Mathematical Philology in the Treatise on Double False Position in an Arabic Manuscript at Columbia University

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Abstract

This article examines an Arabic mathematical manuscript at Columbia University’s Rare Book and Manuscript Library (or. 45), focusing on a previously unpublished set of texts: the treatise on the mathematical method known as Double False Position, as supplemented by Jābir ibn Ibrāhīm al-Ṣābī (tenth century?), and the commentaries by Aḥmad ibn al-Sarī (d. 548/1153–4) and Saʿd al-Dīn Asʿād ibn Saʿīd al-Hamadhānī (12th/13th century?), the latter previously unnoticed. The article sketches the contents of the manuscript, then offers an editio princeps, translation, and analysis of the treatise. It then considers how the Swiss historian of mathematics Heinrich Suter (1848–1922) read Jābir’s treatise (as contained in a different manuscript) before concluding with my own proposal for how to go about reading this mathematical text: as a witness of multiple stages of a complex textual tradition of teaching, extending, and rethinking mathematics—that is, we should read it philologically.

Keywords

Jābir ibn Ibrāhīm al-Ṣābī – Abū l-Futūḥ Aḥmad ibn al-Sarī – Arabic mathematics – History of Science – Heinrich Suter – Double False Position – algebra

“A woman dies, leaving her husband, a son, and three daughters.”¹ You are tasked with dividing the property among her heirs according to Islamic law.

¹ The example is from al-Khwārizmi’s Algebra, trans. Solomon Gandz, “The Algebra of Inheritance: A Rehabilitation of Al-Khuwarizmi,” Osiris 5 (1938): 327.
The rules in this case are that the husband inherits one quarter of the estate, and that in dividing what remains, the son’s share is twice each daughter’s share. Suppose her estate is 100 dinars. How much does the son inherit?

You have never studied basic algebra. Or if you have, pretend for a moment that you have not. You know how to add, subtract, multiply, and divide integers and even integral fractions. So to answer this question, you begin guessing. You guess that the son’s share is 50. Since the husband’s share is \( \frac{100}{4} = 25 \), and each daughter’s share is half of the son’s share, this would yield a total estate of \( 25 + 50 + 3 \times \frac{50}{2} = 150 \). Too large. What about if the son’s share is 20? Then the total estate would be \( 25 + 20 + 3 \times \frac{20}{2} = 75 \). Too small. Maybe 40? But then the estate would be \( 25 + 40 + 3 \times \frac{40}{2} = 125 \). Too large again. You could keep going, but you start to wonder if there is a better way to do this. Fortunately, you come upon a manuscript that includes a treatise that describes a method called “calculation by two errors” (\( \text{ḥisāb al-khaṭaʾayn} \)). You’ve already made three errors, so this seems promising. You read on.

The present article is about just such a manuscript and just such a treatise. The manuscript is New York, Columbia University, or. 45 (ca. thirteenth century).\(^2\) The treatise is the *Explication of the Demonstration of Calculation by Two Errors, Improved Edition* (\( \text{iṣlāḥ} \)) by Abū Saʿd Jābir ibn Ibrāhīm al-Ṣābī. “Calculation by two errors,” known in English as the method of Double False Position, appears as a minor chapter in the history of mathematics, especially when conceived as a linear history of progress from primitive problem-solving and limited understanding to sophisticated techniques and more complete theorems. Double False Position is a somewhat sophisticated technique but one that was at least at first glance entirely superseded by algebra.

To solve the above inheritance problem by Double False Position, we define \( x_1 \) and \( x_2 \) as the first and second guesses, \( y_1 \) and \( y_2 \) as the resulting outputs, where \( y = 100 \) is the desired output. The aim is to find \( x \) such that operating on \( x \) as stipulated in the problem yields the desired output \( y \). Based on the above calculations, we can assign these terms the following values: \( x_1 = 50, y_1 = 150; x_2 = 20, y_2 = 75 \). We further define two “errors” \( e_1 = y_1 - y = 50 \) and \( e_2 = y_2 - y = -25 \). Finally, we plug these values into the formula

\[
x = \frac{x_1 e_2 - x_2 e_1}{e_2 - e_1}
\]  

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\(^2\) See Gūrgīs ʿAwwād, “al-Makhṭūṭāt al-ʿarabiyyah fī dūr al-kutub al-Amīrikiyyah” [Arabic Manuscripts in American Libraries], *Sumer* 7 (1951): 262–63, esp. 262 (date, based on the script).
in order to obtain
\[ x = \frac{50 \cdot (-25) - 20 \cdot 50}{-25 - 50} = \frac{1250 + 1000}{75} = 30. \]

And indeed, \( 25 + 30 + \frac{30}{2} = 55 + 45 = 100 \), which was the deceased’s total estate as stipulated in the problem. So the son’s share is 30 dinars, making each daughter’s share 15 dinars. Double False Position has allowed us to come to this conclusion without knowledge of algebra.

Once a systematic algebra of polynomials (albeit restricted to positive rational numbers and quadratic equations) had been developed by al-Khwārizmī in the ninth century, one might even expect Double False Position to have been abandoned altogether as superfluous.

But it was not. Part of the reason must be that Double False Position can come in handy even if one knows algebra. As Randy Schwartz has pointed out, not only was it an accessible method for a wide range of tradesmen with limited education to use; for some types of problems, it is in fact a quicker and simpler method than first expressing the problem as an algebraic equation and then solving for \( x \). Schwartz’s example is from a twelfth-century Latin treatise and can be summarized as follows: you carry some apples through three gates but at each gate must give up half of what you are carrying plus two apples to the gatekeeper; at the end you have one apple; how many did you start with? In the time that you will take to write out the algebraic expression corresponding to this problem, I can make two guesses, run them through the procedure, and have an answer from Double False Position.

Indeed, the manuscript at the center of the present article contains not only a treatise on Double False Position with commentary but also Omar Khayyam’s treatise on algebra, which built on al-Khwārizmī and his successors to produce a more systematic treatment that included cubic equations. This perhaps

3 Roshdi Rashed, *The Development of Arabic Mathematics: Between Arithmetic and Algebra* (Dordrecht: Kluwer, 1994), 8–21.
4 Indeed, it still has a place in modern mathematics as a standard way to approximate solutions to equations whose algebraic solutions are unknown.
5 Randy K. Schwartz, ”Issues in the Origin and Development of Hisab al-Khata’ayn (Calculation by Double False Position),” in *Actes du huitième colloque maghrébin sur l’histoire des mathématiques arabes: Tunis, les 18–19–20 décembre 2004* (Tunis: Association Tunisienne des sciences mathematiques, 2004), 2–3. In this particular example, one might obtain the answer even faster by “working backwards”: \( [(1 + 2) \cdot 2 + 2] \cdot 2 + 2 = 36 \). But that is beside the point, which is that to solve this problem by algebra is exceedingly cumbersome compared to either of these numerical methods.
6 Rashed, *Development of Arabic Mathematics*, 43–50.
surprising juxtaposition offers the opportunity for the present inquiry, which is primarily focused not on the history of mathematics but on the history of \textit{mathematical philology}: the scholarly and textual practices used to preserve, communicate, and explore mathematics and its history.\footnote{For definitions of philology—or \textit{Zukunftsp hilologie}, to which this journal is devoted—and some of the stakes involved, see, e.g., Sheldon Pollock, “Philology and Freedom,” \textit{Ph ilological Encounters} 1 (2016): 4–30; Sheldon Pollock et al., eds., \textit{World Philology} (Cambridge MA: Harvard University Press, 2015). Such a philology liberates the intellectual historian from the distorted extremes of a seamless but wholly anachronistic history of linear progress on the one hand and a disconnected string of solipsistic “original” texts on the other. In its place, a historical philology, by highlighting the iterated attempts to understand and \textit{make present} with each new generation texts and ideas from the past, offers an intellectual history that emphasizes thinking subjects—whether author, reader, scribe, or commentator—and their interactions, between contemporaries and across time, potentially all the way up to our present day. Not only does such a philology seek to bridge the gap between historicism (meaning of a text when it was first composed) and presentism (what it means for me today). It also conceives of each thinking subject, at least potentially, as a fellow philologist, who likewise might have read texts—for his or her own purposes of course—as a historicist, as a presentist, and also, perhaps especially, with an eye to the intervening tradition.}

This article will move between several chronological layers. As its title makes clear, the treatise on Double False Position in question was written, at least in its current form, by Abū Saʿd Jābir ibn Ibrāhīm al-Ṣābī. In particular, it indicates that Jābir edited a preexisting text to improve its clarity, fill in gaps in its reasoning or exposition, or standardize its technical vocabulary.\footnote{The title indicates that it is a treatise, whose author remains unnamed, that Jābir subsequently revised, producing a new, revised edition (\textit{iṣlāḥ}) of the text. As Mohammed Abattouy has shown, the term \textit{iṣlāḥ} was typically used to describe the product of correcting, clarifying, and filling in the gaps in mathematical texts (often early translations from Greek into Arabic) that were faulty, unclear, or lacunose. This was often carried out by a scholar with technical, rather than linguistic, expertise, although \textit{iṣlāḥ} can also refer to revisions of a primarily stylistic or linguistic nature. See Mohammed Abattouy, “La tradition arabe de \textit{Maqāla fi al-mīzān}, un traité sur la théorie du levier attribué à Euclide,” \textit{Mirror of Heritage} (Ayene-ye Mirās): Quarterly Journal of Book Review, Bibliography and Text Information (Tehran), n.s., 4, no. 4 (2007): 67–104, esp. §1. I owe this reference to an anonymous reviewer.}

To judge from Jābir’s name, he may be the son of the Sabian physician and mathematician Abū Ishāq Ibrāhīm ibn Sinān ibn Thābit ibn Qurra (909–946), grandson of the famous mathematician, physician, astronomer, and translator Thābit ibn Qurra (born in Ḥarrān; active in Baghdad; d. 901).\footnote{Carl Brockelmann, \textit{Geschichte der arabischen Literatur}, 1st ed. (Weimar; Berlin: Emil Felber, 1898–1902), 1:219; Heinrich Suter, \textit{Die Mathematiker und Astronomen der Araber und ihre Werke} (Leipzig: Teubner, 1900), 69 = no. 162; Fuat Sezgin, \textit{Geschichte des arabischen Schrifttums}, 17 vols. (Leiden: Brill, 1967–2015), 5:254. For Abū Ishāq Ibrāhīm ibn Sinān, see Suter, \textit{Die Mathematiker und Astronomen}, 53–54 = no. 113; Sezgin, \textit{GAS}, 5:292–295. On the other hand, I am not aware of anything that would exclude identifying his father with other Sabians named...} If this identification should
prove correct, it would place Jābir in the tenth century. (Otherwise all we have is the *terminus ante quem* provided by the text’s commentators.) Interspersed with the original text is a commentary by the mathematician and philosopher Ibn al-Sarī (also known as Ibn al-Ṣalāḥ; from Hamadān; active in Baghdad; d. 1153–4),10 as well as a brief note by one Sa’d al-Dīn al-Hamadhānī. Works by both of these commentators appear elsewhere in the manuscript. As discussed in the following, Ibn al-Sarī’s commentary points out a fatal flaw in Jābir’s geometrical proof of the validity of the method of Double False Position; al-Hamadhānī explains a single aspect of Ibn al-Sarī’s commentary.

In the early twentieth century, this treatise was studied by the Swiss teacher and historian of mathematics, Heinrich Suter (1848–1922).11 Suter was primarily interested in the text as evidence that the method of Double False Position was known prior to the twelfth century, when it appears in Latin. His secondary interest was to evaluate the mathematical worth of the treatise; concurring with Ibn al-Sarī, he rated it quite low. (Suter subsequently gained access to a different Arabic treatise on Double False Position by the ninth/tenth-century Byzantine Christian scholar Qusṭā ibn Lūqā of Ba’labakk. He concluded that it was of sufficient worth to merit being published in German translation.)12

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10 Suter, *Die Mathematiker und Astronomen*, 120 = no. 287; Theodosius, *Sphaerica: Arabic and Medieval Latin Translations*, ed. and trans. Paul Kunitzsch and Richard Lorch (Stuttgart: Steiner, 2010), 2 (introduction), mentioned because one of the manuscripts of Theodosios’s *Sphairika* says that it was copied from an exemplar that was copied from an exemplar in Ibn al-Sarī’s own hand.

11 Heinrich Suter, “Einige geometrische Aufgaben bei arabischen Mathematikern,” *Bibliotheca Mathematica, 3rd ser.*, 8 (1907–8): 23–36, esp. §1 = 24–27; reprinted (with original pagination) in Heinrich Suter, *Beiträge zur Geschichte der Mathematik und Astronomie im Islam: Nachdruck seiner Schriften aus den Jahren 1892–1922*, ed. Fuat Sezgin, vol. 2, 2 vols. (Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften an der Johann Wolfgang Goethe-Universität, 1986), 217–230. On Suter’s life and work, see Julius Ruska, “Heinrich Suter,” *Isis*, 5, no. 2 (1923): 409–17.

12 Heinrich Suter, “Die Abhandlung Qostā ben Luqās und zwei andere anonyme über die Rechnung mit zwei Fehlern und mit der angenommenen Zahl,” *Bibliotheca Mathematica, 3rd ser.*, 9 (1908–9): 111–22; reprinted in Suter, *Beiträge: Nachdruck*, 231–242. Suter points out a flaw (albeit a less fatal flaw) in this treatise as well, concluding that it is much closer to being a successful proof of Double False Position, such that Jābir cannot have based his treatise on it.—Qusṭā can plausibly be described as a Byzantine Christian because he was a Chalcedonian Christian in communion with the Byzantine church and a native speaker of Greek; see Maria Mavroudi, “Greek Language and Education under Early
Now, in the twenty-first century, I revisit this treatise as a locus for understanding the aims and approaches of premodern mathematical writers, readers of mathematical manuscripts, and twentieth- and twenty-first-century historians of mathematics. First, I sketch the contents of the Columbia manuscript containing Jābir’s treatise (§1). Then I present an editio princeps (§2), translation (§3), and analysis (§4) of the treatise. After considering Suter’s reading of this treatise (§5), I offer my own historical, mathematical, and philological reading of the text, its manuscript context, and its commentators (§6), arguing for the value of such a mathematical treatise—fatally flawed but nevertheless copied and read—for intellectual history, not only the history of mathematics, but also of how mathematical texts were read.

