SINGULAR PERTURBED RENORMALIZATION GROUP THEORY AND ITS APPLICATION TO HIGHLY OSCILLATORY PROBLEMS

WENLEI LI
College of Mathematics, Jilin University, Changchun 130012, China
and
Beijing Computational Science Research Center, ZPark II, No. 10 Dongbeiwang West Road, Haidian District, Beijing 100094, China

SHAOYUN SHI
College of Mathematics & State key laboratory of automotive simulation and control, Jilin University, Changchun 130012, China

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Abstract. Renormalization group method in the singular perturbation theory, originally introduced by Chen et al, has been proven to be very practicable in a large number of singular perturbed problems. In this paper, we will firstly reconsider the Renormalization group method under some general conditions to get several newly rigorous approximate results. Then we will apply the obtained results to investigate a class of second order differential equations with the highly oscillatory phenomenon of highly oscillatory properties, which occurs in many multiscale models from applied mathematics, physics and material science, etc. Our strategy, in fact, can be also used to analyze the same problem for related evolution equations with multiple scales, such as nonlinear Klein-Gordon equations in the nonrelativistic limit regime.

1. Introduction. Renormalization group (RG) method in the singular perturbation theory was originally introduced by Chen et al [4, 5] in 1980s, inspired from the classical renormalization idea in quantum mechanics [10]. The main goal of this method is to compute the effective approximate solution of different kinds of singular perturbation problems in a unified manner. So far, it has been proven to be very practicable in a large number of singular perturbed problems, such as secular problem, boundary layer problem, center manifold problem etc.[5, 7, 8, 9, 12, 15, 16, 19].

In 1999, Ziane [21] considered a system of differential equations

\[
\begin{align*}
\dot{x} + \frac{1}{\varepsilon}Ax &= f(x), \\
x(0) &= x_0,
\end{align*}
\]

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Corresponding author: Shaoyun Shi (shisy@jlu.edu.cn).

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where \( A \) is a complex diagonalizable matrix, \( f(x) \) is a polynomial nonlinear term, and \( \varepsilon \) is a small parameter. After a typically systematic renormalization procedure, he proved two approximate results under certain assumptions.

Recently, Chiba [7] considered a more general class of singular perturbation problems as the following form

\[
\dot{x} = \varepsilon g(x, t, \varepsilon), \quad x \in U,
\]

where \( U \) is an open set in \( \mathbb{C}^n \) and the closure \( \bar{U} \) is compact. \( g(x, t, \varepsilon) \) is a vector field parameterized by \( \varepsilon \in \mathbb{R}_+ \), he presented a higher order RG theory for (1) with a key assumption that the nonlinear terms are almost-periodic in \( t \), and the set of corresponding Fourier exponents having no accumulation on \( \mathbb{R} \). Moreover, his work turns out that, in many cases, RG theory can also lead to the existence of approximate invariant manifolds, inheritance of symmetries from those for the original equation to those for the RG equation, and unify traditional singular perturbation methods, such as the averaging method, the multiple time scale method and the center manifold reduction, etc.

A natural equation is, how about the renormalization group approach to the problems with more general conditions? The first part of the present paper is to deal with this topic, in detail, we will consider equation (1) with the following assumptions on \( g(x, t, \varepsilon) \):

\begin{itemize}
  \item [(G_1)]: \( g(x, t, \varepsilon) \) is \( C^2 \) in \( x \in U \), \( C^1 \) in \( t \) and analytic in \( \varepsilon \in I_0 \), with \( I_0 \subset \mathbb{R} \) an open neighborhood of the origin. Furthermore, \( g(x, t, \varepsilon) \) is Lipschitz continues in \( x \) on \( U \), i.e., there exists a constant \( L_U > 0 \) such that
  \[
  \|g(x, t, \varepsilon) - g(y, t, \varepsilon)\| \leq L_U \|x - y\|,
  \]
  for all \( x, y \in U, t \in \mathbb{R}, \varepsilon \in I_0 \).
  \item [(G_2)]: \( g(x, t, \varepsilon) \) is sufficiently smooth in \( (x,t,\varepsilon) \) and quasi-periodic in \( t \), i.e., it can be rewritten as the following form
  \[
  g(x, t, \varepsilon) = \sum_{\nu \in \mathbb{Z}^d} g_\nu(x, \varepsilon)e^{i(\omega, \nu)t} = \sum_{\nu \in \mathbb{Z}^d} \varepsilon \nu \sum_{l=0}^{\infty} \varepsilon l g_{\nu l}(x)e^{i(\omega, \nu)t},
  \]
  where \( \omega \in \mathbb{R}^d \) is a constant vector, \( \hat{1} = \sqrt{-1} \), and
  \[
  \max\{\|g_{\nu l}(x)\|, \|\nabla g_{\nu l}(x)\|\} \leq \Gamma_0 \Gamma_1^l e^{-\sigma|\nu|}
  \]
  is valid for some \( \Gamma_0, \Gamma_1, \sigma > 0 \) and any \( l \in \mathbb{N}, x \in U \).
\end{itemize}

On the other hand, we are also interested in a class of highly oscillatory second order differential equations

\[
\begin{cases}
\epsilon^2 \ddot{y}(t) + (A + B/\varepsilon)\dot{y}(t) + f(y(t)) = 0, & y \in \mathbb{C}^d, \ t > 0, \\
y(0) = \eta_1, \quad \dot{y}(0) = \eta_2.
\end{cases}
\]

where " denotes the derivatives of \( y \) respective to the time \( t \), \( \varepsilon \) is a small real parameter, \( A, B \in \mathbb{R}^{d \times d} \) are nonnegative definite matrix and positive definite matrix, respectively, \( f : \mathbb{C}^d \to \mathbb{C}^d \) is a vector-valued analytic function with gauge invariance, i.e.,

\[
\mathbf{f}(\epsilon \mathbf{x}) = \mathbf{e}^{i \mathbf{x}} \mathbf{f}(\mathbf{y}), \quad s \in \mathbb{R}.
\]

