EXTENSION OF EUCLIDEAN OPERATOR RADIUS INEQUALITIES

MOHAMMAD SAL MOSLEHIAN\textsuperscript{1}, MOSTAFA SATTARI\textsuperscript{1} AND KHALID SHEBRAWI\textsuperscript{2}

Abstract. To extend the Euclidean operator radius, we define $w_p$ for an $n$-tuples of operators $(T_1, \ldots, T_n)$ in $\mathcal{B}(\mathcal{H})$ by $w_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} (\sum_{i=1}^{n} |\langle T_i x, x \rangle|^p)^{\frac{1}{p}}$ for $p \geq 1$. We generalize some inequalities including Euclidean operator radius of two operators to those involving $w_p$. Further we obtain some lower and upper bounds for $w_p$. Our main result states that if $f$ and $g$ are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$, then

$$w_p^r (A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( [B_i^* f^2 (|T_i|) B_i]^{rp} + [A_i^* g^2 (|T_i^*|) A_i]^{rp} \right) \right\|$$

for all $p \geq 1$, $r \geq 1$ and operators in $\mathcal{B}(\mathcal{H})$.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The numerical radius of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$w(A) = \sup \{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$  

It is well known that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\| \cdot \|$. Namely, we have

$$\frac{1}{2} \|A\| \leq w(A) \leq \|A\|.$$  

for each $A \in \mathcal{B}(\mathcal{H})$. It is known that if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $w(A) = \|A\|$. An important inequality for $w(A)$ is the power inequality stating that $w(A^n) \leq w^n(A)$ for $n = 1, 2, \ldots$. There are many inequalities involving numerical radius; see [2, 3, 4, 10, 11, 12] and references therein.

The Euclidean operator radius of an $n$-tuple $(T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^{(n)} := \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H})$ is defined as

$$\rho_{\text{euclid}}(T_1, \ldots, T_n) := \inf \{ r \geq 0 : w_p(T_1, \ldots, T_n) \leq r \}$$  

for $p \geq 1$. The main result of this paper is to extend the Euclidean operator radius to $w_p$ for an $n$-tuples of operators $(T_1, \ldots, T_n)$ in $\mathcal{B}(\mathcal{H})$ by $w_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} (\sum_{i=1}^{n} |\langle T_i x, x \rangle|^p)^{\frac{1}{p}}$ for $p \geq 1$.
\[ w_e(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left( \sum_{i=1}^{n} \langle T_i x, x \rangle \right)^{\frac{1}{2}}. \]

The particular cases \( n = 1 \) and \( n = 2 \) are numerical radius and Euclidean operator radius. Some interesting properties of this radius were obtained in [9]. For example, it is established that

\[ \frac{1}{2\sqrt{n}} \left( \sum_{i=1}^{n} T_i^* T_i \right)^{\frac{1}{2}} \leq w_e(T_1, \ldots, T_n) \leq \left( \sum_{i=1}^{n} T_i^* T_i \right)^{\frac{1}{2}}. \]  \hspace{1cm} (1.1)

We also observe that if \( A = B + iC \) is the Cartesian decomposition of \( A \), then

\[ w_e^2(B, C) = \sup_{\|x\|=1} \{ |\langle B x, x \rangle|^2 + |\langle C x, x \rangle|^2 \} = \sup_{\|x\|=1} |\langle A x, x \rangle|^2 = w^2(A). \]

By the above inequality and \( A^* A + AA^* = 2(B^2 + C^2) \), we have

\[ \frac{1}{16} \| A^* A + AA^* \| \leq w^2(A) \leq \frac{1}{2} \| A^* A + AA^* \|. \]

We define \( w_p \) for \( n \)-tuples of operators \( (T_1, \ldots, T_n) \in \mathbb{B}(\mathcal{H})^{(n)} \) for \( p \geq 1 \) by

\[ w_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left( \sum_{i=1}^{n} |\langle T_i x, x \rangle|^p \right)^{\frac{1}{p}}. \]

It follows from Minkowski’s inequality for two vectors \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \), namely,

\[ (|a_1 + b_1|^p + |a_2 + b_2|^p)^{\frac{1}{p}} \leq (|a_1|^p + |a_2|^p)^{\frac{1}{p}} + (|b_1|^p + |b_2|^p)^{\frac{1}{p}} \quad \text{for} \quad p > 1 \]

that \( w_p \) is a norm.

