Distributed Adaptive Consensus Protocols for Linear Multi-agent Systems with Directed Graphs and External Disturbances

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Abstract

This paper addresses the distributed consensus design problem for linear multi-agent systems with directed communication graphs and external disturbances. Both the cases with strongly connected communication graphs and leader-follower graphs containing a directed spanning tree with the leader as the root are discussed. Distributed adaptive consensus protocols based on the relative states of neighboring agents are designed, which can ensure the ultimate boundedness of the consensus error and adaptive gains in the presence of external disturbances. The upper bounds of the consensus error are further explicitly given. Compared to the existing consensus protocols, the merit of the adaptive protocols proposed in this paper is that they can be computed and implemented in a fully distributed fashion and meanwhile are robust with respect to external disturbances.

Index Terms

Multi-agent system, cooperative control, consensus, distributed control, adaptive control, robustness.

I. INTRODUCTION

In the past two decades, rapid advances in miniaturizing of computing, communication, sensing, and actuation have made it feasible to deploy a large number of autonomous agents to work cooperatively to accomplish civilian and military missions. This, compared to a single complex agent, has the capability to significantly improve the operational effectiveness, reduce the costs, and provide additional degrees of redundancy [?, ?]. Having multiple autonomous agents to work together to achieve collective behaviors is usually referred to as cooperative control of multi-agent systems. Due to its potential applications in...
various areas such as surveillance and reconnaissance systems, satellite formation flying, electric power systems, and intelligent transportation systems, cooperative control of multi-agent systems has received compelling attention from the systems and control community. In the area of cooperative control, consensus is an important and fundamental problem, which means that a group of agents reaches an agreement on certain quantity of interest by interacting with their local neighbors. Advances of various consensus algorithms have been reported in a quite large body of research papers; see [?], [?], [?], [?], [?], [?], [?], [?], [?], [?] and the references therein.

For a consensus control problem, the main task is to design appropriate protocols to achieve consensus. Due to the large number of agents, limited sensing capability of sensors, and short wireless communication ranges, distributed control, depending only on local information of the agents and their neighbors, appears to be a promising tool for handling multi-agent systems. Note that designing appropriate distributed protocols is generally a challenging task, especially for multi-agent systems with complex dynamics, due to the interplay of the agent dynamics, the communication graph among agents, and the cooperative control laws.

Take the consensus problem for multi-agent systems with general continuous-time linear dynamics for instance. In previous works [?], [?], [?], [?], [?], [?], several static and dynamic consensus protocols based on the local state or output information of neighboring agents have been proposed. A common feature in the aforementioned existing papers is that the design of the consensus protocols needs to use some eigenvalue information of the Laplacian matrix associated with the communication graph. Actually, for the simply case with second-order integrator agent dynamics, the design of the consensus protocols does rely on the smallest real part of the nonzero eigenvalues of the Laplacian matrix [?]. However, the smallest real part of the nonzero eigenvalues of the Laplacian matrix is global information of the communication graph, because every agent has to know the whole communication graph to compute it. Therefore, the consensus protocols in the aforementioned works cannot be designed by the agents in a fully distributed way, i.e., relying on only the local information of neighboring agents. To avoid this limitation, two types of distributed adaptive consensus protocols are proposed in [?], [?], which implement adaptive laws to dynamically update the coupling weights of neighboring agents. Similar adaptive schemes are presented in [?], [?] to achieve consensus for multi-agent systems with second-order nonlinear dynamics. Note that the adaptive protocols in [?], [?], [?], [?] are applicable to only undirected communication graphs or leader-follower graphs where the subgraph among followers is undirected. Because of the asymmetry of the Laplacian matrices of directed graphs, designing fully distributed adaptive consensus protocols for general directed graphs is more much challenging. By introducing monotonically increasing functions to provide additional freedom for design, a distributed adaptive consensus protocol is constructed in [?] to achieve leader-follower consensus for the communication graphs containing a directed spanning tree with the leader as the root node.
Similar adaptive protocols for directed graphs are developed in [?] for the special case where the agents are described by double integrators. Even though the aforementioned advances have been reported on designing distributed adaptive protocols for general linear multi-agent systems with directed graphs, there are still many important open problems awaiting further investigation. For instance, to design distributed adaptive protocols for the case of general directed graphs without a leader, to examine the robustness issue associated with the adaptive protocols, and to propose distributed dynamic adaptive protocols for the case where only local output information is available, to name just a few.

In this paper, we intend to address the first two aforementioned problems. Specifically, we address the distributed adaptive consensus protocol design problem for general linear multi-agent systems with directed graphs and external disturbances. We consider both the cases where the communication graph among the agents is strongly connected and contains a directed spanning tree with the leader as the root. For the case with leader-follower graphs containing a directed spanning tree with the leader as the root, we revisit the distributed adaptive protocol in [?]. It is pointed out that in the presence of external disturbances, the adaptive gains of the adaptive protocol in [?] will slowly grow to infinity, which is the well-known parameter drift phenomenon in the adaptive control literature [?]. To deal with this instability issue associated with the adaptive protocol in [?], we propose a novel distributed adaptive consensus protocol, by using $\sigma$ modification technique [?]. This novel adaptive protocol is designed in fully distributed fashion to ensure the ultimate boundedness of both the consensus error and the adaptive coupling gains. That is, the proposed adaptive protocol is robust in the presence of external disturbances. The upper bound of the consensus error is also explicitly given. The case with strongly connected communication graphs is further studied. A distributed robust adaptive protocol is also presented, which can guarantee the ultimate boundedness of the consensus error and the adaptive coupling gains in the presence of external disturbances. A sufficient condition for the existence of the adaptive protocols proposed in this paper is that each agent is stabilizable.

The rest of this paper is organized as follows. Mathematical preliminaries required in this paper is summarized in Section 2. Distributed robust adaptive consensus protocols are presented in Sections 3 and 4 for multi-agent systems with strongly connected graphs and directed leader-follower graphs, respectively. Simulation examples are presented for illustration in Section 5. Conclusions are drawn in Section 6.

