The Yang–Baxter equation for $\mathcal{PT}$ invariant 19-vertex models

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Abstract
We study the solutions of the Yang–Baxter equation associated with 19-vertex models invariant by the parity-time symmetry from the perspective of algebraic geometry. We determine the form of the algebraic curves constraining the respective Boltzmann weights and find that they possess a universal structure. This allows us to classify the integrable manifolds into four different families reproducing three known models, besides uncovering a novel 19-vertex model in a unified way. The introduction of the spectral parameter on the weights is made via the parameterization of the fundamental algebraic curve which is a conic. The diagonalization of the transfer matrix of the new vertex model and its thermodynamic limit properties are discussed. We point out a connection between the form of the main curve and the nature of the excitations of the corresponding spin-1 chains.

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1. Introduction

An important concept in the theory of soluble two-dimensional lattice models of statistical mechanics is to embed the respective transfer matrices into a family of pairwise commuting operators [1]. Let $T_L(\omega_1^{(1)}, \ldots, \omega_n^{(1)})$ denote the transfer matrix on a chain of size $L$ with Boltzmann weights $\omega_1^{(1)}, \ldots, \omega_n^{(1)}$. This approach requires that the transfer matrix fulfill the following property:

$$[T_L(\omega_1^{(1)}, \ldots, \omega_n^{(1)}), T_L(\omega_1^{(2)}, \ldots, \omega_n^{(2)})] = 0, \quad (1)$$

for arbitrary $L$ and weights $\omega_1^{(1)}, \ldots, \omega_n^{(1)}, \omega_1^{(2)}, \ldots, \omega_n^{(2)}$ for $i = 1, 2$.

At first sight it appears that one needs to verify an infinite number of constraints for the weights since equation (1) should be valid for any values of $L$. This is however not the case since Baxter argued that a finite number of local conditions on the weights are sufficient to ensure the commutativity among distinct transfer matrices for arbitrary $L$ [2]. These conditions
can be written as a matrix equation whose structure depends on the family of the lattice model under consideration. In what follows, we shall discuss them for a relevant class of lattice systems denominated vertex models.

Let us for example consider a vertex model on a square lattice of size \( L \times L \). The respective statistical configurations sit on the edges of the lattice. In the simplest case, the numbers of states living on the horizontal and vertical edges are the same and take values on a set of integers numbers \( 1, 2, \ldots, N \). Furthermore, to each vertex configuration \( a,b,c,d \) is assigned the Boltzmann weight \( \mathcal{L}(a,c|b,d) \) as defined in figure 1.

The sufficient condition for the commutativity of two distinct transfer matrices associated with a given vertex model for arbitrary \( L \) is the celebrated Yang–Baxter equation \([2]\). Considering the notation of figure 1, these sets of functional relations can be written as follows:

\[
\sum_{\gamma_1, \gamma_2, \gamma_3 = 1}^{N} R(a_1, \gamma_1|a_2, \gamma_2) \mathcal{L}^{(1)}(\gamma_1, b_1|a_3, \gamma_3) \mathcal{L}^{(2)}(\gamma_2, b_2|\gamma_3, b_3) = \sum_{\gamma_1, \gamma_2, \gamma_3 = 1}^{N} \mathcal{L}^{(2)}(a_2, \gamma_2|a_3, \gamma_3) \mathcal{L}^{(1)}(a_1, \gamma_1|\gamma_3, b_3) R(\gamma_1, b_1|\gamma_2, b_2), \tag{2}
\]

where \( R(a, c|b, d) \) are the elements of an invertible \( N^2 \times N^2 \) matrix often called an \( R \)-matrix.

In order to classify solvable vertex models with given statistical configurations, one has to find the possible solutions of the corresponding Yang–Baxter equation. In principle, we can consider this problem by following the method devised by Baxter for two-state-vertex models \([2]\). We start by eliminating the matrix elements \( R(a, b|c, d) \) with the help of a suitable subset of relations derived from equation (2). The remaining functional equations will then depend only on the weights \( \mathcal{L}^{(1)}(a, b|c, d) \) and \( \mathcal{L}^{(2)}(a, b|c, d) \) which need to be decoupled using separation of variables. This step is essential to reveal the algebraic invariants constraining the Boltzmann weights’ parameter space. The parameterization of such manifolds provides the dependence of Boltzmann weights on spectral parameters, which is for instance useful to formulate the algebraic Bethe ansatz \([3–5]\).
The implementation of all the steps described above, even for a given fixed vertex state configuration, is in general a tantalizing problem in mathematical physics. The concrete results are mostly concentrated on vertex models having two states per edge, see for instance [6–8]. In this case, the non-trivial models that have been uncovered are mainly associated with the algebraic manifolds of the asymmetric six-vertex model [9], the symmetric eight-vertex model [10] and the so-called free-fermion systems [11]. The difficulties with the problem increase with the number \( N \) of states due to the proliferation of the possible allowed weights that are ultimately fixed by different classes of functional relations. The merit of this approach is that it makes possible an unambiguous classification of integrable vertex models with a given statistical configuration from first principles.

The purpose of this paper is to begin a study of the Baxter method for three-state models satisfying the so-called ice-rule. The respective statistical configuration leads us to the total number of 19 Boltzmann weights. Here we shall consider a relevant subclass of such models whose weights are invariant by the joint action of the parity and time (\( PT \)) reversal symmetry. For this family of 19-vertex models, we show that the dependence of the underlying algebraic curves on the respective weights is rather universal. The possible solvable vertex models are classified by the distinct branches of certain invariant values entering the definition of these manifolds. This general analysis provides the classification of 19-vertex models in four different families. The first three have already been discovered in the context of the integrable spin-1 \( XXZ \) chain [12], the quantum inverse scattering of the Mikhailov–Shabat model [13] and the quantum algebra \( \mathcal{U}_q[SU(2)] \) at roots of unity [14–16]. Interestingly enough, our results reveal that these models have the same underlying algebraic background despite their rather distinct quantum group origin. To the best of our knowledge, the fourth uncovered vertex model is new in the literature.

We have organized this paper as follows. For the sake of completeness, we review in the next section the main characteristics of the 19-vertex models. In section 3, we consider the analysis of the functional relations coming from the Yang–Baxter equation. We develop a systematic way to solve such Yang–Baxter relations leading us to determine the algebraic curves fulfilled by the Boltzmann weights. It turns out that the principal algebraic curve is a conic involving three basic independent weights. The remaining amplitudes of the parameter space are remarkably resolved in terms of the ratios of polynomials depending on such basic weights. This makes it possible to classify the \( PT \) invariant 19-vertex models from a unified perspective. In section 4, we discuss the parameterization of the Boltzmann weights in terms of a spectral parameter and the associated spin-1 Hamiltonians. We note that the geometric form of the fundamental algebraic curve is directly related to the nature of the excitations of the corresponding spin chains. In section 5, we present the eigenvalues of the transfer matrix associated with the new 19-vertex model and its respective Bethe ansatz equation. We investigate the bulk properties of this model providing additional support to the mentioned relationship among the curve geometry and the behavior of the spin-1 chain excitations. Our conclusions are presented in section 6. In the appendices, we summarize some technical details useful for understanding the main text.

2. The 19-vertex model

In this section, we review the main features of the 19-vertex model. This lattice model has three states per edge, which will be denoted here by \( a, b, c, d = 0, \pm \). The allowed statistical configurations compatible with the ice-rule \( a + b = c + d \) lead us to 19 different Boltzmann weights. These weights are represented in figure 2, where the respective subscripts emphasize the non-null charge sectors \( a + b \).
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imposes the following relationship among the weights of figure 2:

\[ b_\pm = b_\pm, \quad f = f, \quad e = d, \quad \tilde{e} = \tilde{d}. \]  

Figure 2. The vertex configurations of a general 19-vertex model on a square lattice.

In this paper, we investigate the integrable manifolds of 19-vertex models whose weights are \( PT \) invariant. This invariance relates some of the weights leading to the subclass of three-state-vertex models to 14 distinct Boltzmann weights. More precisely, the \( PT \) symmetry imposes the following relationship among the weights of figure 2:

\[ b_\pm = b_\pm, \quad \tilde{f} = f, \quad e = d, \quad \tilde{e} = \tilde{d}. \]  

We begin by introducing a common notation for the Boltzmann weights of such family of 19-vertex models. Taking into account the parameter subspace (3), one can represent the content of the \( L^{(i)} \)-operators by the following matrix:

\[
L^{(i)} = a^{(i)}_+ e_{1,1} \otimes e_{1,1} + b^{(i)}_+ [e_{1,1} \otimes e_{2,2} + e_{2,2} \otimes e_{1,1}] + f^{(i)} [e_{1,1} \otimes e_{3,3} + e_{3,3} \otimes e_{1,1}] \\
+ b^{(i)}_+ [e_{2,2} \otimes e_{3,3} + e_{3,3} \otimes e_{2,2}] + g^{(i)} [e_{2,2} \otimes e_{2,2} + a^{(i)}_+ e_{3,3} \otimes e_{3,3} + h^{(i)} e_{1,1} \otimes e_{3,3} \\
+ h^{(i)} e_{3,3} \otimes e_{1,1} + c^{(i)}_+ e_{1,1} \otimes e_{2,2} + z^{(i)} e_{2,2} \otimes e_{1,1} + c^{(i)}_- e_{3,3} \otimes e_{3,3} + \tilde{c}^{(i)} e_{3,3} \otimes e_{2,2} \\
+ d^{(i)} [e_{1,1} \otimes e_{3,3} + e_{3,3} \otimes e_{1,1}] + \tilde{d}^{(i)} [e_{2,2} \otimes e_{1,1} + e_{1,1} \otimes e_{2,2}], \quad i = 1, 2,
\]

where \( a_{n,b} \) denote \( 3 \times 3 \) Weyl matrices.

By substituting the \( L^{(i)} \)-operators (4) into the Yang–Baxter equation (2), we are able to determine the null matrix elements of the \( R \)-matrix. Under the mild assumption that the weights \( a^{(i)}_n \) are in general distinct from \( b^{(i)}_n \) and \( f^{(i)} \), it is not difficult to see that the \( R \)-matrix has the same form of the \( L^{(i)} \)-operators. For convenience, we therefore express the \( R \)-matrix as

\[
R = a^{(0)}_+ e_{1,1} \otimes e_{1,1} + b^{(0)}_+ [e_{1,1} \otimes e_{2,2} + e_{2,2} \otimes e_{1,1}] + f^{(0)} [e_{1,1} \otimes e_{3,3} + e_{3,3} \otimes e_{1,1}] \\
+ b^{(0)}_- [e_{2,2} \otimes e_{3,3} + e_{3,3} \otimes e_{2,2}] + g^{(0)} [e_{2,2} \otimes e_{2,2} + a^{(0)}_- e_{3,3} \otimes e_{3,3} + h^{(0)} e_{1,1} \otimes e_{3,3} \\
+ h^{(0)} e_{3,3} \otimes e_{1,1} + c^{(0)}_+ e_{1,1} \otimes e_{2,2} + z^{(0)} e_{2,2} \otimes e_{1,1} + c^{(0)}_- e_{3,3} \otimes e_{3,3} + \tilde{c}^{(0)} e_{3,3} \otimes e_{2,2} \\
+ d^{(0)} [e_{1,1} \otimes e_{3,3} + e_{3,3} \otimes e_{1,1}] + \tilde{d}^{(0)} [e_{2,2} \otimes e_{1,1} + e_{1,1} \otimes e_{2,2}],
\]

(5)
We emphasize here that we are interested in classifying genuine 19-vertex models and therefore all weights $a(i), b(i), c(i), d(i), \tilde{c}(i), f(i), g(i), h(i)$ and $\tilde{h}(i)$ for $i = 0, 1, 2$ are assumed to be non-null. We recall that a classification of solvable three-state-vertex models has been attempted before in the literature [17]. In [17], more stringent symmetry conditions for the weights were assumed besides the fact that some of them could be null. As a result, the only strict 19-vertex model found was the standard Fateev–Zamolodchikov spin-1 model [12].

3. The Yang–Baxter equation

The purpose of this section is to investigate the functional relations for the Boltzmann weights which are derived by substituting the $L(i)$-operator (4) and the $R$-matrix (5) structures into the Yang–Baxter equation (2). We find that the functional relations can be classified in terms of the number of distinct triple product of weights. It turns out that the minimum number of triple products is 2, while the maximum is 5. In table 1, we summarize the number of different equations having two, three, four and five types of triple product of weights. Clearly, both the number and structure of the functional relations to be analyzed are far more involving than that associated with the $N = 2$ state-vertex model satisfying the ice-rule. We therefore start our discussion by first solving the simplest relations containing two triple products.

3.1. Two-term relations

The six equations possessing only two different types of triple product of weights are given by

\[
\begin{align*}
    c^{(0)}_{\pm} c^{(1)}_{\pm} c^{(2)}_{\pm} - c^{(0)}_{\pm} c^{(1)}_{\pm} c^{(2)}_{\pm} &= 0, \\
    d^{(0)} d^{(1)} c^{(2)}_{\pm} - d^{(0)} d^{(1)} c^{(2)}_{\pm} &= 0, \\
    c^{(0)}_{\pm} d^{(1)} d^{(2)} - c^{(0)}_{\pm} d^{(1)} d^{(2)} &= 0.
\end{align*}
\]

We first note that the apparent difference between $c^{(i)}_{\pm}$ and $\tilde{c}^{(i)}_{\pm}$ can be gauged away by a transformation preserving the Yang–Baxter equation. Without losing generality we can set

\[
\tilde{c}^{(i)}_{\pm} = c^{(i)}_{\pm}, \quad \text{for } i = 0, 1, 2.
\]

By substituting result (9) into equations (7) and (8) we conclude that the weight $\tilde{d}^{(i)}$ becomes proportional to $d^{(i)}$,

\[
\tilde{d}^{(i)} = \Psi, \quad \text{for } i = 0, 1, 2,
\]

where $\Psi$ is our first invariant value.