1 The Columbia Manuscript

New York, Columbia University, or. 45 is a medieval codex containing a significant collection of mathematical texts. It includes texts on astronomy and engineering, but the primary focus is geometry. Most of its texts are in Arabic; one is in Persian. A paper codex, most of it was written by a single scribe (Scribe 1) in a neat naskh (with some Persian features such as the shape of initial hā’), typically with 19 lines per page (texts no. 2–18, on fols. 15v–128v). Text no. 1 was written by a different scribe (Scribe 2), with no dots, also typically with 19 lines per page (fols. 1r–14r). The last portion of the manuscript, apparently added later, was written much more informally, in less regular scripts: one hand (Scribe 3) wrote items nos. 19 and 21 with no margins (fols. 129r–137v, 144r–146v), and another (Scribe 4) wrote no. 20 with slight margins (139v–143r).

Two previous descriptions of the manuscript are known to me: a page devoted to the manuscript’s contents in Gūrgīs ʿAwwād’s catalog of Arabic manuscripts in American libraries,13 and the series of typewritten cards in the unpublished card catalog of Arabic and other Islamicate manuscripts housed at Columbia’s Rare Book and Manuscript Library (RBML).14 ʿAwwād’s list of the

13 ʿAwwād, “Dūr al-kutub al-Amrīkiyyah,” 262–63.
14 I am grateful to Jane Siegel, Rare Books Librarian at RBML, for introducing me to Columbia’s Arabic manuscript collection and pointing me to this card catalog when I first arrived at Columbia in the Fall of 2015. Since then I have benefited from her knowledge of the collection’s history. She kindly provided me with photographs of the manuscript so that I could continue to work on it after my departure from New York.
manuscript’s contents is more complete but still quite limited. In what follows, I present a much improved and elaborated description.

My aim here, I should note, is not simply to improve our bibliographic knowledge by supplementing Sezgin’s handbook on the evidence of this manuscript (that is a happy side-effect). Instead, I include a description of the manuscript as an integral part of the project to construct a philology that treats manuscripts not merely as warehouses to be mined for the texts they contain but also as evidence for the intellectual milieux that produced and studied them. Like the anonymous treatise on Double False Position, Jābir’s attempt to improve it, Ibn al-Sarī’s critique of that attempt, and al-Hamadhānī’s brief gloss on that critique (and, we might be tempted to add, Suter’s account and critique of the whole assemblage), the Columbia manuscript represents one of many historical layers of interest in and engagement with a particular mathematical problem.\(^{15}\) To interpret the manuscript as such, first we must read it.

ʿAwwād dated the script to the seventh Hijrī century (thirteenth century CE). Modern bibliographical notes added to the manuscript itself place it in the thirteenth or fourteenth century CE. The manuscript includes Arabic translations of ancient Greek texts, but many of its texts date from the eleventh and especially twelfth centuries.

Two brief notes, one on the treatise on Double False Position, are ascribed to one Saʿd al-Dīn Asʿad ibn Saʿīd al-Hamadhānī. This may be someone associated with the production of this manuscript or of the collection it contains. The manuscript refers to him as one refers to an acquaintance: “the wise judge (al-qāḍi al-ḥakīm) Saʿd al-Dīn Asʿad ibn Saʿīd al-Hamadhānī, may God preserve his high rank.”

The Persian treatise (no. 8) and features of Scribe 1’s handwriting would tend to situate the manuscript’s production in Iran. Its apparently close connection to al-Hamadhānī, as well as the prominence of another author from the same city, Ibn al-Sarī, points to Hamadān as a possible place of the manuscript’s production and subsequent use.

[Scribe 2: first text only.]

1. Menelaos (fl. ca. 95 CE), *On Spherics (Kitāb Mānālāwus fī l-Kuriyyāt, 1v–14v)*. Greek original not extant, but various Arabic versions survive.\(^{16}\)

\(^{15}\) It thus expands the possibilities of philology on “plane 2,” to use the terminology of Pollock, “Philology and Freedom,” 19–26.

\(^{16}\) The author’s name is partly damaged, which prevented ʿAwwād from naming its author. The undamaged portion of the name (a mīm, the tops of two alifs, the tail of a waw or rāʾ, and a sīn) suggests that Menelaos is the author, and this is confirmed by the close similarity of the beginning of the text (as well as its diagrams) with the beginning of Menelaos’s text in *Kitāb Mānālāwus fī l-Askkāl al-kuriyyah*, in the edition (iṣlāḥ) of Aḥmad ibn Abi
2. Question asked by Shams al-Dīn, the Niẓāmī chief emir, of Sharaf al-Dīn Bahāʾ al-Islām Ḥujjat al-Zamān Muẓaffar ibn Muḥammad al-Muẓaffar al-Ṭūsī in Hamadān AH 50<6> [= 1112–13 CE] about Dividing a Square ...

17 Masʾalah saʾalahā Shams al-Dīn amīr al-umarāʾ al-Niẓāmīyyah ʿan al-imām al-ajall al-awḥad al-ʿālim Sharaf al-Dīn Bahāʾ al-Islām Ḥujjat al-Zamān Muẓaffar ibn Muḥammad al-Muẓaffar al-Ṭūsī ... bi-balad Hamadhān sanata <sittah> wa-khamas-miʾah ḥijrīyyah ʿan murabbaʾa mutasāwī al-aḍlāʾ. The word sittah was omitted by the manuscript's scribe by haplography (since the previous word, sanah, has the same consonantal skeleton). I supply it from the catalog entry of the Leiden manuscript that contains this same text (and was used by Suter): P. de Jong and M.J. de Goeje, eds., Catalogus codicum Orientalium Bibliothecae Academiae Lugduno Batavae, vol. 3 (Leiden: Brill, 1865), 71. Likewise, the word I have printed as ḥujjah is from the Leiden catalog; this word in the Columbia manuscript is spelled ḥujuḥ without vowels or diacritics. The final letter could be understood as a tāʾ, used (in Persian fashion) in place of a tāʾ marbūṭah.

3. Ibn al-Haytham (354–430/965–1039), On Deriving the Altitude of the Pole (Maqālah fi istikhrāj irtifāʿ al-quṭb, 18v–24v).

18 Sezgin, GAS, 5:358–374, esp. 366, work no. 5. Latin and German translations have appeared in print.

4. Abū Rashīd (?), Useful Shapes for the Book of Archimedes (Ashkāl nāfīʿah fī kitāb Arshimīdis, 24v–25v).

19 Sezgin, 5:156, referring only to ʿAwwād’s description of this very manuscript. The text begins:

It ends:

Saʿd al-Harawī (d. ca. 990–1000 CE), appearing after the editor’s preface and the words qāla Mānālāwus al-muhandis, in Leiden, Univ. Library, or. 399, fol. 83v (available at http://hdl.handle.net/1887.1/item:1567441). Further research might establish which of the Arabic versions described by Sezgin (GAS, 5:161–163) is contained in the Columbia manuscript, e.g., whether it is al-Harawi’s edition stripped of its preface, or the otherwise nonextant edition of al-Māhānī upon which al-Harawi’s is based, or one of the others.—In response to this note, an anonymous reviewer kindly alerted me to the recent critical edition and English translation of this very text in Menelaus’ “Spherics”: Early Translation and al-Māhānī, al-Harawī’s Version, ed. and trans. Roshdi Rashed and Athanase Papadopoulos (Berlin: de Gruyter, 2017), 399–483. The editors identify the text in the Columbia manuscript as the fragment of an early anonymous translation of Menelaos’s Spherics, contrasting it with al-Harawi’s iṣlāḥ (and indeed with the original translation upon which al-Māhānī’s iṣlāḥ was itself based), of which they also provide a critical edition and translation (485–777).
5. Archimedes (ca. 287–212f BCE), “ascribed,” *The Circle’s Area and the Relation of the Diameter to the Circumference* (*Qawl mansūb ilā Arshimīdhis fī misāḥat al-dāʾirah wa-nisbat al-quṭr ilā l-muḥīṭ*, 25°–30°).20

6. Aristarchus (fl. 280 BCE), *On the Bulks of the Sun and Moon and Their Distances* (*Kitāb Aristā<r>khus fī jirmay al-shams wa-l-qamar wa-abʿādihimā, 30°–48°*).21

7. ʿAlī ibn Aḥmad al-Nasawī (11th century), *Book of the Brilliant concerning the Examples of the Comprehensive Tables* (*Kitāb al-Lāmiʿ fī amthilat al-Zīj al-Jāmiʿ*, 49°–75°).22 The text begins with a detailed preface describing the purpose of the work.23 The text itself, as advertised in the preface, is a series of examples meant to illustrate and supplement each chapter

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20 Sezgin, 5:130–131, no. 2. Extant in Greek: Κύκλου μέτρησις, ed. J. Heiberg.
21 Sezgin, 6:74–75. Part of the Alexandrian collection known as “the Little Astronomy.” Extant in Greek: Περὶ μεγεθῶν καὶ ἀποστημάτων ἡλίου καὶ σελήνης, ed./trans. Thomas Heath (Oxford, 1913).
22 Sezgin, 6:245–246, no. 1, citing only ‘Awwād’s description of the Columbia manuscript. Note that this work is *not* entitled *Risālah fī maʿrifat al-taqwīm wa-l-asturlāb* as Sezgin, transcribing ‘Awwād’s description of the text, reports. There is no title at the beginning of the text, but al-Nasawi explicitly mentions the title he has given his work at the end of his preface (fol. 49v6). Nor is the work primarily about calendars or astrolabes. Instead, it is a commentary on a set of astronomical tables by Abū l-Ḥasan Kūshyār ibn Labbān al-Jīlī (as al-Nasawī notes in his preface, fol. 49r14), the Comprehensive Tables (*al- Zīj al-Jāmiʿ*, compiled ca. 1000 CE); on which see Edward Stewart Kennedy, “A Survey of Islamic Astronomical Tables,” *Transactions of the American Philosophical Society* 46, no. 2 (1956): 125, no. 9, and further 156–57 = §10; Sezgin, GAS, 6:246–248.
23 Al-Nasawi begins his preface (fol. 49r) by dividing astral scientists (*ʿulamāʾ al-nujūm*) into four cumulative levels (*ṭabaqāt*): those who know (1) calendars and the astrolabe; (2) basic astrology like planetary and zodiacal attributes and the astrological verdicts (*aḥkām*) that result from their combination—these he calls the “verdicticians” (*aḥkāmiyyūn*); (3) basic calculations of astral positions and use of astronomical tables and calendars; (4) mathematical astronomy (*hayʾah*) and geometrical proofs of the validity of such calculations—the province of the “complete astronomer” (*al-munajjim al-tāmm*). Most people of “our time,” continues al-Nasawī, only reach the first two levels. Now, [Abū ʿAbdallāh Muhammad ibn Sinān ibn Ḧabīr al-Šābi] al-Battānī’s astronomical tables (the *Sabian Tables*, ed. Nallino; see Kennedy, “Survey,” 132–33, no. 55) are rightly regarded as the most accurate, but unfortunately they are built upon the Roman (Byzantine) and Hijrī calendars, which are difficult to use in combination with the Persian calendar “because of leap years and fractions” (*bi-sabab al-kabāʾis wa-l-kusūr*). And so the late Kūshyār ibn Labbān made his astronomical tables, the Comprehensive Tables (*al- Zīj al-Jāmiʿ*), using the Persian calendar. This makes them much easier to use. Continuing on the next page (fol. 49v), al-Nasawi explains that he has produced a supplement to these tables (or to an abridgment of the tables in 85 chapters) in which he provides explanatory examples for tricky or unclear chapters as a sort of commentary (*sharḥ*). This material is on the third level of astronomical achievement, he explains, but made possible by his knowledge of *al-hayʾah* (the fourth level).
of *al-Zīj al-Jāmiʿ* by Kūshyār ibn Labbān. Based on internal evidence, al-Nasawī’s text can be dated to 1047 CE. This should lead us to modify Sezgin’s estimate that al-Nasawī was active in the last quarter of the tenth century and the first quarter of the eleventh.

Persian treatise entitled *Treatise of Ornamentation on Calculating the Table of the Thirty* (*Risālat al-tazyīn fi ḥisāb jadwal al-thalāthīn*, 76v–81r, rest of page blank).

9. ‘Umar al-Khayyām (d. 1123), *Algebra* (*al-Jabr wa-l-muqābalah*, 81r–109r, rest of page blank). Ed. Rashed and Jabbār.

10. anonymous and Jābir ibn Ibrāhīm al-Ṣābī [with commentary by Aḥmad <ibn> al-Sarī and al-qāḍī al-ḥakīm Sa‘d al-Dīn As‘ad ibn Sa‘īd al-Hamadhānī], *Explication of the Demonstration of Calculation by Two Errors, Improved Edition by Abū Sa‘d Jābir ibn Ibrāhīm al-Ṣābī* (*Īḍāḥ al-burhān ʿalā ḥisāb al-khaṭaʾayn, islāḥ Abī Sa‘d Jābir ibn Ibrāhīm al-Ṣābī, 109r–113r*).

The beginning of the text is indeed about calendars and conversions between them, but this is presumably because *al-Zīj al-Jāmiʿ* began with this topic. Al-Nasawī’s text soon proceeds to discussing the geometry and mathematical astronomy necessary for the construction of zīj es. Al-Nasawī does not always give an example; for “part 4, chapter 1,” he writes, “You need no example because it is obvious” (*lā taḥtāju ilā mithālin li-annahu ẓāhirun*; fol. 55v6).

At one point (fol. 53v), al-Nasawī gives an example of how to convert a Hijrī date to other formats; the example he gives is of his own present day, which he gives as 12 Ṣafar 439 (8 August 1047 CE). A later hand pointed out that the text was composed in 439 AH, in a note in the top margin on the first page of the text (fol. 49r). The creator of the catalog entry housed at Columbia’s RBML seems to have misread this note as 429 AH (rather than 439 AH) and misinterpreted it as the year when the text was copied rather than when it was written.