II. MATHEMATICAL PRELIMINARIES

In this paper, we use the following notations and definitions: $\mathbb{R}^{n \times m}$ represents the set of $n \times m$ real matrices. $I_N$ denotes the identity matrix of dimension $N$. $\mathbf{1}$ denotes a column vector of appropriate dimension with its entries equal to one. For real symmetric matrices $W$ and $X$, $W \geq X$ means that $W - X$ is positive (semi-)definite. $A \otimes B$ denotes the Kronecker product of the matrices $A$ and $B$. Denote by $\sigma_{\text{max}}(B)$ the largest singular value of a matrix $B$. A matrix $A = [a_{ij}] \in \mathbb{R}^n$ is
A directed communication graph $G$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \cdots, v_N\}$ is a nonempty finite set of vertices (i.e., nodes) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, where an edge is represented by an ordered pair of distinct vertices. For an edge $(v_i, v_j)$, $v_i$ is called the parent vertex, $v_j$ is called the child vertex, and $v_i$ is a neighbor of $v_j$. If a directed graph having the property that $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ for any $v_i, v_j \in \mathcal{V}$, then it is an undirected graph. A directed path from vertex $v_i$ to vertex $v_j$ is a sequence of ordered edges in the form $(v_{i_k}, v_{i_{k+1}})$, $k = 1, \cdots, l - 1$. A directed graph contains a directed spanning tree if there exists a vertex called the root, which has no parent vertex, such that there exist directed paths from the vertex to all other vertices in the graph. A directed graph is strongly connected if there is a directed path between every distinct vertices. A directed graph has a directed spanning tree if it is strongly connected, but not vice versa.

For the directed graph $G$, its adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined such that $a_{ii} = 0$, $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N \times N}$ associated with $G$ is defined as $L_{ii} = \sum_{j \neq i} a_{ij}$ and $L_{ij} = -a_{ij}$, $i \neq j$.

**Lemma 1 (\cite{1}):** Zero is an eigenvalue of $L$ with 1 as a right eigenvector and all the nonzero eigenvalues have positive real parts. Zero is a simple eigenvalue of $L$ if and only if $G$ contains a directed spanning tree.

**Lemma 2 (\cite{1}):** Suppose that $G$ is strongly connected. Let $r = [r_1, \cdots, r_N]$ be the positive left eigenvector of $L$ associated with the zero eigenvalue and $R = \text{diag}(r_1, \cdots, r_N)$. Then, $\hat{L} \triangleq RL + L^T R$ is the symmetric Laplacian matrix associated with an undirected connected graph. Let $\xi$ be any vector with positive entries. Then, $\min_{\xi > 0, \xi \neq 0} \frac{\xi^T \hat{L} \xi}{\xi^T R \xi} \geq \frac{\lambda_2(\hat{L})}{N}$, where $\lambda_2(\hat{L})$ denotes the smallest nonzero eigenvalue of $\hat{L}$.

**Lemma 3 (\cite{1}):** For a nonsingular $M$-matrix $M$, there exists a positive diagonal matrix $G$ such that $GM + MTG > 0$.

Moreover, $G$ can be given by $\text{diag}(q_1, \cdots, q_N)$, where $q = [q_1, \cdots, q_N]^T = (MT)^{-1}1$.

**Lemma 4 (Young’s Inequality, \cite{1}):** If $a$ and $b$ are nonnegative real numbers and $p$ and $q$ are positive real numbers such that $1/p + 1/q = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

**Lemma 5 (\cite{1}):** For a dynamical system $\dot{x} = f(x, t)$, where $f(\cdot, \cdot)$ is locally Lipschitz in $x$ and piecewise continuous in $t$, suppose that there exists a continuous and differentiable function $V(x, t)$ such that along any trajectory of the system,

$$\alpha_1(\|x\|) \leq V(x, t) \leq \alpha_2(\|x\|),$$

$$\dot{V}(x, t) \leq -\alpha_3(\|x\|) + \epsilon,$$

where $\epsilon$ is a positive constant, $\alpha_1$ and $\alpha_2$ are class $\mathcal{K}_\infty$ functions, and $\alpha_3$ is a class $\mathcal{K}$ function. Then, the solution $x(t)$ of $\dot{x} = f(x, t)$ is uniformly ultimately bounded.
III. DISTRIBUTED ROBUST ADAPTIVE PROTOCOLS FOR LEADER-FOLLOWER GRAPHS IN THE PRESENCE OF EXTERNAL DISTURBANCES

A. Problem Formulation and Motivation

In this paper, we consider a group of \(N\) identical agents with general linear dynamics. The dynamics of the \(i\)-th agent are described by

\[
\dot{x}_i = Ax_i + B[u_i + \omega_i], \quad i = 1, \cdots, N, \tag{1}
\]

where \(x_i \in \mathbb{R}^n\) is the state vector, \(u_i \in \mathbb{R}^p\) is the control input vector, \(A\) and \(B\) are known constant matrices with compatible dimensions, and \(\omega_i \in \mathbb{R}^n\) denotes external disturbances associated with the \(i\)-th agent, which satisfies the matching condition and the following assumption.

Assumption 1: There exist positive constants \(\upsilon_i\) such that \(\|\omega_i\| \leq \upsilon_i, \quad i = 1, \cdots, N\).

In this section, we consider the case where there are \(N - 1\) followers and one leader. Without loss of generality, let the agent in (1) indexed by 1 be the leader whose control input is assumed to be zero, i.e., \(u_1 = 0\), and the agents indexed by \(2, \cdots, N\), be the followers. The communication graph \(G\) among the \(N\) agents is assumed to satisfy the following assumption.

Assumption 2: The graph \(G\) contains a directed spanning tree with the leader as the root node.

Since the vertex indexed by 1 is the leader and it does not have any neighbor, the Laplacian matrix \(L\) associated with \(G\) can be written into

\[
L = \begin{bmatrix}
0 & 0_{1 \times (N-1)} \\
L_2 & L_1
\end{bmatrix}, \tag{2}
\]

where \(L_2 \in \mathbb{R}^{(N-1) \times 1}\) and \(L_1 \in \mathbb{R}^{(N-1) \times (N-1)}\). In light of Lemma 1 it is easy to see that \(L_1\) is a nonsingular \(M\)-matrix if \(G\) satisfies Assumption 2.

This intention of this section is to solve the consensus problem for the agent in (1), i.e., to design distributed consensus protocols under which the states of the \(N - 1\) followers converge to the state of the leader in the sense of \(\lim_{t \to \infty} \|x_i(t) - x_1(t)\| = 0, \quad \forall \ i = 2, \cdots, N\).

