A possible quantum fluid-dynamical approach
to vortex motion in nuclei

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Dedicated to the Memory of Toshio Marumori

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Abstract

The essential point of Bohr-Mottelson theory is to assume a irrotational flow. As
was already suggested by Marumori and Watanabe, the internal rotational motion,
i.e., the vortex motion, however, may exist also in nuclei. So, we must take the vor-
tex motion into consideration. In classical fluid dynamics, there are various ways to
treat the internal rotational velocity. The Clebsch representation,
\[ \mathbf{v}(\mathbf{x}) = -\nabla \phi(\mathbf{x}) + \lambda(\mathbf{x}) \nabla \psi(\mathbf{x})(\phi; \text{velocity potential, } \lambda \text{ and } \psi : \text{Clebsch parameters}) \]
is very powerful and allows for the derivation of the equations of fluid motion from a Lagransian. Making
the best use of this advantage, Kronig-Thellung, Ziman and Ito obtained a Hamilto-
nian including the internal rotational motion, the vortex motion, through the term
\[ \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}). \]
Going to quantum fluid dynamics, Ziman and Thellung finally derived
the roton spectrum of liquid Helium II postulated by Landau. Is it possible to follow
a similar procedure in the description of the collective vortex motion in nuclei? The
description of such a collective motion has not been considered in the context of the
Bohr-Mottelson model (BMM) for a long time. In this paper, we will investigate the
possibility of describing the vortex motion in nuclei on the basis of the theories of
Ziman and Ito together with Marumori’s work.

Keywords: Collective motion in nuclei; velocity operator; vortex motion

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1 Introduction

An exact treatment of collective variables in nuclei has been attempted [1, 2, 3]. In [1] and [2], Marumori first gave a foundation of the unified model of collective motion and the independent particle motion in nuclei. Applying Tomonaga’s basic idea in his collective motion theory [4] to nuclei, with the aid of Sunakawa’s integral equation method [5], one of the present authors (S.N.) developed the description of nuclear surface oscillations [6] and of two-dimensional nuclei [7]. These descriptions are considered to provide a possible microscopic foundation of nuclear collective motion derived from the Bohr-Mottelson model (BMM) [8, 9, 10]. In terms of collective variables, these descriptions were formulated within the first quantized language, in contrast with the second quantized approach in Sunakawa’s method. Extending Tomonaga’s idea, Miyazima-Tamura [11, 12] proposed a collective description of nuclear surface oscillation. An alternative attempt was proposed from a different viewpoint, the canonical transformation theory [13]. To approach elementary excitations in a one-dimensional Fermi system, Tomonaga brought a revolutionary idea to the collective motion theory [4]. The Sunakawa’s method, which is applicable also to a Fermi system, may work well for such a problem. It has been considered in the exact canonical momenta approach to a neutron-proton system [14].

According to Bohr-Mottelson [15], a nucleus is considered to be a portion of nuclear matter resembling an incompressible classical fluid but having a sharp surface. Owing to the assumption of a small surface deformation, collective coordinates of the surface oscillations are described as \( R(\theta, \varphi) = R_0 (1 + \sum \alpha_{\lambda \mu} \gamma_{\lambda \mu}(\theta, \varphi)) \) (\( R_0 \): Nuclear equilibrium radius). Expanding the collective coordinates around equilibrium, \( \alpha_{\lambda \mu} = 0 \), the surface Hamiltonian is given as
\[
H = \sum_{\lambda \mu} \left\{ \frac{1}{2} B_\lambda |\alpha_{\lambda \mu}|^2 + \frac{1}{2} C_\lambda |\alpha_{\lambda \mu}|^2 \right\} (\omega_\lambda = \sqrt{\frac{C_\lambda}{B_\lambda}}) \]
which is a set of harmonic oscillators with frequencies \( \omega_\lambda \). The parameters \( B_\lambda \) and \( C_\lambda \) represent the mass parameter associated with the collective flow and the nuclear deformability, respectively. The essential point of their theory is to assume an irrotational flow. Let us introduce a velocity potential \( \phi(x, t) \). The velocity field \( v(x, t) = -\nabla \phi(x, t) \) plays a central role over the whole theory. The BMM, however, gives values of the moment of inertia smaller than the empirical values [16]. To resolve this, Marumori and Watanabe suggested to take into account not only the surface rotation caused by the irrotational flow but also another kind of rotation due to the individual particle motion [13, 3]. They called the latter rotation the internal rotational motion, i.e., a vortex motion. Such a motion may be expected to occur also in nuclei. So, it is necessary to take the vortex motion into consideration. In classical fluid dynamics, there are various ways to treat the velocity field \( v(x, t) \) in connection with the internal rotational motion. Especially, the Clebsch representation [17], \( v(x, t) = -\nabla \phi(x, t) + \lambda(x, t) \nabla \psi(x, t) (\lambda, \psi: \text{Clebsch parameters}) \) is a useful tool and allows for the derivation of the equations of fluid motion from a fluid Lagransian. Making maximum use this advantage, Kronig-Thellung [18], Ziman [19] and Ito [20] obtained a Hamiltonian including the internal rotational motion, the vortex motion, via \( \lambda(x, t) \nabla \psi(x, t) \) which introduces the vorticity, \( \text{rot} v(x, t) = \nabla \lambda(x, t) \times \nabla \psi(x, t) \). The moment of inertia may increase by a cooperation between rotational and irrotational motions. By quantization of fluid dynamics, Ziman and Thellung [21] finally derived the roton spectrum of liquid Helium II postulated by Landau [22]. Is it possible to apply this procedure to a description of collective vortex motion in nuclei? The description of such a collective motion has not been considered in the context of the BMM for a long time. In this
paper, following Ziman, Ito and Marumori, we will investigate the possibility of describing the vortex motion in nuclei.

In Sec. 2 we give a brief recapitulation of classical fluid dynamics in terms of Clebsch variables. In Sec. 3 we show that a unitary transformation of an $A$-particle Hamiltonian leads to particle and collective Hamiltonians. The collective Hamiltonian consists of **irrotational** and **internal rotational** motions and their interactions. Section 4 is devoted to Ziman transformation and to derive the roton Hamiltonian. Section 5 is devoted to determination of Clebsch parameters through a one-form gauge potential. Finally in Sec. 6 some discussions and further outlook are given.
2 Recapitulation of classical fluid dynamics in terms of Clebsch representation

For a velocity field \( \mathbf{v}(\mathbf{x}, t) \) with a vortex component, the Clebsch term \( \lambda(\mathbf{x}, t) \nabla \psi(\mathbf{x}, t) \) is useful to derive the equation of fluid motion from a Hamilton formalism [23, 24, 17]. Such a representation is very effective for passing from classical fluid dynamics to quantum fluid dynamics. The Clebsch representation is given by

\[
\mathbf{v}(\mathbf{x}, t) = - \nabla \phi(\mathbf{x}, t) + \lambda(\mathbf{x}, t) \nabla \psi(\mathbf{x}, t). \tag{2.1}
\]

Then the vorticity \( \mathbf{w}(\mathbf{x}, t) \) becomes

\[
\mathbf{w}(\mathbf{x}, t) = \text{rot} \mathbf{v}(\mathbf{x}, t) = \nabla \lambda(\mathbf{x}, t) \times \nabla \psi(\mathbf{x}, t), \tag{2.2}
\]

which generally does not vanish. For simplicity, we denote \( \mathbf{v}(\mathbf{x}, t) \) etc. simply as \( \mathbf{v} \) etc. In the Clebsch transformation (2.1), it is possible to choose \( \lambda \) and \( \psi \) so that the surfaces \( \lambda = \text{const.} \) and \( \psi = \text{const.} \) [23] move with the fluid, i.e.,

\[
\frac{D \lambda}{Dt} = \dot{\lambda} + \mathbf{v} \cdot \nabla \lambda = 0, \quad \frac{D \psi}{Dt} = \dot{\psi} + \mathbf{v} \cdot \nabla \psi = 0, \tag{2.3}
\]

where \( \frac{D}{Dt} \) is the substantial derivative [25, 24]. Let us start from the Euler equation of fluid dynamics,

\[
\int \rho \frac{D \mathbf{v}}{Dt} d\tau = \int (\rho \mathbf{F}(\mathbf{x}) - \nabla \mathbf{p}) d\tau \rightarrow \frac{D \mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho} \nabla \mathbf{p}, \quad \left( \mathbf{F} = - \nabla \mathbf{U} \right. \text{ and } \nabla \mathbf{p} = - \nabla \mathbf{p} \left. \right). \tag{2.4}
\]

where \( \rho \) and \( p \) are the density and the pressure of the fluid and \( \mathbf{F} \) is the external force. The Lagrange differentiation (substantial derivative) for the velocity, \( \frac{D \mathbf{v}}{Dt} \), is computed as

\[
\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{2} \nabla \mathbf{v}^2 + \dot{\mathbf{v}} - \mathbf{v} \times \text{rot} \mathbf{v} = - \nabla \left( \mathbf{P} + \mathbf{U} \right), \quad \left( \mathbf{P} = \int_{\rho_0}^{\rho} \frac{1}{\rho} \, dp \right), \tag{2.5}
\]

