HIGHER RANK BRILL-NOETHER THEORY ON SECTIONS OF $K3$ SURFACES

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ABSTRACT. We discuss the role of $K3$ surfaces in the context of Mercat’s conjecture in higher rank Brill-Noether theory. Using liftings of Koszul classes, we show that Mercat’s conjecture in rank 2 fails for any number of sections and for any gonality stratum along a Noether-Lefschetz divisor inside the locus of curves lying on $K3$ surfaces. Then we show that Mercat’s conjecture in rank 3 fails even for curves lying on $K3$ surfaces with Picard number 1. Finally, we provide a detailed proof of Mercat’s conjecture in rank 2 for general curves of genus 11, and describe explicitly the action of the Fourier-Mukai involution on the moduli space of curves.

1. INTRODUCTION

The Clifford index $\text{Cliff}(C)$ of an algebraic curve $C$ is the second most important invariant of $C$ after the genus, measuring the complexity of the curve in its moduli space. Its geometric significance is amply illustrated for instance in the statement $K_{p,2}(C, K_C) = 0 \iff p < \text{Cliff}(C)$ of Green’s Conjecture [G] on syzygies of canonical curves. It has been a long-standing problem to find an adequate generalization of $\text{Cliff}(C)$ for higher rank vector bundles. A definition in this sense has been proposed by Lange and Newstead [LN1]: If $E \in \mathcal{U}_C(n, d)$ denotes a semistable vector bundle of rank $n$ and degree $d$ on a curve $C$ of genus $g$, one defines its Clifford index as

$$\gamma(E) := \mu(E) - \frac{2}{n} h^0(C, E) + 2 \geq 0,$$

and then the higher Clifford indices of $C$ are defined as the quantities

$$\text{Cliff}_n(C) := \min \{ \gamma(E) : E \in \mathcal{U}_C(n, d), \ d \leq n(g - 1), \ h^0(C, E) \geq 2n \}.$$

Note that $\text{Cliff}_1(C) = \text{Cliff}(C)$ is the classical Clifford index of $C$. By specializing to sums of line bundles, it is easy to check that $\text{Cliff}_n(C) \leq \text{Cliff}(C)$ for all $n \geq 1$. Mercat [Me] proposed the following interesting conjecture, which we state in the form of [LN1] Conjecture 9.3, linking the newly-defined invariants $\text{Cliff}_n(C)$ to the classical geometry of $C$:

$$(M_n) : \quad \text{Cliff}_n(C) = \text{Cliff}(C).$$

Mercat’s conjecture $(M_2)$ holds for various classes of curves, in particular general $k$-gonal curves of genus $g > 4k - 4$, or arbitrary smooth plane curves, see [LN1]. In [FO] Theorem 1.7, we have verified $(M_2)$ for a general curve $[C] \in \mathcal{M}_g$ with $g \leq 16$. More generally, the statement $(M_2)$ is a consequence of the Maximal Rank Conjecture (see [FO] Conjecture 2.2), therefore it is expected to be true for a general curve $[C] \in \mathcal{M}_g$.

1The invariant $\text{Cliff}_n(C)$ is denoted in the paper [LN1] by $\gamma'_n(C)$. Since the appearance of [LN1], it has become abundantly clear that $\text{Cliff}_n(C)$, defined as above, is the most relevant Clifford type invariant for rank $n$ vector bundles on $C$. Accordingly, the notation $\text{Cliff}_n(C)$ seems appropriate.
on the set of Noether-Lefschetz divisors and in particular it (i) fixes the can be identified with the Noether-Lefschetz divisor on the moduli space of polarized $K$ with Mercat’s conjecture, the action of the Fourier-Mukai involution lies on a unique we study in detail the case $g$ specific to (i) rank $2$ and $4$ is, Mercat’s conjecture holds generically on genus $2$ G.FARKAS AND A. ORTEGA carry rank $2$ vector bundles $E$ with a prescribed (and exceptionally high) number of sections invalidating Mercat’s inequality $\gamma(E) \geq \text{Cliff}(C)$:

**Theorem 1.1.** We fix integers $p \geq 1$ and $a \geq 2p + 3$. There exists a smooth curve $C$ of genus $2a + 1$ and Clifford index $\text{Cliff}(C) = a$, lying on a $K3$ surface $C \subset S \subset \mathbb{P}^{2p+2}$ with $\text{Pic}(S) = \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H$, where $H^2 = 4p + 2$, $H \cdot C = \text{deg}(C) = 2a + 2p + 1$, as well as a stable rank $2$ vector bundle $E \in \text{SU}(2, \mathcal{O}_C(H))$, such that $h^0(C, E) = p + 3$. In particular $\gamma(E) = a - \frac{1}{2} < \text{Cliff}(C)$ and Mercat’s conjecture (M2) fails for $C$.

It is well-known cf. [MV2], [V1], that a curve $[C] \in M_{2a+1}$ lying on a $K3$ surface $S$ possesses a rank $2$ vector bundle $E \in \text{SU}(2, K_C)$ with $h^0(C, F) = a + 2$. In particular, $\gamma(F) = a \geq \text{Cliff}(C)$ (with equality if $\text{Pic}(S) = \mathbb{Z} \cdot C$), hence such bundles satisfy condition (M2). Let us consider the $K3$ locus in the moduli space of curves $K_g := \{[C] \in M_g : C \text{ lies on a } K3 \text{ surface}\}$.

When $g = 11$ or $g \geq 13$, the variety $K_g$ is irreducible and $\text{dim}(K_g) = 19 + g$, see [CLM], Theorem 5. For integers $r, d \geq 1$ such that $d^2 > 4(r - 1)g$ and $2r - 2 \not| d$, we define the Noether-Lefschetz divisor inside the locus of sections of $K3$ surfaces

$$\mathfrak{M}^r_{g,d} := \{[C] \in K_g \mid C \text{ lies on a } K3 \text{ surface } S, \text{ Pic}(S) \supset \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H, H \in \text{Pic}(S) \text{ is nef, } H^2 = 2r - 2, \ C \cdot H = d, \ C^2 = 2g - 2 \}.$$  

A consequence of Theorem 1.1 can be formulated as follows:

**Corollary 1.2.** We fix integers $p \geq 1$ and $a \geq 2p + 3$ and set $g := 2a + 1$. Then Mercat’s conjecture (M2) fails generically along the Noether-Lefschetz locus $\mathfrak{M}^2_{g,2a+2p+1}$ inside $K_g$, that is, $\text{Cliff}_2(C) < \text{Cliff}(C)$ for a general point $[C] \in \mathfrak{M}^2_{g,2a+2p+1}$.

It is wonderful to know that it is necessary to pass to a Noether-Lefschetz divisor in $K_g$, or perhaps, all curves $[C] \in K_g$ give counterexamples to conjecture (M2). To see that this is not always the case and all conditions in Theorem 1.1 are necessary, we study in detail the case $g = 11$. Mukai [M3] proved that a general curve $[C] \in M_{11}$ lies on a unique $K3$ surface $S$ with $\text{Pic}(S) = \mathbb{Z} \cdot C$, thus, $M_{11} = K_{11}$.

**Theorem 1.3.** For a general curve $[C] \in M_{11}$ one has the equality $\text{Cliff}_2(C) = \text{Cliff}(C)$, that is, Mercat’s conjecture holds generically on $M_{11}$. Furthermore, the locus

$$\{[C] \in M_{11} : \text{Cliff}_2(C) < \text{Cliff}(C)\}$$

can be identified with the Noether-Lefschetz divisor $\mathfrak{M}^1_{11,13}$ on $M_{11}$.

