Geometric Hardy and Hardy–Sobolev inequalities on Heisenberg groups

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In this paper, we present geometric Hardy inequalities for the sub-Laplacian in half-spaces of stratified groups. As a consequence, we obtain the following geometric Hardy inequality in a half-space of the Heisenberg group with a sharp constant:

\[ \int_{\mathbb{H}^+} |\nabla_{\mathbb{H}} u|^p d\xi \geq \left( \frac{p-1}{p} \right) \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi, \quad p > 1, \]

which solves a conjecture in the paper [S. Larson, Geometric Hardy inequalities for the sub-elliptic Laplacian on convex domain in the Heisenberg group, Bull. Math. Sci.]

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1. Introduction

Let us recall the Hardy inequality in the half-space of $\mathbb{R}^n$

$$
\int_{\mathbb{R}^n_+} |\nabla u|^p \, dx \geq \left( \frac{p - 1}{p} \right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n} \, dx, \quad p > 1,
$$

(1.1)

for every function $u \in C^\infty_0(\mathbb{R}^n_+)$, where $\nabla$ is the usual Euclidean gradient and $\mathbb{R}^n_+ := \{(x', x_n) | x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}, n \in \mathbb{N}$. There is a number of studies related to inequality (1.1), see e.g. [1] [2] [5] [12].

Filippas et al. in [2] established the Hardy–Sobolev inequality in the following form:

$$
\left( \int_{\mathbb{R}^n_+} |\nabla u|^p \, dx - \left( \frac{p - 1}{p} \right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n} \, dx \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{R}^n_+} |u|^p \, dx \right)^{\frac{1}{p}},
$$

(1.2)

for all function $u \in C^\infty_0(\mathbb{R}^n)$, where $p^* = \frac{np}{n-p}$ and $2 \leq p < n$. For a different proof of this inequality, see Frank and Loss [3] as well as Psaradakis [13].

A Hardy inequality for a half-space of the Heisenberg group was shown by Luan and Young [11] in the form

$$
\int_{\mathbb{H}^n_+} |\nabla_H u|^2 \, d\xi \geq \int_{\mathbb{H}^n_+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 \, d\xi.
$$

(1.3)

An alternative proof of this inequality was given by Larson in [10], where the author generalized it to any half-space of the Heisenberg group

$$
\int_{\mathbb{H}^n_+} |\nabla_H u|^2 \, d\xi \geq \frac{1}{4} \int_{\mathbb{H}^n_+} \sum_{i=1}^n (X_i(\xi), \nu)^2 + (Y_i(\xi), \nu)^2 \frac{|u|^2 \, d\xi}{\text{dist}(\xi, \partial \mathbb{H}^n)^2},
$$

where $X_i$ and $Y_i$ (for $i = 1, \ldots, n$) are left-invariant vector fields on the Heisenberg group, $\nu$ is the Riemannian outer unit normal (see [9]) to the boundary. In the same
Let \( P \) be the following \( L^p \) generalization of the inequality was proved
\[
\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n |\langle X_i(\xi), \nu \rangle|^p + |Y_\nu(\xi)|^p}{\text{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi.
\]

Note also that the authors in [15] have extended this result to general Carnot groups, see [16].

The main aim of this paper is to improve the \( L^p \) version of the geometric Hardy inequality for the sub-Laplacian in the half-spaces of stratified groups (Carnot groups), where the obtained inequality will be a natural extension of the inequality derived by the authors in [10, 15] on Heisenberg and stratified groups, respectively. In particular, we prove an inequality conjectured in [10]. Moreover, we obtain a version of the Hardy–Sobolev inequality in the setting of the Heisenberg group.

The main results of this paper are as follows:

- **Geometric \( L^p \)-Hardy inequality on \( G^+ \):** Let \( G^+ := \{ x \in G : \langle x, \nu \rangle > d \} \) be a half-space of a stratified group \( G \). Then for all \( u \in C_0^\infty(G^+) \), \( \beta \in \mathbb{R} \) and \( p > 1 \), we have
\[
\int_{G^+} |\nabla_G u|^p dx \geq -(p-1) \beta \int_{G^+} \frac{W(x)^p}{\text{dist}(x, \partial G^+)^p} |u|^p dx + \beta \int_{G^+} \mathcal{L}_p(\text{dist}(x, \partial G^+)) \mathcal{L}_p^{-1} |u|^p dx,
\]
where \( W(x) := (\sum_{i=1}^n \langle X_i(x), \nu \rangle^2)^{1/2} \), and \( \mathcal{L}_p \) is the \( p \)-sub-Laplacian on \( G \), see [16].

- **Geometric \( L^p \)-Hardy inequality on \( \mathbb{H}^+ \):** Let \( \mathbb{H}^+ := \{ \xi \in \mathbb{H} : \langle \xi, \nu \rangle > d \} \) be a half-space of the Heisenberg group. Then for all \( u \in C_0^\infty(\mathbb{H}^+) \) and \( p > 1 \) we have
\[
\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi,
\]
where \( W(\xi) := (\sum_{i=1}^n \langle X_i(\xi), \nu \rangle^2 + |Y_\nu(\xi)|^2)^{1/2} \) and the constant is sharp.