Moreover \( w_p, p \geq 1, \) for \( n \)-tuple of operators \( (T_1, \ldots, T_n) \in \mathbb{B}(\mathcal{H})^{(n)} \) satisfies the following properties:

(i) \( w_p(T_1, \ldots, T_n) = 0 \Leftrightarrow T_1 = \ldots = T_n = 0. \)

(ii) \( w_p(\lambda T_1, \ldots, \lambda T_n) = |\lambda| w_p(T_1, \ldots, T_n) \) for all \( \lambda \in \mathbb{C}. \)

(iii) \( w_p(T_1 + T_1', \ldots, T_n + T_n') \leq w_p(T_1, \ldots, T_n) + w_p(T_1', \ldots, T_n') \) for \( (T_1', \ldots, T_n') \in \mathbb{B}(\mathcal{H})^{(n)}. \)

(iv) \( w_p(X^* T_1 X, \ldots, X^* T_n X) \leq \|X\|^2 w_p(T_1, \ldots, T_n) \) for \( X \in \mathbb{B}(\mathcal{H}). \)

Dragomir [1] obtained some inequalities for the Euclidean operator radius \( w_e(B, C) = \sup_{\|x\|=1} \{ |\langle B x, x \rangle|^2 + |\langle C x, x \rangle|^2 \}^{\frac{1}{2}} \) of two bounded linear operators in a Hilbert space. In section 2 of this paper we extend some his results including inequalities for the Euclidean operator radius of linear operators to \( w_p \) \( (p \geq 1). \) In addition, we apply some known inequalities for getting new inequalities for \( w_p \) in two operators.
In section 3 we prove inequalities for $w_p$ for $n$-tuples of operators. Some of our result in this section, generalize some inequalities in section 2. Further, we find some lower and upper bounds for $w_p$.

2. Inequalities for $w_p$ for two operators

To prove our generalized numerical radius inequalities, we need several known lemmas. The first lemma is a simple result of the classical Jensen inequality and a generalized mixed Cauchy–Schwarz inequality [7, 8, 6].

Lemma 2.1. For $a, b \geq 0, 0 \leq \alpha \leq 1$ and $r \neq 0$,

(a) $a^{\alpha}b^{1-\alpha} \leq (1 - \alpha)b \leq \alpha a + (1 - \alpha)b^{\frac{1}{r}}$ for $r \geq 1$,

(b) If $A \in B(H)$, then $|\langle Ax, y \rangle|^2 \leq \langle |A|^2 x, x \rangle \langle |A^*|^2 (1-\alpha)y, y \rangle$ for all $x, y \in H$, where $|A| = (A^*A)^{\frac{1}{2}}$.

(c) Let $A \in B(H)$, and $f$ and $g$ be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$|\langle Ax, y \rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|$$

for all $x, y \in H$.

Lemma 2.2 (McCarthy inequality [5]). Let $A \in B(H)$, $A \geq 0$ and let $x \in H$ be any unit vector. Then

(a) $\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$ for $r \geq 1$,

(b) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for $0 < r \leq 1$.

Inequalities of the following lemma were obtained for the first time by Clarkson[7].

Lemma 2.3. Let $X$ be a normed space and $x, y \in X$. Then for all $p \geq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(a) $2(||x||^p + ||y||^p)^{q-1} \leq ||x + y||^q + ||x - y||^q$,

(b) $2(||x||^p + ||y||^p) \leq ||x + y||^p + ||x - y||^p \leq 2^{p-1}(||x||^p + ||y||^p)$,

(c) $||x + y||^p + ||x - y||^p \leq 2(||x||^q + ||y||^q)^{p-1}$.

If $1 < p \leq 2$ the converse inequalities hold.

Making the transformations $x \rightarrow \frac{x+y}{2}$ and $y \rightarrow \frac{x-y}{2}$ we observe that inequalities (a) and (c) in Lemma 2.3 are equivalent and so are the first and the second inequalities of (b). First of all we obtain a relation between $w_p$ and $w_e$ for $p \geq 1$. 

Proposition 2.4. Let $B, C \in \mathcal{B}(\mathcal{H})$. Then

$$w_p(B, C) \leq w_q(B, C) \leq 2^{\frac{1}{q} - \frac{1}{p}} w_p(B, C)$$

for $p \geq q \geq 1$. In particular

$$w_p(B, C) \leq w_e(B, C) \leq 2^{\frac{1}{2} - \frac{1}{p}} w_p(B, C)$$ \hspace{1cm} (2.1)

for $p \geq 2$, and

$$2^{\frac{1}{q} - \frac{1}{p}} w_p(B, C) \leq w_e(B, C) \leq w_p(B, C)$$

for $1 \leq p \leq 2$.

Proof. An application of Jensen’s inequality says that for $a, b > 0$ and $p \geq q > 0$, we have

$$(a^p + b^p)^{\frac{1}{p}} \leq (a^q + b^q)^{\frac{1}{q}}.$$ 

Let $x \in \mathcal{H}$ be a unit vector. Choosing $a = |\langle Bx, x \rangle|$ and $b = |\langle Cx, x \rangle|$, we have

$$\left( |\langle Bx, x \rangle|^p + |\langle Cx, x \rangle|^p \right)^{\frac{1}{p}} \leq \left( |\langle Bx, x \rangle|^q + |\langle Cx, x \rangle|^q \right)^{\frac{1}{q}}.$$

