Quantum bit commitment and unconditional security

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Abstract

It is generally believed that unconditionally secure quantum bit commitment is impossible, due to widespread acceptance of an impossibility proof that utilizes quantum entanglement cheating. In this paper, we delineate how the impossibility proof formulation misses various types of quantum bit commitment protocols based on two-way quantum communications. We point out some of the gaps in the impossibility proof reasoning, and present corresponding counterexamples. Four different types of bit commitment protocols are constructed with several new protocol techniques. A specific Type 4 protocol is described and proved unconditionally secure. Security analysis of a Type 1 protocol and a Type 2 protocol are also sketched. The security of Type 3 protocols is as yet open. A development of quantum statistical decision theory and quantum games is needed to provide a complete security analysis of many such protocols.

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†Note: this paper analyzes in detail, for the first time, the various gaps in the QBC impossibility proof, many of which I indicated before but few of which seem to be understood. There is clearly a need to focus on these gaps, which is an issue logically distinct from whether any protocol can be proved unconditionally secure. One of the original three protocols I described in the QCM at Capri, July 2000, protocol Y3 that appeared as QBC1 in v2 of this paper, was pointed out to be insecure in the QCM at MIT, July 2002. It is extended, with the same underlying idea, and renamed QBC4 in this v3 with a full security proof. A considerable amount of new material is also added, including a protocol based on cheating detection alone and clarifications on unknown parameters and entanglement purification. It is also stressed that a priori there can be no impossibility proof without a QBC definition.
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1 Introduction

There is a nearly universal acceptance of the general impossibility of secure quantum bit commitment (QBC), taken to be a consequence of the Einstein-Podolsky-Rosen (EPR) type entanglement cheating which supposedly rules out QBC and other quantum protocols that have been proposed for various cryptographic objectives [1]. In a bit commitment scheme, one party, Adam, provides another party, Babe, with a piece of evidence that he has chosen a bit $b$ (0 or 1) which is committed to her. Later, Adam would open the commitment by revealing the bit $b$ to Babe and convincing her that it is indeed the committed bit with the evidence in her possession, which she can verify. The usual concrete example is for Adam to write down the bit on a piece of paper, which is then locked in a safe to be given to Babe, while keeping for himself the safe key that can be presented later to open the commitment. The evidence should be binding, i.e., Adam should not be able to change it, and hence the bit, after it is given to Babe. It should also be concealing, i.e., Babe should not be able to tell from it what the bit $b$ is. Otherwise, either Adam or Babe would be able to cheat successfully.

In standard cryptography, secure bit commitment is to be achieved either through a trusted third party, or by invoking an unproved assumption concerning the complexity of certain computational problems. By utilizing quantum effects, specifically the intrinsic uncertainty of a quantum state, various QBC schemes not involving a third party have been proposed to be unconditionally secure, in the sense that neither Adam nor Babe could cheat with any significant probability of success as a matter of physical laws. In 1995-1996, a supposedly general proof of the impossibility of unconditionally secure QBC, and the insecurity of previously proposed protocols, was presented [2]-[4]. Henceforth it has been generally accepted that secure QBC and related objectives are impossible as a matter of principle [5]-[7].

There is basically just one impossibility proof, which gives the EPR attacks for the cases of equal and unequal density operators that Babe has for the two different bit values. The
proof purports to show that if Babe’s cussessful cheating probability $P^B_c$ is close to the value 1/2, which is obtainable from pure guessing of the bit value, then Adam’s successful cheating probability $P^A_c$ is close to the perfect value 1. This result is stronger than the mere impossibility of unconditional security, namely that it is impossible to have both $P^B_c \sim 1/2$ and $P^A_c \sim 0$. The impossibility proof describes the EPR attack on a specific type of protocols, and then argues that all possible QBC protocols are of this type.

Typically, one would expect that a proof of impossibility of carrying out some thing X would show that any possible way of doing X would entail a feature that is logically contradictory to given principles, as, for example, in the cases of quantum no-cloning [12, 13] and von Neumann’s no-hidden-variable theorem [14]. In the present case, one may expect a proof which shows, e.g., that any QBC protocol that is concealing is necessarily not binding. It is important for this purpose that the framework of QBC protocol formulation is all-inclusive. In the absence of a proof that all possible QBC protocols have been included in its formulation, any impossibility proof is at best incomplete. Indeed, in the QBC impossibility proof, only certain techniques of protocol design, such as the use of classical random numbers in a quantum protocol, are included in its formulation without showing that all possible techniques have been included. In this paper, we will describe several new techniques that are not accounted for in the impossibility proof formulation.

There are two related assertions in the impossibility proof that are crucial to both its claim of universality in general, and its specific claim of covering the use of random numbers in particular. These are the assertions that all measurements in the commitment phase of a quantum protocol can be postponed until the opening and the verification phases, and that classical random numbers can be equivalently described by pure quantum states, via quantum purification or the doctrine of “Church of the Larger Hilbert Space.” In this paper, we will extensively analyze the serious problems associated with these assertions.

The essential argument of the general impossibility proof is described in Section 2, and some of its problems are indicated in Section 3. A proper framework for QBC protocols is
discussed in Section 4. In Section 5, we describe several new protocol techniques that lead to the development of four new types of protocols not covered by the impossibility proof. In Section 6 we describe Type 1 protocols, in which the postponement of a measurement until opening and verification would yield a protocol with different cheating performance. A specific protocol $QBC_1$ is presented with a sketch of the security proof. In Section 7, the logic underlying Type 2 protocols is delineated. A specific protocol, $QBC_2A$, is presented with an outline of the security proof. The security analysis of protocols $QBC_1$ and $QBC_2A$ are not complete in the sense that exact optimality can only be proved with a sequential quantum decision theory yet to be developed, although all essential points are included under the assumption that no party can cheat if it can be detected with a nonvanishing probability before a bit is committed. In Section 8, the widely accepted equivalence between classical randomness and quantum purification is analyzed. We will show that they are not equivalent in bit commitment. We also introduce Type 3 protocols, the security status of which is yet undecided. In Section 9, we introduce Type 4 protocols which involve Babe’s open questions related to Adam’s committed evidence. A specific protocol $QBC_4$ is proved unconditionally secure. The last Section 10 contains a brief summary of the main points. The appendices, especially Appendix B, are an integral part of the paper, being separated for convenient organization of this rather subtle and multi-faceted subject. Also, the different types of protocols in this paper are not mutually exclusive. Again, they are mainly introduced for the purposes of organization.
2 The impossibility proof: Type 0 protocols

The impossibility proof, in its claimed generality, has never been systematically spelled out in one place, but the essential ideas that constitute this proof are generally agreed upon [3-11]. The formulation and the proof can be cast as follows. Adam and Babe have available to them two-way quantum communications that terminate in a finite number of exchanges, during which either party can perform any operation allowed by the laws of quantum physics, all processes ideally accomplished with no imperfection of any kind. During these exchanges, Adam would have committed a bit with associated evidence to Babe. It is argued that, at the end of the commitment phase, there is an entangled pure state $|\Phi_b\rangle$, $b \in \{0, 1\}$, shared between Adam who possesses state space $\mathcal{H}^A$, and Babe who possesses $\mathcal{H}^B$. For example, if Adam sends Babe one of $M$ possible states $\{|\phi_{b_i}\rangle\}$ for bit $b$ with probability $p_{b_i}$, then

$$|\Phi_b\rangle = \sum_i \sqrt{p_{b_i}} |e_i\rangle |\phi_{b_i}\rangle$$

with orthonormal $|e_i\rangle \in \mathcal{H}^A$ and given $|\phi_{b_i}\rangle \in \mathcal{H}^B$. Adam would open by making a measurement on $\mathcal{H}^A$, say $\{|e_i\rangle\}$, communicating to Babe his result $i_0$ and $b$; then Babe would verify by measuring the corresponding projector $|\phi_{b_{i_0}}\rangle \langle \phi_{b_{i_0}}|$ on $\mathcal{H}^B$, accepting as correct only the result 1.

More generally, when classical random numbers known only to one party are used in the commitment, they are to be replaced by corresponding quantum entanglement purification. The commitment of $|\phi_{b_i}\rangle$ with probability $p_{b_i}$ in (1) is, in fact, an example of such purification. An example involving Babe is an anonymous state protocol [13-16] where $|\phi_{b_i}\rangle$ in (1) is to be obtained by Adam applying unitary operations $U_{b_i}$ on state $|\psi_{k}\rangle \in \mathcal{H}^{B_1}$ sent to him by Babe with probability $\lambda_k$, $k \in \{1, \ldots, K\}$. Generally, for any random $k$ used by Babe, it is argued that from the doctrine of the “Church of the Larger Hilbert Space” [16], it is to be replaced by the purification $|\Psi\rangle$ in $\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$,

$$|\Psi\rangle = \sum_k \sqrt{\lambda_k} |\psi_{k}\rangle |f_{k}\rangle,$$
where the $|f_k\rangle$'s are complete orthonormal in $\mathcal{H}^{B2}$ kept by Babe while $\mathcal{H}^{B1}$ would be sent to Adam. With such purification, it is claimed that any protocol involving classical secret parameters would become quantum-mechanically determinate, i.e., the shared state $|\Phi_b\rangle$ at the end of commitment is completely known to both parties. Note that, from (2), this means that both $\{\lambda_k\}$ and $\{|f_k\rangle\}$ are taken to be known exactly to both Babe and Adam.