- **Geometric Hardy–Sobolev inequality on \( \mathbb{H}^+ \):** Let \( \mathbb{H}^+ := \{ \xi \in \mathbb{H} : \langle \xi, \nu \rangle > d \} \) be a half-space of the Heisenberg group. Then for all \( u \in C_0^\infty(\mathbb{H}^+) \) and \( 2 \leq p < Q \) with \( Q = 2n + 1 \), there exists some \( C > 0 \) such that we have
\[
\left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \frac{(p-1)^p}{p} \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{H}^+} |u|^p d\xi \right)^{\frac{1}{p^*}},
\]
where \( \text{dist}(\xi, \partial \mathbb{H}^+) := \langle \xi, \nu \rangle - d \) is the distance from \( \xi \) to the boundary and \( p^* := Qp/(Q-p) \).

### 1.1. Preliminaries on stratified groups

Let \( G = (\mathbb{R}^n, o, \delta) \) be a stratified Lie group (or a homogeneous Carnot group), with dilation structure \( \delta \) and Jacobian generators \( X_1, \ldots, X_N \), so that \( N \) is the
dimension of the first stratum of \( G \). Let us denote by \( Q \) the homogeneous dimension of \( G \). We refer to the recent books [7, 20] for extensive discussions of stratified Lie groups and their properties.

The sub-Laplacian on \( G \) is given by
\[
\mathcal{L} = \sum_{k=1}^{N} X_k^2.
\] (1.4)

We also recall that the standard Lebesgue measure \( dx \) on \( \mathbb{R}^n \) is the Haar measure for \( G \) (see, e.g. [7, Proposition 1.6.6]). Each left invariant vector field \( X_k \) has an explicit form and satisfies the divergence theorem, see e.g. [7] for the derivation of the exact formula: more precisely, we can express
\[
X_k = \frac{\partial}{\partial x_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} \alpha_{k,m}^{(l)}(x^{(l)}, \ldots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}},
\] (1.5)
with \( x = (x', x^{(2)}, \ldots, x^{(r)}) \), where \( r \) is the step of \( G \) and \( x^{(l)} = (x_1^{(l)}, \ldots, x_{N_l}^{(l)}) \) are the variables in the \( l \)th stratum, see also [7, Sec. 3.1.5] for a general presentation.

The horizontal gradient is given by
\[
\nabla_G := (X_1, \ldots, X_N),
\]
and the horizontal divergence is defined by
\[
\text{div}_G v := \nabla_G \cdot v.
\]
The \( p \)-sub-Laplacian has the following form:
\[
\mathcal{L}_p v = \nabla_G (|\nabla_G v|^{p-2} \nabla_G v).
\] (1.6)

Let us define the half-space on the stratified group \( G \) as
\[
G^+ := \{ x \in G : \langle x, \nu \rangle > d \},
\]
where \( \nu := (\nu_1, \ldots, \nu_r) \) with \( \nu_j \in \mathbb{R}^{N_j}, j = 1, \ldots, r \), is the Riemannian outer unit normal to \( \partial G^+ \) (see [9]) and \( d \in \mathbb{R} \). Let us define the so-called angle function
\[
\mathcal{W}(x) := \left( \sum_{i=1}^{N} (X_i(x), \nu)^2 \right)^{1/2};
\]
such function was introduced by Garofalo [9] in his investigation of the horizontal Gauss map. The Euclidean distance to the boundary \( \partial G^+ \) is denoted by \( \text{dist}(x, \partial G^+) \) and defined as
\[
\text{dist}(x, \partial G^+) = \langle x, \nu \rangle - d.
\]
2. Geometric Hardy Inequalities

Theorem 2.1. Let $G^+$ be a half-space of a stratified group $G$. Then for all $\beta \in \mathbb{R}$ and $p > 1$, we have

$$\int_{G^+} |\nabla_G u|^p dx \geq -(p-1)(|\beta|^\frac{p}{p-1} + \beta) \int_{G^+} \frac{W(x)^p}{\text{dist}(x, \partial G^+)^p} |u|^p dx$$

$$+ \beta \int_{G^+} \frac{C_p(\text{dist}(x, \partial G^+))}{\text{dist}(x, \partial G^+)^p} |u|^p dx,$$

(2.1)

for all $u \in C_0^\infty(G^+)$.

