Iterative learning control for fractional order nonlinear system with initial shift

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Abstract In this study, a closed-loop $D^\alpha$-type iterative learning control (ILC) with a proportional $D$-type iterative learning updating law for the initial shift is applied to nonlinear conformable fractional order system. First, the system with the initial shift is introduced. Then, fractional order ILC (FOILC) frameworks that experience the initial shift problem for the path tracking of nonlinear conformable fractional order systems are addressed. Moreover, the sufficient condition for the convergence of tracking errors is obtained in the time domain by introducing $\lambda$-norm and Hölder’s inequality. Lastly, numerical examples are provided to illustrate the effectiveness of the proposed methods.

Keywords Nonlinear conformable fractional differential equations · Iterative learning control · Initial shift · Hölder’s inequality

1 Introduction

Fractional differential calculus was originally developed in the 17th century, however, it has only been applied to the control field for a few decades [1,2]. Many systems in practical fields can be accurately described with the use of fractional derivatives and integrals [3]. Besides, numerous fractional order controllers have been applied in engineering areas [4–6]. Iterative learning control (ILC), which is used in control systems that repeat the same task via an unknown model during a finite duration, was first introduced by Uchiyama in Japanese [7]. Arimoto et al. [8] further developed this method in English. Thereafter, ILC has achieved considerable development [9] and has been found to be a good alternative in practical application [10]. The combination of fractional differential calculus and ILC was first reported by Chen and Moore [11] who proposed a $D^\alpha$-type control law analyzed in the frequency domain. FOILC has currently become a new topic and has received increasing attention [12–14] because it can enhance tracking performance in control systems.

The basic idea of Fo-ILC is illustrated in Fig. 1, where $u_k(t)$ and $y_k(t)$ are, respectively, the system input and output in the $k$th iteration, $u_{k+1}(t)$ is the system input of the $(k + 1)$th trial, and $y_d(t)$ is the given desired trajectory. The goal of ILC is that $\lim_{k \to \infty} y_k(t) = y_d(t)$ for all $t \in [0, T]$, where $T$ is a fixed constant.

To the best of our knowledge, Li et al. [15] described a fractional order linear system in state space form, and the convergent conditions for a $D^\alpha$-type law were provided. In [16], the asymptotic stability of $PD^\alpha$-type ILC for a fractional order linear time invariant (LTI) system was studied. The convergence condition for
open-loop $P$-type ILC for fractional order nonlinear system was investigated in [17]. High-order fractional order $PID$-type ILC strategies for a class of Caputo-type fractional order LTI system was discussed in [18]. FOILC for both linear and nonlinear fractional order multi-agent systems was applied to solve the consensus tracking problem in [19]; for nonlinear and linear conformable fractional differential equations, Wang et al. [20–23] provided standard analysis technique for standard open-loop $P$-type, $D_\alpha$-type, and conformable $PI\alpha D_\alpha$-type ILC; unfortunately, it is supposed that the initial state is coincident with the desired initial state.

However, the aforementioned existing literature, FOILC laws generally assume that the initial state of a system must be strictly in accordance with the desired states. Therefore, the initial shift problem should be involved to extend the application of FOILC [24–26]. For example, Zhao et al. [27] introduced an initial state learning scheme coupled with $D_\alpha$-type ILC updating law to eliminate initial shift states in fractional order linear invariant systems. The possibility of application to the motion control of robot manipulators was discussed under specific conditions. Moreover, the $D_\alpha$-type FOILC scheme was applied to the fractional order linear system with different historical functions under the Riemann–Liouville definition [28], and the relationship between memory and convergent performance was highlighted. Lan [29] presented a $P$-type ILC scheme with initial state learning for a single-input single-output fractional order nonlinear system, and the asymptotic stability of $PD_\alpha$-type ILC was studied by Li et al. [30]. Li and Zhou [31] discussed fractional order nonlinear systems with delay by $P$-type ILC scheme, initial state delay was considered. Li et al. [35] discussed the consensus problem of fractional-order multiagent systems with nonzero initial states, both open- and closed-loop $PD_\alpha$-type fractional-order iterative learning control are presented. For the consensus problem of nonzero initial states of fractional-order multiagent systems, Li et al. [32], Luo et al. [33] and Zhou et al. [34] discussed relevant fractional-order iterative learning control, respectively. In addition, an adaptive generalized FOILC, which expands the practical applications of FOILC, was illustrated in fractional order nonlinear systems [35].