Assuming \( \mathbf{F} = 0 \), with the aid of the vorticity relation (2.2) and the the substantial derivatives for \( \lambda \) and \( \psi \), (2.3), the sum of the second and third terms in the L.H.S. (2.5) is calculated as

\[
\dot{\mathbf{v}} - \mathbf{v} \times \text{rot} \mathbf{v} = - \nabla \dot{\phi} + \lambda \nabla \dot{\psi} + \lambda \nabla \psi - \mathbf{v} \times (\nabla \lambda \times \nabla \psi) = - \nabla \dot{\phi} + \left( \dot{\lambda} + \mathbf{v} \cdot \nabla \lambda \right) \nabla \lambda + \nabla \left( \lambda \dot{\psi} \right) \tag{2.6}
\]

from which and (2.4), if \( \mathbf{U} = 0 \), we have \( \dot{\phi} - \lambda \dot{\psi} = \frac{1}{2} \nabla \mathbf{v}^2 + \mathbf{P} \). Let us define a Lagrangian density

\[
\mathcal{L} = \rho \left( \dot{\phi} - \lambda \dot{\psi} - \frac{1}{2} \nabla \mathbf{v}^2 \right) - E_{\text{pot}}(\rho), \quad E_{\text{pot}}(\rho) \equiv \rho \int_{\rho_0}^{\rho} \frac{p - p_0}{\rho^2} \, dp. \tag{2.7}
\]

The conjugate momentum to \( \phi \) and \( \psi \) are expressed as \( \pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \rho \) and \( \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = - \rho \lambda \). Then the Hamiltonian density is given as

\[
\mathcal{H} = \pi_\phi \dot{\phi} + \pi_\psi \dot{\psi} - \mathcal{L} = \frac{1}{2} \rho \left( - \nabla \phi - \frac{\pi_\psi}{\rho} \nabla \psi \right)^2 - E_{\text{pot}}(\rho). \tag{2.8}
\]

Thus we have the canonical equations of motion \( \dot{\rho} = - \frac{\delta \mathcal{H}}{\delta \rho}, \quad \dot{\phi} = \frac{\delta \mathcal{H}}{\delta \pi_\phi}, \quad \dot{\psi} = \frac{\delta \mathcal{H}}{\delta \pi_\psi}, \) and \( \dot{\pi}_\phi = - \frac{\delta \mathcal{H}}{\delta \phi} \).
From the first equation we can derive the continuity equation of the fluid as follows:

$$\dot{\rho} = \sum_k \frac{\partial}{\partial x_k} \left( \frac{\partial H}{\partial \phi_k} \right) = \sum_k \frac{\partial}{\partial x_k} (-\rho v_k) = -\text{div}(\rho \mathbf{v}), \quad (x_k = x, y, z, \quad \partial_k \phi = \frac{\partial \phi}{\partial x_k}, \quad k = 1, 2, 3). \quad (2.9)$$

The second, third and forth lead to Eq. (2.3). Further the second also gives $\dot{\phi} - \lambda \dot{\psi} = \frac{1}{2} \mathbf{v}^2 + P$. In the following Sections, the classical scalar fields $\phi(x, t)$, $\lambda(x, t)$ and $\psi(x, t)$ are treated as the corresponding quantal field operators of the quantized fluid and then the vector field $\mathbf{v}(x, t)$ becomes the quantal velocity operator.
3 Unitary transformation of an $A$-particle Hamiltonian

In an $A$-particle system with variables $(x_n, p_n)$, the Hamiltonian is given by

$$H = \frac{1}{2} \sum_{n=1}^{A} \frac{p_n^2}{2m} + V(x_1, x_2, \cdots, x_A), \quad (V: \text{Interaction potential}).$$

(3.1)

For convenience, from now on we omit the time argument. So, here, we write $\rho$ etc. as $\rho(x)$ etc. We introduce a unitary operator (UOp) defined as the exponential of a symmetrized form of operators,

$$U = \exp \left[ \frac{i}{\hbar} \int \left( (\rho(x) \cdot \phi(x) + \text{sym.)} - (\rho(x)\lambda(x) \cdot \psi(x) + \text{sym.}) \right) dx \right], \quad \rho(x) = n \sum_{n=1}^{A} \delta(x-x_n),$$

(3.2)

which is an extension of UOp in Ref. [2] to the UOp with the Clebsch variables. By the symbol sym. we mean that a term of the form $Y \cdot X$ is to be added to the operator product $X \cdot Y$. The canonically conjugate variables $\rho(x)$ and $\phi(x)$ are the density and the velocity potentials of the quantized fluid which obey the canonical equations of motion of the quantized fluid and satisfy the commutation relation $[\rho(x), \phi(x')] = i\hbar \delta(x-x')$. The operators $\rho(x)\lambda(x)$ and $\psi(x)$ satisfy a commutation relation of the same form as the above one.

As the operator $p_n$ is given by $p_n = \frac{\hbar}{i} \frac{\partial}{\partial x_n}$, we have the unitary transformations of $p_n$ and $\frac{p_n^2}{2m}$ as

$$\left\{ \begin{array}{l}
U p_n U^{-1} = p_n + \left[ U, \frac{\hbar}{i} \frac{\partial}{\partial x_n} \right] U^{-1} = p_n - \frac{\hbar}{i} \frac{\partial U}{\partial x_n} U^{-1}, \\
U \frac{p_n^2}{2m} U^{-1} = \frac{p_n^2}{2m} + \left[ U, \frac{1}{2m} \frac{\hbar}{i} \frac{\partial}{\partial x_n} \right] U^{-1} = \frac{p_n^2}{2m} + \frac{1}{2m} \frac{\hbar}{i} \left( p_n \frac{\partial U}{\partial x_n} + \frac{\partial U}{\partial x_n} p_n \right) U^{-1}.
\end{array} \right.$$  

(3.3)

Owing to $\frac{\partial}{\partial x_n} \delta(x-x_n) = -\frac{\partial}{\partial x_n} \delta(x-x_n) = \delta(x-x_n) \frac{\partial}{\partial x_n}$, we get the gradient formula for $U$ as

$$\frac{\partial U}{\partial x_n} = - \frac{i}{\hbar} \int \left( (n \sum_{n=1}^{A} \delta(x-x_n) \cdot \partial \phi(x) + \text{sym.)} - (n \sum_{n=1}^{A} \delta(x-x_n) \lambda(x) \cdot \partial \psi(x) + \text{sym.}) \right) dx U,$$

(3.4)

$$= \left[ - \frac{i}{\hbar} \int \left( (m \delta(x-x_n) \cdot \partial \phi(x) + \text{sym.)} - (m \delta(x-x_n) \lambda(x) \cdot \partial \psi(x) + \text{sym.}) \right) dx U. \right.$$ 

Note that the gradient operator acts onto only canonical conjugate variables $\phi$ and $\psi$ of $\rho$ and $-\rho \lambda$, respectively. Namely, we here adopt a special technical operator-action rule. Then, the first and second Eqs. of (3.3) read

$$U p_n U^{-1} = p_n + \int \left( (m \delta(x-x_n) \cdot \nabla \phi(x) + \text{sym.)} - (\lambda(x) m \delta(x-x_n) \cdot \nabla \psi(x) + \text{sym.}) \right) dx,$$

$$\sum_{n=1}^{A} \sum_{n=1}^{A} U \frac{p_n^2}{2m} U^{-1} = \sum_{n=1}^{A} \frac{p_n^2}{2m} + \frac{1}{2} \sum_{n=1}^{A} \int \left[ \{ (p_n \delta(x-x_n) + \delta(x-x_n) \cdot \nabla \phi(x) + \text{sym.)} - (\lambda(x) \cdot \nabla \psi(x) + \text{sym.)} \right) \right] dx + \frac{1}{2} \int \nabla \phi(x) \cdot \rho(x) \nabla \phi(x) dx$$

(3.5)

$$- \frac{1}{2} \int \lambda(x) \nabla \psi(x) \cdot \rho(x) \nabla \phi(x) + \nabla \phi(x) \cdot \rho(x) \lambda(x) \nabla \psi(x) - \lambda(x) \nabla \psi(x) \cdot \rho(x) \lambda(x) \nabla \psi(x) \right] dx.$$

Finally, it turns out that the original Hamiltonian is transformed to a new Hamiltonian

$$\tilde{H}(=U H U^{-1}) = \tilde{H}_0 + \tilde{H}_{\text{int.}} dx + \tilde{H}_{\text{field}}$$

where $\tilde{H}_0 = \sum_{n=1}^{A} \frac{p_n^2}{2m} + V(x_1, \cdots, x_A)$ and $\tilde{H}_{\text{int.}}$ is given by

$$\tilde{H}_{\text{int.}} = \frac{1}{2} \sum_{n=1}^{A} \left[ (p_n \delta(x-x_n) + \delta(x-x_n) \cdot \nabla \phi(x) - \lambda(x) \rho(n \delta(x-x_n) + \delta(x-x_n) \cdot \nabla \psi(x) + \text{sym.)} \right].$$

(3.6)
The Hamiltonian $H_{\text{field}}$ is written as $\int (H_{\text{phon.}} + H_{\text{rot.}} + H_{\text{int.}}) \, dx$ in which each Hamiltonian $H$ is expressed as

$$
\begin{align*}
H_{\text{phon.}} &= \frac{1}{2} \nabla \phi(x) \cdot \rho(x) \nabla \phi(x), \\
H_{\text{rot.}} &= \frac{1}{2} \lambda(x) \nabla \psi(x) \cdot \rho(x) \lambda(x) \nabla \psi(x), \\
H_{\text{int.}} &= -\frac{1}{2} \{ \nabla \psi(x) \cdot \rho(x) \nabla \phi(x) + \nabla \phi(x) \cdot \rho(x) \lambda(x) \nabla \psi(x) \}.
\end{align*}
$$

