Quasilocal first law from local Lorentz transformations

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We construct the Hamiltonian charges corresponding to local Lorentz transformations on a space-time admitting isolated horizon as an internal boundary. From this construction, it arises quite generally that the area of the horizon of a non-extremal black hole is the Hamiltonian charge for the local Lorentz boost on the horizon. Using this further, we confirm the recent results of Frodden-Ghosh-Perez on the local formulation of black hole mechanics.
I. INTRODUCTION

The laws of dynamics of black hole horizons are the laws of classical thermodynamics. This fact also implies that black holes are thermal objects and that there may be a deep connection between the dynamics of spacetime (gravity) and thermodynamics of horizons. Hawking’s result that black holes can be assigned a temperature $T = \kappa/2\pi$ implies from the first law of black hole mechanics that black holes of area $A$ also have thermodynamic entropy given by $S = A/4G$. Immediately question arises as to what is the statistical mechanical origin of this entropy. Such intimate and detailed matching can be done only if a quantum theory of gravity is available. Indeed, any candidate theory of quantum gravity must account for this entropy through statistical mechanical counting of appropriate microstates. However, in absence of any accepted theory, one must try to obtain a deeper understanding of the classical laws of black hole mechanics. Quite recently, it has been shown in the context of stationary black holes, that observers at a proper distance $l_0$ from the horizon define a notion of horizon energy given by $E = A/8\pi G l_0$. Furthermore, it arises that the local surface gravity, given by $\kappa = 1/l_0$, is universal and consequently, the first law can be written in the quasi-local setting as $\delta E = (\kappa/8\pi G)\delta A$. These results are extremely interesting in that it directly relates the black hole area with energy. Our objective is to obtain the above result (and also related ones) using a different approach but still using the perspective of a quasi-local observer. In particular, we show that the quasi-local energy and the first law may also be obtained by constructing the Hamiltonian charge corresponding to local Lorentz transformations. However, since the behaviour of black hole horizons is different for different class of observers and one must fix a notion for use which we describe below.