This model comes from a numerical study made by Bao et al [1] of problem (4) with \( d = 1 \) and \( B = E \), here \( E \) denotes the identity matrix. They are mainly interested in the numerical methods to solve a family of highly oscillation problems
like the following nonlinear Klein-Gordon equation in the nonrelativistic limit regime [2]

\[
\begin{aligned}
\varepsilon^2 \partial^2_t u - \Delta u + \frac{1}{\varepsilon} u + f(u) &= 0, \ x \in \mathbb{R}^n, \ t > 0, \\
u(x, 0) &= \phi(x), \ \partial_t u(x, 0) = \frac{1}{\varepsilon} \gamma(x),
\end{aligned}
\]

where \( u = u(x, t) \) is a real-valued function, \( 0 < \varepsilon \ll 1 \) is scaled to be inversely proportional to the speed of light, \( \phi \) and \( \gamma \) are given real-valued functions, \( f(u) \) is a dimensionless real-valued function independent of \( \varepsilon \) and satisfies \( f(0) = 0 \). The model (4) was proposed from (5) by finite difference or spectral discretization with a fixed mesh size (see details in [2]). And both of these two models have a same oscillatory phenomenon that propagates high oscillatory waves with wavelength at \( O(\varepsilon^2) \) and amplitude at \( O(1) \), which ultimately causes many difficulties in the asymptotic analysis and severe burdens in practical computation, making the numerical approximation extremely challenging and costly in the regime of \( 0 < \varepsilon \ll 1 \).

In their recent work about (4) with \( d = 1, B = E \), Bao et al [1] developed so called multiscale time integrators method give more effective numerical results compared with some classical numerical integrators, such as the finite difference methods and exponential wave integrators used in [2]. Meanwhile, to secure the validity of their above results, one assumption was necessarily proposed, i.e., the exact solution \( y(t) \) of (4) satisfies

\[
\|y^{(k)}(t)\| \leq \frac{M}{\varepsilon^{2k}}, \ x \in [0, T], \ k = 0, 1, 2, \ldots
\]

for some constant \( M > 0 \) and \( 0 < T < T^* \) with \( T^* \) the maximum time, here \( \| \cdot \| \) represents the Euclidean norm. A similar assumption was also given in their work about equation (5).

In fact, the computational difficulties referred above are caused by the existence of secular terms in the naive approximate solution, this lead us to make a complete investigation of (4) from the point of asymptotic analysis, and it would be a lot better if we can get a rigorous statement of (6). The second part of this paper is using our new results in the first part to give a complete discussion of the highly oscillatory problem (4), and the rigorous estimation of (6) will be presented.

The rest of this paper is organized as follows. In section 2, we will introduce a typically strategy of RG method, and present some necessary estimation for the general cases (G_1) and (G_2). Highly oscillatory second order singular perturbed problem (4) will be then investigated in section 3.

2. Renormalization group method. So far, mathematicians have developed many kinds of formulations to the renormalization group theory, each with a intrinsic interest, such as the classical form [5] from the idea of the renormalization in quantum mechanics, or envelope form [8] from the geometric point, etc. In this section, we will present a rigorous strategy of the renormalization group method to (1) in classical procedure in new conditions to get some effective approximate results.

The RG procedure can be naturally presented as flowing three steps.

**Step 1. [Naive expansion]**

Expand \( x \) as \( x = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \cdots \), substitute it into equation (1) and equate the same degrees of \( \varepsilon \), we have

\[
\dot{x}^{(0)} = 0,
\]
\[ \dot{x}^{(1)} = G_1(x^{(0)}, t), \]
\[ \dot{x}^{(2)} = G_2(x^{(0)}, x^{(1)}, t), \]

where
\[ G_1(x^{(0)}, t) = g(x^{(0)}, t, 0), \]
\[ G_2(x^{(0)}, x^{(1)}, t) = \frac{\partial g}{\partial x}(x^{(0)}, t, 0)x^{(1)} + \frac{\partial g}{\partial \epsilon}(x^{(0)}, t, 0). \]

One can naturally solve the equations (7)–(9) as
\[ x^{(0)} = \xi_0, \]
\[ x^{(1)} = \int_0^t G_1(\xi_0, s)ds, \]
\[ x^{(2)} = \int_0^t G_2(\xi_0, \int_0^s G_1(\xi_0, \tau)d\tau, s)ds, \]

where \( \xi_0 \in \mathbb{C}^n \) is a initial constant vector.

Up to now, we have been able to get the formal expansion of the solution to (1).

However, there remains another important task in this step, that is to figure out the singular terms. In order to deal with this point, in general, more information about \( g(x, t, \epsilon) \) should be added. Here we follow a general assumption as referred in [6].

**KBM condition:** We say \( g(x, t, \epsilon) \) satisfies the KBM condition, if for arbitrary \( t_0 \in \mathbb{R}_+ \), the limit
\[ R(y, \epsilon) = \lim_{t \to \infty} \frac{1}{t-t_0} \int_{t_0}^t g(y, s, \epsilon)ds \] (10)
exists uniformly for \( y \in \bar{U}, \epsilon \in I_0 \).

Under the KBM condition, \( x^{(1)} \) and \( x^{(2)} \) can be further expressed as
\[ x^{(1)} = \int_0^t G_1(\xi_0, s)ds = R_1(\xi_0)t + N_1(\xi_0, t), \]
\[ x^{(2)} = \int_0^t G_2(\xi_0, R_1(\xi_0)s + N_1(\xi_0, s), s)ds \]
\[ = (R_2(\xi_0) + \frac{\partial N_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0))t + N_2(\xi_0, t) + \frac{1}{2} \frac{\partial R_2}{\partial \xi_0}(\xi_0)R_1(\xi_0)t^2, \]

where
\[ R_1(\xi_0) = \lim_{t \to \infty} \frac{1}{t} \int_0^t G_1(\xi_0, s)ds, \]
\[ N_1(\xi_0, t) = \int_0^t (G_1(\xi_0, s) - R_1(\xi_0))ds; \]
\[ R_2(\xi_0) = \lim_{t \to \infty} \frac{1}{t} \int_0^t (G_2(\xi_0, N_1(\xi_0, s), s) - \frac{\partial N_1}{\partial \xi_0}(\xi_0, s)R_1(\xi_0))ds, \]
\[ N_2(\xi_0, t) = \int_0^t (G_2(\xi_0, N_1(\xi_0, s), s) - \frac{\partial N_1}{\partial \xi_0}(\xi_0, s)R_1(\xi_0) - R_2(\xi_0))ds, \]

According to (10), \( R_i(\xi_0) \) and \( N_i(\xi_0, t) \) are well-defined. Then we can present the naive perturbed expansion as.
especially, choose the naive expansion such that all the secular terms could be collected as an unified singular perturbation theory. The main idea of the RG method is to renormalize the asymptotic results. How to deal with these terms? It is one of the main problems in the discussion. The so-called secular terms \([11]\) to cause the nonuniformly valid asymptotic results.