(3.7)

In the Hamiltonian for the quantized fluid, the first and the second Hamiltonians in (3.7) contribute to the occurrence of the phonon and roton spectra, respectively. The last one gives their interaction. They are coincident with the classical fluid Hamiltonian (2.8) in the classical limit.
4 Ziman transformation and roton Hamiltonian

To go from classical fluid dynamics to quantum fluid dynamics, Ziman introduced variables
\[ \psi_1, \psi_2, \pi_\psi = -\rho \lambda = -\frac{i\psi_2}{2} \] and field operators \( \Psi = \frac{1}{\sqrt{2\hbar}}(\psi_1 + i\psi_2) \) and \( \Psi^* = \frac{1}{\sqrt{2\hbar}}(\psi_1 - i\psi_2) \) [19]. Introducing these relations into (2.1) the fluid velocity \( \mathbf{v} \) is expressed as
\[
\mathbf{v} = -\nabla \phi - \frac{i\hbar}{2\rho} (\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1) = -\nabla \phi + \frac{i\hbar}{2\rho} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) ,
\]
Similarly, the Hamiltonian \( \mathcal{H}_{\text{rot}} \) in (3.7) is expressed as
\[
\mathcal{H}_{\text{rot}} = \frac{\hbar^2}{8\rho} \left[ (\Psi^* \nabla \Psi \cdot \mathbf{v} \nabla \Psi^* + \Psi \nabla \Psi^* \cdot \mathbf{v} \nabla \Psi) - (\Psi^* \nabla^2 \Psi \cdot \mathbf{v} \nabla \Psi^* + \Psi \nabla^2 \Psi^* \cdot \mathbf{v} \nabla \Psi) \right] \equiv \mathcal{H}_{\text{rot}, I} + \mathcal{H}_{\text{rot}, II} .
\]
For the incompressible fluid, due to the continuity equation of the fluid, we have the condition
\[
\text{div} \mathbf{v} = -\nabla^2 \phi + \frac{i\hbar}{2\rho} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) = 0, \quad (\rho_0 : \text{equilibrium density}).
\]

In the BMM, the collective flow in nuclei is assumed to be irrotational. Namely, the velocity potential satisfies \( \nabla^2 \phi = 0 \) which, using (4.3), is ensured by requiring \( \nabla^2 \Psi = \nabla^2 \Psi^* = 0 \). Then the \( \Psi \) and \( \Psi^* \) can be expanded in terms of spherical harmonics as follows:
\[
\Psi(\mathbf{x}) = \sum_{\lambda \mu} b_{\lambda \mu} \left( \frac{r}{R_0} \right)^\lambda \mathbf{Y}_{\lambda \mu}(\theta, \phi), \quad \Psi^*(\mathbf{x}) = \sum_{\lambda \mu} b_{\lambda \mu}^* \left( \frac{r}{R_0} \right)^\lambda \mathbf{Y}_{\lambda \mu}(\theta, \phi),
\]
where \( b_{\lambda \mu} \) and \( b_{\lambda \mu}^* \) are regarded as boson annihilation and creation operators satisfying \([b_{\lambda \mu}, b_{\lambda' \mu'}^*] = \delta_{\lambda \lambda'} \delta_{\mu \mu'}, [b_{\lambda \mu}, b_{\lambda' \mu'}] = [b_{\lambda \mu}^*, b_{\lambda' \mu'}^*] = 0 \). Canonical commutation relations between \( \Psi \) and \( \Psi^* \) are also assumed, i.e., \([\Psi(\mathbf{x}), \Psi^*(\mathbf{x}')] = 0 (\mathbf{x} \neq \mathbf{x}')\), \([\Psi(\mathbf{x}), \Psi(\mathbf{x}')] = 0\) and \([\Psi^*(\mathbf{x}), \Psi^*(\mathbf{x}')] = 0\).

The corresponding Poisson brackets in fluid dynamics were discussed in detail by Zakharov-Kuznetsov [26]. The operators \( b_{\lambda \mu} \) and \( b_{\lambda \mu}^* \) are the roton operators proposed by Landau [22]. Ziman obtained the roton Hamiltonian in terms of them and derived a roton spectrum of liquid Helium. It also found the roton Hamiltonian by a different approach. Adopting a vector potential \( \mathbf{A} \) satisfying the Poisson equation \( \nabla^2 \mathbf{A} = -\mathbf{w} \) for the vorticity \( \mathbf{w} \), \( \mathbf{A} \) is represented as \( \mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{w}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' \) [20]. There exists an invariant integral
\[
I = \int \mathbf{v} \cdot d\mathbf{x} = \frac{1}{4\pi} \int \int \frac{|\mathbf{w}(\mathbf{x}) \times \mathbf{w}(\mathbf{x}')|}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x} d\mathbf{x}', \quad \text{expressed in terms of circulations on vortex curves} \quad C_i \quad \text{and} \quad C_j \quad \text{with strengths} \quad \kappa_i \quad \text{and} \quad \kappa_j .
\]
The integral \( I \) may be rewritten as
\[
I = \sum_{i,j} \alpha_i \kappa_i \kappa_j, \quad \alpha_i \equiv \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\mathbf{x} - \mathbf{x}') \cdot [d\mathbf{l}_i \times d\mathbf{l}_j]}{|\mathbf{x} - \mathbf{x}'|^3} \quad (\alpha_i : \text{winding number}) \quad [27, 28, 29].
\]

First we consider the angular momentum of an \( \mathbf{A} \)-particle system. The total angular momentum of the system is the sum of individual particle angular momenta. Then we have
\[
\mathbf{J} = \sum_{n=1}^{A} \mathbf{j}_n = \sum_{n=1}^{A} \mathbf{x}_n \times \mathbf{p}_n = \sum_{n=1}^{A} \frac{\hbar}{i} \mathbf{x}_n \times \nabla n
\]
By a unitary transformation of \( \mathbf{J} \), a new total angular momentum \( \mathbf{\bar{J}} (= \mathbf{U} \mathbf{J} \mathbf{U}^{-1}) \) is changed to
\[
\mathbf{\bar{J}} = \sum_{n=1}^{A} \mathbf{j}_n U^{-1} = \sum_{n=1}^{A} [j_n + [U, j_n]] U^{-1} = \sum_{n=1}^{A} j_n - \frac{\hbar}{i} \mathbf{x}_n \times (\nabla U) U^{-1} .
\]
Here, \( \nabla U \) is given by (3.4). Define \( \mathbf{L} \) as \( \mathbf{L} = \frac{\hbar}{i} \mathbf{x} \times \nabla \). Thus, we get
\[
\mathbf{\bar{J}} = \mathbf{J} + \int \frac{\hbar}{i} \{ \rho(\mathbf{x}) \mathbf{L} \phi(\mathbf{x}) - \rho(\mathbf{x}) \lambda(\mathbf{x}) \mathbf{L} \psi(\mathbf{x}) \} d\mathbf{x},
\]
which consists of the previous operator \( \mathbf{J} \) and of the contribution to the angular momentum arising from the irrotational and rotational flows. Thus we can find the angular momentum \( \mathbf{J} \) due to the rotational flow,
$$\mathbf{J} = -\frac{i}{\hbar}\rho\lambda\mathbf{L}\psi = \frac{i}{\hbar}\pi\psi\mathbf{L}\psi = \frac{1}{2}(\Psi\mathbf{L}\Psi^* - \Psi^*\mathbf{L}\Psi), \quad \mathbf{J}_k \equiv \frac{1}{2}(\Psi L_k\Psi^* - \Psi^* L_k\Psi). \quad (4.8)$$

The spherical tensor representation of $L_k(k=\pm,0)$ is given as $L_{\pm 1} = \mp\frac{1}{\sqrt{2}}(L_{\pm} \pm L_0)$, $L_0 = L_z$. Using the Clebsch-Gordan coefficient $\langle l_1 m_1 l_2 m_2 | l_3 m_3 \rangle$, $\Psi L_k\Psi^*$ and $\Psi^* L_k\Psi$ are calculated as

$$\Psi L_k\Psi^* = \sum_{\lambda'\mu'} b_{\lambda'\mu'} \left( \frac{r}{R_0} \right)^{\lambda' + \lambda} \lambda' \mu' \sum_{\lambda\mu} b_{\lambda\mu} \left( \frac{r}{R_0} \right)^{\lambda} \lambda \mu \langle L_k Y_{\lambda\mu} \rangle$$

$$= \sum_{\lambda'\mu'} \sum_{\lambda\mu} b_{\lambda'\mu'} b^*_{\lambda\mu} \left( \frac{r}{R_0} \right)^{\lambda + \lambda'} \lambda' \mu' \langle L_k Y_{\lambda\mu} \rangle (-1)^{\mu + k} \sqrt{\lambda(\lambda+1)} \langle \lambda k - \mu 1 - k | \lambda - \mu \rangle Y_{\lambda'\mu'} Y_{k\mu} \quad (4.9)$$

$$= \sum_{\lambda'\mu'} \sum_{\lambda\mu} b_{\lambda'\mu'} b^*_{\lambda\mu} \left( \frac{r}{R_0} \right)^{\lambda + \lambda'} \lambda' \mu' \langle L_k Y_{\lambda\mu} \rangle (-1)^{\mu + k} \left( \frac{r}{R_0} \right)^{\lambda + \lambda'} \sqrt{\lambda(\lambda+1)} \langle \lambda k - \mu 1 - k | \lambda - \mu \rangle \times \sum_{LM} \sqrt{\frac{(2\lambda+1)(2\lambda'+1)}{4\pi(2L+1)}} \langle \lambda k - \mu \lambda' \mu' | LM \rangle \langle \lambda 0 \lambda' 0 | L0 \rangle Y_{LM} \quad (4.10)$$

