Gradient estimates for singular parabolic $p$-Laplace type equations with measure data

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Abstract
We are concerned with gradient estimates for solutions to a class of singular quasilinear parabolic equations with measure data, whose prototype is given by the parabolic $p$-Laplace equation $u_t - \Delta_p u = \mu$ with $p \in (1, 2)$. The case when $p \in (2 - \frac{1}{n+1}, 2)$ were studied in Kuusi and Mingione (Ann Sc Norm Super Pisa Cl Sci 5 12(4):755–822, 2013). In this paper, we extend the results in Kuusi and Mingione (2013) to the open case when $p \in (\frac{2n}{n+1}, 2 - \frac{1}{n+1}]$ if $n \geq 2$ and $p \in (\frac{5}{7}, \frac{3}{2}]$ if $n = 1$. More specifically, in a more singular range of $p$ as above, we establish pointwise gradient estimates via linear parabolic Riesz potential and gradient continuity results via certain assumptions on parabolic Riesz potential.

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1 Introduction

In this paper, we consider the quasilinear parabolic equation with measure data

$$u_t - \text{div}(a(x, t, Du)) = \mu$$

in a cylindrical domain $\Omega_T = \Omega \times (-T, 0) \subset \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$. Here and in what follows, the operators “$D$” and “div” stand for the gradient and divergence with respect to the space variable $x$. Moreover, $\mu$ is a finite signed Radon measure in $\Omega_T$, namely, $|\mu|(\Omega_T) < \infty$. The vector field $a = (a_1, \ldots, a_n) : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy the following growth, ellipticity, and continuity conditions: there exist constants $0 < v \leq L$, $s \geq 0$, and $p > 1$ such that

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\[ |a(x, t, \xi)| + (s^2 + |\xi|^2)^{1/2} |D_\xi a(x, t, \xi)| \leq L(s^2 + |\xi|^2)^{(p-1)/2}, \quad (1.2) \]

\[ \langle D_\xi a(x, t, \xi) \eta, \eta \rangle \geq \nu(s^2 + |\xi|^2)^{(p-2)/2} |\eta|^2, \quad (1.3) \]

and

\[ |a(x, t, \xi) - a(x_0, t, \xi)| \leq L \omega(|x - x_0|)(s^2 + |\xi|^2)^{(p-1)/2} \quad (1.4) \]

hold for every \( x, x_0 \in \Omega, t \in (-T, 0) \), and \( (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\} \), where \( \omega : [0, \infty) \to [0, 1] \) is a concave non-decreasing function satisfying

\[ \lim_{\rho \to 0^+} \omega(r) = \omega(0) = 0 \]

and the Dini condition

\[ \int_0^1 \omega(\rho) \frac{d\rho}{\rho} < +\infty. \quad (1.5) \]

A typical model equation is given by the (possibly nondegenerate) parabolic \( p \)-Laplace equation with measure data and \( s \geq 0 \):

\[ u_t - \text{div} \left( (|Du|^2 + s^2)^{\frac{p-2}{2}} Du \right) = \mu \quad \text{in} \ \Omega_T. \]

By a (weak) solution to the Eq. (1.1), we mean a function

\[ u \in C^0(-T, 0; \ L^2(\Omega)) \cap L^p(-T, 0; \ W^{1,p}(\Omega)) \]

such that the distributional relation

\[ -\int_{\Omega_T} u \varphi_t \, dx \, dt + \int_{\Omega_T} \langle a(x, t, Du), D\varphi \rangle \, dx \, dt = \int_{\Omega_T} \varphi \, d\mu \]

holds whenever \( \varphi \in C_0^\infty(\Omega_T) \) has compact support in \( \Omega_T \).

The gradient estimates for the super-quadratic case when \( p \geq 2 \) were well studied in the literature. See [12,20,21] and also [10,18,19,24] for estimates for elliptic problems. However, the corresponding results for the singular case when \( p \in (1, 2) \) are still not complete.

In this paper, we are concerned with only the singular case when \( p \in (1, 2) \).

### 1.1 Pointwise gradient estimates

First, we recall the potential estimates of the gradients of solutions to the stationary equations

\[ -\text{div} \ a(x, Du) = \mu \quad \text{in} \ \Omega. \quad (1.6) \]

The following pointwise gradient estimates were established in [11] by Duzaar and Mingione for the case when \( p \in (2 - \frac{1}{n}, 2) \):

\[ |Du(x)| \leq c \left[ I_1^{\mu p}(x, R) \right]^\frac{1}{p-1} + c \int_{B_R(x)} (|Du(y)| + s) \, dy \]

holds for any solution \( u \) to the Eq. (1.6) and any ball \( B_R(x) \subset \Omega \). Here \( B_R(x) \subset \mathbb{R}^n \) denotes the ball centered at \( x \) with radius \( R \), \( f_E \) stands for the integral average over a measurable set \( E \), and

\[ I_1^{\mu p}(x, R) := \int_0^R \frac{\mu(B_{\rho}(x))}{\rho^{p-1}} \frac{d\rho}{\rho} \quad (1.7) \]
is the truncated first-order elliptic Riesz potential. In [9], we extended the results above to include the case when \( p \in \left( \frac{3n-2}{2n-1}, 2 - \frac{1}{n} \right) \) and also derived the following Lipschitz estimates for the case when \( p \in (1, \frac{3n-2}{2n-1}) \):

\[
\|Du\|_{L^\infty(B_{R/2}(x))} \leq c \left\| I_1^{\mu}(\cdot, R) \right\|_{L^\infty(B_R(x))}^{\frac{1}{p-1}} + c R^{-\frac{n}{p-1}} \|Du\|_0 + s \|L^{2-p}(B_R(x))\|.
\]

For more gradient estimates for the elliptic problem in the singular case \( p \in (1, 2) \), we refer the reader to [25–27]. The first gradient potential result for singular parabolic \( p \)-Laplace type equations was obtained by Kuusi and Mingione in [17] for the case when \( p \in (2 - \frac{1}{n+1}, 2) \) using intrinsic geometry and exit time arguments. More precisely, they first showed that there exists a constant \( c = c(n, p, \nu, L, \omega) \) such that if

\[
c \int_{Q_{r}(x_0,t_0)} (|Du| + s) \, dx \, dt + c \int_{0}^{2r \lambda} \frac{|\mu|(\Omega_{\rho}(x_0,t_0)) \, d\rho}{\rho^{n+1}} \leq \lambda
\]

for some constant \( \lambda > 0 \), then \( |Du(x_0, t_0)| \leq \lambda \). Here \( r_\lambda := \lambda^{(p-2)/2} r \) and

\[
\Omega_{\rho}(x_0, t_0) := B_{r_\lambda}(x_0) \times (t_0 - \lambda^{2-p} \rho^2, t_0)
\]

is called an intrinsic cylinder for \( p, \lambda > 0 \). They also used the intrinsic Riesz potential result above to establish the following parabolic Riesz potential bound when \( p \in (2 - \frac{1}{n+1}, 2) \):

\[
|Du(x_0, t_0)| \leq c \left[ I_1^{\mu}(x_0, t_0, 2r) \right]^{2/[(n+1)p-2n]} + c \left( \int_{Q_r(x_0,t_0)} (|Du| + s + 1) \, dx \, dt \right)^{2/[2-2(n-p)]}
\]

holds for any solution \( u \) to the Eq. (1.1) in the standard parabolic cylinder \( Q_{2r}(x_0, t_0) := B_{2r}(x_0) \times (t_0 - 4r^2, t_0) \subset \Omega_T \). Here

\[
I_1^{\mu}(x_0, t_0, r) := \int_{0}^{r} \frac{|\mu|(\Omega_{\rho}(x_0,t_0)) \, d\rho}{\rho^{n+1}}
\]

is the truncated first-order parabolic Riesz potential. For more notation in parabolic (intrinsic) geometry, see Sect. 2.1 below.

In this paper, we extend their results to include the case when \( p \in (p^*(n), 2 - \frac{1}{n+1}) \), where

\[
p^*(n) := \max \left\{ \frac{2n}{n+1}, \frac{3n+2}{2n+2} \right\} = \begin{cases} \frac{5}{4} & \text{when } n = 1, \\ \frac{2n}{n+1} & \text{when } n \geq 2. \end{cases}
\]

Note that \( p^*(n) < 2 - \frac{1}{n+1} \) holds for every integer \( n \geq 1 \). Moreover, it is clear that if \( p \in (p^*(n), 2 - \frac{1}{n+1}) \), then

\[
0 < \max \left\{ \frac{n+2}{2(n+1)}, \frac{(2-p)n}{2} \right\} < p - \frac{n}{n+1} \leq 1
\]

so that we can choose a constant \( q \in (0, 1) \) satisfying

\[
q \in \left( \max \left\{ \frac{n+2}{2(n+1)}, \frac{(2-p)n}{2} \right\}, p - \frac{n}{n+1} \right) \subset (0, 1).
\]

Our first main result is stated as follows.
Theorem 1.1 (Intrinsic Riesz potential estimate) Let \( u \) be a solution to (1.1) with \( p \in (p^*(n), 2 - \frac{1}{p+1}) \), where \( p^*(n) \) is defined in (1.10). Let \( q \in (0, 1) \) satisfy (1.11). Under the assumptions (1.2)–(1.5), there exist constants \( c \geq 1 \) and \( R_0 \in (0, 1/2] \), both depending only on \( n, p, v, L, q, \) and \( \omega \), such that the following holds for a.e. \( (x_0, t_0) \in \Omega_T \): If

\[
c \left( \int_{Q_{r}^\lambda(x_0,t_0)} |Du| + s \right)^{q} dxdt \right)^{1/q} + c \int_0^{2r\lambda} \frac{\mu(\Omega^\lambda_{\rho}(x_0,t_0))}{\rho}\,d\rho \leq \lambda, \tag{1.12}
\]

where \( \lambda > 0 \) is a constant, \( r_\lambda := \lambda^{(p-2)/2} \in (0, R_0] \), \( Q_{2r\lambda}^\lambda(x_0,t_0) \subset \Omega_T \), and \( Q_{\rho}^\lambda(x_0,t_0) \) is the intrinsic cylinder defined in (1.8), then it holds that

\[
|Du(x_0,t_0)| \leq \lambda.
\]

Theorem 1.1 implies pointwise gradient estimates in standard parabolic cylinders as in Theorem 1.2 and Corollary 1.3 below.

Theorem 1.2 (Pointwise gradient estimate via parabolic Riesz potential) Let \( u \) be a solution to (1.1) with \( p \in (p^*(n), 2 - \frac{1}{p+1}) \), where \( p^*(n) \) is defined in (1.10). Let \( q \in (0, 1) \) satisfy (1.11). Under the assumptions (1.2)–(1.5), there exist constants \( c \geq 1 \) and \( R_0 \in (0, 1/2] \), both depending only on \( n, p, v, L, q, \) and \( \omega \), such that

\[
|Du(x_0,t_0)| \leq c \left[ I_\lambda^{[\mu]}(x_0,t_0,2r) \right]^{2/(n+1)p-2n} + c \left( \int_{Q_r(x_0,t_0)} (|Du| + s + 1)^q \, dxdt \right)^{2q/[2q-n(2-p)]} \tag{1.13}
\]

holds for a.e. \( (x_0, t_0) \in \Omega_T \) and every \( Q_{2r}(x_0, t_0) \equiv B_{2r}(x_0) \times (t_0 - 4r^2, t_0) \subset \Omega_T \) with \( r \in (0, R_0] \), where \( I_\lambda^{[\mu]} \) is the truncated parabolic Riesz potential defined in (1.9).

Corollary 1.3 (Pointwise gradient estimate via elliptic Riesz potential) Let \( u \) be a solution to (1.1) with \( p \in (p^*(n), 2 - \frac{1}{p+1}) \), where \( p^*(n) \) is defined in (1.10) and assume that \( \mu = \mu_0 \otimes f \), where \( \mu_0 \) is a finite signed Radon measure on \( \mathbb{R}^n \) and \( f \in L^\infty(-T, 0) \). Let \( q \in (0, 1) \) satisfy (1.11). Under the assumptions (1.2)–(1.5), there exist constants \( c \geq 1 \) and \( R_0 \in (0, 1/2] \), both depending only on \( n, p, v, L, q, \) and \( \omega \), such that

\[
|Du(x_0,t_0)| \leq c \|f\|_L^{1/(p-1)} \left[ I_\lambda^{[\mu]}(x_0,t_0,2r) \right]^{1/(p-1)} + c \left( \int_{Q_r(x_0,t_0)} (|Du| + s + 1)^q \, dxdt \right)^{2q/[2q-n(2-p)]} \tag{1.17}
\]

holds for a.e. \( (x_0, t_0) \in \Omega_T \) and every \( Q_{2r}(x_0, t_0) \equiv B_{2r}(x_0) \times (t_0 - 4r^2, t_0) \subset \Omega_T \) with \( r \in (0, R_0] \), where \( I_\lambda^{[\mu]} \) is the truncated elliptic Riesz potential defined in (1.7).

We remark that a preliminary assumption \( p > \frac{2n}{n+2} \) always occurs in regularity theory for parabolic \( p \)-Laplace type equations. See for instance [5]. In this paper, we need the lower bound \( p > \frac{2n}{n+2} \) for the applications of reverse Hölder type estimates (Theorem 4.2) and Lipschitz estimates (Theorem 4.5). Our proofs of the main theorems require a stronger assumption \( p > \frac{2n}{n+1} \) when \( n \geq 2 \). At the time of this writing, it is not clear to us whether this assumption is optimal for these potential estimates to hold. However, we note that the assumption \( p > \frac{2n}{n+1} \) is also needed for the existence of fundamental solution to the parabolic \( p \)-Laplace equation (the Barenblatt solution; see [1], [29, Chapter 11], [17, Section 1.3]), and the Harnack inequalities (see [5, Chapter 7]).
1.2 Gradient continuity results

In [17], the authors proved a sufficient condition for gradient continuity in the case when $p \in (2 - \frac{1}{n+1}, 2)$, namely, the Riesz potential $I_\mu^\beta (x_0, t_0, r) \to 0$ uniformly with respect to $(x_0, t_0)$ when $r \to 0$. We extend that result to the case when $p \in (p^*(n), 2 - \frac{1}{n+1})$.

**Theorem 1.4** (Gradient continuity via Riesz potential) Let $u$ be a solution to (1.1) with $p \in (p^*(n), 2 - \frac{1}{n+1})$, where $p^*(n)$ is defined in (1.10). Assume that (1.2)–(1.5) are satisfied and that the functions

$$(x, t) \mapsto I_\mu^\beta (x, t, r)$$

converge locally uniformly to zero in $\Omega_T$ as $r \to 0$. (1.14)

Then $Du$ is continuous in $\Omega_T$.

Recall the Lorentz space $L^{n+2,1}$ is the collection of measurable functions $f$ such that

$$\int_0^\infty |\{(x, t) : |f(x, t)| \geq h\}|^{\frac{1}{n+2}} dh < \infty.$$

Theorem 1.4 has the following corollary.

**Corollary 1.5** (Gradient continuity via Lorentz spaces) Let $u$ be a solution to (1.1) with $p \in (p^*(n), 2 - \frac{1}{n+1})$, where $p^*(n)$ is defined in (1.10). Assume that (1.2)–(1.5) are satisfied and that

$$\mu \in L^{n+2,1} \text{ holds locally in } \Omega_T.$$ (1.15)

Then $Du$ is continuous in $\Omega_T$.

A further, actually immediate, corollary of Theorem 1.4 concerns measures with certain density properties.

**Corollary 1.6** (Gradient continuity via density) Let $u$ be a solution to (1.1) with $p \in (p^*(n), 2 - \frac{1}{n+1})$, where $p^*(n)$ is defined in (1.10). Assume that (1.2)–(1.5) are satisfied and that $\mu$ satisfies

$$|\mu|(Q_\rho(x, t)) \leq c_D \rho^{n+1} h(\rho)$$ (1.16)

for every standard parabolic cylinder $Q_\rho(x, t) = B_\rho(x) \times (t_0 - \rho^2, t_0) \subseteq \Omega_T$, where $c_D$ is a positive constant and $h : [0, \infty) \to [0, \infty)$ is a function satisfying the Dini condition

$$\int_0^R h(r) \frac{dr}{r} < \infty \text{ for some } R > 0.$$ (1.17)

Then $Du$ is continuous in $\Omega_T$.

We also establish the following measure density criterion to ensure gradient Hölder continuity, which is a parabolic generalization of Lieberman’s result in [22]. Recall that in parabolic setting, for any $\beta \in (0, 1)$ and any set $C \subseteq \mathbb{R}^{n+1}$, the Hölder space $C^{0, \beta}(C)$ is the collection of measurable functions $f$ such that

$$\|f\|_{C^{0, \beta}(C)} := \sup_{C} |f| + \sup_{(x_1, t_1), (x_2, t_2) \in C, (x_1, t_1) \neq (x_2, t_2)} \frac{|f(x_1, t_1) - f(x_2, t_2)|}{|x_1 - x_2|^\beta_{\text{par}}} < \infty,$$ (1.18)
where

\[ |(x_1, t_1) - (x_2, t_2)|_{\text{par}} := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\} \]

is the parabolic distance between those two points. Moreover, \( C^{0,\beta}_{\text{loc}}(C) \) is defined as the collection of measurable functions \( f \) such that \( f \in C^{0,\beta}(K) \), for every compact set \( K \subset \subset C \).

**Theorem 1.7** (Gradient Hölder continuity via Riesz potential) Let \( u \) be a solution to (1.1) with \( p \in (p^*(n), 2 - \frac{1}{n+1}) \), where \( p^*(n) \) is defined in (1.10). Assume that (1.2)–(1.5) are satisfied and that \( \omega, \mu \) satisfy

\[ \omega(r) \leq c_D r^\delta \quad \text{and} \quad |\mu|(Q_r(x, t)) \leq c_D r^{n+1+\delta} \quad (1.19) \]

for every \( r \in (0, 1) \) and every standard parabolic cylinder \( Q_r(x, t) = B_r(x) \times (t_0 - \rho^2, t_0) \subset \subset \Omega_T \), where \( c_D \geq 1 \) and \( \delta \in (0, 1) \). Then there exists an exponent \( \beta \in (0, 1) \) depending only on \( n, p, \nu, L, c_D, \) and \( \delta \), such that \( Du \in C^{0,\beta}_{\text{loc}}(\Omega_T) \).

