ON ALGEBRAIC STRUCTURES IMPLICIT IN TOPOLOGICAL QUANTUM FIELD THEORIES

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1 INTRODUCTION

In the course of the development of our understanding of topological quantum field theory (TQFT) [1,2], it has emerged that the structures of generators and relations for the construction of low dimensional TQFTs by various combinatorial methods are equivalent to the structures of various fundamental objects in abstract algebra.

Thus, 2D-TQFTs can be constructed from commutative Frobenius algebras [3] or from semisimple associative algebras [4]; while 3D theories can be constructed either from nicely behaved braided monoidal categories [5,6,7,8,9] or from Hopf algebras [10,11].

In [12], a possible method was proposed to extend this picture to D=4. Namely, it was shown how to construct a 4D TQFT from a new type of algebraic structure called a Hopf category.

The purpose of this paper is to show that under physically reasonable hypotheses, the seemingly exotic algebraic structures used in the constructions above arise naturally from 3D and 4D TQFT’s. We shall show that any 3D-TQFT with a property which we call factorizability, which any TQFT which came from a path integral with a topological lagrangian would be expected to satisfy, contains a braided monoidal category Hopf in its structure, and that this category arises from a generalized Hopf algebra by a construction first proposed by Yetter. We shall show moreover that any factorizable 4D-TQFT gives rise to a Hopf category object in a certain reasonably concrete bicategory. This theorem lends weight to the conjecture in [12] that the 4D-TQFT which is believed to be constructable from Donaldson-Floer theory is related to the Hopf category constructed in [12] from the canonical basis of a quantum group [13].

The importance of the procedure we outline in this paper is greatly increased by the recent breakthrough in the understanding of Donaldson-Floer theory made by Witten [14]. The pair of differential equations whose solutions are related to DF theory in this new approach is much more tractible than the self duality equation [15]. In particular, there is a well behaved version of them on manifolds with boundary. Thus, we can take the geometric constructions in this paper as a prescription: consider the space of solutions of Witten’s equations on the manifolds with boundaries or corners we are examining, there must then appear certain algebraic operations on them, from which the TQFT can be reconstructed.

The contents of this paper are as follows: Section 2 gives the definition of a 3D-TQFT with factorizability and physical motivation for the definition. In Section 3, we prove that every 3D-TQFT with factorizability contains a Hopf algebra object, and show the relation between this object and the category used to define factorization. Section 4 recapitulates the definition of a Hopf category. In Section 5, we explain the extension of the definition of factorizable TQFT to D=4, and prove of the main theorem in 4D. Finally, in Section 6 we outline some extensions of our argument and suggest directions for further work.

Throughout the paper all manifolds (with or without boundary or corners) are compact and oriented.
\section{Factorizable 3D-TQFT}

The subject of TQFT began with the study of path integrals for lagrangians with topological invariance. A rigorous treatment of this approach is not within the reach of the mathematics of our time. Nevertheless, it is possible to make formal manipulations of path integrals to deduce that the theories derived from them should have certain properties. We define a TQFT with these properties as factorizable.

The properties come from two aspects of the theory of path integrals. One is the idea that since a path integral is a “sort of” integral for each point in a space, we can separate it into integrals over parts of a space, by a “sort of” Fubini’s theorem. The other is that the topological lagrangians possess a large gauge symmetry, with respect to which physical states must be invariant. If we cut space up along submanifolds of codimensions one and two, we get states with boundary attached to codimension one submanifolds with codimension two boundaries which transform non-trivially under the quantum version of the gauge symmetry on the codimension two submanifolds. It is this gauge symmetry which is responsible for the appearance of tensor categories or Hopf algebras in the structure of a TQFT.

A formal derivation from a path integral would serve no mathematical purpose, since path integrals themselves are not rigorously defined. Let us, then simply write down a set of axioms for a factorizable 3D-TQFT. The argument from the path integral is given in the special case of CSW theory in [14].