where the symmetry $\langle \lambda k | \lambda - \mu \rangle = (-1)^{\mu |k} \langle \lambda 1 k | \lambda \mu + k \rangle$ and the formula for the product of two spherical harmonics are used [30]. Further using the property of the Racah coefficients [30],

$$\langle \lambda 1 k | \lambda \mu + k \rangle \langle \lambda \mu + k \lambda' \mu' | LM \rangle$$

$$= \sum_{L'M'} \sqrt{(2\lambda+1)(2L'+1)} W(\lambda 1 LL'; \lambda L') \langle 1k \lambda' \mu' | L'L' \rangle \langle \lambda \mu L'M' | LM \rangle,$$

and substituting (4.9) and (4.10) into (4.8), the $k$-th component of rotational angular momentum $\mathbf{J}_k$ is derived as

$$\mathbf{J}_k = \frac{1}{2} \sum_{\lambda'\mu'} \sum_{LML'M'} (-1)^{\lambda + \lambda' + 1} (2\lambda + 1) \sqrt{\frac{\lambda(\lambda+1)(2\lambda'+1)(2L'+1)}{12\pi}} \left( \frac{r}{R_0} \right)^{\lambda + \lambda'}$$

$$\times W(\lambda 1 LL'; \lambda L') \langle \lambda 0 \lambda' 0 | L0 \rangle Y_{LM} \langle LM \lambda \mu | L'M' \rangle \langle L'M' \lambda' \mu' | 1k \rangle$$

$$\times [b^*_{\lambda\mu} (-1)^{\mu} b_{\lambda'\mu'} - b_{\lambda'\mu'} (-1)^{\mu} b_{\lambda\mu} + \delta_{\lambda \lambda'} (-1)^{\mu} \delta_{\mu \mu'}], \quad (4.11)$$

whose form, neglecting the constant term, is very similar to the angular momentum given in terms of quadratic surface-phonon operators in the BMM.

Finally, using the gradient formula [31, 30],

$$\nabla_k \left( \frac{r}{R_0} \right)^{\lambda} Y_{\lambda\mu} = -\sqrt{\frac{\lambda}{2\lambda - 1}} \langle \lambda 1 k | \lambda - 1 \mu + k \rangle \frac{2\lambda + 1}{R_0} \left( \frac{r}{R_0} \right)^{\lambda - 1} Y_{\lambda - 1 \mu + k},$$

and noticing the fact that the scalar product of any two first-rank tensors $\mathbf{v}$ and $\mathbf{v}'$ is given in the spherical tensor representation as $\sum_{k} (-1)^{k} v_{k} v'_{k}$, we obtain the roton Hamiltonian $H_{\text{rot.}}(\equiv H_{\text{rot.1}} + H_{\text{rot.2}})$ (4.2) in terms of the roton operators $b_{\lambda\mu}$ and $b^*_{\lambda\mu}$ as,
The roton Hamiltonian $H_{\text{rot}}$ for the case of $\lambda$ sign. It is expressed in terms of the roton operators without the constant term.

It has a quadratic form of surface-phonon operators. This is a remarkable difference between them.

$$H_{\text{rot}} = \frac{\hbar^2}{8\rho_0} \sum_k \frac{\lambda'}{\lambda} \kappa' \mu' \nu' \nu' \sum_{LL'M'M'T'T'AN} \frac{\gamma_{J\kappa}(1)}{R_0^3} \frac{1}{R_0^3} \frac{(2\lambda'+1)(2\kappa'+1)}{3}$$

$$\times \langle L0\lambda' \kappa' \mu' | L0\lambda' \kappa' | \Gamma \Lambda \rangle L.M' \kappa' \nu' | \Gamma' \Lambda' \rangle L M L' M'| J K \rangle Y_{JK} \quad (4.12)$$

$$\times \langle \lambda 0 \lambda' \mu' | \Gamma' \Lambda' \kappa' \nu' | 1-k \rangle W(\lambda' \lambda -1 \Gamma \lambda \kappa' \Gamma' \kappa' \lambda' \mu' \nu' \mu' \nu' \mu' \nu')$$

$$\times \left[ b_{\lambda' \mu' \nu'}^{\ast}(-1)^{\mu} b_{\lambda' \nu'} \delta_{\lambda' \nu'} \mu' \nu' b_{\mu' \nu'}^{\ast}(-1)^{\nu} b_{\lambda' \nu'} + \delta_{\lambda' \nu'} \mu' \nu' b_{\mu' \nu'}^{\ast}(-1)^{\nu} b_{\lambda' \nu'} + \delta_{\lambda' \nu'} \mu' \nu' b_{\mu' \nu'}^{\ast}(-1)^{\nu} b_{\lambda' \nu'} \right] ,$$

$$H_{\text{rot}} = \frac{\hbar^2}{8\rho_0} \sum_k \frac{\lambda'}{\lambda} \kappa' \mu' \nu' \nu' \sum_{LL'M'M'T'T'AN} \frac{\gamma_{J\kappa}(1)}{R_0^3} \frac{1}{R_0^3} \frac{(2\lambda'+1)(2\kappa'+1)}{3}$$

$$\times \langle L0\lambda' \kappa' \mu' | L0\lambda' \kappa' | \Gamma \Lambda \rangle L.M' \kappa' \nu' | \Gamma' \Lambda' \rangle L M L' M'| J K \rangle Y_{JK} \quad (4.13)$$

$$\times \langle \lambda 0 \lambda' \mu' | \Gamma' \Lambda' \kappa' \nu' | 1-k \rangle W(\lambda' \lambda -1 \Gamma \lambda \kappa' \Gamma' \kappa' \lambda' \mu' \nu' \mu' \nu' \mu' \nu')$$

$$\times \left[ (-1)^{\mu} b_{\lambda' \mu' \nu'}^{\ast}(-1)^{\nu} b_{\lambda' \nu'} + b_{\lambda' \nu'}^{\ast} b_{\lambda' \mu' \nu'} + \delta_{\lambda' \nu'} \mu' \nu' b_{\lambda' \nu'} + \delta_{\lambda' \nu'} \mu' \nu' b_{\lambda' \nu'} + \delta_{\lambda' \nu'} \mu' \nu' b_{\lambda' \nu'} \right] .$$

The roton Hamiltonian $H_{\text{rot}}$ consists of normal-ordered quartic and quadratic terms with respect to roton operators and constant terms. On the other hand, the BMM Hamiltonian has a quadratic form of surface-phonon operators. This is a remarkable difference between them.

There exist many multipole degrees of freedom in the collective coordinates of the surface oscillations in the BMM. As was done in the BMM, we also pay special attention to collective excitations with quadrupole degrees of freedom since such degrees of freedom play a fundamental role in almost all nuclei. Then, in Eqs. (4.11), (4.12) and (4.13), we restrict to the case of $\lambda = \lambda' = \kappa = \kappa'$ and they are rewritten in the following forms:

The rotational angular momentum $\vec{J}_k$ (4.11) at the nuclear surface ($r=R_0$) is expressed as

$$\vec{J}_k = \frac{5}{2} \sqrt{\frac{2 \cdot 3 \cdot 5(2L' + 1)}{12\pi}} \sum_{LL'M'M'} W(21LL'; 22) \langle 2020 | L0 \rangle Y_{LM}$$

$$\times \langle L M 2 \mu L' M' | L M' 2 \mu' | 1 k \rangle b_{\mu' \nu'}^{\ast}(-1)^{\mu} b_{2 \mu - \mu} - b_{\mu' \nu'}^{\ast}(-1)^{\mu} b_{2 \mu - \mu} - (-1)^{\mu} \delta_{\mu-\mu'} ,$$

in which we pick up only the term with $L=0$. Then we have a simple formula for $\vec{J}_k$ as

$$\vec{J}_k = - \frac{\sqrt{2 \cdot 5}}{4\pi} \sum_{\mu \mu'} (2 \mu 2 \mu' | 1 k \rangle b_{\mu' \nu'}^{\ast}(-1)^{\mu} b_{2 \mu - \mu'} ,$$

which has the same form as the BMM angular momentum operator except for the minus sign. It is expressed in terms of the roton operators without the constant term.
The roton Hamiltonian $\mathcal{H}_\text{rot.}(\equiv \mathcal{H}_\text{rot.1} + \mathcal{H}_\text{rot.2})$ (4.2) reduces to simpler forms in terms of the roton operators $b_{2\mu}^*$ and $b_{2\mu}$ as,

$$
\mathcal{H}_\text{rot.1} = \frac{\hbar^2}{8\rho_0 R_0^2} \left( \frac{r}{R_0} \right)^6 \frac{5}{3} \frac{2\cdot 3\cdot 5}{4\pi} \sum_k \sum_{\mu'\nu'} \sum_{LL'M'M'T'T'AN} \sum_{JK} (-1)^{k+J} \times \frac{(2L+1)(2L'+1)(2\Gamma+1)(2\Gamma'+1)}{(4\pi(2J+1))} (4.16)
$$

