Non-Gaussian continuous-variable teleportation

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Abstract

We have recently shown that the output field in the Braunstein-Kimble protocol of teleportation is a superposition of two fields: the input one and a field created by Alice’s measurement and by displacement of the state at Bob’s station by using the classical information provided by Alice. We study here the noise added by teleportation and compare its influence in the Gaussian and non-Gaussian settings.

Keywords: teleportation; Gaussian; noise
I. INTRODUCTION

In the Braunstein-Kimble protocol for teleporting one-mode states of the electromagnetic field [1], two distant operators, Alice at a sending station, and Bob at a receiving terminal, share a two-mode entangled quantum state and exploit its nonlocal character as a quantum resource for teleporting a single-mode state whose density operator is denoted by $\rho_{in}$. Mode 1 of the shared resource having the density operator $\rho_{AB}$ is given to Alice and mode 2 is given to Bob. First Alice performs a complete projective measurement on the joint system described by the three-mode product state $\rho_{in} \otimes \rho_{AB}$ and then conveys its outcome to Bob via a classical communication channel. As a consequence of Alice’s measurement, the total state of the three-mode system collapses. Finally, Bob makes use of the information transmitted classically by Alice to transform his state into an output that is a replica of the original unknown input. Unfortunately, a perfect replica is obtained only for an ideally entangled state $\rho_{AB}$. In general, teleportation in the continuous-variable settings is a noise-generating process [2]. A comprehensive account for the role of teleportation in the context of continuous-variable quantum information is given in the reviews [3, 4]. The present paper is a continuation of our work on Braunstein-Kimble protocol in the characteristic-function (CF) description [2]. We focus here on describing the distortion of the teleported state in terms of the properties of the two-mode resource state. In Sec. 2 we review the steps of the Braunstein-Kimble protocol and derive the factorized CF of the teleported state. We discuss in Sec. 3 the properties of the distorting state by using the relation between its normally ordered correlation functions and two-mode correlation functions of the resource state. We find that the conclusions drawn in our paper [2] for a Gaussian resource are valid in general. In Sec. 4 we show that the mean occupancy in the distorting field equals the EPR-uncertainty of the resource state. This gives us reasons for using it as a measure of teleportation accuracy. Finally we discuss the case of pure resource states of given entanglement and show that the two-mode squeezed vacuum state (SVS) generates the minimal noise in the teleportation output.
II. TELEPORTATION REVISITED

Recall the one-to-one correspondence between the density operator $\rho$ of a $n$-mode field state and its CF defined as the expectation value of the $n$-mode displacement operator $\chi(\lambda_1, \lambda_2 \cdots \lambda_n) := \text{Tr}[\rho D(\lambda_1)D(\lambda_2) \cdots D(\lambda_n)]$. Here $D(\alpha) = \exp((\alpha \hat{a}^\dagger - \alpha^* \hat{a}))$ is a Weyl displacement operator and $\hat{a}$ denotes the annihilation operator. The density operators of the states involved in the protocol, $\rho_{in}$ and $\rho_{AB}$, were written as Weyl expansions,

$$\rho_{in} = \frac{1}{\pi} \int d^2 \lambda \chi_{in}(\lambda) D(-\lambda),$$

and

$$\rho_{AB} = \frac{1}{\pi^2} \int d^2 \lambda_1 d^2 \lambda_2 \chi_{AB}(\lambda_1, \lambda_2) D_1(-\lambda_1) D_2(-\lambda_2).$$

Here $\chi_{in}(\lambda)$ is the CF of the one-mode input state and $\chi_{AB}(\lambda_1, \lambda_2)$ describes the two-mode resource state. The steps of the Braunstein-Kimble protocol are summarized as follows.

