Contests with a Non-Convex Strategy Space

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Abstract

We characterize the Nash equilibria of a class of two-player contests with a non-convex strategy space under the usual concavity assumptions. The analysis sheds light on behavior in international conflicts. For instance, it may explain why some attempts to resolve international conflicts have been successful while others have not.

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1. Introduction

The standard assumption in contest theory is that players have a convex choice set of effort. This may be a reasonable assumption when units of effort are sufficiently small, but there are important contexts, such as international conflicts, in which this assumption does not hold. For example, though it may be possible to contain an overseas crisis with a small number of aircraft, transporting and maintaining them may require the use of an aircraft carrier. Thus, the actual choice is between a large operation and no operation at all. Likewise, there may be a large fixed cost in supplementing air power with “boots on the ground”. A possible example is the US war in Afghanistan, in which at one point around 180k soldiers were fighting the Taliban. In 2014, most of this force was withdrawn almost all at once. It is quite possible that a more gradual withdrawal would have been more effective, but that course of action was technically and logistically unfeasible. Furthermore, the choice set may be further constrained by international law and agreements, such as the Geneva Convention which forbids the use of certain weapons. We show that such constraints may substantially change the outcome of a conflict.

We consider two types of simultaneous two-player contests with non-increasing return on effort and a prize with a common value. The first is a symmetric strategy contest in which both players have the same non-convex choice set of effort (as in perhaps the case of the Cold War). The structure of the choice set that we consider is quite general. For instance, it can contain two separate convex subsets or it may be

\[1\] We also discuss the case in which players have a different prize valuation.
entirely discrete. The second is an asymmetric strategy contest in which one player has a convex choice set of effort while the other's is non-convex, such as in the case of the Taliban forces which had more flexible options than the Western coalition.

We present a complete characterization of the Nash equilibria of these contests. Usually there exists a unique Nash equilibrium in pure strategies. However, in the asymmetric contest, multiple equilibria are more common. In equilibrium, a player with a non-convex choice set chooses one of the two efforts that is nearest—either from below or from above—to the equilibrium effort of the corresponding contest, in which both players have convex choice sets.

The main result concerns the relationship between the strategy space and the intensity of effort (or intensity of conflict when the model is applied in the context of international conflicts). Specifically, we show that small changes in the strategy space can produce large changes in effort. On the other hand, we also identify conditions under which large changes in strategy produce no change whatsoever in the intensity of conflict for either side. According to the above description of the player’s behavior, it is straightforward to show that changes within the boundaries of the choice set are irrelevant to effort since in any case such choices are too extreme, that is, they are either too moderate or too costly. On the other hand, changes in the inner choice set are more likely to affect the intensity of a conflict. In particular, when the player is approximately indifferent between keeping the conflict balanced and deviating to some other available but completely different strategy, small changes can have a dramatic effect on the intensity of a conflict. Under these
circumstances, attempts to reduce conflict intensity should be carefully considered. For example, ruling out the use of a moderate weapon may push the player “into a corner” by leaving her no better option than to land a costly but powerful blow. These possible equilibria may explain why some attempts to reduce international conflicts have been successful while others have not.

Consider the Hague Conventions as another example. At the First Hague Convention in 1899, it was agreed to forbid the use of various destructive weapons, such as various kinds of explosive bullets (e.g., Dum-Dum bullets), and the use of balloons to drop explosives. Given the technology available at that time, these constraints were quite effective and apparently helped to maintain peace for a while. Nevertheless, the set of restrictions was incomplete, especially as a result of subsequent technological improvement in the design of weapons. Therefore, the Second Hague Convention was held in 1907 but is viewed by many historians as a failure. At this convention, Britain tried to restrict the use of various types of weapons that were not dealt with in the first convention, but due to Germany’s resistance, only relatively esoteric weapons, such as automatic naval mines, were banned. Given that these restrictions were only marginally effective, other weapons not dealt with at the convention were used a few years later in WWI.

A further example is the Congress of Vienna held after the Napoleonic wars in 1814-1815 which was an attempt to maintain peace and to achieve balance between the great powers in Europe. The congress was criticized as being too conservative for trying to reestablish the status quo that prevailed in Europe before
the French Revolution and thus suppressing a new and more liberal spirit and set of ideas. Nevertheless, for almost forty years, it managed to maintain a peaceful status quo until the Crimean War in 1853-1856 and the Franco-Prussian War in 1870-1871. In both of these conflicts, the winners: (1) had superior technologies and (2) used modern weapons more effectively on the battlefield. In the Crimean War, Britain and France had the telegraph, while in the Franco-Prussian War, the Prussian army effectively exploited the railroads. Indeed, the Crimean War is considered to be the first in which modern technologies were used. According to the model presented here, the Congress of Vienna was sufficient to maintain peace, given the level of technology available at the time; however, given the subsequent technological advances, the agreements reached at the Congress of Vienna could no longer ensure a stable situation in Europe.

Returning to the analysis of the model, the strategic constraints in the asymmetric strategy contest allow the constrained player to make some kind of commitment. However, given that in the symmetric convex contest commitment usually has no value,\(^2\) she does not benefit from such constraints. Nevertheless, we demonstrate that under some circumstances, it is preferable to be the constrained player.

The rest of the paper is structured as follows: In the remainder of this section, we explain our contribution to the literature. In section 2, we describe the general model. Section 3 characterizes equilibria in the symmetric strategy contest, and section 4 analyzes the asymmetric strategy contest. Section 5 concludes.

\(^2\) See Dixit (1987), Baik and Shogren (1992), Baye and Shin (1999) and Possajennikov (2009).
Review of the literature

The extensive and still growing literature on contests has been surveyed by Konrad (2009) and more recently by Corchón and Serena (2018). The assumptions commonly made in the literature are that players have a convex strategy space and that effort exhibits non-increasing returns to scale. Usually, these assumptions are made for the sake of tractability and analytic convenience.

In settings where effort exhibits increasing returns, standard solution methods fail. In this case, researchers approximate the strategy space by considering the limit of a sequence of discretized versions of the game. Baye et al. (1994) were the first to apply this method to characterizing the mixed strategy equilibria in the Tullock contest (Tullock, 1980), in which return on effort is greater than two. Later studies that applied the same method include Amaldoss and Jain (2002) and Dechenaux et al. (2006). In their models, the non-convexity of strategies is merely a tool used to examine the equilibrium rather than a feature of interest on its own. In contrast, non-convex strategy spaces are the present model’s main focus, rather than a convenience. To be precise, we focus on contests with non-increasing return on effort and a linear cost function for effort, which implies that the effective cost of effort is increasing and convex, as in the canonical models presented by Dixit (1987), Tullock

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3 Technically, the equilibrium characterization offered in these works represents the limit of a discrete strategy space rather than a convex one. It is an open question as to whether the limit equilibrium coincides with a real equilibrium. See Dasgupta and Maskin (1986) for a careful analysis of the issue.

4 See Alcade and Dahm (2010) and Ewerhart (2015) for the complete characterization of these equilibria.

5 Without loss of generality, we also assume that marginal cost equals 1.
(1980) and Skaperdas (1996). To the best of our knowledge, the convexity assumption regarding the strategy space has not been relaxed previously.

A close predecessor of our model is Dubey (2013). He analyzed an all-pay auction with asymmetric players and incomplete information, in which players have a binary choice set. The restriction on the strategy space is intended mainly to permit equilibrium characterization with asymmetries and private information rather than being of interest in itself.

We examine more general types of non-convexities and contest success functions. The tradeoff for adopting greater generality is that we must exclude private information as a possibility. Thus, one can view Dubey (2013) as the private information analog to our model.