Let us begin by recalling the structure of a cobordism category. The category of oriented n-dimensional cobordisms has oriented compact n-1 dimensional manifolds as objects and cobordisms as morphisms. (A cobordism from M to N is an oriented n-dimensional manifold P with boundary, together with an oriented diffeomorphism between the boundary of P and M*∪N.) Composition of morphisms comes from gluing of manifolds along shared boundary components.

The category of oriented n-cobordisms has the natural structure of a tensor category with duality. The tensor product is direct sum and the duality is reversal of orientation.

The most elegant definition of an n-dimensional TQFT is that it is a monoidal functor from the category of oriented n-cobordisms with disjoint union as tensor product to the category $\text{VECT}$ of finite dimensional vector spaces with the usual tensor product (i.e. a functor which preserves tensor product up to canonical coherent isomorphism). It is a point often missed that this suffices—that a manifold with opposite orientation is sent to the dual space of the image with the given orientation is an easy theorem, not a necessary part of the definition.

We often modify the definition of TQFTs by modifying the cobordism category. For instance, we can specify a framing of the tangent bundle of the cobordisms and of a formal neighborhood of the closed manifolds. Another possibility is to include insertions of submanifolds in the manifolds and matching insertions in the cobordisms. We also refer to tensor and duality preserving functors from such modified cobordism categories to $\text{VECT}$ as TQFTs.

Let us spell out this definition for the less categorically inclined. An n-dimensional TQFT assigns a vector space to each oriented n-dimensional manifold, and a linear map to each oriented cobordism in such a way that the composition of cobordisms corresponds to the composition of linear maps, the disjoint union of manifolds gets the tensor product of vector spaces, and the manifold with opposite orientation gets assigned the dual space.

Thus, for n=3, a TQFT assigns a vector space to a surface, and a linear map to a 3-dimensional cobordism.

Let us note that since the boundary of a cobordism is a disjoint union of two manifolds, one with reversed orientation, it is equivalent to assign a linear map to a cobordism, or a vector in the vector space on the boundary to a manifold with boundary. It follows from this observation
that the invariant of closed 3-manifolds arising from a TQFT can be viewed as the dual pairing of vectors associated to 3-manifolds with (common, but oppositely oriented) boundary. This is, of course, Atiyah’s original view.

A 3D-TQFT with factorizability has an analogous structure one layer farther down in dimension so that we can cut surfaces along sets of circles and write the vector space on the surface as the hom-space between objects associated to the pieces (the categorical analogue of a dual pairing!).

To set up the formal definition embodying this notion, we must remind the reader that a finitely generated semisimple linear category is one in which each object is isomorphic to a direct sum of irreducible (simple) objects chosen from a finite set of such objects, hom-sets are complex vector spaces, and composition is bilinear. As categories, they are equivalent to $\text{VECT}^n$ for some $n$. For the theory of such categories, also called $\text{VECT}$ modules, see [16]. As shown in [16], these categories form a monoidal bicategory: objects are $\text{VECT}$ modules, 1-arrows are exact $\mathbb{C}$-bilinear functors, 2-arrows are natural transformations, and the tensor product is given up to canonical equivalence by using pairs of the generating simple objects in the tensorands as a set of generating simple objects.

Similarly observe that there is a monoidal bicategory of 3-dimensional cobordisms with corners, $3 - \text{cobord}_2$: its objects are 1-manifolds, its 1-arrows are (2-dimensional) cobordisms of 1-manifolds, and its 2-arrows are cobordisms with corners between pairs of 2-dimensional cobordisms with the same source and target. To be precise, a 3-dimensional cobordism with corners is a 3-manifold with corners, whose boundary is a union along a family of circles joining corresponding boundary components of the two surfaces. The 1-dimensional composition of 1-arrows and 2-dimensional composition of 2-arrows are just given by gluing target to source. The 1-dimensional composition of 2-arrows consists of gluing along the corners and gluing on a “collar” as shown in Figure 1. It is trivial to verify that disjoint union gives this bicategory the structure of a monoidal bicategory.

In what follows, we shall refer to a cobordism (resp. cobordism with corners) as “trivial” if its underlying space is a product of one of its boundaries with the interval (resp. a product of one of its boundary strata with the interval modulo collapsing the product of the bounding corner with the interval back onto the corner). Note that a trivial cobordism or cobordism with corners need not be the identity cobordism—the attaching maps at the ends could be different. However, a trivial cobordism is manifestly invertible.