Let us give a brief description of the proofs. We first prove decay estimates of the \( L^q \)-mean oscillation of the gradient for a solution to the homogeneous equation with \( x \)-independent nonlinearity

\[ v_t - \div(a(x_0, t, Dv)) = 0, \quad (1.20) \]

where \( q \in (0, 1) \) and \( x_0 \in \mathbb{R}^n \) is fixed. Here by the \( L^q \)-mean oscillation of \( Dv \) in a domain \( C \subset \mathbb{R}^{n+1} \), we mean

\[ \inf_{\Theta \in \mathbb{R}^n} \left( \int_C |Dv - \Theta|^q \, dx \, dt \right)^{1/q}. \]

Our proof of the decay estimates adapts the singular iteration scheme in [17, Section 3] to the \( L^q \) setting for \( q \in (0, 1) \). For the precise definition of the \( L^q \)-mean oscillation with \( q \in (0, 1) \) and some of its properties, see Sect. 2.2. We also refer the reader to [3,4,6–9,15] for its applications in other problems.

Our proofs of the pointwise gradients estimates and the gradient continuity results are all based on the decay estimates for \( Dv \) mentioned above and comparison estimates between the original solution \( u \) to (1.1) and a solution \( v \) to (1.20). As a bridge between \( u \) and \( v \), we introduce the solution \( w \) to the homogeneous equation

\[ w_t - \div(a(x, t, Dw)) = 0 \]

in a cylinder \( Q \) with the boundary condition \( w = u \) on \( \partial_{\text{par}} Q \). Under appropriate boundary condition on \( v \), we obtain an \( L^p \) bound for \( Dw - Dv \), which originated from [16, Lemma 4.3]. We also utilize a comparison estimate between \( u \) and \( w \) in [28, lemma 3.1], which provides an \( L^q \) bound for \( Du - Dw \) in terms of \( \mu \) for some \( q \in (0, 1) \). By proving a reverse Hölder type inequality for \( Dw \), we establish an \( L^q \) estimate for \( Du - Dw \) for some \( q \in (0, 1) \).

With the decay estimates of the \( L^q \)-mean oscillation for \( Dv \) and the \( L^q \) estimate for \( Du - Dw \) in hand, we then borrow the idea in [7] by estimating the \( L^q \)-mean oscillation and adapt the exit time argument and iteration argument used, for instance, in [17, Theorem 1.1] to prove the pointwise gradient estimates. For gradient continuity results, we first prove a uniform decay estimate of the \( L^q \)-mean oscillation of \( Du \) in Proposition 5.1. Then for the gradient Hölder continuity result, we show the decay rate of the \( L^q \)-mean oscillation of \( Du \) and adapt Campanato’s idea of characterizing Hölder continuity to the \( L^q \) setting for some \( q \in (0, 1) \). Finally, for the gradient continuity result, we adapt the “maximal iteration chain”
argument introduced in [17, Theorem 1.5] to our \( L^q \) setting and apply the uniform decay of the \( L^q \)-mean oscillation of \( Du \).

The rest of the paper is organized as follows. In the next section, we collect basic notation and give the definition and some basic properties of the \( L^q \)-mean oscillation for \( q \in (0, 1) \). In Sect. 3, we prove some decay estimates for the \( L^q \)-mean oscillation of the gradients of solutions to the homogeneous equations with \( x \)-independent nonlinearities. In Sect. 4, we derive some comparison estimates and give the proofs of Theorem 1.1–Corollary 1.3. Finally, Sect. 5 is devoted to the gradient continuity results Theorems 1.4–1.7.

# 2 Notation and basic inequalities

## 2.1 Notation

In this paper, we adapt the same notation as in [17] for comparison purposes. For completeness, we briefly record the notation that will be used throughout this paper. For any vector \( x = (y_1, \ldots, y_n) \in \mathbb{R}^n \), we define two different norms

\[
|y| := \left( \sum_{i=1}^{n} y_i^2 \right)^{1/2} \quad \text{and} \quad \|y\| := \max_{1 \leq i \leq n} |y_i|.
\]

These two norms are equivalent since

\[
\|y\| \leq |y| \leq \sqrt{n} \|y\|, \quad \forall \, y \in \mathbb{R}^n.
\]

We use

\[
B_r(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \}
\]

to denote the open Euclidean ball in \( \mathbb{R}^n \) with center \( x_0 \) and radius \( r \) and denote

\[
Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)
\]

as the standard parabolic cylinder with center \((x_0, t_0)\) and radius \( r \). For \( \lambda > 0 \), we define the intrinsic cylinders as

\[
Q^\lambda_r(x_0, t_0) := B_r(x_0) \times (t_0 - \lambda^2 r^2, t_0).
\]

Clearly, when \( \lambda = 1 \), an intrinsic cylinder becomes a standard parabolic cylinder, namely, \( Q^1_r(x_0, t_0) = Q_r(x_0, t_0) \). For simplicity, we also denote

\[
\delta Q^\lambda_r(x_0, t_0) := Q^\lambda_{\delta r}(x_0, t_0) = B_{\delta r}(x_0) \times (t_0, \lambda^2 \delta^2 r^2, t_0).
\]

Namely, the parameter \( \delta > 0 \) before an intrinsic cylinder \( Q^\lambda_r(x_0, r_0) \) should be viewed as a dilation factor of the radius \( r \). We often denote \( r_\lambda := \lambda^{(p-2)/2} r \) and therefore

\[
Q^\lambda_{r_\lambda}(x_0, t_0) = Q^\lambda_{\lambda^{(p-2)/2} r}(x_0, t_0) = B_{\lambda^{(p-2)/2} r}(x_0) \times (t_0 - r^2, t_0).
\]

A useful property is that when \( p \in (1, 2) \),

\[
Q^\lambda_{r_{\lambda_2}}(x_0, t_0) \subset Q^\lambda_{r_{\lambda_1}}(x_0, t_0), \quad \text{if} \quad 0 < \lambda_1 \leq \lambda_2.
\]

When there is no confusion and no need to specify the center, we also denote \( Q^\lambda_r := Q^\lambda_r(x_0, t_0) \). For any cylindrical domain \( C = D \times (t_1, t_2) \) with \( D \subset \mathbb{R}^n \), the parabolic
boundary is defined as
\[ \partial_{\text{par}} C := D \times \{ t_1 \} \cup \partial D \times \{ t_1, t_2 \}. \]

The parabolic distance between two points is defined as
\[ |(x_1, t_1) - (x_2, t_2)|_{\text{par}} := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\} \]
and the corresponding parabolic distance between two sets is defined as
\[ \text{dist}_{\text{par}}(A_1, A_2) := \inf\{|(x_1, t_1) - (x_2, t_2)|_{\text{par}} : (x_1, t_1) \in A_1, (x_2, t_2) \in A_2\}. \]

Next, for any measurable mapping \( g : A \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \), we denote its integral average as
\[ \int_A g \, dx \, dt := \frac{1}{|A|} \int_A g(x, t) \, dx \, dt \]
and we denote its oscillation as
\[ \text{osc}_A g := \sup_{(x, t), (x_0, t_0) \in A} |g(x, t) - g(x_0, t_0)|. \]

Finally, throughout the paper, we denote by \( c, c', c'' \) some general constants which may differ from line to line. We also use \( c_1, c_2, c_3, \ldots \) to denote specific constants which may be used later.

### 2.2 Definition of the \( L^q \)-mean oscillation and some basic inequalities

For \( q \in (0, 1) \), we first recall some basic inequalities in one dimension:
\[ (a + b)^q \leq a^q + b^q, \quad (a + b)^{1/q} \leq 2^{1/q-1}(a^{1/q} + b^{1/q}) \quad \forall a, b > 0. \]  \( \quad (2.1) \)

Therefore, for any \( \Theta_1, \Theta_2 \in \mathbb{R}^k \), where \( k \) is a positive integer, we have
\[ |\Theta_1 + \Theta_2|^q \leq (|\Theta_1| + |\Theta_2|)^q \leq |\Theta_1|^q + |\Theta_2|^q. \]  \( \quad (2.2) \)

We now consider a measurable function \( F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^k \) for some integer \( k \geq 1 \). For simplicity, we assume that \( F \in L^1_{\text{loc}}(\mathbb{R}^{n+1} : \mathbb{R}^k) \). For any \( q \in (0, 1) \) and any bounded domain \( C \subset \mathbb{R}^{n+1} \), we define the \( L^q \)-mean oscillation of \( F \) on \( C \) as
\[ \phi_q(F, C) := \inf_{\Theta \in \mathbb{R}^k} \left( \int_C |F(x, t) - \Theta|^q \, dx \, dt \right)^{1/q}. \]

By (2.2) and the fact that \( F \in L^1(C) \), we know that the function
\[ h(\Theta) := \int_C |F(x, t) - \Theta|^q \, dx \, dt \]
is a continuous function of \( \Theta \in \mathbb{R}^k \) satisfying \( \lim_{|\Theta| \to \infty} h(\Theta) = \infty \). Therefore the minimum of \( h(\cdot) \) can be attained in \( \mathbb{R}^k \) and we choose \( \mathbf{m}(F, C) \in \mathbb{R}^k \) such that
\[ \left( \int_C |F(x, t) - \mathbf{m}(F, C)|^q \, dx \, dt \right)^{1/q} = \phi_q(F, C). \]

Next, we prove some useful properties related to the \( L^q \)-mean oscillation. By (2.2), we have
\[ |\mathbf{m}(F, C)|^q \leq |F(x, t) - \mathbf{m}(F, C)|^q + |F(x, t)|^q. \]
By taking the average over \((x, t) \in C\), taking the \(q\)-th root, and using (2.1), we obtain
\[
|m(F, C)| \leq 2^{1/q-1} \phi_q(F, C) + 2^{1/q-1} \left( \int_C F^q \, dx \, dt \right)^{1/q} \leq 2^{1/q} \left( \int_C F^q \, dx \, dt \right)^{1/q}.
\]
(2.3)

For two bounded domains \(C_1 \subset C_2 \subset \mathbb{R}^{n+1}\), using the same argument as above, we also have
\[
|m(F, C_1) - m(F, C_2)| \leq 2^{1/q-1} \phi_q(F, C_1) + 2^{1/q-1} \left( \int_{C_1} |F - m(F, C_2)|^q \, dx \, dt \right)^{1/q}
\]
\[
\leq 2^{1/q-1} \phi_q(F, C_1) + 2^{1/q-1} \left( \frac{C_2}{|C_1|} \right)^{1/q} \phi_q(F, C_2).
\]
(2.4)

### 3 Gradient estimates for homogeneous equations

In this section, we derive decay estimates of the \(L^q\)-mean oscillation of gradients of solutions to the homogeneous equations of the type
\[
v_t - \text{div}(a_0(t, Du)) = 0
\]
(3.1)
in a given cylinder \(Q = B \times (t_1, t_2)\), where \(a_0 = a_0(\tau, \xi)\) is a vector field independent of \(x\) satisfying conditions (1.2) and (1.3) for some \(s \geq 0, L \geq v > 0,\) and \(p \in (1, 2]\). First, we recall an oscillation estimate given in [17, Theorem 3.2].

**Theorem 3.1** Suppose that \(v\) is a solution to (3.1) in a given cylinder \(Q\) under assumptions (1.2) and (1.3). If \(p \in (1, 2]\), \(\lambda > 0\), and
\[
s + \sup_{Q^\lambda} \|Du\| \leq A\lambda
\]
holds for a constant \(A \geq 1\) and an intrinsic cylinder \(Q^\lambda_r \subset Q\), then there exists a constant \(\alpha \in (0, 1)\) depending only on \(n, p, v, L, A\), such that
\[
|Du(x_1, t_1) - Du(x_2, t_2)| \leq 4\sqrt{nA} \lambda^{p-1}
\]
holds for any \((x_1, t_1), (x_2, t_2) \in Q^\lambda_r\), where \(Q^\lambda_r \subset Q^\lambda_r\) is another intrinsic cylinder with the same center. Moreover, if \(p \in \left(\frac{2n}{n+2}, 2\right]\), then \(Du\) is Hölder continuous in \(Q\).

The main goal of this section is to derive the following decay estimate of the \(L^q\)-mean oscillation of the gradient with \(q \in (0, 1)\).

**Theorem 3.2** Let \(p \in (1, 2]\). Suppose that \(v\) is a solution to (3.1) in an intrinsic cylinder \(Q^\lambda_r\) under assumptions (1.2) and (1.3) and that \(A, B, q, \gamma\) are constants satisfying \(A, B \geq 1\) and \(q, \gamma \in (0, 1)\). Then there exist constants \(\delta, s, \xi \in (0, 1/2)\) depending only on \(n, p, v, L, A, B, \gamma, q, \xi \in (0, 1/4)\) depending only on \(n, p, v, L, A, B, \gamma, s\), such that if
\[
\lambda \leq \max \left\{ \frac{s}{\xi \delta}, B \sup_{\delta_r, Q^\lambda_r} \|Du\| \right\}, \quad s + \sup_{Q^\lambda_r} \|Du\| \leq A\lambda,
\]
(3.2)

where \(\lambda > 0\) is a constant, then
\[
\phi_q(Du, \delta_r, Q^\lambda_r) \leq \gamma \phi_q(Du, Q^\lambda_r).
\]
(3.3)
Moreover, there exist constants $\alpha \in (0, 1)$ depending only on $n, p, v, L, A, B, \gamma$, but not on $q$, and $c(A, B) \geq 1$ depending on $n, p, v, L, A, B, \gamma, q$, such that

$$\delta \gamma = \frac{\gamma^{1/\alpha}}{c(A, B)}.$$ 

Also, the estimate (3.3) still holds when replacing $\delta \gamma$ with a smaller number.

In the proof of gradient continuity results in Sect. 5, we need a different version of Theorem 3.2 under a stronger condition as follows, which gives a more precise dependence of $\delta \gamma$ in terms of $A, B, a$, and $\gamma, q$, namely,

$$\delta \gamma = \frac{\gamma^{1/\alpha_2}}{c(A)B^{1/\alpha_1}}$$

for some exponent $\alpha_1 \in (0, 1)$ independent of $B$ and some $\alpha_2 \in (0, 1)$ independent of $B$ and $q$.

**Theorem 3.3** Let $p \in (1, 2]$. Suppose that $v$ is a solution to (3.1) in an intrinsic cylinder $Q^\lambda_r$ under assumptions (1.2) and (1.3) and that $A, B, q, \gamma$ are constants satisfying $A, B \geq 1$ and $q, \gamma \in (0, 1)$. Then there exists a constant $\delta \gamma \in (0, 1/2)$ depending only on $n, p, v, L, A, B, \gamma, q$, such that if

$$\lambda \leq B \sup_{\delta \gamma \subset Q^\lambda_r} \|Dv\|, \ s + \sup_{Q^\lambda_r} \|Dv\| \leq A\lambda,$$

where $\lambda > 0$ is a constant, then

$$\phi_q(Dv, \delta \gamma Q^\lambda_r) \leq \gamma \phi_q(Dv, Q^\lambda_r).$$

(3.4)

Moreover, there exist constants $\alpha_1 \in (0, 1)$ and $c(A) \geq 1$, both depending only on $n, p, v, L, A, B, \gamma, q$, but not on $B$, and $\alpha_2 \in (0, 1)$ depending only on $n, p, v, L, A, B, \gamma, q$, but not on $q$ and $B$, such that

$$\delta \gamma = \frac{\gamma^{1/\alpha_2}}{c(A)B^{1/\alpha_1}}.$$ 

(3.5)

Also, the estimate (3.4) still holds when replacing $\delta \gamma$ with a smaller number.

As in [17], we first assume $s > 0$. The next two De Giorgi-Nash-Moser type estimates for $Dv$ can be found in [17] for vector field $a_0(\cdot)$ with no time dependence. However, they still hold here since their proofs only rely on differentiating the equation with respect to the space variable. See [17, Remark 3.6].

**Proposition 3.4** Assume $s > 0$ and $\lambda > 0$. Suppose that

$$s + \sup_{Q^\lambda_r} \|Dv\| \leq A\lambda$$

(3.6)

holds for some constant $A \geq 1$. There exists a constant $\sigma \in (0, 1/2)$ depending only on $n, p, v, L, A$ such that if either

$$|Q^\lambda_r \cap \{D_{x_i}v < \lambda/2\}| \leq \sigma |Q^\lambda_r|$$

(3.7)

or

$$|Q^\lambda_r \cap \{D_{x_i}v > -\lambda/2\}| \leq \sigma |Q^\lambda_r|$$

(3.8)
holds for some $i \in \{1, \ldots, n\}$, then

$$|D_{x_i}v| \geq \lambda/4 \text{ a.e. in } Q_{\rho/2}^\lambda.$$ 

**Proposition 3.5** Assume $s > 0$ and $\lambda > 0$. Suppose that (3.6) holds and that neither (3.7) nor (3.8) is satisfied for the constant $\sigma$ in Proposition 3.4. Then there exists a constant $\eta \in (1/2, 1)$ depending only on $n, p, q, L, A$, such that

$$\|D v\| \leq \eta \lambda \text{ a.e. in } Q_{\sigma \rho/2}^\lambda.$$  

(3.9)

Next, we prove a decay result for the $L^q$-mean oscillation of gradients of solutions to linear parabolic equations with $q \in (0, 1)$.

**Lemma 3.6** Suppose that $\tilde{u} \in L^2(-1, 0; \ W^{1,2}(B_1(0)))$ is a solution to the following linear parabolic equation

$$\tilde{u}_t - \text{div}(A(x, t) D \tilde{u}) = 0,$$  

(3.10)

where the matrix $A(x, t)$ has measurable entries and satisfies

$$v_0|\xi|^2 \leq \langle A(x, t) \xi, \xi \rangle, \quad |A(x, t)| \leq L_0$$

for any $\xi \in \mathbb{R}^n$, where $0 < v_0 \leq L_0$ are fixed constants. Let $q \in (0, 1)$. Then there exist constants $c_1, c_2 \geq 1$ depending on $n, v_0, L_0, q$, and $\beta_0 \in (0, 1)$, such that

$$\sup_{Q_{1/2}} |\tilde{u}| \leq c_1 \left( \int_{Q_1} |\tilde{u}|^q \, dx \, dt \right)^{1/q}$$  

(3.11)

and

$$\phi_q(\tilde{u}, Q_\delta) \leq c_2 \delta^{\beta_0} \phi_q(\tilde{u}, Q_1)$$  

(3.12)

hold for any $\delta \in (0, 1)$. Here for each $\rho > 0$, $Q_\rho \equiv B_\rho(0) \times (-\rho^2, 0)$ stands for the standard parabolic cylinder centered at 0 with radius $\rho$.

