Spectral analysis of one class perturbed first order differential operators

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Abstract. We use the method of similar operators to study a mixed problem for a first order differential equation with a fractional integration operator. The differential operator defined by the equation is transformed into a similar operator that is an orthogonal direct sum of finite-rank operators. The estimates of eigenvalues, eigenvectors are obtained. The result is used to construct an operator group.

1. Introduction
In this paper we consider a mixed problem for a differential equation with a fractional integration operator in the following form

\[
\begin{align*}
\frac{\partial u(t,s)}{\partial t} = & \frac{\partial u(t,s)}{\partial s} - \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} u(t,x) \, dx, \\
u(t,0) = & u(t,\omega), \\
u(0,s) = & \varphi(s), \\
t \in \mathcal{J}, \ s \in [0,\omega], \ & \alpha > 1/2,
\end{align*}
\]

where \( \mathcal{J} \) is one of the intervals \((-\infty, \infty), \ (-\infty, b], [a, b] \) or \([a, \infty) \). It is always assumed that \( 0 \in \mathcal{J} \).

To formulate the problem we introduce the functional spaces. Let \( \mathcal{H} = L_2 = L_2[0,\omega] \) be the Hilbert space of equivalence classes of square integrable Lebesgue measurable complex functions on closed interval \([0,\omega]\). The inner product on this space is given, as usually, by

\[
(x,y) = \frac{1}{\omega} \int_0^\omega x(t)\overline{y(t)} \, dt, \quad x, y \in \mathcal{H}.
\]

The norm in \( \mathcal{H} \) is induced by this inner product. By \( W^1_2[0,\omega] \) we denote the Sobolev space \( \{ y \in L_2 : y \text{ is absolutely continuous and } y' \in L_2 \} \). By \( C(\mathcal{J}, L_2) \) we shall denote the linear space of all functions \( v : \mathcal{J} \times [0,\omega] \to \mathbb{C} \) such that, for each fixed \( t \in \mathcal{J} \), the function \( s \mapsto v(t,s) \) belongs to \( L_2 \) and the function \( \tilde{v} : \mathcal{J} \to L_2, \ (\tilde{v}(t))(s) = v(t,s), \ t \in \mathcal{J}, \ s \in [0,\omega], \) is continuous. If \( \mathcal{J} \) is a finite interval, then \( C(\mathcal{J}, L_2) \) is a Banach space with the norm \( \|v\|_\infty = \max_{t \in \mathcal{J}} \|\tilde{v}(t)\|_2 \).

The function \( \tilde{v} \) is called associated function to \( v \) and they will be identified.
The problem (1) in the Hilbert space $L^2$ in the operator form is written respectively as
\[ \ddot{u} + Lu = 0, \quad \dot{u}(0) = \varphi. \] (2)

The operator $L : D(L) \subset L^2 \to L^2$ in the equation (2) is defined by
\[ (Ly)(s) = \frac{dy(s)}{ds} - \frac{1}{\Gamma(\alpha)} \int_0^s (s - x)^{\alpha - 1} y(x) \, dx, \quad \alpha > 1/2. \] (3)

The domain $D(L)$ is given by the periodic boundary conditions $D(L) = \{ y \in W^1_2[0, \omega] : y(0) = y(\omega) \}$.

We note that $L = L_0 - B$, where $L_0 = d/dt$, $D(L_0) = D(L)$ and the operator $B$ is the fractional integration operator (see [1–4]) with $\alpha > 1/2$. The operator $L_0$ we call the unperturbed or free operator, the operator $B$ we call the perturbation and the operator $L$ we call the perturbed operator. The operator $B$ is called the Riemann-Liouville fractional integration operator and in general $y \in L^1[0, \omega]$ (see [1–4]). In our case ($\alpha > 1/2$) this operator is a integral operator with square summable kernel on $[0, \omega]$ (see [5]).

