Lower Bounding the AND-OR Tree via Symmetrization

William Kretschmer∗

Abstract

We prove a simple, nearly tight lower bound on the approximate degree of the two-level AND-OR tree using symmetrization arguments. Specifically, we show that $\tilde{\deg}(\text{AND}_m \circ \text{OR}_n) = \Omega(\sqrt{mn})$. We prove this lower bound via reduction to the OR function through a series of symmetrization steps, in contrast to most other proofs that involve formulating approximate degree as a linear program [BT13, She13, BDBGK18]. Our proof also demonstrates the power of a symmetrization technique involving Laurent polynomials (polynomials with negative exponents) that was previously introduced by Aaronson, Kothari, Kretschmer, and Thaler [AKKT20].

1 Introduction

1.1 History of the AND-OR Tree

The two-level AND-OR tree has played an important role in the study of quantum query complexity. Given a set of $mn$ inputs over $\{0, 1\}$, the problem is to compute the function $\text{AND}_m \circ \text{OR}_n = \bigwedge_{i=1}^m \bigvee_{j=1}^n x_{i,j}$. In the query model (see [BdW02]), we assume access to an oracle that on input $(i, j)$ returns the bit $x_{i,j}$, and our goal is to compute $\text{AND}_m \circ \text{OR}_n(x_{1,1}, \ldots, x_{m,n})$ in as few queries as possible.

One can show without too much difficulty that $\Theta(mn)$ queries are necessary and sufficient for any bounded-error randomized classical algorithm that computes $\text{AND}_m \circ \text{OR}_n$, by a standard adversary argument [SW86, Amb00]. But in the quantum query model, we can make queries on superpositions of inputs, and so the same lower bounds do not hold. In fact, one can do better in the quantum setting by using Grover’s algorithm [Gro96], which with high probability finds a marked item in a list of $n$ items using just $O(\sqrt{n})$ queries. By applying Grover’s algorithm recursively, along with an error reduction step on the inner subroutine, one obtains a quantum algorithm that makes just $O(\sqrt{mn})$ queries.

The problem of lower bounding the quantum query complexity of the two-level AND-OR tree proved to be much more challenging. Many of the early lower bounds on quantum query complexity were proved using the polynomial method of Beals et al. [BBC+01], which established a connection between quantum query complexity and approximate degree. The approximate degree of a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$, denoted $\tilde{\deg}(f)$, is defined as the least degree of a polynomial $p(x_1, \ldots, x_n)$ over the reals that pointwise approximates $f$. We focus on constant factor

∗University of Texas at Austin. Email: kretsch@cs.utexas.edu. Supported by a Simons Investigator Award.
approximations: that is, we require that for all \( X \in \{0, 1\}^n \), \( p(X) \in [0, \alpha] \) whenever \( f(X) = 0 \), and \( p(X) \in [\beta, 1] \) whenever \( f(X) = 1 \), for some constants \( 0 < \alpha < \beta < 1 \). Note that the choices of \( \alpha \) and \( \beta \) are arbitrary, in the sense that varying them changes the approximate degree by at most a constant multiplicative factor. Beals et al. [BBC+01] showed that for any quantum query algorithm that makes \( T \) queries to a string \( x \in \{0, 1\}^n \), the acceptance probability of the algorithm can be expressed as a real polynomial of degree at most \( 2T \) in the bits of \( x \). In particular, if the quantum query algorithm computes some function \( f \) with bounded error, then this establishes that \( \deg(f) \) asymptotically lower bounds the number of quantum queries needed to compute \( f \).

The AND-OR tree initially resisted attempts at lower bounds via approximate degree. At the time, one of the only known tools for lower bounding approximate degree was *symmetrization*, a technique that involves transforming a (typically symmetric) multivariate polynomial into a univariate polynomial.\(^1\) One typically appeals to classical results in approximation theory to analyze the resulting univariate polynomial. As an example, Paturi [Pat92] used symmetrization arguments to tightly characterize the approximate degree of all symmetric Boolean functions (i.e. functions that only depend on the Hamming weight of the input). While proofs by symmetrization are usually straightforward and easy to understand, for more complicated functions, symmetrization appears to have limited power in proving lower bounds. Indeed, symmetrization is inherently a lossy technique: a univariate polynomial can only capture part of the behavior of a multivariate polynomial.

Motivated by this difficulty, Ambainis [Amb00] introduced a quantum analogue of the classical adversary method, and used it to prove the first tight lower bound of \( \Omega(\sqrt{mn}) \) for the quantum query complexity of \( \text{AND}_m \circ \text{OR}_n \).\(^2\) Still, the AND-OR tree was the simplest Boolean function for which no tight approximate degree lower bound was known. Aaronson [Aar08] even re-posed the question of resolving \( \overline{\deg(\text{AND}_m \circ \text{OR}_n)} \) as a challenge problem for developing techniques beyond symmetrization.

