Net-baryon number fluctuations with HRG model using Tsallis distribution

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The hadron resonance gas (HRG) model with Tsallis distribution has been used to explain the energy dependence of the product of the moments, $S\sigma$ and $\kappa\sigma^2$ of net-proton multiplicity distributions recently published STAR data at RHIC energies. While excellent agreements are found between model predictions and measurements of $S\sigma\kappa$ and $\kappa\sigma^2$ of most peripheral collisions and $S\sigma$ of most central collisions, the $\kappa\sigma^2$ of most central collisions deviates significantly from the model predictions particularly at $\sqrt{s_{NN}} = 19.6$ GeV and 27 GeV. This could be an indication of the presence of additional dynamical fluctuations not contained in the HRG-Tsallis model.

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Recently, the STAR collaboration has reported the energy dependence of the moments (mean $M$, variance $\sigma$, skewness $S$ and kurtosis $\kappa$) and their products for net-proton multiplicity distribution at RHIC energies \cite{1}. The product of the moments $S\sigma$ and $\kappa\sigma^2$ can be linked to the ratios of susceptibilities ($\chi$) associated with the baryon number conservation \cite{2,3,4}. For example, the product $S\sigma$ can be written as the ratio of third order ($\chi_3$) to second order ($\chi_2$) and the product $\kappa\sigma^2$ as the ratio of fourth order ($\chi_4$) to second order ($\chi_2$) baryon susceptibilities. The recent STAR measurements of $S\sigma$ and $\kappa\sigma^2$ show significant deviations from the predictions of the Skellam distribution (where $\kappa\sigma^2$ should be unity) at all energies indicating presence of large non-statistical fluctuations \cite{5}. The particle production in heavy ion collisions at relativistic energies is well described in terms of the hadron resonance gas (HRG) model where fermions and bosons follow Fermi-Dirac (FD) and Bose-Einstein (BE) distributions respectively. The success of HRG model would mean that thermal system which might have gone through a possible phase transition has (nearly) equilibrated both thermally and chemically at freeze out. It is believed that if the thermal system has retained some memory of the phase transition with finite correlation length at freeze out, it must be reflected in the higher moments of the conserved quantities although not so obvious in the study of thermal abundance of the individual species \cite{6,8}. Therefore, the study of fluctuations in various conserved quantities like: net-charge, net-strangeness and net-baryon number through the higher moments using HRG model is expected to provide a base line to observe the deviation in experimental observables, which may indicate the presence of non statistical fluctuations, if any.

The HRG model in Boltzmann approximation follows an exponential behavior of particle production corresponding to Boltzmann-Gibbs (BG) statistics. Recently, it has been realized that particle production both in heavy ion and proton-proton collisions at RHIC and LHC energies can be described successfully using a power law type distribution rather than the exponential one \cite{9,10}. Therefore, the Tsallis distribution function is being used for particle production with nonextensive parameter $q$ such that it approaches Boltzmann distribution in the limit $q \to 1$. In the context of particle production in heavy ion collisions, Tsallis distribution has been interpreted as the superposition of Boltzmann distributions with different temperatures \cite{12}. Such a situation can occur when the hadronic fireball is not homogeneous in temperature $T$ but fluctuates from point to point around some equilibrium value $T_f$ \cite{13}. The temperature fluctuation which exists in small part of the phase space with respect to the whole is another source of non-statistical fluctuation. This is different from the statistical fluctuations of event by event type and should be properly accounted in the model. In general, Tsallis nonextensive statistics is supposed to include all situations characterized by long range interactions, long range microscopic memory and space time fractal structure of the process \cite{14,15}. Therefore, we consider a hadron resonance gas model using Tsallis nonextensive distribution to study fluctuations of net-baryon production in Au+Au collisions at RHIC energies. In the limit $q \to 1$, we recover the HRG results.