We now turn our attention to the relations involving three types of triple products.

\[\text{Table 1. The dependence of the number of distinct functional relations on the respective number of triple products.}\]

| Number of triple products | Number of equations |
|---------------------------|---------------------|
| Two                       | 6                   |
| Three                     | 36                  |
| Four                      | 57                  |
| Five                      | 24                  |

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3.2. Three-term relations

Using solution (9) and (10), we find that the 36 relations possessing three triple products are pairwise equivalent leading us to 18 distinct functional equations. They can be classified in terms of three different groups of six equations given by

- group $G_{\pm}^{(1)}$
  \[ b^{(0)}_\pm c^{(1)}_\pm c^{(2)}_\pm + c^{(0)}_\pm a^{(1)}_\pm b^{(2)}_\pm - c^{(0)}_\pm b^{(1)}_\pm a^{(2)}_\pm = 0, \]  \[ b^{(0)}_\pm c^{(1)}_\pm b^{(2)}_\pm + c^{(0)}_\pm a^{(1)}_\pm c^{(2)}_\pm - a^{(0)}_\pm c^{(1)}_\pm a^{(2)}_\pm = 0, \]  \[ c^{(0)}_\pm c^{(1)}_\pm b^{(2)}_\pm + b^{(0)}_\pm a^{(1)}_\pm c^{(2)}_\pm - a^{(0)}_\pm b^{(1)}_\pm c^{(2)}_\pm = 0, \]

- group $G_{\pm}^{(2)}$
  \[ d^{(0)}_\pm b^{(1)}_\pm b^{(2)}_\pm + f^{(0)}_\pm d^{(1)}_\pm c^{(2)}_\pm - d^{(0)}_\pm f^{(1)}_\pm a^{(2)}_\pm = 0, \]  \[ f^{(0)}_\pm d^{(1)}_\pm b^{(2)}_\pm + d^{(0)}_\pm b^{(1)}_\pm c^{(2)}_\pm - b^{(0)}_\pm d^{(1)}_\pm a^{(2)}_\pm = 0, \]  \[ \Psi d^{(0)}_\pm d^{(1)}_\pm b^{(2)}_\pm + f^{(0)}_\pm b^{(1)}_\pm c^{(2)}_\pm - b^{(0)}_\pm f^{(1)}_\pm c^{(2)}_\pm = 0, \]

- group $G_{\pm}^{(3)}$
  \[ b^{(0)}_\pm b^{(1)}_\pm d^{(2)}_\pm + c^{(0)}_\pm d^{(1)}_\pm f^{(2)}_\pm - a^{(0)}_\pm f^{(1)}_\pm d^{(2)}_\pm = 0, \]  \[ c^{(0)}_\pm b^{(1)}_\pm d^{(2)}_\pm + b^{(0)}_\pm d^{(1)}_\pm f^{(2)}_\pm - a^{(0)}_\pm b^{(1)}_\pm d^{(2)}_\pm = 0, \]  \[ \Psi c^{(0)}_\pm d^{(1)}_\pm d^{(2)}_\pm + c^{(0)}_\pm b^{(1)}_\pm f^{(2)}_\pm - c^{(0)}_\pm f^{(1)}_\pm b^{(2)}_\pm = 0, \]

where the subscript $\pm$ means that each of them splits into two independent functional relations.

Altogether we have 18 linear homogeneous equations but only eight Boltzmann weights $a^{(0)}_\pm, b^{(0)}_\pm, c^{(0)}_\pm, d^{(0)}_\pm$ and $f^{(0)}_\pm$ are at our disposal to be eliminated. Therefore, we have a high degree of over-determination to overcome, making the solution of equations (11)–(19) far from being trivial. We shall start our analysis by considering the group $G_{\pm}^{(1)}$ of equations. They are similar to the functional equations underlying the symmetric six-vertex model and can be easily handled. We first eliminate the weights $a^{(0)}_\pm, b^{(0)}_\pm$ in terms of $c^{(0)}_\pm$ with the help of equations (11) and (12). As a result, we obtain

\[ \frac{a^{(0)}_\pm}{c^{(0)}_\pm} = \frac{b^{(1)}_\pm a^{(2)}_\pm}{c^{(1)}_\pm a^{(2)}_\pm} - \frac{a^{(1)}_\pm d^{(2)}_\pm}{c^{(1)}_\pm d^{(2)}_\pm}, \]  \[ \frac{b^{(0)}_\pm}{c^{(0)}_\pm} = \frac{b^{(1)}_\pm a^{(2)}_\pm}{c^{(1)}_\pm a^{(2)}_\pm} - \frac{a^{(1)}_\pm b^{(2)}_\pm}{c^{(1)}_\pm b^{(2)}_\pm}. \]

By substituting the above results (20) and (21) into equation (13), we find two separable curves that involve the weights $a^{(i)}_\pm, b^{(i)}_\pm$ and $c^{(i)}_\pm$ for $i = 1, 2$. Their solution leads us to invariants typical of six-vertex models,

\[ \frac{[a^{(i)}_\pm]^2 + [b^{(i)}_\pm]^2 - [c^{(i)}_\pm]^2}{a^{(i)}_\pm b^{(i)}_\pm} = \Delta_\pm, \quad \text{for } i = 1, 2, \]  \[ \Delta_\pm \text{ are free invariant parameters.} \]
We now turn our attention to the relations associated with the group $G^{(2)}_{\pm}$. We start by first eliminating the weights $d^{(0)}$ and $f^{(0)}$ from equations (14) and (15). Because they can be isolated from equations possessing different charge sectors, we need to keep track of their explicit expressions. The compatibility of such distinct solutions will be implemented subsequently. The expressions for $d^{(0)}$ and $f^{(0)}$ are

$$
\frac{d^{(0)}}{c^{(0)}_\pm} = \frac{d^{(1)}_\pm a^{(2)}_\pm}{c^{(1)}_\pm a^{(2)}_\pm} \left\{ \frac{b^{(1)}_\pm a^{(2)}_\pm - a^{(1)} b^{(2)}_\pm}{f^{(1)}_\pm a^{(2)}_\pm b^{(2)}_\pm - b^{(1)}_\pm [b^{(2)}_\pm]^2 + b^{(1)}_\pm [c^{(2)}_\pm]^2} \right\},
$$

$$
f^{(0)} = \left[ \frac{f^{(1)}_\pm a^{(2)}_\pm - b^{(1)}_\pm b^{(2)}_\pm}{d^{(1)}_\pm c^{(2)}_\pm} \right] d^{(0)},
$$

where as before they are given in terms of the common weights $c^{(0)}_\pm$.

We then substitute equations (23) and (24) into equation (16) and by taking into account that the Boltzmann weights are non-null, we find the following relation:

$$
c^{(2)}_\pm \left\{ \left[ (b^{(0)}_\pm)^2 + [f^{(1)}_\pm]^2 \right] a^{(2)}_\pm b^{(2)}_\pm - b^{(1)}_\pm f^{(1)}_\pm \left[ (b^{(2)}_\pm)^2 + [c^{(2)}_\pm]^2 \right] \right\} - \Psi [d^{(1)}_\pm a^{(2)}_\pm b^{(2)}_\pm c^{(2)}_\pm] = 0.
$$

(25)

Using the previous invariant (22) in the second term of equation (25), one is able to split the weights with index $i = 1$ from those labeled by $i = 2$. The solution of equation (25) leads us to the new constraints,

$$
\frac{[b^{(0)}_\pm]^2 + [f^{(1)}_\pm]^2 - \Delta_\pm b^{(1)}_\pm f^{(1)}_\pm}{[d^{(1)}_\pm]^2} = \Psi \frac{b^{(2)}_\pm c^{(2)}_\pm}{b^{(2)}_\pm c^{(2)}_\pm} = \Lambda_\pm,
$$

(26)

where $\Lambda_\pm$ are once again constant parameters. However, due to the consistency of the right-hand side of equation (26) they are related by

$$
\Lambda_+ \Lambda_- = \Psi^2.
$$

(27)

In order to complete the analysis of the group $G^{(2)}_{\pm}$ we still need to impose the compatibility between the two possibilities for $d^{(0)}$ and $f^{(0)}$ derived from relations (23) and (24). The consistency for the weight $d^{(0)}$ is easily resolved by fixing the relation between the amplitudes $c^{(0)}_\pm$ and $c^{(0)}_{\pm}$. In contrast, the compatibility for weight $f^{(0)}$ requires us to identify the expressions for the charge indices $\pm$ of equation (24). The result of such identification is

$$
b^{(1)}_\pm b^{(2)}_\pm c^{(2)}_\pm - b^{(1)}_\pm b^{(2)}_\pm c^{(2)}_\pm + f^{(1)}_\pm a^{(2)}_\pm c^{(2)}_\pm - f^{(1)}_\pm a^{(2)}_\pm c^{(2)}_\pm = 0.
$$

(28)

Fortunately, equation (28) becomes separable once we take into account the right-hand side of equation (26). This allows us to relate the products $b^{(2)}_\pm c^{(2)}_\pm$ to $b^{(2)}_\pm c^{(2)}_\pm$, and as a result, equation (28) can be solved by the method of separation of variables. The solution is

$$
\frac{\Delta_\pm b^{(1)}_\pm - b^{(1)}_\pm}{f^{(1)}_\pm} = \frac{a^{(2)}_\pm c^{(2)}_\pm - a^{(2)}_\pm c^{(2)}_\pm}{b^{(2)}_\pm c^{(2)}_\pm} = \Omega,
$$

(29)

where $\Omega$ is a new invariant value.

Before proceeding with the analysis of group $G^{(3)}_{\pm}$ we should pause to discuss the results obtained so far. The main feature of the last invariant (29) is that it relates weights to different charge index. We expect therefore that the invariant values $\Delta_\pm$, $\Lambda_\pm$, $\Psi$ and $\Omega$ should not be independent of each other. One way to unveil such constraints is to proceed as follows. Using the left-hand side of equations (26) and (29), we can eliminate the weights $d^{(1)}$ and $b^{(1)}$, namely

$$
[d^{(1)}]^2 = \frac{1}{\Lambda_\pm} \left\{ [b^{(1)}_\pm]^2 + [f^{(1)}_\pm]^2 - \Delta_\pm b^{(1)}_\pm f^{(1)}_\pm \right\}, \quad b^{(1)}_\pm = \frac{\Psi}{\Lambda_\pm} b^{(1)}_\pm + \Omega f^{(1)}_\pm
$$

(30)
We now substitute results (30) into the left-hand side of equation (26) with the charge index. After using relation (27), we obtain the expression

\[ \Psi(2\Omega_{\Psi} + \Delta_{\Psi} \Psi - \Delta_{\Delta_{\Psi}}) b^{(2)} + \left( \Lambda^{2}_{+} - \Delta_{\Delta_{\Psi}} \Omega_{\Psi} - \Psi^{2} + \Omega^{2} \Psi^{2} \right) f^{(1)} = 0. \]  

(31)

We next consider analogous approach to the weights having the index \( i = 2 \). We first eliminate the weights \( b^{(2)}_{-} \) and \( c^{(2)}_{-} \) with the help of the right-hand side of equations (26) and (29),

\[ b^{(2)}_{-} = \frac{\Delta_{+} b^{(2)}_{+} c^{(2)}_{+}}{\Psi_{-} c^{(2)}_{+}}, \quad c^{(2)}_{-} = -\frac{a^{(2)}_{+} c^{(2)}_{+}}{a^{(2)}_{+} + \Omega b^{(2)}_{+}}. \]  

(32)

By substituting equation (32) into the invariant connecting the weights \( a^{(2)}_{-}, b^{(2)}_{-} \) and \( c^{(2)}_{-} \) given by equation (22), one easily obtains

\[ \Lambda^{2}_{+} \left[ b^{(2)}_{+} \right]^{2} - \Delta_{-} \Lambda_{+} \Psi b^{(2)}_{+} \left[ a^{(2)}_{+} + \Omega b^{(2)}_{+} \right] + \Psi^{2} \left\{ \left[ a^{(2)}_{+} \right]^{2} + 2\Omega a^{(2)}_{+} b^{(2)}_{+} + \Omega^{2} b^{(2)}_{+}^{2} - \left[ c^{(2)}_{+} \right]^{2} \right\} = 0. \]  

(33)

The above relation can be further simplified by using in the last term of equation (33) the expression for the invariant \( \Delta_{+} \), see equation (22). By performing such simplification, we find

\[ \Psi(2\Omega_{\Psi} + \Delta_{\Psi} \Psi - \Delta_{\Delta_{\Psi}}) a^{(2)}_{+} + \left( \Lambda^{2}_{+} - \Delta_{\Delta_{\Psi}} \Omega_{\Psi} - \Psi^{2} + \Omega^{2} \Psi^{2} \right) b^{(2)}_{+} = 0. \]  

(34)

We have now reached a point in which two different possibilities emerge. The first consists in the assumption that the pair of weights \( \{ b^{(0)}_{+}, f^{(0)} \} \) and \( \{ a^{(0)}_{+}, b^{(0)}_{+} \} \) are considered linearly dependent. From equations (31) and (34), we see that this latter hypothesis implies in the identity \( \frac{b^{(0)}_{+}}{f^{(0)}} = \frac{a^{(0)}_{+}}{b^{(0)}_{+}} \), which together with equation (24), leads us to conclude that the weight \( f^{(0)} \) vanishes. Here we stress that we are looking for genuine 19-vertex models and therefore such a solution is disregarded. Thus, we are left with the second possibility which is simply to set the coefficients of equations (31) and (34) to zero. This condition implies that the invariants \( \Delta_{\pm}, \Lambda_{+}, \Psi \) and \( \Omega \) are constrained by the following equations:

\[ 2\Omega_{\Psi} + \Delta_{\Psi} \Psi - \Delta_{\Delta_{\Psi}} = 0, \]  

(35)

\[ \Lambda^{2}_{+} - \Delta_{-} \Lambda_{+} \Omega_{\Psi} - \Psi^{2} + \Omega^{2} \Psi^{2} = 0. \]  

(36)

Let us now discuss the solution of the functional equations associated with the group \( G^{(3)}_{\pm} \). For this group we see that the weights at our disposal to be eliminated are \( a^{(0)}_{\pm}, b^{(0)}_{\pm} \) and \( c^{(0)}_{\pm} \). However, they have already been computed by means of equations (11)–(13) and therefore our task consists in making the equations of the group \( G^{(3)}_{\pm} \) compatible with those of the group \( G^{(1)}_{\pm} \). The first step to solve this problem is to ensure from the start that all six functional relations of the group \( G^{(3)}_{\pm} \) are indeed satisfied. This is done by eliminating the weights \( a^{(0)}_{\pm} \) and \( b^{(0)}_{\pm} \) from equations (17) and (18),

\[ \frac{a^{(0)}_{\pm}}{c^{(0)}_{\pm}} = \frac{1}{d^{(1)} d^{(2)}} \left\{ \frac{b^{(1)}_{\pm} d^{(2)} - d^{(1)} f^{(1)} f^{(2)}}{b^{(1)}_{\pm} b^{(2)}_{\pm} - f^{(1)} f^{(2)}} \right\}, \]  

(37)

\[ \frac{b^{(0)}_{\pm}}{c^{(0)}_{\pm}} = \frac{1}{d^{(1)} d^{(2)}} \left\{ \frac{b^{(1)}_{\pm} f^{(1)} d^{(2)} - \left[ d^{(1)} \right]^{2} b^{(2)}_{\pm} f^{(2)}}{b^{(1)}_{\pm} b^{(2)}_{\pm} - f^{(1)} f^{(2)}} \right\}. \]  

(38)

By substituting expressions (37) and (38) into the last equation of the group \( G^{(3)}_{\pm} \), i.e. equation (19), we obtain for the charge + sector,

\[ b^{(1)}_{+} f^{(1)} \left\{ b^{(2)}_{+} b^{(2)}_{+} - \Psi \left[ d^{(2)} \right]^{2} \right\} - b^{(2)}_{+} f^{(2)} \left\{ b^{(1)}_{+} b^{(1)}_{+} - \Psi \left[ d^{(1)} \right]^{2} \right\} \]

\[ - \left[ f^{(1)} \right]^{2} b^{(2)}_{+} f^{(2)} + b^{(1)}_{+} f^{(1)} f^{(2)} \right]^{2} = 0, \]  

(39)
while for the charge $-$ sector, one finds
\[ b^{(1)}_i f^{(1)} \left[ b^{(2)} b^{(2)}_+ - \Psi[d^{(2)}]^2 \right] - b^{(2)}_v f^{(2)} \left[ b^{(1)}_i b^{(1)}_+ - \Psi[d^{(1)}]^2 \right] - [f^{(1)}] b^{(2)}_+ f^{(2)} + b^{(1)}_i f^{(1)} [f^{(2)}]^2 = 0. \] (40)

While equations (39) and (40) are not individually separable, it turns out that suitable linear combinations of such equations can be split with the indices $i = 1, 2$. In fact, by adding and subtracting equations (39) and (40) we found that they are solved provided the following constraint is verified:
\[ \frac{[b^{(1)}_i \pm b^{(2)}_v] f^{(i)}}{b^{(1)}_v b^{(1)}_+ - \Psi[d^{(1)}]^2 \pm [f^{(1)}]^2} = \Theta_{\pm}, \text{ for } i = 1, 2, \] (41)
where $\Theta_{\pm}$ are additional invariants.

