Steiner’s formula and a variational proof of the isoperimetric inequality

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Abstract
We give a new proof of the isoperimetric inequality in the plane, based on Steiner’s formula for the area of a convex neighborhood. This proof establishes the isoperimetric inequality directly, without requiring that we separately establish the existence of an optimal domain. In doing so, this proof bypasses the main difficulty in all of the proofs Steiner outlined for the plane isoperimetric inequality.

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1 | INTRODUCTION

The classical isoperimetric inequality states that among all simple closed curves of length $L$ in the plane, the unique curve enclosing the largest area is the circle of circumference $L$:

**Theorem 1.1** (The Isoperimetric Inequality). Let $\gamma$ be a simple closed curve in the plane of length $L$, enclosing a domain $D$ of area $A$.

Then $L^2 \geq 4\pi A$, with equality precisely if $\gamma$ is a circle.

This paper gives a proof of the isoperimetric inequality based on Steiner’s formula, which describes the area of a neighborhood of a convex domain in $\mathbb{R}^2$:

**Theorem 1.2** (Steiner’s Formula, [20]). Let $D$ be a bounded, convex domain in $\mathbb{R}^2$, of area $A$ and perimeter $L$, and let $D_r$ be the $r$-neighborhood of $D$, that is, the points in $\mathbb{R}^2$ whose distance from $D$ is $r$ or less. Then:

(A) $\text{Area}(D_r) = \pi r^2 + Lr + A$;
(B) $\text{Length}(\partial D_r) = 2\pi r + L$. 

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Jakob Steiner (March 18th, 1796 to April 1st, 1863) proved Theorem 1.2 for convex polygons and a similar formula for convex polyhedra in \( \mathbb{R}^3 \). By polygonal approximation, Theorem 1.2 then follows for any compact, convex set in \( \mathbb{R}^2 \), and in fact a version of Steiner’s formula holds in much greater generality — for more about Steiner’s formula, see \([10, 19]\). Steiner was fascinated by the isoperimetric inequality, and he sketched several ideas for proving it — cf. \([3, 21]\). The isoperimetric problem was already ancient when Steiner considered it in the nineteenth century, but Theorem 1.1 had never been proven rigorously. It remained unproven in Steiner’s lifetime, and all of Steiner’s ideas for proving the isoperimetric inequality required the same additional step, which he never provided: one must show that the isoperimetric problem has a solution.

More precisely, we define the isoperimetric ratio of a domain \( D \) with area \( A \) and perimeter \( L \) to be:

\[
\frac{L^2}{4\pi A}. \tag{1.3}
\]

The isoperimetric ratio is scale-invariant — we formulate the isoperimetric inequality in terms of \( L^2 \) and \( A \), as in Theorem 1.1, because \( L^2 \) and \( A \) transform the same under rescalings of the plane. The isoperimetric inequality then states that the isoperimetric ratio of any plane domain is greater than or equal to 1, with equality precisely for disks. Steiner developed many proofs that no domain other than a disk minimizes the isoperimetric ratio, but he did not establish the existence of a domain that minimizes \( (1.3) \).

The first proof of the existence of a domain minimizing the isoperimetric ratio seems to have been in unpublished lecture notes of Weierstrass in 1879, cf. \([3]\). The existence of an optimal isoperimetric domain in the plane is now known to be a consequence of several compactness theorems in metric geometry and geometric measure theory, however the proof below does not require that we establish the existence of a minimizer for the isoperimetric ratio — we show directly that no domain has an isoperimetric ratio less than 1. We believe part of the significance of our proof is that it shows how one of Steiner’s ideas from convex geometry can be used to prove the isoperimetric inequality without separately establishing the existence of an optimal domain.