2. The model

While much of the motivation for the model is based on the effects of various restrictions on the nature of conflicts, the model is nonetheless presented in a broader context. Two risk-neutral players compete for a prize (the objective of success in the context of a conflict) with a common value $v$. Each player $i \in \{1,2\}$ chooses an irreversible non-negative effort $e_i$ from the choice set $S_i$. There are two types of choice sets: 1) a convex interval $S_c = [0,\infty)$ and 2) a non-convex subset of $S_c$, $S_{nc}(\subset S_c)$. Other than non-convexity, $S_{nc}$ is completely arbitrary with no additional restrictions. For example, it may contain two separate convex subsets or, alternatively, may be completely discrete. The usual form of analysis assumes that
both players have the same convex choice set. We instead consider two alternatives:
1) Both players have the same non-convex choice set; and 2) One player’s choice set
is convex, while the other’s is non-convex. We shall refer to the first situation as a
*symmetric strategy contest* and the second as an *asymmetric strategy contest*.

Let $p(e_i, e_j)$ be the probability that player $i \in \{1,2\}$ will win. We shall make
use of the term contest success function (or CSF) to describe $p$. It is well known that
some restrictions are needed to ensure that there is a unique equilibrium when
identical competitors with convex strategy spaces compete.\(^6\) We retain these
restrictions on the CSF throughout the analysis. This is done for both analytic
tractability and, more importantly, to allow for a ready comparison of our results to
those in the literature, thus ensuring that any difference depends purely on
dropping the convexity assumption. To complete the model, we assume that $p$ is
twice continuously differentiable when $e_i + e_j > 0$, and that it also satisfies the usual
properties:

\begin{align*}
(1) \quad & p(e_i, e_j) = 1 - p(e_j, e_i) \quad \text{and} \quad p_i \equiv \frac{\partial p}{\partial e_i} > 0 \quad \text{and} \quad p_{ii} \equiv \frac{\partial^2 p}{\partial e_i^2} < 0 \quad \text{in the interior of} \\
& \text{its domain.}
\end{align*}

To summarize, player $i$’s problem is therefore:

\begin{align*}
(2) \quad & \max_{e_i} E\pi(e_i; e_j) \text{ such that } e_i \in S_i, \text{where } E\pi(e_i, e_j) = p(e_i, e_j)v - e_i \quad \forall i \in \{1,2\}.
\end{align*}

\(^6\) See Dixit (1987) and Baye and Shin (1999).

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The remainder of the paper characterizes the equilibria when we relax the assumption of continuous available strategies. In the next section, we compute the analog of the main results in contest theory, in a setting where the two competitors are identical.

3. Symmetric strategy contests

As a benchmark, Figure 1 illustrates the case in which strategy spaces are convex. It is well-known that, in this case, there is a unique symmetric interior solution which we denote by $e^*_c$.

![Best response curves in the symmetric convex contest](image)

**Figure 1:** Best response curves in the symmetric convex contest

We now compare this to a situation in which the set of available strategies is identical, but non-convex. We reproduce the standard equilibrium analysis with only this modification.
Suppose that the players' strategy spaces are $S_1 = S_2 = S_{nc}$. When $e_c^*$ is contained in $S_{nc}$, it is trivial to show that both players choosing $e_c^*$ remains an equilibrium. But it is unclear whether the deletion of strategies might also produce new equilibria. In Proposition 0, we claim that, regardless of the structure of $S_{nc}$, no new equilibria appear, but to formally demonstrate this will require Lemma 1, which applies regardless of whether or not $e_c^*$ is in $S_{nc}$. For now, we postpone the discussion and merely note the following:

**Proposition 0** *In the symmetric strategy contest, if $e_c^*$ is contained in the strategy space, then it is the unique equilibrium.*

In a sense, Proposition 0 is anticlimactic. It says that the decision to model the strategy space as convex is of no consequence since deleting equilibrium strategies has no effect on the set of equilibria. The rest of the analysis, however, shows that this is the exception rather than the rule when considering non-convexities in players' strategic options. The main result, which we illustrate in several ways, shows that when $e_c^*$ is not in $S_{nc}$, the resulting equilibrium effort does indeed change, sometimes dramatically. Moreover, the equilibrium need no longer be unique.

Herein we assume that $e_c^*$ is not contained in $S_{nc}$ and do not impose any other restrictions on the structure of $S_{nc}$. One might have assumed that when $e_c^*$ is not available, a player will choose some nearby strategy and thus non-convexity merely shifts effort to a pair of nearby strategies. Note, however, that “nearby” strategies need not be especially close to the original equilibrium pair. For instance, suppose that the strategy space consists of the unit interval and the equilibrium point occurs
at \( e^* = \frac{1}{2} \). If the player’s strategy consists instead of \([0,1/3] \cup [2/3,1]\), then the nearest available strategies are not especially close to \( e^* \). Nonetheless, our first result identifies conditions under which the size of the “gap” surrounding \( e^* \) is of no consequence.

To this end, define \( \underline{e} \) and \( \overline{e} \) to be the efforts that bracket the equilibrium point under convex effort. That is, \( \forall e \in S_{nc} | e \in (\underline{e}, \overline{e}) \), where \( 0 \leq \underline{e} \leq e^* \leq \overline{e} \). In the symmetric case, i.e. in which both players have the same set of strategies, all resulting equilibria are “nearby,” as shown by Lemma 1:

**Lemma 1** In a symmetric strategy contest with non-convex effort, each player’s equilibrium effort is an element from the set \( \{\underline{e}, \overline{e}\} \).

Lemma 1 implies that, regardless of the full set of elements of \( S_{nc} \), it suffices to consider only the symmetric 2 \( \times \) 2 contest where choice is limited to \( \{\underline{e}, \overline{e}\} \). This greatly simplifies the analysis. Finally, notice that Proposition 0 follows immediately since, when \( e^* \) is contained in \( S_{nc} \), the set \( \{\underline{e}, \overline{e}\} \) becomes a singleton.

While Lemma 1 significantly reduces the candidates for equilibrium strategies, it remains to characterize the resulting set of equilibria. It proves useful to define a *threshold* effort level \( \hat{e} \) with the following properties: (1) When her rival chooses \( \overline{e} \), a player is indifferent between choosing the threshold and \( \overline{e} \). (2) Likewise, when her rival chooses \( \hat{e} \), the player is again indifferent between choosing \( \hat{e} \) and \( \overline{e} \).

Formally, \( \hat{e} \) solves \( E \pi(\hat{e}, e') = E \pi(\overline{e}, e') \) for \( e' \in \{\hat{e}, \overline{e}\} \). The expression \( \hat{e} \) is well-defined

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7 More precisely, Lemma 1 implies that a necessary condition for an equilibrium in the original game is for it to be an equilibrium in the 2x2 game. However, it can be readily shown that this is also a sufficient condition, which is formally demonstrated in the proof of Proposition 1.
so long as $\bar{e}$ is not too large. Otherwise, no interior solution exists for solving the indifference condition. Define $\delta$ to be the maximal value of $\bar{e}$ such that there exists a (weak) interior threshold, i.e. $\delta$ solves $E\pi(0,e') = E\pi(\delta,e')$ for $e' \in \{0, \delta\}$. Notice that the left-hand side of the indifference condition takes account of the fact that the lowest possible threshold occurs when $\bar{e} = 0$. Moreover, it can readily be seen that $\bar{e}$ is decreasing with respect to $\bar{e}$. Lemma 2 formally establishes the properties of the threshold.

**Lemma 2** There exists a unique $\delta$, which is not greater than $v/2$. If $\bar{e} \leq \delta$, then there exists a unique $\bar{e} \in [0, e^*]$, which is monotonically decreasing in $\bar{e}$; otherwise $\nexists \bar{e}$.

Identifying the bracket points, $e$ and $\bar{e}$, together with the threshold suffices to completely characterize equilibria for the symmetric case, as we show in Proposition 1.

**Proposition 1** Generically, a symmetric strategy contest with non-convex choices has a unique equilibrium in pure strategies:

a. Effort is $\underline{e}$ when $\bar{e} > \delta$ or $\bar{e} < e$

b. Effort is $\bar{e}$ when $\bar{e} > e$.

