Velocity operator approach to quantum fluid dynamics in a three-dimensional neutron-proton system

Seiya NISHIYAMA* and João da PROVIDÊNCIA†
Centro de Física, Departamento de Física, Universidade de Coimbra, P-3004-516 Coimbra, Portugal†

Abstract
In the preceding paper, introducing isospin-dependent density operators and defining exact momenta (collective variables), we could get an exact canonically momenta approach to a one-dimensional (1D) neutron-proton (NP) system. In this paper, we attempt at a velocity operator approach to a 3D NP system. Following Sunakawa, after introducing momentum density operators, we define velocity operators, denoting classical fluid velocities. We derive a collective Hamiltonian in terms of the collective variables.

Keywords: Collective motion of a three-dimensional neutron-proton system; velocity operator; vortex operator; Grassmann variables

PACS Number (s): 21.60.-n, 21.60.Ev

1 Introduction
To approach elementary excitations in a Fermi system, Tomonaga gave basic idea in his collective motion theory [1,2]. On the other hand, it is anticipated that the Sunakawa’s discrete integral equation method for a Fermi system [3] may also work well for a collective motion problem. In the preceding paper [4] (referred to as I), on isospin space ($T, T_z$), introducing density operators $\rho_{T, T_z}^0, T=0$ and associated variables $\pi_{T, T_z}^0$ and defining exact momenta $\Pi_{T, T_z}^0$ (collective variables), we could get an exact canonically momenta approach to a one-dimensional (1D) neutron-proton (NP) system. In 3D quadrupole nuclear collective motions, we also have proposed exact canonically momenta to collective coordinates and given exact canonically momenta- and collective coordinate-dependence of the kinetic part of the Hamiltonian [5]. In this paper, we attempt at a velocity operator approach to a 3D NP system. Following Sunakawa, after introducing momentum density operators $g_{T, T_z}^0$, we define velocity operators $v_{T, T_z}^0$ which denote classical fluid velocities. We derive a collective Hamiltonian in terms of the collective variables $v_{T, T_z}^0$ and $\rho_{T, T_z}^0$ for irrotational motion. Its lowest order is diagonalized and leads us to the Bogoliubov transformation [6].

In Section 2 first we introduce collective variables $\rho_{T, T_z}^0$ and associated variables $\pi_{T, T_z}^0$ and give commutation relations between them. Next the velocity operator $v_{T, T_z}^0$ is defined by a discrete integral equation and commutation relations between the velocity operators are also given. In Section 3 the dependence of the original Hamiltonian on $v_{T, T_z}^0$ and $\rho_{T, T_z}^0$ is determined. This section is also devoted to a calculation of a constant term in the collective Hamiltonian. Finally in Section 4 some discussions and further outlook are given.

1 Corresponding author.
E-mail address: seikoceu@khe.biglobe.ne.jp; nisiyama@teor.fis.uc.pt
2 E-mail address: providencia@teor.fis.uc.pt
2 Collective variable and velocity operator

In I, we have defined the Fourier component of the density operator \((\rho(x) = \psi^\dagger(x)\psi(x))\) on the isospin space \((T=0, T_z=0)\). In the 3D system, they are extended as

\[
\rho_{\mathbf{k}}^{0,0} = \frac{1}{\sqrt{A}} \sum \mathbf{p}_{\mathbf{r}_\tau} a^\dagger \mathbf{p}_{\mathbf{k},\mathbf{r}_\tau} a \mathbf{p}_{\mathbf{k} - \mathbf{P}_{\mathbf{r}_\tau}}, \quad \rho_{\mathbf{k}}^{0,0} = \frac{1}{\sqrt{A}} \sum \mathbf{p}_{\mathbf{r}_\tau} a^\dagger \mathbf{p}_{\mathbf{r}_\tau} a \mathbf{p}_{\mathbf{r}_\tau} = \frac{N + Z}{\sqrt{A}},
\]

where we have used the anti commutation relation (CR)s among \(a \mathbf{k},\mathbf{r}_\tau\) s and \(a^\dagger \mathbf{k},\mathbf{r}_\tau\) s given by

\[
\{a_{\mathbf{k},\mathbf{r}_\tau}, a^\dagger_{\mathbf{k}',\mathbf{r}_\tau}\} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{r}_\tau,\mathbf{r}_\tau'}, \quad \{a_{\mathbf{k},\mathbf{r}_\tau}, a_{\mathbf{k}',\mathbf{r}_\tau'}\} = \{a^\dagger_{\mathbf{k},\mathbf{r}_\tau}, a^\dagger_{\mathbf{k}',\mathbf{r}_\tau'}\} = 0,
\]

where \(N\) and \(Z\) mean the number of neutron \((\tau_z = \frac{1}{2})\) and proton \((\tau_z = -\frac{1}{2})\). We here consider a spin-less fermion. The Hamiltonian \(H\) in a 3D box \(\Omega = L^3\) is given by Eq. (2.10) in I as

\[
H = T + V = \sum_{\mathbf{k},\mathbf{r}_\tau} \frac{\hbar^2 k^2}{2m} a_{\mathbf{k},\mathbf{r}_\tau} a_{\mathbf{k},\mathbf{r}_\tau}^\dagger - \frac{A}{8\Omega} \sum \mathbf{k} \nu_T(\mathbf{k}) \rho_{\mathbf{k}}^{0,0} - \frac{A}{4\Omega} \sum \mathbf{k} \nu_{T=0}(\mathbf{k}),
\]

where \(\nu_{T=0}\), denoted simply as \(\nu\), is a scalar function of \(\mathbf{k}\) which specifies the interaction.