Why should Adam and Babe share a pure state instead of a mixed one at the end of commitment? One key ingredient of the impossibility proof is the use of measurement purification, or quantum computers, in lieu of actually taking macroscopic measurement readings. During commitment, quantum registers holding the measurement results would be passed along instead. Furthermore, any measurement followed by a unitary operation $U_l$ depending on the measurement result $l$ would be equivalently described by an overall unitary operator. Thus, if the orthonormal $\{|g_l\rangle\}$ on $\mathcal{H}^{C1}$ is measured with result $l$, and then $U_l$ operates on $\mathcal{H}^{C2}$, it is equivalent to the unitary operation

$$U = \sum_l |g_l\rangle\langle g_l| \otimes U_l$$

(3)

on $\mathcal{H}^{C1} \otimes \mathcal{H}^{C2}$. It is claimed that any actual measurement during commitment can be postponed until the opening and the verification phases of the protocol without affecting the protocol in any essential way. In order to maintain quantum determinacy, the exact $\{|g_l\rangle\}$ in (3) are taken to be known to both parties, even though the measurement may be chosen by a party among different possible alternatives. Let us use $k$ to denote Babe’s secret parameter, and $i$ to denote Adam’s secret parameter, such as the $i_0$ with probabilities $\{p_i\}$ in (1). These crucial assumptions of openly known $\{p_i\}$, $\{\lambda_k\}$, $\{|f_k\rangle\}$, and $\{|g_l\rangle\}$ are made in the impossibility proof through the use of known fixed quantum computers or quantum machines for data storage and processing by either party [3, 4, Appendix], even though the control of such machines belongs only to one of the parties.

Generally, Babe can try to identify the bit from $\rho_b^B$, the marginal state of $|\Phi_b\rangle$ on $\mathcal{H}^B$, by performing an optimal quantum measurement that yields the optimal cheating probability $P_c^B$ for her. Adam cheats by committing $|\Phi_0\rangle$ and making a measurement on $\mathcal{H}^A$ to open $i_0$
and $b = 1$. His probability of successful cheating is computed through $|\Phi_b\rangle$, his particular measurement, and Babe's verifying measurement; the one optimized over all of his possible actions will be denoted $\bar{P}_c^A$. For a fixed measurement basis, Adam's cheating can be described by a unitary operator $U^A$ on $\mathcal{H}^A$. Thus, his general EPR attack goes as follows. For a general protocol, the shared state $|\Phi_b\rangle$ at the end of commitment is not necessarily of the form (1), but is nevertheless an openly known pure state on $\mathcal{H}^A \otimes \mathcal{H}^B$. If the protocol is perfectly concealing, i.e., $\bar{P}_c^B = 1/2$, then $\rho_0^B = \rho_1^B$. By writing $|\Phi_b\rangle$ as the Schmidt decomposition on $\mathcal{H}^A \otimes \mathcal{H}^B$,

$$|\Phi_b\rangle = \sum_j \sqrt{p_j}|\tilde{e}_{bj}\rangle|\tilde{\phi}_j\rangle,$$

where $|\tilde{\phi}_j\rangle$ are the eigenvectors of $\rho_b^B$ and $\{|\tilde{e}_{bj}\rangle\}$ for each $b$ are complete orthonormal in $\mathcal{H}^A$, it follows that Adam can obtain $|\Phi_1\rangle$ from $|\Phi_0\rangle$ by a local cheating transformation $U^A$ that brings $\{|e_{0j}\rangle\}$ to $\{|e_{1j}\rangle\}$. Whatever operations he needs to perform to open, which may involve identifying his previous operation rather than a state on $\mathcal{H}^B$, can be carried out accordingly after the cheating transformation. Thus his optimum cheating probability is $\bar{P}_c^A = 1$ in this case.

For unconditional, rather than perfect, security, one demands that both cheating probabilities $\bar{P}_c^B - 1/2$ and $\bar{P}_c^A$ can be made arbitrarily small when a security parameter $n$ is increased [3]. Thus, unconditional security is quantitatively expressed as

$$(\text{US}) \quad \lim_n \bar{P}_c^B = \frac{1}{2}, \quad \lim_n \bar{P}_c^A = 0.$$  \hspace{1cm} (5)

The condition (5) says that, for any $\epsilon > 0$, there exists an $n_0$ such that for all $n > n_0$, $\bar{P}_c^B - 1/2 < \epsilon$ and $\bar{P}_c^A < \epsilon$, to which we may refer as $\epsilon$-concealing and $\epsilon$-binding. These cheating probabilities are to be computed purely on the basis of logical and physical laws, and thus would survive any change in technology, including an increase in computational power. In general, one can write down explicitly

$$\bar{P}_c^B = \frac{1}{4} \left(2 + \|\rho_0^B - \rho_1^B\|_1\right),$$  \hspace{1cm} (6)
where $\| \cdot \|_1$ is the trace norm, $\| \tau \|_1 \equiv \text{tr}(\tau^\dagger \tau)^{1/2}$ for a trace-class operator $\tau$, but the corresponding $\bar{P}_c^A$ is more involved. Nevertheless, the impossibility proof shows that Adam can find a cheating $U^A$ that yields

$$(\text{IP}) \quad \lim_{n} \bar{P}_c^B = \frac{1}{2} \Rightarrow \lim_{n} \bar{P}_c^A = 1$$

within its formulation [2, 15]. Note that the impossibility proof makes a stronger statement (IP) than the mere impossibility of (US), i.e., (7) is stronger than (5) not being possible.

There are various gaps and implicit assumptions hidden in the impossibility proof, many of which seem to spring from the idea that a protocol leads to a closed quantum system all by itself, requiring no interaction with external agents or preparers. These gaps render the proof incomplete in several ways. As to be discussed in the following, some of them can be partially justified or closed, but many still remain and cannot be bridged. We will refer to protocols that fit this impossibility proof formulation as Type 0 protocols, and will describe four additional types, 1, 2, 3, and 4, that are clearly not covered by this proof. Before proceeding, we first elaborate on the limited scope of the impossibility proof formulation.
3 Problems of the impossibility proof

A plausible first reaction to the impossibility proof is: why are all possible QBC protocols reducible to the formulation described in the last section? More precisely, how may one characterize quantitatively the necessary feature of an unconditionally secure QBC protocol in order to show it to be impossible? To put this in yet another way, what is the mathematical definition of a QBC protocol, or the mathematical statement of the necessary feature of an unconditionally secure QBC protocol, that is required for any proof of a mathematical theorem that says such protocol is impossible? No such definition is available. The situation is similar to the lack of a definition of an “effectively computable” function. Since nobody calls the Church-Turing thesis the Church-Turing theorem, at best the impossibility proof is a “thesis” which may be found incorrect in the future. This \textit{a priori} logical point is further elaborated in Appendix A.

The crucial starting point of the impossibility proof asserts that, in general, a protocol is equivalent to one with openly known pure states $|\Phi_b\rangle$ on $\mathcal{H}^A \otimes \mathcal{H}^B$ at the end of commitment. Let us explore what this entails. Suppose Adam commits, in a prescribed protocol, one of $M$ possible $|\phi_{bi}\rangle$ for each $b$ without entanglement. Then $\rho_0^B$ is identical or close to $\rho_1^B$ as before, but Adam cannot cheat. This situation is not one where a pure $|\Phi_b\rangle$ is known to Babe, which occurs only when all the randomness on $\rho_0^B$ comes from quantum entanglement. Even then Adam can cheat only if the entanglement is controlled by him. Indeed, quantum entanglement is not a conceptual resource, but rather a physical one, and needs to be physically established. See Appendix B for a discussion on randomness generated by quantum entanglement versus that generated by other means, and the confusion surrounding the doctrine of “Church of the Larger Hilbert Space.” There may exist protocols in which Adam is forced to generate randomness without being able to entangle over it during the course of commitment, so that at the end of commitment one has the situation described

\footnote{In practice, this is what would happen currently due to the difficulties of generating and maintaining entanglement. Some of these difficulties are not merely technical, but are actually inherent in principle.}
above instead of openly known $|\Phi_b\rangle$ on $\mathcal{H}^A \otimes \mathcal{H}^B$. In our Type 4 protocols, the protocol design technique of open questioning of evidence could be used to achieve this situation, as described in Section 3 and Appendix C.

In general, if Babe makes an actual measurement during commitment, there would not be an openly known $|\Phi_b\rangle$ at the end of commitment. The impossibility proof claims that such measurement can be postponed until after commitment with the use of measurement purification and $\mathcal{B}$ in place of an actual measurement. However, no proof is given that both $\bar{P}_c^A$ and $\bar{P}_c^B$ would not be affected. Furthermore, one has to make sure that it is not the microscopic states Babe is thus required to store her measurement results that are being reversed by Adam’s cheating. Otherwise, Babe can take macroscopic readings instead.

In Section 5, we will show how cheating detection during commitment is not incorporated in the impossibility proof formulation with an openly known $|\Phi_b\rangle$. As a consequence, the situation of actual measurements during commitment has to be explicitly included in a general formulation of QBC protocols.

The use of anonymous states $\mathcal{B}$ alone, where $\{\lambda_k\}$ may be unknown to Adam, leads to our Type 3 protocols. As elaborated in Appendix B, the distinction between an unknown and a random parameter is crucial in this situation, and the assertion that $|\Phi_b\rangle$ is openly known cannot be maintained. A theory of statistical quantum games is required for an analysis of protocols of this type.

