Discrete-time approximation for stochastic optimal control problems under the $G$-expectation framework

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Abstract. In this paper, we propose a class of discrete-time approximation schemes for stochastic optimal control problems under the $G$-expectation framework. The proposed schemes are constructed recursively based on piecewise constant policy. We prove the convergence of the discrete schemes and determine the convergence rates. Several numerical examples are presented to illustrate the effectiveness of the obtained results.

Key words. stochastic optimal control, $G$-expectation, discrete scheme, convergence rate

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1 Introduction

Recently, Peng [33–35] systematically introduced the notion of $G$-expectation. Under the $G$-expectation framework, a new kind of Brownian motion, called $G$-Brownian motion, was constructed. The related stochastic calculus of Itô’s type have been established. For a recent account and development of $G$-expectation theory and its applications, we refer the reader to [21, 22, 37, 38, 41] and the references therein.

It is well known that stochastic optimal control theory provides a useful tool for mathematical financial models. However, in some utility models with ambiguity volatility, it cannot be done within a probability space framework (see, e.g., [11, 12]). The $G$-expectation framework does not require a probability space and is convenient to study financial problems involving volatility uncertainty. Motivated by this, for each fixed $T > 0$, we consider a stochastic controlled system driven by $G$-Brownian motion in the following form:

$$\begin{cases} dX_s = b(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dB_s + h(s, X_s, \alpha_s)d\langle B \rangle_s, \quad s \in (t, T], \\ X_t = x. \end{cases}$$

The value function of the optimal control problem is defined by

$$v(t, x) = \sup_{\alpha \in A[t, T]} \mathbb{E}^{\alpha}_{t, x} \left[ g(X_T) + \int_t^T f(s, X_s, \alpha_s)ds \right],$$

for $(t, x) \in [0, T] \times \mathbb{R}^m$. Under some standard conditions, Hu et al. [22] proved that the value function is

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the viscosity solution of the following fully nonlinear Hamilton–Jacobi–Bellman (HJB) equation:

\[
\begin{aligned}
\partial_t v + \sup_{a \in A} F \left( t, x, D_x v, D_x^2 v, a \right) &= 0, \quad (t, x) \in [0, T) \times \mathbb{R}^m, \\
v (T, x) &= g (x),
\end{aligned}
\]

where

\[
\begin{aligned}
F \left( t, x, D_x v, D_x^2 v, a \right) &= G \left( H \left( t, x, D_x v, D_x^2 v, a \right) \right) + \langle b (t, x, a), D_x v \rangle + f (t, x, a), \\
H_{ij} \left( t, x, D_x v, D_x^2 v, a \right) &= \langle D_x^2 v \sigma_i (t, x, a), \sigma_j (t, x, a) \rangle + 2 \langle D_x v, h_{ij} (t, x, a) \rangle.
\end{aligned}
\]

The basic framework for convergence of numerical schemes to viscosity solutions of HJB equations was established by Barles and Souganidis [6]. In particular, the provable order of convergence for the second-order HJB equations was first obtained by Krylov in [28–30], and one of the main ideas is a method named by himself “shaking the coefficients” combined with a mollification procedure. This technique was further developed by Barles and Jakobsen to apply to general monotone approximation schemes (see [3–5] and the references therein). Since then, many results have been done on the numerical schemes for optimal control problems and HJB equations (see, e.g., [7–9, 16, 20, 24, 36] and the references therein). We also mention some important works for solving stochastic optimal control problems by means of forward and backward stochastic differential equations (see, e.g., [2, 14, 15, 18, 19] and the references therein). However, little attention has been paid to the discrete schemes for stochastic optimal control problems under the G-framework (G-SOCPs).

Owing to the sublinear nature of G-expectation, there is no density representation for the G-normal distribution, which leads to the failure of the classical Markov chain approximation (see, e.g., [29, 32]). Note that Peng [35] established the central limit theorem in the G-expectation framework (G-CLT), which provides a theoretical foundation for approximating G-normal distributed random variables. In particular, an outstanding contribution to the convergence rate of G-CLT was by Krylov [31] using stochastic control method under different model assumptions. Let us also note other important works on this topic. In [25], Huang and Liang studied the convergence rate for a more general G-CLT via a monotone approximation scheme and obtained an explicit bound of Berry-Esseen type. Fang et al. [17] and Song [40] proved the convergence rate of G-CLT by using Stein’s method.

In this paper, we devote ourselves to designing a discrete-time approximation scheme for G-SOCPs. With the help of piecewise constant policy, Euler time-stepping, and G-CLT, we propose a general discrete-time approximation scheme for solving G-SOCPs. By choosing the parameter set, two different kinds of numerical schemes are derived. We also establish the convergence rates for the numerical schemes by using the “shaking the coefficients” method. The result shows that our discrete scheme has a 1/6 order rate in the general case and a better rate of order 1/4 in the special case. We remark that a similar convergence result was acquired in [25], but with different problem settings and discrete schemes. Several numerical examples are demonstrated to illustrate the obtained theoretical findings.

The rest of the paper is organized as follows. In Section 2, we review some basic notations and results on the G-expectation theory. We propose the discrete-time approximation scheme for G-SOCPs in Section 3. The convergence rate of the proposed scheme is proved in Section 4. Numerical examples are given in Section 5.
2 Preliminaries

This section briefly introduces some notions and preliminaries in the G-framework. For more details, we refer the reader to [33–35] and the references therein.

Definition 2.1 Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$, satisfies $c \in \mathcal{H}$ for each constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables. A functional $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$ is called a sublinear expectation: if for all $X,Y \in \mathcal{H}$, it satisfies the following properties:

(i) Monotonicity: If $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$;
(ii) Constant preservation: $\hat{E}[c] = c$;
(iii) Sub-additivity: $\hat{E}[X+Y] \leq \hat{E}[X] + \hat{E}[Y]$;
(iv) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for each $\lambda > 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2 Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{b,Lip}(\mathbb{R}^n)$, the space of bounded Lipschitz continuous functions on $\mathbb{R}^n$.

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1,\ldots,Y_n), Y_i \in \mathcal{H}$, is said to be independent from another random vector $X = (X_1,\ldots,X_m), X_i \in \mathcal{H}$ under $\hat{E}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{E}[\varphi(X,Y)] = \hat{E}[\hat{E}[\varphi(x,Y)]_{x=X}]$.

Definition 2.4 A $d$-dimensional random vector $X = (X_1,\ldots,X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-normal distributed if for each $a,b \geq 0$ we have

$$aX + b\tilde{X} = \sqrt{a^2 + b^2}X,$$

where $\tilde{X}$ is an independent copy of $X$, $G : \mathcal{S}(d) \rightarrow \mathbb{R}$ denotes the function

$$G(A) := \frac{1}{2} \hat{E}[\langle AX, X \rangle], \quad A \in \mathcal{S}(d),$$

where $\mathcal{S}(d)$ denotes the collection of $d \times d$ symmetric matrices.

Definition 2.5 Let $\Omega_T = C_0([0,T];\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous paths on $[0,T]$ with $\omega_0 = 0$, endowed with the supremum norm, and $B_t(\omega) = \omega_t$ be the canonical process. Set

$$\text{Lip}(\Omega_T) = \{ \varphi(B_{t_1},\ldots,B_{t_n}) : n \geq 1, t_1,\ldots,t_n \in [0,T], \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n}) \},$$
where $C_{b,\text{Lip}}(\mathbb{R}^{d\times n})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{d\times n}$. $G$-expectation on $(\Omega_T, \text{Lip}(\Omega_T))$ is a sublinear expectation defined by

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[\varphi(t_1, \ldots, t_n)],$$

for all $X = \varphi(B_t^1 - B_{t_0}, \ldots, B_t^n - B_{t_n})$, where $\xi_1, \ldots, \xi_n$ are identically distributed $d$-dimensional $G$-normal distributed random vectors in a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{E}})$ such that $\xi_{i+1}$ is independent from $(\xi_1, \ldots, \xi_i), i = 1, \ldots, n - 1$. The corresponding canonical process $B_t = (B_t^i)_{i=1}^d$ is called a $G$-Brownian motion and $(\Omega_T, \text{Lip}(\Omega_T), \hat{\mathbb{E}})$ is called a $G$-expectation space.

