1. INTRODUCTION

Engineering problems in time-frequency analysis of coherent vector expansions, Gabor bases, wavelets based on scaling and integral translations, and multiresolution algorithms in signal processing are generally not thought to be related to operator algebras. In this paper, we show nonetheless that a fundamental idea of Kolmogorov adds clarity to known constructions in operator algebra theory, and moreover is the key to an extension of recent results in the more applied areas that we enumerated above. Of our original results (see section 5 below) we highlight a new algorithm for the construction of certain orthonormal frames of wavelet type. Our paper proposes a general method of construction of representations of various algebraic structures as operators on Hilbert spaces. Our goal is to show how some well known constructions of representations fit into the same framework and are consequences of a general result. Among the structures considered, we mention $C^*$-algebras, groups, Gabor type unitary systems and wavelet representations.

In operator theory, the GNS construction producing representations of $C^*$-algebras is a fundamental tool (see [BraRo]). In harmonic analysis unitary representations of groups can be constructed when a function of positive type is present (see [Fol]). Representations are ubiquitous also in the theory of wavelets and frames (see [HL], [Jor98]). We will see how these various results have in fact a common ground - a classical theorem of Kolmogorov (theorem 2.2), also known in the literature as the Kolmogorov decomposition of positive definite kernels. We follow here the ideas introduced in [EvLe]. It is shown there that the Kolmogorov theorem gives a unified treatment of several important dilation theorems such as the GNS-Stinespring construction for $C^*$-algebras, the Naimark-Sz. Nagy unitary dilation of positive definite functions on groups, the construction of Fock spaces and the algebras of canonical commutation and anticommutation relations. Kolmogorov’s result was used also by Sz. Nagy and C. Foias in dilation theory, for the commutant lifting theorem ([SzF68], [SzF70]) which in turn was a key idea used by D. Sarason to obtain a solution to the Nevanlinna-Pick interpolation problem ([Sar71]). For a more complete account of the history and applications of Kolmogorov’s result, we refer to [C96].

We will indicate how this technique can be used also for construction of wavelet representations and Gabor type unitary systems.
More general constructions for Hermitian kernels are also possible and they are based on Krein spaces (see [C97]).

In section 2 we review the general result of Kolmogorov and we show how it can be used for the GNS construction and for positive definite maps on groups. Then we apply it to Gabor type unitary systems and we obtain unitary representations and for wavelets we get the cyclic representations introduced in [For98].

Section 3 concerns operators compatible with the representations defined in section 2, called intertwining operators. Again, the starting point is a general theorem (theorem 3.2). We consider some particular cases and study how the intertwining operators will be compatible with the additional structure that appears.

In section 4 we analyze some connections between representations and frames. We recall that a set \(\{x_n \mid n \in \mathbb{N}\}\) of vectors in a Hilbert space \(H\) is called a frame for the Hilbert space \(H\) if there are some positive constants \(A\) and \(B\) such that

\[
A\|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f \mid x_n \rangle|^2 \leq B\|f\|^2, \quad (f \in H).
\]

When \(A = B = 1\) the we call it normalized tight frame.

It is known that any normalized tight frame is the projection of an orthonormal basis of a bigger Hilbert space (see [HL]). We will prove that the normalized tight frames can be dilated to orthonormal bases in a way that is compatible with the representations defined in section 2. We will get as immediate consequences the dilation theorems for groups and Gabor type unitary systems introduced in [HL].

In the last section we consider the case of wavelets obtained from a multiresolution analysis. It is known (see [Dau92]) that, unless some restrictions are imposed on the low-pass filter that starts the MRA construction, the wavelets obtained do not form an orthonormal basis but a normalized tight frame. Since such frames can be dilated to orthonormal bases, a natural question would be if the dilation preserves the multiresolution structure. The answer is affirmative and it is given in theorem [5.2] and more concretely in theorem [5.3]. In this way we obtain "wavelets" in a Hilbert space bigger then \(L^2(\mathbb{R})\).

2. Positive definite maps and representations

We begin this section with a general result of Kolmogorov ([EvLe], [EvKa]). Then we consider several structures and show how to obtain representations from this general theorem.

**Definition 2.1.** Let \(X\) be a nonempty set. We say that a map \(K : X \times X \to \mathbb{C}\) is positive definite, and denote this by \(0 \leq K\), if

\[
\sum_{i,j=1}^{n} K(x_i, x_j) \xi_i \xi_j^* \geq 0, \quad (n \in \mathbb{N}, x_i \in X, \xi_i \in \mathbb{C} \text{ for all } i \in \{1, \ldots, n\}).
\]

**Theorem 2.2.** [Kolmogorov’s theorem] If \(K : X \times X \to \mathbb{C}\) is positive definite then there exists a Hilbert space \(H_K\) and a map \(v_K : X \to H_K\) such that the linear span of \(\{v_K(x) \mid x \in X\}\) is dense in \(H_K\) and

\[
\langle v_K(x) \mid v_K(y) \rangle = K(x,y), \quad (x,y \in X).
\]

Moreover, \(H_K\) and \(v_K\) are unique up to unitary isomorphisms.

**Remark 2.3.** Kolmogorov’s theorem is valid also for operator-valued positive definite maps and in this form it can be applied for the Stinespring construction and the Naimark-Sz.Nagy dilation. For details consult [EvLe] and [EvKa].

In this paper, for the application to wavelets and Gabor frames, we will need only the more particular version of Kolmogorov’s theorem that we mentioned before.

**Definition 2.4.** If \(K : X \times X \to \mathbb{C}\) is positive definite then we call \([H_K, v_K]\) the representation associated to \(K\).
We note that Kolmogorov’s theorem is purely set theoretic; there is no structure on $X$. We expect that, if $X$ has some additional structure on it and if we assume some compatibility between the positive definite map $K$ and this structure, then the representation associated to $K$ will also be in agreement with the structure of $X$. In the next examples we will see that this is indeed the case and we review the technique in the case of $C^*$-algebras and groups.

**Example 2.5. [C*-algebras and the GNS construction]** We consider now the case when $X = A$ is a $C^*$-algebra and prove that we can obtain the well known GNS construction from Kolmogorov’s theorem.

**Theorem 2.6. [The GNS construction]** If $A$ is a $C^*$-algebra and $\varphi$ is a positive linear functional on $A$, then there exists a representation $\pi$ of $A$ on a Hilbert space $H$, that has a cyclic vector $\xi_0 \in H$ such that

$$\langle \pi(x)\xi_0 \mid \xi_0 \rangle = \varphi(x), \quad (x \in A).$$

**Proof.** The idea is to define $K : A \times A \rightarrow \mathbb{C}$ by

$$K(x, y) = \varphi(y^* x), \quad (x, y \in A).$$

We can use Kolmogorov’s theorem to obtain the Hilbert space $H_K$ and the map $v_K : A \rightarrow H_K$.

For a fixed $x \in A$, define the operator $\pi(x)$ as follows:

$$\pi(x)(v_K(y)) = v_K(xy), \quad (y \in A),$$

and extend by linearity. Then everything checks out. \qed

**Example 2.7. [Groups and unitary representations]** Take $X = G$ a group. We call $K : G \times G \rightarrow \mathbb{C}$ a group positive definite map if $0 \leq K$ and

$$K(x, y) = K(zx, zy), \quad (x, y, z \in G).$$

We note that such a positive definite map $K$ is uniquely determined by its restriction $\phi(x) = K(x, 1)$ and $\phi$ is a function of positive type (see [50]). The proof of theorem 2.8 will show how the well known correspondence between functions of positive type and unitary representations of groups can be regarded as a consequence of Kolmogorov’s theorem.

**Theorem 2.8.** Let $G$ be a group and $K$ a group positive definite map on $G$. Then there exists a unitary representation $\pi_K$ of $G$ on a Hilbert space $H_K$ with a cyclic vector $\xi_0 \in H_K$ such that

$$\langle \pi_K(x)\xi_0 \mid \pi_K(y)\xi_0 \rangle = K(x, y), \quad (x, y \in G).$$

**Proof.** The proof works exactly as in the case of $C^*$-algebras: consider $[H_K, v_K]$ the representation associated to $K$ by Kolmogorov’s theorem. Define the operators $\pi_K(x)$ for $x \in G$ as follows:

$$\pi_K(x)(v_K(y)) = v_K(xy), \quad (x, y \in G)$$

and extend by linearity. \qed

**Remark 2.9.** Note that in the proof of theorem 2.8 we used the representation associated to $K$ and we see that, when $K$ is a group positive definite map, this representation has the unitary representation $\pi_K$ attached to it. The same observation can be done for the GNS construction: the representation of the $C^*$-algebra is attached to the representation $v_K$. This confirms our expectation: when the positive definite map has some compatibility with the existent structure on $X$, this compatibility projects a nice structure on the associated representation $[H_K, v_K]$. This is the idea that we use throughout this section.

**Example 2.10. [Gabor type unitary systems]** We recall that a Gabor system is associated to two positive constants $a, b > 0$ and a function $g \in L^2(\mathbb{R})$ and is defined by

$$g_{m,n}(\xi) = e^{2\pi i m b \xi}g(\xi - na), \quad (\xi \in \mathbb{R}).$$

The Gabor systems are one of the major subjects in the study of frames and wavelet theory. If we define the unitary operators $U, V$ on $L^2(\mathbb{R})$,

$$(Uf)(\xi) = e^{2\pi i b \xi}f(\xi), \quad (f \in L^2(\mathbb{R})), \quad (Vf)(\xi) = f(\xi - na),$$

then

$$[UH_K, v_K] = \pi_K,$$
(V f)(ξ) = f(ξ − a), \quad (f \in L^2(\mathbb{R})),
then \(g_{m,n} = U^mV^n g, \quad (m,n \in \mathbb{Z})\), and \(U\) and \(V\) satisfy the relation
\[U V = e^{2\pi iab} V U.\]

Following [HL], if \(U\) and \(V\) are unitary operators on a Hilbert space \(H\) that verify the relation
\[U V = \lambda V U\]
for some unimodular scalar \(\lambda\), we then call \(\{U^mV^n \mid m,n \in \mathbb{Z}\}\) a Gabor type unitary system. We will prove that these systems fit into our general framework and we construct representations for them.

