GLOBAL RIGIDITY AND EXPONENTIAL MOMENTS FOR SOFT AND HARD EDGE POINT PROCESSES

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We establish global rigidity upper bounds for universal determinantal point processes describing edge eigenvalues of random matrices. For this, we first obtain a general result which can be applied to general (not necessarily determinantal) point processes which have a smallest (or largest) point: this allows us to deduce global rigidity upper bounds from the exponential moments of the counting function of the process. Combining our general result with known exponential moment asymptotics for the Airy and Bessel point processes, we improve on the best known upper bounds for the global rigidity of the Airy point process, and we obtain new global rigidity results for the Bessel point process.

Secondly, we obtain exponential moment asymptotics for Wright’s generalized Bessel process and the Meijer-G process, up to and including the constant term. As a direct consequence, we obtain new results for the expectation and variance of the associated counting functions. Furthermore, by combining these asymptotics with our general rigidity theorem, we obtain new global rigidity upper bounds for these point processes.

1. Introduction and statement of results

An important question in recent years in random matrix theory has been to understand how much the ordered eigenvalues of a random matrix can deviate from their typical locations. It has been observed [Johansson 1998; Gustavsson 2005; Arguin et al. 2017; Erdős et al. 2009; 2012; Claeys et al. 2019a] that the individual eigenvalues fluctuate on scales that are only slightly bigger than the microscopic scale. This property is loosely called the rigidity of random matrix eigenvalues. To make this more precise, let us consider the circular unitary ensemble (CUE) which consists of \( n \times n \) unitary Haar distributed matrices. The eigenvalues of such a random matrix lie on the unit circle in the complex plane, and if we denote the eigenangles as \( 0 < \theta_1 \leq \cdots \leq \theta_n \leq 2\pi \), we can expect that \( \theta_j \) will, for typical configurations, lie close to \( \frac{2\pi j}{n} \) because of the rotational invariance of the probability distribution of the eigenvalues. Indeed, it was shown in [Arguin et al. 2017, Theorem 1.5] (see also [Paquette and Zeitouni 2018]) that

\[
\lim_{n \to \infty} \mathbb{P}_{\text{CUE}} \left( (2 - \epsilon) \frac{\log n}{n} < \max_{j=1,\ldots,n} \left| \theta_j - \frac{2\pi j}{n} \right| < (2 + \epsilon) \frac{\log n}{n} \right) = 1
\]

for any \( \epsilon > 0 \). We call this an optimal global rigidity result because the lower and upper bounds of the maximal eigenvalue deviation differ only by a multiplicative factor which can be chosen arbitrarily close to 1. Similar optimal global rigidity results have been obtained in circular \( \beta \)-ensembles [Chhaibi et al.]

MSC2020: 33B15, 33E20, 35Q15, 41A60, 60B20.

Keywords: rigidity, exponential moments, Muttalib–Borodin ensembles, product random matrices, random matrix theory, Asymptotic analysis, large gap probability, Riemann–Hilbert problems.
In unitary invariant random matrix ensembles [Claeys et al. 2019a], and also for the sine \( \beta \)-process [Holcomb and Paquette 2018; Lambert 2021]. In the two-dimensional setting of the Ginibre ensemble, results of a similar nature were obtained in [Lambert 2020].

One of the most important features of random matrix eigenvalues is their universal nature: their asymptotic behavior on microscopic scales is similar for large classes of random matrix models. For instance, in many matrix models of Hermitian \( n \times n \) matrices, like the GUE, Wigner matrices, and unitary invariant matrices, the microscopic large \( n \) behavior of bulk eigenvalues is described by the sine point process (see, e.g., [Erdős and Yau 2017]), whereas the microscopic behavior of edge eigenvalues is described by the Airy point process [Deift and Gioev 2007; Deift et al. 1999; Bourgade et al. 2014; Forrester 1993; Prähofer and Spohn 2002; Soshnikov 2000; Tracy and Widom 1994a]. For ensembles of positive-definite Hermitian matrices, the situation is somewhat more complicated. In the Wishart–Laguerre ensemble and its unitary invariant generalizations, the Bessel point process typically describes the microscopic behavior of the smallest eigenvalues close to the hard edge 0 [Forrester 1993; Kuijlaars and Vanlessen 2002; Tracy and Widom 1994b]. However, in a generalization of the Wishart–Laguerre ensemble known as the Muttalib–Borodin Laguerre ensemble [Borodin 1999; Muttalib 1995], the local behavior of eigenvalues near the hard edge is described by a different determinantal point process known as Wright’s generalized Bessel point process [Borodin 1999; Kuijlaars and Molag 2019]. Another generalization of the Bessel process, known as the Meijer-G point process, arises at the hard edge of Wishart-type products of Ginibre or truncated unitary matrices [Akemann et al. 2013a; 2013b; Kuijlaars and Zhang 2014; Kuijlaars and Stivigny 2014; Kieburg et al. 2016], and in Cauchy multimatrix ensembles [Bertola and Bothner 2015; Bertola et al. 2014].

In this paper, we will establish upper bounds for the global rigidity of the Airy point process, the Bessel point process, and its (determinantal) generalizations arising near the hard edge in Muttalib–Borodin ensembles and in product random matrix ensembles. We do this by combining asymptotics for exponential moments of the counting measures of these point processes, which are Fredholm determinants of certain integral operators, with a global rigidity estimate which can be applied to general point processes which almost surely have a smallest (or largest) point. In the case of the Airy and Bessel point processes, asymptotics for the exponential moments are known (see [Bothner and Buckingham 2018; Charlier and Claeys 2020] for Airy and [Bothner et al. 2019; Charlier 2020] for Bessel) and they allow us to improve on the best known upper bounds for the Airy point process (see [Zhong 2019] and [Corwin and Ghosal 2020, Theorem 1.6]), and to deduce completely new global rigidity results for the Bessel point process.

As global rigidity estimates give an intuitive idea of how the particles in a point process behave, we believe that they may lead to a better understanding of these random matrix point processes. Besides that, global rigidity estimates allow us to control averages of multiplicative statistics; global rigidity estimates in the Airy point process were for instance used in this way in [Corwin and Ghosal 2020] to obtain estimates for the lower tail of the Kardar–Parisi–Zhang equation.

Another main contribution of this paper consists in deriving exponential moment asymptotics for Wright’s generalized Bessel and Meijer-G point processes. We emphasize that we explicitly compute the multiplicative constant in these asymptotics, which is in general very challenging; see, e.g., [Forrester...
2014; Krasovsky 2009] as general references and [Charlier et al. 2019a; Charlier et al. 2019b] in the case of gap probabilities for the processes under consideration in this paper (the exponential moments under consideration here are gap probabilities for thinnings of these processes). As consequences of the exponential moment asymptotics, we obtain asymptotics for the average and variance of the counting functions of these processes, and an upper bound for their global rigidity.

**General rigidity theorem.** Suppose that $X$ is a locally finite random point process on the real line which has almost surely a smallest particle, and denote the ordered random points in the process by $x_1 \leq x_2 \leq \cdots$. We write $N(s)$ for the random variable that counts the number of points $\leq s$. We will work under the following assumptions, which, as we will see later, are fairly easy to verify in practice.

**Assumptions 1.1.** There exist constants $C, a > 0, s_0 \in \mathbb{R}, M > \sqrt{2/a}$ and continuous functions $\mu, \sigma : [s_0, +\infty) \to [0, +\infty)$ such that the following holds:

1. We have
   $$\mathbb{E}[e^{-\gamma N(s)}] \leq C e^{-\gamma \mu(s) + \gamma^2/2\sigma^2(s)}$$
   for any $\gamma \in [-M, M]$ and for any $s > s_0$.

2. The functions $\mu$ and $\sigma$ are strictly increasing and differentiable on $(s_0, +\infty)$, and they satisfy
   $$\lim_{s \to +\infty} \mu(s) = +\infty \quad \text{and} \quad \lim_{s \to +\infty} \sigma(s) = +\infty.$$
   Moreover, $s \mapsto s \mu'(s)$ is weakly increasing and $\lim_{s \to +\infty} s \mu'(s)/\sigma^2(s) = +\infty$.

3. The function $\sigma^2 \circ \mu^{-1} : [\mu(s_0), +\infty) \to [0, +\infty)$ is strictly concave, and
   $$(\sigma^2 \circ \mu^{-1})(s) \sim a \log s \quad \text{as} \quad s \to +\infty.$$

In the above assumptions, $C$ and $s_0$ are auxiliary constants whose values are unimportant, but on the other hand $a, \mu, \sigma$ will turn out to encode information about fundamental quantities of the point process under consideration, like the mean and variance of the counting functions.

**Theorem 1.2 (rigidity).** Suppose that $X$ is a locally finite point process on the real line which almost surely has a smallest particle and which is such that Assumptions 1.1 hold. Let us write $x_k$ for the $k$-th smallest particle of the process $X, k \geq 1$. Then, there are constants $c > 0$ and $s_0 > 0$ such that for any small enough $\epsilon > 0$ and for all $s \geq s_0$,

$$\mathbb{P}\left(\sup_{k \geq \mu(2s)} \frac{|\mu(x_k) - k|}{\sigma^2(\mu^{-1}(k))} > \sqrt{\frac{2}{a}(1 + \epsilon)} \right) \leq \frac{c \mu(s)^{-\frac{3}{2}}}{\epsilon}.$$  (1-2)

In particular, for any $\epsilon > 0$, 

$$\lim_{k_0 \to \infty} \mathbb{P}\left(\sup_{k \geq k_0} \frac{|\mu(x_k) - k|}{\sigma^2(\mu^{-1}(k))} \leq \sqrt{\frac{2}{a}(1 + \epsilon)} \right) = 1.$$

**Remark 1.3.** The above result derives an upper bound for the global rigidity via the asymptotics for the first exponential moment of the counting function. Estimates for the first exponential moment however do
not allow us to obtain a sharp lower bound for the global rigidity. For this, one would need more delicate information, like estimates for higher exponential moments, about more complicated averages in the point process, or about convergence of the counting function to a Gaussian multiplicative chaos measure; see, e.g., [Arguin et al. 2017; Berestycki et al. 2018; Claeyss et al. 2019a; Lambert et al. 2018]. In point processes arising in random matrix theory for which optimal lower bounds for the global rigidity are available (see, e.g., [Arguin et al. 2017; Chhaibi et al. 2018; Claeyss et al. 2019a; Holcomb and Paquette 2018; Paquette and Zeitouni 2018]), it turns out that the upper bounds obtained via the first exponential moment are sharp, and therefore we believe that the upper bound in Theorem 1.2 is, at least for the concrete examples considered below related to random matrix theory, close to optimal.

Outline of the proof of Theorem 1.2. We will prove Theorem 1.2 in Section 2 using elementary probabilistic estimates. The most delicate step in the proof consists of establishing a probabilistic bound for the supremum of the normalized counting function of the point process under consideration. For this, we need to use a discretization argument, a union bound, and Markov’s inequality together with the exponential moment asymptotics from Assumptions 1.1. Next, we prove that the bound on the supremum of the normalized counting function implies rigidity of the points, and we quantify the relevant probabilities to obtain Theorem 1.2. This method is similar to that of [Claeyss et al. 2019a, Section 4].

Global rigidity for the Airy point process. The Airy point process is a determinantal point process on \( \mathbb{R} \) whose correlation kernel is given by

\[
\kappa_{Ai}(x, y) = \frac{A_i(x) A_i'(y) - A_i'(x) A_i(y)}{x - y}, \quad x, y \in \mathbb{R},
\]

where \( A_i \) denotes the Airy function. This point process describes the largest eigenvalues in a large class of random matrix ensembles, and it has almost surely a largest particle. Upper bounds for the fluctuations of the points have been obtained recently in [Zhong 2019] and [Corwin and Ghosal 2020, Theorem 1.6]. A sharper upper bound can be obtained by combining the exponential moment estimates from [Bothner and Buckingham 2018] with Theorem 1.2.

The Airy point process satisfies Assumptions 1.1 only after considering the opposite points \( x_j = -a_j \), where \( a_1 > a_2 > \cdots \) are the random points in the Airy point process. We write \( N_{Ai}(s) \) for the number of points \( x_j \) smaller than or equal to \( s \). It was proved in [Bothner and Buckingham 2018] (see also [Bogatskiy et al. 2016; Charlier and Claeys 2020]) that

\[
\mathbb{E}[e^{-2\pi v N_{Ai}(s)}] = 8v^2 G(1 + iv)G(1 - iv)e^{-2\pi v \mu(s) + 2\pi^2 v^2 \sigma^2(s)}(1 + O(s^{-3/2}))
\]

as \( s \to +\infty \) uniformly for \( v \) in compact subsets of \( \mathbb{R} \), where \( G \) is Barnes’ \( G \) function, and where

\[
\mu(s) = \frac{2}{3\pi} s^{3/2}, \quad \sigma^2(s) = \frac{3}{4\pi^2} \log s.
\]

It is straightforward to verify from this that the Airy point process satisfies Assumptions 1.1 with

\[
M = 10, \quad \gamma = 2\pi v, \quad C = 2 \max_{v \in [-\frac{M}{2\pi}, \frac{M}{2\pi}]} 8v^2 G(1 + iv)G(1 - iv), \quad a = \frac{1}{2\pi^2},
\]

and with \( s_0 \) a sufficiently large constant. Applying Theorem 1.2, we obtain the following result.
Figure 1. Global rigidity for the Airy point process: On the left, the blue dots represent the random points and have coordinates $(k, x_k)$, the blue curves are the upper and lower bounds in (1-5) (with $\epsilon = 0.05$), and the orange curve is parametrized by $(t, (\frac{3\pi}{2} t)^{2/3})$. On the right, the blue dots represent the normalized random points with coordinates $(k, (\frac{2}{3\pi} x_k^{3/2} - k)/\log k)$. The orange lines indicate the heights $\pm \frac{1}{\pi} \pm \epsilon$ with $\epsilon = 0.05$. We observe the presence of points in the bands between the orange lines, indicating that Theorem 1.4 can be expected to be sharp. The points shown in the figure are not exactly the points in the Airy point process: they are sampled as rescaled extreme eigenvalues of a large GUE matrix, which approximate the points in the Airy point process.

**Theorem 1.4** (rigidity for the Airy process). Let $-x_1 > -x_2 > \cdots$ be the points in the Airy point process. There exists a constant $c > 0$ such that

$$P\left(\sup_{k \geq \mu(s)} \left| \frac{2}{3\pi} x_k^{3/2} - k \right| > \frac{\sqrt{1 + \epsilon}}{\pi} \log k \right) \leq cs^{-\frac{3\pi}{2\epsilon}},$$

as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for any $\epsilon > 0$, we have

$$\lim_{k_0 \to \infty} P\left(\sup_{k \geq k_0} \left| \frac{2}{3\pi} x_k^{3/2} - k \right| \leq \frac{1}{\pi} + \epsilon \right) = 1.$$

**Remark 1.5.** This result implies that for any $\epsilon > 0$, the probability that

$$\left(\frac{3\pi}{2}\right)^{2/3} \left(k - \left(\frac{1}{\pi} + \epsilon\right) \log k\right)^{2/3} \leq x_k \leq \left(\frac{3\pi}{2}\right)^{2/3} \left(k + \left(\frac{1}{\pi} + \epsilon\right) \log k\right)^{2/3}$$

tends to 1 as $k_0 \to +\infty$. Figure 1 illustrates this and supports our belief that Theorem 1.4 is close to optimal (see also Remark 1.3). Let us compare the above with known recent results: from [Corwin and Ghosal 2020, Theorem 1.6], it follows that for any $\epsilon > 0$, the probability that

$$\left| x_k - \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3}\right| \leq \epsilon k^{2/3}$$

tends to 1 as $k_0 \to \infty$; in [Zhong 2019], large deviation bounds were obtained which can be used to reduce the right-hand side from $\epsilon k^{2/3}$ to $\log^3 k$, but not to $\log k$ (see, e.g., Proposition 2.2 in [Zhong 2019]).
Global rigidity for the Bessel point process. The Bessel point process is another canonical point process from the theory of random matrices. It models the behavior of the eigenvalues near hard edges in a large class of random matrix ensembles, with the Laguerre–Wishart ensemble as the most prominent example [Forrester 1993]. The Bessel point process is a determinantal point process on $(0, +\infty)$ whose correlation kernel is given by

$$K_{\alpha}^{\text{Be}}(x, y) = \frac{\sqrt{y} J_{\alpha}(\sqrt{x}) J_{\alpha}'(\sqrt{y}) - \sqrt{x} J_{\alpha}(\sqrt{y}) J_{\alpha}'(\sqrt{x})}{2(x - y)}, \quad x, y > 0,$$

where $\alpha > -1$ and $J_{\alpha}$ is the Bessel function of the first kind of order $\alpha$. To the best of our knowledge, there are no global rigidity upper bounds available in the literature for the Bessel process, but the corresponding exponential moment asymptotics have been obtained in [Bothner et al. 2019; Charlier 2020], and they allow us to apply Theorem 1.2. Let us write $N_{\text{Be}}(s)$ for the number of points $x_j$ smaller than or equal to $s$ in the Bessel process. By [Bothner et al. 2019, equation (1.35)], we have

$$E[e^{-2\pi \nu N_{\text{Be}}(s)}] = 4 \nu^2 e^{\pi \nu a G(1 + i \nu) G(1 - i \nu)} e^{-2\pi \nu \mu(s) + 2\pi^2 \nu^2 \sigma^2(s)} (1 + O(s^{-1/2})), \quad s \to +\infty$$

as $s \to +\infty$ uniformly for $\nu$ in compact subsets of $\mathbb{R}$, with

$$\mu(s) = \frac{\sqrt{s}}{\pi}, \quad \sigma^2(s) = \frac{1}{4\pi^2} \log s.$$

We verify from (1-7) that the Bessel point process satisfies Assumptions 1.1 with

$$M = 10, \quad \gamma = 2\pi \nu, \quad C = 2 \max_{\nu \in [-M/2\pi, M/2\pi]} 4\nu^2 e^{\pi \nu a G(1 + i \nu) G(1 - i \nu)} a = \frac{1}{2\pi^2},$$

and with $s_0$ a sufficiently large constant. Applying Theorem 1.2, we obtain the following result.

**Theorem 1.6** (rigidity for the Bessel point process). Let $x_1 < x_2 < \cdots$ be the points in the Bessel point process. There exists a constant $c > 0$ such that

$$\mathbb{P}\left( \sup_{k \geq \mu(s)} \frac{\left| \frac{1}{\pi} x_k^{1/2} - k \right|}{\log k} > \frac{\sqrt{1 + \epsilon}}{\pi} \right) \leq c s^{-\frac{s}{1 + \epsilon}},$$

as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for any $\epsilon > 0$ we have

$$\lim_{k_0 \to \infty} \mathbb{P}\left( \sup_{k \geq k_0} \frac{\left| \frac{1}{\pi} x_k^{1/2} - k \right|}{\log k} \leq \frac{1}{\pi} + \epsilon \right) = 1.$$ 

**Remark 1.7.** The above implies that for any $\epsilon > 0$, the probability that

$$\pi^2 \left( k - \left( \frac{1}{\pi} + \epsilon \right) \log k \right)^2 \leq x_k \leq \pi^2 \left( k + \left( \frac{1}{\pi} + \epsilon \right) \log k \right)^2$$

for all $k \geq k_0$ (1-9) tends to 1 as $k_0 \to +\infty$. Figure 2 illustrates this estimate.

**Exponential moments and rigidity for Wright’s generalized Bessel process.** Wright’s generalized Bessel process appears as the limiting point process at the hard edge of Muttilib–Borodin ensembles [Borodin
Figure 2. Global rigidity for the Bessel point process: On the left, the blue dots represent the random points and have coordinates \((k, x_k)\), the blue curves are the upper and lower bounds in (1-9) (with \(\epsilon = 0.05\)), and the orange curve is parametrized by \((t, \pi^2 t^2)\). On the right, the blue dots represent the normalized random points with coordinates \((k, (\frac{1}{\pi} k^{1/2} - k)/\log k)\). The orange lines indicate the heights ±\(\frac{1}{\pi}\) ± \(\epsilon\) with \(\epsilon = 0.05\). We observe the presence of points in the bands between the orange lines, indicating that Theorem 1.6 can be expected to be sharp. The points shown in the figure are not exactly the points in the Bessel point process: they are sampled as rescaled extreme eigenvalues of a large Laguerre–Wishart random matrix, which approximate the points in the Bessel point process.

1999; Claeys and Romano 2014; Forrester and Wang 2017; Kuijlaars and Molag 2019; Liu et al. 2016; Zhang 2015; 2017]. It is a determinantal point process on \((0, +\infty)\) which depends on parameters \(\theta > 0\) and \(\alpha > -1\). The associated kernel is given by

\[
K_{\text{Wr}}(x, y) = \theta (xy)^{\alpha/2} \int_0^1 J_{\alpha+1, \theta}^{-1}(xt) J_{\alpha+1, \theta}(yt^\theta) t^\alpha dt, \quad x, y > 0,
\]

(1-10)

where \(J_{\alpha, \theta}\) is Wright’s generalized Bessel function,

\[
J_{\alpha, b}(x) = \sum_{m=0}^{\infty} \frac{(-x)^m}{m! \Gamma(a + bm)}, \quad x > 0.
\]

If \(\theta = 1\), this point process reduces (up to a rescaling) to the Bessel point process:

\[
K_{\text{Wr}}(x, y)|_{\theta=1} = 4 K_{\text{Be}}(4x, 4y), \quad x, y > 0.
\]

(1-11)

We obtain asymptotics for the exponential moments in this point process.

**Theorem 1.8.** Let \(\nu \in \mathbb{R}\) and let \(N_{\text{Wr}}(s)\) denote the number of points smaller than or equal to \(s\) in Wright’s generalized Bessel process. As \(s \to +\infty\), we have

\[
\mathbb{E}[e^{-2\pi \nu N_{\text{Wr}}(s)}] = C \exp(-2\pi \nu \mu(s) + 2\pi^2 \nu^2 \sigma^2(s) + O(s^{-1/19})),
\]

(1-12)
where the functions $\mu$ and $\sigma^2$ are given by
\[
\mu(s) = \frac{1 + \theta}{\pi} \theta^{-\frac{\theta}{1 + \theta}} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} s^{\frac{\theta}{1 + \theta}} \right) \quad \text{and} \quad \sigma^2(s) = \frac{\theta}{2\pi^2(1 + \theta)} \log s,
\]
and the values of $C$ by
\[
C = \exp \left( \pi \nu \left( 1 - \frac{\theta + 2\alpha}{1 + \theta} \right) \right) \left[ 4(1 + \theta) \theta^{-\frac{\theta}{1 + \theta}} \sin^2 \left( \frac{\pi \theta}{1 + \theta} \right) \right]^{\nu^2} G(1 + i\nu)G(1 - i\nu),
\]
where $G$ is Barnes’ $G$-function. Furthermore, the error term in (1-12) is uniform for $\nu$ in compact subsets of $\mathbb{R}$.

**Remark 1.9.** By setting $\theta = 1$ in (1-12) and then applying the rescaling $s \mapsto \frac{s}{4}$, we recover the asymptotics (1-7) for the Bessel point process, which is consistent with (1-11).

