Traces of Subharmonic Functions to Fractal Sets

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Abstract
We study traces of a class of subharmonic functions to Ahlfors regular subsets of $\mathbb{C}^n$. In particular, we establish for the traces a generalized BMO-property and the reverse Hölder inequality.

1. Introduction.

1.1. A compact subset $K \subset \mathbb{R}^n$ is said to be (Ahlfors) $d$-regular if there is a positive number $a$ such that for any $x \in K$, $0 < t \leq \text{diam}(K)$

$$\mathcal{H}^d(K \cap \mathbb{D}(x, t)) \leq at^d. \quad (1.1)$$

Here $\mathcal{H}^d(\omega)$ denotes the $d$-Hausdorff measure of $\omega$. This class will be denoted by $\mathcal{A}(d, a)$.

A compact subset $K \in \mathcal{A}(d, a)$ is said to be a $d$-set if there is a positive number $b$ such that for any $x \in K$, $0 < t \leq \text{diam}(K)$

$$bt^d \leq \mathcal{H}^d(K \cap \mathbb{D}(x, t)). \quad (1.2)$$

Denote this class by $\mathcal{A}(d, a, b)$.

The purpose of this paper is to study traces of subharmonic functions to $d$-sets. Let us recall that the class of $d$-sets, in particular, contains Lipschitz $d$-manifolds (with $d$ integer), Cantor type sets and self-similar sets (with arbitrary $d$), see, e.g., [JW], p. 29 and [M], sect. 4.13.

Let us, first, formulate our results in $\mathbb{C}$. In the sequel we denote $\mathbb{D}_s := \{ z \in \mathbb{C} : |z| < s \}$ and $\mathbb{D}(x, t) := \{ z \in \mathbb{C} : |z - x| < t \}$.

Assume that $f$ is a subharmonic in $\mathbb{D}_1$ function satisfying

$$\sup_{\mathbb{D}_1} f \leq M_1 \quad \text{and} \quad \sup_{\mathbb{D}_r} f \geq M_2 \quad (r < 1). \quad (1.3)$$

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Theorem 1.1 Let \( \omega \subset D(x, t) \) be a compact set of \( A(d, a) \) satisfying \( \mathcal{H}^d(\omega) \geq \epsilon > 0 \). Assume that \( D(x, t/r) \subset D_r \). Then there is a constant \( c = c(r) > 0 \) such that inequality
\[
\sup_{D(x, t)} f \leq \sup_{\omega} f + (M_1 - M_2)c \log \frac{4\epsilon a^{1/d}}{r(d\epsilon)^{1/d}}
\]
holds for any subharmonic \( f \) satisfying (1.3).

We use Theorem 1.1 to establish a generalized BMO-property and the reverse Hölder inequality for traces of subharmonic functions to \( d \)-sets. Let us recall

Definition 1.2 Let \( X \) be a complete metric space equipped with a regular Borel measure \( \mu \). A locally integrable on \( X \) function \( f \) belongs to \( \text{BMO}(X, \mu) \) if
\[
|f|_* := \sup \left\{ \frac{1}{\mu(B)} \int_B |f - f_B| d\mu \right\} < \infty;
\]
here supremum is taken over all metric balls \( B \subset X \) and \( f_B = \frac{1}{\mu(B)} \int_B f d\mu \).

In order to formulate the next two results consider a compact \( d \)-set \( K \subset \mathbb{C} \). Assume that \( f \) is a subharmonic function defined in an open neighbourhood of \( K \).

Theorem 1.3 Restriction \( f|_K \) belongs to \( \text{BMO}(K, \mathcal{H}^d) \).

Theorem 1.4 For any \( K_{x,t} := D(x, t) \cap K, x \in K, t > 0 \), and any \( 1 \leq p \leq \infty \) the inequality
\[
\left( \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^{pf} d\mathcal{H}^d \right)^{1/p} \leq C(K, f, d) \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^{f} d\mathcal{H}^d
\]
holds.

1.2. In this section we consider multidimensional generalizations of the previous results for plurisubharmonic (psh) functions in \( \mathbb{C}^n, n \geq 2 \). Note that it is impossible to obtain in this situation the results similar to those in \( \mathbb{C} \). Indeed, let \( N := \{ z \in \mathbb{C}^n : f(z) = 0 \} \neq \emptyset \) for a nontrivial holomorphic function \( f \). Let \( K = K_1 \cup K_2 \) be a connected set where \( K_1, K_2 \) are compact \((2n - 2)\)-Lipschitz manifolds such that \( K_1 \subset N \) and \( K_2 \not\subset N \). Then the psh function \( \log |f| \) equals \(-\infty \) on \( K_1 \) and \( \text{sup}_{K} \log |f| > -\infty \). Hence the results similar to Theorems 1.1, 1.3 and 1.4 are not true in this case.

So we restrict ourselves to a special class of psh functions whose local behavior is similar to that of subharmonic functions in \( \mathbb{C} \).

Consider a family \( F = \{ f_1, ..., f_n \} \) of holomorphic functions defined in Euclidean ball \( B_r \subset \mathbb{C}^n \) of radius \( r > 1/2 \) centered at 0. We will prove the results for psh function \( u = \log |F| \) where \(|F|^2 := |f_1|^2 + ... + |f_n|^2 \). Assume that
\[
\sup_{B_r} u \leq 0 \quad \text{and} \quad \inf_{\partial B_{1/2}} u \geq -M
\]
for some \( M > 0 \). The latter condition means that the system \( F = 0 \) has only discrete zeros in a neighbourhood of the closed ball \( \overline{B_{1/2}} \).
**Definition 1.5** Let $h$ be a nonnegative analytic function with a finite number of zeros defined on an open set $U \subset \mathbb{R}^N$. A zero $x$ of $h$ is said to be elliptic if the Taylor expansion of $f$ at $x$ has the following form

$$h(x + t\omega) = t^d f(\omega) + o(t^d)$$

with

$$\inf_{S^{N-1}} f > 0.$$ 

Here $t > 0$ and $\omega$ belongs to the unit sphere $S^{N-1}$ of $\mathbb{R}^N$.

