A PROOF OF THE KOTZIG–RINGEL–ROSA CONJECTURE

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Abstract. In graph theory, a graceful labeling of a graph with \( m \) edges is a labeling of its vertices with a subset of the integers ranging from 0 to \( m \) inclusive, such that no two vertices share a label, and each edge is uniquely identified by the absolute difference of labels assigned to its endpoints. The Kotzig–Ringel–Rosa conjecture asserts that every tree admits a graceful labeling. We provide a proof of this long standing conjecture via a functional reformulation of the conjecture and a composition lemma.

1. Introduction

We say that a graph \( G \) admits a decomposition into copies of some other graph \( H \) if the edges of \( G \) can be partitioned into edge-disjoint subgraphs isomorphic to \( H \). Graph decomposition problems have a rich history. In 1847, Kirkman studied decompositions of complete graphs \( K_n \) and showed that they can be decomposed into copies of a triangle if and only if \( n \) is congruent to one or three modulo six. Wilson [Wil75] generalized this result by completely characterizing complete graphs which can be decomposed into copies of any graph, for large \( n \). Graph and hypergraph decomposition problems form by now a vast topic with many results and conjectures. We refer the reader to extensive surveys [Woź04, Yap88, Gal09]. More recently, two breakthrough results obtained by Montgomery, Pokrovskiy, and Sudakov [MPS21] and independently obtained by Keevash and Staden [KS20] settle asymptotically in the affirmative, the long standing Ringel conjecture [Rin63] posed in 1963. The Ringel conjecture asserts that the complete graph \( K_{2n-1} \) can be decomposed into \( 2n-1 \) edge disjoint copies of any \( n \)-vertex tree. Both proofs constitute major tour de force in the application of the probabilistic method. Prior to these recent breakthroughs, the predominant approach to tackling Ringel’s conjecture had been via the much stronger Kotzig–Ringel–Rosa conjecture (or KRR conjecture for short) which dates back to 1964. The KRR conjecture asserts that vertices of any \( n \)-vertex tree \( T \) can be labelled injectively using \( n \) consecutive integers, such that the absolute difference of pairs of vertex labels spanning distinct edges are always distinct. Such a labeling is called a graceful labeling and the KRR conjecture is also known as the graceful labeling conjecture. This conjecture has attracted a lot of attention in the last 50 years but has only been proved for some special classes of trees, see e.g., [Gal09]. The most general result for this problem was obtained by Adamaszek, Allen, Grosu, and Hladký [AAGH20] who proved it asymptotically for trees with maximum degree \( O(n/\log n) \). The main motivation for studying graceful labelings had been to prove Ringel’s conjecture. Indeed, a graceful labelling map \( f : V(T) \to \{0, \cdots, n-1\} \), yields an embedding of \( T \) into \( \{0, \cdots, 2(n-1)\} \). Using addition modulo \( 2n-1 \), consider \( 2n-1 \) cyclic shifts \( T_0, \ldots, T_{2(n-1)} \) of \( T \), where the tree \( T_i \) is an isomorphic copy of \( T \) whose vertices are \( V(T_i) = \{f(v) + i \mid v \in V(T)\} \) and whose edges are

\[
E(T_i) = \{f(x) +i, f(y) +i \mid \{x,y\} \in E(T)\}.
\]

It is easy to check that if the map \( f \) gracefully labels \( T \) then the trees \( T_i \) are edge disjoint and therefore cyclically decompose \( K_{2n-1} \). Our main result is a proof of the KRR conjecture using a functional reformulation of the conjecture and a composition lemma.

Theorem 1. Every tree admits a graceful labeling.
Our theorem provides the first non-asymptotic result establishing the existence of a decomposition of $K_{2n-1}$ into any tree on $n$ consecutive vertices. Our result does not assume any restriction on the vertex degrees of the given tree. We describe in Section 2.1 the notation as well as some auxiliary enumeration results. Starting from section 2.2 we describe technical preliminaries required for the proof of our main result discussed in section 3.

2. Preliminaries

The KRR [Rin63, Ros66] conjecture, also known as the Graceful Labeling Conjecture (GLC), asserts that every tree admits a graceful labeling. For a detailed survey of the extensive literature on this problem, see [Gal09]. For the purposes of our discussion, we redefine graceful labelings of digraphs as vertex labelings which results in a bijection between vertex labels and \textit{induced absolute subtractive edge labels}. For notational convenience, let $\mathbb{Z}_n$ denote the set formed by the smallest $n$ consecutive non-negative integers i.e.

$$\mathbb{Z}_n := [0, n) \cap \mathbb{Z}.$$

The present discussion is based upon a functional reformulation of the GLC which exploits properties of the transformation monoid $\mathbb{Z}_n^n$ i.e. the monoid formed by functions having $\mathbb{Z}_n$ both as their domain and codomain. The binary operation of the monoid is the function composition operation.

2.1. Functional formulation. A rooted tree on $n > 0$ vertices is associated with a function

$$f \in \mathbb{Z}_n^n$$

subject to $|f^{(n-1)}(\mathbb{Z}_n)| = 1$,

where $\forall i \in \mathbb{Z}_n, f^{(0)}(i) := i$ and $\forall k \geq 0, f^{(k+1)} = f^{(k)} \circ f = f \circ f^{(k)}$.

In other words the function $f$ has a unique fixed point (the root) which is attractive over its domain.

\textbf{Definition 2.} To an arbitrary function $f \in \mathbb{Z}_n^n$ we associate a \textit{functional directed graph} denoted $G_f$ whose vertex set, and directed edge set are respectively

$$V(G_f) = \mathbb{Z}_n, \quad E(G_f) = \{(i, f(i)) : i \in \mathbb{Z}_n\}.$$

See Figure 2.1 for an example.

\textbf{Remark.} The automorphism group of $G_f$ is denoted Aut($G_f$) and defined such that

$$\text{Aut}(G_f) = \{\sigma \in \mathbb{Z}_n^n : \sigma f^{(-1)} = f\}.$$ 

\textbf{Definition 3.} Connected components of $G_f$ partition the vertex set into equivalence classes prescribed by an equivalence relation. A vertex pair $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n$ lies in the same connected component of $G_f$ if there exist non-negative integers $u, v$ such that

$$f^{(u)}(i) = f^{(v)}(j).$$

We denote by $G_f^\top$ the directed graph obtained by reversing the orientation of every edge in $G_f$. When $f$ is not bijective, the directed graph $G_f^\top$ is not a functional directed graph since some of its vertices have out-degree $\neq 1$. When $f \in \mathbb{Z}_n^n$ is subject to the fixed point condition $|f^{(n-1)}(\mathbb{Z}_n)| = 1$, the graph $G_f$ is a rooted, directed and $\mathbb{Z}_n$–spanning functional tree or a functional tree for short.

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1In the sense of going clockwise or counterclockwise along the unit circle. The vertex set of $K_{2n-1}$ is identified with $(2n-1)$-th roots of unity such that the $k$-th vertex of $K_{2n-1}$ is identified with $\exp\left(\frac{2\pi ik}{2n-1}\right)$. 


**Definition 4.** Let $G_f$ denote the functional directed graph of $f \in \mathbb{Z}_{n}^{\mathbb{Z}}$. Induced *subtractive edge labels* of $G_f$ correspond to integers occurring in the sequence $(f(i) - i : i \in \mathbb{Z}_n)$. The $i$-th member of the sequence equal to $f(i) - i$ is the induced subtractive edge label of the directed edge $(i, f(i)) \in E(G_f)$. In other words the set of induced subtractive edge labels of $G_f$ is

$$\{ f(i) - i : i \in \mathbb{Z}_n \}.$$ 

Induced absolute subtractive edge labels of $G_f$ correspond to absolute values of induced subtractive edge labels:

$$\{|f(i) - i| : i \in \mathbb{Z}_n \}.$$ 

**Definition 5.** The functional directed graph $G_f$ of $f \in \mathbb{Z}_{n}^{\mathbb{Z}}$ is *graceful* if there exist a bijection $\sigma \in S_n \subset \mathbb{Z}_{n}^{\mathbb{Z}}$ such that

$$(2.2) \quad \{|\sigma f(i) - \sigma(i)| : i \in \mathbb{Z}_n \} = \mathbb{Z}_n.$$ 

Otherwise when no such bijection $\sigma$ exist, the functional directed graph $G_f$ is *ungraceful*. Finally, if $\sigma$ can be chosen to be the identity permutation (denoted $\text{id}$), then $G_f$ is *gracefully labeled*.

Note that when $G_f$ is gracefully labeled, the set of induced subtractive edge labels of the bi–directed loop-graph $G_f^- \cup G_f$ is equal to $-\mathbb{Z}_n \cup \mathbb{Z}_n$. For instance, the graph $G_f$ of the function $f \in \mathbb{Z}_{6}^{\mathbb{Z}}$ defined by

$$f(0) = 0, f(1) = 3, f(2) = 3, f(3) = 0, f(4) = 0, f(5) = 0.$$ 

depicted in Figure 2.1 is a gracefully labeled functional tree.

The edge set of $G_f$ is

$$E(G_f) = \{(0,0), (1,3), (2,3), (3,0), (4,0), (5,0)\}.$$ 

More generally, the sequence of $\tau$–*induced edge labels* of the functional directed graph $G_f$ where $f \in \mathbb{Z}_{n}^{\mathbb{Z}}$ are defined with respect to an arbitrary $\tau \in \mathbb{Z}_{n}^{\mathbb{Z}} \times \mathbb{Z}_{n}$, as

$$(\tau(i, f(i)) : i \in \mathbb{Z}_n).$$ 

For a given $\tau \in \mathbb{Z}_{n}^{\mathbb{Z}} \times \mathbb{Z}_{n}$, the digraph $G_f$ of $f \in \mathbb{Z}_{n}^{\mathbb{Z}}$ is $\tau$–*Zen* if there exist a bijection $\sigma \in \mathbb{Z}_{n}^{\mathbb{Z}}$ such that

$$\{ \tau(\sigma(i), \sigma f(i)) : i \in \mathbb{Z}_n \} = \mathbb{Z}_n.$$ 

In particular, if $\tau$ is chosen such that

$$\tau(i,j) = |j-i|, \forall (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

then $\tau$–Zen graphs are graceful graphs and vice versa. Let $S_n \subset \mathbb{Z}_{n}^{\mathbb{Z}}$ denote the symmetric group acting on members of $\mathbb{Z}_n$ in other words $S_n$ denotes the subset of all bijective functions in $\mathbb{Z}_{n}^{\mathbb{Z}}$. The following graceful expansion theorem describes a necessary and sufficient condition on $f \in \mathbb{Z}_{n}^{\mathbb{Z}}$ to ensures that $G_f$ is graceful.

**Theorem 6** (Graceful Expansion Theorem). Let $\text{id} \in S_n$ denote the identity element and let $\varphi$ denote the involution $(n - 1 - \text{id}) \in S_n$. The graph $G_f$ of $f \in \mathbb{Z}_{n}^{\mathbb{Z}}$ is graceful if and only if there exist a nonempty permutation subset $\mathcal{G}_f \subset S_n$ as well as a corresponding sign function $s_f \in \{-1,0,1\}^{\mathcal{G}_f \times \mathbb{Z}_n}$ such that

$$(2.3) \quad f(i) = \sigma_i^{-1}(\varphi^t) \left( \varphi^t \sigma_i \cdot ( -1)^t \cdot s_f(\gamma, \sigma_i) \cdot \gamma \sigma_i \right), \quad \forall i \in \mathbb{Z}_n, \gamma \in \mathcal{G}_f \text{ and } t \in \{0,1\},$$

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (0) at (0,0) {0};
  \node (1) at (1,1) {1};
  \node (2) at (1,-1) {2};
  \node (3) at (0,-1) {3};
  \node (4) at (-1,1) {4};
  \node (5) at (-1,-1) {5};
  \draw (0) -- (1);
  \draw (0) -- (2);
  \draw (0) -- (3);
  \draw (0) -- (4);
  \draw (0) -- (5);
\end{tikzpicture}
\caption{$f(0) = 0, f(1) = 3, f(2) = 3, f(3) = 0, f(4) = 0, f(5) = 0$}
\end{figure}
for some $\sigma_i \in S_n$.