**Definition 2.6** Assume that $X \in \text{Lip}(\Omega_T)$ has the representation $X = \varphi(B_t^1 - B_{t_0}, \ldots, B_t^n - B_{t_n})$. The conditional $G$-expectation $\hat{\mathbb{E}}_t$ of $X$ is defined by, for some $1 \leq i \leq n$,

$$\hat{\mathbb{E}}_t[\varphi(B_t^i - B_{t_0}, \ldots, B_t^n - B_{t_n})] = \hat{\varphi}(B_t^i - B_{t_0}, \ldots, B_t^n - B_{t_n}),$$

where

$$\hat{\varphi}(x_1, \ldots, x_i) = \hat{\mathbb{E}}_i[\varphi(x_1, \ldots, x_i, B_{t+1} - B_t, \ldots, B_{t+n} - B_{t+n})].$$

For each given $p \geq 1$, define $\|X\|_{L^p_G} = (\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $X \in \text{Lip}(\Omega_T)$, and denote by $L^p_G(\Omega_T)$ the completion of $\text{Lip}(\Omega_T)$ under the norm $\|\cdot\|_{L^p_G}$. Then for $t \in [0, T]$, $\hat{\mathbb{E}}_t[\cdot]$ can be extended continuously to the completion $L^1_G(\Omega_T)$ of $\text{Lip}(\Omega_T)$ under the norm $\|\cdot\|_{L^1_G}$.

**Theorem 2.7** ([10], [23]) There exists a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X],$$

for all $X \in L^1_G(\Omega_T)$.

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

In the sequel, let $(B_t)_{t \geq 0}$ be a one dimensional $G$-Brownian motion with $G(a) = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-)$, where $\hat{\mathbb{E}}[B_t^2] = \sigma^2$, $-\hat{\mathbb{E}}[-B_t^2] = \sigma^2$, $0 \leq \sigma \leq \sigma < \infty$. Denis et al. [10] gave the following concrete set $\mathcal{P}_M$ that represents $\hat{\mathbb{E}}$:

$$\mathcal{P}_M := \left\{ P^\theta : P^\theta = P \circ (\int_0^t \theta_s dW_s)^{-1}, \theta_s \in \Sigma \right\},$$

where $W.$ is a one dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = \{ \mathcal{F}_t \}_{t \geq 0}$, and $\Sigma$ is the collection of all $[\sigma, \bar{\sigma}]$-valued $\mathbb{F}$-adapted processes $(\theta_s)_{0 \leq s \leq T}$.

Let $\{t_0, t_1, \ldots, t_N\}$ be a sequence of partitions of $[0, T]$ and set $M^0_G(0, T) = \{ \eta_t(\omega) = \sum_{n=0}^{N-1} \xi_n(\omega)I_{[t_n, t_{n+1})} : \xi_n \in \text{Lip}(\Omega_{t_n}) \}$. For each given $p \geq 1$, denote by $M^p_G(0, T)$ the completion of $M^0_G(0, T)$ with the norm $\|\cdot\|_{M^p_G} = (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p ds])^{1/p}$. Then, for any $\xi_t, \eta_t \in M^2_G(0, T)$ and $\eta_t \in M^1_G(0, T)$, the $G$-Itô integral $\int_0^T \xi_t dB_t$ and $\int_0^T \eta_t dB_t$ are well defined, see [33], [35] for more details.

**Proposition 2.8** For each $\xi_t \in M^2_G(0, T)$ and $\eta_t \in M^1_G(0, T)$, we have

(i) $\hat{\mathbb{E}} \left[ \int_0^T \xi_t dB_t \right] = 0$; $\hat{\mathbb{E}} \left( \left( \int_0^T \xi_t dB_t \right)^2 \right) = \hat{\mathbb{E}} \left[ \int_0^T \xi_t^2 dB_t \right]$;

(ii) $\hat{\mathbb{E}} \left[ \int_0^T \eta_t dB_t \right] \leq \sigma^2 \hat{\mathbb{E}} \left[ \int_0^T |\eta_t| dt \right]$.
We end this section with Krylov’s regularization results. Take a nonnegative \( \zeta \in C_0^\infty(\mathbb{R}^{m+1}) \) with unit integral and support in \( \{(t, x) : -1 < t < 0, |x| < 1\} \) and for \( \varepsilon \in (0, 1) \) let \( \zeta_\varepsilon(t, x) = \varepsilon^{-m-2}\zeta(t/\varepsilon^2, x/\varepsilon) \). For locally integrable \( u(t, x) \), we denote the mollification of \( u \) by

\[
u^{\varepsilon}(t, x) = u(t, x) \ast \zeta_\varepsilon(t, x).\]

Furthermore, if \( u \) is 1/2-Hölder continuity in time and Lipschitz continuity in space, then the standard properties of mollifiers (see Lemma 3.5 in [29] or C.4 in [13]) indicate that

\[
\|u - u^{\varepsilon}\|_\infty \leq C\varepsilon \quad \text{and} \quad \|\partial^l_t D^k_x u^{\varepsilon}\|_\infty \leq C\varepsilon^{1-2l-k} \quad \text{for} \quad k + l \geq 1.
\] (2.1)

3 Discrete-time approximation

In this section, we present a class of discrete-time approximation schemes for solving G-SOCPs.

3.1 Formulation of the problem

We first give the definition of admissible controls.

**Definition 3.1** For each \( t \in [0, T] \), \( \alpha \) is said to be an admissible control on \([t, T]\), if it satisfies the following conditions:

(i) \( \alpha : [t, T] \times \Omega \to A \), where \( A \) is a compact set of \( \mathbb{R}^q \);

(ii) \( \alpha \in M^2_G([t, T]; \mathbb{R}^q) \).

The set of admissible controls on \([t, T]\) is denoted by \( A[t, T] \).

Let \( t \in [0, T] \), \( x \in \mathbb{R}^m \), and \( \alpha \in A[t, T] \). Consider the following \( m \)-dimensional controlled system:

\[
\begin{align*}
dX^{\alpha, t, x} &= b(s, X^{\alpha, t, x}, \alpha_s)ds + \sigma(s, X^{\alpha, t, x}, \alpha_s)dB_s + h(s, X^{\alpha, t, x}, \alpha_s)d\langle B \rangle_s, \quad s \in (t, T], \\
X^{\alpha, t, x} &= x,
\end{align*}
\] (3.1)

and

\[
v(t, x) = \sup_{\alpha \in A[t, T]} \mathbb{E}^\alpha_{t, x} \left[ g(X_T) + \int_t^T f(s, X_s, \alpha_s)ds \right],
\] (3.2)

where \( B \) is a one dimensional G-Brownian motion defined in the G-expectation space \((\Omega_T, L^1_G(\Omega_T), \mathbb{E})\), \((B)\); is the quadratic variation process of the G-Brownian motion, \( b, \sigma, h \) are deterministic functions with values in \( \mathbb{R}^m \), and \( g, f \) are two \( \mathbb{R} \)-valued functions. The indices \( \alpha, t, x \) on the G-expectation \( \mathbb{E} \) indicate that the state process \( X \) depends on the starting point \((t, x)\) and the control \( \alpha \).

We need the following standard assumptions on the optimal control problem:

(H1) The maps \( b, \sigma, h : [0, \infty) \times \mathbb{R}^m \times A \to \mathbb{R}^m \) are continuous, and there exists a constant \( L > 0 \) such that, for any \( t, s \in [0, \infty) \), \( x, y \in \mathbb{R}^m \), and \( a \in A \),

\[
|\varphi(t, x, a) - \varphi(s, y, a)| \leq L \left( |t - s|^{1/2} + |x - y| \right) \quad \text{and} \quad |\varphi(t, a)| \leq L, \quad \text{for} \quad \varphi = b, \sigma, h.
\]
(H2) The maps $f : [0, \infty) \times \mathbb{R}^m \times A \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ are continuous, and there exists a constant $L > 0$ such that, for any $t, s \in [0, \infty)$, $x, y \in \mathbb{R}^m$, and $a \in A$,

$$|g(x) - g(y)| + |f(t, x, a) - f(s, y, a)| \leq L \left( |t - s|^{\frac{1}{2}} + |x - y| \right) \text{ and } |f(t, x, a)| \leq L.$$

Let $N \in \mathbb{N}$ and $\Delta = T/N > 0$. We introduce a time mesh $t_0 = n \Delta$ for $n = 0, 1, \ldots, N$. Let $\mathcal{A}_h[t, T]$ be the subset of $\mathcal{A}[0, T]$ consisting of all processes $\alpha$ which are constant in the intervals $[t_n, t_{n+1})$, $n = 0, 1, \ldots, N-1$. For simplicity, we will identify any $\alpha \in \mathcal{A}_h$ by the sequence of random variables $a_i$ taking values in $A$ and denote $\alpha = (a_0, a_1, \ldots, a_{N-1})$. For any $(t, x) \in [0, T] \times \mathbb{R}^m$, define the value function corresponding to the piecewise constant control set $\mathcal{A}_h[t, T]$

$$v_\Delta(t, x) = \sup_{\alpha \in \mathcal{A}_h[t, T]} \hat{E}_t^\alpha \left[ g(X_T) + \int_t^T f(s, X_s, \alpha_s)ds \right].$$

Since $\mathcal{A}_h[t, T] \subset \mathcal{A}[t, T]$, it is clear that $v(t, x) - v_\Delta(t, x) \geq 0$.

