Light-Ray Radon Transform for Abelianin and Nonabelian Connection in 3 and 4 Dimensional Space with Minkowsky Metric.

M. Zyskin*

Abstract
We have a real manifold of dimension 3 or 4 with Minkovsky metric, and with a connection for a trivial $GL(n, C)$ bundle over that manifold. To each light ray on the manifold we assign the data of parallel transport along that light ray. It turns out that these data are not enough to reconstruct the connection, but we can add more data, which depend now not from lines but from 2-planes, and which in some sense are the data of parallel transport in the complex light-like directions, then we can reconstruct the connection up to a gauge transformation. There are some interesting applications of the construction: 1) in 4 dimensions, the self-dual Yang Mills equations can be written as the zero curvature condition for a pair of certain first order differential operators; one of the operators in the pair is the covariant derivative in complex light-like direction we studied.

2) there is a relation of this Radon transform with the supersymmetry.

3) using our Radon transform, we can get a measure on the space of 2 dimensional planes in 4 dimensional real space. Any such measure give rise to a Crofton 2-density. The integrals of this 2-density over surfaces in $\mathbb{R}^4$ give rise to the Lagrangian for maps of real surfaces into $\mathbb{R}^4$, and therefore to some string theory.

4) there are relations with the representation theory. In particular, a closely related transform in 3 dimensions can be used to get the Plancerel formula for representations of $SL(2, R)$.

*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855
1 Introduction.

Let $M$ be the real 3 or 4 dimensional manifold with Minkowski metric. Let us take some connection $\Omega = A_m(x)dx^m$. Consider the parallel transport equation in the light-like direction:

\[ \alpha^m \left( \frac{\partial}{\partial x^m} - A_m(x) \right) \mu(\alpha, x) = 0, \]

where

\[ \alpha_m \alpha^m = 0. \]

Here $x \in M; A_m(x)$ and $\mu(\alpha, x)$ in the abelian case are with values in numbers, and in the nonabelian case have values in complex $n \times n$ matrices. (For simplicity, we consider the $\text{gl}(n, \mathbb{C})$ case only.)

Remark. We use the summation convention: summation over repeated indexes is always assumed.

$\mu(\alpha, x)$ depends on the point and on the ray we choose. We will consider certain functionals of $\mu$, which depend only on the light ray. We call such functionals spectral data of the light-ray Radon transform. An example of spectral data functional: take the asymptotic of $\mu(\alpha, x)$ along the light ray $x = \alpha t + \beta$ as $t \to -\infty$ to be 1, and compute $\varphi(\alpha, \beta) := \lim_{t \to +\infty} \mu(\alpha, \alpha t + \beta)$.

The spectral data are functionals of the connection $\Omega$. For a given connection $\Omega$, the spectral data are functions on the space of light rays. We will show that it is possible to choose such spectral data functionals that the inverse functional exist, namely, if we are given the values of the spectral data for every light ray, we are able to reconstruct the connection up to a gauge transformation $A_m(x) \to g(x)A_m(x)g^{-1}(x) - g(x) \frac{\partial}{\partial x^m}g^{-1}(x)$ (see Lemma 1.3 and Lemma 2.2). In dimension 4, the space of all light rays has dimension 5, therefore, there is a compatibility condition for scattering data.

The problem to describe the scattering data and to write inversion formulas was suggested by prof. E. Witten.

The motivation to study the light-ray Radon transform came from supersymmetric Yang-Mills theories. It is known that in certain sense supersymmetry transformation is a square root from light-like translations.
Namely, let $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ be the generators of the supersymmetry algebra
\[
\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0
\]
\[
\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^{m}_{\alpha,\dot{\beta}} i \partial \frac{\partial}{\partial x^m},
\]
where $\sigma^{m}_{\alpha,\dot{\alpha}}$ are Pauli matrices. Take complex numbers $\lambda^\alpha, \eta^\dot{\alpha}$. Then
\[
\frac{1}{2} \left(\lambda^\alpha Q_\alpha + \eta^\dot{\alpha} \bar{Q}_{\dot{\alpha}}\right)^2 = \lambda^\alpha \eta^\dot{\alpha} \sigma^{m}_{\alpha,\dot{\alpha}} i \partial \frac{\partial}{\partial x^m}
\]
is a light-like translation: take $v^m = \lambda^\alpha \eta^\dot{\alpha} \sigma^{m}_{\alpha,\dot{\alpha}}$; then $v^m v_m = 0$. If $\eta^\dot{\alpha} = \bar{\lambda}^\alpha$, (where bar means complex conjugate), then $v^m$ is real.

In recent solutions for supersymmetric Yang-Mills theories \[3\] the Yang-Mills connection was written as Gauss-Manin connection, using the known monodromies on an auxiliary complex plane. In our paper, we have a similar story for nonsupersymmetric gauge theories, namely, we express the non-supersymmetric Yang-Mills connection through the solution of some auxiliary Riemann-Hilbert problem on complex plane.

The technique, similar to the technique developed in this paper, is useful for problems in real integral geometry. Also, we expect some modification of these technique will allow to obtain the Plancherel-type formulas for representation of $SL(2, R)$, using methods of integral geometry. Details will be published elsewhere.

2 Spectral data, associated with parallel transport in the light-like direction in dimension 3.

2.0 The transform for the function

Consider the standard Radon transform of the function, that is the integral of the function over 2 dimensional planes:
\[
F(\omega_0, \omega_1, \omega_2; p) := \int \delta(\omega x - p) f(x) d^3x.
\]
This transformation has the inverse:

\[ f(x) = \int \delta''(\omega x - p)F(\omega, p)d\omega dp \]

where \( \omega \) is integrated over the unit sphere.

A plane with \( \omega_1^2 + \omega_2^2 - \omega_3^2 > 0 \) contains two 1-parametric families of parallel light rays, belonging to such plane, thus the integrals of the function over such plane can be written through the integrals of the function over the light rays. Consider a light ray \( x = \alpha t + \beta \), where \( \alpha_1^2 + \alpha_2^2 = \alpha_0^2 \). Let \( \varphi(\alpha, \beta) \) be the integral of the function over the light ray,

\[ \varphi(\alpha, \beta) = \int f(\alpha t + \beta)dt. \]

Then

\[ F(-\alpha_1 \omega_1 - \alpha_2 \omega_2, \omega_1, \omega_2; p) = \int \varphi(1, \alpha_1, \alpha_2; 0, \beta_1, \beta_2) \delta(\beta_1 \omega_1 + \beta_2 \omega_2 - p)d\beta_1 d\beta_2 \]

Thus, in the inversion formula the integrals over 2 dimensional planes with \( \omega_1^2 + \omega_2^2 - \omega_3^2 > 0 \) can be expressed through the integrals over light rays, and the other planes we need to keep. The integrals over the remaining planes are related to the complex light rays in the following way: for real light rays, for pairs \( \alpha, \eta \) such that

\[ (\alpha^1)^2 + (\alpha^2)^2 - (\alpha^0)^2 = 0 \]
\[ \eta_0 \alpha^0 + \eta_1 \alpha^1 + \eta_2 \alpha^2 = 0. \]  

(1)

at fixed \( \alpha \), the function \( \varphi(\alpha, \beta) e^{i\beta^m \eta_m} \), as function of \( \beta \), does not change if we add to \( \beta \) a vector proportional to \( \alpha \), thus it can be restricted to the factor-space of \( \mathbb{R}^3 \) over the 1-dimensional subspace spanned by \( \alpha \). We define

\[ \Phi(\alpha, \eta) = \int_{\mathbb{R}^3/\{\alpha\}} \varphi(\alpha, \beta) e^{i\beta^m \eta_m} d_{\alpha}\beta \]

where the integral is over the factor space \( \mathbb{R}^3/\{\alpha\} \), and \( d_{\alpha}\beta = |\alpha^0 d\beta^1 \wedge d\beta^2 + \alpha^1 d\beta^2 \wedge d\beta^0 + \alpha^2 d\beta^0 \wedge d\beta^1| \) is a nonoriented volume element on this space. It’s easy to see that \( \Phi(\alpha, \eta) = \hat{f}(\eta) \), where \( \hat{f}(\eta) \) is the Fourier transform of
Thus, we have the following:

\[-(\alpha^0)^2 + (\alpha^1)^2 + (\alpha^2)^2 = 0\]

\[\eta_0 \alpha^0 + \eta_1 \alpha^1 + \eta_2 \alpha^2 = 0\]

\[\Phi(\alpha, \eta) = \hat{f}(\eta)\]

For real \(\alpha\), it follows that \(\eta_1^2 + \eta_2^2 \geq \eta_0^2\), otherwise there are no solutions. Thus we take complex \(\alpha\) and consider this system as the definition of what \(\Phi(\alpha, \eta)\) is. Take \(\alpha = (1, \frac{\lambda + \lambda^{-1}}{2}, -i \frac{\lambda - \lambda^{-1}}{2})\). Solving for \(\eta_1, \eta_2\) we will have

\[\Phi(\alpha, \eta) = \hat{f}(\eta_0, -\frac{\lambda + \bar{\lambda}}{1 + \lambda \lambda} \eta_0, i \frac{\lambda - \bar{\lambda}}{1 + \lambda \lambda} \eta_0)\]

