On The Algebraic \( L \)-theory of \( \Delta \)-sets

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Abstract: The algebraic \( L \)-groups \( L_\ast(A,X) \) are defined for an additive category \( A \) with chain duality and a \( \Delta \)-set \( X \), and identified with the generalized homology groups \( H_\ast(X;L_\ast(A)) \) of \( X \) with coefficients in the algebraic \( L \)-spectrum \( L_\ast(A) \). Previously such groups had only been defined for simplicial complexes \( X \).

Keywords: Surgery theory, \( \Delta \)-set, \( L \)-groups.

Introduction

A ‘\( \Delta \)-set’ \( X \) in the sense of Rourke and Sanderson \cite{RS} is a simplicial set without degeneracies. A simplicial complex is a \( \Delta \)-set; conversely, the second barycentric (aka derived) subdivision of a \( \Delta \)-set is a simplicial complex, and the homotopy theory of \( \Delta \)-sets is the same as the homotopy theory of simplicial complexes. However, \( \Delta \)-sets are sometimes more convenient than simplicial complexes: they are generally smaller, and the quotient of a \( \Delta \)-set by a group action is again a \( \Delta \)-set. In this paper we extend the algebraic \( L \)-theory of simplicial complexes of Ranicki \cite{R} to \( \Delta \)-sets.

In the original formulation of Wall \cite{W} the surgery obstruction theory of high-dimensional manifolds involved the algebraic \( L \)-groups \( L_\ast(R) \) of a ring with involution \( R \), which are the Witt groups of quadratic forms over \( R \) and their automorphisms. The subsequent development of the theory in \cite{R} viewed \( L_\ast(R) \) as the cobordism groups of \( R \)-module chain complexes with quadratic Poincaré duality, constructed a spectrum \( L_\ast(R) \) with homotopy groups \( L_\ast(R) \), and also introduced the algebraic \( L \)-groups \( L_\ast(R,X) \) of a

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simplicial complex $X$. An element of $L_n(R, X)$ is a cobordism class of directed systems over $X$ of $R$-module chain complexes with an $n$-dimensional quadratic Verdier-type duality. The groups $L_\ast(R, X)$ were identified with the generalized homology groups $H_\ast(X; \mathbb{L}_\ast(R))$, and the algebraic $L$-theory assembly map $A: L_\ast(R, X) \to L_\ast(R[\pi_1(X)])$ was defined and extended to the algebraic surgery exact sequence

$$\cdots \to L_n(R, X) \xrightarrow{A} L_n(R[\pi_1(X)]) \to S_n(R, X) \to L_{n-1}(R, X) \to \cdots$$

with $S_n(R, X)$ the cobordism groups of the $R[\pi_1(X)]$-contractible directed systems. In particular, the 1-connective version gave an algebraic interpretation of the exact sequence of the topological version of the Browder-Novikov-Sullivan-Wall surgery theory: if the polyhedron $\|X\|$ of a finite simplicial complex $X$ has the homotopy type of a closed $n$-dimensional topological manifold then $S_{n+1}(\mathbb{Z}, X)$ is the structure set of closed $n$-dimensional topological manifolds $M$ with a homotopy equivalence $M \simeq \|X\|$.

The Verdier-type duality of [6] used the dual cells in the barycentric subdivision of a simplicial complex $X$ to define the dual of a directed system over $X$ of $R$-modules to be a directed system over $X$ of $R$-module chain complexes. The $\Delta$-set analogues of dual cells introduced by us in Ranicki and Weiss [8] are used here to define a Verdier-type duality for directed systems of $R$-modules over a $\Delta$-set $X$, which is used to define the generalized homology groups $L_\ast(R, X) = H_\ast(X; \mathbb{L}_\ast(R))$ and an algebraic surgery exact sequence as in the simplicial complex case.

The algebraic $L$-theory of $\Delta$-sets is used in Macko and Weiss [5], and its multiplicative properties are investigated in Laures and McClure [3].

1. Functor categories

In this section, $X$ denotes a category with the following property. For every object $x$, the set of morphisms to $x$ (with unspecified source) is finite; moreover, given morphisms $f : y \to x$ and $g : z \to x$ in $X$, there exists at most one morphism $h : y \to z$ such that $gh = f$.

Let $\mathbb{A}$ be an additive category with zero object $0 \in \text{Ob}(\mathbb{A})$.

**Definition 1.1.** (i) A function $M : \text{Ob}(X) \to \text{Ob}(\mathbb{A})$; $x \mapsto M(x)$ is finite if $M(x) = 0$ for all but a finite number of objects $x$ in $\mathbb{A}$.

The direct sum $\sum_{x \in \text{Ob}(X)} M(x)$ will be written as $\sum_{x \in X} M(x)$.

(ii) A functor $F : X \to \mathbb{A}$ is finite if the function $F : \text{Ob}(X) \to \text{Ob}(\mathbb{A})$ is finite.
Definition 1.2. (i) The contravariant functor category $A^*_X$ is the additive category of finite contravariant functors $F : X \to A$. The morphisms in $A^*_X$ are the natural transformations.

(ii) The covariant functor category $A^*_X$ is the additive category of covariant functors $F : X \to A$. The morphisms in $A^*_X$ are the natural transformations. We write $A^*_f[X]$ for the full subcategory whose objects are the finite functors in $A^*_X$. □

Remark 1.3. We use the terminology $A^*_X$ for the covariant functor category because it behaves contravariantly in the variable $X$. Indeed a functor $g : X \to Y$ induces a functor $A^*_Y \to A^*_X$ by composition with $g$. Our reasons for using the terminology $A^*_X$ for the contravariant functor category are similar, but more complicated. Below we introduce a variation denoted $A^*_s(X)$ which behaves covariantly in $X$. □

For the remainder of this section we shall only consider the contravariant functor category $A^*_X$, but every result also has a version for the covariant functor category $A^*_X$ (or $A^*_f[X]$ in some cases).

Definition 1.4. (i) A chain complex in an additive category $A$

$$
\xymatrix{ \ldots & C_{n+1} \ar[d]^d & C_n \ar[l]_d & C_{n-1} \ar[l] \ar[d] & \ldots \quad (d^2 = 0) }
$$

is finite if $C_n = 0$ for all but a finite number of $n \in \mathbb{Z}$.

(ii) Let $B(A)$ be the additive category of finite chain complexes in $A$ and chain maps. □

A finite chain complex $C$ in $A^*_X$ is just an object in $B(A)_s[X]$, and likewise for chain maps, so that

$$B(A^*_X) = B(A)_s[X].$$

Definition 1.5. A chain map $f : C \to D$ of chain complexes in $A^*_X$ is a weak equivalence if each

$$f[x] : C[x] \to D[x] \quad (x \in X)$$

is a chain equivalence in $A$. □

A morphism $f : C \to D$ in $B(A^*_X)$ which is a chain equivalence is also a weak equivalence, but in general a weak equivalence need not be a chain equivalence – see [111] for a more detailed discussion.

Definition 1.6. Let $x$ be an object in $X$.

(i) The under category $x/X$ is the category with objects the morphisms $f : x \to y$ in $X$, and morphisms $g : f \to f'$ the morphisms $g : y \to y'$ in $X$
such that \( gf = f' \)

\[
\begin{array}{c}
\text{x} \\
\downarrow f \quad \downarrow f' \\
\text{y} \quad \text{g} \quad \text{y'} \\
\end{array}
\]

The open star of \( x \) is the set of objects in \( x/X \)

\[ \text{st}(x) = \text{Ob}(x/X) = \{ x \to y \} . \]

(ii) The over category \( X/x \) is the category with morphisms \( f : y \to x \) in \( X \) as its objects, and so that morphisms \( g : f \to f' \) are the morphisms \( g : y \to y' \) in \( X \) such that \( f = f'g \)

\[
\begin{array}{c}
\text{y} \\
\downarrow g \\
\text{x} \quad \text{f} \quad \text{f'} \\
\end{array}
\]

The closure of \( x \) is the set of objects in \( X/x \)

\[ \text{cl}(x) = \text{Ob}(X/x) = \{ y \to x \} . \]

Because of our standing assumptions on \( X \), the over category \( X/x \) is isomorphic to a finite poset. \( \square \)

In the applications of the contravariant functor category \( \mathbb{A}_*[X] \) to topology we shall be particularly concerned with the subcategory of functors satisfying the following property.

