Influence of Long-range Disorder on Electron Motion in Two Dimensions

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Abstract

We consider a two-dimensional electron gas with long range disorder. Assuming that time reversal symmetry is broken either by an external magnetic field or, as in the case of a delta-correlated random magnetic field, by the disorder itself, we derive a supermatrix $\sigma$-model. As an intermediate step, we provide a microscopic derivation of the ballistic $\sigma$-model, and find that certain corrections to its usual form may become important. We then integrate out degrees of freedom corresponding to short length scales to derive a low-energy supermatrix $\sigma$-model. We find an extra term in the free energy that couples to the correlator of local currents. Use of a proper ultraviolet regularisation procedure that preserves gauge invariance indicates that the contribution of the extra term seems finally to become irrelevant. Within the scope of our analysis, we therefore do not find any deviation of the scaling behaviour of the delta-correlated random magnetic field model from that of the conventional unitary ensemble. We then generalize the discussion to include models of even longer-ranged disorder, plus short-range disorder. When the disorder is sufficiently long-ranged that the local currents become delta-correlated, a new term appears in the free energy that does give rise to logarithmic corrections to the conductivity. A renormalisation group analysis of the free energy yields a scaling form for the diffusion coefficient which contains both a positive correction, that represents classical superdiffusion, and a negative correction, which is the usual weak localization correction. The fact that both corrections are of the same order and opposite sign leads to the interesting possibility of a quantum phase transition at weak disorder in two dimensions, tuned by the relative strengths of the short and long range disorder.

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I. INTRODUCTION

Since the appearance of the scaling theory of disordered metals \cite{1-3}, the theory of localization in 2D systems has become well established. Of course, this phenomenon depends on symmetries. The addition of spin-orbital impurities leads to the violation of central symmetry and hence a situation when the resistivity vanishes at small frequencies. At the same time, the delocalizing effect of a magnetic field is apparent in a number of well-known phenomena, a natural example being the integer quantum Hall effect in 2D \cite{5,6} (for a review see e.g. Ref. \cite{7}). Another example is provided by the random flux model which describes, for example, electron hopping on a bipartite lattice structure with link disorder \cite{8,9}. In this model a tendency towards delocalization is displayed as the band center is approached due to the existence of a chiral symmetry at the band center \cite{8,10}.

A question of long-standing interest in this context has been the influence of a random, static magnetic field on a two-dimensional electron gas. This is an example of a disordered system with a broken time reversal invariance, where the effects of the magnetic field can become very complicated. Interest in this problem has been stimulated by its relevance to a variety of experimental situations. The solution of the problem might help to clarify the behavior of composite fermions for the fractional quantum Hall effect near half-filling \cite{11}. In this model, the interacting electron liquid is replaced by a Fermi gas of quasiparticles, each carrying two flux quanta of a fictitious magnetic field. While the Chern-Simons field exactly cancels the external field at half-filling, variations in the electron density due to screening of the impurity potential leads to fluctuations in the effective magnetic field around the zero value. A similar model applies also in the gauge-field description \cite{12} of doped Mott insulators, where slow fluctuations in the gauge field may be approximated by a static random field. Further experimental realizations have involved the introduction of a random magnet field (with non-zero mean) onto a high-mobility 2D semiconductor through an overlayer containing randomly pinned flux vortices \cite{14} or type-I superconducting grains \cite{13} or through a nearby permanent magnet \cite{15}.

A theoretical analysis of the random magnetic field (RMF) problem is made technically difficult due to the spatially long-ranged nature of the vector potential correlations, which result even for short-ranged magnetic field correlations. For example, a straightforward application of perturbation theory \cite{16,17} readily demonstrates the appearance of infrared divergencies in the Born approximation for the single-particle Green function. Consequently a wide variety of alternative techniques have been applied to this problem. A real-space path integral representation has been introduced by Altshuler and Ioffe \cite{10} and Altshuler et al. \cite{18}, while the eikonal \cite{19} and related paraxial \cite{20} approximations have also been employed. If the correlation length, $d$, of the random magnetic field is sufficiently large that $d \gg l, 1/k_F$, where $l$ is the single-particle mean free path and $k_F$ is the Fermi momentum, then a “classical” regime is reached in which the contribution of classical memory effects has been shown to be significant \cite{21,22}. If in addition the RMF is sufficiently strong that $d$ is greater than the average cyclotron radius, then a description of electron transport in terms of percolation between a network of ‘snake’-states at zero B-field contours becomes appropriate \cite{23,24}.

Numerical investigation of the RMF model (equivalent to the random flux model away from the band center) has continued without apparent consensus, mainly due to the great
difficulty of distinguishing between delocalization and localization of states with the very large localization length that is typical for two dimensions at large conductances. Conclusions for this model are divided between localization of all states [26–28], the existence of a critical region [29–34], and the localization of all states except at precisely zero energy [35,10]. It is also of interest that, even for the case of one dimension, recent analytical [36] and numerical [37] work supports the existence of a metal-insulator transition in the presence of sufficiently long-ranged disorder correlations.

In this paper, we study the 2D electron gas with long range disorder. We assume that the time reversal symmetry is broken. Our approach applies both to the RMF model and also to a model with a long range, potential disorder and a constant magnetic field. The latter model is very close to models with short-range disorder with broken time-reversal invariance (unitary ensemble) for which the localization of all states is well established. The leading order weak localization correction to the conductivity appears, for short range disorder, at two-loop order and is negative [3,39,40]. While this result has been also derived from conventional diagrammatics [38], an equivalent and more convenient procedure is provided by field theoretical methods based on a nonlinear $\sigma$-model. In addition, the use of the $\sigma$-models allows one to prove the existence of the renormalization group (RG) and go beyond perturbation theory.

Aronov et al. [41] have already provided a generalization of the standard field-theoretical approach [3,40] to the model of a delta-correlated RMF, in deriving an appropriate form of a supersymmetric $\sigma$-model. They found that the $\sigma$-model was identical to the one derived previously for short-range disordered systems with broken time-reversal invariance (the unitary ensemble) [40], and therefore concluded that all states are localized as for the unitary ensemble. In an earlier paper, Zhang and Arovas [42] had proposed the existence of an additional term in the $\sigma$-model originating from logarithmic interaction of topological density or, equivalently, current-carrying edge states around magnetic domains; this term was argued to lead to a Kosterlitz-Thouless transition from the localized to extended states with power-law correlations. Unfortunately the calculation of Ref. [42] relies on an incorrect relation between a correlator of Hall conductivities and the average longitudinal conductivity, and the additional term was not found in the analysis of Aronov et al. [41]. As concerns the models in a constant magnetic field with an arbitrary range of disorder correlations, a conventional $\sigma$-model, with a classical diffusion coefficient appropriate to the correlations, has been derived [44] using similar methods to the work of Aronov et al.

Even so, the question has so far remained open of whether the new term written by Zhang and Arovas [42] can exist in principle and, if so, what would be the possible consequences. We examine this possibility more carefully in this paper. First we focus on the case of a delta-correlated RMF model. For this model, we provide a derivation of the appropriate $\sigma$-model, that is somewhat more complete than the original work of Aronov et al. [41]. Our derivation demonstrates the appearance of certain technical subtleties that were not realised in the original work of Ref. [41]. This work is an extension of calculations presented previously in a short letter [43]. As a first step we derive a form of the free energy that has appeared previously under the name of the ballistic $\sigma$-model [51,52]. In fact our derivation (also presented in the Letter [43]) represents the first formally justified derivation of the ballistic $\sigma$-model, since the long-ranged nature of the disorder provides a small parameter (the small ratio of the single-particle and transport lifetimes) to allow a controlled separation
of modes and expansion of the free energy, in contrast to previous derivations \[51,52\]. Indeed this approach allows us to go further and derive further terms to the usual ballistic $\sigma$-model that become important for shorter-range disorder. We remark that our derivation of the ballistic $\sigma$-model is valid for long-range potential disorder as well as for vector potential disorder, with minor modifications.

Integration over the degrees of freedom in this model that correspond to length-scales shorter than the transport mean free path allows us to derive the final form of the free energy that is applicable for the description of low-energy transport. Along with the usual terms of the unitary $\sigma$-model, we find an extra term which couples to the current-current correlator. This term appears also for models of long-range potential disorder, as well as vector potential disorder, as long as time-reversal symmetry is completely broken. The new term is similar to that written by Zhang and Arovas, except for the appearance of an additional factor $\Pi(q)$ that represents the current-current correlator.

For evaluation of the quantity $\Pi(q)$, it is necessary to employ an ultraviolet regularization procedure which should, as shown recently by Gornyi \textit{et al.} \[46\], be formulated carefully in order to preserve gauge invariance. The use of this procedure then indicates that the factor $\Pi(q)$ should vanish as $q \to 0$, as confirmed by diagrammatic arguments. This implies that, within the scope of our analysis, the new term remains irrelevant for the delta-correlated RMF model in the calculation of corrections to the conductivity that are logarithmic in frequency, although it may still lead to higher order corrections known as memory effects (we remark that our earlier letter \[45\] employed a more naive regularisation procedure which led to incorrect conclusions).

As a result we do not find evidence for deviation of the scaling behaviour of the delta-correlated random magnetic field model from that of the conventional unitary ensemble. As such our conclusions coincide with those of the original work of Aronov \textit{et al.}, although for considerably more subtle reasons than were originally realised in that work (which remains a partial analysis since they neglect higher harmonics of the $Q$-matrix). Even so, certain questions remain, concerning the role of the massive modes of the field theory, which deserve further study before a definitive conclusion can be drawn for this problem.

We then generalize the argument to consider the possibility of longer-ranged disorder. We derive the free energy for a model of short-range disorder plus longer-ranged disorder, and showed how the long-range disorder couples to the free energy via a Wess-Zumino term. If the long-range disorder is sufficiently long-ranged that correlator of local currents remains finite in the limit of $q \to 0$, then a new term appears in the free energy which is responsible for new corrections to the conductivity that are logarithmic in frequency. For this model, the new term does takes the same form as the one originally written by Zhang and Arovas \[42\] (although the latter work does not contain a correct microscopic justification). As follows from the preceding discussion, for this term to exist the magnetic field correlations must be longer-ranged than delta-correlated: instead, the Fourier transform of the correlator $\langle B(\mathbf{r})B(\mathbf{r'})\rangle$ must diverge as $1/q^2$ as $q \to 0$, where $B$ is the magnetic field.

At the classical level, this free energy represents particle motion in a field of random velocities: similar classical models have already been considered \[48–50\], assuming the velocities to be delta-correlated. Indeed, Kravtsov \textit{et al.} \[49\] and Fisher \textit{et al.} \[50\] have demonstrated, via a renormalization group analysis, that particle propagation in this purely classical problem (the “diffusion-advection model”) obeys a “superdiffusion”. The electron gas with the
long-range disorder represents a quantum generalisation of this model, in which quantum coherence phenomena are also present and interplay with the above classical effects.

By performing a RG analysis of the free energy, we find that two competing contributions to the scaling form for the diffusion coefficient appear. The positive contribution corresponds to classical superdiffusion in the presence of long-ranged currents, while the negative contribution is the quantum mechanical, weak localization correction for the unitary ensemble. Since the classical and quantum contributions to the conductivity are of opposite sign and potentially of the same order, we have the interesting possibility of a quantum phase transition in two dimensions at weak disorder, tuned by the relative disorder strengths.

The plan for this paper is as follows. In section II, we introduce the model for the delta-correlated RMF. In section III, we derive the free-energy functional for this model in terms of a supermatrix $\sigma$-model. This involves deriving the ballistic $\sigma$-model microscopically for this problem. We then simplify the free energy by integrating out degrees of freedom associated with length-scales shorter than the transport mean free path. The derivation is applicable with minor changes to a model with a potential long range disorder and a constant magnetic field. In section IV, we present direct perturbation theory calculations of the conductivity and the current-current correlator. In section V, we review classical models of diffusion in a field of random velocities, and consider an electron gas in the presence of both short-range and longer-ranged disorder. We show how the local current couples to the free energy through a Wess-Zumino term, and how averaging over this term leads to a term similar to that written by Zhang and Arovas [42]. In section VI we subject the free energy to a renormalisation group analysis and derive the scaling form for the diffusion coefficient. This demonstrates the appearance of a positive contribution to the scaling form for the conductivity in competition with the usual negative, weak localization contribution. Section VII concludes with a summary and discussion. Some technical details are presented in the Appendix.

II. RANDOM MAGNETIC FIELD MODEL

As we have mentioned in the Introduction, the considerations below are applicable both to a RMF model, and to a model with long range potential impurities and a constant magnetic field (to break time-reversal symmetry). While the derivation of the $\sigma$-model is similar for the two types of model, to be specific we carry out calculations for the RMF model. We write the Hamiltonian as follows,

$$\mathcal{H}(\mathbf{r}) = \frac{1}{2m} \left( -i \nabla_{\mathbf{r}} - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 - \epsilon_F$$

where $e$ and $m$ are the electron charge and mass, $c$ is the velocity of light, $\epsilon_F$ is the Fermi energy and $\mathbf{A}$ is the vector potential. We focus below on the case where the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, is delta-correlated,

$$\langle B(\mathbf{r})B(\mathbf{r}') \rangle = 2 \left( \frac{mcv_F}{e} \right)^2 \gamma \delta(\mathbf{r} - \mathbf{r}'),$$

with a strength characterized by the dimensionless parameter $\gamma$ (and with $\langle B \rangle = 0$), although one may as easily consider a more general case with a finite range of the correlations. We choose the London gauge for the vector potential $\mathbf{A}$.
\[ \text{div} \mathbf{A} = 0, \quad A_n = 0 \quad (3) \]

where \( A_n \) is the component taken at, and perpendicular to, the surface of the sample. The sample may be either truly two-dimensional, with the thickness of a single atomic layer, or simply have a two-dimensional geometry. While we write explicit formulae for the former case, extension to thicker samples is trivial.