Indeed, an anonymous reviewer has informed me that a recent study (Abū l-Ḥasan ʿAlī ibn Ahmad al-Nasawi, *Kitāb al-tajrīd fī uṣūl al-handasah*, ed. Muṣṭafā Mawālidī [London: Mu’assasat al-Furqān li-l-Turāth al-Islāmī, Markaz Dirāsāt al-Makhṭūṭāt al-Islāmiyyah, 2016], 17) establishes al-Nasawī’s birth date as 393/1002 and places his death after 473/1080. I have not been able to consult this book.

The title, reported by Awwād, appears at the end of the preface, fol. 78v.

Omar Khayyam, *Rasāʾîl al-Khayyām al-jabriyyah*, ed. and trans. Roshdi Rashed and Ahmad Jabbar (Aleppo: Jāmiʿat Ḥalab, Maʿhad al-Turāth al-ʿArabī, 1981), 1–73, updated (as pointed out by an anonymous reviewer) in three publications by Roshdi Rashed and Bijan Vahabzadeh: *Al-Khayyam mathématicien* (Paris: Albert Blanchard, 2000); *Omar Khayyam, the Mathematician* (New York: Bibliotheca Persica Press, 2000); *Riyāḍiyāt ʿUmar al-Khayyām* (Beirut: Markaz Dirāsāt al-Waḥdah al-ʿArabiyyah, 2005).

See nos. 17 and 18 below.
11. Aḥmad ibn Ibrāhīm al-Sijzī (?), Introduction to Crafting an Instrument by which Distances Can Be Known (Muqaddimah li-ṣanʿat ālatin tuʿraf bihā al-abʿād, 113r–115r).

12. Aḥmad ibn Muḥammad ibnʿ Abd al-Jalīl al-Sijzī (fl. ca. second half of the 10th century), Treatise on the Knowledge of Straight and Bent Lines (Risālah fī maʿrifat al-khaṭṭayn al-mustaqaʿm wa-l-munhaṭnī, 115r–118v).

13. Muḥammad ibn Aḥmad ibn Muḥammad ibn Kashnah(?) al-Qummī (d. ca. first half of the 11th century), On the Possibility of Two Lines Existing Which Become Indefinitely Closer but Do Not Meet (... imkān wujūd al-khaṭṭayn alladhayn yaqtaribān abadan wa-lā yaltaqiyān, 119r–121v).

14. Ibn al-Haytham, Treatise on Obtaining the Altitude of Standing Objects, Mountains, and Clouds (Maqālah li-l-shaykh Abī ‘Alī ibn al-Haytham fī maʿrifat irtifāʿ al-qāʾimah wa-aʿmidat al-jībāl wa-rtifāʿ al-ghuyūm, 121v–122r).

15. comments of al-Ḥakīm al-fāḍil Saʿd al-Dīn {ibn} Ibn al-Sarī (122r1–3). This is the same person who added a brief note to Ibn al-Sarī’s commentary on Jābir’s “improved edition” (iṣlāḥ) of the treatise on Double False Position appearing earlier in this manuscript (no. 10).

16. a problem from Abū Naṣr al-Fārābī (d. ca. 950), Music fann 1, maqālah 1 (Masʾalah dhakarahā Abū Naṣr al-Fārābī fī l-maqālah al-ūlā min al-fann al-awwal fī l-mūsīqā, 122v–125r2).

17. Abū l-Futūḥ ibn al-Sarī (d. 548/1153–4), Problem, on constructing “a triangle whose sides are equal to the diameter” of a given circle (Masʾalah min kalām Abī l-Futūḥ ibn al-Sarī, 125r3–126r1). This is the same Ibn al-Sarī who commented on the treatise on Double False Position (no. 10 above).

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30 Sezgin, GAS, 5:333, no. 26, citing only ‘Awwād’s description. ‘Awwād only mentioned the author’s nisbah, so Sezgin treats this as a work of Aḥmad ibn Muḥammad ibn ‘Abd al-Jalīl al-Sijzī (explicitly named as the author of the next text in the Columbia manuscript), but the scribe here seems deliberately to have named someone else—who is not clear to me. Someone by this name studied with ʿAbd al-Qāhir al-Jurjānī (11th century): Yaḥqūq al-Rūmī, Muʿjam al-udabāʾ, ed. Iḥsān ʿAbbās, 6 vols., continuous pagination (Beirut, 1993), 187, no. 53. This is probably a coincidence.

31 Sometimes called al-Saḥārī, but here (on fol. 119r6) the name is marked with diacritics, as al-Sijzī.

32 Sezgin, GAS, 5:333, no. 25, citing only ‘Awwād’s description of the Columbia manuscript.

33 Sezgin, Geschichte des arabischen Schrifttums, 5:336. Following Suter, Sezgin considers al-Qummī a “younger contemporary” of Aḥmad ibn Muḥammad ibn ‘Abd al-Jalīl al-Sijzī.

34 This “ibn” is a mistaken addition that should be suppressed to make the name correspond with that of the same personage in no. 10.

35 See n. 10 above.
18. Abū l-Futūḥ ibn al-Sarī, *Treatise on Constructing an Equilateral Triangle Inside another Equilateral Triangle of a Given Proportion to the First* (Qawl li-Abī l-Futūḥ Aḥmad ibn Muḥammad ibn al-Sarī fī ʿamal muthallath mutasāwī l-ṣadlaʾ fī dākhil muthallath mutasāwī l-ṣadlaʾ lahu nisbah ilayhi mafriḍah ...). Again, this is the same Ibn al-Sarī.

[Scribe 3: new, messier hand, fills up whole pages with no margin or regular ruling.]

19. Commentary on the First Book of Euclid’s *Uṣūl*.36 Incipit: Ikhtilāf wuqūʿ li-šaḵlī B min al-maqālāh al-ūlā min Kitāb al-Uṣūl li-Uqlīdis ... (129r–137v).

[Scribe 4: new hand, likewise unruled but leaves a slight margin. Begins in brown ink then after a few lines changes to black ink, with figures in brown ink. Much of it is without dots.]

20. Banū Mūsā ibn Shākir (9th century), extracts, *What we need by way of introduction to the Book of Apollonios from the introductory remarks of the Banū Mūsā at the beginning of their improved edition* (Mā naḥtāj ilā taqdimihi ʿalā Kitāb Abuluniyūs mimma qaddamahu Banī [sic] Mūsā ʿalayhi fī ʿadr [islāhi] him, 139v–143r).37

[Scribe 3, again.]

21. Notes, beginning: Idhā ukhrīja quṭru GEB min dāʾirat BGDZ ʿalā istiqāmatihi ... (144v–146v).

[Scribe 3: new, messier hand, fills up whole pages with no margin or regular ruling.]

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36 As ʿAwwād calls it: Taʿlīqāt ʿalā l-maqālāh al-ūlā min Kitāb al-Uṣūl li-Uqlīdis.

37 At the very bottom of fol. 143r, it looks like the scribe has written tammat, signalling the end of the text, suggesting that what follows are other notes. As an anonymous reviewer pointed out to me, this text was extracted from Muqaddamāt Kitāb al-makhrūtāt li-Banī Mūsā, ed./trans. in Apollonius de Perge, Coniques: Texte grec et arabe, ed. Roshdi Rashed, multiple vols. (Berlin: de Gruyter, 2008), 1:500–533, beginning after the preface at Lemma 1 (509). This edition used the Columbia manuscript, among others, to establish the text; see 498. The folio numbers I have indicated do not match Rashed’s; this may be because Rashed’s numbering follows page numbers rather than the folio numbers that have quite recently been added in pencil in the top left corner of each recto.
2 Text

In what follows, I present an *editio princeps* of the treatise on Double False Position as it appears in New York, Columbia University, or. 45 (no. 10; fols. 109r–113v). As with the sketch of the manuscript’s contents in the previous section, I carry out this work, philological in the narrower sense (textual criticism), in the spirit and in the service of a more ambitious philology that shows curiosity for and invests resources in texts preserved by a textual tradition that may at first sight appear irrelevant to modern critics.

Within the text, diagrams (see Figure 1) illustrate the geometrical definitions and proofs on fol. 110v after line 3 (Diagram 1: two-dimensional diagram illustrating the geometric proof in ¶2), fol. 111r in the upper margin extending into line 1 (Diagram 2: vertical line $A—G—D—B$), fol. 111r 8–12 with the lines of text written around it (Diagram 3: vertical line $D—A—G—B$), and fol. 112v in the midst of the last line of the page (Diagram 4: horizontal line $A—D—G—B$).

There is also a space of about 5 lines left blank at the end of fol. 111r after 14 lines

![Figure 1](image-url)

*Figure 1* Diagrams in the treatise on Double False Position in New York, Columbia University, or. 45, no. 10, on fols. 109v, 111r, 112v. Top: *Diagram 1*. Middle left: *Diagram 2*. Middle right: *Diagram 3*. Bottom: *Diagram 4*. 
of text, as if for a diagram that was never added; at the top of the next page a modern hand pencilled in the heading ḥisāb al-khaṭaʾāyn, but the original text simply began at the top of the page (with the beginning of Ibn al-Sarī commentary) with no new heading.

The commentaries by Ibn al-Sarī and al-Hamadhānī were originally copied as a single block of text visually undifferentiated from the main text of the treatise. Marginal and interlinear labels were subsequently added to distinguish commentary from focus text (Figure 2). This suggests that an ancestor of the Columbia manuscript had the commentaries (or at least Ibn al-Sarī’s commentary) in the margin; that a more proximate ancestor that descended from the first then incorporated the marginalia into the body of the text, probably rubricated or visually differentiated from the focus text some other way; and

![Figure 2](image.jpg)

**Figure 2** New York, Columbia University, or. 45, fol. 111v: example of later marginal and interlinear labels distinguishing the commentary (ḥāshiyah) from the focus text (matn).
that a subsequent scribe, perhaps the scribe of the Columbia manuscript, then copied this ancestor without rubrication.

In the Arabic text, I supply *hamzas* where necessary to suit modern orthography. I do not emend the consonant alternation skeleton without indicating it (e.g., with angle brackets), except that wherever the manuscript says "الخائِينَ" I write "الخاءَينَ".

In the translation, I render the Arabic letters used to represent geometric points with the English letter corresponding to the *abjad* order, i.e., "ا" is *A*, "ب" is *B*, "ج" is *G*, "د" is *D*, "ذ" is *Z*, "ح" is *H*, "ط" is *Y*, "ي" is *K*, "ل" is *L*, "م" is *M*, "ن" is *N*, "س" is *S*, "ع" is *O*, "ف" is *F*.

(ص ٩٠١ ب)

إيضاح الأبرهان على حساب الخاطئ

إصلاح أبي سعد جابر بن إبرهيم الصابي

(١) إذا أردت حساب شيء من فنون هذا الباب فافتيق مقداره من الجنس الذي تساندل عنه، أي مقدار كان، كالعدد أو الخط أو السطح أو غير ذلك مما يقع عليه الحساب، وتم ذلك المقدار للذين الأولى. ثم افعل به مثل ما قبل ك في السؤال، فإن أتيك لك أن تصب، فهو الجواب، والإصابة على هذه السبيل لا يعمد بها. وإن أخطأ ما أردت، فقد مقدار ما أخطأت به وسما الخاطئ الأول. وإن كان العمل أنتج لك زيادة بذلك المقدار عما يوجهه السؤال، فسمه الخاطئ الزائد. وإن كان أنتج نقصانًا، فسمه الخاطى الناقص. ثم اقتضب بعد ذلك مقدارًا آخر مكافأة للأسئلة، وسما الخاطئ الثاني، وإن كان أيضًا زائدًا، فسمه الزائد، وإن كان ناقصًا، فسمه الناقص. ثم اقتضب بعد ذلك مقدارًا آخر مكافأة للأسئلة، وسما الخاطئ الثاني. وإن كان أيضًا زائدًا، فسمه الزائد، وإن كان ناقصًا، فسمه الناقص. ثم اقتضب بعد ذلك مقدارًا آخر مكافأة للأسئلة، وسما الخاطئ الثاني. إنه أضلاعه ومسماً ذلك عمله الجزء، وهو الذي عليه القسمة. ثم اقتضب بعد ذلك مقدارًا آخر مكافأة للأسئلة، وسما الخاطئ الأول. وهو الخاطئ من أكثرهما. إن كان الخاطئ من أكثرهما. فَأَنْيَاً أَقلُّ خَطَأً وَأَنْيَاً أَقلُّ خَطَأً.}

(٢) تعليق ذلك: كُل خط يقسم بثلاثة أقسام، فإن ضرب ذلك بَخَطَاء في الأوسط من أقسامه مع <...> ضرب التسمس الأوسط مع أحد القسمين الذي عن جنبيه مجموعين خط في التسمس الأوسط تُبيين مع القسم الآخر مجموعين خط واحد. مثال ذلك أن خط آب مقسم بثلا أقسام، وهي "آخِينَ"، فأقول فإن ضرب آب بِخَطَاء في جهد، وضرب آب آب، وجمع آب، وهو "خالِينَ". برهانه...
أن نرسم على خط أب مربع أب١٠، متساوي الأضلاع، قائم الزوايا، ونخرج فُطره، وهو أب١٠، ونخرج من نقطة تمودا على ضلع أب١٠ يقطع قطر المربع على نقطة ط٦، ويلمِّع ضلع ر٦ على نقطة ج، ونُعبر (٤) على خط ط٦، ضع موازياً لضلع جب١٠، وننجم سطح أب٦ المستطيل، ونخرج خط ر٦ لاستقامة، حتى يبلغ ضلع أب٦ المستطيل على نقطة ل. فن البين أن سطح جب مربع متساوي الأضلاع قائم الزوايا، وكذلك سطح ط٦ مربع متساوي الأضلاع قائم الزوايا، وسطحاً متساويين، لأنهم المنتميان لنذان جنبيَّ قط مربع جب١٠، وإذا كان ذلك كذلك، فإننا نقول أن سطح آ٨ لر٦ جموعين مساويين، لسطح آ٨، برهانه أن سطح ط٦ مساو٦ لسطح ء٦ فتجعل سطحي ط٦ لر٦ مشتركة، فيكون سطحاً آ٨ لر٦ مساوية، لسطح آ٨، لكن سطح آ٨ هو من ضرب خط أب في خط جب١٠ في خط جب١٠، لأن خط جب١٠ مسأخي خط جب١٠، وسطحاً لد٦ هو من ضرب خط آ٨ (ص ١١٠) في خط جب١٠، لأن خط جب١٠ مسأخي خط جب١٠، فنام سطح آ٨، فإنه من ضرب آ٨ في جب١٠. لأن خط جب١٠ مسأخي خط جب١٠. فضرب خط أب٦ في جب١٠ مع ضرب آ٨ (رسم) (١) في جب١٠، مثل ضرب خط ط٦ في خط جب١٠، وذلك ما أردنا أن نبين.