For the case without external disturbances, i.e., \(\omega_i = 0, \ i = 1, \cdots, N\), a distributed adaptive consensus protocol based on the relative states of neighboring agents was proposed for each follower as [?]:

\[
\begin{align*}
\dot{\bar{c}}_i(t) &= \bar{c}_i \rho_i (\xi_i^T Q \xi_i) K \xi_i, \\
\dot{\xi}_i &= \xi_i^T \Gamma \xi_i, \quad i = 2, \cdots, N,
\end{align*} \tag{3}
\]

where \(\xi_i \triangleq \sum_{j=1}^N a_{ij} (x_i - x_j), \quad i = 2, \cdots, N, \bar{c}_i(t)\) denotes the time-varying coupling gain (weight) associated with the \(i\)-th follower with \(\bar{c}_i(0) \geq 1, \ a_{ij}\) is the \((i, j)\)-th entry of the adjacency matrix \(A\) associated with \(G, \ K \in \mathbb{R}^{p \times n}\) and \(\Gamma \in \mathbb{R}^{n \times n}\) are
the feedback gain matrices, $\rho_i(\cdot)$ are smooth and monotonically increasing functions which satisfies that $\rho_i(s) \geq 1$ for $s > 0$, and $Q > 0$ is a solution to the following algebraic Riccati equation (ARE):

$$A^TQ + QA + I - QBB^TQ = 0. \quad (4)$$

**Lemma 6 ([?]):** Suppose that the communication graph $G$ satisfies Assumption 2. Then, the consensus problem of the agents in (1) is solved by the adaptive protocol (3) with $K = -B^TQ$, $\Gamma = QBB^TQ$, and $\rho_i(\xi^T_iQ\xi_i) = (1 + \xi^T_iQ\xi_i)^3$. Moreover, each coupling gain $\tilde{c}_i$ converges to some finite steady-state value.

Lemma 6 shows that the adaptive protocol (3) can achieve consensus for the case where the agents in (1) are not perturbed by external disturbances. The adaptive protocol (3) constructed by Lemma 6, contrary to the consensus protocols in [?], [?], [?], [?], [?], depends on only the agent dynamics and the relative states of neighboring agents, and thereby can be computed and implemented by each agent in a fully distributed way.

Note that in many circumstances, the agents may be subject to certain external disturbances, for which case it is necessary and interesting to investigate whether the adaptive protocol (3) designed by Lemma 6 is still applicable, i.e., whether (3) is robust with respect to external disturbances. Due to the existence of nonzero $\omega_i$ in (1), the relative states will not converge to zero any more but rather can only be expected to converge into some small neighborhood of the origin. Since the derivatives of the adaptive gains $\tilde{c}_i$ are of nonnegative quadratic forms in terms of the relative states, in this case it is easy to see from (3) that $\tilde{c}_i$ will keep growing to infinity, which is known as the parameter drift phenomenon in the classic adaptive control literature [?]. Therefore, the adaptive protocol (3) is not robust in the presence of external disturbances.

**B. Distributed Robust Adaptive Consensus Protocols**

The objective of this subsection is to made modifications on (3) in order to present novel distributed robust adaptive protocols which can guarantee the stability or ultimate boundedness of the consensus error in the presence of external disturbances. By utilizing the $\sigma$ modification technique [?], we propose a new distributed adaptive consensus protocol as follows:

$$u_i = c_i\rho_i(\xi^T_iQ\xi_i)K\xi_i,$$

$$\dot{c}_i = -\varphi_i(c_i - 1) + \xi^T_i\Gamma\xi_i, \quad i = 2, \cdots, N,$$

where $\varphi_i$, $i = 2, \cdots, N$, are small positive constants and the rest of the variables are defined as in (3).

Let $\xi = [\xi^T_2, \cdots, \xi^T_N]^T$, where $\xi_i = \sum_{j=1}^N a_{ij}(x_i - x_j)$. Then, it follows that

$$\xi = (\mathcal{L}_1 \otimes I_n) \begin{bmatrix} x_2 - x_1 \\ \vdots \\ x_N - x_1 \end{bmatrix}, \quad (6)$$
where $\mathcal{L}_1$ is defined as in (2). Because $\mathcal{L}_1$ is a nonsingular matrix for $\mathcal{G}$ satisfying Assumption 2, the consensus problem is solved if and only if $\xi$ asymptotically converges to zero. Hereafter, $\xi$ is referred to as the consensus error. In light of (1) and (5), it is not difficult to verify that $\xi$ and $c_i$ satisfy the following dynamics:

$$\dot{\xi} = [I_{N-1} \otimes A + \mathcal{L}_1 \hat{C} \hat{\rho}(\xi) \otimes BK] \xi + (\mathcal{L}_1 \otimes B) \omega,$$

$$\dot{c}_i = -\varphi_i (c_i - 1) + \xi_i^T \Gamma \xi_i,$$

where $\bar{\omega} \triangleq [\omega_2^T - \omega_1^T, \cdots, \omega_N^T - \omega_1^T]^T$, $\hat{\rho}(\xi) \triangleq \text{diag}(\rho_2(\xi_2^T Q \xi_2), \cdots, \rho_N(\xi_N^T Q \xi_N))$, and $\hat{C} \triangleq \text{diag}(c_2, \cdots, c_N)$.

In light of Assumption 1, it is easy to see that

$$\|\bar{\omega}\| \leq \sqrt{\sum_{i=2}^{N} (v_i + v_1)^2}.$$

By the second equation in (7), we have

$$c_i(t) = c_i(0) e^{-\varphi_i t} + \int_0^t e^{-\varphi_i (t-s)} (\varphi_i + \xi_i^T \Gamma \xi_i) ds$$

$$= (c_i(0) - 1) e^{-\varphi_i t} + 1 + \int_0^t e^{-\varphi_i (t-s)} \xi_i^T \Gamma \xi_i ds$$

$$\geq 1,$$

where we have used the fact that $c_i(0) \geq 1$ to get the last inequality.

The following theorem designs the adaptive protocol (5).