**Theorem 2.11.** Suppose \(\lambda\) is a unimodular scalar and \(K : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}\) is a positive definite map satisfying
\begin{equation}
K((m+1,n),(m'+1,n')) = K((m,n),(m',n')), \quad (m,n,m',n' \in \mathbb{Z}),
\end{equation}
\begin{equation}
K((m,n+1),(m',n'+1)) = \lambda^{m-m'} K((m,n),(m',n')), \quad (m,n,m',n' \in \mathbb{Z}).
\end{equation}
Then, on the Hilbert space \(H_K\), there are unitaries \(U\) and \(V\) and a vector \(\xi_0 \in H_K\) such that
\begin{equation}
UV = \lambda V U
\end{equation}
\begin{equation}
\{U^mV^n \xi_0 \mid m,n \in \mathbb{Z}\}
\end{equation}
is dense in \(H_K\)
\begin{equation}
\langle U^mV^n \xi_0 \mid U^mV^n' \xi_0 \rangle = K((m,n),(m',n')), \quad (m,n,m',n' \in \mathbb{Z}).
\end{equation}
Moreover, this representation is unique up to unitary isomorphism.

**Proof.** Let \(v_K : \mathbb{Z}^2 \to H_K\) be the representation associated to \(K\). Define the operators \(U\) and \(V\) as follows:
\[U(v_K(m,n)) = v_K(m+1,n), \quad (m,n \in \mathbb{Z}),\]
\[V(v_K(m,n)) = \lambda^{-m} v_K(m,n+1), \quad (m,n \in \mathbb{Z}),\]
and then extend by linearity. We check that \(U\) and \(V\) are well defined and isometric. Take \(a_i \in \mathbb{C}, (m_i,n_i) \in \mathbb{Z}^2, (i \in \{1,...,p\})\).
\[
\left\langle V(\sum_{i=1}^{p} a_i v_K(m_i,n_i)) \mid V(\sum_{i=1}^{p} a_i v_K(m_i,n_i)) \right\rangle = \\
= \left(\sum_{i=1}^{p} a_i \lambda^{-m_i} v_K(m_i,n_i+1)\right) \left(\sum_{i=1}^{p} a_i \lambda^{-m_i} v_K(m_i,n_i+1)\right)
= \left(\sum_{i,j=1}^{p} a_i a_j \lambda^{-m_i-m_j} K((m_i,n_i),(m_j,n_j))\right).
\]
A similar calculation shows that \(U\) is well defined and isometric. Since the linear span of the vectors \(v_K(m,n)\) is dense in \(H_K\), we can extend \(U\) and \(V\) to unitaries on \(H_K\).

Next, we check (2.3). Take \((m,n) \in \mathbb{Z}^2\).
\[UV v_K(m,n) = \lambda \lambda^{-m} v_K(m+1,n+1) = \lambda v_K(m+1,n+1) = \lambda V (v_K(m+1,n)) = \lambda V (v_K(m,n))\]
and (2.4) follows by density. Also, note that, if \(\xi_0 = v_K(0,0)\), then
\[U^mV^n \xi_0 = U^mV^n v_k(0,0) = U^m v_K(0,n) = v_K(m,n), \quad (m,n \in \mathbb{Z}).\]
This will imply (2.4) and (2.5). The uniqueness is a consequence of the uniqueness part of Kolmogorov’s theorem. \(\square\)
Remark 2.12. Any Gabor type unitary system $U, V$ on a Hilbert space $H$, that has a vector $\xi_0 \in H$ with the property that the linear span of

$$\{U^mV^n\xi_0 \mid m, n \in \mathbb{Z}\}$$

is dense in $H$, gives rise to a positive definite map $K$ on $\mathbb{Z}^2$ that satisfies (2.1), (2.2) as follows:

$$K((m, n), (m', n')) = \left\langle U^mV^n\xi_0 \mid U^{m'}V^{n'}\xi_0 \right\rangle, \quad (m, n, m', n' \in \mathbb{Z}).$$

(2.1), (2.2) are just immediate consequences of the fact that $U$ and $V$ are unitary and $UV = \lambda VU$.

Example 2.13. [Wavelet representations] We recall briefly some facts about wavelet representations. Wavelet theory deals with two unitary operators $U$ and $T$ on $L^2(\mathbb{R})$, corresponding to the integer $N \geq 2$ called the scale:

$$Uf(x) = \frac{1}{\sqrt{N}} f\left(\frac{x}{N}\right), \quad Tf(x) = f(x - 1), \quad (x \in \mathbb{R}, f \in L^2(\mathbb{R})).$$

A wavelet is a function $\psi \in L^2(\mathbb{R})$ such that

$$\{U^mT^n\psi \mid m, n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. One way to construct wavelets is by multiresolutions and scaling functions (see [Dau92]). Scaling functions satisfy equations of the form

(2.6) $$U\varphi = \sum_{k \in \mathbb{Z}} a_k T^k \varphi,$$

where $a_k$ are complex coefficients.

The scaling equation can be reformulated using representations. There is a representation of $L^\infty(\mathbb{T})$ ($\mathbb{T}$ is the unit circle) on $L^2(\mathbb{R})$ given by

$$(\pi(f)\xi) = f\hat{\xi}, \quad (f \in L^\infty(\mathbb{R}), \xi \in L^2(\mathbb{R}))$$

($\hat{\xi}$ denotes the Fourier transform of $\xi$ and functions on $\mathbb{T}$ are identified with $2\pi$-periodic functions on $\mathbb{R}$). Using this representation, (2.6) can be rewritten as

$$U\varphi = \pi(m_0)\varphi,$$

$m_0(e^{-i\theta}) = \sum_{k \in \mathbb{Z}} a_k e^{-ik\theta}$ is called a low-pass filter. Also, the representation satisfies

$$U\pi(f)U^{-1} = \pi(f(z^N)), \quad (f \in L^\infty(\mathbb{T})).$$

$(U, \pi, L^2(\mathbb{R}), \varphi)$ is called the wavelet representation with scaling function $\varphi$.

The wavelet theory has shown a strong interconnection between properties of the scaling function $\varphi$ and spectral properties of the transfer operator associated to the low-pass filter $m_0$:

$$R_{m_0, m_0} f(z) = \frac{1}{N} \sum_{w^N = z} |m_0|^2(w)f(w), \quad (z \in \mathbb{T}, f \in L^1(\mathbb{T})).$$

where $\mathbb{T}$ is endowed with the normalized Haar measure. For more information on this we refer the reader to [BraJo]. In particular, functions that are harmonic with respect to $R_{m_0, m_0}$, i.e. $R_{m_0, m_0} h = h$, play an important role in the theory.

We recall here a theorem from [Lor98] which establishes the link between functions which are harmonic with respect to $R_{m_0, m_0}$ and wavelet representations, because it is another particularized instance of Kolmogorov’s theorem.

Theorem 2.14. If $m_0 \in L^\infty(\mathbb{T})$ is non-singular (i.e. it doesn’t vanish on a set of positive measure) and $h \in L^1(\mathbb{T})$, satisfies

$$R_{m_0, m_0} h = h, \quad h \geq 0,$$

then there exists a Hilbert space $H_h$, a representation $\pi_h$ of $L^\infty(\mathbb{T})$ on $H_h$, a unitary $U_h$ on $H_h$ and a vector $\varphi_h \in H_h$ such that

$$\text{span}\{U_h^{-n}\pi_h(f)\varphi_h \mid n \in \mathbb{N}, f \in L^\infty(\mathbb{T})\} = H_h;$$

$$U_h\pi_h(f)U_h^{-1} = \pi_h(f(z^N)), \quad (f \in L^\infty(\mathbb{T}));$$
Then, we must have
\[ U_h \varphi_h = \pi_h(m_0) \varphi_h; \]
\[ \langle \pi_h(f) \varphi_h | \varphi_h \rangle = \int_T f h \, d\mu. \]

Moreover, this is unique up to unitary equivalence.

**Proof.** We give here only a sketch of the proof that uses Kolmogorov’s theorem, the rest are calculations which can be found in [Jor98].

Let
\[ X = \{ (f, n) | f \in L^\infty(\mathbb{T}), n \in \mathbb{N} \}. \]

We want to define a positive definite map \( K \) on \( X \) such that in the end \( v_K(f, n) = U_h^{-n} \pi_h(f) \varphi_h. \)

Then, we must have
\[ K((f, n), (g, m)) = \langle U_h^{-n} \pi_h(f) \varphi_h | U_h^{-m} \pi_h(g) \varphi_h \rangle = \langle \pi_h(f(z^m)m_0(n)(z)) \varphi_h | \pi_h(g(z^n)m_0(n)(z)) \varphi_h \rangle = \int_T f(z^m)m_0(n)(z)g(z^n)m_0(n)(z)h \, d\mu, \]

where \( m_0(n)(z) = m_0(z)m_0(z^N)...m_0(z^{N^{m-1}}). \)

So we have to define, for \((f, n), (g, m) \in X, \)
\[ K((f, n), (g, m)) = \int_T f(z^m)m_0(n)(z)g(z^n)m_0(n)(z)h \, d\mu. \]

\( K \) can be checked to be positive definite so it induces a representation \((H_h, v_h), \) according to Kolmogorov’s theorem. Then, define \( \varphi_h = v_h(1, 0), \)
\[ U_h v_h(f, 0) = v_h(f(z^N)m_0, 0), \quad (f \in L^\infty(\mathbb{T})), \]
\[ U_h v_h(f, n) = (f, n - 1), \quad (n \geq 1, f \in L^\infty(\mathbb{T})), \]

and extend by linearity and density.
\[ \pi_h(f)v_h(g, n) = (f(z^N)m_0(g(z)), n), \quad (f, g \in L^\infty(\mathbb{T}), n \in \mathbb{N}), \]

and extend by linearity and density.

