Quantized Compressive Sensing with RIP Matrices: 
The Benefit of Dithering

Chunlei Xu* and Laurent Jacques*

January 19, 2018

Abstract

In Compressive Sensing theory and its applications, quantization of signal measurements, as integrated into any realistic sensing model, impacts the quality of signal reconstruction. In fact, there even exist incompatible combinations of quantization functions (e.g., the 1-bit sign function) and sensing matrices (e.g., Bernoulli) that cannot lead to an arbitrarily low reconstruction error when the number of observations increases.

This work shows that, for a scalar and uniform quantization, provided that a uniform random vector, or random dithering, is added to the compressive measurements of a low-complexity signal (e.g., a sparse or compressible signal, or a low-rank matrix) before quantization, a large class of random matrix constructions known to respect the restricted isometry property (RIP) are made “compatible” with this quantizer. This compatibility is demonstrated by the existence of (at least) one signal reconstruction method, the projected back projection (PBP), whose reconstruction error is proved to decay when the number of quantized measurements increases.

Despite the simplicity of PBP, which amounts to projecting the back projection of the compressive observations (obtained from their multiplication by the adjoint sensing matrix) onto the low-complexity set containing the observed signal, we also prove that given a RIP matrix and for a single realization of the dithering, this reconstruction error decay is also achievable uniformly for the sensing of all signals in the considered low-complexity set.

We finally confirm empirically these observations in several sensing contexts involving sparse signals, low-rank matrices, and compressible signals, with various RIP matrix constructions such as sub-Gaussian random matrices and random partial Discrete Cosine Transform (DCT) matrices.

1 Introduction

Compressive sensing (CS) theory \cite{1,2,3} has shown us how to compressively and non-adaptively sample low-complexity signals, such as sparse vectors or low-rank matrices, in high-dimensional domains. In this framework, accurate estimation of such signals from their compressive measurements is still possible thanks to non-linear reconstruction algorithms (e.g., $\ell_1$-norm minimization, greedy algorithms) exploiting the signal low-complexity nature. In other words, by generalizing the concepts of sampling and reconstruction, CS has somehow extended Shannon-Nyquist theory initially restricted to the class of band-limited signals.

*CX and LJ are with Image and Signal Processing Group (ISPGroup), ICTEAM/ELEN, Université catholique de Louvain (UCL). E-mail: \{chunlei.xu,laurent.jacques\}@uclouvain.be. The authors are funded by the Belgian F.R.S.-FNRS. Part of this study is funded by the project ALTERSENSE (MIS-FNRS).
Specifically, given a sensing (or measurement) matrix $\Phi \in \mathbb{R}^{m \times n}$ with $m \leq n$, CS describes how one can recover a signal $x \in \mathbb{R}^n$ from the $m$ measurements associated with the underdetermined linear model

$$y = \Phi x + n,$$

where $y \in \mathbb{R}^m$ is the measurement vector, $n \in \mathbb{R}^m$ stands for a possible additive measurement noise, and $x$ is assumed restricted to a low-complexity signal set $\mathcal{K} \subset \mathbb{R}^n$, e.g., the set $\Psi \Sigma_k^n := \{\Psi \alpha \in \mathbb{R}^n : \|\alpha\|_0 = |\text{supp}(\alpha)| \leq k, \alpha \in \mathbb{R}^n\}$ of $k$-sparse vectors in an orthonormal basis $\Psi \in \mathbb{R}^{n \times n}$. In particular, it has been shown that the recovery of $x$ is guaranteed if $\frac{1}{\sqrt{m}} \Phi$ respects the Restricted Isometry Property (RIP) over $\mathcal{K}$, which essentially states that $\frac{1}{\sqrt{m}} \Phi$ behaves as an approximate isometry for all elements of $\mathcal{K}$ (see Sec. 3.1). Interestingly many random constructions of sensing matrices have been proved to respect the RIP with high probability (w.h.p.) \cite{4, 3}. For instance, if $\Phi$ is a Gaussian random matrix with entries identically and independently distributed (i.i.d.) as a standard normal distribution $\mathcal{N}(0, 1)$, $\frac{1}{\sqrt{m}} \Phi$ respects the RIP over $\mathcal{K} = \Psi \Sigma_k^n$ with very high probability provided $m = O(k \log(n/k))$. For a more general set $\mathcal{K}$, the RIP is verified as soon as $m$ is sufficiently large compared to the intrinsic complexity of $\mathcal{K}^* := \mathcal{K} \cap \mathbb{R}^n$ in $\mathbb{R}^n$, e.g., as measured by the squared Gaussian mean width $w(\mathcal{K}^*)^2$ or the Kolmogorov entropy of $\mathcal{K}^*$ \cite{7, 5, 4} (see Sec. 3.1 and Sec. 7).

Under the satisfiability of the RIP, many signal reconstruction methods (e.g., Basis Pursuit DeNoise \cite{1}, or greedy algorithms such as Orthogonal Matching Pursuit \cite{8} or Iterative Hard Thresholding [3]) achieve a stable and robust estimate of $x$ from the sensing model (1), e.g., for $\mathcal{K} = \Sigma_k^n$. They typically display the following reconstruction error bound, or $\ell_2 - \ell_1$ instance optimality \cite{9, 1, 2},

$$\|x - ˆx\| \leq C\|x - x_k\|_1 + D\epsilon,$$

where $ˆx$ is the signal estimate, $x_k$ the best $k$-term approximation of $x$, $\epsilon \geq \|n\|$ is an estimator bounding the noise energy, and $C, D > 0$ are only depending on $\Phi$.

In this brief overview of CS theory, we thus see that, at least in the noiseless setting, the measurement vector $y$ is assumed represented with infinite precision. However, any realistic device model imposes digitalization and finite precision data representations, e.g., to store, transmit or process the acquired observations. In particular, (1) must be turned into a Quantized CS formalism where the objective is to reliably estimate a low-complexity signal $x \in \mathcal{K} \subset \mathbb{R}^n$ from the quantized measurements

$$y = \mathcal{Q}^k(\Phi x).$$

In \cite{3}, $\mathcal{Q}^k : u \in \mathbb{R}^m \mapsto \mathcal{Q}^k(u) \in \mathcal{A} \subset \mathbb{R}^m$ is a general quantization function, or quantizer, mapping $m$-dimensional vectors to some vectors in a discrete set, or codebook, $\mathcal{A} \subset \mathbb{R}^m$.

While \cite{10} only studied uniform quantization of CS measurements as an additive, bounded noise in (1), inducing thus a constant error bound in \cite{2}, \cite{11}, \cite{12}, various kind of quantizers have since then been studied more deeply in the context of QCS \cite{12}. Their list includes $\Sigma\Delta$-quantization \cite{11}, non-regular scalar quantizers \cite{13}, non-regular binned quantization \cite{14} \cite{15}, and even vector quantization by frame permutation \cite{16}. These quantizers, when combined with an appropriate signal

\footnote{Some of the mathematical notations and conventions used below are defined at the end of this section.}

\footnote{Hereafter, we will write w.h.p. if the probability of failure of the considered event decays exponentially with respect to the number of measurements.}
reconstruction procedure, achieve different decay rate of the reconstruction error when the number of measurements \( m \) increases. For instance, for a \( \Sigma \Delta \)-quantizer combined with Gaussian or sub-Gaussian sensing matrices \([11]\), or with random partial circulant matrices generated by a sub-Gaussian random vector \([17]\), this error can decay polynomially in \( m \) for an appropriate reconstruction procedure, and, in the case of a 1-bit quantizer, adapting the sign quantizer by inserting in it adaptive thresholds can even lead to exponential decay \([18]\).

In this paper, our objective is, however, not to focus on optimizing the quantizer to achieve the best decay rate for the reconstruction error of some appropriate algorithm when \( m \) increases. Actually, our aim is to show that a simple scalar quantization procedure, \textit{i.e.}, a uniform quantizer, applied componentwise onto vectors (or entry-wise on matrices), is compatible with the large class of sensing matrices known to satisfy the RIP, provided that we combine the quantization with a random, uniform pre-quantization \textit{dithering} \([13,19,20]\). This access to a broader set of sensing matrices for QCS, \textit{i.e.}, not only restricted to unstructured sub-Gaussian random constructions, is indeed desirable in many CS applications where specific, structured sensing matrices are constrained by technology or physics, such as (random) partial Fourier/DCT matrices in magnetic resonance imaging \([21]\), in radio-astronomy \([22]\) or in radar or communication applications \([23,24]\). Moreover, in this context, we focus on the estimation of signals belonging to a general low-complexity set \( \mathcal{K} \) in \( \mathbb{R}^n \), \textit{e.g.}, the set of sparse or compressible vectors, the set of low-rank matrices, or any set having a small Kolmogorov entropy (see Sec. 3.1 and Sec. 7), provided that this set also supports the RIP of \( \frac{1}{\sqrt{m}} \Phi \), \textit{i.e.}, we want the reconstruction guarantees of QCS to reduce to those of CS if the quantization disappears (\textit{e.g.}, when its precision becomes infinite).

Mathematically, our work considers the problem of estimating a signal \( x \) from the QCS model
\[
y = A(x) = A(x; \Phi, \xi) := Q(\Phi x + \xi),
\]
where \( A \) is a quantized random mapping, \textit{i.e.}, \( A : \mathbb{R}^n \mapsto \delta \mathbb{Z}^m \), \( Q(\cdot) := \delta [\cdot] \) is a uniform scalar quantization of resolution \( \delta > 0 \), and \( \xi \) is a uniform random dithering vector whose components are i.i.d. as a uniform distribution over \([0, \delta]\), \textit{i.e.}, \( \xi_i \sim \text{i.i.d. } U([0, \delta]) \) for \( i \in [m] \), or, more briefly, \( \xi \sim U^m([0, \delta]) \).

The compatibility mentioned above between the QCS model \([4]\) and the class of RIP matrices is demonstrated by showing that a simple (often non-iterative) reconstruction method, the Projected Back Projection (PBP) of the quantized measurements \( y \) onto the set \( \mathcal{K} \), \textit{i.e.}, finding the closest point \( \hat{x} \) in \( \mathcal{K} \) to the back projection \( \frac{1}{m} \Phi^\top y = Q(\Phi x + \xi) \) for any \( x \in \mathcal{K} \) (see Sec. 4), achieves a reconstruction error \( \|x - \hat{x}\| \) that decays like \( O(m^{-1/p}) \) when \( m \) increases, for some \( p > 1 \) only depending on \( \mathcal{K} \).

For instance, we prove in Sec. 7 that, given a RIP matrix \( \frac{1}{\sqrt{m}} \Phi \) and a fixed signal \( x \in \mathcal{K} \), if the dithering \( \xi \) is random and uniform over \([0, \delta]^m\), then one achieves, \textit{w.h.p.}, \( \|x - \hat{x}\| = O(m^{-1/2}) \) when \( \mathcal{K} \) is the set of sparse vectors, the set of low-rank matrices \([3]\) or any finite union of low-dimensional subspaces, as with model-based CS schemes \([26]\) or group-sparse signal models \([27]\). Interestingly, for these specific sets, the same error decay rate is proved, up to extra log factors in the involved dimensions, in a uniform setting, \textit{i.e.}, when the randomly generated \( \xi \) allows the estimation of all vectors of \( \mathcal{K} \) \textit{w.h.p..} More generally, if \( \mathcal{K} \) is a convex and bounded set of \( \mathbb{R}^m \), \textit{e.g.}, the set of

---

\(^3\)The term “resolution” does not refer here to the number of bits used to encode the quantization bins \([25]\).

\(^4\)Up to the identification of these matrices with their vector representation.
compressible signals $\Sigma_k^u := \{u \in \mathbb{R}^n : \|u\|_1 \leq \sqrt{k}, \|u\|_2 \leq 1 \} \supset \Sigma_k^u \cap \mathbb{B}^n$, we observe that $p = 8$ and $p = 16$ in the non-uniform and in the uniform setting, respectively.

Knowing if other reconstruction algorithms can reach faster error decay is a matter of future study, in this regard, PBP can be seen as a reconstruction principle providing an initial guide for more advanced reconstruction algorithms, e.g., iteratively enforcing the consistency of the estimate with the observations $y$ from an initial guess provided by PBP [28, 29, 30, 31, 32].

In all our developments, the importance of the random dithering in the QCS model [4] founds its origin in the simple observation that, for $u \sim \mathcal{U}([0,1])$, $\mathbb{E}[\lambda + u] = \lambda$ for all $\lambda \in \mathbb{R}$ (see Lemma A.1 in Appendix A). By the law of large numbers, this thus means that for $m$ different r.v.’s $u_i \sim$ i.i.d. $\mathcal{U}([0,1])$ with $1 \leq i \leq m$ and $m$ increasingly large, an arbitrary projection of the vector $r := |a + u| - a := [(a_1 + u_1) - a_1, \ldots, (a_m + u_m) - a_m]^\top$ for some vector $a \in \mathbb{R}^m$ onto a fixed direction $b \in \mathbb{R}^m$ tends to $b^\top \mathbb{E}r$ that is zero when $m$ increases. Moreover, this effect should persist for all $a$ and $b$ selected in a set whose dimension is small compared to $\mathbb{R}^m$, and, in our case of interest, if these vectors are selected in the image of a low-complexity set $\mathcal{K} \cap \mathbb{B}^n$ by a RIP matrix $\frac{1}{\sqrt{m}} \Phi$.

In order to accurately bound the deviation between these projections and zero, we prove using tools from measure concentration theory (and some extra care to deal with the discontinuities of $Q$) that, given a resolution $\delta > 0$ and for $m$ large before the intrinsic complexity of $\mathcal{K}$ (as measured by its Kolmogorov entropy), the quantized random mapping $A : u \rightarrow \delta^{-1}(\Phi u + \xi)$ associated with a RIP matrix $\frac{1}{\sqrt{m}} \Phi$ and a random dithering $\xi$ respects, w.h.p., the Limited Projection Distortion property over $\mathcal{K}$, or LPD, defined by

$$\frac{1}{m} | \langle A(u), \Phi v \rangle - \langle \Phi u, \Phi v \rangle | \leq \nu, \quad \forall u, v \in \mathcal{K} \cap \mathbb{B}^n,$$

where $\nu > 0$ is a certain distortion depending on $\Phi$, $\delta$, $n$, and $m$. In fact, we will see in Sec. 6 that $\nu = \frac{1}{\sqrt{\log n}}$ if the dithering is random and uniform, and where $\epsilon$ is an arbitrary small distortion impacting the requirement on $m$. For instance, forgetting all other dependencies, $m = O(\epsilon^{-2} \log(1/\epsilon))$ for the set of sparse vectors as classically established for ensuring the RIP of a Gaussian random matrix $[4]$ (see Sec. 6 and Sec. 7.3). Moreover, by localizing the LPD on a fixed $u \in \mathcal{K}$, the impact of quantization is reduced and $\nu = \delta \epsilon$, as deduced in Sec. 5.

Interestingly, the LPD is useful to characterize the reconstruction error of PBP. This is easily understood in the case of the estimation of $k$-sparse signals in $\Sigma_k^u$. Postponing the accurate proof of this fact to Sec. 4 we first observe that if $\frac{1}{\sqrt{m}} \Phi$ respects the RIP over $\Sigma_k^u$ with distortion $\epsilon$ one can show that $\frac{1}{m} | \langle \Phi u, \Phi v \rangle - \langle u, v \rangle | \leq \epsilon$ for all $u, v \in \Sigma_k^u \cap \mathbb{B}^n$ (see Lemma 3.5). Therefore, if $A$ satisfies the LPD property over the distortion $\nu$, then, a simple use of the triangular inequality provides

$$\frac{1}{m} | \langle A(u), \Phi v \rangle - \langle u, v \rangle | \leq (\epsilon + \nu), \quad \forall u, v \in \mathcal{K} \cap \mathbb{B}^n.$$

Therefore, for a bounded sparse signal $x \in \Sigma_k^u \cap \mathbb{B}^n$, its estimate $\hat{x} \in \Sigma_k^n$ provided by the PBP of the quantized observations $y = A(x)$ is such that $\hat{x}$ is the best $k$-sparse approximation of $a = \frac{1}{m} \Phi^\top y$. However, it is also the best $k$-sparse approximation of the $2k$-sparse vector $\hat{a}$ whose entries are equal to those of $a$ if they are indexed in $T := \text{supp}(\hat{x}) \cup \text{supp}(x)$ and to 0 otherwise. Therefore, $\|\hat{x} - x\| \leq \|\hat{x} - \hat{a}\| + \|x - \hat{a}\| \leq 2\|x - \hat{a}\|$. Moreover, by the definition of the $\ell_2$-norm,

$$\|x - \hat{a}\| = \sup_{v \in \mathbb{B}^n} \langle v, x - \hat{a} \rangle = \sup_{v \in \Sigma_k^u \cap \mathbb{B}^n} \langle v, x - a \rangle = \sup_{v \in \Sigma_k^u \cap \mathbb{B}^n} \langle v, x \rangle - \frac{1}{m} \langle \Phi v, A(x) \rangle.$$
with $\Sigma_k$ the set of vectors in $\mathbb{R}^n$ supported on $\mathcal{T}$, i.e., the support of $x - \bar{a}$. Consequently, since $v$ is at most $2k$-sparse, the LPD of $A$ over $\Sigma_{2k}^n$ with distortion $\nu$ provides finally the bound $\|\hat{x} - x\| \leq 2(\epsilon + \nu)$ on the reconstruction error of PBP.

The rest of the paper is structured as follows. We present in Sec. 2 a few works related to our study, namely former usages of the PBP method in 1-bit CS, other definitions in 1-bit CS and in non-linear CS of matrix properties similar to our definition of the LPD, and certain known reconstruction error bounds of PBP and related algorithms for a few QCS and non-linear sensing contexts. In this presentation of the state-of-the-art, we note that all works are based on (sub)Gaussian random projections of signals altered by quantization or other non-linear disturbances, with one noticeable exception using subsampled Gaussian circulant sensing matrix $[33]$. After having introduced a few preliminary concepts in Sec. 3, such as the characterization of low-complexity spaces, the PBP method and the formal definition of the (L)LPD, Sec. 4 establish the reconstruction error bound of PBP when the LPD of $A$ and the RIP of $\frac{1}{\sqrt{m}}\Phi$ are both verified. We realize this analysis for three kinds of low-complexity sets, namely, finite union of low-dimensional spaces (e.g., the set of (group) sparse signals), the set of low-rank matrices, and the (unstructured) case of a general bounded convex set. In Sec. 5, we prove that the L-LPD holds w.h.p. over low-complexity sets for linear sensing model corrupted by additive sub-Gaussian noise. This analysis will later simplify the characterization of PBP of QCS observations in the non-uniform case when the observed signal is fixed prior to the generation of the random dithering. In Sec. 6, we prove that the quantized random mapping $A$ integrating a uniform random dithering is sure to respect the (uniform) LPD w.h.p. provided $m$ is large before the complexity of $K$. From the results of these two last sections, we instantiate in Sec. 7 the general bounds found in Sec. 4 and establish the decay rate of the PBP reconstruction error when $m$ increases for the same low-complexity sets considered in Sec. 4 and for several classes of RIP matrices including sub-Gaussian and structured random sensing matrices. Finally, in Section 8, we numerically validate the error distortions via the PBP over the special sets discussed in Sec. 2.

Conventions and notations: We find it useful to introduce this introduction with the conventions and notations used throughout this paper. We denote vectors and matrices with bold symbols, e.g., $\mathbf{F} \in \mathbb{R}^{m \times n}$ or $\mathbf{u} \in \mathbb{R}^m$, while lowercase light letters are associated with scalar values. The identity matrix in $\mathbb{R}^n$ reads $I_n$, and the zero vector $0 := (0, \ldots, 0)^\top \in \mathbb{R}^n$, its dimension being clear from the context. The $i$th component of a vector (or of a vector function) $\mathbf{u}$ reads either $u_i$ or $(\mathbf{u})_i$, while the vector $\mathbf{u}_i$ may refer to the $i$th element of a set of vectors. The set of indices in $\mathbb{R}^d$ is $[d] := \{1, \ldots, d\}$ and the support of $\mathbf{u} \in \mathbb{R}^d$ is supp $\mathbf{u} \subset [d]$. The Kronecker symbol is denoted by $\delta_{ij}$ and is equal to 1 if $i = j$ and to 0 otherwise, while the indicator $\chi_S(i)$ of a set $S \subset [d]$ is equal to 1 if $i \in S$ and to 0 otherwise. For any $S \subset [d]$ of cardinality $S = |S|$, $\mathbf{u}_S \in \mathbb{R}^S$ denotes the restriction of $\mathbf{u}$ to $S$, while $\mathbf{B}_S$ is the matrix obtained by restricting the columns of $\mathbf{B} \in \mathbb{R}^{d \times d}$ to those indexed by $S$. The complement of a set $S$ reads $S^c$. For any $p \geq 1$, the $\ell_p$-norm of $\mathbf{u}$ is $\|\mathbf{u}\|_p = (\sum_i |u_i|^p)^{1/p}$ with $\|\cdot\| = \|\cdot\|_2$. The $(n - 1)$-sphere in $\mathbb{R}^n$ in $\ell_p$ is $S^{n-1}_{\ell_p} = \{x \in \mathbb{R}^n : \|x\|_p = 1\}$. For $\ell_2$, we write $B^n = B^n_{\ell_2}$ and $S^{n-1} = S_{\ell_2}^{n-1}$. By extension, $B^n_{F}$ is the Frobenius unit ball of $n_1 \times n_2$ matrices $\mathbf{U}$ with $\|\mathbf{U}\|_F \leq 1$, where the Frobenius norm $\|\cdot\|_F$ is associated with the scalar product $\langle \mathbf{U}, \mathbf{V} \rangle = \text{tr}(\mathbf{U}^\top \mathbf{V})$ through $\|\mathbf{U}\|_F^2 = \langle \mathbf{U}, \mathbf{U} \rangle$, for two matrices $\mathbf{U}, \mathbf{V}$. The common flooring operator is denoted $\lfloor \cdot \rfloor$. An important feature of our study is that we do not pay particular attention to constants in the many bounds developed in this paper. For instance, the symbols $C, C', C'', \ldots, c, c', c'', \ldots > 0$ are
positive and universal constants whose value can change from one line to the other. We also use the ordering notations $A \leq B$ (or $A \gtrsim B$), if there exists a $c > 0$ such that $A \leq cB$ (resp. $A \gtrsim cB$) for two quantities $A$ and $B$.

