Inflation with a constant rate of roll

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Received June 1, 2015
Accepted August 12, 2015
Published September 7, 2015

Abstract. We consider an inflationary scenario where the rate of inflaton roll defined by $\dot{\phi}/H\dot{\phi}$ remains constant. The rate of roll is small for slow-roll inflation, while a generic rate of roll leads to the interesting case of ‘constant-roll’ inflation. We find a general exact solution for the inflaton potential required for such inflaton behaviour. In this model, due to non-slow evolution of background, the would-be decaying mode of linear scalar (curvature) perturbations may not be neglected. It can even grow for some values of the model parameter, while the other mode always remains constant. However, this always occurs for unstable solutions which are not attractors for the given potential. The most interesting particular cases of constant-roll inflation remaining viable with the most recent observational data are quadratic hilltop inflation (with cutoff) and natural inflation (with an additional negative cosmological constant). In these cases even-order slow-roll parameters approach non-negligible constants while the odd ones are asymptotically vanishing in the quasi-de Sitter regime.

Keywords: inflation, physics of the early universe, cosmological perturbation theory

ArXiv ePrint: 1411.5021
1 Introduction

The inflationary paradigm based on the assumption of the existence of a quasi-de Sitter stage in the early Universe before the hot radiation dominated Big Bang [1–5] has now become a well established part of modern cosmology. It makes definite predictions about present properties of the Universe which have been confirmed by numerous observations. The most common way to realize this quasi-de Sitter expansion is to employ a scalar field (inflaton) with an approximately flat potential and consider a slow-roll solution of the field equations. It is also possible to realize inflationary expansion without introducing such an inflaton field by upgrading the gravitational action from the Einstein-Hilbert one to a nontrivial function of the Ricci scalar, $f(R)$. However, using the conformal transformation, equations for an $f(R)$ type inflation can be transformed to the Einstein-Hilbert action with a canonical scalar field dubbed scalaron with its potential defined by the form of $f(R)$. In this approach, the $R^2$ inflation [1], including its generalization to accommodate present dark energy [6–8] or to consider a small deviation from it [9–11], can generate primordial scalar and tensor metric fluctuations of the same form as scalar field inflation.

Given that slow-roll inflationary models with an approximately flat inflaton potential yield a simple realization of inflation with viable observational predictions, it is natural to ask what happens if we omit the assumption of the inflaton slow roll. The slow-roll approximation assumes that in the Klein-Gordon equation for the inflaton given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0,$$  \hspace{1cm} (1.1)

the second derivative term $\ddot{\phi}$ is negligible compared to other terms. The necessity to go beyond this approximation and to use some exact solutions instead has been already occurred in a number of cases, in particular, when the inflaton potential $V(\phi)$ has some local non-analytic feature [12] or is near its local extremum [13], if $\partial V/\partial \phi = 0$ holds for an extended period [14–16] (the latter case dubbed the ‘ultra-slow-roll’ inflation) and if the period of fast-roll inflation preceded about 50 e-folds of slow-roll inflation [17–19]. An important new feature appearing in all these cases is that the curvature power spectrum becomes evolving on super-Hubble scales temporarily. Furthermore, the non-Gaussianity consistency relation for single field inflation can be violated [16].
Motivated by these phenomenologically interesting features, the ultra-slow-roll inflation was generalized in [20]. Starting from an assumption of a constant rate of roll with \( \dot{\phi}/H = -3 - \alpha \), where nonzero \( \alpha \) implies deviation from the flat potential, they derived a potential that satisfies the assumption approximately. However, it is actually possible to construct an exact solution under this assumption, which is the main topic of the present paper. We refer to this class of models as the 'constant-roll' inflation. We shall show that the general exact solution for these models includes power-law inflation [21, 22] and other two solutions: one of them is equivalent to a solution previously found in [23] in a different context, and the other one amounts to a particular case of hilltop inflation [24] with cutoff or natural inflation [25] with an additional negative cosmological constant. We shall also investigate the evolution of scalar (curvature) perturbations in these models. In general, these constant-roll models may have super-Hubble evolution of the scalar perturbation, which is the same situation as the ultra-slow-roll inflation. However, we shall show that for the particular cases of hilltop inflation and natural inflation do not suffer from it and they actually remain observationally feasible with the most recent constraints for the tilt of the scalar power spectrum and the tensor-to-scalar ratio.

The rest of the paper is organized as follows. In section 2, we determine the inflaton potential required for constant-roll inflation. We show that it is possible to derive exact solution for inflationary dynamics without assumptions made in the literature. We shall see that in order to make the inflationary regime an attractor, one needs the constant-roll parameter \( \alpha < -3/2 \). In section 3, we explore scalar and tensor perturbations generated during constant-roll inflation. In section 3.1, we calculate power spectrum of the curvature perturbation and show that \( \alpha \gtrless 0 \) or \( \alpha \lesssim -3 \) provides a slightly red-tilted spectrum, the latter of which has an attractor inflationary regime. In section 3.2, we confirm that the super-Hubble evolution of the curvature perturbations in non-attractor model \( \alpha > -3/2 \), whereas \( \alpha < -3/2 \) serves a constant mode and a decaying mode on super-Hubble scales as the standard slow-roll inflation. Therefore, \( \alpha \lesssim -3 \) is observationally viable and analytically solvable constant-roll inflation model, which includes particular cases of hilltop inflation and natural inflation. In section 3.3, we addressed tensor perturbation. We then conclude in section 4. Throughout the paper, we will work in the natural unit where \( c = 1 \), and the metric signature is \((- + + +)\).