After a series of successively tighter lower bounds on the AND-OR tree (see Table 1), the approximate degree of \( \text{AND}_m \circ \text{OR}_n \) was finally resolved in 2013, independently by Sherstov [She13] and Bun and Thaler [BT13]. They showed that \( \overline{\deg(\text{AND}_m \circ \text{OR}_n)} = \Omega(\sqrt{mn}) \), answering a question that had been open for nearly two decades. Both lower bound proofs used the method of “dual witnesses” (or “dual polynomials”), which involves formulating approximate degree as a linear program, then exhibiting a solution to the dual linear program to prove a lower bound on approximate degree. The dual witness method has the advantage that it can theoretically prove tight lower bounds for every Boolean function, and for general Boolean functions, the technique appears unavoidable. Indeed, many of the recent advances in polynomial approximation lower bounds seem to require such machinery [BKT18, SW19, AKKT20]. Nevertheless, dual witness proofs tend to be more complicated, and finding and verifying explicit dual witnesses can be difficult (for example, notice in Table 1 that it took 4 years to improve the lower bound between [She09] and [BT13, She13]!). To this date, essentially all tight lower bounds on \( \overline{\deg(\text{AND}_m \circ \text{OR}_n)} \) rely on the dual formulation of approximate degree in some capacity.\(^3\)

\(^1\)Strictly speaking, the term “symmetrization” originally referred to the process of averaging a multivariate polynomial over permutations of its inputs, giving rise to a symmetric polynomial [MP69]. We follow the convention of more recent work (e.g. [She09, BT16, AKKT20]), and use use “symmetrization” more generally to mean any process of transforming a multivariate polynomial to a univariate polynomial in a symmetry-exploiting, degree non-increasing way.
\(^2\)Note that Ambainis’ \( \Omega(\sqrt{mn}) \) lower bound [Amb00] predated the tight \( O(\sqrt{mn}) \) upper bound [HMcW03], but the \( O(\sqrt{mn \log m}) \) upper bound was known at the time.
\(^3\)Recently, Ben-David, Bouland, Garg, and Kothari [BDBG18] proved that \( \overline{\deg(\text{AND}_m \circ f)} = \Omega\left(\sqrt{m \deg(f)}\right) \) for any total Boolean function \( f \). By a reduction involving polynomials derived from quantum algorithms, they show
Table 1: History of lower bounds for \( \tilde{\deg}(\text{AND}_n \circ \text{OR}_n) \).

| Bound | Primary Technique | Reference |
|-------|-------------------|-----------|
| \( \Omega(\sqrt{n}) \) | Symmetrization | Nisan and Szegedy [NS92] |
| \( \Omega(\sqrt{n \log n}) \) | Symmetrization | Shi [Shi02] |
| \( \Omega(n^{2/3}) \) | Other | Ambainis [Amb05] |
| \( \Omega(n^{3/4}) \) | Dual witnesses | Sherstov [She09] |
| \( \Omega(n) \) | Dual witnesses | Bun and Thaler [BT13]; Sherstov [She13] |
| \( \Omega(n) \) | Other (See Footnote 3) | Ben-David, Bouland, Garg, and Kothari [BDBGK18] |
| \( \Omega\left(\frac{n}{\log n}\right) \) | Symmetrization | This paper |

1.2 Our Contribution: a Simpler Lower Bound

In this work, we prove a nearly tight lower bound on the AND-OR tree using very different techniques. Rather than going through the dual formulation of approximate degree directly, we lower bound the AND-OR tree via reduction to the OR function through a series of symmetrization steps. Thus, our proof technique more closely mirrors some of the oldest lower bound proofs for symmetric Boolean functions [MP69, NS92, Pat92]. Our proof is completely self-contained, with the exception of the lower bound on the OR function (which is easily proved via the classical Markov brothers’ inequality [NS92]). Ultimately, we show that \( \tilde{\deg}(\text{AND}_m \circ \text{OR}_n) = \Omega\left( \frac{\sqrt{mn}}{\log m} \right) \), which is tight up to the log factor.

Crucially, our proof relies on a technique due to Aaronson, Kothari, Kretschmer, and Thaler [AKKT20]: we use \textit{Laurent polynomials} (polynomials that can have both positive and negative exponents) and an associated symmetrization (Lemma 3) that reduces bivariate polynomials to univariate polynomials. In fact, our proof outline is very similar to the lower bound in [AKKT20] on the one-sided approximate degree of \( \text{AND}_2 \circ \text{ApxCount}_{N,w} \). At a high level, our proof begins with what is essentially a lower bound on the degree of a “robust”, partially symmetrized polynomial that approximates \( \text{AND}_m \circ \text{OR}_n \). In particular, we show a generalization of the following:

\[ \textbf{Theorem 1 (Informal).} \quad \text{Suppose that } p(x_1, \ldots, x_m) \text{ is a polynomial with the property that for all } (x_1, \ldots, x_m) \in [0, n]^m: \]

1. \text{If } x_i \leq \frac{1}{3} \text{ for some } i, \text{ then } 0 \leq p(x_1, \ldots, x_m) \leq \frac{1}{3}.

2. \text{If } x_i \geq \frac{2}{3} \text{ for all } i, \text{ then } \frac{2}{3} \leq p(x_1, \ldots, x_m) \leq 1.

Then \( \deg(p) = \Omega(\sqrt{mn}) \).

To give intuition, the variables \( x_i \) roughly correspond to the Hamming weight of the inputs to each OR\(_n\) gate. Indeed, any polynomial that satisfies the statement of the theorem can be turned into one that approximates \( \text{AND}_m \circ \text{OR}_n \) by letting \( x_i \) equal the sum of the \( \{0, 1\} \) inputs to the \( i \)th OR\(_n\) gate. However, the polynomial is also required to be “robust” in the sense the polynomial must behave similarly when \( x_i \) is not an integer.