**Proof** First, the estimate (3.11) follows from standard local boundedness estimates for non-degenerate parabolic equations (see for example (3.26) in [21, Lemma 3.1]) and a standard interpolation and iteration argument. See also [23, Theorems 6.17]. Next we give the proof of (3.12) for $\delta \in (0, 1/4)$. From the classical De Giorgi-Nash-Moser theory for linear parabolic equations (see for example [23, Theorems 6.28-6.29]), there exist constants $c_0 \geq 1$ and $\beta_0 \in (0, 1)$, both depending on $n, v_0, L_0$, such that $\tilde{u} \in C^{\beta_0}(Q_{1/4})$ and that

$$\|\tilde{u}\|_{C^{\beta_0}(Q_{1/4})} \leq c_0 \sup_{Q_{1/2}} |\tilde{u}|,$$

where $\| \cdot \|_{C^{\beta_0}}$ is the parabolic Hölder norm defined in (1.18). Thus, using the last estimate and the assumption that $\delta \in (0, 1/4)$, we have

$$\phi_q(\tilde{u}, Q_\delta) \leq \left( \int_{Q_\delta} |\tilde{u} - \tilde{u}(0, 0)|^q \, dx \, dt \right)^{1/q} \leq (2\delta)^{\beta_0} \|\tilde{u}\|_{C^{\beta_0}(Q_{1/4})} \leq c \delta^{\beta_0} \sup_{Q_{1/2}} |\tilde{u}|,$$  

(3.13)

for some constant $c = c(n, v_0, L_0) > 0$. Combining (3.13) and (3.11), we get

$$\phi_q(\tilde{u}, Q_\delta) \leq c_2 \delta^{\beta_0} \left( \int_{Q_1} |\tilde{u}|^q \, dx \, dt \right)^{1/q}.$$
and therefore (3.12) follows by replacing $\tilde{u}$ with $\tilde{u} - m(\tilde{u}, Q_1)$ since $\tilde{u} - m(\tilde{u}, Q_1)$ is still a solution to (3.10) and $\phi_q(\tilde{u}, Q_1) \equiv \phi_q(\tilde{u} - m(\tilde{u}, Q_1), Q_1).

Finally, for $\delta \in [1/4, 1)$, the proof of (3.12) follows standard manipulations as follows. Recalling the definition of $m$, we have

$$\phi_q(\tilde{u}, Q_1) \leq \left( \frac{Q_1}{Q_4} \right)^{1/q} \int_{Q_1} |\tilde{u} - m(\tilde{u}, Q_1)|^{q} dxdt \right)^{1/q} \leq 4^{(n+2)/q} \phi_q(\tilde{u}, Q_1).$$

The proof is now completed. $\square$

Combining Proposition 3.4 and Lemma 3.6, we obtain the following result by using a scaling argument.

**Proposition 3.7** Suppose that $s > 0$, $\lambda > 0$, and that (3.6) holds. There exist constants $\beta \in (0, 1)$ depending on $n, p, \nu, L, A, q$, and $c_d \geq 1$ depending on $n, p, \nu, L, A, q$, such that if either (3.7) or (3.8) holds for the constant $\sigma$ in Proposition 3.4 and some $i \in \{1, \ldots, n\}$, then

$$\phi_q(Dv, Q^\lambda_r) \leq c_d \delta^\beta \phi_q(Dv, Q^\lambda_r).$$

**Proof** We will only give the proof for $\delta \in (0, 1/2)$ since the proof for $\delta \in [1/2, 1)$ follows by standard manipulations as in Lemma 3.6. From Proposition 3.4, we have

$$\lambda/4 \leq \|Dv(x, t)\| \leq s + \|Dv(x, t)\| \leq A\lambda \quad \forall (x, t) \in Q^\lambda_r/2. \quad (3.15)$$

Then we rescale the solution in the cylinder $Q_1$, namely,

$$\tilde{v}(x, t) := \frac{1}{r_1} v(r_1 x, \lambda^{2-p} r_1^2 t), \quad (x, t) \in Q_1, \quad (3.16)$$

where $r_1 = r/2$. Thus $\tilde{v}$ satisfies

$$\lambda^{p-2} \tilde{v}_t - \text{div} \tilde{a}_0(t, D\tilde{v}) = 0, \quad (3.17)$$

where

$$\tilde{a}_0(t, z) := a_0(\lambda^{2-p} r_1^2 t, z), \quad t \in (-1, 0), \quad z \in \mathbb{R}^n.$$

Also, (3.15) implies

$$\lambda/4 \leq \|D\tilde{v}(x, t)\| \leq s + \|D\tilde{v}(x, t)\| \leq A\lambda \quad \forall (x, t) \in Q_1. \quad (3.18)$$

Since $D\tilde{v}$ is bounded, we know that

$$D\tilde{v} \in L^2_{\text{loc}}(-1, 0; W^{1,2}_{\text{loc}}(B_1, \mathbb{R}^n)) \cap C^0(-1, 0; L^2_{\text{loc}}(B_1, \mathbb{R}^n)).$$

See [5, Chapter 8, Section 3] for details. Therefore we can differentiate (3.17) in $x_i$-direction for each $i \in \{1, \ldots, n\}$ and get

$$(\tilde{v}_x)_t - \text{div}(A(x, t)D\tilde{v}_x) = 0, \quad \text{where } A(x, t) := \lambda^{2-p} \partial_{x_i} \tilde{a}_0(t, D\tilde{v}(x, t)). \quad (3.19)$$

By (3.18), the matrix $A(x, t)$ is uniformly elliptic, namely, for any $\eta \in \mathbb{R}^n$

$$\langle A(x, t)\eta, \eta \rangle \geq v\lambda^{2-p} (|D\tilde{v}(x, t)|^2 + s^2)^{p-2} |\eta|^2 \geq v(\sqrt{n}A)^{p-2} |\eta|^2,$$
Let \( \delta \) hold for any \( \delta \in (0, 1) \) and \( i \in \{1, \ldots, n\} \), where \( \beta = \beta(n, p, v, L, A) \in (0, 1) \) and \( c = c(n, p, v, L, A, q) \geq 1 \). Let \( \Theta_0 := (m(\bar{v}_{x_1}, Q_{\delta_0}), \ldots, m(\bar{v}_{x_n}, Q_{\delta_0})) \) and \( \Theta_1 := (m(\bar{v}_{x_1}, Q_1), \ldots, m(\bar{v}_{x_n}, Q_1)) \). Using the triangle inequality and Jensen’s inequality, we have

\[
\phi_q(D\bar{v}, Q_{\delta_0}) \leq \left( \int_{Q_{\delta_0}} |D\bar{v} - \Theta_0|^q \, dx \, dt \right)^{1/q} \leq n^{1/q - 1} \sum_{i=1}^n \phi_q(\bar{v}_{x_i}, Q_{\delta_0}).
\]  

(3.21)

On the other hand, by the definition of \( \phi_q \), we have

\[
\phi_q(\bar{v}_{x_i}, Q_1) \leq \phi_q(D\bar{v}, Q_1), \quad \forall i \in \{1, \ldots, n\}
\]

and therefore,

\[
\sum_{i=1}^n \phi_q(\bar{v}_{x_i}, Q_1) \leq n \phi_q(D\bar{v}, Q_1).
\]  

(3.22)

By (3.20), (3.21), and (3.22), we obtain

\[
\phi_q(D\bar{v}, Q_{\delta_0}) \leq c' \delta_0^\beta \phi_q(D\bar{v}, Q_1)
\]  

(3.23)

for some \( c' = c'(n, p, v, L, A, q) \geq 1 \). Rescaling back in \( v \), (3.23) becomes

\[
\phi_q(Dv, Q_{\delta_0}^\lambda) \leq c' \delta_0 \phi_q(Dv, Q_{r/2}^\lambda),
\]  

(3.24)

where \( \delta = \delta_0/2 \in (0, 1/2) \). Moreover, by the definition of \( \phi_q \), we see that

\[
\phi_q(Dv, Q_{r/2}^\lambda) \leq \left( \int_{Q_{r/2}^\lambda} |Dv - m(Dv, Q_{r/2}^\lambda)|^q \, dx \, dt \right)^{1/q} \leq 2^{(n+2)/q} \phi_q(Dv, Q_{r}^\lambda).
\]  

(3.25)

Combining (3.24) and (3.25), we obtain (3.14) for any \( \delta \in (0, 1/2) \). The proof is now completed.

Similarly, we have the following result for relatively large \( s \).

**Proposition 3.8** Let \( \lambda, A > 0 \) be constants. Assume that

\[
\sup_{Q_{r}^\lambda} \|Dv\| \leq A \lambda \quad \text{and} \quad 0 < \xi \lambda \leq s \leq \xi_1 A \lambda, \quad \text{where} \quad 0 < \xi < \xi_1 A.
\]  

(3.26)

Then there exist constants \( \beta_1 \in (0, 1) \) depending on \( n, p, v, L, A, \xi, \xi_1, \) and \( \tilde{c}_d \geq 1 \) depending on \( n, p, v, L, A, \xi, \xi_1, q, \) such that

\[
\phi_q(Dv, Q_{r}^\lambda) \leq \tilde{c}_d \delta_0^\beta \phi_q(Dv, Q_{r}^\lambda)
\]  

(3.27)

holds for any \( \delta \in (0, 1) \).
Proof First, we rescale the solution in $Q_1$ as in (3.16), but this time with $r_1 = r$. Then the rescaled solution $\tilde{v}$ stills satisfies (3.17) in $Q_1$ and $\tilde{v}_{x_i}$ solves (3.19). This time we can use (3.26) to get

$$
(A(x, t)\eta, \eta) \geq v\lambda^2 - \nu^2 (|D\tilde{v}(x, t)|^2 + s^2)^{\frac{p-2}{2}}|\eta|^2 \geq v(n + \xi^n_1)^{\frac{p-2}{2}} A^{p-2} |\eta|^2,
$$

and

$$
|A(x, t)| \leq L\lambda^2 - \nu^2 (|D\tilde{v}(x, t)|^2 + s^2)^{\frac{p-2}{2}} \leq \xi^{p-2} L.
$$

Then, we can proceed exactly as in Proposition 3.7 and eventually obtain (3.27).

Now we are ready to give the proofs of Theorems 3.2 and 3.3.

Proof of Theorem 3.2 Our proof closely follows that of [17, Theorem 3.1] using the “singular iteration” argument with Propositions 3.7, 3.8, and 3.5 in place of [17, Propositions 3.8, 3.9, and 3.11]. More specifically, the case when Proposition 3.7 occurs is called the nonsingular alternative and the case when Proposition 3.5 occurs is called the singular alternative. The main idea is to construct a chain of intrinsic cylinders where the singular alternative occurs and show that the chain will stop in a finite time because of the occurrence of the nonsingular alternative. As in [17], we first assume $s > 0$ in Steps 1-5 below.

Step 1: Stopping time for the singular iteration and the choice of $\xi$. Let $\eta_1 := (1 + \eta)/2 \in (0, 1)$, where $\eta$ is given in Proposition 3.5, and therefore $\eta_1$ depends only on $n, p, v, L,$ and $A$. Define $m$ as the smallest integer such that

$$
\eta_1^m A < \frac{1}{2B}.
$$

Thus we know that $m$ depends on $n, p, v, L, A, B$, and we have

$$
\frac{\log(2AB)}{-\log(\eta_1)} < m \leq 1 + \frac{\log(2AB)}{-\log(\eta_1)}.
$$

Next we choose

$$
\xi := \min\{1/8, (\eta_1 - \eta)\eta_1^{2m} A\},
$$

which also depends on $n, p, v, L, A,$ and $B$.

Step 2: The first nonsingular case: $\xi \lambda \leq s$. Proposition 3.8 implies that (3.3) holds whenever

$$
\delta_\gamma \leq (\gamma/\tilde{c}_d)^{1/\beta_1}.
$$

Note that here $\beta_1 \in (0, 1)$ depends only on $n, p, v, L, A, B,$ and $\tilde{c}_d \geq 1$ depends on $n, p, v, L, A, B,$ and $q$. From now on, we assume instead

$$
s < \xi \lambda \leq (\eta_1 - \eta)\eta_1^{2m} A \lambda,
$$

where the second inequality always holds by the definition of $\xi$ in (3.30).

Step 3: The singular iteration. Given a cylinder $Q_1^\lambda$, where (3.6) holds, then by Propositions 3.7 and 3.5, we know that at least one of (3.14) and (3.9) must hold. The case when Proposition 3.7 applies is called the nonsingular alternative, while the case when Proposition 3.5 applies is called the singular alternative. We now let $\sigma_1 = \sigma/2$, where $\sigma$ is the constant in Proposition 3.4, and therefore $\sigma_1$ depends only on $n, p, v, L,$ and $A$. We define the sequences $\lambda_0 := \lambda, \lambda_{i+1} := \eta_1 \lambda_i$ and $R_0 := r, R_{i+1} := \sigma_1 R_i$. Exactly as in the proof of [17, Theorem 3.1], we build the singular iteration scheme by induction. Assume that the singular alternative holds.
in $Q_{R_i}^{\lambda_i - 1}$ whenever $i \in \{1, \ldots, j\}$ for some $1 \leq j \leq 2m$. Arguing exactly as in [17], under the condition (3.32) we obtain
\[
 s + \sup_{Q_{R_j}^{\lambda_j}} \|Dv\| \leq A \lambda_j, \quad (3.33)
\]
which verifies the upper bound (3.6) in the cylinder $Q_{R_j}^{\lambda_j}$. Then we can determine whether the singular alternative or the nonsingular alternative occurs in $Q_{R_j}^{\lambda_j}$.

**Step 4: The second nonsingular case.** Let
\[
 \delta_{\gamma} \leq \delta := \left(\eta_{1}^{(2-p)/2} \sigma_{1}\right)^{m+1}, \quad (3.34)
\]
and define
\[
 \tilde{m} = \min\{k \in \mathbb{N} : \text{the singular alternative does not hold in } Q_{R_k}^{\lambda_k}\}. \quad (3.35)
\]
By (3.2), (3.32), and the definition of $m$ in (3.28), arguing exactly as in [17], we have $\tilde{m} \leq m$. Moreover, by (3.33) and the second inequality in (3.2), it holds that
\[
 s + \sup_{Q_{R_j}^{\lambda_j}} \|Dv\| \leq A \lambda_j, \quad \text{for } j \in \{0, \ldots, \tilde{m}\}. \quad (3.36)
\]
Since (3.36) in particular holds for $j = \tilde{m}$, we can apply Proposition 3.7 in $Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}$ and get
\[
 \phi_q(Dv, Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}) \leq c_d \delta^\beta \phi_q(Dv, Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}) \quad (3.37)
\]
for any $\delta \in (0, 1)$, where $c_d \geq 1$ is a constant depending on $n$, $p$, $\nu$, $L$, $A$, and $q$. Moreover, using the fact that $\tilde{m} \leq m$, we have
\[
 \phi_q(Dv, Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}) \leq \left(\frac{|Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}|}{|Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}|}\right)^{1/q} \left(\int_{Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}} |Dv - m(Dv, Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}})|^q \, dx \, dt\right)^{1/q} \leq \sigma_1^{-m(n+2)/q} \eta_{1}^{-m(2-p)/q} \phi_q(Dv, Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}). \quad (3.38)
\]
Furthermore, by the definition of $\delta$ in (3.34) and again the fact that $\tilde{m} \leq m$, we have $Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}} \subset Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}$ and thus
\[
 \phi_q(Dv, Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}) \leq \left(\frac{|Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}|}{|Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}|}\right)^{1/q} \left(\int_{Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}} |Dv - m(Dv, Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}})|^q \, dx \, dt\right)^{1/q} \leq \delta^{-(n+2)/q} \phi_q(Dv, Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}). \quad (3.39)
\]
Combining (3.37), (3.38), and (3.39), we obtain
\[
 \phi_q(Dv, Q_{R_{\delta R_{\tilde{m}}}}^{\lambda_{\delta R_{\tilde{m}}}}) \leq c_d \delta^\beta \left[\delta^{-(n+2)}/\sigma_1^{-m(n+2)} \eta_{1}^{-m(2-p)}\right]^{1/q} \phi_q(Dv, Q_{R_{\tilde{m}}}^{\lambda_{\tilde{m}}}) \quad (3.40)
\]
for any $\delta \in (0, 1)$. Thus (3.3) holds for any
\[
 \delta_{\gamma} := \delta_{\gamma}, \quad \text{with } \delta \leq \left(\frac{\gamma}{c_d}\right)^{\frac{1}{p}} \left(\delta^{n+2}/\sigma_1^{-m(n+2)} \eta_{1}^{-m(2-p)}\right)^{1/p}. \quad (3.40)
\]
Step 5: Determining the constant $\delta_\gamma$. In view of conditions (3.31) and (3.40), we can define

$$\delta_\gamma = \delta \left( \tilde{\delta}^{n+2} \sigma_1^{m(n+2)} \eta_1^{m(2-p)} \right)^{\frac{1}{\bar{p}^\nu}} \left( \frac{\gamma}{c_d + \tilde{c}_d} \right)^{\frac{1}{\min[p, \eta_1]}},$$

where we recall that $\tilde{\delta} = (\eta_1^{(2-p)/2} \sigma_1)^{m+1}$. The only differences are that $v$ is in place of $w$ and that instead of choosing $\tilde{\gamma} = 2^{-(n+2)\gamma}$ as in [17, Eq. (3.86)], we now take $\tilde{\gamma} = 2^{-(n+2)/q} \gamma$. We also remark that for a strongly convergent sequence in $L^p$-mean oscillation of the sequence also converges for any $\delta' > 0$.