Assume that $F$ satisfies condition (1.5) and all zeros of $|F|^2$ are elliptic. Let $k$ be the number of zeros of $|F|^2$ in $B_{1/2}$ counting with their multiplicities. (By multiplicity we mean the degree of map $F$ at zero.)

**Theorem 1.6** Let $\omega \subset B(x, t)$ be a compact set of $\mathcal{A}(d, a)$ and $\mathcal{H}^d(\omega) \geq \epsilon > 0$. Assume that $B(x, 4r^2t) \subset B_{1/2}$. Then there is a constant $c = c(r, F) > 0$ such that

$$\sup_{B(x,t)} \log |F| \leq \sup_{\omega} \log |F| + k \log \frac{16\epsilon t \Gamma(1/d)}{c(d) \Gamma(1/d)}.$$ 

The following theorem gives the results similar to those of Theorems 1.3 and 1.4.

**Theorem 1.7** Assume that a compact set $K \subset \mathbb{C}^n$ belongs to $\mathcal{A}(d, a, b)$. Then under assumptions of Theorem 1.6, $|F|$ satisfies the reverse Hölder inequality in each ball $B(x, t) \cap K$ and $\log |F| \in BMO(K, H^*)$.

2. **Abstract Version of Cartan’s Lemma.**

Our proofs are based on estimates for psh functions which generalize well-known Cartan’s Lemma for polynomials (see [Ca]). We need a version of the generalized Cartan’s Lemma proved by Gorin (see [GK]).

Let $X$ be a complete metric space and let $\mu$ be a finite Borel measure on $X$. We consider a continuous, strictly increasing, nonnegative function $\phi$ on $[0, +\infty[$, $\phi(0) = 0$, $\lim_{x \to \infty} \phi(x) > \mu(X)$. The function $\phi$ will be called a majorant.

For each point $x \in X$ we set $\tau(x) = \sup\{ t : \mu(B(x, t)) \geq \phi(t)\}$, where $B(x, t)$ is the closed ball in $X$ with center $x$ and radius $t$. It is easy to see that $\mu(B(x, \tau(x)) = \phi(\tau(x))$ and $\sup_x \tau(x) \leq \phi^{-1}(\mu(X)) < \infty$.

A point $x \in X$ is said to be regular (with respect to $\mu$ and $\phi$) if $\tau(x) = 0$, i.e., $\mu(B(x, t)) < \phi(t)$ for all $t > 0$. The next result shows that the set of regular points is sufficiently large for an arbitrary majorant $\phi$.

**Lemma 2.1** (Gorin) Let $0 < \gamma < 1/2$. There exists a sequence of balls $B_k = B(x_k, t_k)$, $k = 1, 2, \ldots$, which collectively cover all the irregular points and which are such that $\sum_{k \geq 1} \phi(\gamma t_k) \leq \mu(X)$ (i.e., $t_k \to 0$).
For the sake of completeness we present Gorin’s proof of the lemma.

**Proof.** Let $0 < \alpha < 1$, $\beta > 2$ but $\gamma < \alpha / \beta$. We set $B_0 = \emptyset$ and assume that the balls $B_0, \ldots, B_{k-1}$ have been constructed. If $\tau_k = \sup \{ \tau(x) : x \notin B_0 \cup \cdots \cup B_{k-1} \}$, then there exists a point $x_k \notin B_0 \cup \cdots \cup B_{k-1}$, such that $\tau(x_k) \geq \alpha \tau_k$. We set $t_k = \beta \tau_k$ and $B_k = B(x_k, t_k)$. Clearly, the sequence $\tau_k$ (and thus also $t_k$) does not increase.

The balls $B(x_k, \tau_k)$ are pairwise disjoint. Indeed, if $l > k$, then $x_l \notin B_k$, i.e., the distance between $x_l$ and $x_k$ is greater than $\beta \tau_k > 2 \tau_k \geq \tau_k + \tau_l$. Then,

$$ \sum_{k=1}^{\infty} \phi(\gamma t_k) \leq \sum_{k=1}^{\infty} \phi(\alpha \tau_k) \leq \sum_{k=1}^{\infty} \phi(\tau(x_k)) = \sum_{k=1}^{\infty} \mu(B(x_k, \tau_k)) \leq \mu(X) ; $$

consequently, $\tau_k \to 0$, i.e., for each point $x$, not belonging to the union of the balls $B_k$, $\tau(x) = 0$, $x$ is a regular point. In addition, $t_k = \beta \tau_k \to 0$. \quad \Box

**Remark 2.2** If $X$ is a locally compact metric space then one can take $\gamma = 1 / 2$ (for similar arguments see, e.g., [L], Th. 11.2.3).

We now apply Lemma 2.1 to obtain estimates for logarithmic potentials of measures.

Assume that $X$ is a locally compact metric space with metric $d(,)$.

**Theorem 2.3** Let

$$ u(z) = \int_X \log d(x, \xi) d\mu(\xi) $$

where $\mu$ is a Borel measure, $\mu(X) = k < \infty$.

Given $H > 0, d > 0$ there exists a system of metric balls such that

$$ \sum_{j=1}^{d} r_j^d \leq \frac{(2H)^d}{d} $$

(2.1)

where $r_j$ are radii of these balls, and

$$ u(z) \geq k \log \frac{H}{e} $$

everywhere outside these balls.