(٣) كل ثلاثة خطوط أو أعداد مختلفة فإن ضرب الأعظم على الأوسط مع الأصغر مع الأصغر في زيادته الأوسط على الأصغر مع الأصغر في زيادته الأوسط مع الأصغر مثل ضرب الأوسط على الأوسط مع الأصغر، لكي تكون ثلاثة خطوط أو أعداد مختلفة عليها آ٨ أو آ٨ أو آ٨، وأعظمها، آ٨ أصغرها. فأقول إن ضرب الأعظم، وهو آ٨، في زيادة الأعظم مع الأصغر، وهو جب١٠. فنام سطح آ٨، كموجعين مثل ضرب الآ٨ الأوسط، هو آ٨، في زيادة الأعظم مع الأصغر، وهو جب١٠. برهانه أن خط آ٨ أو عدد آ٨ قسم ثلثين أقسام، وهي آ٨ تقضي دب١٠، ففعل ما بيننا فيما تقدم يكون ضرب أب٦ بأب٦ في جب١٠ مع ضرب آ٨ (رس٢) (١) في جب١٠، وذلك ما أردنا أن نبين.

(٤) وإذا قدمبوب (٤) هذه المعاني، فتفضع ثلاثة خطوط أو أعداد مختلفة، وهي دب١٠ أب١٠ جب١٠، ولكن أولما دب١٠ أصغر من آ٨، وجب١٠ أصغر من آ٨، فزيادة دب١٠ على آ٨ هي دب١٠، ونقياس دب١٠ عن آ٨ هو أج، مسأخي دب١٠ جب١٠ مختلفة، فضرب الأعظم، وهو دب١٠، في زيادة الأعظم على الأصغر، وهو أب٦، مع ضرب الأصغر، وهو جب١٠، في زيادة الأعظم على الأوسط، وهو آ٨، مثل ضرب الأصغر على الأوسط، وهو آ٨، في زيادة الأعظم على الأصغر، وهو جب١٠. (٥) فصاحب (رس٣) حساب الخطأين لما كان يلتسم متطابقاً بمجلة آ٨ هاهنا، فأخذ: ١دب١٠ في شماعة جب١٠، وأخذ مقداراً ما بينهما، وهو دب١٠، ومقداراً الأول الزائد. ثم رفع فاقتضب مقدارآ آخر، فوق آ٨ جب١٠، فوجدته ناقصاً عن آ٨، فأخذ مقداراً ما بينهما، وهو آ٨، ومقداراً الثاني الناقص. ثم ضرب الذي أخذه أولاً، وهو يسميه المال الأول، وهو في مثالنا هذا دب١٠، في الخطأ الثاني، وهو آ٨، ثم ضرب الذي يسميه المال الثاني، وهو جب١٠. . .
حاشية السري، ج ١: قال أحمد السري إن هذا الكلام الذي يذكره أبو سعد الصافي من أن صاحب حساب الخاطئين لما كان يستمس مطولاً بمثلة آب فأخذ ذهبًا، فوجد زائداً عليه، وأخذ مقداراً ما بينهما، وهو دًا» واكب كتابه إلى آخر كتابه ليس هو الذي يستعمله صاحب حساب الخاطئين، وذلك لأنه إذا طلب آب ووجد ذهب زائداً عليه وأخذ مقداراً ما بينهما، وهو دًا، وجب أن يكون آب المجهول معلوماً من غير ضرب أو قسمة، وذلك لأن دب معلوم وقد أخذ منه دًا المعلوم، فتبقى آب المعلوم...

حاشية السري، ج ٢: لا يحتاج فيه إلى عمل آخر، وهذا الرجل فإنما يستخرج بهذا العمل ١٠ المفروض له في المسألة ١٠ لا المطلوب. مثل ذلك آننا التسنا منه معرفة أي مال إذا زدنا عليه يصفي وثلث كان أحد عشر، فكأنه فرض لأحد عشر آب، ثم ضرب عدداً، وهو تسعة، فزاد عليه يصف وثلث، فأصل ذلك سنة عشر وتسع، وتفسيره عدد دب، ثم قاس دب إلى الأحد عشر، وهو آب، وجد دب زائداً على الأحد عشر خمسة وتسع من غير ضرب أو قسمة. وذلك لأن عدد الأول هو أربعة، وهو أصغر، وهو أصغر من القدر الذي تم فيه.

حاشية السري، ج ٣: فكانت السبعة وتسع فرضه، ثم قاسه إلى الأحد عشر، وجد من خلال ذلك أنهه أحد عشر، وهو أصغر، وهو أصغر، وهو أصغر من القدر الذي تم فيه.

حاشية السري، ج ٤: فلما كا الخاطئ مختل ين أحدهما زائد والآخر ناقص، جمع هذين المضروبين، فكان المجموع مثل ضرب الأوسط، وهو آب، في زيادة الأعظم على الأصغر، وهي جدة، ثم ضربه دب إلى دب، الذي هو جمع الخاطئين، فجعل القسمة عليه، أعني قسمة المضروبين، فخرج له من القسمة مقدار آب، وهو المطلوب.

إلى ١، وأيضاً في يكن على الوجه...

حاشية السري، ج ٥: ولا يحتاج فيه إلى عمل آخر. وهذا الرجل فإنما يستخرج بهذا العمل ١٠ المفروض له في المسألة ١٠ لا المطلوب. مثل ذلك آننا التسنا منه معرفة أي مال إذا زدنا عليه يصفي وثلث كان أحد عشر، فكأنه فرض لأحد عشر آب، ثم ضرب عدداً، وهو تسعة، فزاد عليه يصف وثلث، فأصل ذلك سنة عشر وتسع، وتفسيره عدد دب، ثم قاس دب إلى الأحد عشر، وهو آب، وجد دب زائداً على الأحد عشر خمسة وتسع من غير ضرب أو قسمة. وذلك لأن عدد الأول هو أربعة، وهو أصغر، وهو أصغر من القدر الذي تم فيه.

حاشية السري، ج ٦: فكانت السبعة وتسع فرضه، ثم قاسه إلى الأحد عشر، وجد من خلال ذلك أنهه أحد عشر، وهو أصغر، وهو أصغر، وهو أصغر من القدر الذي تم فيه.
فصاحب حساب الخطأين، لما كان مطلوبه بمثلث أب٨١، فأخذ أولاً دب٨١ ويسمى المثال الأول، فوجد زائداً.

"حاشية السيري، ج٤: "نصف، والعدد الثاني هو الأربعة لا السبعة والتلث، وهذه الأعداد الثلاثة، أغنى على نسبة العدد الثلاثة، وأغنى على نسبة التسعة والأربعة والعدد الجهول المطلوب منا. وله فعلاً طريق الخطأين وقسمنا المتبقي على (ص ١٣١ ب) بعدئن نخرج لنا الجهول: ستة، فسق بأن كلامه ليس يبيان حساب الخطأين، وكذلك في باقي الشكلين الذين بعده، فافهم ذلك، أغنى أنه تلقوه!" يمثل هذه الكلام.

"حاشية الهمذاني": قال القاضي الحكم سعد الدين سعد بن سعيد الهذاني أدام الله علوه: طريق الخطأين أن يضرب بدلاً للتبض التسعة التي هي على نسبته في أب٨١، وبدل بـج٤ الأربعة التي هي على نسبته في أب فيكون مجموعهما ١٠٥، ثم ٢٣ يقسمه على الجزء، هو سبعة وסדس، فيخرج المطلوب.

"النافث الشرعي": Pace the المقدار آخر، فوقع له جدب، فسماه المال الثاني، ووجده أيضاً زائداً على ما يطلب وهو دب، فأخذ المقدار ما أخطأه، وهو جد، وسمي الخطأ الثاني الزائد أيضاً. ثم ضرب المال الثاني، وهو جد في الخطأ الأول، وهو أب، وأسقط من ذلك ضرب المال الأول، وهو دب، في الخطأ الثاني، وهو جد، فاقترض حديثة زائدة. ففي له مقدار مساوٍ لضرب آب في جد، فيجعل قيمة هذه البقية على جد٢٣، الذي هو فضل ما بين الخطأ الأول والخطأ الثاني، فخرج له من القسمة مقدار آب، وهو المطلوب، الذي كان يتبسم عليه.

(٧) وأيضاً فليكن على الوجه الثالث كل واحد من دب جدب أصغر من آب، فيكون أيضاً خطوط أو أعداد آب دب جدب الثلاثة مختلفة، واي أعمدها ودب الأوسط، وجدب الأصغر، فدجاج هو زيادة الأوسط على الأصغر، واي هو زيادة الأعظم على الأوسط، وجا هو زيادة الأعظم على الأصغر (رسم ٤) فعلى ما يرتب فيما تقدم (ص ١٣١) يكون ضرب آب في دب مع ضرب دب في آب مجموع مثل ضرب دب في دب في دب، فليحوز ذلك. وصاحب حساب الخطأين هاهنا أيضاً لما كان يتبسن وجود آب، فأخذ أولاً دب وسماء المال الأول، وأخطأ فيه بدأ، فسمى آب الخطأ الأول الناقص. ثم رجع فاقترض مالاً ثانياً، فافتقه له جدب وأخطأ فيه يدأ، فسمى جد الخطأ الثاني الناقص أيضاً. ثم ضرب دب، وهو المال الثاني، في دب، وهو الخطأ الأول، ففي له مقدار مساوٍ لضرب آب في دب، فإذا قسم هذه البقية على دب الذي هو فضل ما بين الخطأ الأول والخطأ الثاني، خرج له من القسمة آب، وهو المطلوب الذي كان يتبسن، وذلك ما أردنا أن نبين.

فم ولحم الله رب العالمين، وصلواته على سيدنا محمد وآلله أجمعين.
Translation

Explication of the Demonstration of Calculation by Two Errors, Improved Edition by Abū Sa‘d Jābir ibn Ibrāhīm al-Ṣābī

1. If you wish to calculate this sort of thing, then you will come up with an amount of the kind about which you inquire, whatever amount it may be, such as number, line, surface, or other things that can be calculated. That amount becomes the first estate. Then operate upon it as you were instructed in the question. If you happen to be right, then that is the answer. Getting it right this way is unreliable. If it errs from what you were seeking, find the amount by which you erred, and call it the first error. If the operation yielded an excess by that amount above what the question requires, then call it the excessive error; if it yielded a deficit, then call it the deficient error. Then after that come up with another amount different from the first, and call it the second estate. Do the same to it as you did to the first estate. If it errs, find the amount of the error and call it the second error. If it too is excessive, call it excessive; if deficient, call it deficient. Then look, and if the two errors are both excessive or both deficient, displace [i.e., subtract] the lesser from the greater; but if they are different, one excessive and the other deficient, then take their sum. Call the

38 I translate māl here literally as the “estate.” Here it refers to the unknown quantity x that one is seeking to find by Double False Position (not, as is frequent in Arabic algebra, the square of the unknown, x²).
result of either operation the part; this will be the divisor. Then multiply the first estate by the second error, and the second estate by the first error. Look, and if you summed the two errors, then sum these two [products] too; and if you subtracted the lesser of the two errors from the greater of the two, then subtract the lesser of these two [products] from the greater of the two. Either way, divide the result by the part. The result is the answer.

2. The justification of this method: Each line is divisible into three segments. That line in its entirety multiplied by its middle segment, plus \( \text{the product of multiplying the middle segment plus one of the two adjacent segments joined into a line by the middle segment itself plus the other segment joined into a single line} \). For example, the line \( AB \) is divided into three segments, namely \( AG, GD, DB \). I’m saying that \( AB \) in its entirety multiplied by \( GD \) and \( AG \) multiplied by \( DB \), summed together, are congruent with the two lines \( GD, AG \) summed together, which is \( AD \), multiplied by the two lines \( GD, DB \) summed together, which is \( GB \). The proof of this is for us to draw, upon the line \( GB \), an equilateral right quadrangle \( GBEZ \); determine its diagonal, which is \( GE \); extend from point \( D \) a vertical line upon side \( GB \) such that it intersects the square’s diagonal at point \( T \) and meets side \( ZE \) at point \( H \); place line \( YTK \) on \( T \) parallel to side \( GB \); complete the oblong surface \( AE \) and extend line \( YTK \) straight until it meets the side of oblong surface \( AE \) at point \( L \). [See Figures 1 and 3.] It is clear that the surface \( GT \) is an equilateral right quadrangle [a square], and so is the surface \( TE \). The two surfaces \( DY \) and \( TZ \) are equal because they are the two complements that are on either side of the diagonal of the square \( GBEZ \). If that is so, then we say that the two surfaces

39. Literally, “that over which the division” will take place.
40. Call them \( a, b, c \).
41. \( b(a + b + c) = ab + b^2 + bc \).
42. The text here appears corrupt; for \( ma' \) perhaps read \( mithl \), “is like,” i.e., equal to.
43. \((b + c)(b + a) = b^2 + bc + ab + ac \). As the text stands, it is not a complete sentence. If we emend \( ma' \) to \( mithl \), it still isn’t quite right because the first half of the equation would be missing a term: \( ac \).
44. I.e., \( AB \times GD + AG \times DB = (GD + AG) \times (GD + DB) = AD \times GB \).
45. I.e., a square.
46. Here, rectangular in particular.
47. Cf. Euclid, \textit{Elements} 1.43; cited by Linda Hand Noel, “The Fundamental Theorem of Algebra: A Survey of History and Proofs” (EdD diss., Oklahoma State University, 1991), 26 (reading “43” for “4”). In particular, the wording here (لأبهما تمما عن جنبية قار مربع جبهر) is reminiscent of the eighth/ninth-century Arabic translation of Euclid’s \textit{Elements} by al-Ḥajjāj (on which see Sezgin, \textit{GAS}, 590), ed. R.O. Besthorn and J.L. Heiberg (Copenhagen, 1897), 126:166. For the case of a square, this result is visually apparent, but the theorem applies more generally to parallelograms.
AY, LZ taken together are equal to the surface AH. The proof is that surface DY is equal to surface TZ, so we consider\textsuperscript{48} surfaces AT, LZ to be a shared \textlt{portion?}.\textsuperscript{49} Surfaces AY, LZ are thus equal to surface AH. But surface AY is from the multiplication of line AB by line GD because line GD is equal to line YB;\textsuperscript{50} and surface LZ is from the multiplication of line AG by line DB because line LK is equal to line AG and line KZ is equal to line DB.\textsuperscript{51} As for surface AH, it is from the multiplication of AD by GB because line GB equals line DH.\textsuperscript{52} And so the multiplication of line AB in its entirety by line GD, along with the multiplication of AG [diagram 1] by DB is congruent to the multiplication of line AD by line GB.\textsuperscript{53} Q.E.D.

