Students perception on a problem of pattern generalization

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Abstract. The aim of this research is to describe the process of pattern generalization based on the taxonomy for categorizing generalization i.e. relating, searching and extending. The subjects were the fifth and the sixth grade students of elementary school. Qualitative data were obtained from their written work when they solved the problems of patterns and from interviews. First, at the first stage, students were asked to solve a problem of pattern generalization and at the second stage, they were interviewed in order to find out whether they understood the process of generalization or not. The results showed that the students focused on two aspects when they were facing the problem of pattern generalization, that is, they focused on the number of squares in the figure without considering the shape of the figure and they focused on the shape of the figure by paying attention to the square in the middle and to the addition of square on each side of the figure. The result of this study also revealed that students who focused on the shape of the figure found it more easily to formulate its general rule.

1. Introduction

Generalization is considered as an integral part of general mathematical ability and it is an important component of algebraic activities [1], [2] as well as one of the important goals in learning Mathematics [3]. Generalization is the heartbeat of Mathematics [4].

One of the generalization activities for elementary school students is through patterning. Patterning is considered as one way to introduce algebra to students [4]. Patterning activities encourage students to identify the similarities of specific things (i.e. every term in the pattern) and it also motivates students to pay attention to what has to change and what is fixed. The introduction to similarity of making patterns can be extended and generalized and the variable can be used to describe the generalized relationships. Through this process, students develop their own way that enables them to
produce a general rule related to the pattern \[^5\]. Researchers \[^6,7,8\] have shown that students tend to look for recursive similarities by identifying what is changed in each term, \( p_{k+1} \) and from the previous term, namely \( kp \). Their study showed that students tend to use recursive strategies (the next pattern is based on the previous pattern) to describe generalization, not to look for functional relationships between variables. \[^7\] emphasized that recursive strategies enables students to predict the next pattern, but it does not enable them to perceive the structural relationships among the data to find out the expected rules or it does not see the data as the domain or co-domain of the function. Also showed that the activity of observing patterns is able to help students think from arithmetical thinking to algebraic generalization.

\[^9\], \[^10\] describes the types of generalization when students apply algebraic reasoning. The taxonomy differentiates students’ activities when they are making generalization (generalization actions), i.e. relating, searching and extending. First, relating occurs when students form a relationship between two or more mathematical problems, situations, ideas, or objects. At the relating stage, students attempt to relate situations which have been previously faced, find out something new, or focus on the similar features or shapes of the mathematical objects. Second, searching occurs when students get involved in the repeated mathematical actions such as calculating ratios or determining a pattern to be placed in the same elements. In this case, students focus on relations, procedures, patterns or solutions when they are searching. Third, extending refers to the extension of patterns, relations, or rules in a general structure.

When being faced to the problem of pattern generalization, students generally find difficulties, especially at the perceptual level (looking at the intended pattern), at the level of verbalizing the rules (expressing the rules clearly) and at the level of symbolizing (using \( n \) to represent the intended rules) \[^11\]. According to \[^12\], two important aspects in the pattern generalization include perception, i.e. giving meaning to the given figures and incorporating the meaning in the general frame which make the perceived object able to be recognized and interpreted.

In this study the researcher was interested in understanding the process followed by students when they were looking at patterns (students’ perception) and explaining the process based on the taxonomy for categorizing generalization \[^9\], \[^10\] namely relating, searching, and extending.

2. Methods

This study was conducted to explore the process of generalization made by the students when they were being faced with pattern problems. The sample of the study was the students of elementary school with the total number of 18 students (8 students of grade 5, 10 students of grade 6). The qualitative data were obtained from the result of “think aloud” and from the students’ notes during their attempt to solve the problems and from the result of in-depth interviews. The results were then analyzed and classified into two main focuses: the number of squares in the figures and the shape of the figures by paying attention to the addition of one square in each side of the picture and one fixed square in the middle of the picture. The researchers chose one student for each main focus. Based on the result of classification, two subjects were then selected, namely S1 and S2.