Once again invariant (41) connects weights carrying distinct charge sectors $\pm$. We therefore have to repeat the same analysis as we did for the previous invariant $\Omega$, see equation (29). As before, expression (41) for the index $i = 1$ is simplified with the help of the weights $d^{(1)}$ and $b^{(1)}_i$ given by equation (30). This leads us to the following polynomial identity:
\[ [\Lambda_{\pm} \mp \Psi \mp \Theta_{\pm} (\Delta_+ + \Omega)] b^{(1)}_+ + [\pm \Omega \Psi \mp \Theta_{\pm} (\Lambda_+ \mp \Psi)] f^{(1)} = 0. \] (42)

Considering that we are looking for linearly independent solutions for the weights $b^{(1)}_i$ and $f^{(1)}$, we are required to set the coefficients of equation (42) to zero. In other words, we have the additional restrictions
\[ \Lambda_{\pm} \mp \Psi \mp \Theta_{\pm} (\Delta_+ + \Omega) = 0, \] (43)
and
\[ \mp \Omega \Psi \mp \Theta_{\pm} (\Lambda_+ \mp \Psi) = 0. \] (44)

It turns out that by solving equation (43) for the invariants $\Theta_{\pm}$ and substituting the result into the companion relation (44), one finds that it is trivially satisfied once we consider the previous constraint (35) for the invariant $\Delta_-$. This means that equations (43) and (44) are both solved, provided that we choose
\[ \Theta_{\pm} = \frac{\Lambda_{\pm} \mp \Psi}{\Psi (\Delta_+ + \Omega)}. \] (45)

For the index $i = 2$, we once again have to use the expressions for the weights $b^{(2)}_v$ and $c^{(2)}_v$ given by equation (32). By substituting these weights into equation (41), we obtain two linear equations for the weight $a^{(2)}_-$ associated with the charge sectors $\pm$. They are given by
\[ a^{(2)}_- = \Psi [a^{(2)}_+ + \Omega b^{(2)}_+] \left[ \Psi \Theta_{\pm} [d^{(2)}]^2 + b^{(2)}_+ f^{(2)} \mp \Theta_{\pm} [f^{(2)}]^2 \right]. \] (46)

The compatibility of such relations for the sectors $\pm$ and the fact that the weights should be not null lead us to the following relation among the weights $b^{(2)}_v$, $d^{(2)}$ and $f^{(2)}$:
\[ (\Theta_{\pm} - \Theta_{\pm}) \left[ [b^{(2)}_v]^2 + [f^{(2)}]^2 \right] + \Psi (\Theta_{\pm} - \Theta_{\pm}) [d^{(2)}]^2 + 2 (1 - \Theta_{\pm} \Theta_{\pm}) b^{(2)}_v f^{(2)} = 0. \] (47)

Taking into account constraints (35), (36) and (45), one is able to show that equation (47) becomes equivalent to the following expression:
\[ \Psi^2 (\Delta_+ + \Omega) \left[ [b^{(2)}_v]^2 + [f^{(2)}]^2 - \Lambda_{\pm} [d^{(2)}]^2 - \Delta_+ b^{(2)}_v f^{(2)} \right] = 0. \] (48)
The unique solution of equation (48) that does not drive us to null weights is

\[ \left[ a^{(2)}_i \right]^2 + [f^{(2)}_i]^2 - \Delta_+ b^{(2)}_i f^{(2)}_i = \Lambda_+. \] (49)

Interestingly enough, equation (49) generalizes the left-hand side of invariant (26) to the weights with index \( i = 2 \). At this point we observe that this fact also works for the invariant \( \Lambda_- \) involving the weights \( b^{(2)}_i, d^{(2)}_i \) and \( f^{(2)}_i \). In fact, using the expression for the weights \( b^{(2)}_i, c^{(2)}_i, a^{(2)}_i \) coming from equations (32) and (46), besides constraints (27), (35), (36) and (45) among the invariants, as well as equation (49) to eliminate the weight \( [d^{(2)}_i]^2 \), one is able to verify that

\[ \left[ b^{(2)}_i \right]^2 + [f^{(2)}_i]^2 - \Delta_- b^{(2)}_i f^{(2)}_i = \Lambda_- . \] (50)

We can now discuss again the solution of the remaining equations of the group \( G^{(3)}_{\pm} \). At this stage we just have to match the weights \( a^{(0)}_k \) and \( b^{(0)}_k \) obtained from a distinct pair of relations associated with the groups \( G^{(1)}_{\pm} \) and \( G^{(3)}_{\pm} \). In other words, we have to make equation (20) compatible with equation (37), as well as equation (21) consistent with equation (38). This requires us to solve the following functional relations:

\[
    d^{(1)} d^{(2)} \left[ b^{(1)}_\pm a^{(2)}_\pm - a^{(1)}_\pm b^{(2)}_\pm \right] \left[ b^{(1)}_\pm b^{(2)}_\pm - f^{(1)}_\pm f^{(2)}_\pm \right] \\
    = c^{(1)}_\pm c^{(2)}_\pm \left[ b^{(1)}_\pm f^{(1)}_\pm [d^{(2)}_\pm]^2 - [d^{(1)}_\pm]^2 b^{(2)}_\pm f^{(2)}_\pm \right],
\] (51)

\[
    d^{(1)} d^{(2)} \left[ b^{(1)}_\pm a^{(2)}_\pm b^{(2)}_\pm - a^{(1)}_\pm [b^{(2)}_\pm]^2 + a^{(1)}_\pm [b^{(2)}_\pm]^2 \right] \left[ b^{(1)}_\pm b^{(2)}_\pm - f^{(1)}_\pm f^{(2)}_\pm \right] \\
    = c^{(1)}_\pm c^{(2)}_\pm a^{(2)}_\pm \left[ [b^{(1)}_\pm]^2 [d^{(2)}_\pm]^2 - [d^{(1)}_\pm]^2 f^{(2)}_\pm \right].
\] (52)

Further progress is made by squaring both sides of expressions (51) and (52). This operation makes it possible to use the invariants \( \Delta_{\pm} \) (22) to eliminate the weights \( [c^{(1)}_\pm]^2 \), as well as equations (26), (49) and (50) to extract \( [d^{(i)}_\pm]^2 \) for both \( i = 1, 2 \). By performing such a two-step procedure we are able to show that equations (51) and (52) become proportional to the expression

\[
    b^{(2)}_\pm \left[ a^{(2)}_\pm + f^{(2)}_\pm \right] \left[ [b^{(1)}_\pm]^2 - a^{(1)}_\pm f^{(1)}_\pm \right] - b^{(2)}_\pm \left[ [b^{(1)}_\pm]^2 + a^{(1)}_\pm f^{(1)}_\pm \right] \left[ [b^{(2)}_\pm]^2 - a^{(2)}_\pm f^{(2)}_\pm \right] \\
    - \Delta_{\pm} \left[ a^{(1)}_\pm f^{(1)}_\pm [b^{(2)}_\pm]^{2} - a^{(1)}_\pm [b^{(2)}_\pm]^{2} \right] = 0. \] (53)

By inspecting equation (53), we conclude that such a relation can indeed be separated, leading us to our last invariant associated with the three-term relations

\[
    a^{(1)}_\pm b^{(2)}_\pm - \Delta_{\pm} a^{(1)}_\pm f^{(1)}_\pm + b^{(1)}_\pm f^{(1)}_\pm \\
    \left[ b^{(1)}_\pm f^{(2)}_\pm \right]^2 - a^{(1)}_\pm f^{(1)}_\pm = \Gamma_{\pm} \quad \text{for} \quad i = 1, 2.
\] (54)

where \( \Gamma_{\pm} \) are additional variables.

The final step of our analysis consists in matching the different charge sectors \( \pm \) of invariant (54). This task involves the manipulation of cumbersome expressions, and the respective technical details are presented in appendix A. The condition of consistency turns

\[ \Psi = 0 \Rightarrow d^{(i)}_\pm = 0, \quad \text{while} \quad \Delta_{\pm} + \Omega = 0 \quad \text{is equivalent to} \quad b^{(2)}_\pm b^{(2)}_\pm - \Psi [d^{(2)}_\pm]^2 = 0. \] (54)

Note that \( \Psi = 0 \) implies \( d^{(i)}_\pm = 0 \), while \( \Delta_{\pm} + \Omega = 0 \) is equivalent to \( b^{(2)}_\pm b^{(2)}_\pm - \Psi [d^{(2)}_\pm]^2 = 0. \) The latter identity implies that \( f^{(2)}_\pm = 0 \).
out to be a constraint among the invariants $\Gamma_{\pm}, \Lambda_{+}$ and $\Omega$ whose expression is rather simple, namely
\[
\Gamma_{-} \Lambda_{+} - \Psi (\Gamma_{+} + \Omega) = 0. \tag{55}
\]

In addition, an important byproduct of the analysis performed in appendix A is that the only independent weights are $a_{+}^{(i)}, b_{+}^{(i)}$ and $c_{+}^{(i)}$. In fact, the remaining amplitudes entering the three-term relations can be written in terms of such weights. As a result, the weights $a_{-}^{(i)}, b_{-}^{(i)}, c_{-}^{(i)}, d^{(i)}$ and $f^{(i)}$ are given in terms of the ratios of polynomials whose degrees are at most 2. In what follows, we present such relations since they are going to be useful later on. Following appendix A, the simplified expressions for the weights $a_{-}^{(i)}, b_{-}^{(i)}, c_{-}^{(i)}$ are
\[
a_{-}^{(i)} = \frac{\Psi a_{+}^{(i)} \Omega b_{+}^{(i)}}{\Lambda_{+}^2 \Gamma_{+}} \left[ (\Delta_{+} - \Gamma_{+}) a_{+}^{(i)} - (1 + \Gamma_{+}) b_{+}^{(i)} \right], \tag{56}
\]
\[
b_{-}^{(i)} = \frac{\Psi \left[ (\Delta_{+} - \Gamma_{+}) a_{+}^{(i)} - (1 + \Gamma_{+}) b_{+}^{(i)} \right]}{\Lambda_{+} \left[ (\Delta_{+} - \Gamma_{+}) a_{+}^{(i)} - b_{+}^{(i)} \right]}, \tag{57}
\]
\[
c_{-}^{(i)} = \frac{\Psi \left[ (\Delta_{+} - \Gamma_{+}) a_{+}^{(i)} - (1 + \Gamma_{+}) b_{+}^{(i)} \right]}{\Lambda_{+}^2 \left[ (\Delta_{+} - \Gamma_{+}) a_{+}^{(i)} - b_{+}^{(i)} \right]}, \tag{58}
\]

while for the weights $d^{(i)}$ and $f^{(i)}$, we have
\[
d^{(i)} = \pm \sqrt{1 - \Delta_{+} \Gamma_{+} + \frac{b_{+}^{(i)} c_{+}^{(i)}}{\Lambda_{+}} (-\Delta_{+} + \Gamma_{+}) a_{+}^{(i)} + b_{+}^{(i)}}, \quad f^{(i)} = \frac{\left[ a_{+}^{(i)} - \Gamma_{+} b_{+}^{(i)} \right] b_{+}^{(i)}}{(\Delta_{+} - \Gamma_{+}) a_{+}^{(i)} - b_{+}^{(i)}}. \tag{59}
\]

We would like to conclude this section with the following comments. First, it is important to stress that all the invariants obtained so far are also valid for the index $i = 0$. This result is verified using the expressions of the eliminated weights $a_{\pm}^{(0)}, b_{\pm}^{(0)}, c_{\pm}^{(0)}, d^{(0)}$ and $f^{(0)}$ on the form of the respective invariants. As a consequence of that, the above expressions (56)–(59) remain valid for the weights $a_{\pm}^{(0)}, b_{\pm}^{(0)}, c_{\pm}^{(0)}, d^{(0)}$ and $f^{(0)}$. We next note that out of ten possible invariants values $\Delta_{\pm}, \Lambda_{\pm}, \Gamma_{\pm}, \Theta_{\pm}, \Omega$ and $\Psi$, we end up with only four free parameters because of the six constraints (27), (35), (36), (45) and (55). For the sake of completeness, we summarize such conclusions in figure 3.