In Section 5, we describe in detail the divisor $\mathfrak{M}^1_{11,13}$ and discuss, in connection with Mercat’s conjecture, the action of the Fourier-Mukai involution $FM : \mathcal{F}_{11} \to \mathcal{F}_{11}$ on the moduli space of polarized $K3$ surfaces of genus $11$. The automorphism $FM$ acts on the set of Noether-Lefschetz divisors and in particular it (i) fixes the $6$-gonal locus.
and maps the divisor \( \Omega^4_{11,13} \) which corresponds to certain elliptic \( K3 \) surfaces, to the Noether-Lefschetz divisor corresponding to \( K3 \) surfaces carrying a rational curve of degree 3.

Next we turn our attention to the conjecture \((M_n)\) for \( n \geq 3 \). It was observed in [LMN] that Mukai’s description [M4] of a general curve of genus 9 in terms of linear sections of a certain rational homogeneous variety, and especially the connection to rank 3 Brill-Noether theory, can be used to construct, on a general curve \([C] \in M_9\), a stable vector bundle \( E \in SU_3(3, K_C) \) such that \( h^0(C, E) = 6 \). In particular \( \gamma(E) = \frac{10}{3} < \text{Cliff}(C) \), that is, Mercat’s conjecture \((M_3)\) fails for a general curve \([C] \in M_9\). A similar construction is provided in [LMN] for a general curve of genus 11. In what follows we outline a construction illustrating that the results from [LMN] are part of a larger picture and curves on \( K3 \) follow as a corollary, we note that for sufficiently high genus Mercat’s statement \((M_3)\) fails. In particular, Mercat’s conjecture \((M_3)\) fails generically along \( K_g \).

**Theorem 1.4.** For a curve \( C \subset S \) and \( A \in W^g_3(C) \) as above there exists a globally generated vector bundle \( E \) on \( C \) with rank \( E = r + 1 \) and \( \text{det}(E) = K_C \), expressible as an extension

\[
0 \longrightarrow M_A \longrightarrow H^0(C, A) \otimes O_C \xrightarrow{ev} A \longrightarrow 0.
\]

As usual, we set \( Q_A := M_A^2 \), hence \( \text{rank}(Q_A) = r \) and \( \text{det}(Q_A) = A \). Following a procedure that already appeared in [L], [M2], [V1], we note that \( C \) carries a vector bundle of rank \( r + 1 \) with canonical determinant and unexpectedly many global sections:

**Theorem 1.5.** Fix \( C \subset S \) as above with \( g = 7, 9 \) or \( g \geq 11 \) such that \( \text{Pic}(S) = \mathbb{Z} \cdot C \), as well as \( A \in W^d_3(C) \), where \( d := \frac{2g+1}{3} \). Then any globally generated rank 3 vector bundle \( E \) on \( C \) lying non-trivially in the extension

\[
0 \longrightarrow Q_A \longrightarrow E \longrightarrow K_C \otimes A^\vee \longrightarrow 0,
\]

and with \( h^0(C, E) = h^0(C, A) + h^0(C, K_C \otimes A^\vee) = g - d + 5 \), is stable.

As a corollary, we note that for sufficiently high genus Mercat’s statement \((M_3)\) fails to hold for any smooth curve of maximal Clifford index lying on a \( K3 \) surface.

**Corollary 1.6.** We fix an integer \( g = 9 \) or \( g \geq 11 \) and a curve \([C] \in K_g\). Then the inequality \( \text{Cliff}_{3}(C) < \frac{g-1}{2} \) holds. In particular, Mercat’s conjecture \((M_3)\) fails generically along \( K_g \).
We close the Introduction by thanking Herbert Lange and Peter Newstead for making a number of very pertinent comments on the first version of this paper.

2. HIGHER RANK VECTOR BUNDLES WITH CANONICAL DETERMINANT

In this section we treat Mercat’s conjecture \((M_3)\) and prove Theorems \([1.4\) and \([1.5\). We begin with a curve \(C\) of genus \(g\) lying on a smooth \(K3\) surface \(S\) such that \(\text{Pic}(S) = \mathbb{Z} \cdot C\), and fix a linear series \(A \in \mathcal{W}_d^2(C)\) of minimal degree \(d := \left[\frac{2g+8}{4}\right]\). Under such assumptions both \(A\) and \(K_C \otimes A^\vee\) are base point free. From the onset, we point out that the existence of vector bundles of higher rank \(C\) having exceptional Brill-Noether behaviour has been repeatedly used in \([L]\), \([M2]\) and \([V1]\). Our aim is to study these bundles from the point of view of Mercat’s conjecture and discuss their stability.

We define the Lazarsfeld-Mukai sheaf \(\mathcal{F}_A\) via the following exact sequence on \(S\):

\[
0 \rightarrow \mathcal{F}_A \rightarrow H^0(C, A) \otimes \mathcal{O}_S \xrightarrow{e_{\mathcal{F}_A}} A \rightarrow 0.
\]

Since \(A\) is base point free, \(\mathcal{F}_A\) is locally free. We consider the vector bundle \(\mathcal{E}_A := \mathcal{F}_A^\vee\) on \(S\), which by dualizing, sits in an exact sequence

\[
0 \rightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \rightarrow \mathcal{E}_A \rightarrow K_C \otimes A^\vee \rightarrow 0.
\]

Since \(K_C \otimes A^\vee\) is assumed to be base point free, the bundle \(\mathcal{E}_A\) is globally generated. It is well-known (and follows from the sequence \([1]\), that \(c_1(\mathcal{E}_A) = \mathcal{O}_S(C)\) and \(c_2(\mathcal{E}_A) = d\).

**Proof of Theorem 1.4** We write down the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(C, A) \otimes \mathcal{O}_S(-C) \\
\downarrow & & \downarrow \\
\mathcal{F}_A & \rightarrow & H^0(C, A) \otimes \mathcal{O}_S \\
\downarrow & & \downarrow \\
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

from which, if we set \(F_A := \mathcal{F}_A \otimes \mathcal{O}_C\) and \(E_A := \mathcal{E}_A \otimes \mathcal{O}_C\), we obtain the exact sequence

\[
0 \rightarrow M_A \otimes K_C^\vee \rightarrow H^0(C, A) \otimes K_C^\vee \rightarrow F_A \rightarrow M_A \rightarrow 0
\]

(\(\text{use that } \text{Tor}_{\mathcal{O}_S}^1(M_A, \mathcal{O}_C) = M_A \otimes K_C^\vee\)). Taking duals, we find the exact sequence

\[
0 \rightarrow Q_A \rightarrow E_A \rightarrow K_C \otimes A^\vee \rightarrow 0.
\]

Since \(S\) is regular, from \([1]\) we obtain that

\[
h^0(S, \mathcal{E}_A) = h^0(C, A) + h^0(C, K_C \otimes A^\vee)
\]

while \(h^0(S, \mathcal{E}_A \otimes \mathcal{O}_S(-C)) = 0\), that is,

\[
h^0(S, \mathcal{E}_A) \leq h^0(C, E_A) \leq h^0(C, A) + h^0(C, K_C \otimes A^\vee).
\]

Thus the sequence \([2]\) is exact on global sections.

