Singularities of optimal time affine control systems: the limit case

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Abstract

We study the singularities for minimum time control-affine problems in 4D with 2D controls. After regularization, the problem boils down to the study of a bifurcation around some nilpotent equilibrium in the singular locus.

Introduction

Control-affine systems generalize sub-Riemannian geometry by adding a drift, and arise naturally from controlled mechanical systems. We handle the case of minimizing the final time, for an affine control system defined on a connected 4-dimensional manifold $M$ under a generic assumption given below. The control $u$ is in dimension 2 (double input case), and contained in the unitary euclidean ball, $B$:

$$\dot{x} = F_0(x) + u_1F_1(x) + u_2F_2(x), \ x \in M, u \in B, F_i \in \Gamma(TM).$$

In this case, the research of optimal trajectories leads, according to Pontrjagin’s Maximum Principle, to study a singular Hamiltonian system defined on the cotangent bundle of the manifold $M$. We are interested in the local behavior of the flow of this Hamiltonian system, called extremal flow, around its singularities. The singular locus, also called the switching set, is the codimension submanifold where a switch - a discontinuity of the optimal control - is susceptible to occur. We compare this case with single-input control systems where singularities are of codimension one, i.e., when the control is scalar: $\dot{x} = F_0(x) + uF_1(x)$. They have been extensively studied, and many things are known regarding their singularities, see for instance the monograph [3]. In the present paper, we give a precise description of the behavior of the extremal flow around the singular locus. We answer two important questions which remained open:
- Is there trajectories crossing each point of the singular locus?
- What is the regularity of the lifted flow in that case?

It is known since [6] that the singular locus can be partitioned into three subsets, giving three different configurations for the flow. The first two cases where handle in [2] and [4]. The last one, detailed below, was more difficult to handle, being the frontier of a bifurcation phenomenon, and a higher order analysis of the dynamics was necessary.
Those questions have simpler answers when one consider the problem with the control set $U$ being a polyhedron (which makes no difference in the single input case), and have been treated for the minimum time case in [10, 11]. We attempt to give a unify point of view for those three cases, through a formulation with a parameter.

1 Setting

Let $M$ be a 4-dimensional manifold, $x_0, x_f \in M$ and consider the following time optimal control system:

$$
\begin{align*}
\dot{z}(t) &= F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad t \in [0, t_f], \quad u \in U \\
x(0) &= x_0 \\
x(t_f) &= x_f \\
t_f &\to \min.
\end{align*}
$$

(1)

where the $F_i$’s are smooth vector fields. Set $F_{ij} := [F_i, F_j]$, and $H_{ij} = \{H_i, H_j\}$. We make the following generic assumption on the distribution $\mathcal{D} = \{F_0, F_1, F_2\}$.

$$
\det(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) \neq 0, \text{ for almost all } x \in M.
$$

(A)

This is slightly stronger than the assumption that the distribution $\mathcal{D}$ is of step 2, but it is natural in the applications to mechanical systems, for instance. By Pontrjagin’s Maximum Principle, optimal trajectories are the projection of integral curves of the maximized Hamiltonian system defined on the cotangent bundle of $M$ by

$$
H_{\text{max}}(z) := H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)}, \quad z \in T^*M
$$

where we have denoted $H_i(z) := \langle p, F_i(x) \rangle$ in canonical coordinate $(x, p) \in T^*_xM$. See [4] or [1] for more details about Pontrjagin’s Maximum Principle. Besides, it implies the feedback control

$$
u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2)
$$

whenever $(H_1, H_2) \neq (0, 0)$. An integral curve of $H_{\text{max}}$ is called an extremal. The extremal system is smooth outside the singular locus, or switching surface, defined by

$$
\Sigma := \{H_1 = H_2 = 0\}.
$$

Definition 1 (Bang and bang-bang and singular extremals). An extremal $z(t)$ is said to be bang if $(H_1, H_2)(z(t)) \neq (0, 0)$ for all $t$. It is bang-bang if it is a concatenation of bang arcs. We say it is singular if it is contained in the singular locus.

For single input systems, if one consider bounded controls, taking values in $[-1, 1]$, then the maximized Hamiltonian is

$$
H(x, p) = H_0(x, p) + |H_1(x, p)|
$$
with as before $H_i(x,p) = \langle p, F_i(x) \rangle$, $i = 0, 1$, and $u = \text{sign}(H_1)$ when $H_1$ is non-zero. The singular locus for this problem is $\{H_1 = 0\}$. The singular controls can then be easily calculated by differentiating the relation $H_1(z(t)) = 0$. For the so-called order one singular extremals, we have

**Proposition 1.** The singular flow for single input systems is given by the Hamiltonian

$$H^* = H_0 - \frac{H_{01}}{H_{12}} H_1,$$

with the singular control given by $u^* = \frac{\langle \dot{H}_1, H_0 \rangle}{\langle H_1, H_0 \rangle}$. Discontinuities of the control $u$ along an extremal are called switchings, a time $t$ at which a switch occurs is called switching time, and $z(t)$, a switching point. Bang extremals are the one that do not cross $\Sigma$. Now we tackle the double-input system. A singular minimum time extremal is such that $H_{12}(z(t)) \neq 0$, see remark 1 below. One can also see [6].