**Remark 1.10.** It follows from the end of Section 1 in [Borodin 1999] that the left-hand side of (1-12) is invariant under the following changes of the parameters:
\[
s \mapsto s^\theta, \quad \theta \mapsto \frac{1}{\theta}, \quad \text{and} \quad \alpha \mapsto \alpha^* := \frac{1 + \alpha}{\theta} - 1.
\]
It follows that the constant $C$ and the functions $\mu$ and $\sigma^2$ must obey the following symmetry relations for any $\theta > 0$ and $\alpha > -1$:
\[
\mu(s, \theta, \alpha) = \mu \left( s^\theta, \frac{1}{\theta}, \alpha^* \right), \quad \sigma^2(s, \theta, \alpha) = \sigma^2 \left( s^\theta, \frac{1}{\theta}, \alpha^* \right), \quad C(\theta, \alpha) = C \left( \frac{1}{\theta}, \alpha^* \right),
\]
where we have indicated the dependence of the quantities on $\theta$ and $\alpha$ explicitly. These identities can be verified directly from (1-14) and provide a consistency check of our results.

**Remark 1.11.** It is not entirely obvious that the kernel (1-10) defines a point process. To see this, we note first that the kernel (1-10) arises as the large $n$ limit of the correlation kernel $K_n$ in the Muttalib–Borodin Laguerre ensemble with $n$ particles (see [Borodin 1999]). Next, from [Lenard 1973] and [Soshnikov 2000, Theorem 1], we know that a kernel defines a point process if and only if it generates locally integrable correlation functions which are symmetric under permutations of variables and satisfy a certain positivity condition. Since $K_n$ must satisfy the symmetry and positivity conditions, and since these conditions are closed under taking limits, we can conclude that (1-10) also defines a point process. The uniqueness of the point process follows from the fact that the process is characterized by its Laplace transform $\mathbb{E}e^{-\sum_{k=1}^{\infty} f(x_k)}$ for continuous compactly supported functions $f$, where $x_1, x_2, \ldots$ are the points in the process. For a determinantal point process with a kernel $K$ which is trace-class on any compact, the Laplace transform is characterized by $K$ since
\[
\mathbb{E}e^{-\sum_{k=1}^{\infty} f(x_k)} = \det(1 - (1 - e^{-f})K).
\]
The Fredholm determinant at the right is defined since the trace-norm of $(1 - e^{-f})K$ is bounded by $\|1 - e^{-f}\|_\infty$ times the trace-norm of $K$ restricted to the support of $f$. Hence the process defined by $K$ is unique. See also, e.g., [Fenzl and Lambert 2020, Remark 11] for a similar situation.

**Theorem 1.8** has the following immediate consequence.
Corollary 1.12. As $s \to +\infty$, we have
\[
\mathbb{E}[N^{Wr}(s)] = \mu(s) - \frac{1 - \theta + 2\alpha}{2(1 + \theta)} + O(s^{-\frac{\theta}{1+\theta}}), \quad (1-16)
\]
\[
\text{Var}[N^{Wr}(s)] = \sigma^2(s) + \frac{1}{2\pi^2} \log \left[ 4(1 + \theta) \theta^{-\frac{\theta}{1+\theta}} \sin^2 \left( \frac{\pi \theta}{1+\theta} \right) \right] + \frac{1 + \gamma E}{2\pi^2} + O(s^{-\frac{\theta}{1+\theta}}), \quad (1-17)
\]
where $\gamma E \approx 0.5772$ is Euler’s constant and the functions $\mu$ and $\sigma^2$ are given by (1-13).

Proof. The asymptotics (1-12) are valid uniformly for $\nu$ in compact subsets of $\mathbb{R}$. Hence, we obtain (1-16)–(1-17) by first expanding (1-12) as $\nu \to 0$, and then identifying the power of $\nu$ using
\[
\mathbb{E}[e^{-2\pi \nu N^{Wr}(s)}] = 1 - 2\pi \nu \mathbb{E}[N^{Wr}(s)] + 2\pi^2 \nu^2 \mathbb{E}[(N^{Wr}(s))^2] + O(\nu^3), \quad \text{as } \nu \to 0. \quad \square
\]

Remark 1.13. Setting $\theta = 1$ in (1-16)–(1-17) and then rescaling $s \mapsto \frac{s}{\theta}$, we recover the asymptotic formula [Bothner et al. 2019, equation (1.35)].

We verify from Theorem 1.8 that Wright’s generalized Bessel process satisfies Assumptions 1.1 with $s_0$ a sufficiently large constant, and
\[ M = 10, \quad \gamma = 2\pi \nu, \quad C = 2 \max_{\nu \in \left[ -\frac{M}{2\pi}, \frac{M}{2\pi} \right]} C(\nu), \quad a = \frac{1}{2\pi^2}, \]
where $C = C(\nu)$ is given by (1-14). We obtain the following global rigidity result by combining Theorem 1.2 with Theorem 1.8.

Theorem 1.14 (rigidity for Wright’s generalized Bessel process). Let $x_1 < x_2 < \cdots$ be the points in Wright’s generalized Bessel point process. There exists a constant $c > 0$ such that
\[
\mathbb{P} \left( \sup_{k \geq \mu(s)} \left| \frac{1 + \theta}{\pi} \theta^{-\frac{\theta}{1+\theta}} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) x_k^{\frac{\theta}{1+\theta}} - k \right| \log k \right) \leq \frac{cs^{-\frac{\theta}{1+\theta}} + \epsilon}{\epsilon},
\]
as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for any $\epsilon > 0$, we have
\[
\lim_{k_0 \to \infty} \mathbb{P} \left( \sup_{k \geq k_0} \left| \frac{1 + \theta}{\pi} \theta^{-\frac{\theta}{1+\theta}} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) x_k^{\frac{\theta}{1+\theta}} - k \right| \log k \leq \frac{1}{\pi} + \epsilon \right) = 1.
\]

Remark 1.15. This result implies that for any $\epsilon > 0$, the probability that
\[
\left[ \frac{\pi}{1 + \theta} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) \left( k - \left( \frac{1}{\pi} + \epsilon \right) \log k \right) \right]^{1+\theta} \leq x_k \leq \left[ \frac{\pi}{1 + \theta} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) \left( k + \left( \frac{1}{\pi} + \epsilon \right) \log k \right) \right]^{1+\theta}
\]
for all $k \geq k_0$, tends to 1 as $k_0 \to +\infty$. We illustrate this numerically in Figure 3 (left) for $\epsilon = 0.05$ and different values of $\theta$.

Exponential moments and rigidity for the Meijer-G process. The Meijer-G process is the limiting point process at the hard edge of Wishart-type products of Ginibre matrices or truncated unitary matrices and
Figure 3. Global rigidity for Wright’s generalized Bessel and Meijer-G point processes: The blue dots represent the random points for different values of the parameters, the blue curves represent the upper and lower bounds in (1-18) and (1-27) with $\epsilon = 0.05$, and the orange curves correspond to $\epsilon = -\frac{1}{\pi}$. The points were sampled as random matrix eigenvalues which are known to approximate the random points. On the left, we rely for this on a result of Cheliotis [2018, Theorem 4], and on the right, on a result of Kuijlaars and Zhang [2014, Theorem 5.3].
Remark 1.17. If \( q = 0 \) and if the parameters \( \nu_1, \ldots, \nu_r \) form an arithmetic progression, then the kernel \( \mathcal{K}^{\text{Me}} \) defines the same point process (up to rescaling) as Wright’s generalized Bessel point process (for a rational \( \theta \)); see [Kuijlaars and Stivigny 2014, Theorem 5.1]. More precisely, if \( r \geq 1 \) is an integer, \( \alpha > -1 \) and

\[
\theta = \frac{1}{r}, \quad \nu_j = \alpha + \frac{j - 1}{r}, \quad j = 1, \ldots, r,
\]

then the kernels \( \mathcal{K}^{\text{Me}} \) and \( \mathcal{K}^{\text{Wr}} \) are related by

\[
\left( \frac{x}{y} \right)^{\frac{q}{2}} \mathcal{K}^{\text{Me}}(x, y) = r^r \mathcal{K}^{\text{Wr}}(r^r x, r^r y).
\]

Therefore, if the parameters satisfy (1-22), we obtain the relations

\[
\mu^{\text{Me}}(s) = \mu^{\text{Wr}}(r^r s), \quad \sigma^{\text{Me}}(s)^2 = \sigma^{\text{Wr}}(s)^2, \quad C^{\text{Me}} = r^{\frac{q^2}{2}} C^{\text{Wr}},
\]

where the quantities with superscript \( \text{Wr} \) and \( \text{Me} \) are given in Theorems 1.8 and 1.16, respectively. All the identities in (1-24) can be verified by a direct computation; this provides another consistency check of our results.

Remark 1.18. The existence and uniqueness of a point process with correlation kernel (1-19) can be shown in a similar way as outlined in Remark 1.11 for Wright’s generalized Bessel process.

Corollary 1.19. As \( s \to +\infty \), we have

\[
\mathbb{E}[N^{\text{Me}}(s)] = \mu(s) - \frac{1}{1 + r - q} \left[ \sum_{j=1}^{r} \nu_j - q \sum_{j=1}^{q} \mu_j \right] + O(s^{-\frac{1}{1 + r - q}}),
\]

\[
\text{Var}[N^{\text{Me}}(s)] = \sigma^2(s) + \frac{1}{2\pi^2} \log \left[ 4(1 + r - q) \sin^2 \left( \frac{\pi}{1 + r - q} \right) \right] + \frac{1 + \gamma_E}{2\pi^2} + O(s^{-\frac{1}{1 + r - q}}),
\]

where \( \gamma_E \) is Euler’s constant and the functions \( \mu \) and \( \sigma^2 \) are given by (1-21).

Proof. The proof is similar to the proof of Corollary 1.12.

We verify from Theorem 1.16 that the Meijer-G process satisfies Assumptions 1.1 with \( s_0 \) a sufficiently large constant, and

\[
M = 10, \quad \nu = 2\pi \nu, \quad C = 2 \max_{v \leq \frac{M}{2\pi}} C(v), \quad a = \frac{1}{2\pi^2},
\]

where \( C = C(v) \) is given in Theorem 1.16. We obtain the following rigidity result by combining Theorem 1.2 with Theorem 1.16. We emphasize that we do not need the value of the constant \( C \) to prove this result, but that its value might be important in view of other applications.

Theorem 1.20 (rigidity for the Meijer-G process). Let \( x_1 < x_2 < \cdots \) be the points in the Meijer-G point process. There exists a constant \( c > 0 \) such that

\[
P \left( \sup_{k \geq \mu(s)} \left| \frac{1 + r - q}{\pi} \cos \left( \frac{\pi}{2} \frac{r - q - 1}{r - q + 1} x_k \right) - k \right| \log k > \frac{\sqrt{1 + \epsilon}}{\pi} \right) \leq \frac{cs^{-\frac{-c}{2(1 + r - q)}}}{\epsilon},
\]
as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for all $\epsilon > 0$ we have

$$\lim_{k_0 \to +\infty} \mathbb{P} \left( \sup_{k \geq k_0} \left| \frac{1+q-r}{\pi} \cos \left( \frac{\pi}{2} \frac{r-q-1}{r-q+1} \right) \frac{1}{x_k^{1+q-r}} - k \right| \leq \frac{1}{\pi} + \epsilon \right) = 1. \leqno{1.77}$$

**Remark 1.21.** The above means that for any $\epsilon > 0$, the probability that

$$\left[ \frac{\pi}{1+r-q} \frac{1}{\cos \left( \frac{\pi}{2} \frac{r-q-1}{r-q+1} \right)} \left( k - \left( \frac{1}{\pi} + \epsilon \log k \right) \right)^{1+q-r} \leq x_k \right.$$

$$\left. \leq \left[ \frac{\pi}{1+r-q} \frac{1}{\cos \left( \frac{\pi}{2} \frac{r-q-1}{r-q+1} \right)} \left( k + \left( \frac{1}{\pi} + \epsilon \log k \right) \right)^{1+q-r} \right. \right]$$

$$\text{for all } k \geq k_0 \leqno{1.77}$$

tends to 1 as $k_0 \to +\infty$. This has been verified numerically for $q = 0$ and different values of $r$ by generating products of $r$ independent Ginibre matrices [Akemann et al. 2013b; Kuijlaars and Zhang 2014] and is illustrated in Figure 3 (right) for $\epsilon = 0.05$.

**Outline of the proofs of Theorems 1.8 and 1.16.** It is well known [Borodin 2011; Johansson 2006; Soshnikov 2000] that the left-hand sides of (1-12) and (1-20) are (for any determinantal point process generated by one of the kernels (1-10) or (1-19), recall Remark 1.11 and Remark 1.18) equal to the Fredholm determinants

$$\det(1 - (1-t)\mathbb{K}^\text{Wr}_{[0,s]}) \quad \text{and} \quad \det(1 - (1-t)\mathbb{K}^\text{Me}_{[0,s]}), \quad t = e^{-2\pi v} \leqno{1.78}$$

respectively. The kernels $\mathbb{K}^\text{Wr}$ and $\mathbb{K}^\text{Me}$ are known to be integrable in the sense of Its, Izergin, Korepin and Slavnov (IIKS) [Its et al. 1990] only for particular values of the parameters. For example, for $\theta = p/q$, $p, q \in \mathbb{N}_{>0}$, $\mathbb{K}^\text{Wr}$ is integrable of size $p+q$ [Zhang 2017], and there are associated Riemann–Hilbert (RH) problems of size $(p+q) \times (p+q)$. We expect the analysis of these RH problems to be rather complicated (except in the simplest case when $p = q = 1$). Furthermore, for irrational values of $\theta$, $\mathbb{K}^\text{Wr}$ is not known to be integrable at all. To circumvent this problem, we use the ideas from [Claeys et al. 2019b] to rewrite (1-28) in Section 3 in terms of the determinant of an integrable operator of size 2, and to derive a differential identity in $s$, i.e., to express, for all values of the parameters, the derivatives

$$\partial_s \log \det(1 - (1-t)\mathbb{K}^\text{Wr}_{[0,s]}) \quad \text{and} \quad \partial_s \log \det(1 - (1-t)\mathbb{K}^\text{Me}_{[0,s]}), \leqno{1.79}$$

in terms of the solution, denoted $Y$, to a $2 \times 2$ RH problem. We then perform, in Section 4, an asymptotic analysis of this RH problem by means of the Deift–Zhou [Deift and Zhou 1993] steepest descent method. The local analysis requires the use of parabolic cylinder functions. By integrating in $s$ the derivatives (1-29), we obtain

$$\log \det(1 - (1-t)\mathbb{K}_{[0,s]}) = \log \det(1 - (1-t)\mathbb{K}_{[0,M]})$$

$$+ \int_M^s \partial_s \log \det(1 - (1-t)\mathbb{K}_{[0,\tilde{s}]})d\tilde{s}, \quad \mathbb{K} = \mathbb{K}^\text{Wr}, \mathbb{K}^\text{Me}, \leqno{1.80}$$

for a certain constant $M$. By substituting the large $s$ asymptotics of (1-29) in the integrand of (1-30), we
determine the functions $\mu$ and $\sigma^2$ of Theorems 1.8 and 1.16 in Section 5. However, the quantity

$$\log \det(1 - (1 - t)\mathcal{K}|_{[0,M]})$$

is an unknown constant, so this method does not allow for the evaluation of $C$ (the constants of order 1) of Theorems 1.8 and 1.16. Such constants are notoriously difficult to compute explicitly [Krasovsky 2009], and require the use of other differential identities which are more complicated to analyze. To obtain $C$, we will use a differential identity in $t$, i.e., we will express

$$\partial_t \log \det(1 - (1 - t)\mathcal{K}|_{[0,s]}) = \mathcal{K} = \mathcal{K}^{\text{Wr}}, \mathcal{K}^{\text{Me}}, \quad (1-31)$$

in terms of $Y$ in Section 3. Large $s$ asymptotics for the derivatives (1-31) appears to be rather complicated to obtain. In particular, it requires the explicit evaluation of certain (regularized) integrals involving parabolic cylinder functions. The key observation is that these integrals do not depend on any other parameters than $t$. Then we evaluate explicitly these complicated integrals by using the known expansion (1-7) from [Bothner et al. 2019; Charlier 2020]. By integrating (1-31) in $t$, we have the identity

$$\log \det(1 - (1 - t)\mathcal{K}|_{[0,s]}) = \log \det(1 - (1 - t)\mathcal{K}|_{[0,s]})|_{t=1} + \int_1^t \partial_t \log \det(1 - (1 - \tilde{t})\mathcal{K}|_{[0,s]})d\tilde{t}, \quad \mathcal{K} = \mathcal{K}^{\text{Wr}}, \mathcal{K}^{\text{Me}}, \quad (1-32)$$

where $t \in (0, 1]$ is fixed. By substituting the large $s$ asymptotics of (1-31) in the integrand of (1-32) and by using the results of [Bothner et al. 2019] in Section 6, we obtain $C$ (and moreover we recover the same functions $\mu, \sigma^2$ as obtained via the differential identity in $s$), since

$$\log \det(1 - (1 - t)\mathcal{K}|_{[0,s]})|_{t=1} = 0.$$ 

The direct analysis of (1-32) is rather involved. In conclusion, each of the two differential identities has its advantages and disadvantages: the differential identity in $s$ leads to an easier analysis, but does not allow for the evaluation of $C$, while the differential identity in $t$ is significantly more involved but allows us to compute $C$. Another advantage of the differential identity in $s$ is that it allows us, with only limited efforts, to give the optimal estimates $O(s^{-\frac{1}{T+\eta}})$ and $O(s^{-\frac{1}{T+\eta-q}})$ for the error terms of Theorems 1.8 and 1.16; with the differential identity in $t$, we are only able to obtain errors of order $O(s^{-\frac{1}{T+\eta}})$ and $O(s^{-\frac{1}{T+\eta-q}})$.

### 2. Proof of Theorem 1.2

In this section, we suppose that $X$ is a locally finite random point process on the real line which has a smallest particle almost surely, with counting function $N(s)$, and which is such that Assumptions 1.1 hold for certain constants $C, a > 0, s_0 \in \mathbb{R}, M > \sqrt{2/a}$, and for certain functions $\mu, \sigma$.

We start by establishing a bound for the tail of the probability distribution of the extremum of the normalized counting function.

**Lemma 2.1.** There exist $c > 0$ and $s_0 > 0$ such that for any $\epsilon > 0$ sufficiently small and $s > s_0$,

$$\mathbb{P}\left(\sup_{x>s} \left| N(x) - \mu(x) \right| > \sqrt{\frac{2}{a}}(1 + \epsilon) \right) \leq \frac{c \mu(s)^{-\epsilon}}{2\epsilon}. \quad (2-1)$$
In particular, for any $\epsilon > 0$,
\[
\lim_{s \to +\infty} \mathbb{P}\left( \sup_{x > s} \left| \frac{N(x) - \mu(x)}{\sigma^2(x)} \right| \leq \sqrt{\frac{2}{a}}(1 + \epsilon) \right) = 1.
\]

**Proof.** Let us define $\kappa_k = \mu^{-1}(k)$. We start by noting that for $x \in [\kappa_{k-1}, \kappa_k]$, $k \in \mathbb{N}$, we have by monotonicity of $\mu$ and of the counting function $N$ that
\[
N(x) - \mu(x) \leq N(\kappa_k) - \mu(\kappa_{k-1}) = N(\kappa_k) - \mu(\kappa_k) + 1,
\]
and since $\sigma$ is increasing, we also have $\sigma^2(x) \geq \sigma^2(\kappa_{k-1})$. For large enough $s$, it follows that
\[
\sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} \leq \sup_{k: \kappa_k > s} \frac{N(\kappa_k) - \mu(\kappa_k) + 1}{\sigma^2(\kappa_{k-1})}.
\]
Hence, by a union bound, for any $\gamma > 0$ we have
\[
\mathbb{P}\left( \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} > \gamma \right) \leq \sum_{k: \kappa_k > s} \mathbb{P}\left( \frac{N(\kappa_k) - \mu(\kappa_k) + 1}{\sigma^2(\kappa_{k-1})} > \gamma \right)
\]
\[
= \sum_{k: \kappa_k > s} \mathbb{P}(e^{\gamma N(\kappa_k)} > e^{\gamma \mu(\kappa_k)-\gamma+\gamma^2 \sigma^2(\kappa_{k-1})}) \leq \sum_{k: \kappa_k > s} \mathbb{E}(e^{\gamma N(\kappa_k)})e^{-\gamma \mu(\kappa_k)+\gamma-\gamma^2 \sigma^2(\kappa_{k-1})}, \tag{2-2}
\]
where the last inequality is obtained by applying Markov’s inequality on the positive random variable $e^{\gamma N(\kappa_k)}$. Using (1-1) in (2-2), we obtain
\[
\mathbb{P}\left( \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} > \gamma \right) \leq C e^\gamma \sum_{k: \kappa_k > s} e^{-\frac{\gamma^2}{2} \sigma^2(\kappa_k)} e^{\gamma^2 \sigma^2(\kappa_{k-1})} e^{-\gamma \sigma^2(\kappa_k) - \sigma^2(\kappa_{k-1})}.
\]
Because $(\sigma^2 \circ \mu^{-1})$ is strictly concave and behaves as $(\sigma^2 \circ \mu^{-1})(k) \sim a \log k$ as $k \to +\infty$, we have that
\[
e^{\gamma^2 \sigma^2(\kappa_{k-1})} \leq e^{\gamma^2 \sigma^2(\mu^{-1}(k)) - \sigma^2(\mu^{-1}(k-1))}
\]
decreases with $k$ and is uniformly bounded in $k$ by a constant which we denote as $C'$ and which we can choose independently of $\gamma \in [0, M]$. Using also the fact that $\sigma^2$ and $\mu$ are increasing, we obtain
\[
\mathbb{P}\left( \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} > \gamma \right) \leq C C' e^\gamma \sum_{k: \kappa_k > s} e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(k))}
\]
\[
\leq C C' e^\gamma \left( e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(x))} + \int_{\mu(s)}^{\infty} e^{\gamma^2 \sigma^2(x)} \frac{1}{\sigma^2(x)} dx \right). \tag{2-3}
\]
Similarly, we obtain
\[
\mathbb{P}\left( \sup_{x > s} \frac{\mu(x) - N(x)}{\sigma^2(x)} > \gamma \right) \leq \sum_{k: \kappa_k > s} \mathbb{P}\left( \frac{\mu(\kappa_{k-1}) - N(\kappa_{k-1}) + 1}{\sigma^2(\kappa_{k-1})} > \gamma \right)
\]
\[
= \sum_{k: \kappa_k+1 > s} \mathbb{P}(e^{-\gamma N(\kappa_k)} > e^{-\gamma \mu(\kappa_k)-\gamma+\gamma^2 \sigma^2(\kappa_k)}) \leq \sum_{k: \kappa_k+1 > s} \mathbb{E}(e^{-\gamma N(\kappa_k)})e^{-\gamma \mu(\kappa_k)+\gamma-\gamma^2 \sigma^2(\kappa_k)}.
\]
Using again (1-1) and the fact that $\sigma$ and $\mu$ are increasing, we then get
\[
\mathbb{P}\left(\sup_{x > s} \frac{\mu(x) - N(x)}{\sigma^2(x)} > \gamma\right) \leq C e^{\gamma'} \sum_{k: x_{k+1} > s} e^{-\frac{C^2}{2} \sigma^2(\mu^{-1}(k))} 
\leq C e^{\gamma'} \left(2 e^{-\frac{\gamma^2}{2} (\sigma^2(\mu(s)-1))} + \int_{\mu(s)}^{\infty} e^{-\frac{\gamma^2}{2} (\sigma^2(\mu^{-1})(x))} dx\right). \tag{2-4}
\]
By combining (2-3) and (2-4), we obtain
\[
\mathbb{P}\left(\sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| > \gamma\right) \leq C(C' + 2)e^{\gamma'} \left(e^{-\frac{\gamma^2}{2} (\sigma^2(\mu^{-1})(\mu(s)-1))} + \int_{\mu(s)}^{\infty} e^{-\frac{\gamma^2}{2} (\sigma^2(\mu^{-1})(x))} dx\right).
\]
It follows from criteria (2) and (3) of Assumptions 1.1 that the right-hand side converges to 0 as $s \to +\infty$, provided that $\gamma > \sqrt{2/\alpha}$. More precisely, for $\gamma \in (\sqrt{2/\alpha}, M]$ the right-hand side is smaller than
\[
2C(C' + 2)e^M \left((\mu(s) - 1)^{-\frac{\alpha^2}{2}} + \int_{\mu(s)}^{\infty} x^{-\frac{\alpha^2}{2}} dx\right) \leq 3C(C' + 2)e^M \frac{\mu(s)^{1-\frac{\alpha^2}{2}}}{\alpha^2 - 1}
\]
for all sufficiently large $s$. In conclusion, for any $\gamma \in (\sqrt{2/\alpha}, M]$, we have
\[
\mathbb{P}\left(\sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| > \gamma\right) \leq 3C(C' + 2)e^M \frac{\mu(s)^{1-\frac{\alpha^2}{2}}}{\alpha^2 - 1}.
\]
We obtain the claim after setting $c = 6C(C' + 2)e^M$ and $\gamma = \sqrt{\frac{2}{\alpha}}(1 + \epsilon)$.