The basic observation for our proof is the following: if \( D \) is a bounded convex domain in \( \mathbb{R}^2 \), we can use Steiner’s formula to calculate the isoperimetric ratio \( I(r) \) of the \( r \)-neighborhood of \( D \) as a function of \( r \). Letting \( A \) be the area of \( D \) and \( L \) its perimeter, we have

\[
I(r) = \frac{(2\pi r + L)^2}{4\pi (\pi r^2 + Lr + A)} = \frac{4\pi^2 r^2 + 4\pi Lr + L^2}{4\pi^2 r^2 + 4\pi Lr + 4\pi A}. \tag{1.4}
\]

Differentiating with respect to \( r \), we have

\[
I'(r) = \frac{(4\pi A - L^2)(8\pi^2 r + 4\pi L)}{(4\pi^2 r^2 + 4\pi Lr + 4\pi A)^2} = \left(\frac{4\pi A - L^2}{4\pi (\pi r^2 + Lr + A)}\right)^2. \tag{1.5}
\]

This implies that \( I(r) \) is a monotone function of \( r \), decreasing if the isoperimetric ratio of \( D \) is greater than 1 and constant if the isoperimetric function of \( D \) is equal to 1. If \( D \) were a convex domain with an isoperimetric ratio less than 1, \( I(r) \) would increase monotonically to 1, the isoperimetric ratio of the disk, as \( r \) goes to infinity. As \( r \) goes to infinity, the \( r \)-neighborhoods of any convex domain \( D \), when rescaled to have constant area, converge to a disk — see Proposition 3.1. We will see that this gives a variation of the disk, as an argument for the functional on plane domains...
given by the isoperimetric ratio. We will use Steiner’s formula to find its first and second variations — in particular, we will relate them to the isoperimetric ratio of the domain $D$ in question. We will then be able to deduce the isoperimetric inequality from the fact that the disk is a critical point, with non-negative second variation, for the isoperimetric ratio on plane domains. For later reference, the quantity $L^2 - 4\pi A$ whose negative appears in (1.5) is called the \textit{isoperimetric deficit} of a domain.

It will be important in our proof that, in the plane, the convex hull \textit{conv}(D) of a non-convex domain $D$ always has a smaller isoperimetric ratio than $D$ itself: \textit{conv}(D) encloses a larger area than $D$ with a smaller perimeter. Therefore, to prove the isoperimetric inequality, it is enough to show that Theorem 1.1 holds for convex domains. Steiner was aware of this fact and used it in several of his ideas for proving the isoperimetric inequality. In dimensions greater than 2, this is no longer true: the isoperimetric ratio of a 3-dimensional domain with volume $V$ and surface area $A$ is defined to be $\frac{A^3}{36\pi V^2}$. Like (1.3) for plane domains, the isoperimetric ratio of a domain in $\mathbb{R}^3$ is scale-invariant and the ball has isoperimetric ratio equal to 1. The isoperimetric inequality in $\mathbb{R}^3$ states that the isoperimetric ratio of any domain is greater than or equal to 1, with the ball being the unique minimizer. For a ball with a long spike in $\mathbb{R}^3$, both the volume and surface area, and thus the isoperimetric ratio, can be made arbitrarily close to that of the ball by making the spike narrow enough. On the other hand, the convex hull of such a domain will be approximately a cone with a hemispherical cap, with an isoperimetric ratio much greater than 1: for a spike of length $\eta$ on the unit ball, the isoperimetric ratio of its convex hull will be approximately $\frac{\eta+3}{4}$ for $\eta$ very large.

The outline of this paper and our proof of the isoperimetric inequality is as follows. In Section 2, we will calculate the first and second variations of the isoperimetric ratio of the disk. By doing so, we will show that the disk is a stable critical point of the isoperimetric ratio and that any variation has positive second variation unless, to first order, the variation is the sum of a translation and a rescaling of the disk. In Section 3, we will use the $r$-neighborhoods of a compact, convex domain $D$ in the plane to construct a variation of the disk of the type analyzed in Section 2. We will use Steiner’s formula to relate its first and second variations to the isoperimetric deficit of $D$, and in doing so, we will show that the isoperimetric deficit of $D$ is non-negative. Once we know that the isoperimetric inequality $L^2 - 4\pi A \geq 0$ holds, any of Steiner’s arguments then prove that the disk is the only domain for which equality holds. However, we will show in Section 4 that the uniqueness of the disk as a minimizing domain also follows from our proof.