**Definition 1.7.** A contravariant functor

\[ F : X \to \mathbb{A} ; \ x \mapsto F[x] \]

in \( \mathbb{A}_*[X] \) is induced if there exists a finite function \( x \mapsto F(x) \in \text{Ob}(\mathbb{A}) \) and a natural isomorphism

\[ F[x] \cong \bigoplus_{x \to y} F(y) . \]

The sum ranges over \( \text{st}(x) \), and since the function \( x \mapsto F(x) \) is finite, \( F[x] \) is only a sum of a finite number of non-zero objects in \( \mathbb{A} \).

Similarly a covariant functor

\[ F : X \to \mathbb{A} ; \ x \mapsto F[x] \]

in \( \mathbb{A}^*[X] \) is induced if there exists a function \( x \mapsto F(x) \in \text{Ob}(\mathbb{A}) \) and a natural isomorphism

\[ F[x] \cong \bigoplus_{y \to x} F(y) . \]

The full subcategories of the functor categories \( \mathbb{A}_*[X] \), respectively \( \mathbb{A}^*[X] \), with objects the induced functors \( F : X \to \mathbb{A} \) are equivalent, as we shall prove below, to the following categories.
Definition 1.8. Let $\mathcal{A}_s(X)$ be the additive category whose objects are functions $x \mapsto F(x)$ such that $F(x) = 0$ for all but a finite number of objects $x$. A morphism $f : E \to F$ in $\mathcal{A}_s(X)$ is a collection of morphisms $f(\phi) : E(x) \to F(y)$ in $\mathcal{A}$, one for each morphism $\phi : x \to y$ in $X$. The composite of the morphisms

$$f = \{f(\phi)\} : M \to N, \ g = \{g(\theta)\} : N \to P$$

is the morphism

$$gf = \{gf(\psi)\} : M \to P$$

with

$$gf(\psi : x \to z) = \sum_{\phi : x \to y, \theta : y \to z, \theta \phi = \psi} g(\theta)f(\phi) : M(x) \to P(z).$$

We can view an object $F$ of $\mathcal{A}_s(X)$ as an object in $\mathcal{A}_s[X]$ by writing

$$F[x] = \bigoplus_{y \to x} F(y).$$

A morphism $\theta : w \to x$ in $X$ induces a morphism $F[x] \to F[w]$ in $\mathcal{A}$ which maps the summand $F(y)$ corresponding to some $\phi : x \to y$ identically to the summand $F(y)$ corresponding to the composition $\phi \theta : w \to y$.

Let $\mathcal{A}^*(X)$ be the additive category whose objects are functions $x \mapsto F(x)$. A morphism $f : E \to F$ in $\mathcal{A}_s(X)$ is a collection of morphisms $f(\phi) : E(y) \to F(x)$ in $\mathcal{A}$, one for each morphism $\phi : x \to y$ in $X$. Again we can view an object $F$ of $\mathcal{A}^*(X)$ as an object in $\mathcal{A}_s[X]$ by writing

$$F[x] = \bigoplus_{y \to x} F(y).$$

Proposition 1.9. (i) For any object $M$ in $\mathcal{A}_s(X)$ and any object $N$ in $\mathcal{A}_s[X]$

$$\text{Hom}_{\mathcal{A}_s[X]}(M, N) = \sum_{x \in X} \text{Hom}_{\mathcal{A}}(M(x), N[x]).$$

(ii) For any objects $L, M$ in $\mathcal{A}_s(X)$

$$\text{Hom}_{\mathcal{A}_s[X]}(L, M) = \sum_{x \to y} \text{Hom}_{\mathcal{A}}(L(x), M(y)).$$

(iii) The additive category $\mathcal{A}_s(X)$ is equivalent to the full subcategory of the contravariant functor category $\mathcal{A}_s[X]$ with objects the induced functors.

Proof. (i) A morphism $f : M \to N$ in $\mathcal{A}_s[X]$ is determined by the composite morphisms in $\mathcal{A}$

$$M(x) \xrightarrow{\text{inclusion}} M[x] \xrightarrow{f[x]} N[x] \ (x \in X)$$
By (i), a morphism $f : L \rightarrow M$ in $\mathbb{A}_* [X]$ is determined by the composite morphisms in $\mathbb{A}$

$$L(x) \xrightarrow{\text{inclusion}} L[x] \xrightarrow{f[x]} M[x] = \sum_{x \rightarrow y} M(y) \ (x \in X).$$

Every object $M$ in $\mathbb{A}_* (X)$ determines an induced contravariant functor $X \rightarrow \mathbb{A} ; x \mapsto M[x] = \sum_{x \rightarrow y} M(y),$ i.e. an object in $\mathbb{A}_*[X],$ and every induced functor is naturally equivalent to one of this type. □

Proposition 1.10. The following conditions on a chain map $f : C \rightarrow D$ in $\mathbb{A}_*(X)$ are equivalent:

(a) $f$ is a chain equivalence,

(b) each of the component chain maps in $\mathbb{A}$

$$f(1_x) : C(x) \rightarrow D(x) \ (x \in X)$$

is a chain equivalence,

(c) $f : C \rightarrow D$ is a weak equivalence in $\mathbb{A}_*[X],$ that is, $C[x] \rightarrow D[x]$ is a chain equivalence for all $x.$

Proof. The proof given in Proposition 2.7 of Ranicki and Weiss [8] in the case when $\mathbb{A}$ is the additive category of $R$-modules (for some ring $R$) works for an arbitrary additive category. □

Remark 1.11. Every chain equivalence of chain complexes in $\mathbb{A}_*[X]$ is a weak equivalence. By 1.10 every weak equivalence of degreewise induced finite chain complexes in $\mathbb{A}_*[X]$ is a chain equivalence. See Ranicki and Weiss [8, 1.13] for an explicit example of a weak equivalence of finite chain complexes in $\mathbb{A}_*[X]$ which is not a chain equivalence. It is proved in [8, 2.9] that every finite chain complex $C$ in $\mathbb{A}_*[X]$ is weakly equivalent to one in $\mathbb{A}_*(X).$ □

2. $\Delta$-sets

Let $\Delta$ be the category with objects the sets

$$[n] = \{0, 1, \ldots, n\} \ (n \geq 0)$$

and morphisms $[m] \rightarrow [n]$ order-preserving injections. Every such morphism has a unique factorization as the composite of the order-preserving injections
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\[ \partial_i : [k-1] \rightarrow [k] ; \ j \mapsto \begin{cases} 
    j & \text{if } j < i \\
    j + 1 & \text{if } j \geq i .
\end{cases} \]

**Definition 2.1.** (Rourke and Sanderson [9]) A Δ-set is a contravariant functor
\[ X : \Delta \rightarrow \{ \text{sets and functions} \} ; [n] \mapsto X^{(n)} . \]

Equivalently, a Δ-set \( X \) can be regarded as a sequence \( X^{(n)} \) (\( n \geq 0 \)) of sets, together with face maps
\[ \partial_i : X^{(n)} \rightarrow X^{(n-1)} \quad (0 \leq i \leq n) \]
such that
\[ \partial_i \partial_j = \partial_j \partial_i \quad \text{for } i < j . \]
The elements \( x \in X^{(n)} \) are the \( n \)-simplices of \( X \).