Next, assuming a bound for the extremum of the normalized counting function, we derive the global rigidity of the points in the process.

**Lemma 2.2.** Let $\epsilon > 0$. For all sufficiently large $s$, if the event
\[
\sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| = \sup_{x \geq s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| \leq \sqrt{\frac{2}{\alpha}}(1 + \epsilon) \tag{2-5}
\]
holds true, then we have
\[
\sup_{k \geq \mu(2s)} \left|\frac{\mu(x_k) - k}{\sigma^2(\mu^{-1})(k)}\right| \leq \sqrt{\frac{2}{\alpha}}(1 + 2\epsilon). \tag{2-6}
\]

**Proof.** We start by proving that
\[
x_k > s, \quad \text{for all} \; k \geq \mu(2s), \tag{2-7}
\]
for all large enough $s$. Suppose that $x_k \leq s < 2s \leq \kappa_k$, where $\kappa_k = \mu^{-1}(k)$. Then
\[
\mu(2s) \leq \mu(\kappa_k) = k = N(x_k) \leq N(s),
\]
which implies by Assumptions 1.1 that
\[
\frac{N(s) - \mu(s)}{\sigma^2(s)} \geq \frac{\mu(2s) - \mu(s)}{\sigma^2(s)} \geq \frac{s \inf_{s \leq \xi \leq 2s} \mu'(\xi)}{2 \sigma^2(s)} \geq \frac{\inf_{s \leq \xi \leq 2s} \xi \mu'(\xi)}{2 \sigma^2(s)} \geq \frac{s \mu'(s)}{2 \sigma^2(s)}. \tag{2-8}
\]
Again by Assumptions 1.1, the right-hand side of (2-8) tends to $+\infty$ as $s \to +\infty$, so there is a contradiction with (2-5), provided that $s$ is chosen large enough. We conclude that $x_k > s$ for all $k \geq \mu(2s)$, provided that $s$ is large enough.

We split the proof of (2-6) into two parts. We first prove the following upper bound for $\mu(x_k)$:

$$\mu(x_k) \leq k + \sqrt{\frac{2}{a}(1 + 2\epsilon)(\sigma^2 \circ \mu^{-1})(k)}, \quad \text{for all } k \geq \mu(2s). \tag{2-9}$$

Define $m = m(k)$ as the unique integer such that $\kappa_{k+m} < x_k \leq \kappa_{k+m+1}$. If $m < 0$, then (2-9) is immediately satisfied. Let us now treat the case $m \geq 0$. Since $k \geq \mu(2s)$, we have $x_k > s$ by (2-7). Therefore, we use (2-5) together with $m \geq 0$ to conclude that

$$\sqrt{\frac{2}{a}(1 + \epsilon)} \geq \frac{\mu(x_k) - N(x_k)}{\sigma^2(x_k)} \geq \frac{m}{(\sigma^2 \circ \mu^{-1})(k + m + 1)},$$

and it follows that

$$m \leq \sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k + m + 1)} \leq \sqrt{\frac{2}{a}(1 + \epsilon)}((\sigma^2 \circ \mu^{-1})(k) + (m + 1)(\sigma^2 \circ \mu^{-1})(k)),$$

where we used the concavity of $\sigma^2 \circ \mu^{-1}$ from Assumptions 1.1. This inequality can be rewritten as

$$\left(1 - \sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k)}\right)m \leq \sqrt{\frac{2}{a}(1 + \epsilon)}((\sigma^2 \circ \mu^{-1})(k) + (\sigma^2 \circ \mu^{-1})(k)).$$

Since $\sigma \circ \mu^{-1}$ is concave, the derivative $(\sigma \circ \mu^{-1})'$ is decreasing, and thus for $k \geq k_0$ we have

$$(\sigma \circ \mu^{-1})(k) = (\sigma \circ \mu^{-1})(k_0) + \int_{k_0}^{k} (\sigma \circ \mu^{-1})'(\tilde{k})d\tilde{k} \geq (\sigma \circ \mu^{-1})(k_0) + (\sigma \circ \mu^{-1})'(k)(k - k_0). \tag{2-10}$$

Since $(\sigma \circ \mu^{-1})(k) \sim a \log(k)$ as $k \to +\infty$, (2-10) yields $(\sigma \circ \mu^{-1})(k) \to 0$ as $k \to +\infty$. We deduce that, for any fixed $\delta > 0$,

$$(1 - \delta)m \leq (1 + \delta)\sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k)} - (1 - \delta), \quad \text{for all } k \geq \mu(2s),$$

provided that $s$ is large enough. We choose $\delta > 0$ sufficiently small such that

$$\frac{1 + \delta}{1 - \delta} \sqrt{\frac{2}{a}(1 + \epsilon)} < \sqrt{\frac{2}{a}(1 + 2\epsilon)}.$$

Therefore, we achieve the inequality

$$m + 1 \leq \sqrt{\frac{2}{a}(1 + 2\epsilon)(\sigma^2 \circ \mu^{-1})(k)}, \quad \text{for all } k \geq \mu(2s),$$

provided that $s$ is large enough. It follows that

$$\mu(x_k) \leq \mu(\kappa_{k+m+1}) = k + m + 1 \leq k + \sqrt{\frac{2}{a}(1 + 2\epsilon)(\sigma^2 \circ \mu^{-1})(k)}.$$
In the second part of the proof, we show the following lower bound for $\mu(x_k)$:

$$k - \sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k)} \leq \mu(x_k), \quad \text{for all } k \geq \mu(2s)$$  \hfill (2-11)

which is even slightly better than what is required to prove (2-6). Suppose that $\mu(x_k) < k - m$ with $m > 0$. By combining (2-7) with (2-5), we have

$$\sqrt{\frac{2}{a}(1 + \epsilon)} \geq \frac{N(x_k) - \mu(x_k)}{\sigma^2(x_k)} > \frac{m}{\sigma^2(x_k)} > \frac{m}{(\sigma^2 \circ \mu^{-1})(k)}, \quad \text{for all } k \geq \mu(2s),$$

and it follows that $m < \sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k)}$, which proves the lower bound. \hfill $\square$

It now suffices to combine the above two results in order to obtain Theorem 1.2.

**Proof of Theorem 1.2.** It follows from Lemma 2.1 that there exists $c > 0$ such that for all $\epsilon > 0$ sufficiently small and for all $s$ sufficiently large, we have

$$\mathbb{P}\left(\sup_{x > s} \left| \frac{N(x) - \mu(x)}{\sigma^2(x)} \right| \leq \sqrt{\frac{2}{a}\left(1 + \frac{\epsilon}{2}\right)}\right) \geq 1 - \frac{c\mu(s)^{-\frac{1}{2}}}{\epsilon}. \hfill (2-12)$$

Furthermore, Lemma 2.2 implies that

$$\mathbb{P}\left(\sup_{k \geq \mu(2s)} \left| \frac{\mu(x_k) - k}{\sigma^2(\mu^{-1}(k))} \right| \leq \sqrt{\frac{2}{a}(1 + \epsilon)} \sup_{x > s} \left| \frac{N(x) - \mu(x)}{\sigma^2(x)} \right| \leq \sqrt{\frac{2}{a}\left(1 + \frac{\epsilon}{2}\right)}\right) = 1, \hfill (2-13)$$

for all sufficiently large $s$. Theorem 1.2 follows by a direct application of Bayes’ formula, combining (2-12) and (2-13). \hfill $\square$

### 3. RH problem and differential identities

**Double contour integral representation for the kernels.** For convenience, let us write

$$\mathcal{K}^{(1)} = \mathcal{K}^{\text{Me}} \quad \text{and} \quad \mathcal{K}^{(2)} = \mathcal{K}^{\text{Wr}},$$

where $\mathcal{K}^{\text{Me}}$ and $\mathcal{K}^{\text{Wr}}$ have been defined in (1-19) and (1-10), respectively. For our analysis, we will use the following double contour representation for these kernels [Claeys et al. 2019b]:

$$\mathcal{K}^{(j)}(x, y) = \frac{1}{4\pi^2} \int_{\gamma} du \int_{\hat{\gamma}} dv \frac{F^{(j)}(u) x^{-u} y^{v-1}}{u-v}, \quad j = 1, 2, \hfill (3-1)$$

with

$$F^{(1)}(z) = \frac{\Gamma(z) \prod_{k=1}^{q} \Gamma(1 + \mu_k - z)}{\prod_{k=1}^{r} \Gamma(1 + \nu_k - z)}, \quad F^{(2)}(z) = \frac{\Gamma(z + \frac{\alpha}{2})}{\Gamma\left(\frac{\alpha + 1 - z}{\theta}\right)}. \hfill (3-2)$$

For $j = 1$, we require $r, q \in \mathbb{N}$, $r > q \geq 0$, $\nu_1, \ldots, \nu_r \in \mathbb{N}$ and $\mu_1, \ldots, \mu_q \in \mathbb{N} \geq 0$ such that $\mu_k > \nu_k$, $k = 1, \ldots, q$. If $q = 0$, the product in the numerator is understood as 1. For $j = 2$, we require $\alpha > -1$ and $\theta > 0$. The contours $\gamma, \hat{\gamma}$ are both oriented upward, do not intersect each other, and intersect the real
线在区间 \((0, 1 + \nu_{\min})\) 如果 \(j = 1\)，与 \(\nu_{\min} := \min\{\nu_1, \ldots, \nu_r\}\)，和在区间 \((-\frac{\alpha}{2}, 1 + \frac{\alpha}{2})\) if \(j = 2\). Furthermore, \(\gamma\) 趋于无限大在严格左半平面的向量中，而 \(\tilde{\gamma}\) 趋于无限大在严格右半平面的向量中，见 Figure 4.

**Integrable kernels.** As mentioned at the end of Section 1, the kernels \(k^{(j)}\), \(j = 1, 2\) are known to be integrable only for particular values of the parameters. With minor modifications of [Claeys et al. 2019b, Propositions 2.1 and 2.2], we obtain the following.

**Proposition 3.1.** Let \(t \in (0, +\infty)\). For \(j = 1, 2\), we have

\[
\det(1 - (1 - t)k^{(j)}|_{[0, s]}) = \det(1 - M^{(j)}_{s, t}), \quad j = 1, 2, \tag{3-3}
\]

where \(M^{(j)}_{s, t}\) is the trace-class integral operator acting on \(L^2(\gamma \cup \tilde{\gamma})\) with kernel

\[
M^{(j)}_{s, t}(u, v) = \frac{f(u)^T g(v)}{u - v},
\]

where \(f\) and \(g\) are given by

\[
f(u) = \frac{1}{2\pi i} \left( \frac{\chi_{\gamma}(u)}{s^u \chi_{\tilde{\gamma}}(u)} \right), \quad g(v) = \left( \frac{-\sqrt{1 - t} F^{(j)}(v)^{-1} \chi_{\tilde{\gamma}}(v)}{\sqrt{1 - ts^v F^{(j)}(v) \chi_{\gamma}(v)}} \right), \tag{3-4}
\]

and \(\chi_{\gamma}\) and \(\chi_{\tilde{\gamma}}\) are the characteristic functions of \(\gamma\) and \(\tilde{\gamma}\), respectively. The determination of \(\sqrt{1 - t}\) in the definition of \(g\) is unimportant (but the same determination must be chosen for both entries of \(g\)). For definiteness, we require

\[
\begin{align*}
\sqrt{1 - t} &\in [0, 1) \quad \text{if} \ t \in (0, 1], \\
\sqrt{1 - t} &\in [0, i \infty) \quad \text{if} \ t \in [1, +\infty). \tag{3-6}
\end{align*}
\]

**Proof.** The proof for \(t = 0\) (including the fact that \(M^{(j)}_{s, t}\) is trace-class) can be found in [Claeys et al. 2019b, Propositions 2.1 and 2.2] and relies on a conjugation of \(k^{(j)}|_{[0, s]}\) with a Mellin transform. The proof for arbitrary values of \(t \in (0, +\infty)\) only requires minor modifications: the quantity \(H^{(j)}_{s}(v, z)\) of [Claeys...
et al. 2019b, equation (2.10)\(^1\) needs to be modified to

\[
\frac{\sqrt{1-t}}{2\pi i} \int_{\gamma} \frac{du}{2\pi i} \frac{s^{z-u} F^{(j)}(u)}{F^{(j)}(v)(v-u)(z-u)},
\]

and the kernels \(A^{(j)}\) and \(B^{(j)}\) of [Claeys et al. 2019b, equation (2.19)] need to be modified to

\[
A^{(j)}(u, z) = \sqrt{1-t} \frac{s^{z-u} F^{(j)}(u)}{2\pi i (z-u)}, \quad B^{(j)}(v, u) = \frac{\sqrt{1-t}}{2\pi i F^{(j)}(v)(v-u)},
\]

where the determination of \(\sqrt{1-t}\) is unimportant, as long as it is the same for \(A^{(j)}\) and \(B^{(j)}\).

Using a method developed by Its, Izergin, Korepin, and Slavnov [Its et al. 1990], we will establish differential identities in \(s\) and \(t\) for the logarithm of the Fredholm determinants (3-3) in terms of the following RH problem:

**RH problem for** \(Y = Y^{(j)}, \ j = 1, 2:\)

(a) \(Y : \mathbb{C} \setminus (\gamma \cup \tilde{\gamma}) \to \mathbb{C}^{2 \times 2}\) is analytic.

(b) \(Y(z)\) has continuous boundary values \(Y_{\pm}(z)\) as \(z\) approaches the contour \(\gamma \cup \tilde{\gamma}\) from the left (+) and right (−), according to its orientation, and we have the jump relations

\[
Y_+ (z) = Y_- (z) J(z), \quad z \in \gamma \cup \tilde{\gamma},
\]

with jump matrix \(J = J^{(j)}\) given by

\[
J(z) = I - 2\pi i f(z)g(z)^T = \begin{cases} 
\left( \begin{array}{cc} 1 & -\sqrt{1-t}s^{-1} F^{(j)}(z) \\ 0 & 1 \end{array} \right), & z \in \gamma, \\
\left( \begin{array}{cc} 1 & 0 \\ -\sqrt{1-t}s^{-1} F^{(j)}(z)^{-1} & 1 \end{array} \right), & z \in \tilde{\gamma}.
\end{cases}
\]  \hspace{1cm} (3-7)

(c) As \(z \to \infty\), there exists \(Y_1 = Y_1^{(j)}(s, t)\) independent of \(z\) such that

\[
Y(z) = I + \frac{Y_1}{z} + \mathcal{O}(z^{-2}).
\]

**Remark 3.2.** We have some freedom in the choice of \(\gamma\) and \(\tilde{\gamma}\). We choose them symmetric with respect to the real line. This symmetry will be useful later to simplify computations.

**Lemma 3.3.** For \(j = 1, 2\), we have the following differential identities:

\[
\partial_s \log \det(1 - (1-t)\mathbb{K}^{(j)}|_{[0,s]}) = \frac{Y_{1,11}}{s} - \frac{Y_{1,22}}{s}, \tag{3-8}
\]

\[
\partial_t \log \det(1 - (1-t)\mathbb{K}^{(j)}|_{[0,s]}) = -\frac{1}{2(1-t)} \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}(z) Y'(z)(J(z) - I)] \frac{dz}{2\pi i}. \tag{3-9}
\]

\(^1\)There is a factor \(\frac{1}{2\pi i}\) missing in the expressions for \(H_2^{(j)}(v, z)\) and \(B^{(j)}(v, u)\) of [Claeys et al. 2019b, equations (2.10) and (2.19)].
Remark 3.4. We do not mention whether we take the $+$ or $-$ boundary values of $Y$ in the integrand of (3-9). This is without ambiguity, because

$$
\text{Tr}[Y_+^{-1}(z)Y'_+(z)(J(z) - I)] = \text{Tr}[Y_+^{-1}(z)Y'_+(z)(J(z) - I)]
$$

$$
= \text{Tr}[Y_+^{-1}(z)Y'_+(z)(J(z) - I)]
$$

$$
= \text{Tr}[Y_-^{-1}(z)Y'_+(z)(J(z) - I)].
$$

Proof. Both (3-8) and (3-9) are specializations of more general results from [Bertola 2010]. For the proof of (3-8), we refer to [Claeys et al. 2019b, Theorem 1.1], and for the proof of (3 -9), we apply [Bertola 2010, Section 5.1] (with $\partial = \partial_t$) to obtain

$$
\partial_t \log \det(1 - (1 - t)\zeta^{(j)})|_{[0,s]} = \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y_-^{-1}(z)Y'_-(z)\partial_t J(z)J^{-1}(z)] \frac{dz}{2\pi i}.
$$

From (3-7), it is straightforward to verify that

$$
\partial_t J(z)J(z)^{-1} = \frac{-1}{2(1 - t)}(J(z) - I),
$$

which yields (3-9) and finishes the proof.

4. Steepest descent analysis

In this section, we use the Deift–Zhou [Deift and Zhou 1993] steepest descent method to perform an asymptotic analysis of $Y = Y^{(j)}$, $j = 1, 2$, as $s \to +\infty$ uniformly for $t$ in compact subsets of $(0, +\infty)$. The first transformation $Y \mapsto U$ in Section 4A is a rescaling which is identical to the one from [Claeys et al. 2019b]. The rest of the analysis differs drastically from [Claeys et al. 2019b], and we highlight the main ideas for it here. As common in steepest descent analysis of RH problems, we will need to do a saddle points analysis of a phase function appearing in the jump matrix for $U$ (see Section 4B). In contrast to the approach of [Claeys et al. 2019b], the opening of the lenses is done in two steps $U \mapsto \hat{T} \mapsto T$ presented in Section 4C, and we emphasize that this is somewhat unusual, in that it requires two different factorizations of the jump matrix, each on a different part of the jump contour. The global parametrix $P^{(\infty)}$ of Section 4D approximates $T$ everywhere in the complex plane except near the saddle points $b_1$ and $b_2$. In Sections 4E and 4F, we construct local parametrices $P^{(b_k)}$ in terms of parabolic cylinder functions. The local parametrix $P^{(b_k)}$ is defined in a small disk $D_{b_k}$ centered at $b_k$ and satisfies the same jumps as $T$. The global and local parametrices are completely different than the ones in [Claeys et al. 2019b], because of the different jump matrix factorizations used for opening the lenses. The last step $T \mapsto R$ of the steepest descent analysis is completed in Section 4G. A matrix $R$ is built in terms of $T$, $P^{(\infty)}$, $P^{(b_1)}$, and $P^{(b_2)}$, and we show that it satisfies a small norm RH problem. In particular, $R(z)$ is close to the identity matrix as $s \to +\infty$. We also compute the first two subleading terms of $R$ which are needed for the proofs of Theorems 1.8 and 1.16.
4A. **First transformation** $Y \mapsto U$. We first rescale the variable of the RH problem for $Y$ in a convenient way. In the same way as in [Claeys et al. 2019b, Section 3.1], we define

$$U(\zeta) = s^{\frac{1}{2}\sigma_3} Y(is^{\rho} \zeta + \tau)s^{-\frac{1}{2}\sigma_3},$$ \hspace{1cm} (4-1)$$

where $\tau = \tau^{(j)}$ and $\rho = \rho^{(j)}$, $j = 1, 2$, are given by

$$\tau^{(1)} = \frac{\nu_{\min} + 1}{2}, \quad \rho^{(1)} = \frac{1}{r - q + 1},$$ \hspace{1cm} (4-2)$$

$$\tau^{(2)} = \frac{1}{2}, \quad \rho^{(2)} = \frac{\theta}{\theta + 1},$$ \hspace{1cm} (4-3)$$

and $\nu_{\min} := \min\{\nu_1, \ldots, \nu_r\}$. The matrix $U$ satisfies the following RH problem.

**RH problem for $U$:**

(a) $U : \mathbb{C} \setminus (\gamma_U \cup \tilde{\gamma}_U) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where

$$\gamma_U = \{\zeta \in \mathbb{C} : is^{\rho} \zeta + \tau \in \gamma\} \quad \text{and} \quad \tilde{\gamma}_U = \{\zeta \in \mathbb{C} : is^{\rho} \zeta + \tau \in \tilde{\gamma}\}. \hspace{1cm} (4-4)$$

The contour $\gamma_U$ (resp. $\tilde{\gamma}_U$) lies in the upper (resp. lower) half plane and is oriented from left to right.