We will prove the isoperimetric inequality for a plane domain under the assumption that its boundary $\partial D$ is smooth, and we will make the further simplifying assumption that the curvature of $\partial D$ is strictly positive — that is, the curvature vector of $\partial D$ always points into $D$ and never vanishes. However, by approximation (and the reduction to the convex case) this inequality then follows immediately for any plane domain with a rectifiable boundary. The corresponding issue is more difficult in higher dimensions — this is discussed in [16, Section 2]. In all dimensions, however, the boundary of a compact, convex domain can be realized as the Lipschitz image of a round sphere and is therefore rectifiable.

Throughout the paper, we will discuss the relationship between this proof and other known proofs of the isoperimetric inequality. Osserman’s article [16] gives an overview of the isoperimetric inequality, its generalizations and their significance in mathematics. Chavel’s [7] and Santaló’s [18] books both discuss many results and questions in geometry and analysis which are based on the isoperimetric inequality and give several proofs of the classical isoperimetric inequality. Blåsjö
discusses the history of the isoperimetric inequality in [3], and Howards, Hutchings and Morgan in [13] and Treibergs in [21] present several proofs of the classical isoperimetric inequality.

2  THE FIRST AND SECOND VARIATIONS OF THE ISOPERIMETRIC RATIO

We will calculate the first and second variations of the isoperimetric ratio of the disk for variations through families of convex domains — in particular, we will see that the disk is a critical point of the isoperimetric ratio and, infinitesimally, a minimizer.

A compact, convex domain $D$ can be described by its support function $p(\theta) : S^1 \to \mathbb{R}$, defined as follows:

$$p(\theta) = \max\{h_\theta(x) := x_0 \cos(\theta) + x_1 \sin(\theta) \mid x = (x_0, x_1) \in D\}.$$ 

If the boundary $\partial D$ of $D$ is smooth and has strictly positive curvature, then $p(\theta) + p''(\theta)$ is its radius of curvature. In this case, the area $A$ and perimeter $L$ of $D$ are given by

$$A = \left(\frac{1}{2}\right) \int_0^{2\pi} p(\theta)(p(\theta) + p''(\theta))d\theta = \left(\frac{1}{2}\right) \int_0^{2\pi} p(\theta)^2 - p'(\theta)^2d\theta, \quad (2.1)$$

$$L = \int_0^{2\pi} p(\theta)d\theta. \quad (2.2)$$

This is described in [18, Chapter 1]. A variation of the unit disk $D_0$ through a family of such domains $\{D_t\}_{t \geq 0}$ can therefore be described by a smooth function $p(\theta, t)$, with $p(\theta, t)$ the support function of the domain $D_t$. In particular, $p(\theta, 0) \equiv 1$.

**Proposition 2.3.** Let $D_t$ be a family of compact, convex domains in the plane, with the boundary $\partial D_t$ of each domain smooth and with positive curvature, which give a variation of the disk $D_0$ as above. Let $I(t)$ be the isoperimetric ratio of the domain $D_t$.

Then $I'(0) = 0$ and $I''(0) \geq 0$, with equality if and only if, to first order, the family of domains coincides with a rescaling and translation of the disk.