Now the first inequality follows by taking the supremum over all unit vectors in $\mathcal{H}$. A simple consequence of the classical Jensen’s inequality concerning the convexity or the concavity of certain power functions says that for $a, b \geq 0, 0 \leq \alpha \leq 1$ and $p \geq q$, we have

$$(\alpha a^q + (1 - \alpha) b^q)^{\frac{1}{q}} \leq (\alpha a^p + (1 - \alpha) b^p)^{\frac{1}{p}}.$$ 

For $\alpha = \frac{1}{2}$, we get

$$(a^q + b^q)^{\frac{1}{q}} \leq 2^{\frac{1}{q} - \frac{1}{p}} (a^p + b^p)^{\frac{1}{p}}.$$ 

Again let $x \in \mathcal{H}$ be a unit vector. Choosing $a = |\langle Bx, x \rangle|$ and $b = |\langle Cx, x \rangle|$ we get

$$\left( |\langle Bx, x \rangle|^q + |\langle Cx, x \rangle|^q \right)^{\frac{1}{q}} \leq 2^{\frac{1}{q} - \frac{1}{p}} \left( |\langle Bx, x \rangle|^p + |\langle Cx, x \rangle|^p \right)^{\frac{1}{q}}.$$ 

Now the second inequality follows by taking the supremum over all unit vectors in $\mathcal{H}$. \hfill \square

On making use of inequality (2.1) we find a lower bound for $w_p$ ($p \geq 2$).

Corollary 2.5. If $B, C \in \mathcal{B}(\mathcal{H})$, then for $p \geq 2$

$$w_p(B, C) \geq 2^{\frac{1}{q} - 2} \|B^*B + C^*C\|^\frac{1}{2}.$$
Proof. According to inequalities (1.1) and (2.1) we can write
\[ w_e(B, C) \geq \frac{1}{2\sqrt{2}} \|B^*B + C^*C\|^\frac{1}{2} \]
and
\[ w_p(B, C) \geq 2^{\frac{1}{p} - \frac{1}{2}} w_e(B, C), \]
respectively. We therefore get desired inequality. □

The next result is concerned with some lower bounds for \( w_p \). This consequence has several inequalities as special cases. Our result will be generalized to \( n \)-tuples of operators in the next section.

**Proposition 2.6.** Let \( B, C \in \mathbb{B}(\mathcal{H}) \). Then for \( p \geq 1 \)
\[ w_p(B, C) \geq 2^{\frac{1}{p} - \frac{1}{2}} \max (w(B + C), w(B - C)). \quad (2.2) \]
This inequality is sharp.

Proof. We use convexity of function \( f(t) = t^p \ (p \geq 1) \) as follows:
\[
\left( |\langle Bx, x \rangle|^p + |\langle Cx, x \rangle|^p \right)^{\frac{1}{p}} \geq 2^{\frac{1}{p} - 1} \left( |\langle Bx, x \rangle| + |\langle Cx, x \rangle| \right)
\]
\[
\geq 2^{\frac{1}{p} - 1} |\langle Bx, x \rangle \pm \langle Cx, x \rangle|
\]
\[
= 2^{\frac{1}{p} - 1} |\langle (B \pm C)x, x \rangle|.
\]
Taking supremum over \( x \in \mathcal{H} \) with \( \|x\| = 1 \) yields that
\[ w_p(B, C) \geq 2^{\frac{1}{p} - 1} w(B \pm C). \]
For sharpness one can obtain the same quantity \( 2^{\frac{1}{p}} w(B) \) on both sides of the inequality by putting \( B = C \). □

**Corollary 2.7.** If \( A = B + iC \) is the Cartesian decomposition of \( A \), then for all \( p \geq 2 \)
\[ w_p(B, C) \geq 2^{\frac{1}{p} - 1} \max (\|B + C\|, \|B - C\|), \]
and
\[ w(A) \geq 2^{\frac{1}{p} - 2} \max (\|(1 - i)A + (1 + i)A^*\|, \|(1 + i)A + (1 - i)A^*\|) \]

Proof. Obviously by inequality (2.2) we have the first inequality. For the second we use inequality (2.1). □
Corollary 2.8. If \( B, C \in \mathcal{B}(H) \), then for \( p \geq 1 \)
\[
    w_p(B, C) \geq 2^{\frac{1}{p}-1} \max \{ w(B), w(C) \}.
\] (2.3)

In addition, if \( A = B + iC \) is the Cartesian decomposition of \( A \), then for \( p \geq 2 \)
\[
    w(A) \geq 2^{\frac{1}{p}-2} \max \left( \| A + A^* \|, \| A - A^* \| \right).
\]

Proof. By inequality (2.2) and properties of the numerical radius, we have
\[
    2^\frac{1}{p} w_p(B, C) \geq 2 - 2 \max \left\{ w(B + C), w(B - C) \right\}.
\]
So
\[
    w_p(B, C) \geq 2^{\frac{1}{p}-1} w(B).
\]
By symmetry we conclude that
\[
    w_p(B, C) \geq 2^{\frac{1}{p}-1} \max(w(B), w(C)).
\]
While the second inequality follows easily from inequality (2.1). □

Now we apply part (b) of Lemma 2.3 to find some lower and upper bounds for \( w_p \) \((p \geq 1)\).