### 3.3. Four-term relations

After using the two-term solution (9) and (10), the total number of four-term relations is reduced to 51. The majority of these relations depend on the weights $g^{(i)}, h^{(i)}$ and $\tilde{h}^{(i)}$ that are still to be determined. The only exception is a factorizable functional equation given by
\[
d^{(0)} d^{(2)} (\Psi^2 - 1) \left[ c_{+}^{(i)} - c_{-}^{(i)} \right] = 0. \tag{60}
\]

The other 50 equations can be classified in five different groups characterized by the number and the type of unknown weights $g^{(i)}, h^{(i)}, \tilde{h}^{(i)}$ present in such functional relations. We have equations involving only one of the weights, the conjugate pair of amplitudes $h^{(i)}$ and $\tilde{h}^{(i)}$ as well as all the weights $g^{(i)}, h^{(i)}, \tilde{h}^{(i)}$ together. In what follows we list these distinct groups of relations:

- **group $G_{+}^{(4)}$**

\[
- c_{\pm}^{(0)} c_{\pm}^{(2)} + [\Psi d^{(0)} d^{(1)} b_{\pm}^{(2)} + g^{(0)} b_{\pm}^{(1)} c_{\pm}^{(2)} - b_{\pm}^{(0)} g^{(1)} c_{\pm}^{(2)}] = 0, \tag{61}
\]
The invariant structures solving the three-term functional relations (11)–(19) valid for \( i = 0, 1, 2 \), whose values are constrained by equations (27), (35), (36), (45), (55).

\[
\begin{align*}
-a_\pm^{(0)} c_\pm^{(1)} c_\pm^{(2)} + \Psi b_\pm^{(0)} d^{(1)} d^{(2)} - c_\pm^{(0)} g^{(1)} b_\pm^{(2)} + c_\pm^{(0)} b_\pm^{(1)} g^{(2)} &= 0, \\
-d_\pm^{(0)} b_\pm^{(1)} c_\pm^{(2)} + c_\pm^{(0)} b_\pm^{(1)} d^{(2)} - g^{(0)} d^{(1)} b_\pm^{(2)} + b_\pm^{(0)} d^{(1)} g^{(2)} &= 0,
\end{align*}
\]

- **group** \( G_{\pm}^{(5)} \)

\[
\begin{align*}
h^{(0)} (b^{-1}_- b^2_+) - b^{(1)} b^{(2)}_+ + d^{(0)} d^{(1)} (c^{(2)}_+ - c^{(2)}_-) &= 0, \\
h^{(1)} (b^{(0)} b^{(2)}_+) - b^{(0)} b^{(2)}_+ - c^{(0)} c^{(1)} c^{(2)}_+ + c^{(0)} c^{(1)} c^{(2)}_- &= 0, \\
h^{(2)} (b^{(0)} b^{(1)}_+) - b^{(0)} b^{(1)}_+ + d^{(0)} d^{(2)} (c^{(2)}_+ - c^{(2)}_-) &= 0, \\
-d^{(0)} d^{(1)} b^{(2)}_+ + c^{(0)} c^{(1)} b^{(2)}_+ - h^{(0)} b^{(1)}_+ c^{(2)}_+ + b^{(0)} h^{(1)}_+ c^{(2)}_- &= 0, \\
c^{(0)} c^{(1)} b^{(2)}_+ + b^{(0)} c^{(1)} c^{(2)}_- - b^{(0)} d^{(1)} d^{(2)} - c^{(0)} b^{(1)} h^{(2)}_+ &= 0, \\
a^{(0)} c^{(1)} c^{(2)}_- - d^{(0)} c^{(1)} d^{(2)} - f^{(0)} h^{(1)} f^{(2)} - h^{(0)} a^{(1)} h^{(2)}_+ &= 0,
\end{align*}
\]

- **group** \( G_{\pm}^{(6)} \)

\[
\begin{align*}
h^{(0)} b^{(0)} b^{(2)}_+ - b^{(1)} b^{(2)}_+ + \Psi^2 d^{(0)} d^{(1)} (c^{(2)}_- - c^{(2)}_+) &= 0, \\
h^{(1)} b^{(0)} b^{(2)}_+ - b^{(0)} b^{(2)}_+ - c^{(0)} c^{(1)} c^{(2)}_- + c^{(0)} c^{(1)} c^{(2)}_+ &= 0, \\
h^{(2)} (b^{(0)} b^{(1)}_+) - b^{(0)} b^{(1)}_+ + \Psi^2 d^{(1)} d^{(2)} (c^{(2)}_- - c^{(2)}_+) &= 0, \\
\Psi^2 d^{(0)} d^{(1)} b^{(2)}_+ - c^{(0)} c^{(1)} b^{(2)}_+ - b^{(0)} h^{(1)} c^{(2)}_- + h^{(0)} b^{(1)} c^{(2)}_+ &= 0,
\end{align*}
\]
\[-c_{\pm}^{(0)} b_{\pm}^{(1)} (b_{\pm}^{(2)}) - c_{\pm}^{(0)} c_{\pm}^{(2)} + c_{\pm}^{(0)} b_{\mp}^{(0)} d_{\pm}^{(2)} c_{\pm}^{(1)} d_{\pm}^{(2)} + c_{\pm}^{(0)} b_{\mp}^{(1)} c_{\pm}^{(2)} = 0, \tag{74}\]
\[-d_{\pm}^{(0)} h_{\pm}^{(1)} a_{\pm}^{(2)} + f_{\mp}^{(0)} h_{\mp}^{(1)} f_{\pm}^{(2)} + \Psi_{\mp}^{2} d_{\mp}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} + h_{\mp}^{(0)} a_{\pm}^{(1)} f_{\mp}^{(2)} = 0, \tag{75}\]

- group \(G_{\pm}^{(7)}\)

\[(\Psi_{\mp}^{2} - 1) d_{\pm}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} + h_{\pm}^{(0)} h_{\pm}^{(1)} h_{\pm}^{(2)} - h_{\mp}^{(0)} h_{\mp}^{(1)} h_{\mp}^{(2)} = 0, \tag{76}\]
\[\Psi_{\mp}^{(1)} d_{\pm}^{(2)} + d_{\mp}^{(0)} b_{\pm}^{(1)} c_{\pm}^{(2)} - \Psi_{\mp}^{(0)} b_{\pm}^{(0)} d_{\pm}^{(2)} - b_{\mp}^{(0)} d_{\mp}^{(2)} = 0, \tag{77}\]
\[-h_{\pm}^{(0)} d_{\pm}^{(1)} b_{\pm}^{(2)} - \Psi_{\mp}^{(0)} b_{\pm}^{(0)} c_{\pm}^{(2)} + c_{\pm}^{(0)} h_{\pm}^{(1)} d_{\pm}^{(2)} + \Psi_{\pm}^{(0)} d_{\pm}^{(1)} d_{\pm}^{(2)} = 0, \tag{78}\]
\[-h_{\pm}^{(0)} f_{\pm}^{(1)} a_{\pm}^{(2)} - h_{\mp}^{(0)} a_{\pm}^{(1)} f_{\pm}^{(2)} - \Psi_{\mp}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} - f_{\pm}^{(0)} h_{\pm}^{(1)} h_{\pm}^{(2)} = 0, \tag{79}\]
\[-h_{\mp}^{(0)} f_{\mp}^{(1)} a_{\pm}^{(2)} + h_{\mp}^{(0)} a_{\pm}^{(1)} f_{\mp}^{(2)} + \Psi_{\pm}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} + f_{\pm}^{(0)} h_{\pm}^{(1)} h_{\pm}^{(2)} = 0, \tag{80}\]
\[-h_{\mp}^{(0)} f_{\pm}^{(1)} f_{\mp}^{(2)} + \Psi_{\mp}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} + f_{\pm}^{(0)} d_{\pm}^{(1)} h_{\pm}^{(2)} - a_{\pm}^{(0)} f_{\pm}^{(1)} h_{\pm}^{(2)} = 0, \tag{81}\]
\[-h_{\mp}^{(0)} h_{\pm}^{(1)} f_{\pm}^{(2)} - \Psi_{\mp}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} - f_{\pm}^{(0)} a_{\pm}^{(1)} h_{\pm}^{(2)} + a_{\pm}^{(0)} f_{\pm}^{(1)} h_{\pm}^{(2)} = 0, \tag{82}\]

- group \(G_{\pm}^{(8)}\)

\[-\Psi_{\mp}^{(0)} d_{\pm}^{(1)} a_{\pm}^{(2)} + f_{\pm}^{(0)} h_{\pm}^{(1)} d_{\pm}^{(2)} + \Psi_{\mp}^{(0)} d_{\pm}^{(1)} c_{\pm}^{(2)} + \Psi_{\pm}^{(0)} h_{\pm}^{(1)} d_{\pm}^{(2)} = 0, \tag{83}\]
\[c_{\pm}^{(0)} d_{\pm}^{(1)} a_{\pm}^{(2)} - h_{\pm}^{(0)} a_{\pm}^{(1)} d_{\pm}^{(2)} - d_{\mp}^{(0)} c_{\pm}^{(1)} g_{\pm}^{(2)} - \Psi_{\mp}^{(0)} h_{\pm}^{(1)} d_{\pm}^{(2)} = 0, \tag{84}\]
\[d_{\pm}^{(0)} f_{\pm}^{(1)} a_{\pm}^{(2)} - d_{\mp}^{(0)} f_{\pm}^{(1)} c_{\pm}^{(2)} - \Psi_{\mp}^{(0)} h_{\pm}^{(1)} f_{\pm}^{(2)} - d_{\pm}^{(0)} a_{\pm}^{(1)} h_{\pm}^{(2)} = 0, \tag{85}\]
\[d_{\mp}^{(0)} f_{\pm}^{(1)} c_{\pm}^{(2)} - d_{\mp}^{(0)} f_{\pm}^{(1)} f_{\pm}^{(2)} + \Psi_{\pm}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} + \Psi_{\pm}^{(0)} d_{\pm}^{(1)} d_{\pm}^{(2)} = 0, \tag{86}\]
\[h_{\pm}^{(0)} c_{\mp}^{(1)} f_{\mp}^{(2)} - c_{\pm}^{(0)} f_{\pm}^{(1)} c_{\pm}^{(2)} + \Psi_{\mp}^{(0)} g_{\pm}^{(0)} d_{\pm}^{(1)} d_{\pm}^{(2)} + f_{\pm}^{(0)} c_{\pm}^{(1)} h_{\pm}^{(2)} = 0, \tag{87}\]
\[c_{\pm}^{(0)} f_{\mp}^{(1)} c_{\pm}^{(2)} - \Psi_{\mp}^{(0)} c_{\pm}^{(1)} d_{\pm}^{(2)} - \tilde{h}_{\pm}^{(0)} c_{\mp}^{(1)} f_{\mp}^{(2)} - f_{\pm}^{(0)} c_{\pm}^{(1)} h_{\pm}^{(2)} = 0. \tag{88}\]

From equation (60), we see that we have to deal with at least two possible branches. Either \(\Psi_{\mp}^{2} = 1\) or \(c_{\pm}^{(1)} = c_{\pm}^{(1)}\); since the weights \(d_{\pm}^{(1)}\) are assumed non-null. This information, however, is not necessary to obtain the invariant values associated with the weights \(g_{\mp}^{(1)}\). Indeed, we observe that equations (61) and (62) from the group \(G_{\pm}^{(4)}\) contain the triple products \(b_{\pm}^{(0)} c_{\pm}^{(1)} c_{\pm}^{(2)}\), \(c_{\pm}^{(0)} c_{\pm}^{(1)} c_{\pm}^{(2)}\), \(\Psi_{\mp}^{(0)} d_{\pm}^{(0)} d_{\pm}^{(1)} d_{\pm}^{(2)}\), and \(\Psi_{\mp}^{(0)} b_{\pm}^{(1)} d_{\pm}^{(2)}\), which are also present in the previously solved three-term relations, see equations (11), (13), (16) and (19). By eliminating the first triple products of the latter-mentioned relations and substituting them into equations (61) and (62), we obtain the following expressions:

\[b_{\pm}^{(0)} [a_{\pm}^{(1)} - g_{\pm}^{(2)}] - [a_{\pm}^{(0)} - g_{\pm}^{(0)}] b_{\pm}^{(1)} - f_{\mp}^{(0)} b_{\pm}^{(1)} + b_{\pm}^{(0)} f_{\pm}^{(1)} = 0, \tag{89}\]

and

\[d_{\pm}^{(0)} g_{\pm}^{(1)} a_{\pm}^{(2)} - b_{\pm}^{(0)} c_{\pm}^{(2)} + f_{\mp}^{(0)} b_{\pm}^{(1)} - b_{\pm}^{(0)} f_{\pm}^{(2)} = 0. \tag{90}\]

An essential step to separate equations (89) and (90) is to use the invariant \(\Omega\) which links the weights \(b_{\pm}^{(0)}\) and \(b_{\pm}^{(1)}\). Indeed, from the third box of figure 3 one is able to eliminate \(b_{\pm}^{(1)}\) in terms of \(b_{\pm}^{(0)}\) and \(f_{\pm}^{(1)}\). By substituting the result into equations (89) and (90) and after few algebraic manipulations which include the use of identity (27), we find that the invariant associated with the weight \(g_{\pm}^{(1)}\) is

\[-g_{\pm}^{(1)} + a_{\pm}^{(1)} + \frac{\Psi_{\mp}^{2}}{\Psi_{\mp}^{2}} f_{\pm}^{(1)} b_{\pm}^{(1)} = \Delta_{\pm}^{(g)} \quad \text{for } i = 0, 1, 2, \tag{91}\]

where \(\Delta_{\pm}^{(g)}\) are constant parameters.
Our next task is to ensure the consistency of equation (91) as far as the charge sectors \( \pm \) are concerned. This requires us to impose that the expressions for \( g^{(i)} \) coming from the different charge sectors are the same, namely
\[
a^{(i)}_+ - a^{(i)}_- + \left( \frac{\Psi}{\Lambda_+} - \frac{\Psi}{\Lambda_-} \right) f^{(i)} - \Delta^{(e)}_+ b^{(i)}_+ + \Delta^{(e)}_- b^{(i)}_- = 0 \quad \text{for } i = 0, 1, 2. \tag{92}
\]
In order to solve equation (92), we substitute the expressions for the weights \( a^{(i)}_+ \), \( b^{(i)}_- \) and \( f^{(i)} \) derived in the previous section, see equations (56), (57) and (59). After some cumbersome simplifications, we find that equation (92) becomes a polynomial relation on the amplitudes \( a^{(i)}_+ \) and \( b^{(i)}_+ \) having the following form:\3
\[
A_1 \left[ a^{(i)}_+ \right]^2 + A_2 a^{(i)}_+ b^{(i)}_+ + A_3 \left[ b^{(i)}_+ \right]^2 = 0 \quad \text{for } i = 1, 2. \tag{93}
\]
The coefficients \( A_1, A_2 \) and \( A_3 \) depend solely on some combinations of certain invariants and should vanish to ensure the validity of equation (92). Their expressions after using constraint (27) are
\[
A_1 = (\Delta_+ - \Gamma_+) \Lambda^2_+ - (\Delta_+ - \Gamma_+ + \Omega) \Psi^2 \equiv 0,
\]
\[
A_2 = -\Lambda^3_+ \left[ 1 + \Delta^{(e)}_+(\Delta_+ - \Gamma_+) \right] \Lambda^2_+ \Psi + \left[ 1 + \Delta^{(e)}_+(\Delta_+ - \Gamma_+ + \Omega) \right] \Lambda_+ \Psi^2
+ [1 - \Omega(\Delta_+ - 2\Gamma_+ + \Omega)] \Psi^3 \equiv 0,
\]
\[
A_3 = \Delta^{(e)}_+ \Psi \Lambda^2_+ + \Gamma_+ \Lambda^3_+ - \left[ \Delta^{(e)}_+ + \Gamma_+ + \Delta^{(e)}_+ \Gamma_+ \Omega \right] \Lambda_+ \Psi^2 + (1 + \Gamma_+ \Omega) \Psi^3 \equiv 0. \tag{94}
\]
The next step consists in fixing the form of invariants associated with the Boltzmann weights \( h^{(i)} \) and \( \tilde{h}^{(i)} \). To make progress in this direction we have to use explicitly the two branches’ data encoded in equation (60).