After the Fourier transform over the remaining \(\eta_0\) we get

\[\int \hat{f}(\eta_0, -\frac{\lambda + \bar{\lambda}}{1 + \lambda \lambda} \eta_0, i \frac{\lambda - \bar{\lambda}}{1 + \lambda \lambda} \eta_0) e^{i \eta_0 \phi} d\eta_0 = \int \delta(x^0 - \frac{\lambda + \bar{\lambda}}{1 + \lambda \lambda} x^1 + i \frac{\lambda - \bar{\lambda}}{1 + \lambda \lambda} x^2 - p) f(x) d^3 x\]

For \(|\lambda| \neq 1\) this is the integral over those 2-planes which do not contain real light rays, and as we change \(\lambda\), we span all of them. Notice also, that in some sense the plane above contain the complex light ray \((1, \frac{\lambda + \lambda^{-1}}{2}, -i \frac{\lambda - \lambda^{-1}}{2})\), since

\[\left(\frac{\partial}{\partial x^0} + \frac{\lambda + \lambda^{-1}}{2} \frac{\partial}{\partial x^1} - i \frac{\lambda - \lambda^{-1}}{2} \frac{\partial}{\partial x^2}\right) \left( x^0 - \frac{\lambda + \bar{\lambda}}{1 + \lambda \lambda} x^1 + i \frac{\lambda - \bar{\lambda}}{1 + \lambda \lambda} x^2 \right) = 0\]

2.1 Abelian Connection on 3-dimensional manifold.

In the Abelian case we are given a 1-form \(\Omega = A_m(x) dx^m\), with \(\{A_m(x)\}\) taking values in numbers and fast decreasing as \(x \to \infty\), defined up to an exact form.

**Proposition 1.1**

Let us parametrize light rays as follows:

\[x = \alpha t + \beta,\]

where \(\alpha = (\alpha^0, \alpha^1, \alpha^2)\), \(\alpha_m \alpha^m = 0\), \(\alpha^0 > 0\) and \(\alpha, \beta \in \mathbb{R}^3\). Let

\[\varphi(\alpha, \beta) = \int_{x = \alpha t + \beta} \Omega \equiv \int_{-\infty}^{+\infty} \alpha^i A_i(\alpha t + \beta) dt.\]
be the integral of 1-form over oriented light ray. Such integral gives us a transformation from 1-forms \( \Omega = A_m(x)dx^m \) to functions on the space of light rays \( \varphi(\alpha, \beta) \):

\[
\varphi(\lambda \alpha, \beta) = \text{sgn}\lambda \varphi(\alpha, \beta) \\
\varphi(\alpha, \beta + \lambda \alpha) = \varphi(\alpha, \beta)
\]

If we know \( \varphi(\alpha, \beta) \), we can reconstruct the Fourier transform of the 1-form \( \Omega \) outside the light cone, up to an exact form. We will give two ways to do this:

1. For pairs \( \alpha, \eta \) such that

\[
(\alpha^1)^2 + (\alpha^2)^2 - (\alpha^0)^2 = 0 \tag{2}
\]

\[\eta_0 \alpha^0 + \eta_1 \alpha^1 + \eta_2 \alpha^2 = 0.
\]

define

\[
\Phi(\alpha, \eta) = \int_{\mathbb{R}^3/\{\alpha\}} \varphi(\alpha, \beta) e^{i \beta^m \eta_m} d_\alpha \beta
\]

where the integral is over the factor space of \( \mathbb{R}^3 \) over the 1-dimensional subspace spanned by \( \alpha \), and \( d_\alpha \beta = |\alpha^0 d^1 + \alpha^1 d^2 \wedge d^3 + \alpha^2 d^0 \wedge d^3| \) is a nonoriented volume element on this space.

Using homogenous properties, we can choose

\[\alpha^0 = 1\]

For fixed \( \eta \), (2) viewed as an equation for \( \alpha \) has two real-valued solutions \( \alpha^{(1)}, \alpha^{(2)} \), provided \( \eta_1^2 + \eta_2^2 - \eta_0^2 > 0 \),

\[
\alpha^{(1)} = \left( 1, \frac{-\eta_0 \eta_1 - \eta_2 \sqrt{\eta_1^2 + \eta_2^2 - \eta_0^2}}{\eta_1^2 + \eta_2^2}, \frac{-\eta_0 \eta_2 + \eta_1 \sqrt{\eta_1^2 + \eta_2^2 - \eta_0^2}}{\eta_1^2 + \eta_2^2} \right)
\]

\[
\alpha^{(2)} = \left( 1, \frac{-\eta_0 \eta_1 + \eta_2 \sqrt{\eta_1^2 + \eta_2^2 - \eta_0^2}}{\eta_1^2 + \eta_2^2}, \frac{-\eta_0 \eta_2 - \eta_1 \sqrt{\eta_1^2 + \eta_2^2 - \eta_0^2}}{\eta_1^2 + \eta_2^2} \right)
\]
and no real-valued solutions for $\eta_1^2 + \eta_2^2 - \eta_0^2 < 0$.
For $\eta$ such that $\eta_1^2 + \eta_2^2 - \eta_0^2 > 0$, we can reconstruct the 1-form up to the exact form:

$$
\eta_0 \hat{A}_1(\eta) - \eta_1 \hat{A}_0(\eta) = \rho \left( (\alpha^{(1)}_1, \eta) - (\alpha^{(2)}_2, \eta) \right)
$$

$$
\eta_1 \hat{A}_2(\eta) - \eta_2 \hat{A}_1(\eta) = \rho \left( (\alpha^{(2)}_0, \eta) - (\alpha^{(1)}_2, \eta) \right)
$$

$$
\eta_2 \hat{A}_0(\eta) - \eta_0 \hat{A}_2(\eta) = \rho \left( (\alpha^{(2)}_1, \eta) - (\alpha^{(1)}_1, \eta) \right)
$$

where

$$
\rho = \frac{\eta_0}{\alpha^{(1)}_1 \alpha^{(2)}_2 - \alpha^{(1)}_2 \alpha^{(2)}_1} = \frac{\eta_1}{\alpha^{(1)}_1 \alpha^{(2)}_0 - \alpha^{(1)}_0 \alpha^{(2)}_1} = \frac{\eta_2}{\alpha^{(1)}_0 \alpha^{(2)}_1 - \alpha^{(1)}_1 \alpha^{(2)}_0}
$$

2. There is another way to write the inversion formulas. Define

$$
\hat{\varphi}(\alpha, \eta) = \int \varphi(\alpha, \beta) e^{i\beta m_\eta} d^3 \beta
$$

to be the Fourier transform over all $\beta$. Here $\alpha_m \alpha^m = 0$.

We can reconstruct the connection up to the exact form as follows:

Let us parametrise $\alpha$ by

$$
\alpha = (1, \cos \gamma, \sin \gamma).
$$

Then

$$
\frac{\eta_1^2 + \eta_2^2}{2(\eta_1^2 + \eta_2^2 - \eta_0^2)^{1/2}} \int_0^{2\pi} (\eta_2 \cos \gamma - \eta_1 \sin \gamma) \varphi(\alpha(\gamma), \eta) \frac{d\gamma}{2\pi} =
$$

$$
(\eta_2 \hat{A}_1(\eta) - \eta_1 \hat{A}_2(\eta)) \Theta(\eta_1^2 + \eta_2^2 - \eta_0^2)
$$

$$
\frac{1}{2} \left( \frac{\eta_1^2 + \eta_2^2 - \eta_0^2}{\eta_1^2 + \eta_2^2} \right)^{1/2} \int_0^{2\pi} \varphi(\alpha(\gamma), \eta) \frac{d\gamma}{2\pi} =
$$

$$
(\hat{A}_0(\eta) - \frac{\eta_0}{\eta_1^2 + \eta_2^2} (\hat{A}_1(\eta) \eta_1 + \hat{A}_2(\eta) \eta_2)) \Theta(\eta_1^2 + \eta_2^2 - \eta_0^2)
$$

where

$$
\Theta(x) = \begin{cases} 
1, & \text{if } x > 0; \\
0, & \text{if } x < 0
\end{cases}
$$
We see that we can reconstruct the Fourier transform of the connection outside the light cone, that is, for \( \eta_1^2 + \eta_2^2 - \eta_0^2 > 0 \), only. To get the connection inside the cone, we need additional data, namely, we have to use the complex-valued solutions of (2) for \( \alpha \).