$$
\times \langle LML'M'|J\rangle \langle J\rangle \langle \Gamma \Lambda 2\mu|1k\rangle \langle \Gamma' \Lambda' 2\nu|1-k\rangle W(21\Gamma 2; 1L) W(21\Gamma'2; 1L') \times [b_{2\mu}^* b_{2\nu}^*(-1)^{\mu'} b_{2-\mu'}(1)^{\nu'} b_{2-\nu} + (-1)^{\nu'} \delta_{\nu',\nu} b_{2\mu}^*(-1)^{\nu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{\mu',\nu} b_{2\mu}^*(-1)^{\nu'} b_{2-\mu'} + (-1)^{\mu} \delta_{\mu,\nu} b_{2\mu}^*(-1)^{\nu'} b_{2-\nu} + \cdots ] ,
$$

$$
\mathcal{H}_\text{rot.2} = \frac{\hbar^2}{8\rho_0 R_0^2} \left( \frac{r}{R_0} \right)^6 \frac{5}{3} \frac{2\cdot 3\cdot 5}{4\pi} \sum_k \sum_{\mu'\nu'} \sum_{LL'M'M'T'T'AN} \sum_{JK} \times \frac{(2\Gamma+1)}{(4\pi(2J+1))} (4.17)
$$

$$
\times \langle LML'M'|J\rangle \langle J\rangle \langle \Gamma' \Lambda' 1\nu+k|1k\rangle W(21\Gamma'1; 1L') \times [(-1)^{\mu} b_{2-\mu}^*(-1)^{\mu'} b_{2-\nu}^*(-1)^{\nu'} b_{2-\nu} + b_{2\mu}^* b_{2\nu}^* b_{2\nu} b_{2\mu} + \delta_{\mu,\nu} b_{2\mu}^* b_{2\mu} + \cdots ] ,
$$

Introducing $f(r)$ defined as $f(r) = \frac{1}{(4\pi)^2} \frac{\hbar^2}{8\rho_0 R_0^2} \left( \frac{r}{R_0} \right)^6$, finally we reach the final expression for the roton Hamiltonian $\mathcal{H}_\text{rot.}$ given in the following form:

$$
\mathcal{H}_\text{rot.} = -\frac{100}{3} f(r) \sum_k \sum_{\nu} \langle 2-\nu 1\nu+k|1k \rangle + \frac{8}{5} f(r) \sum_{\mu} b_{2\mu}^* b_{2\mu} + \frac{4}{5} f(r) \sum_{\mu} b_{2\mu}^* b_{2\mu} + \sum_{\mu'\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} (2\mu 1M|1\Lambda) \langle 2\nu 1\nu+M|1\Lambda' \rangle \langle 1-k \rangle \times [2(-1)^{\nu'} \delta_{\nu',\nu} b_{2\mu}^*(-1)^{\mu'} b_{2-\mu'} + (1+(-1)^{\Lambda+\Lambda'}) (-1)^{\mu} \delta_{\mu,\nu} b_{2\mu}^*(-1)^{\nu'} b_{2-\nu}] \times \frac{27}{40} f(r) \sum_k \sum_{\mu'\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} (2\mu 1M|1\Lambda) \langle 2\nu 1\nu+M|1\Lambda' \rangle \langle 1-k \rangle \times [b_{2\mu}^* b_{2\nu}^*(-1)^{\mu'} b_{2-\nu} + b_{2\mu}^* b_{2\nu}^* b_{2\mu} b_{2\nu} + \cdots ] \times \frac{27}{40} f(r) \sum_k \sum_{\mu'\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} (2\mu 1M|1\Lambda) \langle 2\nu 1\nu+M|1\Lambda' \rangle \langle 1-k \rangle \times [b_{2\mu}^* b_{2\nu}^*(-1)^{\mu'} b_{2-\nu} + b_{2\mu}^* b_{2\nu}^* b_{2\mu} b_{2\nu} + \cdots ] \cdots ,
$$

where we omit terms with higher rank of angular momenta $L, L'$ and $J$ etc. on the computation of (4.16) and (4.17) because their explicit expressions become too long to write. The roton Hamiltonian at the equilibrium nuclear surface is given by $\mathcal{H}_\text{rot.}|_{r=R_0}$.
The velocity potential $\phi(x)$ is also expanded as $\phi(x) = \sum_\mu \pi_{2\mu} (r/R_0)^2 Y_{2\mu}(\theta, \varphi)$ where $\pi_{2\mu}$ and the collective coordinate $\alpha_{2\mu}$ given previously are related to the BMM boson operator. To get the fluid velocity $v(x)$ at the surface, we also require a surface boundary condition \[
abla R(\theta, \varphi) \bigg|_{r=R_0} = v_r \bigg|_{r=R_0}.
\] Using the fluid velocity $v(4.1)$, this condition is given as follows:

\[
R_0 \sum_\mu \dot{\alpha}_{2\mu} Y_{2\mu} = - \sum_\mu \pi_{2\mu} \left( \frac{\partial}{\partial t} \left( \frac{r}{R_0} \right)^2 Y_{2\mu} \right)_{r=R_0} - \frac{i\hbar}{2\rho_0} \sum_\mu \left( \frac{r}{R_0} \right)^2 Y_{2\mu}^* \frac{\partial}{\partial r} \left( \frac{r}{R_0} \right)^2 Y_{2\mu'} \bigg|_{r=R_0} - \frac{2}{R_0} \sum_\mu \pi_{2\mu} Y_{2\mu} - \frac{i\hbar}{2\rho_0} \sum_\mu \left( \sum_{\mu'}(-1)^\mu b_{2\mu'}^* Y_{2\mu} \right) b_{2\mu} - \left( \sum_{\mu'} b_{2\mu'}^* Y_{2\mu'} \right) (-1)^\mu b_{2-\mu}^* Y_{2\mu},
\]

which gives the relation between the time derivative $\dot{\alpha}_{2\mu}$ and the $\pi_{2\mu}$ and the roton operator $b_{2\mu}$ as

\[
\dot{\alpha}_{2\mu} = - \frac{2}{R_0} \pi_{2\mu} - \frac{i\hbar}{2\rho_0} \frac{2}{R_0} \left( \sum_{\mu'}(-1)^\mu b_{2-\mu}^* Y_{2\mu'} \right) b_{2\mu} - \left( \sum_{\mu'} b_{2\mu'}^* Y_{2\mu'} \right) (-1)^\mu b_{2-\mu}^*.
\]

It should be noticed that in the above there exist bi-linear terms in $b_{2\mu}$ and $(-1)^\mu b_{2-\mu}^*$. This is a remarkable aspect of the relation. However, if we discard these terms, we can reach the well-known result that $\pi_{2\mu}$ is the canonical conjugate variable to $\alpha_{2\mu}$. 

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5 Determination of Clebsch parameters through a one-form gauge potential

Using the equilibrium density $\rho_0$, the Ziman variables $\psi_1$ and $\psi_2$ are expressed through the two Clebsch parameters $\psi$ and $\lambda$ as

$$\psi = \frac{\psi_1}{\psi_2}, \quad \psi_1 = \psi \psi_2 = \psi \sqrt{2 \rho_0 \lambda},$$

$$\rho_0 \lambda = \frac{\psi_2^2}{2}, \quad \psi_2 = \sqrt{2 \rho_0 \lambda}. \quad (5.1)$$

Substituting (5.1) into the Ziman field operators $\Psi = \frac{1}{\sqrt{2\hbar}}(\psi_1 + i\psi_2)$ and $\Psi^* = \frac{1}{\sqrt{2\hbar}}(\psi_1 - i\psi_2)$, then we recover the expression of Alcock-Kuper [32] for $\Psi$ and $\Psi^*$:

$$\Psi = \sqrt{\frac{\rho_0 \lambda}{\hbar}}(\psi + i),$$

$$\Psi^* = \sqrt{\frac{\rho_0 \lambda}{\hbar}}(\psi - i). \quad (5.2)$$

For the incompressible fluid, the continuity equation leads to the condition

$$\text{div} \mathbf{v} = -\nabla^2 \phi + \frac{i\hbar}{2\rho_0} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) = 0. \quad (5.3)$$

The collective flow is assumed to be irrotational, i.e., $\nabla^2 \phi = 0$ which means $\nabla^2 \Psi = \nabla^2 \Psi^* = 0$ where the Laplacian in the spherical polar-coordinate is expressed as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + 2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (5.4)$$