The correlations of the vector potential \( \mathbf{A} \) corresponding Eqs. (2) and (3) are long-ranged and we choose them in the form

\[
\langle A^i(r) A^j(r') \rangle = \frac{2m^2e^2}{\epsilon^2} V^{ij}(r - r'),
\]

\[
V^{ij}(\mathbf{q}) = v_F^2 \frac{1}{q^2 + \kappa^2 p_F^2} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right). \quad (4)
\]

The correlator \( V^{ij} \) shown here is transversal, that is,

\[
\sum_i \frac{\partial}{\partial r_i} V^{ij} = 0,
\]

which corresponds to the transversality of the vector potential \( \mathbf{A} \), Eq. (3). The parameter \( \kappa \ll 1 \) is a cutoff that renders finite the otherwise-infinite range of the disorder correlations. The \( \delta \)-correlated fluctuations of the magnetic field correspond to the limit \( \kappa \to 0 \). The cutoff \( \kappa \) appears [17,41] in perturbation theory for the infrared-divergent single-particle lifetime. For example, the simple Born approximation yields for the mean free time

\[
\tau_{BA}^{-1} = 4 \epsilon_F \gamma \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{\cos^2(\varphi/2)}{4 \sin^2(\varphi/2) + \kappa^2}
\]

\[
\simeq 2 \epsilon_F \gamma / \kappa. \quad (5)
\]

The self-consistent Born approximation (SCBA) displays a weaker divergency,

\[
\tau_{\text{SCBA}}^{-1} = \frac{8}{\pi} \epsilon_F \gamma \tau_{\text{SCBA}} \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_{-\infty}^{\infty} d\xi \sin^2 \varphi \times
\]

\[
\frac{1}{(4 \sin^2(\varphi/2) + \xi^2/\epsilon_F^2)(4 \sin^2(\varphi/2) + \xi^2/\epsilon_F^2 + \kappa^2)}, \quad (6a)
\]

\[
\tau_{\text{SCBA}}^{-1} \simeq 2 \epsilon_F (\gamma \ln(1/\kappa)/\pi)^{1/2}, \quad (6b)
\]

due to off-shell contributions in the collision integral, Eq. (6b). The inverse transport time \( \tau_{tr}^{-1} \) remains convergent however in the limit of \( \kappa \to 0 \),

\[
\tau_{tr}^{-1} = \epsilon_F \gamma \int_0^{2\pi} \frac{d\varphi}{2\pi} (1 - \cos(\varphi)) \cot^2(\varphi/2)
\]

\[
= \epsilon_F \gamma. \quad (7)
\]

Although we focus on the above choice of correlator \( V^{ij}(\mathbf{q}) \) in this paper, we emphasize again that our method remains valid for an arbitrary form of disorder correlations, with minor modifications for scalar rather than vector potential disorder.
III. FREE ENERGY FUNCTIONAL

In this section we derive the free energy functional for the 2D electron gas in the RMF. We employ the supersymmetry method which by now has been extensively developed [40] as a method for the exact evaluation of spectral and wave function properties of metals with delta-correlated disorder. In the conventional case the free energy functional takes the form of a nonlinear $\sigma$-model containing an $8 \times 8$, position-dependent $Q$-matrix. In order to generalize the method to allow for long-ranged disorder correlations, it becomes necessary to employ a $Q$-matrix that depends on two position variables, rather than one. Following a Fourier transformation and a harmonic expansion around the Fermi surface, the $Q$-matrix may be represented as depending on a position variable and an angular harmonic index.

The degrees of freedom contained in the non-zero harmonics of the $Q$-matrix are weakly massive for a long range disorder and we integrate them out rather than neglect them. Although the coupling of the non-zero harmonics to the zeroth harmonic is weak, integration over the nonzero harmonics requires some care to determine the possible renormalization of the effective functional for the zeroth harmonic. One consequence of this renormalization, due to the first harmonics only, amounts to the inclusion of two-particle vertex corrections to the single-particle lifetime, $\tau$, leading to its replacement in the free energy by the transport relaxation time, $\tau_{tr}$. This was the main conclusion of Ref. [44], where arbitrary range potential impurities were considered, and of Ref. [41], devoted to study of the RMF problem.

We find however a further difficulty associated with the integration over non-zero harmonics that was not examined in the previous works [44,41]. Namely, additional terms arise in the free energy whose evaluation requires a careful integration over higher non-zero harmonics, rather than only the first harmonic as considered in the approach of Refs. [44,41]. While the contribution of these terms seem finally to be irrelevant to computation of logarithmic corrections to the conductivity, as we discuss below, the proper demonstration of this fact is far from trivial and beyond the original analysis of Ref. [41].

In order to properly include the potential contribution of the higher harmonics, we derive as an intermediate step a free energy that is similar to the “ballistic $\sigma$-model” of Muzykantskii and Khmelnitskii [51]. The latter model itself represents a generalization of the diffusive $\sigma$-model to the case of ballistic disorder, where the typical energy scale of $Q$-fluctuations may be as large as the inverse scattering time. In fact we also derive certain additional terms to their form of the free energy that become relevant for shorter-range disorder.

Employing standard methods [40], we introduce a supersymmetric $\psi$-field that contains eight components, corresponding to fermion/boson, advanced/retarded and time-reversed sectors. An averaged product of Green’s functions may then be expressed in terms of a functional integral weighted by a Lagrangian that is quadratic in the $\psi$-field:

$$\langle G^A_{\epsilon - \omega/2}(r, 0) G^R_{\epsilon + \omega/2}(0, r) \rangle = -4 \int \psi_\alpha^1(r) \bar{\psi}_\alpha^1(0) \psi_\beta^2(0) \bar{\psi}_\beta^2(r) e^{-L} D\psi D\bar{\psi},$$

$$L = -i \int \bar{\psi}(r) \left( \mathcal{H}_0 + \frac{ie}{mc} A \nabla \tau_3 + \frac{e^2}{2mc^2} A^2 \right) \psi(r) dr,$$

where
\[ \mathcal{H}_0 \equiv -\frac{\nabla^2}{2m} - \epsilon - \epsilon_F + \frac{(\omega + i\delta) \Lambda}{2}, \]

and \( \tau_3 \) is the Pauli matrix in time-reversal space.

We now average over the vector potential, \( \mathbf{A} \). In doing so we neglect the term in \( A^2 \) in Eq. (8). This approximation is standard and has been used in previous analytic works on the RMF model [41,43]. Furthermore, for the case of long range potential disorder, for which our general method is valid, \( A^2 \)-like terms are absent anyway. As for the question of gauge invariance, note that our derivation has assumed already a choice of gauge through the form of the vector potential correlations (see section II). Nevertheless Appendix A contains a discussion of how the \( A^2 \) term may be handled more carefully and found to be negligible.

The term linear in \( \mathbf{A} \) induces after the averaging a term that is quartic in the \( \psi \)-fields and, in contrast to the case of short-range impurities, non-local in position:

\[
\mathcal{L}_{\text{int}} = \int \left( \bar{\psi}(r) \tau_3 \nabla^i r \psi(r) \right) V_{ij}(r - r') \left( \bar{\psi}(r') \tau_3 \nabla^j r' \psi(r') \right) d^2r d^2r'.
\]

Following an integration by parts, \( \mathcal{L} \) may be rewritten

\[
\mathcal{L}_{\text{int}} = \int \bar{\psi}^\alpha(r) \psi^\beta(r') \tau_{3,3}\bar{\psi}^\beta(r') \psi^\alpha(r) \tau_3 \alpha \alpha,
\]

\[
\bar{V}_{r,r'} \equiv -\frac{1}{2} \sum_{ij} V_{ij}(r - r') (\nabla^i r - \nabla^i r') (\nabla^j r - \nabla^j r')
\]

where \( k = \text{diag}(1, -1) \) in boson-fermion space. We now decouple \( \mathcal{L}_{\text{int}} \) via an \( 8 \times 8 \) matrix \( Q(r, r') \). While in the calculation for time reversal invariant systems [40], this step requires a careful identification of the slowly varying modes, in the case of the system with the broken time reversal invariance, all slow modes are easily identified as corresponding to pairs \( \psi^\alpha(r) \bar{\psi}^\beta(r') \). Following an integration over the \( \psi \)-fields, we find the Lagrangian

\[
\mathcal{L} = \int \left[ -\frac{1}{2} \text{Str} \ln \left( i\mathcal{H}_0(r) \delta(r - r') + \bar{V}_{r,r'} Q(r, r') \right) + \frac{1}{4} \text{Str} \left( Q(r, r') \bar{V}_{r,r'} Q(r', r) \right) \right] dr dr'.
\]

Here \( \bar{Q} = Q_n + iQ_\perp \), whereas \( Q = Q_n + Q_\perp \) and \( Q_n (Q_\perp) \) commutes (anticommates) with \( \tau_3 \). The \( Q \)-matrix satisfies the standard symmetries \( Q(r, r') = \bar{Q}(r', r) = KQ^T(r', r)K \), where \( \bar{Q} \equiv CQ^T C^T \) and \( C \) and \( K \) defined as in Ref. [40].

The free energy, Eq. (10), in principle provides an exact description of the system but it is not useful in this form as it allows for all possible energy scales for \( Q \)-fluctuations. As a first step in its simplification we search as usual for the saddle-point value of \( Q \) and perform a gradient expansion around this value to identify the free energy for low energy fluctuations.
A. Saddle-point

To search for the saddle-point to Eq. (10), we take \( Q_\perp = 0 \) and \( Q \) to depend on \( r - r' \) only. This means that the cooperon “degrees of freedom” are suppressed and that the supermatrix \( Q \) at the saddle point is homogeneous in space. After Fourier transforming with respect to \( r - r' \), the saddle-point equation reads \( Q_p = g_p \), where

\[
\left( \epsilon - \xi_p - \frac{\omega}{2} \Lambda - 2i \int \frac{d^2 p_1}{(2\pi)^2} V_p^{ij} p_1^i p_1^j Q_{p_1} \right) g_p = -i, \tag{11}
\]

and \( \xi_p = \frac{p^2}{2m} - \epsilon_F \).

The saddle point of the Lagrangian, Eq. (10), is continuously degenerate and the solution of Eq. (11) can be written generally in the form

\[
Q_p = V_p \Lambda_p \tilde{V}_p, \quad V_p \tilde{V}_p = 1 \tag{12}
\]

where \( \Lambda_p \) is a diagonal matrix depending on \( p \) in a complicated way. The analysis simplifies however in the limits of \( \kappa \gg (\epsilon_F \tau)^{-1} \) (short-range disorder) and \( \kappa \ll (\epsilon_F \tau)^{-1} \) (long-range disorder), the latter case being relevant here, where \( \tau \) is the mean free time to be determined from the solution. For either case we may employ the ansatz

\[
\Lambda_p = \frac{i}{\xi_p - \epsilon + i\Lambda/(2\tau)}, \tag{13}
\]

where \( \Lambda = \text{diag} (1, 1, 1, 1, -1, -1, -1, -1) \), which leads to the condition

\[
\frac{\Lambda}{2\tau} = 2 \int \sum_{ij} V_p^{ij} p_1^i p_1^j \Lambda_p \frac{d^2 p_1}{(2\pi)^2}. \tag{14}
\]

For short-range disorder, \( \kappa \gg (\epsilon_F \tau)^{-1} \), the integral in Eq. (14) is as usual dominated by the on-shell value of \( p - p_1 \) and \( \tau \) takes its simple BA value, as given by Eq. (5), which is equivalent to the SCBA value in this limit.

In the opposite limit of long-ranged disorder, \( \kappa \ll (\epsilon_F \tau)^{-1} \), which is relevant to the RMF model, the off-shell dependence of \( V_p^{ij} \) contributes significantly to the integral. In this limit, substituting Eq. (13) into Eq. (14) we come to Eq. (6a), thus obtaining the SCBA value of \( \tau \), given by Eq. (6b). This value of \( \tau \) is less divergent than the simple BA value, Eq. (5). In the following we will denote this SCBA value as simply \( \tau \). At the same time we ensure that we stay within the weak disorder limit, \( \epsilon_F \tau \gg 1 \) (equivalently, \( \gamma \ll 1 \)). This condition restricts the space of relevant fluctuations in the \( Q \)-matrix, as we see in the next section.

B. Fluctuations

To proceed we expand the free energy in fluctuations of \( Q \) about its saddle-point value. In the case of short range impurities, fluctuations of the eigenvalues of the supermatrix \( Q \) are massive with the characteristic energy \( 1/\tau \) and may be neglected. In this case, what
remains is to consider massless (in the limit $\omega \to 0$) modes with energies smaller than $1/\tau$, which leads to the conventional supermatrix $\sigma$-model.

In the models with long range disorder considered now, however, an additional energy scale appears due the difference between the mean free time $\tau$ and $\tau_{tr}$. In this case fluctuations separate into three types:

(a) hard massive, with a mass of $\tau^{-1}$,

(b) soft massive, with a mass of $\tau_{tr}^{-1}$, and

(c) ‘massless’, that is, with a mass vanishing in the limit $\omega \to 0$.

Type (a) fluctuations are associated with the fluctuations of the eigenvalues of $Q$ and the Cooperons. The weak disorder condition $\epsilon_F \tau \gg 1$ ensures that these fluctuations are irrelevant and hence we may restrict attention to $Q$-matrices of the form $Q^2 = 1$ and $[Q, \tau_3] = 0$. Alternatively, one can use the form of the supermatrices $Q$ given by Eq. (12) with $[V, \tau_3] = 0$. The effective functional for the remaining (b) and (c) modes will then contain energy scales only much smaller than $\tau^{-1}$.

The above separation of scales may also be expressed in terms of length scales: at distances exceeding $l_{tr} = v_F \tau_{tr}$, density relaxation in the classical limit is described by a diffusion equation, whereas for distances between $l = v_F \tau$ and $l_{tr}$ one should use a Boltzmann equation. The possibility to generalize the above classical descriptions to include quantum fluctuation phenomena is afforded by the $\sigma$-model technique.

To derive a free energy functional describing both soft massive and massless modes we proceed as follows. First we introduce the following Fourier transformation for $Q(r, r')$ with respect to $r - r'$:

$$Q(r, r') = \int \frac{d^2p}{(2\pi)^2} Q_p(R) \exp(i\tau_3 p (r - r')),$$

where $R = (r + r')/2$. The appearance of the $\tau_3$ factor in Eq. (15) ensures that $Q_p(R)$ satisfies the symmetries

$$Q_p(R) = Q_p(R) = K Q_p(R) K.$$

Next we introduce the following parametrization for $Q_p(R)$:

$$Q_p(R) = U(R) Q_p^{(0)} \bar{U}(R),$$

$$Q_p^{(0)} = V_n(R) \Lambda_p \bar{V}_n(R),$$

where $n = p/|p|$. In Eq. (16), $U(R)$ is independent of $n$, and $Q_p^{(0)}(R)$ contains only non-zero harmonics in $n$ around the Fermi surface. The matrices $\bar{U}$ and $V_n$ obey the symmetries

$$\bar{U} U = 1, \quad \bar{U} = K U^\dagger K,$$

$$V_n V_n = 1, \quad V_n = K V_n^\dagger K.$$

In what follows we will make expansions in deviations of the supermatrix $Q_p^{(0)}(R)$ from $\Lambda_p$. For this purpose the supermatrix $Q_p^{(0)}(R)$ can be conveniently parametrized, for example, as
\[ Q_p^{(0)} (R) = \Lambda_p \left( \frac{1 + iP_n(R)}{1 - iP_n(R)} \right) \]  

(18)

with \( \int d\mathbf{n} P_n = 0 \), \( \overline{\mathbf{n}} = -P_n \). \( \{P_n, \Lambda\} = 0 \). Both \( U \) and \( V_n \) vary slowly with \( R \), that is, on length scales longer than \( l = v_F \tau \). While \( U(r) \) represents the massless (c) modes, the weakly massive modes (b) are contained in \( V_n \). It is the dependence of the supermatrix \( V_n \) on the vector \( \mathbf{n} \) that leads to the gap in the spectrum of excitations of the (b) modes.