Following Tomonaga [1], first we introduce associated collective momenta \(\pi_{\mathbf{k}}^{0,0}\) defined by

\[
\pi_{\mathbf{k}}^{0,0} \equiv \frac{m \hbar}{k^2} \left(\rho_{\mathbf{k}}^{0,0} - \mathbf{p}_{\mathbf{k}}\right), \quad \left[\rho_{\mathbf{k}}^{0,0}, \pi_{\mathbf{k}}^{0,0}\right] = \frac{m \hbar}{k^2} [T, \rho_{\mathbf{k}}^{0,0}], \quad (\mathbf{k} \neq 0),
\]

where the upper symbol dot \(\cdot\) means a time derivative. Calculating the commutator (2.4), we obtain the explicit expression for the associated collective variables \(\pi_{\mathbf{k}}^{0,0}\) as

\[
\pi_{\mathbf{k}}^{0,0} = -\frac{i\hbar}{\sqrt{A\mathbf{k} \cdot \mathbf{k}}} \sum \mathbf{p}_{\mathbf{r}_\tau} \mathbf{k} \cdot \mathbf{p}_{\mathbf{r}_\tau} a^\dagger \mathbf{p}_{\mathbf{k},\mathbf{r}_\tau} a \mathbf{p}_{\mathbf{r}_\tau} \mathbf{k} - \mathbf{p}_{\mathbf{k},\mathbf{r}_\tau}, \quad \pi_{\mathbf{k}}^{0,0} = \pi_{\mathbf{k}}^{0,0} + \pi_{\mathbf{k}}^{0,0} = \frac{i\hbar}{\sqrt{A\mathbf{k} \cdot \mathbf{k}}} \sum \mathbf{p}_{\mathbf{r}_\tau} \mathbf{k} \cdot \mathbf{p}_{\mathbf{r}_\tau} a^\dagger \mathbf{p}_{\mathbf{k},\mathbf{r}_\tau} a \mathbf{p}_{\mathbf{r}_\tau} \mathbf{k} - \mathbf{p}_{\mathbf{k},\mathbf{r}_\tau}.
\]

Then, it turns out that the quantum mechanical operator \(\mathbf{v}(\mathbf{x})\) which satisfies the famous CR

\[
[\mathbf{v}^{0,0}(i)(\mathbf{x}), \rho^{0,0}(j)(\mathbf{x}')] = \frac{i\hbar}{m} \delta_\mathbf{x}(\mathbf{x} - \mathbf{x}') \rho^{0,0}(\mathbf{x})^{-1} (\text{rot} \mathbf{v}^{0,0}(\mathbf{x}))^{(k)} \quad (i, j, k) \text{ cyclic},
\]

corresponds to the fluid dynamical density operator and the velocity operator. Then, it turns out that the quantum mechanical operator \(\mathbf{v}(\mathbf{x})\), which satisfies the famous CR

\[
[\mathbf{v}^{0,0}(\mathbf{x}), \rho^{0,0}(\mathbf{x}')] = -\frac{i\hbar}{m} \nabla_\mathbf{x} \delta_\mathbf{x}(\mathbf{x} - \mathbf{x}'),
\]
3 \hspace{1cm} \nu_k^i \text{ and } \rho_k^i \text{-dependence of the Hamiltonian }

We derive here a collective Hamiltonian in terms of the \( \nu_k^{0,0} \) and \( \rho_k^{0,0} \). Following Sunakawa, we expand the kinetic operator \( T \) in a power series of the velocity operator \( \nu_k^{0,0} \) as follows:

\[
T = T_0(\rho) + \sum_{p \neq 0} T_1(\rho; p) \cdot \nu_p^{0,0} + \sum_{p \neq 0, q \neq 0} T_2(\rho; p, q) \nu_p^{0,0} \cdot \nu_q^{0,0} + \cdots, \quad T_2(\rho; p, q) = T_2(\rho; q, p),
\]

in which \( T_n(n \neq 0) \) are unknown expansion coefficients. In order to determine their explicit expressions, we take the commutators between \( T \) and \( \rho_k^{0,0} \) as follows:

\[
[T, \rho_k^{0,0}] = \hbar T_1(\rho; k) \cdot k + 2\hbar \sum_{p \neq 0} T_2(\rho; p, k) \nu_p^{0,0} \cdot k + \cdots,
\]

\[
[[T, \rho_k^{0,0}], \rho_k^{0,0}] = 2\hbar^2 T_2(\rho; k, k') k' \cdot k + \cdots.
\]