Black hole horizons are null surfaces and behave like one way membranes. A better way of defining such surfaces is through the notion of a Killing horizon. A Killing horizon is a lightlike submanifold generated by a Killing vector. Familiar examples are the horizon of a Schwarzschild black hole, generated by $(\partial/\partial\tau)^a$ and the horizon of the Kerr black hole, generated by $(\partial/\partial\phi)^a + \Omega_H (\partial/\partial\phi)^a$. Another familiar example is the Rindler horizon in the Minkowski spacetime which is generated by the Lorentz boost vectors which, in the Minkowski coordinates, may be written as $K^a = x(\partial/\partial\tau)^a + t(\partial/\partial x)^a$. This boost Killing vector may be written more suggestively in the polar coordinates as $K^a = (\partial/\partial\eta)^a$ where $\eta$ is the hyperbolic opening angle. In this coordinate system, the Minkowski metric is given by $ds^2 = -l^2 d\eta^2 + dl^2 + dX^2$, and the flow lines of $K^a$ are hyperbolas. For an accelerated or Rindler observer with unit acceleration, the surface generated by $(\partial/\partial\eta)^a$ is a horizon out of which no information can be extracted. For this Rindler horizon, with the boost Killing vector normal to the horizon, the boost is canonically conjugate to the horizon area. [Let us also note that the scaling of the boost parameter is possible to $\tau = l_0\tilde{\eta}$ so that with respect to this observer (with proper time $\tau$), moving in a hyperbola at a distance $l_0$ from the origin, the metric is $ds^2 = -(l/l_0)^2 d\tau^2 + dl^2 + dX^2$. In other words, the surface gravity of the Killing vector $\partial/\partial\tau$, which generates proper time on the hyperbola, is $\kappa = 1/l_0$.] The key similarity between the Rindler horizon and the black hole horizon is that both can block information which however, is observer dependent. In fact, the Rindler horizon does not block information from an inertial observer just as a black hole horizon does not block information from a freely falling observer. However, one would expect that for a black hole horizon too, the Killing time is in some sense canonically conjugate to area. Indeed, it is well known, at least in general relativity, the horizon area is canonically conjugate to the boost time \cite{10,12}. In this paper, we present another proof of this statement although, from a different point of view: Given a spacetime with a non-extremal isolated horizon as an internal boundary, the observer very near the horizon (say at a distance $l_0$), perceives the spacetime to be a Rindler spacetime. The horizon appears to the observer as a Rindler horizon generated by it’s boost Killing vector field. In the inertial coordinates $(t, x, x_\perp)$, the horizon is at $x = t$ and the boost vector field is given by $x(\partial/\partial t) + t(\partial/\partial x)$. However, one may multiply the boost vector field by any scale factor. For our case, this vector field is to be multiplied with $1/l_0$ since this represents the natural non-affine parameter (surface gravity) associated with the null vector fields on the horizon. The value of this constant $l_0$ depends on the proper distance of the observer. All these vector fields are spacetime vector fields and have seemingly no connection with action of local Lorentz transformations which act on internal indices. However, we shall argue that if the spacetime
admits a Killing vector field, the action of spacetime diffeomorphisms may be realised as the action of local Lorentz transformations on internal Lorentz indices. Such notion arises from a improved notion of symmetries where one insists that the coordinate invariance be extended to allow for non-invariance that can be compensated by the gauge transformation of that vector field \[\xi\] \[\ell\]. This is precisely the relation we use. Indeed, in this case, the action of diffeomorphism (generated by a Killing vector field) on the tetrad \(e_a^I\) is given by \(\mathcal{L}_\xi e^I = \epsilon(\xi)^I_J e^J\), with \(\epsilon(\xi)^I_J\) being the infinitesimal transformation matrix generated due to Lie dragging (by the vector field \(\xi^a\)) of the tetrad basis. Note that this change in the tetrad basis is induced on the internal Lorentz indices and this also determines, through the no torsion condition, the action the diffeomorphisms on the spin-connection. We must stress that to understand the action of the Lorentz transformations, we must use the first order connection formalism. The second order metric formalism mixes the action of local Lorentz transformations and spacetime diffeomorphisms. On the other hand, the first order formalism is particularly suitable for our case since we are interested in obtaining the Hamiltonian charges corresponding to the Lorentz transformations. The first order connection formalism contains information regarding the Lorentz index as well as the spacetime indices and hence it is easier to directly obtain the action of the local Lorentz transformations and the Hamiltonian or conserved quantities related to them. We would like to point out that the approach that we develop below to construct the Lorentz charges and hence the quasilocal first law of black hole mechanics has not been dealt with before in the literature, though, there are constructions of Lorentz charges in the context of asymptotically locally AdS spaces [20–22]. Also, in [19], the authors have constructed the horizon Noether charge for a combination of diffeomorphism and local Lorentz transformations to show that it is the black hole entropy. Recently, such constructions have also been considered in the context of black hole mechanics [23] [24]. However, our approach differs substantially both in motive and in methodology.

We proceed as follows: Let us take the same observer in the Minkowski space who is moving with an acceleration \(1/l_0\) and place him at rest at a proper distance \(l_0\), very near to a non-extremal black hole horizon. For this static observer, the spacetime is Rindler and the horizon is generated by the boost vector field of the Rindler spacetime. Since boost is the symmetry of this spacetime, the observer may define a conserved charge. Our objective is to determine this charge seen by the static observer (and show that it is given by \(E = A/8\pi G l_0\)) as in [9]. The spacetime boost isometry will be shown to induce a boost transformation on the internal indices of the tetrad basis. Naturally, on the horizon (which we shall model as an isolated horizon), this boost transformation matrix turns out to be one of the generators of the little group of the full Lorentz group, that is, belongs to \(ISO(2) \times \mathbb{R}\). We then determine the charges due to this boost transformation. This charge then enables one to unambiguously determine the energy as seen by the observer at rest. Thus, we may directly determine the Hamiltonian charge corresponding to the Lorentz boosts on the horizon.

In the next section, we introduce the notion and boundary conditions for isolated horizon action followed by the construction of the symplectic structure appropriate to the given set of boundary conditions. That is followed by the section on construction of charges for Lorentz transformation and their interpretation through the improved notion of invariances. In the discussion section, we deliberate on the result and explore future avenues.