**Theorem 1**: Suppose that the communication graph $\mathcal{G}$ satisfies Assumption 2. Then, both the consensus error $\xi$ and the coupling gains $c_i$, $i = 2, \cdots, N$, in (7), under the adaptive protocol (5) with $K$, $\Gamma$, and $\rho_i(\cdot)$ designed as in Lemma 6 are uniformly ultimately bounded. Furthermore, if $\varphi_i$ is chosen to be small enough such that $\delta \triangleq \min_{i=1, \cdots, N} \varphi_i < \tau \triangleq \frac{1}{\lambda_{\text{max}}(Q)}$, then $\xi$ exponentially converges to the residual set

$$\mathcal{D}_1 \triangleq \left\{ \xi : \|\xi\|^2 \leq \frac{2\Pi}{(\tau - \delta) \lambda_{\text{min}}(Q) \min_{i=2, \cdots, N} q_i} \right\},$$

where

$$\Pi \triangleq \frac{\lambda_0}{24} \sum_{i=2}^{N} \varphi_i (\alpha - 1)^2 + \frac{12}{\lambda_0} \sigma_{\text{max}}(G \mathcal{L}_1) \sum_{i=2}^{N} (v_i + v_1)^2,$$

$$[q_2, \cdots, q_N]^T = (\mathcal{L}_1^T)^{-1} 1, \ G = \text{diag}(q_2, \cdots, q_N), \ \alpha = \frac{q_2}{\lambda_0} \max_{i=2, \cdots, N} q_i^2 + \max_{i=2, \cdots, N} \frac{2q_i}{\lambda_0}, \ \text{and} \ \hat{\lambda}_0 \text{ denotes the smallest eigenvalue of } G \mathcal{L}_1 + \mathcal{L}_1^T G.$$

**Proof**: Consider the following Lyapunov function candidate:

$$V_1 = \sum_{i=2}^{N} \frac{c_i q_i}{2} \int_0^t \xi_i^T Q \xi_i ds + \frac{\hat{\lambda}_0}{24} \sum_{i=2}^{N} \hat{c}_i^2,$$

where $\hat{c}_i = c_i - \alpha$. As mentioned earlier, for a communication graph $\mathcal{G}$ satisfying Assumption 2, $\mathcal{L}_1$ is a nonsingular $M$-matrix, which, by Lemma 3 implies that $G > 0$ and $\hat{\lambda}_0 > 0$. Furthermore, by noting that $\rho_i(\cdot)$ are monotonically increasing functions
satisfying \( \rho_i(s) \geq 1 \) for \( s > 0 \) and that \( c_i(t) \geq 1 \) for \( t > 0 \) as in (ii), it is not hard to get that \( V_1 \) is positive definite with respect to \( \xi_i \) and \( \hat{c}_i \), \( i = 2, \ldots, N \).

The time derivative of \( V_1 \) along the trajectory of (7) can be obtained as

\[
\dot{V}_1 = \sum_{i=2}^{N} c_i q_i \rho_i(\xi_i^T Q \xi_i) \dot{\xi}_i^T \dot{Q} \xi_i
\]

\[
+ \sum_{i=2}^{N} \frac{\hat{c}_i q_i}{2} \int_{\xi}^{\xi_i^T Q \xi_i} \rho_i(s) ds
\]

\[
+ \frac{\lambda_0}{12} \sum_{i=2}^{N} (c_i - \alpha)[-\varphi_i(c_i - 1) + \xi_i^T \Gamma \xi_i].
\]

In the rest of this proof, we will use \( \hat{\rho} \) and \( \rho_i \) instead of \( \hat{\rho}(\xi) \) and \( \rho_i(\xi_i^T Q \xi_i) \), respectively, whenever without causing any confusion.

Observe that

\[
\sum_{i=2}^{N} c_i q_i \rho_i \xi_i^T Q \dot{\xi}_i = \xi^T (\hat{C} \hat{\rho} G \otimes Q) \dot{\xi}
\]

\[
= \frac{1}{2} \xi^T (\hat{C} \hat{\rho} G \otimes (QA + AT Q))
\]

\[
- \hat{C} \hat{\rho}(G \mathcal{L}_1 + L^T_1 G) \hat{C} \hat{\rho} \otimes QBB^T Q) \xi
\]

\[
+ \xi^T (\hat{C} \hat{\rho} G \mathcal{L}_1 \otimes QB) \omega
\]

\[
\leq \frac{1}{2} \xi^T (\hat{C} \hat{\rho} G \otimes (QA + AT Q)) - \lambda_0 \hat{\rho}^2 \otimes QBB^T Q) \xi
\]

\[
+ \xi^T (\hat{C} \hat{\rho} G \mathcal{L}_1 \otimes QB) \omega,
\]

where we have used the fact that \( G \mathcal{L}_1 + L^T_1 G > 0 \) to get the first inequality.

Since \( \rho_i \) are monotonically increasing functions satisfying that \( \rho_i(s) \geq 1 \) for \( s > 0 \), it follows that

\[
\sum_{i=2}^{N} \hat{c}_i q_i \int_{\xi}^{\xi_i^T Q \xi_i} \rho_i(s) ds \leq \sum_{i=2}^{N} \hat{c}_i q_i \int_{\xi}^{\xi_i^T Q \xi_i} \rho_i(s) ds
\]

\[
\leq \sum_{i=2}^{N} \hat{c}_i q_i \rho_i \xi_i^T Q \xi_i
\]

\[
\leq \sum_{i=2}^{N} \frac{\hat{c}_i q_i^2}{3 \lambda_0} + \frac{2}{3} \lambda_0 \hat{c}_i \rho_i^2 (\xi_i^T Q \xi_i)^{\frac{3}{2}}
\]

\[
\leq \sum_{i=2}^{N} \frac{\hat{c}_i q_i^2}{3 \lambda_0} + \frac{2}{3} \lambda_0 \hat{c}_i \rho_i^2 (1 + \xi_i^T Q \xi_i)^{\frac{3}{2}}
\]

\[
\leq \sum_{i=1}^{N} \frac{q_i^2}{3 \lambda_0} + \frac{2}{3} \lambda_0 \rho_i^2 (\xi_i^T QBB^T Q \xi_i),
\]

where we have used the fact that \( \hat{c}_i \leq \hat{c}_i \) (\( \hat{c}_i \) is defined in (ii)) to get the first inequality, used the well-known mean value theorem for integrals to obtain the second inequality, and used Lemma 4 to get the third inequality.
Substituting (14) and (15) into (13) yields

\[
\dot{V}_1 \leq \frac{1}{2} \xi^T [\hat{C}\hat{\rho}G \otimes (QA + A^TQ)] \xi - \sum_{i=1}^{N} [\lambda_0 (\frac{1}{2} c_i^2 \rho_i^2 - \frac{1}{12} c_i - \frac{1}{3} \rho_i^2) + \frac{1}{12} (\lambda_0 \alpha - \frac{2 \hat{\alpha}^3}{\hat{\alpha}^2})] \xi^T QBB^T Q \xi_i
\]

\[
+ \frac{\lambda_0}{24} \sum_{i=1}^{N} \varphi_i [-\hat{c}_i^2 + (\alpha - 1)^2] + \xi^T (\hat{C}\hat{\rho}GL_1 \otimes QB) \tilde{\omega},
\]

where we have used the following fact:

\[- (c_i - \alpha)(c_i - 1) = -\hat{c}_i(\hat{c}_i + \alpha - 1) \leq -\frac{1}{2} \hat{c}_i^2 + \frac{1}{2} (\alpha - 1)^2.\]  