(b) $U$ satisfies the jumps $U_+(\zeta) = U_-(\zeta)J_U(\zeta)$ for $\zeta \in \gamma_U \cup \tilde{\gamma}_U$ with

$$J_U(\zeta) = \begin{cases} 
\begin{pmatrix} 1 & -\sqrt{1 - t} s^{-is^{\rho} \zeta} F(is^{\rho} \zeta + \tau) \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \gamma_U, \\
\begin{pmatrix} 1 & \sqrt{1 - t} s^{is^{\rho} \zeta} F(is^{\rho} + \tau)^{-1} \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \tilde{\gamma}_U.
\end{cases}$$

(c) As $\zeta \rightarrow \infty$, we have

$$U(\zeta) = I + \frac{U_1}{\zeta} + O(\zeta^{-2}),$$

where $U_1 = U_1^{(j)}(s, t)$ is given by

$$U_1 = \frac{1}{is^{\rho}} s^{\frac{1}{2}\sigma_3} Y is^{-\frac{1}{2}\sigma_3}.$$

**Remark 4.1.** Since $\gamma$ and $\tilde{\gamma}$ are symmetric with respect to the real line, the contours $\gamma_U$ and $\tilde{\gamma}_U$ are symmetric with respect to $i\mathbb{R}$. Furthermore, we note that the function

$$\zeta \mapsto f(\zeta) := s^{-is^{\rho} \zeta} F(is^{\rho} \zeta + \tau)$$

satisfies the symmetry relation $f(\zeta) = \overline{f(-\zeta)}$, and thus we also have $J_U(\zeta) = J_U(-\zeta)$ for $\zeta \in \gamma_U \cup \tilde{\gamma}_U$. By uniqueness of the RH solution $U$, we conclude that

$$U(\zeta) = \overline{U(-\zeta)}, \quad \zeta \in \mathbb{C} \setminus (\gamma_U \cup \gamma_{\tilde{U}}).$$
4B. Saddle point analysis. We choose the branch for \( \log F^{(j)} \), \( j = 1, 2 \), such that

\[
\log F^{(1)}(z) = \log \Gamma(z) - \sum_{k=1}^{r} \log \Gamma(1 + v_k - z) + \sum_{k=1}^{q} \log \Gamma(1 + \mu_k - z),
\]

\[
\log F^{(2)}(z) = \log \Gamma\left(z + \frac{\alpha}{2}\right) - \log \Gamma\left(\frac{\alpha}{\theta} + 1 - z\right),
\]

where \( z \mapsto \log \Gamma(z) \) is the log-gamma function, which has a branch cut along \( (-\infty, 0] \). Therefore, \( z \mapsto \log F^{(1)}(z) \) has a branch cut along \( (-\infty, -\frac{q}{2}] \cup [1 + \frac{q}{2}, +\infty) \), and \( z \mapsto \log F^{(2)}(z) \) has a branch cut along \( (-\infty, -\frac{a}{2}] \cup [1 + \frac{a}{2}, +\infty) \). Asymptotics for \( \log(s^{-is^\rho \xi} F(is^\rho \xi + \tau)) \) as \( s \to +\infty \) and simultaneously \( s^\rho \xi \to \infty \), \( |\arg(\xi) + \frac{\pi}{2}| > \epsilon > 0 \) were computed in [Claeys et al. 2019b] and are given by

\[
\log(s^{-is^\rho \xi} F(is^\rho \xi + \tau)) = is^\rho [c_1 \log(i \xi) + c_2 \log(-i \xi) + c_3 \xi] + c_4 \log(s) + c_5 \log(i \xi) + c_6 \log(-i \xi) + c_7 + \frac{c_8}{i s^\rho \xi} + O\left(\frac{1}{s^{2 \rho \xi}}\right),
\]

where the constants \( c_i = c_i^{(j)}_{\; l=1} \), \( j = 1, 2 \) are given by [Claeys et al. 2019b, equations (3.10)–(3.12)].

The values of \( c_7 \) and \( c_8 \) turn out to be unimportant for us. We recall the values of the other constants here.

For \( j = 1 \), we have

\[
c_1 = 1, \quad c_2 = r - q, \quad c_3 = -(r - q + 1),
\]

\[
c_4 = \frac{v_{\min}}{2} - \sum_{k=1}^{r} v_k - \sum_{k=1}^{q} \mu_k \quad \frac{r - q + 1}{r - q},
\]

\[
c_5 = \frac{v_{\min}}{2}, \quad c_6 = \frac{(r - q) v_{\min}}{2} - \sum_{j=1}^{r} v_j + \sum_{k=1}^{q} \mu_k,
\]

and for \( j = 2 \), we have

\[
c_1 = 1, \quad c_2 = \frac{1}{\theta}, \quad c_3 = -\frac{\theta + 1 + \log \theta}{\theta},
\]

\[
c_4 = \frac{(\theta - 1)(1 + \alpha)}{2(\theta + 1)}, \quad c_5 = \frac{\alpha}{2}, \quad c_6 = \frac{\theta - \alpha - 1}{2\theta}.
\]

Following [Claeys et al. 2019b], we define \( h(\xi) = h^{(j)}(\xi) \), \( j = 1, 2 \), by

\[
h(\xi) = -c_1 \xi \log(i \xi) - c_2 \xi \log(-i \xi) - c_3 \xi,
\]

where the principal branch is chosen for the logarithms, and \( G = G^{(j)} \) is defined via

\[
s^{-is^\rho \xi} F(is^\rho \xi + \tau) = e^{-is^\rho h(\xi)} G(\xi; s).
\]

We have

\[
\log G(\xi; s) = c_4 \log s + c_5 \log(i \xi) + c_6 \log(-i \xi) + c_7 + \frac{c_8}{i s^\rho \xi} + O\left(\frac{1}{s^{2 \rho \xi}}\right)
\]

as \( s \to +\infty \) such that \( s^\rho \xi \to \infty \), \( |\arg(\xi) + \frac{\pi}{2}| > \epsilon > 0 \). On the other hand, as \( s \to +\infty \) and simultaneously \( \xi \to 0 \) such that \( s^\rho \xi = O(1) \), and such that \( is^\rho \xi + \tau \) is bounded away from the poles of \( F \), we

\[\text{[Footnote 2]}\]

The superscripts \( j = 1, 2 \) in this paper correspond to the superscripts \( j = 2, 3 \) in [Claeys et al. 2019b]. Also, there is a typo in [Claeys et al. 2019b, equation (3.12)] for the constant \( c_8^{(3)} \). The correct value of \( c_8^{(3)} \) can be found in [Charlier et al. 2019a, equation (2.4)].
Figure 5. The thick dashed curves correspond to \( \text{Re}(ih(\xi)) - \ell = 0 \). These curves divide the complex plane into four regions, and we indicate the sign of \( \text{Re}(ih(\xi)) - \ell \) in each of these regions by the symbols ±. The thin dashed curves represent the real and imaginary axis.

We consider the zero set of \( \text{Re}(ih(\xi)) - \ell \):

\[ \mathcal{N} = \{ \xi \in \mathbb{C} : \text{Re}(ih(\xi)) = \ell \}, \]

which is visualized in Figure 5.
Lemma 4.2. The set $\mathcal{N}$ consists of five simple curves $\Gamma_j, \ j = 1, \ldots, 5$ and satisfies the symmetry $\mathcal{N} = -\overline{\mathcal{N}}$. Three of these curves, say $\Gamma_1, \Gamma_2$ and $\Gamma_3$, join $b_2$ with $b_1$. The curve $\Gamma_4$ starts at $b_2$ and leaves the right half plane in the sector $\arg \xi \in (-\epsilon, \epsilon)$ for any $\epsilon > 0$. The last curve satisfies $\Gamma_5 = -\overline{\Gamma_4}$. In particular, $\mathcal{N}$ divides the complex plane in four regions: two unbounded regions, and two bounded regions. Furthermore, the sign of $\Re(ih(\xi)) - \ell$ in each of these regions is as shown in Figure 5.

Proof: We divide the proof in four steps.

Claim 1: $\mathcal{N}$ intersects the imaginary axis at three distinct points $y_1 < y_2 < y_3$ such that $y_1 < 0$ and $y_3 > 0$.

To prove this, it suffices to inspect the graph of the function $y \mapsto \Re(ih(iy))$ for $y \in \mathbb{R}$. It is a simple computation to verify that

$$\Re(ih(iy)) = (c_1 + c_2)y \log |y| + c_3y, \quad y \in \mathbb{R}.$$

This function is odd in the variable $y$, is equal to 0 at $y = 0$, tends to $+\infty$ as $y \to +\infty$ and admits a local minimum at $y = y_* := \exp(-\frac{c_1+c_2+c_3}{c_1+c_2})$ where it takes the value

$$\Re(ih(iy_*)) = -(c_1 + c_2) \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) < \ell. \quad (4-11)$$

Since $y \mapsto \Re(ih(iy))$ is odd, and since

$$\Re(ih(-iy_*)) = (c_1 + c_2) \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) > \ell,$$

the equation $\Re(ih(iy)) = \ell$ admits three solutions $y_1, y_2, y_3$ satisfying

$$y_1 < -y_*, \quad y_2 \in (-y_*, y_*), \quad y_3 > y_*.$$

Claim 2: For any $\epsilon \in (0, \frac{\pi}{2})$, there exists $\rho_\epsilon > 0$ such that for all $\rho \geq \rho_\epsilon$, $\mathcal{N}$ intersects $\{\rho \ e^{i\phi} : \phi \in (-\epsilon, \epsilon)\}$ at a single point.

This follows from a direct computation using the following expression for $\xi = \rho \ e^{i\phi}, \ \phi \in (-\epsilon, \epsilon)$:

$$\Re(ih(\xi)) = \Re(\xi) \left[(c_1 + c_2)(\tan \phi \log \rho + \phi) + c_3 \tan \phi - \frac{\pi}{2}(c_2 - c_1)\right].$$

Claim 3: There exists no closed curve $\Gamma$ lying entirely in either the left or right half plane such that $\Gamma \subset \mathcal{N}$.

Since $h$ is analytic in $\mathbb{C} \setminus i\mathbb{R}$, $\xi \mapsto \Re(ih(\xi))$ is harmonic in $\mathbb{C} \setminus i\mathbb{R}$. Let $\Gamma \subset \mathbb{C} \setminus i\mathbb{R}$ be a closed curve such that $\Gamma \subset \mathcal{N}$. The maximum principle for harmonic functions implies that $\Re(ih(\xi)) \equiv \ell$ on the interior of $\Gamma$. Since $\Re(ih(\xi))$ is nonconstant on any open disk, we conclude that there exists no such curve $\Gamma$.

Proof of Lemma 4.2. Since $h'(b_2) = 0$ and $h''(b_2) \neq 0$, there are four curves $\{\Gamma_j\}_{j=1}^4$ emanating from $b_2$ that belong to $\mathcal{N}$. From Claim 3, none of these curves is a closed curve lying entirely in the right half plane. We conclude that these curves must leave the right half plane either on $i\mathbb{R}$ or at $\infty$. From Claim 1 and Claim 2, three curves $\Gamma_j, \ j = 1, 2, 3$ leave the right half plane on $i\mathbb{R}$ at $y_1, y_2$ and $y_3$, respectively, and the last curve $\Gamma_4$ leaves the right half plane at $\infty$ in the sector $\arg \xi \in (-\epsilon, \epsilon)$ (for any $\epsilon > 0$ fixed).
By \( \text{Re}(ih(-e)) = \text{Re}(ih(e)) \), \( N \) is symmetric with respect to \( i\mathbb{R} \) and this proves that \( \Gamma_j, \ j = 1, 2, 3 \), join \( b_2 \) and \( b_1 \), and that there exists \( \Gamma_5 \subset N \) in the left half plane satisfying \( \Gamma_5 = -\overline{\Gamma}_4 \). The sign of \( \text{Re}(ih(e)) - \ell \) in the topmost bounded region is negative by (4-11). Since the sign of \( \text{Re}(ih(e)) - \ell \) changes every time a curve \( \Gamma_j, \ j \in \{1, \ldots, 5\} \) is crossed, this determines the sign of \( \text{Re}(ih(e)) - \ell \) in the other regions as well. \( \square \)

4C. Second transformation \( U \mapsto T \). We will now define \( T \) in terms of \( U \) in two steps, \( U \mapsto \hat{T} \) and \( \hat{T} \mapsto T \). The transformation \( U \mapsto \hat{T} \) is similar to the one from [Claeys et al. 2019b, Section 3.2]. Let us define the union of two line segments \( \Sigma_5 := [b_1, 0] \cup [0, b_2] \), as shown in Figure 6. \( \hat{T} \) consists of analytic continuations of \( U \) in different regions, such that it has jumps on \( \bigcup_{j=1}^{5} \Sigma_j \) instead of \( \gamma_U \cup \tilde{\gamma}_U \), where the contours \( \Sigma_1, \ldots, \Sigma_5 \) are shown in Figure 6. More precisely, denote \( U_I \) for the analytic continuation of the function \( U \) as defined in the region above the contour \( \gamma_U \), \( U_{II} \) for the analytic continuation of \( U \) as defined in the region between \( \gamma_U \) and \( \tilde{\gamma}_U \), and \( U_{III} \) for the analytic continuation of \( U \) as defined in the region below \( \tilde{\gamma}_U \); then with the regions I’, II’, III’ as in Figure 6, we define \( \hat{T} = U_I \) in region I’, \( \hat{T} = U_{II} \) in the two regions II’, and \( \hat{T} = U_{III} \) in region III’.

\( \hat{T} \) satisfies the same RH conditions as \( U \), except for a modified jump relation on \( \Sigma_5 \).

**RH problem for \( \hat{T} \):**

(a) \( \hat{T} \) is analytic in \( \mathbb{C} \setminus \bigcup_{j=1}^{5} \Sigma_j \), where the contour \( \bigcup_{j=1}^{5} \Sigma_j \) is shown in Figure 6 and is chosen to be symmetric with respect to \( i\mathbb{R} \).

(b) For \( \zeta \in \bigcup_{j=1}^{5} \Sigma_j \), we have \( \hat{T}_+ (\zeta) = \hat{T}_- (\zeta) J_{\hat{T}} (\zeta) \), where

\[
J_{\hat{T}} (\zeta) = \begin{cases} 
\begin{pmatrix} 1 & -\sqrt{1-t} e^{-i\rho \rho(\zeta)} \mathcal{G}(\zeta; s) \\
0 & 1 \end{pmatrix} & \text{if } \zeta \in \Sigma_1 \cup \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\
\sqrt{1-t} e^{i\rho \rho(\zeta)} \mathcal{G}(\zeta; s)^{-1} & 1 \end{pmatrix} & \text{if } \zeta \in \Sigma_3 \cup \Sigma_4, \\
\begin{pmatrix} 1 & 0 \\
\sqrt{1-t} e^{i\rho \rho(\zeta)} \mathcal{G}(\zeta; s)^{-1} & -\sqrt{1-t} e^{-i\rho \rho(\zeta)} \mathcal{G}(\zeta; s) \end{pmatrix} & \text{if } \zeta \in \Sigma_5.
\end{cases}
\]
Figure 7. The jump contour for the RH problem for $T$.

(c) As $\zeta \to \infty$, we have

$$\hat{T}(\zeta) = I + \frac{U_1}{\zeta} + O(\zeta^{-2}).$$

As $\zeta \to b_1$ and as $\zeta \to b_2$, we have $\hat{T}(\zeta) = O(1)$.

We note that

$$\begin{align*}
\left( \frac{1}{\sqrt{1-t}} e^{i\sigma_3} \frac{1}{\sqrt{1-t}} e^{i\sigma_3} \hat{G}(\zeta; s) \right)^{-1} & = \frac{1}{\sqrt{1-t}} e^{i\sigma_3} \frac{1}{\sqrt{1-t}} e^{i\sigma_3} \hat{G}(\zeta; s), \\
\hat{T}(\zeta) = \begin{pmatrix}
1 & -\sqrt{1-t} e^{-i\sigma_3} \hat{G}(\zeta; s) \\
\sqrt{1-t} e^{i\sigma_3} \hat{G}(\zeta; s) & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} & = \begin{pmatrix}
1 & 0 \\
\frac{1}{\sqrt{1-t}} e^{i\sigma_3} \hat{G}(\zeta; s) & 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{1-t}} e^{i\sigma_3} \hat{G}(\zeta; s) & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
\frac{1}{\sqrt{1-t}} e^{i\sigma_3} \hat{G}(\zeta; s) & 1
\end{pmatrix},
\end{align*}$$

We have used the factorization (4-12) in the transformation $U \mapsto \hat{T}$ to collapse part of the contours on $\Sigma_5$. In the transformation $\hat{T} \mapsto T$, we now use the other factorization (4-13) to open lenses on the other side of $\Sigma_5$.

Let $\Sigma_6$, $\Sigma_7$ be curves as shown in Figure 7, and define $T$ as

$$T(\zeta) = s^{-\frac{\epsilon_3}{2}} \sigma_3 e^{i\sigma_3} \hat{T}(\zeta) H(\zeta) e^{-i\sigma_3} s^{\frac{\epsilon_3}{2}} \sigma_3,$$

where

$$H(\zeta) = \begin{cases}
\begin{pmatrix}
1 & 0 \\
\frac{1}{\sqrt{1-t}} e^{i\sigma_3} \hat{G}(\zeta; s) & 1
\end{pmatrix} & \text{if } \zeta \in \text{int}(\Sigma_5 \cup \Sigma_6), \\
\begin{pmatrix}
1 & -\frac{1}{\sqrt{1-t}} e^{-i\sigma_3} \hat{G}(\zeta; s) \\
\sqrt{1-t} e^{i\sigma_3} \hat{G}(\zeta; s) & 1
\end{pmatrix} & \text{if } \zeta \in \text{int}(\Sigma_5 \cup \Sigma_7), \\
I & \text{otherwise}.
\end{cases}$$
Note $e^{-is^p h(\zeta)} \tilde{G}(\zeta; s)$ is analytic (in particular has no poles) in the lower half plane, while $e^{is^p h(\zeta)} G(\zeta; s)^{-1}$ is analytic in the upper half plane, so that $H(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus (\Sigma_5 \cup \Sigma_6 \cup \Sigma_7)$. Note also that $h(\zeta)$ tends to 0 as $\zeta \to 0$. Then $T$ satisfies the following RH problem:

**RH problem for $T$:**

(a) $T : \mathbb{C} \setminus \bigcup_{j=1}^{\infty} \Sigma_j \to \mathbb{C}^{2 \times 2}$ is analytic. The contour $\bigcup_{j=1}^{7} \Sigma_j$ is shown in Figure 7 and is chosen to be symmetric with respect to $i \mathbb{R}$.

(b) It satisfies the jumps $T_+ (\zeta) = T_- (\zeta) J_T (\zeta)$ for $\zeta \in \bigcup_{j=1}^{7} \Sigma_j$, where

$$J_T (\zeta) = \begin{cases} 
\begin{pmatrix} 1 & -\sqrt{-1} e^{-\text{i} p (\rho(\zeta) - \ell)} \tilde{G}(\zeta; s) \\ \text{i} \\ 1 
\end{pmatrix} & \text{if } \zeta \in \Sigma_1 \cup \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\ \text{i} & 1 
\end{pmatrix} & \text{if } \zeta \in \Sigma_3 \cup \Sigma_4, \\
\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 
\end{pmatrix} & \text{if } \zeta \in \Sigma_5, \\
\begin{pmatrix} 1 & 0 \\ \text{i} & 1 
\end{pmatrix} & \text{if } \zeta \in \Sigma_6, \\
\begin{pmatrix} 0 & -\sqrt{-1} e^{-\text{i} p (\rho(\zeta) - \ell)} \tilde{G}(\zeta; s) \\ 1 
\end{pmatrix} & \text{if } \zeta \in \Sigma_7,
\end{cases}
$$

(4-15)

where

$$\tilde{G}(\zeta; s) = G(\zeta; s) s^{-c_4}.
$$

(4-16)

(c) As $\zeta \to \infty$, we have

$$T(\zeta) = I + \frac{T_1}{\zeta} + \mathcal{O}(\zeta^{-2}),
$$

(4-17)

where

$$T_1 = s^{-\frac{c_1}{2} \sigma_3} e^{\frac{\text{i} p}{2} \sigma_3} U_1 e^{\frac{\text{i} p}{2} \sigma_3} s^{\frac{c_3}{2} \sigma_3} = \frac{1}{i s^p} s^{-\frac{c_1}{2} \sigma_3} e^{\frac{\text{i} p}{2} \sigma_3} s^{\frac{c_3}{2} \sigma_3} Y_1 s^{-\frac{c_3}{2} \sigma_3} e^{-\frac{\text{i} p}{2} \sigma_3} s^{\frac{c_3}{2} \sigma_3}.
$$

(4-18)

As $\zeta \to b_1$ and as $\zeta \to b_2$, we have $T(\zeta) = \mathcal{O}(1)$.

**Remark 4.3.** We choose the jump contour for $T$ to be symmetric with respect to $i \mathbb{R}$ for later use (it will make the analysis simpler). Using this symmetry, we show in a similar way as in Remark 4.1 that $J_T (\zeta) = J_T (-\zeta)$ for $\zeta \in \bigcup_{j=1}^{7} \Sigma_j$. By uniqueness of the solution to the RH problem for $T$, this implies the symmetry

$$T(\zeta) = T(-\zeta), \quad \zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{7} \Sigma_j.
$$

(4-19)

By Lemma 4.2, the jumps for $T$ tend to $I$ exponentially fast as $s \to +\infty$ on $\bigcup_{j=1}^{7} \Sigma_j \setminus \Sigma_5$, and this convergence is uniform outside neighborhoods of $b_1$ and $b_2$.

For convenience, we use the notation

$$\tilde{c} := \text{Im}(i h(b_2)) = - \text{Im}(i h(b_1)) = (c_1 + c_2) \exp \left( - \frac{c_1 + c_2 + c_3}{c_1 + c_2} \right) \cos \left( \frac{\pi}{2} \frac{c_2 - c_1}{c_1 + c_2} \right).
$$

(4-20)
4D. Global parametrix. In this subsection we construct the global parametrix \( P^{(\infty)} \). We will show in Section 4G that \( P^{(\infty)} \) approximates \( T \) outside of neighborhoods of \( b_1 \) and \( b_2 \).

RH problem for \( P^{(\infty)} \):

(a) \( P^{(\infty)} : \mathbb{C} \setminus \Sigma_5 \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) It satisfies the jumps

\[
P^{(\infty)}_+(\zeta) = P^{(\infty)}_-(\zeta) \begin{pmatrix} 1 & 0 \\ i & t \end{pmatrix}, \quad \zeta \in \Sigma_5.
\]

(c) As \( \zeta \to \infty \), we have

\[
P^{(\infty)}(\zeta) = I + \frac{P^{(\infty)}_1}{\zeta} + O(\zeta^{-2}). \tag{4-21}
\]

(d) As \( \zeta \) tends to \( b_1 \) or \( b_2 \), \( P^{(\infty)}(\zeta) \) remains bounded.