**Proposition 2.** There exists a singular flow inside $\Sigma$, on which we have the control feedback: $u_0 = \frac{1}{H_{12}}(-H_{02}, H_{01})$, and the singular flow is smooth. It is solution of the Hamiltonian system given by $\dot{H} = H_0 - \frac{H_{02}}{H_{12}} H_1 + \frac{H_{01}}{H_{12}} H_2$.

**Proof.** The proposition is obtained by differentiating the identically zero switching function $(H_1, H_2)(z(t))$ with respect to the time.

From [6] and [4], we know $\Sigma$ is partitioned into three subsets, leading to three very different local dynamics in their neighborhoods, namely (we use the notation $H_{ij} = \{H_i, H_j\}$)

- $\Sigma_- = \{H_{12}(z)^2 < H_{02}(z)^2 + H_{01}(z)^2\}$
- $\Sigma_+ = \{H_{12}(z)^2 > H_{02}(z)^2 + H_{01}(z)^2\}$
- $\Sigma_0 = \{H_{12}(z)^2 = H_{02}(z)^2 + H_{01}(z)^2\}$.

The behavior of the flow in a neighborhood of $\Sigma_0$ remains open, as the case $\Sigma_-$ and $\Sigma_+$ were settled in [4, 2], but we attempt to provide in the next section a unification of the different viewpoints.

**Remark 1.** Note that, since along a Pontrjagin extremal, the adjoint state $p$ cannot vanish, (A) is equivalent to

$$H_1^2 + H_2^2 + H_{01}^2 + H_{02}^2 > 0.$$
2 Formulation with a parameter

In this section we introduce rather artificially a parameter in the previous dynamical system in order to unify the viewpoints. Thanks to (A), one can make the change of coordinates:

\[ z = (x, p) \in T^* M \mapsto (x, H_1, H_2, H_{01}, H_{02}) \in M \times \mathbb{R}^4. \]

Then use a polar blow up by setting \((H_1, H_2) = \rho(\cos \theta, \sin \theta)\) and \((H_{01}, H_{02}) = r(\cos \phi, \sin \phi)\). The dynamics boils down to the system:

\[
\begin{cases}
\dot{\rho} = r \cos(\theta - \phi) \\
\dot{\theta} = \frac{1}{\rho}(H_{12} - r \sin(\theta - \phi)) \\
\xi = h(\rho, \theta, \xi)
\end{cases}
\]

where \(\xi = (x, r, \phi)\) and \(h\) is a smooth function defined on an open set \(O\) of \(\mathbb{R} \times \mathbb{R} \times D\), \(D\) being a compact domain of \(\mathbb{R}^6\); \(h\) has values in \(\mathbb{R}^6\). We set \(\psi = \theta - \phi\), and rescale the time according to \(dt_1 = rd\bar{t}\): by remark 1, \(r\) is never 0 in a neighborhood of \(\Sigma\), meaning this defines a diffeomorphism and new dynamics is conjugate the one of system 2. This boils down to study a general system with the following structure (the derivation with respect to the time \(t_1\) still being noted \(\ast\ast\)):

\[
\begin{cases}
\dot{\rho} = \cos \psi \\
\dot{\psi} = \frac{1}{\rho}(g(\rho, \psi, \xi) - \sin \psi) \\
\dot{\xi} = \hat{h}(\rho, \psi, \xi)
\end{cases}
\]

where \(g, \hat{h}\) are smooth functions defined on an open set \(O\) of \(\mathbb{R} \times \mathbb{R} \times D\), \(D\) being a compact domain of \(\mathbb{R}^k\), for any \(k \geq 1\) (in our problem \(k = 6\)); \(g\) depends smoothly on \(\rho(\cos \theta, \sin \theta)\). One can set \(s = \psi - \pi/2\). By a small abuse of the notations, we still note \(g(\rho, s, \xi) = g(\rho, s + \pi/2, \xi)\), and we have \(g(\rho, s, \xi) = a(\xi) + O(\rho)\) near \(\rho = 0\).

The three cases \(\Sigma_+, \Sigma_-\) and \(\Sigma_0\) correspond to the values \(g(\bar{z}) > 1, g(\bar{z}) < 1\) and \(g(\bar{z}) = 1\), with \(\bar{z} = (0, \bar{s}, \bar{\xi}) \in \Sigma\), see [4] for more details. We set \(a(\xi) = 1 + \alpha + a_0(\xi)\), with \(a_0(\xi) = 0\). So that the bifurcation parameter giving the three cases is \(\alpha = a(\xi) - 1\) and

\[ g(\rho, s, \xi) = 1 + \alpha + a_0(\xi) + O(\rho). \]

Then (3) in the new time becomes (with slight abuse of notation):

\[
\begin{cases}
\dot{\rho} = -\sin s \\
\dot{s} = \frac{1}{\rho}(1 + \alpha + a_0(\xi) - \cos s + O(\rho)) = \frac{G_0(\rho, s, \xi)}{\rho} \\
\dot{\xi} = \hat{h}(\rho, s, \xi)
\end{cases}
\]

**Remark 2.** We actually have \(a(x, H_{01}, H_{02}) = \frac{H_{12}(x, 0, 0, H_{01}, H_{02})}{\sqrt{H_{01}^2 + H_{02}^2}}\).
2.1 The case $\Sigma_-$. 