Proof of Theorem 2.1. Let us begin with the divergence theorem, then we apply the H"older inequality and the Young inequality, respectively. It follows for a vector field $V \in C^\infty(G^+)$ that

$$\int_{G^+} \text{div}_G V |u|^p dx = -p \int_{G^+} |u|^{p-1} (V, \nabla_G |u|) dx$$

$$\leq p \left( \int_{G^+} |\nabla_G |u| |^{p} dx \right)^{\frac{1}{p}} \left( \int_{G^+} |V|^{\frac{p}{p-1}} |u|^{p} dx \right)^{\frac{p-1}{p}}$$

$$\leq \int_{G^+} |\nabla_G |u| |^{p} dx + (p-1) \int_{G^+} |V|^{\frac{p}{p-1}} |u|^{p} dx.$$

By rearranging the above expression and using $|\nabla_G u| \geq |\nabla_G |u||$, we arrive at

$$\int_{G^+} |\nabla_G u|^p dx \geq \int_{G^+} (\text{div}_G V - (p-1) |V|^\frac{p}{p-1}) |u|^p dx.$$  \hspace{1cm} (2.2)

Now, we choose $V$ in the following form:

$$V = \beta |\nabla_G \text{dist}(x, \partial G^+)|^{p-2} \text{dist}(x, \partial G^+)^{p-1} \nabla_G \text{dist}(x, \partial G^+),$$

(2.3)

that is

$$|V|^\frac{p}{p-1} = |\beta|^\frac{p}{p-1} |\nabla_G \text{dist}(x, \partial G^+)|^p \text{dist}(x, \partial G^+)^{p}.$$  

Also, we have

$$|\nabla_G \text{dist}(x, \partial G^+)|^p = |(X_1 \text{dist}(x, \partial G^+), \ldots, X_N \text{dist}(x, \partial G^+))|^p$$

$$= |(\langle X_1(x), \nu \rangle, \ldots, \langle X_N(x), \nu \rangle)|^p$$

$$= \left( \sum_{i=1}^N \langle X_i(x), \nu \rangle^2 \right)^{\frac{p}{2}} = W(x)^p.$$
Indeed, let us show that $\langle X_i(x), \nu \rangle = X_i(x, \nu)$:

$$X_i(x) = (0, \ldots, 1, \ldots, 0, a_{i,1}^{(r)}(x'), \ldots, a_{i,N_i}^{(r)}(x'), \ldots, a_{i,1}(x', x^{(2)}, \ldots, x^{(r-1)}), \ldots, a_{i,N_i}(x', x^{(2)}, \ldots, x^{(r-1)})),$$

and

$$\langle X_i(x), \nu \rangle = \nu_i' + \sum_{l=2}^r \sum_{m=1}^{N_i} a_{i,m}^{(l)}(x', x^{(2)}, \ldots, x^{(r-1)}) \nu_m^{(l)}.$$

A direct calculation shows that

$$\text{div}_G V = \beta \frac{\nabla_G (|\nabla_G \text{dist}(x, \partial G^+)|^{p-2} \nabla_G \text{dist}(x, \partial G^+) - \beta(p-1) |\nabla_G \text{dist}(x, \partial G^+)|^{p-2} \nabla_G \text{dist}(x, \partial G^+)}{\text{dist}(x, \partial G^+)^{2(p-1)}}$$

$$= \beta L_p(\text{dist}(x, \partial G^+)) \text{dist}(x, \partial G^+)^{p-1} - \beta(p-1) |\nabla_G \text{dist}(x, \partial G^+)|^p \text{dist}(x, \partial G^+)^{p-1}.$$

So, we get

$$\text{div}_G V - (p-1)|V|^{\frac{p}{p-1}} = -(p-1)(\beta |V|^{\frac{p}{p-1}} + \beta |\nabla_G \text{dist}(x, \partial G^+)|^p \text{dist}(x, \partial G^+)^{p-1})$$

$$+ \beta L_p(\text{dist}(x, \partial G^+)) \text{dist}(x, \partial G^+)^{p-1}.$$

Putting the above expression into inequality \cite{22}, we arrive at

$$\int_{G^+} |\nabla_G u|^p dx \geq -(p-1)(\beta |V|^{\frac{p}{p-1}} + \beta |\nabla_G \text{dist}(x, \partial G^+)|^p \text{dist}(x, \partial G^+)^{p-1})$$

$$+ \beta \int_{G^+} L_p(\text{dist}(x, \partial G^+)) \text{dist}(x, \partial G^+)^{p-1} |u|^p dx,$$

completing the proof.
2.1. Preliminaries on the Heisenberg group

Let us give a brief introduction of the Heisenberg group. Let $H^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \tilde{\xi} := (z + \tilde{z}, t + \tilde{t} + 2 \text{Im}(z, \tilde{z})),$$

where $\xi := (x, y, t) \in H^n$, $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, $z := (x, y) \in \mathbb{R}^{2n}$ is identified by $C^n$, and $\xi^{-1} = -\xi$ is the inverse element of $\xi$ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form:

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t)$$

for $\lambda > 0$.

