We derive exact equations that determine the spectra of undirected and directed sparse-regular graphs containing loops of arbitrary length. The implications of our results to the structural and dynamical properties of networks are discussed by showing how loops influence the size of the spectral gap and the propensity for synchronization. Analytical formulas for the spectrum are obtained for specific length of the loops.

In this letter we present a systematic study of the spectra of regular Husimi graphs containing undirected or directed edges, going beyond previous studies on local tree-like networks without short loops. We analyze the influence of loops on some important network properties: the size of the spectral gap and the stability of synchronized states. The simplicity and exactness of our equations, confirmed by direct diagonalization methods, leads to accurate results for arbitrary loop lengths and allows for an extension of the Kesten-McKay law to triangular and square undirected Husimi graphs as well as to directed regular graphs without short loops.

**Sparse regular graphs with loops.** We consider the ensemble of $(\ell,k)$-regular (un)directed Husimi graphs containing $N$ vertices or nodes. Each vertex is incident to $k > 1$ loops composed of $\ell$ nodes, with $k$ and $\ell$ independent of $N$. The indegree and outdegree of any node are equal to each other, and given by $2k$ or $k$ in the case of undirected or directed Husimi graphs, respectively. For $N \to \infty$ the graphs have a local tree-like structure on the level of loops, illustrated in figure 1 for triangular ($\ell = 3$) and square ($\ell = 4$) Husimi graphs. The model allows to interpolate between $\ell = 2$ and $\ell \to \infty$, both cases representing situations where short loops are absent.

We study the spectral density of the $N \times N$ adjacency matrix $J$ for $N \to \infty$, which is trivially related to the
spectrum of the Laplacian matrix in the case of regular graphs. The matrix element $J_{ij}$ assumes 1 if there is a directed edge from node $i$ to node $j$, and zero otherwise. Denoting the eigenvalues of a given instance of $J$ as $\{\lambda_i\}_{i=1,\ldots,N}$, the spectral density is defined as $\rho(\lambda) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$. The matrix $J$ is symmetric or asymmetric depending whether the graph is undirected or directed, respectively. The eigenvalues are real in the former case and complex in the latter. The local tree-like structure shown in figure 1 allows to calculate $\rho(\lambda)$ exactly for $N \to \infty$.

Spectra of undirected Husimi graphs The resolvent $G(z)$ of $J$ is defined through $G(z) = (z - J)^{-1}$, where the complex variable $z = \lambda - i\epsilon$ contains a regularizer $\epsilon$. The spectrum is extracted from the diagonal components of $G(z)$ according to $\rho(\lambda) = \lim_{N \to \infty, \epsilon \to 0^+} (\pi N)^{-1} \text{Im} \text{Tr} G(\lambda - i\epsilon)$.

Due to the absence of disorder, a closed expression can be derived for the diagonal elements $G_{ii}(z) = G(z)$, $\forall i$. For graphs without short loops, either one writes $G_{ii}(z)$ as the variance of a Gaussian function and uses the cavity method (or the replica method) to obtain the local convergence of graphs to a tree. Generalizing these methods to Husimi graphs, we have derived the following equation for $\rho(\lambda)$

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Im} [z - k G_s]^{-1} , \quad (1)$$

where $G_s$ solves

$$G_s = J_s^T \left[ (z - (k - 1) G_s) I_{\ell-1} - L_{\ell-1} - L_{\ell-1}^T \right]^{-1} J_s , \quad (2)$$

with $I_{\ell-1}$ the $(\ell-1) \times (\ell-1)$ identity matrix, $L_{\ell-1}$ the $(\ell-1)$-dimensional matrix with elements $[L_{\ell-1}]_{ij} = \delta_{i,j-1}$, and $J_s^T$ the $(\ell-1)$-dimensional vector $J_s^T = (1 \ 0 \ldots 0\ 1)$. For $\ell = 2$, the solution of eq. (2) yields the Kesten-McKay law [10], where $\rho(\lambda)$ takes the form