Concerning statistical quantities, $X^{m \times n}$ and $X^m$ denote an $M \times N$ random matrix or an $m$-length random vector, respectively, whose entries are identically and independently distributed (or i.i.d.) as the probability distribution $X$, e.g., $N^{m \times n}(0, 1)$ (or $U^m([0, 1])$) is the distribution of a matrix (resp. vector) whose entries are i.i.d. as the standard normal distribution $N(0, 1)$ (resp. the uniform distribution $U([0, 1])$). We also use extensively the sub-Gaussian and sub-exponential characterization of random variables (or r.v.) and of random vectors detailed in \[34\]. The sub-Gaussian and the sub-exponential norms of a random variable $X$ are thus denoted by $\|X\|_{\psi_2}$ and $\|X\|_{\psi_1}$, respectively, with the Orlicz norm $\|X\|_{\psi_2} := \sup_{p \geq 1} p^{-1/\alpha} (\mathbb{E}|X|^p)^{1/p}$ for $\alpha \geq 1$. The random variable $X$ is therefore sub-Gaussian (or sub-exponential) if $\|X\|_{\psi_2} < \infty$ (resp. $\|X\|_{\psi_1} < \infty$).

### 2 Related works

We now provide a comparison of our work with the most relevant literature in the fields of 1-bit and quantized compressive sensing, and in signal recovery from non-linear sensing model. All the results presented below are summarized in Table 1, reporting there, amongst other aspects, the sensing model, the algorithm, the type of admissible sensing matrices and the low-complexity sets chosen in each of the referenced works.

#### PBP in 1-bit CS:
Recently, signal reconstruction via projected back projection has been studied in the context 1-bit compressive sensing (1-bit CS), an extreme QCS scenario where only the sign
of the compressive measurements are retained \([40, 18, 31, 39]\). In this case \((3)\) is turned into

\[
y = \text{sign} (\Phi x).
\]

It has been shown that if the sensing matrix \(\Phi \in \mathbb{R}^{m \times n}\) satisfies the sign product embedding property (SPE) over \(K = \Sigma_k^\mathbb{R} \cap S^{n-1}\) \([39, 31]\), that is, up to some distortion \(\epsilon > 0\) and some universal normalization \(\mu > 0\),

\[
|\mu \langle \text{sign}(\Phi u), \Phi v \rangle - \langle u, v \rangle| \leq \epsilon,
\]

for all \(u, v \in K\), then the reconstruction error of the PBP of \(y\) is bounded by \(2\epsilon\) \([31\text{ Prop. 2}]\). In other words, for a signal \(x \in \Sigma_k^n\) with unknown norm, since the binary measurements are invariant under a positive renormalization of the signal \([40]\), the PBP method allows us to estimate the direction of a sparse signal, but not its norm. This remains true for all methods assuming \(x\) to be of unit norm, as those explained below.

So far, the SPE property has only been proved for Gaussian sensing matrices, with i.i.d. standard normal entries and \(\mu = \sqrt{2/\pi}\). Such matrices respect the SPE with high probability if \(m = O(e^{-6k} \log \frac{n}{k})\), conferring to PBP a (uniform) reconstruction error decay of \(O(m^{-1/6})\) when \(m\) increases for all \(x \in \Sigma_k^n \cap S^{n-1}\). Besides, by localizing the SPE to a given \(u \in \Sigma_k^m \cap S^{n-1}\), a non-uniform variant of the previous result, i.e., where \(\Phi\) is randomly drawn conditionally to the knowledge of \(x\), gives a faster error decay of \(O(m^{-1/2})\) \([31\text{ Prop. 2}]\).

For more general low-complexity set \(K \subset \mathbb{B}^n\) (such that \(K \cap S^{n-1} \neq \emptyset\)) and with \(x \in K \cap S^{n-1}\), provided \(\Phi\) is a random sub-Gaussian matrix (with i.i.d. centered sub-Gaussian random entries of unit variance \([34]\)), a variant of the PBP method, which amounts to finding the vector \(\hat{x} \in K\) maximizing its scalar product with \(\Phi^\top y\), is proved to reach, with high probability, small reconstruction error \([38]\), and this even if the binary sensing model \([5]\) is noisy (e.g., with possible random sign flips on a small percentage of the measurements). In fact, this error decays like \(Cm^{-1/4} + D\|x\|_\infty^{1/4}\) when \(m\) increases, with \(C > 0\) depending only on the level of measurement noise, on the distribution of \(\Phi\), and on \(K\) (actually, on its Gaussian mean width, see Sec. 7), while \(D > 0\) is associated with the non-Gaussian nature of the sub-Gaussian random matrix \(\Phi\) (i.e., \(D = 0\) if it is Gaussian) \([38, 33\text{ Thm 1.1}]\). Therefore, as a paid-off for having a 1-bit sub-Gaussian sensing (e.g., 1-bit Bernoulli sensing), the reconstruction error is not anymore guaranteed to decrease below a certain floor level \(D\), and this level is driven by the sparsity of \(x\) (i.e., it is high if \(x\) is very sparse). In fact, in the case of 1-bit Bernoulli sensing, there exist counterexamples of 2-sparse signals that cannot be reconstructed, i.e., with constant reconstruction error if \(m\) increases, showing that the bound above is tight \([39]\).

More recently, \([33]\) has shown that, if \(\Phi\) is a subsampled Gaussian circulant sensing matrix in the binary observation model \([5]\), PBP can reconstruct the direction of any sparse vector up to an error decaying as \(O(m^{-1/4})\) (see \([33\text{ Thm 4.1}]\)). Moreover, by adding a random dithering to the linear random measurements before their binarization, the same authors proved that a second-order cone program (CP) can fully estimate the vector \(x\), i.e., not only its direction. For the same subsampled circulant sensing matrix, they also proved that their results extend to the dithered, uniformly quantized CS expressed in \([4]\). In fact, with high probability, and for all effectively sparse signals \(x \in \mathbb{B}^n\), i.e., such that \(\|x\|_2^2\) is small, the same CP program achieves a reconstruction error decaying like \(O(m^{-1/6})\) when \(m\) increases. Their only requirement is that, first, the dithering is...
made of the addition of a Gaussian random vector with a uniform random vector adjusted to the quantization resolution, and second, that $\|x\|^2_2$ is smaller than $\sqrt{n}$ [33, Thm 6.2].

In the same order of ideas, [11, 18] have also shown that, for Gaussian random sensing matrices, adding an adaptive or random dithering to the compressive measurements of a signal before their binarization allows accurate reconstruction of this signal, i.e., of both its norm and its direction, using either PBP or the same CP program as in [33]. Additionally, for random observations altered by an adaptive dithering before their 1-bit quantization, i.e., in a process close to noise shaping or ΣΔ-quantizer [11, 12], an appropriate reconstruction algorithm can achieve an exponential decay of its error in terms of the number of measurements. This is only demonstrated, however, in the case of Gaussian sensing matrices and for sparse signals only.

**QCS and other non-linear sensing models:** The (scalar) QCS model [4] can be seen as a special case of the more general, non-linear sensing model $y = f(\Phi x)$, with the random non-linear function $f : u \in \mathbb{R}^m \mapsto (f_1(u_1), \ldots, f_m(u_m))^T \in \mathbb{R}^m$ such that $f_i \sim_{i.i.d.} f$, for some random function $f : \mathbb{R} \rightarrow \mathbb{R}$ [35, 37]. In the QCS context defined in [4], this non-linear sensing model corresponds to setting $f_i(\lambda) = Q(\lambda + \xi_i)$ with $\xi_i \sim_{i.i.d.} U([0, \delta])$.

In [37], the authors proved that, for a Gaussian random matrix $\Phi$ and for a bounded, star-shaped set $K$, provided that $f$ leads to finite moments $\mu := \mathbb{E}(f(g)g)$, $\sigma^2 := \mathbb{E}(f(g)^2 - \mu^2)$, and $\eta^2 := \mathbb{E}(f(g)^2)$ with $g \sim \mathcal{N}(0, 1)$, and provided that $f(g)$ is sub-Gaussian with finite sub-Gaussian norm $\psi := \|f(g)\|\psi$, [34], one can estimate with high probability $\mu \frac{x}{\|x\|} \in K$ from the solution $\hat{x}$ of the PBP of $y$ in [35] (see [37, Thm 9.1]). In the specific case where $f$ matches the QCS model [4], this analysis proves that for Gaussian random matrix $\Phi$, the PBP of QCS observations estimates the direction $x/\|x\|$ with a reconstruction error decaying like $O((1 + \delta)^2 \sqrt{w(K)}m^{-\frac{1}{2}})$ when $m$ increases (the details of this analysis are given in App. [3]).

A similar result is obtained in [35] for the estimate $\tilde{x}$ provided by a K-Lasso program, which finds the element $u \in K$ minimizing the $\ell_2$-cost function $h(u) := \|\Phi u - y\|^2$, when $y = f(\Phi x)$, $x \in \mathbb{S}^{n-1} \cap \frac{1}{\mu}K$ and under the similar hypotheses on the non-linear corruption $f_i \sim f$ than above (i.e., with finite moments $\mu, \sigma$, and $\eta^2 = \mathbb{E}(f(g) - \mu g)^2$). Of interest for this work, [35] introduced a form of the (local) LPD (see Sec. 1 and Sec. 3) in the case where $A = f \circ \Phi$ and $\Phi$ is a Gaussian random matrix (with possibly unknown covariance between rows). The authors indeed analyzed when, for some $\epsilon > 0$,

$$\frac{1}{m} \left( \langle f(\Phi x), \Phi v \rangle - \langle \Phi \mu x, \Phi v \rangle \right) \lesssim \epsilon, \quad \forall v \in D^* = D \cap \mathbb{B}^n, \quad (6)$$

with $D = D(K, \mu x) := \{\tau \mathbf{h} : \tau \geq 0, \mathbf{h} \in K - \mu x\}$ being the tangent cone of $K$ at $x$. An easy rewriting of [35] Proof of Thm 1.4] then essentially shows that the RIP of $\Phi$ over $D^*$ combined with [6] provides $\|x - \mu x\| \lesssim \epsilon$. In particular, thanks to the Gaussianity of $\Phi$, they prove that, with large probability, [6] hold with $\epsilon = (w(D^*) \sigma + \tilde{\eta})/\sqrt{m}$, with $w(D^*)$ the Gaussian mean width of $D^*$ measuring its intrinsic complexity (see Sec. 3.1). For instance, if $K = \Sigma_k$, and if $f$ is such that $\mu, \sigma, \tilde{\eta} = O(1)$, this proves that $\epsilon = O(\sqrt{k \log(n/k)}/\sqrt{m})$. Correspondingly, if the $f_i$’s are thus selected to match the QCS model with a Gaussian sensing $\Phi$, this shows that K-Lasso achieves a non-uniform reconstruction error decay of $O(1/\sqrt{m})$ of $x \in \mathbb{S}^{n-1} \cap \frac{1}{\mu}K$ if the Gaussian mean width of the tangent cone $D^*$ can be bounded (e.g., for sparse or compressible signals, or for low-rank matrices).

---

6The set $K$ is star-shaped if, for any $\lambda \in [0, 1]$, $\lambda K \subset K$.

7As implied by Markovs inequality combined with [35, Lem. 4.3].
In other words, when instantiated to our specific QCS model, but only in the context of a Gaussian random matrix $\Phi$ and with some restrictions on the norm of $x$, the non-uniform reconstruction error decays of PBP and K-Lasso in [37] and [35], respectively, are similar to the one achieved in our work (see Sec. 7).

3 Preliminaries

3.1 Low-complexity spaces

In this work, our ability to estimate a signal $x$ from the QCS model (4) is developed on the hypothesis that this signal belongs to a “low-complexity” set $K \subset \mathbb{R}^n$. In other words, we first suppose that, for any radius $\eta > 0$, the restriction$^8$ of $K$ to the $\ell_2$-ball $\mathbb{B}^n$ can be covered by a relatively small number of translated $\ell_2$-balls of radius $\eta$. In other words, we assume that $K \cap \mathbb{B}^n$ has a small Kolmogorov entropy $H(K \cap \mathbb{B}^n, \eta)$ before $n$ [12], with, for any bounded set $S$,

$$H(S, \eta) := \log \min \{ |\mathcal{G}| : \mathcal{G} \subset S \subset \mathcal{G} + \eta \mathbb{B}^n \},$$

where the addition is the Minkowski sum between sets. Most of the time, e.g., if $K$ is a low-dimensional subspace of $\mathbb{R}^n$ (or a finite union of such spaces, as for the set of sparse vectors) or the set of low-rank matrices, $H(K \cap \mathbb{B}^n, \eta)$ is well controlled by standard covering arguments of $K \cap \mathbb{B}^n$ [43]. In fact, as explained in Sec. 7 and summarized in [20, Table 1], for most of these sets, we can consider that $H(K \cap \mathbb{B}^n, \eta) \leq C \eta^2 (K \cap \mathbb{B}^n) \log(1 + \frac{1}{\eta})$, where $w(S)$ is the Gaussian mean width of a bounded set $S \subset \mathbb{R}^n$ [39, 44] defined by

$$w(S) := \mathbb{E} \sup_{x \in S} |\langle g, u \rangle|, \quad g \sim \mathcal{N}(0, I_n).$$

Interestingly, we have indeed $w^2(K \cap \mathbb{B}^n) \ll n$ for a large number of low-complexity sets $K$, such as those mentioned above. For instance, $w^2(\Sigma_k \cap \mathbb{B}^n) \lesssim k \log(n/k)$, and the square Gaussian mean width of bounded, square rank-$r$ matrices with $n$ entries is bounded by $r \sqrt{n}$ (see e.g., [15, 44] and [20, Table 1]).

When $K$ does not belong to these easy cases, Sudakov minoration provides the (generally) looser bound $H(K \cap \mathbb{B}^n, \eta) \leq C \eta^{-2} w^2(K \cap \mathbb{B}^n)$ [39, 44]. The analysis of both $H$ and $w$ for dithered QCS will be further investigated in Sec. 7.

Another implicit assumption we make on the set $K$, or on its $n$th-multiple $K^{\times n} := \sum_{k=1}^n K$ for some $n \in \mathbb{N}$ (see Sec. 7), is that it is compatible with the RIP of $\Phi$. In other words, given a distortion $\epsilon \in (0, 1)$, we assume $\Phi$ respects the RIP($K, \epsilon$) defined by

$$\left| \frac{1}{m} \| \Phi u \|^2 - \| u \|^2 \right| \leq \epsilon, \quad \forall u \in K \cap \mathbb{B}^n. \quad (7)$$

This assumption is backed up by a growing literature in the field of compressive sensing and we will refer to it in many places. In particular, it is known that sensing matrices with i.i.d. centered sub-Gaussian random entries satisfy the RIP if $m$ is large compared to the typical dimension of $K$, as measured by the square Gaussian mean width of $K$ [7, 5, 3]. Note that in the case where $K = K_0 \cap \mathbb{B}^n$ with $K_0$ a cone, i.e., $\lambda K_0 \subset K_0$ for all $\lambda > 0$, a simple rescaling argument provides the usual formulation of the RIP, i.e., (7) implies

$$(1 - \epsilon) \| u \|^2 \leq \frac{1}{m} \| \Phi u \|^2 \leq (1 + \epsilon) \| u \|^2, \quad \forall u \in K_0. \quad (8)$$

$^8$Note that if $K$ is bounded, all our developments can be rescaled in order to directly analyze $K$ instead of $K \cap \mathbb{B}^n$. 

9
3.2 Projected Back Projection

As announced in the Introduction, the standpoint of this work is to show the compatibility of a RIP matrix $\Phi$ with the dithered QCS model (4), provided that the dithering is random and uniform, through the possibility to estimate $x$ via the Projected Back Projection (PBP) onto $\mathcal{K}$ of the quantized observations $y = Q(\Phi x + \xi)$.

Mathematically, the PBP method is simply defined by

$$\hat{x} := \mathcal{P}_\mathcal{K}(\frac{1}{m} \Phi^\top y),$$

where $\mathcal{P}_\mathcal{K}$ is the (minimal distance) projector on $\mathcal{K}$, i.e.,

$$\mathcal{P}_\mathcal{K}(z) \in \text{arg min}_{u \in \mathcal{K}} \|z - u\|.$$  

Throughout this work, we assume that $\mathcal{P}_\mathcal{K}$ can be computed, i.e., in polynomial complexity with respect to $m$ and $n$. For instance, if $\mathcal{K} = \Sigma_k^n$, $\mathcal{P}_\mathcal{K}$ is the standard best $k$-term hard thresholding operator, and if $\mathcal{K}$ is convex and bounded, $\mathcal{P}_\mathcal{K}$ is the orthogonal projection onto this set.

In the context where we assume the matrix $\Phi$ to be fixed and to satisfy the RIP (i.e., whatever the random construction that led to the generation of $\Phi$) the only random element in the QCS model is the dithering $\xi \in \mathbb{R}^m$. With respect to this randomness, the analysis of the reconstruction errors achieved by PBP is thus divided into two categories: uniform estimation, i.e., with high probability on the choice of the dithering, all signals in the set $\mathcal{K}$ are estimated using the same dithering, and fixed signal estimation (or non-uniform) where, given a fixed signal, the dithering is randomly generated and, with high probability, one can estimate this signal.

3.3 Limited Projection Distortion

We already sketched at the end of the Introduction that a central machinery of our analysis is the combination of the RIP of $\Phi$ with another property jointly verified by ($\Phi, \xi$), or equivalently by the quantized random mapping $A$ defined in (4). As will be clear later, this property, the (local) limited projection distortion, or (L)LPD, and the RIP allow us to bound the reconstruction error of the PBP. We define it as follows.

Definition 3.1 (Limited Projection Distortion). Given a matrix $\Phi \in \mathbb{R}^{m \times n}$ and a distortion $\nu > 0$, we say that a general mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ respects the limited projection distortion property over a set $\mathcal{K} \subset \mathbb{R}^n$ observed by $\Phi$, or $\text{LPD}(\mathcal{K}, \Phi, \nu)$, if

$$\frac{1}{m} |\langle A(u), \Phi v \rangle - \langle \Phi u, \Phi v \rangle| \leq \nu, \quad \forall u, v \in \mathcal{K} \cap \mathbb{R}^n.$$  

In particular, when $u$ is fixed in (10), we say that $A$ respects the local limited projection distortion on $u$, or $L$-LPD($\mathcal{K}, \Phi, u, \nu$).

Remark 3.2. As explained in Sec. 2, the LPD property was (implicitly) introduced in [25] in the special case where $A = f \circ \Phi$ and $f$ is a non-linear function applied componentwise on the image of $\Phi$. The LPD is also connected to the SPE introduced in [39] for the specific case of a 1-bit sign quantizer if we combine the LPD property with the RIP of $\frac{1}{\sqrt{m}} \Phi$ in order to approximate $\frac{1}{m} \langle \Phi u, \Phi v \rangle$ by $\langle u, v \rangle$ in (10) (see Lemma 3.5). This literature was however restricted to the analysis of Gaussian random matrices.

\footnote{Note that there exist cases where $\min_{u \in K} \|z - u\|$ could have several equivalent minimizers, e.g., if $K$ is non-convex. If this happens, we just assume $P_K$ picks one of them, arbitrarily.}
Remark 3.3. In the case where $A$ is the quantized random mapping introduced in [4], a small distortion $\nu$ in $\{10\}$ is meaningful in certain contexts. As proved in Sec. [4], if $\xi \sim \mathcal{U}([0, \delta])$ is a random uniform dithering, an arbitrary low-distortion $\nu > 0$ is expected for large values of $m$ since, in expectation, $E_{\xi}(A(u), \Phi v) - (\Phi u, \Phi v) = 0$ from Lemma [4]. Note also that for such a random dithering, if $\delta$ tends to 0, then the quantizer $A$ tends to the identity operator and $|\langle A(u), \Phi v \rangle - \langle \Phi u, \Phi v \rangle|$ must vanish. In fact, by Cauchy-Schwarz and the triangular inequality, this is sustained by the deterministic bound

$$|\langle A(u), \Phi v \rangle - \langle \Phi u, \Phi v \rangle| = |\langle (A(u) - \Phi u), \Phi v \rangle| \leq \|\Phi v\| (\|A(u) - (\Phi u + \xi)\| + \|\xi\|) \leq 2\delta \sqrt{m} \|\Phi v\|.$$ 

Remark 3.4. The definition of the L-LPD also includes the simple case of linear random observations corrupted by an additive noise, i.e., if $A(u) = \Phi u + \rho$. We have then $A(u) - \Phi u = \rho$ and proving the LPD amounts to showing that $\frac{1}{m}(\rho, \Phi v)$ is small, as for instance in the case where $\rho$ is composed of i.i.d. sub-Gaussian random components. As will be clear later, this includes the situation where $u$ is fixed and where $A$ is the quantized random mapping $[4]$ since then the $m$ i.i.d. r.v.'s $\rho_i := Q((\Phi u)_i + \xi_i) - (\Phi u)_i$ are bounded and thus sub-Gaussian. However, this cannot be easily generalized to a uniform LPD property without more accurately considering the geometrical nature of $A$. In the case of a quantized mapping, we need in particular to control the impact of discontinuities introduced by $Q$ on $\rho$. This will be developed in Sec. [4].