3. For any three different lines or numbers,\textsuperscript{54} the product of the greatest and the excess of the intermediate above the least,\textsuperscript{55} along with the product of the least and the excess of the greatest over the intermediate,\textsuperscript{56} summed together, is congruent to the product of the intermediate and the excess of the greatest above the least.\textsuperscript{57} Let there be three different lines or numbers

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Arabic diagram 1 redrawn using the corresponding Latin letters}
\end{figure}

\textsuperscript{48} najʿal.
\textsuperscript{49} I.e., AH and AY + LZ each include AT + LZ. Their remaining portions are, respectively, TZ and DY.
\textsuperscript{50} I.e., \(AY = AB \times YB = AB \times GD\).
\textsuperscript{51} I.e., \(LZ = LK \times KZ = AG \times DB\).
\textsuperscript{52} I.e., \(AH = AD \times DH = AD \times GB\).
\textsuperscript{53} \(AB \times GD + AG \times DB = [AY + LZ = AH] = AD \times GB\).
\textsuperscript{54} \(a > b > c\).
\textsuperscript{55} \(a(b - c)\).
\textsuperscript{56} \(c(a - b)\).
\textsuperscript{57} \(b(a - c)\).
$AB, AD, AG$, where $AB$ is the greatest of them and $AG$ is the least.\footnote{I am saying that the greatest, $AB$, multiplied by the excess of the intermediate above
the least, namely $GD$, along with the least, $AG$, multiplied by the excess of the
greatest above the intermediate, namely $DB$, summed together, is congruent
with the intermediate, $AD$, multiplied by the excess of the greatest above the
least, namely $GB$. The proof of this is that the line or number $AB$ is divided into
three sections, $AG, GD, DB$. According to what we have shown above \[\S2\],
$AB$ in its entirety multiplied by $GD$, the intermediate, along with \[\text{Diagram 2}\] $AG$ multiplied by $DB$ is congruent to $AD$ multiplied by $GB$. Q.E.D.}

4. Since these concepts have been made clear \(?\), let us posit three different
lines or numbers, $DB, AB, GB$. But first, $DB > AB$, and $GB < AB$, so the
excess of $DB$ above $AB$ is $DA$,\footnote{DB − AB = DA.} and the deficiency of $GB$ from $AB$ is $AG$.\footnote{AB − GB = AG.}
The lines $AB, DB, GB$ are distinct. Thus the greatest, $DB$, multiplied by the excess
of the intermediate over the least, which is $AG$, along with the least, $GB$
multiplied by the excess of the greatest over the intermediate, which is $DA$, is
congruent to the intermediate, $AB$, multiplied by the excess of the greatest over the
least, which is $GD$.\footnote{I.e., $DB \times AG + GB \times DA = AB \times GD.$}

5. The author \[\text{Diagram 3}\] of Calculation by Two Errors, when he sought
the desired result in the place of $AB$ here, he took $DB$ and found it in excess
over the former. Taking the amount between the two, $DA$, he called it the first,
excessive error. Then he went back and came up with another amount, so now
he had $GB$. He found it short of $AB$, so he took the amount between them, $AG$
and called it the second, deficient error. Then he multiplied what he had first
taken—which he calls the first estate—by the second error, $AG$. Then he mul-
tiplied the one that he calls the second estate, namely $GB$ ...

[ca. 5 blank lines]

[\[\text{Fol. 111v}\] \[\text{Comment, part 1.}\] Aḥmad <ibn> al-Sarī said: This wording that Abū Saʿd
al-Ṣābī mentions, that “the author of Calculation by Two Errors, when he sought
the desired result in the place of $AB$ here, he took $DB$ and found it in excess over
the former. Taking the amount between the two, $DA$ ...” and so on to the end of
what he wrote is not what the author of Calculation by Two Errors in fact uses.
This is because if he seeks $AB$ and finds $DB$ in excess over it and chooses some
amount between the two, namely $DA$, then the unknown $AB$ must become known]
without multiplication or division, since $DB$ is known, and the known $DA$ is subtracted from it, leaving the known $AB$ ...

[Jābir, ¶5 continued.] ... by the first error, which is $DA$. Since the two errors are different, one of them excessive and the other deficient, he summed these two products of multiplication ($maḍrūbayn$). The sum was congruent to the intermediate, $AB$, multiplied by the excess of the greatest over the least, namely $GD$, [by which] he meant $DG$, which is the sum of the two errors. Then he performed the division over it—I mean the division of the sum of the two products ($maḍrūbayn$). From the division came out the amount $AB$, which is the desired result.

6. And also, let there be, in the [second] place ...

[Comment, part 2.] ... and for that [i.e., finding $AB$] he needs no other operation. Thus what this man is in fact doing with this operation is computing the amount stipulated for him in the question, not the desired result. Suppose for example we ask him for knowledge of what estate is such that if we increase it by half of itself plus a third of itself, it becomes eleven. Next he stipulates that eleven is $AB$, then stipulates [as his first guess] a number, say nine, and added to it half of itself plus a third of itself, resulting in sixteen and a half. Next he stipulates that this is the number $DB$ then compares $DB$ to eleven, which is $AB$, and finds [it] in excess over eleven by five and a half [which he assigns] to the number $DA$. And so he calls $DA$ the first error—really, I swear! Then he stipulates [another] number, say four, and adds to it half of itself plus a third of itself ...

[Jābir, ¶6 continued.] ... [in the] second [place], $DB$ and $GD$ [read: $GB$], each greater than $AB$, such that the lines or numbers $AB$, $GD$ [read: $GB$], $DB$ ...

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63 The word ‘second’ only appears below, after the intervening comment.
64 Reading bi-hādhā l-ʿamal for bi-hādhā lī-l-ʿamal.
65 I.e., find $x$ where $x + \frac{x}{2} + \frac{x}{3} = 11$. (By algebra: $\Rightarrow \frac{11}{6}x = 11 \Rightarrow x = 6.$)
66 I.e., he says let $AB = 11$, and let, $x_1 = 9$, then calculates $9 + \frac{9}{2} + \frac{9}{3} = 16\frac{1}{2}$.
67 I.e., let $DB = 16\frac{1}{2}$.
68 I.e., $DB - AB = 16\frac{1}{2} - 11 = 5\frac{1}{2}$.
69 I.e., he says let $x_2 = 4$, then calculates $4 + \frac{4}{2} + \frac{4}{3} = 7\frac{1}{3}$.
70 The phrase “in the second place” (ʿalā l-wajhi l-thānī) straddles the intervening comment; this block of focus text begins with al-thānī.
are all three different, with $DB$ the greatest of them and $AB$ the least of them.\footnote{I.e., let $DB > GB > AB$. The reading $GD$, though it appears twice, must be a scribal error for $GB$ (as appears in the next line).} And let the excess of the intermediate, $GB$, above the least, $AB$, be $GA$; and let the excess of the greatest, $DB$, above the least, $AB$, be $AD$.\footnote{I.e., $GD = GB - AB$ and $DB - AB = AD$.} On account of the foregoing, the greatest, $DB$, multiplied by the excess of the intermediate above the least, which is $GA$, along with the product of $AB$ ...

[Comment, part 3.] ... and so it becomes seven and a third. He posits that it is the number $GB$ then compares it to eleven and finds that it’s short of [eleven] by three and two thirds [which he assigns] to the number $GA$, which he calls the second error. He multiplies seven and one third, which is $BG$, by $AD$, the first error, and divides the sum of the two dividends\footnote{This rather interventionist emendation is meant to make the sentence make sense both grammatically and mathematically. On mathematical grounds, this is clearly the meaning that the text originally conveyed; I base the wording of this emendation on how the text describes similar terms in other equations.} by $GD$, the sum of the two errors. Thus he obtains $AB$, which is the eleven posited to begin with, not the desired result. Thus he does not calculate by the method of two errors because the first number is nine, not sixteen ...

[\textit{Jābir, ¶6 continued.}] ... which is the least, multiplied by $DG$, the excess of the greatest over the least, <is congruent to the multiplication-product of the intermediate,>\footnote{I.e., $DB \times GA + AB \times DG = BG \times AD$.} which is $BG$, by the excess of the greatest over the least, which is $AD$.\footnote{\textit{murtafaʿayn}, lit., “raised [numbers].”} Let that [amount] be remembered. Now, the author of \textit{Calculation by Two Errors}, since the result he was seeking was in the place of (\textit{bi-manzilat}) $AB$, he first took $DB$ and called it the \textit{first estate} and found that it was in excess ...

[Comment, part 4.] ... and a half. The second number is four, not seven and a third. These three numbers—I mean eleven, sixteen and a half; and seven and a third—are proportional to the \textit{other} three numbers—I mean proportional to nine, four, and the unknown number asked of us. If we had operated according to the method of two errors and had divided the dividend by \textit{[fol. 112v]} the same $GD$, then we would have obtained the unknown: six. It has thus become clear that what he says tells us nothing about calculation by Two Errors, and the same goes for the two figures that come after it. Therefore understand this, I mean that this kind of talk is a bunch of nonsense (\textit{?}).

[Comment by al-Hamadhānī.] The wise judge Saʿd al-Dīn Asʿad ibn Saʿīd al-Hamadhānī, may God preserve his high rank, said: The method of Two Errors
is to multiply, in place of BD, nine, which is related to it, by AG; and in place of BG, four, which is related to it, by AD, so that the sum of the two is 55.\textsuperscript{76} Then he divides it by the part, which is nine and a sixth, and the desired result emerges.

[Jābir, ¶6 continued.]\textsuperscript{77} ... above AB by the amount DA, and he called DA the first, excessive error. Then he went back and tried another amount, coming up with GB, which he called the second estate. It too was found to be in excess above what was sought, namely AB. So he took the amount by which he had erred, which is AG, and it was called the second error, also excessive. Then he multiplied the second estate, GB, by the first error, AD, and he subtracted from that the product of the first estate, DB, multiplied by the second error, GA, for the errors in this case are both excessive.\textsuperscript{78} So he is left with two amounts equal to the product of AB multiplied by GD [read: GB?] . He carries out the division of this remainder by GD [read: GB?], which is the surplus between the first and second errors, and so he obtains from the division the amount AB, the desired result that he sought to know.

7. Also, in the third place, let DB, GB each be less than AB, such that the lines or numbers AB, DB, GB are also all three different, where AB is the greatest, DB is the intermediate, and GB is the least. Then DG is the excess of the intermediate over the least; AD is the excess of the greatest over the intermediate; and GA is the excess of the greatest over the least. [Diagram 4] According to what we have demonstrated in the foregoing, //fol. 113r// AB × DG + GB × DA summed together is congruent to DB × DA. Let that [amount] be remembered. At this point too, the author of Calculation by Two Errors, since he was seeking to find AB, first took DB and called it the first estate. Using it, he erred by DA, so he called DA the first, deficient error. Then he went back and tried a second estate, and he happened to get GB, in which he erred by GA, so he called GA the second error, also deficient. Then he multiplied DB, the second estate, by DA, the first error, and he was left with an amount equal to AB × DG. When he divided this remainder by DG, which is the surplus between the first error and the second error, he obtained AB from the division, being the result which he was seeking. Q.E.D.

The end. Praise be to God, lord of the worlds, and his blessings upon our sayyid Muhammad and all of his family.

\textsuperscript{76} The manuscript expresses this number using decimal ‘Arabic’ numerals.

\textsuperscript{77} The manuscript indicates the continuity between this text and the previous portion of the matn with a small circle that appears above both the last word of that portion (zāʾidan) and the first word of this continuation (ʿalā).

\textsuperscript{78} I.e., this is why he subtracted here instead of adding.
4 Analysis

The text does not explicitly distinguish between the original (anonymous) text and Jābir’s revisions and additions. Nevertheless, ¶4 begins by indicating that the foregoing text (¶1–3) may require further elaboration, and then ¶5 explicitly refers to the author of the Calculation by Two Errors, which seems to be a short version of the title of the work that Jābir undertook to revise and improve. This strongly suggests that Jābir’s contribution begins there, probably with ¶4 and definitely with ¶5. As for the comments by Ibn al-Sarī and Saʿd al-Dīn al-Hamadhānī, these are clearly labeled in the text.

In the present section, I will provide a rather detailed mathematical paraphrase and analysis of the treatise. Though mathematicians and historians of mathematics may find it excessive to spell out every step, my hope is that this will make explicit more of my interpretive reasoning. In other words, I aim to foreground the philology—the self-critical interpretation of a text and a textual tradition—involving in rendering a medieval mathematical text into modern mathematical modes of expression, rather than elide it.