---

Footnote 3: Ben-David, Bouland, Garg, and Kothari [BDBGK18] construct an explicit dual witness, the lower bound on \( \tilde{\deg}(\text{XOR}_m \circ f) \) relies on
The proof of Theorem 5 works as follows: we group the $m$ variables into $\frac{m}{2}$ pairs and apply the Laurent polynomial symmetrization to each pair. We argue that this has the effect of “switching” the role of AND and OR, in the sense that the resulting polynomial (in $\frac{m}{2}$ variables) looks like a partially symmetrized polynomial that approximates $\text{NOR}_{m/2} \circ \text{OR}_{\Theta(n)} = \text{NOR}_{\Theta(mn)}$, which has approximate degree $\Omega(\sqrt{mn})$ by known lower bounds [NS92].

We then show (Theorem 6) that starting with a polynomial that approximates $\text{AND}_m \circ \text{OR}_n$, we can “robustly symmetrize” to construct a polynomial of the same degree that behaves like the one in the statement of Theorem 5, at the cost of a log $m$ factor in the lower bound on the degree of the polynomial. This polynomial is obtained by applying the “erase-all-subscripts” symmetrization (Lemma 2) to the variables corresponding to each OR$_n$ gate, producing a polynomial in $m$ variables. This immediately implies (Corollary 7) that any polynomial that approximates $\text{AND}_m \circ \text{OR}_n$ has degree $\Omega\left(\frac{\sqrt{mn}}{\log m}\right)$.

2 Preliminaries

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, and use log to denote the natural logarithm. We will need the following two symmetrization lemmas, which were both introduced in these forms in [AKKT20]. However, Lemma 2 is also a folklore result that previously appeared e.g. in [Shi02] under the name “linearization”:

Lemma 2 (Erase-all-subscripts symmetrization [Shi02, AKKT20]). Let $p(x_1, \ldots, x_n)$ be a real multilinear polynomial, and for any real number $\rho \in [0,1]$, let $B(n, \rho)$ denote the distribution over $\{0,1\}^n$ wherein each coordinate is selected independently to be 1 with probability $\rho$. Then there exists a real polynomial $q$ with $\deg(q) \leq \deg(p)$ such that for all $\mu \in [0,n]$:

$$q(\mu) = \mathbb{E}_{(x_1, \ldots, x_n) \sim B(n, \mu)}[p(x_1, \ldots, x_n)].$$

Proof. Write:

$$q(\mu) = p\left(\frac{\mu}{n}, \frac{\mu}{n}, \ldots, \frac{\mu}{n}\right).$$

Then the lemma follows from linearity of expectation, because $p$ is assumed to be multilinear. ■

Lemma 3 (Laurent polynomial symmetrization [AKKT20]). Let $p(x_1, x_2)$ be a real polynomial that is symmetric (i.e., $p(x_1, x_2) = p(x_2, x_1)$ for all $x_1, x_2$). Then there exists a real polynomial $q(t)$ with $\deg(q) \leq \deg(p)$ such that for all real $s$, $q(s + 1/s) = p(s, 1/s)$.

Proof. Write $p(s, 1/s)$ as a Laurent polynomial $\ell(s)$ (a polynomial in $s$ and $1/s$). Because $p$ is symmetric, we have that $\ell(s) = p(s, 1/s) = p(1/s, s) = \ell(1/s)$. This implies that the coefficients of the $s^i$ and $s^{-i}$ terms of $\ell(s)$ are equal for all $i$, as otherwise $\ell(s) - \ell(1/s)$ would not be identically zero. Write $\ell(s) = \sum_{i=0}^d a_i \cdot (s^i + s^{-i})$ for some coefficients $a_i$, where $d \leq \deg(p)$. Then, it suffices to show that $s^i + s^{-i}$ can be expressed as a real polynomial of degree $i$ in $s + 1/s$ for all $0 \leq i \leq d$.

We prove by induction on $i$. The case $i = 0$ is a constant polynomial. For $i > 0$, observe that $(s + 1/s)^i = s^i + s^{-i} + r(s)$, where $r(s)$ is some real Laurent polynomial of degree $i - 1$ satisfying dual witnesses in an essential way.
Lemma 3 gives a polynomial. Theorem 5 can be applied to polynomials with additional variables, and which states that every symmetric polynomial in \( s + 1/s \) (by the induction assumption).

We remark that Lemma 3 can also be viewed as a consequence of the fundamental theorem of symmetric polynomials, which states that every symmetric polynomial in \( n \) variables can be expressed uniquely as a polynomial in the elementary symmetric polynomials in \( n \) variables. In 2 variables, the elementary symmetric polynomials are \( x + y \) and \( xy \). So, if \( p(x, y) = q(x + y, xy) \), then restricting \( p \) to the set \( \{ (s, 1/s) : s \in \mathbb{R} \} = \{ (x, y) : xy = 1 \} \) and writing in terms of \( s \) corresponds to taking \( q(s + 1/s, 1) \), which is just a polynomial in \( s + 1/s \). (Of course, one would also have to show that this transformation can be applied in a degree-preserving way, as the proof of Lemma 3 does).