In Definitions 2.1 and 5.2 below, the non-categorically minded reader is advised on first reading to read only the bold-faced portions of the definitions. These give the essential flavor of the definition, without going into excessive categorical detail.

**Definition 2.1** A 3D TQFT with factorization is a monoidal bifunctor from $3 - \text{cobord}_2$ to $\text{VECT} - \text{mod}$.

Less briefly, but more intelligibly to the non-categorically minded, this entails an assignment of

1. A finitely generated semisimple $\mathbb{C}$-linear category to each compact 1-manifold.

2. An exact $\mathbb{C}$-linear functor to each 2-dimensional cobordism. In particular, since an exact $\mathbb{C}$-linear functor from $\text{VECT}$ to a semisimple $\mathbb{C}$-linear category is completely determined by the image of $\mathbb{C}$, we have a choice of an object in the category associated to the

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1Kapranov and Voevodsky call the structure a (monoidal) weak 2-category. Their notion of a weak 2-category has long been studied by categorists under the name “bicategory” (cf. Benabou [17]). We here adopt the older name. For terminology in this regard, and details of some of the more abstract points of category theory, we refer the reader to a recent paper of Gordon, Power and Street [18] and references found therein.
boundary of each oriented surface with boundary, and more particularly, we have an assignment of a vector space to every closed oriented surface.

3. A natural transformation to each 3-dimensional cobordism with corners. In particular for a 3-manifold with boundary and corners consisting of two surfaces with boundary sharing their common boundary as a corner, we have a map in the category associated to the boundary of the surfaces between the objects associated to the surfaces. Likewise, since the empty surface is assigned \( \mathbb{C} \), a 3-manifold with boundary is assigned a vector in the vector space associated to its boundary, and finally a 3-manifold without boundary is assigned a number.

Moreover, these assignments will satisfy:

1. The disjoint union of two 1-manifolds gets the tensor product in the sense of [16] of the semisimple categories attached to the parts. The empty 1-manifold will be assigned VECT.

2. The 1-manifold with opposite orientation is assigned the dual category. (cf. Yetter [19])

3. The disjoint union of surfaces is assigned the tensor product of the vector spaces on the surfaces.

4. The surface with opposite orientation is assigned the dual vector space.

5. If we cut an oriented surface along a 1-manifold (union of circles), the vector space on the closed surface is naturally isomorphic to the hom set of the two objects in the category corresponding to the cuts which correspond to the two surfaces with boundary. A similar result holds for the case when we cut a surface with boundary and take hom with respect to the “tensor indices” corresponding to the cuts only.

6. If we cut a 3-manifold along a surface with boundary, the number invariant of the manifold is the dual pairing of the vectors associated to the two manifolds with boundary.

7. If we join two cobordisms with corners to form a cobordism, the linear map associated to the cobordism is the hom of the two linear maps. If we join two cobordisms with corners along a surface with boundary to form a new cobordism with the same corner, the map corresponding to the new cobordism is the composite of the old maps.

The reader will no doubt have noticed that these assignments and conditions fall into two analogous tiers, with semisimple linear categories closely paralleling vector spaces. The situation for \( D=4 \) will be closely analogous again, with a third categorical tier.

It follows from these axioms that the category on a circle has an associative tensor product, corresponding to the three holed sphere, or trinion, with associativity constraints given by trivial cobordisms with corners. Moreover, the category must be braided, again with structure maps given by trivial cobordisms with corners. It is the careful working out of analogous arguments one dimension up which gives rise to the Hopf algebra object.
This definition is similar to ones proposed by Kazhdan [20], Walker [8] and Lawrence [21], except that they were not motivated by the gauge group of a lagrangian.

It is straightforward to generalize this definition to various modifications of the cobordism categories as mentioned above. Such an augmented TQFT will also be called factorizable.

3 THE HOPF ALGEBRA OBJECT

Given a factorizable 3D-TQFT, $\mathcal{T}$, we obtain an object in the category on the circle (a tensor category, as we mentioned above), corresponding to the once punctured torus. Let us denote this object as $O_T$.