While Proposition 1 shows that the introduction of non-convexities does not disturb the uniqueness of the equilibrium in the case of pure strategies, whether the

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$^8$ For the non-generic case in which $\bar{e} = e$, there is a continuum of equilibria consisting of arbitrary weights given to $\{e, \bar{e}\}$. For the generic case in which $\bar{e} \neq e$, if by mixing $e$ and $\bar{e}$ the player obtains an expected effort for which $e$ and $\bar{e}$ are the best responses, then it is an equilibrium, in addition to the one obtained by pure strategies. However, for a given $e$ and $\bar{e}$, such effort may not exist. This is demonstrated in the proof.
resulting equilibrium effort increases or decreases depends on both of the bracket points and the resulting threshold. Nevertheless, with further restrictions on the CSF, we obtain a more sharply defined result regarding the equilibrium effort. Thus, under the Tullock CSF (i.e. \( p(e_i, e_j) = e_i^\alpha / (e_i^\alpha + e_j^\alpha) \), \( 0 < \alpha < 1 \)), non-convexities heighten competition relative to \( e_c^* \). To see this, consider a situation in which \( e \) and \( \bar{e} \) are a distance \( \Delta \) away from \( e_c^* \). Then, it can be readily shown that \( e < \bar{e} \) and, therefore, by Proposition 1, effort is \( \bar{e} \). Thus, the equilibrium effort increases as a function of the distance between the bracket points and \( e_c^* \).

As described in the introduction, there are myriad situations in which effort is “lumpy” and hence non-convexities arise. Moreover, the nature of this lumpiness in effort choices varies according to circumstances. For instance, a technological improvement subject to a minimum efficiency scale might reduce (or enlarge) the “jumps” in the set of available strategies. Political considerations might also affect the choice of effort. For instance, a country that adheres to international agreements concerning the use of force, such as the Geneva Convention, may find itself constrained in the amount of force it can apply.

Proposition 1 identifies circumstances in which small changes in the strategy space will produce larger ones in effort, and conversely in which large changes in the strategy space may not affect effort at all. To see this, fix \( e \) in the neighborhood of \( \hat{e} \). Then, small changes in \( e \) (or \( \bar{e} \)) that change the sign of \( e - \hat{e} \) result in a "jump" in

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9 The properties of the Tullock CSF are discussed below in section 4.
10 In the case that \( e \) and \( \bar{e} \) are a distance \( \Delta \) away from \( e_c^* \) and \( \alpha = 1 \), then \( \hat{e} = \bar{e} \). A formal proof for the entire argument in the text appears at the end of the appendix.
effort from $\varepsilon$ to $\bar{e}$ or vice versa. On the other hand, effort will not change when $e$ and $\bar{e}$ remain the same. This is illustrated in the following simple (and somewhat extreme) example:

**Example 1:** Assume that $p(e,0) = 1$ for all $e > 0$, and let $S_{nc} = \{0\} \cup [v/2 + \varepsilon, v]$, where $\varepsilon > 0$. Then there is an effortless peaceful equilibrium. However, as the result of a small change in $S_{nc}$ such that $\varepsilon < 0$, the player now exerts $v/2 + \varepsilon$ and the conflict becomes far more intense since there is almost full rent dissipation.

The basic intuition behind this result is related to the approximately inverted U-shape of $E\pi(e_i,e_j)$, which means that each level of $E\pi(e_i,e_j)$ (apart from its maximum) can be obtained by responding with either of two different (non-tangent) values of $e_i$ to $e_j$. Furthermore, these two values of $e_i$ are negatively correlated, such that the smaller is the smaller of the two efforts, the larger will be the larger of the two. Consequently, even under only slightly different constraints, there may be a large shift in the player’s chosen effort. In other words, a deviation from the status quo need not be large if the circumstances are "right" in order to bring about a major change in effort. This characterization may correspond to the behavior observed in the Cold War, in which a significant event such as the Cuban Missile Crisis did not develop into a conflict, while a relatively unimportant confrontation in Vietnam escalated into a major conflict. Another possible example is the Falklands War, which involved a sovereignty dispute over the Falkland Islands between Britain and Argentina, dating back to the 19th century. Until 1982, the situation remained calm. However, in that year a new leader came to power in Argentina, though there was
no regime change. This was in addition to a reorganization of British forces stationed on the Falkland Islands. These seemingly unimportant events nonetheless triggered the Falklands War.

Up to this point, the focus has been on **symmetric strategy contests** with a common prize value. However, this may not characterize every case. In the following example, we demonstrate that equilibria can change dramatically when players have different prize valuations.

**Example 2:** Assume that 

\[ p(e_i, e_j) = \begin{cases} 
 1 & \text{if } e_i + e_j > 0, \\
 2 & \text{otherwise} 
\end{cases} \]

the prize by players 1 and 2' be 1 and 2, respectively. When both players face a convex strategy space, the equilibrium efforts are \( \{e_{c,1}', e_{c,2}'\} = \{\frac{2}{5}, \frac{4}{9}\} \). Now consider a **symmetric strategy contest** in which \( S_{nc} = \{0, \frac{1}{9}, \frac{1}{5}, \frac{5}{9}\} \) and therefore \( e_1 = e_2 = \frac{1}{5} \) and \( \bar{e}_1 = \bar{e}_2 = \frac{5}{9} \). In this contest, the equilibrium efforts are: \( \{e_1', e_2'\} = \{\frac{1}{9}, \frac{5}{9}\} \) and therefore \( e_1' \notin \{e, \bar{e}\} \). This implies that when players have different prize valuations, Lemma 1 may not hold, even when both are subject to the same bracketing of the strategy space. Furthermore, holding everything else constant, let the largest element in the strategy space be 2/3 instead of 5/9. In this case, there is no Nash equilibrium in pure strategies. Thus, unlike in the common valuation case, Nash equilibrium in pure strategies may not exist when prize valuations differ.

To conclude, a key property of (symmetric) international conflicts is that a change in the status quo does not require a large change in circumstances, but rather
only the "right" circumstances. This may explain why similar constraints on weapons of war have produced different outcomes in different circumstances.

4. **Asymmetric strategy contests with non-convex effort costs**

In this section, we analyze an asymmetric contest in which player 1 has a convex choice set, i.e. $S_1=S_c$, while player 2 has a non-convex choice set, i.e. $S_2=S_{nc}$.

To solve the model, we assume that:

$$p(e_i, e_j) = \begin{cases} \frac{e_i^\alpha}{e_i^\alpha + e_j^\alpha} & \text{if } e_i + e_j > 0, \\ \frac{1}{2} & \text{otherwise} \end{cases},$$

where $\alpha \leq 1$.

Note that in addition to the concavity of $p$ assumed in (1) (which is satisfied when $\alpha \leq 1$), we also assume the Tullock CSF, which is a special case of the logit functional form that is homogenous of degree zero in effort. In this case, it is well known that $e^*_c = \frac{\alpha \bar{c}}{4}$.

Given the structure of player 1’s best response function, $BR_1(e_2)$, pictured in Figure 1 and using the proof of Lemma 1, Corollary 1 is immediate.

**Corollary 1** In any asymmetric strategy Tullock contest with non-convex choices:

(i) **Player 2’s equilibrium effort is an element from the set** $\{\bar{e}, \bar{e}\}$.

(ii) **The equilibrium effort in the case of pure strategies is as follows:** if

$$E\pi(e, BR_1(\bar{e})) > E\pi(\bar{e}, BR_1(\bar{e})) \text{ and } E\pi(e, BR_1(e)) \geq E\pi(\bar{e}, BR_1(e)),$$

then
effort is \((BR_1(e), e)\); if \(E\pi(e, BR_1(\bar{e})) \leq E\pi(\bar{e}, BR_1(\bar{e}))\) and \(E\pi(e, BR_1(\bar{e})) < E\pi(\bar{e}, BR_1(\bar{e}))\), then effort is \((BR_1(\bar{e}), \bar{e})\); if \(E\pi(e, BR_1(\bar{e})) \leq E\pi(\bar{e}, BR_1(\bar{e}))\) and \(E\pi(e, BR_1(\bar{e})) \geq E\pi(\bar{e}, BR_1(\bar{e}))\), then the possible efforts are \((BR_1(\bar{e}), e)\) and \((BR_1(\bar{e}), \bar{e})\); otherwise there does not exist a Nash equilibrium in the case of pure strategies.