\( A_m(x)dx^m \) is a 1-form, that is a linear function on vector fields \( r_n \frac{\partial}{\partial x^n} \), with real \( r_n \). Using linearity, we can extend it to the linear function on vector fields \( c_n \frac{\partial}{\partial x^n} \) with complex \( c_n \). Using the 1-form \( A_m(x)dx^m \) as the connection, consider the parallel transport equation

\[
\left( \frac{\partial}{\partial x^0} + \frac{\lambda + \lambda^{-1}}{2} \frac{\partial}{\partial x^1} - i \frac{\lambda - \lambda^{-1}}{2} \frac{\partial}{\partial x^2} \right) \mu(x, \lambda, \bar{\lambda}) = \\
\left( A_0(x) + \frac{\lambda + \lambda^{-1}}{2} A_1(x) - i \frac{\lambda - \lambda^{-1}}{2} A_2(x) \right) \mu(x, \lambda, \bar{\lambda})
\]

(4)

Here \( \lambda \) is a complex number, \( x \in \mathbb{R}^3 \), \( A_m(x) \) takes values in numbers. We have parametrised the direction of the (complex) light ray by \( \alpha = (1, \frac{\lambda + \lambda^{-1}}{2}, -i \frac{\lambda - \lambda^{-1}}{2}) \).

Let us take the following solution of (4):

\[
\mu(x, \lambda, \bar{\lambda}) = \exp \left( \int G(x - y, \lambda, \bar{\lambda}) \left( 2\lambda A_0(y) + (\lambda^2 + 1) A_1(y) - i(\lambda^2 - 1) A_2(y) \right) d^3 y \right),
\]

(5)

where

\[
G(x, \lambda, \bar{\lambda}) := \int \frac{e^{i \eta x}}{2\lambda \eta_0 + \lambda^2 (\eta_1 - i \eta_2) + (\eta_1 + i \eta_2)} \frac{d^3 \eta}{i (2\pi)^3} = \\
\frac{\bar{\lambda} \text{sgn}(\lambda \bar{\lambda} - 1)}{\pi} \frac{\delta(-x^0(\lambda \bar{\lambda} + 1) + x^1(\lambda + \bar{\lambda}) - ix^2(\lambda - \bar{\lambda}))}{x^0(\lambda \bar{\lambda} - 1) + x^1(\lambda - \bar{\lambda}) - ix^2(\lambda + \bar{\lambda})}.
\]

(6)

Here

\[
\text{sgn}(x) = \begin{cases} 
1, & \text{if } x > 0; \\
-1, & \text{if } x < 0
\end{cases}
\]

8
Lemma 1.1
Define the functions of the light ray $S(x, \lambda, \bar{\lambda}), T(x, \gamma)$ as follows:

$$S(x, \lambda, \bar{\lambda}) := \frac{1}{\mu(x, \lambda, \bar{\lambda})} \frac{\partial}{\partial \bar{\lambda}} \mu(x, \lambda, \bar{\lambda})$$

$$T(x, \gamma) := \frac{\mu(x, \lambda, \bar{\lambda})|_{\lambda=(1-\varepsilon)e^{i\gamma}} - \mu(x, \lambda, \bar{\lambda})|_{\lambda=(1+\varepsilon)e^{i\gamma}}}{\mu(x, \lambda, \bar{\lambda})|_{\lambda=(1-\varepsilon)e^{i\gamma}}}$$

where $\mu(x, \lambda, \bar{\lambda})$ is given by (5), (6).

1). If we are given a connection, we can find $S(x, \lambda, \bar{\lambda}), T(x, \gamma)$, which depend from 3 real parameters, and given by

$$S(x, \lambda, \bar{\lambda}) = \frac{sgn(\lambda \bar{\lambda} - 1)}{2\pi} \int \delta' \left( (\lambda \bar{\lambda} + 1)(x^0 - y^0) - (\lambda + \bar{\lambda})(x^1 - y^1) + i(\lambda - \bar{\lambda})(x^2 - y^2) \right)$$

$$(2 \lambda A_0(y) + (\lambda^2 + 1)A_1(y) - i(\lambda^2 - 1)A_2(y))$$

$$T(x, \gamma) = I - \exp \left( \frac{1}{2i\pi} \int \delta \left( -(x^0 - y^0) + (x^1 - y^1) \cos(\gamma) + (x^2 - y^2) \sin(\gamma) \right) \right)$$

$$(A_0(y) + \cos(\gamma)A_1(y) + \sin(\gamma)A_2(y))$$

The quantities $S(x, \lambda, \bar{\lambda}), T(x, \gamma)$ do not change in the light-like direction,

$$\left( \frac{\partial}{\partial x^0} + \frac{\lambda + \lambda^{-1}}{2} \frac{\partial}{\partial x^1} - i \frac{\lambda - \lambda^{-1}}{2} \frac{\partial}{\partial x^2} \right) S(x, \lambda, \bar{\lambda}) = 0,$$

$$\left( \frac{\partial}{\partial x^0} + \cos(\gamma)\frac{\partial}{\partial x^1} + \sin(\gamma)\frac{\partial}{\partial x^2} \right) T(x, \gamma) = 0.$$
connection, up to a gauge transformation, using the relations

\[ (\eta_1^2 + \eta_2^2) (2\lambda \eta_0 + \lambda^2 (\eta_1 - i\eta_2) + \eta_1 + i\eta_2) \int_0^{2\pi} t(\eta, \gamma) \frac{e^{i\gamma} i d\gamma}{\lambda - e^{i\gamma} 2\pi} = \]

\[ (2\lambda (A_0(\eta_1^2 + \eta_2^2) - \eta_0 (\eta_1 A_1 + \eta_2 A_2) - i\eta_0 (\eta_2 A_1 - \eta_1 A_2)) - 2i(\eta_2 A_1 - \eta_1 A_2)(\eta_1 + i\eta_2)) \Theta(\eta^2 - \eta_0^2), \]

\[ (\eta_1^2 + \eta_2^2) (2\lambda \eta_0 + \lambda^2 (\eta_1 - i\eta_2) + \eta_1 + i\eta_2) \int \hat{S}(\eta, \lambda, \bar{\lambda}) \frac{1}{\lambda - \lambda_1} \frac{d\lambda_1 d\bar{\lambda}_1}{2\pi} = \]

\[ (2\lambda (A_0(\eta_1^2 + \eta_2^2) - \eta_0 (\eta_1 A_1 + \eta_2 A_2) - i\eta_0 (\eta_2 A_1 - \eta_1 A_2)) - 2i(\eta_2 A_1 - \eta_1 A_2)(\eta_1 + i\eta_2)) \Theta(\eta_0^2 - \eta^2). \]

Here

\[ t(\eta, \gamma) = \int \ln (T(x, \gamma) - 1) e^{inx} dx, \quad \hat{S}(\eta, \lambda, \bar{\lambda}) = \int S(x, \lambda, \bar{\lambda}) e^{inv} dx. \]

Proof.

Part 1) can be checked directly, using the definitions. To prove 2), notice that,

\[ \hat{S}(\eta, \lambda, \bar{\lambda}) = (2\pi) sgn(\lambda \bar{\lambda} - 1) \frac{i\eta_0}{(1 + \lambda \bar{\lambda})^2} \delta \left( \eta_1 + \frac{\lambda + \bar{\lambda}}{1 + \lambda \bar{\lambda}} \eta_0 \right) \delta \left( \eta_2 - i \frac{\lambda - \bar{\lambda}}{1 + \lambda \bar{\lambda}} \eta_0 \right) \]

\[ (2\lambda \hat{A}_0(\eta) + (\lambda^2 + 1) \hat{A}_1(\eta) - i(\lambda^2 - 1) \hat{A}_2(\eta)) \]

\[ t(\eta, \gamma) := (ln (T - 1)) (\eta, \gamma) = -4\pi \delta (\eta_0 + \eta_1 \cos(\gamma) + \eta_2 \sin(\gamma)) \]

\[ sgn (-\sin(\gamma)\eta_1 + \cos(\gamma)\eta_2) \left( \hat{A}_0(\eta) + \hat{A}_1(\eta) \cos(\gamma) + \hat{A}_2(\eta) \sin(\gamma) \right) \]

After some computation, we get (7).

Notice that the formulas (7) do not give the connection at \( \eta_1^2 + \eta_2^2 = 0 \).

We need to take \( \eta_1^2 + \eta_2^2 = \epsilon \) and go to the limit \( \epsilon \to 0 \), using the fact that
$A(\eta)$ is continuous. Thus, there are certain conditions on the spectral data, which were obtained from the 1-form, particularly at $\eta_1^2 + \eta_2^2 = 0$. We will not discuss these subtleties here.

Remark The jump $T(x, \gamma)$ is related to the integral of the connection over the real light ray in the following way:

$$(\ln (T - 1)) \cdot (\eta, \gamma) \cdot \text{sgn} (-\sin(\gamma)\eta_1 + \cos(\gamma)\eta_2) =$$

$$-4\pi\delta(\eta_0 + \eta_1 \cos(\gamma) + \eta_2 \sin(\gamma)) \left(\hat{A}_0(\eta) + \hat{A}_1(\eta) \cos(\gamma) + \hat{A}_2(\eta) \sin(\gamma)\right) = -2\varphi(\gamma, \eta),$$

where $\varphi(\gamma, \eta)$ is the Fourier transform of $\varphi(\alpha, \beta)$ over $\beta$, with $\alpha = (1, \cos \gamma, \sin \gamma)$; $\varphi(\alpha, \beta)$ was defined above as the integral of 1-form over the light ray.