An important feature of our parametrization for \( Q_p(R) \) in the form

\[ Q_p (R) = T_n (R) \Lambda_p \tilde{T}_n (R) \]

is the separation of the rotation matrix \( T_n = UV_n \) into the \( U \) and \( V_n \) factors. This step is not only convenient for computation but, more importantly, ensures the preservation of an original symmetry of the initial Lagrangian, Eq. (8). Specifically, it ensures that the final free energy functional is invariant under global rotations \( U(R) \to U_0 U(R) \) with \( U_0 = U_0^{-1} \) independent of coordinates. This invariance follows from the original invariance of the Lagrangian, Eq. (8), under global rotations \( \psi(r) \to U_0 \psi(r) \).

Substitution of the parametrization (16) into expression (10) yields a free energy in terms of only the weakly massive (b) modes and the massless (c) modes. Accordingly this free energy will describe fluctuations over energy scales only much less than \( \tau^{-1} \). In order to investigate the truly low-frequency behavior of transport coefficients, however, we need to reduce this free energy even further by integrating over the weakly massive (b) modes. This procedure will reduce the free energy to a form in terms of only \( U(r) \), representing the massless (c) modes. In this way we obtain the final form of the free energy that is applicable on energy scales much less than \( \tau^{-1} \) and so appropriate for the description of low-energy transport. It is necessary to integrate carefully over the (b) modes, rather than simply neglecting them, because, even though their coupling to the (c) modes is weak, it is sufficiently complicated to lead on integration to a potentially non-trivial renormalization of the bare free energy for the (c) modes. Due to the soft mass of the (b) modes it is sufficient to apply a Gaussian approximation in \( P_n \); higher order terms give a small contribution provided the inequality \( \epsilon_F \tau \gg 1 \) (equivalently, \( \gamma \ll 1 \)) is fulfilled.

In principle, if the correlations of the disorder decay sufficiently slowly (for the form of the correlations given by Eq. (4), this applies in the limit of \( \kappa \to 0 \)), one needs to employ a cutoff at large momentum \( k \) (short distance). The need for an ultra-violet regularization is well-known \([52–54]\) in the context of the ballistic \( \sigma \)-model and originates from the fact that the gradient (Liouvillean) operator in the logarithm of the initial Lagrangian, Eq. (10), is singular in two dimensions. It turns out that in this model one needs to be careful about the precise form of the short-distance cutoff, a point which we discuss further below.

It is helpful at this point to make contact with the conventional calculation for delta-correlated impurities \([40]\). To do so we simply set \( V_n = 1 \) in Eq. (16) so that \( Q \) becomes a function of \( R \) only. A straightforward gradient expansion of the Lagrangian (10) to second-order in the gradients then recovers the conventional \( \sigma \)-model. It is useful also to note that, prior to the gradient expansion, the Lagrangian (10) is invariant under a local “gauge” transformation

\[ U(R) \to U(R) h(R), \quad [h(R), \Lambda] = 0 \]  

(19)
(note the term “gauge” is used here in a separate sense from that of the original vector potential). As we will see later, this invariance remains conserved also for the case of long range disorder and is related to conservation of local currents.

C. Gradient expansion

We proceed with the derivation of the free energy functional by expansion of the logarithm in the Lagrangian (10) in low energy terms. Substituting the parametrization, Eq. (16), into the Lagrangian (10), and cycling factors under the supertrace, we find

\[
\mathcal{L} = \int \left[ -\frac{1}{2} \text{Str} \ln \left( -i\xi_p + \frac{\Lambda}{2\tau} + \mathcal{A}_{\text{kin}} + \mathcal{A}_{\text{coll}} + \mathcal{A}_\omega \right) \\
+ \frac{1}{2} \int \text{Str} \left( Q_p^{(0)}(R) V_{ij}^p p_i n_i^j Q_p^{(0)}(R) \right) \frac{d^2 p_1}{(2\pi)^2} \right] dR d^2 p_1 (2\pi)^2,
\]

(20)

where

\[
\mathcal{A}_{\text{kin}} = v_F T_n(R) n \nabla_R T_n(R) \tau_3,
\]

\[
\mathcal{A}_{\text{coll}} = 2 \int V_{ij}^p p_i n_i^j \left( T_p(R) Q_p(R) T_p(R) - \Lambda p_i \right) \frac{d^2 p_1}{(2\pi)^2},
\]

\[
\mathcal{A}_\omega = -i (\omega + i\delta) \bar{T}_n(R) \Lambda T_n(R),
\]

and \(T_n = UV_n\). The term \(\mathcal{A}_{\text{kin}}\) describes the kinetic energy, \(\mathcal{A}_{\text{coll}}\) is the collision integral and \(\mathcal{A}_\omega\) is the usual frequency term entering the \(\sigma\)-model. Then we perform an expansion of the logarithm in the terms \(\mathcal{A}_{\text{kin}}, \mathcal{A}_{\text{coll}}\) and \(\mathcal{A}_\omega\). If we keep to first order in all of these terms, we find that the contribution of \(\mathcal{A}_{\text{coll}}\) is \((-2)\) times that of the last term in Eq. (20): in this way we recover the usual form \([51,52]\) of the \(\sigma\)-model for ballistic disorder:

\[
F = \frac{\pi \nu}{4} \text{Str} \int \left[ \frac{1}{2} \int w(n, n')(Q_n(r) - Q_{n'}(r))^2 \frac{d\mathbf{n} d\mathbf{n}'}{(2\pi)^2} \\
- 2v_F \int \Lambda \bar{T}_n(r) n \nabla_R T_n(r) \frac{d\mathbf{n}}{2\pi} + i\omega \int \Lambda Q_n(r) \frac{d\mathbf{n}}{2\pi} \right] d\mathbf{r},
\]

(21)

where \(\nu\) is the density of states and \(w(n_1, n_2)\) is defined as

\[
w(n_1, n_2) = 2\pi \nu p_F^2 \sum_{ij} V_{n_1^i n_2^j} n_1^i n_2^j
\]

(22)

so that

\[
\frac{1}{2\tau_{BA}} = \int w(n, n') \frac{d\mathbf{n} d\mathbf{n}'}{(2\pi)^2},
\]

\[
\frac{1}{2\tau_{tr}} = \int w(n, n')(1 - n n') \frac{d\mathbf{n} d\mathbf{n}'}{(2\pi)^2},
\]

(23)

In the limit \(\kappa \to 0\) the function \(w(n_1, n_2)\) takes a simpler form.
\[ w(n_1, n_2) = \frac{v^2_n v_\pi \gamma (1 + n_1 n_2)}{2 (1 - n_1 n_2)} \quad (\kappa \to 0) \quad (24) \]

Notice that the free energy \( F \), Eq. (21), does not contain any divergencies in the limit of \( \kappa \to 0 \), since the singularity in \( w(n, n') \) as \( n \to n' \), Eq. (24), is compensated by the \((Q_n - Q_{n'})^2\) factor in the collision integral of Eq. (21). This is a reflection of the fact that the free energy contains energy scales only less than \( \tau^{-1} \) since we have eliminated the hard massive (a) modes.

We emphasize that, while the form of the ballistic \( \sigma \)-model in Eq. (21) has been presented previously [51,52], the derivation here for the case of long-ranged disorder represents the first formally justified derivation of this model. This has been made possible by the separation of the single-particle and transport lifetimes, \( \tau \) and \( \tau_{\text{tr}} \): this leads to the small parameter \( \tau/\tau_{\text{tr}} \ll 1 \) which separates the (a) type fluctuations from the (b) and (c) type fluctuations, and allows a gradient expansion of the free energy in a controlled manner. The derivation remains valid also for long-range potential disorder, instead of vector potential. In this case, Cooperons become operative unless time-reversal symmetry is completely broken by a constant magnetic field.

An interesting point is that, for the RMF model, the ratio \( \tau/\tau_{\text{tr}} \) formally vanishes as the cutoff \( \kappa \) is taken to zero, at least when the SCBA value of the single-particle lifetime \( \tau \) is taken. This suggests superficially that the above derivation of the ballistic \( \sigma \)-model of Eq. (21) becomes exact for the RMF model in the limit of \( \kappa \to 0 \), and in principle applicable on length-scales all the way down to the Fermi wavelength. A peculiarity of the RMF problem however is that it is not straightforward to specify the precise value of the single-particle lifetime: for example, a gauge-invariant formulation [18] produces a value of the single-particle lifetime that is convergent, rather than vanishing, as \( \kappa \to 0 \), at least when the average magnetic field is nonzero. The precise value of the single-particle lifetime in the RMF model that couples to the (a)-mode fluctuations, and hence the value of the shortest length-scales down to which the ballistic \( \sigma \)-model is applicable for this model, remains an open question.

A further strength of the derivation of the ballistic \( \sigma \)-model shown here is that, for finite values of \( \tau/\tau_{\text{tr}} \), the expansion of the free energy may be continued to find further terms that serve as corrections to the usual form of the ballistic \( \sigma \)-model, Eq. (21), but may nevertheless become relevant, as we discuss further below.

Our aim now is to integrate over the \( P_n \) fields by treating them in a Gaussian approximation. To do so we find it convenient to Fourier transform from \( P_n(r) \) to angular harmonic and momentum space via

\[
P_n(r) = \int \sum_m P_{m,k} \exp(i(kr + m \varphi) \tau_3) \frac{d^2k}{(2\pi)^2},
\]

where \( \varphi \) is the polar angle of \( n \). Performing the harmonic expansion on the collision integral, we see that, as well as the SCBA scattering lifetime described previously, a whole series of lifetimes associated with successive ballstic harmonics appears. We define the \( m \)th lifetime \( \tau^{(m)} \) by

\[
\frac{\Lambda}{2 \tau^{(m)}} = 2 \int \frac{d^2p_1}{(2\pi)^2} V^{ij}_{p-p_1} p_1^i p_1^j \Lambda_{p_1} \cos(m \varphi), \quad (25)
\]
where $\varphi$ is the angle between $\mathbf{p}$ and $\mathbf{p}_1$, so that $\tau^{(0)}$ coincides with the SCBA $\tau$ and $1/\tau_{tr} = 1/\tau - 1/\tau^{(1)}$. For example, the collision term in Eq. (21) becomes
\begin{equation}
F_{\text{coll}} = -\frac{\pi \nu}{4} \text{Str} \int w(\mathbf{n}, \mathbf{n}')Q_n(r)Q_{n'}(r) \frac{d\mathbf{n}d\mathbf{n}'}{(2\pi)^2} d\mathbf{r} = \pi \nu \int \frac{d^2 k}{(2\pi)^2} \sum_{m=1}^{\infty} \text{Str} P_{m,k} P_{m,-k} \left( \frac{1}{\tau} - \frac{1}{\tau^{(m)}} \right) \tag{26}
\end{equation}

While the form of the free energy given by Eq. (21) with the collision integral (26) is sufficient to treat the case of long-ranged disorder correlations (for which $\tau_{tr} \gg \tau$), in the limit of short-range disorder certain further terms are also relevant and should be included for the integration over the nonzero harmonics. These extra terms represent corrections to the usual form of the ballistic $\sigma$-model. These terms are found by continuing the expansion of the logarithm in Eq. (20) to include terms of order $A^2_{\text{coll}}, A_{\text{kin}}, A_{\text{coll}} A_{\text{kin}}, A_{\text{kin}}^2 A_{\text{coll}}$, and $A_{\text{kin}}^2$. In doing so, we ensure that we include all terms in the free energy that contribute to Gaussian order in $P_n$ and lead to no more than two gradient operators in each term in the final free energy. Collecting the various terms in the gradient expansion together, we come to the free energy $F = F_0 + F_n + F_\perp + F_{\text{unit}}$ where
\begin{align*}
F_0 &= \pi \nu \int \frac{d^2 k}{(2\pi)^2} \sum_{m=1}^{\infty} \text{Str} \left[ P_{m,k} P_{m,-k} \frac{\tau^{(m)}}{\tau} \left( \frac{1}{\tau} - \frac{1}{\tau^{(m)}} \right) \\
&\quad + \frac{i \nu_F}{2} \Lambda \left( P_{m,k} P_{m-1,-k} \vec{k}^* + P_{m+1,k} P_{m,-k} \vec{k} \right) \right], \\
F_n &= \pi \nu \nu_F \int d\mathbf{r} \sum_{m=1}^{\infty} \text{Str} \left[ \Phi_x^\nu \tau_3 \Lambda \left( P_{m,m-1} + P_{m+1,m-1} \right) - i \Phi_y^\nu \Lambda \left( P_{m,m-1} - P_{m+1,m-1} \right) \right], \\
F_\perp &= \frac{-i \pi \nu \nu_F}{2} \int d\mathbf{r} \text{Str} \left[ \bar{\Phi}_x^\perp \tau_3 \Lambda \left( P_1 + P_{-1} \right) + i \bar{\Phi}_y^\perp \Lambda \left( P_1 - P_{-1} \right) \right], \\
F_{\text{unit}} &= \frac{\pi \nu}{8} \int d\mathbf{r} \text{Str} \left[ D_0(\nabla Q)^2 + 2i \omega \Lambda Q \right], \tag{27}
\end{align*}

and $\vec{k} = k_x + i k_y \tau_3$. The supermatrices $\Phi$ and $Q$ are defined by
\begin{equation}
\Phi = \bar{U} \nabla U, \quad Q(\mathbf{r}) = U(\mathbf{r}) \Lambda \bar{U}(\mathbf{r}) \tag{28}
\end{equation}
and $\Phi^\nu$ ($\Phi^\perp$) are the components of $\Phi$ that commute (anti-commute) with $\Lambda$. The parameter $D_0$ is equal to $D_0 = \nu_F^2 \tau / 2$. As an intermediate step we have rescaled $P_m \rightarrow P_m \tau^{(m)}/\tau$. Eqs. (27) now represent the free energy that is applicable for energy scales that are much less than $\tau^{-1}$, but may still be as large as $\tau_{tr}^{-1}$.