Motivated by the above-mentioned research, a comfortable closed-loop $D_\alpha$-type ILC with an initial state learning law was proposed in the current study to eliminate the influence of the initial shift in nonlinear comfortable fractional systems. It can expand the application scope of ILC method and eliminate the influence of initial value change on tracking effect in practical engineering. Notably, sufficient conditions in the time domain were derived by introducing $\lambda$-norm and using Hölder’s inequality. Based on this convergence condition, the learning gain of the initial learning and input learning updating law can be determined. Unlike the existing methods, the ILC scheme will not fix the initial value on the expected condition at the beginning of each iteration. And the availability of this contribution is examined using two numerical examples.

The rest of the paper is organized in the following manner. Preliminary knowledge is briefly reviewed in Sect. 2. The nonlinear conformable fractional differential equations and the design of learning control are presented in detail in Sect. 3. Learning convergence is analyzed in Sect. 4. A number of simulations demonstrate the effectiveness of the theory in Sect. 5. Conclusions are drawn and suggestions for future work are provided in Sect. 6.

2 Fractional derivative and preliminary

In this section, some related mathematical definitions and properties are introduced, which will be applied in the following sections.

2.1 The $\lambda$-norm

Suppose $C(J, \mathbb{R}^n)$ is the space of vector-value continuous functions from $J \rightarrow \mathbb{R}^n$. Consider $C(J, \mathbb{R}^n)$ endowed with $\lambda$-norm, and

$$\|x\|_\lambda = \sup_{t \in J} \left\{ e^{-\lambda t} \|x\| \right\}, \quad \lambda > 0$$ (1)
2.2 Hölder’s inequality [36]

Suppose that \( \Omega \) is a measured space, \( p, q \in [1, \infty) \) and satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( f \in L^p(\Omega) \), and \( g^q(\Omega) \), then

\[
\int_\Omega |f(x)g(x)|dx \leq \left( \int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} \times \left( \int_\Omega |g(x)|^q dx \right)^{\frac{1}{q}}
\]

2.3 Conformable fractional order calculus

The definition of Fractional calculus contains Grünwald–Letnikov, Riemann–Liouville, Caputo’s and other fractional derivatives [37,38]. In this paper, conformable fractional order calculus is defined and considered, the derivative of \( \alpha \)-order function \( f(t) : [0, \infty) \rightarrow \mathbb{R}^n \) is defined as follows [39].

\[
D^\alpha f(t) = \lim_{\varepsilon \to 0} D^\alpha f(t) = \lim_{t \to t_0^+} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}
\]

where \( D^\alpha f(t) \) exists and is finite, \( 0 < \alpha \leq 1, t_0 \geq 0 \), \( D^\alpha \) presents fractional order derivative operator in \( [0, t] \).

Besides, the conformable fractional integral is defined as

\[
I^\alpha f(t) = \int_0^t f(\tau)\tau^{\alpha-1}d\tau
\]

where \( I^\alpha \) presents fractional order integral operator in \( [0, t] \).

2.4 Solution of conformable fractional order function

According to the lemma 4 in [40], if the conformable fractional order function \( D^\alpha x(t) = f(t) \) is continuous and given, the solution \( x(t) \) of \( f(t) \) with the initial value \( x(0) = x_0 \) satisfies

\[
x(t) = x_0 + \int_0^t (f(\tau))\tau^{\alpha-1}d\tau
\]

3 ILC design for fractional order nonlinear systems

In order to solve the above problem, a class of fractional order nonlinear system is described. The repetitive nonlinear conformable fractional differential system is considered as

\[
\begin{aligned}
\dot{x}_k(t) &= F(t, x_k(t)) + Q(t)u_k(t) \\
y_k(t) &= P(t)x_k(t)
\end{aligned}
\]

where \( D^\alpha(0 < \alpha < 1) \) denotes conformable fractional derivative, \( x_i(t) \in \mathbb{R}^n, y_i(t) \in \mathbb{R}^n, P(t) \in \mathbb{R}^{m \times n}, u_i(t) \in \mathbb{R}^r, Q(t) \in \mathbb{R}^{n \times r}; F : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous differentiable in \( t \). \( x_k(0) \) is an initial condition of state variable \( x_k(t) \) for the \( k \)th iteration.