Assuming that $|\Phi_b\rangle$ is openly known at the end of commitment, it is still not proved that Adam can cheat in general because special structure or mingling of $\mathcal{H}^A$ and $\mathcal{H}^B$ during commitment may lead to an opening and verification procedure different from Adam and Babe acting on $\mathcal{H}^A$ and $\mathcal{H}^B$ separately. Our Type 2 protocols give one such possibility, but no doubt there are others. Generally in a QBC protocol with a given $|\Phi_b\rangle$ at the end of commitment, different opening and verification strategies are possible, depending on exactly how $|\Phi_b\rangle$ is arrived at. Both our Type 2 and Type 4 protocols may be viewed as ones where these phases are more complex than the one given in the formulation of the impossibility
proof. In the next section we will first elaborate on the issue of what may constitute a QBC protocol and whether we can give it a mathematical definition.
4 Proper framework for protocol formulation

The following two principles, the Intent Principle and the Libertarian Principle, govern the viability and meaningfulness of any bit commitment protocol in a descriptive, not normative, sense. That is, they would be satisfied in what we would take intuitively to be a proper protocol, and are not imposed in a legislative fashion, as discussed in the following.

INTENT PRINCIPLE — Each party would act to achieve the intent of the protocol if no cheating by the other party is (probabilistically) possible.

Thus, each party would cooperate so that the protocol would not be aborted, which happens when one party is found cheating by the other through a possible cheat-detection mechanism during the commitment phase. Since each party can always just abort by noncooperation during any stage of any two-party protocol, the Intent Principle does not exclude any action not otherwise possible. Thus, if the cheating detection probability leads to an overall cheating success probability within the given $\epsilon$, the protocol is a proper one and cannot be declared illegitimate because one party may keep cheating, though keep being detected.

We also have the

LIBERTARIAN PRINCIPLE — At any stage of the protocol, each party can freely perform any possible local operation consistent with the Intent Principle for cooperation.

Thus, no party can be assumed to be honest in anything if the action leads to his/her own advantage and would not get caught. That is, each party can cheat whenever possible, unless it violates the Intent Principle for cooperation. There would be no need for any protocol if the parties can be assumed honest. Similarly, each party can do whatever is possible to thwart the other party’s cheating. Under the Intent Principle, a party is obliged to accept a protocol if he is assured that the probability of cheating against him is within the tolerance
level \( \epsilon \), even though he does not know a secret parameter of the other party. The following Secrecy Principle is a corollary of the above two principles.

**Corollary (SECRECY PRINCIPLE)** — A party does not need to reveal a secret parameter chosen by her in whatever manner, if it does not affect the other party’s security.

On the other hand, if a party has no control or checking on a secret parameter that the other party may use to cheat successfully, she would not accept the protocol.

Any finite sequence of two-way quantum communication exchanges that results in bit commitment under the Intent Principle is evidently a QBC protocol, whose security is to be analyzed under the Libertarian Principle. More importantly, any QBC formulation that fails to include all such sequences does not capture all possible QBC protocols. The present framework is more general than the “Yao model” in that aborting the protocol on the basis of cheating detection is allowed during commitment, and is more specific in the explicit formulation of the above principles. As discussed in the preceding section, the impossibility proof formulation is not complete in that it misses protocols with cheating detection during commitment because such detection would involve actual measurements that may not be postponed until after commitment to yield openly known \( |\Phi_b\rangle \). Also, the Secrecy Principle a priori *contradicts* directly the claim of openly known \( |\Phi_b\rangle \).

The above principles do not constitute a mathematical definition of a QBC protocol. They are too broad and too narrow at the same time — too broad in the sense that bit commitment is not defined, and too narrow in that other possibilities may still exist. My personal suspicion is that the “too broad” problem, or the difficulty of defining bit commitment, is much more serious than the “too narrow” problem.

As in all QBC formulations so far, it is assumed in this paper that Adam opens perfectly on one bit value, say \( b = 0 \). More generally, one may allow QBC protocols that open on one bit with a success probability \( P_0 = 1 - \epsilon' \) for a small \( \epsilon' \). It appears that protocols for which
neither bit can be opened with near-unity probability are of little interest. In conjunction with $\epsilon$-concealing and $\epsilon$-binding, one may then consider the possibility of $(\epsilon,\epsilon')$-protocols, the detailed treatment of which will be given elsewhere.
5 New protocol techniques, or gaps in the impossibility proof

In this section we describe three new techniques for constructing QBC protocols, which are not covered by the impossibility proof formulation. Our Type 1 protocol is based on the first technique, Type 2 on the second and, possibly, additional others, Type 3 on random numbers, and Type 4 on the third technique. Each of these protocol types will be discussed separately in the following sections.

The first technique introduces testing on states of an ensemble, in space or in time, submitted by the other party, in order to check whether only admissible states of the protocol are being used. This was already utilized in QBC2 of Ref. [15]. The protocol is aborted if cheating is detected by a measurement. Such protocols are allowed under the Intent Principle, but not included in the impossibility proof formulation for the following reason. Babe can use many different possible $U_l$ in (3), secretly chosen to be recognized only by her, in order to represent her choice of aborting the protocol. Thus, the resulting $|\Phi_b\rangle$ is not known to Adam. Even if the measurement checking is postponed until verification, there is no proof that the cases of Adam’s successful cheating do not correspond to the ones aborted by Babe. That is, a careful analysis of the overlaps between aborting probabilities by Adam and Babe with $P_c^A$ and $P_c^B$ is required. One also has to rule out the situation where one keeps aborting if he finds the situation not conducive to his cheating. Generally, in accordance with the Intent Principle, a fixed number $N_c$ of cheating detections may be built into the protocol, beyond which the whole attempt at a protocol is aborted. An appropriate theory of statistical quantum games needs to be developed for general analysis of such protocols.

For the second technique, consider a protocol in which Babe forms (3) and sends Adam $\mathcal{H}^{B_1}$, with $|\psi_k\rangle = |\psi_{k1}\rangle|\psi_{k2}\rangle$ in $\mathcal{H}^{B_1} = \mathcal{H}^{B_{11}} \otimes \mathcal{H}^{B_{12}}$. Adam randomly switches the state in $\mathcal{H}^{B_{11}}$ to be that of $|\psi_{k1}\rangle$ or $|\psi_{k2}\rangle$ by the unitary permutation $P_m$, $m \in \{1, 2\}$, modulates the resulting state in $\mathcal{H}^{B_{11}}$ by a single $U_b$ for each $b$, and sends it to Babe. He opens by revealing $b$, his random permutation $P_m$, and returning $\mathcal{H}^{B_{12}}$. Babe verifies by testing the appropriate
states in $H^{B_{11}}$ for checking $b$, and $H^{B_{12}}$ for checking that there is no change. Thus, Adam cannot entangle and use $H^{B_{12}}$. It is possible that the protocol is both concealing and binding because, for the final commitment state $|\Phi_b\rangle$ with Adam entangling the $P_m$ with $|e_i\rangle \in H^{A_1}$, we have $H^A = H^{A_1} \otimes H^{B_{12}}$ and $H^B = H^{B_{11}} \otimes H^{B_{12}}$. Thus, $\rho_0^B$ can be close to $\rho_1^B$ because $H^{B_{12}} \otimes H^{B_{13}}$ is not available to Babe for her cheating. However, only $H^{A_1}$, and not $H^A$, is available to Adam’s cheating, so he cannot apply the required cheating $U^A$ without being found cheating with a nonvanishing probability. There is no impossibility proof covering this situation.

Example 1 (protocol QBCp2)

As a specific example, consider the case $H^B = H^{B_{11}} \otimes H^{B_{12}} \otimes H^{B_{13}} \otimes H^{B_{14}}$ of four qubits, with $\{|\psi_k\rangle\} = \{|1\rangle|2\rangle|3\rangle|4\rangle, |4\rangle|1\rangle|2\rangle|3\rangle, |3\rangle|4\rangle|1\rangle|2\rangle, |2\rangle|3\rangle|4\rangle|1\rangle\}$, where $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ are, e.g., a fixed set $S_0$ of four possible BB84 states on a given great circle of a qubit. Adam permutes each $|\psi_k\rangle$ by one of four possible $P_m$, and returns the first qubit to Babe unchanged for $b = 0$, while shifted by $\pi/2$ in the great circle for $b = 1$. Assume first that Babe either did not entangle, or cannot use her entanglement in $H^{B_2}$, so that Adam receives one of the four possible $|\psi_k\rangle$. It is then easy to see that $\rho_0^{B_{11}}(\psi_k) = \rho_1^{B_{11}}(\psi_k)$ for all $k$. It is also not hard to see that no entanglement of the four possible $P_m$ would produce a rotation on the first qubit while not disturbing the others. Thus, Adam cannot cheat perfectly and has a fixed $\bar{P}_A^c$ for this protocol which is not arbitrarily close to one, even though it is perfectly concealing. In Section[4], we will indicate how Babe can be effectively denied her use of entanglement via $H^{B_2}$.

In the third technique, Babe asks Adam some of infinitely many possible questions concerning the evidence that Adam committed, demanding the answers to be presented to her in a random fashion, specified by her as a quantum code. Since Adam cannot entangle this new random code on top of the entanglement he already formed, he could only cheat successfully if the required presentations of the answers have been pre-entangled by him. However, he
can pre-entangle answers to only a finite number of questions, and thus can only cheat with an arbitrarily small probability.

Each of these techniques will now be elaborated upon in the different types of protocols.
6 Some measurements cannot be postponed: Type 1 protocols

In this Section we will show that the protocol technique of testing for cheating detection alone, with resulting protocols referred to as Type 1, could already lead to unconditional security. The general idea leading to our protocol \textbf{QBC1} would be first described before security analysis and a precise statement of the protocol.