**Proof.** Let $\phi(t) = (pt)^d$ be a majorant with $p = \frac{(kd)^{1/d}}{H}$. We cover all irregular points of $X$ by balls according to Gorin’s Lemma 2.1 and Remark 2.2. It remains to estimate the potential $u$ outside of these balls, i.e., at any regular point $z$. Let $n(t; z) = \mu(\{ \xi : d(z, \xi) \leq t \})$. Clearly, for any $N \geq \max\{1, H\}$

$$ u(z) \geq \int_{d(z, \xi) \leq N} \log d(z, \xi) d\mu(\xi) = \int_0^N \log t \ dn(t; z) = n(t; z) \log t |_{t=0}^{t=N} - \int_0^N \frac{n(t; z)}{t} dt. $$

Since $n(t; z) < (pt)^d$, we then have

$$ u(z) \geq n(N; z) \log N - \int_0^N \frac{n(t; z)}{t} dt. $$
In addition, \( n(t; z) \leq n(N; z) \) for \( t \leq N \). Therefore,

\[
\begin{align*}
u(z) & \geq n(N; z) \log N - \int_0^H \frac{(pt)^d}{t} dt - \int_H^N \frac{n(N; z)}{t} dt = \\
& = n(N; z) \log N - \frac{(pH)^d}{d} - n(N; z) \log N + n(N; z) \log H = -k + n(N; z) \log H
\end{align*}
\]

Letting here \( N \to \infty \) and taking into account that \( \lim_{N \to \infty} n(N; z) = k \) we obtain the required result. 

\[\square\]

3. Proofs of Theorems 1.1, 1.3 and 1.4.

Proof of Theorem 1.1. We begin with

**Proposition 3.1** Let \( u \) be a nonpositive subharmonic function on \( \mathbb{D}_1 \) satisfying

\[
\sup_{\mathbb{D}_r} u \geq -1 \quad \text{for some } r < 1.
\]

Then for any \( H > 0, d > 0 \) there is a set of disks such that

\[
\sum r_j^d \leq \frac{(2H)^d}{d},
\]

where \( r_j \) are radii of these disks, and

\[
u(z) \geq c \log \frac{H}{e}
\]

outside these disks in \( \mathbb{D}_r \). Here \( c = c(r) > 0 \) depends on \( r \) only.

**Proof.** Let \( \kappa \) be a nonnegative radial \( C^\infty \)-function on \( \mathbb{C} \) satisfying

\[
\int \int_\mathbb{C} \kappa(x, y) dxdy = 1 \quad \text{and} \quad \text{supp}(\kappa) \subset \mathbb{D}_1 \quad (z = x + iy).
\]

Let \( u_k \) denote the function defined on \( \mathbb{D}_{1-1/k} \) by

\[
u_k(w) := \int \int_\mathbb{C} \kappa(x, y) u(w - z/k) dxdy.
\]

It is well known, see, e.g., [K], Theorem 2.9.2, that \( u_k \) is subharmonic on \( \mathbb{D}_{1-1/k} \) of the class \( C^\infty \) and that \( u_k(w) \) monotonically decreases and tends to \( u(w) \) for each \( w \in \mathbb{D}_1 \) as \( k \to \infty \). Let \( K := \{ z \in \mathbb{D}_1 : \frac{1+r}{2} \leq |z| \leq \frac{3+r}{4} \} \) be an annulus in \( \mathbb{D}_1 \) and \( k \geq k_0 = \left[ \frac{8}{1+r} \right] + 1 \). We are based on the following result (see, e.g., [Br], Lemma 2.3).

There are a constant \( A = A(r) > 0 \) and numbers \( t_k, k \geq k_0 \), satisfying \( \frac{1+r}{2} \leq t_k \leq \frac{3+r}{4} \) such that \( u_k(z) \geq -A \) for any \( z \in \mathbb{C}, |z| = t_k \).
Then we can construct functions $f_k$ subharmonic on $\mathbb{C}$ by

$$f_k(z) := \begin{cases} 
  u_k(z) & (z \in \mathbb{D}_{t_k}); \\
  \max\left\{ u_k(z), \frac{-2A \log |z|}{\log t_k} \right\} & (z \in \mathbb{D}_1 \setminus \mathbb{D}_{t_k}); \\
  \frac{-2A \log |z|}{\log t_k} & (z \in \mathbb{C} \setminus \mathbb{D}_1).
\end{cases}$$

Without loss of generality we may assume that $t_k \to t \in \left[\frac{1+r}{2}, \frac{3+r}{4}\right]$ as $k \to \infty$. Finally, define

$$f(z) = \left(\lim_{k \to \infty} f_k(z)\right)^*,$$

where $g^*$ denotes upper semicontinuous regularization of $g$. Then $f$ is subharmonic in $\mathbb{C}$ satisfying

$$f(z) = u(z) \quad (z \in \mathbb{D}_1) \quad \text{and} \quad f(z) = \frac{-2A \log |z|}{\log t} \quad (z \in \mathbb{C} \setminus \mathbb{D}_1).$$

