A generalized argument for dominions in varieties of groups

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Abstract. An argument used to show that certain varieties of nilpotent groups have instances of nontrivial dominions is considered, and generalized. The same is done with the argument used to show that there are nontrivial dominions in the variety of metabelian groups, to suggest how this general technique may be used.

Section 1. Introduction

Let $C$ be a full subcategory of the category of all algebras (in the sense of Universal Algebra) of a fixed type, which is closed under passing to subalgebras. Let $A \in C$, and let $B$ be a subalgebra of $A$. Recall that, in this situation, Isbell (see [1]) defines the dominion of $B$ in $A$ (in the category $C$) to be the intersection of all equalizer subalgebras of $A$ containing $B$. Explicitly,

$$\text{dom}^C_A(B) = \{a \in A \mid \forall C \in C, \forall f, g: A \to C, \text{ if } f|_B = g|_B \text{ then } f(a) = g(a)\}.$$ 

Note that $\text{dom}^C_A(B)$ always contains $B$. If $B$ is properly contained in its dominion, we will say that the dominion of $B$ in $A$ is nontrivial, and call it trivial if it equals $B$. A category $C$ has instances nontrivial dominions if there is an algebra $A \in C$, and a subalgebra $B$ of $A$ such that the dominion of $B$ in $A$ (in the category $C$) is nontrivial.

In this work we will restrict our attention to the case where $C$ is a variety of groups. For the basic properties of dominions in the context of varieties of groups, we direct the reader to [4]. We recall the most important properties: for a group $G$, $\text{dom}^C_G(\cdot)$ is a closure operator on the lattice of subgroups of $G$, and normal subgroups are dominion-closed; dominions respect finite direct powers; and dominions respect quotients. That is, if $G \in C$, $H$ is a subgroup of $G$, and $N$ is a normal subgroup of $G$ contained in $H$, then

$$\text{dom}^C_{(G/N)}(H/N) = \left(\text{dom}^C_G(H)\right) / N.$$ 

All groups will be written multiplicatively unless otherwise noted. All maps will be assumed to be group morphisms unless otherwise noted. Given a group $G$, we will denote the identity element of $G$ by $e_G$, although we will omit the subscript.

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if it is clear from context. Given a group $G$, and elements $x$ and $y$ of $G$, we denote their commutator by $[x, y] = x^{-1}y^{-1}xy$.

Given two subsets $A, B$ of $G$ (not necessarily subgroups), we denote by $[A, B]$ the subgroup of $G$ generated by all elements $[a, b]$ with $a \in A$ and $b \in B$. We also define inductively the left-normed commutators of weight $c + 1$:

$$[x_1, \ldots, x_c, x_{c+1}] = [[x_1, \ldots, x_c], x_{c+1}]; \quad c \geq 2.$$ 

We will denote the center of $G$ by $Z(G)$.

The following lemma, which is easily established by direct computation will be useful in subsequent considerations.

**Lemma 1.1.** The following hold for any elements $x$, $y$, $z$, and $w$ of an arbitrary group $G$:

(a) $xy = yx[x, y]$; \quad $x^y = x[x, y]$.
(b) $[x, y]^{-1} = [y, x]$.
(c) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$.
(d) $[x, zw] = [x, w][x, z]^w = [x, w][x, z][x, z, w]$.

Varieties will be denoted by calligraphic letters, $\mathcal{A}$, $\mathcal{V}$, $\mathcal{W}$, etc. We will denote the variety of all groups by $\mathcal{G}$, and the variety consisting only of the trivial group by $\mathcal{E}$. We will denote the variety of nilpotent groups of class two by $\mathcal{N}_2$, and the variety of metabelian groups (that is, groups which are an extension of an abelian group by an abelian group) by $\mathcal{A}^2$.

In Section 2 we will consider the argument used in [3] to prove the existence of nontrivial dominions in $\mathcal{N}_2$ and some of its subvarieties, and we will modify it to a more general context. The ideas can easily be modified to deal with other arguments, and we do this in Section 3, where we consider the argument used in [4] to establish the existence of nontrivial dominions in $\mathcal{A}^2$.