**Theorem 3.2** Assume that (H1)-(H2) hold. Then there exists a constant $C$ independent of $\Delta$ such that

$$v(t, x) - v_\Delta(t, x) \leq C\Delta^{\frac{1}{2}},$$

for any $(t, x) \in [0, T] \times \mathbb{R}^m$.

This theorem is proved in Appendix, and it is based on the “shaking the coefficients” method of Krylov in [29] and the improved result of Jakobsen et al. in [26].

### 3.2 Euler scheme

We now give an approximation of the process $X$ by the Euler method. In what follows, we shall denote $\mathcal{A}_h[t, T]$ as $\mathcal{A}_h$ without causing confusion. For any fixed $\alpha = (a_0, a_1, \ldots, a_{N-1}) \in \mathcal{A}_h$, we denote $\Delta B := B_{t_{n+1}} - B_{t_n}$ and $\Delta(B) := \langle B \rangle_{t_{n+1}} - \langle B \rangle_{t_n}$ and define the Euler approximation process $X^{a, t_0, x}$ recursively by

$$\bar{X}_{t_{n+1}} = \bar{X}_{t_n} + b(t_i, \bar{X}_{t_i}, a_i) \Delta + \sigma(t_i, \bar{X}_{t_i}, a_i) \Delta B + h(t_i, \bar{X}_{t_i}, a_i) \Delta(B),$$

for $i = n, \ldots, N - 1$, with $\bar{X}_{t_n} = x$. Under the assumptions (H1)-(H2), similar to the proof in [27], one can obtain that

$$\hat{E} \left[ \sup_{t \in [t_n, t_{n+1}]} |X^{a, t_n, x}_{t} - \bar{X}_{t_n}^{a, t_n, x}|^2 \right] \leq C\Delta,$$

for any $\alpha \in \mathcal{A}_h$, $n = 0, 1, \ldots, N - 1$, where $C$ is a positive constant independent of $\Delta$.

For any $n = 0, 1, \ldots, N - 1$, $x \in \mathbb{R}^m$, define

$$\bar{v}_\Delta(t_n, x) = \sup_{\alpha \in \mathcal{A}_h} \hat{E}_t^\alpha \left[ g(\bar{X}_T) + \sum_{i=n}^{N-1} f(t_i, \bar{X}_{t_i}, a_i) \Delta \right].$$

Furthermore, $\bar{v}_\Delta$ satisfies the following dynamic programming principle (DPP):

$$\bar{v}_\Delta(t, x) = g(x),$$

$$\bar{v}_\Delta(t_n, x) = \sup_{a \in A} \hat{E}_t^a \left[ \bar{v}_\Delta(t_{n+1}, \bar{X}_{t_{n+1}}) + f(t_n, x, a) \Delta \right],$$

(3.6)
for \( n = 0, \ldots, N - 1, x \in \mathbb{R}^m \). With the help of Theorem 3.2 and 3.4, it is easy to check that for any \( n = 0, \ldots, N - 1, x \in \mathbb{R}^m \),

\[
|v(t_n, x) - v_\Delta(t_n, x)| \leq |v(t_n, x) - v_\Delta(t_n, x)| + |v_\Delta(t_n, x) - v_\Delta(t_n, x)| \leq C \Delta^{\frac{1}{2}}.
\] (3.7)

Define \( B_1 := \{ x \in \mathbb{R}^m : |x| < 1 \} \) and \( B = A \times \{(\mu, \lambda) : \mu \in (-1,0), \lambda \in B_1 \} \). Extend \( b, \sigma, f \) for negative \( t \) following the example \( b(t, x, \alpha) = b(0, x, \alpha) \). For a fixed \( \varepsilon \in (0, 1] \) and any \( \beta = (\alpha, \mu, \lambda) \in B \) let

\[
b(t, x, \beta) = b_x(t, x, \beta) = b(t + \varepsilon^2 \mu, x + \varepsilon \lambda, \alpha),
\]

and similarly define \( \sigma(t, x, \beta) \) and \( f(t, x, \beta) \). We denote by \( B_h \) the set of \( B \)-valued progressively measurable processes which are constant in each time interval. Now we proceed with the regularization of the value function \( \bar{v}_\Delta \). Letting \( S = T + \varepsilon \) and \( N_x = \lfloor S/\Delta \rfloor \), for any \( x \in \mathbb{R}^m \), \( n = 0, \ldots, N_x \), define the following “shaken” value function:

\[
\bar{u}_\Delta(t_n, x) = \sup_{\beta \in B_h} \bar{E}_{t_n,x}^\beta \left[ g(\bar{X}_{t_n,x}) + \sum_{i=n}^{N_x} f(t_i, \bar{X}_{t_i}, \beta_i) \Delta \right].
\]

where \( \bar{X}^{\beta, t_n, x} \) is recursively defined by

\[
\bar{X}_{t+d} = \bar{X}_t + b(t, \bar{X}_t, \beta_t) \Delta + \sigma(t, \bar{X}_t, \beta_t) \Delta \bar{B} + h(t, \bar{X}_t, \beta_t) \Delta \langle \bar{B} \rangle,
\]

for \( i = n, \ldots, N_x - 1 \) with \( \bar{X}_{t_n} = x \). Moreover, the shaken value function satisfies the following DPP:

\[
\bar{u}_\Delta(t_n, x) = \sup_{\beta \in B} \bar{E}_{t_n,x}^\beta \left[ \bar{u}_\Delta(t_{n+1}, \bar{X}_{t_{n+1}}) + f(t_n, x, \beta) \Delta \right],
\] (3.8)

for \( n = 0, 1, \ldots, N_x - 1, x \in \mathbb{R}^m \).

It is convenient to extend \( \bar{v}_\Delta \) and \( \bar{u}_\Delta \) defined on the grid points to the whole of \([0, T]\) and \([0, S]\), respectively, keeping its values on \( \{t_n\}_{0 \leq n \leq N} \), and making it equal to the value at \( t_n \) on each interval \([t_n, t_{n+1}]\). The following standard result is obtained by referring to the proof of Lemma 2.2 of [31] and using the estimate of Proposition 5.3.1 of [35].

**Proposition 3.3** Assume that (H1)-(H2) hold. Then, there exists a constant \( C \geq 0 \) such that for any \( t \in [0, T] \), \( x \in \mathbb{R}^m \),

\[
|\bar{u}_\Delta(t, x) - \bar{v}_\Delta(t, x)| \leq C \varepsilon,
\]

and for any \( t, s \in [0, S] \), \( x, y \in \mathbb{R}^m \)

\[
|\bar{u}_\Delta(t, x) - \bar{u}_\Delta(s, y)| \leq C(|t - s|^{\frac{1}{2}} + \Delta^{\frac{1}{2}} + |x - y|).
\]

Then we give some results about the mollified function \( \bar{u}_\Delta^{(\varepsilon)} \).

