Convergent dynamics of optimal nonlinear damping control

Michael Ruderman

University of Agder, 4604-Norway

Abstract

Following Demidovich’s concept and definition of convergent systems, we analyze the optimal nonlinear damping control, recently proposed [1] for the second-order systems. Targeting the problem of output regulation, correspondingly tracking of $C^1$-trajectories, it is shown that all solutions of the control system are globally uniformly asymptotically stable. The existence of the unique limit solution in the origin of the control error and its time derivative coordinates are shown in the sense of Demidovich’s convergent dynamics. Explanative numerical examples are also provided along with analysis.

Keywords: Convergent dynamics, optimal nonlinear damping, tracking control, convergence analysis, stability analysis

1. Introduction

Unlike the linear feedback systems, the nonlinear controllers and plants require, mostly, some non-universal methods and tools for analysis and design. Depending on the complexity of underlying nonlinear dynamics, the local or global approach, and (not least) the control specification, e.g. whether a set-point or output regulation [2] problem is of interest, the stability and convergence analysis can require a more specific approach than that based on the standard Lyapunov stability theory [3, 4]. For analyzing the behavior of nonlinear control systems subject to external inputs, such as reference or disturbance signals, one needs to prove the existence and global asymptotic stability of a solution, along which the output regulation is ensured. The so-called incremental stability [5] and contraction analysis [6], or more generally contraction theory, see e.g. [7], provide the means to show that all system trajectories converge uniformly to a unique solution for which the output regulation error is zero.

Being motivated by the Demidovich’s [8] concept and definition of the convergent systems, reviewed in [9], this note analyzes and develops the output regulation properties of the optimal nonlinear damping control [1].

2. Preliminaries

2.1. Convergent dynamics

In this section, we briefly recall the main definition and properties of a convergent system dynamics, according to Demidovich [8], while we will closely follow the developments and notations provided in [1].

For a large class of $n$-dimensional nonlinear systems

$$\dot{x} = f(x, t),$$

where the state vector $x \in \mathbb{R}^n$, with $2 \leq n < \infty$, is continuous in time $t$, and $f(\cdot)$ is the vector field which is differentiable in $x$, the following notion of convergent systems can be given according to [8].

**Definition 1.** The system (1) is said to be convergent if for all initial conditions $t_0 \in \mathbb{R}$, $\dot{x}_0 \in \mathbb{R}^n$ there exists a solution $\bar{x}(t) = x(t, t_0, \dot{x}_0)$ which satisfies:

(i) $\bar{x}(t)$ is well-defined and bounded for all $t \in (-\infty, \infty)$;
(ii) $\bar{x}(t)$ is globally asymptotically stable.

Such solution $\bar{x}(t)$ is called a limit solution, to which all other solutions of the system (1) converge as $t \to \infty$. In other words, all solutions of a convergent system 'forget' their initial conditions after some transient time, which is depending on the exogenous values like reference or disturbance signals, and converge asymptotically to $\bar{x}(t)$.

**Remark 1.** If a globally asymptotically stable limit solution exists, it may be non-unique. Yet if $\bar{x}(t)$ is the single solution defined and bounded for all $t \in (-\infty, \infty)$, then the system (1) is said to be uniformly convergent.

The uniform convergence requires the system (1) to have an unique limit solution $\bar{x}(t)$, like in case of linear systems. Otherwise, a non-uniformly convergent system might have also another globally asymptotically stable solutions $\bar{x}(t)$, bounded for all $t \in (-\infty, \infty)$, so that for any such pair of solutions it is valid $\|\bar{x}(t) - \bar{x}(t)\| \to 0$ as $t \to \infty$.

**Remark 2.** The system (1) is, moreover, exponentially convergent if it is uniformly convergent and the limit solution $\bar{x}(t)$ is globally exponentially stable.