- quantum measurement of the variables

$$\hat{Q}_m = \hat{q}_{in} - \hat{q}_1, \quad \hat{P}_m = \hat{p}_{in} + \hat{p}_1$$

with the canonical operators: $\hat{q}_j = (\hat{a}_j + \hat{a}_j^\dagger) / \sqrt{2}$, $\hat{p}_j = (\hat{a}_j - \hat{a}_j^\dagger) / (\sqrt{2}i)$. $\hat{a}_j$ and $\hat{a}_j^\dagger$ ($j = 1, 2$) are the amplitude operators for the entangled modes of the state $\rho_{AB}$.

The common eigenfunctions of the commuting observables $\hat{Q}_m, \hat{P}_m$ have the following expansion

$$|\Phi_{m,A}(q,p)\rangle = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} d\eta e^{i\eta q} |q + \eta\rangle_{in} \otimes |\eta\rangle_A,$$

where the pair $(q, p)$ denotes the outcome of the measurement.

- classical communication of the measurement’s results (performed by Alice to Bob) The state at Bob’s side predicted by quantum mechanics after Alice’s measurement is

$$\rho_B(q,p) \sim \text{Tr}_{in,A} \{[|\Phi(q,p)\rangle \langle \Phi(q,p)| \otimes I_B] (\rho_{in} \otimes \rho_{AB})\}.$$

- displacement of the state at Bob’s location with the parameter $\mu = q + ip$
Following the protocol step by step we have finally reached the following factorization formula for the CF of the teleported state \( \chi_{out}(\lambda) \)

\[
\chi_{out}(\lambda) = \chi_{in}(\lambda)\chi_{AB}(\lambda^*,\lambda).
\]  
(6)

Therefore \( \chi_{out}(\lambda) \) is the product between the CF of the input state \( \chi_{in}(\lambda) \) and a function only depending on the properties of the two-mode resource state and the geometry of measurement \[2\]. We have interpreted Eq. (6) as describing the superposition between the input field \( \chi_{in} \) and a single-mode one reduced from the entangled \( AB \) field by the measurement performed by Alice followed by the phase-space translation performed by Bob. We have identified the function \( \chi_{AB}(\lambda^*,\lambda) \) as the normally ordered CF of a one-mode distorting field whose density operator is denoted by \( \rho_M \):

\[
\chi_M^{(N)}(\lambda) = \chi_{AB}(\lambda^*,\lambda).
\]  
(7)

Equation (6) is valid for arbitrary input and resource states. According to Eq. (7) the properties of the distorting field state \( \rho_M \) are fully determined by the two-mode resource state. Thus, if the entangled state \( \rho_{AB} \) is a two-mode Gaussian state, then any single-mode Gaussian input is teleported as a single-mode Gaussian output. In our paper \[2\] we focused on the Gaussian resource state case and derived several properties of the state \( \rho_M \). We have written the covariance matrix (CM) of the state \( \rho_M \) by making explicitly use of the Gaussian character of the resource state \( \rho_{AB} \). The input state distortion through teleportation was described by the average photon number of the measurement-induced field \( \rho_M \) that we called added noise. In the case of symmetric Gaussian resource states we have found a relation between the optimal added noise and the minimal EPR correlations used to define inseparability. Our principal aim in the present paper is to deepen the relation between the properties of the two-mode resource state and those of the distorting field \( M \).

### III. SECOND-ORDER CORRELATIONS

According to the definition of a normally ordered CF \[5\] we have for the one-mode distorting state (amplitude operators \( \hat{a}, \hat{a}^\dagger \))

\[
\chi_M^{(N)}(\lambda) = \sum_{l,m=0}^{\infty} \frac{1}{l!m!} \lambda^{l}(-\lambda^*)^{m} \langle (\hat{a}^\dagger)^l \hat{a}^m \rangle_M,
\]  
(8)
From the similar expansion of the two-mode CF of the state $\rho_{AB}$ we find

$$\chi_{AB}(\lambda^*, \lambda) = \exp\left(-|\lambda|^2\right) \sum_{s,r,l,m} \left( \begin{array}{c} l \\ r \\ m \\ s \end{array} \right) \frac{(-1)^{r+s}\lambda^{l-m}}{l!m!} \times \langle (\hat{a}_1^\dagger)^s (\hat{a}_2^\dagger)^{l-r} \hat{a}_1^{m-s} \hat{a}_2^{r} \rangle_{AB}. \tag{9}$$