(17)

By noting that \(\rho_i \geq 1\) and \(c_i \geq 1, i = 1, \cdots, N,\) and that \(\alpha = 2\hat{\alpha} + \max_{i=1,\cdots,N} \frac{2 \hat{\alpha}^3}{\lambda_0},\) where \(\hat{\alpha} = \frac{36}{\lambda_0} \max_{i=1,\cdots,N} q_i^2,\) it follows from (16) that

\[
\dot{V}_1 \leq \frac{1}{2} \xi^T [\hat{C}\hat{\rho}G \otimes (QA + A^TQ)] \xi - \frac{\lambda_0}{12} \sum_{i=1}^{N} (c_i^2 \rho_i^2 + 2\hat{\alpha}) \xi^T QBB^T Q \xi_i
\]

\[
+ \frac{\lambda_0}{24} \sum_{i=1}^{N} \varphi_i [-\hat{c}_i^2 + (\alpha - 1)^2] + \xi^T (\hat{C}\hat{\rho}GL_1 \otimes QB) \tilde{\omega}.
\]

(18)

Note that

\[
2\xi^T (\hat{C}\hat{\rho}GL_1 \otimes QB) \omega = 2\xi^T (\sqrt{\frac{\lambda_0}{12}} \hat{C}\hat{\rho} \otimes QB)(\sqrt{\frac{12}{\lambda_0}} GL_1 \otimes I) \tilde{\omega}
\]

\[
\leq \frac{\lambda_0}{12} \xi^T (\hat{C}^2 \hat{\rho}^2 \otimes QBB^T Q) \xi + \frac{12}{\lambda_0} \| (GL_1 \otimes I) \tilde{\omega} \|^2
\]

\[
\leq \frac{\lambda_0}{12} \xi^T (\hat{C}^2 \hat{\rho}^2 \otimes QBB^T Q) \xi + \frac{12}{\lambda_0} \sigma_{\max}^2 (GL_1) \sum_{i=2}^{N} (v_i + v_1)^2,
\]

(19)

where we have used (8) to get the last inequality. Then, substituting (19) into (18) gives

\[
\dot{V}_1 \leq \frac{1}{2} \xi^T [\hat{C}\hat{\rho}G \otimes (QA + A^TQ)] - \frac{\lambda_0}{24} (\hat{C}^2 \hat{\rho}^2 + 4\hat{\alpha}I) \otimes QBB^T Q) \xi - \frac{\lambda_0}{24} \sum_{i=2}^{N} \varphi_i \hat{c}_i^2 + \Pi
\]

\[
\leq \frac{1}{2} W(\xi) - \frac{\lambda_0}{24} \sum_{i=1}^{N} \varphi_i \hat{c}_i^2 + \Pi,
\]

where we have used the assertion that \(\frac{\lambda_0}{12} (\hat{C}^2 \hat{\rho}^2 + 4\hat{\alpha}I) \geq \frac{\lambda_0}{6} \sqrt{\alpha} \hat{C}\hat{\rho} \geq \hat{C}\hat{\rho}G \) if \(\sqrt{\alpha} \geq \frac{6}{\lambda_0} G\) to get the last inequality, \(\Pi\) is defined as in (28), and

\[
W(\xi) \triangleq \xi^T (\hat{C}\hat{\rho}G \otimes (QA + A^TQ - QBB^T Q)) \xi
\]

\[
= -\xi^T (\hat{C}\hat{\rho}G \otimes I) \xi
\]

\[\leq 0.\]

Therefore, we can verify that \(\frac{1}{2} W(\xi) - \frac{\lambda_0}{24} \sum_{i=2}^{N} \varphi_i \hat{c}_i^2\) is negative definite. In virtue of the results in [?], we get that both the consensus error \(\xi\) and the adaptive gains \(c_i\) are uniformly ultimately bounded.
Note that (18) can be rewritten into

$$\dot{V}_1 \leq -\delta V_1 + \delta V_1 + \frac{1}{2} W(\xi) - \frac{\lambda_0}{24} \sum_{i=2}^{N} \varphi_i \xi_i^2 + \Pi. \quad (21)$$

Because $\rho_i$ are monotonically increasing and satisfy $\rho_i(s) \geq 1$ for $s > 0$, as shown in (15), we have

$$\sum_{i=2}^{N} c_i q_i \int_{0}^{\xi_i} Q \rho_i(s) ds \leq \sum_{i=2}^{N} c_i q_i \rho_i \xi_i^T Q \xi_i$$

$$= \xi^T (\hat{C} \hat{\rho} G \otimes Q) \xi. \quad (22)$$

Substituting (22) into (21) yields

$$\dot{V}_1 \leq -\delta V_1 + \frac{1}{2} \tilde{W}(\xi) - \frac{\lambda_0}{24} \sum_{i=2}^{N} (\varphi_i - \delta) \xi_i^2$$

$$- \frac{\tau - \delta}{2} \xi^T (\hat{C} \hat{\rho} G \otimes Q) \xi + \Pi,$$

where $\tilde{W}(\xi) \triangleq \xi^T [\hat{C} \hat{\rho} G \otimes (\hat{\rho} I + \tau Q)] \xi$. Because $\tau = \frac{1}{\lambda_{\max}(Q)}$, we can obtain that $\tilde{W}(\xi) \leq 0$. Then, it follows from (41) that

$$\dot{V}_1 \leq -\delta V_1 - \frac{\tau - \delta}{2} \lambda_{\min}(Q) \sum_{i=2}^{N} \min_{q_i \leq 0} q_i \|\xi\|^2 + \Pi,$$

(24)

where we have used the facts that $\varphi_i \geq \delta$, $\delta < \tau$, $i = 2, \cdots, N$, $\hat{C} I \geq I$, $\hat{\rho} I \geq I$, and $G > 0$. Obviously, it follows from (41) that $\dot{V}_1 \leq -\delta V_1$ if $\|\xi\|^2 > \frac{2\Pi}{(\tau - \delta) \lambda_{\min}(Q) \sum_{i=2}^{N} \min_{q_i \leq 0} q_i}$. Then, we can get that if $\delta < \tau$ then $\xi$ exponentially converges to the residual set $D_1$ in (10) with a convergence rate faster than $e^{-\delta t}$.