According to Jackiw [33, 34], the two Clebsch parameters $\psi$ and $\lambda$ are constructed as

$$\lambda = 2 \left(1 - \sin^2 \frac{f}{2} \sin^2 \theta\right), \quad \frac{\partial \lambda}{\partial \varphi} = 0,$$

$$\psi = \varphi + \tan^{-1} \left(\tan \frac{f}{2} \cos \theta\right), \quad \frac{\partial \psi}{\partial \varphi} = 1, \quad (5.5)$$

where a profile function $f = f(r)$ will be determined later. Denoting by differentiation with respect to $r$ by prime, the differential formulas for $\psi$ and $\lambda$ with respect to $r$ and $\theta$ are given as

$$\frac{\partial \lambda}{\partial r} = \sin f \sin^2 \theta \prime, \quad \frac{\partial \psi}{\partial r} = -\sqrt{\frac{\rho_0 \lambda}{\hbar}} \left\{ \frac{1}{2\lambda} \sin f \sin^2 \theta \prime (\psi + i) - \frac{\cos \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)} \right\}, \quad (5.6)$$

$$\frac{\partial \psi}{\partial r} = \frac{\cos \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)} \prime, \quad \frac{\partial \lambda}{\partial \theta} = -2 \sin^2 \frac{f}{2} \sin 2\theta, \quad \frac{\partial \psi}{\partial \theta} = -\sqrt{\frac{\rho_0 \lambda}{\hbar}} \left\{ \frac{1}{\lambda} \sin \frac{f}{2} \sin 2\theta \prime (\psi + i) + \sin f \sin \theta \right\}, \quad (5.7)$$

$$\frac{\partial \psi}{\partial \theta} = -\frac{\sin f \sin \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)}. \quad (5.8)$$
Substitution of (5.5), (5.6) and (5.7) casts $\nabla^2 \Psi = 0$ into the following equation:

$$\nabla^2 \Psi = -\sqrt{\frac{\rho_0 \lambda}{\hbar}} \left[ \left\{ \frac{1}{2(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( \frac{1}{4} \sin^2 f \sin^4 \theta f'^2 + \frac{4}{r^2} \sin^4 \frac{\theta}{2} \sin^2 \cos^2 \theta \right) \right. \right.$$

$$+ \left. \frac{1}{2(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( \cos \sin^2 \theta f'^2 + \sin \sin^2 \theta f''^2 + 2 \sin \sin^2 \theta f' \frac{1}{r} f' + \frac{4}{r^2} \sin \frac{\theta}{2} (3 \cos^2 \theta - 1) \right) \right] \left( \psi + i \right)$$

$$- \frac{\cos \theta}{2(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( \sin \sin^2 \theta f'^2 + 2 \left( 1 - \sin^2 \frac{\theta}{2} \sin^2 \theta \right) f'^2 + 4 \left( 1 - \sin^2 \frac{\theta}{2} \sin^2 \theta \right) \frac{1}{r} f' - \frac{4}{r^2} \sin f \right)$$

$$+ \frac{\cos \theta}{2(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( \sin \sin^2 \theta f'^2 - \frac{4}{r^2} \sin f \sin^2 \frac{\theta}{2} \sin^2 \right) \right]$$

$$= -\sqrt{\frac{\rho_0 \lambda}{\hbar}} \left[ \left\{ \frac{1}{2(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( (1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} - \cos^2 \theta) f'^2 + \frac{4}{r^2} \sin \frac{\theta}{2} (1 - \sin^2 \frac{\theta}{2}) \sin^4 \theta \right) \right. \right.$$

$$+ \left. \frac{\sin^2 \theta}{4(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( \sin f f''^2 + 2 \sin f \frac{1}{r} f' + \frac{8}{r^2} \sin f \cos^2 \theta \right) \right] \left( \psi + i \right) - \frac{\cos \theta}{2(1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2})} \left( f'' + \frac{2}{r^2} f' - \frac{2}{r^2} \sin f \right)$$

$$= -\sqrt{\frac{\rho_0 \lambda}{\hbar}} \left[ \frac{2 \sin^2 \frac{\theta}{2} (1 - \sin^2 \frac{\theta}{2}) \sin^4 \theta}{r^2 \cos^2 \theta} + \frac{\sin f \sin^2 \theta}{2 \cos \theta} \left( f'' + \frac{2}{r^2} f' - \frac{2}{r^2} \sin f \right) \right] \left( \psi + i \right) - \left( f'' + \frac{2}{r^2} f' - \frac{2}{r^2} \sin f \right) = 0. \quad (5.8)$$

In the above, to eliminate the term $f'^2$, we have assumed the auxiliary condition

$$1 - \sin^2 \frac{\theta}{2} = \cos \theta, \quad (5.9)$$

which implies that either $\cos \theta = 1$ or $\cos \theta = \cot \frac{\theta}{2}$. This condition plays an essential role in the solution of the Laplace equation $\nabla^2 \Psi = 0$. For the present central purpose, from the outset, we demand that the profile function $f = f(r)$ obeys the differential equation $f'' + \frac{2}{r^2} f' - \frac{2}{r^2} \sin f = 0$. This equation to determine $f$ was previously obtained by Jackiw-Pi [34]. It has been integrated numerically by Bergner (Ref. [10] of [34]). Introducing the dimensionless variable $x$ as $x = \frac{r}{R_0}$, the positive solution is plotted in Fig.1 of [34]. They present the solution of $f(x)$ that is regular at the origin, $x = 0$, vanishing linearly with $x(0 < x < 1)$, and tending to $\pi$ in an oscillatory manner for larger $x$. In the region $0 < x < 1$, we have $\sin f \approx 0$ and due to (5.9) $\cos \theta = 1$. The coefficient of the term $(\psi + i)$ in the last line of equation (5.8) vanishes exactly. Thus we obtain an almost complete solution for $\nabla^2 \Psi = 0$ over the whole region of $x$.

By projecting onto a fixed direction $\hat{n}^a$ (constant unit vector; $a = 1 \sim 3$) in the isospin space and using $U^{-1} dU$, consider a Clebsch-parameterized gauge potential 1-form $a$ [33, 34] given as

$$a = i \text{Tr} \hat{n}^a \sigma^a U^{-1} dU, \quad (5.10)$$

where $U = e^{\frac{\omega}{2} \sigma^a \frac{f}{2} - i \omega \sigma^a \sin f}$. $\sigma^a \equiv (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ and $\sigma^a$ is the Pauli matrices. Taking $\hat{n}^a$ to point in the third direction, the 1-form $a$ (5.10) is expressed as

$$a = \cos \theta df - \sin f \sin \theta d\theta - (1 - \cos f) \sin^2 \theta d\varphi, \quad \text{(see Appendix A)}. \quad (5.11)$$
The integral \( \int \frac{1}{24\pi^2} \text{Tr}(U^{-1}dU)^3 \) is an integer, i.e., the winding number \( \alpha_{i,j} \) or the quantized helicity [35, 34] related to the Chern-Simons numbers of a non-Abelian vacuum gauge potential [36]. For the motion of quantum vortices, Nambu set up an Hamilton-Jacobi formalism and reached the conclusion that the corresponding Hamilton-Jacobi functions are the Clebsch potentials [37]. On the other hand, the canonical quantization in terms of the Clebsch parameters has been developed by Rasetti-Regge. Their commutation relations are those of a current algebra [38, 39, 40].
6 Discussions and further outlook

In the preceding sections, we have attempted to formulate a description of the rotational velocity field of the nuclear fluid through the Clebsch transformation. In the framework of quantum fluid dynamics, we have derived the vortex Hamiltonian of the fluid in terms of the roton operators. According to the previous considerations, the quantum fluid-dynamical approach may be applied to the three-dimensional nuclear fluid. We expect that such an approach to nuclei will provide an interesting description of a new kind of elementary energy excitation, namely, "vortex modes" because such an approach is designed to take into account essential many-body effects, which were not considered in the traditional treatment of the rotational collective motion.

On the other hand, extending Tomonaga’s idea and using Sunakawa’s method, one of the present authors (S.N.) has developed the collective description of nuclear surface oscillations and the collective theory of two-dimensional nuclei, in the context of the first quantized language, as opposed to the second quantized approach adopted in the Sunakawa’s method. Applying the Tomonaga’s revolutionary idea and the Sunakawa’s discrete integral equation method for collective theory, we have developed successfully an exact canonical momenta approach to one-dimensional neutron-proton systems and a velocity operator approach to three-dimensional neutron-proton systems [41]. Particularly in the latter approach, we, however, have restricted the Hilbert space to a subspace \(|>\) in which the vortex operator, \(\text{rot} \mathbf{v}(\mathbf{x})\), satisfies \(\text{rot} \mathbf{v}(\mathbf{x}) \geq 0\), where the velocity operator \(\mathbf{v}(\mathbf{x})\) is given as \(\mathbf{v}(\mathbf{x}) = -\nabla \phi(\mathbf{x}) + \lambda(\mathbf{x}) \nabla \psi(\mathbf{x})\), i.e., equation (2.1).

To describe the rotational velocity field of the nuclear fluid, we have introduced the Clebsch transformation. However, we did not yet fully clarify the vortex motion. It is still an important problem to be solved. We present some clues for the solution of such problems:

Firstly, the vorticity \(w\) (2.2) is expressed as \(\frac{i\hbar}{2\rho} (\nabla \Psi \times \nabla \Psi^* - \nabla \Psi^* \times \nabla \Psi)\). Since \(w\) is orthogonal with \(\nabla \lambda\) and \(\nabla \psi\), if we add a condition \((\nabla \lambda) \cdot (\nabla \psi) = 0\), the three vectors \(w, \nabla \lambda\) and \(\nabla \psi\) are orthogonal to each other. Under the condition, the absolute value of \(w\), \(|w|\) becomes maximum. Due to the Ziman transformation applied to the complex field operators \(\Psi\) and \(\Psi^*\), that condition is rewritten as \(\frac{i\hbar}{\rho} (\nabla \Psi - \nabla \Psi^*) \cdot \frac{\nabla \Psi \Psi^* - \Psi \Psi^*}{\Psi - \Psi^*}\). Then, \((\nabla \lambda) \cdot (\nabla \psi) = 0\) is changed to

\[
(\nabla \Psi - \nabla \Psi^*) \cdot (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi^*) = 0
= (\nabla \Psi) \cdot \Psi \nabla \Psi^* - (\nabla \Psi^*) \cdot \Psi \nabla \Psi^* - (\nabla \Psi) \cdot \Psi^* \nabla \Psi + (\nabla \Psi^*) \cdot \Psi^* \nabla \Psi.
\]

The first term \((\nabla \Psi) \cdot \Psi \nabla \Psi^*(= \Xi)\) vanishes, since \(\langle 2020|10\rangle = 0\). Thus \(\Xi\) has a form given below \(\Xi \propto \langle 2020|10\rangle R_0 \sum_{\mu \nu \mu' \nu' k} 2k|1\mu+k\rangle |1\mu+k2\mu'\rangle |1\mu''+k\rangle |2-\mu''1-k\rangle b_{2\mu}b_{2\mu'} b_{2\mu''} = 0\). (6.2)

The other terms also vanish. Then the above orthogonal condition is automatically satisfied. Eckart discussed an extension of the vorticity \(w\) to \(\nu(\nabla \lambda) \times (\nabla \psi)\), where \(\nu\) is some scalar function. The parameter \(\nu\) however, can be taken equal to one without loss of generality [42].

