Comment on “Theory of Impure Superconductors: Anderson versus Abrikosov and Gor’kov”

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In a recent article, Kim and Overhauser have found fault with the theory of impure superconductors by Abrikosov and Gor’kov and have proposed an alternative formalism based on Green’s functions which are not derivable from a Dyson equation. Although the corrections to the Abrikosov-Gor’kov theory found by Kim and Overhauser are correct for non-retarded interactions, I argue that these corrections do not appear when one treats realistic retarded interactions within the field-theoretic approach of Abrikosov and Gor’kov. Direct numerical computation of the impurity-induced suppression of the superconducting transition temperature $T_c$ for both retarded and non-retarded interactions illustrates these points. I conclude that Abrikosov-Gor’kov theory applied to physical electron-electron interactions yields a tractable formalism which accurately predicts the effects of magnetic and non-magnetic impurities on $T_c$.

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A recent paper by Kim and Overhauser (KO) [1] discusses the effect of non-magnetic impurities on the critical temperature $T_c$ of a model superconductor mediated by a non-retarded BCS interaction within the context of the theories of Abrikosov and Gor’kov (AG) [2,3] and Anderson [4]. Literal evaluation [1] of the AG theory leads to the conclusion that $T_c$ should be strongly suppressed by these impurities, in contradiction to experiment. While there are also corrections to Anderson’s theory, a perturbative calculation of these corrections leads to the expected conclusion that the $T_c$ for normal metals is insensitive to small concentrations of normal impurities. In order to reconcile these two approaches, KO propose the use of “projected” Green’s functions, which yield a $T_c$ that is rigorously unaffected by normal scatterers as long as the usual BCS gap equation is valid. “Anderson’s Theorem” is thus regained.

The use of non-retarded interactions to model the behavior of superconducting [5,6] and superfluid [7] systems has produced a variety of important physical insights. The temptation, therefore, is to conclude from the KO results that the entire theory of impure superconductors based on the formalism created by Abrikosov and Gor’kov [2,3] is invalid. In this comment, I will demonstrate that, while the corrections to the Abrikosov-Gor’kov theory derived by Kim and Overhauser are correct for a non-retarded interaction, they do not imply that the conventional treatment of impurities in boson-mediated superconductors needs to be modified [8].

The fundamental point is that all known superconducting and superfluid Cooper pairing is mediated by retarded interactions, whether they be phonons in conventional superconductors, paramagnons in $^3$He, or possibly antiferromagnetic spin fluctuations in the heavy fermion and cuprate superconductors. As such, the pairing potential is frequency-dependent and is nonzero only up to some finite frequency $\omega_D$, but is unrestricted in wavevector space. Conventional BCS models approximate this full interaction with a frequency-independent pairing potential cut-off in wavevector space. While a useful toy model for pure superconductors, this approach leads to the unphysical results obtained by KO when applied to a dirty superconductor.
The most tractable way to treat a fully retarded interaction is to apply the field-theoretic approach of Abrikosov and Gor’kov as contained in strong-coupling Eliashberg theory \[9\]. This formalism is a mean-field theory in which the pairing boson is treated in the single-exchange-graph approximation and the impurities are included by a self-consistent perturbation theory. The advantages of this approach are that it can readily treat retarded interactions, it is easily generalized to include strong electron-boson coupling, and it may be cast in the form of a Dyson equation so that the equations can be solved numerically if not analytically.

Most importantly, however, this formalism produces results consistent with experiment when realistic retarded interactions are employed. The usual equations of Eliashberg theory require that the calculated \(T_c\) be unaffected by the presence of non-magnetic impurities \[9\]. Hence, Anderson’s Theorem \[4\] is obeyed. The slight sensitivity of \(T_c\) to normal defects in real materials can be attributed to anisotropy in the gap function, structure in the density of states, a finite bandwidth, or the change of the pairing potential and the band structure due to the presence of the impurities. All but the last of these have been successfully incorporated into Eliashberg theory and used to quantitatively describe the measured suppression of \(T_c\) by impurities in the conventional \[9,10,11\] and A15 \[12,13\] superconductors.