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**Section 2. Generalizing the $[x, y]^p$ argument**

In [3], we studied subvarieties of $\mathcal{N}_2$, and exhibited a large family of such subvarieties that had nontrivial dominions. The basic idea was as follows: Given a group $G$ in such a variety, and elements $x$ and $y$ in $G$, we looked at $[x, y]$. Since $G$ is nilpotent of class two, the element $[x, y]$ commutes with both $x$ and $y$. It
follows that the identity \([x^n, y] = [x, y]^n\) holds in \(G\) for every \(n \in \mathbb{Z}\). If the commutator subgroup of \(G\) does not have squarefree exponent, then we choose a prime \(p\) whose square divides the order of \([x, y]\). Finally, we let \(H\) be generated by \(x^p\) and \(y^p\). Then the elements \([x, y]^p\) lies in the dominion of \(H\). Some mild conditions on \(G\) guarantee that \([x, y]^p\) does not lie in \(H\), and this gives an instance of a nontrivial dominion; see [3], Section 6 for the details.

In this section, we will generalize this process by replacing \(x\) and \(y\) with words \(v\) and \(w\), and replacing \(N_2\) with a variety in which the commutator of \(v\) and \(w\) commutes with both \(v\) and \(w\).

Let \(V\) be a variety of groups, let \(v\) and \(w\) be two words in \(m\) letters, and let \(F\) be the relatively free \(V\)-group in \(m\) generators, generated by \(z_1, \ldots, z_m\).

**Lemma 2.2.** Assume the notation of the preceding paragraph, and let

\[
\mathbf{z} = (z_1, \ldots, z_m).
\]

Suppose that in \(F\) the following identities hold:

\[
\begin{align*}
[[v(z), w(z)], v(z)] &= e \\
[[v(z), w(z)], w(z)] &= e.
\end{align*}
\]

(2.3)

Then for any group \(G \in V\), and any elements \(g_1, \ldots, g_m \in G\), the subgroup generated by \(v(g_1, \ldots, g_m)\) and \(w(g_1, \ldots, g_m)\) is nilpotent of class at most two.

**Proof:** This follows because

\[
\langle x, y \mid [x, y, x] = [x, y, y] = e \rangle
\]

is a presentation for the relatively free \(N_2\)-group of rank two, and hence the subgroup of \(F\) generated by \(v(z)\) and \(w(z)\) is a quotient of an \(N_2\)-group. Since the relations in question hold in the relatively free group of the variety, it follows that they are inherited by all groups in the variety.

For the remainder of this section, we will assume that (2.3) holds in \(V\). Note that (2.3) implies that the commutator subgroup of \(\langle v(z), w(z) \rangle\) is cyclic and generated by \([v(z), w(z)]\).

Assume that \([v(z), w(z)]\) is not of exponent \(k\) for any square free positive integer \(k > 1\). We note in particular that \([v(z), w(z)]\) cannot be trivial. Let \(k_0\) be the order of \([v(z), w(z)]\) in \(F\), if this order is finite, and \(k_0 = 0\) if \([v(z), w(z)]\) has infinite order. By assumption, \(k_0\) is not square free.

Let \(a_0\) be the order of \(v(z)\) if the latter is finite, and \(a_0 = 0\) if the order is infinite, and \(b_0\) the order of \(w(z)\) if this is finite, and \(b_0 = 0\) if the order of \(w(z)\) is infinite. Note that we necessarily have \(k_0|\gcd(a_0, b_0)\).
Therefore, the group $G = \langle v(z), w(z) \rangle$ is a quotient of the group
\[
K = \langle x, y \mid x^{a_0} = y^{b_0} = [x, y]^{k_0} = [x, y], x = [x, y], y = e \rangle.
\]

Thus, every element $g \in G$ can be written in the form
\[
g = v(z)^a w(z)^b [v(z), w(z)]^c
\]
where $0 \leq a < a_0$ (if $a_0 > 0$, and $a \in \mathbb{Z}$ if $a_0 = 0$), $0 \leq b < b_0$ (if $b_0 > 0$, and $b \in \mathbb{Z}$ if $b_0 = 0$), and $0 \leq c < k_0$ (if $k_0 > 0$, and $c \in \mathbb{Z}$ if $k_0 = 0$).