We are left with proving that the extension \([2]\) is non-trivial. We set \(r = 2\) and then \(\text{rank}(\mathcal{E}_A) = 3\) and place ourselves in the situation when \(\text{Pic}(S) = \mathbb{Z} \cdot C\) (the case \(r = 1\) works similarly). By contradiction we assume that \(E_A = Q_A \oplus (K_C \otimes A^\vee)\) and...
denote by \( s : E_A \to Q_A \) a retract and by \( \tilde{s} : \mathcal{E}_A \to Q_A \) the induced map. We set \( \mathcal{M} := \text{Ker}\{\mathcal{E}_A \to Q_A\} \), hence \( \mathcal{M} \) can be regarded as an elementary transformation of the Lazarsfeld-Mukai bundle \( \mathcal{E}_A \) along \( C \). By direct calculation we find that
\[
c_1(\mathcal{M}) = \mathcal{O}_S(-C) \quad \text{and} \quad c_2(\mathcal{M}) = 2d - 2g + 2,
\]
hence the discriminant of \( \mathcal{M} \) equals
\[
\Delta(\mathcal{M}) := 6c_2(\mathcal{M}) - 2c_1^2(\mathcal{M}) = 4(3d - 4g + 4) < 0.
\]

Thus the sheaf \( \mathcal{M} \) is \( \mathcal{O}_S(C) \)-unstable. Applying [HL] Theorems 7.3.3 and 7.3.4, there exists a subsheaf \( \mathcal{M}' \subset \mathcal{M} \) such that if \( \xi_{\mathcal{M},\mathcal{M}'} := \frac{c_1(\mathcal{M}')}{\text{rank}(\mathcal{M}')} - \frac{c_1(\mathcal{M})}{\text{rank}(\mathcal{M})} \in \text{Pic}(S)_{\mathbb{R}} \), then
\[
(i) \quad \xi_{\mathcal{M},\mathcal{M}'} \cdot C > 0 \quad \text{and} \quad (ii) \quad \xi_{\mathcal{M},\mathcal{M}'}^2 \geq -\frac{\Delta(\mathcal{M})}{18}.
\]

Since \( \text{Pic}(S) = \mathbb{Z} \cdot C \), we may write \( c_1(\mathcal{M}') = \mathcal{O}_S(aC) \) and also set \( r' := \text{rank}(\mathcal{M}') \). The Lazarsfeld-Mukai bundle \( \mathcal{E}_A \) is \( \mathcal{O}_S(C) \)-stable, in particular \( \mu_C(\mathcal{M}') \leq \mu_C(\mathcal{E}_A) \), which yields \( a \leq 0 \). Then from \( (i) \) we write that \( 0 \leq \frac{2}{9} + \frac{1}{3} \leq \frac{1}{3} \), whereas from \( (ii) \) one finds
\[
\frac{1}{9} \geq \frac{4(g - 1) - 3d}{9(g - 1)} \iff d \geq g - 1,
\]
which is a contradiction. It follows that the extension (2) is non-trivial. \( \square \)

It is natural to ask when is the above constructed bundle \( E_A \) stable. We give an affirmative answer under certain generality assumptions, when \( r < 3 \).

We fix a \( K3 \) surface \( S \) such that \( \text{Pic}(S) = \mathbb{Z} \cdot C \) and as before, set \( d := \lceil \frac{2g + 8}{3} \rceil \). Under these assumptions, it follows from [L] that \( C \) satisfies the Brill-Noether theorem. We prove the stability of every globally generated non-split bundle \( E \) sitting in an extension of the form (2) and having a maximal number of sections.

**Proof of Theorem 1.5** We first discuss the possibility of a destabilizing sequence
\[
0 \to F \to E \to B \to 0,
\]
where \( F \) is a vector bundle of rank 2 and \( \text{deg}(F) \geq \frac{4}{3}(g - 1) \). Since \( E \) is globally generated, it follows that \( B \) is globally generated as well, hence \( h^0(C, B) \geq 2 \), in particular \( \text{deg}(B) \geq \frac{(g + 2)}{2} \) and hence \( \text{deg}(F) \leq \frac{2}{3}g - 3 \). Since \( \text{deg}(B) \leq \frac{2}{3}(g - 1) \) and \( C \) is Brill-Noether general, it follows that \( h^0(C, B) = 2 \), therefore \( h^0(C, F) \geq g - d + 3 \). There are two cases to distinguish, depending on whether \( F \) possesses a subpencil or not.

Assume first that \( F \) has no subpencils. We apply [PR] Lemma 3.9 to find that
\[
h^0(C, \text{det}(F)) \geq 2h^0(C, F) - 3 \geq 2g - 2d + 3.
\]
Writing down the inequality
\[
\rho(g, 2g - 2d + 2, \text{deg}(F)) \geq 0
\]
and using that \( \text{deg}(F) < \frac{2}{3}g - 3 \), we obtain a contradiction. If on the other hand, \( F \) has a subpencil, then as pointed out in [FO] Lemma 3.2, \( \gamma(F) \geq \text{Cliff}(C) \), but again this is a contradiction. This shows that \( E \) cannot have a rank 2 destabilizing subsheaf.

We are left with the possibility of a destabilizing short exact sequence
\[
0 \to B \to E \to F \to 0,
\]
where \( B \) is a line bundle with \( \text{deg}(B) \geq \frac{2}{3}(g - 1) \) and \( F \) is a rank 2 bundle. The bundle \( Q_A \) is well-known to be stable and based on slope considerations, \( B \) cannot be a subbundle of \( Q_A \), that is, necessarily \( H^0(C, K_C \otimes A^\vee \otimes B^\vee) \neq 0 \). Since the bundle \( E \) is not decomposable, it follows that \( \text{deg}(B) \leq \text{deg}(K_C \otimes A^\vee) - 1 = 2g - 3 - d \). Furthermore \( h^1(C, B) \geq 3 \).
If \( F \) is not stable, we reason along the lines of \([LMN]\) Proposition 3.5 and pull-back a destabilizing line subbundle of \( F \) to obtain a rank 2 subbundle \( F' \subset E \) such that

\[
\deg(F') \geq \deg(B) + \frac{1}{2} \left( \deg(E) - \deg(B) \right) \geq \frac{4}{3}(g - 1),
\]

which is the case which we have already ruled out. So we may assume that \( F \) is stable. We write \( h^0(C, B) = a + 1 \), hence \( h^0(C, F) \geq g - d - a + 4 \). Assume first that \( F \) admits no subpencils. Then from \([PR]\) Lemma 3.9 we find the following estimate for the number of sections of the line bundle \( \det(F) = K_C \otimes B^\vee \),

\[
h^0(C, K_C \otimes B^\vee) \geq 2h^0(C, F) - 3 \geq 2g - 2d - 2a + 5,
\]

which, after applying Riemann-Roch to \( B \), leads to the inequality

\[
3a \geq g - 2d + 5 + \deg(B).
\]

Combining this estimate with the Brill-Noether inequality \( \rho(g, a, \deg(B)) \geq 0 \) and substituting the actual value of \( d \), we find that \( 3a + 3 \geq g \). On the other hand \( a \leq h^0(C, K_C \otimes A^\vee) = 2 = g - d < \frac{g - 3}{3} \), and this is a contradiction.

Finally, if \( F \) admits a subpencil, then \( \gamma(F) \geq \text{Cliff}(C) \). Combining this with the classical Clifford inequality for \( B \), we find that \( \gamma(E) \geq \text{Cliff}(C) \), which again is a contradiction. We conclude that the rank 3 bundle \( E \) must be stable. \( \square \)

3. Rank 2 Bundles and Koszul Classes

The aim of this section is to prove Theorem \([L1]\). We shall construct rank 2 vector bundles on curves using a connection between vector bundles on curves and Koszul cohomology of line bundles, cf. \([AN]\) and \([V2]\). Let us recall that for a smooth projective variety \( X \), a sheaf \( F \) and a globally generated line bundle \( L \) on \( X \), the Koszul cohomology group \( K_{p,q}(X; F, L) \) is defined as the cohomology of the complex:

\[
\bigwedge^{p+1} H^0(L) \otimes H^0(F \otimes L^{q-1}) \xrightarrow{d_{p+1,q-1}} \bigwedge^p H^0(L) \otimes H^0(F \otimes L^q) \xrightarrow{d_{p,q}} \bigwedge^p H^0(L) \otimes H^0(F \otimes L^{q+1}).
\]

Most of the time \( F = O_X \), and then one writes \( K_{p,q}(X; O_X, L) := K_{p,q}(X, L) \).