2.2 Nonabelian Connection.

In the nonabelian case we have the connection $A_m(x)dx^m, \ m = 0, 1, 2$, over real 3 dimensional space with Minkowski metric. Here $A_m(x)$ are $n \times n$ complex matrices (we consider the $gl(n, C)$ connection only). Two connections $A_m(x)$ and $\tilde{A}_m(x)$, are related by a gauge transformation, iff

$$A_m(x) = g(x)\tilde{A}_m(x)g^{-1}(x) - g(x)\frac{\partial}{\partial x^m}g^{-1}(x).$$

We assume that the connection is fast decreasing as $x \to \infty$.

For $\lambda \in C$ consider the equation,

$$\left(2\lambda \frac{\partial}{\partial x^0} + (\lambda^2 + 1) \frac{\partial}{\partial x^1} - i(\lambda^2 - 1) \frac{\partial}{\partial x^2}\right)\mu(x, \lambda, \bar{\lambda}) =$$

$$(2\lambda A_0(x) + (\lambda^2 + 1)A_1(x) - i(\lambda^2 - 1)A_2(x)) \mu(x, \lambda, \bar{\lambda}).$$

Here $\mu(x, \lambda, \bar{\lambda})$ has values in $n \times n$ complex matrices

Lemma 1.2 Choose the solution of (8), given by the solution of the Fredholm integral equation, (we assume that this integral equation has the
unique solution, which is true at least for the connection of small norm):

$$\mu(x, \lambda, \overline{\lambda}) = I +$$

$$\int G(x - y, \lambda, \overline{\lambda}) \left(2 \lambda A_0(y) + (\lambda^2 + 1)A_1(y) - i(\lambda^2 - 1)A_2(y)\right) \mu(y, \lambda) d^3y,$$

(9)

where $G(x, \lambda, \overline{\lambda})$ is given by (6), and $I$ is the unit matrix.

$\mu(x, \lambda, \overline{\lambda})$, considered as a function of $\lambda$ has a jump on the unit circle $|\lambda| = 1$.

Define $S(x, \lambda, \overline{\lambda}), T(x, \gamma)$ as follows:

$$S(x, \lambda, \overline{\lambda}) = \mu^{-1}(x, \lambda, \overline{\lambda}) \frac{\partial}{\partial \lambda} \mu(x, \lambda, \overline{\lambda}), \quad |\lambda| \neq 1; \quad (10)$$

$$\mu_+(x, \gamma) - \mu_-(x, \gamma) = \mu_+(x, \gamma) (I - T(x, \gamma)), \quad (11)$$

where

$$\mu_\pm(x, \gamma) = \mu(x, \lambda, \overline{\lambda})|_{\lambda=(1 \pm \epsilon)e^{i\gamma}}.$$

Define also

$$t_\pm(x, \gamma) = \lim_{t \to -\infty} \mu_\pm(\alpha t + \beta, \gamma),$$

is the asymptotic of $\mu_\pm$ at $t = -\infty$ along the light ray $x = \alpha t + \beta$, with $\alpha = (1, \cos \gamma, \sin \gamma)$

1. For a given connection, spectral data $S(x, \lambda, \overline{\lambda}), T(x, \gamma)$ are well-defined, depend from 3 real parameters, and given by

$$S(x, \lambda, \overline{\lambda}) = \frac{\text{sgn}(\lambda \overline{\lambda} - 1)}{2\pi} \int \delta' \left((\lambda \overline{\lambda} + 1)(x^0 - y^0) - (\lambda + \overline{\lambda})(x^1 - y^1) + i(\lambda - \overline{\lambda})(x^2 - y^2)\right)$$

$$(2\lambda A_0(y) + (\lambda^2 + 1)A_1(y) - i(\lambda^2 - 1)A_2(y)) \mu(y, \lambda, \overline{\lambda}) d^3y.$$

$$T(x, \gamma) = (t_-(x, \gamma))^{-1} t_+(x, \gamma), \quad (12)$$
where
\[
t_+(x, \gamma) = I + \frac{1}{i\pi} \int \frac{\delta \left( -(x^0 - y^0) + (x^1 - y^1) \cos(\gamma) + (x^2 - y^2) \sin(\gamma) \right)}{(x^1 - y^1) \sin(\gamma) - (x^2 - y^2) \cos(\gamma) + i0} \\
(A_0(y) + \cos(\gamma) + A_1(y) \sin(\gamma) A_2(y)) \mu_+(y, \gamma) d^3 y
\]
\[
t_-(x, \gamma) = I - \frac{1}{i\pi} \int \frac{\delta \left( -(x^0 - y^0) + (x^1 - y^1) \cos(\gamma) + (x^2 - y^2) \sin(\gamma) \right)}{(x^1 - y^1) \sin(\gamma) - (x^2 - y^2) \cos(\gamma) - i0} \\
(A_0(y) + \cos(\gamma) A_1(y) + \sin(\gamma) A_2(y)) \mu_-(y, \gamma) d^3 y.
\]

2. Spectral data \( S(x, \lambda, \bar{\lambda}), T(x, \gamma) \) do not change along the light ray, namely:
\[
\left( 2\lambda \frac{\partial}{\partial x^0} + (\lambda^2 + 1) \frac{\partial}{\partial x^1} - i(\lambda^2 - 1) \frac{\partial}{\partial x^2} \right) S(x, \lambda, \bar{\lambda}) = 0,
\]
\[
\left( \frac{\partial}{\partial x^0} + \cos(\gamma) \frac{\partial}{\partial x^1} + \sin(\gamma) \frac{\partial}{\partial x^2} \right) T(x, \gamma) = 0.
\]

3. The spectral data \( S(x, \lambda, \bar{\lambda}), T(x, \gamma) \) are invariant under the gauge transformation
\[
A_m(x) \rightarrow g^{-1}(x) A_m(x) g(x) - g^{-1}(x) \frac{\partial}{\partial x_m} g(x)
\]

**Proof.** In the proof of 1), we use the uniqueness of solution of (8). The rest can be checked by a straightforward computation.

Define the asymptotic \( \mu(x, \infty) \)
\[
\mu(x, \infty) := \lim_{\lambda \to \infty} \mu(x, \lambda, \bar{\lambda}).
\]
where \( \mu(x, \lambda, \bar{\lambda}) \) is the solution of the integral equation (8).
\( \mu(x, \infty) \) can be obtained as the solution of the Fredholm integral equation
\[
\mu(x, \infty) = I + \frac{1}{2\pi} \int \frac{1}{x^1 - y^1 - i(x^2 - y^2)} \delta(x^0 - y^0) \left( A_1(y) - iA_2(y) \right) \mu(y, \infty) d^3 y.
\]
Remark

1) As $\lambda \to \infty$ the equation (8) becomes
\[
\left( \frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right) \mu(x, \infty) = (A_1(x) - iA_2(x)) \mu(x, \infty),
\]
or
\[
A_1(x) - iA_2(x) = \left( \left( \frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right) \mu(x, \infty) \right) \left( \mu(x, \infty) \right)^{-1}
\]
Thus, for the case of $gl(n, C)$ $\mu(x, \infty)$ is the gauge where $A_1(x) - iA_2(x) = 0$.

2) For other Lie algebras, say for $so(n)$, in general it is not possible to choose the gauge $A_1(x) - iA_2(x) = 0$. Thus, we need to add the normalization $\mu(x, \infty)$ to the list of spectral data. $\mu(x, \infty)$ and $g(x)\mu(x, \infty)$, where $g(x)$ belongs to the appropriate group, give rise to the gauge equivalent connections. There will be certain conditions on spectral data for the Lie algebras other than $gl(n, C)$. We will not investigate it here.

**Proposition 1.2** If $S(x, \lambda, \bar{\lambda})$, $T(x, \gamma)$, and $\mu(x, \infty)$ are given, we can reconstruct $\mu(x, \lambda, \bar{\lambda})$ as the solution of the Fredholm equation
\[
\mu(x, \lambda, \bar{\lambda}) = \mu(x, \infty) + \frac{1}{2\pi i} \int \frac{1}{\lambda_1 - \lambda} \mu(x, \lambda_1, \bar{\lambda}_1) S(x, \lambda_1, \bar{\lambda}_1) d\lambda_1d\bar{\lambda}_1 + \frac{1}{2\pi} \oint_{0}^{2\pi} \frac{1}{e^{i\gamma} - \lambda} \mu(x, \lambda_1, \bar{\lambda}_1)|_{\lambda_1 = (1-\epsilon)e^{i\gamma}} (I - T(x, \gamma)) e^{i\gamma}d\gamma,
\]
(16)

**Proof**
This follows from the Cauchy formula for the $\bar{\partial}$ problem
\[
\frac{\partial}{\partial \lambda} \mu(x, \lambda, \bar{\lambda}) = \mu(x, \lambda, \bar{\lambda}) s(x, \lambda, \bar{\lambda})
\]
with $\mu(x, \lambda, \bar{\lambda})$ having the given jump at the unit circle and the given asymptotic at $\lambda = \infty$