Suppose further that $z$ is a disjoint union of two sets of indeterminates
\[
x = (x_1, \ldots, x_{n_1})
\]
\[
y = (y_1, \ldots, y_{n_2})
\]
and that the word $v$ involves only one of these subsets and $w$ involves only the other, say $x$ and $y$ respectively. Under these extra assumptions, we claim that the expression in (2.5) is unique. Indeed, suppose that we had a relation
\[
v(x)^a w(y)^b [v(x), w(y)]^c = e.
\]

We want to show that $a_0 | a$, $b_0 | b$ and $k_0 | c$.

Consider the endomorphism $\psi$ of $F$ given by sending $y_1, \ldots, y_{n_2}$ to $e$ and leaving the $x_i$ unchanged. Applying $\psi$ to (2.6), we obtain that $v(x)^a = e$. Therefore, $a_0 | a$. Applying the endomorphism of $F$ that sends $x_1, \ldots, x_{n_1}$ to $e$ and leaves the $y_j$ unchanged we obtain, again from (2.6), that $w(y)^b = e$, and hence $b_0 | b$. Combining these two relations with (2.6) we get that $[v(x), w(y)]^c = e$, hence $k_0 | c$. This proves our claim.

The above situation allows us to deduce that certain elements must lie in the dominion of a given subgroup. Namely:

**Lemma 2.7.** Let $\mathcal{V}$ be a variety of groups, and let $v(x)$ and $w(y)$ be two words. Suppose that in $\mathcal{V}$, the relatively free group $F$ on $n_1 + n_2$ generators $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ satisfies the two identities (2.3). If $G \in \mathcal{V}$, $g_1, \ldots, g_{n_1+n_2} \in G$, and $H$ is a subgroup of $G$ which contains
\[
v(g_1, \ldots, g_{n_1})^n, \quad w(g_{n_1+1}, \ldots, g_{n_1+n_2})^n
\]
for some $n \in \mathbb{Z}$, then $\text{dom}_G^\mathcal{V}(H)$ also contains
\[
[v(g_1, \ldots, g_{n_1}), w(g_{n_1+1}, \ldots, g_{n_1+n_2})]^n.
\]

**Proof:** From Lemma 2.2 it follows that all groups in $\mathcal{V}$ satisfy the identities
\[
[a^n, b] = [a, b]^n = [a, b^n]
\]
for all elements $a$ and $b$, and for all $n \in \mathbb{Z}$.

Write $g_1 = (g_1, \ldots, g_{n_1})$, and $g_2 = (g_{n_1+1}, \ldots, g_{n_1+n_2})$.

Let $K \in \mathcal{V}$, and let $f, h : G \to K$ be two morphisms such that $f|_H = h|_H$. Then

$$f \left( [v(g_1), w(g_2)]^n \right) = f \left( [v(g_1)^n, w(g_2)] \right)$$

$$= [f(v(g_1)^n), f(w(g_2))]$$

$$= [h(v(g_1)^n), f(w(g_2))]$$

$$= [h(v(g_1)), f(w(g_2))]^n$$

and by a symmetric argument, this term equals $h([v(g_1), w(g_2)]^n)$. In particular, $[v(g_1), w(g_2)]^n$ lies in $\text{dom}^\mathcal{V}_F(H)$.

Note that in Lemma 2.7 we have made no claims on whether $[v(g_1), w(g_2)]^n$ lies in $H$ or not.

Still under the assumptions of the two paragraphs preceding Lemma 2.7, let $p$ be a prime such that $p^2|k_0$, and let $H$ be the subgroup of $G$ generated by $v(x)^p$ and $w(y)^p$. The elements of $H$ correspond to those elements of $G$ that can be written as in (2.5), with $p|a$, $p|b$, and $p^2|c$.

