(u_m^2 + c_s^2 u_A^2)^2}.$$ \hspace{1cm} (31)

The negative sign in (30b) refers to the relativistic proper slow magnetosonic waves with phase velocity $u_s = \omega/k$, and the positive sign to the fast magnetosonic waves with $u_m = \omega/k$. Together with the Alfvén waves, we have obtained a $6 \times 6$ MHD representation that in the special-relativistic limit reduces to the equations in [13].

In a low plasma-beta ($\beta_{\text{pl}} = 4\pi p/B_0^2$), e.g. strongly magnetized, plasma where $u_A \gg c_s$, the fast mode reduces to the magneto-acoustic mode (20a) slightly altered by the presence of the gas, whereas for a high plasma-beta ($c_s \gg u_A$) it reduces to a sound wave with velocity (13). For the slow mode, the behavior is essentially the other way around.

The angular dependence is similar: for parallel propagation ($\theta = 0$), $u_t \rightarrow u_A$ and $u_n \rightarrow c_s$, whereas for perpendicular propagation ($\theta = \pi/2$) which we studied in Paper I, $u_t \rightarrow u_m$ and $u_n \rightarrow 0$.

C. Coupling

In this section we present a formal derivation of the solutions to the inhomogeneous wave equation (28) fol-
The units polarization vector for a wave mode $M$, $n_M(k)$ with $n_M \cdot n_M = 1$, can be constructed from $\lambda_{ij}$:

$$n_M(k)n_{Mj}(k) = \lambda_{ij}(\omega, k_M) \lambda_{ii}(\omega, k_M)$$ (34)

**D. Alfvén waves driven by $\times$ polarized gw**

The wave solutions can now be evaluated in Laplace space. The $\lambda_{ij}$ component couples to the $\times$-polarized source term and excites Alfvén waves, viz perturbations of the magnetic field perpendicular to the background:

$$v_y^1(k, \omega) = -i \frac{h_x \omega u_A \omega + k}{2 \omega^2 - k^2 u_A^2}$$ (35a)

$$B_y^1(k, \omega) = -v_y^1(k, \omega) \frac{B_0^1 \omega + ku_A^2}{u_A \omega + k}$$ (35b)

and similarly for $E_x^1$ and $j_x^1$. Note that because the Alfvén wave propagates obliquely with respect to the background magnetic field, the electric field is no longer divergence free and consequently the Alfvén wave is accompanied by perturbations in the charge density: $\nabla \cdot E^1 = i k E_z^1 = 4 \pi T^1$.

The polarization of the Alfvén wave components is summarized in Fig. 1.

**E. Slow and fast MSW driven by $\times$ polarized gw**

As we expect from our considerations in Paper I, a $\times$ polarized gw excites slow and fast magneto-acoustic waves in the plasma. The velocity components are:

$$v_s^1(k, \omega) = \frac{i \frac{h_s \omega^2 u_A^2}{2} \omega^2 - k^2 u_A^2} \omega + k$$ (36a)

$$v_f^1(k, \omega) = \frac{-v_s^1(k, \omega)}{\tan \theta} \left(1 - \frac{k^2 u_A^2}{\omega^2}\right)$$ (36b)

and from (22) one can easily find the pressure: $p^1(k, \omega) = \frac{\omega}{k^2} p \nu_s^1(k, \omega)$. The magnetic component can be derived from (38) as:

$$B_z^1 = \frac{v_z^1 \sin \theta - v_z^1 \cos \theta - \omega}{\omega + k} \frac{1 - u_A^2}{u_A^2} \cos \theta$$ (37)

The fact that the magnetic field perturbation is orthogonal to the direction of propagation of the gw is dictated by $\nabla \cdot B = 0$.

The polarizations in the magneto-acoustic modes are illustrated by Fig. 2.