2 Constant-roll inflation

We consider inflation driven by a minimally coupled scalar field defined by the action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2M_{Pl}^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right].
\]  
(2.1)

where \( M_{Pl} = (8\pi G)^{-1/2} \). Working in the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the Friedmann equation and the equation of motion for the scalar field are given by

\[
3M_{Pl}^2 H^2 = \frac{\dot{\phi}^2}{2} + V, \quad 3M_{Pl}^2 \dot{H} = \dot{\phi}^2, \quad \dot{\phi} + 3H \dot{\phi} + \frac{\partial V}{\partial \phi} = 0,
\]  
(2.2, 2.3, 2.4)
where $a$ is the scale factor, a dot denotes the derivative with respect to $t$, $H \equiv \dot{a}/a$ is the Hubble parameter. Inflationary evolution is characterized by the slow roll parameters:

$$
\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_{n+1} = \frac{\dot{\epsilon}_n}{H \epsilon_n}.
$$

(2.5)

In the slow-roll inflation, the slow-roll parameters are small, $|\epsilon_n| \ll 1$, and the first terms of the right-hand side of (2.2) and the left-hand side of (2.4) can be neglected. As a result, an approximately flat spectrum of scalar (curvature) perturbations can be obtained, whose exact form depends on the functional shape of the potential $V(\phi)$ during the quasi-de Sitter expansion regime. Here, instead, we are interested in a different regime when $\ddot{\phi}$ is not negligible in (2.4). Following [20], we adopt an ansatz of a constant rate of roll $\dot{\phi}/H\dot{\phi}$ during inflation and parametrize it as

$$
\ddot{\phi} = -(3 + \alpha)H\dot{\phi},
$$

(2.6)

with an arbitrary constant value of $\alpha$. The standard slow-roll inflation occurs if $\alpha \simeq -3$ while the 'ultra-slow-roll' case corresponds to $\alpha = 0$. Thus, the constant-roll inflation interpolates between these two regimes. By the way, this shows that the term 'ultra-slow-roll inflation' is rather misleading. In fact, this regime has to be considered as a specific case of fast-roll inflation since the slow-roll approximation breaks down during it.

In [20], by neglecting $\dot{\phi}^2/2$ term in (2.2), an inflaton potential which satisfies the ansatz (2.6) was derived. However, in this paper we shall show that it is actually possible to construct an inflaton potential which realizes the relation (2.6) without any approximation and to investigate its dynamics fully analytically. In order to solve the system of equations (2.2)–(2.4) with the condition (2.6), we consider the Hubble parameter as a function of the inflaton field $H = H(\phi)$ and use the Hamiltonian-Jacobi-like formalism [26, 27]. This approach is applicable as long as $t = t(\phi)$ is a single-valued function, i.e. $\dot{\phi} \neq 0$. Since $\ddot{H} = \dot{\phi}dH/d\phi$ holds in this approach, we can rewrite (2.3) as

$$
\dot{\phi} = -2M_{\text{Pl}}^2 \frac{dH}{d\phi}.
$$

(2.7)

Plugging the time derivative of (2.7) to (2.6), we obtain the differential equation for the Hubble parameter with respect to $\phi$,

$$
\frac{d^2H}{d\phi^2} = \frac{3 + \alpha}{2M_{\text{Pl}}^2}H.
$$

(2.8)

The general solution of this equation is

$$
H(\phi) = C_1 \exp \left( \sqrt{\frac{3 + \alpha}{2}} \frac{\phi}{M_{\text{Pl}}} \right) + C_2 \exp \left( -\sqrt{\frac{3 + \alpha}{2}} \frac{\phi}{M_{\text{Pl}}} \right).
$$

(2.9)

For $\alpha < -3$, the exponents become imaginary, and then $C_2 = C_1^*$. Using (2.2) and (2.7), we find the inflaton potential required for the exact solution (2.9):

$$
V(\phi) = M_{\text{Pl}}^2 \left[ 3H^2 - 2M_{\text{Pl}}^2 \left( \frac{dH}{d\phi} \right)^2 \right]
$$

(2.10)

$$
= M_{\text{Pl}}^2 \left[ -\alpha C_1^2 \exp \left( \sqrt{2(3+\alpha)} \frac{\phi}{M_{\text{Pl}}} \right) + 2(6+\alpha)C_1C_2 - \alpha C_2^2 \exp \left( -\sqrt{2(3+\alpha)} \frac{\phi}{M_{\text{Pl}}} \right) \right].
$$
Then we can derive evolution of inflaton $\phi(t)$ by solving (2.7), and obtain $H(t)$ by plugging $\phi(t)$ back into (2.9). However, it should be kept in mind that the Hamilton-Jacobi formalism produces one solution $\phi(t)$ for the derived potential $V(\phi)$ only which needs not be an attractor solution (even an intermediate one). So, its stability with respect to all FLRW solutions with the same potential has to be checked each time.

The most interesting particular solutions for $\alpha > -3$ are

\begin{align}
H &= Me^{\pm \sqrt{\frac{3 + \alpha}{2}} \frac{\phi}{M_{\text{Pl}}}}, \quad (2.11) \\
H &= M \cosh \left( \sqrt{\frac{3 + \alpha}{2}} \frac{\phi}{M_{\text{Pl}}} \right), \quad (2.12) \\
H &= M \sinh \left( \sqrt{\frac{3 + \alpha}{2}} \frac{\phi}{M_{\text{Pl}}} \right), \quad (2.13)
\end{align}

where $M$ is an integration constant which determines the amplitude of the power spectrum of the curvature perturbation. Note that the third solution describes a bounce. For $\alpha < -3$, only the last two solutions have physical sense, and the hyperbolic functions in them have to be substituted by the corresponding trigonometric ones. Consequently, for $\alpha < -3$ two solutions are the same up to a field redefinition.