4.1 Method (¶1)

Let \( f(x) = a_1x \pm a_2x \pm \cdots \pm a_Nx \pm b_1 \pm b_2 \pm \cdots \pm b_P \), where \( N, P \in \mathbb{Z}^+ \) and \( x, a_1, \ldots, a_N, b_1, \ldots, b_P \in \mathbb{R}^+ \).

Find \( x \) such that \( f(x) = y \) for some \( y \in \mathbb{R}^+ \).

Let \( x_1 \) be the first guess (\( māl \), “estate”). Let \( y_1 = f(x_1) \). If \( y_1 = y \), then \( x = x_1 \).

Lucky guess. If not, then let \( e_1 = y_1 - y \) be the first error. If \( y_1 > y \), then \( e_1 \) is an excessive error (i.e., \( e_1 > 0 \)). If \( y_1 < y \), then \( e_1 \) is a deficient error (i.e., \( e_1 < 0 \)).

Let \( x_2 \) be the second guess. Let \( y_2 = f(x_2) \). Assuming \( y_2 \neq y \) (so that the answer \( x \) is not simply \( x_2 \)), let \( e_2 = y_2 - y \) be the second error, excessive if \( y_2 > y \) and deficient if \( y_2 < y \).

Now if \( e_1 < 0 \) and \( e_2 < 0 \) or \( e_1 > 0 \) and \( e_2 > 0 \) (i.e., if \( e_1e_2 > 0 \)), then let the ‘part’ (\( juzʾ \)) be \( j = |e_2 - e_1| \). Otherwise (if \( e_1e_2 < 0 \)), \( j = |e_1| + |e_2| \). (In either case, we can express this as \( j = |e_2 - e_1| \), since when \( e_1e_2 < 0 \), \( |e_2 - e_1| = |e_1| + |e_2| \).) Compute \( x_1e_2 \) and \( x_2e_1 \). If \( e_1e_2 < 0 \), then sum them together: \( |x_1e_2| + |x_2e_1| \). (Since \( x_1 > 0 \) and \( x_2 > 0 \) by assumption, for the text does not employ negative numbers, this is just \( |x_1e_2 - x_2e_1| \).) If \( e_1e_2 > 0 \), then subtract: \( |x_1e_2 - x_2e_1| \). (Thus, either way we are finding \( |x_1e_2 - x_2e_1| \).) Now divide by \( j \) to obtain

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79 Thus the words ‘excessive’ and ‘deficient’ are used to represent a positive and negative result in the absence of the concept of negative numbers.
Using modern algebraic computation (including the concept of negative numbers), it is trivial to justify this method by expressing $x$ in terms of $x_1, x_2, f(x_1), f(x_2)$ then expressing $f(x)$ as $ax + b$ (i.e., reducing it to linear and constant terms) and reducing the result to

$$x = \frac{y - b}{a},$$

which is the algebraic solution to the equation

$$ax + b = y.$$

Perhaps more intuitively, working in the other direction, the method of Double False Position can be derived from a basic result of linear algebra, namely that two points define a line, whose slope is thus known (see Figure 4). Once we know $(x_1, y_1)$ and $(x_2, y_2)$—by choosing guesses $x_1$ and $x_2$ arbitrarily then calculating the corresponding outputs $y_1$ and $y_2$—we can express them in terms of the two (possibly negative) errors $e_1 = y_1 - y$ and $e_2 = y_2 - y$: $(x_1, y + e_1)$ and $(x_2, y + e_2)$. The line's slope is then

$$a = \frac{(y + e_2) - (y + e_1)}{x_2 - x_1} = \frac{e_2 - e_1}{x_2 - x_1}.$$

Now we start at point $(x_1, y_1)$. Since $(x, y)$ is on the same line, we know that

$$\frac{y - y_1}{x - x_1} = \frac{-e_1}{x - x_1},$$

also $= a$, so

$$-e_1 = \frac{e_2 - e_1}{x_2 - x_1} (x - x_1)$$

or

$$x - x_1 = \frac{-e_1 (x_2 - x_1)}{e_2 - e_1}.$$

As a result,
\[ x = \frac{-e_1(x_2 - x_1) + x_1(e_2 - e_1)}{e_2 - e_1} = \frac{x_1 e_2 - x_2 e_1}{e_2 - e_1}. \]

But of course this is not how the text proceeds.

### 4.2 Geometrical Proof of a Relation between Line Segments (¶2)

Let \( AB \) be a line (segment) subdivided by two points along it, \( G \) and \( D \): \( A—G—D—B \). The resulting line segments are related as follows:

\[ AB \times GD + AG \times DB = (GD + AG) \times (GD + DB) = AD \times GB \quad (3) \]

**Proof:** Construct diagram 1 (see Figures 1 and 3). The rectangles \( DY \) and \( TZ \) are equal because they are complements about the diagonal of the square \( GBEZ \) [Euclid, *Elements* 1.43]. This lets us equate the gnomon \( BAMZKY \) [supposing we label the lower-right corner of the diagram \( M \)]—which the text calls \( AY \) plus \( LZ \)—with rectangle \( AH \), since the only difference between the two is that \( AH \) contains \( TZ \) rather than \( DY \), but as we just saw, \( DY = TZ \). The rectangle \( AY \) can be expressed as \( AB \times YB = AB \times GD \), and rectangle \( LZ \) is \( LK \times KZ = AG \times DB \), so the gnomon is \( AB \times GD + AG \times DB \). On the other hand, the rectangle \( AH \) is \( AD \times DH = AD \times GB \). Therefore, \( AB \times GD + AG \times DB = AD \times GB \). Q.E.D.

Suter says that this step is flawed because Jābir unnecessarily restricts his result by using a square rather than a rectangle in the proof’s geometrical construction. But this part of the text is not purporting to be the entire proof; it is simply proving a geometrical relation between lines and the numbers corresponding to their lengths. It is only when we arrive at ¶5 that Suter’s critique hits home. Indeed, it is there that Ibn al-Sarī critiqués Jābir—a critique that I believe amounts to the same one that Suter makes.
4.3 Generalize This Result to Any Three Numbers (¶3)
Let \( a > b > c > 0 \), where \( a, b, c \in \mathbb{R}^+ \). Then

\[
a(b - c) + c(a - b) = b(a - c).
\] (4)

In today's algebra, this is trivial to prove. The text, however, offers a geometric proof that rests on the proof in ¶2:

Assign the three numbers to segments of the original line in Figure 1: \( a = AB \), \( b = AD \), \( c = AG \). In ¶2, we showed that

\[
AB \times GD + AG \times DB = AD \times GB,
\]
where

\[
GD = AD - AG, \ DB = AB - AD, \text{ and } GB = AB - AG.
\]
Thus

\[
AB \times (AD - AG) + AG \times (AB - AD) = AD \times (AB - AG). \quad (5)
\]
Substitute \( a, b, c \) for \( AB, AD, AG \) to obtain

\[
a(b - c) + c(a - b) = b(a - c). \text{ Q.E.D.}
\]

4.4 Restating the Result in Different Terms (¶4)
This result can be restated in terms of three other line segments, called \( DB \), \( AB \), \( GB \), where the points \( A, B, G, D \) do not correspond to those used in ¶2–3. In particular, their relative sizes are different: \( DB > AB > GB \) (whereas in ¶2–3 the relation was \( AB > GB > DB \)). This means that we should draw the points as follows: \( D—A—G—B \). Thus we now have \( a = DB \), \( b = AB \), \( c = GB \). By Equation 4,

\[
DB \times (AB - GB) + GB \times (DB - AB) = AB \times (DB - GB).
\]
Furthermore, we can define all the differences more simply as their own line segments:

\[
DB - AB = DA, \ AB - GB = AG.
\]
(Also, though the text doesn’t mention this relation explicitly, \( DB - GB = GD \).) Substituting in these simpler expressions, we obtain

\[
DB \times AG + GB \times DA = AB \times GD. \quad (6)
\]

4.5 Relating This Result to Double False Position (¶5)
According to Jābir, the method of Calculation by Two Errors (Double False Position) can be mapped onto Equation 6.

In the first case (one error excessive, the other deficient):

\[
x = AB, \ x_1 = DB, \ e_1^+ = x_1 - x = DB - AB = DA, \ x_2 = GB, \ e_2^- = x - x_2 = AB - GB = AG.
\]
[This step, if I have understood it correctly, is Jābir’s misstep: by defining the two errors as differences between the unknown, \( x \), and, respectively, the two
guesses $x_1$ and $x_2$, Jābir has entirely changed their definition as it appears in the
method of Double False Position, namely $e_1 = y_1 - y$ and $e_2 = y_2 - y$.
Thus, continues Jābir, $x_1e_2 = DB \times AG$, and $x_2e_1 = GB \times DA$. Since one error
was excessive and the other deficient, the method says to sum them: $x_1e_2 + x_2e_1 = DB \times AG + GB \times DA$. But (by Equation 6) we know that this equals $AB \times GD$. Since $GD = DG = DB - GB$, we can write $GD = DB - GB = (DB - AB) + (AB - GB) = e_1 + e_2$. Thus $AB \times GD = x(e_1 + e_2)$. Therefore,

$$x_1e_2 + x_2e_1 = x(e_1 + e_2).$$

(7)

Then, Jābir tells us, the author of *Calculation by Two Errors* divided the left side
of this equation by $e_1 + e_2$ to obtain the result

$$\frac{x_1e_2 + x_2e_1}{e_1 + e_2} = AB \left[ = x \right]$$

(8)

4.6 *The Same, for the Case Where Both Errors Are Excessive* (¶6)

[Instead of continuing in the order of the text and translation, I will skip the com-
mments of Ibn al-Sarī and al-Hamadhānī for now and get back to them after finish-
ing the analysis of Jābir.]

In the second case, both guesses produce an output that is greater than the
target: $DB > GB > AB$, so that we can again return to Equation 4, this time mak-
ing the substitutions $a = DB$, $b = GB$, $c = AB$ to obtain

$$DB \times (GB - AB) + AB \times (DB - GB) = GB \times (DB - AB).$$

(Though the text does not offer a diagram at this point, the relations imply that
the points should be arranged like this: $D \rightarrow G \rightarrow A \rightarrow B$.) From our diagram we
observe that $GA = GB - AB$, $AD = DB - AB$ (and, though again the text does not
mention it, $DG = DB - GB$). This allows us to simplify the equation to

$$DB \times GA + AB \times DG = GB \times AD.$$ 

Now again Jābir correlates this with the method of Double False Position, with
the same problem described above.

4.7 *The Same, for the Case Where Both Errors Are Deficient* (¶7)

In the third case, both guesses are low: $AB > DB > GB$. So again, in Equation 4,
we substitute $a = AB$, $b = DB$, $c = GB$, and so obtain
\[AB \times (DB - GB) + GB \times (AB - DB) = DB \times (AB - GB).\]

As Diagram 4 illustrates, the points are arranged like this: \(A\rightarrow D\rightarrow G\rightarrow B\). Therefore, \(DG = DB - GB, AD = AB - DB, GA = AB - GB\). So again we can simplify the equation to

\[AB \times DG + GB \times AD = DB \times GA.\]

And again, Jābir correlates this with the method of Double False Position, with the same fatal flaw.

4.8 Ibn al-Sarī’s Critique of ¶5–7

[The critique focuses on ¶5, but as Ibn al-Sarī points out, it applies just as much to ¶6–7.]

Ibn al-Sarī begins by pointing out that Jābir has misrepresented what the method of Double False Position entails. Jābir, Ibn al-Sarī explains, has defined \(x = AB, x_1 = DB, e_1 = DA = DB - AB = x_1 - x\), and so on. Thus \(x = AB\) is the unknown quantity sought but it is used to calculate \(e_1\); thus it has “become known without multiplication or division,” since \(x_1\) (our first guess) is known and, apparently, \(e_1\) is known as well. Thus all one has to do to find \(x\), as Jābir has defined it, is to calculate \(x = x_1 - e_1\). Therefore, Ibn al-Sarī continues, in effect Jābir is simply “computing the amount stipulated for him in the question” \((y)\), “not the desired result” \((x)\).

Ibn al-Sarī is not saying that Jābir does not know how to use the method of Double False Position in practice, but rather that Jābir’s purported proof implies the faulty method Ibn al-Sarī describes.

Ibn al-Sarī proceeds to explain his critique by means of an example: suppose we want to know \(x\) such that

\[x + \frac{x}{2} + \frac{x}{3} = 11.\]  (9)

The method implied by Jābir’s proof would be to say \(AB[=x] = 11\) (even though in fact \(y\), not \(x\), is supposed to equal 11), then make a guess, 9, then plug it into the left side of Equation 9 to obtain

\[9 + \frac{9}{2} + \frac{9}{3} = 16\frac{1}{2}.\]

But instead of defining \(DB\) (which is supposed to be \(x_1\)) as 9, the first guess, now Jābir would have us define \(DB\) as \(16\frac{1}{2}\) (which is actually the output of
the first guess, $y_1$) and then compares $DB$ to $AB$, finding that as he has defined them $DB - AB = DA = 16 \frac{1}{2} - 11 = 5 \frac{1}{2}$.

Then, Ibn al-Sarī continues, Jābir would have us guess another number $x_2 = 4$. Plugging this second guess into Equation 9 produces

$$4 + \frac{4}{2} + \frac{4}{3} = 7 \frac{1}{3}.$$ 

This result is then called $GB$ (or $BG$), so $BG = 7 \frac{1}{3}$, and then compared to 11; the difference between them, defined as $GA$, is $GA = 11 - 7 \frac{1}{3} = 3 \frac{2}{3}$, which is then called the second error.

To continue our erroneous calculation, we compute $BG \times AD$ (where $AD$ is the same as $DA$) to obtain $7 \frac{1}{3} \cdot 5 \frac{1}{2} = 40 \frac{1}{2}$. [Ibn al-Sarī skips the next step, presumably because it is obvious, namely taking the other product, $DB \times AG$, where $AG$ is $GA$, which is $16 \frac{1}{2} \cdot 3 \frac{2}{3} = 60 \frac{1}{3}$.] Then we sum these two products to obtain $40 \frac{1}{2} + 60 \frac{1}{3} = 100 \frac{2}{3}$, and divide that sum by $GD$, where $GD$ is defined as “the sum of the two errors,” namely $DA + GA = 5 \frac{1}{2} + 3 \frac{2}{3} = 9 \frac{1}{6}$. And so this means that we calculate $100 \frac{2}{3} \div 9 \frac{1}{6} = 605 \times \frac{6}{6} = 11$. This, observes Ibn al-Sarī, is nothing but the desired output initially stipulated in the question ($y$), not the unknown that was to produce it ($x$).