Step 6: The case $s = 0$ via approximation. The approximation scheme introduced in [17, Section 3.3] also works perfectly here. The only differences are that $v$ is in place of $w$ and that instead of choosing $\tilde{\gamma} = 2^{-(n+2)\gamma}$ as in [17, Eq. (3.86)], we now take $\tilde{\gamma} = 2^{-(n+2)/q} \gamma$. We also remark that for a strongly convergent sequence in $L^p$ for some $p \geq 1$, the $L^q$-mean oscillation of the sequence also converges for any $q \in (0, 1)$. Indeed, by the triangle inequality, we have

$$\phi_q(Dv_e, Q_\beta^\lambda)^q \leq \int_{Q_\beta} |Dv_e - m(Dv, Q_\beta^\lambda)|^q \, dx \, dt$$

$$\leq \phi_q(Dv, Q_\beta^\lambda)^q + \int_{Q_\beta} |Dv_e - Dv|^q \, dx \, dt.$$

Therefore if $Dv_e \to Dv$ strongly in $L^p(Q_\beta^\lambda)$, then by H"older’s inequality and the previous estimate, we have $\phi_q(Dv_e, Q_\beta^\lambda) \to \phi_q(Dv, Q_\beta^\lambda)$. □

**Proof of Theorem 3.3** The proof also closely follows that of [17, Theorem 3.5] and therefore we only give an outline and indicate the necessary modifications. As in the proof of Theorem 3.2, we assume $s > 0$ and use the singular iteration. The case $s = 0$ follows by applying the same approximation scheme as in the proof of Theorem 3.2. We still define $\tilde{m}$ as in (3.35). However, in this time since (3.32) is no longer in force, the iteration may stop at $\tilde{m}$ either because the nonsingular alternative occurs or because the upper bound

$$s + \sup_{Q_\tilde{m}^\lambda} \|Dv\| \leq A\lambda_{\tilde{m}}$$

does not hold. In the former case, we proceed as in Theorem 3.2 and we can take $\delta_\gamma$ as in (3.40). In the latter case, exactly as in Step 4 of the proof of [17, Theorem 3.5], we can show that

$$\sup_{Q_\tilde{m}^\lambda} \|Dv\| \leq A\lambda_{\tilde{m}} \quad \text{and} \quad (\eta_1 - \eta) A\lambda_{\tilde{m}} \leq s \leq \frac{A}{\eta_1} \lambda_{\tilde{m}}.$$

Applying Proposition 3.8 in $Q_{\tilde{m}^\lambda}$ with $\xi = (\eta_1 - \eta) A$ and $\xi_1 = 1/\eta_1$, we obtain

$$\phi_q(Dv, Q_{\tilde{m}^\lambda}) \leq c_d^\xi \delta_\gamma^\beta_1 \phi_q(Dv, Q_{\tilde{m}^\lambda}),$$

(3.41)

for any $\delta \in (0, 1)$, where the constant $c_d^\xi \geq 1$ depends on $n, p, \nu, L, A, q$ and the constant $\beta_1 \in (0, 1)$ depends only on $n, p, \nu, L, A$. With the estimate (3.41) in place of (3.37), we
can proceed as in Theorem 3.2, Step 4. In this case, (3.4) holds for any
\[
\delta_{\gamma} := \delta_{\tilde{\gamma}}, \quad \text{with} \quad \delta \leq \left(\frac{\gamma}{\epsilon_d}\right)^{\frac{1}{p_1}} \left(\tilde{\delta}^{n+2} \eta_1^{m(2-p)} \eta_1^{m(2-p)}\right)^\frac{1}{q \beta_1}.
\]

Finally, to ensure the above inequality and (3.40), we can choose
\[
\delta_{\gamma} := \tilde{\delta} \left(\frac{\gamma}{\epsilon_d}\right)^{\frac{1}{p_1}} \left(\tilde{\delta}^{n+2} \eta_1^{m(2-p)} \eta_1^{m(2-p)}\right)^\frac{1}{q \beta_1},
\]
where \(\tilde{\delta} = \left(\eta_1^{(2-p)/2} \sigma_1\right)^{m+1}\). Since the constants \(\sigma_1, \eta_1, \beta, \beta_1\) depend only on \(n, p, v, L, A\), and the constants \(c_d, \tilde{c}_d\) depend only on \(n, p, v, L, A, q\), we know that the only constant depending on \(B\) in (3.42) is \(m\). From the characterization of \(m\) described in (3.29), we can take \(\delta_{\gamma}\) in the form of (3.5) by further decreasing the constant in (3.42). More specifically, we can take
\[
\alpha_1 := \frac{\min\{\beta, \beta_1\} q}{(n+5) + (2n+5) \log(\sigma_1)/\log(\eta_1)}, \quad \alpha_2 := \min\{\beta, \beta_1\},
\]
and
\[
c(A) := (2A)^{1/\alpha_1} \eta_1^{p-2} \sigma_1^{2} \left(\frac{c_d + \tilde{c}_d}{\sigma_1} \eta_1^{3(n+2)/q}\right)^\frac{1}{\alpha_2}.
\]
The proof is now completed. \(\square\)

## 4 Pointwise gradient estimates

This section is devoted to the proofs of the pointwise gradient estimates. We derive some comparison results in Sect. 4.1 and give the proofs of Theorem 1.1–Corollary 1.3 in Sect. 4.2.

### 4.1 Comparison results

Let \(\rho, \lambda > 0\) and \(Q^\lambda_\rho := Q^\lambda_\rho(x_0, t_0) \subset \Omega_T\) be a parabolic cylinder. We consider the unique solution \(w \in C^0([t_0 - \lambda^{2-p} \rho^2, t_0); L^2(B_\rho(x_0))) \cap L^p(t_0 - \lambda^{2-p} \rho^2, t_0); W^{1,p}(B_\rho(x_0))\) to the Cauchy-Dirichlet problem:

\[
\begin{cases}
aw_t - \text{div}(a(x, t, Dw)) = 0 & \text{in } Q^\lambda_\rho, \\
w = u & \text{on } \partial_{\text{par}} Q^\lambda_\rho.
\end{cases}
\]  

(4.1)

We have the following comparison estimate between \(u\) and \(w\) from \([28, \text{Lemma 3.1}]\).

**Lemma 4.1** Let \(w\) be a solution to (4.1) under the assumptions (1.2)–(1.5) and assume that \(p \in \left[\frac{3n+2}{2n+2}, 2 - \frac{1}{n+1}\right]\). Then there exists a constant \(c_3 = c_3(n, p, v, L, q) \geq 1\) such that

\[
\left(\int_{Q^\lambda_\rho} |Dw - Dw|^q \, dx \, dt\right)^{1/q} \leq c_3 \left[\frac{|\mu(Q^\lambda_\rho)}{|Q^\lambda_\rho|^{\frac{n+1}{n+2}}}\right]^{\frac{n+2}{n+1+p}} + c_3 \left[\frac{|\mu(Q^\lambda_\rho)}{|Q^\lambda_\rho|^{\frac{n+1}{n+2}}}\right] \left(\int_{Q^\lambda_\rho} (|Dw| + s)^q \, dx \, dt\right)^{(2-p)(n+1)}
\]
for any constant $q$ such that $\frac{n+2}{2(n+1)} < q < p - \frac{n}{n+1} \leq 1$.

We remark that only the case $s = 0$ was considered in [28, Lemma 3.2], but their proof also works in the case $s > 0$ with $V_s(\xi) := (|\xi|^2 + s^2)^{\frac{p-2}{2}} \xi$ in place of $V(\xi) := |\xi|^\frac{p-2}{2} \xi$ for $\xi \in \mathbb{R}^n$.

We also have a reverse Hölder type inequality for $Dw$.

**Theorem 4.2** Let $\lambda > 0$ and $w$ be a solution to (4.1) under the assumptions (1.2) and (1.3) and assume that $\max\{1, \frac{2n}{n+2}\} < p < 2$. Then there exists a constant $c_4 = c_4(n, p, \nu, L, q) \geq 1$ such that

$$
\int_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dx \, dt \leq c_4 \lambda^p + c_4 s^p + c_4^\lambda \frac{n(p-2)(p-q)}{n(p-2)+2q} \left( \int_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^q \, dx \, dt \right)^{\frac{n(p-2)+2p}{n(p-2)+2q}}
$$

(4.2)

holds for every $q \in \left( \frac{n(2-p)}{2}, \frac{2n}{n+2} \right)$ when $n \geq 2$, and that

$$
\int_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dx \, dt \leq c_4 \lambda^p + c_4 s^p + c_4^\lambda \frac{(p-2)(p-q)}{n(q-2)} \left( \int_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^q \, dx \, dt \right)^{\frac{2p-2}{p-q-2}}
$$

(4.3)

holds for every $q \in (2 - p, 1)$ when $n = 1$.

**Proof** This type of estimates can be deduced from higher integrability results as in [2, Theorem 1] and standard interpolation and iteration arguments as in [13, Remark 6.12]. See also [14], [28, Lemma 3.3]. For completeness, we give a direct proof below. First, we note that it suffices to show (4.2) and (4.3) for the special case when $\lambda = \rho = 1$, $(x_0, t_0) = 0$ by using a standard rescaling argument. Indeed, we can define

$$w_1(x, t) := (\lambda \rho)^{-1} w(x_0 + \rho x, t_0 + \lambda^2 - \rho \lambda^2 t)$$

and

$$a_1(x, t, \xi) := \lambda^{1-p} a(x_0 + \rho x, t_0 + \lambda^2 - \rho \lambda^2 t, \lambda \xi).$$

Then $w_1$ solves

$$\partial_t w_1 - \text{div}(a_1(x, t, Dw_1)) = 0 \quad \text{in} \quad Q_1^1(0) \equiv B_1(0) \times (-1, 0)$$

and $a_1$ satisfies the assumptions (1.2) and (1.3) with $s_1 := \lambda^{-1} s$ in place of $s$. We can see that if (4.2) (or (4.3)) holds for $w_1$ in $Q_1^1(0)$ with $s_1 := \lambda^{-1} s$ in place of $s$, then by rescaling back to $w$, (4.2) (or (4.3)) also holds for $w$ in $Q_\rho^\lambda$.

Next, we prove (4.2) and (4.3) for the case when $\lambda = \rho = 1$, $(x_0, t_0) = 0$. For simplicity, in this proof, we denote $B \equiv B_1(0)$, $Q \equiv Q_1^1(0) \equiv B_1(0) \times (-1, 0)$, and $\frac{1}{2}Q \equiv \frac{1}{2}Q_1^1(0) \equiv B_\frac{1}{2}(0) \times (-\frac{1}{4}, 0)$. First we take $\xi_1 \in C_0^\infty(B)$ such that $0 \leq \xi_1 \leq 1$ in $B$, $\xi_1 = 1$ in $\frac{1}{2}B$ and $|D\xi_1| \leq 4$ in $B$. We then define $\xi := \xi_1 / (\int_B \xi_1(x) \, dx)$ and therefore $\int_B \xi_1(x) \, dx = 1$.

Now we define $\tilde{w}(t) := \int_B w(x, t) \xi(x) \, dx$. If $n \geq 2$, it follows from the Sobolev–Poincaré inequality that

$$
\int_B |w(x, t) - \tilde{w}(t)|^2 \, dx \leq c \left( \int_B |Dw| \, dx \right)^n \frac{2^n}{n+2}
$$

(4.4)

holds for some constant $c = c(n)$. On the other hand, if $n = 1$, then we have

$$
\int_B |w(x, t) - \tilde{w}(t)|^2 \, dx \leq |B| \sup_B |w - \tilde{w}(t)|^2 \leq c \left( \int_B |Dw| \, dx \right)^2.
$$

(4.5)
We now proceed with the case when $n \geq 2$ and we will indicate necessary modifications for the case when $n = 1$ at the end of our proof. By (4.1), we know that
\[
\frac{d}{dt} \tilde{w}(t) = -\int_B a(x, t, Dw) D\zeta dx := g(t)
\]
holds in distribution sense and therefore the Eq. (4.1) also reads
\[
(w - \tilde{w})_t - \text{div}(a(x, t, Dw)) + g(t) = 0 \quad \text{in} \quad Q.
\]
(4.6)
Note that by (1.2), we have
\[
\int_{-1}^{0} |g(t)| dt \leq c \int_Q (|Dw|^2 + s^2)^{\frac{p-1}{2}} dx dt
\]
(4.7)
for some constant $c = c(n, p, L)$. Next, we choose a nondecreasing smooth function $\eta : \mathbb{R} \to [0, 1]$ satisfying $\eta \equiv 1$ on $[-1/4, \infty)$, $\eta \equiv 0$ on $(-\infty, -1]$ and $|\eta'| \leq 2$ in $\mathbb{R}$. We then take $\phi$ as the product of $\zeta$ and $\eta$, namely, $\phi(x, t) = \zeta(x)\eta(t)$ for every $(x, t) \in Q$. We also choose
\[
\tilde{p} := \max\left\{ \frac{2p}{2-p}, \frac{n+2}{n-1} \right\}.
\]
Now formally we test the Eq. (4.6) with $\psi := (w - \tilde{w})\tilde{p}1_{(-\infty, \tau]}(t)$ for $\tau \in (-1, 0)$ and get
\[
0 = \frac{1}{2} \int_B (w(x, \tau) - \tilde{w}(\tau))^2 (\phi(x, \tau))^{\tilde{p}} dx - \frac{\tilde{p}}{2} \int_{B \times (-1, \tau]} (w - \tilde{w})^2 \phi^{\tilde{p}-1} \phi_t dx dt
\]
\[
+ \tilde{p} \int_{B \times (-1, \tau]} a(x, t, Dw)(w - \tilde{w})\phi^{\tilde{p}-1} D\phi dx dt
\]
\[
+ \int_{B \times (-1, \tau]} a(x, t, Dw)D\phi^{\tilde{p}} dx dt + \int_{B \times (-1, \tau]} (w - \tilde{w})\phi^{\tilde{p}} g(t) dx dt
\]
\[
:= I + II + III + IV + V.
\]
(4.8)
Note that the computations above can be justified in a standard way using Steklov averages as in [5]. Then we estimate the terms in (4.8). By Hölder’s inequality, Young’s inequality with conjugate exponents ($\tilde{p}$, $\tilde{p}/(\tilde{p} - 1)$), and (4.4), we have
\[
|II| \leq \frac{\tilde{p}}{2} \int_{-1}^{0} \left( \int_B (w - \tilde{w})^2 \phi^{\tilde{p}} dx \right)^{\frac{\tilde{p}-1}{\tilde{p}}} \left( \int_B (w - \tilde{w})^2 \phi^{\tilde{p}} dx \right)^{\frac{1}{\tilde{p}}} dt
\]
\[
\leq \frac{\tilde{p}}{2} \sup_{t \in (-1, 0)} \left( \int_B (w - \tilde{w}(t))^2 \phi^{\tilde{p}} dx \right)^{\frac{\tilde{p}-1}{\tilde{p}}} \left( \int_{-1}^{0} \left( \int_B (w - \tilde{w})^2 \phi^{\tilde{p}} dx \right)^{\frac{1}{\tilde{p}}} dt \right)
\]
\[
\leq \frac{1}{8} \sup_{t \in (-1, 0)} \int_B (w - \tilde{w}(t))^2 \phi^{\tilde{p}} dx + c \left( \int_{-1}^{0} \left( \int_B (w - \tilde{w})^2 dx \right)^{\frac{1}{\tilde{p}}} dt \right)^{\tilde{p}}
\]
(4.9)
\[
\leq \frac{1}{8} \sup_{t \in (-1, 0)} \int_B (w - \tilde{w}(t))^2 \phi^{\tilde{p}} dx + c \left( \int_{-1}^{0} |Dw|^{\frac{2n}{p+2}} dx \right)^{\frac{n+2}{n}}
\]
\[
\leq \frac{1}{8} \sup_{t \in (-1, 0)} \int_B (w - \tilde{w}(t))^2 \phi^{\tilde{p}} dx + c \left( \int_Q |Dw|^{\frac{2n}{p+2}} dx dt \right)^{\frac{n+2}{n}}
\]
for some constant $c = c(n, p)$. Here we also used the fact that $\frac{n+2}{np} \leq 1$ in the last line.
By (1.2), Young’s inequality with exponents \((p/(p - 1), p)\) and \((2/p, 2/(2 - p))\), and the fact that \(\frac{2}{p} (\bar{p} - p) \geq \bar{p}\), we have

\[
|III| \leq \bar{p} L \int_{B \times (-1, \tau)} (s^2 + |Dw|^2)^{p-1} |w - \bar{w}| |\varphi| |D\varphi| \, dx \, dt
\]

\[
\leq \frac{v}{2} \int_{B \times (-1, \tau)} (s^2 + |Dw|^2)^{\frac{p}{2}} \varphi \bar{p} \, dx \, dt + c \int_{B \times (-1, \tau)} \varphi \bar{p} |w - \bar{w}| |D\varphi| \, dx \, dt
\]

\[
\leq \frac{v}{2} \int_{B \times (-1, \tau)} (s^2 + |Dw|^2)^{\frac{p}{2}} \varphi \bar{p} \, dx \, dt + \frac{1}{8} \int_{B \times (-1, \tau)} |w - \bar{w}|^2 \varphi \bar{p} \, dx \, dt
\]

\[
+ c \int_{B \times (-1, \tau)} |D\varphi|^{\frac{2p}{p+2}} \, dx \, dt
\]

\[
\leq \frac{v}{2} \int_{B \times (-1, \tau)} |Dw|^p \varphi \bar{p} \, dx \, dt + \frac{1}{8} \sup_{t \in (-1, 0)} \int_{B} (w - \bar{w}(t))^2 \varphi \bar{p} \, dx + c(1 + s^p)
\]

for some constant \(c = c(n, p, v, L)\). Here \(v\) is the constant in (1.3).

By (1.3), we know that \(a(x, t, \xi) \xi \leq v(s^2 + |\xi|^2)^{\frac{p-2}{2}} |\xi|^2 \geq v(\xi^p - s^p)\) for every \(\xi \in \mathbb{R}^n\) and therefore

\[
IV \geq v \int_{B \times (-1, \tau)} |Dw|^p \varphi \bar{p} \, dx \, dt - cs^p
\]

for some constant \(c = c(n)\).