On the other hand, from (2.7) and (2.9), we can calculate the commutators between \( T \) and \( \rho_k^{0,0} \) by using the relations (2.4) and (2.7) as follows:

\[
[T, \rho_k^{0,0}] = \frac{\hbar}{m} k \cdot g_{0,-k} - \frac{\hbar}{m} \nu_k^{0,0} = \frac{\hbar}{m} \nu_k^{0,0} + \frac{\hbar}{m\sqrt{A}} \sum_{p \neq 0} \rho_{-k}^{0,0} p \cdot k \cdot \nu_p^{0,0},
\]

and using the relation (2.10) successively, we can easily obtain the following commutators:

\[
[[T, \rho_k^{0,0}], \rho_k^{0,0}] = -\frac{\hbar^2 k^2}{m} \delta_{k, -k}, \quad \frac{\hbar^2}{m\sqrt{A}} k \cdot k' \rho_{k+k'}^{0,0}, \quad k' \neq -k,
\]

\[
[[T, \rho_k^{0,0}], \rho_k^{0,0}], \rho_k^{0,0}] = 0, \quad \cdots.
\]

Comparing the above results with the commutators (3.2), we can determine the coefficients \( T_n(n \neq 0) \). Then we can express the kinetic part \( T \) in terms of the \( \rho_k^{0,0} \) and \( \nu_k^{0,0} \) as follows:

\[
T = T_0(\rho) + \frac{1}{2m} \sum_{k \neq 0} \nu_k^{0,0} \nu_{-k}^{0,0} + \frac{1}{2m\sqrt{A}} \sum_{p \neq 0} \rho_p^{0,0} p \cdot \nu_p^{0,0} \cdot \nu_q^{0,0}, \quad (\nu_0^{0,0} = 0).
\]

Up to the present stage, all the expressions have been derived without any approximation.

Our remaining task is to determine the term \( T_0(\rho) \) in (3.1) which depends only on \( \rho_k^{0,0} \). For this aim, we also expand it in a power series of the collective coordinates \( \rho_k^{0,0} \) in the form

\[
T_0(\rho) = C_0 + \sum_{p \neq 0} C_1(p) \rho_p^{0,0} + \sum_{p \neq 0, q \neq 0} C_2(p, q) \rho_p^{0,0} \rho_q^{0,0} + \cdots, \quad C_2(p, q) = C_2(q, p),
\]

and to get expansion coefficients, we take the commutators between \( T \) and \( \nu_k^{0,0} \) as follows:

\[
[v_i^{(i)}, T_0(\rho)] = \hbar k_i C_1(k) + 2\hbar k_i \sum_{p \neq 0} C_2(p; k) \rho_p^{0,0} + \cdots,
\]

\[
[v_i^{(j)}, T_0(\rho)] = 2\hbar^2 k_i j C_2(k; k') + 6\hbar^2 k_i k' \sum_{p \neq 0} C_3(p; k', k) \rho_p^{0,0} + \cdots.
\]

We here restrict our Hilbert space to subspace in which eigenvalue of the vortex operator satisfies \( \text{rot}\nu^{0,0}(\rho) > 0 \), i.e., \( [\nu_k^{0,0}(i), \nu_k^{0,0}(j)] = 0 \). From (2.9), we have a discrete integral equation

\[
[v_i^{(i)}, T_0(\rho)] = f_i(\rho; k) - \frac{1}{\sqrt{A}} \sum_{p \neq 0} k \cdot \rho_{-k}^{0,0} [v_i^{(i)}, T_0(\rho)].
\]

With the aid of (3.6) and using two CRs of (2.8), the inhomogeneous term \( f_i(\rho; k) \) becomes

\[
f_i(\rho; k) \equiv [g_k^{0,0}(i), T_0(\rho)] = \left[ g_k^{0,0}(i), T - \frac{1}{2m} \sum_{k \neq 0} g_k^{0,0} \cdot \nu_k^{0,0} - \frac{1}{2m\sqrt{A}} \sum_{p \neq 0} q \cdot \rho_p^{0,0} \cdot \nu_p^{0,0} \cdot \nu_q^{0,0} \right]
\]

\[
= [g_k^{0,0}(i), T] - \frac{\hbar}{m\sqrt{A}} \sum_{p \neq 0} q \cdot \rho_p^{0,0} \cdot \nu_p^{0,0} \cdot \nu_q^{0,0} - \frac{\hbar}{m\sqrt{A}} \sum_{q \neq 0} (g_k^{0,0} q - g_k^{0,0} q) \cdot \nu_q^{0,0} \cdot \nu_q^{0,0}
\]

\[
= [g_k^{0,0}(i), T] - \frac{\hbar}{m\sqrt{A}} \sum_{p \neq 0} q \cdot \rho_p^{0,0} \cdot \nu_p^{0,0} \cdot \nu_q^{0,0}, \quad (\text{due to change of the variable } p \text{ to } p-k).
\]
To obtain the explicit formula for \( f^{(i)}(\rho; k) \), first we calculate the CR between \( g^{0,0(i)}_{ \mathbf{k} } \) and \( T \) as

\[
[g^{0,0(i)}_{ \mathbf{k} }, T] = \frac{\hbar^3}{2m^2 A} \sum_{p_{\text{all},\tau}} p_i \left( p + \frac{k^2}{2} - \left( p - \frac{k^2}{2} \right) \right) a^\dagger_{p - k_{\tau}, \tau} a_{p + k_{\tau}, \tau} = \frac{\hbar^3}{m^2 A} \sum_{p_{\text{all},\tau}} p_i \langle \mathbf{p} \cdot \mathbf{k} \rangle a^\dagger_{p - k_{\tau}, \tau} a_{p + k_{\tau}, \tau}. \tag{3.11}
\]