![Figure 2: Orientation of the perturbations in the MSW modes](attachment:figure2.png)

**V. SPACE-TIME SOLUTIONS**

To find the solutions in space-time we apply the inverse Fourier and Laplace transformations to the results of the previous section. We define the phases of the wave modes as $\phi^e = \pm k_A z - \omega t$ and similarly for $\phi_s^e$, $\phi_f^e$, and $\phi_k = \omega(z - t)$.

**A. Alfvén waves**

The most straightforward are the Alfvén waves with wavenumber $k_A = \omega/u_A$:

$$\frac{B_s^1}{B_s^0} = h_s \frac{1 - u_A^2}{1 + u_A^2} e^{i \phi_A} + \frac{1 + u_A^2}{1 - u_A^2} e^{i \phi_A} - \frac{1 + u_A^2}{1 - u_A^2} 2e^{i \phi_k}$$ (38)
and similarly for $v^1_{\phi}(z,t)$, $E^1_{z,\phi}(z,t)$, $j^1_{z,\phi}(z,t)$, and $\tau^1(z,t)$ (see Appendix A). Since the Alfvén waves are non-compressional, we do not find a $p^1$ or $\mu^1$ contribution.

B. Magneto-acoustic waves

Slightly more complicated is the coupled superposition of slow and fast wave modes, polarized in the $x$-magnetic field, but on the other hand, the magnetic field Alfvén wave should be almost parallel to the background ($\propto \frac{\omega}{k}$). Note that for different relative amplitudes, summarized in Appendix 2.

The remaining magneto-acoustic wave components, $B^1_{z}(z,t)$, $E^1_{z}(z,t)$, $j^1_{z}(z,t)$, and $\mu^1(z,t)$, are equivalent superpositions of slow and fast waves but with different relative amplitudes, summarized in Appendix B.

C. Growth

In general, for arbitrary $u_A$, $c_s$, and $\theta$, we always find $u_f > u_s$. In the limit where the phase velocity of the fast mode approaches the speed of light, $u_f \uparrow 1$, coherent interaction with the GW is possible. The amplitude of the retreating fast wave ($\propto (1 + u_f)^{-1}$) is negligible with respect to the forward wave ($\propto (1 - u_f)^{-1}$). The forward wave grows linearly with distance because with $k_f = \omega/u_f = \omega + \Delta k$ to first order in $\Delta k$ we have:

$$e^{ik_{\parallel}z} - e^{i\omega z} = \frac{\omega}{u_f \Delta k} e^{i\omega z} i \Delta k z.$$  

This limiting behavior is the same for all magneto-acoustic components (see Appendix A).

For the Alfvén waves, the conditions for growing solutions are less favorable because of the angular dependence of the resonance condition. To interact coherently with the GW, the phase speed of the Alfvén wave ($v_{\phi} = u_A \cos \theta$) has to approach the speed of light. On the one hand this means that the wave vector of the Alfvén wave should be almost parallel to the background magnetic field, but on the other hand, the magnetic field should also have a transverse component, since the amplitudes are proportional to $B^0 \propto \theta B^0 \phi$ for small angles. Explicitly:

$$B^1_{\phi}(z,t) \sim -v^1_{\phi}(z,t) \sim \frac{\theta h_z \omega z \Im [e^{i\phi_k}] + O(\theta^2]}{2}.$$  

The excited slow magnetosonic wave is a purely oscillatory wave propagating both in the forward and the backward directions.

VI. A WARM RELATIVISTIC PLASMA WIND

We now want to consider a warm relativistic plasma wind flowing out in the $z$ direction with constant (background) velocity $\beta$ and corresponding Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$. We are allowed to use simple Lorentz transformations in this general relativistic treatment, because the whole concept of gravitational waves as small perturbations of the background space-time (linearized theory), relies on the fact that we can treat the GWs as a field living in flat space-time as long as the scale of curvature is much larger than the wavelength of the GWs: $R \gg \lambda_{GW}$, which was verified in Sect. II B.