Everything can be checked out as the reader may see in [Jor98].

\[ \square \]

3. Intertwining operators

In the previous section we saw how positive definite maps induce representations on Hilbert spaces. Now we will show that intertwining operators can be constructed in a similar way from maps \( L : X \times X \rightarrow \mathbb{C} \) which satisfy some boundedness condition. We will also see that, when \( X \) has some structure on it and \( L \) is compatible with this structure, then the intertwining operator induced by \( L \) will be compatible with the extra structure existent on the induced representations, i.e. the operator is indeed intertwining.

The format of this section is similar to the format of the previous one. We begin with a general, set theoretic result and then particularize it to various structures to obtain more information.

**Definition 3.1.** Consider two positive definite maps \( K, K' : X \times X \rightarrow \mathbb{C} \) and \( L : X \times X \rightarrow \mathbb{C} \) (not necessarily positive definite). We say that \( L \) is bounded with respect to \( K \) and \( K' \) if there is a constant \( c > 0 \) such that
\[
\left| \sum_{i=1}^m \sum_{j=1}^n L(x_i, y_j) \xi_i \eta_j \right|^2 \leq c \left( \sum_{i, i' = 1}^m K(x_i, x_{i'}) \xi_i \overline{\xi}_{i'} \right) \left( \sum_{j, j' = 1}^n K'(y_j, y_{j'}) \eta_j \overline{\eta}_{j'} \right),
\]
for all \( x_i, y_j \in X, \xi_i, \eta_j \in \mathbb{C}, i \in \{1, ..., m\}, j \in \{1, ..., n\}. \) We denote this by
\[ L^2 \leq c KK'. \]
Theorem 3.2. Suppose $X$ is a nonempty set and $K,K'$ are positive definite maps on $X$. If $L : X \times X \to \mathbb{C}$ and $L^2 \leq cKK'$ for some $c > 0$, then there exists a unique bounded linear operator $S : H_K \to H_{K'}$ such that

$$(3.2) \quad \langle S v_K(x) \mid v_{K'}(y) \rangle = L(x,y), \quad (x,y \in X).$$

($(H_K, v_K), (H_{K'}, v_{K'})$ are the representation induced by $K$ and $K'$ respectively, according to Kolmogorov’s theorem). Moreover, $\|S\| \leq \sqrt{c}$. Conversely, if $S : H_K \to H_{K'}$ is a bounded linear operator, then there is a unique map $L : X \times X \to \mathbb{C}$ with $L^2 \leq \|S\|^2 cK K'$ that satisfies \((3.3)\).

Proof. Define $B : H_K \times H_{K'} \to \mathbb{C}$ as follows: for $x_i, y_j \in X, \xi_i, \eta_j \in \mathbb{C},$

$$B(\sum_{i=1}^{n} \xi_i v_K(x_i), \sum_{j=1}^{n} \eta_j v_{K'}(y_j)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \overline{\eta}_j L(x_i, y_j).$$

Because $L^2 \leq cK K'$, we have

$$\left| B(\sum_{i=1}^{n} \xi_i v_K(x_i), \sum_{j=1}^{n} \eta_j v_{K'}(y_j)) \right|^2 \leq c \left( \sum_{i=1}^{m} K(x_i, x_i') \xi_i \overline{\xi}_{i'} \right) \left( \sum_{j=1}^{m} K(y_j, y_j') \eta_j \overline{\eta}_{j'} \right) \right.$$

$$\cdot \left( \sum_{j=1}^{m} K'(y_j, y_j') \eta_j \overline{\eta}_{j'} \right) = c \left( \sum_{i=1}^{m} \xi_i v_K(x_i) \right)_{H_K} \left( \sum_{j=1}^{m} \eta_j v_{K'}(y_j) \right)_{H_{K'}}.$$

This shows that $B$ is a well defined bounded sesquilinear map which can be extended (by the density properties of $v_K$ and $v_{K'}$) to a bounded sesquilinear map $B : H_K \times H_{K'} \to \mathbb{C}$. Then there exists a bounded linear operator $S : H_K \to H_{K'}$ such that $\|S\| \leq \sqrt{c}$ and

$$B(v_1, v_2) = \langle Sv_1 \mid v_2 \rangle, \quad (v_1 \in H_K, v_2 \in H_{K'}).$$

In particular, one obtains \((3.2)\).

The uniqueness is clear because the spans of $\{v_K(x) \mid x \in X\}$ and $\{v_{K'}(y) \mid y \in X\}$ are dense. The converse is also easy, one needs to check that the map $L$ defined by \((3.2)\) satisfies $L^2 \leq \|S\|^2 cK K'$, but this is a consequence of Schwarz’s inequality. \(\square\)

Definition 3.3. We call the operator $S$ associated to $L$ in theorem \((3.2)\) the intertwining operator associated to $L$.

We will also be interested in subrepresentations and in the commutant of a representation. In these instances we will work with only one positive definite map $K$. We give here a definition which will be appropriate for these situations.

Definition 3.4. Consider $K, K'$, two positive definite maps on a nonempty set $X$ and a constant $c > 0$. We denote

$$K' \leq cK$$

if, for all $x_i \in X$ and $\xi_i \in \mathbb{C}, \{i \in \{1, \ldots, n\}\},$

$$\sum_{i,j=1}^{n} K'(x_i, x_j) \xi_i \overline{\xi}_j \leq c \sum_{i,j=1}^{n} K(x_i, x_j) \xi_i \overline{\xi}_j.$$

Proposition 3.5. If $K$ and $K'$ are positive definite maps and $c > 0$ then $K' \leq cK$ if and only if $K'^2 \leq c^2 K K'$.

Proof. Suppose $K' \leq cK$. Take $x_i, y_j \in X, \xi_i, \eta_j \in \mathbb{C},$

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{n} K'(x_i, y_j) \xi_i \overline{\eta}_j \right|^2 \leq \left( \sum_{i=1}^{m} \xi_i v_{K'}(x_i) \right)_{H_{K'}} \left( \sum_{j=1}^{m} \eta_j v_{K'}(y_j) \right)_{H_{K'}}.$$
Proof. Using proposition 3.5 and theorem 3.2, we find an operator $\|H\|$ and (3.4) holds with $cH$.

Recall that $\langle S\pi(x) | v_K(y) \rangle = K(x, y)$, $(x, y \in X)$. Moreover, $\|S\| \leq c$. Conversely, for every positive definite operator $S$ on $H_K$ there is a unique positive definite map on $X$ that satisfies (3.3). In addition, $K' \leq H$. 

Proof. Using proposition 3.5 and theorem 3.2 we find an operator $S$ on $H_K$ that satisfies (3.3) and $\|S\| \leq c$. $S$ is positive because 

$$\left\langle S\left(\sum_{i=1}^{n} \xi_i v_K(x_i)\right) | \sum_{i=1}^{n} \xi_i v_K(x_i) \right\rangle = \sum_{i,j=1}^{n} \xi_j K'(x_i, x_j) \geq 0$$

and $\{v_K(x) | x \in X\}$ span a dense subspace of $H_K$.

For the converse, when $S$ is given, theorem 3.2 shows that there is a $K'$ satisfying (3.3) and $K' \leq H$. $K'$ is positive because $S$ is, and proposition 3.5 implies $K' \leq H$. 

In the remainder of this section we apply theorem 3.2 to the situations when $X$ has some additional structure on it and see how the intertwining operators are in compliance with the extra structure of the representations.

Example 3.7. [C*-algebras] Consider now $X = A$, a C*-algebra. We saw in example 2.5 that, when the positive definite map $K : A \times A \to C$ is given by a positive functional $\varphi : A \to C$, 

$$K(x, y) = \varphi(y^* x), \ (x, y \in A),$$

then the representation induced by $K$ has the GNS construction attached to it. We want to see for what functions $L : A \times A \to C$ the associated intertwining operator will intertwine the GNS representations.

Theorem 3.8. Let $A$ be a C*-algebra and $\varphi, \varphi'$ two positive functionals on $A$. Suppose that $\varphi_0 : A \to C$ is linear and $\varphi_0^2 \leq c \varphi \varphi'$ for some $c > 0$ in the sense that 

$$|\varphi_0(y^* x)|^2 \leq c \varphi(x^* x) \varphi'(y^* y), \ (x, y \in A).$$

Then there exists a unique bounded operator $S : H_{\varphi} \to H_{\varphi'}$ such that 

$$\langle S\pi_{\varphi}(x) | \pi_{\varphi'}(y) \xi_0 \rangle = \varphi_0(y^* x), \ (x, y \in A).$$

(Here $(H_{\varphi}, \pi_{\varphi}, \xi_0)$ and $(H_{\varphi'}, \pi_{\varphi'}, \xi_0')$ are the GNS representations associated to $\varphi$ and $\varphi'$ respectively (see theorem 2.0).) Moreover $\|S\| \leq \sqrt{c}$. Conversely, if $S : H_{\varphi} \to H_{\varphi'}$ is a bounded operator that satisfies (3.3) then there is a unique linear map $\varphi_0 : A \to C$ that satisfies (3.3). In addition (3.3) holds with $c = \|S\|^2$.

Proof. Let $K_{\varphi}, K_{\varphi'} : A \times A \to C$, 

$$K_{\varphi}(x, y) = \varphi(y^* x), K_{\varphi'}(x, y) = \varphi'(y^* x), \ (x, y \in A).$$

Recall that $H_{\varphi} = H_{K_{\varphi}}, H_{\varphi'} = H_{K_{\varphi'}}, \pi_{\varphi}(x) \xi_0 = v_K(x), \pi_{\varphi'}(x) \xi_0 = v_{K_{\varphi'}}(x)$ (see the proof of theorem 2.0).
Define $L(x, y) = \varphi_0(y^* x)$ for $x, y \in \mathcal{A}$. Then $(3.8)$ implies $L^2 \leq cK\varphi'$. Theorem $3.2$ gives an operator $S$ with
\[ \langle S\pi\varphi(x) | \pi\varphi(y)\xi_0 \rangle = \varphi'(y^* x), \quad (x, y, z \in \mathcal{A}); \]
then one checks that $S$ satisfies all the requirements.

As a corollary we deduce a basic fact about positive operators in the commutant of the GNS representation (see Remark $2.2$).