Note also that Lemma 2 and Lemma 3 can be applied to polynomials with additional variables, because of the isomorphism between the polynomial rings \( \mathbb{R}[x_1, \ldots, x_n] \) and \( \mathbb{R}[x_1, \ldots, x_k][x_{k+1}, \ldots, x_n] \). For example, the Laurent polynomial symmetrization can be applied more generally to any polynomial \( p(x_1, x_2, \ldots, x_n) \) that is symmetric in \( x_1 \) and \( x_2 \) by rewriting \( p \) as a sum of the form:

\[
p(x_1, x_2, \ldots, x_n) = \sum_i f_i(x_1, x_2) \cdot g_i(x_3, \ldots, x_n)
\]

where \( \{f_i\} \) and \( \{g_i\} \) are sets of polynomials and the \( f_i \)'s are all symmetric. Then, symmetrizing each \( f_i \) according to Lemma 3 gives a polynomial \( q(s + 1/s, x_3, \ldots, x_n) = p(s, 1/s, x_3, \ldots, x_n) \).

Finally, we note the tight characterization of the approximate degree of OR and AND:

**Lemma 4** ([NS92]). \( \deg(\text{OR}_n) = \deg(\text{AND}_n) = \Theta(\sqrt{n}) \).

## 3 Main Result

We begin with the following theorem, which essentially lower bounds the degree of a robust, partially symmetrized polynomial that approximates \( \text{AND}_m \circ \text{OR}_n \). Note that the case with \( a \) and \( b \) constant and \( m = 2 \) was essentially proved in [AKKT20].

**Theorem 5.** Let \( 0 < \alpha < \beta < 1 \) be arbitrary constants, and let \( 0 < a < b < n \) such that \( \frac{b}{a} < \frac{b}{p} \) also holds. Let \( m \geq 2 \), and suppose that \( p(x_1, \ldots, x_m) \) is a polynomial with the property that for all \( (x_1, \ldots, x_m) \in [0, n]^m \):

1. If \( x_i \leq a \) for some \( i \), then \( 0 \leq p(x_1, \ldots, x_m) \leq \alpha \).
2. If \( x_i \geq b \) for all \( i \), then \( \beta \leq p(x_1, \ldots, x_m) \leq 1 \).

Then \( \deg(p) = \Omega \left( \frac{n}{b-a} \sqrt{\frac{mn}{b-a}} \right) \).

We first give a more intuitive, high-level overview of the proof of Theorem 5. For a symmetric polynomial \( p(x_1, x_2) \), the Laurent polynomial symmetrization yields a polynomial \( q(t) \) that captures the behavior of \( p \) restricted to a hyperbola in which \( x_1 \) and \( x_2 \) are inversely proportional. Larger \( t \) correspond to points on the hyperbola that are further from the origin, and thus correspond to points where one of \( x_1 \) or \( x_2 \) is small and the other is large. Smaller \( t \) correspond to points on the hyperbola where both \( x_1 \) and \( x_2 \) are reasonably large.

---

4Indeed, our proof even mirrors the standard proof of the fundamental theorem of symmetric polynomials.
The polynomial \( p(x_1, \ldots, x_m) \) that we start with has the property that if any input \( x_i \) is small (less than \( a \)), then the polynomial should be close to 0. Otherwise, if all inputs are large (greater than \( b \)), then the polynomial should be close to 1. The key observation is that the Laurent polynomial symmetrization essentially reverses the role of small and large inputs. The idea is to group the inputs of \( m/2 \) pairs and apply the Laurent polynomial symmetrization to each pair, resulting in a polynomial \( q(t_1, \ldots, t_{m/2}) \) in half as many variables. Now, when some input \( t_i \) to \( q \) is large, this corresponds to some pair of variables in \( p \) where one is small and one is large. Conversely, when all \( t_i \)'s are small, this corresponds to the case where all inputs to \( p \) are reasonably large. As a result, if any input \( t_i \) to \( q \) is large, then \( q \) should be close to 0, and otherwise, if all inputs are small, then \( q \) should be close to 1—precisely the reverse of \( p \). For additional clarification, a diagram of these steps is shown in Figure 1.

It remains to lower bound the degree of \( q \). We do so by observing that \( q \) can be turned into a polynomial that approximates the NOR function, whose approximate degree we know from Lemma 4.

**Proof.** Assume that \( m \) is even, since we can always set \( x_m = b \) and consider the polynomial \( p(x_1, \ldots, x_{m-1}, b) \) on \( m - 1 \) variables instead. Assume without loss of generality that \( p \) is symmetric in \( x_1, \ldots, x_m \), because we can always replace \( p \) by its average over all \( m! \) permutations of the inputs. Group the variables into \( m/2 \) pairs \( \{(x_{2i-1}, x_{2i}) : i \in [m/2]\} \) and apply the Laurent polynomial symmetrization (Lemma 3) to each pair to obtain a polynomial \( q(t_1, \ldots, t_{m/2}) \) in corresponding variables \( \{t_i : i \in [m/2]\} \) with \( \deg(q) \leq \deg(p) \). We think of \( t_i = s_i + 1/s_i \) as corresponding to the restriction \( (x_{2i-1} = bs_i, x_{2i} = b/s_i) \) (note that this involves rescaling each input by \( b \) in applying Lemma 3). Then we observe that for all \( (t_1, \ldots, t_{m/2}) \in [2, b + \frac{b}{n}]^{m/2} \):

1. \( 0 \leq q(t_1, \ldots, t_{m/2}) \leq \alpha \) when \( t_i \geq \frac{b}{a} + \frac{a}{b} \) for some \( i \), as \( t_i \geq \frac{b}{a} + \frac{a}{b} \) corresponds to either \( s_i \geq \frac{b}{a} \)
or \( s_i \leq \frac{a}{b} \), which corresponds to either \( x_{2i} \leq a \) or \( x_{2i-1} \leq a \), respectively. (This is where we need the assumption \( \frac{n}{b} > \frac{b}{a} \), as otherwise this case never holds).

