A Note on the $c = 1$ Barrier in Liouville Theory

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Abstract. The instability of Liouville theory coupled to $c > 1$ matter fields is shown to persist even when the “spikes” which represent highly singular geometries are allowed to interact in a natural way.

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1 Introduction

Recent years have seen a great deal of work on non-critical bosonic string theories, especially using discrete methods [1]. Most of the efforts have been directed at understanding the nature the $c = 1$ barrier, where the critical exponents predicted by the KPZ formulae [2] become meaningless.

As has been pointed out by Bowick, among others, it is unusual that whatever progress that has been made in understanding this barrier has been obtained via discrete methods, such as discretized random surfaces, rather than by considering continuum models. The recent results of David [3] have their origin in the study of matrix models, and computation of physical quantities in the $c > 1$ regime take as their starting point the branched polymer (BP) ensemble, a collection of simplicial trees which approximate the highly singular geometry expected to dominate such regions.

Work on the continuum version of the BP transition goes back to the heuristic arguments of Cates [4] and the more elaborate calculations of Krzwicki [5], in which they considered a BP-like configuration (a “spike”), and found that such configurations were favoured as soon as the central charge $c$ exceeds unity. This spikes were non-interacting, and Krzwicki in particular noted that it would be interesting to consider interacting spikes with a view to determining whether or not the interaction changes the transition.

The purpose of this paper is to consider one such case. We start by reviewing the relevant arguments from [5] which lead to the computation of the spike free energy. We then introduce a natural spike interaction and investigate its properties, as well as computing the free energy of the interacting spikes in the ”spike gas” regime. We end by speculating on the strongly-coupled spike ensemble and its effect on the BP transition.

2 Spikes in Liouville Theory

In non-critical bosonic string theory (or alternatively, in two-dimensional quantum gravity coupled to matter fields), we integrate over all 2D geometries and over all matter field configurations to obtain quantities of interest, such as correlation functions. Among the more interesting types of matter field we can couple to geometry are the Conformal Field Theories (CFTs), which are characterized by the central charge $c$. Once we integrate out the matter fields, we are left with the Liouville action

$$S_L = \frac{26 - c}{96\pi} \int d^2\xi \left( \phi \partial^2 \phi + ke^\phi \right)$$  \hspace{1cm} (1)
where \( \xi = (\xi_1, \xi_2) \) are the coordinates describing the manifolds, \( \phi(\xi) \) is the Liouville field describing the metric via \( g_{\mu\nu} = e^{\phi} \delta_{\mu\nu} \), and \( k \) is the cosmological constant. From the work carried out in \[2\], we know that the large-area behaviour of the generating functional

\[
Z(A) = e^{k c A} A^{\gamma_{str}(h)-3} \left( 1 + \ldots \right)
\]

is controled by \( c \) via the famous KPZ relation

\[
\gamma_{str}(h) = 2 - \frac{1 - h}{12} \left\{ 25 - c + \sqrt{(1 - c)(25 - c)} \right\}
\]

Clearly, formula (3) makes no sense for \( c > 1 \), and this breakdown has been attributed to the emergence of BP configurations which dominate that region. The continuum analogue of such configurations is Cates' spike

\[
\phi_0 = -\frac{\mu}{2} \log \left\{ (\xi - \xi_0)^2 + \alpha^2 \right\}
\]

where \( \mu > 2 \) and \( \alpha << 1 \). The area of such a spike goes like

\[
A = \int d^2 \xi e^{\phi_0} \sim \alpha^{2-\mu}
\]

up to regular terms in the \( \alpha \to 0 \) limit. The question posed (and answered) by Cates was whether or not such spikes would be favoured in the \( c > 1 \) region; the answer was affirmative \[4\]. The rigorous arguments which confirmed the answer are due to Krzwicki \[5\], and we now give a brief overview of his reasoning. The action corresponding to the spike (4) can be easily computed and is, up to regular terms in \( \alpha \)

\[
S_{\phi_0} = \frac{26 - c}{96} \left\{ \mu^3 \log(1/\alpha) \right\}
\]