Now, one can find that

\[
R_n(\sigma, \epsilon) = \sum_{i=1}^{\infty} \frac{\partial^i R_n(\sigma, \epsilon)}{\partial \sigma^i} \sigma^i + \frac{\partial R_n(\sigma, \epsilon)}{\partial \epsilon} \epsilon + \text{higher order terms}
\]

\(\sigma\), \(\epsilon\) are the so-called secular terms to cause the nonuniformly valid asymptotic results. How to deal with these terms? It is one of the main problems in the singular perturbation theory. The main idea of the RG method is to renormalize the naive expansion such that all the secular terms could be collected as an unified regular term. An usual way to illustrate this idea can be formulated as follows.

Firstly, propose an ansatz free parameter \(\sigma \in \mathbb{R}\) into \(x(\xi_0, t, \epsilon)\) to replace \(t^i\) by \(t^i - \sigma^i + \sigma^i, i = 1, 2, \ldots\), so that

\[
x(\xi_0, t, \epsilon) = \xi_0 + \epsilon(R_1(\xi_0) + N_1(\xi_0, t)) + \epsilon^2((R_2(\xi_0) + \frac{\partial N_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0))t - \sigma + \sigma) + \frac{\partial R_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0)(t^2 - \sigma^2)) + \cdots.
\]

(11)

where \(\cdots\) denotes the collection of terms whose order in \(\epsilon\) are higher than 2.

Step 2. [Renormalization]

Now, one can find that \(R_1(\xi_0)\sigma, (R_2(\xi_0) + \frac{\partial N_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0))\sigma and \(\frac{\partial R_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0)\sigma^2\) to a single term \(\xi(\sigma, \epsilon)\), such that \(x(\xi_0, t, \epsilon)\) can be rewritten as

\[
x(\xi_0, t, \epsilon) = \tilde{x}(\xi(\sigma, \epsilon), t, \sigma, t, \epsilon).
\]

(12)

In details, expand \(\xi(\sigma, \epsilon) = \xi_0 + \epsilon A_1(\sigma) + \epsilon^2 A_2(\sigma) + \cdots\) with \(A_k(\sigma)\) to be determined, take it into (12) and equate the same order terms of \(\epsilon\), we can inductively get

\[
A_1 = R_1(\xi(\sigma, \epsilon))\sigma, A_2 = R_2(\xi(\sigma, \epsilon))\sigma + \frac{\partial R_1}{\partial \xi}(\xi(\sigma, \epsilon))R_1(\xi(\sigma, \epsilon))\sigma^2, \cdots.
\]

Then

\[
\tilde{x}(\xi(\sigma, \epsilon), t - \sigma, t, \epsilon) = \xi(\sigma, \epsilon) + \epsilon(R_1(\xi(\sigma, \epsilon))t - \sigma + N_1(\xi(\sigma, \epsilon), t)) + \epsilon^2((R_2(\xi(\sigma, \epsilon)) + \frac{\partial N_1}{\partial \xi}(\xi(\sigma, \epsilon), t)R_1(\xi(\sigma, \epsilon)))t - \sigma) + \frac{\partial R_1}{\partial \xi}(\xi(\sigma, \epsilon))R_1(\xi(\sigma, \epsilon))(t^2 - \sigma^2)) + \cdots.
\]

At last, by noting the independence of \(x(\xi_0, t, \epsilon)\) with respect to \(\sigma\), we obtain

\[
\frac{\partial}{\partial \sigma} \tilde{x}(\xi(\sigma, \epsilon), t - \sigma, t, \epsilon) = \frac{\partial}{\partial \sigma} x(\xi_0, t, \epsilon) = 0,
\]

especially, choose \(\sigma = t\), we get the so-called renormalization group (RG) equation

\[
\dot{\xi} = \epsilon R_1(\xi) + \epsilon^2 R_2(\xi) + \cdots,
\]

with the initial condition \(\xi(0) = \xi_0\), and the corresponding approximate solution

\[
x(\xi_0, t, \epsilon) = \xi(t) + \epsilon N_1(\xi(t), t)) + \epsilon^2 N_2(\xi(t), t) + \cdots.
\]

Step 3. [Estimation]

Here we only consider the first order RG equation

\[
\dot{\xi} = \epsilon R_1(\xi)
\]

(13)
with \( \xi(0) = \xi_0 \), and the corresponding approximate solution

\[
x(t) = \Phi_1(\xi(t), t)
\]

with \( \Phi_1(\xi, t) = \xi + \varepsilon N_1(\xi, t), \xi \in \mathbb{C}^n \).

Based on the above preparations, we can show that (14) is indeed a first order uniformly valid approximate solution of the corresponding exact solution \( x(\xi_0, t, \varepsilon) \) (briefly denoted by \( x(t) \)) of (1). Let \( \tau = \varepsilon t, \eta(\tau) = \xi(t) \), then (13) becomes

\[
\frac{d\eta}{d\tau} = R_1(\eta).
\]

**Theorem 2.1.** Assume that \( g(x, t, \varepsilon) \) satisfies \((G_1)\) and the KBM condition. Let \( x(t) \) be a solution of (1), \( \eta(\tau) \) be the solution of (15) with \( \eta(0) = x(0) \) and the maximum existence interval \((a, b)\). Then, for any closed interval \([T_1, T_2] \subset (a, b)\) with \( T_1 \geq 0 \), there exists a constant \( \varepsilon_0 > 0 \), such that, for any \( \varepsilon \in (0, \varepsilon_0) \),

\[
\| x(t) - \hat{x}(t) \| < C\delta(\varepsilon),
\]

as long as \( \frac{T_2}{\varepsilon} < t < \frac{T_2}{\varepsilon} \), where \( C \) is a positive constant independent to \( \varepsilon \), and \( \delta(\varepsilon) = o(1) \) as \( \varepsilon \to 0 \).

**Proof.** For any closed interval \([T_1, T_2] \subset (a, b)\), there exists an open set \( U_T \subset U \), such that \( \{ \eta(\tau) \in U | \tau \in [T_1, T_2] \} \subset U_T \) and \( U_T \) is compact. Let

\[
\delta_0(\varepsilon) = \sup_{\xi \in U_T} \sup_{t \in [T_1, T_2]} \varepsilon \| \int_0^t (g(\xi, s, 0) - R_1(\xi)) ds \|.
\]

By Lebovitz’s lemma [18], \( \delta_0(\varepsilon) = o(1) \), which means that the map \( \Phi_1(\xi, t) = \xi + o(1) \) and \( \nabla \Phi_1(\xi, t) = E + o(1) \). Then there exist \( \varepsilon_2 > 0 \) and \( V_T \subset \mathbb{C}^n \) such that \( V_T \) is compact and \( \cup_{0 < \varepsilon < \varepsilon_2} \Phi_1(U_T, t) \subset V_T \subset U \).