**II. ISOLATED HORIZON AS THE INNER BOUNDARY**

The formalism of isolated horizons is useful to model the horizon of a black hole. We first describe the geometrical set-up and the boundary conditions. Let us consider a 4-manifold \(\mathcal{M}\) equipped with a metric \(g_{ab}\) having signature \((-\,+,\,+,\,+)\). Let \(\Delta\) be a null hypersurface in \(\mathcal{M}\) generated by a future directed null vector field \(\ell^a\). The coordinate system on (a coordinate patch of) \(\Delta\) is given as follows. Consider a cross-section \(S_0\) of \(\Delta\) with coordinates \(x^i_\perp (i = 1, 2)\). The tangent on this patch is then given by \(\ell^a = (\partial/\partial \lambda)^a\), with \(\lambda\) being the affine parameter. Let us, without any loss of generalisation, choose the value of the affine parameter on \(S_0\) as \(\lambda = 0\). Further, we denote the spatial cross-sections which foliate the horizon by \(S_\lambda\) which are essentially surfaces of constant \(\lambda\). Thus, if \(P\) is any point on \(S_\lambda\), it’s coordinates are \((\lambda, x^i_\perp)\), where \(\lambda\) is the affine separation of the point \(P\) from \(S_0\). We are however interested in a specific
class of null surfaces which are being generated by the Killing vector field \( l^a = (\partial/\partial v)^a \), existing only on the horizon. Since \( l^a \) is geodetic, \( l^a \nabla_a b = \kappa(l) l^b \), where \( \kappa(l) \) is the acceleration corresponding to the null normal \( l^a \). In the context of black hole, this plays the role of surface gravity. If surface gravity is constant, the horizon generating parameter \( v \) is related to the affine parameter \( \lambda \) through \( \lambda = a e^{\kappa v} + b \).

Note that if \( \xi \) is a positive function on the null surface, then \( \xi l^a \) is also horizon generating. The expansion \( \theta(l) \) of the null normal \( l^a \) is defined by \( \theta(l) = q^{ab} \nabla_a l_b \), where \( \nabla_a \) is the covariant derivative compatible with \( g_{ab} \) and \( q^{ab} \) is the degenerate metric on \( \Delta \). The surface \( \Delta \), equipped with the class \( [\xi l^a] \) of null normals, is called a weak isolated horizon (WIH) in \( (\mathcal{M}, g_{ab}) \) if the following conditions hold \([26–29]\):

1. \( \Delta \) is topologically \( S^2 \times \mathbb{R} \).
2. The expansion \( \theta(l) = 0 \).
3. The equations of motion hold on \( \Delta \) and the vector field \(-T^a l_b \) is future directed and causal on \( \Delta \).
4. There exits an one form \( \omega(l) \) on \( \Delta \) which is lie dragged \( \mathcal{L}_l \omega(l) = 0 \).

All these boundary conditions are intrinsic to \( \Delta \) and also imply the existence of a Killing vector field \( \xi l^a \) on \( \Delta \). The first condition is only a topological restriction while the second condition applies to black hole horizons and also to Rindler horizons. The third condition ensures that equations of motion and energy condition hold. Note that the first three conditions hold true for all vectors in the equivalence class \( [\xi l^a] \) of null normals, is called a weak isolated horizon (WIH) in \( (\mathcal{M}, g_{ab}) \) if the following conditions hold \([26–29]\):