By Lemma 2.7, we know that $[v(x), w(y)]^p$ lies in $\text{dom}^\mathcal{V}_F(H)$. Since elements of $G$ can be written uniquely in the form given in (2.5), $[v(x), w(y)]^p$ does not lie in the subgroup $H$. This gives the following result:

**Theorem 2.8.** Let $\mathcal{V}$ be a variety of groups, and $v(x_1, \ldots, x_{n_1})$, $w(y_1, \ldots, y_{n_2})$ be two words. Suppose that in $\mathcal{V}$, the relatively free group $F$ on $n_1 + n_2$ generators, $x_1, \ldots, x_{n_1}$, $y_1, \ldots, y_{n_2}$ satisfies

$$[[v(x), w(y)], v(x)] = e$$

$$[[v(x), w(y)], w(y)] = e.$$  

Let $k$ denote the order of $[v(x), w(y)]$ if this is finite, and $k = 0$ if the order is infinite. Suppose $k$ is divisible by the square of some prime number $p$. Then there are instances of nontrivial dominions in $\mathcal{V}$. Namely, for $H = \langle v(x)^p, w(y)^p \rangle$, we have $H \not\subseteq \text{dom}^\mathcal{V}_F(H)$.

Recall that if $\mathcal{V}$ and $\mathcal{W}$ are varieties of groups, corresponding to the fully invariant subgroups $V$ and $W$ of $F_\infty$, respectively, we define the variety $[\mathcal{V}, \mathcal{W}]$ to be the variety corresponding to the fully invariant subgroup $[V, W]$ of $F_\infty$. It is not hard to verify that $[\mathcal{V}, \mathcal{W}]$ is defined by the identities $[v, w]$, where $v$ is an identity of $\mathcal{V}$, and $w$ an identity of $\mathcal{W}$.
Also, if we let $E$ denote the variety consisting only of the trivial group, then for a given variety $V$ we define the variety of center-by-$V$ groups to be the variety $[E, V]$. These groups can be described as groups $G$ such that $G/Z(G) \in V$, where $Z(G)$ denotes the center of $G$.

As such, it is not hard to verify that the center-by-abelian groups are the nilpotent groups of class 2 (that is, $[E, A] = N_2$), and in general that $[E, N_c] = N_{c+1}$. Thus, Theorem 2.8 (together with the well-known fact that the relatively free groups in $N_c$ are torsion-free) could be used to prove a result from [3], that there are nontrivial dominions in the variety of nilpotent groups of class $c > 1$.

Also, it is not hard to verify that given a variety $V$, $[V, V]$ is the variety of abelian-by-$V$ groups, that is the variety $AV$; and that the variety $[[V, V], V]$ is the variety of $N_2$-by-$V$ groups, $N_2V$. This variety satisfies the hypothesis of Theorem 2.8, so we obtain the following:

**Corollary 2.9.** If $V$ is any variety of groups, and $V \neq G$, then the variety $N_2V$ has instances of nontrivial dominions.

**Section 3. Other generalizations**

The ideas of Section 2 can be expanded to generalize other arguments used to prove nontriviality of dominions in certain varieties. In this section, we will generalize the argument in [4] as an indication of how this would be done.

In [4] we proved that given a group $G \in A^2$, the variety of metabelian groups, and a subgroup $H$ of $G$, if $x \in H \cap [G, G]$, and $y, z \in G$ are such that $[x, y]$ and $[x, z]$ lie in $H$, then $[x, y, z]$ lies in $\text{dom}_{G}(H)$. See Lemma 3.25 in [4] for the details. We now proceed as in the previous section to lay the groundwork for the generalization.

**Lemma 3.10.** Let $G$ be a group, and $u, v, w \in G$. If

$$[v, [w^{-1}, z^{-1}]] = [[v, w], [v, z]] = e$$

then $[v, w, z] = [v, z, w]$.

