CANONICAL HEIGHTS AND ENTROPY IN ARITHMETIC DYNAMICS

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Abstract. The height of an algebraic number in the sense of Diophantine geometry is known to be related to the entropy of an automorphism of a solenoid associated to the number. An elliptic analogue is considered, which necessitates introducing a notion of entropy for sequences of transformations. A sequence of transformations are defined for which there is a canonical arithmetically defined quotient whose entropy is the canonical height, and in which the fibre entropy is accounted for by local heights at primes of singular reduction, yielding a dynamical interpretation of singular reduction. This system is related to local systems, whose entropy coincides with the local canonical height up to sign. The proofs use transcendence theory, a strong form of Siegel’s theorem, and an elliptic analogue of Jensen’s formula.

These elliptic systems are based upon iteration of the duplication map; the ideas extend to morphisms of projective space, giving examples where the associated entropies coincide with the morphic heights of Call and Goldstine. In particular, the local morphic heights at infinity for polynomials are realized as integrals over an associated Julia set with respect to the maximal measure, giving an analogue of the Jensen formula in that setting also.

1. Introduction

Let $Q$ denote a finite rational point of the projective line $\mathbb{P}^1$. Then $Q$ has an associated dynamical system $T_Q : X_Q \to X_Q$, where $T_Q$ is a continuous map on an underlying compact group $X_Q$ known as a solenoid (defined later). The topological entropy of this dynamical system, an intrinsic invariant measuring orbit complexity, coincides with the Diophantine height $h(Q)$ of $Q$. If $Q = [q, 1]$ corresponds to the rational number $q = a/b$ then this height can be written using

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Jensen’s formula as an integral,

\[
h(Q) = \log \max\{|a|, |b|\} = \int_{\mathbb{T}} \log |bx - a| dm,
\]

where \(\mathbb{T}\) is the unit circle and \(m\) is Haar measure. The number of elements of \(X_Q\) fixed by \(T_Q^n\) is \(|b^n - a^n|\). Writing \(\phi_n(x) = x^n - 1\), the polynomial whose roots form the \(n\)-torsion subgroup of the unit circle, gives \(|b^n - a^n| = |b^n \phi_n(a/b)|\). The main point of reference is the approach taken in [20], where the entropy is calculated by noting that the space \(X_Q\) is covered by the adeles and the dynamics lift nicely. The lifted map restricts to the local components, and the local entropies agree with the local projective heights. In this covering space, the periodic point data is destroyed however.

The arithmetic side of the last paragraph has a direct analogue in which (roughly speaking) \(\mathbb{T}\) is replaced by a complex elliptic curve, and the projective height is replaced by the global canonical height, which is known to decompose as a sum of local canonical heights. The cyclotomic division polynomials carrying knowledge of torsion in the circle are replaced by the elliptic division polynomials. Several attempts have been made to find the fourth corner of this square of ideas, namely a family of elliptic dynamical systems, whose topological entropy is given by the canonical height on the curve, and whose periodic point data is given by expressions involving the elliptic division polynomial (see [8], [11], [12]).

In this paper, we have three objectives. The first is to show that by widening the usual concept of a dynamical system to include sequences of transformations which are not necessarily the iterates of a single transformation, we can construct dynamical systems from rational points on elliptic curves which interpret the known arithmetic properties of heights. The results include a dynamical interpretation of the phenomenon of singular reduction. In this wider concept of dynamics, there is a natural notion of entropy, measuring growth in orbit complexity along the sequence. The maps act on the adeles, just as in [20]; they are built from the duplication map on the underlying elliptic curve. Duplication can be viewed as a morphism on the projective line, which informs our second objective: to construct sequences of transformations built from iterates of morphisms on projective space, and relate the entropy to the canonical height for these morphisms. The underlying space for all of these systems is locally compact (it is the adele ring). The third objective is to argue through examples that if the underlying space is compact this places severe restrictions upon the volume growth rates in our entropy calculations. By this we mean to
imply that our maps on the adeles are natural from the point of view of the systems we seek.

Since we are bringing together two areas (arithmetic and dynamics) with a fair amount of technical detail, the main conclusions are stated with precise definitions later. In [20], the entropy was calculated by showing it is equal to that of the diagonal multiplication by \( q \) on the rational adeles.

**Theorem** (see Section 6) Let \( E \) denote an elliptic curve defined over \( \mathbb{Q} \), and \( Q \) a rational point on \( E \). Then \( Q \) generates a sequence of diagonal transformations \( U \) on the adeles with the following properties.
1. If \( Q \) has non-singular reduction modulo \( p \) for all primes \( p \) then the entropy \( h(U) = h(Q) \), the global canonical height of \( Q \).
2. Let \( S \) denote the set of primes \( p \) for which \( Q \) has singular reduction modulo \( p \); write \( Q_S = \prod_{p \in S} Q_p \), and \( U_S \) for the restriction of \( U \) to \( Q_S \). Then the quotient entropy \( h(U/U_S) = h(Q) \).

Duplication on an elliptic curve provides a natural example of a morphism on \( \mathbb{P}^1 \) of degree 4. Suppose now that \( F \) denotes an arbitrary morphism on \( \mathbb{P}^1 \) defined over \( \mathbb{Q} \), corresponding to the rational function \( f \) in one variable. In [6] a notion of canonical height is attached to \( F \). Write \( \hat{h}_f(Q) \) for the global canonical height of \( Q \in \mathbb{P}^1(\mathbb{Q}) \) and, for each prime \( p \leq \infty \), write \( \lambda_{f,p}(Q) \) for the local canonical height. If \( f \) is a polynomial, we obtain a simple result which matches up local and global heights with the local and global entropies attached to a natural sequence of transformations coming from the iterates of \( f \) on a single rational number. This point of view is closer to dynamical systems in the usual sense: iterates of rational maps provide the raw material for the transformations.

**Theorem** (see Section 7) Suppose \( F : \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{P}^1(\mathbb{Q}) \) corresponds to a polynomial \( f \) in one variable. Let \( Q \in \mathbb{P}^1(\mathbb{Q}) \) denote a finite rational point corresponding to \( q \in \mathbb{Q} \). Then the iterates of \( f \) on \( q \) generate a sequence of transformations \( T \) on the adeles with the following properties:
1. The global entropy \( h(T) = \hat{h}_f(q) \),
2. The restriction \( T_p \) of \( T \) to \( \mathbb{Q}_p \) has entropy \( h(T_p) = \lambda_{f,p}(q) \).

This system emulates an automorphisms of the solenoid. It is natural to ask what can be done for a general morphism – one that does not correspond to a polynomial. The elliptic system provides an answer for a particular morphism of degree 4, and clarifies some problems with the general case. Examples show that these more general results may be
interpreted in terms of known objects. For example, the local heights in the elliptic case, and in the polynomial case of a morphism, arise as integrals over an associated Julia set. In the elliptic case, this is because the Julia set coincides with the elliptic curve. In the case of the second result, it is possible to see this from what is known already. For the more general case, it is an open problem that completes this circle of ideas in a satisfactory way.

2. Definitions and background on entropy

Most of the definitions and results below are straightforward modifications of well-known theory, so the results are simply stated. The interest is in the later examples. Let $X$ be a ‘space’: a standard probability space $(X, B, \mu)$, a compact metric space $(X, \rho)$, or a locally compact metric space $(X, d)$. A sequential action on $X$ is a sequence $T = (T_n)_{n \geq 1}$ of maps $T_n : X \to X$ with the property that each $T_n$ is a $\mu$-preserving $B$-measurable map, a continuous map, or a uniformly continuous map respectively. One of the essential features of the elliptic phenomena we are trying to capture is that the volume grows at some natural rate. Let $r : \mathbb{N} \to \mathbb{R}$ be non-decreasing with $r(n) \to \infty$. A finite partition $\xi$ of $(X, B, \mu)$ is a collection $\{A_1, \ldots, A_k\}$ of $B$-measurable sets with $\mu(\bigcup_{i=1}^k A_i) = 1$ and $\mu(A_i \cap A_j) = 0$ for all $i \neq j$. The entropy of such a partition is $H(\xi) = -\sum_{i=1}^k \mu(A_i) \log \mu(A_i)$ (with the convention that $0 \log 0 = 0$), and the join of $\xi$ with another finite partition $\eta = \{B_1, \ldots, B_\ell\}$ is the partition $\xi \vee \eta = \{A_i \cap B_j \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}$. If $T : X \to X$ is a measurable map, then $T^{-1}\xi$ denotes the partition $\{T^{-1}A_1, \ldots, T^{-1}A_k\}$. 