By (14), one can get

\[
\dot{\hat{x}} = \varepsilon g(\hat{x}, t, \varepsilon) + \varepsilon (g(\xi, t, 0) - g(\hat{x}, t, 0)) - \varepsilon^2 F(\hat{x}, t, \varepsilon) + \varepsilon^2 G(\xi, t), \quad \hat{x} \in V_T,
\]

where \( F(x, t, \varepsilon) = (g(x, t, \varepsilon) - g(x(0, t, \varepsilon))/\varepsilon, G(\xi, t) = \nabla N_1(\xi, t)R_1(\xi) \). It is obviously that \( F(x, t, \varepsilon) \) is Lipschitz continuous in \( x \) on \( U \), and satisfies the KBM condition, therefore \( R_2(\varepsilon) = \lim_{t \to \infty} \frac{1}{t} \int_0^t F(\xi, s, \varepsilon) ds \) is well-defined. Let

\[
\delta_1(\varepsilon) = \sup_{\xi \in V_T \cup U_T} \sup_{t \in [T_1, T_2]} \varepsilon \| \int_0^t (g(\xi, s, \varepsilon) - R(\xi, \varepsilon)) ds \|,
\]

\[
\delta_2(\varepsilon) = \sup_{\xi \in V_T \cup U_T} \sup_{t \in [T_1, T_2]} \varepsilon \| \int_0^t (g(\xi, s, 0) - R_1(\xi)) ds \|,
\]

\[
\delta_3(\varepsilon) = \sup_{\xi \in V_T \cup U_T} \sup_{t \in [T_1, T_2]} \varepsilon \| \int_0^t (F(\xi, s, \varepsilon) - R_2(\xi, \varepsilon)) ds \|,
\]

\[
\delta_0(\varepsilon) = \max \{ \varepsilon, \delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon) \}.
\]

By KBM condition, \( \delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon) \) are well defined, and by Lebovitz’s lemma, \( \delta_0(\varepsilon) = o(1) \). Then, by noting the Lipschitz condition \((G_1)\), we have

\[
\| x - \hat{x} \| = \| \varepsilon \int_0^t (g(x(s), s, \varepsilon) - g(\hat{x}(s), s, \varepsilon)) ds + \varepsilon \int_0^t (g(\hat{x}(s), s, 0)) \| - g(\xi(s), s, 0) \| ds + \varepsilon^2 \int_0^t (F(\hat{x}(s), s, \varepsilon) - G(\xi(s), s)) ds \|
\]
complete the proof by choosing a fixed number $0 < \varepsilon < 1$. It is well-known that a main advantage of RG method is that

By Gronwall inequality, we have

Furthermore, by the definition of $\delta(\varepsilon)$, one can also find that $\varepsilon \parallel N_1(\xi(t), t) \parallel$, $\varepsilon \parallel \int_0^t (F(\hat{x}(0), s, \varepsilon) - R_r(\hat{x}(0), \varepsilon))ds \parallel$, and also $\varepsilon \parallel G(\xi(t), t) \parallel$ (just needing a little modified) are super bounded by $\delta_0(\varepsilon)$. This means there there must exist another two positive constants $C_2, C_3$, such that

By Gronwall inequality, we have

Remark 1. 1. It is well-known that a main advantage of RG method is that it starts with a naive expansion and does not require any further a priori assumptions regarding the structure of the perturbation series, like an anticipation of scales involved in WKB and multiple scale analysis[11]. So far, the RG procedure has also been successfully developed to several kinds of evolution equations, such as singular Navier-Stokes equations [13, 14], Primitive equations [17], Schrödinger equations [10], etc.

2. The smoothness assumption about $g(x, t, \varepsilon)$ is necessary, and it is not difficult to see that one can get more stronger estimations if the smooth condition is better. For example, $\delta(\varepsilon)$ can be always obtained as $\delta(\varepsilon) = O(\varepsilon)$ if $g(x, t, \varepsilon)$ is
Assume the assumption for some $\gamma$.

Proof. By the proof of Theorem 2.1, we need only to prove that $f$ satisfies the GBD condition. Let $x$ be a solution of (1), $\eta(\tau)$ be the solution of (15) with $\eta(0) = x(0)$ and the maximum existence interval $(a, b)$. Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\| x(t) - \hat{x}(t) \| < C \varepsilon,$$

(17)

as long as $\frac{T_1}{\varepsilon} < t < \frac{T_2}{\varepsilon}$, where $C$ is a positive constant independent to $\varepsilon$.

Proof. The proof can be easily concluded by noting the fact $\delta(\varepsilon) = O(\varepsilon)$ based on the assumption, we omit it here.

In what following, we consider the case $(G_2)$ raised in a large number of applications as well as the model (4), it is more general rather than the one in Theorem 2.2. It is also well-known that, for general $\omega \in \mathbb{R}^d$, small divisors may occur in corresponding integrals, such that the KBM condition may not be valid. Therefore, additional constraints should be proposed to overcome this difficulty. Here we introduce the general Diophantine condition.

GBD condition: A fixed vector $\eta \in \mathbb{R}^l$ is said to satisfy the GBD condition, if it can be decomposed as $\eta = (a_1 \eta^{(1)}, \ldots, a_j \eta^{(j)}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_j}$, where $\eta^i \in \mathbb{Z}^{l_i}$, $l_i \geq 0$, $i = 1, \ldots, j$, $l_1 + \cdots + l_j = l$, such that the vector $a = (a_1, \ldots, a_j)$ satisfies the Bryuno-Diophantine condition.

Here, a fixed vector $\eta \in \mathbb{R}^l$ is said to satisfy the Bryuno-Diophantine condition, if the Bryuno function\cite{3}

$$\mathfrak{B}(\eta) = \sum_{m=0}^{\infty} \frac{1}{2^m} \ln \frac{1}{\alpha_m^l(\eta)} < \infty,$$

where $\alpha_m^l(\eta) = \inf\{|\langle \eta, \nu \rangle|\nu \in \mathbb{Z}^l, \text{such that } 0 < |\nu| < 2^m\}$.