Corollary 1 implies that in an asymmetric strategy Tullock contest, and regardless of player 2's full set of elements, it is sufficient to identify the signs of \(E\pi(e, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e}))\) and \(E\pi(e, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e}))\). We restrict what follows to the non-trivial case, in which \(\bar{e} < \nu\).^{11}

In order to characterize the equilibria in the asymmetric strategy Tullock contest, we need to define three threshold effort levels: (1) \(e_R\) which solves \(E\pi(e_R, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e})) = E\pi(e_R, BR_1(e_R)) - E\pi(\bar{e}, BR_1(e_R))\); (2) \(e_0\) which solves \(E\pi(e_0, BR_1(e_0)) = E\pi(\bar{e}, BR_1(e_0))\); and (3) \(e_{00}\) which solves \(E\pi(e_{00}, BR_1(\bar{e}))) = E\pi(\bar{e}, BR_1(\bar{e}))\).

For a given \(\bar{e}\), the terms: \(E\pi(e, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e}))\) and \(E\pi(e, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e}))\) and the resulting thresholds defined above are pictured in Figure 2 (and formally described by Lemma 3 and Corollary 2 in the appendix).

\(^{11}\) If \(\bar{e} \geq \nu\), then \(E\pi(\bar{e}, BR_1(\bar{e})) = E\pi(\bar{e}, 0) \leq 0\) and thus equilibrium effort is the pair \((e, BR_1(\bar{e}))\).
Figure 2: The shape of $E\pi(e, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e}))$ and $E\pi(e, BR_1(e)) - E\pi(\bar{e}, BR_1(e))$ when $\alpha < 1^{12}$.

We are now in a position to characterize the set of equilibria in an asymmetric strategy Tullock contest with a non-convex choice space:

**Proposition 2** Given that $\bar{e} < v$, in any asymmetric strategy Tullock contest with non-convex choices, the Nash equilibria in pure strategies are as follows:

a. Effort is $(BR_1(\bar{e}), \bar{e})$ when $\underline{e} < e_0$

b. Effort is $(BR_1(e), e)$ when $e > e_{00}$

c. Otherwise, both $(BR_1(\bar{e}), \bar{e})$ and $(BR_1(e), e)$ are equilibrium efforts.$^{13}$

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$^{12}$ Notice that $e_R = e_0 = e_{00}$ when $\alpha = 1$.

$^{13}$ The structure of the equilibria in which player 2 mixes pure strategies is as follows: Given that $\alpha < 1$, when $e \in (e_0, e_{00})$ there exists $q \in (0,1)$ such that $(BR_1(qe_0 + (1 - q)\bar{e}), q\bar{e} + (1 - q)\bar{e})$ is a pair of expected equilibrium efforts, but when $e \notin (e_0, e_{00})$ such an equilibrium may not exist. When $\alpha = 1$ and $e = e_R(1)(BR_1(e_R), qe_R + (1 - q)\bar{e})$ is a pair of expected equilibrium efforts for all $0 < q < 1$. 

Proposition 2 implies that as in the case of the symmetric strategy contest, efforts are sensitive to small changes in player 2’s strategy space when $\bar{e}$ is in the neighborhood of $e_0$ or $e_{00}$. However, unlike the symmetric strategy contest, in which equilibrium in pure strategies is generically unique, there are two different Nash equilibria in pure strategies when $\bar{e}$ is contained in the interval $[e_0, e_{00}]$, and $[e_0, e_{00}]$ is a singleton only when the exponent $\alpha$ is equal to 1.

Notice that player 2 will always prefer to have a convex choice set of effort, or more precisely, to be able to choose $e_{c'}$. This is because in the symmetric convex Tullock contest, commitment has no value, and therefore a player does not benefit from any type of bracketing in the strategy space (Dixit, 1987; Baik and Shogren, 1992; Possajennikov 2009). Nevertheless, in an asymmetric Tullock strategy contest, a player will prefer to be constrained under some circumstances. To see this, consider the following: When the equilibrium effort is $(BR_1(\bar{e}), \bar{e})$, player 1’s expected net payoff is larger than her rival’s (and even larger than the payoff in the standard symmetric convex contest). 14 Indeed, in this case, player 1 benefits from the handicap of her rival. However, this may not be the case when equilibrium effort is $(BR_1(\bar{e}), \bar{e})$. We demonstrate this in example 3.

14 Given that $BR_1(\bar{e}) > \bar{e}$ and $\frac{\partial \pi(e, e)}{\partial e} < 0$, by the definition of $BR_1(\bar{e})$, $E\pi(BR_1(\bar{e}), \bar{e}) > E\pi(\bar{e}, \bar{e}) > E\pi(\bar{e}, BR_1(\bar{e}))$ and $E\pi(BR_1(\bar{e}), \bar{e}) > E\pi(\bar{e}, \bar{e}) > E\pi(e_{c'}, e_{c'})$. 
Example 3: Consider the CSF described in example 2, and let $v=1$ and $S_2 = (0.9/16)$.

Then $(e^*_1, e^*_2) = \left(\frac{3}{16}, \frac{9}{16}\right)$ and $(E\pi^*_1, E\pi^*_2) = \left(\frac{1}{16}, \frac{3}{16}\right)$. Thus, player 2’s expected net payoff is larger than player 1’s.\(^{15}\)

5. Conclusions

Nash equilibria are characterized in the case of contests in which the convexity assumption of the strategy space has been relaxed and are applied to actual international conflicts. The main implication of the results is that a marginal change in circumstances can either generate a peaceful equilibrium or alternatively trigger a war. We also show that a constraint on a player’s strategy space can benefit the player when her rival is not constrained.

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\(^{15}\) Notice that $BR_1\left(\frac{9}{16}\right) = \sqrt{\frac{9}{16}} - \frac{9}{16} = \frac{3}{16}$. Furthermore, it is also possible that $(e^*_1, e^*_2) = (BR_1(\bar{e}), \bar{e})$ and player 1’s expected net payoff will be larger than player 2’s. In particular, this can happen when $a$, the exponent in the CSF, is sufficiently small (such that $p \approx p_2$ and $e_1 < e_2$).
Appendix:

Proof of Lemma 1: It is well known that in the standard symmetric strategy convex contest, player $i$’s best response function, $BR_i(e_j)$ (pictured in Figure 1), has the following property:\(^{16}\)

\[(A.1) \quad BR_i(e_j) \overset{>}{{\overset{<}{\leftarrow}}} e_j \iff e_j \overset{<}{\overset{>}{\leftarrow}} e^*_j \forall e_j > 0,\]

which implies that:

\[(A.2) \quad \text{Max}BR_i(e_j) = BR_i(e^*_j) = e^*_j.\]

Thus, it is immediate that in a symmetric strategy non-convex contest, equilibrium effort $e$ satisfies:

\[(A.3) \quad e \leq \bar{e}.\]

Given that $E\pi(e_i; e_j)$ is concave in $e_i$ in the interior of its domain and therefore for a given $e_j > 0$, $E\pi(BR_i(e_j); e_j)$ is the unique maximum of $E\pi(e_i; e_j)$, it is immediate from (A.1) that:

\[(A.4) \quad E\pi(x, x) > E\pi(y, x) \forall e^*_j \geq x > y \geq 0.\]

We now present the following important identity:\(^{17}\)

\[(A.5) \quad E\pi(x, y) - E\pi(y, y) = p(x, y)v + y - \left(\frac{v}{x} + x\right)\]

\(^{16}\) For example, see equation (8) in Dixit (1987) and Baik and Shogren (1992).

\(^{17}\) Note that (A.5) implies that in a symmetric $2 \times 2$ contest where choice is limited to a set of two effort levels, there always exists a dominant (possibly weak) strategy. As far as we know, this result is presented here for the first time and it may have wider applications that are beyond the scope of this paper.
\[
\frac{v}{2} - x - (v(1 - p(x, y)) - y) = E\pi(x, x) - E\pi(y, x).
\]

Applying (A.5) to (A.4) results in:

(A.6) \[ E\pi(x, y) > E\pi(y, y) \forall e^*_x \geq x > y \geq 0. \]

The rest of the proof is carried out in two steps: In the first, we restrict the analysis to pure strategies, and in the second we extend the analysis to mixed strategies.