In the system above, $\bar{z} \in \Sigma_-$ if and only if $\alpha < 0$. That case was settled in [4] by theorem 1 recalled below:

**Theorem 1.** In a neighborhood $O_\bar{z}$ with $\bar{z} \in \Sigma_-$, existence and uniqueness hold, all extremal are bang-bang, with at most one switch. The extremal flow $z : (t, z_0) \in [0, t_f] \times O_\bar{z} \mapsto z(t, z_0) \in M$ is piecewise smooth. More precisely, $O_\bar{z}$ can be stratified as follows:

$$O_\bar{z} = S_0 \cup S^s \cup \Sigma$$

where $S^s$ is the codimension-one submanifold of initial conditions leading to the switching surface, $S_0 = O_\bar{z} \setminus (S^s \cup \Sigma)$. Both are stable by the flow, which is smooth on $[0, t_f] \times S_0$, and on $[0, t_f] \times S^s \setminus \Delta$ where $\Delta = \{(t(0), z_0), z_0 \in S^s\}$, and $t(0)$ is the switching time of the extremal initializing at $z_0$, and continuous on $O_\bar{z}$.

In [4], the authors also studied the regular-singular transition between the strata, and exhibited log-type singularities.

**Remark 3.** By proposition 2, there is no admissible singular flow contained in $\Sigma_-$, otherwise $\|u_s\|^2 = H^2_0 + H^2_1 > 1$ which violates our constraint on $u$.

2.2 The case $\Sigma_+$. 

This corresponds to $\alpha > 0$. We prove the following, see also [2], theorem 3.5:

**Proposition 3.** In a neighborhood of a point $\bar{z}$ in $\Sigma_+$, there is no switch, and the extremal flow is smooth, i.e., $\Sigma_+$ is never crossed. In other words, $\rho$ does not vanish in (4).

**Proof.** By the analysis above, this boils down to prove that, if $\alpha > 0$, along an extremal $z$, $\rho$ never vanishes in a (relatively compact) neighborhood $\bar{O}$ of $(0, 0, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k$.

Set $f$ such that $\rho \cdot s = 1 + \alpha + f(\rho, s, \xi) - \cos s := \Theta$ in such a neighborhood, the differential of $f$ is bounded by below by a negative constant $-a < 0$.

$$\frac{d}{dt}(\rho \Theta) = \dot{\rho}(1 + \alpha - \cos s + f(z)) + \rho(\sin s \dot{s} + df(z) \cdot \dot{z}) = \rho df(z) \cdot Z(z)$$

We get $\frac{d}{dt}(\rho \Theta) > -a \rho = \rho \Theta(-a/\Theta)$. Eventually, since $\alpha > 0$ and $u(0, 0, 0) = 0$, if $\bar{O}$ is small enough, there exists two positive constant $K$, $k$ with $K > \Theta > k > 0$. So that, along an extremal

$$\frac{d}{dt}(\rho \Theta) > -\frac{a}{k} \rho \Theta.$$

By integration on an arbitrary time interval $[0, t]$, we end up with

$$\rho(t) > \frac{\rho_0 \Theta_0}{K} e^{-\frac{a}{k} t},$$

and the proposition follows. \qed

Despite the absence of switch, there exists a singular flow inside $\Sigma_+$, however, singular extremal lying in $\Sigma_+$ cannot be optimal by the Goh condition, [6].
2.3 The bifurcation $\alpha = 0$: case $\Sigma_0$.

This is the main topic of this paper. In this case, the two equilibria considered in $\Sigma$ merge, and we obtain one nilpotent equilibrium that needs desingularization. Nevertheless, under the generic condition

$$
\frac{\partial H_{12}}{\partial x}(F_0(x) - \sin \phi F_1(x) + \cos \phi F_2(x)) + \frac{\partial H_{12}}{\partial H_{01}}(H_{001}(\bar{z}) + H_{101}(\bar{z})) + \frac{\partial H_{12}}{\partial H_{02}}(H_{002}(\bar{z}) + H_{102}(\bar{z})) \neq 0
$$

where $\bar{\phi} = \arg(H_{01}, H_{02})(\bar{z})$, we will prove

**Theorem 2.** For generic systems (1), meaning, if assumption (A) and (5) holds: Let $\bar{z}$ be in $\Sigma_0$ either: there exists a unique trajectory passing through $\bar{z}$, or there exist unique trajectory going out of $\Sigma_0$ at $\bar{z}$.

This result contradicts the last part of theorem 3.5, in [2], a counter example in a particular case was given in [5] (nilpotent case). The next figure is a scheme of the whole behavior.

**Remark 4.** In the first case, this trajectory can be connected to the singular flow in $\Sigma_0$.

The regularity of the extremal flow is of theoretical and numerical importance, and we have:

**Theorem 3.** In a neighborhood $O_{\bar{z}}$ of a point $\bar{z} \in \Sigma_0$, the flow is well defined, and continuous.

In the process we also obtain the jumps occurring on the extremal control, a switching on the control is called a $\pi$-singularity if it is an instant rotation of angle $\pi$, see [4].

**Proposition 4.** Consider the extremal $z(t)$ entering the singular locus in $z(t) = \bar{z} \in \Sigma_0$,

- If $H_{12}(\bar{z}) = r(\bar{z})$, the extremal control is continuous,
- If $H_{12} = -r(\bar{z})$, the extremal control has a $\pi$-singularity at time $\bar{t}$.