If carried out honestly, the protocol would work as follows. Adam sends Babe a large number \(n\) of qubits named by their temporal position with states selected randomly and independently from the set \(S_0\) of four \textbf{BB84} states on a given fixed great circle \(C\) of the qubits. Babe randomly selects \(n - n_0\) qubits, tests them by asking Adam what these states are, and verifies them, with \(n\) and \(n_0\) large so that the remaining \(n_0\) states would also be distributed nearly uniformly on \(S_0\). She then picks randomly one of the remaining \(n_0\) states and sends it Back to Adam who would modulate it by \(U_0 = I\) or \(U_1 = R(\pi)\), rotation by \(\pi\) radians on the circle \(C\), depending on \(b = 0\) or \(1\). He opens by revealing \(b\) and all the \(n_0\) qubit states. Babe verifies by checking all the qubits in her possession. This protocol \textbf{QBC1} is \(\epsilon\)-concealing and \(\epsilon\)-binding for the following reasons.

Adam may entangle each individual qubit he sends in the form (2) with \(|\psi_k\rangle \in S_0\), and then measure \(\{|f_k\rangle\}\) when asked to reveal by Babe. If he sends in other qubit states, the chance \(\epsilon_1^A\) he would escape detection is arbitrarily small for large \(n - n_0\). If he entangles across qubits, that merely reduces his freedom in response to Babe’s testing. When he accepts the qubit sent back by Babe, he would have to measure \(\{|f_k\rangle\}\) in all the remaining qubits before his modulation, or else he could not commit because there would be no difference between his two \(U_b\) actions. If he measures on the qubit sent back by Babe, he would not be able to open perfectly for \(b = 0\). More significantly, the information is of little use to him since he does not know the name of that qubit. He can only cheat by declaring \(b = 1\) and switching the names of some of the qubits, hoping that it would fit his cheating \(b = 1\) opening. However, the chance that would succeed without being detected can be seen to be arbitrarily small.
for large $n_0$.

This protocol is $\epsilon$-concealing because all of Babe’s possible cheatings would be unsuccessful as follows. With a high probability, Adam checks the qubit sent back by Babe with a question on its name, and verifies it is correct. He would accept the qubit at some point. If Babe sends in a state different from one in Adam’s ensemble, the probability that would not get detected is arbitrarily small when Adam tests a large number $m \ll n_0$ of times. Assuming that both Adam and Babe employ a randomized strategy applied independently from qubit to qubit during Adam’s testing, it can be readily shown that the protocol is $\epsilon$-concealing and $\epsilon$-binding for sufficiently large $n$ and $n_0$. This different-state attack by Babe includes her possible entanglement, even though it can be shown independently that her entanglement would not help. She can also try to determine the qubit state she sends back by measuring the other qubits in her possession, but these are not correlated to the qubit she sends back. We have the following protocol QBC1.

**PROTOCOL QBC1**

(i) Adam sends Babe a large number $n$ of independent qubit states drawn randomly from $S_0$, a set of openly known BB84 states on a given great circle $C$ of the qubits. The qubits are named by their temporal positions as received by Babe.

(ii) Babe randomly picks a large number $n_0$ of these qubits, sets them aside, and asks Adam to open the remaining ones. She verifies them to be correct and distributed nearly uniformly, as prescribed in (i). Otherwise the protocol is aborted.

(iii) Babe sends back one of the $n_0$ remaining qubits to Adam, who checks it a sufficient number of times in a game with Babe, accepts one, and modulates it by either $U_0 = I$ or $U_1 = R(\pi)$, rotation by $\pi$ radians on $C$, and sends it back to Babe.

(iv) Adam opens by revealing $b$ and all the remaining qubit states. Babe verifies by measuring the corresponding projectors.

In addition to being an outline, the above security analysis is incomplete because the optimal sequential decision in both Adam’s and Babe’s testing have not been analyzed. A
new development of quantum sequential decision theory and quantum games is needed for such an analysis. In general, a fixed number $N_c$ of cheating by each party is allowed as a protocol design parameter in a quantum game situation. A party is not permitted, i.e., loses the game, if found cheating more than $N_c$ times. It is evident that QBC1 is unconditionally secure if $N_c$ is taken to be zero or a small number. I believe that, for any given $N_c$, $n$ and $n_0$ can be chosen so that the protocol is unconditionally secure for any $\epsilon > 0$.

Note that QBC1 is not just a cheat-sensitive protocol. In particular, the cheat detection is done before the bit is committed. As shown in the preceding section, it would not be equivalent to a protocol with an openly known $|\Phi_b\rangle$ at the end of commitment.
7 Who has which space: Type 2 protocols

The use of the first technique in Section 5, test for cheating via measurement, has the effect of changing and pinning down the \( \epsilon \)-concealing condition of the protocol, as compared to one without the test. Generally, the condition

\[
\rho_0^B(\Psi) \sim \rho_1^B(\Psi) \quad \text{for one } |\Psi\rangle \in \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2},
\]

while weaker than

\[
\rho_0^B(\Psi) \sim \rho_1^B(\Psi) \quad \text{for every } |\Psi\rangle \in \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2},
\]

is not equivalent to

\[
\rho_0^B(\psi_k) \sim \rho_1^B(\psi_k) \quad \forall |\psi_k\rangle \in \mathcal{H}^{B_1}.
\]

Specifically, (8) does not imply (10) because there can be a \( |\psi_1\rangle \) for which \( \rho_0^B(\psi_1) \) and \( \rho_1^B(\psi_1) \) are far apart under (8) with \( \lambda_1 \) small \([16]\). Also, it is easy to check that, in Example 1 of Section 5, (10) holds with equality, but there is a finite gap for \( \|\rho_0^B - \rho_1^B\|_1 \) upon entanglement with \( \mathcal{H}^{B_2} \). This renders false the claim that the use of random numbers as in (10) can be equivalently described by their quantum purifications as in (8). Further discussion on this point is given in Section 9. Here we note that (9) is, in general, a sufficient but not necessary (at least not having been proved necessary) condition for the protocol to be concealing, again to be further discussed in Section 9. It is rather a severe restriction on the protocol that can be relaxed to (8) with test for cheating.

A Type 2 protocol involving also the first technique of cheating detection may work as follows. Similar to QBC1, a large \( n \)-sequence \((n\)-fold tensor product\) of qubit states, drawn independently with probability \( \lambda_k \) from a fixed set \( S_0 = \{|\psi_k\rangle\} \), would be sent from Babe to Adam, each state named by its position in the sequence. Adam puts aside randomly chosen \( n_0 \) of them, and asks Babe to reveal the remaining \( n - n_0 \) ones for testing. For large enough \( n - n_0 \), Babe cannot use any \( |\Psi\rangle \in \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2} \) other than that of the form
(4) without getting caught with probability arbitrarily close to one, so that the concealing condition is (8), and not (9). If Adam randomly picks one of the remaining $n_0$, or $m$ for full unconditional security, modulates it by a single $U_b$ for each $b$, and return it without the name to Babe, she would not be able to use her entanglement (8) effectively on any qubit. This technique is similar to the use of decoy states from Adam to Babe in [16], and results in an effective concealing condition (10) in place of (8), although (8) still applies overall. While the use of a single $U_b$ does not allow Adam to cheat successfully on a fixed qubit, the freedom from the $n_0$-ensemble still allows him to entangle and launch an EPR attack. This attack is thwarted via the second technique of Section 5, which demands that Adam return the remaining $n_0 - 1$ qubits so Babe can verify that they have not been disturbed. Example 1, our protocol QBCp2, can be extended in this way to become an unconditionally secure protocol QBC2, which is a modified version of a protocol with the same name in Ref. [15]. Alternatively, the same logic applies to the following protocol, which is somewhat simpler.

PROTOCOL QBC2A

(i) Babe sends Adam $n$ qubits named by their temporal position, each drawn independently with equal probability from $S_0$, a fixed set of four possible BB84 states.

(ii) Adam randomly picks $n_0$ of these qubits and sets them aside, and asks Babe to open the remaining $n - n_0$ ones. He verifies them to be correct in that they are distributed as prescribed in step (i). Otherwise the protocol is aborted.

(iii) Adam randomly picks $m$ out of the $n_0$ remaining ones, modulates each by the same $U_0 = I$ or $U_1 = R(\pi)$, rotation by $\pi$ on the great circle containing $S_0$, and sends them back to Babe.

(iv) Adam opens by revealing $b$ and returning the remaining $n_0 - m$ qubits. Babe verifies by measuring the corresponding projectors.

By proper choice of $m$, $n_0$, and $n$, this protocol can be made both $\epsilon$-concealing and $\epsilon$-binding for any $\epsilon > 0$, given that Adam opens perfectly on $b = 0$. The main steps of the proof may be outlined as follows. Babe can cheat by entangling over each individual qubit.
and also by using a distribution of qubits more biased than the one presented in step (i). To defeat her qubit entanglement cheating, let $n_0/n = \epsilon_1$. The probability that she would pair $H^{B_{11}}$ with the correct $H^{B_{21}}$, where $H^B = H^{B_1} \otimes H^{B_2}$, $H^{B_m} = H^{B_{m1}} \otimes \ldots \otimes H^{B_{mn}}$, $m \in \{1, 2\}$, is thus $\epsilon_1$. If the pairing is incorrect, the trace distance in (7) is not affected because, for any three general states $\rho, \rho', \sigma$,

$$\|(\rho - \rho') \otimes \sigma\|_1 = \|\rho - \rho'\|_1.$$  \hspace{1cm} (11)

If the pairing is correct, we take the upper bound value of two for the trace distance. By making both $n_0$ and $n$ large and testing on the arbitrary $n - n_0$ qubits, one may guarantee, to within any $\epsilon_2 > 0$ for the resulting $P^B_c = 1/2 + \epsilon_2$ with $\epsilon_2 \to 0$ in the limit $n_0 \to \infty$ and $n \to \infty$, that the distribution of states in the two sets of qubits is indeed the one prescribed. Accordingly, Babe can only get $P^B_c = 1/2 + \epsilon_3$ for $\epsilon_3 \to 0$ from the $m$ committed qubits for any fixed $m$. This situation has been analyzed for QBC2 in Ref. [15]. From the union bound on probability, one may take $\epsilon_1 + \epsilon_2 + \epsilon_3 \leq \epsilon$, and the protocol becomes $\epsilon$-concealing. The asymptotic situation at $m, n_0, n \to \infty$ is quite apparent even in the absence of any quantification with respect to the $\epsilon$’s. The protocol is binding on Adam, because $m$ can be chosen large enough so that Adam’s optimum one-qubit cheating probability $p^m_A$ becomes $p^m_A \leq \epsilon$.