Consider now $\mu = \Delta f$. Then $\mu$ is a finite Borel measure on $\mathbb{C}$ supported in $\overline{\mathbb{D}}_1$. According to F. Riesz’s theorem (see, e.g., [HK], Th. 3.9)

$$\tilde{f}(z) := \frac{1}{2\pi} \int \int_{\mathbb{C}} \log |z - \xi| d\mu(\xi)$$

is subharmonic in $\mathbb{C}$ and satisfies $\Delta \tilde{f} = \Delta f = \mu$. Thus $h = \tilde{f} - f$ is a real-valued harmonic in $\mathbb{C}$ function. Moreover, $h$ goes to infinity as

$$\left(\frac{\mu(\mathbb{C})}{2\pi} - \frac{2A}{\log t} \log |z|\right).$$

This immediately implies $h = 0$ and $\Delta(\mathbb{C}) = \frac{-2A}{\log t}$. Now according to Theorem 2.3 applied to $f (= \tilde{f})$, for any $H > 0, d > 0$ there is a system of disks with radii $r_j$ satisfying

$$\sum r_j^d \leq \frac{(2H)^d}{d}$$

such that

$$f \geq \frac{-2A}{\log t} \log \frac{H}{e} \geq \frac{-2A}{\log r} \log \frac{H}{e}$$

outside these disks. It remains to set $c = \frac{-2A}{\log r}$.

The proof of the proposition is complete. \qed

Assume now that $f$ is subharmonic and satisfies (1.3). Then by the main theorem in [Br] there is a constant $C = C(r) > 0$ such that the inequality

$$\sup_{\mathbb{D}(x,t/r)} f \leq C(M_1 - M_2) + \sup_{\mathbb{D}(x,t)} f$$

holds for any pair of disks $\mathbb{D}(x,t) \subset \mathbb{D}(x,t/r) \subset \mathbb{D}_r$. Applying inequality of Proposition 3.1 to the function

$$u(z) = \frac{f(tz/r) - \sup_{\mathbb{D}(x,t/r)} f}{C(M_1 - M_2)} \quad (z \in \mathbb{D}_1)$$

and then going back to $f$ we obtain the following
**Proposition 3.2** There is a constant \( c = c(r) > 0 \) such that for any disk \( \mathbb{D}(x, t) \) satisfying \( \mathbb{D}(x, t) \subset \mathbb{D}(x, t/r) \subset \mathbb{D}_r \) and any \( H > 0, d > 0 \) there is a system of disks such that
\[
\sum r_j^d \leq \frac{(2tH/r)^d}{d},
\]
where \( r_j \) are radii of these disks, and
\[
f(z) \geq \sup_{\mathbb{D}(x, t)} f + c(M_1 - M_2) \log \frac{H}{e}
\]
outside these disks in \( \mathbb{D}(x, t) \).

**Remark 3.3** A particular case of Proposition 3.2 for functions \( u = \log |f| \) with holomorphic \( f \) and for \( d = 1 \) was proved in [L].

We proceed to the proof of Theorem 1.1. First we show that \( \omega \) can not be covered by a system of disks such that
\[
\sum r_j^d \leq \frac{(1 - 1/n)\epsilon}{2^d a} \quad (n \geq 1)
\]
(3.4)
where \( r_j \) are radii of these disks. Assume to the contrary that there exist \( \epsilon > 0 \) such that \( \omega \) is covered by a system of disks \( \{ \mathbb{D}(x_j, r_j) \} \) whose radii satisfy (3.4) which covers \( \omega \). For any \( x_j \) choose \( y_j \in \omega \) so that \( |x_j - y_j| \leq r_j \). Then the system of disks \( \{ \mathbb{D}(y_j, 2r_j) \} \) also covers \( \omega \). Since \( \omega \in A(d, a) \), we obtain inequality
\[
\mathcal{H}^d(\omega) \leq \sum \mathcal{H}^d(\omega \cap \mathbb{D}(y_j, 2r_j)) \leq 2^d a \sum r_j^d < \epsilon
\]
which contradicts to \( \mathcal{H}^d(\omega) \geq \epsilon \).

We now apply Proposition 3.2 with \( H_n = \frac{(d(1-1/n)a)^{1/d}}{4ad^{1/d}} \). Since any system of disks with \( \sum r_j^d \leq \frac{(2H_n/r)^d}{d} \) can not cover \( \omega \), Proposition 3.2 implies that there is a point \( x_n \in \omega \) such that
\[
\sup_{\omega} f \geq f(x_n) \geq \sup_{\mathbb{D}(x, t)} f + c(M_1 - M_2) \log \frac{H_n}{e}
\]
Letting \( n \to \infty \) we get the required inequality.

Theorem 1.1 is proved. \( \Box \)

Our next result shows that \( d \)-regularity is a necessary condition for the set to satisfy the inequality of Theorem 1.1.

**Proposition 3.4** Let \( K \subset \mathbb{D}_{1/2} \) be a compact set with \( \mathcal{H}^d(K) < \infty \). Assume that the inequality
\[
\sup_{\mathbb{D}(x, t)} f \leq \sup_{\omega} f + L + C \log \frac{t}{\epsilon^{1/d}}
\]
holds for any \( \omega \subset K \cap \mathbb{D}(x, t') \subset \mathbb{D}(x, 3t/2) \subset \mathbb{D}_{2/3} \), \( x \in K \), with \( \mathcal{H}^d(\omega) = \epsilon \) and any \( f \) subharmonic in \( \mathbb{D}_1 \) satisfying (1.3) with \( r = 2/3 \) and some \( M_1, M_2 \). Here \( L \) and \( C > 0 \) depend on \( K, d, M_1, M_2 \). Then \( K \in A(d, c) \) for some \( c > 0 \).
Lemma 3.5

Let \( f : D \to \mathbb{K} \) of \( f \) now we have

for \( t > \omega \) where we choose \( \tilde{\omega} \).