2.3.1 Proof of theorem 2

To that end, we give a precise description of the behavior of the flow around such an equilibrium. Make the following change of time $dt_1 = \rho dt_2$ to regularize the vector field $Z$, and denote $' = \frac{d}{dt_2}$ to get ($\alpha = 0$):

$$
\begin{cases}
\rho' = -\rho \sin s \\
s' = 1 + a_0(\xi) - \cos s + O(\rho) = G_0(\rho, s, \xi) \\
\xi' = \rho \tilde{h}(\rho, s, \xi)
\end{cases}
$$
Figure 1: The stable and unstable manifold of $\Sigma_-$ merging on $\Sigma_0$. 

$\Sigma = \{\rho = 0\}$
Since $\xi$ is constant when $\rho = 0$, for the sake of clarity, we will work in a neighborhood of $\tilde{\xi} = 0$, keeping in mind that we don’t lose generality (and the results holds for any point). Assumption (5) is equivalent to

$$da(0).\tilde{h}(0,0,0) \neq 0 \quad (7)$$

Then, we can order the coordinates of $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$ such that $\frac{\partial a}{\partial \xi_i}(0) \neq 0$. This implies $\Gamma = \{G_0(0,s,\xi) = 0\}$ is a dimension $k$ manifold around $(s,\xi) = (0,0) \in S^1 \times \mathbb{R}^k$. We can then chose coordinates $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_k) = \Phi(\xi)$ such that $\tilde{\xi}_1 = a_0(\xi)$ meaning, $\Gamma = \{\xi_1 + 1 - \cos s = 0\}$. Then, set $\zeta = \tilde{\xi}_1$ to simplify the notations. We obtain

$$\left\{ \begin{array}{l}
\rho' = -\rho s + O(\rho s^3) \\
\zeta' = c\rho + \rho O(\rho + |s| + |\tilde{\xi}|) \\
s' = \zeta + s^2/2 + O(\rho) + O(|s|^4)
\end{array} \right. \quad (8)$$

We do not write the dynamics of the other components of $\tilde{\xi}$. They do not influence the dynamics of $(\rho, s)$ as we explain below in the paragraph "Back to the original system". Actually, as we will exhibit below, only the first order terms in the derivative of $\zeta$ are relevant for the local dynamics around $0$. In the equation (8), $c = \tilde{h}(0,0,0)$, so that assumption (7) prevent it from being $0$. It will be clear from what follows that the terms of higher order are useless for the local analysis. On $\{\rho = 0\}$, the field has two lines of zero with a quadratic contact with $\{\zeta = 0\}$. Futhermore the circles $\{\rho = \text{cst}\}$ are tangent to the vector field in $\rho = 0$. Those lines are normally hyperbolic except at $(\rho, s) = (0,0)$, which is a nilpotent equilibrium.

**Blow up.** To study the nilpotent equilibrium $(\rho, s, \zeta) = (0,0,0)$, we will use a specific blow-up process, called quasi homogeneous blow-up, see [7] chap. 1:

$$\left\{ \begin{array}{l}
\rho = u^3 \tilde{\rho} \\
s = u \tilde{s} \\
\zeta = u^2 \tilde{\zeta}
\end{array} \right. \quad \text{with} \quad (\tilde{\rho}, \tilde{s}, \tilde{\zeta}) \in S^2_+ \quad \text{the hemisphere} \quad \rho \geq 0, 
\text{u} \in \mathbb{R}_+. \quad \text{We will study the dynamics in the two following charts given a vector field smoothly equivalent to the global one we could obtain on } \mathbb{R} \times S^2_+:

(i) For the interior of $S^2_+$, $\tilde{\rho} = 1$, $(\tilde{s}, \tilde{\zeta})$ in a disc $D^2$, $u \geq 0$ in a neighborhood of the critical locus of the blow up $u = 0$.

(ii) For the the boundary of $S^2_+$, $(\tilde{s}, \tilde{\zeta}) \in S^1$, in a neighborhood of $(\tilde{\rho}, u) = (0,0)$.
The charts (i). Let us write the dynamics in the blown up coordinates \( \varphi(\rho, s, \zeta) = (u^3, us, u^2\zeta) \). The blown up vector field \( \tilde{X} = \frac{1}{\rho} \varphi_{\rho} X \) writes

\[
\tilde{X} : \begin{cases}
    u' = -\frac{1}{3} us + O(u^2) \\
s' = \frac{5}{6} s^2 + \tilde{c} + O(u) \\
\zeta' = \frac{2}{3} \tilde{s}\zeta + c + O(u).
\end{cases}
\]

(9)

Thus, there is a unique equilibrium depending on \( \omega \), which is solution of

\[
\begin{cases}
    u = 0 \\
    \zeta + \frac{5}{6} s^2 = 0 \\
\frac{2}{3} \tilde{s}\zeta + c = 0
\end{cases}
\]