**Remark 1:** It is well known that there exists a unique solution $Q > 0$ to the ARE (4) if $(A, B)$ is stabilizable [7]. Therefore, a sufficient condition for the existence of an adaptive protocol (5) satisfying Theorem 1 is that $(A, B)$ is stabilizable. The consensus protocol (5) also be equivalently designed by solving the linear matrix inequality: $AP + PA^T - 2BB^T < 0$, as in [7], [5]. In this case, the parameters in (5) can be chosen as $K = -B^T P^{-1}$, $\Gamma = P^{-1} BB^T P^{-1}$, and $\rho_i = (1 + \xi_i^T P^{-1} \xi_i)^3$.

Similar to the adaptive protocol (3) in Lemma 6, the adaptive protocol (5) in Theorem 1 depending only on the agent dynamics and the relative states of neighboring agents, can be constructed and implemented in a fully distributed fashion.

**Remark 2:** Theorem 1 shows that the modified adaptive protocol (5) can ensure the ultimate boundedness of the consensus error $\xi$ and the adaptive gains $c_i$ for the agents in (1), implying that (5) is indeed robust in the presence of bounded external disturbances. From (10), it can be observed that the upper bound of the consensus error $\xi$ depends on the communication graph, the upper bounds of the external disturbances, and the parameters $\varphi_i$ of the adaptive protocol (5). Roughly speaking, $\varphi_i$ should be chosen to be relatively small in order to ensure a smaller bound for $\xi$.

**IV. DISTRIBUTED ROBUST ADAPTIVE PROTOCOLS FOR STRONGLY CONNECTED GRAPHS IN THE PRESENCE OF EXTERNAL DISTURBANCES**

The results in the previous section are applicable to the case where there exists a leader. In this section, we extend to consider the case where the communication graph among the agents is directed and does not contain a leader.
The dynamics of the $N$ agents are still described by (1). The communication graph $\mathcal{G}$ among the $N$ agents is assumed to be strongly connected in this section.

Based on the relative states of neighboring agents, we propose the following distributed adaptive consensus protocol:

$$ u_i = d_i \rho_i(\zeta_i^T Q \zeta_i)K \zeta_i, \quad (25) $$
$$ \dot{d}_i = -\varphi_i(d_i - 1) + \zeta_i^T \Gamma \zeta_i, \quad i = 1, \ldots, N, $$

where $\zeta_i \triangleq \sum_{j=1}^N a_{ij}(x_i - x_j)$, $d_i(t)$ denotes the time-varying coupling gain associated with the $i$-th agent with $d_i(0) \geq 1$, $\varphi_i, i = 1, \ldots, N$, are small positive constants, and the rest of the variables are defined as in (5).

Let $\zeta = [\zeta_1^T, \ldots, \zeta_N^T]^T$ and $x = [x_1^T, \ldots, x_N^T]^T$. Then, $\zeta = (\mathcal{L} \otimes I_n)x$, where $\mathcal{L}$ denotes the Laplacian matrix associated with $\mathcal{G}$. Since $\mathcal{G}$ is strongly connected, it is well known via Lemma [1] that the consensus problem is solved if and only if $\zeta$ asymptotically converges to zero. Hereafter, we refer to $\zeta$ as the consensus error. In virtue of (1) and (25), it is not difficult to get that $\zeta$ and $d_i$ satisfy the following dynamics:

$$ \dot{\zeta} = [I_N \otimes A + \mathcal{L} \hat{D} \hat{\rho}(\zeta) \otimes BK] \zeta + (\mathcal{L} \otimes B) \omega, \quad (26) $$
$$ \dot{d}_i = -\varphi_i(d_i - 1) + \zeta_i^T \Gamma \zeta_i, $$

where $\hat{\rho}(\zeta) \triangleq \text{diag}(\rho_1(\zeta_1^T Q \zeta_1), \ldots, \rho_N(\zeta_N^T Q \zeta_N)), \omega \triangleq [\omega_1^T, \ldots, \omega_N^T]^T$, and $\hat{D} \triangleq \text{diag}(d_1, \ldots, d_N)$.

**Theorem 2:** Suppose that the communication graph $\mathcal{G}$ is strongly connected and Assumption 1 holds. Then, both the consensus error $\zeta$ and the coupling gains $d_i, i = 1, \ldots, N$, in (26), under the adaptive protocol (25) with $K$, $\Gamma$, and $\rho_i$ designed as in Theorem [1] are uniformly ultimately bounded. Furthermore, if $\psi_i$ is chosen to be small enough such that $\varepsilon \triangleq \min_{i=1,\ldots,N} \psi_i < \tau \triangleq \frac{1}{\lambda_{\max}(Q)}$, then $\zeta$ exponentially converges to the residual set

$$ D_2 \triangleq \left\{ \zeta: \|\zeta\|^2 \leq \frac{2\Xi}{(\tau - \varepsilon)\lambda_{\min}(Q)} \min_{i=1,\ldots,N} r_i \right\}, \quad (27) $$

where $[r_1, \ldots, r_N]^T$ is the positive left eigenvector of $\mathcal{L}$ associated with the zero eigenvalue, $\lambda_2(\hat{\mathcal{L}})$ denotes the smallest nonzero eigenvalue of $\hat{\mathcal{L}} \triangleq R \mathcal{L} + \mathcal{L}^T R$, $R \triangleq \text{diag}(r_1, \ldots, r_N) > 0$, $\beta = \frac{72N^2}{\lambda_2(\hat{\mathcal{L}})} \max_{i=1,\ldots,N} r_i^2 + \max_{i=1,\ldots,N} \frac{2r_i^3N^3}{\lambda_2(\hat{\mathcal{L}})^3}$, and

$$ \Xi \triangleq \frac{\lambda_2(\hat{\mathcal{L}})}{24N} \sum_{i=1}^N \varphi_i(\alpha - 1)^2 + \frac{12N}{\lambda_2(\hat{\mathcal{L}})} \sigma_{\max}(R) \sum_{i=1}^N v_i^2. \quad (28) $$

**Proof:** Consider the following Lyapunov function candidate:

$$ V_2 = \sum_{i=1}^N \frac{d_i r_i}{2} \int_0^t \rho_i(s) ds + \frac{\lambda_2(\hat{\mathcal{L}})}{24N} \sum_{i=1}^N \int_0^t \dot{d}_i^2 d\tau, \quad (29) $$

where $\dot{d}_i = d_i - \beta$. Similarly as shown in (9), it is easy to see that $d_i(t) \geq 1$ for $t > 0$. Furthermore, by noting that $\rho_i(\cdot)$ are monotonically increasing functions satisfying $\rho_i(s) \geq 1$ for $s > 0$, it is not difficult to see that $V_2$ is positive definite.
The time derivative of $V_2$ along the trajectory of (26) is given by

$$
\dot{V}_2 = \sum_{i=1}^{N} d_i r_i \rho_i (\zeta_i^T P \zeta_i) \zeta_i^T Q \dot{\zeta}_i
+ \sum_{i=1}^{N} \frac{d_i r_i}{2} \int_{0}^{s} \rho_i(s) ds
+ \frac{\lambda_2(\hat{\mathcal{L}})}{12N} \sum_{i=1}^{N} (d_i - \beta)[-\varphi_i(d_i - 1) + \zeta_i^T \Gamma \zeta_i].
$$