2. \( \beta \leq q(t_1, \ldots, t_{m/2}) \leq 1 \) when \( t_i = 2 \) for all \( i \), as \( t_i = 2 \) corresponds \( s_i = 1 \), which corresponds to \( x_{2i-1} = x_{2i} = b \).

Perform an affine shift of \( q \) with \( \bar{t}_i = (t_i - 2) \frac{ab}{(b-a)^2} \) to obtain \( \bar{q}(\bar{t}_1, \ldots, \bar{t}_{m/2}) = q(t_1, \ldots, t_{m/2}) \).

The reason for this choice is for convenience, so that the cutoffs on \( t_i \) for the inequalities above become 1 and 0, respectively. It is easiest to see this by using the identity \( \frac{n}{b} + \frac{b}{a} - 2 = \frac{(b-a)^2}{ab} \).

Let \( k = \left( \frac{n}{b} + \frac{b}{a} - 2 \right) \frac{ab}{(b-a)^2} = \frac{2(n-b)^2}{n(b-a)^2} \), so that \( t_i = \frac{n}{b} + \frac{b}{a} \) corresponds to \( \bar{t}_i = k \). Note that \( k \geq 1 \) because the affine transformation is monotone, and \( t_i = \frac{n}{b} + \frac{b}{a} \) corresponds to \( \bar{t}_i = 1 \). Then we observe that for all \((\bar{t}_1, \ldots, \bar{t}_{m/2}) \in [0, k]^{m/2}\):

1. \( 0 \leq \bar{q}(\bar{t}_1, \ldots, \bar{t}_{m/2}) \leq \alpha \) when \( \bar{t}_i \geq 1 \) for some \( i \), as \( \bar{t}_i \geq 1 \) corresponds to \( t_i \geq \frac{n}{b} + \frac{a}{b} \).

2. \( \beta \leq \bar{q}(\bar{t}_1, \ldots, \bar{t}_{m/2}) \leq 1 \) when \( \bar{t}_i = 0 \) for all \( i \), as \( \bar{t}_i = 0 \) corresponds \( t_i = 2 \).

Notice that \( \bar{q} \) approximates a partially symmetrized NOR function. Now, we “un-symmetrize” \( \bar{q} \).

Let \( \bar{t}_i = \bar{t}_{i,1} + \bar{t}_{i,2} + \cdots + \bar{t}_{i,k} \).

Then \( \bar{q}(\bar{t}_{1,1} + \cdots + \bar{t}_{1,k}, \bar{t}_{2,1} + \cdots + \bar{t}_{2,k}, \ldots, \bar{t}_{m/2,1} + \cdots + \bar{t}_{m/2,k}) \)

approximates \( \text{NOR}_{m[k]/2} \) over the variables \((\bar{t}_{1,1}, \ldots, \bar{t}_{m/2,1}) \in \{0,1\}^{m[k]/2}\). Since we know that \( \text{deg}(\text{NOR}_{m[k]/2}) = \text{deg}(\text{OR}_{m[k]/2}) = \Omega(\sqrt{mk}) \) (Lemma 4), and since this construction satisfies \( \text{deg}(\bar{q}) = \text{deg}(q) \leq \text{deg}(p) \), we conclude that \( \text{deg}(p) = \Omega\left( \frac{n-k}{b-a} \sqrt{\frac{mk}{n}} \right) \).

Next, we show that a polynomial that approximates \( \text{AND}_m \circ \text{OR}_n \) can be “robustly symmetrized” like in the statement of Theorem 5 with \( a = O(1) \) and \( b = O(\log m) \).

**Theorem 6.** Let \( p(x_1, \ldots, x_{m, n}) \) be a polynomial in variables \{\(x_{i,j} : (i, j) \in [m] \times \{0, 1\} \}\) that \( \frac{1}{3}\)-approximates \( \text{AND}_m \circ \text{OR}_n \), where \( n > 2\log m \) and \( m \geq 10 \). Specifically, we assume that \( 0 \leq p(x_1, \ldots, x_{m, n}) \leq \frac{1}{3} \) on a 0-instance, and \( \frac{2}{3} \leq p(x_1, \ldots, x_{m, n}) \leq 1 \) on a 1-instance. Then there exists a polynomial \( q(x_1, \ldots, x_m) \) with \( \text{deg}(q) \leq \text{deg}(p) \) such that for all \((x_1, \ldots, x_m) \in [0, n]_m\):

1. \( \text{If } x_i \leq \frac{1}{6} \text{ for some } i, \text{ then } 0 \leq q(x_1, \ldots, x_m) \leq \frac{1}{2} \).

2. \( \text{If } x_i \geq 2\log m \text{ for all } i, \text{ then } \frac{2}{3} \leq q(x_1, \ldots, x_m) \leq 1 \).