(10)
i.e., \( m_0 = (0, \tilde{s}_0, \tilde{c}_0) = (0, \text{sign}(c)(\frac{9}{4}|c|)^{1/3}, -\text{sign}(c)(\frac{9}{4}|c|)^{2/3}) \). The Jacobian matrix of \( \tilde{X} \) at \( m_0 \) is

\[
\begin{pmatrix}
    -\frac{1}{3} \tilde{s}_0 & 0 & 0 \\
    -\frac{5}{6} \tilde{s}_0 & 1 & \frac{2}{3} \tilde{s}_0 \\
    \frac{2}{3} \tilde{s}_0 & -\frac{2}{3} \tilde{s}_0 & \frac{2}{3} \tilde{s}_0
\end{pmatrix}
\]

giving the eigenvalue \( -\frac{1}{3} \tilde{s}_0 \) in the direction of \( u \). On \( \{u = 0\} \) we get the two conjugate eigenvalues \( \tilde{s}_0(\frac{7}{4} \pm \frac{\sqrt{7}}{4} i) \). Thus \( m_0 \) is a hyperbolic equilibrium point, if \( c > 0 \) (implying \( \tilde{s}_0 > 0 \)), it has one dimension stable manifold transverse to \( S^2_+ \), and a two dimensional unstable one, contained in \( S^2_+ \). If \( c < 0 \), the situation is symmetric.

The chart (ii). Along \( \partial S^2_+ \) we set \( \bar{\zeta} = \cos \omega \) and proceed to the blow up \( \rho = u^3 \bar{\rho}, \ s = u \sin \omega, \ \zeta = u^2 \cos \omega \). The pulled-back dynamics is

\[
Y : \begin{cases}
    u' = \frac{u}{1+\cos^2 \omega}(\sin \omega(\cos \omega + \frac{1}{2} \sin^2 \omega) + c\bar{\rho} \cos \omega) + O(u^2) \\
\omega' = \frac{\omega}{1+\cos^2 \omega}(\cos \omega(2 \cos \omega + \sin^2 \omega) - c\bar{\rho} \sin \omega) + O(\bar{\rho} u) \\
\bar{\rho}' = -\frac{\bar{\rho}}{1+\cos^2 \omega}(\sin \omega(1 + \cos^2 \omega + \cos \omega + 1/2 \sin^2 \omega) + c\bar{\rho} \cos \omega) + O(\bar{\rho} u)
\end{cases}
\]

(11)

and is equivalent to

\[
\tilde{Y} : \begin{cases}
    u' = u(\sin \omega(\cos \omega + \frac{1}{2} \sin^2 \omega) + c\bar{\rho} \cos \omega) + O(u^2) \\
\omega' = \cos(2 \cos \omega + \sin^2 \omega) - c\bar{\rho} \sin \omega + O(\bar{\rho} u) \\
\bar{\rho}' = -\bar{\rho}(\sin(1 + \cos^2 \omega + \cos \omega + 1/2 \sin^2 \omega) + c\bar{\rho} \cos \omega) + O(\bar{\rho} u)
\end{cases}
\]

(12)

In restriction to \( \{\bar{\rho} = u = 0\} \), we obtain 4 equilibrium points, namely, the solutions of \( \cos \omega(\sin^2 \omega + 2 \cos \omega) = 0 \). In addition to the trivial \( \pm \pi/2 \), we end up with \( \cos \omega = 1 - \sqrt{2} \), this last equation gives two solutions \( \omega_0 \in ]\pi/2, \pi[ \) and \( -\omega_0 \). All this zeros are simple (in the direction of \( \omega \)) so the dynamics on \( \partial S^2_+ \) can be deduced by the sign of \( \cos \omega(2 \cos \omega + \sin^2 \omega) - c\bar{\rho} \sin \omega + O(\bar{\rho} u) \) on \( \{\bar{\rho} = \omega = u = 0\} \), which is positive. Actually,
from $\omega_0$ and $-\omega_0$ we get two lines of zero in the plane $\{\bar{\rho} = 0\}$, which are the blow up of the parabola $\zeta = -s^2/2$ (corresponding $\Gamma$).

Let us write the Jacobian matrix of $\bar{Y}$ in the plane $\bar{\rho} = 0$:

$$J = \begin{pmatrix} U(\omega) & * & * \\ 0 & \Omega(\omega) & * \\ 0 & 0 & R(\omega) \end{pmatrix}$$

with $U(\omega) = \frac{\sin \omega}{\hat{\tau}}(2 \cos \omega + \sin^2 \omega)$, $\Omega(\omega) = -\sin \omega(2 \cos \omega + \sin^2 \omega - 2 \cos^2 \omega + 2 \cos \omega)$, and $R(\omega) = -\sin \omega(\cos \omega + \frac{3}{2} \sin^2 \omega + 2 \cos^2 \omega)$. We still need two informations:

- the eigenvalues of $\pm \pi/2$ in the radial direction, given by $U(\pm \pi/2) = \mp 1$.
- the eigenvalues of the 4 equilibria in the direction of $\bar{\rho}$, given by $R(\pm \pi/2) = \mp \frac{3}{2}$ and $R(\omega_0) < 0$, $R(-\omega_0) > 0$. Now we have a clear description of the phase portrait in a neighborhood of $\partial S^2_+ \cap \bar{\rho}$ in Figure 2. $\pm \pi/2$ are hyperbolic equilibria and $\pm \omega_0$ are hyperbolic in restriction to $S^2_+$ (but not in dimension 3). The dynamics of (8) is also stable by perturbation by higher order terms.
Global dynamics. We restrict ourselves to the case $c > 0$, the case $c < 0$ being symmetric. We are now going to glue the studies in both charts to obtain the phase portrait on a whole neighborhood of the hemisphere. The main tool in that regard will be the following theorem from Poincaré and Bendixson, [8].