**Theorem 3.1** For any factorizable 3D-TQFT $\mathcal{T}$, the object $O_T$ admits a natural structure as a Hopf algebra object in the category on the circle. There is a natural isomorphism with the dual object, which is also a Hopf algebra object.

**Proof:** The proof is essentially pictorial. In each case we show a cobordism with corners (or composition of such) with a neighborhood of the collar removed for clarity: in all cases the bottom end of the drawing should have the annulus shrunk to a circle. The Hopf algebra object is the image of the torus with a disk removed portrayed in Figure 2. Its tensor with itself as an object in the category on the circle is given by Figure 3. The cobordisms (with corners) which give the product, unit, coproduct, and counit are shown in Figure 4. Figure 5 shows the antipode. Figures 6 and 7 give the proof of associativity, observe that the inner and outer shells of the two figures are identical with respect to the markings (which give the attaching map for the boundary), as is the space between, only the division into composed cobordisms differs. The proof of coassociativity is obtained by turning Figures 6 and 7 inside-out. The proof of (half of) the antipode axiom is given in Figure 8 (the other half being exactly similar). The reader should note that the inner-most shell in the bottom portion of Figure 8 is not linked with the hole in the outer shell, and that tracing the stripe shows that the attaching map is the same as in the upper portion of Figure 8. The proofs of the unit and counit axioms can be readily drawn by the reader. The most difficult axiom to verify is, as usual the connecting axiom. Observe that for an object in a braided monoidal category, the connecting axiom must be taken in the sense of Majid’s braided Hopf algebras [22]. The one side of the connecting axioms (multiply then comultiply) is shown in Figure 9. Figures 10 through 12 give the maps which are composed to swap factors in the comultiply then multiply side of the axiom, while Figure 13 gives that side of the axiom (with the space between the second and third shells reading inward given by the composition of Figures 10 though 12. Again careful perusal shows that the space between the outer-most and inner-most shells in Figures 9 and 13 is identical, as are the markings indicating the attaching maps.

The isomorphism with the dual is depicted in Figure 14. $\square$

We note that the quantum group associated with a Borel subalgebra is isomorphic to its dual.

It is tempting now, to identify the category associated to the circle with the representations (say modules) of the Hopf algebra. This however would be a mistake. First, there is not necessarily an “underlying vector-space” or fibre functor from the category to $\text{VECT}$. Even were there such a functor, however, this would be a mistake. In fact, the relationship between the category and the object is more subtle. To state it we need to recall one of the variants of Yetter’s notion of crossed bimodule [23] given by Radford and Towber [24].

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2 In Radford and Towber’s terminology [24] what was defined in Yetter [23] were “left-left Yetter-Drinfel’d structures”, while what we use here are “left-right Yetter-Drinfel’d structures.”
Definition 3.2 A left-right crossed bimodule over a Hopf-algebra $A$ is a vector-space equipped with a left $A$-comodule structure and a right $A$-module structure, moreover satisfying the condition (in modified Sweedler notation):

$$\Sigma a_{(1)} \cdot m_{<1>} \otimes a_{(2)} \cdot m_{(2)} = \Sigma (a_{(2)} \cdot m)_{<1>} \otimes (a_{(2)} \cdot m)_{(2)} a_{(1)}$$

Now, in terms of commutative diagram, this is given in Figure 15. Observe that in two places, the symmetry map occurs. Unfortunately, the category associated to the circle in a factorizable 3D-TQFT is only braided, so there is a question what the correct generalization is. It turns out that it is given by

Definition 3.3 A left-right crossed bimodule over a Hopf-algebra object $H$ in a braided tensor category is an object equipped with both a left $H$-comodule structure and a right $H$-module structure such that the diagram in Figure 16 commutes. A map of left-right crossed bimodules is a map in the ambient category between crossed bimodules which is both a left comodule map and a right module map.

The reader should note that in both Figures 15 and 16 we have used the coherence theorem of Mac Lane [25] to suppress mention of the associativity natural transformation.