$$\rho(\lambda) = \frac{k}{2\pi} \frac{\sqrt{4(k-1) - \lambda^2}}{k^2 - \lambda^2} \quad (3)$$

for $|\lambda| < 2\sqrt{k-1}$, and zero otherwise. For $\ell > 2$, we have inverted the matrix in eq. (2) [19], leading to

$$G_s = \frac{2\alpha_{\ell-2} + 2}{\alpha_{\ell-1}} , \quad (4)$$

where the coefficients $\alpha_2, \ldots, \alpha_{\ell-1}$ follow from the recurrence relation $\alpha_i = \alpha_1 \alpha_{i-1} - \alpha_{i-2}$, with $\alpha_0 = 1$ and $\alpha_1 = z - (k - 1) G_s$. Equation (4) leads to a polynomial in the variable $G_s$ and can be solved analytically for smaller values of $\ell$, extending the Kesten-McKay law to regular graphs containing short loops. For larger values of $\ell$ a straightforward numerical solution can be obtained, giving sharp results for $\rho(\lambda)$. Equation (4) is one of the main results of our work, allowing to compute exactly the spectrum for increasing values of $\ell$.

For $\ell = 3$ we recover the analytical expression for $\rho(\lambda)$ presented in [16]. For $\ell = 4$ eq. (4) becomes a cubic polynomial with discriminant

$$D(\lambda) = -\frac{2}{3} \lambda^4 - \frac{2}{3} (k^2 - 22k + 13) + \frac{8}{3} (k - 2)^3 . \quad (5)$$

Defining the functions $s_{\pm}(\lambda) = 9\lambda(k+1) - \lambda^3 \pm 9\sqrt{D(\lambda)}$ and $q_{\pm}(\lambda) = s_{\pm}^{1/3} \pm s_{\pm}^{1/3}$, the spectrum of square Husimi graphs reads

$$\rho(\lambda) = \frac{6 \sqrt{3} k(k - 1) q_{-}(\lambda)}{\pi \left[ 2(k - 3) \lambda + k q_{+}(\lambda) \right]^2 + 3 \pi k^2 q_{-}^2(\lambda)} \quad (6)$$

for $D(\lambda) > 0$, and $\rho(\lambda) = 0$ otherwise. The edges of $\rho(\lambda)$ solve the equation $D(\lambda) = 0$. The analytic expression for some higher values of $\ell$ is given elsewhere [17].

In figure 2 we compare direct diagonalization results of finite matrices with the solution to eq. (4), for $k = 2$ and several values of $\ell$. The agreement is excellent, following from the exactness of eq. (4) for $N \to \infty$. When rescaling the matrix elements $J_{ij} \to J_{ij}/\sqrt{2k-1}$ we find analytically the convergence of $\rho(\lambda)$ to the Wigner semicircle law for $k \to \infty$ and arbitrary $\ell$ [18]. Interestingly, the spectrum of a square Husimi graph exhibits a striking similarity with the spectrum of the two-dimensional square Bravais lattice and the Kesten-McKay law are presented for comparison.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Spectrum of ($\ell,k$) undirected Husimi graphs with $k = 2$ and $J_{ij} \to J_{ij}/\sqrt{2k-1}$, obtained by solving eqs. (1) and (4). The symbols are direct diagonalization results of adjacency matrices of size $N = 10^4$. The spectrum of the two-dimensional square Bravais lattice and the Kesten-McKay law are presented for comparison.}
\end{figure}
not to the dimensional nature of lattices [20]. For \( \ell \to \infty \), the spectrum converges to the Kesten-Mckay law with degree \( 2k \) [4], as illustrated in figure 2 for \( \ell = 10 \). Therefore, loops composed of ten nodes can be neglected and the graph can be considered locally tree-like [10, 11].