There exists a \( L_F > 0 \) such that satisfies

\[
\| F(t, x_1(t)) - F(t, x_2(t)) \| \leq L_F \| x_1(t) - x_2(t) \|
\]

where \( \forall t \in J, \exists x_1, x_2 \in \mathbb{R}^n \).

According to Sect. 2.4, the solution of \( x_k(t) \) in Eq. (5) can be derived as follows

\[
x_k(t) = x_k(0) + \int_0^t (F(\tau, x_k(\tau)) + Q(\tau)u_k(\tau))\tau^{\alpha-1}d\tau
\]

Let \( y_d(t) \) be a continuous differentiable desired function on \( J \) and \( u_d(t) \) be an expectation control variable. If the desired initial value is \( x_d(0) = x_{d0} \), and \( y_d(t) = P(t)x_d(t) \), where

\[
x_d(t) = x_{d0} + \int_0^t (F(\tau, x_d(\tau)) + Q(\tau)u_d(\tau))\tau^{\alpha-1}d\tau
\]

For simplification, it is defined

\[
\begin{aligned}
\delta u_k(t) &= u_d(t) - u_k(t) \\
\delta x_k(t) &= x_d(t) - x_k(t) \\
\epsilon_k(0) &= y_d(0) - y_k(0)
\end{aligned}
\]

The ILC updating law with initial state learning is defined as follows,

\[
x_{k+1}(0) = x_k(0) + L\epsilon_{k+1}(0)
\]
where the subscript $k$ is iterative index, $L \in R^{n \times r}$ and $A \in R^{r \times r}$ are the learning gains to be designed based on prior-knowledge about the system under investigation.

### 4 Convergence proof and analysis

**Lemma 1** Consider the system (5) with a bounded reference $y_d(t)$ and arbitrary continuous initial state $x_k(0)$. Assuming that $\|[(I + P(t)L)^{-1}]\| < 1$ holds for all $t \in [0, T]$, then the initial learning law guarantees that $\lim_{k \to \infty} \|e_k(0)\|_\lambda = 0$

**Proof** It follows Eqs. (9) and (10), it obtains

$$x_{k+1}(0) - x_{d+1}(0) = \delta x_{k}(0) - Le_{k-1}(0)$$

Multiplying both sides of the above Eq. (12) by $P(t)$ (defined in Eq. (5)),

$$P(t)\delta x_{k+1}(0) = P(t)\delta x_{k}(0) - P(t)Le_{k+1}(0)$$

It gets

$$e_{k+1}(0) = [(I + P(t)L)^{-1}]e_k(0)$$

where $I$ denotes identity matrix. Taking the norm $\|\cdot\|$ of Eq. (15), it yields

$$\|e_{k+1}(0)\| \leq \left\|[P(t)L]^{-1}\right\| \|e_k(0)\|$$

Multiply $e^{-\lambda t}$ both sides of the above Eq. (16), it derives

$$e^{-\lambda t} \|e_{k+1}(0)\| \leq e^{-\lambda t} \left\|[P(t)L]^{-1}\right\| \|e_k(0)\|$$

Taking the $\lambda$-norm the above Eq. (17), it yields

$$\|e_{k+1}(0)\|_\lambda \leq \left\|[P(t)L]^{-1}\right\| \|e_k(0)\|_\lambda$$

Therefore, if $\|[(I + P(t)L)^{-1}]\| < 1$ is satisfied, $\lim_{k \to \infty} \|e_{k+1}(0)\|_\lambda = 0$ can be concluded.

