Large Scale Structures, Symmetry, and Universality in Sandpiles

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We introduce a sandpile model where, at each unstable site, all grains are transferred randomly to downstream neighbors. The model is local and conservative, but not Abelian. This does not appear to change the universality class for the avalanches in the self-organized critical state. It does, however, introduce long-range spatial correlations within the metastable states. We find large scale networks of occupied sites whose density vanishes in the thermodynamic limit, for \(d \geq 1\).

One of the puzzling questions about macroscopic complex phenomena concerns the mechanisms responsible for the large spatially correlated structures that are often seen in Nature. It has been proposed that Self-Organized Criticality (SOC) \([1]\) may be one mechanism, where the bursty, scale-free threshold dynamics of a slowly driven system is intimately linked to the emergence of long-range spatial (and temporal) correlations in it \([2]\). An obvious candidate for this picture would be the stick-slip dynamics of earthquakes, described by the Gutenberg Richter power law distribution for seismic moments, and faults, which form a fractal pattern in the crust of the earth. However, the simple sandpile, or earthquake models do not clearly show large scale structures. Furthermore, although many macroscopic systems show bursty transport phenomenology, a general feature of SOC, the link is not yet established because questions of robustness and universality are not yet resolved. Since the number of possible models that may be studied numerically is inexhaustible, it is essential to determine the symmetry (or other) criteria for universality \([3]\) and robustness of SOC. For this purpose, we consider simple models that may be related to analytically solvable ones.

Here we propose what may be the simplest sandpile model that gives large spatially correlated structures. For \(d \geq 1\) the avalanches in the model have a scale-free distribution with critical coefficients in the same universality class as the Abelian Stochastic Directed Sandpile Model (A-SDM) \([4–6]\). There it has been proven that no spatial correlations exist in the steady state metastable configurations \([5]\). The model we introduce is closely related to the A-SDM. However, a change in the rule for updating unstable sites breaks the Abelian symmetry. (The Abelian symmetry refers to the fact that the order for updating the unstable sites has no effect on the final state that is reached.) This symmetry breaking introduces obvious large scale structures, consisting of networks of occupied sites, within the metastable states that are reached in the steady state, as shown in Figs. \([4]\) and \([5]\). These spatial correlations are not present when the symmetry is restored. The avalanches change the network configuration slowly, just as earthquakes change the configuration of faults slowly. During a single or a few events it might appear falsely that the configuration is static or “pre-existing”.

Breaking the Abelian symmetry, however, has no effect on the critical exponents for the avalanches, for \(d \geq 1\). There is universality and robustness for the bursty phenomena with respect to breaking the Abelian symmetry. Thus two systems in the same universality class with respect to the scaling behavior of avalanches show totally different structures of the metastable states, with one being completely uncorrelated and the other having channels, or networks at large scales.

Consider a two dimensional square lattice as shown in Fig. \([6]\). The direction of propagation is labelled by \(t\), with \(0 \leq t < T\). The transverse direction is labelled by \(x\), with periodic boundary conditions. On each site, an integer variable \(z(x,t)\) is assigned. The \(i\)'th grain is added to a randomly chosen site \(x_i\) on the top row \(t = 0\). There \(z(x_i,0) \rightarrow z(x_i,0) + 1\). When any site acquires a height greater than \(z_c\) it topples, transferring all the grains at

\[\text{FIG. 1. The model in } d = 1. \text{ All grains from unstable sites in row } t \text{ are thrown randomly onto neighboring sites in the next row } t + 1.\]
that site, i.e. \( z(x, t) \to 0 \) for \( z(x, t) > z_c \). Each grain from a toppling site is given equal probability to go to either downstream nearest neighbor, independent of where the other grains from the toppling site are placed. For each toppling event, the total number of grains are conserved. This is true except at the open boundary \( t = T \) where toppling sites simply discharge their grains out of the system.

Sites are relaxed according to a parallel update until there are no more unstable sites, and the properties of the resulting avalanche are recorded. Then a new avalanche is initiated by adding a single grain to a randomly chosen site on the top row, \( t = 0 \). An avalanche can be characterized by its longitudinal extent, \( t_c \), the largest \( t \) row affected, its width, \( x_c \), the largest transverse distance from the avalanche origin to any site affected by the avalanche, its area, \( a \), the total number of sites affected, its size, \( s \), the total number of grains thrown in toppling events, and the maximum number of grains thrown at a single site which topples, \( n_c \). The fact that \( z \) is set to zero at a toppling site makes the model non-Abelian. The corresponding rule for the A-SDM for an unstable site is e.g. \( z(x, t) \to z(x, t) - 2 \).