**Theorem 4** (Poincaré-Bendixson). Let $X$ be a vector field in the plane, any maximal solution of $\dot{x} = X(x)$ contained in a compact set, is either converging to an equilibrium point, a limit cycle, or a graphic, i.e., a close invariant curve union of a finite number of orbits connecting equilibria.

The equilibria of the flow restricted to $S^2_+$ (i.e., $u = 0$) are as followed: $\pi/2 \in S^1 \cong \partial S^2_+$ is a stable node, likewise, $-\pi/2$ is an unstable node. The equilibrium $m_0$ is an unstable focus. $\omega_0$ and $-\omega_0$ are saddles: the stable manifold of $\omega_0$ (separatrix) and the unstable manifold of $-\omega_0$ are transverse to $S^2_+$ which contains their other invariant manifolds. Its unidimensional unstable manifold is on $\partial S^2_+$. Besides, for $\omega_0$, the opposite happens: It has a one dimensional stable manifold, and its unstable manifold is along $\partial S^2_+$. Now we will prove that $\tilde{X}$ does not have any periodic orbits. This, according to Poincaré-Bendixson, will allow us to link the trajectory coming from unstable directions to the stable manifolds belonging to other singular points in $S^2_+$.

**Lemma 1.** $\tilde{X}$ does not have a periodic orbit on $S^2_+$.

**Proof.** Between the charts $(i)$ and $(ii)$, we have the following change of coordinates:

$$
\begin{align*}
\bar{s} &= \sin \omega \bar{p}^2 - \bar{q}^2, \\
\bar{\zeta} &= \cos \omega \bar{p}^2 - \bar{q}^2.
\end{align*}
$$

We can now define the two orthogonal axis $(O\bar{\zeta})$ and $(O\bar{s})$ in $S^2_+$, $\partial S^2_+$ included. In the chart $(ii)$, $(O\bar{\zeta})$ is going from $\omega = \pi$ to $\omega = 0$. Consider the convex domain $A$, such that $\partial A = (O\bar{s})_+ \cup \pi/2, \pi \cup (O\bar{\zeta})_-$. Then $m_0 \in \text{Int} A$ is the only equilibrium of $\tilde{X}$ in $S = \text{Int} S^2_+$. The field $\tilde{X}$ is positively collinear to $(0\bar{\zeta})_+$ in $(0, 0)$, and transverse to those axis everywhere else. Thus, we can smooth the boundary of $A$ corresponding to the part $(O\bar{\zeta})_- \cup (O\bar{s})_+$ by a curve $\alpha$ in order to make $\tilde{X}$ transverse to $\partial A$. See figure 3.

Denote $\tilde{A}$ the part of $S^2_+$ such that $\tilde{A} \subset A$ and $\partial \tilde{A} = ]\pi/2, \pi \cup \alpha$. Now $X$ is transverse to $\tilde{A}$ and pointing outside $\tilde{A}$. In $\tilde{A}$, we have div$(X) = 2\bar{s} > 0$. Now assume $\gamma$ is a periodic orbit of $\tilde{X}$. By Jordan’s theorem, $\gamma$ is the boundary of a compact set $D \subset S^2_+$, diffeomorphic to a disk. The result is then a consequence of the Poincaré-Hopf formula:

**Theorem 5** (Poincaré-Hopf). Let $M$ be a compact manifold, and $X$ a vector field that has isolated zeros on $M$. Then $\sum_{i=1}^m \text{Index}(x_i) = \chi(M)$, where the $x_i$ are all the zeros of $X$ in $M$, and $\chi$ denotes the Euler characteristic.

$D$ being contractile, $\chi(D) = 1$, hence $D$ contains at least one equilibrium point, and since $m_0$ is the only one in $S$, $m_0 \in D$. As a result, either $\gamma$ lies in $A$ or intersects $\alpha$. Let us consider the first alternative: $\gamma \subset A$. We have

$$
0 < \int_D \text{div}(\tilde{X})d\bar{\zeta} \wedge d\bar{s} = \int_D d(\iota_{\tilde{X}}(d\bar{\zeta} \wedge d\bar{s})) = \int_{\gamma} \iota_{\tilde{X}}(d\bar{\zeta} \wedge d\bar{s}) = 0
$$

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by Stokes formula, which excludes that case. Now, note that all intersection points between $\gamma$ and $\alpha$ are transverse, since $\dot{X}$ is transverse to $\alpha$: Thus, there is no tangency, and $\gamma$ intersects $\alpha$ twice. But this is also excluded because $\dot{X}$ being transverse, it is only pointing outside $\bar{A}$.

\[ \square \]

By the Poincaré-Bendixson theorem above, since there is no periodic orbits, in $\text{Int}\bar{A}$ every trajectory converges to $m_0$ when the time tends to $-\infty$. $\omega_0 \in \partial\bar{A}$ has a stable manifold of dimension one, and this stable manifold lies inside $\text{Int}\bar{A}$ (at least close to $\omega_0$). This implies that the stable manifold from $\omega_0$ converges to $m_0$ in negative infinite time. Apart from the equilibrium $\pi/2$, it is the only stable direction in $S$. That means all the other trajectories converge to the stable node (restricted to $S_+^2$) $\pi/2$, leading to the phase portrait of figure 4. As shown in [?], the stable stratum of theorem 1, $S^s$, is the disjoint union of on dimensional stable manifolds to the equilibria lines from $\omega_0$. This submanifold expends on the critical locus and join $m_0$. The blow down leads to the local-global picture of figure 1