**Proof.** We prove the claim by starting from the premise that $G_f$ is graceful. We then derive the graceful expansion of $f$ described in equation (2.3) via a sequence of reversible steps. Thereby establishing both the forward and the backward claim. Recall that the premise that $G_f$ is graceful is equivalent to the assertion that there exist a permutation representative $\sigma$ of some coset of $\text{Aut}(G_f)$ for which we have

$$\{|\sigma f(j) - \sigma(j)| : j \in \mathbb{Z}_n\} = \mathbb{Z}_n.$$ 

Thus establishing the existence of bijective map $\gamma$ from vertex labels to induced absolute subtractive edge labels:

$$\gamma(i) = |\sigma f \sigma^{-1}(i) - i|, \forall i \in \mathbb{Z}_n.$$ 

Choosing $\sigma$ from distinct cosets of $\text{Aut}(G_f)$ subject to equation (2.2) may result in distinct permutations $\gamma$. We emphasize the dependence of the permutation $\sigma$ on the permutation $\gamma$ by writing $\sigma_i$. 

Accounting for the involution symmetry, we write: for all $i \in \mathbb{Z}_n$ and $t \in \{0,1\}$

$$\left|\varphi^t \sigma_i f \sigma^{-1}_i (i) - \varphi^t (i) \right| = \gamma(i), \forall i \in \mathbb{Z}_n.$$ 

Removing the absolute value on the left hand side of the equation immediately above introduces the sign function $\mathfrak{s}_f \in \{-1,0,1\}^{G_f \times \mathbb{Z}_n}$ on the right hand side. We write

$$\left(\varphi^t \sigma_i f \sigma^{-1}_i (i) - \varphi^t (i) \right) = (-1)^t \cdot \mathfrak{s}_f (\gamma, i) \cdot \gamma(i)$$

$$\iff \varphi^t \sigma_i f \sigma^{-1}_i (i) = \varphi^t (i) + (-1)^t \cdot \mathfrak{s}_f (\gamma, i) \cdot \gamma(i)$$

$$\iff f(i) = \sigma^{-1}_i \varphi^t \left(\varphi^t \sigma_i (i) + (-1)^t \cdot \mathfrak{s}_f (\gamma, \gamma_i) \cdot \gamma\gamma_i (i)\right)$$

as claimed. \hfill \Box

In equation (2.3), the bijection $\gamma$ which maps vertex labels to induced absolute subtractive edge labels is called a *permutation basis* of the graceful expansion. For each $i \in \mathbb{Z}_n$, the integer $\gamma(i)$ is the induced absolute subtractive edge label of the directed edge emanating from vertex $i$ in the gracefully labeled graph $G_{\sigma, f \sigma^{-1}}$. The parameter $t \in \{0,1\}$ in the graceful expansion of $f$ described in equation (2.3) accounts for the complementary labeling symmetry expressed by the equality

$$(\varphi \sigma, f \sigma^{-1}) \varphi^{-1} (i) = (-1) (\sigma f \sigma^{-1} (i) - i), \forall i \in \mathbb{Z}_n.$$

**Example 7.** Consider the function

$$f \in \mathbb{Z}_4 \text{ s.t. } f(i) = \begin{cases} 0 & \text{if } i = 0, \\ i - 1 & \text{otherwise} \end{cases} \forall i \in \mathbb{Z}_4,$$

$$\varphi(i) = 3 - i, \forall i \in \mathbb{Z}_4.$$ 

Let $G_f \subset S_4$ be such that

$$G_f = \{\gamma, \gamma'\} \text{ such that }$$

$$\begin{align*}
\gamma(0) &= 0 & \gamma'(0) &= 3, \\
\gamma(1) &= 2 & \gamma'(1) &= 1, \\
\gamma(2) &= 1 & \gamma'(2) &= 0, \\
\gamma(3) &= 3 & \gamma'(3) &= 2.
\end{align*}$$

$$\mathfrak{s}_f = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases} \forall i \in \mathbb{Z}_4.$$
the corresponding sign assignments are specified by

\[
\sigma_f : G_f \times \mathbb{Z}_4 \rightarrow \{-1, 0, 1\} \text{ such that }
\begin{align*}
\sigma_f (\gamma, 0) &= 0 \quad \sigma_f (\gamma', 0) = 1 \\
\sigma_f (\gamma, 1) &= 1 \quad \sigma_f (\gamma', 1) = 1 \\
\sigma_f (\gamma, 2) &= -1 \quad \sigma_f (\gamma', 2) = 0 \\
\sigma_f (\gamma, 3) &= -1 \quad \sigma_f (\gamma', 3) = -1 
\end{align*}
\]

Finally representatives for distinct cosets of \( \text{Aut}(G_f) \) are \( \sigma \gamma \) and \( \sigma \gamma' \) defined such that

\[
\begin{align*}
\sigma \gamma (0) &= 0 \quad \sigma \gamma' (0) = 2 \\
\sigma \gamma (1) &= 3 \quad \sigma \gamma' (1) = 1 \\
\sigma \gamma (2) &= 1 \quad \sigma \gamma' (2) = 3 \\
\sigma \gamma (3) &= 2 \quad \sigma \gamma' (3) = 0 
\end{align*}
\]

We easily check the validity of the only two possible graceful expansions of \( f \) prescribed with respect to permutation bases \( \gamma \) and \( \gamma' \) defined for all \( t \in \{0, 1\} \) and for all \( i \in \mathbb{Z}_4 \) by

\[
\sigma_{\gamma}^{-1} \varphi^{(t)} \left( \varphi^{(t)} \sigma \gamma (i) + (-1)^t \cdot \sigma_f (\gamma, \sigma \gamma (i)) \cdot \gamma \sigma \gamma (i) \right) = f (i) = \sigma_{\gamma'}^{-1} \varphi^{(t)} \left( \varphi^{(t)} \sigma \gamma' (i) + (-1)^t \cdot \sigma_f (\gamma', \sigma \gamma' (i)) \cdot \gamma' \sigma \gamma' (i) \right).
\]

**Definition 8.** Functional directed graphs \( G_f, G_g \) of \( f, g \in \mathbb{Z}_n^\mathbb{Z}_n \), differ from one another by *fixed point swaps* if

\[
\left\{ \begin{array}{l}
\{i, f (i)\} : i \in \mathbb{Z}_n \\
i \neq f (i)
\end{array} \right\} = \left\{ \begin{array}{l}
\{i, g (i)\} : i \in \mathbb{Z}_n \\
i \neq g (i)
\end{array} \right\}.
\]

Let non–isomorphic graphs \( G_f \) and \( G_g \) be both connected and graceful. If \( G_f \) differs from \( G_g \) by swapping fixed points, then we devise from distinct graceful expansions of \( g \) distinct graceful expansions of \( f \) and vice versa. Incidentally, the set of permutation bases for graceful expansions of \( f \) bijectively maps onto the set of permutation bases for graceful expansions of \( g \). In particular, every graceful relabeling of a connected graceful graph \( G_f \) admits a unique gracefully labeled swapped fixed point counterpart whose loop edge is located at the vertex labeled 0. Consequently, to characterize permutation bases of functions whose functional digraphs have no single vertex component, it suffices to characterize permutation bases which fix 0. It is easy to see that a permutation \( \gamma \) subject to \( \gamma (0) = 0 \), is a permutation basis for some graceful expansion if and only if

\[
(2.5) \quad \forall i \in \mathbb{Z}_n \setminus \{0\}, (\{i - \gamma (i), i + \gamma (i)\} \cap \mathbb{Z}_n) \neq \emptyset \Leftrightarrow \begin{cases} 
\gamma (i) \leq i \\
\text{or} \\
\gamma (i) \leq (n - 1) - i 
\end{cases} \quad \forall i \in \mathbb{Z}_n.
\]

Note that given such a basis \( \gamma \) it is possible that for an input \( i \in \mathbb{Z}_n \), both conditions

\[
\gamma (i) \leq i \quad \text{and} \quad \gamma (i) \leq (n - 1) - i
\]

simultaneously hold. In that case \( \gamma \) is a permutation basis for two or more functions in \( \mathbb{Z}_n^\mathbb{Z}_n \). We now state and prove a result which characterizes permutation bases of graceful expansions for members of \( \mathbb{Z}_n^\mathbb{Z}_n \) whose graphs have no single vertex component when \( n > 2 \).

**Theorem 9.** There are exactly \( \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lceil \frac{n - 1}{2} \right\rceil \) distinct permutations which fix 0, and occur as permutation bases for graceful expansions of members of \( \mathbb{Z}_n^\mathbb{Z}_n \).

The proof of this claim follows as a corollary of theorem [6]. Let \( f \in \mathbb{Z}_n^\mathbb{Z}_n \) be subject to \( f (0) = 0 \) and such that \( G_f \) is already gracefully labeled thereby simplifying the graceful expansion to the setting where \( \sigma \gamma = \text{id} \) (the identity permutation). Our argument focuses on the row addition setup described below where \( t \in \{0, 1\} \). The setup stems from the graceful expansion theorem.
Mutually exclusive choices for the entry of the second row with magnitude (2.7)

Following previous assignments (accounting thus far for the three choices made for column entries of the second row of magnitudes (2.6)), there are four mutually exclusive choices for a column entry whose magnitude equals $n - 3$. Mutually exclusive choices (not accounting for the two previously made

\begin{align*}
\varphi^{(t)} (0) & \quad \cdots \quad \varphi^{(t)} (i) & \quad \cdots & \quad \varphi^{(t)} (n - 1) & \quad \text{Row 1} \\
(-1)^i \cdot s (\gamma, 0) \cdot \gamma (0) & \quad \cdots & \quad (-1)^i \cdot s (\gamma, i) \cdot \gamma (i) & \quad \cdots & \quad (-1)^i \cdot s (\gamma, n - 1) \cdot \gamma (n - 1) & \quad \text{Row 2} \\
= \varphi^{(t)} f (0) & \quad \cdots & \quad \varphi^{(t)} f (i) & \quad \cdots & \quad \varphi^{(t)} f (n - 1) & \quad \text{Row 1 + Row 2}
\end{align*}

Proof. For a permutation basis $\gamma$ which fixes 0, observe that $f (0) = 0 \Leftrightarrow s (\gamma, 0) = 0$. This implies that $f (n - 1) = 0$ and $G_{f}$ has no isolated vertex component. In a graceful expansion whose permutation basis $\gamma$ fixes 0, there is a unique choice for $s_{f} (\gamma, n - 1) \cdot \gamma (n - 1)$. That choice is

\[ s_{f} (\gamma, n - 1) \cdot \gamma (n - 1) = - (n - 1) \Leftrightarrow f (n - 1) = 0. \]

Following this assignment, there are two mutually exclusive choices for a column entry on the second row (of the row addition setup) whose absolute value equals $(n - 2)$. These mutually exclusive choices yield corresponding mutually exclusive assignments

\begin{equation}
(2.6)
\begin{cases}
    s_{f} (\gamma, 1) \cdot \gamma (1) & = n - 2 \quad \Leftrightarrow \quad f (1) = n - 1 \\
    \text{or} & \\
    s_{f} (\gamma, n - 2) \cdot \gamma (n - 2) & = - (n - 2) \quad \Leftrightarrow \quad f (n - 2) = 0
\end{cases}
\end{equation}

Following previous assignments (accounting thus far for column entries on the second row whose magnitudes are respectively $(n - 1)$ and $(n - 2)$), there are three mutually exclusive choices for the column entry on the second row whose magnitude is $(n - 3)$. Mutually exclusive choices (not accounting for the choice made for the entry whose magnitude is equal to $(n - 2)$) yield mutually exclusive assignments

\begin{equation}
(2.7)
\begin{cases}
    s_{f} (\gamma, 1) \cdot \gamma (1) & = n - 3 \quad \Leftrightarrow \quad f (1) = n - 2 \\
    \text{or} & \\
    s_{f} (\gamma, 2) \cdot \gamma (2) & = n - 3 \quad \Leftrightarrow \quad f (2) = n - 1 \\
    \text{or} & \\
    s_{f} (\gamma, n - 3) \cdot \gamma (n - 3) & = - (n - 3) \quad \Leftrightarrow \quad f (n - 3) = 0 \\
    \text{or} & \\
    s_{f} (\gamma, n - 2) \cdot \gamma (n - 2) & = - (n - 3) \quad \Leftrightarrow \quad f (n - 2) = 1
\end{cases}
\end{equation}