This gives the estimate of the BMO-norm in each ball

3.5. From inequality 3.5 it follows

The proof follows straightforwardly from the inequality of Theorem 1.1

Proof. For any \( f, \omega, t \leq 1/9 \) satisfying assumptions of the proposition the inequality

\[
-C \log \frac{t}{\epsilon_1/d} \leq \sup_{D(x,t)} f - \sup_{D(x,t)} f - C \log \frac{t}{\epsilon_1/d} \leq L < \infty
\]

holds. For a point \( x \in K \) we set \( f_x(z) = \log |z-x| \) and \( \epsilon_t := \mathcal{H}^d(D(x,t) \cap K) \).

Clearly the family \( \{ f_x \} \) satisfies inequality (1.3) with \( r = 2/3 \), \( M_1 = 3/2 \) and \( M_2 = 1/6 \).

Then the above inequality applied to \( f_x \) gets

\[
L \geq -C \log \frac{t}{\epsilon_1/d},
\]

that is equivalent to \( \epsilon_t \leq \tilde{L}^d \) for \( \tilde{L} = e^{d \epsilon_1} \). So we checked the definition of \( d \)-regularity for \( t \leq 1/9 \). For \( t > 1/9 \) the inequality is obvious. \qed

Assume that \( f \) satisfies (1.3) and \( K \subset \mathbb{D}_r \) is a compact from \( \mathcal{A}(d, a) \). For a pair \( \mathbb{D}(x,t) \subset \mathbb{D}(x,t/r) \subset \mathbb{D}_r \) we set \( K_{x,t} := \mathbb{D}(x,t) \cap K \) and \( f_{x,t} = \sup_{\mathbb{D}(x,t)} f \).

Further, set \( f' = f_{x,t} - f \). In the proofs of Theorem 1.3 and 1.4 we use

Lemma 3.5

Let \( D_f(\lambda) = \mathcal{H}^d\{ y \in K_{x,t} : f'(y) \geq \lambda \} \) be the distribution function of \( f' \). Then

\[
D_f(\lambda) \leq \frac{(4et)^d a}{r^d d} e^{-\lambda d/((c(M_1 - M_2))}.
\]

(3.5)

Proof. The proof follows straightforwardly from the inequality of Theorem 1.2 where we choose \( \omega := D_f(\lambda) \). We leave the details to the reader. \qed

Proof of Theorem 1.3. First, we prove a local version of the theorem. Assume that \( K \subset \mathbb{D}_r \) is a compact from \( \mathcal{A}(d, a, b) \) and \( f, \mathbb{D}(x,t) \) satisfy conditions of Lemma 3.3. From inequality 3.3 it follows

\[
\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} f'd\mathcal{H}^d \leq \frac{1}{\mathcal{H}^d(K_{x,t})} \int_0^\infty D_f'(x)dx \leq \frac{1}{bt^d} \frac{c(M_1 - M_2)}{d} \frac{(4et)^da}{r^d d} = \frac{ca(4e)^d(M_1 - M_2)}{br^d d^2}.
\]

(3.6)

Now we have

\[
\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |f - f_{K_{x,t}}|d\mathcal{H}^d \leq \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |(f - f_{x,t}) - (f - f_{x,t})_{K_{x,t}}|d\mathcal{H}^d \leq \frac{2}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} f'd\mathcal{H}^d \leq \frac{2ca(4e)^d(M_1 - M_2)}{br^d d^2}.
\]

This gives the estimate of the BMO-norm in each ball \( K(x,t) = \mathbb{D}(x,t) \cap K \) with \( \mathbb{D}(x,t) \subset \mathbb{D}(x,t/r) \subset \mathbb{D}_r \). In the general case, we cover \( K \) by a finite number of open disks \( \mathbb{D}(x_i, R), i = 1, ..., N \) such that \( f \) is defined in the union of these disks, the set \( \bigcup_i^N \mathbb{D}(x_i, R/2) \) also covers \( K \) and any disk of radius \( \leq R/4 \) centered at a point of \( K \) belongs to one of \( \mathbb{D}(x_i, R/2) \). Then the estimate of the BMO-norm in
any $\mathbb{D}(x,t) \cap K$, $x \in K$, $t \leq R/4$, follows from Theorem 1.1 and inequality (3.6). To estimate BMO-norms for $\mathbb{D}(x,t) \cap K$ with $t \geq R/4$ we write

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |f - f_{K_{x,t}}| d\mathcal{H}^d \leq \frac{4^d}{bR^d} \int_{K_{x,t}} 2|f| d\mathcal{H}^d < C \int_{K} |f| d\mathcal{H}^d.$$ 

To complete the proof note that (3.5) implies $f_{K} |d\mathcal{H}^d < \infty$. □

We now formulate another corollary of Theorem 1.1. Assume that a subharmonic function $f$ defined on $\mathbb{C}$ satisfies

$$f(z) \leq c' + \log(1 + |z|) \quad (z \in \mathbb{C})$$

for some $c' \in \mathbb{R}$. Assume also that $S \in \mathcal{A}(d,a,b)$. Then $f|_S \in BMO(S, \mathcal{H}^d)$ and the BMO norm $|f|_S \leq \frac{C}{bR^d}$ with an absolute constant $c$.