Let us begin with the first solution (2.11). Since two solutions are equivalent under $\phi \to -\phi$, we focus on $H = Me^{\pm \sqrt{\frac{3 + \alpha}{2}} \frac{\phi}{M_{\text{Pl}}}}$. From (2.10), we then obtain

\begin{align}
V(\phi) &= -\alpha M^2 M_{\text{Pl}}^2 \exp \left( \sqrt{2(3 + \alpha)} \frac{\phi}{M_{\text{Pl}}} \right), \quad (2.14) \\
\phi &= -\sqrt{\frac{2}{3 + \alpha}} M_{\text{Pl}} \ln [(3 + \alpha)Mt], \quad (2.15) \\
H &= \frac{1}{3 + \alpha}, \quad (2.16) \\
a &\propto t^{\frac{1}{3 + \alpha}}. \quad (2.17)
\end{align}

The positivity of $V(\phi)$ requires $\alpha < 0$, and for $-2 < \alpha < -3$ this is nothing but the power-law inflation [21, 22]. As is well known, it leads to the constant slopes $n_s - 1 = n_t = 2(3 + 2\alpha)/(2 + \alpha) < 0$ of the power spectra of primordial scalar and tensor perturbations. Of course, to obtain an exit from inflation, one has to assume that the potential (2.14) changes its form and quickly approaches zero at some value of $\phi$. But from the observational point of view, power-law inflation is certainly excluded because of the absence of the large amount of primordial tensor perturbations (gravitational waves) that would be $r = 8(1 - n_s) \approx 0.28$ in this case.

Plugging the second solution (2.12) to (2.10), we obtain for $\alpha > -3$

\begin{align}
V(\phi) &= 3M^2 M_{\text{Pl}}^2 \left[ 1 + \frac{\alpha}{6} \left\{ 1 - \cosh \left( \sqrt{2(3 + \alpha)} \frac{\phi}{M_{\text{Pl}}} \right) \right\} \right], \quad (2.18) \\
\phi &= M_{\text{Pl}} \sqrt{\frac{2}{3 + \alpha}} \ln \left[ \coth \left( \frac{3 + \alpha}{2} Mt \right) \right], \quad (2.19) \\
H &= M \coth [(3 + \alpha)Mt], \quad (2.20) \\
a &\propto \sinh^{1/(3 + \alpha)} [(3 + \alpha)Mt]. \quad (2.21)
\end{align}
This solution is equivalent to a solution found in [23] in a different context. Note also that (2.21) with \( \alpha = 3(w-1)/2 \) mimics the evolution of the FLRW model with a cosmological constant filled by an ideal fluid with the equation of state \( p = w \rho \) with \( w = \text{const.} \) This case encompasses a more special case of mimicking the ΛCDM model with \( w = 0 \), which is found in [28]. For \(-3 < \alpha < 0\), \( V(\phi) \) has a minimum at \( \phi = 0 \). Thus, to end inflation, a kind of phase transition at this point has to be assumed additionally, e.g. similar to that occurs in hybrid inflation [29].

Contrary, for \( \alpha < -3 \) with (2.12) we obtain

\[
V(\phi) = 3M^2M_{\text{Pl}}^2 \left[ 1 + \frac{\alpha}{6} \left\{ 1 - \cos \left( \sqrt{2|3+\alpha|} \frac{\phi}{M_{\text{Pl}}} \right) \right\} \right],\tag{2.22}
\]

\[
\phi = 2\sqrt{\frac{2}{|3+\alpha|}} M_{\text{Pl}} \arctan(e^{|3+\alpha|M_t}),\tag{2.23}
\]

\[
H = -M \tanh \left( |3 + \alpha|M_t \right),\tag{2.24}
\]

\[
a \propto \cos^{-1/|3+\alpha|} \left( |3 + \alpha|M_t \right).\tag{2.25}
\]

which is a particular case of hilltop inflation [24] with the inverted quadratic potential near the origin. As usual in this case, we assume that \( \phi \rightarrow +0 \) at \( M_t \rightarrow -\infty \), so the Hubble parameter remains positive and approaches \( M \). Then inflation takes place as the inflaton rolls down from the origin, and ends near the point where the potential crosses zero. In fact, the latter has to be changed somewhere before this point and the initial condition for \( \phi \) may be made less restrictive. We shall also see that this case includes natural inflation [25] with an additional negative cosmological constant. We discuss the above points in section 3.2.

Finally, the third solution (2.13) yields

\[
V(\phi) = -3M^2M_{\text{Pl}}^2 \left[ 1 + \frac{\alpha}{6} \left\{ 1 + \cosh \left( \sqrt{2(3+\alpha)} \frac{\phi}{M_{\text{Pl}}} \right) \right\} \right],\tag{2.26}
\]

\[
\phi = -2M_{\text{Pl}} \sqrt{\frac{2}{3 + \alpha}} \arctanh \left[ \tan \left( \frac{3 + \alpha}{2} M_t \right) \right],\tag{2.27}
\]

\[
H = -M \tan [(3 + \alpha)M_t],\tag{2.28}
\]

\[
a \propto \cos^{1/(3+\alpha)} [(3 + \alpha)M_t].\tag{2.29}
\]

Although this is a mathematically allowed solution, it has \( \dot{a}(t) < 0 \). Therefore, it cannot describe an inflationary model in the usual sense.

For the following, we investigate the second solution (2.12), i.e. (2.18)--(2.21) with \( \alpha > -3 \) and (2.22)--(2.25) with \( \alpha < -3 \), and clarify if they serve observationally feasible inflationary scenario. In particular, we shall check the potential and the analytic solutions for the former case (2.18) below as the potential for the latter case (2.22) amounts to a particular case of natural inflation or hilltop inflation with \( V \propto \text{const} - \phi^2 \) around \( \phi = 0 \). We present the potential (2.18) in figure 1 (a) for \( \alpha = -0.5 \) (blue solid), 0 (red dashed), 0.5 (yellow dotted). Clearly, \( \alpha = 0 \) yields a flat potential and it reduces to the case of the original ultra-slow-roll inflation. The evolution of inflaton is depicted in figure 1 (b). The field value monotonically decreases and approaches to the origin. For the case of \( \alpha = 0.5 \), the inflaton climbs up the potential and approaches to the top of the potential at the origin. We note that the Hubble parameter in figure 1 (c) approaches constant as time goes by, which leads to de Sitter expansion of the universe, i.e. \( a(t) \propto e^{M_t} \) for \( M_t \gg 1 \), which is displayed in figure 1 (d).
Figure 1. (a) Potential ($2.18$), and its exact solution for (b) inflaton $\phi$, (c) Hubble parameter $H$, (d) scale factor $a$, and slow-roll parameters (e) $\epsilon_1 \equiv -\dot{H}/H^2$, (f) $\epsilon_2 \equiv \dot{\epsilon}_1/H\epsilon_1$, for $\alpha = -0.5$ (blue solid), 0 (red dashed), 0.5 (yellow dotted). While $\epsilon_2$ approaches to $-2(3 + \alpha)$, which is not necessarily negligible, the Hubble parameter approaches constant and the scale factor evolves as exponentially. The higher slow-roll parameters are given by $\epsilon_{2n+1} = 2\epsilon_1$ and $\epsilon_{2n} = \epsilon_2$.