Proposition 2.9. Let \( B, C \in \mathcal{B}(H) \). Then for all \( p \geq 2 \),

(i) \( 2^\frac{1}{p}-1 w_p(B + C, B - C) \leq w_p(B, C) \leq 2^{-\frac{1}{p}} w_p(B + C, B - C) \);
(ii) \( 2^\frac{1}{p}-1 (w^p(B + C) + w^p(B - C))^\frac{1}{p} \leq w_p(B, C) \leq 2^{-\frac{1}{p}} (w^p(B + C) + w^p(B - C))^\frac{1}{p} \).

If \( 1 < p \leq 2 \) these inequalities hold in the opposite direction.

Proof. Let \( x \in H \) be a unit vector. Part (b) of Lemma 2.3 implies that for any \( p \geq 2 \)
\[
    2^{1-p}(|a + b|^p + |a - b|^p) \leq |a|^p + |b|^p \leq \frac{1}{2}(|a + b|^p + |a - b|^p) .
\]
Replacing \( a = |\langle Bx, x \rangle| \) and \( b = |\langle Cx, x \rangle| \) in above inequalities we obtain the desired inequalities. □

Remark 2.10. In inequality (2.3), if we take \( B + C \) and \( B - C \) instead of \( B \) and \( C \), then for \( p \geq 1 \)
\[
    w_p(B + C, B - C) \geq 2^{\frac{1}{p}-1} \max \{ w(B + C), w(B - C) \} .
\]

By employing the first inequality of part (i) of Proposition 2.9, we get
\[
    w_p(B, C) \geq 2^{\frac{2}{p}-2} \max \{ w(B + C), w(B - C) \} .
\]
for $p \geq 1$.

Taking $B + C$ and $B - C$ instead of $B$ and $C$ in the second inequality of part (ii) of Proposition 2.9, we reach

$$w_p(B + C, B - C) \leq 2^{1 - \frac{1}{p}} (w^p(B) + w^p(C))^\frac{1}{p}.$$ 

for all $p \geq 1$.

Now by applying the second inequality of part (i) of Proposition 2.9, we infer for $p \geq 1$ that

$$w_p(B, C) \leq 2^{1 - \frac{2}{p}} (w^p(B) + w^p(C))^{\frac{1}{p}}.$$ 

So

$$2^{\frac{2}{p} - 2} \max\{w(B + C), w(B - C)\} \leq w_p(B, C) \leq 2^{1 - \frac{2}{p}} (w^p(B) + w^p(C))^{\frac{1}{p}}.$$ 

Moreover if $B$ and $C$ are self-adjoint, then

$$2^{\frac{2}{p} - 2} \max\{\|B + C\|, \|B - C\|\} \leq w_p(B, C) \leq 2^{1 - \frac{2}{p}} (\|B\|^{p} + \|C\|^{p})^{\frac{1}{p}}$$

for all $p \geq 1$.

In the following result we find another lower bound for $w_p$ ($p \geq 1$).

**Theorem 2.11.** Let $B, C \in \mathbb{B}(\mathcal{H})$. Then for $p \geq 1$

$$w_p(B, C) \geq 2^{\frac{1}{p} - 1} w^\frac{1}{p} (B^2 + C^2).$$

**Proof.** It follows from (2.2) that

$$2^{\frac{2}{p} - 2} w^2(B \pm C) \leq w^2_p(B, C).$$

Hence

$$2w_p^2(B, C) \geq 2^{\frac{2}{p} - 2} \left[ w^2(B + C) + w^2(B - C) \right] \geq 2^{\frac{2}{p} - 2} \left[ w \left( (B + C)^2 \right) + w \left( (B - C)^2 \right) \right] \geq 2^{\frac{2}{p} - 2} \left[ w \left( (B + C)^2 + (B - C)^2 \right) \right] = 2^{\frac{2}{p} - 1} w(B^2 + C^2).$$

It follows that

$$w_p(B, C) \geq 2^{\frac{1}{p} - 1} w^\frac{1}{p} (B^2 + C^2).$$

□
Corollary 2.12. If $A = B + iC$ is the Cartesian decomposition of $A$, then
\[ w_p(B, C) \geq 2^{\frac{1}{p} - 1} \|B^2 + C^2\|^{\frac{1}{p}}. \]

And
\[ w(A) \geq 2^{\frac{1}{p} - \frac{3}{2}} \|A^*A + AA^*\|^{\frac{1}{p}}. \]
for any $p \geq 2$.