In a recent work, Dhar \cite{7} has shown that the stochastic Manna model \cite{8}, where a fixed number of grains are removed from toppling sites, exhibits the Abelian property and is a special case of the Abelian Distributed Processors Model. This property was used to solve analytically for the critical state properties of the A-SDM, since in that case the Abelian property makes the appropriately mapped dynamics invertible \cite{45}. It is only necessary to realize that for stochastic models, instead of associating probabilities with each toppling to determine where the grains will be thrown, we can assign to each site an infinite stack of independent, identically distributed random numbers. The quenched random numbers in each site’s stack then determine the allocation of grains during each toppling event. Thus, for a given quenched array, and initial state of the system, the order of updates will not change the final configuration reached when the number of grains thrown from an unstable site is fixed. This Abelian property makes the directed model invertible, which leads to the product measure property of the metastable states, and the solvability of the A-SDM. However, when all grains from unstable sites are removed in toppling then the model is not Abelian anymore.

It is straightforward to generalize the definition of our non-Abelian sandpile model to higher dimensions, with the number of directions transverse to the direction of propagation being \( d \). The threshold \( z_c \) can be chosen either to scale with dimension as \( z_c = 2d - 1 \) (for \( d \geq 1 \)) or it can remain constant at \( z_c = 1 \). The same behaviour and scaling exponents are recovered under both conditions. Below, unless stated otherwise, we refer to the model with \( z_c = 1 \).

First, we discuss the case \( d = 1 \). From the dynamical rules it is clear that the avalanches must, themselves, be essentially compact. Thus each avalanche sweeps out areas of the lattice leaving empty sites. At the edges of the avalanche, sites occupied with grains may remain. Thus in the stationary state the structure of the sandpile will consist roughly of empty areas bounded by wandering paths of occupied sites, which can branch and recombine. At the top of the sandpile, where the grains are added, the network of grains is dense, but pushing into the sandpile it becomes coarser and coarser. This coarsening reflects the fact that avalanches that reach further into the system are bigger and wider and thus leave traces at their edges that are further apart. A steady-state sandpile configuration is shown in Fig. 2.

In fact the average density of sites occupied with grains scales with distance from the top of the pile as \( \rho(t) \sim t^{-\alpha} \), with \( \alpha = 0.45 \pm 0.02 \). In spite of the vanishing density in the thermodynamic limit, this network of grains is essential for maintaining the steady state of SOC, providing a drainage outlet for grains to be transported from the top to the bottom of the system. The situation for the A-SDM is completely different. In that case, the density of occupied sites is 1/2 for \( d = 1 \), and the occupation numbers for sites are completely uncorrelated, being described by a product measure in the steady state \cite{46}.

\[ \text{FIG. 2. A steady-state configuration of our } d = 1 \text{ sandpile model showing sites occupied with grains forming a network that can transport grains from one end of the sandpile to the other. The shaded area indicates the sites which toppled in the preceding avalanche.} \]
have been determined analytically and numerically. Avalanches correspond to an absorbing state phase transition. They are described by a variant of the Edwards-Wilkinson stochastic interface equation where the noise amplitude is a threshold function of the height (occupancy in the sandpile model). Using these analytically determined values for critical exponents, good data collapse is also obtained for our non-Abelian model, where no analytic solution exists at present. Thus, within our numerical accuracy, the critical exponents for the avalanches are the same in the two cases. It appears as though the non-Abelian sandpile has organized large scale structures in such a way as to maintain the universality class, governed by a stochastic continuum equation, for the avalanches.

A configuration in the steady state of our sandpile model in two dimensions is shown in Fig. 4. We observe a type of domain tube structure with walls separating the different domains. The tube domains get larger as they go into the system. Again, a numerical analysis of the avalanche size distribution using finite size scaling gives critical exponents \( \tau_s = 3/2 \) and \( D = 2 \), the same values as determined analytically for the A-SDM. Similarly for the distribution of avalanche times, \( t \), we find finite size scaling with exponents \( \tau_t = D = 2 \). There is no such tube structure, though, in the A-SDM.