By using Eq. (6) we could find relations between the correlation functions of the distorting state and two-mode correlation functions of the resource state. Let us now suppose that the two-mode resource state is undisplaced. Our first goal is to write the CM of the one-mode distorting field $\rho_M$ defined as

$$\mathcal{V}_M = \begin{pmatrix} \sigma(qq) & \sigma(qp) \\ \sigma(qp) & \sigma(pp) \end{pmatrix}. \tag{10}$$

In Eq. (10) we have denoted by $\sigma(qq) = \langle \hat{q}_1^2 \rangle_M$, $\sigma(pp) = \langle \hat{p}_2^2 \rangle_M$, $\sigma(qp) = \langle \hat{q}\hat{p} \rangle_M$, the second-order correlations of the canonical operators $\hat{q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$, $\hat{p} = (\hat{a} - \hat{a}^\dagger)/(\sqrt{2}i)$, where $\hat{a}$ and $\hat{a}^\dagger$, are the amplitude operators of the state $M$. Equation (6) gives us via Eqs. (8) and (9)

$$\langle \hat{a}_1^2 \rangle_M = \langle \hat{a}_2^2 \rangle_{AB} + \langle (\hat{a}^\dagger_1)^2 \rangle_{AB} - 2\langle \hat{a}_1^\dagger \hat{a}_2 \rangle_{AB}, \tag{11}$$

$$\langle \hat{a}_1^\dagger \hat{a} \rangle_M = 1 + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle_{AB} + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle_{AB} - \langle \hat{a}_1 \hat{a}_2 \rangle_{AB} - \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle_{AB}, \tag{12}$$

such that the entries of the CM of the distorting state $M$ are

$$\sigma(qq) = \frac{1}{2} + \sigma(q_2q_2) + \sigma(q_1q_1) - 2\sigma(q_1q_2). \tag{13}$$

$$\sigma(qp) = \sigma(q_2p_2) - \sigma(q_1p_1) + \sigma(q_2p_1) - \sigma(q_1p_2). \tag{14}$$

$$\sigma(pp) = \frac{1}{2} + \sigma(p_2p_2) + \sigma(p_1p_1) + 2\sigma(p_1p_2). \tag{15}$$

We have thus expressed the CM of the state $\rho_M$ in terms of the correlations of the canonical operators $\hat{q}_j$, $\hat{p}_j$, $(j = 1, 2)$ of the two-mode resource state $\rho_{AB}$. Equations (13)–(15) are valid for arbitrary two-mode resource state (Gaussian and non-Gaussian). For later convenience let us introduce two commuting operators $\hat{Q}$, $\hat{P}$ closely related to those measured by Alice:

$$\hat{Q}(\hat{q}_1, \hat{q}_2) := \hat{Q}_m(\hat{q}_1 \rightarrow \hat{q}_1, \hat{q}_m \rightarrow \hat{q}_2) = \hat{q}_2 - \hat{q}_1 \tag{16}$$

$$\hat{P}(\hat{p}_1, \hat{p}_2) = \hat{P}_m(\hat{p}_1 \rightarrow \hat{p}_1, \hat{p}_m \rightarrow \hat{p}_2) = \hat{p}_1 + \hat{p}_2. \tag{17}$$
The correlations (13)–(15) are written now

\[ \sigma(qq) = \frac{1}{2} + \langle \hat{Q}^2 \rangle, \quad \sigma(qp) = \langle \hat{Q}\hat{P} \rangle, \quad \sigma(pp) = \frac{1}{2} + \langle \hat{P}^2 \rangle. \]  

(18)

Two conclusions arise from the new aspect of the entries of the CM (10):

1. The Robertson-Schrödinger uncertainty relation,

\[ \det V_M = \sigma(qq)\sigma(pp) - (\sigma(qp))^2 \geq \frac{1}{4}, \]  