**Lemma 2** For the system (5) with the $D^a$-type ILC schemes (11), suppose the assumption in Lemma 1 is satisfied, and if

$$\rho_1 = (\|I + AP(t)Q(t)\|^{-1} < 1$$

holds for all $t \in [0, T]$. Given an arbitrary initial input

$$u_k(0), \lim_{k \to \infty} \|ek\|_\lambda = 0$$

**Proof** It follows from the aforementioned Eq. (11), the $u_{k+1}(t)$ can be yield

$$\delta u_{k+1}(t) = \delta u_k(t) - AD^a_k e_{k+1}(t)$$

$$= \delta u_k(t) - AD^a_k (y_d(t) - y_{k+1}(t))$$

$$= \delta u_k(t) - AD^a_k (P(t)\delta x_{k+1}(t))$$

$$= \delta u_k(t) - AD^a_k (P(t)\delta x_{k+1}(t) - P(t)D^0_t \delta x_{k+1}(t))$$

$$= \delta u_k(t) - AD^a_k (P(t)\delta x_{k+1}(t) - P(t)(F(t,x_d(t))$$

$$- F(t,x_{k+1}(t))) + Q(t)\delta u_{k+1}(t))$$

Taking the norm $\|\cdot\|$ on both sides of the above Eq. (18), it obtains

$$\|I + AP(t)Q(t)\| \|\delta u_{k+1}(t)\|$$

$$\leq \|\delta u_k(t)\| + \|\hat{A}^\alpha P(t)\| \|\delta x_{k+1}(t)\|$$

$$+ \|\hat{A}P(t)\| \|\delta x_{k+1}(t)\|$$

$$\leq \|\delta u_k(t)\| + \|\hat{A}^\alpha P(t)\| \|\delta x_{k+1}(t)\|$$

$$\leq \|\delta u_k(t)\| + \delta x_{k+1}(t)$$

where

$$h = \sup_{t \in [0,T]} \|\hat{A}^\alpha P(t)\| \|\delta x_{k+1}(t)\|$$

and

$$\delta x_{k+1}(t)$$

$$= \delta x_{k+1}(0) + \int_0^t (F(\tau, x_d(\tau)) - F(\tau, x_{k+1}(\tau))$$

$$+ Q(\tau)\delta u_{k+1}(\tau))\tau^{\alpha-1}d\tau$$

$$\leq \delta x_{k+1}(0) + LF \int_0^t \delta x_{k+1}(\tau)\tau^{\alpha-1}d\tau$$

$$+ \int_0^t Q(\tau)\delta u_{k+1}(\tau)\tau^{\alpha-1}d\tau$$
It yields
\[
\|\delta x_{k+1}(t)\| \leq \|\delta x_{k+1}(0)\| + L_F \int_0^t \|\delta x_{k+1}(\tau)\| \tau^{a-1} \, d\tau
\]
\[+ \int_0^t \int_0^t \|Q(\tau) \delta u_{k+1}(\tau)\| \tau^{a-1} \, d\tau \, d\tau \leq \|\delta x_{k+1}(0)\| + \int_0^t \|Q(\tau) \delta u_{k+1}(\tau)\| \tau^{a-1} \, d\tau \leq \|\delta x_{k+1}(0)\| + \int_0^t \|Q(\tau) \delta u_{k+1}(\tau)\| \tau^{a-1} \, d\tau \leq \|\delta x_{k+1}(0)\| + \int_0^t e^{\lambda t} \tau^{-1} \, d\tau \|Q(\tau)\| \|\delta u_{k+1}\| \lambda e^{L_F \tau^{a-1}} \]
\[ (21) + h \frac{1}{\lambda} e^{\lambda t} \frac{T^{a-1}}{\sqrt{q} \alpha - q + 1} \|Q(\tau)\| \|\delta u_{k+1}\| \lambda e^{L_F \tau^{a-1}} \]