A Koszul class \( [\zeta] \in K_{p,1}(X, L) \) is said to have rank \( \leq n \), if there exists a subspace \( W \subset H^0(X, L) \) with \( \dim(W) = n \) and a representative \( \zeta \in \bigwedge^p W \otimes H^0(X, L) \). The smallest number \( n \) with this property is the rank of the syzygy \( [\zeta] \).

Next we discuss a connection due to Voisin \([V2]\) and expanded in \([AN]\), between rank 2 vector bundles on curves and syzygies. Let \( E \) be a rank 2 bundle on a smooth curve \( C \) with \( h^0(C, E) \geq p + 3 \geq 4 \) and set \( L := \det(E) \). Let

\[
\lambda : \bigwedge^2 H^0(C, E) \to H^0(C, L)
\]

be the determinant map, and we assume that there exists linearly independent sections \( e_1 \in H^0(C, E) \) and \( e_2, \ldots, e_{p+3} \in H^0(C, E) \), such that the map

\[
\lambda((e_1 \land -) : (e_2, \ldots, e_{p+3}) \to H^0(C, L)
\]

in injective onto its image. Such an assumption is automatically satisfied for instance if \( E \) admits no subpencils. We introduce the subspace

\[
W := \langle e_2 := \lambda(e_1 \land e_2), \ldots, e_{p+3} := \lambda(e_1 \land e_{p+3}) \rangle \subset H^0(C, L).
\]
By assumption, $\dim(W) = p + 2$. Following [AN] and [V2], we define the tensor
$$\zeta(E) := \sum_{i<j} (-1)^{i+j} s_2 \wedge \ldots \wedge s_i \wedge \ldots \wedge s_j \wedge \ldots \wedge s_{p+3} \otimes \lambda(e_i \wedge e_j) \in \wedge^p W \otimes H^0(C, L)$.

One checks that $d_{p,1}(\zeta(E)) = 0$, hence $[\zeta(E)] \in K_{p,1}(C, L)$ is a non-trivial Koszul class of rank at most $p + 2$. Conversely, starting with a non-trivial class $[\zeta] \in K_{p,1}(C, L)$ represented by an element $\zeta$ of $\wedge^p W \otimes H^0(C, L)$ where $\dim(W) = p + 2$, Aprodu and Nagel [AN] Theorem 3.4 constructed a rank 2 vector bundle $E$ on $C$ with $h^0(C, E) \geq p + 3$ and such that $[\zeta(E)] = [\zeta]$. This correspondence sets up a dictionary between the Brill-Noether loci in $\{ E \in SU_C(2, L) : h^0(C, E) \geq p + 3 \}$ and Koszul classes of rank at most $p + 2$ in $K_{p,1}(C, L)$.

Let us now fix integers $p \geq 1$ and $a \geq 2p + 3$. Using the surjectivity of the period mapping, see e.g. [K] Theorem 1.1, one can construct a smooth $K3$ surface $S \subset \mathbb{P}^{2p+2}$ of degree $4p + 2$ containing a smooth curve $C \subset S$ of degree $d := 2a + 2p + 1$ and genus $g := 2a + 1$. The surface $S$ can be chosen with $\text{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$, where $H^2 = 4p + 2$, $H \cdot C = d$ and $C^2 = 4a$. The smooth curve $H \subset C$ is the hyperplane section of $S$ and has genus $g(H) = 2p + 2$. The following observation is trivial:

**Lemma 3.1.** Keeping the notation above, we have that $H^0(S, \mathcal{O}_S(H - C)) = 0$.

**Proof.** It is enough to notice that $H$ is nef and $(H - C) \cdot H = 2p - 2a + 1 < 0$. □

We consider the decomposable rank 2 bundle $K_H = A \oplus (K_H \otimes A^\vee)$ on $H$, where $A \in W_{p+2}^+(H)$. Via the Green-Lazarsfeld non-vanishing theorem [GL1] (or equivalently, applying [AN]), one obtains a non-zero Koszul class of rank $p + 1$
$$\beta := \left[ \zeta(A \oplus (K_H \otimes A^\vee)) \right] \in K_{p,1}(H, K_H).$$

Since $S$ is a regular surface, there exist an exact sequence
$$0 \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow H^0(S, \mathcal{O}_S(H)) \longrightarrow H^0(H, K_H) \longrightarrow 0,$$

which induces an isomorphism [G] Theorem (3.b.7)
$$\text{res}_H : K_{p,1}(S, \mathcal{O}_S(H)) \cong K_{p,1}(H, K_H).$$

By construction, the non-trivial class $\alpha := \text{res}_H^{-1}(\beta) \in K_{p,1}(S, \mathcal{O}_S(H))$ has rank at most $\text{rank}(\beta) + 1 = p + 2$. Using [G] Theorem (3.b.1), we write the following exact sequence in Koszul cohomology:
$$\cdots \longrightarrow K_{p,1}(S; -C, H) \rightarrow K_{p,1}(S, H) \rightarrow K_{p,1}(C, H \otimes \mathcal{O}_C) \rightarrow K_{p-1,2}(S; -C, H) \rightarrow \cdots.$$

Since $H^0(S, \mathcal{O}_S(H - C)) = 0$, it follows that $K_{p,1}(S; -C, H) = 0$, in particular the non-zero class $\alpha \in K_{p,1}(S, H)$ can be viewed as a Koszul class of rank at most $p + 2$ inside the group $K_{p,1}(C, \mathcal{O}_C(H))$. This class corresponds to a stable rank 2 bundle on $C$.

**Proposition 3.2.** Let $C \subset S \subset \mathbb{P}^{2p+2}$ as above and let $L := \mathcal{O}_C(1) \in \text{Pic}^{2a+2p+1}(C)$. Then there exists a stable vector bundle $E \in SU_C(2, L)$ with $h^0(C, E) = p + 3$.

**Proof.** From [AN] we know that there exists a rank 2 vector bundle $E$ on $C$ with $\text{det}(E) = L$ such that $[\zeta(E)] = \alpha \in K_{p,1}(C, L)$, in particular $h^0(C, E) \geq p + 3$. Geometrically, $E$ is the restriction to $C$ of the Lazarsfeld-Mukai bundle $E_A$ on $S$ corresponding to a pencil $A \in W_{p+2}^+(H)$. In particular, $E$ is globally generated, being the restriction of a globally generated bundle on $S$. We also know that $\text{Cliff}(C) = a$ (to be proved in Proposition
Theorem 3 in [F], with the difference that we must also consider curves with self-intersection \( D^2 \geq 2 \) because \( E \) is globally generated. It is easily verified that both \( B \) and \( L \otimes B^\vee \) contribute to \( \text{Cliff}(C) \), which brings about a contradiction. Assume now that

\[
0 \rightarrow B \rightarrow E \rightarrow L \otimes B^\vee \rightarrow 0
\]

is a destabilizing sequence, where \( B \in \text{Pic}(C) \) has degree at least \( a + p + 1 \). As already pointed out, \( h^0(C, B) \leq 1 \), hence \( h^0(C, L \otimes B^\vee) \geq 2 + p \). If \( h^1(C, L \otimes B^\vee) \leq 1 \), then \( p + 2 \leq h^0(C, L \otimes B^\vee) \leq 1 + \deg(L \otimes B^\vee) - 2a \), which leads to a contradiction. If on the other hand \( h^1(C, L \otimes B^\vee) \geq 2 \), then \( \text{Cliff}(L \otimes B^\vee) \leq a - p - 2 < a \), which is impossible. Thus \( E \) is a stable vector bundle. \( \square \)

We are left with showing that the curve \( C \subset S \) constructed above has maximal Clifford index \( a \). Note that the corresponding statement when \( p = 1 \) has been proved in [FO] Theorem 3.6.