From now on, we make an approximation for \( \rho^{0,0}_0 = \sqrt{A} \), \( v^{0,0}_k \) and \( \rho^{0,0}_k \) as

\[
v^{0,0}_k \approx \frac{\hbar k}{2} \sum_{\kappa, \tau} \left( \bar{\theta} a_{k, \kappa, \tau} - a^\dagger_{-k, \kappa, \tau} \right), \quad \rho^{0,0}_k \approx \sum_{\kappa, \tau} \left( \bar{\theta} a_{k, \kappa, \tau} + a^\dagger_{-k, \kappa, \tau} \right). \tag{3.12}
\]

The operators \( a_{0, \tau} \) and \( a^\dagger_{0, \tau} \) are regarded as c-numbers but with the Grassmann variables which play crucial roles to compute the inhomogeneous term \( f^{(i)}(\rho; k) \) \([3, 9]\), as shown in 1. The explicit forms of the operators are simply given as

\[
a_{0, \tau} \equiv \sqrt{A} \theta, \quad a^\dagger_{0, \tau} \equiv \sqrt{A} \bar{\theta}, \tag{3.13}
\]

where the \( \theta \) and \( \bar{\theta} \) are the Grassmann variables and anti-commute with \( a_{k, \tau} \) and \( a^\dagger_{k, \tau} \) \([11, 12, 13]\).

Then the second term in the last line of \((3.10)\) is approximately computed as

\[
-\frac{\hbar^3}{m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot q \sum_{a_{\text{all}}} \rho^{0,0}_0 p + q - k \cdot (\kappa \cdot v^{0,0}_p) v_q^{0,0(i)}
\]

\[
= -\frac{\hbar^3}{m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{2m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{4m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{4m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{4m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

\[
= -\frac{\hbar^3}{4m^2 A} \sum_{p_{\text{all},\tau'}} p \cdot \left( \frac{\kappa^2}{2} - \frac{k^2}{2} \right) \left( p_i - \frac{k_i}{2} \right) a^\dagger_{p - k_{\tau'}, \tau'} a_{p + k_{\tau'}, \tau'}
\]

The last term of which is obtained by extracting the terms with \( p = \frac{k}{2} \) or \( p = -\frac{k}{2} \) in the last term in the second line from the bottom and it evidently vanishes.

Substituting the resultant formula of \((3.11)\), \( [g^{0,0(i)}_{ \mathbf{k} }, T] = \frac{\hbar^3 k^2 \kappa^2}{4m^2} \rho^{0,0}_k \), and the calculated result \((3.14)\) into \( f^{(i)}(\rho; k) \), i.e., \((3.10)\), we get an approximate formula for the \( f^{(i)}(\rho; k) \) up to the order of \( \frac{1}{\sqrt{A}} \) in the following form:

\[
f^{(i)}(\rho; k) = \frac{\hbar^3 k^2}{4m} \rho^{0,0}_k - \frac{\hbar^3}{4m^2 A} \sum_{p_{\text{all}}} \rho^{0,0}_k \left( \mathbf{p} \cdot \mathbf{k} \right) (k_i - p_i) \rho^{0,0}_k \rho^{0,0}_p \cdot \mathbf{p} \cdot \mathbf{k}. \tag{3.15}
\]

Further substituting \((3.15)\) into \((3.9)\) and bringing the next leading term, we can rewrite the R.H.S. of the discrete integral equation \((3.9)\) as

\[
[v^{0,0(i)}_{ \mathbf{k} }, T_0(\rho)] = \frac{\hbar^3 k^2}{4m} \rho^{0,0}_k - \frac{\hbar^3}{4m^2 A} \sum_{p_{\text{all}}} \rho^{0,0}_k \left( \mathbf{p} \cdot \mathbf{k} \right) (k_i - p_i) \rho^{0,0}_k \rho^{0,0}_p \cdot \mathbf{p} \cdot \mathbf{k}
\]

\[
\approx \frac{\hbar^3 k^2}{4m} \rho^{0,0}_k - \frac{\hbar^3 k^2}{8m^2 A} \sum_{p_{\text{all}}} \rho^{0,0}_k \left( \mathbf{p} + \mathbf{k} \right) \rho^{0,0}_k \rho^{0,0}_p \cdot \mathbf{p} \cdot \mathbf{k}, \quad \text{(under the assumption of } p_i = \frac{k_i}{2}). \tag{3.16}
\]
From (3.16) and the CRs (2.6) and (2.10), we get the following commutation relations:

\[
\begin{align*}
[v_{k'}^0, [v_{k'}^0, T_0(\rho)]] & = -\frac{\hbar^4 k_k^2}{4m} \delta_{k', -k} - \frac{\hbar^4 k_k^2}{4m} \frac{(k^2 + k' + k'^2) \rho_{-k, -k'}}{4m\sqrt{A}} \\
[v_{k''}^0, [v_{k''}^0, [v_{k''}^0, T_0(\rho)]]] & = -\frac{\hbar^5 k_k^2}{4m\sqrt{A}} \frac{(k^2 + k' + k'^2)}{4m\sqrt{A}},
\end{align*}
\]