This QBC2A utilizes the first technique and denial of entanglement matching, in addition to its use of the second technique, which makes it Type 2. Even though there is yet no example, one cannot a priori rule out the possibility that the use of the second technique alone, as described in Section 3, would lead to an unconditionally secure protocol. Even if that turns out to be impossible, the impossibility proof formulation does not cover such situation, and need to be extended for a proof.
8 Classical randomness and quantum purification: Type 3 protocols

A cornerstone of the general impossibility proof is the assertion that classical randomness can be equivalently described as quantum determinacy via purification, say by (2), through the doctrine of “Church of the Larger Hilbert Space,” a technique also widely used in quantum coin tossing. But equivalent for what? In the following, we analyze the ways in which they are not equivalent for use by Babe in a QBC protocol. The best argument I know for their equivalence would be given alongside. Appendix B is essential for clarification of this issue.

First of all, it is clearly not true that all classical randomness can be reduced to that arising from quantum description of a system. After all, there were many scenarios for the occurrence of classical randomness before the rise of quantum physics, including especially classical statistical mechanics. Even if one grants a determinate quantum description for the underlying classical randomness involved, it is unreasonable to assume that any party would possess the detailed knowledge to write down the complete quantum description. However, in the context of QBC protocols, it is not only reasonable, but, in fact, mandatory to consider such purification (2) for which a party can form and use such purification for cheating. Thus it is a consideration of entanglement cheating, not the “Church of the Larger Hilbert Space,” that compels one to consider (2).

The following argument, in the spirit of the impossibility proof, appears to show that the exact \{|f_k\rangle\} in (2) need not be known by Adam for finding his cheating transformation. Let the protocol be \(\epsilon\)-concealing as a consequence of \(\rho_0^B(\Psi)\) being close to \(\rho_1^B(\Psi)\) for one \(|\Psi\rangle\) generated by Babe in the form (2). Assume Babe verifies by first measuring \{\(|f_k\rangle\}\} and then checking Adam’s opening. The commutativity of Adam’s and Babe’s operations shows that the protocol performance is the same whether Babe measures \{\(|f_k\rangle\}\} during commitment or after Adam opens. The fact that Adam can cheat after Babe measures \{\(|f_k\rangle\}\} shows that the cheating must be independent of the specific \{\(|f_k\rangle\}\}, even though it is obtained for a known \{\(|f_k\rangle\}\}. Note that this argument does not extend to the knowledge of \{\(\lambda_k\)\}.

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Nevertheless, even just for \(|f_k\rangle\) this argument contains a major gap, which is, in fact, a general gap in the impossibility proof: it is not guaranteed that there is only one verifying measurement for the protocol. In the particular case of randomness described above, it means that the split measurement of \(|f_k\rangle\) of \(\mathcal{H}^{B_1}\), and then a measurement on \(\mathcal{H}^{B_2}\), is not the verifying measurement of \(\Pi\) that has been proved susceptible to cheating as prescribed by the impossibility proof. It is not true that whenever \(\Pi\) is verified on a cheating state, then so is the split measurement. The cheating probability \(\bar{P}_c\) depends on the verifying measurement. For an arbitrary protocol, the impossibility proof formulation does not, and, in fact, cannot specify what the possible verifying measurements could be. There is no proof given that there cannot be more than one verifying measurement, for which different cheating transformations are needed. It turns out that for several types of protocols, though not for all, I can prove that this is indeed the case in the sense (IP) of (7) for all perfectly verifying measurements, i.e., measurements that yield the “yes” result with probability one corresponding to the opening bit value.

However, condition (8), which is taken to be the \(\epsilon\)-concealing condition in the impossibility proof, is not a proper concealing condition due to the Libertarian Principle. Indeed, while (8) implies that Adam can cheat according to the impossibility proof, the situation is misrepresented in that it may be Babe who can actually cheat by using a different \(|\Psi\rangle\). Two examples are given in Ref. [16]. It makes no sense to insist that Babe has to stick to a prescribed \(\{\lambda_k\}\), in contradiction to the Secrecy Principle, so that the protocol is concealing and Adam can cheat, while Babe can actually use a different \(\{\lambda_k\}\) and instead cheat successfully herself. There is no reason for Babe to commit such bit suicide. For any protocol, one cannot simply say that a protocol is now taken to be \(\epsilon\)-concealing. One has to describe quantitatively a necessary \(\epsilon\)-concealing condition for the protocol before any meaningful performance analysis can be made, which is something the impossibility proof fails to do in general. Thus, there is no impossibility proof whenever anonymous states \(|\Psi\rangle\) are used in a protocol.
Suppose that condition (3) is to be used, which is a sufficient condition that has not been shown to be necessary for concealing, as to be discussed later. For a class of anonymous-state protocols that are perfectly concealing, it may be shown [16, 19, 20] that the cheating $U^A$ is independent of any $\{\lambda_k\}$ and $\{|f_k\rangle\}$ in (2). The reason why secure protocols based on classical random numbers alone are hard to construct is not necessarily because one forgets quantum purification. It is because concealing under quantum purification is often more restrictive than concealing under classical randomness, as in protocol QBCp2 or Example 1. We say that a protocol is of Type 3 whenever states of the form (2) are used by Babe. An example is QBC3 of Ref. [16]. Under $\epsilon$-concealing (3) for such protocols, it is not known whether $P^A_c$ is close to one independently of $\{\lambda_k\}$. Thus, there is no impossibility proof if (8) in a Type 0 protocol with (2) is replaced by (9) or just (10). For such Type 3 protocols, unconditional security may arise in the following way. Since Adam does not know $\{\lambda_k\}$, one may consider first a fixed $\{\lambda_k\}$ and then average over all possible cheating $U^A$. Such an average cannot produce $P^A_c \sim 1$. The performance analysis for the overall situation seems rather involved, and new approaches may be needed to see whether security is actually provable. A direct approach to the analysis of such protocols is given in Ref. [19]. However, what we have here is actually a game-theoretic situation involving freedom on both sides with opposing objectives with regard to the performance criteria $P^A_c$ and $P^B_c$. It is most appropriate to regard $\{\lambda_k\}$, $\{p_i\}$, etc. as unknown with no meaningful distribution on them, a situation that happens in many problems of classical statistics whenever there is a lack of statistical regularity or meaningful ensemble, the situation we have here. See Appendix B for further elaboration.

It is argued in [21] that $\{\lambda_k\}$ has to be taken openly known in a meaningful protocol, because there is no guarantee that it can be kept secret. In any cryptographic protocol, one has to assume that anything one party does on her locality is not known to another party in a distant locality, relativity or not, or else nothing can be a secret, including a secret key. The issue is not why Adam does not know $\{\lambda_k\}$. It is why he would know. Indeed, one may
use the same reasoning and assume Babe knows $U^A$ so she can defeat Adam's cheating. The actual situation is that $\{\lambda_k\}$ is an unknown parameter in an infinite-dimensional space over $\mathbb{R}$ or $\mathbb{C}$, as discussed in Appendix B. The conclusion arrived at above can also be repeated in this regard. Under proper concealing (9), there is no need for Adam to know $\{\lambda_k\}$ in accordance with the Secrecy Principle. There is no way Adam can find out which particular $\{\lambda_k\}$ Babe uses. It is entirely her private affair. Without (9), Babe is not going to commit bit suicide with (8). She would use a different $\{\lambda_k\}$ instead.

Generally, it is difficult to pin down a necessary condition for $\epsilon$-concealing for an arbitrary protocol without utilizing specific information about the protocol details. In fact, the very meaning of concealing in an arbitrary protocol has to be decided upon. Thus, (9) may be too strong because Babe in general does not know the distribution $\{p_i\}$ on Adam's secret parameter $i$. It may not be necessary for $\epsilon$-concealing that $\rho^B_0 \sim \rho^B_i$ holds for any $\{p_i\}$ due to averaging or to the game situation involving $\{\lambda_k\}$ just discussed. Thus, a general impossibility proof for Type 3 protocols would face the immediate obstacle of not being able to specify quantitatively either a necessary $\epsilon$-concealing or $\epsilon$-binding condition. One the other hand, security proof for a particular protocol is much easier because sufficient conditions and protocol mechanism can be specifically exploited.

We summarize the main points concerning random numbers.

1. Classical randomness is not generally reducible to quantum uncertainty.

2. The condition of $\epsilon$-concealing with random numbers is not equivalent to its quantum purification version, i.e., (8) is not equivalent to (10).

3. The coefficients $\{\lambda_k\}$ in the quantum purification (2) are generally not known to the other party.

4. The concealing condition (8) used in the impossibility proof is, in general, neither necessary nor sufficient for concealing.

5. There is no general impossibility proof when anonymous states are involved in a pro-
tocol.

6. With random $k$ and $i$, it is difficult to formulate a necessary $\epsilon$-concealing or $\epsilon$-binding condition in order to start an impossibility proof.