Proof. The first inequality is obvious. For the second we have $A^*A + AA^* = 2(B^2 + C^2)$. Now by using inequality (2.1) the proof is complete. □

Corollary 2.13. If $B, C \in B(H)$, then for $p \geq 2$
\[ w_p(B, C) \geq 2^{\frac{2}{p} - \frac{3}{2}} w^{\frac{1}{2}}(B^2 + C^2). \]

Proof. By choosing $B + C$ and $B - C$ instead of $B$ and $C$ in Theorem 2.11 and employing part (i) of Proposition 2.9 we conclude that the desired inequality. □

The following result providing other bound for $w_p (p > 1)$ may be stated as follows:

Proposition 2.14. Let $B, C \in B(H)$. Then
\[ w_p(B, C) \leq w_q \left( \frac{B + C}{2}, \frac{B - C}{2} \right). \]
for any $p \geq 2, 1 < q \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $1 < p \leq 2$, the reverse inequality holds.

Proof. Let $x \in H$ be a unit vector. Part (a) of Lemma 2.3 implies that
\[ |a|^p + |b|^p \leq 2^{\frac{1}{p}} \left( |a + b|^q + |a - b|^q \right)^{\frac{1}{q - 1}}. \]
So
\[ (|a|^p + |b|^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p(p + q)}} \left( |a + b|^q + |a - b|^q \right)^{\frac{1}{q - 1}}. \]
Now replacing $a = \langle Bx, x \rangle$ and $b = \langle Cx, x \rangle$ in the above inequality we conclude that
\[ (\|Bx, x\|^p + \|Cx, x\|^p)^{\frac{1}{p}} \leq \left( \left( \left( \frac{B + C}{2} \right) x, x \right)^q + \left( \left( \frac{B - C}{2} \right) x, x \right)^q \right)^{\frac{1}{q}}. \]
By taking supremum over $x \in H$ with $\|x\| = 1$ we deduce that
\[ w_p(B, C) \leq w_q \left( \frac{B + C}{2}, \frac{B - C}{2} \right) \]
for any $p \geq 2, 1 < q \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. □
Corollary 2.15. Inequality (2.4) implies that
\[ w_p(B, C) \leq \left( w^q \left( \frac{B + C}{2} \right) + w^q \left( \frac{B - C}{2} \right) \right)^{\frac{1}{q}}. \]
for any \( 1 < q \leq 2, p \geq 2 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Further, if \( B \) and \( C \) are self-adjoint, then
\[ w_p(B, C) \leq \frac{1}{2} (\|B + C\|^q + \|B - C\|^q)^{\frac{1}{q}}. \]
If \( 1 < p \leq 2 \), the converse inequalities hold.

Corollary 2.16. If \( B, C \in \mathcal{B}(\mathcal{H}) \), then
\[ w_q \left( \frac{B + C}{2}, \frac{B - C}{2} \right) \leq 2^\frac{1}{p} w_p \left( \frac{B + C}{2}, \frac{B - C}{2} \right). \]
for all \( 1 < p \leq 2 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( p \geq 2 \), the above inequality is valid in the opposite direction.

Proof. By Proposition 2.14 we have
\[ w_q \left( \frac{B + C}{2}, \frac{B - C}{2} \right) \leq w_p(B, C). \]
for all \( 1 < p \leq 2 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Proposition 2.9 follows that
\[ w_p(B, C) \leq 2^\frac{1}{p-1} w_p(B + C, B - C) = 2^\frac{1}{p} w_p \left( \frac{B + C}{2}, \frac{B - C}{2} \right). \]
We therefore get the desired inequality. \( \square \)

3. Inequalities of \( w_p \) for \( n \)-tuples of Operators

In this section, we are going to obtain some numerical radius inequalities for \( n \)-tuples of operators. Some generalization of inequalities in the previous section are also established. According to the definition of numerical radius, we immediately get the following double inequality for \( p \geq 1 \)
\[ w_p(T_1, \ldots, T_n) \leq \left( \sum_{i=1}^{n} w^p (T_i) \right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} w (T_i). \]
An application of Holder’s inequality gives the next result, which is a generalization of inequality (2.2).
Theorem 3.1. Let \((T_1, \ldots, T_n) \in B(\mathcal{H})^{(n)}\) and \(0 \leq \alpha_i \leq 1\), \(i = 1, \ldots, n\), with \(\sum_{i=1}^{n} \alpha_i = 1\). Then

\[ w_p(T_1, \ldots, T_n) \geq w \left( \alpha_1^{\frac{1}{p}} T_1 \pm \alpha_2^{\frac{1}{p}} T_2 \pm \cdots \pm \alpha_n^{\frac{1}{p}} T_n \right) \]

for any \(p > 1\).