**Definition 2.1.** The (measure-theoretic) sequential entropy of $T$ on $(X, B, \mu)$ is given by

$$h^r_\mu(T) = \sup_{\xi} \limsup_{n \to \infty} \frac{1}{r(n)} \left( \sum_{j=1}^n H(T^{-1}\xi) \right),$$

where the supremum is taken over all finite partitions.

**Example 2.2.**

1. Let $r(n) = n$, and let $T_j = T^j$ for all $j \geq 1$ where $T$ is a single measure-preserving transformation. Then $h^r_\mu(T) = h_\mu(T)$, the usual measure-theoretic entropy of $T$.

2. Let $r(n) = n$ again, and let $T_j = T^{a_j}$ for a fixed increasing sequence $A = (a_1, a_2, \ldots)$. Then $h^r_\mu(T) = h_A(T)$ the ‘$A$-entropy’ or sequence-entropy introduced by Kushnirenko [18] as an invariant of measure-preserving transformations not reducible to entropy or spectral invariants unless $T$ has positive entropy (see [17]).
It follows from the second example that $h^r_\mu$ cannot be more functorial than the $A$-entropy $h_A$. In particular, the relation $h^r_\mu(T \times S) = h^r_\mu(T) + h^r_\mu(S)$ does not always hold (by [19, Example 7]), and writing $T^k$ for the sequence $(T^k_j)_{j \geq 1}$, the relation $h^r_\mu(T^k) = k h^r_\mu(T)$ does not always hold (by [19, Example 1]).

Following Bowen, we next define a topological entropy and a volume-growth entropy for the topological context. Let $X$ be a compact metric space $(X, \rho)$, write $N(U)$ for the least cardinality of a finite subcover of an open cover $U$, and use $\lor$ to denote the common refinement of two open covers.

**Definition 2.3.** The (topological) entropy of $T$ on $(X, \rho)$ is

$$h^r_{\text{top}}(T) = \sup_U \limsup_{n \to \infty} \frac{1}{r(n)} \log N \left( \lor_{j=1}^n T^{-1}U \right),$$

where the supremum is taken over all open covers $U$ of $X$.

**Example 2.4.**
1. Let $r(n) = n$, and let $T_j = T^j$ for all $j \geq 1$ where $T$ is a single continuous map on $(X, \rho)$. Then $h^r_{\text{top}}(T) = h_{\text{top}}(T)$, the topological entropy of $T$ introduced in [1].
2. Let $r(n) = n$, and let $T_j = T^{a_j}$ for a fixed increasing sequence $A = (a_1, a_2, \ldots)$. Then $h^r_{\text{top}}(T) = h^{A}_{\text{top}}(T)$ is the topological sequence entropy (see [10]).
3. The directional entropy introduced by Milnor coincides with the entropy in this sense, with $r(n) = n$, for the sequence of transformations seen in a strip along the chosen direction (see [23]).

As in the measure-theoretic case, it follows that $h^r_{\text{top}}$ cannot be better-behaved than the topological sequence entropy. In particular, the relation $h^r_{\text{top}}(T \times S) = h^r_{\text{top}}(T) + h^r_{\text{top}}(S)$ does not always hold (by [19, Example 5]), writing $T^k$ for the sequence $(T^k_j)_{j \geq 1}$, the relation $h^r_{\text{top}}(T^k) = k h^r_{\text{top}}(T)$ does not always hold (by [19, Example 2]), and the variational principle $h^r_{\text{top}}(T) = \sup_\mu h^r_\mu(T)$ (where the supremum is taken over all probabilities $\mu$ invariant under all the $T_j$’s) does not always hold (see [10, Example 3]).

Definition 2.3 is less than easy to work with, and the calculation of topological entropy is facilitated by Bowen’s introduction of spanning and separated sets, homogeneous measures, and volume growth. Let now $X$ be a locally compact metric space $(X, d)$, and assume that each $T_j$ is uniformly continuous.
Definition 2.5. Let $K \subset X$ be compact. A set $E \subset K$ is $(n, \epsilon)$-separated under $T$ if for any distinct points $x, y$ in $E$, there is a $j$, $1 \leq j \leq n$, for which $d(T_jx, T_jy) > \epsilon$. A set $F \subset X$ is $(n, \epsilon)$-spans $K$ if, for every $x \in K$ there is a $y \in F$ for which $d(T_jx, T_jy) \leq \epsilon$ for $1 \leq j \leq n$. Let $r_n(\epsilon, K)$ (resp. $s_n(\epsilon, K)$) denote the largest (smallest) cardinality of a separating (spanning) set for $K$ under $T$. Then define

$$h^r_{Bowen}(T) = \sup_K \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{r(n)} \log r_n(\epsilon, K)$$

and

$$h^s_{Bowen}(T) = \sup_K \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{r(n)} \log s_n(\epsilon, K),$$

where the supremum is taken over all compact sets $K \subset X$, and the coincidence of the two limits is shown as in [4, Lemma 1].

As in the usual case, it may be shown that $h^r_{Bowen}(T) = h^r_{top}(T)$ (see [4, Sect. 7.2]) when $(X, d)$ is compact, and that $h^r_{Bowen}(T)$ depends only on the uniform equivalence class of the metric $d$ (see [4, Proposition 3]).

Definition 2.6. Assume that each $T_j$ is a uniformly continuous map on the locally compact metric space $(X, d)$; write

$$D_n(x, \epsilon, T) = \bigcap_{k=1}^n T_k^{-1}B_\epsilon(T_kx)$$

with $B_\epsilon$ a metric open ball of radius $\epsilon$. Just as in [4, Definition 6], call a Borel measure $\mu$ on $X$ homogeneous for $T$ if $\mu$ is finite on compact sets, positive on some compact set, and, for every $\epsilon > 0$ there exist a $\delta > 0$ and a $C > 0$ such that $\mu(D_n(y, \delta, T)) \leq C\mu(D_n(x, \epsilon, T))$ for all $n \geq 1$ and $x, y \in X$. For such a measure, the volume-growth entropy is defined to be

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{r(n)} \log \mu(D_n(x, \epsilon, T)),$$

which is independent of $x$ by homogeneity, and (see [4, mutatis mutandis]) it coincides with $h^r_{Bowen}(T)$.

Example 2.7. To see Definition 2.6 in an arithmetic setting, let $X$ be the locally compact ring $\mathbb{Q}_\Lambda$ (see [34, Chap. IV] for details on the adele ring). Write elements of the adele ring as $x = (x_\infty, x_2, x_3, \ldots)$, then define (for $\alpha \in \mathbb{Q}$) $\alpha x = (\alpha x_\infty, \alpha x_2, \alpha x_3, \ldots)$. Let $\mu_p$ be the Haar measure on $\mathbb{Q}_p$ ($p \leq \infty$) normalized to have $\mu_p(\mathbb{Z}_p) = 1$ ($p < \infty$) and $\mu_{\infty}([0,1]) = 1$, and write $\mu = \prod_{p \leq \infty} \mu_p$. It is enough to consider the neighbourhood $B = (-1, 1) \times \prod_{p < \infty} \mathbb{Z}_p$ in the following examples,
since any $\epsilon$-ball around the identity contains the image of $B$ under an automorphism of $\mathbb{Q}_A$.

1. Let $p_1, p_2, p_3, \ldots$ be the rational primes in their usual order, let $T_j(x) = p_1 \ldots p_j x$, and let $r(n) = n \log n$. Then it is clear that

$$\mu \left( \bigcap_{j=1}^n T_j^{-1}B \right) = \frac{1}{p_1 \ldots p_n},$$

so $h_{\text{Bowen}}^r(T) = 1$ (this follows from the estimate $n \log n \ll p_n \ll n \log n$ in [3, Theorem 4.7]).