**Corollary 3.9.** Let $\varphi, \varphi'$ be two positive functionals on a $C^*$-algebra $\mathcal{A}$, $\varphi' \leq c\varphi$ for some $c > 0$ (i.e. $\varphi'(x) \leq c\varphi(x)$ for all positive $x \in \mathcal{A}$). There exists a unique positive linear operator $S$ in the commutant of the GNS representation corresponding to $\varphi$ such that
\[ (3.7) \quad \langle S\pi\varphi(x) | \pi\varphi(y)\xi_0 \rangle = \varphi'(y^* x), \quad (x, y, z \in \mathcal{A}). \]
Conversely, for any positive operator $S$ in the commutant of $\pi\varphi(\mathcal{A})$, there is a unique positive functional $\varphi'$ on $\mathcal{A}$ such that $(3.7)$ holds and $\varphi' \leq \|S\|\varphi$.

**Example 3.10.** [Groups] Take now $\mathcal{X} = G$ a group. We know from theorem $2.8$ that, if $K : G \times G \to \mathbb{C}$ is positive definite and satisfies
\[ K(x, y) = K(zx, zy), \quad (x, y, z \in G), \]
then $K$ induces a unitary representation of $G$ on $H_K$. In the next theorem we look at operators that intertwine these representations.

**Theorem 3.11.** Suppose $G$ is a group and $K, K'$ are positive definite maps on $G$ satisfying
\[ K(x, y) = K(zx, zy), K'(x, y) = K'(zx, zy), \quad (x, y, z \in G). \]
Let $L : G \times G \to \mathbb{C}$ with $L^2 \leq cKK'$ for some $c > 0$. If
\[ (3.8) \quad L(x, y) = L(zx, zy), \quad (x, y, z \in G), \]
then there is a unique operator $S : H_K \to H_{K'}$ such that
\[ (3.9) \quad S\pi_K(x) = \pi_{K'}(x)S, \quad (x \in G), \]
\[ ((H_K, \pi_K, \xi_0), (H_{K'}, \pi_{K'}, \xi_0')) \text{ are the unitary representations of } G \text{ associated to } K \text{ and } K' \text{ respectively (see theorem } 2.8). \]
Moreover $\|S\| \leq \sqrt{c}$. Conversely, if $S : H_K \to H_{K'}$ satisfies $(3.9)$, then there is a unique $L$ that satisfies $(3.8)$ and $L^2 \leq \|S\|^2 KK'$.

**Proof.** Recall that, if $v_{K'} : G \to H_{K'}$ and $v_{K} : G \to H_K$ are the representations associated to $K$ by Kolmogorov’s theorem, then
\[ \pi_K(x)\xi_0 = v_K(x), \pi_{K'}(x)\xi_0' = v_{K'}(x), \quad (x \in G) \]
(see the proof of theorem $2.8$).

Theorem $3.2$ implies the existence of an operator $S : H_K \to H_{K'}$ with $\|S\| \leq \sqrt{c}$ and
\[ \langle S\pi_K(x) | v_{K'}(y) \rangle = L(x, y), \quad (x, y \in G). \]
The rest follows.

**Corollary 3.12.** Let $K, K'$ be two positive definite maps on the group $G$ that satisfy
\[ K(x, y) = K(zx, zy), K'(x, y) = K'(zx, zy), \quad (x, y, z \in G), \]
and $K' \leq cK$ for some $c > 0$. Then there exists a unique positive operator $S$ on $H_K$ in the commutant of the unitary representation $\pi_K(G)$, such that
\[ (3.11) \quad \langle S\pi_K(x) | \pi_K(y)\xi_0 \rangle = K'(x, y), \quad (x, y \in G). \]
Conversely, for every positive operator $S$ in the commutant of $\pi_K(G)$, there is a unique positive definite map $K'$ on $G$ that satisfies $(3.11)$ and
\[ K'(x, y) = K'(zx, zy), \quad (x, y, z \in G). \]
Proof. It is an immediate consequence of theorem 3.11. It can also be proved from corollary 3.6. □

Example 3.13. [Gabor type unitary systems] We proved in theorem 2.11 that, given a unimodular \( \lambda \in \mathbb{C} \) and a positive definite map \( K \) on \( \mathbb{Z}^2 \) that satisfies
\[
K((m+1,n),(m'+1,n')) = K((m,n),(m',n')) \quad (m,n,m',n' \in \mathbb{Z}),
\]
(3.12)
the necessary and sufficient condition is that
\[
K((m,n+1),(m',n'+1)) = \lambda^{m-m'}K((m,n),(m',n')) \quad (m,n,m',n' \in \mathbb{Z}),
\]
(3.13)
there is a Gabor type unitary system on \( H_K \) generated by two unitaries \( U_K \) and \( V_K \). As the reader probably expects, we look at the operators that intertwine these systems.

**Theorem 3.14.** Let \( \lambda \in \mathbb{C} \), \( |\lambda| = 1 \) and \( K, K' \) positive definite maps on \( \mathbb{Z}^2 \) satisfying the corresponding relations (3.12) and (3.13). Let \( L : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C} \) be the property that \( L^2 \leq cKK' \) for some \( c > 0 \). If \( L \) satisfies the relations (3.12) and (3.13), (with \( K \) replaced by \( L \), of course), then there is a unique operator \( S : H_K \to H_{K'} \) such that
\[
SU_K = U_{K'}S, \quad SV_K = V_{K'}S,
\]
(3.14)
\[
\langle SU_K^n V_K^m | V_{K'}^{m'} S^n \rangle = L((m,n),(m',n')) \quad (m,n,m',n' \in \mathbb{Z}).
\]
(3.15)
\[\langle SU_K^n V_K^m | V_{K'}^{m'} S^n \rangle \quad \text{are given by theorem 2.11}. \]
Proof. We recall that \( U_K^n V_K^m \|0\rangle = v_K(m,n) \) and similarly for \( K' \), \( (m,n \in \mathbb{Z}) \) (see the proof of theorem 2.11). Theorem 3.12 shows that there is an operator \( S : H_K \to H_{K'} \) with \( \|S\| \leq \sqrt{c} \) and such that (3.15) holds. We need to check (3.14). Take \( m,n,m',n' \in \mathbb{Z} \) and compute:
\[
\langle SU_K^n V_K^m | V_{K'}^{m'} S^n \rangle = L((m+1,n),(m',n'))
\]
\[
= \langle SU_K^n V_K^m | V_{K'}^{m'} S^n \rangle = \langle SU_K^n V_K^m | V_{K'}^{m'} S^n \rangle
\]
\[
= L((m,n),(m'-1,n')) = L((m+1,n),(m',n')).
\]
The density of the linear spans of \( \{U_K^n V_K^m | m,n \in \mathbb{Z}\} \) and \( \{U_K^n V_K^m \|0\rangle \quad (m,n,m',n' \in \mathbb{Z}\} \) implies \( SU_K = U_{K'}S \). A similar calculation shows that \( SV_K = V_{K'}S \).

The converse follows from theorem 3.12 if \( L \) is defined by (3.10), the only thing that remains to be verified is that \( L \) satisfies (3.12) and (3.13), but this is a consequence of (3.11) and \( U_K V_K = \lambda V_K U_K, U_K V_{K'} = \lambda V_{K'} U_{K'} \).

**Corollary 3.15.** If \( K, K' \) are positive definite maps on \( \mathbb{Z}^2 \) satisfying the relations (3.12) and (3.13) and \( \lambda \leq cK \) then \( S \) is a unique positive definite operator \( S \) on \( H_K \) that commutes with \( U_K \) and \( V_K \) and
\[
\langle SU_K^n V_K^m | V_{K'}^{m'} S^n \rangle = K((m,n),(m',n')) \quad (m,n,m',n' \in \mathbb{Z}).
\]
(3.16)
Conversely, if \( S \) is a positive operator that commutes with \( U_K \) and \( V_K \) then \( K' \) defined by (3.10) satisfies (3.12) and (3.13).

**Proof.** The proof follows the same lines as before. □

**Remark 3.16.** Theorem 3.12 gives us a general existence result for intertwining operators. The next theorem answer the question what conditions should be imposed on \( L \) to obtain that its associated operator \( S \) intertwines the extra structure existent on \( H_K \)? We saw that for \( C^* \)-algebras the necessary and sufficient condition is that \( L(x,y) = \varphi_0(y^*x) \) for some linear \( \varphi_0 \), for groups we must have \( L(x,y) = L(zx,zy) \) and for Gabor type unitary systems, \( L \) must satisfy the relations (3.14) and (3.13).
Example 3.17. [Intertwiners of wavelet representations] We mentioned in example 2.13 and theorem 2.14 how wavelet representations can be associated to positive functions \( h \in L^1(\mathbb{T}) \) with \( R_{m_0,m_0}(h) = h \). In [Dut1] and [Dut2] we studied the operators that intertwine these representations. We indicate now how these can be connected to Kolmogorov’s theorem. So we will recall the results from [Dut1] and we sketch the proof based on theorem 3.2.

Given \( h \) as in theorem 2.14 call \((U_h, \pi_h, H_h, \phi_h)\) the cyclic representation of \( \mathfrak{A}_N \) associated to \( h \). Also, define the transfer operator associated to a pair \( m_0, m_0’ \in L^\infty(\mathbb{T}) \) by

\[
R_{m_0,m_0’}(z) = \frac{1}{N} \sum_{w \in m_0^{-1}(1)} m_0(w)\overline{m_0’(w)} f(w), \quad (z \in \mathbb{T}, f \in L^1(\mathbb{T})).
\]

Theorem 3.18. Let \( m_0, m_0’ \in L^\infty(\mathbb{T}) \) be non-singular and \( h, h’ \in L^1(\mathbb{T}) \), \( h, h’ \geq 0 \), \( R_{m_0,m_0}(h) = h, R_{m_0,m_0’}(h’) = h’ \). Let \((U, \pi, H, \phi), (U’, \pi’, H’, \phi’)\) be the cyclic representations corresponding to \( h \) and \( h’ \) respectively.