The (L)LPD characterizes the proximity of scalar products between distorted and undistorted random observations in the compressed domain $\mathbb{R}^m$. In order to assess how $\frac{1}{m}(A(u), \Phi v)$ approximates $\langle u, v \rangle$ we can consider this standard lemma from the CS literature (see e.g., [10]).

**Lemma 3.5.** Given two symmetric subsets $K_1, K_2 \subset \mathbb{R}^n$ and $K_+ \supset \frac{1}{2}(K_1 + K_2)$, if $\frac{1}{\sqrt{m}}\Phi$ is RIP($K_+, \epsilon$) with $0 < \epsilon < 1$, then

$$|\frac{1}{m} \langle \Phi u, \Phi v \rangle - \langle u, v \rangle| \leq 2\epsilon, \quad \forall u \in K_1 \cap \mathbb{B}^n, \forall v \in K_2 \cap \mathbb{B}^n. \quad (11)$$

In particular, if $K_1$ and $K_2$ are two cones, we have

$$|\frac{1}{m} \langle \Phi u, \Phi v \rangle - \langle u, v \rangle| \leq 2\epsilon \|u\| \|v\|, \quad \forall u \in K_1, \forall v \in K_2. \quad (12)$$

**Proof.** Note that since $K_1$ and $K_2$ are symmetric, $\pm K_1 \pm K_2 = K_1 + K_2 \subset 2K_+$. Given $u \in K_1 \cap \mathbb{B}^n$ and $v \in K_2 \cap \mathbb{B}^n$, if $\frac{1}{\sqrt{m}}\Phi$ is RIP($K_+, \epsilon$) with $0 < \epsilon < 1$, then, from the polarization identity, the fact that $\frac{1}{2}(u \pm v) \in K_+ \cap \mathbb{B}^n$ and from (7),

$$\frac{1}{m} \langle \Phi u, \Phi v \rangle = \frac{1}{m} \|\Phi(u + v)\|^2 - \frac{1}{m} \|\Phi(u - v)\|^2 \leq \frac{1}{4} \|u + v\|^2 - \frac{1}{4} \|u - v\|^2 + 2\epsilon.$$

The lower bound is obtained similarly. A simple rescaling argument provides (12).

Therefore, applying the triangular identity, it is easy to verify the following corollary.

**Corollary 3.6.** Given $\mathcal{L} \subset \mathbb{R}^n$, two symmetric subsets $K_1, K_2 \subset \mathcal{L}$, and $K_+ \supset \frac{1}{2}(K_1 + K_2)$. If $\frac{1}{\sqrt{m}}\Phi$ respects the RIP($K_+, \epsilon$) and $A$ verifies the LPD($\mathcal{L}, \Phi, \nu$) for $\epsilon, \nu > 0$, then

$$|\frac{1}{m} \langle A(u), \Phi v \rangle - \langle u, v \rangle| \leq 2\epsilon + \nu, \quad \forall u \in K_1 \cap \mathbb{B}^n, \forall v \in K_2 \cap \mathbb{B}^n. \quad (13)$$

The same observation holds $u$ is fixed when the L-LPD is invoked instead of the LPD.

Note that we recover Lemma 3.5 if $A$ is identified with $\Phi$. 

11
4 PBP reconstruction error in Distorted CS

In this section, we provide a general analysis of the reconstruction error of the estimate provided by the PBP of some general *distorted* CS model

$$\mathbf{y} = \mathbf{D}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{K} \cap \mathbb{B}^n. \quad (14)$$

This is achieved in the context where $\mathbf{D} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is only assumed to respect the (L)LPD property, which, in a certain sense, characterizes the proximity of this (possibly non-linear) mapping with a RIP matrix $\frac{1}{\sqrt{m}}\Phi$.

Note that the results of this general study can (and will) be applied of course to the quantized, random mapping introduced in (4) (as explained in Sec. 5 and Sec. 6), but it could potentially concern other distorted sensing models, provided that the associated mapping meets the (L)LPD property.

Hereafter, we analyze the cases where the low-complexity set $\mathcal{K}$ of the vector is a union of low-dimensional subspaces, the set of low-rank matrices, or a convex set of $\mathbb{B}^n$. Sec. 7 will later analyze these general results when $\mathbf{D}$ is the quantized random mapping $\mathbf{A}$ introduced in (4).

4.1 Union of low-dimensional subspaces

Let us first consider the reconstruction error of PBP for estimating vectors belonging to a union of $\mathcal{K} \subseteq \mathbb{R}^n$ low-dimensional subspaces, or ULS. In a ULS model we can write $\mathcal{K} := \bigcup_{i \in [K]} \mathcal{K}_i$, where each $\mathcal{K}_i$ is a low-dimensional subspace of $\mathbb{R}^n$ for $i \in [K]$. This model encompasses, e.g., sparse signals in an orthonormal basis or in a dictionary [47, 44], co-sparse signal models [48], group-sparse signals [27], and model-based sparsity [26].

The next theorem states that the PBP reconstruction error is bounded by the addition of the distortion induced by the RIP of $\Phi$ (as in CS) and the one provided by (L)LPD of $\mathbf{D}$.

**Theorem 4.1** (PBP for ULS). Let us consider the ULS model $\mathcal{K} := \bigcup_{i \in [K]} \mathcal{K}_i \subseteq \mathbb{R}^n$. Given two distortions $\epsilon, \nu > 0$, if $\frac{1}{\sqrt{m}}\Phi \in \mathbb{R}^{m \times n}$ respects the RIP$(\mathcal{K} - \mathcal{K}, \epsilon)$ and if the mapping $\mathbf{D} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the LPD$(\mathcal{K} - \mathcal{K}, \Phi, \nu)$, then, for all $\mathbf{x} \in \mathcal{K} \cap \mathbb{B}^n$, the estimate $\hat{\mathbf{x}}$ obtained by the PBP of $\mathbf{y} = \mathbf{D}(\mathbf{x})$ onto $\mathcal{K}$ satisfies

$$\|\mathbf{x} - \hat{\mathbf{x}}\| \leq 4\epsilon + 2\nu.$$  

Moreover, if $\mathbf{x}$ is fixed, then the same result holds if $\mathbf{D}$ respects the L-LPD$(\mathcal{K} - \mathcal{K}, \Phi, \mathbf{x}, \nu)$.

**Proof.** The proof generalizes the proof sketch given at the end of the Introduction for the reconstruction error of PBP in the case where $\mathcal{K} = \Sigma^n_K$. Since $\mathbf{x} \in (\bigcup_{i \in [K]} \mathcal{K}_i) \cap \mathbb{B}^n$ and $\hat{\mathbf{x}} \in \mathcal{K}$, there must exist two subspaces $\mathcal{K}_x := \mathcal{K}_i$ and $\hat{\mathcal{K}} := \mathcal{K}_{i'}$, for some $i, i' \in [K]$ such that $\mathbf{x} \in \mathcal{K}_x$ and $\hat{\mathbf{x}} \in \hat{\mathcal{K}}$. Let us define $\mathbf{a} = \frac{1}{m}\Phi^\top \mathbf{y}$, the subspace $\hat{\mathcal{K}} := \mathcal{K}_x + \hat{\mathcal{K}}$ and the orthogonal complement $\hat{\mathcal{K}}^\perp$ of $\hat{\mathcal{K}}$. We can always decompose $\mathbf{a}$ as $\mathbf{a} = \bar{\mathbf{a}} + \bar{\mathbf{a}}^\perp$ with $\bar{\mathbf{a}} := \mathcal{P}_{\mathcal{K}}(\mathbf{a})$ and $\bar{\mathbf{a}}^\perp := \mathcal{P}_{\mathcal{K}^\perp}(\mathbf{a})$, with the projector $\mathcal{P}_{\mathcal{K}}$ defined in Sec. 4. Since $\hat{\mathbf{x}} = \mathcal{P}_{\mathcal{K}}(\mathbf{a}) \in \mathcal{K}$, we have

$$\|\hat{\mathbf{x}} - \mathbf{a}\|^2 = \|\hat{\mathbf{x}} - \bar{\mathbf{a}} - \bar{\mathbf{a}}^\perp\|^2 \leq \|\mathbf{x} - \mathbf{a}\|^2 = \|\mathbf{x} - \bar{\mathbf{a}} - \bar{\mathbf{a}}^\perp\|^2.$$ 

Moreover, since both $\hat{\mathbf{x}} - \bar{\mathbf{a}}$ and $\mathbf{x} - \bar{\mathbf{a}}$ belong to $\hat{\mathcal{K}}$, the last inequality is equivalent to $\|\hat{\mathbf{x}} - \bar{\mathbf{a}}\|^2 + \|\bar{\mathbf{a}}^\perp\|^2 \leq \|\mathbf{x} - \bar{\mathbf{a}}\|^2 + \|\bar{\mathbf{a}}^\perp\|^2$, which implies $\|\hat{\mathbf{x}} - \bar{\mathbf{a}}\| \leq \|\mathbf{x} - \bar{\mathbf{a}}\|$. Consequently, the triangular inequality gives

$$\|\mathbf{x} - \hat{\mathbf{x}}\| \leq \|\mathbf{x} - \bar{\mathbf{a}}\| + \|\hat{\mathbf{x}} - \bar{\mathbf{a}}\| \leq 2\|\mathbf{x} - \bar{\mathbf{a}}\|.$$
From the assumptions of the theorem, \( \Phi \) and \( D \) respect the RIP(\( K-K, \epsilon \)) and the LPD(\( K-K, \Phi, \nu \)), respectively. Therefore, from the symmetry of \( K \), using Cor. \([3.6]\) with \( K_+ = L = K-K = \frac{1}{2}(K+K) \) and \( K_1 = K_2 = \bar{K} \), since \( x \in \bar{K} \) and \( u + x \in \bar{K} \subset K-K \) for all \( u \in \bar{K} \subset K-K \), we find

\[
\|x-\hat{x}\| \leq 2\|x-\bar{a}\| = 2\sup_{u \in \mathbb{B}} \langle u, x-\bar{a} \rangle = 2\sup_{u \in \mathbb{K} \cap \mathbb{B}} \langle u, x-a \rangle
\]

\[
= 2\sup_{u \in \mathbb{K} \cap \mathbb{B}} \{ \langle u, x \rangle - \langle u, \frac{1}{m} \Phi \top y \rangle \}
\]

\[
= 2\sup_{u \in \mathbb{K} \cap \mathbb{B}} \{ \langle u, x \rangle - \frac{1}{m} (\Phi u, D(x)) \}
\]

\[
\leq 4\epsilon + 2\nu,
\]

which gives the result. Moreover, if \( x \) is fixed, we clearly see that only the L-LPD(\( K-K, \Phi, x, \nu \)) is required.

\[\square\]

Note that, if \( D \equiv \Phi \), only the RIP distortion in \( \epsilon \) remains. This is consistent with classical results of the CS literature, \( e.g. \), as observed from the bound obtained on the first iteration of the iterative hard thresholding algorithm, \( i.e. \), the PBP of \( \Phi x \), in the case where \( K = \Sigma_r^a \) \([3, \text{Thm 6.15}]\).

### 4.2 Bounded low-rank matrices

The (L)LPD and the RIP also allow to bound the reconstruction error of PBP in the estimation of low-rank matrices observed by the distorted CS model \([14]\), up to an easy adaptation of this model to matrix sensing. In fact, given a bounded rank-\( r \) matrix \( X \in K \cap \mathbb{B}_r^{n_1 \times n_2} \), with

\[
K := C_r^{n_1 \times n_2} = \{ Z \in \mathbb{R}^{n_1 \times n_2} : \text{rank} Z \leq r \}, \quad r \leq \min(n_1, n_2),
\]

and introducing the Frobenius ball \( \mathbb{B}_r^{n_1 \times n_2} := \{ Z \in \mathbb{R}^{n_1 \times n_2} : \| Z \|_F^2 := \text{tr}(Z \top Z) \leq 1 \} \), the (linear) CS model reads \( y = \Theta(X) \), where \( \Theta : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m \) is the linear measurement operator defined by \( m \) scalar products of a \( n_1 \times n_2 \)-matrix with \( m \) pre-defined \( n_1 \times n_2 \)-matrices \( \{ \Phi_1, \cdots, \Phi_m \} \), \( i.e. \),

\[
[\Theta(X)]_i = \langle X, \Phi_i \rangle := \text{tr}(X \top \Phi_i).
\]

Correspondingly, the adjoint of \( \Theta \) is \( \Theta^\top : w \in \mathbb{R}^m \to \Theta^\top(w) = \sum_{i=1}^m w_i \Phi_i \in \mathbb{R}^{n_1 \times n_2} \).

Equivalently, by vectorizing any matrix \( Z \in \mathbb{R}^{n_1 \times n_2} \) into its vector representation \( \text{vec}(Z) \in \mathbb{R}^n \) with \( n = n_1 n_2 \), \( i.e. \), stacking up all its columns on top of one another, the CS model can be rewritten as \( \Phi \text{vec}(X) \), where \( \Phi := [\text{vec}(\Phi_1), \cdots, \text{vec}(\Phi_m)]^\top \in \mathbb{R}^{m \times n} \), \( i.e. \), \( \mathbb{R}^{n_1 \times n_2} \) and \( \mathbb{B}_r^{n_1 \times n_2} \) are thus identified with \( \mathbb{R}^n \) and \( \mathbb{B}^n \), respectively. Moreover, requesting \( \Theta \) to satisfy the RIP over some subset \( K \subset \mathbb{R}^{n_1 \times n_2} \) is thus equivalent to ask \( \Phi \) to respect it over \( \text{vec}(K) \subset \mathbb{R}^n \) as defined in \([7]\), with also \( \Phi^\top w = \text{vec}(\Theta^\top w) \) for \( w \in \mathbb{R}^m \).

For simplicity, we thus consider that the distorted model \([14]\) is also defined over a vectorization of the matrix domain, \( i.e. \), for a mapping \( D : \mathbb{R}^n \to \mathbb{R}^m \), and the definition of the (L)LPD is thus considered in the same sense, \( i.e. \), for sensing matrix \( \Phi \) related to the vectorized form of \( \Theta \).

Before establishing the main result of this section, let us specify two useful properties for our developments. First, concerning \( \Phi \), since the rank of a matrix is subadditive, \( C_s^{n_1 \times n_2} \pm C_s^{n_1 \times n_2} \subset C_{2s}^{n_1 \times n_2} \), the RIP(\( C_{2s}^{n_1 \times n_2}, \epsilon \)) involves the RIP(\( C_s^{n_1 \times n_2} - C_s^{n_1 \times n_2}, \epsilon \)) for any \( s > 0 \).

Second, the projector \( \mathcal{P}_r := \mathcal{P}_r^{C_r^{n_1 \times n_2}} \) of any matrix \( Z \in \mathbb{R}^{n_1 \times n_2} \) onto the set of rank-\( r \) matrices \( \mathcal{K} = C_r^{n_1 \times n_2} \) is given by \([49, 50]\)

\[
\mathcal{P}_r(Z) := \arg \min_{U \in C_r^{n_1 \times n_2}} \| U - Z \|_F = U \mathcal{M}_r(\Sigma)V^\top.
\]

(15)
In [15], $\mathcal{M}_r(D)$ is the $r$-thresholding operator setting all but the $r$-first diagonal entries of $D$ to zero, and $U\Sigma V^\top$ is the singular value decomposition of $Z$, where $U \in \mathbb{R}^{n_1 \times n_1}$ and $V \in \mathbb{R}^{n_2 \times n_2}$ are the unitary matrices formed by the left and right singular vectors of $Z$, respectively, and $\Sigma \in \mathbb{R}^{n_1 \times n_2}$ is the (rectangular) diagonal matrix formed by the (decreasing) singular values $\{\sigma_i : 1 \leq i \leq \min(n_1, n_2)\}$ of $Z$ (i.e., $\sigma_{ij} = \sigma_i\delta_{ij}$, $\sigma_1 \geq \sigma_{i+1}$). In other words, (15) provides the best rank-$r$ matrix approximate of $Z$ in the Frobenius norm.

As in the previous section, we are now ready to leverage both the (L)LPD of a mapping $D$ and the RIP of $\frac{1}{\sqrt{m}} \Phi$ for proving that PBP provides a controllable error distortion to estimate a common subspace for both $X$ and its PBP estimate.

**Theorem 4.2** (PBP for bounded low-rank matrices). Let us consider the low-rank model $\mathcal{K} := \mathcal{C}_{r}^{n_1 \times n_2} \subset \mathbb{R}^{n_1 \times n_2}$ with $0 < r < \min(n_1, n_2)$. Given two distortions $\epsilon, \nu > 0$, if $\frac{1}{\sqrt{m}} \Phi : \mathbb{R}^{n} \to \mathbb{R}^{m}$ (i.e., $\frac{1}{\sqrt{m}} \Phi : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{m}$) respects the RIP($\mathcal{C}_{r}^{n_1 \times n_2}, \epsilon$) and if the mapping $D : \mathbb{R}^{n} \to \mathbb{R}^{m}$ satisfies the LPD($\mathcal{C}_{r}^{n_1 \times n_2}, \Phi, \nu$), then, for all $X \in \mathcal{K} \cap \mathbb{R}^{n_1 \times n_2}$, the estimate $\hat{X}$ obtained by the PBP of $y$ on $\mathcal{C}_{r}^{n_1 \times n_2}$, i.e.,

$$\hat{X} := \mathcal{P}_r(\frac{1}{m} \Phi^*(y)),$$

satisfies

$$\|X - \hat{X}\|_F \leq 4\epsilon + 2\nu.$$

Moreover, if $X$ is fixed, then the same result holds if $D$ respects the L-LPD($\mathcal{C}_{r}^{n_1 \times n_2}, \Phi, X, \nu$).

**Proof.** Let $U\Sigma V^\top$ and $U'\Sigma' V'^\top$ be the SVD decompositions of $X$ and $W = \frac{1}{m} \Phi^*(y)$, respectively. We have thus $\hat{X} = U' \mathcal{M}_r(\Sigma') V'^\top$. Let $Q \in \mathbb{R}^{n_1 \times 2r}$ be an orthonormal matrix whose columns span a subspace containing the subspace spanned by the first $r$ columns of both $U$ and $U'$, and let $R \in \mathbb{R}^{n_2 \times 2r}$ be defined similarly from $V$ and $V'$.

We form the subspace

$$\mathcal{S} = \{QB^\top + CR^\top \in \mathbb{R}^{m \times n} : B \in \mathbb{R}^{n_2 \times 2r}, C \in \mathbb{R}^{n_1 \times 2r}\}.$$

Note that $\dim \mathcal{S} \leq 2r(n_1 + n_2)$, $\mathcal{S} \subset \mathcal{C}_{r}^{n_1 \times n_2}$ and $X, \hat{X} \in \mathcal{S}$. This space actually includes the tangent space of the smooth manifold of rank-$2r$ matrices at $QR^\top$ in $\mathbb{R}^{n_1 \times n_2}$ [51, 52].

Furthermore, for any matrix $Z \in \mathbb{R}^{n_1 \times n_2}$, the projection of $Z$ onto $\mathcal{S}$ is simply written as $\mathcal{P}_\mathcal{S}(Z) = FZ + ZG^\top - FZG^\top$, where $F := QQ^\top$ and $G := RR^\top$, and the projection on the orthogonal complement $\mathcal{S}^\perp$ reads $\mathcal{P}_{\mathcal{S}^\perp}(Z) = (I_{n_1} - F)Z(I_{n_2} - G^\top) = Z - \mathcal{P}_\mathcal{S}(Z)$.

Since $\hat{X} = \mathcal{P}_r(W)$ and $W = \bar{W} + W^\perp$, with $\bar{W} = \mathcal{P}_\mathcal{S}(W)$ and $W^\perp = \mathcal{P}_{\mathcal{S}^\perp}(W)$, we have

$$\|\hat{X} - (\bar{W} + W^\perp)\|_F^2 \leq \|X - (\bar{W} + W^\perp)\|_F^2.$$

However, both $\hat{X} - \bar{W}$ and $X - \bar{W}$ belong to $\mathcal{S}$ and are thus orthogonal to $W^\perp$. Decomposing both sides of the last inequality by Pythagoras' theorem and simplifying the common terms, we thus find $\|\hat{X} - \bar{W}\|_F \leq 2\|X - \bar{W}\|_F$, which gives

$$\|\hat{X} - X\|_F \leq 2\|X - \bar{W}\|_F$$

by the triangular inequality.
We now proceed as in the proof of Theorem 4.1. We note first that for any $T \in S$, $T \pm X \in S$ since $S \subset C_{4r}^{n_1 \times n_2}$ is a subspace including $X$. Second, by assumption, $\Theta$ respects the RIP($C_{4r}^{n_1 \times n_2}, \epsilon$) and the random mapping $D$ satisfies the LPD($C_{4r}^{n_1 \times n_2}, \Phi, \nu$). Therefore, from the symmetry of $K = C_{4r}^{n_1 \times n_2}$ and using (the matrix form of) Cor. 3.6 with $K_+ = \mathcal{L} = C_{4r}^{n_1 \times n_2} \supset S$ and $K_1 = K_2 = S$, we get

$$\|\hat{x} - X\|_F \leq 2\|X - W\|_F = 2 \sup_{T \in S \cap B_{p_1}^{n_1} \times n_2} \langle T, \mathcal{P}_S(X - W) \rangle$$

$$= 2 \sup_{T \in S \cap B_{p_1}^{n_1} \times n_2} \langle T, X - W \rangle = 2 \sup_{T \in S \cap B_{p_1}^{n_1} \times n_2} \{\langle T, X \rangle - \frac{1}{m}\langle T, \Theta^*(y) \rangle\} \leq 4\epsilon + \nu.$$ 

Moreover, if $X$ is fixed, we also see that only the L-LPD($C_{4r}^{n_1 \times n_2}, \Phi, X, \nu$) is required on $D$. □

4.3 Bounded convex sets

The PBP method can also achieve a small reconstruction error for any signal belonging to a symmetric, bounded convex set provided that both the RIP of $\frac{1}{\sqrt{m}}\Phi$ and the (L)LPD hold on $K - K$. However, this error is amplified compared to ones observed for the more structured sets analyzed in the previous section.