**Proposition 3.3.** We fix integers \( p \geq 1, a \geq 2p + 3 \) and a K3 surface \( S \) with Picard lattice \( \text{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C \) where \( C^2 = 4a, H^2 = 4p + 2 \) and \( C \cdot H = 2a + 2p + 1 \). Then \( \text{Cliff}(C) = a \).

**Proof.** First note that \( C \) has Clifford dimension 1, for curves \( C \subset S \) of higher Clifford dimension have even genus. Observe also that \( h^0(C, \mathcal{O}_C(1)) = 2p + 3 \) and \( h^1(C, \mathcal{O}_C(1)) = 2 \), hence \( \mathcal{O}_C(1) \) contributes to the Clifford index of \( C \) and

\[
\text{Cliff}(C) \leq \text{Cliff}(C, \mathcal{O}(1)) = C \cdot H - 2(2p + 2) = 2a - 2p - 3 \geq a.
\]

Assume by contradiction that \( \text{Cliff}(C) < a \). According to [GL2], there exists an effective divisor \( D \equiv mH + nC \) on \( S \) satisfying the conditions

\[
(3) \quad h^0(S, \mathcal{O}_S(D)) \geq 2, \quad h^0(S, \mathcal{O}_S(C - D)) \geq 2, \quad C \cdot D \leq g - 1,
\]

and with \( \text{Cliff}(\mathcal{O}_C(D)) = \text{Cliff}(C) \). By [Ma] Lemma 2.2, the dimension \( h^0(C', \mathcal{O}_{C'}(D)) \) stays constant for all smooth curves \( C' \in \{ C \} \) and its value equals \( h^0(S, D) \). We conclude that \( \text{Cliff}(C) = \text{Cliff}(\mathcal{O}_C(D)) = D \cdot C - 2 \dim |D| \). We summarize the numerical consequences of the inequalities (3):\n
\[
(i) \quad md + 2n(g - 1) \leq g - 1 \\
(ii) \quad (2p + 1)m^2 + mnd + n^2(g - 1) \geq 0 \\
(iii) \quad (4p + 2)m + dn > 2,
\]

We claim that for any divisor \( D \subset S \) verifying (i)-(iii), the following inequality holds:

\[
\text{Cliff}(\mathcal{O}_C(D)) = D \cdot C - D^2 - 2 \geq H \cdot C - H^2 - 2 = 2a - 2p - 3 \geq a.
\]

This will contradict the assumption \( \text{Cliff}(C) < a \). The proof proceeds along the lines of Theorem 3 in [F], with the difference that we must also consider curves with \( D^2 = 0 \), that is, elliptic pencils which we now characterize. By direct calculation, we note that there are no \((-2)\)-curves in \( S \). Equality holds in (ii) when \( m = -n \) or \( m = -un \) with \( u := 2a/(2p + 1) \).

First, we describe the effective divisors \( D \subset S \) with self-intersection \( D^2 = 0 \). Consider the case \( m = -un \). If \( 2p + 1 \) does not divide \( a \), then \( D \equiv 2aH - (2p + 1)C \) and \( D \cdot C = 2a(2a - 2p - 1) > g - 1 \), that is, \( D \) does not verify condition (i). If \( a = k(2p + 1) \), for \( k \geq 2 \), then \( D \equiv 2kH - C \). Notice that \( D \cdot C = a(4k - 4) + 2k(2p + 1) > 2a \) for \( k \geq 2 \),
that is, $D$ does not satisfy (i).

In the the case $m = -n$, the effective divisor $D \equiv C - H$, satisfies (i)-(iii) and

$$\text{Cliff}(\mathcal{O}_C(C - H)) = 2a - 2p - 3 \geq a.$$\hspace{2cm}(5)

Case $n < 0$. From (ii) we have either $m < -n$ or $m > -un$. In the first case, by using inequality (iii), we obtain $2 < -(4p + 2)n + dn = n(2a - 2p - 1)$, which is a contradiction since $n < 0$ and $2a > 2p + 1$. Suppose $m > -un > 0$. Inequality (i) implies that

$$(-n)\frac{2ad}{2p + 1} < (g - 1)(2n - 1) = -2a(2n - 1),$$

then $(-n)(d - (4p + 2)) < 2p + 1$ and since $d > 4p + 2$, this yields $2a + 2p + 1 = d < 6p + 3$ which contradicts the hypothesis $a \geq 2p + 3$.

Case $n > 0$. Again, by condition (ii), we have either that $m < -un$ or $m > -n$. In the first case, using (iii) we write that

$$0 < (4p + 2)m + dn < n \left(d - (4p + 2)\frac{2a}{2p + 1}\right),$$

but one can get easily check that $d(2p + 1) < 2a(4p + 2)$, which yields a contradiction. Suppose now $-n < m < 0$. By (i) we have $2a(2n - 1) \leq -md < nd$, so $n < \frac{2a}{4a - d} = \frac{2a}{2a + 2p + 1} < 2$, since $a \geq 2p + 1$. This implies $n = 1$, therefore for $n > 0$ there are no divisors $D \subset S$ with $D^2 > 0$ satisfying the inequalities (i)-(iii).

Case $n = 0$. From (i), one writes $m \leq -\frac{a - 1}{a} = \frac{2a}{2a + 2p + 1} < 1$, but this yields to a contradiction since by (iii) it follows that $m > 0$. The proof is thus finished.

\[\square\]

4. Curves with prescribed gonality and small rank 2 Clifford index

The equality $\text{Cliff}_2(C) = \text{Cliff}(C)$ is known to be valid for arbitrary $k$-gonal curves $[C] \in \mathcal{M}_{g,k}$ of genus $g > (k - 1)(2k - 4)$. It is thus of some interest to study Mercat’s question for arbitrary curves in a given gonality stratum in $\mathcal{M}_g$ and decide how sharp is this quadratic bound. We shall construct curves $C$ of unbounded genus and relatively small gonality, carrying a stable rank 2 vector bundle $E$ with $h^0(C, E) = 4$ such that $\gamma(E) < \text{Cliff}(C)$. In order to be able to determine the gonality of $C$, we realize it as a section of a $K3$ surface $S$ in $\mathbb{P}^4$ which is special in the sense of Noether-Lefschetz theory. The pencil computing the gonality is the restriction of an elliptic pencil on the surface. The constraint of having a Picard lattice of rank 2 containing, apart from the hyperplane class, both an elliptic pencil and a curve $C$ of prescribed genus, implies that the discriminant of $\text{Pic}(S)$ must be a perfect square. This imposes severe restrictions on the genera for which such a construction could work.