We can now show the following:

Theorem 3.4 Let $T$ be a factorizable 3D-TQFT and let $C$ be the category associated with the circle, and $H$ be the Hopf-algebra object in $C$ associated to the torus with a disk removed, then every object associated to a surface with a single boundary component is a left-right crossed bimodule over $H$, and the map between objects associated to any cobordism with a single circle as corner is a crossed bimodule map. Moreover, the finite set of generating objects used to define factorizability may be taken without loss of generality to be left-right crossed bimodules.

Proof: Except for the last statement the proof is again pictorial. Figure 17 (resp. 18) shows the action (resp. coaction). That the action is associative in the appropriate sense is verified in Figure 19. Verification of coassociativity, unitalness and counitalness of the action are similar and left to the reader. The verification of the left-right crossed bimodule axiom is given in Figures 20 through 27. Figures 20 and 21 are composed to give Figure 22, giving the top way around the diagram in Figure 16. Figures 23, 24, 25 and the two parts of Figure 26 are composed to give Figure 27 (in Figure 26, the curves labelled 1 in each part correspond), giving the bottom way around the diagram in Figure 16.

The remaining statement is a matter of algebra. First, notice that we may discard from the description of a factorizable 3D-TQFT any of the simple generating objects that do not occur as direct summands of any object associated to a surface with a single boundary component. Now, observe that each simple object is a summand of the object associated to a surface with as single boundary component inherits a comodule structure. In general it would not inherit a module structure. In this case, however, the Hopf-algebra object is self-dual, so a module structure is a comodule structure for the dual Hopf-algebra object, and is thus inherited also. Of course the restrictions necessarily satisfy the same compatibility condition as the original action and coaction did, so the subobject is a crossed bimodule. □
4 HOPF CATEGORIES

The passage from D=3 to D=4 will have the effect of lifting our constructions by one categorical level. The analog of Theorem 3.1 will accordingly produce a Hopf category. The notion of Hopf category was introduced in [12]. For completeness, we repeat the definition here.

Let us note that there actually exist Hopf categories associated to the quantum groups. This highly nontrivial fact follows from work of Lustig [13].

4.1 Categories and Algebras

We are now going to construct an analog of the structure of a Hopf algebra on a tensor category of a very special type.

To explain the idea of an analog of an algebraic structure on a category, let us think briefly about the category $\text{VECT}$ of finite dimensional vector spaces. This category possesses two product operations, $\oplus$ and $\otimes$ and special objects $0$ and $1$ with the properties

\[
(A \otimes B) \otimes C \cong A \otimes (B \otimes C)
\]
\[
(A \oplus B) \oplus C \cong A \oplus (B \oplus C)
\]
\[
A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)
\]
\[
A \oplus 0 \cong A
\]
\[
A \otimes 1 \cong A
\]
\[
A \oplus B \cong B \oplus A
\]

These isomorphisms satisfy certain equations, called coherence relations. (cf. Laplaza [26]) This is completely parallel to the definition of a ring. We describe this by saying that $\text{VECT}$ is a ring category. This structure is a categorical analog of a ring.

Other ring categories include various categories of (differential) graded vector-spaces or modules.

What has happened is that equations in the ring correspond to isomorphisms in the category. There are then natural equations that the isomorphisms should satisfy, so that combining them in different orders to produce a larger isomorphism always gives consistent results. These were termed “coherence relations” by MacLane [25]. The coherence relations corresponding to the commutative and associative laws are the Stasheff pentagons and hexagons (cf. [16]).

Thus, if we replace an algebraic structure by a categorical analog, its axioms will hold only up to natural isomorphisms, which in turn must satisfy a new set of more complex equations, which are its coherence relations. One of the fundamental ideas of the dimensional ladder is that if we start with an algebraic structure which can be used to construct a TQFT, than the coherence relations of a categorical analog of it are just right to construct a TQFT in one higher dimension. Our use of a Hopf category in 4D-TQFT is an application of this idea.

4.2 Hopf Categories

Now let us describe the structure of a Hopf category.