If \( h_0 \in L^1(\mathbb{T}) \), \( R_{m_0,m_0’}(h_0) = h_0 \) and \( |h_0|^2 \leq c h h’ \) for some \( c > 0 \) then there exists a unique operator \( S : H \to H’ \) such that

\[
SU = U’ S, \quad S\pi(f) = \pi’(f) S, \quad (f \in L^\infty(\mathbb{T}))
\]

(3.17)\( S\pi(f) \phi | \phi’ = \int_T f h_0 \, d\mu, \quad (f \in L^\infty(\mathbb{T})). \)

Moreover \( \|S\| \leq \sqrt{c} \). Conversely, if \( S \) is an operator that satisfies (3.17), then there is a unique \( h_0 \in L^1(\mathbb{T}) \) with \( R_{m_0,m_0’}(h_0) = h_0 \) such that (3.8) holds. Moreover, \( |h_0|^2 \leq \|S\|^2 h h’ \).

Proof. Define \( X \) as in the proof of theorem 2.14. For all \((f, n), (g, m) \in X\), we want to obtain

\[
L((f, n), (g, m)) = \langle S^U \pi^n(f) \phi | U’^{-m} \pi’^n(g) \phi’ \rangle
\]

\[
= \langle SU^m \pi(f) \phi | U’^{-m} \pi’(g) \phi’ \rangle
\]

\[
= \langle S \pi(f^m) \phi | U’(g^m) \phi’ \rangle
\]

\[
= \int_T f(z^m) \overline{g(z^m)} d\mu(z). \quad (f \in H).
\]

Keep the first and the last terms of the equality and this defines \( L \). \( L \) will give rise to \( S \) by theorem 3.2. For the details of the required computations, see [Dut1]. The converse can be also obtained from theorem 3.2 but here the generality of theorem 3.2 isn’t really needed. \( \square \)

4. Frames and Dilations

Recall that a set \( \{x_i | i \in I\} \) of vectors in a Hilbert space \( H \) is called a frame if there are two constants \( A, B > 0 \) such that

\[
A\|f\|^2 \leq \sum_{i \in I} |\langle f, x_i \rangle|^2 \leq B\|f\|^2, \quad (f \in H).
\]

If \( A = B = 1 \) the set \( \{x_i | i \in I\} \) is called a normalized tight frame.

Frames have been used extensively in applied mathematics for signal processing and data compression. They play a central role in wavelet theory and the analysis of Gabor systems.

In [Hil1] the normalized tight frames are interpreted as projections of orthonormal bases and it is proved there that Gabor type normalized tight frames can be dilated to Gabor type orthonormal bases, and normalized tight frames generated by groups can be dilated to orthonormal bases generated by the same group (see theorem 3.8 and 4.8 in [Hil1]). We will revisit these theorems and show that they are immediate consequences of a general result which proves that any normalized tight frame can be dilated to an orthonormal basis in such a way that the extra structure that may exists is preserved under the dilation.

We begin with a proposition that establishes what positive definite maps give rise to normalized tight frames when represented on a Hilbert space.
Proposition 4.1. Let $K$ be a positive definite map on a set $X$. Then \( \{v_K(x) \mid x \in X\} \) is a normalized tight frame if and only if for all \( x, y \in X, \xi, \eta \in \mathbb{C} \), (\( i \in \{1, \ldots, n\} \)):

\[
\sum_{i,j=1}^{n} K(x_i, x_j) \xi_i \overline{\eta}_j = \sum_{x \in X} \left| \sum_{i=1}^{n} K(x, x_i) \xi_i \right|^2.
\]

Proof. If \( \{v_K(x) \mid x \in X\} \) is a normalized tight frame then take \( f = \sum_{i=1}^{n} \xi_i v_K(x_i) \). The fact that

\[
\|f\|^2 = \sum_{x \in X} | \langle f \mid v_K(x) \rangle |^2,
\]

translates into (4.1). For the converse, we only need to verify Proposition 1.1 for \( \delta \) in a dense subset of \( H_K \) (see [HeVe] lemma 1.10). Since the linear span of \( \{v_K(x) \mid x \in X\} \) is dense in \( H_K \), we can take \( f = \sum_{i=1}^{n} \xi_i v_K(x_i) \) and \( \|f\|^2 \) follows from (4.1).

Definition 4.2. A positive definite map \( K \) on \( X \) is called a NTF if and only if \( \{v_K(x) \mid x \in X\} \) is a normalized tight frame for \( H_K \).

Before we prove our general result we note that, if \( \delta : X \times X \to \mathbb{C} \) is defined by

\[
\delta(x, y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise}
\end{cases}
\]

then \( \delta \) is a positive definite map and \( \{v_\delta(x) \mid x \in X\} \) is an orthonormal basis for \( H_\delta \).

Proposition 4.3. If \( K \) is a NTF positive definite map on \( X \) then \( K \leq \delta \).

Proof. By definition, \( \{v_K(x) \mid x \in X\} \) is a normalized tight frame for \( H_K \). Then, by Proposition 1.1, there exists a Hilbert space \( H \) containing \( H_K \) as a subspace and an orthonormal basis \( \{e(x) \mid x \in X\} \) such that, if \( P \) is the projection onto \( H_K \), then \( Pe(x) = v_K(x) \) for all \( x \in X \).

Now take \( x_i \in X, \xi_i \in \mathbb{C}, (i \in \{1, \ldots, n\}) \):

\[
\sum_{i,j=1}^{n} K(x_i, x_j) \xi_i \overline{\eta}_j = \left( \sum_{i=1}^{n} \xi_i v_K(x_i) \right) \overline{\left( \sum_{i=1}^{n} \xi_i v_K(x_i) \right)}
\]

\[
= \left( P\left( \sum_{i=1}^{n} \xi_i e(x_i) \right) \right) \overline{\left( P\left( \sum_{i=1}^{n} \xi_i e(x_i) \right) \right)}
\]

\[
\leq \|P\| \left( \sum_{i=1}^{n} \xi_i e(x_i) \right)^2 = \sum_{i=1}^{n} |\xi_i|^2
\]

\[
= \sum_{i,j=1}^{n} \delta(x_i, x_j) \xi_i \overline{\eta}_j.
\]

Therefore \( K \leq \delta \).

Theorem 4.4. If \( K, K' \) are NTF positive definite maps on a countable set \( X \), \( K \leq cK' \) for some \( c > 0 \), then there exists an isometry \( W : H_K \to H_{K'} \), \( W \) is induced by \( K' \), that is

\[
\langle Wu_K(x) \mid v_{K'}(y) \rangle = K(x, y), \quad (x, y \in X),
\]

the projection \( P \) onto \( WH_K \) is also induced by \( K' \), i.e.

\[
\langle Pv_K(x) \mid v_{K'}(y) \rangle = K(x, y), \quad (x, y \in X),
\]

and \( PV_{K'}(x) = Wv_K(x) \).

Proof. Since \( K \leq cK' \), by corollary 3.10 there exists a positive operator \( S \) on \( H_{K'} \) such that

\[
\langle S_{v_{K}}(x) \mid v_{K'}(y) \rangle = K(x, y), \quad (x, y \in X).
\]

Since \( S \) is positive, it has a positive square root \( S^\frac{1}{2} \). Then

\[
\langle S^\frac{1}{2} v_{K'}(x) \mid S^\frac{1}{2} v_{K'}(y) \rangle = \langle Sv_{K'}(x) \mid v_{K'}(y) \rangle = K(x, y), \quad (x, y \in X).
\]
Take
\[ H = \overline{\text{span}}\{S^*v_K(x) \mid x \in X\}. \]
By the uniqueness part of Kolmogorov’s theorem, there is a unitary \( W : H_K \to H \) such that
\[ Wv_K(x) = S^*v_K(x), \quad (x \in X). \]
But then \( \{S^*v_K(x) \mid x \in X\} \) is a normalized tight frame for \( H \). Also, we know that \( \{v_K(x) \mid x \in X\} \) is a normalized tight frame for \( H_K \). So \( S^* : H_K \to H \) maps a normalized tight frame to a normalized tight frame, therefore it must be a co-isometry (see [HL] proposition 1.9). It follows that \( S^* (S^*)^* : H \to H \) is the identity on \( H \) so \( S \) is the identity on \( H \).

We also know that \( \text{range}(S^*) = \text{range}(S) \). This implies that \( S(Sv) = Sv \) for all \( v \in H_K \) and, as \( S \geq 0 \), \( S \) is the projection onto \( H \). Consequently, we also have \( S = S^* \) and everything follows now by an easy computation:
\[ Sv_K(x) = S^*v_K(x) = Wv_K(x), \quad (x \in X). \]
\[ \langle Wv_K(x) \mid v_K(y) \rangle = \langle Sv_K(x) \mid v_K(y) \rangle = K(x, y), \quad (x, y \in X). \]

\[ \square \]

**Remark 4.5.** Theorem [HL] can be used to construct dilation theorems for unitary systems (the reader should have in mind the specific examples of groups and Gabor type unitary systems).

Recall some definitions from [HL]. If \( \mathcal{U} \) is a countable set of unitaries on a Hilbert space \( H \), then \( \xi \in H \) is called a complete wandering vector (complete normalized tight frame vector) if \( \{U\xi \mid U \in \mathcal{U}\} \) is an orthonormal basis (normalized tight frame) for \( H \). A dilation theorem will take the following form:

If \( \mathcal{U} \) is a unitary system on a Hilbert space \( H \) that has a complete normalized tight frame vector \( \eta \), then there is a Hilbert space \( H_1 \) that contains \( H \) and a unitary system \( U_1 \) on \( H_1 \) such that \( U_1 \) has a complete wandering vector \( \xi \) and if \( P \) is the projection onto \( H \) then \( P\xi = \eta \), \( P \) commutes with \( U_1 \) and \( U_1 \mapsto U_1|_H \) is an isomorphism of \( U_1 \) onto \( \mathcal{U} \).