In the present work, we show that the HRG-Tsallis model with a temperature dependent nonextensive $q$ parameter reproduces the energy dependence of $S\sigma$ and $\kappa\sigma^2$ for most peripheral collisions as well as $S\sigma$ for central collisions. However, the energy dependence of $\kappa\sigma^2$ of central collision deviate significantly from the HRG-Tsallis model predictions particularly at energies 19.6 GeV and 27 GeV. We argue here that the predictions of HRG-Tsallis characterized by a temperature dependent $q$ parameter should be taken as the base line to study (experimentally) fluctuations of dynamical origin if any, which is still not contained in the Tsallis non-extensive thermodynamics.

The Tsalli’s form of FD and BE distribution can be
written as,

\[ f = \frac{1}{\exp_q (E - \mu) + 1} \]  \hspace{1cm} (1)

where "+" and "-" signs are used for fermions and bosons respectively and \( \exp_q (x) \) is given by,

\[ \exp_q (x) = \begin{cases} \left[ 1 + (q - 1)x \right]^{1/(q-1)} & \text{if } x > 0 \\ \left[ 1 + (1 - q)x \right]^{1/(1-q)} & \text{if } x \leq 0 \end{cases} \]  \hspace{1cm} (2)

where \( x = (E - \mu)/T \). The above distribution approaches standard FD and BE distributions in the limit \( q \to 1 \). Using above nonextensive distribution function, the average number density can be written as,

\[ n_q = \sum_i q_i X_i \int \frac{d^3 k}{(2\pi)^3} f_q^i (E_i, T_f, \mu_i), \]  \hspace{1cm} (3)

where \( T_f \) is the chemical freeze-out temperature, \( \mu_i \) is the chemical potential and \( g_i \) is the degeneracy factor of the \( i \)-th particle. The total chemical potential \( \mu = B_i \mu_i + Q_i \mu_Q + S_i \mu_S \), where \( B_i, Q_i \) and \( S_i \) are the baryon, electric charge and strangeness number of the \( i \)-th particle, with corresponding chemical potentials \( \mu_B, \mu_Q \) and \( \mu_S \), respectively. The term \( X_i \) represents either B, Q or S of the \( i \)-th particle depending on whether the computed \( n_q \) represents baryon density, electric charge density or strangeness density respectively [11]. Note that the exponent \( q \) in \( f_q \) has been introduced as a constraint for thermodynamical consistency [12]. The factor \( d^3 k \) can be expressed in terms of phase space variables, \( d^3 k = p_T \sqrt{p_T^2 + m^2} \cos \eta \, dp_T \, d\eta \, d\phi \) where energy \( E \) is expressed as, \( E = \sqrt{p_T^2 + m^2} \cos \eta \). Since the average density and the pressure \( P \) are shown to be thermodynamically consistent i.e. \( n_q = \frac{\partial P}{\partial T} \), we can define a generalizability susceptibility as,

\[ \chi_q^n = \frac{\partial^n [P(T_f, \mu)]}{\partial \mu^n} \bigg|_T = \frac{\partial^{n-1} [n_q(T_f, \mu)]}{\partial \mu^{n-1}}. \]  \hspace{1cm} (4)

Using Eq(3) and Eq(4), we have estimated \( S\sigma \, (\chi^3/\chi^2) \) and \( \kappa \sigma^2 \, (\chi^4/\chi^2) \) for net-baryon density within STAR detector acceptance. For comparison with experiments, it would have been more appropriate to estimate net-proton density which includes primordial as well as yields from resonance decays. However, we have noticed from an earlier study that within STAR acceptance, the difference between two methods is negligible (less than 2%) [13]. Therefore, in the present study we consider net-baryon density within STAR acceptance. We parametrized the freeze-out temperatures and chemical potentials using the relations, \( \mu_B(\sqrt{s_{NN}}) = -1.3 \varepsilon_{NN} \) and \( T(\mu_B) = a - b\mu_B^2 + c\mu_B^4 \). The parameters are taken from Ref. [12] for most central collision and are listed in Table I. For peripheral collision, we extract these parameters from STAR preliminary data [14] for most peripheral (70-80)% centrality collisions and the extracted parameters are listed in Table I. We use similar parametrization for \( \mu_S \) and \( \mu_Q \) as that of \( \mu_B \) and the corresponding parameters are listed in the table. We set \( \mu_Q \) to zero for peripheral collision as its contribution is very small as compared to \( \mu_B \) and \( \mu_S \).