Remark 2. With a little computation, on can find that the Bryuno-Diophantine condition is a generalization of the classical Diophantine inequality $|\langle \eta, \nu \rangle| > \gamma_0 |\nu|^{-\tau_0}$ for some $\gamma_0, \tau_0 > 0$.

Theorem 2.3. Assume the assumption $(G_2)$ holds, and the corresponding $\omega$ satisfies the GBD condition. Let $x(t)$ be a solution of (1), $\eta(\tau)$ be the solution of (15) with $\eta(0) = x(0)$ and the maximum existence interval $(a, b)$. Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\| x(t) - \hat{x}(t) \| < C \varepsilon,$$

(18)

as long as $\frac{T_1}{\varepsilon} < t < \frac{T_2}{\varepsilon}$, where $C$ is a positive constant independent to $\varepsilon$.

Proof. By the proof of Theorem 2.1, we need only to prove that $g(x, t, \varepsilon)$ satisfies the KBM condition and $\delta(\varepsilon) = O(\varepsilon)$.

Firstly, we assume that $\omega$ satisfies the Bryuno-Diophantine condition. By (2) and (3) and the dimension of $k$th order homogeneous polynomial space of $d$ variables,
there exist some constants $\Gamma_2 > \Gamma_0$ and $\bar{\sigma} \in (0, \sigma)$, such that,

$$\left\| \sum_{\nu \in \mathbb{Z}^{d}/\{0\}} \frac{1}{4(\omega, \nu)} g_\nu(x, \varepsilon) e^{i(\omega, \nu)t} \right\| \leq \sum_{\nu \in \mathbb{Z}^{d}/\{0\}} \sum_{l=0}^\infty \frac{1}{\|\nu\|} \left\| g_\nu(x) \right\| \varepsilon^l \leq \Gamma_0 \sum_{l=0}^\infty \frac{1}{\alpha_m^d(\omega)} e^{-2^{m-1}\sigma(\varepsilon\Gamma_1)^l} \leq \Gamma_2 \sum_{l=0}^\infty (\varepsilon\Gamma_1)^l \left( \sum_{m=1}^\infty \frac{1}{\alpha_m^d(\omega)} e^{-2^{m-1}\bar{\sigma}} \right).$$

Choose $\sigma_0 \in (0, \frac{\bar{\sigma}}{2})$, since $\omega$ satisfies the Bryuno-Diophantine condition, there exists $N \in \mathbb{Z}$, such that when $m > N$, 

$$\frac{1}{2^m \ln \alpha_m^d(\omega)} < \frac{\bar{\sigma}}{2} - \sigma_0,$$

therefore 

$$\sum_{m=N+1}^\infty \frac{1}{\alpha_m^d(\omega)} e^{-2^{m-1}\bar{\sigma}} = \sum_{m=N+1}^\infty e^{2^m\left(\frac{\bar{\sigma}}{2} - \frac{1}{2^m \ln \alpha_m^d(\omega)}\right)} < \sum_{m=N+1}^\infty e^{-2^m\sigma_0}. \quad (20)$$

On the other hand,

$$\sum_{m=1}^N \frac{1}{\alpha_m^d(\omega)} e^{-2^{m-1}\bar{\sigma}} \leq \Gamma_3 \sum_{m=1}^N e^{-2^m\sigma_0}, \quad (21)$$

for some constant $\Gamma_3 > 0$. By (19), (20) and (21), we obtain that

$$\left\| \sum_{\nu \in \mathbb{Z}^{d}/\{0\}} \frac{1}{4(\omega, \nu)} g_\nu(x, \varepsilon) e^{i(\omega, \nu)t} \right\| \leq \Gamma \left( \sum_{l=0}^\infty (\varepsilon\Gamma_1)^l \right) \left( \sum_{m=0}^\infty e^{-\sigma_0 2^m} \right) < +\infty, \quad (22)$$

where $\Gamma_3 > 0$ is a constant.

By (22) we can conclude easily that

$$\sup_{\mathcal{U}, t \in \mathbb{R}} \left\| \int_0^t \sum_{\nu \in \mathbb{Z}^{d}/\{0\}} g_\nu(x, \varepsilon) e^{i(\omega, \nu)s} ds \right\| < +\infty,$$

therefore $g(x, t, \varepsilon)$ satisfies the KBM condition and $\delta(\varepsilon) = O(\varepsilon)$.

Secondly, we consider the case when $\omega$ satisfies the GBD condition, i.e., it can be decomposed as $\omega = (a_1\omega^{(1)}, \ldots, a_k\omega^{(k)}) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$ with $\omega_j \in \mathbb{Z}^{d_j}$, $d_j \geq 0$, $j = 1, \ldots, k$, $d_1 + \cdots + d_k = d$, and $\mathbf{a} = (a_1, \ldots, a_k)$ satisfies the Bryuno-Diophantine condition. For $\nu \in \mathbb{Z}^{d}$, we decompose it as $\nu = (\nu^{(1)}, \ldots, \nu^{k}) \in \mathbb{Z}^{d_1} \times \cdots \times \mathbb{Z}^{d_k}$, and $\mathbb{Z}^{d} = \mathbb{Z}_r \cup \mathbb{Z}_p \cup \mathbb{Z}_n$ with

$$\mathbb{Z}_r = \{ \nu \in \mathbb{Z}^{d} | (\omega, \nu) = 0 \} = \bigcap_{j=1}^k \{ \nu \in \mathbb{Z}^{d} | \langle (\omega^{(j)}, \nu^{(j)}) \rangle = 0 \},$$

$$\mathbb{Z}_p = \{ \nu \in \mathbb{Z}^{d} | (\omega, \nu) = \langle (\omega, \nu) \rangle \},$$

and

$$\mathbb{Z}_n = \{ \nu \in \mathbb{Z}^{d} | (\omega, \nu) = \langle (\omega, \nu) \rangle + \langle (\omega^{(1)}, \nu^{(1)}) \rangle + \cdots + \langle (\omega^{(k)}, \nu^{(k)}) \rangle \}.$$
\[ Z_p = \bigcup_{j=1}^{k} \{ \nu \in Z \mid \langle \omega(j), \nu(j) \rangle \neq 0 \}, \]
\[ Z_n = \{ \nu \in Z^d \mid \nu \notin Z_r, \nu \notin Z_p \}, \]
\[ Z_{rj} = \bigcap_{l=1, l \neq j}^{k} \{ \nu \in Z^d \mid \langle \omega(l), \nu(l) \rangle = 0 \}, \quad j = 1, \cdots, k. \]