2. Let $r(n) = n \log n$, $T_j(x) = (1/p_1 \ldots p_j)x$, and let $r(n) = n$. Then (2) holds again, so $h_{\text{Bowen}}^r(T) = 1$ as before. However, in this example each ‘local’ entropy contribution

$$\limsup_{n \to \infty} -\frac{1}{r(n)} \log \mu_p \left( \bigcap_{j=1}^n T_j^{-1}A_p \right),$$

where $A_p = \mathbb{Z}_p$ for $p < \infty$ and $A_\infty = (-1, 1)$, is zero. This should be contrasted with the usual setting, where the local entropies sum to the global entropy (see [21]).

3. Let $T_j(x) = \prod_{p \leq j} p x$, where the product is over all primes less than or equal to $j$ and $r(n) = n$. As before,

$$\mu \left( \bigcap_{j=1}^n T_j^{-1}B \right) = \frac{1}{\prod_{p \leq n} p},$$

so $h_{\text{Bowen}}^r(T)$ is positive and no larger than $2 \log 2$ (see [16, Theorem 414]).

4. Let $T_j(x) = jx$, and $r(n) = \log n$. Then it is easy to see that $h_{\text{Bowen}}^r(T) = 1$.

5. Let $T_j(x) = j! x$ and $r(n) = n \log n$; then in a similar way one sees that $h_{\text{Bowen}}^r(T) = 1$ by Stirling’s formula.

It is well-known that ‘there is only one entropy’, which in our context means that $r(n)$ should always be sub-linear or linear in $n$ in order to allow $h^r$ to be positive. However, we have already seen in Example 2.7 that other rates are possible: the point is that in those examples the underlying space is not compact.

**Lemma 2.8.** If $X$ is compact (or of finite measure), and $\frac{r(n)}{n} \to \infty$ as $n \to \infty$, then, for any sequence of transformations $T$,

$$h_{\text{top}}^r(T) = h_{\text{Bowen}}^r(T) = h_{\mu}^r(T) = 0.$$  

**Proof.** Consider the measure-theoretic case first: let $\xi$ be a finite partition. Then $H \left( \bigcup_{j=1}^n T_j^{-1}\xi \right) \leq nH(\xi)$ since by [12, Theorem 4.3]
\[ H(\xi \vee \eta) \leq H(\xi) + H(\eta) \text{ for any finite partitions } \xi \text{ and } \eta, \text{ and } H(T_j^{-1}\xi) = H(\xi) \text{ for all } j. \]

Turning to the topological case, it is convenient to work with \( h_{\text{top}} \). Fix an open cover \( U \) of \( X \); then

\[
N \left( \bigvee_{j=1}^{n} T_j^{-1}U \right) \leq \prod_{j=1}^{n} N(T_j^{-1}U) = N(U)^n,
\]

which gives the result. \( \square \)

3. Solenoids

Suppose first that \( Q = [q, 1] \in \mathbb{P}^1(\mathbb{Q}) \) is a finite point on the algebraic projective line. Then the map \( x \mapsto qx \) on \( \mathbb{Z}[q] \) or \( \mathbb{Z}[q^{\pm 1}] \) determines a dual map \( T_Q : X_Q \to X_Q \) on the compact abelian dual group. Identify \( X_Q \) with the dual of a subgroup of \( \mathbb{Q}^d \) for some \( d \); then \( T_Q \) becomes the map dual to a rational \( d \times d \) matrix \( A \). The topological entropy of \( T_Q \) is given by Yuzvinskii’s formula, \( h_{\text{top}}(T_Q) = \log |s| + \sum |\lambda_i| \), where \( s \) is the g.c.d. of the denominators of the coefficients of the characteristic polynomial of \( A \), and \( \{\lambda_i\} \) are the eigenvalues of \( A \) counted with multiplicity (see [35] for the original derivation of this result). A more suggestive ‘local-to-global’ formulation of this result is given in [20]:

\[ h_{\text{top}}(T_Q) = \sum_{p \leq \infty} \sum_{i} \log^+ |\lambda_{i,p}|_p, \]

where the inner sum is taken over the eigenvalues of \( A \) in the algebraic closure of \( \mathbb{Q}_p \), and \( |·|_p \) denotes the usual extension of the \( p \)-adic valuation.

Example 3.1. In each case the solenoid \( X_Q \) is described, and periodic points – points whose orbit under the map \( T_Q \) is finite – are also discussed.

1. If \( q \notin \{-1, 0, 1\} \) is integral, then \( \mathbb{Z}[q] = \mathbb{Z} \), so the dual group \( X_Q \) is the circle \( \mathbb{T} \). The map \( T_Q \) is \( x \mapsto qx \mod 1 \), and it is easy to see that \( h_{\text{top}}(T) = \log |q| \).

2. If \( q \) is an algebraic integer (non unit-root) of degree \( d \) whose minimal polynomial has constant coefficient \( \pm 1 \), then \( X_Q \) is the \( d \)-torus \( \mathbb{T}^d \), and we may take for \( A \) the companion matrix to the minimal polynomial of \( q \). A similar argument shows that \( h_{\text{top}}(T_Q) = \sum |\lambda_i| \), and \( |f_n(T)| = \prod |\lambda_i^n - 1| \). It is still the case that \( (1/n) \log |f_n(T)| \to h_{\text{top}}(T_Q) \), but this is non-trivial because of the possibility of eigenvalues with unit modulus (see [12] for a detailed discussion).

3. If \( q = a/b \) is a rational in lowest terms, and \( X_Q \) is dual to the group \( \mathbb{Z}[q^{\pm 1}] = \mathbb{Z}[\frac{1}{ab}] \), then \( h_{\text{top}}(T_Q) = \sum_{p \leq \infty} \log^+ |q|_p = \log \max\{|a|, |b|\} \) is.
the usual projective height of the point \([q, 1]\). Here \(f_n(T_Q) = |a^n - b^n| = |b^n\phi_n(a/b)|\), and again \((1/n) \log |f_n(T_Q)| \to h_{\text{top}}(T_Q)\).

Notice that the topological entropy in each case is given by an integral over the circle by Jensen’s formula. Yuzvinskii’s formula is proved in [20] using an adelic covering space: Example 3.1.3 is a natural quotient of the map \(x \mapsto qx\) on \(\mathbb{R} \times \prod_{\mathfrak{p} \mid ab} \mathbb{Q}_\mathfrak{p}\), and the entropy may be calculated in the covering space using the following results.

Firstly, the topological entropy of the action of \(A \in M_d(\mathbb{Q})\) on \(\mathbb{Q}_d\) is given by

\[
A \Rightarrow h_{\text{Bowen}}(A) = \sum \log |\lambda_{i,p}|^{1/p},
\]

where the sum is taken over the eigenvalues of \(A\) in the algebraic closure of \(\mathbb{Q}_p\). Secondly, the covering map has the same topological entropy as the quotient map:

\[
h_{\text{Bowen}}(\mathbb{Q}_p \xrightarrow{\times q} \mathbb{Q}_p) = h_{\text{top}}(X \xrightarrow{\times q} X).
\]

We therefore pursue an elliptic analogue of Yuzvinskii’s formula by considering actions on the adele ring. Notice that Lemma 2.8 shows that the quadratic growth rates found on elliptic curves preclude the possibility of a single homeomorphism of a compact metric space realizing ‘elliptic dynamics’ in a non-trivial way.

This section was deliberately formulated to reveal an underlying genus 0: passing to genus 1 brings us to elliptic curves.