We thus begin by assuming that the player restricts her actions to pure strategies. In what follows, we show that the equilibrium effort cannot be smaller than \(e\). Let \(e \geq x \geq y \geq 0\). Equations (A.4) and (A.6) imply that unless \(y = x = e\), \((x,y)\) cannot be a pair of equilibrium efforts. Moreover, if \(y < e\), then \((\bar{e}, y)\) cannot be a pair of equilibrium efforts either. This can be shown by contradiction: Assume for now that \((\bar{e}, y)\) is a pair of equilibrium efforts. Then, by (A.5), \(E\pi(y, y) = E\pi(\bar{e}, y)\), but by (A.6), \(E\pi(y, y) < E\pi(e, y)\) and thus \(E\pi(e, y) > E\pi(\bar{e}, y)\), which contradicts the initial assumption that \((\bar{e}, y)\) is a pair of equilibrium efforts. Thus, \((y,e)\) cannot be a pair of equilibrium efforts for all \(e \in \{S_{nc} \mid e \leq \bar{e}\}\) and \(y < e\). Given (A.3), this implies that in a symmetric strategy non-convex contest equilibrium:

(A.7) \[ e \geq \bar{e}. \]

By (A.3) and (A.7), when the player is restricted to pure strategies, her equilibrium effort can only be an element from the set \([e, \bar{e}]\).

We now proceed to the second step of the proof, in which the player can mix pure strategies. We show that such an equilibrium can only exist when the player mixes \(e\) and \(\bar{e}\). Assume that the pair of expected equilibrium efforts is \((qa + (1-a)\bar{e})\).
$q)b, q'c+(1-q')d$, where $0<q, q'<1$, and without loss of generality, $a>b$ and $c>d$. Given that for $e_i>0$, $E\pi(e_i;e_i)$ is concave in $e_i$ with a unique maximum at $E\pi(BR_i(e_i);e_i)$ (henceforth summarized as "the structure of $E\pi_i$"), it is immediate that:

$a>BR_i(q'c+(1-q')d)>b$

and

$c>BR_i(qa+(1-q)b)>d$.

Notice that when $a=c$ (and therefore $b=d$), all effort pairs other than $(b,a)=(\bar{e},\bar{e})$ can be deleted. First, it must be that $a>e\bar{e}$, since otherwise, by (A.6), $E\pi(a,qa+(1-q)b)>E\pi(b,qa+(1-q)b)$ when $a>b$. Therefore, given (3), it is only possible that $a=\bar{e}$. We now show by contradiction that it is only possible that $b=e$ when $a=c$ and $b=d$.

Assume for now that $b<e$. Given that $\bar{e}>BR_i(q\bar{e}+(1-q)b)>b$, by the structure of $E\pi_i(e_i;e_i)$, either $BR_i(q\bar{e}+(1-q)b)>e>b$ and thus $E\pi(e, q\bar{e}+(1-q)b)>E\pi(b, q\bar{e}+(1-q)b)$ or $e\bar{e}>BR_i(q\bar{e}+(1-q)b)>b$ and thus $E\pi(e, q\bar{e}+(1-q)b)>E\pi(\bar{e}, q\bar{e}+(1-q)b)$. In the first option, $b$ is not a best response to $q\bar{e}+(1-q)b$ and in the second $\bar{e}$ is not a best response to $q\bar{e}+(1-q)b$.

Now let $a\neq c$ (and therefore $b\neq d$). Then, either $b>c$ or $d>a$; otherwise, given the structure of $E\pi_i$, it is impossible that both $a$ and $b$ are the best responses to $q'c+(1-q')d$, and $c$ and $d$ are the best responses to $qa+(1-q)b$.\(^{18}\)

Without loss of generality, let $b>c$. Then:

\(^{18}\) For instance, if $d<b<a<c$, then it is not possible that both $c$ and $d$ are the best responses to $qa+(1-q)b$; rather, either $c>BR_i(qa+(1-q)b)>a$ and therefore $E\pi(a, qa+(1-q)b)>E\pi(d, qa+(1-q)b)$ or $a>BR_i(qa+(1-q)b)>d$ and therefore $E\pi(b, qa+(1-q)b)>E\pi(c, qa+(1-q)b)$ or $a>BR_i(qa+(1-q)b)>b$ and therefore both $E\pi(a, qa+(1-q)b)>E\pi(c, qa+(1-q)b)$ and $E\pi(b, qa+(1-q)b)>E\pi(d, qa+(1-q)b)$. A similar argument can be shown for any partition of $a,b,c$ and $d$ such that neither $b>c$ nor $d>a$. 

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Thus, by the structure of $E\pi_i$,

\[ E\pi(a, q'c+(1-q')d) = (b, q'c+(1-q')d) > E\pi(q'c+(1-q')d, q'c+(1-q')d) \]

\[ \rightarrow \]

\[ E\pi(qa+(1-q)b, q'c+(1-q')d) > E\pi(q'c+(1-q')d, q'c+(1-q')d), \]

where by (A.5),

\[ E\pi(qa+(1-q)b, q'c+(1-q')d) > E\pi(q'c+(1-q')d, q'c+(1-q')d) \]

\[ \leftrightarrow \]

\[ E\pi(qa+(1-q)b, qa+(1-q)b) > E\pi(q'c+(1-q')d, qa+(1-q)b). \]

However, the structure of $E\pi_i$ when $b > c$ also implies that:

\[ E\pi(d, qa+(1-q)b) = E\pi(c, qa+(1-q)b) > E\pi(qa+(1-q)b, qa+(1-q)b) \]

\[ \rightarrow \]

\[ E\pi(q'c+(1-q')d, qa+(1-q)b) > E\pi(qa+(1-q)b, qa+(1-q)b), \]

a contradiction.\(^{19}\) QED

Proof of Lemma 2: Given (A.5),

(A.8) \[ E\pi(\hat{e}, \hat{e}) = E\pi(\bar{e}, \bar{e}) \]

\[ \leftrightarrow \]

\[ E\pi(\hat{e}, \hat{e}) = E\pi(\bar{e}, \bar{e}) \]

\[ \leftrightarrow \]

\[ \frac{\nu}{2} - \hat{e} = p(\bar{e}, \hat{e})\nu - \bar{e}. \]

\(^{19}\) Notice that this argument holds also when $q$ or $q'$ equals zero or one.
Substituting $\hat{e} = 0$ into (A.8) results in:

\[(A.9) \quad E\pi(0,0) = E\pi(\bar{e}, 0) \iff \bar{e} = v \left( p(\bar{e}, 0) - \frac{1}{2} \right).\]

Notice that $p(\bar{e}, 0) \leq 1$ and thus the RHS of the second equality in (A.9) is bounded from above by $\frac{v}{2}$. Also, by (A.5), $E\pi(e^*_c, e^*_c) > E\pi(0, e^*_c) \iff E\pi(e^*_c, 0) > E\pi(0,0)$, which implies that $e^*_c < v \left( p(e^*_c, 0) - \frac{1}{2} \right)$. Therefore, since the LHS in the second equality of (A.9) is increasing and linear in $\bar{e}$ and the RHS is non-decreasing and weakly concave in $\bar{e}$, by the Intermediate Value Theorem, there exists a unique $\bar{e} \equiv \delta \in (e^*_c, \frac{v}{2}]$ that solves (A.9) such that $\bar{e} \geq \delta \iff E\pi(0,0) \geq E\pi(\bar{e}, 0)$. Thus, since for a given $\bar{e}$ both sides of the third equality of (A.8) are decreasing in $\hat{e}$ while the LHS is linear and the RHS is convex, and by definition $E\pi(e^*_c, e^*_c) > E\pi(\bar{e}, e^*_c)$ which implies that $\frac{v}{2} - e^*_c > p(\bar{e}, e^*_c)v - \delta$, there exists a unique $\hat{e}$ over the interval $[0, e^*_c)$ such that $e^*_c \geq \hat{e} \iff E\pi(\hat{e}, \hat{e}) \geq E\pi(\bar{e}, \hat{e})$ iff $\bar{e} \leq \delta$. Given that, $p(\bar{e}, \hat{e})v - \bar{e}$ is decreasing in $\bar{e}$ and $\frac{v}{2} - \hat{e}$ is independent of $\bar{e}$, $\hat{e}$ is monotonically decreasing in $\bar{e}$. QED