**Definition 2.2.** (Rourke and Sanderson [9])
(i) The **realization** of a Δ-set \( X \) is the CW complex
\[ \| X \| = \prod_{n=0}^{\infty} (X^{(n)} \times \Delta^n) / \sim \]
with
\[ \Delta^n = \{ (s_0, s_1, \ldots, s_n) \in \mathbb{R}^n \mid 0 \leq s_i \leq 1, \sum_{i=0}^{n} s_i = 1 \} , \]
\[ \partial_i : \Delta^{n-1} \hookrightarrow \Delta^n ; (s_0, s_1, \ldots, s_{n-1}) \mapsto (s_0, s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n) , \]
\[ (x, \partial_i s) \sim (\partial_i x, s) \quad (x \in X^{(n)}, s \in |\Delta^{n-1}|) . \]
(ii) There is one \( n \)-cell \( x(\Delta^n) \subseteq \| X \| \) for each \( n \)-simplex \( x \in X \), with characteristic map
\[ x : \Delta^n \rightarrow \| X \| ; (s_0, s_1, \ldots, s_n) \mapsto (x, (s_0, s_1, \ldots, s_n)) . \]
The **boundary** \( x(\partial \Delta^n) \subseteq \| X \| \) is the image of
\[ \partial \Delta^n = \bigcup_{i=0}^{n} \partial_i |\Delta^{n-1}| \]
\[ = \{ (s_0, s_1, \ldots, s_n) \in \mathbb{R}^n \mid 0 \leq s_i \leq 1, \sum_{i=0}^{n} s_i = 1, s_i = 0 \text{ for some } i \} \]
and the **interior** \( x(\Delta^n) \subseteq \| X \| \) is the image of
\[ \Delta^n = \Delta^n \setminus \partial \Delta^n \]
\[ = \{ (s_0, s_1, \ldots, s_n) \in \mathbb{R}^n \mid 0 < s_i \leq 1, \sum_{i=0}^{n} s_i = 1 \} \subseteq \Delta^n . \]
The characteristic map \( x : \Delta^n \rightarrow \| X \| \) is injective on \( \Delta^n \subseteq \Delta^n . \) \qed
Example 2.3. Let $\Delta^n$ be the $\Delta$-set with 

$$(\Delta^n)^{(m)} = \{ \text{morphisms} \ [m] \to [n] \text{ in } \Delta \} \ (0 \leq m \leq n).$$

The realization $\|\Delta^n\|$ is the geometric $n$-simplex $\Delta^n$ (as in the above definition). It should be clear from the context whether $\Delta^n$ refers to the $\Delta$-set or the geometric realization. □

We regard a $\Delta$-set $X$ as a category, whose objects are the simplices, writing the dimension of an object $x \in X$ as $|x|$, i.e. $|x| = m$ for $x \in X^{(m)}$. A morphism $f : x \to y$ from an $m$-simplex $x$ to an $n$-simplex $y$ is a morphism $f : [m] \to [n]$ in $\Delta$ such that

$$f^*(y) = x \in X^{(m)}.$$

In particular, for any $x \in X^{(m)}$ with $m \geq 1$ there are defined $m + 1$ distinct morphisms in $X$

$$\partial_i : \partial_i x \to x \ (0 \leq i \leq m).$$

Example 2.4. (i) Let $X$ be a $\Delta$-set. An object $M$ of $\mathbb{A}_*(X)$ is just an object $M$ of $\mathbb{A}$ with a direct sum decomposition $M = \bigoplus_{x \in X} M(x)$. A morphism $f : M \to N$ in $\mathbb{A}_*(X)$ is a collection of morphisms $f_{xy,\lambda} : M(x) \to N(y)$, one such for every pair of simplices $x, y$ and face operator $\lambda$ such that $\lambda^* y = x$. We like to think of a morphism $f : M \to N$ in $\mathbb{A}_*(X)$ as a morphism in $\mathbb{A}$ with additional structure. Source and target of that morphism in $\mathbb{A}$ are $M(X) = \bigoplus_x M(x)$ and $N(X) = \bigoplus_x N(x)$, respectively. For simplices $x$ and $y$, the $xy$-component of the morphism $M(X) \to N(X)$ determined by $f$ is

$$\sum_{\lambda} f_{xy,\lambda}$$

where the sum runs over all $\lambda$ such that $\lambda^* y = x$.

(ii) If $X$ is a simplicial complex then a morphism in $\mathbb{A}_*(X)$ is just a morphism $f : M \to N$ in $\mathbb{A}$ between objects with finite direct sum decompositions

$$M = \sum_{x \in X} M(x) , \ N = \sum_{y \in X} N(y)$$

such that the components $f(x, y) : M(x) \to N(y)$ are 0 unless $x \leq y$.

(iii) The description of $\mathbb{A}_*(X)$ in (ii) also applies in the case of a $\Delta$-set $X$ where, for any two simplices $x$ and $y$, there is at most one morphism from $x$ to $y$. In particular it applies when $X = Y'$ is the barycentric subdivision of another $\Delta$-set $Y$, to be defined in the next section. □

Definition 2.5. Let $X$ be a $\Delta$-set, and let $R$ be a ring.

(i) The $R$-coefficient simplicial chain complex of $X$ is the free (left) $R$-module chain complex $\Delta(X; R)$ with

$$d = \sum_{i=0}^n (-)^i \partial_i : \Delta(X; R)_n = R[X^{(n)}] \to \Delta(X; R)_{n-1} = R[X^{(n-1)}].$$
The *R-coefficient homology* of $X$ is the homology of $\Delta(X; R)$

$$H_* (X; R) = \text{H}_*(\Delta(X; R)) = \text{H}_*(\|X\|; R),$$

noting that $\Delta(X; R)$ is the $R$-coefficient cellular chain complex of $\|X\|$.

(ii) Suppose that $R$ is equipped with an involution $R \to R; r \mapsto \overline{r}$ (e.g. the identity for a commutative ring), allowing the definition of the *dual* of an $R$-module $M$ to be the $R$-module $M^* = \text{Mod}_R(\Delta(X; R), R)$

$$\text{Hom}_R(\Delta(X; R), R)$$

The *R-coefficient simplicial cochain complex* of $X$

$$\Delta(X; R)^* = \text{Hom}_R(\Delta(X; R), R)$$

is the $R$-module cochain complex with

$$d^n = \sum_{i=0}^{n+1} (-1)^i \partial_i^n : \Delta(X; R)^n = R[X^{(n)}]^* \to \Delta(X; R)^{n+1} = R[X^{(n+1)}]^*,$$

The *R-coefficient cohomology* of $X$ is the cohomology of $\Delta(X; R)^*$

$$H^*(X; R) = H^*(\Delta(X; R)^*) = H^*(\|X\|; R),$$

noting that $\Delta(X; R)^*$ is the $R$-coefficient cellular cochain complex of $\|X\|$.

A simplicial complex $X$ is *ordered* if the vertices in any simplex are ordered, with faces having compatible orderings. From now on, in dealing with simplicial complexes we shall always assume an ordering.

