A new method for solving the linear recurrence relation with nonhomogeneous constant coefficients in some cases

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Abstract. Recursive relation mainly describes the unique law satisfied by a sequence, so it plays an important role in almost all branches of mathematics. It is also one of the main algorithms commonly used in computer programming. This paper first introduces the concept of recursive relation and two common basic forms, then starts with the solution of linear recursive relation with non-homogeneous constant coefficients, gives a new solution idea, and gives a general proof. Finally, through an example, the general method and the new method given in this paper are compared and verified.

1. Introduction

Recurrence relation, which mainly characterizes the unique laws satisfied by a sequence, plays an important role in almost all branches of mathematics. This is especially true for combinatorial mathematics, because each combinatorial counting problem has its combinatorial characteristics and laws. In many cases, recursive relationship is one of the most appropriate tools to describe this combinatorial law. Because of its importance, it has been deeply studied and applied in many aspects of engineering and life [1-4].

How to establish the recurrence relation, what are the properties of the known recurrence relation and how to solve the recurrence relation are several basic problems in the recurrence relation. Consider a sequence \( \{a_n : n \geq 0\} \), where \( n \) is the number of schemes to be calculated in a specific mathematical model. Actually, it's a function of \( n \). For such an unknown sequence, in order to calculate the number of schemes, a natural idea is to directly find its general term expression and then find all its terms. However, when the general term \( a_n \) cannot be obtained directly, if we can find a relationship between \( a_n \) and its several adjacent terms, and this relationship is always true when \( n \) is greater than or equal to a fixed positive integer, we have the possibility of finding the sequence \( a_n \) by using this relationship.

Depending on whether there is a nonhomogeneous term, recursive relations are generally divided into two types: r-order homogeneous constant coefficient linear recursive relations and r-order non-homogeneous constant coefficient linear recursive relations. The solution of r-order homogeneous constant coefficient linear recursive relation is relatively simple. As long as the characteristic polynomial is listed to solve the characteristic root, and then the formula is applied to solve the undetermined coefficient, the solution of the recursive relation can be obtained. This paper mainly studies a special
case of non-homogeneous recurrence relation, gives a new solution method from another point of view, and gives a general proof. Finally, an example is given to verify the correctness of the method.

2. Common solving methods and new ideas of recursive relations

The general form of r-order non-homogeneous linear recursive relation with constant coefficients is:

\[ u_n = \sum_{j=1}^{r} c_j u_{n-j} + g(n) \]  

(1)

For the r-order linear recursive relation with non-homogeneous constant coefficients, the general solution is divided into three steps. The first step is to find the special solution of the non-homogeneous linear recursive relation; The second part is to solve the general solution of its corresponding derived recurrence relationship; Finally, the third step is to add the two solutions, and the result of the addition is the general solution of the original r-order non-homogeneous linear recurrence relationship with constant coefficients. The general solution of deducing the recurrence relation is actually to solve an r-order homogeneous linear recurrence relation. There are specific solutions in the corresponding textbooks [5,6]. Next, we mainly study the solution method of the special solution of the non-homogeneous recurrence relation. However, in the common solution, this method can only find the special solution for some special \( g(n) \), which refers to the part of the linear recursive relationship with non-homogeneous constant coefficients excluding the derived recursive relationship.

In order to facilitate the solution, formula (1) is transformed into r-order homogeneous constant coefficient linear recursive relationship, and the transformed form is

\[ u_n = \sum_{j=1}^{r} c_j u'_{n-j} \]  

(2)

where \( u'_{n-j} = u_{n-i} + g(n-i) \) \( (i = 1, 2, \cdots, r) \). Through observation and analysis, it is known that polynomials \( g(n) \) and \( g'(n) \) have the same degree. It can be obtained from the combination of formulas (1) and (2) that

\[ g'(n) - \sum_{j=1}^{r} c_j g(n-j) = -g(n) \]  

(3)

Next, a brief proof of the feasibility of this method is given.

Case 1: \( g(n) \) is polynomial

Suppose \( g(n) = A_m n^m + A_{m-1} n^{m-1} + \cdots + A_1 n + A_0 \), \( g'(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0 \).

In this case, it is required to prove the existence of \( a_i (i = 1, 2, \cdots, m) \). It can be obtained from equation (1-3),

\[
\left( a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0 \right) - \sum_{j=1}^{r} c_j \left[ a_m (n-j)^m + a_{m-1} (n-j)^{m-1} + \cdots + a_1 (n-j) + a_0 \right] \\
= A_m n^m + A_{m-1} n^{m-1} + \cdots + A_1 n + A_0
\]

Comparing the coefficients of \( n^m \), we can get \( a_m - \sum_{j=1}^{r} c_j a_m = A_m \), then

\[ a_m = \frac{A_m}{1 - \sum_{j=1}^{r} c_j} \]  

(4)
Comparing the coefficients of \( n^{m-1} \), we can get
\[
a_{m-1} - \sum_{j=1}^r c_j [a_m C_m^i (-j) + a_{m-1}] = A_{m-1}.
\]