Thus, Ibn al-Sarī concludes, Jābir’s proof is not about Double False Position at all because “the first number” (i.e., $x_1$) should be 9, not $16 \frac{1}{2}$ (the first output $y_1$); and “the second number” (i.e., $x_2$) should be 4, not $7 \frac{1}{3}$ (the second output $y_2$). These numbers stand in a relation of proportionality to each other.

$$16 \frac{1}{2} : 9 = 7 \frac{1}{3} : 4 = 11 : x,$$

or, more generally,

$$y_1 : x_1 = y_2 : x_2 = y : x. \quad (10)$$

(As Ibn al-Sarī seems to be pointing out here, this problem does not require double false position: a single guess would have sufficed, since then by Equation 10, $x = \frac{x}{y}$, or, in this example,
\[ x = \frac{9}{16} \cdot 11 = 6. \]

In any case, this is not the main point he is trying to make.)

If we had used Double False Position properly and calculated \( x_1e_2 + x_2e_1 = 9 \cdot \frac{2}{3} + 4 \cdot \frac{1}{2} = 55 \) and then divided by the same \( GD = e_1 + e_2 = 9 \frac{1}{6} \), we would have arrived at the correct answer: \( x = \frac{55}{9} \cdot \frac{1}{6} = \frac{55 \cdot 6}{55} = 6. \)

The next two parts of Jābir’s proof (for the cases where the two errors have the same sign, positive or negative) follow the example of the first part, so Ibn al-Sarī doesn’t deal with them individually; instead, he dismisses Jābir’s proof as insufficient to tell us anything about Double False Position.

4.9 **Al-Hamadhānī’s Comment**

Here a brief comment by one Saʿd al-Dīn Asʿad ibn Saʿīd al-Hamadhānī appears, spelling out the calculation implied by Ibn al-Sarī’s statement (indeed, al-Hamadhānī does precisely what I just did in my paraphrase of Ibn al- Sarī). He says that in the formula \( BD \times AG + BG \times AD \) one should replace \( BD \) (as Jābir had defined it) with the proportional number 9 and \( BG \) (as Jābir had defined it) with the proportional number 4, in order to obtain \( 9 \cdot \frac{2}{3} + 4 \cdot \frac{1}{2} = 55 \). Then divide that by the part, which is \( \frac{1}{6} \), to get “the desired result.” (Al-Hamadhānī doesn’t spell out what that result is, presumably leaving it to the reader to perform the calculation.)

5 **Suter as a Reader of the Treatise**

Suter did not have a high opinion of Jābir’s treatise. He consulted the text contained in Leiden, Univ. Library, or. 14, nos. 3–4 (218–223).\(^80\) To judge from his description of the text, it was very similar to the version contained in the Columbia manuscript, including the intermingled commentary of (Ibn) al-Sari.\(^81\) Suter did not deign to publish the text or a translation: “Since \[the

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80 Suter, “Einige geometrische Aufgaben,” 23–24.
81 Item no. 4 of the Leiden manuscript, Suter describes, “contains not only … the commentary but also the text’s continuation mixed together with glosses” (“Nr. 4 enthält nämlich nicht nur … den Kommentar, sondern die Fortsetzung des Textes mit Glossen.
Suter remarks, namely his statement and geometrical proof of a relation between three arbitrary, consecutive segments of a line: given the line \( AB \) and two points \( G \) and \( D \) between \( A \) and \( B \), in the order \( A—G—D—B \), Jābir shows (¶2) that

\[
AB \times GD + AG \times BD = AD \times BG. \tag{11}
\]

But after that he starts to go wrong.

“Now,” continues Suter, “Jābir sets \( AG \) equal to the first guess \( \alpha_1 = x_1 \) and \( GD \) equal to the first error \( f(\alpha_1) = e_1 \), and further \( AB = \alpha_2 = x_2 \) and \( BD = f(\alpha_2) = e_2 \); then from the equation above [Equation 11] he obtains the following expression for the unknown magnitude \( AD \):

\[
AD = x = \frac{\alpha_1 f(\alpha_2) + \alpha_2 f(\alpha_1)}{f(\alpha_1) + f(\alpha_2)} \left[ = \frac{x_e e_2 + x_1 e_1}{e_1 + e_2} \right],
\]

which is correct in the case where the errors \( f(\alpha_1) \) and \( f(\alpha_2) \) [\( e_1 \) and \( e_2 \)] have different signs but is here taken absolutely.”84 Here Suter suggests that Jābir has already gone astray by claiming generality for a result that only applies in a special case. But in fact, this is only the first of the three parts of Jābir’s proof, each addressing one of the three possible cases: the errors have opposite signs, i.e., \( e_1 e_2 < 0 \) (¶5); both errors are “excessive,” or positive, i.e., \( e_1 > 0, e_2 > 0 \) (¶6); or both errors are “deficient,” or negative, i.e., \( e_1 < 0, e_2 < 0 \) (¶7).

Suter’s next critique should be taken more seriously: as he mentions, it is the same critique that Ibn al-Sarī himself undertook to make. It is not clear how closely Suter read Jābir’s text, but we can be sure that Ibn al-Sarī had read it carefully. As Suter puts it:

82 Suter, “Einige geometrische Aufgaben,” 24: “Da der Beweis selbst etwas verfehlt ist, so wäre es eine unnütze Mühe, eine vollständige wörtliche Übersetzung desselben geben zu wollen ...”

83 Suter, “Einige geometrische Aufgaben,” 24: “... und bediene mich so oft als möglich unserer heutigen Darstellungsweise.”

84 Suter, “Einige geometrische Aufgaben,” 24–25.
Jābir seems not to have recognized, however, that this proof is valid only for a very special case, namely for the case where \( e_1 + e_2 \) is exactly equal to \( BG = x_2 - x_1 \). This was also recognized by the glossator Ahmad ibn al-Surri [i.e., Ibn al-Sarī] when he remarks that for calculating the unknown here there would of course be no need at all for any multiplication or division, since \( x \) would of course be simply \( AG + DG = x_1 + e_1 \), or \( AB - BD = x_2 - e_2 \).\(^{85}\)

Indeed, this is precisely the point that Ibn al-Sarī makes.

But Suter’s next remark seems to misread the rest of Ibn al-Sarī’s commentary. Suter writes:

> Another error that the glossator [Ibn al-Sarī] accuses the author [Jābir] of making is, however, unfounded [i.e., the accusation is unfounded]. He seems to have overlooked the fact that when Jābir al-Ṣābī applies his geometrical equation to the Rule of Two Errors, he adopts different letters from those in the figure accompanying the proof [of that geometrical equation] ...\(^{86}\)

This suggests that Suter did not realize that Ibn al-Sarī’s entire commentary (assuming it is the same in the Leiden and Columbia manuscripts) is devoted to addressing the same fatal flaw in Jābir’s proof that Suter identified. As described in §4 above, Ibn al-Sarī begins by noting this fatal flaw then devotes the rest of his note to illustrating that flaw with a numerical example. So Ibn al-Sarī is not pointing out “another error” at all, as Suter thought, but simply seeking to make clear to his reader why Jābir’s proof fails. Suter’s explanation of Ibn al-Sarī’s purported error indicates that Suter must have read Ibn al-Sarī’s commentary very cursorily, since he imagines that Ibn al-Sarī was confused by Jābir’s repeated redefinition of the line segments corresponding to the underlying quantities in question (¶4–5, 6, and 7).

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\(^{85}\) Suter, “Einige geometrische Aufgaben,” 25: “Ǧâbir scheint aber nicht erkannt zu haben, daß dieser Beweis nur für einen ganz speziellen Fall zutrifft, nämlich für den Fall, wo \( f(a_1) + f(a_2) \) genau gleich \( bg = a_2 - a_1 \) ist. Das hat auch der Glossator Ahmed b. el-Surri eingesehen, indem er bemerkt, daß es hier zur Berechnung der Unbekannten ja gar keiner Multiplikation und Division bedürfe, denn \( x \) wäre ja einfach \( = ag + dg = a_1 + f(a_1) \), oder \( = ab - bd = a_2 - f(x_2) \).”

\(^{86}\) Suter, “Einige geometrische Aufgaben,” 25: “Ein anderer Fehler, den der Glossator dem Verfasser vorwirft, ist aber unbegründet, er scheint übersehen zu haben, daß Ǧâbir el-Ṣâbī bei der Anwendung seines geometrischen Satzes auf die Regel der beiden Fehler andere Buchstaben annimmt als in der Beweisfigur ...”
Suter’s subsequent remark confirms his cursory reading not only of Ibn al-Sarī but of Jābir’s text as well. According to Suter, Ibn al-Sarī “also seems not to have correctly construed the sense of some admittedly obscure passages.” Here Suter opens the possibility that he himself has overlooked something, continuing,

we at least found no other error than the one already discussed, including in the continuation of the treatise, where the author [Jābir] gives the proofs for the cases where the errors $e_1$ and $e_2$ have the same sign, so that $x_1$ and $x_2$ are either both greater or both smaller than $AD = x$.\(^87\)

It is not clear which passages of Jābir’s text Suter found “obscure,” since he correctly understood that the rest of Jābir’s treatise repeats the proof for the other two cases (¶6–7). It is even less clear, then, which part of Ibn al-Sarī he thought might be misinterpreting those obscurities. To his credit, Suter does not claim here to have a full understanding of either Jābir’s or Ibn al-Sarī’s text. In spite of this, he is nonetheless inclined to view Ibn al-Sarī’s commentary as flawed.

Suter’s subsequent discussion embraces the assumption that he, Suter, has understood the texts in question sufficiently to be able to evaluate Jābir’s (and presumably also Ibn al-Sarī’s) worth as a mathematician. He introduces his own corrections to Jābir’s proof with the words

Jābir certainly cannot have been much of a mathematical mind; otherwise, he would have recognized his own error and would easily have figured out how to come to his own aid: he could have generalized his proof in the following way ...\(^88\)

What then follows is Suter’s revised version of the geometrical proof that omits certain constraints. In particular, he constructs the same diagram with the same labels (see Figure 3), but without requiring the rectangles $BZ (= BGZE)$,

\(^87\) Suter, “Einige geometrische Aufgaben,” 25: “... und scheint auch den Sinn einiger andern, dings undeutlicher Stellen nicht richtig aufgefaßt zu haben; wir wenigstens haben keinen anderen Fehler als den eben besprochenen gefunden, auch nicht in der Fortsetzung der Abhandlung, wo der Verfasser die Beweise für die Fälle gibt, wo die Fehler $f(x_1)$ und $f(x_2)$ beide gleiches Zeichen haben, also $x_1$ und $x_2$ entweder beide größer oder beide kleiner als $ad$ sind; auf diese Beweise treten wir hier aber nicht mehr ein, sie sind leicht aus dem ersten abzuleiten.”

\(^88\) Suter, “Einige geometrische Aufgaben,” 25: “Ein bedeutender mathematischer Kopf kann Ġābir allerdings nicht gewesen sein, sonst hätte er seinen Fehler erkannt und sich leicht zu helfen gewußt, er hätte seinen Beweis in folgender Weise verallgemeinern können ...”
DK (= DGKT), or YH (= YTHE) to be squares (i.e., he does not set BG = GZ, DG = GK, or YT = TH). He then makes the figure correspond to Double False Position as follows (using my rather than Suter’s notation): \( x_1 = AG = KL \) (first guess), \( e_1 = GK = BY \) (first error), \( x_2 = AB = LY \) (second guess), and \( e_2 = YE = KZ \) (second error). [Furthermore, \( x = AD \) (the quantity we wish to find)].89 Finally, he points out that the gnomon \( BAMZKY \) (i.e., the sum of the rectangles \( B Aly \) and \( KLMZ \)) is still equal to the rectangle \( DAMH \) (because \( BDTY = TKZH \), since the two rectangles are complements about the diagonal).90 As a result, \( AB \times BY + KL \times KZ = AD \times DH \); since \( DH = BY + YE \), this becomes \( x_2 e_1 + x_1 e_2 = x(e_1 + e_2) \), or

\[
x = \frac{x_2 e_1 + x_1 e_2}{e_1 + e_2},
\]

Q.E.D.

Having completed the proof, Suter then notes that like Jābir, Ibn al-Sarī too failed to recognize the generalized version of Jābir’s proof.91 In a certain sense, this is true: Ibn al-Sarī does not provide the generalized proof offered by Suter, or indeed any other proof, in place of Jābir’s flawed proof. But perhaps Ibn al-Sarī’s only aim in the commentary was to show why Jābir’s proof fails to work. It seems a bit hasty to say that Ibn al-Sarī failed to come up with a correct version of the proof when he may simply have chosen not to present it here.

### 6 Mathematical Philology

Suter was a prolific historian and philologist of Arabic mathematics and did much to advance the field. The son of a farmer and postmaster in a village outside of Zurich, he was remembered as an “unpretentious man,” a hardworking and humble scholar who resolved to learn Arabic at the age of forty out of

89 The geometrical diagram does constrain these quantities such that \( EH : HT = TK : GK \) (to ensure that the diagonal goes through point \( T \)). This means that \( EH = BD = AB - AD, HT = YE, TK = DG, \) and \( GK = BY \); and so \( (AB - AD) : YE = DG : BY = (AD - AG) : BY, \) or \( (x_2 - x) : e_2 = (x - x_1) : e_1 \), i.e., that there is a fixed proportion between how far off the guess is (where we are dealing with the case in which \( x_2 > x \) and \( x_1 < x \)) and how far off the resulting output is. Since we are dealing only with linear functions, this constraint poses no problem; the inverse of that fixed proportion is (the magnitude of) the line’s slope.

90 This exploits Euclid, Elements 1.43, to make a point that is less obvious than the one Jābir had used that theorem to make; see n. 47 above.