7. The general situation of an unspecified protocol, even the simple case (I), is game-theoretic.
9 Too many possible questions to entangle: Type 4 protocols

In Type 4 protocols, the unentangled state $|\phi_{bi}\rangle$ is brought about from the entangled openly known $|\Phi_{b}\rangle$ through the asking of questions related to the evidence by Babe. As a consequence, one arrives at the situation discussed at the beginning of Section 3, where the randomness that makes up $\rho_{b}^{B}$ is not entangled under Adam. The ideas and procedure are best explained for the specific case of protocol QBC4 in the following.

Adam sends Babe a sequence of $n$ qubits, each in either one of $\{|\phi\rangle, |\phi'\rangle\}$, such that an even number of $|\phi'\rangle$ corresponds to $b = 0$, and an odd number to $b = 1$. As shown in Appendix C, the protocol is $\epsilon$-concealing for large $n$ for any $|\langle \phi | \phi' \rangle|^{2} = \epsilon_{1}$, and Adam has the usual EPR cheat with the entanglement

$$|\Phi_{0}\rangle = \sum_{i} \sqrt{p_{i}}|e_{i}\rangle|\phi_{0i}\rangle$$

(12)

for $p_{i} = 1/2^{n-1}$. It was suggested in v2 of this paper (quant-ph/0207089v2) that Babe now asks Adam to reveal to her $n - n_{0}$ qubits, randomly selected out of $n$, with $n_{0}$ remaining ones sufficient to ensure $\epsilon$-concealing. The idea is to force him to measure the $\{|e_{i}\rangle\}$ in (12) to pin down a specific $|\phi_{0i}\rangle$, thus destroying the entanglement. However, Adam can respond as follows. Let $i = (i_{1}, \ldots, i_{n})$, $i_{l} \in \{0, 1\}$, $l \in \{1, \ldots, n\}$, $|\phi_{0i}\rangle = |\phi_{0i_{1}}\rangle \cdots |\phi_{0i_{n}}\rangle$, $|\phi_{0i_{l}}\rangle \in \{|\phi\rangle, |\phi'\rangle\}$ in each $\mathcal{K}_{l2}$, $\mathcal{H}^{B} = \bigotimes_{l} \mathcal{K}_{l2}$. Then $|\Phi_{0}\rangle$ can be extended through local operation to

$$|\Phi'_{0}\rangle = \sum_{i} \sqrt{p_{i}}|i_{1}\rangle \cdots |i_{n}\rangle|e_{i}\rangle|\phi_{0i}\rangle$$

(13)

in $\mathcal{H}^{A'} \otimes \mathcal{H}^{A} \otimes \mathcal{H}^{B}$, $\mathcal{H}^{A'} = \bigotimes_{l=1}^{n} \mathcal{H}_{l2}$ a product of qubits, and $\langle i_{l} = 0|i_{l} = 1 \rangle = 0$ for each $l$ are two orthogonal states from the BB84 state set $S_{0}$ on a fixed great circle $C$ of each qubit, with $|i_{l} = 0\rangle$ corresponding to $|\phi\rangle$ and $|i_{l} = 1\rangle$ to $|\phi'\rangle$. In response to Babe’s question on a subset $S \subset \{i_{1}, \ldots, i_{n}\}$, $|S| = n - n_{0}$, Adam sends Babe the state spaces $\mathcal{H}_{l2}$ for all $l \in S$. Babe can measure on $\bigotimes_{l \in S} \mathcal{H}_{l2}$ to find the answer to her question and verify on the committed $\{|\phi_{0i_{l}}\rangle\}$
in $\mathcal{K}_{i_2}$’s. Since the protocol is concealing with Babe possessing $\mathcal{H}^B \otimes (\bigotimes_{l \in S} \mathcal{H}_{i_2})$ and Adam possessing $\mathcal{H}^{A'} \equiv \mathcal{H}^A \otimes (\bigotimes_{l \in S} \mathcal{H}_{i_2})$, $S^c = \{1, \ldots, n\} - S$, Adam can cheat successfully by finding the proper cheating transformation $U^{A'}$ on $\mathcal{H}^{A'}$.

Following our same idea, the protocol is now extended as follows. Babe may ask Adam to do the following instead. He is going to provide Babe with the $|i_l\rangle$ for all $l$, but with each $|i_l\rangle$ turned with probability $1/2$ on the great circle $C$ to the other two orthogonal BB84 states for $b = 0$ and $b = 1$. That is, with $S_0 = \{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ where $\langle 1|3\rangle = \langle 2|4\rangle = 0$, the $|i_l\rangle$ is equally probable to be $\{|1\rangle, |2\rangle\}$ for a $|\phi\rangle$, and $\{|3\rangle, |4\rangle\}$ for a $|\phi'\rangle$. The distribution would be across the $n$ $|i_l\rangle$’s. It is easy to see, as shown in Appendix C, that the protocol remains $\epsilon$-concealing for any $\epsilon_1$ by making $n_0$ sufficiently large. Now Babe asks Adam to reveal a random set of $n - n_0$ $|i_l\rangle$’s and verifies them on her corresponding $|\phi_{0i_l}\rangle$’s, making sure that all states in $S_0$ appear within, say, the Chernov bound limit. Since Adam has not entangled over the above randomness, from his point of view $\rho_0^B$ and $\rho_1^B$ are not close at all. Let $F$ be the Uhlmann fidelity $\text{tr} \sqrt{(\rho_0^B)^{1/2}\rho_1^B(\rho_0^B)^{1/2}}$ between $\rho_0^B$ and $\rho_1^B$ under (12), and $F'$ for the situation just described. It is difficult to evaluate $F'$, but one can bound it loosely by $F' \leq \frac{1}{\sqrt{2}}F$ from the fact that one has (13) now with a single tensor product extension $\bigotimes_l |i_l\rangle$ for each $|\phi_{0i}\rangle$ constructed from $S_0$. The factor $\frac{1}{\sqrt{2}}$ comes from the largest value of a single overlap $\langle i_l = 0|i_l = 1\rangle$, the minimum between $|\phi_{0i}\rangle$ and $|\phi_{1j}\rangle$. From (C.7) in Appendix C, Adam’s optimal cheating probability satisfies $\bar{P}_c^A \leq F_A(\rho_0^B, \rho_1^B)$ with $F_A$ computed from his point of view. Thus, we have arrived at $\bar{P}_c^A \leq \frac{1}{\sqrt{2}}$, contradicting the impossibility proof assertion (IP) of (7). By utilizing this scheme repeatedly, one may obtain an unconditionally secure protocol, as in the way described in QBC2 of (13). However, we can also achieve unconditional security as follows.

Babe now asks Adam to present $|i_l\rangle$ in two equally probable states $\{|1\rangle, |5\rangle\}$ with $|5\rangle = R(\theta)|1\rangle$ for $|\phi\rangle$ or $i_l = 0$, and in $\{|3\rangle, |6\rangle\}$ with $|6\rangle = R(\theta)|3\rangle$ for $|\phi'\rangle$ or $i_l = 1$. The angle $\theta$ is chosen so small that the overlap $\langle 5|3\rangle$ is $\epsilon$ instead of $\frac{1}{\sqrt{2}}$ in the case of $S_0$. Thus $\bar{P}_c^A \leq \epsilon$, while $\bar{P}_c^B$ can still be made $\leq \epsilon$ by having $n_0$ sufficiently large. Babe would ask Adam to reveal
of these $|i_i\rangle$’s and verify them as above. Adam’s probability of successful cheating by whatever action is exponentially small in $n - n_0$, but we do not need to quantify it under the assumption he needs to open perfectly for $b = 0$. That would only occur if he measures $\{|e_i\rangle\}$ in (L3) to answer. Babe can even postpone the measurement to the verification phase.

Adam cannot change (L3) to one that allows him to cheat from pre-entanglement on answers to the questions, due to local state invariance as follows. Consider just the pair of qubit states at the same $l$th position $|i_i\rangle|\phi_{0i}\rangle$ with all other qubits fixed. When a measurement of $\{|1\rangle, |3\rangle\}$ is performed on $H_{l2}$, the state on $K_{l2}$ is a superposition of $|\phi\rangle\langle\phi|$ and $|\phi'\rangle\langle\phi'|$ under (L3) when averaged over the measurement results. If the $|i_i\rangle$ is entangled to $|\phi_{0i}\rangle$ over all four states in $S_0$ or $\{|1\rangle, |5\rangle, |3\rangle, |6\rangle\}$ in the required form

$$\sum_{i_i=1}^{4} |i_i\rangle|\phi_{0i}\rangle$$

(14)

with the proper matching $|\phi_{0i}\rangle$, such a measurement would produce a state on $K_{l2}$ with non-vanishing interference terms $|\phi\rangle\langle\phi'|, |\phi'\rangle\langle\phi|$ that can be easily computed. No local operation on $H_{A'}$ can change the state of $K_{l2}$ this way, as a consequence of local state invariance stated in Appendix C. Intuitively, it is clear that Adam cannot entangle on top of entanglement. Mathematically, he can extend (L3) as a tensor product, but not as a direct sum.

More generally, Babe can ask Adam to present his answers for each qubit in any coded form in one qubit, two qubits, or $m$ qubits. She can adjoin an integer $m$ to precede $i$ in the binary representation with $m + i$ bits, present Adam with her secretly chosen computable and invertible arithmetical function $f : \mathbb{N} \rightarrow \{0, 1\}$, and ask for the answer $f(\{m, i\})$. If Adam is honest and submitted unentangled $|\phi_{0i}\rangle$, he could answer straightforwardly. With his entanglement, he could do the quantum computation in

$$|\Phi'_0\rangle = \sum_{i} \sqrt{p_i}|f(\{m, i\})\rangle|e_i\rangle|\phi_{0i}\rangle,$$

(15)

passing $H_{A'}$ with state $|f(\{m, i\})\rangle$ to Babe. Babe can ask for presentations as in the parity case above. Indeed, the possibilities of presenting the individual parity answers in different spaces alone lead to an infinite set of possible questions of arbitrarily large cardinality.
To be able to cheat under such open questioning, Adam has to pre-entangle the answer to every possible question. However, he can only pre-entangle the answers to a finite number of questions. Indeed, even if he pre-entangles an infinite number, he cannot locate the answer by an algorithm, say the set of Turing-computable functions alone is already not recursively enumerable. For the parity function, Babe can ask questions involving subset parities on the n qubits, already generating $2^n$ types of questions that is too many to entangle for $n > 410$ even if one can use all the physical resources in the Universe. See Appendix D for this physical limit.