**Lemma 1.3 (Inverse Transform)**
Suppose that we are given the spectral data $s(x, \lambda, \bar{\lambda})$, $t_{\pm}(x, \gamma)$. From these spectral data, we can reconstruct $A_m(x)$ up to the gauge transformation:
Let \( \psi(x, \lambda, \bar{\lambda}) \) be the solution of the integral Fredholm equation

\[
\psi(x, \lambda, \bar{\lambda}) = I + \frac{1}{2\pi i} \int_{\lambda_1}^{\bar{\lambda}_1} \frac{1}{\lambda_1 - \lambda} \psi(x, \lambda_1, \bar{\lambda}_1) S(x, \lambda_1, \bar{\lambda}_1) \, d\lambda_1 d\bar{\lambda}_1 + \frac{1}{2\pi} \oint_{\gamma=0}^{2\pi} \frac{1}{e^{i\gamma} - \lambda} \psi(x, \lambda_1, \bar{\lambda}_1)|_{\lambda_1=(1-\varepsilon)e^{i\gamma}} (I - T(x, \gamma)) \, e^{i\gamma} d\gamma,
\]

There exist some \( g(x) \in GL(n, C) \) such that connection is given by

\[
g^{-1}(x) (A_1(x) + iA_2(x)) g(x) - g^{-1}(x)(\frac{\partial}{\partial x^1} + i \frac{\partial}{\partial x^2}) g(x) = \left( \left(2\lambda \frac{\partial}{\partial x^0} + (\lambda^2 + 1) \frac{\partial}{\partial x^1} - i(\lambda^2 - 1) \frac{\partial}{\partial x^2} \right) \psi(x, \lambda, \bar{\lambda}) \right) \left( \psi(x, \lambda, \bar{\lambda}) \right)^{-1} \bigg|_{\lambda=0}
\]

\[
g^{-1}(x) (A_0(x)) g(x) - g^{-1}(x) \frac{\partial}{\partial x^0} g(x) = \frac{1}{2} \left( \frac{\partial}{\partial \lambda} \left( 2\lambda \frac{\partial}{\partial x^0} + (\lambda^2 + 1) \frac{\partial}{\partial x^1} - i(\lambda^2 - 1) \frac{\partial}{\partial x^2} \right) \psi(x, \lambda, \bar{\lambda}) \right) \left( \psi(x, \lambda, \bar{\lambda}) \right)^{-1} \bigg|_{\lambda=0}
\]

\[
g^{-1}(x) (A_1(x) - iA_2(x)) g(x) - g^{-1}(x)(\frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2}) g(x) = 0
\]

(18)

\[
Proof
\]
Comparing with Prop. 1.2, we see that for \( g(x) = \mu(x, \infty) \) (18) follows from (16), (17).

3 Spectral data in dimension 4.
### 3.1 Abelian connection.

#### Proposition 2.1

Consider the light ray
\[ x = \alpha t + \beta, \]
where \( \alpha = (\alpha^0, \alpha^1, \alpha^2, \alpha^3), \) \( \alpha_m \alpha^m = 0, \) \( \alpha^0 > 0 \) and \( \alpha, \beta \in \mathbb{R}^4. \) Let
\[ \varphi(\alpha, \beta) = \int_{x=\alpha t+\beta} \Omega \equiv \int_{-\infty}^{+\infty} \alpha^i A_i(\alpha t + \beta) dt. \]
be the integral of 1-form over oriented light ray. It is a function on the space of light rays.
\[ \varphi(\lambda \alpha, \beta) = \text{sgn} \lambda \varphi(\alpha, \beta) \]
\[ \varphi(\alpha, \beta + \lambda \alpha) = \varphi(\alpha, \beta) \]
The space of all light rays has dimension 5, therefore, there should be some differential equation for \( \varphi(\alpha, \beta). \) This equation can be written as
\[ \left( \varepsilon^{iklm} n_i \frac{\partial}{\partial \beta^k} n_l \frac{\partial}{\partial \alpha^m} \right)^3 \varphi(\alpha, \beta) = \]
\[ \left( n^p n_p \frac{\partial}{\partial \beta^r} \frac{\partial}{\partial \beta^r} - n^p n^r \frac{\partial}{\partial \beta^p} \frac{\partial}{\partial \beta^r} \right) \left( \varepsilon^{iklm} n_i \frac{\partial}{\partial \beta^k} n_l \frac{\partial}{\partial \alpha^m} \right) \varphi(\alpha, \beta) \]
(19)
where \( n^i \) is an arbitrary vector in \( \mathbb{R}^4, \) and \( \varepsilon^{iklm} \) is totally antisymmetric tensor with \( \varepsilon^{0123} = 1 \)

**Proof**

This can be checked by the straightforward computation.

#### Proposition 2.2

If we know \( \varphi(\alpha, \beta), \) we can reconstruct the Fourier transform of the 1-form \( \Omega \) outside the light cone, up to an exact form. We will give two ways to do this:

1. For pairs \( \alpha, \eta \) such that
\[ (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 - (\alpha^0)^2 = 0 \]
\[ \eta_0 \alpha^0 + \eta_1 \alpha^1 + \eta_2 \alpha^2 + \eta_3 \alpha^3 = 0, \]

(20)
Define
\[ \Phi(\alpha, \eta) = \int_{\mathbb{R}^4/\alpha} \varphi(\alpha, \beta) e^{i\eta \cdot \beta} d\alpha d\beta \]
to be the Fourier transform of the function \( \varphi(\alpha, \beta) \) over the factor space \( \beta \in \mathbb{R}^4/\{\alpha\} \). Here \( d\alpha d\beta = |\alpha^0 d\beta^1 \wedge d\beta^2 \wedge d\beta^3 + \alpha^1 d\beta^2 \wedge d\beta^3 \wedge d\beta^0 + \alpha^2 d\beta^3 \wedge d\beta^0 \wedge d\beta^1 + \alpha^3 d\beta^0 \wedge d\beta^1 \wedge d\beta^2| \) is a nonoriented volume element on this space. From homogenous property, we can choose
\[ \alpha^0 = 1 \]
For fixed \( \eta \), (20) viewed as an equation for \( \alpha \), with \( \alpha^0 = 1 \), has a 1-parametric family of real-valued solutions, provided
\[ \eta_1^2 + \eta_2^2 - \eta_0^2 > 0 : \]
and no real-valued solutions for
\[ \eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_0^2 < 0 : \]
For \( \eta \) such that \( \eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_0^2 > 0 \), we can reconstruct the connection up to the exact form. Let
\[ \alpha^{(i)}, i = 1, 2, 3. \]
be any three solutions of (20), linearly independent as vectors in \( \mathbb{R}^4 \). (Such 3 solutions always exist for \( \eta_1^2 + \eta_2^2 - \eta_0^2 > 0 \).) Notice that for pairs \( (\alpha, \eta) \), solving (20),
\[ \Phi(\alpha, \eta) = \alpha^0 A_0(\eta) + \alpha^1 A_1(\eta) + \alpha^2 A_2(\eta) + \alpha^3 A_3(\eta). \quad (21) \]
Thus, from the linear algebra it follows that
\[ \eta_i \hat{A}_j(\eta) - \eta_j \hat{A}_i(\eta) = \frac{\rho}{2} \sum_{k \neq i, j} \text{sgn}(p(i, j, k, l)) \]
\[ \left( \begin{array}{ccc} \alpha^{(2),k} & \alpha^{(2),l} \\ \alpha^{(3),k} & \alpha^{(3),l} \end{array} \right) \Phi(\alpha^{(1)}, \eta) \right) + \left( \begin{array}{ccc} \alpha^{(3),k} & \alpha^{(3),l} \\ \alpha^{(1),k} & \alpha^{(1),l} \end{array} \right) \Phi(\alpha^{(2)}, \eta) \right) + \left( \begin{array}{ccc} \alpha^{(1),k} & \alpha^{(1),l} \\ \alpha^{(2),k} & \alpha^{(2),l} \end{array} \right) \Phi(\alpha^{(3)}, \eta) \right) \]
where, for each \( p(i, j, k, l) \) to be a permutation of numbers \( (0, 1, 2, 3) \)
\[ \rho = \text{sgn}(p(i, j, k, l)) \]
\[ \eta_i \]
\[ \begin{array}{ccc} \alpha^{(1),j} & \alpha^{(1),k} & \alpha^{(1),l} \\ \alpha^{(2),j} & \alpha^{(2),k} & \alpha^{(2),l} \\ \alpha^{(3),j} & \alpha^{(3),k} & \alpha^{(3),l} \end{array} \]
2. There is another way to write the inversion formulas. Define

\[ \hat{\varphi}(\alpha, \eta) = \int \varphi(\alpha, \beta) e^{i\beta^m \eta^m} d^4 \beta \]

to be the Fourier transform over all \( \beta \). Here \( \alpha_m \alpha^m = 0 \).