In the limit of short-range (delta-correlated) disorder ($\tau^{(m)} \rightarrow \infty$ for $m \neq 0$), we see that the $P_m$ fields in Eq. (27) are now infinitely massive. This observation requires the presence of the extra terms that we have included in addition to those of usual ballistic $\sigma$-model, Eq. (21). These terms lead to the appearance of the extra ($\tau^{(m)}/\tau$) factor in $F_0$, as compared to expression (26). Due to the infinite mass of the $P_m$ fields, only the terms
in $F_{\text{unit}}$ remain in the free energy and the conventional unitary $\sigma$-model is recovered in this limit, as required.

As a technical point we remark that our form of the free energy (27) differs from the ballistic $\sigma$-model as written in Eq. (21) in a further sense: the lifetimes $\tau^{(m)}$ that appear in our expressions are defined through a self-consistent Born approximation, according to Eq. (25). This expression allows in general for off-shell contributions from the momenta $p_1$. The collision integral in the model of Eq. (21), by contrast, amounts to only a simple Born approximation (see Eq. (23)) as it includes only on-shell contributions through the factor $w(n, n')$. For the RMF model with a finite $\kappa$, the distinction is not important for sufficiently low angular harmonics, as then the relevant momentum integrals are restricted to the energy shell anyway due to the compensation of the singularity in $w(n, n')$ as $n \to n'$ by the $(Q_n - Q_{n'})^2$ factor in Eq. (21). As an example, for $m = 1$ this compensation has been encountered already in Eq. (7) for $\tau_{tr}$. For harmonics $m$ of the order of $\kappa^{-1}$, however, the simple Born approximation becomes unreliable and instead the full momentum dependence of $V_{p - p_1}$ must be accounted for. For such high harmonics, the factor of $\tau^{(m)}/\tau$ in $F_0$ becomes much larger than unity and hence the contribution of these harmonics is strongly damped. The end result of these considerations is that for the RMF model we may apply a cutoff to the angular harmonic summation in Eq. (27) to $m \ll \kappa^{-1}$.

D. Integration over nonzero harmonics

Having derived the free energy functional of Eq. (27) for fluctuations at energy scales much less than $\tau^{-1}$, the next step is to reduce this form, by an integration over the non-zero harmonics $P_n(r)$, to one that is applicable at the lowest energy scales, which are much less than $\tau_{tr}^{-1}$. In other words, we need to average the terms in the free energy that couple the (b) and (c) modes ($F_{\parallel}$ and $F_{\perp}$) with respect to the bare (b) mode free energy ($F_0$) to produce the required renormalization of the bare (c) mode free energy ($F_{\text{unit}}$). Equivalently, we aim to determine the influence of ballistic electron motion on distances smaller than $l_{tr}$, at which distances the classical limit is described by the Boltzmann equation, on quantum interference processes at large distances.

Since relevant terms in the free energy will contain no more than two gradient operators, the relevant contribution from this averaging comes from the second-order cumulant of $F_{\parallel} + F_{\perp}$:

$$F_0 + F_{\parallel} + F_{\perp} \to -\frac{1}{2}\langle(F_{\parallel} + F_{\perp})^2\rangle F_0.$$  \hspace{1cm} (29)

While the contribution from the cross term $F_{\parallel}F_{\perp}$ vanishes, the terms in $F_{\perp}$ may be eliminated by a simple shift in $P_{\perp}$ which leads to a dressing of the bare diffusion coefficient, $D_0$, appearing in $F_{\text{unit}}$: $D_0 \to D = D_0^2/\tau_{tr}/2$. In terms of perturbation theory, the replacement of the bare $D_0$ by the classical diffusion coefficient $D$ corresponds as usual to the inclusion of two-particle vertex corrections. Taking into account only the contributions from $F_{\parallel}$ and $F_{\perp}$ corresponds to the calculation of Refs. [44,41] and gives the conventional unitary $\sigma$-model with the free energy functional $F_{\text{unit}}$, Eq. (27), and the classical coefficient $D$.

What remains is to evaluate the contribution from the terms in $F_{\parallel}$. In contrast to the contributions from $F_{\perp}$ and $F_{\text{unit}}$, which involve only the zeroth and first harmonics, the
contribution from $F_0$ involves correlations between higher harmonics. To calculate this contribution we need to perform the set of Gaussian integrals that correspond to the integration over $P_m$ with the free energy functional $F_0$. This step is no more than an application of Wick’s theorem in the space of the angular harmonics (see also Aleiner and Larkin [53] for a further example) and so requires inversion of the quadratic form in $P_m$ contained in the bare free energy $F_0$.

The full form of the Gaussian integration over the $P_m$ fields is made clear if we write out the components of $P_m$ as

$$P_m, k = (\hat{a}_{\pm, k} \hat{B}, \dot{\hat{a}}_{\pm, k}, \hat{b}_{\pm, k} \hat{L}, \dot{\hat{b}}_{\pm, k}, \hat{\sigma}_{\pm, k} \hat{L}, \dot{\hat{\sigma}}_{\pm, k}, \hat{\rho}_{\pm, k} \hat{L}, \dot{\hat{\rho}}_{\pm, k})$$

We see from the form of $F_0$ that there are no correlations between negative and positive harmonics of $B_m, k$; defining the column vectors $\vec{a}_\pm = (a_{\pm 1}, a_{\pm 2}, \ldots)$, $\vec{b}_\pm = (b_{\pm 1}, b_{\pm 2}, \ldots)$ and similarly for $\vec{\sigma}$ and $\vec{\rho}$, we come to

$$F_0 = F_0^+ + F_0^-$$

where

$$F_0^\pm = \frac{2\pi \nu}{\tau} \int \frac{d^2k}{(2\pi)^2} [\hat{a}^*_\pm, k \hat{L} (\hat{s}_\pm) \hat{a}^*_\pm, k + \hat{b}^*_\pm, k \hat{L} (\hat{s}_\pm) \hat{b}^*_\pm, k] + \hat{\sigma}^*_\pm, k \hat{L} (\hat{s}_\pm) \hat{\sigma}^*_\pm, k - \hat{\rho}^*_\pm, k \hat{L} (\hat{s}_\pm) \hat{\rho}^*_\pm, k]$$

Here $\hat{L}$ is a tridiagonal, semi-infinite matrix with the entries

$$(\hat{L} (\hat{s}))_{m, m'} = \frac{\tau m}{\tau_{m}} \left( \frac{1}{\tau} - \frac{1}{\tau_{m}} \right) \delta_{m, m'} + i \hat{s} \delta_{m+1, m'} + i \hat{s}^* \delta_{m, m'+1},$$

where $m$ and $m'$ are positive and

$$\hat{s} = \frac{l_{Tr}}{2} (k_x + i k_y)$$

After a similar reexpression of $F_c$ in terms of the vectors $\vec{a}$, $\vec{b}$, $\vec{\sigma}$ and $\vec{\rho}$, one can perform the averaging. In the process we use the relations $\text{Str} \Phi = \text{Str} (\Lambda \Phi) = \text{Str} (\tau_3 \Phi) = 0$ that follow from the symmetries of $U$ given by Eq. (17): hence only the combination $\text{Str} (\tau_3 \Lambda \Phi)$ can enter the final formulae. Integrating over the supermatrices $P_m$ in $\langle F_0^2 \rangle$, Eq. (29), with the free energy $F_0$, Eq. (27), we obtain a term quadratic in $\text{Str} (\tau_3 \Lambda \Phi)$ with a coefficient determined by an integral over $k$. Indeed we reduce the additional term $F_c$ in the free energy to the form

$$F_c = -\frac{l^2}{16} \int \text{Str} (\tau_3 \Lambda \Phi_i (r)) \text{Str} (\tau_3 \Lambda \Phi_j (r')) \Pi^{ij} (r, r') drdr'$$

where

$$\Pi^{ij} (r, r') = \int n^i n'^j \Gamma (r, r'; n, n') \Gamma (r', r; n', n) \frac{dndn'}{(2\pi)^2}.$$
The function $\Gamma (\mathbf{r}, \mathbf{r}'; \mathbf{n}, \mathbf{n}')$ depends only on the coordinate difference $\mathbf{r} - \mathbf{r}'$ and may be written in the momentum representation as

$$
\Gamma (\mathbf{k}; \mathbf{n}, \mathbf{n}') = \sum_{m,m' > 0} \left[ (\hat{L}^{-1} (\mathbf{s}))_{m,m'} e^{i(m\phi - m'\phi')} + (\hat{L}^{-1} (-\mathbf{s}))_{m,m'} e^{-i(m\phi - m'\phi')} \right],
$$

where $\phi$ and $\phi'$ are the polar angles of the vectors $\mathbf{n}$ and $\mathbf{n}'$. An alternative formulation is that $\Gamma (\mathbf{r}, \mathbf{r}'; \mathbf{n}, \mathbf{n}')$ satisfies the following Boltzmann-like equation:

$$
l_{tr} \mathbf{n} \nabla \mathbf{r} \Gamma (\mathbf{r}, \mathbf{r}'; \mathbf{n}, \mathbf{n}') + \int W (\mathbf{n}, \mathbf{n}'') \Gamma (\mathbf{r}, \mathbf{r}'; \mathbf{n}'', \mathbf{n}') \frac{d\mathbf{n}''}{2\pi} = \delta (\mathbf{r} - \mathbf{r}') \delta (\mathbf{n} - \mathbf{n}'),
$$

where $W (\mathbf{n}, \mathbf{n}')$ is a function of $\mathbf{nn}'$, such that its eigenvalues are equal to the diagonal entries of $\hat{L}$, i.e., $\tau_{tr} \tau^{(m)} \tau^{-1} \left( \tau^{-1} - (\tau^{(m)})^{-1} \right)$. As we want to derive a free energy functional for supermatrices $\Phi$, that vary slowly on the scale of $l_{tr}$, we assume that the functions $\Phi (\mathbf{r})$ depend more slowly on $\mathbf{r}$ than $\Pi^{ij} (\mathbf{r}, \mathbf{r}')$.

Using Eqs. (33), one can check without difficulty that the function $\Pi^{ij} (\mathbf{r}, \mathbf{r}')$, Eq. (33), satisfies a transversality condition

$$
\nabla_{\mathbf{r} i} \Pi^{ij} (\mathbf{r}, \mathbf{r}') = \nabla_{\mathbf{r}' j} \Pi^{ij} (\mathbf{r}, \mathbf{r}') = 0.
$$

As the function $\Pi^{ij} (\mathbf{r}, \mathbf{r}')$ depends only on $\mathbf{r} - \mathbf{r}'$ we perform the Fourier transformation in this variable and using Eq. (33) write it in the form

$$
\Pi^{ij} (\mathbf{q}) = \Pi_{0} (\mathbf{q}) \left( \delta^{ij} - \frac{\mathbf{q}^i \mathbf{q}^j}{q^2} \right).
$$

Using Eqs. (32), we write the final form of the $\sigma$-model as

$$
F [Q] = \frac{\pi \nu}{8} \int \text{Str} \left[ D (\nabla Q (\mathbf{r}))^2 + 2i\omega \Lambda Q (\mathbf{r}) \right] d\mathbf{r}
- \frac{l_{tr}^2}{16} \int \text{Str} \left( \tau_3 \Lambda \Phi_i (\mathbf{r}) \right) \text{Str} \left( \tau_3 \Lambda \Phi_j (\mathbf{r}') \right) \Pi^{ij} (\mathbf{r}, \mathbf{r}') d\mathbf{r} d\mathbf{r}'.
$$

The invariance of the free energy functional $F$ under the local gauge transformations means that it can be written in terms of the supermatrix $Q$ only (without $\Phi$). This may be done by employing the transversal form for $\Pi^{ij} (\mathbf{q})$, as given by Eq. (37), in the second term of Eq. (38). Then we obtain

$$
F [Q] = \frac{\pi \nu}{8} \int \text{Str} \left[ D (\nabla Q (\mathbf{r}))^2 + 2i\omega \Lambda Q (\mathbf{r}) \right] d\mathbf{r}
- \frac{l_{tr}^2}{256} \int \frac{\Pi_{0} (\mathbf{q})}{q^2} M (\mathbf{q}) M (-\mathbf{q}) \frac{d^2q}{(2\pi)^2},
$$

where

$$
M (\mathbf{r}) = \text{Str} \left( \tau_3 Q (\mathbf{r}) \left[ \nabla_x Q (\mathbf{r}), \nabla_y Q (\mathbf{r}) \right] \right).
$$
represents the local topological density \( \mathcal{Q} \). The free energy functional \( F[\mathcal{Q}] \), Eq. (39), has a similar form to that written by Zhang and Arovas [42] (although the latter reference does not contain a proper microscopic justification), up to the presence of the factor \( \Pi_0(q) \).

What is crucial is the behaviour of \( \Pi_0(q) \) as \( q \to 0 \). If \( \Pi_0(q) \) were to remain finite as \( q \to 0 \), then the extra term in the free energy, Eq. (39), would remain relevant and give rise to logarithmic corrections to the conductivity. A vanishing value of \( \Pi_0(q) \) means that the new term is not relevant, and gives rise to only higher than logarithmic corrections to the conductivity. These statements are made clear in section IV, where we show how the quantity \( \Pi^{ij}(r, r') \) appears in a direct calculation of the conductivity. Moreover we show in section \( \nabla \beta \) that the quantity \( \Pi^{ij}(r, r') \) has a direct physical meaning: it is proportional the mesoscopic current-current correlation function. Thus the question of the potential relevancy of the new term in Eq. (39) amounts to whether or not the Fourier transform of the current-current correlator remains nonzero in the limit of \( q \to 0 \).

In fact the evaluation of the quantity \( \Pi_0(q) \) as \( q \to 0 \) is a delicate procedure within the framework of the \( \sigma \)-model. As shown very recently by Gornyi et al. [46], a proper evaluation requires a careful regularisation procedure that ensures the preservation of gauge invariance. Their proposal is to evaluate \( \Gamma (r, r'; n, n') \) by a semiclassical approximation and to impose a short-distance cutoff via a minimum path length. Regularising in this way, the value of \( \Pi_0(q) \) turns out to vanish as \( q \to 0 \). Their proposal was supplemented by a diagrammatic calculation for the current-current correlator which seems to verify the use of regularisation procedure; we refer to Ref. [46] for further details (we remark that our earlier letter [45] employed a more naive regularisation procedure which led to incorrect conclusions).

This demonstrates that, within the scope of our analysis, the new term seems to remain irrelevant in the calculation of corrections to the conductivity that are logarithmic in frequency. As a result we do not find evidence for deviation of the localization behaviour of the delta-correlated random magnetic field model from that of the conventional unitary ensemble. As such our conclusions coincide with those of the original work of Aronov et al., although for rather more subtle reasons than were originally realised in that work.