Conditions (a)–(c) for the RH problem for \( P^{(\infty)} \) are obtained from the RH problem for \( T \) by ignoring the jumps on \((\bigcup_{j=1}^7 \Sigma_j) \setminus \Sigma_5\). Condition (d) has been added to ensure uniqueness of the solution of the RH problem for \( P^{(\infty)} \). This solution can be easily obtained by using Cauchy’s formula and is given by

\[
P^{(\infty)}(\zeta) = D(\zeta)^{-\sigma_3}, \tag{4-22}
\]

where

\[
D(\zeta) = \exp \left( i \nu \int_{\Sigma_5} \frac{d\xi}{\xi - \zeta} \right) = \exp \left( i \nu \log \left[ \frac{\zeta - b_2}{\zeta - b_1} \right] \right), \quad \nu := -\frac{1}{2\pi} \log t \in \mathbb{R}, \tag{4-23}
\]

where the branch for the log is taken along \( \Sigma_5 \). The function \( D \) satisfies

\[
D_+(\zeta) = D_-(\zeta)t, \quad \zeta \in \Sigma_5,
\]

\[
D(\zeta) = 1 + \frac{D_1}{\zeta} + O(\zeta^{-2}), \quad \text{as} \ \zeta \to \infty,
\]

where \( D_1 = -i\nu(b_2 - b_1) = -2i\nu \Re b_2 \). From (4-21) and (4-22), we obtain

\[
P^{(\infty)}_1 = -D_1 \sigma_3. \tag{4-24}
\]

We will also need asymptotics for \( P^{(\infty)}(\zeta) \) as \( \zeta \to b_2 \). From (4-23), as \( \zeta \to b_2 \) we have

\[
D(\zeta) = \left( \frac{\zeta - b_2}{b_2 - b_1} \right)^{i\nu} \left( 1 - i\nu \frac{\zeta - b_2}{b_2 - b_1} + O((\zeta - b_2)^2) \right),
\]

which implies by (4-22) that

\[
P^{(\infty)}(\zeta) = \left( \frac{\zeta - b_2}{b_2 - b_1} \right)^{-i\nu \sigma_3} \left( I + i\nu \frac{\zeta - b_2}{b_2 - b_1} \sigma_3 + O((\zeta - b_2)^2) \right), \quad \text{as} \ \zeta \to b_2, \tag{4-25}
\]

it is also straightforward to verify from (4-22) and (4-23) that \( P^{(\infty)} \) satisfies the symmetry relation

\[
P^{(\infty)}(\zeta) = P^{(\infty)}(-\overline{\zeta}), \quad \zeta \in \mathbb{C} \setminus \Sigma_5. \tag{4-26}
\]
4E. Local parametrix at $b_2$. We construct the local parametrix $P^{(b_2)}$ in a small disk $D_{b_2}$ around $b_2$ with radius independent of $s$. We require $P^{(b_2)}$ to satisfy the same jumps as $T$ inside $D_{b_2}$, to remain bounded as $\zeta \to b_2$, and to match with $P^{(\infty)}$ on the boundary of $D_{b_2}$, in the sense that

$$P^{(b_2)}(\zeta) = (I + o(1))P^{(\infty)}(\zeta), \quad \text{as } s \to +\infty,$$

uniformly for $\zeta \in \partial D_{b_2}$. The solution can be constructed in terms of the solution $\Phi_{PC}$ to the parabolic cylinder model RH problem presented in the Appendix. This model RH problem depends on a parameter $q$; in our case we need to choose $q = \sqrt{1-t}$. Let us define

$$f(\zeta) = \sqrt{-2(h(\zeta) - h(b_2))}.$$  \hfill (4-27)

This is a conformal map from $D_{b_2}$ to a neighborhood of 0 satisfying $f(b_2) = 0$ and

$$f'(b_2) = \sqrt{c_1 + c_2 \over b_2} = \frac{\sqrt{c_1 + c_2}}{c_1 + c_2} \exp\left(-\frac{c_1 + c_2}{2(c_1 + c_2)}\right) \exp\left(i\frac{\pi}{4}c_2-c_1\right) \quad \text{and} \quad f''(b_2) = -\frac{1}{3b_2}f'(b_2).$$ \hfill (4-28)

In small neighborhoods of $D_{b_2}$ and $D_{b_1}$, we slightly deform the contour $\bigcup_{j=1}^{7} \Sigma_j$ such that it remains symmetric with respect to $i\mathbb{R}$ and such that it satisfies

$$f\left(\bigcup_{j=1}^{7} \Sigma_j \cap D_{b_2}\right) \subset \Sigma_{PC}, \quad f(\Sigma_5) \subset (-\infty, 0],$$ \hfill (4-29)

where $\Sigma_{PC}$ is shown in Figure 8. The local parametrix is given by

$$P^{(2)}(\zeta; s) = E(\zeta; s)\Phi_{PC}(s^{\frac{\nu}{2}}f(\zeta); \sqrt{1-t})e^{\frac{2}{\nu}(i(h(\zeta)-\ell))\sigma_3}\tilde{G}(\zeta; s)^{-\frac{\sigma_3}{2}},$$ \hfill (4-30)

where $E$ is analytic in $D_{b_2}$ and given by

$$E(\zeta; s) = P^{(\infty)}(\zeta)\tilde{G}(\zeta; s)^{\sigma_1} e^{-\frac{2}{\nu}i\ell\sigma_3}(s^{\frac{\nu}{2}}f(\zeta))^{i\nu\sigma_3},$$ \hfill (4-31)

where $\nu = \nu(t) \in \mathbb{R}$ is given by (4-23) and the branch cut for $(s^{\frac{\nu}{2}}f(\zeta))^{i\nu\sigma_3}$ is taken along $\Sigma_5 \cap D_{b_2}$. Note that $\tilde{G}(\zeta; s)$ depends on $s$, but by (4-7) and (4-16) it is bounded as $s \to +\infty$ uniformly for $\zeta \in D_{b_2}$ (see also (4-36) below). Since $\nu \in \mathbb{R}$ and $\tilde{\ell} \in \mathbb{R}$ (see (4-20)), we thus have $E(\zeta; s) = O(1)$ as $s \to +\infty$,.
uniformly for \( \zeta \in \mathcal{D}_{b_2} \). Using \((4-25)\) and \((4-28)\), we infer that

\[
E(\zeta; s) = \alpha(s)^{\nu_3} (1 + \beta(s) \sigma_3 (\zeta - b_2) + O((\zeta - b_2)^2)), \quad \zeta \to b_2, \tag{4-32}
\]

\[
\alpha(s) = [(b_2 - b_1) f'(b_2) s^2]^{i \nu} \tilde{G}(b_2; s) s^{1/2} e^{i \nu/2 \bar{\zeta}}, \tag{4-33}
\]

\[
\beta(s) = \frac{iv}{b_2 - b_1} + \frac{1}{2} (\log \tilde{G})'(b_2; s) - \frac{iv}{6b_2}. \tag{4-34}
\]

As \( s \to +\infty \), for any \( N \in \mathbb{N} \), we have

\[
P^{(b_2)}(\zeta) P^{(\infty)}(\zeta)^{-1} = I + E(\zeta; s) \left( \sum_{j=1}^{N} \frac{\Phi_{PC,j}(\zeta)}{s^{\nu_2} f(\zeta)^j} \right) E(\zeta; s)^{-1} + O(s^{-N+1}b) \tag{4-35}
\]

uniformly for \( \zeta \in \partial \mathcal{D}_{b_2} \), where the matrices \( \Phi_{PC,1} \) and \( \Phi_{PC,2} \) are defined by \((A-2)\). In particular, the matrix \( \Phi_{PC,1} \) is expressed in terms of the quantities \( \beta_{12} \) and \( \beta_{21} \) defined in \((A-3)\). Furthermore, the matrices \( \Phi_{PC,2k} \) are diagonal for every \( k \geq 1 \) and the matrices \( \Phi_{PC,2k-1} \) are off-diagonal for every \( k \geq 1 \); see again \((A-2)\).

We need to expand \( E(\zeta; s) \) as \( s \to \infty \). By the expansion \((4-7)\) of \( \tilde{G} \) and the definition \((4-16)\) of \( \tilde{G} \), we get

\[
\log \tilde{G}(\zeta; s) = c_5 \log(i \xi) + c_6 \log(-i \xi) + c_7 + \frac{c_8}{is^\rho \xi} + O(s^{-2\rho}) \quad \text{as} \quad s \to +\infty, \tag{4-36}
\]

uniformly for \( \zeta \in \mathcal{D}_{b_2} \), where the error term can be expanded in a full asymptotic series in integer powers of \( s^{-\rho} \). We deduce from this that

\[
\tilde{G}(b_2; s) = (ib_2)^{c_5} (-ib_2)^{c_6} e^{c_7} \left( 1 + \frac{c_8}{is^\rho b_2} + O(s^{-2\rho}) \right), \tag{4-37}
\]

\[
(\log \tilde{G})'(b_2; s) = \frac{c_5 + c_6}{b_2} - \frac{c_8}{is^\rho b_2^2} + O(s^{-2\rho}), \tag{4-38}
\]

as \( s \to \infty \), and by \((4-31)\), we can write

\[
E(\zeta; s) = \sum_{j=0}^{N} E_j(\zeta; s) s^{-\rho j} + O(s^{-(N+1)\rho}), \quad \text{as} \quad s \to +\infty, \tag{4-39}
\]

for any \( N \in \mathbb{N} \), uniformly for \( \zeta \in \mathcal{D}_{b_2} \), and where the diagonal matrices \( E_j(\zeta; s) \) depend on \( s \) but are bounded; in particular

\[
E_0(b_2; s) = [(b_2 - b_1) f'(b_2) s^2]^{i \nu} s^{2i / \rho_1} e^{i \nu / 2 \bar{\zeta}} e^{-\nu / 2 \bar{\zeta}}, \tag{4-40}
\]

\[
E'_0(b_2; s) = \beta_0(s) E_0(b_2; s) \sigma_3, \tag{4-41}
\]

\[
\beta_0(s) = \frac{iv}{b_2 - b_1} + \frac{c_5 + c_6}{2b_2} - \frac{iv}{6b_2}. \tag{4-42}
\]

For later use, we note that this implies

\[
\epsilon(s) := \frac{E_0(b_2; s)_{11}}{E_0(b_2; s)_{11}} = \exp(2i \arg(E_0(b_2; s)_{11})) \]

\[
= [(b_2 - b_1) f'(b_2) s^2]^{2i} \exp(ic_5 \arg(ib_2) + ic_6 \arg(-ib_2)) e^{-i \bar{\zeta} s^\rho}. \tag{4-43}
\]
4F. Local parametrix at $b_1$. We construct the local parametrix $P^{(b_1)}$ in a small disk $D_{b_1}$ around $b_1$ in a similar way as we defined $P^{(b_2)}$ in $D_{b_2}$. More precisely, we require $P^{(b_1)}$ to satisfy the same jumps as $T$ inside $D_{b_1}$, to remain bounded as $\zeta \to b_1$, and to satisfy the matching condition

$$P^{(b_1)}(\zeta) = (I + o(1))P^{(\infty)}(\zeta), \quad \text{as } s \to +\infty,$$

uniformly for $\zeta \in \partial D_{b_1}$. It is possible to construct $P^{(b_1)}(\zeta)$ in a similar way as $P^{(b_2)}(\zeta)$ in terms of parabolic cylinder functions. To avoid unnecessary analysis and computations, we choose $D_{b_1} = -D_{b_2}$, and we rely on the symmetry $J_T(\zeta) = J_T(-\zeta)$ for $\zeta \in \bigcup_{j=1}^7 \Sigma_j$ (see Remark 4.3) to conclude directly that the function

$$P^{(b_1)}(\zeta) = P^{(b_2)}(-\zeta), \quad \zeta \in D_{b_1} \setminus \bigcup_{j=1}^7 \Sigma_j \quad (4.44)$$

satisfies the required conditions for the local parametrix.

4G. Small norm RH problem. In this section we show that, as $s$ becomes large, $P^{(\infty)}(z)$ approximates $T(z)$ for $z \in \mathbb{C} \setminus \bigcup_{j=1}^2 D_{b_j}$ and $P^{(b_j)}(z)$ approximates $T(z)$ for $z \in D_{b_j}, \ j = 1, 2$. We define

$$R(\zeta) = \begin{cases} 
T(\zeta)P^{(\infty)}(\zeta)^{-1} & \text{if } \zeta \in \mathbb{C} \setminus (D_{b_1} \cup D_{b_2}), \\
T(\zeta)P^{(b_1)}(\zeta)^{-1} & \text{if } \zeta \in D_{b_1}, \\
T(\zeta)P^{(b_2)}(\zeta)^{-1} & \text{if } \zeta \in D_{b_2}.
\end{cases} \quad (4.45)$$

Since $P^{(b_j)}, \ j = 1, 2,$ have the exact same jumps as $T$ inside the disks, $R$ is analytic in $\bigcup_{j=1}^2 D_{b_j} \setminus \{b_j\}$. Furthermore, since $T(z)$ and $P^{(b_j)}(z)^{-1}$ remain bounded as $z \to b_j, \ j = 1, 2$, we conclude that $R(z)$ is also bounded as $z \to b_j, \ j = 1, 2$. Thus the singularities of $R$ at $b_1$ and $b_2$ are removable and $R$ is analytic in the entire open disks. $R$ satisfies the following RH problem.

RH problem for $R$:

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2 \times 2}$ is analytic, where

$$\Sigma_R = \partial D_{b_1} \cup \partial D_{b_2} \bigcup_{j=1}^7 \Sigma_j \setminus (D_{b_1} \cup D_{b_2} \cup \Sigma_5).$$
The contour \( \Sigma_R \) is oriented as shown in Figure 9. In particular, we orient the circles \( \partial D_{b_1} \) and \( \partial D_{b_2} \) in the clockwise direction.

(b) For \( \zeta \in \Sigma_R \), \( R \) satisfies the jumps \( R_+ (\zeta) = R_-(\zeta) J_R(\zeta) \), where

\[
J_R(\zeta) = P^{(b_1)}(\zeta) P^{(\infty)}(\zeta)^{-1}, \quad \zeta \in \partial D_{b_1},
\]

\[
J_R(\zeta) = P^{(b_2)}(\zeta) P^{(\infty)}(\zeta)^{-1}, \quad \zeta \in \partial D_{b_2},
\]

\[
J_R(\zeta) = P^{(\infty)}(\zeta) J_T(\zeta) P^{(\infty)}(\zeta)^{-1}, \quad \zeta \in \Sigma_R \setminus (\partial D_{b_1} \cup \partial D_{b_2}).
\]

(c) \( R(\zeta) \) remains bounded as \( \zeta \) tends to the points of self-intersection of \( \Sigma_R \).

As \( \zeta \to \infty \), there exists \( R_1 = R_1(s) \) such that

\[
R(\zeta) = I + \frac{R_1}{\zeta} + O(\zeta^{-2}). \tag{4-46}
\]

**Remark 4.4.** The contour \( \Sigma_R \) is symmetric with respect to \( i \mathbb{R} \). Furthermore, by (4-26) and (4-44), the jumps \( J_R \) satisfy the symmetry relation \( J_R(\zeta) = J_R(\overline{\zeta}) \) for \( \zeta \in \Sigma_R \). Hence, by uniqueness of the solution to the RH problem for \( R \), we conclude that

\[
R(\zeta) = R(\overline{\zeta}), \quad \zeta \in \mathbb{C} \setminus \Sigma_R. \tag{4-47}
\]

From Lemma 4.2 and the fact that \( P^{(\infty)} \) is independent of \( s \) and uniformly bounded outside \( D_{b_1} \cup D_{b_2} \), we have

\[
J_R(\zeta) = I + O(e^{-cs|\zeta|}), \quad \text{as } s \to +\infty \tag{4-48}
\]

uniformly for \( \zeta \in \Sigma_R \setminus (\partial D_{b_1} \cup \partial D_{b_2}) \), and uniformly for \( t \) in compact subsets of \((0, \infty)\). By substituting the expansion (4-39) in (4-35), we infer that, for any \( N \in \mathbb{N} \), \( J_R \) has an expansion in the form

\[
J_R(\zeta) = J_R(\zeta; s) = I + \sum_{j=1}^{N} J^{(j)}_R(\zeta; s) s^{-\frac{j\rho}{2}} + O(s^{-\frac{(N+1)\rho}{2}}), \quad \text{as } s \to +\infty, \tag{4-49}
\]

where all coefficients \( J^{(j)}_R(\zeta; s) \) satisfy the symmetry \( J^{(j)}_R(\zeta; s) = J^{(j)}_R(\overline{\zeta}; s) \) and are bounded as \( s \to +\infty \), uniformly for \( \zeta \in \partial D_{b_1} \cup \partial D_{b_2} \) and for \( t \) in compact subsets of \((0, \infty)\). The first two coefficients \( J^{(j)}_R(\zeta; s) \), for \( j = 1, 2 \) are given by

\[
J^{(j)}_R(\zeta; s) = E_0(\zeta; s) \frac{\Phi_{PC,j}}{f(\zeta)} E_0(\zeta; s)^{-1}, \quad j = 1, 2. \tag{4-50}
\]

The jump relation for \( R \) can also be written in the additive form \( R_+ = R_- + R_-(J_R - I) \), and together with the asymptotics for \( R \), this implies the integral equation

\[
R(\zeta) = R(\zeta; s) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(\zeta; s) (J_R(\zeta; s) - I)}{\xi - \zeta} d\xi. \tag{4-51}
\]

We conclude from (4-48) and (4-49) that \( R \) satisfies a small norm RH problem as \( s \to +\infty \), and by standard theory [Deift et al. 1999], it follows that \( R \) exists for sufficiently large \( s \). Moreover, substituting
we prove that
\[
\partial (4-48) \quad \text{and} \quad (4-49) \quad \text{in} \quad (4-51) \quad \text{and} \quad \text{expanding as} \quad s \to +\infty, \quad \text{we obtain}
\]
\[
R(\xi; s) = I + \sum_{j=1}^{N} \frac{R^{(j)}(\xi; s)}{s^{\frac{N+1-j}{2}}} + O(s^{-\frac{(N+1)\rho}{2}}), \quad \text{as} \quad s \to +\infty, \quad (4-52)
\]
\[
R'(\xi; s) = \sum_{j=1}^{N} \frac{R^{(j)}(\xi; s)}{s^{\frac{N+1-j}{2}}} + O(s^{-\frac{(N+1)\rho}{2}}), \quad \text{as} \quad s \to +\infty,
\]
uniformly for \( \zeta \in \mathbb{C} \setminus \Sigma_R \), and uniformly for \( t \) in compact subsets of \((0, \infty)\). All the coefficients \( R^{(j)} \) can in principle be computed iteratively. In particular, a substitution of \( (4-49) \) and \( (4-52) \) in \( R_+ = R_- J_R \) yields
\[
R_+^{(1)}(\xi; s) = R_-^{(1)}(\xi; s) + J_R^{(1)}(\xi; s), \quad R_-^{(2)}(\xi; s) = R_-^{(1)}(\xi; s) J_R^{(1)}(\xi; s) + J_R^{(2)}(\xi; s)
\]
for \( \zeta \in \partial D_{b_1} \cup \partial D_{b_2} \). These jumps, together with the asymptotics \( R^{(1)}(\xi; s) = O(\xi^{-1}) \) and \( R^{(2)}(\xi; s) = O(\xi^{-1}) \) as \( \zeta \to \infty \), imply that
\[
R^{(1)}(\xi; s) = \frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{J_R^{(1)}(\xi; s)}{\xi - \zeta} d\xi + \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{J_R^{(1)}(\xi; s)}{\xi - \zeta} d\xi, \quad (4-53)
\]
and
\[
R^{(2)}(\xi; s) = \frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{R_-^{(1)}(\xi; s) J_R^{(1)}(\xi; s) + J_R^{(2)}(\xi; s)}{\xi - \zeta} d\xi + \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{R_-^{(1)}(\xi; s) J_R^{(1)}(\xi; s) + J_R^{(2)}(\xi; s)}{\xi - \zeta} d\xi. \quad (4-54)
\]
where we recall that \( \partial D_{b_1} \) and \( \partial D_{b_2} \) are oriented clockwise.

In the rest of this section, we evaluate \( R^{(1)}(\xi; s) \) and \( R^{(2)}(\xi; s) \) explicitly for \( \zeta \in \mathbb{C} \setminus (D_{b_1} \cup D_{b_2}) \), and we prove that \( R^{(k)}(\xi; s) \) can be chosen diagonal for \( k \) even and off-diagonal for \( k \) odd.

The expression \( (4-50) \) for \( J_R^{(1)} \) can be analytically continued from \( \partial D_{b_2} \) to the punctured disk \( D_{b_2} \setminus \{b_2\} \), and we note that \( J_R^{(1)}(\xi; s) \) has a simple pole at \( \zeta = b_2 \). Therefore, for \( \zeta \) outside the disks, a residue calculation gives
\[
\frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{J_R^{(1)}(\xi; s)}{\xi - \zeta} d\xi = \frac{A^{(1)}(s)}{\xi - b_2}, \quad \text{with} \quad A^{(1)}(s) = \text{Res}(J_R^{(1)}(\xi; s), \xi = b_2). \quad (4-55)
\]
To evaluate the first integral that appears at the right-hand side of \( (4-53) \), we appeal to the symmetries of Remark 4.4 to write
\[
\frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{J_R^{(1)}(\xi)}{\xi - \zeta} d\xi = \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{J_R^{(1)}(\xi)}{\xi + \zeta} d\xi = \frac{-A^{(1)}(s)}{\xi - b_1}. \quad (4-56)
\]
Therefore, it remains to evaluate \( A^{(1)}(s) = \text{Res}(J_R^{(1)}(\xi; s), \xi = b_2) \). From \( (4-40) \), \( (4-50) \), and \( (A-2) \), we
immediately obtain that
\[ A^{(1)}(s) = \frac{1}{f'(b_2)} E_0(b_2; s) \Phi_{PC,1} E_0(b_2; s)^{-1} = \frac{1}{f'(b_2)} \begin{pmatrix} 0 & \beta_{12} E_0(b_2; s)^2 \\ \beta_{21} E_0(b_2; s)_1 & 0 \end{pmatrix}, \]  
which is an off-diagonal matrix, and \( E_0(b_2; s) \) has been explicitly evaluated in (4-40). Combining (4-55) with (4-56), we obtain
\[ R^{(1)}(\xi; s) = \frac{A^{(1)}}{\xi - b_2} - \frac{A^{(1)}}{\xi - b_1}, \quad \text{for } \xi \in \mathbb{C} \setminus (\mathcal{D}_{b_1} \cup \mathcal{D}_{b_2}), \]  
where \( A^{(1)} \) is given by (4-57).