The non-extensive parameter \( q \) characterizes the degree of non-equilibrium in the system which depends on both collision energy as well as on the centrality of collisions. In fact, \( q \) depends on \( T_f, \mu \) as well as on the type of identified particles [20]. From the previous study of STAR data, it is noticed that \( q \) value decreases with increasing centrality indicating an evolution from a highly non-equilibrated system towards a more thermalized one [3]. It is found that \( (q - 1) \) has a parabolic dependence on temperature [12]. Similarly, from the analysis of SPS and RHIC data, it is noticed that \( q \) value is maximum at SPS energy of 80 AGeV (which corresponds to 12.3 GeV in center of mass energy) and decreases on either side i.e. at 158 AGeV as well as at 40 AGeV [21]. The \( q \) value is also found nearly unity at 200 GeV RHIC energy. In term of freeze out temperature, these two observations suggest that the \( q \) parameter may have a maximum value at a particular temperature \( T_m \) and will decrease on either side. The maximum temperature \( T_m \) is fixed at 0.154 GeV corresponding to the freeze out temperature (for central collision) at \( \sqrt{s_{NN}} = 12.3 \) GeV which corresponds to 80 AGeV SPS collision energy. Therefore, we use the parametrization \( q = 1 + [\alpha(T_m - T_f)]^{2/3} \) for \( T_m \leq T_f \leq T_0 \) and allow it to decrease to unity linearly with a slope \( (q_m - 1)/(T_m - T_f) \) for \( T_f < T_m \) where \( q_m \) is the \( q \) value at \( T_m \). We set \( q = 1 \) for \( T_f \leq T_1 \). In the above, \( T_0 \) is a reference temperature fixed at 0.167 GeV which is close to the freeze-out temperature for \( \sqrt{s_{NN}} = 200 \) GeV so that \( q = 1 \) for \( T_f > T_0 \). The parameter \( T_1 \) is taken 0.120 GeV corresponding to \( \sqrt{s_{NN}} = 5 \) GeV. The inset in Fig. I shows a typical dependence of \( q \) on freeze-out temperature. The parameter \( \alpha \) is kept free and is adjusted to fit the data. When \( \alpha = 0 \), we get back the HRG results. Note that we use the same parametrization irrespective of whether the collision is central or peripheral except different centrality will have different chemical freeze-out parameters.

Figure 1 shows the energy dependence of \( S\sigma \) (top panel) and \( \kappa \sigma^2 \) (middle panel) estimated using parameters as listed in Table I for most peripheral (70-80)% centrality collisions. As can be seen, the HRG results (\( \alpha = 0 \)) do not explain the experimental values [1], and \( \kappa \sigma^2 \) in HRG model is always close to unity where as data points are about 20% below the HRG values. On the other hand, HRG-Tsallis with a nominal \( \alpha = 0.015 \) (corresponds to a maximum \( q \) value of 1.02) can explain both \( S\sigma \) and \( \kappa \sigma^2 \) extremely well. The excellent agreement between data and model predictions can be seen from the ratio plot in the bottom panel of Fig. I. The corresponding results for most central (0-5)% centrality collisions are shown in Fig. 2. Also for the central collisions, the HRG predictions do not explain the experimental data. It is also interesting to note that \( \alpha = 0.015 \) explains...
FIG. 1: (Color online) The energy dependence of product of moments, $S\sigma$ and $\kappa\sigma^2$ of net-baryon distribution calculated using standard HRG (solid curve) and Tsallis-HRG with $\alpha = 0.015$ (dotted curve). The filled circles are STAR data points for most peripheral collision corresponding to (70–80)% collision centrality in Au+Au collisions \[1\]. The bottom panel shows the ratio of data and our model calculation for $S\sigma$ and $\kappa\sigma^2$. The figure inset at the top shows the temperature dependence of the $q$ parameter.