Even so, certain questions remain which deserve further study before a definitive conclusion can be drawn for this problem. In particular, the role of the massive (a) modes, describing fluctuations of the eigenvalues of \( Q \) and the Cooperons, has been neglected in our analysis and should be considered more carefully. For example, there is some ambiguity regarding the precise value of the single-particle lifetime, \( \tau \), in the RMF model. Although the SCBA for the RMF model yields a value of \( \tau^{-1} \) that is divergent as \( \kappa \to 0 \), it is likely that the true value of \( \tau^{-1} \) that couples to the massive (a) modes should remain finite; in a similar manner, a gauge-invariant formulation [18] (beyond the SCBA) has been shown to produce a finite single-particle inverse lifetime, at least when the average magnetic field is nonzero. In the formalism of the field theory, this suggests that an improved saddle-point may be found which does not contain any divergencies in the limit of \( \kappa \to 0 \). Since the single-particle mean free path determines the value of the shortest lengthscales down to which the ballistic \( \sigma \)-model is applicable for this model, and the evaluation of the new term in the free energy depends crucially on the regularization procedure at such short distances, further study of the role of the massive (a) modes in this problem seems desirable.

Although the new term in the free energy, Eq. (39), appears not to give rise to logarithmic corrections to the conductivity for the delta-correlated RMF, it is still responsible for cor-
rections that are higher than logarithmic. Such corrections are known as “memory effects” and have been examined recently by Wilke et al. [22]. The derivation of such corrections is contained in the calculation of the conductivity in next section.

We remark that a free energy of the form of Eq. (39) arises also for a model of long-range potential disorder, as well as vector potential, as long as time-reversal symmetry is broken by a constant magnetic field (in which case, a further topological term associated with Hall quantization appears). If time-symmetry symmetry is preserved, then the additional term in Eq. (39) is absent for trivial symmetry reasons.

The question remains whether the new term in the free energy, Eq. (39), is always irrelevant or if such irrelevancy has been specific to our original choice of model. In section V we show that, indeed, for other models of disorder such a new term can remain relevant, and give logarithmic corrections to the conductivity. What is required is disorder of even longer range than has been envisaged so far, that leads to a finite value of the current-current correlator as \( q \to 0 \). This situation leads to the interesting possibility of a quantum phase transition at weak disorder.

IV. PERTURBATION THEORY

In this section we supplement the derivation of the free energy in the previous sections by direct perturbative calculations within the field theory formalism. In section VA we calculate the conductivity. We identify the appearance in this calculation of the quantity \( \Pi^{ij}(r, r') \), that appears in the free energy, Eq. (39), and show how it leads to higher order than logarithmic corrections to the conductivity. We also calculate in section VB the mesoscopic current-current correlation function, and find it is directly proportional to \( \Pi^{ij}(r, r') \), thus providing a physical interpretation for this quantity.

While we operate within the framework of a field theory, we emphasize that the calculations of this section are equivalent to conventional diagrammatics. We remark also that the results for the conductivity of section VA may also be derived via an RG analysis for the diffusion coefficient, similar to that of section VI: the equivalence of the two approaches follows from the Einstein relation between the conductivity and the diffusion coefficient.

A. Conductivity

In this section, we demonstrate a direct calculation of the conductivity within the framework of the field theory. According to standard linear response theory (see e.g. [53,40]), the response \( R \) is defined by the relation of the current density \( J \) to the oscillating part of an applied vector potential \( A_\omega \),

\[
J^i(\omega) = \frac{i\omega}{c} R^{ij}(\omega) A^j_\omega.
\]

Following Ref. [40], we write the response as \( R^{ij}(\omega) = \int d\mathbf{r}' R^{ij}(\mathbf{r}, \mathbf{r}') \), where

\[
R^{ij}(\mathbf{r}, \mathbf{r}') = -\frac{e^2}{16\pi} \sum_{\gamma, \delta = 1}^4 \langle (1 - \tau_3)_{\gamma \gamma} \pi^j_\gamma \psi^1_\gamma(\mathbf{r}') \bar{\psi}^1_\gamma(\mathbf{r})(1 - \tau_3)_{\delta \delta} \pi^j_\delta \psi^2_\delta(\mathbf{r}) \bar{\psi}^2_\delta(\mathbf{r}') \rangle, \tag{41}
\]
and where the averaging is with respect to the Lagrangian $L$ in Eq. (8). Here

$$\hat{n}^i_r \equiv (-i\nabla^i_r - (e/c) A) / m$$

is the velocity operator ($A$ is the static vector potential). On averaging the right-hand side of Eq. (41), we may pair the $\psi$ fields, according to Wick’s theorem, in three different ways. This leads to $R = R_1 + R_2 + R_3$ where

$$R_1^{ij}(r, r') = -\frac{e^2}{16\pi} \text{Str} [k(1 - \tau_3)^3 \hat{n}^i_r g^{11}(r', r)] \text{Str} [k(1 - \tau_3)^3 \hat{n}^j_r g^{22}(r, r')] ,$$

$$R_2^{ij}(r, r') = \frac{e^2}{16\pi} \text{Str} [k(1 - \tau_3)^3 \hat{n}^i_r g^{12}(r', r') k(1 - \tau_3)^3 \hat{n}^j_r g^{21}(r, r)] ,$$

$$R_3^{ij}(r, r') = \frac{e^2}{16\pi} \text{Str} [k(1 - \tau_3)^3 \hat{n}^i_r g^{12}(r', r) \hat{n}^j_r k(1 + \tau_3)^3 g^{21}(r', r)] ,$$

and where the symbol $\hat{n}$ indicates a velocity operator whose derivative acts to the left. The Green function is defined by $g^{\alpha\beta}(r, r') = 2\langle \psi^\alpha(r) \bar{\psi}^\beta(r') \rangle$ and, following the introduction of the $Q$-matrix, satisfies (c.f. with Eq. (41))

$$\left( \epsilon + \epsilon_F + \frac{\nabla^2_r}{2m} - \frac{\omega}{2} \right) g(r, r') - i \int dr'' \hat{V}_{r,r''} \bar{Q}(r, r'') g(r'', r') = -i\delta(r - r'),$$

where the function $\hat{V}_{r,r'}$ is determined by Eq. (9). In the limit of short-range disorder, $\tau = \tau_{tr}$, the fluctuations of the nonzero harmonics of $Q_n(r)$ are strongly suppressed, as demonstrated in the derivation of the free energy in section III C. In this case, $Q(r, r')$ becomes a function of $R = (r + r')/2$ only and Eq. (43) reduces to the relations already found in the direct computation of the conductivity for short-range disorder (see Ref. [10], Chapter 8, and Ref. [50]). However, in the limit of long range disorder, when non-zero harmonics are not suppressed, an additional term appears that contains the quantity $\Pi^{ij}(r, r')$.

In the limit of long-ranged disorder, for which $\tau_{tr} \gg \tau$, we may perform an expansion of the Green function $g(p, R)$ in the collision term $\sum_{p_1} V^{ij}_{p-p_1} p^i_1 p^j_1 (Q_{p_1} - Q_p)$, where the momentum $p$ arises from a Fourier transform with respect to $r - r'$: this expansion corresponds directly to the expansion of the free energy in section III C in the quantity $A_{coll}$, and corresponds to an expansion in powers of $\tau/\tau_{tr} \ll 1$. Therefore to leading order in $\tau/\tau_{tr}$, we may take

$$g(p, R) = i \left[ \frac{p^2}{2m} - \epsilon + \frac{i}{2\tau} Q_n \right]^{-1} + \ldots .$$

Since we neglect the hard massive $(a)$ modes (see section III B), we have also that $[g, \tau_3] = 0$ and hence $R_3 = 0$. Performing the necessary integrals over the modulus of the momentum, we find

$$R_1^{ij} = 2e^2 \nu_F^2 \tau \int \frac{dn}{2\pi} n^i n^j \left[ 1 - \frac{1}{32} \langle \text{Str} (k(1 - \tau_3)(Q_n(r) - \Lambda)^{11}) \times \text{Str} (k(1 - \tau_3)(Q_n(r) - \Lambda)^{22}) \rangle \right] ,$$

$$R_2^{ij} = \frac{\pi (e\nu_F)^2}{16} \int dr \frac{dn}{(2\pi)^2} n^i n^j \langle \text{Str} [k(1 - \tau_3) [Q_n(r)(1 + v_F\tau n\nabla_r Q_n(r))]^{12} \times k(1 - \tau_3) [Q_{n'}(r')(1 + v_F\tau n'\nabla_r Q_{n'}(r'))]^{21}] \rangle ,$$

$$R_3^{ij} = \frac{\pi (e\nu_F)^2}{16} \int dr \frac{dn}{(2\pi)^2} n^i n^j \langle \text{Str} [k(1 - \tau_3) [Q_n(r)(1 + v_F\tau n\nabla_r Q_n(r))]^{12} \times k(1 - \tau_3) [Q_{n'}(r')(1 + v_F\tau n'\nabla_r Q_{n'}(r'))]^{21}] \rangle .$$
where now the averaging is with respect to the free energy (27). In the limit of \( \tau \ll \tau_\text{tr} \), to leading order Eq. (17) reduces to include only the following term \( \tilde{R} \), originating from \( R_2 \):
\[
\tilde{R}^{ij} = \frac{(e
u
\nu')^2}{16} \int \! dr^i \! \frac{d \mathbf{n} \! \cdot \! d \mathbf{n}'}{2(2\pi)^2} \left( \langle \text{Str} \left[ k(1 - \tau_3)\mathbf{Q}_n^{12} (\mathbf{r}) k(1 - \tau_3)\mathbf{Q}_{n'}^{21} (\mathbf{r}') \right] \rangle \right) .
\] (46)

The response \( \tilde{R}^{ij} \), Eq. (46), is the contribution that was absent in the standard treatment of models with short range disorder [56,40]. It involves integration over non-zero harmonics and may be computed by averaging with respect to the free energy (27).

As is consistent with the derivation of the final form of the free energy in section III D, we employ the parametrization of Eqs. (16) and (18), and expand each of the \( Q_n \) matrices in Eq. (16) up to second order in \( P_n \). To reproduce the leading-order quantum correction to the conductivity, we also expand in the deviation of the \( U \) matrix from unity, although we delay this step for convenience. In other words, averaging over \( P_n (\mathbf{r}) \) we derive as the first step an expression for the conductivity in a form of an integral over supermatrices \( Q (\mathbf{r}) \) depending on \( \mathbf{r} \) only. Then, one can compute the integral and obtain the final formulae.

To zeroth order in \( P_n \), the expression (13) for \( R \) vanishes due to the integrations over \( \mathbf{n} \) and \( \mathbf{n}' \). On expanding both of the \( Q_n \) matrices in Eq. (14) to first order in \( P_n \), averaging over \( P_n \) from Eq. (15) we find the following contributions to \( R \):
\[
R_a^{ij} = 2e^2 \nu D \delta^{ij} \left[ 1 - \frac{1}{32} \langle \text{Str} (k(1 - \tau_3)(Q - \Lambda)^{11}(\mathbf{r})) \rangle \times \right.
\]
\[
\left. \text{Str} (k(1 - \tau_3)(Q - \Lambda)^{22}(\mathbf{r})) \rangle \right],
\] (47)
\[
R_b^{ij} = \frac{\pi}{16} e^2 \nu^2 D^2 \delta^{ij} \int \! dr' \langle \text{Str} \left[ k(1 - \tau_3)(Q \nabla_r Q)^{21}(\mathbf{r}') \times \right. \rangle
\]
\[
\left. k(1 - \tau_3)(Q \nabla_r Q)^{12}(\mathbf{r}) \rangle \right),
\] (48)

where \( Q (\mathbf{r}) = U (\mathbf{r}) \Lambda \bar{U} (\mathbf{r}) \) and the averaging is now with respect to the final form of the free energy, Eq. (38). We remark that to derive the contribution \( R_b \) it is necessary to include a contraction with the terms in the free energy, Eq. (27), that are linear in \( P_n \).

In the derivation of both the contributions \( R_a \) and \( R_b \) of Eqs. (17) and (48), the coupling of only the first harmonics, \( P_{\pm 1} \), to the \( U \) matrix is involved. As was found in the derivation of the free energy in section III D, the role of the first harmonics, beyond that of the higher harmonics, is to dress the bare diffusion coefficient, \( D_0 = \nu_F^2 \tau / 2 \), with classical vertex corrections, so that it is renormalized according to \( D_0 \rightarrow D = \nu_F^2 \tau_{\text{tr}} / 2 \). For this reason we see that the expressions for \( R_a \) and \( R_b \) include a renormalized diffusion coefficient, \( D \). Moreover, up to this renormalization, they coincide precisely with the corresponding expressions that are found [41,54] for the conductivity in the case of short-range disorder.

An additional contribution, \( R_c \), comes from expanding both of the \( Q_n \) matrices in Eq. (16) to second order in \( P_n \). On averaging over \( P_n \) with the free energy functional (27), we find the new contribution that can be written in the form
\[
R_c^{ij} = \frac{1}{32\pi} (e\nu_{\text{tr}})^2 \int \! d \mathbf{r} \langle \text{Str} \left[ k(1 - \tau_3)\mathbf{Q}_n^{12}(\mathbf{r}) k(1 - \tau_3)\mathbf{Q}_{n'}^{21}(\mathbf{r}') \right] \rangle d \mathbf{r}',
\] (49)
where the function $\Pi^{ij}$ is given by Eq. (33). In contrast to the contributions $R_a$ and $R_b$, the term $R_c$ does not have a counterpart in the calculation for short-range disorder, since its presence relies on the long-range disorder correlations that pertain to the RMF model. In contrast to the derivation of the terms $R_a$ and $R_b$, the derivation of $R_c$ requires consideration of all harmonics, rather than only the zeroth and first harmonics.

The final step to compute the quantum correction to the conductivity is to expand $U$ around unity, according to, for example,

$$U = \left( \frac{1 - iP}{1 + iP} \right)^{1/2},$$

in similarity with Eq. (78). In the same time, one should also expand in $P$ the free energy, Eq. (38). It is important that the new term $F_c$ leads to terms starting from $P^4$ and may contribute to the conductivity only in higher orders than considered here. Expanding $R_a$, $R_b$ and $R_c$ to fourth, sixth and second order in $P$ respectively, and averaging over $P$, we obtain the conductivity

$$\sigma^{ij} = \sigma_0 \left[ \delta^{ij} (1 + \Delta_a + \Delta_b) + \Delta_c^{ij} \right],$$

where

$$\Delta_a = \frac{t^2}{128} I_d, \quad \Delta_b = -\frac{t^2}{64d} I_d,$$

$$\Delta_c^{ij} = \frac{t^2 l_{tr}^2}{128} \int \frac{\Pi^{ij}(q)}{(q^2 + \lambda^2)} \frac{d^d q}{(2\pi)^d},$$

and

$$I_d = \int \frac{1}{(q^2 + \lambda^2)} \frac{d^d q}{(2\pi)^d}.$$  

Here $\sigma_0 = 2e^2 \nu D$ is the classical conductivity and $t = 8/(\pi \nu D)$. The contributions $\Delta_{a-c}$ originate from the expressions $R_{a-c}$ respectively.