**Back to the original system** The initial problem lies in dimension $k+2$ ($k = 6$ in our motivating control affine problem). From (6), we see that when $\rho = 0$, the $\xi$-component
Figure 4: Phase portrait around $S^2_\pi$. 
of (6) vanishes. Thus, in the blown up coordinates, when \( u = 0 \) (on \( S^2 \)) or when \( \bar{\rho} = 0 \), the spaces \( \{ \bar{\xi}_2 = \text{const.}, \ldots, \bar{\xi}_k = \text{const} \} \) are preserved. Let us write their dynamics in the chart (i) (with obvious notation with respect to (6)):

\[
\begin{align*}
\bar{\xi}_2' &= u^2 \bar{h}_2(u, \bar{\rho}, \bar{s}, \bar{\xi}) \\
\vdots \\
\bar{\xi}_k' &= u^2 \bar{h}_k(u, \bar{\rho}, \bar{s}, \bar{\xi}).
\end{align*}
\]

The blown up space is \( S^2 \times \mathbb{R}^{k-1} \), and the linear part of the total dynamics is the same as in (9), completed with zeros to obtain a \( k + 2 \) matrix. As a result, in the initial system, the hyperbolic equilibrium point \( m_0 \) is replaced by a \( k - 1 \) manifold of equilibria, denoted \( N \), parametrized by \( (\bar{\xi}_1, \ldots, \bar{\xi}_k) \). Each of these points have a stable one dimensional manifold in the direction of \( u \), when \( c > 0 \), (resp. unstable hen \( c < 0 \)): there exists a trajectory of (6) converging to each of this points. The stable manifolds of \( m_0 = m_0(\bar{\xi}_1, \ldots, \bar{\xi}_k) \) will allow us to prove theorem 2.

It remains to show that the trajectory coming from the stable manifold to \( m_0 \) is actually going to \( m_0 \) in finite time, for the original time \( t \). We have been doing the following changes of times: \( dt = \rho dt_1, \, dt_2 = r dt_1, \, dt_3 = u dt_2 \) (\( t_3 \) is the time in which we study the blown up system (9)), so that \( dt = \frac{\rho}{r^2} dt_3 \). We will show that the interval of time from a point of the stable manifold to \( m_0 \) is finite. Assumption (A) implies among other things: \( \rho = 0 \Rightarrow r > 0 \), so that in a neighborhood \( O \) of \( \Sigma \), we have \( r > 0 \). Then in \( O \), \( r \) is bounded below and above by two positive constant \( A > r > B > 0 \). In the blown up coordinates, \( \rho = u^3 \bar{\rho} \), so that the previously mentioned interval of time is

\[
\Delta t = \int_{t^0_3}^{t^0_3} \frac{u^2(t_3)\bar{\rho}}{r(t_3)} dt_3 < \frac{1}{B} \int_{t^0_3}^{t^0_3} u^2(t_3)\bar{\rho}(t_3)dt_3.
\]

Notice that \( \bar{\rho} \) is bounded by above by a positive constant \( K \) along the trajectory in the stable manifold, since it converges to \( m_0 \).

Now, the first line of system (9) is \( u' = -\frac{1}{3} u \bar{s} \). Since \( m_0 \in \{ \bar{s} > 0 \} \), if \( O \) is small enough, \( u' < -cu \) for a constant \( c > 0 \). Then as along the stable manifold to \( m_0 \), we have \( u(t_3) < u_0 e^{-ct_3} \) by integration between a time \( t_3 \) and \( t^0_3 \). So that finally,

\[
\Delta t < K/B \int_{t^0_3}^{t^0_3} u_0^2 e^{-2ct_3} dt_3 < +\infty.
\]

So \( \bar{z} \) is reached in finite time. From figure 4, and the fact that the \( \bar{\xi}_i \)'s, \( i > 1 \) are constant on \( S^2_+ \) and when \( \bar{\rho} \) vanishes, one can make the same time estimates to prove that the extremal goes out of \( S^2_+ \) in finite time, and as such is connected to the singular flow.

### 2.3.2 Proof of theorem 3

In the process of proving theorem 2, we obtained a good description of the singular flow around a point of \( \Sigma_0 \). We will make the proof when \( c > 0 \), the opposite case being similar. The continuity is obtained by the same proof than in the \( \Sigma_- \) case, see [4].
2.3.3 Proof of proposition 4

Depending on the sign of $H_{12}$, the control does not have the same regularity. In the coordinates of system (2), when $t < \hat{t}$, $u(t) = (\cos \theta(t), \sin \theta(t))$, but from proposition 2, when $t > \hat{t}$, $u(t) = u_*(t) = \frac{(-H_{02}, H_{01})}{H_{12}} = \frac{r(\sin \phi, \cos \phi)}{H_{12}} = \frac{r}{H_{12}}(\cos(\phi + \pi/2), \sin(\phi + \pi/2))$.