**Spectra of directed Husimi graphs** In the case of directed Husimi graphs, the density of states \( \rho(\lambda) \) at a certain point \( \lambda = x + iy \) of the complex plane can be written as \( \rho(\lambda) = \lim_{N \to \infty} (N\pi)^{-1} \partial^* \text{Tr} G(\lambda) \), where \( \partial^* = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \) and \( G(\lambda) = (\lambda - J)^{-1} \). The operation \((\cdot)^*\) denotes complex conjugation. Due to the non-analytic behavior of \( G_{ii}(\lambda) \) in the complex plane [21], it is convenient to define the \( 2N \times 2N \) block matrix [22]

\[
H_{\ell}(\lambda) = \begin{pmatrix}
\mathbf{i}I_N \\
-\mathbf{i}(-\mathbf{J}^T \mathbf{J})^T
\end{pmatrix}.
\]

The \( N \times N \) lower-left block of \( \lim_{\ell \to \infty} (-1)^{\ell} H_{\ell}^{-1}(\lambda) \) is precisely the matrix \( G(\lambda) \). Thus, the problem reduces to calculating the matrix elements \( G_{ij}(\lambda, \epsilon) \) of the directed Husimi graphs [17], from which the spectrum is determined according to \( \rho(\lambda) = -\frac{1}{N\pi} \lim_{N \to \infty, \epsilon \to 0+} \sum_{j=1}^{N} \partial^* G_{ij}(\lambda, \epsilon) \).

By representing \( (-1)^{\ell} H_{\ell}^{-1}(\lambda) \) as a Gaussian integral one can generalize the cavity method, as developed for sparse non-Hermitian random matrices [22], to calculate the spectrum of directed Husimi graphs [17]. Due to the absence of disorder we have that \( G_{ij}(\lambda, \epsilon) = G(\lambda, \epsilon), \forall j \), and \( \rho(\lambda) \) is given by

\[
\rho(\lambda) = \frac{1}{i\pi} \lim_{\epsilon \to 0} \partial^* \left[ S(\lambda) + k G_{\ell} \right]^{-1},
\]

where \( S(\lambda) = \{\mathbf{i} \mathbf{I}_2 - \mathbf{i}(x\sigma_x - y\sigma_y)\} \) and \( \{\sigma_x, \sigma_y\} \) are Pauli matrices. For \( \ell > 2 \), the two-dimensional matrix \( G_{\ell} \) is obtained by

\[
G_{\ell} = J_{2}^{\ell} \left[ (S(\lambda) + (k - 1) G_{\ell}) \mathbf{I} - \mathbf{I}_{1} \right] \\
+ i J_{\ell - 1} \mathbf{J}^{T} \mathbf{J}_{\ell - 1}^{-1} \mathbf{J}_{\ell - 1}
\]

where \( J_{\ell} \) is the \( 2 \times 2(\ell - 1) \) block matrix \( J_{\ell} = (\mathbf{J} 0 \ldots 0 \mathbf{J}^{T}) \), with \( J = \frac{1}{2}(\sigma_x + i\sigma_y) \). The derivative of eq. (9) allows to derive the number of critical points for the spectrum of directed Husimi graphs as a function of \( \ell \).

In figure 3 we present the spectrum \( \rho(\lambda) \) for \( \ell = 3 \) and \( k = 2 \), comparing the solution to eqs. (8,9) with direct diagonalization results. The agreement is excellent. A prominent feature of \( \rho(\lambda) \) is the \( \ell \)-fold rotational symmetry due to the transformation properties of \( G_{\ell} \) under rotations of \( 2\pi/\ell \). By rescaling \( J_{ij} \to J_{ij}/\sqrt{k - 1} \), we find analytically the convergence of \( \rho(\lambda) \) to Girko’s circular law for \( k \to \infty \) and arbitrary \( \ell \) [18].