For further details on uniform, asymptotic, and exponential properties of stability of the system solutions we refer to seminal literature, e.g. [3, 4]. The existence and
uniqueness of a limit solution of system (1) has an essential application to the output regulation problems [2,10]. Here, for a given reference signal of the closed-loop system, one can seek for demonstrating the control system (1) is convergent, i.e. has an asymptotically stable limit solution along which the regulated output control error is zero. The property of a system to be convergent follows from the sufficient condition, given by Demidovich [5], which is formulated in the following theorem [3]:

**Theorem 1.** Consider the system (1). Suppose, for some positive definite matrix \( P = P^T > 0 \) the matrix

\[
J(x,t) := \frac{1}{2} \left( P \frac{\partial f}{\partial x}(x,t) + \left[ \frac{\partial f}{\partial x}(x,t) \right]^T P \right)
\]

is negative definite uniformly in \((x,t) \in \mathbb{R}^n \times \mathbb{R}\) and \(|f(0,t)| \leq \text{const} < +\infty \) for all \( t \in \mathbb{R} \). Then the system (1) is convergent.

The proof of the Theorem 1 can be found in [9]. Note that a particular case \( f(0,t) \equiv 0 \) of the Theorem 1 implies the well celebrated Krasovskii stability theorem [11]. The uniform negative definiteness of (2) implies a vanishing difference between any two solutions \( x_i(t) \) and \( x_{ii}(t) \) of the system (1), while for an exponentially convergent system (cf. Remark 2) that means

\[
|x_i(t) - x_{ii}(t)| < \alpha \exp(-\beta(t - t_0)) |x_i(t_0) - x_{ii}(t_0)|
\]

for all \( t > t_0 \), where \( \alpha, \beta > 0 \) have the same values for all pairs of the solution \([x_i(t), x_{ii}(t)]\).

2.2. Optimal nonlinear damping control

In the following, we will summarize the optimal nonlinear damping control, insofar as necessary for the main results presented in Section 3 while for that recently introduced control and its properties we refer to [1].

![Figure 1: Phase portrait of the control system (4), (5).](image)

The second-order closed-loop control system with an optimal nonlinear damping is written as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -kx_1 - x_2^3|x_1|^{-1}\text{sign}(x_2),
\end{align*}
\]

where \( k > 0 \) is the single arbitrary design parameter. The system (4), (5), where \( x_1 \) is the output of interest, is globally asymptotically stable and converges to the unique equilibrium in the origin: (i) along the attractor

\[ x_2 + \sqrt{k}x_1 = 0 \]

of trajectories in vicinity of the origin, and (ii) without crossing the \( x_2 \)-axis, see Figure 1. Note that the (ii)-nd property prevents singularity (owing to \( x_1 = 0 \)) in the solutions of (4), (5). and that for all initial conditions \( x_1(t_0) \neq 0 \land x_2(t_0) \in \mathbb{R} \) and all trajectories outside the origin i.e. \( \|x_1, x_2\|(t) \neq 0 \). The control system (4), (5) assumes an unperturbed system dynamics, so that the robustness against external perturbations is the subject of future research. Other, here relevant remarks follow.

**Remark 3.** The output convergence of the control system (4), (5) is quadratic on the logarithmic scale \( \log |x_1| \). Hence, the control constitutes a fast alternative to the standard proportional-derivative controller which, with the same proportional gain factor \( k \), converges only linearly on the logarithmic scale \( \log |x_1| \), cf. [1], Fig. 5.

It is also worth noting that the nonlinear damping law is completely independent of the proportional feedback gain \( k \). The latter scales solely the state trajectories in the \((t, x_2)\)-coordinates, see examples depicted in Figure 2.