4. BACKGROUND ON HEIGHTS AND ELLIPTIC CURVES

Let \(E\) be an elliptic curve defined over the rationals, given by a generalized Weierstrass equation

\[
y^2 + c_1xy + c_3y = x^3 + c_2x^2 + c_4x + c_6,
\]

where \(c_1, \ldots, c_6 \in \mathbb{Z}\). For each rational prime \(p\), there is a continuous function \(\lambda_p : E(\mathbb{Q}_p) \to \mathbb{R}\) which satisfies the parallelogram law

\[
\lambda_p(Q + P) + \lambda_p(Q - P) = 2\lambda_p(Q) + 2\lambda_p(P) - \log |x(Q) - x(P)|_p.
\]

If it is required that the expression \(\lambda_p(Q) - \frac{1}{2} \log |x(Q)|_p\) be bounded as \(Q \to 0\) (the identity of \(E\)), then there is only one such map, called the local canonical height. Note that in [31], local heights are normalized to make them invariant under isomorphisms: this involves adding a constant which depends on the discriminant of \(E\), the local heights in [31] then satisfy a different form of the parallelogram law. For a discussion of local heights in the form used here, see [29]. On \(E(\mathbb{Q})\) the global height \(\hat{h}\) can be written as a sum of local heights – see [8] below – which is remarkable since there is a more direct definition using limits of projective heights. If \(0 \neq Q = [x(Q), y(Q)] \in E(\mathbb{Q})\) has \(x(Q) = \frac{a}{b}\), define \(h_E(Q)\) to be \(\frac{1}{2} \log \max\{|a|, |b|\}\). Then \(h_E(Q)\) coincides
with $\frac{1}{2}h([x(Q),1])$ in the usual sense of Diophantine geometry. Taking the logarithmic height of the identity to be zero gives the alternative definition

$$\hat{h}(Q) = \lim_{n \to \infty} 4^{-n}h_{E}(2^n Q).$$

There are explicit formulæ for each of the local heights (see [28], and [31], [13] for an alternative approach). For a prime $p$ where $Q$ has non-singular reduction,

$$\lambda_p(Q) = \frac{1}{2} \log^+ |x(Q)|_p. \quad (5)$$

Notice in particular that if $x(Q)$ is integral at $p$ and $Q$ has non-singular reduction at $p$ then $\lambda_p(Q) = 0$. The singular reduction case is more involved, and to avoid a major digression we deal only with split multiplicative reduction (see [31, p. 362] for details on this): the results all hold more generally but require passage to extension fields. In the split multiplicative case, the points on the curve are isomorphic to the group $\mathbb{Q}^*_p/\ell \mathbb{Z}$ where $\ell \in \mathbb{Q}^*_p$ has $|\ell|_p < 1$. The explicit formulæ for the $x$ and $y$ coordinates of a non-identity point on the Tate curve are given in terms of the parameter $u \in \mathbb{Q}^*_p$ by

$$x = x_u = \sum_{n \in \mathbb{Z}} \frac{\ell^n u}{(1 - \ell^n u)^2} - 2 \sum_{n \geq 1} \frac{n\ell^n}{(1 - \ell^n)^2},$$

$$y = y_u = \sum_{n \in \mathbb{Z}} \frac{\ell^{2n} u^2}{(1 - \ell^n u)^3} + \sum_{n \geq 1} \frac{n\ell^n}{(1 - \ell^n)^2}.$$ 

It is clear that $x_u = x_{u\ell}$ and $x_u = x_{u^{-1}}$. Suppose $Q$ corresponds to the point $u \in \mathbb{Q}^*_p$ and assume, by invariance under multiplication by $\ell$, that $u$ lies in the fundamental domain $\{ u \mid p^{-k} = |\ell|_p < |u|_p \leq 1 \}$. Then (by [13] or [31]),

$$\lambda_p(Q) = \begin{cases} 
- \log |1 - u|_p & \text{if } |u|_p = 1, \\
- \frac{k}{2} \left( \frac{r}{k} - \left( \frac{r}{k} \right)^2 \right) & \text{if } |u|_p = p^{-r} < 1.
\end{cases}$$

Notice that for $|u|_p = 1$, the local height is non-negative, while if $|u|_p < 1$ the local height is negative. Also, these formulæ extend to all of $E(\Omega_p)$ by [31] ($\Omega_p$ is a fixed algebraic closure of $\mathbb{Q}_p$). In [7], [8], [11] and [12], attempts have been made to define dynamical systems whose topological entropy is given by $\hat{h}(Q)$, the global canonical height of $Q$. In the spirit of the algebraic case, and to reflect the fact that the global canonical height is a sum of local canonical heights, one looks to realize each local height as the entropy of a corresponding local component. In [7] and [8] the elliptic adeles are used. D’Ambros works
over function fields and assumes that the point $Q$ has everywhere non-singular reduction. In [8] a similar non-singular reduction assumption is made, together with an assumption that $Q$ lies in a neighbourhood of the identity; there is also an artificiality in the construction. Of particular interest is the fact that the coincidence between periodic point counts and division polynomials seen above holds asymptotically (cf. Remark 6.1). The extra freedom of sequential actions allows a different approach to these problems, and gives a very clear dynamically motivated description of the global canonical height and the phenomenon of singular reduction. Now the simple arithmetic structure of $P^1(Q)$ is replaced by the richer arithmetic of $E$.

5. Duplication on elliptic curves

To fix notation, let $E$ be given in generalized Weierstrass form as in (3). It follows from the shape of this equation that the denominator of the $x$-coordinate of any rational point is a square. Write $x(2^nQ) = \theta_n = a_n/b_n^2$, $b_n > 0$ as a rational in lowest terms.

Remark 5.1. It follows from the explicit formulæ for duplication that the sequence of integers $(b_j)$ satisfies the strong divisibility property $b_i|b_j$ for $i < j$, that ensures the existence of well-defined transitional maps $T_k^j : \mathbb{T} \to \mathbb{T}$ for $k \geq j$ with the property that $T_k(x) = T_k^j(T_j(x))$ where $T_j(x) = b_jx \text{ mod } 1$. This brings the family of maps $(T_j)$ closer to the iterates of a single transformation.

According to Lemma 2.8, if we are to realize the canonical height as the entropy of a sequence of transformations on a compact space (say the circle $\mathbb{T}$) then the rate must be $r(n) = n$ essentially. However, such systems cannot exhibit interesting entropies unless they are of a very special shape.

Lemma 5.2. Let $X = \mathbb{T}$, and $T_j(x) = b_jx \text{ mod } 1$ for $j > 1$, where $b_j|b_{j+1}$ for all $j \geq 1$, and $b_{j+1}/b_j \to \infty$ as $j \to \infty$. Then $h_{\text{Bowen}}^r(T) = \infty$ for $r(n) = n$ and is zero for $r(n)/n \to \infty$.

Proof. Let $B_\varepsilon = (-\varepsilon, \varepsilon)$ and think of the circle $\mathbb{T}$ as $[-1/2, 1/2)$. Notice that it is not possible to use the set $B$ of Example 2.7 since the group is compact. For small $\varepsilon$ (specifically, for $eb_1 < 1$), $B_\varepsilon \cap T_1^{-1}B_\varepsilon$ is a single interval, so

$$\mu(B_\varepsilon \cap T_1^{-1}B_\varepsilon) = b_1^{-1}(2\varepsilon).$$

However, for large $k$ the pre-image under $T_k$ of $B_\varepsilon$ meets $B_\varepsilon$ in a union of intervals:

$$\mu(B_\varepsilon \cap T_k^{-1}B_\varepsilon) = [b_\varepsilon \varepsilon]b_k^{-1}(2\varepsilon) + O(\varepsilon/b_k).$$
It follows that for large \( k \),
\[
\mu \left( \bigcap_{j=1}^{k} T_j^{-1} B_\epsilon \cap T_k^{-1} B_\epsilon \right) = \epsilon \mu \left( \bigcap_{j=1}^{k} T_j^{-1} B_\epsilon \right) + O(\epsilon b_{k-1}/b_k)
\]
which shows the entropy is at least \(- \log(\epsilon)\). It follows that \( h^r_{\text{Bowen}}(T) = \infty \) for \( r(n) = n \). The last case follows from Lemma 2.8.

In the arithmetically simple case where the \( b_j \) are all powers of a single number, Lemma 5.3 has a simpler proof. For example, if \( b_j = 2^{c_j} \), then the factor map \( x \mapsto \sum_{n=1}^{\infty} x_n 2^{-n} \) from the shift space \( \Sigma = \{0, 1\}^\mathbb{N} \) onto \( \mathbb{T} \) intertwines doubling with the left shift. The maps \( T_j \) are then factors of powers of the shift: \( T_j = \sigma_c \), where \( \sigma \) is the left shift on \( \Sigma \). If \( B \) is now a cylinder set defined on finitely many coordinates in \( \Sigma \), then it is clear that the sets \( T_j^{-1} B \) for distinct large \( j \) are independent, which shows that the topological entropy is infinite.