**Theorem 4.3** (PBP for bounded convex sets). Let $K \subset \mathbb{R}^n$ be a symmetric convex set. Given two distortions $\epsilon, \nu > 0$, if $\frac{1}{\sqrt{m}}\Phi \in \mathbb{R}^{m \times n}$ respects the RIP($K, \epsilon$) and if the mapping $D : \mathbb{R}^n \to \mathbb{R}^m$ satisfies the LPD($K, \Phi, \nu$), then, for all $x \in K$, the estimate $\hat{x}$ obtained by the PBP of $y = D(x)$ onto $K$ satisfies

$$\|x - \hat{x}\| \leq (4\epsilon + 2\nu)^\frac{1}{2}. \quad (16)$$

Moreover, if $x$ is fixed, then the same result holds if $D$ respects the L-LPD($K, \Phi, x, \nu$).

**Proof.** Since $x \in K$ and $K$ is symmetric and convex, the nonexpansivity of the orthogonal projector $\mathcal{P}_K$ onto $K$ [53] gives

$$\|x - \hat{x}\|^2 = \|\mathcal{P}_K(x) - \mathcal{P}_K(a)\|^2 \leq \langle x - a, \mathcal{P}_K(x) - \mathcal{P}_K(a) \rangle,$$

with $a := \frac{1}{m}\Phi^T y$. Therefore, since $K$ is symmetric,

$$\|\mathcal{P}_K(x) - \mathcal{P}_K(a)\|^2 \leq |\langle x - a, \mathcal{P}_K(x) \rangle| + |\langle x - a, \mathcal{P}_K(a) \rangle|$$

$$\leq 2 \sup_{u \in K} \{\langle u, x - a \rangle - \frac{1}{m}\langle \Phi u, D(x) \rangle\} = 2 \sup_{u \in K} \{\langle u, x \rangle - \frac{1}{m}\langle \Phi u, D(x) \rangle\}. $$

Therefore, Cor. 3.6 with $\mathcal{L} = K_1 = K_2 = K$ and $K_+ = K \supset \frac{1}{2}(K + K)$ (by convexity of $K$) provides

$$\sup_{u \in K} \{\langle u, x \rangle - \frac{1}{m}\langle \Phi u, D(x) \rangle\} \leq 2\epsilon + \nu,$$

which gives the result. Moreover, if $x$ is fixed, only the L-LPD($K, \Phi, x, \nu$) is required on $D$. □

Note that the Theorem 4.3 presents a worst case analysis of the reconstruction error of the PBP method for bounded convex sets. In particular, (16) displays only a reconstruction error which is the square root of those presented in Thm. 4.2 and Thm. 4.1. We will see, however, that at least for the convex set of compressible signals $\Sigma_a := \{u \in \mathbb{R}^n : \|u\|_1 \leq \sqrt{s}, \|u\| \leq 1\}$ and if $D$ is the quantized random mapping $A$ given in [41], the numerical reconstruction errors of the PBP presented in Sec. 8 behaves similarly to the ones predicted in the case of structured sets, if the random sensing matrix is either Gaussian, Bernoulli or a random partial Fourier/DCT sensing.
5 Local limited projection distortion for noisy linear mapping

The previous sections have focused on exploring the implications of the (L)LPD when this property holds for a mapping $D$ inducing the distorted CS model \[14\]. The question is, however, to understand for which mapping $D$ and under which conditions on the space $K$ and on the number of observations $m$ we can expect this property to be verified, i.e., with high probability.

In this section, we first prove that the L-LPD holds when the mapping $D$ amounts to the corruption of a linear sensing with an additive sub-Gaussian noise $\rho \in \mathbb{R}^m$. Below, the condition that the linear sensing must respect is less constraining than the RIP, i.e., we ask it to be Lipschitz continuous over $K$ in the following sense.

**Definition 5.1 (Lipschitz).** A linear mapping $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be $(\eta, L)$-Lipschitz continuous over $K \subset \mathbb{R}^n$ for some $\eta > 0$ if $\|\Gamma(K \cap B^n)\| \leq L$ and

$$\|\Gamma u\| \leq L\eta, \quad \forall u \in (K - K) \cap \eta B^n.$$ (17)

Interestingly, in the case where $K$ is a cone and if $\frac{1}{\sqrt{m}} \Phi$ respects the RIP($K - K, \epsilon$), then it is also $(\eta, \sqrt{1 + \epsilon})$-Lipschitz continuous for any $\eta > 0$. This is easily observed from (8), which gives

$$\frac{1}{\sqrt{m}} \|\Phi(u - v)\| \leq \sqrt{1 + \epsilon} \|u - v\| \leq \sqrt{2} \eta, \quad \text{for all } u, v \in K \text{ with } u - v \in \eta B^n.$$ Moreover, we will see in Sec. 7.2 that for some non-conic low-complexity sets and certain random sensing matrices $\frac{1}{\sqrt{m}} \Phi$ verifying the RIP condition \[7\], one can also prove their Lipschitz continuity (as defined in Def. 5.1).

The key element that allows us to prove the (L)LPD of $A$ is that a Lipschitz continuous mapping preserves the low-complexity nature of a set in its image, as measured by its Kolmogorov entropy.

**Lemma 5.2 (Adapted from \[54\] Lemma 4)).** Given a bounded subset $K \subset \mathbb{R}^n$, a radius $\eta > 0$ and a linear mapping $\Phi \in \mathbb{R}^{m \times n}$, if $\frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n}$ is $(\eta, L)$-Lipschitz continuous over $K$ for some $L > 0$, then

$$\mathcal{H}(\Phi K, L\eta \sqrt{m}) \leq \mathcal{H}(K, \eta).$$ (18)

**Proof.** Let $K_\eta$ be an optimal $\eta$-covering of $K$ for some $\eta > 0$. Then, all $a \in \Phi K$ can be rewritten as $a = \Phi x = \Phi x_0 + \Phi r$ for some $x \in K$, with $x_0$ the closest point to $x$ in $K_\eta$, and $r \in (K - K) \cap \eta B^n$. Therefore, $\Phi K_\eta$ is a $(L\eta \sqrt{m})$-covering of $\mathcal{J} := \Phi K$ since, from the Lipschitz continuity of $\frac{1}{\sqrt{m}} \Phi$, $\|\Phi r\| \leq L\eta \sqrt{m}$. In particular, $\mathcal{H}(\mathcal{J}, L\eta \sqrt{m}) \leq \log |\Phi K_\eta| \leq \log |K_\eta| = \mathcal{H}(K, \eta)$, since the covering $\Phi K_\eta$ of $\mathcal{J}$ is not necessarily optimal.

We can now state the main result of this section.

**Proposition 5.3 (L-LPD for noisy linear sensing).** Given a set $K \subset \mathbb{R}^n$, a distortion $\epsilon > 0$ and a matrix $\frac{1}{\sqrt{m}} \Phi$ that is $(\epsilon, L)$-Lipschitz continuous over $K$ with $L = O(1)$, if $D(u) := \Phi u + \rho$, with $\rho \in \mathbb{R}^m$ a vector with i.i.d. centered sub-Gaussian random components, i.e., $\|\rho_i\|_{\psi_2} \leq R$ for $1 \leq i \leq m$ and $R \geq 1$, then, given $u \in K \cap B^n$ and provided

$$m \geq \epsilon^{-2} \mathcal{H}(K \cap B^n, \epsilon \epsilon),$$

the mapping $D$ respects the L-LPD($K, \Phi, u, R\epsilon$) with probability exceeding $1 - C \exp(-c' \epsilon^2 m)$. 

16
Proof. We note first that since \( \|\rho_i\|_{\psi_2} \leq R \), then, for fixed \( u, v \in K \cap B^n \), \( \langle D(u) - \Phi u, \Phi v \rangle = \sum_{i=1}^{m} \rho_i(\Phi u - v_i) \) is a weighted sum of i.i.d. centered sub-Gaussian random variables. Therefore, from [34] Prop. 5.10,

\[
\mathbb{P}(\|D(u) - \Phi u, \Phi v\| > t) \leq C \exp(-c \frac{t^2}{\|\Phi v\|^2}).
\]

Applying the change of variable \( t = \epsilon R \|\Phi v\| \sqrt{m} \), this shows that

\[
\frac{1}{m} \|D(u) - \Phi u, \Phi v\| \leq \epsilon R \frac{1}{\sqrt{m}} \|\Phi v\|,
\]

with probability exceeding \( 1 - C \exp(-c \epsilon^2 m) \). Given an optimal \( \epsilon \)-covering \( K_\epsilon \) of \( K \cap B^n \), i.e., with \( \log |K_\epsilon| = \mathcal{H}(K, \epsilon) \), a standard union bound argument then provides that (19) holds for all \( v' \in K_\epsilon \) with probability exceeding \( 1 - C \exp(-\frac{c}{2} \epsilon^2 m) \) provided \( m \geq \frac{1}{c} \epsilon^{-2} \mathcal{H}(K, \epsilon) \).

Moreover, since \( \|\rho_i^2\|_{\psi_1} \leq 2 \|\rho_i\|_{\psi_2}^2 \leq 2R^2 \) for \( 1 \leq i \leq m \) [34] Lemma 5.14, \( \|\rho\|^2 = \sum_{i=1}^{m} (\rho_i)^2 \) is a sum of sub-exponential i.i.d. random variables so that from [34] Cor. 5.17 (with setting there \( \epsilon = 1 \)) proves that \( \|\rho\|^2 \leq 2R^2 m \) with probability exceeding \( 1 - 2 \exp(-cm) \).

Therefore, conditionally to the last two random events, which occur jointly with probability exceeding \( 1 - C \exp(-\frac{1}{2} \epsilon^2 m) \) (by union bound) under the same requirement on \( m \), we assumed \( \frac{1}{\sqrt{m}} \Phi \) to be \( L \)-Lipschitz continuous over \( K \) with \( L = O(1) \), we have \( \frac{1}{\sqrt{m}} \|\Phi u\| = O(1) \) and \( \frac{1}{\sqrt{m}} \|\rho\| \leq 2R \). Consequently, for any \( v \in K \cap B^n \) whose closest point in \( K_\epsilon \) is \( v' \), i.e., \( \|v - v'\| \leq \epsilon \), we have

\[
\frac{1}{m} \|\langle \rho, \Phi \rangle\| \leq \frac{1}{m} \|\langle \rho, \Phi v'\rangle\| + \frac{1}{m} \|\langle \rho, \Phi (v - v')\rangle\| \leq \epsilon R \frac{1}{\sqrt{m}} \|\Phi v'\| + \frac{1}{m} \|\langle \rho, \Phi (v - v')\rangle\|.
\]

where we used Cauchy-Schwartz in the last line. A simple rescaling of \( \epsilon \) then provides the result.

The previous proposition enables us to characterize the L-LPD property of the quantized and dithered mapping \( A \) introduced in (4). This is straightforwardly obtained by observing that, given the resolution \( \delta > 0 \) defining both the quantizer \( Q \) and the range of the random uniform dithering \( \xi \sim U^m([0, \delta]) \), for any \( u \in \mathbb{R}^n \), we have

\[
y = A(u) = \Phi u + \rho,
\]

with \( \rho = A(u) - \Phi u \) and

\[
\|\rho_i\|_{\psi_2} = \|Q((\Phi u)_i + \xi_i) - (\Phi u)_i\|_{\psi_2} \leq \|Q((\Phi u)_i + \xi_i) - (\Phi u + \xi)_i\|_{\psi_2} + \|\xi_i\|_{\psi_2} \leq 2\delta,
\]

since for any r.v. \( X \) such that \( |X| \leq s \) for some \( s > 0 \), \( \|X\|_{\psi_2} \leq s \).

Corollary 5.4 (L-LPD for quantized, dithered mapping). Given a low-complexity set \( K \subset \mathbb{R}^n \), a distortion \( \epsilon > 0 \), a quantization resolution \( \delta > 0 \) and a matrix \( \frac{1}{\sqrt{m}} \Phi \) that is \( (\epsilon, L) \)-Lipschitz continuous over \( K \) with \( L = O(1) \), if \( A \) is defined from (4) with a uniform random dithering \( \xi \sim U^m([0, \delta]) \), then, given \( u \in K \cap B^n \) and provided

\[
m \geq \epsilon^{-2} \mathcal{H}(K \cap \mathbb{B}^n, \epsilon c),
\]

the mapping \( A \) respects the L-LPD\((K, \Phi, u, \delta \epsilon)\) with probability exceeding \( 1 - C \exp(-c \epsilon^2 m) \).
6 Limited projection distortion for a quantized, dithered random mapping

The previous section and Cor. 5.4 have shown that the L-LPD almost trivially holds for the quantized, dithered random mapping $A$ introduced in [55] by observing that the corresponding sensing model is equivalent to a noisy linear sensing corrupted by an additive sub-Gaussian noise with i.i.d. components.

However, by analyzing more carefully the interplay between the quantizer discontinuities and the dithering in $A$, we can prove that the uniform LPD hold. Moreover, for structured sets for which the Kolmogorov entropy only increases logarithmically with the involved radius (e.g., as for conic low-complexity sets such as the set of sparse vectors or the set of low-rank matrices), the sample complexity ensuring the LPD is very similar to the one guaranteeing the LPD, up to few logarithmic factors.

We will proceed in two steps. First, we analyze in Sec. 6.1 the geometric properties of the random mapping $A : a \in \mathbb{R}^m \mapsto Q(a + \xi) \in \delta \mathbb{Z}^m$ associated with a random vector $\xi \sim U^m([0, \delta])$ and to the quantizer $Q(\cdot) := \delta [\cdot / \delta]$ for some quantization resolution $\delta > 0$. Second, in Sec. 6.2 the characteristics of $A = A \circ Q$ will be then explained by the Lipschitz embedding realized by any RIP matrix $\Phi \in \mathbb{R}^{m \times n}$ between a low-complexity set $\mathcal{K} \subset \mathbb{R}^n$ and its (still low-complexity) image $\mathcal{J} := \Phi \mathcal{K}$, and from the limited inner product distortions induced by $A(\mathcal{J}) \subset \delta \mathbb{Z}^m$ in $\mathbb{R}^m$.

6.1 Analysis of the dithered quantizer

As a first positive impact of the random dithering over the discontinuous quantizer $Q$, the following Lemma essentially shows that the number of components of $A$ that are truly discontinuous in a $\ell_\infty$-neighborhood of an arbitrary point of $\mathbb{R}^m$, i.e., components for which a threshold of the quantizer belongs to the interval of the $i$th coordinates of this neighborhood, is controlled by the size of this neighborhood. This analyses of the continuity of $A$ is inspired by a strategy developed in [55] for more general discontinuous mappings.

Lemma 6.1. Given $a \in \mathbb{R}^m$, $\epsilon > 0$ and $0 < \rho < \delta$, denote by $Z = Z(a + \rho \ell_\infty) \in \{0, \ldots, m\}$ the discrete random variable associated with the number of components of $A$ displaying a discontinuity (i.e., with at least two different values) over $a + \rho \ell_\infty$. The random variable $Z$ has a binomial distribution with $m$ trials and a probability of success $p = \frac{2\rho}{\delta}$, i.e., $Z \sim \text{Bin}(m, p)$. Therefore, $E[Z] = mp = m \frac{2\rho}{\delta}$, and $\frac{1}{m}(EZ^2 - (EZ)^2) = p(1-p) =: \sigma^2 < p$ and

$$\mathbb{P}[Z > m(\frac{2\rho}{\delta} + \epsilon)] \leq \exp(-\frac{1}{2} \frac{3m\sigma^2}{\rho^2}).$$

(21)

In particular, setting $\epsilon = \frac{2\rho}{\delta} = p > \sigma^2$ provides

$$\mathbb{P}[Z > m \frac{4\rho}{\delta}] \leq \exp(-m \frac{\sigma^2}{\delta}).$$

(22)

Proof. Let $Z_i$ be the random variable equal to 1 if $A_i(\cdot)$, the $i$th component of $A(\cdot)$, is discontinuous over $a + \rho \ell_\infty$, and to 0 otherwise, for $i \in [m]$. We have $Z = \sum_i Z_i$. Moreover, $\mathbb{P}[Z_i = 1] = \frac{2\rho}{\delta}$ from the uniformity of $\xi$ and the fact that the discontinuities of $Q$ are given by $\delta \mathbb{Z}$. The rest of the Lemma is then simply obtained by observing that the r.v.’s $\{Z_i : 1 \leq i \leq m\}$ are i.i.d. and by a simple application of Bernstein’s inequality to $Z - EZ$. 

18
In a similar argument to the one used at the end of Sec. 3, the next lemma indicates that, on a fixed vectors $\mathbf{a} \in \mathbb{R}^m$ and in a fixed direction induced by a vector $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A}(\mathbf{a})$ concentrates around its expectation $\mathbf{a}$, and this concentration improves exponentially with $m$.

**Lemma 6.2.** Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ and $\epsilon > 0$, we have

$$
P[|\langle \mathbf{A}(\mathbf{a}) - \mathbf{a}, \mathbf{b} \rangle| \geq \delta \epsilon \sqrt{m} \|\mathbf{b}\|] \leq 2 \exp(-2\epsilon^2 m).$$

**Proof.** We define the i.i.d. random variables $\{X_i = \mathbf{A}(a_i) - a_i \in [-\delta, \delta], \; i \in [m]\}$. From Lemma A.1, $\mathbb{E}X_i = 0$. Moreover $|X_i| \leq \delta$ for all $i \in [m]$. Therefore, Hoeffding’s inequality\(^{10}\) provides $P[|\langle \mathbf{A}(\mathbf{a}) - \mathbf{a}, \mathbf{b} \rangle| \geq t] = P[|\sum_i X_i b_i| \geq t] \leq 2 \exp(-\frac{2t^2}{\delta^2 \|\mathbf{b}\|^2})$ for $t > 0$. Operating the change of variable $t = \epsilon \delta \sqrt{m} \|\mathbf{b}\|$ gives the result. \qed

We are now ready to show that projecting a vector onto the image of any vector of a set $\mathcal{J} \subset \mathbb{R}^m$ by $\mathbf{A}$ is close to the inner product of both vectors, i.e., $\mathbf{A}$ induces limited inner product distortion provided that $\mathcal{J}$ has bounded Kolmogorov entropy \(^{42}\). In fact, given a fixed direction $\mathbf{b}/\|\mathbf{b}\|$, this amounts to extending Lemma 6.2 and to analyzing how $\langle \mathbf{A}(\mathbf{a} + \xi) - \mathbf{a} \rangle$ concentrates in this direction for all $\mathbf{a} \in \mathcal{J}$.

**Proposition 6.3** (Limited inner product distortion in a fixed direction). Given a distortion $0 < \epsilon < 1$, a fixed $\mathbf{b} \in \mathbb{R}^m$, a value $0 < \delta, \delta \leq 1$, and a subset $\mathcal{J} \subset \mathbb{R}^m$ with bounded Kolmogorov entropy $\mathcal{H}(\mathcal{J}, \cdot)$, provided

$$m \geq C \epsilon^{-2} \mathcal{H}(\mathcal{J}, c \delta^3 \epsilon^2 \sqrt{m}),$$

with $C \leq 96$ and $c \geq 1/6^3$, we have, with probability exceeding $1 - 3 \exp(-C^{-1} \epsilon^2 m)$,

$$|\langle \mathbf{A}(\mathbf{a}), \mathbf{b} \rangle - \langle \mathbf{a}, \mathbf{b} \rangle| \leq (\delta + \delta^s)\epsilon \sqrt{m} \|\mathbf{b}\|, \; \forall \mathbf{a} \in \mathcal{J}. \quad (25)$$

Before proving this proposition we can already analyze the nature of the distortion in (25). First, in the case where $s = 1$, we see that this distortion is proportional to $\delta \epsilon \sqrt{m}$ and it could be made arbitrarily small if $\delta \to 0$. However, this reduction is limited by the requirement (24) that is harder to satisfy when $\delta$ is small, at least for sets $\mathcal{J}$ whose Kolmogorov entropy increases rapidly when the radius decreases. For instance, for structured sets (see Sec. 7), $\mathcal{H}(\mathcal{J}, \eta) = O(\log(1 + 1/\eta))$ so that the requirement (24) is still manageable for relatively low values of $\delta$, even for $s = 1$. However, for other sets, e.g., if $\mathcal{J}$ is only known to be bounded, we only know that $\mathcal{H}(\mathcal{J}, \eta) = O(1/\eta^2)$ (from Sudakov inequality). In this case, setting $s \neq 0$ quickly leads to a requirement on $m$ that does not allow a small distortion when $\delta$ decreases. In this case, it is easier to set $s = 0$ and to restrict to a distortion proportional to $\epsilon(1 + \delta)$.

Thanks to Lemmata 6.1 and 6.2, the proof of this proposition is relatively simple and inspired by some methodology defined in \(^{55}\). We observe that the number of components of $\mathbf{A}$ that are continuous (and thus constant) over each ball of a dense covering of $\mathcal{J}$ are close to $m$ when this value is large. Then, we can basically bound the discontinuous components, which are in minority, with deterministic bounds known on the quantizer. This method is considerably simpler, in this case, than the softening strategy of the discontinuous mapping $\mathbf{A}$ proposed in \(^{19} 20\), as inspired by the softening of the one-bit (“sign”) quantizer introduced in \(^{56}\).