**Example 2.6.** A simplicial complex $X$ can be regarded as a $\Delta$-set, with $X^{(n)}$ the set of $n$-simplices and

$$\partial_i : X^{(n)} \to X^{(n-1)}; (v_0v_1 \ldots v_n) \mapsto (v_0v_1 \ldots v_{i-1}v_{i+1} \ldots v_n).$$

There is one morphism $x \to y$ in $X$ for each face inclusion $x \leq y$. The realization $\|X\|$ of $X$ regarded as a $\Delta$-set is the polyhedron of the simplicial complex $X$, with the characteristic maps $x : \Delta^{[x]} \to \|X\|$ ($x \in X$) injections. The simplicial chain complex $\Delta(X; R)$ is just the usual $R$-coefficient simplicial chain complex of $X$, and $\Delta(X; R)^*$ is the $R$-coefficient simplicial cochain complex of $X$.

**Example 2.7.** Let $X$ be a $\Delta$-set, and let $x \in X$ be a simplex.

(i) In general, the canonical map

$$\text{Ob}(x/X) = \text{st}(x) \to \text{Ob}(X); (x \to y) \mapsto y$$

is not injective. The simplices $y \in \text{Ob}(X) \setminus \text{im} (\text{st}(x))$ are the objects of a sub-$\Delta$-set $X \setminus \text{im} (\text{st}(x)) \subset X$. If $X$ is a simplicial complex then $\text{st}(x) \to \text{Ob}(X)$ is injective, and $X \setminus \text{st}(x) \subset X$ is the subcomplex with simplices $y \in X$ such
that $x \not\leq y$.

(ii) The over category $X/x = \{ y \to x \}$ \textsuperscript{(1.6)} is a $\Delta$-set with

$$(X/x)^{(n)} = \{ y \to x \mid y \in X^{(n)} \} \quad (n \geq 0).$$

It is isomorphic as a $\Delta$-set to $\Delta^{|x|}$. The forgetful functor

$$X/x \to X \; ; \; (y \to x) \mapsto y$$

is a $\Delta$-map, inducing the characteristic map $\Delta^{|x|} \to \|X\|$. If $X$ is a simplicial complex then $X/x \to X$ is injective, and so is the induced characteristic map. \hfill \Box

\textbf{Example 2.8.} (i) If a group $G$ acts on a $\Delta$-set $X$ the quotient $X/G$ is again a $\Delta$-set, with realization $\|X/G\| = \|X\|/G$. However, if $X$ is a simplicial complex and $G$ acts on $X$, then $X/G$ is not in general a simplicial complex. See (ii) for an example.

(ii) Suppose $X = \mathbb{R}$, the $\Delta$-set with

$$X^{(0)} = X^{(1)} = \mathbb{Z} , \quad \partial_0(n) = n , \quad \partial_1(n) = n + 1 ,$$

and let the infinite cyclic group $G = \mathbb{Z} = \{ t \}$ act on $X$ by $tn = n + 1$.

The quotient $\Delta$-set $S^1 = \mathbb{R}/\mathbb{Z}$ is the circle, with one 0-simplex $x_0$ and one 1-simplex $x_1$

$$(S^1)^{(0)} = \{ x_0 \}, \quad (S^1)^{(1)} = \{ x_1 \}, \quad \partial_0(x_1) = \partial_1(x_1) = x_0 .$$

\hfill \Box

\textbf{Example 2.9.} For any space $M$ use the standard $n$-simplices $\Delta^n$ and face inclusions $\partial_i : \Delta^{n-1} \hookrightarrow \Delta^n$ to define the \textit{singular} $\Delta$-set $X = M^{\Delta}$ by

$$X^{(n)} = M^{\Delta^n} , \quad \partial_i : X^{(n)} \to X^{(n-1)} ; \; x \mapsto x \circ \partial_i .$$

We shall say that a singular simplex $x : \Delta^n \to X$ is a face of a singular simplex $y : \Delta^m \to X$ if $x = y \circ \partial_{i_1} \circ \cdots \circ \partial_{i_{m-n}}$ for a given face inclusion

$$\partial_{i_1} \circ \cdots \circ \partial_{i_{m-n}} : \Delta^n \hookrightarrow \Delta^m ,$$

writing $x \leq y$ (and $x < y$ if $x \neq y$). The simplicial chain complex $\Delta(X; R) = S(M; R)$ is just the usual $R$-coefficient singular chain complex of $M$, so that

$$H_*(\|X\|; R) = H_*(X; R) = H_*(M; R) .$$

Also $\Delta(X; R)^* = S(M; R)^*$ is the $R$-coefficient singular cochain complex of $M$, and

$$H^*(\|X\|; R) = H^*(X; R) = H^*(M; R) .$$

\hfill \Box
3. The barycentric subdivision

The Δ-set analogue of the barycentric subdivision $X'$ of a simplicial complex $X$ and the dual cells $D(x, X) \subset X'$ ($x \in X$) makes use of the following standard categorical construction.

**Definition 3.1.** (i) The nerve of a category $\mathcal{C}$ is the simplicial set with one $n$-simplex for each string $x_0 \to x_1 \to \cdots \to x_n$ of morphisms in $\mathcal{C}$, with
\[
\partial_i(x_0 \to x_1 \to \cdots \to x_n) = (x_0 \to x_1 \to \cdots \to x_{i-1} \to x_{i+1} \to \cdots \to x_n).
\]
(ii) An $n$-simplex $x_0 \to x_1 \to \cdots \to x_n$ in the nerve is non-degenerate if none of the morphisms $x_i \to x_{i+1}$ is the identity.

If the category $\mathcal{C}$ has the property that the composite of non-identity morphisms is a non-identity, then the non-degenerate simplices in the nerve define a Δ-set, which we shall also call the nerve and denote by $\mathcal{C}$. **Definition 3.2.** (Rourke and Sanderson [9, §4], Ranicki and Weiss [8, 1.6, 1.7]) Let $X$ be a Δ-set.
(i) The barycentric subdivision of $X$ is the Δ-set $X'$ defined by the nerve of the category $X$.
(ii) The dual $x^\perp$ of a simplex $x \in X$ is the nerve of the under category $x/X$ [1.6]. An $n$-simplex in the Δ-set $x^\perp$ is thus a sequence of morphisms in $X$
\[
x \to x_0 \to x_1 \to \cdots \to x_n
\]
such that $x_0 \to x_1 \to \cdots \to x_n$ is non-degenerate. In particular
\[
(x^\perp)^{(0)} = \{x \to x_0\} = \text{st}(x).
\]
(iii) The boundary of the dual $\partial x^\perp$ is the sub-Δ-set of $x^\perp$ consisting of the $n$-simplices $x \to x_0 \to x_1 \to \cdots \to x_n$ such that $x \to x_0$ is not the identity.

The under category $x/X$ has an initial object, so that the nerve $x^\perp$ is contractible. The rule $x \to x^\perp$ is contravariant, i.e. every morphism $x \to y$ induces a Δ-map $y^\perp \to x^\perp$.

**Lemma 3.3.** The realizations $\|X\|, \|X'\|$ of a Δ-set $X$ and its barycentric subdivision $X'$ are homeomorphic, via a homeomorphism $\|X'\| \to \|X\|$ sending the vertex $x \in X = (X')^{(0)}$ to the barycentre
\[
\hat{x} = x\left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \in x(\Delta^n) \subseteq \|X\|.
\]

**Proof.** It suffices to consider the special case $X = \Delta^n$, so that $X$ and $X'$ are simplicial complexes, and to define a homeomorphism $\|X'\| \to \|X\|$ by $x \mapsto \hat{x}$ and extending linearly. □
Definition 3.4. Let $X$ be a $\Delta$-set, and let $x \in X$ be a simplex.

(i) The open star space

\[ \| \text{st}(x) \| = \bigcup_{y \in x^\perp \setminus \partial x^\perp} \Delta^{|y|} \subseteq \| X' \| = \| X \| \]

is the subspace of the realization $\| X' \|$ of the barycentric subdivision $X'$ defined by the union of the interiors of the simplices $y \in x^\perp \setminus \partial x^\perp$, i.e.

\[ y = (x \to x_0 \to \cdots \to x_n) \in X' \]

with $x \to x_0 = x$ the identity.