**Theorem 4.1.** We fix integers $a \geq 3$ and $b = 4, 5, 6$. There exists a smooth curve $C \subset \mathbb{P}^4$ with

$$\deg(C) = 6a + b, \quad g(C) = 3a^2 + ab + 1 \quad \text{and gonality} \quad \text{gon}(C) = ab,$$

such that $C$ lies on a $(2, 3)$ complete intersection $K3$ surface. In particular $K_{1,1}(C, \mathcal{O}_C(1)) \neq 0$ and conjecture $(M_2)$ fails for $C$. 
Before presenting the proof, we discuss the connection between Theorem 3.1 and conjecture \((M_2)\). For \(C \subset S \subset \mathbf{P}^4\) as above, we construct a vector bundle \(E\) with \(\text{det}(E) = \mathcal{O}_C(1)\) and \(h^0(C, E) = 4\), lying in an exact sequence

\[
0 \rightarrow E \rightarrow W \otimes \mathcal{O}_C(1) \rightarrow \mathcal{O}_C(2) \rightarrow 0,
\]

where \(W \in G(3, H^0(C, \mathcal{O}_C(1)))\) has the property that the quadric \(Q \in \text{Sym}^2 H^0(C, \mathcal{O}_C(1))\) induced by \(S\) is representable by a tensor in \(W \otimes H^0(C, L)\). This construction is a particular procedure of associating vector bundles to non-trivial syzygies, cf. [AN].

The proof that \(E\) is stable is standard and proceeds along the lines of e.g. [GMN] Theorem 3.2. Next we compute the Clifford invariant:

\[
\gamma(E) = 3a + \frac{b}{2} < ab - 2 = \text{Cliff}(C),
\]

since \(b \geq 4\), so not only \(\text{Cliff}_2(C) < \text{Cliff}(C)\), but the difference \(\text{Cliff}(C) - \text{Cliff}_2(C)\) becomes arbitrarily positive.

**Proof.** By means of [K] Theorem 6.1, there exist a smooth complete intersection surface \(S \subset \mathbf{P}^4\) of type \((2, 3)\) such that \(\text{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C\), where \(H^2 = 6\), \(H \cdot C = d = 6a + b\) and \(C^2 = 2(g - 1)\). (Note that such a surface exists when \(d^2 > 12g\), which is satisfied when \(b \geq 4\)). The divisor \(E := C - aH\) verifies \(E^2 = 0\), \(E \cdot H = b\) and \(E \cdot C = ab\). In particular \(E\) is effective. The class \(E\) is primitive, hence it follows that \(h^0(S, E) = h^0(C, \mathcal{O}_C(E)) = 2\), where the last equality follows by noting that \(H^1(S, \mathcal{O}_S(E - C)) = 0\) by Kodaira vanishing. Furthermore, \(h^1(C, \mathcal{O}_C(E)) \geq 3a^2 + 2\), that is, \(\mathcal{O}_C(E)\) contributes to \(\text{Cliff}(C)\) and then we write that

\[
\text{gon}(C) = \text{Cliff}(C) + 2 \leq \text{Cliff}(C, \mathcal{O}_C(E)) + 2 = ab.
\]

We shall show that \(\mathcal{O}_C(E)\) computes the Clifford index of \(C\).

First, we classify the primitive effective divisors \(F \equiv mH + nC \subset S\) having self-intersection zero. By solving the equation \((mH + nC)^2 = 0\), where \(m, n \in \mathbb{Z}\), we find the following primitive solutions: \(E_1 \equiv (3a + b)H - 3C\) for \(b \neq 6\) (respectively \(E_2 \equiv (a + 2)H - C\) for \(b = 6\)), and \(E_3 = E \equiv C - aH\). A simple computation shows that \(E_i \cdot C > ab\) for \(i = 1, 2\).

Since \(\text{Cliff}(C) \leq ab - 2 < \left(\frac{a^2}{2}\right)\), the Clifford index of \(C\) is computed by a bundle defined on \(S\). Following [GL2], there exists an effective divisor \(D \equiv mH + nC\) on \(S\), satisfying the following numerical conditions:

\[
h^0(S, D) = h^0(C, \mathcal{O}_C(D)) \geq 2, \quad h^0(S, C - D) \geq 2, \quad D^2 \geq 0 \quad \text{and} \quad D \cdot C \leq g - 1,
\]

and such that

\[
f(D) := \text{Cliff}(\mathcal{O}_C(D)) + 2 = D \cdot C - D^2 = \text{Cliff}(C) + 2.
\]

Furthermore, \(D\) can be chosen such that \(h^1(S, D) = 0\), cf. [Ma]. To bound \(f(D)\) and show that \(f(D) \geq ab\), we distinguish two cases depending on whether \(D^2 > 0\) or \(D^2 = 0\).

By a complete classification of curves with self-intersection zero, we have already seen that for any elliptic pencil \(|D|\) satisfying (4), one has \(f(D) \geq ab = f(E)\). We are left with the case \(D^2 > 0\) and rewrite the inequalities (4):

\[
\begin{align*}
(i) \quad (6a + b)m + (2n - 1)(3a^2 + ab) &\leq 0 \\
(ii) \quad (m + an)(3an + 3m + bn) &> 0 \\
(iii) \quad 6m + (6a + b)n &> 2,
\end{align*}
\]
where (ii) comes from the assumption $D^2 > 0$ and (iii) from the fact that $D \cdot H > 2$. Furthermore,

$$f(m, n) := D \cdot C - D^2 = -6m^2 + m(d - 2nd) + (n - n^2)(2g - 2).$$

We prove that for any divisor $D$ satisfying (i) - (iii), the inequality $f(m, n) \geq ab$ holds, from which we conclude that $\text{Cliff}(C) = ab - 2$.

**Case** $n < 0$. From (iii) we find that $m > 0$. Then $m < -an$ or $3m > -(3a + b)n$. When $m < -an$, from (iii) we have that $2 < 6m + dn < -6an + dn = nb < 0$, which is a contradiction. Suppose $(3a + b)n + 3m > 0$. For a fixed $n$ the function $f(m, n)$ reaches its maximum at $m_0 := \frac{d(1 - 2n)}{12}$. So when $3m_0 + (3a + b)n \leq 0$, we have $f(m, n) \geq f\left(\frac{(1 - 2n)(g - 1)}{d}, n\right)$, since by condition (i), $m \leq \frac{(1 - 2n)(g - 1)}{d}$. A simple computation gives that whenever $n < 0$, one has the inequality:

$$f\left(\frac{(1 - 2n)(g - 1)}{d}, n\right) = (2n^2 - 2n)(g - 1) \frac{b^2}{d^2} + (g - 1) \left(1 - \frac{6(g - 1)}{d^2}\right) \geq 4(g - 1) \frac{b^2}{d^2} + \frac{g - 1}{d^2} (18a^2 + b^2 + 6ab) \geq \frac{3a^2 + ab}{2} \geq ab.$$

Assume now that $3m_0 + (3a + b)n > 0$. Since $m \in \left(-\frac{(3a + b)n}{3}, \frac{(1 - 2n)(g - 1)}{d}\right]$, we have

$$f(m, n) \geq \min \left\{ f\left(-\frac{(3a + b)n}{3}, n\right), f\left(\frac{(1 - 2n)(g - 1)}{d}, n\right)\right\}.$$

A direct computation yields

$$f\left(-\frac{(3a + b)n}{3}, n\right) = -n \left(ab + \frac{b^2}{3}\right) \geq ab + \frac{b^2}{3} \geq ab.$$

**Case** $n > 0$. If $m \geq 0$ we get a contradiction to (i). Suppose $m < 0$, then we have either $3m + (3a + b)n < 0$, or else $m > -an$. The first case contradicts (iii), so it does not appear. Suppose $m > -an$. Reasoning as before, observe that $m_0 < (1 - 2n)(g - 1)/d$, where $m_0$ is the maximum of $f(m, n)$ for a fixed $n$, and $m$ takes values in the interval $(-an, \frac{(1 - 2n)(g - 1)}{d}]$. If $-an \geq m_0$, then $f(m, n) \geq f\left(\frac{(1 - 2n)(g - 1)}{d}, n\right)$. Since we are assuming $-an < \frac{(1 - 2n)(g - 1)}{d}$, we have that $n < \frac{3a}{b} + 1$. We use this bound to directly show, like in the previous case, that $f\left(\frac{(1 - 2n)(g - 1)}{d}, n\right) \geq ab$. When $-an < m_0$ we have that

$$f(m, n) \geq \min\left\{ f(-an, n), f\left(\frac{(1 - 2n)(g - 1)}{d}, n\right)\right\}.$$

In this case it is enough to note that $f(-an, n) = nab \geq ab$.