**Proposition 3.4** Assume that (H1)-(H2) hold. Then,

(i) there exists a constant \( C \geq 0 \) such that for any \( t \in [0, S] \), \( x \in \mathbb{R}^m \),

\[
|\bar{u}_\Delta(t, x) - \bar{u}_\Delta^{(\varepsilon)}(t, x)| \leq C \varepsilon;
\]
(ii) the function $\bar{u}_\Delta^{(c)} \in C^0_0([0, T] \times \mathbb{R}^m)$ and

$$\|D^k \bar{u}_\Delta^{(c)}\|_\infty \leq C \varepsilon^{1-2l-k}, \text{ for } k + l \geq 1;$$

(iii) $\bar{u}_\Delta^{(c)}$ satisfies the following super-dynamic programming principle: for any $n = 0, \ldots, N-1, x \in \mathbb{R}^m$,

$$\bar{u}_\Delta^{(c)}(t_n, x) \geq \sup_{a \in A} \mathbb{E}_{t_n}^{\hat{a}} \left[ \bar{u}_\Delta^{(c)}(t_{n+1}, \bar{X}_{t_{n+1}}) + f(t_n, x, a)\Delta \right].$$

Proof. Properties (i)-(ii) are immediate from the properties of mollifiers (2.1). Next, we only prove (iii). In view of (3.8) and noting that $\bar{X}_{t_{n+1}}^{\varepsilon_1, t_n} = \bar{X}_{t_{n+1}}^{\varepsilon_1, t_n, x} - \varepsilon \lambda$, for each $\beta = (a, \mu, \lambda) \in B$, we obtain that for $n = 0, \ldots, N-1$,

$$\bar{u}_\Delta^{(c)}(t_n - \varepsilon^2 \mu, x - \varepsilon \lambda) \geq \sup_{a \in A} \mathbb{E}_{t_n}^{\hat{a}} \left[ \bar{u}_\Delta^{(c)}(t_{n+1} - \varepsilon^2 \mu, \bar{X}_{t_{n+1}} - \varepsilon \lambda) + f(t_n, x, a)\Delta \right].$$

Taking convolutions of both sides of the above equality with $\zeta$, from the convexity of $\sup_{a \in A} \mathbb{E}[-]$, we obtain the desired result. ■

3.3 Discrete approximation scheme

According to Theorem 2.7 we rewrite the Euler scheme (3.6) as

$$\check{v}_\Delta(T, x) = g(x),$$

$$\check{v}_\Delta(t_n, x) = \sup_{a \in A} \mathbb{E}^{a, \theta}_{t_n} \left[ \check{v}_\Delta(t_{n+1}, \check{X}_{t_{n+1}}) + f(t_n, x, a)\Delta \right],$$

for $n = N - 1, \ldots, 0$, where $\check{X}_{t_{n+1}}^{a, \theta, t_n, x}$ satisfies

$$\check{X}_{t_{n+1}} = \check{X}_{t_n} + b(t_n, \check{X}_{t_n}, a)\Delta + \sigma(t_n, \check{X}_{t_n}, a)\Delta B^\theta + h(t_n, \check{X}_{t_n}, a)\Delta (B)^\theta,$$

with $\Delta B^\theta := \int_{t_n}^{t_{n+1}} \theta ds$ and $\Delta (B)^\theta := \int_{t_n}^{t_{n+1}} \theta^2 ds$.

We now consider a general discrete-time approximation scheme to further calculate the value function $\check{v}_\Delta$. Let $\Theta$ be a collection of $\mathbb{R}$-valued random variables, given perhaps on different probability spaces such that $\mathbb{E}[\xi] = 0, \forall \xi \in \Theta$ and

$$\inf_{\xi \in \Theta} \mathbb{E}[\xi^2] = \sigma^2, \quad \sup_{\xi \in \Theta} \mathbb{E}[\xi^2] = \sigma^2.$$

Let $\{\xi_n\}_{n=0}^\infty \in \Theta$ be a sequence of i.i.d. $\mathbb{R}$-valued random variables. For any $\alpha = (a_0, \ldots, a_{N-1}) \in A_h$, we denote by $\check{X}_{t_{n+1}}^{\alpha, \xi, t_n, x}$ the discrete approximation of the process $\check{X}_{t_{n+1}}^{a, \theta, t_n, x}$ recursively defined by

$$\check{X}_{t_{i+1}} = \check{X}_{t_i} + b(t_i, \check{X}_{t_i}, a)\Delta + \sigma(t_i, \check{X}_{t_i}, a)\sqrt{\Delta} \xi_i + h(t_i, \check{X}_{t_i}, a)\Delta (\xi_i)^2,$$

for $i = n, \ldots, N-1$, with $\check{X}_{t_n} = x$. Then define the following discrete-time approximation scheme:

$$\check{v}_\Delta(T, x) = g(x),$$

$$\check{v}_\Delta(t_n, x) = \sup_{a \in A} \sup_{\xi \in \Theta} \mathbb{E}^{a, \xi}_{t_n} \left[ \check{v}_\Delta(t_{n+1}, \check{X}_{t_{n+1}}) + f(t_n, x, a)\Delta \right].$$

(3.10)
Moreover, we derive that for \( n = 0, 1, \ldots, N - 1 \) and \( x \in \mathbb{R}^m \),

\[
\hat{v}_\Delta(t_n, x) = \sup_{\alpha \in A_n} \sup_{\xi \in \Theta} \mathbb{E}^{\alpha, \xi}_{n,x} \left[ g(\hat{X}_T) + \sum_{i=n}^{N-1} f(t_i, \hat{X}_i, a_i) \Delta \right].
\]

**Remark 3.5** Indeed, the equation (3.10) is a general discrete scheme for the G-SOCP (3.1)-(3.2). By choosing the appropriate parameter set, we can derive different kinds of numerical schemes.

**Example 3.6** Let \( \Theta \) be a collection of \( \mathbb{R} \)-valued random variables, such that for any \( \xi \in \Theta \),

\[
P(\xi = p_i) = \omega_i^\sigma, \quad \text{for } \sigma \in \{\sigma, \bar{\sigma}\}, \quad i = 1, 2, 3,
\]

where

\[
\begin{aligned}
p_1 &= -1, \quad \omega_1^\sigma = \sigma^2/2; \\
p_2 &= 0, \quad \omega_2^\sigma = 1 - \sigma^2; \\
p_3 &= 1, \quad \omega_3^\sigma = \sigma^2/2.
\end{aligned}
\]

It is easy to verify that for any \( \xi \in \Theta \), \( k \in \mathbb{N}^+ \), \( \mathbb{E}[\xi^{2k-1}] = 0 \),

\[
\inf_{\xi \in \Theta} \mathbb{E}[\xi^{2k}] = \sigma^2, \quad \sup_{\xi \in \Theta} \mathbb{E}[\xi^{2k}] = \bar{\sigma}^2.
\]

From the scheme (3.10), we can define the following discrete approximation scheme

\[
\hat{v}_\Delta(T, x) = g(x),
\]

\[
\hat{v}_\Delta(t_n, x) = \sup_{\alpha \in A_n} \sup_{\sigma \in \{\sigma, \bar{\sigma}\}} \left[ \sum_{i=1}^{3} \omega_i^\sigma \hat{v}_\Delta(t_{n+1}, \zeta_{n+1}^\sigma,i) + f(t_n, x, a) \Delta \right], \tag{3.11}
\]

for \( n = N - 1, \ldots, 0 \), where

\[
\zeta_{n+1}^\sigma,i = x + b(t_n, x, a) \Delta + \sigma(t_n, x, a) \sqrt{\Delta} p_i + h(t_n, x, a) \Delta (p_i)^2.
\]

**Example 3.7** Let \( \Theta \) be a collection of \( \mathbb{R} \)-valued random variables, such that for given integer \( L \geq 2 \) and any \( \xi \in \Theta \),

\[
P(\xi = p_i^\sigma) = \omega_i, \quad \text{for } \sigma \in \{\sigma, \bar{\sigma}\}, \quad i = 1, \ldots, L,
\]

where \( \omega_i = \frac{A_{2L}}{\sqrt{\pi}}, \quad p_i^\sigma = \sigma \sqrt{2x_i} \), and \( \{(A_i, x_i)\}_{i=1}^{2L} \) are the weights and roots of the Gauss-Hermite quadrature rule (see e.g., [\( \Pi \) ]). Noting that the Gauss-Hermite quadrature rule is exact for any polynomial of degree \( 2L - 1 \), one can check that for any \( \xi \in \Theta \), \( k = 1, 2, \ldots, L \), \( \mathbb{E}[\xi^{2k-1}] = 0 \),

\[
\inf_{\xi \in \Theta} \mathbb{E}[\xi^{2k}] = \sigma^{2k}(2k-1)!!, \quad \sup_{\xi \in \Theta} \mathbb{E}[\xi^{2k}] = \bar{\sigma}^{2k}(2k-1)!!.
\]