**Proof.** For functions $f$ satisfying conditions of the corollary the Bernstein-Walsh inequality

$$\sup_{\mathbb{D}(x,q)} f \leq \log q + \sup_{\mathbb{D}(x,t)} f$$

holds for any $x \in \mathbb{C}$, $t \geq 0$, $q \geq 1$. (The proof is based on the classical Bernstein inequality for polynomials and the polynomial representation of the $L$-extremal function of the disk (see, e.g. [K]).) Then the estimate of the BMO-norm in $f|_{\mathbb{D}(x,t) \cap S}$ follows from inequality (3.6) with $r = 1/2$ and $M_1 - M_2 = \log 2$. □

**Proof of Theorem 1.4.** As in the proof of Theorem 1.3 we, first, consider a local version of the theorem. Assume that $K \subset \mathbb{D}_r$ is a compact from $\mathcal{A}(d,a,b)$ and $f, \mathbb{D}(x,t)$ satisfy conditions of Lemma 3.5. Denote $g_t = e^{-f''} = e^f/e^{f_{x,t}}$. Consider the distribution function $d_g(\lambda) := \mathcal{H}^d \{y \in K_{x,t} : g_t(y) \leq \lambda\}$. Then from the inequality of Lemma 3.5 for $D_f$ we deduce

$$d_g(\lambda) \leq \frac{(4et)^d a}{r^d d} (\lambda)^{d/(c(M_1 - M_2))}.$$ 

Let $g_s(s) = \inf \{\lambda : d_g(\lambda) \geq s\}$. From the previous inequality we obtain

$$g_s(s) \geq \left( \frac{st^d d}{(4et)^d a} \right)^{c(M_1 - M_2)/d}.$$ 

In particular,

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} g_t d\mathcal{H}^d = \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{0}^{\mathcal{H}^d(K_{x,t})} g_s(s) ds \geq \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{0}^{\mathcal{H}^d(K_{x,t})} \left( \frac{st^d d}{(4et)^d a} \right)^{c(M_1 - M_2)/d} ds \geq \frac{1}{1 + c(M_1 - M_2)/d} \left( \frac{r^d d b}{(4e)^d a} \right)^{c(M_1 - M_2)/d}.$$ 

Here we used inequality $\mathcal{H}^d(x,t) \geq bt^d$. Thus we obtain

$$\sup_{K_{x,t}} e^f \leq (1 + c(M_1 - M_2)/d) \left( \frac{(4e)^d a}{r^d d b} \right)^{c(M_1 - M_2)/d} \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^f d\mathcal{H}^d \quad (3.8)$$
which implies the required local reverse Hölder inequality. In the general case, we cover again $K$ by a finite number of open disks $D(x_i, R)$, $i = 1, ..., N$ such that $f$ is defined in the union of these disks, the set $\bigcup_{i=1}^{N} D(x_i, R/2)$ also covers $K$ and any disk of radius $\leq R/4$ centered at a point of $K$ belongs to one of $D(x_i, R/2)$. Then the reverse Hölder inequality of the form (3.8) holds for any $K_{x,t} = D(x,t) \cap K$, $x \in K$, $t \leq R/4$. Assume now that $t > R/4$ and set

$$m := \inf_{x \in K, t > R/4} \left\{ \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^f d\mathcal{H}^d \right\}.$$  

Then $m > 0$. Indeed, let $x_i, t_i > R/4$, be a sequence for which the expression on the right above converges to $m$. Without loss of generality we may assume also that $x_i$ tends to $x \in K$ and $t_i$ tends to $t \geq R/4$. Then there is $i_0$ such that for any $i \geq i_0$, the ball $K_{x_i,t_i}$ contains $K_{x,R/8}$. Note that $\sup_{K_{x,R/8}} e^f > 0$ because $K_{x,R/8}$ is not a polar set. Then inequality (3.8) applied to $K_{x,R/8}$ and the $d$-regularity of $K$ show that

$$m \geq \frac{C}{\mathcal{H}^d(K_{x,R/8})} \int_{K_{x,R/8}} e^f d\mathcal{H}^d > 0$$

for a constant $C := C(K)$. Finally, since $\sup_{K_{x,t}} e^f \leq M := \sup_K e^f < \infty$, inequality (3.8) for $t > R/4$ is valid with the constant $M/m$.

The proof of the theorem is complete. \hfill $\Box$

**Corollary 3.7** Assume that a subharmonic function $f$ defined on $\mathbb{C}$ satisfies

$$f(z) \leq c' + \log(1 + |z|) \quad (z \in \mathbb{C})$$

for some $c' \in \mathbb{R}$. Assume also that $S \in A(d,a,b)$. Then for $e^f|_S$ the reverse Hölder inequality (3.8) holds with the constant $C(\frac{a}{db})^{c_2/d}$, where $c_1, c_2$ are absolute positive constants.

**Proof.** The proof follows directly from the Bernstein-Walsh inequality (3.7) and Theorem 1.4. \hfill $\Box$

### 4. Multidimensional Case.

In this part we prove the results of section 1.2. Let $k$ be the number of zeros of $e^u$ in $B_{1/2}$ counting with their multiplicities (see definition in section 1.2). Here $u = \frac{1}{2} \log(|f_1|^2 + ... + |f_n|^2)$ satisfies inequalities (1.3). Below we estimate $k$ by $M, n, r$ only.

**Theorem 4.1** Given $H > 0, d > 0$ there exists a system of Euclidean balls such that

$$\sum r_j^d \leq \left( 2H \right)^d$$

where $r_j$ are radii of these balls, and

$$u(z) \geq -M + k \log \left( \frac{H}{e} \right)$$

everywhere outside these balls in $B_{1/2}$. 