(3.17)

By a procedure similar to the previous one, we can determine the coefficients \(C_n(\tau \neq 0)\) in (3.7) and then get an approximate form of \(T_0(\rho)\) in terms of variables \(\rho_{k, k}^{0,0}\) in the following form:

\[
T_0(\rho) = C_0 + \frac{\hbar^2}{8m} \sum_{k \neq 0} k^2 \rho_{k, k}^{0,0} - \frac{\hbar^2}{24mn\sqrt{A}} \sum_{p \neq 0, q \neq \pm p, q \neq 0} (p^2 + p \cdot q + q^2) \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q}^{0,0} + O \left(\frac{1}{A}\right).
\]

(3.18)

With the aid of the underlying identities

\[
\begin{align*}
\sum_{p \neq 0, q \neq 0, p+q \neq 0} (p^2 + p \cdot q + q^2) \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q}^{0,0} & = \sum_{p \neq 0, q \neq 0, p+q \neq 0} (p+q)^2 \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q}^{0,0}, \\
\sum_{p \neq 0, q \neq 0, p+q \neq 0} (p^2 + p \cdot q + q^2) \rho_{-p, -q}^{0,0} & = -2 \sum_{p \neq 0, q \neq 0, p+q \neq 0} p \cdot q \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q}^{0,0},
\end{align*}
\]

(3.19)

the lowest kinetic term of \(T_0(\rho)\) (3.18) is rewritten as

\[
T_0(\rho) = C_0 + \frac{\hbar^2}{8m} \sum_{k \neq 0} k^2 \rho_{k, k}^{0,0} + \frac{\hbar^2}{8m\sqrt{A}} \sum_{p \neq 0, q \neq 0} p \cdot q \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q}.
\]

(3.20)

From now on, we calculate the constant term \(C_0\), the first term in the R.H.S. of (3.18). Substituting (3.18) into (3.6), the constant term \(C_0\) is computed up to the order of \(\frac{1}{A}\):

\[
C_0 = T - \frac{\hbar^2}{8m} \sum_{k \neq 0} k^2 \rho_{k, k}^{0,0} - \frac{1}{2m} \sum_{k \neq 0} k^2 \rho_{k, k}^{0,0} v_{k, k}^{0,0} + \frac{1}{2m\sqrt{A}} \sum_{p+q \neq 0} \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} v_{p, q}^{0,0} v_{q, p}^{0,0} + O \left(\frac{1}{A}\right)
\]

(3.21)

Using \(\rho_{k, k}^{0,0} \approx \sum_{\tau} (\bar{a}_{-k, \tau} + a_{k, \tau}^+)\) and \(v_{k, k}^{0,0} \approx \frac{\hbar k}{2} \sum_{\tau} (\bar{a}_{k, \tau} - a_{k, \tau}^+)\), we can calculate the third term in (3.21) very similarly to the calculation of (3.11) and then reach to a result such as

\[
-\frac{1}{2m} \sum_{k} k^2 \rho_{k, k}^{0,0} v_{k, k}^{0,0} = \frac{\hbar^2}{8m} \sum_{k} k^2 \sum_{\tau} \left(\bar{a}_{k, \tau} - a_{k, \tau}^+ \right)^2 \sum_{\tau'} \left(\bar{a}_{k, \tau'} - a_{k, \tau'}^+ \right)^2.
\]

(3.22)

As for the forth and last terms in (3.21), due to the relations \(\theta \theta = 0\) and \(\bar{\theta} \bar{\theta} = 0\), we simply have

\[
\sum_{p+q \neq 0} \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} v_{p, q}^{0,0} v_{q, p}^{0,0} \approx \frac{\hbar^2}{8m\sqrt{A}} \sum_{p \neq 0, q \neq 0, p+q \neq 0} p \cdot q \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q}.
\]

(3.23)

\[
\sum_{p \neq 0, q \neq 0, p+q \neq 0} p \cdot q \rho_{p, p}^{0,0} \rho_{q, q}^{0,0} \rho_{-p, -q} = 0.
\]

(3.24)