(19)

is verified because

\[ \langle \hat{Q}^2 \rangle\langle \hat{P}^2 \rangle - \langle \hat{Q}\hat{P} \rangle^2 \geq 0. \]  

(20)

2. The distorting state is not squeezed: \( V_M \geq \frac{1}{2}I_2 \).

IV. ACCURACY OF TELEPORTATION. ADDED NOISE

Originally, the quality of the teleportation protocol was quantified by the overlap of the \textit{in} and \textit{out} states for pure \textit{in} states [1] or the Uhlmann fidelity for mixed Gaussian states [6, 7]. So defined, the \textit{fidelity of teleportation} depends on the input state:

\[ F(in, out) = \frac{1}{\pi} \int d^2\lambda \chi^*_in(\lambda) \chi_{out}(\lambda). \]

In particular, the fidelity of teleportation for a coherent state is written via Eq. (6)

\[ F_{coh} = \frac{1}{\pi} \int d^2\lambda \exp (-|\lambda|^2) \chi_{AB}(\lambda^*, \lambda), \]  

(21)

or, equivalently

\[ F_{coh} = Q_M(0). \]  

(22)

In Eq. (22), \( Q_M(0) \) is the expectation value of the density operator of the state \( M \) in vacuum, namely is the \( Q(\alpha) \)-function [5] of this state at \( \alpha = 0 \).

Following Refs. [8, 9, 10] we evaluate the teleportation quality in terms of the mean occupancy in the remote field \( \rho_M \) which can be seen as the amount of noise distorting the properties of the input field state. From Eq. (12) via Eq. (18) we get

\[ \langle \hat{a}^\dagger \hat{a} \rangle_M = \frac{1}{2} \left[ \langle \hat{Q}^2 \rangle + \langle \hat{P}^2 \rangle \right]. \]  

(23)
The r.h.s. of Eq. (23) is in fact the EPR-uncertainty generally defined in the undisplaced two-mode case as

\[ \Delta_{EPR}(\rho) := \frac{1}{2} \left[ (\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle)^2 + \langle (\hat{p}_1 + \hat{p}_2)^2 \rangle \right]. \]  

(24)

We can now formulate the main results of this paper.

Theorem 1: The amount of noise distorting the properties of the input field state is rigourously equal to the EPR–uncertainty of the resource state \( \rho_{AB} \).

\[ \langle \hat{a}^\dagger \hat{a} \rangle_M = \Delta_{EPR}(\rho_{AB}). \]  

(25)

Note that this is valid for an arbitrary two-mode resource state. When the condition \( \Delta_{EPR}(\rho_{AB}) < 1 \) is met the state \( \rho_{AB} \) presents non-local correlations. Therefore the teleportation process generates less noise in the output state when the non-locality of the resource state expressed by the EPR-uncertainty (24) is stronger (\( \Delta_{EPR}(\rho_{AB}) \) is smaller).

We discuss now the case of pure two-mode resource states. According to Giedke et al. [11], there is a direct relation between the amount of entanglement and the EPR-uncertainty of pure two-mode states. Thus, among all pure two-mode states (Gaussian and non-Gaussian), the squeezed vacuum state (SVS) has the minimal amount of entanglement at a prescribed EPR–uncertainty. Otherwise said, among all pure states with the same entanglement the SVS has minimal \( \Delta_{EPR} \) (maximal EPR–correlations). Application of this important result leads us to a strong interpretation of Eq. (25):

Theorem 2: The minimal noise added in teleportation with pure two-mode resource states having the same entanglement is realized by the SVS.

To conclude, in this paper we have shown that the quality of the continuous-variable teleportation is determined by the amount of non-locality measured by the EPR-uncertainty of the resource state. As a consequence of the factorization formula \( \text{(6)} \) this characterization of the efficiency of the teleportation process is valid for arbitrary input one-mode states. We could apply the strong theorem proved in Ref. [11] to show that SVS generates the minimal noise in the teleportation output when comparing all pure resource states of given entanglement.
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