Proof. Let \( \xi_1, \ldots, \xi_k \) be zeros of \( e^u \) in \( B_{1/2} \). We begin with the following

**Lemma 4.2**

\[
-M + \sum_{i=1}^{k} \log |z - \xi_i| \leq u(z) \quad (z \in B_{1/2}).
\]

**Proof.** Without loss of generality we may assume that each of zeros of the system \( F = 0 \) is of multiplicity 1. In fact, according to our assumptions image \( F(B_{1/2}) \subset \mathbb{C}^n \) is of complex dimension \( n \). In particular, by Sard’s theorem we can approximate \( F \) by maps \( F_c = F - c \) where \( c \) is a regular value of \( F \) close to 0 in \( \mathbb{C}^n \) and \( F_c^{-1}(0) \) is a family of zeros of multiplicity 1. Then we prove the lemma for \( \log |F_{n_i}| \) and going to the limit as \( c \to 0 \) obtain the required statement. Further, observe that \( u \) satisfies the complex Monge-Ampere equation everywhere in \( B_{1/2} \setminus \{\xi_1, \ldots, \xi_k\} \). In fact, \( u = \frac{1}{2} F^*U \), where \( U = \log(\sum_{i=1}^{n} |z|^2) \) satisfies the Monge-Ampere equation in \( \mathbb{C}^n \setminus \{0\} \). Since \( F \) is holomorphic, \( u \) satisfies the required equation on \( B_{1/2} \setminus F^{-1}(0) \).

We recall the following result from [BT]:

Assume that \( u_1, u_2 \) are continuous plurisubharmonic functions in a bounded domain \( D \) with a compact boundary \( K \). Assume also that \( u_1 \geq u_2 \) on \( K \) and \( u_1 \) satisfies the complex Monge-Ampere equation in an open neighbourhood of \( D \). Then \( u_1 \geq u_2 \) everywhere on \( D \).

Let \( g_n = -M + (1 + 1/n) \sum_{i=1}^{k} \log |z - \xi_i| \). Since by the assumption \( \xi_i \) is a simple zero of \( F \), for any \( i \) there is a ball \( B_{r_{n,i}} \) of small radius \( r_{n,i} \) centered at \( \xi_i \) such that \( g_n \leq u \) on its boundary. Without loss of generality we may assume that these balls are pairwise disjoint and \( r_{n,i} \to 0 \) as \( n \to \infty \). Moreover, by definition \( g_n \leq u \) on \( S_{1/2} \). Then according to the above maximal principle, \( g_n \leq u \) in \( B_{1/2} \setminus (\cup_i B_{r_{n,i}}) \). It remains to take the limit as \( n \to \infty \) to obtain by continuity \( g \leq u \) in \( B_{1/2} \) where \( g = -M + \sum_{i=1}^{k} \log |z - \xi_i| \).

The lemma is proved. \( \square \)

We now apply Theorem 2.3 to the function \( g \) with \( X = \mathbb{C}^n \), \( d(x, y) = |x - y| \) and \( \mu = \sum_{i=1}^{k} \delta_{\xi_i} \). Then we obtain

Given \( H > 0 \), \( d > 0 \), there exists a system of Euclidean balls such that

\[
\sum r_j^d \leq \frac{(2H)^d}{d}
\]

where \( r_j \) are radii of these balls, and

\[
g(z) \geq -M + k \log \frac{H}{e}
\]

everywhere outside these balls. Taking into account that \( u \geq g \) in \( B_{1/2} \) we obtain the required statement.

The proof of Theorem 4.1 is complete. \( \square \)

**Remark 4.3** In the inequality of Theorem 4.1, we can take any \( p \geq k \) instead of \( k \). We obtain this replacing inequality of Lemma 4.2 by

\[
-M + \frac{p}{k} \sum_{i=1}^{k} \log |z - \xi_i| \leq u(z) \quad (z \in B_{1/2})
\]

and then repeating the arguments of the proof of Theorem 4.1 applied to \( \frac{k}{p}u \).
We now estimate the number of zeros $k$.

**Lemma 4.4** Under assumptions of Theorem 4.1

\[ k \leq c(r, n)e^{(2n-1)M}. \]  

**Proof.** Let $h = \log(|z_1|^2 + ... + |z_n|^2)$. Consider the differential form $\omega = C(n)(\overline{\partial}h) \wedge (\overline{\partial} \partial h)^{n-1}$. Then we have $d\omega = 0$ on $\mathbb{C}^n \setminus \{0\}$ and for some $C(n) \in \mathbb{C}$ the Bochner-Martinelli formula is valid

\[ \phi(0) = \int_{\partial D} \phi(\xi) \omega. \]

Here $D$ is a domain containing 0 with a smooth boundary $\partial D$ and $\phi$ is holomorphic in an open neighbourhood of 0. Consider now the form $F^* \omega$ in $B_r$. Since $F$ is a holomorphic map, $F^* \omega = C(n)(\overline{\partial}F^*h) \wedge (\overline{\partial} \partial F^*h)^{n-1}$ and $d(F^*\omega) = 0$ on $B_{1/2} \setminus F^{-1}(0)$. In particular, by Stocks' theorem $\int_{S_{1/2}} F^*\omega = \sum_{i=1}^k \int_{S_i} F^*\omega$, where $S_i$ is a sphere of a small radius centered at $\xi_i$ and $S_{1/2} = \partial B_{1/2}$. Assume without loss of generality that 0 is a regular value of $F$, i.e., there are small neighbourhoods of $\xi_1, ..., \xi_k$ such that $F$ maps them biholomorphically to a ball centered at 0. Assume also that these neighbourhoods contain $S_1, ..., S_k$. Doing in each of these neighbourhoods a holomorphic change of variables and then applying the Bochner-Martinelli formula we obtain $\int_{S_{1/2}} F^*\omega = k$. Note that

\[ F^*\omega = C'(n) \frac{F^*\sigma}{|F|^{2n}}, \]

where $\sigma = \sum_{i=1}^n (-1)^{i-1} z_i d\overline{z}_1 \wedge ... \wedge \overline{z}_n \wedge dz = \overline{dz}_1 \wedge ... \wedge \overline{dz}_n$ (see [GH]). From here, Cauchy’s inequalities for the first derivatives of a holomorphic function and the estimate $|F| \geq e^{-M}$ on $S_{1/2}$ we finally get

\[ k \leq c(r, n)e^{(2n-1)M}. \]

**Remark 4.5** Unlike the one-dimensional case, the global Cartan’s estimate of Theorem 4.1 do not imply similar local estimates in each ball inside of $B_{1/2}$ (consider, e.g., function $\log(|z_1|^2 + |z_2|^4)$). However, under assumptions of Theorem 1.6 the multidimensional case is similar to one-dimensional.