The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $H^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}$$

for $1 \leq i \leq n$,

$$Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

for $1 \leq i \leq n$,

and with their (nonzero) commutator

$$[X_i, Y_i] = -4 \frac{\partial}{\partial t}.$$

The horizontal gradient of $H^n$ is given by

$$\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n),$$

so the sub-Laplacian on $H^n$ is given by

$$\mathcal{L} := \sum_{i=1}^n (X_i^2 + Y_i^2).$$

Let us define the half-space of the Heisenberg group by

$$H^+ := \{ \xi \in H^n : \langle \xi, \nu \rangle > d \},$$

where $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x, \nu_y \in \mathbb{R}^n$ and $\nu_t \in \mathbb{R}$ is the Riemannian outer unit normal to $\partial H^+$ (see $\mathbb{H}$) and $d \in \mathbb{R}$. Let us define the so-called angle function

$$W(\xi) := \sqrt{\sum_{i=1}^n (X_i(\xi), \nu)^2 + (Y_i(\xi), \nu)^2}.$$

The Euclidean distance to the boundary $\partial H^+$ is denoted by $\text{dist}(\xi, \partial H^+)$ and defined by

$$\text{dist}(\xi, \partial H^+) := \langle \xi, \nu \rangle - d.$$
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2.2. Consequences on the Heisenberg group

As a consequence of Theorem 2.1 we have the following inequality.

Corollary 2.2. Let $H^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$, we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi,$$

where the constant is sharp.

**Remark 2.3.** Note that inequality (2.4) was conjectured in [10] which is a natural extension of inequality (1.3) in [11]. Also, the sharpness of inequality (2.4) was proved by choosing $\nu := (1, 0, \ldots, 0)$ and $d = 0$ in the paper [10].

**Proof of Corollary 2.2.** Let us rewrite the inequality in Theorem 2.1 in terms of the Heisenberg group as follows:

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p-1)(\beta \frac{1}{p} + \beta) \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi$$

$$+ \beta \int_{\mathbb{H}^+} \mathcal{L}_p(\text{dist}(\xi, \partial \mathbb{H}^+)) |u|^p d\xi.$$

In the case of the Heisenberg group, we need to show that the last term vanishes to prove Corollary 2.2. Indeed, we have

$$\mathcal{L}_p(\text{dist}(\xi, \partial \mathbb{H}^+)) = \nabla_G(\nabla_G(\langle \xi, \nu \rangle - d)^{p-2}\nabla_G(\langle \xi, \nu \rangle - d))$$

$$= \nabla_G(\langle \nabla_G(\langle \xi, \nu \rangle - d)^{p-2}\nabla_G(\langle \xi, \nu \rangle - d) \rangle$$

$$\sum_{i=1}^n (X_i(\langle \nabla_G(\langle \xi, \nu \rangle - d)^{p-2}\nabla_G(\langle \xi, \nu \rangle - d) \rangle, \nu \rangle)$$

$$= \sum_{i=1}^n X_i \left( \sum_{j=1}^n \left( \langle X_j(\langle \xi, \nu \rangle)^{p-2}\nabla_G(\langle \xi, \nu \rangle - d) \rangle \langle Y_j(\langle \xi, \nu \rangle, \nu) \rangle \right)^{\frac{p-2}{2}} \langle X_i(\langle \xi, \nu \rangle, \nu) \rangle \right)$$

$$+ \sum_{i=1}^n Y_i \left( \sum_{j=1}^n \left( \langle X_j(\langle \xi, \nu \rangle)^{p-2}\nabla_G(\langle \xi, \nu \rangle - d) \rangle \langle Y_j(\langle \xi, \nu \rangle, \nu) \rangle \right)^{\frac{p-2}{2}} \langle Y_i(\langle \xi, \nu \rangle, \nu) \rangle \right)$$

$$= (p-2) \sum_{i=1}^n \left( \sum_{j=1}^n \left( \langle X_j(\langle \xi, \nu \rangle)^{p-2}\nabla_G(\langle \xi, \nu \rangle - d) \rangle \langle Y_j(\langle \xi, \nu \rangle, \nu) \rangle \right)^{\frac{p-2}{2}} \right)$$

$$\times \langle Y_i(\langle \xi, \nu \rangle, X_i(\langle \xi, \nu \rangle, \nu) \rangle \langle X_i(\langle \xi, \nu \rangle, \nu) \rangle \rangle.$$
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\[ (p - 2) \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \left( (X_j(\xi), \nu)^2 + (Y_j(\xi), \nu)^2 \right)^{\frac{p}{2} - 1} \right) \]
\[ \times \langle X_i(\xi), \nu \rangle (Y_i(X_i(\xi), \nu))(Y_i(\xi), \nu) = 0, \]

since
\[ \langle X_i(\xi), \nu \rangle = \nu_{x,i} + 2y_i \nu_t, \quad \langle Y_i(\xi), \nu \rangle = \nu_{y,i} - 2x_i \nu_t, \]
\[ X_i(X_i(\xi), \nu) = 0, \quad Y_i(Y_i(\xi), \nu) = 0, \]
\[ X_i(X_i(\xi), \nu) = 2 \nu_t, \quad X_i(Y_i(\xi), \nu) = -2 \nu_t, \]
where \( \xi := (x, y, t) \) with \( x, y \in \mathbb{R}^n \) and \( t \in \mathbb{R} \), \( \nu := (\nu_x, \nu_y, \nu_t) \) with \( \nu_x := (\nu_{x,1}, \ldots, \nu_{x,n}) \) and \( \nu_y := (\nu_{y,1}, \ldots, \nu_{y,n}) \). Then, we have
\[ X_i(\xi) = (0, \ldots, 1, \ldots, 0, \ldots, 0, 2y_i), \]
\[ Y_i(\xi) = (0, \ldots, 0, \ldots, 1, \ldots, 0, -2x_i). \]