From now on, all quantities derived in the previous sections for the comoving frame will be denoted by primes. Since the phase of plane waves, $\phi = k_o x^\parallel \approx (1 + \beta)$, is invariant under Lorentz transformations, a boost mostly implies addition of velocities. We then have $u_f^\prime = \frac{u_f^\prime - 1}{1 - \beta u_f}$, $u_s^\prime = \frac{u_s^\prime - 1}{1 - \beta u_s}$, and $u_{A^\parallel}^\prime = \frac{u_{A^\parallel}^\prime - 1}{1 - \beta u_{A^\parallel}}$ for the phase velocities and:

$$v^1_{z} \approx \frac{(v^1_{z})^\prime}{\gamma^2}, \quad v^1_{x,y} \approx \frac{(v^1_{x,y})^\prime}{\gamma}.$$  

The boosted background magnetic field is $B^0 = (\gamma B^0_{z0})$, and in the laboratory frame one finds a zeroth order electric field $E^0 = -\beta \times B^0 = (0, -\beta \gamma B^0_{z0}, 0)$.

A. Relativistic Alfvén waves

We define $\gamma_{A^\parallel}^2 = 1/(1 - u_{A^\parallel}^2) + O(\theta^2)$ as the Lorentz factor associated with the Alfvén speed for small $\theta$ and boost (A.1) to the laboratory frame as an example:

$$v^1_{y} = \frac{h_x u_{A^\parallel} \gamma_{A^\parallel}^2}{4} \left[ \left( \frac{4}{1 - u_{A^\parallel}^2} - (1 + u_{A^\parallel}^2) \frac{e^{i\phi_k}}{\gamma^2} \right) \right].$$  

Or in the ultra-relativistic limit $u_A^\parallel \uparrow 1$ (and $\beta \simeq 1$):

$$v^1_{y}(z,t) \approx -\frac{h_x u_{A^\parallel}}{4 \gamma^2} \omega z \Im [e^{i\phi_k}].$$
B. Relativistic MSW

In a Poynting flux dominated force-free plasma wind, where the magnetic energy density strongly dominates the matter density, the plasma flows out at ultrarelativistic velocities. In this regime the phase velocity of the fast mode approaches the Alfvén velocity, \( u_t \simeq u_A \uparrow 1 \) and the phase velocity of the slow mode becomes negligible, \( u_s \downarrow 0 \). The solutions are therefore quite similar to those found in Paper I, the main difference being the angular dependence for \( H \)

\[
v_x = v'_x \cos \theta \simeq \frac{\sin 2 \theta}{4 \gamma^3} \, \omega \, \omega_z \, \omega_x \, e^{i \phi_x} \tag{46a}
\]

\[
v_z = v'_z \cos \theta \simeq \frac{\sin 2 \theta}{4 \gamma^3} \, \omega \, \omega_z \, \omega_x \, e^{i \phi_x} \tag{46b}
\]

Appendix [H] lists the limiting behavior of all the remaining MSW components in a relativistic wind.

VII. DAMPING OF THE GW

To find an evolution equation for the gravitational waves, we project the transverse traceless part of the stress-energy perturbation as in [2] (as we discussed at the end of Sect. II E and [2] also hold in the of).

The perturbation of the matter part only has \( \delta T_{TT}^{TT} = \delta T_{YY}^{YY} = P_1(z, t) \) which is purely gauge (the trace can be removed by a gauge transformation). Only the magnetic field interactions directly with the gravitational waves. All the other perturbations are excited through the MHD processes in the plasma. In the general case where oblique \( h^+ \) GW excite slow and fast magneto-acoustic waves and \( h^- \) GW excite Alfvén waves, [8] is replaced with:

\[
\square h^+ = 4B_y^0 B_x^1, \quad \square h^- = 4B_x^0 B_y^1. \tag{47}
\]

As we discussed in Sect. II B we have assumed that the GW have a slowly varying amplitude \( H(z) \). We have neglected this variation in studying the interaction with the plasma since the short-wavelength approximation all the derivatives have \( \partial^2 \omega H(z) \ll \omega H(z) \). It is however this slowly varying amplitude that describes the damping of the GW. We could find an order of magnitude expression for \( H(z) \) by integrating [17] using [A2] and [S8], but this wouldn’t be entirely self-consistent since in deriving [A2] and [S8] we already assumed that \( \omega = k \) for the GW.