For the computation of \( R^{(2)} \), we recall that \( J^{(1)}_R \) and \( J^{(2)}_R \) are given by (4-50), and we note that \( J^{(2)} \) can be simplified as
\[ J^{(2)}_R(\xi) = \frac{1}{f(\xi)^2} E_0(\xi; s) \Phi_{PC,2} E_0(\xi; s)^{-1} = \frac{1}{f(\xi)^2} \Phi_{PC,2}, \quad \xi \in \partial \mathcal{D}_{b_2}, \]  
where we have used that both \( E_0(\xi; s) \) and \( \Phi_{PC,2} \) are diagonal matrices. We note that \( J^{(2)}_R \) can also be analytically continued from \( \partial \mathcal{D}_{b_2} \) to the punctured disk \( \mathcal{D}_{b_2} \setminus \{b_2\} \). Let us start by evaluating the integral over \( \partial \mathcal{D}_{b_2} \) which appears at the right-hand side of (4-54). For \( \xi \in \mathbb{C} \setminus (\partial \mathcal{D}_{b_1} \cup \partial \mathcal{D}_{b_2}) \), since \( R^{(1)}_z \) is analytic on \( \mathcal{D}_{b_1} \cup \mathcal{D}_{b_2} \), and since \( J^{(j)}_R \) admits a pole of order \( j \) at \( b_2 \), \( j = 1, 2 \), we have
\[ \frac{1}{2\pi i} \int_{\partial \mathcal{D}_{b_2}} \frac{R^{(1)}_z(\xi; s) J^{(1)}_R(\xi; s) + J^{(2)}_R(\xi; s)}{\xi - \xi} d\xi = \frac{A^{(2)}(s)}{\xi - b_2} + \frac{B^{(2)}(s)}{(\xi - b_2)^2}, \]
\[ A^{(2)}(s) = R^{(1)}(b_2; s) A^{(1)} + \text{Res}(J^{(2)}_R(\xi; s), \xi = b_2), \]
\[ B^{(2)}(s) = \text{Res}((\xi - b_2) J^{(2)}_R(\xi; s), \xi = b_2). \]

We again appeal to the symmetry \( \xi \mapsto -\bar{\xi} \) of Remark 4.4 to evaluate the integral over \( \partial \mathcal{D}_{b_1} \):
\[ \frac{1}{2\pi i} \int_{\partial \mathcal{D}_{b_1}} \frac{R^{(1)}_z(\xi; s) J^{(1)}_R(\xi; s) + J^{(2)}_R(\xi; s)}{\xi - \xi} d\xi = \frac{1}{2\pi i} \int_{\partial \mathcal{D}_{b_2}} \frac{R^{(1)}_z(\xi; s) J^{(1)}_R(\xi; s) + J^{(2)}_R(\xi; s)}{\xi + \xi} d\xi \]
\[ = -\frac{A^{(2)}(s)}{\xi - b_1} + \frac{B^{(2)}(s)}{(\xi - b_1)^2}. \]

To evaluate \( A^{(2)} \) and \( B^{(2)} \) explicitly, it remains to compute \( R^{(1)}(b_2; s) \), and the two residues
\[ \text{Res}(J^{(2)}_R(\xi; s), \xi = b_2) \quad \text{and} \quad \text{Res}((\xi - b_2) J^{(2)}_R(\xi; s), \xi = b_2). \]  
It is fairly easy to compute the residues (4-60) from the expression (4-50) for \( J^{(2)}_R(\xi; s) \). We obtain
\[ B^{(2)}(s) = \text{Res}((\xi - b_2) J^{(2)}_R(\xi; s), \xi = b_2) = \frac{1}{f'(b_2)^2} \begin{pmatrix} (1+i\nu)^2 & 0 \\ 0 & (1-i\nu)^2 \end{pmatrix}, \]
\[ \text{Res}(J^{(2)}_R(\xi; s), \xi = b_2) = -\frac{f''(b_2)}{2 f'(b_2)^3} \begin{pmatrix} (1+i\nu)^2 & 0 \\ 0 & (1-i\nu)^2 \end{pmatrix}. \]
where \( f'(b_2) \) and \( f''(b_2) \) are given by (4-28). Since

\[
R^{(1)}_+ (\xi; s) = R^{(1)}_+ (\xi; s) - J^{(1)}_R (\xi; s),
\]

and since \( R^{(1)}_+ (\xi; s) \) has already been computed in (4-58), we obtain

\[
R^{(1)}_+ (b_2; s) = -\frac{A^{(1)}(s)}{b_2 - b_1} - \text{Res}
\left( \frac{J^{(1)}_R (\xi; s)}{\xi - b_2}, \xi = b_2 \right)
\]

\[
= -\frac{A^{(1)}(s)}{b_2 - b_1} + \frac{1}{f'(b_2)} \begin{pmatrix} 0 & \left(-\frac{1}{6b_2} - 2\beta_0(s)\right)\beta_{12}E_0(b_2; s)_{11}^2 \\ -\frac{1}{6b_2} - 2\beta_0(s) \end{pmatrix},
\]

and the constant \( \beta_0(s) \) is given by (4-42). Summarizing, we have

\[
R^{(2)}_+ (\xi; s) = A^{(2)}(s) \frac{(\xi - b_2)}{\xi - b_1} + B^{(2)}(s) \frac{(\xi - b_2)}{(\xi - b_1)^2}, \quad \beta^{(2)}_{12} = \beta_0(s),
\]

for \( \xi \in \mathbb{C} \setminus (D_{b_1} \cup D_{b_2}) \), (4-61)

where \( A^{(2)} \) and \( B^{(2)} \) are diagonal matrices given by

\[
B^{(2)}_{11} (s) = \frac{(1 + iv)v}{2f'(b_2)^2}, \quad B^{(2)}_{22} (s) = \frac{(1 - iv)v}{2f'(b_2)^2},
\]

\[
A^{(2)}_{11} (s) = \frac{1}{f'(b_2)^2} \left( \left(-\frac{1}{6b_2} - 2\beta_0(s)\right)\beta_{12} + \frac{1}{6b_2} (1 + iv)v - \frac{f'(b_2)}{f'(b_2)} (b_2 - b_1)^2 \right),
\]

\[
A^{(2)}_{22} (s) = \frac{1}{f'(b_2)^2} \left( \left(-\frac{1}{6b_2} + 2\beta_0(s)\right)\beta_{12} + \frac{1}{6b_2} (1 - iv)v - \frac{f'(b_2)}{f'(b_2)} (b_2 - b_1)^2 \right),
\]

where \( e = e(s) \) depends on \( s \), but satisfies \( |e(s)| = 1 \) for all values of \( s \). Its precise expression is given by (4-43). The formula (A-7) allows the simplification

\[
A^{(2)}_{11} (s) = \frac{1}{f'(b_2)^2} \left( \left(-\frac{1}{6b_2} - 2\beta_0(s)\right) + \frac{1}{6b_2} (1 + iv)v - \frac{f'(b_2)}{f'(b_2)} \beta_{12} \right), \quad \beta^{(2)}_{12} = \beta_0(s),
\]

and similarly for \( A^{(2)}_{22} \). We end this section with a lemma about the structure of the matrices \( R^{(j)}_+ \), \( j \geq 1 \).

**Lemma 4.5.** For any \( j \geq 1 \), the matrix \( R^{(2j-1)}_+ \) is off-diagonal and the matrix \( R^{(2j)}_+ \) is diagonal.

**Proof.** By (4-51), the matrices \( R^{(j)}_+ \) can be computed recursively as follows:

\[
R^{(j)}_+ (\xi; s) = \frac{1}{2\pi i} \int_{\partial D_{b_1} \cup \partial D_{b_2}} \frac{\sum_{\ell=1}^{j} R^{(j-\ell)}_+ (\xi; s) J^{(\ell)}_R (\xi; s)}{\xi - \xi} d\xi, \quad j \geq 1.
\]

The result follows by induction, provided that the matrices \( J^{(2j)}_R \) are diagonal and \( J^{(2j-1)}_R \) are off-diagonal. To prove this claim, consider (4-35) and (4-39). These imply that \( J^{(j)}_R (\xi; s) \) from (4-50) is composed of terms of the form

\[
\frac{1}{f^{j-2k}_j} E_m \Phi_{PC, j-2k} (E^{-1})_{k-m},
\]

for \( m = 0, \ldots, k \) and \( k = 0, 1, \ldots, \left[ \frac{j-1}{2} \right] \), and where \( E_m \) and \( (E^{-1})_{k-m} \) are diagonal matrices. All these terms are diagonal if \( j \) is even and off-diagonal if \( j \) is odd. \( \square \)
5. Proofs of Theorems 1.8 and 1.16: part 1

In this section, we use the analysis of Section 4 to prove part of Theorems 1.8 and 1.16 via the differential identity in $s$

$$\partial_s \log \det (1 - (1 - t) \mathbb{K}(j)|_{[0,s]}) = \frac{Y_{1,11}}{s},$$  \hspace{1cm} (5-1)

which was derived in (3-8). As mentioned in the introduction, the advantage of this differential identity is that it leads to a significantly simpler analysis than the one carried out in Section 6 and that it allows us to prove the optimal bound $O(s^{-\rho})$ for the error terms of (1-12) and (1-20). The main disadvantage is that it does not allow for the evaluation of the constants $C$ of (1-12) and (1-20). These constants will be obtained in Section 6.

By (4-18), we have

$$T_{1,11} = \frac{1}{i s^\rho} Y_{1,11}.$$

On the other hand, for $\zeta$ outside the lenses and outside the disks, we know from (4-45) that

$$T(\zeta) = R(\zeta) P(\infty)(\zeta),$$

from which we deduce, by (4-17), (4-21), (4-46), and (4-52) that

$$T_1 = T_1(s) = P_1(\infty) + R_1(s) = P_1(\infty) + \sum_{j=1}^{2N+1} R_1^{(j)}(s) s^{-\frac{j\rho}{2}} + O(s^{-(N+1)\rho}), \hspace{1cm} \text{as } s \to +\infty,$$

uniformly for $t$ in compact subsets of $\mathbb{R}$, where $N \in \mathbb{N}$ is arbitrary, and where the coefficients $R_1^{(j)}(s)$, $j \geq 1$, are defined via the expansion

$$R^{(j)}(\zeta) = \frac{R_1^{(j)}(s)}{\zeta} + O(\zeta^{-2}), \hspace{1cm} \text{as } \zeta \to \infty.$$

We know from Lemma 4.5 that $R_1^{(2j-1)}$ is off-diagonal for all $j \geq 1$. Thus, using (5-1), we find

$$\partial_s \log \det (1 - (1 - t) \mathbb{K}(j)|_{[0,s]}) = i s^{\rho-1} \left( P_1(\infty) + \sum_{j=1}^{2N+1} R_1^{(j)}(s) s^{-\frac{j\rho}{2}} + O(s^{-(N+1)\rho}) \right)_{11}$$

$$= i s^{\rho-1} \left( P_{1,11} + \sum_{j=1}^{N} R_1^{(2j)}(s) s^{-j\rho} + O(s^{-(N+1)\rho}) \right), \hspace{1cm} (5-2)$$

as $s \to +\infty$. After integrating (5-2), we obtain

$$\log \det (1 - (1 - t) \mathbb{K}(j)|_{[0,s]}) = \frac{i}{\rho} P_{1,11} s^\rho + \int_{M} \frac{i R_1^{(2)}(s)}{s} ds + \log C_1 + O(s^{-\rho}), \hspace{1cm} (5-3)$$

as $s \to +\infty$, where $C_1$ is an unknown constant of integration and $M$ is a sufficiently large constant (i.e., $M$ is independent of $s$). An explicit expression for $P_1(\infty)$ has been computed in (4-24). Then, the leading
We conclude that the integral in (5-3) has the asymptotics
\[
\frac{i}{\rho} P_{1,11}^{(\infty)} = -\frac{i D_1}{\rho} = -2 \frac{v^2}{c_1 + c_2} \rho.
\] (5-4)

We now turn to the computation of the second term of (5-3). Using (4-62) and (4-61), we obtain
\[
i R^{(2)}_{1,11}(s) = i (A_{11}^{(2)}(s) - \overline{A_{11}^{(2)}(s)}) = -2 \text{Im} A_{11}^{(2)}(s) = \frac{v^2}{c_1 + c_2} + \frac{1}{|f'(b_2)|^2} \text{Re} b_2 \text{Im}(\beta_{21}\beta_{12}\epsilon(s)^{-2}).
\]

We recall that \(\epsilon(s)\) is given by
\[
\epsilon(s) = [(b_2 - b_1)|f'(b_2)|s^2]^{2i\nu} \exp(ic_5 \arg(b_2) + ic_6 \arg(-ib_2)) e^{-i\tilde{\ell}s^\rho}.
\]

In particular, it satisfies \(|\epsilon(s)| = 1\) and it oscillates rapidly as \(s \to +\infty\), since \(\tilde{\ell} \neq 0\), see (4-20). Therefore,
\[
\int_M e^{-s} s^{-1} ds = \tilde{\ell} \int_M e^{2i\tilde{\ell}s^\rho - 2i\log s^\rho} s^{-1} ds = \frac{\tilde{\ell}}{\rho} \int_M e^{2i\tilde{\ell}u - 2i\log u} u^{-1} du
\]
\[
= \log C_2 + C_3(s).
\]

where \(\tilde{\ell}\) and \(C_2\) are constants whose exact values are unimportant for us, and \(C_3(s)\) is bounded by
\[
|C_3(s)| = O\left(\frac{1}{s^\rho}\right) \quad \text{as} \quad s \to +\infty.
\]

We conclude that the integral in (5-3) has the asymptotics
\[
\int_M \frac{i R^{(2)}_{1,11}(s)}{s} ds = \frac{v^2}{c_1 + c_2} \log s + \log(C_2) + O(s^{-\rho}), \quad \text{as} \quad s \to +\infty.
\] (5-5)

Since \(\rho = \frac{1}{c_1 + c_2}\), by combining (5-3), (5-4), and (5-5), we get
\[
\log \det(1 - (1-t)\kappa^{(j)}|_{[0,s]}) = -\frac{2v}{\rho} \text{Re} b_2 s^\rho + v^2 \log s^\rho + \log(C) + O(s^{-\rho}),
\] (5-6)

as \(s \to +\infty\), where \(C = C_1C_2\). The values of \(\rho^{(1)}\) and \(\rho^{(2)}\) are given by (4-2) and (4-3), respectively, and \(\text{Re}b_2^{(j)}\), \(j = 1, 2\), can be evaluated by using (4-10) together with the coefficients \(c_1, c_2\) and \(c_3\) (given above (4-5)). Recalling also that
\[
\log \det(1 - (1-t)\kappa^{(j)}|_{[0,s]}) = \mathbb{E}[e^{-2\pi v^2N(s)}],
\]
we have now completed the proofs of Theorems 1.8 and 1.16, up to the determination of the constants \(C = C^{(j)}, \quad j = 1, 2\).

6. Proofs of Theorems 1.8 and 1.16: part 2

In this section, we will compute \(C\) via the differential identity in \(t\)
\[
\partial_t \log \det(1 - (1-t)\kappa^{(j)}|_{[0,s]}) = -\frac{1}{2(1-t)} \int_{y \cup \tilde{y}} \text{Tr}[Y^{-1}(z)Y'(z)(J(z) - I)] \frac{dz}{2\pi i},
\] (6-1)

which was derived in Lemma 3.3.
We divide the proofs as a series of lemmas. First, we use the analysis of Section 4 to expand the right-hand side of (6-1) as $s \to +\infty$.

**Lemma 6.1.** As $s \to +\infty$, we have

$$\partial_t \log \det(1 - (1 - t)U^{(j)}_{[0,s]}) = I_1 + I_2 + 2 \Re I_{b_2} + O(e^{-cs^\rho}),$$  

where $c > 0$ and

$$I_1 = -\frac{1}{t} \int_{\Sigma_5} \left( \log(e^{-is^\rho h(\xi)}G(\xi; s))' \right) \frac{d\xi}{2\pi i},$$

$$I_2 = -\frac{2}{t} \Re \left[ \int_{[0,b_2]} (-is^\rho h'(\xi) + \log G(\xi; s))' \frac{d\xi}{2\pi i} \right],$$

$$I_{b_2} = \frac{1}{2\sqrt{1 - t}} \int_{\Sigma_2 \cap \mathbb{D}_{b_2}} e^{-s^\rho (ih(\xi) - \ell)} \tilde{G}(\xi; s) \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i},$$

$$+ \frac{2 - t}{2t^2 \sqrt{1 - t}} \int_{\Sigma_1 \cap \mathbb{D}_{b_2}} e^{-s^\rho (ih(\xi) - \ell)} \tilde{G}(\xi; s) \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_+] \frac{d\xi}{2\pi i},$$

$$+ \frac{-1}{2\sqrt{1 - t}} \int_{\Sigma_4 \cap \mathbb{D}_{b_2}} e^{s^\rho (ih(\xi) - \ell)} \tilde{G}(\xi; s) - \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_-] \frac{d\xi}{2\pi i},$$

$$+ \frac{-(2 - t)}{2t^2 \sqrt{1 - t}} \int_{\Sigma_6 \cap \mathbb{D}_{b_2}} e^{s^\rho (ih(\xi) - \ell)} \tilde{G}(\xi; s) - \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_-] \frac{d\xi}{2\pi i},$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

**Proof.** Using the change of variables $z = is^\rho \xi + \tau$ in (6-1), we obtain

$$\partial_t \log \det(1 - (1 - t)U^{(j)}_{[0,s]}) = \frac{-1}{2(1 - t)} \int_{\gamma_U \cup \tilde{\gamma}_U} \text{Tr}[U^{-1}(\xi)U'(\xi)(J_U(\xi) - I)] \frac{d\xi}{2\pi i} = I_\gamma + I_\tilde{\gamma},$$

$$I_\gamma = \frac{1}{2\sqrt{1 - t}} \int_{\gamma_U} e^{-is^\rho h(\xi)} \tilde{G}(\xi; s) \text{Tr}[U^{-1}(\xi)U'(\xi)\sigma_+] \frac{d\xi}{2\pi i},$$

$$I_\tilde{\gamma} = \frac{-1}{2\sqrt{1 - t}} \int_{\tilde{\gamma}_U} e^{is^\rho h(\xi)} \tilde{G}(\xi; s) \text{Tr}[U^{-1}(\xi)U'(\xi)\sigma_-] \frac{d\xi}{2\pi i},$$

where $U$ is defined in (4-1), $\gamma_U$ and $\tilde{\gamma}_U$ are defined in (4-4), and where we have used (4-8). Note that we do not specify whether we take the $+$ or $-$ boundary values of $U$ in (6-6) and (6-7), which is without ambiguity, see Remark 3.4. Now, we deform the contours of integration by using the analytic continuation.
of $U$ (denoted $\hat{T}$ and defined in Section 4C). We obtain

$$I_Y = \frac{1}{2\sqrt{1 - t}} \int_{\Sigma_1 \cup \Sigma_2} e^{-is^p h(\zeta)} G(\zeta; s) \text{Tr}[\hat{T}^{-1}(\zeta) \hat{T}'(\zeta) \sigma_+] \frac{d\zeta}{2\pi i}$$

$$+ \frac{1}{2\sqrt{1 - t}} \int_{\Sigma_5} e^{-is^p h(\zeta)} G(\zeta; s) \text{Tr}[\hat{T}^{-1}(\zeta) \hat{T}'(\zeta) \sigma_+] \frac{d\zeta}{2\pi i}, \quad (6-8)$$

$$I_\gamma = \frac{-1}{2\sqrt{1 - t}} \int_{\Sigma_1 \cup \Sigma_4} e^{is^p h(\zeta)} G(\zeta; s) \text{Tr}[\hat{T}^{-1}(\zeta) \hat{T}'(\zeta) \sigma_-] \frac{d\zeta}{2\pi i}$$

$$+ \frac{-1}{2\sqrt{1 - t}} \int_{\Sigma_5} e^{is^p h(\zeta)} G(\zeta; s) \text{Tr}[\hat{T}^{-1}(\zeta) \hat{T}'(\zeta) \sigma_-] \frac{d\zeta}{2\pi i}, \quad (6-9)$$

where the contours $\Sigma_1, \ldots, \Sigma_5$ are shown in Figure 6. Once more, we have not specified the boundary values of $\hat{T}$ for the integrals over $\Sigma_j$, $j = 1, \ldots, 4$ of (6-8) and (6-9); again, this is without ambiguity. Note however that this is not the case for the integrals over $\Sigma_5$. For $\zeta \in \Sigma_5$, by (4-14) we have

$$\text{Tr}[\hat{T}_\pm \hat{T}'_\mp \sigma_+] = \text{Tr}[H_\pm (H_\pm^{-1})' \sigma_+] + \text{Tr} \left[ T_\pm^{-1} T'_\pm s \frac{\sigma_1}{2} \sigma_3 e^{i\rho \ell} \sigma_3 H_\pm^{-1} \sigma_\pm H_\pm e^{-i\rho \ell} \frac{\sigma_1}{2} \sigma_3 \right],$$

$$e^{-is^p h(\zeta)} G(\zeta; s) \text{Tr}[H_\pm (H_\pm^{-1})' \sigma_+] = -\frac{\sqrt{1 - t}}{t} \left( \frac{\log(e^{-is^p h(\zeta)} G(\zeta; s))}{\sigma_1} \right)' \frac{d\zeta}{2\pi i},$$

$$e^{is^p h(\zeta)} G(\zeta; s) \text{Tr}[H_\pm (H_\pm^{-1})' \sigma_-] = \frac{\sqrt{1 - t}}{t} \left( \frac{\log(e^{is^p h(\zeta)} G(\zeta; s))}{\sigma_1} \right)' \frac{d\zeta}{2\pi i},$$

$$e^{-is^p h(\zeta)} G(\zeta; s) s \frac{\sigma_1}{2} \sigma_3 e^{i\rho \ell} \sigma_3 H_\pm^{-1} \sigma_\pm H_\pm e^{-i\rho \ell} \frac{\sigma_1}{2} \sigma_3 \frac{\sigma_3}{2} \sigma_3 = \begin{pmatrix} \sqrt{1 - t} & e^{s^p (ih(\zeta) - \ell)} \tilde{G}(\zeta; s) \frac{d\zeta}{2\pi i} \\ -\frac{1 - t}{t^{2}} e^{s^p (ih(\zeta) - \ell)} \tilde{G}(\zeta; s) \frac{d\zeta}{2\pi i} & -\sqrt{1 - t} \right),$$

$$e^{is^p h(\zeta)} G(\zeta; s) s \frac{\sigma_1}{2} \sigma_3 e^{i\rho \ell} \sigma_3 H_\pm^{-1} \sigma_\pm H_\pm e^{i\rho \ell} \frac{\sigma_1}{2} \sigma_3 \frac{\sigma_3}{2} \sigma_3 = \begin{pmatrix} \frac{\sqrt{1 - t}}{t^{2}} e^{s^p (ih(\zeta) - \ell)} \tilde{G}(\zeta; s) \frac{d\zeta}{2\pi i} \\ -\frac{1 - t}{t^{2}} e^{s^p (ih(\zeta) - \ell)} \tilde{G}(\zeta; s) \frac{d\zeta}{2\pi i} & -\sqrt{1 - t} \right).$$

Therefore, using also the jumps for $T$ given by (4-15), we obtain

$$\frac{1}{2\sqrt{1 - t}} \int_{\Sigma_5} e^{-is^p h(\zeta)} G(\zeta; s) \text{Tr}[\hat{T}_+^{-1}(\zeta) \hat{T}_+'(\zeta) \sigma_+] \frac{d\zeta}{2\pi i}$$

$$= -\frac{1}{2t} \int_{\Sigma_5} \left( \frac{\log(e^{-is^p h(\zeta)} G(\zeta; s))}{\sigma_1} \right)' \frac{d\zeta}{2\pi i}$$

$$- \frac{1}{2t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_3] \frac{d\zeta}{2\pi i}$$

$$- \sqrt{1 - t} \int_{\Sigma_6} e^{s^p (ih(\zeta) - \ell)} \tilde{G}(\zeta; s) \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_+] \frac{d\zeta}{2\pi i}$$

$$+ \frac{1}{2t^{2}} \int_{\Sigma_7} e^{-s^p (ih(\zeta) - \ell)} \tilde{G}(\zeta; s) \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_+] \frac{d\zeta}{2\pi i}.$$
and
\[
\frac{-1}{2\sqrt{1 - t}} \int_{\Sigma_5} e^{is\rho(h(\zeta))} G(\zeta; s)^{-1} \text{Tr}[T'\zeta^{-1}(\zeta)T'(\zeta)\sigma_{-}] d\zeta 2\pi i \\
= -\frac{1}{2t} \int_{\Sigma_5} (\log(e^{-is\rho(h(\zeta))} G(\zeta; s)))' \frac{d\zeta}{2\pi i} \\
-\frac{1}{2t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{3}] \frac{d\zeta}{2\pi i} - \frac{1}{2t^2\sqrt{1 - t}} \int_{\Sigma_6} e^{s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{-}] \frac{d\zeta}{2\pi i} \\
+ \frac{\sqrt{1 - t}}{2t^2} \int_{\Sigma_7} e^{-s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{+}] \frac{d\zeta}{2\pi i}.
\]

Thus, using again (4-14) to rewrite the integrals over \(\Sigma_j\), \(j = 1, 2, 3, 4\), in terms of \(T\), and collecting the above computations, we rewrite (6-8)–(6-9) as
\[
I_{\gamma} = \frac{1}{2\sqrt{1 - t}} \int_{\Sigma_1 \cup \Sigma_2} e^{-s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{+}] \frac{d\zeta}{2\pi i}, \\
-\frac{1}{2t} \int_{\Sigma_5} (\log(e^{-is\rho(h(\zeta))} G(\zeta; s)))' \frac{d\zeta}{2\pi i} \\
-\frac{1}{2t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{3}] \frac{d\zeta}{2\pi i} - \frac{1}{2t^2\sqrt{1 - t}} \int_{\Sigma_6} e^{s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{-}] \frac{d\zeta}{2\pi i} \\
+ \frac{1}{2t^2\sqrt{1 - t}} \int_{\Sigma_7} e^{-s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{+}] \frac{d\zeta}{2\pi i},
\]
\[
(6-10)
\]
\[
I_{\tilde{\gamma}} = \frac{-1}{2\sqrt{1 - t}} \int_{\Sigma_3 \cup \Sigma_4} e^{s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{-}] \frac{d\zeta}{2\pi i}, \\
-\frac{1}{2t} \int_{\Sigma_5} (\log(e^{-is\rho(h(\zeta))} G(\zeta; s)))' \frac{d\zeta}{2\pi i} \\
-\frac{1}{2t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{3}] \frac{d\zeta}{2\pi i} - \frac{1}{2t^2\sqrt{1 - t}} \int_{\Sigma_6} e^{s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{-}] \frac{d\zeta}{2\pi i} \\
+ \frac{\sqrt{1 - t}}{2t^2} \int_{\Sigma_7} e^{-s\rho\left(h(\zeta) - \ell\right)} G(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_{+}] \frac{d\zeta}{2\pi i}.
\]
\[
(6-11)
\]
We note from (4-22) that \(P^{(s)}(\zeta)\) is independent of \(s\). From (4-30) and (4-44), we also note that \(P^{(b_j)}(\zeta), j \in \{1, 2\}\), depends on \(s\) but is bounded as \(s \to +\infty\) uniformly for \(\zeta \in D_{b_j}\), and that \(P^{(b_j)'}(\zeta) = O(s^0)\) as \(s \to +\infty\) uniformly for \(\zeta \in D_{b_j}\). Using (4-45) and (4-52), we infer that
\[
T(\zeta) = O(1), \quad T'(\zeta) = O(s^0), \quad \text{as} \quad s \to +\infty,
\]
uniformly for \(\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \Sigma_j\). Since \(\text{Re}(ih(\zeta) - \ell) > 0\) for \(\zeta \in \Sigma_1 \cup \Sigma_2\) and \(\text{Re}(ih(\zeta) - \ell) < 0\) for \(\zeta \in \Sigma_3 \cup \Sigma_4\) (see Figures 5 and 6), we have
\[
I_{\gamma} + I_{\tilde{\gamma}} = I_1 + I_2 + I_{b_2} + I_{b_1} + O(e^{-cs^0}), \quad \text{as} \quad s \to +\infty,
\]
where \( I_1, I_2 \) and \( I_{b_2} \) are defined in (6-3), (6-4) and (6-5), respectively, and \( I_{b_1} \) is defined in a similar way to \( I_{b_2} \). Using the symmetry \( \zeta \mapsto -\overline{\zeta} \) (see in particular (4-19)), we obtain

\[
I_{b_1} = \overline{I_{b_2}},
\]

which finishes the proof.