The proof will be guided by the following steps:
1. Construct \( K : \mathcal{U} \times \mathcal{U} \to \mathbb{C}, K(x, y) = \langle x\eta \mid y\eta \rangle \); then \( K \) is an NTF positive definite map and \( H_K = H, v_K(x) = x\eta, (x \in \mathcal{U}) \) and \( \mathcal{U} \) is the extra structure \( U_K \) induced by \( K \).
2. Verify that \( \delta : \mathcal{U} \times \mathcal{U} \to \mathbb{C} \) satisfies the required compatibility conditions with \( \mathcal{U} \).
3. Construct \( H_\delta, v_\delta \) and the additional structure \( \mathcal{U}_\delta \) with cyclic vector \( \xi_\delta \) which is a complete wandering vector for \( \mathcal{U}_\delta \).
4. Since \( K \leq \delta \) (proposition [HL]), according to theorem [HL] there is an isometry \( W : H \to H_\delta \) which is induced by \( K \); the projection \( P \) onto \( WH \) is also induced by \( K \) and \( P\xi_\delta = \eta \). As \( K \) is compatible with the structure \( \mathcal{U} \), \( W \) will intertwine \( \mathcal{U} \) and \( \mathcal{U}_\delta \) and \( P \) commutes with \( \mathcal{U}_\delta \). So \( WH \) is invariant for \( \mathcal{U}_\delta \) and \( WUUW^{-1} = U_\delta|_H \) for all \( U \in \mathcal{U} \) (\( U_\delta \) is the unitary in \( \mathcal{U}_\delta \) that corresponds to \( U \) in the representation). 5. Identify \( H \) with \( WH \) and everything will follow.

We will use the guidelines of remark [HL] to show how one can obtain the dilation theorems 3.8 and 4.8 from [HL] for groups and Gabor type unitary systems.

**Theorem 4.6.** [HL] Suppose \( \mathcal{U} \) is a unitary group on \( H \) with a complete normalized tight frame vector \( \eta \). Then there is a Hilbert space \( H_1 \) containing \( H \) and a unitary group \( U_1 \) such that \( U_1 \) has a complete wandering vector \( \xi \), if \( P \) is the projection onto \( H \) then \( P \) commutes with \( U_1 \), \( P\xi = \eta \) and \( U_1 \mapsto U_1|_H \) is an isomorphism of \( U_1 \) onto \( \mathcal{U} \). Consequently, \( PU_1\xi = U_1|_H\eta \) for all \( U_1 \in U_1 \) (that is the normalized tight frame \( \{U\eta \mid U \in \mathcal{U}\} \) can be dilated to the orthonormal basis \( \{U_1\xi \mid U_1 \in U_1\} \)).

**Proof.** Define \( K : \mathcal{U} \times \mathcal{U} \to \mathbb{C}, K(x, y) = \langle x\eta \mid y\eta \rangle \) for \( x, y \in \mathcal{U} \). It is clear that \( K \) is an NTF positive definite map with
\[ K(xz, yz) = K(x, y), \quad (x, y, z \in \mathcal{U}) \]
and \( H_K = H, v_K(x) = x\eta \) and the representation \( \pi_K \) given by theorem [HL] is \( \pi_K(x) = x \) for \( x \in \mathcal{U} \).
It is also clear that $\delta$ satisfies a relation of type (5.3) so it is compatible with the group structure and by theorem (2.3) it induces a cyclic representation $(H_0, \pi_0, \xi_0)$ of $U$ with $\xi_0 = v_0(1)$ a complete wandering vector. By proposition (4.3) $K \leq \delta$. By theorem (4.2) there is an isometry $W : H \to H_0$ which is induced by $K$, the projection $P$ onto $WH$ is also induced by $K$ and $P\xi_0 = \eta$. Then, by theorem (3.11) $W$ is intertwining that is

$$Wx = \pi(x)W, \quad (x \in U),$$

and $P$ is in the commutant of $\pi_0(U)$. So $WH$ is invariant for all $\pi_0(x), x \in U$ and $WxW^{-1} = \pi_0(x)$ for $x \in U$.

Identify $H$ with $WH$ and define $U_1 = \pi_0(U), \xi = \xi_0$ and everything follows. \hfill \Box

**Theorem 4.7.** [H] Let $U = \{U^mV^n \mid m, n \in \mathbb{Z}\}$ be a Gabor type unitary system associated to $\lambda$ on a Hilbert space $H$. Suppose $U$ has a complete normalized tight frame vector $\eta \in H$. Then there is a Gabor type unitary system $U_1(= \{U_1^mV_1^n \mid m, n \in \mathbb{Z}\}$ associated to $\lambda$ on a Hilbert space $H_1$ containing $H$, such that $U_1$ has a complete wandering vector $\xi$ and if $P$ is the projection onto $H$ then $P$ commutes with $U_1$ and $V_1$, $P\xi = \eta$ and $U = U_1|_H \ V = V_1|_H$.

**Proof.** The proof is analogous to the proof of theorem (4.6), the only difference is to verify that $\delta$ satisfies the compatibility relations (5.12) and (5.13) and this is trivial. \hfill \Box

5. A DILATION THEOREM FOR WAVELETS

Let us recall the algorithm for the construction of compactly supported wavelets. For details we refer the reader to [Dau92] for the scale $N = 2$ and to [BraJo97] for arbitrary scale $N$.

One starts with the low-pass filter $m_0 \in L^2(\mathbb{T})$ which is a trigonometric polynomial that satisfies $m_0(1) = \sqrt{N}$ and the quadrature mirror filter condition

$$\frac{1}{N} \sum_{w \in \mathbb{Z}} |m_0|^2(w) = 1, \quad (z \in \mathbb{T}).$$

Then define the scaling function $\varphi \in L^2(\mathbb{R})$ by taking the inverse Fourier transform of

$$\hat{\varphi}(x) = \prod_{k=1}^{\infty} \frac{m_0(z^k)}{\sqrt{N}}, \quad (x \in \mathbb{R}).$$

To construct wavelets one needs the high-pass filters $m_1, ..., m_{N-1} \in L^\infty(\mathbb{T})$ such that the matrix

$$\frac{1}{\sqrt{N}} \begin{pmatrix} m_0(z) & m_0(\rho z) & \cdots & m_0(\rho^{N-1} z) \\ m_1(z) & m_1(\rho z) & \cdots & m_1(\rho^{N-1} z) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1}(z) & m_{N-1}(\rho z) & \cdots & m_{N-1}(\rho^{N-1} z) \end{pmatrix}$$

is unitary for a.e $z \in \mathbb{T}$.

($\rho = e^{2\pi i/N}$).

When $N = 2$ a choice for $m_1$ can be

$$m_1(z) = z \overline{m_0(-z)} f(z^2), \quad (z \in \mathbb{T}),$$

where $|f(z)| = 1$ on $\mathbb{T}$.

The wavelets are defined as follows:

$$\hat{\psi}_i(x) = \frac{m_i(z)}{\sqrt{N}} \hat{\varphi} \left( \frac{x}{N} \right), \quad (x \in \mathbb{R}, i \in \{1, ..., N-1\}),$$

or, in terms of the wavelet representation,

$$\psi_i = U^{-1} \pi(m_i) \varphi, \quad (i \in \{1, ..., N-1\}).$$

It is known that, in order to achieve orthogonality, extra conditions must be imposed on $m_0$. If $R_{m_0,m_0}$ has only one continuous fixed point (up to a multiplicative constant), the set

$$\{U^mT^n \psi_i \mid m, n \in \mathbb{Z}, i \in \{1, ..., N-1\}\}$$
is an orthonormal basis for $L^2(\mathbb{R})$.

However, when this extra condition is not satisfied, one still gets good properties, namely, the fact that the above set is a normalized tight frame for $L^2(\mathbb{R})$. In the sequel, we show how one can dilate this normalized tight frame to an orthonormal basis in such a way that the multiresolution structure is preserved so that "wavelets" in a space bigger then $L^2(\mathbb{R})$ are obtained.

We begin with a proposition that explains the multiresolution structure of the cyclic representations presented in example 2.13.

In the sequel we define $m_0 \in L^\infty(\mathbb{T})$ to be non-singular if the set \{ $z \in \mathbb{T}$ | $m_0(z) = 0$ \} has zero measure and $|m_0|$ is not constant 1 a.e.

**Proposition 5.1.** Let $m_0 \in L^\infty(\mathbb{T})$ be non-singular, $h \in L^1(\mathbb{T})$, $h \geq 0$ and $R_{m_0, m_0} h = h$. Let $(U_h, \pi_h, H_h, \varphi_h)$ be the cyclic representation associated to $h$. Define $T_h = \pi_h(z)$,

\[
V_0^h = \text{span}(T_h \varphi_h | k \in \mathbb{Z}),
\]

\[
V_j^h = U^{-j}_h V_0^h, \quad (j \in \mathbb{Z}).
\]

Then
\[
(5.6) \quad U_h T_h U_h^{-1} = T_h^N,
\]

\[
(5.7) \quad V_j^h \subset V_{j+1}^h, \quad (j \in \mathbb{Z}),
\]

\[
(5.8) \quad \bigcup_{j \in \mathbb{Z}} V_j^h = H_h,
\]

\[
(5.9) \quad \cap_{j \in \mathbb{Z}} V_j^h = \{0\}
\]

Assume $h = 1$. Then
\[
(5.10) \quad \{T_k^h \varphi_h | k \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0^h.
\]

If $m_1, \ldots, m_{N-1}$ satisfy (5.3) and
\[
(5.11) \quad \psi_i^h = U_h^{-1} \pi_h(m_i) \varphi_h, \quad (i \in \{1, \ldots, N-1\}),
\]

then
\[
(5.12) \quad \{T_k^h \psi_i^h | k \in \mathbb{Z}, i \in \{1, \ldots, N-1\}\} \text{ is an orthonormal basis for } V_1^h \oplus V_0^h
\]

and
\[
(5.13) \quad \{U_m^h T_n^h \psi_i^h | m, n \in \mathbb{Z}, i \in \{1, \ldots, N-1\}\} \text{ is an orthonormal basis for } L^2(\mathbb{R}).
\]

**Proof.** (5.6) follows from $U_h \pi_h(f(z)) U_h^{-1} = \pi_h(f(z^N))$ with $f(z) = z$.

If $f(z) = \sum_{k=-p}^{p} a_k z^k$ is a trigonometric polynomial then

\[
\pi_h(f) \varphi_h = \sum_{k=-p}^{p} a_k T_k^h \varphi_h \in V_0^h.
\]

Each $f \in L^\infty(\mathbb{T})$ is the pointwise limit of a uniformly bounded sequence of trigonometric polynomial, hence, by lemma 2.8 in [10], $\pi_h(f) \varphi_h \in V_0^h$. Then

\[
U_h \pi_h(f) \varphi_h = \pi_h(f(z^N)) U_h \varphi_h = \pi_h(f(z^N) m_0(z)) \varphi_h \in V_0^h
\]

so $V_1^h \subset V_0^h$ and this implies (5.7).