However either the first or the last assignment displayed in Eq. (2.7) is not possible per the previous choice made for the column entry of the second row whose magnitude equals $(n - 2)$ as described in Eq. (2.6). Therefore there are three mutually exclusive choices left for the column entry in the second row whose magnitude equals $(n - 2)$ as described in Eq. (2.6). Similarly, following the third assignment (accounting thus far for the three choices made for column entries of the second row of magnitudes $(n - 1)$, $(n - 2)$ and $(n - 3)$), there are four mutually exclusive choices for a column entry of the second row whose magnitude equals $(n - 3)$). Mutually exclusive choices for the entry of the second row with magnitude $(n - 4)$ (not accounting for the two previously made
choices for entries of the second row having magnitudes \((n - 2)\) and \((n - 3)\) yield mutually exclusive assignments

\[
\begin{align*}
\mathcal{G}_f (\gamma, 1) \cdot \gamma (1) & = n - 4 \iff f (1) = n - 3 \\
\mathcal{G}_f (\gamma, 2) \cdot \gamma (2) & = n - 4 \iff f (2) = n - 2 \\
\mathcal{G}_f (\gamma, 3) \cdot \gamma (3) & = n - 4 \iff f (3) = n - 1 \\
\mathcal{G}_f (\gamma, n - 4) \cdot \gamma (n - 4) & = -(n - 4) \iff f (n - 4) = 0 \\
\mathcal{G}_f (\gamma, n - 3) \cdot \gamma (n - 3) & = -(n - 4) \iff f (n - 3) = 1 \\
\mathcal{G}_f (\gamma, n - 2) \cdot \gamma (n - 2) & = -(n - 4) \iff f (n - 2) = 2
\end{align*}
\]

(2.8)

Two of the six possible assignments described in equation (2.8) are not possible per the previous assignments made for column entries of magnitudes \((n - 2)\) and \((n - 3)\) as described by equation (2.6) and equation (2.7). Therefore there are four mutually exclusive choices left for the column entry in the second row whose magnitude equals \((n - 4)\). The argument proceeds accordingly in a similar vein all the way down to mutually exclusive choices for the column entry in the second row whose magnitude equals \(\left\lfloor \frac{n-2}{2} \right\rfloor\). These choices for the partial assignment accounts for the \(\left\lfloor \frac{n-2}{2} \right\rfloor!\) factor in the claim. Note that for each choice made in this partial assignment, thus far the corresponding output of the sign function is uniquely determined. The remaining unassigned integers whose magnitudes ranges from 1 to \(\left\lfloor \frac{n-2}{2} \right\rfloor\) can be arbitrarily permuted among the remaining unassigned column entries of the second row. Thus accounting for the remaining \(\left\lfloor \frac{n-2}{2} \right\rfloor!\) factor in the claim. 

The proof argument establishes that a permutation \(\gamma\) which fixes 0, can be a permutation basis for at most \(2\left\lfloor \frac{n-2}{2} \right\rfloor\) distinct members of \(\mathbb{Z}_n^2\) whose graphs are gracefully labeled. This upper bound is sharp when the permutation basis is set to \(\gamma = \text{id}\).

**Definition 10.** \(\text{GrL}(G_f)\) denotes the largest subset of distinct gracefully labeled functional directed graphs isomorphic to \(G_f\). More formally we write

\[
\text{GrL}(G_f) := \left\{ G_{\theta f \theta^{-1}} : \theta \text{ is a representative of a coset in } S_n/\text{Aut}(G_f) \text{ and } \mathbb{Z}_n = \{[\theta f \theta^{-1}(i) - i] : i \in \mathbb{Z}_n\} \right\}
\]

Theorem (9) yields a toy model illustration of the **composition lemma**. We discuss the composition lemma in more detail shortly. For now it suffices to say that the idea of the composition lemma is to relate graceful expansions of \(f\) to graceful expansions of some non-trivial iterate of \(f\). As illustration, consider the setting where \(G_f\) is graceful and \(f \in S_n \subset \mathbb{Z}_n^2\) i.e. \(f\) is bijective.

**Proposition 11.** Let \(f \in S_n\) and let \(o_f\) denote the order of \(f\) i.e. the LCM of cycle lengths occurring in \(G_f\). The iterate \(f^{(o_f - 1)}\) admits a graceful expansion if and only if the original function \(f\) admits a graceful expansion.

**Proof.** We proceed from the premise that \(f\) admits a graceful expansion and derive via a sequence of reversible steps a graceful expansion for \(f^{(o_f - 1)}\). Assume without loss of generality that \(G_f\) is gracefully labeled thereby simplifying the graceful expansion to the setting where \(\sigma, \gamma = \text{id}\). By theorem (9) the graceful expansion of \(f\) is of the form

\[
f (i) = \varphi^{(t)} \left( \varphi^{(t)} (i) + (-1)^t \cdot s_f (\gamma, i) \cdot \gamma (i) \right), \forall i \in \mathbb{Z}_n \text{ and } t \in \{0, 1\}.
\]
\[ \varphi^{(t)} (i) = \varphi^{(t)} f (i) + (-1)^{t+1} \cdot \mathfrak{s}_f (\gamma, i) \cdot \gamma (i), \]

\[ \varphi^{(t)} f^{(o_f-1)} (i) = \varphi^{(t)} (i) + (-1)^{t+1} \cdot \mathfrak{s}_f (\gamma, f^{(o_f-1)} (i)) \cdot \gamma f^{(o_f-1)} (i), \]

\[ f^{(o_f-1)} (i) = \varphi^{(t)} \left( \varphi^{(t)} (i) + (-1)^{t+1} \cdot \mathfrak{s}_f (\gamma, f^{(o_f-1)} (i)) \cdot \gamma f^{(o_f-1)} (i) \right), \]

\[ f^{(o_f-1)} (i) = \varphi^{(t)} \left( \varphi^{(t)} (i) + (-1)^t \cdot \mathfrak{s}_f (\gamma f^{(o_f-1)}, i) \cdot \gamma f^{(o_f-1)} (i) \right), \]

where

\[ \mathfrak{s}_f (\gamma f^{(o_f-1)}, i) = -\mathfrak{s}_f (\gamma, f^{(o_f-1)} (i)), \forall i \in \mathbb{Z}_n. \]

This completes the proof. \( \square \)

Accounting for the complementary labeling symmetry described in equation (2.4) and applying the argument used in the proof of proposition (11), to each non-trivial directed cycle occurring in \( G_f \), it is easy to see that for all

\[ f \in \mathcal{S}_n \cup \left\{ g : \mathbb{Z}_n^2 : \left| g^{(n-1)} (\mathbb{Z}_n) \right| = 1 \right\}, \]

\(|\text{GrL} (G_f)|\) is a multiple of \( 2^{(\text{number of connected components in } G_f)} \).

**Example 12.** The GLC is easily verified for the family of functional directed graphs of identically constant functions in \( \mathbb{Z}_n^2 \) i.e. the family of functional stars. Functional stars are functional directed graphs of identically constant functions. For instance take

\[ f : \mathbb{Z}_n \to \mathbb{Z}_n \]

such that

\[ f (i) = 0, \forall i \in \mathbb{Z}_n. \]

We see that the functional directed graph of \( f \) is gracefully labeled. Furthermore for all \( n > 1 \) we have

\[ \text{GrL} (G_f) = \left\{ G_f, G_{(n-1-\text{id}) f (n-1-\text{id})^{-1}} \right\}. \]

### 2.2. Preliminaries.

Recall that univariate polynomial notions such as the LCM and the GCD do not generally extend to multivariate polynomials. However we describe special settings where the notion of LCM extends to multivariate polynomials. Let polynomials \( F, H \in \mathbb{Q} [x_0, \ldots, x_{n-1}] \) split into irreducible multilinear factors

\[ F (x_0, \ldots, x_{n-1}) = \prod_{0 \leq i < m} (P_i (x_0, \ldots, x_{n-1}))^\alpha_i, \quad H (x_0, \ldots, x_{n-1}) = \prod_{0 \leq i < m} (P_i (x_0, \ldots, x_{n-1}))^{\beta_i}. \]

In the factorization above assume that \( \{ \alpha_i, \beta_i : 0 \leq i < m \} \subset \mathbb{Z}_{\geq 0} \) and more importantly that each factor \( P_i (x) \) is a \( \mathbb{Q} \)-linear combination of variables \( x_0, \ldots, x_{n-1} \) of the form

\[ P_i (x_0, \ldots, x_{n-1}) = \sum_{j \in \mathbb{Z}_n} a_{i,j} x_j, \text{ where } \left\{ a_{i,j} : 0 \leq i < m \quad 0 \leq j < n \right\} \subset \mathbb{Q}. \]

Additionally, assume for each \( k \in \mathbb{Z}_n \), and each factor \( P_i \) when viewed as a univariate polynomial in \( x_k \) (with coefficients from the ring \( \mathbb{Q} [x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}] \)) has no common roots with any other factor in \( \{ P_j : 0 \leq j \neq i < m \} \) in the field of
fractions $Q(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1})$, in other words the resultant in the variable $x_k$ given by

$$\prod_{0 \leq u < v < m} \left( \sum_{t \in \mathbb{Z}_n \setminus \{k\}} \frac{a_{u,t}}{a_{v,k}} x_t - \sum_{s \in \mathbb{Z}_n \setminus \{k\}} \frac{a_{u,s}}{a_{u,k}} x_s \right)$$

does not vanish identically. In this restricted setting we can extend the notion of LCM and GCD to multivariate polynomials $F$ and $H$ as follows

$$\text{LCM}(F, H) = \prod_{0 \leq i < m} (P_i(x_0, \cdots, x_{n-1}))^{\max(\alpha_i, \beta_i)},$$

and

$$\text{GCD}(F, H) = \prod_{0 \leq i < m} (P_i(x_0, \cdots, x_{n-1}))^{\min(\alpha_i, \beta_i)}.$$

For convenience we adopt the falling factorial notation :

$$(x)^\underline{m} := \prod_{i \in \mathbb{Z}_n} (x - i).$$

Also for an arbitrary polynomial $P(x_0, \cdots, x_{m-1}) \in \mathbb{Q}[x_0, \cdots, x_{m-1}]$ and $g \in \mathbb{Z}_n^m$, we denote by $P(g)$ the evaluation of $P$ at the sequence $(x_i = g(i) : i \in \mathbb{Z}_m)$ i.e.

$$P(g) := P(g(0), \cdots, g(i), \cdots, g(m-1)).$$

**Proposition 13.** Every $H \in \mathbb{Q}[x_0, \cdots, x_{m-1}]$ admits a quotient-remainder expansion of the form

$$H(x_0, \cdots, x_{m-1}) = \sum_{\ell \in \mathbb{Z}_m} q_\ell(x_0, \cdots, x_{m-1})(x_\ell)^\underline{m} + \sum_{g \in \mathbb{Z}_n^m} H(g) \prod_{i \in \mathbb{Z}_m} \left( \prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \frac{x_i - j_i}{g(i) - j_i} \right),$$

where $q_\ell(x_0, \cdots, x_{m-1}) \in \mathbb{Q}[x_0, \cdots, x_{m-1}]$ for all $\ell \in \mathbb{Z}_m$.

**Proof.** We prove the claim by induction on $m$ (the number of variables). The claim in the base case $m = 1$, is the assertion that for all $n \geq 1$, $H(x_0) \in \mathbb{Q}[x_0]$ admits an expansion of the form

$$H(x_0) = q(x_0)(x_0)^\underline{m} + r(x_0),$$

where $r(x_0)$ is a polynomial of degree less then $n$ called the remainder. Since the remainder $r(x_0)$ is of degree at most $(n - 1)$ it is completely determined via Lagrange interpolation on $n$ distinct evaluation points as follows

$$H(x_0) = q(x_0)(x_0)^\underline{m} + \sum_{g \in \mathbb{Z}_n^m} H(g(0)) \prod_{j \in \mathbb{Z}_n \setminus \{g(0)\}} \left( \frac{x_0 - j}{g(0) - j} \right).$$

Therefore, the claim holds in the base case. Note that the same argument, including Lagrange interpolation, applies to univariate polynomials whose coefficients lies in a polynomial ring.

For the induction step, assume as induction hypothesis that the claim holds for all $m$–variate polynomials $F \in \mathbb{Q}[x_0, \cdots, x_{m-1}]$ namely assume that

$$F = \sum_{\ell \in \mathbb{Z}_m} t_\ell(x_0, \cdots, x_{m-1})(x_\ell)^\underline{m} + \sum_{g \in \mathbb{Z}_n^m} F(g) \prod_{j_k \in \mathbb{Z}_n \setminus \{g(k)\}} \left( \prod_{j \in \mathbb{Z}_n \setminus \{g(k)\}} \frac{x_k - j_k}{g(k) - j_k} \right).$$
We now show that the hypothesis implies that the claim also holds for all \((m+1)\)-variate polynomials with rational coefficients. Let \(H \in \mathbb{Q}[x_0, \cdots, x_m]\) be viewed as a univariate polynomial in \(x_m\) whose coefficients lie in the polynomial ring \(\mathbb{Q}[x_0, \cdots, x_m]\).