**Case** $n = 0$. From inequalities (i) and (iii) with $n = 0$, we have $1 \leq m \leq \frac{d}{12}$. Note that $f(m, 0) = -6m^2 + md$ reaches its maximum at $\frac{d}{12}$. So, since $\frac{a - 1}{d} \leq \frac{d}{12}$, we conclude that $f(m, 0) \geq f(1, 0) = 6a + b - 6$. Finally, we observe that $6a + b - 6 \geq ab$ if and only if $b \leq 6$. This finishes the proof. \qed
5. The Fourier-Mukai involution on $\mathcal{F}_{11}$

The aim of this section is to provide a detailed proof of Mercat’s conjecture ($M_2$) in one non-trivial case, that of genus 11, and discuss the connection to Mukai’s work [M1], [M3]. We denote as usual by $F_g$ the moduli space parametrizing pairs $[S, \ell]$, where $S$ is a smooth $K3$ surface and $\ell \in \text{Pic}(S)$ is a primitive nef line bundle with $\ell^2 = 2g - 2$. Furthermore, we introduce the parameter space

$$
\mathcal{P}_g := \{ [S, C] : S \text{ is a smooth } K3 \text{ surface, } C \subset S \text{ is a smooth curve, } [S, O_S(C)] \in \mathcal{F}_g \}
$$

and denote by $\pi : \mathcal{P}_g \to \mathcal{F}_g$ the projection map $[S, C] \mapsto [S, O_S(C)]$. If $S$ is a $K3$ surface, following [M1], we set $\tilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ and $\tilde{N}_S(S) := H^0(S, \mathbb{Z}) \oplus NS(S) \oplus H^4(S, \mathbb{Z})$.

We recall the definition of the Mukai pairing on $\tilde{H}(S, \mathbb{Z})$:

$$(\alpha_0, \alpha_2, \alpha_4) \cdot (\beta_0, \beta_2, \beta_4) := \alpha_0 \cup \beta_2 - \alpha_4 \cup \beta_0 - \alpha_0 \cup \beta_4 \in H^4(S, \mathbb{Z}) = \mathbb{Z}.$$ 

Let now $r, s \geq 1$ be relatively prime integers such that $g = 1 + rs$. For a polarized $K3$ surface $[S, \ell] \in \mathcal{F}_g$ one defines the Fourier-Mukai dual $\tilde{S} := \text{mot}_{r, \ell, s}$, where

$$M_{S}(r, \ell, s) = \{ E : E \text{ is an } \ell - \text{stable sheaf on } S, \text{rk}(E) = r, c_1(E) = \ell, \chi(S, E) = r + s \}.$$ 

Setting $v := (r, \ell, s) \in \tilde{H}(S, \mathbb{Z})$, there is a Hodge isometry, see [M1] Theorem 1.4:

$$\psi : H^2(M_S(r, \ell, s), \mathbb{Z}) \rightarrow v^\perp / \mathbb{Z}.$$ 

We observe that $\hat{\ell} := \psi^{-1}((0, 0, 2s))$ is a nef primitive vector with $(\hat{\ell})^2 = 2g - 2$, and in this way the pair $(\tilde{S}, \hat{\ell})$ becomes a polarized $K3$ surface of genus $g$. The Fourier-Mukai involution is the isomorphism $FM : \mathcal{F}_g \to \mathcal{F}_g$ defined by $FM([S, \ell]) := [\tilde{S}, \hat{\ell}]$.

We turn to the case $g = 11$, when we set $r = 2$ and $s = 5$. For a general curve $[C] \in \mathcal{M}_{11}$, the Lagrangian Brill-Noether locus

$$SU_C(2, K, 7) := \{ E \in U_C(2, 20) : \text{det}(E) = K, h^0(C, E) = 7 \}$$

is a smooth $K3$ surface. The main result of [M3] can be summarized as saying a general $[C] \in \mathcal{M}_{11}$ lies on a unique $K3$ surface which moreover can be realized as $SU_C(2, K, 7)$. Furthermore, there is a birational isomorphism

$$\phi_{11} : \mathcal{M}_{11} \to \mathcal{F}_{11}, \quad \phi_{11}([C]) := [SU_C(2, K, 7), C]$$

and we set $q_{11} := \pi \circ \phi_{11} : \mathcal{M}_{11} \to \mathcal{F}_{11}$. On the moduli space $\mathcal{M}_{11}$ there exist two distinct irreducible Brill-Noether divisors

$$\mathcal{M}_{11,6} := \{ [C] \in \mathcal{M}_{11} : W_6^1(C) \neq \emptyset \} \text{ and } \mathcal{M}_{11,9} := \{ [C] \in \mathcal{M}_{11} : W_9^3(C) \neq \emptyset \}. $$

Via the resuduation morphism $W_6^1(C) \ni L \mapsto K_C \otimes L \in W_{14}^0(C)$, the Hurwitz divisor is the pull-back of a Noether-Lefschetz divisor on $\mathcal{F}_{11}$, that is, $\mathcal{M}_{11,6} = q_{11}^*(D^1_6)$ where

$$D_6^1 := \{ [S, \ell] \in \mathcal{F}_{11} : \exists H \in \text{Pic}(S), H^2 = 8, H \cdot \ell = 14 \}.$$ 

Similarly, via the resuduation map $W_9^3(C) \ni L \mapsto K_C \otimes L \in W_{11}^3(C)$, one has the equality of divisors $\mathcal{M}_{11,9} = q_{11}^*(D^2_9)$, where

$$D_9^2 := \{ [S, \ell] \in \mathcal{F}_{11} : \exists H \in \text{Pic}(S), H^2 = 4, H \cdot \ell = 11 \}.$$ 

Next we establish Mercat’s conjecture for general curves of genus 11.
Theorem 5.1. The equality $\text{Cliff}_2(C) = \text{Cliff}(C)$ holds for a general curve $[C] \in \mathcal{M}_{11}$.