3.3.1. Branch 1. This branch is chosen by setting \( \Psi = \pm 1 \). This constraint allows us to eliminate the weight \( \tilde{h}^{(0)} \) from equation (76),
\[
\tilde{h}^{(0)} = \frac{\tilde{h}^{(1)} h^{(2)}_+}{h^{(1)} h^{(2)}_-} = h^{(0)}. \tag{95}
\]
By substituting result (95) into the pairs of equations (69) and (75), and (67) and (73), we find that their linear combination fixes a relation between \( h^{(i)} \) and \( \tilde{h}^{(i)} \) which is
\[
h^{(i)} = \tilde{h}^{(i)} \quad \text{for } i = 0, 1, 2. \tag{96}
\]
As a consequence of equation (96), the equations of the groups \( G^{(5)}_\pm \) and \( G^{(6)}_\pm \) become the same and the number of independent relations of the groups \( G^{(7)}_\pm \) and \( G^{(8)}_\pm \) is reduced to half. At this point we can apply the same strategy as used for solving the weight \( g^{(i)} \). We first eliminate the triple products of the weights \( d^{(0)} b^{(1)}_+ c^{(2)}_+ \) and \( c^{(0)}_+ b^{(1)}_+ d^{(2)}_+ \) from equations (15) and (18) and by substituting the results into equation (77) we find a separable expression for \( h^{(0)} \) and \( h^{(2)} \), namely
\[
-\Psi h^{(0)} + \Psi a^{(0)}_+ + f^{(0)}_b = -h^{(2)} + a^{(2)}_+ + \Psi f^{(2)}_b = \Delta^{(h)}_\pm, \tag{97}
\]
where \( \Delta^{(h)}_\pm \) are constants of separability.

Once again we have to implement the compatibility among relations associated with different charge sectors \( \pm \). This matching is performed in the same way as for the weight \( g^{(i)} \). \3

\footnote{For \( i = 0 \), the compatibility equation (92) is automatically satisfied once \( A_1 = A_2 = A_3 = 0 \) and provided that we take into account relations (27), (35) and (36) among the invariants.}
As a result, we find that \( h^{(0)} \) and \( h^{(2)} \) derived from the charge sectors \( \pm \) agree, provided that the following relations between invariants are satisfied:

\[
\Delta_+^{(h)} = \frac{1}{\Lambda_+^2} (1 + \Gamma_+ \Omega)(\Delta_-^{(h)} \Lambda_+ \Psi - \Omega),
\]

\[
\Omega = (\Delta_+ - \Gamma_+)(\Lambda_+^2 - 1),
\]

\[
(\Lambda_+^2 - 1)[1 + (\Delta_+ - \Gamma_+)[(\Delta_+ - \Gamma_+)(\Lambda_+^2 - 1) - \Delta_-^{(h)} \Lambda_+ \Psi)] = 0.
\]

From equation (100) we observe that branch 1 splits into two distinct families since this relation admits two possible solutions. We shall denominate such possible branches 1A and 1B. For branch 1A, the value of \( \Lambda_+ \) is fixed by

\[
\Lambda_+^2 = 1,
\]

while branch 1B is defined by solving equation (100) for \( \Delta_-^{(h)} \).

\[
\Delta_-^{(h)} = \frac{1 + (\Lambda_+^2 - 1)(\Delta_+ - \Gamma_+)^2}{\Lambda_+ \Psi(\Delta_+ - \Gamma_+)}. \tag{102}
\]

It turns out that for both branches we are able to manipulate equations (67) and (68) in order to determine the only remaining weight \( h^{(1)} \). It is given by an expression similar to that found for \( h^{(0)} \) and \( h^{(2)} \), which is

\[
-\frac{h^{(1)} + a_+^{(1)} + \Psi f^{(1)}}{b_+^{(1)}} = \Delta_+^{(h)}. \tag{103}
\]

We now have reached a point in which all the Boltzmann weights have been determined in terms of the invariant values and the weights \( a_+^{(1)} \), \( b_+^{(1)} \) and \( c_+^{(1)} \). By substituting them into the four-term relations not used so far and using invariant (22) to eliminate the weight \( |c_+^{(1)}|^2 \), we obtain polynomial equations depending on the variables \( a_+^{(1)} \) and \( b_+^{(1)} \) whose coefficients are the functions of the invariant values. By setting these coefficients to zero and considering the previous constraints (27), (35), (36), (45) and (55) as well as equations (94), (98)–(100) we are able to compute the invariants values. It turns out that once equations (63) and (83) are satisfied, all other remaining relations involving four terms are automatically fulfilled. The final result of such analysis is summarized in table 2.

Here we remark that the case \( \Psi = -1 \) or \( \Lambda_+ = -1 \) leads us to specific invariant values which can be reproduced in the context of solution 1B and another branch that is going to be discussed in the next subsection. The technical details concerning such special situations have been collected in appendix B. We finally observe that branches 1A and 1B have a unique free parameter chosen to be \( \Lambda_+ \). However, only branch 1A is invariant under the charge symmetry + \( \leftrightarrow - \).

### 3.3.2. Branch 2

Another possibility of satisfying equation (60) is to set \( c_+^{(1)} = c_-^{(1)} \). This condition, together with the previous relation (58), leads us to a polynomial equation for the weights \( a_+^{(1)} \) and \( b_+^{(1)} \).

\[
[(\Lambda_+^2 - \Psi^2)(\Delta_+ - \Gamma_+) - \Omega \Psi^2]a_+^{(1)} + [- \Lambda_+^2 + (1 + \Gamma_+ \Omega) \Psi^2]b_+^{(1)} = 0. \tag{104}
\]

The linear combination (104) is fulfilled for arbitrary \( a_+^{(1)} \) and \( b_+^{(1)} \) by imposing that its coefficients are null. This fixes the values of the invariants \( \Omega \) and \( \Lambda_+ \).

\[
\Omega = 0 \quad \text{and} \quad \Lambda_+ = \pm \Psi. \tag{105}
\]

Note that the possible solution \( \Delta_+ = \Gamma_+ + \Gamma_-^{-1} \) is disregarded since it leads us to \( d^{(1)} = 0 \), see equation (59).
By substituting (105) back into the expressions for the weights \( a^{(i)} \), \( b^{(i)} \) and \( c^{(i)} \) given in equations (56)–(58), we find the rather simple relations

\[
\frac{a^{(i)}}{a^{(i)}} = 1, \quad \frac{b^{(i)}}{b^{(i)}} = \pm, \quad \frac{c^{(i)}}{c^{(i)}} = 1 \quad \text{for } i = 0, 1, 2. \tag{106}
\]

An immediate consequence of equation (106) is that we can implement several simplifications on the four-term equations. First, we clearly see that equations (64)–(66) and (70)–(72) are automatically satisfied, besides that the number of independent equations reduces dramatically. For the groups \( G^{(5)} \) and \( G^{(6)} \) we just have three distinct relations, while for \( G^{(7)} \) and \( G^{(8)} \) the number of equations is reduced to half. We are now in a position to determine the weights \( \h^{(i)} \) and \( \tilde{h}^{(i)} \) using the same method that fixed \( g^{(i)} \). In fact, by eliminating the triple products \( b^{(i)} c^{(i)} c^{(i)} + b^{(i)} c^{(i)} c^{(i)} \) and \( \Psi b^{(i)} d^{(i)} d^{(i)} \) with the help of equations (11), (13), (16), (19) and by substituting them into equations (67) and (68), we are able to obtain separable functional relations for the weight \( \h^{(i)} \). The solution of such relations is

\[
-\Psi h^{(i)} + \Psi a^{(i)} + f^{(i)} = \Delta^{(h)} \quad \text{for } i = 0, 1, 2. \tag{107}
\]

By the same token the weight \( \tilde{h}^{(i)} \) can also be calculated. By using the above approach but now for equations (73) and (74), one finds

\[
-\tilde{h}^{(i)} + a^{(i)} + \Psi f^{(i)} = \Delta^{(h)} \quad \text{for } i = 0, 1, 2. \tag{108}
\]

At this point all the Boltzmann weights for branch 2 have been determined. The remaining task is to substitute them into equations (69) and (75) as well as in the relations of the groups

| Invariants | Branch 1A | Branch 1B |
|------------|----------|----------|
| \( \Delta_\ast \) | Free | Free |
| \( \Delta_+ \) | \( \Delta_\ast \) | \(-\frac{\Delta_+ + \sqrt{\Delta_+^2 - \Delta_\ast^2}}{2\Delta_\ast} \)
| \( \Delta_- \) | 1 | \( \frac{2\Delta_+ \sqrt{\Delta_+^2 - \Delta_-^2}}{3\Delta_+ + \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Delta_- \) | 1 | \( \frac{\sqrt{3}\Delta_+ \sqrt{\Delta_+^2 - \Delta_-^2}}{2\Delta_+ \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Psi \) | 1 | 1 |
| \( \Omega \) | 0 | \(-\frac{6\Delta_+^2 - \Delta_\ast^2}{3\Delta_+ + \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Gamma_+ \) | \( \Delta_\ast + \epsilon_1 \) | \(-\frac{3\Delta_+ \sqrt{\Delta_+^2 - \Delta_-^2}}{\Delta_+ + \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Gamma_- \) | \( \Delta_\ast + \epsilon_1 \) | \( \frac{1}{\Delta_\ast + \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Theta_\ast \) | \( \frac{1}{\Delta_\ast + \sqrt{\Delta_+^2 - \Delta_-^2}} \) | \( \frac{\Delta_\ast + \epsilon_1}{\Delta_\ast + \epsilon_1 + \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Delta_\ast \) | \( \Delta_\ast - \epsilon_1 \) | \( \sqrt{\Delta_\ast^2 - \epsilon_1^2} \)
| \( \Delta_\ast \) | \( \Delta_\ast - \epsilon_1 \) | \( \frac{\epsilon_1}{\Delta_\ast + \epsilon_1 + \sqrt{\Delta_+^2 - \Delta_-^2}} \)
| \( \Delta_\ast \) | \( \Delta_\ast \) | \( \Delta_\ast \)
| \( \Delta_\ast \) | \( \Delta_\ast \) | \( \frac{\sqrt{3}\Delta_\ast \sqrt{\Delta_+^2 - \Delta_-^2}}{2\Delta_\ast} \)
Table 3. The invariant values for branch 2 where $\omega = \exp \left( \frac{i \pi \epsilon_2}{3} \right)$ and the discrete variables $\epsilon_1 = \epsilon_2 = \pm 1$.

| Invariants          | Branch 2A | Branch 2B |
|---------------------|-----------|-----------|
| $\Delta_+^+$        | $\epsilon_1 \sqrt{3}$ | $-\epsilon_1 \sqrt{3}$ |
| $\Delta_-^+$        | $\omega$ | $\omega$ |
| $\Lambda_+^+$       | $\frac{2 - \Delta_+^+ \Delta_-^+ \sqrt{\Delta_+^+ - 4}}{2}$ | $\frac{2 - \Delta_+^+ \Delta_-^+ \sqrt{\Delta_+^+ - 4}}{2}$ |
| $\Psi$              | $0$ | $0$ |
| $\Omega$            | $\epsilon_1$ | $\epsilon_1$ |
| $\Gamma_+$          | $\frac{2}{\sqrt{3}}$ | $\frac{2}{\sqrt{3}}$ |
| $\Theta_+$          | $0$ | $-\frac{2}{\sqrt{3}}$ |
| $\Delta_+^{(2)}$    | $\Delta_+ - \epsilon_1$ | $0$ |
| $\Delta_-^{(2)}$    | $\Delta_+ - \epsilon_1$ | $0$ |
| $\Delta_+^{(3)}$    | $0$ | $0$ |
| $\Delta_-^{(3)}$    | $0$ | $0$ |

$G_{\pm}^{(7)}$ and $G_{\pm}^{(8)}$ to fix the invariants values. Once again we have to consider two possible branches, since from the very beginning, equation (105) has two allowed solutions. Taking into account all previous constraints among the invariants, our final results are summarized in table 3.

Note that only branch 2A has a free parameter and is invariant under charge conjugation. We would like to conclude this section with the following remark. Although the algebraic curves for the weights $h^{(i)}$ and $\tilde{h}^{(i)}$ come from different equations for branches 1 and 2, their final expressions can be put in a unified form. In fact, they can be written as

$$\frac{-\Psi h^{(i)} + \Psi a^{(i)} + f^{(i)}}{b^{(i)}_{\pm}} = \Delta^{(\pm)}_{\pm}, \quad \frac{-\tilde{h}^{(i)} + a^{(i)} + \Psi f^{(i)}}{b^{(i)}_{\pm}} = \Delta^{(\tilde{h})}_{\pm} \text{ for } i = 0, 1, 2. \tag{109}$$

The general result (109) recovers branch 1 by simply making the identification $\Delta^{(h)}_{\pm} = \Delta^{(\tilde{h})}_{\pm}$, while for branch 2 we have to consider equation (106) and therefore have to impose $\frac{\Delta^{(h)}_{\pm}}{\Delta^{(\tilde{h})}_{\pm}} = \frac{\Delta^{(\tilde{h})}_{\pm}}{\Delta^{(h)}_{\pm}} = \pm 1$. In figure 4 we summarized the invariants obtained in the analysis of the four-term functional relations.

### 3.4. Five-term relations

The number of the five-term functional equations remains unchanged after the solution of the two-term relations. Below we present the structure of the corresponding 24 equations,

$$b^{(0)}_{\pm} c^{(1)}_{\pm} + d^{(0)}_{\pm} \tilde{h}^{(1)} + d^{(2)}_{\pm} - g^{(0)}_{\pm} c^{(1)}_{\pm} + e^{(0)}_{\pm} g^{(1)} + e^{(2)}_{\pm} - \Psi^2 d^{(0)}_{\pm} a^{(1)}_{\pm} d^{(2)}_{\pm} = 0 \tag{110}$$

$$b^{(0)}_{\pm} c^{(1)}_{\pm} + d^{(0)}_{\pm} \tilde{h}^{(1)} + d^{(2)}_{\pm} - g^{(0)}_{\pm} c^{(1)}_{\pm} + e^{(0)}_{\pm} g^{(1)} + e^{(2)}_{\pm} - d^{(0)}_{\pm} a^{(1)}_{\pm} d^{(2)}_{\pm} = 0 \tag{111}$$

$$\Psi b^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} + d^{(0)}_{\pm} c^{(1)}_{\pm} \tilde{h}^{(2)} - d^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} - \Psi f^{(0)}_{\pm} a^{(1)}_{\pm} d^{(2)} - h^{(0)}_{\pm} \tilde{h}^{(1)} d^{(2)} = 0 \tag{112}$$

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can be rewritten in the form of a conic,

\[
\frac{-g^{(i)} + \alpha^{(i)} + \frac{\Psi}{\lambda^{(i)}} f}{b^{(i)}} = \Delta^{(g)}
\]

\[
\frac{-\Psi h^{(i)} + \Psi \alpha^{(i)} + f^{(i)}}{b^{(i)}} = \Delta^{(h)}
\]

\[
\frac{\mp \tilde{h}^{(i)} + \alpha^{(i)} + \Psi f^{(i)}}{b^{(i)}} = \Delta^{(\tilde{h})}
\]

Figure 4. The invariant structure associated with the four-term equations valid for \(i = 0, 1, 2\). The expressions for the invariants \(\Delta^{(g)}, \Delta^{(h)}\) and \(\Delta^{(\tilde{h})}\) are given in tables 2 and 3.