Proof. In the Euclidean space \(\mathbb{R}^n\) with the standard inner product, Holder’s inequality

\[ \sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}} \]

holds, where \(p\) and \(q\) are in the open interval \((1, \infty)\) with \(\frac{1}{p} + \frac{1}{q} = 1\) and \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n\). For \((y_1, \ldots, y_n) = (\alpha_1^{\frac{1}{p}}, \ldots, \alpha_n^{\frac{1}{p}})\) we have

\[ \sum_{i=1}^{n} |\alpha_i^{\frac{1}{p}} x_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |\alpha_i^{\frac{1}{p}}|^q \right)^{\frac{1}{q}} \]

Thus

\[ \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \geq \sum_{i=1}^{n} |\alpha_i^{\frac{1}{p}} x_i| \]

Choosing \(x_i = |\langle T_i x, x \rangle|, i = 1, \ldots, n\), we get

\[ \left( \sum_{i=1}^{n} |\langle T_i x, x \rangle|^p \right)^{\frac{1}{p}} \]

\[ \geq \sum_{i=1}^{n} \left| \langle \alpha_i^{\frac{1}{p}} T_i x, x \rangle \right| \]

\[ \geq \left| \langle \alpha_1^{\frac{1}{p}} T_1 x, x \rangle \pm \langle \alpha_2^{\frac{1}{p}} T_2 x, x \rangle \pm \cdots \pm \langle \alpha_n^{\frac{1}{p}} T_n x, x \rangle \right| \]

\[ = \left| \langle \alpha_1^{\frac{1}{p}} T_1 \pm \alpha_2^{\frac{1}{p}} T_2 \pm \cdots \pm \alpha_n^{\frac{1}{p}} T_n \rangle x, x \rangle \right| \]

Now the result follows by taking the supremum over all unit vectors in \(\mathcal{H}\). \(\square\)

Now we give another upper bound for the powers of \(w_p\). This result has several inequalities as special cases, which considerably generalize the second inequality of \(1.1\).
Theorem 3.2. Let \((T_1, \ldots, T_n), (A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathbb{B}(\mathcal{H})^{(n)}\) and let \(f\) and \(g\) be nonnegative continuous functions on \([0, \infty)\) satisfying \(f(t)g(t) = t\) for all \(t \in [0, \infty)\). Then

\[
w_p^r (A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^n \left[ B_i^* f^2(|T_i|) B_i \right]^{rp} + \left[ A_i^* g^2(|T_i^*|) A_i \right]^{rp} \right\|
\]

for \(p \geq 1\) and \(r \geq 1\).

Proof. Let \(x \in \mathcal{H}\) be a unit vector.

\[
\sum_{i=1}^n |\langle A_i^* T_i B_i x, x \rangle|^p
\]

\[
= \sum_{i=1}^n |\langle T_i B_i x, A_i x \rangle|^p
\]

\[
\leq \sum_{i=1}^n \|f(|T_i|) B_i x\| \|g(|T_i^*|) A_i x\|^p \quad \text{(by Lemma 2.1(c))}
\]

\[
= \sum_{i=1}^n \langle f(|T_i|) B_i x, f(|T_i|) B_i x \rangle^{\frac{p}{2}} \langle g(|T_i^*|) A_i x, g(|T_i^*|) A_i x \rangle^{\frac{p}{2}}
\]

\[
= \sum_{i=1}^n \langle B_i^* f^2(|T_i|) B_i x, x \rangle^{\frac{p}{2}} \langle A_i^* g^2(|T_i^*|) A_i x, x \rangle^{\frac{p}{2}}
\]

\[
\leq \sum_{i=1}^n \langle (B_i^* f^2(|T_i|) B_i)^p x, x \rangle^{\frac{p}{2}} \langle (A_i^* g^2(|T_i^*|) A_i)^p x, x \rangle^{\frac{p}{2}}
\]

(by Lemma 2.2(a))

\[
\leq \sum_{i=1}^n \left( \frac{1}{2} \left( \langle (B_i^* f^2(|T_i|) B_i)^p x, x \rangle + \langle (A_i^* g^2(|T_i^*|) A_i)^p x, x \rangle \right) \right)^{\frac{1}{p}}
\]

(by Lemma 2.1(a))

\[
\leq \sum_{i=1}^n \left( \frac{1}{2} \left( \langle (B_i^* f^2(|T_i|) B_i)^{rp} + (A_i^* g^2(|T_i^*|) A_i)^{rp} \rangle x, x \right) \right)^{\frac{1}{p}}
\]

(by Lemma 2.2(a))

\[
\leq \left( \frac{1}{2} \left( \sum_{i=1}^n \left( \langle (B_i^* f^2(|T_i|) B_i)^{rp} + (A_i^* g^2(|T_i^*|) A_i)^{rp} \rangle x, x \right) \right) \right)^{\frac{1}{p}}
\]
Thus
\[
\left( \sum_{i=1}^{n} |(A_i^* T_i B_i x, x)|^p \right)^\frac{r}{p} \leq \frac{1}{2} \left( \sum_{i=1}^{n} \left( (B_i^* f^2 (|T_i|) B_i)^{rp} + (A_i^* g^2 (|T_i^*|) A_i)^{rp} \right) x, x \right)
\]

Now the result follows by taking the supremum over all unit vectors in \( \mathcal{H} \).

Choosing \( A = B = I \), we get.