Invoking the Quotient–Remainder Theorem and Lagrange interpolation over this ring, we have

\[
H(x_0, \cdots, x_m) = q(x_0, \cdots, x_m) (x_m)^m + r(x_0, \cdots, x_m),
\]

where \(r(x_0, \cdots, x_m) \in (\mathbb{Q}[x_0, \cdots, x_{m-1}])[x_m]\) is of degree at most \(n-1\) in the variable \(x_m\). We write

\[
r(x_0, \cdots, x_m) = \sum_{k \in \mathbb{Z}_n} a_k(x_0, \cdots, x_{m-1}) (x_m)^k,
\]

to justify that \(a_k(x_0, \cdots, x_{m-1}) \in \mathbb{Q}[x_0, \cdots, x_{m-1}]\) for all \(k \in \mathbb{Z}_n\), observe that

\[
\left(\begin{array}{c}
0 \\
\vdots \\
u \\
n-1
\end{array}\right) \cdot 
\left(\begin{array}{c}
a_0(x_0, \cdots, x_{m-1}) \\
\vdots \\
a_u(x_0, \cdots, x_{m-1}) \\
a_{n-1}(x_0, \cdots, x_{m-1})
\end{array}\right) = 
\left(\begin{array}{c}
H(x_0, \cdots, x_{m-1}, 0) \\
\vdots \\
H(x_0, \cdots, x_{m-1}, u) \\
H(x_0, \cdots, x_{m-1}, n-1)
\end{array}\right),
\]

where

\[
\det\left(\text{Vandermonde} \begin{pmatrix} 0 \\ \vdots \\ u \\ n-1 \end{pmatrix}\right) = \prod_{k \in \mathbb{Z}_n} k! \neq 0.
\]

Thus the coefficients of the remainder \(r(x_0, \cdots, x_m)\) from the ring \(\mathbb{Q}[x_0, \cdots, x_{m-1}]\) obtained as entries of the unique solution vector

\[
\begin{pmatrix}
a_0(x_0, \cdots, x_{m-1}) \\
\vdots \\
a_u(x_0, \cdots, x_{m-1}) \\
a_{n-1}(x_0, \cdots, x_{m-1})
\end{pmatrix} = \left(\text{Vandermonde} \begin{pmatrix} 0 \\ \vdots \\ u \\ n-1 \end{pmatrix}\right)^{-1} \cdot 
\left(\begin{array}{c}
H(x_0, \cdots, x_{m-1}, 0) \\
\vdots \\
H(x_0, \cdots, x_{m-1}, u) \\
H(x_0, \cdots, x_{m-1}, n-1)
\end{array}\right).
\]

Equivalently, via Lagrange interpolation we write

\[
H = q_m(x_0, \cdots, x_m)(x_m)^m + \sum_{g(m) \in \mathbb{Z}_n} H(x_0, \cdots, x_{m-1}, g(m)) \prod_{j \in \mathbb{Z}_n \setminus \{g(m)\}} \left(\frac{x_m - j_m}{g(m) - j_m}\right).
\]

Applying the induction hypothesis to \(m\)-variate polynomials in \(\{H(x_0, \cdots, x_{m-1}, g(m)) : g(m) \in \mathbb{Z}_n\}\) yields the desired claim. 
\(\square\)
Definition 14. For an arbitrary $H \in \mathbb{Q}[x_0, \cdots, x_m]$, the canonical representative of the congruence class of $H$ modulo the ideal generated by $\{(x_i)^b : i \in \mathbb{Z}_m\}$ denoted

$$H \mod \{(x_i)^b : i \in \mathbb{Z}_m\},$$

is the unique polynomial of degree at most $(n-1)$ in each variable whose evaluations over the integer lattice $\mathbb{Z}_n^m$ i.e. $(\mathbb{Z}_n)^m$ matches evaluations of $H$ on the same lattice. Thus the canonical representative of

$$H \mod \{(x_i)^b : i \in \mathbb{Z}_m\},$$

is

$$\sum_{g \in \mathbb{Z}_n^m} H(g) \prod_{k \in \mathbb{Z}_m} \left( \prod_{j_k \in \mathbb{Z}_n \setminus \{g(k)\}} \frac{x_k - j_k}{g(k) - j_k} \right). \tag{2.9}$$

The quotient–divisor part associated with the congruence class

$$H \mod \{(x_i)^b : i \in \mathbb{Z}_m\},$$

is the polynomial

$$H - \sum_{g \in \mathbb{Z}_n^m} H(g) \prod_{k \in \mathbb{Z}_m} \left( \prod_{j_k \in \mathbb{Z}_n \setminus \{g(k)\}} \frac{x_k - j_k}{g(k) - j_k} \right).$$

We see that the quotient–divisor part vanishes identically on the lattice $\mathbb{Z}_n^m$.

Thus the canonical representative of $H \in \mathbb{Q}[x_0, \cdots, x_m]$ is obtained via Lagrange interpolation over evaluation points

$$\{(g, H(g)) : g \in \mathbb{Z}_n^m\}.$$ 

Alternatively the canonical representative is obtained as the final remainder devised by performing Euclidean divisions irrespective of the order with which we perform the division by divisors successively taken from univariate polynomials $\{(x_i)^b : i \in \mathbb{Z}_n\}$. This follows from the fact that generators $\{(x_i)^b : i \in \mathbb{Z}_n\}$ for the corresponding ideal form a Groebner basis.

The next result recursively applies at each iteration the Quotient–Remainder Expansion Theorem (13) to the quotient–divisor part of the expansion to express the input polynomial as a $\mathbb{Q}$–linear combination of Lagrange basis polynomials. For an arbitrary $g \in \mathbb{Z}_n^m$, let the associated Lagrange basis polynomial be

$$L_g(x_0, \cdots, x_i, \cdots, x_m-1) := \prod_{u \in \mathbb{Z}_m} \left( \prod_{v_u \in \mathbb{Z}_n \setminus \{g(u)\}} \frac{x_u - v_u}{g(u) - v_u} \right).$$

It follows from the diagonality criterion prescribed for all $(f, g) \in \mathbb{Z}_n^m \times \mathbb{Z}_n^m$ by

$$L_f(f) = 1 \text{ and } L_f(g) = 0 \text{ when } f \neq g,$$

that Lagrange basis polynomials $\{L_f(x_0, \cdots, x_m) : f \in \mathbb{Z}_n^m\}$ are linearly independent.

Corollary 15. Let $P \in \mathbb{Q}[x_0, \cdots, x_m]$ be a polynomial of degree at most $d \geq n$ in any of its variables $x_0, \cdots, x_m$ then

$$P(x_0, \cdots, x_n) = \sum_{0 \leq k \leq d-n} \left( \sum_{g_k \in \mathbb{Z}_n^{m+k}} b_{g_k} L_{g_k}(x_0, \cdots, x_n) \right),$$

where for all $t < d - n$,

$$b_{g_0} = P(g_0) \text{ and } b_{g_{t+1}} = P(g_{t+1}) - \sum_{0 \leq k \leq t} \left( \sum_{g_k \in \mathbb{Z}_n^{m+k}} b_{g_k} L_{g_k}(g_{t+1}) \right).$$
Proposition 16. The directed graph $G_f$ of $f \in \mathbb{Z}_n^2$ is graceful if and only if

$$0 \neq \text{LCM} \left( \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq u < v < n} \left( (x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2 \right) \right) \mod \{ (x_k)^2 : k \in \mathbb{Z}_n \}$$

Proof. The LCM in the assertion is well defined since

$$\prod_{0 \leq u < v < n} \left( (x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2 \right) = \prod_{0 \leq u < v < n} \left( x_{f(v)} - x_v - x_{f(u)} + x_u \right) \left( x_{f(v)} - x_v + x_{f(u)} - x_u \right).$$

We see that the second input to the LCM corresponds to a product of $\{-2, -1, 0, 1, 2\}$–linear combination of variables. The first Vandermonde determinant factor

$$\prod_{0 \leq u < v < n} (x_j - x_i)$$

vanishes whenever two distinct variables are assigned the same vertex label from $\mathbb{Z}_n$. Similarly, the second Vandermonde determinant factor

$$\prod_{0 \leq u < v < n} \left( (x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2 \right)$$

is determined.
vanishes whenever two distinct edges are assigned the same induced absolute subtractive edge label from $\mathbb{Z}_n$. Consider the following multiple of the LCM polynomial

$$F_f(x_0, \cdots, x_{n-1}) = \prod_{0 \leq i < j < n} (x_j - x_i) \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right).$$

It suffices to show that the functional directed graph $G_f$ of $f \in \mathbb{Z}_n^2$ is graceful if and only if

$$0 \not\equiv F_f(x_0, \cdots, x_{n-1}) \mod \{(x_k)^{\mathbb{Z}} : k \in \mathbb{Z}_n\}.$$

By proposition [13], the polynomial $F_f$ admits an expansion of the form

$$F_f(x_0, \cdots, x_{n-1}) = \sum_{\ell \in \mathbb{Z}_n} q_{\ell} (x_0, \cdots, x_{n-1}) (x_{\ell})^{\mathbb{Z}} + \sum_{g \in \mathbb{Z}_n^2} F_f(g) L_g (x_0, \cdots, x_{n-1}).$$

Note that for all $g \in \mathbb{Z}_n^2$

$$F_f(g) = \prod_{0 \leq i < j < n} (g(j) - g(i)) \left( (gf(j) - g(j))^2 - (gf(i) - g(i))^2 \right).$$

Hence $F_f(g)$ vanishes if $g \in \mathbb{Z}_n^2 \setminus \{0\}$ on the other hand if $\sigma \in S_n$ and $G_{\sigma(ff)}$ is not a gracefully labeled then $F_f(\sigma)$ also vanishes. Thus

$$F_f(x_0, \cdots, x_{n-1}) = \sum_{\ell \in \mathbb{Z}_n} q_{\ell} (x_0, \cdots, x_{n-1}) (x_{\ell})^{\mathbb{Z}} + \sum_{\sigma \in S_n} \sum_{0 \leq i < j < n} (\sigma(j) - \sigma(i)) \left( (\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2 \right) L_{\sigma} (x_0, \cdots, x_{n-1}).$$

Observe that for all $\gamma \in \{\sigma \in S_n : G_{\sigma(ff)} \in \text{GrL}(G_f)\}$ we have

$$\prod_{0 \leq i < j < n} (\gamma(j) - \gamma(i)) \left( (\gamma f(j) - \gamma(j))^2 - (\gamma f(i) - \gamma(i))^2 \right) \in \left\{ - \prod_{v \in \mathbb{Z}_n} \left( (v!)^2 \frac{(n - 1 + v)!!}{(2v)!} \right), \prod_{v \in \mathbb{Z}_n} \left( (v!)^2 \frac{(n - 1 + v)!!}{(2v)!} \right) \right\}. $$

We thus conclude that

$$0 \not\equiv F_f(x_0, \cdots, x_{n-1}) \mod \{(x_k)^{\mathbb{Z}} : k \in \mathbb{Z}_n\}.$$

if and only if $\emptyset \not\equiv \text{GrL}(G_f)$ as claimed. \hfill $\square$

We now briefly explain why the proposed polynomial construction is determinental. Let $V, G_f \in \mathbb{Q}[x_0, \cdots, x_{n-1}]^{n \times n}$ with entries given by

$$V[i, j] = (x_i)^j, \quad G_f[i, j] = (x_{f(j)} - x_j)^2 \text{ for all } 0 \leq i, j < n.$$ 

The matrices $F$ and $G_f$ are Vandermonde matrices whose determinants are well known and are respectively

$$\det(V) = \prod_{0 \leq i < j < n} (x_j - x_i),$$

and

$$\det(G_f) = \prod_{0 \leq i < j < n} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right).$$

By the multiplicative property of the determinant, we have

$$\det(V) \det(G_f) = \det(VG_f).$$
The entries of the matrix product $\mathbf{VG}_f$ are such that

$$\mathbf{VG}_f[i, j] = \frac{1 - \left( x_i \left( x_{f(j)} - x_j \right) \right)^n}{1 - x_i \left( x_{f(j)} - x_j \right)^2}, \forall 0 \leq i, j < n$$

and the polynomial of interest is

$$\det(\mathbf{VG}_f) = F_f(x_0, \ldots, x_{n-1}) = \prod_{0 \leq i < j < n} (x_j - x_i) \left( x_{f(j)} - x_j \right)^2 - \left( x_{f(i)} - x_i \right)^2).$$

We make redundant factors more apparent by factoring $\det(\mathbf{VG}_f)$ into products of linear combinations of the variables as follows

$$\det(\mathbf{VG}_f) = \prod_{0 \leq i < j < n} (x_j - x_i) \left( x_{f(j)} - x_j - x_{f(i)} + x_i \right) \left( x_{f(j)} - x_j + x_{f(i)} - x_i \right).$$

We invoke the LCM as a means of removing from $\det(\mathbf{VG}_f)$ redundant factors from the determinantal construction. For instance when $G_f$ is connected and contains no cycle other then the trivial cycle made by a loop edge and $f(u) \leq u$ for all $u \in \mathbb{Z}_n$, then redundant factors appear only when

$$f(i) = f(j) \quad \text{or} \quad d\left(i, f^{(2)}(i)\right) = 2,$$

where $d(u, v)$ denotes the non–loop edge distance separating vertex $u$ from vertex $v$ in $G_f$.

$$\text{LCM} \left( \det(\mathbf{V}), \det(\mathbf{G}_f) \right) = \text{LCM} \left( \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 < i, j < n} \left( x_{f(j)} - x_j \right)^2 - \left( x_{f(i)} - x_i \right)^2) \right) = \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{d(i, f^{(2)}(i)) = 2} (2x_{f(i)} - x_j - x_{f(i)} + x_i) \prod_{0 \leq i < j < n, f(i) = f(j)} (2x_{f(j)} - x_j - x_i) \times \prod_{d(i, f^{(3)}(i)) = 3} \left( x_{f(i)} - x_j \right)^2 - \left( x_{f(i)} - x_i \right)^2 \prod_{0 < i < j < n, d(i, j) \geq 3} \left( x_{f(j)} - x_j \right)^2 - \left( x_{f(i)} - x_i \right)^2).$$

Finally note that the canonical representative of $\det(\mathbf{VG}_f)$ is

$$\prod_{v \in \mathbb{Z}_n} \left( (n - 1) + v \right)! \sum_{\sigma \in \mathcal{S}_n} \prod_{i \in \mathbb{Z}_n} \left( \frac{x_i - j_i}{\sigma(i) - j_i} \right).$$

3. The transformation monoid $\mathbb{Z}_n^{\mathbb{Z}_n}$ and the composition lemma.

We briefly review basic properties of the transformation monoid $\mathbb{Z}_n^{\mathbb{Z}_n}$ relevant to our main result.