\(^{10}\)As at the end of Sec. 3, we could also use here the sub-Gaussianity of the r.v.’s $X_i$ to study the concentration of $\langle \mathbf{A}(\mathbf{a}), \mathbf{b} \rangle$ around $\langle \mathbf{a}, \mathbf{b} \rangle$. Hoeffding’s inequality is simply more accurate with respect to the multiplicative constants of the tail bound.

19
Proof of Prop.6.3. From the homogeneity of (\ref{eq:6.3}) we can fix \(b \in \mathbb{R}^m\) and assume \(\|b\| = 1\). Let \(\mathcal{J}_{\eta \sqrt{m}}\) be an optimal \((\eta \sqrt{m})\)-covering in the \(\ell_2\)-metric of \(\mathcal{J}\) for some \(\eta > 0\) to be fixed later, i.e., 

\[
\log |\mathcal{J}_{\eta \sqrt{m}}| = \mathcal{H}(\mathcal{J}, \eta \sqrt{m}).
\]

Let us first consider an arbitrary vector \(a' \in \mathcal{J}\). By definition of the covering above, we can always write \(a' = a + r\) for some \(a \in \mathcal{J}_{\eta \sqrt{m}}\) and \(r \in (\eta \sqrt{m})\mathbb{B}^m\). Given a value \(P > 0\) whose value will be fixed later, we define the set 

\[
T = T(r) := \{i \in [m] : |r_i| \leq \eta \sqrt{P}\}.
\]

We observe that \(|T^c| \leq \frac{m}{P}\) independently of \(r\), since \(\eta^2 m \geq \|r\|_2^2 \geq \|r_{T^c}\|_2^2 \geq \eta^2 P |T^c|\). Writing \(\sigma_i = \chi_T(i)\) for \(i \in [m]\) and \(\tilde{r} := \sigma \odot r \in (\eta \sqrt{P})\mathbb{B}_\ell^m\), we now develop the LHS of \(\ref{eq:6.3}\):

\[
|\langle \mathbb{A}(a'), b \rangle - \langle a', b \rangle| = |\sum_i b_i(\mathbb{Q}(a_i + r_i + \xi_i) - (a_i + r_i))|
\]

\[
\leq |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + \tilde{r}_i))|
\]

\[
+ |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + r_i)) - \mathbb{Q}(a_i + r_i + \xi_i) + (a_i + r_i))|
\]

\[
\leq |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + \tilde{r}_i))|
\]

\[
+ |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + r_i)) + \|b\|_{\ell^c}\|
\]

\[
\leq |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + r_i)) + \delta\|b\|_{\ell^c}\|
\]

\[= I + \delta(\frac{1}{P})^{1/2} \sqrt{m}, \tag{26}\]

where we have used the fact that \(|\langle [a] + b \rangle - ([a] + b)| \leq 1\) for all \(a, b \in \mathbb{R}\), and \(\chi_T(i) = 1\) if \(i \in T\) and 0 otherwise.

Notice that the term \(I = |\langle \mathbb{A}(a + \tilde{r}) - (a + \tilde{r}), b \rangle|\) in \(\ref{eq:26}\) has thus to be characterized for a neighborhood \((\eta \sqrt{P})\mathbb{B}^m_{\ell_\infty}\) of \(a\) since \(\|\tilde{r}\|_\infty \leq \eta \sqrt{P}\). Interestingly, from Lemma 6.1 and \(\ref{eq:22}\), a union bound argument allows us to bound the number \(Z\) of discontinuous components\(^{11}\) of \(\mathbb{A}\) over all \(\ell_\infty\)-balls of \(\mathcal{J}_{\eta \sqrt{m}} + (\eta \sqrt{P})\mathbb{B}^m_{\ell_\infty}\) so that

\[
P\left[ \exists u \in \mathcal{J}_{\eta \sqrt{m}} : (u + (\eta \sqrt{P})\mathbb{B}^m_{\ell_\infty}) > m \frac{4\eta \sqrt{P}}{\delta} \right] \leq \exp \left( \mathcal{H}(\mathcal{J}, \eta \sqrt{m}) - \frac{3}{4} m \frac{\eta \sqrt{P}}{\delta} \right).
\]

Equivalently,

\[
\max\{Z(u + (\eta \sqrt{P})\mathbb{B}^m_{\ell_\infty}) : u \in \mathcal{J}_{\eta \sqrt{m}}\} \leq m \frac{4\eta \sqrt{P}}{\delta},
\]

with probability exceeding \(1 - \exp(\mathcal{H}(\mathcal{J}, \eta \sqrt{m}) - \frac{3}{4} m \frac{\eta \sqrt{P}}{\delta})\).

Let us denote \(C = C(a) \subset [m]\) the set of components of \(\mathbb{A}\) that are continuous over \(a + (\eta \sqrt{P})\mathbb{B}^m_{\ell_\infty}\), with \(\sigma'_i := \chi_C(i)\) for \(i \in [m]\). We also write \(\tilde{r} = \sigma \odot \sigma' \odot \tilde{r} = \sigma' \odot \tilde{r}\). By construction, we have thus \(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) = \mathbb{Q}(a_i + \xi_i)\) for \(i \in [m]\). The bound above shows that with the same probability

\[
I = |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + \tilde{r}_i))|
\]

\[
\leq |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + \tilde{r}_i))|
\]

\[
+ |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + r_i)) - \mathbb{Q}(a_i + r_i + \xi_i) + (a_i + r_i))|
\]

\[
\leq |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + \tilde{r}_i))|
\]

\[
+ |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + r_i)) + |\sum_i b_i \tilde{r}_i|
\]

\[
+ |\sum_i b_i(\mathbb{Q}(a_i + \tilde{r}_i + \xi_i) - (a_i + r_i)) + \mathbb{Q}(a_i + \xi_i) + \tilde{r}_i - (\mathbb{Q}(a_i + \xi_i) + \tilde{r}_i)).|
\]

\(^{11}\)Remark that by bounding \(Z\) we do not forbid \(\mathbb{A}\) to have different discontinuous components for different balls of \(\mathcal{J}_{\eta \sqrt{m}} + (\eta \sqrt{P})\mathbb{B}^m_{\ell_\infty}\).
Note that in the last term, since \( \tilde{r}_i = \sigma_i r_i \) is equal to \( r_i = \sigma_i r_i \) if \( i \in \mathcal{C} \) or if \( i \in \mathcal{T}^c \), all terms of the sum are zero but those for which \( i \in (\mathcal{C} \cup \mathcal{T}^c)^c = \mathcal{C}^c \cap \mathcal{T} \subset \mathcal{C}^c \). For these non-zero terms, \(|(\mathcal{Q}(a_i + \xi_i + \tilde{r}_i) + r_i - (\mathcal{Q}(a_i + \xi_i + \tilde{r}_i))| \leq \delta \). Therefore,

\[
I \leq |\sum_i b_i(\mathcal{Q}(a_i + \xi_i) - a_i)| + \eta \sqrt{\mathcal{T}} \|b\|_1 + \delta \|b_{c\cdot}\|_1 \\
\leq |\sum_i b_i(\mathcal{Q}(a_i + \xi_i) - a_i)| + \eta \sqrt{\mathcal{T}} \sqrt{m} + 2\delta (\frac{\eta \sqrt{\mathcal{T}}}{\delta})^{1/2} \sqrt{m} \\
=: II + \eta \sqrt{\mathcal{T}} \sqrt{m} + 2\delta (\frac{\eta \sqrt{\mathcal{T}}}{\delta})^{1/2} \sqrt{m}. \tag{27}
\]

Applying Lemma 6.2 with a union bound on all \( a \in \mathcal{J}_{\eta \sqrt{m}} \) ensures that

\[
II = |\sum_i b_i(\mathcal{Q}(a_i + \xi_i) - a_i)| \leq \delta \sqrt{m}, \tag{28}
\]

holds with probability exceeding \( 1 - 2 \exp(\mathcal{H}(\mathcal{J}, \eta \sqrt{m}) - 2\epsilon^2 m) \).

Gathering all the bounds from \(26\), \(27\) and \(28\) and applying a last union bound on the failure of the different events shows that

\[
|\langle \mathcal{A}(a'), b \rangle - \langle a', b \rangle| \leq \delta \epsilon \sqrt{m} + \eta \sqrt{\mathcal{T}} \sqrt{m} + 2\delta (\frac{\eta \sqrt{\mathcal{T}}}{\delta})^{1/2} \sqrt{m} + \delta (\frac{1}{\epsilon})^{1/2} \sqrt{m},
\]

with probability exceeding

\[
1 - \exp(\mathcal{H}(\mathcal{J}, \eta \sqrt{m}) - \frac{3}{4} m \frac{\eta \sqrt{\mathcal{T}}}{\delta}) - 2 \exp(\mathcal{H}(\mathcal{J}, \eta \sqrt{m}) - 2\epsilon^2 m).
\]

We can now set our free parameters and fix \( P = \delta^{2(1-s)} \epsilon^{-2}, \eta = \delta^{s^3} \) for some \( 0 \leq s \leq 1 \), so that \( \frac{\eta \sqrt{\mathcal{T}}}{\delta} = \epsilon^2 \) and \( (\frac{1}{\epsilon})^{1/2} = \epsilon \delta^{s-1} \). Therefore, from a few simplifications and for \( \epsilon < 1 \), we have

\[
|\langle \mathcal{A}(a'), b \rangle - \langle a', b \rangle| \leq 4\delta \epsilon \sqrt{m} + \epsilon \delta^{s} \sqrt{m}
\]

with probability exceeding \( 1 - 3 \exp(-\frac{3}{8} m \epsilon^2) \) provided

\[
m \geq \frac{8}{3} \epsilon^{-2} \mathcal{H}(\mathcal{J}, \delta^s \epsilon^3 \sqrt{m}).
\]

Finally, a rescaling of \( \epsilon \) provides the result. \( \square \)

Note that if \( \mathcal{J} \) is reduced to a single point in Prop. 6.3, then \( \mathcal{H}(\mathcal{J}, \cdot) = 0 \) and we recover Lemma 6.2 as no condition is imposed on \( m \) anymore.

We prove now a (global) limited inner product distortion property of \( \mathcal{A} \) for scalar products of all elements of a subset \( \mathcal{J} \subset \mathbb{R}^m \), hence extending the previous proposition to all elements of \( b \in \mathcal{J} \) provided that this set has bounded Kolmogorov entropy.

**Proposition 6.4** (Limited inner product distortion). Given a subset \( \mathcal{J} \subset \mathbb{R}^m \) with bounded Kolmogorov entropy \( \mathcal{H}(\mathcal{J}, \cdot) \), and another subset \( \mathcal{J}' \subset \mathbb{R}^m \), possibly reduced to a single point, such that, given \( b' \in \mathcal{J} \) and \( 0 < \epsilon < 1 \),

\[
P\{\forall a \in \mathcal{J}': |\langle \mathcal{A}(a), b' \rangle - \langle a, b' \rangle| \leq \epsilon \nu_\delta \sqrt{m} \|b'\| \} \geq 1 - C \exp(-c\epsilon^2 m), \tag{29}
\]

for some \( \nu_\delta \geq \delta \) only depending on \( \delta \). Provided that

\[
m \geq \frac{1}{2c} \epsilon^{-2} \mathcal{H}(\mathcal{J}, \epsilon \sqrt{m}), \tag{30}
\]

we have with probability exceeding \( 1 - C \exp(-\frac{c}{2} \epsilon^2 m) \),

\[
|\langle \mathcal{A}(a), b \rangle - \langle a, b \rangle| \leq \epsilon \nu_\delta (\sqrt{m} \|\mathcal{J}\| + m), \quad \forall a \in \mathcal{J}', \forall b \in \mathcal{J}, \tag{31}
\]

with \( \|\mathcal{J}\| = \sup \{\|u \in \mathcal{J}\|\} \).
Proof. We consider $\delta > 0$ since $[31]$ trivially holds if $\delta = 0$. Let $J_{\sqrt{m}}$ be an optimal $(\epsilon \sqrt{m})$-covering of $J$. From $[29]$, provided
\[ m \geq \frac{1}{2\epsilon^2} \mathcal{H}(J, \epsilon \sqrt{m}), \]
we have, with probability exceeding $1 - C \exp(-\frac{1}{2} \epsilon^2 m)$,
\[ |\langle A(a), b' \rangle - \langle a, b' \rangle| \leq \epsilon \nu \sqrt{m} \|b'\|, \quad \forall a \in J, \forall b' \in J_{\epsilon \sqrt{m}}. \]
Therefore, for all $b \in J$, $b = b_0 + r$ with $r \in (\sqrt{m})^B$ and $b_0 \in J_{\epsilon \sqrt{m}}$,
\[ |\langle A(a), b \rangle - \langle a, b \rangle| = |\langle A(a) - a, b \rangle| \leq |\langle A(a) - a, b_0 \rangle| + |\langle A(a) - a, r \rangle| \]
\[ \leq \epsilon \nu \sqrt{m} \|J\| + \|A(a) - a\| \|r\| \]
\[ \leq \epsilon \nu \sqrt{m} \|J\| + \delta \epsilon m, \]
where we used Cauchy-Schwartz several times.

6.2 Limited projection distortion

We can now determine under which conditions $A$ has limited projection distortion by bridging the analysis of $A$ with the action of a RIP matrix $\sqrt{\epsilon \epsilon} \Phi \in \mathbb{R}^{m \times n}$.

Proposition 6.5 (LPD for dithered and quantized random projections). Given a subset $K \subset \mathbb{R}^n$, a distortion $\epsilon > 0$, a quantization resolution $\delta > 0$, a matrix $\frac{1}{\sqrt{m}} \Phi$ that is $(\epsilon^3, L)$-Lipschitz continuous over $K$ with $L = O(1)$, and a random dithering $\xi \sim \mathcal{U}^m([0, \delta])$, provided
\[ m \geq C \epsilon^{-2} \mathcal{H}(K \cap \mathbb{R}^n, \epsilon \epsilon^3), \]
the random mapping $A$ defined in $[4]$ respects the LPD($K, \epsilon(1 + \delta)$), i.e.,
\[ \frac{1}{m} |\langle A(u), \Phi v \rangle - \langle \Phi u, \Phi v \rangle| \leq \epsilon (1 + \delta), \quad \forall u, v \in K \cap \mathbb{R}^n, \] (33)
with probability exceeding $1 - C' \exp(-c\epsilon^2 m)$.

We note that there is a price to pay for the uniformity of $[33]$, i.e., for its verification over all $u, v \in K \cap \mathbb{R}^n$. First, the factor $\epsilon (1 + \delta)$ defining the LPD distortion does not vanish when $\delta \to 0$, conversely to the non-uniform L-LPD distortion in $\delta \epsilon$ established in Cor. 5.4 for the same mapping $A$. This effect is however mitigated by the deterministic bound $|\langle A(u), \Phi v \rangle - \langle \Phi u, \Phi v \rangle| \leq \delta \|\Phi v\| \sqrt{m} \leq L \delta m$ proved in Rem. 3.2. Second, as stated in Cor. 5.4, the sample complexity $[32]$ is markedly different from the one required by the L-LPD in Cor. 5.4, i.e., the Kolmogorov entropy radius display here a cubic exponent and is not linear in $\epsilon$ anymore. As described in Sec. 7, this will have anyway a limited impact for structured low-complexity sets $K$ where the Kolmogorov entropy only depends logarithmically on its radius. However, the impact will be stronger for other sets where we only dispose of Sudaakov’s bound to observe that this entropy is bounded by a function scaling like the inverse-square of its radius.

Proof. The proof consists in applying Prop. 6.4 in the case where $J' = J = \Phi(K \cap \mathbb{R}^n)$, and to use Lemma 5.2 to bound the Kolmogorov entropy of $\Phi(K \cap \mathbb{R}^n)$ with the one of $K \cap \mathbb{R}^n$.

In this setting, from $[24]$ in Prop. 6.3, with $s = 0$, we have for a given $v' \in K \cap \mathbb{R}^n$,
\[ \mathbb{P}(\forall u \in K : |\langle A(u), \Phi v' \rangle - \langle \Phi u, \Phi v' \rangle| \leq \epsilon (1 + \delta) \sqrt{m} \|\Phi v'\|) \geq 1 - C \exp(-c\epsilon^2 m), \]
\[ (34) \]
provided that

\[ m \geq C\epsilon^{-2} \mathcal{H}(\Phi(K \cap \mathbb{B}^n), c'\sqrt{m}). \]

The bound (34) is the probability bound required in (29) of Prop. 6.4. Additionally, Prop. 6.4 requires in (30) to have

\[ m \geq C\epsilon^{-2} \mathcal{H}(\Phi(K \cap \mathbb{B}^n), \epsilon\sqrt{m}). \]

From the fact that \( 0 < \epsilon < \epsilon < 1 \), all conditions on \( m \) are thus guaranteed if

\[ m \geq C\epsilon^{-2} \mathcal{H}(\Phi(K \cap \mathbb{B}^n), ce^3\sqrt{m}), \]

However, if \( \frac{1}{\sqrt{m}} \Phi \) is \((\epsilon^3, L)\)-Lipschitz continuous with \( L = O(1) \), Lemma 5.2 provides

\[ \mathcal{H}(\Phi(K \cap \mathbb{B}^n), ce^3\sqrt{m}) \leq \mathcal{H}(K \cap \mathbb{B}^n, c\epsilon^3), \]

for appropriate constants \( c, c' > 0 \) only depending on \( L \), which justifies (32).

Consequently, by union bound over the events covered by Prop. 6.3 and Prop. 6.4, we have that (31) in Prop. 6.4 holds with \( a = \Phi u \) and \( b = \Phi v \), for all \( u, v \in K \), and with probability exceeding \( 1 - C\exp(-cm\epsilon^2) \). Finally, since \( \|\Phi(K \cap \mathbb{B}^n)\| \leq \sqrt{mL} \) from the assumed Lipschitz continuity of \( \Phi \) (see Def. 5.1), a final rescaling of \( \epsilon \) provides the result.

\[ \square \]

7 PBP reconstruction error decay in a few special cases

In this section, we focus on the reconstruction error of the PBP method when the general distorted sensing model (14) is the dithered, quantized random sensing of signals in a low-complexity set, as defined in (4). In particular, we combine the general analysis provided in Sec. 4 with the known conditions that ensure, with high probability, the verification of the RIP of a random sensing matrix \( \frac{1}{\sqrt{m}} \Phi \), and the (L)LPD property of the associated quantized random mapping \( A \) as determined in Sec. 5 and Sec 6.

To this analysis, we must also add the conditions guaranteeing the Lipschitz continuity of the random matrix \( \frac{1}{\sqrt{m}} \Phi \), as imposed by Prop. 6.5 and Cor. 5.4. Fortunately, we will show that most known random matrix constructions proved to respect, w.h.p., the RIP over a set \( K \subset \mathbb{R}^n \), are also Lipschitz in the sense of Def. 5.1. This straightforwardly holds if \( K \) is conic, but it is also verified if \( K \) is bounded and star-shaped (see Prop. 7.4).

As a result, we establish the decay rate of the PBP reconstruction error when \( m \) increases for specific examples of random sensing matrices and of low-complexity spaces for which the conditions enabling the RIP in (7) have been characterized in the CS literature. The list of examples given below is of course not exhaustive. Our purpose is mainly to instantiate, in a few representative contexts, the general results provided in Sec. 4 and hence to illustrate their practicality in our QCS setting.

7.1 Conditions ensuring the RIP over low-complexity sets

As reviewed below, the CS literature provides now many random sensing matrix constructions, structured or not, which are proved to respect the RIP w.h.p.. The following definition summarized the general statements and conditions surrounding these constructions.

**Definition 7.1 (Low-complexity Set Random Embedding - LSRE).** Given a low-complexity set \( K \subset \mathbb{R}^n \), a distortion \( \epsilon > 0 \) and a failure probability \( 0 < \zeta < 1 \), a random matrix construction
\( \frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n} \) is said to be a Low-complexity Set Random Embedding construction, or LSRE, if this mapping respects the RIP(\( \mathcal{K}, \epsilon \)) defined in (7) with probability exceeding \( 1 - \zeta \) provided
\[
m \gtrsim \epsilon^{-2} w(\mathcal{K} \cap \mathbb{B}^{n})^2 \mathcal{P}_{\log}(m, n, 1/\zeta),
\]
with \( \mathcal{P}_{\log} \) a low-degree polynomial of logarithms (or polylogarithmic function) in its arguments.

Here are a few examples of known LSRE constructions. First, for sensing \( k \)-sparse signals in an orthonormal basis \( \Psi \in \mathbb{R}^{n \times n} \), i.e., signals of \( \mathcal{K} = \Psi \Sigma_k^n \), many studies have proved that, given a distortion \( \epsilon > 0 \) and as soon as
\[
m \gtrsim \epsilon^{-2} k \log(n/k),
\]
a sub-Gaussian random matrix \( \frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n} \), i.e., with random entries i.i.d. from a centered sub-Gaussian distribution with unit variance (e.g., Gaussian, Bernoulli) \([57, 5, 4, 3]\), satisfies the RIP(\( \Psi \Sigma_k^n, \epsilon \)) with probability exceeding \( 1 - C \exp(-c\epsilon^2 m) \). Consequently, this proves also that we can lower bound this probability by \( \epsilon^{-2} k \log(n/k) \log(1/\zeta) \) with \( 0 < \zeta < 1 \) since then \( m \epsilon^2 \gtrsim \log(1/\zeta) \) and \( 1 - C \exp(-c\epsilon^2 m) \gtrsim 1 - \zeta \), for an appropriate constant \( C' > 0 \). We will see below that a sub-Gaussian random matrix (SGRM) can actually embed more general sets than the case of sparse signals, thanks to a generalization of (36) compatible with (35).