**Proof of Theorem 1.6.** Assume that all zeros of $|F|^2$ are elliptic.

**Proposition 4.6** There is a constant $c = c(r, F) > 0$ such that for any ball $B(x, t)$ satisfying $B(x, t) \subset B(x, 4r^2 t) \subset B_{1/2}$ and any $H > 0, d > 0$ there is a system of balls such that

\[ \sum r_j^d \leq \frac{(8rtH)^d}{d}, \]

where $r_j$ are radii of these balls, and

\[ \log |F| \geq \sup_{B(x, t)} \log |F| + k \log \frac{cH}{e} \]

outside these balls in $B(x, t)$. 

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Proof. Set

\[ f_{x,t}(z) = \log |F(z)| - \sup_{B(x,4r^2t)} \log |F| \]

and

\[
H_{x,t} = \sup_{t \leq p \leq 2rt \in S(x,p)} \inf_{z \in S(x,t)} f_{x,t}(z),
\]

where \( S(x,p) := \{ z \in \mathbb{C}^n : |z - x| = p \} \). Let \( K := \{(x,t)\} \) be the set of centers and radii of balls satisfying conditions of Proposition 4.6.

**Lemma 4.7** \( C := \sup_{(x,t) \in K} H_{x,t} > -\infty \).

**Proof.** Assume that \( \{(x_n,t_n)\}_{n \geq 1} \subset K \) is a sequence satisfying

\[
\lim_{n \to \infty} H_{x_n,t_n} = C.
\]

Without loss of generality we may assume that \( B(x_n,t_n) \to B(x^*,t^*) \) in the Hausdorff metric. Further, consider the following cases.

1. \( t^* > 0 \). Then \( C = H_{x^*,t^*} \) by continuity. Clearly, \( C > -\infty \) because \( F \) has only finite number of zeros in \( B(x^*,2rt^*) \).

2. \( t^* = 0 \). If \( x^* \) is not a zero of \( F \) then \( H_{x^*,t^*} = 0 \) by continuity. Assume now that \( x^* \) is a zero of \( F \). Then by ellipticity of \( x^* \) the equality

\[
\log |F(x^* + t\omega)| = s \log t + \log f(\omega) + o(t) \quad (0 \leq t \leq t_0, \ \omega \in S^{2N-1}, s \leq k)
\]

holds for a sufficiently small \( t_0 \). Here \( 0 < \inf_{S^{2N-1}} f \leq \sup_{S^{2N-1}} f < \infty \). From this representation it follows that it suffices to check the lemma in this case for \( |F| = t \), \( t \leq t_0 \), and \( x^* = 0 \). Then a straightforward computation gets

\[
\sup_{B(x_n,2rt_n)} \log t = \log(|x_n| + 4r^2t_n)
\]

and so

\[
H_{x_n,t_n} = \begin{cases} 
\log \frac{|x_n| - t_n}{|x_n| + 4r^2t_n} & (0 \notin B(x_n,\frac{(2r+1)t_n}{2})); \\
\log \frac{2rt_n - |x_n|}{|x_n| + 4r^2t_n} & (0 \in B(x_n,\frac{(2r+1)t_n}{2})).
\end{cases}
\]

In the first case \( H_{x_n,t_n} \) is a monotonically increasing in \( x_n \) function. Thus

\[
H_{x_n,t_n} \geq H_{(2r+1)t_n/2,t_n} = \log \frac{2r - 1}{8r^2 + 2r + 1} > -\infty
\]

because \( r > 1/2 \). In the second case \( H_{x_n,t_n} \) is a monotonically decreasing in \( x_n \) function. This implies

\[
H_{x_n,t_n} \geq H_{(2r+1)t_n/2,t_n} = \log \frac{2r - 1}{8r^2 + 2r + 1} > -\infty.
\]
The proof of the lemma is complete. □

We proceed with the proof of Proposition 4.6. According to Lemma 4.7 there is a sphere $S(x, p)$, $t \leq p \leq 2rt$, such that $\inf_{S(x, p)} f_{x, t} \geq C > -\infty$. In addition, by conditions of the theorem $\sup_{B(x, 2rp)} f_{x, t} \leq 0$. We set $F'(z) = f_{x, t}(2zp)$, $|z| \leq r$. Then $F'$ satisfies inequalities (1.5). Applying Theorem 4.1 to $F'$ and going back to the ball $B(x, t) \subset B(x, p)$ we obtain

Given $H > 0, d > 0$, there exists a system of Euclidean balls such that

$$\sum r_j^d \leq \frac{(8rtH)^d}{d}$$

where $r_j$ are radii of these balls, and

$$\log |F| \geq \sup_{B(x, 4r^2t)} \log |F| + C + k \log \frac{H}{e} \geq \sup_{B(x, t)} \log |F| + k \log \frac{e^CH}{e}$$

outside these balls in $B(x, t)$. We used here that $C \leq 0$.

The proposition is proved. □

Proofs of Theorems 1.6 and 1.7. Proofs of these results repeat word-for-word proofs of Theorems 1.1, 1.3 and 1.4 and might be left to the reader. □

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