By means of the scheme (3.10), we obtain the following discrete approximation scheme

\[
\hat{v}_\Delta(T, x) = g(x),
\]

\[
\hat{v}_\Delta(t_n, x) = \sup_{\alpha \in A_n} \sup_{\sigma \in \{\sigma, \bar{\sigma}\}} \left[ \sum_{i=1}^{L} \omega_i \hat{v}_\Delta(t_{n+1}, \zeta_{n+1}^{\sigma,i}) + f(t_n, x, a) \Delta \right], \tag{3.12}
\]

for \( n = N - 1, \ldots, 0 \), where

\[
\zeta_{n+1}^{\sigma,i} = x + b(t_n, x, a) \Delta + \sigma(t_n, x, a) \sqrt{\Delta} p_i^\sigma + h(t_n, x, a) \Delta (p_i^\sigma)^2.
\]
Consider “shaken” coefficients and use again the argument defined above. Following the definitions of \(b(t, x, \beta), \sigma(t, x, \beta),\) and \(f(t, x, \beta),\) we define the “shaken” value function as follows:

\[
\hat{u}_\Delta(t_n, x) = \sup_{\beta \in \mathcal{B}_x} \sup_{\xi \in \Theta} \mathbb{E}_{t_n, x}^{\beta, \xi} \left[ g(\hat{X}_{t_{N_s}}) + \sum_{i=n}^{N_s} f(t_i, \hat{X}_{t_i}, \beta_i) \Delta \right],
\]

where \(\hat{X}^{\beta, \xi, t_n, x}\) is recursively defined by

\[
\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + b(t_i, \hat{X}_{t_i}, \beta_i) \Delta + \sigma(t_i, \hat{X}_{t_i}, \beta_i) \sqrt{\Delta} \xi_i + h(t_i, \hat{X}_{t_i}, \beta_i) A(\xi_i),
\]

for \(i = n, \ldots, N_s - 1\) with \(\hat{X}_{t_n} = x.\)

We also extend \(\hat{v}_\Delta\) and \(\hat{u}_\Delta\) defined on the grid points to the whole of \([0, T]\) and \([0, S]\), respectively. In the similar way above, we give the following properties.

**Proposition 3.8** Assume that (H1)-(H2) hold. Then, there exists a constant \(C \geq 0\) such that for any \(t \in [0, T], x \in \mathbb{R}^m,\)

\[
|\hat{u}_\Delta(t, x) - \hat{v}_\Delta(t, x)| \leq C \varepsilon,
\]

and for any \(t, s \in [0, S], x, y \in \mathbb{R}^m\)

\[
|\hat{u}_\Delta(t, x) - \hat{u}_\Delta(s, y)| \leq C(|t - s|^2 + \Delta^2 + |x - y|).
\]

**Proposition 3.9** Assume that (H1)-(H2) hold. Then,

(i) there exists a constant \(C \geq 0\) such that for any \(t \in [0, S], x \in \mathbb{R}^m,\)

\[
|\hat{u}_\Delta(t, x) - \hat{u}^{(\varepsilon)}(t, x)| \leq C \varepsilon;
\]

(ii) the function \(\hat{u}^{(\varepsilon)}_\Delta \in C^\infty([0, S] \times \mathbb{R}^m)\) and

\[
\left\| \partial^l_t D^k_x \hat{u}^{(\varepsilon)}_\Delta \right\|_\infty \leq C \varepsilon^{1-2l-k}, \quad \text{for } k + l \geq 1;
\]

(iii) \(\hat{u}^{(\varepsilon)}_\Delta\) satisfies the following super-dynamic programming principle: for any \(n = 0, \ldots, N - 1, x \in \mathbb{R}^m,\)

\[
\hat{u}^{(\varepsilon)}_\Delta(t_n, x) \geq \sup_{a \in A} \mathbb{E}_{t_n, x}^{a, \xi} \left[ \hat{u}^{(\varepsilon)}_\Delta(t_{n+1}, \hat{X}_{t_{n+1}}) + f(t_n, x, a) \Delta \right].
\]

4 Convergence rate of the discrete approximation scheme

In this section, we derive an error bound for the convergence of the solution of our discrete approximation scheme (3.10). In the sequel, \(C\) represents a generic constant which does not depend on the time partition and may be different from line to line. We also need the following assumption:

(H3) For any \(\xi \in \Theta, \mathbb{E}[|\xi|^3] < \infty.\)

The main result in this paper is the following:
Theorem 4.1 Assume (H1)-(H3) hold. Let v and $\hat{v}_\Delta$ be the solution of (3.2) and (3.10), respectively. Then, (i) for any $x \in \mathbb{R}^m$, $n = 0, 1, \ldots, N$,

$$|v(t_n, x) - \hat{v}_\Delta(t_n, x)| \leq C\Delta^{\frac{1}{2}};$$

(ii) moreover, if $\mathbb{E}[\xi^3] = 0$ for any $\xi \in \Theta$ and $\sup_{\xi \in \Theta} \mathbb{E}[|\xi|^4] < \infty$, then for any $x \in \mathbb{R}^m$, $n = 0, 1, \ldots, N$,

$$|v(t_n, x) - \hat{v}_\Delta(t_n, x)| \leq C\Delta^{\frac{1}{2}}.$$  

**Proof.** Without loss of generality, we suppose that $m = 1$ and $h = 0$. (i) Step 1. Lower bound on $v - \hat{v}_\Delta$. From Proposition 3.4 (iii) and Theorem 2.7 we have for $n = 0, \ldots, N - 1$,

$$\bar{u}_\Delta^{(c)}(t_n, x) \geq \sup_{a \in A} \sup_{\theta \in [\bar{a}, \bar{\sigma}]} \mathbb{E}_{t_n, x}^{a, \theta} \left[ \bar{u}_\Delta^{(c)}(t_{n+1}, \bar{X}_{t_{n+1}}) + f(t_n, x, a)\Delta \right].$$

Denote

$$L^c_\Delta u(t, x) = \partial_t u(t, x) + b(t, x, a)\partial_x u(t, x) + \frac{1}{2}\lambda^2 \sigma^2(t, x, a)\partial_x^2 u(t, x).$$

Using Taylor's formula and Proposition 2.8, we have

$$\mathbb{E}_{t_n, x}^{a, \theta} \left[ \bar{u}_\Delta^{(c)}(t_{n+1}, \bar{X}_{t_{n+1}}) \right] = \bar{u}_\Delta^{(c)}(t_n, x) + L^c_\Delta \bar{u}_\Delta^{(c)}(t_n, x)\Delta + \bar{R}_\Delta^c,$$

where

$$\bar{R}_\Delta^c = \frac{1}{2}\Delta^2 \left[ \partial_t^2 \bar{u}_\Delta^{(c)}(t_n, x) + \partial_x^2 \bar{u}_\Delta^{(c)}(t_n, x) b^2(t_n, x, a) + 2\partial_x^2 \bar{u}_\Delta^{(c)}(t_n, x)b(t_n, x, a) \right]$$

$$+ \frac{1}{3!} \mathbb{E} \left[ (\Delta \partial_t + \Delta \bar{X} \partial_x)^3 \bar{u}_\Delta^{(c)}(t_n + \tau\Delta, x + \tau\Delta \bar{X}) \right],$$

with $\Delta \bar{X} := \bar{X}_{t_{n+1}}^{\bar{a}, \bar{t}, \bar{x}, t_{n+1}} - x$ and $\tau \in [0, 1]$. By Proposition 3.4 (ii), one can check that

$$|\bar{R}_\Delta^c| \leq C \sum_{i=0}^3 \varepsilon^{-2-i} \Delta^{\frac{i+3}{2}}.$$  

(4.3)

Together with (4.1), (4.2), and (4.3), we obtain that for $n = 0, \ldots, N - 1$,

$$\sup_{a \in A} \sup_{\theta \in [\bar{a}, \bar{\sigma}]} \mathbb{E}_{t_n, x}^{a, \theta} \left[ L^c_\Delta \bar{u}_\Delta^{(c)}(t_n, x) + f(t_n, x, a) \right] \leq C \sum_{i=0}^3 \varepsilon^{-2-i} \Delta^{\frac{i+1}{2}}.$$  

(4.4)

We claim that if $1 \leq k \leq N$ and $\eta := \sup_{x \in \mathbb{R}} [\bar{v}_\Delta(t_k, x) - \bar{u}_\Delta^{(c)}(t_k, x)]$, then for any $x \in \mathbb{R}$, $n \leq k$,

$$\bar{v}_\Delta(t_n, x) \leq \bar{u}_\Delta^{(c)}(t_n, x) + \eta + \bar{I}_\Delta(t_k - t_n).$$  