On the other hand, the non-compact analogue of this system does exhibit interesting volume growth.

**Theorem 5.3.** Let \( r(n) = 4^n \), \( X = \mathbb{R} \), and \( T_j(x) = b_j x \) for \( j > 1 \) with the sequence \( (b_n) \) defined by \( x(2^n Q) = a_n/b_n^2 \). Then \( h^r_{\text{Bowen}}(T) = \hat{h}(Q) \) for non-torsion \( Q \).

**Proof.** By a strong form of Siegel’s theorem (see [28, p. 250]),
\[
\lim_{n \to \infty} \frac{\log |a_n|}{2 \log |b_n|} = 1.
\]
Also,
\[
\lim_{n \to \infty} \frac{1}{r(n)} \log \frac{1}{2} \max\{|a_n|, |b_n^2|\} = \hat{h}(Q)
\]
by [28] Chap. VIII, Sect. 9. Thus \( |b_n| \to \infty \) and \( \lim_{n \to \infty} \frac{1}{r(n)} \log b_n = \hat{h}(Q) \) by (3) and (7). It follows that
\[
\log \mu \left( \bigcap_{j=1}^{n} T_j^{-1} B_\epsilon \right) = - \log \max_{1 \leq j \leq n} \{|b_j|\} + \log 2\epsilon.
\]
For any real sequence \( (d_n) \) with \( \frac{d(n)}{r(n)} \to \omega \geq 0 \),
\[
\max_{1 \leq j \leq n} \left\{ \frac{d(j)}{r(n)} \right\} \to \omega \geq 0.
\]
It follows that \( \frac{\max_{1 \leq j \leq n} \{|b_j|\}}{r(n)} \to \hat{h}(Q) \) as required.

**Theorem 5.4.** Let \( r(n) = 4^n \), \( X = \mathbb{Q}_K \), and \( T_n(x) = \theta_n x \) where \( \theta_n = a_n/b_n^2 = x(2^n Q) \). Then \( h^r_{\text{Bowen}}(T) = 2 \hat{h}(Q) \).
Proof. It is enough to measure the volume growth of the open set $B = (-1, 1) \times \prod_{p<\infty} \mathbb{Z}_p$. At the infinite place, we need a bound on $\max_{1 \leq n \leq N} \{|\theta_n|\}$, and this is provided by elliptic transcendence theory (see [9]). The minimum distance of $nQ$ from the identity on $\mathbb{C}/\mathcal{L}$ is bounded below by $n^{-A}$ for some $A = A(E, Q) > 0$. The size of the $x$-coordinate is approximately the inverse square of this quantity. Since we are running through the powers of 2 only, this gives an upper bound for $\max_{1 \leq n \leq N} \{|\theta_n|\}$ of the shape $CN$. Thus, if
\[
\bigcap_{j=1}^{N} T_j^{-1} B = B_{N,\infty} \times \prod_{p<\infty} B_{N,p},
\]
the measure of $B_{N,\infty}$ is $O(C^N)$. For the finite places, the sequence $(b_n)$ – and hence $(b_n^2)$ – has a very strong divisibility property: $b_i|b_{i+1}$ for all $i \geq 1$ (by the duplication formula). Thus
\[
\mu(B_{N,p}) = \mu\left( \bigcap_{n=1}^{N} \left( a_n/b_n^2 \right)^{-1} \mathbb{Z}_p \right) = \min_{1 \leq n \leq N} \left\{|a_n/b_n^2|^{-1}_p\right\} = |b_N|_p^2.
\]
It follows that
\[
\log \mu\left( \bigcap_{j=1}^{N} T_j^{-1} B \right) = 2 \log \prod_{p<\infty} |b_N|_p + O(\log C^N)
\]
\[
= -2 \log |b_N| + O(N).
\]
So $h_{Bowen}(T) = 2\hat{h}(Q)$ as in the proof of Theorem 5.3.

6. A DYNAMICAL INTERPRETATION OF SINGULAR REDUCTION

The systems described in Example 3.1 have local entropies which sum to the global topological entropy. Example 2.7 shows that the entropy of simple examples of sequences of transformations on the adeles may not add up in an analogous way. In pursuit of the connection between heights and entropy on elliptic curves, a more substantial problem appears, preventing Theorems 5.3 and 5.4 from decomposing into local contributions. On the height side, it is still the case that the global canonical height is a sum of local heights,
\[
\hat{h}(Q) = \sum_{p \leq \infty} \lambda_p(Q),
\]
(see [28, App. C, Sect. 18]). When $p$ is a prime of singular reduction for the curve, or $p = \infty$, it is possible for the local height $\lambda_p(Q)$
to be strictly negative. This means that it certainly cannot represent the topological entropy of anything, even in the sense of Definition 2.1. In [8], an approach to interpreting the global height as the entropy of a dynamical system is presented. Roughly speaking, since (9) decomposes into an expression for the global canonical height as the difference of two non-negative quantities, it was suggested there that a global system on the adeles might have a canonical factor, whose quotient has the canonical height as entropy, and whose fibres carry the other component of the entropy.

If \( P = [x(P), y(P)] \) denotes a generic point on the curve \( E \), described by a generalized Weierstrass equation as before, then \( x(nP) \) is a rational function of \( x \) and \( y \). In particular, the denominator of that rational function is a polynomial which vanishes on the \( n \)-torsion of \( E \). This polynomial can be used to generate a sequence of transformations with more arithmetical subtlety. Let \( \psi_n \) denote the \( n \)th division polynomial of \( E \) for \( n \geq 1 \) (see [12, App. C], [28]). Thus, \( \psi_n \) is an integral polynomial of degree \( n^2 - 1 \) and leading coefficient \( n^2 \) whose roots are exactly the \( x \)-coordinates of all the non-identity points of order dividing \( n \) on \( E \). It is well-known that \( \psi_n(x) \) is always the square of a polynomial in both \( x \) and \( y \) and, for odd \( n \), it is the square of a polynomial in \( x \) alone (see [28, p. 105]). Writing \( q = a/b = x_Q \) for the \( x \)-coordinate of a fixed rational point \( Q \), define

\[
q_n = \left| b^{n^2-1} \psi_n(a/b) \right| \in \mathbb{Z}.
\]

The remarks above show that \( q_n \) is a square for all \( n \geq 1 \). Additionally, the sequence \( (q_n) \) is a divisibility sequence in the usual sense: \( m|n \) implies \( q_m|q_n \).

Remark 6.1. In the broad analogy being pursued, the obvious candidate for the cardinality of periodic points is the sequence \( q_n \). However, if \( f : X \to X \) is any bijection, then the periodic points of \( f \) must satisfy the combinatorial congruence

\[
0 \leq \sum_{d|n} \mu(n/d) \times \# \{ x \in X \mid f^d(x) = x \} \equiv 0 \mod n, \quad (10)
\]

for each \( n \geq 1 \). Taking \( E : y^2 - y = x^3 - x \) as the curve, and \( Q = (0,0) \) as the point, the sequence \( |\psi_n(0)| \) begins 1, 1, 1, 1, 5, \ldots which does not satisfy (10).

These elliptic divisibility sequences \( (q_n) \) were studied in an abstract setting by Morgan Ward in a sequence of papers - see [33] for the details. Shipsey’s thesis [27] contains more recent applications of these sequences.
Define a sequence of non-negative integers by \( u_n^2 = q_{2^n} \). If \( Q \) is not a torsion point then the terms of the sequence \((u_n)\) are always non-zero. The divisibility of the sequence \((q_n)\) implies that
\[
u_1 | u_2 | u_3 | \ldots.
\]
Define a sequence of transformations on \( \mathbb{Q}_\Lambda \) by
\[
U_j(x) = u_j^{-1}x \tag{11}
\]
for \( j \geq 1 \). In Theorems 5.3 and 5.4 the denominator of \( \theta_n \) is responsible for the volume growth, and hence the entropy. These denominators may be thought of as evaluations of the division polynomial (though in practice a large amount of cancellation takes place). Let \( S \) denote the set of primes for which the point \( Q \) has singular reduction, and define the \( S \)-adeles to be \( \mathbb{Q}_S = \prod_{p \in S} \mathbb{Q}_p \). Write \( U_S \) for restriction of \( U \) to \( \mathbb{Q}_S \).