**Proof.** Let $p(\theta, t)$ be the support function of $D_t$ as above. Then letting $A(t)$ be the area and $L(t)$ the perimeter of $D_t$, by (2.1) and (2.2), we have

$$A(t) = \left(\frac{1}{2}\right) \int_0^{2\pi} p(\theta, t)^2 - \frac{\delta}{\delta t}(\theta, t)^2d\theta, \quad (2.4)$$

$$L(t) = \int_0^{2\pi} p(\theta, t)d\theta. \quad (2.5)$$

Because $p(\theta, 0) \equiv 1$ and $\frac{\delta p}{\delta t}(\theta, 0) \equiv 0$, $A'(0)$ and $L'(0)$ are both equal to $\int_0^{2\pi} \frac{\delta p}{\delta t}(\theta, 0)d\theta$. 

We then have that $I'(0) = \frac{2A(0)L'(0) - L(0)^2A'(0)}{4\pi A(0)^2}$ is equal to
\[
2 \times \pi \times \frac{2\pi}{4\pi^3} \left( \frac{2\pi}{\int_0^2 \frac{\partial p}{\partial t}(\theta, 0) d\theta} \right) - 2 \times \frac{2\pi}{\int_0^2 \frac{\partial p}{\partial t}(\theta, 0) d\theta} = 0.
\]

$L''(0)$ is equal to $\int_0^{2\pi} \frac{\partial^2 p}{\partial t^2}(\theta, 0) d\theta$ and, using again that $p(\theta, 0) \equiv 1$ and $\frac{\partial p}{\partial \theta}(\theta, 0) \equiv 0$, we have
\[
A''(0) = \int_0^{2\pi} \left[ \frac{\partial p}{\partial t}(\theta, 0)^2 + \frac{\partial^2 p}{\partial t^2}(\theta, 0) - \frac{\partial^2 p}{\partial t \partial \theta}(\theta, 0)^2 \right] d\theta. \tag{2.6}
\]

We then have that $I''(0) = \frac{(2A'(0) - L'(0))^2 + 2\pi(L''(0) - A''(0))}{2\pi^2}$ is equal to
\[
\frac{\left( \int_0^{2\pi} \frac{\partial p}{\partial t}(\theta, 0) d\theta \right)^2}{2\pi^2} + 2\pi \frac{\left( \int_0^{2\pi} \frac{\partial^2 p}{\partial t^2}(\theta, 0)^2 - \frac{\partial p}{\partial t}(\theta, 0)^2 d\theta \right)}{2\pi^2}. \tag{2.7}
\]

Wirtinger’s inequality states that if $\varphi(\theta)$ is a $2\pi$-periodic, continuously differentiable function with $\int_0^{2\pi} \varphi(\theta) d\theta = 0$, then
\[
\int_0^{2\pi} \varphi'(\theta)^2 d\theta \geq \int_0^{2\pi} \varphi(\theta)^2 d\theta.
\]

Equality holds if and only if $\varphi(\theta) = a_0 \cos(\theta) + a_1 \sin(\theta)$ for some constants $a_0, a_1$. Wirtinger’s inequality thus implies by (2.7) that $I''(0) \geq 0$ and is strictly positive unless $\frac{\partial p}{\partial t}(\theta, 0) = a_0 \cos(\theta) + a_1 \sin(\theta) + 2\pi \hat{p}$, where $\hat{p} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial p}{\partial t}(\theta, 0) d\theta$. The variation corresponding to $a_0 \cos(\theta) + a_1 \sin(\theta)$ gives a translation of the disk, in the direction whose argument is $\arctan\left( \frac{a_1}{a_0} \right)$ at speed $\sqrt{a_0^2 + a_1^2}$, and the variation corresponding to $2\pi \hat{p}$ rescales the disk, by a factor $1 + t_0 2\pi \hat{p}$ when $t = t_0$. □

Wirtinger’s inequality can be proved by comparing the Fourier series of a $2\pi$-periodic function with that of its derivative, cf. [9]. Wirtinger’s inequality also implies the isoperimetric inequality directly. This was discovered by Hurwitz, who gave the first proof of the isoperimetric inequality based on Fourier analysis and Wirtinger’s inequality in [14]. A variant of this proof, in which the role of Wirtinger’s inequality is made explicit, can be found in [16] and [1]. As with our proof, Hurwitz’s proof of the isoperimetric inequality does not require that one separately establish the existence of a minimizing domain — his argument shows directly that $L^2 \geq 4\pi A$ for any plane domain, with equality precisely when the domain is a disk.