Then we only need to test that
\[
\sup_{x \in U, t \in \mathbb{R}} \left\| \sum_{\nu \in \mathbb{Z}^d / Z_r} \frac{1}{1(\omega, \nu)} g_{\nu}(x, \varepsilon) e^{i \langle \omega, \nu \rangle t} \right\| < +\infty. \tag{23}
\]
In fact,
\[
\begin{aligned}
\left\| \sum_{\nu \in \mathbb{Z}^d / Z_r} \frac{1}{1(\omega, \nu)} g_{\nu}(x, \varepsilon) e^{i \langle \omega, \nu \rangle t} \right\|
\leq \sum_{\nu \in Z_p} \frac{1}{1(\omega, \nu)} \| g_{\nu}(x, \varepsilon) \| + \sum_{\nu \in Z_n} \frac{1}{1(\omega, \nu)} \| g_{\nu}(x, \varepsilon) \|
\leq \sum_{\nu \in Z_p} \| g_{\nu}(x, \varepsilon) \| + \Gamma_0 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\nu \in \mathbb{Z}^d / \{0\}} \frac{1}{\alpha_m^2(a)} e^{-\sigma 2^m (\varepsilon \Gamma_1)^l}
\leq C + \Gamma \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\alpha_m^2(a)} e^{-\sigma 2^m (\varepsilon \Gamma_1)^l},
\end{aligned}
\]
with certain positive constants \( C \) and \( \Gamma \) independent to \( \varepsilon \), then, we can get the inequality (23) with the same discussion in the first case. And we have, in fact, completed the whole proof by the definitions of the KBM condition and \( \delta(\varepsilon) \). \( \square \)

3. **Highly oscillatory problems.** In this section, we turn back to consider the asymptotic properties of the highly oscillatory problem (4). Let us make the change of variables
\[
\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} E & iE \\ i\Lambda & \Lambda \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -A y_1(0) - f(y_1(0)) \end{pmatrix}, \tag{24}
\]
where \( \cdot \) denotes the derivation with respect to \( x \), then (4) becomes the following singular perturbed problem
\[
\begin{cases}
\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} O & E \\ -B & O \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -A y_1 - f(y_1) \end{pmatrix}, \\
\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},
\end{cases}
\tag{25}
\]
with \( O \) the \( d \times d \) zero matrix. Without loss of generality, we assume that \( B \) is a diagonal matrix, and \( B = \Lambda^2 \) with \( \Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_d\} \), \( \lambda_j > 0, j = 1, \cdots, d \). Furthermore, we assume that \( (\lambda_1, \cdots, \lambda_d) \) satisfies the GBD condition.

Make the change of variables
\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} E & iE \\ i\Lambda & \Lambda \end{pmatrix} \begin{pmatrix} e^{i\Lambda x} & 0 \\ 0 & e^{-i\Lambda x} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{26}
\]
then (25) becomes
\[
\begin{align*}
\begin{cases}
Z' = \mu g(Z, x), \\
Z(0) = \left( \frac{n_1 - i\Lambda^{-1}n_2}{2}, -\frac{1}{1n_1 + A^{-1}n_2} \right),
\end{cases}
\end{align*}
\tag{27}
\]
where \( Z = (z_1^T, z_2^T)^T \),

\[
g(Z, x) = \frac{1}{2} \left( A\Lambda^{-1}(i z_1 - e^{-2i\Lambda x}z_2) + i\Lambda^{-1}e^{-i\Lambda x}f(e^{i\Lambda x}z_1 + 1)e^{-i\Lambda x}z_2 \right),
\]
and we can naturally get the corresponding approximate solution
\[
Z(x) = \Phi_1(W(x), x),
\]
where \( \Phi_1(W, x) = W + \mu N_1(W, x), N_1(W, x) = \int_0^t (g(W, s) - R_1(W))ds, \) and \( W(x) = (w_1^T(x), w_2^T(x))^T \) is the solution of the first order RG equation with initial condition
\[
\begin{align*}
\begin{cases}
W' = \mu R_1(W), \\
W(0) = \left( \frac{n_1 - i\Lambda^{-1}n_2}{2}, -\frac{1}{1n_1 + A^{-1}n_2} \right),
\end{cases}
\end{align*}
\tag{28}
\]
with
\[
R_1(W) = \frac{1}{2} A\Lambda^{-1} \left( w_1 \right) + \frac{1}{2} \lim \frac{1}{x} \int_0^x \left( i\Lambda^{-1}e^{-i\Lambda x}f(e^{i\Lambda x}w_1 + 1)e^{-i\Lambda x}w_2 \right) ds.
\]
Let
\[
\begin{pmatrix}
\hat{y}_1(t) \\
\hat{y}_2(t)
\end{pmatrix} = \begin{pmatrix} E & iE \\
\Lambda & -\Lambda \end{pmatrix} \begin{pmatrix} e^{i\Lambda t/\varepsilon^2} & O \\
O & e^{-i\Lambda t/\varepsilon^2} \end{pmatrix} \Phi_1(V(t), t/\varepsilon^2),
\]
where \( V(t) \) is the solution of initial value problem
\[
\begin{align*}
\begin{cases}
dV/dt = R_1(V), \\
V(0) = \left( \frac{n_1 - i\Lambda^{-1}n_2}{2}, -\frac{1}{1n_1 + A^{-1}n_2} \right)
\end{cases}
\end{align*}
\tag{29}
\]
with the maximum existence interval \((a, b)\). Obviously, \( W(x) = V(\mu x) \).

**Theorem 3.1.** Assume vector \((\lambda_1, \cdots, \lambda_d)\) satisfies the GBD condition. Let \( y(t) \) be the solution of (4), and \( V(t) \) be the solution of (29). Then, for any closed interval \([T_1, T_2] \subset (a, b)\), there exists a constant \( \varepsilon_0 > 0 \), such that, for any \( 0 < \varepsilon < \varepsilon_0 \),
\[
|y(t) - \hat{y}_1(t)| < C\varepsilon^2,
\]
\[
|y(t) - \hat{y}_2(t)| < C/\varepsilon^2,
\]
as long as \( T_1 \leq t \leq T_2 \), where \( C \) is a positive constant independent to \( \varepsilon \).