Analogously to undirected Husimi graphs, \( \rho(\lambda) \) converges to the spectrum of a directed regular graph without short loops for \( \ell \to \infty \). In this case, we find a remarkable extension of the Kesten-Mckay law, Eq. (3), to directed graphs, where \( \rho(\lambda) \) takes the form

\[
\rho(\lambda) = \frac{k - 1}{k^{2} - |\lambda|^{2}},
\]

for \( |\lambda|^{2} < k \), and zero otherwise. A comparable equation appeared in [22], but with a different support of \( \rho(\lambda) \).

In inset (b) of figure 3 we plot the boundary of \( \rho(\lambda) \) for \( k = 2 \) and increasing values of \( \ell \). In accordance with eq. (10), the boundary converges to the circle \( |\lambda|^{2} = k \) in the limit \( \ell \to \infty \). For \( \ell = 10 \) we have obtained numerically that \( \rho(\lambda) \) is given approximately by eq. (10) and the graph becomes locally tree-like [22].

**Structural and dynamical properties** Let us order the eigenvalues of a regular undirected Husimi graph as \( \lambda_{1} < \lambda_{2} < \cdots < \lambda_{N} \), where \( \lambda_{N} = 2k \). The spectral gap \( g \) and the eigenratio \( Q \) are, respectively, defined by \( g \equiv (\lambda_{N} - \lambda_{N-1})/2k \) and \( Q \equiv (\lambda_{N} - \lambda_{1})/(\lambda_{N} - \lambda_{N-1}) \). Analogously, for regular directed Husimi graphs, the eigenvalues can be ordered according to their real parts \( \text{Re} \lambda_{1} < \text{Re} \lambda_{2} < \cdots < \text{Re} \lambda_{N} \), with \( \text{Re} \lambda_{N} = k \). In this case, the spectral gap \( g \) and the eigenratio \( Q \) are given by \( g \equiv (\text{Re} \lambda_{N} - \text{Re} \lambda_{N-1})/k \) and \( Q \equiv (\text{Re} \lambda_{N} - \text{Re} \lambda_{1})/(\text{Re} \lambda_{N} - \text{Re} \lambda_{N-1}) \). The spectral gap \( g \) controls the speed of convergence to the stationary state of diffusion processes on the graph [1]. Designing communication networks with a large \( g \) is known to be important due to improved robustness and communication properties [2, 3], for undirected networks. The eigenratio \( Q \) measures the propensity for
synchronization in networks of oscillators. A linear stability analysis shows that synchronized states are more stable for smaller values of Q.

Figure 4 depicts g and Q as functions of ℓ for regular Husimi graphs, showing that g increases while Q decreases for increasing values of ℓ. For undirected Husimi graphs, g and Q converge, respectively, to \((k - \sqrt{2k - 1})/k\) and \(2k/(k - \sqrt{2k - 1})\) as \(\ell \to \infty\), consistent with the Alon-Boppana bound for the second largest eigenvalue. For directed Husimi graphs g and Q converge to \((k - \sqrt{k})/k\) and \(2k/(k - \sqrt{k})\), respectively. In summary, short loops have a negative influence on the synchronization properties and on the size of the spectral gap, which is more pronounced at low connectivities.

Conclusions We have determined the spectrum of sparse regular random graphs with short loops through a set of exact equations, including extensions of the Kesten-McKay law to triangular and square undirected Husimi graphs as well as to directed regular graphs without short loops. We find that short loops in directed and undirected networks have a negative influence on the stability of synchronized states, they also worsen the communication properties due to a decrease of the spectral gap. Our spectral results make the absence of loops in network construction apparent, while neural networks are under-short looped. For the square Husimi graph we recover a singularity at the origin, which is also present in a square Bravais lattice. Overall, we find that the spectra of Bravais lattices are similar to the spectra of Husimi graphs with suitable neighborhoods, indicating that Husimi graphs serve as good toy models for Bravais lattices. Our results on spectra of sparse random matrices are of wide interest to diverse fields including the study of Markov chains, dynamics of spin-glasses, etc. Since our work is mainly based upon the cavity method, it allows for an extension to e.g. irregular graphs with loops and eigenvector localization studies.

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