In this limit, the effective inflaton mass squared becomes $m^2(\phi) \equiv \partial^2 V/\partial \phi^2 = -\alpha(3 + \alpha)M^2$. Then the generic solution for the inflaton is the sum of two exponents $e^{\alpha Mt}$ and $e^{-(3+\alpha)Mt}$. However, our constant-roll solution (2.19) chooses the second exponent only due to the definition (2.6). The same situation happens for $\alpha < -3$ in the limit of $Mt \to -\infty$. This is an illustration of the remark above that the Hamilton-Jacobi method of finding exact solutions produces only one solution for the potential $V(\phi)$ determined by its use at the same time.
We can also express the conformal time \( \tau = \int \frac{dt}{a} \) analytically in terms of the Gauss’ hypergeometric function \( _2F_1 \). For (2.18) with \( \alpha > -3 \), we obtain

\[
\tau = \left(-1\right)\frac{\cosh[(3 + \alpha)Mt]}{M(3 + \alpha)} \left[ _2F_1 \left( \frac{1}{2}, \frac{4 + \alpha}{2(3 + \alpha)}; \frac{3}{2}; \cosh^2[(3 + \alpha)Mt] \right) - \left(-1\right)^{-1/2} \right],
\]

where we fixed the integration constant to make \( \tau \to -0 \) as \( Mt \to \infty \). On the other hand, for (2.22) with \( \alpha < -3 \), we obtain

\[
\tau = i \frac{\cosh \frac{2\alpha}{3 + \alpha} [(3 + \alpha)Mt]_2F_1 \left( \frac{1}{2}, \frac{2 + \alpha}{2(3 + \alpha)}, \frac{8 + 3\alpha}{2(3 + \alpha)}; \cosh^2[(3 + \alpha)Mt] \right)}{\sqrt{\pi} \Gamma \left( \frac{8 + 3\alpha}{2(3 + \alpha)} \right) \Gamma \left( \frac{5 + 2\alpha}{2(3 + \alpha)} \right)},
\]

where the integration constant is determined by \( \tau \to -0 \) as \( Mt \to -0 \). Of course, to get a realistic model, we have to cut the potential somewhere before it becomes negative. In that case the end of the inflation should be defined accordingly and we would choose integration constant to set \( \tau \to -0 \) at the end of inflation.

The above derivation has been done fully analytically. Let us check that the slow-roll approximation indeed breaks down in these models. It is easy to show that the second slow-roll parameter is non-negligible by plugging its definition \( \epsilon_2 = 2\epsilon_1 + 2\ddot{\phi}/(H\dot{\phi}) \) and the condition (2.6). This fact has been pointed out from the beginning of this class of models [14]. Our exact solution allows us to see the violation in higher order slow-roll parameters. For (2.18) with \( \alpha > -3 \), the first slow-roll parameter \( \epsilon_1 \) is given by

\[
\epsilon_1 = \frac{3 + \alpha}{\cosh^2[(3 + \alpha)Mt]} = \frac{3 + \alpha}{a^{2(3+\alpha)} + 1},
\]

and for higher order slow-roll parameters \( \epsilon_n \) with \( n \geq 1 \) are given by

\[
\epsilon_{2n} = -2(3 + \alpha) \tanh^2[(3 + \alpha)Mt], \quad \epsilon_{2n+1} = \frac{2(3 + \alpha)}{\cosh^2[(3 + \alpha)Mt]}. \quad (2.33)
\]

We show their evolution in figure 1 (e) and (f). In particular, for \( Mt \gg 1 \),

\[
2\epsilon_1 = \epsilon_{2n+1} \simeq 2(3 + \alpha) a^{-2(3+\alpha)}, \quad \epsilon_{2n} \simeq -2(3 + \alpha). \quad (2.34)
\]

Thus, the odd-order slow-roll parameters asymptotically approach to zero, while the even ones approach to \( -2(3 + \alpha) \), which is not necessarily negligible, rather, \( \epsilon_{2n} \) can be of order unity. Thus the slow-roll approximation clearly breaks down. Nevertheless, the Hubble parameter approaches to a constant \( H \simeq M \) and the scale factor grows as exponentially, so this is still inflation. Likewise, for (2.22) with \( \alpha < -3 \), the slow-roll parameters are given by

\[
2\epsilon_1 = \epsilon_{2n+1} = -\frac{2(3 + \alpha)}{\sinh^2[(3 + \alpha)Mt]}, \quad \epsilon_{2n} = -\frac{2(3 + \alpha)}{\tanh^2[(3 + \alpha)Mt]}. \quad (2.35)
\]
and their asymptotic behaviors are $2\epsilon_1 = \epsilon_{2n+1} \to 0$ and $\epsilon_{2n} \to -2(3 + \alpha)$ as $Mt \to -\infty$, which are the same with the case for $\alpha > -3$.