FIG. 2: (Color online) The energy dependence of product of moments, $S\sigma$ and $\kappa\sigma^2$ of net-baryon distribution calculated using standard HRG (solid curve) and Tsallis-HRG with $\alpha = 0.015$ (dotted curve). The dot-dashed curve shows the corresponding values for $\alpha = 0.3$. The filled circles are STAR data points for most central collision corresponding to (0–5)% collision centrality in Au+Au collisions \[1\]. The bottom panel shows the ratio of data to model calculation for $S\sigma$ only.

The energy dependence of $S\sigma$ rather well except at 200 GeV (Although small, the disagreement between measurement and prediction for $S\sigma$ is also found for peripheral collisions at 200 GeV). However, the model fails to explain $\kappa\sigma^2$ as the deviations are more significant for lower energy data ($\sqrt{s_{NN}} < 39$ GeV). We have also increased $\alpha$ from 0.015 to 0.3 which results in the dashed-dot curves in Fig.2. Although use of a higher $\alpha$ parameter can explain the general energy dependence of $\kappa\sigma^2$, it fails to explain the $S\sigma$ data. In short, it is not possible to explain both $S\sigma$ and $\kappa\sigma^2$ simultaneously by varying $\alpha$.

In the frame work of Tsallis non-extensive thermodynamics, the definition of generalized susceptibility as defined in Eq.2 can be interpreted as follows. Assuming $f^q \approx \exp_q(x)$ which is true when $q$ is close to 1 (maximum $q$ value used in this work is about 1.02), the Tsallis-Boltzmann distribution Eq.2 can be written as superposition of Boltzmann distribution and the average baryon number density as

$$n^B_q(T_f, \mu) = \int_0^\infty f(T, T_f, q)n^B(T, \mu)d(1/T),$$

where the weight factor $f(T, T_f, q)$ represents a gamma distribution as defined in \[12\] and $n^B$ is the average number density obtained using Boltzmann statistics. Using above relation, Eq.4 can be expressed as weighted sum of the susceptibilities estimated using standard Boltzmann statistics and can be interpreted as an average taken over the whole phase space which is inhomogeneous in temperature. Probably, this is what will be measured experimentally if the temperature fluctuation exists in the hadronizing medium corresponding to a nonextensive value of $q$ which deviates from unity. Therefore, it is reasonable to argue that Eq.4 can be used to estimate temperature average higher moments. It may be mentioned here that the above argument is strictly correct when quantum statistics is not important and Tsallis distribution can be written as the superposition of Boltzmann distributions. This is mostly true for baryons.
where quantum statistics are not important. It may be mentioned here that, we have not considered the van der Waals (VDW) type excluded volume effect which could be another source of deviation of $S\sigma$ and $\kappa\sigma^2$ from the HRG predictions. However, the VDW type deviation increases with decreasing energies where as the STAR experimental values have maximum deviation at $\sqrt{s_{NN}} = 19.6$ GeV and less deviation at the two lowest RHIC energies. Therefore, we have not considered VDW type effect in the present calculation. The transport model like UrQMD within STAR acceptance also produces similar results for $S\sigma$ and $\kappa\sigma^2$ which decrease with decreasing energies. In the frame work of grand canonical HRG model, the $n$-th order cumulant for net-proton density can be written as $C_n = C_n^1 + (-1)^n C_n^2$ where $C_n^1$ and $C_n^2$ are the $n$-th cumulants for proton and anti-proton respectively. This is the same expression which has been used in Ref. [1] except for the fact that proton and anti-proton productions are independent. Since $C_n^1$ and $C_n^2$ are taken from the experiments, it is not surprising that the independent production model explains the STAR data. However, in the HRG model, the proton and anti-proton productions are not independent rather constraints through baryon number conservation. Further, we would like to add here that the present HRG-Tsallis model is different from the grand canonical HRG model as $C_n^1$ in our model represents a weighted average over the distribution of temperatures. The primary motivation of the STAR measurements of $S\sigma$ and $\kappa\sigma^2$ is to look for the existence of a critical end point in the QCD phase diagram where the product of these moments may show large non monotonic variations. The Tsallis distribution also includes quantum effect which is important at lower collision energies. Therefore, in this work we explored, how much non-statistical fluctuations inherently present in the Tsallis non-extensive distribution, which can explain the experimental observations without considering other physics effects.