**Proof.** Obviously, \( g(Z, x) \) is quasi-periodic with respect to \( x \) with the exponents \((\lambda_1, \cdots, \lambda_d)\). Since \((\lambda_1, \cdots, \lambda_d)\) satisfies the GBD condition, by Theorem 2.3, we know that for any closed interval \([T_1, T_2] \subset (a, b)\), there exists a constant \( \mu_0 > 0 \), such that
\[
\| Z(x) - \hat{Z}(x) \| < C_0\mu,
\]
as long as \( 0 < \mu < \mu_0 \) and \( \frac{T_1}{\mu} \leq x \leq \frac{T_2}{\mu} \), where \( C_0 \) is a positive constant independent to \( \mu \).
Let $\mathbf{Y}(x) = (\mathbf{y}_1(x)^T, \mathbf{y}_2(x)^T)^T$, then by (26) and (30), we obtain
\[
\| \mathbf{Y}(x) - \hat{\mathbf{Y}}(\mu x) \| = \| \left( \begin{array}{c} E \ \ iE \\ i\Lambda \ \ A \end{array} \right) \left( e^{i\Lambda x} \ \ O \ \ e^{-i\Lambda x} \right) (\mathbf{Z}(x) - \Phi_1(\mathbf{V}(\mu x), x)) \| \leq C_1 \| \mathbf{Z}(x) - \hat{\mathbf{Z}}(x) \| < C\mu,
\]
where $C_1$ and $C$ are some constants. Now by the inverse of the transformation (24) and (31), we get
\[
\| \left( \begin{array}{c} \mathbf{y}(t) \\ \varepsilon^2 \mathbf{y}(t) \end{array} \right) - \hat{\mathbf{y}}(t) \| < C\varepsilon^2,
\]
as long as $0 < \varepsilon < \varepsilon_0 = \sqrt{\mu_0}$ and $T_1 \leq t \leq T_2$. The proof is completed.

**Theorem 3.2.** Assume vector $(\lambda_1, \cdots, \lambda_d)$ satisfies the GBD condition. Let $\mathbf{y}(t)$ be the solution of (4). Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that, for any $k \in \mathbb{N}$, $0 < \varepsilon < \varepsilon_0$ and $T_1 \leq t \leq T_2$,
\[
\| \mathbf{y}^{(k)}(t) \| \leq \frac{M}{\varepsilon^{2k}},
\]
where $M$ is a positive constant independent to $\varepsilon$.

**Proof.** Let us investigate (27) as
\[
\mathbf{Z}' = \mu \mathbf{g}(\mathbf{Z}, x).
\]

Then, by inductive calculation, we get
\[
\frac{d^2}{dx^2} \mathbf{Z}(x) = \mu \frac{\partial \mathbf{g}}{\partial x}(\mathbf{Z}, x) + \mu^2 \frac{\partial^2 \mathbf{g}}{\partial^2 x}(\mathbf{Z}, x) \mathbf{g}(\mathbf{Z}, x),
\]
\[
\frac{d^3}{dx^3} \mathbf{Z}(x) = \mu \frac{\partial^2 \mathbf{g}}{\partial x^2}(\mathbf{Z}, x) + 2\mu^2 \frac{\partial^2 \mathbf{g}}{\partial^2 x}(\mathbf{Z}, x) \mathbf{g}(\mathbf{Z}, x) + \mu^3 \frac{\partial^3 \mathbf{g}}{\partial^3 x}(\mathbf{Z}, x) \mathbf{g}^2(\mathbf{Z}, x),
\]
\[
\frac{d^k}{dx^k} \mathbf{Z}(x) = \mu \frac{\partial^k \mathbf{g}}{\partial x^k}(\mathbf{Z}, x) + \sum_{i=2}^k \mu^i \mathbf{H}^{(k,i)}(\mathbf{Z}, x), \quad k > 3,
\]
where $\mathbf{H}^{(k,i)}(\mathbf{Z}, x)$ is composed by $\mathbf{g}(\mathbf{Z}, x)$ and its partial derivative in $\mathbf{Z}$ and $x$ up to order $l$. Noting that $\mathbf{g}(\mathbf{Z}, x)$ is analytic in $\mathbf{Z}$, and quasi-periodic in $x$, we have
\[
\| \frac{d^k}{dx^k} \mathbf{Z}(x) \| = O(\| \mu \frac{\partial^k \mathbf{g}}{\partial x^k}(\mathbf{Z}, x) \|) \quad \text{as } \mu \rightarrow 0.
\]

By (26) and (32),
\[
\| \mathbf{y}^{(k)}(t) \| = \| \frac{d^k}{dt^k} (e^{i\Lambda x} \mathbf{z}_1 + i e^{-i\Lambda x} \mathbf{z}_2) \| = \frac{1}{\varepsilon^{2k}} \| \frac{d^k}{dt^k} (e^{i\Lambda x} \mathbf{z}_1 + i e^{-i\Lambda x} \mathbf{z}_2) \| = \frac{1}{\varepsilon^{2k}} \| \sum_{i=0}^{k} \frac{k!}{l!(k-l)!} (i\Lambda)^l e^{i\Lambda x} \frac{d^{k-l}}{dx^{k-l}} \mathbf{z}_1 + i(-i\Lambda)^l e^{-i\Lambda x} \frac{d^{k-l}}{dx^{k-l}} \mathbf{z}_2 \| = O(\frac{1}{\varepsilon^{2k}} \| (i\Lambda)^k e^{i\Lambda x} \mathbf{z}_1 + i(-i\Lambda)^k e^{-i\Lambda x} \mathbf{z}_2 \|) \quad \text{as } \varepsilon \rightarrow 0.
\]

This completes the proof of Theorem 3.2.
Remark 3. With the same strategy, one can get similar asymptotic results to the following two models:

\[
\begin{aligned}
\begin{cases}
\ddot{y}(t) + \frac{1}{\varepsilon^2} A y(t) + f(y(t)) = 0, \quad y \in \mathbb{C}^d, \quad t > 0, \\
y(0) = \eta_1, \quad \dot{y}(0) = \frac{\eta_1}{\varepsilon}
\end{cases}
\end{aligned}
\]  \tag{33}

and

\[
\begin{aligned}
\begin{cases}
\dot{y}(t) + \frac{1}{\varepsilon} A y(t) + f(y(t)) = 0, \quad y \in \mathbb{C}^d, \quad t > 0, \\
y(0) = \eta_1,
\end{cases}
\end{aligned}
\]  \tag{34}

where \(\Lambda \in \mathbb{R}^{d \times d}\) is a positive definite matrix, \(f : \mathbb{C}^d \to \mathbb{C}^d\) is a vector-valued analytic function. (33) and (34) are obtained from some evolution equations with multiscale behavior as nonlinear Klein-Gordon equation (5).