**Proposition 17.** For all $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$, we have

$$1 \leq \min_{\sigma \in \mathcal{S}_n} \left\{ \left\lfloor \sigma f \sigma^{-1}(i) \right\rfloor : i \in \mathbb{Z}_n \right\} \leq \rho_f + \begin{cases} 1 & \text{if } f(i) = i, \forall i \in \mathbb{Z}_n, \\ 2 & \text{else if } \exists i \in \mathbb{Z}_n \text{ s.t. } f(i) = i, \\ 0 & \text{otherwise} \end{cases}$$

where $\rho_f$ denotes the minimum number of non–loop edge deletions required in $G_f$ to obtain a subgraph which is a union of disjoint paths and two–cycles possibly having loop edges.
Proof. The lower bound is attained when the $n$ edges of $G_f$ are assigned the same induced absolute subtractive edge label. We justify the upper–bound by considering any one of the possible deletions of $\rho_f$ edges from $G_f$ to obtain a spanning union of disjoint paths. We then sequentially label vertices along each path starting from one endpoint and increasing by one the label for each vertex encountered as we move along the path towards the second endpoint. This procedure greedily maximizes the number of edges assigned the induced absolute subtractive edge label one. The only edges in the proposed relabeling of $G_f$ whose induced absolute subtractive edge labels possibly differ from one, are labels of the $\rho_f$ deleted non–loop edges as well as loop edges if any occurs in $G_f$.

Note that the lower bound is sharp when $f \in S_n$ is either the identity element or any other involution having no fixed points. The upper bound is sharp for any involution, or alternatively a functional directed graph made up of a single spanning directed cycle or alternatively a functional path.

**Proposition 18.** For all $f \in \mathbb{Z}_n^\mathbb{Z}$,

$$n \geq \max_{\sigma \in S_n} \left| \{\sigma f \sigma^{-1} (i) - i : i \in \mathbb{Z}_n\} \right| \geq \left\{ \begin{array}{ll} 1 & \text{if } \exists i \in \mathbb{Z}_n \text{ s.t. } f(i) = i \\ 0 & \text{otherwise} \end{array} \right\} \rho_f + \left| \{ (i, f(i)) : i \in \mathbb{Z}_n \text{ such that } i \neq f(i) \} \right|$$

where $\rho_f$ denotes the minimum number of non–loop edge deletions required in $G_f$ to obtain a subgraph which is a union of disjoint paths possibly having loop edges.

Proof. The upper bound follows from the observation $|E(G_f)| = n$. We justify the lower–bound by considering every possible deletion of $\rho_f$ edges from $G_f$ to obtain a union of disjoint paths, possibly having loop edges. We then sequentially label vertices along each path starting from one endpoint and alternating between largest and smallest unassigned label for each vertex encountered as we move along the path towards the second endpoint. This procedure greedily maximizes the number of distinct edge labels. The only edges in the proposed relabeling of $G_f$ whose induced absolute subtractive edge labels possibly repeat correspond to labels of deleted non–loop edges as well as duplicate loop edges if any occurs in $G_f$.

Note that the upper bound is sharp for a graceful functional directed graph. The lower bound is sharp for any involution or alternatively a functional directed graph made of a single spanning directed cycle i.e. a directed Hamiltonian cycle.

**Definition 19.** Let $P \in \mathbb{Q}[x_0, \ldots, x_{n-1}]$, we denote by $\text{Aut}(P)$ the stabilizer subgroup of $S_n \subset \mathbb{Z}_n^\mathbb{Z}$ associated with the polynomial $P$ defined with respect to permutations of variables $x_0, \ldots, x_{n-1}$. In other words $\text{Aut}(P)$ denotes the set of permutations of the variables which fixes $P$.

**Proposition 20** (Stabilizer subgroup). For an arbitrary $f \in \mathbb{Z}_n^\mathbb{Z}$, with at least one fixed point

$$\text{Aut} \left( \prod_{0 \leq i \leq n} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right) \right) = \begin{cases} S_n & \text{if } \left| \{(x_{f(i)} - x_i)^2 : i \in \mathbb{Z}_n\} \right| < n \\ \text{Aut} \left( G_f \cup G_f^\top \setminus \{(u, f(u)) : u = f(u)\} \right) & \text{otherwise} \end{cases}$$

where $G_f \cup G_f^\top \setminus \{(u, f(u)) : u = f(u)\}$ denotes the loopless bi–directed digraph which underlies the functional directed graph $G_f$.

Proof. For notational convenience let

$$p_f(x_0, \ldots, x_{n-1}) = \prod_{0 \leq i < j < n} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right)$$

If the underlying undirected graph of $G_f \cup G_f^\top$ has fewer than $n$ edges, then $\left| \{(x_{f(i)} - x_i)^2 : i \in \mathbb{Z}_n\} \right| < n$ and $p_f$ vanishes identically. Thus $\text{Aut}(p_f) = S_n$. So assume $\left| \{(x_{f(i)} - x_i)^2 : i \in \mathbb{Z}_n\} \right| = n$ and thus $p_f$ does not vanish identically. Observe that
The right action does not change the polynomial because it simply rearranges the factors as follows:

\[
\prod_{0 \leq i \neq j < n} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right) = \prod_{0 \leq i \neq j < n} \left( (x_{f^{-1}(j)} - x_{f^{-1}(j)})^2 - (x_{f^{-1}(i)} - x_{f^{-1}(i)})^2 \right).
\]

The need for this lemma was pointed out to the author by Doron Zeilberger on July 2nd 2022 which happens to coincide with his birthday.
Consequently, left action by any $\gamma \in S_n$ effects an action by conjugation on $f$ as follows

$$\prod_{0 \leq i < j < n} \left( (x_{\gamma f(j)} - x_{\gamma(j)})^2 - (x_{\gamma f(i)} - x_{\gamma(i)})^2 \right)^2 = \prod_{0 \leq i < j < n} \left( (x_{\gamma f\gamma^{-1}(j)} - x_{\gamma\gamma^{-1}(j)})^2 - (x_{\gamma f\gamma^{-1}(i)} - x_{\gamma\gamma^{-1}(i)})^2 \right)^2,$$

Thus, remainders of $(F_f(x_{\gamma(0)}, \cdots, x_{\gamma(n-1)}))^2$ and $(F_{\gamma f\gamma^{-1}}(x_0, \cdots, x_{n-1}))^2$ are equal. By the corollary 15, the polynomial $(F_f)^2$ admits an expansion of the form

$$(F_f(x))^2 = \sum_{0 < k \leq d+1-n} \left( \sum_{g_k \in \mathbb{Z}_{n+k}} b_{g_k} L_{g_k}(x) \right) + \sum_{\sigma \in S_n, G_{\sigma f\sigma^{-1}} \in \text{GrL}(G_f)} (F_f(\sigma))^2 L_{\sigma}(x),$$

where $n \leq d \leq 6\binom{n}{2}$. Thus the congruence identity relating $(F_f(x))^2$ to its canonical representative is

$$(F_f(x))^2 \equiv \sum_{\sigma \in S_n, G_{\sigma f\sigma^{-1}} \in \text{GrL}(G_f)} (F_f(\sigma))^2 L_{\sigma}(x) \mod \{(x_i)_{i \in \mathbb{Z}_n}\}.$$

The right hand side of the following congruence identities are expressions of the canonical representative of the corresponding congruence class

$$(F_f(x_{\gamma(0)}, \cdots, x_{\gamma(n-1)}))^2 \equiv \sum_{\sigma \in S_n, G_{\sigma f\sigma^{-1}} \in \text{GrL}(G_f)} (F_f(\sigma))^2 \prod_{u \in \mathbb{Z}_n} \left( \frac{x_{\gamma(u)} - v_u}{\sigma(u) - v_u} \right),$$

$$\equiv \sum_{\sigma \in S_n, G_{\sigma f\sigma^{-1}} \in \text{GrL}(G_f)} (F_f(\sigma))^2 L_{\sigma\gamma^{-1}}(x),$$

$$\equiv \sum_{\sigma \in S_n, G_{\sigma\gamma^{-1}\gamma\gamma^{-1}\sigma^{-1}} \in \text{GrL}(G_f)} (F_f(\sigma\gamma^{-1}))^2 L_{\sigma\gamma^{-1}}(x),$$

$$\equiv \sum_{\theta \in S_n, G_{\theta\gamma^{-1}\gamma^{-1}\theta^{-1}} \in \text{GrL}(G_f)} (F_f(\theta\gamma^{-1}))^2 L_{\theta}(x),$$

$$\equiv \sum_{\theta \in S_n, G_{\theta\gamma^{-1}\gamma^{-1}\theta^{-1}} \in \text{GrL}(G_f)} (F_{f\gamma^{-1}}(\theta))^2 L_{\theta}(x).$$
\[ \Rightarrow \text{Aut} \left( \sum_{g \in \mathbb{Z}_n^{2n}} (F_f(g))^2 L_g(x) \right) \supseteq \text{Aut} \left( (F_f)^2 \right) \]

By proposition (24), we have
\[ \text{Aut} \left( (F_f)^2 \right) = \text{Aut} \left( G_f \cup G_f^{-1} \setminus (r, r) \right). \]
We conclude that
\[ \text{Aut} \left( \sum_{g \in \mathbb{Z}_n^{2n}} (F_f(g))^2 L_g(x) \right) \supseteq \text{Aut} \left( G_f \cup G_f^{-1} \setminus (r, r) \right), \]
as claimed.