We proceed to check whether our exact solution is an attractor solution or not. We numerically solved inflationary dynamics under the assumption (2.6) with various initial conditions and obtained the phase space diagram as depicted in figure 2 for different $\alpha$. The top left panel of figure 2 for $\alpha = -0.02$ shows a typical attractor behavior where the phase space flow converges to $\dot{\phi} = 0$ and $\phi = 0$. This implies that the inflaton approaches to the global minimum of the potential at $\phi = 0$. Thus the analytic solution with negative $\alpha$ is an attractor. The top right panel for $\alpha = 0$ of figure 2 shows phase space flow converges various points depending on initial conditions. This is not surprising because we note that for $\alpha = 0$ the potential is exactly constant $V = 3M^2M_{Pl}^2$, so it is invariant under translation $\phi \to \phi + \text{const}$. The scalar field moves on the flat potential with the Hubble friction, and it eventually stops at a point which depends on an initial condition. In this case, we need to recover the integration constant for our analytic solution (2.19) and determine it according to an initial condition. Finally, the bottom panel of figure 2 for $\alpha = 0.02$ shows that the inflaton goes to $\dot{\phi} \to \pm \infty$ and $\phi \to \pm \infty$, respectively. Therefore the analytic solution for positive $\alpha$ is not an attractor. This is obvious because the potential is not bounded from below. The scalar field rolls down to $\phi \to \pm \infty$ since $V \to -\infty$. Hence, we need to make fine-tuning of the initial condition for scalar field obeying the analytical solution (2.19) for positive $\alpha$.
For a general $\alpha > -3$, this special constant-roll solution for which $\phi(t) \propto \exp[-(3+\alpha)Mt]$ in the limit $Mt \gg 1$ is stable, i.e. it is an attractor during expansion, if it grows faster with time than the second linearly independent solution for the same potential which is $\phi(t) \propto \exp(\alpha Mt)$. Therefore, the stability condition is $-(3+\alpha) > \alpha$, or $\alpha < -3/2$ that includes the slow-roll case $\alpha \approx -3$. The same formally refers to the $\alpha < -3$ case which is stable as far as inflation goes on.

3 Evolution of scalar and tensor perturbation

3.1 Scalar perturbation

We consider the gauge invariant curvature perturbation $\zeta_k$ in our models (2.18) and (2.22). It relates to the metric perturbation through $\delta g_{ij} = a^2 (1 - 2\zeta) \delta_{ij}$ in a gauge $\delta \phi = 0$. The evolution of the mode function $v_k \equiv \sqrt{2} M_{Pl} z \zeta_k$ with $z \equiv a \sqrt{\epsilon_1}$ is governed by the Mukhanov-Sasaki equation [30, 31]:

$$v_k'' + \left( k^2 - \frac{z''}{z} \right) v_k = 0,$$

(3.1)

where a prime denotes a derivative with respect to the conformal time $\tau$. The potential term $z''/z$ is exactly expressed in terms of slow-roll parameters:

$$\frac{z''}{z} = a^2 H^2 \left( 2 - \epsilon_1 + \frac{3}{2} \epsilon_2 + \frac{1}{4} \epsilon_2^2 - \frac{1}{2} \epsilon_1 \epsilon_2 + \frac{1}{2} \epsilon_2 \epsilon_3 \right).$$

(3.2)

We can solve (3.1) by the standard treatment of the slow-roll inflation except that $\epsilon_2$ is not negligible whereas $\epsilon_2 n_{-1}$ is. Starting from the sub-Hubble regime where $k^2 \gg z''/z$, (3.1) reduces to $v_k'' + k^2 v_k = 0$. We choose the adiabatic vacuum initial condition, i.e. no particles and minimum energy at $\tau \to -\infty$:

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}.$$

(3.3)

As the mode $k$ approaches to the Hubble radius crossing, the potential term $z''/z$ becomes non-negligible. Since during generation of the curvature perturbation $2\epsilon_1 = \epsilon_{2n+1} \to 0$ and $\epsilon_{2n} \to -2(3+\alpha)$ for both cases with $\alpha > -3$ and $\alpha < -3$, we can simplify the potential term as

$$\frac{z''}{z} \simeq \frac{(1+\alpha)(2+\alpha)}{\tau^2} = \frac{\nu^2 - 1/4}{\tau^2},$$

(3.4)

where

$$\nu \equiv \sqrt{(1+\alpha)(2+\alpha) + 1/4} = |\alpha + 3/2|.$$

(3.5)

With (3.3) as the boundary condition, the solution is given in terms of Hankel function as usual,

$$v_k(\tau) = \frac{\sqrt{-\pi \tau}}{2} H_{\nu}^{(1)}(-k\tau).$$

(3.6)

The power spectrum of the curvature perturbation is then given by

$$\Delta^2(k) \equiv \frac{k^3}{2\pi^2} |\zeta_k|^2 = \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_1} \left( \frac{k}{aH} \right)^3 \frac{\pi}{2} |H_{\nu}^{(1)}(-k\tau)|^2.$$

(3.7)
Using the asymptotic formula $\lim_{x \to 0} H_\nu^{(1)}(x) \simeq -\frac{i}{\pi} \Gamma(\nu) \left( \frac{x}{2} \right)^{-\nu}$, we obtain

$$
\Delta^2_s(k) = \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_1} 2^{2\nu-1}[\Gamma(\nu)]^2 \left( \frac{k}{aH} \right)^{3-2\nu},
$$

(3.8)

and the spectral index is then $n_s - 1 = 3 - 2\nu$. For given $n_s$ from observations, the constant-roll parameter $\alpha$ is given by

$$
\alpha = \frac{1 - n_s}{2} \quad \text{or} \quad \frac{n_s - 7}{2}. \quad (3.9)
$$

For instance, for $n_s = 0.96$, we can choose $\alpha = 0.02$ or $-3.02$. However, as we discussed in the previous section, the analytic solution for $\alpha = 0.02$ is not an attractor solution. In addition, we should be careful for a possible evolution of the curvature perturbation on super-Hubble scales [20]. Indeed, from (3.7) we confirmed that $\Delta^2_s(k) \propto H^{-1+2\nu} \epsilon_1^{-1} a^{-3+2\nu} \propto a^{2n+3+|2n+3|}$ evolves for $\alpha = 0.02$, but approaches to a constant for $\alpha = -3.02$. Therefore $\alpha = -3.02$ is viable, and the standard slow-roll approximation works in this case. However, our exact solution gives a possibility to sum all higher-order corrections to it, at least for the background evolution.