It is, however, possible to derive a dispersion relation for the damped GW and the excited MHD waves simultaneously in a self-consistent way. If we assume that the GW oscillates at a fixed frequency but leaves the spatial dependence unspecified (e.g. a boundary value problem with \( h(z, t) \propto h(z) e^{-i \omega t} \) we can still solve the full set of MHD equations in Laplace space. For the magnetic field we find:

\[
B_y^1(k, \omega) = \frac{B_y^0 h_+(k, \omega) \omega_2 + k^2 u_A^2}{2 \omega_2 - k^2 u_A^2}, \tag{48a}
\]

\[
B_x^1(k, \omega) = \frac{B_x^0 h_+(k, \omega) \omega_2 + k^2 u_A^2}{2 \omega_2 - k^2 u_A^2} \tag{48b}
\]

where the second expression is in the limit of a Poynting flux dominated wind with \( u_A > c_0 \downarrow 0 \) and \( h_{+, \times}(k, \omega) \) are the Laplace transforms of \( h_{+, \times}(z, t) \) (the reason that [18] look different from [38b] and [37] is that derivatives of \( h_{+, \times} \) depend on both \( \omega \) and \( k \) in this more general case). If we insert [18] in the Laplace transform of [17], \( h_{+, \times}(k, \omega) \) drop out and we find the self-consistent dispersion relation for the coupled fast magnetosonic - gravitational (\( h_+ \)) mode:

\[
\omega^2 - k^2 = 2(B_0^0)^2 \omega_2 + k^2 u_A^2 \tag{49}
\]

and the coupled Alfvén - gravitational (\( h_- \)) mode which looks the same but with \( u_A \) replaced with \( u_A \).

The two modes allowed by [40] are given by:

\[
k_{1,2}^2 u_A^2 = \frac{\omega_2}{2} \left[ \frac{1 + u_A^2}{1 - u_A^2} \pm \sqrt{\frac{1 + u_A^2}{1 - u_A^2} + \epsilon^2} \right] \tag{50}
\]

in terms of \( \epsilon = \left( \frac{B_0^0}{\omega} \right)^2 \frac{2}{1 - u_A^2} - \frac{B_0^0}{\omega} \Delta k \). For small \( \epsilon \), [50] reduces to:

\[
k_1^2 = \omega^2 \left( 1 - \frac{1 + u_A^2}{u_A^2} \right) + \mathcal{O}[\epsilon^2], \tag{51a}
\]

\[
k_2^2 u_A^2 = \omega^2 \left( 1 + 2 \epsilon \right) + \mathcal{O}[\epsilon^2]. \tag{51b}
\]

Mode 1 corresponds to a superluminal mode with phase velocity \( \omega/k_1 > 1 \) and mode 2 is subluminal:

\[
v_{ph,1} = 1 + 2(\frac{B_0^0}{\omega})^2 \frac{1 + u_A^2}{1 - u_A^2}, \quad v_{ph,2} = \frac{1 - 2(\frac{B_0^0}{\omega})^2 \frac{1 + u_A^2}{1 - u_A^2}}{\omega^2} \tag{52a}
\]

\[
u_{ph,1}^2 = 1 + \frac{2(\frac{B_0^0}{\omega})^2 \frac{1 + u_A^2}{1 - u_A^2}}{\omega^2}, \quad v_{ph,2}^2 = \frac{1 - 2(\frac{B_0^0}{\omega})^2 \frac{1 + u_A^2}{1 - u_A^2}}{\omega^2} \tag{52b}
\]

with \( v_{ph,1} v_{ph,2} = 1 \) and \( v_{ph,2} v_{ph,2} = u_A^2 \). Both modes have group velocity \( \partial \omega / \partial k_{1,2} < 1 \) as long as (consistent with [17]):

\[
\frac{8 \pi G (\frac{B_0^0}{\gamma})^2}{\mu_0} \ll \omega (\Delta k) c. \tag{52}
\]