So, we have
\[ \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq -(p - 1)(|\beta|^{\frac{p}{p-1}} + \beta) \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\operatorname{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi. \]

Now, we optimize by differentiating the above inequality with respect to \( \beta \), so that we have
\[ \frac{p}{p-1} |\beta|^{\frac{p}{p-1}} + 1 = 0, \]
which leads to
\[ \beta = - \left( \frac{p - 1}{p} \right)^{p-1}. \]

Using this value of \( \beta \), we arrive at
\[ \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left( \frac{p - 1}{p} \right)^p \int_{\mathbb{H}^+} \frac{W(\xi)^p}{\operatorname{dist}(\xi, \partial \mathbb{H}^+)^p} |u|^p d\xi. \]

We have finished the proof of Corollary 2.2. \hfill \square

3. Geometric Hardy–Sobolev Inequalities
In this section, we present the geometric Hardy–Sobolev inequality in the half space on the Heisenberg group.
3.1. A lower estimate for the geometric Hardy type inequalities

We start with an estimate for the remainder.

**Lemma 3.1.** Let \( \mathbb{H}^n \) be a half-space of the Heisenberg group \( \mathbb{H}^n \). Then for \( p \geq 2 \), there exists a constant \( C_p > 0 \) such that

\[
E_p[u] = \int_{\mathbb{H}^n} |\nabla H u|^p \, d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^n} \frac{|\nabla H \text{dist}(\xi, \partial \mathbb{H}^n)|^p}{\text{dist}(\xi, \partial \mathbb{H}^n)^p} |u|^p \, d\xi \\
\geq C_p \int_{\mathbb{H}^n} |\text{dist}(\xi, \partial \mathbb{H}^n)|^{p-1} |\nabla H v|^p \, d\xi,
\]

for all \( u \in C_0^\infty(\mathbb{H}^n) \), where \( \text{dist}(\xi, \partial \mathbb{H}^n) := |\xi, \nu| - d \) is the distance from \( \xi \) to the boundary, \( C_p = (2^{p-1} - 1)^{-1} \), and \( u(\xi) = \text{dist}(\xi, \partial \mathbb{H}^n)^{\frac{p}{p-1}} v(\xi) \).

The Euclidean version of such a lower estimate to the Hardy inequality was established by Barbaris et al. [4].

**Proof of Lemma 3.1.** Let us begin by recalling once again the angle function, denoted by \( \mathcal{W} \),

\[
|\nabla H \text{dist}(\xi, \partial \mathbb{H}^n)|^p = |(X_1(\xi, \nu), \ldots, X_n(\xi, \nu), Y_1(\xi, \nu), \ldots, Y_n(\xi, \nu))|^p \\
= \left( \sum_{i=1}^n |X_i(\xi, \nu)|^2 + |Y_i(\xi, \nu)|^2 \right)^{\frac{p}{2}} = \mathcal{W}(\xi)^p.
\]

Note that \( X_i(\xi, \nu) \) is equal to \( \langle X_i(\xi), \nu \rangle \), see the proof of Theorem 2.1. This expression \( |\nabla H \text{dist}(\xi, \partial \mathbb{H}^n)|^p = \mathcal{W}(\xi)^p \) will be used later. For now, we will estimate the following form:

\[
E_p[u] := \int_{\mathbb{H}^n} |\nabla H u|^p \, d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^n} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^n)^p} |u|^p \, d\xi.
\]

To estimate this, we introduce the following transformation:

\[
u(\xi) = \text{dist}(\xi, \partial \mathbb{H}^n)^{\frac{p}{p-1}} v(\xi).
\]

By inserting it into (3.3) and using (3.2), we have

\[
E_p[u] = \int_{\mathbb{H}^n} \left( \frac{p-1}{p} \right)^p \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^n)^p} |\nabla H \nu|^p \, d\xi \\
- \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^n} \frac{\mathcal{W}(\xi)^p}{\text{dist}(\xi, \partial \mathbb{H}^n)^p} |\text{dist}(\xi, \partial \mathbb{H}^n)^{-\frac{p}{p-1}} v|^p \, d\xi \\
\geq \int_{\mathbb{H}^n} \left( \frac{p-1}{p} \right)^p \text{dist}(\xi, \partial \mathbb{H}^n)^{-\frac{p}{p-1}} v + \text{dist}(\xi, \partial \mathbb{H}^n)^{-\frac{p}{p-1}} \frac{\nabla H v}{\nabla H \text{dist}(\xi, \partial \mathbb{H}^n)} |\mathcal{W}(\xi)|^p \\
- \left( \frac{p-1}{p} \right)^p \text{dist}(\xi, \partial \mathbb{H}^n)^{-\frac{p}{p-1}} v |\mathcal{W}(\xi)|^p \, d\xi.
\]
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Then for $p \geq 2$ and $A, B \in \mathbb{R}^n$, we have that