As elaborated in Appendix B and in Sections 3-5, Babe can automatize a secretly chosen rule to specify the questions she may ask. If one wants to talk about probability, it is fair to say that the probability her question was pre-entangled by Adam is arbitrarily small. Alternatively, to avoid unfruitful terminological debate, one may just say that Adam can cheat if Babe’s questions drawn from an infinite set fall under Adam’s finite set of pre-entangled questions. This situation was not realized before, and serves effectively to produce an unconditionally secure protocol. It may be emphasized that if $|\Phi_b\rangle$ is openly known, Babe can always ask a further question that is not pre-entangled in it, thus rendering it unentangled, as discussed above.

### PROTOCOL QBC4

(i) Adam sends Babe a sequence of n qubits, each being either one of $\{|\phi\rangle, |\phi'\rangle\}$, such that an even number of $|\phi'\rangle$'s corresponds to $b = 0$, and an odd number to $b = 1$.

(ii) Babe asks Adam questions that are sufficient to pin down the total committed state, requiring him to present his answers in a specific randomized form.

(iii) Babe verifies some of the answers by further questions on how Adam actually randomized them.

(iv) Adam opens by revealing all the unknown states to Babe. She verifies by corresponding measurements.
This protocol bears a resemblance to a Type 3 protocol, in which Babe uses a parameter unknown to Adam. The difference is that an additional technique, an open questioning of evidence, is used to guarantee $\epsilon$-concealing for each unknown value that would require a different cheating arrangement by Adam, something that is difficult to achieve by means of anonymous states alone. Also, there is no discrete approximation to the infinite set of possibilities in this case, in contrast to the probabilities $\{\lambda_k\}$. Furthermore, when Adam misses the value, his cheating probability is vanishingly small, in contrast to a mistmatch between $U^A$ and $\{\lambda_k\}$.

To recapitulate the logic of Type 4 protocols: by asking open questions concerning the evidence with answers presented in a specific randomized form chosen secretly by her, Babe ensures that Adam can only cheat successfully by pre-entangling the whole question correctly. However, he can do that only with a vanishingly small probability. It is appropriate to emphasize that this type of protocols shows that, other than unknown parameters, the specifics of a protocol may play a significant role in rendering untenable the assertion that an openly known $|\Phi_b\rangle$ is obtained at the end of commitment. This issue has not been adequately addressed in the impossibility proof.
10 Summary and conclusion

If there is a general impossibility proof for secure QBC, one should be able to apply it schematically to any proposed QBC protocol to show that it is insecure. This often cannot be done. The reason is that the impossibility proof formulation is quite restrictive, and many nontrivial details in a systematic proof have not been spelled out. Some such criticisms have already been discussed in Ref. [15], but they are analyzed quantitatively in this paper.

We introduced several new techniques for protocol design, not covered by the impossibility proof formulation which only applies to what we call Type 0 protocols. We presented three new types of protocols:

- Type 1 — measurement for cheating detection,
- Type 2 — shifting of evidence state spaces,
- Type 3 — utilization of anonymous states,
- Type 4 — open questioning of evidence.

A specific Type 4 protocol, QBC4, is proved unconditionally secure. We indicated how a Type 1 protocol, QBC1, and a Type 2 protocol, QBC2A, may be proved secure. The situation is yet undecided for Type 3 protocols. There is no impossibility proof, but there is no protocol which is clearly secure either. A general theory of quantum statistical games needs to be developed for addressing many such QBC problems in a satisfactory manner.

The content of this paper hopefully makes clear the vast richness of this subject yet to be uncovered, especially for protocols that can be practically implemented in a realistic environment.
Appendix A: no impossibility theorem without QBC definition

It is generally believed by mathematicians that a mathematical theorem can only be obtained from precise mathematical definitions. In the impossibility proof of trisecting the angle $\pi/3$ by straightedge and compass only, for example, the action of these two instruments is precisely captured mathematically by a quadratic extension field of the rational numbers. Do we need a definition of a QBC protocol to have a theorem which says that unconditionally secure QBC is impossible? After all, E. Witten got a Fields Medal in mathematics for work that made essential use of the Feynman path integral that M. Atiyah, a former Fields medalist and judge on the medal’s decision panel, commented: “... provided one believes that the integral makes sense,” to which Witten had the reply: ”We have forty years of experience of computing these types of integrals” [22]. Regardless of one’s opinion concerning the Feynman path integral (which, I think, is one of the greatest scientific creations), it is not similar to a QBC protocol which, unlike the path integral, has no definite expression that could serve as a starting point.

A closer analogy to a QBC protocol is an “effectively computable” function, a function whose value for any specific argument can be “mechanically” obtained in a finite number of steps without the intervention of “intelligence.” The well-known Church-Turing thesis says that any effectively computable function can be computed recursively or by a Turing machine. It can be cast as an impossibility statement: there is no effective procedure that cannot be simulated by a Turing machine. It was found that a function that can be computed by a method that is clearly effective, such as Post machines and Markov algorithms, is indeed also Turing-computable. However, nobody calls the Church-Turing thesis the Church-Turing theorem. This is because there is no mathematical definition of an effective procedure. The logical possibility is open that someday a procedure is found that is intuitively or even physically effective, but which can compute a nonrecursive arithmetical function.

Thus, in the absence of a precise definition of a QBC protocol, one would have at best
an “impossibility thesis,” not an impossibility theorem. (This view was emphasized to the author by Masanao Ozawa.) This concern about definition is not scholasticism. There is no definition that would characterize all classical cryptographic protocols, say for bit commitment, partly because, I believe, of the open possibilities described in Section 5 of this paper. It is at least not clear why a definition in the more general quantum case can ever be found. Just as there appear to be many different forms of effective procedures, there are many different QBC protocol types that appear not to be captured by the impossibility proof formulation. To uphold just the “impossibility thesis,” one would need to prove that unconditionally secure QBC is impossible in each of these types — four of them are given in this paper. My contention is that not only is there no impossibility proof for these four types, but in fact unconditional security can be obtained in at least three of them.
Appendix B: unknown versus random parameters and “Church of the Larger Hilbert Space”

A considerable amount of confusion surrounds the equivalence between the use of classical random numbers and their quantum entanglement purification via the doctrine of “Church of the Larger Hilbert Space,” which is employed in various subareas of quantum information and quantum cryptography. There is also confusion about whether there exists a secret parameter with no probability distribution that can nevertheless be automatized by a machine. These questions are tangled up with a basic assumption or assertion of the impossibility proof that, in any QBC protocol, there is a publicly known pure state $|\Phi\rangle$ to start with, which results in a publicly known $|\Phi_b\rangle$ at the end of commitment, connected to $|\Phi\rangle$ by a publicly known unitary transformation. In this Appendix, we will show that in the making of a QBC cryptographic system, some external agent is always involved, and the system is always open; thus, the above assertion is untenable. In the process, we hope to bring out some clear demarcations that would dispel various confusions.

To begin with, not every unknown parameter can be, or should be, modelled as a random variable for different reasons, which is well-known in classical statistics. One reason is the impossibility of assigning probabilities to an infinite sample space in some situations, such as a uniformly distributed random variable with the values in the positive integers $\mathbb{N}$ or the real numbers $\mathbb{R}$, and similarly on general countably infinite or uncountable spaces. This situation occurs in QBC4 of Section 9, when the space of all possible actions is of arbitrarily large cardinality, or, say, just $|\mathbb{N}|$, even though everything is finite to start with. A second reason is that there may be no meaningful ensemble for the parameter $r$, which should be just left as an unknown parameter to be drawn from a given set, finite or infinite. This happens in various circumstances, such as the measurement of a physical (say, astronomical) characteristic that takes on a fixed value to be estimated. Such estimation of an unknown parameter without the use of a priori information on its distribution is, in fact, very common. In quantum teleportation, one talks about the fidelity of receiving a state $|\psi\rangle$ of a qubit that is just
unknown, not with respect to any uniform distribution, so that an ensemble is described by the density operator $I/2$. The ensemble is, rather, $|\psi\rangle, |\psi\rangle, \ldots$, and the scheme is supposed to work for any $|\psi\rangle$, not on $I/2$. A third reason is that often $r$ is subjected to the control and decision of an agent, and no probability distribution has a meaning in terms of relative frequency, as it may have in other cases. Indeed, the frequency interpretation of probability has been rejected from the beginning in decision theory [23], for applications to which the “subjective” interpretations are more meaningful. Not only may $r$ be used only once, but also the controlling agent may use it repeatedly (i.e., $r, r, \ldots$) in an actual ensemble once it is decided upon. There is no actual ensemble that yields $r_1, r_2, \ldots$ according to whatever probability distribution.