Still in the context of sparse signals embedding, the RIP can also be proved for certain structured random matrix constructions, i.e., with reduced memory storage and low matrix-to-vector multiplication complexity. This is the case of Partial Random Orthonormal Matrix (PROM) or Bounded Orthonormal Systems (BOS) constructions \([57, 6]\). For instance, if \( U \in \mathbb{R}^{n \times n} \) is an orthonormal basis such as the Discrete Cosine Transform (DCT) or the Hadamard transform, then a sub-Gaussian random matrix \( \Phi \in \mathbb{R}^{m \times n} \) with bounded Gaussian mean width \( w(\mathcal{K} \cap \mathbb{B}^{n}) \) (e.g., for signals displaying some structured form of sparsity, for compressible vectors or for low-rank models) there exist also several key results showing similar requirements on \( m \) to ensure the RIP of random matrix constructions w.h.p. \([27, 44, 58, 50]\) (see also \([20, \text{Table 1}]\)).

In the case of a more general low-complexity set \( \mathcal{K} \subset \mathbb{R}^n \) with bounded Gaussian mean width \( w(\mathcal{K} \cap \mathbb{B}^{n}) \) (e.g., for signals displaying some structured form of sparsity, for compressible vectors or for low-rank models) there exist also several key results showing similar requirements on \( m \) to ensure the RIP of random matrix constructions w.h.p. \([27, 44, 58, 50]\) (see also \([20, \text{Table 1}]\)).

First, given a subset \( \mathcal{S} \subset \mathbb{R}^{n-1} \) and some \( 0 < \epsilon < 1 \), provided \( m \gtrsim \epsilon^{-2} w(\mathcal{S})^2 \), if \( \Phi \in \mathbb{R}^{m \times n} \) is again a sub-Gaussian random matrix, then
\[
\sup_{v \in \mathcal{S}} \frac{1}{m} \| \Phi v \|^2 - 1 \leq \epsilon,
\]
with probability exceeding \( 1 - \exp(-c\epsilon^2 m) \) \([5, 7]\), and \( \frac{1}{\sqrt{m}} \Phi \) then satisfies the RIP(\( \mathcal{S}, \epsilon \)) from (7).

\[\text{\footnotesize{\textsuperscript{12}Or the discrete Fourier transform provided the results of this work are extended to the complex domain.}}\]

\[\text{\footnotesize{\textsuperscript{13}This result is an easy rewriting of \([6, \text{Theorem 8.1}]\) for incoherent discrete bases of \( \mathbb{R}^{n \times n} \).}}\]
Interestingly, we can extend this result to any bounded subset of $\mathbb{R}^n$ thanks to a “lifted strategy” developed in [56] (see also in [19, App. E] for another use of this trick). This is established in the next proposition proved in App. [C].

**Proposition 7.2.** Let $\mathcal{K} \subset \mathbb{R}^n$ be a bounded subset of $\mathbb{R}^n$ and $0 \in \mathcal{K}$. Given a distortion $\epsilon > 0$ and a probability of failure $0 < \zeta < 1$, if

$$m \geq \epsilon^{-2} \frac{w(K)^2}{\|K\|^2} \log(1/\zeta), \quad (40)$$

and if $\Phi \in \mathbb{R}^{m \times n}$ is a random matrix whose entries are i.i.d. from a centered sub-Gaussian distribution with unit variance, then

$$\left| \frac{1}{m} \| \Phi u \|_2^2 - \| u \|_2^2 \right| \leq \epsilon \|K\|^2, \quad \forall u \in \mathcal{K}, \quad (41)$$

with probability exceeding $1 - \zeta$.

In particular, this proposition states that for any set $\mathcal{K} \subset \mathbb{R}^n$ such that $\| \mathcal{K} \cap \mathbb{B}^n \| = 1$, if

$$m \geq \epsilon^{-2} w(\mathcal{K} \cap \mathbb{B}^n)^2 \log(1/\zeta), \quad (42)$$

for some $0 < \zeta < 1$, a sub-Gaussian random matrix $\frac{1}{\sqrt{m}} \Phi$ satisfies the RIP($\mathcal{K}, \epsilon$), as defined in (7), with probability exceeding $1 - \zeta$. We will see in the next section that Prop. 7.2 also specifies that this random mapping is Lipschitz continuous with high probability, even if $\mathcal{K}$ is not a cone.

A similar result has been recently proved for Subsampled Orthogonal with Random Sign (SORS) sensing matrices [59]. This construction essentially combines a PROM construction (see above) with a pre-modulation matrix set with random signs, i.e., $\Phi = \sqrt{n} R_{\Omega} U D$ where $R_{\Omega} \in \{0, 1\}^{\Omega \times n}$ is the selection matrix such that $R_{\Omega} u = u_\Omega$ for some $\Omega \subset [n]$, $U \in \mathbb{R}^{n \times n}$ is an orthonormal matrix with $\max_{ij} |U_{ij}| = O(1/\sqrt{n})$, and $D$ is a $n \times n$ random diagonal matrix with the diagonal entries i.i.d. $\pm 1$ with equal probability.

**Proposition 7.3** (Adapted from [59, Thm 3.3]). Let $\mathcal{K} \subset \mathbb{R}^n$ be a bounded subset of $\mathbb{R}^n$. Given a distortion $\epsilon > 0$ and a probability of failure $0 < \zeta < 1$, if

$$m \geq \epsilon^{-2} \max(1, \frac{w(\mathcal{K})^2}{\|K\|^2})(\log n)^4 \log(2/\zeta)^2, \quad (43)$$

and if $\Phi \in \mathbb{R}^{m \times n}$ is a SORS random matrix, then, with probability exceeding $1 - \zeta$,

$$\left| \frac{1}{m} \| \Phi u \|_2^2 - \| u \|_2^2 \right| \leq \max(\epsilon, \epsilon^2) \|K\|^2, \quad \forall u \in \mathcal{K}, \quad (44)$$

**Proof.** In [59, Thm 3.3], the requirement on the sample complexity reads

$$m \geq \epsilon^{-2} \max(1, \frac{w(\mathcal{K})^2}{\|K\|^2})(\log n)^4 (1 + \varpi)^2,$$

which allows (44) to hold with probability exceeding $1 - 2e^{-\varpi}$. Setting $\zeta = 2e^{-\varpi}$, i.e., $\varpi = \log(2/\zeta)$ and observing that $\log(2/\zeta)^2 \geq (1 + \log(2/\zeta))^2$ for $\zeta \in [0, 1]$ gives the result.

25
Consequently, for any set $K \subset \mathbb{R}^n$ with $\|K\| = 1$, there exists at least one unit vector $v \in K$, and $w(K \cap \mathbb{B}^n) \geq \mathbb{E}|\langle g, v \rangle| = \sqrt{2/\pi}$ for $g \sim \mathcal{N}(0, I_n)$. In this case, the last proposition shows that, for $0 < \epsilon < 1$, if
\[
m \gtrsim \epsilon^{-2}w(K \cap \mathbb{B})^2(\log n)^4 \log(2/\zeta)^2,\]
then a SORS matrix $1/\sqrt{m} \Phi$ respects the RIP($K, \epsilon$) with probability exceeding $1 - \zeta$.

In summary, according to the requirements (36), (38), (42) and (45), we have thus seen that all the examples covered above (i.e., SGRM, PROM, BOS, and SORS for $0 < \epsilon < 1$) respect the context of an LSRE construction defined in Def. [7.1]

### 7.2 Conditions for Lipschitz continuity

As explained in Sec. [1], when $K$ is a cone and if $1/\sqrt{m} \Phi$ respects the RIP($K - K, \epsilon$) for some $0 < \epsilon < 1$, then this mapping is trivially $(\eta, \sqrt{2})$-Lipschitz continuous from an easy use of (8) for any $\eta > 0$. Interestingly, as explained in the next proposition, this fact remains true if $K$ is not conic but bounded and star-shaped, and if $\Phi \in \mathbb{R}^{m \times n}$ is an LSRE construction.

**Proposition 7.4.** Given a bounded, star-shaped subset $K \subset \mathbb{B}^n$, with $0 \in K$ and $\|K\| = 1$, a radius $\eta > 0$ and a probability of failure $0 < \zeta < 1$, if $\Phi \in \mathbb{R}^{m \times n}$ is an LSRE construction, and if
\[
m \gtrsim \min(1, \eta)^{-2} w(K)^2 \mathcal{P}_{\log}(m, n, 1/\zeta),\]
then $1/\sqrt{m} \Phi$ is $(\eta, \sqrt{2})$-Lipschitz continuous over $K$ with probability exceeding $1 - \zeta$, i.e., $\|\Phi K\| \leq \sqrt{2m}$ and
\[
\|\Phi u\| \leq \eta \sqrt{2m}, \quad \forall u \in (K - K) \cap \eta \mathbb{B}^n.\]

**Proof.** The proof is easily established from the context of Def. [7.1]. We first note that if $S \subset \mathbb{R}^n$ is a bounded set then $S = \|S\|(S^*) = \|S\|(S^* \cap \mathbb{B}^n)$, with $S^* = S/\|S\| \subset \mathbb{B}^n$, and $w(S) = \|S\|w(S^*)$ from the positive homogeneity of the Gaussian mean width. Therefore, provided that $1/\sqrt{m} \Phi$ is an LSRE construction, a rescaling $\epsilon \leftarrow \epsilon \|S\|^2$ in (35) shows that if
\[
m \gtrsim \epsilon^{-2}w(S)^2 \mathcal{P}_{\log}(m, n, 1/\zeta),\]
then
\[
\left| \frac{1}{m} \|\Phi u\|^2 - \|u\|^2 \right| \leq \epsilon \|S\|^2, \quad \forall u \in S,
\]
with probability exceeding $1 - \zeta$. Let us now set $\epsilon = 1$ and $S = (K - K) \cap \eta \mathbb{B}^n$. We then observe that $\|K - K\| \geq \|K\| = 1$ if $0 \in K$, so that $\|(K - K) \cap \eta \mathbb{B}^n\| = \min(\eta, \|K - K\|) \geq \min(\eta, 1)$. This last bound is verified by analyzing the value of $\|(K - K) \cap \eta \mathbb{B}^n\|$ for the three possible cases $(K - K) \subset \eta \mathbb{B}^n$, $(K - K) \supset \eta \mathbb{B}^n$ and $(K - K) \cap (\eta \mathbb{B}^n)^c = \emptyset$, where, in this last possibility, $\|(K - K) \cap \eta \mathbb{B}^n\| = \eta$ since $K$ is star-shaped. Moreover, $w((K - K) \cap \eta \mathbb{B}^n) \leq w(K - K) \leq 2w(K)$, from the monotonicity and positive homogeneity of the Gaussian mean width, and, as explained after Prop. [7.3], $w(K) \geq \sqrt{2/\pi}$ if $\|K\| = 1$.

Consequently, (48) is verified if
\[
m \gtrsim \min(1, \eta)^{-2} w(K)^2 \mathcal{P}_{\log}(m, n, 1/\zeta),\]

26
and, in this case,
\[ \left| \frac{1}{m} \| \Phi u \|^2 - \| u \|^2 \right| \leq \eta^2, \quad \forall u \in (K - K) \cap \eta \mathbb{B}^n, \]
with probability exceeding 1 - \( \zeta \).

This involves \( \left| \frac{1}{\sqrt{m}} \| \Phi u \|^2 \right| \leq 2\eta^2 \), for all \( u \in (K - K) \cap \eta \mathbb{B}^n \), as requested in \( \text{(17)} \) in the definition of \((\eta, \sqrt{2})\)-Lipschitz continuity. Finally, since \((K - K) \cap \eta \mathbb{B}^n \supset K\) if \( \mathbf{0} \in K \), setting above \( \eta = 1 \) provides \( \| \Phi K \| \leq \sqrt{2m} \) with the same probability, and the result is proved by union bound over this last event and the case \( \eta \neq 1 \).

### 7.3 Conditions for (L)LPD

As a last ingredient of our error analysis, we need now to characterize the conditions ensuring the (L)LPD of the quantized random mapping \( A \) for the low-complexity sets considered above. There are basically two important cases that differentiate themselves from the behavior of the Kolmogorov entropy of the involved sets.

**A. Structured low-complexity sets:** The low-complexity set \( K \) can be first structured, meaning that \( K \) is a cone and the Kolmogorov entropy \( \mathcal{H}(K \cap \mathbb{B}^n, \eta) \) is tightly bounded with a function only depending in the logarithm of \( 1/\eta \) and of the complexity of \( K \cap \mathbb{B}^n \) \cite{44,28}.

For instance, for the set of sparse signals in an orthonormal basis or a dictionary \( \Psi \) and for the set of rank-\( r \) matrix of size \( n_1 \times n_2 \), we have respectively
\[
\mathcal{H}(\Psi \Sigma_k \cap \mathbb{B}^n, \eta) \lesssim k \log(n/k) \log(1 + 1/\eta) \quad \text{(see e.g., \cite{4})},
\]
\[
\mathcal{H}({C}_{r}^{n_1 \times n_2} \cap \mathbb{B}^n, \eta) \lesssim r(n_1 + n_2) \log(1 + 1/\eta) \quad \text{(see e.g., \cite{15,14})}.
\]

Interestingly, for these examples, we have also that
\[
w(\Psi \Sigma_k \cap \mathbb{B}^n)^2 \lesssim k \log(n/k) \quad \text{and} \quad w({C}_{r}^{n_1 \times n_2} \cap \mathbb{B}^{n_1 \times n_2})^2 \lesssim r(n_1 + n_2),
\]
and we note that these Gaussian mean width bounds are involved in the Kolmogorov entropy bounds above. In fact, this happens to be true for many other structured sets of low Kolmogorov entropy (see e.g., \cite{20, Table 1} for other examples).

Consequently, inspired by this correspondence, and as introduced in \cite{28,20}, we consider that a set \( K \subset \mathbb{R}^n \) is structured if it is conic and if the following bound holds:
\[
\mathcal{H}(K \cap \mathbb{B}^n, \eta) \lesssim w(K \cap \mathbb{B}^n)^2 \log(1 + 1/\eta). \quad (49)
\]

**Proposition 7.5 ((L)LPD for structured sets and LSRE constructions).** Given a structured subset \( K \subset \mathbb{R}^n \) for which (49) holds, a distortion \( \epsilon > 0 \), a quantization resolution \( \delta > 0 \), a random matrix \( \frac{1}{\sqrt{m}} \Phi \) generated from an LSRE construction, and a random dithering \( \xi \sim U^m([0, \delta]) \), the random mapping \( A \) defined in \( \text{(4)} \) respects the LPD\((K, \epsilon(1 + \delta))\), or the L-LPD\((K, \Phi, \mathbf{x}, \epsilon \delta)\) for a given \( \mathbf{x} \in \mathbb{R}^n \), with probability exceeding \( 1 - \zeta \) provided the following requirement is satisfied
\[
m \gtrsim \epsilon^{-2} w((K - K) \cap \mathbb{B}^n)^2 \log(1 + \frac{1}{\delta}) \mathcal{P}_\log(m, n, 1/\zeta), \quad (50)
\]
where \( p = 3 \) for the LPD and \( p = 1 \) for the L-LPD property, and \( \mathcal{P}_\log \) is a polylogarithmic function specified by the LSRE construction.
Proof. For the value \( p \) specified above, to establish the (L)LPD property of \( A \), we have simply to verify that the requirements of Prop. 6.5 and Cor. 5.4 are verified if (50) holds in the context of Prop. 7.5.

First, both Prop. 6.5 and Cor. 5.4 impose that \( \frac{1}{\sqrt{m}} \Phi \) is a \((\epsilon, L)\)-Lipschitz continuous embedding with \( L = O(1) \). Since this mapping is an LSRE construction, we know that if (50) holds then \( \frac{1}{\sqrt{m}} \Phi \) satisfies the RIP\((K - K, \epsilon)\) with probability exceeding \( 1 - \zeta \). From (8), this ensures that this mapping is \((\epsilon, \sqrt{2})\)-Lipschitz continuous.

Second, we note that both (32) and (20) require \( m \gtrsim \epsilon_2^{-2} w(K \cap B^n, \epsilon^p) \). However, for a structured set \( K \), we have

\[
H(K \cap \B^n, \epsilon^p) \lesssim \frac{1}{\eta^2} w(K \cap \B^n)^2.
\]

Moreover, since \( K \) is a cone, \( 0 \in K \subseteq K - K \) and \( w(K \cap \B^n, \epsilon^p) \lesssim w((K - K) \cap \B^n, \epsilon^p) \), which proves that (32) and (20) are satisfied if (50) holds. This concludes the proof.

**B. Bounded, star-shaped sets:** In the case of a bounded, star-shaped set \( K \subseteq \B^n \) that cannot be obtained as the restriction of a structured set to the ball \( \B^n \), we have seen in Sec. 7.1 that LSRE random matrix constructions are proved to verify w.h.p. the RIP\((K, \epsilon)\) with controlled distortion \( \epsilon > 0 \) provided \( \epsilon^2 m \) is larger than \( w(K)^2 \), up to a polylogarithmic factor. Moreover, thanks to Prop. 7.4, the same random constructions are shown to be \((\eta, \sqrt{2})\)-Lipschitz continuous with high probability as soon as \( \eta^2 m \) is also larger than \( w(K)^2 \), up to a polylogarithmic factor. Therefore, following Prop. 6.5 or Cor. 5.4, it remains to bound the Kolmogorov entropy \( H(K, \eta) \) for any \( \eta > 0 \) in order to determine when the quantized random mapping \( A \) satisfies the (L)LPD.

Unfortunately, in the case where \( K \) is not structured, we cannot bound the evolution of \( H \) with respect to \( \eta \) as tightly as in (49). Instead, we can invoke a looser bound induced by Sudakov inequality [60, 39]:

\[
H(K \cap \B^n, \eta) \lesssim \frac{1}{\eta^2} w(K \cap \B^n)^2.
\]

This bound then leads to the following result.

**Proposition 7.6** ((L)LPD for bounded sets and LSRE constructions). Given a bounded, star-shaped set \( K \subseteq \B^n \) with \( K \ni 0 \) and \( \|K\| = 1 \), a distortion \( 0 < \epsilon \ll 1 \), a quantization resolution \( \delta > 0 \), a matrix \( \frac{1}{\sqrt{m}} \Phi \in \R^{m \times n} \) generated from an LSRE construction, and a random dithering \( \xi \sim \U^m([0, \delta]) \), the random mapping \( A \) defined in (4) respects the LPD\((K, \epsilon(1 + \delta))\), or the L-LPD\((K, \Phi, x, e\delta)\) for a given \( x \in \R^n \), with probability exceeding \( 1 - \zeta \) provided the following requirement is satisfied

\[
m \gtrsim \frac{1}{\epsilon^{2p} \log \epsilon} w(K)^2 \mathcal{P}_\log(m, n, 1/\zeta), \tag{51}
\]

where \( p = 3 \) or \( 1 \) for the LPD or the L-LPD property, respectively.

**Proof.** For the values \( p \) specified above, to establish the (L)LPD property of \( A \), we have simply to verify that the requirements of Prop. 6.5 and Cor. 5.4 hold if (51) holds in the context of Prop. 7.6.

First, both Prop. 6.5 and Cor. 5.4 impose that \( \frac{1}{\sqrt{m}} \Phi \) is \((\epsilon^p, L)\)-Lipschitz continuous over \( K \) with \( L = O(1) \). Since this mapping is an LSRE construction, Prop. 7.4 states that if

\[
m \gtrsim \epsilon^{-2p} w(K)^2 \mathcal{P}_\log(m, n, 1/\zeta),
\]

then
then \( \frac{1}{\sqrt{m}} \Phi \) is \((\epsilon^p, \sqrt{2})\)-Lipschitz continuous with probability exceeding \(1 - \zeta\).

Second, we note that both (32) and (20) require \( m \gtrsim \epsilon^{-2} \mathcal{H}(\mathcal{K} \cap \mathbb{B}^n, c\epsilon^p) \). Therefore, since Sudakov’s inequality gives \( \mathcal{H}(\mathcal{K} \cap \mathbb{B}^n, c\epsilon^p) \lesssim \frac{1}{c^p} w(\mathcal{K} \cap \mathbb{B}^n) \), (32) and (20) are verified if (51) holds. This concludes the proof.

### 7.4 Analysis of the decay rate of the PBP reconstruction error

Thanks to the three previous sections, we are now ready to analyze the decay of the reconstruction error of PBP of dithered QCS observations when \( m \) increases. Following the general analysis of Sec. 4, our analysis is split between the cases associated with structured low-complexity sets, namely for the union of low-dimensional subspaces and for low-rank matrices, and the case of bounded convex sets.

#### A. Decay of the reconstruction error for ULS and low-rank-models:

In the case where \( x \in \mathcal{K} \cap \mathbb{B}^n \), with a low-complexity set \( \mathcal{K} \subset \mathbb{R}^n \) being a ULS or the set of rank-\( r \) matrices (up to its vectorization), which are both associated with structured sets, Theorems 4.1 and 4.2 establish that the reconstruction error of the estimate \( \hat{x} \) provided by the PBP of the QCS observations of \( x \) is bounded as

\[
\|x - \hat{x}\| \leq 4\epsilon + 2\nu.
\]

This occurs if \( \frac{1}{\sqrt{m}} \Phi \) and the quantized mapping \( A \) are RIP(\( \mathcal{K}^s \), \( \epsilon \)) and LPD(\( \mathcal{K}^s \cap \mathbb{B}^n, \nu \)) (or its localized form) for some \( \nu > 0 \), respectively, where \( \mathcal{K}^s := \sum_{i=1}^s \mathcal{K} \) is the \( s \)-th multiple of the symmetric set \( \mathcal{K} \) (with \( s \in \mathbb{N}_0 \)), and with \( s = 2 \) if \( \mathcal{K} \) is a ULS, and \( s = 4 \) (i.e., \( \mathcal{K}^4 = C(4r) \)) if \( \mathcal{K} = C_{r_1 \times r_2} \) is the set of rank-\( r \) matrices.