\[
b^{(0)}_{\pm} b^{(1)}_{\pm} d^{(2)}_{\pm} + \Psi c^{(0)}_{\pm} d^{(1)}_{\pm} h^{(2)} - \Psi d^{(0)}_{\pm} c^{(1)}_{\pm} g^{(2)} - f^{(0)}_{\pm} \alpha^{(1)}_{\pm} d^{(2)} - \Psi \tilde{h}^{(0)}_{\pm} h^{(1)}_{\pm} d^{(2)} = 0
\]

\[
\Psi d^{(0)}_{\pm} b^{(1)}_{\pm} b^{(2)}_{\pm} - \Psi d^{(0)}_{\pm} a^{(1)}_{\pm} f^{(2)} - g^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} - d^{(0)}_{\pm} h^{(1)}_{\pm} h^{(2)} + \tilde{h}^{(0)}_{\pm} d^{(1)}_{\pm} c^{(2)}_{\pm} = 0
\]

\[
d^{(0)}_{\pm} b^{(1)}_{\pm} b^{(2)}_{\pm} - d^{(0)}_{\pm} a^{(1)}_{\pm} f^{(2)} - \Psi d^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} - \Psi d^{(0)}_{\pm} h^{(1)}_{\pm} c^{(2)}_{\pm} + \Psi \tilde{h}^{(0)}_{\pm} d^{(1)}_{\pm} c^{(2)}_{\pm} = 0
\]

\[
b^{(0)}_{\pm} c^{(1)}_{\pm} b^{(2)}_{\pm} + c^{(0)}_{\pm} h^{(1)}_{\pm} c^{(2)}_{\pm} - f^{(0)}_{\pm} c^{(1)}_{\pm} f^{(2)} - h^{(0)}_{\pm} c^{(1)}_{\pm} h^{(2)} - d^{(0)}_{\pm} g^{(1)}_{\pm} d^{(2)} = 0
\]

\[
b^{(0)}_{\pm} c^{(0)}_{\pm} b^{(2)}_{\pm} + c^{(0)}_{\pm} h^{(1)}_{\pm} c^{(2)}_{\pm} - f^{(0)}_{\pm} c^{(1)}_{\pm} f^{(2)} - h^{(0)}_{\pm} c^{(1)}_{\pm} h^{(2)} - \Psi d^{(0)}_{\pm} g^{(1)}_{\pm} d^{(2)} = 0
\]

\[
\Psi d^{(0)}_{\pm} b^{(1)}_{\pm} c^{(2)}_{\pm} + \Psi g^{(0)}_{\pm} d^{(1)}_{\pm} c^{(2)}_{\pm} - f^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} - \Psi d^{(0)}_{\pm} g^{(1)}_{\pm} g^{(2)} - \Psi \tilde{h}^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} = 0
\]

\[
d^{(0)}_{\pm} b^{(1)}_{\pm} b^{(2)}_{\pm} + g^{(0)}_{\pm} d^{(1)}_{\pm} c^{(2)}_{\pm} - \Psi f^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} - d^{(0)}_{\pm} g^{(1)}_{\pm} g^{(2)} - h^{(0)}_{\pm} c^{(1)}_{\pm} d^{(2)} = 0
\]

\[
b^{(0)}_{\pm} b^{(1)}_{\pm} d^{(2)}_{\pm} - \Psi d^{(0)}_{\pm} c^{(1)}_{\pm} f^{(2)} + c^{(0)}_{\pm} d^{(1)}_{\pm} g^{(2)} - g^{(0)}_{\pm} d^{(1)}_{\pm} d^{(2)} = 0
\]

\[
\Psi b^{(0)}_{\pm} b^{(1)}_{\pm} d^{(2)}_{\pm} - d^{(0)}_{\pm} c^{(1)}_{\pm} f^{(2)} + \Psi c^{(0)}_{\pm} d^{(1)}_{\pm} g^{(2)} - \Psi g^{(0)}_{\pm} g^{(1)}_{\pm} d^{(2)} - \Psi d^{(0)}_{\pm} c^{(1)}_{\pm} h^{(2)} = 0.
\]

The above relations do not involve any new Boltzmann weights. As a consequence of that, we can substitute the weight expressions obtained in the two previous sections into equations (110)–(121) and search for further restrictions on the invariant values for branches 1 and 2. Remarkably enough, after the three and four-term relations are solved, all the above five-term equations are automatically satisfied. Therefore, the four families of vertex models defined by the invariants given in tables 2 and 3 are exactly integrable.

4. Parameterization and Hamiltonian

We shall present here the parameterization of the weights associated with the four distinct integrable manifolds of the previous section. As discussed in section 3, we have only one fundamental algebraic curve involving the weights \(a_{\pm}^{(i)}, b_{\pm}^{(i)}\) and \(c_{\pm}^{(i)}\). The corresponding curve (22) can be rewritten in the form of a conic,

\[
(x_i - \frac{\Lambda_+}{2} y_i)^2 - \left(\frac{\Lambda_+^2}{4} - 1\right) y_i^2 = 1.
\]

(122)
where the new variables \(x_i\) and \(y_i\) are related to the weights by

\[
x_i = a^{(i)}_+ \frac{c^{(i)}_-}{c^{(i)}_+}, \quad y_i = b^{(i)}_+ \frac{c^{(i)}_-}{c^{(i)}_+},
\]

(123)

The rational parameterization of conics involves only one spectral parameter \(\lambda_i\) and can be done through hyperbolic functions, namely

\[
x_i - \frac{\Delta_+}{2} y_i = \cosh(\lambda_i), \quad y_i = \frac{\sinh(\lambda_i)}{\sqrt{\Delta_+^2 - 1}}
\]

(124)

For convenience we introduce the following definition for the invariant \(\Delta_+\):

\[
\Delta_+ = 2 \cosh(\gamma).
\]

(125)

By substituting equation (123) into equation (124) and after using definition (125), one finds

\[
a^{(i)}_+ = 1, \quad b^{(i)}_+ = \frac{\sinh(\lambda_i)}{\sinh(\lambda_i + \gamma)}, \quad c^{(i)}_+ = \frac{\sinh(\gamma)}{\sinh(\lambda_i + \gamma)}.
\]

(126)

where \(a^{(i)}_+\) has been fixed by freedom of an overall normalization.

Before proceeding we stress that under the above parameterization the Yang–Baxter equation is additive as far as the spectral parameters \(\lambda_i\) are concerned,

\[
\lambda_0 = \lambda_1 - \lambda_2.
\]

(127)

The remaining weights can be determined in terms of \(a^{(i)}_+, b^{(i)}_+\) and \(c^{(i)}_+\) with the help of equations (56)–(59), (91) and (109). In what follows, we list the simplified expressions of the weights associated with the four different integrable vertex models defined by the invariant values given in tables 2 and 3.

• Branch 1A

This vertex model falls in the family of the factorized spin-1 \(S\)-matrix found by Zamolodchikov and Fateev [12] with the additional presence of the discrete variables \(\epsilon_1 = \pm 1\). We remark that the transfer matrix spectrum for \(\epsilon_1 = +1\) can be related to that with \(\epsilon_1 = -1\) only for \(L\) even. The respective weights are given by

\[
a^{(i)}_\pm = 1, \quad b^{(i)}_\pm = \frac{\sinh(\lambda_i)}{\sinh(\lambda_i + \gamma)}, \quad c^{(i)}_\pm = \frac{\sinh(\gamma)}{\sinh(\lambda_i + \gamma)}.
\]

(128)

\[
d^{(i)} = d^{(i)}_\pm = \pm \frac{\sinh(\lambda_i)}{\sinh(\lambda_i + \gamma)}, \quad f^{(i)} = \frac{\sinh(\lambda_i - \frac{\lambda}{2} + \tilde{\gamma}) \sinh(\lambda_i + \gamma)}{\sinh(\lambda_i + \frac{\lambda}{2} + \tilde{\gamma}) \sinh(\lambda_i + \gamma)},
\]

(129)

\[
g^{(i)} = -2 \cosh(\tilde{\gamma}) + \cosh(\frac{3\lambda}{2} + \tilde{\gamma}) + \cosh(2\lambda_i + \frac{\lambda}{2} - \tilde{\gamma})
\]

\[
= \frac{2 \cosh(\tilde{\gamma})}{\sinh(\lambda_i + \frac{\lambda}{2} + \tilde{\gamma}) \sinh(\lambda_i + \gamma)},
\]

(130)

\[
h^{(i)} = h^{(i)}_\pm = \frac{2 \cosh(\tilde{\gamma})}{\sinh(\lambda_i + \frac{\lambda}{2} + \tilde{\gamma}) \sinh(\lambda_i + \gamma)}.
\]

(131)

where \(\tilde{\gamma} = \frac{\lambda}{4}(1 - \epsilon_1)\).
\[ a_+^{(i)} = 1, \quad b_+^{(i)} = \frac{\sinh(\lambda_i)}{\sinh(\lambda_i + \gamma)}, \quad c_+^{(i)} = \frac{\sinh(\gamma)}{\sinh(\lambda_i + \gamma)}, \]

\[ a_-^{(i)} = \frac{\sinh(\lambda_i) \sinh(\lambda_i - \gamma)}{\sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \quad b_-^{(i)} = \frac{\sinh(\gamma - \lambda_i) \sinh(\lambda_i)}{\sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \]

\[ c_-^{(i)} = \frac{\sinh(\gamma - \gamma_0) \sinh(\lambda_i - \gamma)}{\sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \quad d_-^{(i)} = \frac{\sinh(\lambda_i - \gamma) \sinh(\lambda_i)}{\sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \]

\[ f^{(i)} = \frac{\sinh(\lambda_i - \gamma_0) \sinh(\lambda_i)}{\sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \quad g^{(i)} = -1 + \cosh(2\lambda_i + \gamma_0) \cosh(2\gamma - \gamma_0) - \frac{1}{2 \sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \]

\[ \tilde{h}^{(i)} = \frac{\sinh(\gamma - \gamma_0) \sinh(\lambda_i - \gamma)}{\sinh(\lambda_i + \gamma - \gamma_0) \sinh(\lambda_i + \gamma)}, \]

where \( \gamma_0 = \frac{\pi}{\lambda} \epsilon_1. \)

- Branch 2A

Apart from the presence of the extra discrete variables \( \epsilon_1 = \epsilon_2 = \pm 1, \) this vertex model is equivalent to the so-called Izergin–Korepin R-matrix [13]. The transfer matrix spectrum of this model is however independent of the parameter \( \epsilon_2. \) The explicit forms of the weights are

\[ a_\pm^{(i)} = 1, \quad b_\pm^{(i)} = \frac{\sinh(\lambda_i)}{\sinh(\lambda_i + \gamma)}, \quad c_\pm^{(i)} = \frac{\sinh(\gamma)}{\sinh(\lambda_i + \gamma)}, \]

\[ d^{(i)} = \frac{\exp(e_2\gamma) \sinh(\gamma_0) \sinh(\lambda_i)}{\cosh(\lambda_i + \frac{\gamma_0}{\lambda} + \frac{\gamma}{2}) \sinh(\lambda_i + \gamma)}, \quad \tilde{d}^{(i)} = -\exp(-2e_2\gamma) d^{(i)}, \]

\[ f^{(i)} = \frac{\cosh(\lambda_i + \frac{\gamma}{2}) \sinh(\lambda_i)}{\cosh(\lambda_i + \frac{\gamma_0}{\lambda} + \frac{\gamma}{2}) \sinh(\lambda_i + \gamma)}, \]

\[ g^{(i)} = -\sinh\left(\frac{\gamma}{2} + \tilde{\gamma}\right) - \sinh\left(\frac{\gamma_0}{\lambda} - \tilde{\gamma}\right) + \sinh\left(\frac{\gamma}{2} + \tilde{\gamma}\right) + \sinh\left(2\lambda_i + \frac{\gamma}{2} - \tilde{\gamma}\right), \]

\[ h^{(i)} = \frac{\cosh(\lambda_i + \frac{\gamma}{2} + \tilde{\gamma}) \sinh(\lambda_i + \gamma) - \exp(2e_2\gamma) \cosh(\lambda_i + \frac{\gamma}{2} + \tilde{\gamma}) \sinh(\lambda_i)}{\cosh(\lambda_i + \frac{\gamma_0}{\lambda} + \frac{\gamma}{2}) \sinh(\lambda_i + \gamma)}, \]

\[ \tilde{h}^{(i)} = \frac{\cosh(\lambda_i + \frac{\gamma}{2} + \tilde{\gamma}) \sinh(\lambda_i + \gamma) - \exp(-2e_2\gamma) \cosh(\lambda_i + \frac{\gamma}{2} + \tilde{\gamma}) \sinh(\lambda_i)}{\cosh(\lambda_i + \frac{\gamma_0}{\lambda} + \frac{\gamma}{2}) \sinh(\lambda_i + \gamma)}, \]

where as before \( \tilde{\gamma} = \frac{\pi}{\lambda} (1 - \epsilon_1). \)
• Branch 2B

The 19-vertex model defined by the weights given below appears to be new in the literature. This model violates charge conjugation symmetry since $b^{(i)} = -b^{(i)}$. The respective weights are

\begin{align}
a^{(i)}_\pm &= 1, \\
b^{(i)}_\pm &= \pm \frac{\sinh(\lambda_i)}{\sinh(\lambda_i + \gamma)}, \\
c^{(i)}_\pm &= \frac{\sinh(\gamma)}{\sinh(\lambda_i + \gamma)}, \\
d^{(i)} &= \pm \epsilon_1 \epsilon_2 \frac{\exp \left( \frac{i \pi \epsilon_2}{3} \right) \sinh(\lambda_i) \cosh(\lambda_i - 2\gamma) \cosh(\lambda_i)}{2 \cosh(\lambda_i - 2\gamma) \cosh(\lambda_i)}, \\
f^{(i)} &= -\frac{\sinh(\lambda_i + 2\gamma) \sinh(\lambda_i)}{\cosh(\lambda_i - 2\gamma) \cosh(\lambda_i)}, \\
g^{(i)} &= \frac{\cosh(2\lambda_i)}{2 \cosh(\lambda_i - 2\gamma) \cosh(\lambda_i)}, \\
h^{(i)} &= \frac{\cosh(\lambda_i - 2\gamma) \cosh(\lambda_i) + \exp \left( \frac{i \pi \epsilon_2}{3} \right) \sinh(\lambda_i + 2\gamma) \sinh(\lambda_i)}{\cosh(\lambda_i - 2\gamma) \cosh(\lambda_i)}, \\
n^{(i)} &= \frac{\cosh(\lambda_i - 2\gamma) \cosh(\lambda_i) + \exp \left( \frac{i \pi \epsilon_2}{3} \right) \sinh(\lambda_i + 2\gamma) \sinh(\lambda_i)}{\cosh(\lambda_i - 2\gamma) \cosh(\lambda_i)},
\end{align}

where $\gamma = \frac{i \pi}{6} (1 - \epsilon_1) + \frac{i \pi}{6} \epsilon_1$ such that $\epsilon_1 = \epsilon_2 = \pm 1$.

For the sake of completeness, we also present the expressions of the one-dimensional spin chains associated with these vertex models. At this point, we observe that the matrix representation of the weights (4) at $\lambda = 0$ is exactly the operator of permutations on $C^3 \otimes C^3$. This property tells us that the respective Hamiltonians are given as a sum of next-neighbor terms,

\begin{equation}
H = J_0 \sum_{i=1}^{L} H_{i,i+1},
\end{equation}

where $J_0$ is an arbitrary normalization.