Recently, interest in the mixed problem for the hyperbolic equation with an involution has intensified (see, for example, [6–18] and references therein). In these works the problem of justification of Fourier method was studied. There was also studied the asymptotics of eigenvalues and equiconvergence of spectral resolution. In [6–12] the resolvent method was used to justify the Fourier method. Another alternative research method for study this problem is the similar operators method. In [13–18] the modification of the similar operators method for the first order differential operators with an involution was given. This modification is fully suitable for the study of the problem (1) (without preliminary similary transform), which this paper is devoted to.

In this paper we study the spectral properties of the operator $L$, in particular, the problem of bi-invariant subspaces (see Definition 5). Our primary focus is describing the group generated by the operator $L$.

The similar operators method was pioneered by Friedrichs [19] and then extensively developed and used, for example, in [20–22]. This method has many modifications (one can see in [13,20,23,25,26,28]). We note, that this method can be used for the various classes of perturbed linear operators: for the differential operators second order [20,21], for the nonquasianalytic operators [23], for the integro-differential operators [24], for the difference operator with growing potential [25,26]. We use the modification first proposed in [13] then used in [14–18]. This version of the similar operators method is fully formed in [27].

2. Materials and methods
The main tools for studying the problem under consideration are similar operators and direct sums.

The notation and terminology used herein agree almost completely with that of [14,15,27].

By symbol $\text{End} \mathcal{H}$ we denote the Banach algebra of all bounded linear operators in $\mathcal{H}$ with the norm $\|X\|_\infty = \sup_{\|x\| \leq 1} \|Xx\|, \quad x \in \mathcal{H}, \quad X \in \text{End} \mathcal{H}$.

We begin with the definition of the similar operators.

**Definition 1.** Two linear operators $A_i : D(A_i) \subset \mathcal{H} \to \mathcal{H}, \quad i = 1, 2$, are called similar if there exists a continuously invertible operator $U \in \text{End} \mathcal{H}$ such that
\[ A_1 U x = U A_2 x, \quad x \in D(A_2), \quad UD(A_2) = D(A_1). \]

The operator $U$ is called the operator of transformation operator $A_1$ into $A_2$ or intertwining operator [2].
The similar operator method is one of the transmutation methods. The history, state of the art and of the transmutation theory one can be found in [2, 29–31].

**Definition 2.** A nontrivial linear subspace \( M \subset \mathcal{H}, M \neq \mathcal{H}, \) from the Hilbert space \( \mathcal{H} \) is called invariant for a linear operator \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) if \( Ax \in M \) for all \( x \in M \cap D(A) \).

Similar operators possess a series properties:

1) \( \text{Im} \, A_1 = U(\text{Im} \, A_2) \);
2) \( \sigma(A_1) = \sigma(A_2) \);
3) if \( \lambda_0 \) is an eigenvalue of the operator \( A_2 \) and \( x \) is the corresponding eigenvector, \( A_2x = \lambda_0x \), then \( Ux \) is the eigenvector of the operator \( A_1, A_1Ux = \lambda_0Ux \);
4) if the operator \( A_2 \) admits the expansion \( A_2 = A_{21} \oplus A_{22} \), where \( A_{2k} = A_2|_{\mathcal{H}_k}, k = 1, 2, \) is the restriction of \( A_2 \) on \( \mathcal{H}_k \) with respect of the direct sum \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) of subspaces \( \mathcal{H}_k, k = 1, 2, \) invariant with respect to \( A_2 \), then subspaces \( \tilde{\mathcal{H}}_k = U(\mathcal{H}_k), k = 1, 2, \) are invariant with respect to the operator \( A_1 \) and \( A_1 = A_{11} \oplus A_{12} \), where \( A_{1k} = A|_{\mathcal{H}_k}, k = 1, 2, \) at that, \( \mathcal{H} = \tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}_2 \).