The contributions $\Delta_a$ and $\Delta_b$ correspond to the usual weak localization corrections to the conductivity for short-range disorder in the unitary ensemble. Since $\Pi^{ij}(q) \to 0$ as $q \to 0$ (see Ref. [10]), the contribution $R_c$ is of higher-order than logarithmic. In fact, since $\Pi^{ij}(q) \propto q^2$, $R_c$ corresponds to a “memory effect” of the type studied recently in Ref. [22], and gives rise to a nonanalytic correction $\propto |\omega|$ to the conductivity as $\omega \to 0$. The contributions $\Delta_{a-c}$ may all be derived via an RG approach for the diffusion coefficient (see section VII), as follows by the Einstein relation.

While we have used the $\sigma$-model approach to compute all of the above diagrams, we emphasize that in principle it is possible to evaluate all of the above diagrams directly using the bare Green functions and without reference to an effective Lagrangian. Such a calculation would be useful as a check for the regularisation procedure used for the ballistic $\sigma$-model. The calculation is not simple as, for example, it becomes necessary to take into account a proper dressing of the Hikami boxes and vertices with impurity lines [11]. In addition
extra, mutually cancelling diagrams would appear that correspond to the hard massive modes of the field theory. The diagrammatic arguments of Gornyi et al. [46] do not fulfill the purpose of such a check: although their calculation of the current-current correlation function seems valid (see the next subsection), their calculation of the conductivity relies on a relation between the conductivity and the current-current correlation function which contains the quantity \( \tau \). As we mentioned above, this quantity formally vanishes within the SCBA for the delta-correlated RMF model. Thus their arguments are questionable for the delta-correlated RMF model, although they may be applicable when the RMF is a weak perturbation to sufficiently strong, short-range disorder, in which case a well-defined \( \tau \) does exist. Instead, what is required is a direct diagrammatic calculation of the first logarithmic correction to the conductivity, a task which is not simple even for short-range disorder and has not been attempted so far for the RMF model.

### B. Mesoscopic Current Correlations

In this section we continue the perturbative analysis to a calculation of the current-current correlation function for long-ranged disorder, such as a delta-correlated RMF. This enables us to provide a physical interpretation of the quantity \( \Pi^{ij}(r, r') \), as being directly proportional to the current-current correlation function.

We employ the definition

\[
J^i(r) = \frac{ie}{4\pi} \lim_{r' \to r} (\hat{\pi}^i_r - \hat{\pi}^i_{r'}) (G^B_r(r, r') - G^A_r(r, r')) ,
\]

for the local current, where \( \hat{\pi}^i_r \) is the velocity operator, Eq. (42). The current-current correlator

\[
I^{ij}(r - r') \equiv \langle J^i(r) J^j(r') \rangle_{\text{imp}}
\]

is then given by

\[
I^{ij}(r - r') = -\frac{e^2}{32 \pi^2} \lim_{s \to r, s' \to r'} \langle (1 - \tau_3)_{\gamma\gamma} (\pi^i_r - \pi^i_s) \psi^j_\gamma(r) \bar{\psi}^j_\gamma(s) \times \rangle \times (1 - \tau_3)_{\delta\delta} (\pi^j_{r'} - \pi^j_{s'}) \psi^j_\delta(r') \bar{\psi}^j_\delta(s') \rangle_L
\]

where a summation over \( \gamma \) and \( \delta \) is implied. The angle brackets in Eq. (54) stand for averaging over impurities, while the angle brackets in Eq. (53) stand for averaging with the Lagrangian \( L \), Eq. (8).

There are now three possible ways of making pairings of the \( \psi \) and \( \bar{\psi} \) fields. Note that the calculation is following very similar lines to those of the calculation of the conductivity in section [IV A]. As in that case, only one of the pairings need be considered at leading order, where the first \( \psi \) and \( \bar{\psi} \) are paired, and the second \( \psi \) and \( \bar{\psi} \) are paired. Following Fourier transformations with respect to \( r - s \) and \( r' - s' \), we find

\[
I^{ij}(r - r') = -\frac{e^2}{128 \pi^2} \lim_{s \to r, s' \to r'} \text{Str} \langle k(1 - \tau_3)(\pi^i_r - \pi^i_s) g^{11}(r, s) \rangle \times \text{Str} \langle k(1 - \tau_3)(\pi^j_{r'} - \pi^j_{s'}) g^{22}(r', s') \rangle,
\]
where $g(r, s)$ is defined as in section E, Eq. (44). Following Fourier transformations with respect to $r - s$ and $r' - s'$, integrating over the modulus of the momentum, and keeping the leading-order term in the limit of $\tau \ll \tau_{tr}$, we arrive at

$$I^{ij}(r - r') = -\frac{e^2}{128}(\nu v_F)^2 \int \frac{dndn'}{(2\pi)^2} n^i n'^j \text{Str}(k(1 - \tau_3)Q_n(r)(1 + \Lambda)) \times$$

$$\times \text{Str}(k(1 - \tau_3)Q_{n'}(r')(1 - \Lambda)).$$ (57)

Eq. (57) represents the required formula from which the current-current correlation function may be computed (compare with Eq. (42) for the conductivity). As for the conductivity, we employ the parametrization of Eqs. (16) and (18), and expand each of the $Q_n$ matrices in Eq. (57) up to second-order in $P_n$. If we expand both of the $Q_n$ terms in Eq. (57) to second-order in $P_n$, we find the contribution

$$I^{ij}_0(r - r') = \frac{e^2}{32(\nu v_F)^2} \int \frac{dndn'}{(2\pi)^2} n^i n'^j \text{Str}(k(1 - \tau_3)P_n(r)^2(1 + \Lambda)) \times$$

$$\times \text{Str}(k(1 - \tau_3)P_{n'}(r')^2(1 - \Lambda)).$$ (58)

Performing the contractions in Eq. (58), we find

$$I^{ij}_0(r - r') = \frac{e^2}{2\pi^2} l_{tr}^2 \Pi^{ij}(r, r')$$ (59)

where the function $\Pi^{ij}(r, r')$ is given by Eq. (33). We see from Eq. (59) that the correlator of mesoscopic currents $I^{ij}_0$ depends on the same function $\Pi^{ij}$ that enters the action, Eq. (32), and the conductivity, Eq. (49).

Strictly speaking, the function $\Pi^{ij}$ that appears in the free energy represents not precisely the current-current correlation function, but the contribution of only the nonzero harmonics to this quantity: this is reflected by the fact that $I^{ij}$ contains further contributions, beyond $I^{ij}_0$, which are zeroth order in $P_n$, but second-order in the generator $P$ of $U$ (see the parametrization of Eq. (50)). In a similar way, the function $\Gamma(k; n, n')$ of Section III D is defined in Eq. (34) through a projection onto only nonzero harmonics $m > 0$. An alternative definition of $\Gamma(k; n, n')$ may be written which treats the zeroth and nonzero harmonics on an equal footing, as in Ref. [46], although the difference becomes inessential for momenta $k \gg l_{tr}^{-1}$; the deviation of the function $\Pi^{ij}$ from the true value of the current-current correlation function is therefore minor. The separation of zeroth and nonzero harmonics is in fact inessential in the computation of correlation functions, as performed in this section and in Ref. [46]; however, it is unavoidable for the derivation of a free energy in terms of a local $Q(r)$ matrix that represents only the zeroth harmonics, as has been performed in section III.

V. CLASSICAL SUPERDIFFUSION

We now generalise the discussion to include the possibility of longer-ranged disorder than has been represented so far by the delta-correlated RMF. The question that arises...
from the preceding analysis is whether a new term, of the type represented in the free energy of Eq. (39), may exist and still remain relevant to give new, logarithmic corrections to the conductivity. As has been demonstrated in section [III D], the potential relevancy of such a term is related to the question of whether the current-current correlator remains nonzero in the limit of \( q \rightarrow 0 \). Although such a situation does not arise for the model of a delta-correlated RMF, it is not ruled out for models of longer ranged disorder.

In fact, the problem of diffusion in a field of currents that are correlated in precisely such a way, ie. so that the current-current correlator remains nonzero as \( q \rightarrow 0 \), has already been examined on a classical level [48–50]. These authors considered a model of classical 2D diffusion in a random stationary velocity field, as governed by a classical Fokker-Planck equation

\[
\frac{\partial}{\partial t} + \nabla (v - D_0 \nabla) P(r, t) = 0.
\] (60)

Here \( P(r, t) \) is the distribution function of a randomly walking particle, where \( D_0 \) is the classical diffusion coefficient. The randomness of the velocity \( v \) makes the problem in 2D quite non-trivial. One can imagine different forms of the velocity-velocity correlations and so, different models leading to different types of diffusive behavior were considered. In one of the models (Model II in classification of Ref. [49]) the following correlation was assumed:

\[
\langle v_i(r) v_j(r') \rangle = \gamma_0 K^{ij}(r - r'),
\] (61)

where the function \( K^{ij}(r - r') \) is defined as

\[
K^{ij}(r - r') = \delta^{ij} \delta(r - r') - \frac{1}{2\pi} \nabla_i r \nabla_j r' \ln |r - r'|,
\] (62)

so that in momentum space, it takes the form

\[
K^{ij}(q) = \delta^{ij} - \frac{q_i q_j}{q^2}.
\] (63)

The model of Eqs. (60) and (61) may describe, for example, Brownian motion of a particle in an incompressible liquid with random stationary flow. Although roughly speaking the function \( K^{ij} \) describes short-range correlations of the liquid, more precisely long ranged correlations are present in Eq. (61) due to the incompressibility.

In Refs. [49,50] a Green function of Eq. (60) was introduced and expressed as a functional integral over a vector field \( \phi \) (in Ref. [49] a replica formulation was used). In this way calculation of the diffusion coefficient \( D \) is reduced to a study of \( \phi^4 \) theory. In 2D the corrections to the bare diffusion coefficient are logarithmic, and hence a renormalization group treatment may be employed, up to the two loop approximation in Ref. [49]. The dependence of the diffusion coefficient \( D \) on frequency \( \omega \) was obtained in the form

\[
D(\omega) = D_0 \left[ 1 + g_0 \ln \left( \frac{D_0 k_0^2}{\omega} \right) \right]^{1/2}
\] (64)

where \( k_0 \) is an ultraviolet cutoff and \( g_0 = \gamma_0 / (4\pi D_0^2) \). The dependence \( D(\omega) \) given by Eq. (64) is classified as superdiffusion because the diffusion coefficient grows as the frequency \( \omega \) vanishes.
With this background in mind, we return to the problem of electron motion in the presence of long-ranged disorder. To make closer contact with the diffusion-advection problem, we in fact consider electron motion in the presence of both short range and long range disorder. We assume that at short distances the electron motion is determined by the short range potential and is diffusive. The long range potential, which may be scalar or vector potential or a combination of both, influences the diffusion at large distances and may be considered as a random force.

Remarkably, we find a direct relation of this electron model with the diffusion-advection model, described by Eq. (60), as far as certain terms in the solution for the diffusion coefficient are concerned (see Eq. (64)). In the electron model, we find two different contributions, of opposite sign, in the solution for the diffusion coefficient. The positive part of the contribution corresponds precisely to the term appearing in Eq. (64) and hence corresponds to classical superdiffusion. It derives from a new term in the free energy, which is of the precisely the form written in Eq. (39), although with a finite value of the current-current correlator \( \Pi_0(q) \) as \( q \to 0 \); the term therefore is of the same form as that written originally by Zhang and Arovas [42]. The electron model differs however from the diffusion-advection model in that it is quantum mechanical: hence an additional negative contribution appears in the solution for the diffusion coefficient. This negative contribution is the usual unitary weak localization correction due to quantum interference, and competes with the above classical correction. This competition leads to the interesting possibility of a quantum phase transition at weak disorder, tuned by the relative disorder strengths.

In order to derive a field theory for the electron model we apply the formalism developed in the preceding sections. We proceed by averaging first over the short-range impurities, but not over the long range potential. A local, quartic term appears in the Lagrangian in the usual way, which may be decoupled with a local \( Q(r) \) matrix. We find the Lagrangian

\[
\mathcal{L} = \int \left[ -\frac{1}{2} \text{Str} \ln \left( \frac{1}{2m} \left( p - \frac{e}{c} A(r) - i\tau_3 \nabla_r \right)^2 + V(r) - \epsilon_F + \frac{\omega A}{2} + \frac{i}{2\tau} Q(r) \right) \right] \, dr,
\]

where \( \tau \) is the mean free time for scattering from the short-range impurities (corresponding to \( D_0 \) in Eq. (60)). The vector potential \( A(r) \) and the long range potential \( V(r) \), both of which may be random, are not specified now.

Next we write \( Q(r) = U(r) \Lambda \bar{U}(r) \), cycle the \( U \) and \( \bar{U} \) factors under the supertrace and perform a gradient expansion in \( \bar{U} \nabla U \tau_3 \). To first order in the expansion, we find the term

\[
\delta F = \frac{1}{2m} \int dr \frac{d^2 p}{(2\pi)^2} \text{Str} \left( \bar{U} \nabla U(r) \tau_3 \hat{g}_p(r) \left( p - \frac{e}{c} A \right) \right),
\]

where the matrix function \( \hat{g}_p \) is defined as

\[
\left[ \frac{1}{2m} \left( p - \frac{e}{c} A(r) - i\tau_3 \nabla_r \right)^2 + V(r) - \epsilon_F + \frac{i\Lambda}{2\tau} \right] \hat{g}_p(r) = i.
\]

Note that in the absence of the long range potentials, the term \( \delta F \) would vanish by symmetry on integration over \( p \). We notice further that the term \( \delta F \) may be expressed in terms of
the local classical current density \( J \), which has been defined in Eq. (53). Following standard transformations, \( J \) may be reexpressed in terms of \( \hat{g} \) as

\[
J_i(r) = \frac{e}{2\pi m} \int \frac{d^2 p}{(2\pi)^2} \left( p^i - \frac{e}{c} A^i(r) \right) (\hat{g}^{11}_p(r) - \hat{g}^{22}_p(r))_{\alpha\alpha},
\]

where there is no summation over \( \alpha \), which allows us to rewrite Eq. (66) in the form

\[
\delta F = \frac{\pi e}{2c} \int d r J_i(r) \text{Str}(\Lambda \tau_3 \Phi_i(r)).
\]

(67)

where \( \Phi(r) = \bar{U}(r) \nabla U(r) \). Eq. (67) clarifies immediately the physical meaning of the supermatrix \( \Phi_{\parallel} \), showing that it plays the role of an effective vector potential. If the particle number is conserved, the current is transversal (\( \text{div} J = 0 \)), which means that only the transversal part of \( \Phi \) may enter the final free energy functional.