(ii) The homology of the open star is

\[ H_*(\text{st}(x)) = H_*(\Delta(\text{st}(x))) \]

with $\Delta(\text{st}(x))$ the chain complex defined by

\[ \Delta(\text{st}(x)) = \Delta(x^\perp, \partial x^\perp)_{* - |x|} . \]

Lemma 3.5. For any simplex $x \in X$ of a $\Delta$-set $X$ the characteristic $\Delta$-map

\[ i : x^\perp \to X' ; (x \to x_0 \to \cdots \to x_n) \mapsto (x_0 \to \cdots \to x_n) \]

is injective on $x^\perp \setminus \partial x^\perp$. The images $i(\partial x^\perp), i(x^\perp) \subseteq X'$ are sub-$\Delta$-sets such that

\[ \| i(x^\perp) \| \cap \| i(\partial x^\perp) \| = \| \text{st}(x) \| \subseteq \| X' \| \]

and there are homology isomorphisms

\[ H_*(\text{st}(x)) = H_{*-|x|}(x^\perp, \partial x^\perp) \]

\[ \cong H_{*-|x|}(i(x^\perp), i(\partial x^\perp)) \]

\[ \cong H_*(\| X \|, \| X \| \setminus \| \text{st}(x) \|) \]

\[ \cong H_*(\| X \|, \| X \| \setminus \{ \hat{x} \}) . \]

Proof. The inclusion $\| X \|, \| X \| \setminus \| \text{st}(x) \|) \mapsto (\| X \|, \| X \| \setminus \{ \hat{x} \})$ is a deformation retraction, and the open star subspace $\| \text{st}(x) \| \subset \| X \|$ has an open regular neighbourhood

\[ \| \text{st}(x) \| \times \Delta^{|x|} \subset \| X \| \]

with one-point compactification

\[ \| \text{st}(x) \| \times \Delta^{|x|}\infty = \| i(x^\perp) \| \cap \| i(\partial x^\perp) \| \setminus \Delta^{|x|} / \partial \Delta^{|x|} , \]

so that

\[ H_*(\| X \|, \| X \| \setminus \{ \hat{x} \}) \cong H_*(\| X \|, \| X \| \setminus \| \text{st}(x) \|) \]

\[ \cong \tilde{H}_*(\| i(x^\perp) \| \cap \| i(\partial x^\perp) \| \setminus \Delta^{|x|} / \partial \Delta^{|x|}) \]

\[ \cong H_{*-|x|}(i(x^\perp), i(\partial x^\perp)) . \]
Example 3.6. Let $X$ be a simplicial complex. The barycentric subdivision of $X$ is the ordered simplicial complex $X'$ with one $n$-simplex for each sequence of proper face inclusions $x_0 < x_1 < \cdots < x_n$. By definition, the *dual cell* of a simplex $x \in X$ is the subcomplex $D(x,X) \subseteq X'$ consisting of all the simplices $x_0 < x_1 < \cdots < x_n$ with $x \leq x_0$. The *boundary* of the dual cell is the subcomplex $\partial D(x,X) \subseteq D(x,X)$ consisting of all the simplices $x_0 < x_1 < \cdots < x_n$ with $x < x_0$. The $\Delta$-sets associated to $X, X', D(x,X), \partial D(x,X)$ are just the $\Delta$-sets $X, X', x^\perp, \partial x^\perp$ of 3.2, with the characteristic map $i : x^\perp = D(x,X) \to X'$ injective. Moreover, $X \setminus \text{st}(x) \subset X$ is a subcomplex such that $$\|X \setminus \text{st}(x)\| = \|X\|\|\text{st}(x)\|$$ and $$\Delta(\text{st}(x)) = \Delta(D(x,X), \partial D(x,X)) \simeq \Delta(X, X \setminus \text{st}(x)).$$

Example 3.7. Let $X$ be the $\Delta$-set (2.8) with one 0-simplex $x_0$ and one 1-simplex $x_1$, with non-identity morphisms

and realization $\|X\| = S^1$. The barycentric subdivision $X'$ is the $\Delta$-set with 2 0-simplices and 2 1-simplices:

$$X'^{(0)} = \{ x_0, x_1 \}, \quad X'^{(1)} = \{ \xymatrix{ x_0 \ar@{<-}[r] & x_1 } \}. $$

The duals and their boundaries are given by

$$x_0^\perp = \{ \xymatrix{ x_0 \ar@{<-}[r] & x_0, \quad x_0 \ar@{<-}[r] & x_1 } \} \cup \{ \xymatrix{ x_0 \ar@{<-}[r] & x_0, \quad x_0 \ar@{<-}[r] & x_1 } \},$$

$$\partial x_0^\perp = \{ \xymatrix{ x_0 \ar@{<-}[r] & x_1 } \} = \{ 0, 1 \},$$

$$x_1^\perp = \{ \xymatrix{ x_1 \ar@{<-}[r] & x_1 } \}, \quad \partial x_1^\perp = \emptyset.$$

The characteristic map $i : x_0^\perp \to X'$ is surjective but not injective, and

$$H_n(x_0^\perp, \partial x_0^\perp) = H_n(i(x_0^\perp), i(\partial x_0^\perp)) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

Example 3.8. Let $X$ be the contractible $\Delta$-set with one 0-simplex $x_0$, one 1-simplex $x_1$ and one 2-simplex $x_2$, with non-identity morphisms

$$\xymatrix{ x_0 \ar@{<-}[r] & x_1, \quad x_1 \ar@{<-}[r] & x_2, \quad x_0 \ar@{<-}[r] & x_2 \}$$
The realization $\|X\|$ is the dunce hat (Zeeman [14]). The barycentric subdivision $X'$ is the $\Delta$-set with three 0-simplices, eight 1-simplices and six 2-simplices:

- $X'(0) = \{x_0, x_1, x_2\}$,
- $X'(1) = \{x_0 \rightarrow x_1\} \cup \{x_1 \rightarrow x_2\} \cup \{x_0 \rightarrow x_2\}$
- $X'(2) = \{x_0 \rightarrow x_1 \rightarrow x_2\}$

The duals and their boundaries are given by

- $x_0^\perp = \{x_0 \rightarrow x_0 \rightarrow x_1 \rightarrow x_0 \rightarrow x_2\}$
- $\partial x_0^\perp = \{x_0 \rightarrow x_1 \rightarrow x_2\} \cup \{x_0 \rightarrow x_1 \rightarrow x_2\}$, $\|\partial x_0^\perp\| \simeq S^1 \vee S^1$,
- $x_1^\perp = \{x_1 \rightarrow x_1 \rightarrow x_2\} \cup \{x_1 \rightarrow x_1 \rightarrow x_2\}$
- $\partial x_1^\perp = \{x_1 \rightarrow x_2\}$, $\|\partial x_1^\perp\| \simeq \{0, 1, 2\}$
- $x_2^\perp = \{x_2 \rightarrow x_2\}$, $\partial x_2^\perp = \emptyset$.

The characteristic map $i : x_0^\perp \rightarrow X'$ is surjective but not injective, with

$$\|i(x_0^\perp)\| \simeq \{\ast\}, \|i(\partial x_0^\perp)\| \simeq S^1 \vee S^1$$

and

$$H_n(x_0^\perp, \partial x_0^\perp) = H_n(i(x_0^\perp), i(\partial x_0^\perp)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$$
The characteristic map $i : x^\perp_1 \to X'$ is neither surjective nor injective, with
$$\|i(x^\perp_1)\| \simeq S^1 \vee S^1, \quad \|i(\partial x^\perp_1)\| \simeq \{*\}$$
and
$$H_n(x^\perp_1, \partial x^\perp_1) = H_n(i(x^\perp_1), i(\partial x^\perp_1)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}.$$

Definition 3.9. Given a ring $R$ let Mod($R$) be the additive category of left $R$-modules. For $R = \mathbb{Z}$ write Mod($\mathbb{Z}$) = Ab, as usual.