As a concrete example of the effect retardation can have on the response of a superconductor to impurities, Fig. 1 shows the calculated suppression of \(T_c\) by normal and magnetic impurities for both a non-retarded pairing potential given by the BCS model used in KO and a retarded pairing potential modeled by Einstein phonons. In all cases, the transition temperature of the pure material \(T_{c0}\) was set to 7.2 K, \(\omega_D = 88\) K, and \(E_F = 9.5\) eV in order to model Pb \[14,15\]. One can see that the non-retarded interaction shows a strong suppression of \(T_c\) due to normal impurities, in agreement with the analysis of Kim and Overhauser. However, the retarded interaction produces no such decrease in \(T_c\); in fact, to the numerical precision to which the Eliashberg equations were solved (about 2 %), \(T_c\) is completely unchanged by the presence of normal impurities \[16\]. In contrast, one can see from the inset to Fig. 1 that the Abrikosov-Gor’kov theory of magnetic impurities \[3\] is
qualitatively but not quantitatively obeyed for both the retarded and non-retarded pairing potentials. Essentially all of these results have appeared in the literature in one form or another; the numerical calculations in Fig. 1 simply present these results in a form that allows ready comparison between retarded and non-retarded interactions and magnetic and non-magnetic impurities.

If one still insists on using a non-retarded interaction and would like to keep Anderson’s Theorem, there are three possible approaches, the first two of which were pointed out by KO. The first is to transform the full electron-electron interaction from wavevector space into the space of exact eigenstates of the non-interacting, impure electron gas, apply the cut-off to the exact eigenenergies, and then use this potential in the appropriately modified BCS equations. The second method is to use these exact wave functions to construct electronic Green’s functions which contain only those wavefunctions whose eigenenergies are within $\omega_D$ of the Fermi energy and then to insert them into the BCS gap equation. The final option is to transform the Matsubara sum in the gap equation into a frequency integral and to truncate the frequency integral at $\omega_D$. Neither of the first two choices allow a computation of the properties of real materials, whereas the last is easily evaluated and is directly related to the actual retarded nature of the interaction which is neglected in this approximation. Clearly, the last option is preferred.

In conclusion, although the unphysical results obtained from the application of the Abrikosov-Gor’kov theory to non-retarded interactions pointed out by Kim and Overhauser are present, the AG theory yields the correct behavior when used with a physical retarded interaction. If one still wishes to consider the toy model of a non-retarded interaction in an impure superconductor, the most useful procedure is to apply the BCS cut-off in frequency rather than in wavevector space.
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For the Einstein-phonon-mediated superconductor, $\omega_D$ is taken to be the phonon frequency, while for the BCS model, it is the cut-off in wavevector space. Also, the density of states is taken to be flat with a band width $2E_F$ as in Ref. [1] and is fully accounted for when computing $T_c$. The non-retarded equations are those appearing in Ref. [16] generalized to include a cut-off less than the band width and a fully self-consistent treatment of the pairing interaction; the retarded equations are those from Refs. [12] and [13] modified for a flat density of states. Finally, the chemical potential is chosen so that the interacting band is half-filled.

This result is not as trivial as it sounds. F. Marsiglio, Phys. Rev. B 45, 956 (1992) has shown that $T_c$ is sensitive to normal impurities if the band width is on the order of or smaller than $T_c$. The band width here is too large to show such effects, but they do appear at smaller band widths, although the suppression of $T_c$ is not as strong as in the non-retarded calculations.
FIGURES

FIG. 1. Critical temperature relative to the critical temperature in the absence of impurities $T_c/T_{c0}$ vs. impurity scattering rate $\tau_{imp}^{-1}$ in meV for a non-retarded, BCS superconductor with normal (long dashed line) and magnetic (dot dashed line) impurities and a retarded, Einstein-phonon-mediated superconductor with normal (short dashed line) and magnetic (triple-dot dashed line) impurities. In all cases, the coupling strength is chosen to give $T_{c0} = 7.2$ K, the BCS cut-off and the Einstein phonon frequency are 88 K, and the Fermi energy is 9.5 eV in order to model Pb. Inset: $T_c/T_{c0}$ vs. the pair-breaking parameter $\alpha = 1/(1 + \lambda)\tau_{imp}T_{c0}$ for the magnetic impurity data from the main figure. The solid line is the predicted curve from the Abrikosov-Gor’kov theory [consult Ref. [3] and [9] for details].