Power-Law Scaling for Communication Networks with Transmission Errors

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A generic communication model of a boolean network with transmission errors is proposed to explore the power-law scaling of states’ evolution in small-world networks. In the model, the power spectrum of the population difference between agents with “1” and “-1” exhibits a power-law tail: $P(f) \sim f^{-\alpha}$. The exponent $\alpha$ does not depend explicitly on the error rate but on the structure of the networks. Error rate enters only into the frequency scales and thus governs the frequency range over which these power laws can be observed. On the other hand, the exponent $\alpha$ reveals the intricate internal structure of networks and can serve as a structure factor to classify complex networks. The finite transmission error is shown to provide a mechanism for the common occurrence of the power-law relation in complex systems, such as financial markets and opinion formation.

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The dynamics of complex networks are associated with their structures and the interactions among their elements. In complex systems, elements are often organized by communicating information to one another with their specific links. For example, society is organized as a social web, whose nodes are individuals and links represent various social interactions and the WWW forms a complex web whose nodes are documents and links are URLs. In our daily life, we communicate with each other via language, whose nodes are words, and the links between words are established by grammar. The webs in which the constituents interact bring out the complexity of those systems. Pioneering work done by Watts and Strogatz has demonstrated that many natural, social, and technological networks have topological properties in between those of regular lattices and random graphs. These complex networks exhibit a certain degree of clustering like regular lattices and a logarithmic increase in the average shortest path between two nodes with the number of nodes, as in random graphs. Watts and Strogatz proposed the small-world model - an interpolation between regular lattices and random graphs - to model these networks. The two features of small-world networks lead to cluster formation and the spread of global information through these complex systems.

Besides the network structure, another important factor attributed to the behavior of complex systems is how correctly information can be transmitted between nodes. For example, in a social structure, the individual’s attitude toward some issue is often influenced by his/her friends and public opinion. Most people are inclined to the opinion of the majority, which inclination is also called herd behavior. In a perfect communication network without information error, the states of nodes will interact and finally evolve to a steady state. However, information transmitted between nodes is often liable to error due to inevitable noises. Thus, the imperfection of communication in networks always perturbs the whole system into non-equilibrium state. Consequently, it is natural to ask how transmission errors impact the dynamical behavior of networks. Communication errors can be generally classified into three groups: (a) Technical problem: how accurately can the symbols of communication be transmitted? (b) Semantic problem: how precisely do the transmitted symbols convey the desired meaning? and (c) Effectiveness problem: how effectively does the received meaning affect conduct in the desired way? This work focuses on the effect of the technical problem on communication networks; namely, how accurately information can be transmitted and what is its influence?

In this study, a system of $N$ agents is considered. The agents are represented by nodes in a small-world structure. For simplicity, the network model adopted here is a boolean network in which the state of agent $i$ is represented by $S_i(t) = \{-1, 1\}$. The evolution of the $i_{th}$ agent’s state, $S_i(t)$, follows the updating rule,

$$S_i(t+1) = sgn\left(\sum_{j=1}^{N} w_{ij} P_{j \rightarrow i} [S_j(t), \phi]\right),$$

(1)