Substituting these into (3.21), we have

\[
C_0 \approx (1 - \theta \bar{\theta}) T + \theta \bar{\theta} \frac{\hbar^2}{2m} \sum_{k} k^2 - \theta \bar{\theta} \sum_{k, \tau \neq \tau'} \frac{\hbar^2 k^2}{2m} a_{k, \tau}^+ a_{k, \tau'} \frac{\hbar^2 k^2}{2m} k_{\tau, \tau'} a_{k, \tau} \approx \frac{\hbar^2}{2m} \sum_{k} k^2.
\]
Here we have used the relation $\tilde{\theta} = 1$ and neglected the term $\sum_{k, \tau \neq \tau'} \frac{\hbar^2 k^2}{2m} a_{k, \tau} \sigma_{k, \tau}'$ which does not exist in the case of the isospin-less Fermion system. The result (3.24) is not identical with the Sunakawa’s result [3] for a Bose system. Then we get a result which is considered as a natural consequence for a Fermi system. It is surprising to see that the $C_0$ (3.24) coincides with the constant term in the resultant ground state energy given by the Tomonaga’s method [1]. Using (3.6), (3.18) and (3.24) and separating the term $C_0 \approx \sum \frac{\hbar^2 k^2}{2m}$ into two parts $-\sum \frac{\hbar^2 k^2}{4m}$ and $\frac{3}{2} \sum \frac{\hbar^2 k^2}{2m}$, we reach final goal of expressing the Hamiltonian $H$ (2.3) in terms of $\rho_{k}^{0}$ and $\nu_{k}^{0}$ as follows:

$$
H = -\frac{A(A+2)}{8\Omega} \nu(0) - \frac{A}{4\Omega} \sum_{k \neq 0} \nu(k) - \sum_{k} \frac{\hbar^2 k^2}{4m} + \frac{3}{2} \sum_{k} \frac{\hbar^2 k^2}{2m} + \sum_{k \neq 0} \left\{ \frac{1}{2m} \nu_{k}^{0} \nu_{k}^{0} \left( \frac{\hbar^2 k^2}{8m} - \frac{A}{8\Omega} \nu(k) \right) \rho_{k}^{0} \rho_{-k}^{0} \right\} - \frac{1}{2m\nu A} \sum_{p \neq 0, q \neq 0, \rho \neq 0, \nu} q_{p}^{0} q_{q}^{0} p_{p}^{0} p_{q}^{0} q_{q}^{0} p_{q}^{0} + \frac{\hbar^2}{8\Omega} \sum_{k \neq 0, p} q_{q}^{0} p_{q}^{0} p_{q}^{0} p_{q}^{0} q_{q}^{0} - \rho_{p}^{0} \rho_{-p}^{0} q_{q}^{0} - \rho_{p}^{0} \rho_{-p}^{0} q_{q}^{0} - \rho_{p}^{0} \rho_{-p}^{0} q_{q}^{0} - \rho_{p}^{0} \rho_{-p}^{0} q_{q}^{0} - \rho_{p}^{0} \rho_{-p}^{0} q_{q}^{0} - \rho_{p}^{0} \rho_{-p}^{0} q_{q}^{0},
$$

in which the standard expression for the interaction $V$ in isospin $T = 0$ nuclei is given by (2.9) and (2.10) in I. In the case of 3D nuclei, there appear volume $\Omega$ instead of length $L$. Then we have the terms such as $-\frac{A}{4\Omega}$ and $-\frac{A}{8\Omega}$. The expression (3.25) is just the Sunakawa’s form up to the order of $\frac{1}{A}$ [3], except the last term $\frac{3}{2} \sum \frac{\hbar^2 k^2}{2m}$ in the first line of the R.H.S. of (3.23). The second term $-\frac{A}{4\Omega} \nu(k)$ also in the first line is separated into two parts $\frac{A}{8\Omega} \nu(k)$ and $-\frac{3A}{8\Omega} \nu(k)$. These differences arise due to the fact that we deal with a isospin $T = 0$ Fermi system but not a Bose system. At the present moment, we discard the underlined term. In (3.25), the sum of some terms below is considered as the lowest order Hamiltonian $H_0$,

$$
H_0 = -\frac{A(A+2)}{8\Omega} \nu(0) + \sum_{k \neq 0} \left\{ -\frac{\hbar^2 k^2}{4m} + \frac{A}{8\Omega} \nu(k) + \frac{1}{2m} \nu_{k}^{0} \nu_{-k}^{0} + \frac{\hbar^2 k^2}{8m} - \frac{A}{8\Omega} \nu(k) \right\} \rho_{k}^{0} \rho_{-k}^{0}.
$$

Now, let us introduce the Boson annihilation and creation operators defined as

$$
\alpha_{k} \equiv \sqrt{\frac{m E_{k}}{2\hbar^2 k^2 + \rho_{k}^{0} \rho_{-k}^{0}}} \nu_{k}^{0}, \quad \alpha_{k}^{\dagger} \equiv \sqrt{\frac{m E_{k}}{2\hbar^2 k^2 + \rho_{k}^{0} \rho_{-k}^{0}}} \nu_{-k}^{0}, \quad (k \neq 0).
$$