Definition 4.1 A category is semisimple if each object is a direct sum of simple objects (objects with no nontrivial sub- or quotient objects). A semisimple category is finitely generated if it has only finitely many inequivalent irreducible objects. In this paper, we will only consider finitely generated categories (in order to make all sums finite).

Definition 4.2 A category is linear if the set of morphisms has the structure of a vector space, and composition is bilinear.
Definition 4.3 If \( R \) is a ring category, then \( M \) is a module category over \( R \) if \( M \) has an associative direct sum and we are given a functor \( R \times M \to M \) (denoted as multiplication) such that

\[
A_1 \otimes (A_2 \otimes R) \cong (A_1 \otimes A_2) \otimes R
\]
\[
(A_1 \oplus A_2) \otimes R \cong (A_1 \otimes R) \oplus (A_2 \otimes R)
\]
\[
A \otimes (R_1 \oplus R_2) \cong (A \otimes R_1) \oplus (A \otimes R_2)
\]

and the isomorphisms satisfy the natural coherence relations (cf. [16]).

The concept of module category is the categorical analog of the concept of a module.

Definition 4.4 An algebra category is a ring category which is also a \( \text{VECT} \) module such that \( \oplus \) is a module map and \( \otimes \) is a module map in each variable separately.

Note that a \( \text{VECT} \) module must be linear.

\( \text{VECT} \) modules are a categorical analog of vector spaces, so algebra categories are categorical analogs of algebras.

Now recall that the dual \( C^\text{op} \) of a category \( C \) has the same objects as the category, but with morphisms reversed. Similarly, the dual of any algebraic construction has diagrams corresponding to the first one, but with arrows reversed. The dual of a \( \text{VECT} \) module has a natural structure as a \( \text{VECT} \) module, including a natural direct sum. Less familiar is the fact (cf. Yetter [19]) that in the case of \( \text{VECT} \) modules, the dual category \( C \) is equivalent to the hom-category \( [C, \text{VECT}] \), and thus gives rise to a contravariant functor from \( \text{VECT} \) modules to \( \text{VECT} \) modules.

Definition 4.5 A coalgebra category is a \( \text{VECT} \) module category whose dual is an algebra category when dual category is understood in the contravariant sense.

This is equivalent to a \( \text{VECT} \) module category with a comultiplication functor \( \Delta : A \to A \otimes A \) satisfying the dual of the axioms of an algebra category (coassociativity, etc.). Here, \( \otimes \) is the bicategorical tensor product of \( \text{VECT} \) modules (cf. [16]).

Definition 4.6 A (not-necessarily unital or counital) bialgebra category is a \( \text{VECT} \) module category which is both an algebra and a coalgebra category, together with a consistency natural isomorphism as in Figure 28.

The associativity and coassociativity isomorphisms and the consistency map \( \alpha : \Delta(A) \otimes \Delta(B) \cong \Delta(A \otimes B) \) must satisfy the following four commuting cubes given in Figure 29 as coherence relations.

The first two of these are the Mac Lane pentagon and the dual relation for comultiplication. The latter two first appear in [12].

Finally, a Hopf category is a semisimple bialgebra category together with the categorical analogs of a unit object, counit functor, and an antipode functor. For brevity, we only sketch the of the corresponding coherence relations: the unit object is the identity for the monoidal (algebra) category structure, and dually for the counit functor, the counit functor is a monoidal functor, and finally the antipode functor, satisfies the usual equations up to natural isomorphism, with coherence conditions given by requiring that it be a monoidal functor between the category with the given algebra structure, and the category with its monoidal struture reversed, and dually for the coalgebra structure.

Note, this is a slightly different (and apparently more restrictive) notion of Hopf category than was given in [12], there the categorical analogues of structures equivalent to an antipode in the
finite-dimensional case, but easier to construct concretely in terms of categories were used as the definition. We take the a priori more restrictive notion, because our theorem for 4D TQFT’s holds with this notion, and is thus a priori a stronger result than the corresponding theorem with the definition of [12] would be.