![Figure 2: State trajectories \( x_1(t) \) in (a) and \( x_2(t) \) in (b) for the varying values of the control gain \( k = [10, 100, 1000] \) of (4), (5).](image)

**Remark 4.** The closed-loop control system (4), (5) allows, in addition, for a bounded control action \( |\dot{x}_2| < S \), with \( S = \text{const} > 0 \) to be the controller saturation (see [1]), which is affecting yet neither the stability nor convergence performance of the state trajectories.
Although a saturated control of \([4, 5]\) enables bypassing singularity in the solutions of a set-point problem, the output regulation problem requires us to allow for \(x_1 = 0 \land x_2 \neq 0\) at all times \(t_0 \leq t \to \infty\). For ensuring non-singularity of the solutions in the entire state-space \((x_1, x_2) \in \mathbb{R}^2\), we will next make a necessary regularization of the nonlinear damping term in \([5]\). With keeping this in mind, we are now in the position to formulate the main results of this work.

3. Main results

For output tracking of a reference trajectory \(r(t) \in \mathcal{C}^1\), we introduce the error state \(e_1 = x_1 - r\). Its time derivative is \(e_2 = x_2 - \dot{r}\), respectively. Note that for output tracking of \(\mathcal{C}^1\)-trajectories, one can assume \(\dot{r}(t) = 0\) for \(t > \tau\), while \(t \leq \tau\) characterizes some transient phase where \(\dot{r} \neq \text{const}\). In the sense of a motion control, for instance, all \(t \leq \tau\) will correspond to the transient acceleration or deceleration phases of a reference trajectory. If a reference trajectory \(r(t)\) contains multiple, but finite in time, transient phases with \(\dot{r}(t) \neq 0\), these will act as temporary perturbations upon which the convergent dynamics of the control error, i.e. \(\|e_1, e_2\| \to 0\), must be guaranteed.

With the introduced above states of the control error and the steady-state reference (i.e. \(\dot{r} = 0\)), the closed-loop control system \([4, 5]\) can be rewritten as

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= -\mu e_1 - \frac{|e_2| e_2}{|e_1| + \mu}.
\end{align*}
\]

Note that the newly introduced regularization term \(0 < \mu \ll k\) does not act as an additional design parameter, yet it prevents singularity in the solutions of the system \([4, 5]\), cf. Section \[3\]. Evaluating the Jacobian of \(f(x, t)\), with \(x = [e_1, e_2]^T\) cf. \([4, 5]\) and \([4]\), one obtains

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
0 & 1 \\
-k|e_1|\text{sign}(e_1)/(|e_1| + \mu)^2 & -2|e_2|/(|e_1| + \mu)
\end{bmatrix}.
\]

Then, suggesting the positive definite matrix

\[
P = \frac{1}{2} \begin{bmatrix}
k & 0 \\
0 & 1
\end{bmatrix},
\]

one can show that the matrix \(J(x, t)\), which is the solution of \([2]\), is negative definite and, correspondingly, the Theorem \[1\] holds. For proving it, we substitute \([9]\) and \([10]\) into \([2]\) and evaluate the matrix definiteness as

\[
x^TJ(x, t)x = -\frac{3}{4} \frac{|e_2| e_2^2 (|e_1| + 2\mu)}{(e_1 + \mu \text{sign}(e_1))^2} \leq 0 \ \forall \ x \neq 0. \tag{11}
\]

Note that the obtained inequality \([11]\) proves only the negative semi-definiteness of \(J(x, t)\), since it is visible that \(x^T J(x, t) x = 0\) for \(e_2 = 0 \land e_1 \neq 0\). Yet it appears possible to show that \([e_1, e_2] = 0\) is the unique limit solution by evaluating the \(e_2\) dynamics at \(e_2 = 0\). Substituting \(e_2 = 0\) into \([3]\) results in \(\dot{e}_2 = -\mu e_1\). It implies that \(e_1 \neq 0, e_2 = 0(t)\) cannot be a limit (correspondingly steady-state) solution, since any trajectory will be repulsed away from \(e_2 = 0\) as long as \(e_1 \neq 0\). Hence, the closed-loop control system \([4, 5]\) reveals as uniformly convergent. Respectively, the origin of the control error \([e_1, e_2](t) = 0 \equiv \bar{x}\) is the unique limit solution, for all times \(t_0 < \tau < t\) and independent of the initial conditions \(t_0\) and \([e_1, e_2](t_0)\).