Proof of Proposition 1: By (A.5), the player’s equilibrium effort in the symmetric 2x2 contest where choice is limited to $[e, \bar{e}]$ is determined by the following inequality:

\[(A.10) \quad E\pi(e, \bar{e}) \geq E\pi(\bar{e}, \bar{e}) \iff \]

\[\text{since } p(\bar{e}, e) \text{ is increasing and concave in } e \text{ for } e > 0 \text{ and continuous in } e, p(\bar{e}, 0) \text{ must be non-decreasing in } \bar{e}.\]

\[\text{This is due to the Intermediate Value Theorem.}\]
\[ E\pi(e, e) \gtrless E\pi(\bar{e}, e). \]

According to the proof of Lemma 2, if \( \bar{e} > \delta \) then \( E\pi(e, e) > E\pi(\bar{e}, e) \) for all \( e \); otherwise \( e \sim \bar{e} \Leftrightarrow E\pi(e, e) \gtrless E\pi(\bar{e}, e) \). Lemma 1 implies that a necessary condition for an equilibrium in the original contest is for it to be an equilibrium in the 2x2 contest. In the following, it is shown that this is also a sufficient condition. By (A.3) and (A.6), if \((e, e)\) is an equilibrium in the 2x2 contest, then it is an equilibrium in the original contest. In addition, if \((\bar{e}, \bar{e})\) is an equilibrium in the 2x2 contest, then \(E\pi(\bar{e}, \bar{e}) \geq E\pi(e, \bar{e})\) and therefore by the structure of \( E\pi_i \), \( \bar{e} > BR_i(\bar{e}) > e \) which also implies that \( E\pi(\bar{e}, \bar{e}) \geq E\pi(e, \bar{e}) > E\pi(y, \bar{e}) \) for all \( y < e \) and therefore given (A.3), if \((\bar{e}, \bar{e})\) is an equilibrium in the 2x2 contest, then it is also one in the original contest. \(^{22}\)

QED

Proof of Example 2: In the first case, the matrix of the players' net payoffs is:

| player 1/ player 2 | 1/9 | 1/5 | 5/9 |
|---------------------|-----|-----|-----|
| 1/9                 | 7/18, 8/9 | 37/70, 38/63 | 5/18, 1/18 |
| 1/5                 | 31/126, 38/35 | 3/10, 4/5 | 55/306, 28/85 |
| 5/9                 | 2/9, 10/9 | 11/170, 140/153 | -1/18, 4/9 |

\(^{22}\) Notice that, in addition to the equilibrium in the case of pure strategies determined by the inequality in (A.10), there may exist an equilibrium in mixed strategies. More precisely, if there exists a convex combination \( q\tilde{e} + (1-q)e \), \( 0 < q < 1 \) for which the player's best responses are \( e \) and \( \bar{e} \), then it is an equilibrium in the case of mixed strategies. Such an equilibrium can be constructed as follows: Pick a tuple \((a, b, c)\) such that \( a < e, c < b \) and \( E\pi(a, c) = E\pi(b, c) \), and let \( a = \tilde{e} \) and \( b = \bar{e} \). Then, \( c = qa + (1-q)b \) is the player's expected equilibrium effort. However, for a given \( e \) and \( \bar{e} \), such an equilibrium may not exist. For instance, by the continuity of \( E\pi_i \) in the interior of its domain, the player always prefers responding with \( e \) rather than \( \bar{e} \) to any convex combination of \( e = c, c \in [\tilde{e}, \bar{e}] \) with \( \bar{e} \) sufficiently large.
Notice that 5/9 is a dominant strategy for player 2 and player 1's best response to 5/9 is 1/9.

In the second case, the matrix is:

| player 1/ player 2 | 1/9   | 1/5   | 2/3   |
|--------------------|-------|-------|-------|
| 1/9                | 7/18, 8/9 | 37/70, 38/63 | 4/21, 11/63 |
| 1/5                | 31/126, 38/35 | 3/10, 4/5 | 4/39, 17/65 |
| 2/3                | 2/63, 22/21 | 2/65, 34/39 | -1/16, 1/3 |

Notice that for player 2, 1/9 is dominated by 2/3, and for player 1, 1/5 dominates 2/3. However, there is no Nash equilibrium in the case of pure strategies but only in the case of mixed strategies. QED

Proof of Corollary 1: The behavior of player 1 described in (A.1) and the behavior of player 2 described in (A.3), (A.4) and (A.6) result in Corollary 1. QED

Proof of Proposition 2: First, we present an auxiliary lemma and a corollary that follows from it to describe the structure of the terms $E\pi(e, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e}))$ and $E\pi(e, BR_1(e)) - E\pi(\bar{e}, BR_1(e))$.

**Lemma 3** For a given $\bar{e} < v$: 
(i) \( E\pi\left(e, BR_1(\bar{e})\right) - E\pi\left(\bar{e}, BR_1(\bar{e})\right) \) is concave with an interior maximum point over the interval \( e \in [0, \frac{av}{4}] \), where \( E\pi\left(0, BR_1(\bar{e})\right) - E\pi\left(\bar{e}, BR_1(\bar{e})\right) = \bar{e} - vp(\bar{e}, BR_1(\bar{e})) \) and \( E\pi\left(\frac{av}{4}, BR_1(\bar{e})\right) - E\pi\left(\bar{e}, BR_1(\bar{e})\right) \geq 0 \).

(ii) \( E\pi\left(e, BR_1(e)\right) - E\pi\left(\bar{e}, BR_1(\bar{e})\right) \) is monotonically increasing over the interval \( e \in (0, \frac{av}{4}] \), where \( E\pi\left(0, BR_1(0)\right) - E\pi\left(\bar{e}, BR_1(0)\right) \approx \bar{e} - v \) and

\[
E\pi\left(\frac{av}{4}, BR_1\left(\frac{av}{4}\right)\right) - E\pi\left(\bar{e}, BR_1\left(\frac{av}{4}\right)\right) \geq 0.
\]

(iii) There exists a unique \( e_R \in (0, \frac{av}{4}] \). If \( \alpha < 1 \) (\( \alpha = 1 \)), then \( E\pi\left(e_R, BR_1(e')\right) < E\pi\left(\bar{e}, BR_1(e')\right) \) (where \( e' \in \{ e_R, \bar{e} \} \)).

Proof of Lemma 3i: For a given \( \bar{e}, E\pi\left(\bar{e}, BR_1(\bar{e})\right) \) is constant and \( E\pi\left(e, BR_1(\bar{e})\right) \) is concave over the interval \( e \in [0, \frac{av}{4}] \), where

\[
\max E\pi\left(e, BR_1(\bar{e})\right) = E\pi\left(BR_1(BR_1(\bar{e})), BR_1(\bar{e})\right).\]

Also, (A.2) implies that

\[
E\pi\left(\frac{av}{4}, BR_1(\bar{e})\right) > E\pi\left(\bar{e}, BR_1(\bar{e})\right).
\]

Proof of Lemma 3ii: Notice that:

\[
(A.11) \ E\pi\left(e_2, BR_1(e)\right) = \frac{ve_2^n}{e_2^n + (BR_1(e))} - e_2.
\]

Notice that \( BR_1(e_2) \) satisfies player 1’s FOC for an interior maximum and therefore:

\[
(A.12) \ \frac{\partial E\pi_1(BR_1(e)e)}{\partial e_1} = \frac{\alpha(BR_1(e))^{\alpha-1}e^v}{(e^\alpha + (BR_1(e))^{\alpha})^2} v - 1 = 0
\]
\[
e^{a}v = \frac{(e^{a} + (BR_{1}(e))^{a})^{2}}{\alpha(BR_{1}(e))^{a-1}}.
\]

Substituting the RHS of (A.12) into the denominator of the first term in the RHS of (A.11) results in:

(A.13) \[E\pi\left(\bar{e}, BR_{1}(\bar{e})\right) = \frac{1}{\alpha}(BR_{1}(\bar{e}) + (e^{a}(BR_{1}(\bar{e}))^{1-a}) - e.\]

Taking the derivative of (A.13) with respect to \(e\) yields:

(A.14) \[\frac{dE\pi(e, BR_{1}(e))}{de} = \frac{1}{\alpha}\left(\frac{dBR_{1}(e)}{de} + \alpha\left(\frac{BR_{1}(e)}{e}\right)^{1-a} + (1 - \alpha)\left(\frac{e}{BR_{1}(e)}\right)^{\alpha}\frac{dBR_{1}(e)}{de}\right) - 1
\]

\[= \left(\frac{BR_{1}(e)}{e}\right)^{1-a} + \frac{1}{\alpha}\left(1 + (1 - \alpha)\left(\frac{e}{BR_{1}(e)}\right)^{\alpha}\right)\frac{dBR_{1}(e)}{de} - 1.\]

By (A.1), \(BR_{1}(\bar{e}) > \bar{e}\) and \(\frac{dBR_{1}(\bar{e})}{de} > 0\), and thus, by (A.14), \(\frac{dE\pi(e, BR_{1}(e))}{de} \geq 0\). Given that \(BR_{1}(\bar{e})\) is increasing in \(e\), \(E\pi\left(\bar{e}, BR_{1}(\bar{e})\right)\) is decreasing in \(e\). Therefore, \(E\pi\left(\bar{e}, BR_{1}(\bar{e})\right) - E\pi\left(\bar{e}, BR_{1}(\bar{e})\right)\) is increasing in \(e > 0\). Notice that, \(BR_{1}(0) = e\) and thus \(E\pi(0, BR_{1}(0)) = 0\) while \(E\pi(e, BR_{1}(e)) > 0\) for \(e > 0\). This implies that, \(E\pi\left(\bar{e}, BR_{1}(\bar{e})\right) - E\pi\left(\bar{e}, BR_{1}(\bar{e})\right)\) is increasing at \(e = 0\) despite the discontinuity of \(BR_{1}(\bar{e})\) at \(e = 0\). Notice also that

\[E\pi(0, BR_{1}(0)) - E\pi\left(\bar{e}, BR_{1}(\bar{e})\right) = \bar{e} - \frac{\bar{e}}{\bar{e}+v}v = \bar{e} - v.\]

Proof of Lemma 3iii: We prove Lemma 3iii in two steps: First, we show that there exists a unique \(e_{R} \in \left[0, \frac{w_{0}}{4}\right]\) and then we find the sign of \(E\pi(\bar{e}, BR_{1}(e_{R})) - E\pi(e_{R}, BR_{1}(e_{R})).\)

We thus begin by proving that there exists a unique \(e_{R}\):
(A.15) \[ \begin{align*} E\pi(e_R, BR_1(\bar{e})) - E\pi(\bar{e}, BR_1(\bar{e})) &= E\pi(e_R, BR_1(e_R)) - E\pi(\bar{e}, BR_1(e_R)) \\ \iff \quad p(e_R, BR_1(\bar{e})) - p(\bar{e}, BR_1(\bar{e})) &= p(e_R, BR_1(e_R)) - p(\bar{e}, BR_1(e_R)) \\ \iff \quad 1 - p(\bar{e}, BR_1(\bar{e})) - (1 - p(e_R, BR_1(\bar{e}))) &= 1 - p(\bar{e}, BR_1(e_R)) - (1 - p(e_R, BR_1(e_R))) \\ \iff \quad p(BR_1(\bar{e}), \bar{e}) - p(BR_1(\bar{e}), e_R) &= p(BR_1(e_R), \bar{e}) - p(BR_1(e_R), e_R) \\ \iff \quad E\pi(BR_1(\bar{e}), \bar{e}) - E\pi(BR_1(e_R), \bar{e}) &= E\pi(BR_1(\bar{e}), e_R) - E\pi(BR_1(e_R), e_R) \\ \iff \quad BR_1(\bar{e}) &= BR_1(e_R). \end{align*} \]

By the structure of $BR_i$ described in (A.1),\(^{23}\) for any given $\bar{e} \in (\frac{\alpha v}{4}, v)$, there exists a unique $e_R \in [0, \frac{\alpha v}{4})$ that solves $BR_1(\bar{e}) = BR_1(e_R)$.

We now proceed to find the sign of $E\pi(\bar{e}, BR_1(e_R)) - E\pi(e_R, BR_1(e_R))$. By the FOC for maximization by player 1:

(A.16) \[ \frac{\partial p(BR_1(\bar{e}), \bar{e})}{\partial e_1} = \frac{\partial p(BR_1(e_R), e_R)}{\partial e_1} \]

\[ \iff \quad \bar{e}^\alpha \left( \bar{e}^\alpha + (BR_1(e_R))^{\alpha} \right)^{-\frac{\alpha}{2}} = \frac{e_R^\alpha}{(e_R^\alpha + (BR_1(e_R))^{\alpha})^{\frac{\alpha}{2}}} \]

\[ \iff \quad (e_R^\alpha + (BR_1(e_R))^{\alpha})\sqrt{\bar{e}^\alpha} = (\bar{e}^\alpha + (BR_1(e_R))^{\alpha})\sqrt{e_R^\alpha} \]

\(^{23}\) Over the interval $(0, v)$, $BR_i$ is positive and has an approximate inverted U-shape with a unique maximum at $av/4$. See Pérez-Castrillo and Verdier (1992) for a detailed analysis of the reaction functions in the Tullock contest.
\[(BR_1(e_R))^\alpha - \sqrt{\bar{e}e_R})^\alpha)(\sqrt{\bar{e}^\alpha} - \sqrt{e_R^\alpha}) = 0\]

\[\Rightarrow\]

\[BR_1(e_R) = \sqrt{\bar{e}e_R} \cdot\]

Substituting \(\bar{e} = e_R\) and (A.16) into (A.12) results in:

(A.17) \[
\frac{\alpha(\bar{e}e_R)^{\alpha-1} e_R^\alpha}{(e_R^\alpha + \sqrt{(\bar{e}e_R)^\alpha})} v = 1
\]

\[\Rightarrow\]

\[
\frac{\alpha(\sqrt{\bar{e}e_R})^{\alpha-1}}{(\sqrt{e_R^\alpha} + \sqrt{\bar{e}^\alpha})^2} v = 1
\]

\[\Rightarrow\]

\[
\alpha(\sqrt{\bar{e}e_R})^{\alpha-1} v = (\sqrt{\bar{e}^\alpha} + \sqrt{e_R^\alpha})^2.
\]

Also, substituting (A.16) into \(E\pi(\bar{e}, BR_1(e_R)) - E\pi(e_R, BR_1(e_R))\) results in:

(A.18) \[
E\pi(\bar{e}, BR_1(e_R)) - E\pi(e_R, BR_1(e_R))
\]

\[
= \left(\frac{\bar{e}^\alpha}{\bar{e}^\alpha + \sqrt{\bar{e}e_R})^\alpha} - \frac{e_R^\alpha}{e_R^\alpha + \sqrt{(\bar{e}e_R)^\alpha}}\right) v - (\bar{e} - e_R)
\]

\[
= \left(\frac{\sqrt{e_R^\alpha}}{\sqrt{\bar{e}^\alpha} + \sqrt{e_R^\alpha}} - \frac{\sqrt{\bar{e}^\alpha}}{\sqrt{\bar{e}^\alpha} + \sqrt{e_R^\alpha}}\right) v - (\bar{e} - e_R)\]

\[
= \frac{\bar{e}^\alpha - e_R^\alpha}{(\sqrt{\bar{e}^\alpha} + \sqrt{e_R^\alpha})^2} v - (\bar{e} - e_R).
\]