Lemma 6.2.

\[
I_1 = \frac{\text{Re} \, b_2}{\pi \rho t} s^\rho + \frac{c_1 c_6 - c_2 c_5}{t (c_1 + c_2)} + \mathcal{O}(s^{-\rho}), \quad \text{as } s \to +\infty.
\]

Proof. By the definition (6-3) of \( I_1 \), we have

\[
I_1 = \frac{s^\rho}{\pi t} \text{Im} \int_{[0,b_2]} i h'(\zeta) d\zeta - \frac{1}{\pi t} \text{Im} \int_{[0,b_2]} (\log \mathcal{G}(\zeta; s))' d\zeta.
\]

For the first integral, we use (4-5), (4-20), and (4-10), to obtain

\[
\text{Im} \int_{[0,b_2]} i h'(\zeta) d\zeta = \text{Im}(i h(b_2)) = \overline{\ell} = (c_1 + c_2) \exp \left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) \cos \left(\frac{\pi c_2 - c_1}{2 (c_1 + c_2)}\right) = \frac{\text{Re} \, b_2}{\rho}.
\]

For the second integral, we find

\[
\int_{[0,b_2]} (\log \mathcal{G}(\zeta; s))' d\zeta = \log \mathcal{G}(b_2; s) - \log \mathcal{G}(0; s).
\]

Using (4-7) and (4-6), as \( s \to +\infty \) we have

\[
\int_{[0,b_2]} (\log \mathcal{G}(\zeta; s))' d\zeta = c_4 \log s + c_5 \log(i b_2) + c_6 \log(-i b_2) + c_7 - \log F(\tau) + \mathcal{O}(s^{-\rho}),
\]

and thus, by (4-10), we get

\[
\text{Im} \int_{[0,b_2]} (\log \mathcal{G}(\zeta; s))' d\zeta = c_5 \text{arg}(i b_2) + c_6 \text{arg}(-i b_2) + \mathcal{O}(s^{-\rho}),
\]

\[
= c_5 \frac{\pi}{2} \left(\frac{c_2 - c_1}{c_2 + c_1} + 1\right) + c_6 \frac{\pi}{2} \left(\frac{c_2 - c_1}{c_2 + c_1} - 1\right) + \mathcal{O}(s^{-\rho}). \quad \square
\]

We split \( I_{b_2} \) into four parts

\[
I_{b_2} = I_{b_2,1} + I_{b_2,2} + I_{b_2,3} + I_{b_2,4},
\]

where \( I_{b_2,j}, \ j = 1, 2, 3, 4, \) are given by

\[
I_{b_2,1} = \frac{1}{2 \sqrt{1 - t}} \int_{\Sigma_2 \cap \mathbb{D}_{b_2}} e^{-s^\rho (i h(\zeta) - \ell)} \overline{\mathcal{G}(\zeta; s)} \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_+ \sigma_+] \frac{d\zeta}{2 \pi i},
\]

\[
I_{b_2,2} = \frac{2 - t}{2 t^2 \sqrt{1 - t}} \int_{\Sigma_2 \cap \mathbb{D}_{b_2}} e^{-s^\rho (i h(\zeta) - \ell)} \overline{\mathcal{G}(\zeta; s)} \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_+ \sigma_-] \frac{d\zeta}{2 \pi i},
\]

\[
I_{b_2,3} = \frac{-1}{2 \sqrt{1 - t}} \int_{\Sigma_4 \cap \mathbb{D}_{b_2}} e^{s^\rho (i h(\zeta) - \ell)} \mathcal{G}(\zeta; s) - \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_- \sigma_+] \frac{d\zeta}{2 \pi i},
\]

\[
I_{b_2,4} = \frac{-(2 - t)}{2 t^2 \sqrt{1 - t}} \int_{\Sigma_4 \cap \mathbb{D}_{b_2}} e^{s^\rho (i h(\zeta) - \ell)} \mathcal{G}(\zeta; s) - 1 \text{Tr}[T^{-1}(\zeta) T'(\zeta) \sigma_-] \frac{d\zeta}{2 \pi i}.
\]
Lemma 6.3. As \( s \to +\infty \), we have

\[
I_{b_2,1} = \frac{1}{2\sqrt{1-i}} \int_{e^{\frac{\pi}{4}[0, +\infty)}} e^{\frac{\pi}{4}(z)} \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_+ + O(s^{-\frac{\rho}{2}})],
\]

\[
I_{b_2,2} = \frac{2-\imath}{2t^2\sqrt{1-t}} \int_{e^{-\frac{3\pi i}{4}(-\infty, 0]}} e^{\frac{3\pi i}{4}} \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_+ + O(s^{-\frac{\rho}{2}})],
\]

\[
I_{b_2,3} = \frac{-1}{2\sqrt{1-t}} \int_{e^{-\frac{3\pi i}{4}(-\infty, 0]}} e^{\frac{3\pi i}{4}} \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_- + O(s^{-\frac{\rho}{2}})],
\]

\[
I_{b_2,4} = \frac{-(2-\imath)}{2t^2\sqrt{1-t}} \int_{e^{\frac{3\pi i}{4}[0, +\infty)}} e^{\frac{3\pi i}{4}} \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_- + O(s^{-\frac{\rho}{2}})].
\]

Proof: From (4-27) and (4-28), we have

\[
s^\rho(i\hbar(\zeta) - \ell) = s^\rho\left(i\ell - i\frac{f(\zeta)^2}{2}\right) = i\ell s^\rho - \frac{i s^\rho}{2} f'(b_2)^2(\zeta - b_2)^2(1 + O(\zeta - b_2)), \quad \text{as } \zeta \to b_2,
\]

so the main contribution as \( s \to +\infty \) in the integrals for \( I_{b_2,j}, \ j = 1, \ldots, 4 \) comes from the integrand as

\[
s^\frac{\rho}{2}(\zeta - b_2) = O(1).
\]

We first obtain an expansion for \( \text{Tr}[T^{-1}T'\sigma_\pm] \) as \( s^\frac{\rho}{2}(\zeta - b_2) \to 0 \) and simultaneously \( s \to +\infty \). For \( \zeta \) inside the disk \( D_{b_2} \), by (4-45) we have

\[
T(\zeta) = R(\zeta)P^{(b_2)}(\zeta).
\]

Thus for \( \zeta \in D_{b_2} \), we have

\[
\text{Tr}[T^{-1}T'\sigma_\pm] = \text{Tr}[(P^{(b_2)})^{-1}(P^{(b_2)})'\sigma_\pm] + \text{Tr}[(P^{(b_2)})^{-1}R^{-1}R'P^{(b_2)}\sigma_\pm].
\]

We recall from (4-30) that \( P^{(b_2)} \) is given by

\[
P^{(b_2)}(\zeta) = E(\zeta; s)\Phi_{PC}\left(s^\frac{\rho}{2}f(\zeta); \sqrt{1-t}\right)e^{\frac{\rho}{2}(ih(\zeta) - \ell)\sigma_3}\tilde{G}(\zeta; s)^{-\frac{\sigma_3}{4}}, \quad \zeta \in D_{b_2} \setminus \bigcup_{j=1}^7 \Sigma_j,
\]

and thus

\[
\text{Tr}[(P^{(b_2)})^{-1}(P^{(b_2)})'\sigma_\pm] = e^{\pm s^\rho(ih(\zeta) - \ell)}\tilde{G}(\zeta; s)^{-\frac{\sigma_3}{4}}(s^\frac{\rho}{2}f'(\zeta) + O(s^{-\frac{\rho}{2}})),
\]

\[
\text{Tr}[(P^{(b_2)})^{-1}R^{-1}R'P^{(b_2)}\sigma_\pm] = e^{\pm s^\rho(ih(\zeta) - \ell)}\tilde{G}(\zeta; s)^{-\frac{\sigma_3}{4}}\text{Tr}[\Phi_{PC}^{-1}E^{-1}R^{-1}R'E\Phi_{PC}\sigma_\pm],
\]

where \( \Phi_{PC} \) and \( \Phi_{PC}' \) are evaluated at \( s^\frac{\rho}{2}f(\zeta) \) and the other functions are evaluated at \( \zeta \). We also recall from (4-31) that

\[
E(\zeta; s) = P^{(\infty)}(\zeta)\tilde{G}(\zeta; s)^{-\frac{\sigma_3}{4}}e^{-\frac{\rho}{2}i\sigma_3}(s^\frac{\rho}{2}f(\zeta))^{i\sigma_3}
\]

is analytic for \( \zeta \in D_{b_2} \), and thus

\[
E^{\pm}(\zeta; s) = O(1), \quad E'(\zeta; s) = O(1),
\]

(6-17)
as \( s \to +\infty \) uniformly for \( \zeta \in \mathcal{D}_{b_2} \). By (6-13), (6-15), and (6-16), we have

\[
I_{b_2,1} = \frac{1}{2\sqrt{1-t}} \int_{\Sigma_2 \cap \mathcal{D}_{b_2}} (s^\frac{\rho}{2} f' \text{Tr}[\Phi_{PC}^{-1} \Phi_{PC} \sigma_+]) \frac{d\zeta}{2\pi i} + \frac{1}{2\sqrt{1-t}} \int_{\Sigma_2 \cap \mathcal{D}_{b_2}} \text{Tr}[\Phi_{PC}^{-1} E^{-1} R' E \Phi_{PC} \sigma_+] \frac{d\zeta}{2\pi i}.
\]

Let us now perform the change of variables

\[
z = s^\frac{\rho}{2} f(\zeta),
\]

where we recall that \( f \) is injective on \( \mathcal{D}_{b_2} \). Then we have

\[
e^{-s^\rho (ih(\zeta) - t)} = e^{-is^\rho e^{i\frac{\rho}{2} z^2}} \quad \zeta = f^{-1}(s^{-\frac{\rho}{2}} z), \quad dz = s^\frac{\rho}{2} f'(\zeta) d\zeta.
\]

Since \( \Sigma_2 \cap \mathcal{D}_{b_2} \) is mapped by \( f \) to a subset of \( e^{\frac{\rho}{2}} [0, +\infty) \), see (4-29), this change of variables allows us to rewrite \( I_{b_2,1} \) as

\[
I_{b_2,1} = \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, s^\frac{\rho}{2} r]} \text{Tr}[\Phi_{PC}^{-1}(z) \Phi_{PC}(z) \sigma_+] \frac{dz}{2\pi i} + \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, s^\frac{\rho}{2} r]} \text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) E'(\zeta; s) \Phi_{PC}(z) \sigma_+] \frac{dz}{2\pi i} + \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, s^\frac{\rho}{2} r]} \frac{d\zeta}{2\pi i},
\]

where \( r := |f(r_*)| \) with \( r_* \) defined by \( \mathcal{D}_{b_2} \cap \Sigma_2 = \{r_*\} \). We note from (A-1) that

\[
\Phi_{PC}(z) \sigma_+ \Phi_{PC}^{-1}(z) = O(e^{\frac{\rho}{2}}) \quad \text{as} \quad z \to \infty,
\]

and we conclude from (4-52) and (6-17) that

\[
\frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, s^\frac{\rho}{2} r]} \frac{\text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) E'(\zeta; s) \Phi_{PC}(z) \sigma_+]}{s^\frac{\rho}{2} f'(\zeta)} \frac{dz}{2\pi i} = O(s^{-\frac{\rho}{2}}),
\]

\[
\frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, s^\frac{\rho}{2} r]} \frac{\text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) R^{-1}(\zeta) R'(\zeta) E(\zeta; s) \Phi_{PC}(z) \sigma_+]}{s^\frac{\rho}{2} f'(\zeta)} \frac{dz}{2\pi i} = O(s^{-\rho}),
\]

\[
\frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, s^\frac{\rho}{2} r]} \frac{\text{Tr}[\Phi_{PC}(z) \Phi_{PC}'(z) \sigma_+]}{2\pi i} \frac{dz}{2\pi i} = \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{\rho}{2}} [0, +\infty)} \frac{d\zeta}{2\pi i} + O(e^{-cs^\rho})
\]

as \( s \to +\infty \), for a certain \( c > 0 \). This finishes the proof for \( I_{b_2,1} \). The proofs of the expressions for the other integrals are similar.

**Lemma 6.4.** We have

\[
I_{b_2} = \mathcal{I}_{b_2} + O(s^{-\frac{\rho}{2}}), \quad \text{as} \quad s \to +\infty,
\]

where \( \mathcal{I}_{b_2} \) depends on \( t \) but is independent of the other parameters. More precisely, for \( j = 1 \) (the Meijer-G process), \( \mathcal{I}_{b_2} \) is independent of \( r, q, v_1, \ldots, v_r, \mu_1, \ldots, \mu_q \), and for \( j = 2 \) (Wright’s generalized Bessel process), \( \mathcal{I}_{b_2} \) is independent of \( \alpha \) and \( \theta \).
Proof. This follows from (6-12), Lemma 6.3, and the fact that $\Phi_{PC}$ only depends on $q = \sqrt{1 - t}$. \hfill \Box

Let $b_* := \Sigma_5 \cap \partial D_{b_2}$. We split $I_2$ into two parts:

\[ I_2 = -\frac{2}{t} \text{Re} \left[ \int_{[0,b_2]} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right] = I_{2,1} + I_{2,2}, \]

\[ I_{2,1} = -\frac{2}{t} \text{Re} \left[ \int_{[0,b_*]} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right], \]

\[ I_{2,2} = -\frac{2}{t} \text{Re} \left[ \int_{\Sigma_5 \cap \partial D_{b_2}} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right]. \]

**Lemma 6.5.**

\[ I_{2,1} = \frac{2\nu}{\pi t} \log \left| \frac{b_* - b_2}{b_* - b_1} \right| + O(s^{-\frac{\nu}{2}}), \quad \text{as } s \to +\infty. \]

Proof. For $\xi \in [0, b_*) \subset \mathbb{C} \setminus D_{b_2}$, by (4-45) we have $T(\xi) = R(\xi) P^{(\infty)}(\xi)$, and thus

\[ \text{Tr}[T^{-1}T' \sigma_3] = \text{Tr}[(P^{(\infty)})^{-1}(P^{(\infty)})' \sigma_3] + \text{Tr}[(P^{(\infty)})^{-1} R^{-1} R' P^{(\infty)} \sigma_3] \]

\[ = \text{Tr}[(P^{(\infty)})^{-1} (P^{(\infty)})' \sigma_3] + O(s^{-\frac{\nu}{2}}), \quad \text{as } s \to +\infty, \]

uniformly for $\xi \in [0, b_*)$, where we have used (4-52). We recall that $P^{(\infty)}$ is given by

\[ P^{(\infty)}(\xi) = D(\xi)^{-\sigma_3}, \quad \text{where } D(\xi) = \exp \left( i\nu \int_{\Sigma_5} \frac{d\xi}{\xi - \xi} \right) = \exp \left( i\nu \log \left[ \frac{\xi - b_2}{\xi - b_1} \right] \right) \]

and where the branch of the logarithm is taken along $\Sigma_5$. Thus

\[ \text{Tr}[(P^{(\infty)})^{-1} (P^{(\infty)})' \sigma_3] = -2(\log D)', \]

and we find as $s \to +\infty$,

\[ I_{2,1} = -\frac{2}{t} \text{Re} \left[ \int_{[0,b_*]} \text{Tr}[(P^{(\infty)})^{-1} (P^{(\infty)})' \sigma_3] \frac{d\xi}{2\pi i} \right] + O(s^{-\frac{\nu}{2}}), \]

\[ = \frac{4}{t} \text{Re} \left[ \int_{[0,b_*]} (\log D)' \frac{d\xi}{2\pi i} \right] + O(s^{-\frac{\nu}{2}}) = \frac{2\nu}{\pi t} \log \left| \frac{b_* - b_2}{b_* - b_1} \right| + O(s^{-\frac{\nu}{2}}). \quad \Box \]

**Lemma 6.6.** As $s \to +\infty$, we have

\[ I_{2,2} = I_{2,2}^{(1)} + I_{2,2}^{(2)} + I_{2,2}^{(3)} + O(s^{-\frac{\nu}{2}}), \]

\[ I_{2,2}^{(1)} = -\frac{2}{t} \text{Re} \left[ \int_{\Sigma_5 \cap \partial D_{b_2}} (s^\nu h'(\xi) - (\log \widetilde{G})'(\xi; s)) \frac{d\xi}{2\pi i} \right], \]

\[ I_{2,2}^{(2)} = -\frac{2}{t} \text{Re} \left[ \int_{\Sigma_5 \cap \partial D_{b_2}} f'(\xi) \text{Tr}[\Phi_{PC}(s^\nu f_{b_2}(\xi)) \Phi_{PC}(s^\nu f_{b_2}(\xi)) \sigma_3] \frac{d\xi}{2\pi i} \right], \]

\[ I_{2,2}^{(3)} = -\frac{2}{t} \text{Re} \left[ \int_{\Sigma_5 \cap \partial D_{b_2}} \text{Tr}[\Phi_{PC}(s^\nu f_{b_2}(\xi)) E^{-1}(\xi; s) E'(\xi; s) \Phi_{PC}(s^\nu f_{b_2}(\xi)) \sigma_3] \frac{d\xi}{2\pi i} \right]. \]
Proof. For \( \zeta \in \Sigma_5 \cap D_{b_2} \), by (4-45) we have \( T(\zeta) = R(\zeta) P^{(b_2)}(\zeta) \), and thus
\[
\text{Tr}[T^{-1}T'\sigma_3] = \text{Tr}[(P^{(b_2)})^{-1}(P^{(b_2)})'\sigma_3] + \text{Tr}[(P^{(b_2)})^{-1}R^{-1}R'P^{(b_2)}\sigma_3].
\]
We recall that \( P^{(b_2)} \) is given by
\[
P^{(b_2)}(\zeta) = E(\zeta; s) \Phi_{PC}(\zeta) e^{\frac{s}{2} (\arg(\tilde{G}(\zeta; s)) - \zeta)} \]
and thus
\[
\text{Tr}[(P^{(b_2)})^{-1}(P^{(b_2)})'\sigma_3] = (s^\rho h' - (\log(\tilde{G})))' + s^\rho f' \text{Tr}[\Phi_{PC}'\Phi_{PC} \sigma_3] + \text{Tr}[\Phi_{PC}' E^{-1} E' \Phi_{PC} \sigma_3],
\]
\[
\text{Tr}[(P^{(b_2)})^{-1}R^{-1}R'P^{(b_2)}\sigma_3] = \text{Tr}[\Phi_{PC}' E^{-1} R^{-1} R' E \Phi_{PC} \sigma_3],
\]
where \( \Phi_{PC} \) and \( \Phi_{PC}' \) are evaluated at \( z = s^\frac{\rho}{2} f_2(\zeta) \) and the other functions are evaluated at \( \zeta \). Thus
\[
\int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[T^{-1}T'\sigma_3] \frac{d\zeta}{2\pi i} = \int_{\Sigma_4 \cap D_{b_2}} (s^\rho h' - (\log(\tilde{G})))' \frac{d\zeta}{2\pi i} + s^\rho \int_{\Sigma_5 \cap D_{b_2}} f' \text{Tr}[\Phi_{PC}'\Phi_{PC} \sigma_3] \frac{d\zeta}{2\pi i} + \int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[\Phi_{PC}' E^{-1} E' \Phi_{PC} \sigma_3] \frac{d\zeta}{2\pi i} + \int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[\Phi_{PC}' E^{-1} R^{-1} R' E \Phi_{PC} \sigma_3] \frac{d\zeta}{2\pi i}.
\]
From the Appendix, we know that
\[
\Phi_{PC}(z)\sigma_3 \Phi_{PC}(z)^{-1} = O(1), \quad (6-19)
\]
uniformly for \( z \in \mathbb{C} \). Therefore, by the cyclic property of the trace, and using also the estimates (4-52) and (6-17), we conclude that
\[
\int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[\Phi_{PC}' E^{-1} R^{-1} R' E \Phi_{PC} \sigma_3] \frac{d\zeta}{2\pi i} = O(s^{-\frac{\rho}{2}}), \quad \text{as} \ s \to +\infty.
\]
\[\square\]

Lemma 6.7. As \( s \to +\infty \), we have
\[
I^{(1)}_{2,2} = -\frac{\text{Im}(ih(b_2)) - \text{Im}(ih(b_2))}{\pi t} s^\rho + \frac{c_5 + c_6}{\pi t} \arg\left(\frac{b_2}{b_*}\right) + O(s^{-\rho}).
\]

Proof. By definition of \( I^{(1)}_{2,2} \), we have
\[
I^{(1)}_{2,2} = \int \frac{ih'(\zeta) d\zeta}{\pi t} + \frac{1}{\pi t} \int \frac{\text{Im}(\log(\tilde{G}')(\zeta; s))' d\zeta}{\pi t},
\]
and
\[
\text{Im} \int \frac{ih'(\zeta) d\zeta}{\pi t} = \text{Im}(ih(b_2)) - \text{Im}(ih(b_*)),
\]
\[
\int \frac{(\log(\tilde{G}')(\zeta; s))' d\zeta}{\pi t} = \log(\tilde{G}(b_2; s)) - \log(\tilde{G}(b_*; 0)). \quad (6-20)
\]
The right-hand side of (6-20) can be expanded as \( s \to +\infty \) using (4-7), and we find

\[
\int_{\Sigma s^{\rho} \cap D_{b_2}} (\log \tilde{G}(\zeta; s))' d\zeta = (c_5 + c_6) \log \left( \frac{b_2}{b_*} \right) + \mathcal{O}(s^{-\rho}),
\]

and the result follows. \( \square \)