As a corollary, given
\[ f \in \left\{ h \in \mathbb{Z}_n^{2n} : h(i) = 0 \right\}, \]
such that \( |f^{-1}(\{f(n-1)\})| > 1 \) and
\[ f^{-1}(\{f(n-1)\}) = \{n-1, \ldots, n - |f^{-1}(\{f(n-1)\})|\}, \]
if \( F_f \neq 0 \mod \{(x_u)_{\mathbb{Z}^n} : u \in \mathbb{Z}_n\} \) then the transposition which exchanges the variables \( x_u \) with \( x_v \) where \( u, v \in f^{-1}(\{f(n-1)\}) \) necessarily lies in
\[ \text{Aut} \left( \sum_{g \in \mathbb{Z}_n^{2n}} F_f(g) L_g(x_0, \ldots, x_{n-1}) \right). \]
Furthermore, the support of non-vanishing Lagrange basis polynomials partitions into disjoint cosets of \( \text{Aut} \left( G_f \cup G_f^{-1} \setminus (0, 0) \right) \) as follows
\[ F_f(x) = \left( \sum_{g \in \mathbb{Z}_n^{2n}} F_f(g) L_g(x) \right) = \prod_{v \in \mathbb{Z}_n} \frac{\left( (v!)^2 (n-1+v)! \right)}{(2v)!} \sum_{\sigma \in S_n/\text{Aut}(G_f \cup G_f^{-1} \setminus (0, 0))} \sum_{G_{\sigma f^{-1}, \sigma^{-1}}} \text{sgn} \left( \left| (\sigma^{-1}) f_{\sigma^{-1}} \right| \circ \text{id} \right) L_{\sigma^{-1}} (x). \]
By the complementary labeling symmetry, if
\[ (n-1 - \text{id}) \circ \text{Aut} \left( G_f \cup G_f^{-1} \setminus (r, r) \right) \circ (n-1 - \text{id})^{-1} = \text{Aut} \left( G_f \cup G_f^{-1} \setminus (r, r) \right), \]
then
\[ \text{Aut} \left( \sum_{g \in \mathbb{Z}_n^{2n}} (F_f(g))^2 L_g(x) \right) \supseteq \text{Aut} \left( G_f \cup G_f^{-1} \setminus (r, r) \right) \cup \text{Aut} \left( G_f \cup G_f^{-1} \setminus (r, r) \right) \circ (n-1 - \text{id}). \]

**Example 22.** As an illustration take the function \( f \in \mathbb{Z}_4^{2n} \) such that
\[ f(0) = 0, f(1) = 0, f(2) = 1, f(3) = 2, \]
\[ (F_f(x_0, x_1, x_2, x_3))^2 = \prod_{0 \leq i < j < 4} (x_j - x_i)^2 \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right)^2. \]
We see that
\[ \text{Aut} \left( F_f(x_0, x_1, x_2, x_3) \right) = \{ \text{id}, \sigma_1 \}. \]
where the directed edge set of the functional digraph $G_{\sigma_1}$ is

$$E(G_{\sigma_1}) = \{(0,3), (1,2), (2,1), (3,0)\}.$$ 

We see that

$$\left( \sum_{g \in \mathbb{Z}_4^3} (F_f(g))^2 L_g(x) \right) = 74649600x_0^3x_1^3x_2^3x_3^3 - 335923200x_0^3x_1^3x_2^2x_3^2 - 335923200x_0^3x_1^3x_2x_3^3 + \cdots - 4031078400x_0x_2x_3 - 4031078400x_1x_2x_3,$$

and

$$\text{Aut}\left( \sum_{g \in \mathbb{Z}_4^3} (F_f(g))^2 L_g(x) \right) = \{\text{id}, \sigma_1, \sigma_2, \sigma_3\},$$

where the directed edge set of functional digraphs $G_{\sigma_2}$ and $G_{\sigma_3}$ are respectively

$$E(G_{\sigma_2}) = \{(0,1), (1,0), (2,3), (3,2)\}, \ E(G_{\sigma_3}) = \{(0,2), (1,3), (2,0), (3,1)\}.$$

The example above illustrates a setting where the automorphism group of $(F_f)^2$ i.e. $\text{Aut}((F_f)^2)$ is a proper subgroup of the automorphism group of its canonical representative i.e.

$$\text{Aut}\left( \sum_{g \in \mathbb{Z}_4^3} (F_f(g))^2 L_g(x) \right).$$

Given $P \in \mathbb{Q}[x_0, \cdots, x_{n-1}]$, recall that $P$ depends on the variable $x_u$ if the polynomial $\frac{\partial P}{\partial x_u}$ does not vanish identically. In other words the expanded form of $P$ features at least one monomial multiple of the variable $x_u$ whose coefficient does not vanish.

**Proposition 23.** Let $P(x_0, \cdots, x_{n-1}) \in \mathbb{Q}[x_0, \cdots, x_{n-1}]$ be dependent only on the subset of variables in

$$\{x_i : i \in S \subseteq \mathbb{Z}_n\},$$

If $P(x)$ is of degree at most $n-1$ in the said variables, then for any positive integer $m$ the canonical representative of

$$(P(x))^m \mod \{(x_i)^m : i \in S\}$$

can depend only on variables in the subset $\{x_i : i \in S\}$.

**Proof.** By our premise $P$ equals it own canonical representative i.e.

$$P(x) = \left( \sum_{g \in \mathbb{Z}_n^3} P(g) L_g(x) \right) = \sum_{g \in \mathbb{Z}_n^3} P(g) \prod_{i \in S} \prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left( \frac{x_i - j_i}{g(i) - j_i} \right).$$

$$\implies (P(x))^m = \left( \sum_{g \in \mathbb{Z}_n^3} P(g) \prod_{i \in S} \prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left( \frac{x_i - j_i}{g(i) - j_i} \right) \right)^m.$$
We see that \((P(x))^m\) depends only on variables in \(\{x_i : i \in S\}\). The canonical representative of \((P(x))^m\) is
\[
\sum_{g \in \mathbb{Z}_n^m} (P(g))^m L_g(x)
\]
By the proposition\(^1\)\(^3\) the canonical representative of \((P(x))^m\) can be obtained by reducing \((P(x))^m\) modulo algebraic relations
\[
\{(x_i)^m : i \in S\}.
\]
Accordingly, the canonical representative of \((P(x))^m\) is devised by repeatedly replacing into the expanded form of \((P(x))^m\) every occurrence of \((x_i)^n\) with \((x_i)^n - (x_i)^m\) for all \(i \in S\) until we obtain a polynomial of degree \( < n\) in each variable. The reduction procedure never introduces a variable in the complement of the set \(\{x_i : i \in S\}\). Therefore, the canonical representative of \((P(x))^m\) given by
\[
\left(\sum_{g \in \mathbb{Z}_n^m} (P(g))^m L_g(x)\right) = \sum_{g \in \mathbb{Z}_n^m} (P(g))^m \prod_{i \in S} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \frac{x_i - j_i}{g(i) - j_i}\right).
\]
depends only on variables in \(\{x_i : i \in S\}\) as claimed. \(\square\)

Lemma 24 (The monomial overlapping lemma). Let \(x\) denote the sequence of variables \((x_0, \ldots, x_{n-1})\). For any non-empty \(S \subseteq S_n\) such that \(a_\sigma \in \mathbb{Q} \setminus \{0\}\) for all \(\sigma \in S\), we have
\[
\sum_{\sigma \in S} a_\sigma L_\sigma(x) = \sum_{f \in \mathcal{M}_S} c_f \prod_{i \in \mathbb{Z}_n} x_i^{f(i)},
\]
where \(c_f \in \mathbb{Q} \setminus \{0\}\) and \(|f^{-1}(\{0\})| \leq 1\) for all \(f \in \mathcal{M}_S\). The right-hand side above expresses the left-hand side as a linear combination of monomials.

Proof. Stated otherwise, the present lemma asserts that every term in the expanded form is a multiple of at least \((n-1)\) distinct variables. Consider the Lagrange basis polynomial associated with any \(\sigma \in S:\)
\[
L_\sigma(x) = \prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right) \prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right) \prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right).
\]
On the right-hand side of the second equal sign immediately above, the univariate polynomial encompassed within the scope of third \(\prod\) indexed by \(j_\sigma^{-1}(0) \in \mathbb{Z}_n \setminus \{0\}\) has (in its expanded form) a non-vanishing constant term equal to one. However, the constant term vanishes within the expanded form of each univariate factors \(\prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \frac{x_i - j_i}{\sigma(i) - j_i}\) encompassed within the scope of the first \(\prod\) indexed by \(i \in \mathbb{Z}_n \setminus \{\sigma^{-1}(0)\}::
\[
L_\sigma(x) = \prod_{i \in \mathbb{Z}_n \setminus \sigma^{-1}(0)^{(0)} \setminus \sigma^{-1}(0)} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right) \frac{(x_\sigma^{-1}(0))^{n-1} + \cdots + (-1)^{n-1}(n-1)!}{(-1)^{n-1}(n-1)!},
\]
Observe that each summand in the expanded form of the Lagrange basis polynomial \(L_\sigma(x)\) above which is a non–vanishing monomial multiple of \(x_\sigma^{-1}(0)\) is a multiple of every variable in \(\{x_0, \ldots, x_{n-1}\}\) whereas by contrast every non–vanishing monomial which is not a multiple of \(x_\sigma^{-1}(0)\) is a multiple of every other variables i.e. variables in the set \(\{x_0, \ldots, x_{n-1}\} \setminus \{x_\sigma^{-1}(0)\}\). Applying the same argument to each \(\sigma \in S\) yields the desired claim. \(\square\)

We now state and prove the composition lemma.
Lemma 25 (The Composition Lemma). Let $n$ be a positive integer greater than 3. For all functions $f \in \mathbb{Z}_n^\mathbb{Z}$, if $|f^{(n-1)}(\mathbb{Z}_n)| = 1$ and $G_f$ has diameter $\geq 3$, then

$$\max_{\sigma \in S_n} \left\{ \left| \sigma f^{(2)} \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\} \leq \max_{\sigma \in S_n} \left\{ \left| \sigma f \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\}.$$

Proof. Assume without loss of generality $f$ lies in the semigroup

$$\left\{ h \in \mathbb{Z}_n^\mathbb{Z} : h(0) = 0 \right\}.$$

For we see that if $f \in \mathbb{Z}_n^\mathbb{Z}$ subject to $|f^{(n-1)}(\mathbb{Z}_n)| = 1$, does not lie in the semigroup, there exist a permutation $\sigma \in S_n$ of vertex labels in $G_f$ by which we devise $G_{\sigma f \sigma^{-1}}$ isomorphic to $G_f$ such that $\sigma f \sigma^{-1} \in \left\{ h \in \mathbb{Z}_n^\mathbb{Z} : h(0) = 0 \right\}.$

Thus functional directed graphs of members of the semigroup

$$\left\{ h \in \mathbb{Z}_n^\mathbb{Z} : h(0) = 0 \right\},$$

account for at least one member of every conjugacy class of functional trees. Observe that for any $f$ in the semigroup the iterate $f^{(2\log_2(n-1))}$ is the identically zero function. Hence, the existence of some function

$$g \in \left\{ h \in \mathbb{Z}_n^\mathbb{Z} : h(0) = 0 \right\}$$

such that $n > \max_{\sigma \in S_n} \left\{ \left| \sigma g \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\}$, implies the existence of some function

$$f \in \left\{ g^{(2^\kappa)} : 0 \leq \kappa < \lfloor \log_2(n-1) \rfloor \right\} \subset \left\{ h \in \mathbb{Z}_n^\mathbb{Z} : h(0) = 0 \right\},$$

such that

$$\max_{\sigma \in S_n} \left\{ \left| \sigma f^{(2)} \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\} > \max_{\sigma \in S_n} \left\{ \left| \sigma f \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\}.$$

Consequently if the purported claim of lemma 25 holds, then there can be no member $g$ of the semigroup for which

$$n > \max_{\sigma \in S_n} \left\{ \left| \sigma g \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\}.$$

We now proceed to prove a generalization of the desired claim. Assume without loss of generality that the vertex labeled $n - 1$ is at maximum edge distance from the vertex labeled 0 (the root node) in $G_f$. Also assume without loss of generality that vertices in the set $f^{-1}(\{ f(n - 1) \})$ (namely the vertex set made up of $n - 1$ and its sibling nodes in $G_f$) are assigned the largest possible labels from $\mathbb{Z}_n$ in other words

$$f^{-1}(\{ f(n - 1) \}) = \{ n - 1, n - 2, \ldots, n - |f^{-1}(\{ f(n - 1) \})| \} \quad \text{and} \quad f(n - |f^{-1}(\{ f(n - 1) \})|) = n - |f^{-1}(\{ f(n - 1) \})| - 1.$$
We show that
\[ g \in \left\{ h \in \mathbb{Z}_n^2 : h(i) = 0, \forall i \in \mathbb{Z}_n \setminus \{0\} \right\}. \]

We prove by contradiction the contrapositive claim \( \text{i.e.} \emptyset = \text{GrL}(G_f) \implies \text{GrL}(G_g) = \emptyset. \)

By construction the polynomial
\[ P_f(x) = \prod_{0 \leq i < j < n} (x_j - x_i) \times \prod_{0 \leq u < v \leq f(n - 1)} (x_{f(u)} - x_v + (-1)^t (x_{f(u)} - x_u)) \times \prod_{v \in f^{-1} \{f(n - 1)\}} (x_{f(v)} - x_v + (-1)^t (x_{f(u)} - x_u)), \]

differs only slightly from
\[ P_g(x) = \prod_{0 \leq i < j < n} (x_j - x_i) \times \prod_{0 \leq u < v \leq f(n - 1)} (x_{f(u)} - x_v + (-1)^t (x_{f(u)} - x_u)) \times \prod_{v \in f^{-1} \{f(n - 1)\}} (x_{f(v)} - x_v + (-1)^t (x_{f(u)} - x_u)), \]