This is a first-order equation containing two unknowns \( a_m \) and \( a_{m-1} \). If we substitute a solved \( a_m \) by formula (4) into it, we can solve \( a_{m-1} \). Similarly, by comparing the coefficients of all \( n^{m-i} \), we can get
\[
a_{m-i} - \sum_{j=1}^r c_j [a_m C_m^i (-j) + a_{m-1} C_m^{i-1} (-j)^{-1} + \cdots + a_{m-i}] = A_{m-i}.
\]

The value of \( a_m, a_{m-1}, \ldots, a_{m-i+1} \) can be solved by substituting \( a_{m-i} \) solved by the previous equation into this system of equations, which contains \( i+1 \) univariate equations.

From the above, we can know the value of \( a_m, a_{m-1}, \ldots, a_1, a_0 \), that is, we can prove that these \( m+1 \) coefficients do exist, so this new method is feasible when \( g(n) \) is a polynomial.

Case 2. \( g(n) = \alpha^n \)

Similar to the above idea, we can draw the conclusion of \( g'(n) = k \alpha^n \). Furthermore, the following conclusions can be obtained
\[
g'(n) = -\frac{\alpha^n}{\alpha^r - \sum_{i=1}^r c_i \alpha^{r-i}} \quad (5)
\]

Case 3. \( g'(n) = n^i \alpha^n \)

First, through the analysis, it can be seen that the form of \( g'(n) \) probably has the appearance of \( g'(n) = kn^i \alpha^n \). Furthermore, the following conclusions can be obtained
\[
k = -\frac{\alpha^r}{\alpha^r - \sum_{i=1}^r c_i \alpha^{r-i}} \quad (6)
\]

3. Specific examples

Example 1: find the special solution of the following recursive relationship
\[
u_{n+2} + u_{n+1} + u_n = n^2 + n + 1
\]

Solution 1 (classical solution method):

It is easy to find that the characteristic equation corresponding to the recursive relationship is \( c(x) = x^2 + x + 1 \), so 1 is not the root of the characteristic equation, and its special solution can be solved directly by the difference method.

\[
u_n = (\Delta^2 + 3\Delta + 3I)(n^2 + n + 1) = 3^{-1} (3^{-1} \Delta^2 + \Delta + I)^{-1} (n^2 + n + 1)
\]
\[
= 3^{-1} [I - (\Delta + 3^{-1} \Delta^2) + (\Delta + 3^{-1} \Delta^2)^{-2} \cdots] (n^2 + n + 1)
\]
\[
= 3^{-1} [I - \Delta + 2 \cdot 3^{-1} \Delta^2 + \cdots] (n^2 + n + 1)
\]
\[
= 3^{-1} [I - \Delta + 2 \cdot 3^{-1} \Delta^2 + \cdots] (n)_2 + 2(n)_1 + 1
\]
\[
= 3^{-1} [n(n-1) + 2n + 1 - 2n - 2 + 4 \cdot 3^{-1}]
\]
\[
= 3^{-1} (n^2 - n + 3^{-1})
\]

So \( 3^{-1} (n^2 - n + 3^{-1}) \) is the special solution of this recursive relation.
Solution 2 (new solution):

Because \( g(n) = n^2 + n + 1 \), it can be assumed that \( g'(n) = an^2 + bn + c \), and then the equations can be listed and solved to obtain \( a = -\frac{1}{3}, b = \frac{1}{3}, c = -\frac{1}{9} \). Then

\[
[u_{n+2} + (-\frac{1}{3}(n+2)^2 + \frac{1}{3}(n+2) - \frac{1}{9})] + [u_{n+1} + (-\frac{1}{3}(n+1)^2 + \frac{1}{3}(n+1) - \frac{1}{9})] + [u_n + (-\frac{1}{3}n^2 + \frac{1}{3}n - \frac{1}{9})] = 0
\]

Suppose if \( \frac{b_n}{b_{n+2}} + \frac{b_{n+1}}{b_n} = 0 \), then the recursive relationship of order 2 homogeneous constant coefficients holds: \( b_n + b_{n+1} = 0 \), of which \( b_n = 0 \), is a special solution. Therefore, the special solution of the above linear recursive relationship with non-homogeneous constant coefficients is

\[
u(n) = \frac{1}{3}n^2 - \frac{1}{3}n + \frac{1}{9} = 3^{-1}(n^2 - n + 3^{-1})
\]

Obviously, the special solutions obtained by the two methods are the same. The new method avoids the calculation of difference. In fact, in other cases, it can also be solved by this method.

4. Conclusion

It can be found that the new method is a little troublesome in determining the undetermined coefficients at the beginning, but there is no too much difference operation in the later steps of finding the special solution, and the calculation process is simplified, which is obviously simpler than the conventional method. But here, for some special functions, it is also difficult to determine the undetermined coefficients to solve, so this method still has room for improvement.

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