5 FACTORIZABLE 4D-TQFT AND THE HOPF CATEGORY

The definition of a factorizable 4D-TQFT is closely analogous to the 3D concept. The new element is that we can consider the gauge groups not only on codimension 2 manifolds, i.e. on surfaces, but also on surfaces with boundary. Thus, the category of representations of the gauge group of the surface with boundary is acted on functorially by the category of representations of the gauge group on the union of circles, since there is a restriction map on gauge groups. This means that the category of representations of the gauge group of a surface with boundary is an element of the category of representations of the category of representations of the gauge group on the one dimensional boundary, which has the natural structure of a bicategory.

(We apologise that it is not practical to make this paper self contained as regards category theory. See [16] for the discussion of module categories over a tensor category. The reader who does not know the definition of a bicategory can probably just ignore it on first reading.)

With that as physical motivation, let us proceed to the axiomatization of a factorizable 4D-TQFT.

From one point of view one would like a clean formulation as was given in the 3D case. Unfortunately (or fortunately, depending on one’s tastes) the notion of a monoidal tricategory has not been formalized, so we will be obliged to describe the highest codimension assignments in a more concrete fashion. To do this, we require:

Definition 5.1 A (finitely generated) linear bicategory is a bicategory which is a finite product of copies of the bicategory $\text{VECT-mod}$. We identify the indices of the product with generating objects which are $\text{VECT}$ in the indexed coordinate and the trivial category in all others. The tensor product of two linear bicategories $A$ and $B$ has as generating objects ordered pairs of a generating object from $A$ and $B$, and will be denoted $A \odot B$.

Definition 5.2 A 4D-TQFT with factorizability is the following collection of assignments:

1. To each 1-manifold a linear bicategory.

2. To each 2-dimensional cobordism a bifunctor from the linear bicategory assigned to the source to the linear bicategory assigned to the target. In particular to a surface with boundary we have assigned an object in the linear bicategory associated to the boundary. (Recall that this object will in fact be an $n$-tuple of $\text{VECT}$ modules.) More particularly, to every closed surface, there will be assigned a $\text{VECT}$ module.

3. To each 3-dimensional cobordism with corners a dinatural transformation between the bifunctors assigned to the surfaces meeting at the corners. In particular, to a 3-dimensional manifold whose boundary is a pair of surfaces meeting on corners, there will be assigned a map between the objects (in the bicategory assigned to the corners) assigned to the surfaces. Likewise to a 3-dimensional cobordism between closed surfaces there will be assigned a functor between the $\text{VECT}$ modules associated
to the surfaces, while to a 3-manifold with boundary there will be associated an object in the VECT module on the surface. And finally to a 3-manifold without boundary will be assigned an object in VECT (i.e. a vector space).

4. To each 4-dimensional cobordism with corners (in codimensions 2 and 3) a modification between the binatural transformations assigned to the 3-manifolds with corners meeting at the codimension 3 corners. In particular, to each 4-dimensional cobordism with corners (in codimension 2) between 3-manifolds with boundary (with common boundary—the codim 2 corner) is assigned a map in the VECT module assigned to the corner surface between the objects assigned to the 3-manifolds with boundary. In the case of a 4-dimensional cobordism between closed 3-manifolds, this is a linear map between the vector spaces assigned to the 3-manifolds. Most particularly 4-manifolds with boundary are assigned elements in the vector space assigned to the bounding 3-manifold and closed 4-manifolds are assigned numbers.

Moreover, these assignments satisfy

1. Disjoint unions are assigned the \( \odot \) product of what is assigned to the parts.

2. The assignment is a trifunctor from the tricategory whose objects are 1-manifolds, 1-arrows are 2-dimensional cobordisms, 2-arrows are 3-dimensional cobordisms with corners and 3-arrows are 4-dimensional cobordisms with corners in codimensions 2 and 3 (with the obvious compositions) to the tricategory whose objects are linear bicategories, 1-arrows are bifunctors, 2-arrows are binatural transformations, and 3-arrows are modifications (cf. [18]). In particular, the assignment carries gluing along 1-dimensional strata (codimension 3 corners) of boundaries to composition of bifunctors, carries gluing along 2-dimensional strata (corners) to composition of binatural transformations, and carries gluing along 3-dimensional strata of boundaries to composition of modifications, and in each of the first two cases to the induced composition on binatural transformations and modifications. More particularly given a closed surface obtained by gluing two surfaces with boundary, the VECT module assigned to the surface is the hom-category in the linear bicategory assigned to the cut between the objects therein assigned to the surfaces with boundary; given a closed 3-manifold obtained by gluing two 3-manifolds with boundary, the vector-space associated to the 3-manifold is the hom-space in the VECT module assigned to the cut between the objects therein assigned to the 3-manifolds with boundary; and given an closed 4-manifold obtained by gluing two 4-manifolds with boundary, the number assigned to the 4-manifold is the dual pairing between in the vector space assigned to the cut between the vectors therein assigned to the 4-manifolds with boundary.