Remark 5. When assuming a quadratic Lyapunov function candidate

\[
V(x) = x^TPx = \frac{1}{2} k e_1^2 + \frac{1}{2} e_2^2,
\]

which represents the entire energy level (i.e. potential energy plus kinetic energy) of the system \([7, 8]\), its time derivative results in

\[
\frac{d}{dt} V(x) = -\frac{|e_2| e_2^2}{|e_1| + \mu}. \tag{13}
\]

Thus, the rate at which the control system \([7, 8]\) reduces its energy is cubic in the error rate, i.e. \(\sim |e_2|^3\), and hyperbolic in the error size, i.e. \(\sim |e_1|^{-1}\), cf. Figure 3

![Figure 3: Energy reduction rate |V| of the system \(7, 8\): depending on \(e_1\) in (a), depending on \(e_2\) in (b), and as the overall function, according to \(13\), in (c).](image)

It becomes apparent, cf. Figure 3 (a), that the regularization factor \(\mu\) prevents an infinite energy rate and, thus, ensure a finite control action as \(|e_1| \to 0\). At the same time, a hyperbolic energy rate allows to accelerate the convergence as \(|e_1| \to 0\). The cubic dependence of energy rate
on the error rate enables the control to react faster to the error dynamics. It reveals as relevant in case of, for example, non-steady trajectory phases (i.e. \( \hat{r}(t) \neq 0 \)) or sudden external perturbations which can provoke high \(|e_2|\) values.

Numerical examples
Following numerical examples are provided for the implemented system (7), (8), assuming \( k = 100 \) and \( \mu = 0.0001 \) parameters and \( r(t) \in C^1 \) reference trajectories.

First, the output regulated trajectories are shown for the different initial values \( x_0 \equiv [x_1, x_2](t_0) \):

\[
x_0 = \{(0.5, 50), (0.1, 20), (1, 0), (1.5, -30), (0.3, -20)\}.
\]

The assigned reference trajectory is a linear slope \( r(t) = t \).

The output response \( x_1(t) \) under control is depicted in Figure 4 (a). The corresponding phase portrait of the error states, i.e. \((e_1, e_2)\), is depicted Figure 4 (b).

Next, we demonstrate the control performance of the output tracking when \( r(t) \) is only piecewise \( C^1 \) and contains the finite phases where \( \hat{r}(t) \neq \text{const} \). Furthermore, in order to emphasize a practical control applicability, both \( x_1(t) \) and \( x_2(t) \) signals, used for the feedback control in (8), are subject to a non-correlated bandlimited white-noise. The output response \( x_1(t) \) under control is depicted in Figure 5 (a) over the reference trajectory. The \( x_2(t) \) state, which corresponds to the relative velocity if (for example) a motion control is intended, is depicted in Figure 5 (b). For the sake of comparison, the case without regularization, i.e. \( \mu = 0 \), is also shown. Moreover, the \( x_2(t) \)-response of a critically damped proportional-derivative linear control is also demonstrated, for the sake of comparison. The assigned linear controller, with the same \( k = 100 \), features the \( e_2 = -100e_1 - 20e_2 \) error dynamics, correspondingly.

4. Conclusions
The Demidovich’s concept of convergent system dynamics was used for analyzing the convergence properties of the optimal nonlinear damping control, introduced in [1]. The existence of unique limit solution in the coordinate origin of the output tracking error and its time derivative was demonstrated. A regularization term was introduced, comparing to [1], which prevents singularity in the solutions when an output zero crossing occurs outside of origin. The provided analysis and results can be relevant for using the optimal nonlinear damping control as an alternative to a standard proportional-derivative controller. The optimal nonlinear damping control performs as significantly faster converging and is, moreover, robust against the measurement noise of the feedback states.

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