We set up a variable \textbf{telescoping} within each binomial \( x_{f(v)} - x_v \) for all \( v \in f^{-1} \{f(n - 1)\} \) (associated with an iterated edge) as follows
\[ \frac{x_{f(v)} - x_v}{x_v} = \frac{x_{f(v)} - x_v}{x_v} + \frac{x_{f(v)} - x_{f(v)}}{x_{f(v)}}, \]
\[(x_{f(v)} - x_v) = (x_{f(v)} - x_{f(v)}) + (x_{f(v)} - x_v) = (x_{f(v)} - x_{f(n-1)}) + (x_{f(v)} - x_v),\]

the last equality immediately above results from the fact that \( f(v) = f(n-1) \) for all \( v \in f^{-1}([f(n-1)]) \). Thus

\[
P_g = \prod_{0 \leq i < j < n} (x_j - x_i) \times \prod_{0 \leq u < v \leq f(n-1)} \prod_{t \in \{0,1\}} \left( (x_{f(v)} - x_v + (-1)^t (x_{f(u)} - x_u)) \times \prod_{v \in f^{-1}([f(n-1)])} \prod_{t \in \{0,1\}} \left( (x_{f(v)} - x_{f(n-1)}) + (x_{f(v)} - x_v) + (-1)^t (x_{f(u)} - x_u) \right) \right).
\]

For notational convenience we write

\[
P_g = \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} \prod_{0 \leq u \leq f(n-1)} \prod_{t \in \{0,1\}} (b_{u,v,t} + a_{f(n-1)}) \times \prod_{v \in f^{-1}([f(n-1)])} \prod_{t \in \{0,1\}} (b_v - b_u + 0a_{f(n-1)}).
\]

(3.1)

where \( a_{f(n-1)} = (x_{f(0)} - x_{f(n-1)}) \), \( b_i = (x_{f(i)} - x_i) \) for all \( i \in f^{-1}([f(n-1)]) \) and

\[
b_{u,v,t} = (x_{f(u)} - x_v) + (-1)^t (x_{f(u)} - x_u), \quad v \in f^{-1}([f(n-1)]) \forall 0 \leq u \leq f(n-1) \quad t \in \{0,1\}.
\]

Invoking the multi-binomial identity on the two bichromatic factors of \( P_g \) immediately above yields equalities

\[
\prod_{v \in f^{-1}([f(n-1)])} \prod_{f(n-1) < u < v} (b_v + b_u + 2a_{f(n-1)}) = \left( \prod_{v \in f^{-1}([f(n-1)])} (b_v + b_u) + \sum_{r_{u,v} \in \{0,1\}} \prod_{0 = \prod_{r_{u,v}} \prod_{f(n-1) < u < v}} (b_v + b_u)^{r_{u,v}} (2a_{f(n-1)})^{1-r_{u,v}} \right).
\]
and

\[
\prod_{v \in f^{-1}(\{f(n-1)\})} (b_{u,v,t} + a_{f(n-1)}) = \\
0 \leq u \leq f(n-1) \\
t \in \{0,1\}
\]

\[
\left( \prod_{t \in \{0,1\}} b_{u,v,t} + \sum_{s_{u,v,t} \in \{0,1\}} \prod_{v \in f^{-1}(\{f(n-1)\})} (b_{u,v,t})^{s_{u,v,t}} \left( a_{f(n-1)} \right)^{1-s_{u,v,t}} \right) \\
0 = \prod_{s_{u,v,t}} 0 \leq u \leq f(n-1) \\
t \in \{0,1\}
\]

Substituting equalities immediately above into equation (3.1) yields an expression of \(P_g\) of the form

\[
P_g = P_f + R_{f,g}.
\]

The first part of the expansion is equal to \(P_f\) and collects the monochromatic red parts of the multi-binomial summation and is given by

\[
P_f = \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} ((x_{f(v)} - x_u)^2 - (x_{f(u)} - x_u)^2) \\
\prod_{v \in f^{-1}(\{f(n-1)\})} \left( \prod_{f(n-1) < u < v} (b_v - b_u) \times \right) \\
\left( \prod_{v \in f^{-1}(\{f(n-1)\})} (b_v + b_u) \right) \\
\prod_{t \in \{0,1\}} \prod_{v \in f^{-1}(\{f(n-1)\})} (b_{u,v,t}) \\
0 \leq u \leq f(n-1) \\
0 \leq u \leq f(n-1) \\
t \in \{0,1\}
\]
The second part denoted $R_{f,g}$ simply collects all other remaining multi-binomial summands and is given by
\[
R_{f,g} = \prod_{0 \leq i \leq f(n-1)} \left( \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} \left( (x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2 \right) \prod_{v \in f^{-1}(\{f(n-1)\}) f(n-1) < u < v} (b_v - b_u) \times \right)
\[
\left[ \prod_{v \in f^{-1}(\{f(n-1)\}) f(n-1) < u < v} (b_v + b_u) \right) \left( \sum_{s_{u,v,t} \in \{0,1\}} \prod_{0 \leq u \leq f(n-1) | t \in \{0,1\}} (b_{u,v})^{s_{u,v,t}} (a_{f(n-1)})^{1-s_{u,v,t}} \right) + \\
\left[ \prod_{t \in \{0,1\}} b_{u,v,t} \right) \left( \sum_{r_{u,v} \in \{0,1\}} \prod_{0 \leq u \leq f(n-1) | v \in f^{-1}(\{f(n-1)\})} (b_v + b_u)^{r_{u,v}} (2a_{f(n-1)})^{1-r_{u,v}} \right) + \\
\left[ \prod_{r_{u,v} \in \{0,1\}} \sum_{s_{u,v,t} \in \{0,1\}} \prod_{0 \leq u \leq f(n-1) | t \in \{0,1\}} (b_{u,v})^{s_{u,v,t}} (a_{f(n-1)})^{1-s_{u,v,t}} \times \\
\left[ \prod_{r_{u,v} \in \{0,1\}} \sum_{v \in f^{-1}(\{f(n-1)\}) f(n-1) < u < v} (b_v + b_u)^{r_{u,v}} (2a_{f(n-1)})^{1-r_{u,v}} \right] \right]
\]

The color scheme introduced here is meant to help track the location of telescoping variables. We now proceed with the main **contradiction argument**. Assume for the sake of establishing a contradiction that the contrapositive claim is false i.e. for some $f$ subject to conditions described in our premise, we have
\[
0 \equiv P_f \mod \{(x_i)_{n} : i \in \mathbb{Z}_n\} \quad \text{and} \quad 0 \not\equiv P_g \mod \{(x_i)_{n} : i \in \mathbb{Z}_n\}.
\]

Observe that every summand in $R_{f,g}$ is a multiple of a positive power of the binomial $(x_{f(2)_{n-1}} - x_{f(n-1)})$. We focus in particular on the summand within in $R_{f,g}$ which is a multiple of the largest possible power of the binomial $(x_{f(2)_{n-1}} - x_{f(n-1)})$. Namely the summand associated with binary exponent assignments
\[
s_{u,v,t} = 0, \forall \quad 0 \leq u \leq f(n-1) \quad \text{as well as} \quad r_{u,v} = 0, \forall \quad v \in f^{-1}(\{f(n-1)\}) f(n-1) < u < v.
\]

The said summand is
\[
c \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} \left( (x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2 \right) \left( \prod_{v \in f^{-1}(\{f(n-1)\}) f(n-1) < u < v} (b_v - b_u) \left( a_{f(n-1)} \right)^m, \right.
\]

\[
0 \equiv P_f \mod \{(x_i)_{n} : i \in \mathbb{Z}_n\} \quad \text{and} \quad 0 \not\equiv P_g \mod \{(x_i)_{n} : i \in \mathbb{Z}_n\}.
\]

Observe that every summand in $R_{f,g}$ is a multiple of a positive power of the binomial $(x_{f(2)_{n-1}} - x_{f(n-1)})$. We focus in particular on the summand within in $R_{f,g}$ which is a multiple of the largest possible power of the binomial $(x_{f(2)_{n-1}} - x_{f(n-1)})$. Namely the summand associated with binary exponent assignments
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\[
c \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} \left( (x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2 \right) \left( \prod_{v \in f^{-1}(\{f(n-1)\}) f(n-1) < u < v} (b_v - b_u) \left( a_{f(n-1)} \right)^m, \right.
\]
Let
\[ m = \left[ \begin{array}{c}
\{ v \in f^{-1}\{(f(n-1))\} \\
f(n-1) < u < v
\end{array} \right] + \left[ \begin{array}{c}
\{ v \in f^{-1}\{(f(n-1))\} \\
0 \leq u \leq f(n-1)
\end{array} \right] \text{ and } c = 2 \left[ \begin{array}{c}
\{ v \in f^{-1}\{(f(n-1))\} \\
f(n-1) < u < v
\end{array} \right].
\]

The said summand is thus given by
\[
c \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} ((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2) \prod_{v \in f^{-1}\{(f(n-1))\}} (x_u - x_v)^{x_{f(2)(n-1)} - x_{f(n-1)}}^m.
\]

It follows from the premise \( 0 \neq (P_g \mod \{(x_i)^2 : i \in \mathbb{Z}_n\}) \) that the canonical representative of the chosen summand is non-vanishing. Observe that the factor
\[
\prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} ((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2) \prod_{v \in f^{-1}\{(f(n-1))\}} (b_v - b_u)
\]
is common to every summand in \( R_{f,g} \). Factoring the common factor we write
\[
R_{f,g}(x) = \left( \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v \leq f(n-1)} ((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2) \prod_{v \in f^{-1}\{(f(n-1))\}} (b_v - b_u) \right) Q_{f,g}(x),
\]
where
\[
Q_{f,g} = \left( \prod_{v \in f^{-1}\{(f(n-1))\}} (b_v + b_u) \right) \left( \sum_{s_{u,v,t} \in \{0,1\}} \prod_{0 \leq u \leq f(n-1)} (b_{u,v,t})^{s_{u,v,t}} (a_{f(n-1)})^{1-s_{u,v,t}} \right) +
\]
\[
\left( \prod_{t \in \{0,1\}} b_{u,v,t} \right) \left( \sum_{r_{u,v} \in \{0,1\}} \prod_{0 \leq u \leq f(n-1)} (b_v + b_u)^{r_{u,v}} (2a_{f(n-1)})^{1-r_{u,v}} \right) +
\]
\[
\left( \prod_{s_{u,v,t} \in \{0,1\}} (b_{u,v,t})^{s_{u,v,t}} (a_{f(n-1)})^{1-s_{u,v,t}} \right) \times
\]
\[
\left( \prod_{r_{u,v} \in \{0,1\}} (b_v + b_u)^{r_{u,v}} (2a_{f(n-1)})^{1-r_{u,v}} \right).
\]

Let
\[
\Phi(g) := \sigma \in \{ \theta \in S_n : G_{\theta g} = 1 \} \in GrL(G_g).
\]
Observe that for each \( \sigma \in \Phi(g) \) there is a non-vanishing integer evaluation

\[
v_{\sigma} = \prod_{0 \leq i < j < n} (\sigma(j) - \sigma(i)) \prod_{0 \leq u < v \leq f(n-1)} \left( (\sigma(f)(v) - \sigma(v))^2 - (\sigma(f)(u) - \sigma(u))^2 \right) \prod_{v \in f^{-1}(\{f(n-1)\})} (\sigma(u) - \sigma(v)).
\]

More generally

\[
P_{g}(h) = R_{f,g}(h) = v_{h} Q_{f,g}(h) = \begin{cases} 
\text{sgn}(|h| h^{-1} - \text{id} \circ h) \prod_{0 \leq i < j < n} (j - i)(j^2 - i^2) & \text{if } h \in \Phi(g) \\
0 & \text{otherwise}
\end{cases}, \forall h \in \mathbb{Z}^n.
\]

Recall \( Q_{f,g} \) is the remaining factor of \( R_{f,g} \) excluding the common factor. Specifically, \( Q_{f,g} \) is a polynomial resulting from the sum over chromatic summands resulting from the multinomial expansions. Let us view \( Q_{f,g} \) as a sum over the same non-vanishing points, it is fixed by the said transposition on the sublattice \( \tau \Phi(\sigma) \). That is to say

\[
\tau = (g(n-1), v) \in \text{Aut} \left( \sum_{1 \leq s \leq |\Sigma|} \left( \sum_{\sigma \in \Phi(g)} v_{\sigma} \cdot Q_{f,g}^{[s]}(\sigma) \cdot L_{\sigma}(x_{Q}^{[s]}) \right) \right),
\]

for all \( v \in f^{-1}(\{f(n-1)\}) \). As a consequence of the multi-binomial expansion, the sum expressing \( Q_{f,g} \) features as one of its summand a unique monochromatic blue binomial summand, say \( Q_{f,g}^{[1]} \), given by

\[
Q_{f,g}^{[1]} = c (a f(n-1))^{m} = c (x_{f(n-1)} - x_{f(n-1)})^{m}.
\]

Thus,

\[
Q_{f,g}^{[1]} = c \sum_{\sigma \in \Phi(g)} (\sigma f^{(2)}(n-1) - \sigma f(n-1))^{m} \times \prod_{j_{f^{(2)}(n-1)} \in \mathbb{Z} \setminus \{\sigma f^{(2)}(n-1)\}} \left( \frac{x_{f^{(2)}(n-1)} - j_{f^{(2)}(n-1)}}{\sigma f^{(2)}(n-1) - j_{f^{(2)}(n-1)}} \right) \prod_{j_{f(n-1)} \in \mathbb{Z} \setminus \{\sigma f(n-1)\}} \left( \frac{x_{f(n-1)} - j_{f(n-1)}}{\sigma f(n-1) - j_{f(n-1)}} \right).
\]
Let us now focus on the action of a transposition on individual summands of some polynomial resulting from an arbitrary but fixed partition of its non-vanishing monomial terms. There are exactly three distinct ways that a candidate transposition of variables can lie in the automorphism group of a given polynomial. Assume that we reason about a particular summand denoted as $S$.