(4.5)

where

$$\bar{I}_\Delta = C \sum_{i=0}^3 \varepsilon^{-2-i} \Delta^{\frac{i+4}{2}}.$$  

Taking $k = N$ in (4.5) and using the following estimates that by Propositions 3.3 and 3.4

$$|\bar{u}_\Delta^{(c)}(T, x) - \bar{u}_\Delta(T, x)| + |\bar{u}_\Delta(T, x) - g(x)| \leq C(|S - T|^{1/2} + \varepsilon) \leq C\varepsilon,$$

$$|\bar{u}_\Delta^{(c)}(t_n, x) - \bar{u}_\Delta(t_n, x)| + |\bar{u}_\Delta(t_n, x) - \bar{v}_\Delta(t_n, x)| \leq C\varepsilon.$$
we can deduce that for any \( x \in \mathbb{R}, n = 0, \ldots, N, \)
\[
\hat{v}_\Delta(t_n, x) \leq \bar{v}_\Delta(t_n, x) + \bar{I}_{\Delta}(T - t_n) + C\varepsilon \leq \bar{v}_\Delta(t_n, x) + \bar{J}_{\Delta},
\]
where
\[
\bar{J}_{\Delta} = C(\varepsilon + \sum_{i=0}^{3} \varepsilon^{-2-i} \Delta^{\frac{14+3}{2}}).
\]

Now we prove the assertion \([4.5]\) by induction. For \( n = k, \) the inequality \([4.5]\) obviously holds. Assume that for \( 1 \leq n \leq k, \) the assertion \([4.5]\) holds. Then
\[
\hat{v}_\Delta(t_{n-1}, x) = \sup_{a \in A} \sup_{\xi \in \Theta} \mathbb{E}_{t_{n-1}, x}^{a, \xi} \left[ \hat{v}_\Delta(t_n, \hat{X}_{t_n}) + f(t_{n-1}, x, a) \Delta \right]
\]
\[
\leq \sup_{a \in A} \sup_{\xi \in \Theta} \mathbb{E}_{t_{n-1}, x}^{a, \xi} \left[ \bar{u}_\Delta^{(\varepsilon)}(t_n, \hat{X}_{t_n}) + f(t_{n-1}, x, a) \Delta \right] + \eta + \bar{I}_{\Delta}(t_k - t_{n-1}).
\]
By Taylor’s formula, for any \( \xi \in \Theta, \) we can deduce
\[
\mathbb{E}_{t_{n-1}, x}^{a, \xi} \left[ \bar{u}_\Delta^{(\varepsilon)}(t_n, \hat{X}_{t_n}) \right] = \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) + \mathbb{E}[L_\xi^{(\varepsilon)} \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) \Delta] + \bar{H}_\Delta,
\]
where
\[
\bar{H}_\Delta = \Delta^2 \left[ \partial_\xi^2 \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) b(t_{n-1}, x, a) \right]
\]
\[
+ \frac{1}{2} \Delta^2 \left[ \partial_\xi^2 \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) + \partial_\xi^2 \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) b^2(t_{n-1}, x, a) \right]
\]
\[
+ \frac{1}{3!} \mathbb{E} \left[ (\Delta \partial_t + \Delta \hat{X} \partial_x)^3 \bar{u}_\Delta^{(\varepsilon)}(t_{n-1} + \tau \Delta, x + \tau \Delta \hat{X}) \right],
\]
with \( \Delta \hat{X} := \hat{X}_{t_n}^{a, \xi, t_{n-1}, x} - x \) and \( \tau \in (0, 1). \) Under the assumption \((H3), \) from the regularity of \( \partial_\xi^k \hat{u}_\Delta^{(\varepsilon)} \) given in Proposition 3.4 (ii), it follows that
\[
|\bar{H}_\Delta| \leq C \sum_{i=0}^{3} \varepsilon^{-2-i} \Delta^{\frac{14+3}{2}}.
\]
Together with \([4.4], [4.7], \) and \([4.8], \) we can derive that
\[
\hat{v}_\Delta(t_{n-1}, x) \leq \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) + \eta + \bar{I}_{\Delta}(t_k - t_{n-1}) + 2|\bar{H}_\Delta|
\]
\[
\leq \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) + \eta + \bar{I}_{\Delta}(t_k - t_{n-1}),
\]
where we have used the fact that
\[
\sup_{\xi \in \Theta} \mathbb{E} \left[ L_\xi^{(\varepsilon)} \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) \right] = \sup_{\theta \in \mathbb{Z}, \varepsilon} \mathbb{E} \left[ L_\theta \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) \right].
\]
By the principle of induction the assertion is true for all \( 0 \leq n \leq k \) and \( x \in \mathbb{R}. \)

Step 2. Upper bound on \( v(t, x) - \hat{v}_\Delta(t, x), \) From Proposition 3.9 (iii), by a similar analysis as step 1, we obtain that for \( n = 0, \ldots, N - 1, \)
\[
\sup_{\theta \in \Xi} \mathbb{E} \left[ L_\theta \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) + f(t_{n-1}, x, a) \right] = \sup_{\xi \in \Theta} \mathbb{E} \left[ L_\xi \bar{u}_\Delta^{(\varepsilon)}(t_{n-1}, x) + f(t_{n-1}, x, a) \right] \leq C \sum_{i=0}^{3} \varepsilon^{-2-i} \Delta^{\frac{14+1}{2}}.
\]

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Similarly, by induction, it follows from (4.9) that if $1 \leq k \leq N$ and $\hat{\eta} := \sup_{x \in \mathbb{R}} |\bar{u}_\Delta(t_k, x) - \bar{u}_\Delta(t_k, x)|$, then for any $x \in \mathbb{R}$, $n \leq k$,

$$\bar{v}_\Delta(t_n, x) \leq \bar{v}_\Delta(t_n, x) + \hat{\eta} + \hat{I}_\Delta(t_k - t_n),$$

(4.10)

where $\hat{I}_\Delta = C \sum_{i=0}^{3} \varepsilon^{-2-i} \Delta^{\frac{i+1}{2}}$. Taking $k = N$ in (4.10), by Propositions 3.8, 3.9 one knows that

$$|\bar{u}_\Delta(T, x) - \bar{u}_\Delta(T, x)| + |\bar{u}_\Delta(T, x) - g(x)| \leq C|S - T|^{1/2} + \varepsilon \leq C\varepsilon,$$

$$|\bar{u}_\Delta(t_n, x) - \bar{u}_\Delta(t_n, x)| + |\bar{u}_\Delta(t_n, x) - \bar{u}_\Delta(t_n, x)| \leq C\varepsilon.$$

Thus, we have for any $n = 0, \ldots, N$, $x \in \mathbb{R}$,

$$\bar{v}_\Delta(t_n, x) \leq \bar{v}_\Delta(t_n, x) + C(\varepsilon + \sum_{i=0}^{3} \varepsilon^{-2-i} \Delta^{\frac{i+1}{2}}).$$

(4.11)

Step 3. Conclusion. Together with (4.6) and (4.11), by taking $\varepsilon = \Delta^{1/6}$, then we obtain that for any $n = 0, \ldots, N$, $x \in \mathbb{R}$,

$$|\bar{v}_\Delta(t_n, x) - \bar{v}_\Delta(t_n, x)| \leq C\Delta^{\frac{1}{6}};$$

(4.12)

Combining (3.7) and (4.12), we conclude (i).

(ii) We follow the proof of (i) and take more terms in Taylor’s formula. Then the remainder $R_\Delta$ in (4.2) takes the form

$$R_\Delta = \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n, x) + \partial^2 u_\Delta(t_n, x) \partial^2(t_n, x, a) + 2\partial^2 u_\Delta(t_n, x) b(t_n, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n, x) b(t_n, x, a) \sigma^2(t_n, x, a) \partial^3 u_\Delta(t_n, x) \sigma^2(t_n, x, a) \theta^2 \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n, x) b(t_n, x, a) \partial^3 u_\Delta(t_n, x) \sigma^2(t_n, x, a) \theta^2 \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n, x) \partial^3 u_\Delta(t_n, x) b^2(t_n, x, a) \right]$$

$$+ \frac{1}{6}\Delta \left[ \partial^2 u_\Delta(t_n, x) \partial^3 u_\Delta(t_n, x) b^3(t_n, x, a) \right]$$

$$+ \frac{1}{4!} \left[ (\Delta \partial_t + \Delta \tilde{X}, \partial_x) \partial^4 u_\Delta(t_n + \tau \Delta, x + \tau \Delta \tilde{X}) \right],$$

with $\Delta \tilde{X} = \tilde{X}_{t_n+1}^{a, b, t_n, x} - x$ and $\tau \in [0, 1]$. In addition, take again more terms in Taylor’s formula, then the remainder $H_\Delta$ in (4.8) can be expressed as

$$H_\Delta = \Delta \left[ \partial^2 u_\Delta(t_n-1, x) b(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) + \partial^2 u_\Delta(t_n-1, x) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) \partial^2(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) b(t_n-1, x, a) \partial^3 u_\Delta(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) b(t_n-1, x, a) \partial^3 u_\Delta(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) \partial^3 u_\Delta(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) \partial^3 u_\Delta(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) \partial^3 u_\Delta(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ \partial^2 u_\Delta(t_n-1, x) \partial^3 u_\Delta(t_n-1, x, a) \right]$$

$$+ \frac{1}{2}\Delta \left[ (\Delta \partial_t + \Delta \tilde{X}, \partial_x) \partial^4 u_\Delta(t_n + \tau \Delta, x + \tau \Delta \tilde{X}) \right],$$