Finally, by Hölder’s inequality, Young’s inequality with conjugate exponents 2 and 2, and the estimate (4.7), we obtain

\[
|V| \leq \int_{-1}^{\tau} \left( \int_{B} |w - \bar{w}|^2 \varphi \bar{p} \, dx \right)^{\frac{1}{2}} \left( \int_{B} |g(t)|^2 \varphi \bar{p} \, dx \right)^{\frac{1}{2}} \, dt
\]

\[
\leq c \sup_{t \in (-1, 0)} \left( \int_{B} (w - \bar{w}(t))^2 \varphi \bar{p} \, dx \right)^{\frac{1}{2}} \int_{-1}^{\tau} |g(t)| \, dt
\]

\[
\leq \frac{1}{8} \sup_{t \in (-1, 0)} \int_{B} (w - \bar{w}(t))^2 \varphi \bar{p} \, dx + c \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^{\frac{1}{2}} + c(1 + s^p)
\]

for some constant \(c = c(n, p, L)\). Here we also used the fact that 2 \((p - 1) \leq p\) and therefore \(s^{2(p - 1)} \leq 1 + s^p\) in the last inequality. Using the estimates (4.9)–(4.12) in (4.8), we get

\[
\frac{1}{2} \int_{B} (w(x, \tau) - \bar{w}(\tau))^2 (\varphi(x, \tau))^p \, dx + \frac{v}{2} \int_{B \times (-1, \tau)} |Dw|^p \varphi \bar{p} \, dx \, dt
\]

\[
\leq \frac{3}{8} \sup_{t \in (-1, 0)} \int_{B} (w - \bar{w}(t))^2 \varphi \bar{p} \, dx + c \left( \int_{Q} |Dw|^{\frac{2n}{p+2}} \, dx \, dt \right)^{\frac{n+2}{n}}
\]

\[
+ c \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^{\frac{1}{2}} + c(1 + s^p)
\]

for another constant \(c = c(n, p, v, L)\). By taking the supremum over \(\tau \in (-1, 0)\), we have

\[
\frac{1}{8} \sup_{t \in (-1, 0)} \int_{B} (w - \bar{w}(t))^2 \varphi \bar{p} \, dx
\]

\[
\leq c \left( \int_{Q} |Dw|^{\frac{2n}{p+2}} \, dx \, dt \right)^{\frac{n+2}{n}} + c \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^{\frac{1}{2}} + c(1 + s^p),
\]
and therefore
\[ \int_{\frac{1}{2}Q} |Dw|^p \, dx \, dt \leq c \int_{B \times (-1,0)} |Dw|^p \varphi^\beta \, dx \, dt \]
\[ \leq c \left( \int_{Q} |Dw|^{\frac{2n}{n+2}} \, dx \, dt \right)^{\frac{n+2}{n}} + c \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^2 + c(1 + s^p) \]  
(4.13)
for some constant \( c = c(n, p, \nu, L) \). Next, we interpolate the terms in (4.13). We recall that
\( p \in \left( \frac{2n}{n+2}, 2 \right) \) and \( q \in \left( \frac{n(2-p)}{2}, \frac{2n}{n+2} \right) \) for \( n \geq 2 \). Since \( q < \frac{2n}{n+2} < p \), by Hölder’s inequality, we have
\[ \left( \int_{Q} |Dw|^{\frac{2n}{n+2}} \, dx \, dt \right)^{\frac{n+2}{n}} \leq \left( \int_{Q} |Dw|^p \, dx \, dt \right)^{\alpha \frac{n+2}{n}} \left( \int_{Q} w^n \, dx \, dt \right)^{(1-\alpha) \frac{n+2}{n}}. \]  
(4.14)
where \( \alpha \in (0, 1) \) is a constant such that \( \alpha p + (1 - \alpha)q = \frac{2n}{n+2} \). Namely,
\[ \alpha = \left( \frac{2n}{n+2} - q \right)/(p - q). \]
Since \( q > \frac{n(2-p)}{2} \), we know that \( \alpha \frac{n+2}{n} < 1 \). Therefore, by (4.14) and Young’s inequality with exponents \( 1/(\alpha \frac{n+2}{n}) \) and \( 1/(1 - \alpha \frac{n+2}{n}) \), we obtain
\[ \left( \int_{Q} |Dw|^{\frac{2n}{n+2}} \, dx \, dt \right)^{\frac{n+2}{n}} \leq \epsilon \int_{Q} |Dw|^p \, dx \, dt + c \epsilon \left( \int_{Q} |Dw|^q \, dx \, dt \right)^{\frac{n(p-2)+2p}{n(p-2)+2q}} \]  
(4.15)
for any \( \epsilon \in (0, 1) \), where \( c \epsilon \) is a constant depending only on \( \epsilon, n, p, \) and \( q \). We then estimate the term \( \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^2 \). Since \( p - 1 < 1 \leq \frac{2n}{n+2} \), by Hölder’s inequality, we have
\[ \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^2 \leq c \left( \int_{Q} |Dw|^{\frac{2n}{n+2}} \, dx \, dt \right)^{\frac{n(p-2)+2p}{n(p-2)+2q}} \]  
(4.16)
for some constant \( c = c(n, p, q) \). Therefore, it follows by using (4.15) and (4.16) in (4.13), and a standard iteration argument that
\[ \int_{\frac{1}{2}Q} (|Dw| + s)^p \, dx \, dt \leq c(1 + s^p) + c \left( \int_{Q} (|Dw| + s)^q \, dx \, dt \right)^{\frac{n(p-2)+2p}{n(p-2)+2q}} \]
holds for every \( q \in \left( \frac{(2-p)n}{2}, \frac{2n}{n+2} \right) \) and some constant \( c = c(n, p, \nu, L, q) \). Note that the last estimate is exactly (4.2) in the case when \( \lambda = \rho = 1, (x_0, t_0) = 0 \). The proof for the case when \( n \geq 2 \) is now completed. Finally, briefly indicate the modifications needed for the case when \( n = 1 \). In this case, we test the Eq. (4.6) with \( (w - \bar{w})\varphi^\beta \) \( 1_{(-\infty, \tau)}(t) \) for \( \tau \in (-1, 0) \), where
\[ \varphi^\beta := \max \left\{ \frac{2p}{2-p}, 2 \right\}. \]
Arguing exactly as in the case when \( n \geq 2 \), with the estimate (4.5) in place of (4.4), we obtain
\[ \int_{\frac{1}{2}Q} |Dw|^p \, dx \, dt \leq c \left( \int_{Q} |Dw| \, dx \, dt \right)^2 + c \left( \int_{Q} |Dw|^{p-1} \, dx \, dt \right)^2 + c(1 + s^p). \]  
(4.17)
Finally, it follows from the last estimate and similar interpolation and iteration arguments as in the case when \( n \geq 2 \) that
\[
\int_{\frac{1}{2}Q^\lambda} (|Dw| + s)^p \, dx \, dt \leq c(1 + s^p) + c\left( \int_{Q} (|Dw| + s)^q \, dx \, dt \right)^{\frac{2p-2}{p+q-2}}
\]
holds for every \( q \in (2 - p, 1) \) and some constant \( c = c(p, v, L, q) \). This proves (4.3) in the case when \( \lambda = \rho = 1 \) and \((x_0, t_0) = 0\). The Theorem is now proved. \( \square \)

**Remark 4.3** It can be seen from the proof of Theorem 4.2 that when \( \max\{1, \frac{2n}{n+2}\} < p < 2 \), reverse Hölder type estimates actually hold for every \( q > \frac{n(2-p)}{2} \) if \( n \geq 2 \) and for every \( q > 2 - p \) if \( n = 1 \). Indeed, by (4.13), (4.17), Hölder’s inequality, and similar rescaling arguments, there exists a constant \( c = c(n, p, v, L, q) \geq 1 \) such that
\[
\int_{\frac{1}{2}Q^\lambda} (|Dw| + s)^p \, dx \, dt \leq c\lambda^p + cs^p + c\lambda^{p-2}\left( \int_{Q^\lambda} (|Dw| + s)^q \, dx \, dt \right)^{\frac{2}{q}}
\]
holds for every \( q \geq \frac{2n}{n+2} \) if \( n \geq 2 \), and for every \( q \geq 1 \) if \( n = 1 \).

Now we consider the unique solution \( v \in C^0([t_0 - \lambda^2 - p \rho^2, t_0]; L^2(B_\rho(x_0))) \cap L^p(t_0 - \lambda^2 - p \rho^2, t_0; W^{1,p}(B_\rho(x_0))) \) to the Cauchy-Dirichlet problem:
\[
\begin{cases}
  v_t - \text{div}(a(x_0, t, Dv)) = 0 & \text{in } \frac{1}{2}Q^\lambda, \\
  v = w & \text{on } \partial \text{par}(\frac{1}{2}Q^\lambda). 
\end{cases}
\]

(4.18)

We have the following comparison result between \( w \) and \( v \).

**Lemma 4.4** Let \( w \) be a solution to (4.1) and \( v \) be a solution to (4.18) under the assumptions (1.2)–(1.5). Assume that \( p \in (1, 2) \). Then there exists a constant \( c_5 = c_5(n, p, v, L) \geq 1 \), such that
\[
\int_{\frac{1}{2}Q^\lambda} |Dw - Dv|^p \, dx \, dt \leq c_5[\omega(\rho)]^p \int_{\frac{1}{2}Q^\lambda} (|Dw| + s)^p \, dx \, dt.
\]

(4.19)

**Proof** By [16, Remark 4.1], we know that
\[
\int_{\frac{1}{2}Q^\lambda} |V_\delta(Dw) - V_\delta(Dv)|^2 \, dx \, dt \leq c[\omega(\rho/2)]^2 \int_{\frac{1}{2}Q^\lambda} (|Dw| + s)^p \, dx \, dt.
\]

(4.20)

where \( V_\delta(\xi) := (|\xi|^2 + \delta^2)^{-\frac{p-2}{2}} \xi \) for \( \xi \in \mathbb{R}^n \) and \( c > 0 \) is a constant depending only on \( n, p, v, \) and \( L \). Also similar to [16, Eq. (4.11)], we have the following inequality
\[
|Dw - Dv| \leq c(|Dw| + s)^{(2-p)/2}|V_\delta(Dw) - V_\delta(Dv)| + c|V_\delta(Dv) - V_\delta(Dw)|^{2/p},
\]

and therefore by Young’s inequality with exponents \( 2/(2 - p) \) and \( 2/p \), and the fact that \( \omega(\rho) \in [0, 1] \) for every \( \rho > 0 \), we obtain
\[
|Dw - Dv|^p \leq c[\omega(\rho)]^p (|Dw| + s)^p + c[\omega(\rho)]^{p-2}|V_\delta(Dw) - V_\delta(Dv)|^2.
\]

(4.21)

Thus (4.19) follows from (4.20), (4.21) and the fact that \( \omega \) is a nondecreasing function. \( \square \)

We also have a Lipschitz estimate for \( v \) from [5, Chapter 8, Theorem 5.2'].
Lemma 4.6 Assume that
\[ \sup_{\frac{1}{2} Q^R_\rho} \| Dv \| \leq c_6(\lambda + s) + c_6\lambda \frac{n(p-2)}{(n+2)(p-2)+2q} \left( \int_{\frac{1}{2} Q^R_\rho} |Dv| + s \right) \frac{q}{q+1} \frac{1}{\rho^{n+1}}. \]

In the rest of this section, we always assume \( p \in (p^*(n), 2 - \frac{1}{n+1}] \) and \( q \in \left( \max\left\{ \frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}, \frac{n}{n+1} \right\} \right), \)

where \( p^*(n) \) is defined in (1.10) so that all of the assumptions in Lemma 4.1–Theorem 4.5 are satisfied. In particular, we have \( q \in \left( \frac{1}{2}, 1 \right) \) and therefore
\[ (a+b)^q \leq a^q + b^q, \quad (a+b)^{1/q} \leq 2a^{1/q} + 2b^{1/q}, \quad \forall a, b > 0. \] (4.22)

Lemma 4.6 Assume that
\[ \left( \int_{\frac{1}{2} Q^R_\rho} |Du| + s \right)^\frac{1}{q} \leq \frac{|\mu|(Q^R_\rho)}{\rho^{n+1}} \leq \lambda. \] (4.23)

Then there exists a constant \( c_7 = c_7(n, p, v, L, q) \geq 1 \), such that
\[ \left( \int_{\frac{1}{2} Q^R_\rho} |Du - Dv|^q \right)^\frac{1}{q} \leq c_7 \omega(\rho)\lambda + c_7 \frac{|\mu|(Q^R_\rho)}{\rho^{n+1}}. \] (4.24)

Proof First, by (4.23) and Lemma 4.1, we can argue as in the proof of [17, Corollary 4.4] and get
\[ \int_{Q^R_\rho} |Du - Dv|^q \right) \right)^\frac{1}{q} \leq \frac{c_7 |\mu|(Q^R_\rho)}{\rho^{n+1}}. \] (4.25)

Indeed, we know that
\[ \frac{|\mu|(Q^R_\rho)}{\rho^{n+1}} = |B_1|^{\frac{n+1}{p+2} \frac{(n+1)(p-2)}{p+2} \frac{|\mu|(Q^R_\rho)}{\rho^{n+1}}}, \] (4.26)

where \( |B_1| \) is the volume of a unit ball in \( \mathbb{R}^n \). Moreover, by (4.26) and the second inequality in (4.23), we also have
\[ \left( \frac{|\mu|(Q^R_\rho)}{\rho^{n+1}} \right)^{\frac{n+1}{p+2} \frac{(n+1)(p-2)}{p+2} \frac{|\mu|(Q^R_\rho)}{\rho^{n+1}}}, \] (4.27)

The estimate (4.25) now follows by applying Lemma 4.1 and using (4.27), (4.26), and the first inequality in (4.23).

Next, using (4.23), (4.25), and the triangle inequality, we have
\[ \int_{Q^R_\rho} (|Dw| + s)^q \right) \leq c_\lambda^q \cdot \] (4.28)
From (4.23) we also have \( s \leq \lambda \), which together with (4.28) and Theorem 4.2 implies
\[
\int_{\frac{1}{2}Q_\rho} (|Dw| + s)^p \, dx \, dt \leq c \lambda^p.
\]
Thus, Lemma 4.4, the last inequality, and Hölder’s inequality imply
\[
\int_{\frac{1}{2}Q_\rho} |Dw - Dv|^q \, dx \, dt \leq c \omega(\rho)^{q^2} \lambda^q.
\] (4.29)
The estimate (4.24) now follows using (4.25), (4.29) and the triangle inequality.

**Lemma 4.7** Let \( \delta, \theta \in (0, 1/2) \). Assume that
\[
\left( \int_{\frac{1}{2}Q_\rho} (|Du| + s)^q \, dx \, dt \right)^{\frac{1}{q}} \leq \lambda, \quad \frac{|\mu| (Q_\lambda^\rho)}{\rho^{n+1}} \leq \frac{\delta^{\frac{n+2}{2}} \theta^\frac{1}{2}}{2c_7} - \lambda, \quad \omega(\rho) \leq \frac{\delta^{\frac{n+2}{2}} \theta^\frac{1}{2}}{2c_7},
\] (4.30)
where \( c_7 \) is the same constant as in Lemma 4.6. Then there exists a constant \( c_8 = c_8(n, p, v, L, q) \geq 1 \), such that
\[
s + \sup_{\frac{1}{2}Q_\rho} \|Dv\| \leq c_8 \lambda.
\]
Moreover, it holds that
\[
\int_{\delta Q_\rho^\lambda} |Du|^q \, dx \, dt - \theta \lambda^q \leq \int_{\delta Q_\rho^\lambda} |Dv|^q \, dx \, dt \leq \sqrt{n} \left( \sup_{\delta Q_\rho^\lambda} \|Dv\| \right)^q.
\]

**Proof** Using (4.30) and Lemma 4.6, we have
\[
\int_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dx \, dt \leq \delta^{n+2} \theta \lambda^q. \tag{4.31}
\]
This together with the first inequality in (4.30) and the triangle inequality implies
\[
\int_{\frac{1}{2}Q_\rho^\lambda} (|Dv| + s)^q \, dx \, dt
\leq \int_{\frac{1}{2}Q_\rho^\lambda} (|Du| + s)^q \, dx \, dt + \int_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dx \, dt \leq (2^{n+2} + 1) \lambda^q.
\]
Thus Theorem 4.5 and the last inequality imply
\[
s + \sup_{\frac{1}{2}Q_\rho^\lambda} \|Dv\| \leq c_8 \lambda
\]
for some constant \( c_8 = c_8(n, p, v, L, q) \geq 1 \). Moreover, using (4.31) and the triangle inequality, we also have
\[
\int_{\delta Q_\rho^\lambda} |Du|^q \, dx \, dt \leq \int_{\delta Q_\rho^\lambda} |Dv|^q \, dx \, dt + \int_{\delta Q_\rho^\lambda} |Du - Dv|^q \, dx \, dt
\leq \int_{\delta Q_\rho^\lambda} |Dv|^q \, dx \, dt + (2\delta)^{-(n+2)} \int_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dx \, dt
\leq \int_{\delta Q_\rho^\lambda} |Dv|^q \, dx \, dt + \theta \lambda^q.
\]
The lemma is proved. \( \Box \)
Lemma 4.8 Let $\delta \in (0, 1/4)$ and $\varepsilon \in (0, 1]$. Suppose that the bounds
\[
\left( \frac{1}{Q_\rho^\lambda} \int_{Q_\rho^\lambda} |(Du) + s|^q \, dx \, dt \right)^{\frac{1}{q}} \leq \lambda, \quad \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \leq \lambda
\]
hold, and that $Dv$ satisfies
\[
\phi_q(Dv, \delta Q_\rho^\lambda) \leq \frac{\varepsilon}{24(n+3)} \phi_q(Dv, \frac{1}{4} Q_\rho^\lambda).
\] (4.32)
Then
\[
\phi_q(Du, \delta Q_\rho^\lambda) \leq \phi_q(Dv, \delta Q_\rho^\lambda) + c_\gamma \delta^{-(n+2)/q} \left[ \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \right] + c_\gamma \delta^{-(n+2)/q} \omega(\rho) \lambda,
\] (4.33)
where $c_\gamma = c_\gamma(n, p, v, L, q)$ is the same constant as in Lemma 4.6.