$$|A + B|^p - |A|^p \geq C_p |B|^p + p|A|^{p-2}A \cdot B,$$

where $C_p = (2^{p-1} - 1)^{-1}$ (see \[3.3\]). By taking

$$A := \frac{p-1}{p} \text{dist}(\xi, \partial \mathbb{H}^+) \frac{1}{\nu} \nabla_H u \quad \text{and} \quad B := \text{dist}(\xi, \partial \mathbb{H}^+) \frac{p-1}{p} \nabla_H u,$$

then we have the following lower estimate:

$$E_p[u] \geq \int_{\mathbb{H}^+} |W(\xi)|^p (|A + B|^p - |A|^p) d\xi$$

$$\geq C_p \int_{\mathbb{H}^+} \text{dist}(\xi, \partial \mathbb{H}^+) |\nabla_H u|^p \frac{|W(\xi)|^p}{|\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)|^p} d\xi + \left( \frac{p-1}{p} \right)^{p-1}$$

$$\times \int_{\mathbb{H}^+} |W(\xi)|^p |\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)|^p - 2 \left( \nabla_H \text{dist}(\xi, \partial \mathbb{H}^+) \cdot \nabla_H |u|^p \right) d\xi$$

$$\geq C_p \int_{\mathbb{H}^+} \text{dist}(\xi, \partial \mathbb{H}^+) |\nabla_H u|^p d\xi.$$

In the last line, we have used \[3.2\] and we dropped the last term on the right-hand side. This completes the proof of Lemma \[3.1\].

\[3.2\] Geometric Hardy–Sobolev inequality in the half-space on $\mathbb{H}^n$

Now, we are ready to obtain the geometric Hardy–Sobolev inequality in the half-space on the Heisenberg group $\mathbb{H}^n$.

**Theorem 3.2.** Let $\mathbb{H}^+ \subset \mathbb{H}^n$ be a half-space of the Heisenberg group $\mathbb{H}^n$. For $2 \leq p < Q$ with $Q = 2n + 1$ there exists some $C > 0$ such that for every function $u \in C_0^p(\mathbb{H}^+)$ we have

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - \left( \frac{p-1}{p} \right)^p \int_{\mathbb{H}^+} \frac{|W(\xi)|^p}{|\text{dist}(\xi, \partial \mathbb{H}^+)|^p} |u|^p d\xi \right)^{\frac{1}{p}} \geq C \left( \int_{\mathbb{H}^+} |u|^{p^*} d\xi \right)^{\frac{1}{p^*}},$$

where $p^* := Qp/(Q-p)$ and $\text{dist}(\xi, \partial \mathbb{H}^+) := (\xi, \nu) - d$ is the distance from $\xi$ to the boundary.

**Remark 3.3.** Note that for $p = 2$ we have the Hardy–Sobolev–Maz’ya inequality in the following form:

$$\left( \int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \frac{1}{4} \int_{\mathbb{H}^+} \frac{|W(\xi)|^2}{|\text{dist}(\xi, \partial \mathbb{H}^+)|^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left( \int_{\mathbb{H}^+} |u|^{2^*} d\xi \right)^{\frac{1}{2^*}}$$

where $2^* := 2Q/(Q-2)$. 

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Proof of Theorem 3.2. Our key ingredient of proving the Hardy–Sobolev inequality in the half-space of $\mathbb{H}^n$ is the $L^1$-Sobolev inequality, or the Gagliardo–Nirenberg inequality. It has been established on the Heisenberg group by Baldi et al. in [3].

The $L^1$-Sobolev inequality on the Heisenberg group follows in the form:

$$c \left( \int_{\mathbb{H}^n} |g|^{Q-1} d\xi \right)^{\frac{Q-1}{Q}} \leq \int_{\mathbb{H}^n} |\nabla_H g| d\xi,$$

for some $c > 0$, for every function $g \in W^{1,1}(\mathbb{H}^n)$. Now, let us set $g = |u|^{p^*(1-1/Q)}$, then we obtain

$$c \left( \int_{\mathbb{H}^n} |u|^{p^*} d\xi \right)^{\frac{Q-1}{Q}} \leq \left| \int_{\mathbb{H}^n} \frac{p(Q-1)}{Q} \left| u \right|^{\frac{Q(p-1)}{p(Q-p)}} |\nabla_H u| d\xi, \right.$$

$$\leq \left| \int_{\mathbb{H}^n} \frac{p(Q-1)}{Q} \left| u \right|^{\frac{Q(p-1)}{p(Q-p)}} |\nabla_H u| d\xi, \right.$$ 

$$= \left| \int_{\mathbb{H}^n} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi, \right.$$

We have used $|\nabla_H u| \leq |\nabla_H u|$ (see [4] Proof of Theorem 2.1)). Then, we arrive at

$$C_1 \left( \int_{\mathbb{H}^n} |u|^{p^*} d\xi \right)^{\frac{Q-1}{Q}} \leq \int_{\mathbb{H}^n} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi, \tag{3.7}$$

where $C_1 := c \left| \frac{Q-p}{p(Q-Q-1)} \right| > 0$. Let us estimate the right-hand side of inequality (3.7).