The local height of \( Q \) is non-positive for each prime in \( S \), while for any prime \( p \) dividing \( b \), \( Q \) has non-singular reduction and the local height there is
\[
-\frac{1}{2} \log |b|_p.
\]

**Theorem 6.2.** For the sequence of transformations \((11)\) and \( r(n) = 4^n \),
\begin{enumerate}
  \item \( h^r_{\text{Bowen}}(U) = \lambda_{\infty}(Q) + \frac{1}{2} \log |b| \),
  \item \( h^r_{\text{Bowen}}(U_S) = -\sum_{p \in S} \lambda_p(Q) \geq 0 \), and
  \item \( h^r_{\text{Bowen}}(\bar{U}) = \hat{h}(Q) = \lambda_{\infty}(Q) + \frac{1}{2} \log |b| + \sum_{p \in S} \lambda_p(Q) \) where \( \bar{U} \) is the quotient sequence of transformations induced by \( U \) on \( \mathbb{Q}_\Lambda / \mathbb{Q}_S \).
\end{enumerate}

Notice that the first formula is an analogue of Yuzvinskii’s formula. Theorem 6.2 will be proved later.

**Corollary 6.3.** If \( Q \) has everywhere non-singular reduction then
\[
h^r_{\text{Bowen}}(U) = \hat{h}(Q).
\]
If \( Q \) has singular reduction at \( p \in S \) then, with \( \bar{U} \) as before,
\[
h^r_{\text{Bowen}}(\bar{U}) = \hat{h}(Q).
\]

Define \( \epsilon_p(Q) \) to be 1 if \( \lambda_p(Q) \geq 0 \) and \(-1 \) if \( \lambda_p(Q) < 0 \). This map has the following properties.
1. If \( Q \) is integral, then \( \epsilon_{\infty}(Q) = 1 \) (see comments after (12)).
2. The set of primes \( p \) for which \( \epsilon_p(Q) = -1 \) is finite.
3. There is a finite-index subgroup in \( E(\mathbb{Q}) \) on which \( \epsilon_p(Q) = 1 \) for all \( p \in S \) (and therefore for all finite \( p \)) – see [12, Sect. 6.2] or [8, Sect. 5].
4. For all \( Q \) in a neighbourhood of the identity, \( \epsilon_p(Q) = 1 \).

**Theorem 6.4.** For the sequence of transformations on \( \mathbb{Q}_p \) defined by \( T_j(x) = q_j^{r(n)} x \) for \( j \geq 1 \), where \( Q \in E(\mathbb{Q}) \) is a non-torsion point, \( q = x(Q), q_j^2 = |\psi_j(q)| \) and \( r(n) = n^2 \), \( h^r_{\text{Bowen}}(T) = \epsilon_p(Q) \lambda_p(Q) \).
Proof. There are three cases to consider. If $p = \infty$, we claim firstly that
\[ \lim_{N \to \infty} N^{-2} \log |\psi_N(q)| = 2\lambda_\infty(Q). \] (12)
Notice that this explains the first of the properties of $\epsilon_\infty$ above: if $Q$ is integral, then the left-hand side of (12) is non-negative for all $N$.

Formula (12) was proved in [12, Theorem 6.18]; the proof is sketched here because it is similar to the singular reduction case. Take $G = E(\mathbb{C})$ and consider the elliptic Jensen formula
\[ \int_G \log |x(P) - x(Q)| d\mu_G(P) = 2\lambda_\infty(Q) \] (13)
where $\mu_G$ is the normalized Haar measure on $G$ (see [14]). The points of $N$-torsion are dense and uniformly distributed in $E(\mathbb{C})$ as $N \to \infty$, so the limit sum over the torsion points will tend to the integral when the integrand is continuous. The only potential problem arises from torsion points close to $Q$: by [9], for $x = x(P)$ with $NP = 0$, $|x - x(Q)| > N^{-C}$ for some $C > 0$ which depends on $E$ and $Q$ only. This inequality is enough to imply that the Riemann sum given by the $N$-torsion points for $\log |x(P) - x(Q)|$ converges, which gives (12). Now $q_n^2 = |\psi_n(q)|$, so
\[ \log \mu \left( \bigcap_{j=1}^N T_j^{-1}B_\epsilon \right) = -\log e_N + \log \epsilon, \]
where $e_N = \max_{1 \leq j \leq N} \{ q_n^p(Q) \}$, so using (12) gives
\[ \lim_{N \to \infty} N^{-2} \log e_N = \epsilon_\infty(Q)\lambda_\infty(Q). \]

Assume that $p$ is a prime of singular reduction. If $|x(Q)|_p > 1$ then $Q$ has non-singular reduction at $p$ and the result follows from the final case below. Assume therefore that $|x|_p \leq 1$, and use the parametrisation of the curve described in Section 4. The explicit formulae of that section show that the local height is non-positive. The points of order dividing $N$ on the Tate curve are precisely those of the form $\zeta^i\bar{\zeta}^{j/N}$, $1 \leq i,j \leq N$, where $\zeta \in \Omega_p$ denotes a fixed, primitive $N$th root of unity in $\Omega_p$. We claim that
\[ \lim_{N \to \infty} N^{-2} \log |\psi_N(q)|_p = 2\lambda_p(Q); \] (14)
this gives another proof that the local height is non-positive at a point which is $p$-integral, where $p$ is a prime of singular reduction. Let $G$ denote the closure of the torsion points: $G$ is not compact, so the $p$-adic elliptic Jensen formula cannot be used. Instead we use a variant of the
Shnirelman integral: for $f : E(\Omega_p) \to \mathbb{R}$ define the elliptic Shnirelman integral to be

$$
\int_G f(Q) dQ = \lim_{N \to \infty} N^{-2} \sum_{N\tau = 0} f(\tau)
$$

whenever the limit exists.

We claim firstly that for any $P \in E(\mathbb{Q}_p)$, the Shnirelman integral

$$
\int_G \lambda_p(P + Q) dQ = S(E)
$$

exists and is independent of $P$. \hfill (15)

First assume that $P$ is the identity. Using the explicit formula for the local height gives

$$
-N^{-2} \sum_{i=1}^{N-1} \log |1 - \zeta|^p - N^{-2} \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \frac{k}{2} \left( \frac{j}{N} - \left( \frac{j}{N} \right)^2 \right).
$$

The first sum is bounded by $\log |N|^p/N$, which vanishes in the limit; the second sum converges to $-k/12$.

For the general case, let $P$ correspond to the point $v$ on the multiplicative Tate curve. If for some large $N$ no $j$ has $|\ell^{j/N} v|^p = 1$ then the analogous sum to (13) is close to $-k/12$ by the same argument. Assume therefore that there is a $j$ with this property. Then the first sum in (16) is replaced by

$$
-N^{-2} \sum_{i=0}^{N-1} \log |1 - \ell^{j/N} v \zeta|^p - N^{-2} \log |1 - (\ell^r v)^N|^p,
$$

where $r = j/N$ only depends on $v$. By $p$-adic elliptic transcendence theory (see \cite{9}), there is a lower bound for $\log |1 - (\ell^r v)^N|^p$ of the form $-(\log N)^A$, where $A$ depends on $E$ and $v = v(P)$ only. It follows that the first sum vanishes in the limit as before. The second sum in (16) is simply rearranged under rotation by $v$, so converges to $-k/12$ as before. This proves (15).

The claimed limit (14) now follows by taking the elliptic Shnirelman integral of both sides of the parallelogram law (4) and noting that equation (15) shows that three terms cancel to leave the required limit. Consider

$$
\log \mu \left( \bigcap_{j=1}^{N} T_j^{-1} \mathbb{Z}_p \right) = - \log f_N + \log \epsilon,
$$

where $f_N = \max_{1 \leq j \leq N} \{|q_j|_p\}$. Dividing by $N^2$ and taking the limit gives the result as in the case $p = \infty$. \hfill (17)
Finally, assume that $Q$ has non-singular reduction at $p$. This is the easiest case. If $|x(Q)|_p = |q|_p > 1$, then
\[
\log \mu \left( \bigcap_{j=1}^N T_j^{-1}B_\epsilon \right) = -\log f_N + \log \epsilon,
\]
where $f_N = \max_{1 \leq j \leq N}\{|q_n|_p\}$.