Another important aspect which has not been considered here is the effect of non-extensivity on the freeze-out parameters which are generally extracted from the experiments using HRG model. It is observed in Ref. [21] that while freeze-out temperature decreases, chemical potential increases with increasing $q$ parameter. However, for $q < 1.02$ which has been used in the present study, we notice that the increase in $\mu_B$ is not significant although $T_f$ decreases by about 10% from the value when $q = 1$. As argued in [21], since Tsallis distribution is broader than the Boltzmann distribution, temperature needs to be decreased in order to conserve the particle density. Therefore, we have estimated the moments keeping $\mu_B$ unchanged but allowing freeze-out temperature to decrease up to 10%. Interestingly, we notice that while $S\sigma$ increases slightly, $\kappa\sigma^2$ remains unchanged. This suggests that $S\sigma$ is sensitive to both temperature and chemical potential while $\kappa\sigma^2$ is sensitive to $q$ parameter.

In conclusion, we have studied the energy dependence of the fluctuations of net-baryon productions through higher moments namely $S\sigma$ and $\kappa\sigma^2$ using HRG with Tsallis non-extensive distribution function. When the non-extensive parameter $q$ is close to unity, the moments obtained using HRG-Tsallis model can be interpreted as the weighted average of the moments estimated using many Boltzmann distribution corresponding to the distribution of temperatures over the whole phase space. It is shown that this HRG-Tsallis model can explain the energy dependence of $S\sigma$ and $\kappa\sigma^2$ measurements of recently published STAR data corresponding to the most peripheral collisions which is otherwise impossible to explain using normal HRG model which predicts $\kappa\sigma^2$ close to unity for net-baryon productions. The HRG-Tsallis model also explains the energy dependence of $s\sigma$ data of central collision. However, the model can not explain the corresponding $\kappa\sigma^2$ values of the central collision particularly at energies 19.6 GeV and 27 GeV. This deviation is so significant that it is an indication of the presence of additional fluctuations at around 20 GeV which may have some dynamical origin not contained in the Tsallis HRG model.

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
$a$ (GeV) & $b$ (GeV$^{-1}$) & $c$ (GeV$^{-2}$) \\
\hline
$T$ & $0.166 \pm 0.002$ & $0.139 \pm 0.016$ & $0.053 \pm 0.021$ \\
\hline
$d$ (GeV) & $e$ (GeV$^{-1}$) \\
$\mu_B$ & $1.308 \pm 0.028$ & $0.273 \pm 0.008$ \\
$\mu_S$ & $0.214$ & $0.161$ \\
$\mu_Q$ & $0.0211$ & $0.106$ \\
\hline
\end{tabular}
\caption{TABLE I: Parametrization of chemical potentials and freeze-out temperature for most central collisions taken from [17].}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
$a$ (GeV) & $b$ (GeV$^{-1}$) & $c$ (GeV$^{-2}$) \\
\hline
$T$ & $0.158 \pm 0.002$ & $0.159 \pm 0.034$ & $0.500 \pm 0.001$ \\
\hline
$d$ (GeV) & $e$ (GeV$^{-1}$) \\
$\mu_B$ & $0.900 \pm 0.039$ & $0.251 \pm 0.008$ \\
$\mu_S$ & $0.239 \pm 0.001$ & $0.300 \pm 0.001$ \\
\hline
\end{tabular}
\caption{TABLE II: Parametrization of chemical potentials and freeze-out temperature extracted from the STAR experimental data for (70-80)% centrality [19].}
\end{table}

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The Skellam distribution is the discrete probability distribution of the difference $N_1 - N_2$ where $N_1$ and $N_2$ are two random variables each following a Poisson distribution.