From Sec. 7.1, since \( \mathcal{K}^s \) is itself a structured set for any integer \( s > 0 \), we know that any random matrix \( \frac{1}{\sqrt{m}} \Phi \) provided by an LSRE construction (e.g., sub-Gaussian, PROM, BOS or SORS) satisfies the RIP(\( \mathcal{K}^s \), \( \epsilon \)) with probability exceeding \( 1 - \zeta \) if

\[
m \gtrsim \epsilon^{-2} w(\mathcal{K}^s \cap \mathbb{B}^n)^2 \mathcal{P}_{\log}(m, n, 1/\zeta),
\]

for some polylogarithmic function \( \mathcal{P}_{\log} \) depending on the selected LSRE construction.

Moreover, from Prop. 7.5 in Sec. 7.3 if

\[
m \gtrsim \epsilon^{-2} w(\mathcal{K}^{2s} \cap \mathbb{B}^n)^2 \log(1 + \frac{1}{\epsilon^p}) \mathcal{P}_{\log}(m, n, 1/\zeta),
\]

then the mapping \( A \), defined from the combination of \( \Phi \) with the random dithering \( \xi \sim \mathcal{U}^m([0, \delta]) \), satisfies the LPD(\( \mathcal{K}^s \), \( \nu = \epsilon(1 + \delta) \)) for \( p = 3 \), or the L-LPD(\( \mathcal{K}^s \), \( \Phi, x, \nu = \epsilon\delta \)) for \( p = 1 \), with probability exceeding \( 1 - \zeta \).

Consequently, since \( \mathcal{K}^s \subset \mathcal{K}^{2s} \), we see that the last condition involves the first and the event where both the LPD and the RIP hold occurs with probability exceeding \( 1 - 2\zeta \) by union bound. Taking the minimal \( m \) that satisfies this last condition and inverting the relation between \( m \) and \( \epsilon \) provide \( \epsilon = O(m^{-\frac{1}{2}} w(\mathcal{K}^{2s} \cap \mathbb{B}^n)) \), up to some log factors in \( m, n, 1/\delta \) and \( 1/\zeta \).

Therefore, from Theorems 4.1 and 4.2 and replacing \( \nu \) by its dependence in \( \epsilon \) and \( \delta \), we can conclude the following fact:
For all vectors $x \in K \cap \mathbb{B}^n$, with $K$ being either a ULS or a low-rank-model, if $\frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n}$ is an LSRE random construction and if $\xi \sim \mathcal{U}^m([0, \delta])$, then the PBP of the dithered QCS observations $y = A(x)$ produces (with high probability and up to some missing log factors) an estimate $\hat{x}$ whose reconstruction error decays like

$$||x - \hat{x}|| = O\left(\frac{1+\delta}{\sqrt{m}} w(K^{2s} \cap \mathbb{B}^n)\right),$$

(52)

when $m$ increases, where $s = 2$ or $4$ if $K$ is a ULS or a low-rank models, respectively. Moreover, up to minor changes in the log factors, this decay rate is preserved if $x$ is fixed and the random dithering $\xi \sim \mathcal{U}^m([0, \delta])$ is generated conditionally to that knowledge.

In other words, up to some log factors and up to a multiplicative factor depending on the structure of $K$, the reconstruction error decreases like $O((1 + \delta) m^{-1/2})$ when $m$ increases, non-uniformly or uniformly for all elements of $K \cap \mathbb{B}^n$.

**B. Reconstruction error for bounded, star-shaped convex sets:** For the case of a signal $x$ taken in a bounded, star-shaped convex set $K \subset \mathbb{B}^n$ with $K \ni 0$ and $||K|| = 1$, one can still estimate this signal by the PBP of its dithered QCS observations $y = A(x) = Q(\Phi x + \xi)$ with $\xi \sim \mathcal{U}^m([0, \delta])$ and $\frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n}$ an LSRE random construction. In this case, the PBP method still provides reconstruction error decaying when $m$ increases. However, as made clear below, the decaying rate is slower than in the case of signals belonging to a structured set, with also a clear difference between the rate of the uniform reconstruction guarantees, i.e., valid for all low-complexity signals given one couple $(\Phi, \xi)$, and the rate of the non-uniform error decay determined on a fixed signal $x$.

Actually, provided $\frac{1}{\sqrt{m}} \Phi$ respects the RIP($K, \epsilon$) and the quantized mapping $A$ verifies the LPD($K, \Phi, \nu$) (or L-LPD($K, \Phi, x, \nu$)) for some $\nu > 0$, Theorem 4.3 shows that

$$||x - \hat{x}|| \leq (4\epsilon + 2\nu)^{1/2},$$

with $\hat{x}$ the PBP estimate.

Concerning the RIP, if $\frac{1}{\sqrt{m}} \Phi$ is a LSRE random construction (Sec. 7.1) and provided that

$$m \gtrsim \epsilon^{-2} w(K)^2 \mathcal{P}_\log(m, n, 1/\zeta),$$

(53)

for some polylogarithmic function $\mathcal{P}_\log$ depending on the selected LSRE construction, then this mapping is RIP($K, \epsilon$) with probability exceeding $1 - \zeta$.

To get the (L)LPD property from the same mapping $\Phi$ combined with a random dithering $\xi \sim \mathcal{U}^m([0, \delta])$, Prop. 7.6 explains that $A$ respects the LPD($K, \Phi, \nu = \epsilon(1+\delta)$) or the L-LPD($K, \Phi, x, \nu = \epsilon\delta$) with probability exceeding $1 - \zeta$ provided

$$m \gtrsim \frac{1}{\log(m)} w(K)^2 \mathcal{P}_\log(m, n, 1/\zeta),$$

(54)

with $p = 3$ or $1$ for the LPD or the L-LPD property, respectively. This last condition dominating clearly over (53), we can thus achieve both the RIP and the (L)LPD with probability exceeding $1 - 2\zeta$ (by union bound) provided (54) is respected.

By saturating (54) and inverting the relation between $m$ and $\epsilon$, we find $\epsilon = O\left(\frac{w(K)^2}{m}\frac{1}{\log(m)}\right)$, up to some log factors in $n$ and $1/\zeta$. Replacing $\nu$ by its dependence in $\epsilon$ and $\delta$, we can thus conclude the following fact from Theorem 4.3.
For all vectors $\mathbf{x}$ selected in a convex, star-shaped set $\mathcal{K} \subset \mathbb{B}^n$ with $\mathcal{K} \ni 0$ and $\|\mathcal{K}\| = 1$, if $\frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n}$ is an LSRE random construction and if $\xi \sim \mathcal{U}^m([0, \delta])$, the PBP of the dithered QCS observations $\mathbf{y} = A(\mathbf{x})$ for $\mathbf{x} \in \mathcal{K}$ produces with high probability an estimate $\hat{\mathbf{x}}$ whose reconstruction error decays like

$$\|\mathbf{x} - \hat{\mathbf{x}}\| = O\left((1 + \delta)^{\frac{3}{2}} \left(\frac{\log(\mathcal{K})^2}{m}\right)^{\frac{1}{4(p+1)}}\right)$$

when $m$ increases, with $p = 3$. In the case of a non-uniform reconstruction guarantee where $\mathbf{x}$ is fixed, then the decay rate is defined by setting $p = 1$ above, with minor changes in the hidden log factors.

In other words, up to some log factors and up to a multiplicative factor depending on the structure of $\mathcal{K}$, the reconstruction error decreases like $O\left(m^{-\frac{1}{2}} m^{-\frac{1}{4(p+1)}}\right)$ when $m$ increases, with $p = 3$ if this must hold for all elements of $\mathcal{K} \cap \mathbb{B}^n$, and $p = 1$ if $\mathbf{x}$ is fixed (non-uniform case).

### 8 Numerical verifications

In this section, we validate numerically the behavior of the PBP reconstruction error when $m$ or $\delta$ increase on three kind of low-complexity sets discussed in the Sec. 7, namely, the set of sparse vectors, the set of compressible signals and the set of low-rank matrices. This study is carried out for several sensing matrices respecting the RIP property, e.g., for sub-Gaussian random matrices and for one PROM random construction. Moreover, we empirically demonstrate the importance of the dithering by observing how the reconstruction error is impacted when this dithering is removed.

![Figure 1: Reconstruction error of PBP on dithered QCS observations of low-complexity signals in $\mathbb{R}^n$. For each plot, we present the reconstruction error evolution for $\delta = 0.5$ (blue diamonds), $\delta = 1$ (orange circles) and $\delta = 2$ (pink triangles). The two guiding dashed lines represent the two decaying rate $m^{-1/2}$ and $m^{-1}$.](image)

**A. Comparison between three low-complexity sets:** Our first set of experiments tests the relationship between the PBP reconstruction error of low-complexity signals in $\mathbb{R}^n$. For different values of the quantization resolution $\delta$, in the case where $\frac{1}{\sqrt{m}} \Phi \in \mathbb{R}^{m \times n}$ is a Gaussian random matrix and for $\xi \sim \mathcal{U}^m([0, \delta])$.

For the QCS of $k$-sparse signals in $\Sigma_k^n$ and of compressible signals in $\Sigma_k^n$, we have set $n = 512$, $k = 4$ with a number of measurements selected in $m \in [4k \log_2(n/k), n]$, where $4k \log_2(n/k)$ would
lead to perfect (w.h.p.) reconstruction of $k$-sparse signals in the absence of quantization and in
our specific setting (e.g., with BPDN [10]). Each $k$-sparse signal $x$ has been generated by picking
one $k$-length support uniformly at random amongst the $\binom{n}{k}$ available $k$-length supports of $[n]$, and
then drawing each element of the signal support i.i.d. from a standard normal distribution.
In the case of a compressible signal $x$, we set $x_i = c \epsilon_i u_i$ with $\epsilon_i$ equal to $\pm 1$ with probability
$1/2$, $u = (u_1, \ldots, u_n)\top$ a random permutation of $(1, 2^{-\alpha_k}, \ldots, n^{-\alpha_k})\top$ with $\alpha_k > 0$ such that
$\|x\|_1/\|x\|_2 \leq \sqrt{k}$, and $c > 0$ finally ensuring that $\|x\|_2 = 1$. 

Fig. 1a and Fig. 1b display the reconstruction error of PBP on quantized observations of 4-sparse
and compressible signals in $\Sigma_k$, respectively, in function of $m$ (in a log-log plot). For each figure,
nine curves are given for $\delta \in \{0.5, 1, 2\}$. Two guiding dashed lines are also given to represent the
decay rates $m^{-1/2}$ and $m^{-1}$. For every $\delta$ and $m$, the PBP reconstruction was tested over 100
trials of the random generation of $\Phi$, $\xi$ and $X$. When $m$ increases, we clearly see that the decay
rate of the reconstruction of sparse signals is slightly faster than $O(m^{-1/2})$ (e.g., the curve at $\delta = 1$ is
well fitted by $O(m^{-0.67})$), as predicted by (52), while the error reconstruction decreasing closely
matches the rate $O(m^{-1/2})$ for compressible signals. We note that this decay is much faster than the
pessimistic decreasing announced by (55), and this even for the faster non-uniform decay rate
in $O(m^{-1/8})$. This is not particularly astonishing since this last bound is valid for any convex sets
$K \subset \mathbb{R}^n$, not only the set of compressible signals, which, in the case of our simulations above, is
possibly close to $\Sigma_k$.

Concerning the sensing of low-rank matrices, we followed the description given in Sec. 4.2 and
have analyzed the reconstruction error of PBP achieved for rank-$r$ $n_1 \times n_2$-matrices with $n_1 = n_2 =
64$, i.e., $n = n_1 n_2 = 4096$, and $r = 2$. Each rank-2 matrix $X$ was generated from the model
$X = c BC\top$ where $B, C \in \mathbb{R}^{\sqrt{n} \times 2}$ are two random matrices with entries i.i.d. as a standard normal
distribution, and $c > 0$ ensures that $\|X\|_F = 1$. In this context, the random Gaussian sensing
matrix $\frac{1}{\sqrt{m}}\Phi$ thus operated over the vectorized form $x = \text{vec}(X)$ of each low-rank matrix $X$,
before the dithered quantization defining the QCS sensing in [4]. Fig. 1c shows the decay of the
reconstruction error of the PBP estimate $\hat{x} = \text{vec}(\hat{X})$ when $m \in [n/16, n]$ increases (in a log-log
plot) for $\delta \in \{0.5, 1, 2\}$ and with an average over 50 trials for each curve points (over the generation
of $\Phi$, $\xi$ and $X$). As predicted in (52) for structured sets, we observe that the rate of the decay
closely matches $O(m^{-1/2})$.

B. Evolution of the error with $\delta$: In this experiment, sparse and compressible signals are
generated in the same setting as above in the objective of testing the evolution of the PBP recon-
struction error in the function of $\log_2 \delta \in [-3, 5]$ and with a fixed $m = n/2$. The results are plotted in
Fig. 2a and Fig. 2b in the case of sparse and compressible signals, respectively. Interestingly, we
observe that both curves are compatible with the bound $\|x - \hat{x}\| = O(1 + \delta)$, with some error floor
at very small $\delta$ corresponding to the error achieved by PBP in the CS regime where quantization
distortion can be neglected.

C. Analysis of different random sensing matrices: In this new experiment, the reconstruction
error of PBP is tested on the reconstruction of 4-sparse vectors in the case where $A$ is induced by
the combination of a dithering $\xi \sim U^n([0, \delta])$ with two different non-Gaussian sensing matrices,
$i.e.,$ a Bernoulli random matrix, with i.i.d. entries equal to $\pm 1$ with equal probability, and a
partial DCT random matrix obtained by picking $m$ rows uniformly at random amongst the $n$ rows
of an $n \times n$ orthonormal DCT matrix, $i.e.,$ a member of PROM matrix construction described
Figure 2: Evolution of the reconstruction error of PBP over the QCS observations of sparse and compressible vectors in function of $\log_2 \delta \in [-3, 5]$.

Figure 3: Reconstruction error of PBP for QCS observations obtained with a Bernoulli random matrix and a partial DCT sensing matrix. For each plot, we present the reconstruction error evolution for $\delta = 0.5$ (blue diamonds), $\delta = 1$ (orange circles) and $\delta = 2$ (pink triangles). The two guiding dashed lines represent the two decaying rate $m^{-1/2}$ and $m^{-1}$.

in Sec. 7.1 and in [57, 6]. The values of $n$, $m$, $\delta$ as well as the number of averaged trials in this experiment are the ones used in Sec. 8A. Fig. 3a and Fig. 3b show the reconstruction error achieved for the Bernoulli and the Partial DCT random matrices, respectively. As predicted by theory, we clearly see that, for the tested values of $\delta$, these two sensing matrices enjoy similar performances compared to the reconstruction error of a Gaussian random matrix. Moreover, we can verify numerically that the decay rate of the reconstruction error as $m$ increases is again bounded by the decay rate $O(m^{-1/2})$.

D. On the importance of dithering: In this last test, we retake exactly the same context than the one defined in the previous experiment except for one point: the dithering has been removed from the quantized mapping $A$ and we observe how the reconstruction error of PBP over sparse signals is impacted in the case of a Bernoulli and a Partial DCT random matrix. The results
are presented in Fig. 4a and Fig. 4b. We clearly see that the decay of the reconstruction error when $m$ increases is much slower than if the dithering is present (as presented in Fig. 3a) in the case of random Bernoulli sensing, while this error seems to reach a constant floor in the case of Partial DCT sensing. In fact, even for the Bernoulli sensing, in absence of dithering, it is possible to construct two distinct and 2-sparse vectors providing the same QCS observations for any value of $m$ [19, Sec. 5], hence showing that the reconstruction error of any algorithm is necessarily lower bounded by a constant. This experiments thus confirms the importance of the dithering in QCS with non-Gaussian sensing matrices respecting the RIP.

9 Conclusion and Perspectives

In this work, we considered one possibility to combine the compressive sensing of low-complexity signals, as supported by the numerous random matrix constructions now available in the CS literature, with the non-linear distortion induced by a uniform quantizer applied on the compressive signal observations, e.g., for transmission or storage purposes. A key enabler of this combination is a uniform random dithering that is added to the linear observations before their quantization. Thanks to it, we have proved that it is possible to estimate all or one signal selected in a low-complexity set from its dithered, quantized observations. This estimation is ensured by at least one reconstruction method, the projected back projection (PBP), whose reconstruction error is provably decaying when the number of measurements increases. For instance, we have characterized this phenomenon for several “structured” low-complexity sets, e.g., the set of sparse signals, any union of low-dimensional subspaces, the set of low-rank matrices, or more advanced group-sparse models. In this case, given a quantization resolution $\delta > 0$, the decay rate of the reconstruction error is w.h.p. $O((1 + \delta)/\sqrt{m})$ (up to log factors) when the number of measurements $m$ increases. For convex and star-shaped sets, e.g., for the set of compressible signals, the error is still decaying with $m$ but with the less favorable decay rate of $O((1 + \delta)/m^{4p+1})$, with $p = 3$ or 1 for uniform or non-uniform reconstruction error, respectively.
On the side, we have also established in Sec. 4 that for more general distorted sensing models, i.e., beyond the quantized mapping described above, the reconstruction error of the PBP method can be bounded provided that the associated distorted mapping respects a certain Limited Projected Distortion (LPD) property. This one bounds its discrepancy with a linear mapping assumed to respect the RIP. For instance, as shown in Sec. 5, linear sensing models corrupted by an additive, sub-Gaussian random noise are quickly shown to satisfy the LPD property.

Lastly, numerical tests have validated the theoretical reconstruction error bounds in several sensing scenarios for several low-complexity sets, and with a varying number of measurements and different quantization resolutions. In particular, we have empirically demonstrated the positive impact of the dithering in the quantization, especially for non-Gaussian sensing matrices, e.g., Bernoulli random matrix and partial DCT random matrix. In fact, we have observed numerically that the impact of quantization on the PBP reconstruction error is indeed lessened by the presence of a random dithering, with apparent limitations in the absence of dithering for specific sensing matrices (e.g., for partial DCT random sensing).

As we have mentioned in the introduction, PBP can be seen as an initial guide for any advanced reconstruction algorithms in dithered QCS. For instance, PBP can undoubtedly be improved in the sense that its estimate is not consistent with the quantized observations, i.e., re-observing this estimate with the same quantized sensing model than the one that observed the true signal is not guaranteed to match the initial quantized observations. Therefore, one interesting future work would be to characterize consistent iterative reconstruction methods whose initialization is equal (or related) to the PBP estimate. For instance, the Quantized Iterative Hard Thresholding (QIHT) \cite{31} and the QCOSAMP \cite{61} algorithms enter in this category of methods, even if no reconstruction error guarantees have been proved for them so far. Speaking of QIHT, given a signal $x$ in a low-complexity set $K \subset \mathbb{R}^n$, this algorithm relies merely on the induction

$$x^{(t+1)} = \mathcal{P}_K \left( x^{(t)} + \frac{\mu}{m} \Phi^T (A(x) - A(x^{(t)})) \right), \quad \text{with } x^{(1)} = \mathcal{P}_K \left( \frac{\mu}{m} \Phi^T A(x) \right),$$

i.e., $x^{(1)}$ is the PBP estimate. A potential proof strategy could be to split the problem into two steps: first, proving that the QIHT algorithm is sure to converge w.h.p. to a consistent solution $x^*$ when it is initialized from the PBP estimate $x^{(1)}$, and second, showing that any pair of signals in $K$ that are consistent with respect to the quantized random mapping $A$, as for $x$ and $x^*$, have a distance bounded by, e.g., $O(w(K)/m)$. This last bound, coined the consistency width in \cite{32}, is known to decay like $O(1/m)$ when $m$ increases if $\Phi$ is a Gaussian random matrix and if $K = \Sigma_k$ (see \cite{32} Theorem 2). Unfortunately, the consistency width decays more slowly when $m$ increases for more general RIP matrices\footnote{This is easily observed by enforcing consistency in Prop. 1 and Prop. 2 in \cite{20}.} and knowing if the rate $O(1/m)$ holds for these is an open problem.

Another study could be carried out on the question of allowing other distributions for the generation of the dithering, e.g., the Gaussian distribution as in \cite{33}. As made clear in our work, the uniform distribution reveals its value in the fact that it cancels out the uniform quantization by expectation (Lemma A.1). However, it should be possible to admit other distribution $D$ such that, if $\xi \sim D$, there exist two constants $0 < \mu_0 < \mu_1$ for which

$$\mu_0 \lambda \leq \mathbb{E}_\xi Q(\lambda + \xi) \leq \mu_1 \lambda, \quad \forall \lambda \in \mathbb{R},$$

(56)
with \( \mu_0 = \mu_1 = 1 \) if \( \mathcal{D} \sim \mathcal{U}([0, \delta]) \). For a distribution \( \mathcal{D} \) compatible with \((56)\), it should be possible to show that extra distortions impact the reconstruction performance of PBP, with a reduced influence if \( \mu_0 \approx m_1 \).