### 3.2 Super-Hubble evolution

Alternatively, we can examine the behavior of curvature perturbations in the super-Hubble regime $k^2 \ll z''/z$ by directly solving $v''_k - (z''/z) v_k = 0$. The formal solution for this equation is given by a linear combination of $z$ and $z \int dz'/z^2$. Since $\zeta_k \propto v_k/z$, we thus obtain

$$
\zeta_k = A_k + B_k \int \frac{dt}{a^3 \epsilon_1},
$$

(3.10)

where $A_k$ and $B_k$ are integration constants. The first term expresses the constant mode of the curvature perturbations. For slow-roll inflation, the second term yields decaying mode, thus only first term remains, and as a result the curvature perturbation is conserved outside the Hubble radius. However, this is not necessarily the case for constant-roll inflation, as we shall see below.

For the model (2.18) with $\alpha > -3$, we can perform the integral of the second term of (3.10) analytically using (2.21) and (2.32). The result is

$$
\int \frac{dt}{a^3 \epsilon_1} = 2M_{Pl}^2 \int \frac{H^2}{a^3 \epsilon_1^2} dt
$$

$$
\propto \int dt \frac{\cosh^2[(3 + \alpha)Mt] \sinh^{3+\alpha}[(3 + \alpha)Mt]}{3 + \alpha}
$$

$$
= \frac{(-1)^{\frac{6+\alpha}{2(3+\alpha)}}}{3M(3+\alpha)^2} \cosh^3[(3 + \alpha)Mt] \left( \frac{3}{2} \right) \frac{6 + \alpha}{2(3 + \alpha)} \cosh^2[(3 + \alpha)Mt],
$$

(3.11)

where $u(a, b; c; x)$ is a solution for the hypergeometric differential equation:

$$
x(1 - x) \frac{d^2 u}{dx^2} + [c - (a + b + 1)x] \frac{du}{dx} - abu = 0.
$$

(3.12)

It has two independent solutions and the solutions that converge for $x \gg 1$ are expressed in terms of the hypergeometric function. (i) For $\text{Re}(a + b - c) > 0$, two solutions are

$$
u_1 = (-x)^{b-c}(1-x)^{-a-b} _2F_1(1-b, c-b, a-b+1; 1/x) \simeq x^{-a},
$$

(3.13)

$$
u_2 = (-x)^{a-c}(1-x)^{-a-b} _2F_1(1-a, c-a, b-a+1; 1/x) \simeq x^{-b},
$$

(3.14)
and (ii) for $\text{Re}(a + b - c) < 0$,
\begin{align}
    u_3 &= (-x)^a \text{F}_1(a, a - c + 1, a - b + 1; 1/x) \simeq x^{-a}, \\
    u_4 &= (-x)^b \text{F}_1(b, b - c + 1, b - a + 1; 1/x) \simeq x^{-b}.
\end{align}
\noindent
For the case of (3.11), $-3 < \alpha < 0$ amounts to the case (i), and the integral is evaluated by $x^{3/2}u_1 \simeq \text{const}$ and $x^{3/2}u_2 \simeq x^{\frac{3 + 2\alpha}{2(3 + \alpha)}}$, where $x = \cosh^2[(3 + \alpha)Mt]$. Therefore, the first solution gives a constant mode, which is absorbed into the definition of $A_k$, whereas the second solution is a decaying mode for $-3 < \alpha < -3/2$, or a growing mode for $-3/2 < \alpha < 0$. The remaining region $\alpha > 0$ amounts to the case (ii), and two solutions behave as $x^{3/2}u_3 \simeq \text{const}$ and $x^{3/2}u_4 \simeq x^{\frac{3 + 2\alpha}{2(3 + \alpha)}}$. These asymptotic behavior of two solutions are the same with the case (i). The first solution gives a constant mode while the second one is a growing mode as $\alpha > 0$.

On the other hand, for the model (2.22) with $\alpha < -3$, we can prove that the second term of (3.10) yields sum of constant mode and decaying mode as $Mt$ increases from $-\infty$. The integral is given by
\begin{equation}
    \int \frac{dt}{a^2c_1} \propto -\frac{i}{\alpha(3 + \alpha)} \cosh^{\alpha/(3 + \alpha)} [(3 + \alpha)Mt] \left\{ -\frac{1}{2} \frac{\alpha}{2(3 + \alpha)} \frac{3(2 + \alpha)}{2(3 + \alpha)} ; \cosh^{2[(3 + \alpha)Mt]} \right\}. \tag{3.17}
\end{equation}

As $a + b - c = -3/2 < 0$ for the arguments of $u(a, b, c; x)$, its two independent solutions are given by $u_3$ and $u_4$. Thus the asymptotic behavior of the above integral is $x^b u_3 \simeq x^{b - a} = x^{\frac{3 + 2\alpha}{2(3 + \alpha)}} = x^{1 + \frac{\alpha}{3 + \alpha}}$ and $x^b u_4 \simeq \text{const}$. The latter mode is a constant mode, and the former mode is a decaying mode as $Mt$ increases from $-\infty$, namely, as $x = \cosh^2[(3 + \alpha)Mt]$ decreases from $+\infty$.

In summary, in the super-Hubble regime a curvature perturbation has two modes which asymptotically approach to
\begin{equation}
    \text{const}, \quad \text{and} \quad \cosh^{\frac{\alpha + 2\alpha}{3 + \alpha}} [(3 + \alpha)Mt], \tag{3.18}
\end{equation}
respectively. The latter one is a decaying mode for $\alpha < -3/2$, but it is a growing mode for $\alpha > -3/2$. The condition for the decaying mode coincides with the attractor condition for background evolution derived in the previous section.