Definition 3.10. (Ranicki and Weiss [8, 1.9] for simplicial complexes)
(i) The $R$-coefficient simplicial chain complex $\Delta(X'; R)$ of the barycentric subdivision $X'$ of a finite $\Delta$-set $X$ is the chain complex in Mod($R$)$_\ast(X)$ with
$$\Delta(X'; R)(x) = \Delta(x^\perp, \partial x^\perp; R) , \quad \Delta(X'; R)[x] = \Delta(x^\perp; R).$$
Compare example 2.4 case (iii).
(ii) Let $f : Y \to X'$ be a $\Delta$-map from a finite $\Delta$-set $Y$ to the barycentric subdivision $X'$ of a $\Delta$-set $X$. The $R$-coefficient simplicial chain complex $\Delta(Y; R)$ is the chain complex in Mod($R$)$_\ast(X)$ with
$$\Delta(Y; R)(x) = \Delta(x/f, \partial(x/f); R) , \quad \Delta(Y; R)[x] = \Delta(x/f; R) \quad (x \in X)$$
with $x/f, \partial(x/f)$ the $\Delta$-sets defined to fit into strict pullback squares of $\Delta$-sets

\[
\begin{array}{ccc}
\partial(x/f) & \longrightarrow & x/f \\
\downarrow & & \downarrow f \\
\partial x^\perp & \longrightarrow & x^\perp \\
\downarrow & & \downarrow \\
x^\perp & \longrightarrow & i \\
\end{array}
\]

\[
\begin{array}{ccc}
Y & \longrightarrow & X' \\
\downarrow & & \downarrow \\
f & \longrightarrow & \\
\end{array}
\]

□

4. THE TOTAL COMPLEX

For a finite chain complex $C$ in $\mathbb{A}_\ast[X]$, there is defined a chain complex in $\mathbb{A}_\ast(X)$, called the total complex of $C$.

Definition 4.1. The total complex $\text{Tot}_\ast C$ of a finite chain complex $C$ in $\mathbb{A}_\ast[X]$ is the finite chain complex in $\mathbb{A}_\ast(X)$ given by
$$(\text{Tot}_\ast C)(x)_n = C[x]_{n-|x|}$$
with differential $d = d_{C[x]} + \sum_{i=0}^{\lfloor x \rfloor} (-i+|x|)C(\partial_i x \to x).$ The construction is natural, defining a covariant functor
$$\mathbb{B}(\mathbb{A}_\ast[X]) \to \mathbb{B}(\mathbb{A}_\ast)^\ast(X) ; \ C \mapsto \text{Tot}_\ast C.$$
Remark 4.2. There is a forgetful functor $\mathbb{B}(\mathbb{A})^* f(X) \to \mathbb{B}(\mathbb{A})$ taking $C$ in $\mathbb{B}(\mathbb{A})^* f(X)$ to
\[ C(X) = \bigoplus_{x \in X} C(x) . \]

Compare example 2.4. The chain complex $(\text{Tot}_* C)(X)$ in $\mathbb{A}$ is the ‘realization’
\[ \left( \sum_{x \in X} \Delta(\Delta^{|x|}) \otimes \mathbb{Z} C[x] \right) / \sim \]
with $\sim$ the equivalence relation generated by $a \otimes \lambda^* b \sim \lambda_* a \otimes b$ for a morphism $\lambda : y \to z$ in $X$, with $\lambda \in \Delta(\Delta^{|y|})$, $b \in C[z]$. \hfill \square

Example 4.3. The simplicial chain complex $\Delta(X)$ of a finite $\Delta$-set $X$ is $(\text{Tot}_* C)(X)$ for the chain complex $C$ in $\text{Ab}_\ast [X]$ defined by $C[x] = \mathbb{Z}$ for all $x$ (a constant functor). \hfill \square

Remark 4.4. There are evident forgetful functors
\[ \mathbb{B}(\mathbb{A})_* (X) \to \mathbb{B}(\mathbb{A}) ; \ C \mapsto C(X) , \]
\[ \mathbb{B}(\mathbb{A})^*_f (X) \to \mathbb{B}(\mathbb{A}) ; \ C \mapsto C(X) . \]

The diagram
\[ \mathbb{B}(\mathbb{A})_* (X) \xrightarrow{\text{Tot}_*} \mathbb{B}(\mathbb{A})_* [X] \xrightarrow{\text{Tot}_*} \mathbb{B}(\mathbb{A})^*_f (X) \]
\[ \xrightarrow{\mathbb{B}(\mathbb{A})} \]
commutes up to natural chain homotopy equivalence: for any finite chain complex $C$ in $\mathbb{A}_\ast (X)$
\[ (\text{Tot}_* C)(X)_n = \sum_{x \in X} \sum_{x \to y} C(y)_{n-|x|} = \sum_{y \in X} \left( \Delta(X/y) \otimes \mathbb{Z} C(y) \right)_n \]
with $X/y$ the $\Delta$-set defined in 2.7, which is contractible. \hfill \square

Proposition 4.5. (i) For any objects $M, N$ in $\mathbb{A}_\ast (X)$ the abelian group $\text{Hom}_{\mathbb{A}_\ast (X)}(M, N)$ is naturally an object in $\text{Ab}_\ast f(X)$, with
\[ \text{Hom}_{\mathbb{A}_\ast (X)}(M, N)(x) = \text{Hom}_{\mathbb{A}}(M(x), [N][x]) \]
\[ = \sum_{x \to y} \text{Hom}_{\mathbb{A}}(M(x), N(y)) \ (x \in X) . \]

If $f : M' \to M$, $g : N \to N'$ are morphisms in $\mathbb{A}_\ast (X)$ there is induced a morphism in $\text{Ab}_\ast (X)$
\[ \text{Hom}_{\mathbb{A}_\ast (X)}(M, N) \to \text{Hom}_{\mathbb{A}_\ast (X)}(M', N') ; \ h \mapsto ghf . \]
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(ii) For any objects $M, N$ in $\mathbb{A}^*_*(X)$ the abelian group $\text{Hom}_{\mathbb{A}^*_*(X)}(M, N)$ is naturally an object in $\text{Ab}_*(X)$, with

$$\text{Hom}_{\mathbb{A}^*_*(X)}(M, N)(x) = \text{Hom}_{\mathbb{A}^*}(M(x), [N][x]) = \sum_{y \rightarrow x} \text{Hom}_{\mathbb{A}^*}(M(x), N(y)) \quad (x \in X).$$

Naturality as in (i).

Proof. Immediate from 1.9.