Multiplying both sides of the above Eq. (24) by $e^{-\lambda t}$ and taking the $\lambda$-norm, it has
\[
e^{-\lambda t} \|I + AP(t)Q(t)\| \|\delta u_{k+1}(t)\| \leq e^{-\lambda t} \|\delta u_k(t)\| + e^{-\lambda t} \|\delta x_{k+1}(0)\|
\]
\[+ h \frac{1}{\lambda} \frac{T^{a-1}}{\sqrt{q} \alpha - q + 1} \|Q(\tau)\| \|\delta u_{k+1}\| \lambda e^{L_F \tau^{a-1}} \]
and
\[
\left(\|I + AP(t)Q(t)\| - h \frac{1}{\lambda} e^{L_F \tau^{a-1}} \right) \frac{\frac{T^{a-1}}{\sqrt{q} \alpha - q + 1}}{\|Q(\tau)\|} \|\delta u_{k+1}\| \lambda
\]
\[ \leq \|\delta u_k\| + h \|\delta x_{k+1}(0)\| \lambda
\]
where
\[l = h \frac{1}{\lambda} e^{L_F \tau^{a-1}} \frac{T^{a-1}}{\sqrt{q} \alpha - q + 1} \|Q(\tau)\|
\]
and
\[\|\delta u_{k+1}\| \lambda \leq \|[I + AP(t)Q(t)]^{-1}\| \|\delta u_k\| \lambda + h \|[I + AP(t)Q(t)]^{-1}\| \|I + P(t)L\|^{-1} \|P(t)^{-1}\| \|\delta e_k\| \lambda \]
\[+ \|[I + AP(t)Q(t)]^{-1}\| \|\delta u_k\| \lambda \]

where $\rho_1 = \|[I + AP(t)Q(t)]^{-1}\|$, $\rho_2 = \|[I + AP(t)Q(t)]^{-1}\| \|[I + P(t)L]^{-1}\| \|P(t)^{-1}\|$. Based on Lemma 1, the $\|\delta e_k\| \lambda$ is convergence and bounded. Thus, it possible to exist a constant $\zeta$ sufficiently small enough, it satisfies that $\|\delta e_k\| \lambda \leq \zeta \|\delta u_k\| \lambda$, and the equation can be written as
\[
\|\delta u_{k+1}\| \lambda \leq (\rho_1 + \zeta) \|\delta u_k\| \lambda
\]
where $\zeta = \rho_2 \zeta$. It exists $\lambda$ sufficiently large that satisfies $\rho_2 \zeta \rightarrow 0$; hence it exists $\max \sup_{0 \leq t, \tau \leq T} \rho_1 < 1$, together
with Lemma 1, it satisfies $\|\delta u_{k+1}\|_\lambda \leq \rho_1 \|\delta u_k\|_\lambda$. Therefore, $\lim_{k \to \infty} \|\delta u_k\|_\lambda = 0$ can be concluded.

Then the system output error is

$$e_{k+1}(t) = P(t)\delta x_{k+1}(t)$$  \hfill (29)

Taking the norm $\|\cdot\|$ on both sides of the above Eq. (29) and multiply by $e^{-\lambda}$, it has

$$e^{-\lambda} \|e_{k+1}(t)\| \leq e^{-\lambda} \|P(t)\| \|\delta x_{k+1}(t)\|$$  \hfill (30)

Now, substituting (23) into (30), it gets

$$\|e_{k+1}\|_\lambda \leq \|e_{k+1}(0)\|_\lambda + l \|P(t)\| \|\delta u_{k+1}\|_\lambda$$  \hfill (31)

$$\leq \left\|\left[I + P(t)L\right]^{-1}\|e_k(0)\|_\lambda + l \|P(t)\| \|\delta u_{k+1}\|_\lambda$$

It exists $\lambda$ sufficiently large that satisfies $l \to 0$; together with Lemmas 1 and 2, it can be concluded $\lim_{k \to \infty} \|e_k\|_\lambda = 0$ \hfill \Box

5 Numerical examples

In this section, numerical examples are presented to test the effectiveness of the designed methods. The following simulations are performed for the fractional order nonlinear system.