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where $\Delta \hat{X} = \hat{X}_{t_{n}}^{n,\xi_{n-1},x} - x$, $\tau \in (0,1)$. Seeing that $E[\xi^3] = 0$, $\forall \xi \in \Theta$ and sup $E[|\xi|^4] < \infty$, by Proposition 3.4 (ii), we deduce that $|\hat{H}_\Delta| \leq C \sum_{i=1}^{5} \varepsilon^{-2-i} \Delta^{\frac{i}{2}}$. This indicates that for any $n = 0, \ldots, N$, $x \in \mathbb{R}$,

$$\hat{v}_\Delta(t_n, x) \leq \hat{v}_\Delta(t_n, x) + C(\varepsilon + \sum_{i=1}^{5} \varepsilon^{-2-i} \Delta^{\frac{i+1}{2}}).$$

Analogously, we can also obtain the upper bound, for any $n = 0, \ldots, N$, $x \in \mathbb{R}$,

$$\hat{v}_\Delta(t_n, x) \leq \hat{v}_\Delta(t_n, x) + C(\varepsilon + \sum_{i=1}^{5} \varepsilon^{-2-i} \Delta^{\frac{i+1}{2}}).$$

Taking $\varepsilon = \Delta^{1/4}$, by (3.7), we conclude the proof. ■

**Remark 4.2** According to Theorem 4.1, we know that the discrete schemes (3.11) and (3.12) admit a $1/4$ order rate of convergence.

# 5 Numerical example

In this section, we present several numerical simulations to illustrate the obtained theoretical results. In the following, we apply the scheme (3.11) and scheme (3.12) with $L = 6$ to test the examples. In our tables, we shall denote by CR the convergence rate, TR the Scheme (3.11), and GH the Scheme (3.12), respectively.

**Example 5.1** We first consider the following G-heat equation

$$\begin{cases}
\partial_t v + G(\partial_{xx}^2 v) = 0, & (t, x) \in (0, T) \times \mathbb{R}, \\
v(T, x) = g(x),
\end{cases}$$

where

$$g(x) = \begin{cases}
\frac{2}{1+\beta} \cos\left(\frac{1+\beta}{2} x\right), & x \in \left[-\frac{\pi}{1+\beta} + 2k\pi, \frac{\pi}{1+\beta} + 2k\pi\right), \\
\frac{2}{1+\beta} \cos\left(\frac{1+\beta}{2} x + \frac{\pi-1}{2\beta}\right), & x \in \left[\frac{\pi}{1+\beta} + 2k\pi, \frac{(2\beta+1)\pi}{1+\beta} + 2k\pi\right),
\end{cases},$$

$k \in \mathbb{Z}$, with $\beta = \frac{\sigma}{2}$. Set $T = 1$, $x_0 = 0$, $\sigma = 0.1$ and $\tau = 1$. The exact solution of the G-heat equation given in [39] is $u(t, x) = e^{-\frac{\xi^2(1-\xi)}{2}} g(x)$, where $\rho = \frac{\pi_{1,\beta}}{2}$. We solve this example by the scheme (3.11) and scheme (3.12), and the errors and convergence rates are listed in Table 1. In our test, the value function and data are often more regular than the assumption, which leads to a higher convergence rate than the theoretical result.

### Table 1: Errors and convergence rates for Example 5.1

| Scheme | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | CR |
|--------|----------|----------|----------|----------|----------|----|
| TR     | 3.217E-03| 1.684E-03| 8.693E-04| 4.405E-04| 2.226E-04| 0.964|
| GH     | 2.605E-03| 1.331E-03| 6.663E-04| 3.336E-04| 1.665E-04| 0.993|
Table 2: Errors and convergence rates for Example 5.2

| N     | 16    | 32    | 64    | 128   | 256   | CR   |
|-------|-------|-------|-------|-------|-------|------|
| $|v(0, x_0) - v_\Delta(0, x_0)|$ | 1.599E-02 | 8.016E-03 | 4.011E-03 | 2.005E-03 | 1.000E-03 | 1.000 |

Example 5.2 We now test a fully nonlinear HJB equation

$$\begin{cases}
\partial_t v + \sup_{a \in A} \left[ G(a^2 D_x^2 v) + (\kappa x - a) D_x v + 2\sqrt{\alpha_t} e^{-r_0 t} \right] = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\
v(T, x) = x,
\end{cases}$$

where the control set $A = [\frac{1}{5}, 1]$. This can be interpreted as the value function of a control problem as follows

$$v(t, x) = \sup_{a \in A} E_{t,x}^\alpha [X_T + \int_t^T 2\sqrt{\alpha_t} e^{-r_0 s} \, ds],$$

with

$$\begin{cases}
dX_s = (\kappa X_s - \alpha_s) \, ds + \alpha_s \, dB_s, & s \in (t, T], \\
X_t = x,
\end{cases}$$

(5.1)

Then, the optimal control is $\alpha_t^* = e^{2(\kappa - r_0) - 2\kappa T}$ and the corresponding value function is

$$v^*(t, x) = e^{\kappa(T-t)} x + \frac{e^{-\kappa T}}{r_0 - \kappa} \left[ e^{(\kappa - 2r_0)T} - e^{(\kappa - 2r_0)t} \right].$$

We test this example by the scheme (3.11). The errors and convergence rates for $T = 1$, $x_0 = 0$, $\sigma = 0.5$, $\bar{\sigma} = 1$, $\kappa = 0.5$, and $r_0 = 0.03$ are shown in Table 2.

Example 5.3 Consider the following fully nonlinear HJB equation

$$\begin{cases}
\partial_t v + \sup_{a \in A} \left[ G(\sin^2(t + x) D_x^2 v) + (2a \sin^2(t + x) - 1) D_x v + 2\cos^2(t + x) - \cos^4(t + x) - a^2 \right] = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\
v(T, x) = x,
\end{cases}$$

where the control set $A = [0, 1]$. One can express the solution of the above equation as the value function of the following control problem

$$v(t, x) = \sup_{a \in A} E_{t,x}^\alpha [X_T + \int_t^T (2\cos^2(s + X_s) - \cos^4(s + X_s) - \alpha_s^2) \, ds],$$

with

$$\begin{cases}
dX_s = (2\alpha_s \sin^2(s + X_s) - 1) \, ds + \sin^2(s + X_s) \, dB_s, & s \in (t, T], \\
X_t = x,
\end{cases}$$

where $B_1 \sim N(0, [\sigma^2, \bar{\sigma}^2])$. It can be checked that the optimal control is $\alpha_t^* = \sin^2(t + X_t)$ and the corresponding value function is $v^*(t, x) = x$. The numerical results by using the scheme (3.12) are reported in Table 3 with $T = 1$, $x_0 = 0$, $\sigma = 0.5$, and $\bar{\sigma} = 1$. It can be seen that the numerical simulation is consistent with our theoretical results.
6 Appendix

Proof of Theorem 3.2. Let $\hat{X}$ be the solution of (3.1) with coefficients replaced by $b^{(\varepsilon)}, \sigma^{(\varepsilon)}$ and $h^{(\varepsilon)}$, and $\hat{v}$ be the value function of the optimal control problems (3.1)-(3.2) with $X, f, g$ replaced by $\hat{X}, f^{(\varepsilon)}, g^{(\varepsilon)}$ satisfying the estimate

$$|v(t, x) - \hat{v}(t, x)| \leq C\varepsilon, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^m. \quad (6.1)$$