The zero dimensional model is a chain of sites. Since each site has only one downstream nearest neighbor, the dynamical rules of the model must be specified in a slightly different way. We allow grains to be distributed to both the nearest and next-nearest neighbours down the chain, and consider two ways in which the relaxation of critical sites can be ordered. With a parallel update rule all critical sites are relaxed simultaneously. In this case, the time in terms of the parallel update at which a site can become critical is not equal to its row number and sites may topple many times during an avalanche. (This does not occur for \( d > 0 \).) We can also define a single-site update rule in which the top-most unstable site is relaxed each time step. Multiple toppings cannot occur in that case. These two cases lead to different sets of critical exponents for the avalanches in \( d = 0 \).

The parallel update dynamics yields the same avalanche exponents, e.g. \( D = 3/2, \tau_s = 4/3 \), as the Abelian model that was studied by Kloster et al (see Fig. 5). We found good data collapse for system sizes ranging from \( T \approx 10^3 \) to \( T \approx 3 \times 10^4 \), for both the size, \( s \), and time extent, \( t \), of the avalanches. However the average occupancy of sites in the steady state does not decay to zero as the distance from the top site, where grains are added, increases, as in our model in higher dimensions. Instead it is constant \( \rho(t) = 1/4 \), apart from small \( t \) where the average occupancy adjusts exponentially from the initial value of \( 1/2 \). Occupied sites do not appear to be spatially correlated. Thus the behavior of the model is very similar to its Abelian relative and appears to be in the same universality class.

In the parallel update dynamics, as more than one site can topple at each time step and the location of the topplings is allowed to vary, the avalanches themselves have structure. On average the active front moves through the system with velocity 1.5 (i.e. advances 3 lattice spaces in 2 parallel updates, on average). Around this average the active sites are split into a series of smaller fronts which
spread, branch and recombine (see inset in Fig. 5).

Since our model is not Abelian, a change in the rule for the order in which unstable sites are updated can have a pronounced effect. We tried a variety of update rules in higher dimensions $d > 0$, but all the versions we tried appeared to give the same universality class for the avalanches, although the structure of metastable states did change drastically.

However, in $d = 0$ the critical behaviour is less robust. A finite size scaling analysis of the avalanche size distribution using the single site update rule described above does not collapse with the same exponents as the $d = 0$ A-SDM. A reasonable data collapse can be produced by changing the cutoff dimension to $D = 1.1$ and keeping $\tau_s = 4/3$, but it is not completely convincing. A multifractal data collapse \[9\] did not yield noticeably better results. The multifractal collapse was performed with parameters $s_0 = 0.5$, $l_0 = 0.2$. For the single site model, the density of occupied sites does decay going into the system from the top where the grains are added. It behaves approximately as $\rho(s) \simeq 1/t$.

This work was supported by the EPSRC. We thank P. Bak and S. Lise for comments on the manuscript.

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FIG. 5. Finite size scaling of the distribution of avalanche sizes for the A-SDM and our model with parallel update in $d = 0$, using $\tau_s = 4/3$ and $D = 3/2$. Inset shows the position of active sites away from the average at each time step.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987).
[2] For a review see P. Bak, How Nature Works: The Science of Self-Organized Criticality (Copernicus, New York, 1996).
[3] M. Paczuski and S. Boettcher, Phys. Rev. Lett. 77, 111 (1996).
[4] R. Pastor-Satorras and A. Vespignani, J. Phys. A 33, L33 (2000).
[5] M. Paczuski and K. Bassler, Phys. Rev. E 62, 5347 (2000); the first version of the paper was published by mistake. The correct version is web publication xxx.lanl.gov/abs/cond-mat/0005340. This version includes a reference to the independent work by Kloster et al.\[6\].
[6] M. Kloster, S. Maslov, and C. Tang, Phys. Rev. E, art. no. 026111 (2001).
[7] D. Dhar, Physica A 270, 69 (1999); preprint cond-mat/9902137 (1999).
[8] S. S. Manna, J. Phys. A 24 L363, (1991).
[9] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. M. Zhou, Phys. Rev. A 39, 6524 (1989).
[10] S. Lübeck, B. Tadić, and K. D. Usadel, Phys. Rev. E 53, 2182 (1996); B. Tadić and R. Ramaswamy, Phys. Rev. E 54, 3157 (1996); B. Tadić and D. Dhar, Phys. Rev. Lett. 79, 1519 (1997).
[11] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London Ser. A 381, 17 (1982).
[12] A. Vazquez, cond-mat preprint 0003420.
[13] M. Paczuski, S. Maslov, and P. Bak, Europhys. Lett. 28, 295 (1994).