We can reconstruct the connection up to the exact form as follows: Let

\[ \alpha = (1, \bar{\alpha}), \; |\bar{\alpha}| = 1 \]

\[
|\bar{\eta}| \int \hat{\varphi}(\alpha, \eta) \frac{d\omega_\alpha}{(2\pi)^2} = \left( A_0 - \frac{\eta_0}{\bar{\eta}^2} \right) \Theta(\bar{\eta}^2 - \eta_0^2) 
\]

\[
\frac{2|\bar{\eta}|^3}{\eta^2} \int \hat{\varphi}(\alpha, \eta) \sum_{\nu=1}^3 \left( \delta_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{\bar{\eta}^2} \right) \alpha_\nu \frac{d\omega_\alpha}{(2\pi)^2} = \sum_{\nu=1}^3 \left( \delta_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{\bar{\eta}^2} \right) A_\nu \Theta(\bar{\eta}^2 - \eta_0^2)
\]

where \( \int d\omega_\alpha \) is the integral over the unit sphere.

We see that the formulas above give the Fourier transform of the connection inside the light cone only. To reconstruct the Fourier transform of the connection inside the light cone, we need additional data. For the abelian connection, there is a nice way to give supplementary data, using the quaternions. It is not clear how to use the quaternions for the nonabelian problem, and we will use different methods there.

### 3.2 Supplementary Data for the Abelian Connection using Quaternions.

Consider purely imaginary quaternions

\[ q = q_1 i + q_2 j + q_3 k, \]

where \( q_1, q_2, q_3 \) are real numbers, and \( i^2 = j^2 = k^2 = -1, \; ij = -ji = k \) and so on. Define \( |q| := q_1^2 + q_2^2 + q_3^2 \) For \( q \) such that \( |q| + \frac{1}{|q|} < 6 \) define the 'quaternionic light ray' \( \alpha(q) \):

\[ \alpha(q) = \frac{1}{2} \left( (6 + q^2 + q^{-2})^{\frac{1}{2}}, -iq + q^{-1}i, -jq + q^{-1}j, -kq + q^{-1}k \right) \]

(23)
For \( q \) with \(|q| = 1\), \( \alpha(q) \) is just the usual real light ray. For \( q \) with \(|q| \neq 1\), \( \alpha(q) \) is a vector with values in quaternions such that \( \alpha(q)^m \alpha(q)_m = 0 \).

**Proposition**

Suppose that in addition to \( \Phi(\alpha, \eta) \), (21) defined for pairs \((\alpha, \eta)\) satisfying (20), with \( \alpha \) real, we are given

\[
\Phi(\alpha(q), \eta) := \alpha(q)^m A_m(\eta),
\]

which is defined for pairs \((\alpha(q), \eta)\), with real \( \eta \) and quaternionic \((\alpha(q), \eta)\), (23) such that

\[
\alpha(q)^m \eta_m = 0
\]

for \(|q| \neq 1\). Then we can reconstruct the Fourier transform of the connection inside the light cone as well.

Indeed, from (25) it follows that

\[
(q_1, q_2, q_3) = \rho(\eta_1, \eta_2, \eta_3)
\]

where \( \rho \) should be a real number, and

\[
\rho^2 \bar{\eta}^2 = \frac{3 \eta_0^2 - \bar{\eta}^2 \pm \sqrt{8|\eta_0|(|\eta_0^2 - \bar{\eta}^2)^\frac{1}{2}}}{\eta_0^2 + \bar{\eta}^2}
\]

For \( \eta_0^2 - \bar{\eta}^2 > 0 \) such real \( \rho \) always exist. Substituting in \( \Phi(\alpha(q), \eta) \) we have

\[
\Phi(\alpha(q), \eta) = \left( A_0 - \eta_0 \frac{\bar{A}\eta}{\bar{\eta}^2} \right) \frac{\bar{\eta}^2 \rho}{\eta_0} (1 - \frac{1}{\rho^2 \bar{\eta}^2}) + \\
\rho(1 + \frac{1}{\rho^2 \bar{\eta}^2}) \left( (A_1 \eta_2 - A_1 \eta_1)k + (A_2 \eta_3 - A_3 \eta_2)i + (A_3 \eta_1 - A_1 \eta_3)j \right)
\]

Thus, we can reconstruct \( A(\eta) \) for \( \eta_0^2 - \bar{\eta}^2 > 0 \) up to an exact form.

### 3.3 Nonabelian Connection

For the spectral data we need to use complex light rays. In dimension 4 is highly nontrivial, since the dimension of the space of all real light rays
is 5, (that is higher than the dimension of space-time). Also, the direction of the light-ray is parametrized by the unit sphere $S^2$, and there is no canonical way to introduce the complex directions, thus we need to make choices. (It would be nice to use quaternions, but in the nonlinear case it is not yet clear how to write the inversion formula with quaternions). Thus we accept a different approach. The main idea is to reduce the 4-dimensional case to the 3-dimensional, considered before. The direction of the light ray is parametrized by a point on a sphere. We will choose the spherical coordinates for the direction of real light rays, and for each fixed azimuthal angle we can go to complex direction in the same way as we have done in the 3-dimensional case. Thus, we have made the following choice to go to complex light-rays:

\[
\left(2\lambda \frac{\partial}{\partial x^0} + \sin \Theta \left((\lambda^2 + 1) \frac{\partial}{\partial x^1} - i(\lambda^2 - 1) \frac{\partial}{\partial x^2}\right) + 2\lambda \cos \Theta \frac{\partial}{\partial x^3}\right) \mu(x, \lambda, \bar{\lambda}, \Theta) = \\
(2\lambda A_0(x) + \sin \Theta ((\lambda^2 + 1) A_1(x) - i(\lambda^2 - 1) A_2(x)) + 2\lambda \cos \Theta A_3(x)) \mu(x, \lambda, \bar{\lambda}, \Theta)
\]

Lemma 2.1 Choose the solution of (26), given by the solution of the Fredholm integral equation, (we assume that this integral equation has the unique solution, which is true at least for the connection of small norm):

\[
\mu(x, \lambda, \bar{\lambda}, \Theta) = I + \int G(x - y, \lambda, \bar{\lambda}, \Theta) \\
\left(2\lambda A_0(y) + \sin \Theta ((\lambda^2 + 1) A_1(y) - i(\lambda^2 - 1) A_2(y)) + 2\lambda \cos \Theta A_3(x)) \mu(y, \lambda) d^3y,
\]

where $G(x, \lambda, \bar{\lambda}, \Theta)$ is given by

\[
G(x, \lambda, \bar{\lambda}, \Theta) := \frac{\bar{\lambda} \text{sgn}(\lambda \bar{\lambda} - 1)}{\pi} \delta \left(\frac{-(\lambda \bar{\lambda} + 1) x^+ \sin \Theta + x^1 (\lambda + \bar{\lambda}) - ix^2 (\lambda - \bar{\lambda})}{(\lambda \bar{\lambda} - 1) x^+ \sin \Theta + x^1 (\lambda - \bar{\lambda}) - ix^2 (\lambda + \bar{\lambda})}\right) \delta(x^-)
\]

where $x^+ = \frac{x^0 + \cos \Theta x^3}{1 + \cos \Theta^2}, x^- = -\cos \Theta x^0 + x^3$.

$\mu(x, \lambda, \bar{\lambda}, \Theta)$, considered as a function of $\lambda$ has a jump on the unit circle $|\lambda| = 1$.

Define $S(x, \lambda, \bar{\lambda}, \Theta), T(x, \gamma, \Theta)$ as follows:

\[
S(x, \lambda, \bar{\lambda}, \Theta) = \mu^{-1}(x, \lambda, \bar{\lambda}, \Theta) \frac{\partial}{\partial \lambda} \mu(x, \lambda, \bar{\lambda}, \Theta), \quad |\lambda| \neq 1;
\]

\[
T(x, \gamma, \Theta) = \mu^{-1}(x, \gamma, \Theta) \frac{\partial}{\partial \gamma} \mu(x, \gamma, \Theta), \quad |\gamma| \neq 1.
\]
\[ \mu_+(x, \gamma, \Theta) - \mu_-(x, \gamma, \Theta) = \mu_+(x, \gamma, \Theta) (I - T(x, \gamma, \Theta)), \] 

(30)

where 

\[ \mu_\pm(x, \gamma, \Theta) = \mu(x, \lambda, \bar{\lambda}, \Theta)|_{\lambda = (1 \pm \varepsilon)e^{i\gamma}}. \]

1) The scattering data \( S(x, \lambda, \bar{\lambda}, \Theta) \), \( T(x, \gamma, \Theta) \) are given by

\[ S(x, \lambda, \bar{\lambda}, \Theta) = \frac{\text{sgn}(\lambda \bar{\lambda} - 1)}{2\pi} \int \delta' \left( (\lambda \bar{\lambda} + 1) \sin \Theta (x^+ - y^+) - (\lambda + \bar{\lambda})(x^1 - y^1) + i(\lambda - \bar{\lambda})(x^2 - y^2) \right) \delta(x^- - y^-) \]

\[ (2\lambda A_0(y) + \sin \Theta (\lambda^2 + 1)A_1(y) - i(\lambda^2 - 1)A_2(y)) + \cos \Theta A_3(y) \mu(y, \lambda, \bar{\lambda}, \Theta) d^4y, \]

\[ T(x, \gamma, \Theta) = (t_-(x, \gamma, \Theta))^{-1} t_+(x, \gamma, \Theta), \]