where $w_{ij} = 1(0)$ if agents $i$ and $j$ are (not) connected. The probability function, $P_{j \rightarrow i} [S_j(t), \phi]$, determines how accurately information can be transmitted from agent $j$ to $i$. Thus, the transmitted information from agent $j$ to $i$ is $S_i(t)$ with probability $(1-\phi)$ where $\phi$ is the transmission error rate. Function $sgn(x)=+1 (-1)$ if the argument $x > 0 \quad (0 < x)$. If $x=0$, $S_i(t+1) = S_i(t)$, that is, the transmitted information remains unchanged. The updating rule followed here corresponds to the majority rule, that individual opinion often follow public opinion. For simplicity, this study does not consider the free will of the individual, thus the effect of transmission errors on the dynamics of information spreading in networks can be fully retrieved. The evolution of the system is characterized by a succession of discrete events $S(1), S(2), S(3), ...$, where $S(t) = \sum_{j=1}^{N} S_j(t)$ corresponds to the population difference between the states +1 and -1. When
randomness adopted here varies from the definition in the small-world network. Although the definition of randomness can be divided into two classes: (a) \( S_{\text{internal}} \) between clusters of different states and (b) \( S_{\text{boundary}} \) with coordination number \( z = 2k \). Next, \( m \) shortcuts are added between randomly selected pairs of nodes. The randomness, \( p \), of the small-world network is defined as \( p = \frac{m}{N} \) where \( N \) is the population of the agents in the small-world network. Although the definition of randomness adopted here varies from the definition of randomness, \( R \), in Ref. \[1\], these two definitions \((p \text{ and } R)\) are approximately equivalent in the small-world regime. The simulation mainly used Newman and Watts \[10\], to construct the small-world network. Initially, each node is connected to all of its neighbors up to some fixed \( k \) range to make a network with coordination number \( z = 2k \). Next, \( m \) shortcuts are added between randomly selected pairs of nodes.

The simplest network topology, with a range distance \( k=1 \) and no short cuts, \( m=0 \), was first studied to elucidate the main factors in determining the evolution caused by transmission errors. Such a topology is true of a 1D regular lattice, yielding nontrivial dynamics and thus uncovering the underlying features. Some clusters consisting of agents with the same state begin to develop during the evolution, as shown in Fig. 1(a). Here, the transmission error, \( \phi \), is set to 0.05. By increasing the value of \( \phi \) to be 0.1, as shown in Fig. 1(b), clusters converge and split as time evolves in the plot of time-space distribution of agents’ states. The clusters developed more rapidly in the space direction, while their persistence over time was lower than in part (a). Further increasing \( \phi \) to 0.2, sizes of the clusters were much reduced, as shown in Fig. 1(c). Indeed, increasing the transmission error amplifies the uncertainty of information from neighbors to nodes, causing low coherence among the states of the connected nodes. Fig. 2(a) displays the time series \( S(t) \) for \( \phi = 0.1 \). \( S(t) \) shows an irregular fluctuation after the transition from the “all +1” state. A Fourier transform was performed on \( S(t) \) to determine the power spectrum \( P(f) \), and thereby analyze the non-stationary dynamical behavior, as shown in Fig. 2(b). In the low frequency regime, the power spectrum has the quasi-exponential form,

\[
\ln P(f) \propto -f.
\]  

(2)

Surprisingly, in the high frequency regime, the tail of \( P(f) \) exhibited a novel power-law dependence on frequency,

\[
P(f) \sim f^{-\alpha}.
\]  

(3)

The dotted line represented the cumulative amplitude, \( \bar{P}(f) \), of the power spectrum, \( P(f) \), defined as follows,

\[
\bar{P}(f) = \int_{f}^{\infty} P(f')df'.
\]  

(4)
If $P(f)$ displays a power-law scaling as in Eq. (3), then $\overline{P}(f)$ also possesses the property of power-law scaling but with a different exponent, $\beta = 1 - \alpha$. Fitting to the result showed $\beta \approx -0.9$ and therefore $\alpha \approx 2.0$ which, indeed, is the prediction for the power spectrum of a random walk, $P(f) \propto 1/f^2$.

Simulations with different error rates were performed to further examine the effect of the transmission error. Figure 3 shows that the exponent of power-law scaling is almost constant over different error rates, whereas the range of quasi-exponential decay in the low frequency regime expands with the error rate increases, as shown in the inset of Fig. 3. The inevitable transmission error $\phi$ yields a cutoff frequency, $f_c$, of the power-law scaling in the low frequency regime. Fitting to the simulation data shows that $f_c$ scales exponentially with $\phi$: $f_c \propto \exp(14.38\phi)$. As stated above, the correlation between neighboring nodes is destroyed when $\phi$ increases. Thus, the power-law scaling in the lower frequency regime perishes first due to the disappearance of the long range correlation in $S(t)$. For large $\phi$, power-law scaling is entirely destroyed by noise (errors), leading to a uniform distribution of the power spectrum. However, the simulation results presented here indicate that the exponent $\alpha$ in the power-law scaling is almost independent of the transmission error $\phi$, implying the existence of a short time correlation in a noisy environment and thus the possibility of predicting the behavior of the system within short time.