However, we can get further contributions to the free energy from the functional integration over spike configurations. As shown in \[5\], by considering a suitably regularized Laplacian operator on a manifold \( M \), the functional integration reduces to a product of integrations over the centers of the spikes and of integrations of small fluctuations about the spike configuration, which we write symbolically as \( D\chi \prod_i d^2 \xi_i \). The expression for \( D\chi \) can be computed, and it renormalizes the free energy which becomes

\[
S_{\phi_0} = \mu \log(1/\alpha) \left\{ \frac{(25 - c)\mu^2}{96} - 1 - \frac{2(2 - \mu)}{\mu} \right\}
\]

Notice the factor of \( 25 - c \), instead of the \( 26 - c \) of (3), as well as the extra term.

By setting \( \mu = 2 + \eta \) it is easy to see that (7) becomes large negative if \( c > 1 \) for \( \eta \) small; since \( \eta \) is arbitrary (because we expect all spikes to be present in the
functional integration), we conclude that the formation of spikes with $\mu$ close to 2 is favoured as soon as $c$ becomes greater than unity.

Due to the decoupling of the functional integration, the results above extend trivially to a gas of $N$ non interacting spikes, which can be represented by

$$\phi = -\sum_{i=1}^{N} \frac{\mu}{2} \log \left\{ (\xi_i - \xi_{0_i})^2 + \alpha^2 \right\}$$  \hspace{1cm} (8)

One could, however, consider interacting spikes, and if we make the restriction that such interaction is only dependent on the distance between the centers of the the spikes and not on the shape of the spike configuration, then the free energy will be changed by an amount which can be computed from

$$Z_{int} = \int \prod_i d^2 \xi_i e^{-S_{int}(\xi_0_1, \ldots, \xi_0_N)}$$  \hspace{1cm} (9)

3 Interacting Spikes

We consider the case of spikes interacting via a Coulomb-like potential in two dimensions

$$S_{int} = \prod_{i \neq j} G|\xi_{0_i} - \xi_{0_j}|$$  \hspace{1cm} (10)

The coupling constant $G$ is chosen to be the product of the areas of the spikes, which gives an indication of the “mass” of each configuration

$$G = 4\alpha^{2(2-\mu)},$$  \hspace{1cm} (11)

the factor of 4 being inserted for later convenience.

With this choice of interaction, we are left with the task of computing integrals of the type

$$Z_{int}(G) = \int \prod_l d^2 \xi_{0_l} \prod_{i \neq j} |\xi_{0_i} - \xi_{0_j}|^{-G}$$  \hspace{1cm} (12)

which must be regularized. One possibility is to consider the area of the integration region to be large but limited, another is to prolong the definition of the integrals (12) to the complex plane and use the conformal symmetry which they then possess to evaluate them, by fixing three of the spikes at coordinates 0, 1 and $\infty$.

With this proviso, and introducing $\rho = -G/4$ we find

$$Z_{int}(\rho) \simeq (\Delta(1 - \rho))^N \prod_{j=1}^{N} \Delta(j\rho) \prod_{l=0}^{N-1} (\Delta(1 + (l + 1/2)\rho))^2 \Delta(-1 - (N + l)\rho)$$  \hspace{1cm} (13)

\[2\] While this choice of interaction might appear somewhat restrictive, the same qualitative results would be obtained by considering any analytic function of $\alpha$ and $\eta$ which obeys $f = 1$ for $\eta \to 0^+$, $\alpha > 0$. 

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where
\[ \Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)} = \frac{1}{\pi} (\Gamma(x))^2 \sin(\pi x) \] (14)
and \( N \) is the number of free spikes, i.e. those not fixed at special positions; the total number of spikes is \( N + 3 \).

From (14) it is easy to classify the zeros and singularities of (13); this structure and its origins have been discussed at length in [7]. The main point to retain is that the singularity at \( \rho = -2 \) signals a transition from an unclumped phase \( (\rho > -2) \) where the spikes are distant from each other, to a clumped phase \( (\rho < -2) \), where they are very close. Using (13) and the properties of the Gamma function, it is now easy to write down the free energy for some cases of interest: small coupling \( (\rho \sim 0) \), and close to the clumping transition \( (\rho = -2 + \epsilon \text{ with } \epsilon \text{ small but positive}) \).