Using (3.27) and (3.12), the collective variables $\rho_{k}^{0}$ and $\nu_{k}^{0}$ are expressed as

$$
\rho_{k}^{0} = \sqrt{\frac{\hbar^2 k^2}{2m E_{k}}} \frac{1}{2} \left( \alpha_{-k} + \alpha_{k}^{\dagger} \right) = \sum_{\tau} \bar{\theta}_{a_{k, \tau}} + \sum_{\tau} a_{k, \tau}^{\dagger} \theta, \quad (k \neq 0), \quad \nu_{k}^{0} = -i \sqrt{\frac{2m E_{k}}{\hbar^2 k^2}} \frac{1}{2} \left( \alpha_{-k} - \alpha_{k}^{\dagger} \right) = \frac{\hbar k}{2} \left( \sum_{\tau} \bar{\theta}_{a_{k, \tau}} - \sum_{\tau} a_{k, \tau}^{\dagger} \theta \right), \quad (k \neq 0),
$$

and substituting which into (3.26), the lowest order Hamiltonian $H_0$ is diagonalized as

$$
H_0 = E_0^{G} + \sum_{k \neq 0} E_{k} \alpha_{k}^{\dagger} \alpha_{k}, \quad E_0^{G} \equiv -\frac{A(A+2)}{8\Omega} \nu(0) - \frac{1}{2} \sum_{k \neq 0} \left( E_{k} - \frac{\hbar^2 k^2}{2m} - \frac{A}{2\Omega} \nu(k) \right),
$$

$$
E_{k} \equiv \sqrt{\left( \varepsilon_{k} \right)^2 - \frac{\hbar^2 k^2}{m} \frac{A}{4\Omega} \nu(k)}, \quad \varepsilon_{k} \equiv \frac{\hbar^2 k^2}{2m}, \quad (E_{k}: \text{Quasi-particle energy}),
$$

where we have used the commutation relation $[\nu_{k}^{0}, \nu_{k}^{0}^{\dagger}] = \hbar k$ given by (2.10). In this sense, the zero point energy of the collective mode is included in the above diagonalization. Since $E_{k}$ is approximated as $\frac{\hbar^2 k^2}{2m} - \frac{A}{4\Omega} \nu(k)$, the term $\frac{1}{2} \left( E_{k} - \frac{\hbar^2 k^2}{2m} - \frac{A}{2\Omega} \nu(k) \right)$ becomes $-\frac{3A}{8\Omega} \nu(k)$. 

6
This is why we must separate $-\frac{A}{\hbar^2}\nu(k)$ into $\frac{A}{\hbar^2}\nu(k)$ and $-\frac{3A}{\hbar^2}\nu(k)$ but not arbitrary. These facts lead us to $H_0$ (3.26). The quantity $E_0^G$ in (3.29) corresponds to the ground state energy. Thus we have a Bogoliubov transformation for Boson-like operators $\sum_{\tau_z} \theta a_{k,\tau_z}$ and $\sum_{\tau_z} a^\dagger_{-k,\tau_z} \theta$ as

$$\alpha_k = \frac{(E_k + \varepsilon_k) \sum_{\tau_z} \theta a_{k,\tau_z} + (E_k - \varepsilon_k) \sum_{\tau_z} a^\dagger_{-k,\tau_z} \theta}{2\sqrt{\varepsilon_k E_k}}$$

$$\alpha^\dagger_{-k} = \frac{(E_k - \varepsilon_k) \sum_{\tau_z} \theta a_{k,\tau_z} + (E_k + \varepsilon_k) \sum_{\tau_z} a^\dagger_{-k,\tau_z} \theta}{2\sqrt{\varepsilon_k E_k}}, \tag{3.30}$$

which is the same as the famous Bogoliubov transformation for the usual Bosons [6]. The diagonalization (3.29) has been given similarly to the usual Bose system by Sunakawa [9].
4 Discussions and further outlook

In the preceding sections, we have proposed a velocity operator approach to a 3D NP system. After introducing collective variables, the velocity operator approach to the 3D NP system could be provided. Particularly, an interesting problem of describing excitations occurring in nuclei, isospin $T = 0$ surface vibrations, may be possible to treat as an elementary exercise. For this problem, for example, see textbooks [13,15]. By applying the velocity operator approach to such a problem, an excellent description of the excitations in isospin $T = 0$ nuclei will be expected to reproduce various correct behaviors including excited energies. Because the present theory is constructed to take into account important many-body correlations, which have not been investigated sufficiently for a long time in the historical ways for such a problem. In this context, it is said that a new field of exploration of excitations in a 3D Fermi system may open with aid of the velocity operator approach whose new achievement may be appeared elsewhere. By the way, connection of the present theory with the fluid dynamics was been mentioned briefly by Sunakawa. He transformed the quantum-fluid Hamiltonian (3.25) to the one in the configuration space and obtained the classical-fluid Hamiltonian for the case of the irrotational flow [16,9].

There also exist isospin $T=1$ excitations in nuclei. As stressed in I, the structures of the commutators among $\rho^{0,0}_k, \rho^{1,0}_k, g^{0,0}_k = ik\pi^{0,0}_k$ and $g^{1,0}_k = ik\pi^{1,0}_k$ have the twisted property in the isospin space $(T, T_z)$. Due to this fact, commutators $[g^{0,0}_k, \rho^{1,0}_k]$ and $[g^{1,0}_k, g^{1,0}_k]$ are not closed. The velocity operator $v^{1,0}_k$ defined in the same way as (2.9) and density operator $\rho^{1,0}_k$ do not satisfy an important commutator $[v^{1,0}_k, \rho^{1,0}_k] = \hbar \delta_{kk'}$. Therefore, the $\rho^{1,0}_k$ and $v^{1,0}_k$ are not a suitable pair of collective operators for our object. Then it turns out that the isovector $T=1$ surface vibrations [14,15] can’t be treated in the present approach.