**Lemma 6.8.** Let \( m \in \mathbb{C} \setminus \mathbb{R}^- \). As \( s \to +\infty \), we have

\[
I^{(2)}_{2,2} = -\frac{s^\rho}{\pi t} \left[ \text{Im}(ih(b_*)) - \text{Im}(ih(b_2)) \right] - \frac{v \rho}{\pi t} \log s - \frac{2v}{\pi t} \log r + \frac{2v}{\pi t} \log |m| + \mathcal{I}^{(2)}_{2,2}(m) + \mathcal{O}(s^{-\frac{\rho}{2}}),
\]

where \( r = |f(b_*)| = -f(b_*) \) and

\[
\mathcal{I}^{(2)}_{2,2}(m) = -\frac{2}{t} \Re \left[ \int_{(-\infty,0]} \left( \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] - \left( iz - \frac{2iv}{z-m} \right) \right) \frac{dz}{2\pi i} \right].
\]

**Proof.** Using the change of variables \( z = s^{\frac{\rho}{2}} f_{b_2}(\xi) \) and denoting \( r = |f_{b_2}(b_*)| = -f_{b_2}(b_*) \), we rewrite \( I^{(2)}_{2,2} \) as

\[
I^{(2)}_{2,2} = -\frac{2}{t} \Re \left[ \int_{[-s^{\frac{\rho}{2}} r,0]} \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] \frac{dz}{2\pi i} \right]. \tag{6-21}
\]

From the expansion (A-1), we get

\[
\text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] = iz - \frac{2iv}{z} + \mathcal{O}(z^{-2}), \quad \text{as } z \to -\infty.
\]

Let \( m \in \mathbb{C} \setminus \mathbb{R}^- \). We have

\[
\int_{[-s^{\frac{\rho}{2}} r,0]} \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] \frac{dz}{2\pi i} = \int_{[-s^{\frac{\rho}{2}} r,0]} \left( iz - \frac{2iv}{z-m} \right) \frac{dz}{2\pi i} + \int_{[-s^{\frac{\rho}{2}} r,0]} \left( \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] - \left( iz - \frac{2iv}{z-m} \right) \right) \frac{dz}{2\pi i}. \tag{6-22}
\]

Since

\[
\text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] - \left( iz - \frac{2iv}{z-m} \right) = \mathcal{O}(z^{-2}), \quad \text{as } z \to -\infty,
\]

we have

\[
\int_{[-s^{\frac{\rho}{2}} r,0]} \left( \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] - \left( iz - \frac{2iv}{z-m} \right) \right) \frac{dz}{2\pi i} = \int_{(-\infty,0]} \left( \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_3] - \left( iz - \frac{2iv}{z-m} \right) \right) \frac{dz}{2\pi i} + \mathcal{O}(s^{-\frac{\rho}{2}}), \quad \text{as } s \to +\infty.
\]
On the other hand, the first integral on the right-hand side of \((6-22)\) can be easily expanded as follows:

\[
\int_{[-s, s]_{r, 0}} \left( i z - \frac{2i v}{\nu - m} \right) \frac{dz}{2\pi i} = -\frac{s^2 r^2}{4\pi} + \frac{v}{\pi} \log\left(\frac{s^2 r}{m} + 1\right) + O(s^{-\frac{2}{3}}), \quad \text{as } s \to +\infty.
\]

Therefore, as \(s \to +\infty\) we have

\[
I_{2,2}^{(2)} = -\frac{2}{t} \text{Re} \left[ -\frac{s^2 r^2}{4\pi} + \frac{v}{\pi} \log s + \frac{v}{\pi} \log \frac{r}{m} \right] + O(s^{-\frac{2}{3}}),
\]

and the claim follows by noticing that

\[
r^2 = f(b_*)^2 = -2(h(b_*) - h(b_2)) = -2(\text{Im}(ih(b_*)) - \text{Im}(ih(b_2))).
\]

\[\square\]

**Lemma 6.9.** We have

\[
I_{2,2}^{(3)} = O(1), \quad \text{as } s \to +\infty,
\]

\[
I_{2,2}^{(3)} = O(b_* - b_2), \quad \text{as } b_* \to b_2.
\]

**Proof.** This follows from the previous estimates \((6-17)\) and \((6-19)\), and the cyclic property of the trace. \[\square\]

**Lemma 6.10.** As \(s \to +\infty\), we have

\[
I_2 = -\frac{v}{\pi t} \log s - \frac{2v}{\pi t} \log |(b_2 - b_1) f'(b_2)| + I_{2,2}^{(2)}(1) + O(s^{-\frac{2}{3}}).
\]

**Proof.** By combining Lemmas 6.5, 6.6, 6.7, 6.8 and 6.9, as \(s \to +\infty\) we have

\[
I_2 = I_{2,1} + I_{2,2} = I_{2,1}^{(1)} + I_{2,2}^{(2)} + I_{2,2}^{(3)} + O(s^{-\frac{2}{3}})
\]

\[
= \frac{2v}{\pi t} \log \left| \frac{b_* - b_2}{b_* - b_1} \right| - \frac{\text{Im}(ih(b_2)) - \text{Im}(ih(b_*))}{\pi t} s^2 + \frac{c_5 + c_6}{\pi t} \arg \left( \frac{b_2}{b_*} \right) \]

\[
- s^2 \left[ \text{Im}(ih(b_*)) - \text{Im}(ih(b_2)) \right] - \frac{v}{\pi t} \log s - \frac{2v}{\pi t} \log |f(b_*)| + \frac{2v}{\pi t} \log |m| + I_{2,2}^{(2)}(m) + I_{2,2}^{(3)} + O(s^{-\frac{2}{3}})
\]

\[
= \frac{v}{\pi t} \log s + \frac{2v}{\pi t} \log \left| \frac{b_* - b_2}{(b_* - b_1) f(b_*)} \right| + \frac{c_5 + c_6}{\pi t} \arg \left( \frac{b_2}{b_*} \right) + \frac{2v}{\pi t} \log |m| + I_{2,2}^{(2)}(m) + I_{2,2}^{(3)} + O(s^{-\frac{2}{3}}). \quad (6-23)
\]

The term of order \(O(1)\) as \(s \to +\infty\) in this expansion is given by

\[
\frac{2v}{\pi t} \log \left| \frac{b_* - b_2}{(b_* - b_1) f(b_*)} \right| + \frac{c_5 + c_6}{\pi t} \arg \left( \frac{b_2}{b_*} \right) + \frac{2v}{\pi t} \log |m| + I_{2,2}^{(2)}(m) + I_{2,2}^{(3)}.
\]

\[\text{We simplify this term by noticing that the disks can be chosen arbitrarily small (though independent of } s).\]
Therefore it is possible to evaluate (6.24) simply by taking the limit \( b_* \to b_2 \). As \( b_* \to b_2 \), we have

\[
\frac{b_* - b_2}{(b_* - b_1) f(b_*)} = \frac{1}{(b_2 - b_1) f'(b_2)} + \mathcal{O}(b_* - b_2), \quad \arg\left(\frac{b_2}{b_*}\right) = \mathcal{O}(b_* - b_2), \quad I^{(3)}_{2,2} = \mathcal{O}(b_* - b_2),
\]

where we have used Lemma 6.9. Therefore, taking the limit \( 6.23 \), we obtain

\[
I_2 = -\frac{\nu \rho}{\pi t} \log s - \frac{2\nu}{\pi t} \log((b_2 - b_1) f'(b_2)) + \frac{2\nu}{\pi t} \log |m| + I^{(2)}_{2,2}(m) + \mathcal{O}(s^{-\frac{\nu}{2}}), \quad s \to +\infty. \quad (6.25)
\]

We have the freedom to choose \( m \in \mathbb{C} \setminus \mathbb{R}^- \). The claim follows after setting \( m = 1 \) in (6.25). \( \square \)

**Lemma 6.11.** For \( j = 1, 2 \), we have

\[
\partial_t \log \det(1 - (1 - t)\mathcal{K}^{(j)}_{[0,s]}) = \frac{\Re b_2}{\pi \rho t} s^\rho - \frac{\nu \rho}{\pi t} \log s + \frac{c_1 c_6 - c_2 c_5}{t(c_1 + c_2)} - \frac{2\nu}{\pi t} \log((b_2 - b_1) f'(b_2)) + \partial_t[\log(G(1 + i\nu)G(1 - i\nu))] + \mathcal{O}(s^{-\frac{\nu}{2}}), \quad s \to +\infty, \quad (6.26)
\]

where \( G \) is Barnes’ \( G \)-function.

**Proof.** It follows from Lemmas 6.2, 6.4 and 6.10 that

\[
\partial_t \log \det(1 - (1 - t)\mathcal{K}^{(j)}_{[0,s]}) = I_1 + I_2 + 2 \Re I_{b_2} + \mathcal{O}(e^{-\nu})
\]

\[
= \frac{\Re b_2}{\pi \rho t} s^\rho - \frac{\nu \rho}{\pi t} \log s + \frac{c_1 c_6 - c_2 c_5}{t(c_1 + c_2)} - \frac{2\nu}{\pi t} \log((b_2 - b_1) f'(b_2)) + \chi(t) + \mathcal{O}(s^{-\frac{\nu}{2}}). \quad (6.27)
\]

where

\[
\chi(t) := 2 \Re I_{b_2} + I^{(2)}_{2,2}(1).
\]

It is rather difficult to obtain an explicit expression for \( \chi(t) \) from a direct analysis. However, it follows from Lemmas 6.4 and 6.10 that \( \chi(t) \) depends on \( t \) but is independent of the other parameters. More precisely, for \( j = 1 \), \( \chi(t) \) is independent of \( r, q, v_1, \ldots, v_r, \mu_1, \ldots, \mu_q \), and for \( j = 2 \), \( \chi(t) \) is independent of \( \alpha \) and \( \theta \). We will take advantage of that by using the known result from [Bochner et al. 2019] for the Bessel point process given by (1-7). If \( j = 1 \), then we set \( r = 1, q = 0 \) and \( v_1 = 0 \), and if \( j = 2 \), we set \( \theta = 1 \) and \( \alpha = 0 \). In these cases, \( \Re b_2 = 1, \rho = \frac{1}{2}, c_1 = c_2 = 1, c_5 = c_6 = 0, f'(b_2) = \sqrt{2} \) and (6.27) becomes

\[
\partial_t \log \det(1 - (1 - t)\mathcal{K}_{[0,s]}^{(j)}) = \frac{2}{\pi t} \sqrt{s} - \frac{\nu}{2\pi t} \log s + \frac{\nu}{\pi t} \log 8 + \chi(t) + \mathcal{O}(s^{-\frac{1}{2}}). \quad (6.28)
\]

On the other hand, the asymptotics (1-7) can be differentiated with respect to \( t \) (this follows from the analysis done in [Bochner et al. 2019] and [Charlier 2020]), and we get as \( s \to +\infty \),

\[
\partial_t \log \det(1 - (1 - t)\mathcal{K}_{\mathbb{B}e|[0,4s]}) = \partial_t \left( -4\nu \sqrt{s} + \nu^2 \log(8\sqrt{s}) + \log(G(1 + i\nu)G(1 - i\nu)) + \mathcal{O}\left(\frac{1}{\sqrt{s}}\right) \right)
\]

\[
= \frac{2}{\pi t} \sqrt{s} - \frac{\nu}{2\pi t} \log s - \frac{\nu}{\pi t} \log 8 + \partial_t \left( \log(G(1 + i\nu)G(1 - i\nu)) + \mathcal{O}\left(\frac{1}{\sqrt{s}}\right) \right). \quad (6.29)
\]
By (1-11) and (1-23), the left-hand sides of (6-28) and (6-29) are equal, and this yields the relation
\[ \chi(t) = \partial_t (\log(G(1 + i\nu)G(1 - i\nu))). \]

Lemma 6.12. As \( s \to +\infty \), we have
\[
\log \det(1 - (1 - t)K^{(j)}_{(0,s)}) = -\frac{2v \Re b_2}{\rho} s^{\rho} + v^2 \rho \log s - 2\pi v \frac{c_1 c_6 - c_2 c_5}{c_1 + c_2} + 2v^2 \log|\Gamma_2 - \Gamma_1| f'(b_2) | + \log(G(1 + i\nu)G(1 - i\nu)) + O(s^{-\frac{2}{3}})
\]

Proof. It suffices to integrate (6-26) in \( t \).

Thus the constants \( C = C^{(j)}, \ j = 1, 2 \) of Theorems 1.8 and 1.16 are given by
\[
\log C = -2\pi v \frac{c_1 c_6 - c_2 c_5}{c_1 + c_2} + 2v^2 \log|\Gamma_2 - \Gamma_1| f'(b_2) | + \log(G(1 + i\nu)G(1 - i\nu)).
\]

This expression can be computed more explicitly by substituting the values for the constants \( c_1, c_2, c_5, c_6 \) given at the beginning of Section 4B, and the values (4-10) and (4-28) for \( b_2, b_1, \) and \( f'(b_2) \).

Appendix: Parabolic cylinder model RH problem

Let \( q \in \mathbb{T} = [0, 1) \cup i[0, +\infty) \) and let
\[ \nu := -\frac{1}{2\pi} \log(1 - q^2) \in \mathbb{R}. \]

Consider the following model RH problem.

**RH problem for \( \Phi_{PC} \):**

(a) \( \Phi_{PC} : \mathbb{C} \setminus \Sigma_{PC} \to \mathbb{C}^{2 \times 2} \) is analytic, where
\[
\Sigma_{PC} = \mathbb{R}^- \cup \bigcup_{j=0}^{3} e^{\frac{\pi i}{2} + j\frac{\pi i}{4}} \mathbb{R}^+,
\]

as shown in Figure 8.

(b) With the contour \( \Sigma_{PC} \) oriented as in Figure 8, \( \Phi_{PC} \) satisfies the jumps
\[
\Phi_{PC,+}(z) = \Phi_{PC,-}(z)
\]
\[
\begin{pmatrix}
1 & -q \\
0 & 1
\end{pmatrix}, \quad z \in e^{\frac{\pi i}{2}} \mathbb{R}^+, \]
\[
\begin{pmatrix}
1 & 0 \\
q & 1
\end{pmatrix}, \quad z \in e^{\frac{3\pi i}{4}} \mathbb{R}^+, \]
\[
\begin{pmatrix}
q & 0 \\
1 & 1
\end{pmatrix}, \quad z \in e^{-\frac{3\pi i}{4}} \mathbb{R}^+, \]
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad z \in e^{-\frac{\pi i}{2}} \mathbb{R}^+, \]
\[
\begin{pmatrix}
1-q^2 & 0 \\
0 & 1-q^2
\end{pmatrix}, \quad z \in \mathbb{R}^-.
\]
(c) As $z \to 0$, we have $\Phi_{PC}(z) = \mathcal{O}(1)$.

As $z \to \infty$, $\Phi_{PC}$ admits an asymptotic series of the form

$$
\Phi_{PC}(z) \sim \left( I + \sum_{k=1}^{\infty} \frac{\Phi_{PC,k}(q)}{z^k} \right) z^{-i\nu \sigma_3} e^{\frac{i\pi}{4} \sigma_3},
$$

(A-1)

where the principal branch is taken for $z^{\pm i\nu}$, and where

$$
\Phi_{PC,1}(q) = \begin{pmatrix} 0 & \beta_{12}(q) \\ \beta_{21}(q) & 0 \end{pmatrix},
$$

$$
\Phi_{PC,2}(q) = \begin{pmatrix} (1+i\nu)q & 0 \\ 0 & (1-i\nu)q \end{pmatrix},
$$

$$
\Phi_{PC,2k-1}(q) = \begin{pmatrix} 0 & \Phi_{PC,2k-1}(q)_{12} \\ \Phi_{PC,2k-1}(q)_{21} & 0 \end{pmatrix}, \quad k \geq 2,
$$

$$
\Phi_{PC,2k}(q) = \begin{pmatrix} \Phi_{PC,2k}(q)_{11} & 0 \\ 0 & \Phi_{PC,2k}(q)_{22} \end{pmatrix}, \quad k \geq 2,
$$

(A-2)

where

$$
\beta_{12}(q) = \frac{e^{-\frac{3\pi i}{4}} e^{-\frac{\pi v}{2} \sqrt{2\pi} q}}{q \Gamma(i\nu)} \quad \text{and} \quad \beta_{21}(q) = \frac{e^{\frac{3\pi i}{4}} e^{-\frac{\pi v}{2} \sqrt{2\pi} q}}{q \Gamma(-i\nu)}.
$$

(A-3)

The solution $\Phi_{PC}(z) = \Phi_{PC}(z; q)$ can be expressed in terms of the parabolic cylinder function $D_a(z)$ (see [Olver et al. 2010, Chapter 12] for a definition). RH problems related to parabolic cylinder functions were first studied in [Its 1981], and first used in a steepest descent analysis in [Deift and Zhou 1993]. We also refer to [Fokas et al. 2006, Chapter 9, §4], [Bothner 2017, Section 5.2], and [Lenells 2017, Appendix B] for more recent works using $D_a$ to construct certain model RH problems. The solution to the above RH problem for $q \in [0, 1)$ is the same as in [Lenells 2017]; however for $q \in i(0, +\infty)$ it differs from the one of [Lenells 2017] and, for the convenience of the reader, we construct its explicit solution here.

**Lemma A.1.** The unique solution to the model RH problem for $\Phi_{PC}$ is given by

$$
\Phi_{PC}(z) = \Psi(z) B(z)^{-1},
$$

(A-4)

where

$$
B(z) = \begin{cases} 
\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}, & \text{arg } z \in (0, \frac{\pi}{4}), \\
\begin{pmatrix} \frac{1}{q} & 0 \\ 0 & 1 \end{pmatrix}, & \text{arg } z \in (\frac{\pi}{4}, \pi), \\
\begin{pmatrix} 1 & q \nu \\ 0 & 1 \end{pmatrix}, & \text{arg } z \in (-\pi, -\frac{3\pi}{4}), \\
\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}, & \text{arg } z \in (-\frac{\pi}{4}, 0), \\
I, & \text{elsewhere},
\end{cases}
$$
and

\[
\Psi(z) = \begin{pmatrix}
\psi_{11}(z) & (\frac{-i}{2} + \frac{z}{2}) \psi_{22}(z) \\
(i \frac{z}{2} - \frac{z}{2}) \psi_{11}(z) & \psi_{22}(z)
\end{pmatrix}
\begin{pmatrix}
\frac{\beta_{21}(q)}{\beta_{12}(q)} \\
\frac{\beta_{21}(q)}{\beta_{12}(q)}
\end{pmatrix},
q \in \mathbb{T}, z \in \mathbb{C} \setminus \mathbb{R},
\]

(A-5)

where the functions \(\psi_{11}\) and \(\psi_{22}\) are defined by

\[
\psi_{11}(z) = \begin{cases}
e^{-\frac{z}{2}} D_{-i\nu}(e^{-\frac{z}{2}}), & \text{Im} z > 0, \\
e^{\frac{3\pi i}{4}} D_{-i\nu}(e^{\frac{3\pi i}{4}} z), & \text{Im} z < 0,
\end{cases}
\]

(A-6a)

\[
\psi_{22}(z) = \begin{cases}
e^{-\frac{3\pi i}{4}} D_{i\nu}(e^{-\frac{3\pi i}{4}} z), & \text{Im} z > 0, \\
e^{\frac{\pi i}{2}} D_{i\nu}(e^{\frac{\pi i}{2}} z), & \text{Im} z < 0.
\end{cases}
\]

(A-6b)

Proof. It is a classical fact (see, e.g., [Olver et al. 2010, Chapter 12]) that \(D_{\alpha}(z)\) is an entire function in both \(\alpha\) and \(z\), which satisfies the second order ODE in \(z\),

\[
D''_{\alpha}(z) = \left(\frac{z^2}{4} - \frac{1}{2} - \alpha\right) D_{\alpha}(z).
\]

Therefore, we verify that the function \(\Psi\) defined by (A-5) satisfies the first order matrix differential equation

\[
\Psi'(z) = \frac{iz}{2} \sigma_3 \Psi(z) - i \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix} \Psi(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

Since \(\Psi_+\) and \(\Psi_-\) satisfy the same linear differential equation, there exists \(J_{\Psi}\) independent of \(z\) such that \(\Psi_+(z) = \Psi_-(z) J_{\Psi}\) for \(z \in \mathbb{R}\). Using

\[
D_{i\nu}(0) = \frac{2i\nu}{\Gamma(\frac{1-i\nu}{2})}, \quad D'_{i\nu}(0) = -\frac{\frac{1+i\nu}{\Gamma(\frac{-1-i\nu}{2})}},
\]

we obtain after a computation that

\[
J_{\Psi} = \Psi_-(0)^{-1} \Psi_+(0) = \begin{pmatrix} 1 & -q \\ q & 1 - q^2 \end{pmatrix}
\]

where we have also used (A-3). Using the jumps for \(\Psi\), it is easy to verify that \(\Phi_{\text{PC}}\) defined by (A-4) satisfies the jumps of the RH problem for \(\Phi_{\text{PC}}\). For each \(\delta > 0\), the parabolic cylinder function satisfies the asymptotic formula

\[
D_{\alpha}(z) = z^{\alpha} e^{-\frac{z^2}{4} \left(1 - \frac{a(a - 1)}{2z^2} + O(z^{-4})\right)} \\
-\tilde{c}(z) \frac{\sqrt{2\pi} e^{\frac{z^2}{2} - a - 1}}{\Gamma(-a)} \left(1 + \frac{(a + 1)(a + 2)}{2z^2} + O(z^{-4})\right), \quad z \to \infty, \quad \alpha \in \mathbb{C},
\]

where the error terms are uniform with respect to \(\alpha\) in compact subsets and \(z\) in the given ranges.
Using this formula and the identity
\[ D'_a(z) = \frac{z}{2} D_a(z) - D_{a+1}(z), \]
the asymptotic equation (A-1) follows from a tedious but straightforward computation. This shows that \( \Phi_{\text{PC}} \) given by A.1 satisfies all the conditions of the RH problem for \( \Phi_{\text{PC}} \). \( \square \)

**Formula for \( \beta_{12}\beta_{21} \).** Since \( \nu \in \mathbb{R} \), we note from [Olver et al. 2010, formula 5.4.3] that
\[
|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{\nu(e^{\pi\nu} - e^{-\pi\nu})}} = \frac{\sqrt{2\pi}}{\sqrt{\nu} q^2 e^{\pi\nu}},
\]
from which we deduce the identity
\[
\beta_{12}\beta_{21} = \nu. \tag{A-7}
\]

**Acknowledgements**

Charlier was supported by the European Research Council, Grant Agreement No. 682537. Claeys was supported by the Fonds de la Recherche Scientifique-FNRS under EOS project O013018F. The authors are grateful to Gaultier Lambert and Christian Webb for useful discussions, in particular related to Remark 1.11.

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Received 7 Oct 2020. Revised 1 Dec 2020. Accepted 15 Dec 2020.

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Spectral radius of random matrices with independent entries    221
    Johannes Alt, László Erdős and Torben Krüger

Nonconvex interactions in mean-field spin glasses    281
    Jean-Christophe Mourrat

Convergence of asymptotic costs for random Euclidean matching problems    341
    Michael Goldman and Dario Trevisan

Global rigidity and exponential moments for soft and hard edge point processes    363
    Christophe Charlier and Tom Claeys