**Corollary 3.3.** Let \((T_1, \ldots, T_n) \in \mathfrak{B} (\mathcal{H})^{(n)}\) and let \( f \) and \( g \) be nonnegative continuous functions on \([0, \infty)\) satisfying \( f (t) g (t) = t \) for all \( t \in [0, \infty) \). Then
\[
w_p^{(r \alpha)} (T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( f^{2rp} (|T_i|) + g^{2rp} (|T_i^*|) \right) \right\|
\]
for \( p \geq 1 \) and \( r \geq 1 \).

Letting \( f (t) = g (t) = t^{\frac{1}{2}} \), we get.

**Corollary 3.4.** Let \((T_1, \ldots, T_n), (A_1, \ldots, A_n), (B_1, \ldots, B_n)\) are in \( \mathfrak{B} (\mathcal{H})^{(n)} \). Then
\[
w_p^{(r \alpha)} (A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( B_i^* |T_i| B_i)^{rp} + (A_i^* |T_i^*| A_i)^{rp} \right) \right\|
\]
for \( p \geq 1 \) and \( r \geq 1 \).

**Corollary 3.5.** Let \((A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathfrak{B} (\mathcal{H})^{(n)} \). Then
\[
w_p^{(r \alpha)} (A_1^* B_1, \ldots, A_n^* B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |B_i|^{2rp} + |A_i|^{2rp} \right) \right\|
\]
for \( p \geq 1 \) and \( r \geq 1 \).

**Corollary 3.6.** Let \((T_1, \ldots, T_n) \in \mathfrak{B} (\mathcal{H})^{(n)} \). Then
\[
w_p (T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |T_i|^{2p} + |T_i^*|^{2(1-\alpha)p} \right) \right\|
\]
for \( 0 \leq \alpha \leq 1 \), and \( p \geq 1 \). In particular,
\[
w_p (T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |T_i|^p + |T_i^*|^p \right) \right\|.
\]
Corollary 3.7. Let $B, C \in \mathbb{B}(\mathcal{H})$. Then
\[
wp(B, C) \leq \frac{1}{2} \left\| |B|^{2\alpha} + |B^*|^{2(1-\alpha)} + |C|^{2\alpha} + |C^*|^{2(1-\alpha)} \right\|
\]
for $0 \leq \alpha \leq 1$, and $p \geq 1$. In particular,
\[
wp(B, C) \leq \frac{1}{2} \left\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \right\|.
\]

The next results are related to some different upper bounds for $w_p$ for $n$-tuples of operators, which have several inequalities as special cases.

Proposition 3.8. Let $(T_1, \ldots, T_n) \in \mathbb{B}(\mathcal{H})^n$. Then
\[
w_p(T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \right) \right\|^{\frac{1}{p}}
\]
for $0 \leq \alpha \leq 1$, and $p \geq 1$.

Proof. By using the arithmetic-geometric mean, for any unit vector $x \in \mathcal{H}$ we have
\[
\sum_{i=1}^{n} |\langle T_i x, x \rangle|^p \leq \sum_{i=1}^{n} \left( |\langle T_i^{2\alpha} x, x \rangle|^{\frac{1}{2}} \langle |T_i^{2(1-\alpha)} x, x \rangle^{\frac{1}{2}} \right)^p
\]
(by Lemma 2.1(b))
\[
\leq \frac{1}{2p} \sum_{i=1}^{n} \left( |\langle T_i^{2\alpha} x, x \rangle| + |\langle T_i^{2(1-\alpha)} x, x \rangle| \right)^p
\]
\[
= \frac{1}{2p} \sum_{i=1}^{n} \left( |\langle T_i^{2\alpha} + |T_i^{2(1-\alpha)}| x, x \rangle| \right)^p.
\]
\[
\leq \frac{1}{2p} \sum_{i=1}^{n} \left( |\langle T_i^{2\alpha} + |T_i^{2(1-\alpha)}| x, x \rangle| \right)^p.
\]
(by Lemma 2.2(a))

Now the result follows by taking the supremum over all unit vectors in $\mathcal{H}$. \qed

Proposition 3.9. Let $(T_1, \ldots, T_n) \in \mathbb{B}(\mathcal{H})^n$. Then
\[
w_p(T_1, \ldots, T_n) \leq \left\| \sum_{i=1}^{n} (\alpha |T_i|^p + (1 - \alpha) |T_i^*|^p) \right\|^{\frac{1}{p}}
\]
for $0 \leq \alpha \leq 1$, and $p \geq 2$. 
Proof. For every unit vector $x \in \mathcal{H}$, we have

$$\sum_{i=1}^{n} |\langle T_i x, x \rangle|^p = \sum_{i=1}^{n} (|\langle T_i x, x \rangle|^2)^{\frac{p}{2}} \leq \sum_{i=1}^{n} \left( \langle |T_i|^{2\alpha} x, x \rangle \langle |T_i^*|^{2(1-\alpha)} x, x \rangle \right)^{\frac{p}{2}} \text{ (by Lemma 2.1(b))}$$