(30)

By using (26) and making some mathematical manipulations, we can get that

$$
\sum_{i=1}^{N} d_i r_i \rho_i \zeta_i^T Q^{-1} \dot{\zeta}_i = \zeta^T (\hat{D} \rho R \otimes Q) \dot{\zeta}
= \frac{1}{2} \zeta^T [\hat{D} \rho R \otimes (QA + AT) Q]
- D \rho [L \hat{D} \rho \otimes QBB^T Q] \zeta + \zeta^T (\hat{D} \rho R L \otimes PB) \nu.
$$

(31)

Let $\zeta = (\hat{D} \rho \otimes I_n) \zeta$. By the definitions of $\zeta$ and $\zeta$, we have

$$
\zeta^T (\hat{D}^{-1} \rho^{-1} r \otimes I_n) = \zeta^T (r \otimes I_n)
= x^T (\mathcal{L}^T r \otimes I_n) = 0,
$$

where we have used fact that $r^T \mathcal{L} = 0$. Since every entry of $\hat{D}^{-1} \rho^{-1} r \otimes I_n$ is also positive. In light of Lemma (22) we get that

$$
\zeta^T (\hat{D} \rho \otimes I_n) \zeta > \frac{\lambda_2(\hat{\mathcal{L}})}{N} \zeta^T \zeta
= \frac{\lambda_2(\hat{\mathcal{L}})}{N} \zeta^T (\hat{D}^2 \rho^2 \otimes I_n) \zeta.
$$

(32)

Substituting (32) into (31) gives

$$
\sum_{i=1}^{N} d_i r_i \rho_i \zeta_i^T Q \dot{\zeta}_i \leq \frac{1}{2} \zeta^T [\hat{D} \rho R \otimes (QA + AT) Q]
- \frac{\lambda_2(\hat{\mathcal{L}})}{N} \hat{D}^2 \rho^2 \otimes QBB^T Q \zeta
+ \zeta^T (\hat{D} \rho R L \otimes QB) \nu.
$$

(33)

Similar to (15), we can obtain that

$$
\sum_{i=1}^{N} d_i r_i \int_{0}^{s} \rho_i(s) ds \leq \sum_{i=1}^{N} \left[ \frac{r_i^2 N^2}{3 \lambda_2(\hat{\mathcal{L}})^2} + \frac{2 \lambda_2(\hat{\mathcal{L}})}{3N} \rho_i^2 \right] \zeta_i^T QBB^T Q \zeta_i.
$$

(34)

Substituting (33) and (34) into (30) yields

$$
\dot{V}_2 \leq \frac{1}{2} \zeta^T [\hat{D} \rho G \otimes (QA + AT) Q] \zeta
- \sum_{i=1}^{N} \frac{\lambda_2(\hat{\mathcal{L}})}{N} \left( \frac{1}{12} d_i \rho_i^2 - \frac{1}{3} \rho_i^2 \right)
+ \frac{1}{12N} \left( \beta \lambda_2(\hat{\mathcal{L}}) - \frac{2 r_i^3 N^3}{\lambda_2(\hat{\mathcal{L}})^2} \right) \zeta_i^T QBB^T Q \zeta_i
+ \frac{\lambda_2(\hat{\mathcal{L}})}{24N} \sum_{i=1}^{N} \varphi_i [-d_i^2 + (\beta - 1)^2] + \zeta^T (\hat{D} \rho R L \otimes QB) \nu,
$$

(35)
where we have used (17). Similar to (19), it is easy to verify that
\[
2\xi^T(\hat{D}\hat{\rho}R \otimes QB)\omega \leq \frac{\lambda_2(\hat{L})}{12N} \xi^T(\hat{D}^2 \hat{\rho}^2 \otimes QBB^T \hat{Q})\xi + \frac{12N}{\lambda_2(\hat{L})} \sigma_{\text{max}}^2(R\mathcal{L}) \sum_{i=1}^N \nu_i^2.
\] (36)

Choose \( \beta = 2\hat{\beta} + \max_{i=1,\ldots,N} \frac{2\gamma_i N^3}{\lambda_2(\hat{L})^3} \), where \( \hat{\beta} = \frac{36N^2}{\lambda_2(\hat{L})} \max_{i=1,\ldots,N} r_i \). Substituting (36) into (35) gives
\[
\dot{V}_2 \leq \frac{1}{2} \xi^T[\hat{D}\hat{\rho}R \otimes (QA + A^T Q)]\xi
- \frac{\lambda_2(\hat{L})}{24N}(\hat{D}^2 \hat{\rho}^2 + 4\hat{\beta}I) \otimes \xi^T QBB^T \xi_i
- \frac{\lambda_2(\hat{L})}{24N} \sum_{i=1}^N \varphi_i d_i^2 + \Xi
\leq \frac{1}{2} Z(\xi) - \frac{\lambda_2(\hat{L})}{24N} \sum_{i=1}^N \varphi_i d_i^2 + \Xi,
\] (37)
where \( \Xi \) is defined as in (28).

\[
Z(\xi) \triangleq \xi^T[\hat{D}\hat{\rho}R \otimes (QA + A^T Q - QBB^T Q)]\xi
= -\xi^T(\hat{D}\hat{\rho}R \otimes I)\xi \leq 0,
\]
and to get the last inequality, we have used the assertion that if \( \sqrt{\beta} I \geq \frac{6N}{\lambda_2(\hat{L})} R \), then \( \frac{\lambda_2(\hat{L})}{12N}(\hat{D}^2 \hat{\rho}^2 + 4\hat{\beta}I) \geq \frac{\lambda_2(\hat{L})}{6N} \sqrt{\beta} \hat{D}\hat{\rho} \geq \hat{D}\hat{\rho}R \). Therefore, we can verify that \( \frac{1}{2} Z(\xi) - \frac{\lambda_2(\hat{L})}{24N} \sum_{i=1}^N \varphi_i d_i^2 \) is negative definite. In virtue of Lemma 5, we get that both the consensus error \( \xi \) and the adaptive gains \( d_i \) are uniformly ultimately bounded.