Moreover, if \( P \) is a projection corresponding to the expansion \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), that is, \( \mathcal{H}_1 = \text{Im} \, P \) is the image of the projection \( P, \mathcal{H}_2 = \text{Im} \, (I - P) \) is the image of the additional projection \( I - P \), then the projection \( \tilde{P} \in \text{End} \, \mathcal{H} \) corresponding to the expansion \( \mathcal{H} = \tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}_2 \) is defined by

\[ \tilde{P} = UPU^{-1}. \]

5) if the operator \( A_2 \) is a generator of a strongly continuous group of the operators \( T_2 : \mathbb{R} \rightarrow \text{End} \, \mathcal{H} \) (of class \( C_0 \)), then the operator \( A_1 \) is a generator of a strongly continuous group of operators

\[ T_1(t) = UT_2(t)U^{-1}, \quad t \in \mathbb{R}, \quad T_1 : \mathbb{R} \rightarrow \text{End} \, \mathcal{H}. \]

We shall need to extend property 4) to the case of countable direct sum (see also [14, 15]). Let the Hilbert space \( \mathcal{H} \) represented as the direct sum of orthogonal non-zero closed subspaces \( \mathcal{H}_n, n \in \mathbb{Z}, \) that is

\[ \mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n, \tag{4} \]

where \( \mathcal{H}_n \) is orthogonal to \( \mathcal{H}_j \) as \( i \neq j, i, j \in \mathbb{Z}, \) and \( x = \sum_{n \in \mathbb{Z}} x_n, x_n \in \mathcal{H}_n, \|x\|^2 = \sum_{n \in \mathbb{Z}} \|x_n\|^2. \)

In other word, we have a disjunctive resolution of identity

\[ \mathcal{P} = \{P_n, n \in \mathbb{Z}\} \tag{5} \]

that is five properties hold:

1) \( P_n^* = P_n, n \in \mathbb{Z}; \)
2) \( P_nP_j = 0 \) if \( i \neq j, i, j \in \mathbb{Z}; \)
3) the series \( \sum_{n \in \mathbb{Z}} P_nx \) unconditionally converges to \( x \in \mathcal{H} \) and

\[ \|x\|^2 = \sum_{n \in \mathbb{Z}} \|P_nx\|^2; \]

4) the identities \( P_kx = 0, k \in \mathbb{Z}, \) imply that the vector \( x \) is zero;
5) \( \mathcal{H}_k = \text{Im} \, P_k, x_k = P_kx, k \in \mathbb{Z}. \)

According to [5, Ch. 5] the system of subspaces \( \mathcal{H}_k, k \in \mathbb{Z}, \) is an orthogonal basis of subspaces in \( \mathcal{H}. \)

**Definition 3.** [14, 15] A linear operator \( \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \) is called an orthogonal direct sum of bounded operators \( \mathcal{A}_n \in \text{End} \, \mathcal{H}_n, n \in \mathbb{Z}, \) with respect to resolution (4) and it is written as

\[ \mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n, \]
if
1) \( \mathcal{H}_n \subset D(\mathcal{A}) = \{ x \in \mathcal{H} : \sum_{k \in \mathbb{Z}} \| \mathcal{A}_k x_k \|^2 < \infty, x_k = P x, k \in \mathbb{Z} \} \) for all \( n \in \mathbb{Z} \);
2) each subspace \( \mathcal{H}_n, n \in \mathbb{Z} \), is invariant with respect to the operator \( \mathcal{A} \) and \( \mathcal{A}_n, n \in \mathbb{Z} \), is the restriction of the operator \( \mathcal{A} \) on \( \mathcal{H}_n, n \in \mathbb{Z} \);
3) \( \mathcal{A} x = \sum_{k \in \mathbb{Z}} \mathcal{A}_k x_k, x \in D(\mathcal{A}) \), where \( x_k = P x, k \in \mathbb{Z} \).