**Proof:** Since $v$ commutes with $[w^{-1}, z^{-1}]$, we have that

$$v^{wzw^{-1}z^{-1}} = v$$

and therefore, $v^{wz} = v^{zw}$. Left-multiplying by $v^{-1}$ we get $[v, wz] = [v, zw]$. Expanding each side using commutator identities, we have

$$[v, w][v, z][v, w, z] = [v, z][v, w][v, z, w]$$

and since $[v, w]$ commutes with $[v, z]$ by hypothesis, we may cancel the first two terms from both sides to get

$$[v, w, z] = [v, z, w]$$

as claimed. \qed
Lemma 3.11. Suppose that a variety of groups $V$ satisfies the identities

\begin{equation}
[v(x), [[w(y)^{-1}, z(x)^{-1}]]] = e \\
[[v(x), w(y)], [v(x), z(x)]] = e
\end{equation}

for certain group-theoretic words $v$, $w$ and $z$ in tuples of variables $x$ and $y$. Then $V$ also satisfies the identity

$$[v(x), w(y), z(x)] = [v(x), z(x), w(y)].$$

Proof: This follows from Lemma 3.10. \hfill \square

Now we may apply the argument from [4] to prove:

Theorem 3.13. Let $V$ be a variety and suppose that $V$ satisfies the identities (3.12). If $G \in V$, then for any particular tuple $x$ of elements of $G$, the dominion of the subgroup generated by $v(x)$, $[v(x), w(x)]$ and $[v(x), z(x)]$ contains

$$[v(x), w(x), z(x)].$$

Proof: Let $H$ be the subgroup of $G$ generated by the elements $v(x)$, $[v(x), w(x)]$, and $[v(x), z(x)]$. Let $K \in V$, and let $f, g: G \to K$ be two group morphisms such that $f|_H = g|_H$. Then:

$$f([v(x), w(x), z(x)]) = f[[v(x), w(x)], f(z(x))]
= \left[ g([v(x), w(x)]), f(z(x)) \right] \quad \text{(since $[v(x), w(x)] \in H$)}
= \left[ g(v(x)), g(w(x)), f(z(x)) \right]
= \left[ f(v(x)), f(z(x)), g(w(x)) \right]
\quad \text{(using Lemma 3.11 and the fact that $v(x) \in H$)}
= \left[ f([v(x), z(x)]), g(w(x)) \right]
= \left[ g([v(x), z(x)]), g(z(x)) \right] \quad \text{(since $[v(x), z(x)] \in H$)}
= g([v(x), z(x), w(x)])
= g([v(x), w(x), z(x)]) \quad \text{(by Lemma 3.11)}
$$

so $[v(x), w(x), z(x)] \in \text{dom}^V_G(H)$, as claimed. \hfill \square
From this, we can deduce the following strengthening of Lemma 3.25 in [4]:

**Corollary 3.14.** Let \( d \geq 2 \), and let \( A^d \) be the variety of all solvable groups of solvability length at most \( d \). Let \( G \in A^d \), and let \( H \) be a subgroup of \( G \). If \( x \in G^{(d-1)} \), \( y, z \in G^{(d-2)} \) are such that \( x, [x,y] \), and \( [x,z] \) lie in \( H \), then \( [x,y,z] \in \text{dom}_{G}^{A^d}(H) \).

Similar arguments may be used to generalize other results about specific varieties, which are done through word-theoretic arguments such as the above. A related general result on dominions in the context of Universal Algebra can be found in [5].

**References**

[1] Isbell, J. R. *Epimorphisms and dominions* in Proc. of the Conference on Categorical Algebra, La Jolla 1965, pp. 232–246. Lange and Springer, New York 1966. MR:35#105a (The statement of the Zigzag Lemma for rings in this paper is incorrect. The correct version is stated in [2].)

[2] Isbell, J. R. *Epimorphisms and dominions IV.* Journal London Math. Society (2), 1 (1969) pp. 265–273. MR:41#1774

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