$$\leq \sum_{i=1}^{n} \langle |T_i|^{\alpha p} x, x \rangle \langle |T_i^*|(1-\alpha)^p x, x \rangle \text{ (by Lemma 2.2(a))}$$

$$\leq \sum_{i=1}^{n} \langle |T_i|^p x, x \rangle^\alpha \langle |T_i^*|^p x, x \rangle^{(1-\alpha)} \text{ (by Lemma 2.2(b))}$$

$$\leq \sum_{i=1}^{n} \left( \alpha \langle |T_i|^p x, x \rangle + (1-\alpha) \langle |T_i^*|^p x, x \rangle \right) \text{ (by Lemma 2.1(a))}$$

$$\leq \sum_{i=1}^{n} \left( \alpha |T_i|^p + (1-\alpha) |T_i^*|^p \right) x, x \right)$$

$$= \left\langle \left( \sum_{i=1}^{n} \left( \alpha |T_i|^p + (1-\alpha) |T_i^*|^p \right) \right) x, x \right\rangle.$$

Now the result follows by taking the supremum over all unit vectors in $\mathcal{H}$. □

Remark 3.10. As special cases,

1. For $\alpha = \frac{1}{2}$, we have

$$w_p^p (T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \left( |T_i|^p + |T_i^*|^p \right) \right\|.$$

2. For $B, C \in \mathcal{B}(\mathcal{H}), 0 \leq \alpha \leq 1$, and $p \geq 1$, we have

$$w_p^p (B, C) \leq \|\alpha |B|^p + (1-\alpha) |B^*|^p + \alpha |C|^p + (1-\alpha) |C^*|^p \|.$$

In particular,

$$w_p^p (B, C) \leq \frac{1}{2} \left\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \right\|.$$

The next result reads as follows.

Proposition 3.11. Let $(T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n, 0 \leq \alpha \leq 1, r \geq 1$ and $p \geq 1$. Then

$$w_p^p (T_1, \ldots, T_n) \leq \left( \sum_{i=1}^{n} \|\alpha |T_i|^{2r} + (1-\alpha) |T_i^*|^{2r} \|^\frac{p}{2r} \right)^\frac{1}{p}.$$
Proof. Let \( x \in \mathcal{H} \) be a unit vector.

\[
\sum_{i=1}^{n} |\langle T_i x, x \rangle|^p = \sum_{i=1}^{n} \left( |\langle T_i x, x \rangle|^2 \right)^{\frac{p}{2}} \leq \sum_{i=1}^{n} \left( |\langle T_i^2 x, x \rangle|^{2(1-\alpha)} x, x \rangle \right)^{\frac{p}{2}} \tag{by Lemma 2.1(b)}
\]

\[
\leq \sum_{i=1}^{n} \left( \alpha |\langle T_i^2 x, x \rangle|^{r} + (1 - \alpha) \langle |T_i^*|^2 x, x \rangle \right)^{\frac{p}{2r}} \tag{by Lemma 2.1(a)}
\]

\[
\leq \sum_{i=1}^{n} \langle |T_i|^2 + (1 - \alpha) |T_i^*|^2 \rangle x, x \rangle^{\frac{p}{2r}}. \tag{by Lemma 2.2(a)}
\]

Now the result follows by taking the supremum over all unit vectors in \( \mathcal{H} \). \( \square \)

Remark 3.12. Some special cases can be stated as follows:

1. For \( \alpha = \frac{1}{2} \), we have

\[
w_p (T_1, \ldots, T_n) \leq \left( \frac{1}{2^{\frac{1}{2r}}} \sum_{i=1}^{n} \| |T_i|^2 + |T_i^*|^2 \|^{\frac{r}{p}} \right)^{\frac{p}{r}}.
\]

2. For \( B, C \in \mathbb{B}(\mathcal{H}), 0 \leq \alpha \leq 1 \), and \( p \geq 1 \), we have

\[
w_p (B, C) \leq \left( \| \alpha |B|^2 + (1 - \alpha) |B^*|^2 \| \right)^{\frac{r}{p}} + \| \alpha |C|^2 + (1 - \alpha) |C^*|^2 \| \right)^{\frac{r}{p}}.
\]

In particular,

\[
w_p (B, C) \leq \frac{1}{2^{\frac{1}{2r}}} \left( \| |B|^2 + |B^*|^2 \| \right)^{\frac{r}{p}} + \| |C|^2 + |C^*|^2 \| \right)^{\frac{r}{p}}.
\]
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1 Department of Pure Mathematics, Center Of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran

E-mail address: moslehian@um.ac.ir
E-mail address: msattari.b@gmail.com

2 Department of Applied Sciences, Al-Balqa’ Applied University Salt, Jordan; Department of Mathematics, College of Science, Qassim University, Qassim, Saudi Arabia

E-mail address: khalid@bau.edu.jo