## A Vanishing dithered quantization

**Lemma A.1.** For \( \mathcal{Q}(\cdot) := \delta \cdot / \delta \), any \( a \in \mathbb{R} \) and \( \xi \sim \mathcal{U}([0, \delta]) \), we have

\[
\mathbb{E}_\xi \mathcal{Q}(a + \xi) = \mathbb{E}_\xi \delta \lfloor \frac{a + \xi}{\delta} \rfloor = a. \tag{57}
\]

**Proof.** Without loss of generality, we set \( \delta = 1 \) and denote \( a' = |a| \) and \( a'' = a - a' \in [0, 1) \). We can always write \( a + \xi = a' + a'' + \xi = a' + X \), with \( X = a'' + \xi \). Since \( \xi \sim \mathcal{U}([0, 1]) \), \( 0 \leq a'' + \xi < 2 \) and \( \mathbb{P}(X = 0) = \mathbb{P}(a'' + \xi < 1) = \mathbb{P}(\xi < 1 - a'') = 1 - a'' \). Therefore,

\[
\mathbb{E}(a' + X) = a' + (a' + 1) \mathbb{P}(X = 1) = a'(1 - a'') + (a' + 1)a'' = a,
\]

and \( \mathbb{E}_\xi \lfloor \frac{a + \xi}{\delta} \rfloor = a \) holds by a simple rescaling argument for \( \delta > 0 \). \( \square \)

## B Estimation of the moments of the dithered QCS model seen as a non-linear sensing

Writing \( \Phi = (\varphi_1, \ldots, \varphi_m)^T \in \mathbb{R}^{m \times n} \), the (scalar) QCS model \((4)\) can be seen as a special case of the more general, non-linear sensing model

\[
y_i = f_i((\varphi_i, x)), \quad i \in [m], \tag{58}
\]

with the random functions \( f_i : \mathbb{R} \to \mathbb{R} \) being i.i.d. as a random function \( f : \mathbb{R} \to \mathbb{R} \) for \( i \in [m] \). This is observed by setting \( f_i(\lambda) = \mathcal{Q}(\lambda + \xi) \).

In [37], the authors proved that, for a Gaussian random matrix and a bounded star-shaped set \( \mathcal{K} \), provided \( f \) leads to finite \( \mu := \mathbb{E} f(g) g, \eta = \mathbb{E} f(g)^2 \) and \( \psi \) is the sub-Gaussian norm of \( f(g) \) with \( g \sim \mathcal{N}(0, 1) \), i.e., \( \psi := \| f(g) \|_{\psi_2} \) [34], one can estimate the direction of \( x \in \mathcal{K} \) from the solution \( \hat{x} \) of the PBP of \( y \) defined in \((58)\).

In particular, [37] Theorem 9.1 shows that, given \( x \in \mathcal{K}, t > 0, 0 < s < \sqrt{m} \), the estimate of PBP satisfies, with probability exceeding \( 1 - 2 \exp(-cs^2/\psi^4) \),

\[
\| \hat{x} - \mu \| x \|^{-1} x \| \leq t + \frac{\eta}{\sqrt{m}} (s + \frac{w_t(K)}{t}).
\]

This result can be turned into the flavor of those given in this work. In particular, given some distortion \( 0 < \epsilon < \frac{\eta}{\psi^2} \), setting above \( s = \frac{\epsilon \psi^2}{9\eta^2} \sqrt{m}, t = \frac{4\psi^2}{9\eta^2} \epsilon \), and using the fact that \( w_t(K) \leq w(K) \), provided

\[
m \geq \frac{9\psi^6}{\epsilon^4 w^2(K)},
\]

it is easy to see that the same estimate satisfies, with probability exceeding \( 1 - 2 \exp(-c\epsilon^2 m) \),

\[
\| \hat{x} - \mu \| x \|^{-1} x \| \leq \frac{\psi^2}{\eta} \epsilon.
\]
In the particular context of the scalar QCS model \([4]\) where \(f(\lambda) := \delta(\lambda + \xi)/\delta\) for \(\xi \sim \mathcal{U}([0, \delta])\), using the law of total expectation and the fact that, from Lemma \(4.1\) \(\mathbb{E}f(\lambda) = \mathbb{E}\delta(\lambda + \xi)/\delta = \lambda\), we compute easily that, for \(g \sim \mathcal{N}(0, 1)\),

\[
\begin{align*}
\mu &:= \mathbb{E}f(g)g = \delta\mathbb{E}_g\mathbb{E}_\xi[(g + \xi)/\delta]g = \mathbb{E}g^2 = 1, \\
\psi &:= \|\delta[(g + \xi)/\delta]\|_{\psi_2} \leq \|g\|_{\psi_2} + \|\delta[(g + \xi)/\delta] - (g + \xi)\|_{\psi_2} + \|\xi\|_{\psi_2} \leq 1 + \delta, \\
\eta &:= (\mathbb{E}f(g)^2)^{1/2} = (\mathbb{E}f(g^2)^{1/2}(\mathbb{E}g^2)^{1/2}) \geq \mathbb{E}f(g)g = 1, \\
\eta &\leq \sqrt{2}\sup_{p \geq 1} p^{-1/2}(\mathbb{E}f(g)^p)^{1/p} \leq \sqrt{2}\psi \leq 1 + \delta,
\end{align*}
\]

where the second line used the triangular inequality of the sub-Gaussian norm, and the third one is based on Hölder’s inequality. Therefore, we have \(\psi^2/\eta \leq \psi^2 \leq (1 + \delta)^2\) and \(\eta^2/\psi^2 \leq 1/\psi^2 \leq 1\), which proves that, provided \(m \geq \epsilon^{-4}w^2(\mathcal{K})\) with \(0 < \epsilon < \frac{\epsilon}{(1 + \delta^2)^2}\), the reconstruction error is bounded like

\[
\|\hat{x} - \mu\|\|\hat{x}\|^{-1}\|x\| \leq (1 + \delta^2)\epsilon,
\]

with probability exceeding \(1 - 2\exp(-c\epsilon^2 m)\). Roughly speaking, saturating the condition on \(m\), this shows that one can estimate, with high probability, the direction of any vector \(x\) with an error decaying like \(\mathcal{O}((1 + \delta^2)^2\sqrt{w(\mathcal{K})}m^{-1/4})\).

\section{Proof of Prop. 7.2}

\textit{Proof.} Since \([40]\) and \([41]\) are invariant under any rescaling of \(\mathcal{K}\), we can assume \(\mathcal{K} \subset \mathbb{B}^n\) and \(\|\mathcal{K}\| = 1\) without loss of generality.

First, for a fixed \(t = 1/2\), we form the lifted set \(\mathcal{K}' := \{u/\|u\| : \ u \in \mathcal{K} \oplus t\} \subset \mathbb{S}^n\) with \(\mathcal{K} \oplus t := \{(\frac{\lambda}{\sqrt{t}}) : \ x \in \mathcal{K} \subset \mathbb{B}^n\} \subset \mathbb{R}^{n+1}\). From \([19]\) App. E], the Gaussian mean with of \(\mathcal{K}'\) is easily bounded by

\[
w(\mathcal{K}') \leq \frac{1}{t}w(\mathcal{K}) + w(\mathcal{K}) \leq w(\mathcal{K}).
\]

Since \(\mathcal{K}' \subset \mathbb{S}^n\), given some distortion \(\epsilon > 0\), we know from \([39]\) established in \([5, 7]\) that if \(m \geq \epsilon^{-2}w^2(\mathcal{K}) \geq \epsilon^{-2}w^2(\mathcal{K}')\), a random matrix \(1/m\Phi' = \frac{1}{\sqrt{m}}[\Phi, \phi] \in \mathbb{R}^{n \times n+1}\) whose entries are i.i.d. from a centered sub-Gaussian distribution with unit variance, satisfies

\[
\sup_{v \in \mathcal{K}'} |\frac{1}{m}\|\Phi'v\|^2 - 1| \leq \epsilon,
\]

with probability exceeding \(1 - \exp(-c\epsilon^2 m)\).

Let us assume from now on that this event holds. Since \(v \in \mathcal{K}'\) iff there exists a \(x \in \mathcal{K}\) such that \(v = (\frac{\lambda}{\sqrt{t}})/\|\frac{\lambda}{\sqrt{t}}\|\), \([59]\) is equivalent to

\[
\sup_{x \in \mathcal{K}} |\frac{1}{m}\|\Phi x + \phi t\|^2 - \|x\|^2 - t^2| \leq \epsilon(\|x\|^2 + t^2),
\]

which amounts to writing

\[
(1 - \epsilon)(\|x\|^2 + t^2) \leq \frac{1}{m}(\|\Phi x\|^2 + \|\phi t\|^2 + 2\langle \Phi x, \phi \rangle) \leq (1 + \epsilon)(\|x\|^2 + t^2), \quad \forall x \in \mathcal{K}.
\]

Note that \(\langle \phi \rangle \in \mathcal{K}'\) since, by assumption, \(0 \in \mathcal{K}\). Therefore, from \([59]\), \(\|\phi\|^2 - m \leq m\epsilon\), using the fact that \(\|\phi\| = \|\Phi'\langle \phi \rangle\|\). Moreover, from the symmetry of \(\mathcal{K}\), using the polarization identity and applying \([60]\) on \(|\pm x\rangle\) gives

\[
4|\langle \Phi x, \phi \rangle| = \|\phi + \Phi x\|^2 - \|\phi - \Phi x\|^2 \leq 2\epsilon(t^2 + \|x\|^2)m \leq 4em.
\]
Therefore, from the RHS of (61),
\[
\|\Phi x\|_2^2 \leq m(1 + \epsilon)(\|x\|_2^2 + t^2) - \|t\phi\|_2^2 - 2\langle \Phi x, t\phi \rangle \\
\leq m(1 + \epsilon)\|x\|_2^2 + m(1 + \epsilon)t^2 - t^2m(1 - \epsilon) + 2\epsilon m \\
= m(1 + \epsilon)\|x\|_2^2 + 2\epsilon m(1 + t^2) \leq m(\|x\|_2^2 + 4\epsilon).
\]

Since we get similarly that \(\|\Phi x\| \geq m(\|x\|_2^2 - 4\epsilon)\), this shows finally that,
\[
\left| \frac{1}{m} \|\Phi x\|^2 - \|x\|^2 \right| \leq 4\epsilon, \quad x \in \mathcal{K}.
\]

References

[1] E. J. Candes and T. Tao, “Decoding by linear programming,” IEEE transactions on information theory, vol. 51, no. 12, pp. 4203–4215, 2005.
[2] D. L. Donoho, “Compressed sensing,” IEEE Transactions on information theory, vol. 52, no. 4, pp. 1289–1306, 2006.
[3] S. Foucart and H. Rauhut, A mathematical introduction to compressive sensing. Birkhäuser Basel, 2013, vol. 1, no. 3.
[4] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, “A simple proof of the restricted isometry property for random matrices,” Constructive Approximation, vol. 28, no. 3, pp. 253–263, 2008.
[5] S. Mendelson, A. Pajor, and N. Tomczak-Jaegermann, “Uniform uncertainty principle for Bernoulli and subgaussian ensembles,” Constructive Approximation, vol. 28, no. 3, pp. 277–289, 2008.
[6] H. Rauhut, “Compressive sensing and structured random matrices,” Theoretical foundations and numerical methods for sparse recovery, vol. 9, pp. 1–92, 2010.
[7] B. Klartag and S. Mendelson, “Empirical processes and random projections,” Journal of Functional Analysis, vol. 225, no. 1, pp. 229–245, 2005.
[8] J. A. Tropp and A. C. Gilbert, “Signal recovery from random measurements via orthogonal matching pursuit,” IEEE Transactions on information theory, vol. 53, no. 12, pp. 4655–4666, 2007.
[9] A. Cohen, W. Dahmen, and R. DeVore, “Compressed sensing and best k-term approximation,” Journal of the American mathematical society, vol. 22, no. 1, pp. 211–231, 2009.
[10] E. J. Candes, J. K. Romberg, and T. Tao, “Stable signal recovery from incomplete and inaccurate measurements,” Communications on pure and applied mathematics, vol. 59, no. 8, pp. 1207–1223, 2006.
[11] C. S. Güntürk, M. Lammers, A. M. Powell, R. Saab, and Ö. Yılmaz, “Sobolev duals for random frames and ΣΔ quantization of compressed sensing measurements,” Foundations of Computational mathematics, vol. 13, no. 1, pp. 1–36, 2013.
[12] P. T. Boufounos, L. Jacques, F. Krahmer, and R. Saab, “Quantization and compressive sensing,” in Compressed Sensing and its Applications: MATHEON Workshop 2013. Springer, 2015, pp. 193–237.
[13] P. T. Boufounos, “Universal rate-efficient scalar quantization,” IEEE transactions on information theory, vol. 58, no. 3, pp. 1861–1872, 2012.
[14] R. J. Pai, “Nonadaptive lossy encoding of sparse signals,” Ph.D. dissertation, Massachusetts Institute of Technology, 2006.
[15] U. S. Kamilov, V. K. Goyal, and S. Rangan, “Message-passing de-quantization with applications to compressed sensing,” IEEE Transactions on Signal Processing, vol. 60, no. 12, pp. 6270–6281, 2012.
[16] H. Q. Nguyen, V. K. Goyal, and L. R. Varshney, “Frame permutation quantization,” Applied and Computational Harmonic Analysis, vol. 31, no. 1, pp. 74–97, 2011.
[17] J.-M. Feng, F. Krahmer, and R. Saab, “Quantized Compressed Sensing for Partial Random Circulant Matrices,” arXiv preprint arXiv:1702.04711, 2017.
R. G. Baraniuk, S. Foucart, D. Needell, Y. Plan, and M. Wootters, “Exponential decay of reconstruction error from binary measurements of sparse signals,” IEEE Transactions on Information Theory, vol. 63, no. 6, pp. 3368–3385, 2017.

L. Jacques, “Small width, low distortions: quantized random embeddings of low-complexity sets,” IEEE Transactions on Information Theory, vol. 63, no. 9, pp. 5477–5495, 2015.

L. Jacques and V. Cambareri, “Time for dithering: fast and quantized random embeddings via the restricted isometry property,” Information and Inference: A Journal of the IMA, p. iax004, 2017.

B. Adcock, A. C. Hansen, C. Poon, and B. Roman, “Breaking the coherence barrier: A new theory for compressed sensing,” Forum of Mathematics, Sigma, vol. 5, 2017.

R. E. Carrillo, J. McEwen, and Y. Wiaux, “Sparsity averaging reweighted analysis (sara): a novel algorithm for radio-interferometric imaging,” Monthly Notices of the Royal Astronomical Society, vol. 426, no. 2, pp. 1223–1234, 2012.

J. H. Ender, “On compressive sensing applied to radar,” Signal Processing, vol. 90, no. 5, pp. 1402–1414, 2010.

L. Anitori, A. Maleki, M. Otten, R. G. Baraniuk, and P. Hoogeboom, “Design and analysis of compressed sensing radar detectors,” IEEE Transactions on Signal Processing, vol. 61, no. 4, pp. 813–827, 2013.

R. M. Gray and D. L. Neuhoff, “Quantization,” IEEE transactions on information theory, vol. 44, no. 6, pp. 2325–2383, 1998.

R. G. Baraniuk, V. Cevher, M. F. Duarte, and C. Hegde, “Model-based compressive sensing,” IEEE Transactions on Information Theory, vol. 56, no. 4, pp. 1982–2001, 2010.

U. Ayaz, S. Dirksen, and H. Rauhut, “Uniform recovery of fusion frame structured sparse signals,” Applied and Computational Harmonic Analysis, vol. 41, no. 2, pp. 341–361, 2016.

S. Oymak and B. Recht, “Near-optimal bounds for binary embeddings of arbitrary sets,” arXiv preprint arXiv:1512.04433, 2015.

H.-J. M. Shi, M. Case, X. Gu, S. Tu, and D. Needell, “Methods for quantized compressed sensing,” in Information Theory and Applications Workshop (ITA), 2016. IEEE, 2016, pp. 1–9.

L. Jacques, J. N. Laska, P. T. Boufounos, and R. G. Baraniuk, “Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors,” IEEE Transactions on Information Theory, vol. 59, no. 4, pp. 2082–2102, 2013.

L. Jacques, K. Degraux, and C. D. Vleeschouwer, “Quantized Iterative Hard Thresholding: Bridging 1-bit and High-Resolution Quantized Compressive Sensing,” in Proc. of SAMPTA2013 (July 1st-5th, Bremen, Germany). IEEE, 2013, pp. 105–108.

L. Jacques, “Error decay of (almost) consistent signal estimations from quantized Gaussian random projections,” IEEE Transactions on Information Theory, vol. 62, no. 8, pp. 4696–4709, 2016.

S. Dirksen, H. C. Jung, and H. Rauhut, “One-bit compressed sensing with partial Gaussian circulant matrices,” arXiv preprint arXiv:1710.03287, 2017.

R. Vershynin, Introduction to the non-asymptotic analysis of random matrices. Cambridge University Press, 2012, ch. 5, p. 210268.

Y. Plan and R. Vershynin, “The generalized lasso with non-linear observations,” IEEE Transactions on Information theory, vol. 62, no. 3, pp. 1528–1537, 2016.

X. Gu, S. Tu, H.-J. M. Shi, M. Case, D. Needell, and Y. Plan, “Optimizing quantization for Lasso recovery,” arXiv preprint arXiv:1606.03055, 2016.

Y. Plan, R. Vershynin, and E. Yudovina, “High-dimensional estimation with geometric constraints,” Information and Inference: A Journal of the IMA, vol. 6, no. 1, pp. 1–40, 2017.

A. Ai, A. Lapanowski, Y. Plan, and R. Vershynin, “One-bit compressed sensing with non-Gaussian measurements,” Linear Algebra and its Applications, vol. 441, pp. 222–239, 2014.

Y. Plan and R. Vershynin, “Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach,” IEEE Transactions on Information Theory, vol. 59, no. 1, pp. 482–494, 2013.

P. T. Boufounos and R. G. Baraniuk, “1-bit compressive sensing,” in Information Sciences and Systems, 2008. CISS 2008. 42nd Annual Conference on. IEEE, 2008, pp. 16–21.
[41] K. Knudson, R. Saab, and R. Ward, “One-bit compressive sensing with norm estimation,” *IEEE Transactions on Information Theory*, vol. 62, no. 5, pp. 2748–2758, 2016.

[42] V. Tikhomirov, “ε-entropy and ε-capacity of sets in functional spaces,” in *Selected works of AN Kolmogorov*. Springer, 1993, pp. 86–170.

[43] G. Pisier, *The volume of convex bodies and Banach space geometry*. Cambridge University Press, 1999, vol. 94.

[44] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, “The convex geometry of linear inverse problems,” *Foundations of Computational mathematics*, vol. 12, no. 6, pp. 805–849, 2012.

[45] E. J. Candès and Y. Plan, “Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements,” *IEEE Transactions on Information Theory*, vol. 57, no. 4, pp. 2342–2359, 2011.

[46] S. Foucart and H. Rauhut, *A mathematical introduction to compressive sensing*. Birkhäuser Basel, 2013, vol. 1, no. 3.

[47] H. Rauhut, K. Schnass, and P. Vandergheynst, “Compressed sensing and redundant dictionaries,” *IEEE Transactions on Information Theory*, vol. 54, no. 5, pp. 2210–2219, 2008.

[48] S. Nam, M. E. Davies, M. Elad, and R. Gribonval, “The cosparse analysis model and algorithms,” *Applied and Computational Harmonic Analysis*, vol. 34, no. 1, pp. 30–56, 2013.

[49] M. Fazel, “Matrix rank minimization with applications,” Ph.D. dissertation, PhD thesis, Stanford University, 2002.

[50] B. Recht, M. Fazel, and P. A. Parrilo, “Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization,” *SIAM Review*, vol. 52, no. 3, pp. 471–501, 2010.

[51] K. Wei, J.-F. Cai, T. F. Chan, and S. Leung, “Guarantees of Riemannian optimization for low rank matrix recovery,” *SIAM Journal on Matrix Analysis and Applications*, vol. 37, no. 3, pp. 1198–1222, 2016.

[52] B. Vandereycken, “Low-rank matrix completion by Riemannian optimization,” *SIAM Journal on Optimization*, vol. 23, no. 2, pp. 1214–1236, 2013.

[53] J. Dattorro, “Convex Optimization and Euclidean Distance Geometry,” *Palo Alto*, 2005.

[54] N. Keriven, A. Bourrier, R. Gribonval, and P. Pérez, “Sketching for large-scale learning of mixture models,” in *Acoustics, Speech and Signal Processing (ICASSP), 2016 IEEE International Conference on*. IEEE, 2016, pp. 6190–6194.

[55] P. T. Boufounos, S. Rane, and H. Mansour, “Representation and coding of signal geometry,” *Information and Inference: A Journal of the IMA*, p. iax002, 2017.

[56] Y. Plan and R. Vershynin, “Dimension reduction by random hyperplane tessellations,” *Discrete & Computational Geometry*, vol. 51, no. 2, pp. 438–461, 2014.

[57] E. J. Candés and T. Tao, “Near-optimal signal recovery from random projections: Universal encoding strategies?” *IEEE transactions on information theory*, vol. 52, no. 12, pp. 5406–5425, 2006.

[58] M. Golbabaee and P. Vandergheynst, “Hyperspectral image compressed sensing via low-rank and joint-sparse matrix recovery,” in *Acoustics, Speech and Signal Processing (ICASSP)*, 2012 IEEE International Conference on. Ieee, 2012, pp. 2741–2744.

[59] S. Oymak, B. Recht, and M. Soltanolkotabi, “Isometric sketching of any set via the Restricted Isometry Property,” arXiv preprint arXiv:1506.03521, 2015.

[60] M. Ledoux and M. Talagrand, *Probability in Banach Spaces: Isoperimetry and Processes*. New York: Springer, 1991.

[61] H.-J. M. Shi, M. Case, X. Gu, S. Tu, and D. Needell, “Methods for quantized compressed sensing,” in *Information Theory and Applications Workshop (ITA)*, 2016. IEEE, 2016, pp. 1–9.