Next, consider the continuity equation, \(\dot{\rho} + \text{div} (\rho \mathbf{v}) = \dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla \rho) + \rho \text{div} \mathbf{v} = 0\), in which \(\text{div} \mathbf{v} = 0\) does not imply that both \(\dot{\rho} = 0\) and \(\nabla \rho = 0\). If the Lagrange differentiation for density satisfies \(\frac{D\rho}{Dt} = \dot{\rho} + (\mathbf{v} \cdot \nabla) \rho = 0\), it leads to \(\text{div} \mathbf{v} = 0\). In the BMM, the expression for the velocity
field \( v(x, t) = -\nabla \phi(x, t) \) plays a central role. Adding to this, the collective coordinates are expanded around the nuclear equilibrium and then the expansion coefficient of \( \phi \) reduces to \( \alpha \), i.e., \( \pi \). To go from classical fluid dynamics to quantum fluid dynamics by quantization, through \( v = -\nabla \phi \), what is essential is a canonical commutation relation for \( \phi \) and \( \rho \), 

\[
[\phi(x), \rho(x')] = i\hbar \delta(x-x').
\]

Since we assume a constant density, \( \rho = \rho_0 \), we can not see an apparent role for the above commutation relation. To make further development of the BMM, we naturally arrive at the idea of the expansion of \( \rho \) around \( \rho_0 \), i.e., \( \rho = \rho_0 + \rho' \) [43, 18, 19, 21].

Lastly, we prove the gauge invariance of the velocity operator \( \mathbf{v} \) expressed in terms of the Clebsch parameters as \( \mathbf{v} = -\nabla \phi - \frac{1}{\rho} \pi \nabla \psi \), i.e., (2.1) where we have used the relation \( \pi = -\rho \lambda \).

We consider the following gauge transformation with the generating function \( \omega(\pi', \psi) = \omega(\frac{\pi'}{\rho}, \psi) \):

\[
\phi' = \phi + \pi' \frac{\partial \omega}{\partial \pi'} - \omega, \quad \pi' = \pi + \rho \frac{\partial \omega}{\partial \psi}, \quad \psi' = \psi - \rho \frac{\partial \omega}{\partial \pi'}, \quad \rho \frac{\partial \omega}{\partial \rho} + \omega + \pi' \frac{\partial \omega}{\partial \pi'} - \omega = 0. \quad (6.3)
\]

This kind of the gauge transformation was proposed by Ito [44, 20]. Kambe also has formulated a variation of ideal fluid flows according to the gauge principle in the Clebsch solution [45]. Substituting the gauge transformation (6.3) into \( \mathbf{v}' = -\nabla \phi' - \frac{1}{\rho} \pi' \nabla \psi' \),

\[
\mathbf{v}' = -\nabla \left( \phi' + \pi' \frac{\partial \omega}{\partial \pi'} - \omega \right) - \frac{1}{\rho} \left( \pi + \rho \frac{\partial \omega}{\partial \psi} \right) \nabla \left( \psi - \rho \frac{\partial \omega}{\partial \pi'} \right) = -\nabla \phi - \frac{1}{\rho} \pi \nabla \psi = \mathbf{v}, \quad (6.4)
\]

where we have used the second and last relations of (6.3) and the gradient formula for \( \omega \) [20],

\[
\nabla \omega = \frac{\partial \omega}{\partial \psi} \nabla \psi + \frac{\partial \omega}{\partial \pi'} \nabla \pi' - \frac{1}{\rho} \pi' \frac{\partial \omega}{\partial \pi'} \nabla \rho. \quad (6.5)
\]

Thus we prove the gauge invariance of the velocity \( \mathbf{v} \). Following [20], let us separate \( \mathbf{v} \) as \( \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \). Putting (6.3) into the velocity \( \mathbf{v}_0' = -\nabla \phi' - \frac{1}{\rho_0} \pi' \nabla \psi' \) and using (6.5),

\[
\mathbf{v}_0' = -\nabla \left( \phi' + \pi' \frac{\partial \omega}{\partial \pi'} - \omega \right) - \frac{1}{\rho_0} \left( \pi + \rho \frac{\partial \omega}{\partial \psi} \right) \nabla \left( \psi - \rho \frac{\partial \omega}{\partial \pi'} \right)
\]

\[
= -\nabla \phi - \frac{1}{\rho_0} \pi \nabla \psi - \rho' \frac{\partial \omega}{\rho_0} \nabla \psi + \pi' \frac{\partial \omega}{\partial \pi'} \nabla \pi' - \frac{1}{\rho_0} \pi' \frac{\partial \omega}{\partial \pi'} \nabla \rho
\]

\[
= -\nabla \phi - \frac{1}{\rho_0} \pi \nabla \psi - \rho' \frac{\partial \omega}{\rho_0} \nabla \psi - \pi' \frac{\partial \omega}{\partial \pi'} \nabla \pi' + \nabla \left( \omega - \pi' \nabla \frac{\partial \omega}{\partial \pi'} \right) = \mathbf{v}_0 - \rho' \frac{\partial \omega}{\rho_0} \nabla \left( \phi - \phi' \right), \quad (6.6)
\]

where linearization of \( \frac{1}{\rho} \) is made as \( \frac{1}{\rho} \approx \frac{1}{\rho_0} \left( 1 - \frac{\rho'}{\rho_0} + \frac{\rho'^2}{\rho_0^2} + \cdots \right) \) and the first relation of (6.3) is used. The other velocity component \( \mathbf{v}_1 \) is expressed as \( \mathbf{v}_1 = \rho' \frac{1}{\rho_0} + \cdots \pi \nabla \psi \). Since the gauge invariance of the velocity \( \mathbf{v} \) is guaranteed, the velocities \( \mathbf{v}_0 \) and \( \mathbf{v}_1 \) must be invariant, respectively. From the right-hand side in the last line of Eq. (6.6), we must demand \( \nabla (\phi - \phi') = 0 \) which means \( \nabla \omega - \pi' \frac{\partial \omega}{\partial \pi'} - \pi' \nabla \frac{\partial \omega}{\partial \pi'} = 0 \). Further, using this relation and (6.5), \( \pi' \nabla \psi' \) is calculated as

\[
\pi' \nabla \psi' = \pi \nabla \psi + \rho' \frac{\partial \omega}{\rho_0} \nabla \psi = \pi \nabla \psi. \quad \text{Then, } \mathbf{v}_1' = \mathbf{v}_1 \text{ is proved.}
\]

Thus, we can also prove the gauge invariance of the velocity \( \mathbf{v} \), even if we make the separation of \( \mathbf{v} \) as \( \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \) under the linearization of \( \frac{1}{\rho} \) around \( \frac{1}{\rho_0} \) as given in the above linearization procedure.
In contrast to the present approach to the vortex motion in nuclei, we notice the papers [46, 47] in which Holtzwarth, Schütte and Eckart have derived fluid-dynamical equations of motion allowing for velocity fields with vorticity. They have parametrized the amplitude and the phase of a many-body wave function of a fermion system and have given a time-dependent variational derivation of nuclear fluid dynamics. It is may be interesting to compare the approaches of Holztzwarth and collaborators and of Ziman for deeper understanding of nuclear fluid dynamics.
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Appendix

A Detailed derivation of Eq. (5.11)

Using \( U = \cos \frac{f}{2} - i\sigma^a \omega^a \sin \frac{f}{2} \), \( \omega^a \equiv (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) given in the previous Section 5, \( dU \) is calculated as

\[
dU = -\frac{1}{2} \left( \sin \frac{f}{2} + i\sigma^a \omega^a \cos \frac{f}{2} \right) df - i\sigma^a \sin \frac{f}{2} d\omega^a,
\]

\[
d\omega^a = (\cos \theta d\theta \cos \varphi - \sin \theta \sin \varphi d\varphi, \cos \theta d\theta \sin \varphi + \sin \theta \cos \varphi d\varphi, -\sin \theta d\varphi).
\]

Further using \( U^{-1} = U^\dagger \) and (A.1), \( U^{-1}dU \) is computed as

\[
U^{-1}dU = \left( \cos \frac{f}{2} + i\sigma^a \omega^a \sin \frac{f}{2} \right) \left\{ -\frac{1}{2} \left( \sin \frac{f}{2} - i\sigma^a \omega^a \cos \frac{f}{2} \right) df - i\sigma^b \sin \frac{f}{2} d\omega^b \right\}.
\]

(A.2)