As described in Section 3, hitherto, we have restricted Hilbert space to subspace in which the vortex operator satisfies $\text{rot}v^{0,0}(x)| > 0$, i.e., $\{v^{0,0}_k, v^{0,0}_k, v^{0,0}_k, v^{0,0}_k'\} = 0$. While, in the classical fluid dynamics, the velocity field $v(x)$ is given as $v(x) = -\nabla \phi(x) - \lambda(x) \nabla \psi(x)$, where $\phi(x)$ is the velocity potential and $\lambda(x)$ and $\psi(x)$ are Clebsch parameters [17]. This fact was already been pointed out by Sunakawa [3]. As was suggested long time ago by Marumori et al. [18] and Watanabe [19], the internal rotational motion, i.e., the vortex motion, however, may exist also in nuclei. So, we have something worthwhile in taking the vortex motion into consideration. In the very near future, we will attempt at a description of rotational velocity field of a fluid in nuclei through a Clebsch transformation. Contrary to the above ways to the vortex motion in nuclei, we should notice the paper in which Holtzwarth and Schütte have attempted at a derivation of fluid-dynamical equations of motion which allow for velocity fields with vorticity, starting from a time-dependent variational principle for a many-fermion system. They have derived an interesting relation between the vorticity and the two-body correlations [20]. Standing on the above Clebsch viewpoint and Ziman’s [21], going from classical fluid dynamics to quantum fluid dynamics, we will derive a vortex Hamiltonian of the fluid in terms of roton operators. The quantum fluid-dynamical approach may be applied to a realistic nuclei. Such an application to nuclei will provide an excellent description of another kind of elementary energy excitation, so-called the ”vortex excitation” occurred in nuclei because the quantum fluid-dynamical manner may approach various features of many-body effects, which have been discarded in the traditional treatments of the problem of rotational collective motion. This work will be presented elsewhere in a forthcoming paper.
Acknowledgements

One of the authors (S.N.) would like to express his sincere thanks to Professor Constança Providência for kind and warm hospitality extended to him at the Centro de Física, Universidade de Coimbra. This work was supported by FCT (Portugal) under the project CERN/FP/83505/2008.
References

[1] S. Tomonaga, *Prog. Theor. Phys.* 5, 544 (1950).

[2] V.J. Emery, Theory of the One-Dimensional Electron Gas in *Highly Conducting One-Dimensional Solids*, Editors, J.T. Devreese, R.P. Evrard and V.E. van Dore, Physics of Solids and Liquids, Springer US, 1979, pp 247-303.

[3] S. Sunakawa, Y. Yoko-o and H. Nakatani, *Prog. Theor. Phys.* 27, 589 (1962).

[4] S. Nishiyama and J. da Providência, *Int. J. Mod. Phys.* E24, 1550045 (2015).

[5] S. Nishiyama and J. da Providência, *Nucl. Phys.* A923, 51 (2014).

[6] N.N. Bogoliubov, *J. Phys.* 11, 23 (1947).

[7] H.J. Lipkin, *Lie Groups for Pedestrians* (North-Holland Publ. Co., Amsterdam, 1965).

[8] D.J. Rowe and J.L. Wood, *Fundamentals of Nuclear Models, Foundational Models*, World Scientific Publishing Co. Pte. Ltd. 2010.

[9] S. Sunakawa, Y. Yoko-o and H. Nakatani, *Prog. Theor. Phys.* 27, 600 (1962).

[10] L. Landau, *J. Phys.* 5, 71 (1941).

[11] F.A. Berezin, *The Method of Second Quantization*, Academic Press, New York and London, 1966.

[12] R. Casalbuoni, *Nuovo Cimento* 33A, 115 (1976).

[13] R. Casalbuoni, *Nuovo Cimento* 33A, 389 (1976).

[14] P. Ring and P. Schuck, *The nuclear many body problem*, Texts and monographs in Physics (Springer-Verlag, Berlin, Heidelberg and New York 1980).

[15] J.M. Eisenberg and W. Greiner, *Nuclear Models*, North-Holland Physics Publishing, Elsevier Science Publisher Company, Inc. 1987.

[16] S. Sunakawa, Y. Yoko-o and H. Nakatani, *Prog. Theor. Phys.* 28, 127 (1962).

[17] A. Clebsch, *J. Reine Angew. Math.* 54, 293 (1857); 56, 1 (1859).

[18] S. Nagata, R. Tamagaki, S. Amai and T. Marumori, *Prog. Theor. Phys.* 19, 495 (1955).

[19] Y. Watanabe, *Prog. Theor. Phys.* 16, 1 (1956).

[20] G. Holtzwarth and D. Schütte, *Phys. Lett.* B73, 255 (1978).

[21] J.M. Ziman, *Proc. Roy. Soc.* A219, 257 (1953).