**Example 1** Consider the first fractional order nonlinear system

$$D_0^{0.5}x_k(t) = 0.6[x_k(t)]^2 + 0.5u_k(t)$$

$$y_k(t) = x_k(t), \quad t \in [0, 1]$$

The iterative learning control laws are chosen

$$x_{k+1}(0) = x_k(0) + 0.5e_k(0)$$

$$u_{k+1}(t) = u_k(t) + D_0^{0.5}e_{k+1}(t)$$

where the system state is $x(t)$, and the desired trajectory is $y_d(t) = 12t^2(1-t)$, the initial control is $u_0(t) = 0$ and with initial condition $x_k(0) = 0.5$.

In this case, it can be calculated that

$$\left\|\left[I + P(t)L\right]^{-1}\right\| = \frac{2}{3} < 1;$$

$$\rho_1 = \left\|[I + L\Gamma P(t)Q(t)]^{-1}\right\| = 0.5 < 1.$$
It concludes that closed-loop $D^\alpha$-type ILC updating law performs better in convergence rate during the learning process.

**Example 2** Consider the second fractional order nonlinear system

$$\begin{cases}
D_0^{0.5}x_k(t) = 0.5[\sin x_k(t)]^2 + u_k(t) \\
y_k(t) = x_k(t), \ t \in [0, 1],
\end{cases} \tag{37}$$

The iterative learning control laws are chosen

$$x_{k+1}(0) = x_k(0) + 0.3e_k(0) \tag{38}$$

and

$$u_{k+1}(t) = u_k(t) + 2D_0^{0.6}e_{k+1}(t) \tag{39}$$

where the system state is $x(t)$, and the desired trajectory is $y_d(t) = 6\sin(t)$, the initial control is $u_0(t) = 0$ and with initial condition $x_k(0) = 0.6$.

In this case, it can be calculated that

$$\left\|\left[I + P(t)L\right]^{-1}\right\| = \frac{10}{13} < 1;$$

$$\rho_1 = \left\|I + \Lambda P(t)Q(t)\right\|^{-1} = \frac{5}{8} < 1.$$ 

The simulation results are demonstrated in Figs. 5, 6, and 7.

**Example 3** Consider the Third fractional order nonlinear system

$$\begin{cases}
D_0^{0.4}x_k(t) = x_k(t) + \sin(x_k(t)) + u_k(t) \\
y_k(t) = 0.3x_k(t), \ t \in [0, 1]
\end{cases} \tag{40}$$
The iterative learning control laws are chosen

\[ x_{k+1}(0) = x_k(0) + 0.8e_k(0) \]  \hspace{1cm} (41) \]

and

\[ u_{k+1}(t) = u_k(t) + 0.4D^{-0.4}e_{k+1}(t) \]  \hspace{1cm} (42) \]

where the system state is \( x(t) \), and the desired trajectory is \( y_d(t) = \cos(2\pi t)\sin(4\pi t) \), the initial control is \( u_0(t) = 0 \) and with initial condition \( x_k(0) = -2 \).

In this case, it can be calculated that

\[ \left\| (I + P(t)L)^{-1} \right\| = \frac{25}{31} < 1; \]

\[ \rho_1 = \left\| (I + LP(t)Q(t))^{-1} \right\| = \frac{25}{28} < 1. \]

The simulation results are demonstrated in Figs. 8, 9, 10.

Therefore, from the aforementioned simulations, it concluded that the proposed laws with initial state laws perform well. Moreover, it can be seen that after iteration, they all arrive at the reference trajectory under the desired precision.

6 Conclusions and future work

This study presents \( D^\alpha \)-type ILC with \( D \)-type initial learning strategy for a class of nonlinear conformable fractional order systems with the initial shift. Its major feature is that disturbance in the initial state at each iteration is eliminated by introducing an initial state learning scheme. Furthermore, the robust convergent analysis of tracking errors with respect to initial errors is derived by introducing Hölder’s inequality. Lastly, numerical simulations are provided to validate the obtained theoretical results.

In the future, \( PI^\lambda D^\alpha \)-type ILC for general nonlinear fractional order systems with repetitive properties will be researched. Moreover, when \( PI^\lambda D^\alpha \)-type ILC is applied to track the nonlinear fractional order system, nonrepetitive uncertainties (such as time delay, input saturation, and nonrepetitive desired trajectory) should be considered.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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