Note that (37) can be rewritten into
\[
\dot{V}_2 \leq -\epsilon V_2 + \epsilon V_2 + \frac{1}{2} Z(\xi) - \frac{\lambda_2(\hat{L})}{24N} \sum_{i=1}^N \varphi_i d_i^2 + \Xi.
\] (38)
As shown in (29), we have
\[
\sum_{i=1}^N d_i r_i \int_0^t \rho_i(s) ds \leq \xi^T(\hat{D}\hat{\rho}R \otimes Q)\xi.
\] (39)
Substituting (39) into (38) yields
\[
\dot{V}_2 \leq -\epsilon V_2 + \frac{1}{2} Z(\xi) - \frac{\lambda_2(\hat{L})}{24N} \sum_{i=1}^N (\varphi_i - \epsilon) d_i^2
- \frac{\tau - \epsilon}{2} \xi^T(\hat{D}\hat{\rho}R \otimes Q)\xi + \Xi,
\] (40)
where \( Z(\xi) \triangleq \xi^T[\hat{D}\hat{\rho}R \otimes (-I + \tau Q)]\xi \). Because \( \tau = \frac{1}{\lambda_{\text{max}}(Q)} \), we can obtain that \( Z(\xi) \leq 0 \). Then, it follows from (40) that
\[
\dot{V}_2 \leq -\epsilon V_2 - \frac{\tau - \epsilon}{2} \lambda_{\text{min}}(Q)( \min_{i=1,\ldots,N} r_i ) \| \xi \|^2 + \Xi,
\] (41)
where we have used the facts that \( \varphi_i \geq \epsilon, \epsilon < \tau, \hat{D} \geq I, \hat{\rho} \geq I, \) and \( R > 0 \). Obviously, it follows from (41) that \( \dot{V}_2 \leq -\epsilon V_2 \) if \( \| \xi \|^2 > \frac{2\epsilon}{(\tau - \epsilon) \lambda_{\text{min}}(Q) \min_{i=1,\ldots,N} r_i} \). Then, we can get that if \( \epsilon \leq \tau \) then \( \xi \) exponentially converges to the residual set \( D_2 \) in (10) with a convergence rate faster than \( e^{-\tau t} \).
In the robust adaptive protocol (25), the term $-\varphi_i(d_i - 1)$ is inspired by the $\sigma$ modification technique, which is vital to ensuring the ultimate boundedness of the consensus error $\zeta$ and the adaptive gains $d_i$ in the presence of external disturbances. For the case where the external disturbances in (1) do not exist, the adaptive protocol (25) with the term $-\varphi_i(d_i - 1)$ removed, i.e., the following adaptive protocol

$$u_i = d_i \rho_i (\zeta_i^T Q \zeta_i) K \zeta_i,$$

$$\dot{d}_i = \zeta_i^T \Gamma \zeta_i, \quad i = 1, \cdots, N,$$

(42)
can ensure the asymptotical convergence of the consensus error $\zeta$. This is summarized in the following corollary.

**Corollary 1:** For the $N$ agents described by $\dot{x}_i = Ax_i + Bu_i$, $i = 1, \cdots, N$, whose communication graph $G$ is strongly connected, the consensus error $\zeta$ under the adaptive protocol (25) with $K$, $\Gamma$, and $\rho_i$ designed as in Theorem 1 asymptotically converges to zero. Moreover, each coupling gain $d_i$ converges to some finite steady-state value.

The above corollary can be proved by following similar steps in the proof of Theorem 2. The adaptive protocol (42) complements the adaptive protocol (3) in [?] which are applicable to directed graphs with a leader.

**Remark 3:** Compared to the previous works [?], [?] which also present distributed adaptive protocols for directed graphs, the main contribution of this paper is that distributed robust adaptive protocols are presented, which can exclude the parameter drift phenomenon encountered by the adaptive protocols in [?], [?] in the presence of external disturbances. Besides, the agents are restricted to be second-order integrators in [?]. The ultimate boundedness of both the consensus errors and the adaptive gains is shown and the upper bounds of the consensus errors is given, which are far from being easy.

V. SIMULATION EXAMPLE

![Fig. 1: The directed communication graph.](image-url)
In this section, a simulation example is provided for illustration.

Consider a network of double integrators, described by (1), with

\[ x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

For illustration, the disturbances associated with the agents are assumed to be \( \omega_1 = 0, \omega_2 = 0.2 \sin(t), \omega_3 = 0.1 \sin(t), \omega_4 = 0.2 \cos(2t), \omega_5 = -0.3 \exp^{-2t}, \omega_6 = -0.2 \sin(x_{51}), \) and \( \omega_7 = 0. \) The communication graph is given as in Fig. 1, where the vertex indexed by 1 is the leader which is only accessible to the vertex indexed by 2. It is easy to verify that the graph in Fig. 1 satisfies Assumption 2.

Solving the ARE (4) by using MATLAB gives a solution \( Q = \begin{bmatrix} 1 & 1.7321 \\ 1 & 1.7321 \end{bmatrix}. \) Thus, the feedback gain matrices in (5) are obtained as

\[ K = - \begin{bmatrix} 1 \\ 1.7321 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 1.7321 \\ 1.7321 & 3 \end{bmatrix}. \]

To illustrate Theorem 1, let \( \varphi_i = 0.02 \) in (5) and the initial states \( c_i(0) \) be randomly chosen within the interval [1, 3]. The consensus errors \( \xi_i, i = 2, \cdots, 7, \) of the double integrators, defined as in (6), and the coupling weights \( c_i \) associated with the followers, under the adaptive protocol (5) with \( K, \Gamma, \) and \( \rho_i \) chosen as in Theorem 1 are depicted in in Figs. 2 and 3, respectively, both of which are clearly bounded.

![Graph](image-url)

**Fig. 2:** The consensus errors \( \xi_i, i = 2, \cdots, 7, \) of double integrators under the protocol (5).

**VI. Conclusion**

In this paper, we have presented distributed adaptive consensus protocols to achieve consensus for linear multi-agent systems with directed graphs which are strongly connected or contain a directed spanning tree with a leader as the root node. Specifically,
distributed robust adaptive protocols are designed, which can guarantee the ultimate boundedness of both the consensus error and the adaptive gains in the presence of external disturbances. Note that the design of these adaptive protocols depends only on the agent dynamics and the relative state information of neighboring agents, which thereby can be done by each agent in a truly distributed fashion. Interesting future works include designing distributed adaptive consensus protocols using only relative output information.