We introduce a two-sided ideal of Hilbert-Schmidt operators \( \mathfrak{S}_2(\mathcal{H}) \) in the algebra \( \text{End} \mathcal{H} \). By \( \| X \|_2 \) we denote the norm of Hilbert-Schmidt operator \( X \in \mathfrak{S}_2(\mathcal{H}) \), that is \( \| X \|_2 = (\text{tr} XX^*)^{1/2} \).

Here \( \text{tr} XX^* \) is the trace of the operator \( XX^* \) belonging to a two-sided ideal \( \mathfrak{S}_1(\mathcal{H}) \) of nuclear operators in \( \text{End} \mathcal{H} \), with the \( \| X \|_1 = \text{tr} XX^* = \sum_{n \in \mathbb{Z}} s_n \), where \( (s_n) \) is the sequence of s-numbers of an operator \( X \) (see [5]).

**Definition 4.** [14,15] The decomposition

\[
\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} U \mathcal{H}_k
\]

of the Hilbert space \( \mathcal{H} \), where \( U \) is an invertible operator in \( \text{End} \mathcal{H} \) and \( \mathcal{H} \) is the orthogonal direct sum (4) of subspaces \( \mathcal{H}_k, k \in \mathbb{Z} \), will be said to be quasiorthogonal or \( U \)-orthogonal. The quasiorthogonal decomposition of the space \( \mathcal{H} \) is called also a Riesz decomposition.

If the operator \( U \) can be represented in the form \( U = I + W \), where \( W \in \mathfrak{S}_2(\mathcal{H}) \), then the quasiorthogonal decomposition of the space \( \mathcal{H} \) is called a Bari decomposition. A linear closed operator \( A : D(A) \subset \mathcal{H} \to \mathcal{H} \) will be called the quasiorthogonal (\( U \)-orthogonal) direct sum of bounded operators \( \mathcal{A}_k, k \in \mathbb{Z} \), with respect to the quasiorthogonal decomposition (6) of the space \( \mathcal{H} \) if \( A_k = UA_k U^{-1}, k \in \mathbb{Z} \), for some invertible operator \( U \in \text{End} \mathcal{H} \). In this case, we write \( A = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k \).

Further, according to [5, Ch. 5], the system of subspaces \( U \mathcal{H}_k, k \in \mathbb{Z} \), is a basis of subspaces equivalent to an orthogonal one, or a rectifiable basis [32].

**Definition 5.** A nontrivial closed subspace \( M \subset \mathcal{H} \) is called bi-invariant if it is invariant for the operator \( A : D(A) \subset \mathcal{H} \to \mathcal{H} \) with its complements.

The following lemma takes place.

**Lemma 1.** Let an operator \( A : D(A) \subset \mathcal{H} \to \mathcal{H} \) commutes with a projection \( P \). Then \( \text{Ker} P \) and \( \text{Im} P \) is bi-invariant subspaces for \( A \).

We go back to the Definition 5. Let the operator \( A_2 \) have the bi-invariant subspace \( M \subset \mathcal{H} \). Then the operator \( A_1 \) have the bi-invariant subspace \( UM \). This subspace is called the Riesz bi-invariant subspace. If the transformation operator \( U \) (see Definition 2) can be represented in the form \( U = I + W, W \in \mathfrak{S}_2(\mathcal{H}) \), then the bi-invariant subspace \( UM \) is called the Bari bi-invariant subspace.

3. Results and discussion

In this section \( \mathcal{H} = L_2[0, \omega] \).