Proposition 2.3 is an infinitesimal version of a family of results pioneered by Bernstein [2], Liebmann [15] and Bonnesen [4–6] which give stronger versions of the isoperimetric inequality of the form $L^2 - 4\pi A \geq B(D) \geq 0$, where $B(D)$ is an invariant of plane domains $D$ which vanishes if and
only if $D$ is a disk. A result of this type implies the isoperimetric inequality $L^2 \geq 4\pi A$ and characterizes the equality case. The invariant $B(D)$ in many of these results has a geometric significance which gives additional insight into the isoperimetric problem. Fejes Tóth proved several results of this type using Steiner’s formula [8] and Wallen gave a proof of one of Bonnesen’s inequalities using Wirtinger’s inequality [22]. Osserman discusses these results and their history in [17]. He notes that versions of Bonnesen’s inequalities, and in particular the isoperimetric inequality, can be generalized from $\mathbb{R}^2$ to several larger classes of Riemannian surfaces. Several recent results of this type in surfaces can be found in [12].

3 STEINER’S FORMULA AND THE MONOTONICITY OF THE ISOPERIMETRIC RATIO

To prove Theorem 1.1, we begin by confirming that the $r$-neighborhoods of a bounded, convex domain $D$, when rescaled to have constant area, give a variation of the disk of the type considered in Proposition 2.3:

Proposition 3.1. Let $D$ be a compact, convex domain in the plane whose boundary is smooth and has positive curvature. For $t > 0$, let $D_t$ be the $r = \frac{1}{t}$-neighborhood of $D$, rescaled to have the same area as $D$, and let $D_0$ be a disk with the same area as $D$.

Then $\{D_t\}_{t \geq 0}$ gives a variation of the disk $D_0$, as in Proposition 2.3. More precisely, if $q(\theta)$ is the support function of $D$, this variation is described by

$$p(\theta, t) = \sqrt{\frac{A}{\pi t^2 + L + \pi}} (q(\theta)t + 1), \quad (3.2)$$

where $A$ is the area and $L$ is the perimeter of $D$.

Proof. Let $D$ be as above — without loss of generality, suppose $D$ has area $\pi$. Note first that each $r$-neighborhood of $D$ is also convex, cf. Remark 3.4 below, so that the variation in question is through a family of convex sets. If $q(\theta)$ is the support function of $D$, then $q(\theta) + r$ is the support function of $D_r$ and, by Theorem 1.2, $\sqrt{\frac{\pi}{\pi t^2 + L + \pi}}(q(\theta) + r)$ is the support function of the rescaling of $D_r$ whose area is equal to that of $D$. Rewriting this in terms of $t = \frac{1}{r}$ for $r > 0$, we have

$$p(\theta, t) = \sqrt{\frac{\pi}{\pi (\frac{1}{t})^2 + L + \frac{1}{t}}} (q(\theta) + \frac{1}{t}) = \sqrt{\frac{\pi}{\pi t^2 + L + \pi}} (q(\theta)t + 1). \quad (3.3)$$

We then have

$$p(\theta, t) + \frac{\partial^2 p}{\partial \theta^2}(\theta, t) = \sqrt{\frac{\pi}{\pi t^2 + L + \pi}} (t(q(\theta) + q''(\theta)) + 1).$$

Because the curvature of $\partial D$ is positive, $q(\theta) + q''(\theta) > 0$, so $p(\theta, t) + \frac{\partial^2 p}{\partial \theta^2}(\theta, t) > 0$, which implies that $\partial D_t$ has positive curvature for all $t > 0$. $p(\theta, t)$ extends smoothly to $t = 0$, where it is equal to the support function of the unit disk, and gives a variation of the disk as in Proposition 2.3. □
Remark 3.4. The \( r \)-neighborhood \( D_r \) of a compact, convex set \( D \) is the Minkowski sum of \( D \) with a disk of radius \( r \) in \( \mathbb{R}^2 \). Minkowski summation of convex sets is discussed extensively in [19] and many other texts on convex and integral geometry.