If $(p, n) = 1$ then $|\psi_n(q)|_p = |q|_p^{n^2-1}$. If $(p, n) \neq 1$ then $(p, n-1) = 1$. It follows that
\[
|q|_p^{(N-1)^2-1} \leq f_N \leq |q|_p^{N^2-1}.
\]
Therefore
\[
\lim_{N \to \infty} N^{-2} \log \mu \left( \bigcap_{j=1}^N T_j^{-1}B_\epsilon \right) = \frac{1}{2} \log |q|_p = \lambda_p(Q)
\]
by the explicit formula (2). If $|q|_p \leq 1$ then $q$ is a $p$-adic integer and $\epsilon_p(Q) = 1$. In this case $\bigcap_{j=1}^N T_j^{-1}B_\epsilon = B_\epsilon$, so there is no contribution to the entropy. \hfill \square

Proof. (of Theorem 6.2) For $p = \infty$ there can be no entropy contribution for the sequence $U_j(x) = u_j^{-1}x$, since $u_n$ is an integer sequence. For $p$ finite, recall that $u_1|u_2|u_3|\ldots$. It follows that
\[
\bigcap_{j=1}^N U_j^{-1} \left( \prod_{p<\infty} \mathbb{Z}_p \right) = u_N \left( \prod_{p<\infty} \mathbb{Z}_p \right),
\]
which has measure $u_N^{-1}$. The first result follows, since $\log u_N = \frac{1}{2}(4^N - 1) \log |b| + \frac{1}{2} \log |\psi_N(q)|$, by (12). The second follows at once by using $Q_S$ and (14). For the third part of the theorem, the calculation is the same except that we are adrift by $\frac{1}{2} \sum_{p \in S, (p,b) = 1} \log |u_N|_p$. It follows from the proof of Theorem 6.4 that the entropy is adjusted by the contribution of the local heights where $Q$ has singular reduction. \hfill \square

7. Morphisms

Following the results of Call, Goldstine, Morton and Silverman (see [10], [24]), we are able to place some of the results above in a broader arithmetic-geometric context. Recall that $F : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism of degree $d$ if it is given by 2 homogeneous polynomials of degree $d$, having only the point $(0, \ldots, 0)$ as a common zero. Write $[x, y]$ for the projective coordinates in $\mathbb{P}^1$ and $z = x/y$. Then we can think of a morphism $F$ as a single rational function $f$ in $z$. In this case, we will
identify $F([z, 1])$ with $[f(z), 1]$ in the sequel. For example, $f(z) = z^2$ is a morphism of degree 2; for $a, b \in \mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$,

$$f(z) = \frac{z^4 - 2az^2 - 8bz + a^2}{z^3 + az + b}$$  \hspace{1cm} (18)

is a morphism of degree 4.

The second example is the doubling map on an elliptic curve. Iteration of rational maps has been widely studied (see [3] for an overview) and we will show how our approach to dynamics can be developed in this wider context. The rational map (18) on the Riemann sphere already appears in both contexts: in [3, pp. 73-79], Beardon gives this map as the classic example of a rational function with empty Fatou set. The map appears in our work as the map generating the sequential actions having entropies which agree with classical notions of height.

Now recall the connection between heights and complex dynamics. If $F : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree $d$ defined over $\mathbb{Q}$ then it has an associated canonical height $\hat{h}_f$ with the properties

1. $\hat{h}_f(f(q)) = d\hat{h}_f(q)$ for any $q \in \mathbb{P}^N(\mathbb{Q})$;
2. $q$ is pre-periodic if and only if $\hat{h}_f(q) = 0$.

A point $q$ is called pre-periodic if the orbit $\{f^n(q)\}$ is finite. In the elliptic example, and working over $\mathbb{C}$, the set of $\mathbb{C}$-pre-periodic points is precisely the set of $\mathbb{C}$-torsion points. This accounts for the example in [3] because the set of $\mathbb{C}$-torsion points is dense in the complex elliptic curve. Put differently, the set of pre-periodic points in $\mathbb{P}^1(\mathbb{C})$ under the rational map is dense in $\mathbb{P}^1(\mathbb{C})$. Recent work in [3] and [24] has decomposed the global canonical height $\hat{h}_f$ into a sum of local heights,

$$\hat{h}_f(q) = \sum_p \lambda_{f,p}(q).$$

These results, besides yielding beautiful formulæ, have been used to give good bounds for the number of pre-periodic points in certain cases.

**Theorem 7.1.** Let $F$ denote a morphism on $\mathbb{P}^1$ defined over $\mathbb{Q}$. Let $[q, 1]$ denote a finite point of $\mathbb{P}^1(\mathbb{Q})$. Assume that the corresponding rational function $f$ is actually a polynomial. Then the iterates of $f$ on $q$ give a sequence of rational numbers $f_n = f^n(q)$. The diagonal sequential action on the adeles $T_n(x) = f_n x$ has the following properties:

1. $h(T) = \hat{h}_f(q)$
2. If $T_p$ denotes the restriction of $T$ to $\mathbb{Q}_p$ then $h(T_p) = \lambda_{f,p}(q)$.

**Proof.** Recall the following facts from [3]:

$$\lambda_{f,p}(q) = \lim_{n \to \infty} \frac{1}{d^n} \lambda_p(f^n(q)), \hspace{1cm} (19)$$
where $\lambda_p(a/b)$ is the local projective height $\lambda_p(a/b) = \log^+ |a/b|_p$. The local height $\lambda_{f,p}(q)$ vanishes if and only if $|f^n(q)|_p$ is bounded, for all $n$. Finally, $q$ is pre-periodic if and only if

$$\lambda_{f,p}(q) = 0 \text{ for all } p \leq \infty.$$  \hspace{1cm} (20)

Note that properties (19) and (20) only hold in the case when $f$ is a polynomial: it is these properties which makes the local heights so much easier to recognize as entropies.

Write $f_n = a_n/b_n$, a rational in lowest terms. Suppose first that $|f^n(q)|_p$ is bounded for some $p < \infty$. Then both the local height and the local entropy are zero. If $|f^n(q)|_p$ is unbounded then $|b_n|_p$ is eventually decreasing. It follows that the volume growth rate

$$\log \mu \left( \bigcap_{n=1}^{N} f_n^{-1} \mathbb{Z}_p \right) = \log |b_N|_p + O(1) = -\log^+ |f_N|_p + O(1).$$

Dividing by $d^N$ and letting $N \to \infty$ shows this tends to the local canonical height, $\lambda_{f,p}(q)$.

For $p = \infty$,

$$\log \mu \left( \bigcap_{n=1}^{N} f_n^{-1} B_\epsilon \right) = -\max_{1 \leq n \leq N} \log |f_n| - \log 2\epsilon,$$

which gives the result as in (8) since $\log^+ |f_n|/d^n \to \lambda_{f,\infty}(q) \geq 0$.

The global case follows by combining the local ones: $\bigcap_{n=1}^{N} f_n^{-1} B = B_{N,\infty} \times \prod_{p < \infty} B_{N,p}$, and for all but finitely many $p$, $B_{N,p} = \mathbb{Z}_p$ for all $N$. The volume growth rates of each of the (now finitely many) local terms is the local height as shown above. These sum to the global height. \hfill \Box

**Remark 7.2.** Using a deeper form of Siegel’s Theorem due to Silverman [30], it may be shown that the action $T_j(x) = \theta_j x$ on $\mathbb{Q}_\ell$, where $\theta_j = f^j(q)$, has global entropy equal to the global height for any rational function $f$ fixing $\infty$.

This result is much closer to the solenoid case we started with. It also helps to put our elliptic results in a better context. These show that the theorem is not true in general when $f$ is a rational function; for example, the local heights do not necessarily match up with the local entropies. Nonetheless, there is a dynamical interpretation of the values of the local heights. It would be of interest to work out the general rational case along the lines of the elliptic examples.