Again, we use a ground transform $u(\xi) = \text{dist}(\xi, \partial \mathbb{H}^n)^{\frac{p-1}{p}} v(\xi)$ which leads to

$$\int_{\mathbb{H}^n} |u|^{p^*(1-1/p)} |\nabla_H u| d\xi$$

$$\leq \int_{\mathbb{H}^n} |u|^{p^*(1-1/p)} \text{dist}(\xi, \partial \mathbb{H}^n)^{\frac{p-1}{p}} |\nabla_H v| d\xi$$

$$+ \frac{p-1}{p} \int_{\mathbb{H}^n} \text{dist}(\xi, \partial \mathbb{H}^n)^{p^*(1-1/p)^2-1/p}$$

$$\times |\nabla_H \text{dist}(\xi, \partial \mathbb{H}^n)||v|^{p^*(1-1/p)+1} d\xi$$

$$= I_1 + \frac{p-1}{p} I_2.$$

In the last line we have denoted two integrals by $I_1$ and $I_2$, respectively. Also, for simplification we denote $\alpha := p^*(1-1/p)^2 + 1 - 1/p$. First, we estimate $I_2$ using
integrations by parts

\[
I_2 = \int_{\mathbb{H}^+} \text{dist}(\xi, \partial \mathbb{H}^+)^{\alpha-1} |\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)| |v|^{\alpha p/(p-1)} d\xi
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{H}^+} (\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+), \nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)) \frac{|v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)|} d\xi
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial \mathbb{H}^+)^{\alpha} \nabla_H \left( \frac{\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)|v|^{\alpha p/(p-1)}}{|\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)|} \right) d\xi
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial \mathbb{H}^+)^{\alpha} \nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)|v|^{\alpha p/(p-1)} d\xi
\]

\[
\leq \frac{p}{p-1} \int_{\mathbb{H}^+} \text{dist}(\xi, \partial \mathbb{H}^+)^{\alpha} |v|^{\alpha p/(p-1)-1} |\nabla_H v| d\xi
\]

\[
= \frac{p}{p-1} \int_{\mathbb{H}^+} |u|^{p' (1-1/p)} \text{dist}(\xi, \partial \mathbb{H}^+) \frac{1}{p-1} |\nabla_H v| d\xi \leq \frac{p}{p-1} I_1.
\]

We have used \(|\nabla_H| u| \leq |\nabla_H u|\), and

\[
\mathcal{L}(\text{dist}(\xi, \partial \mathbb{H}^+)) = \sum_{i=1}^{n} (X_i(\text{dist}(\xi, \partial \mathbb{H}^+), \nu) + Y_i(\text{dist}(\xi, \partial \mathbb{H}^+), \nu)) = 0,
\]

since

\[
\langle X_i(\xi), \nu \rangle = \nu_{x,i} + 2y_i \nu_{t}, \quad \langle Y_i(\xi), \nu \rangle = \nu_{y,i} - 2x_i \nu_{t},
\]

\[
X_i(\text{dist}(\xi, \partial \mathbb{H}^+), \nu) = 0, \quad Y_i(\text{dist}(\xi, \partial \mathbb{H}^+), \nu) = 0,
\]

where \(\xi := (x, y, t)\) with \(x, y \in \mathbb{R}^n\) and \(t \in \mathbb{R}\), \(\nu := (\nu_x, \nu_y, \nu_t)\) with \(\nu_x := (\nu_{x,1}, \ldots, \nu_{x,n})\) and \(\nu_y := (\nu_{y,1}, \ldots, \nu_{y,n})\). Also, we have

\[
\langle \nabla_H \text{dist}(\xi, \partial \mathbb{H}^+), \nabla_H |\nabla_H \text{dist}(\xi, \partial \mathbb{H}^+)| \rangle
\]

\[
= \frac{2\nu_t}{\text{dist}(\xi, \partial \mathbb{H}^+)}
\]

\[
\times \left( \frac{(2x_1 \nu_t - \nu_{y,1})(\nu_{x,1} + 2y_1 \nu_t) + \cdots + (2x_n \nu_t - \nu_{y,n})(\nu_{x,n} + 2y_n \nu_t)}{\sqrt{n}^{\alpha}} \right)
\]

\[
+ \frac{(\nu_{y,1} - 2x_1 \nu_t)(\nu_{x,1} + 2y_1 \nu_t) + \cdots + (\nu_{y,n} - 2x_n \nu_t)(\nu_{x,n} + 2y_n \nu_t)}{\sqrt{n}^{\alpha}} = 0,
\]