Let us describe the spectral properties of the free operator \( L_0 : D(L) \subset \mathcal{H} \to \mathcal{H} \). The spectrum \( \sigma(L_0) \) of the free operator \( L_0 \) can be represented as

\[
\sigma(L_0) = \bigcup_{n \in \mathbb{Z}} \sigma_n = \bigcup_{n \in \mathbb{Z}} \{ \lambda_n \} = \bigcup_{n \in \mathbb{Z}} \{ i2\pi n/\omega \},
\]

where \( \lambda_n, n \in \mathbb{Z} \), are isolated simple eigenvalues. The functions \( e_n(s) = e^{i2\pi ns/\omega}, s \in [0, \omega], n \in \mathbb{Z} \), are corresponding eigenvectors. The spectral Riesz projections \( P_n = P(\{ \lambda_n \}, L_0), n \in \mathbb{Z} \), are defined by

\[
(P_n y)(s) = \frac{1}{\omega} \left( \int_0^\omega y(\tau) e^{-i2\pi n \tau/\omega} d\tau \right) e^{i2\pi ns/\omega} = \hat{y}(n) e^{i2\pi ns/\omega},
\]
for $x \in \mathcal{H}$, $s \in [0, \omega]$. Here $\hat{y}(n), n \in \mathbb{Z}$, are the Fourier coefficients of a function $y \in \mathcal{H}$:

$$\hat{y}(n) = \frac{1}{\omega} \int_0^\omega y(\tau) e^{-i2\pi n \tau / \omega} d\tau.$$ 

We denote $P_{(k)} = \sum_{|i| \leq k} P_i$, $k \in \mathbb{Z}_+$. Let $I_n$, $n \in \mathbb{Z}$, are the identity operators in the one dimensional space $\mathcal{H}_n = \text{Im} P_n$, $n \in \mathbb{Z}$, and $I_{(k)}$ is the identity operator in the subspace $\mathcal{H}_{(k)} = \text{Im} P_{(k)}$, $k \in \mathbb{Z}_+$.

The unperturbed operator $L_0$ is the orthogonal direct sum of the operators $(L_0)_n = L_0|_{\mathcal{H}_n} = \frac{i2\pi n}{\omega} I_n = \lambda_n I_n$. All the operators $(L_0)_n$ have rank one. The free operator $L_0$ is also the orthogonal direct sum of the operators $(L_0)_{(k)} = L_0|_{\mathcal{H}_{(k)}}, k \in \mathbb{Z}$, and $(L_0)_n = L_0|_{\mathcal{H}_n}, |n| > k$. The operator $(L_0)_{(n)}$ has rank $2k + 1$. We have

$$L_0 = \bigoplus_{n \in \mathbb{Z}} (L_0)_n = \bigoplus_{n \in \mathbb{Z}} \frac{i2\pi n}{\omega} I_n = (L_0)_{(k)} \oplus \left( \bigoplus_{n > k} (L_0)_n \right)$$

$$= (L_0)_{(k)} \oplus \left( \bigoplus_{|n| > k} \frac{i2\pi n}{\omega} I_n \right), \quad k \in \mathbb{Z}_+, \quad n \in \mathbb{Z},$$

with respect of the representations of the space $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n, \quad \mathcal{H} = \mathcal{H}_{(k)} \oplus \left( \bigoplus_{n > k} \mathcal{H}_n \right), \quad k \in \mathbb{Z}_+.$$ 

Note that $(L_0)_{(k)} = \bigoplus_{|j| \leq k} (L_0)_j = \bigoplus_{|j| \leq k} \frac{i2\pi j}{\omega} I_j$ with respect to the orthogonal expansion $\mathcal{H}_{(k)} = \bigoplus_{|j| \leq k} \mathcal{H}_j$.

The perturbation $B$ belongs to the ideal $S_2(\mathcal{H})$,

$$(B y)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} y(x) dx$$

and

$$\|B\|_2^2 \leq \frac{1}{\Gamma^2(\alpha)} \omega^{2(\alpha-1)} (2\alpha - 1) 2\alpha.$$ 

Let $b_j = (B e_j, e_j) = \frac{1}{\omega^2 |\alpha|} \int_0^\omega \int_0^\omega (s-x)^{\alpha-1} e^{i2\pi j(x-s)/\omega} dx ds$.