**Proof** Recalling the definition of $\phi_q$ and using (4.32), (4.22), and the triangle inequality, we obtain
\[
\phi_q(Du, \delta Q_\rho^\lambda) \leq \phi_q(Du - m(Dv, \delta Q_\rho^\lambda), \delta Q_\rho^\lambda) dx dt \right)^{1/q} \\
\leq 2 \phi_q(Dv, \delta Q_\rho^\lambda) + 2 \left( \frac{1}{Q_\rho^\lambda} \int_{Q_\rho^\lambda} |Du - Dv| dx dt \right)^{1/q} \\
\leq \varepsilon 2^{-(4n+1)} \phi_q(Dv, \frac{1}{4} Q_\rho^\lambda) + 2(2\delta)^{-(n+2)/q} \left( \frac{1}{Q_\rho^\lambda} \int_{Q_\rho^\lambda} |Du - Dv| dx dt \right)^{1/q}.
\] (4.34)
Again using the definition of $\phi_q$ and the triangle inequality, we have
\[
\phi_q(Dv, \frac{1}{4} Q_\rho^\lambda) \leq 2 \phi_q(Dv, \frac{1}{4} Q_\rho^\lambda) + 2 \left( \frac{1}{Q_\rho^\lambda} \int_{Q_\rho^\lambda} |Du - Dv| dx dt \right)^{1/q} \\
\leq 2 \cdot 4^{(n+2)/q} \phi_q(Dv, Q_\rho^\lambda) + 2 \cdot 4(2^{(n+2)/q}) \left( \frac{1}{Q_\rho^\lambda} \int_{Q_\rho^\lambda} |Du - Dv| dx dt \right)^{1/q}.
\] (4.35)
Note that $q \in (1/2, 1)$. Therefore, the estimate (4.33) follows by combining (4.34) and (4.35) and applying Lemma 4.6. \qed

4.2 Proof of pointwise gradient estimates

**Proof of Theorem 1.1.** First, for $\lambda > 0$, we define the Lebesgue set
\[
L_\lambda := \left\{(x_0, t_0) \in \Omega_T : \lim_{\rho \to 0} \frac{1}{Q_\rho^\lambda(x_0,t_0)} |Du - Du(x_0, t_0)| dx dt = 0 \right\}.
\] (4.36)
Direct calculations imply that $L := L_1 = L_\lambda$ for all $\lambda > 0$. Moreover, by the Lebesgue differentiation theorem, $\Omega_T \setminus L$ has zero lebesgue measure. We will prove Theorem 1.1 for every $(x_0, t_0) \in L$.

**Step 1: Choices of constants and basic inequalities.** First, we take the constant $c_8 = c_8(n, p, v, L, q)$ in Lemma 4.7 and define
\[
A := c_8, \quad B := 2560000n, \quad \gamma := 2^{-4(n+3)}.
\] (4.37)
We fix the constants $\delta_\gamma$ and $\alpha$ in Theorem 3.3 with the choices of $A, B, \gamma$ in (4.37). We now define

$$\delta_1 = \delta_\gamma / 4, \quad r_i = \delta_1^i r_\lambda, \quad Q_i = Q_{r_i}^\lambda$$

for any integer $i \geq 0$. Next, we choose $k$ as the smallest integer larger than or equal to 2 such that

$$4\sqrt{n}A_1^{(k-1)/q} \leq \delta_1^{(n+2)/q}/1600,$$  \hspace{1cm} (4.38)

and we define

$$H_1 := 400\delta_1^{-(n+2)/q}, \quad H_2 := 5120000c_7\delta_1^{-(k+2)(n+2)/q}, \quad c := \max\{H_1, H_2\},$$  \hspace{1cm} (4.39)

where $c_7 = c_7(n, p, \nu, L, q)$ is the constant in Lemma 4.6. We then choose $R_0 = R_0(n, p, \nu, L, q) \in (0, \nu]$ to be the largest number in $(0, \nu]$ such that

$$\int_{\rho}^{2R_0} \omega(\rho) \frac{d\rho}{\rho} \leq \frac{\delta_1^{(k+2)(n+2)/q}}{5120000c_7}.$$  \hspace{1cm} (4.40)

The constants $c$ and $R_0$ defined in (4.39) and (4.40) are the same constants we choose in the statement of Theorem 1.1. The choice of $H_1$ in (4.39) and (1.12) imply that

$$\int_{Q_0} (|Du| + s)^q \, dx \, dt \leq 1 + \frac{\delta_1^{-(n+2)/q} \phi_q(Du, Q_0)}{q} \leq \frac{\lambda}{100}$$  \hspace{1cm} (4.41)

and that

$$s \leq \frac{\lambda}{400}. \hspace{1cm} (4.42)$$

Since $\delta_1 \in (0, 1/4)$, by using the comparison principle for the Riemann integrals, we have

$$\int_0^{2r_\lambda} \frac{|\mu|(Q_{r_\lambda}^\rho)}{\rho^{n+1}} \, d\rho \geq \sum_{i=0}^{\infty} \int_{r_i}^{2r_i} \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \, d\rho \geq \frac{1}{n+1} \left(1 - \frac{1}{2^{n+1}}\right) \sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}} \geq \delta_1^{n+2} \sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}}.$$  \hspace{1cm} (4.43)

The estimate (4.43) together with the choice of $H_2$ in (4.39) and (1.12) imply that

$$\sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}} \leq \frac{\delta_1^{(k+1)(n+2)/q}}{5120000c_7}.$$  \hspace{1cm} (4.44)

Similarly, since we have $r_\lambda \leq R_0$, the inequality (4.40) implies that

$$\sum_{i=0}^{\infty} \omega(r_i) \leq \delta_1^{-(n+2)} \int_0^{2r_\lambda} \omega(\rho) \frac{d\rho}{\rho} \leq \frac{\delta_1^{(k+1)(n+2)/q}}{5120000c_7}. \hspace{1cm} (4.45)$$

Step 2: Exit time argument.
For any integer $i \geq 0$, we define
\[
C_i := \left( \int_{Q_i} (|Du| + s)^q \, dx \, dt \right)^{\frac{1}{q}} + \delta_1^{-\frac{(n+2)}{q}} \phi_q(Du, Q_i).
\]
Thus $C_0 \leq \lambda/100$ by (4.41). We can now assume without loss of generality that there exists an exit time $i_e \geq 0$, satisfying
\[
C_{i_e} \leq \frac{\lambda}{100}, \quad C_j > \frac{\lambda}{100} \quad \forall \, j > i_e.
\]
In fact, if this is not true, we can always find an increasing subsequence $j_i$ such that $C_{j_i} \leq \frac{\lambda}{100}$ for every integer $i \geq 0$ and therefore,
\[
|Du(x_0, t_0)| \leq \limsup_{i \to \infty} \left( \int_{Q_{j_i}} |Du|^q \, dx \, dt \right)^{1/q} \leq \frac{\lambda}{100},
\]
since $(x_0, t_0) \in L$ is a Lebesgue point.

**Step 3: Estimates after the exit time.** The core of the proof is the following iteration lemma.

**Lemma 4.9** If $i \geq i_e$ and
\[
\left( \int_{Q_i} (|Du| + s)^q \, dx \, dt \right)^{\frac{1}{q}} \leq \lambda,
\]
then
\[
\phi_q(Du, Q_{i+1}) \leq \frac{1}{4} \phi_q(Du, Q_i) + c_7 \delta_1^{-\frac{(n+2)}{q}} \left[ \frac{[|\mu|(Q_i)]}{r_{i+1}^{n+1}} \right] + c_7 \delta_1^{-\frac{(n+2)}{q}} \omega(r_i) \lambda.
\]

**Proof** Let $w$ and $v$ be the solutions to (4.1) and (4.18) respectively, with $\rho = r_i$. We then take $\rho = r_i$ and $\theta = 1/1600$ in Lemma 4.7. By (4.46), (4.44), and (4.45), we can apply Lemma 4.7 both with $\delta = \delta_1$ and with $\delta = \delta_k$. We obtain
\[
s + \sup_{Q_{i+1}} \|Dv\| \leq s + \sup_{Q_i} \|Dv\| \leq c_8 \lambda = A \lambda
\]
and
\[
\int_{Q_{i+k}} |Du|^q \, dx \, dt - \frac{\lambda^q}{1600} \leq \int_{Q_{i+k}} |Dv|^q \, dx \, dt \leq \sqrt{n} \left( \sup_{Q_{i+k}} \|Dv\| \right)^q.
\]
Next, by Theorem 3.1 with $\rho = r_{i+k}$ and $r = r_{i+1}$, (4.48), and (4.38), we have
\[
\text{osc}_{Q_{i+k}} Dv \leq 4\sqrt{n} A \delta_1^{(k-1)\alpha} \leq \frac{\delta_1^{(n+2)/q}}{1600} \lambda.
\]
Again by (4.46), (4.44), and (4.45), we can apply Lemma 4.6 with $\rho = r_i$ and get
\[
\left( \int_{Q_{i+k}} |Du - Dv|^q \, dx \, dt \right)^{\frac{1}{q}} \leq \frac{\left[ \frac{1}{2} Q_i \right]}{|Q_{i+k}|} \left( \int_{Q_i} |Du - Dv|^q \, dx \, dt \right)^{\frac{1}{q}} \leq \frac{c_7}{2} \delta_1^{-(n+2)/q} \omega(r_i) \lambda + \frac{c_7}{2} \delta_1^{-(n+2)/q} \left[ \frac{[|\mu|(Q_i)]}{r_i^{n+1}} \right]
\]
\[
\leq \frac{\delta_1^{(n+2)/q}}{1600} \lambda.
\]
Therefore, from the above two inequalities,
\[
\phi_q(Du, Q, i + k) \leq 2^{1/q-1} \phi_q(Dv, Q, i + k) + 2^{1/q-1} \left( \int_{Q, i + k} |Du - Dv|^q \, dx \, dt \right)^{1/q} \\
\leq 2 \text{ osc } Dv + 2 \left( \int_{Q, i + k} |Du - Dv|^q \, dx \, dt \right)^{1/q} \\
\leq \frac{\delta_1^{(n+2)/q}}{400} \lambda.
\]

The last estimate and (4.42) imply that
\[
C_{i+k} = \left( \int_{Q, i + k} (|Du| + s)^q \, dx \, dt \right)^{1/q} + \delta_1^{-(n+2)/q} \phi_q(Du, Q, i + k) \\
\leq 2 \left( \int_{Q, i + k} |Du|^q \, dx \, dt \right)^{1/q} + 2s + \frac{\lambda}{400} \\
\leq 2 \left( \int_{Q, i + k} |Du|^q \, dx \, dt \right)^{1/q} + \frac{3\lambda}{400}.
\]

Therefore, by (4.49) and the fact that \( C_{i+k} > \lambda/100 \), we have
\[
\sup_{Q, i+1} \|Dv\| \geq \sup_{Q, i+1} \|Dv\| \geq \left( \frac{1}{1600 \sqrt{n}} \right)^{1/q} \lambda \geq \frac{\lambda}{2560000n} = \frac{\lambda}{B}. \tag{4.50}
\]

By (4.48) and (4.50), we can apply Theorem 3.3 with \( Q^\lambda = \frac{1}{4} Q_i \) and the constants \( A, B, \gamma \) chosen in (4.37) and get
\[
\phi_q(Dv, Q, i+1) = \phi_q(Dv, \frac{\delta_\gamma}{4} Q_i) \leq 2^{-4(n+3)} \phi_q(Dv, \frac{1}{4} Q_i).
\]

Finally, using (4.46), (4.44), and the last inequality, we can apply Lemma 4.8 with \( \varepsilon = 1 \) to obtain (4.47). \( \square \)

**Step 4: Iteration and conclusion.** For any integer \( j \geq 0 \), we denote
\[
\Phi_j := \phi_q(Du, Q, j), \quad m_j := m(Du, Q, j).
\]

Since we have
\[
(|Du(x, t)| + s)^q \leq |Du(x, t) - m_j|^q + (|m_j| + s)^q,
\]
by taking the average over \((x, t) \in Q_j\) and then taking the \( q \)-th root, we obtain
\[
\left( \int_{Q_j} (|Du| + s)^q \, dx \, dt \right)^{1/q} \leq 2\Phi_j + 2|m_j| + 2s. \tag{4.51}
\]

Here we also used the fact that \( q \in (1/2, 1) \). Moreover, by (2.3), we have
\[
|m_j| \leq 2\Phi_j + 2\left( \int_{Q_j} |Du(x, t)|^q \, dx \, dt \right)^{1/q} \leq 2C_j. \tag{4.52}
\]

Using (2.4), we also have
\[
|m_{j+1} - m_j| \leq 2\Phi_{j+1} + 2\delta_1^{-(n+2)/q} \Phi_j. \tag{4.53}
\]
We now prove by induction that

$$\Phi_j + |m_j| + s \leq \frac{\lambda}{2}$$

(4.54)

holds for any $j \geq i_e$. First, by the definition of exit time $i_e$ and (4.52), we know that

$$C_{i_e} \leq \frac{\lambda}{100}, \quad |m_{i_e}| \leq \frac{\lambda}{50},$$

and thus

$$\Phi_{i_e} + |m_{i_e}| + s \leq \frac{\lambda}{2}.$$ 

Assume that (4.54) holds for any $j \in \{i_e, \ldots, i\}$. Then by (4.51) we have

$$\left( \int_{Q_j} (|Du| + s)^q \, dx \, dt \right)^{\frac{1}{q}} \leq \lambda$$

for any $j \in \{i_e, \ldots, i\}$. Therefore, we can apply Lemma 4.9 to get

$$\Phi_{j+1} \leq \frac{1}{4} \Phi_j + c_7 \delta_1^{-(n+2)/q} \left[ \frac{|\mu| Q_j}{r_{j+1}^n + 1} \right] + c_7 \delta_1^{-(n+2)/q} \omega(r_j) \lambda$$

(4.55)

for any $j \in \{i_e, \ldots, i\}$. Thus, using (4.54) with $j = i$, (4.55), (4.44) and (4.45), we have

$$\Phi_{i+1} \leq \frac{\lambda}{8} + c_7 \delta_1^{-(n+2)/q} \left[ \frac{|\mu| (Q_j)}{r_{i+1}^n + 1} \right] + c_7 \delta_1^{-(n+2)/q} \omega(r_i) \lambda \leq \frac{\lambda}{4}. \tag{4.56}$$

Summing up (4.55) in $j \in \{i_e, \ldots, i\}$, we also have

$$\sum_{j=i_e}^{i+1} \Phi_j \leq \Phi_{i_e} + \frac{1}{4} \sum_{j=i_e}^i \Phi_j + c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^i \left[ \frac{|\mu| (Q_j)}{r_{j+1}^n + 1} \right] + c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^i \omega(r_j) \lambda$$

and thus

$$\sum_{j=i_e}^{i+1} \Phi_j \leq \frac{4}{3} \Phi_{i_e} + \frac{4}{3} c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^i \left[ \frac{|\mu| (Q_j)}{r_{j+1}^n + 1} \right] + \frac{4}{3} c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^i \omega(r_j) \lambda. \tag{4.57}$$

Using (4.53), (4.57), (4.44), and (4.45), we obtain

$$|m_{i+1} - m_i| \leq \sum_{j=i_e}^i |m_{j+1} - m_j| \leq 4 \delta_1^{-(n+2)/q} \sum_{j=i_e}^{i+1} \Phi_j \leq 8 \delta_1^{-(n+2)/q} \Phi_{i_e} + 8 c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^i \left[ \frac{|\mu| (Q_j)}{r_{j+1}^n + 1} \right] + 8 c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^i \omega(r_j) \lambda$$

$$\leq 8 \delta_1^{-(n+2)/q} \Phi_{i_e} + \frac{\lambda}{100}.$$ 

Therefore, it follows from (4.52) and the previous inequality that

$$|m_{i+1}| \leq |m_{i_e}| + 8 \delta_1^{-(n+2)/q} \Phi_{i_e} + \frac{\lambda}{100} \leq 10 C_{i_e} + \frac{\lambda}{100} \leq \frac{\lambda}{8}. \tag{4.58}$$
By (4.42), (4.56) and (4.58), we obtain
\[ \Phi_{i+1} + |m_{i+1}| + s \leq \frac{\lambda}{4} + \frac{\lambda}{8} + \frac{\lambda}{400} \leq \frac{\lambda}{2}, \]
which completes the induction. Since we have
\[ |m_j - Du(x_0, t_0)|^q \leq |Du(x, t) - m_j|^q + |Du(x, t) - Du(x_0, t_0)|^q, \]
by taking the average over \((x, t) \in Q_j\) and then taking the \(q\)-th root, we obtain
\[ |m_j - Du(x_0, t_0)| \leq 2\phi_q(Du, Q_j) + 2\left( \int_{Q_j} |Du - Du(x_0, t_0)|^q \, dx \, dt \right)^{1/q} \]
\[ \leq 4\left( \int_{Q_j} |Du - Du(x_0, t_0)|^q \, dx \, dt \right)^{1/q}. \tag{4.59} \]

Since \((x_0, t_0) \in \mathcal{L}\) is a Lebesgue point, using (4.59) and (4.54), we obtain
\[ |Du(x_0, t_0)| = \lim_{j \to \infty} |m_j| \leq \frac{\lambda}{2}. \]
The proof of Theorem 1.1 is completed. \( \Box \)