This last situation happens in the use of anonymous states in a QBC protocol. Suppose Babe generates (2) instead of $|\psi_k\rangle$ in an entanglement purification. The state of $\mathcal{H}^{B_1}$ or $\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$ is anonymous to Adam because he does not know exactly what it is — Babe has the freedom to choose $\lambda_k$ in (2). According to the Secrecy Principle of Section 5, under a proper concealing condition she can pick $\{\lambda_k\}$ with any rule made up by her and unknown to Adam, either for one use or in repeated uses of (2) for a sequence of different bit commitments. In such a sequence, she can use exactly the same value, or use different values generated according to the parameters decided by another value. For example, assuming all possible $\{\lambda_k\}$ form a finite set, she can pick one randomly and stick to it in a sequence of commitments. This can clearly be automatized, and the result does not appear to Adam as an ensemble with a distribution $\{\lambda_k\}$, but rather as an “ensemble” with one fixed unknown $\{\lambda_k\}$.

Now we are led to the equivalence between random numbers and their quantum purifications. I assert that the following sequence of states generated by random numbers with probability $\{\lambda_k\}$ in $\mathcal{H}^{B_1}$,

$$\phi_1, \ldots, \phi_l, \ldots \quad |\phi_l\rangle \in \{|\psi_k\rangle\}$$

(B.1)
is different from the $\mathcal{H}^{B_1}$ states in the sequence obtained by the purification $\Psi$ of (2):

$$
\Phi_1, \ldots, \Phi_l, \ldots \quad |\Phi\rangle = \sum_k \sqrt{\lambda_k} |\psi_k\rangle |f_k\rangle,
$$

where $|\Phi_l\rangle = |\Phi\rangle$ for every $l$. In a specific instance $\phi_l$ of (B.1) there is no average over $\{|\psi_k\rangle\}$, but there is always such an average for each $\Phi_l$. Under (B.2), which can be used for cheating in the form (1), the agent controlling $\mathcal{H}^{B_2}$ of (2) or $\mathcal{H}^A$ of (1) can select a preferred ensemble in $\mathcal{H}^{B_1}$ or $\mathcal{H}^B$, which makes EPR cheating possible. On the other hand, the ensemble in (B.1) is fixed and cannot be changed. Quantum entanglement is a physical resource that needs to be established. Not all randomness is reducible to that of quantum entanglement\(^\dagger\). Indeed, (B.1) does not allow EPR cheating. This is the situation in QBC4, created through Babe’s questioning on Adam’s measurement purification states. Note that the agent controlling (B.1) or (B.2) may choose to generate any $\psi_k$ on the $\phi_l$’s or $\Phi_l$’s. This situation with unknown parameters is also relevant to our Type 3 protocols.

The above difference can be rephrased as follows. In the density operator expansion

$$
\rho = \sum_k \lambda_k |\psi_k\rangle \langle \psi_k|,
$$

the randomness in $k$ may come from a variety of sources. If all of it comes from quantum entanglement, then (2) applies, and the agent controlling $\mathcal{H}^{B_2}$ can select the ensemble in $\mathcal{H}^{B_1}$. If some of it comes from elsewhere, it would not be equivalent to (2), and ensemble selection or entanglement cheating is limited or becomes impossible, depending on the exact form of the joint state. The occurrence of such non-entanglement randomness is always possible because the system is subject to intervention by agents. In any meaningful and realistic formulation of the problem, the agents’ possible actions are infinitely varying and open. They cannot be described as a public $|\Phi\rangle$ being transformed in a closed system to another public $|\Phi_b\rangle$. Indeed, for QBC there is in general a game-theoretic situation, where both parties can choose actions unknown to the other party.

\(^\dagger\)Note that even if it is, an agent can cheat only if he controls $\mathcal{H}^{B_2}$ in (2).
Appendix C: protocol QBC4

Here we fill in certain mathematical details on QBC4. We consider first the case when Babe asks no question on the evidence.

Adam can guarantee concealing by using uniform probability \(1/2^{n-1}\) for each sequence of either parity. In that case, \(\rho_0^B - \rho_1^B\) factorizes into products of individual qubit parts. Let \(j = \{j_1, \ldots, j_n\} \in \{0, 1\}^n\), \(P_{l0} = |\phi\rangle \langle \phi|\), \(P_{l1} = |\phi'\rangle \langle \phi'|\), \(l \in \{1, \ldots, n\}\). Let \(\Lambda_0 = \{j|\bigoplus_{l=1}^n j_l = 0\}\), \(\Lambda_1 = \{j|\bigoplus_{l=1}^n j_l = 1\}\) be the even- and odd-parity \(n\)-bit sets. Then

\[
\rho_b^B = \frac{1}{2^{n-1}} \sum_{j \in \Lambda_b} \bigotimes_{l=1}^n P_{lj_l}, \quad b \in \{0, 1\},
\]

and so

\[
\rho_0^B - \rho_1^B = \frac{1}{2^{n-1}} \bigotimes_{l=1}^n (P_{l0} - P_{l1}).
\]

Thus, Babe’s optimum quantum decision reduces to optimally discriminating between \(|\phi\rangle\) and \(|\phi'\rangle\) for each qubit individually, and then seeing whether there is an even or odd number of \(|\phi'\rangle\)'s. the optimum error probability \(p_e\) for each qubit is well-known \[24, 15\],

\[
p_e = \frac{1}{2} - \frac{1}{2} \sqrt{1 - |\langle \phi | \phi' \rangle|^2}.
\]

The optimum error probability \(\bar{P}_c^B\) of correct bit decision on the sequence is, from the even and odd binomial sums, given by

\[
\bar{P}_c^B = \frac{1}{2} + \frac{1}{2}(1 - 2p_e)^n.
\]

Thus \(\bar{P}_c^B\) is close to 1/2 exponentially in \(n\) independently of \(1/2 \geq p_e > 0\).

After committing \(|\Phi_0\rangle\), Adam can still try to cheat with the \(|e_i\rangle\) measurement by declaring one qubit to be in a state different from the actual one. the probability of success is

\[
P_A^A = |\langle \phi | \phi' \rangle|^2 \equiv \epsilon_1, \quad \text{a design parameter of the protocol. It can be made}\ \epsilon\text{-concealing by choosing}
\]

\[
|\langle \phi | \phi' \rangle|^2 \equiv \epsilon_1 \leq \epsilon
\]
and, from (C.4), choosing $n_0$ to satisfy

$$(1 - \epsilon_1)^{n_0} \leq 4\epsilon^2. \quad (C.6)$$

When Adam presents the additional $|i_l\rangle$’s in $S_0$ or other sets, Babe’s density operator $\rho_0^B$ is diagonal, similar to (B.2), in the basis that diagonalizes each pair $\mathcal{H}_{l2} \otimes \mathcal{K}_{l2}$. Her optimum decision reduces to optimally discriminating between the two density operators corresponding to $i_l = 0$ and $i_l = 1$ for each of these $n$ pairs, and then choosing the total resulting parity from the $n$ decisions. Thus, $\tilde{P}_c^B$ is given by (C.4) with $p_e$ given by the optimum pair decision, which just yields a different function of $\epsilon_1$ from (C.3). For any fixed $\epsilon_1$, $P_c^B$ can be made smaller than any $\epsilon$ as in (C.6) with a large enough $n_0$.

The following theorem characterizes Adam’s optimal probability of cheating $\tilde{P}_c^A$ when $|\Phi_0\rangle$ of (11) is used with resulting $\rho_b^B$ and $F$ between them.

**Theorem**

$$F^2 \leq \tilde{P}_c^A \leq F. \quad (C.7)$$

The bounds (C.7) are identical to (21) for $\tilde{P}_c^A$ in Ref. [16]. It can be seen from Appendix A, (6), and (18) of [16] that actually $\tilde{P}_c^A = \tilde{P}_c^A$.

The following theorem [15] is also used in Section 9.

**Theorem** (local state invariance). Let $\rho^{AB}$ be a state on $\mathcal{H}^A \otimes \mathcal{H}^B$ with $\rho^B \equiv \text{tr}_A \rho^{AB}$. The state $\rho^B$ remains invariant under any quantum operation on $\mathcal{H}^A$ alone.
Appendix D: physical limits and unconditional security

In cryptography, a system is typically called unconditionally secure if it cannot be broken with infinite computational power, i.e., its security is not based on computational complexity of any kind. In quantum cryptography, the system’s security depends on the validity of the laws of quantum physics and not on the limits of computational power, so this security is unconditional. More broadly, one can say that in physical cryptography, the system’s security is based on facts of our physical world which are immutable, and hence is unconditional also. Indeed, the laws of physics are part of the facts of Nature, which include both the laws and the initial conditions of the Universe that give rise to the world we live in. For example, we can exploit and utilize the background radiation from the sun or even the cosmos, because they are always there, not removable by any technological advance. Such physical limits are fundamentally different from ones that arise from computational complexity, quantum or classical.

Similarly, there are facts of nature that impose physical limits on the possible number of qubits one may use in entanglement. What may be surprising is that the number so limited is small on just an exponential scale. By various estimates, the total number of elementary fermions in the world is \( \lesssim 10^{89} < 2^{400} \) [23]. If \( E_1 \) is the energy range available with separation \( \Delta E \sim \hbar/\Delta t \), taking \( \Delta t \) to be the age of the Universe (\( < 2^{40} \) sec.), the total number of qubits available with energy \( E_1 \) is \( 2^{400} \log \frac{E_1 \Delta t}{\hbar} < 2^{410} \). For the boson electromagnetic field with total energy \( E_2 \), the number of qubits available is \( \Delta t \log \frac{E_2}{\Delta \tau \hbar} \) from the bit capacity of a boson field [26, 27]. Taking \( \tau \) to be the Planck time \( 10^{-44} \) sec and \( E_2 \) the total electromagnetic radiation energy in the Universe [23], this yields \( < 2^{380} \) qubits. Thus, one can entangle no more than \( \sim 2^{410} \) binary possibilities.
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