The assigned generalization of patterns were as follows:
Figure 1. Problem of patterns generalization

Based on the figure above:

a. Determine the pattern of existing shapes from term 1 to term 3 and to term 3.
b. Based on point a, draw figures of the next pattern of shape from the 4\textsuperscript{th} term to the 7\textsuperscript{th} and to the 10\textsuperscript{th} term.
c. Determine the number of squares found in the 12\textsuperscript{th} term and in the 49\textsuperscript{th} term. Explain your answer!
d. How to determine the number of square in the n\textsuperscript{th} term? Explain your answer!

3. Result and discussion

This section explains the thinking process followed by the subjects based on the taxonomy for categorizing generalization by [9], [10] which includes relating, searching, and extending.

3.1 Focus on the number of squares on the figure

At the stage of relating, S1 focused on the same features of the displayed shape, in which there were an addition of 4 squares to each total number of squares existing in the previous term. The subject focused on the number of entire squares without considering the shape of the figures. The following is the result of S1’s work.

The result of interview explained the work of S1:

\begin{itemize}
  \item \textit{P}: “How did you determine the pattern?
  \item \textit{S1} : “It is from the number of squares.
  \item \textit{P} : “What does it mean?
  \item \textit{S1} : “It means that if the first term has 5 squares, so every term is always added by 4 squares.
\end{itemize}
The following result confirmed the explanation of S1’s answer:

\[
\text{One square is always added to the previous term}
\]

**Figure 3.** The explanation of S1 about the addition of square to each term

When the subject perceived the figure, she tended to perceive the figure as a collection of squares without considering the specific configuration of the figure. According to [13] this kind of students tends to merely focus on calculating the total number of squares.

Next, S1 drew the 4th term by making 9 squares horizontally then making 4 squares upwards, and 4 squares downwards.

**Figure 4.** The 4th Figure drawn by S1

The data from interview explained the work of S1:

P: “How did you draw the 4th term?

S1: “The 4th term is always added by 4 also. Because the 3rd term has the total of 13 squares, the 4th term is added by 4 so that the total of entire squares is 17.

P: “Is there any rule for drawing the 4th term?

S1: “If I am not mistaken, the number of horizontal squares is 9. Four squares upwards, 4 squares to the left, 4 squares to the right, 5 squares downwards.

P: “five squares downwards? Why?

S1: “I counted them from here (pointing the square at the far end, in the right side of figure.

The next step, S1 drew the 7th term. The following is the excerpt of the data from “think aloud” :

S1: “for the 7th term, 17….21….25…..29 (the subject counted the number of squares on terms 4, 5, 6 and 7).

Then S1 drew the 7th term, starting by making 14 squares horizontally, 7 squares upwards, and 7 squares downwards. The following is the work of S1:
At the stage of searching, S1 determined the number of squares on the 12th term.

The following excerpt of interview explains the answer of S1:

P: “how many squares are there on the 12th term?
S1: “term 10 has 21 horizontal squares, then term 11 is added by 4, term 12 is added by 4, there are 25 horizontal squares, therefore, the total number of squares is 49 (the subject counted the number of squares verbally, namely 25 plus 24).

When being asked to determine the number of squares on the 49th term, S1 found it difficult to count them. According to [14] even individuals could claim they knew what was taking place in their patterns but were relatively unsuccessful in describing them in a representational form that [11] refers to as algebraically useful that is, patterns whose structures could be conveyed by a direct formula. [15] notion of algebraic generalization involves patterns necessitates closure in the form of a direct expression.

3.2 Focus on the shape of the figure
At the stage of relating, S2 focused on the similar features of the displayed shapes, that is, there was an addition of square in each side of the previous terms. S2 had realized that term 1, term 2 and term 3 had formed a pattern.
The following is the work of S2:

The following is the work of S2:

Figure 7. Addition of square in each side of the previous terms

The following is the work of S2:

Figure 8. Picture created S2 to assert pattern 1

S2 gave a mark on the term he drew (the mark given on the figure indicated the addition of one square in each side of the shape).

Then S2 drew the 7th term. The following is the excerpt of interview with S2.