Proof. We fix a curve $[C] \in \mathcal{M}_{11}$ such that (i) $W^1_7(C)$ is a smooth curve, (ii) $W^2_3(C) = \emptyset$ (in particular, any Petri general curve will satisfy these conditions) and (iii) the rank 2 Brill-Noether locus $SU_C(2, K_C, 7)$ is a smooth $K3$ surface of Picard number 1. As discussed in both [LMN] Proposition 4.5 and [FO] Question 3.5, in order to verify (M$_2$), it suffices to show that $C$ possesses no bundles $E \in U_C(2, 13)$ with $h^0(C, E) = 4$. Suppose $E$ is such a vector bundle. Then $L := \det(E) \in W^4_3(C)$ is a linear series such that the multiplication map $\nu_2(L) : \text{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is not injective. For each extension class

$$e \in P_L := P(\text{Coker } \nu_2(L))^\vee \subset P(\text{H}^0(C, L^{\otimes 2}))^\vee = \text{PEX}^1(L, K_C \otimes L^\vee),$$

one obtains a rank 2 vector bundle $F$ on $C$ sitting in an exact sequence

$$0 \to K_C \otimes L^\vee \to F \to L \to 0,$$

such that $h^0(C, F) = h^0(C, L) + h^0(C, K_C \otimes L^\vee) = 7$. We claim that any non-split vector bundle $F$ with $h^0(C, F) = 7$ and which sits in an exact sequence (6), is semistable. Indeed, let us assume by contradiction that $M \subset F$ is a destabilizing line subbundle with $\text{deg}(M) \geq 11$. Since $\text{deg}(M) > \text{deg}(K_C \otimes L^\vee)$, the composite morphism $M \to L$ is non-zero, hence we can write that $M = L(-D)$, where $D$ is an effective divisor of degree 1 or 2. Because $W^2_3(C) = \emptyset$, one finds that $h^0(C, K_C \otimes L^\vee(D)) = 2$ and $L$ must be very ample, that is, $h^0(C, L(-D)) = h^0(C, L) - \text{deg}(D)$. We obtain that $h^0(L) + h^0(K_C \otimes L^\vee) = h^0(L) \leq h^0(M) + h^0(K_C \otimes M^\vee) = h^0(L) - \text{deg}(D) + h^0(K_C \otimes L^\vee)$, a contradiction. Thus one obtains an induced morphism $u : P_L \to SU_C(2, K_C, 7)$. Since $SU_C(2, K_C, 7)$ is a $K3$ surface, this also implies that $\text{Coker } \nu_2(L)$ is 2-dimensional, hence $P_L = P^1$.

We claim that $u$ is an embedding. Setting $A := K_C \otimes L^\vee \in W^4_3(C)$, we write the exact sequence $0 \to H^0(C, O_C) \to H^0(C, F^\vee \otimes L) \to H^0(C, K_C \otimes A^{\otimes(-2)})$, and note that the last vector space is the kernel of the Petri map $H^0(C, A) \otimes H^0(C, L) \to H^0(C, K_C)$, which is injective, hence $h^0(C, F^\vee \otimes L) = 1$. This implies that $u$ is an embedding. But this contradicts the fact that $\text{Pic } SU_C(2, K_C, 7) = \mathbb{Z}$, in particular $SU_C(2, K_C, 7)$ contains no $(-2)$-curves. We conclude that $\nu_2(L)$ is injective for every $L \in W^4_3(C)$. □

This proof also shows that the failure locus of statement (M$_2$) on $\mathcal{M}_{11}$ is equal to the Koszul divisor

$$\mathfrak{S}_{11, 13}^4 := \{ [C] \in \mathcal{M}_{11} : \exists L \in W^4_3(C) \text{ such that } K_{1, 1}(C, L) \neq 0 \}.$$ 

Suppose now that $[C] \in \mathfrak{S}_{11, 13}^4$ is a general point corresponding to an embedding $C \xrightarrow{[L]} \mathbb{P}^4$ such that $C$ lies on a $(2, 3)$ complete intersection $K3$ surface $S \subset \mathbb{P}^4$. Then $S = SU_C(2, K_C, 7)$ and $p(S) = 2$ and furthermore $\text{Pic}(S) = \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H$, where $H^2 = 6, C \cdot H = 13$ and $C^2 = 20$. In particular we note that $S$ contains no $(-2)$-curves, hence $S$ and $\tilde{S}$ are not isomorphic.

Let us define the Noether-Lefschetz divisor

$$D_{13}^4 := \{ [S, \ell] \in \mathcal{F}_{11} : \exists H \in \text{Pic}(S), \ H^2 = 6, H \cdot \ell = 13 \},$$

therefore $\mathfrak{S}_{11, 13}^4 = q_{11}^*(D_{13}^4)$. 
Proposition 5.2. The action of the Fourier-Mukai involution $FM: \mathcal{F}_{11} \to \mathcal{F}_{11}$ on the three distinguished Noether-Lefschetz divisors is described as follows:

(i) $FM(D_6^1) = D_6^1$.
(ii) $FM(D_5^2) = \{[S, \ell] \in \mathcal{F}_{11} : \exists R \in \text{Pic}(S) \text{ such that } R^2 = -2, R \cdot \ell = 1\}$.
(iii) $FM(D_{13}^4) = \{[S, \ell] \in \mathcal{F}_{11} : \exists R \in \text{Pic}(S) \text{ such that } R^2 = -2, R \cdot \ell = 3\}$.

Proof. For $[S, \ell] \in \mathcal{F}_{11}$, we set $v := (2, \ell, 5) \in \tilde{H}(S, \mathbb{Z})$ and $\hat{\ell} := (0, \ell, 10) \in \tilde{H}(S, \mathbb{Z})$ for the class giving the genus 11 polarization. We describe the lattice $\psi(\text{NS}(\tilde{S})) \subset \text{NS}(S)$.

In the case of a general point of $D_6^1$ with lattice $\text{NS}(S) = \mathbb{Z} \cdot \ell \oplus \mathbb{Z} \cdot H$, by direct calculation we find that $\psi(\text{NS}(\tilde{S}))$ is generated by the vectors $\hat{\ell}$ and $(2, \ell + H, 12)$. Furthermore, $(2, \ell + H, 12)^2 = 8$ and $(2, H + \ell, 12) \cdot \hat{\ell} = 14$, that is, $\text{Pic}(\tilde{S}) \cong \text{Pic}(S)$, hence $D_6^1$ is a fixed divisor for the automorphism $FM$.

A similar reasoning for a general point of the divisor $D_5^2$ shows that the Neron-Severi groups $\psi(\text{NS}(\tilde{S}))$ is generated by $\ell$ and $(-1, H - \ell, -2)$, where $(-1, H - \ell, -2)^2 = -2$ and $(-1, H - \ell, -2) \cdot \ell = 1$. In other words, the class $(-1, H - \ell, -2)$ corresponds to a line in the embedding $\tilde{S} \to \mathbb{P}^{11}$. Finally, for a general point of $D_{13}^4$ corresponding to a lattice $\mathbb{Z} \cdot \ell \oplus \mathbb{Z} \cdot H$, the Picard lattice of the Fourier-Mukai partner is spanned by the vectors $\hat{\ell}$ and $(-1, H - \ell, -1)$, where $(-1, H - \ell, -1)^2 = -2$ and $(-1, H - \ell, -1) \cdot \ell = 3$. □

Remark 5.3. The fact that the divisor $D_6^1$ is fixed by the automorphism $FM$ is already observed and proved with geometric methods in [M3 Theorem 3].

Remark 5.4. It is instructive to point out the difference between a general element of $D_{13}^4$ and its Fourier-Mukai partner. As a polarized $K3$ surface, $SU_C(2, K_C, 7)$ is characterized by the existence of a degree 3 rational curve $u(P_L) \subset SU_C(2, K_C, 7)$. On the other hand, the complete intersection surface $S \subset \mathbb{P}^4$ containing $C \subset \mathbb{P}^4$, where $L \in W_{13}^3(C)$, carries no smooth rational curves. It contains however elliptic curves in the linear system $|O_S(C - H)|$. Thus the involution $FM$ assigns to a $K3$ surface with a degree 7 elliptic pencil, a $K3$ surface containing a $(-2)$-curve. Since $S = SU_C(2, K_C, 7)$, it also follows that the complete intersection $S$ is a smooth $K3$ surface, which a priori is not at all obvious.

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