In the limit \( \epsilon \ll 1 \), [51b] reduces to a fast magnetosonic wave in flat space-time (or, equivalently, to the Alfvén mode for \( h_+ \)). Mode [51a] reduces to a vacuum GW, justifying our approximation \( h_{+, \times} \propto e^{i \omega(z-t)} \) in deriving the plasma perturbations in Sect. IV.

VIII. INTERPRETATION

In this section we present an intuitive interpretation of our main results: that a + polarized GW excites magneto-acoustic waves and a × polarized GW excites Alfvén waves in a uniform magnetic field.
The driving force exerted by a GW on test particles is described by (12). These equations can be illustrated by force lines in the plane orthogonal to the propagation of the GW. Integrating (12) twice with respect to time results in the well known equations of spatial deviations of test masses in an interferometer detector such as LIGO:

\[ \delta x = \frac{1}{2}(h_+ x_0 + h \times y_0), \quad \delta y = \frac{1}{2}(h \times x_0 - h_+ y_0). \] (53)

In the MHD limit these test particles are ‘glued’ to the magnetic field lines, so the magnetic field will exhibit the same behavior (in fact the presence of the plasma is not required, the magnetic field lines can just be viewed as parametrized by (53)). Since the action of the GW is only in the \(x-y\) plane, and the magnetic field lies in the \(x-z\) plane we expect that a \(+\) polarized GW results in:

\[ \delta B_x \propto \frac{1}{2} h_+ B^0_x, \quad \delta B_y \propto \frac{1}{2} h_+ B^0_y = 0, \] (54)

and a \(\times\) polarized GW excites:

\[ \delta B_x \propto \frac{1}{2} h \times B^0_y = 0, \quad \delta B_y \propto \frac{1}{2} h \times B^0_x. \] (55)

This is exactly what we found in the mathematical treatment of the previous section. Perturbations in the other directions are higher order effects.

![Figure 3: The MSW (left) and Alfvén mode (right) illustrated as an oscillating vector field. The \(x-y\) axes in the right figure are rotated by \(\pi/2\) with respect to those in the left figure.](image)

Fig. 3 gives a schematic illustration of the perturbed magnetic field in the two wave modes. The left figure shows the vector field (in arbitrary units) for the oblique magneto-acoustic wave propagating in the \(z\)-direction. The perturbations are highly exaggerated to emphasize the effect. Amplification of the magnetic field occurs when \(B^0_y\) is amplified and dilution when \(B^0_x\) is suppressed. Since \(B^0_z\) is constant the total magnetic field has an overall wavy pattern.

Alfvén waves are non-compressional and only set up a vibration in the field lines perpendicular to the background field \((B^0_y)\). This is illustrated in the right figure, where the axes are rotated to emphasize the \(y-z\) plane.

![Figure 4: Merging neutron star binary](image)

In a pulsar environment the plasma initially flows out \(along\) the open field lines but develops into a force-free wind outside the light-cylinder in which the toroidal component of the magnetic field dominates the poloidal one. Here the magnetic field is predominantly perpendicular to the radial propagation of the wind as illustrated by Fig. 4. Gravitational waves would mainly excite Alfvén waves in the former region, whereas in the latter case the magneto-acoustic waves are favored. In the relativistic wind the GW frequency is red-shifted and the interaction is suppressed by \(\gamma^{-2}\), but the interaction length scale becomes very large.