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Now, given that the internal boundary is an isolated horizon, one may construct the space of solutions which admit an isolated horizon as an internal boundary. For this, we work with the first order Palatini Lagrangian and the covariant phase space formalism. Given a Lagrangian, the on-shell variation gives \( \delta L = d\Theta(\delta) \) where \( \Theta \) is called the symplectic potential. It is a 3-form in space-time and a 0-form in phase space. Using this symplectic potential, one constructs the symplectic current \( J(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2) - \delta_2 \Theta(\delta_1) \), which, by definition, is closed on-shell. The symplectic structure is then defined to be:

\[
\Omega(\delta_1, \delta_2) = \int_M J(\delta_1, \delta_2)
\]

where \( M \) is a space-like hypersurface. Since \( dJ = 0 \) provided the equations of motion and linearized equations of motion hold, this implies that when integrated over a closed region of spacetime bounded by \( M_+ \cup M_- \cup \Delta \) (where \( \Delta \) is the inner boundary considered),

\[
\int_{M_+} J - \int_{M_-} J + \int_{\Delta} J = 0,
\]

where \( M_+, M_- \) are the initial and the final space-like slices, respectively. For the case when WIH is an internal boundary, third term is exact, \( \int_{\Delta} J = \int_{\Delta} \delta J \), and the hypersurface independent symplectic structure is given by:

\[
\Omega(\delta_1, \delta_2) = \int_M J - \int_{S_{\Delta}} \delta J
\]

where \( S_{\Delta} \) is the 2-surface at the intersection of the hypersurface \( M \) with the boundary \( \Delta \). The quantity \( j(\delta_1, \delta_2) \) is called the boundary symplectic current and symplectic structure.
As we have said before, we are interested in constructing the space of solutions of general relativity, and we shall use the first order formalism in terms of tetrads and connections. This formalism is naturally adapted to the nature of the problem in the sense that the boundary conditions are easier to implement, construction of the covariant phase space becomes simpler and the action of local Lorentz transformations are easier to disentangle. For the first order theory, we take the fields on the manifold to be $(e^a_I, A_{aI}^J)$, where $e_a^I$ is the co-tetrad, $A_{aI}^J$ is the gravitational connection. The Palatini action in first order gravity is given by \[ S_G = -\frac{1}{16\pi G} \int_M (\Sigma^{IJ} \wedge F_{IJ}) \] (4)

where $\Sigma^{IJ} = \frac{1}{2} \epsilon^{IJ} K_L e^K \wedge e^L$, $A_{IJ}$ is a Lorentz $SO(3,1)$ connection and $F_{IJ}$ is a curvature two-form corresponding to the connection given by $F_{IJ} = dA_{IJ} + A_{IK} \wedge A^K_J$. Our strategy shall be to construct the symplectic structure for the action given in eqn. (4). Let us first look at the Lagrangian for gravity. The symplectic potential in this case is given by, $16\pi G \Theta(\delta) = -\Sigma^{IJ} \wedge \delta A_{IJ}$. The symplectic structure is given by \[ \Omega(\delta_1, \delta_2) = -\frac{1}{8\pi G} \int_M (\delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ}) - \frac{1}{4\pi G} \int_{S_\Delta} \delta_1 \psi \delta_2 \epsilon. \] (5)

The function $\psi$ is a potential for the surface gravity $\kappa(l)$ and is defined by $L(l) = \kappa$, and $\epsilon$ is the area two form on the spherical cross sections $S_\Delta$ of the horizon. The field $\psi$ is assumed to satisfy the boundary condition that $\psi = 0$ at some initial cross section.

III. LORENTZ TRANSFORMATIONS AND CHARGES

As we have said before, we shall work in the first order formulation of gravity. In the previous section, we have used this first order theory to construct the symplectic structure. To evaluate the charges arising due to local Lorentz transformations, we take a local basis consisting of the co-tetrads $e^I$. The co-tetrads and the connection transform under a Lorentz transformation in the following way.

\[ e^I \rightarrow \Lambda^I_J e^J \] (6)
\[ A^{IJ} \rightarrow (\Lambda^{-1})^I_K A^{KL} \Lambda_L^J + (\Lambda^{-1})^I_K d\Lambda^{KL} \] (7)

where $\Lambda^I_J$ is the Lorentz transformation matrix. The variations of the co-tetrads and the connection due to infinitesimal Lorentz transformations, $\delta e^I = (\delta^I_J + \epsilon^I_J)$, are given by (note that $\epsilon^I_J$ are the generators of the Lorentz transformations),

\[ \delta e^I = \epsilon^I_J e^J \] (8)
\[ \delta A^{IJ} = d\epsilon^{IJ} + A^{IK} \epsilon^K_J + A^{JK} \epsilon^I_K. \] (9)