**Proof of Theorem 1.2.** Without loss of generality, we assume that \( I^{|\mu|}_1(x_0, t_0, 2r) < \infty \). We consider the function
\[
h(\lambda) := \lambda - c \left( \int_{Q^\lambda_{(p-2)/2, \rho}} (|Du| + s + 1)^q \, dx \, dt \right)^{1/q} \int_0^{2\rho} |\mu|(Q^\lambda_{\rho}(x_0, t_0)) \frac{d\rho}{\rho^{n+1}} - c \lambda A(\lambda) - c \lambda^{\frac{(n+1)(p-2)}{4}} B(\lambda),
\]
where \( r_\lambda := \lambda^{(p-2)/2}r \) and
\[
A(\lambda) := \frac{1}{|Q_\rho(x_0, t_0)|^{1/q}} \left( \int_{Q^\lambda_{\rho}} (|Du| + s + 1)^q \, dx \, dt \right)^{1/q}
\]
and
\[
B(\lambda) := \int_0^{2\rho} |\mu|(Q^\lambda_{(p-2)/2, \rho}(x_0, t_0)) \frac{d\rho}{\rho^{n+1}}.
\]
Here \( c \) is the same constant as in Theorem 1.1. Since \( p \in (1, 2) \), we have \( Q^\lambda_{(p-2)/2, \rho} \subset Q^\lambda_{1/2} \) for every \( \lambda > 0 \) and \( \rho > 0 \) and, therefore, \( A \) and \( B \) are nonincreasing functions of \( \lambda \). Moreover, the functions \( A, B \), \( h \) are well defined for \( \lambda \in [1, \infty) \) since \( Q^\lambda_{(p-2)/2, \rho} \subset \Omega \) for any \( \lambda \in [1, \infty) \). Clearly, \( h \) is a continuous function on \([1, \infty)\) and \( h(1) \leq 0 \) since \( c \geq 1 \) and \( A(1) \geq 1 \). On the other hand, since \( q > \frac{(2-p)}{2} \) and \( p > \frac{2n}{n+1} \), we have
\[
\lim_{\lambda \to \infty} h(\lambda) \geq \lim_{\lambda \to \infty} \left( \lambda - c \lambda^{\frac{(n+1)(2-p)}{4}} B(1) \right) = \infty.
\]
Thus there exists some \( \lambda \geq 1 \) such that \( h(\lambda) = 0 \) and therefore (1.12) holds for such \( \lambda \). Since \( \lambda \geq 1 \) and \( r \in (0, R_0) \), we have \( r_\lambda \equiv \lambda^{(p-2)/2}r \in (0, R_0) \). Applying Theorem 1.1 and using the fact that \( h(\lambda) = 0 \), we obtain
\[
\lambda + |Du(x_0, t_0)| \leq 2\lambda = 2c\lambda^{\frac{(n+1)(2-p)}{4}} A(\lambda) + 2c\lambda^{\frac{(n+1)(2-p)}{4}} B(\lambda). \tag{4.60}
\]
Using the fact that $A$, $B$ are nonincreasing functions and Young’s inequality with conjugate exponents \(\left(\frac{2q}{n(2-p)}, \frac{2q}{n(2-p)}\right)\) and \(\left(\frac{2}{(p+1)(2-p)}, \frac{2}{(n+1)p-2n}\right)\), we obtain
\[
2\lambda^{\frac{n(2-p)}{2q}} A(\lambda) \leq 2\lambda^{\frac{n(2-p)}{2q}} A(1) \leq \frac{\lambda}{4} + c'[A(1)]^{\frac{2q}{2-n(2-p)}},
\]
and
\[
2\lambda^{\frac{(n+1)(2-p)}{2}} B(\lambda) \leq 2\lambda^{\frac{(n+1)(2-p)}{2}} B(1) \leq \frac{\lambda}{4} + c'[B(1)]^{\frac{2q}{(n+1)p-2n}}.
\]
(4.61)
Therefore, (1.13) follows by using the last two inequalities and (4.60).

**Proof of Corollary 1.3.** Corollary 1.3 follows similarly as in the proof of Theorem 1.2. The only difference is that we need to replace (4.61) with the following estimate
\[
2\lambda^{\frac{(n+1)(2-p)}{2}} B(\lambda) \leq \lambda/4 + c''\|f\|_{L^\infty}^{\frac{1}{(p-1)}[\mathbf{I}^0_1(t_0, x_0, 2r)]^{1/(p-1)},}
\]
which was already proved in [17, Corollary 1.3].

## 5 Gradient continuity results

### 5.1 Preliminary choices of constants and the geometry

In this section, we always assume $p \in (p^*(n), 2 - \frac{1}{n+1}]$, where $p^*(n)$ is defined in (1.10). We also choose $q := q(n, p) \in (1/2, 1)$ as a fixed constant depending only on $n$ and $p$. For instance, we can take
\[
q := \frac{1}{2}\left(\max\left\{\frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}\right\} + p - \frac{n}{n+1}\right) \in \left(\frac{1}{2}, 1\right).
\]
The choices of geometry in this section are essentially the same as in [17, Section 5.1]. For completeness, we shall still briefly report the choices. First, we fix an open cylinder $\tilde{Q} \subset \subset \Omega_T$ and take another cylinder $Q'$ such that $\tilde{Q} \subset \subset Q' \subset \subset \Omega_T$. Let $R_1 := \text{dist}_{\text{par}}(\tilde{Q}, \partial_{\text{par}} Q') > 0$. Under the assumptions of Theorem 1.4 or Theorem 1.7, it always holds that the Riesz potential $\mathbf{I}^0_1(x, t; r)$ is locally bounded in $\Omega_T$ for some $r > 0$. Therefore, we can apply Theorem 1.2 to obtain that $Du$ is locally bounded in $\Omega_T$ and that in particular, $Du$ is bounded in $\tilde{Q}'$. Thus we can choose
\[
M := 1 + s + \sup_{\tilde{Q}'} |Du| < \infty.
\]
Let
\[
\lambda_M := M \quad \text{and} \quad R_1 := \frac{1}{4} \lambda_M^{(p-2)/2} \tilde{R}_1.
\]
(5.1)
Then we have $Q^M_r(x_0, t_0) \subset \tilde{Q}'$ whenever $(x_0, t_0) \in \tilde{Q}$ and $r \in (0, R_1]$, and therefore,
\[
s + \sup_{Q^M_r(x_0, t_0)} |Du| \leq \lambda_M, \quad \forall r \in (0, R_1].
\]
(5.2)
By (5.1) and (5.2), we have
\[
\left(\int_{Q^M_r(x_0, t_0)} (|Du| + s)^q dx dt\right)^{1/q} \leq M \equiv \lambda_M
\]
(5.3)
for any \((x_0, t_0) \in \tilde{Q}\) and any \(\rho \in (0, R_1]\).

### 5.2 Proof of gradient continuity results

First, we prove the following proposition.

**Proposition 5.1** Let \(\varepsilon \in (0, 1]\). Assume that the Riesz potential \(I_1^{[\mu]}(x, t; r)\) is locally bounded in \(\Omega_T\) for some \(r > 0\) and that
\[
\lim_{r \to 0} \frac{|\mu|(Q_r(x, t))}{r^{n+1}} = 0 \quad \text{locally uniformly in } (x, t) \in \Omega_T.
\]
(5.4)

Then there exist constants \(\alpha = \alpha(n, p, v, L) \in (0, 1), c_0 = c_0(n, p, v, L) \geq 1\), and \(R_\varepsilon = R_\varepsilon(n, p, v, L, \mu, \varepsilon) \in (0, R_1)\) such that
\[
\phi_q(Du, Q_\rho^M(x_0, t_0)) < \varepsilon \lambda_M
\]
(5.5)

holds for every \((x_0, t_0) \in \tilde{Q}\) and every \(\rho \in (0, \rho_\varepsilon]\), where
\[
\rho_\varepsilon = \frac{\varepsilon^{2/\alpha}}{c_0} R_\varepsilon.
\]

More specifically, the constant \(R_\varepsilon\) is determined in (5.8)–(5.9) below.

**Proof** First we take
\[
\lambda = \lambda_M, \quad A = c_8, \quad B = \frac{400n}{\varepsilon}, \quad \gamma = \frac{\varepsilon}{2^{(n+3)}},
\]
(5.6)

where \(c_8 = c_8(n, p, v, L)\) is the same constant as in Lemma 4.7. Then we choose \(\delta_\gamma = \delta_\gamma(n, p, v, L, \varepsilon) \in (0, 1/2)\) as in Theorem 3.3 with the choices of \(A, B, \gamma\) in (5.6) and we set \(\delta_1 = \delta_\gamma/4\). Then by (3.5), we have
\[
\delta_1 = \frac{\varepsilon^{2/\alpha}}{c_9}
\]
(5.7)

for some constants \(\alpha \in (0, 1)\) and \(c_9 \geq 1\) both depending on \(n, p, v, L\). Next, we take \(R_\varepsilon \in (0, R_1)\) such that
\[
\omega(R_\varepsilon) \leq \frac{n+2}{\delta_1^q \varepsilon} \leq \frac{n+2}{800c_7 \delta_1^q} \varepsilon
\]
(5.8)

and that
\[
\sup_{(x_0, t_0) \in \tilde{Q}} \sup_{0 < \rho \leq \lambda_{M}^{(2-p)/2} R_\varepsilon} \frac{|\mu|(Q_\rho(x_0, t_0))}{\rho^{n+1}} \leq \frac{n+2}{800c_7 \lambda_{M}^{(n+1)(2-p)/2}}
\]
(5.9)

where \(c_7 = c_7(n, p, v, L)\) is the same constant as in Lemma 4.6. Thus we have
\[
\sup_{(x_0, t_0) \in \tilde{Q}} \sup_{0 < \rho \leq R_\varepsilon} \frac{|\mu|(Q_\rho^M(x_0, t_0))}{\rho^{n+1}} \leq \sup_{(x_0, t_0) \in \tilde{Q}} \sup_{0 < \rho \leq R_\varepsilon} \frac{|\mu|(Q_\rho^{(2-p)/2} \rho(x_0, t_0))}{\rho^{n+1}} \leq \frac{n+2}{800c_7} \delta_1^q \varepsilon \leq \frac{n+2}{800c_7} \lambda_M.
\]
(5.10)
For $i \in \mathbb{N}$, we define

$$Q_i := Q_{r_i}^\lambda(x_0, t_0), \quad r_i := \delta_1 r, \quad r \in (\delta_1 R_\varepsilon, R_\varepsilon].$$

We will prove that for every $i \geq 1$, it holds that

$$\phi_q(Du, Q_i) < \varepsilon \lambda_M. \quad (5.11)$$

Let $i \geq 1$. We consider two different cases. First, suppose that

$$\left(\int_{Q_i} |Du|^q \, dx \, dt\right)^{1/q} \leq \frac{\varepsilon}{10} \lambda_M. \quad (5.12)$$

In this case, the definition of $\phi_q$ implies that

$$\phi_q(Du, Q_i) \leq \left(\int_{Q_i} |Du|^q \, dx \, dt\right)^{1/q} \leq \frac{\varepsilon}{10} \lambda_M$$

and therefore (5.11) holds. On the other hand, suppose that (5.12) does not hold. Then we have

$$\left(\int_{Q_i} |Du|^q \, dx \, dt\right)^{1/q} \geq \frac{\varepsilon}{10} \lambda_M. \quad (5.13)$$

Let $w$ and $v$ be the solutions defined in (4.1) and (4.18) respectively, with the choices $\rho = r_{i-1}$ and $\lambda = \lambda_M$. By (5.3), (5.8), and (5.10), we can apply Lemma 4.7 with the parameters $\rho = r_{i-1}, \lambda = \lambda_M, \delta = \delta_1, and \theta = \varepsilon^q/20$. Thus using (5.13) we obtain

$$\lambda_M B = \frac{\varepsilon \lambda_M}{400n} \leq \sup_{Q_i} \|Dv\|, \quad s + \sup_{\frac{1}{4} Q_{i-1}} \|Dv\| \leq c_8 \lambda_M = A \lambda_M.$$

Recalling that $\delta_1 = \delta_\gamma / 4$ and applying Theorem 3.3 in $\frac{1}{4} Q_i = \frac{1}{4} Q_{r_{i-1}}$, we have

$$\phi_q(Dv, Q_i) = \phi(Dv, \frac{\delta_\gamma}{4} Q_{i-1}) \leq \frac{\varepsilon}{24(n+3)} \phi_q(Dv, \frac{1}{4} Q_{i-1}). \quad (5.14)$$

By (5.3), (5.10), and (5.14), we can apply Lemma 4.8 and get

$$\phi_q(Du, Q_i) \leq \frac{\varepsilon}{4} \phi_q(Du, Q_{i-1}) + c_7 \delta_1^{-(n+2)/q} \left[\frac{\|D(v)(Q_{i-1})\|}{r_{n+1}}\right] + c_7 \delta_1^{-(n+2)/q} \omega(r_{i-1}) \lambda_M \leq \frac{\varepsilon}{4} \lambda_M + \frac{\varepsilon}{400} \lambda_M < \varepsilon \lambda_M.$$

The proof of (5.11) is completed. Now we take $\rho_\varepsilon = \delta_1 R_\varepsilon$, where $\delta_1$ has the form in (5.7). Since for any $\rho \in (0, \rho_\varepsilon]$, there exist $r \in (\delta_1 R_\varepsilon, R_\varepsilon]$ and an integer $k \geq 1$ such that $\rho = \delta_1^k r$, (5.5) follows directly from (5.11). \qed

A corollary of Proposition 5.1 is Theorem 1.7.

**Proof of Theorem 1.7.** We are now able to determine the exact form of $R_\varepsilon$ in Proposition 5.1 for any $\varepsilon \in (0, 1)$ thanks to the assumption (1.19). By (5.7), to verify (5.8) and (5.9), we need to show that

$$\omega(R_\varepsilon) \leq \frac{\varepsilon^{2(n+2)/q} + 1}{800c_7 c_9^{(n+2)/q}}.$$
and that
\[
\sup_{(x_0, t_0) \in \tilde{Q}} \sup_{0 < \rho \leq \lambda_M^{2-q}/2} \frac{|\mu|(Q_\rho(x_0, t_0))}{\rho^{n+1}} \leq \frac{\varepsilon^{2(a+2)+1}}{800c_7c_9^{(n+2)/q}\lambda_M^{(n+1)(2-p)/2}}.
\]
Thus using (1.19), it is sufficient to take \( R_\varepsilon \in (0, R_1) \) such that
\[
R_\varepsilon \leq \left( \frac{\varepsilon^{2(a+2)+1}}{800c_Dc_7c_9^{(n+2)/q}\lambda_M^{(n+2)(2-p)/2}} \right)^{1/\delta} := \frac{\varepsilon^{1/\theta}}{c_{10}},
\]
where \( \theta = \theta(n, p, v, L, \delta) \in (0, 1) \) and \( c_{10} = c_{10}(n, p, v, L, \delta, c_D, M) \geq 1 \). Now we take
\[
R_\varepsilon = \min\{1, R_1\} \frac{\varepsilon^{1/\theta}}{c_{10}} \quad \text{and} \quad \rho_\varepsilon = \min\{1, R_1\} \frac{\varepsilon^{2(a+1)+1}}{c_9c_{10}}.
\]
Thus we can apply Lemma 5.1 and obtain
\[
\phi_q(Du, Q^{\lambda_M}_{\rho_\varepsilon}(x_0, t_0)) < \varepsilon\lambda_M.
\]
Since \( \varepsilon \) is an arbitrary number in \((0, 1)\), the last inequality implies that
\[
\phi_q(Du, Q^{\lambda_M}_{\rho_\varepsilon}(x_0, t_0)) < c\rho^\beta \quad (5.15)
\]
holds for every \((x_0, t_0) \in \tilde{Q}\) and every \( \rho \in (0, R_2] \), where \( R_2 := \min\{1, R_1\}/(c_9c_{10}) \in (0, 1) \), \( c > 0 \) is a constant depending on \( n, p, v, L, \delta, c_D, M \), and \( R_1 \), and
\[
\beta := \frac{1}{2 + \frac{1}{\theta}} \in (0, 1)
\]
depends only on \( n, p, v, L, \) and \( \delta \).

We are ready to prove \( Du \in C^{0, \beta}(\tilde{Q}) \) using (5.15). Let \((x_1, t_1), (x_2, t_2) \in \tilde{Q} \cap \mathcal{L}\), such that \( \rho := |(x_1, t_1) - (x_2, t_2)|_{\text{par}} \leq R_2/2 \). Here \( \mathcal{L} \equiv \mathcal{L}^{\lambda_M} \) is the set of Lebesgue points of \( Du \) defined in (4.36). Without loss of generality, we assume that \( t_1 \leq t_2 \). Arguing exactly as in (4.59), we know that
\[
\lim_{r \to 0} |m(Q^{\lambda_M}_{\rho}(x_1, t_1) \setminus Du(x_1, t_1))| = 0. \quad (5.16)
\]
Using (2.4), (5.16), (5.15), and the fact that \( q \in (1/2, 1) \), we obtain
\[
|m(Du, Q^{\lambda_M}_{\rho}(x_1, t_1) \setminus Du(x_1, t_1))|
\]
\[
\leq \sum_{j=0}^{\infty} |m(Du, Q^{\lambda_M}_{2^{-j+1}\rho}(x_1, t_1)) - m(Du, Q^{\lambda_M}_{2^{-j+1}\rho}(x_1, t_1))|
\]
\[
\leq 2^{2n+6} \sum_{j=0}^{\infty} \phi_q(Du, Q^{\lambda_M}_{2^{-j+1}\rho}(x_1, t_1)) \leq c 2^{2n+6} \sum_{j=0}^{\infty} (2^{-j} \rho)^\beta \leq c' \rho^\beta. \quad (5.17)
\]
Similarly, we have
\[
|m(Du, Q^{\lambda_M}_{2\rho}(x_2, t_2) \setminus Du(x_2, t_2))| \leq c'' \rho^\beta. \quad (5.18)
\]
By (5.17), (5.18), and the triangle inequality, it holds that

\[
|Du(x_1, t_1) - Du(x_2, t_2)| \\
\leq |Du(x_1, t_1) - m(Du, Q^\lambda_\rho^\vee (x_1, t_1))| + |m(Du, Q^\lambda_\rho^\vee (x_2, t_1)) - Du(x_2, t_2)| \\
+ |m(Du, Q^\lambda_\rho^\vee (x_1, t_1)) - m(Du, Q^\lambda_\rho^\vee (x_2, t_2))| \\
\leq c \rho^\beta + |m(Du, Q^\lambda_\rho^\vee (x_1, t_1)) - m(Du, Q^\lambda_\rho^\vee (x_2, t_2))|.
\]

(5.19)

Recalling the definition of the parabolic distance and the assumption that \( t_1 \leq t_2 \), we know that \( Q^\lambda_\rho^\vee (x_1, t_1) \subset Q^\lambda_\rho^\vee (x_2, t_2) \) and therefore, by (2.4), (5.15) and the fact that \( q \in (\frac{1}{2}, 1) \), we obtain

\[
|m(Du, Q^\lambda_\rho^\vee (x_1, t_1)) - m(Du, Q^\lambda_\rho^\vee (x_2, t_2))| \\
\leq 2\phi_q(Du, Q^\lambda_\rho^\vee (x_1, t_1)) + 2^{2n+5}\phi_q(Du, Q^\lambda_\rho^\vee (x_2, t_2)) \leq c \rho^\beta.
\]

(5.20)

Combining (5.19), (5.20) and using the fact that \( \rho = |(x_1, t_1) - (x_2, t_2)|_{\text{par}} \), we have

\[
|Du(x_1, t_1) - Du(x_2, t_2)| \leq c \rho^\beta = c |(x_1, t_1) - (x_2, t_2)|_{\text{par}},
\]

where \( c \) is a constant depending only on \( n, p, \nu, L, \delta, c_D, M, \) and \( R_1 \). Recall that by Theorem 1.2, \( Du \) is bounded in \( \hat{Q} \). Therefore the last estimate implies that \( Du \in C^{0,\beta}(\hat{Q}) \) since \( |\hat{Q} \setminus (\mathcal{L} \cap \hat{Q})| = 0 \). The proof is now completed. \( \square \)

Next, we turn to the proofs of Theorem 1.4 and its corollaries. We start with the following proposition.