Also $U_h^{-n} \pi_h(f) \varphi_h \in V_n^h$ for all $f \in L^\infty(\mathbb{T})$ and $n \in \mathbb{Z}$ and (5.8) follows by density. (5.9) is proved in theorem 5.6 from [10].

If $h = 1$ then, for $k \in \mathbb{Z}$,

\[
\langle T_k^h \varphi_h \rangle = \langle \pi_h(z^k) \varphi_h \rangle = \int_\mathbb{T} z^k d\mu = \delta_{k,0},
\]

so (5.10) is valid.
It remains to prove (5.12) because (5.13) follows from this immediately. The argument is essentially the one in [Bra16] theorem 10.1. We will include it here to make sure everything works.

For \( k, l \in \mathbb{Z} \) and \( i, j \in \{1, \ldots, N - 1\} \) we have
\[
\langle T^k_h \psi_i^h | T^l_h \psi_j^h \rangle = \langle \pi_h(z^k) U^{-1}_h \pi_h(m_i) \varphi_h | \pi_h(z^l) U^{-1}_h \pi_h(m_j) \varphi_h \rangle
\]
\[
= \langle U^{-1}_h \pi_h(z^{Nk} m_i(z)) \varphi_h | U^{-1}_h \pi_h(z^{Nl} m_j(z)) \varphi_h \rangle
\]
\[
= \int_{\mathbb{T}} z^{N(k-l)} m_i(z) \overline{m_j(z)} \, d\mu
\]
\[
= \int_{\mathbb{T}} \frac{1}{N} \sum_{w \in \mathbb{Z}^N = z} m_i(w) \overline{m_j(w)} \, d\mu = \delta_{i,j} \delta_{k,l},
\]
for the last equality we used (5.3). So
\[
\{T^k_h \psi_i^h | k \in \mathbb{Z}, i \in \{1, \ldots, N - 1\}\}
\]
is an orthonormal set.

Take, \( U^{-1}_h \pi_h(m) \varphi_h \in V^h \), \( m \in L^\infty(\mathbb{T}) \) (vectors of this form are dense in \( V^h \)).

For \( \pi_h \psi_i^h \perp V^0 \) is equivalent to, for all \( f \in L^\infty(\mathbb{T}) \):
\[
0 = \langle U^{-1}_h \pi_h(m) \varphi_h | \pi_h(f) \varphi_h \rangle = \langle \pi_h(m) \varphi_h | \pi_h(f(z^N) m_0(z)) \varphi_h \rangle
\]
\[
= \int_{\mathbb{T}} m(z) \overline{f(z^N)} m_0(z) \, d\mu
\]
\[
= \int_{\mathbb{T}} \frac{1}{N} \sum_{w \in \mathbb{Z}^N = z} m(w) \overline{m_0(w)} \overline{f(z)} \, d\mu
\]
which is equivalent to
\[
\frac{1}{N} \sum_{w \in \mathbb{Z}^N = z} m(w) \overline{m_0(w)} = 0 \text{ a.e. on } \mathbb{T}.
\]

This shows in particular that \( \psi_i^h \perp V^0 \) for all \( i \in \{1, \ldots, N - 1\} \). Also, the vector
\[
\tilde{m}(z) = (m(z), m(\rho z), \ldots, m(\rho^{N-1} z))
\]
\((\rho = e^{-2\pi i/N})\) must be perpendicular to the vector
\[
\tilde{m}_0(z) = (m_0(z), m_0(\rho z), \ldots, m_0(\rho^{N-1} z))
\]
for almost all \( z \), so
\[
\tilde{m}(z) = \sum_{k=1}^{N-1} \mu_k(z) \tilde{m}_k(z)
\]
where \( \mu_k(z) = \langle \tilde{m}(z) | \tilde{m}_k(z) \rangle \) (which shows that \( \mu_k \in L^\infty(\mathbb{T}) \)).

Since \( \tilde{m}(\rho z) \) is a circular permutation of \( \tilde{m}(z) \), it follows that we must have \( \mu_k(\rho z) = \mu_k(z) \), that is \( \mu_k(z) = \lambda_k(z^N) \) for some \( \lambda_k \in L^\infty(\mathbb{T}) \).

Then
\[
m(z) = \sum_{k=1}^{N-1} \lambda_k(z^N) m_k(z), \quad (z \in \mathbb{T}),
\]
and we compute
\[
U^{-1}_h \pi_h(m) \varphi_h = U^{-1}_h \pi_h \left( \sum_{k=1}^{N-1} \lambda_k(z^N) m_k(z) \right) \varphi_h = \sum_{k=1}^{N-1} \pi_h(\lambda_k) \psi_k^h.
\]
and this shows that
\[
U^{-1}_h \pi_h(m) \varphi_h \in \text{span}\{T^k_h \psi_i^h | k \in \mathbb{Z}, i \in \{1, \ldots, N - 1\}\}
\]
by an argument similar to the one used in the beginning of the proof (now for \( \psi_i^h \) instead of \( \varphi_h \)). This completes the proof of (5.12). \( \square \)
Motivated by the discussion in the beginning of this section, we give a dilation theorem for wavelets. The theorem describes how one can dilate a normalized tight frame wavelet to an orthonormal wavelet in a bigger space.

**Theorem 5.2.** Let \( m_0 \in L^\infty(\mathbb{T}) \) be a non-singular filter with \( R_{m_0,m_0} 1 = 1 \), \( h \in L^\infty(\mathbb{T}) \), \( h \geq 0 \), \( R_{m_0,m_0} h = h \), and consider \((U_h, \pi_h, H_h, \varphi_h)\), the cyclic representation associated to \( h \). Assume also there are given filters \( m_1,...,m_{N-1} \in L^\infty(\mathbb{T}) \) such that (5.13) holds, define \( \psi_i \) as in (5.14), and suppose
\[
\{U_h^m T_h^i \psi_i \mid m,n \in \mathbb{Z}, i \in \{1,...,N-1\}\}
\]is a normalized tight frame for \( H_h \) (\( T_h = \pi_h(z) \)).

Then, if \((U_1, \pi_1, H_1, \varphi_1)\) is the cyclic representation associated to the constant function 1, then there exists an isometry \( W: H_h \to H_1 \) with the following properties:

(i) \( WU_h = U_1 W, \ W \pi_h(f) = \pi_1(f)W \) for all \( f \in L^\infty(\mathbb{T}) \);

(ii) If \( P \) is the projection onto \( WH_h \) then
\[
(5.14) \quad PU_1 = U_1 P, \quad P \pi_1(f) = \pi_1(f)P, \quad (f \in L^\infty(\mathbb{T}));
\]

(iii) If \( \psi_i = U_1^{-1} \pi_1(m_i) \varphi_1, i \in \{1,...,N-1\} \) then
\[
(5.15) \quad P \varphi_1 = W \varphi_h.
\]

Also there are given filters \( \{W \} \) with \( T_h = \pi_1(z) \);
\[
(5.16) \quad \{U_h^m T_h^i \psi_i \mid m,n \in \mathbb{Z}, i \in \{1,...,N-1\}\} \text{ is an orthonormal basis for } H_1,
\]
where \( T = \pi_1(z) \);
\[
(5.17) \quad P \psi_i = W T_h^i, \quad (i \in \{1,...,N-1\}.
\]

**Proof.** The proof is similar to the one of theorem 1.3 but some additional arguments are needed. Since \( h \in L^\infty(\mathbb{T}) \), we have \( |h|^2 \leq \|h\|_1 \) so, by theorem 3.18 there is a positive operator \( S \) on \( H_1 \) that commutes with \( U_1 \) and \( \pi_1 \) and
\[
(5.18) \quad \langle S \pi_1(f) \varphi_1 \mid \varphi_1 \rangle = \int_\mathbb{T} fh \, d\mu, \quad (f \in L^\infty(\mathbb{T})).
\]

\( S \) has a positive square root \( S^{1/2} \) that commutes with \( U_1 \) and \( \pi_1 \). Also the projection \( P \) onto the range \( H \) of \( S^{1/2} \) must commute with \( U_1 \) and \( \pi_1 \). Then we can restrict \( \pi_1 \) and \( U_1 \) to \( H \) and
\[
(5.19) \quad \langle \pi_1(f)S^{1/2} \varphi_1 \mid S^{1/2} \varphi_1 \rangle = \int_\mathbb{T} fh \, d\mu,
\]
\[
U_1 S^{1/2} \varphi_1 = \pi_1(m_0)S^{1/2} \varphi_1.
\]
The uniqueness part of theorem 3.3 implies that there is a unitary \( W \) from \( H_h \) to \( H \) with \( W \varphi_h = S^{1/2} \varphi_1 \), \( WU_h = U_1 W \) and \( W \pi_h(f) = \pi_1(f)W \) for \( f \in L^\infty(\mathbb{T}) \). From these commuting properties of \( W \) and \( P \) it follows that
\[
W(U^m_h T_h^i \psi_i) = S^{1/2}(U_1^m T_1^i \psi_i), \quad (m,n,i \in \{1,...,N-1\}).
\]

Hence, \( S^{1/2} \) maps an orthonormal basis to a normalized tight frame so it must be a co-isometry. Then, proceeding as in the proof of theorem 3.3 we get that \( S = S^{1/2} = P \) and everything follows.

When \( m_0 \) is a regular filter (we will give the precise meaning of that in a moment), we can really get our hands on the abstract cyclic representation associated to the constant function 1 so that we obtain a very concrete dilation theorem for non-orthogonal wavelets in \( L^2(\mathbb{R}) \). The construction is given in the next theorem and it is based on the results presented in [1,2].