**IX. CONCLUSIONS**

We have studied the propagation properties of a plane polarized gravitational wave in a magnetized astrophysical plasma in the most general case. Both polarizations of the GW have been taken into account. Oblique propagation with respect to the background magnetic field was studied including pressure terms, and relativistic velocities. The only approximations in this treatment are the MHD conditions for the plasma, the linear perturbative approach, the geometric optics or small wavelength limit for the gravitational waves, and our assumed geometry of the interaction region.

The result is a very rich astrophysical problem, where all three fundamental plasma wave modes are excited, two of which can interact coherently with the driving gravitational waves, and as a result grow linearly with distance, dissipating GW energy into the plasma.

Alfvén waves are excited already in linearized theory by \(\times\) polarized gravitation waves propagating at an angle with respect to an ambient magnetic field, a result that as far as we know has not been found before (note that both conditions, oblique GW propagation and \(\times\) polarization,
are required). The Alfvén waves are non-compressional shear waves that have orthogonal electromagnetic components with a corresponding drift velocity in the plasma, a current flowing along the electric field and a deviation from charge neutrality caused by the divergence of the electric field, but no pressure or density components. To interact coherently with a GW one has to find the optimum of longest interaction length scale and significant amplitude as a function of the angle between the Alfvén wave vector and background magnetic field as was discussed in Sect. V C.

As we already derived in a Paper I, + polarized gravitational waves excite magneto-acoustic waves propagating parallel to the gravitational waves. In this paper we have generalized our previous treatment to include the oblique magnetic field and a relativistic equation of state with non-vanishing pressure. As a result we find both the slow and the fast magneto-acoustic waves with phase velocities that depend on both the electromagnetic and the matter properties of the plasma.

The magneto-acoustic wave modes are compressional waves, that excite pressure, density, and magnetic field gradients along their wave vector direction, but no perturbation of charge neutrality. The electric and the magnetic field perturbations and the wave vector are mutually orthogonal \( (E^1, B^1, k_{s,t}) \), but the drift velocity is no longer exactly perpendicular to the magnetic field due to the pressure. Of course, the plasma motion in a GW is non-compressional but it generates magnetic field compression if it propagates across a magnetic field either in a vacuum or in an ideal frozen-in plasma, and hence couples to the magneto-acoustic wave.

As to the slow MSW, no coherent interaction can occur with the GW since the phase velocity of the slow mode is always much smaller than that of the gravitational waves. Therefore the amplitude does not grow in time or with distance and cannot become significant.

The most effective interaction occurs between the GW and the fast magneto-acoustic wave. The phase velocity of those waves can easily approach the speed of light in a strongly magnetized plasma (in the limit \( u_A \gg c_s \)) as we derived in Paper I and therefore the waves will grow linearly with distance. As the plasma waves grow, the amplitude of the gravitational waves decays correspondingly but as long as the total length of the interaction region is smaller than the background curvature produced by the plasma itself, this will only be a small fraction of the total gravitational wave energy.

Astrophysical applications of the above interactions lie in sources of strong gravitational waves embedded in strongly magnetized plasmas. Examples are non-spherical rotating neutron stars, fast rotating neutron stars that are unstable to torsional oscillations, non-spherical supernovae collapses, magnetars, and the progenitors of gamma-ray bursts (non-spherical collapse of a massive star or a merging neutron star binary). Most of these sources are probably accompanied by an extended strongly magnetized and force-free plasma wind, flowing out at ultra-relativistic velocities. Since the coupling constant of gravitational waves is exceedingly small, only in the most extreme sources will the interaction with the plasma be significant.

Fig. 4 shows as an example a binary neutron star as a gamma-ray burst progenitor. For such a merging binary with a magnetar class surface magnetic field of \( 10^{12} \text{T} \), an angular frequency at the end of the spiral-in phase of the order of \( 1 \text{kHz} \) and a force-free wind flowing out with a Lorentz factor of \( \Gamma \sim 100 \) up to a fraction of a light year, a total energy of \( 10^{43} \text{J} \) can be transferred from the gravitational waves to the wind.