We also require the expression for the variation of $\Sigma_{IJ}$. After a bit of algebra, one can show that,

\[ \delta \Sigma_{IJ} = \varepsilon_{IJKL} e^K_M e^L \wedge e^L = e^K_J \Sigma_I^K - e^K_I \Sigma_J^K. \] (10)

The action of the Lorentz transformations on the fields in the bulk symplectic structure is obtained as follows:

\[ \Omega_B(\delta, \delta) = -\frac{1}{16\pi G} \int_M (\epsilon^K_J \Sigma_{IK} - \epsilon^K_K \Sigma_{IJ} \wedge (d\epsilon^{IJ} + A^{IK} \epsilon^K_J + A^{JK} \epsilon^I_K) \] (11)
The third term may be rewritten as
\[ \delta \Sigma_{IJ} \wedge d\epsilon^{IJ} = d(\delta \Sigma_{IJ} \epsilon^{IJ}) - \delta (d\Sigma_{IJ}) \epsilon^{IJ} \]
\[ = d(\delta \Sigma_{IJ} \epsilon^{IJ}) + \delta (A_j^K \wedge \Sigma_{KJ} + A_j^K \wedge \Sigma_{IK}) \epsilon^{IJ} \] (12)

Therefore, the contribution only survives on the cross-sections of the horizon
\[ \Omega_B(\delta_\epsilon, \delta) = \frac{1}{16\pi G} \int_{S_\Delta} \delta \Sigma_{IJ} \epsilon^{IJ}. \] (13)

To evaluate the above expression, we must determine the form of \( \epsilon_{IJ} \). Note that the WIH reduces the local Lorentz group \( SL(2, C) \) to \( ISO(2) \times R \), the little group of the Lorentz group \( [30] \). More precisely, the WIH boundary conditions are invariant under a subgroup of the local Lorentz group. Explicitly, the Lorentz matrices associated with the transformations which keep the WIH boundary conditions invariant are given by

\[ \Lambda_{IJ} = - \xi_l n_j - \xi^{-1} l_1 n_j + 2m_{(l} \bar{m}_{n])}, \] (14)
\[ \Lambda_{IJ} = - 2 l_{(l} n_j + (e^{\theta} m_{l} \bar{m}_{j} + c.c.),} \] (15)
\[ \Lambda_{IJ} = - l_1 n_j - (n_j - c m_{l} - \bar{c} m_{j} + |c|^2 I_l) l_j \]
\[ + (m_l - \bar{c} l_j) \bar{m}_{j} + (\bar{m}_{l} - c l_j) m_{j} \] (16)

The generators corresponding to these transformations are obtained to be:

\[ B_{IJ} = (\partial \Lambda_{IJ} / \partial \xi)_{\xi=1} = -2l_{[l} m_{j]}, \] (17)
\[ R_{IJ} = (\partial \Lambda_{IJ} / \partial \theta)_{\theta=0} = 2im_{[l} \bar{m}_{j]}, \] (18)
\[ P_{IJ} = (\partial \Lambda_{IJ} / \partial \text{Re} c)_{c=0} = 2m_{[l} l_{j]} + 2 \bar{m}_{[l} l_{j]}, \] (19)
\[ Q_{IJ} = (\partial \Lambda_{IJ} / \partial \text{Im} c)_{c=0} = 2im_{[l} l_{j]} - 2i \bar{m}_{[l} l_{j]}, \] (20)

where \( B \) generates boost on \( \Delta \), \( R \) generates rotation on the spherical cross-sections of \( \Delta \). The generators \( P, Q \) generate transformations which keep the direction of \( l \) and \( n \) invariant respectively. The \( \epsilon_{IJ} \) in equation (13) can thus be either of these above four generators. We recall that the expression of \( \Sigma_{IJ} \) in terms of a null basis adapted to \( \Delta \) is given in the following form:

\[ \Sigma^{IJ} = 2l_{[l} n_{j]} 2 \epsilon + 2n \wedge (im_{[l} \bar{m}_{j]} - i \bar{m}_{[l} m_{j]}) \] (21)