The influence of network structures on the dynamical behavior of our model is now further analyzed. Simulations were performed with different levels of randomness of the small-world networks: $p=0.01, 0.05, 0.1, 0.2,$ and $0.4$; the other parameters were fixed: $N = 500, k = 1$, and $\phi = 0.1$. The simulation results in Fig. 4 show that the value of $\alpha$ depends linearly on the complexity of network structures, obeying

$$\alpha(p) \approx 2(1 - p)$$

for $p$ less than $0.4$. For large $p(>0.4)$, the networking structure is no longer a small-world network, but like a random one. The exponent $\alpha$ saturates to zero, reflecting the disappearance of the ebb and flow of clusters. Equation (3) was also confirmed to hold for large $N$, $N = 5000$. The size effect is weak in the model due to the fact that the power-law scaling in the high frequency regime is substantially contributed by the boundary interaction between clusters, causing large fluctuation. Equations (3), (4), and (5) suggest the hypothesis that $P(f)$ approximately follows the form,

$$P(f) \propto \frac{1}{1 + [f/g(\phi)]^\alpha}$$

FIG. 2. (a) Time evolution of system states $S(t)$ and (b) its power spectrum $P(f)$ (continuous line) and cumulative amplitude $\overline{P}(f)$ (dotted line) defined in Eq. (4). In the regime of power-law scaling, the exponent of $\overline{P}(f)$, $\beta \sim -0.9$. Thus, $\alpha = 1 - \beta \sim 1.9$. Data used here is the same as in Fig. 1(b).

FIG. 3. Cumulative power spectral densities of $S(t)$ with different levels of transmission error $\phi$. From up to down: $\phi = 0.7, 0.6, 0.5, 0.4, 0.3, 0.2,$ and $0.1$. Inset: plot of the cut-off frequency $f_c$ versus the error rate $\phi$. The fitting line (continuous) to the data follows the form, $f_c \propto \exp(14.38\phi)$.
where \( \ln g(\phi) \propto \phi \). The power spectrum \( P(f) \) approaches a constant for \( f \ll g(\phi) \), and exhibits power-law scaling in the high frequency regime. The two different scaling regimes are distinguished by the cut-off frequency, \( f_c \sim g(\phi) \).

Previous research into real social networks \[1,4\] has revealed that the randomness of social structures range from 0.01 to 0.1, corresponding to a decrease in the value of \( \alpha \) from 2.0 to 1.5 in the simulation presented here. Interestingly, recent studies \[12–15\] of financial markets show that the fluctuation of prices also exhibits power-law scaling with the exponent values ranging from 2.0 to 1.5. The remarkable agreement of our results with empirical data suggests that the information errors ignored in previous studies of social phenomena may importantly participate in the appearance of power-law scaling in complex systems. Furthermore, the power-law scaling’s independence of the system size in this model may offer a possible explanation of the similar behavior of financial markets with different sizes but similar social structures \[13,14\]. The model has been applied to examine the scaling and fluctuating behavior of financial prices. Preliminary results successfully resemble empirical results in which frequent large events occur abruptly over time evolution of price turns. Detailed results will be published elsewhere.

In summary, in investigating the communication networks with respect to the accuracy of transmitted information, the power spectrum of the population difference between agents with “+1” and “−1” has been shown to exhibit a power-law tail: \( P(f) \sim f^{-\alpha} \). An important feature is that the exponent \( \alpha \) is independent of the error rate, \( \phi \), but does depend on the structure of networks. Instead, the error rate enters only into the frequency scales and thereby determines the frequency range over which these power laws apply. However, the exponent, \( \alpha \), depends on the intricate internal structure of networks and can serve as a structure factor to classify the complexity of networks. Since, in practice, transmission error is inevitable, finite transmission error has shown to be an important factor responsible for the common appearance of power-law relation in complex systems, such as financial markets and opinion formation.

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