For simplicity, we still use $(b, \sigma, f, g)$ instead of $(b^{(\varepsilon)}, \sigma^{(\varepsilon)}, f^{(\varepsilon)}, g^{(\varepsilon)})$. By Propositions 2.3-2.4 in [26], one can find a family of smooth functions $\omega_{\varepsilon}(t, x) \in C_0^\infty([0, T] \times \mathbb{R}^m)$ with $\varepsilon \in (0, 1)$, such that for any $t, s \in [0, T]$ and $x, y \in \mathbb{R}^m$,

$$|\omega_{\varepsilon}(t, x) - \omega_{\varepsilon}(s, y)| \leq C(\varepsilon + |t-s|^\frac{1}{2} + |x-y|), \quad (6.2)$$

and

$$\|\omega_{\varepsilon} - \hat{v}\|_{\infty} \leq C\varepsilon \quad \text{and} \quad \|\partial_t^k D_x^l \omega_{\varepsilon}\|_{\infty} \leq C\varepsilon^{1-2l-k}, \quad \text{for } k + l \geq 1, \quad (6.3)$$

where $C$ is a positive constant independent of $\Delta$. Moreover, one can obtain that, for any $(t, x) \in [0, T - \Delta] \times \mathbb{R}^m, a \in A$,

$$\omega_{\varepsilon}(t, x) \geq \hat{E}_{t,x}^a \left[ \omega_{\varepsilon}(t + \Delta, \hat{X}_{t+\Delta}) + \int_t^{t+\Delta} f(s, \hat{X}_s, a)ds \right]. \quad (6.4)$$

Step 1. Upper bound on $L^0_\alpha \omega_{\varepsilon} + 2G(L^1_\alpha \omega_{\varepsilon}) + f$. For any $(t, x) \in [0, T - \Delta] \times \mathbb{R}^m, a \in A$, applying G-Itô’s formula to $\omega_{\varepsilon}$ (twice) and $f$, we have

$$\omega_{\varepsilon}(t + \Delta, \hat{X}_{t+\Delta}) + \int_t^{t+\Delta} f(s, \hat{X}_s, a)ds$$

$$= \omega_{\varepsilon}(t, x) + L^0_\alpha \omega_{\varepsilon}(t, x)\Delta + \int_t^{t+\Delta} L^1_\alpha \omega_{\varepsilon}(t, x)d(B) + f(t, x, a)\Delta$$

$$+ \int_t^{t+\Delta} \int_t^s L^0_\alpha L^0_\alpha \omega_{\varepsilon}(r, \hat{X}_r)drd(B) + \int_t^{t+\Delta} \int_t^s L^1_\alpha L^0_\alpha \omega_{\varepsilon}(r, \hat{X}_r)d(B) + \int_t^{t+\Delta} \int_t^s L^1_\alpha L^1_\alpha \omega_{\varepsilon}(r, \hat{X}_r)d(B), \quad (6.5)$$

where

$$L^0_\alpha u(t, x) = \partial_t u(t, x) + \sum_{i=1}^m b(t, x, a)\partial_{x_i} u(t, x),$$

$$L^1_\alpha u(t, x) = \sum_{i=1}^m h_i(t, x, a)\partial_{x_i} u(t, x) + \frac{1}{2} \sum_{i,j=1}^m [\sigma\sigma^T]_{i,j} (t, x, a)\partial^2_{x_i x_j} u(t, x).$$

Table 3: Errors and convergence rates for Example 5.3

| $N$ | 16   | 32   | 64   | 128  | 256  | CR  |
|-----|------|------|------|------|------|-----|
| $|v(0, x_0) - \hat{v}_\Delta(0, x_0)|$ | 2.598E-05 | 2.303E-05 | 1.465E-05 | 4.165E-06 | 1.058E-06 | 1.170 |
Noting that \( \hat{\mathbb{E}}[L^0_\alpha \omega_{\varepsilon}(t,x)\Delta(B)] = 2G(L^0_\alpha \omega_{\varepsilon}(t,x))\Delta \), by inserting (6.5) into (6.4), we deduce that for any \( a \in A, (t,x) \in [0,T-\Delta] \times \mathbb{R}^m \),
\[
L^0_\alpha \omega_{\varepsilon}(t,x) + 2G(L^0_\alpha \omega_{\varepsilon}(t,x)) + f(t,x,a) \leq C\varepsilon^{-3} \Delta, \tag{6.6}
\]
where we have used the bound of the term \( L^i_\alpha f, L^i_\alpha L^j_\alpha \omega_{\varepsilon}, i,j = 1,2 \).

Step 2. Upper bound on \( \bar{v}(t,x) - v_\Delta(t,x) \). For any \( (t,x) \in [0,T-\Delta] \times \mathbb{R}^m, \alpha \in A \), by Itô’s formula, we have
\[
\hat{\mathbb{E}}_{t,x}^\alpha [\omega_{\varepsilon}(T-\Delta, \hat{X}_{T-\Delta})] = \omega_{\varepsilon}(t,x) + \hat{\mathbb{E}}_{t,x}^\alpha \left[ \int_t^{T-\Delta} L^0_\alpha \omega_{\varepsilon}(s, \hat{X}_s)ds + \int_t^{T-\Delta} L^1_\alpha \omega_{\varepsilon}(s, \hat{X}_s)d\langle B \rangle_s \right].
\]
In view of Corollary 3.5.8 in Peng [35], we obtain that, for each \( \eta \in M^1_\mu(0,T) \),
\[
\int_t^{T-\Delta} \eta_s d\langle B \rangle_s \leq 2 \int_t^{T-\Delta} G(\eta_s)ds. \tag{6.7}
\]
Combining with (6.3), (6.6), and (6.7), from the definition of \( G(\cdot) \), we can deduce
\[
\hat{\mathbb{E}}_{t,x}^\alpha [\omega_{\varepsilon}(T-\Delta, \hat{X}_{T-\Delta})] \leq \omega_{\varepsilon}(t,x) + \hat{\mathbb{E}}_{t,x}^\alpha \left[ \int_t^{T-\Delta} \{ L^0_\alpha \omega_{\varepsilon}(s, \hat{X}_s) + 2G(L^0_\alpha \omega_{\varepsilon}(s, \hat{X}_s)) \} ds \right]
\leq v_\Delta(t,x) - \hat{\mathbb{E}}_{t,x}^\alpha \left[ \int_t^{T} f(s, \hat{X}_s, \alpha_s)ds \right] + C(\varepsilon + \Delta^{3/2} + \varepsilon^{-3}\Delta),
\]
Then, by (6.2), we have
\[
\hat{\mathbb{E}}_{t,x}^\alpha [g(X_T)] \leq \hat{\mathbb{E}}_{t,x}^\alpha [\omega_{\varepsilon}(T-\Delta, \hat{X}_{T-\Delta})] + \hat{\mathbb{E}}_{t,x}^\alpha [\omega_{\varepsilon}(T, X_T) - \omega_{\varepsilon}(T-\Delta, \hat{X}_{T-\Delta})]
\leq v_\Delta(t,x) - \hat{\mathbb{E}}_{t,x}^\alpha \left[ \int_t^{T} f(s, \hat{X}_s, \alpha_s)ds \right] + C(\varepsilon + \Delta^{3/2} + \varepsilon^{-3}\Delta),
\]
which implies
\[
\bar{v}(t,x) \leq v_\Delta(t,x) + C(\varepsilon + \Delta^{3/2} + \varepsilon^{-3}\Delta).
\]

For any \( (t,x) \in [T-\Delta,T] \times \mathbb{R}^m, \alpha \in A[t,T] \),
\[
\bar{v}(t,x) - g(x) \leq \hat{\mathbb{E}}_{t,x}^\alpha [|X_T - x|] + C\Delta \leq C\Delta^{3/2},
\]
and similarly, we have \( v_\Delta(t,x) - g(x) \leq C\Delta^{3/2} \). Thus
\[
\bar{v}(t,x) - v_\Delta(t,x) \leq \bar{v}(t,x) - g(x) - (v_\Delta(t,x) - g(x)) \leq C(\varepsilon + \Delta^{3/2}).
\]
To sum up, for any \( (t,x) \in [0,T] \times \mathbb{R}^m \), we get
\[
\bar{v}(t,x) - v_\Delta(t,x) \leq C(\varepsilon + \Delta^{3/2} + \varepsilon^{-3}\Delta). \tag{6.8}
\]

Step 3. Together with (6.1) and (6.8), we have for any \( (t,x) \in [0,T] \times \mathbb{R}^m \),
\[
v(t,x) - v_\Delta(t,x) \leq C(\varepsilon + \Delta^{1/2} + \varepsilon^{-3}\Delta).
\]
By taking \( \varepsilon = \Delta^{1/4} \), the desired result follows.
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