We now prove the inequality in Theorem 1.1: for a compact domain in \( \mathbb{R}^2 \) with perimeter \( L \) and area \( A, L^2 \geq 4\pi A \). We will then address the characterization of the equality case in Section 4.

**Proof of Theorem 1.1, Part 1.** Let \( D \) be a compact, convex domain in the plane with area \( A \) and boundary length \( L \), and suppose \( \partial D \) is smooth and has positive curvature as above. By (1.4), for \( t > 0 \), the isoperimetric ratio \( I(t) \) of the \((\frac{1}{t})\)-neighborhood of \( D \) is

\[
I(t) = \frac{L^2 t^2 + 4\pi L t + 4\pi^2}{4\pi A t^2 + 4\pi L t + 4\pi^2}.
\] (3.5)

Letting \( \delta \) be the least absolute value of the roots of \( f(t) = 4\pi A t^2 + 4\pi L t + 4\pi^2 \), the denominator of (3.5) (see Remark 3.8), the function of \( t \) defined by (3.5) extends smoothly to \((-\delta, \infty)\). In particular, (3.5) extends smoothly to \( t = 0 \) to give the isoperimetric ratio of the variation \( \{D_t\}_{t \geq 0} \) of the disk described in Proposition 3.1. \( I(t) \) is a monotone function of \( t \geq 0 \), with the sign of \( I'(t) \) determined by the isoperimetric deficit of \( D \):

\[
I'(t) = \frac{(L^2 - 4\pi A)(Lt^2 + 2\pi t)}{4\pi (At^2 + Lt + \pi)^2}.
\] (3.6)

Therefore, \( I'(0) = 0 \) (which also follows from Propositions 2.3 and 3.1) and for \( t > 0 \), \( I'(t) \) has the same sign as the isoperimetric deficit of \( D \). To show that \( L^2 \geq 4\pi A \), we calculate the second derivative of \( I(t) \):

\[
I''(t) = \left( \frac{L^2 - 4\pi A}{2\pi} \right) \left( \frac{\pi^2 - 3\pi A t^2 - A L t^3}{(A t^2 + L t + \pi)^3} \right).
\] (3.7)

In particular, \( I''(0) = \frac{L^2 - 4\pi A}{2\pi^2} \). The sign of \( L^2 - 4\pi A \) is the same as that of \( I''(0) \), which by Proposition 2.3 is greater than or equal to 0.

\[\square\]

Remark 3.8. The roots of \( f(t) = 4\pi A t^2 + 4\pi L t + 4\pi^2 \), the denominator of (3.5), are

\[
-L \pm \frac{\sqrt{L^2 - 4\pi A}}{2\pi}.
\] (3.9)

The isoperimetric inequality is equivalent to the statement that the roots of this polynomial are real, and thus negative, and are distinct unless the domain is a disk. For our purposes, it is enough simply to note that any real roots of \( f(t) \) are negative since \( f(t) \geq 4\pi^2 \) when \( t \geq 0 \). The roots of the Steiner polynomial were studied by Green and Osher in [11] (the Steiner polynomial of a domain \( D \) with area \( A \) and perimeter \( L \) is \( \pi r^2 + L r + A \), with roots \( -L \pm \sqrt{L^2 - 4\pi A} \)). They note that Steiner’s formula implies the isoperimetric deficit of the \( r \)-neighborhood of \( D \) is equal to that of \( D \).
4 | THE UNIQUENESS OF THE DISK