Example 4.6. (i) For a chain complex $C$ in $\text{Ab}_*(X)$ the total complex in $\text{Ab}_*(X)$ of the corresponding chain complex $[C]$ in $\text{Ab}_*[X]$ is given by

$$[C]^*[X] = \text{Hom}_{\text{Ab}_*[X]}(\Delta(X)^{-*}, C).$$

(ii) For a chain complex $D$ in $\text{Ab}^*(X)$ the total complex in $\text{Ab}_*(X)$ of the corresponding chain complex $[D]$ in $\text{Ab}^*[X]$ is given by

$$[D]^*[X] = \text{Hom}_{\text{Ab}^*[X]}(\Delta(X), D).$$

5. Chain duality in $L$-theory

In general, it is not possible to extend an involution $T : \mathbb{A} \rightarrow \mathbb{A}$ on an additive category $\mathbb{A}$ to the functor category $\mathbb{A}_*(X)$ for an arbitrary category $X$. An object in $\mathbb{A}_*(X)$ is an induced contravariant functor $F : X \rightarrow \mathbb{A}$ and the composite of the contravariant functors

$$X \xrightarrow{F} \mathbb{A} \xrightarrow{T} \mathbb{A}$$

is a covariant functor, not a contravariant functor, let alone an induced contravariant functor. A ‘chain duality’ on $\mathbb{A}$ is essentially an involution on the derived category of finite chain complexes and chain homotopy classes of chain maps; an involution on $\mathbb{A}$ is an example of a chain duality. Given a chain duality on $\mathbb{A}$ we shall now define a chain duality on the induced functor category $\mathbb{A}_*(X)$, for any $\Delta$-set $X$, essentially in the same way as was carried out for a simplicial complex $X$ in [6].

Definition 5.1. (Ranicki [6, 1.1]) A chain duality $(T, e)$ on an additive category $\mathbb{A}$ is a contravariant additive functor

$$T : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$$

together with a natural transformation

$$e : T^2 \rightarrow 1 : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$$

such that for each object $M$ in $\mathbb{A}$
(i) \( e(T(M)) \circ T(e(M)) = 1 : T(M) \to T^3(M) \to T(M) \),
(ii) \( e(M) : T^2(M) \to M \) is a chain equivalence.

\[\square\]

A chain duality \((T, e)\) on \(\mathbb{A}\) extends to a contravariant functor on the bounded chain complex category

\[T : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A}) ; C \mapsto T(C)\]

using the double complex construction with

\[T(C)_n = \sum_{p+q=n} T(C_{-p})_q, \quad d_{T(C)} = d_{T(C_{-p})} + (-)^q T(d : C_{-p+1} \to C_{-p})\],

and \(e(C) : T^2(C) \to C\) a chain equivalence. For any objects \(M, N\) in an additive category \(\mathbb{A}\) there is defined a \(\mathbb{Z}\)-module \(\text{Hom}_\mathbb{A}(M, N)\). Thus for any chain complexes \(C, D\) in \(\mathbb{A}\) there is defined a \(\mathbb{Z}\)-module chain complex \(\text{Hom}_\mathbb{A}(C, D)\), with

\[\text{Hom}_\mathbb{A}(C, D)_n = \sum_{q-p=n} \text{Hom}_\mathbb{A}(C_p, D_q), \quad d_{\text{Hom}_\mathbb{A}(C, D)}(f) = df + (-)^q fd_C\].

If \((T, e)\) is a chain duality on \(\mathbb{A}\) there is defined a \(\mathbb{Z}\)-module chain map

\[\text{Hom}_\mathbb{A}(TC, D) \to \text{Hom}_\mathbb{A}(TD, C) ; f \mapsto e(C)T(f)\]

which is a chain equivalence for finite \(C\).

**Example 5.2.** An involution \((T, e)\) on \(\mathbb{A}\) is a contravariant functor \(T : \mathbb{A} \to \mathbb{A}\) with a natural equivalence \(e : T^2 \to 1\) such that for each object \(M\) in \(\mathbb{A}\)

\[e(T(M)) = T(e(M)^{-1}) : T^3(M) \to T(M)\).

This is essentially the same as a chain duality \((T, e)\) such that \(T(M)\) is a 0-dimensional chain complex for each object \(M\) in \(\mathbb{A}\). \(\square\)

**Definition 5.3.** A chain product \((\otimes_\mathbb{A}, b)\) on an additive category \(\mathbb{A}\) is a natural pairing

\[\otimes_\mathbb{A} : \text{Ob}(\mathbb{A}) \times \text{Ob}(\mathbb{A}) \to \{\text{\(\mathbb{Z}\)-module chain complexes}\} ; (M, N) \mapsto M \otimes_\mathbb{A} N\]

together with a natural chain equivalence

\[b(M, N) : M \otimes_\mathbb{A} N \to N \otimes_\mathbb{A} M\]

such that up to natural isomorphism

\[(M \oplus M') \otimes_\mathbb{A} N = (M \otimes_\mathbb{A} N) \oplus (M' \otimes_\mathbb{A} N),\]

\[M \otimes_\mathbb{A} (N \oplus N') = (M \otimes_\mathbb{A} N) \oplus (M \otimes_\mathbb{A} N')\]

and

\[b(N, M) \circ b(M, N) \simeq 1 : M \otimes_\mathbb{A} N \to M \otimes_\mathbb{A} N.\]

\(\square\)
Remark 5.4. The notion of chain product is a linear version of an ‘SW-product’ in the sense of Weiss and Williams [12], where SW = Spanier-Whitehead.

Given an additive category $\mathcal{A}$ with a chain product $(\otimes_{\mathcal{A}}, b)$ and chain complexes $C, D$ in $\mathcal{A}$ let $C \otimes_{\mathcal{A}} D$ be the $\mathbb{Z}$-module chain complex defined by

$$(C \otimes_{\mathcal{A}} D)_n = \sum_{p+q+r=n} (C_p \otimes_{\mathcal{A}} D_q)_r ,$$

$$d_{C \otimes_{\mathcal{A}} D} = d_{C_p \otimes_{\mathcal{A}} D_q} + (-)^r (1 \otimes_{\mathcal{A}} d_D + (-)^q d_{C} \otimes_{\mathcal{A}} 1) .$$

By the naturality of $b$ there is defined a natural chain equivalence

$$b(C, D) : C \otimes_{\mathcal{A}} D \rightarrow D \otimes_{\mathcal{A}} C .$$

Proposition 5.5. Let $\mathcal{A}$ be an additive category.

(i) A chain duality $(T, e)$ on $\mathcal{A}$ determines a chain product $(\otimes_{\mathcal{A}}, b)$ on $\mathcal{A}$ by

$$M \otimes_{\mathcal{A}} N = \text{Hom}_{\mathcal{A}}(TM, N) ,$$

$$b(M, N) : M \otimes_{\mathcal{A}} N \rightarrow N \otimes_{\mathcal{A}} M ;$$

$$(f : TM \rightarrow N) \mapsto (e(M) \circ T(f) : TN \rightarrow T^2 M \rightarrow M) .$$

(ii) If $(\otimes_{\mathcal{A}}, b)$ is a chain product on $\mathcal{A}$ such that

$$M \otimes_{\mathcal{A}} N = \text{Hom}_{\mathcal{A}}(TM, N) , b(M, N)(f) = e(M) \circ T(f)$$

for some contravariant additive functor $T : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{A})$ and natural transformation $e : T^2 \rightarrow 1 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{A})$, then $(T, e)$ is a chain duality on $\mathcal{A}$.