As the end, let us further consider a more concrete case for problem (4): \(B = E, A\) is a semi-positive definite matrix, and \(f(y) = \frac{\partial}{\partial v} h(|y|^2)\) with \(h(r) = \sum_{k=0}^N h_k r^k\) is an \(N\)-th order real polynomial function in \(\mathbb{R}\). In this case, we can get the corresponding approximate solution

\[
\begin{aligned}
\begin{pmatrix}
\tilde{y}_1(t) \\
\tilde{y}_2(t)
\end{pmatrix}
&= \begin{pmatrix} E & iE \\
i\Lambda & -\Lambda \end{pmatrix} \begin{pmatrix} e^{i\Lambda t/\varepsilon^2} & O \\
o & e^{-i\Lambda t/\varepsilon^2} \end{pmatrix} \Phi_1(V, t/\varepsilon^2),
\end{aligned}
\]

where \(\Phi_1(V, x) = V + \varepsilon^2 N_1(V, x), V(t)\) is the solution of the initial value problem

\[
\begin{aligned}
\begin{cases}
\frac{dV}{dt} = R_1(V), \\
V(0) = \left( \frac{m - i\Lambda^{-1} \eta_2}{2} \right) \\
\end{cases}
\end{aligned}
\]  \tag{35}

with

\[
R_1(V) = \frac{iA}{2} \begin{pmatrix} v_1 \\
-\bar{v}_2 \end{pmatrix}
\]

\[
+ \frac{1}{2} \sum_{k=1}^N \sum_{l=0}^{k-1} \frac{k! h_k}{l!(k-1-2l)!} \left( \begin{pmatrix} |v_1|^2 + |v_2|^2 & k-2l \end{pmatrix} (\bar{v}_1, \bar{v}_2) (\bar{v}_2, v_1) \right) \bar{v}_1^{2l} v_2^{2l-2} \bar{v}_2^{2l-2} (v_1, \bar{v}_2) v_2^{2l-2} (v_1, v_1) v_1^{2l-2} (v_2, \bar{v}_2) v_2^{2l-2} (v_2, v_2) v_2^{2l-2} (v_1, v_1),
\]

\[
+ \sum_{l=1}^{\lfloor \frac{d}{2} \rfloor} \frac{k! h_k}{l!(l-1)!(k-2l)!} \left( \begin{pmatrix} |v_1|^2 + |v_2|^2 & k-2l \end{pmatrix} (\bar{v}_1, \bar{v}_2) (\bar{v}_2, v_1) \right) \bar{v}_1^{2l-2} v_2^{2l-2} (v_1, \bar{v}_2) v_2^{2l-2} (v_1, v_1) v_1^{2l-2} (v_2, \bar{v}_2) v_2^{2l-2} (v_2, v_2) v_2^{2l-2} (v_1, v_1),
\]

and

\[
N_1(V, x)
= \frac{iA}{4} \begin{pmatrix} (1 - e^{-2ix}) v_2 \\
(e^{2ix} - 1) v_1 \end{pmatrix}
\]

\[
+ \frac{1}{4} \sum_{k=1}^N \sum_{l_1 + l_2 + l_3 = k} \frac{k! h_k}{l_1! l_2! l_3!} \left( \begin{pmatrix} |v_1|^2 + |v_2|^2 & l_1 + l_2 - l_3 \end{pmatrix} (\bar{v}_1, \bar{v}_2) (\bar{v}_2, v_1) \right) \bar{v}_1^{l_1} v_2^{l_2} (v_1, \bar{v}_2) v_2^{l_3} (v_1, v_1) v_1^{l_1} (v_2, \bar{v}_2) v_2^{l_2} (v_2, v_1) v_1^{l_3}.
\]
Remark 4.

1. For where $M$ is a positive constant independent to $\varepsilon$. Furthermore, by noticing that

\[
\left( \frac{\partial}{\partial t} \right)^2 + 2 - t_0 \left( e^{-2(t_0 - t_1) \varepsilon} - 1 \right) \left| v_1 \right|^2 + |v_2|^2 \right) + (v_2, \bar{v}_1)v_1 \right).
\]

Furthermore, by noticing that

\[
\frac{d}{dt} |V|^2 = \langle R_1(V), \bar{V} \rangle + \langle \bar{R}_1(V), V \rangle = 0,
\]

$V(t)$ exists on $\mathbb{R}$. By Theorem 3.1 and 3.2, we obtain the following statements.

**Corollary 1.** Assume that $B = E$, $A$ is a semi-positive definite matrix, and $f(y) = \frac{d}{dy} h(|y|^2)$ with $h(r)$ is a polynomial function in $\mathbb{R}$. Let $y(t)$ be the solution of the singular perturbed problem (4), $V(t)$ be the solution of (35). Then, for any $T > 0$, there exists a constant $\varepsilon_0 > 0$, such that, for any $k \in \mathbb{N}, 0 < \varepsilon < \varepsilon_0$ and $-T \leq t \leq T$,

\[
|y(t) - \bar{y}_1(t)| < C \varepsilon^2,
\]

\[
|\dot{y}(t) - \frac{\dot{y}_2(t)}{\varepsilon^2}| < M,
\]

\[
|y^{(k)}(t)| \leq \frac{M}{\varepsilon^k},
\]

where $M$ is a positive constant independent to $\varepsilon$.

**Remark 4.**

1. For $d = 1$, by Corollary 1 we have, in fact, obtained the positive answer to the assumption (6) proposed in [1].

2. If $\eta_1, \eta_2 \in \mathbb{R}^d$, the equation (35) is equivalent to

\[
\begin{cases}
\dot{v}_1 + \frac{1}{2} Av_1 + \frac{1}{2} \sum_{k=1}^{N} \left( \sum_{i=1}^{k} C_i^k C_{k-i}^{l-1} \right) k h_k |v_1|^{2k-2} v_1 = 0, \\
v_1(0) = \frac{\eta_1 - \eta_2}{2},
\end{cases}
\]

which is, in fact, a Schrödinger-Hamiltonian system.

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E-mail address: lwlei@jlu.edu.cn
E-mail address: shisy@jlu.edu.cn