91 Suter, “Einige geometrische Aufgaben,” 26: “Dies hat auch der Glossator Ahmed b. el-Surrī nicht erkannt.”
a fascination with the Islamic world, and a “free thinker” who believed in the similarity of world religions and the common humanity of all. He worked at a time when even less (much less) of the relevant source material was available outside of manuscripts and when photographs of distant manuscripts were much harder to come by. He read many mathematical texts attentively and with great discernment. His publications on the topic are a vast repository of information and astute analysis and remain key references today. In the case of Jābir’s text and Ibn al-Sarī’s response, he does not pretend to have dwelt on it at length or captured every nuance of the text.

For all these reasons, it would be rash, unproductive, and entirely unfair to hurl back Suter’s insults at him, calling him not much of a philological mind, just as he called Jābir not much of a mathematical mind, and thus generalize about Suter based on a single section of a single scholarly article. Nevertheless, the rapidity of Suter’s reading of the treatise was driven by the overarching priorities and methodological principles embraced by Suter and his fellow historians of mathematics. For this reason—combined with his warm and conscientious attitude toward Arabic mathematical texts, which rules out any facile dismissal of his work—it is perhaps worth dwelling for a moment on the characteristics of Suter’s reading of the treatise before considering what alternative mode might best suit a different set of priorities.

Suter was working within the tradition that approaches the history of science and mathematics by asking who first discovered things that we now know to be true and when. Information not pertaining to this line of inquiry was accordingly unimportant—hence his decision to refrain from publishing Jābir’s faulty proof verbatim.

Tellingly, after he mentions the treatise’s (correct) proof of Equation 11 in ¶2, Suter continues by regretting that he could not answer the question that presumably he could expect his reader to be asking: who, and in particular which nation, first came up with that correct proof? Suter writes: “Whether this theorem belongs properly to the Greeks or the Arabs, we cannot decide; it is not to be found in Euclid to our knowledge.” Likewise, to conclude his discussion of Jābir’s treatise and Ibn al-Sarī’s commentary, Suter writes that he cannot help but mention that the existence of this treatise and Ibn al-Nadīm’s references to other works on Double False Position refute the view expressed by some of his

92 Ruska, “Heinrich Suter,” esp. 409 (“diesem anspruchslosen Manne”), 411 (“ein freier Denker”), 411–412.
93 Suter, “Einige geometrische Aufgaben,” 25: “Ob dieser Satz griechisches oder arabisches Eigentum sei, können wir nicht entscheiden, bei Euklid findet er sich unseres Wissens nicht.”
contemporaries that the method of Double False Position was first discovered in the twelfth century by European mathematicians.94

To avoid any misunderstanding, it is worth emphasizing here that the response to Suter that I propose is not a critique of “Orientalist thought,” a vindication of Arabic or Islamic mathematics in the face of European bias or ignorance. That vindication is precisely what Suter was eager to carry out. Instead, Suter’s blind spot is connected to the approach to the history of science and mathematics that he embraced, one in which the wheat must be separated from the chaff—according to simple scientific or mathematical criteria, not hermeneutically recursive historical or textual criteria—so that the historian could avoid wasting too much effort on the chaff. In other words, it is a historical approach in which the historian adopts the criteria of his own contemporaries in the natural and mathematical sciences and uses them as historical criteria.95 The question of most interest to historians embracing this approach is when each aspect of modern science or mathematics was first “discovered.” In addressing an individual scholar of the past, the questions then become how much he knew and understood of (modern) science or mathematics, and how much credit he deserves for uncovering some part of that modern body of knowledge.96

While quite powerful in its own way, this approach tends to downplay or omit altogether an account of how mathematicians of the past thought about, discussed, arrived at, and communicated their results. With an emphasis on what they knew and when they knew it, in other words, it tends to skip over false starts, flawed proofs, and critiques of such errors, thus suppressing valuable evidence for the aims of mathematicians and the conceptual frameworks that conditioned those aims and how they were pursued and that were in turn shaped by all aspects of mathematical production, not only the statements and proofs admired by modern mathematicians.

94 Suter, “Einige geometrische Aufgaben,” 26–27: “Man entschuldige uns, wenn wir hier folgende Bemerkung nicht unterlassen können ...”

95 This should not be confused with using one’s contemporary scientific criteria as scientific criteria for assessing the past, which entails using what we know or think we know today in order to gain a perspective on past scientific work that might not have been available to past scientists themselves. The difference is crucial: when these scientific criteria are used as a substitute for historical criteria, we allow present-day scientific concerns to warp our understanding of how and why ideas developed. See further Richard Rorty et al., Philosophy in History: Essays on the Historiography of Philosophy (Cambridge: Cambridge University Press, 1984), 1–14, who frame similar debates in terms of “history of philosophy” versus “intellectual history.”

96 See Thomas S. Kuhn, The Structure of Scientific Revolutions, 3rd ed. (Chicago: University of Chicago Press, 1996), chs. 1–2.
A philology of mathematics that takes such evidence into account will be the best equipped to produce the kind of deeper history of mathematics that Roshdi Rashed has advocated, a history not only of methods available and theorems proven (or at least exploited) but also of conceptual framing, modes of understanding, and notions of the possible directions available to a given field of mathematics in a given time and place.\footnote{Rashed, \textit{Development of Arabic Mathematics}, ch. 1, esp. 14–16. Such an approach is related more broadly to the methods practiced and advocated, for example, by Kuhn and the historians and sociologists who have taken inspiration from aspects of his approach; see Barry Barnes, \textit{T.S. Kuhn and Social Science} (London: Macmillan, 1982).} In the case of Jābir’s treatise on Double False Position and the Columbia manuscript, such a mode of reading, applicable to flawed and flawless mathematical texts alike, allows us to return to the juxtaposition with which this article began. What was a flawed proof of a numerical method that today’s mathematicians would regard as hopelessly elementary doing in the same manuscript, copied by the same scribe, as Omar Khayyam’s pathbreaking treatise on algebra and, indeed, coming right after it? As already mentioned, Double False Position could be very useful in practice. But this manuscript was not a manual for traders; clearly this collection was produced by and for mathematicians, focused on theoretical texts and demonstration of theorems, not practical numerical methods and their applications. Why, then, include the treatise on Double False Position? The answer, I propose, lies in precisely what Suter found unsatisfactory about the text: the faulty proof that Jābir added to the basic description of the method of Double False Position, along with Ibn al-Sarī’s critique of that proof. This may seem like an odd proposal: why would working mathematicians wish to preserve and even study a misguided, incorrect proof? But the Columbia manuscript is evidence of just that wish: mathematicians and students of mathematics in the medieval Islamic world—in particular Iran, probably Hamadān—were interested in understanding what was wrong with Jābir’s proof.\footnote{A single scribe (Scribe 1) copied texts no. 2–18 in the manuscript, including the treatise on Double False Position (no. 10). Even if the compilation represented by this subset of the manuscript had already been compiled piecemeal over time (such that Scribe 1 would not be the compiler of this compilation, only its copyist), nevertheless it was still the scribe’s choice to copy a pre-existing compilation \textit{in its entirety}—a choice that suggests an interest in studying the text on Double False Position alongside the other texts.} This would have offered them a lesson in how to catch a proof’s fault while preserving an episode in the history of their discipline.

Nor was this episode necessarily lodged exclusively in the past from the perspective of the scholars who used this manuscript. After all, there were plenty of other treatises on Double False Position. As already mentioned, that...
of the Arabophone Byzantine Christian scholar Qusṭā ibn Lūqā (d. ca. 912–13) sparked Suter’s interest because it contained a more nearly valid geometric proof of why Double False Position works. Various other treatises on the topic are known today only by their titles. Sezgin lists treatises entitled (Hisāb) al-Khaṭaʾāyn, or (Calculation by) Two Errors, by Abū Kāmil (whom Sezgin tentatively places in the second half of the ninth century), Abū Yūsuf al-Rāzī and Abū Yūsuf al-Miṣṣīṣī (both probably active in the first half of the tenth century according to Sezgin), al-Karaṇji (active ca. tenth/eleventh century), and Ibn al-Haytham (965–1039). (There is no significance to the fact that Ibn al-Haytham is the latest author in this list; Sezgin’s multivolume biobibliographical reference work stops at ca. 430 AH/1038 CE, so it would automatically have excluded any treatises on Double False Position that might have been composed after the mid-eleventh century.) In other words, there seems to have been enduring interest in this algorithm and its mathematical justification. Further research into such treatises—especially if any of them should turn up in the vast number of uncatalogued and undercatalogued Arabic manuscripts around the world—might help us understand the context of Jābir’s treatise.

For example, if indeed he was working later than Qusṭā, as Suter thought, we might imagine that Jābir was seeking to produce a simpler proof, or else that he sought to reproduce Qusṭā’s proof from memory and ended up getting it wrong without realizing his mistake. Similarly, if indeed Jābir did not have much of a head for math, as Suter claimed, it would be interesting to know what social and cultural incentives impelled him to take up the task of proving Double False Position nevertheless. Or, if other works by the same Jābir turn up showing him to be more of a mathematical mind than Suter thought, we might ask what led him astray in this one treatise—or we might reconsider what he was

99 See n. 12 above. As Suter points out (Suter, “Die Abhandlung Qosṭā ben Lūqās,” 119–21), Qusṭā’s treatise (at least as translated by Suter) sets up the correspondence between the line segments in its geometric proof and Double False Position’s parameters in such a way as to assume implicitly that the equation in question is of the form \( ax = y \), i.e., that the \( y \)-intercept is zero. Suter is puzzled that a mathematician like Qusṭā would have missed this and suggests that the attribution may be false. But if the correspondence is tweaked, the proof is successful; indeed Suter also suggests that an error of transmission could have introduced the error into the text. To be sure of what is going on, it will be necessary to consult the original Arabic of Qusṭā’s treatise anew.

100 Abū Kāmil: Sezgin, GAS, 5:277–281, esp. 277 (date) and 281 (al-Khaṭaʾāyn). Abū Yūsuf al-Rāzī (Hisāb al-khaṭaʾāyn): Sezgin, 5:300. Abū Yūsuf al-Miṣṣīṣī (al-Khaṭaʾāyn): Sezgin, 5:297. Al-Karaṇji (who dedicated one of his works to a patron who died in 1014), al-Khaṭaʾāyn: Sezgin, 5:329; on whether his name was al-Karaṇji or al-Karkhī, see Rashed, Development of Arabic Mathematics, 22. Ibn al-Haytham, Hisāb al-khaṭaʾāyn: Sezgin, GAS, 5:374.
trying to do in this treatise and ask why subsequent readers from Ibn al-Sarī to Suter to the present author misunderstood his aims.¹⁰¹

In any case, we must still contend with the widespread interest in proofs of Double False Position. Jābir’s purported proof was clearly something that Ibn al-Sarī considered worth his time to refute in the twelfth century, and his refutation was still being studied closely when Sa’d al-Dīn al-Hamadhānī subsequently explained it (presumably to students) and when the Columbia manuscript was produced. This concern for refuting a bad proof of Double False Position might have stemmed in part from the numerical method’s widespread use, but ultimately it must have been part of medieval Arabic mathematicians’ broader project. Perhaps it was precisely because Double False Position was clearly applicable to many of the same problems that the new algebra subsumed that it was important to study it not simply as a handy numerical method but as a theorem to be demonstrated by a satisfactory and revealing geometric proof and thus properly integrated into the new mathematics.¹⁰²

Suter’s observation that Jābir’s treatise attests to the existence of the method of Double False Position already in early Arabic mathematics, then, is only the beginning of the historian’s task. Rather than stop there and dismiss the

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¹⁰¹ As an anonymous reviewer generously informed me, Jābir is named as the author of astronomical works in Oxford, Bodleian, Thurston 3 (13th century; José Bellver, on Ptolemaeus Arabus et Latinus, https://ptolemaeus.badw.de/ms/672, entry updated 10 November 2018) and Oxford, Bodleian, Marsh 720 (17th century; Bellver, http://ptolemaeus.badw.de/ms/685, entry updated 10 November 2018): Maqālah fī hayʾat aflāk ʿUṭārid wa-khtilāf marākizihā wa-masīrihā (Benno van Dalen, Ptolemaeus Arabus et Latinus, https://ptolemaeus.badw.de/work/225; Thurston 104r, Marsh 207r–v, entry updated 8 November 2018) and «Muqaddimāt fī bidʾ ashkāl min al-Majisṭī» (van Dalen, https://ptolemaeus .badw.de/work/221; Thurston 105⁷–107⁷, Marsh 208⁸–211v, entry updated 8 November 2018), of which Burhān mā qālahu Baṭlamiyūs fī l-shakl al-rābiʿ min al-maqālah al-thāniyah ʿashar <min al-Majisṭī> (not explicitly ascribed to Jābir: Bellver, https://ptolemaeus.badw.de/work/226; Thurston 107⁶, Marsh 212r, entry updated 8 November 2018) is probably a continuation (according to Bellver). These manuscripts also contain Jābir’s treatise on Double False Position and Ibn al-Sarī’s commentary on it (Thurston 136⁶–137⁵, Marsh 271v–272v), followed by Qusṭā’s treatise on Double False Position. Bellver, in his entry on the Thurston manuscript, mentions that “a note on f. 105r indicates that this group of works by Jābir b. Ibrāhīm al-Ṣābī was copied from a first generation copy from an autograph by Jābir b. Ibrāhīm al-Ṣābī.” All this suggests that these manuscripts, especially the Thurston manuscript, would be a promising avenue for future research.

¹⁰² J. Murdoch asked a question of Roshdi Rashed at a conference after the latter’s talk on the social context of algebra’s development. As the discussion continued, Murdoch asked an open-ended question about “false position” and its place in this history of algebra, as an example of a topic for future historical research. See Rashed, Development of Arabic Mathematics, 61. Texts like Jābir’s would presumably be at the heart of such an inquiry.
treatise as otherwise useless because mathematically incorrect, philologically-minded historians of mathematics might ask how the treatise, its commentary, its subsequent study, and other treatises like it on Double False Position can be reconciled and integrated into the picture of medieval mathematics that continues to emerge, one newly edited mathematical text at a time.

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