Before we state the result, some definitions are needed. A vector \((z_1,z_2,...,z_p)\) is called an \( m_0 \)-cycle if \( z_1^N = z_2, z_2^N = z_3, ..., z_p^N = z_1 \), \( z_i \) are distinct and \( |m_0(z_i)| = \sqrt{N} \) for all \( i \in \{1,...,p\} \). For \( f \in L^\infty(\mathbb{T}) \) and \( z_0 \in \mathbb{T} \) define \( \alpha_{z_0}(f)(z) = f(zz_0) \) for \( z \in \mathbb{T} \). For \( n \in \mathbb{N} \)
\[
m^{(n)}_0(z) = m_0(z)m_0(z^N)...m_0(z^{N^{n-1}}), \quad (z \in \mathbb{T}).
\]
Theorem 5.3. Let \( m_0 \) be a Lipschitz function on \( T \) with finitely many zeroes \( m_0(1) = \sqrt{N} \), \( R_{m_0,m_0} = 1 \). Let \( C_j = (z_{j,1},...,z_{j,p_j}) \) be the \( m_0 \)-cycles, \( j \in \{1,...,n\} \), \( m_0(z_{j,k}) = \sqrt{N} e^{i \theta_{j,k}} \) for all \( k \in \{1,...,p_j\}, j \in \{1,...,n\} \), \( \theta_j = \theta_{j,1} + ... + \theta_{p_j,j} \).

For each \( j \in \{1,...,n\} \) define: \( H_j = L^2(\mathbb{R}^{p_j}) \), \( U_j : H_j \rightarrow H_j \)

\[
U_j(\xi_1,...,\xi_{p_j}) = \left(e^{i \theta_{j,1}} U_\xi_1,...,e^{i \theta_{p_j,j}} U_\xi_{p_j},e^{i \theta_{p_j,j}} U_\xi_1\right),
\]

where \( U_\xi(x) = \frac{1}{\sqrt{N}} f \left( \frac{x}{\sqrt{N}} \right) \) for \( \xi \in L^2(\mathbb{R}) \).

For \( f \in L^\infty(T) \)

\[
\pi_j(f)(\xi_1,...,\xi_{p_j}) = \left( \pi \left( \alpha_{z_{j,1}}(f) \right) (\xi_1),...\pi \left( \alpha_{z_{p_j,j}}(f) \right) (\xi_{p_j}) \right),
\]

where \( \pi \) is the representation on \( L^2(\mathbb{R}) \) defined in example 2.13.

Finally, define \( H_0 = H_1 \oplus ... \oplus H_n \), \( U_0 = U_1 \oplus ... \oplus U_n \), \( \pi_0(f) = \pi_1(f) \oplus ... \oplus \pi_n(f) \) for \( f \in L^\infty(T) \) and \( \varphi_0 = \varphi_1 \oplus ... \oplus \varphi_n \). Then \( (U_0,\pi_0, H_0, \varphi_0) \) is the cyclic representation associated to the constant function 1.

Also if \( C_1 \) is the trivial \( m_0 \)-cycle \( C_1 = (1) \) then \( H_1 = L^2(\mathbb{R}) \), \( U_1 = U \), \( \pi_1 = \pi \) and

\[
\tilde{\varphi}_1(x) = \prod_{i=1}^{\infty} m_0 \left( \frac{x}{\sqrt{N}} \right), \quad (x \in \mathbb{R}),
\]

so \( (U_1,\pi_1, H_1, \varphi_1) \) is the usual wavelet representation on \( L^2(\mathbb{R}) \).

If \( T_j = \pi_j(z) \), for \( j \in \{1,...,n\} \) then

\[
T_j(\xi_1,...,\xi_{p_j})(x) = (z_{j,1} T_1 \xi_1,...,z_{p_j,j} T_p \xi_{p_j}),
\]

where \( T_\xi(x) = \xi(x-1) \) for \( \xi \in L^2(\mathbb{R}) \).

Assume \( m_1,...,m_{N-1} \in L^\infty(T) \) satisfy 5.3. Define

\[
\psi_i^1 = U_1^{-1} \pi_1(m_i) \varphi_1(\in L^2(\mathbb{R})), \psi_i^0 = U_0^{-1} \pi_0(m_i) \varphi_0, \quad i \in \{1,...,N-1\}
\]

and let \( P_1 \) be the projection from \( H_0 \) onto \( H_1 \) and \( T_0 = T_1 \oplus ... \oplus T_n \). Then

\[
P_1 U_0 = U_0 P_1, \quad P_1 T_0 = T_0 P_1, \quad P_1 \pi_0(f) = \pi_0(f) P_1, \quad (f \in L^\infty(T));
\]

\[
U_0|_{H_1} = U_1(= U), \quad T_0|_{H_1} = T_1(= T), \quad \pi_0(f)|_{H_1} = \pi_1(f)(= \pi(f)), \quad (f \in L^\infty(T));
\]

\[
P_1 \varphi_0 = \varphi_1, \quad P_1 \psi_i^0 = \psi_i^1, \quad (i \in \{1,...,N-1\});
\]

\[
U_0 \varphi_0 = \pi_0(m_0) \varphi_0, \quad U_1 \varphi_1 = \pi_1(m_0) \varphi_1;
\]

\[
\{T_0^k \varphi_0 \mid k \in \mathbb{Z}\} \text{ is an orthonormal set};
\]

\[
\{U_0^n T_0^n \psi_i^0 \mid m, n \in \mathbb{Z}, i \in \{1,...,N-1\}\} \text{ is an orthonormal basis for } H_0;
\]

\[
\{U_1^n T_1^n \psi_i^1 \mid m, n \in \mathbb{Z}, i \in \{1,...,N-1\}\} \text{ is a normalized tight frame for } H_1 = L^2(\mathbb{R}).
\]
Proof. The fact that the cyclic representation associated to the constant function 1 is proved in [Dan2], one needs only to take the inverse Fourier transform of the representation presented there to obtain the one described here. Then (5.18) and (5.19) follow trivially from the definition, (5.20) follows from the definition and the commuting properties of $P_1$, (5.21) is included in the definition of the cyclic representation, (5.22) and (5.23) are consequences of proposition 5.1 and (5.24) (which is also well known, see [Dau92] or [BraJo97]) follows from the fact that the projection of an orthonormal basis is a normalized tight frame (see [HL]).

□

Example 5.4. We apply theorem 5.3 to the low-pass filter

$$m_0(z) = \frac{1 + z^3}{\sqrt{2}} = \sqrt{2}e^{-\frac{3\theta}{2}} \cos \left(\frac{3\theta}{2}\right), \quad (z = e^{-i\theta} \in \mathbb{T})$$

which is known to give non-orthogonal wavelets. The scale $N = 2$. Some short computations show that $m_0(1) = \sqrt{2}$, $R_{m_0,m_0}1 = 1$. The $m_0$-cycles are

$$C_1 = (z_{1,1} = 1), C_2 = (z_{2,1} = e^{2\pi i/3}, z_{2,2} = e^{\pi i/3})$$

$p_1 = 1, p_2 = 2$, $m_0(z_{1,1}) = m_0(z_{2,1}) = m_0(z_{2,2}) = \sqrt{2}$ so $\theta_{1,1} = \theta_{2,1} = \theta_{2,2} = 0$ and $\theta_1 = \theta_2 = 0$.

$$U_0 : L^2(\mathbb{R})^3 \to L^2(\mathbb{R})^3, U_0(\xi_1, \xi_2, \xi_3) = (U\xi_1, U\xi_3, U\xi_2).$$

$$T_0 : L^2(\mathbb{R})^3 \to L^2(\mathbb{R})^3, T_0(\xi_1, \xi_2, \xi_3) = (T\xi_1, e^{2\pi i/3}T\xi_2, e^{\pi i/3}T\xi_3).$$

Then, as $\alpha_{z_{1,1}}(m_0) = m_0$, $\alpha_{z_{2,1}}(m_0) = \alpha_{z_{2,2}}(m_0) = m_0$,

$$\tilde{\varphi}_{1,1}(x) = \prod_{l=1}^{\infty} e^{-\frac{3\pi i}{2l+1}} \cos \left(\frac{3x}{2l+1}\right) = e^{-\frac{3\pi i}{2}} \sin \left(\frac{3x}{2}\right), \quad (x \in \mathbb{R}),$$

$$\tilde{\varphi}_{2,1}(x) = \prod_{l=1}^{\infty} \frac{m_0^{(2)}(\frac{x}{2l})}{\sqrt{2}} \prod_{l=1}^{\infty} \frac{m_0^{(2)}(\frac{x}{2l})}{\sqrt{2}} = \tilde{\varphi}_{1,1}(x),$$

and similarly for $\tilde{\varphi}_{2,2}$. Hence, $\varphi_{1,1} = \varphi_{2,1} = \varphi_{2,2} =: \varphi = \frac{1}{3}\chi_{[0,3]},$ and $\varphi_0 = (\varphi, \varphi, \varphi)$.

To construct the wavelet we can pick

$$m_1(z) = \frac{1 - z^3}{\sqrt{2}}, \quad (z \in \mathbb{T}).$$

Then the wavelet $\psi_0 = (\psi_1, \psi_2, \psi_3)$ is given by

$$U_0\psi_0 = \frac{1}{\sqrt{2}}(\varphi_0 - T_0^3\varphi_0)$$

so

$$\psi_1 = \psi_2 = \psi_3 =: \psi = \frac{1}{3}(\chi_{[0,\frac{1}{3}]} - \chi_{[\frac{1}{3},1]}),$$

and

$$\{U_0^m T_n^0 \psi_0 | m, n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})^3$ which dilates the normalized tight frame of $L^2(\mathbb{R})$

$$\{U^m T^n \psi | m, n \in \mathbb{Z}\}.$$

Acknowledgements. The author wants to thank professor Şerban Strâtilă for pointing out the connection between the GNS construction and Kolmogorov’s theorem. This was the starting point and the key idea of this paper. Also many thanks to professor Palle Jorgensen for his suggestions and his constant support.
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