**Proposition 5.2** Let \( \varepsilon \in (0, 1] \). Under the same assumptions as in Theorem 1.4, there exists a radius \( R_\varepsilon' \in (0, R_1) \) depending only on \( n, p, \nu, L, \omega, \varepsilon, M, \) and \( R_1 \), such that

\[
|m(Du, Q^\lambda_{\rho_1}(x_0, t_0)) - m(Du, Q^\lambda_{\rho_2}(x_0, t_0))| \leq 3\varepsilon \lambda_M
\]

(5.21)

holds for every \( \rho_1, \rho_2 \in (0, R_\varepsilon'] \) and \( (x_0, t_0) \in \hat{Q} \).

**Proof** We still use an exit time argument similar to the proof of [17, Theorem 1.6].

**Step 1: Choices of constants.** First, we take

\[
\lambda = \lambda_M, \quad A = c_8, \quad B = \frac{400n}{\varepsilon}, \quad \gamma = 2^{-4(n+3)},
\]

(5.22)

where \( c_8 = c_8(n, p, \nu, L) \) is the same constant as in Lemma 4.7. Then we choose \( \delta_\gamma = \delta_\gamma(n, p, \nu, L, \varepsilon) \in (0, 1/2) \) as in Theorem 3.3 with the choices of \( A, B, \gamma \) in (5.22) and we set \( \delta_1 = \delta_\gamma/4 \).
Next, we take $R'_\varepsilon \in (0, R_1)$ depending only on $n, p, v, L, \omega, \varepsilon, M,$ and $R_1,$ such that
\begin{equation}
\int_{0}^{2R'_\varepsilon} \omega(\rho) \frac{d\rho}{\rho} \leq \frac{\delta_1^{\frac{q}{q}} \varepsilon}{800c_7}, \quad (5.23)
\end{equation}

\begin{equation}
\omega(R'_\varepsilon) \leq \frac{\delta_1^{\frac{q}{q}} \varepsilon}{800c_7}, \quad (5.24)
\end{equation}

\begin{equation}
\sup_{(x_0,t_0) \in \bar{Q}} \int_{0}^{2\lambda_M^{(2-p)/2} R'_\varepsilon} |\mu|\left(Q_{\rho}(x_0,t_0)\right) \frac{d\rho}{\rho^{n+1}} \leq \frac{\delta_1^{\frac{q}{q}} \varepsilon}{800c_7 \lambda_M^{(n+1)(2-p)/2}}, \quad (5.25)
\end{equation}

\begin{equation}
\sup_{(x_0,t_0) \in \bar{Q}} \sup_{0<\rho \leq \lambda_M^{(2-p)/2} R'_\varepsilon} \frac{|\mu|\left(Q_{\rho}(x_0,t_0)\right)}{\rho^{n+1}} \leq \frac{\delta_1^{\frac{q}{q}} \varepsilon}{800c_7 \lambda_M^{(n+1)(2-p)/2}}, \quad (5.26)
\end{equation}

and
\begin{equation}
\sup_{(x_0,t_0) \in \bar{Q}} \sup_{0<\rho \leq R'_\varepsilon} \phi_q(Du, Q_{\rho}^\lambda_M(x_0,t_0)) \leq \frac{\delta_1^{\frac{q}{q}} \varepsilon}{800}, \quad (5.27)
\end{equation}

where $c_7 = c_7(n, p, v, L)$ is the same constant as in Lemma 4.6. Let us briefly explain why we can choose such an $R'_\varepsilon.$ First, (5.23) and (5.24) are possible since $\omega$ is a nondecreasing function satisfying the Dini condition (1.5). Moreover, (5.25) and (5.26) are possible by using the assumption (1.14). Finally, (5.27) is possible by Proposition 5.1, which is applicable since the assumption (1.14) directly implies (5.4).

Arguing exactly as in (5.10), the bound (5.26) implies that
\begin{equation}
\sup_{(x_0,t_0) \in \bar{Q}} \sup_{0<\rho \leq R'_\varepsilon} \frac{|\mu|\left(Q_{\rho}^\lambda_M(x_0,t_0)\right)}{\rho^{n+1}} \leq \frac{\delta_1^{\frac{q}{q}} \varepsilon}{800c_7 \lambda_M}. \quad (5.28)
\end{equation}

Now we fix $(x_0,t_0) \in \bar{Q}$ and define a sequence of shrinking intrinsic cylinders for $i \in \mathbb{N},$ namely,

\[ Q_i := Q_{\rho_i}^\lambda_M(x_0,t_0), \quad r_i := \delta_1^{\frac{q}{q}} R'_\varepsilon. \]

**Step 2: The iteration step.** We have the following lemma.

**Lemma 5.3** Assume that
\begin{equation}
\left( \int_{Q_{i+1}} |Du|^q \, dx \, dt \right)^{1/q} \geq \frac{\varepsilon}{10} \lambda_M. \quad (5.29)
\end{equation}

Then we have
\begin{equation}
\phi_q(Du, Q_{i+1}) \leq \frac{1}{4} \phi_q(Du, Q_i) + c_7 \delta_1^{-(n+2)/q} \frac{|\mu|(Q_i)}{r_i^{n+1}} + c_7 \delta_1^{-(n+2)/q} \omega(r_i) \lambda_M. \quad (5.30)
\end{equation}

**Proof** Let $w$ and $v$ be the solutions defined in (4.1) and (4.18) respectively, with the choices $\rho = r_i$ and $\lambda = \lambda_M.$ By (5.3) and (5.28), we can apply Lemma 4.7 with choices of parameters $\rho = r_i,$ $\lambda = \lambda_M,$ $\delta = \delta_1,$ and $\theta = \varepsilon^q / 20.$ Thus using (5.29) we obtain

\[ \frac{\lambda_M}{B} \leq \frac{\varepsilon \lambda_M}{400n} \leq \sup_{Q_{i+1}} \|Dv\|, \quad s + \sup_{\frac{1}{4}Q_i} \|Du\| \leq c_8 \lambda_M = A \lambda_M. \]
Recalling that $\delta_1 = \delta_r/4$ and applying Theorem 3.3 in $1/4 Q_i = 1/4 Q_i^{\lambda_M}$, we have

$$\phi_q(Dv, Q_i+1) = \phi(Dv, \frac{\delta_r}{4} Q_i) \leq 2^{-4(n+3)} \phi_q(Dv, 1/4 Q_i). \quad (5.31)$$

By (5.3), (5.24), (5.28), and (5.31), we can apply Lemma 4.8 (with $\varepsilon = 1$) and get (5.30). $\square$

**Step 3: Exit time argument.** The main result we want to prove is as follows.

**Lemma 5.4** It holds that

$$|m(Du, Q_j) - m(Du, Q_k)| < \varepsilon \lambda_M, \quad \forall 0 \leq j \leq k.$$  

**Proof** For simplicity, we still denote $m_i := m(Du, Q_i)$ and $\Phi_i := \phi_q(Du, Q_i)$ for $i \geq 0$. We denote the set

$$L := \left\{ i \in \mathbb{N} : \left( \int_{Q_i} |Du|^q \, dx \, dt \right)^{1/q} < \frac{\varepsilon}{10} \lambda_M \right\}. \quad (5.32)$$

We can assume $0 \leq j < k$ and there are two different cases:

$$L \cap \{ j + 1, \ldots, k \} = \emptyset, \quad \text{or} \quad L \cap \{ j + 1, \ldots, k \} \neq \emptyset.$$

**Case 1:** $L \cap \{ j + 1, \ldots, k \} = \emptyset$. By the definition of the set $L$ in (5.32), we can apply Lemma 5.3 for $i \in \{ j, \ldots, k - 1 \}$ and obtain

$$\Phi_{i+1} \leq \frac{1}{4} \Phi_i + c_7 \delta_1^{-(n+2)/q} \frac{1}{r_i^{n+1}} \sum_{j=1}^{k-1} \omega(r_i) \lambda_M \quad (5.33)$$

Summing up (5.33) for $i \in \{ j, \ldots, k - 1 \}$, and using standard manipulations as in the proof of (4.57), we have

$$\sum_{i=j}^{k} \Phi_i \leq 2 \Phi_j + 2c_7 \delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \frac{|\mu|(Q_i)}{r_i^{n+1}} + 2c_7 \delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \omega(r_i) \lambda_M \leq \frac{\delta_1^{4(n+2)/q} \varepsilon}{400} + 2c_7 \delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \frac{|\mu|(Q_i)}{r_i^{n+1}} + 2c_7 \delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \omega(r_i) \lambda_M. \quad (5.34)$$

Here we also used (5.27) in the last inequality. Using (4.43) (with $\lambda = 1$), (5.25), and the fact that $Q_i \subset Q_{\lambda_{M}^{(2-p)/2} r_i}(x_0, t_0)$, we have

$$\sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}} \leq 2 \lambda_{M}^{(2-p)/2} \delta_1^{-(n+2)} \int_0^{2\lambda_{M}^{(2-p)/2} r_i} \frac{|\mu|(Q_{\rho}(x_0, t_0))}{\rho^{n+1}} d\rho \quad (5.35)$$

Similarly, from (5.23) we have

$$\sum_{i=0}^{\infty} \omega(r_i) \lambda_M \leq \frac{\delta_1^{3(n+2)/q} \varepsilon}{800c_7}. \quad (5.36)$$
Combining (5.34), (5.35), and (5.36), we obtain
\[
\sum_{i=j}^{k} \Phi_i \leq \frac{\delta_1^{2(n+2)/q} \epsilon}{100}.
\]

By (2.4), the triangle inequality, and the previous inequality, it holds that
\[
|m_k - m_j| \leq \sum_{i=j}^{k-1} |m_{i+1} - m_i| \leq 4 \delta_1^{-(n+2)/q} \sum_{i=j}^{k} \Phi_i \leq \frac{\epsilon \lambda_M}{25}.
\]

**Case 2:** \( L \cap \{j + 1, \ldots, k\} \neq \emptyset \). We prove in this case that \( |m_j| < \frac{\epsilon \lambda_M}{2} \) and \( |m_k| < \frac{\epsilon \lambda_M}{2} \).

We only give the proof of the former inequality and the proof for the latter is similar. By the assumption that \( L \cap \{j + 1, \ldots, k\} \neq \emptyset \), we define \( j' := \min\{l \in L : l \geq j + 1\} \) and we have \( j' \in L \). Thus by (2.3) and the fact that \( q \in (1/2, 1) \), we obtain
\[
|m_{j'}| \leq 4 \left( \int_{Q_j} |Du(x, t)|^q \, dx \, dt \right)^{1/q} < \frac{2 \epsilon \lambda_M}{5}.
\] (5.37)

There are two possibilities, namely, \( j' = j + 1 \) or \( j' > j + 1 \). First, we assume \( j' = j + 1 \). Using (2.4) and (5.27), we have
\[
|m_j - m_{j+1}| \leq 2 \Phi_{j+1} + 2 \delta_1^{-(n+2)/q} \Phi_j \leq \frac{\epsilon}{200} \leq \frac{\epsilon \lambda_M}{200}.
\] (5.38)

Therefore, by the triangle inequality, (5.37), and (5.38),
\[
|m_j| \leq |m_j - m_{j+1}| + |m_{j'}| \leq \frac{\epsilon \lambda_M}{25} + \frac{2 \epsilon \lambda_M}{5} < \frac{\epsilon \lambda_M}{2}.
\]

Otherwise, we have \( j' > j + 1 \). Then by the definition of \( j' \), we know that \( L \cap \{j + 1, \ldots, j' - 1\} = \emptyset \). Therefore, we can apply Lemma 5.3 for \( i \in \{j, \ldots, j' - 2\} \). From now on, we can argue exactly as in Case 1 to get
\[
|m_j - m_{j'-1}| \leq \frac{\epsilon \lambda_M}{25}.
\] (5.39)

Again using (2.4) and (5.27), we have
\[
|m_{j'-1} - m_{j'}| \leq 2 \Phi_{j'} + 2 \delta_1^{-(n+2)/q} \Phi_{j'-1} \leq \frac{\epsilon}{200} \leq \frac{\epsilon \lambda_M}{200}.
\] (5.40)

Therefore, by the triangle inequality, (5.37), (5.39), and (5.40),
\[
|m_j| \leq |m_j - m_{j'-1}| + |m_{j'-1} - m_{j'}| + |m_{j'}| \leq \frac{\epsilon \lambda_M}{25} + \frac{\epsilon \lambda_M}{200} + \frac{2 \epsilon \lambda_M}{5} < \frac{\epsilon \lambda_M}{2}.
\]

The proof of the lemma is now completed. \( \Box \)

**Step 4: Conclusion.** For any \( \rho_1, \rho_2 \in (0, R'_e] \), there exist two integers \( j, k \geq 0 \) such that
\[
\delta_{1}^{j+1} R'_e < \rho_1 \leq \delta_{1}^{j} R'_e \quad \text{and} \quad \delta_{1}^{k+1} R'_e < \rho_2 \leq \delta_{1}^{k} R'_e.
\]
By (2.4), we have
\[
|m(Du, Q_{\rho_1}^{\lambda M}(x_0, t_0)) - m(Du, Q_j)|
\leq 2\phi_q(Du, Q_{\rho_1}^{\lambda M}(x_0, t_0)) + 2\left(\frac{|Q_j|}{|Q_{\rho_1}^{\lambda M}(x_0, t_0)|}\right)^{1/q}\phi_q(Du, Q_j) \leq \frac{\varepsilon}{2^{100}}.
\]

Here we used (5.27) in the last line. Similarly, it holds that
\[
|m(Du, Q_{\rho_2}^{\lambda M}(x_0, t_0)) - m(Du, Q_k)| \leq \frac{\varepsilon}{2^{100}}.
\]

Thus, the estimate (5.21) follows by using Lemma 5.4, the triangle inequality, and the last two inequalities. The proposition is proved. \(\square\)

**Proof of Theorem 1.4.** For any \((x_0, t_0) \in \tilde{Q} \cap \mathcal{L}\), where \(\mathcal{L} \equiv \mathcal{L}_{\lambda M}\) is the set of Lebesgue points of \(Du\) defined in (4.36), arguing exactly as in (4.59), we know that
\[
\lim_{r \to 0} m(Q_r^{\lambda M}(x_0, t_0) - Du(x_0, t_0)) = 0.
\]

Hence by Proposition 5.2,
\[
\lim_{r \to 0} |m(Q_r^{\lambda M}(x_0, t_0) - Du(x_0, t_0)| = 0 \quad \text{uniformly in } (x_0, t_0) \in \tilde{Q} \cap \mathcal{L}. \tag{5.41}
\]

Let \((x_1, t_1), (x_2, t_2) \in \tilde{Q} \cap \mathcal{L}\) and \(\rho := |(x_1, t_1) - (x_2, t_2)|_{\text{par}}.\) Without loss of generality, we assume that \(t_1 \leq t_2\). Therefore \(Q_{\rho M}^\lambda(x_1, t_1) \subset Q_{2\rho}^{\lambda M}(x_2, t_2)\) and by (2.4) and the fact that \(q \in (\frac{1}{2}, 1)\), we have
\[
|m(Du, Q_{\rho}^{\lambda M}(x_1, t_1)) - m(Du, Q_{2\rho}^{\lambda M}(x_2, t_2))|
\leq 2\phi_q(Du, Q_{\rho}^{\lambda M}(x_1, t_1)) + 2^{2n+5}\phi_q(Du, Q_{2\rho}^{\lambda M}(x_2, t_2)).
\]

By the triangle inequality and the previous inequality, we obtain
\[
|Du(x_1, t_1) - Du(x_2, t_2)|
\leq |Du(x_1, t_1) - m(Du, Q_{\rho}^{\lambda M}(x_1, t_1))| + |m(Du, Q_{\rho}^{\lambda M}(x_1, t_1)) - m(Du, Q_{2\rho}^{\lambda M}(x_2, t_2))|
+ |m(Du, Q_{\rho}^{\lambda M}(x_1, t_1)) - m(Du, Q_{2\rho}^{\lambda M}(x_2, t_2))|.
\tag{5.42}
\]

Using (5.41), (5.42), and Proposition 5.1, it follows that for \((x_1, t_1), (x_2, t_2) \in \tilde{Q} \cap \mathcal{L},
|Du(x_1, t_1) - Du(x_2, t_2)| \to 0 \quad \text{when } \rho := |(x_1, t_1) - (x_2, t_2)|_{\text{par}} \to 0.
\]

Since \(|\tilde{Q} \setminus \mathcal{L}| = 0\), we conclude that \(Du\) is continuous in \(\tilde{Q}\). \(\square\)

**Proof of Corollary 1.5.** By [17, Lemma 2.1], we know that the assumption (1.15) implies (1.14). Therefore, Corollary 1.5 is a direct consequence of Theorem 1.4. \(\square\)

**Proof of Corollary 1.6.** Corollary 1.6 follows directly from Theorem 1.4 since the assumptions (1.16) and (1.17) directly imply (1.14). \(\square\)
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