**Proof.** Assume without loss of generality that \( p \) is multilinear (because \( x^2 = x \) over \([0, 1]\)), so that we can apply the erase-all-subscripts symmetrization (Lemma 2) separately to the inputs of each OR gate. This erases all of the “j” subscripts, giving a polynomial \( q(x_1, \ldots, x_m) \) with \( \text{deg}(q) \leq \text{deg}(p) \) such that:

\[
q(x_1, \ldots, x_m) = \mathbb{E}_{(x_{i,1}, \ldots, x_{i,n}) \sim B(n, \frac{\rho}{n})} [p(x_{i,1}, \ldots, x_{m,n})]
\]

where \( B(n, \rho) \) denotes the distribution over \([0, 1]^n\) where each coordinate is selected from an independent Bernoulli distribution with probability \( \rho \).

Suppose \( x_i \in [2\log m, n] \) for all \( i \in [m] \). Then we have that:

\[
\Pr_{(x_{i,1}, \ldots, x_{i,n}) \sim B(n, \frac{\rho}{n})}[\text{AND}_m \circ \text{OR}_n(x_{1,1}, \ldots, x_{m,n}) = 0] \leq \sum_{i=1}^{m} \Pr[(x_{i,1}, \ldots, x_{i,n}) = 0^n]
\]

(1)
\[ \leq m \cdot \left(1 - \frac{2 \log m}{n}\right)^n \]  \hspace{1cm} (2)
\[ \leq \frac{1}{m} \]  \hspace{1cm} (3)

where (1) follows from a union bound, (2) follows by expanding each term, and (3) follows from
the exponential inequality.

On the other hand, suppose \( x_i \in [0, n] \) for all \( i \in [m] \), and that \( x_i^* \in [0, \frac{1}{6}] \) for some \( i^* \in [m] \). Then we have that:
\[ \Pr_{(x_{i,1}, \ldots, x_{i,n}) \sim B(n, \frac{x_i}{n})} [\text{AND}_m \circ \text{OR}_n(x_{1,1}, \ldots, x_{m,n}) = 0] \geq \Pr [(x_{i^*,1}, \ldots, x_{i^*,n}) = 0^n] \]
\[ \geq \left(1 - \frac{1}{6n}\right)^n \]
\[ \geq \frac{5}{6} \]

by similar inequalities.

Since in general we can bound \( q(x_1, \ldots, x_m) \) by
\[ q(x_1, \ldots, x_m) \geq \frac{2}{3} \cdot \Pr_{(x_{i,1}, \ldots, x_{i,n}) \sim B(n, \frac{x_i}{n})} [\text{AND}_m \circ \text{OR}_n(x_{1,1}, \ldots, x_{m,n}) = 1] \]

and
\[ q(x_1, \ldots, x_m) \leq \frac{1}{3} \cdot \Pr [\text{AND}_m \circ \text{OR}_n(x_{1,1}, \ldots, x_{m,n}) = 0] + 1 \cdot \Pr [\text{AND}_m \circ \text{OR}_n(x_{1,1}, \ldots, x_{m,n}) = 1] \]

we can conclude that for all \((x_1, \ldots, x_m) \in [0, n]^m\):

1. \( 0 \leq q(x_1, \ldots, x_m) \leq \frac{1}{3} \cdot 1 + 1 \cdot \frac{1}{6} = \frac{1}{2} \) when \( x_i \leq \frac{1}{6} \) for some \( i \).
2. \( \frac{3}{5} \leq \frac{2}{3} \left(1 - \frac{1}{m}\right) \leq q(x_1, \ldots, x_m) \leq 1 \) when \( x_i \geq 2 \log m \) for all \( i \).

Putting these two theorems together gives:

**Corollary 7.** \( \widetilde{\text{deg}}(\text{AND}_m \circ \text{OR}_n) = \Omega \left(\sqrt{mn \log m}\right) \).

**Proof.** If \( n \leq 24 \log^2 m \), then the theorem holds trivially by the lower bound on \( \widetilde{\text{deg}}(\text{AND}_m) \) (Lemma 4), since \( \text{deg}(\text{AND}_m \circ \text{OR}_n) \geq \widetilde{\text{deg}}(\text{AND}_m) = \Omega(\sqrt{m}) \). Likewise, if \( m < 10 \), then the theorem holds trivially by the lower bound on \( \text{deg}(\text{OR}_n) \). Otherwise, putting Theorem 5 and Theorem 6 together gives \( \widetilde{\text{deg}}(\text{AND}_m \circ \text{OR}_n) = \Omega \left(\sqrt{mn \log m}\right) \).

**4 Discussion**

Though our lower bound on \( \widetilde{\text{deg}}(\text{AND}_m \circ \text{OR}_n) \) is not tight, it suggests a natural way to obtain a tighter lower bound via the same method: either tighten the “robust symmetrization” argument in Theorem 6 to eliminate the log factor (i.e. construct a polynomial of the same degree with \( a = \Omega(1) \)
and $b = O(1)$), or show that the lower bound in Theorem 5 can be tightened by a log $m$ factor. We conjecture that Theorem 5 is tight, and that Theorem 6 can be improved.

We remark that nevertheless, it is an open problem to exhibit polynomials of minimal degree that satisfy the statement of Theorem 5 with $a$ and $b$ constant, even in the case where $n = 1!$ We highlight this below:

**Problem 8.** What is the minimum degree of a polynomial $p(x_1, \ldots, x_m)$ with the property that for all $(x_1, \ldots, x_m) \in [0, 1]^m$:

1. If $x_i \leq \frac{1}{3}$ for some $i$, then $0 \leq p(x_1, \ldots, x_m) \leq \frac{1}{3}$.
2. If $x_i \geq \frac{2}{3}$ for all $i$, then $\frac{2}{3} \leq p(x_1, \ldots, x_m) \leq 1$.