Derivation of the remaining terms in the free energy continues as usual: the gradient expansion is continued up to terms second-order in \( \Phi \), and first order in \( \omega \). If we include, for the sake of generality, the possibility of a nonzero average component \( (B_0) \) to the magnetic field, we come to the free energy

\[
F[U] = \frac{\pi \nu}{8} \int d r \text{Str}(D[\Phi, \Lambda]^2 + 2D_{xy} \tau_3 \Lambda \Phi_x \Phi_y^{-1} + 2i\omega \Lambda Q)
\]

\[
+ \frac{\pi}{2c} \int d r J_i(r) \text{Str}(\Lambda \tau_3 \Phi_i(r)),
\]

(68)

where \( D = v_F^2 \tau / 2 \) and \( D_{xy} = eB_0 \tau D/mc \) (and we have assumed that the magnetic field \( B_0 \) is classically weak, i.e. \( eB_0 \tau/mc \ll 1 \)). The free energy may then be rewritten in terms of the \( Q \)-matrix, the last term in Eq. (68) by means of a reexpression as a Wess-Zumino term [51,52] (using the identity \( \text{div} J = 0 \):

\[
F[Q] = \frac{\pi \nu}{8} \int d r \text{Str}(D[\nabla Q]^2 - D_{xy} \tau_3 Q[\nabla_x Q, \nabla_y Q] + 2i\omega \Lambda Q)
\]

\[
+ \frac{\pi}{8e} \int d r J \int_0^1 du \text{Str} \left( \tau_3 Q \left[ \frac{\partial Q}{du}, \nabla Q \right] \right)
\]

(69)

The free energy (69) represents the most general free energy for electron diffusion in the presence of an external magnetic field and nonzero local currents.

The relation of the free energy (68) to the classical models of diffusion in a field of random velocities [48–50] is made clear if we consider the corresponding classical diffusion propagator. To derive this propagator, we consider \( Q \) to be close to \( \Lambda \) and expand to quadratic order around \( \Lambda \), via \( Q = \Lambda(1 + 2iP - 2P^2 + \ldots) \). The free energy reduces to

\[
F[P] = \frac{\pi \nu}{2} \int d r \text{Str} \left( P \left[ -D \nabla^2 - i\omega - \frac{1}{e\nu} \Lambda \tau_3 \nabla \right] P \right).
\]

(70)

Comparison of the square brackets in Eq. (70) with the Fokker-Planck equation (60), and use of the classical relation between the current and velocity,
shows that the free energy $F$ coincides with the diffusion-advection model on the classical level. The free energy however contains additional physics to the diffusion-advection model, since it describes also quantum interference phenomena as represented by interaction of the diffusion modes and contained in higher powers of $P$.

We now average over the random potentials. As we want in this section only to make the connection with Refs. [48–50], we do not require a microscopic specification of the long-range disorder correlations, but may simply assume that the current densities $J(r)$ are random and delta-correlated, so that

$$\langle J'(r)J'(r') \rangle = e^2 G_0 K^{ij}(r - r'),$$  \hspace{1cm} (71)

where $K^{ij}(r - r')$ is defined as in Eq. (62). In particular, $K^{ij}(q)$, as given by Eq. (63), remains nonzero in the limit of $q \to 0$ and takes the required transversal form. The term in the free energy, Eq. (68), that couples to the currents becomes

$$\delta F \to -\frac{\pi^2}{6e^2} \int dr dr' \langle J'(r)J'(r') \rangle \text{Str} (\tau_3 \Lambda \Phi^i(r)) \text{Str} (\tau_3 \Lambda \Phi^j(r')) \hspace{1cm} (72)$$

In this way, we arrive at the free energy

$$F[Q] = \frac{\pi \nu}{8} \int \text{Str} [D(\nabla Q(r))^2 - D_{xy} \tau_3 Q[\nabla_x Q, \nabla_y Q](r) + 2i\omega \Lambda Q(r)] dr$$

$$- \frac{\beta}{16\pi} \int dr dr' K^{ij}(r - r') \text{Str} (\tau_3 \Lambda \Phi^i(r)) \text{Str} (\tau_3 \Lambda \Phi^j(r')),$$  \hspace{1cm} (73)

where the dimensionless coefficient $\beta$ is defined by

$$\beta = 2\pi^3 G_0.$$  \hspace{1cm} (74)

An alternative reexpression of the free energy of Eq. (73) is

$$F[Q] = \frac{\pi \nu}{8} \int \text{Str} [D(\nabla Q(r))^2 - D_{xy} \tau_3 Q[\nabla_x Q, \nabla_y Q](r) + 2i\omega \Lambda Q(r)] dr$$

$$- \frac{2\beta}{(32\pi)^2} \int M(r) \ln |r - r'| M(r') dr dr',$$  \hspace{1cm} (75)

where $M(r)$ is defined as in Eq. (40).

So far we have specified the long-range disorder only through the resultant current-current correlations, requiring that the correlator (71) remains nonzero in the limit of $q \to 0$. Although such a formulation has the advantage of generality, it is useful to identify its relation to specific microscopic models. It is clear from the preceding sections that a delta-correlated RMF does not provide the required form of current-current correlations. Instead, the magnetic field correlations need to be of longer range: the Fourier transform of the correlator $\langle B(r)B(r') \rangle$ must diverge as $1/q^2$ as $q \to 0$.

We see that the final term of Eq. (73) is of the same form as that written by Zhang and Arovas [42], although the latter work does not contain a correct microscopic derivation;
specifically, a proper treatment along the lines of Ref. [42], since they deal with short-range magnetic field correlations, would result in a replacement of the function $K^{ij}(q)$ appearing in the free energy of Eq. (73) by $q^2 K^{ij}(q)$. Indeed we may apply the methods of this section to a model of short-range disorder plus a delta-correlated RMF, to find just such a replacement in the free energy of Eq. (73). It is straightforward then to verify that this free energy reproduces precisely the results of Wilke et al. [22] for the memory-effect correction to the conductivity in this model.

In contrast with the final term of the free energy, Eq. (39), for the delta-correlated RMF problem, here the new term does contribute logarithmic corrections to the conductivity. In the next section we sum all such logarithmic corrections by subjecting the free energy of Eq. (75) to an RG analysis.

VI. RENORMALIZATION GROUP ANALYSIS

In this section we subject the free energy of Eqs. (73) and (75) to a renormalisation group analysis. In this way we find that the new term in the free energy (the final term of Eq. (73)) leads to a new contribution in the scaling form of the diffusion coefficient with respect to frequency. The new contribution is positive in sign, and hence competes with the usual negative contribution that represents weak localization. Moreover the new contribution has precisely the same form as that which appears in scaling form in the problem of classical superdiffusion [48–50], according to Eq. (64).

Our approach is entirely analogous to the standard procedure applied in the case of unitary disorder [39,3,40]. The only differences are caused by the presence of the additional term in the free energy (73). The latter term leads to an extra ‘effective charge’, $\beta$, whose flow is coupled to that of the usual charge $t \equiv 8/(\pi \nu D)$. In the standard case, the first order (one-loop) contribution in the conventional unitary $\sigma$-model vanishes and the localizing behavior originates from the two-loop diagrams that are the next order in $t^{-1}$. For the free energy (73), the $\beta$-term contributes to the effective charge $t$ already in the first order and therefore this contribution can be larger than the conventional one. While the calculation of the conventional two-loop diagrams is rather involved and needs a special regularization (a dimensional regularization is usually used), the contribution of the $\beta$-term is much simpler and does not need any such regularization.

Following a standard procedure (see e.g. Ref. [40]) we separate the unitary supermatrix $U(r)$ into slow $\tilde{U}(r)$ and fast $U_0(r)$ parts:

$$U(r) = \tilde{U}(r)U_0(r).$$

(76)

The fast part $U_0(r)$ contains in the momentum representation momenta in the interval $\lambda < k < k_0$, where $k_0$ is the upper cutoff momentum. To guarantee that the rotational symmetry is not violated after this integration we impose an infrared cutoff in the momentum integrals by adding the following term in the free energy

$$F_{\text{reg}} = -\frac{2\lambda^2}{t} \int \text{Str} (\Lambda Q) \, dr,$$

where $\lambda^2 \gg \tilde{\omega} \equiv \omega / D$. 29
new free energy functional. This term does not contribute to the RG flow equations to any perturbative order. Then we integrate over \( P \) and renormalize and the coefficient \( \beta \) keeps its bare value.

Our task is now to integrate \( \exp (-F) \) over \( U_0 \) using perturbative theory and obtain a new free energy functional \( \tilde{F} \) containing second powers of \( \tilde{\Phi}_\alpha (r) \). We parametrize \( U_0 \) by

\[
U_0 = \left( \frac{1 - iP}{1 + iP} \right)^{1/2},
\]

expand the free energy to the required order in \( P \) and average over \( P \) to obtain the required renormalized free energy.

It is important to mention that the bare free energy, which is quadratic in \( P \) and with which the subsequent averaging will be done, originates from \( F^t \). As concerns the contribution from \( F^\beta \), its expansion in \( P \) begins with \( P^4 \) terms. Therefore the new term does not modify the diffusion equation itself but changes interaction between the diffusion modes.

Terms coming to the renormalized functional from \( F^t \) are well known and we simply use the results written in Ref. [40]. It is important that one does not obtain from \( F^t \) any contribution to the renormalized \( \beta \)-term. As concerns \( F^\beta \), we write \( F^\beta = F^\beta_1 + \tilde{F}^\beta \), where

\[
F^\beta_1 = \frac{\beta}{4\pi} \int dr dr' K^{ij} (r - r') \{ \text{Str}(\tau_3 \Lambda P^t (r) \tilde{\Phi}_i (r)) \text{Str}(\tau_3 \Lambda P (r') \tilde{\Phi}_j (r')) \\
+ \text{Str}(\tau_3 \Lambda \tilde{\Phi}_i (r)) \text{Str}(\tau_3 \Lambda \tilde{\Phi}_j (r') + \text{Str}(\tau_3 \Lambda \tilde{\Phi}_i (r)) \text{Str}(\tau_3 \Lambda \nabla_{r'} P (r') P (r')) \}
\]

Then we integrate over \( P \) using the contraction rule [40]

\[
< \text{Str}(PM_1) \text{Str}(PM_2) >_0 = \frac{t}{8(k^2 + \lambda^2)} \text{Str}(M_1 M_2),
\]

and \( < PP > = 0 \), where the supermatrices \( M_1 \) and \( M_2 \) anticommute with \( \Lambda \) and are self-conjugate.

We see that in the first order approximation, there is no contribution to the renormalized \( \beta \)-term coming from \( F^\beta \). So, we conclude that in the one-loop order the \( \beta \)-term is not renormalized and the coefficient \( \beta \) keeps its bare value.
The first order contribution to the diffusion (first) term in the action, Eq. (38), comes from only the first term in $F_\beta^1$ in Eq. (79), while the contributions from the second and the third terms in Eq. (79) vanish. The contribution from the first term in $F_\beta^1$ contains the integral
\[
- \frac{\beta t}{32\pi} \int \text{Str} \left( \Phi_+^i (r) \Phi_+^j (r) \right) dr \int \frac{d^2 k}{(2\pi)^2} \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right) \frac{1}{k^2 + \lambda^2}.
\] (81)

After performing the above integral, the renormalized free energy $\tilde{F}$ can be written in $d = 2 + \varepsilon$ dimensions (for small $\varepsilon$) as
\[
\tilde{F} = \frac{1}{t} \int d\mathbf{r} \left[ \text{Str}(\nabla \tilde{Q})^2 \left\{ 1 + \frac{t^2}{64} \left( \frac{1}{2} - \frac{1}{d} \right) I_d^2 + \frac{\beta t^2}{256\pi} I_2 \right\} + 2i\omega \text{Str}(\Lambda \tilde{Q}) \right]
- \frac{\beta}{16\pi} \int d\mathbf{r} d\mathbf{r}' K^{ij}(\mathbf{r} - \mathbf{r}') \text{Str}(\tau_3 \Lambda \tilde{\Phi}_i(\mathbf{r})) \text{Str}(\tau_3 \Lambda \tilde{\Phi}_j(\mathbf{r}')).
\] (82)

The second term in the figure brackets in Eq. (82) is the conventional two-loop contribution of the unitary $\sigma$-model. To write it properly one may use a dimensional regularization, which is the reason this term is written for $\varepsilon \neq 0$. At the same time, there are no ambiguities in the calculation of the third term in the figure brackets and it is written for $\varepsilon \ll 1$ already in $2D$.

The Gell-Man-Low function for $\tilde{t} = t \frac{16\pi}{32\pi}$ becomes, after continuation to 2 dimensions,\[
\frac{d\tilde{t}}{d\ln \lambda} = \frac{1}{2} \tilde{t}^2 (\beta - 1) + O(\tilde{t}^4).
\] (83)

The solution to the flow equation, Eq. (83), may be found by integrating over $\lambda$ up to the ultraviolet cutoff of $1/l_{tr}$, and we obtain
\[
\tilde{t}(\omega) = \tilde{t}_0 \left[ 1 + \frac{1}{2}(\beta - 1)\tilde{t}_0^2 \ln \left( \frac{1}{\omega \tau_{tr}} \right) \right]^{-1/2},
\] (84)

where $\tilde{t}_0$ is the bare value of $t$ proportional to the classical resistivity. Rewriting Eq. (84) in terms of the diffusion coefficient and reintroducing $G_0$ (see Eq. (71)), we find
\[
D(\omega) = D_0 \left[ 1 + \left\{ \frac{G_0}{4\pi \nu^2 D_0^2} - \frac{1}{8\pi^4 \nu^2 D_0^2} \right\} \ln \left( \frac{1}{\omega \tau_{tr}} \right) \right]^{1/2}.
\] (85)

The classical relation $\mathbf{J} = e\nu \mathbf{v}$ between the current density $\mathbf{J}$ and the velocity of a single particle, $\mathbf{v}$, leads to $G_0 = \nu^2 \gamma_0$. The coefficient of the positive correction to the diffusion coefficient can then be reexpressed as
\[
\frac{G_0}{4\pi \nu^2 D_0^2} = g_0,
\]

where \( g_0 = \frac{\gamma_0}{4\pi D_0^2} \). Comparison with the scaling equation (64) shows that this coefficient coincides precisely with that found previously [49, 50] in the scaling form for the problem of classical superdiffusion.