Given the generators of the Lorentz transformations in eqn. (17)-(20), it immediately follows that the bulk term survives only for null boosts \(-2l_{[l} m_{j]} \) and the Hamiltonian charge \( \delta Q \) is given by

\[ \Omega_B(\delta_\eta, \delta) = \delta Q = \delta \left( \frac{\eta A}{8\pi G} \right) \] (22)

where \( \eta \) is an infinitesimal parameter of transformation and \( A \) is the area of the cross-section. If the phase-space is restricted such that the boost parameter \( \eta \) is a phase-space constant, then the Hamiltonian is obtained to be:

\[ \delta Q = \frac{\eta}{8\pi G} \delta A \] (23)

We shall show in the next section that the natural value of \( \eta \) is \( 1/l_0 \) and then, the Hamiltonian charge turns out to be \( A/8\pi GL_0 \).

To complete the argument, we show below that the boundary symplectic structure does not contribute to the Hamiltonian. Let \( \psi \) be a potential for \( \kappa_{(l)} \) defined by \( L_{1l} \psi = \kappa_{(l)} \) and hence, \( d\psi = -\kappa_{(l)} n + \alpha_{(l)} m + \bar{\alpha}_{(l)} \bar{m} \). Also, note that a variation \( \delta \) may not act on the boundary variables in a fashion similar to its action in the bulk. This is due to the fact that on \( \Delta \), the variation has to satisfy the constraint from
the zeroth law that on the horizon, $\kappa_{(l)}$ is a constant. The variations on the surface symplectic structure may arise from different transformations. First, let us consider the null boosts $\delta_\eta$. For null boosts, such that, $\eta$ is a constant on $\Delta$, $L_l^j \delta_\eta \psi_{(l)} = 0$. Therefore $\delta_\eta \psi_{(l)}$ is a function only of coordinates on $S^2$ and can be set equal to zero at the initial cross-section. On the other hand $\delta_\eta \epsilon^2 = 0$. Therefore the boundary contribution vanishes i.e $j(\delta_\eta, \delta) = 0$. We also look at contributions from spacelike rotations given by, $l \rightarrow l, n \rightarrow n, m \rightarrow e^{ij} m_i$. Under these transformations $\delta \kappa = \delta n = 0$ which implies that one can set $\delta \theta = 0$. Moreover $\delta_\theta \epsilon^2 = 0$. Therefore $j(\delta_\theta, \delta) = 0$. Finally, one may also look at null rotations keeping $l$ fixed, given by $l \rightarrow l, n \rightarrow n - cm - \tilde{c}m + \tilde{c}l, m \rightarrow m - \tilde{c}l$, for which it follows that $\delta_c \epsilon^2 = 0$ and $j(\delta_c, \delta) = 0$.

IV. LORENTZ TRANSFORMATIONS ON HORIZON

In this section, we shall show that the natural value of $\eta$ is $\frac{1}{l_0}$ and in the process, try to decipher the meaning of the Hamiltonian charge obtained in the previous section in eqn. (22). We show below that one needs a better notion of derivatives, called the Lie-Lorentz derivatives, to handle fields which are Lorentz-Lie algebra valued spacetime fields. To exemplify, let us consider the electromagnetic one form $A$ and find how it behaves under a diffeomorphism under a vector field $\chi^a$. The result is:

$$L_\chi A = \chi \cdot F + d(\chi \cdot A).$$

This Lie derivative is not gauge invariant but can be made so by subtracting the gauge transformation term $d(\chi \cdot A)$, if we define a new Lie derivative by

$$L_\chi' A = L_\chi A - d(\chi \cdot A).$$

Invariance of the electromagnetic vector field may be taken to be that $L_\chi' A = 0$. This is modified form of invariance requirement where one insists that the coordinate invariance be extended to allow for non-invariance that can be compensated by the gauge transformation of that vector field, that is the invariant gauge potential is vanishing up to a total derivative, $L_\chi A = d(\lambda \chi)$. The modified derivative $L_\chi'$ is sometimes called the Lie-Maxwell derivative.