In particular, do such polynomials exist of degree $O(\sqrt{m})$?

While degree $\Omega(\sqrt{m})$ polynomials are clearly necessary (Lemma 4), the best upper bound we are aware of for Problem 8 is $O(\sqrt{m} \log^* m)$ for some constant $c$, where $\log^*$ denotes the iterated logarithm. Such polynomials can be derived from a quantum algorithm for search on bounded-error inputs described in the introduction of [HMdW03], using the connection between quantum query algorithms and approximating polynomials [BBC+01].

On the other hand, we note that the conditions of Theorem 5 are stronger than necessary. Upon closer observation, it suffices to have a polynomial that satisfies the following weaker conditions:

1. If $x_i \in [0, a] \cup [b, n]$ for all $i$ and $x_i \leq a$ for some $i$, then $0 \leq p(x_1, \ldots, x_m) \leq \alpha$.
2. If $x_i = b$ for all $i$, then $\beta \leq p(x_1, \ldots, x_m) \leq 1$.

And indeed, under these weaker conditions, the existence of such polynomials of degree $O(\sqrt{m})$ for $n = 1, a = \frac{1}{3}, b = \frac{2}{3}$ follows from the main result of Høyer, Mosca, and de Wolf [HMdW03]. Alternatively, the existence of such polynomials follows from Sherstov’s result on making polynomials robust to noise [She12].

Beyond the results that we (re-)proved, we wonder if the techniques introduced will find applications elsewhere. For example, we observed a connection between Lemma 3 and the fundamental theorem of symmetric polynomials. Is the Laurent polynomial symmetrization a special case of a more general class of symmetrizations that involve analyzing symmetric polynomials in the basis of elementary symmetric polynomials? If so, do any of these other types of symmetrizations have applications in proving new lower bounds on approximate degree?

### 4.1 Followup work

After this work was published, we became aware of a recent work, due to Huang and Viola [HV20], that gives yet another lower bound proof for the approximate degree of the AND-OR tree. The proof is based around the notion of $k$-wise indistinguishability, which captures distributions over $n$-bit strings for which the marginal distributions on any $k$ bits are the same.

---

5 Note that the main result of [HMdW03] is of no use here, as it requires all inputs to be in $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, whereas case 1 of Problem 8 allows inputs in $[\frac{1}{3}, \frac{2}{3}]$. 

9
Acknowledgements

This paper originated as a project in Scott Aaronson’s Spring 2019 Quantum Complexity Theory course; I am grateful for his guidance. I thank Robin Kothari for bringing Problem 8 and [BDBGK18] to my attention. Thanks also to Justin Thaler for helpful discussions and feedback that were the inspiration for this paper.

References

[Aar08] Scott Aaronson. The polynomial method in quantum and classical computing. In 49th Annual IEEE Symposium on Foundations of Computer Science, pages 3–3, Oct 2008. doi:10.1109/FOCS.2008.91. [p. 2]

[AKKT20] Scott Aaronson, Robin Kothari, William Kretschmer, and Justin Thaler. Quantum Lower Bounds for Approximate Counting via Laurent Polynomials. In Shubhangi Saraf, editor, 35th Computational Complexity Conference (CCC 2020), volume 169 of Leibniz International Proceedings in Informatics (LIPIcs), pages 7:1–7:47, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/12559, doi:10.4230/LIPIcs.CCC.2020.7. [pp. 1, 2, 3, 4, 5]

[Amb00] Andris Ambainis. Quantum lower bounds by quantum arguments. In Proceedings of the Thirty-second Annual ACM Symposium on Theory of Computing, STOC ’00, pages 636–643, New York, NY, USA, 2000. ACM. URL: http://doi.acm.org/10.1145/335305.335394, doi:10.1145/335305.335394. [pp. 1, 2]

[Amb05] Andris Ambainis. Polynomial degree and lower bounds in quantum complexity: Collision and element distinctness with small range. Theory of Computing, 1(3):37–46, 2005. URL: http://www.theoryofcomputing.org/articles/v001a003, doi:10.4086/toc.2005.v001a003. [p. 3]

[BBC+01] Robert Beals, Harry Buhrman, Richard Cleve, Michele Mosca, and Ronald de Wolf. Quantum lower bounds by polynomials. J. ACM, 48(4):778–797, July 2001. URL: http://doi.acm.org/10.1145/502090.502097, doi:10.1145/502090.502097. [pp. 1, 2, 9]

[BCW98] Harry Buhrman, Richard Cleve, and Avi Wigderson. Quantum vs. classical communication and computation. In Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing, STOC ’98, pages 63–68, New York, NY, USA, 1998. ACM. URL: http://doi.acm.org/10.1145/276698.276713, doi:10.1145/276698.276713. [p. 1]