The two-body term $H_{i,i+1}$ is obtained by the action of the permutator on the derivative of the respective $C^{(i)}$-operators at $\lambda_i = 0$. Its general expression in terms of spin-1 matrices is

\begin{equation}
H_{i,i+1} = J_1 S_i^+ S_{i+1}^- + J_2 S_i^+ (S_i^-)^2 (S_{i+1}^-) + J_3 (S_i^+)^2 (S_i^-)^2 (S_{i+1}^-) + J_4 (S_i^+)^2 (S_i^-) (S_{i+1}^-) + J_5 (S_i^+ (S_i^-)^2 (S_{i+1}^-) + \delta_i (S_i^+ S_{i+1}^+ S_{i+1}^-) + \delta_1 (S_i^- S_{i+1}^- (S_{i+1}^-)^2)
\end{equation}

where $S^\pm$ and $S^c$ are spin-1 matrices satisfying the $SU(2)$ algebra.

We note that in equation (49) we have omitted any chemical potential terms that are canceled under periodic boundary conditions. In table 4, we show the values of the couplings entering equation (49) for all the four branches. At this point we remark that the branch 2B Hamiltonian is a particular case of an integrable spin-1 chain discovered in [18]. This equivalence occurs when the parameters $t_1$ and $t_p$ defined in [18] take the values $t_1 = -1$ and $t_p = \pm \sqrt{2}$.

We conclude this section with the following comment. Some of the physical properties of the first three one-dimensional spin chains have already been investigated in the literature, see for instance [19–22]. This knowledge includes the nature of their excitation spectrum over the anti-ferromagnetic ground state. Considering these results, we conclude that the spectrum is massless when the free parameter $\Delta_1$ belongs to the interval $-2 < \Delta_1 < 2$. In contrast, outside this region the spectrum is expected to have a nonzero gap whose value
the Hilbert space can be separated in 2

\[ \delta J \]

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Table 4. The Hamiltonian couplings for the four branches where \( \gamma = \frac{\omega}{2} (1 - \varepsilon), \gamma_0 = \frac{\pi a}{2}, \omega = \exp (\frac{3\pi i}{2}) \) and \( \varepsilon_1 = \varepsilon_2 = \pm 1 \). The variable \( r \) is defined as \( r = \sinh(\gamma) \sinh(\gamma - \gamma_0) \).

| Branch | 1A | 1B | 2A | 2B |
|--------|----|----|----|----|
| \( J_1 \) | \( \pm \frac{1}{2} \frac{\sinh(\gamma)}{\sinh(\gamma + \frac{\gamma_0}{2})} \) | \( \pm \frac{1}{\sqrt{2}} \) | \( \pm \frac{1}{2} \exp(\gamma) \cosh(\frac{\gamma}{2} + \frac{\gamma_0}{2}) \) | \( \pm \frac{\gamma}{2} \sinh^2 \frac{\gamma}{2} \) |
| \( J_2 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( J_3 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( J_4 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( J_5 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( J_6 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( J_7 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( J_8 \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) | \( \frac{1}{\sinh(\gamma + \gamma_0)} \) |
| \( \delta_1 \) | \( \frac{2\cosh(2\gamma) + \gamma_0}{2\sinh(\gamma + \frac{\gamma_0}{2})} \) | \( 0 \) | \( \frac{2\cosh(2\gamma) + \gamma_0}{2\sinh(\gamma + \frac{\gamma_0}{2})} \) | \( \frac{2\cosh(2\gamma) + \gamma_0}{2\sinh(\gamma + \frac{\gamma_0}{2})} \) |
| \( \delta_2 \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) |
| \( \delta_3 \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) |
| \( \delta_4 \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\cosh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) |
| \( \omega \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) | \( \frac{2\sinh(2\gamma - \gamma_0)}{2\sinh(2\gamma - \gamma_0)} \) |

depends on \( \Delta_+ \). Remarkably enough, such a change of physical behavior is directly related to the variation of the geometrical form of the fundamental algebraic curve (122). In fact, in the massless regime the corresponding curve is closed and it is described by an ellipse, while in the massive region the form of the respective curve is that of an open hyperbola. This relationship between the geometric form of the principal algebraic curve and the nature of the excitations suggests that the anti-ferromagnetic regime of the fourth spin-1 chain should be gapless since in this case \( \Delta_+ = \pm \sqrt{3} \). In the next section, we present evidence supporting such prediction.

5. Exact solution of branch 2B

We present here the exact diagonalization of the transfer matrix associated with weights of the novel vertex model defined as branch 2B. As usual, the corresponding row-to-row transfer matrix \( T(\lambda) \) can be written as the trace of an ordered product of \( L^{(s)} \)-operators parameterized by the spectral parameter \( \lambda \). Recall that the expression for such operators in terms of Weyl matrices is given by equation (4), while the corresponding weights \( f^{(s)}, g^{(s)}, h^{(s)} \) and \( q^{(s)} \) are given in equations (143)–(147).

The eigenspectrum of the transfer matrix \( T(\lambda) \) can be determined by using the algebraic Bethe ansatz framework developed in [5]. Here we present only the main results that are necessary to unveil the thermodynamic limit properties of the spin-1 chain associated with the vertex model 2B. Its transfer matrix commutes with the total spin operator \( \sum_i S_i^z \) where \( S_i^z \) denotes the azimuthal component of a spin-1 matrix acting on the \( i \)-th site. Consequently, the Hilbert space can be separated in \( 2L + 1 \) sectors labeled by an integer number \( n = L - \gamma \).
where \( r = 0, \pm 1, \ldots, \pm L \). We denote by \( \Lambda_n(\lambda) \) the eigenvalue of the transfer matrix \( T(\lambda) \) on a given sector \( n \). By applying the method [5], we find that the expression for \( \Lambda_n(\lambda) \) is

\[
\Lambda_n(\lambda) = \prod_{j=1}^{n} \sinh \left[ \frac{\lambda_j - \lambda + \frac{i\pi}{12}}{2} \right] + \sum_{j=1}^{n} \sinh \left[ \frac{2(\lambda - \lambda_j) + \frac{i\pi}{6}}{2} \right] \sinh \left[ \frac{\lambda - \lambda_j - \frac{i\pi}{12}}{2} \right] + \sum_{j=1}^{n} \sinh \left[ \frac{\lambda - \lambda_j - \frac{i\pi}{12}}{2} \right] \cosh \left[ \frac{\lambda - \lambda_j + \frac{i\pi}{12}}{2} \right] \sinh \left[ \frac{2(\lambda - \lambda_j) - \frac{i\pi}{6}}{2} \right] \cosh \left[ \frac{\lambda - \lambda_j + \frac{i\pi}{12}}{2} \right],
\]

(150)

provided that the rapidities \( \{\lambda\} \) satisfy the following Bethe ansatz equations:

\[
\left( \frac{\sinh \left[ \frac{\lambda_j + \frac{i\pi}{12}}{2} \right]}{\sinh \left[ \frac{\lambda_j - \frac{i\pi}{12}}{2} \right]} \right)^L = \prod_{k \neq j} \sinh \left[ \frac{2(\lambda_j - \lambda_k) + \frac{i\pi}{6}}{2} \right] \cosh \left[ \frac{\lambda_j - \lambda_k - \frac{i\pi}{12}}{2} \right] \cosh \left[ \frac{2(\lambda_j - \lambda_k) - \frac{i\pi}{6}}{2} \right] \cosh \left[ \frac{\lambda_j - \lambda_k - \frac{i\pi}{12}}{2} \right] \cosh \left[ \frac{2(\lambda_j - \lambda_k) + \frac{i\pi}{6}}{2} \right] \cosh \left[ \frac{\lambda_j - \lambda_k - \frac{i\pi}{12}}{2} \right] \cosh \left[ \frac{2(\lambda_j - \lambda_k) - \frac{i\pi}{6}}{2} \right] \cosh \left[ \frac{\lambda_j - \lambda_k - \frac{i\pi}{12}}{2} \right],
\]

(151)

We now have the basic ingredients to investigate the ground state and nature of the excitations of the corresponding spin chain. The corresponding eigenvalue \( E_n(L) \) is obtained by taking the logarithmic derivative of \( \Lambda_n(\lambda) \) at \( \lambda = 0 \) and using as normalization \( J_0 = -i \).

The result is

\[
E_n(L) = \epsilon_1 \sum_{j=1}^{n} \frac{2 \sin \left( \frac{\pi}{2} \right)}{\cos \left( \frac{\pi}{2} \right) - \cosh(2\lambda_j)},
\]

(152)

From now on, we focus our attention on studying the properties of the anti-ferromagnetic regime \( \epsilon_1 = +1 \). Further progress can be made after the identification of the distribution of roots \( \{\lambda_j\} \) that are able to reproduce the low-lying energies of the spin Hamiltonian. This step is done by numerically solving the Bethe ansatz equation (151) and substituting the roots into equation (152). We then compare the results for \( E_n(L) \) with the exact diagonalization of the Hamiltonian up to \( L = 12 \). We observe that though the Hamiltonian is not Hermitian, we find that its eigenvalues are all real. By performing this analysis, we find that the ground state for \( L \) even sits on the sector with zero magnetization \( n = L \). Interestingly enough, the shape of the roots on the complex plane depends whether \( \frac{L}{2} \) is even or odd. In figure 5, we exhibit the Bethe roots for \( L = 8, 10, 12 \) and 14 scaled by the factor \( \frac{3}{4} \).

We then solve the Bethe equations (151) for larger values of \( L \) to figure out the pattern of the roots in the thermodynamic limit \( L \to \infty \). In figure 6, we show the corresponding Bethe roots for \( L = 40 \) and 42. For a better display of the roots’ curvature, we show the positive and the negative imaginary parts separately.

The above analysis leads us to conclude that as \( L \to \infty \), the roots \( \lambda_j \) cluster in a complex 2-strings. Each 2-string has the same real part \( \mu_j \) and equally spaced imaginary parts,

\[
\lambda_j = \mu_j \pm \frac{i\pi}{3}, \quad \mu_j \in \mathbb{R}.
\]

(153)

We now substitute the structure (153) into the Bethe equations (151) and by taking its logarithm we find that the resulting relations for \( \mu_j \) are

\[
L \left[ \phi \left( \mu_j, \frac{5\pi}{12} \right) - \phi \left( \mu_j, \frac{\pi}{4} \right) \right] = -2\pi Q_j + \sum_{k \neq j} \phi \left( 2\mu_j - 2\mu_k, \frac{\pi}{3} \right), \quad \text{for} \quad j = 1, \ldots, n
\]

(154)
Figure 5. The ground state roots $z_j = \frac{3\lambda_j}{\pi}$ for (a) $L = 8$, (b) $L = 10$, (c) $L = 12$, (d) $L = 14$.

Figure 6. The ground state roots $z_j = \frac{3\lambda_j}{\pi}$ for $L = 40$ and $L = 42$ where (a) $z_j = x_j + i y_j$ and (b) $z_j = x_j - i y_j$.

where the function $\phi(x, y) = 2 \arctan[\tanh(x) \cot(y)]$ and $Q_j$ are integer or semi-integer numbers characterizing the logarithm branches.

Considering our numerical analysis, we find that the low-lying spectrum can be described by the following sequence of $Q_j$ numbers:

$$Q_j = -\frac{1}{2} \left[ \frac{L}{2} - n - 1 \right] + j - 1, \quad j = 1, \ldots, \frac{L}{2} - n. \quad (155)$$

For large $L$, the roots approach toward a continuous distribution with density $\rho(\mu)$ given by

$$\rho(\mu) = \frac{d}{d\lambda} [Z_L(\mu)], \quad (156)$$
where the counting function is
\[
Z_L(\mu_j) = \frac{1}{2\pi} \left[ \phi \left( \mu_j, \frac{\pi}{4} \right) - \phi \left( \mu_j, \frac{5\pi}{12} \right) + \frac{1}{L} \sum_{k \neq j} \phi \left( 2\mu_j - 2\mu_k, \frac{\pi}{3} \right) \right].
\]

Strictly in \( L \to \infty \), equations (156) and (157) go into a linear integral equation for the density \( \rho(\mu) \) which can be solved by the standard Fourier transform method. The final result for \( \rho(\mu) \) is
\[
\rho(\mu) = \frac{1}{\pi \cosh(2\mu)}.
\]
The ground state energy per site \( e_\infty \) can now be computed by taking the infinity volume limit of equation (152) and with the help of equation (158). By writing the result in terms of its Fourier transform, we find
\[
e_\infty = \int_0^\infty d\omega \left[ \sinh \left( \frac{\pi \omega}{12} \right) - \sinh \left( \frac{\pi \omega}{4} \right) \cosh \left( \frac{\pi \omega}{4} \right) \sinh \left( \frac{\pi \omega}{2} \right) \right] = -\frac{2}{\pi} + \frac{\sqrt{3}}{9}.
\]

Let us turn our attention to the behavior of the low-lying excitations. These states are obtained from the Bethe equations (154) by introducing vacancies on the sequence of numbers \( Q_j \). This method is by now standard to integrable models and for technicalities see for instance [23]. It turns out that the energy \( \epsilon(\mu) \) and the momenta \( \rho(\mu) \), measured from the ground state, of a hole excitation are given by
\[
\epsilon(\mu) = 2\pi \rho(\mu) \quad \text{and} \quad \rho(\mu) = \int_{-\infty}^{\infty} \epsilon(x) dx.
\]

In order to calculate the low-lying dispersion relation \( \epsilon(p) \) we first have to compute the integral in equation (160) and afterward eliminate the variable \( \mu \) from the function \( \epsilon(\mu) \). By performing these steps, we find that the dispersion relation is
\[
\epsilon(p) = 2 \sin(p).
\]

From equation (161), we conclude that the low momenta excitations have a linear behavior on the respective momenta. Therefore, the excitations are massless corroborating the connection between the geometric form of the curve and the nature of the spin-1 chain excitations made at the end of section 4.

6. Conclusion

In conclusion, we have investigated the solutions of the Yang–Baxter equation for 19-vertex models invariant by the action of parity-time reversal symmetry. We have developed a method to solve the corresponding functional equations from an algebraic point of view without the need of \textit{a priori} spectral parameterization assumption. The structure of the algebraic manifolds constraining the Boltzmann weights follows a rather universal pattern allowing us to classify the possible integrable vertex models in four different families. They share the same fundamental algebraic curve which turns out to be a conic depending on three basic weights. Using the standard parameterization of conics, we are able to obtain the dependence of the Boltzmann weights on the spectral parameter from a unified perspective. Three such vertex models were already known, but the fourth model appears to be new in the literature.

We have observed an intriguing relation between the form of the main algebraic curve and the nature of the low-lying excitations of the related spin-1 chains. In the regime in which the geometric form of such curve is an ellipse, the excitations are massless, while when we have
a hyperbolic structure, the excitations have a mass gap. This fact is supported by previous
knowledge on the physical properties of the first three spin-1 chains and also by the exact
solution of the novel 19-vertex model.

It seems interesting to investigate whether or not the scenario described above remains
valid for other families of \( PT \) invariant vertex models with larger number of states. In principle,
the systematic method developed here can be extended to tackle other models whose statistical
configurations preserve at least one \( U(1) \) symmetry. Another interesting problem is to study
the algebraic invariants underlying vertex models that are not invariant by the \( PT \) symmetry
and in particular to unveil the form of their principal algebraic curve. We hope to address
these questions in future publications.