A deep uniformity in the behaviour of pre-periodic points has been conjectured by Morton and Silverman.
Conjecture 7.3 (Morton and Silverman). Let \( F : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C}) \) be a morphism of degree \( d \) defined over \( \mathbb{Q} \). The number of pre-periodic points in \( \mathbb{P}^1(K) \), where \( K \) denotes an algebraic number field, is bounded by a constant which depends on \( N, d \) and \( [K : \mathbb{Q}] \) only.

Proving this conjecture would have far-reaching consequences: for example, it implies the ‘Uniform Boundedness Conjecture’ of Mazur and Kamienny for torsion on elliptic curves (and, more generally, on abelian varieties: see [22] for a proof of the former and [25] for some discussion of the latter). In the examples we gave earlier, the ‘morphic’ heights correspond to the well known heights as follows.

1. For \( f(z) = z^2 \), \( \hat{h}_f(q) = \log^+ |q| \), the projective height. The local canonical height, \( \hat{\lambda}_{f,p}(q) = \log^+ |q|_p \), in agreement with the local component for the projective height.

2. For \( f(z) = \frac{z^4 - 2az^2 - 8b}{z^4 + az^2 + b} \), the morphic height is precisely the global canonical height and the local morphic heights agree with the local canonical heights.

Notice now that the ‘circle’ systems at the start can be interpreted morphically. Given a rational \( q \), the sequence of squares generates a sequential action on the adeles by defining \( T_j(x) = q^{2^j}x \) (from repeated iteration of \( z \mapsto z^2 \)). The sequential entropy agrees with the global morphic height, and the local sequential entropies agree with the local morphic heights.

8. Heights, Periodic Points and the Julia set

This paper has been about realizing elliptic heights – and some morphic heights – as entropies of sequential transformations, in analogy with known circle results. The most convincing elliptic examples rely upon recognizing local elliptic heights as integrals. The space of integration turns out to be the local curve, and this coincides with the local Julia set of the associated rational function. These examples give a good indication of how to approach the more general problem of recognizing morphic heights (both local and global) as entropies of sequences of transformations; namely, by recognizing the heights as integrals over the Julia set. We finish with an example and a theorem which illustrate this connection between the Julia set and the morphic height. They suggest that Jensen’s formula (1) could be a fundamental stepping stone between heights and periodic points in the morphic examples also.
Theorem 8.1. If \( f(z) = az^d + \cdots + a_0 \) is a polynomial, then for any \( q \in \mathbb{C} \),
\[
\lambda_{f,\infty}(q) = \frac{1}{d-1} \log |a| + \int_{J(f)} \log |x - q| dm(x),
\]
where \( m \) is the maximal measure for \( f \) on \( J(f) \).

Proof. Assume first that \( q \) is not in the Julia set of \( f \). The zeros of the polynomial \( f_n(x) = f^n(x) - x \) are precisely the solutions of the equation \( f^n(x) = x \). Note that \( d_n = \deg(f_n) = d^n \), where \( d = \deg(f) \). If \( |f^n(q)| \to \infty \), \( \frac{1}{d_n} \log |f_n(q)| \) is approximately \( \frac{1}{d^n} \log |f^n(q)| \), which converges to \( \lambda_{f,\infty}(q) \) (the archimedean local height of \( q \) for the morphism associated to \( f \)). If \( |f^n(q)| \) is bounded, then the same is true for \( |f_n| \), so both expressions tend to \( \lambda_{f,\infty}(q) \). Since \( q \) lies in the open Fatou set, \( \log |x - q| \) is continuous on \( J(f) \). Now
\[
\frac{1}{d_n} \log |f_n(q)| = \frac{1}{d_n} \sum_{f^n(x) = x} \log |x - q| + \frac{1}{d_n} \log |B_n|,
\]
where the sum is over the \( n \)th ‘division points’ and
\[
B_n = a^{1+d+d^2+\cdots+d^{n-1}}
\]
is the leading coefficient of \( f^n(x) \). Thus
\[
\frac{1}{d_n} \log |B_n| = \frac{1}{d^n} \left( \frac{d^n - 1}{d - 1} \right) \log |a| \to \frac{1}{d-1} \log |a|.
\]

Now it is known that
\[
\frac{1}{d_n} \sum_{f^n(x) = x} \log |x - q| \to \int_{J(f)} \log |x - q| dm(x),
\]
where \( m \) is the maximal invariant measure for \( f \) restricted to the Julia set (see [22]; [13]).

It remains to show that the formula holds for \( q \in J(f) \). Without loss of generality, assume that \( a = 1 \) (if not, we may conjugate by a linear map to ensure this). Since \( J(f) \) has no interior, there is a sequence \( q_n \to q \) with \( q_n \notin J(f) \). Then \( \log |x - q_n| \to \log |x - q| \) for all \( x \in J(f) \setminus \{q\} \). Since \( J(f) \) is bounded, \( \log |x - q_n| \) and \( \log |x - q| \) are uniformly bounded above by \( M \) say for \( x \in J(f) \setminus \{q\} \). So by Fatou’s lemma
\[
0 = \lim_{n \to \infty} \int_{J(f)} \log |x - q_n| dm(x) \leq \int_{J(f)} \log |x - q| dm(x) \leq M.
\]

This shows that \( x \mapsto \log |x - q| \) is in \( L^1(m) \).
Now $|x - f(q)| = \prod_{f(t)=x} |t - q|$, so

$$\int_{J(f)} \log |x - f(q)| \, dm(x) = \int_{J(f)} \sum_{f(t)=x} \log |t - q| \, dm(x)$$

$$= d \int_{J(f)} \log |x - q| \, dm(x)$$

(the last equality follows from [21] or [13, Theorem (d)]). If $\int_{J(f)} \log |x - q| \, dm(x) > 0$, then the last equation contradicts (23). \qed

Example 8.2. Consider the Tchebycheff polynomial of degree $d$, $f(z) = T_d(z) = \cos(d \arccos(z))$. The Julia set is the interval $J(f) = [-1, 1]$. The map $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = \frac{1}{d}(z + z^{-1})$ is a semi-conjugacy from $g : z \mapsto z^d$ onto $z \mapsto f(z)$, in other words, $f(\phi(z)) = \phi(z^d)$. Write $\psi$ for the branch of the inverse of $\phi$ defined on $\{z \in \mathbb{C} \mid |z| > 1\}$. The canonical morphic height at the infinite place is (for $q \notin J(f)$)

$$\lambda_{f, \infty}(q) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(q)|$$

$$= \lim_{n \to \infty} \frac{1}{d^n} \log^+ |\phi^n \psi(q)|$$

$$= \lim_{n \to \infty} \frac{1}{d^n} \log^+ \left\{ \frac{1}{2} \left( g^n \psi(q) + \frac{1}{g^n \psi(q)} \right) \right\}$$

$$= \lim_{n \to \infty} \max \left\{ 0, \frac{1}{d^n} \log |g^n \psi(q)| \right\}$$

$$= \log^+ |\psi(q)|.$$  

For $q \in J(f)$, the same formula holds since $\hat{h}_{f, \infty}(q) = 0$ there by \[\footnote{\text{[3]}}\] and $\log^+ |\psi(q)| = 0$ there by a direct calculation.

Now by Jensen’s formula, for any $q \in \mathbb{C}$, [26, Theorem 15.18],

$$\log^+ |\psi(q)| = \log 2 + \int_{S^1} |\phi(y) - q| \, dy$$

$$= \log 2 + \int_{J(f)} \log |t - q| \, dm(t)$$

since $m$ is the image under $\phi$ of the maximal measure (Lebesgue) on the circle. That is,

$$\hat{h}_{f, \infty}(q) = \log 2 + \int_{J(f)} \log |t - q| \, dm(t).$$

The constant $\log 2$ in $\hat{h}_\infty(q)$ may be explained in accordance with Theorem 8.1. The leading coefficient of $T_d$ is $2^{d-1}$, so $\frac{1}{d-1} \log |a|$ in this case is exactly $\log 2$.\[\footnote{\text{[8]}}\]
A similar approach can be adopted in the case of polynomials with connected Julia sets. There the local conjugacy near $\infty$ extends to the whole domain of attraction of $\infty$, which is the complement of the filled Julia set.

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