The main result of this paper is that the perturbed operator $L$ is similar to the operator $L_0 - B_0$, which is the orthogonal direct sum of the finite rank operators.

Let us apply a similarity transformation from [27] to the original operator (3). The perturbed operator $L$ satisfied for all condition of [27, Theorem 4.5]. We apply this theorem and imply the main result of the present paper.

**Theorem 1.** There exists a number $k \in \mathbb{Z}_+$, such that the operator $L$ is similar to the operator $L_0 - B_0$, where the operator $B_0$ belongs to the ideal $S_2(\mathcal{H})$ and the subsets $\mathcal{H}_{(k)} = \text{Im} P_{(k)}$, $\mathcal{H}_i = \text{Im} P_i$, $k \in \mathbb{Z}_+,$ $|i| > k$, are invariant with respect to the operators $L_0, B_0, L_0 - B_0$. One has

$$L(I + W) = (I + W)(L_0 - B_0)$$

and the operator $L$ is the $(I + W)$-orthogonal sum. We have

$$L = (I + W) \left( L_0 - B_0 \right)_{(k)} \oplus \left( \bigoplus_{|j| > k} (L_0 - B_0)_j \right) (I + W)^{-1},$$

and
(L_0 - B_0)(k) = (L_0 - B_0)|\mathcal{H}(k)|, (L_0 - B_0)_j = (L_0 - B_0)|\mathcal{H}_j = (i2\pi j/\omega + b_0)I_j. The operator \(I + W\) is bounded and invertible, \(W \in G_2(\mathcal{H})\) and the \((I + W)\)-orthogonal decomposition \(\mathcal{H} = (I + W)\mathcal{H}(k) \oplus (\oplus_{|j| > k}(I + W)\mathcal{H}_j)\) is the Bari decomposition.

Note that the operator \(B_0 \in G_2(\mathcal{H})\) has a block-diagonal matrix with respect to the system (5) of spectral projections of the unperturbed operator \(L_0\). We don’t need the preliminary similar transform [27, § 4.2] of the similar operators method. On our case the perturbation \(B\) belongs to \(G_2(\mathcal{H})\).

The next theorem is formulated under the assumptions of Theorems 1 and 2 in terms of notations introduced in Theorem 2 (see also [27, § 4.2]).

**Theorem 2.** The spectrum \(\sigma(L)\) of the operator \(L\) can be represented in the form

\[
\sigma(L) = \tilde{\sigma}(k) \cup \bigcup_{|j| > k} \tilde{\sigma}_j,
\]

where the set \(\tilde{\sigma}(k)\) contains at most \(2k + 1\) eigenvalues, the sets \(\tilde{\sigma}_j = \{\tilde{\lambda}_j\}, |j| > k\), are singletons and the representation

\[
\tilde{\lambda}_j = i\frac{2\pi j}{\omega} - b_j + \beta_j, \quad |j| > k,
\]

holds with the sequence \((\beta_j, |j| > k)\), such that \(\sum_{|j| > k} |\beta_j| < \infty\). The corresponding eigenvectors \(\tilde{e}_j, j \in \mathbb{Z}\), \(\tilde{e}_j = (I + W)e_j, |j| > k\), of the operator \(L\) form the Bari basis in \(\mathcal{H}\) and

\[
\sum_{|j| > k} \|\tilde{e}_j - e_j\|_2^2 < \infty.
\]

The next theorem is formulated under the assumptions of Theorems 1 and 2 in terms of notations introduced in Theorem 2 (see also [27]).

**Theorem 3.** The limiting relation

\[
\lim_{n \to \infty} \left\| \sum_{|j| \leq n} P_j - \tilde{P}_k \right\|_2 = 0
\]

holds, where \(\tilde{P}_j = P(\tilde{\sigma}(k), L), \tilde{P}_j = P(\{\tilde{\lambda}_j\}, L), j \in \mathbb{Z}\).