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Appendix: ALL SPACE-TIME SOLUTIONS

1. Comoving Alfvén wave components

The components of the Alfvén wave can be derived most easily by starting with it’s velocity component:

\[
\frac{v^1_y}{u_{A\perp}} = \frac{h}{4} \left[ \frac{1 + u_{A\parallel}}{1 - u_{A\parallel}} e^{i\phi_A} - \frac{1 - u_{A\parallel}}{1 + u_{A\parallel}} e^{-i\phi_A} - \frac{4u_{A\parallel}}{1 - u_{A\parallel}} \right].
\]

(A.1)

From \( E^1 = -v \times B^0 \) we find the electric field:

\[
\frac{E^1_z(z,t)}{B^0_z} = -\frac{E^1_z(z,t)}{B^0_z} = -u^1_y(z,t),
\]

whereas the current density follows from Ampère’s law:

\[
j^1_x(z,t) = -\frac{i\omega}{4\pi} \frac{1 - u^2_{A\parallel}}{u_{A\parallel}^2} E^1_z(z,t), \quad j^1_z(z,t) = \frac{i\omega}{4\pi} E^1_z(z,t).
\]

The charge density is given by \( \nabla \cdot E^1 = 4\pi\tau^1 \):

\[
\frac{\tau^1(z,t)}{B^0_y \tan \theta} = \frac{i\omega h_x}{4} \left[ \frac{1 + u_{A\parallel}}{1 - u_{A\parallel}} e^{i\phi_A} + \frac{1 - u_{A\parallel}}{1 + u_{A\parallel}} e^{-i\phi_A} - \frac{4u_{A\parallel}^2}{1 - u_{A\parallel}} \right].
\]

2. Comoving MSW components

The most complicated MSW component is the magnetic field that clearly betrays the mixed nature (gas and electromagnetic fields) of this mode.

\[
B^1_z(z,t) = \frac{B^0_y h_x}{4(u_{z\perp}^2 - u_{z\parallel}^2)} [C(z,t) + D(z,t)] \quad (A.2)
\]
with:

\[ C(z, t) = - \frac{2(u_0^2 - u_z^2)(1 + u_A^2)e^{i\phi_z}}{(1 - u_A^2)} - \frac{u_t(u_t + u_z^2)e^{i\phi_z}}{1 - u_t} \]

\[ - \left( u_t - \frac{u_t^2}{u_A^2} \right) e^{i\phi_r} + \frac{u_t(u_t - u_z^2)e^{i\phi_r}}{1 - u_t} + \frac{u_x(u_x - u_z^2)e^{i\phi_r}}{1 + u_x} \right) \]

and:

\[ D(z, t) \equiv -u_t u_x \left\{ \frac{2(u_0^2 - u_z^2)\gamma_{\text{A}}}{(1 - u_A^2)} \right\} \]

\[ - \left[ \frac{1 - u_A^2}{u_A^2} + \frac{1 - u_t}{u_t} \right] e^{i\phi_r} - \left[ \frac{1 - u_A^2}{u_A^2} - \frac{1 - u_t}{u_t} \right] e^{i\phi_r} \]

\[ + \left[ \frac{1 - u_A^2}{u_A^2} + \frac{1 - u_t}{u_t} \right] e^{i\phi_r} \]

Note that in the relativistic limit \( (u_x \downarrow 0) \) \( D \) vanishes. From \( E = -v^1 \times B^0 \) we have:

\[ E^1_y(z, t) = B_x^0 y^1(z, t) - B_z^0 x^1(z, t). \]

The current density is most easily found as:

\[ j^1_y(z, t) = -\frac{i\omega - \gamma_p^0}{B_0 \cos \theta} e^{i\phi_z} \]

and the pressure is derived from \[ \mu^1(z, t) = \frac{p^1(z, t)}{c_s^2}. \]

which readily leads to the energy density:

\[ \mu^1(z, t) = \frac{p^1(z, t)}{c_s^2}. \]