In comparison with slow-roll inflation, where we always have a constant mode and a decaying mode outside the Hubble radius, we may have a growing mode in constant-roll inflation. This is because $\epsilon_1$ decays as $a^{-2(3 + \alpha)}$, which is faster than $a^{-3}$ for $\alpha > -3/2$. As a result, $\zeta \simeq \int dt/(a^2\epsilon_1)$ grows in this case. Any model with background evolution with $\epsilon_2 = d \ln \epsilon_1/d \ln a < -3$ possesses the super-Hubble evolution of the curvature perturbation. This situation is similar to what happens in the chaotic new inflation model where curvature perturbation grows anomalously in between chaotic and new inflation stages [32]. However, as was shown at the end of the previous section, $\alpha > -3/2$ is just the condition that our particular constant-roll solution for the given potential is not an attractor during expansion. Thus, in this case such solution can typically occur for no more than a few e-folds.

To confirm our discussion above, we numerically integrated (3.1) with analytic solution for the background quantities derived in the previous section. The result is shown in figure 3. In the left and right panel, we present the result for (2.18) with $\alpha = 0.02$, and the result for (2.22) with $\alpha = -3.02$, respectively. The solid blue line represents the amplitude of the curvature perturbation $|\zeta_k|$, the red dashed line expresses $(k/aH)^2$ and yellow dotted line is $(z''/z)/(aH)^2$. We normalized the e-folds $N = \ln a$ so that $k^2 = z''/z$ at $N = 0$, which
The situation for $\alpha = 0.02$ above is similar to the generalized ultra-slow-roll inflation [20], which also has a growing mode and results in an unwanted amplification of the curvature perturbation outside the Hubble radius. Combined with the observational value of the amplitude of the scalar power spectrum, this growth of the curvature perturbation should be compensated by imposing extremely small energy scale of inflation, at least $M \approx 10^{-12} M_{Pl}$. This is much smaller than the BBN bound $M > O(\text{MeV})$. The same occurs for constant-roll inflation. Thus, the case $\alpha \gtrsim 0$ with (2.18) does not give us a possibility to construct a physically relevant inflationary model.

Much more perspective appears the case $\alpha \lesssim -3$ with (2.22). However, in this case the potential has to be modified at some value $\phi = \phi_0$ which is less than the critical value $\phi_c = \pi M_{Pl}/\sqrt{2|3+\alpha|}$ where $V(\phi) < 0$ and $H$ changes the sign. The aim of this modification is to finish inflation at $\phi = \phi_0$. This situation is completely analogous to what happens in the case of power-law inflation. So, then our solution can be applied not to the whole evolution of the Universe but to its evolution during inflation only (apart from its very end) including the observable range of e-folds. The best-fit value of the constant-roll parameter $\alpha$ and the prediction for the tensor-to-scalar ratio $r$ will depend on $\phi_0$. We shall see below that $|3+\alpha|$ has to be small for any reasonable value of $\phi_0$, so $\phi_c \gg M_{Pl}$ anyway. Thus, depending on the value of $\phi_0$, such a model can describe both small-field and large-field inflation.
Let us first consider the case when $\phi_0 \ll \phi_c$. Then $V(\phi)$ can be approximated as

$$V(\phi) = 3M^2M_{\text{Pl}}^2 \left( 1 - \frac{\alpha(3 + \alpha)\phi^2}{6M_{\text{Pl}}^2} \right), \quad (3.19)$$

during the whole inflation. In this case $n_s - 1$ is constant and equal to $-2\alpha(3 + \alpha)/3$. So, the best fit from the recent CMB data is $\alpha \approx -3.02$. As was shown above, the curvature perturbation remains constant on super-Hubble scales in this case. The number of e-folds from the end of inflation is

$$N = M_{\text{Pl}}^{-2} \int_0^{\phi_0} \frac{V}{dV/d\phi} d\phi = \frac{3}{\alpha(3 + \alpha)} \ln \left( \frac{\phi_0}{\phi} \right) \approx 50 \ln \left( \frac{\phi_0}{\phi} \right). \quad (3.20)$$

Thus, due to numerical coincidence between the measured value of $1 - n_s$ and $2/N_{H_0}$ where $N_{H_0} = 50 - 60$ corresponds to the present Hubble radius $H_0^{-1}$, the inflaton field value in the observable range of e-folds during inflation is $\approx \phi_0 e^{-1}$. This provides a possibility to significantly enlarge possible range for an initial condition for the inflaton field at the local beginning of inflation: it is sufficient to have its any value less than $\approx \phi_0 e^{-1}$.

If now $\phi_0 \lesssim \phi_c$, but not specifically close to it, the potential (2.22) and the corresponding exact constant-roll solution (2.23)–(2.25) may be taken as they are up to $\phi = \phi_0$. Then the model looks like the so called ‘natural inflation’ [25] but with the additional negative cosmological constant $\Lambda = M^2(3 + \alpha) < 0$. Also, it is always large-field inflation. In this case, the measured value of $n_s - 1$ does not determine $\alpha$ unambiguously, it provides only an upper bound on $|3 + \alpha|$. Still $|3 + \alpha|$ has to be small, so $|\Lambda| \ll M^2$. As a result, this cosmological constant is subdominant during inflation. However, it affects higher-order slow-roll corrections. Therefore, the model (2.22) represents a novel example of viable and exactly analytically solvable (for the evolution of background and perturbations in the super-Hubble regime) model of inflation.