Similar to the the case for electromagnetic fields, one may also demand the existence of a derivative for the Lorentz transformations. Note that such kind of derivatives are useful since in many cases, even if the spacetime has a Killing vector $(\chi^a)$, $L_\chi g_{ab} = 0$, but the co-tetrad is not lie dragged, $L_\chi e_a^l \neq 0$, which happens since the tetrad also suffers Lorentz transformations in the process of being Lie dragged. To take care of this additional Lorentz transformations that the tetrad frame undergoes, one usually defines a Lie-Lorentz derivative, which is a derivative $L_\chi e_a^l = 0$. If the spacetime contains a Killing vector, one can show that such a choice of derivative is always possible [18, 19]. Then, for these spacetimes, in these tetrad frames, the action of a diffeomorphisms can be realised as local Lorentz transformations. Particularly, consider the case that $\chi^a$ is a vector field and that it’s action on $e^l$ is given by:

$$L_\chi e^l = \epsilon(\chi)^l_j e^j,$$

where $\epsilon(\chi)^l_j$ is an infinitesimal Lorentz transformation (associated to $\chi^a$) and hence is antisymmetric in the Lorentz indices. It is simple to show that $L_\chi g_{ab} = 0$ [18, 19].

Now, since the horizon is that of a non-extremal black hole, the metric as perceived by an observer at a distance $l_0$ from the horizon is given by the Rindler metric given by the form

$$ds^2 = -(l/l_0)^2 d\tau^2 + d\tau^2 + dx_\perp^2,$$

where, $\tau$ is the proper time of the hyperbola located at the distance $l_0$ from the horizon. The acceleration of this hyperbolic worldline is $1/l_0$. Also note that, the Killing vector field in the Rindler frame is $(\partial/\partial \tau)^a$
and has surface gravity of $1/l_0$. This generator of time translation, given in the Rindler frame by $(\partial/\partial t)^a$, is essentially the boost Killing vector field in the inertial coordinates. The horizon appears to the observer as a Rindler horizon generated by its boost Killing vector field. In the inertial coordinates $(t, x, x_\perp)$, where the Rindler metric is given by $ds^2 = -dt^2 + dx^2 + dx_\perp^2$, the boost vector field is given by

$$\chi^a = \alpha \left[ x \left( \frac{\partial}{\partial t} \right)^a + t \left( \frac{\partial}{\partial x} \right)^a \right],$$

where $\alpha$ is any arbitrary scaling factor which becomes $1/l_0$ on the horizon since it represents the natural non-affine parameter (surface gravity) associated with the null vector fields on the horizon. The horizon appears in the inertial coordinates to be given by $x = t$ and it is straightforward to see that the boost vector field approaches the value: $\chi^a = (x/l_0) [(\partial/\partial t)^a + (\partial/\partial x)^a]$. Given a Killing vector field, the infinitesimal components $\epsilon_{IJ}$ are given by:

$$\epsilon^{IJ}_{(\chi)} = e^{[I}_a \xi^J_a].$$

For the Killing vector eqn. (28), the infinitesimal Lorentz transformations are, $\epsilon^1_2 = 1 = \epsilon^2_1$, which are the infinitesimal boosts multiplied by the parameter $\eta = 1/l_0$. On, the horizon, the quantity $\epsilon_{IJ}$ may be written suggestively as $\epsilon_{IJ} = -(2/l_0) l_{[I} n_{J]}$. Using this value of $\epsilon_{IJ}$, we get instead of the equation (22),

$$\delta Q = \delta \left( \frac{A}{8\pi G l_0} \right)$$

The above equation then immediately indicates the Hamiltonian and implies that the first law corresponding to the generator $(\partial/\partial \eta)$ is given by $\delta H = \delta A/8\pi G l_0$.