P : How did you draw the 7th term?
S2: “if term 1 in this side has one (pointing at the square on the left side of the figure), term 2 here has 2, 2, 2 and 2 (pointing at the squares on the top, left, right and bottom of figure 2), then the 7th term here has 1, 2, 3, 4, 5, 6, 7 (drawing squares upwards; it is given a dot in the middle, here there are 7; 1, 2, 3, 4, 5, 6, 7 (drawing squares to the left and again counting the number of squares she/he had drawn); there are 7 squares here; 1, 2, 3, 4, 5, 6, 7 (drawing squares to the right side and counting the number of squares she/he had drawn once more); and here there are other seven squares, 1, 2, 3, 4, 5, 6, 7 (drawing squares downwards and counting the number of squares he had made once again).”

The following is the work of S2:

Figure 9. The 7th terms
At the stage of searching, S2 found out the number of squares on the 12th term and 49th term. The following is the excerpt of the interview with S2:

S2: “So, for the 12th term, because the total of the same boxes here is 4 and one square in the middle (pointing at the square on the left, on the top, and at the bottom of the 4th term), then the 12th term will be 12 multiplied by 4, then added by 1, namely 48 plus 1 is 49.

P: ”How did you determine the number of squares on the 49th term?

S2: “The way is that because this square is not included in the pattern (pointing at the middle part of the 4th term), and because it does not have 4 squares, and there are only 4 parts which have 4 squares (pointing to the left, the top, the right and the bottom of the 4th term) and the one in the middle is not counted, then it is added by 1; therefore 4 x 4 +1 for the 4th term, 12 x 4 + 1 for the 12th term and 49 x 4 + 1 for the 49th term.

Furthermore, S2 formulated the general pattern of the requested term (extending stage).

\[ N \times 4 + 1 \]

Figure 10. The nth rule made by S2

The following data from the interview explained the formula made by S2

S2: “Because the total of the nth term is unknown, if it is the 12th term, the number of its squares in each side will be 12 (pointing to the left side of the 3rd term); therefore, if it is the 90th term, the number of its squares in each side will be 90 (pointing to the top of the 3rd term); whatever the number of squares a term has, the formula is multiplied, the number of squares in each term is multiplied by 4 and then added by 1.

Based on the result of the study above, it is obvious that S1 saw the displayed shapes merely as an object (see an object as being a mere object in itself [16] and this is called sensory perception. The displayed shapes are not more than a series of numbers 5, 9, 13, ….. The form of the displayed figure was not meaningful to S1. Therefore, when being asked to determine the total number of squares in the 12th term, the subject had to construct the preceding 11 stages. On the contrary, from the beginning, S2 focused his attention to the shape of the figure. S2 exhibited cognitive perception. Cognitive perception goes beyond the sensory; it happens when individual sees or recognizes a fact or a property in relation to the object. Therefore, when being asked to determine the number of squares on the 12th term, the subject immediately wrote $12 \times 4 + 1$. When being asked to write the general rule, S2 was able to write it easily.

4. Conclusions

In solving the problem of pattern generalization, generally the students encountered difficulties at the perceptual level (that is, looking at the designated pattern), at the level of verbalizing the rule (expressing the pattern clearly) and at the level of symbolizing (using n to represent the requested rule) [11]. In the patterning activities, students attempted to find out the order and structure mathematically. In a study on patterning activities, [17] stated that students need coordination of two skills, perceptual and inferential symbolic skills. The coordination of these two skills enables the students to pay attention to the similarities of all patterns [14], [15]. Consequently, the students need to build and justify algebraic structure existing in the formula they are making [17]. At this stage, the focus of the students is no longer on the existing pattern, but on the relationship among the existing patterns [2].

Pattern generalization can be a powerful instrument in bridging the thinking process from arithmetic to algebraic thinking. The way students pay attention to the pattern will affect the formation
of a general rule. The research finding above showed that the students who had been aware of the changes of the given terms, would easily determine the general rule. On the contrary, the students who did not pay attention to the shape of the terms often found it difficult to determine the general rule.

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