Proof. Immediate from the definitions. $\square$

Example 5.6. Let $R$ be a ring with an involution $R \rightarrow R; r \mapsto \overline{r}$. Regard a (left) $R$-module $M$ as a right $R$-module by

$$M \times R \rightarrow M ; (x, r) \mapsto \overline{r} x .$$

Thus for any $R$-modules $M, N$ there is defined a $\mathbb{Z}$-module

$$M \otimes_R N = (M \otimes_{\mathbb{Z}} N)/\{\overline{r} x \otimes y - x \otimes \overline{r} y \mid x \in M, y \in N, r \in R\}$$

with a natural isomorphism

$$b(M, N) : M \otimes_R N \rightarrow N \otimes_R M ; x \otimes y \mapsto y \otimes x$$

defining a (0-dimensional) chain product $(\otimes_R, b)$ on the $R$-module category $\text{Mod}(R)$. As in 2.3 use the involution on $R$ to define the contravariant duality functor

$$T : \text{Mod}(R) \rightarrow \text{Mod}(R) ; M \mapsto M^* = \text{Hom}_R(M, R)$$

with

$$R \times M^* \rightarrow M^* ; (r, f) \mapsto (x \mapsto f(x)\overline{r}) .$$
The natural $\mathbb{Z}$-module morphism defined for any $R$-modules $M, N$ by
\[
M \otimes_R N \to \text{Hom}_R(M^*, N) ; \quad x \otimes y \mapsto (f \mapsto f(x)y)
\]
is an isomorphism for f.g. projective $M$. The $R$-module morphism defined for any $R$-module $M$ by
\[
e'(M) : M \to M^{**} ; \quad x \mapsto (f \mapsto f(x))
\]
is an isomorphism for f.g. projective $M$. Let $\text{Proj}(R) \subset \text{Mod}(R)$ be the full subcategory of f.g. projective $R$-modules. The natural isomorphisms
\[
e(M) = e'(M)^{-1} : M^{**} \to M
\]
define an involution $(T, e)$ on $\text{Proj}(R)$, corresponding to the restriction to $\text{Proj}(R)$ of the chain product $(\otimes_R, b)$ on $\text{Mod}(R)$. □

Proposition 5.7. (Ranicki [6, 5.1, 5.9, 7], Weiss [11, 1.5])
A chain duality $(T_A, e_A)$ on an additive category $A$ extends to a chain duality $(T_A^\ast(X), e_A^\ast(X))$ on $A^\ast(X)$, for any $\Delta$-set $X$.

\[
T_{A^\ast(X)} : A_*^\ast(X) \xrightarrow{T_*} \text{Tot}_A^\ast(X) \xrightarrow{T_A} B(A)_e^\ast(X)
\]
where $T_A : B(A)^\ast_e(X) \to B(A)_e^\ast(X)$ is the extension of the contravariant functor
\[
T_A : A^\ast_e(X) \to B(A)_e^\ast(X) ; \quad M = \sum_{x \in X} M(x) \mapsto T_A(M) = \sum_{x \in X} T_A(M(x)) .
\]

More explicitly, the chain dual of a finite chain complex $C$ in $A_*^\ast(X)$ is given by
\[
T_{A_*^\ast(X)}(C) = T_A(T_*C) ,
\]
so that
\[
T_{A_*^\ast(X)}(C)(x) = T_A(C[x]_{s-|x|})
= \sum_{x \to y} T_A(C(y)_{s-|x|}) (x \in X) .
\]

Example 5.8. Let $A = A(\mathbb{Z})$, the additive category of f.g. free abelian groups.
(i) For any finite chain complex $C$ in $A_*^\ast(X)$, which we also view as a (degreewise) induced chain complex $C$ in $A_*^\ast(X)$, the total complex $\text{Tot}_A^\ast(C)$ is given by [4,6] to be
\[
\text{Hom}_{A_*^\ast(X)}(\Delta(X)^{-s}, C) ,
\]
so that the chain dual of $C$ is given by
\[
T_{A_*^\ast(X)}(C) = \text{Hom}_A(\text{Hom}_{A_*^\ast(X)}(\Delta(X)^{-s}, C), \mathbb{Z}) .
\]
(ii) As in [3,10] regard the simplicial chain complex $\Delta(X')$ of the barycentric subdivision $X'$ of a finite $\Delta$-set $X$ as a chain complex in $A_*^\ast(X)$ or in $A_*[X]$ with
\[
\Delta(X')(x) = \Delta(x^+, \partial x^+) , \quad \Delta(X')[x] = \Delta(x^+)
\]
for $x \in X$. The chain dual $T(\Delta(X'))$ is the chain complex in $\mathbb{A}_*(X)$ with

$$T(\Delta(X'))(x) = \Delta(x^{\perp})|x|^{-*} \quad (x \in X).$$

\[ \square \]

**Remark 5.9.** See Fimmel [2] and Woolf [13] for Verdier duality for local coefficient systems on simplicial sets and simplicial complexes. In particular, [13] relates the chain duality of [6, Chapter 5] defined on $\text{Proj}(R)_*(X)$ for a simplicial complex $X$ to the Verdier duality for sheaves of $R$-module chain complexes over the polyhedron $\|X\|$. \[ \square \]

For any additive category with chain duality $\mathbb{A}$ let $L_\bullet(\mathbb{A})$ be the quadratic $L$-theory $\Omega$-spectrum defined in Ranicki [6], with homotopy groups

$$\pi_n(L_\bullet(\mathbb{A})) = L_n(\mathbb{A}).$$

It was shown in [6, Chapter 13] that the covariant functor

$$\{\text{simplicial complexes}\} \to \{\Omega-\text{spectra}\} ; X \mapsto L_\bullet(B(\mathbb{A}_*(X)))$$

is an unreduced homology theory, i.e. a covariant functor which is homotopy invariant, excisive and sends arbitrary disjoint unions to wedges. More generally:

**Proposition 5.10.** ([6, 13.7] for simplicial complexes)

(i) If $\mathbb{A}$ is an additive category with chain duality and $X$ is a $\Delta$-set then $\mathbb{A}_*(X)$ is an additive category with chain duality.

(ii) The functor

$$\{\Delta-\text{sets}\} \to \{\Omega-\text{spectra}\} ; X \mapsto L_\bullet(\mathbb{A},X) = L_\bullet(B(\mathbb{A}_*(X)))$$

is an unreduced homology theory, that is $L_\bullet(\mathbb{A},X) = H_*(X;L_\bullet(A))$.

(iii) Let $R$ be a ring with involution, so that $\mathbb{A} = \text{Proj}(R)$ is an additive category of f.g. projective $R$-modules with the duality involution. If $X$ is a $\Delta$-set and $p : \tilde{X} \to X$ is a regular cover with group of covering translations $\pi$ (e.g. the universal cover with $\pi = \pi_1(X)$) the assembly functor

$$A : B(R)_*(X) \to B(R[\pi]) ; C \mapsto C(\tilde{X})$$

$$(C(\tilde{X}) = \sum_{x \in \tilde{X}} C(p(x)))$$

is a functor of additive categories with chain duality. The assembly maps $A$ induced in the $L$-groups fit into an exact sequence

$$\cdots \to H_n(X;L_\bullet(R)) \xrightarrow{A} L_n(R[\pi_1(X)]) \to S_n(R,X) \to H_{n-1}(X;L_\bullet(R)) \to \cdots$$

with $S_n(R,X)$ the cobordism group of the $R[\pi_1(X)]$-contractible $(n-1)$-dimensional quadratic Poincaré complexes in $\mathbb{A}_*(X)$. 
Proof. Exactly as for the simplicial complex case, but using the ∆-set duals instead of the dual cells! □

Example 5.11. Let $X = S^1$ be the ∆-set of the circle (2.8, 3.7) with one 0-simplex and one 1-simplex. Given a ring with involution $R$ let the Laurent polynomial extension ring $R[z, z^{-1}]$ have the involution $\overline{z} = z^{-1}$. An $n$-dimensional quadratic Poincaré complex in $\text{Proj}(R_\bullet(S^1))$ is an $n$-dimensional fundamental quadratic Poincaré cobordism over $R$, with assembly the union $n$-dimensional quadratic Poincaré complex over $R[z, z^{-1}]$, and the assembly maps

$$A : H_n(S^1; \mathbb{L}_\bullet(R)) = L_n(R) \oplus L_{n-1}(R) \to L_n(R[z, z^{-1}])$$

are isomorphisms modulo the usual $K$-theoretic decorations (Ranicki [7, Chapter 24]. □

Remark 5.12. Proposition 5.10 has an evident analogue for the symmetric $L$-groups $L^*$. □

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