In the first alternative, the extremal reaches the singular locus at the equilibrium point in the time $\hat{t} = \int_0^\infty \rho dt_1$, and we have $\theta(\hat{t}) - \phi(\hat{t}) = \pi/2$: the control is continuous when the connection with the singular flow occurs. In the second one, $\theta(\hat{t}) - \phi(\hat{t}) = -\pi/2$, so that $\theta(\hat{t}^-) = \theta(\hat{t}^+) + \pi$.

\[\square\]

Remark 5. From the phase portrait of figure 4, we can actually retrieve all three cases. Indeed, one can make a change of coordinates to integrate the parameters $\alpha$. More precisely, set $\tilde{\xi}_1 = a(\xi) - 1$. Furthermore, the cases $\Sigma_-$ can be seen as the West part of the phase portrait above the sphere $S^2_\Sigma$, the two lines of zeros corresponding to the partially hyperbolic equilibrium of [4], to retrieve the global phase portrait one has to quotient the $s$ axis to keep $s$ in $S^1$. The Est part above $S^2_+$, being the $\Sigma_+$ case. The dynamics is actually structurally stable, and the whole situation is contained in the nilpotent case $\Sigma_0$.

2.4 Example

The following example is close to the nilpotent approximation of the minimum time Kepler problem proposed in [3].

Example 1. Let us exhibit a control-affine system with the kind of trajectory describe in theorem 2 when the final time is minimized.

Consider

$$\begin{align*}
\dot{x}(t) &= F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad t \in [0, t_f], \quad u \in U \\
x(0) &= x_0 \\
x(t_f) &= x_f \\
t_f &\to \min.
\end{align*}$$

(13)

on $\mathbb{R}^4$ with

$$\begin{align*}
F_0(x) &= x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_1}, \\
F_1(x) &= x_2 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1}, \\
F_2(x) &= \frac{\partial}{\partial x_2}.
\end{align*}$$

Then

$$\text{rank}(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) = 4, \quad \forall x \in \mathbb{R}^4 \setminus \{x_2 = 0\}.$$ 

The maximized Hamiltonian is $H_{12}^\text{max}(x, p) = p_3 x_1 + p_4 x_2 + \sqrt{(p_1 x_2 + p_3)^2 + p_2^2}$ and we have

$$\begin{align*}
\dot{x}_1 &= \frac{(p_1 x_2 + p_3)x_2}{\sqrt{(p_1 x_2 + p_3)^2 + p_2^2}}, \quad \dot{x}_3 = x_1, \\
\dot{x}_2 &= \frac{p_2}{\sqrt{(p_1 x_2 + p_3)^2 + p_2^2}}, \quad \dot{x}_4 = x_2.
\end{align*}$$

(14)
The coordinates $x_3$ and $x_4$ are cyclic, so $p_3$ and $p_4$ are constant. Denote $p_3 = -a$, $p_4 = -c$, we get $p_1(t) = at + b$, $p_2(t) = ct + d$, where $b = p_1(0)$, $d = p_4(0)$. Eventually:

$$\dot{x}_2 = \frac{ct+d}{\sqrt{((at+b)x_2-a)^2+(ct+d)^2}}. \quad \text{(15)}$$

We also have $\Sigma = \{p_2 = p_1x_2 + p_3 = 0\}$, and the condition $H_{p_1}^2 + H_{p_2}^2 = H_{x_2}^2$ gives $\Sigma_0 = \Sigma \cap \{p_1^2 = p_2^3x_2^2 + p_2^4\}$. The contact time with $\Sigma$ has to be $t = -\frac{d}{b}$. At $t$, we must have $x_2(t) = -p_3/p_1(t) = -\frac{ac}{ad-bc}$. In order to reach $\Sigma_0$, we shall have $x_2^2(t) = \frac{1}{a^2}(p_1^2(t) - p_2^4) = \frac{(ad-bc)^2-c^2}{a^2c^2} := \bar{x}$. This gives an equation on the initial conditions $(a, b, c, d)$:

$$a^2x_2^2[(ad-bc)^2-c^2] - a^4c^4 = 0, \quad \text{(16)}$$

choosing $a$ and $c$ non-zero. This condition imposes $z(t, z_0) \in \Sigma \Rightarrow z(t, z_0) \in \Sigma_0$. Now note that $x_2$ verifies a real ordinary differential equation (though, time dependent) $\dot{x}_2 = f(t, x_2)$ with $f$ defined by (15). $f$ is regular on $\mathbb{R}^2 \setminus \{(t, \bar{x})\}$. To regularize the dynamics of $x_2$ set $dt = \sqrt{((at+b)x_2-a)^2+(ct+d)^2}ds$ to obtain a continuous dynamical system in the plane:

$$\begin{cases}
    x'_2 = ct + d \\
    t' = \sqrt{((at+b)x_2-a)^2+(ct+d)^2}
\end{cases} \quad \text{(15)}$$

$(\bar{x}, \bar{t})$ is its only equilibrium. Outside of it, $t' > 0$. Choosing $c > 0$, there exists a one dimensional stable manifold going to $(\bar{x}, \bar{t})$, and thus a trajectory converging to it in infinite time $s$. This implies the existence of a trajectory for (15) such that $x_2(t) = \bar{x}$. Hence, together with condition (16), there exists an extremal reaching $\Sigma_0$.

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