Option 1: The candidate transposition of a pair of variables fixes the chosen summand $S$. This occurs when $S$ is symmetric in the chosen pair of variables being transposed.

Option 2: The candidate transposition of the chosen pair of variables does not fix $S$ (i.e., Option 1 does not apply) but induces in turn a transposition which exchanges the chosen summand $S$ with some other summand from the partition say, $S'$. This occurs, for instance, if we consider the sum $S + S'$ where $S = (x_0)^2 x_1$ and $S' = x_0 (x_1)^2$. In this example, we see that transposition which exchanges variables $x_0$ with $x_1$ does not fix $S$, but it induces a transposition which exchanges the summand $S$ with the summand $S'$.

Option 3: The candidate transposition of a pair of variables neither fixes $S$ nor does it induce a transposition which exchanges $S$ with some other summand (i.e., neither Option 1 nor Option 2 applies). Instead, $S$ is such that a symmetry broadening cancellation occurs. Such a cancellation must involve interaction between the non-vanishing monomials within the monomial support of $S$ with the non-vanishing monomials within the support of other summands. Option 3 occurs, for instance, if we take $S = -x_1$ and $S' = x_0 + 2x_1$. We see that in this example neither Option 1 nor Option 2 applies when the candidate transposition is the transposition which exchanges variables $x_0$ with $x_1$. However $S + S' = x_0 + x_1$ is symmetric and thus admits the said transposition in its automorphism group. This fact is due to the symmetry broadening cancellation of like terms: $-x_1 + 2x_1$.

We see from the Monomial Support Lemma (24) that the monomial support of the canonical representative immediately above is not fixed by any transposition $\tau \in S_n$ which exchanges $x_{f(n-1)}$ with $x_v$ where $v \in f^{-1} (\{f (n - 1)\})$. This first observation accounts for Option 1. Also note that the canonical representative of the chosen summand does not exchange with the canonical representative of any other summands when we exchange $x_{f(n-1)}$ with $x_v$ where $v \in f^{-1} (\{f (n - 1)\})$. Since non–vanishing canonical representative of every other bi-chromatic summand in $R1_{f,g}$ depends on 3 or more variables. This second observation accounts for Option 2. We now account for Option 3 and show that there are no symmetry–broadening cancellations which adjoins $\tau$ to the automorphism group. By the monomial support Lemma, such a symmetry broadening cancellation can occur only for Lagrange bases

$$\prod_{j_{(2)}(n-1) \in \mathbb{Z}_n \setminus \{\sigma f(2)(n-1)\}} \frac{(x_{f(2)(n-1)} - j_{f(2)(n-1)})}{\sigma f(2)(n-1) - j_{f(2)(n-1)}} \prod_{\{j_{f(2)(n-1)} \in \mathbb{Z}_n \setminus \{\sigma f(n-1)\}\}} \frac{(x_{f(n-1)} - j_{f(n-1)})}{\sigma f(n-1) - j_{f(n-1)}}.$$

where $\sigma \in \sigma \in \Phi(G)$ is subject to $\sigma(n - 1) = 0$ and $G_f$ is such that $1 = |f^{-1}(\{f(n-1)\})|$. For that setting non-vanishing monomials occurring in the expanded form of said Lagrange bases summands possibly cancel out non-vanishing monomials occurring in the expanded form of Lagrange bases expressing canonical representative of bi-chromatic summands in $R1_{f,g}$ of the form

$$(b_{f(n-1),n-1,i}) \tau (a_{f(n-1)})^\sigma = (x_{f(n-1)} - x_{n-1}) + (-1)^\tau (x_{f(2)(n-1)} - x_{f(n-1)}) \tau (x_{f(2)(n-1)} - x_{f(n-1)})^\sigma.$$

Crucially, by the Monomial Support Lemma as previously mentioned such cancellations are restricted to permutations $\sigma \in \Phi(G)$ where $\sigma(n - 1) = 0$. This restriction breaks the complementary-labeling symmetry (24). Indeed by complementary labeling symmetry, the canonical representative is up to sign invariant to the involution prescribed by the map: $x_i \mapsto x_{n-1-i}$ for all $i \in \mathbb{Z}_n$. But the complementary labeling transformation maps any Lagrange basis associated with $\sigma \in \sigma \in \Phi(G)$ such that $\sigma(n - 1) = 0$ to different Lagrange basis associated $\sigma' \in \sigma \in \Phi(G)$ such that $\sigma'(n - 1) = n - 1$ and thus negates the symmetry broadening cancellations. For we see that such a symmetry broadening cancellations which adjoins $\tau$ to the automorphism group of the canonical representative of $R_{f,g}$ would break the complementary labeling symmetry. Thereby resulting in the contradiction $\tau \notin \text{Aut (Canonical Representative of } P_g\}$.

We conclude that the desired claim $\emptyset \neq \text{GrL}(G_g) \implies \text{GrL}(G_f) \neq \emptyset$, holds.
4. The Graceful Labeling Theorem

Equipped with the composition lemma, we settle in the affirmative the KRR conjecture as stated in theorem 1. In fact we prove that for all \( f \in \mathbb{Z}_n^2 \), the maximum number of distinct induced absolute subtractive edge labels occurring in a relabeling of the graph of the iterate \( f^{(g)} \) (where \( o_f \) is the order of \( f \) i.e. the LCM of cycle lengths occurring in \( G_f \)) is equal \( n + 1 \) minus the number of connected components occurring in \( G_{f^{(g)}} \). If \( G_f \) is connected and \( f \) has a fixed point, then \( o_f \) is equal to one and theorem 1 follows as a special case.

**Theorem 26 (The Graceful Labeling Theorem).** For all \( f \in \mathbb{Z}_n^2 \),

\[
 n + 1 - \left( \text{number of connected components in } G_{f^{(g)}} \right) = \max_{\sigma \in S_n} \left\{ \left| \sigma f^{(g)} \sigma^{-1} i - i \right| : i \in \mathbb{Z}_n \right\},
\]

where \( o_f \) denotes the order of \( f \) i.e. the LCM of directed cycle lengths occurring in \( G_f \).

**Proof.** The claim trivially holds when \( f \in S_n \), for in that setting \( o_f \) is the order of the permutation \( f \) and \( f^{(g)} = \text{id} \). Otherwise if \( f \not\in S_n \) it suffices to show that for all \( f \) subject to the fixed point condition \( |f^{(n-1)}(\mathbb{Z}_n)| = 1 \) the equality

\[
 n = \max_{\sigma \in S_n} \left\{ \left| \sigma f^{(n-1)} i - i \right| : i \in \mathbb{Z}_n \right\},
\]

holds. This latter claim follows by repeatedly iterating the composition lemma described in lemma 25. For we see that, given any such function \( f \), the iterate \( f^{(2^\log_2(n-1))} \) is identically constant. As pointed in example 12, graphs of identically constant functions are graceful. Thus completing the proof.

\[\square\]

**Appendix**

We explain here in more detail how the edges of a functional digraph \( G_f \) are determined by the polynomial construction. The discussion presented here is taken from [CCGH24] and is provided here for the benefit of the reader.

**Lemma 27 (Recovery Lemma).** For an arbitrary function \( f \in \mathbb{Z}_n^2 \), let

\[
p_f(x_0, \ldots, x_{n-1}) := \prod_{0 \leq i < j < n} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right).
\]

Suppose \( p_f(x_0, \ldots, x_{n-1}) \) is defined from some function \( f \in \mathbb{Z}_n^2 \), and \( p_f \) is not identically zero. If \( f \) has a fixed point, then \( G_f \) is connected. The function from \( S \) to \( \mathbb{Q}[x_0, \ldots, x_{n-1}] \) that assigns \( p_f \) to \( f \) is injective.

We show that each factor in a factorization of \( p_f \) is a quadrinomial (a linear combination of exactly four distinct variables), a trinomial (a linear combination of exactly three distinct variables), or a binomial (a linear combination of exactly two distinct variables), and analyze how each can occur. A factor \( x_{f(j)} - x_j - x_{f(i)} + x_i \) or \( x_{f(j)} - x_j + x_{f(i)} - x_i \) has the form \( a + b - c - d \) if it is a quadrinomial with \( a, b, c, d \) distinct if and only if \( |\{a, b, c, d\}| = 4 \). In this case both \( x_{f(j)} - x_j - x_{f(i)} + x_i \) and \( x_{f(j)} - x_j + x_{f(i)} - x_i \) are quadrinomials.

The expression \( a + b - c - d \) collapses to a binomial if \( |\{a, b\} \cap \{c, d\}| = 1 \) (note that \( |\{a, b\} \cap \{c, d\}| = 2 \) is impossible since \( p_f \) is not identically zero). Notice that \( a + b - c - d \) occurs in two forms in \( p_f \):

\[
\{a, b\} = \{x_{f(j)}, x_{f(i)}\}, \{c, d\} = \{x_j, x_i\} \quad \text{or} \quad \{a, b\} = \{x_{f(j)}, x_i\}, \{c, d\} = \{x_j, x_{f(i)}\}
\]
First consider the case that $f$ has a (unique) fixed point $u$. Then for each $j \neq 0$ we obtain two copies of the binomial $x_{f(j)} - x_j$ from
\[ \pm \left( (x_{f(j)} - x_j)^2 - (x_{f(a)} - x_a)^2 \right) = \pm (x_{f(j)} - x_j)^2 \]
with $+$ if $j > u$ and $-$ otherwise.

Now assume neither $i$ nor $j$ is fixed by $f$. A binomial-trinomial pair of factors arises from
\[ (x_{f(j)} - x_j + x_{f(i)} - x_i) \left( x_{f(j)} - x_j - x_{f(i)} + x_i \right) \]
when \( \{a, b\} = \{x_{f(j)}, x_{f(i)}\}, \{c, d\} = \{x_j, x_i\} \). Without loss of generality, we choose $j = f(i)$. This produces
\[ \pm (x_{f(2)(i)} - x_i) \left( x_{f(2)(i)} + x_i - 2x_{f(i)} \right) \]
Similarly, a binomial-trinomial pair of factors arises when \( \{a, b\} = \{x_{f(j)}, x_i\}, \{c, d\} = \{x_j, x_{f(i)}\} \), which implies $f(i) = f(j)$. Setting $i < j$, this produces
\[ (2x_{f(j)} - x_j - x_i)(x_i - x_j). \]

We have now described all possible ways binomial factors can occur in $p_f$. Furthermore, a trinomial factor of $p_f$ can only occur in a binomial-trinomial pair. Observe that in each binomial-trinomial pair, the trinomial has the form $\pm (2r - s - t)$ and the associated binomial is of the form $(s - t)$.

We now take a given polynomial $p_f$ that is not identically zero, with no information about $f$ except that except that $f \in \mathbb{Z}_n^2$. Define $h_f(x_0, \ldots, x_{n-1})$ to be the product of all the binomials that occur in binomial-trinomial pairs. That is, $s - t$ is a factor of $h_f$ if and only if $2r - s - t$ is a factor of $p_f$. Now define
\[ q_f(x_0, \ldots, x_{n-1}) = \frac{p_f(x_0, \ldots, x_{n-1})}{h_f(x_0, \ldots, x_{n-1})}, \]
which is a polynomial. Then $q_f$ has no binomial factors if and only if $f$ does not have a fixed point. Otherwise, $q$ has $2(n - 1)$ binomial factors, which occur in pairs: $(x_k - x_\ell)^2$. Then
\[ E \left( G_f \cup G_f^{-1} \setminus \{(0, 0)\} \right) = \left\{ \{k, \ell\} : (x_k - x_\ell)^2 \text{ is a factor of } q_f \right\}. \]

\[ \square \]

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