(31)

where

\[ t_+(x, \gamma, \Theta) = I + \frac{1}{i\pi} \int \frac{\delta (-\sin \Theta (x^+ - y^+) + (x^1 - y^1) \cos \gamma + (x^2 - y^2) \sin \gamma)}{(x^1 - y^1) \sin \gamma - (x^2 - y^2) \cos \gamma + i0} \delta(x^- - y^-) \]

\[ (A_0(y) + \sin \Theta (\cos \gamma A_1(y) \sin \gamma A_2(y)) + A_3(y) \cos \Theta) \mu_+(y, \gamma, \Theta) d^4y \]

\[ t_-(x, \gamma, \Theta) = I - \frac{1}{i\pi} \int \frac{\delta (-\sin \Theta (x^+ - y^+) + (x^1 - y^1) \cos \gamma + (x^2 - y^2) \sin \gamma)}{(x^1 - y^1) \sin \gamma - (x^2 - y^2) \cos \gamma - i0} \delta(x^- - y^-) \]

\[ (A_0(y) + \sin \Theta (\cos \gamma A_1(y) \sin \gamma A_2(y)) + A_3(y) \cos \Theta) \mu_+(y, \gamma, \Theta) d^4y, \]

(32)

Scattering data \( S(x, \lambda, \bar{\lambda}, \Theta) \), \( T(x, \gamma, \Theta) \) depend from 5 real parameters. They do not change along the light-line:

\[ \left( 2\lambda \frac{\partial}{\partial x^0} + \sin \Theta \left( (\lambda^2 + 1) \frac{\partial}{\partial x^1} - i(\lambda^2 - 1) \frac{\partial}{\partial x^2} \right) + 2\lambda \cos \Theta \frac{\partial}{\partial x^3} \right) S(x, \lambda, \bar{\lambda}, \Theta) = 0, \]

\[ \left( \frac{\partial}{\partial x^0} + \sin \Theta \left( \cos \gamma \frac{\partial}{\partial x^1} + \sin \gamma \frac{\partial}{\partial x^2} \right) + \cos \Theta \frac{\partial}{\partial x^3} \right) T(x, \gamma, \Theta) = 0. \]

2) Scattering data \( S(x, \lambda, \bar{\lambda}, \Theta) \), \( T(x, \gamma, \Theta) \) do not change under the gauge transformations
The asymptotic as $\lambda \to \infty$ is given by

$$\mu(x, \infty) = I + \frac{1}{2\pi} \int \frac{1}{x^1 - y^1 - i(x^2 - y^2)} \delta(x^0 - y^0) \delta(x^3 - y^3) \left( A_1(y) - iA_2(y) \mu(y, \infty) \right) d^4y.$$ 

**Lemma**

1. If we are given the scattering data $S(x, \lambda, \bar{\lambda}, \Theta)$, $T(x, \gamma, \Theta)$, and we know that the scattering data were obtained from some connection, we can reconstruct the connection $A(x)$ up to a gauge transformation, as follows:

   Define $\psi(x, \lambda, \bar{\lambda}, \Theta)$ to be the solution of the integral Fredholm equation

   $$\psi(x, \lambda, \bar{\lambda}, \Theta) = I + \frac{1}{2\pi} \int \frac{1}{\bar{\lambda}_1 - \lambda} \psi(x, \lambda_1, \bar{\lambda}_1, \Theta) s(x, \lambda_1, \bar{\lambda}_1, \Theta) d\lambda_1 d\bar{\lambda}_1 + \frac{1}{2\pi} \oint \frac{1}{e^{i\gamma} - \lambda} \psi(x, \lambda_1, \bar{\lambda}_1, \Theta)|_{\lambda_1=(1-\varepsilon)e^{i\gamma}} (I - T(x, \gamma, \Theta)) e^{i\gamma} d\gamma.$$ (33)

   Define

   $$W(x, \gamma, \Theta) := \frac{1}{2} e^{-i\gamma} \left( \left( 2\lambda \frac{\partial}{\partial x^0} + \sin \Theta \left( \lambda^2 \left( \frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right) \right) + 2\lambda \cos \Theta \frac{\partial}{\partial x^3} \right) \psi(x, \lambda, \bar{\lambda}, \Theta) \right) \psi^{-1}(x, \lambda, \bar{\lambda}, \Theta) \right|_{\lambda = (1-\varepsilon)e^{i\gamma}}$$ (34)

   Then there is some $g(x) \in GL(n, C)$ such that

   $$g^{-1}(x) \left( A_0(x) + \sin \Theta \left( \cos \gamma A_1(x) + \sin \gamma A_2(x) \right) + \cos \Theta A_3(x) \right) g(x) -$$

   $$g^{-1}(x) \left( \frac{\partial}{\partial x^0} + \sin \Theta \left( \cos \gamma \frac{\partial}{\partial x^1} + \sin \gamma \frac{\partial}{\partial x^2} \right) + \cos \Theta \frac{\partial}{\partial x^3} \right) g(x) = W(x, \gamma, \Theta).$$ (35)

2. There compatibility conditions for the scattering data $s(x, \lambda, \bar{\lambda}, \Theta)$, $t_{\pm}(x, \gamma, \Theta)$ are the following: $W(x, \gamma, \Theta)$, which is a functional of the
scattering data, defined by (33),(34) should solve the equation

\[
(u(\gamma, \Theta) \frac{\partial}{\partial \gamma} + v(\gamma, \Theta) \frac{\partial}{\partial \Theta}) \left( -\frac{1}{\sin^2 \Theta} \frac{\partial^2}{\partial \gamma^2} + \frac{1}{\sin \Theta} (\sin \Theta \frac{\partial}{\partial \Theta}) \right) W(x, \gamma, \Theta) = \\
2(u(\gamma, \Theta) \frac{\partial}{\partial \gamma} + v(\gamma, \Theta) \frac{\partial}{\partial \Theta}) W(x, \gamma, \Theta)
\]

(36)

where \(u(\gamma, \Theta), v(\gamma, \Theta)\) are arbitrary functions.

4 Discussion

Consider the following pair of linear differential equations:

\[
\begin{cases}
-2i\lambda \left( \frac{\partial}{\partial y_0} - A_0(y) \right) + \sin \Theta \left( \lambda^2 + 1 \right) \left( \frac{\partial}{\partial y_1} - A_1(y) \right) - i(\lambda^2 - 1) \left( \frac{\partial}{\partial y_2} - A_2(y) \right) + \\
2\lambda \cos \Theta \left( \frac{\partial}{\partial y_3} - A_3(y) \right) \mu(x, \lambda, \bar{\lambda}, \Theta) = 0
\end{cases}
\]

(37)

The first equation in the pair coincides with the equation (5.9) at purely imaginary time \(x_0 = -iy_0\).

The compatibility conditions of the system (5.10) are the following:

\[
F_{mn}(y) = \frac{1}{2} \varepsilon^{mnpq} F_{pq}(y),
\]

(38)

where \(F_{mn}(y)\) is the field strength,

\[
F_{mn}(y) = -\frac{\partial}{\partial y_m} A_n(y) + \frac{\partial}{\partial y_n} A_m(y) + [A_m(y), A_n(y)],
\]

(39)

and \(\varepsilon^{mnpq}\) is the totally antisymmetric tensor with \(\varepsilon^{0123} = 1\). A connection \(A(y)\) is called a self-dual, if the field strength (5.12) satisfies equations (5.11). Thus, we have proved the following
Proposition 1  The compatibility condition of the system of equations (5.10) is that the connection $A(y)$ is a self-dual connection.

Remark 2. Light-Line Radon Transform and Crofton-Nambu Lagrangians. The Light-Line Radon transform, as described in Chapter 4, gives a transformation from a nonabelian connection on a 4 dimensional real Minkowski space to (matrix-valued) functions on the space of 2 dimensional planes in the 4 dimensional space.

Indeed, the functions on the space of such planes which do not contain a light line are present in our construction explicitly: they correspond to the complex values of parameter, characterising the direction of a light line, see (4.31). For planes which do contain a light line, and therefore contain a family of parallel light lines, we can construct a function of such planes by integrating the function on the space of light lines over the family of parallel light lines, contained in the plane.

Proposition 2  Any function $\mu$ on the space of two dimensional planes in the 4 dimensional space give rise to an action for a string theory with the following properties:

0) the action is reparametrization-invariant on a world sheet; its target is the 4 dimensional Euclidean space

1) for a world sheet which is topologically a plane, maps to 2 dimensional planes in the 4 dimensional space are geodetic

2) if the function $\mu$ is positive, the "holomorphic" maps $X(\sigma-\tau)$ are geodetic, and, moreover, give a surface with the minimal value of the action.