Now we can prove our main theorem for 4D-TQFTs:

**Theorem 5.3** In any factorizable 4D-TQFT the category associated to the torus with a disk removed is a Hopf-category object in the tensor bicategory associated to the circle. Moreover, the category associated to every surface with a single boundary component admits both an action and a coaction of this Hopf-category object which satisfies the left-right crossed bimodule axiom up to canonical isomorphism.

**Proof:** The proof is essentially a restatement of the proofs of Theorems 3.1 and 3.4. The once punctured torus now is assigned a semisimple linear category lying in some concrete finitely generated
tensor bicategory, and the cobordisms which gave the structure maps of the Hopf algebra before
now give the corresponding structure functors of the Hopf category. All of the natural isomorphisms
in the coherence structure for the Hopf category are given by (obvious) trivial cobordisms, and the
coherence equations for the category are just equations between compositions of trivial cobordisms.

6 ADDITIONAL STRUCTURES AND CONCLUSIONS

The Hopf category we have constructed possesses some additional very canonical structure. To
begin, there are two natural 4D cobordisms between the 3-manifold with corners representing the
inclusion of the identity object (resp. representing the counit functor) and itself, corresponding to
adding a 1- or 2- handle on a small loop near the edge. These maps would have to satisfy some
identities which would correspond to the laws of the Kirby calculus.

Although we will not attempt to complete the procedure here, there appears to be a natural way
to reverse the argument we are giving and reconstruct a 4D-TQFT from a Hopf category. We would
begin by giving a combinatorial description of 4D manifolds slightly different from but related to
the Kirby calculus. A 4D manifold or cobordism can be represented as a one parameter family of
3-manifolds equipped with Heegaard splittings. The family changes at a discrete set of levels, where
the Heegaard splitting is either stabilized by one handle or has its surface map changed by a simple
Dehn twist. The 3D structures would be represented by combinations of the Hopf category and
its identity, while the 4D segments would be represented by the canonical maps we have indicated.
Identities on the canonical maps would contain the information necessary for 4D invariance. This
would probably be a more elegant construction than the one proposed in [12].

The methods outlined in this paper also could be used to construct a 4D-TQFT by a different
line of attack. It was suggested in [12], and discussed in more detail in [27], that there would be
three natural algebraic constructions of 4D-TQFT, using monoidal bicategories, Hopf categories,
and conjectural structures called “trialgebras.” The bicategory lives on the circle, while the Hopf
category lives on the punctured torus as we have explained in this paper.

The trialgebra is also not hard to locate. It lives on a 3-torus from which a solid torus with a
corner has been removed. The three algebraic operations correspond to the three obvious generators
for the homology of the 3-torus in a way analogous to the correspondence between the two generators
and the product and coproduct (product in the dual) of the Hopf algebra associated to the 2-torus
with a disk removed.

Of course, we still obtain only a trialgebra object in a category. Even if Witten’s monopole
equations allow us to construct our trialgebra object, finding an honest trialgebra would be of
comparable difficulty to finding a quantum group from the category of representations of a loop
group at a given central extension. Motivated by the success of that effort, we state the following
conjecture:

**Conjecture 6.1** There exist a family of trialgebras from whose representations the quantum groups
can be recovered. It is also possible to reproduce Donaldson-Floer theory from them.

The existence is the hard part here, since geometric arguments of the sort described in this
paper will make the reconstructions fairly straightforward.

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