This correspondence provides immediately an classical physical interpretation for the positive part of the correction to the diffusion coefficient in Eq. (85). For the flow to be correlated according to Eqs. (61) and (71), random stationary flow ‘cycles’ should exist in the system. A tendency towards ballistic motion along the cycles of local current (see Fig.1) leads to the positive, or superdiffusive, correction to the diffusion coefficient.
The difference of the electron model we consider here with the models of classical superdiffusion \cite{48,51} is that for the electron model quantum interference processes are also important and give rise to negative corrections to the diffusion coefficient. This is the negative contribution in the scaling form of Eq.\,(85), and it coincides with the usual contribution known for the unitary ensemble for short-range disorder \cite{39,3}.

The presence of both classical and quantum corrections to the conductivity, which are both logarithmic and therefore in competition with each other, provides the interesting possibility of a quantum phase transition in two dimensions at weak disorder (at $\beta = 1$), tuned by the relative strengths of the short-range and long-range disorder.

In the case when the positive contribution in the scaling form for the diffusion coefficient dominates over the negative contribution ($\beta > 1$), Eq. \,(84) can be used for an arbitrarily low frequencies as soon as $\tilde{t}_0 \ll 1$. This is because the effective charge becomes even smaller when $\omega \to 0$ and one may safely neglect higher order terms in Eq. \,(83). This type of behavior is similar to that known in the symplectic symmetry class that describes short-range, spin-orbit impurities \cite{4}, although the final forms of the scaling functions in the two systems are different. In such cases, the type of the renormalization group equations is known as a ‘zero-charge’ situation. The possibility to use Eq. \,(84) down to zero frequencies means that this represents the complete solution. This contrasts with, for example, the localization behavior for the orthogonal ensemble where the effective charge (resistivity) $\tilde{t}$ grows when decreasing the frequency and the solution of the RG equations loses its applicability as $\tilde{t}$ becomes of the order of unity.

The result for the scaling of the diffusion coefficient with frequency, Eqs. \,(85), may be obtained also perturbatively order-by-order, by calculating the conductivity directly, as in section \ref{sec:scaling}. In such a calculation a correction, $\delta\sigma^{ij}$, appears that is similar to $R_c^{ij}$ of Eq.\,(19): it is proportional to the current-current correlator and has no counterpart in the conventional calculation \cite{50,10} for short-range disorder:

$$\delta\sigma^{ij} = \frac{\pi}{16} \int d(r - r') \text{Str}(k(1 - \tau_3)Q^{12}(r')k(1 - \tau_3)Q^{21}(r))\langle J^i(r)J^j(r') \rangle.$$  \hspace{1cm} (86)
Expanding each $Q$ in the right-hand side of Eq. (86) to first-order in $P$, where $Q = \Lambda(1 + iP)(1-iP)^{-1}$, and averaging over $P$ leads to a correction to the conductivity that corresponds to the positive contribution in the scaling form for the diffusion coefficient of Eq. (85), after a perturbative expansion. The correction to the conductivity corresponding to the negative contribution in Eq. (85) can also be derived diagrammatically in the usual way \cite{56,40}. The equivalence of the direct calculation of the conductivity with the RG approach for the diffusion coefficient follows naturally from the Einstein relation, which must hold for the electron problem. There does not seem to be a connection between the mobility studied in Refs. \cite{48–50} and the conductivity of the electron gas with long-range disorder, and its dependence on the frequency differs from that for the diffusion coefficient.

In summary, in this section we have subjected the free energy (75) to a RG procedure to derive to scaling equation for the diffusion coefficient. As showed in section V, the free energy (75) represents a quantum generalisation of the diffusion-advection problem and applies when the current-current correlator due to long-range disorder remains finite in the limit of $q \rightarrow 0$. The scaling form for the diffusion coefficient contains a positive contribution, which corresponds to classical superdiffusion, and a negative contribution, which is the usual weak localization correction for the unitary ensemble. Since they are of opposite sign, the two contributions may compete and this leads to the possibility of a quantum phase transition in two dimensions at weak disorder, tuned by the relative disorder strengths.

VII. SUMMARY AND DISCUSSION

In this paper, we have studied the 2D electron gas with long range disorder and broken time-reversal symmetry. We have provided a careful derivation of the free energy for this system. We have shown how the the ballistic $\sigma$-model may be derived in a controlled manner for this problem, due to the presence of the small parameter $\tau/\tau_r$, thus providing the first formal derivation of this model. We also derived certain additional terms which are beyond the usual form of the ballistic $\sigma$-model but become important for shorter-range disorder. Integration over nonzero harmonics of the $Q$-matrix leads to a free energy with with an extra term beyond those of the conventional unitary $\sigma$-model. The evaluation of the extra term requires the use of a proper regularisation procedure of the ballistic $\sigma$-model that ensures the preservation of gauge invariance and that has been verified by diagrammatic means.

This demonstrates that, within the scope of our analysis of the delta-correlated RMF model, the extra term in the free energy seems to remain irrelevant in the calculation of corrections to the conductivity that are logarithmic in frequency. As a result we do not find evidence for deviation of the localization behaviour of the delta-correlated random magnetic field model from that of the conventional unitary ensemble. As such our conclusions coincide with those of the original work of Aronov et al., although for considerably more subtle reasons than were originally realised in that work (which remains a partial analysis due to their neglect of the higher harmonics of the $Q$-matrix).

Certain questions for this model remain regarding the role of the massive (a) modes, which describe fluctuations of the eigenvalues of $Q$ and the Cooperons, and which have been neglected in our analysis. For example, the precise value of the single-particle lifetime that
couples to the massive modes (a) remains to be determined, and is likely to remain finite as the infrared cutoff \( \kappa \) is taken to zero. This suggests that an improved saddle-point may be found which does not contain any divergencies in the limit of \( \kappa \to 0 \). A proper treatment of the massive (a) modes would allow the application of the ballistic \( \sigma \)-model down to distances of the order of the single-particle mean free path, which is desirable since the evaluation of the new term in the free energy depends crucially on the the regularization procedure at such short length-scales.

The derivation of the free energy was supplemented by direct perturbative calculations within the field theory formalism. We calculated the conductivity directly and showed the appearance of an additional correction, that is analogous to the new term in the free energy and is higher order than logarithmic. We also calculated the current-current correlation function, thus providing a physical interpretation for the coefficient of the new term in the free energy.

We then generalised the discussion to include the possibility of longer ranged disorder. We derived the free energy for a model containing both short-range disorder and longer-ranged disorder, and showed how the long-range disorder couples to the free energy via a Wess-Zumino term. If the long-range disorder is sufficiently long-ranged that the correlator of local currents remains finite in the limit of \( q \to 0 \), then a new term appears in the free energy which is responsible for new corrections to the conductivity that are logarithmic in frequency. We have demonstrated how this free energy may be described as a quantum generalisation of the diffusion-advection model. By performing a RG analysis of the free energy, we have shown how two competing contributions to the scaling form for the diffusion coefficient appear. The positive contribution corresponds to classical superdiffusion in the presence of long-ranged currents, while the negative contribution is the conventional, weak localization correction for the unitary ensemble. The competition between the classical and quantum contributions to the conductivity leads to the interesting possibility of a quantum phase transition in two dimensions at weak disorder, tuned by the relative disorder strengths.

We remark that a direct diagrammatic calculation of the leading logarithmic correction to the conductivity would be a useful check of the predictions of the field theory for the delta-correlated RMF model. The diagrammatic arguments of Gornyi et al. do not fulfill the purpose of such a check: although their calculation of the current-current correlation function seems valid, their calculation of the conductivity relies on a relation between the conductivity and the current-current correlation function that seems questionable for the delta-correlated RMF model, since it depends on the quantity \( \tau \) that formally vanishes within the SCBA. Instead their arguments would seem to be applicable when the RMF is a weak perturbation to sufficiently strong, short-range disorder, in which case a well-defined \( \tau \) does exist. A direct diagrammatic calculation of the first logarithmic correction to the conductivity, which is not simple even for short-range disorder, therefore remains to be attempted for the delta-correlated RMF model.

We remark also that further analysis of the free energy, with its new term, is deserved. For example, the influence of the new term on the scaling of the conductivity at small conductance, and its interplay with the usual topological term, remains an interesting open question. The general expression for the free energy of a disordered conductor in the presence of equilibrium currents also presents new possibilities for further investigation, possibly in different dimensionalities.
The method of derivation of the ballistic $\sigma$-model in this paper, by which a controlled expansion of the free energy becomes possible through the smallness of the parameter $\tau/\tau_{tr}$, opens a promising avenue for further development of ballistic field-theories. It is possible that progress in this direction may help to resolve the problem of “repetitions” \cite{58} in the ballistic $\sigma$-model for chaotic systems.

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**APPENDIX A: NEGLECT OF A^2 TERM**

In this appendix we justify more carefully the neglect of the $A^2$ term in the original Lagrangian, Eq. (8). In fact this approximation is standard and has been used in previous analytic works on the subject [11,13]. Indeed our approach applies also to the case of long-range impurities (scalar disorder) for which $A^2$-like terms are completely absent.

Even so we investigate here more carefully the influence of the $A^2$ term. In fact its neglect was made only for convenience and is not necessary in the early stages of the calculation: instead, it may be included exactly in the disorder average. In doing so, $L_{int}$ of Eq. (9) is replaced by
Choosing a typical pairing of the $A$ and $\psi$ fields gives an estimate of a typical resulting contribution to the free energy:

$$\delta L_{\text{int}} = \int \bar{\psi}^\alpha(r) \psi^\beta(r') \tau_3 \beta \beta k_{B \beta} \tilde{J}_{r,r'} \bar{\psi}^\beta(r') \psi^\alpha(r) \tau_3 \alpha \alpha,$$

where

$$\tilde{J}_{r,r'} = -\frac{1}{2} \sum_{ij} \nabla^i_{r-r'} [V^{-1}(r-r') - 4i m \delta(r-r') \mathbb{I} \bar{\psi}(r) \psi(r)]_{ij}^{-1} \nabla^j_{r-r'} ,$$

where $\mathbb{I}$ is a unit matrix in $ij$-space. The decoupling of the interaction term, $L_{\text{int}}$, by a $Q$-matrix follows as before, as does the integration over the $\psi$-fields. The Lagrangian of Eq. (10) is now replaced by

$$L = \int \left[ -\frac{1}{2} \text{Str} \ln \left( iH_0(r) \delta(r-r') + \tilde{V}_{r,r'} \tilde{Q}(r,r') \right) \right. \left. + \int d\mathbf{r}'' d\mathbf{r}''' \text{Str} \left( Q(\mathbf{r}''', \mathbf{r}') V^k_{\mathbf{r}''', \mathbf{r}''} V^k_{\mathbf{r}'' \mathbf{r}'''} \nabla^i_{\mathbf{r}'''} \nabla^j_{\mathbf{r}''} Q(\mathbf{r}', \mathbf{r}''') \right) \right] d\mathbf{r} d\mathbf{r}' .$$

(A1)

We see that an additional term has arisen inside the logarithm (second line of Eq. (A1)). This term is second-order in the scattering amplitude, $V^{ij}$, and acts to renormalize the chemical potential. Indeed, due to the presence of the Str operator, we see that, due to the presence of the $\mathbb{S}$ operator, it commutes with the rotation matrices $U$ and $V_n$. Performing an expansion of the logarithm in this term will therefore not give any additional contribution to the free energy to any perturbative order. Instead, the effect of the additional term is comparable to that of the $(a)$-modes, corresponding to fluctuations in $Q^2_p$, and hence may be neglected consistently.

An alternative argument for the neglect of the $A^2$ term may be formulated as follows: we may treat the $A^2$ term in the Lagrangian (8) perturbatively and check that it gives only a parametrically small correction to the previously-derived terms in the free energy.

A leading-order contribution of the $A^2$ term, that is fourth-order in $A$, may be estimated via a third-order cumulant expansion: this gives rise to the contribution to the Lagrangian of

$$\delta L_1 \sim \int d\mathbf{r} d\mathbf{r}' d\mathbf{r}'' \left\langle \left( \bar{\psi}(\mathbf{r}) \frac{e}{m c} A(\mathbf{r}) \nabla r \tau_3 \psi(\mathbf{r}) \right) \left( \bar{\psi}(\mathbf{r}') \frac{e}{m c} A(\mathbf{r}') \nabla r \tau_3 \psi(\mathbf{r}') \right) \times \left( \bar{\psi}(\mathbf{r}'') \frac{e^2}{m c^2} A^2(\mathbf{r}'') \psi(\mathbf{r}'') \right) \right\rangle$$

Choosing a typical pairing of the $A$ and $\psi$ fields gives an estimate of a typical resulting contribution to the free energy:

$$\delta F_1 \sim \frac{1}{\epsilon_F} \int d\mathbf{r} \sum_{n_1, n_2, n_3} V_{pF}^{ik}(n_2-n_1) V_{pF}^{kj}(n_3-n_1) n_1^i n_2^j \text{Str}(Q_{n_1}(\mathbf{r}) Q_{n_2}(\mathbf{r}) Q_{n_3}(\mathbf{r})).$$
In fact due to the combination of three $Q$’s in the structure of $\delta F_1$, this terms does not contribute at all to the free energy of the form of Eq. (27), upon expanding in $P_m$. In addition we note that it is second-order in the scattering amplitude $V^{ij}$: for this reason it may be checked that in any case it appears with a small factor $(\epsilon_F \tau_{tr})^{-1} \ll 1$ with respect to the usual collision term (see e.g. Eq. (21)).

The other leading-order contribution, that is fourth-order in $A$, appears from the second-order cumulant:

$$\delta \mathcal{L}_2 \sim \int dr dr' \left\langle \left( \frac{\psi(r)}{mc^2} A^2(r) \psi(r) \right) \left( \frac{\psi(r')}{mc^2} A^2(r') \psi(r') \right) \right\rangle.$$

Choosing a typical pairing as before gives an estimate of a typical resulting contribution to the free energy as

$$\delta F_2 \sim \frac{1}{\epsilon_F} \int dr \sum_{n_1, n_2} \tilde{w}_{n_1-n_2} \text{Str}(Q_{n_1}(r)Q_{n_2}(r)),
$$

$$\tilde{w}_{n_1-n_2} = p_F^2 \int d^2 p_3 V^{ik}_{p_3} V^{ki}_{p_3 + p_F(n_1-n_2)}.$$

We notice that the term $\delta F_2$ is again second-order in the scattering amplitude $V^{ij}$, as compared to the usual collision term in the ballistic $\sigma$-model (Eq. (21)) which is only first-order in $V^{ij}$. As a result it is straightforward to check that the term $\delta F_2$ produces a renormalization of the collision term by a parametrically small factor $(\epsilon_F \tau_{tr})^{-1} \ll 1$. Higher-order cumulants of the $A^2$ term in the original Lagrangian will clearly produce only smaller renormalizations of the collision term corresponding to higher powers of this small parameter. In this way we conclude that it is justified to neglect the $A^2$ term as claimed in the main text.