We are now in the stage to compute \( a = iTrn^a\sigma^a U^{-1}dU \) explicitly. The trace formulas \( Tr\sigma^a = 0 \) and \( Tr(\sigma^a\sigma^b) = 2\delta_{ab} \) are useful. For our aim, let us prepare the following trace formulas:

\[
-\frac{1}{4} iTrn^a\sigma^a (\sin f + 2i\sigma^b \omega^b) df = \frac{1}{2} Tr(\sigma^a\sigma^b) n^a \omega^b df = \delta_{ab} \hat{n}^a \omega^b df = \hat{n}^3 df = \cos \theta df,
\]

\[
\frac{1}{2} Trn^a\sigma^b sin f \omega^b = \frac{1}{2} Tr(\sigma^a\sigma^b) \hat{n}^a \sin f \omega^b df = \delta_{ab} \hat{n}^a \sin f \omega^b df = \sin f \hat{n}^3 = -\sin f \sin \theta d\theta,
\]

\[
Tr(\sigma^a\sigma^b\sigma^c) \hat{n}^a \omega^b \omega^c = Tr(\sigma^a \delta^{bc} + \sigma^a i\epsilon^{bcd} \sigma^d) \hat{n}^a \omega^b \omega^c = i\epsilon^{bcd} Tr(\sigma^a \sigma^d) \hat{n}^a \omega^b \omega^c = 2i\epsilon^{bcd} \hat{n}^a \omega^b \omega^c = 2i(\hat{\omega}^1 \hat{\omega}^2 - \hat{\omega}^2 \hat{\omega}^3) = 0,
\]

(A.3)

\[
-\frac{1}{2} iTrn^a\sigma^b\omega^c \delta_{bc} (1-\cos f) \omega^c = (1-\cos f) \frac{1}{2} i\cdot 2i\epsilon^{bca} \hat{n}^a \omega^b \omega^c = (1-\cos f) \frac{1}{2} i\cdot 2i\epsilon^{bca} \hat{n}^a \omega^b \omega^c = (1-\cos f) \frac{1}{2} \{ 2(-i)(1)(\hat{\omega}^1 \hat{\omega}^2 - \hat{\omega}^2 \hat{\omega}^3) \}
\]

\[
= (1-\cos f) \frac{1}{2} \{ -\sin \theta \cos \varphi (\cos \theta d\theta \sin \varphi) - \sin \theta \sin \varphi (\sin \theta \cos \varphi d\varphi) + \sin \theta \sin \varphi (\cos \theta d\theta \cos \varphi) + \sin \theta \sin \varphi (-\sin \theta \sin \varphi d\varphi) \}
\]

\[
= (1-\cos f) \sin^2 \varphi d\varphi,
\]

in which we take \( \hat{n}^a \) to point in the third direction. Gathering the above all formulas of (A.3), at last we acquire the desired expression for \( a \) (5.11) as

\[
a = \cos \theta df - \sin f \sin \theta d\theta - (1-\cos f) \sin^2 \theta d\varphi.
\]

(A.4)

As described by Jackiw-Pi [34], another formula for (A.4) in the Clebsch representation for the velocity field \( \mathbf{v}, \ \mathbf{v} = -\nabla \phi + \lambda \nabla \psi \), is given by

\[
a = d(-2\varphi) + 2 \left( 1 - \sin^2 \frac{f}{2} \sin^2 \theta \right) d\left\{ \varphi + \tan^{-1} \left( \tan \frac{f}{2} \cos \theta \right) \right\}.
\]

(A.5)

Inversely, from (A.5), we can easily derive (A.4) by noticing the differentiation given below

\[
d\left\{ \varphi + \tan^{-1} \left( \tan \frac{f}{2} \cos \theta \right) \right\} = d\varphi + \frac{1}{1 + \tan^2 \frac{f}{2} \cos^2 \theta} \left\{ \sec^2 \frac{f}{2} \left( \frac{f}{2} \right) \cos \theta dr - \tan \frac{f}{2} \sin \theta d\theta \right\}.
\]

(A.6)
References

[1] T. Marumori, J. Yukawa and R. Tanaka, Prog. Theor. Phys. 13 (1955) 442.
[2] T. Marumori and E. Yamada, Prog. Theor. Phys. 13 (1955) 557 (L).
[3] S. Nagata, R. Tamagaki, S. Amai and T. Marumori, Prog. Theor. Phys. 19 (1955) 495.
[4] S. Tomonaga, Prog. Theor. Phys. 5 (1950) 544; 13 (1955) 467; 482.
[5] S. Sunakawa, Y. Yoko-o and H. Nakatani, Prog. Theor. Phys. 27 (1962) 589; 600.
[6] S. Nishiyama and J. da Providência, Nucl. Phys. A 923 (2014) 51.
[7] S. Nishiyama and J. da Providência, Nucl. Phys. A 935 (2015) 1.
[8] A. Bohr and B.R. Mottelson, Nuclear Structure, Vol.II, Benjamin, Reading, Mass., 1975.
[9] P. Ring and P. Schuck, The nuclear many body problem, Texts and monographs in Physics (Springer-Verlag, Berlin, Heidelberg and New York 1980).
[10] D.J. Rowe and J.L. Wood, Fundamentals of Nuclear Models, Foundational Models, (World Scientific Publishing Co. Pte. Ltd., Singapore, 2010).
[11] T. Miyazima and T. Tamura, Prog. Theor. Phys. 15 (1956) 255.
[12] T. Tamura, Nuovo Cimento 4 (1956) 713.
[13] Y. Watanabe, Prog. Theor. Phys. 16 (1956) 1.
[14] S. Nishiyama and J. da Providência, Int. J. Mod. Phys. E24, 1550045 (2015).
[15] A. Bohr, Kgl. Danske Videnskab. Selskab. Mat.-fys., 26 (14) (1952) 1-40.
A. Bohr and B. R. Mottelson, ibid. 27 (16) (1953) 1-174.
[16] K. Alder, A. Bohr, T. Huus, B.R. Mottelson and A. Winther, Rev. Mod. Phys. 28 (1956) 432.
[17] A. Clebsch, Über eine allgemeine Transformation der hydrodynamischen Gleichungen, J. Reine Angew. Math. 54, 293-312 (1857); Über die Integration der hydrodynamischen Gleichungen, 56, 1-10 (1859).
[18] R. Krönig and A. Thellung, Physica 18 (1952) 749.
[19] J. M. Ziman, Proc. Roy. Soc. A219 (1953) 257.
[20] H. Ito, Prog. Theor. Phys. 13 (1955) 543.
[21] A. Thellung, Physica A. 19 (1953) 217; Helv. Phys. Acta. 29 (1956) 103.
[22] L. Landau, Journal of Physics 5 (1941) 71.
[23] M. A. Lamb, Hydrodynamics, Cambridge University Press, Cambridge, 1932.
[24] C.C. Lin, *Hydrodynamics of Helium II* in *Proceedings of the International School of Physics "Enrico Fermi,"* Course 21, edited by G. Careri (Academic, New York, 1963) 93-146.

[25] Aris Rutherford, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics* (Prentice-Hall, Inc. Englewood, Cliffs, N.J., 1962).

[26] V.A. Zakharov and E.A. Kuznetsov, *Physics-Uspekihi* 40 (1977) 1087.

[27] H.K. Moffatt, *J. Fluid Mech.* 35 (1969) 117.

[28] M.A. Berger and G.B. Field, *J. Fluid Mech.* 147 (1984) 133.

[29] V. Penna and M. Spera, *J. Math. Phys.* 30 (1989) 2778.

[30] M.E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, New York and London, 1957).

[31] A.R. Edmonds, *Angular Momentum in Quantum Mechanics,* Investigations in Physics (Princeton, New Jersey, Princeton University Press 1960).

[32] G.R. Allcock and C.G. Kuper, *Proc. Roy. Soc.* A231 (1955) 226.

[33] A. Jackiw, CRM Series in Mathematical Physics, *Lectures on fluid dynamics,* A Particle Theorist’s View of Supersymmetric, Non-Abelian, Noncommutative Fluid Mechanics and d-Branes, 2002 Springer-Verlag, New York, Inc.

[34] A. Jackiw and So-Young Pi, *Phys. Rev.* D61 (2000) 105015.

[35] E.Z. Kuznetsov and A.V. Mikhailov, *Phys. Lett.* A77 (1980) 47.

[36] S. Deser, A. Jackiw and S. Templeton, *Ann. Phys.* (N.Y.) 140 (1982) 372, 185 (1985) 406 (E) ; Reprinted from Volume 140 (1982) 372-411, 281 (2000) 409-449.

[37] Y. Nambu, *Phys. Lett.* 92B (1980) 327.

[38] M. Rasetti and T. Regge, *Physica* 80A (1975) 217.

[39] M. Rasetti and T. Regge, *Quantum vortices and Diff (R³)* in *Lecture Notes in Physics,* edited by H. Araki et al. (New York, Springer, 1984) vol. 201 311-320.

[40] M. Rasetti and T. Regge, *Quantum Vortices* in *Highlights of Condensed-Matter Theory,* edited by F. Bassani et al. (Compositori, Bologna, 1985) 748-766.

[41] S. Nishiyama and J. da Providência, *Int. J. Mod. Phys.* E25, 1650057 (2016).

[42] C. Eckart, *Phys. Fluids* 3 (1960) 421.

[43] D.D.H. Yee, *Phys. Rev.* 184 (1969) 196.

[44] H. Ito, *Prog. Theor. Phys.* 9 (1953) 117.

[45] T. Kambe, *Fluid Dyn. Res.* 40 (2008) 399.
[46] G. Holtzwarth and D. Schütte, *Phys. Lett.* **B73** (1978) 255.

[47] G. Holtzwarth and G. Eckart, *Z. Phys.* **A283** (1977) 219.