[BDBGK18] Shalev Ben-David, Adam Bouland, Ankit Garg, and Robin Kothari. Classical lower bounds from quantum upper bounds. In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), pages 339–349, Los Alamitos, CA, USA, oct 2018. IEEE Computer Society. doi:10.1109/FOCS.2018.00040. [pp. 1, 2, 3, 10]
[BdW02] Harry Buhrman and Ronald de Wolf. Complexity measures and decision tree complexity: a survey. *Theoretical Computer Science*, 288(1):21 – 43, 2002. Complexity and Logic. URL: http://www.sciencedirect.com/science/article/pii/S030439750100144X, doi:https://doi.org/10.1016/S0304-3975(01)00144-X. [p. 1]

[BKT18] Mark Bun, Robin Kothari, and Justin Thaler. The polynomial method strikes back: Tight quantum query bounds via dual polynomials. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2018, page 297–310, New York, NY, USA, 2018. Association for Computing Machinery. doi:10.1145/3188745.3188784. [p. 2]

[BT13] Mark Bun and Justin Thaler. Dual lower bounds for approximate degree and Markov-Bernstein inequalities. In *Proceedings of the 40th International Conference on Automata, Languages, and Programming - Volume Part I*, ICALP’13, pages 303–314, Berlin, Heidelberg, 2013. Springer-Verlag. URL: http://dx.doi.org/10.1007/978-3-642-39206-1_26, doi:10.1007/978-3-642-39206-1_26. [pp. 1, 2, 3]

[BT16] Mark Bun and Justin Thaler. Dual polynomials for collision and element distinctness. *Theory of Computing*, 12(16):1–34, 2016. URL: http://www.theoryofcomputing.org/articles/v012a016, doi:10.4086/toc.2016.v012a016. [p. 2]

[Gro96] Lov K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing*, STOC ’96, pages 212–219, New York, NY, USA, 1996. ACM. URL: http://doi.acm.org/10.1145/237814.237866, doi:10.1145/237814.237866. [p. 1]

[HMdW03] Peter Høyer, Michele Mosca, and Ronald de Wolf. Quantum search on bounded-error inputs. In *Proceedings of the 30th International Conference on Automata, Languages and Programming*, ICALP’03, pages 291–299, Berlin, Heidelberg, 2003. Springer-Verlag. URL: http://dl.acm.org/citation.cfm?id=1759210.1759241. [pp. 1, 2, 3]

[HV20] Xuanguis Huang and Emanuele Viola. Approximate degree-weight and indistinguishability. *Electronic Colloquium on Complexity Theory*, TR19-085 (Revision 2), 2020. URL: https://eccc.weizmann.ac.il/report/2019/085/. [p. 9]

[MP69] Marvin Minsky and Seymour Papert. *Perceptrons: An Introduction to Computational Geometry*. MIT Press, 1969. [pp. 2, 3]

[NS92] Noam Nisan and Mario Szegedy. On the degree of Boolean functions as real polynomials. In *Proceedings of the Twenty-fourth Annual ACM Symposium on Theory of Computing*, STOC ’92, pages 462–467, New York, NY, USA, 1992. ACM. URL: http://doi.acm.org/10.1145/129712.129757, doi:10.1145/129712.129757. [pp. 3, 4, 5]

11
[Pat92] Ramamohan Paturi. On the degree of polynomials that approximate symmetric Boolean functions. In Proceedings of the Twenty-fourth Annual ACM Symposium on Theory of Computing, STOC ’92, pages 468–474, New York, NY, USA, 1992. ACM. URL: http://doi.acm.org/10.1145/129712.129758, doi:10.1145/129712.129758. [pp. 2, 3]

[She09] Alexander A. Sherstov. The intersection of two halfspaces has high threshold degree. In Proceedings of the 2009 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’09, pages 343–362, Washington, DC, USA, 2009. IEEE Computer Society. doi:10.1109/FOCS.2009.18. [pp. 2, 3]

[She11] Alexander A. Sherstov. Strong direct product theorems for quantum communication and query complexity. In Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing, STOC ’11, pages 41–50, New York, NY, USA, 2011. ACM. URL: http://doi.acm.org/10.1145/1993636.1993643, doi:10.1145/1993636.1993643. [p. 3]

[She12] Alexander A. Sherstov. Making polynomials robust to noise. In Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing, STOC ’12, pages 747–758, New York, NY, USA, 2012. ACM. URL: http://doi.acm.org/10.1145/2213977.2214044, doi:10.1145/2213977.2214044. [p. 9]

[She13] Alexander A. Sherstov. Approximating the AND-OR tree. Theory of Computing, 9(20):653–663, 2013. URL: http://www.theoryofcomputing.org/articles/v009a020, doi:10.4086/toc.2013.v009a020. [pp. 1, 2, 3]

[Shi02] Yaoyun Shi. Approximating linear restrictions of Boolean functions, 2002. URL: https://web.eecs.umich.edu/~shiyy/mypapers/linear02-j.ps. [pp. 3, 4]

[SW86] Michael Saks and Avi Wigderson. Probabilistic Boolean decision trees and the complexity of evaluating game trees. In Proceedings of the 27th Annual Symposium on Foundations of Computer Science, SFCS ’86, pages 29–38, Washington, DC, USA, 1986. IEEE Computer Society. doi:10.1109/SFCS.1986.44. [p. 1]

[SW19] Alexander A. Sherstov and Pei Wu. Near-optimal lower bounds on the threshold degree and sign-rank of AC0. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, page 401–412, New York, NY, USA, 2019. Association for Computing Machinery. doi:10.1145/3313276.3316408. [p. 2]

12