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**Appendix A. Three-term relations**

This appendix is devoted to the presentation of some technical details entering the solution
of the group \( G_\pm^{(i)} \) relations. An immediate consequence of the last invariant (54) is that the
weight \( f^{(i)} \) can be easily written either in terms of the weights \( a^{(i)}_\pm \) and \( b^{(i)}_\pm \) or by means of the
weights \( a^{(i)}_\pm \) and \( b^{(i)}_\pm \), namely

\[
f^{(i)} = \frac{a^{(i)}_\pm - \Gamma \pm b^{(i)}_\pm}{(\Delta \pm - \Gamma \pm) a^{(i)}_\pm - b^{(i)}_\pm} \quad \text{for } i = 1, 2. \tag{A.1}
\]

It turns out that equation (A.1) can be used to obtain a similar expression for the weight \( d^{(i)} \). In fact, by substituting equation (A.1) into equations (26), (49), (50) and by carrying on
some simplification, one finds

\[
\Lambda_\pm [d^{(i)}]^2 = \frac{(1 - \Delta_\pm \Gamma \pm + \Gamma \pm^2) [b^{(i)}_\pm]^2 \{a^{(i)}_\pm - \Delta_\pm a^{(i)}_\pm b^{(i)}_\pm + [b^{(i)}_\pm]^2\}}{(-\Delta_\pm + \Gamma \pm) a^{(i)}_\pm + b^{(i)}_\pm} \quad \text{for } i = 1, 2. \tag{A.2}
\]

By using the form of the invariants \( \Delta_\pm \) given by equation (22) in expression (A.2), one is able
to take the square root of the weight \([d^{(i)}]^2\). The final result is

\[
d^{(i)} = \pm \sqrt{\frac{1 - \Delta_\pm \Gamma \pm + \Gamma \pm^2}{\Lambda_\pm}} \frac{b^{(i)}_\pm a^{(i)}_\pm}{(-\Delta_\pm + \Gamma \pm) a^{(i)}_\pm + b^{(i)}_\pm} \quad \text{for } i = 1, 2. \tag{A.3}
\]

This means that the compatibility among the invariants \( \Gamma \pm \) is equivalent to the matching of the expressions for the weights \( f^{(i)} \) and \( d^{(i)} \) coming from the distinct charge sectors \( \pm \). By
imposing such consistency, one finds that equation (A.1) requires us to solve

\[
[\Delta_\pm - \Gamma \pm] a^{(i)}_\pm - b^{(i)}_\pm \right] a^{(i)}_\pm - \Gamma \pm b^{(i)}_\pm \right] b^{(i)}_\pm - \left[ (\Delta_\pm - \Gamma \pm) a^{(i)}_\pm - b^{(i)}_\pm \right] a^{(i)}_\pm - \Gamma \pm b^{(i)}_\pm \right] b^{(i)}_\pm = 0,
\]

for \( i = 1, 2 \). \tag{A.4}

while equation (A.3) implies that the compatibility for \( d^{(i)} \) is

\[
\sqrt{\Lambda_\pm (1 - \Delta_\pm \Gamma \pm + \Gamma \pm^2)} \left[ (\Delta_\pm + \Gamma \pm) a^{(i)}_\pm + d^{(i)}_\pm \right] a^{(i)}_\pm + b^{(i)}_\pm \right] \frac{b^{(i)}_\pm}{d^{(i)}_\pm} = \sqrt{\Lambda_\pm (1 - \Delta_\pm \Gamma \pm + \Gamma \pm^2)} \times \left[ (\Delta_\pm + \Gamma \pm) a^{(i)}_\pm + b^{(i)}_\pm \right] a^{(i)}_\pm \right] b^{(i)}_\pm \right] = 0 \quad \text{for } i = 1, 2. \tag{A.5}
\]
The analysis of equations (A.4) and (A.5) for the index $i = 1$ is immediate since the weights $a_{+-}^{(1)}$ and $c_{+-}^{(1)}$ are still free. Therefore, equations (A.4) and (A.5) for $i = 1$ are easily solved by eliminating the weights $a_{+-}^{(1)}$ and $c_{+-}^{(1)}$, namely

$$a_{+-}^{(1)} = \frac{\Gamma_-(\Delta_+ - \Gamma_+ a_{+}^{(1)} b_+^{(1)} - a_{-}^{(1)} b_+^{(1)} - \Gamma_- b_+^{(1)} b_+^{(1)} + \Gamma_+ [b_+^{(1)}]^2}{(\Delta_+ - \Gamma_+)a_{+}^{(1)} b_+^{(1)} - (\Delta_- - \Gamma_-)a_{-}^{(1)} b_+^{(1)} - b_+^{(1)} b_+^{(1)} + \Gamma_+ (\Delta_+ - \Gamma_+) [b_+^{(1)}]^2} b_+^{(1)} \quad (A.6)$$

and

$$c_{+-}^{(1)} = -\frac{\sqrt{\Lambda_-(1 - \Delta_+ \Gamma_+ + \Gamma_+^2)(1 - \Delta_+ \Gamma_+ + \Gamma_+^2) b_+^{(1)} b_+^{(1)} c_+^{(1)}}{\sqrt{\Lambda_+ [(\Delta_+ - \Gamma_+)a_{+}^{(1)} b_+^{(1)} - (\Delta_- - \Gamma_-)a_{-}^{(1)} b_+^{(1)} - b_+^{(1)} b_+^{(1)} + \Gamma_+ (\Delta_+ - \Gamma_+) [b_+^{(1)}]^2}}} \quad (A.7)$$

On the other hand, the solution of equations (A.4) and (A.5) for the index $i = 2$ is more involving. This is the case because the weights $a_{+-}^{(2)}$, $b_{+-}^{(2)}$, and $c_{+-}^{(2)}$ have already been determined in terms of the amplitudes $a_{+}^{(2)}$, $b_{+}^{(2)}$, $c_{+}^{(2)}$, $f^{(2)}$, and $d^{(2)}$ thanks to the previous relations (32) and (46). In addition to that we also recall that the weights $f^{(2)}$ and $d^{(2)}$ are also determined in terms of $a_{+}^{(2)}$, $b_{+}^{(2)}$, and $c_{+}^{(2)}$ through equations (A.1) and (A.3). Considering all this information together with the fact that the weight $c_{+-}^{(2)}$ can be eliminated with the help of the invariant $\Delta_+$ (22), one concludes that equations (A.4) and (A.5) are in fact a polynomial relation on the remaining weights $a_{+-}^{(2)}$ and $b_{+-}^{(2)}$. It turns out that the expression of this polynomial associated with equation (A.4), after using constraints (35) and (45), is

$$B_1 [a_{+-}^{(2)}]^3 + B_2 a_{+-}^{(2)} [b_+^{(2)}]^2 + B_3 [a_{+-}^{(2)}] b_+^{(2)} + B_4 [b_+^{(2)}]^3 = 0, \quad (A.8)$$

where its coefficients are given only in terms of the invariant values by

$$B_1 = \Psi [\Lambda_+^2 \Gamma_- (-\Delta_+ + \Gamma_+ \Gamma_+) + \Lambda_+ \Psi (\Delta_+ \Gamma_- + \Gamma_- \Gamma_+ + \Gamma_+^2 + \Delta_+ \Omega + \Gamma_- \Omega - \Gamma_+ \Omega)] - \Psi^2 (\Gamma_+^2 + \Delta_+ \Omega + 2 \Gamma_- \Omega + 2 \Omega^2)], \quad (A.9)$$

$$B_2 = \Lambda_+^2 \Gamma_+ (\Delta_+ + 2 \Gamma_- - \Gamma_+) + \Lambda_+^2 \Psi (1 - 2 \Delta_+ \Gamma_- - \Delta_+ \Gamma_+ + \Gamma_- \Gamma_+ - 4 \Gamma_+ \Omega) + \Lambda_+ \Psi^2 (-1 - \Delta_+ \Gamma_- - \Delta_+ \Gamma_+ - \Gamma_- \Gamma_+ + \Gamma_+^2 + \Delta_+ \Omega + \Gamma_- \Omega - \Gamma_+ \Omega - \Delta_+ \Omega - \Delta_+ \Omega + \Delta_+ \Omega + \Gamma_- \Omega - \Gamma_+ \Omega) - \Delta_+^2 \Omega - \Delta_+ \Omega - \Gamma_- \Omega + \Delta_+ \Omega^2 - \Delta_+ \Omega - \Omega^2 - \Delta_+ \Omega + \Omega^2 + \Gamma_-^2 \Omega^2 - \Psi^3 (1 - 2 \Delta_+ \Gamma_+^2 + \Gamma_- \Omega^2 - 2 \Gamma_+ \Omega - \Gamma_- \Omega^2 + 2 \Omega^2 - 4 \Delta_+ \Omega - \Omega^2 - 4 \Omega^2), \quad (A.10)$$

$$B_3 = -\Lambda_+^3 \Gamma_+ (\Delta_+ + \Gamma_- - \Gamma_+) + \Lambda_+^3 \Psi (\Gamma_- + \Delta_+^2 \Gamma_- - \Gamma_+ - \Delta_+ \Gamma_- - 2 \Omega) + \Lambda_+ \Psi^2 (\Delta_+ + 2 \Gamma_- - 2 \Gamma_+ - \Delta_+ \Gamma_- - 2 \Delta_+ \Gamma_+ + \Delta_+ \Omega - \Gamma_- \Omega + \Delta_+ \Omega^2 + \Gamma_-^2 \Omega^2) + \Psi^2 (2 \Delta_+ \Gamma_+ - 3 \Omega + 4 \Delta_+ \Omega + \Gamma_- \Omega - \Delta_+ \Omega^2 + 2 \Gamma_+ \Omega^2 - 2 \Omega^2), \quad (A.11)$$

$$B_4 = -\Lambda_+^3 \Gamma_+ (1 + \Gamma_- \Gamma_+) + \Lambda_+^3 \Psi (\Gamma_- + \Delta_+^2 \Gamma_- + 2 \Gamma_+^2 \Omega + \Delta_+ \Psi^2 (\Gamma_- - \Delta_+ \Gamma_- \Gamma_+ + \Gamma_+^2 \Omega - \Delta_+ \Omega - \Gamma_- \Omega + \Delta_+ \Gamma_- \Omega - \Gamma_- \Omega + \Delta_+ \Gamma_- \Omega + \Gamma_- \Omega^2 + \Gamma_- \Omega^2) + \Psi^3 (\Omega - 2 \Delta_+ \Gamma_+ \Omega + \Gamma_- \Omega^2 + \Delta_+ \Gamma_- \Omega \Psi^3 - 2 \Gamma_+ \Omega \Psi^3 + 3 \Delta_+ \Gamma_- \Omega \Psi^3 + 2 \Gamma_- \Omega \Psi^3), \quad (A.12)$$

As argued in the main text, we are searching for solutions in which $a_{+-}^{(2)}$ and $b_{+-}^{(2)}$ are independent of each other. This means that we have to set all the coefficients of the polynomial
Table A1. Invariant values for the special branches 1S and 2S.

| Invariants | Branch 1S | Branch 2S |
|------------|-----------|-----------|
| $\Delta_+$ | $\pm \sqrt{3}$ | $\pm 2$ |
| $\Delta_-$ | $\mp \sqrt{3}$ | $\pm 2$ |
| $\Lambda_+$ | $-1$ | $-1$ |
| $\Lambda_-$ | $-1$ | $-1$ |
| $\Psi$ | $1$ | $-1$ |
| $\Omega$ | 0 | 0 |
| $\Gamma_+$ | $\pm \frac{2}{\sqrt{3}}$ | $\mp 1$ |
| $\Gamma_-$ | $\mp \frac{2}{\sqrt{3}}$ | $\pm 1$ |
| $\Theta_+$ | 0 | $\pm 1$ |
| $\Theta_-$ | $\mp \frac{2}{\sqrt{3}}$ | 0 |
| $\Delta_+^{(i)}$ | 0 | $\pm 3$ |
| $\Delta_-^{(i)}$ | 0 | $\pm 3$ |
| $\Lambda_+^{(i)}$ | $\pm \sqrt{3}$ | 0 |
| $\Lambda_-^{(i)}$ | $\mp \sqrt{3}$ | 0 |
| $\Delta_+^{(i)}$ | $\pm \sqrt{3}$ | 0 |
| $\Delta_-^{(i)}$ | $\mp \sqrt{3}$ | 0 |

(A.8) to zero. By imposing this condition for $B_1$, one is able to eliminate the invariant $\Gamma_-$ whose expression can be further simplified with the help of constraint (36). The final result is

$$\Gamma_- = \frac{\Psi}{\Lambda_+} (\Gamma_+ + \Omega). \quad (A.13)$$

By substituting result (A.13) into the expressions of the remaining coefficients (A.10)–(A.12) and by once again taking into account constraint (36), we find that $B_2$, $B_3$ and $B_4$ vanish. The same reasoning can be repeated for equation (A.5) when $i = 2$. We conclude that it does not impose additional restrictions besides equation (A.13).

We conclude by observing that the weights $a_+^{(i)}$, $b_-^{(i)}$ and $c_-^{(i)}$ can also be written in terms of the amplitudes $a_+^{(i)}$, $b_+^{(i)}$ and $c_+^{(i)}$. In order to see that, we first substitute the weights $f_+^{(i)}$ and $d_+^{(i)}$, considering the index + of equations (A.4) and (A.5), into the expressions of the weights $b_+^{(i)}$, $b_-^{(i)}$, $c_-^{(i)}$, $a_+^{(i)}$, $a_-^{(i)}$ and $c_+^{(i)}$, see equations (30), (32), (46), (A.6) and (A.7). We next use the constraint $\Delta_+$ (22) to eliminate the amplitude $[c_+^{(i)}]^2$, besides the explicit form of the relations between invariants (27), (35), (36), (45) and (A.13). By performing such steps, we are able to write rather simple expressions for the weights $a_+^{(i)}$, $b_-^{(i)}$, $c_-^{(i)}$ for $i = 1, 2$. They are given in the main text, see equations (56)–(58).

Appendix B. Particular invariant solutions

We describe particular solutions for the invariants coming from branch 1A. Besides $\Psi = \Lambda_+ = 1$ we have two other possibilities that provide us non-null Boltzmann weights. We shall denominate such special branches as follows:

- branch 1S : $\Psi = -\Lambda_+ = 1$ \hspace{1cm} (B.1)
- branch 2S : $\Psi = \Lambda_+ = -1$ \hspace{1cm} (B.2)
It turns out that choices (B.1) and (B.2) lead us to the situation in which we do not have any free parameter at our disposal. In table A1, we present the corresponding invariant values.

By comparing table A1 with table 2, it is not difficult to see that branch 1S is a particular case of branch 1B once we choose \( \Delta_1 = \pm \sqrt{3} \). By the same token, comparison with table 3 reveals that branch 2S becomes equivalent to branch 2A provided we set \( \Delta_1 = \pm 2 \).

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