Assuming that \(\alpha \neq 0\), substituting the LHS of (A.17) into the denominator of the first term of (A.18) results in:
(A.19) \[ E\pi(\bar{e}, BR_1(e_R)) - E\pi(e_R, BR_1(e_R)) \]
\[ = \frac{\bar{e}^{a} - e_R^{a}}{\alpha \sqrt{(\bar{e} e_R)^{a-1}}} - (\bar{e} - e_R), \text{ where} \]

\[ (A.20) \text{sign} \left( \frac{\bar{e}^{a} - e_R^{a}}{\alpha \sqrt{(\bar{e} e_R)^{a-1}}} - (\bar{e} - e_R) \right) = \text{sign} \left( \frac{\bar{e}^{a} - e_R^{a}}{(\bar{e} - e_R)(\bar{e} e_R)^{\frac{a}{2}}} - \alpha \right) \]
\[ = \text{sign} \left( \frac{\frac{\bar{e}}{e_R} - e_R}{(\bar{e} - e_R)\frac{1}{2}(\bar{e} e_R)^{\frac{a}{2}}} - \alpha \right) \]
\[ = \text{sign} \left( \frac{\frac{\bar{e}}{e_R} - e_R}{(\bar{e} - e_R)\frac{1}{2}(\bar{e} e_R)^{\frac{a}{2}}} - \alpha \right). \]

Notice that \( \frac{\bar{e}}{e_R} \) is increasing and convex with respect to \( \alpha \) over the interval \( \alpha \in (0,1) \), if \( \alpha \in (0,1) \) then \( E\pi(\bar{e}, BR_1(e_R)) - E\pi(e_R, BR_1(e_R)) < 0 \). \( \text{QED} \)

Lemma 3 implies the following corollary:

**Corollary 2** Given \( \bar{e} < v \), there exists a unique tuple \( (e_0, e_{00}, e_R) \). If \( a<1 \) (\( a=1 \)), then \( e_{00} \)

\[ e_0 > e_R \quad (e_{00} = e_0 = e_R). \]

---

\( ^{24} \) Since \( \bar{e} > e_R \), \( \frac{\bar{e}}{e_R} \) is independent of \( \alpha \), and is also positive. Define \( Z(\alpha) = \left( \frac{\bar{e}}{e_R} \right)^{\alpha} - \left( \frac{e_R}{\bar{e}} \right)^{\alpha} \), which is increasing and convex with respect to \( \alpha \) over the interval \( \alpha \in (0,1) \), then \( \frac{dZ}{d\alpha} = \ln \left( \frac{\bar{e}}{e_R} \right) \left( \frac{\bar{e}}{e_R} \right)^{\alpha} + \left( \frac{e_R}{\bar{e}} \right)^{\alpha} > 0 \) and \( \frac{d^2Z}{d\alpha^2} = \frac{1}{4} \ln \left( \frac{\bar{e}}{e_R} \right)^{2} \left( \left( \frac{\bar{e}}{e_R} \right)^{\alpha} - \left( \frac{e_R}{\bar{e}} \right)^{\alpha} \right) > 0. \)

\( ^{25} \) Notice that by (A.18), \( E\pi(\bar{e}, BR_1(e_R)) - E\pi(e_R, BR_1(e_R)) = e_R - \bar{e} < 0 \) at \( \alpha=0. \)
By Corollary 1, Lemma 3 and Corollary 2, the Nash equilibria in the case of pure strategies shown in Proposition 2 is straightforward. Regarding equilibria in the case of mixed strategies: At $q = 1$, $E\pi \left( e, BR_1(qe + (1 - q)\bar{e}) \right) - E\pi \left( \bar{e}, BR_1(qe + (1 - q)\bar{e}) \right) = E\pi \left( e, BR_1(e) \right) - E\pi \left( \bar{e}, BR_1(\bar{e}) \right)$, and at

$q = 0, E\pi \left( e, BR_1(qe + (1 - q)\bar{e}) \right) - E\pi \left( \bar{e}, BR_1(qe + (1 - q)\bar{e}) \right) = E\pi \left( e, BR_1(e) \right) - E\pi \left( \bar{e}, BR_1(\bar{e}) \right)$. Since when $\alpha < 1$ and $e \in (e_0, e_0)$, by Lemma 3 and Corollary 2, $E\pi \left( e, BR_1(e) \right) < E\pi \left( \bar{e}, BR_1(\bar{e}) \right)$ and $E\pi \left( e, BR_1(e) \right) > E\pi \left( \bar{e}, BR_1(\bar{e}) \right)$, by the Intermediate Value Theorem there exists $q \in (0,1)$ such that $E\pi \left( e, BR_1(qe + (1 - q)\bar{e}) \right) = E\pi \left( \bar{e}, BR_1(qe + (1 - q)\bar{e}) \right)$. When $e$ is not contained in $(e_0, e_0)$, the two expressions above do not have opposite signs and therefore an equilibrium in mixed strategies may not exist. The proof for the case in which $\alpha = 1$ is trivial. QED

Proof for the argument after Proposition 1: Let $\bar{e} - e_\alpha^* = 2\beta \Delta$ and $e_\alpha^* - \bar{e} = 2(1 - \beta)\Delta$, where $0 < \beta < 1$. Then, by the definition of $\bar{e}$:

(A.21) $E\pi(\bar{e}, \bar{e}) = E\pi(e, \bar{e}) \iff \nu \left( p(e, \bar{e}) - \frac{1}{2} \right) = 2\Delta \iff p(e, \bar{e}) = \frac{\nu + 4\Delta}{2\nu}$.

Substituting (A.21) into $\frac{p(e, \bar{e})}{p(\bar{e}, e)} = \frac{1-p(e, \bar{e})}{p(e, \bar{e})} = \frac{1}{p(e, \bar{e})} - 1$ results in:

(A.22) $\frac{p(e, \bar{e})}{p(\bar{e}, e)} = \frac{\nu - 4\Delta}{\nu + 4\Delta}$

Also, notice that when $p(e_i, e_j) = e_i^a/(e_i^a + e_j^a)$, $0 < a \leq 1$, then:

26 Notice that player 2 always prefers to respond with $e$ rather than $\bar{e}$ to $BR_1(qe + (1 - q)\bar{e})$ when $e = e^*_\alpha - e$ and $\bar{e}$ is sufficiently large. Thus, in this case, for instance, there does not exist an equilibrium in the case of mixed strategies.
\[ (A.23) \quad \frac{p(\delta, \bar{\delta})}{p(\bar{\delta}, \bar{\delta})} = \left( \frac{\delta}{\bar{\delta}} \right)^a = \left( \frac{\delta^2 - 2(1-\beta)\Delta}{\varepsilon_1^{+2} + 2\beta\Delta} \right)^a = \left( \frac{\frac{\alpha v}{\delta} - 2(1-\beta)\Delta}{\frac{\alpha v}{\delta} + 2\beta\Delta} \right)^a = \left( \frac{\alpha v - 8(1-\beta)\Delta}{\alpha v + 8\beta\Delta} \right)^a. \]

Assuming that $\beta = 1/2$ and by (A.23), we obtain:

\[ (A.24) \quad \frac{p(\delta, \bar{\delta})}{p(\bar{\delta}, \bar{\delta})} = \left( \frac{\alpha v - 4\Delta}{\alpha v + 4\Delta} \right)^a. \]

Notice that at $\alpha = 1$, (A.22) is equal to (A.24) and thus, in this case, $\beta = 1/2$ for all $\Delta > 0$.

For $\alpha < 1$, both $\frac{\alpha v - 4\Delta}{\alpha v + 4\Delta}$ and $\left( \frac{\alpha v - 4\Delta}{\alpha v + 4\Delta} \right)^a$ are: (1) decreasing and strictly convex in $\Delta$, and (2) equal to 1 at $\Delta = 0$. However, at $\alpha v/4$, (A.22) is positive while (A.24) is equal to zero.

Therefore, for $\alpha < 1, \frac{\alpha v - 4\Delta}{\alpha v + 4\Delta} > \left( \frac{\alpha v - 4\Delta}{\alpha v + 4\Delta} \right)^a$ for all $\Delta \in (0, \frac{\alpha v}{4})$. Thus, since $\left( \frac{\alpha v - 8(1-\beta)\Delta}{\alpha v + 8\beta\Delta} \right)^a$ is increasing in $\beta$, $\beta > 0.5$ for $\Delta > 0$ and $\alpha < 1$.

QED

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