Let us construct the operator group generated by \(L\) (see also [15, § 7], [14, § 6]).

**Theorem 4.** The operator \(L\) is the generator a strongly continuous operator group \(\tilde{T} : \mathbb{R} \to \text{End} \mathcal{H}\). The group \(\tilde{T} : \mathbb{R} \to \text{End} \mathcal{H}\) is similar to the group \(T : \mathbb{R} \to \text{End} \mathcal{H}\) that admits the orthogonal decomposition

\[
T(t) = e^{i(L_0 - B_0)(k)t} \oplus \left( \bigoplus_{|j| > k} e^{i\tilde{\lambda}_jt}I_j \right), \quad t \in \mathbb{R}.
\]

The group \(\tilde{T}(t), t \in \mathbb{R}\), admits a \((I + W)\)-orthogonal decomposition with respect to the Bari decomposition \(\mathcal{H} = (I + W)\mathcal{H}(k) \oplus (\oplus_{|j| > k} (I + W)\mathcal{H}_j)\) of the space \(\mathcal{H} = L_2\).

The analogous theorem for the difference operator with growing potential was proved in [33].
Theorem 5. The group $T : \mathbb{R} \to \text{End} \mathcal{H}$ has the representation

$$T(t) = e^{i(L_0 - B_0)t}P + \sum_{|j| > k} e^{i\lambda_j t}P_j.$$  

Let us pass to bi-invariant subspaces.

**Theorem 6.** The subspaces $\mathcal{H}(k) = (I + W)\mathcal{H}(k)$, $\tilde{\mathcal{H}}_j = (I + W)\mathcal{H}_j$ are the bi-invariant Bari subspaces for the perturbed operator $L$.

**Definition 6.** A classical solution to problem (1) if there exists a sequence of functions $\varphi_n \in W_2^1$, $n \geq 1$, such that $\lim_{n \to \infty} \varphi_n = \varphi$ in $L_2$ and $u$ is a uniform limit on compact subsets $J \times [0, \omega]$ of a sequence of classical solutions $(u_n)$, $n \geq 1$, of problem (1) with $u_n(0, s) = \varphi_n(s)$, $s \in [0, \omega]$.

**Theorem 7.** Every classical solution $u \in C(J, L_2)$ of (1) is given by

$$u(t, s) = (\tilde{T}(t)\varphi)(s), \quad s \in [0, \omega], \quad t \in J,$$

where $\varphi \in W_2^1$ and $\varphi(0) = \varphi(\omega)$. Every mild solution is also given by (7) with $\varphi \in L_2$.

**Remark.** In the theory of operator simigroups, a mild solution of problem (1) is defined using the group $T : \mathbb{R} \to \text{End} H$ without using Definition 7 namely, a mild solution is side to be a function of the from $u(t, s) = (T(t)\varphi)(s), s \in [0, \omega], t \in \mathbb{R}, \varphi \in H$. Therefore, a theorem that states that this operator is the generator of a strongly continous group of operator $\tilde{T} : \mathbb{R} \to \text{End} H$ is a theorem on the existence of a mild solution of problem (1).

The existens of the group of operators $\tilde{T}$ makes it possible to correctly define of operator $d/dt - L$ in the Banach space $C_b(\mathbb{R}, L) \subset C(\mathbb{R}, L)$ of continuous and $\mathbb{R}$- bounded functions and in other functional spaces. This enables the use of results obtainet in the works [20,33–37].

4. Conclusion

We consider the perturbed differencial operator first order $L$ (3) which connected with the mixed problem (1). The main method for this exploration was the method of similar operators, which modification from [27] allowed one to investigate the various classes of perturbed differential operators first order. This method allowed us to reduce the operator to one with a block-diagonal matrix. The version of the similar operator method from [27] can be applied for many other operators. In this article the asymptotic estimates of eigenvalues, eigenvectors and spectral projections for the operator $L$ (3) were obtained.

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