Let us parametrise a 2 dimensional plane in a 4 dimensional space $X \in \mathbb{R}^4$ as follows:

$$
\begin{pmatrix}
X_3 \\
X_4
\end{pmatrix} = A \begin{pmatrix}
X_1 \\
X_2
\end{pmatrix} + B,
$$

$$
A = \begin{pmatrix}
a_{31} & a_{32} \\
a_{41} & a_{42}
\end{pmatrix}, \quad B = \begin{pmatrix}
b_3 \\
b_4
\end{pmatrix}
$$

We will write a function on the space of 2 dimensional planes in the 4 dimensional space as

$$
\mu(A, B)
$$
Then the string action for maps $X \equiv X(\sigma, \tau)$ can be written as

$$
S = \int \mu \left( A, \begin{pmatrix} X_3 \\ X_4 \end{pmatrix} - A \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right) \det \begin{pmatrix}
X_{1,\sigma} & X_{1,\tau} & 1 & 0 \\
X_{2,\sigma} & X_{2,\tau} & 0 & 1 \\
X_{3,\sigma} & X_{3,\tau} & a_{31} & a_{32} \\
X_{4,\sigma} & X_{4,\tau} & a_{41} & a_{42}
\end{pmatrix} dA d\sigma d\tau
$$

where $dA$ is an invariant measure on the Grassmanian.

A particular case of such action is the Nambu action, which is just the area of the surface $X(\sigma, \tau)$.

The Lagrangians as above were studied in [11]. In [11] such Lagrangians were called Crofton Lagrangians, due to a classical theorem of Crofton, that if we have a curve in two dimensions, and we count the number of intersection points (without sign) of a curve with a line, and than average over all lines with some measure, the result is proportional to the length of the curve. Similar approach was used here.

Although it is not yet clear, how to use the Lagrangians above in the quantum case, the fact that Yang-Mills turns out to be related with a string theory through a Radon transform is absolutely new and totally unexpected. It is a step towards solving a problem to relate Yang-Mills theory with a string theory, advocated by A.Polyakov.

Remark 3. Light-Line Radon transform, representation theory of the group $SL(2, \mathbb{R})$ in terms of integral geometry, and $SL(2, \mathbb{R})$-type representation theory without a group.

Recall that the simplest version of the Light-Line Radon transform, namely the Light-Line Radon transform of a function $f(x)$ on a real Minkowski space of dimension 3, is given by the integral of the function $f(x)$ over all light lines. We parametrise a light line as follows:

$$
x = \alpha t + \beta, \\
\alpha = (1, \frac{\lambda+\lambda^{-1}}{2}, \frac{\lambda-\lambda^{-1}}{2i}), \lambda = e^{i\gamma}, 0 \leq \gamma \leq 2\pi \\
\beta = (0, \beta_1, \beta_2)
$$

Then the light line Radon transform $J$ of a function $f$, $Jf \equiv \phi_f(\lambda, \beta_1, \beta_2)$ is given by

$$
\phi_f(\lambda, \beta_1, \beta_2) = \int_{-\infty}^{+\infty} f(t, \frac{\lambda+\lambda^{-1}}{2}t + \beta_1, \frac{\lambda-\lambda^{-1}}{2i}t + \beta_2) dt
$$

(42)
The formula (5.15) is similar to a formula in the representation theory of
the group $SL(2, R)$. Let me explain this point.

The group $SL(2, R)$ is the group of real $2 \times 2$ matrices with the deter-
mominant 1. The continuous series of unitary irreducible representations
of the group $SL(2, R)$ is labelled by a multiplicative unitary character $\pi(t) = |t|^\epsilon \text{sgn}(t)$, $\epsilon = 0, 1$. It can be realized on the space of Schwarz class functions
of one real variable $\varphi(x)$. An operator $T_\pi(g)$ corresponding to the element
$g \in G$, $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, $g_{11}g_{22} - g_{12}g_{21} = 0$ in representation $\pi$ is given by:

$$(T_\pi(g)\varphi)(x) = \varphi \left( \frac{g_{11}x + g_{21}}{g_{12}x + g_{22}} \right) \pi \left( \frac{g_{12}x + g_{22}}{|g_{12}x + g_{22}|} \right).$$

Thus, $T_\pi(g)$ is an integral operator with kernel $K_\pi(g; x, y)$ given by a general-
alized function

$$K_\pi(g; x, y) = \delta \left( \frac{g_{11}x + g_{21}}{g_{12}x + g_{22}} - y \right) \pi \left( \frac{g_{12}x + g_{22}}{|g_{12}x + g_{22}|} \right).$$

Let us now take a function on the group, $\Phi(g)$. To any such function corre-
sponds an operator with the kernel

$$(K_\pi \Phi)(x, y) = \int K_\pi(g; x, y) \Phi(g) dg,$$

where $dg = \frac{dg_{12}dg_{22}dg_{21}}{|g_{22}|}$ is an invariant measure on the group. Let us choose
the coordinates on the group as follows:

$$a_1 = g_{12}, a_2 = g_{22}, a_3 = g_{11},$$

and let us write the function on the group as $\Phi(a_1, a_2, a_3)$. Then it is easy
to check that

$$(K_\pi \Phi)(x, y) = \int \Phi(t, \lambda - xt, \lambda^{-1} + yt) \pi(\lambda) \lambda^{-1} d\lambda dt$$

If we know $(K_\pi \Phi)(x, y)$ for all unitary multiplicative characters on the real
line $\pi$, we know also

$$L(\lambda, x, y) = \int_{-\infty}^{\infty} \Phi(t, \lambda + xt, \lambda^{-1} + yt) dt, \lambda \in \mathbb{R},$$

(43)
using the Mellin transform.

Compare this with the formula for the light-line Radon Transform.

\[ \phi_f(\lambda, \beta_1, \beta_2) = \int_{-\infty}^{+\infty} f(t, \frac{\lambda+\lambda^{-1}}{2}t + \beta_1, \frac{\lambda-\lambda^{-1}}{2i}t + \beta_2) dt, \lambda = e^{i\gamma}, 0 \leq \gamma \leq 2\pi \]

Let us introduce the function \( F \) as follows:

\[ f(x_1, x_2, x_3) = x_1^2 F\left(\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right). \]

Then

\[ \phi_f(\lambda, \beta_1, \beta_2) = -\int_{-\infty}^{+\infty} F(t, \frac{\lambda+\lambda^{-1}}{2} + \beta_1 t, \frac{\lambda-\lambda^{-1}}{2i} + \beta_2 t) dt, \lambda = e^{i\gamma}, 0 \leq \gamma \leq 2\pi \] (44)

Thus, for the group \( SL(2, R) \), we have integrals of a function of 3 real variables over all lines intersecting a hyperbola \( a_1 = 0, a_2 a_3 = 1 \), and for a light-line Radon transform, we have integrals of a function of 3 real variables over all lines intersecting a circle \( x_1 = 0, x_2^2 + x_3^2 = 1 \).

In representation theory of the group \( SL(2, R) \), there are Plancherel-type formulas, which allow to reconstruct a function on the group from the kernels of operators of its unitary irreducible representations. The representation theory without a group is the problem to reconstruct a function from the integrals of the function over lines intersecting an arbitrary algebraic curve (this problem makes sense in any dimension \( n = 2, 3, 4, \ldots \)). The idea to have a representation theory without a group is due to I. Gelfand. For a large class of curves, this problem was solved in [12]. The light-line Radon Transform, described above, is, in fact, a nonlinear version of the problem of a representation theory without a group.
Acknowledgements The author is grateful to E.Witten for suggesting the problem and discussions. This work is part of the project with I.Gelfand and M.Graev in integral geometry and its applications. Discussions with A. Fokas, A. Migdal, S. Shenker, A. Zamolodchikov were very valuable. Graduate Fellowship from the High Energy Theory group at Rutgers Univ. is appreciated.

References

[1] E. Witten. An interpretation of the classical Yang-Mills theory. Phys. Lett. 77b, 394-398, 1978.
E.Witten. Twistor-like transform in ten dimensions. Nucl.Phys. B266, 245-264, 1986.

[2] N.Seiberg. The power of holomorphy.[hep-th/9408013]

[3] N.Seiberg, E.Witten. Electric-Magnetic Duality, Monopole Condensation, and Confinement in N=2 Supersymmetric Yang-Mills Theory. Nucl Phys. B431, 484, (1994)

[4] E. Witten, private communication.

[5] I. Gelfand, M. Graev, Y.Shapiro. Sov. Math.

[6] M.Atiyah,N.Hitchin. The geometry and dynamics of Magnetic Monopoles. Princeton Press,1988.

[7] I.Gelfand, M.Zyskin. Light-Ray Radon Transform. The case of Abelian Connection.

[8] I.Gelfand, M.Graev , M.Zyskin. (In preparation).

[9] A.S. Fokas, I.M. Gelfand, M.V. Zyskin. Nonlinear Integrable Equations and Nonlinear Fourier Transform, [hep-th/9504042].

[10] A.Fokas, I.Gelfand, M.Zyskin. Nonlinear Fourier, Radon and Abel Transform. (In preparation).

[11] I.M. Gelfand, M.M. Smirnov. Adv. in Math., 109, 188-227, 1994.
[12] I.M. Gelfand, M.I. Graev. M. Zyskin. Prepint RU 64-95.