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since

\[
\nabla H |\nabla H \text{dist}(\xi, \partial H^+) | = (X_1 |\nabla H \text{dist}(\xi, \partial H^+) |, \ldots, X_n |\nabla H \text{dist}(\xi, \partial H^+) |, \\
Y_1 |\nabla H \text{dist}(\xi, \partial H^+) |, \ldots, Y_n |\nabla H \text{dist}(\xi, \partial H^+) |)
\]

\[
= 2\nu_t \frac{|\nabla H \text{dist}(\xi, \partial H^+) |}{|\nabla H \text{dist}(\xi, \partial H^+) |} \\
\times (2x_1\nu_t - \nu_y, 1, \ldots, 2x_n\nu_t - \nu_y, \nu_y, \nu_x, 1 + 2y_1\nu_t, \ldots, \nu_y + 2y_n\nu_t,)
\]

and

\[
\nabla H \text{dist}(\xi, \partial H^+) = (\nu_x, 1, \ldots, 2x_n\nu_t - \nu_y, \nu_y, \nu_x, 1 + 2y_1\nu_t, \ldots, \nu_y + 2y_n\nu_t).
\]

As we see that integral \(I_2\) can be estimated by integral \(I_1\). From this estimation, we know that

\[
\int_{H^+} |u|^{p^*(1-1/p)} |\nabla H u| d\xi \leq 2I_1.
\]

(3.8)

Now, it comes to estimate \(I_1\) by using the Hölder inequality

\[
I_1 = \int_{H^+} \{ |u|^{p^*(1-1/p)} \} \{ \text{dist}(\xi, \partial H^+)^{(p-1)/p} |\nabla H v| \} d\xi
\]

\[
\leq \left( \int_{H^+} |u|^{p^*} d\xi \right)^{1-1/p} \left( \int_{H^+} \text{dist}(\xi, \partial H^+)^{(p-1)/p} |\nabla H v|^p d\xi \right)^{1/p}
\]

\[
\leq C_p^{1-1/p} \left( \int_{H^+} |u|^{p^*} d\xi \right)^{1-1/p}
\]

\[
\times \left( \int_{H^+} |\nabla H u|^{p^*} d\xi - \left( \frac{p-1}{p} \right)^p \int_{H^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial H^+)^p} |u|^{p} d\xi \right)^{1/p}.
\]

In the last line we have used Lemma \ref{lemma4.3}. Inserting the estimate of \(I_1\) in (3.8), we arrive at

\[
\int_{H^+} |u|^{p^*(1-1/p)} |\nabla H u| d\xi
\]

\[
\leq 2C_p^{1-1/p} \left( \int_{H^+} |u|^{p^*} d\xi \right)^{1-1/p}
\]

\[
\times \left( \int_{H^+} |\nabla H u|^{p^*} d\xi - \left( \frac{p-1}{p} \right)^p \int_{H^+} \frac{W(\xi)^p}{\text{dist}(\xi, \partial H^+)^p} |u|^{p} d\xi \right)^{1/p}.
\]
Proof of Corollary 3.4. Let us demonstrate our result in a particular case when $p = 2$.

Consequence of Theorem $3.2$ Let us demonstrate our result in a particular case when $p = 2$.

**Corollary 3.4.** Let $\mathbb{H}^+ := \{ \xi = (x, y, t) \in \mathbb{R}^n | t > 0 \}$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$ taking $d = 0$, we have

$$
\left( \int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left( \int_{\mathbb{H}^+} |u|^2 d\xi \right)^{\frac{1}{2}},
$$

where $2^* := 2Q/(Q - 2), Q = 2n + 2$, with $C > 0$ independent of $u$.

**Proof of Corollary 3.4** We have the following left-invariant vector fields:

$$
X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \quad \text{and} \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},
$$

with the commutator

$$
[X_i, Y_i] = -4 \frac{\partial}{\partial t}.
$$

Then for $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ and $\nu = (0, \ldots, 0, 0, \ldots, 0, 1)$, we get

$$
\langle X_i(\xi), \nu \rangle = 2y_i \quad \text{and} \quad \langle Y_i(\xi), \nu \rangle = -2x_i,
$$

where

$$
X_i(\xi) = (0, \ldots, 1, \ldots, 0, 0, \ldots, 0, 2y_i),
$$

$$
Y_i(\xi) = (0, \ldots, 0, \ldots, 1, \ldots, 0, -2x_i).
$$

Thus, we arrive at

$$
\frac{W(\xi)^2}{\text{dist}(\xi, \partial\mathbb{H}^+)^2} = 4 \frac{|x|^2 + |y|^2}{t^2}.
$$

Plugging the above expression into inequality (3.9), we obtain

$$
\left( \int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2 + |y|^2}{t^2} |u|^2 d\xi \right)^{\frac{1}{2}} \geq C \left( \int_{\mathbb{H}^+} |u|^2 d\xi \right)^{\frac{1}{2}},
$$

showing (3.9).
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