Once we have shown that \( L^2 \geq 4\pi A \) for all plane domains with perimeter \( L \) and area \( A \), and thus that the disk minimizes the isoperimetric ratio, any of Steiner’s arguments then show that it is the unique minimizer. The uniqueness of the disk as a minimizing domain for the isoperimetric ratio also follows from our argument, subject to some mild technical assumptions:

**Proof of Theorem 1.1, Part 2.** Let \( D \) be a bounded domain in the plane with smooth (or \( C^2 \)) boundary whose area \( A \) and boundary length \( L \) satisfy \( L^2 = 4\pi A \). We can suppose \( A = \pi \) and \( L = 2\pi \). Suppose in addition that the curvature of \( \partial D \) is positive, as above. By (3.5), in the variation \( \{ D_t \}_{t \geq 0} \) of the disk constructed from \( D \) as in Section 3, the isoperimetric ratio of \( D_t \) is equal to 1 for all \( t \geq 0 \), and therefore \( L(t) \equiv 2\pi \). Therefore,

\[
L'(t) = \int_0^{2\pi} \frac{\partial p}{\partial t}(\theta, t) d\theta \equiv 0, \tag{4.1}
\]

\[
L''(t) = \int_0^{2\pi} \frac{\partial^2 p}{\partial t^2}(\theta, t) d\theta \equiv 0. \tag{4.2}
\]

By (2.6) and (4.2), we then have

\[
\int_0^{2\pi} \left[ \frac{\partial p}{\partial t}(\theta, t)^2 - \frac{\partial^2 p}{\partial t \partial \theta}(\theta, t)^2 \right] d\theta = A''(t) \equiv 0. \tag{4.3}
\]

By (4.1), (4.3) and Wirtinger’s inequality, \( \frac{\partial p}{\partial t}(\theta, t) = c_0(t) \cos(\theta) + c_1(t) \sin(\theta) \) for some functions \( c_0(t), c_1(t) \) of \( t \). Letting \( q(\theta) \) be the support function of \( D \), by (3.2),

\[
q(\theta) - \frac{1}{(t + 1)^2} = c_0(t) \cos(\theta) + c_1(t) \sin(\theta). \tag{4.4}
\]

This then implies that \( c_0(t) = \frac{d_0}{(t+1)^2}, c_1(t) = \frac{d_1}{(t+1)^2} \) for some constants \( d_0, d_1 \), and that \( q(\theta) = d_0 \cos(\theta) + d_1 \sin(\theta) + 1 \). \( D \) is therefore the unit disk centered at \((d_0, d_1)\). \( \square \)

We conclude with a few remarks about the technical assumptions in our proof of the characterization of equality in the isoperimetric inequality:

We have assumed the domain \( D \) to be convex and have \( C^2 \) boundary whose curvature is strictly positive, so that it can be described by a \( C^2 \) support function \( q(\theta) \). However, by the reduction to the convex case, any domain realizing equality in the isoperimetric inequality must be convex. Moreover, for any compact, convex set \( D \) and \( r > 0 \), the \( r \)-neighborhood \( D_r \) of \( D \) has \( C^{1,1} \) boundary, which is therefore twice-differentiable almost everywhere. If \( D \) realizes equality in the isoperimetric inequality, then by (1.4) each of its \( r \)-neighborhoods \( D_r \) does as well, and by the convexity of \( D_r \), the curvature of \( \partial D_r \) is non-negative at all points where it is defined. Thus, if one can show that a domain which realizes equality in the isoperimetric inequality, whose boundary is twice-differentiable almost everywhere, and has non-negative curvature at all points where its curvature
is defined is a disk, one will have shown that $D_r$ is a disk for all $r > 0$, and thus that $D$ is a disk as well.