In the limit of a Poynting flux dominated wind where \( u_t \approx u_A \uparrow 1 \) and \( p_0^0 \downarrow 0 \) so \( u_x \approx c_s \downarrow 0 \) the fast mode can interact coherently with the GW and we find growing amplitudes for all components:

\[ \frac{B^1_x(z, t)}{B_0} = \frac{v^1_x(z, t)}{\sin \theta} = -\frac{v^1_x(z, t)}{\cos \theta} = \frac{\mu^1(z, t)}{\mu^0 \sin \theta} = \frac{E^1_y(z, t)}{B_0} \approx \frac{h_1}{2} \sin \theta \omega \Im\{e^{i\phi_x}\}, \]

\[ \frac{B^0_0 j_y(z, t)}{\mu^0 \omega} \approx \frac{h_1}{2} \sin^2 \theta \omega \Re\{e^{i\phi_x}\}. \]

3. All ultra-relativistic Alfvén wave components

First we evaluate \( B'_y \) in terms of laboratory quantities (boosted phase velocities etc. [21]):

\[ B'_y = \frac{h_1 \gamma_{\text{A}}^2 B_0^0}{4\gamma} \left\{ 4\left[(1 + \beta^2)(1 + u_{A}^2) - 4\beta u_{A}\right] e^{i\phi_x} \right\} \]

Now we can express all other components of the Alfvén wave in terms of \[ \left[ A, 3 \right] \text{ and } \left[ A, 1 \right]. \]

\[ B_y = \gamma(B'_y + \beta E'_y) = \gamma B'_y - \beta \gamma B_0 0 \gamma v_y, \]

\[ E_x = \gamma(E'_x + \beta B'_y) = \gamma B'_y - \gamma B_0 v_y, \]

\[ E_z = E'_z = -B_x v_y = -B_x v_y, \]

\[ j_x = j'_x = \frac{i\omega}{8\pi \gamma_{\text{A}}^2 (u_A - \beta)^2}, \]

\[ j_z = \gamma(j'_z + \beta \tau) = \frac{B_0^0}{8\pi} \left[ i\omega v_y - 2\beta v_y \frac{\partial v_y}{\partial z} \right], \]

\[ \tau = \gamma(\tau' + \beta \tau') = \frac{B_0^0}{8\pi} \left[ i\omega v_y - 2\gamma v_y \frac{\partial v_y}{\partial z} \right]. \]

4. All ultra-relativistic MSW components

We will only consider the limit of a Poynting flux dominated ultra-relativistic wind and Lorentz transform \[ \left[ A, 3 \right] \] (with \( \omega = \gamma(\omega' + \beta k') \approx 2\gamma \omega' \)):

\[ B_x = \gamma(B'_x - \beta E'_y) \approx 2\gamma B'_x = \frac{h_1 \gamma_{\text{A}}^2 B_0^0 \omega z}{2\gamma^2} \Im\{e^{i\phi_x}\}, \]

\[ E_y = \gamma(1 + \beta^2)(1 + u_{A}^2) - 4\beta u_{A} \}

\[ \Re\{e^{i\phi_x}\}, \]

\[ \mu = \gamma \mu' = \frac{h_1}{4\gamma^2} \frac{1}{1 - \beta}, \]

\[ \omega z \Im\{e^{i\phi_x}\}. \]

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[19] The rest-space volume element is related to the 4D element \( \epsilon^{abcd} = \epsilon^{[abcd]} \), \( \epsilon^{0123} = \sqrt{|\det g|} \) by \( \epsilon^{abc} \equiv \epsilon^{abkd} \).
[20] However, the commutation functions are defined in such a way as to give the same permutation of indices in the connection coefficients.
[21] We will omit the superscript on first order quantities to avoid confusion with the primes.