We begin by considering \( N = 1 \), the first case for which (13) is applicable, and we find
\[ Z_{\text{int}} = \left[ \frac{\Gamma(1 + \rho/2)}{\Gamma(\rho/2)} \right] \frac{\Gamma(-1 - \rho)}{\Gamma(2 + \rho)} \] (15)
which, for small \( \rho \) has the expansion
\[ Z_{\text{int}} = \frac{\rho}{4} + O(\rho^2) \] (16)
while for \( \rho = -2 + \epsilon \) it behaves as
\[ Z_{\text{int}} = 4\epsilon^{-1} + O(\epsilon^0) \] (17)
In both cases the free energy, written in terms of \( \alpha \) and \( \mu \) reads, retaining only the leading terms,
\[ F = \mu \log(1/\alpha) \left\{ \frac{(25-c)\mu^2}{96} - 1 - \frac{4(2-\mu)}{\mu} \right\} \] (18)
which should be compared with [7]; we see that the only change comes in the last term. This change however does not alter the previous conclusion, and (18) becomes large and negative for \( c > 1 \) and small \( \eta \). This was to be expected, since at small coupling the results are not expected to differ dramatically from the no-coupling regime, as the spikes are widely separated.

The following case of interest is when we allow \( N \) to be large. Using Stirling’s formula and the formula for the behaviour of the Gamma function close to s singularity we find, for small \( \rho \)
\[ Z_{\text{int}} = \frac{2 \Gamma(N + 1/2)4^{-N}}{\sqrt{\pi(\Gamma(N + 1))}} \rho^N \] (19)
leading to a free energy
\[ F = -N \log(\rho) - N \log(4) + O(\log(N)) \] (20)
The $O(\log(N))$ order terms are of no consequence to the overall behaviour of the free energy in the large $N$ limit, the $\log(4)$ term is irrelevant in the small $\rho$ limit, and hence

$$F = N\mu \log(1/\alpha) \left\{ \frac{(25 - c)\mu^2}{96} - 1 - \frac{4(2 - \mu)}{\mu} \right\} + O(\log(N)) \quad (21)$$

A similar calculation holds for $\rho = -2 + \epsilon$; yet again the free energy takes the form of eq. (21). When $c > 1$, spike formation is still favoured for small $\eta$.

4 Conclusions and Discussion

In this paper we discussed the stability of a random surface with interacting spikes, and found that for a particular (unclumped) phase, the transition to a spike dominated phase occurs as soon as $c$ becomes greater than unity. Together with the matrix-model computations of David [3] this provides strong evidence for a sudden transition, devoid of an interpolating region.

A full proof that the $c > 1$ region is dominated by branched polymers is still lacking. While a simple-minded extension of the above calculations suggests that the general structure of the free energy remains the same as in (18) and (21) (because the Gamma function goes like $\Gamma(-n + \epsilon) \sim \epsilon^{-1}$, this is especially apparent close to singularities of (13)) we must be careful about such extrapolations, given that for $\rho < -2$ the spikes are no longer separated, but form clumps, casting doubts on the spike-gas picture.

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References

[1] M. Bowick, “Random Surfaces and Lattice Gravity”, Talk given at the 15th International Symposium on Lattice Field Theory, Nucl. Phys. Proc. Suppl. 63 (1998), 77.

[2] V.G.Knizhnik, A.M.Polyakov and A.B.Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819;
F. David, Mod. Phys. Lett. A3 (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.

[3] F. David, Nucl. Phys. B487 (1997) 633.
[4] M. E. Cates, Phys. Lett. B251 (1990) 553.

[5] A. Krzywicki, Phys. Rev. D41 (1990), 3086.

[6] V. Dotsenko and V. Fateev, Nucl. Phys. B240 (1984) 312.

[7] E. Abdalla and M. R. Tabar, hep-th/9803161