The regularity of $\partial D_r, r > 0$ for a convex domain $D$ and the relationship between the regularity of a domain and the regularity of its support function are both discussed in [19]. Osserman discusses the significance of the regularity of the domains which one allows in the isoperimetric inequality in [16, Section 2]. He notes that one can modify a smooth domain by adding ‘wiggles’ to its boundary, increasing its perimeter while leaving its area unchanged — thus, ‘one has the ironic situation that the more irregular the boundary, the stronger will be the isoperimetric inequality, but the harder it is to prove’. He concludes: ‘The fact is, the isoperimetric inequality holds in the greatest generality imaginable, but one needs suitable definitions even to state it.’

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REFERENCES
1. M. Berger, B. Gostiaux, Differential geometry: manifolds, curves and surfaces, Springer, Berlin, 1988.
2. F. Bernstein, Über die Isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene, Math. Ann. 60 (1905), no. 1, 117–136.
3. V. Blåsjö, The isoperimetric problem, Amer. Math. Monthly 112 (2005), no. 6, 526–566.
4. T. Bonnesen, Über eine Verschärfung der Isoperimetrischen Ungleichheit des Kreises in der Ebene und auf der Kugeloberfläche nebst einer Anwendung auf eine Minkowskische Ungleichheit für konvexe Körper, Math. Ann. 84 (1921), no. 3, 216–227.
5. T. Bonnesen, Über das Isoperimetrische Defizit ebener Figuren, Math. Ann. 91 (1924), no. 3, 252–268.
6. T. Bonnesen, Quelques Problèmes Isopérimétriques, Acta Math. 48 (1926), no. 1-2, 123–178.
7. I. Chavel, Isoperimetric inequalities: differential geometric and analytic perspectives, Cambridge Univ. Press, Cambridge, 2001.
8. L. F. Tóth, Elementarer beweis einer isoperimetrischen ungleichung, Acta Math. Acad. Sci. Hungarica 1 (1950), no. 2-4, 273–276.
9. G. Folland, Real analysis: modern techniques and their applications, John Wiley and Sons, Hoboken, N.J., 1999.
10. A. Gray, Tubes, Birkhauser (2004).
11. M. Green, S. Osher, Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves, Asian J. Math. 3 (1999), no. 3, 659–676.
12. J. A. Hoisington, P. J. McGrath, Symmetry and isoperimetry for Riemannian surfaces, Calc. Variations Partial Diff. Equ. 61 (2022), no. 1.
13. H. Howards, M. Hutchings, and F. Morgan, The isoperimetric problem on surfaces, Amer. Math. Monthly 106 (1999), no. 5, 430–439.
14. A. Hurwitz, Sur Le Problème des Isopérimètres, Comp. Rend. Acad. Sci. 132 (1901), 401–403.
15. H. Liebmann, Das Frobeniussche Kappendreieck und die isoperimetrische Eigenschaft des Kreises, Math. Z. 4 (1919), no. 3, 288–294.
16. R. Osserman, The Isoperimetric Inequality. Bull. Amer. Math. Soc. 84 (1978), no. 6, 1182–1238.
17. R. Osserman, *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly **86** (1979), no. 1, 1–29.
18. L. Santaló, *Integral geometry and geometric probability*, Cambridge Univ. Press, Cambridge, 2004.
19. R. Schneider, *Convex Bodies: The Brunn-Minkowski theory*, Cambridge Univ. Press, Cambridge, 2014.
20. J. Steiner, *Über Parallele Flächen*, Monatsbericht der Akademie der Wissenschaften zu Berlin **114** (1840), 114–118.
21. A. Treibergs, *Inequalities that imply the isoperimetric inequality*. https://www.math.utah.edu/~treiberg/isoperim/isop.pdf, 2002.
22. L. J. Wallen, *All the way with Wirtinger: a short proof of Bonnesen’s inequality*, Amer. Math. Monthly **94** (1987), no. 5, 440–442.