The limiting, although somewhat fine-tuned case, occurs if $\phi_0 = \phi_c - O(M_{\text{Pl}})$. Then at the last stage of inflation, when $\dot{\phi} \gg M_{\text{Pl}}$ and $\phi \gg M_{\text{Pl}}$ where $\ddot{\phi} = \phi_c - \phi$, we get

$$H \approx M \sqrt{\frac{|3 + \alpha|}{2}} \frac{\ddot{\phi}}{M_{\text{Pl}}}, \quad V \approx \frac{\alpha(3 + \alpha)}{2} M_{\text{Pl}}^2 \ddot{\phi}^2, \quad (3.21)$$

i.e. just quadratic chaotic inflation. Then $n_s - 1 = -2/N$, and we obtain its correct value independently of the value of $\alpha$. However, the upper limit on $\alpha$ follows from the condition that the approximation (3.21) still works for $N = N_{H_0}$ when $\phi \approx 15M_{\text{Pl}}$. From the condition that the latter quantity should be $\ll \phi_c$, it follows that $|3 + \alpha| \ll 0.02$, i.e. much less than in the hilltop case $\phi_0 \ll \phi_c$ considered above.

### 3.3 Tensor perturbation

Finally, we consider the tensor perturbation $\delta g_{ij} = a^2 h_{ij}$ for our models. The evolution equation for the mode function of the tensor perturbation $u_{k,\lambda} \equiv aM_{\text{Pl}} h_{k,\lambda}/2$ is given by

$$u_{k,\lambda}'' + \left( k^2 - \frac{a''}{a} \right) u_{k,\lambda} = 0, \quad (3.22)$$

where $\lambda = +, \times$ denotes the two polarization modes of the gravitational waves. As $a''/a = (aH)^2(2 - \epsilon_1) \simeq (2 + 3\epsilon_1)/\tau^2 \simeq 2/\tau^2$, we obtain (in agreement with [33])

$$\Delta_l^2 = \frac{2H^2}{\pi^2 M_{\text{Pl}}^2}, \quad (3.23)$$
and thus the standard consistency relation for the tensor-to-scalar ratio

$$r = \frac{\Delta_t^2}{\Delta_s^2} \approx 16c_1 = 32M_{Pl}^2 \left( \frac{d \ln H(\phi)}{d\phi} \right)^2 \approx 8M_{Pl}^2 \left( \frac{d \ln V(\phi)}{d\phi} \right)^2$$

(3.24)

holds with both \approx signs becoming = in the leading order of the slow-roll approximation.

For the quadratic hilltop case $\phi_0 \ll \phi_c$, we, therefore, get $r = 8(3 + \alpha)^2 \phi^2 / M_{Pl}^2$. So, parametrically by powers of $|3 + \alpha|$, $r$ is of the order of $N^{-2}$. However, actually $r$ can be much less than the latter quantity if $\phi_0 \ll M_{Pl}$. To get $r \sim N^{-2}$ as, e.g. in the $R + R^2$ inflationary model [1], one needs $\phi_0 \sim M_{Pl}$. Larger values of $r$ in constant-roll inflation, of the order $N^{-1}$ parametrically, can be obtained if $M_{Pl} \ll \phi_0 \sim \phi_c$. Then the exact background solution (2.23)–(2.25) has to be used. Finally, in the limiting case $\phi_0 = \phi_c - O(M_{Pl})$, $r$ reaches the value $8/N (\sim 0.14$ for $N = N_{H_0}$) as for the quadratic chaotic inflation, but this model lies just beyond the $2\sigma$ CL contour for the recent Planck data [34].

4 Conclusion

We have investigated an inflationary scenario where the rate of roll defined by $\ddot{\phi}/H\dot{\phi} = -3-\alpha$ remains constant. This class includes slow-roll inflation with negligible rate of roll and fast-roll inflation, in particular, the so called ‘ultra-slow-roll’ one. We find all exact solutions satisfying the constant-rate-of-roll ansatz. They include power-law inflation, the solution found in [23] in a different context, and particular (and somewhat modified) cases of hilltop inflation and natural inflation. In this class of models, even-order slow-roll parameters can be of unity while odd-order slow-roll parameters are asymptotically negligible. It turns out that it is difficult for $\alpha > -3$ to use it to explain the observed Universe due to the anomalous super-Hubble evolution of curvature perturbations. That is, for the model parameter $\alpha$ which yields a slightly red-tilted power spectrum of curvature perturbations, they grow outside the Hubble radius. In order to reproduce the observed amplitude of fluctuations in the presence of such growth, we need an extremely low energy scale of inflation, which is much smaller than BBN bound. Therefore, the case with $\alpha > -3$ is not observationally feasible. On the other hand, for the constant-roll model (2.22) for $\alpha < -3$, the curvature perturbation has a constant mode and a decaying mode on super-Hubble scales, as the standard slow-roll inflation does. Therefore, the model (2.22) with $\alpha \lesssim -3$ is a novel analytically solvable and observationally viable inflationary model with a constant rate of roll, which possesses an attractor background evolution, slightly red-tilted scalar spectrum, and conservation of the curvature perturbation on super-Hubble scales. For a realistic model, we have to cut the potential (2.22) somewhere before it becomes negative, and it has to be changed after that in order to have subsequent reheating and radiation-dominated regimes. For the best-fit choice of its parameters, this model can reproduce the measured value of the slope of the primordial power spectrum of scalar (density) perturbations $n_s \approx 0.96$. The prediction for the tensor-to-scalar ratio $r$ depends on the potential cutoff location, and can be both less than 1%, and in the range $1\% \lesssim r \lesssim 10\%$, too, that is interesting for future search of primordial gravitational waves from inflation.
Acknowledgments

The authors thank J. D. Barrow and T. Suyama for useful discussions. H.M. was partially supported by Japan Society for the Promotion of Science (JSPS) Postdoctoral Fellowships for Research Abroad and J.Y. by JSPS Grant-in-Aid for Scientific Research No. 23340058. A.S. acknowledges RESCEU hospitality as a visiting professor. He was also partially supported by the grant RFBR 14-02-00894 and by the Scientific Programme “Astronomy” of the Russian Academy of Sciences.

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