We now argue that the surface gravity is universal for static observers outside the black hole. Let us consider an observer who is at rest at some distance $\chi$ from the horizon since it represents the natural non-affine parameter (surface gravity) associated with the null vector fields on the horizon. The horizon appears to the observer at rest at a distance $\chi/l_0$ that $\chi$ on the horizon implies that surface gravity as perceived by an observer at a small distance $\epsilon$ away from the horizon of any non-extremal black hole is universal and given by $1/l_0$. We may also understand this intuitively as follows. Note that the Noether charge corresponding to boosts is given by $(E x_i - p_i t)$. For the observer at rest at a distance $l_0$, the Noether charge must be $E l_0$ since the momentum $p_i$s vanish. However, the charge as obtained for the transformation $\epsilon_{IJ} = -2l_{[I} n_{J]}$ is $A/8\pi G$. Comparing, we get that $E = A/8\pi G l_0$ which is exactly the result of [9].

V. DISCUSSION

In this paper, we have shown that (a) the energy of a black hole as determined by an observer at a distance $l_0$ from the horizon is $A/8\pi G l_0$, and (b) that the first law for these observers is given by $\delta H = \kappa \delta A/8\pi G$, with $\kappa$ being a universal constant for these observers with value $1/l_0$. Some further comments are in order. Firstly, we emphasize that these results are obtained as boundary charges arising out of invariance of boundary conditions under Lorentz transformations. It must also be pointed out that the quasi-local energy may also be derived from the boundary terms which survive on the horizon in the second-order metric formulation. This may be seen as follows: the boundary Hamiltonian on a cross-section of a null surface due to an infinitesimal coordinate transformation $x_a \rightarrow x_a + \epsilon_a(x)$ is given by:

$$H = \frac{1}{8\pi G} \int d^2\Sigma^{ab} \nabla_a \epsilon_b,$$

where $d^2\Sigma^{ab}$ is an element of the cross-section and $\epsilon_a$ is the diffeomorphism generated by the Killing vector $\chi^a$, which in the inertial frame of the Rindler spacetime is given by $\chi^a = (x/l_0) [(\partial/\partial t)^a + (\partial/\partial x)^a]$. In the inertial coordinates, the value of the surface element is $1/2\sqrt{\eta} d^2x_\perp$ and $\nabla_a \epsilon_b$ gives a value of $1/l_0$. Thus, the Hamiltonian is $A/8\pi G l_0$. Note that since this is a second-order metric formulation, it does not sense the action of Lorentz transformations. On the other hand, our derivation is in the first order formalism and gives an alternate perspective to the quasi-local first law.
Secondly, if electromagnetic fields are present, there are two additional terms in the symplectic structure, one from the bulk \( \frac{1}{4} \int \delta_1 A \wedge \delta_2 F \) and one from the surface \( \frac{1}{4} \int \delta_1 \Psi \delta_2^* F \). Here, the scalar \( \Psi \) is defined as \( \mathcal{E} \chi \Psi = (\chi \cdot A) \). Using these, we get that the contribution of the electromagnetic Hamiltonian to be \( \delta Q_{em} = (\chi \cdot A - \lambda) \delta q \equiv \Phi \delta q \), where \( q = (-1/4\pi) \int F \). Moreover, on the horizon, one usually sets \( \mathcal{L}_\chi \Lambda = d(\lambda \chi) \) so that \( \Phi \) is a constant. This immediately implies that the energy is to be added by \(-\Phi q \), but the quasi-local first law retains the same form \( \delta E = \kappa \delta A / 8\pi G \). There is an important caveat though: if the condition \( \mathcal{L}_\chi \Lambda = d(\lambda \chi) \) holds not only on the horizon or the near horizon, but to the entire bulk, then, the contribution from the electromagnetic field does not arise. Thirdly, the limiting case of extremal black holes is not easy to treat using the method described here since the Rindler description of the near horizon structure itself breaks down in that case. However, it may still be argued that the first law is vacuous since the proper distance from the black hole is infinite and hence the surface gravity vanishes. Fourth, the result of [9] assumes the existence of the first law in a certain form, which indeed arises if one considers general relativity (their subsequent construction, however, is independent of specific nature of the theory of gravity), but may be entirely different if one considers other theories of gravity, for example a scalar tensor theory. Then, it remains to see how the arguments of [9] may change and what new results may arise. Furthermore, if one considers the additional Holst term, one may also get contributions from the rotation part of the Little group, and may have important implications for the simplicity constraints and the quantisation. These and other related matters will be discussed in a subsequent paper.

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